Mapping between the classical and pseudoclassical models of a relativistic spinning particle in external bosonic and fermionic fields

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Abstract

The problem on mapping between two Lagrangian descriptions (using a commuting c-number spinor $\psi_\alpha$ or anticommuting pseudovector $\xi_\mu$ and pseudoscalar $\xi_5$ variables) of the spin degrees of freedom of a color spinning massive particle interacting with background non-Abelian gauge and fermion fields, is considered. A general analysis of the mapping between a pair of Majorana spinors $(\psi_\alpha, \theta_\alpha)$ ($\theta_\alpha$ is some auxiliary anticommuting spinor) and a real anticommuting tensor system $(S, V_\mu, T_{\mu\nu}, A_\mu, P)$, is presented. A complete system of bilinear relations between the tensor quantities, is obtained. This general analysis was used for the above problem of the equivalence of two different ways of describing the spin degrees of freedom of the relativistic particle. The mapping of the kinetic term $(i\hbar/2)(\bar{\psi}\psi - \bar{\psi}\psi)$, the term $(1/e)(\bar{\theta}\theta)\dot{x}_\mu(\bar{\psi}\gamma^\mu\psi)$ which provides a couple of the spinning variable $\psi$ and the particle velocity $\dot{x}_\mu$, the interaction term $\hbar(\bar{\theta}\theta)Q_aF^a_{\mu\nu}(\bar{\psi}\sigma_{\mu\nu}\psi)$ with an external non-Abelian gauge field, are considered in detail. In the former case a corresponding system of bilinear identities including both the tensor variables and their derivatives $(\dot{S}, \dot{V}_\mu, \dot{T}_{\mu\nu}, \dot{A}_\mu, \dot{P})$, is defined. The exact solution of a system of bilinear identities for the case of Grassmann-odd tensors, is obtained. The solution was used in constructing the mapping of the interaction terms of spinning particle with a background (Majorana) fermion field. A way of the extension of the obtained results in the case when the spinors $\psi_\alpha$ and $\theta_\alpha$ are Dirac ones, is suggested. It is shown that for the construction of one-to-one correspondence between the most general spinors and tensor variables, it is necessary a four-fold increase of the number of the tensor ones. The approach of obtaining an supersymmetric Lagrangian in terms of the even $\psi_\alpha$ and odd $\theta_\alpha$ spinors, is offered. A connection with the higher-order derivative Lagrangians for a point particle, is proposed.

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1 Introduction

In our previous paper [1] the model Lagrangian describing the interaction of a relativistic spinning color-charged classical particle with background non-Abelian gauge and fermion fields was suggested. The spin degrees of freedom have been presented in [1] by a c-number Dirac spinor $\psi_{\alpha}$, $\alpha = 1, \ldots, 4$. By virtue of the fact that the background fermion field $\Psi_{\alpha}(x)$ (which within the classical description is considered as a Grassmann-odd function) has, by definition, the spinor index $\alpha$, the description of the spin degrees of freedom of the particle in terms of the spinor $\psi_{\alpha}$ is a very natural and simplest in technical respect. There is some vagueness with respect to Grassmann parity of this spinor. In our papers [2, 3] in application to an analysis of dynamics of a spinning color particle moving in a hot quark-gluon plasma, the spinor $\psi_{\alpha}$ was thought as the Grassmann-even parity one (although it is not improbable that simultaneous using spinors of the different Grassmann parity may be required for a complete classical description of the spin dynamics in external fields of different statistics, i.e. in other words, it requires introducing a superspinor, see Conclusion). Furthermore, for simplicity throughout our previous works [2–6], we have neglected a change of the spin state of the particle, i.e. we believed $\psi_{\alpha}$ to be a spinor independent of the evolution parameter $\tau$. As a result we have completely neglected an influence of the spin of particle on the general dynamics of the interaction of the particle with background fields. However, for a more detailed study of the motion of a particle in external fields of different statistics and comparing the suggested model with the other approaches known in the literature, it is necessary to account for a change in time of the spin variable $\psi_{\alpha}$. At present there exist a few approaches to the description of the spin degrees of freedom of a particle within the (semi)classical approximation. Below only two approaches closely related to the subject of our subsequent investigation are outlined.

Notice that the description of the spin degrees of freedom by means of a classical commuting spinor is not new. Such a way of the description arises naturally in determining the connection of relativistic quantum mechanics of an electron with relativistic classical mechanics [4]. In particular, it was shown [8–10] that within the WKB-method extended to the relativistic case, the relativistic wave Dirac equation results in a system of equations incorporating not only the classical canonical equations of motion, but also yet another equation for the spin degrees of freedom. This equation is connected directly with the Schrödinger equation (we put throughout $c = 1$ for the speed of light)

$$i\hbar \frac{d\psi(\tau)}{d\tau} = -\frac{q\hbar}{4m} \sigma^{\mu\nu} F_{\mu\nu}(x) \psi(\tau), \quad \sigma^{\mu\nu} \equiv \frac{1}{2i} [\gamma^\mu, \gamma^\nu]$$

for the commuting spinor function $\psi_{\alpha}$. Here, $q$ is an electric charge. This equation describes the motion of the spin of the electron in a given electromagnetic field $F_{\mu\nu}(x)$. The field in (1.1) is defined along the path of particle $x_\mu = x_\mu(\tau, x_0, \tau_0)$ in four-dimensional Minkowski space (‘mostly minus’ metric), as a function of the proper time $\tau$.

Further, Bohm et al. [11] have introduced two-component spinor in classical (non-relativistic) hydrodynamics in view of obtaining a causal model for the Pauli equation. Their method consists in associating the spinor with the rotation of an element of the fluid. Unfortunately, this method breaks down in point mechanics and it is difficult to extend it to the relativistic case. Another line of thought is due to Proca [12] who attaches a bispinor to a point particle. He then proceeds to find the equation of motion for the bispinor which would lead to the correct
equation for the velocity 4-vector and the spin antisymmetrical tensor. He has considered the case of a free point particle and the particle in an external gauge field. Based on the most general heuristic considerations he has suggested the Lagrangian, which in our notations is

\[ L = \frac{1}{2} i \Lambda \left( \frac{d\tilde{\psi}}{d\tau} \psi - \tilde{\psi} \frac{d\psi}{d\tau} \right) + p_\mu (\dot{x}^\mu - \tilde{\psi} \gamma^\mu \psi) + q A_\mu (\tilde{\psi} \gamma^\mu \psi) - \Lambda \frac{q}{4m} F_{\mu\nu}(x) (\tilde{\psi} \sigma^{\mu\nu} \psi). \] (1.2)

Here, \( \Lambda \) is a constant with the dimension of action (in this case the spinor \( \psi_\alpha(\tau) \) is a dimensionless function) and the momentum \( p_\mu \) is considered as a Lagrange multiplier for the constraint

\[ \dot{x}_\mu = \tilde{\psi} \gamma_\mu \psi, \] (1.3)

where the dot denotes differentiation with respect to \( \tau \). Within the framework of the classical model the whole phase space consists of the usual pair of conjugate variables \((x_\mu, p_\mu)\) and of another pair of conjugate classical spinor variables \((\psi, -i \tilde{\psi})\) representing the internal degree of freedom. The configuration space is thus \( M_4 \otimes \mathbb{C}_4 \), \( \psi \in \mathbb{C}_4 \) and the Lagrangian (1.2) describes a symplectic system.

Independently of Bohm and Proca, Gürsey [13] has developed the spinor formulation of relativistic kinematics readily applicable to a free point particle by introducing in classical theory a bispinor with a precise geometrical meaning showing its relation to the wave function of a Dirac particle. In the paper by Takabayasi [14] the bispinor was used for the description of general kinematical and dynamical aspects of relativistic particles possessing internal angular velocity together with internal angular momentum. In the above-mentioned papers the foundation for the theory of the proper bispinor (or simply spinor) associated with the relativistic motion of a point particle has been laid. Such a way of the description of the spin degrees of freedom of elementary particle has been used extensively by Barut with co-workers [15,16]. They employed for this purpose the Lagrangian (1.2) without the last term. The total Lagrangian (1.2) together with the last term was reproduced by Barut and Pašić [17] within the five dimensional Barut-Zanghi model [15] treated à la Kaluza-Klein. Finally, we note that the model Lagrangian (1.2) is similar (in the free case) to the model discussed by Plyushchay in [18].

Further, in the papers [19,20] by Berkovits the BRST invariant actions for a ten-dimensional superparticle moving in super-Maxwell and super-Yang-Mills backgrounds, respectively, have been written out. The equation (1.1) for the commuting spinor \( \psi_\alpha \) in [19,20] is complemented by the appropriate equation for an odd spinor variable \( \theta_\alpha \) (Grassmann coordinate). A more detailed discussion of Berkovits’s approach will be given in Conclusion. Finally, we note that two commuting complex two-component spinor wavefunctions have been used in [21] for describing the massive and massless particles of half-integer spin.

In approach suggested in the present work we give up the constraint (1.3). Next, we define the interaction term with an external non-Abelian gauge field in such a way that the term was in agreement with the equation of motion (1.1). Under these circumstances we suggest the following model Lagrangian that takes into account a change both in the color and in the spinning degrees of freedom of a classical particle propagating in the background non-Abelian gauge field

\[ L = L_0 + L_m + L_\theta, \] (1.4)
where
\[
L_0 = -\frac{1}{2\epsilon} \left( \dot{x}_\mu - \lambda(\bar{\psi}\gamma_\mu\psi) \right)^2 + \hbar\lambda\frac{i}{2} \left( \bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi \right) = \\
= -\frac{1}{2\epsilon} \dot{x}_\mu\dot{x}^\mu + \frac{1}{e} \lambda\dot{x}_\mu(\bar{\psi}\gamma_\mu\psi) - \frac{1}{2\epsilon}\lambda^2(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) + \hbar\lambda\frac{i}{2} \left( \bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi \right),
\]
(1.5)
\[
L_m = -\frac{e}{2} m^2,
\]
(1.6)
\[
L_\theta = i\hbar \left( \theta^i\dot{D}^{ij}(A)\theta^j \right) - \hbar\lambda\frac{\epsilon g}{4} Q^a F^a_{\mu\nu}(\bar{\psi}\sigma^{\mu\nu}\psi).
\]
(1.7)

The Lagrangian that is closely similar to the one (1.4) without taking into account the interaction with an external gauge field, was discussed by Hasiewic et al. in [22]. The authors have shown that in the case when \( \psi \) is a commuting Majorana spinor, the quantization of this model gives a unified quantum-mechanical description for massive and massless particles of arbitrary spin and helicity. The classical Lagrangian (1.4) (in the free case) was obtained from the one with so-called doubly supersymmetry [23] after putting all Grassmann variables equal to zero and adding a kinetic term for the commuting spinor. We note that the last but one term in (1.5) vanishes for the Majorana spinor \( \psi \) due to the Fierz identity.

In (1.4), in contrast to (1.2), we have set \( \Lambda \equiv \hbar\lambda \), where \( \lambda \) is some dimensionless constant, whose explicit form will be defined in the next section; \( D^{ij}(A) = \delta^{ij}\partial/\partial\tau + i(g/\hbar)\dot{x}^\mu A^a_\mu(t^a)^{ij} \) is the covariant derivative along the direction of motion; the self-conjugate pair \( (\theta^i, \theta^j) \) is a set of Grassmann variables belonging to the fundamental representation of the \( SU(N_c) \) color group, i.e. \( i, j, \ldots = 1, 2, \ldots, N_c \), (while \( a, b, \ldots \) run from 1 to \( N_c^2 - 1 \)); the commuting color charge \( Q^a \) is defined by
\[
Q^a \equiv \theta^i(t^a)^{ij}\theta^j.
\]

By virtue of the fact that we have introduced the Planck constant \( \hbar \) as a factor in the first term in (1.7), the color charges \( \theta^i(\tau) \) and \( \theta^i(\tau) \) are dimensionless variables like the spinor function \( \psi_a(\tau) \). Besides, it is worthy of special emphasis that we have kept \( \hbar \) in denominator in the second term of the covariant derivative \( D^{ij}(A) \) (see the definition above), as is generally accepted in the field theory. In this case the group generators \( t^a \) are dimensionless quantities and the non-Abelian gauge field \( A^a_\mu(x) \) has the canonical dimension. The disadvantage of such an approach is that \( \hbar \) will enter in an explicit form into the classical equations of motion for the color charges \( \theta^a \) and \( Q^a \), Eqs. (A.9), (A.11), and also into the covariant derivative \( D_{\mu}^a(A) \) in the Lorentz equation (A.10) and the Yang-Mills equation. Here we follow the papers by Arodz [24][25]. Recall that in the original paper by Wong [26] the group generators \( t^a \) are dimensional quantities, i.e. \( t^a \equiv \frac{1}{2}\hbar\lambda^a \), where \( \lambda^a \) are the Gell-Mann matrices and thus
\[
[t^a, t^b] = i\hbar f^{abc}t^c.
\]

In Wong’s approach the classical color charge \( Q^a \) was identified with the expectation value of the operator \( \hat{t}^a = \frac{1}{\hbar}\hat{\lambda}^a \), i.e. \( Q^a \equiv \langle \hat{t}^a \rangle \), by analogy with the spin vector \( \mathbf{S} \equiv \langle \frac{1}{2}\hbar\hat{\sigma} \rangle \). Such point

\[1\] Here, one can draw some interesting analogy to (super)string theory for the interacting terms in (1.7). In our case the term \( \dot{x}^\mu A^a_\mu(\theta^i(t^a)^{i}\theta^j) \) is similar to \( \dot{x}^\mu A^a_\mu(\bar{\psi}\gamma^\mu\psi) \) defining the interaction with the so-called Neveu-Schwarz (NSNS) gauge fields [24]. Another term of the form \( F_{\mu\nu}(\bar{\psi}\sigma^{\mu\nu}\psi) \) represents analog of the term \( S_{\mu_1\ldots\mu_n}S_{\nu_1\ldots\nu_n} \) for \( n = 2 \), where \( S_a \) and \( S_b \) are the spin fields. The latter term in string theory defines the interaction with the Ramond-Ramond (RR) gauge fields [39]. NSNS and RR gauge fields are quite different in string theory in contrast to the theory of point particles.
of view was also accepted in a number of the subsequent papers concerning a given subject (see, for example, [27, 28]). In this case the Planck constant $\hbar$ disappears in the equations of motion (A.9), (A.11) etc. However the dimension of gauge field in this case will not already be the canonical one as it is accepted in the field theory.

The alternative approach most generally employed for the description of spin for a massive point particle is connected with introduction into consideration of the pseudovector and pseudoscalar dynamical variables $\xi_\mu$ and $\xi_5$ that are elements of the Grassmann algebra [31–34]. For these variables an appropriate Lagrangian of the first order time derivative, was defined. In view of its great importance for a further discussion and for convenience of future references we give in Appendix A a complete form of this Lagrangian. It is these Grassmann-valued variables that appear in the representation of the one-loop effective action in quantum chromodynamics in terms of the path integral over world lines of a hard particle moving in external non-Abelian gauge field [35–37]. We notice also that the $\xi_\mu$ variable in two-dimensional case is used in the description of a spinning string within the Ramond-Neveu-Schwarz formalism, and in particular in the construction of the covariant string vertex operators describing emission (absorption) of gauge vector particles [38] and fermions [39, 40] by the string.

The description of the spin degrees of freedom in terms of the odd pseudovector and pseudoscalar quantities $(\xi_\mu, \xi_5)$ is to some extent more fundamental in comparison with the description in terms of the commuting spinor $\psi_\alpha$. For this reason the interesting question arises as to whether it is possible to define a mapping between these variables, and, finally, the possibility of constructing the mapping between the Lagrangians (1.4) and (A.1). The mapping of this type was first considered by Barut and Pašić [41] (see also Scholtz et al. [42]).

It is pertinent at this point to make one remark which is completely analogous to that made in Introduction of the paper [1]. This remark was concerned with introducing into consideration the Grassmann color charges $\theta^i$ and $\theta^i$. If we closely look at the equations of motion (A.9) – (A.11), and at the expression for the color current (A.12), then we may observe that the odd pseudovector $\xi_\mu$ enters these equations only in the following even combination

$$S^{\mu\nu} = -i\xi^\mu D^\nu,$$  \hspace{1cm} (1.8)

as well as the Grassmann color charges enter these equations in the combination $t^a \theta^i (\equiv Q^a)$. By virtue of (A.8) the function $S^{\mu\nu}$ obeys the equation of motion

$$\frac{dS^{\mu\nu}}{d\tau} = \frac{g}{m} Q^a (F^{a\mu\nu}_\lambda S^{\lambda\nu} - F^{a\nu\lambda}_\mu S^{\lambda\mu}).$$  \hspace{1cm} (1.9)

One notices that a similar tensor of spin can be defined also in terms of the $\psi_\alpha$ spinor

$$S^{\mu\nu} = \frac{1}{2} \hbar \lambda (\bar{\psi}\sigma^{\mu\nu}\psi).$$  \hspace{1cm} (1.10)

\footnote{In the paper [28] a qualitative argument on this matter has been given. If we take $t^a \sim \hbar$ and at the same time believe that a gauge field is of order $1/\hbar$, i.e.

$$A^a_\mu(x) \sim 1/\hbar,$$

then it leads to the fact that the interaction of a color particle with the non-Abelian gauge field has the same $\hbar$-dependence as in quantum electrodynamics. In QCD such a strong field is called an external color field. Just that case we mean in the present paper. However, in the situation which was accepted in [26, 28] the original field $A^a_\mu(x)$ is of order 1. Such the color fields are microscopic or dynamical fields, that falls into purely quantum branch.}
By virtue of (1.1) this tensor of spin obeys the same equation (1.9). In addition, the antisymmetric tensor $S^\mu\nu$ can be considered as a semiclassical limit of quantum-mechanical average of the spinning matrix $\sigma_{\mu\nu}$, i.e.

$$S^\mu\nu = \frac{1}{2} \langle \sigma^\mu\nu \rangle_{\hbar \to 0},$$

where the angular brackets denote the average operation.

Thus in the actual dynamics of a classical color spinning particle one may quite manage with the usual commuting function $S^\mu\nu$. The odd pseudovector variable $\xi_\mu$ give merely the possibility of a classical Lagrangian formulation without any dynamical effects. One can expect that the situation can qualitatively change only if the system is subjected to background (non-)Abelian fermion field which as it were ‘splits’ the combination $S^\mu\nu = -i\xi^\mu \xi^\nu$ into two independent Grassmann-odd parts (see Section 5). Here, the necessity of introducing the Grassmann pseudovector $\xi_\mu$ as a dynamical variable enjoying full rights should be manifested in full.

Further, by virtue of the fact that we have the even spinor $\psi_\alpha$ on the one hand and the odd pseudovector $\xi_\mu$ (and pseudoscalar $\xi_5$) on the other hand, for the construction of the desired mapping we must introduce some auxiliary odd spinor $\theta_\alpha$. The idea of the construction of such a mapping is not new. In due time this problem has been studied extensively in view of analysis of a classical correspondence of theories of relativistic massless spin-1/2 particles [31–33] and superparticles [43–47] and in a more general context between spinning strings and superstrings.

In the paper by Sorokin et al. [48] within the superfield formalism it was noted that such a classical correspondence can be defined by the following relation

$$\xi_\mu \sim \bar{\theta} \gamma_\mu \psi + (\text{conj. part}). \quad (1.11)$$

In [48] the commuting spinor $\psi_\alpha$ played the role of a twistor-like variable which is not dynamical one. In our paper we use the relation (1.11). The only difference is that by virtue of initial statement of the problem, the anticommuting spinor $\theta_\alpha$ will play a role of the auxiliary variable rather than $\psi_\alpha$.

The paper is organized as follows. In Section 2 a general analysis of the mapping between a pair of the Dirac spinors $(\psi_\alpha, \theta_\alpha)$ and the real anticommutative tensor system $(S, V_\mu, *T^\mu\nu, A_\mu, P)$ is proposed. The required initial general expressions of the mapping are written out. An important special case when the spinors $\psi_\alpha$ and $\theta_\alpha$ are the Majorana ones, is considered. The algebraic relations between the tensor quantities are defined with the help of the Fierz identities.

Section 3 is devoted to the discussion of mapping the kinetic term (1.5). Here, the required general expression connecting the kinetic term with the derivative of the tensor quantities: $(\dot{S}, \dot{V}_\mu, *\dot{T}^\mu\nu, \dot{A}_\mu, \dot{P})$, is defined. A limiting case of the Majorana spinors is considered. The procedure of deriving the algebraic relations including aside from the tensor quantities itself, also their derivatives, is described in full.

In Section 4 a detailed analysis of the local bosonic symmetry of the free Lagrangian (1.4) is carried out and a connection of this symmetry with the reduced local supersymmetry transformations presented in Appendix A, is considered.

Section 5, 6 and 7 are concerned with the construction of the explicit solutions for a system of bilinear algebraic equations defined in Section 2 and in Appendix C. In Section 5 as an example the well-known case of the commutative tensor variables $(S, V_\mu, *T^\mu\nu, A_\mu, P)$ is reproduced. In Section 6 this analysis is extended to a qualitative new case of a anticommuting
tensor set. The exact solution of the corresponding system of equations is presented and also an analysis of solving a system of equations including the derivatives of the tensor variables \((S, V_\mu, ^*T_{\mu\nu}, A_\mu, P)\), is given. Finally, in Section 7 the alternative approach in solving the problem based on introducing complex tensor quantities, is considered. Introducing the complex tensor quantities enables us to provide also more insight into the structure of an algebraic system of identities and to see that the system under consideration in principle admits the existence of the second independent solution.

Section 8 provides a detailed discussion of the possibility of the existence of the second solution. Here we give conditions under which the second solution is permissible. It was shown that there exists the only algebraic identity which results in unremovable contradiction at least in a class of Majorana spinors.

In Section 9 a qualitative consideration of supersymmetric generalization of our initial Lagrangian (1.4) is performed.

In Section 10, one possible way of extension of the results obtained in the previous sections to a more complicated case of the Dirac spinors \(\psi_D\) and \(\theta_D\), is suggested. On the basis of a general analysis it is concluded that for the existence of (one-to-one) correspondence between the Dirac spinors \((\psi_D, \theta_D)\) and tensor variables \((S, V_\mu, ^*T_{\mu\nu}, A_\mu, P)\) it is necessary a fourfold increase of the number of the real tensor ones \((S_{ij}, V_{ij\mu}, ^*T_{ij\mu\nu}, A_{ij\mu}, P_{ij})\), where \(i, j = 1, 2\).

Section 11 is concerned with a discussion of the possibility of a connection of the approach stated in the previous sections, with the models of a spinning particle based on the higher-order derivative Lagrangians.

In the concluding Section 12 we briefly discuss the question of further generalization of the ideas of this work, namely, the generalization of the original classical Lagrangians, the construction of the mapping between the systems obtained after quantization of these classical models.

In Appendix A the complete form of the Lagrangian for a spin-\(\frac{1}{2}\) color particle is given and the local SUSY \(n = 1\) transformations, constraints and equations of motion are written out.

In Appendix B all of the necessary formulas of spinor algebra are listed. In Appendix C a complete list of all 15 sets of the bilinear relations connecting the real currents \((S, V_\mu, ^*T_{\mu\nu}, A_\mu, P)\) among themselves is set out. The above-mentioned list is introduced in such a way that it simultaneously covers both the commutative and anticommutative cases of the current variables.

In Appendix D it is given the proof of independence of mapping the kinetic term (1.5) from the fact whether the auxiliary term \(\theta_\alpha\) is constant or variable quantity, provided the commutative spinor \(\psi_\alpha\) and anticommuting auxiliary spinor \(\theta_\alpha\) are the Majorana ones.

In Appendix E the parameter representation of orthogonal tetrad \(h^{(s)}_\mu\) used in deriving solutions of the algebraic equations, is given.

In Appendix F a way of the construction of a system of bilinear identities for the complex tensor variables \((S, V_\mu, ^*T_{\mu\nu}, \ldots)\), is suggested. A system of identities of Appendix C results as a special case.
2 General analysis of a connection between a pair of spinors \((\psi_\alpha, \theta_\alpha)\) and anticommuting tensor system

The problem of defining the mapping between the commuting spinor \(\psi_\alpha\) and anticommuting pseudovector and pseudoscalar variables \((\xi_\mu, \xi_5)\) is in fact a part of a more general analysis of the relation between spinors (Dirac, Majorana or Weyl ones) and Lorentz-invariant real or complex tensor systems. In the case of a commuting c-number Dirac spinor and 16 real commuting bilinear tensor quantities that are formed by the given spinor, such a problem has been studied by Takahashi and Okuda [49, 50], Kaempffer [51], Crawford [52] and from a somewhat different viewpoint by Zhelnorovich [53–55]. The latter author has also considered the special cases of Majorana and Weyl spinors, and the most important for us problem of the relation between a pair of two commuting spinors \((\psi_\alpha, \varphi_\alpha)\) and appropriate tensor set is continued discussion we will follow essentially Zhelnorovich [54, 55].

In the problem under consideration we also have at hand two, in the general case Dirac spinors \(\psi_\alpha\) and \(\theta_\alpha\) (although in our case the latter plays an auxiliary role). However, the second spinor is now classical anticommuting one as distinct from the works [54, 55].

The heart of our subsequent considerations is the expansion of the spinor structure \(\tilde{h}^{1/2} \tilde{\theta}_\beta \psi_\alpha\) in the basis of the Dirac \(\gamma\)-matrices:

\[
\tilde{h}^{1/2} \tilde{\theta}_\beta \psi_\alpha = \frac{1}{4} \left\{ -iS\delta_{\alpha\beta} + V_\mu(\gamma^\mu)_{\alpha\beta} - \frac{i}{2} T_{\mu\nu}(\sigma^{\mu\nu}\gamma_5)_{\alpha\beta} + iA_\mu(\gamma^\mu\gamma_5)_{\alpha\beta} + P(\gamma_5)_{\alpha\beta} \right\}, \tag{2.1}
\]

The expansion for the Hermitian conjugate expression is

\[
\tilde{h}^{1/2} \tilde{\psi}_\beta \theta_\alpha = \frac{1}{4} \left\{ iS^*\delta_{\alpha\beta} + V^*_\mu(\gamma^\mu)_{\alpha\beta} - \frac{i}{2} (T^*_{\mu\nu})(\sigma^{\mu\nu}\gamma_5)_{\alpha\beta} - iA^*_\mu(\gamma^\mu\gamma_5)_{\alpha\beta} - P^*(\gamma_5)_{\alpha\beta} \right\}. \tag{2.2}
\]

Here, the complex anticommuting tensor variables on the right-hand side are defined as follows:

\[
S \equiv i\tilde{h}^{1/2}(\tilde{\theta}\psi), \quad V_\mu \equiv \tilde{h}^{1/2}(\tilde{\theta}\gamma_\mu\psi), \quad T_{\mu\nu}^* \equiv \tilde{h}^{1/2}(\tilde{\theta}\sigma_{\mu\nu}\gamma_5\psi),
\]

\[
A_\mu \equiv i\tilde{h}^{1/2}(\tilde{\theta}\gamma_\mu\gamma_5\psi), \quad P \equiv \tilde{h}^{1/2}(\tilde{\theta}\gamma_5\psi). \tag{2.3}
\]

The multiplies on the right-hand side of expressions in (3.3) have been chosen such that for Majorana spinors \(\psi_\alpha\) and \(\theta_\alpha\) the tensor quantities (2.3) are real, i.e.

\[
S = S^*, \quad V_\mu = V^*_\mu, \quad T_{\mu\nu} = (T^*_{\mu\nu}), \quad \ldots. \tag{2.4}
\]

The symbol * denotes complex conjugation. On the left-hand side of the expressions (2.1) and (2.2) we have introduced the dimensional factor \(\tilde{h}^{1/2}\), since the spinors \(\psi_\alpha\) and \(\theta_\alpha\) are considered as dimensionless variables (see Introduction), while the dimension of the functions \(S, V_\mu, T_{\mu\nu}, \) etc. is \([S] \sim [V_\mu] \sim [T_{\mu\nu}] \sim \ldots \sim [h]^{1/2}\).

Write out next an important formula for the product of two expressions (2.1) and (2.2)

\[
\tilde{h}(\tilde{\theta}_\beta \psi_\alpha)(\tilde{\psi}_\gamma \theta_\delta) = \frac{1}{16} \left\{ -iS\delta_{\alpha\beta} + V_\mu(\gamma^\mu)_{\alpha\beta} - \frac{i}{2} T_{\mu\nu}(\sigma^{\mu\nu}\gamma_5)_{\alpha\beta} + iA_\mu(\gamma^\mu\gamma_5)_{\alpha\beta} + P(\gamma_5)_{\alpha\beta} \right\} \times
\]

\[
\times \left\{ iS^*\delta_{\gamma\delta} + V^*_\mu(\gamma^\mu)_{\gamma\delta} - \frac{i}{2} (T^*_{\mu\nu})(\sigma^{\mu\nu}\gamma_5)_{\gamma\delta} - iA^*_\mu(\gamma^\mu\gamma_5)_{\gamma\delta} - P^*(\gamma_5)_{\gamma\delta} \right\}. \tag{2.5}
\]

\(\text{It seems likely that the decomposition of the direct product of two commuting spinors in terms of tensors was first discussed by Case [56].}\)
Let us consider for example the simplest crossed contraction of the expression (2.5) with \( \delta_{\beta\delta}\delta_{\gamma\alpha} \). As a result we obtain

\[
\hbar(\bar{\theta}\theta)(\bar{\psi}\psi) = \frac{i}{4} \left\{ SS^* + V_\mu(V^\mu)^* - \frac{1}{2} T_{\mu\nu}(T_{\mu\nu})^* - A_\mu(A^\mu)^* - PP^* \right\}.
\]  

(2.6)

Let the commuting spinor \( \psi \) and the anticommuting spinor \( \theta \) be Majorana (M) ones. As is known in this case the following relations hold

\[
(\bar{\psi}_M\psi_M) = 0, \quad (\bar{\psi}_M\gamma_5\psi_M) = 0, \quad (\bar{\psi}_M\gamma_\mu\gamma_5\psi_M) = 0,
\]

(2.7)

By virtue of (2.7), the left-hand side of (2.6) vanishes. The right-hand side of this expression is equal to zero by the condition (2.4) and by nilpotency of the tensor quantities.

A more nontrivial expression can be obtained from (2.5) by its contracting with \( \delta_{\beta\delta}(\sigma^{\mu\nu})_{\gamma\alpha} \). The calculation of the traces of the product of \( \gamma \)-matrices by employing the formulae in Appendix B leads to the expression

\[
\hbar(\bar{\theta}\theta)(\bar{\psi}\sigma^{\mu\nu}\psi) = \frac{i}{4} \left\{ -\left[ S(T_{\mu\nu})^* - T_{\mu\nu}S^* \right] + \left[ T_{\mu\nu}P^* - P(T_{\mu\nu})^* \right] + \left[ V_\mu(V^\nu)^* - V^\nu(V^\mu)^* \right] - \left[ A_\mu(A^\nu)^* - A^\nu(A_\mu)^* \right] - \epsilon^{\mu\nu\lambda\sigma} \left[ V_\lambda(A_\sigma)^* + A_\lambda(V_\sigma)^* \right] - \epsilon^{\mu\nu\lambda\sigma} \left[ \gamma_\lambda(\gamma_\sigma)^* \right] \right\}.
\]

(2.8)

Hereafter, we make use of the formulae for going over from an arbitrary antisymmetric tensor of second rank to its dual tensor and conversely,

\[
\gamma_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} T_{\lambda\sigma}, \quad T_{\mu\nu} = -\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \gamma_{\lambda\sigma}.
\]

The expression (2.8) is just the one entered in the Lagrangian (1.7) for the specific choice

\[
\lambda = (\bar{\theta}\theta).
\]

(2.9)

In the special case of Majorana spinors the expression (2.8) turns to

\[
\hbar(\bar{\theta}_M\theta_M)(\bar{\psi}_M\sigma^{\mu\nu}\psi_M) = \frac{i}{2} \left\{ -\left[ ST_{\mu\nu} + P(T_{\mu\nu})^* \right] + \left[ V^\mu V^\nu - A^\mu A^\nu \right] - \epsilon^{\mu\nu\lambda\sigma} V_\lambda(A_\sigma)^* + \epsilon^{\mu\nu\lambda\sigma} A_\lambda(V_\sigma)^* \right\}.
\]

(2.10)

It is very important as a self-check to consider the contraction of (2.5) with a similar structure \( (\sigma^{\mu\nu})_{\beta\delta}(\gamma_{\gamma\alpha}) \). Here, instead of (2.8), we have

\[
\hbar(\bar{\psi}_M\psi_M)(\bar{\theta}\sigma^{\mu\nu}\theta) = \frac{i}{4} \left\{ -\left[ S(T_{\mu\nu})^* - T_{\mu\nu}S^* \right] + \left[ T_{\mu\nu}P^* - P(T_{\mu\nu})^* \right] - \left[ V^\mu (V^\nu)^* - V^\nu (V^\mu)^* \right] - \left[ A^\mu (A^\nu)^* - A^\nu (A^\mu)^* \right] - \epsilon^{\mu\nu\lambda\sigma} \left[ V_\lambda(A_\sigma)^* + A_\lambda(V_\sigma)^* \right] - \epsilon^{\mu\nu\lambda\sigma} \left[ \gamma_\lambda(\gamma_\sigma)^* \right] \right\},
\]

and in particular for the Majorana spinors, instead of (2.10), we get

\[
\hbar(\bar{\psi}_M\psi_M)(\bar{\theta}_M\sigma^{\mu\nu}\theta_M) = \frac{i}{2} \left\{ -\left[ ST_{\mu\nu} + P(T_{\mu\nu})^* \right] - \left[ V^\mu V^\nu - A^\mu A^\nu \right] - \epsilon^{\mu\nu\lambda\sigma} V_\lambda(A_\sigma)^* - \epsilon^{\mu\nu\lambda\sigma} A_\lambda(V_\sigma)^* \right\}.
\]

(2.11)

4 In our choice of the constant \( \lambda \), the definition of the tensor spin (1.10) takes the form

\[
S^{\mu\nu} = \frac{1}{2} \hbar(\bar{\theta}\theta)(\bar{\psi}\sigma^{\mu\nu}\psi).
\]

In this case only the expression acquires the nilpotency property: the product of its any five components equals zero, i.e. \( \prod_{i=1}^{5} S^{\mu_i\nu_i} = 0 \), as it takes place for the definition \( S^{\mu\nu} \equiv -i \xi^{\mu} \xi^{\nu} \).
However, here by virtue of (2.7) in contrast to (2.10), the left-hand side equals zero\(^5\) while the right-hand side represents a rather complicated algebraic expression. As distinct from (2.6) it is not evident in advance that it must be also equal to zero. The proof of this fact serves a good test for correctness of a system of algebraic equations connecting the tensor quantities \(S, V_\mu, T_\mu \ldots\) among themselves (see below).

Not all of the quantities (2.3) are independent. There exist certain algebraic relations between them. As is known, such relations are provided by the Fierz identities. According to Zhelnorovich [54, 55], the required bilinear equations can be obtained by multiplying out both Majorana spinors and Dirac spinors. Recall that in the latter case the tensor quantities (2.3) are complex. Nevertheless the system (C.1) – (C.15) is unsuitable for analysis of the right-hand side of expression (2.8), since here in the general Dirac case we have a product of tensor quantities (2.3) and its complex conjugation. By virtue of this fact we restrict our consideration only to the important special case of Majorana spinors (real currents (2.3)), when (2.8) goes over into the simpler relation (2.10).

We need equations (C.3), (C.6), (C.8), (C.10), (C.12) and (C.13). For the case of anticommuting currents the equations take the form (we give these equations in the different order)

\[
S \ast T^{\mu \nu} = -\frac{1}{4} \epsilon^{\mu \nu \lambda \sigma} V_\lambda V_\sigma - \frac{1}{4} \epsilon^{\mu \nu \lambda \sigma} A_\lambda A_\sigma - \frac{1}{4} \epsilon^{\mu \nu \lambda \sigma} \ast T_{\lambda \rho} \ast T^{\rho \sigma},
\]

\[
P \ast T^{\mu \nu} = \frac{1}{2} V^\mu V^\nu + \frac{1}{2} A^\mu A^\nu - \frac{1}{2} \ast T^{\mu \lambda} \ast T_\lambda \nu,
\]

\[
V^\mu V^\nu = -\frac{1}{2} ST^{\mu \nu} + \frac{1}{2} P \ast T^{\mu \nu} - \frac{1}{2} \epsilon^{\mu \nu \lambda \sigma} V_\lambda A_\sigma,
\]

\[
A^\mu A^\nu = -\frac{1}{2} ST^{\mu \nu} + \frac{1}{2} P \ast T^{\mu \nu} + \frac{1}{2} \epsilon^{\mu \nu \lambda \sigma} V_\lambda A_\sigma,
\]

\[
V^\mu A^\nu = -\frac{1}{2} g^{\mu \nu} SP + \frac{1}{4} \epsilon^{\mu \nu \lambda \sigma} V_\lambda V_\sigma - \frac{1}{4} A^\mu A^\nu + \frac{1}{4} (\ast T^{\mu \lambda} T_\lambda \nu - T^{\mu \lambda} \ast T_\lambda \nu).
\]

Here, we have omitted equation (C.10) because of the awkwardness. The simple analysis of this system has shown that only three equations are independent and they can be chosen in the following form:

\[
P \ast T^{\mu \nu} - ST^{\mu \nu} = V^\mu V^\nu + A^\mu A^\nu, \quad (2.12)
\]

\[
P \ast T^{\mu \nu} + ST^{\mu \nu} = -\ast T^{\mu \lambda} \ast T_\lambda \nu, \quad (2.13)
\]

\[
-\epsilon^{\mu \nu \lambda \sigma} V_\lambda A_\sigma = V^\mu V^\nu - A^\mu A^\nu. \quad (2.14)
\]

By using (2.13) and (2.14), we obtain instead of (2.10)

\[
\hbar(\bar{\theta}_M \theta_M)(\bar{\psi}_M \sigma^{\mu \nu} \psi_M) = i \left\{ -[ST^{\mu \nu} + P \ast T^{\mu \nu}] + [V^\mu V^\nu - A^\mu A^\nu] \right\}.
\]

---

\(^5\)The fact that the tensor expression for Majorana spinors turns to zero can be considered as indirect evidence for choosing the commuting spinor \(\psi_\alpha\) (rather than anticommuting \(\theta_\alpha\)) as a basic dynamical variable for the classical description of the spin degrees of freedom of a particle that, generally speaking, a priori it is not at all obvious (see Introduction).
The remaining equation (2.12) enables us to get rid either of \((V\mu V\nu - P^* T^{\mu\nu})\) or of \((A^\mu A^\nu + ST^{\mu\nu})\). In the first case we have

\[
\hbar \langle \bar{\theta}_M \theta_M \rangle (\bar{\psi}_M \sigma^{\mu\nu} \psi_M) = -2i \{ A^\mu A^\nu + ST^{\mu\nu} \}.
\] (2.15)

The last expression is the most important result of this section. From a comparison of the force term in different representations

\[-\hbar \langle \bar{\theta} \theta \rangle \frac{ie g}{4} Q^\alpha F^{\alpha}_{\mu\nu} (\bar{\psi} \sigma^{\mu\nu} \psi) \sim \frac{i e g}{2} Q^\alpha F^{\alpha}_{\mu\nu} \xi^\mu \xi^\nu + \ldots ,\]

it follows at once that the relation

\[
\hbar \langle \bar{\theta}_M \theta_M \rangle (\bar{\psi}_M \sigma^{\mu\nu} \psi_M) = -2i \xi^\mu \xi^\nu
\] (2.16)

must hold. A comparison with (2.15) shows that we must set

\[A^\mu = \pm \xi^\mu .\] (2.17)

In this case for the first term on the right-hand side of (2.15) we have the ideal coincidence with (2.16). The second term here can be put equal to zero in the case\(^6\) when \(S = 0\) or \(T^{\mu\nu} = 0\). Further, by using equations (2.13) and (2.14) it is not difficult to verify that the right-hand side of (2.11) vanishes, as it should be.

In conclusion of this section we note that among the tensor structures of the type

\[
h \langle \bar{\theta} \theta \rangle (\bar{\psi} \Gamma^A \psi), \quad \Gamma^A = I, \gamma^\mu, \gamma^\mu \gamma_5, \sigma^{\mu\nu}, \gamma_5,
\]

apart from \(\Gamma^A \equiv \sigma^{\mu\nu}\), the \textit{vector} structure with \(\Gamma^A \equiv \gamma^\mu\) is also different from zero for the case of Majorana spinors. This structure enters into the Lagrangian (1.5) in the form \((1/e) \bar{\chi}_\mu(x) (\bar{\theta} \theta) (\bar{\psi} \gamma^\mu \psi)\). We shall analyze the mapping of this term. This mapping has a specific feature.

For this purpose let us contract the initial expression (2.5) with \(\delta_{\beta\delta}(\gamma^\mu) \gamma_\alpha\). Simple calculations result in

\[
h \langle \bar{\theta} \theta \rangle (\bar{\psi} \gamma^\mu \psi) = \frac{i}{4} \left\{ \left[ S(V^\mu)^* - V^\mu S^* \right] + \left[ P(A^\mu)^* - A^\mu P^* \right] + \left[ V_\nu (T^{\mu\nu})^* - T^{\mu\nu} (V_\nu)^* \right] \right. \]

\[\left. + \left[ A_\nu (T^{\mu\nu})^* - T^{\mu\nu} (A_\nu)^* \right] \right\}, \] (2.18)

and, in particular, for the Majorana spinors we have

\[
h \langle \bar{\theta}_M \theta_M \rangle (\bar{\psi}_M \gamma^\mu \psi_M) = -\frac{i}{2} \left\{ SV^{\mu} - PA^{\mu} - V^\mu T^{\mu\nu} - A_\nu (T^{\mu\nu}) \right\}.
\] (2.19)

Further, let us define a system of algebraic identities that have to satisfy the functions on the right-hand side of (2.19). Here we need the equations of the “vector” type (C.2) and (C.14). For the Grassmann-valued currents they take the form

\[
SV^{\mu} = \frac{1}{2} PA^{\mu} - \frac{1}{2} V_\nu T^{\mu\nu}, \quad PA^{\mu} = \frac{1}{2} SV^{\mu} + \frac{1}{2} A_\nu (T^{\mu\nu})
\] (2.20)

or in a slightly different form they are

\[
3(SV^{\mu} - PA^{\mu}) = -V^\mu T^{\mu\nu} - A_\nu (T^{\mu\nu})
\]

\[
SV^{\mu} + PA^{\mu} = -V^\mu T^{\mu\nu} + A_\nu (T^{\mu\nu}).
\] (2.21)

\(^6\)At this stage, however, we lose a possibility to construct one-to-one correspondence between two classical Lagrangians (1.4) and (A.11). To achieve the one-to-one correspondence we must add additional dynamic variables to the Lagrangian (A.11) (or to its reduced form (A.14), (A.15)).
Moreover, one can obtain two more additional vector equations from (C.7) and (C.11) contracting them with $\epsilon_{\mu\nu\lambda\sigma}$ and $g_{\lambda\nu}$, respectively. A somewhat cumbersome, but simple analysis shows that such obtained two additional equations are a consequence of (2.20).

Equations (2.21) enables us to eliminate in (2.19) either a pair of variables $(V_{\nu}T^{\mu\nu}, A_{\nu}^*T^{\mu\nu})$ or $(SV_{\mu}, V_{\nu}T^{\mu\nu})$:

$$\hbar(\bar{\theta}_M\theta_M)(\bar{\psi}_M\gamma^{\mu}\psi_M) = 2i(PA^\mu - SV^\mu) = 2i(PA^\mu + A_{\nu}^*T^{\mu\nu}).$$

(2.22)

If we take into account (2.17) and naively set

$$P = \pm \xi_5$$

(2.23)

in terms of the variables of the Lagrangian (A.1), then for the first term on the right-hand side of the above expression we obtain rather unusual correspondence

$$\frac{1}{e} \dot{x}_\mu(x)(\bar{\theta}_M\theta_M)(\bar{\psi}_M\gamma^{\mu}\psi_M) = \frac{1}{\hbar} \frac{2i}{e} \dot{x}_\mu \xi_5 \xi^{\mu}.$$  

(2.24)

In contrast to the tensor contribution (2.16), the mapping of the term $(1/e)\dot{x}_\mu(x)(\bar{\theta}\gamma^{\mu}\psi)$ results in the expression containing Planck's constant in the denominator. However, there is another problem. If one compares the expression obtained (2.24) with the last term in (A.14), namely with

$$\frac{i}{m\hbar} \dot{x}_\mu \dot{\xi}_5 \xi^{\mu},$$

(2.25)

then it is seen that this expression is different from (2.24) by the fact that instead of $\xi_5$ we have here $\dot{\xi}_5$. One can overcome these difficulties if instead of (2.23) to use more nontrivial identification

$$P = \pm \frac{\hbar}{2m} \dot{\xi}_5.$$  

(2.26)

By this means as in the case of the tensor contribution considered above, by discarding in (2.22) the terms which have no counterparts in expression (A.14), one can achieve a good agreement with the reduced Lagrangian (A.1) (see the previous footnote).

3 Mapping the kinetic term

Let us consider a mapping of the kinetic term in (1.5), more exactly, of the term

$$\frac{1}{2} i\hbar(\bar{\theta}\theta)\left(\frac{d\bar{\psi}}{d\tau} - \bar{\psi}\frac{d\psi}{d\tau}\right).$$

(3.1)

Here we have taken into account the choice of the constant $\lambda$, Eq. (2.9). Note also that the necessity of introducing the nilpotent factor for the construction of the correct mapping was first pointed out by Barut and Pašič [41]. We shall assume for the moment that in the general case the auxiliary spinor $\theta_\alpha$ may be a function of $\tau$. Differentiating (2.22) over $\tau$, we obtain

$$\hbar^{1/2}\left(\frac{d\bar{\psi}_\beta}{d\tau} \theta_\alpha + \bar{\psi}_\beta \frac{d\theta_\alpha}{d\tau}\right) =$$

$$= \frac{1}{4} \left\{i\hat{S}^*\delta_{\alpha\beta} + V_{\mu}^* (\gamma^{\mu})_{\alpha\beta} - \frac{i}{2} (\hat{T}^{\mu\nu})^* (\sigma^{\mu\nu}\gamma_5)_{\alpha\beta} - i\hat{A}_{\mu}^* (\gamma^{\mu}\gamma_5)_{\alpha\beta} - \hat{P}^* (\gamma_5)_{\alpha\beta}\right\}. $$
We contract the left-hand side of the expression with $\hbar^{1/2}\bar{\psi}_\alpha\bar{\theta}_\alpha$, whereas its right-hand side is contracted with the appropriate expression (2.1). As a result, we have

$$\hbar\left[\bar{\theta} \left(\frac{d\bar{\psi}}{d\tau}\right) \psi \right] - \left(\bar{\psi} \psi \right) \left(\frac{d\theta}{d\tau}\right) = -\frac{1}{4} \left\{ \bar{S} \bar{S} + \bar{V}_\mu V^\mu - \frac{1}{2} (\bar{T}^\mu \gamma_5 \gamma_\mu - \dot{A}_\mu A^\mu - \dot{P}^\mu P) \right\}.$$  

Subtracting from the last expression its complex conjugation, we finally obtain

$$\hbar\left(\bar{\theta} \left(\frac{d\bar{\psi}}{d\tau}\right) \psi \right) - \left(\bar{\psi} \psi \right) \left(\frac{d\theta}{d\tau}\right) = \frac{1}{4} \left\{ \bar{S} \bar{S} - S^2 \right\} + \left(\bar{V}_\mu V^\mu - V^\mu \bar{V}_\mu \right) - \frac{1}{2} \left\{ \left(\bar{T}^\mu \gamma_5 \gamma_\mu - \frac{1}{2} \bar{T}^\mu \gamma_5 \gamma_\mu \right) \right\} \right.$$

This kinetic term is greatly simplified for Majorana spinors. Taking into account the fact that $(\bar{\psi}_M \psi_M) = 0$ and the conditions (2.4) hold, the general expression (3.2) results in

$$\hbar(\bar{\theta}_M \theta_M) \left[\left(\frac{d\bar{\psi}_M}{d\tau}\right) \psi_M \right] - \left(\bar{\psi}_M \psi_M \right) \left(\frac{d\theta}{d\tau}\right) = \frac{1}{2} \left\{ S \frac{dS}{d\tau} + V^\mu \frac{dV^\mu}{d\tau} - \frac{1}{2} \bar{T}^\mu \gamma_5 \gamma_\mu \frac{dT^\mu}{d\tau} - \frac{1}{2} \bar{T}^\mu \gamma_5 \gamma_\mu \right\} \right.$$ 

We note that on the left-hand side the term with the derivative of $\theta_\alpha$ vanishes whether the auxiliary spinor is a function of $\tau$ or not. In Appendix D it is shown how this circumstance manifests on the right-hand side of (3.3).

As in the case of the spin structure (2.10) not all of the terms on the right-hand side of the expression (3.3) are independent. The quadratic terms with the derivatives satisfy a certain algebraic system which is similar to the one (C.1) - (C.15). In particular, this system defines the relationships between the terms in (3.3).

For obtaining the required system of relations we follow the same reasoning as was used in the previous section. Our first step is to multiply the expression (2.1) by its derivative

$$\hbar(\bar{\theta}_\beta \bar{\psi}_\alpha) \frac{d}{d\tau}(\bar{\psi}_\beta \psi_\gamma) = \frac{1}{16} \left\{ -iS \delta_{\alpha\beta} + V^\alpha \gamma_5 \delta_{\alpha\beta} + \frac{i}{2} \bar{T}^\mu \gamma_5 \gamma_\mu \delta_{\alpha\beta} + iA_\mu \gamma_5 \gamma_\mu P + P \right\} \right.$$ 

On the right-hand side we will have every possible products of the functions $S, V^\mu, \bar{T}^\mu, \ldots$ and their derivatives $\dot{S}, \dot{V}_\mu, \dot{T}^\mu, \ldots$. Let us consider in more detail the left-hand side of the above expression. For this purpose we contract the left-hand side of (3.3) with the simplest spinor structure $\delta_{\beta\gamma} \delta_{\delta\alpha}$

$$\hbar(\bar{\theta}_\beta \bar{\psi}_\alpha) \frac{d}{d\tau}(\bar{\psi}_\beta \psi_\gamma) \delta_{\beta\gamma} \delta_{\delta\alpha} = \hbar \left[ \left(\bar{\theta} \psi\right) \left(\bar{\psi} \psi\right) \right] = -i\hbar^{1/2} \left\{ \left(\bar{\psi} \psi\right) \left(\bar{\psi} \psi\right) \right\}.$$ 

By virtue of the anticommutative character of the auxiliary spinor $\theta_\alpha$ and scalar function $S$, on the right-hand side of the last expression we have

$$i\hbar^{1/2} \left[ \left(\bar{\theta} \psi\right) \left(\bar{\psi} \psi\right) \right] = -i\dot{S}.$$ 

Thus, the terms in square brackets do not collect into the required combination

$$\hbar^{1/2} \left[ \left(\bar{\theta} \psi\right) + \left(\bar{\psi} \psi\right) \right] \equiv -i\dot{S},$$
thereby essentially complicates further analysis. For the sake of simplicity, in this section we restrict our consideration to the particular case

\[ \theta_\alpha = \text{const}. \]

In Appendix D we briefly discuss a way for overcoming the difficulty in the general case (at least for the Majorana spinors).

In this way, for the special case \( \theta_\alpha = \text{const} \) from (3.5) we have

\[
h(\bar{\theta}_\beta \psi_\alpha) \frac{d}{d\tau} (\bar{\theta}_\beta \psi_\alpha) \delta_\beta \gamma \delta_\delta = -\dot{SS}.
\]

Further, the contraction of the right-hand side of (3.4) with \( \delta_\beta \gamma \delta_\delta \), is trivial calculated and we obtain the first desired algebraic equation containing the derivatives

\[ 4\dot{SS} = SS - P\dot{P} - V_\mu \dot{V}^\mu - A_\mu \dot{A}^\mu + \frac{1}{2} *T_{\mu\nu} *T^{\mu\nu}. \tag{3.7} \]

The equation is a peculiar analog of Eq. (C.1) with a fundamental distinction that (C.1) vanishes for the Grassmann-valued functions, whereas (3.7) represents a rather nontrivial relation.

We specially have not collected similar terms with \( SS \) on the left- and right-hand sides to show how they arise in the original form from (3.4). In particular, this enables us to understand how all remaining equations of a similar type can be directly obtained from the system (C.2) – (C.15) without recourse to the general formula (3.4). For deriving the required equations it is sufficient on the left-hand side of each bilinear identity from (C.1) – (C.15) to make the replacement

\[ AB \to \dot{A}B, \]

where \( A \) and \( B \) are any functions from the tensor set \( (S,V_\mu, *T_{\mu\nu}, A_\mu, P) \) (i.e the function with derivative must stand on the left), whereas on the right-hand side of identities (C.1) – (C.15) the function with derivative must stand on the right, as it takes place in (3.7). Meanwhile we have to take into consideration the contributions both from anticommutators and from commutators by the following rules

\[ [A,B] = AB - BA \to A\dot{B} - B\dot{A}, \]
\[ \{A,B\} = AB + BA \to A\dot{B} + B\dot{A}. \]

Recall that the anticommutator is equal to zero for arbitrary Grassmann-odd functions \( A \) and \( B \) by definition. For the concrete expression (3.3) in addition to equation (3.7) we need the equations for \( V_\mu \dot{V}^\mu, *T_{\mu\nu} *T^{\mu\nu}, A_\mu \dot{A}^\mu \) and \( P\dot{P} \). These follow from (C.6), (C.13), (C.15) and (C.10) by the scheme given above

\[
\dot{V}^\mu V_\mu = -S\dot{S} - P\dot{P} - \frac{1}{2} V_\mu \dot{V}^\mu + \frac{1}{2} A_\mu \dot{A}^\mu,
\]
\[
\dot{A}^\mu A_\mu = -S\dot{S} - P\dot{P} + \frac{1}{2} V_\mu \dot{V}^\mu - \frac{1}{2} A_\mu \dot{A}^\mu,
\]
\[
4\dot{P}\dot{P} = -S\dot{S} + P\dot{P} - V_\mu \dot{V}^\mu - A_\mu \dot{A}^\mu - \frac{1}{2} *T_{\mu\nu} *T^{\mu\nu}, \tag{3.8}
\]
\[
*T^{\mu\nu} *T_{\mu\nu} = 3(S\dot{S} - P\dot{P}) - \frac{1}{2} *T_{\mu\nu} *T^{\mu\nu}.
\]

Among the equations (3.7) and (3.8) only two ones are independent. It is convenient to write them in the form

\[ 2(S\dot{S} + P\dot{P}) = V_\mu \dot{V}^\mu + A_\mu \dot{A}^\mu, \tag{3.9} \]
\(3(\dot{S} - P\dot{\bar{P}}) = -\frac{1}{2} \ast T_{\mu\nu} \ast \dot{T}^{\mu\nu}. \quad (3.10)\)

These equations allow us to eliminate the terms \((-\frac{1}{2}) \ast T_{\mu\nu} \ast \dot{T}^{\mu\nu}\) and \(V_\mu \dot{V}^\mu\) from the right-hand side of (3.3). Multiplying (3.3) by the factor \((i/2)\), we finally obtain

\[\frac{i}{2} \hbar (\bar{\theta}_M \theta_M) \left[ \left( \frac{d\bar{\psi}_M}{d\tau} \psi_M \right) - \left( \bar{\psi}_M \frac{d\psi_M}{d\tau} \right) \right] = -\frac{i}{2} A_\mu \dot{A}^\mu - \frac{i}{2} P\dot{P} + \frac{3i}{2} S\dot{S}. \quad (3.11)\]

We need to compare the right-hand side of this mapping with appropriate kinetic terms in the reduced expressions (A.14) and (A.15), namely with

\[-\frac{i}{2} \xi_\mu \dot{\xi}^\mu - \frac{i}{2} \xi_5 \dot{\xi}_5. \quad (3.12)\]

The identification of the pseudovector \(A_\mu\) with \(\xi_\mu\) is exactly the same as that in the previous section, Eq. (2.17). One needs to identify the pseudoscalar variable \(P\) with \(\xi_5\). At first sight the identification (2.23) is the most natural and exactly reproduces the second term in (3.12). However, such an identification contradicts (2.25). From the other hand, in choosing (2.26) we will have

\[P\dot{P} = \frac{\hbar^2}{4m^2} \xi_5 \dot{\xi}_5. \]

In this case in order to obtain the correct expression (3.12) we must require the fulfilment of the following equation for the variable \(\xi_5\):

\[\ddot{\xi}_5 = -\frac{4m^2}{\hbar^2} \xi_5. \quad (3.13)\]

A similar equation was first considered in the paper by Barut and Paˇvsic [41].

Further, we can eliminate the additional term with the \(S\) function by setting \(S = 0\) (or in the more general case \(S = \text{const}\)). Note that the condition \(S = 0\) coincides with one of the conditions of vanishing the “excess” term in the tensor expression (2.15) (see footnote 6 though). By this means we see that in the case of Majorana spinors it is also possible to achieve the almost perfect mapping between the kinetic terms in the Lagrangians (1.4) and (A.1) (after eliminating the \(\chi\)-field) as in the case of mapping the force term.

At the close of Section 6 we discuss the possibility of identification of the \(P\) and \(\xi_5\) variables without using the constraint (A.13).

4 Mapping the bosonic symmetry

In this section we would like to discuss the symmetry transformations of the Lagrangian (1.4) and to consider the problem of connection of this symmetry with the symmetry (A.16) – (A.20) of the reduced Lagrangian (A.1). The bosonic invariance for Lagrangians of the (1.4) type was first discussed by Kowalski-Glikman et al. in [23] in the free massless case and then in the paper by Barut and Paˇvsic [41] with an extension to the massive particle. Let us consider the
transformation in the form suggested in the latter paper (Eq. (39)) in the free case:

\[ \delta x_\mu = \lambda (\bar{\beta} \gamma_\mu \psi + \bar{\psi} \gamma_\mu \beta), \]  
\[ \delta e = -\frac{2i}{\hbar} \lambda (\bar{\beta} \psi - \bar{\psi} \beta), \]  
\[ \delta \psi = -\frac{i}{\hbar} \frac{1}{e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] \gamma^\mu \beta. \]  

Hereafter, for brevity we set \( \lambda \equiv (\bar{\theta} \theta) \); the function \( \beta = \beta(\tau) \) is a commuting infinitesimal spinor parameter. For the time being we do not make any restrictions on the type of spinors \( \psi, \theta \) and \( \beta \) considering generally that they are Dirac ones. We shall demand the only condition \( \dot{\theta} = 0 \). The free part of Lagrangian (1.4) transforms as follows

\[ \delta L \equiv \delta L_0 + \delta L_m \]  
\[ = -\lambda \frac{1}{e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] \frac{d}{d\tau} (\bar{\psi} \gamma^\mu \beta) + \lambda \frac{1}{2e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] (\bar{\psi} \gamma^\mu \beta - \bar{\psi} \gamma^\mu \dot{\beta}) \]  
\[ - \lambda (\bar{\psi} \gamma^\mu \beta) \frac{d}{d\tau} \left( \frac{1}{2e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] \right) - \lambda \frac{im^2}{\hbar} (\bar{\psi} \beta) + (\text{compl. conj.}). \]

From an explicit form of the variation \( \delta L \) we see that the transformations (4.1) – (4.3) generally speaking, do not result in the desired invariance. Firstly, the first three terms on the right-hand side of (4.4) do not collect in the total derivative. This is connected with the fact that in the second term the expression in parentheses has incorrect sign. Secondly, the last term in (4.4) connected with the variation \( \delta L_m \) remains uncompensated. To correct the situation, we consider a minimal modification of the transformations (4.1) – (4.3), more exactly the transformation for the commuting spinor \( \psi \):

\[ \delta \psi = -\frac{i}{\hbar} \frac{1}{e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] \gamma^\mu \beta + \rho \dot{\beta}. \]  

Here \( \rho \) is some numerical parameter. The last term with \( \dot{\beta} \) leads to appearance of two additional terms in the variation (4.4)

\[ \lambda \rho \frac{1}{e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] (\bar{\psi} \gamma^\mu \dot{\beta}) + \frac{1}{2} i\hbar \lambda \rho \left( \frac{d}{d\tau} (\bar{\psi} \dot{\beta}) - 2 (\bar{\psi} \ddot{\beta}) \right) + (\text{compl. conj.}). \]

Let us choose the parameter \( \rho \) so that one could collect the total derivative from the first three terms in (4.4). To do this it is sufficient to set

\[ \rho = 1. \]

To eliminate the last term in (4.4) some restriction on the \( \tau \)-dependence of the spinor \( \beta \) is required, namely, we demand the fulfillment of the following equation:

\[ \ddot{\beta} + \left( \frac{m}{\hbar} \right)^2 \beta = 0. \]  

It is similar to the condition for the pseudoscalar variable \( \xi_5 \), which has arisen at the end of the previous section, Eq. (3.13).

Now we turn to the problem of a connection of symmetry transformations (4.1), (4.2) and (4.5) with the reduced transformations of local supersymmetry (A.16) – (A.19). Here we restrict our attention to Majorana spinors only (in what follows, for brevity, we omit the symbol \( M \) for the Majorana spinors). Let us consider the mapping of transformation (4.1). Taking into
account that for Majorana spinors the equality $\bar{\beta}\gamma_\mu\psi = \bar{\psi}\gamma_\mu\beta$ holds, we have

$$\delta x_\mu = 2\lambda(\bar{\psi}\gamma_\mu\beta) \equiv 2(\bar{\theta}\theta)(\bar{\psi}\gamma_\mu\beta).$$

From the other hand, by virtue of (A.16), the definitions (2.3) and identification (2.17), we can write

$$\delta x_\mu = i\alpha\xi_\mu \equiv \pm i\alpha A_\mu = \pm\hbar^{1/2}\alpha(\bar{\psi}\gamma_\mu\gamma_5\theta).$$

Comparing these two expressions, we obtain a connection between the commuting spinor $\beta$ of bosonic transformations and the Grassmann scalar parameter $\alpha$ of the supertransformations

$$(\bar{\theta}\theta)\beta = \pm\frac{1}{\hbar}h^{1/2}\alpha(\gamma_5\theta). \quad (4.7)$$

For the constant auxiliary spinor $\theta$ the equation (4.6) for spinor parameter $\beta$ turns to a similar equation for scalar parameter $\alpha$.

Let us consider further the mapping of transformation of the einbein field $e$, Eq. (4.2). With allowance for (4.7), the property $(\bar{\beta}\psi) = - (\bar{\psi}\beta)$ for Majorana spinors and the definitions (2.3), we have here the chain of equalities

$$\delta e = \frac{4i}{\hbar}\lambda(\bar{\psi}\beta) = \pm\frac{2i}{\hbar^{1/2}}\alpha(\bar{\psi}\gamma_5\theta) \equiv \mp\frac{2i}{\hbar}\alpha P.$$

From the other hand, for the reduced transformation of supersymmetry, Eq. (A.18), we have

$$\delta e = -\frac{2i}{\hbar}\alpha\dot{\xi}_5.$$

Comparing these two expressions we obtain that

$$P = \pm\frac{\hbar}{2m}\dot{\xi}_5. \quad (4.8)$$

It coincides in exact with our choice for the representation of the function $P$, Eq. (2.26).

We proceed now to analysis of mapping the transformation for the commuting spinor $\psi$, Eq. (4.5). First we multiply the expression (4.5) from the left by $i\lambda h^{1/2}\bar{\gamma}_\mu\gamma_5$. Taking into account the connection (4.7) and the identity $\gamma_\mu\gamma_\nu = I \cdot g_{\mu\nu} + i\sigma_{\mu\nu}$, we get

$$i\lambda h^{1/2}\delta(\bar{\gamma}_\mu\gamma_5\psi) =$$

$$= \mp\frac{1}{e}\lambda\alpha\left[\dot{x}_\mu - (\bar{\theta}\theta)(\bar{\psi}\gamma_\mu\psi)\right] \mp\frac{i}{e}\alpha(\bar{\theta}\sigma_{\mu\nu}\theta)[\dot{x}_\mu - \lambda(\bar{\psi}\gamma_\mu\psi)] \pm\hbar\dot{\alpha}(\bar{\theta}\gamma_\mu\theta).$$

In deriving the above expression we have considered that the parameter $\alpha$ and the spinor $\beta$ commute with one another. For Majorana spinors the last two terms become zero by virtue of the properties (2.7). Further we use the exact relation (2.22), where for the function $P$ we take (4.8). Dropping the contribution with $SV^\mu$ (or with $A_\nu^* T^{\mu\nu}$) in (2.22) and taking into account that

$$A_\mu \equiv ih^{1/2}(\bar{\theta}\gamma_\mu\gamma_5\psi) = \pm\xi_\mu,$$

we finally obtain

$$\lambda\delta\xi_\mu = -\lambda\alpha\left(\dot{x}_\mu - \frac{i}{m}\dot{\xi}_5\xi_\mu\right) / e. \quad (4.10)$$

Thus, by simple dropping excess terms in the mapping (2.22), one can achieve exact coincidence with the reduced transformation of supersymmetry (A.17) (more exactly, within the overall nilpotent factor $\lambda \equiv (\bar{\theta}\theta)$).
It remains to reproduce the transformation (A.19). For this purpose we multiply the transformation (4.5) from the left by \( \lambda \frac{\hbar}{1/2} \bar{\theta} \gamma^5 \):

\[
\lambda \frac{\hbar}{1/2} \delta (\bar{\theta} \gamma^5 \psi) = -i \lambda \frac{1}{\hbar^{1/2}} \frac{1}{e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] (\bar{\theta} \gamma^\mu \gamma^5 \beta) + \lambda \hbar^{1/2} (\bar{\theta} \gamma^5 \dot{\beta}).
\]

Taking into account the relation for the spinor \( \beta \), Eq. (4.7), we get

\[
\lambda \frac{\hbar}{1/2} \delta (\bar{\theta} \gamma^5 \psi) = \pm i \alpha \frac{1}{2} e (\bar{\theta} \gamma^\mu \theta) \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] \pm \frac{1}{2} \lambda \dot{\alpha} \hbar.
\] (4.11)

For Majorana spinors the first term on the right hand side is zero. Recalling the definition of the function \( P \equiv \hbar^{1/2} (\bar{\theta} \gamma^5 \psi) \) and identification (4.8), we finally obtain

\[
\lambda \delta \dot{\xi}^5 = \lambda \dot{m} \alpha.
\]

Such approach enables us to reproduce not the transformation (A.19) itself, but its derivative (up to the overall nilpotent factor \( \lambda \), as in the previous case). We specially note that in the mapping (4.11) only the term connected with the last one in the modified transformation (4.5) survives. This is used for additional confirmation of the necessity of the presence of the term with \( \dot{\beta} \) in (4.5) for correct reproduction of supersymmetry transformation.

Next it would be necessary to consider the transformations of bosonic symmetry and its mapping for the model with the interaction, i.e. with allowance made for the Lagrangian (1.7). Here, we restrict our consideration to a few remarks of the general character.

The transformation for the Grassmann-valued color charge \( \theta^i \)

\[
\delta \theta^i = -\frac{i}{\hbar} \lambda g (\bar{\beta} \gamma^\mu \psi + \bar{\psi} \gamma^\mu \beta) A^a_\mu (x) (t^a)^i \theta^j
\]

needs to be added to the symmetry transformations (4.1), (4.2) and (4.5). However, it is not enough to provide the invariance of Lagrangian (1.4). The additional contributions to the symmetry transformations considered above, involving the background Yang-Mills field are required. For example, the transformation for the commuting spinor \( \psi \), Eq. (4.5), should be replaced now by

\[
\delta \psi = -\frac{i}{\hbar} \frac{1}{e} \left[ \dot{x}_\mu - \lambda (\bar{\psi} \gamma_\mu \psi) \right] \gamma^\mu \beta + \dot{\beta} - \frac{i e g}{4} Q^a F^a_{\mu \nu} \sigma^{\mu \nu} \beta.
\] (4.12)

These additional contributions to the symmetry transformations have to provide not only the invariance of the complete Lagrangian (1.4). They must vanish under the mapping into the supertransformations (A.16) – (A.20). Here, it is required a more accurate consideration of what one should understand as the mapping of the transformations. For instance, in the case of the mapping of transformation (4.12), in the form of Eq. (4.9), the additional term

\[
\alpha \frac{e g}{8} Q^a F^{a \nu \lambda} (\bar{\theta} \gamma_\mu \sigma_{\nu \lambda} \theta),
\] (4.13)

which is not zero even for the Majorana spinors, appears. However, if on the left hand side of (4.9) one uses a more accurate expression

\[
\frac{1}{2} i \lambda \hbar^{1/2} \left[ (\bar{\theta} \gamma_\mu \gamma^5 \delta \psi) - (\delta \bar{\psi} \gamma_\mu \gamma^5 \theta) \right],
\]

then, instead of (4.13), we will already have

\[
\alpha \frac{e g}{16} Q^a F^{a \nu \lambda} (\bar{\theta} [\gamma_\mu, \sigma_{\nu \lambda}] \theta).
\]
Taking into account that by virtue of formulae (B.2) the following relation holds
\[ [\gamma_\mu, \sigma_{\nu\lambda}] = \frac{2}{i} (g_{\mu\nu} \gamma_\lambda - g_{\mu\lambda} \gamma_\nu), \]
and for the Majorana anticommuting spinors the equality \((\bar{\theta} \gamma_\mu \theta) = 0\) is true, we obtain that really under the mapping of transformation (4.12) the last term in (4.12) does not give a contribution and ipso facto we return again to the expression (4.10).

5 Solving the algebraic equations. The commutative case

In our considerations so far we have dealt only with mapping bilinear combinations in the form \(\bar{\psi} \hat{O} \psi\), where \(\hat{O}\) is a certain (differential) operator or matrix, to the quadratic combinations \(\xi_\mu \xi_\nu; \xi_\mu \xi^\nu\) etc. In our paper [1] the simplest model classical Lagrangian of the interaction of a color spinning particle with an external non-Abelian fermion field \(\Psi_{\alpha}(x)\), has been suggested. This Lagrangian has the form
\[
L_{\Psi} = -\frac{eg}{\hbar} \sqrt{\frac{m}{2}} \left\{ \theta^i (\bar{\psi}_\alpha \Psi_{\alpha}(x)) + (\bar{\Psi}_{\alpha}(x) \psi_\alpha) \theta^i \right\} + \sqrt{\frac{C_F}{2T_F}} Q^a \left\{ \theta^i (t^a)^i_j (\bar{\psi}_\alpha \Psi_{\alpha}(x)) + (\bar{\Psi}_{\alpha}(x) \psi_\alpha) (t^a)^i_j \theta^j \right\}.
\]

Here, in contrast to [1], we have separated out in an explicit form the dependence of \(\hbar\), introduced the dimensional factor \(m^{1/2}\) and dimensionless nilpotent one \((\bar{\theta} \theta)\). It is easy to see that the Lagrangian has the proper dimension for the canonical dimension of external fermion field (in units \(c = 1\))
\[ [\Psi_{\alpha}(x)] \sim \frac{1}{[\text{time}]^{3/2}}. \]

The commuting spinor \(\psi_\alpha(\tau)\) enters linearly into the expression (5.1). This spinor is connected with a set of the anticommuting tensor quantities \((S, V_\mu, T_{\mu\nu}, \ldots)\) by means of the general relation (2.1), if we preliminarily contract the latter with the auxiliary spinor \(\theta_\beta\)
\[
\hbar^{1/2}(\bar{\theta} \theta) \psi_\alpha = \frac{1}{4} \left\{ -iS \theta_\alpha + V_\mu (\gamma^\mu \theta)_\alpha - \frac{i}{2} T_{\mu\nu} (\sigma^{\mu\nu} \gamma_5 \theta)_\alpha + iA_\mu (\gamma^\mu \gamma_5 \theta)_\alpha + P (\gamma_5 \theta)_\alpha \right\}.
\]

Such an approach of recovering a spinor (by means of an arbitrary constant auxiliary spinor) was considered in the papers by Zhelnorovich [54] and Crawford [52] (see, also Klauder [57]) in the commutative case. If we substitute the representation (5.2) into the Lagrangian (5.1), then in contrast to the mapping of the above-mentioned bilinear combinations, the auxiliary spinor \(\theta_\alpha\) will already enter explicitly into the Lagrangian \(L_{\Psi}\) as an independent entity. Here, a difficult and subtle question of the dependence of the mapped Lagrangian (5.1) on a concrete choice of \(\theta_\alpha\) arises. Whether it is possible to give a physical meaning of the auxiliary spinor. A similar problem was discussed in the paper [52].

Moreover, in the general expansion (5.2) not all of the functions \((S, V_\mu, T_{\mu\nu}, \ldots)\) are independent by virtue of the relations (C.1) – (C.15). Although we have already used some of these relations in the analysis of mapping the bilinear combinations, however, we have not explicitly

\footnote{In [1] the commutative spinor \(\psi_\alpha\) was considered as a dimensional quantity with the dimension \([\text{mass}]^{1/2}\). In the present work the \(\psi_\alpha\) spinor is dimensionless.}
resolved them. Solutions of these equations would define an explicit connection between the quantities in tensor set \((S, V_{\mu}, *T_{\mu\nu}, \ldots)\) and ipso facto would reduce the number of quantities in (5.2). This and two subsequent sections are concerned with the problem.

It is rather useful at the beginning to deal with the case of commuting tensor variables, as was first considered in the papers by Takahashi et. al. [49, 50]. In these papers three independent equations of a complete system were analyzed. In our case these equations follow from (C.14), (C.4), (C.6) and (C.13)

\[
T^{\mu\nu}V_{\nu} = PA^\mu, \tag{5.3}
\]

\[
*T^{\mu\nu}V_{\nu} = SA^\mu, \tag{5.4}
\]

\[
V_{\mu}V^{\mu} = A_{\mu}A^\mu = -(S^2 + P^2). \tag{5.5}
\]

Assuming \(S \neq 0\), we obtain the pseudovector \(A^\mu\) from equation (5.4) and substitute it into (5.3)

\[
(ST^{\mu\nu} - P*T^{\mu\nu})V_{\nu} = 0. \tag{5.3}
\]

Next we consider a particular representation of the tensor \(T_{\mu\nu}\) and of its dual one in terms of two independent functions \(S\) and \(P\)

\[
T^{\mu\nu} = \omega^{\mu\nu}S + *\omega^{\mu\nu}P, \tag{5.6}
\]

\[
*T^{\mu\nu} = *\omega^{\mu\nu}S - \omega^{\mu\nu}P,
\]

where \(\omega^{\mu\nu}\) is a certain antisymmetric tensor, an explicit form of which will be written just below. Substituting these expressions into the equation above, we result in homogeneous algebraic equation for the vector variable \(V^\mu\)

\[
\omega^{\mu\nu}V_{\nu} = 0.
\]

A condition for the existence of a nontrivial solution (\(\det \omega^{\mu\nu} = 0\)) places a restriction on the tensor \(\omega_{\mu\nu}\)

\[
\omega_{\mu\nu} *\omega^{\mu\nu} = 0. \tag{5.7}
\]

One more restriction on the \(\omega_{\mu\nu}\) tensor can be obtained if we consider the pseudoscalar equation (C.5). In the commutative case it takes the form

\[
SP = \frac{1}{4} T_{\mu\nu} *T^{\mu\nu}. \tag{5.8}
\]

Substituting the expressions (5.6) into the above equation, we result in the relation

\[
SP = \frac{1}{4} \omega_{\mu\nu} *\omega^{\mu\nu} (S^2 - P^2) - \frac{1}{2} \omega_{\mu\nu}\omega^{\mu\nu}SP.
\]

To convert this relation into the identity, it is necessary to add to the condition (5.7) another condition

\[
\omega_{\mu\nu}\omega^{\mu\nu} = -2. \tag{5.9}
\]

In the papers [49, 50] the parametrical solution of algebraic system (5.3) – (5.5) was given in terms of the Euler angles \(\alpha, \beta, \gamma\) and pseudoangles \(\chi_i, i = 1, 2, 3\) for a Lorentz boost. Following the papers [49, 50], if we express the original commutative spinor \(\psi_\alpha\) in terms of the same

\[^8\]Our variables in the commutative case are connected with analogous those in the paper [50] by the relations

\[
S \rightarrow iJ, \ P \rightarrow -iJ_5, \ V_{\mu} \rightarrow J_{\mu}, \ A_{\mu} \rightarrow -iJ_{5\mu}, \ *T_{\mu\nu} \rightarrow *J_{\mu\nu}, \ T_{\mu\nu} \rightarrow J_{\mu\nu}.
\]

In addition, our definitions of the matrix \(\sigma^{\mu\nu}\) and antisymmetric tensor \(\epsilon^{\mu\nu\lambda\sigma}\) differ from those of [50] by signs.
parameters \((\alpha, \beta, \gamma, \chi_i)\), then we ipso facto specify a parametrical connection between the \(\psi_\alpha\) and tensor set \((S, V_\mu, *T_{\mu\nu}, A_\mu, P)\) (see, Crawford \[52\]). In the case of \((5.2)\) at the cost of introducing the auxiliary spinor \(\theta_\alpha\) such a connection is more direct.

A quite compact and explicit representation of solutions for the system \((5.3) - (5.5)\) is obtained by using so-called tetrad \((h^{(s)}_\mu)\), \(s = 0, 1, 2, 3,\) i.e. a set of four linear independent orthogonal unit 4-vectors \(h^{(s)}_\mu\) (numbered by the index \(s\)) subject to the relations
\[
\begin{align*}
  h^{(s)}_\mu h^{(s')}_\mu &= g^{ss'} = \text{diag}(1, -1, -1, -1), \\
  h^{(s)}_\mu h^{(s')}_{\nu} g_{ss'} &= g_{\mu\nu} = \text{diag}(1, -1, -1, -1).
\end{align*}
\]
(5.10)

An explicit form of the tetrad vectors is given in Appendix E. These vectors (more exactly, different approach to the construction of an explicit form of real and mutually orthogonal tetrad can be found in [59]).

In the paper by Takabayasi \[59\] a different approach to the construction of an explicit form of real and mutually orthogonal tetrad \(h^{(s)}_\mu\) was given. For constructing the tetrad in \[59\] an arbitrary commuting Dirac spinor \(\varphi\) was used. In terms of the spinor four mutually orthogonal real 4-vectors are defined as follows:
\[
\begin{align*}
  s_\mu &\equiv \bar{\varphi}\gamma_\mu\varphi, & a_\mu &\equiv \bar{\varphi}\gamma_\mu\gamma_5\varphi, \\
  a^{(1)}_\mu &\equiv \frac{1}{2}(\bar{\varphi}\gamma_\mu\varphi - \bar{\varphi}\gamma_\mu\varphi^c), & a^{(2)}_\mu &\equiv \frac{1}{2}(\bar{\varphi}\gamma_\mu\varphi + \bar{\varphi}\gamma_\mu\varphi^c),
\end{align*}
\]
where \(\varphi^c\) denotes the charge conjugated spinor: \(\varphi^c = C\bar{\varphi}^T\). The required tetrad \(h^{(s)}_\mu\) is connected with these vectors by the simple relations
\[
\begin{align*}
  \varrho h^{(0)}_\mu &= s_\mu, & \varrho h^{(1,2)}_\mu &= a^{(1,2)}_\mu, & \varrho h^{(3)}_\mu &= a_\mu,
\end{align*}
\]
(5.13)
\[
\begin{align*}
  \varrho &\equiv (|\varphi|^2 + (\varphi_5\bar{\varphi})^2)^{1/2}.
\end{align*}
\]

Finally, we will also discuss another alternative presentation of tetrad \(h^{(s)}_\mu\) in Section \[11]
6 Solving the algebraic equations. The anticommutative case

Now we turn our attention to solving the algebraic equations for the case of anticommutative quantities \((S, V^\mu, T^{\mu\nu}, \ldots)\). By virtue of nilpotency of the quantities, equation (5.5) vanishes. The following equations

\[
T^{\mu\nu}V^\nu = -PA^\mu + 2SV^\mu, \\
* T^{\mu\nu}V^\nu = -SA^\mu - 2PV^\mu
\]  

(6.1)

are analogous to equations (5.3) and (5.4), correspondingly. The first of them follows from (C.14), and the second does from (C.4). The right-hand side of these equations has a somewhat more complicated structure in comparison with the commutative case (5.3), (5.4) and one would expect that it will be reflected in the structure of their solutions.

Let us present the anticommuting tensor quantity \(T^{\mu\nu}\) (and its dual one) in the form similar to (5.6)

\[
T^{\mu\nu} = \omega^{\mu\nu}S + *\omega^{\mu\nu}P, \\
* T^{\mu\nu} = *\omega^{\mu\nu}S - \omega^{\mu\nu}P
\]

(6.2)

where the commutative antisymmetric tensor \(\omega^{\mu\nu}\) and its dual one are

\[
\omega^{\mu\nu} = -\epsilon^{\mu\nu\lambda\sigma}h^{(1)}_{\lambda}h^{(2)}_{\sigma}, \\
*\omega^{\mu\nu} = h^{(1)}_{\mu}h^{(2)}_{\nu} - h^{(1)}_{\nu}h^{(2)}_{\mu}.
\]

(6.3)

Recall that in (6.2) the functions \(P\) and \(S\) are now nilpotent. It is not difficult to see by using the first property in the definition of tetrad (5.10) that in the case under consideration the following functions are the solution of system (6.1)

\[
A^\mu = Ph^{(1)}_{\mu} + Sh^{(2)}_{\mu}, \\
V^\mu = Sh^{(1)}_{\mu} - Ph^{(2)}_{\mu}.
\]

(6.4)

One can consider an alternative system of equations, “dual” to (6.1), also having one external vector index,

\[
T^{\mu\nu}A^\nu = PV^\mu + 2SA^\mu, \\
* T^{\mu\nu}A^\nu = SV^\mu - 2PA^\mu
\]

(6.5)

(the first equation follows from (C.9), and the second one does from (C.2)) and verify that the solution (6.2) – (6.4) also satisfies this system.

In Section 2 we have obtained three independent equations with two external vector indices (2.12) – (2.14). By direct substitution of (6.2) – (6.4) into (2.12) we get the identity

\[-*\omega^{\mu\nu}SP = -*\omega^{\mu\nu}SP.

Because of the structure of solution (6.2) – (6.4) the following equalities hold

\[V^\mu V^\nu = A^\mu A^\nu, \quad P * T^{\mu\nu} = -ST^{\mu\nu}.
\]

(6.6)

Further, due to the last equality in (6.6) the left-hand side of equation (2.13) is equal to zero and, in its turn, on the right-hand side we have

\[* T^{\mu\lambda} * T^{\lambda\nu} = (\omega^{\mu\lambda} * \omega^{\nu}_{\lambda} - *\omega^{\mu\lambda} \omega^{\nu}_{\lambda})SP.
\]
The right-hand side vanishes by virtue of the identity
\[ \ast \omega^{\mu \lambda} \omega_\lambda^\nu = \omega^{\mu \lambda} \ast \omega_\lambda^\nu = -\frac{1}{4} (\omega^{\lambda \sigma} \ast \omega_{\lambda \sigma}) g^\mu^\nu, \]
which holds for any antisymmetric tensor of second rank.

Finally, the right-hand side of equation (2.4) vanishes by virtue of the first equality in (6.6), and the left-hand side
\[ \varepsilon^{\mu \nu \lambda \sigma} V_\lambda A_\sigma = (S^2 + P^2) \varepsilon^{\mu \nu \lambda \sigma} \ast \omega_{\lambda \sigma} + S P \varepsilon^{\mu \nu \lambda \sigma} (h_\lambda^{(1)} h_\sigma^{(1)} + h_\lambda^{(2)} h_\sigma^{(2)}) \]
equals zero by nilpotency of $S$ and $P$, and by the antisymmetry of $\varepsilon^{\mu \nu \lambda \sigma}$.

In addition to equations of the vector and tensor type let us consider equation of the (pseudo)scalar type (C.5). In the anticommutative case it takes the form
\[ S P = -\frac{1}{2} V_\mu A^\mu. \] 
(6.7)

Note that in the commutative case we have the condition $V_\mu A^\mu = 0$, as it follows from (C.8). It is easy to see that the solution (6.4) satisfies equation (6.7).

On examination of mapping the kinetic term in Section 3, we derived a certain algebraic relations between the basic quantities $(S, V_\mu, \ast T_{\mu \nu}, \ldots)$ and their derivatives. In particular, we obtain two independent relations in the form
\[ 2(S \dot{S} + P \dot{P}) = V_\mu \dot{A}^\mu + A_\mu \dot{V}^\mu, \]
(6.8)
\[ 3(S \dot{S} - P \dot{P}) = -\frac{1}{2} \ast T_{\mu \nu} \ast \dot{T}^\mu^\nu. \] 
(6.9)

The question may now be raised whether the parametric solution (6.2) - (6.4) identically satisfies equations (6.8) and (6.9) or additional restrictions can be arisen. Before we turn to this problem we shall supplement a system of the “scalar” equations (6.8) and (6.9) by two other independent equations of the “pseudoscalar” type. For this purpose we make use of equation (C.5) from the complete system in Appendix C. By means of the rule described in Section 3, from this equation one can define two required equations of the pseudoscalar type
\[ \dot{S} P = \frac{1}{4} (S \dot{P} + P \dot{S}) - \frac{1}{4} (V_\mu \dot{A}^\mu - A_\mu \dot{V}^\mu) + \frac{1}{8} T_{\mu \nu} \ast \dot{T}^\mu^\nu, \]
\[ \dot{P} S = \frac{1}{4} (S \dot{P} + P \dot{S}) + \frac{1}{4} (V_\mu \dot{A}^\mu - A_\mu \dot{V}^\mu) + \frac{1}{8} T_{\mu \nu} \ast \dot{T}^\mu^\nu. \]

By adding and subtracting these two equations we obtain the desired system
\[ 2(P \dot{S} - S \dot{P}) = V_\mu \dot{A}^\mu - A_\mu \dot{V}^\mu, \]
(6.10)
\[ 3(P \dot{S} + S \dot{P}) = -\frac{1}{2} T_{\mu \nu} \ast \dot{T}^\mu^\nu \] 
(6.11)
in addition to equations (6.8) and (6.9).

Let us first look at equation (6.10). Considering that the 4-vectors $h_\mu^{(1)}$ and $h_\mu^{(2)}$ in the general case are functions of the evolution parameter $\tau$ and using the presentation (6.4), we obtain
\[ V_\mu \dot{A}^\mu = (P \dot{S} - S \dot{P}) + S P (h_\mu^{(1)} h_\mu^{(1)} + h_\mu^{(2)} h_\mu^{(2)}). \]

Here, the last term vanishes by virtue of the normalization $(h^{(1)})^2 = (h^{(2)})^2 = -1$. A similar expression with the opposite sign holds for $A_\mu \dot{V}^\mu$, i.e.
\[ A_\mu \dot{V}^\mu = -(P \dot{S} - S \dot{P}). \]
Thus, equation (6.10) goes over into the identity.

Further, we consider equation (6.8). For the presentation (6.4) we have

\[ V_\mu \dot{V}^\mu = A_\mu \dot{A}^\mu = - (S \dot{S} + P \dot{P}) - SP(\dot{h}_\mu^{(1)} h^{(2)\mu} - \dot{h}_\mu^{(1)} h^{(2)\mu}). \]  \hspace{1cm} (6.12)

Here, however, the second term containing the derivatives of \( h^{(1,2)}_\mu \) in the general case does not vanish (the orthogonality condition \( h^{(1)}_\mu h^{(2)\mu} = 0 \) involves the equality \( h^{(1)}_\mu \dot{h}^{(2)\mu} + \dot{h}^{(1)}_\mu h^{(2)\mu} = 0 \)). By this means equation (6.8) after substituting (6.12) and collecting similar terms takes the form of nontrivial relation between the derivatives of commutative and anticommutative quantities

\[ 2(S \dot{S} + P \dot{P}) = - SP(\dot{h}_\mu^{(1)} h^{(2)\mu} - \dot{h}_\mu^{(1)} h^{(2)\mu}). \]  \hspace{1cm} (6.13)

Finally, let us consider two remaining equations (6.9) and (6.11). Substituting the explicit form of tensor \( T^{\mu\nu} \) (and its dual one \( *T^{\mu\nu} \)), Eq. (6.2), into the equations and collecting similar terms, we lead to

\[ \begin{cases} \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} - 3 \{ S \dot{S} - P \dot{P} \} = - \frac{1}{2} \omega_{\mu\nu} * \omega^{\mu\nu} (P \dot{S} + S \dot{P}), \\ \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} - 3 \{ P \dot{S} + S \dot{P} \} = + \frac{1}{2} \omega_{\mu\nu} * \omega^{\mu\nu} (S \dot{S} - P \dot{P}). \end{cases} \]

This system can be presented in a more visual matrix form

\[ \begin{pmatrix} \frac{1}{2} x - 3 & - \frac{1}{2} y \\ - \frac{1}{2} y & \frac{1}{2} x - 3 \end{pmatrix} \begin{pmatrix} S \dot{S} - P \dot{P} \\ P \dot{S} + S \dot{P} \end{pmatrix} = 0, \]  \hspace{1cm} (6.14)

where we have introduced the following notations

\[ x \equiv \omega_{\mu\nu} \omega^{\mu\nu}, \quad y \equiv \omega_{\mu\nu} * \omega^{\mu\nu}. \]  \hspace{1cm} (6.15)

Equation (6.14) has a nontrivial solution if the corresponding matrix on the left-hand side is singular. This results in the relation

\[ \left( \frac{1}{2} x - 3 \right)^2 + \frac{1}{2} y^2 = 0. \]

We consider that the antisymmetric tensor \( \omega_{\mu\nu} \) is real, therefore the relation above leads to the conditions

\[ x = 6, \quad y = 0. \]

For given representation of the tensor \( \omega_{\mu\nu} \), Eq. (6.3), the second condition holds, but the first one does not. For the case of (6.3), we have

\[ x = -2. \]

For obtaining the value \( x = 6 \) we need to introduce a factor

\[ \pm i \sqrt{3}, \]  \hspace{1cm} (6.16)

into the matrix \( \omega_{\mu\nu} \). This violates the requirement of its reality and ipso facto of the reality of the tensor quantities \( (S, V_{\mu}, *T_{\mu\nu}, \ldots) \) under conditions when the spinors \( \psi_\alpha \) and \( \theta_\alpha \) are the Majorana ones. Another way of solving this problem is to take into account only the trivial solution of system (6.14), i.e. to consider that

\[ S \dot{S} - P \dot{P} = 0, \]  \hspace{1cm} (6.17)

\[ P \dot{S} + S \dot{P} = 0. \]  \hspace{1cm} (6.18)
The first equation can be used, for example, in the relation (6.13) for its simplification. Furthermore, if we return to consideration of the right-hand side of the kinetic term (3.11), then the last two terms there, in view of (6.17), take the form

\[-\frac{i}{2} P\dot{P} + \frac{3i}{2} S\dot{S} = iP\dot{P},\]

i.e., the kinetic term with the pseudoscalar variable \( P \) itself changes sign. Thus, for its comparison with the kinetic terms in (A.2) and (A.3) there is no need to use the constraint (A.13).

### 7 Introducing complex tensor quantities

In this section we would like to consider the problem of deriving the solution of a system of algebraic equations by a slightly different way. The approach is based on introducing complex tensor variables instead of real ones. This will enable us to better understand the structure of the equations under consideration and also to clear up the question of whether the solution obtained in the previous section, is unique.

At first we concentrate on a system of equations of the vector type (i.e. having one external vector index), namely on the systems (6.1) and (6.5). Let us rewrite these equations, by analogy with the system (2.21), in a more suitable form

\[ 3(SV^\mu - PA^\mu) = -V^\nu T^{\mu\nu} - A^\nu T^{\mu\nu}, \quad (7.1) \]
\[ 3(SA^\mu + PV^\mu) = -A^\nu T^{\mu\nu} - V^\nu T^{\mu\nu}, \quad (7.2) \]
\[ SV^\mu + PA^\mu = -V^\nu T^{\mu\nu} + A^\nu T^{\mu\nu}, \quad (7.3) \]
\[ SA^\mu - PV^\mu = -A^\nu T^{\mu\nu} - V^\nu T^{\mu\nu}. \quad (7.4) \]

For the sake of convenience of future references we give here once more three independent equations of the tensor type listed in Section 2

\[ P^* T^{\mu\nu} - ST^{\mu\nu} = V^\mu V^\nu + A^\mu A^\nu, \quad (7.5) \]
\[ P^* T^{\mu\nu} + ST^{\mu\nu} = -T^\lambda T^{\nu\lambda}, \quad (7.6) \]
\[ -\epsilon^{\mu\nu\lambda\sigma} V^\lambda A^\sigma = V^\mu V^\nu - A^\mu A^\nu, \quad (7.7) \]

and equation of the pseudoscalar type

\[ SP = -\frac{1}{2} V^\mu A^\mu. \quad (7.8) \]

Our first step is to consider equations (7.1) and (7.2). Multiplying the second equation by \( 1/i \), subtracting and adding it with the first one, we will have

\[ 3(S + iP)(V^\mu + iA^\mu) = -2P^+ T^{\mu\nu\lambda\sigma}(A^\nu T^{\lambda\sigma} + V^\nu T^{\lambda\sigma}), \]
\[ 3(S - iP)(V^\mu - iA^\mu) = +2P^- T^{\mu\nu\lambda\sigma}(A^\nu T^{\lambda\sigma} + V^\nu T^{\lambda\sigma}). \quad (7.9) \]

Here, we have introduced into consideration operators

\[ P^\pm T^{\mu\nu\lambda\sigma} = \frac{1}{2} \left[ \frac{i}{2} \left( g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\rho\nu} \right) \pm \frac{1}{2i} \epsilon^{\mu\nu\lambda\sigma} \right], \]

which project any two-form onto its self-dual and anti-self-dual two-forms [60]. By virtue of the specified representation of the tensor quantities \( T^{\mu\nu} \) and \( *T^{\mu\nu} \), Eq. (5.2), and the definition
of the projectors \( P^{\pm \mu \nu \lambda \sigma} \), we have
\[
P^{\pm \mu \nu \lambda \sigma} T_{\lambda \sigma} = -\omega^{(\pm)\mu \nu} P + *\omega^{(\pm)\mu \nu} S = \pm i(S + iP)\omega^{(\pm)\mu \nu},
\]
(7.10)

etc. Here,
\[
\omega^{(\pm)\mu \nu} \equiv P^{\pm \mu \nu \lambda \sigma} \omega_{\lambda \sigma} = \frac{1}{2} \left[ \omega^{\mu \nu} \mp i \star \omega^{\mu \nu} \right]
\]
(7.11)

and we have taken into account that by virtue of the definition of (anti-)self-dual two-forms the following equalities hold
\[
*\omega^{(+)\mu \nu} = i\omega^{(+)\mu \nu}, \quad *\omega^{(-)\mu \nu} = -i\omega^{(-)\mu \nu}.
\]

From equations (7.9) and projections (7.10) we see that at this stage it is very natural to introduce into consideration the complex variables
\[
Z^{\mu} \equiv V^{\mu} + iA^{\mu}, \quad \bar{Z}^{\mu} = V^{\mu} - iA^{\mu},
\]
\[
D \equiv S + iP, \quad \bar{D} = S - iP.
\]
(7.12)

In terms of these quantities Eqs. (7.9) take a very compact form
\[
3DZ^{\nu} = 2\omega^{(+)}_{\mu \nu} DZ^{\nu},
\]
\[
3\bar{D}\bar{Z}^{\nu} = 2\omega^{(-)}_{\mu \nu} \bar{D}\bar{Z}^{\nu}.
\]
(7.13)

Thus in such rewriting, these equations are complex conjugation of each other! It is possible to make a step forward in this direction. Note first that the equalities
\[
D^2 = \bar{D}^2 = 0
\]
are true by virtue of nilpotency. The function \( D \) (or \( \bar{D} \)) can be canceled from the left- and the right-hand sides of (7.13), and thus with allowance made for the previous equalities, we obtain
\[
3Z^{\nu} = 2\omega^{(+)}_{\mu \nu} Z^{\nu} + a^{\mu} D,
\]
\[
3\bar{Z}^{\nu} = 2\omega^{(-)}_{\mu \nu} \bar{Z}^{\nu} + \bar{a}^{\mu} \bar{D},
\]
where \( a^{\mu} \) is some arbitrary (commutative) complex 4-vector.

Completely similar transformations for the remaining equations (7.3) and (7.4) lead to a system of equations
\[
\bar{D}Z^{\nu} = 2\omega^{(-)}_{\mu \nu} \bar{D}Z^{\nu},
\]
\[
D\bar{Z}^{\nu} = 2\omega^{(+)}_{\mu \nu} D\bar{Z}^{\nu}.
\]
(7.14)

Here, we can also cancel the nilpotent function \( \bar{D} \) (or \( D \)) and thereby we need to introduce into consideration another arbitrary vector \( b^{\mu} \) in addition to \( a^{\mu} \).

By analogy with (6.11) we will seek a solution of a system of equations (7.13) and (7.14) in the form of decomposition
\[
Z^{\mu} = \alpha^{\mu} D + \beta^{\mu} \bar{D},
\]
(7.15)

where \( \alpha^{\mu} \) and \( \beta^{\mu} \) are some unknown commuting (complex) vectors. Substituting (7.15) into (7.13) and (7.14), we result in homogeneous algebraic equations for the vectors \( \alpha^{\mu} \) and \( \beta^{\mu} \), respectively
\[
(g_{\mu \nu} - 2\omega^{(-)}_{\mu \nu}) \alpha^{\nu} = 0,
\]
\[
(3g_{\mu \nu} - 2\omega^{(+)}_{\mu \nu}) \beta^{\nu} = 0.
\]
(7.16)
(7.17)
The condition of existence of nontrivial solutions for the equations is singularity of the corresponding matrices

\[ \det(g_{\mu\nu} - 2\omega^{(-)\mu\nu}) = 0, \]  
(7.18)

\[ \det(3g_{\mu\nu} - 2\omega^{(+\mu\nu}) = 0. \]  
(7.19)

For analysis of these determinants we make use of the following general formula \[58, 61\]:

\[ \det(\lambda g_{\mu\nu} - F_{\mu\nu}) = -\lambda^4 - \frac{1}{2}\lambda^2(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{16}(F_{\mu\nu}\ast F^{\mu\nu})^2, \]  
(7.20)

where \( \lambda \) is some eigenvalue and \( F_{\mu\nu} \) is an arbitrary antisymmetric tensor (generally speaking, it is complex). Let us consider the first condition (7.18). In this special case we have

\[ F_{\mu\nu} \equiv 2\omega^{(-)\mu\nu}. \]

By virtue of anti-self-duality of the form \( \omega^{(-)\mu\nu} \) it follows that

\[ \ast F_{\mu\nu} = -iF_{\mu\nu}. \]

In the notations (6.15) the tensor contractions on the right-hand side of (7.20) take a simple form

\[ F_{\mu\nu}F^{\mu\nu} = 2(x + iy) \equiv 2z, \]

\[ F_{\mu\nu}\ast F^{\mu\nu} = -2iz, \]

and the characteristic equation (7.18), (7.20) goes over into the equation

\[ \lambda^4 + z\lambda^2 + \frac{1}{4}z^2 = 0. \]  
(7.21)

The discriminant of this equation relative to the variable \( \lambda^2 \) equals zero. Therefore, it has two, twofold degenerate, roots which are defined by the formula

\[ \lambda^2 = -\frac{1}{2}z. \]  
(7.22)

For algebraic equation (7.16) we have the explicit value for \( \lambda \), namely

\[ \lambda = 1, \]

and thus the formula (7.22) should be considered as a condition on the variable \( z \) and finally, by virtue of the definitions (6.15) as a condition on the tensor \( \omega_{\mu\nu} \). In this specific case we obtain

\[ y (\equiv \omega_{\mu\nu}\ast\omega^{\mu\nu}) = 0, \quad x (\equiv \omega_{\mu\nu}\omega^{\mu\nu}) = -2. \]  
(7.23)

We have already faced with these restrictions in analysis of algebraic equations in the commutative case, Eqs. (5.7) and (5.9). The restrictions identically hold for a special choice of the antisymmetric tensors \( \omega_{\mu\nu} \) and \( \ast\omega_{\mu\nu} \) in the presentation by means of tetrad \( h^{(s)}_{\mu} \), Eq. (6.3).

Let us consider the second determinant (7.19). In the characteristic equation (7.20) it should be set

\[ F_{\mu\nu} \equiv 2\omega^{(+\mu\nu}, \quad \ast F^{\mu\nu} = iF^{\mu\nu}, \]

and in terms of quantities (6.15) it takes the form

\[ \lambda^4 + \bar{z}\lambda^2 + \frac{1}{4}\bar{z}^2 = 0, \quad \bar{z} = x - iy. \]

Twofold degenerate roots of this equation are defined from

\[ \lambda^2 = -\frac{1}{2}\bar{z}. \]

27
By virtue of the original equation (7.19), here it should be considered that
\[ \lambda = 3. \]

This leads in turn to the following restrictions on the tensor \( \omega_{\mu\nu} \):
\[ y = 0, \quad x = -18. \] (7.24)

The latter condition contradicts (7.23) and thereby contradicts the choice of the representation (6.3). Thus, two conditions (7.18) and (7.19) of the existence of nontrivial solution in the most general form (7.15) contradicts each other. To overcome the difficulty we must set
\[ \beta_{\mu} \equiv 0 \] (7.25)
that reduces (7.15) to the form
\[ Z_{\mu} = \alpha_{\mu} D. \]

In this case the system (7.13) is identically equal to zero and we remain only with (7.14). It is easy to verify that it is the same solution obtained in the previous section, Eq. (6.4). Really, if we recall the definition of the function \( Z_{\mu} \), Eq. (7.12), and substitute (6.4) instead of \( V_{\mu} \) and \( A_{\mu} \), then we obtain
\[ Z_{\mu} = (h^{(1)}_{\mu} + ih^{(2)}_{\mu}) D, \]
i.e.
\[ \alpha_{\mu} \equiv h^{(1)}_{\mu} + ih^{(2)}_{\mu}. \] (7.26)

By straightforward computation we can verify that the 4-vector is really nontrivial zeroth mode of equation (7.16).

### 8 Possibility of the existence of one further solution

We now consider the following question: whether it is possible to construct the second solution of algebraic system (7.1) – (7.8), if in the decomposition (7.15), instead of (7.25), we set the following condition:
\[ \alpha_{\mu} \equiv 0, \]
i.e. we seek a solution in the form
\[ Z_{\mu} = \beta_{\mu} \bar{D}. \] (8.1)

In this case the second system (7.14) identically vanishes and the 4-vector \( \beta_{\mu} \) satisfies homogeneous equation (7.17). As was shown above, this equation will have a non-trivial solution when the tensor \( \omega_{\mu\nu} \) satisfies the two conditions (7.24). As was already mentioned, the first condition \( \omega_{\mu\nu} * \omega^{\mu\nu} = 0 \) arises also in the case of commuting tensor quantities, Eq. (5.7). In regard to the second condition \( \omega_{\mu\nu} \omega^{\mu\nu} = -2 \), in the commutative case it must be held by virtue of the equation of the pseudoscalar type, Eq. (5.8). The overall coefficient in the tensor \( \omega_{\mu\nu} \) in the presentation (5.11) is chosen in such a way that the \( \omega_{\mu\nu} \) satisfies the latter condition.

It is successful that in the anticommutative case the algebraic equations\(^9\) do not impose such a severe restriction on the contraction \( \omega_{\mu\nu} \omega^{\mu\nu} \). One can minimally redefine the form of

\(^9\)Here Eq. (6.7), on the right-hand side of which the vector quantities are contracted instead of the tensor ones, is analog of Eq. (5.8).
antisymmetric tensor $\omega_{\mu\nu}$ with the aim of constructing a nontrivial solution of Eq. (7.17). At first we enter the factor 3 into the initial definition (6.3)

$$\omega_{\mu\nu} = -3\epsilon_{\mu\nu\lambda\sigma}h^{(1)}_{\lambda}h^{(2)}_{\sigma},$$

$$^{*}\omega_{\mu\nu} = 3(h^{(1)}_{\nu}h^{(2)}_{\nu} - h^{(1)}_{\mu}h^{(2)}_{\mu}).$$

Further, by analogy with (7.26) we will seek a solution of Eq. (7.17) in the following form

$$\beta_{\mu} = C(h^{(1)}_{\mu} + ih^{(2)}_{\mu}),$$

where $C$ is an arbitrary constant. Let us substitute (8.3) into equation (7.17). Taking into account the new definition (8.2), we have

$$(3g^{\mu\nu} - 2\omega^{(+)}_{\mu\nu})\beta_{\nu} = (3g^{\mu\nu} - [\omega^{\mu\nu} - i{^{*}\omega^{\mu\nu}}])C(h^{(1)}_{\nu} + ih^{(2)}_{\nu}) = 3C(h^{(1)}_{\mu} + ih^{(2)}_{\mu}) - 3C((h^{(2)})^2h^{(1)}_{\mu} + i(h^{(1)})^2h^{(2)}_{\mu}).$$

We see that the right-hand side reduces to zero when the 4-vectors $h^{(1)}_{\mu}$ and $h^{(2)}_{\mu}$ satisfy the different normalization conditions

$$(h^{(1)})^2 = (h^{(2)})^2 = 1,$$  \hfill (8.4)

i.e. these vectors must be time-like in contrast to the original ones (5.10). By using the solution (8.1), (8.3) it is not difficult to recover an explicit form of the real vector quantities $V_{\mu}$ and $A_{\mu}$:

$$V_{\mu} = C(Sh^{(1)}_{\mu} + Ph^{(2)}_{\mu}),$$

$$A_{\mu} = C(-Ph^{(1)}_{\mu} + Sh^{(2)}_{\mu}).$$

By direct substitution of these expressions into a system of the “vector” equations (7.1) – (7.4) with the use of (6.2) for the anticommuting tensor $T_{\mu\nu}$ (where as $\omega^{\mu\nu}$ one means (8.2) with the normalization (8.4)) we verify that this system holds (for arbitrary $C$), as it should be. Further, in equations of the “tensor” type (7.5) – (7.7) two last equations hold also for arbitrary $C$. The pseudoscalar equation (7.8) will be true only if we set

$$C^2 = 1.$$  \hfill (8.6)

It only remains to consider Eq. (7.5). We write out it once more

$$P^{*}T^{\mu\nu} - ST^{\mu\nu} = V^{\mu}V^{\nu} + A^{\mu}A^{\nu}.$$  \hfill (8.7)

Upon substitution of the functions (6.2) with (8.2) into the left-hand side we will have

$$P^{*}T^{\mu\nu} - ST^{\mu\nu} = -2{^{*}\omega^{\mu\nu}}SP,$$

while on the right-hand side we have

$$V^{\mu}V^{\nu} + A^{\mu}A^{\nu} = \frac{2}{3}C^2{^{*}\omega^{\mu\nu}}SP.$$  

The condition for the fulfilment of (8.7) demands that

$$C^2 = -3, \quad \text{or} \quad C = \pm i\sqrt{3}.$$  \hfill (8.8)

This contradicts (8.6) and, in addition, the vector variables (8.5) become purely imaginary. It is interesting to note that the same factor (8.8) has already arisen in Section 6, Eq. (6.16). Thus, the algebraic equation of the “tensor” type (7.5) is the only one of the total system, which does not give the possibility for the construction of a further independent solution. The reason of
such preferability of the first solution (6.2) – (6.4) is unclear for us.

We note also that another problem here, is the determination of an explicit form of the 4-vectors $h^{(1)}_\mu$ and $h^{(2)}_\mu$. It is easy to construct vectors satisfying (8.4) from the analogous space-like vectors (E.1) and (E.2). For this it is sufficient to replace $S_1 \equiv C_1$ in (E.1), and $S_2 \equiv C_2$ in (E.2), i.e. instead of (E.1) and (E.2) to take

$$h^{(1)}_\mu = (C_1, -S_1 a^{(1)}),$$
$$h^{(2)}_\mu = (C_1 C_2, -S_1 C_2 a^{(1)} - S_2 a^{(2)}).$$

However, such simple approach leads to the violation of the orthogonality condition

$$h^{(1)}_\mu h^{(2)\mu} = C_2 \neq 0,$$

and here, most likely a more subtle consideration is necessary.

9 Mapping into Lagrangian with a local supersymmetry

In Sections 2, 3 and 4 it was considered the mapping of the original Lagrangian (1.5) possesses (local) bosonic invariance into the Lagrangian obtained after the elimination of the variable $\chi$ from the Lagrangian (A.1). The terms containing the fermion counterpart $\chi$ to the einbein field $e$, namely

$$\frac{i}{2e} \chi \hat{x}_\mu \xi^\mu, \quad \frac{im}{2} \chi \xi_5 \quad (9.1)$$

cannot appear in principle under any map because there are no counterparts in the Lagrangian (1.4). These terms are important for local supersymmetry of the Lagrangian (A.1), and its counterparts a priori must be contained in the initial Lagrangian (1.4). In this section we would like to show how terms of this kind may really appear.

The basic idea in determining such terms is that of using an extended Hamiltonian or superHamiltonian for the construction of the “spinning” equation (1.1). Hamiltonian of this type have been considered in a few papers for various reasons. Thus in the papers by Borisov, Kulish [35] and Fradkin, Gitman [62] it has been used in the construction of the Green’s function of a Dirac particle in background non-Abelian gauge field. Within the framework of operator formalism this superHamiltonian has the form

$$-2m \hat{H}_{SUSY} = \left( \hat{D}_\mu \hat{D}^\mu + \frac{1}{2} g \hat{\sigma}_{\mu\nu} F^{a\mu\nu} \hat{T}^a - m^2 \right) + i\chi (\hat{\gamma}_\mu \hat{D}^\mu + m). \quad (9.2)$$

All quantities with hats above represent operators acting in appropriate spaces of representations of the spinor, color and coordinate algebras; $\chi$ is an odd variable. Analog of introducing such a superHamiltonian in the massless limit can be also found in the work of Friedan and Windey [63]. The superHamiltonian was used in the construction of the Green’s function of a Dirac particle in background non-Abelian gauge field. The latter has been used in calculating the chiral anomaly. In the monograph by Thaller [64] within the supersymmetric quantum mechanics a notion of the Dirac operator with supersymmetry has been defined in the most general abstract form. The expression (9.2) is just its special case.

Before studying the general case of the Dirac operator with supersymmetry it is necessary to recall briefly the fundamental points of deriving the equation of motion for the commuting spinor $\psi_\alpha$, Eq. (1.1). The equation arises when we analyze the connection of the relativistic quantum mechanics with the relativistic classical mechanics first performed by Pauli [7] within
the so-called first-order formalism for fermions (see, also [10]). In the paper by Fock [8] and in the book by Akhiezer and Beresteskii [9] this analysis was performed on the basis of the second-order formalism. Here, we will follow the latter line.

In the second-order formalism the original QCD Dirac equation for a wave function \( \Psi \) is replaced by its quadratic form

\[
-2m \hat{H} \Phi = \left( \cancel{D_\mu D^\mu} + \frac{1}{2} g \sigma_{\mu\nu} F^{\mu\nu} - m^2 \right) \Phi = 0,
\]

where a new spinor \( \Phi \) is connected with the initial one by the relation

\[
\Psi = \frac{1}{m} (\gamma_\mu D^\mu + m) \Phi.
\]

In this section we restore Planck’s constant \( \hbar \) in all formulae. Since we are interested in the interaction of the spin degrees of freedom of a particle with an external gauge field most, then for the sake of simplicity we will consider equation (9.3) for the case of the interaction with an Abelian background field (with the replacement of the coupling \( g \) by \( q \)). The presence of the color degree of freedom can result in qualitatively new features, one of them is appearing a mixed spin-color degrees of freedom [24]. In this respect our original model Lagrangian (1.4) is the simplified one. It corresponds to perfect factorization of the spin and color degrees of freedom of the particle. The non-Abelian case also requires appreciable complication of the usual WKB-method in the analysis of Eq. (9.3) that is beyond the scope of this work (see, for example, [25, 28, 65]).

A solution of equation (9.3) in the semiclassical limit is defined as a series in powers of \( \hbar \)

\[
\Phi = e^{iS/\hbar} (f_0 + \hbar f_1 + \hbar^2 f_2 + \ldots),
\]

where \( S, f_0, f_1, \ldots \) are some functions of coordinates and of time. Substituting this series into (9.3) and collecting terms of the same power in \( \hbar \), we obtain the following equations correct to the first order in \( \hbar \)

\[
\hbar^0 : \quad \left( \frac{\partial S}{\partial x_\mu} + q A_\mu \right)^2 - m^2 = 0,
\]

\[
\hbar^1 : \quad \left[ \frac{1}{i} \frac{\partial}{\partial x_\mu} \left( \frac{\partial S}{\partial x_\mu} + q A_\mu \right) \right] f_0 + \frac{2}{i} \left( \frac{\partial S}{\partial x_\mu} + q A_\mu \right) \frac{\partial f_0}{\partial x_\mu} + \frac{q}{2} \sigma_{\mu\nu} F^{\mu\nu} f_0 = 0.
\]

Further, we introduce into consideration the flux fermion density

\[
s_\mu \equiv \bar{\Psi}_0 \gamma_\mu \Psi_0,
\]

where as \( \Psi_0 \) we take the following expression:

\[
\Psi_0 = \frac{1}{m} (\gamma_\mu D^\mu - m) f_0 e^{iS/\hbar} \approx \frac{1}{m} e^{iS/\hbar} [\pi_\mu \gamma_\mu - m] f_0,
\]

\[
\pi_\mu \equiv \frac{\partial S(x, \alpha)}{\partial x_\mu} + q A_\mu(x).
\]

Here, \( \alpha \) designates three arbitrary constants defining a solution for the action \( S \), Eq. (9.5). In terms of the spinor \( f_0 \) the flux density (9.7) takes the form

\[
s_\mu = \frac{2}{m^2} \pi_\mu \left[ \bar{f}_0 (\gamma_\nu \pi^\nu - m) f_0 \right]
\]

and, by virtue of Eqs. (9.5) and (9.6), it satisfies the equation of continuity

\[
\frac{\partial s_\mu}{\partial x_\mu} = 0.
\]
Equation (1.1) arises from an analysis of the Eq. (9.6) for the spinor $f_0$. The equation in terms of the function $\pi_\mu$ can be written in a more compact form

$$\frac{\partial \pi_\mu}{\partial x_\mu} f_0 + 2\pi_\mu \frac{\partial f_0}{\partial x_\mu} + \frac{i q}{2} \sigma_\mu F^{\mu\nu} f_0 = 0.$$  \hspace{1cm} (9.8)

At this point we introduce a new variable

$$\eta = \frac{2}{m^2} \left[ \bar{f}_0 (\gamma_\nu \pi_\nu - m) f_0 \right],$$

such that $s_\mu = \pi_\mu \eta$. Owing to the continuity equation we have an important relation for the $\eta$ function

$$\frac{\partial \pi_\mu}{\partial x_\mu} \eta = -\pi_\mu \frac{\partial \eta}{\partial x_\mu}. \hspace{1cm} (9.9)$$

At the final stage we substitute $f_0 = \sqrt{\eta} \varphi_0$ into Eq. (9.8). In terms of a new spinor function $\varphi_0$ with allowance for (9.9) this equation takes the following form

$$\pi_\mu \frac{\partial \varphi_0}{\partial x_\mu} = -\frac{i q}{4} \sigma_\mu F^{\mu\nu} \varphi_0.$$

In the book [9] a solution of the equation obtained just above, was expressed in terms of the solution of Schrodinger’s equation for the wave function $\psi_\alpha(\tau)$, Eq. (1.1). The latter describes the motion of a spin in a given gauge field $F_{\mu\nu}(x)$. This field is defined along the trajectory of the particle $x_\mu = x_\mu(\tau, \alpha, \beta)$, which in turn is defined from a solution of the equation

$$m \frac{d x_\mu}{d \tau} = \pi_\mu(x, \alpha)$$

with the initial value given by a vector $\beta$.

Let us now discuss the question of a modification of the above equations in the case when instead of the usual Hamilton operator in equation (9.3) one takes its supersymmetric extension, i.e., considers the equation in the form

$$-2m \hat{H}_{\text{SUSY}} \Phi \equiv \left\{ \left( D_\mu D^\mu + \frac{q}{2} \sigma_\mu F^{\mu\nu} - m^2 \right) + i \chi \left( \gamma_\mu \gamma_5 D^\mu + m \gamma_5 \right) \right\} \Phi = 0. \hspace{1cm} (9.10)$$

Here, following [62] in second parentheses we have introduced the $\gamma_5$ matrix into the definition of the linear Dirac operator. This operator $(\gamma_\mu \gamma_5 D^\mu + m \gamma_5)$ should be believed as an odd function. We will seek a solution of equation (9.10) also in the form of a power series (9.4) only under condition that the function $S$ is considered as an usual commuting function, whereas the spinor functions $f_0, f_1, \ldots$ should be considered as those containing both Grassmann even and odd parts. The equations (9.5) and (9.6) are modified as follows

$$h^0: \quad (\pi^2 - m^2) f_0 + i \chi \left( \pi_\mu \gamma^\mu \gamma_5 + m \gamma_5 \right) f_0 = 0,$$

$$h^1: \quad (\pi^2 - m^2) f_1 + i \chi \left( \pi_\mu \gamma^\mu \gamma_5 + m \gamma_5 \right) f_1 + \left[ \frac{1}{i} \frac{\partial \pi_\mu}{\partial x_\mu} f_0 + \frac{2}{i} \pi_\mu \frac{\partial f_0}{\partial x_\mu} + \frac{q}{2} \sigma_\mu F^{\mu\nu} f_0 \right] + \chi \gamma_5 \frac{\partial f_0}{\partial x_\mu} = 0.$$

The next step is to present the spinors $f_0$ and $f_1$ as a sum of even and odd parts

$$\begin{cases} f_0 = f_0^{(0)} + \chi f_0^{(1)}, \\ f_1 = f_1^{(0)} + \chi f_1^{(1)}. \end{cases} \hspace{1cm} (9.11)$$

In the decomposition (9.11) we believe the functions $(f_0^{(0)}, f_1^{(0)})$ to be even ones, and $(f_0^{(1)}, f_1^{(1)})$ to be odd ones. The opposite case of the partition into the Grassmann parity will be mentioned
at the end of this section. By using (9.11) the equation of zeroth order in $\hbar$ is decomposed into two equations

\[
(\pi^2 - m^2)f_0^{(0)} = 0,
\]

\[
(\pi^2 - m^2)f_0^{(1)} + i\left(\pi_\mu \gamma^\mu \gamma_5 + m\gamma_5\right)f_0^{(0)} = 0,
\]

the first of which defines the Hamilton-Jacobi equation for the action $S$, Eq. (9.5), and the second one is reduced to the matrix algebraic equation for the spinor $f_0^{(0)}$

\[
(\pi_\mu \gamma^\mu \gamma_5 + m\gamma_5)f_0^{(0)} = 0.
\]

Further, the equation of first order in $\hbar$ is also decomposed into two equations which with the use of (9.5) take the form

\[
\frac{1}{i} \left(\frac{\partial f_0^{(0)}}{\partial x_\mu}\right) + \frac{2}{i} \pi_\mu \frac{\partial f_0^{(0)}}{\partial x_\mu} + \frac{q}{2} \sigma_{\mu\nu} F^{\mu\nu} f_0^{(0)} = 0,
\]

\[
\frac{1}{i} \left(\frac{\partial f_0^{(1)}}{\partial x_\mu}\right) + \frac{2}{i} \pi_\mu \frac{\partial f_0^{(1)}}{\partial x_\mu} + \frac{q}{2} \sigma_{\mu\nu} F^{\mu\nu} f_0^{(1)} + \gamma_\mu \gamma_5 \frac{\partial f_0^{(0)}}{\partial x_\mu} = \left(\pi_\mu \gamma^\mu \gamma_5 + m\gamma_5\right)f_1^{(0)}.
\]

(9.12)

Notice that the term on the right-hand side of Eq. (9.12) represents the contribution of the quantum correction in contrast to the other terms. The first equation for the even spinor $f_0^{(0)}$ is analyzed similar to Eq. (9.8) by the replacement

\[
f_0^{(0)} = \sqrt{\eta} \varphi_0^{(0)}, \quad \eta \equiv \frac{2}{m^2} \left[\tilde{f}_0^{(0)}(\gamma_\mu \pi^\mu - m)f_0^{(0)}\right].
\]

(9.13)

For the odd spinor $f_0^{(1)}$ we define an analogous replacement by introducing a new odd spinor $\theta_0^{(1)}$ by the rule

\[
f_0^{(1)} = \sqrt{\eta} \theta_0^{(1)},
\]

(9.14)

with the same scalar function $\eta$ as it was defined in (9.13). Taking into account the continuity equation in the form (9.9) and replacement (9.14), we obtain instead of (9.12)

\[
\pi_\mu \frac{\partial \theta_0^{(1)}}{\partial x_\mu} + \frac{i q}{4} \sigma_{\mu\nu} F^{\mu\nu} \theta_0^{(1)} + i\gamma_\mu \gamma_5 \frac{1}{2\sqrt{\eta}} \frac{\partial \sqrt{\eta} \varphi_0^{(0)}}{\partial x_\mu} = \frac{1}{2} \left(\pi_\mu \gamma^\mu \gamma_5 + m\gamma_5\right) \varphi_0^{(1)},
\]

(9.15)

where on the right-hand side we have also set $f_1^{(0)} \equiv \sqrt{\eta} \varphi_0^{(0)}$. The equation obtained can be related to the equation of motion of a spin in external field in the form (1.1), but instead of the even spinor $\psi_\alpha(\tau)$, here we have the odd one $\theta_0^{(1)}(\tau)$. The latter can be identified with the auxiliary Grassmann spinor $\theta_\alpha(\tau)$ we have used throughout this work. Next, the spinor $\varphi_1^{(0)}$ on the right-hand side of (9.15) is even and it can be related to our commuting spinor $\psi_\alpha$ by setting

\[
\varphi_1^{(0)} \equiv m\psi.
\]

The expression in parentheses on the right-hand side of (9.15) should be considered as Grassmann-odd by virtue of the property of being odd of the original operator expression which correlates with it (see the text after formula (9.10)). The Grassmann-odd parity of this expression can be made explicitly if we reintroduce the Grassmann scalar $\chi$ as a multiplier. Taking into account all the above-mentioned and the relation $\dot{x}_\mu = \pi_\mu/m$, we obtain the final equation for the odd spinor $\theta_\alpha$:

\[
\frac{1}{i} \frac{d\theta}{d\tau} + \frac{q}{4m} \sigma_{\mu\nu} F^{\mu\nu} \theta + \cdots = \frac{m}{2i} \chi \dot{x}_\mu \left(\gamma^\mu \gamma_5 \psi\right) + \frac{m}{2i} \chi \left(\gamma_5 \psi\right).
\]

(9.16)
Here, the dots denotes the contribution of the last term on the left-hand side of Eq. (9.15). Its physical meaning is not clear. The terms on the right-hand side of (9.16) can be obtained by varying with respect to \( \bar{\theta} \) from the terms which must be added to the Lagrangian (1.1):

\[
L = \ldots + \left\{ \left( \frac{im}{2} \chi \bar{x}_\mu (\bar{\theta} \gamma^\mu \gamma_5 \psi) + \frac{im}{2} \chi (\bar{\theta} \gamma_5 \psi) \right) + \text{(conj. part)} \right\}
\]

(9.17)

Finally, under the mapping of the Lagrangian (1.1) into (A.1) the expressions in braces should be identified with the Grassmann pseudovector \( \xi_\mu \) and pseudoscalar \( \xi_5 \) by the rule

\[
(\bar{\theta} \theta) \xi_\mu \sim (\bar{\theta} \gamma_\mu \gamma_5 \psi) + \text{(conj. part)},
\]

\[
(\bar{\theta} \theta) \xi_5 \sim (\bar{\theta} \gamma_5 \psi) + \text{(conj. part)},
\]

and, thereby, we can obtain the missing terms (9.1) in our map. Although we have obtained here the equation of motion for the odd spinor, Eq. (9.16), a similar equation can be obtained for the even spinor \( \psi_\alpha \) by changing Grassmann parity of the spinors \( (f_0^{(0)}, f_1^{(0)}) \) and \( (f_0^{(1)}, f_1^{(1)}) \) in the decomposition (9.11) to the opposite one.

## 10 The case of Dirac spinors \( \psi_\alpha \) and \( \theta_\alpha \)

In the preceding sections we have concentrated on mapping the Lagrangian (1.1) in which the classical commuting spinor \( \psi_\alpha \) was considered as Majorana one (and correspondingly, the auxiliary spinor \( \theta_\alpha \) was also Majorana one). In this connection, it is worth noting that in the case of Majorana spinors the four-component formalism is not technically optimal. Here, it is more adequately to use the two-component Weyl formalism

\[
\psi_\text{M} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix}, \quad \theta_\text{M} = \begin{pmatrix} \theta_\alpha \\ \bar{\theta}_{\dot{\alpha}} \end{pmatrix},
\]

(10.1)

where now \( \alpha, \dot{\alpha} = 1, 2 \) are the standard \( SU(2) \) Weyl indices. In the two-component notations the mapping (5.2) can be written in the form

\[
\hbar^{1/2} (\bar{\theta}_\text{M} \psi_\text{M}) = \frac{1}{4} \left\{ -iS \theta_\alpha + V_\mu (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} + \ast T_{\mu\nu} (\sigma_{\mu\nu})_{\alpha\beta} \theta_\beta + i A_\mu (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} - P \theta_\alpha \right\}
\]

(10.2)

\[
\equiv \frac{1}{4} \left\{ -iD \theta_\alpha + Z_\mu (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} + \ast T_{\mu\nu} (\sigma_{\mu\nu})_{\alpha\beta} \theta_\beta \right\}.
\]

Here, in the notations of the textbook by Bailin and Love [66], we have

\[
\sigma^{\mu} \equiv (I, \sigma), \quad \bar{\sigma}^{\mu} \equiv (I, -\sigma), \quad \sigma^{\mu\nu} \equiv \frac{1}{4} (\sigma^{\mu} \bar{\sigma}^{\nu} - \sigma^{\nu} \bar{\sigma}^{\mu}), \quad \bar{\sigma}^{\mu\nu} \equiv \frac{1}{4} (\bar{\sigma}^{\mu} \sigma^{\nu} - \bar{\sigma}^{\nu} \sigma^{\mu}).
\]

From the last line of (10.2) we see that the complex quantities \( D \) and \( Z_\mu \) introduced in Section 7 Eq. (7.12) arise naturally within the Weyl representation.

The term defining the interaction of a spin with background gauge field in these variables takes the form

\[
\frac{i e g}{2} \hbar (\bar{\theta}_\text{M} \psi_\text{M}) Q^\mu F_{\mu\nu}^a(x) \left[ (\psi_\alpha (\sigma^{\mu\nu})_{\alpha\beta} \psi_\beta) + (\bar{\psi}^{\dot{\alpha}} (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}) \right],
\]

and the kinetic term is

\[
\frac{\hbar}{i} (\bar{\theta}_\text{M} \psi_\text{M}) \left( \psi_\alpha \frac{d\psi_\alpha}{d\tau} + \bar{\psi}_{\dot{\alpha}} \frac{d\bar{\psi}_{\dot{\alpha}}}{d\tau} \right).
\]

34
case we have a set of real tensor quantities corresponding to the
and the (one-to-one) mapping into the real tensor variables can be defined by (10.2). In this
case we have a set of real tensor quantities corresponding to the $n = 1$ local supersymmetric spinning particle.

The situation qualitatively changes in the presence of an external fermion field $\Psi_i(x)$ that
in the general case should be considered as a Dirac spinor. The simplest expression for the
interaction of a color spinning particle with the external non-Abelian fermion field has been
given in Section 5, Eq. (5.1). It is clear that such a field inevitable violates the representation
(10.1) for Majorana spinors and ipso facto it is necessary from the outset to deal with the Dirac
spinors $\psi_D$ and $\theta_D$. Attempt at constructing an mapping of the Dirac spinor $\psi_D$ results in the
complex tensor quantities $(S, V_{\mu}, *T_{\mu\nu}, A_{\mu}, P)$, i.e. the conditions (2.4) greatly facilitating
our task, will not take place any more. In the general case both the tensor quantities and
their complex conjugation enter into the mapping of blinear combinations of the type
$\bar{\psi}_D \hat{O} \psi_D$, where $\hat{O}$ is a differential operator or matrix, (see Eqs. (2.6), (2.8), (2.18), (3.2)). In spite of the
fact that a system of identities (C.1) – (C.15) has been obtained without any restrictions on the
spinors $\psi_\alpha$ and $\theta_\alpha$, it is unsuitable for answer the matter which of the terms on the right-hand
side of the mapping (2.8), (3.2) are independent.

This raises the question as to whether it is possible to construct a system of identities similar
to (C.1) – (C.15), on the left- and on right-hand sides of which the tensor quantities with their
complex conjugation enter. Here, we would like to outline on a qualitative level one possible
way of attacking this problem.

As is known, a general Dirac spinor $\psi_D$ can be always written in terms of two Majorana
ones
$$\psi_D = \psi_M^{(1)} + i \psi_M^{(2)},$$
(10.3)
where
$$\psi_M^{(1)} = \frac{1}{2} (\psi_D + \psi_D^c), \quad \psi_M^{(2)} = \frac{1}{2i} (\psi_D - \psi_D^c)$$
and $\psi_D^c$ is the charge-conjugate spinor. Such a decomposition can be performed both for the
background fermion field $\Psi_i(x)$ and for the spinors $\psi_\alpha$ and $\theta_\alpha$. The starting bilinear expressions,
for example, the spin tensor
$$\frac{1}{2} \hbar (\bar{\theta}_D \theta_D) (\bar{\psi}_D \sigma_{\mu\nu} \psi_D)$$
(10.4)
can be presented as a product of expressions of the type
$$(\bar{\theta}_D \theta_D) = \left[ (\bar{\psi}_M^{(1)} \theta_M^{(1)}) + (\bar{\psi}_M^{(2)} \theta_M^{(2)} \right] + i \left[ (\bar{\psi}_M^{(1)} \theta_M^{(2)}) - (\bar{\psi}_M^{(2)} \theta_M^{(1)}) \right] \equiv (\bar{\theta}_M^{(1)} \theta_M^{(1)}) + (\bar{\theta}_M^{(2)} \theta_M^{(2)})$$
(10.5)
and
$$(\bar{\psi}_D \sigma_{\mu\nu} \psi_D) = \left[ (\bar{\psi}_M^{(1)} \sigma_{\mu\nu} \psi_M^{(1)}) + (\bar{\psi}_M^{(2)} \sigma_{\mu\nu} \psi_M^{(2)} \right] + i \left[ (\bar{\psi}_M^{(1)} \sigma_{\mu\nu} \psi_M^{(2)}) - (\bar{\psi}_M^{(2)} \sigma_{\mu\nu} \psi_M^{(1)}) \right]$$
$$\equiv (\bar{\psi}_M^{(1)} \sigma_{\mu\nu} \psi_M^{(1)}) + (\bar{\psi}_M^{(2)} \sigma_{\mu\nu} \psi_M^{(2)}).$$
(10.6)
Here, on the most right-hand sides we have used the properties of commuting and anticommuting
Majorana spinors. By analogy with (2.1) let us consider the expansion of the spinor
structure of the mixed type
\[
\hbar^{1/2} \tilde{\theta}^{(i)j} \psi_{M\alpha}^{(i)} = \frac{1}{4} \left\{ -i S^{ij} \delta_{\alpha\beta} + V_\mu^{ij} (\gamma^\mu)_{\alpha\beta} - \frac{i}{2} T^{ij}_\mu (\sigma^{\mu\nu} \gamma_5)_{\alpha\beta} + i A^{ij}_\mu (\gamma^\mu \gamma_5)_{\alpha\beta} + P^{ij} (\gamma_5)_{\alpha\beta} \right\} \tag{10.7}
\]
and the expansion for the conjugate expression
\[
\hbar^{1/2} \tilde{\psi}^{(i)j} \theta_{M\alpha}^{(i)} = \frac{1}{4} \left\{ i S^{ij} \delta_{\alpha\beta} + V_\mu^{ij} (\gamma^\mu)_{\alpha\beta} - \frac{i}{2} T^{ij}_\mu (\sigma^{\mu\nu} \gamma_5)_{\alpha\beta} - i A^{ij}_\mu (\gamma^\mu \gamma_5)_{\alpha\beta} - P^{ij} (\gamma_5)_{\alpha\beta} \right\}, \tag{10.8}
\]
where the real anticommuting tensor variables on the right-hand side are defined as follows:
\[
S^{ij} \equiv i \hbar^{1/2} (\bar{\theta}^{(i)j} \psi_{M}^{(i)}), \quad V_\mu^{ij} \equiv \hbar^{1/2} (\bar{\theta}^{(i)j} \gamma_\mu \psi_{M}^{(i)}),
\]
\[
A^{ij}_\mu \equiv i \hbar^{1/2} (\bar{\theta}^{(i)j} \gamma_5 \psi_{M}^{(i)}), \quad P^{ij} \equiv \hbar^{1/2} (\bar{\theta}^{(i)j} \gamma_5 \psi_{M}^{(i)}),
\]
i, j = 1, 2. Multiplication of these two expansions \((10.7)\) and \((10.8)\) and the contraction of the obtained expression with \(\delta_{\beta\gamma} (\sigma_{\mu\nu})_{\gamma\alpha}\), yield the following generalization of the expression \((2.10)\):
\[
\hbar (\bar{\theta}^{(i)j} \psi_{M}^{(i)})(\bar{\psi}_{M\gamma}^{(i)}) = \frac{i}{4} \left\{ -[S^{ij} T^{kl}_\mu - T^{ij}_\mu S^{kl}] + [T^{ij}_\mu P^{kl} - P^{ij} T^{kl}_\mu] + [V_\mu^{ij} V_\nu^{kl} - V_\nu^{ij} V_\mu^{kl}] - A^{ij}_\mu A^{kl}_\nu + A^{ij}_\nu A^{kl}_\mu - \epsilon^{\mu\nu\lambda\sigma} [V_\mu^{ij} A^{kl}_\sigma + A^{ij}_\nu V_\sigma^{kl}] + g^{\lambda\sigma} [T^{ij}_\mu \lambda T^{kl}_\nu \sigma - T^{ij}_\mu \lambda T^{kl}_\nu \sigma] \right\} \tag{10.9}
\]
This formula enables us to represent the expression for the spin tensor \((10.4)\) in a rather compact and obvious form which coincides essentially with the expression \((2.10)\) with a slight modification. Actually, multiplying out \((10.5)\) and \((10.6)\), and taking into account \((10.9)\), we obtain
\[
\hbar (\bar{\theta}^{(i)j} \psi_{M}^{(i)})(\bar{\psi}_{M\gamma}^{(i)}) = \frac{i}{2} \text{tr} \left\{ -[S T^{t}_\mu + P * T^{t}_\mu] + [V_\mu V_\nu - A_\mu A_\nu] - \epsilon_{\mu\nu\lambda\sigma} V_\lambda (A^\sigma)^t + g^{\lambda\sigma} * T^{t}_\mu \lambda T^{t}_\nu \sigma \right\} \tag{10.10}
\]
Here, we have introduced into consideration the matrices
\[
S \equiv \begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}, \quad V_\mu \equiv \begin{pmatrix} V_\mu^{11} & V_\mu^{12} \\ V_\mu^{21} & V_\mu^{22} \end{pmatrix}, \quad T^{t}_\mu \equiv \begin{pmatrix} T^{t\mu}_{11} & T^{t\mu}_{12} \\ T^{t\mu}_{21} & T^{t\mu}_{22} \end{pmatrix},
\]
and so on. The symbol \(t\) denotes the transpose of a matrix and \(\text{tr}\) does the trace of the 2 \(\times\) 2 matrices (the symbol \(S_p\) is used for the trace over the spinor indices, Appendix B). In deriving \((10.10)\) we have used the property of the transpose of a product of two matrices
\[
(AB)^t = \pm B^t A^t, \tag{10.11}
\]
where the signs \(\pm\) relate to the matrices composed of commuting or anticommuting elements, correspondingly.

One can obtain a system of bilinear identities which will be a generalization of the identities \((C.1) - (C.15)\). For this purpose let us consider the product of two expansion \((10.7)\)
\[
\hbar (\bar{\theta}^{(j)j} \psi_{M\alpha}^{(j)})(\bar{\theta}^{(i)j} \psi_{M\gamma}^{(i)})(\bar{\theta}^{(i)j} \psi_{M\alpha}^{(i)}) = \frac{1}{16} \left\{ -i S^{ij} \delta_{\alpha\beta} + V_\mu^{ij} (\gamma^\mu)_{\alpha\beta} - \frac{i}{2} T^{ij}_\mu (\sigma^{\mu\nu} \gamma_5)_{\alpha\beta} + \ldots \right\}
\times \left\{ -i S^{kl} \delta_{\gamma\delta} + V_\nu^{kl} (\gamma^\nu)_{\gamma\delta} - \frac{i}{2} T^{kl}_\lambda (\sigma^{\lambda\sigma} \gamma_5)_{\gamma\delta} + \ldots \right\}.
\]
Contracting this expression with \(\delta_{\beta\gamma} \delta_{\alpha\delta}\) and \(\delta_{\beta\gamma} (\gamma^\mu)_{\delta\alpha}\), we obtain correspondingly
\[
S^{kj} S^{il} = \frac{1}{4} S^{ij} S^{kl} - \frac{1}{4} P^{ij} P^{kl} - \frac{1}{4} g^{\mu\nu} V^{ij}_\mu V^{kl}_\nu - \frac{1}{4} g^{\mu\nu} A^{ij}_\mu A^{kl}_\nu + \frac{1}{8} g^{\mu\lambda} g^{\nu\sigma} T^{ij}_\mu T^{kl}_\nu + \ldots, \tag{10.12}
\]
\[
S^{kl} V_\mu^{ij} = \frac{1}{4} (S^{ij} V^{kl}_\mu + V^{ij} S^{kl}_\mu) + \frac{1}{4} (P^{ij} A^{kl} - A^{ij} P^{kl}) + \frac{1}{8} g^{\mu\rho} \epsilon^{\rho\mu\lambda\sigma} (V^{ij}_\nu T^{kl}_\lambda T^{t\mu}_\sigma - T^{ij}_\mu V^{kl}_\nu)
\]
36
These equations are a straightforward generalization of equations (C.1) and (C.2). In a similar way one can derive a generalization of the remaining equations (C.3)–(C.5). A system of bilinear identities thus obtained is rather cumbersome with large amount of variables. It is successful that by virtue of the structure of the right-hand side of (10.10), we need not a complete system, but its rather special case. Contracting (10.12) with \( \delta^{ki} \delta^{jl} \), we obtain the following identities in the terms of the matrices introduced just above

\[
\text{tr}(SS^t) = \frac{1}{4} \text{tr}(SS^t) - \frac{1}{4} \text{tr}(PP^t) - \frac{1}{4} g^{\mu\nu} \text{tr}(V_\mu V_\nu^t) - \frac{1}{4} g^{\mu\nu} \text{tr}(A_\mu A^t_\nu) + \frac{1}{8} g^{\mu\lambda} g^{\nu\sigma} \text{tr}(\gamma_{\mu\nu} T^t_{\lambda\sigma}),
\]

\[
\text{tr}(SY^t_\mu) = \frac{1}{4} \left[ \text{tr}(SY^t_\mu) + \text{tr}(V_\mu S^t) \right] + \frac{1}{4} \left[ \text{tr}(P A^t_\mu) - \text{tr}(A^t_\mu P^t) \right]
\]

\[
+ \frac{1}{8} g_{\mu\nu} \epsilon^{\mu\nu\lambda\sigma} \left[ \text{tr}(V_\mu T^t_{\lambda\sigma}) - \text{tr}(T^t_{\lambda\sigma} V_\nu^t) \right] - \frac{1}{4} g^{\mu\lambda} \left[ \text{tr}(A_\lambda T^t_{\mu\nu}) - \text{tr}(T^t_{\mu\nu} A_\lambda^t) \right]
\]

and similar for the remaining identities. The transpose property (10.11) permits to obtain easily the corresponding systems of identities in the case of commuting or anticommuting tensor variables. Analysis similar to that in Section 2 leads to three independent relations of the form (2.12)–(2.14) with the appropriate replacements

\[
P^t T^t_{\mu\nu} \rightarrow \text{tr}(P^t T^t_{\mu\nu}), \quad S T^t_{\mu\nu} \rightarrow \text{tr}(S T^t_{\mu\nu}), \quad V_\mu V_\nu \rightarrow \text{tr}(V_\mu V_\nu^t)
\]

and so on. From the preceding, we can write out at once the final expression for the tensor of spin (10.10) (compare with (2.15))

\[
h(\bar{\psi}_D \sigma_{\mu\nu} \psi_D) = -2i \left\{ \text{tr}(A_\mu A^t_\nu) + \text{tr}(S T^t_{\mu\nu}) \right\}.
\]
of the following expression

\( h(\bar{\theta}_D \theta_D) \left\{ \left( \bar{\Psi}_D^i(x) \psi_D \right) + \left( \bar{\psi}_D \Psi_D^i(x) \right) \right\} \). \hspace{1cm} \text{(10.16)}

The factor in front of the brace in terms of Majorana spinors is given by the expression \( \text{(10.5)} \).

If we will also represent the background Dirac fermion field in terms of two Majorana spinors

\[
\Psi_D^i = \Psi_M^{(1)i} + i\Psi_M^{(2)i},
\]

then, as it is not difficult to show, the following relations hold

\[
\left( \bar{\Psi}_D^i(x) \psi_D \right) = +2(\delta^i_j + i\varepsilon^{jk})(\bar{\Psi}_M^{(1)i}(x) \psi_M^{(k)}),
\]

\[
\left( \bar{\psi}_D \Psi_D^i(x) \right) = -2(\delta^i_j - i\varepsilon^{jk})(\bar{\Psi}_M^{(1)i}(x) \psi_M^{(k)}),
\]

where \( \varepsilon^{jk} \) is the unit antisymmetric tensor (with components \( \varepsilon^{12} = -\varepsilon^{21} = 1 \)) and the sum over repeated indices is implied. Taking into account these relations and \( \text{(10.5)} \), the expression \( \text{(10.16)} \) takes the initial form for the mapping

\[
h(\bar{\theta}_D \theta_D) \left\{ \left( \bar{\Psi}_D^i(x) \psi_D \right) + \left( \bar{\psi}_D \Psi_D^i(x) \right) \right\} = 2i\hbar \varepsilon^{jk}(\bar{\theta}_M^{(1)j}(x) \psi_M^{(k)}).
\]

The next step is to use the expansion of the spinor structure \( \text{(10.7)} \). Contracting the expansion with the auxiliary spinor \( \theta_M^{(2)\beta} \), we obtain the relation connecting the commuting Majorana spinor \( \psi_M^{(k)} \) with the tensor variables \( (S^{ij}, V^{ij}, T^{ij}_{\mu\nu}, A^{ij}_{\mu}, P^{ij}) \)

\[
h^{1/2}(\bar{\theta}_M^{(1)j}(x)) \psi_M^{(k)} =
\]

\[
\frac{1}{4} \left\{ -iS^{ks}\theta_M^{(s)} + V^{ks}(\gamma^\mu\theta_M^{(s)})_\alpha - \frac{i}{2} T^{k\mu}(\sigma^{\mu\nu}\gamma_5\theta_M^{(s)})_\alpha + iA^{ks}_{\mu}(\gamma^\mu\gamma_5\theta_M^{(s)})_\alpha + P^{ks}(\gamma_5\theta_M^{(s)})_\alpha \right\}.
\]

This expression represents a direct extension of the expansion \( \text{(5.2)} \). As in the case of \( \text{(5.2)} \), not all tensor variables on the right-hand side of \( \text{(10.17)} \) are independent. Here, we are faced again with the problem of constructing the explicit solutions of a system of bilinear algebraic identities to which the functions \( (S^{ij}, V^{ij}, T^{ij}_{\mu\nu}, A^{ij}_{\mu}, P^{ij}) \) satisfy. In contrast to the problem of the motion of a particle in an external gauge field, for the problem with an external fermion field we should analyze a complete system of algebraic equations of the form \( \text{(10.12)} \) rather than the reduced system of the form \( \text{(10.13)} \). The presence of additional indices \( (ij) \) for the tensor variables makes the solution of the problem appreciably more difficult unlike a similar problem considered in Sections \( \text{5} \) and \( \text{6} \) and it will be the subject of separate studying. The mapping of the kinetic term \( \text{(5.1)} \) (or more exactly \( \text{(3.1)} \)) for the case of Dirac spinors will also be considered in a separate paper. The construction of such a mapping is a more subtle and intricate problem. In particular, this would require an appropriate extension of a system of identities \( \text{(3.7)}, \text{(3.8)} \) and most likely a substantial increase of the number of the tensor variables as it is seen from formulas \( \text{(D.1)}, \text{(D.2)} \) for the Majorana case.

Another way to deal with the Dirac spinors \( \psi_D \) and \( \theta_D \) without recourse to the decomposition \( \text{(10.3)} \) is discussed in Appendix F.

### 11 Higher-order derivative Lagrangian for spinning particle

In the final section we would like to consider yet another possible variant for the choice of tetrad \( h_\mu^{(s)} \) which has been introduced in Section \( \text{5} \) in constructing an exact solution of a system
of algebraic bilinear equations. In subsequent discussion we will essentially follow Gürsey [13].

As a basic element in the definition of the tetrad we choose the 4-velocity of algebraic bilinear equations. In subsequent discussion we will essentially follow Gürsey [13].

\[ h^{(0)}_\mu \equiv u_\mu, \]
\[ h^{(1)}_\mu \equiv \rho h^{(0)}_\mu = \rho \dot{u}_\mu, \]
\[ h^{(2)}_\mu \equiv \sigma \left( h^{(1)}_\mu - \rho^{-1} h^{(0)}_\mu \right) = \sigma \left( \rho \ddot{u}_\mu + \rho \dot{u}_\mu - \rho^{-1} u_\mu \right), \]
\[ h^{(3)}_\mu \equiv \kappa h^{(1)}_\mu, \]  \hspace{1cm} (11.1)

where \( \rho^{-1} \), \( \sigma^{-1} \) and \( \kappa^{-1} \) denote the first, second and third curvatures of the world-line, respectively. An explicit form of the first and second ones is given by the expressions

\[ \rho^{-2} = -\dot{u}_\mu \dot{u}^\mu, \]
\[ \sigma^{-2} = -\rho^2 \ddot{u}_\mu \dot{u}^\mu + \rho^{-2} (1 - \rho^2). \]

A system of normals \( h^{(0)} \) obeys the relations \( (5.10) \) and thereby defines the needed tetrad. In terms of tetrad the commuting antisymmetric tensor \( \omega_{\mu\nu} \), Eq. (5.11), takes the following form

\[ \omega_{\mu\nu} = h^{(1)}_\mu h^{(2)}_\nu - h^{(1)}_\nu h^{(2)}_\mu = \rho^2 \sigma \left( \dot{u}_\mu \ddot{u}_\nu - \dot{u}_\nu \ddot{u}_\mu \right) - \sigma \left( \dot{u}_\mu u_\nu - \dot{u}_\nu u_\mu \right). \]  \hspace{1cm} (11.2)

Having at hand the antisymmetric tensor \( \omega_{\mu\nu} \), we can define the following additional contributions to the original Lagrangian (1.4)

\[ h(\bar{\theta}) \omega_{\mu\nu} (\bar{\psi} \sigma_{\mu\nu} \psi), \quad h(\bar{\theta}) \omega^{\mu\nu} Q^a F^a_{\nu\lambda}(x) (\bar{\psi} \sigma^\lambda \psi). \]

The first expression has already been suggested in the paper [10] by Gürsey [13] in the case of zero third curvature \( \kappa^{-1} \). Under the mapping (2.15) the first expression (up to a numerical factor) turns to

\[ \rho^2 \sigma \left( \dot{u}_\mu \ddot{u}_\nu - \dot{u}_\nu \ddot{u}_\mu \right) \xi^\mu \xi^\nu - \sigma \left( \dot{u}_\mu u_\nu - \dot{u}_\nu u_\mu \right) \xi^\mu \xi^\nu + \ldots . \]  \hspace{1cm} (11.4)

The terms of a similar type really arise in some models of the Lagrangians of a relativistic spinning particle with higher derivatives, and in particular in the model presented by Polyakov [67]. Let us consider Polyakov’s approach in more detail.

We write out the initial functional integral in which the action is defined by the Lagrangian (11.1) without regard for the interaction terms

\[ Z = \int \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \mathcal{D}x_\mu \exp \left\{ -\int\limits_0 \frac{1}{2e} \dot{x}_\mu \dot{x}^\mu - \frac{i}{2} \xi_\mu \dot{\xi}^\mu - \frac{e}{2} m^2 \right. \]
\[ \left. + \frac{i}{2e} \chi \dot{x}_\mu \xi^\mu + \frac{i}{2} \xi_\lambda \xi_\sigma + \frac{i}{2} m \chi \xi_5 \right\}. \]  \hspace{1cm} (11.5)

\[ ^{10} \text{Instead of the } \omega_{\mu\nu}, \text{ Gürsey considered the antisymmetric tensor } \Omega_{\mu\nu} \text{ in a more general form} \]
\[ \Omega_{\mu\nu} = \rho^{-1} (h^{(1)}_\mu h^{(0)}_\nu - h^{(1)}_\nu h^{(0)}_\mu) + \sigma^{-1} (h^{(1)}_\mu h^{(2)}_\nu - h^{(1)}_\nu h^{(2)}_\mu) + \kappa^{-1} (h^{(2)}_\mu h^{(3)}_\nu - h^{(2)}_\nu h^{(3)}_\mu) = \]
\[ = \rho^2 (\dot{u}_\mu \ddot{u}_\nu - \dot{u}_\nu \ddot{u}_\mu) - \rho \kappa^{-1} \epsilon_{\mu\nu\lambda\sigma} \dot{u}^\lambda u^\sigma . \]

The tensor describes the rotation of the proper frame (tetrad) \( h^{(\mu)}_\mu \) attached to a world-line.
Further we follow the arguments in Polyakov \cite{Polyakov} closely. Our first step is the functional integrating over $\xi_5$ according to the formula

$$\int D\xi_5 \exp \left\{ -\frac{1}{16} \int_0^1 d\tau_1 d\tau_2 \sign(\tau_1 - \tau_2) \chi(\tau_1) \chi(\tau_2) \right\} = \exp \left\{ -\frac{i m^2}{16} \int_0^1 d\tau_1 d\tau_2 \sign(\tau_1 - \tau_2) \chi(\tau_1) \chi(\tau_2) \right\}. $$

The integral over the gravitino field $\chi$ is also Gaussian one. Performing the $\chi$ integration with allowance for the last equality, we obtain the following expression for the functional integral, instead of (11.5),

$$Z = \int Dx_\mu D\xi_\mu D\chi \exp \left\{ -\int_0^1 d\tau \left[ -\frac{1}{2e} \mathring{x}_\mu \mathring{x}^\mu - \frac{i}{2} \xi_\mu \mathring{\xi}^\mu - \frac{e}{2} m^2 + \frac{i}{m^2} \left( \frac{1}{e} (\mathring{x} \cdot \xi) \right) \frac{d}{d\tau} \left( \frac{1}{e} (\mathring{x} \cdot \xi) \right) \right] \right. $$

$$- \left. \frac{i}{4 m^2} \int_0^1 d\tau_1 d\tau_2 \sign(\tau_1 - \tau_2) \frac{d}{d\tau_1} \left( \frac{1}{e} (\mathring{x} \cdot \xi) \right) \frac{d}{d\tau_2} \left( \frac{1}{e} (\mathring{x} \cdot \xi) \right) \right\} $$

$$= \int Dx_\mu D\xi_\mu D\chi \exp \left\{ -\int_0^1 d\tau \left[ -\frac{1}{2e} \mathring{x}_\mu \mathring{x}^\mu - \frac{i}{2} \xi_\mu \mathring{\xi}^\mu - \frac{e}{2} m^2 \right. $$

$$- \left. \frac{i}{2 m^2 e^2} \omega_{\mu\nu}[x(\tau)] \xi^\mu \xi^\nu - \frac{i}{4 m^2 e^2} \left( \xi^\mu \mathring{\xi}^\nu + \xi^\nu \mathring{\xi}^\mu + \xi^\nu \mathring{\xi}^\mu \right) \mathring{x}_\mu \mathring{x}_\nu \right] \right\} (11.6)$$

Here, the function

$$\omega_{\mu\nu}[x(\tau)] \equiv \frac{1}{2} \left( \mathring{x}_\mu \mathring{x}_\nu - \mathring{x}_\nu \mathring{x}_\mu \right) $$

$$= \frac{1}{2} \left( u_\mu \mathring{u}_\nu - u_\nu \mathring{u}_\mu \right)$$

was introduced by Polyakov. It is the rotation of the tangent vector to the trajectory. An expression similar to (11.6) was also considered in the different context in works \cite{Polyakov}.

According to the obtained expression (11.6), when the boundary term is dropped, we can choose as the Lagrangian for a spinning particle the following expression

$$L = -\frac{1}{2e} \mathring{x}_\mu \mathring{x}^\mu - \frac{e}{2} m^2 $$

$$- \frac{i}{2} \xi_\mu \mathring{\xi}^\mu - \frac{i}{4 m^2 e^2} \left\{ (\mathring{x}_\mu \mathring{x}_\nu - \mathring{x}_\nu \mathring{x}_\mu) \xi^\mu \xi^\nu + (\xi^\mu \mathring{\xi}^\nu + \xi^\nu \mathring{\xi}^\mu) \mathring{x}_\mu \mathring{x}_\nu \right\} + \ldots. $$

Formally, it is obviously independent of the pseudoscalar $\xi_5$ and the gravitino $\chi$. Nevertheless, the Lagrangian (11.7) is still SUSY-invariant\footnote{The residual supersymmetric transformation is}

$$\delta x_\mu = i \alpha \xi_\mu, \quad \delta e = \frac{i}{m^2} \frac{d}{d\tau} \left( \frac{1}{e} (\mathring{x} \cdot \xi) \right), \quad \delta \xi_\mu = -\alpha \mathring{x}_\mu \frac{1}{e} + i \alpha m^2 \frac{d}{d\tau} \left( \frac{1}{e} (\mathring{x} \cdot \xi) \right).$$

(11.4) exactly coincides with the first term in braces (11.7). However, the expression (11.4) contains one further contribution with higher (third) derivative with respect to $\tau$. Its physical interpretation is not clear.
For a spinless particle in the $D$-dimensional space-time the Lagrangian with the third order derivatives of variable $x^\mu(\tau)$ (the relativistic particle with curvature and torsion) has been considered in the papers by Plyushchay [69–71] and Nesterenko [72]. Inclusion into the Lagrangian of higher derivatives with respect to the proper time of dynamical variables entails the account of additional degrees of freedom. As a result, it turns out well in this approach to describe particles with a nonzero spin without introducing additional variables (in particular, anticommuting Grassmann ones). The approach suggested in this section based on introducing tetrad (11.1) and additional contributions (11.3) with the mapping (11.4), can be considered as a “hybrid” approach to the classical description of the spin degrees of freedom simultaneously involving both the Grassmann-odd variable $\xi^\mu(\tau)$ and the position variable $x^\mu(\tau)$ with higher derivatives.

In closing this section we note that the idea of considering the higher-derivative Lagrangians for a point classical particle with the spin (with the proper rotation) is not new and originated in the classical papers by Frenkel [73] and Thomas [74]. Actually, the expressions of the type (11.4) are already contained in the above-mentioned papers if instead of the variable $\xi^\mu$ in (11.4) one introduces the tensor of spin $S_{\mu\nu}$ by the rule (1.8). Besides the second contribution in (11.3) for the case of background Abelian gauge field has also been given in [74] (Eq. (9.8)).

12 Conclusion

In this paper we have presented further analysis of the interaction of a classical color spinning particle with background non-Abelian bosonic and fermionic fields. Here, we confined close attention to the spin sector of the interaction. An explicit form of the interaction terms with an external Majorana fermion field $\Psi^\dagger_{\alpha\mu}(x)$ in terms of the real Grassmann-odd current variables $(S, V_\mu, T^\mu_\nu, A_\mu, P)$ can be obtained by substituting the expression (5.2) in the interaction Lagrangian (5.1). In the particular case the variables $V_\mu, A_\mu$ and $T^\mu_\nu$ are expressed through two independent those $S, P$ and tetrad $h_\mu^{(5)}$ with the aid of the relations (6.4) and (6.2). An explicit form of the tetrad can be chosen in any representation (E.1) – (E.4), (5.13) or (11.1). Further, in Section 10 we suggested a way to extend the above result to a more general case of an external Dirac fermion field $\Psi^\dagger_{D\alpha}(x)$. It was shown that to cover this case one needs to introduce into consideration a similar tensor set $(S, V_\mu, T^\mu_\nu, A_\mu, P)$, where each of these variables represent a $(2 \times 2)$ matrix consisting of real components. Unfortunately, an important problem of defining relations between these matrix variables has remained unsolved.

In addition, a question remains to be answered as to weather it is possible to construct a mapping of the Lagrangian (1.4) into (A.1) without the constrain equation (2.12). In Section 9 it was shown that to construct the map into a complete Lagrangian (A.1) possessing $n = 1$ local proper-time supersymmetry, the initial Lagrangian (1.4) must also possess some supersymmetry (fermion symmetry). To accomplish these ends, we must add the terms of the form (9.17) in (1.4) containing the auxiliary anticommuting spinor $\theta_\alpha$. Moreover, the obtained equation (9.18) for the odd spinor serves as a hint that the spinor should be considered as an independent dynamical variable subject to own dynamical equation. And finally, this odd spinor $\theta_\alpha$ should be related to its superpartner – the even spinor $\psi_\alpha$, and thus we have to consider a single
\[ \Theta_\alpha = \theta_\alpha + \eta \psi_\alpha, \]

as was done in the paper [48]. Here, \( \eta \) is a real odd scalar. The next step forward in this direction is to use at the outset all considered variables \( (\psi_\alpha, \theta_\alpha, \xi_\mu) \) for a description of the spin degrees of freedom, assuming them to be equivalent. This approach is known in literature as the construction of Lagrangians with \textit{doubly supersymmetry}, i.e. possessing both the local (world-time) and the global (space-time) SUSY [75].

However, it is worth noting that the Lagrangians suggested in [48, 75] containing the super-spinor \( \Theta_\alpha \) as a variable, can hardly be considered as the desired extension of the Lagrangian (1.4). The commuting spinor \( \psi_\alpha \) in these models always plays a role of only auxiliary non-dynamical quantity that is unacceptable for us. This circumstance was already mentioned in Introduction.

The Lagrangians suggested in the papers by Berkovits [19, 20] are closely related to the required generalization of the Lagrangian (1.4). Berkovits’ Lagrangians have been formulated within the framework of the so-called \textit{pure spinor formalism} [76] in which kappa (Siegel) symmetry is replaced by a BRST-like invariance, and describe the ten-dimensional superparticle coupling to a super-Maxwell or a super-Yang-Mills background fields. One of the key point in this approach is introducing into consideration the commuting pure spinor ghost variable \( \varphi_\alpha \) (in our present notation) satisfying \( \gamma^m \varphi = 0 \) for \( m = 1 \) to \( 10 \) along with the dynamical anticommuting spinor \( \theta_\alpha \). Thus, for example, the BRST invariant action with the first order Lagrangian for the \( N = 1, D = 10 \) superparticle in the super-Maxwell background is

\[
S_{\text{pure}} = S_0 + S_{\text{int}},
\]

where

\[
S_0 = \int d\tau (P_m \dot{x}^m - \frac{1}{2} P_m P^m + p_\alpha \dot{\varphi}^\alpha + \varphi_\alpha \dot{\psi}^\alpha),
\]

\[
S_{\text{int}} = q \int d\tau \left[ \dot{\varphi}^\alpha A_\alpha(x, \theta) + 2 \gamma^{\alpha m} \varphi_\alpha F_{mn}(x, \theta) \right].
\]

Here, the commuting spinor \( \varphi_\alpha \) is the canonical momentum for \( \psi_\alpha \) and the anticommuting spinor \( p_\alpha \) is the canonical one for \( \theta_\alpha \); \( F_{mn}(x, \theta) \) and \( W_\alpha(x, \theta) \) are the super-Maxwell superfield strengths (see [19] for the definition of the other notations). The equations of motion for the spinors \( \psi_\alpha \) and \( \theta_\alpha \) have the form

\[
\begin{align*}
\frac{id\psi_\alpha}{d\tau} &= -q(\gamma^{mn}\psi)\alpha F_{mn}(x, \theta), \\
\frac{id\theta_\alpha}{d\tau} &= -qW_\alpha(x, \theta).
\end{align*}
\]

The leading terms in the \( \theta \)-expansion of the superfields \( F_{mn}(x, \theta) \) and \( W_\alpha(x, \theta) \) are

\[
F_{mn}(x, \theta) = F_{mn}(x) + \left[ (\theta \gamma_n \partial_m \chi(x)) - (\theta \gamma_m \partial_n \chi(x)) \right] + \ldots, \\
W_\alpha(x, \theta) = \chi_\alpha(x) - \frac{1}{4} (\gamma^{mn})\alpha F_{mn}(x) + \ldots.
\]

The lowest component of the superfield \( F_{mn}(x, \theta) \) is the vector field strength \( F_{mn}(x) \) and the lowest component of \( W_\alpha(x, \theta) \) is the spinor background field \( \chi_\alpha(x) \). We see that equation (1.1) is contained in (12.2) as its integral part (in the action (12.1) the parametrization gauge \( e = -1/2 \) is chosen rather than the proper time one \( e = 1/m \)). Further, the second term in
the expansion of $W_\alpha(x, \theta)$ by substitution into (12.3), enables us to reproduce equation (9.16). Notice that, although the action (12.1) was suggested for the ten-dimensional superparticle, this approach can be also extended to the low-dimension pure-spinor superparticles [77, 78], in particular, for $D = 4$ one. Thus the action (12.1) is the best candidate for the desired extension of the action with Lagrangian (1.4) (in the paper [20] the action (12.1) has been defined for the case of the interaction with a super-Yang-Mills background field, that correspondingly requires introducing the complex Grassmann color charge $\theta^i$). However, here we are faced with different problem. In the action (12.1) we have the interaction of a particle with the supersymmetric gauge field, whereas in (1.4) the usual vector field is presented. The question arises whether it is possible to define analog of the action (12.1) for nonsupersymmetric background field, for example, by simply setting $\chi_\alpha(x) \equiv 0$ in (12.1) and (12.4).

There is a further point to be made here. Throughout this work we have dealt with the mapping of the (pseudo)classical models. Naturally, a more deep and principle question is connected with the construction of a mapping between the dynamical systems obtained after quantization of these classical models. Is it possible to construct such a (one-to-one) mapping of these quantized models and to what extent it will be complete? This problem is worth a careful look. The results reported in this paper are put forward as a classical starting point for the subsequent analysis of mapping the quantized systems.

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Appendix A  Lagrangian of a spinning particle

Here, for convenience of future reference we write out the Lagrangian for a spinning massive particle in external non-Abelian gauge field given in the paper [34]. We also write out the local $n = 1$ super-transformation under which this Lagrangian is invariant, the constrain equations and the equations of motion for dynamical variables.

The most general Lagrangian for a classical relativistic spin-$\frac{1}{2}$ particle moving in the background non-Abelian gauge field is (we put $c = 1$ for the speed of light)

$$L = L_0 + L_m + L_\theta,$$

where

$$L_0 = -\frac{1}{2e} \dot{x}_\mu \dot{x}^\mu - \frac{i}{2} \xi_\mu \dot{\xi}^\mu + \frac{i}{2e} \chi \dot{x}_\mu \xi^\mu,$$

$$L_m = -\frac{e}{2} m^2 + \frac{i}{2} \xi_5 \dot{\xi}_5 + \frac{i}{2} m \chi \xi_5,$$

$$L_\theta = i\hbar \theta^i D_i^j \theta^j + \frac{i}{2} eg Q^a F_{\mu \nu}^a \xi^\mu \xi^\nu.$$

Here, $\xi_\mu, \mu = 0, 1, 2, 3, \text{and} \xi_5$ are dynamical variables describing the relativistic spin dynamics of the massive particle. These variables are elements of the Grassmann algebra [31]. The Lagrangian is invariant up to a total derivative under the following infinitesimal supersymmetry transformation

$$\delta x_\mu = i\alpha \xi_\mu,$$

$$\delta \xi_\mu = -\alpha \left( \dot{x}_\mu - \frac{1}{2} i \chi \xi_\mu \right) / e,$$

$$\delta e = -i\alpha \chi,$$

$$\delta \chi = 2\dot{\alpha},$$

$$\delta \xi_5 = m\alpha,$$

$$\delta \theta^i = (g/\hbar) \alpha \xi^n A^a_\mu (t^n)^{ij} \theta^j,$$

where $\alpha = \alpha(\tau)$ is an arbitrary Grassmann-valued function.

Varying the variables $e, \chi$ and $\xi_5$, we obtain the constraint equations

$$\dot{x}^2 - i\chi \dot{x}_\mu \xi^{\mu}/e^2 - m^2 + ig Q^a F_{\mu \nu}^a \xi^\mu \xi^\nu = 0,$$

$$\dot{x}_\mu \xi^\mu + me \xi_5 = 0,$$

$$2\dot{\xi}_5 - m\chi = 0,$$

which by the specific choice of the proper time gauge $e = 1/m, \chi = 0$ and $\xi_5 = 0$ are reduced to

$$m^2 \dot{x}^2 - m^2 + ig Q^a F_{\mu \nu}^a \xi^\mu \xi^\nu = 0,$$

$$\dot{x}_\mu \xi^\mu = 0.$$

---

12 Here, in contrast to [34], as the notation of spin variable we use the letter $\xi$ instead of generally accepted notation $\psi$, since the latter is used throughout the present work for the notation of bispinor $\psi_\alpha$. 

44
Finally, variation over the remaining dynamical variables gives the equations of motion
\begin{align}
\dot{\xi}_\mu - \frac{g}{m} Q^a F^a_{\mu\nu} \xi^\nu &= 0, \\
\dot{\theta}^i + \frac{ig}{\hbar} \left( A^a_\mu \dot{x}^\mu - \frac{i}{2m} F^a_{\mu\nu} \xi^\nu \right) (t^a)^{ij} \theta^j &= 0, \\
m \ddot{x}_\mu - g Q^a \left( F^a_{\mu\nu} \dot{x}^\nu - \frac{i}{2m} D^{ab}_\mu (x) F^b_{\nu\lambda} \xi^\nu \xi^\lambda \right) &= 0.
\end{align}

(A.8)  
(A.9)  
(A.10)

Here, \( D^{ab}_\mu (x) = \delta^{ab} \partial / \partial x^\mu + i (g/\hbar) A^c_\mu (x) (T^c)^{ab} \) is the covariant derivative in the adjoint representation, where \((T^c)^{ab} \equiv -if^{cab}\). In deriving (A.10) we have made use of the equation of motion for the commuting color charge \( Q^a \equiv \theta^\dagger t^a \theta \)

\begin{align}
\dot{Q}^a + \frac{ig}{\hbar} \left( A^b_\mu \dot{x}^\mu - \frac{i}{2m} F^b_{\mu\nu} \xi^\nu \xi^\lambda \right) (T^b)^{ac} Q^c &= 0.
\end{align}

(A.11)

This equation follows from the equation of motion for the Grassmann color charge \( \theta^i \), Eq. (A.9).

The color current of the particle, which enters as the source into the equation of motion for the gauge field,
\[ D^{ab}_\mu (x) F^{b\mu\nu} (x) = j^{a\nu} (x), \]

is
\begin{align}
j^{a\mu} (x) &= g \int d\tau \left( Q^a \dot{x}^\mu - i \xi^\mu \xi^\nu \frac{1}{m} D^{ab}_\nu (x) Q^b \right) \delta^{(4)} (x - x(\tau)).
\end{align}

(A.12)

Besides the initial complete expression (A.1) we need a somewhat reduced form of the Lagrangian. For this purpose we use the last constraint equation in (A.6), which expresses the one-dimensional gravitino field \( \chi \) in terms of the quantity \( \xi_5 \):
\[ \chi = \frac{2}{m} \dot{\xi}_5. \]

(A.13)

Substituting (A.13) into (A.2) and (A.3) we obtain the required form of \( L_0 \) and \( L_m \), respectively
\begin{align}
L_0 &= -\frac{1}{2e} \dot{x}_\mu \dot{x}^\mu - \frac{i}{2} \xi^\mu \xi^\nu \frac{1}{m} \dot{\xi}_5 \xi^\mu, \\
L_m &= -\frac{e}{2} \frac{\dot{x}_5^2}{m^2} - \frac{i}{2} \xi_5 \dot{\xi}_5.
\end{align}

(A.14)  
(A.15)

The supersymmetry transformations (A.5) in this case take the form:
\begin{align}
\delta x_\mu &= i \alpha \xi_\mu, \\
\delta \xi_\mu &= -\alpha \left( \dot{x}_\mu - \frac{i}{m} \xi_5 \dot{\xi}_5 \right) / e, \\
\delta e &= -\frac{2i}{m} \alpha \dot{\xi}_5, \\
\delta \xi_5 &= m \alpha, \\
\delta \theta^i &= (g/\hbar) \alpha \xi^\mu A^a_\mu (t^a)^{ij} \theta^j.
\end{align}

(A.16)  
(A.17)  
(A.18)  
(A.19)  
(A.20)

We note that after the elimination (A.13), the kinetic term \( \xi_5 \dot{\xi}_5 \) in (A.3) changes its sign to opposite one.
Appendix B  Spinor matrix algebra

In this Appendix we give some necessary formulae of the spinor matrix algebra, which are used in the text. The first basic formula is

$$\gamma^\mu \gamma^\nu = I \cdot g^{\mu \nu} + i \sigma^{\mu \nu}, \quad \sigma^{\mu \nu} = \frac{1}{2i} [\gamma^\mu, \gamma^\nu],$$

where $I$ is the identity spinor matrix. We use the metric $g^{\mu \nu} = \text{diag}(1, -1, -1, -1)$. The useful identity is also

$$\sigma^{\mu \nu} \gamma_5 = \frac{1}{2i} \epsilon^{\mu \nu \lambda \sigma} \sigma_{\lambda \sigma}, \quad (B.1)$$

where $\gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$; $\epsilon^{\mu \nu \lambda \sigma}$ is the totally antisymmetric tensor so that $\epsilon^{0123} = +1$.

The expansion of the product of the $\gamma$- and $\sigma$-matrices reads

$$\sigma^{\mu \nu} \gamma^\lambda = \frac{1}{i} \left[ (g^{\mu \lambda} \gamma^\mu - g^{\nu \lambda} \gamma^\nu) - \epsilon^{\mu \nu \lambda \sigma} \gamma_5 \gamma^\sigma \right], \quad (B.2)$$

The formula of the expansion for the product of two $\sigma$-matrices has the following form:

$$\sigma^{\mu \nu} \sigma^\lambda = I \cdot (g^{\mu \lambda} g^{\nu \sigma} - g^{\mu \sigma} g^{\nu \lambda}) + \frac{1}{i} \left[ (g^{\nu \lambda} \sigma^{\mu \sigma} - g^{\rho \lambda} \sigma^{\mu \rho} - g^{\sigma \nu} \sigma^{\mu \rho} + g^{\mu \sigma} \sigma^{\lambda \rho}) + \frac{1}{i} \epsilon^{\mu \nu \lambda \sigma} \gamma_5 \right]. \quad (B.3)$$

Finally, for the product of three $\sigma$-matrices we have

$$\sigma^{\mu \nu} \sigma^\lambda \sigma^\rho = \frac{1}{i} \left\{ g^{\lambda \mu} (g^{\rho \sigma} \gamma^\delta - g^{\rho \sigma} \gamma^\delta) - g^{\mu \lambda} (g^{\rho \sigma} \gamma^\delta - g^{\rho \sigma} \gamma^\delta) - g^{\nu \sigma} (g^{\rho \sigma} \gamma^\lambda - g^{\rho \sigma} \gamma^\lambda) + g^{\lambda \sigma} (g^{\rho \sigma} \gamma^\mu - g^{\rho \sigma} \gamma^\mu) \right\} \cdot I$$

$$+ \left[ (g^{\lambda \rho} g^{\sigma \delta} - g^{\rho \sigma} g^{\lambda \delta}) \sigma^{\mu \nu} + (g^{\mu \sigma} g^{\rho \delta} - g^{\rho \sigma} g^{\mu \delta}) \sigma^{\nu \sigma} + (g^{\mu \rho} g^{\sigma \delta} - g^{\rho \sigma} g^{\mu \delta}) \sigma^{\delta \nu} \right] \cdot \epsilon^{\mu \nu \lambda \sigma} \gamma_5 \right\}.$$ \quad (B.4)

It is worthy of special emphasis that in the expansion (B.4) an explicit form of the last term is not uniquely defined by virtue of the fact that there are exist the identities relating the metric tensor $g^{\mu \nu}$ and the antisymmetric tensor $\epsilon^{\mu \nu \lambda \sigma}$:

$$g^{\lambda \nu} \epsilon^{\rho \delta \mu \sigma} - g^{\mu \lambda} \epsilon^{\rho \delta \nu \sigma} - g^{\nu \sigma} \epsilon^{\rho \delta \mu \lambda} + g^{\lambda \sigma} \epsilon^{\rho \delta \nu \lambda} =$$

$$g^{\rho \sigma} \epsilon^{\lambda \nu \delta \mu} - g^{\rho \nu} \epsilon^{\lambda \sigma \delta \mu} - g^{\sigma \nu} \epsilon^{\lambda \rho \delta \mu} =$$

$$g^{\mu \lambda} \epsilon^{\nu \rho \delta \sigma} - g^{\nu \mu} \epsilon^{\rho \lambda \delta \sigma} - g^{\rho \nu} \epsilon^{\lambda \mu \delta \sigma} + g^{\lambda \nu} \epsilon^{\rho \mu \delta \sigma} = \frac{1}{i} \epsilon^{\mu \nu \lambda \sigma} \gamma_5 \gamma^\rho.$$\quad (B.5)

These relations arise, for example, in calculating the following trace:

$$\text{Sp}(\sigma^{\mu \nu} \sigma^\lambda \sigma^\rho \gamma_5)$$

by making use\textsuperscript{13} of the formula (B.1). Another useful identity of such a kind \textsuperscript{50} is

$$\epsilon^{\mu \beta \gamma \delta} \gamma^{\alpha \nu} - \epsilon^{\mu \gamma \alpha \delta} \gamma^{\beta \nu} + \epsilon^{\mu \delta \alpha \beta} \gamma^{\gamma \nu} - \epsilon^{\mu \alpha \beta \gamma} \gamma^{\delta \nu} = \epsilon^{\alpha \beta \gamma \delta} \gamma^{\mu \nu}.$$ \quad (B.6)

\textsuperscript{13} Calculation of the trace for a product of six $\gamma$-matrices and $\gamma_5$, has been considered, for instance, in \textsuperscript{50}. However, the ambiguity of representation for the trace has not been discussed there.
Appendix C  A complete list of the bilinear identities

Here, we give a complete list of all 15 sets of the bilinear relations. These relations are given in the same consequence as in the paper [51]. Our list is introduced in such a form that it simultaneously covers both the commutative and non-commutative cases (here, [.] designates commutator, and { , } is anticommutator). Such writing, in particular, enables us to define immediately the correct expressions for bilinear relations containing the derivatives of the currents $(S,V,^*T, A, P)$ (see Section 3). Note that some expressions, namely (C.3), (C.7), (C.8), (C.10) and (C.11), can be introduced in somewhat different but equivalent forms. This fact is connected with ambiguity in calculating the trace $Sp(\sigma^{\mu\nu}\sigma^{\rho\sigma}\gamma_{5})$, as was mentioned in closing the previous Appendix. The equivalence of the different representations can be directly proved with using the identities (B.5) and (B.6).

\[
SS = \frac{1}{4} SS - \frac{1}{4} PP - \frac{1}{4} V^\mu V^\mu - \frac{1}{4} A^\mu A^\mu + \frac{1}{8} T^\mu T^\mu, \tag{C.1}
\]

\[
SV^\mu = \frac{1}{4} \{S, V^\mu\} + \frac{1}{4} \{P, A^\mu\} + \frac{1}{8} \epsilon^{\mu\nu\lambda}\sigma[V^\nu, ^*T^\lambda] - \frac{1}{4} \{A^\mu, ^*T^\mu\}, \tag{C.2}
\]

\[
S^*T^\mu = \frac{1}{4} \{S, ^*T^\mu\} + \frac{1}{8} \epsilon^{\mu\nu\lambda}\sigma\{P, ^*T^\lambda\} - \frac{1}{4} \{A^\mu, V^\nu\} + \frac{1}{4} \{V^\mu, A^\nu\} - \frac{1}{4} \epsilon^{\mu\nu\lambda}\sigma V^\nu V^\sigma \tag{C.3}
\]

\[
SA^\mu = \frac{1}{4} \{S, A^\mu\} - \frac{1}{4} \{P, V^\mu\} + \frac{1}{8} \epsilon^{\mu\nu\lambda}\sigma[A^\nu, ^*T^\lambda] + \frac{1}{4} \{V^\nu, ^*T^\mu\}, \tag{C.4}
\]

\[
SP = \frac{1}{4} \{S, P\} - \frac{1}{4} \{V^\mu, A^\mu\} - \frac{1}{16} \epsilon^{\mu\nu\lambda}\sigma T^\mu T^\nu T^\lambda, \tag{C.5}
\]

\[
V^\mu V^\nu = \frac{1}{4} \{V^\mu, V^\nu\} - \frac{1}{4} \{A^\mu, A^\nu\} - \frac{1}{4} \epsilon^{\mu\nu\lambda}\sigma(S + PP + V^\lambda A^\lambda - A^\lambda A^\lambda - \frac{1}{2} ^*T^\lambda ^*T^\lambda) \tag{C.6}
\]

\[
V^\mu T^\nu T^\lambda = \frac{1}{4} g^{\mu\nu}\{S, A^\nu\} - \frac{1}{4} g^{\mu\nu}\{S, A^\lambda\} + \frac{1}{4} \epsilon^{\mu\nu\lambda}\sigma[S, V^\nu] \tag{C.7}
\]

\[
V^\mu A^\nu = \frac{1}{4} \{S, ^*T^\mu\} - \frac{1}{8} \epsilon^{\mu\nu\lambda}\sigma\{P, ^*T^\lambda\} - \frac{1}{4} \epsilon^{\mu\nu}\sigma[S, P] + \frac{1}{4} \epsilon^{\mu\nu\lambda}\sigma V^\lambda V^\nu - \frac{1}{4} \epsilon^{\mu\nu\lambda}\sigma A^\nu A^\lambda \tag{C.8}
\]

\[
V^\mu P = \frac{1}{4} \{S, A^\nu\} + \frac{1}{4} \{P, V^\mu\} + \frac{1}{8} \epsilon^{\mu\nu\lambda}\sigma\{A^\nu, ^*T^\lambda\} - \frac{1}{4} \{V^\nu, ^*T^\mu\}. \tag{C.9}
\]
\[ *T^{\mu\nu} *T^{\lambda\sigma} = \frac{1}{4} (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\lambda}) (SS - PP + V_\rho V^\rho + A_\rho A^\rho + \frac{1}{2} *T_{\rho\delta} *T^{\rho\delta}) \] (C.10)

\[ - \frac{1}{4} \epsilon^{\mu\nu\lambda\sigma} \{ S, P \} + \frac{1}{8} (g^{\nu\sigma} \epsilon^{\rho\mu\lambda} - g^{\mu\sigma} \epsilon^{\rho\mu\lambda} + g^{\nu\lambda} \epsilon^{\rho\mu\sigma} + g^{\mu\lambda} \epsilon^{\rho\nu\sigma}) [S, *T_{\rho\delta}] \]

\[ + \frac{1}{4} [P, (g^{\nu\sigma} *T^{\mu\lambda} - g^{\mu\sigma} *T^{\nu\lambda} + g^{\nu\lambda} *T^{\mu\sigma} + g^{\mu\lambda} *T^{\nu\sigma})] \]

\[ - \frac{1}{4} (g^{\mu\lambda} \{ V^\nu, V^\nu \} - g^{\nu\lambda} \{ V^\sigma, V^\mu \} - g^{\mu\sigma} \{ V^\lambda, V^\nu \} + g^{\nu\sigma} \{ V^\lambda, V^\mu \}) \]

\[ - \frac{1}{4} (g^{\mu\lambda} \{ A^\sigma, A^\nu \} - g^{\nu\lambda} \{ A^\sigma, A^\mu \} - g^{\mu\sigma} \{ A^\lambda, A^\nu \} + g^{\nu\sigma} \{ A^\lambda, A^\mu \}) \]

\[ - \frac{1}{4} \left( \epsilon^{\mu\lambda\rho\sigma} P_{\nu} A^\sigma - \epsilon^{\nu\lambda\rho\sigma} P_{\nu} A^\mu \right) + \frac{1}{4} \left( \epsilon^{\mu\nu\lambda\rho} V_\sigma A_\rho - \epsilon^{\mu\nu\sigma\rho} V_\lambda A_\rho \right) \]

\[ + \frac{1}{4} \left( \epsilon^{\mu\nu\lambda\rho} A_\sigma P_{\rho} - \epsilon^{\nu\lambda\rho\sigma} A_\mu P_{\rho} \right) - \frac{1}{4} \left( \epsilon^{\mu\nu\lambda\rho} A_\sigma V_\rho - \epsilon^{\mu\nu\sigma\rho} A_\lambda V_\rho \right) \]

\[ + \frac{1}{4} \left( \{ *T^{\mu\nu}, *T^{\lambda\sigma} \} + \{ *T^{\mu\lambda}, *T^{\nu\sigma} \} - \{ *T^{\mu\sigma}, *T^{\nu\lambda} \} \right) \]

\[ + \frac{1}{4} (g^{\mu\lambda} \{ *T^{\nu\rho}, *T^{\mu\rho} \} - g^{\nu\sigma} \{ *T^{\mu\rho}, *T^{\mu\rho} \} + g^{\nu\lambda} \{ *T^{\mu\rho}, *T^{\mu\rho} \}) \]

\[ *T^{\mu\nu} A^\lambda = \frac{1}{4} g^{\mu\lambda} \{ S, V^\nu \} - \frac{1}{4} g^{\nu\lambda} \{ S, V^\mu \} + \frac{1}{4} \epsilon^{\mu\nu\lambda\sigma} \{ P, V_\sigma \} \]

\[ + \frac{1}{4} g^{\mu\lambda} [P, A^\nu] - \frac{1}{4} g^{\nu\lambda} [P, A^\mu] - \frac{1}{4} \epsilon^{\mu\nu\lambda\sigma} [S, A_\sigma] \]

\[- \frac{1}{4} \{ A^\mu, *T^{\nu\lambda} \} + \frac{1}{4} \{ A^\lambda, *T^{\mu\nu} \} + \frac{1}{4} \left( A^\nu, *T^{\mu\lambda} \right) + \frac{1}{4} g^{\nu\lambda} \{ A_\sigma, *T^{\sigma\mu} \} - \frac{1}{4} g^{\mu\lambda} \{ A_\sigma, *T^{\sigma\nu} \} \]

\[ \left( \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} [V_\lambda, *T_{\sigma\rho}] + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} [V_\sigma, *T_{\rho\nu}] - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} [V_\sigma, *T_{\rho\mu}] \right), \]

\[ *T^{\mu\nu} P = \frac{1}{4} \{ P, *T^{\mu\nu} \} - \frac{1}{8} \epsilon^{\mu\nu\lambda\sigma} \{ S, *T_{\lambda\sigma} \} + \frac{1}{4} \epsilon^{\mu\nu\lambda\sigma} \{ V_\lambda, A_\sigma \} - \frac{1}{4} [V^\nu, V^\nu] - \frac{1}{4} [A^\mu, A^\nu] \]

\[ + \frac{1}{4} \left( *T^{\mu\lambda}, *T^{\lambda\nu} \right), \]

\[ A^\mu A^\nu = \frac{1}{4} \{ A^\mu, A^\nu \} - \frac{1}{4} \left( V^\mu, V^\nu \right) - \frac{1}{4} g^{\mu\nu} (SS + PP - V_\lambda V^\lambda + A_\lambda A^\lambda - \frac{1}{2} *T_{\lambda\sigma} *T^{\lambda\sigma}) \]

\[ + \frac{1}{8} \epsilon^{\mu\nu\lambda\sigma} [S, *T_{\lambda\sigma}] + \frac{1}{4} [P, *T^{\mu\nu}] + \frac{1}{4} \epsilon^{\mu\nu\lambda\sigma} [V_\lambda, A_\sigma] + \frac{1}{4} \left( *T^{\mu\lambda}, *T^{\lambda\nu} \right), \]

\[ A^\mu P = - \frac{1}{4} [S, V^\mu] + \frac{1}{4} \{ P, A^\mu \} - \frac{1}{8} \epsilon^{\mu\nu\lambda\sigma} \{ V_\nu, *T_{\lambda\sigma} \} - \frac{1}{4} [A_\nu, *T^{\mu\nu}], \]

\[ PP = \frac{1}{4} PP - \frac{1}{4} SS - \frac{1}{4} V_\mu V^\mu - \frac{1}{4} [A_\mu, A^\nu] - \frac{1}{8} *T_{\mu\nu} *T^{\mu\nu}. \]

48
Appendix D  Mapping the kinetic term

In Section 3 we produced a set of identities containing the tensor variables and their derivatives. We have shown by the example of the "scalar" equation (see Eqs. (3.5), (3.6)) that on the left-hand side of these identities one can not collect the required expressions with the derivative of tensor variables \( \dot{S}, \dot{V}_\mu, \ldots \) when \( \dot{\theta}_a \neq 0 \). For this reason in Section 3 we have restricted our attention only to the case when the auxiliary spinor \( \dot{\theta}_a \) is independent of \( \tau \). In this appendix we would like to show that the condition \( \dot{\theta}_a = \text{const} \) is not needed in deriving the mapping of the kinetic term (3.11) in a class of Majorana spinors.

Let us introduce two types of expansions for the spinor structures containing the derivatives of spinors. The first of them is

\[
\hbar^{1/2} \langle \dot{\hat{\theta}}_\beta \psi_\alpha \rangle = \frac{1}{4} \left\{ -i \dot{S}_1 \delta_{a\beta} + \dot{V}_1^\mu (\gamma_\mu)_{a\beta} - \frac{i}{2} \dot{T}_1^{\mu\nu} (\sigma_{\mu\nu})_{a\beta} + i \dot{A}_1^\mu (\gamma_{\mu\gamma})_{a\beta} + \dot{P}_1 (\gamma_5)_{a\beta} \right\}, \tag{D.1}
\]

where \( \dot{S}_1 = \langle \dot{\hat{\theta}}_\gamma \rangle, \dot{V}_1^\mu \equiv i \langle \dot{\hat{\theta}}_{\gamma\mu} \psi \rangle \), etc., and the second is

\[
\hbar^{1/2} \langle \ddot{\theta}_\beta \dot{\psi}_\alpha \rangle = \frac{1}{4} \left\{ -i \ddot{S}_2 \delta_{a\beta} + \ddot{V}_2^\mu (\gamma_\mu)_{a\beta} - \frac{i}{2} \ddot{T}_2^{\mu\nu} (\sigma_{\mu\nu})_{a\beta} + i \ddot{A}_2^\mu (\gamma_{\mu\gamma})_{a\beta} + \ddot{P}_2 (\gamma_5)_{a\beta} \right\}, \tag{D.2}
\]

where in turn, \( \ddot{S}_2 = \langle \ddot{\hat{\theta}}_\gamma \dot{\psi} \rangle, \ddot{V}_2^\mu \equiv i \langle \ddot{\hat{\theta}}_{\gamma\mu} \dot{\psi} \rangle \) etc. We specially note that \( \langle \dot{S}_{1,2}, \dot{V}_{1,2}, \dot{T}_{1,2}^{\mu\nu}, \ldots \rangle \) are merely symbols and the dot over the tensor variables is not the derivative over \( \tau \). Only the sum of such two expressions

\[
\dot{S} = \dot{S}_1 + \dot{S}_2, \quad \dot{V}^\mu = \dot{V}_1^\mu + \dot{V}_2^\mu, \ldots
\]

will be the "actual" derivative. In terms of these quantities the right-hand side of the expression for derivative (3.3) is divided into two pieces

\[
\hbar \langle \ddot{\theta}_M \dot{\psi}_M \rangle \left[ \left( \frac{d\dot{\psi}_M}{d\tau} \dot{\psi}_M \right) - \left( \ddot{\psi}_M \frac{d\dot{\psi}_M}{d\tau} \right) \right] = \frac{1}{2} \left\{ \dot{S} \dot{S}_1 + V_\mu \dot{V}_1^\mu - \frac{1}{2} T^{\mu\nu} \dot{T}_1^{\mu\nu} - A_\mu \dot{A}_1^\mu - \dot{P}_1 \right\} \tag{D.3}
\]

\[
= \frac{1}{2} \left\{ S \ddot{S}_1 + V_\mu \ddot{V}_1^\mu - \frac{1}{2} T^{\mu\nu} \ddot{T}_1^{\mu\nu} - A_\mu \ddot{A}_1^\mu - \ddot{P}_2 \right\}.
\]

The systems of algebraic equations containing the functions \( \dot{S}_{1,2}, \dot{V}_{1,2}, \ldots \) are obtained by the scheme described in Section 3. First, it is necessary to multiply the expression (2.1) by the expressions (D.1) and (D.2), respectively (by analogy with Eq. (3.1)) and then perform a crossed contraction with different spinor structures of the type \( \delta_{\beta\gamma}, \delta_{\delta\alpha}, \delta_{\beta\gamma}(\gamma_5)_{\delta\alpha}, \ldots \). It is not difficult to see that a system of equations containing the functions \( \dot{S}_2, \dot{V}_2^\mu, \ldots \) will be completely similar to the system (3.7), (3.8) with appropriate replacements \( \dot{S} \rightarrow \dot{S}_2, \dot{V}_\mu \rightarrow \dot{V}_2^\mu \) etc.

For a system of equations containing \( \dot{S}_1, \dot{V}_1^\mu, \ldots \) the situation is somewhat involved. Here, the right-hand side of equations will be completely similar to the right-hand side of corresponding equations for the functions \( \dot{S}_2, \dot{V}_2^\mu, \ldots \). On the left-hand side the functions with the "derivatives" and without derivatives should be rearranged among themselves with no change.
of a sign. By this means, instead of (3.7) and (3.8), we will now have a system of identities

$$4S\dot{S}_1 = S\dot{S}_1 - P\dot{P}_1 - (V_\mu \dot{V}_\mu^\mu + A_\mu \dot{A}_1^\mu) + \frac{1}{2} \ast T_{\mu\nu} \ast \dot{T}_{\mu\nu},$$

$$V_\mu \dot{V}_1^\mu = -(S\dot{S}_1 + P\dot{P}_1) - \frac{1}{2} (V_\nu \dot{V}_1^\nu - A_\nu \dot{A}_1^\nu),$$

$$A_\mu \dot{A}_1^\mu = -(S\dot{S}_1 + P\dot{P}_1) + \frac{1}{2} (V_\nu \dot{V}_1^\nu - A_\nu \dot{A}_1^\nu),$$

$$4P\dot{P}_1 = -(S\dot{S}_1 - P\dot{P}_1) - (V_\nu \dot{V}_1^\nu + A_\nu \dot{A}_1^\nu) - \frac{1}{2} \ast T_{\mu\nu} \ast \dot{T}_{\mu\nu},$$

$$\ast T_{\mu\nu} \ast \dot{T}_{\mu\nu} = 3(S\dot{S}_1 - P\dot{P}_1) - \frac{1}{2} \ast T_{\mu\nu} \ast \dot{T}_{\mu\nu}. $$

Inspection of these five equations has shown that only three of them are independent (in contrast to a similar system for $S_2, V_{2\mu}$, ..., where we had dealt with only two independent equations (3.9) and (3.10)). It is convenient to represent these equations in the following form:

$$V_\mu \dot{V}_1^\mu = A_\mu \dot{A}_1^\mu,$$

$$S\dot{S}_1 - P\dot{P}_1 = \frac{1}{2} \ast T_{\mu\nu} \ast \dot{T}_{\mu\nu},$$

$$-2(S\dot{S}_1 + P\dot{P}_1) = V_\mu \dot{V}_1^\mu + A_\mu \dot{A}_1^\mu. $$

It is easy to see that by virtue of these equations all the contribution in [D.3] containing the function $\dot{S}_1, \dot{V}_1^\mu$, ..., vanishes and thus all terms containing $\dot{\theta}_\alpha$ are completely excluded from consideration for the Majorana spinors as it occurs on the left-hand side of Eq. (3.3). This circumstance can be considered as a good test for the correctness of the equations under examination and of the approach as a whole.

Appendix E  Parameter representations of triad and tetrad

In this appendix we give an explicit form of the parameter representations of orthogonal triad and tetrad [50,58]. The orthogonal triad $(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)})$ satisfying the orthonormality relation

$$\mathbf{a}^{(i)} \cdot \mathbf{a}^{(j)} = \delta^{ij}, \quad i, j = 1, 2, 3$$

and the completeness relation

$$a_i^{(k)} a_j^{(k)} = \delta_{ij},$$

can be expressed, for example, in terms of three Euler angles $\alpha, \beta$ and $\gamma$

$$\mathbf{a}^{(1)} = \begin{pmatrix} \cos \alpha \sin \beta \\
\sin \alpha \sin \beta \\
\cos \beta \end{pmatrix},$$

$$\mathbf{a}^{(2)} = \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma \\
\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma \\
- \sin \beta \cos \gamma \end{pmatrix},$$

$$\mathbf{a}^{(3)} = \begin{pmatrix} - \cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma \\
- \sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma \\
\sin \beta \sin \gamma \end{pmatrix}. $$
As is known the components of these vectors constitute the rotation matrix \((a_{ij}) \equiv (a_i^j)\) in \(\mathbb{C}^3\). The other parameterizations of the triad are also possible: in terms of the angles \(\theta, \phi\) which describe the direction of rotation axes and the rotation angle \(\omega\) or in terms of the Cayley-Klein parameters \(\psi, \theta\) [81].

Further, the orthogonal tetrad \(h_\mu^s, s = 0, 1, 2, 3\) in Minkowski space is subject to the orthonormality relation

\[
h_\mu^s h^{(s')\mu} = g^{ss'} = \text{diag}(1, -1, -1, -1),
\]

and the completeness relation

\[
h_\mu^s g_{ss'} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)
\]

can be parameterized as follows

\[
h_\mu^1 = (S_1, -C_1 a^{(1)}), \tag{E.1}
\]
\[
h_\mu^2 = (C_1 S_2, -S_1 S_2 a^{(1)} - C_2 a^{(2)}), \tag{E.2}
\]
\[
h_\mu^3 = (C_1 C_2 S_3, -S_1 C_2 S_3 a^{(1)} - S_2 S_3 a^{(2)} - C_3 a^{(3)}), \tag{E.3}
\]
\[
h_\mu^0 = (C_1 C_2 C_3, -S_1 C_2 C_3 a^{(1)} - S_2 C_3 a^{(2)} - S_3 a^{(3)}), \tag{E.4}
\]

where \(C_i \equiv \cosh \chi_i, S_i \equiv \sin \chi_i, i = 1, 2, 3\); \(\chi_i\) are the Eulerian pseudoangles for the Lorentz boost. The 4-vectors \(h_\mu^k, k = 1, 2, 3\) are space-like, whereas \(h_\mu^0\) is time-like. Similar to the triad \(a^{(i)}\) the components of the tetrad \(h_\mu^s\) constitute the rotation matrix in Minkowski space [88]. Here, the other parameterizations of the 4-vectors \(h_\mu^s\) are permissible. One of them is mentioned at the end of Section 5.

### Appendix F  An extended system of bilinear identities

Here we represent another way to deal with the Dirac spinors \(\psi_D\) and \(\theta_D\) without recourse to their decomposition in terms of Majorana spinors \(\psi_M^{(i)}\) and \(\theta_M^{(i)}, i = 1, 2, \) Eq. (10.3). For this purpose we shall use our basic formula of a product of the two expansions \((2.1)\) and \((2.2)\), namely Eq. (10.5). In addition to the expansions \((2.1)\) and \((2.2)\) we need also the ones of the spinor structure\(^{14}\) \(h^{1/2} \bar{\psi}_\beta \psi_\alpha\) and \(h^{1/2} \bar{\theta}_\beta \theta_\alpha\), correspondingly:

\[
h^{1/2} \bar{\psi}_\beta \psi_\alpha = \frac{1}{4} \left\{ S_\psi \delta_\alpha_\beta + V_\psi^{\mu} (\gamma_\mu)_{\alpha_\beta} - \frac{i}{2} T_\psi^{\mu\nu} (\sigma_{\mu\nu})_{\alpha_\beta} - A_\psi^{\mu} (\gamma_\mu \gamma_5)_{\alpha_\beta} - i P_\psi (\gamma_5)_{\alpha_\beta} \right\}, \tag{F.1}
\]
\[
h^{1/2} \bar{\theta}_\beta \theta_\alpha = \frac{1}{4} \left\{ S_\theta \delta_\alpha_\beta + V_\theta^{\mu} (\gamma_\mu)_{\alpha_\beta} - \frac{i}{2} T_\theta^{\mu\nu} (\sigma_{\mu\nu})_{\alpha_\beta} - A_\theta^{\mu} (\gamma_\mu \gamma_5)_{\alpha_\beta} - i P_\theta (\gamma_5)_{\alpha_\beta} \right\}. \tag{F.2}
\]

Here, the \textit{real commuting} tensor variables on the right-hand side are defined as follows

\[
S_\psi \equiv h^{1/2} (\bar{\psi} \psi), \quad V_\psi^{\mu} \equiv h^{1/2} (\bar{\psi} \gamma^\mu \psi), \quad * T_\psi^{\mu\nu} \equiv i h^{1/2} (\bar{\psi} \sigma_{\mu\nu} \gamma_5 \psi),
\]
\[
A_\psi^{\mu} \equiv h^{1/2} (\bar{\psi} \gamma^\mu \gamma_5 \psi), \quad P_\psi \equiv i h^{1/2} (\bar{\psi} \gamma_5 \psi), \tag{F.3}
\]

and similarly for \((S_\theta, V_\theta^{\mu}, T_\theta^{\mu\nu}, A_\theta^{\mu}, P_\theta)\) with the replacement \(\psi \rightarrow \theta\). Recall that the latter tensor variables are nilpotent. In particular, in terms of these variables the conditions under which the spinors \(\psi_\alpha\) and \(\theta_\alpha\) are Majorana ones, Eq. (2.7), take the form

\[
S_\psi = A_\psi^{\mu} = P_\psi = 0, \quad V_\theta^{\mu} = * T_\theta^{\mu\nu} = 0. \tag{F.4}
\]

---

\(^{14}\) For simplicity of notations we shall drop the symbol D for the Dirac spinors.
We find the first system of identities from the expression (2.5) by contracting the left- and right-hand sides with every possible combinations of the type
\[ \delta_{\beta\delta}\gamma_{\alpha}, \quad \delta_{\beta\delta}(\gamma_{5})\gamma_{\alpha}, \quad (\gamma_{5})\delta_{\beta}\gamma_{\alpha}, \quad \delta_{\beta\delta}(\gamma_{\mu})\gamma_{\alpha}, \quad (\gamma_{\mu})\delta_{\beta}\gamma_{\alpha}, \quad \delta_{\beta\delta}(\gamma_{\lambda})\gamma_{\alpha}, \quad (\gamma_{\lambda})\delta_{\beta}\gamma_{\alpha}, \] etc. The equations (2.6), (2.8) and (2.18) are just special case of these contractions. In terms of quantities (F.3) the equations (2.6) and (2.18) become
\[ S_{\theta}S_{\psi} = \frac{1}{4} \left\{ SS^{*} + V_{\mu}(V^{\mu})^{*} - \frac{1}{2} T_{\mu\nu}(T^{\mu\nu})^{*} - A_{\mu}(A^{\mu})^{*} - PP^{*} \right\}, \quad (F.6) \]
\[ S_{\theta}V_{\mu} = \frac{1}{4} \left\{ -[S(V^{\mu})^{*} - V^{\mu}S^{*}] + [P(A^{\mu})^{*} - A^{\mu}P] - \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} [V_{\nu}(T_{\lambda\sigma})^{*} - T_{\lambda\sigma}(V_{\nu})^{*}] + [A_{\nu}(T^{\mu\nu})^{*} - T^{\mu\nu}(A_{\nu})^{*}] \right\}, \]

etc. Further, by analogy with (2.5) we define a product of the expansions (F.1) and (F.2) between the various functions equalities (2.4) and (F.4) hold, we have to reproduce the system (C.1) – (C.15).

These systems follow from the expansions (F.1) and (F.2). This enables us to close a system (virtue of the explicit form of the right-hand side of the expression (2.8)), namely
\[ S(T^{\mu\nu})^{*} - T^{\mu\nu}S^{*}, \quad *T^{\mu\nu}P^{*} - P(*T^{\mu\nu})^{*}, \quad V^{\mu}(V^{\nu})^{*} - V^{\nu}(V^{\mu})^{*}, \quad A_{\mu}(A^{\nu})^{*} - A^{\nu}(A_{\mu})^{*}, \]
\[ \epsilon_{\mu\nu\lambda\sigma} [V_{\nu}(A_{\lambda})^{*} + A_{\lambda}(V_{\nu})^{*}], \quad *T^{\mu\lambda}(T_{\mu\nu})^{*} - T^{\nu\lambda}(T_{\mu\nu})^{*}. \]

This circumstance should essentially facilitate the problem of a search for the independent identities.
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