Equivariant torsion and $G$-CW-complexes

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1 Introduction

1.1

In this note we consider equivariant Reidemeister and analytic torsion invariants of closed oriented \(G\)-manifolds, where \(G\) is any compact Lie group.

Equivariant analytic torsion for closed oriented odd-dimensional \(G\)-manifolds for arbitrary compact Lie groups \(G\) and equivariant Reidemeister torsion of closed oriented \(G\)-manifolds for finite \(G\) were introduced in [7] and further studied in [10] and [1]. In the present paper we generalize the definition of equivariant Reidemeister torsion to general compact Lie groups \(G\) and address the question of equality with equivariant analytic torsion.

If \(G\) is finite, then equivariant Reidemeister torsion is in fact an invariant of \(G\)-equivariant locally constant sheaves \(F\) of finite-dimensional Hilbert spaces over \(G\)-CW-complexes. We extend this invariant to general compact Lie groups \(G\).

If \(M\) is a closed \(G\)-manifold (\(G\) a compact Lie group), then there is a natural equivalence class of \(G\)-homotopy equivalences \(f : X \to M\) called simple structure (see section 2.2 for details), where \(X\) is a \(G\)-CW-complex. If \(F\) is a \(G\)-equivariant locally constant sheaf \(F\) of finite-dimensional Hilbert spaces, then the equivariant Reidemeister torsion of \((M, F)\) is defined using \((X, f^*F)\), and it is independent of choices.

Viewing a \(G\)-CW complex \(X\) as a filtered \(G\)-space, we express the equivariant Reidemeister torsion of \((X, F)\) in terms of the equivariant Reidemeister torsion of the restriction of \(F\) to the \(G\)-cells and a contribution of the spectral sequence induced by the filtration. There is a clear separation into an invariant which only depends on restrictions of \(F\) to the \(G\)-cells, and an invariant which depends on the way the cells are glued together. If \(G\) is connected, then the latter invariant is trivial.
We compute the equivariant Reidemeister torsion and the equivariant analytic torsion in terms of contributions of $G$-cells. The contribution of a $G$-cell can be further evaluated by restricting to one-dimensional subgroups of $G$. In particular we compute the equivariant Reidemeister torsion and the equivariant analytic torsion of compact symmetric spaces by topological means and recover the result of \[6\]

1.2

Let $G$ be a compact Lie group, and $M$ be a closed odd-dimensional oriented $G$-manifold. Let $F \to M$ be a $G$-equivariant flat hermitean vector bundle and $\mathcal{F}$ be the associated $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces.

If we choose a $G$-equivariant Riemannian metric $g^M$, then one defines the equivariant analytic torsion $\rho_{an}(M, g^M, \mathcal{F}) : G \to \mathbb{C}$ (see \[4\], §X) as a spectral invariant of the Laplace operator $\Delta_{g^M}$ acting on $F$-valued forms:

$$\rho_{an}(M, g^M, \mathcal{F})(g) := \frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}_s N g e^{-t \Delta_{g^M}} - \chi'(M, \mathcal{F})(g)) t^{s-1} dt,$$

where $\chi'(M, \mathcal{F}) := \sum_{i=0}^\infty (-1)^i \text{Tr} g_i H^i(M, \mathcal{F})$, $N$ denotes the $\mathbb{Z}$-grading of the bundle of $F$-valued forms, and the integral converges for $\text{Re}(s) > > 0$ and has a meromorphic continuation to all of $\mathbb{C}$.

By definition $\rho_{an}(M, g^M, \mathcal{F})$ is a class function on $G$. If $\mathcal{F}$ is acyclic, i.e. $H^*(M, \mathcal{F}) = 0$, then $\rho_{an}(M, g^M, \mathcal{F})$ is independent of $g^M$, and we write $\rho_{an}(M, g^M, \mathcal{F}) =: \rho_{an}(M, \mathcal{F})$.

1.3

Let $G$ be finite, let $M$ be a closed oriented $G$-manifold, and let $\mathcal{F}$ be a $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We choose a $G$-Hilbert module structure on $H^*(M, \mathcal{F})$. Using a smooth $G$-equivariant triangulation of $M$ one can define
equivariant Reidemeister torsion $\rho(M, \mathcal{F}) : G \to \mathbb{C}$ which is again a class function on $G$ (see [7], Sec. 5). Equivalently, there is a natural simple structure $f : X \to M$ and we can define $\rho(M, \mathcal{F}) := \rho(X, f^*\mathcal{F})$. We present the details of this definition in Section 3.

Assume that $M$ is odd-dimensional. If $\mathcal{F}$ is acyclic, then by [7], Prop.16, we have the equality of class functions

$$\rho(M, \mathcal{F}) = \rho_{an}(M, \mathcal{F}) .$$

In case that $\mathcal{F}$ is not acyclic we fix a $G$-invariant Riemannian metric $g^M$. Let $\mathcal{H}(M, g^M, \mathcal{F})$ denote the space of harmonic $F$-valued forms. Then $\mathcal{H}(M, g^M, \mathcal{F})$ is a $G$-Hilbert module. On $H^*(M, \mathcal{F})$ we choose the $G$-Hilbert module structure such that the de Rham isomorphism $\mathcal{H}(M, g^M, \mathcal{F}) \cong H^*(M, \mathcal{F})$ becomes an isometry. Again by [7], Prop.16, we have the equality

$$\rho(M, \mathcal{F}) = \rho_{an}(M, g^M, \mathcal{F})$$

of class functions.

1.4

Let $G$ be any compact Lie group, $M$ be a closed oriented $G$-manifold, and $F \to M$ be a $G$-equivariant flat hermitean vector bundle. We fix a $G$-Hilbert module structure on $H^*(M, \mathcal{F})$.

In the present subsection we define equivariant Reidemeister torsion of $(M, \mathcal{F})$ and discuss the validity of (1) and (2).

Let $FG := \{ g \in G \mid (\exists n \in \mathbb{N} \mid g^n = 1) \}$ denote the set of elements of finite order. $FG$ is a dense subset of $G$ which is invariant under conjugation. Let $C(FG) := \{ f : FG \to \mathbb{C} \mid f(g^h) = f(g), \forall g \in FG, h \in G \}$ denote the space of all real-valued functions on $FG$, which are invariant under conjugation, where $g^h := hgh^{-1}$.

Let $\text{Or}_f(G)$ denote the full subcategory of the orbit category (see [8], 8.16) consisting of all homogeneous spaces $G/\Gamma$, where $\Gamma \subset G$ is finite.
We have a contravariant functor $C : \text{Or}_f(G) \to C - \text{vect}$ associating to the object $G/\Gamma$ the space of class functions $C(\Gamma)$ on $\Gamma$. If $f : G/\Gamma \to G/\Gamma'$ is a morphism in $\text{Or}_f(G)$, then there is a $g \in G$ such that $\{g\gamma g^{-1} | \gamma \in \Gamma\} = \Gamma^g \subset \Gamma'$ and $f(h\Gamma) = hg^{-1}\Gamma'$. The functor $C$ associates to $f$ the map $C(f) : C(\Gamma') \to C(\Gamma^g) \to C(\Gamma)$, where the first arrow $\text{res}^G_{\Gamma'} : C(\Gamma') \to C(\Gamma^g)$ is the restriction of class functions and the second is induced by the map $\Gamma \to \Gamma^g, \gamma \mapsto g\gamma g^{-1}$.

There is a natural bijection

$$C(FG) \cong \lim_{\text{Or}_f(G)} C(\Gamma),$$

which is induced by the restrictions $\text{res}^G_{\Gamma} : C(FG) \to C(\Gamma), \Gamma \subset G$ finite.

For $\Gamma \subset G$ let $\text{res}^G_{\Gamma}M$ denote the $\Gamma$-manifold obtained from $M$ by restricting the $G$-action to $\Gamma$. If $V$ is a $G$-module, then let $\text{res}^G_{\Gamma}V$ denote the $\Gamma$-module obtained by restriction. Note that $H^*(\text{res}^G_{\Gamma}M, \mathcal{F}) = \text{res}^G_{\Gamma}H^*(M, \mathcal{F})$ canonically, and the latter has a natural $\Gamma$-Hilbert module structure.

If $G/\Gamma \in \text{Or}_f(G)$, then $\rho(\text{res}^G_{\Gamma}M, \mathcal{F}) \in C(\Gamma)$ is well defined by \[\ref{eq:rho} \]. By Proposition \ref{prop:section} the collection $\{\rho(\text{res}^G_{\Gamma}M, \mathcal{F})\}_{G/\Gamma \in \text{Or}_f(G)}$ is a section of the functor $C$ and thus defines an element

$$\tilde{\rho}(M, \mathcal{F}) \in \lim_{\text{Or}_f(G)} C(\Gamma) .$$

**Definition 1.1** Let $\rho(M, \mathcal{F}) \in C(FG)$ be the element which corresponds to $\tilde{\rho}(M, \mathcal{F})$ under the bijection \[\ref{eq:bijection} \].

With this definition equality of equivariant Reidemeister and analytic torsion for arbitrary Lie groups becomes a formal consequence of the corresponding result for finite groups.
Lemma 1.2 Let $G$ be a compact Lie group, $M$ be a closed odd-dimensional oriented $G$ manifold, $g^M$ be a $G$-invariant Riemannian metric and $F \to M$ be a $G$-equivariant flat hermitean vector bundle. We equip $H^*(M,F)$ with the $G$-Hilbert module structure such that the de Rham isomorphism $\mathcal{H}^*(M,g^M,F) \cong H^*(M,F)$ becomes an isometry. Then

$$\rho(M,F) = \rho_{an}(M,g^M,F)|_{FG}. $$

Proof. Let $g \in FG$ and $G/\Gamma \in \text{Or}_f(G)$ be such that $g \in \Gamma$. Then obviously $\rho_{an}(M,g^M,F)(g) = \rho_{an}(\text{res}_\Gamma^G M,g^M,F)(g)$. By (2) we have

$$\rho(M,F)(g) \overset{\text{def}}{=} \rho(\text{res}_\Gamma^G M,F)(g) = \rho_{an}(\text{res}_\Gamma^G M,g^M,F)(g) = \rho_{an}(M,g^M,F)(g).$$

\[\square\]

1.5

Note that equivariant Reidemeister torsion depends on the choice of a $G$-Hilbert module structure on the cohomology $H^*(M,F)$. In the present subsection we show how one can define an invariant that is independent of this choice.

Let $G^0 \subset G$ denote the component of the identity of $G$. Then we have an exact sequence

$$0 \to G^0 \to G \to \pi_0(G) \to 0.$$  

Note that the restriction $q_{FG} : \pi_0(G) \to \pi_0(G)$ is still surjective. Thus the pull-back in the sequence below defining $\hat{C}(FG)$ is injective.

$$0 \to \hat{C}(\pi_0(G)) \to \hat{C}(FG) \to 0$$

For any $G/\Gamma \in \text{Or}_f(G)$ let $\Gamma^0 := \Gamma \cap G^0$. Then we have exact sequences

$$0 \to \Gamma^0 \to \Gamma \to \Gamma/\Gamma^0 \to 0$$

$$0 \to \hat{C}(\Gamma/\Gamma^0) \to \hat{C}(\Gamma) \to 0.$$
where \( \hat{C}(\Gamma) \) is defined by the second sequence.

We can consider the functor \( \hat{C} : \text{Or}_f(G) \to \mathbf{C} - \text{vect} \), which associates to \( G/\Gamma \in \text{Or}_f(G) \) the space \( \hat{C}(\Gamma) \). If \( f : G/\Gamma \to G/\Gamma' \) is a morphism in \( \text{Or}_f(G) \), then the map \( \hat{C}(f) : \hat{C}(G/\Gamma') \to \hat{C}(G/\Gamma) \) is represented by \( C(f) : C(\Gamma) \to C(\Gamma') \) which maps \( C(\Gamma/\Gamma_0) \) to \( C(\Gamma'/\Gamma'_0) \).

Since the natural map \( C(\pi_0(G)) \to \lim_{\text{Or}_f(G)} C(\Gamma/\Gamma^0) \) is an isomorphism we conclude from

\[
\begin{array}{cccc}
0 & \to & C(\pi_0(G)) & \to & C(FG) & \to & \hat{C}(FG) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \hat{I} \\
0 & \to & \lim_{\text{Or}_f(G)} C(\Gamma/\Gamma^0) & \to & \lim_{\text{Or}_f(G)} C(\Gamma) & \to & \lim_{\text{Or}_f(G)} \hat{C}(\Gamma) & \to & \lim_{\text{Or}_f(G)}^1 C(\Gamma/\Gamma^0)
\end{array}
\]

that \( \hat{I} \) is injective.

Note that \( G^0 \) acts trivially on \( H^*(M, \mathcal{F}) \). Thus if \( G/\Gamma \in \text{Or}_f(G) \), then \( \Gamma^0 \) acts trivially on \( H^*(\text{res}_G^\Gamma M, \mathcal{F}) \), too. By Lemma 3.1 the class \( \hat{\rho}(\text{res}_G^\Gamma M, \mathcal{F}) \in \hat{C}(\Gamma) \) of \( \rho(\text{res}_G^\Gamma M, \mathcal{F}) \in C(\Gamma) \) is independent of the choice of a \( \Gamma \)-Hilbert module structure on \( H^*(\text{res}_G^\Gamma M, \mathcal{F}) \).

The collection \( \{\hat{\rho}(\text{res}_G^\Gamma M, \mathcal{F})\}_{G/\Gamma \in \text{Or}_f(G)} \) defines a section of the functor \( \hat{C} \) and therefore an element

\[ \tilde{\hat{\rho}}(M, \mathcal{F}) \in \lim_{\text{Or}_f(G)} \hat{C}(\Gamma) . \]

It is easy to see that \( \tilde{\hat{\rho}}(M, \mathcal{F}) \) is in the range of \( \hat{I} \).

**Definition 1.3** Let \( \hat{\rho}(M, \mathcal{F}) \in \hat{C}(FG) \) be the unique element such that \( \hat{I}(\hat{\rho}(M, \mathcal{F})) = \tilde{\hat{\rho}}(M, \mathcal{F}) \).

The class \( \hat{\rho}_{an}(M, g^M, \mathcal{F}) \in \hat{C}(FG) \) of \( \rho_{an}(M, g^M, \mathcal{F})|_{FG} \) is independent of the choice of the \( G \)-invariant Riemannian metric \( g^M \) (see [7], §X). We thus write \( \hat{\rho}_{an}(M, \mathcal{F}) := \hat{\rho}_{an}(M, g^M, \mathcal{F}) \). The proof of the following Lemma is similar to that of Lemma 1.2.
Lemma 1.4 Let $G$ be a compact Lie group, $M$ be a closed odd-dimensional $G$ manifold, and $F \to M$ be a $G$-equivariant flat hermitean vector bundle. Then

$$\hat{\rho}(M, F) = \hat{\rho}_{an}(M, F).$$

(4)

1.6

It is natural to ask what kind of differential-topological information about the $G$-manifold $M$ and the flat bundle $F$ is encoded in the invariants $\rho(M, F)$ and $\hat{\rho}(M, F)$. While $\rho(M, F)$ contains global information about $M$ it turns out that $\hat{\rho}(M, F)$ only depends on the type of $G$-cells of $X$ and the restriction of $f^*F$ to the cells, where $f : X \to M$ represents the preferred simple structure of the smooth closed $G$-manifold $M$. In particular it is independent of the way the cells are glued together.

We now formulate the result in detail. A $G$-space $G/H \times D^n$ is called a $n$-dimensional $G$-cell of type $H$, where $H \subset G$ is a closed subgroup. Let $X$ be a finite $G$-CW-complex and $\mathcal{F}$ be a $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces over $X$. For any $G$-cell $E = G/H_E \times D^n \hookrightarrow X$ of dimension $\dim(E) := n$ let $\mathcal{F}_E \to G/H_E$ denote the restriction of $\mathcal{F}$ to $G/H_E \times \{0\}$. Then $\hat{\rho}(G/H_E, \mathcal{F}_E) \in \hat{\mathcal{C}}(FG)$ is defined.

Proposition 1.5 (Corollary 4.4) Let $X$ be a finite $G$-CW-complex, and $\mathcal{F}$ be a $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces over $X$. Then we have

$$\hat{\rho}(X, \mathcal{F}) = \sum (-1)^{\dim(E)} \hat{\rho}(G/H_E, \mathcal{F}_E),$$

where the sum is taken over all $G$-cells of $X$.

1.7

The equivariant torsion $\hat{\rho}(M, \mathcal{F})$ is determined by its restrictions to all Cartan subgroups $T$ of $G$. If we apply Proposition 1.5 to $\text{res}_T^G M$, then we can compute the equivariant
torsion of $M$ in terms of the $T$-cells. In Section 2.2 we study the equivariant torsion of a $T$-cell. It vanishes iff the isotropy group has codimension zero or greater than one, and it is explicitly computable, if the isotropy group has codimension one.

Let $f : X \to \text{res}_T^G M$ represent the preferred simple structure. Consider $t \in T$. Let $I$ be the collection of $T$-cells $E \cong T/S_E \times D^{n_E}$ of $X$ with $\dim(T/S_E) = 1$. For $E \in I$ let $J_E(t)$ be the collection of connected components $E_i, i \in J_E(t), \text{of } E$ (note that $E_i \cong S^1 \times D^{n_E}$), such that $tE_i = E_i$. Let $m = \dim(\mathcal{F})$ and $U_i \in U(m)$ be the holonomy of $f^* \mathcal{F}|_{E_i}$. Then $U_i$ is determined uniquely by the choice of a base point $o_i \in E_i$, identification of the fibre $\mathcal{F}o_i$ with $\mathbb{C}^m$, and the choice of an orientation of $E_i$. The matrix $U_i$ can be written as $e^{2\pi ia_i}$ for a selfadjoint $a_i \in \text{Mat}(m, \mathbb{C})$. The element $t$ acts as rotation of the circle-part of $E_i$ by the angle $2\pi \tau_i$. Moreover there are unitary isomorphisms $\lambda_i$ of $\mathcal{F}_{o_i}$ given by the action of $t$ composed with parallel transport back to $o_i$ in direction opposite to the orientation. Note that $\lambda_i$ and $a_i$ commute. Then

Lemma 1.6 The equivariant torsion $\hat{\rho}(\text{res}_T^G M, \mathcal{F})$ is given by the class of

$$T \ni t \mapsto \sum_{E \in I, i \in I_E} (-1)^{n_E} \text{Tr} \psi(\lambda_i, a_i, \tau_i),$$

where an explicit formula for $\psi$ is given in 5.4.

For the purpose of illustration in 5.6 we compute the equivariant Reidemeister torsion of odd-dimensional symmetric spaces. Employing (4) we essentially recover the results of [6] about equivariant analytic torsion of symmetric spaces.

2 Restriction of simple structures

In this section we recall some basic results in equivariant topology.
2.1

The $G$-space $G/H \times D^n$ is called a $n$-dimensional $G$-cell of type $H$, where $H \subset G$ is a closed subgroup. A finite relative $G$-CW-complex is a pair of $G$-spaces $(X, A)$ together with a finite filtration by $G$-spaces

$$A = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_N = X, \quad \bigcup_{n=-1}^{N} X_n = X,$$

and a collection of $G$-subspaces $e^n_i \subset X_n$, $i \in I_n$, $n \geq 0$, $\#I_n < \infty$, with the following properties

(1) : $A/G$ is Hausdorff
(2) : $X$ has the weak topology with respect to the filtration $\{X_n\}$
(3) : for $n \geq 0$ there are $G$-push outs

$$\bigsqcup_{i \in I_n} G/H_i \times S^{n-1} \xrightarrow{\bigsqcup Q_i} X_{n-1} \xrightarrow{\downarrow} \bigsqcup_{i \in I_n} G/H_i \times D^n \xrightarrow{\bigsqcup Q_i} X_n,$$

such that $e^n_i = Q_i(G/H_i \times \text{int } D^n)$ (see [8], Def. 1.2.).

2.2

If $X$ is a $G$-space, then a simple structure on $X$ is given by a pair $(Z, f)$, where $Z$ is a finite $G$-CW-complex and $f : Z \to X$ is a $G$-homotopy equivalence. A second pair $(Z', f')$ defines the same simple structure on $X$, if $f'_* r^G((f')^{-1} \circ f) = 0$ holds true in the Whitehead group $Wh^G(X)$, where $(f')^{-1}$ denotes any homotopy inverse of $f'$ (see [8], 4.27).

Let

$$X_0 \to X_1 \xrightarrow{\downarrow} X_2 \to X$$

be a push-out of $G$-spaces and each of the $X_i$, $i = 0, 1, 2$ be equipped with a simple structure. Then $X$ has a preferred simple structure [8], p75.
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2.3

If $X$ is a closed smooth $G$-manifold (possibly with boundary), then it has a preferred simple structure \( [8], 4.36 \). It is obtained by induction over the number of orbit types. If $X$ has one orbit type, then $X/G$ is a smooth manifold which has a smooth triangulation. Lifting this triangulation to $X$ we obtain a $G$-CW-decomposition of $X$ representing the preferred simple structure. If $X$ has several orbit types then we write $X$ as a push-out of two $G$-manifolds with less orbit types. We apply the induction hypothesis and [2.2] in order to obtain the preferred simple structure of $X$.

2.4

Let $G$ be a compact Lie group and $X$ be a finite $G$-CW complex. If $H \subset G$ is a closed subgroup, then in general there is no natural $H$-CW structure on $\text{res}_H^G X$. But we have the following result.

**Proposition 2.1 (1):** Let $X$ be a finite $G$-CW complex. If $H \subset G$ is a closed subgroup, then there exists a preferred simple structure $f : Z \to \text{res}_H^G X$.

**2:** Let $X$ be a finite $G$-CW complex. If $K \subset H \subset G$ are closed subgroups, $f : Z \to \text{res}_H^G X$ and $g : Y \to \text{res}_K^H Z$ represent the preferred simple structures given in (1), then $f \circ g : Y \to \text{res}_K^G X$ represents the preferred simple structure, too.

**3:** Let $M$ be a smooth closed $G$-manifold, and let $f : X \to M$ represent the preferred simple structure. Let $H \subset G$ be a closed subgroup, and let $g : Z \to \text{res}_H^G X$ represent the preferred simple structure. Then $f \circ g : Z \to \text{res}_H^G M$ represents the preferred simple structure of the closed smooth $H$-manifold $\text{res}_H^G M$.

**4:** Let $X$ be a finite $G$-CW complex, $H \subset G$ be a closed subgroup, and let $f : Z \to \text{res}_H^G X$ represent the preferred simple structure. Let $g \in G$, $H^g := g H g^{-1}$, $Z^g$ be the $H^g$-CW complex which is obtained from $Z$ by letting $u \in H^g$ act by $g^{-1} u g$, and define $f^g : Z^g \to \text{res}_{H^g}^G X$ by $f^g = g \circ f$. Then $f^g : Z^g \to \text{res}_{H^g}^G X$ represents the preferred simple structure.
Proof. (1) The homogeneous spaces $G/H$ for all closed subgroups $L \subset G$ are smooth $H$-manifolds. Thus we have prefered simple structures on the $G$-cells of $X$ considered as $H$-spaces. Writing $X$ as a push-out over its $G$-cells and using 2.2 we obtain a prefered simple structure on $\text{res}^G_H X$. Note that the technical assumptions [8], 7.3 and 7.23 for this procedure are satisfied (see [8], 7.27, see also [3],[4],[5]).

(2) It suffices to show this for the homogeneous spaces $G/L$. In this case we can apply (3) (Transitivity of the restriction was also announced in [5]).

(3) This is [8], Lemma 7.4.5.

(4) It again suffices to verify this assertion for the homogeneous spaces $G/L$. In this case we can apply [5], Lemma 1.4. \qed

3 Equivariant torsion

3.1

Let $\Gamma$ be a finite group, and let $R(\Gamma)$ denote the representation ring of $\Gamma$ with real coefficients. If $\pi$ is a finite-dimensional representation of $\Gamma$, then $\chi_\pi$ denotes its character. We have $\chi_\pi \in C(\Gamma)$, and the map $R(\Gamma) \ni \pi \mapsto \chi_\pi \in C(\Gamma)$ induces an isomorphism of $C$-vector spaces $\mathcal{X} : R(\Gamma) \otimes_{\mathbb{R}} C \cong C(\Gamma)$.

3.2

Consider a finite group $\Gamma$. Let $f : V \to W$ be an isomorphism of finite-dimensional $\Gamma$-Hilbert modules. If $\pi$ is an irreducible representation of $\Gamma$, then let $V(\pi), W(\pi)$ denote the $\pi$-isotypical components and $f(\pi) : V(\pi) \to W(\pi)$ the induced isomorphism. We define
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\[ [[f]] : \in R(\Gamma) \text{ by } \]
\[ [[f]](\pi) := \frac{1}{2 \dim(\pi)} \log | \det (\pi)^* f(\pi) | . \]

Let
\[ C : \ldots \to C^0 \xrightarrow{\partial^0} C^1 \to \ldots \]
be an acyclic finite cochain complex of finite-dimensional \( \Gamma \)-Hilbert modules. Then there exists a chain contraction \( \kappa^* : C^* \to C^{* - 1} \), and \( \epsilon^{ev} + \kappa^{ev} : C^{ev} \to C^{odd} \) is an isomorphism, where \( C^{ev} := \oplus_{k \in \mathbb{Z}} C^{2k} \), \( C^{odd} := \oplus_{k \in \mathbb{Z}} C^{2k+1} \). We define
\[ \rho(C) := \mathcal{X}[[\epsilon^{ev} + \kappa^{ev} : C^{ev} \to C^{odd}]] . \]

Note that \( \rho(C) \) does not depend on the choice of the chain contraction \( \kappa \).

Let \( f : C \to D \) be a chain homotopy equivalence. Then we consider the complex \( \text{cone}(f) \) with \( \text{cone}(f)^n := C^n \oplus D^{n-1} \) and the differential
\[ \begin{pmatrix} c^n & 0 \\ f^n & -d^{n-1} \end{pmatrix} . \]

We define \( t(f) = \rho(\text{cone}(f)) \in C(\Gamma) \). Note that if \( g : C \to D \) is homotopy equivalent to \( f \), then \( t(f) = t(g) \). If \( g : D \to E \) is a second chain homotopy equivalence, then \( t(g \circ f) = t(f) - t(g) \).

If \( C \) is a finite complex of finite-dimensional \( \Gamma \)-Hilbert modules, then we consider the complex \( H(C) \) with \( p \)th space \( H^p(C) \) and trivial differential. If \( f : C \to D \) is a homotopy equivalence, then we obtain a homotopy equivalence \( f_* : H(C) \to H(D) \).

Let \( C, D \) be finite complexes of finite-dimensional \( \Gamma \)-Hilbert modules with preferred \( \Gamma \)-Hilbert module structures on \( H^*(C), H^*(D) \). A homotopy equivalence \( f : C \to D \) is called simple if \( t(f) + t(f_*) = 0 \).

We call two finite complexes of finite-dimensional \( \Gamma \)-Hilbert modules \( C, D \) with preferred \( \Gamma \)-Hilbert module structures on \( H^*(C), H^*(D) \) equivalent, if there exists a simple homotopy equivalence \( f : C \to D \). We write \([C]\) for the equivalence class.
If $C$ is a finite complex of finite-dimensional $\Gamma$-Hilbert modules with preferred $\Gamma$-Hilbert module structure on $H^*(C)$, and if $i : \mathcal{H}(C) \to C$ is any embedding such that $i_* = \text{id}$, then we set $\rho(C) := -t(i)$. If $C$ and $D$ are equivalent, then $\rho(C) = \rho(D)$, hence we can write $\rho([C]) := \rho(C)$, where $C$ is any representative of $[C]$.

Let $\Gamma^0 \subset \Gamma$ be a subgroup such that $\Gamma^0$ acts trivially on $\mathcal{H}(C)$. Let $\hat{C} := C(\Gamma) / C(\Gamma / \Gamma^0)$ and $\hat{\rho}(C)$ be the class of $\rho(C)$ in $\hat{C}(\Gamma)$.

**Lemma 3.1** $\hat{\rho}(C)$ is independent of the choice of the $\Gamma$-Hilbert module structure on $H^*(C)$.

**Proof.** Let $\mathcal{H}_j(C)$, $j = 1, 2$, be the complex $\mathcal{H}(C)$ equipped with two $\Gamma$-Hilbert module structures. Let $\rho_j(C)$ be the corresponding torsion. Then we have $\rho_2(C) = \rho_1(C) - t(\text{id} : \mathcal{H}_2(C) \to \mathcal{H}_1(C))$. It is easy to see that $t(\text{id} : \mathcal{H}_2(C) \to \mathcal{H}_1(C)) \in C(\Gamma / \Gamma^0)$. 

### 3.3

Let $\Gamma$ be a finite group and $X$ be a $\Gamma$-space of the homotopy type of a finite $\Gamma$-CW-complex. Let $\mathcal{F}$ be a $\Gamma$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We fix a $\Gamma$-Hilbert module structure on $H^*(X, \mathcal{F})$.

Let $(Z, f)$ represent a simple structure on $X$. Then we form the cellular cochain complex $\mathcal{C}(Z, f^*\mathcal{F})$, which is a finite complex of finite-dimensional $\Gamma$-Hilbert modules. We equip $H^*(\mathcal{C}(Z, f^*\mathcal{F}))$ with the $\Gamma$-Hilbert module structure such that the canonical map $f^* : H^*(X, \mathcal{F}) \to H^*(\mathcal{C}(Z, f^*\mathcal{F}))$ becomes an isometry.

If $(Z', f')$ and $(Z, f)$ represent the same simple structure of $X$, then $\mathcal{C}(Z, f^*\mathcal{F})$ and $\mathcal{C}(Z', (f')^*\mathcal{F})$ are equivalent chain complexes. If $X$ is a $\Gamma$-space with distinguished simple structure and $\Gamma$-Hilbert module structure on $H^*(X, \mathcal{F})$, then we write $[\mathcal{C}(X, \mathcal{F})] := \hat{\rho}(\mathcal{C}(X, \mathcal{F}))$. 

[\mathcal{C}(Z, f^* \mathcal{F})], where \((Z, f^* \mathcal{F})\) is any representative of the distinguished simple structure of \(X\). We define
\[
\rho(X, \mathcal{F}) := \rho([\mathcal{C}(X, \mathcal{F})]) \in C(\Gamma).
\]

Let \(\Gamma^0 \subset \Gamma\) act trivially on \(H^*(X, \mathcal{F})\). Then the class \(\hat{\rho}(X, \mathcal{F}) \in \hat{C}(\Gamma)\) does not depend on the choice of the \(\Gamma\)-Hilbert module structure on \(H^*(X, \mathcal{F})\).

### 3.4

Let \(\Gamma\) be a finite group and \(\Gamma' \subset \Gamma\). If \(Z\) is a \(\Gamma\)-CW complex, then since \(\Gamma\) is finite \(\text{res}_{\Gamma'}^\Gamma Z\) carries a natural \(\Gamma'\)-CW structure. Moreover \(\text{id} : \text{res}_{\Gamma'}^\Gamma Z \rightarrow \text{res}_{\Gamma'}^\Gamma Z\) represents the preferred simple structure given in Proposition 2.1 (1).

Let \(X\) be a \(\Gamma\)-space of the homotopy type of a finite \(\Gamma\)-CW complex. If \((Z, f)\) represents a simple structure for \(X\), then \((\text{res}_{\Gamma'}^\Gamma Z, f)\) represents a simple structure of \(\text{res}_{\Gamma'}^\Gamma X\).

We choose a \(\Gamma\)-Hilbert module structure on \(H^*(X, \mathcal{F})\) which we also use for \(H^*(\text{res}_{\Gamma'}^\Gamma X, \mathcal{F}) = \text{res}_{\Gamma'}^\Gamma H^*(X, \mathcal{F})\).

**Lemma 3.2**

\[
\text{res}_{\Gamma'}^\Gamma \rho(X, \mathcal{F}) = \rho(\text{res}_{\Gamma'}^\Gamma X, \mathcal{F}) .
\]

**Proof.** We use \((\text{res}_{\Gamma'}^\Gamma Z, f)\) to represent the preferred simple structure of \(\text{res}_{\Gamma'}^\Gamma X\). Then
\[
\mathcal{C}(\text{res}_{\Gamma'}^\Gamma Z, f^* \mathcal{F}) = \text{res}_{\Gamma'}^\Gamma \mathcal{C}(Z, f^* \mathcal{F})
\]
\[
[\mathcal{C}(\text{res}_{\Gamma'}^\Gamma Z, f^* \mathcal{F})] = [\text{res}_{\Gamma'}^\Gamma \mathcal{C}(Z, f^* \mathcal{F})] .
\] (5)

Let \(h : V \rightarrow W\) be an isomorphism of finite-dimensional Hilbert-\(\Gamma\)-modules and \(\text{res}_{\Gamma'}^\Gamma h : \text{res}_{\Gamma'}^\Gamma V \rightarrow \text{res}_{\Gamma'}^\Gamma W\). Then for any irreducible representation \(\tau\) of \(\Gamma'\) we have
\[
[[\text{res}_{\Gamma'}^\Gamma h]](\tau) = \frac{1}{2 \dim(\tau)} \log |\det(\text{res}_{\Gamma'}^\Gamma h(\tau)^* \text{res}_{\Gamma'}^\Gamma h(\tau))| .
\]
\[ = \sum_{\pi \in \hat{\Gamma}} \frac{[\pi : \tau]}{2 \dim(\pi)} \log |\det h^*(\pi)h(\pi)| \]
\[ = \sum_{\pi \in \hat{\Gamma}} [\pi : \tau] [[h]](\pi) \]

Note that \( \text{res}_{\Gamma}^\Gamma \chi_\pi = \sum_{\tau \in \hat{\Gamma}} [\pi : \tau] \chi_\tau \). Thus

\[ X([\text{res}_{\Gamma}^\Gamma h]) = \sum_{\tau \in \hat{\Gamma}} ([\text{res}_{\Gamma}^\Gamma h](\tau)) \chi_\tau \]
\[ = \sum_{\tau \in \hat{\Gamma}} \sum_{\pi \in \hat{\Gamma}} [\pi : \tau] [[h]](\pi) \chi_\tau \]
\[ = \sum_{\pi \in \hat{\Gamma}} [[h]](\pi) \text{res}_{\Gamma}^\Gamma \chi_\pi \]
\[ = \text{res}_{\Gamma}^\Gamma X([\mathcal{E}_\Gamma]) . \]

The Lemma now follows from (3) and (4). \( \square \)

### 3.5

Let \( G \) be a compact Lie group, \( X \) be a finite \( G \)-CW-complex and \( \mathcal{F} \) be a \( G \)-equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We choose a \( G \)-Hilbert module structure on \( H^*(X, \mathcal{F}) \) which induces a \( \Gamma \)-Hilbert module structure on \( H^*(\text{res}_\Gamma^\Gamma X, \mathcal{F}) = \text{res}_\Gamma^\Gamma H^*(X, \mathcal{F}) \) for all \( G/\Gamma \in \text{Or}_f(G) \).

If \( G/\Gamma \in \text{Or}_f(G) \), then \( \text{res}_G^\Gamma X \) has a preferred simple structure by Proposition 2.1 (1) and \( \rho(\text{res}_G^\Gamma X, \mathcal{F}) \in \mathcal{C}(\Gamma) \) is defined. If \( h \in G \), then let \( \Gamma^h := h\Gamma h^{-1} \), \( G/\Gamma^h \in \text{Or}_f(G) \), and set \( g^h := hgh^{-1} \) for \( g \in G \).

**Lemma 3.3** If \( g \in \Gamma \) and \( h \in G \), then \( \rho(\text{res}_G^\Gamma X, \mathcal{F})(g) = \rho(\text{res}_G^{\Gamma^h} X, \mathcal{F})(g^h) \).

**Proof.** Let \((Z, f)\) represent the preferred simple structure of \( \text{res}_G^\Gamma X \). Then by Proposition 2.1 (4) (we employ the notation introduced there) the pair \((Z^h, f^h)\) represents the preferred
simple structure of \( \text{res}_{\Gamma}^G X \). We have an isomorphism of complexes of \( \Gamma^h \)-Hilbert modules 
\[
C(Z, f^*F)^h = C(Z^h, (f^h)^*F),
\]
where the underlying space of \( C(Z, f^*F)^h \) is \( C(Z, f^*F) \) and \( \Gamma^h \) acts by \( g^h \mapsto g \). Similarly we have isomorphisms of complexes \( \Gamma^h \)-Hilbert modules 
\[
\mathcal{H}(\text{res}_{\Gamma}^G X, F)^h = \mathcal{H}(\text{res}_{\Gamma}^G X, F).
\]
If \( i, i^h, j \) denote the inclusions \( i : \mathcal{H}(\text{res}_{\Gamma}^G X, F) \hookrightarrow C(Z, f^*F), \)
\( i^h : \mathcal{H}(\text{res}_{\Gamma}^G X, F)^h \hookrightarrow C(Z, f^*F)^h, \) \( j : \mathcal{H}(\text{res}_{\Gamma}^G X, F) \hookrightarrow C(Z^h, (f^h)^*F), \) then we have 
\[
t(i)(g) = t(i^h)(g^h) = t(j)(g^h).
\]
This proves the Lemma.

\[\square\]

3.6

We keep the assumptions of 3.5 Then the Lemmas 3.3 and 3.2 imply

**Corollary 3.4** The collection \( \{\rho(\text{res}_{\Gamma}^G X, F)\}_G/\Gamma \in \text{Or}_f(G) \), defines a section of the functor \( \mathcal{C} \) (see Subsection 1.4).

Let now \( M \) be a closed \( G \)-manifold and \( F \to M \) be a \( G \)-equivariant flat hermitean vector bundle. We fix \( G \)-Hilbert module structures on \( H^*(M, F) \). If \( G/\Gamma \in \text{Or}_f(G) \), then \( \text{res}_{\Gamma}^G M \) has a preferred simple structure. We employ this structure in order to define \( \rho(\text{res}_{\Gamma}^G M, F) \) as explained in Section 3.3.

**Proposition 3.5** The collection \( \{\rho(\text{res}_{\Gamma}^G M, F)\}_G/\Gamma \in \text{Or}_f(G) \), defines a section of the functor \( \mathcal{C} \).

**Proof.** Let \( f : X \to M \) represent the preferred simple structure. Then by Proposition 2.1, (3), we have 
\[
\rho(\text{res}_{\Gamma}^G M, F) = \rho(\text{res}_{\Gamma}^G X, f^*F).
\]
Thus the Proposition is implied by Corollary 3.4. \(\square\)
4 Computations

4.1

Let $G$ be a compact Lie group and $X$ be a finite $G$-CW-complex. Let $\mathcal{F}$ be a $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We choose a $G$-Hilbert module structure on $H^\ast(X, \mathcal{F})$. Being a $G$-CW complex $X$ has a natural filtration $\emptyset = X_0 \subset X_1 \subset \ldots \subset X_N = X$. Consider $G/\Gamma \in \text{Or}_f(G)$.

**Lemma 4.1** There exists a representative $f : Z \to \text{res}_{G}^\mathcal{G}X$ of the preferred simple structure such that

1. $Z$ is filtered by $\Gamma$-CW subcomplexes $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset \ldots \subset Z_N = Z$ and $f|_{Z_p} : Z_p \to \text{res}_{G}^\mathcal{G}X_p$ represents the preferred simple structure for all $p \in \{0, 1, \ldots, N\}$.

2. If $\ldots \subset F_{p+1}\mathcal{C}(Z, f^\ast \mathcal{F}) \subset F_p\mathcal{C}(Z, f^\ast \mathcal{F}) \subset \ldots$ denotes the decreasing filtration of the associated cochain complexes (we write $F_p := F_p\mathcal{C}(Z, f^\ast \mathcal{F})$), then $F_p/F_{p+1}$ is the cochain complex associated to a representative of the preferred simple structure of $\sqcup_{i \in I_p} \text{res}_{G}^\mathcal{G}G/H_i \times (D^p, S^{p-1})$ and the local system $Q^\ast \mathcal{F}$, where $Q$ is given by

$$Q := \sqcup_{i \in I_p} Q_i : \sqcup_{i \in I_p} G/H_i \times D^p \to X_p$$

(recall that $I_p$ is the indexing set of the $p$-dimensional $G$-cells of $X$ and $Q_i$ denote characteristic maps).

**Proof.** The construction of $f : Z \to \text{res}_{G}^\mathcal{G}X$ goes by induction and is based on [8], 4.29-4.32. For $X_{-1}$ the assertion is trivial. Assume that we have constructed the simple structure $(Z_{n-1}, f_{n-1})$ of $\text{res}_{G}^\mathcal{G}X_{n-1}$ together with the filtration by $\Gamma$-subspaces $(Z_{n-1})_m$. Then we have to construct a simple structure $(Z_n, f_n)$ of $\text{res}_{G}^\mathcal{G}X_n$ together with the filtration by $\Gamma$-subspaces $(Z_n)_m$. 
Let $X_n$ be given by the $G$-push out

\[
\bigsqcup_{i \in I_n} G/H_i \times S^{n-1} \rightarrow_{\bigsqcup q_i} X_{n-1} \\
\downarrow \\
\bigsqcup_{i \in I_n} G/H_i \times D^n \rightarrow_{\bigsqcup Q_i} X_n
\]

We choose representatives $h_i : V_i \rightarrow \res^G_{\Gamma} G/H_i$ of the preferred simple structures. Then $h_i \times \id : U_i := V_i \times D^n \rightarrow \res^G_{\Gamma} G/H_i \times D^n$ is a simple structure on the cell $\res^G_{\Gamma} G/H_i \times D^n$. We choose a $\Gamma$-equivariant cellular map $p : \bigsqcup_{i \in I_n} U_i \times S^{n-1} \rightarrow Z_{n-1}$ such that $f_{n-1} \circ p \sim_\Gamma \bigsqcup_{i \in I_n} q_i \circ l_i$ and $\sim_\Gamma$ stands for $\Gamma$-homotopic. We now replace $(Z_{n-1}, f_{n-1})$ by $(\Cyl(p), f'_{n-1})$, which represents the same simple structure on $X_{n-1}$ ($f'_{n-1}$ has still to be constructed). The filtration of $\Cyl(p) = \bigsqcup_{i \in I_n} U_i \times S^{n-1} \times I \cup_p Z_{n-1}$ is given by $\Cyl(p)_m = (Z_{n-1})_m \subset \Cyl(p)$ for $m \leq n - 2$. Let $pr : \Cyl(p) \rightarrow Z_{n-1}$ be the projection. In order to construct $f'_{n-1}$ we consider the following $\Gamma$-homotopy commutative diagram

\[
\bigsqcup_{i \in I_n} \res^G_{\Gamma} G/H_i \times S^{n-1} \bigsqcup X_{n-2} \rightarrow_{\bigsqcup q_i \cup j} \res^G_{\Gamma} X_{n-1} \\
\uparrow \bigsqcup l_i \bigsqcup f_{n-2} \\
\bigsqcup_{i \in I_n} U_i \times S^{n-1} \bigsqcup \Cyl(p)_{n-2} \rightarrow \uparrow J \bigsqcup_{i \in I_n} \Cyl(p)
\]

where $j : X_{n-2} \hookrightarrow X_{n-1}$ is the inclusion, $l_i := (h \times \id)|_{U_i \times S^{n-1}}$, and $J|_{\bigsqcup_{i \in I_n} U_i \times S^{n-1}}$ is the natural identification with the closed subspace $\bigsqcup_{i \in I_n} U_i \times S^{n-1} \times \{0\} \subset \bigsqcup_{i \in I_n} U_i \times S^{n-1} \times I$. Since $J$ is a cofibration we can find $f'_{n-1} \sim_\Gamma f_{n-1} \circ pr$ such that $f_{n-2} \circ J|_{\Cyl(p)_{n-2}} = (f'_{n-1})|_{\Cyl(p)_{n-2}}$ and the following diagram commutes:

\[
\bigsqcup_{i \in I_n} \res^G_{\Gamma} G/H_i \times D^n \leftrightarrow \bigsqcup_{i \in I_n} \res^G_{\Gamma} G/H_i \times S^{n-1} \rightarrow_{\bigsqcup q_i} \res^G_{\Gamma} X_{n-1} \\
\uparrow \\
\bigsqcup_{i \in I_n} U_i \times D^n \leftrightarrow \bigsqcup_{i \in I_n} U_i \times S^{n-1} \rightarrow \uparrow J \bigsqcup_{i \in I_n} \Cyl(p)
\]

Let $Z_n$ be the $\Gamma$-push out

\[
\bigsqcup_{i \in I_n} U_i \times S^{n-1} \rightarrow \Cyl(p) \\
\downarrow \\
\bigsqcup_{i \in I_n} U_i \times D^n \rightarrow Z_n
\]

and $f_n : Z_n \rightarrow \res^G_{\Gamma} X_n$ be the natural map of push outs. Then $f_n : Z_n \rightarrow \res^G_{\Gamma} X_n$ represents the preferred simple structure, and $Z_n$ is filtered by $\Gamma$-subspaces $(Z_n)_m$ such that $(f_n)|_{(Z_n)_m} : (Z_n)_m \rightarrow \res^G_{\Gamma} X_m$ represents the preferred simple structure for all $m \leq n$. This finishes the proof of $(1)$. Assertion $(2)$ is an easy consequence of the construction. \[\square\]
4.2

For \( i \in I_p \) define
\[
\bar{Q}_i : G/H_i \cong G/H_i \times \{0\} \hookrightarrow G/H_i \times D^p Q_i X
\]
and set \( F_i := \bar{Q}_i F \). For any \( p \in \mathbb{N}_0 \) and \( i \in I_p \) we fix \( G\)-Hilbert module structures on \( H^*(G/H_i, \mathcal{F}_i) \). This induces \( G\)-Hilbert module structures on the cohomology complexes \( \mathcal{H}(G/H_i, \mathcal{F}_i) \). If \( \mathcal{C} \) is a cochain complex, then let \( \mathcal{C}[p] \) be the cochain complex with \( \mathcal{C}[p]^n := C^{n+p} \) obtained from \( \mathcal{C} \). We equip \( H^*(F_p/F_{p+1}) \) with \( \Gamma\)-Hilbert module structures such that the natural isomorphism \( \mathcal{H}(F_p/F_{p+1}) \cong \oplus_{i \in I_p} \mathcal{H}(G/H_i, \mathcal{F}_i)[p] \) becomes an isometry. Then we have
\[
[F_p/F_{p-1}] = \oplus_{i \in I_p} [\mathcal{C}(U_i, h_i^* \mathcal{F}_i)[p]] .
\]

We are going to express \( \rho(\text{res}_1^G X, \mathcal{F}) \) in terms of \( \rho(\text{res}_1^G G/H_i, \mathcal{F}_i) \) and a contribution of the spectral sequence \( \mathcal{E} := (E_r^{p,q}, d_r) \) associated to the filtration of \( X \). For trivial \( \Gamma \) this was worked out in [9]. But [9], Lemmas 4.6 and 4.7, extend immediately to the case of a finite group \( \Gamma \). We recall the result. Let
\[
Z_r^{p,q} := \text{im}(H^{p+q}(F_p/F_{p+r}) \to H^{p+q}(F_p/F_{p+1}))
\]
\[
B_r^{p,q} := \text{im}(H^{p+q-1}(F_{p-r+1}/F_p) \to H^{p+q}(F_p/F_{p+1}))
\]
\[
E_r^{p,q} := Z_r^{p,q}/B_r^{p,q}
\]
\[
Z_\infty^{p,q} := \text{im}(H^{p+q}(F_p) \to H^{p+q}(F_p/F_{p+1}))
\]
\[
B_\infty^{p,q} := \text{im}(H^{p+q-1}(\mathcal{C}(Z, f^* \mathcal{F})) \to H^{p+q}(F_p/F_{p+1}))
\]
\[
E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}
\]
\[
F^{p,q} := \text{im}(H^{p+q}(F_p) \to H^{p+q}(\mathcal{C}(Z, f^* \mathcal{F}))) .
\]

There are natural isomorphisms \( \psi^{p,q} : F^{p,q}/F^{p+1,q-1} \to E_\infty^{p,q} \).

Note that \( H^*(\mathcal{C}(Z, f^* \mathcal{F})) \cong H^*(X, \mathcal{F}) \) has a preferred \( \Gamma\)-Hilbert module structure. We equip \( Z_r^{p,q}, B_r^{p,q}, E_r^{p,q}, Z_\infty^{p,q}, B_\infty^{p,q}, E_\infty^{p,q}, F^{p,q} \) with the corresponding (sub)quotient \( \Gamma\)-Hilbert module structures. For any \( p, q, r \) we have a complex of finite \( \Gamma\)-Hilbert modules
\[
\mathcal{E}_r^{p,q} : \ldots \to E_r^{p+nr,q-(r-1)n} \to E_r^{p+(n+1)r,q-(r-1)(n+1)} \to \ldots ,
\]
and $H^n(\mathcal{E}_{r}^{p,q}) = \mathcal{E}_{r+1}^{p+nr,q-(r-1)n}$ has a preferred $\Gamma$-Hilbert module structure. Thus $\rho(\mathcal{E}_{r}^{p,q})$ is well defined. Furthermore note that $\rho(F_p/F_{p+1}) = (-1)^{p} \sum_{i \in I_p} \rho(\text{res}_{i}^{G/H_i,F_i})$. The following Proposition can be proved by repeating the argument of the proof of \cite{9}, Thm. 4.4.

**Proposition 4.2**

$$\rho(\text{res}_{i}^{G,X,F}) = \sum_{p} (-1)^{p} \sum_{i \in I_p} \rho(\text{res}_{i}^{G/H_i,F_i}) + \sum_{r \geq 1} \sum_{p=0}^{r-1} (-1)^{p+q} \rho(\mathcal{E}_{r}^{p,q}) - \sum_{p,q} (-1)^{p+q} \mathcal{X}[[\psi_{p,q}]].$$

Note that all three terms of the right-hand side may depend on the choice of the preferred $\Gamma$-Hilbert module structures on $H^*(G/H_i,F_i)$, while the left hand side does not.

**4.3**

Let $X$ be a finite $G$-CW-complex and $\mathcal{F}$ a $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces. Let $G/\Gamma \in \text{Or}_f(G)$. Then $\hat{\rho}(\text{res}_{i}^{G,X,F}) \in \hat{C}(\Gamma)$ is well defined.

**Proposition 4.3**

$$\hat{\rho}(\text{res}_{i}^{G,X,F}) = \sum_{p} (-1)^{p} \sum_{i \in I_p} \hat{\rho}(\text{res}_{i}^{G/H_i,F_i}).$$

**Proof.** We fix $G$-Hilbert module structures on $H^*(X,F)$ and $H^*(G/H_i,F_i)$. Let $q: \Gamma \to \Gamma/\Gamma^0$ be the projection. We have to show that

$$\rho(\mathcal{E}_{r}^{p,q}) \in q^*C(\Gamma/\Gamma^0), \quad \mathcal{X}[[\psi_{p,q}]] \in q^*C(\Gamma/\Gamma^0).$$

(7)

Viewing $X$ as a filtered $G$-space we see that there is a spectral sequence $\mathcal{E} := (\mathcal{E}_{r}^{p,q}, d_r)$ of $G$-modules with

$$\mathcal{Z}_{r}^{p,q} := \text{im}(H^{p+q}(X_{p+r-1}, X_{p-1}) \to \oplus_{i \in I_p} H^{p+q}(G/H_i,F_i)).$$
\[ \tilde{B}_{p,q} := \text{im}(H^{p+q-1}(X_{p-1}/X_{p-r}) \to \bigoplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) , \]
\[ \tilde{E}_{r}^{p,q} := \tilde{Z}_{r}^{p,q}/\tilde{B}_{r}^{p,q} \]
\[ \tilde{Z}_{r}^{p,q} := \text{im}(H^{p+q}(X, X_{p-1}, \mathcal{F}) \to \bigoplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) \]
\[ \tilde{B}_{\infty}^{p,q} := \text{im}(H^{p+q-1}(X, \mathcal{F}) \to \bigoplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) \]
\[ \tilde{E}_{\infty}^{p,q} := \tilde{Z}_{\infty}^{p,q}/\tilde{E}_{\infty}^{p,q} \]
\[ \tilde{E}_{p,q} := \text{im}(H^{p+q}(X, X_{p-1}, \mathcal{F}) \to H^{p+q}(X, \mathcal{F})) \]

and $G$-equivariant maps $\tilde{\psi}_{p,q}: \tilde{E}_{p,q}/\tilde{E}_{p+1,q-1} \to \tilde{E}_{\infty}^{p,q}$ such that $\mathcal{E} = \text{res}_1^G \tilde{\mathcal{E}}$, $\psi_{p,q} = \text{res}_1^G \tilde{\psi}_{p,q}$.

Consider the exact sequence
\[ 0 \to G^0 \to G \to \pi_0(G) \to 0 . \]

Observe that the representation of $G$ on $\tilde{\mathcal{E}}$ and $\tilde{E}_{p,q}$ factors over $q$. In particular $\Gamma^0$ is represented trivially. Thus (7) follows. \hfill \Box

Fix $G$-Hilbert module structures on $H^*(X, \mathcal{F})$ and $H^*(G/H_i, \mathcal{F}_i)$. Then Proposition 4.3 has the

**Corollary 4.4**
\[ \hat{\rho}(X, \mathcal{F}) = \sum_{p} (-1)^p \sum_{i \in I_p} \hat{\rho}(G/H_i, \mathcal{F}_i) . \]

5 Reduction to Cartan subgroups

5.1

Let $G$ be a compact Lie group, $X$ be a finite $G$-CW complex, and let $F \to X$ be an equivariant flat hermitean vector bundle.
Recall that a Cartan subgroup $T$ of $G$ is a topologically cyclic closed subgroup such that the Weyl group $N_G(T)/T$ is finite, where $N_G(T)$ is the normalizer of $T$ in $G$. If $T \subset G$ is a Cartan subgroup then it is isomorphic to the product of a torus and a finite cyclic group (see [2], p.177 ff, for more details about Cartan subgroups).

If $g \in G$, then there exists a Cartan subgroup containing $g$. The conjugacy classes of Cartan subgroups are in natural bijection with the cyclic subgroups of $\pi_0(G)$.

Let $\{T_i\}_{i \in C}$ be a set of representatives of conjugacy classes of Cartan subgroups of $G$. Then restriction defines inclusions

$$
\bigoplus_{i \in C} \text{res}^G_{T_i} : C(FG) \to \bigoplus_{i \in C} C(FT_i), \quad \bigoplus_{i \in C} \hat{\text{res}}^G_{T_i} : \hat{C}(FG) \to \bigoplus_{i \in C} \hat{C}(FT_i).
$$

In order to determine $f \in \hat{C}(FG)$ it is thus sufficient to compute $\hat{\text{res}}^G_{T_i} f \in \hat{C}(FT_i)$ for all $i \in C$.

Let $T$ be any Cartan subgroup of $G$ and let $f : Z \to \text{res}^G_T X$ represent the preferred simple structure. Then $\hat{\text{res}}^G_T \hat{\rho}(X, \mathcal{F}) = \hat{\rho}(Z, f^* \mathcal{F})$.

### 5.2

We further study the contribution of the $T$-cells of $Z$. Let $S \subset T$ be any closed subgroup, and let $\mathcal{F}$ be a $T$-equivariant local system on $T/S$. Then $\rho(T/S, \mathcal{F}) \in \hat{C}(FT)$ is well defined.

**Lemma 5.1** If $\dim(T/S) \neq 1$, then $\hat{\rho}(T/S, \mathcal{F}) \in \hat{C}(FT) = 0$.

**Proof.** Assume first that $T/S$ is even-dimensional. Then $T/S$ is orientable. Let $t \in FT$ generate the finite group $H \subset \text{Aut}(T/S)$. Then $H$ acts by orientation-preserving diffeomorphisms on $T/S$. We employ [10], Prop. 3.23, which says that $\hat{\rho}(\text{res}^T_H T/S, \mathcal{F})$ can be derived from the Poincaré torsion $\rho^H_{pd}(T/S, \mathcal{F})$ of the $H$-manifold $\text{res}^T_H T/S$ (see loc.cit.
Definition 3.19). Since $T$ is abelian, we see that $H$ acts freely or trivially on $T/S$. In both cases $\rho_{pd}(T/S, \mathcal{F})$ vanishes (see loc.cit. Prop. 3.20), and thus $\text{res}_H^T \hat{\rho}(T/S, \mathcal{F}) = 0$. Since $t$ was arbitrary we conclude that $\hat{\rho}(T/S, \mathcal{F}) = 0$.

Now we consider the case that $T/S$ is odd-dimensional. We fix a $T$-invariant Riemannian metric $g^{T/S}$ on $T/S$ and equip $H^*(T/S, \mathcal{F})$ with the $T$-module structure such that the de Rham isomorphism becomes an isometry. Then $\rho(T/S, \mathcal{F}) \in C(FT)$ is well-defined.

By Lemma 1.2 we have $\rho(T/S, \mathcal{F}) = \rho_{an}(T/S, g^{T/S}, \mathcal{F})|_{FT}$. If $\dim(T/S) > 1$, then we find two everywhere linearly independent unit-length Killing vector fields in $C^\infty(T/S, T(T/S))$. Using the induced decomposition of the de Rham complex by a standard argument $\rho_{an}(T/S, g^{T/S}, \mathcal{F}) = 0$. This implies the lemma. $\square$

5.3

Let $S \subset T$ be any closed subgroup such that $\dim(T/S) = 1$, and let $\mathcal{F}$ be a $T$-equivariant locally constant sheaf of Hilbert spaces. Since $T$ is abelian and $\rho(T/S, \mathcal{F})$ is additive with respect to $\mathcal{F}$ without loss of generality we can assume that $\mathcal{F}$ is one-dimensional.

We fix an orientation on $T/S$. The space $T/S$ is a disjoint union of oriented circles $C_1, \ldots, C_s$. Let $e^{2\pi i a_i} \in S^1$ denote the holonomy of $\mathcal{F}$ on $C_i$. We choose the Riemannian metric $g^{T/S}$ such that the circles have length 1.

Let $t \in T$. We distinguish two cases
a) that $tC_i = C_i$ for all $i$
b) that $tC_i \neq C_i$ for all $i$.
In case b) we have $\rho_{an}(T/S, g^{T/S}, \mathcal{F})(t) = 0$. In case a) we have $\rho_{an}(T/S, g^{T/S}, \mathcal{F})(t) = \sum_{i=1}^s \rho_{an}(C_i, g^{C_i}, \mathcal{F}_i)(t)$, where $\mathcal{F}_i$ is the restriction of $\mathcal{F}$ to $C_i$. 
Thus we are reduced to the following situation. Let $C_i$ be a circle, $F \to C_i$ be a one-dimensional flat hermitean bundle over $C_i$, and $t$ be an automorphism of $F \to C_i$, which acts as rotation on $C_i$. We parametrize $C_i = \mathbb{R}/\mathbb{Z}$ with coordinate $x$. Then $t(x) = x + \tau_i$, $\tau_i \in \mathbb{R}/\mathbb{Z}$. Sections of $F|_{C_i}$ are identified with functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $f(x + 1) = e^{2\pi i a} f(x)$. Then $(tf)(x) = \lambda_i f(x - \tau_i)$ for certain $\lambda_i \in S^1$.

We have $\rho(C_i, g^{C_i}, F)_{an}(t) = \psi(\lambda_i, a_i, \tau_i)$, and $\psi$ will be determined in Subsection 5.4. We obtain $\rho_{an}(T/S, g^{T/S}, F)(t) = \sum_{i=1}^s \psi(\lambda_i, a_i, \tau_i)$. If $F$ is higher-dimensional, then $\lambda_i$ and $a_i$ become commuting diagonalizable matrices, and we have $\rho_{an}(T/S, g^{T/S}, F)(t) = \sum_{i=1}^s \text{Tr} \psi(\lambda_i, a_i, \tau_i)$.

5.4

In this Subsection we derive a formula for $\psi(\lambda, a, \tau)$. Let $C$ be a circle, $F \to C$ be a one-dimensional flat hermitean bundle over $C$, and $t$ be an automorphism of $F \to C$, which acts as rotation on $C$.

We parametrize $C = \mathbb{R}/\mathbb{Z}$ with coordinate $x$. Then $t(x) = x + \tau$, $\tau \in \mathbb{R}/\mathbb{Z}$. Sections of $F$ are identified with functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $f(x + 1) = e^{2\pi i a} f(x)$. Then $(tf)(x) = \lambda f(x - \tau)$ for certain $\lambda \in S^1$.

We identify one-forms on $C$ with functions using the basis $dx \in C^\infty(C, T^*C)$. Let $\Delta_F = -(d/dx)^2$ be the Laplace operator. The eigenvectors of $\Delta_F$ are $f_n(x) = \exp(2\pi i (n + a)x)$, and the corresponding eigenvalue is $\mu_n = 4\pi^2 (n + a)^2$. The action of $t$ on the eigenspace spanned by $f_n$ is multiplication by $\lambda e^{-2\pi i (n + a)\tau}$.

First assume that $a \in (0, 1)$. Let

$$F_n(s) := \frac{\lambda e^{-2\pi i (n + a)\tau}}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-4\pi^2 (n + a)^2 t} dt = \frac{\lambda e^{-2\pi i (n + a)\tau}}{4^s \pi^{2s} (n + a)^{2s}},$$

then

$$\psi(\lambda, a, \tau) = -\frac{1}{ds} |_{s=0} \sum_{n \in \mathbb{Z}} F_n(s),$$
where we employ a meromorphic continuation of the sum $\sum_{n \in \mathbb{Z}} F_n(s)$ which converges for $\Re(s) > 1$.

Let $\phi(y, a, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i y n}}{(n+a)^s}$. Then

$$
\psi(\lambda, a, \tau) = -\frac{d}{ds}_{|s=0} \frac{\lambda}{4^s \pi^{2s}} \left( e^{-2\pi i a} \phi(-\tau, a, 2s) + e^{-2\pi i(a-1)} \phi(\tau, 1-a, 2s) \right).
$$

If $a = 0$ and $\tau \in (0, 1)$, then with $\phi(y, s) := \sum_{n=1}^{\infty} \frac{e^{2\pi i y n}}{n^s}$ we obtain

$$
\psi(\lambda, 0, \tau) = -\frac{d}{ds}_{|s=0} \frac{\lambda}{4^s \pi^{2s}} \left( \phi(\tau, 2s) + \phi(-\tau, 2s) \right).
$$

Using [7], Prop. 31, one can show that

$$
\psi(\lambda, 0, \tau) = \lambda D + \lambda \left( \frac{\Gamma(\tau)'}{\Gamma(\tau)} + \frac{\Gamma(1-\tau)'}{\Gamma(1-\tau)} \right)
$$

for some explicitly known constant $D \in \mathbb{C}$. In the case that $a = 0$ and $\tau = 0$ we have

$$
\psi(\lambda, 0, 0) = -\frac{d}{ds}_{|s=0} \frac{2\lambda \zeta_R(2s)}{4^s \pi^{2s}},
$$

where $\zeta_R$ denotes the Riemann zeta function.

### 5.5

In this Subsection we combine the results of Subsections 5.1, 5.2, and 5.3. For simplicity we assume that $G$ is a connected compact Lie group. Let $X$ be a finite $G$-CW complex, and let $\mathcal{F} \to X$ be a $G$-equivariant locally constant sheaf of finite-dimensional Hilbert spaces. Let $T \subset G$ be a maximal torus, and let $f : Z \to \text{res}^G_T X$ represent the preferred simple structure.

Let $I$ be the index set for the $T$-cells $E = T/S_E \times D^n u_E$ of $Z$ with $\dim(T/S_E) = 1$. Since $T$ is connected the quotient $T/S$ is a circle $S^1$. If $t \in T$, then for each $E \in I$ we fix an orientation of $T/S_E$, and we define the rotation number $\tau_E(t) \in \mathbb{R}/\mathbb{Z}$, the constant $\lambda_E(t)$, and the holonomy $e^{2\pi i a_E}$ of $\mathcal{F}_E$. We obtain the following Proposition.
Proposition 5.2 $\hat{\rho}(X, F)$ is uniquely determined by its restriction to $T$, which is represented by the function

$$FT \ni t \mapsto \sum_{E \in \mathcal{I}} (-1)^{n_E} \text{Tr} \psi(\lambda_E(t), a_E, \tau_E(t)).$$

5.6

In this section we compute the equivariant Reidemeister torsion of odd-dimensional symmetric spaces. We recover results of [6] using a completely different method. Let $\theta$ be the constant sheaf with fibre $\mathbb{C}$, and let $\hat{\rho}(M) := \hat{\rho}(M, \theta)$ for any closed $G$-manifold $M$, where we equip $\theta$ with the obvious $G$-action.

Let $G/K$ be a compact, irreducible, odd-dimensional symmetric space. We assume that $G$ is connected.

Lemma 5.3 If $\hat{\rho}(G/K) \neq 0$, then $\text{rank}G = \text{rank}K + 1$, and

$$G/K = \begin{cases} 
SO(2m)/SO(2p - 1) \times SO(2m - 2p + 1) & \text{or} \\
SU(3)/SO(3)
\end{cases}.$$

Proof. Let $T$ be a maximal torus of $G$ and $f : X \to \text{res}_T^G(G/K)$ be a representative of the preferred simple structure. Using the construction given in [6], 4.36, we can choose $X$ such that it has the same set of $T$-orbit types as $G/K$. If $\hat{\rho}(G/K) = \hat{\rho}(X) \neq 0$, then by Proposition 5.2 there exists a one-dimensional $T$-orbit $TgK \subset G/K$. Then $Tg^{-1} \cap K$ is a rank $G - 1$-dimensional torus in $K$. Hence $\text{rank}G \geq \text{rank}K \geq \text{rank}G - 1$. If $\text{rank}K = \text{rank}G$, then $G/K$ is even-dimensional. Thus $\text{rank}K = \text{rank}G - 1$. The second assertion follows from the classification of irreducible compact symmetric spaces. \qed

Now assume that $G/K$ is a compact, irreducible, odd-dimensional symmetric space with $\text{rank}K = \text{rank}G - 1$. 
Lemma 5.4 There is a one-to-one correspondence of one-dimensional $T$-orbits in $G/K$ and $W_G(T)/W_K(T)$ given by $N_G(T) \ni g \mapsto Tg^{-1}K$, where $W_G(T) := N_G(T)/T$ and $W_K(T) := N_K(T)/T \cap K$.

Proof. We can assume that $T \cap K = S$ is a maximal torus of $K$. If $T' \subset G$ is a second maximal torus of $G$ with $T \cap K = S$, then $T' = T$. Indeed, on the level of Lie algebras we have $t' = s \oplus (k^\perp)^S = t$.

Let $TgK$ be a one-dimensional $T$-orbit in $G/K$. Then $Tg^{-1} \cap K$ is a maximal torus in $K$, and hence $Tg^{-1} \cap K = S^k$ for a suitable $k \in K$. Replacing $g$ by $gk^{-1}$ we obtain $Tg^{-1} \cap K = S$, and thus $g^{-1} \in N_G(T)$. If $g \in T \cup K$, then $TgK = TK$. Thus the correspondence of $W_G(T)/W_K(T)$ with the set of one-dimensional orbits is established by $g^{-1} \in N_G(T) \mapsto TgK$. \hfill \Box

Let $s \subset t$ be the Lie algebra of $S \subset T$. Let $a := t/s$, and let $L \subset a$ be the lattice of those $[l] \in t/s$, $l \in t$, which satisfy $\exp(l) \in S$. We identify $a \cong \mathbb{R}$ such that $L$ is identified with $\mathbb{Z}$. The exponential map yields an identification $i : \mathbb{R}/\mathbb{Z} \cong a/L \cong T/S$. If $t \in T$, then let $\alpha(t) \in \mathbb{R}/\mathbb{Z}$ be such that $ti(x) = i(\alpha(t) + x)$, $\forall x \in \mathbb{R}$.

For $\tau \in \mathbb{R}/\mathbb{Z}$, $\tau \neq 0$, define

$$
\psi(\tau) := D + \left( \frac{\Gamma'(\hat{\tau})'}{\Gamma(\hat{\tau})} + \frac{\Gamma(1 - \hat{\tau})'}{\Gamma(1 - \hat{\tau})} \right),
$$

where $\hat{\tau} \in (0, 1)$ represents $\tau$. If $\tau = 0$, we put

$$
\psi(\tau) := -\frac{d}{ds}_{s=0} \frac{2\lambda \zeta_R(2s)}{4^s \pi^{2s}}.
$$

Because of the equality (4) the following proposition recovers the computation of equivariant analytic torsion of compact symmetric spaces [4], Thm. 11, up to a constant function.
Proposition 5.5  Let $G/K$ be a compact, irreducible, odd-dimensional symmetric space. If $\text{rank} K \neq \text{rank} G - 1$, then $\hat{\rho}(G/K) = 0$. If $\text{rank} K = \text{rank} G - 1$, then let $T \subset G$ be a maximal torus such that $S := T \cap K$ is a maximal torus of $K$. Then the restriction of $\hat{\rho}(G/K)$ to $T$ is represented by

$$FT \ni t \mapsto \frac{1}{\sharp W_K(T)} \sum_{w \in W_G(T)} \psi(\alpha(t^w)) .$$

Proof. Let $f : X \to \text{res}_{N_G(T)}^G(G/K)$ be a representative of the natural simple structure. By Lemma 5.4 there is exactly one isolated one-dimensional $N_G(T)$-orbit in $G/K$. Constructing $X$ by the inductive procedure given in [8], 4.36, we can assume that $X$ has exactly one cell $E = N_G(T)/N_K(T) \times D^{n_E}$ with one-dimensional $N_G(T)$-orbits. Moreover $n_E = 0$.

The $T$-space $\text{res}_{T}^{N_G(T)}(X)$ has a natural $T$-CW structure (since $N_G(T)/T$ is finite), and $\text{res}_{T}^{N_G(T)}(E)$ is the only $T$-cell with one-dimensional $T$-orbits. But $\text{res}_{T}^{N_G(T)}(E)$ is the disjoint union of spaces $T/S^g$, $g \in W_G(T)/W_K(T)$. The proposition now follows from Proposition 5.2. \qed

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