Optimal quantum-state tomography with known parameters

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Abstract
It is a well-known fact that the optimal positive operator valued measure (POVM) for quantum-state tomography is the symmetric, informationally complete POVM (SIC-POVM). We investigate the same problem only in the case when there is some a priori information about the state, specifically when some parameters are known. In this paper, we mainly focus on solving a three-dimensional optimization problem, which gives us a non-trivial example for the so-called conditional SIC-POVMs, a straightforward generalization of the concept of SIC-POVMs. We also present other special cases to show further applications of the proposed numerical methods and to illustrate the complexity of this topic.

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(Some figures may appear in colour only in the online journal)

1. Introduction
State estimation is a fundamental problem in the field of quantum information theory and can be considered as one of the foundational issues in quantum mechanics [9, 11]. The problem may be traced back to the 1970s [5], and interest in a thorough mathematical analysis of the quantum tomography procedures has been flourishing recently [2, 10, 17, 19].

In statistics, the accuracy of the estimation can be quantified by the covariance matrix. The matrices are typically not comparable by the positive semi-definiteness; hence, if different estimation schemes are compared, the determinant of the covariance matrix can be used instead. This approach can be found in [13, 14]; their main result was that the complementary von Neumann measurements are optimal. When all parameters of the density matrix are obtained from a single measurement, a symmetric informationally complete positive operator valued measure (SIC-POVM) turns out to be optimal [16]. A similar result was obtained earlier by
Wootters and Fields [21] for von Neumann measurements and by Scott [17] for POVMs but optimality had a different formulation in both cases.

Some \textit{a priori} information about the state can be given in various ways; the most popular subject in this field is state discrimination: when we know that the system is in one of several given states, we should figure out which one it is [3]. Besides knowing the possible states, we can have an \textit{a priori} probability distribution on the true state, too. This idea was used in [4] to obtain the optimal phase estimation. There the given states do not construct a discrete set; instead, they are searching among all pure states. In our setup, we use a similar assumption: we know that the state is in a given subset of the whole state space (some parameters of the state are given) and optimizes a quantity introduced in [15].

The problem is examined thoroughly through a three-dimensional example, but we give results for other examples, too. The optimization is hard to handle analytically; therefore, numerical methods are proposed to find the optimal POVM. We achieved quite fast convergence and can even give analytic solutions of the state-estimation problems. The results suggest a strong pattern for optimal POVMs, which leads us to the generalization of the concept of SIC-POVMs.

The rest of the paper is organized in the following way. In section 2, we give a short overview of the used concepts. Then, in section 3, we define the efficiency of our estimation that we want to minimize. In section 4, we give an algorithm that solves the minimization problem numerically. In section 5, we give the results for various examples. Finally, in section 6, we discuss the results, introduce the definition of conditional SIC-POVMs and draw the conclusions.

2. Basic concepts

In this section, we will give an overview of the concepts used in this paper; for a more detailed description, see [9, 11].

2.1. Quantum states

An $n$-dimensional quantum state can be described using the density matrix $\rho \in M_n(\mathbb{C})$ fulfilling the following conditions:

$$\rho \geq 0 \quad \text{and} \quad \text{Tr} \, \rho = 1.$$ 

Let us denote the $n$-dimensional generalized Pauli matrices with $\sigma_i$, \quad $(i \in \{0, 1, \ldots, n^2 - 1\})$, where $\sigma_i \geq 0$, $\text{Tr}(\sigma_i \sigma_j) = \delta_{i,j}$, i.e. we have an orthonormal basis on the positive matrices and use the abbreviations $\sigma = \{\sigma_i : 1 \leq i \leq n^2 - 1\}$ and $\sigma_0 = I/\sqrt{n}$.

Then, the density matrix $\rho \in M_n(\mathbb{C})$ will have $n^2 - 1$ real parameters, namely the elements of $\theta \in \mathbb{R}^{n^2-1}$, which can be referred to as the generalized Bloch vector:

$$\rho = \frac{I}{n} + \theta \cdot \sigma = \frac{I}{n} + \sum_{i=1}^{n^2-1} \theta_i \sigma_i.$$ 

In our paper, we will use the assumption that there are exactly $N$ unknown parameters (we can suppose that the unknown parameters are $\theta_1, \theta_2, \ldots, \theta_N$) and our aim is to obtain the most efficient estimation of these parameters.
2.2. Quantum measurements

For measurements, we will use a single finite POVM: \( E = \{E_1, E_2, \ldots, E_m\} \), which satisfies the conditions

\[
\sum_{j=1}^{m} E_j = I \quad \text{and} \quad E_j \geq 0.
\]

The first equality shows that the \( E_j \)'s are not completely independent; so, if we want to have \( N \) independent POVM elements to estimate the \( N \) parameters of the state, the POVM \( E \) has to have at least \( m = N + 1 \) elements.

Then, we use a similar parameterization as in the case of the quantum states:

\[
E_1 = a_0^{(1)} (I + a^{(1)} \cdot \sigma), \quad E_2 = a_0^{(2)} (I + a^{(2)} \cdot \sigma), \quad \ldots, \quad E_{N+1} = a_0^{(N+1)} (I + a^{(N+1)} \cdot \sigma),
\]

where \( a_0^{(j)} \in \mathbb{R}, a^{(j)} \in \mathbb{R}^{n^2-1} \).

The positivity conditions for \( E_j \) are \( a_0^{(j)} \geq 0 \) and \( a^{(j)} \in \mathcal{P} \), with \( \mathcal{P} \) denoting the set of \( a \) vectors that satisfy \( I + a \cdot \sigma > 0 \).

Then, the probability of obtaining an outcome related to \( E_j \) is

\[
p_j = \text{Tr}(E_j \rho) = a_0^{(j)} + a_0^{(j)} (a^{(j)} , \theta)
\]

In matrix notation, we have

\[
\begin{bmatrix}
  p_1 \\
  \vdots \\
  p_N
\end{bmatrix} =
\begin{bmatrix}
  a_0^{(1)} \\
  \vdots \\
  a_0^{(N)}
\end{bmatrix} +
\begin{bmatrix}
  \theta_1 \\
  \vdots \\
  \theta_N
\end{bmatrix}
\]

with the matrix

\[
T :=
\begin{bmatrix}
  a_0^{(1)} a_1^{(1)} & \cdots & a_0^{(1)} a_N^{(1)} \\
  \vdots & \cdots & \vdots \\
  a_0^{(N)} a_1^{(N)} & \cdots & a_0^{(N)} a_N^{(N)}
\end{bmatrix}.
\]

2.3. Quantum state estimation and its efficiency

Let us assume that we have many identical copies of \( \rho \) and repeat the previously described measurement on each of them. If \( \nu_1, \ldots, \nu_N \) denote the relative frequencies of the outcomes related to \( E_1, E_2, \ldots, E_N \) respectively, then from (1) we can obtain the state estimate

\[
\begin{bmatrix}
  \hat{\theta}_1 \\
  \vdots \\
  \hat{\theta}_N
\end{bmatrix} = T^{-1} \begin{bmatrix}
  \nu_1 - a_0^{(1)} \\
  \vdots \\
  \nu_N - a_0^{(N)}
\end{bmatrix}.
\]

It is easy to see that this is an unbiased and efficient estimator gained from the measurement outcomes on the unknown parameters of the quantum state.

The covariance matrix of the random variable \( \hat{\theta} \) is

\[
V(\rho) = [E(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j)]_{i,j=1,\ldots,N} = T^{-1} W(T^{-1})^*,
\]

where \( W \) is the covariance matrix of the random variable \( (\nu_1, \ldots, \nu_N) \). If \( M \) is the number of measurements and \( M_j \) is the number of outcomes related to \( E_j \), then \( (\nu_1, \ldots, \nu_N) = \)
and since \((M_1, M_2, \ldots, M_N)\) have a multinomial distribution, we have

\[
W = \frac{1}{r} \begin{bmatrix}
p_1(1 - p_1) & -p_1p_2 & \cdots & -p_1p_N \\
-p_1p_2 & p_2(1 - p_2) & \cdots & -p_2p_N \\
\vdots & \vdots & \ddots & \vdots \\
-p_1p_N & -p_2p_N & \cdots & p_N(1 - p_N)
\end{bmatrix}.
\]

2.4. Complementarity and symmetric measurements

The heuristic concept of complementarity was born together with quantum theory. A mathematical definition is due to Accardi [1] and Kraus [6]. Let \(\mathcal{H}\) be an \(n\)-dimensional Hilbert space. Let the observables \(A\) and \(B\) have the eigenvectors \(e_1, e_2, \ldots, e_n\) and \(f_1, f_2, \ldots, f_m\) that are orthonormal bases. Then, \(A\) and \(B\) are complementary if

\[
|\langle e_i, f_j \rangle|^2 = \frac{1}{n} \quad (1 \leq i, j \leq n).
\]

(3)

If this condition holds, then the two bases are also called mutually unbiased. Complementarity can be generalized to the case of POVMs. The POVMs \(\{E_1, E_2, \ldots, E_k\}\) and \(\{F_1, F_2, \ldots, F_m\}\) are complementary if

\[
\text{Tr} E_i F_j = \frac{1}{n} \text{Tr} E_i \text{Tr} F_j \quad (1 \leq i \leq k, \quad 1 \leq j \leq m).
\]

(4)

This is equivalent to the orthogonality of the traceless parts

\[
E_i = \frac{\text{Tr} E_i}{n} I \quad \perp \quad F_j = \frac{\text{Tr} F_j}{n} I.
\]

So, we will use the expression quasi-orthogonal, when we use this property of complementary operators. An overview of complementarity can be found in [12].

The SIC-POVM is a popular subject in quantum tomography [7, 16, 22]. A SIC-POVM \(\{E_i : 1 \leq i \leq k\}\) of an \(n\)-level system is described by a set of projections \(P_i = |h_i\rangle\langle h_i|\) \((1 \leq i \leq k)\), such that

\[
\sum_{i=1}^{k} P_i = \lambda I \quad \text{and} \quad \text{Tr} P_i P_j = \mu \quad (i \neq j),
\]

(5)

where \(k = n^2\), \(\lambda = n\) and \(\mu = 1/(n + 1)\). The existence of a SIC-POVM is not known for a general dimension \(n\) [18].

3. The optimization problem

To obtain the optimal measurement setup, we need to quantify the error of our estimation and want to minimize it over the all possible POVMs. In section 2.3, we defined an estimator and calculated its covariance matrix, but the problem is that matrices are in general not comparable, so we need a function from \(f : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}\). A common technique is to take the integral with respect to the Haar measure on the unitaries (e.g., [17]); this way we can in some sense symmetrize the result and handle the a priori knowledge about the state. From [15], we can conclude that the best choice of \(f\) is to take the determinant of the covariant matrix, and the correct order is to take the average first and the determinant after that.

Let \(H\) be the set of all possible states with the known parameters of the quantum state. To obtain the average mean quadratic error matrix, we integrate \(V\) with respect to the Haar
measure on the unitaries ($\mu$), but restricted to $H$, in this way we only average the states with the given known parameters. So, the average of the covariance matrix is

$$\int \chi_H(U \rho U^*)V(U \rho U^*)d\mu(U) = T^{-1}W_0(T^*)^{-1},$$

where $\chi$ is the characteristic function, and

$$W_0 = \int \chi_H(U \rho U^*)W(U \rho U^*) d\mu(U).$$

The next step is to take the determinant

$$ DACM = \det(T^{-1}W_0(T^*)^{-1}) = \frac{\det(W_0)}{\det^2(T)}$$

and we want to minimize the determinant of the average covariance matrix (DACM) for all the possible POVMs to obtain the optimal measurement.

The essence of [15] is that this quantity can be applied successfully for different optimization problems. For instance, it is shown for the $n$-dimensional case that if there are no known parameters, i.e. we want a full state estimation, then the optimal POVM is the SIC-POVM. This is not a surprising result; however, the following theorem is non-trivial and contains some interesting questions for the general case.

**Theorem 1.** In the qubit case the optimal POVM for the unknown $\theta_1$ and $\theta_2$ can be described by the projections $P_i$ ($i = 1, 2, 3$):

$$E_i = 2P_i, \quad \sum_{i=1}^3 P_i = \frac{3}{2}I, \quad \text{Tr}\sigma_3 P_i = 0, \quad \text{Tr}P_iP_j = \frac{1}{4} \quad \text{for} \quad i \neq j.$$

So, this result is in some sense the combination of symmetry and complementarity. Both this result and the SIC-POVM case are proven analytically, but interestingly, the latter was more difficult to prove, since the total symmetry made it easier to prove even in the general $n$-dimensional case. But the simplicity of $M_2(\mathbb{C})$ made the calculations feasible; so the question is whether there is a similar object in higher dimensions or not. Actually, even in the three-dimensional case, the precise mathematical argument is rather complicated and so the present approach is based on numerical methods.

### 4. The algorithm

In this section, we will show a method for solving the previously described optimization problem numerically using an example setting. Specifically, let us assume that we have a three-dimensional system, i.e. a qutrit ($n = 3$), and we know the diagonal entries of the density matrix $\rho$.

The first problem that emerges when implementing this minimization problem is the calculation of the objective function (7) for a given POVM. This is difficult because formula (6) is quite abstract, but we can give a good approximation. We parameterize $M_3(\mathbb{C})$ using the Gell-Mann matrices, use a dense enough grid on the parameter space $\mathbb{R}^8$ and check for each grid point whether it is an element of $H$. Actually, the Bloch vector has only six parameters, since the diagonal entries of the states are known. The actual calculation consists simply of checking for all grid points the positive definiteness of the matrix determined by the actual generalized Bloch vector. Then, we cluster the grid points of $H$ according to their eigenvalues: we partition the interval $[0, 1]$, and two states will belong to the same cluster if their eigenvalues are in the same cells. We choose one cluster, which means all the states with the ‘same’ eigenvalues (i.e.
achievable states using unitary transformations), and we take the sum of $W$ in these points. Let us note that we do not use a normalized measure either here or in (6), since it is not necessary: we obtain the same optimization problem up to a constant factor. Another remark is that if we choose a small cluster, the computation will be less precise than for a large one, but much faster.

The second problem is how to select new POVMs to obtain better and better estimations. We choose an arbitrary initial point in the interior of the state space, and in each step, we take a new random POVM by perturbing the parameters using a normal distribution with a given variance. This means that for each $E_j$, ($j = 1, \ldots, 6$), we calculate $\hat{a}^{(j)} = a^{(j)} + N(0, s(t))$, we repeat the random realization of normal vectors for $E_1$, while $\hat{a}^{(1)}$ will determine a positive matrix, and we continue the realization with $E_2$, and so on. The variance of normal distribution $(s(t))$ is decreasing in time; first, we need a larger variance for faster convergence, but near the boundary of the state space of POVMs, we will easily obtain negative eigenvalues if the disturbance is too high. If we have a new Bloch vector for all the six POVM elements, we take all the variation of $a^{(j)}$ and $\hat{a}^{(j)}$ ($\hat{a}^{(j)} \in [a^{(j)}, \hat{a}^{(j)}]$), and we check for all the $2^6 = 64$ cases whether the correlated $E_7 = I - E_1 - E_2 - \ldots - E_6$ will be a physically possible state or not. Then, we go through the valid POVMs and use simulated annealing [8] for this series of POVMs. Let the current best POVM be $E$ and the next in the line to check is $\hat{E}$; then, we change the best POVM to $\hat{E}$ with the probability

$$P(E \rightarrow \hat{E}) = \frac{1}{1 + \exp \left( \frac{\log(DACM(\hat{E})) - \log(DACM(E))}{T} \right)},$$

where $T$ is the so-called temperature. For high temperatures, the probabilities are close to $1/2$ and so the optimal POVM can roam freely, but for low temperatures, we change the current best POVM only if the new POVM is really better. This transition probability determines a special kind of Glauber dynamics; so, there is a good chance that it will converge to the global optimum. The reason why we use simulated annealing instead of simply selecting the best POVM from the line is that otherwise the algorithm tends to set in one direction and it only converges to the boundary of the state space. The simulated annealing is useful here because it can change this path by overcoming potential barriers. Also, we increase the temperature from time to time to help escape from local optima.

5. Results

5.1. The three-dimensional example

A typical result of the previously described algorithm can be seen in figures 1 and 2; the programming was made with Mathematica [20] and the CPU time of implementation is a few minutes. Besides the DACM, we keep track of the following quantities during the optimization:

$$\sigma = \sum_i \text{(second largest eigenvalue of } E_i)$$

$$\delta = \sum_i (\text{Tr}(E_i E_i) - \langle \text{Tr}(E_i E_i) \rangle)^2$$

$$\Delta = \sum_{i \neq j} (\text{Tr}(E_i E_j) - \langle \text{Tr}(E_i E_j) \rangle)^2.$$
We can conclude the following.

- The DACM converges to the same value, independently from the initial state and from the particular realization of the annealing process; hence, we are close to the optimum (figure 1(left)).
- $\sigma$ converges to 0; so, the optimal POVM contains rank-1 projections (figure 1(right)).
- $\delta$ converges to 0; so for the optimal POVM, $\text{Tr}(E_i E_i)$ is a constant (figure 2(left)).
- $\Delta$ converges to 0; so, for the optimal POVM, $\text{Tr}(E_i E_j)$ ($i \neq j$) is a constant (figure 2(right)). This convergence is the slowest, which we can see from the jumps when the algorithm pushes the process to a different path with different angles.

Thus, all the conditions in (5) are fulfilled, but here we have different constants than in the SIC-POVM case. We can also obtain that the optimal POVM is quasi-orthogonal to the diagonal matrices.

5.2. Obtaining an analytic result

From the previous section, we know that the optimal POVM contains rank-1 projectors, so we can use this property to achieve faster convergence. Let us parameterize the POVM in the
following way:

\[ E_i = c|h_i\rangle\langle h_i|, \]

where \( c = 3/7 \), because \( \sum_{i=1}^{7} E_i = I \). So, this way instead of parameterizing positive \( 3 \times 3 \) matrices, we should parameterize three-dimensional complex vectors:

\[ |h_i\rangle = (z_i^{(1)}, z_i^{(2)}, z_i^{(3)}), \]

where \( ||h_i||_2 = 1 \).

On the other hand, we know from the previous section that \( E_i \) is complementary to the diagonal subalgebra, i.e. it will have the same numbers in the diagonal; so, we know that

\[ |z_i^{(1)}|^2 = |z_i^{(2)}|^2 = |z_i^{(3)}|^2 = 1/3, \quad (i = 1, 2, \ldots, 7). \]  \hspace{1cm} (8)

Since \( \Delta \) can be calculated easily directly from vectors \( \langle E_i, E_j \rangle = |\langle h_i, h_j \rangle|^2 \), we can minimize \( \Delta \) numerically, using condition (8). The convergence to zero is very fast, so we can conclude that there exists a POVM that is highly symmetric (satisfies the conditions in (5)) and minimizes (7).

**Example 1.** By fixing some elements of \( |h_i\rangle \)'s, we can obtain the following analytic result for the conditional SIC-POVM:

\[
E_1 = \frac{1}{7} \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix},
E_2 = \frac{1}{7} \begin{bmatrix}
1 & \varepsilon^6 & \varepsilon^2 \\
\varepsilon & 1 & \varepsilon^3 \\
\varepsilon^5 & \varepsilon^4 & 1 \\
\end{bmatrix},
E_3 = \frac{1}{7} \begin{bmatrix}
1 & \varepsilon^2 & \varepsilon^3 \\
\varepsilon^5 & 1 & \varepsilon \\
\varepsilon & \varepsilon^6 & 1 \\
\end{bmatrix},
E_4 = \frac{1}{7} \begin{bmatrix}
1 & \varepsilon^4 & \varepsilon^5 \\
\varepsilon & 1 & \varepsilon^3 \\
\varepsilon^5 & \varepsilon & 1 \\
\end{bmatrix},
E_5 = E_2, E_6 = E_3, E_7 = E_4,
\]

where \( \varepsilon = \exp \left( \frac{2\pi i}{7} \right) \).

5.3. Other examples

We can use the algorithm described in section 4 for other examples of state estimation problems, too.

It is clear that the same method can be used in four dimensions, the main difference is that there will be more grid points, therefore the clustering part will be longer. The second part of the algorithm, i.e. the search of the optimal POVM mainly depends on the number of unknown parameters; so, if we do not have many unknown parameters, the problem is not much more difficult than in the three-dimensional case.

In the four-dimensional case, we can use the orthonormal basis

\[ \left\{ \frac{1}{\sqrt{2}} \sigma_i \otimes \sigma_j \right\}, \quad i, j \in \{0, 1, 2, 3\}, \]

for parameterization, where \( \sigma_i (i \in \{0, 1, 2, 3\}) \) are the two-dimensional Pauli matrices.

**Example 2.** If \( \rho \in M_4(C) \) and we do not know the parameters related to \( \{ I \otimes \sigma_3, \sigma_3 \otimes I, \sigma_1 \otimes \sigma_3 \} \), i.e. we want to estimate the diagonal elements of the density matrix, then the optimal POVM is \( E_i = \{A\}|_{d,i} = \delta_{i,i} \cdot \delta_{i,k} \} \), \( i \in \{1, 2, 3, 4\} \), i.e. the diagonal matrix units (i.e. rank-1 projections).

**Example 3.** If \( \rho \in M_4(C) \) and we do not know the parameters related to \( \{ \sigma_1 \otimes I, \sigma_3 \otimes I, \sigma_3 \otimes \sigma_3 \} \), i.e. we want to estimate \( M_2 \otimes I \), then the optimal POVM is \( E_i = F_i \otimes I \), \( i \in \{1, 2, 3, 4\} \), where \( F_i \)'s are the elements of the two-dimensional SIC-POVM. In this case, \( E_i = 1/2P_i \), where \( P_i \) is a projection of rank 2.
Example 4. If $\rho \in M_4(\mathbb{C})$ and we do not know the parameters related to $\{\sigma_1 \otimes I, \sigma_2 \otimes I, \sigma_3 \otimes I, I \otimes \sigma_1, I \otimes \sigma_2, I \otimes \sigma_3\}$, then the result of the algorithm indicates that the optimal POVM has the following properties: $E_1, E_2, E_3$ are in $M_2 \otimes I$, while $E_4, E_5, E_6$ are in $I \otimes M_2$ and all of them have the eigenvalues: $(\frac{2}{7}, \frac{2}{7}, 0, 0)$, and so they are the multiples of rank-2 projections. But $E_7$ has the eigenvalues $(\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, 0)$, and so the optimal POVM does not contain only multiples of projections, and although one can observe some kind of symmetry, $\text{Tr} E_i E_j$ is not a constant.

6. Conclusions and discussion

The results from theorem 1 and examples 1–3 show us that if some parameters of the quantum states are known, then the optimal POVM for state estimation has some appealing properties:

(i) $E_i = c P_i$, where $P_i$ is projection, $i \in \{1, 2, \ldots , N + 1\}$,
(ii) $\text{Tr} E_i E_j = d$, $i, j \in \{1, 2, \ldots , N + 1\}$ and $i \neq j$,
(iii) $\text{Tr} \sigma_j E_i = 0$, $i \in \{1, 2, \ldots , N + 1\}$, $j \in \{N + 1, N + 2, \ldots , n^2 - 1\}$,

using the abbreviations from section 2. This means that the elements of the POVM are constant multiples of some projectors, symmetrical and complementary to the known part of the state.

We will call the POVMs fulfilling conditions (i)–(iii) the conditional SIC-POVMs, because they are symmetrical and informationally complete with the condition of knowing some parameters of the state. (Hence, we are not interested in distinguishing in those directions.) The conditional SIC-POVM is always defined with respect to the known parameters, since they define the characteristics of the optimal measurements through (iii). For instance, in examples 2 and 3, there are four-dimensional states and the same number of unknown parameters, so one can expect the same result; however, the optimal POVMs contain projections with different ranks (ranks 1 and 2, respectively).

This is a generalization of SIC-POVMs, since we can obtain them as a special case using $N = n^2 - 1$ (i.e. all parameters are unknown) and rank-1 projections. Note that using this information, constants $c$ in (i) and $d$ in (ii) can be determined, while condition (iii) will result in an empty set.

The existence of such measurements is a difficult question even in the unconditional case; the conjecture is that for every dimension there exists a SIC-POVM. The proposed algorithm shows us (example 4) that for the conditional version, the existence is not always true; however, our results suggest that if the conditional SIC-POVM exists, then it is optimal.

We proposed an algorithm (section 4) that can be a useful tool to gain a better understanding of the optimal POVMs. It can be applied to higher dimensions problems, too; the only problem is the increased number of parameters, which will slow down the convergence, making it unable to handle large systems. If we are only interested in the existence of a rank-1 conditional SIC-POVM, we can use the algorithm in section 5.2, which converges much faster.

Finally, we want to note that example 1 shows us that the class of conditional SIC-POVMs is certainly not trivial, and the existence of conditional SIC-POVMs can be a fundamental question in different quantum tomography problems; therefore, further investigations are suggested.

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