On the structure of the set of positive maps

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Abstract
The full description of the set of positive maps \( T : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) (\( \mathcal{A} \) a C*-algebra) is given. The approach is based on the simple prescription for selecting various types of positive maps. This prescription stems from the Grothendieck theory of projective tensor products complemented by the theory of tensor cones. In particular, the origin of non-decomposable maps is clarified.

Keywords Positive maps · Projective tensor product · Tensor cones

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1 Introduction

Despite the facts that positive maps are an essential ingredient in a description of quantum systems and that they play an important role in mathematics [1,2], a characterization of the structure of the set of all positive maps has been a long standing challenge in mathematical physics. The key reason behind that is the complexity of this structure—the structure of positive maps is drastically nontrivial even for the finite dimensional case. To illustrate this point it is enough to note that even the convex structure of the positive maps, \( \Phi_1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \), is highly complicated even in very low dimensions of the Hilbert space \( \mathcal{H} \).

In the sixties it was shown [3] (see also Størmer’s book [2] and references given there) that every positive map for the 2D case, i.e. for \( \dim \mathcal{H} = 2 \), is decomposable. The first example of non-decomposable map was given by Choi [4], see also [5] and [6], for 3D case, i.e. for \( \dim \mathcal{H} = 3 \). Since then, other examples of non-decomposable maps were constructed. In particular, by results of Woronowicz [6] and Størmer [3], if \( \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2 \leq 6 \), all positive maps \( T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2) \) are decomposable but this is not true in higher dimensions. On the other hand, the emergence of non-
decomposable maps may be considered as a huge obstacle in getting a canonical form for a positive map.

It seems that there exist two attempts towards mastering the positive map theory. The first one is examining the structure of positive maps which are defined on infinite dimensional spaces. The strong motivation for such approach is provided by Quantum Theory. In particular, in Quantum Mechanics ‘The observables correspond one-to-one to the self-adjoint operators in a separable infinite dimensional Hilbert space $\mathcal{K}$’… see page 76 in [7]; see also [8,9]. Furthermore, it is worth pointing out that the canonical quantization can be carried out in infinite dimensional Hilbert spaces only, see [10,11].

The aim of the paper is to proceed with the study of positive maps within the above framework. In that way one gets a possibility of discussing positive (so also quantum) maps for a genuine quantum system.

The second attempt is based on finite dimensional structures. It is motivated by matrix algebras as well as by Quantum Information Theory. In that way, the Choi matrix corresponding to a map is very important concept. In particular, this concept reduces the study of a map to that of skillfully defined matrix. The important point to note here is that analogues of Choi matrix were proposed only for special classes of maps on von Neumann algebras see [12,13]. In other words, for general infinite dimensional spaces the concept of Choi matrix is not available. Consequently, to speak about positive maps defined on a infinite dimensional space we will not use the Choi matrix concept in any essential way.

We emphasize that adding the extra assumption that underlying Hilbert spaces are finite dimensional, the results presented in Sect. 3 can be rewritten in terms of dual cones and Choi matrices. In particular, under this additional assumption one can use the theory of dual cones, see Chapter 6 in [2]. Making use of dual cones theory one will get more complete description of cones examined in Sect. 3. In particular, the cone $C_p$ is dual to $C_i$, $C_{cp}$ is self-dual, etc.

The present work being a continuation of our previous papers [14–19] and [20], provides an analysis as well as a description of the structure of the set of positive maps. These results stem from the observation that the linear tensor product structure is not compatible with either topology (there are many cross-norms) or an order (there are many tensor cones).

The deep Grothendieck’s result gives the relation between linear maps on Banach spaces $\mathcal{L}(X, Y)$ and linear continuous functionals on the projective tensor product $(X \otimes_\pi Y_*)$. That is a landmark, a crucial ingredient of our approach. An analysis of various orders in $(X \otimes_\pi Y_*)$ yields essential information about the structure of positive maps. In particular, as a byproduct, we will get an explanation of the origin of non-decomposable maps.

This paper is organized as follows: first we give necessary preliminaries in Sect. 2. Next, in Sect. 3, a description of various types of positive maps will be given. As a result, the structure of the set of all positive maps will be characterized. Conclusions and final remarks are given in Sect. 4.
2 Definitions and notations

For any $C^*$-algebra $\mathfrak{A}$, we denote the set of all self-adjoint (positive) elements of $\mathfrak{A}$ by $\mathfrak{A}_h(\mathfrak{A}^+)$. As $\mathfrak{A}^+$ is a cone in $\mathfrak{A}_h$, $(\mathfrak{A}_h, \mathfrak{A}^+)$ is an ordered Banach space. If $\mathfrak{A}$ is a unital $C^*$-algebra then a state on $\mathfrak{A}$ is a linear functional $\phi: \mathfrak{A} \to \mathbb{C}$ such that $\phi(a) \geq 0$ for every $a \geq 0$ ($a \in \mathfrak{A}^+$) and $\phi(1) = 1$, where $1$ is the unit of $\mathfrak{A}$. The set of all states on $\mathfrak{A}$ will be denoted by $S_{\mathfrak{A}}$.

A linear map $T: \mathfrak{A}_1 \to \mathfrak{A}_2$ between $C^*$-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is called positive if $T(\mathfrak{A}_1^+) \subseteq \mathfrak{A}_2^+$. The set of all linear, bounded (unital) positive maps $T: \mathfrak{A}_1 \to \mathfrak{A}_2$ will be denoted by $L^+_{\mathfrak{A}_1}(\mathfrak{A}_2)$, respectively. For $k \in \mathbb{N}$ we consider a map $T_k : M_k(\mathbb{C}) \otimes \mathfrak{A}_1 \to M_k(\mathbb{C}) \otimes \mathfrak{A}_2$ where $M_k(\mathbb{C})$ denotes the algebra of $k \times k$ matrices with complex entries and $T_k = id_{M_k} \otimes T$. We say that $T$ is $k$-positive if the map $T_k$ is positive. The map $T$ is said to be completely positive (cp for short) if $T$ is $k$-positive for every $k \in \mathbb{N}$.

From now on we make the assumption that $\mathfrak{A}_2$ is equal to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The important class of positive maps is formed by decomposable maps. They are defined as follows. Let $T \in L^+(\mathfrak{A}_1, \mathcal{B}(\mathcal{H}))$. We say that $T$ is co-positive if $t \circ T \equiv T^t$ is completely positive, where $t$ is the transpose map on $\mathcal{B}(\mathcal{H})$ with respect to some orthonormal basis. $T$ is decomposable if $T = T_1 + T_2$ is the sum of a completely positive map $T_1$ and a co-positive map $T_2$. Otherwise, $T$ is called an indecomposable map.

The presented description of $L^+_{\mathfrak{A}_1}(\mathfrak{A}_2)$ is based on the theory of tensor product of Banach algebras. For the convenience of the reader we review some of the standard facts of this theory. Firstly, we note that in his pioneering work on Banach spaces, Grothendieck [21] observed the links between tensor products and mapping spaces. To describe this, we will select certain results from the theory of tensor products of Banach spaces. The point is that the synthesis of the linear structure of tensor products with a topology is not unique - namely, there are many “good” cross-norms (cf [22]) (the same can be said about the synthesis of the linear structure of tensor product and an order, see the next Section). However, among them, there is the projective norm which gives rise to the projective tensor product and this tensor product linearizes bounded bilinear mappings just as the algebraic tensor product linearizes bilinear mappings (see [23]).

Let $X$, $Y$ be Banach algebras. We denote by $X \odot Y$ the algebraic tensor product of $X$ and $Y$ (algebraic tensor product of two $^*$-Banach algebras is defined as tensor product of two vector spaces with $^*$-algebraic structure determined by the two factors; so the topological questions are not considered). The (projective) norm on $X \odot Y$ is defined as

$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$  \hspace{1cm} (2.1)

We denote by $X \otimes_\pi Y$ the completion of $X \odot Y$ with respect to the projective norm $\pi$ and this Banach space will be referred to as the projective tensor product of the Banach
spaces $X$ and $Y$. Denote by $\mathfrak{B}(X \times Y)$ the Banach space of bounded bilinear mappings $B$ from $X \times Y$ into $\mathbb{C}$ with the norm given by $||B|| = \sup\{|B(x, y)|; \|x\| \leq 1, \|y\| \leq 1\}$. Note (for all details see [23]), that with each bounded bilinear form $B \in \mathfrak{B}(X \times Y)$ there is an associated operator $L_B \in \mathfrak{L}(X, Y^*)$ defined by $\langle y, L_B(x) \rangle = B(x, y)$. The important point to note here is that the mapping $B \mapsto L_B$ is an isometric isomorphism between the spaces $\mathfrak{B}(X \times Y)$ and $\mathfrak{L}(X, Y^*)$. Hence, there is an identification

$$
(X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*). \tag{2.2}
$$

We emphasize that the action of an operator $S : X \to Y^*$ as a linear functional on $X \otimes_\pi Y$ is given by

$$
\left\langle \sum_{i=1}^n x_i \otimes y_i, S \right\rangle = \sum_{i=1}^n \langle y_i, Sx_i \rangle. \tag{2.3}
$$

We wish to complete the presented compilation by recalling another of Størmer’s result (see [24]) which will be the crucial in our work. Moreover, it can serve as an illustration on the given material as well as to indicate that relation (2.2) is very relevant to an analysis of positive maps.

Let $\mathfrak{A}$ be a $C^*$-algebra. $\mathfrak{T}$ will denote the set of trace class operators on a Hilbert space $\mathcal{H}$. As before, we denote by $\mathfrak{A} \otimes \mathfrak{T}$ the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{T}$ while $\mathfrak{A} \otimes_\pi \mathfrak{T}$ means its Banach space closure under the projective norm. Now, we can quote (see [24])

**Lemma 2.1** There is an isometric isomorphism $\phi \mapsto \tilde{\phi}$ between $\mathfrak{L}(\mathfrak{A}, \mathfrak{B}(\mathcal{H}))$ and $(\mathfrak{A} \otimes_\pi \mathfrak{T})^*$ given by

$$
(\tilde{\phi}) \left( \sum_{i=1}^n a_i \otimes b_i \right) = \sum_{i=1}^n \text{Tr} (\phi(a_i)b_i^*), \tag{2.4}
$$

where $\sum_{i=1}^n a_i \otimes b_i \in \mathfrak{A} \otimes \mathfrak{T}$.

Furthermore, $\phi \in \mathfrak{L}^+(\mathfrak{A}, \mathfrak{B}(\mathcal{H}))$ if and only if $\tilde{\phi}$ is positive on $\mathfrak{A}^+ \otimes_\pi \mathfrak{T}^+$.

### 3 Structure of $\mathfrak{L}^+(\mathfrak{A}, \mathfrak{B}(\mathcal{H}))$

We begin with some preliminaries on tensor products of ordered Banach spaces. We already note in the previous section that the synthesis of the linear structure of a tensor product with an order is not unique. Following Wittstock [25] we start with,

**Definition 3.1** ([25]) Let $E$, $F$ be ordered linear spaces with proper cones $E_+$, $F_+$. We call a cone $C_\alpha \subset E \otimes F$ a tensor cone and write $E \otimes_\alpha F \equiv (E \otimes F, C_\alpha)$ if the canonical bilinear mappings $\omega : E \times F \to E \otimes_\alpha F$ and $\omega^* : E^* \times F^* \to (E \otimes_\alpha F)^*$ are positive. Thus, $x \otimes y \in C_\alpha$ for all $x \in E_+$, $y \in F_+$; and $x^* \otimes y^* \in C_\alpha^*$ for all $x^* \in E_+^*$, $y^* \in F_+^*$. where $E_+^*$, $F_+^*$, and $C_\alpha^*$ denote the dual cones.

There are two distinguished cones:
Definition 3.2 ([25]) The projective cone $C_p$:

$$C_p = \text{conv}(E_+ \otimes F_+) = \left\{ \sum_{i=1}^n x_i \otimes y_i; x_i \in E_+, y_i \in F_+, n \in \mathbb{N} \right\}$$  (3.1)

and

Definition 3.3 ([25]) The injective cone $C_i$:

$$C_i = \{ t \in E \otimes F; \langle t, E^*_+ \otimes F^*_+ \rangle \geq 0 \}. $$  (3.2)

One has, see Proposition 1.14 in [25]:

Proposition 3.4 ([25]) If $C_\alpha$ is a tensor cone, then

$$C_p \subset C_\alpha \subset C_i.$$  (3.3)

Combining Grothendieck’s result, (2.3), (2.2) and Lemma 2.1 with the concept of tensor cones, we get:

$C_\alpha$-positivity selection rule 3.5

$$P_\alpha = \left\{ T \in \mathcal{L}(\mathfrak{A}, \mathcal{B}(\mathcal{H})); \quad \tilde{T} \left( \sum_{i=1}^n a_i \otimes b_i \right) = \sum_{i=1}^n \text{Tr} T(a_i)b_i^* \geq 0 \right\} ,$$  (3.4)

for any $\sum_{i=1}^n a_i \otimes b_i \in C_\alpha$,

where $C_\alpha$ is a tensor cone in $\mathfrak{A} \otimes \pi \mathcal{F}$.

Remark 3.6 We emphasize that $P_\alpha$ is a result of selecting such linear mappings which have “nice” positive functionals, where positivity is defined by a selected tensor cone. It is worth pointing out that there are, in general, many tensor cones. Consequently, there are many classes of positive maps. Furthermore, Proposition 3.4 implies:

$$P_i \subset P_\alpha \subset P_p.$$  (3.5)

To get a characterization of the structure of $\mathcal{L}^+(\mathfrak{A}, \mathcal{B}(\mathcal{H}))$ we proceed to a description of some selected cones $C_\alpha$ and positive maps in $P_\alpha$ using the $C_\alpha$-positivity selection rule. This will be done in the next subsections.
3.1 Positive maps determined by \( C_p \)

According to the rule given by (3.4) we are interested in the following maps:

\[
P_p = \left\{ T \in \mathcal{L}(\mathfrak{A}, \mathcal{B}(\mathcal{H})); \quad \tilde{T} \left( \sum_{i=1}^{n} a_i \otimes b_i \right) = \text{Tr} T(a_i)b_i^t \geq 0 \right. \\
\left. \quad \text{for any } \sum_{i=1}^{n} a_i \otimes b_i \in C_p \right\}. \tag{3.6}
\]

It is a simple matter to observe that in (3.6) a linear bounded map \( T \) should satisfy:

\[
\text{Tr} T(a)b \geq 0, \tag{3.7}
\]

for any \( a \in \mathfrak{A}_+ \) and \( b \in \mathfrak{T}_+ \). But, it is nothing else but the definition of a positive map. Consequently, the smallest tensor cone \( C_p \) defines the largest class of positive maps - just the set of all positive maps. We note that this result can be inferred from Stormer’s paper, see Lemma 2.1 and/or [24].

3.2 Positive maps determined by \( C_{cp} \)

We define

\[
C_{cp} = (\mathfrak{A} \otimes_\pi \mathfrak{T})_+. \tag{3.8}
\]

Firstly, we note that as \( \mathfrak{A} \) and \( \mathfrak{T} \) are *-Banach algebras, it is easy to check that then \( \mathfrak{A} \otimes_\pi \mathfrak{T} \) is also *-Banach algebra under the natural involution \( (x \otimes y)^* = x^* \otimes y^* \), cf [22], Sect. IV.4. Thus \( C_{cp} \) is a cone. Clearly \( x \otimes y \in (\mathfrak{A} \otimes \mathfrak{T})_+ \) if \( x \in \mathfrak{A}_+ \) and \( y \in \mathfrak{T}_+ \). Furthermore, for \( \varphi \in \mathfrak{A}_+^* \) and \( \text{Tra}(\cdot) \in \mathfrak{T}_+^* \), \( a \in \mathcal{B}(\mathcal{H})_+ \), one has

\[
\left\langle \varphi \otimes \text{Tra}(\cdot), \sum_{i,j} x_i^*x_j \otimes \varrho_i^* \varrho_j \right\rangle = \sum_{i,j} \varphi(x_i^*x_j)\text{Tr} \varrho_i^* \varrho_j a \geq 0, \tag{3.9}
\]

where \( x_i \in \mathfrak{A}, \varrho_i \in \mathfrak{T} \), and where we have used the Schur product theorem for the Hadamard product. Thus, \( C_{cp} \) is a tensor cone, and we can use the rule (3.4). So, for \( a_i \in \mathfrak{A}, b_i \in \mathfrak{T} \):

\[
\tilde{T} \left( \sum_{i,j}^k (a_i \otimes b_i)^* (a_j \otimes b_j) \right) = \sum_{i,j}^k \text{Tr} T(a_i^*a_j)(b_i^*b_j)^t \\
= \sum_{i,j}^k \sum_{n=1}^{\infty} ((b_j)^te_n, T(a_i^*a_j)(b_j)^te_n) \tag{3.10}
\]
Let \((b_j) = \{ f_j > x \} (\in \mathcal{S})\), where \(f_j \in \mathcal{H}, x \in \mathcal{H}\) such that \(||x|| = 1\). Then,

\[
(3.10) = \sum_{i,j} \sum_k \langle x, e_n \rangle (f_i, T(a_i^* a_j) f_j)(x) = \sum_{i,j} \langle f_i, T(a_i^* a_j) f_j \rangle.
\]

(3.11)

Now, put \(f_i = c_i g\), where \(c_i \in B(\mathcal{H})\), \(g \in \mathcal{H}\). Then,

\[
(3.10) = \sum_{i,j} (c_i g, T(a_i^* a_j) c_j g) \geq 0
\]

(3.12)

It follows immediately from Corollary IV.3.4 in [22] that \(T\) is a completely positive map. Consequently, the cone \(C_{cp}\) is selecting cp maps \(P_{cp}\). Again, we note that a similar characterization of cp maps could be inferred from Størmer’s paper [24].

### 3.3 Positive maps determined by \(C_i\)

We have seen that the tensor cone \(C_i\) is the largest tensor cone. It was already noted by Stinespring [26]—see the nicely elaborated the Stinespring example in [25]—that in general, \(C_i\) is not equal to \(C_{cp}\). Now we wish to describe the corresponding maps which will be denoted by \(P_i\). We note that the condition defining the cone \(C_i\) can be written as

\[
\sum_{i=1}^n \varphi(a_i) \text{Tr} \varrho_i a \geq 0,
\]

(3.13)

for any \(\varphi \in \mathcal{A}_+^*, a \in B(\mathcal{H})_+\), where we took \(\mathcal{A} \otimes \mathcal{T} \ni z = \sum_{i=1}^n a_i \otimes \varrho_i\).

We say that \(z = \sum_{i=1}^n a_i \otimes \varrho_i\) satisfying condition (3.13) is st-positive (st stands for simple tensor) and denote \(z \geq^{st} 0\). We note that the property \(z \geq^{st} 0\) is, in general, weaker than standard positivity \(z \geq 0\), see the example below. Taking into account that \(P_i \subset P_{cp}\), cf Remark 3.6 and the previous subsection, we infer that the rule (3.4) leads to very regular completely positive maps which will be denoted by

\[
P_i = \left\{ T : \sum_i \text{Tr} T(a_i) \varrho_i \geq 0 \text{ for } \sum_i a_i \otimes \varrho_i \geq^{st} 0 \right\}.
\]

(3.14)

To see this regularity explicitly as well as the difference between \(\geq 0\) and \(\geq^{st} 0\), we give an example which can be considered as a continuation of the above mentioned Stinespring’s example.

**Example 3.7** Let \(\mathcal{A}\) be equal to \(B(\mathcal{H})\) with \(\mathcal{H}; \dim \mathcal{H} = n < \infty\). From the condition (3.13) we have

\[
\sum_i \text{Tr} \varrho_i a_i \text{Tr} \varrho_i a \geq 0,
\]

(3.15)
for any \( \varrho_\varphi \in \mathcal{B}(\mathcal{H})_+ \), \( a \in \mathcal{B}(\mathcal{H})_+ \), where \( z \) was taken to be of the form \( z = \sum_i a_i \otimes \varrho_i \) with \( a_i, \varrho_i \in \mathcal{B}(\mathcal{H}) \). Put \( \varrho_\varphi = |f><f| \) and \( a = |g><g| \) for \( f, g \in \mathcal{H} \). Then, (3.15) leads to

\[
\sum_i (f, a_i f)(g, \varrho_i g) = \sum_i (f \otimes g, a_i \otimes \varrho_i f \otimes g) \geq 0,
\]

for any \( f, g \in \mathcal{H} \). Thus \( \sum_i a_i \otimes \varrho_i \geq 0 \) implies the positivity of \( \sum_i a_i \otimes \varrho_i \) only on simple tensors of \( \mathcal{H} \otimes \mathcal{H} \). We emphasize that this condition is weaker than the standard positivity \( \sum_i a_i \otimes \varrho_i \geq 0 \).

To explain the regularity imposed by the cone \( C_i \) we recall that \( P_i \subset P_{cp} \), cf Remark 3.6. In particular, \( T \in P_i \) can be written in the form (usually called the Kraus decomposition),

\[
T(a) = \sum_k V_k^* a V_k,
\]

where \( V_k \in \mathcal{B}(\mathcal{H}) \), see Theorem 4.1.8 in [2]. As the first observation, we consider the case: \( V_k = |f_k><g_k| \) with \( f_k, g_k \in \mathcal{H} \). Then,

\[
\sum_{i,k} \text{Tr} V_k^* a_i V_k \varrho_i = \sum_{i,k,l} (e_l, V_k^* a_i V_k \varrho_l e_l) = \sum_{i,k,l} (|f_k><g_k| a_i |f_k><g_k| \varrho_l e_l) = \sum_{i,k,l} (f_k, a_i f_k)(\varrho_l^* g_k, e_l)(e_l, g_k) = \sum_k (f_k \otimes g_k, \left( \sum_i a_i \otimes \varrho_i \right) f_k \otimes g_k). \tag{3.18}
\]

Consequently, \( T(\cdot) = \sum_i V_k^*(\cdot) V_k \) with \( V_k = |f_k><g_k| \) is a regular cp map which is in \( P_i \). It is worth pointing out that one-rank operators \( V_k \) ensures applicability of st positivity and that such maps \( T(\cdot) = \sum_i V_k^*(\cdot) V_k \) are sometimes called super positive maps, cf Definition 5.1.2 in [2]. Now, we turn to the case \( V_k = \sum_{m=1}^M |f_m^k><g_m^k| \) with \( M \geq 2 \).

\[
\sum_{k,l,i} (e_l, V_k^* a_i V_k \varrho_l e_l) = \sum_{i,k,l} \left( \sum_m |f_m^k><g_m^k| e_l, a_i \sum_n |f_n^k><g_n^k| \varrho_l e_l \right) = \sum_{i,k,l,m,n} \left( e_l, g_m^k \right) \left( f_m^k, a_i f_n^k \right) \left( g_n^k, \varrho_l e_l \right) = \sum_{k,m,n} \left( f_m^k \otimes g_m^k, \left( \sum_i a_i \otimes \varrho_l \right) f_m^k \otimes g_m^k \right). \tag{3.19}
\]

We see at once that st. positivity of \( \sum_i a_i \otimes \varrho_i \) does not ensure the positivity of (3.19). Consequently, such cp maps are not, in general, in \( P_i \).
3.4 Positive maps determined by $C_{cp} \cap id \otimes t(C_{cp})$

We begin by an examination of the set $C_d \equiv C_{cp} \cap id \otimes t(C_{cp})$, where $t$ as before, stands for the transposition. We first note that $C_d$ is a cone. To deduce that $C_d$ is a tensor cone, we note that $x \otimes y \in C_d$ for all $x \in \mathcal{A}_+$ and $y \in \mathcal{X}_+$. Subsequently, we observe that one has $x^* \otimes y^* \in C_d^*$ for all $x^* \in \mathcal{A}_+^*$ and $y^* \in \mathcal{X}_+^*$. Thus, $C_d$ is a tensor cone. Clearly, $C_p \subset C_d \subset C_{cp}$ and $P_{cp} \subset P_d \subset P_p$, (3.20)

where $P_d$ stands for those positive maps which are determined by the cone $C_d$, i.e.

$$P_d = \left\{ T : \sum_{i,j=1}^n \text{Tr} T (a_i^* a_j) (b_i^* b_j) \geq 0 \text{ and } \sum_{i,j=1}^n \text{Tr} T (a_i^* a_j) (b_i^* b_j) \geq 0 \right\},$$

(3.21)

where $a_i \in \mathcal{A}$, and $b_i \in \mathcal{X}$.

Further, let us consider the set $C_{ccp} \equiv id \otimes t(C_{cp})$. It follows by similar arguments as those employed in Subsection 3.2 that $C_{ccp}$ is a tensor cone. Hence, it is easy to check that $C_p \subset C_d \subset C_{ccp}$ and $P_{ccp} \subset P_d \subset P_p$, (3.22)

where $P_{ccp}$ stands for the set of all co-positive maps. Consequently, (3.22) and (3.20) lead to $P_d \supseteq P_{cp} \cup P_{ccp}$.

(3.23)

In other words, the cone $C_d$ determines decomposable maps $P_d$.

To finish this subsection we have to examine the question whether $C_d$ is always non-trivial, i.e. whether the inclusion $C_p \subset C_d$ is the proper one. To answer this question we give:

**Example 3.8** We assume that $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and that dim $\mathcal{H} \leq 3$. To study the non-triviality of $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}))_+ \cap id \otimes t((\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}))_+)$ it is enough to examine one dimensional (orthogonal) projectors of the form $|f >< f|$ with $f \in \mathcal{H} \otimes \mathcal{H}$. To see this we begin with two observations:

**Observation 3.9** For $\mathcal{H} \otimes \mathcal{H} \ni h = \sum_i f_i \otimes g_i$ one has

$$|h >< h| = |\sum_i f_i \otimes g_i >< \sum_j f_j \otimes g_j| = \sum_{i,j} |f_i >< f_j| \otimes |g_i >< g_j|. \quad (3.24)$$

**Observation 3.10** Any $h \in \mathcal{H} \otimes \mathcal{H}$ can be written in the form $h = \sum_i v_i \otimes e_i (= \sum e_i \otimes w_i)$ where $v_i, w_i \in \mathcal{H}$ and $\{e_i\}$ is a basis in $\mathcal{H}$.

Proofs of the above observations are easy and they are left to the reader.

Let $|f >< f|$ be given. Then, the above Observations and (3.4) imply

$$\tilde{T}(|f >< f|) = \sum_{k,l} \text{Tr} T(|e_k >< e_l|)|f_i >< f_k|.$$

(3.25)
where \( f = \sum_k e_k \otimes f_k, \{ e_i \} \) a basis in \( \mathcal{H} \). To see that that the Woronowicz scheme for description of positive maps of low dimensional matrix algebras is reproduced, \([6]\), we note, in the “matrix terms”, that

\[
\left\{ \sum_l T(|e_k><e_l|) f_l><f_k \right\}_{kk} = \sum_l \{ T(|e_k><e_l|) \}_k \{ f_l><f_k \}_l ,
\]

and

\[
\text{Tr}_{\mathcal{H} \otimes \mathcal{H}} \sum_l [T(|e_k><e_l|)] Q_{lk} = \sum_k \text{Tr}_{\mathcal{H}} T(|e_k><e_l|) \cdot |f_l><f_k| ,
\]

where, using Woronowicz’s notation, \( Q \) stands for the operator \( \{|f_k><f_l|\}_k \). Consequently, to infer the triviality of the cone \( \text{C}_d \) for dimension 2 from the Woronowicz result, Theorem 1.1 and Sect. 2 in \([6]\), it is enough to reproduce, in the considered context, Woronowicz’s argument leading to “simple vectors”. To this end, we proceed to show that, for \( \mathcal{H}; \dim \mathcal{H} = 2 \), \( \text{id} \otimes t(h><h) \) is positive only if \( h = f \otimes g, f, g \in \mathcal{H} \), so when \( |h><h| \in B(\mathcal{H}_+ \otimes B(\mathcal{H}_+) \). Let \( x = \sum m w_m \otimes e_m \in \mathcal{H} \otimes \mathcal{H} \), and \( h = \sum i v_i \otimes e_i \in \mathcal{H} \otimes \mathcal{H} \). Then,

\[
(x, \text{id} \otimes t(h><h)x) = \sum_{i,j,m,n} (w_n \otimes e_n, |v_i><v_j| \otimes |e_j><e_i| w_m \otimes e_m )
\]

\[
= \sum_{i,j,m,n} (w_n, v_i)(e_n, e_j)(v_j, w_m)(e_i, e_m )
\]

\[
= \sum_{i,j} (w_j, v_i)(v_j, w_i) ,
\]

(3.27)

where both observations were used.

For, \( \mathcal{H}; \dim \mathcal{H} = 2 \) one gets

\[
(w_1, v_1)(v_1, w_1) + (w_2, v_1)(v_2, w_1) + (w_1, v_2)(v_1, w_2) + (w_2, v_2)(v_2, w_2) \geq 0 ,
\]

(3.28)

while for \( \mathcal{H}; \dim \mathcal{H} = 3 \)

\[
(w_1, v_1)(v_1, w_1) + (w_2, v_1)(v_2, w_1) + (w_1, v_2)(v_1, w_2) + (w_2, v_2)(v_2, w_2) + (w_1, v_3)(v_1, w_3)
\]

\[
+ (w_2, v_3)(v_2, w_3) + (w_3, v_3)(v_3, w_3) + (w_3, v_1)(v_3, w_1)
\]

\[
+ (w_3, v_2)(v_3, w_2) \geq 0 .
\]

(3.29)

We are looking for \( \{ v_i \} \) such that (3.28) [respectively (3.29)] are satisfied for arbitrary \( \{ w_i \} \).

Let’s consider (3.28) in detail. As \( \{ w_i \} \) was arbitrary we may change \( w_1 \) for \( \lambda w_1, \lambda \in \mathbb{R} \) to arrive at

\[
\forall \lambda \ F(\lambda) = \lambda^2 a + \lambda b + c \geq 0 ,
\]

(3.30)
where $a = (w_1, v_1)(v_1, w_1)$, $b = (w_2, v_1)(v_2, w_1) + (w_1, v_1)(v_2, w_1)$, $c = (w_2, v_2)(v_2, w_2)$. (3.30) holds if $c \leq \frac{b^2}{a}$. As $\{w_i\}$ are arbitrary, we may take $w_2 \perp v_1$. Then, $b = 0$. So $c = 0$. As $w_1$ is arbitrary, then $v_1 = \lambda v_2$, where $\lambda \in \mathbb{C}$. However, this implies that $h$ is a simple tensor.

We now turn to the case $\dim \mathcal{H} = 3$. It is easily seen that the above arguments fail when we have the 3 dimensional case. Fortunately, Choi gave an explicit construction of a matrix $U \in (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+$ such that $U \neq M_3(\mathbb{C})^+ \otimes M_3(\mathbb{C})^+$, see [27]; another example was given in [28]. So, it is enough to see that the cone $C_d$ is not trivial for $\mathcal{H}$; $\dim \mathcal{H} = 3$.

3.5 $k$-positive maps

To complete this section we wish to examine $k$-positive maps. Our first observation is that we can apply the strategy described in subsection 3.1 with the following modifications:

$$\mathfrak{A} \rightarrow M_n(\mathfrak{A}) \cong M_n(\mathfrak{G}) \otimes \mathfrak{A},$$

(3.31)

and

$$B(\mathcal{H}) \rightarrow M_n(B(\mathcal{H})) \cong M_n(\mathfrak{G}) \otimes B(\mathcal{H}).$$

(3.32)

Consequently, we will examine

$$\mathcal{L}(M_n(\mathfrak{A}), M_n(B(\mathcal{H}))) \cong (M_n(\mathfrak{A}) \otimes_{\Pi} M_n(B(\mathcal{H})*)^{\ast}.$$  

(3.33)

To proceed with the analysis of $k$-positive maps we remind the reader that any $A \in M_n(\mathfrak{A})$ has the unique representation:

$$A = \sum_{i,j=1}^{n} \epsilon_{ij} \otimes a_{ij},$$

(3.34)

where $a_{ij} \in \mathfrak{A}$, and $\{\epsilon_{ij}\}_{i,j=1}^{n}$ is a matrix unit, ie $(\epsilon_{ij})^* = \epsilon_{ji}$, $\epsilon_{ij}\epsilon_{kl} = \delta_{jk}\epsilon_{il}$, $\sum_{i=1}^{n} \epsilon_{ii} = 1$ cf [29], Appendix T. The above representation of $A \in M_n(\mathfrak{A})$ combined with Kaplan’s arguments, see Proposition 1.1 in [30], plus obvious modifications, give an order isomorphism $\mathfrak{F} : M_n(B(\mathcal{H})*) \rightarrow M_n(B(\mathcal{H})*)$ given by

$$(\mathfrak{F}([\varrho_{ij}])) = \sum_{ij=1}^{n} \varrho_{ij}(a_{ij}),$$

(3.35)

where $[\varrho_{ij}] \in M_n(B(\mathcal{H})*)$, $[a_{ij}] \in M_n(B(\mathcal{H}))$. Then, using the rule analogous to that given by 3.4 one has the following prescription for selection of $k$-positive maps:
\((\ast)\) \quad \{ T_k \equiv i d_{M_k(\mathcal{I})} \otimes T \in \mathcal{L}(M_k(\mathcal{A}), M_k(\mathcal{B}(\mathcal{H}))) \};

\[ \tilde{T}_k \left( \sum_{p=1}^{p_0} [a_{ij}^p] \otimes [\varrho_{ij}^p] \right) = \sum_{p=1}^{p_0} \sum_{ij} \text{Tr} T \left( a_{ij}^p \right) \varrho_{ij}^{p,i} \geq 0, \]

for any \( \sum_{p=1}^{p_0} [a_{ij}^p] \otimes [\varrho_{ij}^p] \) in \( M_k(\mathcal{A})^+ \otimes M_k(\mathcal{B}(\mathcal{H}_*))^+ \).

To verify the above procedure, \((\ast)\), it is enough to note that \([\varrho_{ij}] \in M_k(\mathcal{B}(\mathcal{H}_*))^+ = B(\mathcal{H}^n \otimes \mathcal{H}_*)^+ \) iff \( \varrho_{ij} = \sum_{l=1}^k \sigma_{li}^* \sigma_{lj} \), where the unique representation (3.34) together with properties of trace class and Hilbert-Schmidt operators were used. Hence

\[
0 \leq \sum_{i,j=1}^k \text{Tr} \sum_{m=1}^k T \left( a_{ij}^m \right) \sigma_{mj}^* \sigma_{mi}^* = \sum_{i,j=1}^k \text{Tr} \sum_{m=1}^k \left( \sigma_{mi}^* \right)^* T \left( a_{ij}^m \right) \sigma_{mj}^i. \tag{3.36}
\]

So \( T(a_{ij}^p)_{i,j=1}^k \geq 0 \) and \( T \) is \( k \)-positive.

Finally we note that dropping the restriction for the upper bound of summation in (3.36) one gets

\[
0 \leq \sum_{i,j,m=1}^l \text{Tr} T \left( a_{ij} \right) \sigma_{mj}^* \left( \sigma_{mi}^* \right)^* = \sum_{i,j,m} \text{Tr} \left( \sigma_{mi}^* \right)^* T \left( a_{ij} \right) \sigma_{mj}^i, \tag{3.37}
\]

where \( l \) is an arbitrary natural number. It is easily seen that this is exactly the condition imposed by the cone \( C_{cp} \). In other words, this observation sheds some new light on the origin of the definition of CP-maps.

4 Conclusions and remarks

The principal significance of the Example 3.8 is that it clarifies the appearance of non-decomposable maps, i.e. if \( C_d \neq C_p \) then there is a room for non-decomposable maps. We have seen that both cones \( C_d \) and \( C_p \) are equal to each other for \( \mathcal{H} \); \( \dim \mathcal{H} = 2 \).

The next important point to note here is that when studying the structure of the set of positive maps we restrict ourselves to a few special cones. In general, there could be other tensor cones. Thus, in principle, there could be other classes of interesting positive maps.

In [20] the structure of the set of positive maps was examined using the concept of elementary maps. Although this concept was to some extent vague, it was indicated that general structural properties should play a significant role. Here, we have seen that the synthesis of Grothendieck’s idea with the order on tensor product is playing a crucial role.

It is worth pointing out that the presented scheme is offering another approach to the concept of elementary maps. To see this, from now on we make the assumption:
\( \mathfrak{A} = B(\mathcal{H}) \) with \( \mathcal{H} \); \( \dim \mathcal{H} = n < \infty \), cf [20]. We emphasize that this case is essential for Quantum Information Theory.

We first note that in the rule (3.4) one can restrict oneself to extremal functionals. As an extreme point in a convex set can be treated as an elementary constituent, we are getting another concept of an elementary map. Define

\[
P^e_\alpha = \left\{ T \in \mathcal{L}(B(\mathcal{H}), B(\mathcal{H})); \quad \tilde{T} \left( \sum_{i=1}^n a_i \otimes b_i \right) = \sum_{i=1}^n \text{Tr} T(a_i)b_i^t \geq 0 \right\} \quad (4.1)
\]

for any \( \sum_{i=1}^n a_i \otimes b_i \in C_\alpha \), where \( \sum_{i=1}^n a_i \otimes b_i \in B(\mathcal{H}) \otimes B(\mathcal{H}) \), \( C_\alpha \) is a tensor cone in \( B(\mathcal{H}) \otimes B(\mathcal{H}) \), and \( \tilde{T} \) is assumed to be an extremal functional. Consequently, elementary maps can be written as:

\[
P^e_\alpha = \left\{ T \in \mathcal{L}(B(\mathcal{H}), B(\mathcal{H})); \quad (h', \left( \sum_{i=1}^n a_i \otimes b_i \right) h) = \sum_{i=1}^n \text{Tr} T(a_i)b_i^t \geq 0 \right\} \quad (4.2)
\]

for any \( \sum_{i=1}^n a_i \otimes b_i \in C_\alpha \), where \( \sum_{i=1}^n a_i \otimes b_i \in B(\mathcal{H}) \otimes B(\mathcal{H}) \), \( C_\alpha \) is a tensor cone in \( B(\mathcal{H}) \otimes B(\mathcal{H}) \) and \( h \in \mathcal{H} \otimes \mathcal{H} \).

The interest of this remark is that it provides a recipe for constructing “elementary” maps with specified positivity with respect to the selected cone \( C_\alpha \). Furthermore, in an application of the above scheme to 2-positive maps one recovers the concept of \textit{atomic maps}.

The next important point to note here is that the relation between a cp-map \( T \) and the associated \( C_{cp} \)-positive functional \( \tilde{T} \) can be considered as the starting point for a generalization of Krauss (Stinespring) decomposition, for the corresponding order structures see Sect. 1.4 in [31].

To see this we recall that \( (M_n(\mathbb{C}) \otimes_{\pi} M_n(\mathbb{C})^\ast) \), \( C_{cp} \) is an involutive Banach algebra and the cone \( C_{cp} \) can be used to define the concept of states, cf Chapter I in [22]. Therefore, there is a possibility to speak about GNS construction. Then employing the decomposition theory, as it was given in Chapter 4, in [32], see Sect. 4.1.1, one can decompose the \( C_{cp} \)-positive functional \( \tilde{T} \), so also the corresponding map \( T \). In other words, we are getting a generalized decomposition of a cp-map.

Our next remark is clarifying a general form of a positive map (for finite dimensional case). We note that for any self-adjoint (hermitian) functional \( \tilde{T} \) on a C*-algebra \( \mathfrak{A} \) there is a \textit{unique} pair \( \tilde{T}_+ \) and \( \tilde{T}_- \) of positive functionals such that \( \tilde{T} = \tilde{T}_+ - \tilde{T}_- \), cf Theorem 3.2.5 in [33]. We can use this for unique decomposition of a self-adjoint functional on \( M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \). Subsequently, one can transfer this decomposition to \( M_n(\mathbb{C}) \otimes_{\pi} M_n(\mathbb{C}) \). Thus, we arrive at: a \( C_p \)-positive functional \( \tilde{T} \) has \textit{unique} decomposition \( \tilde{T} = \tilde{T}_+ - \tilde{T}_- \), where \( \tilde{T}_+, \tilde{T}_- \) are \( C_{cp} \)-positive functionals.

Consequently we got, using our scheme, the following decomposition of a positive map \( T; T = T_+ - T_- \), where \( T_+, T_- \) are cp-maps.

**Example 4.1** Let us consider the transposition map \( \tau \), i.e. 1-positive map. The functional \( \tilde{\tau} \) is determined by the (flip) operator \( \varrho_\tau \), so \( \varrho_\tau(f \otimes g) = g \otimes f \), where \( f, g \in \mathfrak{A}^\ast \).
Then, the spectral resolution of $\varrho_t$ reads $\varrho_t = \frac{1}{2}(id + \varrho_t) - \frac{1}{2}(id - \varrho_t)$. Consequently, we arrive at $t(a) = \frac{1}{2}[\text{Tr}_a \cdot I + t(a)] - \frac{1}{2}[\text{Tr}_a \cdot I - t(a)]$.

To elaborate more fully the decomposition formula of $T$ we need some preliminaries, cf. [34].

Let $a_1, \ldots, a_k$, and $c_1, \ldots, c_l$ be in $B(\mathcal{H}, \mathcal{K})$ ($\mathcal{H}, \mathcal{K}$ stand for Hilbert spaces). If for each $x \in \mathcal{H}$, there exists an $l \times k$ complex matrix $\{(\alpha_{i,j}(x))\}$ such that

$$c_i x = \sum_{j=1}^{k} \alpha_{i,j}(x) a_j x, \quad i = 1, \ldots, l,$$

(4.3)

($c_1, \ldots, c_l$) is said to be a locally linear combination of $(a_1, \ldots, a_k)$. If coefficients $\{(\alpha_{i,j}(x))\}$ can be taken in such way that the norm $\|\alpha_{i,j}(x)\| \leq 1$, for every $x$, then ($c_1, \ldots, c_l$) is said to be a contractive linear combination of $(a_1, \ldots, a_k)$. Employing Hou’s result (see Corollary 2.6 in [34]) we arrive at

**Proposition 4.2** A positive map $T : M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ can be uniquely written in the form $T = T_+ - T_-$ where $T_+, T_-$ are cp-maps. In particular $T_+(a) = \sum r c_r a c_r^*$ where $c_r \in M_n(\mathbb{C})$. Furthermore, there exists $(d_p) \subset M_n(\mathbb{C})$ such that $(d_1, \ldots, d_l)$ is a contractive locally linear combination of $(c_1, \ldots, c_k)$ and

$$T(a) = T_+(a) - T_-(a) = \sum_r c_r v c_r^* - \sum_p d_j v d_j^*, \quad (4.4)$$

for all $a \in M_n(\mathbb{C})$.

Finally we note that in [18] the identification of the injective cone $C_i$ with the cone $C_{cp}$ was incorrect.

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