Impact of Eigenstate Thermalization on the Route to Equilibrium

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The eigenstate thermalization hypothesis (ETH) and the theory of linear response (LRT) are celebrated cornerstones of our understanding of the physics of many-body quantum systems out of equilibrium. While the ETH provides a generic mechanism of thermalization for states arbitrarily far from equilibrium, LRT extends the successful concepts of statistical mechanics to situations close to equilibrium. In our work, we connect these cornerstones to shed light on the route to equilibrium for a class of properly prepared states. We unveil that, if the off-diagonal part of the ETH applies, then the relaxation process can become independent of whether or not a state is close to equilibrium. Moreover, in this case, the dynamics is generated by a single correlation function, i.e., the relaxation function in the context of LRT. Our analytical arguments are illustrated by numerical results for idealized models of random-matrix type and more realistic models of interacting spins on a lattice. Remarkably, our arguments also apply to integrable quantum systems where the diagonal part of the ETH may break down.

Introduction. Both, equilibration and thermalization are omnipresent phenomena in nature. Simple examples in everyday life are a cup of hot coffee which cools down to room temperature, or an inkblot in water which spreads in the entire liquid. Even though the irreversible route to equilibrium occurs in any macroscopic and ordinary situation, the underlying microscopic laws of physics are reversible. In fact, the emergence of phenomenological relaxation from truly microscopic principles such as the Schrödinger equation is not satisfactorily understood up to date. While this fundamental question has a long and fertile history, it has been under intense scrutiny in the last decade \[1–5\]. This upsurge of interest is also related to the advent of experiments on cold atomic gases \[6, 7\], the development of sophisticated numerical methods \[8\], as well as the introduction of fresh theoretical concepts such as quantum typicality of pure states \[9–11\] and the eigenstate thermalization hypothesis (ETH) \[12–14\]. In particular, the ETH has become a cornerstone of our understanding of the mere existence of equilibration, but much less is known on the route to equilibrium as such \[15, 16\].

An obvious problem in this context is the absence of an universal approach to the time evolution of quantum many-body systems out of equilibrium. Of course, close to equilibrium, a powerful strategy is provided by the theory of linear response (LRT) \[17\]. Further away from equilibrium, however, this theory is naturally expected to break down and the dynamics might drastically change for states far from equilibrium. In this respect, our work reports an unexpected and intriguing picture. Following earlier ideas developed by Srednicki \[18\], we establish a link between LRT and ETH for specific non-equilibrium setups introduced below in detail. We unveil that, if the off-diagonal part of the ETH holds, then the relaxation process can become independent of whether or not a state is close to equilibrium. Moreover, in this case, the time evolution is generated by a single correlation function, i.e., the relaxation function in the context of LRT. Our analytical arguments are also confirmed numerically for two different models.

Framework. We will study a physical situation like the one illustrated in Fig. 1. A quantum system, described by a Hamiltonian \( \mathcal{H} \), is affected by an external static force of strength \( \varepsilon \). This force gives rise to an additional potential, described by an operator \( \mathcal{O} \). Because of the presence of a heat bath at inverse temperature \( \beta = 1/T \), thermalization to the density matrix \( \rho = \exp[-\beta(\mathcal{H} - \varepsilon \mathcal{O})]/Z \) occurs. (b) Static force and heat bath are both removed and \( \rho \) undergoes unitary time evolution with respect to \( \mathcal{H} \). We discuss the dependence of the relaxation curve \( \langle \mathcal{O}(t) \rangle \) on the perturbation \( \varepsilon \) outside the regime of small \( \varepsilon \).

\[
\rho = e^{-\beta(\mathcal{H} - \varepsilon \mathcal{O})}/Z \tag{1}
\]

FIG. 1. (Color online) Sketch of the setup. (a) A static force of strength \( \varepsilon \) gives rise to an additional potential, described by an operator \( \mathcal{O} \). Because of the presence of a heat bath at inverse temperature \( \beta = 1/T \), thermalization to the density matrix \( \rho = \exp[-\beta(\mathcal{H} - \varepsilon \mathcal{O})]/Z \) occurs. (b) Static force and heat bath are both removed and \( \rho \) undergoes unitary time evolution with respect to \( \mathcal{H} \). We discuss the dependence of the relaxation curve \( \langle \mathcal{O}(t) \rangle \) on the perturbation \( \varepsilon \) outside the regime of small \( \varepsilon \).
and the partition function $Z_{\text{eq}} = \text{Tr}[e^{-\beta H}]$. The static susceptibility $\chi(0)$ is given by a Kubo scalar product \[\chi = \beta \langle \Delta O; O \rangle = \int_0^\beta d\lambda \text{Tr}[e^{-\beta H} \Delta O(\lambda - t) O] \quad (2)\] with $\Delta O = O - \langle O \rangle_{\text{eq}}$ and $\Delta O(\lambda - t) = e^{\lambda H} \Delta O e^{-\lambda H}$. If $\varepsilon$ becomes large enough, this linear relationship breaks down. In particular, given an operator $O$ with bounded spectrum, $\lim_{\varepsilon \to -\infty} \lim_{t \to \infty} \langle O(0) \rangle = O_{\text{max}}$. As a consequence, $\lim_{\varepsilon \to -\infty} \langle O(0) \rangle = O_{\text{max}}$. Hence, a convenient quantity is the non-equilibrium parameter

$$\zeta(\varepsilon) = \frac{\langle \Delta O(0) \rangle}{\Omega_{\text{max}} - \langle O \rangle_{\text{eq}}} \quad (3)$$

This quantity becomes $\zeta(\varepsilon) = 0$ for $\varepsilon = 0$ and $\zeta(\varepsilon) = 1$ for $\varepsilon \to \infty$. While $\zeta(\varepsilon)$ is a natural measure, it might not always be justified to decide solely on this measure if a state is close to or far away from equilibrium [31].

Let us now consider a “sudden quench”, where both the external static force and the heat bath are removed at time $t = 0$. Then, at times $t > 0$, the density matrix $\rho$ is no equilibrium state of the remaining Hamiltonian $H$ and evolves in time according to the von-Neumann equation, $\rho(t) = e^{-iHt} \rho e^{iHt}$ ($\hbar = 1$). Although this setup might be difficult to realize in concrete experiments, it can be still interpreted as a special case of other non-equilibrium scenarios [21, 31].

The central goal of this paper is to investigate the time-dependent expectation value $\langle O(t) \rangle = \text{Tr}[\rho(t)O]$ as a function of the non-equilibrium parameter $\zeta(\varepsilon)$. In the regime of sufficiently small $\zeta(\varepsilon)$, we can certainly expect $\langle O(t) \rangle = \varepsilon \chi(t)$, where $\chi(t)$ denotes the linear-response relaxation function $\chi(t) = \beta \langle \Delta O; O(t) \rangle$ [20, 31]. However, as the theory of linear response is generally restricted to this regime, an intriguing questions is: How does the time dependence of $\langle O(t) \rangle$ change outside this regime? While it is surely challenging to provide a general answer, two of us have recently shown [21] that the time evolution of $\langle O(t) \rangle$ can be become completely independent of $\zeta(\varepsilon)$ if the involved operator $O$ is binary, i.e., if it only has two different eigenvalues. In this paper, we unveil that such an independence can also occur for other observables, if their matrix structure is in accord with the ETH. This impact of the ETH on the route to equilibrium is the main result of our work and demonstrates its relevance beyond the mere existence of equilibrium.

**Illustration: Idealized model.** We begin by illustrating our main result for an idealized model of random-matrix type. Its Hamiltonian $H_I$ is already given in the diagonal form $H_I = \sum_{n=1}^D E_n |n\rangle \langle n|$, where $D$ is the dimension of the Hilbert space and $|n\rangle$ are the eigenstates of $H_I$. The corresponding eigenenergies $E_n$ are simply chosen to be equidistant levels $E_n = n \Delta E/D$ in a spectrum of width $\Delta E$, i.e., the density of states $\Omega = D/\Delta E$ is constant. In

The eigenbasis $|n\rangle$ of $H_I$, is an observable $O_I$ is *constructed* as $O_I = \sum_{D,n=1}^D O_{I,mn} |m\rangle \langle n|$ with matrix elements

$$O_{I,mn} = \frac{\Gamma}{\sqrt{\gamma^2 + \omega_{mn}^2}} (1 - \delta_{mn}) r_{mn}, \quad r_{mn} = r_{nm}^*, \quad (4)$$

where the frequency $\omega_{mn}$ denotes the energy difference $E_m - E_n$ and $\gamma$ is a Lorentzian line width. Thus, for $\gamma \ll \Delta E$, $O_I$ is a banded matrix. Moreover, as $\delta_{mn}$ is the Kronecker symbol, $\langle O_I \rangle_{\text{eq}} = 0$. Within the constraint of a banded matrix, the complex coefficients $r_{mn}$ are free parameters and might be chosen (i) identically or (ii) randomly, e.g., their real and imaginary part might be drawn according to a Gaussian distribution with zero mean. Note that, in the latter case (ii), $O_I$ is an ideal realization of the ETH ansatz discussed in more detail below. The (in principle irrelevant) prefactor $\Gamma$ in Eq. (4) is chosen such that $\text{Tr}[O_I^2] = \text{Tr}[O_I^4]$.

For the case (ii), we show in Fig. 2(a) the expectation value $\langle O_I(t) \rangle$, as resulting for the specific initial state $\rho$ in Eq. (1) and a high temperature $T$. Remarkably, when $\langle O_I(t) \rangle$ is normalized to its initial value $\langle O_I(0) \rangle$, then the relaxation curve is independent of the non-equilibrium parameter $\zeta(\varepsilon) = \langle O_I(0) \rangle / O_{I,\text{max}}$ and coincides with the linear-response relaxation function $\chi(t) \propto \text{Tr}[O_I(t)O_I]$ in the entire range of $\zeta(\varepsilon) \in [0,1]$ possible. This numerical simulation illustrates our main result: In certain cases, $\chi(t)$ might capture the dynamics at arbitrarily strong perturbations. But are such cases generic? This question is non-trivial as counterexamples exist. For instance, for the case (i), the relaxation curve in Fig. 2(c) depends on $\zeta(\varepsilon)$ and agrees with $\chi(t)$ for small $\zeta(\varepsilon) \ll 1$ only.

**Analytical arguments and ETH.** To work towards an answer to this question, let us discuss the initial state $\rho$ in Eq. (1) for high temperatures $T$. For such $T$, we can use the approximation $\rho \approx e^{\beta e^0}/\text{Tr}[e^{\beta e^0}]$ and a Taylor

![FIG. 2. (Color online) Numerical simulations for the idealized model. Relaxation curve of the normalized expectation value $\langle O_I(t)/\langle O_I(0) \rangle$ for various $\zeta(\varepsilon) = \langle O_I(0) \rangle / O_{I,\text{max}}$ as well as the linear-response relaxation function $\chi(t)$. (a) random (Gaussian) $r_{mn}$ and (c) non-random (constant) $r_{mn}$. Other parameters: $D = 1000$, $\Delta E = 1$, $\gamma = \Delta E/100$, $T = 100$. (b) and (d): $M_N(t) = \text{Tr}[O_I(t)O_I^\dagger]$ for exponents $N \leq 4$.](https://example.com/fig2.png)
expansion of this approximation yields

$$\langle O(t) \rangle \approx \sum_{N=0}^{\infty} \alpha_N(\varepsilon) \text{Tr}[O(t)O^N]$$

(5)

with some \( \varepsilon \)-dependent coefficients \( \alpha_N(\varepsilon) \). Now, consider the assumption

$$M_N(t) = \text{Tr}[O(t)O^N] \left\{ \begin{array}{ll} \propto \text{Tr}[O(t)O] & \text{odd } N \\ 0 & \text{even } N \end{array} \right.$$  

(6)

If this assumption was justified, the expansion in Eq. (5) would directly imply \( \langle O(t) \rangle \propto \text{Tr}[O(t)O] \). And in fact, as depicted in Fig. 2 (b), this assumption holds for the idealized model with random \( r_{mn} \). In contrast, as shown in Fig. 2 (d), it breaks down for non-random \( r_{mn} \). Equipped with these prerequisites, we will show that Eq. (6) is closely related to the ETH ansatz

$$O_{mn} = F_d(\bar{E}) \delta_{mn} + \Omega(\bar{E})^{1/2} F_{ad}(\bar{E}, \omega_{mn}) r_{mn} \, ,$$

(7)

where \( \bar{E} = (E_n + E_m)/2 \) and \( F_d \) and \( F_{ad} \) are smooth functions of their arguments. We will argue that Eq. (6) holds if (qualitatively) \( F_{ad} \) varies slowly with \( \bar{E} \) and falls of quickly for larger \( |\omega_{mn}| \). Furthermore, the diagonal elements \( O_{mm} \) do not need to be smooth functions of \( \bar{E} \) as claimed by the ETH, variations that are uncorrelated with the off-diagonal elements \( O_{mn} \), leave Eq. (6) valid. We dub this form of \( F_d \) and \( F_{ad} \) the rigged ETH. As the proof is quite involved for large exponents \( N \), we restrict ourselves to \( N = 2 \) and \( N = 3 \) in the following. For clarity, we also restrict the present consideration to a uniform DOS, \( \Omega(\bar{E}) = \text{const.} \), and \( F_{ad} \) that are independent of \( \bar{E} \). A full derivation for arbitrary \( N \), as well as more general \( \Omega(\bar{E}) \) and \( F_{ad} \), can be found in the supplemental material [31].

We start by writing out the correlation function for \( N = 2 \) explicitly, \( \text{Tr}[O(t)O^2] = \sum_{abc} O_{ab} O_{bc} O_{ca} e^{i\omega_{abc}t} \), and consider a Fourier component of this correlation function at fixed frequency \( \omega \),

$$\text{Tr}[O(t)O^2]_\omega = \sum_{\omega_{ab}=\omega} \sum_c |O_{ab}|^2 O_{ca} \, .$$

(8)

Given the matrix structure in Eq. (7), the biggest part of the addends in the sum are by construction (products of) independent random numbers with zero mean. Thus, to an accuracy set by the law of large numbers, summing the latter yields zero as well. There are, however, index combinations where not all factors within the addends have vanishing mean, namely, \( c = a \). Focusing on these terms, we can write

$$\text{Tr}[O(t)O^2]_\omega \approx \sum_{\omega_{ab}=\omega} |O_{ab}|^2 O_{aa} \, .$$

(9)

While the numbers \( |O_{ab}|^2 \) do not have mean zero, we can, without loss of generality, assume that the numbers \( O_{aa} \) have zero mean. Because both numbers are independent stochastic variables [cf. below Eq. (7)], the sum in Eq. (9) becomes \( \text{Tr}[O(t)O^2]_\omega \approx 0 \). Since this finding does not depend on \( \omega \), we get \( \text{Tr}[O(t)O^2] \approx 0 \), i.e., Eq. (6) for the even case \( N = 2 \).

Now we turn to \( N = 3 \). Here, a Fourier component at fixed frequency \( \omega \) reads

$$\text{Tr}[O(t)O^3]_\omega = \sum_{\omega_{ab}=\omega} \sum_c |O_{ab}|^2 |O_{bc}|^2 + |O_{ac}|^2 \, .$$

(10)

Again, the contributions of most addends approximately cancel each other upon summation. But again, there are also exceptions, namely, the index combinations \( a = c \) or \( d = b \). Focusing on these terms, we find

$$\text{Tr}[O(t)O^3]_\omega \approx \sum_{\omega_{ab}=\omega} |O_{ab}|^2 |O_{bc}|^2 + |O_{ac}|^2 \, .$$

(11)

To proceed, recall the matrix structure (Eq. (7) and below) and consider the above sum over \( c \) without the diagonal elements, i.e., \( \sum_{c \neq a} |O_{bc}|^2 + \sum_{c \neq a} |O_{ac}|^2 \). While these sums do not vanish, they are practically independent of \( a, b \), if \( F_{ad}(\bar{E}, \omega_{mn}) \) is independent of \( \bar{E} \) and vanishes quickly enough for larger \( |\omega_{mn}| \). Thus, the respective sums may be replaced by a constant \( C \), i.e., \( \sum_{c \neq a} |O_{bc}|^2 + |O_{ac}|^2 \approx C + |O_{aa}|^2 + |O_{bb}|^2 \). The \( |O_{aa}^2| \) may be split into their mean and variations: \( |O_{aa}^2| = \bar{O}_{aa}^2 + \delta_{a(b)} \). Inserting the above findings into Eq. (11) and exploiting the assumed “uncorrelated-ness” of the \( \delta_{a(b)} \) with the \( |O_{ab}|^2 \) yields \( \text{Tr}[O(t)O^3]_\omega \approx (C + 2\bar{O}_{aa}^2) \sum_{\omega_{ab}=\omega} |O_{ab}|^2 \). Comparing this to the exact relation \( \text{Tr}[O(t)O]_\omega = \sum_{\omega_{ab}=\omega} |O_{ab}|^2 \) and realizing that all findings are independent of \( \omega \), eventually yields \( \text{Tr}[O(t)O^3]_\omega \propto \text{Tr}[O(t)O] \), i.e., Eq. (6) for the odd case \( N = 3 \). Noting that the calculations for \( N > 3 \) are in principle analogous, we have shown that the assumption in Eq. (6) essentially follows from the ansatz in and below Eq. (6). Hence, we have identified the rigged ETH as the physical mechanism responsible for the numerical observation \( \langle O(t) \rangle \propto \text{Tr}[O(t)O] \), even at strong perturbations.

Illustration: Generic quantum many-body systems. Let us now illustrate the relevance of our results to generic quantum many-body systems. A prototype model in this context is the spin-1/2 XXZ chain. We hence consider the Hamiltonian \( \mathcal{H}_{XXZ} = J \sum_{l=1}^L h_l \) (with periodic boundary conditions),

$$h_l = S_{l}^x S_{l+1}^x + S_{l}^y S_{l+1}^y + \Delta S_{l}^z S_{l+1}^z + \Delta' S_{l+1}^z S_{l+2}^z \, ,$$

(12)

where \( S_{l}^{x,y,z} \) are spin-1/2 operators at lattice site \( l \) and \( J = 1 \) is the antiferromagnetic exchange coupling. For vanishing next-nearest-neighbor interaction \( \Delta' = 0 \), this model is integrable in terms of the Bethe ansatz, whereas integrability is broken for any \( \Delta' \neq 0 \). As an observable,
we choose the well-known spin current [22]

\[ J = \Gamma \sum_{l=1}^{L} S_l^x S_{l+1}^y - S_l^y S_{l+1}^x, \]  

an important quantity in the context of transport. This quantity we study for \( \Delta' = 0 \) and \( \Delta = 0.5 \), where it is partially conserved [22, 23], as well as for \( \Delta = \Delta' = 0.5 \), where it is expected to fully decay. Generally, \( \langle J \rangle_{eq} = 0 \), and again, \( \text{Tr}[J^2] = \text{Tr}[J^4] \) by a corresponding choice of the prefactor \( \Gamma \).

The matrix representation of \( J \) in the eigenbasis of \( H_{XXZ} \) is summarized in Fig. 3. The general structure is visualized in Figs. 3 (a) and (b) by the use of a suitable coarse graining according to

\[ g(E, E') = \frac{\sum_{mn} J_{mn}^2 D(E)}{D(E) D(E')}, \]

where the sum runs over matrix elements \( J_{mn} \) in two energy shells of width \( \delta E \), \( E_n \in [E - \delta E, E + \delta E] \) and \( E_m \in [E' - \delta E, E' + \delta E] \). \( D(E) \), \( D(E') \), and \( D(E') \) denote the number of states in these energy windows. Note that the coarse grained quantity \( g(E, E') \) can be interpreted as a measure of the distribution function \( F_{od} \) in Eq. (7), i.e., \( g(E, E') \propto |F_{od}(E, \omega_{mn})|^2 \). Apparently, the situation is very similar for the integrable and non-integrable case: Weight is concentrated around the diagonal and quickly vanishes further away from the diagonal. However, the close-up of matrix elements \( J_{mn} \) in Figs. 3 (c) and (d) unveils clear differences. On the hand, in the integrable case, substantial weight lies directly on the diagonal and the vast majority of all off-diagonal matrix elements are exactly zero, e.g., due to conservation laws. On the other hand, for the non-integrable case, the matrix elements appear to be randomly distributed and it is difficult to recognize any structure at all.

Let us finally turn to the dynamics of \( J \). In Fig. 4 we depict the expectation value \( \langle J(t) \rangle / \langle J(0) \rangle \) for various \( \zeta(\varepsilon) = \langle J(t) \rangle / J_{\text{max}} \) as well as the linear-response relaxation function \( \chi(t) \). (a) integrable case \( (\Delta' = 0, \Delta = 0.5) \) and (c) non-integrable case \( (\Delta' = 0.5, \Delta = 0.5) \). Other parameters: \( L = 26, T = 100 \). (b) and (d): \( M_N(t) = \text{Tr}[J(t) J^N] \) for exponents \( N \leq 4 \) and \( L = 20 \).

FIG. 3. (Color online) Matrix elements of \( J \) in the eigenbasis of \( H_{XXZ} \) for (a), (c) integrable case \( (\Delta' = 0, \Delta = 0.5) \) and (b), (d) non-integrable case \( (\Delta' = 0.5, \Delta = 0.5) \) in a single symmetry subspace \( (S^z = 1 \text{ and } k = 1) \). Upper row: Coarse grained structure \( g(E, E') \). Lower row: Close-up of \( 200 \times 200 \) matrix elements around the diagonal. In all cases, \( L = 20 \).

FIG. 4. (Color online) Numerical simulations for the XXZ spin-1/2 chain. Decay curve of the normalized expectation value \( \langle J(t) \rangle / \langle J(0) \rangle \) for various \( \zeta(\varepsilon) = \langle J(t) \rangle / J_{\text{max}} \) as well as the linear-response relaxation function \( \chi(t) \). (a) integrable case \( (\Delta' = 0, \Delta = 0.5) \) and (c) non-integrable case \( (\Delta' = 0.5, \Delta = 0.5) \). Other parameters: \( L = 26, T = 100 \). (b) and (d): \( M_N(t) = \text{Tr}[J(t) J^N] \) for exponents \( N \leq 4 \) and \( L = 20 \).
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Methods. In our paper, we use numerical methods to calculate expectation values $\langle O(t) \rangle$. While we use exact diagonalization (ED) to obtain the data for the idealized model in Fig. 1, ED becomes costly for the XXZ spin-1/2 chain because (i) the Hilbert-space dimension $D = 2^L$ grows exponentially with the number of spins $L$ and (ii) the calculation of the expectation value $\langle O(t) \rangle$ requires ED of both, the pre-quench Hamiltonian $\mathcal{H} - \varepsilon O$ and the post-quench Hamiltonian $\mathcal{H}$. Thus, to obtain the data in Figs. 4 (a) and (c), we proceed differently and employ the concept of dynamical quantum typicality as a numerical method [21,28]. Specifically, we construct a non-equilibrium pure state of the form [21]

$$|\psi(0)\rangle = \sqrt{D} |\varphi\rangle / \sqrt{\langle \varphi | \rho | \varphi \rangle},$$

(15)

to mimic the density matrix $\rho$. Here, the reference pure state $|\varphi\rangle$ is prepared according to the unitary invariant Haar measure, i.e.,

$$|\varphi\rangle = \sum_{k=1}^{D} c_k |\varphi_k\rangle,$$

(16)

where the real and imaginary part of the coefficients $c_k$ are both drawn from a Gaussian distribution with zero mean and $|\varphi_k\rangle$ denote the states of our working basis, i.e., the Ising basis. Then, the expectation value $\langle O(t) \rangle$ can be written as [21]

$$\langle O(t) \rangle = \langle \psi(t) | O | \psi(t) \rangle + f(\varphi)$$

(17)

with the statistical error $f(|\varphi\rangle) \propto 1/\sqrt{D}$ in the limit of high temperatures $\beta \to 0$. Thus, $f(|\varphi\rangle)$ is negligibly small for medium-sized lattice sizes $L$ already. The main advantage of Eq. (17) stems from the fact, that the action of the exponentials $e^{-\beta (\mathcal{H} - \varepsilon O)}$ and $e^{-\beta Ht}$ can be conveniently evaluated by a forward propagation of pure states in imaginary time $\beta$ or real time $t$. For this forward propagation, various sophisticated methods can be used, e.g., Trotter decompositions [23] or Chebyshev polynomials [30]. In the present paper, it is sufficient to apply a fourth-order Runge-Kutta scheme [27,28] with a small time step $\delta t$. Since this scheme does not require ED and the involved operators usually feature a sparse matrix representation, we can easily calculate data for $L = 26$ sites, as done in Figs. 1 (a) and (c).

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See supplemental material for details.

\textbf{SUPPLEMENTAL MATERIAL}

Relation to other non-equilibrium scenarios

In the main part of this paper, we have studied the relaxation dynamics \( \langle \mathcal{O}(t) \rangle \), as resulting from an initial state \( \rho = \exp[-\beta(\mathcal{H} - \varepsilon \mathcal{O})]/Z \). While this setup might not be the most common preparation scheme, it can be related to other non-equilibrium scenarios in the regime of linear response, i.e., small perturbations \( \varepsilon \). To this end, let us consider a thermal initial state \( \rho(-\infty) = \rho_{\text{eq}} \) and an external field which is turned on at \( t = -\infty \) and switched off at \( t = 0 \), i.e.,

\[
\mathcal{H}(t) = \begin{cases} 
\mathcal{H} - \varepsilon(t) \mathcal{O}, & t < 0 \\
\mathcal{H}, & t \geq 0
\end{cases}.
\]

In this case, the expectation value \( \langle \mathcal{O}(t) \rangle \) can be written as

\[
\langle \mathcal{O}(t) \rangle = \int_{-\infty}^{0} dt' \phi(t - t') \varepsilon(t'),
\]

where we have introduced the response function \( \phi(t) \). If we now assume a weak and quasi-static external field \( \varepsilon(t) \approx \varepsilon \), then this setup translates into the relaxation experiment considered in the main part of this paper. Specifically, the relaxation function \( \chi(t) \) and the response function \( \phi(t) = \beta(\Delta \mathcal{O}; \mathcal{O}(t)) \) can be related according to \([S1]\)

\[
\chi(t) = \Theta(t) \left[ \chi(0) - \int_{0}^{t} dt' \phi(t') \right].
\]

Hence, knowledge of either \( \chi(t) \) or \( \phi(t) \) is sufficient to describe both scenarios.

Characterization of initial states

In the main text, we have investigated initial states \( \rho = \exp[-\beta(\mathcal{H} - \varepsilon \mathcal{O})]/Z \) and argued that the tuning of the perturbation \( \varepsilon \) allows for a preparation of \( \rho \) close to as well as far away from equilibrium. While it seems to be natural that a \( \rho \) with larger \( \varepsilon \) has to be considered as further away from equilibrium, it is still somewhat ambiguous without reasonable criteria to characterize \( \rho \) as close to or far away from equilibrium. In the following, let us discuss this point in more detail.

As already mentioned, the initial expectation value \( \langle \mathcal{O}(0) \rangle \) can be used as a natural criterion to decide whether a state is far away from equilibrium. Recall that \( \langle \mathcal{O}(0) \rangle \) is limited by the maximum eigenvalue \( \mathcal{O}_{\text{max}} \) of the operator \( \mathcal{O} \). Therefore, in the main text, we have defined the relative deviation from equilibrium

\[
\zeta(\varepsilon) = \frac{\langle \Delta \mathcal{O}(0) \rangle}{\mathcal{O}_{\text{max}} - \langle \mathcal{O}(0) \rangle},
\]

with \( \langle \mathcal{O}(0) \rangle = 0 \) in the two case studies. Thus, one might call a state \( \rho \) close to equilibrium if \( \zeta \approx 0 \) and far from equilibrium if \( \zeta \approx 1 \).

Let us discuss the dependence of \( \zeta(\varepsilon) \) on the strength of the perturbation \( \varepsilon \). In Figs. \([S1]\) (a), (c), and (e), \( \zeta(\varepsilon) \) is shown for the idealized operator \( \mathcal{O}_{I} \) with random and non-random \( r_{mn} \) as well as for the spin-current operator \( \mathcal{J} \). In all cases, the temperature is set to \( T = 100 \). Since the static curves at such high \( T \) practically do not depend on \( \mathcal{H} \), it is sufficient to show \( \langle \mathcal{J}(0) \rangle \) only for one choice of the anisotropies \( \Delta \) and \( \Delta' \). For illustration, Figs. \([S1]\) (a), (c), and (e) indicate those values of \( \varepsilon \) which are chosen in the main text to study the actual dynamics. While we observe that, for all observables, \( \zeta(\varepsilon) \) monotonically increases with increasing \( \varepsilon \) until it eventually saturates at \( \langle \mathcal{O}(0) \rangle = \mathcal{O}_{\text{max}} \), we also find that the values of \( \varepsilon \) to reach this maximum significantly depend on the specific operator. The latter fact becomes clear if one takes into account the different scaling of the horizontal \( \varepsilon \) axis for the three operators. Apparently, a specific value of \( \varepsilon \) may cause a large response for one operator but only a small response for another operator. Thus, compared to the bare value of \( \varepsilon \), the parameter \( \zeta(\varepsilon) \) yields much better information on whether \( \rho \) is close to or far away from equilibrium. However, in addition to \( \zeta(\varepsilon) \), it might be even more insightful to consider the whole spectrum and analyze the density of states (DOS) of the respective operators.

The DOS of some operator \( \mathcal{O} \) is defined as

\[
\Omega(E) = \sum_{n=1}^{D} \delta(E - \mathcal{O}_{n}),
\]

where the \( \mathcal{O}_{n} \) denote the eigenvalues of \( \mathcal{O} \). Even though it is straightforward to calculate \( \Omega(E) \) by means of exact
diagonalization, we additionally use a typicality approach to evaluate the DOS of $\mathcal{J}$. Specifically, we exploit

$$\text{Tr}[e^{-i\mathcal{O}t}] \approx \langle \Phi | e^{-i\mathcal{O}t} | \Phi \rangle$$  \hspace{1cm} (S6)

for a pure state $|\Phi\rangle$ drawn at random. Using the integral representation of the $\delta$-function,

$$\delta(E - O_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(E - O_n)}$$  \hspace{1cm} (S7)

we then have

$$\Omega(E) \approx C \int_{-\Theta}^{\Theta} dt \ e^{i\mathcal{O}t} \langle \Phi | e^{-i\mathcal{O}t} | \Phi \rangle,$$  \hspace{1cm} (S8)

where $C$ is a normalization constant and the spectral resolution $\delta E$ depends on the cutoff time, $\delta E = \pi/\Theta$. In Figs. S1 (b), (d) and (f) we depict the DOS for $\mathcal{O}_1$ with random and non-random $r_{mn}$ and for $\mathcal{J}$. Also in the case of $\Omega(E)$, we observe clear differences between the three operators. First, for $\mathcal{O}_1$ with random $r_{mn}$, $\Omega(E)$ follows the well-known semi-circle from random matrix theory and second, for $\mathcal{O}_1$ with non-random $r_{mn}$, $\Omega(E)$ exhibits a strong degeneracy around $E = 0$ and a long tail up to a quite large maximum eigenvalue $\Omega_{\text{max}}$. Third, in case of $\mathcal{J}$, $\Omega(E)$ has a Gaussian shape.

As an orientation, we indicate also in Figs. S1 (b), (d) and (f) the location of the initial states from the text with their different values of $\zeta(\epsilon)$. While this location is roughly at the maximum of $\Omega(E)$ for small $\zeta(\epsilon)$, it shifts to the borders of the spectrum as $\zeta(\epsilon)$ is increased to larger values. In particular, for $\zeta(\epsilon) \approx 1$, we find that the initial states are located in a region with a very low DOS and it is certainly justified to consider such states as far from equilibrium. Hence, our choices of $\zeta(\epsilon)$ cover the whole range of initial states close to and far away from equilibrium.

To begin with, consider a Fourier component of $\text{Tr}[\mathcal{O}(t)\mathcal{O}^N]$ at fixed frequency $\omega$:

$$\text{Tr}[\mathcal{O}(t)\mathcal{O}^N] = \sum_{\omega_{ab} = \omega} \sum_{i,j,k,l,...} (\mathcal{O}_{bi} \mathcal{O}_{aj} \cdots \mathcal{O}_{kl} \mathcal{O}_{la}),$$  \hspace{1cm} (S9)

where the second sum is just an expanded representation of $(\mathcal{O}^N)_{ba}$ and the first sum runs over all indices $a, b$ with $E_b - E_a = \omega_{ab}$. Since $\mathcal{O}$ is essentially a random matrix (see Eq. [4]), most addends in Eq. [S9] are products of independent random numbers. As such they will be random numbers themselves, with random phases (or signs, in case $\mathcal{O}$ should be real). Hence, to an accuracy set by the law of large numbers, these addends will sum up to zero. However, there are index combinations for which the respective addends are not just products of independent random numbers but necessarily real and positive. (These are also the only addends that would “survive” an averaging of Eq. [S9] over concrete implementations of $\mathcal{O}$.) For the remainder we focus exclusively on these addends.

A first necessary, but by no means sufficient condition on the indices to render the respective addends surely positive, may be stated as follows: The indices of one of the $\mathcal{O}$’s in the second sum on the r.h.s. of Eq. [S9] must be chosen as $\mathcal{O}_{ba}$. A second necessary but not sufficient condition on “surely positive” addends is that the remaining products of $\mathcal{O}$’s must consist of even numbers of factors (elements of $\mathcal{O}$). This means that the addends of the second sum on the r.h.s. of Eq. [S9] must be of the following form

$$\mathcal{O}_{b_1} \cdots \mathcal{O}_{b_{N-P-1}} \mathcal{O}_{a_{P-1}} \mathcal{O}_{a_k} \cdots \mathcal{O}_{a_l},$$  \hspace{1cm} (S10)

where the expressions below the underbraces indicate the respective numbers of multiplied elements of $\mathcal{O}$, $N - P - 1$ and $P$ are even integers. Since $N - P - 1$ and $P$ both must be even, it follows that $N$ must be odd. This already establishes the lower line of Eq. [10]. However, without any further condition on the remaining indices $i, j, k, l$, etc.,
the products given in (S10) are not yet surely positive. A third condition which (together with the previous two conditions) is sufficient for sure positiveness of the addends in Eq. (S9) is that the remaining indices of the underbraced products must all be “paired”. This means that if such a product has some factor $O_{ij}$ it must also have the factor $O_{ji}$ such as to form $|O_{ij}|^2$. (This principle also underlies the first and second necessary condition.) Such a pairing can be achieved in many ways. For example, for the first product in (S10) some of them may be schematically written as

\begin{align}
O_{bi}O_{ij}O_{jk} \cdots O_{ki}O_{ib}, \tag{S11} \\
O_{ba}O_{ib}O_{bj}O_{ji} \cdots O_{ji}O_{jb}, \tag{S12} \\
O_{bi}O_{ij} \cdots O_{ji}O_{ib}O_{bi}O_{ib}. \tag{S13}
\end{align}

The “building blocks” of these paired index combinations are sequences of indices with a mirror symmetry, such that the second part repeats the first part in reversed order, i.e., $b, i, j, k, \ldots, k, j, i, b$. The first example, i.e., (S11), consists only of one single building block. The second and the third example, i.e., (S12) and (S13), consist of two building blocks of different lengths in different order (S12: short first; S13: long first). For large $N - P = 1$ and respectively P, there are very many possibilities to create different building blocks of different lengths and arrange them in different orders. It is very hard to organize these possibilities in a reasonable manner. Fortunately, we do not need to do this here. To proceed further, we first make an assumption on a property of the building blocks and show that the validity of Eq. (4) may be inferred from this assumption. Then we justify the assumption, thereby elucidating its limitations. In order to clearly formulate the assumption we consider a “summed building block”, $f(c, Q)$, given as

\begin{equation}
\nonumber f(c, Q) := \sum_{i,j,k,l, \ldots} O_{ci}O_{ij}O_{jk} \cdots O_{kj}O_{ji}O_{ic}, \tag{S14}
\end{equation}

where $Q$ (integer, even) indicates the numbers of $O$’s in the addends. Note that the summation is over all indices except for the “end index”, in this case $c$. Now, the assumption is as follows: Assume that the $f(c, Q)$ are actually independent of $c$, regardless of $Q$. To this end, consider a sum over the terms in (S10) as

\begin{equation}
\nonumber g(a, b, N, P) := \left( \sum_{i, \ldots, j} O_{bi} \cdots O_{aj} \right)_{N-P-1} O_{ba} \left( \sum_{k, \ldots, l} O_{ak} \cdots O_{al} \right)_{P}. \tag{S15}
\end{equation}

Note that all relevant contributions of the second sum in Eq. (S9), i.e., to $(O^N)_{ba}$, are of the form $g(a, b, N, P)$. Since the paired contributions to the two sums of $g(a, b, N, P)$ all consist of summed building blocks of the form $f(c, Q)$, it follows that, given the above assumption on the $f(c, Q)$, the two sums themselves do not depend on $a, b$ respectively. Hence $g(a, b, N, P)$ is of the approximate (up to unpaired contributions) form

\begin{equation}
\nonumber g(a, b, N, P) \approx C(N, P) \cdot O_{ba}, \tag{S16}
\end{equation}

where $C(N, P)$ is a constant w.r.t. $a, b$. Since, as mentioned below Eq. (S14), $(O^N)_{ba}$ is a sum of $g(a, b, N, P)$’s over the allowed $P$, it follows that

\begin{equation}
\nonumber (O^N)_{ba} \approx O_{ba} + \text{non-surely pos. contribution}. \tag{S17}
\end{equation}

If Eq. (S17) holds, then Eq. (6) from the main text also holds, in the “law of large numbers” sense described below Eq. (S9).

Now, the remaining crucial question is if and to what extend $f(c, Q)$ is indeed independent of $c$. In order to analyze this, consider a version of the ETH ansatz as suggested, e.g., in (S3) and given by Eq. (7) with some additional conditions on the functions $\Omega(E)$, $F_{ad}(E, \omega)$ contained in that ansatz (“rigged ETH”). Let $\Omega(E)$ be a function that is piecewise nicely described by exponentials, i.e., $\Omega(E) \approx \Omega_0 \exp(\beta E)$. Let the energy intervals to which the respective mono-exponential form applies be not too small fractions of the full width of the energy spectrum, such as, e.g., for a Gaussian DOS. Let furthermore $F_{ad}(E, \omega)$ be approximately independent of $E$ within such energy intervals, i.e., $F_{ad}(E, \omega) \approx F_{ad}(\omega)$ for all $E$ from an interval. Eventually $F_{ad}(\omega)$ must be suitably narrow: Let $\delta \omega$ be the typical width of $F_{ad}(\omega)$. Then we require $Q \cdot \delta \omega$ to be still smaller than the interval on which the DOS is mono-exponential. We dub these specifications of Eq. (7) the “rigged ETH”. This eventually sets a limit to the maximum power of $O$, i.e., $N$.

As one last brutal simplification we set $F_{ad}(E) = 0$. This condition can be relaxed substantially as we will demonstrate in a forthcoming publication. Here, we employ it for simplicity and clarity of presentation.

Equipped with these specifications, we now embark on a concrete calculation of $f(c, Q)$,

\begin{equation}
\nonumber f(c, Q) = \sum_{i,j, \ldots, k} |O_{ci}|^2 |O_{ij}|^2 \cdots |O_{kl}|^2. \tag{S18}
\end{equation}

We plug Eq. (7) together with the aforementioned specifications into Eq. (S15). Furthermore, we entirely rely on the law of large numbers, i.e., we replace $|r_{ij}|^2 \to 1$. This yields

\begin{equation}
\nonumber f(c, Q) \approx \sum_{i,j, \ldots, k} \Omega_0^{-Q} e^{-\frac{Q}{2} (E_{ci} + 2E_{li} + \cdots + 2E_{ai} + E_i)} \tag{S19}
\end{equation}

\begin{equation}
\nonumber \cdot F_{ad}^2(E_i - E_i) F_{ad}^2(E_i - E_i) \cdots F_{ad}^2(E_k - E_i). \tag{S20}
\end{equation}

Now, we go from sums to integrals, essentially by plugging in the respective DOS’s

\begin{equation}
\nonumber f(c, Q) \approx \int \Omega_0^{-Q} e^{-\frac{Q}{2} (E_{ci} + 2E_{li} + \cdots + 2E_{ai} + E_i)} \tag{S21}
\end{equation}

\begin{equation}
\nonumber \cdot F_{ad}^2(E_i - E_i) F_{ad}^2(E_i - E_i) \cdots F_{ad}^2(E_{Q-1} - E_i) \tag{S22}
\end{equation}

\begin{equation}
\nonumber \cdot \Omega_0^Q e^{(E_{i+1} + E_2 + \cdots + E_{Q-1} + E_Q)} dE_i dE_2 \cdots dE_Q. \tag{S23}
\end{equation}
Here, we of course heavily rely on the above specifications. A closer look reveals that most of the DOS’s from the ETH ansatz cancel nicely with the DOS’s from the integrations:

\[ f(c, Q) \approx e^{-\frac{\beta}{2} E_c} \int e^{\frac{\beta}{2} E_Q} \cdot F_{od}(E_c - E_1) F_{od}(E_1 - E_2) \cdots F_{od}(E_{Q-1} - E_Q) \cdot dE_1 dE_2 \cdots dE_Q. \]  

(S21)

We apply the following linear change of variables,

\[ E_1, E_2 \cdots E_{Q-1}, E_Q \rightarrow \omega_1 := E_c - E_1, \omega_2 := E_1 - E_2, \cdots \omega_Q := E_{Q-1} - E_Q, \]  

(S22)

which features a Jacobian determinant which equals unity. (The matrix has lower triangle form and the diagonal elements are all \(-1\).) Furthermore, we realize

\[ E_Q = E_c - \sum_{i=1}^{Q} \omega_i. \]  

(S23)

Thus, in the new variables the integral from Eq. (S21) reads

\[ f(c, Q) \approx \int e^{\frac{\beta}{2} \sum_{i=1}^{Q} \omega_i} \cdot F_{od}^2(\omega_1) F_{od}^2(\omega_2) \cdots F_{od}^2(\omega_Q) \cdot d\omega_1 d\omega_2 \cdots d\omega_Q \]

\[ = \left( \int e^{\frac{-\beta}{2} \sum_{i=1}^{Q} F_{od}^2(\omega)} d\omega \right)^Q. \]

Displayed in this form, \( f(c, Q) \) is manifestly independent of \( c \). This completes the justification for the assumption made before Eq. (S15).

Thus, to sum up, the validity of Eq. (6) from the main text can be expected, more or less pronounced, if the following specifications, in addition to the standard ETH ansatz, apply: (i) DOS’s that are in accord with exponentials over substantial energy ranges, (ii) \( F_{od}(\bar{E}, \omega) \)'s that vary slowly or/and weakly with \( \bar{E} \), (iii) \( F_{od}(\bar{E}, \omega) \)'s that are narrow in \( \omega \) direction, (iv) small \( N \). This appears to be in accord with the numerical findings: For the idealized random-matrix model, (i)-(iii) apply very well. Thus, the above arguments apply even for large \( N \) or exponential initial states featuring large exponents. For the lattice spin-model, (i)-(iii) apply also, but not as strictly as for the random-matrix model. Thus, Eq. (6) from the main text breaks down for exponential initial states featuring large exponents.

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