Measure and Hausdorff dimension of randomized Weierstrass-type functions

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Received 11 July 2013, revised 10 February 2014
Accepted for publication 14 February 2014
Published 21 March 2014

Recommended by J Marklof

Abstract

In this paper, we consider functions of the type

$$f(x) = \sum_{n=0}^{\infty} a_n g(b_n x + \theta_n),$$

where $a_n$ are independent random variables such that $a_n$ is uniformly distributed on the interval $(-a^n, a^n)$ for some $0 < a < 1$ and $g$ is a $C^1$ periodic real function with a finite number of critical points in every bounded interval. We prove that if $\lim_{n \to \infty} b_{n+1}/b_n = b > 1$ and $ab > 1$ then the Hausdorff dimension of the graph of $f$ is almost surely equal to $D = 2 + \log a/\log b$. Furthermore, the occupation measure for $f$ is absolutely continuous with $L^2$ density almost surely, provided $b_{n+1}/b_n \geq b > 1$ and $a^2 b > 1$.

Keywords: Hausdorff dimension, Weierstrass function, occupation measure
Mathematics Subject Classification: 28A80, 28A78, 37A45

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we study a family of functions, among which probably the most famous example is the nowhere differentiable Weierstrass function (1872):

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(2\pi b^n x).$$

Weierstrass proved that the function is nowhere differentiable for some class of $a$ and $b$, later Hardy [6] extended the result for all $a, b$ such that $0 < a < 1 < b$ and $ab > 1$. Functions of the Weierstrass type were considered by Besicovitch and Ursell [4] in the 1930s.
and later in the 1980s by Berry–Lewis [3], Mauldin–Williams [13], Przytycki–Urbański [15]
and Ledrappier [11] as examples of fractal curves, questions about dimension were raised. As
the graph of $W(x)$ is roughly self-affine in the sense that $a W(bx)$ differs from $W(x)$ by a smooth
function $\cos(2\pi x)$, it suggests that the dimension should be equal to $D = 2 - \alpha$ for $\alpha = -\log a \log b$ 
(notice that under the previous conditions, $1 < D < 2$). Kaplan et al [8] in 1984 proved that the
box-counting dimension is equal to $D$. However, the question of determining the Hausdorff
dimension of the graph is still not completely solved. Przytycki and Urbański [15] in 1989
proved that the Hausdorff dimension of the graph is bigger than 1. Mauldin and Williams [13]
in 1986 considered functions $w_b(x) = \sum_{n=-\infty}^{\infty} b^{-\alpha n} \phi(b^n x + \theta_n) - \phi(\theta_n)$ for $b > 1, 0 < \alpha < 1,$
arbitrary $\theta_n \in \mathbb{R}$ and $\phi$ piecewise $C^1$ 1-periodic. They proved that for sufficiently large $b$ the
Hausdorff dimension of the graph of $w_b$ has a lower bound $2 - \alpha - C \ln b$.

Recently, in [1] it was proved that for every $b \in \mathbb{N}, b \geq 2$ there exists (explicitly given)
$a_0 \in (\frac{1}{b}, 1)$ such that the Hausdorff dimension of the graph of $W$ is equal to $D$ for every $a \in (a_0, 1)$.

In [7] Hunt considered functions of the form

$$H(x) = \sum_{n=0}^{\infty} a_n g(b_n x + \theta_n)$$

for some 1-periodic functions $g$ (including the case where $g(x) = \cos(2\pi x)$) with random
phases $\theta_n$ (independent random variables with uniform distribution on $[0, 1]$). Using potential
theory methods he proved that the Hausdorff dimension of the graph is almost surely equal to
$D$. Other dynamical systems with random phases were studied by e.g. Kifer [9].

In this paper, following the approach used by Hunt [7], we perturb randomly the parameter $a$
in the Weierstrass-type functions and obtain that the Hausdorff dimension of the graph is $D$
almost surely. The result is formulated as the following theorem.

**Theorem A.** Assume that $f(x) = \sum_{n=0}^{\infty} a_n g(b_n x + \theta_n)$ for $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is $T$-periodic,
$C^1$, with a finite number of critical points in every bounded interval. Moreover, the following
conditions are satisfied:

1. $(a_n)_{n=0}^{\infty}$ is a sequence of real independent random variables defined on some probabilistic
space $(\Omega, \mathbb{P})$ with uniform distribution on the interval $(-a^n, a^n)$ for some $0 < a < 1$,
2. $b_n \in \mathbb{R}, \liminf_{n \to \infty} \frac{\ln a_{b_n}}{b_n} = b$ for some $b > 1, ab > 1$,
3. $\theta_n \in \mathbb{R}$ for $n \geq 0$.

Then the Hausdorff and box dimension of the graph of $f$ are equal to

$$\dim_H \text{graph } f = D = 2 + \frac{\log a}{\log b}$$

almost surely.

**Remark.** The conditions on $g$ are satisfied e.g. if $g$ is a non-constant periodic analytic function
on $\mathbb{R}$.

We also examine the occupation measure $\mu$ for $f$ defined by

$$\mu(S) = \mathcal{L} \left( \{x \in J : f(x) \in S\} \right) ,$$

where $J = [0, T]$ and $\mathcal{L}$ is the Lebesgue measure. Many papers, such as e.g. [10], support the
hypothesis that when $f$ is roughly self-affine and $\mu$ is absolutely continuous with respect to
the Lebesgue measure, then the Hausdorff and box-counting dimension coincide. However, if
$\frac{\ln a_{b_n}}{b_n} \to \infty$, then in many cases the Hausdorff dimension is strictly smaller than the upper
box-counting dimension, see [2,4]. In this paper, we show that the occupation measure for the
Weierstrass-type function of the form
\[ f(x) = \sum_{n=0}^{\infty} a_n g(b_n x + \theta_n) \]
for randomly chosen \( a_n \) has \( L^2 \) density with respect to Lebesgue measure almost surely, which
is stated as theorem B.

**Theorem B.** Let \( f(x) = \sum_{n=0}^{\infty} a_n g(b_n x + \theta_n) \) for \( g : \mathbb{R} \rightarrow \mathbb{R} \), \( T \)-periodic, \( C^1 \) with a finite
number of critical points in every bounded interval, satisfy the following conditions:

1. \( (a_n)_{n=0}^{\infty} \) is a sequence of independent random variables defined on some probabilistic
   space \((\Omega, \mathbb{P})\) with uniform distribution on \((-a^n, a^n)\), \( 0 < a < 1 \),
2. \( b_n \in \mathbb{R} \) and there exists \( b > 1 \) such that \( \frac{b_n}{b_{n+1}} \geq b \) for all \( n \geq 0 \) and \( a^2 b > 1 \)
3. \( \theta_n \in \mathbb{R} \).

Then the occupation measure for the function \( f \) is absolutely continuous with respect to the
Lebesgue measure and has \( L^2 \) density almost surely.

A result of this kind (theorem B) was announced in [7], but to our knowledge it has never
been published. Theorem B immediately implies the following corollary.

**Corollary C.** For the Weierstrass-type function of the form
\[ w_{ab}(x) = \sum_{n=0}^{\infty} a_n \cos(2\pi b^n x) \]
where

1. \( (a_n)_{n=0}^{\infty} \) is a sequence of independent random variables with uniform distribution on
   \((-a^n, a^n)\), \( 0 < a < 1 \),
2. \( b \in \mathbb{N} \) and \( b > 1 \)

then if \( ab > 1 \) the Hausdorff dimension of the graph of \( w_{ab} \) is equal to \( D = 2 + \frac{\log a}{\log 6} \) almost
surely and if \( a^2 b > 1 \) the occupation measure for \( w_{ab} \) is absolutely continuous with respect to the
Lebesgue measure and has \( L^2 \) density almost surely.

The paper is organized as follows: in section 2 we present basic notation and discussion
on the assumptions which should be made on the function \( g \). Section 3 provides the proof of
theorem A, in section 4 we state theorem B, in the following section 5 some technical lemmas
are proved, and finally section 6 discusses an example—the Weierstrass function.

### 2. Preliminaries

For basic definitions and properties of the Hausdorff dimension, we refer to books by Falconer
[5] and Mattila [12]. By \( \mathcal{L} \) we denote an appropriate Lebesgue measure (on \( \mathbb{R} \) or \( \mathbb{R}^2 \)) and for
a given set \( A \) we denote its complement by \( A^c \). The Hausdorff dimension and box dimension
are denoted respectively as \( \dim_H, \dim_B \).

Now we will present some technical lemmas which will be used in the proofs of the main
results.
**Lemma 2.1.** Let $g : \mathbb{R} \to \mathbb{R}$ be a periodic $C^1$ function of period 1 with a finite number of critical points in $[0, 1]$. Then there exists $C > 0$ such that for all sufficiently small $\epsilon > 0$ there is $\delta > 0$ such that $\delta \to 0$ and one can cover the set

$$A = \{(x, y) \in [0, 1]^2 : |g(x) - g(y)| < \epsilon\}$$

by $N \leq \frac{C}{\epsilon^m}$ squares with vertical and horizontal sides of length $\delta$.

**Proof.** Let $m$ be the number of critical points of $g$ in $[0, 1]$. Since $g \in C^1$, for all $\rho > 0$ there exists $\delta(\rho) > 0$ such that $\delta(\rho) \to 0$ and the set $\{x \in [0, 1] : |g'(x)| < \rho\}$ can be covered by $m$ intervals $I_1, \ldots, I_m$ of length $\delta(\rho)$. Since $\delta(\rho) \to 0$, for every sufficiently small $\epsilon > 0$ there exists $\rho_\epsilon > 0$ such that

$$\epsilon \leq \rho_\epsilon \delta(\rho_\epsilon) \quad \text{and} \quad \delta(\rho_\epsilon) \to 0. \quad (1)$$

Fix a small $\epsilon > 0$ and set $\rho = \rho_\epsilon$, $\delta = \delta(\rho_\epsilon)$. It is obvious that the set

$$\bigcup_{i=1}^m (I_i \times [0, 1]) \cup ([0, 1] \times I_i) \quad (2)$$

can be covered by $\frac{C}{\epsilon^m}$ squares of side $\delta$, for some constant $C$ independent of $\epsilon$.

Let $J_j \subseteq [0, 1]$, $j = 1, \ldots, M$, $M \in \{m - 1, m + 1\}$, be the gaps between intervals $I_i$. Now, take $j, k \leq M$. Suppose $g''|_{J_j} \geq \rho$ and $g''|_{J_k} \geq \rho$ (the cases when $g$ is decreasing on $J_j$ or $J_k$ can be proved analogously). By the definition of $A$, if $A \cap (J_j \times J_k) \neq \emptyset$ then

$$A \cap (J_j \times J_k) \subseteq \{(x, y) : x \in J_j, k, h_k^- (x) < \epsilon < h_k^+ (x)\},$$

where

$$h_k^- (x) = (g|_{J_k})^{-1} (\max \{g(x) - \epsilon, \inf g(J_k)\})$$

and

$$h_k^+ (x) = (g|_{J_k})^{-1} (\min \{g(x) + \epsilon, \sup g(J_k)\})$$

are defined on some interval $J_j, k \subseteq J_j$ and are continuous and increasing. It is easy to check that the graph of $h_k^-$ can be covered by $\frac{C(1/\epsilon^2)}{\delta}$ squares of side $\delta$. By the mean value theorem and (1) we have $|h_k^-(x) - h_k^+(x)| \leq \frac{2\epsilon}{\delta} \leq 2\delta$. Thus we obtain that $A \cap (J_j \times J_k)$ can be covered by $\frac{C(1/\epsilon^2)}{\delta}$ squares of side $\delta$. Summing over $j$ and $k$ we obtain that the set $A \setminus \bigcup_{j, k=1}^M (I_j \times [0, 1]) \cup ([0, 1] \times I_k)$ can be covered by $\frac{C}{\epsilon^m}$ squares of side $\delta$, for $C$ independent of $\epsilon$. Using estimations for the set (2) we conclude the proof. \qed

**Definition 2.2.** For $A \subseteq [0, 1]^2$ and $\Theta = (\theta_1, \theta_2, \ldots)$, where $\theta_j \in \mathbb{R}^2$, we define:

1. $A = \sum_{n \in \mathbb{Z}} (A + (n, m))$,
2. $A_n(\Theta) = [0, 1]^2 \setminus A \cap \bigcup_{n=1}^m \bigcap \left(\frac{A - \theta_n}{b_n}\right) \cap \left(\frac{A - \theta_n}{b_n}\right) \cap \cdots \cap \left(\frac{A - \theta_n}{b_n}\right)$, where

$$A_n(\Theta) = \left\{(x, y) : (b_j x, b_j y) + \theta_j \in A_n\right\}.$$

Now we will state an important geometric lemma.
Lemma 2.3. Let \((b_n)_{n=0}^{\infty}\) such that \(b_n > 0, b_0 = 1\) and \(\frac{b_n}{b_{n+1}} \geq b\) for some \(b > 1\). Fix \(C > 0\).

Suppose that \(A \subset [0, 1]^2\) such that for some \(\delta > 0\) the set \(A\) can be covered by \(N \leq \frac{C}{\delta}\) squares of vertical and horizontal sides of length \(\delta\).

Then for sufficiently small \(\delta\) there exists \(\tilde{C} > 0\) such that for every \(\Theta\) and every \(n > 0\)

\[ L(A_n(\Theta)) < \tilde{C}\gamma^n, \]

where \(0 < \gamma < \frac{1}{b} + \epsilon_0 < 1\) and \(\epsilon_0 = \epsilon_0(\delta) > 0\) is arbitrarily small if \(\delta\) is small enough.

**Proof.** For sufficiently small \(\delta\) we can take \(k > 0\) such that

\[ \frac{2}{\delta b^k} < 1 \leq \frac{2}{\delta b^{k-1}} \tag{3} \]

and let us take

\[ \gamma = \sqrt{N \left( \delta + \frac{2}{\delta b^k} \right)^2}. \]

By (3) we can estimate \(\gamma\):

\[ \gamma < \sqrt{C\delta \left( 1 + \frac{2}{\delta b^k} \right)^2} < \sqrt{4C\delta} \]

and as \(b^{k-1} \leq \frac{2}{\delta}\):

\[ k \leq \frac{\log \frac{2}{\delta}}{\log b} + 1 = \frac{\log \frac{2}{\delta} + \log b}{\log b}. \]

Thus,

\[ \gamma < (4C\delta)^{\frac{\log b}{\log b} + 1} \xrightarrow{\delta \to 0} \frac{1}{b}. \]

Hence,

\[ \gamma < \frac{1}{b} + \epsilon_0 < 1 \]

with \(\epsilon_0\) arbitrarily small for sufficiently small \(\delta\). For simplicity we write \(A_n = A_n(\Theta)\). Take \(m = 0, 1, \ldots\). Let \(N_0 = N, \delta_0 = \delta\). Let

\[ \delta_m = \frac{\delta}{b^{mk}}. \]

Proceeding by induction on \(m\) we construct a sequence of coverings of \(A_{mk}\) by squares of horizontal and vertical sides of length \(\delta_m\). Note that \(A_0 = A\) can be covered by \(N_0\) squares of side length \(\delta_0\).

Suppose that \(A_{mk}\) can be covered by \(N_m\) squares of side length \(\delta_m\) for some \(N_m\).

We will cover \(A_{(m+1)k}\) with squares with sides \(\delta_{m+1}\) and calculate their number \(N_{m+1}\). To do it easily we will cover the plane with a square grid with side \(\frac{1}{\delta_{m+1}}\) and translate the grid by \(\theta_{(m+1)k}\). Each square \(Q\) of side \(\delta_m\) is now covered by new squares from the grid, some of them possibly sticking out of \(Q\), see figure 1. The total number of grid squares in each row can be estimated by:

- \(\delta_m^2\) squares which are completely inside \(Q\) in each row,
- there may be at most two squares which stick out of \(Q\) (horizontally).
The set $A_{\delta m}$

(a) The set $A$ and the square $Q$ from the $(m+1)$-th step with $\frac{n_{(m+1)k}}{b_{(m+1)k}}$ grid moved by $\theta_{(m+1)k}$ and a copy of covering of $A$ inside $A_{n(\Theta_1)}$.

Figure 1. Construction of coverings of the set $A_{n(\Theta)}$.

So we obtain $\delta_m b_{(m+1)k} + 2$ grid squares in each row. As we may have at most $\delta_m b_{(m+1)k} + 2$ rows (again, at most 2 rows may stick out vertically of $Q$), the total number of squares of new generation in $Q$ is at most $N(\delta_m b_{(m+1)k} + 2)^2$ (the set $A$ is covered by $N$ squares). We have $N_m$ different squares $Q$ of side $\delta_m$, hence the number $N_{m+1}$ of squares of sides $\delta_{m+1}$ covering $A_{(m+1)k}$ satisfies:

$$N_{m+1} \leq N_m N \left( b_{(m+1)k} \delta_m + 2 \right)^2.$$

Now, let $\mathcal{L}_m = N_m \delta_m^2$. We get a measure ratio:

$$\frac{\mathcal{L}_{m+1}}{\mathcal{L}_m} = \frac{N_{m+1} (\delta_{m+1})^2}{N_m (\delta_m)^2} \leq \frac{N (\delta_m b_{(m+1)k} + 2)^2 (\delta_{m+1})^2}{(\delta_m)^2} = N (\delta + \frac{\delta_{m+1}}{\delta_m})^2 \leq N (\delta + \frac{2}{b_k})^2 = \gamma^k,$$

since by assumption

$$\frac{\delta_m}{\delta_{m+1}} = \frac{b_{mk+1}}{b_{mk}} \cdots \frac{b_{mk+k}}{b_{mk+k-1}} \geq b_k^k.$$

This implies

$$\mathcal{L}(A_{mk}) \leq \mathcal{L}_m \leq \mathcal{L}_0 \gamma^{mk} \leq \gamma^{mk}.$$
Now, let \( n \in \mathbb{N} \), take \( m \) such that \( mk \leq n < (m + 1)k \). Then
\[
L(A_n) \leq L(A_{mk}) \leq \gamma^{mk} < \gamma^{n-k} = \tilde{C} \gamma^n,
\]
for \( \tilde{C} = \frac{1}{\gamma^k} \). Thus the proof is finished. \( \square \)

3. Hausdorff dimension

In this section we will prove, similarly to [7] that the Hausdorff dimension of the graph of \( f \) is equal to \( D = 2 + \frac{\log a}{\log b} \) almost surely.

**Theorem A.** Assume that \( f(x) = \sum_{n=0}^{\infty} a_n g(b_n x + \theta_n) \) for \( x \in \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) is \( T \)-periodic, \( C^1 \), with a finite number of critical points in every bounded interval. Moreover, the following conditions are satisfied:

1. \( \{a_n\}_{n=0}^{\infty} \) is a sequence of real independent random variables defined on some probabilistic space \( (\Omega, \mathbb{P}) \) with uniform distribution on the interval \((-a^n, a^n)\) for some \( 0 < a < 1 \),
2. \( b_n \in \mathbb{R} \), \( \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = b \) for some \( b > 1 \), \( ab > 1 \),
3. \( \theta_n \in \mathbb{R} \) for \( n \geq 0 \).

Then the Hausdorff and box dimension of the graph of \( f \) are equal to
\[
\dim_H \text{graph} f = D = 2 + \frac{\log a}{\log b}
\]
almost surely.

We can obviously assume that the period of \( g \) is 1 and \( b_0 = 1 \). In the estimations we consider the graph of \( f \) over the interval \( J = [0, 1] \).

3.1. Upper bound

We would like to estimate the upper box dimension of the graph of the function, which is defined by
\[
\overline{\dim}_B \text{graph} f = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{-\log \epsilon},
\]
where \( N(\epsilon) \) denotes the minimal number of balls of radius \( \epsilon \) which cover \( \text{graph} (f|_J) \). Fix \( \epsilon > 0 \), let \( n \) be the minimal number such that \( \frac{1}{b_n} < \epsilon \). We will estimate the number \( N(\frac{1}{b_n}) \).

Let us divide the interval \( J \) into intervals of length \( \frac{1}{b_n} \) (the last interval may be shorter) and denote one of such intervals as \( I \). Fix \( x, y \in I \). We have \( |x - y| \leq \frac{1}{b_n} \) and we obtain
\[
|f(x) - f(y)| \leq L (|a_0| b_0 + \ldots + |a_n| b_n) |x - y| + 2M \sum_{k=n+1}^{\infty} |a_k|, \quad (4)
\]
where \( L \) is the Lipschitz constant of \( g \) and \( M = \sup_{x \in J} |g(x)| \). As \( |a_k| \leq a^k \) almost surely we can assume that
\[
\sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=n+1}^{\infty} a^k = \frac{a^{n+1}}{1-a}. \quad (5)
\]
Fix $\frac{1}{a} < b' < b$. Then $\frac{b_{n+1}}{b_n} > b'$ for every $n \geq n_0$ for some $n_0$ and

$$|a_0|b_0 + \cdots + |a_n|b_n \leq b_0 + ab_1 + \cdots + a^n b_n \leq c + \frac{a^{n+1} b_n}{(b')^n b_{n-1}} + \cdots + \frac{a^{n-1} b_n b_0 + a^n b_n}{(b')^n b_{n-1}},$$

where $c = b_0 + ab_1 + \cdots + a^n b_{n_0}$. Using this together with (5) and (4) we obtain

$$|f(x) - f(y)| \leq \frac{Lc}{b_n} + \left(\frac{ab'}{ab' - 1} + \frac{2Ma}{1-a}\right) a^n.$$  

We have $b_n > c_1 (b')^n > c_1$ for some $c_1$. Thus

$$|f(x) - f(y)| \leq c_2 a^n$$

for some $c_2$. Since we have at most $b_n + 1$ intervals $I$,

$$N \left(\frac{1}{b_n}\right) \leq c_2 a^n b_n (b_n + 1) \leq c_3 a^n b_n^2$$

for some $c_3$. Therefore,

$$\dim_B \text{graph } f \leq \lim_{n \to \infty} \frac{\log N \left(\frac{1}{b_n}\right)}{-\log \left(\frac{1}{b_n}\right)} \leq \lim_{n \to \infty} \frac{\log c_2 a^n b_n^2}{\log b_n} \leq 2 + \lim_{n \to \infty} \frac{n \log a}{\log b_n} = 2 + \frac{\log a}{\log b} \leq 2 + \log \frac{a}{b}.$$  

3.2. Lower bound

In the proof we follow a method used in [7]. Let $\nu$ be the Lebesgue measure lifted to the graph of $f|J$. We will use potential theory methods and estimates of the $t$-energy of the measure $\nu$, which by definition is equal to

$$I_t(\nu) = \iint_{J \times J} \frac{d\nu(x) d\nu(y)}{|x-y|^t}. \quad (6)$$

We will show that the integral (6) is finite for every $1 < t < D$. As $\dim_H(A) = \sup \{ t : I_t(\nu) < \infty \text{ for some measure } \nu \text{ supported on } A \}$ (see e.g. [5]) then this will imply that the Hausdorff dimension is at least $D$. This, together with the upper bound and the inequality $\dim_H \text{graph } f \leq \dim_B \text{graph } f \leq \dim_B \text{graph } f \leq D$, will show the equality $\dim_H \text{graph } f = \dim_B \text{graph } f = D$.

We obtain

$$I_t(\nu) = \iint_{J \times J} \frac{dx \, dy}{((x-y)^2 + (f(x) - f(y))^2)^t}. \quad (7)$$

Let us fix $t \in (1, D)$. To show that (7) is finite almost surely we will show that

$$E_t = \int I_t(\nu) dP = \int J \iint_{J \times J} \frac{dx \, dy}{((x-y)^2 + (f(x) - f(y))^2)^t} dP$$

is finite. By the Fubini theorem:

$$E_t = \int \int \int \frac{dP}{((x-y)^2 + (f(x) - f(y))^2)^t} dx \, dy.$$
Now, let \( z_{x,y} = f(x) - f(y) \) for some \( x, y \in J \). As \( z_{x,y} \) is a sum of independent random variables we may write its density \( h_{x,y} \) as an infinite convolution \( h_{x,y} = h_{x,y}^{(0)} * h_{x,y}^{(1)} * \cdots \) of densities

\[
h_{x,y}^{(n)} = \frac{1}{2a^n} |g(b_n x + \theta_n) - g(b_n y + \theta_n)|.
\]

Furthermore,

\[
\int \Omega \left( (x - y)^2 + (f(x) - f(y))^2 \right)^{t/2} = \int_{-\infty}^{\infty} h_{x,y}(s) \, ds \leq C \sup h_{x,y} \left| x - y \right|^{t-1}
\]

for some \( C > 0 \), because \( t > 1 \).

**Definition 3.1.** For \( n \geq 0 \) and \( \epsilon > 0 \) define the sets

\[
A_n = \{(x, y) \in J \times J : |g(b_n x + \theta_n) - g(b_n y + \theta_n)| \geq \epsilon \}.
\]

Let us define the set \( B_n \):

\[
B_n = A_0^c \cap A_1^c \cap \ldots \cap A_{n-1}^c \cap A_n,
\]

where the complements are taken with respect to \( J \times J \). We can write that \( J \times J = \bigcup_n B_n \cup C \), where \( C = (J \times J) \setminus \bigcup_n B_n \).

Take a small \( \epsilon > 0 \) and set

\[
A = A_0^c, \quad \theta_n = (\theta_n, \theta_n).
\]

Then the set \( A_0(\Theta) \) from definition 2.2 is equal to \( A_0^c \cap \ldots \cap A_n^c \). Applying lemmas 2.1 and 2.3 we obtain

\[
\mathcal{L}(A_0^c \cap \ldots \cap A_n^c) < \tilde{C} y^n
\]

for \( y < \frac{1}{b} + \epsilon_0 \) where \( \epsilon_0 > 0 \) is arbitrarily small for small \( \epsilon \).

**Lemma 3.2.** \( \mathcal{L}(C) = 0 \).

**Proof.** From the fact that \( C = \bigcap_{n=0}^{\infty} A_n^c \) using (10) we get that \( \mathcal{L}(C) = 0 \). \( \square \)

By (10) we obtain

\[
\mathcal{L}(B_n) \leq \mathcal{L}(A_0^c \cap A_1^c \cap \ldots \cap A_{n-1}^c) < \tilde{C} y^{n-1}.
\]

Take \((x, y) \in B_n\). We have

\[
\epsilon \leq |g(b_n x + \theta_n) - g(b_n y + \theta_n)| \leq L b_n \left| x - y \right|,
\]

where \( L \) is a Lipschitz constant of \( g \). Since \( h_{x,y} \) is the convolution of \( h_{x,y}^{(n)} \) we have

\[
\sup h_{x,y} \leq \sup h_{x,y}^{(n)} \leq \frac{1}{2a^n} \epsilon.
\]

On the other hand, taking \( b' > b \) arbitrarily close to \( b \), we obtain \( b_n \leq c(b')^n \) for some \( c > 0 \) and

\[
\left| x - y \right|^{1-t} \leq \left( \frac{\epsilon}{L b_n} \right)^{1-t} \leq \left( \frac{\epsilon}{c L (b')^n} \right)^{1-t}.
\]
By this and (8) and (9),

\[ E_t \leq C \sum_n \int_{B_n} |x - y|^{1-t} \sup h^{(n)}_{1/2} \, dx \, dy \]

\[ \leq C \sum_n \int_{B_n} \left( \frac{\epsilon}{cL(b')}^n \right)^{1-t} \frac{1}{2a^n \epsilon} \, dx \, dy = C_1 \sum_n \int_{B_n} \frac{1}{a^n (b')^{m(1-t)}} \, dx \, dy \]

\[ \leq C_2 \sum_n \frac{\mathcal{L}(B_n)}{(a(b')^{1-t})^n} \leq C_3 \sum_n \left( \frac{\gamma}{a(b')^{1-t}} \right)^n < \infty \]

if only \( \gamma < a(b')^{1-t} \). The last step is to check this condition.

Since \( \gamma < \frac{1}{t} + \epsilon_0 \) for arbitrarily small \( \epsilon_0 \) and \( b' \) can be chosen arbitrarily close to \( b \) it is sufficient to check that

\[ \frac{1}{b} < ab^{1-t} \]

which holds because \( t < D \). Hence we obtain the finiteness of (8), which concludes the proof.

4. Proof of theorem B

We state theorem B once again.

**Theorem B.** Let \( f(x) = \sum_{n=0}^\infty a_n g(b_n x + \theta_n) \) for \( g : \mathbb{R} \to \mathbb{R}, T \)-periodic, \( C^1 \) with a finite number of critical points in every bounded interval, satisfy the following conditions:

1. \( (a_n)_{n=0}^\infty \) is a sequence of independent random variables defined on some probabilistic space \( (\mathcal{F}, \mathbb{P}) \) with uniform distribution on \((-a^b, a^b)\), \( 0 < a < 1 \),
2. \( b_n \in \mathbb{R} \) and there exists \( b > 1 \) such that \( \frac{b_n}{b_n} \geq b \) for all \( n \geq 0 \) and \( a^2 b > 1 \)
3. \( \theta_n \in \mathbb{R} \).

Then the occupation measure for the function \( f \) is absolutely continuous with respect to the Lebesgue measure and has \( L^2 \) density almost surely.

**Proof.** In the proof we will use methods used by Peres and Solomyak, see e.g. [14].

We would like to prove that \( ||\mu||_2 < \infty \) almost surely. By the Parseval formula it is sufficient to prove that \( ||\hat{\mu}||_2 < \infty \) almost surely, where \( \hat{\mu} \) is the Fourier transform of the measure \( \mu \). As previously, we can assume \( T = 1, b_0 = 1 \) and consider the graph over the interval \( J = [0, 1] \).

The Fourier transform of \( \mu \) is defined as

\[ \hat{\mu}(u) = \int_{-\infty}^{\infty} e^{iux} \, d\mu(x) = \int_{0}^{1} e^{iuf(x)} \, dx, \]

for \( u \in \mathbb{R} \). We have

\[ ||\hat{\mu}||_2^2 = \int_{-\infty}^{\infty} |\hat{\mu}(u)|^2 \, du = \int_{-\infty}^{\infty} \hat{\mu}(u) \overline{\hat{\mu}(u)} \, du = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iuf(x)} \, dx \, e^{-iuf(y)} \, dy \, du \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iuf(x)-f(y)} \, dx \, dy \, du. \] (12)

We will integrate this expression over the probabilistic space \( \Omega \).

\[ I = \int_{\Omega} ||\hat{\mu}||_2^2 \, d\mathbb{P} = \int_{\Omega} \int_{-\infty}^{\infty} |\hat{\mu}(u)|^2 \, du \, d\mathbb{P} = \int_{\Omega} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iuf(x)-f(y)} \, dx \, dy \, du \, d\mathbb{P} \]

\[ = \lim_{\mu \to \infty} \int_{\Omega} \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} e^{iuf(x)-f(y)} \, dx \, dy \, du = \lim_{\mu \to \infty} \mu^3. \]
If \( I \) is finite, the integral (12) (so that our norm) is also finite almost surely.

Applying the Fubini theorem to \( I_{u_0} \) we may change the integration order and get:

\[
I_{u_0} = \int_{-u_0}^{u_0} \int_{J} \int_{\Omega} e^{iu(f(x) - f(y))} \, dP \, dy \, du.
\]

Let us denote

\[
Z_n = a_n(g(b_n x + \theta_n) - g(b_n y + \theta_n)),
\]

where \( (Z_n)_{n=0}^{\infty} \)—independent random variables with uniform distribution on \((-|\alpha_n|, |\alpha_n|)\) for \( \alpha_n = a^n(g(b_n x + \theta_n) - g(b_n y + \theta_n)) \).

Since \( f(x) \) is a series of independent random variables we obtain

\[
I_{u_0} = \int_{-u_0}^{u_0} \int_{J} \int_{\Omega} e^{iu(f(x) - f(y))} \, dP \, dy \, du = \int_{-u_0}^{u_0} \int_{J} \int_{\Omega} e^{iuZ_n} \, dP \, dy \, du
\]

\[
= \int_{-u_0}^{u_0} \int_{J} \int_{\Omega} \prod_{n=0}^{\infty} \frac{1}{2\alpha_n} e^{iu} \, dx \, dy \, du = \int_{-u_0}^{u_0} \int_{J} \int_{\Omega} \prod_{n=0}^{\infty} \frac{\sin(\alpha_n u)}{u} \, dx \, dy \, du
\]

Now, let us denote

\[
x_n = a_n u = a^n(g(b_n x + \theta_n) - g(b_n y + \theta_n))u.
\]

Then

\[
I_{u_0} = \int_{-u_0}^{u_0} \int_{J} \int_{\Omega} \prod_{n=0}^{\infty} \frac{\sin(x_n)}{x_n} \, dx \, dy \, du.
\]

To conclude the proof of theorem B it is sufficient to show that the integral (13) is finite and the estimations are independent from \( u_0 \). This will be done in the following proposition.

**Proposition 4.1.** Let \( x_n = a^n(g(b_n x + \theta_n) - g(b_n y + \theta_n))u \) for some \( u \in \mathbb{R} \), where \( x, y \in [0, 1] \) and \( a, b_n, g \) are defined in theorem B. Then

\[
\int_{-\infty}^{\infty} \int_{J} \prod_{n=0}^{\infty} \left| \frac{\sin(x_n)}{x_n} \right| \, dx \, dy < \infty.
\]

\( \square \)

### 5. Proof of proposition 4.1

Fix \( M > 0 \). We can divide our integral into three parts, which will be estimated separately

\[
\int_{-\infty}^{\infty} \int_{J} \prod_{n=0}^{\infty} \left| \frac{\sin(x_n)}{x_n} \right| \, dx \, dy
\]

\[
= \int_{-M}^{M} \int_{J} \prod_{n=0}^{\infty} \left| \frac{\sin(x_n)}{x_n} \right| \, dx \, dy + \int_{-\infty}^{-M} \int_{J} \prod_{n=0}^{\infty} \left| \frac{\sin(x_n)}{x_n} \right| \, dx \, dy
\]

\[
+ \int_{M}^{\infty} \int_{J} \prod_{n=0}^{\infty} \left| \frac{\sin(x_n)}{x_n} \right| \, dx \, dy
\]

\( = I_1 + I_2 + I_3. \)
Using the fact that \( |\sin x| \leq 1 \):
\[
I_1 \leq 2M < \infty.
\]
As \( I_2 \) and \( I_3 \) can be estimated in the same way, we will estimate only \( I_3 \).

Fix \( \epsilon > 0 \). Consider the set
\[
A_n = \{(x, y) \in J \times J : |g(b_n x + \theta_n) - g(b_n y + \theta_n)| \geq \epsilon \}
\]
as in definition 3.1.

**Definition 5.1.** For \( 0 \leq n_0 < n_1 \) define
\[
B_{n_0, n_1} = A_{n_0}^c \cap A_{n_1}^c \cap \ldots \cap A_{n_0-1}^c \cap A_{n_0} \cap A_{n_0+1}^c \cap \ldots \cap A_{n_1-1}^c \cap A_{n_1}
\]
where the complements are taken with respect to \( J \times J \).

It means that \( B_{n_0, n_1} \) is a set where the condition \( |g(b_n x + \theta_n) - g(b_n y + \theta_n)| \geq \epsilon \) holds the first time for \( n_0 \) and the next time for \( n_1 \).

Now we divide \( J \times J = \bigcup_{n_0} \bigcup_{n_1>n_0} B_{n_0, n_1} \cup C \), where \( C = (J \times J) \setminus \bigcup_{n_0} \bigcup_{n_1>n_0} B_{n_0, n_1} \).

We would like to prove the following lemma.

**Lemma 5.2.** \( \mathcal{L}(C) = 0 \).

**Proof.** By definition
\[
C = \bigcap_{n=0}^{\infty} A_{n}^c \cup \bigcup_{n_0=0}^{\infty} \left( A_{n_0} \cap \bigcap_{n_1>n_0} A_{n_1}^c \right).
\]
By (10) we have that \( \mathcal{L}(\bigcap_{n=0}^{\infty} A_{n}^c) = 0 \) and \( \mathcal{L}(\bigcap_{n_1>n_0} A_{n_1}^c) = 0 \), hence \( \mathcal{L}(C) = 0 \).

**Fact 5.3.** For \( 0 \leq n_0 < n_1 \)
\[
\int \int_{B_{n_0, n_1}} \prod_{n=0}^{\infty} \left| \frac{\sin x_n}{x_n} \right| \, dx \, dy \leq \frac{\mathcal{L}(B_{n_0, n_1})}{a^{n_0+n_1} u^2 \epsilon^2}.
\]

**Proof.**
\[
\int \int_{B_{n_0, n_1}} \prod_{n=0}^{\infty} \left| \frac{\sin x_n}{x_n} \right| \, dx \, dy = \int \int_{B_{n_0, n_1}} \left| \frac{\sin x_0 \sin x_{n_1}}{x_0 \sin x_{n_1}} \right| \prod_{n \in \mathbb{N} \setminus \{n_0, n_1\}} \left| \frac{\sin x_n}{x_n} \right| \, dx \, dy.
\]
Since we can estimate \( \prod_{n \in \mathbb{N} \setminus \{n_0, n_1\}} \left| \frac{\sin x_n}{x_n} \right| \leq 1 \) and \( |\sin x_i| \leq 1 \), \( i = 0, 1 \) and use the definition of the set \( B_{n_0, n_1} \), we obtain
\[
\left| \frac{\sin x_0 \sin x_{n_1}}{x_0 \sin x_{n_1}} \right| \leq \frac{1}{\epsilon u a^{n_1}},
\]
so that
\[
\int \int_{B_{n_0, n_1}} \left| \frac{\sin x_0 \sin x_{n_1}}{x_0 \sin x_{n_1}} \right| \prod_{n \in \mathbb{N} \setminus \{n_0, n_1\}} \left| \frac{\sin x_n}{x_n} \right| \, dx \, dy
\]
\[
\leq \int \int_{B_{n_0, n_1}} \frac{1}{\epsilon^2 u^2 a^{n_0+n_1}} \, dx \, dy
\]
\[
= \frac{\mathcal{L}(B_{n_0, n_1})}{\epsilon^2 u^2 a^{n_0+n_1}}.
\]

**Observation 5.4.**
\[
\int \int_{J \times J} \prod_{n=0}^{\infty} \left| \frac{\sin x_n}{x_n} \right| \, dx \, dy \leq \frac{1}{\epsilon^2 u^2} \sum_{n_0} \sum_{n_1>n_0} \frac{\mathcal{L}(B_{n_0, n_1})}{a^{n_0+n_1}}.
\]

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Proof.
\[
\int \int_{J \times J} \prod_{n=0}^{\infty} \frac{\sin x_n}{x_n} \, dx \, dy = \int \int_{J \times J} \prod_{n=0}^{\infty} \frac{\sin x_n}{x_n} \, dx \, dy
\]
\[
\leq \sum_{n_0} \sum_{n_1 > n_0} \int \int_{B_{n_0}, n_1} \prod_{n=0}^{\infty} \frac{\sin x_n}{x_n} \, dx \, dy
\]
\[
\leq \sum_{n_0} \sum_{n_1 > n_0} \frac{\mathcal{L}(B_{n_0}, n_1)}{\epsilon^2 u^2 a^{n_0+n_1}}.
\]
\[
\leq \sum_{n_0} \sum_{n_1 > n_0} \frac{\mathcal{L}(B_{n_0}, n_1)}{\epsilon^2 u^2 a^{n_0+n_1}} < \infty.
\]

To complete the proof we now need to show the following lemma.

Lemma 5.5. We have
\[
\sum_{n_0} \sum_{n_1 > n_0} \frac{\mathcal{L}(B_{n_0}, n_1)}{a^{n_0+n_1}} < \infty.
\]

Proof. Fix \(0 \leq n_0 < n_1\). We need to estimate the measures \(\mathcal{L}(B_{n_0}, n_1)\). By definition
\[
\mathcal{L}(B_{n_0}, n_1) \leq \mathcal{L}(A_{n_0}^c \cap A_{n_1}^c \cap \ldots \cap A_{n_0-1}^c \cap A_{n_0+1}^c \cap A_{n_1+1}^c \cap \ldots \cap A_{n_1-1}^c). \tag{14}
\]

Now set \(A = \cap A_{n_0}^c\),
\[
\tilde{b}_n = \begin{cases} b_n & \text{for } n = 0, \ldots, n_0 - 1 \\ b_{n+1} & \text{for } n \geq n_0. \end{cases}
\]

We have \(\frac{\tilde{b}_n}{b_n} \geq b\) or \(\frac{\tilde{b}_{n_0}}{b_{n_0}} \geq b^2 \geq b\) (as \(b > 1\)) and
\[
\tilde{\theta}_n = \begin{cases} \theta_n & \text{for } n = 0, \ldots, n_0 - 1 \\ \theta_{n+1} & \text{for } n \geq n_0. \end{cases}
\]

Then the set \(A_{n_0-2}(\Theta)\) from definition 2.2 is equal to \(A_{n_0}^c \cap A_{n_1}^c \cap \ldots \cap A_{n_0-1}^c \cap A_{n_0+1}^c \cap \ldots \cap A_{n_1-1}^c\). We may apply lemmas 2.1 and 2.3 to obtain
\[
\mathcal{L}(A_{n_0}^c \cap A_{n_1}^c \cap \ldots \cap A_{n_0-1}^c \cap A_{n_0+1}^c \cap A_{n_1+1}^c \cap \ldots \cap A_{n_1-1}^c) \leq \tilde{C} \gamma^{n_1-2}
\]
for \(\gamma < \frac{1}{b} + \epsilon_0\), where \(\epsilon_0\) can be arbitrarily small. From this and (14) we get
\[
\sum_{n_0} \sum_{n_1 > n_0} \frac{\mathcal{L}(B_{n_0}, n_1)}{a^{n_0+n_1}} \leq \tilde{C} \sum_{n_0} \sum_{n_1 > n_0} \gamma^{n_1-2}
\]
\[
= \tilde{C} \gamma^2 \sum_{n_0} \sum_{n_1 > n_0} \gamma^{n_1-2} = \frac{\tilde{C}}{\gamma(a - \gamma)(1 - \frac{1}{a^2})} < \infty
\]
as \(\frac{\gamma}{a} < \frac{1}{b} + \frac{\epsilon_0}{a} < 1\), because \(a^2 b > 1\). Therefore (5.5) is satisfied and lemma 5.5 is proved.

Using lemma 5.2, fact 5.3, observation 5.4 and lemma 5.5 we conclude that the integral \(I_3\) is finite, which ends the proof of proposition 4.1.
Figure 2. Set $A_n^x$ for $g(x) = \cos(2\pi x)$, $a = 0.8$, $b = 2$, $N = 2$ and $\epsilon = 0.05$.

6. Example—the Weierstrass function

Let us consider the function

$$W(x) = \sum_{n=0}^{\infty} a_n \cos(2\pi b^n x),$$

where $(a_n)_{n=0}^{\infty}$—independent random variables with uniform distribution on $(-a^n, a^n)$, for $0 < a < 1 < b$, $ab > 1$ and $b > 1$. Here $g(x) = \cos(2\pi x)$.

The sets $A_n^x$ have the form:

$$A_n^x = \{(x, y) : |\cos(2\pi b^n x) - \cos(2\pi b^n y)| < \epsilon\} = \left\{\left(\frac{x}{2\pi b^n}, \frac{y}{2\pi b^n}\right) : |\cos x - \cos y| < \epsilon\right\}$$

$$= \frac{1}{2\pi b^n} \left\{(x, y) : 2 \cos \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right) \right\} < \epsilon\right\}$$

$$= \frac{1}{2\pi b^n} \left\{(u + v, u - v) : |2 \cos u \cos v| < \epsilon\right\}.$$

The inequality $|\cos u \cos v| < \frac{\epsilon}{2}$ is true when both $|\cos u| \leq \sqrt{\frac{\epsilon}{2}}$ and $|\cos v| \leq \sqrt{\frac{\epsilon}{2}}$. The cosine function near its zeros behaves nearly like a linear function, so we can approximate the set $A_n^x$ by a sum of rectangles with width at most $C\sqrt{\epsilon}$ for constant $C > 0$. It is illustrated in figure 2.

From the second part of theorem B we obtain that the occupation measure on the graph of $W(x)$ has $L^2$ density almost surely and from theorem A we obtain that the Hausdorff dimension of the graph is almost surely equal to $D = 2 + \frac{\log a}{\log b}$.

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