A Kolmogorov-type theorem for stochastic fields

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ABSTRACT

We generalize the Kolmogorov continuity theorem and prove the continuity of a class of stochastic fields with the parameter. As an application, we derive the continuity of solutions for nonlocal stochastic parabolic equations driven by non-Gaussian Lévy noises.

1. Introduction

In 1933, Kolmogorov [1] proved the celebrate result as the following:

Proposition 1.1. If \( \{X(t), t \in [0, 1]\} \) is a stochastic process, such that

\[
\mathbb{E}[|X(t) - X(s)|^2] \leq C|t - s|^{1+\varepsilon}, \tag{1.1}
\]

where \( \gamma, \varepsilon \) and \( C \) are positive constants independent of \( t \), then the trajectories of the process are continuous with probability 1.

The above result is one of the central aspects of stochastic analysis with many applications in the study of the asymptotic distributions of certain tests [1], the tightness of a sequence of stochastic processes [2, 3], the existence of strong solutions and stochastic flows to stochastic differential equations [4], and the regularization of local time for continuous semimartingale [5]. Now, Proposition 1.1 is well known for Kolmogorov’s continuity theorem/criteron. Since then, Kolmogorov’s result was strengthened in different forms [6–10]. To be precise, we give a standard and simple type for the Kolmogorov theorem.

Proposition 1.2. [4, Theorem 3.3.8, p31] and [11, Theorem 2.1, p25] Let \( \{X(t), t \in [0, 1]\} \) be a stochastic process defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Assume that there exist three strictly positive constants \( \gamma, C \) and \( \varepsilon \) such that for all \( 0 \leq t, s \leq 1 \), (1.1) holds. Then \( X \) has a continuous realization \( \tilde{X} \), namely, there exists \( \Omega_0 \) such that \( \mathbb{P}(\Omega_0) = 1 \) and...
for each \( \omega \in \Omega_0, X(t, \omega) = \tilde{X}(t, \omega) \) and \( \tilde{X}(t, \omega) \) is a continuous function of \( t \). Moreover, for every \( x \in [0, e/\gamma) \),

\[
\mathbb{E} \left[ \sup_{t \neq s} \left( \frac{|\tilde{X}(t) - \tilde{X}(s)|}{|t - s|^x} \right)^\gamma \right] < +\infty .
\]

In the present paper, we will generalize Proposition 1.2 to the following form.

**Theorem 1.3.** Let \( (H, \| \cdot \|_H) \) be a Banach space and let \( \{X_t(x), x \in [0,1]^d, t \in [0,1]\} \) be an \( H \)-valued stochastic field. Let \( \phi \) be a nonnegative and nondecreasing continuous function on \( \mathbb{R}_+ = (0, +\infty) \) such that \( \lim_{r \to 0+} \phi(r) = 0 \). Suppose that there exist two strictly positive constants \( c \) and \( C \) such that

\[
\mathbb{E} \sup_{0 \leq t \leq 1} \|X_t(x) - X_t(y)\|_H^\gamma \leq C|x - y|^d \phi(|x - y|), \quad x, y \in [0,1]^d , \quad (1.2)
\]

and there exists another constant \( 0 < \vartheta < 1/\gamma \) such that when \( \gamma \geq 1 \),

\[
\sum_{i=0}^\infty \phi^\vartheta(2^{-i}) < +\infty , \quad (1.3)
\]

and when \( \gamma < 1 \),

\[
\sum_{i=0}^\infty \phi^{\vartheta i}(2^{-i}) < +\infty . \quad (1.4)
\]

Further, we assume that there is a constant \( \lambda \geq 1 \) such that for all sufficiently large natural number \( n \),

\[
\lambda^{-1} \leq \frac{\phi(2^{-n})}{\phi(2^{-n+1})} \leq \lambda . \quad (1.5)
\]

Then, there is a realization \( \tilde{X} \) of \( X \) such that \( \tilde{X} \) is continuous in \( x \) and for every \( x \in (0, 1/\gamma - \vartheta] \)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq 1} \left[ \tilde{X}_t(\cdot) \right]_{x, \varphi}^\gamma \right] < +\infty , \quad (1.6)
\]

where

\[
\left[ \tilde{X}_t(\cdot) \right]_{x, \varphi} := \sup_{x \neq y} \left\{ \frac{\|\tilde{X}_t(x) - \tilde{X}_t(y)\|_H}{\varphi^\vartheta(|x - y|)}, \quad x, y \in [0,1]^d \right\} .
\]

**Remark 1.4.** (i) Let \( T \) and \( c_1 \) be two positive real numbers, and let \( B_{c_1} \) be the ball centered at 0 with radius \( c_1 \). By scaling transformations, the conclusions in Theorem 1.3 are true if one replaces \([0,1]\) and \([0,1]^d\) by \([0, T]\) and \( B_{c_1} \), respectively.

(ii) Theorem 1.3 holds true as well if one uses a complete metric space \((S, \rho)\) instead of the Banach space \((H, \| \cdot \|_H)\).
(iii) When \( \varphi(|x-y|) = |x-y|^e \), then (1.5) is true. Moreover, for every \( \vartheta > 0 \),
\[
\sum_{i=0}^{\infty} \varphi^\vartheta(2^{-i}) = \sum_{i=0}^{\infty} 2^{-i\vartheta} < +\infty \quad \text{and} \quad \sum_{i=0}^{\infty} \varphi^\vartheta(2^{-i}) = \sum_{i=0}^{\infty} 2^{-i\vartheta} < +\infty.
\]
Therefore, (1.3) and (1.4) hold. By Theorem 1.3, there is a continuous realization \( \tilde{X} \) of \( X \) such that for every \( x \in [0, 1/\gamma) \)
\[
\mathbb{E} \sup_{0 \leq t \leq 1} \sup_{x \neq y} \frac{\|\tilde{X}_t(x) - \tilde{X}_t(y)\|_H^2}{|x-y|^\alpha} < +\infty.
\]
This is the result of Proposition 1.2, so we extend Kolmogorov’s continuity theorem.

(iv) The present result is optimal in the sense that if \( \varphi \) has a lower bound, then there is a random field such that (2.1) holds but there is no continuous realization. For simplicity and without loss of generality, we assume the random field is time independent, \( d = 1 \) and \( x \in [0, 1] \), and also write it by \( \{X(t), \ t \in [0, 1]\} \). Here, \( \{X(t), \ t \in [0, 1]\} \) is a Poisson stochastic process defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with parameter \( \lambda \). So for every \( 0 \leq s < t < +\infty \), \( X(t) - X(s) \) is a Poisson random variable with parameter \( \lambda(t-s) \), i.e.
\[
\mathbb{P}\{X(t) - X(s) = k\} = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}, \quad k = 0, 1, 2, \ldots
\]
Therefore, we have
\[
\mathbb{E}[|X(t) - X(s)|] = \lambda|t-s|.
\]
However, the random process \( \{X(t), \ t \in [0, 1]\} \) has nowhere continuous path for almost all \( \omega \in \Omega \).

Let \( \gamma \) be given in Theorem 1.3 and let \( \beta > 1 \) be a real number such that \( \beta > \gamma \) if \( \gamma \geq 1 \) and \( \beta > 1/\gamma \) if \( \gamma < 1 \). Set
\[
\varphi(r) = \begin{cases} (-\log (r))^{-\beta}, & \text{when } r \leq \frac{1}{2}, \\ f(r), & \text{when } r > \frac{1}{2}, \end{cases}
\]
where \( f(r) \) is an arbitrary nondecreasing continuous function defined on \( [1/2, +\infty) \) such that \( f\left(\frac{1}{2}\right) = (\log 2)^{-\beta} \). If \( \gamma \geq 1 \), we choose \( 1/\beta < \vartheta < 1/\gamma \), then
\[
\sum_{i=0}^{\infty} \varphi^\vartheta(2^{-i}) = f^\vartheta(1) + \sum_{i=1}^{\infty} \left(-\log (2^{-i})\right)^{-\beta\vartheta} = f^\vartheta(1) + \frac{1}{(\log 2)^{\beta\vartheta}} \sum_{i=1}^{\infty} \frac{1}{i^{\beta\vartheta}} < +\infty.
\]
If \( \gamma < 1 \), we fetch \( 1/(\beta\gamma) < \vartheta < 1/\gamma \), then
\[
\sum_{i=0}^{\infty} \varphi^{\gamma\vartheta}(2^{-i}) = f^{\gamma\vartheta}(1) + \sum_{i=1}^{\infty} \left(-\log (2^{-i})\right)^{-\beta\gamma\vartheta} = f^{\gamma\vartheta}(1) + \frac{1}{(\log 2)^{\beta\gamma\vartheta}} \sum_{i=1}^{\infty} \frac{1}{i^{\beta\gamma\vartheta}} < +\infty.
\]
Further, for every \( 1 \leq n \in \mathbb{N} \),
\[ \frac{\varphi(2^{-n})}{\varphi(2^{-n-1})} = \frac{(-\log(2^{-n}))^{-\beta}}{(-\log(2^{-n-1}))^{-\beta}} = \left(1 + \frac{1}{n}\right)^\beta. \]

So (1.5) holds true with \( \lambda = 2^\beta \).

Therefore, we have the following corollary.

**Corollary 1.4.** Let \((H, \| \cdot \|_H)\) be a Banach space and let \(\{X_t(x), x \in [0,1]^d, t \in [0,1]\}\) be an \(H\)-valued stochastic field. Let \(\gamma\) and \(\beta\) be two real number such that \(\beta > \gamma\) if \(\gamma \geq 1\) and \(\beta > 1/\gamma\) if \(\gamma < 1\). Suppose that (1.2) holds with \(\varphi\) given by (1.7). Then there is a realization \(X\) of \(X\) such that \(X\) is continuous in \(x\) and for every \(x \in (0,1/\gamma + \vartheta]\), (1.6) holds.

**Remark 1.2.** If one replaces the metric space \(([0,1]^d, | \cdot |)\) (| \cdot | denotes the Euclidian metric) by a \(d\)-dimensional pseudo-metric space \((T, \rho)\) (\(\rho\) is a pseudo-metric) such that the random field \(\{X(x), x \in T\}\) satisfies some moment conditions (see [8, Theorem 3.1] or [7]), then \(\{X(x), x \in T\}\) exists a modification, whose sample paths are continuous with probability one. This result does not contain Corollary 1.4, since \(\rho(x,y) = |x-y|\varphi(|x-y|)\) is not a pseudo-metric. To illustrate this claim, we take \(d=1\) and \(T = [0,1]\), then on \([0,1]\), \(\rho\) is strict increasing. For every \(0 < x_1 < x_2 < 1\) and \(x_3 := x_1 + x_2 < 1\), we have

\[ \rho(x_1, x_2) + \rho(x_2, x_3) < \rho(x_1, x_3). \]

The present result is also different from the results established in [12–15].

Let \(\gamma\) and \(\beta\) be given in Corollary 1.4. We choose \(\varphi\) as the following form

\[ \varphi(r) = \begin{cases} \frac{\log(-k_0 \log(r))}{(-\log(r))^\beta}, & \text{when } r \leq r_0, \\ g(r), & \text{when } r > r_0, \end{cases} \]  

(1.8)

where \(k_0, r_0\) are two positive real numbers such that

\[ r_0 = e^{-\frac{1}{k_0} - \beta}. \]

\(g(r)\) is an arbitrary nondecreasing continuous function defined on \([r_0, +\infty)\) such that \(g(r_0) = \log(-k_0 \log(r_0)) / (-\log(r_0))^\beta > 0\). For \(r \in (0, r_0)\), then

\[ \varphi'(r) = \frac{1}{(-\log(r))^{2\beta}} \left[ (\log(-k_0 \log(r)))'(-\log(r))^\beta - \log(-k_0 \log(r)) \left((-\log(r))^\beta\right)' \right] \]

\[ = \frac{1}{(-\log(r))^{2\beta}} \left[ -r^{-1}(-\log(r))^{\beta-1} + r^{-1}\beta \log(-k_0 \log(r)) \left((-\log(r))^\beta\right)' \right] \]

\[ = \frac{1}{r(-\log(r))^{\beta+1}} [\beta \log(-k_0 \log(r)) - 1] > 0, \]

so \(\varphi\) is nonnegative and nondecreasing on \(\mathbb{R}_+\). Observing that \(0 < r_0 < 1\), there is a positive natural number \(N_0\), such that \(2^{-N_0} \leq r_0 < 2^{-N_0+1}\).
If \( \gamma \geq 1 \), we choose \( 1/\beta < \vartheta < 1/\gamma \), then

\[
\sum_{i=0}^{\infty} \phi^\vartheta(2^{-i}) = \sum_{i=0}^{N_0-1} \phi^\vartheta(2^{-i}) + \sum_{i=N_0}^{\infty} \left[ \frac{\log (-k_0 \log (2^{-i}))}{(-\log (2^{-i}))^\beta} \right]^{\vartheta} \\
\leq \sum_{i=0}^{N_0-1} \phi^\vartheta(2^{-i}) + \frac{1}{(\log 2)^{\beta \vartheta}} \sum_{i=N_0}^{\infty} \left( \frac{\log i}{i^\beta} \right)^{\vartheta} + \left( \frac{\log (k_0 \log 2)}{i^\beta} \right)^{\vartheta} \\
< +\infty.
\]

If \( \gamma < 1 \), we fetch \( 1/(\beta \gamma) < \vartheta < 1/\gamma \), then

\[
\sum_{i=0}^{\infty} \phi^\gamma(2^{-i}) = \sum_{i=0}^{N_0-1} \phi^\gamma(2^{-i}) + \sum_{i=N_0}^{\infty} \left[ \frac{\log (-k_0 \log (2^{-i}))}{(-\log (2^{-i}))^\beta} \right]^{\gamma \vartheta} \\
\leq \sum_{i=0}^{N_0-1} \phi^\gamma(2^{-i}) + \frac{1}{(\log 2)^{\beta \gamma \vartheta}} \sum_{i=N_0}^{\infty} \left( \frac{\log i}{i^\beta} \right)^{\gamma \vartheta} + \left( \frac{\log (k_0 \log 2)}{i^\beta} \right)^{\gamma \vartheta} \\
< +\infty.
\]

Moreover, for every \( \log_2(r_0^{-1}) \leq n \in \mathbb{N}, \)

\[
\frac{\phi(2^{-n})}{\phi(2^{-n-1})} = \frac{\log (-k_0 \log (2^{-n}))}{(-\log (2^{-n}))^\beta} \times \frac{(-\log (2^{-n-1}))^\beta}{\log (-k_0 \log (2^{-n-1}))} = \left( 1 + \frac{1}{n} \right)^\beta \frac{\log (k_0 \log 2) + \log (n)}{\log (k_0 \log 2) + \log (n+1)}.
\]

So (1.5) holds true with

\[
\lambda = \max \left\{ 2^\beta, \frac{\log (k_0 \log 2 (\log_2(r_0^{-1}) + 1))}{\log (k_0 \log 2 \log_2(r_0^{-1}))} \right\}.
\]

Hence, we have the following result.

**Corollary 1.5.** Let \((H, \| \cdot \|_H), \{X_t(x), x \in [0,1]^d, t \in [0,1]\}\), \( \gamma \) and \( \beta \) be stated in Corollary 1.4. Suppose that (1.2) holds with \( \phi \) given by (1.8). Then there is a realization \( \tilde{X} \) of \( X \) such that \( \tilde{X} \) is continuous in \( x \) and for every \( x \in (0,1/\gamma - \vartheta] \), (1.6) holds.

We will give the proof details of Theorem 1.3 in Section 2, and as an application, we prove the continuity of solutions for nonlocal stochastic parabolic equations in Section 3.

### 2. Proof of Theorem 1.3

Let \( D_m \) be the set of points in \([0,1]^d\) whose components are equal to \( 2^{-m}i \) for some integral \( i \in [0,2^m] \). Then, \([0,1]^d = \cup_m D_m\). Let further \( \Delta_m \) be the set of pairs \( (x, y) \) in \( D_m \) such that \( |x - y| = 2^{-m} \). There are no more than \( d2^{(m+1)d} \) such pairs in \( D_m \).
By setting \( K_i(t) = \sup_{(x,y) \in \Delta} \| X_t(x) - X_t(y) \|_H \), then there is a constant \( C_1 \) such that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq 1} K_i(t) \right] \leq \sum_{(x,y) \in \Delta_i} \mathbb{E} \left[ \sup_{0 \leq t \leq 1} \| X_t(x) - X_t(y) \|_H^2 \right] \leq C d 2^{(i+1)d} 2^{-id} \phi(2^{-i}) = C_1 \phi(2^{-i}).
\]

(2.1)

For two points \( x_1 = (x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(d)}) \) and \( x_2 = (x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(d)}) \) in \([0,1]^d\), if \( x_1^{(i)} \leq x_2^{(i)} (1 \leq i \leq d) \), we write it as \( x_1 \leq x_2 \). Let \((x,y) \in [0,1]^d \times [0,1]^d\). Then, there is a natural number \( m_0 \geq 0 \), and two increasing sequences \( \{x_m\} \) and \( \{y_m\}\) such that for each \( m < m_0, x_m \leq x, y_m \leq y \) and from each \( m \geq m_0, x_m = x, y_m = y \).

If \( |x - y| \leq 2^{-m} \), then either \( x_m = y_m \) or \((x_m, y_m) \in \Delta_m\) and in any case
\[
X_t(x) - X_t(y) = \sum_{i=m}^{\infty} \left( X_t(x_{i+1}) - X_t(x_i) \right) + X_t(x_m) - X_t(y_m) - \sum_{i=m}^{\infty} \left( X_t(y_{i+1}) - X_t(y_i) \right).
\]

(2.2)

Since points \( x_{i+1} \) and \( x_i \) can be connected by a piecewise line involving, for any \( i \geq m \), at most \( d \) steps between the nearest neighbors in \( D_{i+1} \) such that each line segment’s length is no more that \( 2^{-i-1} \), we conclude from (2.2) that
\[
\| X_t(x) - X_t(y) \|_H \leq K_m(t) + 2d \sum_{i=m}^{\infty} K_i(t) \leq 2d \sum_{i=m}^{\infty} K_i(t).
\]

(2.3)

Let \( \alpha = 1/\gamma - \theta \). Then
\[
[X_t(\cdot)]_{x,y} \phi \\
= \sup_{x \neq y} \left\{ \frac{\| X_t(x) - X_t(y) \|_H \phi^\alpha(|x-y|)}{\phi^\alpha(|x-y|)} , \ x, y \in [0,1]^d \right\} \\
\leq \sup_{x \neq y, m \in \mathbb{N}} \left\{ \phi^{-\alpha}(2^{-m-1}) \sup_{2^{-m-1} \leq |x-y| < 2^{-m}} \| X_t(x) - X_t(y) \|_H, \ x, y \in [0,1]^d \right\} \\
+ \sup_{x \neq y} \left\{ \phi^{-\alpha}(1) \sup_{|x-y| \geq 1} \| X_t(x) - X_t(y) \|_H, \ x, y \in [0,1]^d \right\}.
\]

(2.4)

Observing that for any \((x,y) \in [0,1]^d \times [0,1]^d\) with \(|x-y| \geq 1\), can be connected by a piecewise linear path at most \( d \) steps such that every line segment’s length is no more than 1. Therefore
\[
\sup_{x \neq y} \left\{ \phi^{-\alpha}(1) \sup_{|x-y| \geq 1} \| X_t(x) - X_t(y) \|_H, \ x, y \in [0,1]^d \right\} \\
\leq d \sup_{x \neq y} \left\{ \phi^{-\alpha}(1) \sup_{|x-y| \leq 1} \| X_t(x) - X_t(y) \|_H, \ x, y \in [0,1]^d \right\}.
\]

(2.5)

Combining (2.4) and (2.5), with the aid of (2.3), we conclude that
\[ [X_t(\cdot)]_{x, \varphi} \leq \sup_{x \neq y, m \in \mathbb{N}} \left\{ \varphi^{-z}(2^{-m-1}) \sup_{|x-y| \leq 2^{-m}} \|X_t(x) - X_t(y)\|_H, \; x, y \in [0, 1]^d \right\} \\
+ d \sup_{x \neq y, m \in \mathbb{N}} \left\{ \varphi^{-z}(1) \sup_{|x-y| \leq 1} \|X_t(x) - X_t(y)\|_H, \; x, y \in [0, 1]^d \right\} \\
\leq (d + 1) \sup_{x \neq y, m \in \mathbb{N}} \left\{ \varphi^{-z}(2^{-m-1}) \sup_{|x-y| \leq 2^{-m}} \|X_t(x) - X_t(y)\|_H, \; x, y \in [0, 1]^d \right\} \\
\leq 2d(d + 1) \sup_{x \neq y, m \in \mathbb{N}} \left\{ \varphi^{-z}(2^{-m-1}) \sum_{i=m}^{\infty} K_i(t) \right\} \\
\leq 2d(d + 1) \sup_{m \in \mathbb{N}} \frac{\varphi^{-z}(2^{-m-1})}{\varphi^{-z}(2^{-m})} \sup_{x \neq y, m \in \mathbb{N}} \left\{ \varphi^{-z}(2^{-m}) \sum_{i=m}^{\infty} K_i(t) \right\} \\
\leq 2d(d + 1) \sup_{m \in \mathbb{N}} \frac{\varphi^{-z}(2^{-m-1})}{\varphi^{-z}(2^{-m})} \sum_{i=0}^{\infty} \varphi^{-z}(2^{-i}) K_i(t). \tag{2.6} \]

Thanks to (1.5), there is a constant \( C > 0 \) such that
\[
\sup_{m \in \mathbb{N}} \frac{\varphi^{-z}(2^{-m-1})}{\varphi^{-z}(2^{-m})} \leq C.
\]

Hence, we achieve from (2.6) that
\[
[X_t(\cdot)]_{x, \varphi} \leq 2d(d + 1) C \sum_{i=0}^{\infty} \varphi^{-z}(2^{-i}) K_i(t) := C(d) \sum_{i=0}^{\infty} \varphi^{-z}(2^{-i}) K_i(t). \tag{2.7}
\]

For \( \gamma \geq 1 \), by using Minkowski’s inequality, we get from (2.1) and (2.7) that
\[
\left[ \mathbb{E} \sup_{0 \leq t \leq 1} [X_t(\cdot)]_{x, \varphi}^\gamma \right]^\frac{1}{\gamma} \leq C(d) \sum_{i=0}^{\infty} \varphi^{-z}(2^{-i}) \left[ \mathbb{E} \sup_{0 \leq t \leq 1} K_i^\gamma(t) \right]^\frac{1}{\gamma} \\
\leq C(d) C_1 \sum_{i=0}^{\infty} \varphi^{-z}(2^{-i}) \varphi^\gamma(2^{-i}) \tag{2.8}
\]
\[
= C(d) C_1 \sum_{i=0}^{\infty} \varphi^\gamma(2^{-i}).
\]

Since the series in (2.2) are actually finite sums, for \( \gamma < 1 \), we gain an analogue of (2.3)
\[
\|X_t(x) - X_t(y)\|_H^\gamma \leq K_\gamma^\gamma(t) + 2d \sum_{i=m+1}^{\infty} K_i^\gamma(t) \leq 2d \sum_{i=m}^{\infty} K_i^\gamma(t). \tag{2.9}
\]

The same calculations applied to \([X_t(\cdot)]_{x, \varphi} \) in (2.4) is adapted to \([X_t(\cdot)]_{x, \varphi}^\gamma \) yields that: there is a constant \( C(d) > 0 \) such that
\[
[X_t(\cdot)]_{x, \varphi}^\gamma \leq C(d) \sum_{i=0}^{\infty} \varphi^{-z}(2^{-i}) K_i^\gamma(t). \tag{2.10}
\]

Combining (2.9) and (2.10), we arrive at
\[
\left[ \mathbb{E} \sup_{0 \leq t \leq 1} |X_t(\cdot)|^2_{2, \varphi} \right] \leq C(d) \sum_{i=0}^{\infty} \varphi^{-\gamma}(2^{-i}) \left[ \mathbb{E} \sup_{0 \leq t \leq 1} K_i^2(t) \right] 
\leq C(d) C_1 \sum_{i=0}^{\infty} \varphi^{-\gamma}(2^{-i}).
\] (2.11)

From (2.8) and (2.11), in view of (1.3) and (1.4), then
\[
\mathbb{E} \sup_{0 \leq t \leq 1} |X_t(\cdot)|^2_{2, \varphi} \leq +\infty.
\] (2.12)

It follows from (2.12) that for almost every \( \omega \), \( X_t(\cdot) \) is uniformly continuous on \([0,1]^d\) and it is uniformly in \( t \), so it make sense to set
\[
\tilde{X}_t(x, \omega) = \lim_{y \to x} X_t(y, \omega).
\]

By Fatou’s lemma and the hypothesis,
\[
\mathbb{P}\left\{ \tilde{X}_t(x, \omega) = X_t(x, \omega), \ t \in [0,1] \right\} = 1.
\]

Therefore, \( \tilde{X} \) is the desired realization.

For general \( \alpha \in (0,1/\gamma - \vartheta) \), we have
\[
[X_t(\cdot)]_{2, \varphi} = \sup_{x \neq y} \left\{ \frac{\|X_t(x) - X_t(y)\|_H}{\varphi^\gamma(|x - y|)}, \ x, y \in [0,1]^d \right\} 
= \sup_{x \neq y} \left\{ \frac{\|X_t(x) - X_t(y)\|_H}{\varphi^{-\vartheta}(|x - y|)} \times \frac{\varphi^{1-\vartheta}(|x - y|)}{\varphi^\gamma(|x - y|)}, \ x, y \in [0,1]^d \right\}. \tag{2.13}
\]

Notice that \( \varphi \) is continuous and non-decreasing, if \( \varphi(\sqrt{d}) \leq 1 \), then
\[
\sup_{x \neq y} \left\{ \frac{\varphi^{1-\vartheta}(|x - y|)}{\varphi^\gamma(|x - y|)} \right\}, \ x, y \in [0,1]^d \leq 1.
\]

If \( \varphi(\sqrt{d}) > 1 \), then there is \( 0 < \tilde{r} < \sqrt{d} \) such that \( \varphi(\tilde{r}) = 1 \). Therefore,
\[
\sup_{x \neq y} \left\{ \frac{\varphi^{1-\vartheta}(|x - y|)}{\varphi^\gamma(|x - y|)} \right\}, \ x, y \in [0,1]^d \leq \sup_{\tilde{r} \leq |x - y| \leq \sqrt{d}} \left\{ \frac{\varphi^{1-\vartheta}(|x - y|)}{\varphi^\gamma(|x - y|)} \right\} + 1 \leq \sup_{\tilde{r} \leq \sqrt{d}} \frac{\varphi^{1-\vartheta}(\tilde{r})}{\inf_{\tilde{r} \leq \tilde{r} \leq \sqrt{d}} \varphi^\gamma(\tilde{r})} + 1 < +\infty. \tag{2.14}
\]

In view of (2.13) and (2.14), (2.12) is true, thus we complete the proof. \( \square \)
3. Application to nonlocal stochastic heat equations

Definition 3.1. ([16]) Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered complete probability space with the right continuous filtration \(\mathcal{F}_t\). Let \(E\) be a ball \(B_c(0)\) in \(\mathbb{R}^d\), of radius \(c\) without the center. A time homogeneous Poisson random measure \(N\) on \((E, \mathcal{B}(E))\) over the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with an intensity measure \(\nu \times \lambda\), is a measurable function \(N : (\Omega, \mathcal{F}) \rightarrow (\mathcal{M}_+(E \times \mathbb{R}_+), \mathcal{M}_+(E \times \mathbb{R}_+))\), such that

(i) for each \(B \times I \in \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}_+)\), if \(\nu(B) < \infty\), \(N(B \times I)\) is a Poisson random variable with parameter \(\nu(B) \lambda(I)\);

(ii) \(N\) is independently scattered, i.e. if the sets \(E_j \times I_j \in \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}_+)\), \(j = 1, \ldots, n\) are pairwise disjoint, then the random variables \(N(E_j \times I_j), \ j = 1, \ldots, n\) are mutually independent;

(iii) for each \(U \subset \mathcal{B}(E)\), the \(\tilde{N} \ (= \mathbb{N} \cup \{+\infty\}\)-valued process \(\{N((0, t], U)\}_{t > 0}\) is \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted and its increments are independent of the past.

Remark 3.1. In the above definition, \(\mathcal{M}_+(E \times \mathbb{R}_+)\) is the family of all \(\sigma\)-finite positive measures on \(E \times \mathbb{R}_+\), \(\lambda\) is the Lebesgue measure on \(\mathbb{R}_+\), \(\nu\) is a Lévy measure which satisfies

\[
\int_E 1 \wedge |\nu|^2 \nu(dv) < \infty.
\]

Definition 3.2. Let \(N\) be a homogeneous Poisson random measure on \((E, \mathcal{B}(E))\) over the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The \(\mathbb{R}\)-valued process \(\{\tilde{N}((0, t], A)\}_{t > 0}\) defined by

\[
\tilde{N}((0, t], A) = N((0, t], A) - \nu(A) t, \quad t > 0, \ A \in \mathcal{B}(E),
\]

is called a compensator Poisson random measure. And now \(\{\tilde{N}((0, t], A)\}_{t > 0}\) is a martingale on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\).

Let \(x \in (0, 2]\) and let \(g \in L^1(\Omega; L^1_{loc}(0, \infty); L^1(E, \nu; L^\infty(\mathbb{R}^d)))\) such that \(\{g(t, v, x, \cdot)\}_{t \geq 0}\) as a family of \(L^1(\Omega, \mathcal{F}, \mathbb{P})\) valued random variables is \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted. Consider the following Cauchy problem:

\[
\begin{cases}
du(t, x) = \Delta^x u(t, x) dt + \int_E g(t, x, v) \tilde{N}(dt, dv), \ t > 0, \ x \in \mathbb{R}^d, \\
u(0, x) = 0, \ x \in \mathbb{R}^d,
\end{cases}
\tag{3.1}
\]

where \(\Delta^x := -(-\Delta)^x\) and \((-\Delta)^x\) is the fractional Laplacian on \(\mathbb{R}^d\), and for \(x \in (0, 2]\) defined by

\[
(-\Delta)^x \phi(x) = c(d, x) \text{P.V.} \int_{\mathbb{R}^d} \frac{\phi(y) - \phi(z + x)}{|z|^{d+x}} dy, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d), \ x \in \mathbb{R}^d,
\]

with \(c(d, x) = 2^{d-1} \pi^{-d/2} \Gamma(\frac{d+x}{2}) \Gamma(\frac{2-d}{2})\).

We call \(u\) a mild solution of (3.1) if the measurable function given by

\[
u(t, x) = \int_{(0, t]} \int_E K(t - r, \cdot) * g(r, \cdot, v)(x) \tilde{N}(dr, dv), \tag{3.2}
\]
Let $g$ be defined by (3.2). Then, $u$ admits a realization $\tilde{u}$ which is continuous in $x$, and for every $\beta \in (0, 1/p - \delta]$, every $T > 0$ and every $c_1 > 0$,

$$E \sup_{0 \leq t \leq T} \sup_{x \neq y} \left\{ \frac{|\tilde{u}(t,x) - \tilde{u}(t,y)|^p}{\varphi^{\beta p}(|x-y|)} \right\} < +\infty. \quad (3.3)$$

To prove Theorem 3.4, we need a lemma.

**Lemma 3.4.** Let $E = B_c(0) \setminus \{0\}$ ($0 < c \in \mathbb{R}$) and $p \geq 1$. Suppose $\psi \in L^p(\Omega; L^p_{\text{loc}}([0, \infty); L^p(E, \nu)))$ is an $\{F_t\}_{t \geq 0}$-adapted stochastic process and

$$I_t = \int_{(0,t]} \int_E \psi(r,v)\mathcal{N}(dr, dv).$$

(i) (Kunita’s first inequality [16, Theorem 4.4.23]) If $p \geq 2$ and $\psi \in L^p(\Omega; L^p_{\text{loc}}([0, \infty); L^2(E, \nu)))$ further. Then for every $t > 0$, there exists a positive constant $C_1 > 0$, such that

$$E \left[ \sup_{0 \leq s \leq t} |I_t|^p \right] \leq C_1 \left\{ E \left[ \int_0^t \int_E |\psi(r,v)|^p \nu(dv)dr \right] + E \int_0^t \int_E |\psi(r,v)|^p \nu(dv)dr \right\}. \quad (3.4)$$

(ii) ([18, Proposition 2.2]) If $1 \leq p < 2$, then for every $t > 0$, there exists a positive constant $C_2 > 0$, such that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |I_s|^p \right] \leq C(p) \mathbb{E} \int_0^t |\psi(r, \nu)|^p \nu(d\nu)dr.
\]

**Proof of Theorem 3.3.** Let \( p \) be given in Theorem 3.3. For every \( x, y \in B_{c_1} \) \((c_1 > 0 \text{ is any given real number})\), and every \( T > 0 \), from (3.2) we have

\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} |u(t, x) - u(t, y)|^p \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{(0, t]} \left[ K(t - r, \cdot) * g(r, \cdot, x) - K(t - r, \cdot) * g(r, \cdot, y) \right] N(dr, dv) \right|^p \\
&\quad - \int_{(0, t]} \left[ K(t - r, \cdot) * g(r, \cdot, x) - K(t - r, \cdot) * g(r, \cdot, y) \right] \nu(dv) dr \\
&\leq 2^{p-1} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{(0, t]} \left[ K(t - r, \cdot) * g(r, \cdot, x) - K(t - r, \cdot) * g(r, \cdot, y) \right] N(dr, dv) \right|^p \\
&\quad + 2^{p-1} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{(0, t]} \left[ K(t - r, \cdot) * g(r, \cdot, x) - K(t - r, \cdot) * g(r, \cdot, y) \right] \nu(dv) dr \right|^p.
\end{align*}
\]

We estimate the convolution in (3.5) by

\[
\begin{align*}
|K(t - r, \cdot) * g(r, \cdot, x) - K(t - r, \cdot) * g(r, \cdot, y)| \\
&= \left| \int_{\mathbb{R}^d} K(t - r, z) \left[ g(r, v, x - z) - g(r, v, y - z) \right] dz \right| \\
&\leq \int_{\mathbb{R}^d} K(t - r, z) |g(r, v, x - z) - g(r, v, y - z)| dz \\
&\leq h(r, v) |x - y|^\beta \phi^\frac{1}{\beta}(|x - y|).
\end{align*}
\]

Combining (3.5) and (3.6), we get
Observing that

Remark 3.2. The main ingredient in proving Theorem 3.3 is to estimate the stochastic evolution

If \( p \geq 2 \), by using Lemma 3.5 (i), we conclude from (3.7) that

If \( 1 \leq p < 2 \), with the aid Lemma 3.5 (ii), we conclude from (3.7) that

which implies the estimate (3.9). In view of Remark 1.1 (i), we finish the proof. \( \square \)

Remark 3.2. The main ingredient in proving Theorem 3.3 is to estimate the stochastic evolution

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_{(0, t]} \int h(r, v) \tilde{N}(dr, dv) \right\|_{L^p(p^\alpha)}^p
\]
for \( f \in L^p(\Omega; L^1_{\text{loc}}([0, \infty); L^1(E, \nu; L^\infty(\mathbb{R}^d))) \cap L^p_{\text{loc}}([0, \infty); L^p(E, \nu; L^\infty(\mathbb{R}^d)))) \). When the integrand and \( L^\infty(\mathbb{R}^d) \) are replaced by \( U(t, r)h(r-) \) (\( U \) is an evolution operator) and a Hilbert space, respectively, the estimate for (3.10) was established by Kotelenez [19]. Since then, Kotelenez’s result was strengthened by Hamedani and Zangeneh [20], Ichikawa [21]. We refer to [22–27] for some other extensions. Observing that all extensions are concentrated on stochastic evolution taking values in martingale type \( (1 < p < \infty) \) Banach spaces, and as noticed in [24, Remark 2.11], \( L^\infty(\mathbb{R}^d) \) is not a Banach space of martingale type \( p \) for any \( p > 1 \), so the following estimates:

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_{[0, t]} \int_{E} f(r, \nu)N(dr, d\nu) \right\|_{L^p(\mathbb{R}^d)}^p \leq C(p) \left\{ \mathbb{E} \left[ \int_0^T \left\| f(r, \nu) \right\|^2_{L^\infty(\mathbb{R}^d)} \nu(d\nu) dr \right] \right\}^{\frac{p}{2}} + \mathbb{E} \int_0^T \left\| f(r, \nu, \cdot) \right\|^p_{L^\infty(\mathbb{R}^d)} \nu(d\nu) dr, \quad p \geq 2,
\]

and

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_{[0, t]} \int_{E} f(r, \nu)N(dr, d\nu) \right\|_{L^p(\mathbb{R}^d)}^p \leq C(p) \mathbb{E} \int_0^T \left\| f(r, \nu, \cdot) \right\|^p_{L^\infty(\mathbb{R}^d)} \nu(d\nu) dr, \quad 1 < p < 2,
\]

for Poisson random measure in general will be not true. Fortunately, if we assume further that \( f \in L^p(\Omega; L^1_{\text{loc}}([0, \infty); L^1(E, \nu; L^\infty(\mathbb{R}^d))) \rangle \), then we gain

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_{[0, t]} \int_{E} f(r, \nu)N(dr, d\nu) \right\|_{L^\infty(\mathbb{R}^d)}^p \leq C(p) \left\{ \mathbb{E} \left[ \int_0^T \left\| f(r, \nu, \cdot) \right\|_{L^\infty(\mathbb{R}^d)} \nu(d\nu) dr \right] \right\}^p + \mathbb{E} \int_0^T \left\| f(r, \nu, \cdot) \right\|^p_{L^\infty(\mathbb{R}^d)} \nu(d\nu) dr \right\}.
\]

(3.11)

The above estimate will play a central role in stochastic partial equations when we prove the boundedness of solutions.

From (3.11), for the Cauchy problem (3.1), we have the following corollary.

**Corollary 3.5.** Let \( g \in L^1(\Omega; L^1_{\text{loc}}([0, \infty); L^1(E, \nu; L^\infty(\mathbb{R}^d))) \) such that \( \{g(t, v, x, \cdot)\}_{t \geq 0} \) as a family of \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \)-valued random variables is \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted, and let \( u \) be given by (3.2). Then \( u \in L^1(\Omega; L^1_{\text{loc}}([0, \infty); L^1(E, \nu; L^\infty(\mathbb{R}^d))) \).
Proof. Let $T > 0$, from (3.2) we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} \esssup_{x \in \mathbb{R}^d} \int_{(0, t]} \int_{E} K(t - r, \cdot) * g(r, \nu, \cdot)(x) \tilde{N}(dr, dv) \leq \mathbb{E} \sup_{0 \leq t \leq T} \esssup_{x \in \mathbb{R}^d} \int_{(0, t]} \int_{E} |K(t - r, \cdot) * g(r, \nu, \cdot)(x)| N(dr, dv) + \mathbb{E} \sup_{0 \leq t \leq T} \esssup_{x \in \mathbb{R}^d} \int_{(0, t]} \int_{E} \left| K(t - r, \cdot) * g(r, \nu, \cdot)(x) \right| \nu(dv) dr \leq \mathbb{E} \sup_{0 \leq t \leq T} \int_{(0, t]} \int_{E} \|g(r, \nu)\|_{L^\infty_{x}(\mathbb{R}^d)} N(dr, dv) + \mathbb{E} \sup_{0 \leq t \leq T} \int_{(0, t]} \int_{E} \|g(r, \nu)\|_{L^\infty_{x}(\mathbb{R}^d)} \nu(dv) dr \leq 2\mathbb{E} \int_{0}^{T} \int_{E} \|g(r, \nu)\|_{L^\infty_{x}(\mathbb{R}^d)} \nu(dv) dr < +\infty.
$$

(3.12)

Remark 3.3. (i) Let $q > 1$ and let $u$ be given by (3.2). If $g \in L^1(\Omega; L^1_{loc}([0, \infty); L^1(E, \nu; L^q(\mathbb{R}^d))))$, we also prove that $u \in L^1(\Omega; L^\infty_{loc}([0, \infty); L^1(E, \nu; L^1(\mathbb{R}^d))))$.

(ii) For regularities of solutions to equation (3.2), one also consults to [28–31].

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References

[1] Kolmogorov, A. N. (1933). Sulla determinazione empirica di una legge distribuzione. Giorn. Ist. Ital. Attuari. 4:83–91.
[27] Wei, J., Duan, J., Lv, G. (2019). Schauder estimates for stochastic transport-diffusion equations with Lévy processes. *J. Math. Anal. Appl.* 474(1):1–22. DOI: 10.1016/j.jmaa.2018.12.066.

[28] Kim, I., Kim, K. (2016). An $L^p$-theory for stochastic partial differential equations driven by Lévy processes with pseudo-differential operators of arbitrary order. *Stochastic Process. Appl.* 126(9):2761–2786. DOI: 10.1016/j.spa.2016.03.001.

[29] Kim, K., Kim, P. (2012). An $L^p$-theory of a class of stochastic equations with the random fractional Laplacian driven by Lévy processes. *Stochastic Process. Appl.* 122(12):3921–3952. DOI: 10.1016/j.spa.2012.08.001.

[30] Lv, G., Gao, H., Wei, J., Wu, J. (2019). BMO and Morrey-Campanato estimates for stochastic convolutions and Schauder estimates for stochastic parabolic equations. *J. Differ. Equations.* 266(5):2666–2717. DOI: 10.1016/j.jde.2018.08.042.

[31] Marinelli, C., Prévôt, C., Röckner, M. (2010). Regular dependence on initial data for stochastic evolution equations with multiplicative poisson noise. *J. Funct. Anal.* 258(2): 616–649. DOI: 10.1016/j.jfa.2009.04.015.