On the Self-Propulsion of a Rigid Body in a Viscous Liquid
by Time-Periodic Boundary Data

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Abstract

Consider a rigid body, \( B \), constrained to move by translational motion in an unbounded viscous liquid. The driving mechanism is a given distribution of time-periodic velocity field, \( v_* \), at the interface body-liquid, of magnitude \( \delta \) (in appropriate function class). The main objective is to find conditions on \( v_* \) ensuring that \( B \) performs a non-zero net motion, namely, \( B \) can cover any given distance in a finite time. The approach to the problem depends on whether the averaged value of \( v_* \) over a period of time is (case (b)) or is not (case (a)) identically zero. In case (a) we solve the problem in a relatively straightforward way, by showing that, for small \( \delta \), it reduces to the study of a suitable and well-investigated time-dependent Stokes (linear) problem. In case (b), however, the question is much more complicated, because we show that it cannot be brought to the study of a linear problem. Therefore, in case (b), self-propulsion is a genuinely nonlinear issue that we solve directly on the nonlinear system by a contradiction argument approach. In this way, we are able to give, also in case (b), sufficient conditions for self-propulsion (for small \( \delta \)). Finally, we demonstrate, by means of counterexamples, that such conditions are, in general, also necessary.

Introduction

The rigorous mathematical analysis of self-propulsion of an either rigid or shape-changing “bulky” body in a viscous fluid is a relatively new area of investigation. Actually, a systematic and consistent study of this problem has began about two decades ago; see, e.g., [3, 5, 2, 15, 17, 13, 19, 20, 21, 23] and the references therein. In the time-dependent case, the main, and by no means trivial, contribution of these papers consists in establishing well-posedness of the corresponding initial-boundary value problem in different functional settings – weak and strong solutions, different situations – one or more bodies, and different flow regions – bounded and unbounded.

However, it should also be noted that the above works seem to leave out what, we believe, is a rather fundamental question. Precisely, to establish sufficient conditions on the shape-changes or (in the case of a rigid body) boundary velocity distributions, ensuring that the body performs a non-zero net motion, that is, the body is indeed able to self-propel. This question was only considered in [21, Section 7], and resolved by numerical simulation of the relevant system of equations in the two-dimensional case. As a consequence, to the best of our knowledge, there is no rigorous analytical condition on the driving mechanism securing self-propulsion of the body. The main objective of this article is to furnish a first contribution to such a question.

Specifically, we shall consider the motion of the coupled system constituted by a rigid body, \( B \), moving in an otherwise quiescent viscous liquid, \( L \), that fills the whole space outside \( B \). The driving mechanism is a distribution of velocity, \( v_* \), at the boundary of \( B \), that is, at the interface body-liquid.

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We also suppose that on $\mathcal{B}$ a suitable torque is applied preventing $\mathcal{B}$ from spinning. Thus, the motion of $\mathcal{B}$ will be merely translational, and we shall denote by $\gamma$ the corresponding translational velocity.

We assume that $v^\ast$ is a given periodic function of time with period $T > 0$ (in short: $T$-periodic function), and look for a special class (to be defined shortly) of corresponding $T$-periodic motions of the system body-liquid. Under the stated assumptions, this question leads us to investigate $T$-periodic solutions to the following set of equations [5]

\[
\begin{align*}
\partial_t v + (v - \gamma) \cdot \nabla v &= \nu \Delta v - \nabla p \\
\text{div } v &= 0 \\
v(x, t) &= v^\ast(x, t) + \gamma(t), \quad \text{on } \partial \Omega \equiv \partial \mathcal{B} \times \mathbb{R}, \\
M \dot{\gamma} + \int_{\partial \Omega} \left[ T(v, p) - (v^\ast + \gamma) \otimes v^\ast \right] \cdot n &= 0 \quad \text{in } \mathbb{R}.
\end{align*}
\]

(0.1)

Here $v = v(x, t)$ and $\rho p$, are velocity and pressure fields of $\mathcal{L}$, respectively, $\rho$ is the (constant) density of $\mathcal{L}$, $\nu := \mu/\rho$ with $\mu$ shear viscosity coefficient of $\mathcal{L}$ and, finally, $M := m/\rho$, where $m$ is the mass of $\mathcal{B}$. We observe that equation (0.1) translates into mathematical terms the requirement that $\mathcal{B}$ self-propels, namely, $\mathcal{B}$ moves without the action of external forces. [5, Section 6].

The main focus of this paper is to address the following question: Suppose the boundary velocity distribution $v^\ast$ is sufficiently smooth. Then, under which conditions on $v^\ast$ the body $\mathcal{B}$ performs a non-zero net motion or, equivalently, $\mathcal{B}$ self-propels?

From the mathematical viewpoint, this question is equivalent to look for a class of $T$-periodic solutions $(v, p, \gamma)$ to (0.1) such that the translational velocity $\gamma$ possesses a nonzero average:

\[
\xi := \frac{1}{T} \int_0^T \gamma(t) \, dt \neq 0.
\]

(0.2)

The physical problem that motivated our study is the self-propulsion of a “fish,” or any mechanical device where the net motion is produced by the continuous periodic movement of parts of its body. Even though modeling a fish as a rigid body and its moving parts as a boundary velocity distribution might look a bit coarse at first sight, it should be observed that, as is well known, the motion of a shape-changing object in a liquid can mathematically be reduced –by an appropriate transformation– to that of an object of fixed shape with a suitable distribution of velocity at its boundary; see, for example, [11] and the reference therein.

Our analysis is divided into two parts, defined by the following mutually exclusive properties of the average of $v^\ast$ over a period:

(a) \[ \overline{v^\ast}(x) := \frac{1}{T} \int_0^T v^\ast(x, t) \, dt \neq 0; \]

(b) \[ \overline{v^\ast}(x) = 0 \]

Let us denote by $\delta$ the magnitude of $v^\ast$ (in a suitable functional class) and set

\[ v^\ast = \delta V^\ast. \]

(0.3)

In case (a) we show that a sufficient condition for the validity of (0.3) is that, for $\delta \leq \delta_0$, the $L^2$-projection, $\mathcal{G}$, of $V^\ast$ onto a three-dimensional subspace, $\mathcal{S}$, of $L^2(\partial \Omega)$ is not zero. The space $\mathcal{S}$ depends only on the geometric properties of $\mathcal{B}$, such as size or shape, but it is otherwise independent of its
mass and the physical properties of $\mathcal{L}$. Thus, $\mathcal{G}$ is the thrust for self-propulsion in case (a). Such a result, proved in the class of weak solutions to (0.1) (see Theorem 2.1), is obtained by showing that, in the limit of “small” $\delta$ the averaged (over a period) solution must tend to the uniquely determined solution of the corresponding linear (time-independent) Stokes problem (see Lemma 3.2). The latter is obtained from (0.1) by formally setting equal to 0 all time derivatives, disregarding all nonlinear terms and taking the time average of the resulting equations (see (3.5)). We also show that the average velocity $\xi$ in (5.2) in the limit of small $\delta$ has the following expression:

$$\xi = \delta \mathcal{A} \cdot \mathcal{G} + o(\delta),$$  

where $\mathcal{A}$ is an invertible matrix depending only on $B$.

We next consider case (b). In this situation, the boundary velocity distribution defines a purely oscillatory regime. From what we have already proved and, in particular, from (0.4) we infer that, being now $\mathcal{G} = 0$, self-propulsion must be searched at an order in $\delta$ higher than 1. Moreover, since the linearized approximation (Stokes problem) possesses in this case only the identically vanishing solution (see Lemma 3.1), the solution to the nonlinear problem that would ensure self-propulsion cannot be obtained by a perturbation argument around its linear counterpart as in case (a). In other words, in case (b) self-propulsion becomes a strictly nonlinear phenomenon. We then use a completely different strategy that will be described next. In the first place, we split the solution $(v, \gamma)$ to (0.1) into its averaged and purely oscillatory components:

$$v = (v - \overline{v}) + \overline{v} := \mathcal{w} + u; \quad \gamma = (\gamma - \overline{\gamma}) + \overline{\gamma} := \chi + \xi$$

where the bar indicates average over a period. As a result, the problem (0.1) splits into a coupled nonlinear problem of “elliptic-parabolic” type in the unknowns $(u, \xi)$ for the “elliptic” part, and $(w, \chi)$ for the “parabolic” part (see (6.12)–(6.15)). If $\delta \leq \delta_0$, we then show that the problem, in this new form, has a unique “strong” solution (see Theorem 6.1). This result is obtained by a successive approximation scheme around an appropriate nonlinear problem (see (6.21)–(6.22)). The fundamental aspect of this finding is that, thanks to the self-propelling condition (0.1), we can show that the averaged component $u$ belongs to a certain function class where, in general, classical boundary-value problems associated to Stokes and Navier-Stokes equations (namely, without the self-propelling condition) do not have a solution if $\xi = 0$ [16, 6]. This space is the homogeneous Sobolev space $D^{1, 2}(\Omega)$ of locally integrable functions with spatial derivatives belonging to the Lebesgue space $L^{\frac{3}{2}}(\Omega)$. In the next step, we solve the “parabolic” problem for $(w, \chi)$, considering $u$ as assigned in its function class, thus implicitly getting $(w = w(u, v_*), \chi = \chi(u, v_*))$, and replace the latter into the “elliptic” problem. This can be done, provided $\delta$ is sufficiently “small.” The final step is then a contradiction argument on the nonlinear problem satisfied by $u$. Precisely, with the help of the previous step, and assuming ad absurdum $\xi = 0$, we show that $u$ must solve the following problem

$$\nu \Delta u - \nabla p = \mathcal{F}(u, v_*) \quad \text{in } \Omega,$$

$$\text{div } u = 0,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

with $\mathcal{F}$ a sufficiently smooth, nonlinear function of its arguments. However, $u \in D^{1, \frac{2}{3}}(\Omega)$ and so, as previously noticed, the problem (5.6) does not, in general, admit solutions in that class. Actually, a solution in $D^{1, \frac{2}{3}}(\Omega)$ may exists if and only if $\mathcal{F}$ satisfies a nonlocal compatibility condition [1]
Section V.5. Our self-propulsion problem reduces then to find requirements on \( \mathbf{v}_* \) that violate this condition, thus implying \( \xi \neq 0 \). We then prove that, for \( \delta \) sufficiently “small”, there is a vector, \( \mathbf{G} \), in \( \mathbb{R}^3 \) depending only on the “shape” of \( \mathcal{B} \), its mass and \( \mathbf{V}_* \) (see (7.1)) such that if \( \mathbf{G} \neq 0 \), then the compatibility condition mentioned above is violated, and hence \( \xi \neq 0 \) (see Theorem 7.1). Thus, \( \mathbf{G} \) is the thrust, in case (b). Furthermore, we prove (see Theorem 8.1)

\[
\xi = \delta^2 \mathbf{A} \cdot \mathbf{G} + O(\delta^{\frac{12}{7}}). 
\]

This formula shows, as expected, that if \( \mathbf{G} \neq 0 \) self-propulsion does occur at an order in \( \delta \) higher than 1. The natural question to ask is then whether, if \( \mathbf{G} = 0 \), could self-propulsion occur at an order in \( \delta \) higher than 2. We show that, in general, the answer is negative (see Section 9). In fact, we give an example showing that given a body \( \mathcal{B} \) of any shape and mass, and an arbitrary period \( T > 0 \), for any any \( \delta > 0 \) there is always a \( T \)-periodic boundary velocity \( \mathbf{v}_* \) such that, if \( \mathbf{G} = 0 \), the averaged velocity field is identically vanishing, thus implying that \( \mathcal{B} \) can only “oscillate,” with zero net motion.

The outline of the paper is as follows. After some preliminary results collected in Section 1, in Section 2 we give a weak formulation of problem (0.1) and prove in Theorem 2.1 existence of corresponding weak solutions under suitable assumptions on \( \mathbf{v}_* \). In the following Section 3, we show in Theorem 3.1 sufficient conditions for self-propulsion when \( \mathbf{v}_* \neq 0 \) in the class of weak solutions. The next two sections are dedicated to preparatory results necessary for the investigation of the case \( \mathbf{v}_* \equiv 0 \). Specifically, in Section 4, we prove existence and uniqueness of a steady-state nonlinear problem in a function class contained in the homogeneous Sobolev space \( D^{1,\frac{7}{2}}(\Omega) \) mentioned earlier on; see Lemma 4.3. This lemma plays a key role in all our subsequent analysis. The main result of the following Section 5 is the proof of existence and uniqueness in the maximal regularity class of solutions to the linearized version of (0.1), in the case when the data have zero average; see Lemma 5.2. With the help of these findings we show, in Section 6, existence and uniqueness of \( T \)-periodic solutions to (0.1) in a rather regular function class, provided \( \delta \) is sufficiently restricted; see Theorem 6.1. By using the contradiction argument mentioned earlier on, we provide in Section 7 sufficient conditions for self-propulsion when \( \mathbf{v}_* \equiv 0 \) (see Theorem 7.1), whereas in Section 8 we furnish the expression of the velocity of propulsion; see Theorem 8.1. In the final Section 9, we prove by means of counterexamples that the self-propelling condition determined in Section 7 is also necessary.

1 Preliminary Results

In this section we shall recall and/or introduce the main notation, and collect some basic results that will be frequently used later on in the paper.

By \( \Omega \) we indicate a domain of \( \mathbb{R}^3 \), complement of the closure of a bounded domain \( \Omega_0 \) (\( \equiv \) the body \( \mathcal{B} \)). We assume \( \Omega \) of class \( C^2 \). Moreover, with the origin in the interior of \( \Omega_0 \), we set \( \Omega_R := \Omega \cap \{ |x| < R \} \) and \( \Omega^R := \Omega \cap \{ |x| > R \} \), for \( R > R_* := \text{diam} \Omega_0 \). As customary, for a domain \( A \subseteq \mathbb{R}^3 \), \( L^q = L^q(A) \) is the Lebesgue space with norm \( \| \cdot \|_{q,A} \), and \( W^{m,q} = W^{m,q}(A) \) denotes Sobolev space, \( m \in \mathbb{N}, q \in [1, \infty) \), with norm \( \| \cdot \|_{m,q,A} \). Corresponding trace norms at \( \partial A \) are denoted by \( \| \cdot \|_{m-\frac{2}{q},q,\partial A} \). Furthermore, \( D^{m,q} = D^{m,q}(A) \) are homogeneous Sobolev spaces with semi-norm \( |u|_{m,q,A} := \sum_{|\alpha|=m} \| D^\alpha u \|_{q} \), whereas \( D^{0,q}_0 = D^{0,q}_0(A) \) is the completion of \( C^\infty_0(A) \) in the norm \( | \cdot |_{1,q,A} \).

In all the above norms, the subscript "A" will be omitted, unless confusion arises. An important (1) A detailed analysis of homogeneous Sobolev spaces including their main properties can be found in [2 Section II.6].
embedding property of the spaces $D^{1,q}$ is recalled in the following lemma, for whose proof we refer to [7, Theorem II.6.1(i) and Theorem II.7.3].

**Lemma 1.1** Let $u \in D^{1,q}(\Omega) \cap L^r(\Omega)$, for some $q \in [1,3)$, $r \in [1,\infty)$. Then $u \in L^{\frac{2q}{3-q}}(\Omega)$ and there is $c = c(\Omega,q)$ such that
\[
\|u\|_{\frac{2q}{3-q}} \leq c |u|_{1,q} .
\]
Suppose, in addition, $u \in D^{2,2}(\Omega)$. Then
\[
\|u\|_s + |u|_{1,\sigma} \leq c \left( |u|_{1,q} + |u|_{2,2} \right), \quad \text{for all } s \in \left[ \frac{3q}{3-q} , \infty \right] \text{ and } \sigma \in [q,6].
\]

Let
\[
\mathcal{C} = \mathcal{C}(\Omega) := \{ \varphi \in C_0^\infty(\Omega) : \text{div} \varphi = 0 \text{ in } \Omega; \varphi(x) = \xi \varphi, \text{ some } \xi \varphi \in \mathbb{R}^3, \text{ in a neighborhood of } \partial \Omega \},
\]
and define
\[
\mathcal{H} = \mathcal{H}(\Omega) \equiv \{ \text{completion of } \mathcal{C}(\Omega) \text{ in the norm } \|D(\cdot)\|_2 \}.
\]
The essential properties of the space $\mathcal{H}$ are collected in the next lemma, whose proof is given in [4, Lemmas 9–11].

**Lemma 1.2** $\mathcal{H}$ is a Hilbert space endowed with the scalar product
\[
[u,w] := \int_{\Omega} D(u) : D(w), \quad u,w \in \mathcal{H}, \quad (1.7)
\]
and the following characterizations hold
\[
\mathcal{H} = \mathcal{H}(\Omega) := \{ u \in W^{1,2}_{\text{loc}}(\Omega) : u \in L^6(\Omega), D(u) \in L^2(\Omega) ; \text{div} u = 0 \text{ in } \Omega ; u(y) = \xi_u, \; y \in \partial \Omega \}. \quad (1.8)
\]
Moreover, we have
\[
\|\nabla u\|_2 \leq \sqrt{2}\|D(u)\|_2 \leq 2\|\nabla u\|_2, \quad (1.9)
\]
and
\[
\|u\|_6 \leq c_0 \|D(u)\|_2, \quad u \in \mathcal{H}, \quad (1.10)
\]
for some constant $c_0 > 0$. Finally, there is another positive constant $c_1$ such that
\[
|\xi_u| \leq c_1 \|D(u)\|_2. \quad (1.11)
\]

We shall need also the following “local” version of the above spaces:
\[
\mathcal{C}(\Omega_R) := \{ \varphi \in C_0^\infty(\Omega_R) : \text{div} \varphi = 0 \text{ in } \Omega_R; \varphi(x) = \xi \varphi, \text{ around } \partial \Omega, \varphi = 0 \text{ around } \partial B_R \},
\]
and
\[
\mathcal{H}(\Omega_R) := \{ u \in W^{1,2}(\Omega_R) : \text{div} u = 0 \text{ in } \Omega_R ; u(y) = \xi_u, \; y \in \partial \Omega, \; u = 0 \text{ at } \partial B_R \}. \quad (1.12)
\]
For each $R > R_*$, $\mathcal{H}(\Omega_R)$ becomes a Hilbert space when endowed with the scalar product defined in (1.7), by setting $\Omega \equiv \Omega_R$. We denote by $\mathcal{H}^{-1}(\Omega_R)$ its dual space. Also, $\mathcal{C}(\Omega_R)$ is dense in $\mathcal{H}(\Omega_R)$. 
Lemma 1.3 Let \( q \in (1, \infty) \) we introduce the following Banach spaces
\[
D^{2,q} := D^{2,2}(\Omega) \cap D^{1,q}(\Omega) \cap \mathcal{H}(\Omega) ; \quad D^{1,q} := D^{1,2}(\Omega) \cap L^q(\Omega), \\
L^q := \{ f = \text{div} \mathcal{F} \in L^2(\Omega) : \mathcal{F} \in L^2(\Omega) \cap L^q(\Omega) \},
\]
endowed with the norms
\[
\| u \|_{D^{2,q}} := |u|_{2,2} + |u|_{1,q} + \| u \|_6 , \quad \| u \|_{D^{1,q}} := |u|_{1,2} + \| u \|_q \\
\| f \|_{L^q} := \| f \|_2 + \| \mathcal{F} \|_2 + \| \mathcal{F} \|_q .
\]
From Lemma 1.2, Lemma 1.1, and [7, Theorem II.9.1] we deduce the following result.

**Lemma 1.3** Let \( q \in (1, 3) \), and set \( q_1 := \min\{ \frac{3q}{3-q}, 6 \} \), \( q_2 := \min\{ q, 2 \} \). Then, the following continuous embedding holds
\[
D^{2,q} \subset \begin{cases} 
L^s(\Omega) , & s \in [q_1, \infty) \\
D^{1,\sigma}(\Omega) , & \sigma \in [q_2, 6].
\end{cases}
\]
Moreover, if \( u \in D^{2,q} \), then
\[
\lim_{|x| \to \infty} u(x) = 0.
\]

If \( A \subset \mathbb{R}^3 \), a function \( u : A \times \mathbb{R} \to \mathbb{R}^3 \) is \( T \)-periodic, \( T > 0 \), if \( u(\cdot, t+T) = u(\cdot, t) \), for a.a. \( t \in \mathbb{R} \), and we set \( \overline{u} := \frac{1}{T} \int_0^T u(t)dt \). Let \( B \) be a function space endowed with seminorm \( \| \cdot \|_B \), \( r \in [1, \infty] \), and \( T > 0 \). Then, \( L^r(0, T ; B) \) is the class of functions \( u : (0, T) \to B \) such that
\[
\| u \|_{L^r(B)} \equiv \begin{cases} 
\left( \int_0^T \| u(t) \|_B^r \right)^{\frac{1}{r}} < \infty , & \text{if } r \in [1, \infty) ; \\
\text{ess sup}_{t \in [0,T]} \| u(t) \|_B < \infty \quad \text{if } r = \infty .
\end{cases}
\]
Likewise, we put
\[
W^{m,r}(0, T ; B) = \left\{ u \in L^r(0, T ; B) : \partial_t^k u \in L^r(0, T ; B), k = 1, \ldots, m \right\}.
\]
Unless confusion arises, we shall simply write \( L^r(B) \) for \( L^r(0, T ; B) \), etc. Moreover, for \( q \in (1, \infty) \), we introduce the following Banach spaces
\[
L^2_q(0, T) = \{ \chi \in L^q(0, T), \ \chi \text{ is } T \text{-periodic with } \overline{\chi} = 0 \}
\]
\[
W^{1,q}_q(0, T) = \{ \chi \in L^q(0, T), \ \dot{\chi} \in L^q(0, T) \}
\]
\[
L^q_\sharp := \{ u \in L^q(L^q) ; \ u \text{ is } T \text{-periodic, with } \overline{u} = 0 \}
\]
\[
W^{2}_\sharp := \{ u \in W^{1,2}(L^2) \cap L^2(W^{2,2} \cap \mathcal{H}); \ u \text{ is } T \text{-periodic, with } \overline{u} = 0 \}
\]
\[
W^{2,q}_\sharp := \{ u \in W^{1,q}(L^q) \cap L^q(W^{2,q}); \ u \text{ is } T \text{-periodic, with } \overline{u} = 0 \}
\]
endowed with natural norms, and define
\[
L^{2,q}_\sharp := L^2_\sharp \cap L^q(L^q) ; \quad W^{2,q}_\sharp := W^2_\sharp \cap W^{2,q} ; \quad \tilde{W}^{2,q}_\sharp := \tilde{W}^2_\sharp \cap \tilde{W}^{2,q}_\sharp
\]
with associated norms

\[ \|u\|_{L^2,q} := \|u\|_{L^2(L^2)} + \|u\|_{L^2(L^q)} \]

\[ \|u\|_{\hat{W}^2,q} := \|u\|_{W^{1,2}(L^2) \cap W^{1,q}(L^q)} + \|u\|_{L^2(W^{2,2}) \cap L^q(W^{2,q})}. \]

Finally, we define

\[ \mathcal{P}^{1,q} := L^2(D^{1,2}) \cap L^q(D^{1,q}). \]

We recall some embedding properties of the spaces \( \hat{W}^2_q \) which are a particular case of \([22, \text{Theorem 2.1}]\).

**Lemma 1.4** The following continuous embedding holds, for all \( r, s \in [q, \infty) \):

\[ \hat{W}^q_r \subset \begin{cases} L^r(L^s), & \frac{3}{2} + \frac{2}{r} > \frac{5}{q} - 2, \\ L^r(D^{1,s}), & \frac{3}{2} + \frac{2}{r} > \frac{5}{q} - 1. \end{cases} \]

We conclude this section with a couple of further notational remarks. The first one regards the standard Landau notation. Precisely, by \( O(\delta^\alpha) \), [respectively, \( o(\delta^\alpha) \)], \( \alpha \geq 0 \), we indicate a generic function \( f \) (say) depending on \( \delta \) and such that \( |f(\delta)| \leq c \delta^\alpha \), \( \delta \leq \delta_0 \), for some positive constants \( c, \delta_0 \) [respectively, \( \lim_{\delta \to 0} f(\delta) \delta^{-\alpha} = 0 \)]. Finally, by \( c, c_0, c_1, \text{etc.} \), we denote positive constants, whose particular value is unessential to the context. When we wish to emphasize the dependence of \( c \) on some parameter \( \zeta \), we shall write \( c(\zeta) \) or \( c_\zeta \).

## 2 Weak Formulation and Existence of of Weak Solutions

In this section we shall prove existence to problem \((0.1)\) in the very general class of weak solutions, under the assumption that the boundary velocity distribution has zero net flux through the boundary of \( \mathcal{B} \) (see \((2.2)\)) and it is sufficiently “small” in appropriate norm. Probably, both conditions could be weakened, but this would not be relevant to our main objective of finding sufficient conditions for self-propulsion that will be discussed in the next section.

To reach the goal above, we begin to put \((0.1)\) in a weak form. For \( A \equiv \Omega_R, \Omega \), we denote by \( \mathcal{C}_A(A) := C_4(A \times [0, T]) \) the space of vector functions obtained by restriction to \([0, T]\) of functions \( \psi \in C^1(A \times \mathbb{R}) \), satisfying:

1. \( \text{div } \psi = 0 \text{ in } A \times \mathbb{R} \);
2. There exists \( \gamma_\psi \in C^1(\mathbb{R}) \) such that \( \psi(x, t) = \gamma_\psi(t) \) for \( x \) in a neighborhood of \( \partial \Omega \), and \( t \in \mathbb{R} \);
3. For each \( \psi \), there exists \( \rho = \rho(\psi) > R_* \) such that \( \psi(x, t) = 0 \) for \( |x| \geq \rho \) and \( t \in \mathbb{R} \), with \( \rho < R \) if \( A \equiv \Omega_R \);
4. \( \psi \) is \( T \)-periodic.

Multiplying formally \((0.1)\) by the test function \( \psi \in \mathcal{C}_A(\Omega) \), integrating by parts over \( \Omega \times [0, T] \), and taking into account \( T \)-periodicity, we find

\[ - \int_0^T \int_\Omega \partial_t \psi \cdot v = \int_0^T \int_{\partial \Omega} \gamma_\psi \cdot [\mathbb{T}(v, p) \cdot n - (v_\gamma + \gamma)v_\gamma \cdot n] + \int_0^T \int_\Omega [(v - \gamma) \cdot \nabla \psi \cdot v - 2 \nu \mathbb{D}(\psi) : \mathbb{D}(v)]. \]
Then, imposing in the latter the self-propelling condition \((0.1)\), we get

\[
-\int_0^T \int_\Omega \partial_t \psi \cdot v + M \dot\gamma \psi \cdot \gamma = \int_0^T \int_\Omega \left[ (v - \gamma) \cdot \nabla \psi \cdot v - 2\nu \mathbb{D}(\psi) : \mathbb{D}(v) \right]
\]  

(2.1)

Following [12], we give the following definition.

**Definition 2.1** The pair \(\{v, \gamma\}\) is a \(T\)-periodic weak solution to \((0.1)\) if the following conditions hold:

(i) \(v\) and \(\gamma\) are both \(T\)-periodic with \(v \in L^2(0, T; D^{1,2}(\Omega))\), \(\gamma \in L^2(0, T)\);

(ii) \(\text{div} v(\cdot, t) = 0\) in \(\Omega\), for a.a. \(t \in [0, T]\);

(iii) \(v = v_* + \gamma\) at \(\partial\Omega\times (0, T)\) (in the trace sense);

(iii) \(\{v, \gamma\}\) verifies (2.1), for all \(\psi \in C^\infty(\Omega)\).

We need a preparatory result concerning suitable extension of the boundary data.

**Lemma 2.1** Let \(V_* \in W^{1,2}(W^{1,2}_{\partial\Omega})\) be \(T\)-periodic with

\[
\int_{\partial\Omega} V_*(x, t) \cdot n = 0, \quad \text{for all } t \in [0, T].
\]

(2.2)

Then, there exists a \(T\)-periodic field \(V \in W^{1,2}(W^{1,2}(\Omega))\) such that

(a) \(V(x, t) = V_*(x, t)\), for all \((x, t) \in \partial\Omega \times [0, T]\);

(b) \(\text{div} V = 0\) in \(\Omega \times [0, T]\);

(c) \(V(x, t) = 0\), for all \((x, t) \in \Omega^{2R_*} \times [0, T]\).

(d) \(\|V(t)\|_{1,2} \leq c \|V_*(t)\|_{1,2(\partial\Omega)}\), for a.a. \(t \in [0, T]\);

(e) \(\|V\|_{W^{1,2}(W^{1,2}(\Omega))} \leq c \|V_*\|_{W^{1,2}(W^{1,2}(\partial\Omega))}\).

**Proof.** For each \(t \in [0, T]\), let \(U = U(x, t)\) be the solution to the following Stokes problem

\[
\begin{align*}
\Delta U &= \nabla P, \quad \text{in } \Omega \\
\text{div} U &= 0 \\
U &= V_*(t) \quad \text{at } \partial\Omega, \quad U = 0 \quad \text{at } \partial B_{3R_*}.
\end{align*}
\]

From classical results [7, Theorem IV.6.1], we know that there exists a unique solution \((U(t), P(t)) \in W^{1,2}(\Omega) \times L^2(\Omega)\) satisfying, in addition,

\[
\|U(t)\|_{1,2} \leq c \|V_*(t)\|_{1,2(\partial\Omega)}.
\]

(2.3)

Clearly, \(U(t)\) is \(T\)-periodic. Moreover, \(U\) is time-differentiable and since \(\partial_t U\) satisfies the same problem as \(U\) with \(V_*\) replaced by \(\partial_t V_*\), we recover that \(\partial_t U(t) \in W^{1,2}(\Omega)\) and

\[
\|\partial_t U(t)\|_{1,2} \leq c \|\partial_t V_*(t)\|_{1,2(\partial\Omega)}.
\]

(2.4)
Let $\varphi = \varphi(|x|)$ be a smooth function such that $\varphi = 1$, for $|x| \leq R_*$, and $\varphi = 0$, for $|x| \geq 2R_*$, and consider the problem

\[
\text{div } w(t) = -\nabla \varphi \cdot U := f(t) \text{ in } \Omega; \quad w, \partial_t w \in W^{1,2}_0(\Omega_{2R_*});
\]

\[
\|w(t)\|_{1,2} \leq c \|f(t)\|_2, \quad \|\partial_t w(t)\|_{1,2} \leq c \|\partial_t f(t)\|_2.
\]

(2.5)

Since, by (2.2), $\int_\Omega f(t) = 0$ for all $t \in [0, T]$, in view of [7, Exercise III.3.6], we may assert that such function $w$ exists and is also $T$-periodic. Thus, if we set $V := \varphi U + w$, extend $w$ to 0 outside $\Omega_{2R_*}$, and employ the regularity assumption on $V_*$ along with (2.3)–(2.5), we recognize at once that $V$ satisfies all the stated properties.

\[\Box\]

The following lemma shows, in particular, a further property of a weak solution that also furnishes a more specific way in which it satisfies the periodicity property.

**Lemma 2.2** Let $(v, \gamma) \in L^2(L^2(\Omega)) \times L^2(0, T)$ satisfy (2.1) for all $\psi \in C_c(\Omega_R)$. Moreover, suppose that $v - \gamma = v_*$ at $\partial \Omega \times [0, T]$, with $v_* \in W^{1,2}(W^{1,2}(\partial \Omega))$, and that $v - V \in L^2(H)$, where $V$ is the extension of $v_*$ given in Lemma 2.1 Then, $\partial_t v \in L^1(0, T; H^{-1}(\Omega_R))$, and there is a constant $c = c(R, \nu)$ such that

\[
\|\partial_t v\|_{L^1(H^{-1}(\Omega_R))} \leq c \left(\|v\|_{L^2(H)}^2 + \|v_*\|_{W^{1,2}(W^{1,2}(\partial \Omega))}^2\right).
\]

So, in particular,

\[v \in C([0, T]; H^{-1}(\Omega_R)).\]

**Proof.** Let us choose in (2.1) $\psi = \chi \varphi$, where $\varphi \in C(\Omega_R)$ and $\chi \in C_0^\infty(0, T)$. We then obtain

\[
\int_0^T (v, \varphi) \chi = -\int_0^T G_\varphi(t) \chi(t), \quad \text{for all } \chi \in C_0^\infty(0, T),
\]

where

\[
G_\varphi(t) := \int_\Omega [(v - \gamma) \cdot \nabla \varphi \cdot v - 2\nu \mathbb{D}(\varphi) : \mathbb{D}(v)]
\]

We now write $v = u + V$. Thus, clearly, for a.a. $t \in [0, T]$, $u \in H(\Omega)$ and $u = \gamma$ at $\partial \Omega$. Observing that, by (1.10) and (1.11),

\[
\|u\|_{4, \Omega_R} \leq C(R) \|u\|_6 \leq c(R) \|\mathbb{D}(u)\|_2; \quad |\gamma| \leq c \|\mathbb{D}(u)\|_2
\]

and that, by Lemma 2.1 and embedding theorems,

\[
\|V\|_4 \leq c \|V\|_{1,2} \leq c \|v_*\|_{\frac{3}{2}, 2(\partial \Omega)},
\]

with the help of Schwarz inequality and (1.9), we obtain for a.a. $t \in [0, T]$

\[
|G_\varphi(t)| \leq c(R, \nu) \left(\|\mathbb{D}(u)\|_2^2 + \|\mathbb{D}(u)\|_2 + \|v_*\|_{\frac{3}{2}, 2(\partial \Omega)}^2 + \|v_*\|_{\frac{3}{2}, 2(\partial \Omega)}^2\right) \|\mathbb{D}(\varphi)\|_{2, \Omega_R}.
\]
Thus, since \( u \in L^2(\mathcal{H}) \), we find \( G_{\varphi}(t) = (g(t), \varphi) \) with \( g \in L^1(0, T; H^{-1}(\Omega_R)) \) and \( \langle \cdot, \cdot \rangle \) duality pairing, and, moreover,
\[
\frac{d}{dt} (v, \varphi) = (g, \varphi),
\]
in the sense of distributions on \([0, T]\). The desired property is then proved.

\[
\square
\]

We are now in a position to prove the following existence result.

**Theorem 2.1** Let \( v_* = \delta V_* \), with \( V_* \) as in Lemma 2.1. Then, there is \( \delta_0 > 0 \) such that for each \( \delta \in (0, \delta_0) \), the problem (2.1) has at least one weak solution. Moreover, the following estimate holds
\[
\|\gamma\|_{L^2(0, T)} + \|\nabla \gamma\|_{L^2(L^2)} \leq c(\Omega) \delta.
\]

**Proof.** The method we use is close to the one employed in [11, Section 3], [12, Section 3] in a similar context. We shall therefore restrict ourselves to provide the main steps, referring the reader to the cited work for details. The basic idea is to couple the classical Galerkin method with the “invading domains” procedure. Thus, let \( \{\Omega_{R_k}\} \) be a sequence with \( R_k \to \infty \) as \( k \to \infty \), and \( \Omega_{R_k} \supset \text{supp} (V) \), where \( V \) is the extension field constructed in Lemma 2.1. For each fixed \( k \), we look for a weak solution to (0.1) in \( \Omega_{R_k} \), namely, a pair \((v^{(k)}, \gamma^{(k)})\) such that
\[
-\int_0^T \int_{\Omega_{R_k}} (\partial_t \psi \cdot v^{(k)} + M \gamma \cdot \gamma^{(k)}) = \int_0^T \int_{\Omega} \left[ (v^{(k)} - \gamma^{(k)}) \cdot \nabla \psi \cdot v^{(k)} - 2 \nu D(\psi) : D(v^{(k)}) \right],
\]
for all \( \psi \in C^0_c(\Omega_{R_k}) \), and satisfying property (i) and (ii) of Definition 2.1 with \( \Omega = \Omega_{R_k} \), property (iii) and, moreover, \( v^{(k)} = 0 \) at \( \partial B_{R_k} \times [0, T] \). Such a solution is then searched as suitable limit of a sequence of functions \( v^{(k)}_m := u^{(k)}_m + \mathcal{V}, \mathcal{V} := \delta \mathcal{V} \), solving the following sequence of “approximating” problems, \( \ell = 1, \ldots, m, m \in \mathbb{N} \):
\[
(\partial_t u^{(k)}_m, \varphi_\ell) + M \gamma^{(k)}_m \cdot \gamma \varphi_\ell + 2 \nu [u^{(k)}_m, \varphi_\ell] - ((u^{(k)}_m + \mathcal{V} - \gamma^{(k)}_m) \cdot \nabla \varphi_\ell, (u^{(k)}_m + \mathcal{V})) + 2 \nu [\mathcal{V}, \varphi_\ell] + (\partial_t \mathcal{V}, \varphi_\ell) = 0,
\]
where \( \{\varphi_\ell\} \subset \mathcal{C}(\Omega_{R_k}) \) is base of \( \mathcal{H}(\Omega_{R_k}) \) normalized by the condition
\[
(\varphi_\ell, \varphi_{\ell'}) + M \gamma \varphi_\ell \cdot \gamma \varphi_{\ell'} = \delta_{\ell, \ell'},
\]
whereas
\[
u^{(k)}_m := \sum_{i=1}^m c^{(k)}_{mi} \varphi_i, \quad \gamma^{(k)}_m := \sum_{i=1}^m c^{(k)}_{mi} \gamma \varphi_i,
\]
where \( c^{(k)}_{mi}(t) \) are functions of time only, requested to solve (2.8). If we multiply both sides of (2.8) by \( c^{(k)}_{mi}(t) \), sum over \( \ell \) from 1 to \( m \), and integrate by parts as necessary we get the following relation where, in order to alleviate the notation, we suppress the superscript \( (k) \):
\[
\frac{1}{2} \frac{d}{dt} \|u_m\|^2_H + M |\gamma_m|^2 + 2 \nu \| \mathcal{D}(u_m) \|^2_H = -((u_m - \gamma_m) \cdot \nabla \mathcal{V}, u_m) + (\mathcal{V} \cdot \nabla u_m, u_m) - 2 \nu [\mathcal{V}, u_m] + (\partial_t \mathcal{V}, u_m).
\]
Denote by $\mathcal{R}_m$ the right-hand side of (2.10). Since $\Omega_{R_1}$ strictly contains the support of $\mathbf{v}$, with the help of Lemma 1.2 and classical embedding theorems we show

$$|\mathcal{R}_m| \leq c \left[ \|\mathbf{v}\|_{L^\infty(\mathbf{V}, \mathbf{H})} \|\mathbb{D}(\mathbf{u}_m)\|_2 + (\|\mathbf{v}\|_{L^\infty(\mathbf{V}, \mathbf{H})} + \|\mathbf{v}\|_{L^\infty(\mathbf{V}, \mathbf{H})}) \|\mathbb{D}(\mathbf{u}_m)\|_2 \right],$$

with the constant $c$ independent of $m$. We now employ in this relation Cauchy-Schwarz inequality along with Lemma 1.2 (e), take $\delta$ sufficiently small and replace the resulting expression back into (2.10) to infer

$$\frac{d}{dt} \|\mathbf{u}_m\|^2 + \nu \|\mathbb{D}(\mathbf{u}_m)\|^2 \leq c \|\mathbf{v}\|^2_{L^\infty(\mathbf{V}, \mathbf{H})} ,$$

(2.11)

By Lemma 1.2

$$\|\mathbb{D}(\mathbf{u}_m)\|^2 \geq c(R)(\|\mathbf{u}_m\|^2 + M |\gamma_m|^2),$$

which, once replaced in (2.11) in conjunction with Gronwall lemma, allows us to deduce

$$\|\mathbf{u}_m(t)\|^2 + M |\gamma_m(t)|^2 \leq (\|\mathbf{u}_m(0)\|^2 + M |\gamma_m(0)|^2) e^{-ct} + c \|\mathbf{v}\|^2_{L^\infty(\mathbf{V}, \mathbf{H})},$$

for some positive constant $c_0$. Employing this inequality we deduce, on the one hand, by (2.9), that $|c_{m_i} |$ are uniformly bounded, which implies the existence of a global, unique solution to the approximating problem (2.8); and, on the other hand, that the map $\{c_{m_i}(0)\} \mapsto \{c_{m_i}(T)\}$ must have a fixed point, which implies that (2.8) admits a $T$-periodic solution for each $m \in \mathbb{N}$; see [11, Lemma 3.1] for details. Furthermore, from (2.11), (1.11) and Lemma 2.1 we deduce the uniform estimate (in both $m$ and $k$)

$$\|\gamma_m\|_{L^2(0,T)} + \|\mathbb{D}(\mathbf{u}_m)\|_{L^2(L^2)} \leq c \delta .$$

(2.12)

The latter together with Lemma 1.2 implies, in particular, the existence of $t_m \in (0,T)$ such that

$$\|\mathbf{u}_m(t_m)\|^2 + M |\gamma(t_m)| \leq c(R) \delta .$$

as a result, integrating both sides of (2.11) from $t_m$ to $t_m + T$ and using the periodicity of $\mathbf{u}_m$, we get, in particular,

$$\|\mathbf{u}_m\|_{L^\infty(L^2)} \leq c(R) \delta .$$

(2.13)

Combining (2.12)–(2.13) with well-known procedures (see e.g. [4, Section 3]), we prove the existence of a field $\mathbf{u}^{(k)}$ and of a subsequence, again denoted by $\{\mathbf{u}_m\}$, such that

$$\mathbf{u}^{(k)} \in L^2(0,T; \mathcal{H}(\Omega_R)) \cap L^\infty(0,T; L^2(\Omega_R)) ;$$

$$\mathbf{u}_m \rightarrow \mathbf{u}^{(k)} \text{ weakly in } L^2(0,T; \mathcal{H}(\Omega_R)) ,$$

$$\mathbf{u}_m \rightarrow \mathbf{u}^{(k)} \text{ strongly in } L^2(0,T; L^2(\Omega_R)) ,$$

$$\mathbf{u}_m(t) \rightarrow \mathbf{u}^{(k)}(t) \text{ weakly in } L^2(\Omega_R) , \text{ for all } t \in [0,T] ,$$

$$\gamma_m \rightarrow \gamma^{(k)} \text{ strongly in } L^2(0,T) .$$

(2.14)

Recalling that $\mathbf{u}_m(0) = \mathbf{u}_m(T)$, $m \in \mathbb{N}$, the last condition in (2.14) furnishes that $\mathbf{u}^{(k)}(0) = \mathbf{u}^{(k)}(T)$, namely, that $\mathbf{u}^{(k)}$ is $T$–periodic. Moreover, in view of (2.12), we find

$$\|\gamma^{(k)}\|_{L^2(0,T)} + \|\mathbb{D}(\mathbf{u}^{(k)})\|_{L^2(L^2)} \leq c \delta ,$$

(2.15)

with $c$ independent of $k$. Integrating (2.8) over $(0,T)$ and using the convergence properties in (2.14), one can show that $\mathbf{v}^{(k)} := \mathbf{u}^{(k)} + \mathbf{v}$ is a solution to (2.7) satisfying the requirements of Definition 2.1.
with $\Omega \equiv \Omega_{R_k}$; see [11] Section 3] for details. The final step is to let $k \to \infty$ in (2.17) and show that $v^{(k)}$ converges, in a suitable sense, to a weak solution to problem (0.1). To this end, one extends every element of the sequence $\{v^{(k)}\}$ to be identically 0 outside $\Omega_{R_k}$. In this way the extended fields, again denoted by $v^{(k)}$, satisfy (2.11), for any arbitrarily fixed $\psi \in C_c^1(\Omega)$, provided $k$ is chosen sufficiently large. From (2.15) it immediately follows the existence of $u \in L^2(0,T;\mathcal{H}(\Omega))$ such that

$$u^{(k)} \to u, \text{ weakly in } L^2(0,T;\mathcal{H}(\Omega)),$$

which, in particular, by the properties of $\mathcal{V}$, gives

$$\|\gamma u\|_{L^2(0,T)} + \|\nabla u\|_{L^2(L^2)} \leq c \delta, \tag{2.16}$$

However, as is well known, the type of convergence in (2.16) is not enough to ensure the convergence of the nonlinear term. Nevertheless, this property can be established by means of Lions-Aubin lemma, provided one shows appropriate uniform bounds (in $k$) for $\partial_t v^{(k)}$. In turn, the latter follows by combining Lemma 2.2 and (2.15), which delivers

$$\|\partial_t v^{(k)}\|_{L^2(\mathcal{H}^{-1}(\Omega_\rho))} + \|v^{(k)}\|_{L^2(\mathcal{W}^{1,2}(\Omega_\rho))} \leq c(\rho) \delta, \text{ for each fixed } \rho \in (R_*,R_k).$$

Since the embedding $W^{1,2}(\Omega_\rho) \subset L^q(\Omega_\rho)$, $q \in [1,6)$, is compact and $L^q(\Omega_\rho) \subset \mathcal{H}^{-1}(\Omega_\rho)$, by Aubin-Lions lemma we deduce, in particular, that there exists a subsequence, again denoted by $\{v^{(k)}\}$ such that

$$v^{(k)} \to v, \text{ strongly in } L^2(0,T;L^2(\Omega_\rho)). \tag{2.18}$$

This subsequence may depend on $\rho$. However, covering $\Omega$ with an increasing sequence of bounded domains and using Cantor diagonalization method, we may select a subsequence for which (2.18) holds for all $\rho$. With (2.16)$_1$ and (2.16) in hand, it is then routine to show that the limiting field $v$ is a weak solution in the sense of Definition 2.1 which, by (2.17), satisfies the estimate stated in the theorem; see [11] Section 3] or [12] Section 3] for details.

3 Sufficient Conditions for Self-Propulsion when $\overline{\nu}_* \neq 0$

The results proved in the previous section allow us to find conditions on the boundary velocity distribution $\nu_*$ guaranteeing a non-zero net motion of the body, at least when $\nu_*$ has a non-zero average over a period. More precisely, provided $\overline{\nu}_* \neq 0$, we show that, in the class of weak solutions, $\mathcal{B}$ can propel itself whenever $\overline{\nu}_*$ has a non-zero projection on a suitable three-dimensional subspace of $L^2(\partial \Omega)$. The more involved case $\overline{\nu}_* = 0$, will be instead addressed in the following sections.

To this end, we begin to introduce the following auxiliary fields $(h^{(i)}, p^{(i)})$, $i = 1, 2, 3$,

$$\begin{align*}
\nu \Delta h^{(i)} &= \nabla p^{(i)} & \text{in } \Omega \\
\text{div } h^{(i)} &= 0 & \\
h^{(i)} &= e_i & \text{at } \partial \Omega.
\end{align*} \tag{3.1}$$

From [7] Lemma V.4.4] it follows that there exists one and only one solution to (3.1) such that

$$(h^{(i)}, p^{(i)}) \in [D^{1,r}(\Omega) \cap L^q(\Omega)] \times L^r(\Omega), \text{ all } r \in (3/2, \infty) \text{ and } q \in (3, \infty). \tag{3.2}$$
We define the quantities

\[ g_i := n \cdot T(h^{(i)}, p^{(i)}) \mid_{\partial \Omega}, \quad i = 1, 2, 3, \]  

which represent stress vectors associated to the flows \((h^{(i)}, p^{(i)})\), evaluated at the boundary, and introduce the matrix \(A\) with components

\[ A_{ki} = \int_{\partial \Omega} (g_i)_k, \quad k, i = 1, 2, 3. \]  

It is well known that \(A\) is symmetric and non-singular \[13, \text{Sections 5.2–5.4}\]. The proof of the following result is given in \[3, \text{Theorem 2.1}\].

**Lemma 3.1** Let \( V_* \) be as in Theorem \[2.1\]. Then, the problem

\[ [u_0, \psi] = 0, \quad \text{for all } \psi \in \mathcal{H}(\Omega) \]

\[ \text{div } u_0 = 0, \]

\[ u_0 = V_* + \xi_0, \quad \text{at } \partial \Omega, \]

has one and only one solution \((u_0, \xi_0) \in D^{1,2}(\Omega) \times \mathbb{R}^3\). Moreover,

\[ \xi_0 = A^{-1} \cdot \mathcal{G}, \]  

where

\[ \mathcal{G} := \sum_{i=1}^{3} \left( \int_{\Omega} V_* \cdot g_i \right) e_i. \]  

Next, let \((v, \gamma)\) be the weak solution determined in Theorem \[2.1\] and define the scaled quantities \((u, \xi)\) and \((w, \chi)\) as follows

\[ \overline{u} = \delta u, \quad \overline{\gamma} = \delta \xi; \quad \delta w = v - \overline{u}, \quad \delta \chi = \gamma - \overline{\gamma}. \]  

Then, by taking, in particular, in \[2.1\] \( \psi \in C(\Omega) \), we get that \(u\) satisfies

\[ 2 \nu [u, \psi] = \delta \left[ ((u - \xi) \cdot \nabla \psi, u) + ((w - \chi) \cdot \nabla \psi, w) \right], \quad \text{for all } \psi \in C(\Omega), \]  

and, in addition,

\[ u = V_* + \xi \quad \text{at } \partial \Omega. \]

**Lemma 3.2** Let \((v, \gamma)\) be the weak solution constructed in Theorem \[2.1\]. Then, as \(\delta \to 0\),

\[ u - u_0 \to 0 \quad \text{weakly in } \mathcal{H}(\Omega); \quad \xi \to \xi_0 \quad \text{in } \mathbb{R}^3, \]

where \((u_0, \xi_0)\) is the solution to the problem \[3.5\] given in Lemma \[3.1\].
Proof. Let \( \{ \delta_n \} \subset (0, \delta_0) \) with \( \delta_n \to 0 \) as \( n \to \infty \), and let \((u_n, \gamma_n)\) be the corresponding weak solutions of Theorem 2.1. From (2.6), (3.8), observing that \((u_n - \bar{V}) \in Y(\Omega)\), and also recalling that \(D^{1,2}(\Omega) \subset W^{1,2}(\Omega_R)\), we readily deduce that there is \((\tilde{u}, \tilde{\xi}) \in D^{1,2}(\Omega) \times \mathbb{R}^3\) such that (possibly, along a subsequence)
\[
\nabla u_n \to \nabla \tilde{u} \text{ weakly in } L^2(\Omega); \quad \xi_n \to \tilde{\xi} \text{ in } \mathbb{R}^3,
\]
with the constant \(c\)
\[
\tilde{u} = V_* + \tilde{\xi} \text{ at } \partial \Omega.
\]
In view of (3.9), we at once deduce that \((u_n, \xi_n, w_n, \nu_n)\) satisfies
\[
2 \nu [u_n, \psi] = \delta_n \left[ \left( (u_n - \xi_n) \cdot \nabla \psi, u_n \right) + \left( (w_n - \nu_n) \cdot \nabla \psi, w_n \right) \right], \quad \text{for all } \psi \in C(\Omega),
\]
Denote by \(I_n\) the quantity in brackets on the right-hand side of (3.11). By applying Schwarz inequality, we show
\[
|I_n| \leq \| \nabla \psi \|_\infty \left[ \| u_n \|_{2, \Omega_p} (|\xi_n| + \| u_n \|_{2, \Omega_p}) + \| w_n \|_{L^2(\Omega_p)} (|\nu_n|_{L^2(0, T)} + \| w_n \|_{L^2(\Omega_p)}) \right],
\]
where \(\Omega_p \supset \text{supp}(\psi)\). From (2.6) and the scaling (3.8) we get
\[
|\xi_n| + \| \nu_n \|_{L^2(0, T)} \leq c.
\]
with \(c\) independent of \(n\). Furthermore, we observe that, for a.a. \(t \in [0, T]\), \(u_n(\cdot, t) - \delta V(\cdot, t)\) is in \(Y(\Omega)\), and so, by (1.10), Lemma 2.1 and (2.6) we infer
\[
\| u_n \|_{L^2(\Omega_p)} \leq c \delta_n,
\]
which implies
\[
\| u_n \|_{2, \Omega_p} + \| w_n \|_{L^2(\Omega_p)} \leq c,
\]
with the constant \(c\) independent of \(n\). Replacing the above information back in (3.12) and using again (2.6), we conclude that \(I_n\) is uniformly bounded in \(n\). Therefore, passing to the limit \(n \to \infty\) in (3.11) and employing (3.10) we conclude
\[
[u_n, \psi] = 0, \quad \text{for all } \psi \in C(\Omega); \quad \text{div } \tilde{u} = 0; \quad \tilde{u} = V_* + \tilde{\xi} \text{ at } \partial \Omega,
\]
that is, \((\tilde{u}, \tilde{\xi}) \in D^{1,2}(\Omega) \times \mathbb{R}^3\) is a solution to (3.5). However, by Lemma 3.1, the solution is unique, which gives, on the one hand, \((\tilde{u}, \tilde{\xi}) \equiv (u_0, \xi_0)\) and, on the other hand, that the convergence in (3.10) holds not only along a subsequence, but as long as \(\delta \to 0\). Finally, since \(u - u_0 \in Y\), (3.10) gives \(u - u_0 \to 0\), weakly in \(Y(\Omega)\), as claimed. The proof of the lemma is completed.

We are now in a position to show the main result of this section.

**Theorem 3.1** Let \((v, \gamma)\) be the weak solution to (1.1) given in Theorem 2.1. If \(V_*\) is such that the corresponding vector \(g\) in (3.7) does not vanish, then also \(\gamma \neq 0\) and we have
\[
\gamma = \delta A^{-1} \cdot g + o(\delta), \quad \text{as } \delta \to 0.
\]
In other words, under the above assumption, the body \(B\) can self-propel.

*Proof.* Recalling that \(\gamma = \delta \xi\), the theorem is an immediate corollary of the previous lemma and (3.6).
Remark 3.1 Theorem 3.1 shows, in particular, that, if \( \nabla u = 0 \), self-propulsion is a phenomenon to be searched at an order in \( \delta \) higher than 1. It also shows that, since the linearized approximation possesses in this case only the identically vanishing solution (see Lemma 3.1), the solution to the nonlinear problem that would ensure self-propulsion cannot be obtained by a perturbation argument around its linear counterpart. In other words, when \( \nabla u = 0 \) self-propulsion is a strictly nonlinear phenomenon. As we will show later on, its resolution will require a suitable contradiction argument directly applied on the full set of nonlinear equations. In order to reach this goal, however, we need to prove well-posedness of the problem (4.1) in a class of solutions more regular than weak solutions. For such a purpose, we will establish some key results on certain steady-state and time-periodic problems, which will be the main object of the following two sections.

4 On the Resolution of a Nonlinear Steady-State Problem

The main objective of this section is to show existence, uniqueness and corresponding estimates for solutions to the following boundary-value problem in the unknowns \( u, p \) and \( \xi_u \):

\[
\begin{align*}
\nu \Delta u + \xi_u \cdot \nabla u - \nabla p &= f \quad \text{in } \Omega, \\
\operatorname{div} u &= 0 \quad \text{on } \partial \Omega, \\
\lim_{|x| \to \infty} u(x) &= 0, \\
\int_{\partial \Omega} [\nabla u(x) - \mathcal{F}] \cdot n &= 0,
\end{align*}
\]

(4.1)

where \( f = \operatorname{div} \mathcal{F} \). Notice that (4.1) is a nonlinear problem. Precisely, we have the following.

Lemma 4.1 Let \( f \in L^q, \quad q \in (1, 2]. \) Then, there is \( (u, p) \in D^{2/q} \times D^{1/q} \) solving (4.1) that, in addition, satisfies the estimate

\[
\|u\|_{D^{2/q}} + \|p\|_{D^{1/q}} \leq C_1 \left( \|f\|_2 + \|\mathcal{F}\|_{q} + \|\mathcal{F}\|_2 + \|\mathcal{F}_x\|_{\frac{3q}{3-q}} \right)
\]

(4.2)

Suppose, next, \( q \in \left(1, \frac{6}{5}\right)\) and let \( (u_1, p_1) \in D^{2/q} \times D^{1/q} \) be another solution to (4.1) corresponding to \( f_1 = \operatorname{div} \mathcal{F}_1 \in L^q \) with \( (\mathcal{F}_1 - \mathcal{F}) \in L^{\frac{3q}{3-q}}(\Omega) \). Then, there exists \( c_0 > 0 \) such that if

\[
\|f\|_{L^q} < c_0,
\]

(4.3)

it follows that \( U := u_1 - u_2 \in D^{2/q} \times D^{1/q}, \) \( P := p_1 - p \in D^{1/q} \) and, in addition, (with \( \mathcal{H} := \mathcal{F}_1 - \mathcal{F} \))

\[
\begin{align*}
\|U\|_{2/q} + |U|_{1, 2/q} + \|P\|_{1/q} &\leq c_1(\|\mathcal{H}\|_2 + \|\mathcal{H}\|_{\frac{3q}{3-q}}), \\
|U|_{2,2} + \|P\|_{1,2} &\leq c \left( \|\operatorname{div} \mathcal{H}\|_2 + \|\mathcal{H}\|_{2} + \|\mathcal{H}\|_{\frac{3q}{3-q}} \right)
\end{align*}
\]

(4.4)

Thus, in particular, if \( f \equiv f_1 \), under the assumption (4.3) the solution \( (u, p) \) is unique in the class \( D^{2/q} \times D^{1/q}, \quad q \in \left(1, \frac{6}{5}\right)\).
Proof. Since the actual value of $\nu$ is irrelevant, we set, for simplicity, $\nu = 1$. We begin to put (4.1) in a weak form. To this end, we dot-multiply both sides of (4.1) by $\varphi \in C$ and integrate by parts as necessary. We get

$$-[u, \varphi] + \xi \cdot \int_{\partial \Omega} [T(u, p) - \mathcal{F}] \cdot n + (\xi u \cdot \nabla u, \varphi) = -(\mathcal{F}, \nabla \varphi),$$

so that, reinforcing (4.1) in the preceding relation, we find

$$-[u, \varphi] + (\xi u \cdot \nabla u, \varphi) = -(\mathcal{F}, \nabla \varphi), \quad \text{for all } \varphi \in C. \quad (4.5)$$

It is also easy to see, conversely, that if $(u, p)$ satisfies (4.5) and is sufficiently smooth, then it satisfies (4.1). Now, by formally replacing $u$ for $\varphi$ in (4.6), and then using Schwarz inequality along with (1.9), we obtain

$$\|D(u)\|_2 \leq \sqrt{2} \|\mathcal{F}\|_2,$$

(4.6)

which, in particular, by (4.11) implies (4.12). Combining (4.6) with the Galerkin method, one can show that (4.5) has at least one solution $u \in H$. To such a solution, we can then associate a pressure field $p$ with $p \in L^2_{\text{loc}}(\Omega)$ and such that

$$\|p\|_{2, \Omega_R} \leq c_R (\|\mathcal{F}\|_2 + |\xi u| \|u\|_{2, \Omega_R});$$

(4.7)

see [7, Lemma V.1.1 and Lemma VII.1.1]. Since $u \in H$, by (4.1) and (1.11) we infer, in a distributional sense,

$$\text{div} [T(u, p) - \mathcal{F}] = -\xi u \cdot \nabla u \in L^2(\Omega)$$

and, by (4.6), (4.7) and the assumption,

$$(T(u, p) - \mathcal{F}) \in L^2(\Omega_R).$$

As a consequence, e.g., [7, Theorem III.3.2],

$$\|T(u, p) - \mathcal{F}\|_{W^{-1/2, 2} (\partial \Omega)}, \quad \text{div} [T(u, p) - \mathcal{F}] = -\xi u \cdot \nabla u \in L^2(\Omega)$$

(4.8)

and, by (4.6), (4.7) and the assumption,

$$(T(u, p) - \mathcal{F}) \in L^2(\Omega_R).$$

Following [16], we shall next make a suitable extension of problem (4.1) to the whole space $\mathbb{R}^3$. Consider the vector-valued Neumann problem:

$$\Delta P = 0 \text{ in } \Omega_0; \quad \frac{\partial P}{\partial n} = (T(u, p) - \mathcal{F}) \cdot n \text{ at } \partial \Omega.$$  

(4.10)

In view of (4.9), and the fact that, by (4.5), we have

$$\int_{\partial \Omega} (T(u, p) - \mathcal{F}) \cdot n = 0,$$

in distributional sense, we may assert that (4.10) admits one and only one (up to a constant) solution $P \in W^{1,2}(\Omega_0)$, and that, setting $\mathcal{G} := \nabla P$, the following estimate holds

$$\|\mathcal{G}\|_2 \leq c \|T(u, p) + \xi u \otimes \xi u - \mathcal{F}\|_{W^{-1/2, 2} (\partial \Omega)} \leq c (\|\mathcal{F}\|_2 + |\xi u| (\|u\|_{2, \Omega_R} + \|\nabla u\|_2)), \quad (4.11)$$

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where we have used (1.9) and (1.11). Define

$$\widetilde{\mathcal{F}}(x) = \begin{cases} \mathcal{F}(x) & \text{if } x \in \Omega \\ \mathcal{G}(x) & \text{if } x \in \Omega_0 \end{cases}.$$  

Taking into account that, by Lemma L2, \(\|u\|_{2,\Omega} \leq c\|u\|_6 \leq c\|u\|_{1,2}\), we deduce with the help of (4.6), (1.11) and, again, Lemma L2

$$\|\mathcal{F}\|_q + \|\widetilde{\mathcal{F}}\|_2 \leq c \left(\|\mathcal{F}\|_q + (1 + |\xi_u|)\|\mathcal{F}\|_2\right) \quad (4.12)$$

Consider now the problem

$$\begin{cases} \Delta v + \xi_u \cdot \nabla v - \nabla p = \text{div} \mathcal{F} \\ \text{div} v = 0 \end{cases} \quad \text{in } \mathbb{R}^3. \quad (4.13)$$

It is well known that (4.13) has one and only one distributional solution \((v, p) \in [D^{1,2}(\mathbb{R}^3) \cap D^{1,\infty}(\mathbb{R}^3)] \times \mathcal{L}_2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)]\) such that

$$|v|_{1,2} + |v|_{1,q} + \|p\|_2 \leq c \left(\|\mathcal{F}\|_2 + \|\widetilde{\mathcal{F}}\|_q\right) \quad (4.14)$$

see [7] Theorem IV.2.2 and Theorem VII.4.2. We now extend \((u, p)\) to the whole of \(\mathbb{R}^3\) by setting

$$\tilde{u} = \begin{cases} u(x) & \text{if } x \in \Omega \\ \xi_u & \text{if } x \in \Omega_0 \end{cases} \quad \tilde{p} = \begin{cases} p(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega_0 \end{cases}.$$  

It is easy to see that \(\tilde{u}, \tilde{p}\) is a solution to (4.13) in the sense of distribution. In fact, for all \(\psi \in C_0^\infty(\mathbb{R}^3)\),

$$\int_{\mathbb{R}^3} \left[\mathcal{L}(\tilde{u}, \tilde{p}) + \xi_u \otimes u - \mathcal{F}\right] \cdot \nabla \psi = \int_{\Omega} \left[\mathcal{L}(u, p) + \xi_u \otimes u - \mathcal{F}\right] \cdot \nabla \psi + \int_{\Omega_0} (\xi_u \otimes \xi_u - \mathcal{G}) \cdot \nabla \psi$$

$$= \int_{\Omega} \left[\mathcal{L}(u, p) + \xi_u \otimes \xi_u - \mathcal{F}\right] \cdot n\psi - \int_{\partial \Omega} (\xi_u \otimes \xi_u - \mathcal{G}) \cdot n\psi = 0$$

We then deduce that \(w := \tilde{u} - v\) satisfies the following equation in distributional sense

$$\begin{cases} \Delta w + \xi_u \cdot \nabla w - \nabla \phi = 0 \\ \text{div} w = 0 \end{cases} \quad \text{in } \mathbb{R}^3.$$  

Therefore, since \(w \in D^{1,2}_0(\mathbb{R}^3)\), it must be \(w \equiv 0\) [7] Theorem IV.2.2 and Theorem VII.4.2] which implies, in particular, \(u \equiv v\) in \(\Omega\). Thus, we have \((u, p) \in D^{1,q}(\Omega) \times [L^q(\Omega) \cap L^2(\Omega)]\) and also, from (1.11), (4.12),

$$|u|_{1,2} + |u|_{1,q} + \|p\|_2 + \|p\|_q \leq c \left(\|\mathcal{F}\|_q + (1 + |\xi_u|)\|\mathcal{F}\|_2\right) \quad (4.15)$$

We next observe that, from (4.11), we obtain that \((u, p)\) can be viewed as a (distributional) solution to the following Stokes problem

$$\begin{cases} \Delta u - \nabla p = F \\ \text{div} u = 0 \end{cases} \quad \text{in } \Omega, \quad u(x) = \xi_u \quad \text{on } \partial \Omega, \quad (4.16)$$
where \( F := -\xi_u \cdot \nabla u + f \). By assumption and what we have shown so far, \( F \) is in \( L^2(\Omega) \), and so by [7, Theorem V.5.3] we deduce, on the one hand, \((u,p) \in D^{2,2}(\Omega^R) \times D^{1,2}(\Omega^R) \) for all \( R > R_* \), and, on the other hand, by local estimates for the Stokes problem [7, Theorem IV.5.1 and Remark IV.5.1]

\[
|u|_{2,2,\Omega_{2R}} \leq c_R (\|F\|_{2,\Omega_{3R}} + \|u\|_{2,\Omega_{3R}} + |\xi u|).
\]

As a consequence, since by (1.10) and (4.6)

\[
\|u\|_{2,\Omega_{R}} \leq c_R \|u\|_{6,\Omega_{R}} \leq c_R \|\mathcal{F}\|_2,
\]

from the latter, Lemma 1.2 and (4.1), we infer

\[
(u,p) \in D^{2,2}(\Omega) \times D^{1,2}(\Omega).
\]

We may then use [7, Lemma V.4.3], (4.2) and (4.17), to show the estimate

\[
|u|_{2} + |p|_{1,2} \leq c (\|f\|_{2} + (1 + |\xi u|)\|\mathcal{F}\|_2).
\]

The proof of existence is thus completed. Let us next prove the validity of (4.4). Setting \( \zeta := \xi u_1 - \xi u \), it follows that

\[
\begin{cases}
\Delta U + \xi u_1 \cdot \nabla U = -\zeta \cdot \nabla u + \nabla P + \text{div} \mathcal{H} \\
\text{div} U = 0
\end{cases}
\]

in \( \Omega \),

\[
U = \zeta \text{ at } \partial \Omega,
\]

\[
\int_{\partial \Omega} [\mathbb{T}(U,P) - \mathcal{H}] \cdot n = 0.
\]

We now dot-multiply both sides of (4.19) by \( U \) and integrate by parts over \( \Omega \). In view of the summability properties of \( U \) and \( u \), and (4.19), we thus get

\[
\|\mathcal{D}(U)\|_2^2 = \int_{\Omega} [\zeta \cdot \nabla U \cdot u - \mathcal{H} \cdot \nabla U],
\]

which, in turn, with the help of Schwarz inequality, (1.9) and (1.11) leads to

\[
|U|_{1,2} \leq c_1 (\|u\|_{2} |U|_{1,2} + \|\mathcal{H}\|_2).
\]

Since \( q \in (1, \frac{6}{5}) \), we may use the embedding in Lemma 1.3 along with (4.2) to show

\[
\|u\|_2 \leq c_2 (\|f\|_2 + \|\mathcal{F}\|_q + \|\mathcal{F}\|_2 + \|\mathcal{F}\|_2^2) \leq c_2 (\|f\|_{L^q} + \|f\|_{L^q}^2).
\]

From the latter and (4.20) we deduce that there is a constant \( c_0 > 0 \) such that if (4.3) holds, then \( \|u\|_2 \leq 1/2c_1 \), which once replaced in (4.20), with the help of (1.11) allows us to conclude

\[
|\zeta| \leq c |U|_{1,2} \leq c_3 \|\mathcal{H}\|_2.
\]

Set \( s := 3q/(3 - q) \). Clearly, \( s \in (\frac{3}{2}, 3) \) and, as a result, we can apply [7, Theorem V.5.1 and Theorem VII.5.2] to the boundary-value problem (4.19)_{1,2,3} and obtain the following inequality

\[
\|U\|_{\frac{3q}{3-q}} + |U|_{1,s} + |P|_s \leq c (|\zeta| \|u\|_s + \|\mathcal{H}\|_s + |\zeta|).
\]

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Our next step will be to give a suitable estimate of the right-hand side of (4.22). From Lemma 1.1 and (1.2) we find
\[
\|u\|_s \leq c (\|f\|_{L^q} + \|f\|_{L^q}^2) .
\] (4.23)
Furthermore, setting
\[
\tilde{U} = \begin{cases} 
U(x) & \text{if } x \in \Omega \\
\zeta & \text{if } x \in \Omega_0 
\end{cases},
\]
we have
\[
|\zeta|_{\Omega_0}^{\frac{2}{3s}} = \left( \int_{\Omega_0} |\zeta|^{\frac{2}{3s}} \right)^{\frac{3s}{2}} \leq \|\tilde{U}\|_s^{\frac{3s}{2}},
\]
which, by Lemma 1.1 entails
\[
|\zeta| \leq c |\tilde{U}|_1,1 = c |U|_{1,s} .
\] (4.24)
Combining (4.21)–(4.24), we thus prove that there exists a constant \(c_0\) such that if (4.3) holds, then
\[
\|U\|_{s,\Omega} + \|U\|_{1,1,s} + \|P\|_s \leq c (\|\mathcal{H}\|_2 + \|\mathcal{H}\|_s) ,
\] (4.25)
which proves (4.4)1. We now pass to the estimate of the second derivatives of \(U\). Applying [7, Lemma V.4.3] to (4.19) we infer
\[
|U|_{2,2} + |P|_{1,2} \leq c (\|\text{div}\mathcal{H}\|_2 + |\xi_{u_1}| |U|_{1,2} + |\zeta| |u|_{1,2} + \|U\|_{2,\Omega} + \|P\|_{2,\Omega}) .
\] (4.26)
From (4.2), (4.3), (4.24), and (4.25) it follows that
\[
|\zeta| |u|_{1,2} \leq c |U|_{1,s} \leq c (\|\mathcal{H}\|_2 + \|\mathcal{H}\|_s) ,
\] (4.27)
whereas by (1.9), (1.10) and (4.21)
\[
\|U\|_{2,\Omega} \leq c \|U\|_{6,\Omega} \leq c |U|_{1,2} \leq c \|\mathcal{H}\|_2 .
\] (4.28)
Recalling that \(q \in (1, \frac{6}{5})\), we have \(s = 2\) if \(q = \frac{6}{5}\), otherwise \(s \in (\frac{6}{5}, 2)\). In the first case, from (4.25) we have
\[
\|P\|_{2,\Omega} \leq c (\|\mathcal{H}\|_2 + \|\mathcal{H}\|_s) .
\] (4.29)
In the second case, we observe that since \(P \in D^{1,2}(\Omega) \cap L^s(\Omega)\), by Lemma 1.1 we get \(P \in L^6(\Omega)\) with
\[
\|P\|_6 \leq c |P|_{1,2} .
\]
Thus, if \(s < 2\), by elementary interpolation, the latter inequality and (4.25) we show, for arbitrary \(\varepsilon > 0\),
\[
\|P\|_{2,\Omega} \leq c_\varepsilon \|P\|_s + \varepsilon \|P\|_6 \leq c_\varepsilon \|P\|_s + \varepsilon C |P|_{1,2} \leq c (\|\mathcal{H}\|_2 + \|\mathcal{H}\|_s) + \varepsilon C |P|_{1,2} .
\] (4.30)
From (4.26)–(4.30) and (4.21) we then deduce
\[
|U|_{2,2} + |P|_{1,2} \leq c \left[ \|\text{div}\mathcal{H}\|_2 + (|\xi_{u_1}| + 1) \|\mathcal{H}\|_2 + \|\mathcal{H}\|_s \right] .
\] (4.31)
Finally, from
\[
|\xi_{u_1}| \leq |\zeta| + |\xi_u| ,
\]
and (4.2), (4.3) and (4.21), we show
\[
|\xi_{u_1}| \leq c (1 + \|\mathcal{H}\|_2) ,
\]
which, once replaced in (4.31), proves (4.4)2. The proof of the lemma is thus completed. □
5 On the Resolution of a Time-Periodic Linear Problem

Our next task is to study the well-posedness of the following problem

\[ \begin{align*}
\partial_t w - \nu \Delta w &= -\nabla p + f \\
\text{div} w &= 0 \\
w &= \chi w \quad \text{on } \partial \Omega \times \mathbb{R}, \\
M \dot{\chi} w + \int_{\partial \Omega} T(w, p) \cdot n &= F
\end{align*} \]

in a suitable function class of $T$-periodic solutions, with $f = f(x, t)$ and $F(t)$ given $T$-periodic functions. In order to reach this goal, we need to premise a preparatory result.

**Lemma 5.1** Consider the boundary-value problems, $i = 1, 2, 3, k \in \mathbb{Z}$:

\[ \begin{align*}
i k \omega H_k^{(i)} &= \Delta H_k^{(i)} - \nabla \gamma_k^{(i)} \\
\text{div} H_k^{(i)} &= 0 \\
H_k^{(i)}|_{\partial \Omega} &= e_i, \quad H_0^{(i)} = 0,
\end{align*} \]

with $\omega := 2\pi/T$. The following properties hold.

(i) There is one and only one solutions $(H_k^{(i)}, \gamma_k^{(i)}) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. This solution satisfies the estimates

\[ \|H_k^{(i)}\|_2 \leq c; \quad \|\nabla H_k^{(i)}\|_2 \leq c (|k| + 1^{1 \over 2}) ; \quad \|H_k^{(i)}\|_{2,2} \leq c (|k| + 1) ; \quad c \leq \|\nabla H_k^{(i)}\|_2, \]

where $c$ is a constant independent of $k$.

(ii) The matrix $\mathbb{B}$ defined by components

\[ (\mathbb{B})_{ij} = \int_{\partial \Omega} T_{ij}(H_k^{(i)}, \gamma_k^{(i)}) n_j \]

satisfies the condition (with $^* \equiv c.c.$)

\[ \zeta^* \cdot \mathbb{B} \cdot \zeta = i k \omega \|\zeta_i H^{(i)}\|_2^2 + \|\zeta_i \nabla H^{(i)}\|_2^2, \]

for all $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3$, and it is, therefore, invertible.

**Proof.** We begin to prove (i). Since the proof is the same for $i = 1, 2, 3$, we chose $i = 1$ and, for simplicity, omit the superscript. Let $\phi = \phi(|x|)$ be a (smooth) cut-off function such that

\[ \phi(x) = \begin{cases} 
1 & \text{in } \Omega_R \\
0 & \text{in } \Omega^2_R,
\end{cases} \]

and set

\[ \Phi(x) = \text{curl} \left( x_2 \phi(x) e_3 \right). \]
Clearly, \( \text{div} \Phi = 0 \) and since \( (\partial_i \equiv \partial/\partial x_i) \)
\[
\Phi(x) = -e_3 \times \nabla(x_2 \phi(x)) = x_2(\partial_2 \phi(x) e_1 - \partial_1 \phi(x) e_2) + \phi(x) e_1,
\]
by the property of \( \phi \) we deduce \( \Phi(x) = e_1 \) for all \( x \in \partial \Omega \). Therefore, \( \Phi \) is a solenoidal extension of \( e_1 \) with support contained in \( \Omega_{2R} \). Setting \( v_k := H_k - \Phi \), from (5.2) we deduce that \( v_k \) satisfies the following boundary-value problem, for all \( |k| \geq 1 \):
\[
\left\{ \begin{array}{l}
 i k \omega v_k = \Delta v_k - \nabla \gamma_k - i k \omega \Phi + \Delta \Phi \\
 \text{div} v_k = 0 \\
 v_k|_{\partial \Omega} = 0
\end{array} \right. \quad \text{in } \Omega, \tag{5.5}
\]
Existence to (5.5) in the stated function class can be easily obtained by the Galerkin method combined with the estimate that we are about to derive. Let us dot-multiply both sides of (5.5) by \( v_k^\ast \), and integrate by parts as necessary. We obtain
\[
i k \omega \|v_k\|_2^2 + \|\nabla v_k\|_2^2 = -i k \omega (\Phi, v_k^\ast) + (\Delta \Phi, v_k^\ast), \tag{5.6}
\]
which, after taking its imaginary part, and employing Schwarz inequality and the properties of \( \Phi \), gives
\[
|k| \|v_k\|_2 \leq c (\|\nabla v_k\|_2 + |k| + 1). \tag{5.7}
\]
Similarly, by taking the real part of (5.6) and using Schwarz inequality, the properties of \( \Phi \) and (5.9) it follows that
\[
\|\nabla v_k\|_2^2 \leq c (|k| + 1)\|v_k\|_2. \tag{5.8}
\]
Replacing (5.8) into (5.7) and then employing Cauchy-Schwarz inequality furnishes
\[
\|v_k\|_2 \leq c, \tag{5.9}
\]
with \( c \) independent of \( k \), which, in turn, by (5.8) implies
\[
\|\nabla v_k\|_2 \leq c (|k| + 1)^{1/2}. \tag{5.10}
\]
Furthermore, from classical estimates for the Stokes problem \[14\], Lemma 1], from (5.5) and (5.10) we get
\[
|v_k|_{2,2} \leq c (|k| \|v_k\|_2 + |k| + 1)
\]
which combined with (5.9) furnishes
\[
|v_k|_{2,2} \leq c (|k| + 1). \tag{5.11}
\]
Recalling that \( v_k = H_k - \Phi \), from (5.9)–(5.11) we readily conclude that \( H_k \) satisfies all the properties listed in \[5.13]_1,2,3. Finally, from the trace inequality
\[
\int_{\partial \Omega} |e_1|^2 \leq c \|\nabla H_k\|_2^2 \tag{5.12}
\]
we prove also \((5.13)\), thus completing the proof of (i). In order to show (ii), set \(u := \zeta H^{(i)}\), \(\phi = \zeta \gamma^{(i)}\) where, for simplicity, the subscript "\(k\)" is omitted. From \((5.2)\) we thus get

\[
\begin{align*}
\begin{aligned}
  i k \omega u &= \Delta u - \nabla \phi \\
  \text{div } u &= 0 \\
  u|_{\partial \Omega} &= \zeta
\end{aligned}
\end{align*}
\]

in \(\Omega\),

\[
\tag{5.13}
\]

If we dot-multiply both sides of \((5.13)\) by \(u^*\) and integrate over \(\Omega\) we show

\[
\zeta^* \cdot B \cdot \zeta = i k \omega \|u\|^2 + \|D(u)\|^2,
\]

and so if 0 is an eigenvalue of \(B\), we must have \(u \equiv 0\), which, once evaluated at \(\partial \Omega\) produces \(\xi = 0\), and the proof of the lemma is completed.

\[
\square
\]

**Lemma 5.2** Let \(q \in (1, \infty)\), \(s = 2\) if \(q \in (1, 2]\), and \(s = q\) if \(q \in (2, \infty)\). Then, for any \((f, F) \in L^2_{q, \sharp} \times L^2_{s, \sharp}(0, T)\), problem \((5.1)\) has one and only one solution \((w, p, \chi w) \in W^2_{q, \sharp} \times P^1_{q, \sharp} \times W^1_{s, \sharp}(0, T)\). This solution satisfies the estimate

\[
\|w\|_{W^2_{q, \sharp}} + \|p\|_{P^1_{q, \sharp}} + \|\chi w\|_{W^1_{s, \sharp}(0, T)} \leq C_2 \left( \|f\|_{L^2_{q, \sharp}} + \|F\|_{L^2_{s, \sharp}(0, T)} \right).
\]

\[
\tag{5.14}
\]

**Proof.** Since the actual value of \(\nu\) and \(M\) is irrelevant to the proof, we set, for simplicity, \(\nu = M = 1\).

We write \(w = z + u\) where \(z\) and \(u\) satisfy the following set of equations

\[
\begin{align*}
\begin{aligned}
  \partial_t z - \Delta z &= -\nabla \tau + f \\
  \text{div } z &= 0
\end{aligned}
\end{align*}
\]

in \(\Omega \times \mathbb{R}\)

\[
\tag{5.15}
\]

and

\[
\begin{align*}
\begin{aligned}
  \partial_t u - \Delta u &= -\nabla q \\
  \text{div } u &= 0
\end{aligned}
\end{align*}
\]

in \(\Omega \times \mathbb{R}\)

\[
\tag{5.16}
\]

From [10, Theorem 12], it follows that, under the stated assumptions, there exists a unique solution \((z, \tau) \in W^2_{q, \sharp} \times P^1_{q, \sharp}, q \in (1, \infty)\) that, in addition, obeys the inequality

\[
\|z\|_{W^2_{q, \sharp}} + \|\tau\|_{P^1_{q, \sharp}} \leq c \|f\|_{L^2_{q, \sharp}}, \quad q \in (1, \infty).
\]

\[
\tag{5.17}
\]

Furthermore, by trace theorem \([3]\)

\[
\|\int_{\partial \Omega} T(z, \tau) \cdot n \|_{L^q(L^q)} \leq c \left( \|z\|_{W^2_{q, \sharp}} + \|\tau\|_{P^1_{q, \sharp}} \right).
\]

\[
\tag{5.18}
\]

\(\text{(2)}\) We use summation convention over repeated indices, unless confusion may arise.

\(\text{(3)}\) Possibly, by modifying \(\tau\) by adding to it a suitable function of time.
By (5.18) we infer that the function $F$ in (5.16) is in $L^2_\tau(0,T)$. In order to find solutions to (5.16), we formally expand $u$, $q$, and $\chi_u (\equiv \chi)$, in Fourier series:

$$
u(x,t) = \sum_{k \in \mathbb{Z}} u_k(x) e^{ik \cdot t}, \quad q(x,t) = \sum_{k \in \mathbb{Z}} q_k(x) e^{ik \cdot t}, \quad \chi(t) = \sum_{k \in \mathbb{Z}} \chi_k e^{ik \cdot t}, \quad u_0 \equiv \nabla q_0 \equiv \chi_0 \equiv 0,$$

(5.19)

where $(u_k, q_k, \chi_k)$ solve the problem $k \neq 0$

$$
i k \cdot \omega \cdot u_k = \Delta u_k - \nabla q_k \quad \text{div} u_k = 0 \quad \left. u_k \right|_{\partial \Omega} = \chi_k,$$

(5.20)

with the further condition

$$i k \cdot \omega \cdot \chi_k + \int_{\partial \Omega} \mathbb{T}(u_k, q_k) \cdot n = \mathcal{F}_k,$$

(5.21)

where $\{\mathcal{F}_k\}$ are Fourier coefficients of $\mathcal{F}$ with $\mathcal{F}_0 \equiv 0$. For each fixed $k \in \mathbb{Z} - \{0\}$, a solution to (5.20)–(5.21) is given by

$$
u_k = \sum_{i=1}^{3} \chi_{ki} H^{(i)}_k, \quad q_k = \sum_{i=1}^{3} \chi_{ki} \gamma^{(i)}_k,$$

(5.22)

with $(H^{(i)}_k, \gamma^{(i)}_k)$ given in Lemma 5.1 and where $\chi_k$ solve the equations

$$i k \cdot \omega \cdot \chi_k + \sum_{i=1}^{3} \chi_{ki} \int_{\partial \Omega} \mathbb{T}(H^{(i)}_k, \gamma^{(i)}_k) \cdot n = \mathcal{F}_k.$$

(5.23)

The latter, with the notation of Lemma 5.1(i), can be equivalently rewritten as

$$(i k \cdot \omega \cdot \mathbb{I} + \mathbb{B}) \cdot \chi_k = \mathcal{F}_k,$$

(5.24)

with $\mathbb{I}$ identity matrix. The matrix $i k \cdot \omega \cdot \mathbb{I} + \mathbb{B}$ is invertible. In fact, using (5.4), for all $\xi \in \mathbb{C}^3$, we obtain the relation

$$\xi^* \cdot (i k \cdot \omega \cdot \mathbb{I} + \mathbb{B}) \cdot \xi = i k \cdot \omega \left( |\xi|^2 + \| \xi H^{(i)}_k \|_2^2 \right) + \| \xi \|_2^2,$$

that shows that 0 is not an eigenvalue. As a result, for the given $\mathcal{F}_k$, (5.24) has one and only one solution $\chi_k$. If we dot-multiply both sides of (5.24) by $\chi_k^*$ and use (5.4) we deduce

$$i k \cdot \omega \left( M |\chi_k|^2 + \| \chi_k H^{(i)}_k \|_2^2 \right) + \| \chi_k H^{(i)}_k \|_2^2 = (\mathcal{F}_k, \chi_k^*),$$

that entails, in particular, the estimate

$$|\chi_k| \leq \frac{1}{|k| \cdot \omega \cdot |\mathcal{F}_k|}, \quad |k| \geq 1,$$

from which we conclude

$$|\chi|_{W^{1,2}(0,T)}^2 = \sum_{|k| \geq 1} (|k|^2 + 1) |\chi_k|^2 \leq c \| \mathcal{F} \|_{L^2(0,T)}^2.$$

(5.25)
Combining (5.22), (5.25) and (5.33) we thus infer
\[
\sum_{|k|\geq 1} \left( (|k|^2 + 1)\|u_k\|^2_2 + \|\nabla u_k\|^2_2 + |u_k|_{2,2}^2 \right) \leq c \sum_{|k|\geq 1} (|k|^2 + 1)|\chi_k|^2 \leq c \|F\|^2_{L^2(0,T)} .
\]  
(5.26)

From (5.23), (5.25) and (5.26) and it follows that the vector functions \(u\) and \(\chi\) defined in (5.19) satisfy (5.16) and, in addition
\[ (u, q, \chi) \in W^2_x \times \mathcal{P}^{1,2} \times W^{1,2}_x (0, T) . \]
(5.27)

We next extend \(\chi\) to a solenoidal function \(h \in W^{2,q}_x\) [7, Section III.3] and write the solution \(u\) to (5.16) as \(u = v + h\). From [10] Theorem 12] and (5.27) it then follows that \(u \in W^{2,q}_x\) and \(q \in \mathcal{P}^{1,q}\). We may thus conclude that \((w := z + u, p := \tau + q, X_w := \chi)\) is a solution to (5.1) in the class \(W^{2,q}_x \times \mathcal{P}^{1,q} \times W^{1,2}_x (0, T)\), that satisfies in addition (5.14). This completes the proof of the existence property when \(q \in (1, 2)\). We shall next show that, if \(F \in L^q_x(0, T)\), with \(q \in (2, \infty)\), the solution \((w, p, X_w)\) just constructed satisfies the other properties stated in the existence part of the lemma. Actually, in view of (5.14) and (5.13), it is enough to show that if \(F \in L^q_x(0, T)\), the solution \((u, \chi)\) to (5.16) that we proved to be in the class \(W^{2,q}_x \times W^{1,2}_x (0, T)\) is, in fact, in \(W^{2,q}_x \times W^{1,q}_x (0, T)\) and that \((w, p, X_w)\) satisfies the estimate given in (5.14) for \(q \in (2, \infty)\). In order to reach our goal, we recall the following inequalities:
\[
\|v\|_{r, \partial \Omega} \leq c \left( \|v\|_{r, \Omega_R} + \|\nabla v\|_{r, \Omega_R} \|v\|_{1, r, \Omega_R} \right) , \quad r \in (1, \infty) , \quad v \in W^{1,2}(\Omega_R) ,
\]  
(5.28)

and
\[
\|q\|_{r, \partial \Omega} \leq c \left( \|\nabla u\|_{r, \Omega_R} + \|\nabla q\|_{r, \Omega_R} \|q\|_{1, r, \Omega_R} \right) , \quad r \in (1, \infty) ,
\]  
(5.29)

where \((u, q)\) is the solution to (5.16) constructed previously. The first is a well known trace inequality (e.g. [7, Theorem II.4.1]), whereas the second one is proved in [9, Lemma 2.5]. We now employ (5.28) with \(v \equiv \nabla u\) and \(r = 2\). Since \(W^{2}_x \subset C([0, T]; W^{1,2}(\Omega))\) and \(u \in W^{2}_x\), we obtain
\[
\nabla u \in L^4(L^2(\partial \Omega)) .
\]  
(5.30)

Likewise, from (5.29), we deduce
\[
q \in L^4(L^2(\Omega_R)) ,
\]
and so, applying (5.28) with \(v \equiv q\) and \(r = 2\), it follows that
\[
q \in L^4(L^2(\partial \Omega)) .
\]  
(5.31)

Therefore, from (5.30), (5.31) and (5.16), we find \(\chi \in W^{1,q}_x(0, T)\), for all \(q \in (2, 4]\), provided \(F \in L^q_x(0, T)\) and \(f \in L^{2,q}_x\). Extending the boundary data \(\chi\) to the smooth function \(h\) as done previously, by [10] Theorem 12] we get that \((u, q) \in W^{2,q}_x \times \mathcal{P}^{1,q}\) as well. With this improved regularity, by an argument entirely analogous to that used before we show that
\[
\nabla u, q \in L^q(L^2(\partial \Omega)), \quad q \in (4, 8]
\]
which gives \((u, q) \in W^{2,q}_x \times \mathcal{P}^{1,q}\), provided \(F \in L^q_x(0, T)\) and \(f \in L^{2,q}_x\). With the help of a simple boot-strap procedure, we then show that \((u, q) \in W^{2,q}_x \times \mathcal{P}^{1,q}\), provided \(F \in L^q_x(0, T)\) and \(f \in L^{2,q}_x\),
for all $q \in (2, \infty)$. Therefore, we conclude that for any $(f, F) \in L^2_\partial \times L^q_\partial (0, T)$, $q \in (2, \infty)$ there exists at least one corresponding solution $(w, p, \chi w) \in W^{2,q}_\partial \times P^{1,q} \times W^{1,q}_\partial (0, T)$ to problem (5.1). Therefore, the validity of the estimate (5.14) for $q \in (2, \infty)$ is a consequence of the Open Mapping Theorem, provided we prove uniqueness in the class $W^{2,q}_\partial \times P^{1,q} \times W^{1,q}_\partial (0, T)$. The latter amounts to show that the problem

$$
\begin{align*}
\partial_t w - \Delta w &= -\nabla p \quad \text{in } \Omega \times \mathbb{R} \\
\operatorname{div} w &= 0 \\
w|_{\partial \Omega} &= \chi w; \\
\dot{\chi} w + \int_{\partial \Omega} T(w, p) \cdot n &= 0
\end{align*}
$$

has only the zero solution in the above function class. This is easily established. In fact, if we dot-multiply (5.32) by $w$, integrate by parts over $\Omega$ and use (5.32), we get

$$
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|_2^2 + \|\chi w(t)\|^2 \right) + \|D(w(t))\|_2^2 = 0.
$$

Integrating both sides of this equation from 0 to $T$ and employing the $T$-periodicity leads us to $\|D(w(t))\|_2 \equiv 0$ which, in turn, by the characterization of the space $H$ given in Lemma 1.2, immediately furnishes $w \equiv \nabla p \equiv 0$. The proof of the lemma is completed.

\[ \square \]

6 On the Strong Solvability of the Nonlinear Problem

The main objective of this section is to show that, if the data $v^*$ are sufficiently small (in certain norms), then the problem (0.1) possesses a unique “strong” solution in a suitable function class. In order to achieve this goal, we need some further preparatory results.

Set

$$
Y^q_\partial := \{ v \in W^{1,q}(0, T; W^{2-\frac{1}{q},q}_\partial (\partial \Omega)) : v \text{ is } T\text{-periodic with } \overline{v} = 0 \},
$$

$$
Y^{2,q}_\partial := Y^q_\partial \cap Y^{q}_\partial
$$

and

$$
\|v\|_{Y^{2,q}_\partial} := \left( \int_0^T \left[ \|v\|_{W^{2,\frac{q}{q-1}}(\partial \Omega)}^2 + \|\partial_t v\|_{W^{1,\frac{q}{q-1}}(\partial \Omega)}^2 \right] \, dt \right)^{\frac{1}{2}} + \left( \int_0^T \left[ \|v\|_{W^{2,q}_\partial (\partial \Omega)}^q + \|\partial_t v\|_{W^{1,q}_\partial (\partial \Omega)}^q \right] \, dt \right)^{\frac{1}{q}}.
$$

The following lemma holds.

**Lemma 6.1** Suppose $v^* \in Y^{2,q}_\partial$, $q \in (1, \infty)$. Then, there exists a solenoidal extension $V$ of $v^*$ to $\Omega$ such that $V \in \tilde{W}^{2,q}_\partial$, and

$$
\|V\|_{\tilde{W}^{2,q}_\partial} \leq c \|v^*\|_{Y^{2,q}_\partial}.
$$

**Proof.** We construct the extension $V$ in a way similar to that given in [8, Theorem 1]. For fixed $t \in [0, T]$ consider the following boundary-value problem

$$
\begin{align*}
V &= \nu \Delta V - \nabla \phi \quad \text{in } \Omega, \\
\operatorname{div} V &= 0 \quad \text{in } \Omega, \\
V &= v^*(t) \quad \text{on } \partial \Omega.
\end{align*}
$$

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By the results of §VII.5, we know that, for each \( t \in [0, T] \) there exists a unique solution \((V(t), \phi(t))\) to (6.2) such that
\[
V(t) \in W^{2,q}(\Omega), \quad \phi(t) \in D^{1,q}(\Omega), \quad \text{all } q \in (1, \infty)
\]
which satisfies, in addition,
\[
\|V(t)\|_{2,q} + \|\nabla \phi(t)\|_q \leq c_1 \|v_*(t)\|_{q, 2 - \frac{4}{q}(\partial \Omega)}.
\]
In view of the uniqueness property and the periodicity of \( v_* \) it follows that \( V \) is \( T \)-periodic with \( V \equiv 0 \). Furthermore, again by uniqueness, the regularity assumptions on \( v_* \) and (6.3) and (6.4), we easily show that \( V \) is time-differentiable in the sense of distribution with \( \partial_t V \) satisfying
\[
\begin{align*}
\partial_t V &= \nu \Delta \partial_t V - \nabla \partial_t \phi \\
\text{div } \partial_t V &= 0 \\
\partial_t V &= \partial_t v_* \quad \text{on } \partial \Omega,
\end{align*}
\]
from which it follows \( \partial_t V \in L^q(\Omega) \) and
\[
\|\partial_t V\|_q \leq c \|\partial_t v_*\|_{q, 1 - \frac{4}{q}(\partial \Omega)}.
\]
The proof of the lemma is thus completed.

Next, consider the following linear problem in the unknowns \((V_0, P_0, \chi_0)\):
\[
\begin{align*}
\partial_t V_0 &= \nu \Delta V_0 - \nabla P_0 \\
\text{div } V_0 &= 0 \\
V_0(x, t) &= V_*(x, t) + \chi_0(t) \quad \text{at } \partial \Omega \times \mathbb{R} \\
M \chi_0 + \int_{\partial \Omega} T(V_0, P_0) \cdot n &= 0
\end{align*}
\]

**Lemma 6.2** Suppose \( V_* \in \mathcal{V}^{2,q}_2, \ q \in (1, \infty). \) Then, problem (6.5) has one and only one solution \((V_0, P_0, \chi_0) \in \mathcal{W}^{2,q}_2 \times \mathcal{P}^{1,q} \times (W^{1,2}_2(0, T) \cap W^{1,q}_2(0, T))\). This solution satisfies the estimate:
\[
\|V_0\|_{\mathcal{W}^{2,q}_2} + \|P_0\|_{\mathcal{P}^{1,q}} + \|\chi_0\|_{W^{1,q}(0, T)} \leq c \|V_*\|_{\mathcal{V}^{2,q}_2},
\]
where \( s = 2 \) if \( q \leq 2 \), and \( s = q \) otherwise.

**Proof.** We look for a solution of the form \( V_0 = V_1 + V_2 \), where \( V_2 \) is the extension of \( V_* \) constructed in Lemma 6.1 and
\[
\begin{align*}
\partial_t V_1 &= \nu \Delta V_1 - \nabla P_0 + h \\
\text{div } V_1 &= 0 \\
V_1(x, t) &= \chi_0(t) \quad \text{at } \partial \Omega \times \mathbb{R} \\
M \chi_0 + \int_{\partial \Omega} T(V_1, P) \cdot n &= -\nu \int_{\partial \Omega} D(V_2) \cdot n.
\end{align*}
\]
where \( \mathbf{h} = -\partial_t V_2 + \nu \Delta V_2 \).

From Lemma \ref{lem:1} we have
\[
\|\mathbf{h}\|_{L^2_q} \leq c \|V_*\|_{W^2_q},
\]
and by the trace theorem and Lemma \ref{lem:6.1},
\[
\|\int \nabla (V_2) \cdot n\|_{L^3(L^q)} \leq c \|V_2\|_{W^{1,2}_q} \leq c \|V_*\|_{W^2_q}.
\]

Therefore, the result follows by applying Lemma \ref{lem:5.2} to \ref{eq:6.7}, and using \ref{eq:6.8}–\ref{eq:6.9}.

\( \square \)

For \( \delta > 0 \), we set
\[
\hat{V} = \delta V_0, \quad \hat{P} = \delta \bar{P}_0, \quad \hat{\chi} = \delta \bar{\chi}_0, \quad v_* = \delta V_*,
\]
and look for a solution to \ref{eq:0.1}–with \( v_* \) as in \ref{eq:6.10}–“around” \( (\hat{V}, \hat{P}, \hat{\chi}) \), namely
\[
v = (v - \overline{v}) + \overline{v} := w + \hat{V} + u; \quad p = (p - \overline{p}) + \overline{p} := \tau + \hat{P} + p;
\]
\[
\gamma = (\gamma - \overline{\gamma}) + \overline{\gamma} := \chi + \hat{\chi} + \xi; \quad \overline{\omega} \equiv \overline{\chi} \equiv 0, \quad \overline{\tau} \equiv 0.
\]

From \ref{eq:0.1} it immediately follows that \((u, p)\) and \((w, \tau)\) solve the following coupled system of nonlinear equations:
\[
\begin{align*}
\nu \Delta u + \xi_u \cdot \nabla u - \nabla p &= \text{div} \mathcal{F}(u, w, \chi_w) \quad \text{in } \Omega, \\
\text{div} u &= 0 \\
\int_{\partial \Omega} [\mathbb{T}(u, p) - \mathcal{F}(u, w, \chi_w)] \cdot n &= 0,
\end{align*}
\]
\( \text{div } w = 0 \)
\[
\begin{align*}
\partial_t w - \nu \Delta w &= -\nabla \tau + f(u, w, \chi_w) \quad \text{in } \Omega \times \mathbb{R}, \\
\text{div } w &= \chi_w \quad \text{at } \partial \Omega \times \mathbb{R}
\end{align*}
\]

with
\[
\mathcal{F}(u, w, \chi_w) := u \otimes u + (w - \chi_w) \otimes w + (\hat{V} - \hat{\chi}) \otimes w + w \otimes \hat{V} + (\hat{V} - \chi_w - \hat{\chi}) \otimes \hat{V}.
\]

and
\[
\begin{align*}
f(u, w, \chi_w) :=& -(w - \chi_w) \cdot \nabla w - (\hat{V} - \hat{\chi}) \cdot \nabla w - w \cdot \nabla \hat{V} + (w - \chi_w) \cdot \nabla w + (\hat{V} - \hat{\chi}) \cdot \nabla w \\
+w \cdot \nabla \hat{V} + \chi_w \cdot \nabla \hat{V} - \chi_w \cdot \nabla \hat{V} - u \cdot \nabla (w + \hat{V}) - (w - \chi_w) \cdot \nabla u - (\hat{V} - \hat{\chi}) \cdot \nabla u \\
-(\hat{V} - \hat{\chi}) \cdot \nabla \hat{V} + (\hat{V} - \hat{\chi}) \cdot \nabla \hat{V}
\end{align*}
\]
\[
F(\chi_w, \xi_u) := \int_{\partial \Omega} [(v_* + \xi_u + \chi_w + \hat{\chi}) \otimes v_* - (v_* + \chi_w + \hat{\chi}) \otimes v_*] \cdot n
\]
\( \text{in } \overline{\Omega} \times \mathbb{R} \).
Existence and uniqueness of a solution to (6.12)–(6.15) for sufficiently small \( \delta > 0 \) will be shown by a suitable successive approximation scheme. To this end, it is convenient to define first another function class. For \( q \in (1, \infty) \), set
\[
\mathfrak{B}^q := \left\{ (\varphi, \psi, \chi_\psi) \in D^{2,q} \times W^{2,q}_0 \times [W^{1,2}_p(0,T) \cap W^{1,q}_p(0,T)] \right\}.
\]
Plainly, \( \mathfrak{B}^q \) becomes a Banach space endowed with the norm
\[
\| (\varphi, \psi, \chi_\psi) \|_{\mathfrak{B}^q} := \| \varphi \|_{D^{2,q}} + \| \psi \|_{W^{2,q}} + \| \chi_\psi \|_{W^{1,2}_p(0,T) \cap W^{1,q}_p(0,T)}.
\]
Moreover, also with the help of [7] Exercise II.6.2, we show that \( \mathfrak{B}^q \) is a reflexive space. Next, define
\[
\mathcal{F}_0 := (V_0 - \chi_0) \otimes V_0, \quad f_0 := (V_0 - \chi_0) \otimes (V_0 - \chi_0) \otimes V_0
\]
\[
F_0 := \int_{\partial \Omega} \left[ (V_* + \chi_0) \otimes V_* - (V_* + \chi_0) \otimes V_* \right] \cdot n.
\]
(6.16)
and
\[
\tilde{\mathcal{F}}(u, w, \chi_w) := \mathcal{F}(u, w, \chi_w) - \delta^2 \mathcal{F}_0; \quad \tilde{f}(u, w, \chi_w) := f(u, w, \chi_w) - \delta^2 f_0,
\]
\[
\tilde{F}(\chi_w, \xi_u) := F(\chi_w, \xi_u) - \delta^2 F_0.
\]
(6.17)
The following lemma holds.

**Lemma 6.3** Assume \( q \in (1, \frac{3}{2}] \), and let \( U := (u, w, \chi_w) \in \mathfrak{B}^q \), and \( \hat{V} \) as in [6.10]. Then, there is a positive constant \( c = c(\Omega, q) \) such that
\[
\| \text{div} \tilde{\mathcal{F}}(U) \|_2 + \| \tilde{\mathcal{F}}(U) \|_2 + \| F(U) \|_{L^{2,q}} \leq c \left( \| U \|_{\mathfrak{B}^q}^2 + \delta \| U \|_{\mathfrak{B}^q} \right),
\]
\[
\| \tilde{F}(\chi_w, \xi_u) \|_{L^{2,q}(0,T) \cap L^q(0,T)} \leq c \delta \| U \|_{\mathfrak{B}^q},
\]
\[
\| \text{div} \mathcal{F}_0(U) \|_2 + \| \mathcal{F}_0(U) \|_2 + \| \mathcal{F}_0(U) \|_q \leq c.
\]
(6.18)

**Proof.** By assumption, \( U \in \mathfrak{B}^q \), and, by Lemma [6.1], \( \hat{V}_0 \in W^{2,q}_0 \). Thus, using (1.11), it follows at once \( \tilde{F} \in L^{2,q}(0,T) \cap L^q(0,T) \) and the validity of (6.18)_2. Moreover, with the help of Lemma [1.3] and Hölder inequality, we get
\[
\| u \otimes u \|_2 + \| u \otimes u \|_q \leq c \| u \|_1^2 + \| u \|_{\frac{3}{2}, q} \| u \|_3 \leq c \| u \|_{D^{2,q}}^2 \leq c \| U \|_{\mathfrak{B}^q}^2,
\]
and
\[
\| \text{div} (u \otimes u) \|_2 = \| u \cdot \nabla u \|_2 \leq \| u \|_6 \| \nabla u \|_3 \leq c \| u \|_{D^{2,q}}^2 \leq c \| U \|_{\mathfrak{B}^q}^2.
\]
In analogous fashion, using this time Lemma [1.4] we get
\[
\| (w - \chi_w) \otimes w \|_2 + \| (w - \chi_w) \otimes w \|_q \leq c \left[ \| \chi_w \|_{L^\infty(0,T)} \left( \| w \|_{L^2(\Omega)} + \| w \|_{L^q(\Omega)} \right) + \| w \|_{L^2(\Omega)} \| w \|_{L^\infty(\Omega)} + \| w \|_{L^q(\Omega)} \| w \|_{L^\infty(\Omega)} \right] \leq c \| U \|_{\mathfrak{B}^q},
\]
and also
\[
\|\text{div} (w - \chi w) \otimes w\|_2 = \|(w - \chi w) \cdot \nabla w\|_2 \leq c \left( \|\chi w\|_{L^\infty(0,T)} \|\nabla w\|_{L^2(L^2)} + \|w\|_{L^6(L^6)} \|\nabla w\|_{L^3(L^3)} \right)
\]
\[
\leq c \|U\|_{B_q}^3 .
\]
By an entirely similar procedure, we show
\[
\|(\bar{V} - \bar{\chi}) \otimes w + w \otimes \bar{V} - \chi w \otimes \bar{V}\|_2 + \|(\bar{V} - \bar{\chi}) \otimes w + w \otimes \bar{V} - \chi w \otimes \bar{V}\|_q \leq c\delta \|U\|_{B_q}
\]
and
\[
\|\text{div} (\bar{V} - \bar{\chi}) \otimes w + w \otimes \bar{V} - \chi w \otimes \bar{V}\|_2 \leq c\delta \|U\|_{B_q} .
\]
The proof of the stated estimate for \(\mathcal{F}_0\) and \(f\) is very much alike, and we leave it to the reader.

We also have the following result.

**Lemma 6.4** Assume \(q \in (1, \frac{3}{2})\), and let

\[
U_i := (u_i, w_i, \chi u_i) \in B_q \cap B^{\frac{aq}{3-q}} , \quad i = 1, 2,
\]
with \(\|U_i\|_{B_q} \leq \delta\), \(i = 1, 2, \delta \in (0, \delta_0]\), suitable \(\delta_0 > 0\). Moreover, let \(\bar{V} \in W^{2,q}_T \cap W^{2,3q}_{3-q} .\) Then, there exists a positive constant \(c(\Omega, q)\) such that

\[
\|\text{div} [\tilde{\mathcal{F}}(U_1) - \tilde{\mathcal{F}}(U_2)]\|_2 + \|\tilde{\mathcal{F}}(U_1) - \tilde{\mathcal{F}}(U_2)\|_2 + \|\tilde{\mathcal{F}}(U_1) - \tilde{\mathcal{F}}(U_2)\|_{L^2(0,T) \cap L^{\frac{3q}{3-q}}(0,T)} \leq c\delta \|U_1 - U_2\|_{B_q^{\frac{aq}{3-q}}}.
\]
(6.19)

**Proof.** Again, under the given assumption, the proof of (6.19)_2 is straightforward. Next, setting for simplicity \(u := u_1 - u_2\), we get

\[
\|u_1 \otimes u_1 - u_2 \otimes u_2\|_2 + \|u_1 \otimes u_2 - u_2 \otimes u_2\|_{L^2(0,T)} \leq \|u_1 \otimes u_2\|_2 + \|u_1 \otimes u_2\|_{L^2(0,T)} \leq c\delta \|U_1 - U_2\|_{B_q^{\frac{aq}{3-q}}}.
\]

Observing that, under the given assumption, we have \(3q/(3 - q) < 3\), by the Hölder inequality and Lemma [1.3] we show

\[
\|u_1 \otimes u_2\|_{L^2(0,T)} \leq c \left( \|u_1\|_3 \|u_2\|_6 + \|u_1\|_3 \|u_2\|_{L^6(L^6)} \right) \leq c\delta \|U_1 - U_2\|_{B_q^{\frac{aq}{3-q}}}
\]
and, likewise,

\[
\|u \otimes u_2\|_{L^2(0,T)} \leq c\delta \|U_1 - U_2\|_{B_q^{\frac{aq}{3-q}}}
\]
so that

\[
\|u_1 \otimes u_1 - u_2 \otimes u_2\|_2 + \|u_1 \otimes u_1 - u_2 \otimes u_2\|_{L^2(0,T)} \leq c\delta \|U_1 - U_2\|_{B_q^{\frac{aq}{3-q}}}.
\]
Moreover,
\[
\| \text{div} (u_1 \otimes u_1 - u_2 \otimes u_2) \|_2 \leq \| u_1 \cdot \nabla u_2 \|_2 + \| u \cdot \nabla u_2 \|_2 \leq c \left( \| u_1 \|_{\infty} \| \nabla u_2 \|_2 + \| u \|_{\infty} \| \nabla u_2 \|_2 \right),
\]
which, with the help of Lemma \ref{lemma1.3}, furnishes
\[
\| \text{div} (u_1 \otimes u_1 - u_2 \otimes u_2) \|_2 \leq c \delta \| U_1 - U_2 \|_{2^3 \frac{3q}{3q}}.
\]
Next, setting \( w := w_1 - w_2, \chi := \chi_1 - \chi_2, \chi_i := \chi_{w_i}, i = 1, 2, \) by Hölder inequality we have
\[
\| (w_1 - \chi_1) \otimes w_1 - (w_2 - \chi_2) \otimes w_2 \|_2 \leq \| (w_1 - \chi_1) \otimes w_2 \|_2 + \| (w_1 - \chi_1) \otimes w_2 \|_2 \leq \| w_1 - \chi_1 \|_{L^2(L^\infty)} \| w_2 \|_{L^\infty(L^2)} + \| w_1 - \chi_1 \|_{L^\infty(L^{\frac{3q}{3q}})} \| w_2 \|_{L^\infty(L^{\frac{3q}{3q}})}.
\]
Thus, observing that \( 3q/(3 - q) > 3/2, \) from Lemma \ref{lemma1.3} and the assumption we derive
\[
\| (w_1 - \chi_1) \otimes w_1 - (w_2 - \chi_2) \otimes w_2 \|_2 \leq c \delta \| U_1 - U_2 \|_{2^3 \frac{3q}{3q}}.
\]
Furthermore, with \( r \in (2, 4), \) we have
\[
\| \text{div} ((w_1 - \chi_1) \otimes w_1 - (w_2 - \chi_2) \otimes w_2) \|_2 \leq \| (w_1 - \chi_1) \cdot \nabla w_2 \|_2 + \| (w_1 - \chi_1) \cdot \nabla w_2 \|_2 \leq \| w_1 - \chi_1 \|_{L^r(L^\infty)} \| \nabla w_2 \|_{L^r(L^2)} + \| w_1 - \chi_1 \|_{L^r(L^\infty)} \| \nabla w_2 \|_{L^r(L^2)}.
\]
which, again by Lemma \ref{lemma1.3} and the assumption, furnishes
\[
\| \text{div} ((w_1 - \chi_1) \otimes w_1 - (w_2 - \chi_2) \otimes w_2) \|_2 \leq c \delta \| U_1 - U_2 \|_{2^3 \frac{3q}{3q}}.
\]
In an entirely (and simpler) fashion one shows
\[
\| (\hat{V} - \hat{\chi}) \otimes w + w \otimes \hat{V} - \chi \otimes \hat{V} \|_2 + \| (\hat{V} - \hat{\chi}) \otimes w + w \otimes \hat{V} - \chi \otimes \hat{V} \|_{\frac{3q}{3q}} \leq c \delta \| U \|_{2^3 \frac{3q}{3q}}
\]
and
\[
\| \text{div} (\hat{V} - \hat{\chi}) \otimes w + w \otimes \hat{V} - \chi w \otimes \hat{V} \|_2 \leq c \delta \| U \|_{2^3 \frac{3q}{3q}}.
\]
Finally, the proof of the stated estimate of \( f \) is performed by exactly the same token, and we leave it to the reader.

\[\Box\]

We are now in a position to show the following existence and uniqueness result.
**Theorem 6.1** Let \( v_* \) be as in (6.10), with \( V_* \in \mathcal{V}^{2\frac{3q}{3q-4}}_q \), \( q \in (1, \frac{4}{3}) \). Then, there is \( \delta_0 > 0 \) such that if \( \delta \leq \delta_0 \), the problem (6.12) - (6.15) has one and only one solution \( (u, w, \chi_w) \in \mathcal{B}^q \cap \mathcal{B}^{\frac{3q}{3q-4}} \) satisfying
\[
\|(u, w, \chi_w)\|_{\mathcal{B}^q} + \|(u, w, \chi_w)\|_{\mathcal{B}^{\frac{3q}{3q-4}}} \leq c \delta^2. \tag{6.20}
\]

**Proof.** We shall employ an approximating procedure in suitable spaces. Precisely, let's define a sequence \( \{u_n, \xi_n, w_n, \chi_n\} \) by recurrence, as follows:
\[
\begin{align*}
\nu \Delta u_n + \xi_n \cdot \nabla u_n - \nabla p_n &= \text{div} \mathcal{F}(u_{n-1}, w_{n-1}, \chi_{n-1}) \quad \text{in } \Omega, \\
\text{div } u_n &= 0 \\
u \Delta u_n + \xi_n &= \text{at } \partial \Omega \\
\int_{\partial \Omega} [\mathcal{T}(u_n, p_n) - \mathcal{F}(u_{n-1}, w_{n-1}, \chi_{n-1})] \cdot n &= 0,
\end{align*}
\]
and
\[
\begin{align*}
\partial_t w_n - \nu \Delta w_n &= -\nabla \tau_n + f(u_{n-1}, w_{n-1}, \chi_{n-1}) \quad \text{in } \Omega \times \mathbb{R}, \\
\text{div } w_n &= 0 \\
\partial_t \tau_n &= n \text{ at } \partial \Omega \times \mathbb{R} \\
M \dot{\chi}_n + \int_{\partial \Omega} T(w_n, \tau_n) \cdot n &= F(\chi_{n-1}, \xi_{n-1}),
\end{align*}
\]
where \( n \geq 1 \) and \( u_0 \equiv w_0 \equiv 0 \). Our first objective is to show that the sequence is bounded in \( \mathcal{B}^q \cap \mathcal{B}^{\frac{3q}{3q-4}} \), provided \( \delta \) is sufficiently small. From (6.10), (6.14) and (6.15) we get
\[
\begin{align*}
\nu \Delta u_1 + \xi_1 \cdot \nabla u_1 - \nabla p_1 &= \delta^2 \text{div} (V_0 - \chi_0) \otimes V_0 \quad \text{in } \Omega, \\
\text{div } u_1 &= 0 \\
\nu \Delta u_1 + \xi_1 &= \text{at } \partial \Omega \\
\int_{\partial \Omega} [\mathcal{T}(u_1, p_1) - \delta^2 (V_* + \chi_0) \otimes V_*] \cdot n &= 0,
\end{align*}
\]
and
\[
\begin{align*}
\partial_t w_1 - \nu \Delta w_1 &= -\nabla \tau_1 - \delta^2 \left( (V_0 - \chi_0) \cdot V_0 - (\overline{V_0 - \chi_0} \cdot V_0) \right) \quad \text{in } \Omega \times \mathbb{R}, \\
\text{div } w_1 &= 0 \\
\partial_t \tau_1 &= \text{at } \partial \Omega \times \mathbb{R} \\
M \dot{\chi}_1 + \int_{\partial \Omega} T(w_1, \tau_1) \cdot n &= \delta^2 \int_{\partial \Omega} [(V_* + \chi_0) \otimes V_* - (V_* + \chi_0) \otimes V_*] \cdot n.
\end{align*}
\]
In view of the regularity assumption made on \( V_* \) \(^{(4)}\), by Lemma 6.2 we see that all the terms involving \( V_0 \), and \( \chi_0 \) meet the hypotheses of Lemma 4.1 and Lemma 5.2. Therefore, we conclude that for
\[
(4) \text{For simplicity, we set } \xi_{u_n} \equiv \xi_n, \chi w_n \equiv \chi_n.
\]
\[
(5) \text{Notice that } \mathcal{V}^{2\frac{3q}{3q-4}}_q \subset \mathcal{V}^{2q}_q.
\]
sufficiently small \( \delta > 0 \) there exists a unique solution such that

\[
(u_1, w_1, \chi_1) \in \mathcal{B}^q \cap \mathcal{B}^{\frac{3q}{2}}.
\]

For fixed \( q \), let \( M \) be any positive constant satisfying

\[
M \geq \| \text{div} \mathcal{F}_0 \|_q + \| \mathcal{F}_0 \|_2 + \| \mathcal{F}_0 \|_q + \| f_0 \|_{L^2_q} + \| F_0 \|_{L^2(0,T) \cap L^2_q(0,T)}
\]

with \( \mathcal{F}_0, f_0 \), and \( F_0 \) given in (6.16), and let \( C_0 = \max \{ C_1, C_2 \} \), with \( C_1, C_2 \) the constants defined in (4.2) and (5.14). Then, again from Lemma 4.1 and Lemma 5.2 we deduce

\[
\| (u_1, w_1, \chi_1) \|_{\mathcal{B}^{\frac{3q}{2}}} + \| (u_1, w_1, \chi_1) \|_{\mathcal{B}^q} \leq \kappa_0 \delta^2,
\]

with \( \kappa_0 := M C_0 \), provided \( \delta \) is small enough. In view of this result, we can employ a classical induction argument to show that

\[
(u_n, w_n, \chi_n) \in \mathcal{B}^q, \quad \text{for all } n \geq 1,
\]

and that

\[
\| (u_n, w_n, \chi_n) \|_{\mathcal{B}^q} \leq 2 \kappa_0 \delta^2, \quad \text{for all } n \geq 1.
\]

Thus, let us assume that (6.26) holds for \( n - 1 \), and show that it is true also for \( n \). To this end, set

\[
U_n := (u_n, w_n, \chi_n), \quad n \geq 1,
\]

and

\[
\mathcal{F}_{n-1} := \mathcal{F}(U_{n-1}) - \mathcal{F}_0; \quad f_{n-1} := f(U_{n-1}) - f_0;
\]

\[
F_{n-1} := F(\chi_{n-1}, \xi_{n-1}) - F_0.
\]

From Lemma 6.3 and (6.26) evaluated at \( n - 1 \) it follows that

\[
\| \text{div} \mathcal{F}_{n-1} \|_2 + \| \mathcal{F}_{n-1} \|_2 + \| \mathcal{F}_{n-1} \|_q + \| f_{n-1} \|_{L^2_q} \leq c \left( \| U_{n-1} \|_{\mathcal{B}^q}^2 + \| \mathcal{F}_{n-1} \|_{\mathcal{B}^q} + \| f_{n-1} \|_{\mathcal{B}^q} \right) \leq c_1 \delta^3,
\]

\[
\| F_{n-1} \|_{L^2(0,T) \cap L^2_q(0,T)} \leq c \delta \| U_{n-1} \|_{\mathcal{B}^q} \leq c_2 \delta^3,
\]

for \( \delta \leq 1 \). Thus, applying Lemma 4.1 and Lemma 5.2 to (6.21), (6.22), with the help of the above estimates we deduce for small \( \delta \)

\[
\| U_n \|_{\mathcal{B}^q} \leq c_0 \delta^3 + \kappa_0 \delta^2,
\]

which, in turn, again by taking \( \delta \) below a certain positive constant, implies (6.25). We shall next show that the sequence \( \{ U_n \} \) is Cauchy in \( \mathcal{B}^{\frac{3q}{2}} \) by proving that it satisfies the following inequality

\[
\| U_{n+1} - U_n \|_{\mathcal{B}^{\frac{3q}{2}}} \leq C \delta \| U_n - U_{n-1} \|_{\mathcal{B}^{\frac{3q}{2}}} \quad \text{all } n \geq 1,
\]

and then choosing \( \delta < C^{-1} \). However, (6.28) is an immediate consequence of the second part of Lemma 4.1, Lemma 5.2, and Lemma 6.3. Thus, denoting by \( \hat{U} := (u, w, \chi_w) \in \mathcal{B}^{\frac{3q}{2}} \) the limiting point of the sequence, we obtain that \( \hat{U} \) is the solution to the problem (6.12)–(6.14) in the above class, for sufficiently small \( \delta > 0 \). Furthermore, from (6.28) and recalling that \( U_0 \equiv 0 \), we infer, again by standard arguments, that

\[
\| U_{n+k} - U_n \|_{\mathcal{B}^{\frac{3q}{2}}} \leq \frac{(C \delta)^n}{1 - C \delta} \| U_n \|_{\mathcal{B}^{\frac{3q}{2}}}, \quad \text{all } n \geq 1, k \geq 0.
\]

Thus, letting \( k \to \infty \) in this relation and choosing \( n = 1 \), with the help of (6.25) we arrive at (6.20). Finally, since \( \mathcal{B}^q \) is reflexive, from (6.27) we deduce that there is \( U^* \in \mathcal{B}^q \) such that \( U_n \to U^* \) weakly in \( \mathcal{B}^q \), along a subsequence, with \( U^* \) satisfying (6.27). As a result, we show that it must be \( U^* = U \), which completes the proof of the theorem.
7 Sufficient Conditions for Self-Propulsion in the case $\overline{v}_* = 0$

As Theorem 2.1 also Theorem 6.1 furnishes a generic existence result of $T$-periodic solutions and, as such, it does not ensure $\xi_u \neq 0$, namely, that the body will perform a non-zero net motion. Our next task is to give sufficient conditions on $v_*$ that, in fact, ensure this property, when $v_*$ has zero average. In other words, we want to provide a sufficient condition on the boundary data that—under fixed geometrical and physical properties of the body $B$ and the liquid—ensure that $B$ is able to self-propel; see (7.3). As will be shown in Section 9, this condition is also necessary, in the sense that we will prove, by means of an explicit example, that if the boundary data are not such as to satisfy that condition, the body $B$ may just “oscillate” without performing any non-zero net motion, regardless of its shape and physical properties.

To accomplish all the above, we begin to define the vector $G$ characterized as follows:

$$G = G(B, V_*, M, \nu) := \sum_{i=1}^{3} \left( \int_{\Omega} (V_0 - \chi_0) \cdot \nabla h(i) \cdot V_0 \right) e_i,$$

(7.1)

where $V_0$ is the velocity field of the solution given in Lemma 6.2, and the fields $(h(i), p(i))$ are given in (3.1). Observe that from Lemma 6.2 and (3.2), we deduce that $G$ is well defined. Notice also that, for a given body and liquid, $G$ depends only on the boundary velocity distribution.

In order to show our main finding, we need the following lemma.

**Lemma 7.1** Let the assumption of Theorem 6.1 be satisfied, and let $U := (u, w, \chi_w) \in \mathcal{B}^q \cap \mathcal{B}^{\frac{3q}{3-q}}$ be the solution to (6.12)–(6.15) there constructed. Further, let $\mathcal{G}$ be a tensor field with $\mathcal{G} \in L^3(\Omega)$. Then, the following inequality holds

$$\left| \int_{\Omega} \tilde{\mathcal{F}}(U) : \mathcal{G} \right| \leq c \delta^3 \| \mathcal{G} \|_3,$$

where $\tilde{\mathcal{F}}$ is defined in (6.17).

**Proof.** From Hölder inequality, we have

$$\left| \int_{\Omega} \tilde{\mathcal{F}}(U) : \mathcal{G} \right| \leq \| \tilde{\mathcal{F}} \|_\frac{3}{2} \| \mathcal{G} \|_3.$$  

(7.2)

Next, we observe that, since $q \in (1, \frac{6}{5})$, by simple interpolation we get

$$\| \tilde{\mathcal{F}} \|_\frac{3}{2} \leq \| \tilde{\mathcal{F}} \|_q \| \tilde{\mathcal{F}} \|_2^{1-\theta},$$

with $\theta = \frac{1}{3} \frac{q}{2-q}$. From the latter and Lemma 6.3, we thus deduce

$$\| \tilde{\mathcal{F}} \|_\frac{3}{2} \leq c \left( \| U \|_{\mathcal{B}^q}^2 + \delta \| U \|_{\mathcal{B}^q} \right),$$

which, in view of (6.20) and (7.2), completes the proof of the lemma.

Our next result shows that the vector $G$ acts as “thrust” for self-propulsion. More precisely, we have the following theorem.
Theorem 7.1 Let $v_*=\delta V_*$ be as in Theorem 6.1. Suppose that

$$G \neq 0.$$  \hfill (7.3)

Then, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the unique solution to (0.1) constructed in Theorem 6.1 and corresponding to $v_*$, must have $\xi_u \neq 0$.

Proof. Let $(u, w, \chi_w) \in \mathcal{B}_q \cap \mathcal{B}^{3-q}_3$ be the solution given in Theorem 6.1 corresponding to $v_*$, and assume, ad absurdum, that $\xi_u = 0$. If we then dot-multiply both sides of (6.12) (with $\xi_u = 0$) by $h^{(i)}$, integrate by parts over $\Omega$ and use (6.12), we get

$$\nu \int_\Omega D(u) : D(h^{(i)}) = \int_\Omega \mathcal{F} : \nabla h^{(i)}. \tag{7.4}$$

Likewise, if we dot-multiply both sides of (3.1) by $u$, integrate by parts over $\Omega$ and assume $\xi_u = 0$, we show

$$\int_\Omega D(u) : D(h^{(i)}) = 0,$$

which, combined with (7.4), furnishes

$$\int_\Omega \mathcal{F} : \nabla h^{(i)} = 0, \quad \text{for all } i = 1, 2, 3. \tag{7.5}$$

Let $G \equiv G_j \neq 0$, for some $j \in \{1, 2, 3\}$. From (7.5) we get

$$\left| \int_\Omega (\mathcal{F} : \nabla h^{(j)} - \widetilde{\mathcal{F}} : \nabla h^{(j)}) \right| = \delta^2 |G| \tag{7.6}$$

where $\widetilde{\mathcal{F}}$ is given in (6.17). Since, in particular, by (3.2) we have $h^{(i)} \in D^{1,r}(\Omega)$, for all $r \in (\frac{3}{2}, \infty)$, thanks to Lemma 7.1 we show that

$$\left| \int_\Omega \widetilde{\mathcal{F}} : \nabla h^{(j)} \right| \leq c_0 \delta^3,$$

which, once replaced in (7.6) allows us to infer, in particular,

$$\left| \int_\Omega \mathcal{F} : \nabla h^{(j)} \right| \geq \delta^2 (|G| - c_0 \delta). \tag{7.7}$$

However, if we choose $\delta < |G|/c_0$ in (7.7), we contradict (7.5) and thus, in turn, the assumption $\xi_u = 0$. The proof of the theorem is therefore completed. \hfill \Box

Remark 7.1 It is clear, from Theorem 6.1 and Theorem 7.1, that the set of boundary distributions assuring the validity of the self-propulsion condition (7.3) is open in the space $\mathcal{V}_q^{2,q}$. As a result, also the set of “thrusts” $G$ is open in $\mathbb{R}^3$. 34
8 On the Velocity of Self-Propulsion in the case $\overline{u}_* = 0$

In the previous section we have furnished sufficient conditions ensuring self-propulsion of the body $B$. Our objective now is to give an estimate of the velocity of self-propulsion, $\xi_u$, and, in particular, its relation to the “thrust” $G$ defined in (7.1).

To this end, we begin to show the following result.

**Lemma 8.1** Let the assumptions of Theorem 7.1 be satisfied, and consider the boundary-value problem:

\[
\begin{aligned}
&\nu \Delta \tilde{h}^{(i)} - \xi_u \cdot \nabla \tilde{h}^{(i)} = \nabla \tilde{p}^{(i)} \\
&\text{div} \tilde{h}^{(i)} = 0 \\
&\tilde{h}^{(i)} = e_i \text{ at } \partial \Omega, \quad \lim_{|x| \to \infty} \tilde{h}^{(i)}(x) = 0.
\end{aligned}
\]

Then, there exists a unique solution

\[
\tilde{h}^{(i)} \in L^{\frac{2}{1-\sigma}}(\Omega) \cap D^{1,\frac{2}{1-\sigma}}(\Omega) \cap D^{2,\sigma}(\Omega), \quad p \in D^{1,\sigma}(\Omega), \quad \text{all } \sigma \in (1, 2)
\]

that, in addition, satisfies the following properties ($i = 1, 2, 3$):

\[
\begin{aligned}
&\int_{\partial \Omega} \mathcal{T}(\tilde{h}^{(i)}, \tilde{p}^{(i)}) \cdot \mathbf{n} = \int_{\partial \Omega} \mathcal{T}(h^{(i)}, p^{(i)}) \cdot \mathbf{n} + g_i(\delta), \quad |g_i(\delta)| \leq c_0 \delta^{\frac{3}{4}} , \\
&\tilde{h}^{(i)} = h^{(i)} + h^{(i)}(\delta), \quad |h^{(i)}|_{1,3} \leq c_0 \delta^{\frac{3}{4}} .
\end{aligned}
\]

**Proof.** Existence (and uniqueness) of the pair $(\tilde{h}^{(i)}, \tilde{p}^{(i)})$ in the class (8.2) is well known [7, Theorem VII.7.1]. In particular, such a solution satisfies the estimate

\[
|\xi_u|^\frac{1}{2} |\tilde{h}^{(i)}|_{1,r} \leq C, \quad \text{all } r \in (\frac{4}{3}, 4), \quad (8.4)
\]

with $C = C(\Omega, r)$. Setting $h^{(i)} := \tilde{h}^{(i)} - h^{(i)}$, $p^{(i)} = \tilde{p}^{(i)} - p^{(i)}$, from (8.1) and (3.1), we deduce that

\[
\begin{aligned}
&\nu \Delta h^{(i)} = \nabla p^{(i)} + \xi_u \cdot \nabla h^{(i)} \\
&\text{div} h^{(i)} = 0 \\
&h^{(i)} = 0 \text{ at } \partial \Omega.
\end{aligned}
\]

Therefore, from (8.4), (8.5) and classical estimates on the Stokes problem, we get [7, Theorem V.4.8]

\[
\begin{aligned}
&|h^{(i)}|_{\frac{3}{4}, r} + |h^{(i)}|_{1, \frac{3}{4}} + |h^{(i)}|_{2,r} + |p^{(i)}|_{1,r} \leq c |\xi_u| |\tilde{h}^{(i)}|_{1,r} \leq c \delta^{\frac{3}{4}}, \quad \text{all } r \in (\frac{4}{3}, \frac{3}{2}), \\
&|h^{(i)}|_{2,s} \leq c |\xi_u| |\tilde{h}^{(i)}|_{1,s} \leq c \delta^{\frac{3}{4}}, \quad \text{all } s \in (3, 4).
\end{aligned}
\]

The first inequality in (8.6) along with trace theorems, proves (8.3).1. Moreover, by classical embedding results for homogeneous Sobolev spaces [7, Theorem II.9.1], we have

\[
|h^{(i)}|_{1,3} \leq c \left( |h^{(i)}|_{1, \frac{3}{4}} + |h^{(i)}|_{2,s} \right),
\]

so that (8.3)2 is a consequence of the latter and (8.6).

We are now in a position to show the main result of this section.
Theorem 8.1 Under the assumptions of Theorem 7.1 the velocity of self-propulsion \( \xi_u \) is (non-zero and) given by the following formula

\[
\xi_u = \delta^2 A^{-1} \cdot G + \sigma(\delta),
\]

where \( A \) is the symmetric, nonsingular matrix defined in (3.4), and \( \sigma \) is vector function depending also on \( \delta \) such that

\[
\sigma(\delta) = O(\delta^{\frac{11}{4}}).
\]

**Proof.** We dot-multiply both sides of (6.12) by \( \tilde{h}^{(i)} \), integrate by parts over \( \Omega \) and employ (6.12) to show

\[
\int_{\Omega} \left[ \nu \mathbb{D}(u) : \mathbb{D}(\tilde{h}^{(i)}) + \xi_u \cdot \nabla \tilde{h}^{(i)} \cdot u - \mathcal{F} : \nabla \tilde{h}^{(i)} \right] = 0.
\]

(8.8)

In a similar way, dot-multiplying both sides of (8.1) by \( u \) and proceeding as before, we get

\[
\xi_u \cdot \int_{\partial\Omega} \mathbb{T}(\tilde{h}^{(i)}, \tilde{p}^{(i)}) \cdot n - \int_{\Omega} \left[ \nu \mathbb{D}(u) : \mathbb{D}(\tilde{h}^{(i)}) + \xi_u \cdot \nabla \tilde{h}^{(i)} \cdot u \right],
\]

(8.9)

so that, summing (8.8) and (8.9) side-by-side, we conclude

\[
\xi_u \cdot \int_{\partial\Omega} \mathbb{T}(\tilde{h}^{(i)}, \tilde{p}^{(i)}) \cdot n = \int_{\Omega} \mathcal{F} : \nabla \tilde{h}^{(i)} , \quad i = 1, 2, 3.
\]

(8.10)

From Lemma 8.1, (1.11) and (6.20) it immediately follows that

\[
\xi_u \cdot \int_{\partial\Omega} \mathbb{T}(\tilde{h}^{(i)}, \tilde{p}^{(i)}) \cdot n = \xi_{uk} \hat{A}_{ki} + O(\delta^{\frac{11}{4}}).
\]

(8.11)

Further, we have

\[
\int_{\Omega} \mathcal{F} : \nabla \tilde{h}^{(i)} = \delta^2 G_i + \int_{\Omega} \mathcal{F} : \nabla h^{(i)} + \int_{\Omega} \mathcal{F} : \nabla h^{(i)} + \delta^2 \int_{\Omega} \mathcal{F}_0 : \nabla h^{(i)} ,
\]

(8.12)

where, we recall, the quantities \( \mathcal{F} \) and \( \mathcal{F}_0 \) are defined in (6.16) and (6.17). From Lemma 7.1 and Lemma 8.1

\[
\int_{\Omega} \mathcal{F} : \nabla h^{(i)} + \delta^2 \int_{\Omega} \mathcal{F}_0 : \nabla h^{(i)} = O(\delta^{\frac{11}{4}}),
\]

(8.13)

and,

\[
\int_{\Omega} \mathcal{F} : \nabla h^{(i)} = O(\delta^3).
\]

(8.14)

The theorem then follows from (8.10)–(8.14).

\[\square\]
9 Some Comments on the Self-Propelling Condition (7.3)

From Theorem 8.1 it turns out that if \( \mathbf{v}_* = \mathbf{0} \), self-propulsion manifests itself at the second order in \( \delta \), provided \( G \neq 0 \). The natural question to ask is then the following one. Suppose \( G = 0 \). Can then self-propulsion occur at an order in \( \delta \) higher than 2? The aim of this section is to prove that, in general, the answer is negative. In fact, we shall provide an example of boundary data, \( \mathbf{v}_* \), of arbitrary magnitude \( \delta > 0 \), for which the corresponding solution to (6.5) does not satisfy (7.3) and the averaged field \( \mathbf{u} \) in problem (6.12) is identically zero. More precisely, our example shows that given a body \( \mathcal{B} \) of any shape and mass, and an arbitrary period \( T > 0 \), we can always find a \( T \)-periodic boundary velocity \( \mathbf{v}_* \) such that (7.3) is violated and the net motion of \( \mathcal{B} \) is zero, that is, \( \mathcal{B} \) can only “oscillate.”

In order to show all the above, we premise the following result.

**Lemma 9.1** Let \( \psi \in D^{1,2}(\Omega) \) be the solution to the Neumann problem:

\[
\Delta \psi = 0 \quad \text{in} \ \Omega; \quad \frac{\partial \psi}{\partial n} = f \quad \text{at} \ \partial \Omega; \quad \lim_{|x| \to \infty} \psi(x) = 0
\]

where \( f \) is a given smooth function satisfying

\[
\int_{\partial \Omega} f = 0.
\]

Moreover, let \( a \in W^{1,q}(0,T) \), \( q \in (1, \infty) \), be \( T \)-periodic with \( \mathbf{a} = 0 \). Then, the triple

\[
\mathbf{V} = a(t) \nabla \psi(x), \quad \mathbf{P} = -\dot{a} \psi, \quad \mathbf{z} = \frac{a(t)}{M} \int_{\partial \Omega} \psi \mathbf{n}
\]

is a solution to (6.5) in the class \( \hat{W}^{2,q}_{\mathbf{r}} \times P^{1,q} \times W^{1,q}_{\mathbf{r}}(0,T) \), corresponding to the boundary data

\[
\mathbf{V}_* := a(t) \left( \nabla \psi|_{\partial \Omega} - \frac{1}{M} \int_{\partial \Omega} \psi \mathbf{n} \right).
\]

Finally, for this solution we have

\[
G = 0.
\]

**Proof.** We begin to observe that, from classical results on the exterior Neumann problem, we have \( \psi \in C^\infty(\Omega) \cap W^{2,q}(\Omega_R) \), for all \( q \in (1, \infty) \) and \( R > R_* \). Furthermore, in view of (6.2), it follows that

\[
D^\alpha \psi = O(|x|^{-2-\alpha}), \quad |\alpha| = 0, 1, \ldots;
\]

see [7, Exercise V.3.6]. As a consequence, the fields (9.3) are in the class \( \hat{W}^{2,q}_{\mathbf{r}} \times P^{1,q} \times W^{1,q}_{\mathbf{r}}(0,T) \). Moreover, since \( \psi \) is harmonic, it immediately follows that \( (\mathbf{V}, \mathbf{P}) \) is a solution to (6.5). Next, we have (in the trace sense)

\[
\int_{\partial \Omega} (\mathbf{V}, \mathbf{P}) \cdot \mathbf{n} = \int_{\partial \Omega} \left[ 2a \nabla(\nabla \psi) \cdot \mathbf{n} + \dot{a} \psi \mathbf{n} \right] := 2aI + \int_{\partial \Omega} \dot{a} \psi \mathbf{n}.
\]

By integrating by parts and using the fact that \( \psi \) is harmonic, we get, for \( i = 1, 2, 3 \),

\[
I_i = \int_{\partial \Omega} \partial_i \partial_j \psi n_j = \int_{\Omega_R} \partial_i (\partial_j \partial_j \psi) - \int_{\partial B_R} \partial_i \partial_j \psi n_j = -\int_{\partial B_R} \partial_i \partial_j \psi n_j,
\]
so that, letting $R \to \infty$ into this relation and using the asymptotic properties of $\psi$, we conclude

$$I = 0.$$ 

Therefore, (6.5) are also satisfied if we choose $z$ as in (9.3), and $\mathbf{V}_*$ as stated. Next, we have

$$G^{(i)} = \frac{1}{T} \int_0^T a(t) \left( \int_\Omega (a(t) \partial_k \psi - z_k(t)) \partial_k h^{(i)}_\ell \partial_\ell \psi \right) \, dt := \frac{1}{T} \int_0^T a(t) I(t) \, dt.$$ 

Thus, integrating by parts over $\Omega$, we get

$$I(t) = \int_\partial \Omega [(a \partial_k \psi - z_k) e_i \cdot \nabla \psi - \int_\partial \Omega e_i \cdot n \left( \frac{1}{2} a |\nabla \psi|^2 - z \cdot \nabla \psi \right) := I_1(t) - I_2(t).$$ 

However, using the fact that $\psi$ is harmonic in conjunction with Gauss theorem, we deduce

$$I_1(t) = \int_\Omega \partial_k [(a \partial_k \psi - z_k) e_i \cdot \nabla \psi] = \int_\Omega \partial_i [\frac{1}{2} a |\nabla \psi|^2 - z \cdot \nabla \psi] = I_2(t),$$

which thus proves (9.5) \(\square\)

Before proceeding further, we believe it is important to emphasize that, by Lemma 5.2, for the indicated boundary data, the solution provided in (9.3) is the only one in the relevant function class. We shall now show, with the help of Lemma 9.1, that the functions $v := \mathbf{V}, \quad p := a(t) \gamma(t) \cdot \nabla \psi - \frac{1}{2} a^2(t) |\nabla \psi|^2 + P, \quad \gamma := z$ satisfy (0.1) and $v_* := \mathbf{V}_*$ defined in (9.4). In fact, observing that

$$(v - \gamma) \cdot v = \nabla \left[ (a(t) \gamma(t) \cdot \nabla \psi - \frac{1}{2} a^2(t) |\nabla \psi|^2 \right],$$

it is immediately checked that the fields $v$ and $p$ thus defined satisfy (0.1)1,2,4. Let us now turn to the surface integral in (0.1)5 that, with $v$ and $p$ defined above, becomes

$$\int_{\partial_\Omega} \left[ 2a \nu \nabla (\nabla \psi) \cdot \mathbf{n} - \dot{a} \psi \mathbf{n} - a (\gamma \cdot \nabla \psi \mathbf{n} - \nabla \psi \gamma \cdot \mathbf{n}) + a^2 \left( \frac{1}{2} |\nabla \psi|^2 \mathbf{n} - \nabla \psi \nabla \psi \gamma \cdot \mathbf{n} \right) \right] := 2a \nu I_1 - \dot{a} I_2 + a I_3 + a^2 I_4.$$ 

As in the proof of Lemma 9.1, we obtain

$$I_1 = 0.$$ 

Next, we have

$$\gamma \cdot \nabla \psi \mathbf{n} - \nabla \psi \gamma \cdot \mathbf{n} = (\mathbf{n} \times \nabla \psi) \times \gamma,$$

and

$$\int_{\partial_\Omega} \mathbf{n} \times \nabla \psi = - \int_{\partial B_R} \mathbf{n} \times \nabla \psi + \int_{\Omega_R} \nabla \times (\nabla \psi) = - \int_{\partial B_R} \mathbf{n} \times \nabla \psi.$$
As a result, letting $R \to \infty$ in the latter and employing (9.6) we deduce
\[ I_3 = 0. \quad (9.10) \]

Finally, again by integration by parts, $i = 1, 2, 3$,
\[
I_{4i} = \int_{\Omega_R} \left[ \frac{1}{2} \partial_i |\nabla \psi|^2 - \partial_t (\partial_i \psi \partial_t \psi) \right] - \int_{\partial B_R} \left( \frac{1}{2} |\nabla \psi|^2 n_i - \partial_i \psi \partial_t \psi n_\ell \right)
\]
\[= - \int_{\partial B_R} \left( \frac{1}{2} |\nabla \psi|^2 n_i - \partial_i \psi \partial_t \psi n_\ell \right) \]
and again by (9.6), we may let $R \to \infty$ in this relation and infer
\[ I_4 = 0. \quad (9.11) \]

Collecting (9.8)–(9.11), we then infer that the functions $v, p, \chi$ defined by (9.7) solve (0.1). However, for such a solution we have $u := \nabla \equiv 0$, that is, \textit{self-propulsion cannot occur}. We also notice that the solution (9.7) can be written in such a form as to satisfy the hypotheses of Theorem 6.1 thus, in this case, providing the \textit{only solution} corresponding to the given $v_\ast$, for “small” $\delta$. To show this, it suffices to define the fields
\[
V_0 := \frac{1}{\delta} V, \quad P_0 := \frac{1}{\delta} P, \quad \chi_0 := \frac{1}{\delta} z, \quad V_\ast := \frac{1}{\delta} V_\ast,
\]
and rewrite (9.7) as follows
\[
v := \delta V_0, \quad p := a(t) \gamma(t) \cdot \nabla \psi - \frac{1}{2} a^2(t) |\nabla \psi|^2 + \delta P_0, \quad \gamma := \delta \chi_0,
\]
which, of course, is in the form (6.11) for arbitrary $\delta > 0$, with $w \equiv \chi \equiv u \equiv p \equiv 0$ and $\tau := a(t) \gamma(t) \cdot \nabla \psi - \frac{1}{2} a^2(t) |\nabla \psi|^2$.

\textbf{Remark 9.1} The example furnished in the previous remark also shows that, in general, if the boundary data have zero average, then a non-zero thrust ($G \neq 0$) is expected to be produced by a boundary velocity distribution possessing a non-vanishing mass flow-rate through the body-liquid interface.

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