Spaces of unbounded Fredholm operators. I. Homotopy equivalences

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Abstract

This paper is devoted to the space of unbounded Fredholm operators equipped with the graph topology, the subspace of operators with compact resolvent, and their subspaces consisting of self-adjoint operators. Our main results are the following: (1) Natural maps between these four spaces and classical spaces of bounded operators representing K-theory are homotopy equivalences. This provides an alternative proof of a particular case of results of Joachim. (2) The subspace of unbounded essentially positive Fredholm operators represents odd K-theory. (3) The subspace of invertible operators in each of these spaces of unbounded operators is contractible.

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1 Introduction

Let \( H \) be a separable complex Hilbert space of infinite dimension. We denote by \( \mathcal{B}(H) \) the space of bounded linear operators on \( H \) with the norm topology; by \( \mathcal{K}(H) \),

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\( \mathcal{U}(H) \), and \( \mathcal{P}(H) \) the subspaces of \( \mathcal{B}(H) \) consisting of compact operators, unitary operators, and projections respectively (by projections we always mean self-adjoint idempotents).

**Regular operators.** An unbounded operator \( A \) on \( H \) is a linear operator defined on a subspace \( \text{dom}(A) \) of \( H \) and taking values in \( H \). Such an operator \( A \) is called closed if its graph is closed in \( \hat{H} = H \oplus H \), and densely defined if its domain \( \text{dom}(A) \) is dense in \( H \). It is called regular if it is closed and densely defined. The adjoint \( A^* \) of a regular operator is itself a regular operator.

Let \( \mathcal{R}(H) \) denote the set of all regular operators on \( H \) and \( \mathcal{R}^{sa}(H) \subset \mathcal{R}(H) \) denote the subset of self-adjoint operators.

**Two topologies on regular operators.** The two most useful topologies on the set \( \mathcal{R}(H) \) of regular operators are the Riesz topology and the graph topology.

The Riesz topology on \( \mathcal{R}(H) \) is induced by the inclusion \( \chi : \mathcal{R}(H) \hookrightarrow \mathcal{B}(H) \) from the norm topology on the space \( \mathcal{B}(H) \) of bounded operators, where \( \chi \) is the so called “bounded transform map”, \( \chi(A) = A(1 + A^*A)^{-1/2} \). The image of this inclusion lies in the closed unit ball \( \mathcal{D}(H) \) of \( \mathcal{B}(H) \).

The graph topology on \( \mathcal{R}(H) \) is induced by the inclusion \( \mathcal{p} : \mathcal{R}(H) \hookrightarrow \mathcal{P}(\hat{H}) \) from the norm topology on the space \( \mathcal{P}(\hat{H}) \) of projections in \( \hat{H} \), where \( \mathcal{p} \) is the map taking a regular operator to the orthogonal projection onto its graph.

Bounded operators are regular\(^1\). This defines the natural inclusion \( \mathcal{B}(H) \hookrightarrow \mathcal{R}(H) \), which is continuous and, moreover, an embedding with respect to both Riesz and graph topology on \( \mathcal{R}(H) \).

Let \( ^r\mathcal{R}(H) \), resp. \( ^g\mathcal{R}(H) \) denote the space of regular operators on \( H \) equipped with the Riesz, resp. graph topology. The identity map \( ^r\mathcal{R}(H) \rightarrow ^g\mathcal{R}(H) \) is continuous, so we have a sequence of continuous maps

\[
\mathcal{B}(H) \hookrightarrow ^r\mathcal{R}(H) \xrightarrow{\text{Id}} ^g\mathcal{R}(H) \xrightarrow{\mathcal{P}} \mathcal{P}(\hat{H}).
\]

Moreover, \( \mathcal{p} \) factors through a continuous map \( \hat{\mathcal{p}} : \mathcal{D}(H) \rightarrow \mathcal{P}(\hat{H}) \), making the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{B}(H) & \hookrightarrow & ^r\mathcal{R}(H) \\
\xrightarrow{\chi} & & \xrightarrow{\text{Id}} ^g\mathcal{R}(H) \\
\mathcal{D}(H) & \xrightarrow{\hat{\mathcal{p}}} & \mathcal{P}(\hat{H})
\end{array}
\]

(1.1)

**Fredholm operators and operators with compact resolvent.** Similar to the bounded case, a regular operator is called Fredholm if its image is closed and its kernel and

\(^1\)and unbounded, but we will refrain from using the term “unbounded” when it may lead to confusion.
cokernel are finite-dimensional. We denote by $B_F(H)$ and $R_F(H)$ the subsets of $B(H)$ and $R(H)$ respectively consisting of Fredholm operators.

A regular operator is said to have **compact resolvent** if both $(1 + AA^*)^{-1}$ and $(1 + A^*A)^{-1}$ are compact operators. We denote by $R_K(H)$ the subset of $R(H)$ consisting of regular operators with compact resolvents,

$$R_K(H) = \{ A \in R(H) \mid (1 + A^*A)^{-1}, (1 + AA^*)^{-1} \in \mathcal{K}(H) \}.$$  

Every operator with compact resolvent is Fredholm.

A self-adjoint regular operator $A$ has compact resolvent if and only if $(A + i)^{-1}$ is a compact operator. Such an operator has a discrete real spectrum.

**Homotopy type.** It is a classical result of Atiyah and Jänich [At] that the space $B_F(H)$ of bounded Fredholm operators is a classifying space for the functor $K^0$. It can be easily seen that in the Riesz topology both $R_F(H)$ and $R_K(H)$ have the same homotopy type as $B_F$. In contrast with that, the homotopy type of the spaces $R_F(H)$ and $R_K(H)$ equipped with the graph topology remained unknown for a long time.

In 2003 Joachim [Jo] showed, using results of [BJS], that these spaces are classifying spaces for the functor $K^0$, similar to the bounded case. He also proved the $K^*$-analog of this result for regular self-adjoint operators. Moreover, he proved these results in a more general situation, for the Hilbert module $H_a = A \otimes H$ over a unital $C^*$-algebra $A$ (the Hilbert space case corresponds to $A = C$).

However, Joachim’s proofs are based on a fairly advanced machinery of Kasparov KK-theory, even in the case $A = C$. In this paper we give a transparent proof of this particular case of Joachim’s results. Moreover, we show that natural maps between these spaces of unbounded operators and classical spaces of bounded operators representing K-theory are homotopy equivalences, the embedding $B_F \hookrightarrow R_F$ being one example.

Our proof of homotopy equivalence is based on covering of spaces (both the source and the target of a map) by open subsets in such a way that the “gluing pattern” of these coverings is exactly the same, up to contractible factors, and is preserved by the map. The same method allows to prove that the spaces of essentially positive and negative Fredholm operators equipped with the graph topology represent odd $K$-theory, in contrast with the bounded case where they are contractible.

**Spaces and maps.** A regular operator $A$ is Fredholm if and only if its bounded transform $\chi(A)$ is Fredholm; $A$ has compact resolvent if and only if $a = \chi(A)$ is essentially unitary (that is, both $1 - a^*a$ and $1 - aa^*$ are compact operators). Let $D_F(H)$, resp. $D_K(H)$ denote the subspace of the unit ball $D(H)$ consisting of Fredholm, resp. essentially unitary operators. Then the bounded transform provides the embeddings $\tau_R F(H) \hookrightarrow D_F(H)$ and $\tau_R K(H) \hookrightarrow D_K(H)$.

---

2If the resolvent set of $A$ is non-empty, then this definition agrees with the usual sense of the words “compact resolvent”: $(A - \lambda)^{-1}$ is compact for every $\lambda \in \text{Res}(A)$. However, a regular operator may have an empty resolvent set; the definition we use covers such operators as well.
Replacing the Riesz topology by the graph topology and the bounded transform by the embedding \( p: \mathcal{R}_F(H) \to \mathcal{P}(\hat{H}) \), one gets a similar picture. Namely, let \( p_0 \) and \( p_\infty \) be the orthogonal projections of \( \hat{H} = H \oplus \hat{o} \) onto the “horizontal” subspace \( H \oplus o \) and the “vertical” subspace \( o \oplus H \), respectively. Then \( p \) provides the embeddings

\[
\mathcal{R}_K(H) \hookrightarrow \mathcal{P}_K(\hat{H}) = \{ p \in \mathcal{P}(\hat{H}) | p - p_\infty \text{ is compact} \}, \\
\mathcal{R}_F(H) \hookrightarrow \mathcal{P}_F(\hat{H}) = \{ p \in \mathcal{P}(\hat{H}) | p - p_0 \text{ is Fredholm} \}.
\]

The space \( \mathcal{P}_F \) is called the Fredholm Grassmanian and \( \mathcal{P}_K \) is called the restricted Grassmanian.

Taking this all together, we obtain a commutative diagram of continuous maps

\[
\begin{array}{ccc}
\mathcal{R}_K(H) & \xrightarrow{\mathcal{R}_F(H)} & \mathcal{P}_K(\hat{H}) \\
\downarrow & & \downarrow \\
\mathcal{B}_F(H) & \xrightarrow{\mathcal{R}_F(H)} & \mathcal{P}_F(\hat{H})
\end{array}
\]

(1.2)

Our first main result is the following theorem.

**Theorem A.** All the maps on Diagram (1.2) are homotopy equivalences. Consequently, all the spaces on the diagram are classifying spaces for the functor \( K^0 \).

In fact, we prove slightly stronger result in Section 7 adding \( \mathcal{D}_K(H) \) and \( \mathcal{D}_F(H) \) to the picture. See Diagram (7.1) and Theorem A’. We do not include it in the Introduction in order to avoid three-dimensional diagrams here.

**Self-adjoint operators.** By a classical result of Atiyah and Singer [AS], the space \( \mathcal{B}_{sa}^F(H) \) of bounded self-adjoint Fredholm operators has three connected components; two of them, \( \mathcal{B}_{sa}^+(H) \) and \( \mathcal{B}_{sa}^-(H) \), are contractible, while the third component \( \mathcal{B}_{sa}^r(H) \) is a classifying space for the functor \( K^1 \). Here \( \mathcal{B}_{sa}^+(H) \), resp. \( \mathcal{B}_{sa}^-(H) \), are the subspaces of \( \mathcal{B}_{sa}(H) \) consisting of essentially positive, resp. essentially negative operators. Recall that a bounded self-adjoint operator is called essentially positive (resp. negative) if it is positive (resp. negative) on some invariant subspace of \( H \) of finite codimension. The definition of essential positivity/negativity for regular self-adjoint operators is exactly the same\(^3\). Equivalently, \( A \in \mathcal{R}_{sa}(H) \) is essentially positive (resp. negative) if its bounded transform \( \chi(A) \) is essentially positive (resp. negative). Let \( \mathcal{R}_{sa}^+(H) \), \( \mathcal{R}_{sa}^-(H) \), and \( \mathcal{R}_{sa}^r(H) \) denote the corresponding subsets of \( \mathcal{R}_{sa}(H) \), that is, the inverse images of \( \mathcal{B}_{sa}^+(H) \), \( \mathcal{B}_{sa}^-(H) \), and \( \mathcal{B}_{sa}^r(H) \) under \( \chi \).

Similarly to the bounded case, the space \( \mathcal{R}_{sa}^r(H) \) equipped with the Riesz topology has three connected components; two of them, \( \mathcal{R}_{sa}^+\mathcal{F}(H) \) and \( \mathcal{R}_{sa}^\mathcal{F}(H) \), are contractible, while the third component \( \mathcal{R}_{sa}^r\mathcal{F}(H) \) is a classifying space for the functor \( K^1 \).

\(^3\)In the bounded case, there is an equivalent definition: a bounded operator is called essentially positive if its essential spectrum is contained in the positive ray \([0, +\infty) \). However, this definition is no longer meaningful for unbounded operators; for example, the essential spectrum of a regular self-adjoint operator with compact resolvent is always empty.
same holds for the subspace \( r\mathcal{R}_K^{sa}(H) \) of operators with compact resolvent: it has three components; two of them, \( r\mathcal{R}_K^+(H) \) and \( r\mathcal{R}_K^-(H) \) are contractible, while the third component \( r\mathcal{R}_K^0(H) \) is a classifying space for the functor \( K^1 \). The natural embedding \( \mathcal{B}_F^{sa}(H) \hookrightarrow r\mathcal{R}_F^{sa}(H) \) is a homotopy equivalence \([\text{Le}, \text{Theorem 5.10}]\), and the bounded transform \( \chi: r\mathcal{R}_F^{sa}(H) \to \mathcal{B}_F^{sa}(H) \) is a homotopy inverse to it. Both maps preserve division of the spaces into the three connected components.

However, with the graph topology the situation changes drastically. The space \( \mathcal{R}_K^{sa}(H) \) is path connected \([\text{BLP}, \text{Theorem 1.10}]\), as well as \( \mathcal{R}_K^{sa}(H) \). We show that each of the subspaces \( \mathcal{R}_K^{sa}(H) \), \( \mathcal{R}_K^{+}(H) \), and \( \mathcal{R}_K^{0}(H) \) is dense in \( \mathcal{R}_K^{sa}(H) \), see Theorem 8.1.

Booss-Bavnbek, Lesch, and Phillips asked in \([\text{BLP}, \text{Remark 1.11}]\) whether the two “trivial parts”, \( \mathcal{R}_F^{-}(H) \) and \( \mathcal{R}_F^{0}(H) \), are contractible in the graph topology. The following theorem, together with Theorem 8.1 below, gives a negative answer to this question and shows that each of these subspaces is a classifying space for the functor \( K^1 \).

**Theorem B.** In the following diagrams, all the embeddings are homotopy equivalences in the graph topology:

\[
\begin{array}{ccc}
\mathcal{R}_K^{-} & \hookrightarrow & \mathcal{R}_K^{sa} \\
\mathcal{R}_F^{-} & \hookrightarrow & \mathcal{R}_F^{sa} \\
\mathcal{R}_K^{+} & \hookrightarrow & \mathcal{R}_K^{sa} \\
\mathcal{R}_F^{+} & \hookrightarrow & \mathcal{R}_F^{sa} \\
\mathcal{R}_K^{0} & \hookrightarrow & \mathcal{R}_K^{sa} \\
\mathcal{R}_F^{0} & \hookrightarrow & \mathcal{R}_F^{sa} \\
\end{array}
\]

We illustrate this result in Section 8 by an example of a loop of elliptic boundary conditions for the differential operator \(-d^2/dt^2\) on the interval \([0,1]\); the corresponding loop of unbounded self-adjoint operators on \( H = L^2[0,1] \) has non-vanishing spectral flow, although all the operators are essentially positive.

**Self-adjoint operators and the Cayley transform.** The space \( \hat{H} = H \oplus H \) has a natural (complex) symplectic structure given by the symplectic form \( \omega(\xi,\eta) = \langle 1\cdot \xi,\eta \rangle \), where \( I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) is a symmetry (that is, a self-adjoint unitary). We call a projection \( p \in \mathcal{P}(\hat{H}) \) Lagrangian (with respect to \( \omega \) or \( I \)) if its range is a Lagrangian subspace of \( \hat{H} \) (equivalently, the symmetry \( 2p - I \) anticommutes with \( I \), or \( Ip = (1 - p)I \)). Let \( \mathcal{P}(\hat{H};I) \) denote the subspace of \( \mathcal{P}(\hat{H}) \) consisting of Lagrangian projections. There is a natural homeomorphism from \( \mathcal{P}(\hat{H};I) \) to the unitary group \( U(H) \); we discuss it in Section 2 in more detail.

A regular operator \( A \) is self-adjoint if and only if its graph is a Lagrangian subspace in \( \hat{H} \) with respect to this symplectic structure, that is, \( p(A) \in \mathcal{P}(\hat{H};I) \). This gives a natural embedding \( p: \mathcal{R}_K^{sa}(H) \to \mathcal{P}(\hat{H};I) \). Its composition with the homeomorphism \( \mathcal{P}(\hat{H};I) \cong U(H) \) provides the natural embedding \( \kappa: \mathcal{R}_K^{sa}(H) \to U(H) \), which is given by the formula \( \kappa(A) = (A - i)(A + i)^{-1} \) and is called the Cayley transform.

Restricting the map \( \tilde{p}: \mathcal{D}(H) \to \mathcal{P}(\hat{H}) \) to the subspace \( \mathcal{D}^{sa}(H) \subset \mathcal{D}(H) \) of self-adjoint operators, we obtain the continuous map \( \tilde{\kappa}: \mathcal{D}^{sa}(H) \to U(H) \), \( \tilde{\kappa}(a) = (a - i\sqrt{1 - a^2})^2 \), which makes the following diagram commutative:
The Cayley transform provides the embeddings
\[ g^\text{sa}_K(H) \hookrightarrow U_K(H) = \{ u \in U(H) \mid u - 1 \text{ is compact} \}, \]
\[ g^\text{sa}_F(H) \hookrightarrow U_F(H) = \{ u \in U(H) \mid u + 1 \text{ is Fredholm} \}. \]

Equivalently, in the Grassmanian picture the map \( p \) provides the embeddings of \( g^\text{sa}_K(H) \) to the restricted Lagrangian Grassmanian \( \mathcal{P}_k^1 = \mathcal{P}^1 \cap \mathcal{P}_K \cong U_K \) and of \( g^\text{sa}_F(H) \) to the Fredholm Lagrangian Grassmanian \( \mathcal{P}_F^1 = \mathcal{P}^1 \cap \mathcal{P}_F \cong U_F \).

Combining the maps discussed above and restricting them to the homotopically non-trivial connected components of \( B^\text{sa}_F(H), r^\text{sa}_K(H), \) and \( r^\text{sa}_F(H) \), we obtain the following commutative diagram of continuous maps:

\[
\begin{array}{c}
\mathcal{B}^\text{sa} \longrightarrow r^\text{sa}_K \longrightarrow U_K \\
\downarrow \kappa \quad \downarrow \\
\mathcal{D}^\text{sa} \longrightarrow U_F
\end{array}
\]

Theorem C. All the maps on Diagram (1.5) are homotopy equivalences. Consequently, all the spaces on the diagram are classifying spaces for the functor \( K^1 \).

We prove slightly stronger result in Section 6, adding \( D^\kappa_K(H) \) and \( D^F_F(H) \) to the picture. See Diagram 6.1 and Theorem C.

Invertible operators. Our proof of Theorems \( A, B, C \) is based on the following key observation. Recall that a regular operator \( A \) is called invertible if \( A \colon \text{dom}(A) \to H \) is bijective and has a bounded inverse \( A^{-1} \in \mathcal{B}(H) \).

Theorem D.

1. Let \( X \) be one of the spaces \( r^\kappa, g^\text{sa}_K, g^\text{sa}_F, g^\text{sa}_R, g^\text{sa}_R \), and let \( X_K = X \cap \mathcal{R}_K \) be the subspace of \( X \) consisting of operators with compact resolvent. Then the subspaces of \( X \) and \( X_K \) consisting of invertible operators are contractible. Moreover, the subspaces

\[ X[-\lambda, \lambda] = \{ A \in X \mid \text{Spec}(A) \cap [-\lambda, \lambda] = \emptyset \} \quad \text{and} \quad X_K[-\lambda, \lambda] = X_K \cap X[-\lambda, \lambda] \]

of \( X \) are contractible for every \( \lambda \geq 0 \).

2. Let \( X = r^\kappa \) or \( g^\text{sa}_F \). Then the subspaces of \( X \) and \( X_K = X \cap \mathcal{R}_K \) consisting of invertible operators are contractible. Moreover, the subspaces

\[ X[-\lambda, \lambda] = \{ A \in X \mid \text{Spec}(\hat{A}) \cap [-\lambda, \lambda] = \emptyset \}, \quad \text{where} \ \hat{A} = (\lambda \ A^\dagger_0^\dagger) \in \mathcal{R}^\text{sa}(\hat{H}), \]

and \( X_K[-\lambda, \lambda] = X_K \cap X[-\lambda, \lambda] \) are contractible for every \( \lambda \geq 0 \).
We consider various spaces of invertible operators in Sections \ref{section1}–\ref{section3} and prove different parts of Theorem \ref{thm1} in Propositions \ref{prop1}–\ref{prop4} and \ref{prop5}. Along with it, we prove contractibility of other spaces that we will need in the proofs of Theorems \ref{thm1}–\ref{thm3}.

**Proof of Theorems \ref{thm1}–\ref{thm3}**. Our proof handles all the maps on Diagrams (1.2), (1.3), and (1.5) at once and gives an alternative proof even for those maps for which other proofs are known.

The proof is based on a theorem of tom Dieck that says, roughly, that a map is a homotopy equivalence if it is locally a homotopy equivalence. We need a particular case of \cite{tomDieck} Theorem 1, which is stated as follows. Let \( \varphi: X \to Y \) be a continuous map. Let \( (X_τ) \), resp. \( (Y_τ) \), be a numerable covering of \( X \), resp. \( Y \), indexed by the same index set \( T \). Assume that \( \varphi(X_τ) \subset Y_τ \) and that for every finite \( T \subset T \) the restriction map \( \bigcap_{τ \in T} X_τ \to \bigcap_{τ \in T} Y_τ \) is a homotopy equivalence. Then \( \varphi \) itself is a homotopy equivalence.

Let us describe an idea of the proof on the example of the inclusion map

\[ \varphi: \tau \mathcal{R}_K^*(H) \to ^9\mathcal{R}_K^*(H). \]

Theorem \ref{thm1} suggests to choose coverings of \( X = \tau \mathcal{R}_K^*(H) \) and \( Y = ^9\mathcal{R}_K^*(H) \) by contractible open subspaces indexed by real numbers \( \lambda \) and consisting of operators \( A \) such that \( A - \lambda \) is invertible.

We prefer to use finer coverings \( (X_τ) \) and \( (Y_τ) \) which are closed under finite intersections and are suitable for the proof of Theorem A as well. We index them by finite symmetric (with respect to zero) non-empty subsets \( τ = \{τ_1 < \ldots < τ_n\} \) of \( \mathbb{R} \), with \( X_τ \), resp. \( Y_τ \), being the subspace of \( X \), resp. \( Y \), consisting of operators whose resolvent set contains \( τ \). We need only to show that the restriction \( \varphi_τ: X_τ \to Y_τ \) of \( \varphi \) is a homotopy equivalence for every such \( τ \).

With every operator \( A \in X_τ \) we associate its “finite part” \( A' = A|_V \) and the “infinite part” \( A'' = A|_{V^\perp} \), where \( V \) is the finite-dimensional range of the spectral projection \( 1_τ(A) \) corresponding to the convex hull \( \bar{τ} = [τ_1, τ_n] \) of \( τ \) (here \( 1_S \) denotes the characteristic function of a subset \( S \subset \mathbb{C} \)). In such a way, we provide \( X_τ \) with the natural structure of a fiber bundle over the base space \( X'_τ \) consisting of pairs \( (V, A') \), where \( V \) is a finite-dimensional subspace of \( H \) and \( A' \in \mathcal{B}^{sa}(V) \) has the spectrum contained in the finite union \( \bar{τ} \setminus τ = \bigcup_{i=1}^{n-1}(τ_i, τ_{i+1}) \) of open intervals. The fiber of \( X_τ \) over \( (V, A') \) is the subspace of \( \tau \mathcal{R}_K^*(V^\perp) \) consisting of operators \( A \) with \( \text{Spec}(A) \cap \bar{τ} = \emptyset \).

The space \( Y_τ \) is equipped with a fiber bundle structure over the same base space in exactly the same manner; the only difference is that \( \tau \mathcal{R}_K^*(V^\perp) \) is replaced by \( ^9\mathcal{R}_K^*(V^\perp) \) in the description of the fibers.

The fiber bundles \( X_τ \to X'_τ \) and \( Y_τ \to X'_τ \) are locally trivial; by Theorem \ref{thm1} their fibers are contractible. It follows that \( \varphi_τ \) is a homotopy equivalence.

All the other spaces on our diagrams are handled similarly, with an appropriate choice of coverings.
2 Preliminaries: operators, spaces, and maps

In this section we recall basic notions and facts about regular operators and prove several simple facts that we will use further. In addition, we explain how the Cayley transform \( \kappa : \mathcal{R}^{sa}(H) \to \mathcal{U}(H) \) is related to the projection map \( p : \mathcal{R}(H) \to \mathcal{P}(H) \).

**Adjoint operators.** The adjoint operator of a regular operator \( A \) is an unbounded operator \( A^* \) on \( H \) with the domain

\[
\text{dom}(A^*) = \{ x \in H | \text{ there exists } y \in H \text{ such that } \langle Az, x \rangle = \langle z, y \rangle \text{ for all } z \in H \}.
\]

For \( x \in \text{dom}(A^*) \) such an element \( y \) is unique and \( A^* x = y \) by definition. The adjoint of a regular operator is itself a regular operator. An operator \( A \in \mathcal{R}(H) \) is called self-adjoint if \( A^* = A \) (in particular, \( \text{dom}(A^*) = \text{dom}(A) \)).

**Bounded transform.** For a regular operator \( A \), the operator \( 1 + A^* A \) is regular, self-adjoint, and has a dense range. Its densely defined inverse \( (1 + A^* A)^{-1} \) is bounded and hence can be extended to a bounded operator defined on the whole \( H \). The bounded transform (or the Riesz map)

\[
\chi : \mathcal{R}(H) \to \mathcal{B}(H), \quad \chi(A) = A(1 + A^* A)^{-1/2},
\]

defines the inclusion of the set \( \mathcal{R}(H) \) of regular operators to the closed unit ball

\[
\mathcal{D}(H) = \{ a \in \mathcal{B}(H) | ||a|| \leq 1 \}
\]
in the space \( \mathcal{B}(H) \) of bounded operators. The image

\[
\hat{\mathcal{D}}(H) = \chi(\mathcal{R}(H)) = \{ a \in \mathcal{D}(H) | 1 - a^* a \text{ has dense range} \}.
\]

of this inclusion is dense in \( \mathcal{D}(H) \). The inverse map \( \chi^{-1} : \hat{\mathcal{D}}(H) \to \mathcal{R}(H) \) is given by the formula \( \chi^{-1}(a) = a(1 - a^* a)^{-1/2} \).

If a regular operator \( A \) is self-adjoint, then so is \( \chi(A) \); more generally, \( \chi(A^*) = \chi(A)^* \).

For \( a = \chi(A) \) the following identity holds:

\[
(1 + A^* A)^{-1} = 1 - a^* a.
\]

In particular, \( A \) has compact resolvent if and only if \( \chi(A) \) is essentially unitary.

**Proposition 2.1.** The image \( \hat{\mathcal{D}}(H) = \chi(\mathcal{R}(H)) \) is a convex subset of \( \mathcal{D}(H) \).

**Proof.** A bounded self-adjoint operator has dense range if and only if it is injective. Since \( 1 - a^* a \) is positive for all \( a \in \mathcal{D}(H) \), (2.1) can be written equivalently as

\[
\hat{\mathcal{D}}(H) = \{ a \in \mathcal{D}(H) : ||a \xi|| < ||\xi|| \text{ for every non-zero } \xi \in H \}.
\]

Let \( a_0, a_1 \in \hat{\mathcal{D}}(H) \) and \( s \in [0, 1] \). Then \( a_s = (1 - s)a_0 + sa_1 \) satisfies the inequality

\[
||a_s \xi|| \leq (1 - s)||a_0 \xi|| + s ||a_1 \xi|| < ||\xi||
\]

for every non-zero \( \xi \in H \), and thus \( a_s \in \hat{\mathcal{D}}(H) \) as well. \( \square \)
Proposition 2.2. The image under \( \chi \) of the subspaces of positive operators in \( \mathcal{R}(H) \) and \( \mathcal{R}_{K}(H) \) are convex subsets of \( \mathcal{D}^{sa}(H) \).

Proof. The first image is the intersection of two convex sets, namely \( \chi(\mathcal{R}(H)) \) and the cone of positive operators. The second image is the intersection of two convex sets, namely \( \chi(\mathcal{R}(H)) \) and

\[
\{ a \in \mathcal{D}^{sa}(H) | 1 - a^2 \in \mathcal{K}(H) \text{ and } a \geq 0 \} = \{ a \in \mathcal{D}^{sa}(H) | 1 - a \in \mathcal{K}(H) \text{ and } a \geq 0 \}.
\]

Therefore, the intersections are also convex. □

Projections. The (orthogonal) projection onto the graph of a regular operator \( A \) is given by the formula

\[
p(A) = \left( \frac{(1 + A^*A)^{-1}}{A(1 + A^*A)^{-1}} \frac{(1 + A^*A)^{-1}A^*}{1 - (1 + AA^*)^{-1}} \right) \in \mathcal{P}(\hat{H}).
\]

Recall that we denoted by \( \mathcal{R}(H) \) and \( \mathcal{R}_{K}(H) \) the space of regular operators on \( H \) equipped with the Riesz and the graph topology respectively. The identity map \( \mathcal{R}(H) \to \mathcal{R}_{K}(H) \) is continuous, that is, \( p: \mathcal{R}(H) \to \mathcal{P}(\hat{H}) \) factors through a continuous map \( \hat{p}: \mathcal{D}(H) \to \mathcal{P}(\hat{H}) \), \( p = \hat{p} \circ \chi \). Moreover, \( \hat{p} \) can be (uniquely) continuously extended to the whole closed unit ball \( \mathcal{D}(H) \), making the following square commutative:

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\text{Id}} & \mathcal{R} \\
\downarrow{\chi} & & \downarrow{\hat{p}} \\
\mathcal{D} & \longrightarrow & \mathcal{P}
\end{array}
\]

Such an extension \( \hat{p}: \mathcal{D}(H) \to \mathcal{P}(\hat{H}) \) is given by the formula

\[
\hat{p}(a) = \left( \frac{1 - a^*a}{a\sqrt{1 - a^*a}} \frac{\sqrt{1 - a^*a} a^*}{aa^*} \right) .
\]

We will use two special projections in \( \hat{H} = H \oplus H \), onto the “horizontal” subspace \( H \oplus 0 \) and the “vertical” subspace \( 0 \oplus H \):

\[
p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_{\infty} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Proposition 2.3. For every \( a \in \mathcal{D}(H) \) the following hold:

1. \( a \) is essentially unitary if and only if \( \hat{p}(a) - p_{\infty} \) is compact.
2. \( a \) is Fredholm if and only if \( \hat{p}(a) - p_0 \) is Fredholm.

It follows that \( A \in \mathcal{R}(H) \) has compact resolvent if and only if \( p(A) - p_{\infty} \) is compact; \( A \) is Fredholm if and only if \( p(A) - p_0 \) is Fredholm.

Proof. The first part is obvious from formula (2.3). The second part follows from the decomposition

\[
\hat{p}(a) - p_0 = \left( \frac{-a^*a}{a\sqrt{1 - a^*a}} \frac{a\sqrt{1 - aa^*}}{aa^*} \right) = \left( \frac{-a^*}{0} \frac{0}{a} \right) \cdot \left( \frac{a}{\sqrt{1 - aa^*}} \frac{-\sqrt{1 - aa^*}}{a^*} \right) .
\]
and the fact that the second factor in it is unitary and thus invertible. □

**Cayley transform.** The Cayley transform

\[
A \mapsto \kappa(A) = (A - i)(A + i)^{-1}
\]

is a continuous embedding of \( \mathcal{D}^{sa}(H) \) into the unitary group \( \mathcal{U}(H) \). For \( a = \chi(A) \) the identity \((1 + A^2)^{-1} = 1 - a^2 \) implies

\[
\kappa(A) = \frac{A - i}{A + i} = \frac{a - i\sqrt{1 - a^2}}{a + i\sqrt{1 - a^2}} = (a - i\sqrt{1 - a^2})^2 = \tilde{\kappa}(a),
\]

where \( \tilde{\kappa} : [-1, 1] \to \mathcal{U}(\mathbb{C}) = \{ z \in \mathbb{C} \mid |z| = 1 \} \) is a continuous function given by the formula

\[
(2.4) \quad \tilde{\kappa}(a) = (a - i\sqrt{1 - a^2})^2.
\]

Thus the Cayley transform factors through the bounded transform: \( \kappa = \tilde{\kappa} \circ \chi \). Moreover, \( \tilde{\kappa} \) can be (uniquely) continuously extended to the whole \( \mathcal{D}^{sa}(H) \); the corresponding map is given by the same formula (2.4).

An operator \( a \in \mathcal{D}^{sa}(H) \) is essentially unitary (that is, \( 1 - a^2 \in \mathcal{K}(H) \)) if and only if

\[
\kappa(a) \in \mathcal{U}_K(H) = \{ u \in \mathcal{U}(H) \mid u - 1 \in \mathcal{K}(H) \},
\]

as can be seen from the identity \( 1 - \tilde{\kappa}(a) = 2(1 - a^2) - 2ia\sqrt{1 - a^2} \).

An operator \( a \in \mathcal{D}^{sa}(H) \) is Fredholm if and only if

\[
\kappa(a) \in \mathcal{U}_F(H) = \{ u \in \mathcal{U}(H) \mid u + 1 \text{ is Fredholm} \},
\]

as can be seen from the identity \( \tilde{\kappa}(a) + 1 = 2a(1 - i\sqrt{1 - a^2}) \) and the fact that \( a - i\sqrt{1 - a^2} \in \mathcal{U}(H) \).

**Lagrangian projections.** As we discussed in the Introduction, the space \( \hat{H} \) has a natural symplectic structure given by the symmetry \( I = \left( \begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix} \right) \). A regular operator \( A \) is self-adjoint if and only if its graph is a Lagrangian subspace in \( H \) with respect to \( I \), that is,

\[
\mathcal{P}(A) \in \mathcal{P}^1(\hat{H}; I) = \{ p \in \mathcal{P}(\hat{H}) \mid I(2p - I) + (2p - I)I = 0 \}.
\]

If \( a \in \mathcal{D}(H) \) is self-adjoint, then \( \mathcal{P}(a) \) is Lagrangian with respect to \( I \) (but the converse is no longer true, in contrast with regular operators).

Let us consider the grading symmetry \( J = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \) of \( \hat{H} \). A symmetry \( r \) anticommutes with \( J \) if and only if it has the form \( r = \left( \begin{smallmatrix} 0 & u^t \\ u & 0 \end{smallmatrix} \right) \) for some \( u \in \mathcal{U}(H) \). Therefore, \( \mathcal{P}^1(\hat{H}; J) \) is naturally homeomorphic to \( \mathcal{U}(H) \).

Choose a unitary \( v \in \mathcal{U}(\hat{H}) \) such that \( J = vlv^* \). Then the conjugation by \( v \) takes \( \mathcal{P}^1(\hat{H}; I) \) to \( \mathcal{P}^1(\hat{H}; J) \) and thus determines a homeomorphism

\[
\psi_v : \mathcal{P}^1(\hat{H}; I) \to \mathcal{U}(H).
\]
If both $v, v' \in \mathcal{U}(\hat{H})$ conjugate $I$ with $J$, then $v' = (w \circ w') \cdot v$ for some unitaries $w, w' \in \mathcal{U}(H)$. It follows that

\[(2.5) \quad \psi_{v'}(p) = w' \cdot \psi_v(p) \cdot w^* \quad \text{for every } p \in \mathcal{P}(\hat{H}; I).\]

Conversely, for a fixed $v$, every pair $w, w' \in \mathcal{U}(H)$ gives rise to $v'$ satisfying (2.5). Since the unitary group is path connected, the isotopy class of the homeomorphism $\psi_v$ does not depend on the choice of $v$.

For $v = \frac{1}{\sqrt{2}} \left( \begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix} \right)$ the composition

\[
\mathcal{D}^{sa}(H) \xrightarrow{\hat{p}} \mathcal{P}(\hat{H}; I) \xrightarrow{\psi_v} \mathcal{U}(H)
\]

coinsides with $\tilde{\kappa}$. In particular, $\psi_v(p(A)) = \kappa(A)$ for every $A \in \mathcal{R}^{sa}(H)$. Such a homeomorphism $\psi_v$ takes $p_{\infty}$ to $1 \in \mathcal{U}(H)$ and $p_0$ to $-1 \in \mathcal{U}(H)$ and thus provides homeomorphisms

\[(2.6) \quad \mathcal{P}^1_\kappa = \mathcal{P}(\hat{H}; I) \cap \mathcal{P}_\kappa(\hat{H}) \to \mathcal{U}_\kappa(H) \quad \text{and} \quad \mathcal{P}^1_\tilde{\kappa} = \mathcal{P}(\hat{H}; I) \cap \mathcal{P}_{\tilde{\kappa}}(\hat{H}) \to \mathcal{U}_{\tilde{\kappa}}(H).
\]

The space $\mathcal{P}^1$ is called the Lagrangian Grassmanian, $\mathcal{P}^1_\kappa$ the Fredholm Lagrangian Grassmanian, and $\mathcal{P}^1_{\tilde{\kappa}}$ the restricted Lagrangian Grassmanian.

Composition of $\hat{p}$ with homeomorphisms (2.6) provides the maps $\tilde{\kappa}: \mathcal{D}^{sa}_\kappa(H) \to \mathcal{U}_\kappa(H)$ and $\tilde{\kappa}: \mathcal{D}^{sa}_{\tilde{\kappa}}(H) \to \mathcal{U}_{\tilde{\kappa}}(H)$ discussed above.

### 3 Proof of Theorem [D]: Riesz and norm topology

In this section we prove the first part of Theorem [D] for the case $X = \mathcal{R}^n$, along with its analogue for $X = \mathcal{B}^\ast, \mathcal{D}^\ast$, and $\mathcal{U}$ that will be used in Section [6].

**Canonical decomposition.** Recall a standard construction that we will use throughout the paper. The trivial Hilbert bundle over $\mathcal{P}(H)$ with the fiber $H$ is canonically decomposed into the direct sum

\[(3.1) \quad H \times \mathcal{P}(H) = \mathcal{H}' \oplus \mathcal{H}''\]

of two vector bundles, whose fibers are $\mathcal{H}'_p = \text{Im}(p)$ and $\mathcal{H}''_p = \text{Ker}(p)$. This decomposition is locally trivial in the following sense: for every $p_0 \in \mathcal{P}(H)$ there is a continuous map $g: W \to \mathcal{U}(H)$, where $W$ is the open ball of radius $1$ around $p_0$, such that $g_p \circ p_{p_0} = p_0$ for every $p \in W$ [WO, Proposition 5.2.6].

Let $\mathcal{P}^\ast(H)$ be the subspace of $\mathcal{P}(H)$ consisting of projections of infinite rank and corank. The restrictions of $\mathcal{H}'$ and $\mathcal{H}''$ to $\mathcal{P}^\ast(H)$ are locally trivial bundles over a paracompact space, and their structure group is the unitary group $\mathcal{U}$ with the norm topology, which is contractible by Kuiper’s theorem [Ku]. Thus these restrictions are
trivial as Hilbert bundles with the structure group $U$. This can be stated as follows: there is a map

\[(3.2) \quad g: \mathcal{P}^*(H) \to \mathcal{U}(H) \quad \text{such that} \quad g_p g^*_p \equiv p_o \quad \text{for} \quad p \in \mathcal{P}^*(H),\]

where $p_o \in \mathcal{P}^*(H)$ is some fixed projection.

Riesz topology. The following proposition proves the first part of Theorem [D] for the case $X = \mathcal{R}^+$. 

**Proposition 3.1.** The spaces $\mathcal{R}^+[-\lambda, \lambda]$ and $\mathcal{R}^+_K[-\lambda, \lambda]$ are contractible for every $\lambda \geq 0$.

**Proof.** Let $\mathcal{P}^*(H)$ be as above and $g$ satisfies (3.2). The positive spectral projection $1_{[0, +\infty)}$ provides the structure of a fiber bundle over $\mathcal{P}^*(H)$ to both $\mathcal{R}^+[-\lambda, \lambda]$ and $\mathcal{R}^+_K[-\lambda, \lambda]$. These bundles are trivial, with trivialization maps $\mathcal{R}^+[-\lambda, \lambda] \to F \times \mathcal{P}^*$ and $\mathcal{R}^+_K[-\lambda, \lambda] \to F_K \times \mathcal{P}^*$ given by the formula

\[A \mapsto (g_p A g^*_p, p), \quad p = 1_{(0, +\infty)}(A),\]

and with the fibers

\[F = \{ A \in \mathcal{R}^+[-\lambda, \lambda] \mid 1_{[0, +\infty)}(A) = p_o \} = \{ A \in \mathcal{R}^sa(H') \mid A > \lambda \} \times \{ -A \in \mathcal{R}^sa(H'') \mid A > \lambda \},\]

\[F_K = F \cap F_K(H) = \{ A \in \mathcal{R}^sa(H') \mid A > \lambda \} \times \{ -A \in \mathcal{R}^sa(H'') \mid A > \lambda \}\]

equipped with the Riesz topology. Here $H' = \text{Im}(p_o)$, $H'' = \text{Ker}(p_o)$, and by $A > \lambda$ we mean that the spectrum of $A$ is contained in the open ray $(\lambda, +\infty)$.

By Proposition [2.2] the bounded transform $\chi$ provides a convex structure on $F$ and $F_K$, so both $F$ and $F_K$ are contractible. The base space $\mathcal{P}^*(H)$ is contractible as well, see [AS] proof of Lemma 3.6]. Therefore, the total spaces $\mathcal{R}^+[-\lambda, \lambda]$ and $\mathcal{R}^+_K[-\lambda, \lambda]$ are also contractible. $\Box$

Norm topology. In the proof of Theorems [B] and [C] we will need the following analogue of Theorem [D] for bounded operators.

**Proposition 3.2.** The following spaces are contractible in the norm topology for every $\lambda \geq 0$:

- $\mathcal{B}^+[-\lambda, \lambda] = \{ A \in \mathcal{B}^*(H) \mid \text{Spec}(A) \cap [-\lambda, \lambda] = \emptyset \}$,
- $\mathcal{D}^+[-\lambda, \lambda] = \{ A \in \mathcal{D}^*(H) \mid \text{Spec}(A) \cap \chi([-\lambda, \lambda]) = \emptyset \}$,
- $\mathcal{U}[-\lambda, \lambda] = \{ A \in \mathcal{U}(H) \mid \text{Spec}(A) \cap \kappa([-\lambda, \lambda]) = \emptyset \}$,
- $\mathcal{D}^+_K[-\lambda, \lambda] = \mathcal{D}_K(H) \cap \mathcal{D}^+[-\lambda, \lambda]$,
- $\mathcal{U}_K[-\lambda, \lambda] = \mathcal{U}_K(H) \cap \mathcal{U}[-\lambda, \lambda]$.

**Proof.** For $\mathcal{B}^+[-\lambda, \lambda]$, $\mathcal{D}^+[-\lambda, \lambda]$, and $\mathcal{D}^+_K[-\lambda, \lambda]$ the contractibility is proved in exactly the same manner as in Proposition [3.1] with the only difference that we have a convex structure on the fibers from the start, without applying the bounded transform.

A deformation retraction of $\mathcal{U}[-\lambda, \lambda]$ to $\{1\}$ is given by the homotopy

\[h: \mathcal{U}[-\lambda, \lambda] \times [0, 1] \to \mathcal{U}[-\lambda, \lambda], \quad h_t(u) = \exp(t \log(u)),\]

where $\log: \{ z \in \mathbb{C}: |z| = 1, z \neq -1 \} \to (-i\pi, i\pi) \subset i\mathbb{R}$ is the branch of the natural logarithm. It preserves $\mathcal{U}_K[-\lambda, \lambda]$ and thus defines a contraction of $\mathcal{U}_K[-\lambda, \lambda]$ to $\{1\}$ as well. $\Box$
4 Proof of Theorem \( \square \): graph topology

This section is devoted to the proof of the first part of Theorem \( \square \) for spaces \( X \) from the following list:

\[
\begin{align*}
g\mathcal{R}^{\text{sa}}, \ g\mathcal{R}^{-}, & \ g\mathcal{R}^{+}, \ g\mathcal{R}^{*}.
\end{align*}
\]

We complete this task in the end of the section, see Propositions 4.4 and 4.5.

**Invertible self-adjoint operators.** Consider first the case \( \lambda = 0 \). Then \( X[-\lambda, \lambda] = X \cap \mathcal{R}_{\text{inv}}(H) \), where \( \mathcal{R}_{\text{inv}}(H) \) denotes the subset of \( \mathcal{R}(H) \) consisting of invertible operators.

The goal of this subsection is to prove contractibility in the graph topology of the space \( \mathcal{R}_{\text{inv}}^{\text{sa}}(H) \) of invertible self-adjoint regular operators, as well as its subspace \( \mathcal{R}_{\text{K, inv}}^{\text{sa}}(H) \) consisting of operators with compact resolvent.

This result may be seen as a combination of two facts:

- contractibility of the space \( \mathcal{P}(H) \) of projections in the strong topology (which can be proven in exactly the same way as contractibility of the unitary group \( \mathcal{U}(H) \) in the strong topology [DD, Proposition 3]), and
- the fact that the positive spectral projection \( 1_{[0, +\infty)} \) provides homotopy equivalences between \( \mathcal{R}_{\text{K, inv}}^{\text{sa}}(H) \) and \( \mathcal{R}_{\text{inv}}^{\text{sa}}(H) \) with the graph topology, on one side, and the space \( \mathcal{P}(H) \) of projections with the strong topology, on the other side.

We choose a different way of proof, incorporating the method of the proof of [DD, Proposition 3] directly into our proof of contractibility of \( \mathcal{R}_{\text{K, inv}}^{\text{sa}}(H) \) and bypassing the use of \( \mathcal{P}(H) \).

**Proposition 4.1.** The map \( \mu: g\mathcal{R}_{\text{inv}}^{\text{sa}}(H) \to \mathcal{B}(H), A \mapsto A^{-1} \), is continuous. It provides homeomorphisms

\[
g\mathcal{R}_{\text{inv}}^{\text{sa}}(H) \to Z := \{ a \in \mathcal{B}(H) \mid a \text{ is injective} \} \quad \text{and} \quad g\mathcal{R}_{\text{K, inv}}^{\text{sa}}(H) \to Z_{\mathcal{K}} := Z \cap \mathcal{K}(H),
\]

and takes \( \mathcal{R}_{\text{sa}}^{\text{sa}}[-\lambda, \lambda] \) onto \( \{ a \in Z \mid \| a \| < \lambda^{-1} \} \).

**Proof.** Clearly, \( A^{-1} \) is injective. For every \( a \in Z \), the range of \( a \) is dense and the graph of \( a \) is closed, so \( a^{-1} \in \mathcal{R}_{\text{inv}}^{\text{sa}}(H) \). The bounded linear transformation \( (x, y) \mapsto (y, x) \) of \( H \oplus H \) takes the graph of \( A \) to the graph of \( A^{-1} \), so \( \mu \) is graph continuous. Since the graph topology on \( \mathcal{B}(H) \) coincides with the norm topology, \( \mu: g\mathcal{R}_{\text{inv}}^{\text{sa}}(H) \to Z \) is a homeomorphism. The rest of the proposition is obvious. \( \square \)

**Proposition 4.2.** The space \( Z_{\mathcal{K}} \) is contractible, so \( \mathcal{R}_{\text{K, inv}}^{\text{sa}}(H) \) is contractible in the graph topology.

**Proof.** Let \( (u_t)_{t \in [0, 1]} \) and \( (v_t)_{t \in [0, 1]} \) be two families of isometries on \( H \) such that \( u_t, u_t^*, v_t, \) and \( v_t^* \) continuously depend on \( t \) in the strong operator topology,

\[
\begin{align*}
p_t &= u_t u_t^* \to 0 \text{ strongly as } t \to 0, \\
q_t &= v_t v_t^* \to 0 \text{ strongly as } t \to 1, \\
u_1 &= v_0 = 1, \quad p_t + q_t = 1.
\end{align*}
\]
To construct such a pair, one can identify $H$ with $L^2[0, 1]$ and take
\[
\begin{align*}
    u_t(f)(s) = \begin{cases} 
        \frac{1}{\sqrt{t}} f\left(\frac{s}{t}\right) & \text{for } s \leq t \\
        0 & \text{for } s > t 
    \end{cases} \\
    v_t(f)(s) = \begin{cases} 
        0 & \text{for } s \leq t \\
        \frac{1}{\sqrt{1-t}} f\left(\frac{s-1}{1-t}\right) & \text{for } s > t 
    \end{cases}
\end{align*}
\]
Then $p_t$ and $q_t$ are the projections onto subspaces $L^2[0, t]$ and $L^2[t, 1]$, respectively. (In terms of Proposition 2, $u_t^* = U_tP_t$.)

Fix $b \in Z_K$. We define a deformation retraction of $Z_K$ to $\{b\}$, $h: [0, 1] \times Z_K \to Z_K$, by the formulae
\[
(4.2) \quad h_0(a) = a, \quad h_1(a) = b, \quad h_t(a) = tu_tau_t^* + (1-t)v_tv_t^* \quad \text{for } 0 < t < 1.
\]

Note that the sum in the last expression is the direct sum corresponding to the orthogonal decomposition $H = H'_t \oplus H''_t$, where $H'_t$ is the range of $p_t$ and $H''_t$ is the range of $q_t$. For $a$ injective, both $h'_t(a) = tu_tau_t^*$ and $h''_t = (1-t)v_tv_t^*$ are injective as operators on $H'_t$ and $H''_t$, respectively. Therefore, $h_t$ preserves the subspace of injective operators. Similarly, $h_t$ preserves the subspace of compact operators, so $h$ takes $[0, 1] \times Z_K$ to $Z_K$.

Since $a, b$ are compact and $u_t, v_t$ are uniformly bounded, the maps $h'$ and $h''$ are norm continuous on $[0, 1] \times Z_K$ and $[0, 1]$, respectively. In addition, $\|h'_t(a)\| \leq t \|a\| \to o$ as $t \to o$, and similarly $\|h''_t\| \leq (1-t) \|b\| \to o$ as $t \to 1$. Therefore, $h$ is norm continuous on the whole domain $[0, 1] \times Z_K$ and thus defines a deformation retraction of $Z_K$ to $\{b\}$.

The second part of the proposition follows from the first part and Proposition 4.1. The corresponding deformation retraction $H$ of $g_{R^sa_{K, \text{inv}}}$ to an arbitrary point $B \in R^sa_{K, \text{inv}}$ is given by the formula
\[
(4.3) \quad H_0(A) = A, \quad H_1(A) = B, \quad H_t(A) = t^{-1}u_tA u_t^* + (1-t)^{-1}v_tBv_t^* \quad \text{for } 0 < t < 1.
\]

This completes the proof of the proposition. $\square$

**Proposition 4.3.** The space $Z$ is contractible, so $R^sa_{\text{inv}}(H)$ is contractible in the graph topology.

 Cf. [Le], Proposition 5.8, where the path connectedness of $g_{R^sa_{\text{inv}}}(H)$ is shown in a completely different manner.

**Proof.** Let $k \in K(H)$ be a positive compact injective operator. Then
\[
(4.4) \quad c_t = (1-t) + tk
\]
is also a positive injective operator for all $t \in [0, 1]$. The formula $h'_t(a) = c_tac_t$ defines the norm continuous map $h': [0, 1] \times Z \to Z$ such that $h'_0 = \text{Id}$ and $h'_t(Z) \subset Z_K$. The corresponding deformation retraction $H'$ of $g_{R^sa_{\text{inv}}}$ to a subspace of $g_{R^sa_{K, \text{inv}}}$ is given by the formula
\[
(4.5) \quad H'_t(A) = C_tC_{t}AC_t, \text{ where } C_t = (c_t)^{-1}.
\]
It follows from Proposition 4.2 that both $Z$ and $g_{R^sa_{\text{inv}}}$ are contractible. $\square$

**Contractibility of $X_K[-\lambda, \lambda]$ and $X[-\lambda, \lambda]$.** Now we are ready to fulfill the main goal of this section.
Proposition 4.4. The space \( X_K[-\lambda, \lambda] \) is contractible for every \( \lambda \geq 0 \) and every space \( X \) from the list (4.1).

**Proof.** Choose \( B \in X_K[-\lambda, \lambda] \); then the homotopy \( H_t \) defined by (4.3) preserves \( X[-\lambda, \lambda] \) and defines a deformation retraction of \( X[-\lambda, \lambda] \) to the singleton \( \{B\} \). Indeed, for every \( A, B \in X_K[-\lambda, \lambda] \) and \( t \in (0, 1) \) we have \( t^{-1}u_tAu_t^* \in X_K(\text{Im} P_t)[-\lambda, \lambda] \), \((1 - t)^{-1}v_tBv_t^* \in X_K(\text{Im} Q_t)[-\lambda, \lambda] \), and thus \( H_t(A) \in X_K[-\lambda, \lambda] \). □

Proposition 4.5. The space \( X[-\lambda, \lambda] \) is contractible for every \( \lambda \geq 0 \) and every space \( X \) from the list (4.1).

**Proof.** Choose a positive injective compact operator \( k \) of norm \( < 1 \). We will show that, for such a \( k \), the homotopy \( H'_t \) defined by (4.5) preserves \( X[-\lambda, \lambda] \) and thus provides a deformation retraction of \( X[-\lambda, \lambda] \) to a subspace of \( X_K[-\lambda, \lambda] \). Contractibility of \( X[-\lambda, \lambda] \) then follows from Proposition (4.4).

Consider first the case \( \lambda = 0 \). Let \( \pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H) \) be the natural projection to the Calkin algebra. If \( A \in \mathcal{R}^{sa}_{inv} \), then \( a = A^{-1} \) is essentially positive, \( \pi(a) \geq 0 \), \( \pi(c_1)t\pi(a)t(c_1) \geq 0 \), so \( c_1ac_1 \) is essentially positive and \( H'_t(A) \in \mathcal{R}^{sa}_{inv} \). By the same reasoning, \( H'_t \) preserves \( \mathcal{R}^{sa}_{inv} \) and \( \mathcal{R}^+ \).

If \( \lambda > 0 \), then \( \|k\| < 1 \) implies \( \|c_1ac_1\| \leq \|a\| < \lambda^{-1} \) for every \( a \) of norm less than \( \lambda^{-1} \), and thus \( H'_t(A) \in \mathcal{R}^{sa}[-\lambda, \lambda] \) for every \( A \in \mathcal{R}^{sa}[-\lambda, \lambda] \). Therefore, \( H'_t \) preserves \( \mathcal{X}_{inv} \cap \mathcal{R}^{sa}[-\lambda, \lambda] = X[-\lambda, \lambda] \). □

## 5 Proof of Theorem [D]: odd case

In this section we prove the second part of Theorem [D] along with its analogues for bounded operators, that we will need in the proof of Theorem A.

We will use the standard embedding

\[
\tau_R : \mathcal{R}(H) \hookrightarrow \mathcal{R}^+(\hat{H}), \quad A \mapsto \hat{A} = \left( \begin{array}{cc} A^* & 0 \\ 0 & A \end{array} \right).
\]

This embedding is a homeomorphism (both in the Riesz and the graph topology on these two spaces) of \( \mathcal{R}(H) \) onto the subspace \( \mathcal{R}^+(\hat{H}) \) of \( \mathcal{R}^{sa}(\hat{H}) \) consisting of odd operators,

\[
\mathcal{R}^+(\hat{H}) = \{ A \in \mathcal{R}^{sa}(\hat{H}) | JA + JA = 0 \}, \quad \text{where} \quad J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)
\]

is the grading symmetry of \( \hat{H} \). The map \( A \mapsto \hat{A} \) takes \( \mathcal{B}(H) \) onto \( \mathcal{B}^1(\hat{H}) \) and \( \mathcal{D}(H) \) onto \( \mathcal{D}^1(\hat{H}) \), where \( \mathcal{B}^1(\hat{H}) \) and \( \mathcal{D}^1(\hat{H}) \) are subspaces of odd operators in \( \mathcal{B}^{sa}(\hat{H}) \) and \( \mathcal{D}^{sa}(\hat{H}) \) respectively.

We will keep designations of Proposition [3.2] with \( H \) replaced by \( \hat{H} \).

**Proposition 5.1.** Let \( X \) be one of the spaces \( \mathcal{R}^{sa}(\hat{H}), \mathcal{B}^{sa}(\hat{H}), \mathcal{D}^{sa}(\hat{H}) \). Then the subspace \( X^1[-\lambda, \lambda] \) consisting of odd operators is contractible for every \( \lambda \geq 0 \). If \( X \) is one of the spaces \( \mathcal{R}^{sa}(\hat{H}), \mathcal{B}^{sa}(\hat{H}), \mathcal{D}^{sa}(\hat{H}) \), then \( X_K[-\lambda, \lambda] = X_K \cap X^1[-\lambda, \lambda] \) is also contractible for every \( \lambda \geq 0 \).
Proof. For $X = \mathbb{R}^1(\hat{\mathbb{H}})$, $\mathcal{B}^1(\hat{\mathbb{H}})$, or $\mathcal{D}^1(\hat{\mathbb{H}})$ the proof is completely similar to the proof of Proposition 3.4. First note that $\mathcal{R}^1(\hat{\mathbb{H}}) \subset \mathcal{R}^*(\hat{\mathbb{H}})$, $\mathcal{B}^1(\hat{\mathbb{H}}) \subset \mathcal{B}^*(\hat{\mathbb{H}})$, and $\mathcal{D}^1(\hat{\mathbb{H}}) \subset \mathcal{D}^*(\hat{\mathbb{H}})$. The positive spectral projection $1_{[0, +\infty)}$ provides $X^1[-\lambda, \lambda]$ and $X^\kappa_k[-\lambda, \lambda]$ with the structure of a trivial fiber bundle over

$$
\mathcal{P}^1(\hat{\mathbb{H}}; \mathcal{F}) = \left\{ p \in \mathcal{P}(\hat{\mathbb{H}}) \mid J(2p - 1) + (2p - 1)J = 0 \right\}.
$$

Their fibers are affine subspaces of convex spaces and thus also convex and contractible. The base space $\mathcal{P}^1(\hat{\mathbb{H}}; \mathcal{F})$ is homeomorphic to $\mathcal{U}(\hat{\mathbb{H}})$ (see Section 2) and thus contractible by Kuiper’s theorem. It follows that $X^1[-\lambda, \lambda]$ and $X^\kappa_k[-\lambda, \lambda]$ are also contractible.

For $X = g^\mathbb{R}^1(\hat{\mathbb{H}})$ we use the homotopies constructed in Sections 3 and 4 with their parameters chosen appropriately. In the proofs of Propositions 4.2 and 4.4 we take $u_t \oplus v_t$ instead of $u_t$, $v_t \oplus v_t$ instead of $v_t$, and choose an operator $B \in X^1[-\lambda, \lambda]$. In the proofs of Propositions 4.3 and 4.5 we take $k \oplus k$ instead of $k$. Then these homotopies preserve the subspaces of odd operators and thus provide contractibility of $g^\mathbb{R}^1[-\lambda, \lambda]$ and $g^\mathbb{R}^\kappa_k[-\lambda, \lambda]$.

Unitary operators. For the unitary group, the role of the subspace of odd operators plays the space

$$
\mathcal{U}^1 = \left\{ u \in \mathcal{U}(\hat{\mathbb{H}}) \mid Ju = u^* \right\}.
$$

The maps $u \mapsto Ju$ and $u \mapsto uJ$ are both homeomorphisms from $\mathcal{U}^1$ to the space of symmetries in $\hat{\mathbb{H}}$.

**Proposition 5.2.** The subspaces $\mathcal{U}^1[-\lambda, \lambda] = \mathcal{U}^1 \cap \mathcal{U}[-\lambda, \lambda]$ and $\mathcal{U}^\kappa_k[-\lambda, \lambda] = \mathcal{U}^1 \cap \mathcal{U}^\kappa_k[-\lambda, \lambda]$ of $\mathcal{U}(\hat{\mathbb{H}})$ are both contractible for every $\lambda \geq 0$.

**Proof.** The deformation retraction of $\mathcal{U}[-\lambda, \lambda]$ to a point used in the proof of Proposition 3.2 preserves the subspaces $\mathcal{U}^1[-\lambda, \lambda]$ and $\mathcal{U}^\kappa_k[-\lambda, \lambda]$, so these subspaces are also contractible.

6 Proof of Theorems $\mathcal{B}$ and $\mathcal{C}$

This section is devoted to the proof of Theorems $\mathcal{B}$ and the following stronger version of Theorem $\mathcal{C}$ (it contains Theorem $\mathcal{C}$ as a part).

**Theorem $\mathcal{C}$.** All the maps on Diagram (6.1) below are homotopy equivalences. Consequently, all the spaces on the diagram are classifying spaces for the functor $K^1$.

$$
\begin{align*}
\mathcal{B}^*_{\mathcal{F}} & \xrightarrow{\mathcal{R}_{\mathcal{F}}^*} \mathcal{R}_{\mathcal{F}}^* & \xrightarrow{\mathcal{R}_{\mathcal{F}}^*} \mathcal{D}_{\mathcal{F}}^* & \xrightarrow{\mathcal{D}_{\mathcal{F}}^*} \mathcal{U}_{\mathcal{F}}^* \\
\mathcal{K} & \xrightarrow{\mathcal{D}_{\mathcal{F}}^*} \mathcal{D}_{\mathcal{F}}^* & \xrightarrow{\mathcal{D}_{\mathcal{F}}^*} \mathcal{U}_{\mathcal{F}}^* & \xrightarrow{\mathcal{D}_{\mathcal{F}}^*} \mathcal{U}_{\mathcal{F}}^*
\end{align*}
$$

(6.1)
Here $\tilde{\kappa} : \mathcal{D}^{sa}(H) \to \mathcal{U}(H)$ is the map defined by formula (2.4). As was explained in Section 2 it takes $\mathcal{D}^{sa}_K$ to $\mathcal{U}_K$ and $\mathcal{D}^{sa}_F$ to $\mathcal{U}_F$.

**Plan of the proof.** Recall first that the space $\mathcal{U}_K(H)$ is well known to be a classifying space for the functor $K^1$. Therefore, the last statement of Theorem C' follows from the first one.

The proof of Theorem B and the first part of Theorem C' follows the idea described in the Introduction. We handle all the maps on Diagrams (1.3) and (6.1) at once.

Let $\mathcal{T}$ to be the set of all symmetric (with respect to zero) finite non-empty subsets of $\mathbb{R}$. The finite union of elements of $\mathcal{T}$ is again an element of $\mathcal{T}$. For every space $X$ on Diagrams (1.3) and (6.1) we will define an open covering $(X_\tau)_\tau \in \mathcal{T}$ of $X$ indexed by $\mathcal{T}$, so that

$$X_\tau \cap X_{\tau'} = X_{\tau \cup \tau'}.$$  

(6.2) We will show that these coverings satisfy the following two properties for every $\tau \in \mathcal{T}$ and every arrow $\varphi : X \to Y$ on the diagrams:

(i) $\varphi$ takes $X_\tau$ to $Y_\tau$.

(ii) The restriction $\varphi_{\tau} : X_\tau \to Y_\tau$ of $\varphi$ is a homotopy equivalence.

Since every space on the diagrams is metric and thus paracompact, Theorems B and C' will follow immediately from these two properties and [Di, Theorem 1].

**Construction of coverings.** First notice that every space on Diagrams (1.3) and (6.1) arises as a subspace of one of the spaces on Diagram (1.4). The spaces on Diagram (1.4) are divided naturally into the three groups:

(R) $\mathcal{B}^{sa}(H)$, $\mathcal{R}^{sa}(H)$ and $\mathcal{gR}^{sa}(H)$ on the top.

(D) The unit ball $\mathcal{D}^{sa}(H)$ on the bottom left.

(U) The unitary group $\mathcal{U}(H)$ on the bottom right.

This determines division of the spaces on Diagrams (1.3) and (6.1) into the three groups:

(R) Subspaces of $\mathcal{B}^{sa}(H)$, $\mathcal{R}^{sa}(H)$ and $\mathcal{gR}^{sa}(H)$; these are the spaces on the top part of (6.1) and all the spaces of (1.3).

(D) Subspaces of $\mathcal{D}^{sa}(H)$; these are the spaces on the left bottom part of (6.1).

(U) Subspaces of $\mathcal{U}(H)$; these are the spaces on the right bottom part of (6.1).

The arrows inside each group are natural inclusions. The arrows from a space of type (R) to a space of type (D) are given by the bounded transfrom $\chi$, from (R) to (U) by the Cayley transform $\kappa$, and from (D) to (U) by $\tilde{\kappa}$.

Let $\bar{\tau}$ denote the convex hull of $\tau \in \mathcal{T}$; it is a closed symmetric interval in $\mathbb{R}$. In order to simplify notations, we will use the symbols $\sigma$ and $\bar{\sigma}$ as replacements for:

- $\tau$ and $\bar{\tau}$ if we deal with a space of type (R),
- $\chi(\tau)$ and $\chi(\bar{\tau})$ if we deal with a space of type (D),
- $\kappa(\tau)$ and $\kappa(\bar{\tau})$ if we deal with a space of type (U).
We will also consider the spaces on the diagrams as depending on a Hilbert space \( H \); we will write \( X(H) \) when the argument \( H \) is not fixed.

If \( X \) is one of the spaces from Diagram (1.3) or (6.1), then we define \( X_\tau \) by the formula

\[
X_\tau = \{ A \in X | \text{Spec}(A) \cap \sigma = \emptyset = \text{Spec}_{\text{ess}}(A) \cap \sigma \},
\]

where \( \text{Spec}_{\text{ess}}(A) \) is the essential spectrum of \( A \).

**Remark.** Of course, if \( X \) is a subspace of \( \mathcal{P}_K(H) \), then the condition for the essential spectrum in (6.3) is void. We prefer to write the definition of \( X_\tau \) in the same form for all possible spaces \( X \), though, because we will work with all these spaces simultaneously.

**Properties of coverings.** Clearly, \( \cup X_\tau = X \), each \( X_\tau \) is open in \( X \), and \( X_\tau \cap X_{\tau'} = X_{\tau \cup \tau'} \) for every space \( X \) on Diagrams (1.3) and (6.1). Moreover, \( \varphi(X_\tau) \subset Y_\tau \) for every arrow \( \varphi: X \to Y \) on these diagrams. To fulfill the program described in the beginning of the section, it remains to prove the following statement:

**Proposition 6.1.** For every \( \tau \in \mathcal{T} \) and every arrow \( \varphi: X \to Y \) on Diagrams (1.3) and (6.1), the restriction \( \varphi_\tau: X_\tau \to Y_\tau \) of \( \varphi \) is a homotopy equivalence.

The rest of the section is devoted to the proof of this proposition.

**Fiber bundle structure.** Let \( X \) be one of the spaces on the diagrams and \( \tau \in \mathcal{T} \). The spectral projection \( \Lambda \mapsto p = 1_\sigma(\Lambda) \) defines the continuous map from \( X_\tau \) to the subspace \( \mathcal{P}_{\text{fin}} \) of \( \mathcal{P}(H) \) consisting of projections of finite rank. Since the canonical decomposition (3.1) is locally trivial and the action of \( \mathcal{U}(H) \) on \( X \) by conjugations is continuous and preserves \( X_\tau \), the map \( X_\tau \to \mathcal{P}_{\text{fin}} \) is a locally trivial fiber bundle.

Let \( H \times \mathcal{P}_{\text{fin}} = \mathcal{K}' \oplus \mathcal{K}'' \) be the restriction of (3.1) to \( \mathcal{P}_{\text{fin}} \). We define \( X'_\tau \) and \( X''_\tau \) to be the (locally trivial) fiber bundles over \( \mathcal{P}_{\text{fin}} \) associated with \( \mathcal{K}' \), resp. \( \mathcal{K}'' \), whose fibers over \( p \in \mathcal{P}_{\text{fin}} \) are given by the formulae

\[
X'_{\tau,p} = \{ A \in X(\mathcal{K}_p') | \text{Spec}(A) \subset \sigma \setminus \sigma \} \quad \text{and} \quad X''_{\tau,p} = \{ A \in X(\mathcal{K}_p'') | \text{Spec}(A) \cap \sigma = \emptyset \}.
\]

**Lemma 6.2.** The bundle \( X_\tau \to \mathcal{P}_{\text{fin}} \) is naturally decomposed as the fiber product

\[
X_\tau = X'_\tau \times_{\mathcal{P}_{\text{fin}}} X''_\tau,
\]

with the bundle maps \( X_\tau \to X'_\tau \) and \( X_\tau \to X''_\tau \) over \( \mathcal{P}_{\text{fin}} \) given by the formulas \( \Lambda \mapsto \Lambda|_{\mathcal{K}_p'} \) and \( A \mapsto A|_{\mathcal{K}_p''}, \quad p = 1_\sigma(\Lambda) \). The product (6.4) is functorial by \( X \), that is, every arrow \( \varphi: X \to Y \) on Diagrams (1.3) and (6.1) induces a commutative diagram of fiber bundles over \( \mathcal{P}_{\text{fin}} \):

\[
\begin{array}{ccc}
X'_{\tau} & \xrightarrow{\phi'_\tau} & X_{\tau} \\
\Downarrow \phi''_\tau & & \Downarrow \phi_\tau \\
Y'_{\tau} & \xrightarrow{\phi''_\tau} & Y_{\tau}
\end{array}
\]

The induced map \( \phi'_\tau: X'_\tau \to Y'_\tau \) is a bundle isomorphism over \( \mathcal{P}_{\text{fin}} \).
**Proof.** Since all the bundles in (4.4) and (6.5) are locally trivial, we only need to check these statements fiberwise, over each \( p \in \mathcal{P}_{\text{fin}} \). But fiberwise the first two statements are trivial. For the last statement, note that

\[
X'_{T,p} = \begin{cases} 
\{ A \in B^{sa}(\mathcal{H}'_p) | \text{Spec}(A) \subset \bar{\tau} \setminus \tau \} & \text{for } X \text{ of type (R)}, \\
\{ A \in D^{sa}(\mathcal{H}'_p) | \text{Spec}(A) \subset \chi(\tau \setminus \tau) \} & \text{for } X \text{ of type (D)}, \\
\{ A \in U(\mathcal{H}'_p) | \text{Spec}(A) \subset \kappa(\tau \setminus \tau) \} & \text{for } X \text{ of type (U)}.
\end{cases}
\]

If \( X \) and \( Y \) are of the same type, then \( X'_T \) and \( Y'_T \) coincide and \( \varphi'_T \) is the identity. If \( X \) is of type (R) and \( Y \) is of type (D), then \( \varphi = \chi \) and the map \( \chi : X'_T \to Y'_T \) is obviously a homeomorphism. The other two cases, \( \varphi = \kappa \) and \( \varphi = \bar{\kappa} \), are similar. \( \square \)

**Lemma 6.3.** The fiber bundle \( X''_T \to \mathcal{P}_{\text{fin}} \) is trivial (that is, isomorphic to the trivial bundle \( X''_{T,0} \times \mathcal{P}_{\text{fin}} \to \mathcal{P}_{\text{fin}} \)) for every space \( X \) on Diagrams (1.3) and (6.1) and every \( \tau \in \mathcal{T} \).

**Proof.** The locally trivial Hilbert bundle \( \mathcal{H}'' \) over the paracompact space \( \mathcal{P}_{\text{fin}} \) has infinite-dimensional separable fibers and thus is trivial by Kuiper’s theorem. Therefore, the fiber bundle \( X''_T \to \mathcal{P}_{\text{fin}} \) associated with \( \mathcal{H}'' \) is also trivial. \( \square \)

**Proof of Proposition 6.1** Lemmas 6.2 and 6.3 imply that the map \( \varphi_T : X_T \to Y_T \) is the product of \( \varphi'_T : X'_T \to Y'_T \) and \( \varphi''_T : X''_{T,0} \to Y''_{T,0} \). The first factor \( \varphi'_T \) is a homeomorphism. The spaces \( X''_{T,0} = X[-\lambda, \lambda] \) and \( Y''_{T,0} = Y[-\lambda, \lambda] \), where \( [-\lambda, \lambda] = \bar{\lambda} \), are contractible by Propositions 3.2, 3.3, 4.3, and 4.5, and thus the second factor \( X''_{T,0} \to Y''_{T,0} \) is a homotopy equivalence. Therefore, the product \( \varphi_T : X_T \to Y_T \) is also a homotopy equivalence. This completes the proof of the proposition and Theorems B and C’. \( \square \)

## 7 Proof of Theorem A

This section is devoted to the proof of the following theorem, which contains Theorem \( \square \) as a part.

**Theorem A’.** All the maps on the Diagram (7.1) below are homotopy equivalences. Consequently, all the spaces on the diagram are classifying spaces for the functor \( K^0 \).

\[
\begin{array}{cccccc}
B_F & \xrightarrow{rK} & \mathcal{P}_K & \xleftarrow{gK} & \mathcal{P}_K \\
\downarrow{\chi} & & & & \\
\mathcal{D}_F & \xrightarrow{p} & \mathcal{P}_F \\
\end{array}
\]

(7.1)

Here \( p : \mathcal{D}(H) \to \mathcal{P}(H) \) is the map defined by formula (2.3). As was explained in Section 2, it takes \( \mathcal{D}_K \) to \( \mathcal{P}_K \) and \( \mathcal{D}_F \) to \( \mathcal{P}_F \).
Proof. Let us describe briefly the plan of the proof. Let \( D' \) denote Diagram (7.1) and \( D \) denote Diagram (6.1). Consider the diagram \( D(\mathcal{H}) \) corresponding to the \( \mathbb{Z}/2 \)-graded Hilbert space \( \mathcal{H} = H \oplus H \). There is a natural morphism of the diagrams \( \iota: D'(H) \to D(\mathcal{H}) \), which induces an embedding \( \iota: X' \hookrightarrow X \) of each space \( X' \) of \( D'(H) \) to the corresponding space \( X \) of \( D(\mathcal{H}) \). This embedding takes \( X' \) to the subspace \( X^i = \iota(X') \) of \( X \). Theorem A' is equivalent to the following statement: for every arrow \( \varphi: X \to Y \) of \( D(\mathcal{H}) \), its restriction \( \varphi^i: X^i \to Y^i \) is a homotopy equivalence. By [Dil Theorem 1], it is sufficient to show that \( X^i = X^i \cap X_\tau \to Y^i = Y^i \cap Y_\tau \) is a homotopy equivalence for every \( \tau \in \mathcal{T} \). The proof of this mostly follows the proof of Theorem C, and we keep designations of Sections 3 and 6. Once we show that all the maps are homotopy equivalences, the last statement of the theorem follows from a result of Atiyah and Jänich about the homotopy type of \( B_F(H) \) [At Theorem A1].

Let us describe this construction in more detail. We will use the standard embedding \( \iota_2: R(H) \to R^s(\mathcal{H}) \), which is a homeomorphism (both in the Riesz and graph topology on these two spaces) of \( R(H) \) onto the subspace \( R^s(\mathcal{H}) \) of \( R^a(\mathcal{H}) \) consisting of odd operators. The bounded transform commutes with \( \iota_2 \). The same map \( A \mapsto \mathcal{A} \) takes \( B(H) \) onto \( B^1(\mathcal{H}) \) and \( D(H) \) onto \( D^1(\mathcal{H}) \), where \( B^1(\mathcal{H}) \) and \( D^1(\mathcal{H}) \) are the subspaces of odd operators in \( B^*(\mathcal{H}) \) and \( D^*(\mathcal{H}) \) respectively. We will denote the corresponding embedding \( D(H) \hookrightarrow D^*(\mathcal{H}) \) by \( \iota_2 \).

The restriction of \( \kappa: D^a(\mathcal{H}) \to U(\mathcal{H}) \) to \( D^1(\mathcal{H}) \) defines a map
\[
\kappa: D^1(\mathcal{H}) \to U^1(\mathcal{H}), \quad \text{where } U^1(\mathcal{H}) = \{ u \in U(\mathcal{H}) | JuJ = u^* \}.
\]

A homeomorphism \( \iota_\varphi: P(\mathcal{H}) \to U^1(\mathcal{H}) \) making the following diagram commutative:
\[
\begin{array}{ccc}
D(H) & \xrightarrow{\bar{P}} & P(\mathcal{H}) \\
\downarrow_{\iota_2} & & \downarrow_{\iota_\varphi} \\
D^1(\mathcal{H}) & \xrightarrow{\kappa} & U^1(\mathcal{H})
\end{array}
\]
is uniquely defined and given by the formula
\[
(7.2) \quad \iota_\varphi(p) = v(1 - 2p)v, \quad \text{where } v = (1 0) \in U(\mathcal{H}), \quad v^2 = J.
\]

We have \( \iota_\varphi(p_\infty) = 1 \) and \( \iota_\varphi(p_o) = -1 \), so \( \iota_\varphi \) takes \( P_K \) to \( U^1_K = U_K \cap U^1 \) and \( P_F \) to \( U^1_F = U_F \cap U^1 \).

For each space \( X \) of \( D(H) \), let
\[
(7.3) \quad X^i_\tau \to P^i_{\text{fin}} = P^i_{\text{fin}}(\mathcal{H}) \cap P^i(\mathcal{H})
\]
be the restriction of the map \( X_\tau \to P^i_{\text{fin}} \) defined in Section 6 to the subspace \( X^i_\tau \). It is a locally trivial fiber bundle over \( P^i_{\text{fin}} \). The trivial Hilbert bundle \( \mathcal{H} \times P^i_{\text{fin}} \to P^i_{\text{fin}} \) has the canonical decomposition into the direct sum \( \mathcal{H}' \oplus \mathcal{H}'' \), as in (3.1). \( \mathcal{H}' \) is locally trivial and \( \mathcal{H}'' \) is trivial as \( \mathbb{Z}/2\mathbb{Z} \)-graded Hilbert bundles. The fiber bundle (7.3) is decomposed into the fiber product as in (6.4). The map \( \varphi^i: X^i \to Y^i \) respects these
bundle and fiber product structures and induces a bundle isomorphism on the first factor of the fiber product. The second factor is a trivial fiber bundle over $\mathcal{P}_{\text{fin}}$, with the fiber $X^i[-\lambda, \lambda] = X^i \cap X[-\lambda, \lambda]$. It only remains to apply Propositions 5.1 and 5.2 where contractibility of $X^i[-\lambda, \lambda]$ is proven for every $\lambda > 0$, $t \in \mathcal{T}$, and every space $X$ of the diagram $D(H)$. This completes the proof of the theorem. □

8 Essentially positive / negative operators

Recall that a regular self-adjoint operator is called essentially positive (resp. negative) if it is positive (resp. negative) on some invariant subspace of $H$ of finite codimension.

**Theorem 8.1.** Each of the subspaces $\mathcal{R}_c^-(H)$, $\mathcal{R}_c^+(H)$, and $\mathcal{R}_c^0(H)$ is dense in $\mathcal{R}_{\text{fin}}^0(H)$ in the graph topology.

**Proof.** Let $A \in \mathcal{R}^\text{sa}_c(H)$ and $\varepsilon > 0$. Choose $c > 0$ large enough, so that $|\kappa(c) - 1| < \varepsilon/2$. The projection $p = \mathbb{1}_{[-c, c]}(A)$ has finite rank, so the range $H'$ of $1 - p$ is infinite-dimensional. Suppose that the resolvent set of an operator $B \in \mathcal{R}^\text{sa}_c(H')$ contains $[-c, c]$. Then $A' = Ap + B(1 - p)$ is an element of $\mathcal{R}_c^\text{sa}(H)$ and $\|\kappa(A') - \kappa(A)\| < \varepsilon$. Moreover, $A'$ coincides with $B$ on the invariant subspace $H'$ of finite codimension. Choosing such a $B$ from $\mathcal{R}_c^-(H')$, $\mathcal{R}_c^+(H')$, or $\mathcal{R}_c^0(H')$, we get $A' \in \mathcal{R}_c^-(H)$, $\mathcal{R}_c^+(H)$, or $\mathcal{R}_c^0(H)$ respectively. Since $A$ and $\varepsilon$ were chosen arbitrarily, this completes the proof of the theorem. □

**Remark.** The proof of Theorem 8.1 uses essentially the fact that $A$ has compact resolvent. This theorem has no analogue for the space $\mathcal{R}^\text{sa}_c(H)$ of Fredholm operators. This can be explained loosely as follows. Roughly speaking, a self-adjoint operator with compact resolvent has all its essential spectrum at the infinity (that is, its Cayley transform has all its essential spectrum at 1). This part of the spectrum can be smuggled through the infinity from a positive half-neighbourhood of the infinity to a negative one and vice versa. In contrast with that, a Fredholm operator may have a non-empty essential spectrum in a finite part of $R$. When such an operator is slightly deformed, this part of the essential spectrum remains near its original position and thus cannot be moved through the infinity.

**Example 8.2 (Non-zero spectral flow for essentially positive operators).** Let $A = -d^2/dt^2$ be the Laplace operator acting on complex-valued functions $\psi: [0, 1] \to \mathbb{C}$. We consider a family of boundary value problems for $A$ parametrized by points of the real projective line $\mathbb{RP}^1 \cong S^1$. For $\bar{x} = [x_0 : x_1] \in \mathbb{RP}^1$, let $A_{\bar{x}}$ be the operator $A$ with the domain

$$D_{\bar{x}} = \{\psi \in H^2([0, 1]; \mathbb{C}) | \psi(0) = 0, x_0\psi(1) - x_1\psi'(1) = 0\}$$

given by local boundary conditions. Here $\psi' = d\psi/dt$, the type of the boundary condition at $t = 1$ is Dirichlet for $\bar{x} = [1 : 0]$, Neumann for $\bar{x} = [0 : 1]$, and Robin for all the other values of parameter $\bar{x}$.4

4The family $A$ restricted to the affine line $\mathbb{R} = \{[1 : x] \subset \mathbb{RP}^1$ coincides with Rellich’s example [Ka]. Example V-4.14]. It is instructive to look at [Ka], Fig. V-4.1].
Every $A_\tilde{x}$ considered as an unbounded operator on $H = L^2(M, C)$ is a self-adjoint operator with compact resolvent. As we will see below, all $A_\tilde{x}$ are essentially positive.

The family $A = (A_\tilde{x})$ of regular operators on $H$ is graph continuous. Indeed, the map $H^2(M; C) \to C^3$, $\psi \mapsto (\psi(0), \psi(1), \psi'(1))$, is continuous, so the domain $\mathcal{D}_\tilde{x}$ is a closed subspace of $H^2(M; C)$ depending continuously on $\tilde{x}$ in the gap topology on $\text{Gr}(H^2(M; C))$. It follows from $[\text{P}1]$, Proposition A.9 that the map $A : \mathbb{RP}^1 \to \mathcal{K}^a(H)$ is graph continuous.

Let us look at the spectral graph

$$\Gamma = \{ [(\tilde{x}, \lambda) \mid \lambda \in \text{Spec}(A_\tilde{x})] \} \in \mathbb{RP}^1 \times \mathbb{R}$$

of this operator family. $\Gamma$ intersects the level $\lambda = 0$ only at $\tilde{x} = [1 : 1]$; the corresponding eigenfunction is $\psi(t) = t$.

For every $\lambda = -\mu^2 < 0$, the space of solutions of the equation $A\psi = \lambda\psi$ satisfying the first boundary condition $\psi(0) = 0$ is one-dimensional and is spanned by $\psi(t) = e^{\mu t} - e^{-\mu t}$. It satisfies the second boundary condition if and only if

$$x_0(e^{2\mu} - 1) = x_1\mu(e^{2\mu} + 1).$$

(8.2)

For $\lambda$ running the ray $(-\infty, 0)$, $\mu$ runs the positive half of the real axis from $+\infty$ to $0$. For $\lambda$ running the ray $(0, +\infty)$, $\mu$ runs the half of the imaginary axis from $i \cdot 0$ to $i \cdot \infty$. (We can consider only half of the real and imaginary axes, since the values $\mu$ and $-\mu$ give the same eigenspace $\langle \psi \rangle$ of $A_\tilde{x}$.) Equation (8.2) has exactly one solution $\tilde{x} (\mu) \in \mathbb{RP}^1$ for every such value $\mu \neq 0$.

A positive $\lambda = \mu^2 > 0$ is an eigenvalue of $A_\tilde{x}$ for exactly one value of $\tilde{x}$, namely $\tilde{x} = [e^{2i\mu} + 1 : e^{2i\mu} - 1]$; the corresponding eigenfunction is $\psi(t) = e^{\mu t} - e^{-\mu t}$ and has multiplicity 1.

Combining all this together, we see that the part of $\Gamma$ corresponding to positive values of $\lambda$ is an infinite spiral line starting at the point $([1 : 1], 0)$ and making an infinite number of rotations in the upward direction over the circle $\mathbb{RP}^1$. The fiber of $\Gamma$ over $\tilde{x} = [0 : 1]$ is $\{ \pi^2(n + 1/2)^2 \mid n \in \mathbb{Z}, n \geq 0 \}$, while the fiber over $\tilde{x} = [1 : 0]$ is $\{ \pi^2 n^2 \mid n \in \mathbb{Z}, n > 0 \}$. In contrast with that, the projection of the negative part of $\Gamma$ to $\mathbb{RP}^1$ is the interval $\{ [1 : x] \mid x \in (0, 1) \}$; the negative eigenvalue $\lambda$ goes to $-\infty$ as $x$ goes to $+\infty$.

In other words, the graph $\Gamma$ is a single line, which has a vertical asymptotics $\lambda \to -\infty$ for $\tilde{x}$ approaching $[1 : +0]$ and the spiral behavior for positive $\lambda$. The projection to the $\lambda$-axis monotonically increases along the whole line $\Gamma$. Every operator $A_\tilde{x}$ is bounded from below and thus is essentially positive; it has a negative eigenvalue if and only if $\tilde{x} = [1 : x]$ with $x \in (0, 1)$.

This description implies that the spectral flow of the graph continuous loop $A : \mathbb{RP}^1 \to \mathcal{R}^+_K(H)$ is equal to 1. In particular, this loop is not contractible in the graph topology, not only in $\mathcal{R}^+_K(H)$ but also in $\mathcal{R}^a_H$. In contrast with this, every Riesz continuous loop of essentially positive operators is contractible. It follows that our family $A$ is not Riesz continuous.
The Riesz discontinuity of $A$ can be also observed in a more direct way. The negative eigenvalue of $a_{x} = \chi(A_{x})$ goes to $-1$ as $x \to [1 : +0]$. However, $A_{[1:0]}$ and thus $a_{[1:0]}$ is positive. Therefore, $a = f \circ A$ is discontinuous at $x = [1:0]$ and thus $A$ is Riesz discontinuous at the same point. It can be easily seen, for example using [P2 Proposition 13.3], that $A$ is Riesz continuous at the rest of $\mathbb{R}^{1}$.

Let us look more closely at what happens with the eigenfunction when $x \to [1 : +0]$ and $\mu \to +\infty$. The normalized eigenfunction is $\psi(t) = c(e^{i\mu t} - e^{-i\mu t})$, where the constant $c$ is defined by the condition $\|\psi\|_{L^{2}} = 1$. When $\mu \to +\infty$, the asymptotics are $c \approx \sqrt{2\mu}e^{-\mu}$ and $\psi(t) \approx \sqrt{2\mu}e^{\mu(t-1)}$. Therefore, $\psi$ is more and more concentrating near the right end $t = 1$ of the interval $[0, 1]$ with the increasing of $\mu$. In the limit, this eigenfunction collapses to the delta-function $\delta(1)$, which is not an element of $L^{2}[0, 1]$.

### Conventions and notations

Throughout the paper, a “Hilbert space” always means a separable complex Hilbert space of infinite dimension.

- $\mathcal{B}(H)$ is the space of bounded linear operators on a Hilbert space $H$ with the norm topology.
- $\mathcal{K}(H)$ is the subspace of $\mathcal{B}(H)$ consisting of compact operators.
- $\mathcal{P}(H)$ is the subspace of $\mathcal{B}(H)$ consisting of projections (that is, self-adjoint idempotents).
- $\mathcal{U}(H)$ is the subspace of $\mathcal{B}(H)$ consisting of unitary operators.
- $\mathcal{D}(H)$ is the closed unit ball in $\mathcal{B}(H)$.
- $\mathcal{R}(H)$ is the set of regular (that is, closed and densely defined) operators on $H$.

All the subspaces of $\mathcal{B}(H)$ are supposed to be equipped with the norm topology, if the inverse is not stated explicitly. We use the left superscript for pointing to topology on $\mathcal{R}(H)$:

- r – Riesz topology
- g – graph topology

The right sub- and superscript point to the type of a subspace. In particular,

- $\mathcal{R}_{r}(H)$ is the subspace of $\mathcal{R}(H)$ consisting of Fredholm operators.
- $\mathcal{R}_{K}(H)$ is the subspace of $\mathcal{R}(H)$ consisting of operators with compact resolvents.
- $\mathcal{D}_{K}(H)$ is the subspace of $\mathcal{D}(H)$ consisting of essentially unitary operators.
- $\mathcal{B}_{sa}(H)$, $\mathcal{B}_{sa}(H)$, $\mathcal{R}_{sa}(H)$, ... are the subspaces of self-adjoint operators.
- $\mathcal{B}^{1}(H)$, $\mathcal{D}^{1}(H)$, $\mathcal{R}^{1}(H)$ are the subspaces of odd self-adjoint operators in $\hat{H} = H \oplus H$.

### References

[At] M. F. Atiyah. K-theory, Lecture notes by DW Anderson. Harvard, Benjamin, New York (Fall, 1964) (1967).

[AS] M. F. Atiyah and I. M. Singer. Index theory for skew-adjoint Fredholm operators. Publications mathématiques de l’IHÉS, 37 (1969), no.1, 5–26.
[BLP] B. Booss-Bavnbek, M. Lesch, and J. Phillips. Unbounded Fredholm operators and spectral flow. Canadian Journal of Mathematics, 57 (2005), no.2, 225–250; arXiv:math/0108014 [math.FA].

[BJS] U. Bunke, M. Joachim, and S. Stolz. Classifying spaces and spectra representing the K-theory of a graded C*-algebra. High-dimensional manifold topology (2003), 80–102.

[DD] J. Dixmier and A. Douady. Champs continus d’espaces hilbertiens et de C*-algèbres. Bulletin de la Société mathématique de France, 91 (1963), 227–284.

[Di] T. tom Dieck. Partitions of unity in homotopy theory. Compositio Mathematica, 23 (1971), no.2, 159–167.

[Jo] M. Joachim. Unbounded Fredholm operators and K-theory. High-dimensional Manifold Topology, World Sci. Publ., River Edge, NJ (2003) 177–199.

[Ka] T. Kato. Perturbation theory for linear operators. A Series of Comprehensive Studies in Mathematics, 132 (1980).

[Ku] N.H. Kuiper. The homotopy type of the unitary group of Hilbert space. Topology, 3 (1965), no. 1, 19–30.

[Le] M. Lesch. The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators. In: Spectral geometry of manifolds with boundary and decomposition of manifolds (B. Booss-Bavnbek, G. Grubb, and K.P. Wojciechowski, eds.), AMS Contemporary Math Proceedings, 366 (2005), 193–224; arXiv:math/0401411 [math.FA].

[P1] M. Prokhorova. Self-adjoint local boundary problems on compact surfaces. I. Spectral flow. Journal of Geometric Analysis, 31 (2021), no.2, 1510–1554; arXiv:1703.06105 [math.AP].

[P2] M. Prokhorova. Spectral sections. arXiv:2008.04672 [math.SP] (2020), 37 pp.

[WO] N.E. Wegge-Olsen. K-theory and C*-algebras: a Friendly Approach. Vol. 1050. Oxford: Oxford university press, 1993.