Optimal stability estimates for a magnetic Schrödinger operator with local data

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Abstract

In this paper we prove identifiability and stability estimates for a local-data inverse boundary value problem for a magnetic Schrödinger operator in dimension \( n \geq 3 \). We assume that the inaccessible part of the boundary is part of a hyperplane. We improve the identifiability result obtained by Krupchyk et al (2012 Commun. Math. 313 87–126) and also derive the corresponding stability estimates. We obtain log-estimates for magnetic and electric potentials.

Keywords: Magnetic Schrödinger operator, Dirichlet–Neumann map, Carleman estimates, complex geometric optics solutions, fourier transform, Riemann–Lebesgue lemma

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set with boundary denoted by \( \partial \Omega \). We work with the following magnetic Schrödinger operator:

\[
\mathcal{L}_{A,q}(x,D) := \sum_{j=1}^{n} (D_j + A_j(x))^2 + q(x) = D^2 + A \cdot D + D \cdot A + A^2 + q, \tag{1.1}
\]

where \( D = -i\nabla \), \( A = (A_j)_{j=1}^{n} \in L^\infty(\Omega; \mathbb{C}^n) \) and \( q \in L^\infty(\Omega; \mathbb{C}) \) represent a magnetic and electric potentials, respectively. Here \( A^2 = \sum_{k=1}^{n} A_j^2 \). Throughout this article we are considering the operator \( \mathcal{L}_{A,q} : H^1(\Omega) \rightarrow H^{-1}(\Omega) \) in a weak sense as follows:

\[
\langle \mathcal{L}_{A,q} u, v \rangle = \int_{\Omega} D u \cdot \overline{D v} + A \cdot (D u \overline{v} + u \overline{D v}) + (A^2 + q) u \overline{v},
\]

for any \( u \in H^1(\Omega) \) and \( v \in H^1_0(\Omega) \). The space \( H^{-1}(\Omega) \) denotes the dual space of \( H^1_0(\Omega) \). The existence of a solution for the Dirichlet problem associated to this operator is ensured by theorem 9.15 in [6]. The inverse boundary value problem (IBVP for short) under consideration in...
this paper is to recover information (inside $\Omega$) about the magnetic and electric potentials from measurements on subsets of the boundary. For a treatment of a more general case we refer the reader to [1], where Bellassoued studied a similar IBVP for the dynamical Schrödinger operator in the context of Riemannian manifolds.

We start by describing the geometric framework and some earlier results. We divide the boundary $\partial \Omega$ in two subsets $\Gamma_0$ and its complementary $\Gamma := \partial \Omega \setminus \Gamma_0$. We shall call $\Gamma_0$ the inaccessible part of the boundary and $\Gamma$ the accessible part. When $\Gamma = \partial \Omega$, we say that the IBVP has global data. In other cases, we say that the IBVP has local data. In the latter case, by assuming that zero is not a Dirichlet eigenvalue of $L_{A,q}$ and if $\Omega$ and $A$ are smooth enough then we can define the local Dirichlet–Neumann map (DN map for short) by

$$
\Lambda_{A,q}^\Gamma : H^1(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega)
$$

$$
f \mapsto (\partial_n + iA \cdot \nu)u|\Gamma.,
$$

(1.2)

Here, by abuse of notation, $|\Gamma$ denotes the local trace map of functions in $H^1(\Omega)$ onto the accessible part $\Gamma$ of the boundary. The vector $\nu$ denotes the exterior unit normal of $\partial \Omega$, the set $H^1(\partial \Omega)$ consists of all $f \in H^1(\partial \Omega)$ such that $\text{supp} f \subset \Gamma$ (we describe this condition as 'support constraint on $\Gamma$'), and $u \in H^1(\Omega)$ is the unique solution of the Dirichlet problem:

$$
\begin{align*}
\{ \mathcal{L}_{A,q} u = 0 & \quad \text{in } \Omega \\
|u|_{\partial \Omega} = f & \quad .
\end{align*}
$$

(1.3)

In this case, the local boundary data $B_{A,q}^\Gamma$ is a subset of $H^\frac{1}{2}(\partial \Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$ and it is given by the graph of $\Lambda_{A,q}^\Gamma$:

$$
B_{A,q}^\Gamma = \{(u|_{\partial \Omega}, \Lambda_{A,q}^\Gamma(u|_{\partial \Omega})) : u \in H^1(\Omega), \mathcal{L}_{A,q} u = 0 \text{ in } \Omega\}.
$$

(1.4)

As it was noted by Sun [16], the global DN map $\Lambda_{A,q}^{\partial \Omega}$ has a gauge invariance: if $\varphi \in C^1(\Omega)$ is a real valued function with $\varphi|_{\partial \Omega} = 0$ then $\Lambda_{A,q}^{\partial \Omega} = \Lambda_{A+\nabla \varphi,q}^{\partial \Omega}$. Because of this and for the global case, the identity $\Lambda_{A,q_1}^{\partial \Omega} = \Lambda_{A,q_2}^{\partial \Omega}$ implies $dA_1 = dA_2$ and $q_1 = q_2$, see for instance [16] and [12]. For this case, stability estimates were derived by Tzou [17]. It easily follows that the local DN map has the same gauge invariance and so from the knowledge of $\Lambda_{A,q}^{\Gamma}$ we only expect to recover $dA$ and $q$. Actually, it was proved by Krupchyk, Lassas and Uhlmann [9] by assuming that the inaccessible part of the boundary $\Gamma_0$ is contained in a hyperplane, the magnetic potential $A \in W^{1,\infty}(\Omega)$ and $q \in L^\infty(\Omega)$, see theorem 1.4 in [9] for more details. Here, if $A := (A^{(1)}, A^{(2)}, \ldots, A^{(n)})$ then $dA = 0$ means that $\partial_n A^{(j)} = \partial_n A^{(k)}$ for all $j, k = 1, 2, \ldots, n$.

The main goal of this paper is to improve the previous local identifiability result [9] and also derive the corresponding stability estimates when recovering $dA$ and $q$. A priori, we are not assuming any smoothness condition neither for the magnetic potential and the domain $\Omega$, except that the inaccessible part of the boundary $\Gamma_0$ is again contained in a hyperplane. More precisely, we assume that:

$$
\begin{align*}
\Omega & \subset \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}, \\
\Gamma_0 & = \partial \Omega \cap \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0 \} \neq \emptyset.
\end{align*}
$$

(1.5)

Throughout this paper, unless otherwise indicated, the domain $\Omega$ and the inaccessible part of the boundary $\Gamma_0$ satisfy (1.5). Recall that we have assumed that zero is not a Dirichlet eigenvalue of $L_{A,q}$. It might be unnatural because to be an eigenvalue depends strongly on the coefficients of the operator $L_{A,q}$, which at the same time depends on $A$ and $q$, just the unknown parameters. In our case, we leave out this eigenvalue assumption. In this lack of
smoothness, we could not ensure, a priori, the existence and uniqueness of solutions for (1.3). As consequence, the local trace map \( | \Gamma | \) has no sense and so the local DN map defined by (1.2) is not well-defined. For these reasons, we extend their definitions by the boundary local map \( T_r^\Gamma \) and the local linear map \( N_{A,q}^\Gamma \). Firstly, we introduce the space \( H^1(\Omega, \Gamma) \) defined by density as follows:

\[
H^1(\Omega, \Gamma) := \left\{ u \in C^\infty(\Omega) : u|_{\Gamma_0} = 0 \right\}^{H^1(\Omega)},
\]

where \( E^{H^1(\Omega)} \) denotes the closure of the set \( E \) in the topology of \( H^1(\Omega) \), with the norm \( H^1(\Omega) \) restricted to \( E \). Thus, we define the local boundary map \( T_r^\Gamma \) as follows:

\[
T_r^\Gamma : H^1(\Omega, \Gamma) \to H^1(\Omega, \Gamma)/H^1_0(\Omega), \quad T_r^\Gamma u = [u]^\Gamma,
\]

where \([\cdot]^\Gamma\) denotes the equivalence class in the quotient space \( H^1(\Omega, \Gamma)/H^1_0(\Omega) \). The local linear map \( N_{A,q}^\Gamma : H^1(\Omega, \Gamma)/H^1_0(\Omega) \to \langle H^1(\Omega, \Gamma)/H^1_0(\Omega) \rangle \) (\( \langle \cdot \rangle \) denotes the dual space of \( J \)) is defined by

\[
\langle N_{A,q}^\Gamma [u]^\Gamma, [g]^\Gamma \rangle = \int_{\Omega} Du \cdot Dv + A \cdot (Du \cdot Dv + uDv) + (A^2 + q)u\overline{v},
\]

for all \( u \in H^1(\Omega, \Gamma) \) satisfying \( \mathcal{L}_{A,q}u = 0 \) in \( \Omega \), and for any \( v \in [g] \) with \( g \in H^1(\Omega, \Gamma) \). With these definitions at hand, we define the local Cauchy data set \( C_{A,q}^\Gamma \) instead of the boundary local data \( B_{A,q}^\Gamma \), as follows

\[
C_{A,q}^\Gamma = \left\{ ([u]^\Gamma, N_{A,q}^\Gamma [u]^\Gamma) : u \in H^1(\Omega, \Gamma), \mathcal{L}_{A,q}u = 0 \text{ in } \Omega \right\}.
\]

Analogously to the global DN map \( A_{\lambda q}^{\Omega \Gamma} \), the global Cauchy data set \( C_{\lambda q}^{\Omega \Gamma} \) has a gauge invariance: if \( \varphi \) is a real valued Lipschitz continuous function on \( \overline{\Omega} \) with \( \varphi|_{\partial \Omega} = 0 \) then \( C_{\lambda q}^{\Omega \Gamma} = C_{\lambda + \nabla \varphi, q}^{\Omega \Gamma} \), see lemma 3.1 in [10]. Krupchyck and Uhlmann proved the identifiability result for the global data case, see theorem 1.1 in [10]. For this case, Caro and Pohjola [3] have derived the corresponding estability estimates. In the local data case, we prove the following identifiability result.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( A_1, A_2 \in L^\infty(\Omega; \mathbb{C}^n) \) be two magnetic potentials and \( q_1, q_2 \in L^\infty(\Omega; \mathbb{C}) \) be two electric potentials. If \( C_{A_1, q_1}^\Gamma = C_{A_2, q_2}^\Gamma \), then \( dA_1 = dA_2 \) and \( q_1 = q_2 \) in \( \Omega \).

In order to obtain stability estimates corresponding to these identifiability results, we need a priori bounds for \( A \) and \( q \), in order to control their oscillations. To do that, we introduce the Besov spaces and the notion of admissible class for both the magnetic and electric potentials.

**Remark 1.2.** According to proposition 3.6 in [14], see also lemma 1.1 in [5], if \( \Omega \) is a Lipschitz domain then the characteristic function \( \chi_\Omega \) belongs to \( H^s(\mathbb{R}^n) \) with \( s \in (0, 1/2) \). Motivated by this fact, we introduce the Besov space \( B^s_{2, \infty}(\mathbb{R}^n; \mathbb{C}) \) and \( B^s_{2, \infty}(\mathbb{R}^n; \mathbb{C}^n) \) with \( s \in (0, 1/2) \).

Following [3], for given \( s > 0 \) we define the Besov space \( B^s_{2, \infty}(\mathbb{R}^n; \mathbb{C}) \) as the space consisting of all functions \( f \in L^2(\mathbb{R}^n; \mathbb{C}) \) for which the norm

\[
\|f\|_{B^s_{2, \infty}(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} + \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{|f(\cdot + y) - f(\cdot)|_{L^2(\mathbb{R}^n)}}{|y|^s}.
\]

(1.10)
is finite. A vectorial function $F = (F_1, \ldots, F_n)$ belongs to the Besov space $B^s_{1,1}(\mathbb{R}^n; \mathbb{C}^n)$ if each one of $F_j$ belongs to $B^s_{1,1}(\mathbb{R}^n; \mathbb{C})$ with $j = 1, \ldots, n$.

**Definition 1.3.** Given $M > 0$ and $s \in (0, 1/2)$, we define the class of admissible magnetic potentials $\mathcal{A}(\Omega, M, s)$ by

$$\mathcal{A}(\Omega, M, s) = \left\{ F \in L^\infty \cap B^s_{1,1}(\mathbb{R}^n; \mathbb{C}^n) : \text{supp } F \subset \overline{\Omega}, \| F \|_{L^\infty \cap B^s_{1,1}} \leq M \right\}.$$  

**Definition 1.4.** Given $M > 0$ and $s \in (0, 1/2)$, we define the class of admissible electric potentials $\mathcal{E}(\Omega, M, s)$ by

$$\mathcal{E}(\Omega, M, s) = \left\{ G \in L^\infty \cap B^s_{1,1}(\mathbb{R}^n; \mathbb{C}) : \text{supp } G \subset \overline{\Omega}, \| G \|_{L^\infty \cap B^s_{1,1}} \leq M \right\}.$$  

Here $\| \cdot \|_{L^\infty \cap B^s_{1,1}}$ denotes $\| \cdot \|_{L^\infty(\mathbb{R}^n)} + \| \cdot \|_{B^s_{1,1}(\mathbb{R}^n)}$.

In a regular framework the proximity between two local boundary data sets is given by the norm operator $\| A_{k, q_j}^C - A_{k, q_j}^C \|_{H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)}$. Following [3], in our non regular framework, instead of the norm operator, we introduce the notion of pseudo-distance between two local-data Cauchy sets, denoted by $\text{dist}(\cdot, \cdot)$. Let $A_1, A_2 \in L^\infty(\Omega; \mathbb{C}^n)$ be two magnetic potentials and let $q_1, q_2 \in L^\infty(\Omega)$ be two electric potentials. Given $(f, g) \in C_{A_1, q_j}^r$ with $j = 1, 2$, we set

$$I((f, g); C_{A_1, q_j}^r) = \inf_{(h, q_k) \in C_{A_1, q_j}} \left[ \| f - h \|_{H^1(\Omega)} + \| g - q_k \|_{H^1(\Omega)} \right],$$

with $k = 1, 2$. Then the pseudo-distance between $C_{A_1, q_j}^r$ and $C_{A_2, q_j}^r$ is defined by

$$\text{dist}(C_{A_1, q_j}^r, C_{A_2, q_j}^r) = \max_{j = 1, 2} \sup_{(f, g) \in C_{A_1, q_j}^r} I((f, g); C_{A_1, q_j}^r).$$ (1.11)

In the remainder of this paper, we denote the local Cauchy data set $C_{A, q_j}^r$ by $C_j^r$, for $j = 1, \ldots, n$. For an open bounded set $E \subset \mathbb{R}^n$ and a function $h : E \to \mathbb{C}$ (or $\mathbb{C}^n$), we denote by $\chi_E h$ the extension by zero of $h$ out of $E$. We can now formulate our stability results.

**Theorem 1.5 (Stability for the magnetic field).** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Consider two constants $M > 0$ and $s \in (0, 1/2)$. Then there exist $C > 0$ (depending on $n, \Omega, M, s, \| q_1 \|_{L^\infty}, \| q_2 \|_{L^\infty}$) and an universal constant $\lambda \in (0, 1)$ such that the following estimate

$$\|d(A_1 - A_2)\|_{H^{-1}(\Omega)} \leq C \| \log \text{dist}(C_{A_1, q_j}^r, C_{A_2, q_j}^r)\|^{-\lambda},$$

holds true for all $\chi_\Omega A_1, \chi_\Omega A_2 \in \mathcal{A}(\Omega, M, s)$ and for all $q_1, q_2 \in L^\infty(\Omega)$, whenever

$$\text{dist}(C_{A_1, q_j}^r, C_{A_2, q_j}^r) \leq e^{-C}.$$  

**Theorem 1.6 (Stability for the electric potential).** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Consider two constants $M > 0$ and $s \in (0, 1/2)$. Then there exist $C > 0$ (depending on
and an universal constant $\lambda \in (0, 1)$ such that the following estimate

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \log \text{dist} (C_{A_1}, C_{A_2})^{-\lambda},$$

holds true for all $\chi_\Omega A_1, \chi_\Omega A_2 \in \mathcal{A}(\Omega, M, s)$ and for all $\chi_\Omega q_1, \chi_\Omega q_2 \in \mathcal{D}(\Omega, M, s)$, whenever

$$\text{dist} (C_{A_1}, C_{A_2}) \leq e^{-C}.$$

The proofs of these theorems will be carried out by deriving and integral inequality relating the local Cauchy data sets with the unknown magnetic and electric potentials in $\Omega$, by means of solutions $u \in H^1(\Omega)$ for the magnetic Schrödinger equation $L_{A_0}u = 0$ with $u|_{\Gamma_0} = 0$. These solutions can be constructed by combining the solutions constructed in [10] and a reflection argument across $\Gamma_0$. It is here that the shape of $\Omega$ plays a crucial role. The original idea of reflecting goes back to Isakov’s work on identifiability for $A_0$ and, see [8]. The integral inequality with these solutions lead us to Fourier transforms, one for the difference of the magnetic fields $dA_1 - dA_2$ and another for the difference of the electric potentials $q_1 - q_2$, plus some error terms. At this point, we use the Riemann–Lebesgue lemma to control the error terms and the invariance of the Cauchy data sets in order to use the already established stability estimate for the magnetic potentials in theorem 1.5. Our results imply log-estimates which seems to be the best stability modulus that one can expect as it was proved by Mandache [11] in the case $A_0$.

Now we turn to mention the results about the IBVP with local data when the boundary $\partial \Omega$ is smooth enough, that is taking into account the local DN map (1.2). In this case, the first identifiability result in the absence of a magnetic potential ($A \equiv 0$) was obtained by Isakov [8]. He proved that if $A_0| q_1 = A_0| q_2$ then $q_1 = q_2$, by assuming that the inaccessible part of the boundary is either part of a plane or a sphere. The corresponding stability estimate to Isakov’s result was derived by Heck and Wang [7], obtaining a log-estimate. Similar arguments were employed by Caro [2] to study an IBVP with local data for the Maxwell operator under the same flatness condition on $\Gamma_0$, obtaining a stability estimate of log type.

As far as we know, there are no results about identifiability for an IBVP with local data for an arbitrary open set. Nevertheless, the analogous of Calderón’s result for the linearization of the DN map in the case of a conductivity, has been proved by Dos Santos Ferreira, Kening, Sjöstrand and Uhlmann [4], in the case of local data for the Schrödinger equation in the absence of a magnetic potential.

Throughout this paper we denote by $C_i (i \in \mathbb{Z}^*)$ a positive constants which might change from formula to formula. These constants depend only on $n, \Omega$ and the priori bounds for magnetic and electric potentials.

This paper is organized as follows. In section 2 we prove theorem 1.5. In section 3 we prove theorem 1.6. Finally, in section 4 we prove theorem 1.1.

## 2. Stability for the magnetic field

### 2.1. CGO solutions for a magnetic Schrödinger operator

The main result in this section is theorem 2.1, which ensures the existence of solutions $u \in H^1(\Omega)$ for the magnetic Schrödinger equation $L_{A_0}u = 0$.

**Theorem 2.1.** Let $V \subset \mathbb{R}^n$ be a bounded open set. Consider $s \in (0, 1/2)$. Let $A \in L^\infty \cap B^{\infty}_2(\mathbb{R}^n, \mathbb{C})$ and $q \in L^\infty(\mathbb{R}^n, \mathbb{C})$ such that $\text{supp}A \subset \overline{V}$ and $\text{supp}q \subset \overline{V}$.
Consider \( \rho \in \mathbb{C}^n \) such that \( \rho \cdot \rho = 0 \) and \( \rho = \rho_0 + \rho_r \) with \( \rho_0 \) being independent of some large parameter \( \tau > 0 \), \( |\mathcal{R}\rho_0| = |\mathcal{R}\rho_0| = 1 \) and \( \rho_r = \mathcal{O}(\tau^{-1}) \) as \( \tau \to \infty \). Then there exist two positive constants \( C \) and \( \tau_0 \) (both depending on \( n, V, s, \|A\|_{L^{\infty}} \cdot \|q\|_{L^{\infty}} \) ; and a solution \( u \in H^1(V) \) to the equation \( \mathcal{L}_{A,q} u = 0 \) in \( V \) of the form
\[
  u(x, \rho; \tau) = e^{\tau r x} \left( e^{\tau i(x, \rho; \tau)} + r(x, \rho; \tau) \right)
\]
with the following properties:

(i) The function \( \Phi^\tau(\cdot, \rho_0; \tau) \in C^\infty(\mathbb{R}^n) \) and satisfies for all \( \alpha \in \mathbb{N}^n \)
\[
  \|\partial^\alpha \Phi^\tau(\cdot, \rho_0; \tau)\|_{L^\infty(\mathbb{R}^n)} \leq C \tau^{(|\alpha|/2 + 2)} \|A\|_{L^\infty(\mathbb{R}^n)}, \quad \tau \geq \tau_0. \tag{2.1}
\]

(ii) The function \( r(\cdot, \rho_0; \tau) \in H^1(V) \) and satisfies
\[
  \|\partial^\alpha r(\cdot, \rho_0; \tau)\|_{L^2(V)} \leq C \tau^{-|\alpha|-s/2}, \quad |\alpha| \leq 1. \tag{2.2}
\]

(iii) If we define by \( \kappa := \sup_{x \in \mathbb{R}^n} |x| \) then the solution \( u \) satisfies
\[
  \|u\|_{H^1(V)} \leq C e^{\tau \kappa |\rho|}. \tag{2.3}
\]

Moreover, if we denote by \( \Phi(\cdot; \rho_0) = (\rho_0 \cdot \nabla)^{-1}(-i\rho_0 \cdot A) \in L^\infty(\mathbb{R}^n) \) the function satisfying the following equation in \( \mathbb{R}^n \)
\[
  \rho_0 \cdot \nabla \Phi + i\rho_0 \cdot A = 0 \tag{2.4}
\]
then
\[
  \|\Phi(\cdot; \rho_0)\|_{L^\infty(\mathbb{R}^n)} \leq C \|A\|_{L^\infty(\mathbb{R}^n)}. \tag{2.5}
\]

Finally, for every \( \chi \in C_0^\infty(\mathbb{R}^n) \) we have
\[
  \|\chi(\Phi^\tau(\cdot, \rho_0; \tau) - \Phi(\cdot; \rho_0))\|_{L^2(\mathbb{R}^n)} \leq C \tau^{-s/2} \|A\|_{L^\infty(\mathbb{R}^n)}, \tag{2.6}
\]
where the constant \( C \) also depends on \( \chi \).

**Remark 2.2.** This theorem is a brief summary of two known results. When \( A \in L^\infty \) and \( q \in L^\infty \), the existence of solutions \( u \in H^1(V) \)of the equation \( \mathcal{L}_{A,q} u = 0 \) it was proved by Krupchyk and Uhlmann, see proposition 2.6 in [10]. When \( A \in L^\infty \cap B_2^{\infty, \infty} \) and \( q \in L^\infty \), the estimates for \( \Phi^\tau(\cdot, \rho_0; \tau) \), \( \Phi(\cdot; \rho_0) \) and \( r(\cdot, \rho_0; \tau) \) have been derived by Caro and Pohjola, see section 3 in [3]. For these reasons we only give the main ideas of the proof with the repetition of the relevant material from [3] and [10], thus making our exposition self-contained.

**Proof.** From [10], it follows that there exists \( u \in H^1(V) \) satisfying \( \mathcal{L}_{A,q} u = 0 \) of the form
\[
  u(x) = e^{\tau r x}(a + r), \tag{2.7}
\]
where the functions \( a \) and \( r \) satisfy (2.10) and (2.11), respectively. The construction of such a solution involves basically two arguments. The first argument concerns a mollification procedure for the magnetic potential \( A \). More precisely: consider \( \varphi \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \varphi \leq 1 \) and \( \text{supp} \varphi \subseteq B_1(0) \), where \( B_1(0) \) denotes the closure of the unit ball in \( \mathbb{R}^n \). Given \( \delta > 0 \) we define \( \varphi_\delta(x) = \delta^{-n}\varphi(x/\delta) \) and the smooth compact supported function \( A_\delta^\tau = A * \varphi_\delta \), where * denotes the convolution. Then there exist two positive constants \( C_1 \) and \( C_2 \) such that
\[ \left\| A - A^\delta \right\|_{L^r(R^n)} \leq C_1 \delta^\tau \| A \|_{L^{r\infty}(R^n)} \] (2.8)

and

\[ \left\| \partial^\alpha A^\delta \right\|_{L^\infty(R^n)} \leq C_2 \delta^{1-|\alpha|} \| A \|_{L^n(R^n)}, \quad \alpha \in \mathbb{N}^n. \] (2.9)

See section 3 in [3] for more details. The second argument involves the use of a Carleman estimate (for the Laplacian operator) between \( H^1(V) \) and its dual space \( H^{-1}(V) \). By a straightforward computation, \( u \) defined by (2.7) is a solution of \( \mathcal{L}_{\lambda_0} u = 0 \) if \( a \) and \( r \) satisfy the following identity in \( H^{-1}(V) \):

\[
0 = \tau^{-2} \mathcal{L}_{\lambda_0} a - \tau^{-1} \left( 2i \rho_r \cdot Da + 2i \rho_0 \cdot (A - A^\delta) a + 2i \rho_r \cdot A^\delta a \right) - \tau^{-1} \left( 2i \rho_0 \cdot Da + 2i \rho_0 \cdot A^\delta a \right) + e^{-\tau^\rho \cdot \tau^{-2}} \mathcal{L}_{\lambda_0} (e^{\tau^\rho \cdot r}).
\]

We impose \( a \) being a solution of the transport equation:

\[ \rho_0 \cdot \nabla a + i \rho_0 \cdot A^\delta a = 0 \] (2.10)

and \( r \) satisfying

\[
e^{-\tau^\rho \cdot \tau^{-2}} \mathcal{L}_{\lambda_0} (e^{\tau^\rho \cdot r}) = -\tau^{-2} \mathcal{L}_{\lambda_0} a + 2i \tau^{-1} \left( \rho_r \cdot Da + \rho_0 \cdot (A - A^\delta) a + \rho_r \cdot A^\delta a \right).
\]

The transport equation (2.10) can be solved as follows. We are looking for solutions of the form \( a = e^{\Phi^\delta} \). Then \( \Phi^\delta \) satisfies the equation

\[ \rho_0 \cdot \nabla \Phi^\delta + i \rho_0 \cdot A^\delta = 0. \] (2.12)

Now this equation can be solved from the invertibility (in suitable spaces) of the operator \( \rho_0 \cdot \nabla \). Thus, the function \( \Phi^\delta := (\rho_0 \cdot \nabla)^{-1}(-i \rho_0 \cdot A^\delta) \) satisfies (2.12). Moreover, by combining lemma 2.1 in [16] with (2.9), we obtain

\[ \left\| \partial^\alpha \Phi^\delta \right\|_{L^\infty(R^n)} \leq C_3 \delta^{-|\alpha|} \| A \|_{L^n(R^n)}, \quad \alpha \in \mathbb{N}^n. \] (2.13)

For more details about the invertibility of the operator \( (\rho_0 \cdot \nabla)^{-1} \), see for instance section 4.3 in [13]. For similar reasons as above, the function \( \Phi(\cdot; \rho_0) := (\rho_0 \cdot \nabla)^{-1}(-i \rho_0 \cdot A) \in L^{\infty}(R^n) \) solves the following equation

\[ \rho_0 \cdot \nabla \Phi + i \rho_0 \cdot A = 0. \]

and satisfies

\[ \| \Phi(\cdot; \rho_0) \|_{L^\infty(R^n)} \leq C_4 \| A \|_{L^\infty(R^n)}. \]

From (2.8), it follows that for every \( \chi \in C^\infty_0(R^n) \) there exist a positive constant \( C_5 \) such that

\[ \| \chi(\Phi^\delta(\cdot; \rho_0) - \Phi(\cdot; \rho_0)) \|_{L^2(R^n)} \leq C_5 \delta^\tau \| A \|_{L^{2\infty}(R^n)}. \]
See section 3 in [3] for more details. We now turn to discuss the existence of \( r \) satisfying (2.11). We start by setting
\[
g := -\tau^{-2}L_{A,a} + 2i\tau^{-1} \left( \rho_\tau \cdot Da + \rho_0 \cdot (A - A_0^2) a + \rho_\tau \cdot A a \right).
\]
Then by proposition 2.3 in [10], there exists \( r \in H^1(V) \) satisfying (2.11) and two positive constants \( C_0 \) and \( \tau_0 \) such that
\[
\|r\|_{H^1_\tau(V)} \leq C_0 \tau \|g\|_{H^{-1}_\tau(V)},
\]
for all \( \tau \geq \tau_0 \). Here the semi-classical norms are defined by
\[
\|r\|^2_{H^1_\tau(V)} = \|r\|_{L^2(V)}^2 + \tau^{-1} \|\nabla r\|_{L^2(V)}^2,
\]
\[
\|g\|^2_{H^{-1}_\tau(V)} = \sup_{\theta \neq \phi \in C^\infty_0(V)} \frac{\langle w, \phi \rangle_{L^2(V)}}{\|\phi\|_{H^1_\tau(V)}}.
\]
We define \( \kappa := \sup_{x \in \Omega} |x| \). From (2.13) and taking \( \delta = \tau^{-1/(\epsilon+2)} \) into (2.9), we get
\[
\|g\|_{H^{-1}_\tau(V)} \leq C_\epsilon e^{\kappa||A||_{L^\infty} \tau^{-2(\epsilon+2)/\epsilon+2}} \times \left( 1 + ||A||_{L^\infty} + ||A||_{L^\infty}^2 + ||q||_{L^\infty} + ||A||_{L^\infty}^2 \right).
\]
By combining the above inequality with (2.14), we obtain
\[
\|r\|_{H^1_\tau(V)} \leq C_\delta e^{\kappa||A||_{L^\infty} \tau^{-2/\epsilon+2}} \times \left( 1 + ||A||_{L^\infty} + ||A||_{L^\infty}^2 + ||q||_{L^\infty} + ||A||_{L^\infty}^2 \right).
\]
Finally, by similar computations as above, we obtain
\[
\|u\|_{H^1(V)} \leq C_\delta e^{\kappa||A||_{L^\infty} \tau^{-2/\epsilon+2}} \times \left( 1 + ||A||_{L^\infty} + ||A||_{L^\infty}^2 + ||q||_{L^\infty} + ||A||_{L^\infty}^2 \right).
\]
These inequalities will be useful in our approach.

**Remark 2.3 (Estimates to prove theorem 1.1).** To prove theorem 1.1 we are only assuming that \( A \in L^\infty(\Omega; \mathbb{C}^n) \) and \( q \in L^\infty(\Omega; \mathbb{C}) \). In this case, we still have similar estimates to (2.1)–(2.6), see proposition 2.6 in [10]. These estimates can be stated as follows. There exist two positive constants \( C \) and \( \tau_0 \) such that the function \( \Phi^\sharp(\cdot, \rho_0; \tau) := (\rho_0 \cdot \nabla)(-i\rho_0 \cdot A^\sharp_0) \) satisfies
\[
\|\partial^\alpha \Phi^\sharp(\cdot, \rho_0; \tau)\|_{L^\infty(\mathbb{R}^n)} \leq C \tau^{-|\alpha|}, \quad \alpha \in \mathbb{N}^n,
\]
for all \( \tau \geq \tau_0 \). By combining proposition 2.3 and the estimate (2.16) both in [10], with \( V \) instead of \( \Omega \), we get
\[
\|r(\cdot, \rho_0; \tau)\|_{H^1_\tau(V)} \leq C \tau^{-1/\sigma} \|g\|_{H^{-1}_\tau(V)} \leq C, \quad 0 < \sigma < 1/2,
\]
which implies
\[ \| \partial^{\alpha} r(\cdot, \rho_0; \tau) \|_{L^2(\Omega)} \leq C \tau^{1/2}, \quad \tau \geq \tau_0, \ |\alpha| \leq 1. \] (2.18)

The estimate (2.5) is the same. Finally, for every \( \chi \in C_0^\infty(\mathbb{R}^n) \) we have
\[ \lim_{\tau \to \infty} \| \chi(\Phi_r(\cdot, \rho_0; \tau) - \Phi_r(\cdot, \rho_0)) \|_{L^2(\mathbb{R}^n)} = 0. \] (2.19)

where the constant \( C \) also depends on \( \chi \). We will not use these estimates until section 5.

2.2. From the boundary to the interior

In this section we state an integral inequality which relates the magnetic and electric potential with the distance between their corresponding Cauchy data sets, see corollary 2.5. We start by stating the following integral identity which was implicitly proved during the proof of proposition 3.2 in [10].

**Lemma 2.4.** Let \( \Omega \) be an open bounded set. Let \( A_1, A_2 \in L^\infty(\Omega; \mathbb{C}^n) \) and \( q_1, q_2 \in L^\infty(\Omega; \mathbb{C}) \). Let \( U_1, U_2 \in H^1(\Omega) \) such that \( \mathcal{L}_{A_1, q_1} U_1 = 0 \) and \( \mathcal{L}_{A_2, q_2} U_2 = 0 \). Then the following identity holds true:
\[
\left\langle N_{A_1, q_1}^\Omega, U_1 \right\rangle_{\partial \Omega} - \left\langle N_{A_2, q_2}^\Omega, U_2 \right\rangle_{\partial \Omega} = \int_\Omega \left| (A_1 - A_2) \cdot (D U_1 U_2^\top + U_1 D U_2^\top) + (A_1^2 - A_2^2 + q_1 - q_2) U_1 U_2 \right| \, dx.
\]

**Corollary 2.5.** Let \( \Omega \) be an open bounded set. Consider \( s \in (0, 1/2) \). Assume that \( A_1, A_2 \in L^\infty(\Omega; \mathbb{C}^n) \) and \( q_1, q_2 \in L^\infty(\Omega; \mathbb{C}) \). Let \( U_1, U_2 \in H^1(\Omega) \) such that \( \mathcal{L}_{A_1, q_1} U_1 = 0 \) with \( U_1 \rvert_{\Gamma_s} = 0 \) and \( \mathcal{L}_{A_2, q_2} U_2 = 0 \) with \( U_2 \rvert_{\Gamma_s} = 0 \). Then there exists a positive constant \( C \) (depending on \( s, \Omega, ||A_j||_{L^\infty(\Omega)} \), \( ||q_j||_{L^\infty(\Omega)} \), \( j = 1, 2 \)) such that
\[
\left( \bigg| \int_\Omega \left| (A_1 - A_2) \cdot (D U_1 U_2^\top + U_1 D U_2^\top) + (A_1^2 - A_2^2 + q_1 - q_2) U_1 U_2 \right| \, dx \bigg| \right)^{1/2} \leq C \text{ dist} (C^\Gamma_1, C^\Gamma_2) \| U_1 \|_{H^1(\Omega)} \| U_2 \|_{H^1(\Omega)},
\] (2.20)

where \( C_j^\Gamma \) denotes the local Cauchy data set \( C_{A_j, q_j}^\Gamma, j = 1, 2 \).

**Proof.** By lemma 2.4 and according to the definition of \( T_j^\Gamma, N_{A_j, q_j}^\Omega \), and \( N_{A_j, q_j}^{\partial \Omega} \), we get
\[
\left( \bigg| \int_\Omega \left| (A_1 - A_2) \cdot (D U_1 U_2^\top + U_1 D U_2^\top) + (A_1^2 - A_2^2 + q_1 - q_2) U_1 U_2 \right| \, dx \bigg| \right)^{1/2} \leq C \text{ dist} (C^\Gamma_1, C^\Gamma_2) \| U_1 \|_{H^1(\Omega)} \| U_2 \|_{H^1(\Omega)}.
\]

Only for this proof, we denote by \( \| \cdot \| \) the norm in the space \( H^1(\Omega, \Gamma) / H^1_0(\Omega) \). Thus, from the definition of \( C_{A_j, q_j}^\Gamma \), it follows that
\[
\left\langle g_2, \left[ U_2^\Gamma \right] \right\rangle_{\partial \Omega} = \left\langle N_{A_2, q_2}^{\partial \Omega}, \left[ U_2^\Gamma \right], [f_2]^\Gamma \right\rangle_{\partial \Omega},
\]

for every \( \left( [f_2]^\Gamma, g_2 \right) \in C_{A_2, q_2}^\Gamma \). As a consequence, we obtain
\[
\left\langle N_{A,q}^T [U_1]^T, [U_2]^T \right\rangle - \left\langle N_{A,q}^T [U_2]^T, [U_1]^T \right\rangle \\
= \left\| [U_1]^T \right\| \left\{ \left\langle N_{A,q}^T [U_1]^T \left\| [U_1]^T \right\|^{-1} - g_2, [U_2]^T \right\rangle - \left\langle N_{A,q}^T [U_2]^T \left\| [U_1]^T \right\|^{-1} - [f_2]^T \right\rangle \right\}. 
\]

From this identity and since the maps \( T_r^T \) and \( N_{A,q}^T \) are bounded, we deduce
\[
\left| \left\langle N_{A,q}^T [U_1]^T, [U_2]^T \right\rangle - \left\langle N_{A,q}^T [U_2]^T, [U_1]^T \right\rangle \right| \\
\leq C \| U_1 \|_{H^1(\Omega)} \| U_2 \|_{H^1(\Omega)} \\
\times I \left( \left\| [U_1]^T \right\| \left\| [U_1]^T \right\|^{-1} - \left\| [U_2]^T \right\| \left\| [U_2]^T \right\|^{-1} \right) N_{A,q}^T \left( C_{A,q,r} \right), 
\]
which, from the definition of \( \text{dist}(C_1^T, C_2^T) \), implies the desired conclusion. \( \square \)

### 2.3. Solutions vanishing on \( \Gamma_0 \)

In this section we use theorem 2.1 to construct solutions \( U \in H^1(\Omega) \) of the equation \( L_{A,q} U = 0 \) with \( U|_{\Gamma_0} = 0 \). To achieve this boundary condition we will use a reflection argument as in [9].

The main result of this section is proposition 2.7.

We start by introducing some notations. For \( x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \), we set \( x^* = (x', -x_n) \). For a function \( g \) with complex or vectorial values, we set \( g^*(x) = g(x^*) \). For a subset \( E \subset \mathbb{R}^n \), we set \( E^* = \{ x^* : x \in E \} \). Finally, for \( \rho \in \mathbb{C}^n \), we define \( \rho^* = (\bar{\rho})^* + i(3\rho)^* \).

Similarly to [9], we extend the magnetic and electric potentials from \( \Omega \) to \( \Omega^* \) by reflection with respect to the plane \( \{ x \in \mathbb{R}^n : x_n = 0 \} \) as follows: We denote a magnetic potential by \( A = (A^{(1)}, A^{(2)}, \ldots, A^{(n-1)}, A^{(n)}) \). Then for \( A^{(k)} \) with \( k = 1, 2, \ldots, n-1 \) we make an even extension \( \widetilde{A}^{(k)} \), and for \( A^{(n)} \) we make an odd extension \( A^{(n)} \).

We denote the extension of \( A \) by \( \widetilde{A} := (\widetilde{A}^{(1)}, \ldots, \widetilde{A}^{(n)}) \).

More precisely, for all \( k = 1, 2, \ldots, n-1 \) we have:
\[
\widetilde{A}^{(k)}(x) = \begin{cases} A^{(k)}(x', x_n), & x \in \Omega \\ A^{(k)}(x', -x_n), & x \in \Omega^* 
\end{cases}
\]

and
\[
\widetilde{A}^{(n)}(x) = \begin{cases} A^{(n)}(x', x_n), & x \in \Omega \\ -A^{(n)}(x', -x_n), & x \in \Omega^*. 
\end{cases}
\]

In the same way, for the electric potential \( q \) we make an even extension \( \widetilde{q} \). More precisely, we have:
\[
\widetilde{q}(x) = \begin{cases} q(x', x_n), & x \in \Omega \\ q(x', -x_n), & x \in \Omega^*. 
\end{cases}
\]

with \( j = 1, \ldots, n \). The following lemma gives us the smoothness properties of these extensions.

**Lemma 2.6.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded set. Let \( A \in L^\infty(\Omega; \mathcal{C}^n) \) and \( q \in L^\infty(\Omega; \mathcal{C}) \). Consider \( M > 0 \) and \( s \in (0, 1/2) \). If \( \chi_{\Omega} A \in \mathcal{S}(\Omega, M, s) \) and \( \chi_{\Omega} q \in \mathcal{D}(\Omega, M, s) \) then \( \chi_{\Omega^*} \widetilde{A} \in \mathcal{S}(\Omega \cup \Omega^*, 2M, s) \) and \( \chi_{\Omega^*} \widetilde{q} \in \mathcal{D}(\Omega \cup \Omega^*, 2M, s) \). Moreover, we have
\[
\left\| \chi_{\Omega \cap \Omega^*} \tilde{A} \right\|_{L^\infty(\mathbb{R}^n)} \leq 2 \left\| \chi_{\Omega} A \right\|_{L^\infty(\mathbb{R}^n)} ,
\]
\[
\left\| \chi_{\Omega \cap \Omega^*} \tilde{q} \right\|_{L^\infty(\mathbb{R}^n)} \leq 2 \left\| \chi_{\Omega} q \right\|_{L^\infty(\mathbb{R}^n)} ,
\]

**Proof.** The proof is based on the following observation. If \( f \in B^{s}_{2,\infty}(\mathbb{R}^n; \mathbb{C}) \) then by Plancherel’s theorem, we have the identity
\[
\left\| f(\cdot + y) - f(\cdot) \right\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|^2 \left| e^{-2\pi i \xi \cdot y} - 1 \right|^2 d\xi ,
\]
for every \( y \in \mathbb{R}^n \). Thus, from (1.10), we have
\[
\left\| f \right\|_{B^{s}_{2,\infty}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|^2 d\xi + \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \left| e^{-2\pi i \xi \cdot y} - 1 \right|^2 d\xi . \tag{2.21}
\]
On the other hand, for any \( j = 1, 2, \ldots, n-1 \), we have
\[
\mathcal{F} \left[ \chi_{\Omega \cap \Omega^*} \tilde{A}^{(j)} \right] (\xi) = \int_{\mathbb{R}^n} e^{i \xi \cdot x} \chi_{\Omega \cap \Omega^*} \tilde{A}^{(j)} dx = \int_{\mathbb{R}^n} e^{i \xi \cdot x} \tilde{A}^{(j)} dx + \int_{\mathbb{R}^n} e^{i \xi \cdot x} \tilde{A}^{(j)} dx = \mathcal{F} \left[ \chi_{\Omega} A^{(j)} \right] (\xi) + \mathcal{F} \left[ \chi_{\Omega} A^{(j)} \right] (\xi^*).
\]
Thus, according to (2.21), we get
\[
\left\| \chi_{\Omega \cap \Omega^*} \tilde{A}^{(j)} \right\|_{B^{s}_{2,\infty}(\mathbb{R}^n)} \leq 2 \left\| \chi_{\Omega} A^{(j)} \right\|_{B^{s}_{2,\infty}(\mathbb{R}^n)} .
\]
Analogously, we obtain
\[
\left\| \chi_{\Omega \cap \Omega^*} \tilde{A}^{(\alpha)} \right\|_{B^{s}_{2,\infty}(\mathbb{R}^n)} \leq 2 \left\| \chi_{\Omega} A^{(\alpha)} \right\|_{B^{s}_{2,\infty}(\mathbb{R}^n)} .
\]
Moreover, since \( A \in L^\infty(\Omega; \mathbb{C}^n) \) it follows that \( \chi_{\Omega \cap \Omega^*} \tilde{A} \in L^\infty(\mathbb{R}^n; \mathbb{C}^n) \). Hence, \( \chi_{\Omega \cap \Omega^*} \tilde{A} \in \mathcal{A}(\Omega \cup \Omega^*; 2M, s) \). The proof for \( \chi_{\Omega \cap \Omega^*} \tilde{q} \) is analogous. So our proof is completed. \( \square \)

The following proposition is an immediate consequence of theorem 2.1.

**Proposition 2.7.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( A \in L^\infty(\Omega; \mathbb{C}^n) \) and \( q \in L^\infty(\Omega, \mathbb{C}) \). Consider \( M > 0 \) and \( s \in (0, 1/2) \). Assume that \( \chi_{\Omega} A \in \mathcal{A}(\Omega, M, s) \). Consider \( \rho \in \mathbb{C}^n \) such that \( \rho \cdot \rho = 0 \) and \( \rho = \rho_0 + \rho_\tau \) with \( \rho_0 \) being independent of some large parameter \( \tau > 0, |\rho_0| = |\tau \rho_0| = 1 \) and \( \rho_\tau = O(\tau^{-1}) \) as \( \tau \to \infty \). Then there exist two positive constants \( C \) and \( \gamma_0 \) (both depending on \( n, \Omega, M, s, \|q\|_{L^\infty(\Omega)} \)) and a solution \( U \in H^1(\Omega) \) to the equation \( L_{A,q} U = 0 \) in \( \Omega \) with \( U|_{\Gamma_0} = 0 \), of the form
\[
U(x, \rho; \tau) = u(x, \rho; \tau) - u(x^*, \rho; \tau), \quad x \in \Omega , \tag{2.22}
\]

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where \( u \in H^1(\Omega \cup \Omega^\ast) \) satisfies \( L_{\lambda_0} u = 0 \) in \( \Omega \cup \Omega^\ast \), and has the form:

\[
u(x, \rho; \tau) = e^{\tau \rho x} \left( e^{\Phi^\ast(x, \rho_0; \tau)} + r(x, \rho; \tau) \right), \quad x \in \Omega \cup \Omega^\ast, \tag{2.23}\]

with the following properties:

(i) The function \( \Phi^\ast(\cdot, \rho_0; \tau) \in C^\infty(\mathbb{R}^n) \) and satisfies for all \( \alpha \in \mathbb{N}^n \)

\[
\| \partial^\alpha \Phi^\ast(\cdot, \rho_0; \tau) \|_{L^\infty(\mathbb{R}^n)} \leq C \tau^{\frac{|\alpha|}{s} + 2}, \quad \tau \geq \tau_0. \tag{2.24}\]

(ii) The function \( r(\cdot, \rho_0; \tau) \in H^1(\Omega \cup \Omega^\ast) \) and satisfies

\[
\| \partial^\alpha r(\cdot, \rho_0; \tau) \|_{L^2(\Omega \cup \Omega^\ast)} \leq C \tau^{\frac{|\alpha|}{s} + 2}, \quad |\alpha| \leq 1. \tag{2.25}\]

(iii) If we define by \( \kappa := \sup_{s \in \mathbb{R}^n} |s| \) then \( u \) satisfies

\[
\| u \|_{H^1(\Omega \cup \Omega^\ast)} \leq C e^\kappa \tau^{\alpha}. \tag{2.26}\]

The function \( \Phi(\cdot; \rho_0) = (\rho_0 \cdot \nabla)^{-1}( -i \rho_0 \cdot (\chi_{\Omega \cup \Omega^\ast} \tilde{A}) \) satisfy in \( \mathbb{R}^n \)

\[
\rho_0 \cdot \nabla \Phi + i \rho_0 \cdot (\chi_{\Omega \cup \Omega^\ast} \tilde{A}) = 0, \tag{2.27}\]

\[
\| \Phi(\cdot; \rho_0) \|_{L^\infty(\mathbb{R}^n)} \leq C. \tag{2.28}\]

Finally, for every \( \chi \in C_0^\infty(\mathbb{R}^n) \) we have

\[
\| \chi(\Phi^\ast(\cdot, \rho_0; \tau) - \Phi(\cdot; \rho_0)) \|_{L^2(\mathbb{R}^n)} \leq C \tau^{\frac{s|\alpha|}{s} + 2}, \tag{2.29}\]

where the constant \( C \) also depends on \( \chi \).

**Proof.** Let \( B \) be a ball centered at some fixed point on \( \Gamma_0 \) and such that \( \Omega \cup \Omega^\ast \subset B \). By hypothesis \( \chi_{\Omega \cup \Omega^\ast} \tilde{A} \) belongs to \( \mathcal{A}(\Omega, M, s) \), thus by lemma 2.6, it follows that \( \chi_{\Omega \cup \Omega^\ast} \tilde{A} \in \mathcal{A}(\Omega \cup \Omega^\ast, 2M, s) \). Since the function \( \chi_{\Omega \cup \Omega^\ast} \tilde{A} \) is zero out of \( B \), we deduce that \( \chi_{\Omega \cup \Omega^\ast} \tilde{A} \) also belongs to \( \mathcal{A}(B, 2M, s) \), which in turn implies \( \chi_{\Omega \cup \Omega^\ast} \tilde{A} \in L^\infty \cap B_1^1 \cap \mathbb{C}^n \) and \( \text{supp} (\chi_{\Omega \cup \Omega^\ast} \tilde{A}) \subset B \). Notice that \( \chi_{\Omega \cup \Omega^\ast} \tilde{q} \in L^\infty(\mathbb{R}^n, \mathbb{C}) \) and \( \text{supp}(\chi_{\Omega \cup \Omega^\ast} \tilde{q}) \subset B \). Thus, we are in position to apply theorem 2.1 to the functions \( \chi_{\Omega \cup \Omega^\ast} \tilde{A}, \chi_{\Omega \cup \Omega^\ast} \tilde{q} \) and the set \( V = B \).

Hence, there exist two positive constants \( C \) and \( \tau_0 \) (both depending on \( n, \Omega, M, \|q\|_{L^\infty(\Omega)} \)) and a function \( u \in H^1(B) \) of the form

\[
u(x, \rho; \tau) = e^{\tau \rho x} \left( e^{\Phi^\ast(x, \rho_0; \tau)} + r(x, \rho; \tau) \right), \quad \tau \geq \tau_0, \tag{2.30}\]

satisfying in \( B \):

\[
L_{\chi_{\Omega \cup \Omega^\ast} \tilde{A}, \chi_{\Omega \cup \Omega^\ast} \tilde{q}} u = 0, \tag{2.30}\]

with the corresponding estimates (2.1)–(2.6). These estimates together with the estimates from lemma 2.6, imply the estimates (2.24)–(2.29). By a straightforward computation we have

\[
L_{\chi_{\Omega \cup \Omega^\ast} \tilde{q}, \chi_{\Omega \cup \Omega^\ast} \tilde{q}} u(x) = 0, \quad x \in B^+ \]

and

\[
L_{\chi_{\Omega \cup \Omega^\ast} \tilde{q}, \chi_{\Omega \cup \Omega^\ast} \tilde{q}} u(x) = 0, \quad x \in B^+. \tag{2.29}\]

where \( B^+ = \{ x \in B : x_n > 0 \} \) denotes the upper half part of \( B \). Thus, from two above equations, we immediately deduce that \( U(x) := u(x) - u(x^*) \) satisfies \( \mathcal{L}_{\chi_0 \mathcal{A} \chi_0} U = 0 \) in \( B^+ \). It is easy to check that \( U(x) = 0 \) on \( \partial B^+ \cap \{ x_n = 0 \} \). Finally, it is clear that \( U \) restricted to \( \Omega \), still denoted by \( U \), satisfies the assertion of this proposition. So the proof is completed. 

**Remark 2.8.** To prove that \( \mathcal{L}_{A,q} U = 0 \) with \( U|_{\Gamma_0} = 0 \), we can use integration by parts in \( B \). This works because \( B \) has a smooth boundary. In contrast to this, since \((a \text{ priori})\) we are not assuming any regularity for \( \partial \Omega \), we can not use integration by parts in \( \Omega \cup \Omega^* \). This is the main reason of why we first construct solutions in \( B \) instead of \( \Omega \cup \Omega^* \).

The next step will be to use proposition 2.7 for suitables \( \rho_1 \) and \( \rho_2 \) belong to \( \mathbb{C}^n \), in order to construct \( U_1, U_2 \in H^1(\Omega) \) satisfying \( \mathcal{L}_{A,q} U_1 = 0 \) with \( U_1|_{\Gamma_0} = 0 \) and \( \mathcal{L}_{A,q} U_2 = 0 \) with \( U_2|_{\Gamma_0} = 0 \). By replacing these solutions into (2.20) we shall obtain information about \( A_1 - A_2 \). First, we give the motivation behind the choice of \( \rho_1 \) and \( \rho_2 \). Given \( \xi \in \mathbb{R}^n \), we choose two unit vectors \( \mu_1, \mu_2 \in \mathbb{R}^n \) such that
\[
\xi \cdot \mu_1 = \xi \cdot \mu_2 = \mu_1 \cdot \mu_2 = 0. \tag{2.31}
\]

Following [10], for a large parameter \( \tau > 0 \), we set
\[
\begin{align*}
|\rho_1| &= \rho_{1,0} + O(\tau^{-1}), & |\rho_1| &= |\rho_1^\ast| = \sqrt{2}, \\
|\rho_2| &= \rho_{2,0} + O(\tau^{-1}), & |\rho_2| &= |\rho_2^\ast| = \sqrt{2},
\end{align*}
\]
where
\[
\rho_{1,0} = i\mu_1 + \mu_2, \quad \rho_{2,0} = i\mu_1 - \mu_2. \tag{2.34}
\]

By proposition 2.7 applied to \( A_1, q_1 \) and \( \rho_1 \), it follows that there exists \( U_1 \in H^1(\Omega) \) satisfying \( \mathcal{L}_{A_1,q_1} U_1 = 0 \) with \( U_1|_{\Gamma_0} = 0 \) of the following form:
\[
U_1 = e^{\tau \rho_{1,x}} \left( e^{\rho_{1}^* x} + r_1 \right) - e^{\tau \rho_{1,x}} \left( e^{\rho_{1}^* x} + r_1 \right).
\]

Analogously, for \( A_2, q_2 \) and \( \rho_2 \), there exists \( U_2 \in H^1(\Omega) \) satisfying \( \mathcal{L}_{A_2,q_2} U_2 = 0 \) with \( U_2|_{\Gamma_0} = 0 \) of the form:
\[
U_2 = e^{\tau \rho_{2,x}} \left( e^{\rho_{2}^* x} + r_2 \right) - e^{\tau \rho_{2,x}} \left( e^{\rho_{2}^* x} + r_2 \right).
\]

Now, we can use corollary 2.5 in order to get information about \( A_1 - A_2 \). This involves the computation of \( DU_1 U_2 + U_1 DU_2 \) and \( U_1 U_2 \). which in turn involves expressions of the form:
\[
\begin{align*}
e^{\tau (\rho_{1} + \overline{\rho_{1}}) x} f_1(x) &= e^{i\xi \cdot x} f_1(x), \\
e^{\tau (\rho_{1} + \overline{\rho_{1}}) x} f_2(x) &= e^{i\xi \cdot x} f_2(x), \\
e^{\tau (\rho_{1} + \overline{\rho_{1}}) x} f_3(x) &= e^{\left( i(\xi + \xi^\ast) + i\sqrt{\tau^2 - ||\rho_{1} - \rho_{1}^\ast|| + (\mu_2 - \mu_2^\ast)} \right) x} f_3(x), \\
e^{\tau (\rho_{1} + \overline{\rho_{1}}) x} f_4(x) &= e^{\left( i(\xi + \xi^\ast) - i\sqrt{\tau^2 - ||\rho_{1} - \rho_{1}^\ast|| - (\mu_2 - \mu_2^\ast)} \right) x} f_4(x),
\end{align*}
\]
for suitable functions $f_1, f_2, f_3$ and $f_4$. The expressions involving $f_1$ and $f_2$ will give us information about the difference of the magnetic potentials $A_1 - A_2$. To estimate the expressions involving $f_3$ and $f_4$ we will use a quantitative version of the Riemann–Lebesgue lemma. To do that, the vectors $\mu_1$ and $\mu_2$ have to satisfy the following properties:

$$\mu_2 = \mu_2^*, \quad (2.35)$$

$$\lim_{\tau \rightarrow \infty} \left| \frac{1}{2}(\xi + \xi^*) + \sqrt{\tau^2 - \frac{\|\xi\|^2}{4}(\mu_1 - \mu_1^*)} \right| = +\infty, \quad (2.36)$$

and

$$\lim_{\tau \rightarrow \infty} \left| \frac{1}{2}(\xi + \xi^*) - \sqrt{\tau^2 - \frac{\|\xi\|^2}{4}(\mu_1 - \mu_1^*)} \right| = +\infty. \quad (2.37)$$

To fix such $\mu_1$ and $\mu_2$ satisfying (2.35)–(2.37), we proceed as in [8].

Given a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n) \in \mathbb{R}^n$, we denote $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1})$. Thus, we can write $\xi = (\xi', \xi_n)$. For each one of $l = 1, 2, \ldots, n - 1$, we set

$$E_l := \left\{ \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n : 0 < \sum_{i=1}^{n-1} \xi_i^2 \right\}. \quad (2.38)$$

Now, for every $\xi \in \bigcap_{l=1}^{n-1} E_l$, we consider the following unit vectors in $\mathbb{R}^n$:

$$e(1) := \frac{1}{|\xi'|}(\xi', 0), \quad e(2), \quad e_n \quad (2.39)$$

with

$$e(2) \in (\text{span } \{e(1), e_n\})^\perp \quad \text{and} \quad e(2) = e(2)^*, \quad (2.40)$$

where $e_n$ denotes the $n$th canonical unit vector in $\mathbb{R}^n$. Notice that every $\xi \in \bigcap_{l=1}^{n-1} E_l$ can be written as $\xi = |\xi'| e(1) + \xi_n e_n$.

**Lemma 2.9.** For each $\xi \in \bigcap_{l=1}^{n-1} E_l$ and given $j, k = 1, \ldots, n$; there exist constants $\alpha, \beta$ and unit vectors in $\mathbb{R}^n, \mu_1$ and $\mu_2$, satisfying (2.31), (2.35)–(2.37), such that

$$\xi_j e_k - \xi_k e_j = \alpha \mu_1 + \beta \mu_2, \quad j, k = 1, 2, \ldots, n, \quad (2.41)$$

where $e_l$ denotes the $l$th canonical unit vector in $\mathbb{R}^n$. Moreover, $\mu_1$ can be chosen independent of $j$ and $k$ of the following form:

$$\mu_1 = -\frac{\xi_n}{|\xi'|} e(1) + \frac{|\xi'|}{|\xi|} e_n. \quad (2.42)$$

**Proof.** From the definition of $E_l$, for every $\xi \in \bigcap_{l=1}^{n-1} E_l$ we have $|\xi'| > 0$ and $|\xi| > 0$. Then $\mu_1$ defined by (2.42) is well-defined. It is easy to check that for every $j, k = 1, 2, \ldots, n - 1$; the unit vectors $\mu_1$ and $\mu_2$ defined by

$$\mu_1 := -\frac{\xi_n}{|\xi'|} e(1) + \frac{|\xi'|}{|\xi|} e_n \quad (2.43)$$
and 
\[ \mu_2 = (\mu_2)_{1:k} := (\xi_j e_k - \xi_k e_j)/\xi_j^2 + \xi_k^2 \]
are well-defined and satisfy (2.31) and (2.35)–(2.37). Moreover, we have the following identity
\[ \xi_j e_k - \xi_k e_j = 0\mu_1 + (\xi_j^2 + \xi_k^2)\mu_2, \quad j, k = 1, 2, \ldots, n - 1. \] (4.44)
Thus in these cases, \( \alpha = 0 \) and \( \beta := \beta_{1:k} = \xi_j^2 + \xi_k^2 \) with \( j, k = 1, 2, \ldots, n - 1 \). It remains to prove (2.41) for vectors of the form \( \xi_j e_n - \xi_n e_j \) with \( j = 1, 2, \ldots, n - 1 \). In these cases, we consider again \( \mu_1 \) as in (2.43), from which we deduce
\[ e_n = \frac{|\xi|}{|\xi'|} \mu_1 + \frac{\xi_n}{|\xi'|} e(1). \]
In these latter cases, we would like to find two constants \( \alpha \) and \( \beta \), and an unit vector \( \mu_2 \) in \( \mathbb{R}^n \) satisfying (2.31), (2.35)–(2.37); such that the equality
\[ \xi_j e_n - \xi_n e_j = \frac{\xi_j}{|\xi'|} |\mu_1| + \frac{\xi_n}{|\xi'|} e(1) - \xi_n e_j = \alpha \mu_1 + \beta \mu_2, \quad j = 1, 2, \ldots, n - 1. \]
holds true. From the orthogonality of \( \mu_1 \) and \( \mu_2 \), we easily see that
\[ \alpha = \frac{\xi_j}{|\xi'|}, \quad \beta := \beta_{j:n} = \frac{\xi_n}{|\xi'|} \left(|\xi'|^2 - \xi_j^2\right)^{1/2}, \quad j = 1, 2, \ldots, n - 1 \]
and
\[ \mu_2 = (\mu_2)_{j:n} := \left(|\xi'|^2 - \xi_j^2\right)^{-1/2} (\xi_j e(1) - |\xi'| e_j), \quad j = 1, 2, \ldots, n - 1. \]
The proof is complete.

From now on, unless otherwise stated, we consider \( \rho_1 \) and \( \rho_2 \) defined by (2.32) with \( \mu_1 \) defined by (2.43) and \( \mu_2 \) as in the proof of lemma 2.9. Hence, we have the following equalities:
\[
\begin{align*}
\tau(\rho_1 + \rho_2) \cdot x = i|\xi| \cdot x, \\
\tau(\rho_1^* + \rho_2^*) \cdot x = i|\xi'| \cdot x, \\
\tau(\rho_1 + \rho_2^*) \cdot x = i \left(\xi', 2 \sqrt{\tau^2 - |\xi'|^2} \right) \cdot x, \\
\tau(\rho_1^* + \rho_2) \cdot x = i \left(\xi', -2 \sqrt{\tau^2 - |\xi'|^2} \right) \cdot x.
\end{align*}
\] (2.45)

### 2.4. A Fourier estimate for the magnetic potentials

This section is devoted to prove proposition 2.10. In the remainder of this article, the symbol \( \mathcal{F} \) denotes the Fourier transform for both scalar and vectorial functions.

**Proposition 2.10.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n) \) and \( q_1, q_2 \in L^\infty(\Omega, \mathbb{C}) \). Consider \( M > 0 \) and \( s \in (0, 1/2] \). Suppose that \( \chi_{\Omega A_1} \) and \( \chi_{\Omega A_2} \) belong to \( \mathcal{S}(\Omega, M, s) \). Then there exist three positive constants \( C, \tau_0 \) and \( \epsilon_0 \) (all depending on \( \Omega, n, M, s, \|q_1\|_{L^\infty}, \|q_2\|_{L^\infty} \)) such that the following estimate:
Then there exist two positive constants $K$ and $e_0$, \( A(y - q) = C \) \( \leq \tau (2.32) \), proposition 2.7 ensures the existence of Lemma 2.12.

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\[ \Lambda \in \mathbb{R}^n \] refer the reader to proposition the inequality holds true with \( \mu \leq 16 \), such that

\[ \|f(\mu + y) - f(\mu)\|_{L^1(\mathbb{R}^n)} \leq C_0 |y|^\sigma \] whenever $|y| < \sigma$. Then there exist two positive constants $K$ and $\epsilon_0$ such that for any $0 < \epsilon < \epsilon_0$ the inequality

\[ |f(\xi)| \leq C_0 K (e^{-\pi \epsilon^2 |\xi|^2} + \epsilon^2), \]

holds true with $K = K(\|f\|_1, \sigma, \epsilon, \mu)$. The second result is a well known result on nonlinear Fourier transform. For the proof we refer the reader to proposition 3.3 in [10] and also lemma 2.6 in [17].

Lemma 2.11. Let $f \in L^1(\mathbb{R}^n)$. Assume that there exist three positive constants $\sigma$, $C_0$ and $s \in (0, 1)$, such that

\[ \|f(\mu + y) - f(\mu)\|_{L^1(\mathbb{R}^n)} \leq C_0 |y|^\sigma \] whenever $|y| < \sigma$. Then there exist two positive constants $K$ and $\epsilon_0$ such that for any $0 < \epsilon < \epsilon_0$, the inequality

\[ |f(\xi)| \leq C_0 K (e^{-\pi \epsilon^2 |\xi|^2} + \epsilon^2), \]

holds true with $K = K(\|f\|_1, \sigma, \epsilon)$. We are now in position to prove proposition 2.10.

Proof. We start by computing the right-hand side of (2.20) multiplied by $\tau^{-1}$, i.e. we compute the expression

\[ \tau^{-1} \int_\Omega [(A_1 - A_2) \cdot (DU_1 U_2 + U_1 DU_2) + (A_1^2 - A_2^2 + q_1 - q_2)U_1 U_2] \, dx, \]

by using solutions given by proposition 2.7. More precisely, for $A_1, q_1$ and $\rho_1$ defined by (2.32), proposition 2.7 ensures the existence of $U_1 \in H^1(\Omega)$ satisfying $L_{A_1, q_1} U_1 = 0$ in $\Omega$ with $U_1|\Gamma_0 = 0$ of the form:

\[ U_1(x) = e^{\sigma \rho_1 x} \left( e^{\theta_1} + r_1 \right) - e^{\sigma \rho_1^* x} \left( e^{\theta_1^*} + r_1^* \right). \] (2.48)

Analogously, by proposition 2.7 applied to $A_2, q_2$ and $\rho_2$ defined by (2.32), there exists $U_2 \in H^1(\Omega)$ satisfying $L_{A_2, q_2} U_2 = 0$ in $\Omega$ with $U_2|\Gamma_0 = 0$ of the form:
\[
U_2(\lambda) = e^{r \rho_1 \cdot (e^{\rho_1} + r_2)} - e^{r \rho_2 \cdot (e^{\rho_2} + r_2)}.
\] (2.49)

Both solutions have the following properties. The functions \( \Phi_1^\dagger(\cdot, \rho_{0,1}; \tau) \) and \( \Phi_2^\dagger(\cdot, \rho_{0,2}; \tau) \) belong to \( C^\infty(\mathbb{R}^n) \) and satisfy for all \( \alpha \in \mathbb{N}^n \)
\[
\left\| \partial^\alpha \Phi_i^\dagger \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \partial^\alpha \Phi_i^{\dagger*} \right\|_{L^\infty(\mathbb{R}^n)} \leq C_T |\alpha|/(r+2), \quad i = 1, 2.
\] (2.50)

For \( i = 1, 2 \), the functions \( r_i \) and \( r_i^{\dagger} \) belong to \( H^1(\Omega \cup \Omega^*) \) and satisfy
\[
\left\| \partial^\alpha r_i \right\|_{L^2(\Omega \cup \Omega^*)} + \left\| \partial^\alpha r_i^{\dagger*} \right\|_{L^2(\Omega \cup \Omega^*)} \leq C_T |\alpha| - s/(r+2), \quad |\alpha| \leq 1.
\] (2.51)

Moreover, from (2.33), we get
\[
\left\| U \right\|_{H^r(\Omega)} \leq C e^{\tau|\rho|} \leq C e^{\tau}, \quad i = 1, 2.
\] (2.52)

If we denote by \( \Phi_1(\cdot; \rho_{1,0}) = (\rho_{1,0} \cdot \nabla)^{-1}(-i \rho_{1,0} \cdot (\chi_{\Omega \cup \Omega^*} \cdot \overline{A}_1)) \in L^\infty(\mathbb{R}^n) \) the function satisfying the equation in \( \mathbb{R}^n \)
\[
\rho_{1,0} \cdot \nabla \Phi_1 + i \rho_{1,0} \cdot (\chi_{\Omega \cup \Omega^*} \cdot \overline{A}_1) = 0
\] (2.53)
and \( \Phi_2(\cdot; \rho_{2,0}) = (\rho_{2,0} \cdot \nabla)^{-1}(-i \rho_{2,0} \cdot (\chi_{\Omega \cup \Omega^*} \cdot \overline{A}_2)) \in L^\infty(\mathbb{R}^n) \) the function satisfying the equation in \( \mathbb{R}^n \)
\[
\rho_{2,0} \cdot \nabla \Phi_2 + i \rho_{2,0} \cdot (\chi_{\Omega \cup \Omega^*} \cdot \overline{A}_2) = 0,
\] (2.54)
then from (2.28), both functions satisfy the estimate
\[
\left\| \Phi_i(\cdot; \rho_{i,0}) \right\|_{L^\infty(\mathbb{R}^n)} \leq C, \quad i = 1, 2.
\] (2.55)

Finally, from (2.29), for every \( \chi \in C_0^\infty(\mathbb{R}^n) \) we have
\[
\left\| \chi(\Phi_i^\dagger(\cdot, \rho_{0,\tau}) - \Phi_i(\cdot; \rho_{0,0})) \right\|_{L^2(\mathbb{R}^n)} \leq C_1 \tau^{-s/(r+2)}, \quad i = 1, 2.
\] (2.56)

With these solutions at hand and by an immediate computation, we get
\[
\tau^{-1} \int_{\Omega} (A_1 - A_2) \cdot (DU_1 \overline{U_2} + U_1 \overline{DU_2})
= i \int_{\Omega} (\overline{r_2} - \rho_1) \cdot (A_1 - A_2) e^{i(\rho_1 + \overline{\rho_2})} e^\Phi_1^{\dagger*} + \overline{\Phi_2^{\dagger*}}
+ i \int_{\Omega} (\overline{r_2}^\dagger - \rho_1^\dagger) \cdot (A_1 - A_2) e^{i(\rho_1^\dagger + \overline{\rho_2^\dagger})} e^\Phi_1^{\dagger*} + \overline{\Phi_2^{\dagger*}}
+ i \int_{\Omega} (\rho_1 - \overline{r_2}) \cdot (A_1 - A_2) e^{i(\rho_1 + \overline{\rho_2})} e^\Phi_1^{\dagger*} + \overline{\Phi_2^{\dagger*}}
+ i \int_{\Omega} (\rho_1^\dagger - \overline{r_2}) \cdot (A_1 - A_2) e^{i(\rho_1^\dagger + \overline{\rho_2})} e^\Phi_1^{\dagger*} + \overline{\Phi_2^{\dagger*}} + \int_{\Omega} R \cdot (A_1 - A_2).
\] (2.57)

Here \( R \) denotes the following expression:
By the extensions of \( \hat{A}_i \) with \( i = 1, 2 \), denoted by \( \tilde{A}_j \) and defined in section 2.3, and replacing (2.32)–(2.34) and (2.45) into (2.57); we get

\[
\tau^{-1} \int_{\Omega} (A_1 - A_2) \cdot (DU \omega \nabla U + U \omega \nabla D) = i \int_{\Omega} (p_2 - p_1) \cdot \left[ \chi_{\Omega,\Omega} \cdot (A_1 - A_2) \right] e^{\xi \cdot x} e^{\Phi_1 + \Phi_2^*} + i \int_{\Omega} R \cdot (A_1 - A_2) \\
+ i \int_{\Omega} (p_1 - p_2) \cdot (A_1 - A_2) e^{i \left( \xi \cdot e^{2} \sqrt{1 - \tau^2} \frac{|\xi|}{|\nabla|} \right) \cdot x} e^{\Phi_1 + \Phi_2^*} \\
+ i \int_{\Omega} (p_2 - p_1) \cdot (A_2 - A_1) e^{i \left( \xi \cdot e^{2} \sqrt{1 - \tau^2} \frac{|\xi|}{|\nabla|} \right) \cdot x} e^{\Phi_1^* + \Phi_2^*} \\
= i \int_{\Omega} (p_2 - p_1) \cdot \left[ \chi_{\Omega,\Omega} \cdot (A_1 - A_2) \right] e^{\xi \cdot x} e^{\Phi_1 + \Phi_2^*} \\
+ \int_{\Omega} \mathcal{O}(\tau^{-1}) \cdot \left[ \chi_{\Omega,\Omega} \cdot (A_1 - A_2) \right] e^{\xi \cdot x} e^{\Phi_1 + \Phi_2^*} + i \int_{\Omega} R \cdot (A_1 - A_2) \\
+ i \int_{\Omega} (p_1 - p_2) \cdot (\chi_{\Omega} (A_1 - A_2)) e^{i \left( \xi \cdot e^{2} \sqrt{1 - \tau^2} \frac{|\xi|}{|\nabla|} \right) \cdot x} e^{\Phi_1 + \Phi_2^*} \\
+ i \int_{\Omega} (p_2 - p_1) \cdot (\chi_{\Omega} (A_1 - A_2)) e^{i \left( \xi \cdot e^{2} \sqrt{1 - \tau^2} \frac{|\xi|}{|\nabla|} \right) \cdot x} e^{\Phi_1^* + \Phi_2^*}.
\]

From this identity, by adding and subtracting terms, we get

\[
i \int_{\Omega} (p_2 - p_1) \cdot \left[ \chi_{\Omega,\Omega} \cdot (A_1 - A_2) \right] e^{\xi \cdot x} e^{\Phi_1 + \Phi_2^*} \\
:= I + II + III + IV + V + VI + VII,
\]

where

\[
I = i \int_{\Omega} (p_2 - p_1) \cdot \left[ \chi_{\Omega,\Omega} \cdot (A_1 - A_2) \right] e^{\xi \cdot x} (e^{\Phi_1 + \Phi_2^*} - e^{\Phi_1^* + \Phi_2^*}),
\]

(2.58)
\[ II = \tau^{-1} \int_{\Omega} \left[ (A_1 - A_2) \cdot (DU_1U_2 + U_1DU_2) + (A_1^2 - A_2^2 + q_1 - q_2)U_1U_2 \right], \]  
\[ III = -\tau^{-1} \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2)U_1U_2, \]  
\[ IV = -\int_{\mathbb{R}^n} \mathcal{O}(\tau^{-1}) \cdot \left[ \chi_{\Omega, \Omega^*}(A_1 - A_2) \right] e^{\imath \rho_1 \cdot x} e^{\imath \Phi_1^* + \imath \Phi_2^*}, \]  
\[ V = \int_{\Omega} R \cdot (A_1 - A_2), \]  
\[ VI = -\imath \int_{\mathbb{R}^n} (\rho_1 - \overline{\rho_2}) \cdot (\chi_{\Omega}(A_1 - A_2)) e^{\imath \left( \sqrt{1 - \tau^{-2}\left| r \right|^2} \left| \Omega \right| / \pi \right) \cdot x} e^{\imath \Phi_1^* + \imath \Phi_2^*}, \]  
\[ VII = -\imath \int_{\mathbb{R}^n} (\rho_1^* - \overline{\rho_2^*}) \cdot (\chi_{\Omega}(A_1 - A_2)) e^{\imath \left( \sqrt{1 - \tau^{-2}\left| r \right|^2} \left| \Omega \right| / \pi \right) \cdot x} e^{\imath \Phi_1^* + \imath \Phi_2^*}. \]  

The task now is to estimate each one of the above terms. To estimate the term \( I \), we use the following fact:
\[ |e^{z_1} - e^{z_2}| \leq |z_1 - z_2| e^{\max\{|Rz_1|, |Rz_2|\}}, \quad z_1, z_2 \in \mathbb{C}. \]  
Thus, from (2.34), the boundedness of \( \Omega \cup \Omega^* \) and (2.56), we obtain
\[ |I| \leq C_1 \left\| \Phi_1 - \Phi_1^* + \overline{\Phi_2} - \Phi_2^* \right\|_{L^2(\mathbb{R}^n)} \leq C_2 \tau^{-s/(s+2)}. \]  

By corollary 2.5 and the estimate (2.52), we obtain
\[ |II| \leq C_3 \tau^{-1} \text{dist}(C_1^\Omega, C_2^\Omega) \left\| U_1 \right\|_{H^s(\Omega)} \left\| U_2 \right\|_{H^s(\Omega)} \leq C_4 \tau^{-1} e^{\tau\kappa} \text{dist}(C_1^\Omega, C_2^\Omega). \]  

We continue in this fashion to estimate the other terms. From the identities (2.48), (2.49), (2.32), (2.43), the estimates (2.50), (2.51) and the triangular inequality, we get
\[ |III| = \tau^{-1} \left\| \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2)e^{\imath \rho_1 \cdot x} U_1 \right\|_{L^2(\Omega)} \leq C_6 \tau^{-1} \left\| e^{\tau^2 \rho_1 \cdot x} U_1 \right\|_{L^2(\Omega)} \leq C_6 \tau^{-1} \left\| e^{\tau^2 \rho_2 \cdot x} U_2 \right\|_{L^2(\Omega)} \]  
\[ = C_6 \tau^{-1} \left\| e^{\tau^2 \rho_1 \cdot x} \left( \Phi_1^* + r_1 \right) - e^{\tau^2 \rho_2 \cdot x} \left( \Phi_2^* + r_2 \right) \right\|_{L^2(\Omega)} \]  
\[ \times \left\| e^{-\tau^2 \rho_1 \cdot x} \left( e^{\tau^2 \rho_1 \cdot x} \left( \Phi_1^* + r_1 \right) - e^{\tau^2 \rho_2 \cdot x} \left( \Phi_2^* + r_2 \right) \right) \right\|_{L^2(\Omega)} \]  
\[ \leq C_6 \tau^{-1} \left( \left\| e^{\Phi_1^*} + r_1 \right\|_{L^2(\Omega)} + \left\| e^{\Phi_2^*} + r_1^* \right\|_{L^2(\Omega)} + \left\| e^{\Phi_2^*} + r_2^* \right\|_{L^2(\Omega)} \right) \leq C_7 \tau^{-1}. \]
where in the last inequality, we have used the identity

\[ \rho_2^2 - \rho_2^2 = i \left( -\frac{1}{2}(\xi^2 + \xi^n) - 2\sqrt{1 - \tau^{-2}\frac{\xi^2}{4}} e(n) \right). \]

From (2.50) and the boundedness of \( \Omega \cup \Omega^* \), it follows easily that

\[ |IV| = \left| \int_{\Omega} O(\tau^{-1}) \cdot \left[ \chi_{\Omega} : \chi_{\Omega^*} (A_1 - A_2) \right] e^{i \xi \cdot x} e^{\Phi_1 + \Phi_2^*} \right| \leq C_8 \tau^{-1}. \]  

(2.70)

From (2.32), (2.45) and (2.50), (2.51), we get

\[ |V| = \left| \int_{\Omega} R \cdot (A_1 - A_2) \right| \leq C_{10} \tau^{-x/2}. \]  

(2.71)

From (2.33), we have

\[ |VI| \leq |\rho_1 - \rho_2| \int_{\Omega} \left( \chi_{\Omega} (A_1 - A_2) \right) e^{i \xi \cdot x} \sqrt{1 - \tau^{-2}\frac{\xi^2}{4}} e^{\Phi_1 + \Phi_2^*} \leq 2\sqrt{2} \left[ F \left( \chi_{\Omega} (A_1 - A_2) e^{\Phi_1 + \Phi_2^*} \right) \left( \xi \cdot x + \tau \sqrt{1 - \tau^{-2}\frac{\xi^2}{4}} \right) \right]. \]  

(2.72)

To estimate this term, we use lemma 2.11 for the function \( \chi_{\Omega} (A_1 - A_2) e^{\Phi_1 + \Phi_2^*} \). First, we verify that it satisfies (2.47). Indeed, for convenience we denote by \( \Phi = \Phi_1 + \Phi_2^* \) and \( A := \chi_{\Omega} (A_1 - A_2) \). Then, from (2.50) and by a standard interpolation between the spaces \( C(\mathbb{R}^n) \) and \( C^1(\mathbb{R}^n) \), we get

\[ \| \Phi \|_{C(\mathbb{R}^n)} \leq C_{10} \| \Phi \|_{C(\mathbb{R}^n)}^{1 - \frac{1}{s}} \| \Phi \|_{C^1(\mathbb{R}^n)}^{\frac{1}{s}} \leq C_{11} \tau^{-x/2}. \]  

(2.73)

Now, for any \( y \in \mathbb{R}^n \), the Cauchy-Schwarz inequality and the estimates (2.50) and (2.66), imply that

\[ \left\| (A e^{\Phi}) (\cdot + y) - (A e^{\Phi}) (\cdot) \right\|_{L^1(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \left| A(x + y) - A(x) \right| e^{\Phi(x + y)} + \left| e^{\Phi(x + y)} - e^{\Phi(x)} \right| A(x) \, dx \leq C_{13} \left( \| A(\cdot + y) - A(\cdot) \|_{L^1(\mathbb{R}^n)} + \| \Phi \|_{L^\infty(\mathbb{R}^n)} \right) \leq C_{14} \left( \| A \|_{B^{s\infty}} + \| \Phi \|_{C(\mathbb{R}^n)} \right) \| y \| \tau^s. \]

Then, by combining this inequality with (2.73), we obtain

\[ \left\| (A e^{\Phi}) (\cdot + y) - (A e^{\Phi}) (\cdot) \right\|_{L^1(\mathbb{R}^n)} \leq C_{15} \tau^{-x/2} \| y \|^s, \quad |y| < 1. \]

Hence, by lemma 2.11 applied to \( f = \chi_{\Omega} (A_1 - A_2) e^{\Phi_1 + \Phi_2^*} \), \( C_0 = C_{15} \tau^{-x/2} \) and \( \sigma = 1 \), there exist two positive constants \( C_{15} \) and \( \epsilon_0 \) such that the following inequality
\[ |\mathcal{F} \left[ \chi_{\Omega} (A_1 - A_2) e^{i \Phi_1 + \Phi_2} \right] (\eta) | \leq C_{15} \tau^{\gamma/2} \left( e^{-\pi \tau^2 |\eta|^2} + \epsilon^\prime \right), \]

holds true for all \( 0 < \epsilon < \epsilon_0 \) and for all \( \eta \in \mathbb{R}^n \). In particular, for

\[ \eta = \left( \epsilon', 2\pi \sqrt{1 - \tau^{-2} \left| \frac{\xi'}{\xi} \right|^2} \right) \]

and from (2.72), we get

\[ |V| \leq C_{16} \tau^{\gamma/2} \left( e^{-4\pi \tau^2 \gamma^2 \left| \frac{\epsilon'}{\xi} \right|^2} + \epsilon^\prime \right). \tag{2.74} \]

In the same manner, we can obtain

\[ |VII| \leq C_{17} \tau^{\gamma/2} \left( e^{-4\pi \tau^2 \gamma^2 \left| \frac{\epsilon'}{\xi} \right|^2} + \epsilon^\prime \right). \tag{2.75} \]

By combining (2.67)–(2.71) and (2.74)–(2.75) into (2.58), and taking into account the identity \( \overline{\rho_{2,0} - \rho_{1,0}} = -2(\overline{\mu_1} + \mu_2) \), we get

\[ \left| \int_{\mathbb{R}^n} (i\mu_1 + \mu_2) \cdot (\chi_{\Omega} \chi_{\Omega^*} (\overline{A_1 - A_2}) e^{i \Phi_1 + \Phi_2} \right| \]

\[ = \frac{1}{2} \left| \int_{\mathbb{R}^n} (\overline{\rho_{2,0} - \rho_{1,0}}) \cdot (\chi_{\Omega} \chi_{\Omega^*} (\overline{A_1 - A_2}) e^{i \Phi_1 + \Phi_2} \right| \]

\[ \leq C_{18} \tau^{-\gamma/2} \left( e^{2\pi \epsilon^\prime \text{dist} (C_1^\prime, C_2^\prime)} + \epsilon^\prime \right). \tag{2.76} \]

The next step will be to remove \( e^{\Phi_1 + \Phi_2} \) from the left-hand side of the above inequality. It can be done by using lemma 2.12. Indeed, from (2.53) and (2.54), it follows easily that

\[ (\overline{\rho_{2,0} - \rho_{1,0}}) \cdot \nabla (\Phi_1 + \Phi_2) - i(\overline{\rho_{2,0} - \rho_{1,0}}) \cdot (\chi_{\Omega} \chi_{\Omega^*} (\overline{A_1 - A_2}) = 0, \]

which implies that

\[ (i\mu_1 + \mu_2) \cdot \nabla (\Phi_1 + \Phi_2) - i(\mu_1 + \mu_2) \cdot (\chi_{\Omega} \chi_{\Omega^*} (\overline{A_1 - A_2}) = 0. \tag{2.77} \]

From this equation, (2.31) and (2.55) we can remove the exponential functions by applying lemma 2.12 to \( \Phi = \Phi_1 + \Phi_2 \) and \( W = -i\chi_{\Omega} \chi_{\Omega^*} (\overline{A_1 - A_2}) \), and then we obtain

\[ \left| \int_{\mathbb{R}^n} (i\mu_1 + \mu_2) \cdot (\chi_{\Omega} \chi_{\Omega^*} (\overline{A_1 - A_2}) e^{i \Phi} \right| \]

\[ \leq C_{19} \tau^{-\gamma/2} \left( e^{2\pi \epsilon^\prime \text{dist} (C_1^\prime, C_2^\prime)} + \epsilon^\prime \right). \tag{2.78} \]
By similar arguments now with $\rho_i$ instead of $\rho_i$, $i = 1, 2$, we also get
\[
\left| \int_{\mathbb{R}^n} (-i\mu_1 + \mu_2) \cdot (\chi_{\mathbb{R}^n} \cdot (\tilde{A}_1 - \tilde{A}_2)) e^{i \xi \cdot x} \right| \\
\leq C_{20} \left[ \tau^{-s/(s+2)} + e^{2\tau \kappa} \text{dist} (C_1^T, C_2^T) \\
+ \tau^{s/(s+2)} \left( e^{-4\pi e^2 \frac{|\xi'|^2}{|\xi|^2} + \epsilon^s} \right) \right].
\] (2.79)

Thus, by combining the estimates (2.78) and (2.79), we have
\[
\left| \int_{\mathbb{R}^n} \mu \cdot (\chi_{\mathbb{R}^n} \cdot (\tilde{A}_1 - \tilde{A}_2)) e^{i \xi \cdot x} \right| \\
\leq C_{21} |\mu| \left[ \tau^{-s/(s+2)} + e^{2\tau \kappa} \text{dist} (C_1^T, C_2^T) \\
+ \tau^{s/(s+2)} \left( e^{-4\pi e^2 \frac{|\xi'|^2}{|\xi|^2} + \epsilon^s} \right) \right].
\] (2.80)

for all $\mu \in \text{span} \{\mu_1, \mu_2\}$ and for all $\xi \in \bigcap_{j=1}^{n-1} E_j$. Finally, lemma 2.9 ensures that for every $j, k = 1, 2, \ldots, n$ the vectors defined by $(\mu_j) := \xi_k e_j$ belong to $\text{span} \{\mu_1, \mu_2\}$. Hence, by replacing these vectors into (2.80), the proof is completed. \(\square\)

2.5. Proof of theorem 1.5

By proposition 2.10 and since the set $\bigcap_{j=1}^{n-1} E_j$ is dense in $\mathbb{R}^n$, it follows that the following estimate
\[
\left| \mathcal{F} \left[ (\chi_{\mathbb{R}^n} \cdot \tilde{A}_1 - \tilde{A}_2) \right] (\xi) - \mathcal{F} \left[ (\chi_{\mathbb{R}^n} \cdot \tilde{A}_2) \right] (\xi) \right| \\
\leq C |\xi| \left[ \tau^{-s/(s+2)} + e^{2\tau \kappa} \text{dist} (C_1^T, C_2^T) \\
+ \tau^{s/(s+2)} \left( e^{-4\pi e^2 \frac{|\xi'|^2}{|\xi|^2} + \epsilon^s} \right) \right].
\] (2.81)

holds true for all $\xi \in \mathbb{R}^n$. Now we consider $R \geq 1$ (which will be fixed later) and we denote by $B_R(0)$ the open ball in $\mathbb{R}^n$ centered at 0 of radius $R$. For convenience we denote $\tilde{A} := \chi_{\mathbb{R}^n} \cdot (\tilde{A}_1 - \tilde{A}_2)$. Then
\[
\left\| d\tilde{A} \right\|_{H^{-1}(\mathbb{R}^n)}^2 = E_1 + E_2,
\] (2.82)

where
\[
E_1 = \int_{B_R(0) \setminus \{0\}} \left( 1 + |\xi|^2 \right)^{-1} \left| \mathcal{F} \left[ d\tilde{A} \right] (\xi) \right|^2 d\xi,
\]
and
\[ E_2 = \int_{\mathbb{R}^n \setminus B_0(0)} (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ d\tilde{A} \right] (\xi) \right|^2 \, d\xi. \]

From (2.81) and taking \( \epsilon = \tau^{-3/2(s+2)} \), we get
\[ E_1 = \int_{B_0(0) \setminus \{0\}} (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ d\tilde{A} \right] (\xi) \right|^2 \, d\xi \]
\[ \leq C_1 R^m \left( \tau^{-2s/(s+2)} + e^{4\pi \epsilon} \text{dist} (C_1, C_2^T)^2 + \tau^{2s/(s+2)} \epsilon^{2s} \right) \]
\[ + C_1 \tau^{2s/(s+2)} \int_{B_0(0) \setminus \{0\}} e^{-8\pi \epsilon^2 |\xi|^2} \, d\xi \]
\[ \leq C_1 R^m \left( \tau^{-2s/(s+2)} + e^{4\pi \epsilon} \text{dist} (C_1, C_2^T)^2 + \tau^{2s/(s+2)} \epsilon^{2s} \right) \]
\[ + C_1 \tau^{2s/(s+2)} R^n \epsilon^{-2} \tau^{-2} \]
\[ \leq C_2 R^m \left( \tau^{-s/(s+2)} + e^{4\pi \epsilon} \text{dist} (C_1, C_2^T)^2 \right). \quad (2.83) \]

To estimate \( E_2 \), we use a mollification argument as it was done in the proof of theorem 2.1. By lemma 2.6, we have \( \tilde{A} \in L^\infty \cap B_2^{\infty}(\mathbb{R}^n) \). Thus, consider \( \varphi \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \varphi \leq 1 \) and \( \text{supp} \, \varphi \subset \bar{B}_1(0) \), where \( \bar{B}_1(0) \) denotes the closure of the ball in \( \mathbb{R}^n \) of radius 1 centered at the origin. For each \( \delta > 0 \), we define \( \varphi_\delta(x) = \delta^{-n} \varphi(x/\delta) \) and we set \( \tilde{A}_\delta = \tilde{A} * \varphi_\delta \), which belongs to \( C_0^\infty(\mathbb{R}^n; \mathbb{C}^n) \). Then, there exists a positive constant \( C_3 \) (depending on \( \Omega \) and \( n \)) such that
\[ \left\| \tilde{A} - \tilde{A}_\delta \right\|_{L^2(\mathbb{R}^n)} \leq C_3 \delta^\alpha \left\| \tilde{A} \right\|_{L^2(\mathbb{R}^n)}, \quad (2.84) \]
and for each \( \alpha \in \mathbb{N}^n \), we have
\[ \left\| \partial^\alpha \tilde{A}_\delta \right\|_{L^\infty(\mathbb{R}^n)} \leq C_3 \delta^{-|\alpha|} \left\| \tilde{A} \right\|_{L^\infty(\mathbb{R}^n)}. \quad (2.85) \]

Now, by Plancherel’s identity and (2.84)–(2.85), we have
\[ E_2 = \int_{\mathbb{R}^n \setminus B_0(0)} (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ d\tilde{A} \right] (\xi) \right|^2 \, d\xi \]
\[ \leq C_4 \int_{\mathbb{R}^n \setminus B_0(0)} (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ d\tilde{A}_\delta \right] (\xi) \right|^2 \, d\xi \]
\[ + C_4 \int_{\mathbb{R}^n \setminus B_0(0)} (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ d\tilde{A}_\delta \right] (\xi) - \mathcal{F} \left[ d\tilde{A} \right] (\xi) \right|^2 \, d\xi \]
\[ \leq C_5 \int_{\mathbb{R}^n \setminus B_0(0)} |\xi|^2 (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ A - A_\delta \right] (\xi) \right|^2 \, d\xi \]
\[ + C_5 \int_{\mathbb{R}^n \setminus B_0(0)} |\xi|^2 (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ A - A_\delta \right] (\xi) \right|^2 \, d\xi \]
\[ \leq C_6 \left( R^{-2} \| dA \|^2_{L^2(\mathbb{R}^n)} + \left\| \tilde{A} - \tilde{A}_\delta \right\|^2_{L^2(\mathbb{R}^n)} \right) \]
\[ \leq C_7 \left( R^{-2} \delta^{-2} + \delta^{2s} \right). \]
By equating the terms $R^{-2\delta-2}$ and $\delta^2$, i.e. taking $\delta = R^{-1/(s+1)}$, we get
\[
\int_{\mathbb{R}^{n}\setminus B_\varepsilon(0)} (1 + |\xi|^2)^{-1} \left| \mathcal{F} \left[ \tilde{d}A \right] (\xi) \right|^2 \, d\xi \leq C_8 R^{-2s/(s+1)}.
\] (2.86)

Then, by combining (3.54) and (2.86) into (2.82), we obtain
\[
\| dA \|_{H^{-1}(\mathbb{R}^n)}^2 \leq C_9 \left( R^n e^{\lambda \tau} \text{dist} \left( C_1^\Gamma, C_2^\Gamma \right)^2 + R^n \tau^{-s/(s+2)} + R^{-2s/(s+1)} \right),
\] (2.87)

for all $\tau \geq \tau_1$. In order to equal the two last terms from the right-hand side, we express $R$ in terms of $\tau$ as follows
\[
R = \tau \frac{s}{s+1} \left( s + \frac{2}{5n} \right) \left( n + ns + 2s \right).
\]

On the other hand, there exists a positive constant $C_{10}$ such that
\[
R^n = \tau \frac{n(s+1)}{s+1} \leq C_{10} e^{\lambda \tau}, \quad \tau \geq \tau_2.
\]
Thus, by replacing these facts into (2.87), we have
\[
\| \tilde{d}A \|_{H^{-1}(\mathbb{R}^n)}^2 \leq C_{11} \left( e^{\lambda \tau} \text{dist} \left( C_1^\Gamma, C_2^\Gamma \right)^2 + \tau^{-\frac{2s}{(s+1)(n+ns+2s)}} \right).
\] (2.88)

While on the other hand, there exists $\tau_0 > 0$ such that
\[
e^{-\lambda \tau} \leq \tau^{-\frac{2s}{(s+1)(n+ns+2s)}}, \quad \tau \geq \tau_3.
\]

Now we consider $\tau_0 \geq \max (\tau_1, \tau_2, \tau_3)$ such that $3\lambda \tau_0 \geq 1$, then it is easy to check that
\[
\tau := \frac{1}{3\lambda} \left| \log \, \text{dist} \left( C_1^\Gamma, C_2^\Gamma \right) \right| \geq \tau_0,
\] (2.89)

whenever
\[
\text{dist} \left( C_1^\Gamma, C_2^\Gamma \right) \leq e^{-3\lambda \tau_0}.
\]

Since $\text{dist} \left( C_1^\Gamma, C_2^\Gamma \right) \leq e^{-3\lambda \tau_0}$ and from (2.88), we get
\[
\| \tilde{d}A \|_{H^{-1}(\mathbb{R}^n)} \leq C_{10} \left| \log \, \text{dist} \left( C_1^\Gamma, C_2^\Gamma \right) \right|^{-\frac{2s}{(s+1)(n+ns+2s)}}.
\] (2.90)

Since $s \in (0, 1/2)$, we immediately deduce that
\[
\frac{s^2}{(s+2)(n+ns+2s)} \geq \frac{s^2}{5n}.
\]

Hence, we conclude the proof by considering this inequality into (2.90), taking $C = \max \{ 3\lambda \tau_0, C_{10} \}$, $\lambda = 1/5$ and the obvious inequality
\[
\| d(A_1 - A_2) \|_{H^{-1}(\Omega)} \leq \| \tilde{d}A \|_{H^{-1}(\mathbb{R}^n)}.
\]
3. Stability for the electric potential

The goal of this section is to prove theorem 1.6. We combine the gauge invariance of the Cauchy data sets, see lemma 3.1 in [10], the stability result already proved for the magnetic potentials in section 2 and the Hodge decomposition derived by Caro and Pohjola, see lemma 6.2 in [3]. Our starting point is the following lemma, which the analogous of lemma 5.1 in [3]. For this reason we only give the main ideas of the proof.

**Lemma 3.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $B \subset \mathbb{R}^n$ be an open ball with $\overline{\Omega} \subset B$ and $\Gamma_0 \subset B \cap \{x \in \mathbb{R}^n : x_n = 0\}$. Let $A_1, A_2 \in L^\infty(\Omega; \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega; \mathbb{C})$. Consider $M > 0$ and $s \in (0, 1/2)$. Suppose that $\chi_M A_1$ and $\chi_M A_2$ belong to $s$ of $(\Omega, M, s)$. Let $\varphi \in W^{1,s}(B) \cap L^\infty(B)$ with $\varphi|_{\partial B} = 0$. Then for any $U_1, U_2 \in H^1(B)$ satisfying in $B$:

$$\mathcal{L}_{\chi_M A_1, \tilde{A}_1} U_1 = 0, \quad U_1|_{\partial B \cap \{x \in \mathbb{R}^n : x_n = 0\}} = 0 \quad (3.1)$$

and

$$\mathcal{L}_{\chi_M A_2, \tilde{A}_2} U_2 = 0, \quad U_2|_{\partial B \cap \{x \in \mathbb{R}^n : x_n = 0\}} = 0, \quad (3.2)$$

there exists a positive constant $C$ (depending on $\Omega, n, M, s, \|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}$) such that

$$\left| \int_B e^{i\varphi} \left\{ (\chi_M A_1 - (\chi_M \tilde{A}_1 + \nabla \varphi)) \cdot (DU_1 U_2 + U_1 D U_2) + \left[ (\chi_M \tilde{A}_1^2 - (\chi_M \tilde{A}_1 + \varphi)^2 + \chi_M \tilde{A}_1 \cdot (\tilde{q}_1 - \tilde{q}_2) - (\chi_M \tilde{A}_1 - \tilde{A}_1 - \nabla \varphi) \cdot \nabla \varphi \right] U_1 U_2 \right\} \right| \leq C \text{ dist} \left( C_1^T, C_2^T \right) \|U_1\|_{H^1(\Omega; \mathbb{R}^n)} \|U_2\|_{H^1(\Omega; \mathbb{R}^n)}. \quad (3.3)$$

**Proof.** We start by restricting the equation (3.1) to $\Omega$. Thus, we obtain

$$\mathcal{L}_{\tilde{A}_1, \tilde{q}_1} (U_1|_{\Omega}) = 0 \quad \text{in} \, \Omega, \quad (U_1|_{\Omega})|_{\Gamma_0} = 0, \quad (3.4)$$

$$\mathcal{L}_{\tilde{A}_2, \tilde{q}_2} (U_2|_{\Omega}) = 0 \quad \text{in} \, \Omega, \quad (U_2|_{\Omega})|_{\Gamma_0} = 0. \quad (3.4)$$

Hence, by corollary 2.5 applied to $\tilde{A}_1, \tilde{q}_1, \tilde{q}_2$ and $U_1|_{\Omega}, U_2|_{\Omega} \in H^1(\Omega)$; we get

$$\left| \int_{\Omega^*} (\tilde{A}_1 - \tilde{A}_2) \cdot (DU_1 U_2 + U_1 D U_2) + \left( \tilde{A}_1^2 - \tilde{A}_2^2 + \tilde{q}_1 - \tilde{q}_2 \right) U_1 U_2 \right| \leq C_1 \text{ dist} \left( C_1^T, C_2^T \right) \|U_1\|_{H^1(\Omega)} \|U_2\|_{H^1(\Omega)} \leq C_1 \text{ dist} \left( C_1^T, C_2^T \right) \|U_1\|_{H^1(\Omega; \mathbb{R}^n)} \|U_2\|_{H^1(\Omega; \mathbb{R}^n)}. \quad (3.5)$$

Now, we restrict (3.1) to $\Omega^*$, to obtain

$$\mathcal{L}_{\tilde{A}_1, \tilde{q}_1} (U_1|_{\Omega^*}) = 0 \quad \text{in} \, \Omega^*, \quad (U_1|_{\Omega^*})|_{\Gamma_0} = 0, \quad (3.6)$$

$$\mathcal{L}_{\tilde{A}_2, \tilde{q}_2} (U_2|_{\Omega^*}) = 0 \quad \text{in} \, \Omega^*, \quad (U_2|_{\Omega^*})|_{\Gamma_0} = 0.$$

Now we denote by $\text{dist}^* \left( C_1^T, C_2^T \right)$ the quantity defined by $\text{dist} \left( C_1^T, C_2^T \right)$ with $\Omega^*$ instead of $\Omega$. It is easy to check that $\text{dist}^* \left( C_1^T, C_2^T \right) = \text{dist} \left( C_1^T, C_2^T \right)$. Then, by corollary 2.5 applied now to $\tilde{A}_1, \tilde{A}_2, \tilde{q}_1, \tilde{q}_2$ and $U_1|_{\Omega^*}, U_2|_{\Omega^*} \in H^1(\Omega^*)$; we get
\[
\int_{\Omega^*} (\tilde{A}_1 - \tilde{A}_2) \cdot (DU_1 \nabla U_2 + U_1 DU_2) + (\tilde{A}_1 - \tilde{A}_2 + \tilde{q}_1 - \tilde{q}_2) U_1 U_2 \leq C_2 \text{ dist}^* (C^T_1, C^T_2) \left\| U_1 \right\|_{H^1(\Omega^*)} \left\| U_2 \right\|_{H^1(\Omega^*)} \\
\leq C_2 \text{ dist}^* (C^T_1, C^T_2) \left\| U_1 \right\|_{H^1(\Omega,\Omega^*)} \left\| U_2 \right\|_{H^1(\Omega,\Omega^*)} .
\]  

(3.7)

Then, from (3.5) and (3.7), we obtain

\[
\int_B \left[ \chi_{\Omega,\Omega^*} (\tilde{A}_1 - \tilde{A}_2) \right] \cdot (DU_1 \nabla U_2 + U_1 DU_2) + \chi_{\Omega,\Omega^*} (\tilde{A}_1 - \tilde{A}_2 + \tilde{q}_1 - \tilde{q}_2) U_1 U_2 \leq 2C_3 \text{ dist}^* (C^T_1, C^T_2) \left\| U_1 \right\|_{H^1(\Omega,\Omega^*)} \left\| U_2 \right\|_{H^1(\Omega,\Omega^*)} .
\]

(3.8)

On the other hand, it is a simple matter the following facts. If \( U_1 \in H^1(B) \) satisfies \( L_{\chi_{\Omega,\Omega^*}, \tilde{A}_1, \chi_{\Omega,\Omega^*}, \tilde{q}_1} U_1 = 0 \) in \( B \) then \( e^{-i\varphi} U_1 \in H^1(B) \) and satisfies \( L_{\chi_{\Omega,\Omega^*}, \tilde{A}_2 + \nabla \varphi, \chi_{\Omega,\Omega^*}, \tilde{q}_2} (e^{-i\varphi} U_1) = 0 \) in \( B \). Analogously, if \( U_2 \in H^1(B) \) satisfies \( L_{\chi_{\Omega,\Omega^*}, \tilde{A}_2 + \nabla \varphi, \chi_{\Omega,\Omega^*}, \tilde{q}_2} U_2 = 0 \) in \( B \) then \( e^{-i\varphi} U_2 \in H^1(B) \) satisfies \( L_{\chi_{\Omega,\Omega^*}, \tilde{A}_1, \chi_{\Omega,\Omega^*}, \tilde{q}_1} (e^{-i\varphi} U_2) = 0 \) in \( B \).

By using these facts and the gauge invariance of the Cauchy data sets (see lemma 3.1 in [10]), we get

\[
\int_B \left[ \chi_{\Omega,\Omega^*} (\tilde{A}_1 - \tilde{A}_2) \right] \cdot (DU_1 \nabla U_2 + U_1 DU_2) + \chi_{\Omega,\Omega^*} (\tilde{A}_1 - \tilde{A}_2 + \tilde{q}_1 - \tilde{q}_2) U_1 U_2 \leq \int_B e^{i\varphi} \left\{ \left( \chi_{\Omega,\Omega^*} \tilde{A}_1 - \left( \chi_{\Omega,\Omega^*} \tilde{A}_2 + \nabla \varphi \right) \right) \cdot (DU_1 \nabla U_2 + U_1 DU_2) \right. \\
\left. + \left[ \chi_{\Omega,\Omega^*} \tilde{A}_1 - \left( \chi_{\Omega,\Omega^*} \tilde{A}_2 + \nabla \varphi \right)^2 + \chi_{\Omega,\Omega^*} \tilde{q}_1 - \tilde{q}_2 \right] U_1 U_2 \right\} .
\]

(3.9)

We conclude the proof by combining (3.8) and (3.9).

We use estimate (3.3), by means of solutions given by proposition 2.7, in order to isolate now \( \tilde{q}_1 - \tilde{q}_2 \). To do that, first we have to deal with the term \( \chi_{\Omega,\Omega^*} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi \). It can be dealt by applying a Hodge decomposition derived in [3] to the function \( \chi_{\Omega,\Omega^*} (\tilde{A}_1 - \tilde{A}_2) \).

**Lemma 3.2.** Let \( B \subseteq \mathbb{R}^n \) be an open ball satisfying \( \overline{\Omega} \subseteq B \). Let \( A_1 \) and \( A_2 \) belong to \( L^\infty(\Omega, \mathbb{C}^n) \). Consider \( p \geq 2 \). Then there exist \( \psi \in W^{1,p}(B) \) and \( C > 0 \) satisfying the following conditions:
\[ \|\psi\|_{W^{1,p}(B)} \leq C \left\| \chi_{\Omega \cup \Omega'} (A_1 - A_2) \right\|_{L^p(\mathbb{R}^n)} \]  

(3.10)

and

\[ \left\| \chi_{\Omega \cup \Omega'} (A_1 - A_2) - \nabla \psi \right\|_{L^2(B)} \leq C \left\| d(\chi_{\Omega \cup \Omega'} (A_1 - A_2)) \right\|_{H^{-1}(B)}, \]  

(3.11)

Moreover, if \( B' \) is another open ball with \( \Omega \subset B' \subset B \) then

\[ \| \psi - \psi^{\Omega} \|_{H^{1}(B', \mathbb{R}^n)} \leq C \left\| d(\chi_{\Omega \cup \Omega'} (A_1 - A_2)) \right\|_{H^{-1}(B)}, \]  

(3.12)

where \( \psi^{\Omega} \) denotes the average of \( \psi \) in \( B \setminus B' \).

3.1 A Fourier estimate for the electric potentials

This section is devoted to obtain an estimate relating the Fourier transform of \( \chi_{\Omega \cup \Omega'} \tilde{q}_1 \) and \( \chi_{\Omega \cup \Omega'} \tilde{q}_2 \) with \( (C_1^p, C_2^p) \), see proposition 3.3.

**Proposition 3.3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( B' \subset \mathbb{R}^n \) be two open balls with \( \Omega \cup \Omega' \subset B' \subset B \). Let \( A_1, A_2 \subset L^\infty(\Omega, \mathbb{C}^n) \) and \( q_1, q_2 \subset L^\infty(\Omega, \mathbb{C}) \). Consider \( M > 0 \) and \( s < 1 \). Suppose that \( \chi_{\Omega} A_1, \chi_{\Omega} A_2 \subset C^s(\Omega, M, s) \) and \( \chi_{\Omega} q_1, \chi_{\Omega} q_2 \subset C^s(\Omega, M, s) \). Then there exist three positive constants \( C, \tau_0, \epsilon_0 \) (all depending on \( \Omega, n, M, s \)) such that the following estimate holds true for every \( \xi \in \bigcap_{i=1}^{n-1} \mathbb{E}_i \) (see (2.38)), for all \( \tau \geq \tau_0 \) and for all \( 0 < \epsilon < \epsilon_0 \). Here \( \theta \in (0, 2/n) \). Recall that \( \kappa := \sup_{\xi \in \bigcap_{i=1}^{n-1} \mathbb{E}_i} |\xi| \).

**Proof.** The proof is similar to the proof of lemma 5.3 in [3]. We first consider the function \( \psi \) given by lemma 3.2 with \( p > n \). This function does not necessarily satisfy the vanishing condition on \( \partial B \). We remedy this obstruction by using a cutoff argument. Let \( \chi \in C^\infty_0(B) \) be a smooth function such that \( \chi(\xi) = 1 \) in \( B' \) and we set \( \varphi = \chi(\psi - \psi^{\Omega}) \). Note that \( \varphi|_{\partial B} = 0 \). Thus, by (3.10), Morrey’s inequality and the boundedness of \( B \), we get

\[ \|\varphi\|_{L^\infty(B)} + \|\nabla \varphi\|_{L^p(B)} + \|\nabla \psi\|_{L^p(B)} \leq C_1. \]  

(3.14)

Now we divide the proof into three steps. As a first step, we prove the following claim.

**Claim 1** If \( \theta \in (0, 2/n) \) then there exist a constant \( C_2 > 0 \) such that the following estimate

\[ \left| \int_B e^{i\varphi} \chi_{\Omega \cup \Omega'} (\tilde{q}_1 - \tilde{q}_2) U_1 U_2 \right| \leq C_2 \left( e^{2\tau_0} \text{dist}(C_1^p, C_2^p) \right) \]

\[ + \|\log \text{dist}(C_1^p, C_2^p)\|_{L^p(B)} \]  

(3.15)
holds true for every $\xi \in \bigcap_{i=1}^{n-1} E_i$. Here $E_i$ is defined by (2.38). Indeed, by adding and subtracting the same terms, we have the identity:

$$\int_B e^{i\varphi} \chi_{\Omega \cup \Omega^c} \cdot (\tilde{q}_1 - \tilde{q}_2) U_1 U_2$$

$$= \int_B e^{i\varphi} (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)) \cdot (DU_1 U_2^* + U_1 DU_2^*)$$

$$+ \int_B e^{i\varphi} (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1^2 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)^2 + \chi_{\Omega \cup \Omega^c} \cdot (q_1 - q_2))$$

$$- (\chi_{\Omega \cup \Omega^c} \cdot (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi) \cdot \nabla \varphi) U_1 U_2$$

$$- \int_B e^{i\varphi} (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)) \cdot (DU_1 U_2 + U_1 DU_2)$$

$$- \int_B e^{i\varphi} (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1^2 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)^2$$

$$- (\chi_{\Omega \cup \Omega^c} \cdot (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi) \cdot \nabla \varphi) U_1 U_2.$$  (3.16)

We set $\varphi' := (1 - \chi)(\psi - \tilde{\psi})$. Then $\psi - \tilde{\psi} = \chi(\psi - \tilde{\psi}) + (1 - \chi)(\psi - \tilde{\psi}) = \varphi + \varphi'$, which implies that

$$\nabla \psi = \nabla \varphi + \nabla \varphi'.$$  (3.17)

From this identity, we deduce

$$\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \psi) = \chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi) - \nabla \varphi'.$$  (3.18)

and

$$\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1^2 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \psi)^2 = \chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1^2 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)^2$$

$$+ \left[\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)\right] \cdot \nabla \varphi'$$

$$- \left[\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 + (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \varphi)\right] \cdot \nabla \varphi' - \nabla \varphi' \cdot \nabla \varphi'.$$  (3.19)

Hence, by replacing (3.17)–(3.19) into (3.16), we obtain

$$\int_B e^{i\varphi} \chi_{\Omega \cup \Omega^c} \cdot (\tilde{q}_1 - \tilde{q}_2) U_1 U_2 := I + II + III + IV,$$

where

$I := - \int_B e^{i\varphi} \left(\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1^2 - (\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 + \nabla \psi)^2\right) U_1 U_2.$  (3.20)

$II := \int_B e^{i\varphi} \left(\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - \chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 - \nabla \psi\right) \cdot \nabla \varphi U_1 U_2.$  (3.21)

$III := - \int_B e^{i\varphi} \left(\chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_1 - \chi_{\Omega \cup \Omega^c} \cdot \tilde{A}_2 - \nabla \psi\right) \cdot (DU_1 U_2 + U_1 DU_2).$  (3.22)
and

\[ IV := \int_B e^{i\varphi} (\chi_{\Omega\cup\Omega^*} \tilde{A}_1 - \chi_{\Omega\cup\Omega^*} \tilde{A}_2 - \nabla \psi) \cdot (DU_1 U_2 + U_1 DU_2) + \int_B e^{i\varphi} \left[ \chi_{\Omega\cup\Omega^*} \tilde{A}_1^2 - (\chi_{\Omega\cup\Omega^*} \tilde{A}_2 + \nabla \psi)^2 + \chi_{\Omega\cup\Omega^*} (\tilde{q}_1 - \tilde{q}_2) \right] (\chi_{\Omega\cup\Omega^*} \tilde{A}_1 - \chi_{\Omega\cup\Omega^*} \tilde{A}_2 - \nabla \psi) \cdot \nabla \varphi \right] U_1 U_2. \tag{3.23} \]

By the triangle inequality, it is immediate to see that

\[ \left| \int_B e^{i\varphi} \chi_{\Omega\cup\Omega^*} (\tilde{q}_1 - \tilde{q}_2) U_1 U_2 \right| \leq |I| + |II| + |III| + |IV|. \tag{3.24} \]

Now, proposition 2.7 with \( B \) instead of \( \Omega \), ensures the existence of functions \( U_1, U_2 \in H^1(B) \) satisfying \( L_{\chi_{\Omega\cup\Omega^*}, \tilde{A}_1, \chi_{\Omega\cup\Omega^*}, \tilde{A}_2} U_1 = 0 \) with \( U_1|_{\partial B \cap \{x_n=0\}} = 0 \) and \( L_{\chi_{\Omega\cup\Omega^*}, \tilde{A}_1, \chi_{\Omega\cup\Omega^*}, \tilde{A}_2} U_2 = 0 \) with \( U_2|_{\partial B \cap \{x_n=0\}} = 0 \); of the form

\[ U_1(x) = e^{\tau \rho_1 x} \left( e^{\rho_1 x} + r_1 \right) - e^{\tau \rho_1 x} \left( e^{\rho_1 x} + r_1 \right), \]

and

\[ U_2(x) = e^{\tau \rho_2 x} \left( e^{\rho_2 x} + r_2 \right) - e^{\tau \rho_2 x} \left( e^{\rho_2 x} + r_2 \right). \]

Here \( \rho_1 \) and \( \rho_2 \) are defined by (2.32). The functions \( \Phi^j, \Phi^j, r_j \) and \( r_j^* \) with \( j = 1, 2 \) satisfy similar estimates to (2.50)–(2.56) with \( B \) instead of \( \Omega \) and \( \Omega \cup \Omega^* \). By the Sobolev’s embedding, we deduce that

\[ \|r_j\|_{L^{2n/(n-1)}(B)} + \|r_j^*\|_{L^{2n/(n-1)}(B)} \leq C_2 \left( \|r_j\|_{W^{1,2}(B)} + \|r_j^*\|_{W^{1,2}(B)} \right) \leq C_3 \tau^{2/(r+2)}. \tag{3.25} \]

From (2.50) and the boundedness of \( B \), we get

\[ \|e^{\Phi^j}\|_{L^{2n/(n-1)}(B)} + \|e^{\Phi^j}^*\|_{L^{2n/(n-1)}(B)} \leq C_4. \tag{3.26} \]

From (3.14) and the boundedness of \( B \), it follows that \( \chi_{\Omega\cup\Omega^*} (\tilde{A}_1 - \tilde{A}_2) - \nabla \psi \) and \( \chi_{\Omega\cup\Omega^*} (\tilde{A}_1 - \tilde{A}_2) + \nabla \psi \) belong to \( L^q(B) \). On the other hand, an easy computations shows that

\[ U_1 U_2 = e^{i\xi} \left( e^{\Phi^j + \Phi^j} + e^{\Phi^j + \Phi^j} \right) + e^{i\xi} \left( e^{\Phi^j + \Phi^j \tau} + e^{\Phi^j + \Phi^j \tau} \right) - e^{i\xi} \left( e^{\Phi^j + \Phi^j \tau} + e^{\Phi^j + \Phi^j \tau} \right) + e^{i\xi} \left( e^{\Phi^j + \Phi^j \tau} + e^{\Phi^j + \Phi^j \tau} \right) + e^{i\xi} \left( e^{\Phi^j + \Phi^j \tau} + e^{\Phi^j + \Phi^j \tau} \right) + e^{i\xi} \left( e^{\Phi^j + \Phi^j \tau} + e^{\Phi^j + \Phi^j \tau} \right) \tag{3.27} \]
Hence, from (3.14), (3.25) and (3.26), the boundedness of $B$ and applying the Hölder’s inequality \((1/n + 1/n + (n - 2)/(2n) + (n - 2)/(2n)) = 1\), we get

\[
|I| = \left| \int_B e^{i\varphi} (\chi_{\Omega^*} \cdot \tilde{A}_1 - (\chi_{\Omega^*} \cdot \tilde{A}_2 + \nabla \psi)) - (\chi_{\Omega^*} \cdot \tilde{A}_1 + (\chi_{\Omega^*} \cdot \tilde{A}_2 + \nabla \psi)) U_1 U_2 \right|.
\]

\[
\leq C_5 \left\| \chi_{\Omega^*} \cdot (\tilde{A}_1 - \tilde{A}_2) - \nabla \psi \right\|_{L^p(B)} \tau^{4/(r+2)}. \tag{3.28}
\]

From (3.14), we have that \(\nabla \varphi \in L^n(B)\). Thus, by applying the Hölder’s inequality \((1/n + 1/n + (n - 2)/(2n) + (n - 2)/(2n) = 1)\), we get

\[
|II| \leq C_6 \left\| \chi_{\Omega^*} \cdot (\tilde{A}_1 - \tilde{A}_2) - \nabla \psi \right\|_{L^n(B)} \tau^{4/(r+2)}. \tag{3.29}
\]

From (3.25), the estimate (2.51) with $B$ instead of $\Omega \cup \Omega^*$, the Hölder’s inequality applied to $1/n + (n - 2)/(2n) + 1/2 = 1$, we obtain

\[
|III| \leq C_7 \left\| \chi_{\Omega^*} \cdot (\tilde{A}_1 - \tilde{A}_2) - \nabla \psi \right\|_{L^n(B)} \tau^{(r+4)/(r+2)}. \tag{3.30}
\]

The term $IV$ requires a more delicate analysis. Replacing (3.18) and (3.19) into (3.23), we get the identity:

\[
IV := \int_B e^{i\varphi} (\chi_{\Omega^*} \cdot \tilde{A}_1 - (\chi_{\Omega^*} \cdot \tilde{A}_2 + \nabla \varphi)) \cdot (DU_1 U_2 + U_1 DU_2)
\]

\[
+ \int_B e^{i\varphi} \left[ (\chi_{\Omega^*} \cdot \tilde{A}_1 - (\chi_{\Omega^*} \cdot \tilde{A}_2 + \nabla \varphi)) \cdot (\tilde{q}_1 - \tilde{q}_2)ight.
\]

\[
- (\chi_{\Omega^*} \cdot \tilde{A}_1 - (\chi_{\Omega^*} \cdot \tilde{A}_2 + \nabla \varphi)) \cdot \nabla \varphi \right] U_1 U_2
\]

\[
- \int_B e^{i\varphi} \nabla \varphi' \cdot (DU_1 U_2 + U_1 DU_2)
\]

\[
- \int_B e^{i\varphi} \left( 2\chi_{\Omega^*} \cdot \tilde{A}_2 \cdot \nabla \varphi' + \nabla \psi \cdot \nabla \varphi' \right) U_1 U_2.
\]

Since the supports of the functions $\chi_{\Omega^*} \cdot \tilde{A}_2$ and $\nabla \varphi'$ are disjoint, it follows that $\int_B e^{i\varphi} (\chi_{\Omega^*} \cdot \tilde{A}_2) \cdot \nabla \varphi' = 0$. Hence, lemma 3.1 and the triangular inequality imply that

\[
|IV| \leq C_8 \left( \text{dist} (C_1^1, C_2^1) \left\| U_1 \right\|_{H^1(\Omega^*)} \left\| U_2 \right\|_{H^1(\Omega^*)}^2 \right.
\]

\[
+ \left| \int_B e^{i\varphi} \nabla \varphi' \cdot (U_1 DU_2 + DU_1 U_2) \right| \left\| \nabla \psi \cdot \nabla \varphi' U_1 U_2 \right\|_{L^n(B)} \right),
\]

then by similar arguments used in section 2.4, we obtain

\[
|IV| \leq C_9 \left( \text{dist} (C_1^1, C_2^1) \left\| U_1 \right\|_{H^1(\Omega^*)} \left\| U_2 \right\|_{H^1(\Omega^*)}^2 \right.
\]

\[
+ \left\| \nabla \varphi' \right\|_{L^n(B)} \tau^{(r+4)/(r+2)} \right), \tag{3.31}
\]

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Now, let $\theta \in (0, 2/n)$ and we fix $p$ such that $\theta/2 + (1 - \theta)/p = 1/n$. Then, the boundedness of $B$, (3.11), the estimate (2.90) with $B$ instead of $\Omega \cup \Omega^{*}$ and an elementary interpolation, give us
\[
\|\chi_{\Omega \cup \Omega^{*}}(\tilde{A}_1 - \tilde{A}_2) - \nabla \psi\|_{L^{\theta}(B)} \lesssim \|\chi_{\Omega \cup \Omega^{*}}(\tilde{A}_1 - \tilde{A}_2) - \nabla \phi\|_{L^{\theta}(B)}^{\theta} \lesssim C_{10} \left\| d(\chi_{\Omega \cup \Omega^{*}}(\tilde{A}_1 - \tilde{A}_2)) \right\|_{H^{-1}(B)}^{\theta} \lesssim C_{10} \left\| d(\chi_{\Omega \cup \Omega^{*}}(\tilde{A}_1 - \tilde{A}_2)) \right\|_{H^{-1}(R^{n})}^{\theta} \lesssim C_{11} \left\| \log \text{ dist} (C_{1}^{T}, C_{2}^{T}) \right\|_{\frac{\tilde{s}^{2}}{\gamma^{(r+1)/(s+2)}}}.
\] (3.32)

Since $\frac{1}{n} - \chi \equiv 0$ in $B'$ and from the identity $\varphi' = (1 - \chi)(\psi - \psi)$, we deduce
\[
\|\nabla \varphi'\|_{L^{\theta}(B)} \lesssim C_{11} \|\psi - \psi\|_{W^{r/(r+1)}(B')}.
\]
which in turn together with theorem 1.5, imply
\[
\|\nabla \varphi'\|_{L^{\theta}(B)} \lesssim C_{12} \|\psi - \psi\|_{H^{r/(r+1)}(B')} \|\psi - \psi\|_{W^{r/(r+1)}(B')}^{1 - \theta} \lesssim C_{13} \left\| d(\chi_{\Omega \cup \Omega^{*}}(\tilde{A}_1 - \tilde{A}_2)) \right\|_{H^{-1}(B)}^{\theta} \lesssim C_{13} \left\| d(\chi_{\Omega \cup \Omega^{*}}(\tilde{A}_1 - \tilde{A}_2)) \right\|_{H^{-1}(R^{n})}^{\theta} \lesssim C_{14} \left\| \log \text{ dist} (C_{1}^{T}, C_{2}^{T}) \right\|_{\frac{\tilde{s}^{2}}{\gamma^{(r+1)/(s+2)}}}.
\] (3.33)

Hence, by using (3.32) into (3.28)–(3.30), we get
\[
|I| + |II| + |III| \lesssim C_{15} \left\| \log \text{ dist} (C_{1}^{T}, C_{2}^{T}) \right\|_{\frac{\tilde{s}^{2}}{\gamma^{(r+1)/(s+2)}}}.
\]
Now by using (3.33) into (3.31), we obtain
\[
|IV| \lesssim C_{16} (e^{2\pi k} \text{ dist} (C_{1}^{T}, C_{2}^{T}) + \left\| \log \text{ dist} (C_{1}^{T}, C_{2}^{T}) \right\|_{\frac{\tilde{s}^{2}}{\gamma^{(r+1)/(s+2)}}}).
\]

We conclude the proof of claim 1 by combining the two above estimates into (3.24).

Claim 2: There exists $C_{17} > 0$ such that
\[
\left| \int_{B} e^{i \varphi} \chi_{\Omega \cup \Omega^{*}}(\tilde{q}_1 - \tilde{q}_2)e^{i \xi \cdot x} \left( e^{i \xi_{1} - \xi_{2}^{*}} + e^{i \xi_{1} + \xi_{2}^{*}} \right) \right| \lesssim C_{17} \left( \left\| \int_{B} e^{i \varphi} \chi_{\Omega \cup \Omega^{*}}(\tilde{q}_1 - \tilde{q}_2)U_{1}U_{2} \right\|_{\gamma^{-1/(s+2)}} + \tau^{-1/(s+2)} \right.
\]
\[
\left. + \left\| \log \text{ dist} (C_{1}^{T}, C_{2}^{T}) \right\|_{\frac{\tilde{s}^{2}}{\gamma^{(r+1)/(s+2)}}} + e^{-4\pi \epsilon \tau^{1/(n+1)}} + e^{\epsilon} \right),
\] (3.34)
\[
\left| \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \xi_\ast} \left( e^{\phi_1 + \phi_2^*} + e^{\phi_1^* + \phi_2} \right) \right| \\
\leq \left| \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) U_1 U_2 \right| \\
+ |V + VI + VII + VIII + IX + X|, \\
(3.35)
\]

where

\[
V = \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \xi_\ast} \left( e^{\phi_1 + \phi_2^*} + e^{\phi_1^* + \phi_2} \right),
\]

\[
VI = \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \xi_\ast} \left( e^{\phi_1 + \phi_2^*} + e^{\phi_1^* + \phi_2} \right),
\]

\[
VII = \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} \times \left( e^{\phi_1 + \phi_2^*} + e^{\phi_1^* + \phi_2} \right),
\]

\[
VIII = \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} \times \left( e^{\phi_1 + \phi_2^*} + e^{\phi_1^* + \phi_2} \right),
\]

\[
IX = \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} e^{\phi_1 + \phi_2^*},
\]

\[
X = \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} e^{\phi_1 + \phi_2^*}.
\]

The task now is to estimate these terms. By (2.51) with \(B\) instead of \(\Omega \cup \Omega^*\), the estimate (3.14), the boundedness of \(B\) and the H"older’s inequality in \(L^2(B)\), we get

\[
|V + VI + VII + VIII| \leq C_\theta \tau^{-\frac{1}{r+2}}.
(3.36)
\]

To estimate \(IX\) and \(X\), we require a more delicate analysis. By adding and subtracting the same terms, it is easy to check that

\[
|IX| \leq \left| \int_B \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} \left( e^\phi - e^{\phi_1 + \phi_2} \right) \right|
\]

\[
+ \int_B \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} (1 - e^{\phi_1 + \phi_2})
\]

\[
+ \int_B e^{i\varphi} \chi_{\Omega \cup \Omega^*} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \xi_\ast 2 \tau \sqrt{1 + \frac{2i|\xi|}{\tau}} \right)} (e^{\phi_1 + \phi_2} - e^{\phi_1 + \phi_2^*}) \left| e^{\phi_1 + \phi_2} - e^{\phi_1 + \phi_2^*} \right|.
(3.37)
\]
On the one hand, by lemma 2.6 we have \( \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) \in B_{2, \infty}^* (\mathbb{R}^n) \). Thus, by lemma 2.11 applied to \( f = \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) \), \( C_0 = \| \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) \|_{B_{2, \infty}^*} \) and \( \sigma = 1 \), we obtain
\[
\left| \int_B \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \epsilon \cdot 2\tau \sqrt{1 - \tau^2 - |\xi|^2} \right)} \right| \leq C_19 \left( e^{-4\pi \epsilon^2 \tau^2 |\xi|^2 / |\tau|^2} + \epsilon^4 \right).
\] (3.38)

On the other hand, from (2.77), we deduce that
\[
(i\mu_1 + \mu_2) \cdot \nabla (\Phi_1 + \overline{\Phi}_2 + i\varphi) = (\mu_1 - i\mu_2) \cdot (\chi_{\Omega_1 \cup \Omega_2} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi).
\] (3.39)

Thus, by the boundeness of \( (i\mu_1 + \mu_2) \cdot \nabla \)^{-1} in weighted \( L^2 \) spaces, and (3.11) and (3.12), we obtain
\[
\| \Phi_1 + \overline{\Phi}_2 + i\varphi \|_{L^2(\Omega)} \leq C_19 \left( \| \chi_{\Omega_1 \cup \Omega_2} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi \|_{L^2(\Omega)} \right)
\leq C_20 \left( \| \chi_{\Omega_1 \cup \Omega_2} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi \|_{L^2(\Omega)} + \| \psi - \psi^* \|_{H^1(\Omega, \mathbb{R})} \right)
\leq C_21 \left( \log \text{dist} (C_1^T, C_2^T) \right)^{-\frac{2}{2 + \sigma}} .
\] (3.40)

This inequality, (2.66) and the boundedness of \( B \), imply that
\[
\left| \int_B \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) e^{i \left( \epsilon \cdot 2\tau \sqrt{1 - \tau^2 - |\xi|^2} \right)} \right| \left( 1 - e^{\Phi_1 + \overline{\Phi}_2 + i\varphi} \right)
\leq C_22 \| \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) \|_{L^2(\Omega)} \| \Phi_1 + \overline{\Phi}_2 + i\varphi \|_{L^2(\Omega)}
\leq C_23 \left( \log \text{dist} (C_1^T, C_2^T) \right)^{-\frac{2}{2 + \sigma}} .
\] (3.41)

From (2.56), (2.66) and (3.14), we get
\[
\left| \int_B e^{i\varphi} \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) \right| e^{i \left( \epsilon \cdot 2\tau \sqrt{1 - \tau^2 - |\xi|^2} \right)} \left( e^{\Phi_1 + \overline{\Phi}_2} - e^{\Phi_1^* + \overline{\Phi}_2^*} \right)
\leq \| e^{i\varphi} \chi_{\Omega_1 \cup \Omega_2} (\tilde{q}_1 - \tilde{q}_2) \|_{L^2(\Omega)} \| \Phi_1 - \Phi_1^* + \overline{\Phi}_2 - \Phi_2^* \|_{L^2(\Omega)}
\leq C_24 \tau^{-(2/(r+2))} .
\] (3.42)

Hence, by using (3.38)–(3.42) into (3.37), we obtain
\[
|X| \leq C_{25} \left( \tau^{-(2/(r+2))} + \frac{\| \log \text{dist} (C_1^T, C_2^T) \|^{-\frac{2}{2 + \sigma}}}{(2 + \sigma)^{\frac{2}{2 + \omega}} + \epsilon^4} \right).
\] (3.43)

Now we estimate the term \( X \). From (2.77) and (3.39), we deduce
\[
(i\mu_1^* + \mu_2^*) \cdot \nabla (\Phi_1^* + \overline{\Phi}_2^* + i\varphi) = (\mu_1^* - i\mu_2^*) \cdot (\chi_{\Omega_1 \cup \Omega_2} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi).
\] (3.44)

which implies that
\[
\| \Phi_1^* + \overline{\Phi}_2^* + i\varphi \|_{L^2(\Omega)} \leq C_{19} \left( \| \chi_{\Omega_1 \cup \Omega_2} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi \|_{L^2(\Omega)} \right)
\leq C_{20} \left( \| \chi_{\Omega_1 \cup \Omega_2} (\tilde{A}_1 - \tilde{A}_2) - \nabla \varphi \|_{L^2(\Omega)} + \| \psi - \psi^* \|_{H^1(\Omega, \mathbb{R})} \right)
\leq C_{21} \left( \log \text{dist} (C_1^T, C_2^T) \right)^{-\frac{2}{2 + \sigma}} .
\] (3.45)
Thus, by similar arguments used to estimate $IX$, we deduce that

$$|X| \leq C_{26} \left( \tau^{-s/(s+2)} + \left| \log \, \text{dist} (C_f^1, C_f^2) \right|^{\frac{4}{(s+2)(s+4)}} \tau^{(s+4)/(s+2)} + e^{-4\pi\tau + \frac{|\xi|^2}{16\tau}} + e^\epsilon \right).$$

Finally, by combining (3.36) and (3.43)–(3.46) into (3.35), we conclude the proof of claim 2.

On the other hand, by a straightforward computation, for every $\xi \in \bigcap_{i=1}^{n-1} E_i$, we deduce

$$|\mathcal{F}[\chi_{\Omega,\Gamma^i} \hat{q}_1] (\xi) - \mathcal{F}[\chi_{\Omega,\Gamma^i} \hat{q}_2] (\xi)| \leq |M_1 + M_2 + M_3 + M_4 + M_5|,$$

where

$$M_1 = \int_B e^{i\xi \chi_{\Omega,\Pi^i} \hat{q}_1} e^{\Phi_1 + i\eta} + \int_B e^{i\xi \chi_{\Omega,\Pi^i} \hat{q}_2} e^{\Phi_2 + i\eta},$$

$$M_2 = \int_B e^{i\xi \chi_{\Omega,\Pi^i} \hat{q}_1} e^{\Phi_1 + i\eta} (1 - e^{\Phi_1 + i\eta}),$$

$$M_3 = \int_B e^{i\xi \chi_{\Omega,\Pi^i} \hat{q}_2} e^{\Phi_2 + i\eta} (1 - e^{\Phi_1 + i\eta}),$$

$$M_4 = \int_B e^{i\xi \chi_{\Omega,\Pi^i} \hat{q}_1} e^{\Phi_1 + i\eta} e^{\Phi_1 - \Phi_2 + i\eta},$$

$$M_5 = \int_B e^{i\xi \chi_{\Omega,\Pi^i} \hat{q}_2} e^{\Phi_2 + i\eta} e^{\Phi_1 - \Phi_2 + i\eta}.$$

From the claims 1 and 2, we obtain

$$|M_1| \leq C_{26} \left( \left| \log \, \text{dist} (C_f^1, C_f^2) \right|^{\frac{4}{(s+2)(s+4)}} \tau^{(s+4)/(s+2)} + | \log \, \text{dist} (C_f^1, C_f^2) |^{\frac{2}{(s+2)(s+4)}} \tau^{s/(s+2)} + e^{-4\pi\tau + \frac{|\xi|^2}{16\tau}} + e^\epsilon \right).$$

From the boundedness of $B$, (2.66) and (3.40), we get

$$|M_2| \leq C_{27} \left| \chi_{\Omega,\Pi^i}(\hat{q}_1 - \hat{q}_2) \right|_{L^2(B)} \left| \Phi_1 + \overline{\Phi_2} + i\eta \right|_{L^2(B)} \leq C_{28} \left| \log \, \text{dist} (C_f^1, C_f^2) \right|^{\frac{4}{(s+2)(s+4)}}.$$

From (3.45), we have

$$|M_3| \leq C_{29} \left| \log \, \text{dist} (C_f^1, C_f^2) \right|^{\frac{4}{(s+2)(s+4)}}.$$

From (2.56), (2.66) and (3.14), we get

$$|M_4 + M_5| \leq C_{30} \tau^{-s/(s+2)}.$$

We conclude the proof of proposition 3.3 by replacing (3.48)–(3.51) into (3.47).
### 3.2. Proof of theorem 1.6

The proof of theorem 1.6 is similar to the proof of theorem 1.5. By proposition 3.3 and since the set $\bigcap_{i=1}^{n-1} E_i$ is dense in $\mathbb{R}^n$, it follows that the following estimate

$$\|F[\chi_{\Omega}\cdot \tilde{q}]\|_{L^r(\mathbb{R}^n)} \leq C \left( \left| \log \text{dist} (C^1, C^2) \right|^{ \frac{s}{2(r+4)}} \tau^{(s+4)/(s+2)} + e^{2\gamma \cdot \text{dist} (C^1, C^2)} + \tau^{-s/(s+2)} + e^{-4\pi \varepsilon^2 (s+1)/\tau^2} \right).$$

(3.52)

holds true for all $\xi \in \mathbb{R}^n$. Now we consider $R \geq 1$ (which will be fixed later) and for convenience we denote $\hat{q} := \chi_{\Omega\cdot \tilde{q}}$. By Plancherel’s theorem, we have

$$\|\hat{q}\|_{L^r(\mathbb{R}^n)}^2 = \int_{B_R(0) \setminus \{0\}} |F[\hat{q}]| \xi \|_r^2 \text{d} \xi + \int_{\mathbb{R}^n \setminus B_R(0)} |F[\hat{q}]| \xi \|_r^2 \text{d} \xi.$$  

From (3.52), we get

$$\int_{B_R(0) \setminus \{0\}} |F[\hat{q}]| \xi \|_r^2 \text{d} \xi \leq C_1 R^n \left( \tau^{-2s/(s+2)} + e^{2s} + e^{4\gamma \cdot \text{dist} (C^1, C^2)} \right)^2$$

$$+ \left| \log \text{dist} (C^1, C^2) \right|^{ \frac{s}{2(r+4)}} \tau^{(s+4)/(s+2)} \right) + C_1 \int_{B_R(0) \setminus \{0\}} e^{-4\pi \varepsilon^2 (s+1)/\tau^2} \left( \tau^{-2s/(s+2)} + e^{4\gamma \cdot \text{dist} (C^1, C^2)} \right)^2 + e^{2s} + e^{-2 \cdot \tau^{-2}}$$

$$+ \left| \log \text{dist} (C^1, C^2) \right|^{ \frac{s}{2(r+4)}} \tau^{(s+4)/(s+2)} \right).$$

Thus, by equating $e^{2s}$ and $e^{-2 \cdot \tau^{-2}}$, i.e. by taking $\varepsilon = \tau^{-1/(s+1)}$, we obtain

$$\int_{B_R(0) \setminus \{0\}} |F[\hat{q}]| \xi \|_r^2 \text{d} \xi \leq C_1 R^n \left( \tau^{-2s/(s+2)} + e^{4\gamma \cdot \text{dist} (C^1, C^2)} \right)^2$$

$$+ \left| \log \text{dist} (C^1, C^2) \right|^{ \frac{s}{2(r+4)}} \tau^{(s+4)/(s+2)} \right).$$

(3.54)

We now turn to estimate the integral term on $\mathbb{R}^n \setminus B_R(0)$. By lemma 2.6 and since $\chi_{\Omega\cdot \tilde{q}}$ and $\chi_{\Omega\cdot \tilde{q}}$ belong to $Q(\Omega, M, s)$, it follows that $\tilde{q}$ belong to $Q(\Omega \cup \Omega^*, 2M, s)$. In particular, $\tilde{q} \in B^s_{1/2}(\mathbb{R}^n)$ and $\|\tilde{q}\|_{B^s_{1/2}} \leq 2M$. By proposition 10 and theorem 5 in [15], we obtain the following chain of embeddings:

$$B^s_{1/2}(\mathbb{R}^n) \subset B^s_{s/2}(\mathbb{R}^n) \subset H^{s/2}(\mathbb{R}^n).$$

Hence, we deduce that $\tilde{q} \in H^{s/2}(\mathbb{R}^n)$ and its norm in $H^{s/2}(\mathbb{R}^n)$ only depends on the priori bounds for the magnetic and electric potentials. Then, we have
\[
\int_{\mathbb{R}^n \setminus B_\delta(0)} |\mathcal{F}[\tilde{q}] (\xi)|^2 \, d\xi
\]
\[
= \int_{\mathbb{R}^n \setminus B_\delta(0)} (1 + |\xi|^2)^{-s/2}(1 + |\xi|^2)^{s/2} |\mathcal{F}[\tilde{q}] (\xi)|^2 \, d\xi
\]
\[
\leq R^{-s} \int_{\mathbb{R}^n \setminus B_\delta(0)} (1 + |\xi|^2)^{s/2} |\mathcal{F}[\tilde{q}] (\xi)|^2 \, d\xi
\]
\[
\leq R^{-s} \|\tilde{q}\|^2_{L^2(\mathbb{R}^n)} \leq C_4 R^{-s}. \tag*{(3.55)}
\]

Thus, by replacing (3.54) and (3.55) into (3.53) we have that there exist two positive constants \(C_3\) and \(\tau_1\) such that the estimate

\[
\|\tilde{q}\|^2_{L^2(\mathbb{R}^n)} \leq C_5 \left( R^{n-2s/((n+s)(s+2))} + R^{-s} + R^0 e^{4s\kappa} \text{ dist } (C_1^T, C_2^T)^2 + R^0 |\log \text{ dist } (C_1^T, C_2^T)|^{-2(n+4s)/((n+s)(s+2))} \right). \tag*{(3.56)}
\]

holds true for all \(\tau \geq \tau_1\). In order to equal the two first term from the left-hand side of this inequality, we express \(R\) as follows \(R = \tau^{2s/((n+s)(s+2))}\). On the other hand, there exist two positive constants \(C_6\) and \(\tau_2\) such that

\[
R^0 = \tau^{2s/((n+s)(s+2))} \leq C_6 e^\kappa, \quad \tau \geq \tau_2.
\]

Hence, by using (3.54) and (3.56) into (3.53), we have

\[
\|\tilde{q}\|^2_{L^2(\mathbb{R}^n)} \leq C_7 \left( \tau^{-4s/((n+s)(s+2))} + e^{5\kappa} \text{ dist } (C_1^T, C_2^T)^2 + |\log \text{ dist } (C_1^T, C_2^T)|^{-2(n+4s)/((n+s)(s+2))} \right). \tag*{(3.57)}
\]

Now, we take \(\tau_0 \geq \max \{\tau_1, \tau_2\}\) such that \(5\kappa \tau_0 \geq 1\). Then, it is easy to check that

\[
\tau := \frac{1}{5\kappa} |\log \text{ dist } (C_1^T, C_2^T)|^{-2(n+4s)/((n+s)(s+2))} \geq \tau_0, \tag*{(3.58)}
\]

whenever

\[
\text{dist } (C_1^T, C_2^T) \leq e^{-\frac{2(n+4s)}{(n+s)(s+2)} \cdot \left(\frac{\tau_0}{\tau}\right)^{2(n+4s)/((n+s)(s+2))}}. \tag*{(3.59)}
\]

From (3.58), it follows that

\[
\tau^{-4s/((n+s)(s+2))} \leq C_8 |\log \text{ dist } (C_1^T, C_2^T)|^{-2(n+4s)/((n+s)(s+2))} \cdot \tau^{-\frac{2(n+4s)}{(n+s)(s+2)} \cdot \left(\frac{\tau_0}{\tau}\right)^{2(n+4s)/((n+s)(s+2))}}. \tag*{(3.60)}
\]

Since \(\text{dist } (C_1^T, C_2^T) \leq e^{-1}\) and from (3.58), we deduce that

\[
e^{5\kappa} \text{ dist } (C_1^T, C_2^T)^2 \leq e^{5\kappa} \frac{2(n+4s)}{(n+s)(s+2)} \cdot \left(\frac{\tau_0}{\tau}\right)^{2(n+4s)/((n+s)(s+2))} \leq e^{-5\kappa}. \tag*{(3.61)}
\]

Then

\[
e^{5\kappa} \text{ dist } (C_1^T, C_2^T)^2 \leq |\log \text{ dist } (C_1^T, C_2^T)|^{-2(n+4s)/((n+s)(s+2))} \cdot \left(\frac{\tau_0}{\tau}\right)^{2(n+4s)/((n+s)(s+2))}. \tag*{(3.61)}
\]
The last term from the right-hand side of (3.57) satisfies
\[
\left| \log \operatorname{dist}(C_1^n, C_2^n) \right| \lesssim \frac{2\theta^3}{(s+2)(n+ns+2s)(2ns+4n+s^2+4s)} \lesssim \frac{\theta^3}{n^2}.
\]
Then, by replacing (3.60)–(3.62) into (3.57), we obtain
\[
\|q\|_{L^2(\mathbb{R}^n)} \lesssim C_{11} \left| \log \operatorname{dist}(C_1^n, C_2^n) \right|^{\frac{n+9}{2}} \lesssim \frac{\theta^3}{n^{\frac{3}{2}}}.
\]
Since \( n \geq 3 \) and \( s \in (0, 1/2) \), we have
\[
n + \frac{n}{2} + 1 \leq 2n, \quad 5n + \frac{9}{4} \leq 6n,
\]
and
\[
\frac{\theta^3}{(s+2)(n+ns+2s)(2ns+4n+s^2+4s)} \geq \frac{2\theta^3}{5(n+\frac{s}{2}+1)(5n+\frac{9}{4})} \geq \frac{\theta^3}{30n^2}.
\]
By replacing this inequality into (3.63), we get
\[
\|q\|_{L^2(\mathbb{R}^n)} \lesssim C_{11} \left| \log \operatorname{dist}(C_1^n, C_2^n) \right|^{\frac{n}{6}}.
\]
We conclude the proof by taking \( \lambda = \theta/30 \) and \( C \) as follows
\[
C = \max \left\{ \left( 5\kappa \tau_0 \right)^{\frac{2(s+2)(n+ns+2s)(2ns+4n+s^2+4s)}{9\theta^3(n+\frac{s}{2}+1)(5n+\frac{9}{4})}}, C_{11} \right\},
\]
and taking into account that
\[
\|q_1 - q_2\|_{L^1(\Omega)} \leq \|\tilde{q}\|_{L^1(\mathbb{R}^n)}.
\]

4. Identifiability for the magnetic field and the electric potential

4.1. Proof of theorem 1.1

We only give the main ideas to prove the identifiability for the magnetic field and electric potential since it is just the qualitative version of what we have proved in the previous sections. We consider \( \rho_1 \) and \( \rho_2 \) defined by (2.32). Now let \( U_1, U_2 \in H^1(\Omega) \) satisfying \( \mathcal{L}_{A_1, q_1} U_1 = 0 \) with \( U_1|_{\Gamma_0} = 0 \) and \( \mathcal{L}_{A_2, q_2} U_2 = 0 \) with \( U_2|_{\Gamma_0} = 0 \). The existence of such functions are given by proposition 2.7, except that we replaced the estimates (2.24)–(2.29) for the estimates (2.17)–(2.19) given in remark 2.3. From corollary 2.5 and since \( C_{A_1, q_1}^\dagger = C_{A_2, q_2}^\dagger \), we deduce that
\[
\int_{\Omega} (A_1 - A_2) \cdot (DU_1 U_2^* + U_1 D U_2^*) + (A_1^2 - A_2^2 + q_1 - q_2) U_1 U_2^* = 0.
\]
From this integral identity and following the proof of the proposition 2.10, we can prove the identifiability result for the magnetic potentials by applying the Riemann–Lebesgue lemma to the function \( \chi_{\Omega}(A_1 - A_2)e^{i\theta_1 + i\theta_2} \) and taking into account lemma 2.12 in order
to remove the term $e^{\Phi_1 + \Phi_2}$ in the left-hand side of (2.76). At this point, since the Fourier transform is analytic, proposition 2.10 and (2.81) give us the following equality in the sense of the distributions in $\mathbb{R}^n$:

$$d(\chi_{\Omega\cup\Omega^* - A_1}) = d(\chi_{\Omega\cup\Omega^* - A_2}),$$

which implies that $dA_1 = dA_2$ in $\Omega$.

The proof of the identifiability for the electric potential is as follows. We consider the Hodge decomposition for $\chi_{\Omega\cup\Omega^*} (\tilde{A}_1 - \tilde{A}_2)$ in a ball $B$ satisfying $\Omega \cup \Omega^* \subset \subset B$. We also take into account the estimates from remark 2.3 and the Riemann–Lebesgue lemma now applied to the function $\chi_{\Omega\cup\Omega^*} (\tilde{q}_1 - \tilde{q}_2)$. From proposition 3.3, the estimate (3.52) and since the Fourier transform is analytic, we deduce that

$$\chi_{\Omega\cup\Omega^*} \tilde{q}_1 = \chi_{\Omega\cup\Omega^*} \tilde{q}_2,$$

which implies that $q_1 = q_2$ in $\Omega$.

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