A CHARACTERIZATION OF THE DEGENERATE COMPLEX HESSIAN EQUATIONS FOR FUNCTIONS WITH BOUNDED $(p,m)$-ENERGY

PER ÅHAG AND RAFAL CZYŻ

Abstract. By proving an estimate of the sublevel sets for $(\omega, m)$-subharmonic functions we obtain a Sobolev type inequality that is then used to characterize the degenerate complex Hessian equations for such functions with bounded $(p,m)$-energy.

March 16, 2020

1. Introduction

Ever since the 1930s when the interest in Kähler geometry gained momentum with the publication of Erich Kähler’s article [13], the attention has been immense both from mathematicians and physicists. Take, for example, the works of Aubin [2] and Yau [20], as well as the highly regarded Seiberg-Witten theory [18, 19] of physics. In the mentioned work of Aubin and Yau they showed how geometric information of a Kähler manifold can be retrieved by solving certain partial differential equation of Monge-Ampère type. This is part of the explanation of why these equations and associated methods have been of great interest in recent decades. Our motivation is instead from a pluripotential theoretical background and the highly influential work of Bedford and Taylor [3, 4], and Cegrell [7, 8].

Combining the ideas of Cegrell’s energy classes with globally defined plurisubharmonic functions known as $\omega$-plurisubharmonic functions Guedj and Zeriahi introduced and studied weighted energy classes of $\omega$-plurisubharmonic functions ([14]). In particular, they proved the existence of solutions to the Dirichlet problem for the complex Monge-Ampère operator, and later Dinew proved the uniqueness ([15]).

Here we shall also use the idea of energy classes, but for the interpolation spaces of $m$-subharmonic functions. These spaces interpolate between subharmonic and plurisubharmonic functions, and the differential operator is the complex Hessian operator. The idea of these interpolation spaces goes back to Caffarelli et al. [6], and pluripotential methods were introduced by Blocki in [5].

The general setting of this paper is that $n \geq 2$, $p > 0$, and $1 \leq m \leq n$. Furthermore, we shall use $(X, \omega)$ to denote a connected and compact Kähler manifold of complex dimension $n$, where $\omega$ is a Kähler form on $X$ such that $\int_X \omega^n = 1$. The energy classes of $(\omega, m)$-subharmonic functions with bounded $(p,m)$-energy that is

2010 Mathematics Subject Classification. Primary 32U05, 31C45, 46E35; Secondary 53C55, 35J60.

Key words and phrases. compact Kähler manifolds, complex Hessian equation, $(\omega, m)$-subharmonic functions, Sobolev type inequalities.
central for this paper is defined by
\[ \mathcal{E}_m^p(X, \omega) := \{ u \in \mathcal{E}_m(X, \omega) : u \leq 0, e_{p,m}(u) < \infty \}, \]
where
\[ e_{p,m}(u) = \int_X (-u)^p H_m(u), \]
and \( H_m \) denote the complex Hessian operator (see Section \( 2 \) for details). For a historical account and references see e.g. \cite{1, 17}.

By proving in Lemma \( 1.1 \) an estimate of the sublevel sets for \((\omega, m)\)-subharmonic functions we obtain the following Sobolev type inequality.

**Theorem 5.1.** Let \( n \geq 2, p > 0, \) and let \( 1 \leq m \leq n \). Assume that \((X, \omega)\) is a connected and compact Kähler manifold of complex dimension \( n \), where \( \omega \) is a Kähler form on \( X \) such that \( \int_X \omega^n = 1 \). Furthermore, assume that \( \mu \) is a Borel measure defined on \( X \). Fix a constant \( \beta \) such that \( 1 > \beta > \max \left( \frac{pn-n}{pn-n+m}, \frac{p}{p+1} \right) \), for \( p > 1 \), and \( \beta = \frac{p}{p+1} \) for \( p \leq 1 \). The following conditions are then equivalent:

1. \( \mathcal{E}_m^p(X, \omega) \subset L^q(X, \mu); \)
2. there exists a constant \( C > 0 \) such that for all \( u \in \mathcal{E}_m(X, \omega) \cap L^\infty(X) \) with \( \sup_X u = -1 \) it holds
\[ \int_X (-u)^q \, d\mu \leq C e_{p,m}(u)^{\frac{q}{p}}; \]
3. there exists a constant \( C > 0 \) such that for all \( u \in \mathcal{E}_m^p(X, \omega) \) with \( \sup_X u = -1 \) it holds
\[ \int_X (-u)^q \, d\mu \leq C e_{p,m}(u)^{\frac{q}{p}}. \]

Theorem \( 5.1 \) is then used in Theorem \( 5.2 \) to characterize the degenerate complex Hessian equation for \((\omega, m)\)-subharmonic functions with bounded \((p, m)\)-energy. This equation was first considered for smooth solutions, and later for continuous functions (see e.g. \cite{11, 16, 17} and references therein). In \cite{16}, Lu and Nguyen recently solved the Dirichlet problem for the complex Hessian equation in \( \mathcal{E}_m(X, \omega) \). In their paper, they used the variational method. By instead using our Sobolev type inequality we can in Theorem \( 5.2 \) generalize Lu and Nguyen results to \( p > 0 \).

**Theorem 5.2.** Let \( n \geq 2, p > 0, \) and let \( 1 \leq m \leq n \). Assume that \((X, \omega)\) is a connected and compact Kähler manifold of complex dimension \( n \), where \( \omega \) is a Kähler form on \( X \) such that \( \int_X \omega^n = 1 \). Furthermore, assume that \( \mu \) is a Borel probability measure defined on \( X \). The following conditions are then equivalent:

1. \( \mathcal{E}_m^p(X, \omega) \subset L^p(X, \mu); \)
2. there exists unique \((\omega, m)\)-subharmonic function \( u \) in \( \mathcal{E}_m^p(X, \omega) \) such that \( \sup_X u = -1 \) and \( H_m(u) = \mu \).

2. **Preliminaries**

Let \( \Omega \subset \mathbb{C}^n \), \( n \geq 2 \), be a bounded domain, \( 1 \leq m \leq n \), and define \( \mathcal{C}_{(1,1)} \) to be the set of \((1,1)\)-forms with constant coefficients. We then define
\[ \Gamma_m = \{ \alpha \in \mathcal{C}_{(1,1)} : \alpha \wedge \beta^{n-1} \geq 0, \ldots, \alpha^m \wedge \beta^{n-m} \geq 0 \}, \]
where \( \beta = dd^c |z|^2 \) is the canonical Kähler form in \( \mathbb{C}^n \).
**Definition 2.1.** Let \( n \geq 2 \), and \( 1 \leq m \leq n \). Assume that \( \Omega \subset \mathbb{C}^n \) is a bounded domain, and let \( u \) be a subharmonic function defined on \( \Omega \). Then we say that \( u \) is \( m \)-subharmonic if the following inequality holds

\[
\dd^c u \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0, 
\]

in the sense of currents for all \( \alpha_1, \ldots, \alpha_{m-1} \in \Gamma_m \). With \( \mathcal{SH}_m(\Omega) \) we denote the set of all \( m \)-subharmonic functions defined on \( \Omega \).

Let \( \sigma_k \) be \( k \)-elementary symmetric polynomial of \( n \)-variable, i.e.,

\[
\sigma_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k},
\]

It can be proved that a function \( u \in C^2(\Omega) \) is \( m \)-subharmonic if, and only if,

\[
\sigma_k(u(z)) = \sigma_k(\lambda_1(z), \ldots, \lambda_n(z)) \geq 0,
\]

for all \( k = 1, \ldots, m \), and all \( z \in \Omega \). Here, \( \lambda_1(z), \ldots, \lambda_n(z) \) are the eigenvalues of the complex Hessian matrix \( \left[ \frac{\partial^2 u}{\partial z_j \partial z_k}(z) \right] \).

Next we shall consider compact Kähler manifold.

**Definition 2.2.** Let \( n \geq 2 \), and \( 1 \leq m \leq n \). Assume that \( (X, \omega) \) is a connected and compact Kähler manifold of complex dimension \( n \), where \( \omega \) is a Kähler form on \( X \) such that \( \int_X \omega^n = 1 \). A function \( u : X \to \mathbb{R} \cup \{-\infty\} \) is called \((\omega, m)\)-subharmonic if in any local chart \( \Omega \) of \( X \), the function \( f + u \) is \( m \)-subharmonic, where \( f \) is a local potential of \( \omega \). We shall denote by \( \mathcal{SH}_m(X, \omega) \) the set of all \((\omega, m)\)-subharmonic functions on \( X \).

The following notation is convenient: for any \( u \in \mathcal{SH}_m(X, \omega) \) let

\[
\omega_u := \dd^c u + \omega.
\]

With this notation we have that a smooth function \( u \) is \((\omega, m)\)-subharmonic if, and only if,

\[
\omega_u^k \wedge \omega^{n-k} \geq 0, \quad \text{for all } k = 1, \ldots, m.
\]

In the following proposition we list useful properties of \((\omega, m)\)-subharmonic functions. For proofs see e.g. [14] and the references therein.

**Proposition 2.3.** Let \( (X, \omega) \) be a compact Kähler manifold. Then

1. if \( u, v \in \mathcal{SH}_m(X, \omega) \), \( t \in [0, 1] \), then \( tu + (1 - t)v \in \mathcal{SH}_m(X, \omega) \);
2. if \( u \in \mathcal{SH}_m(X, \omega) \), \( t \in [0, 1] \), then \( tu \in \mathcal{SH}_m(X, \omega) \);
3. if \( u, v \in \mathcal{SH}_m(X, \omega) \), then \( \max(u, v) \in \mathcal{SH}_m(X, \omega) \);
4. if \( u_j \in \mathcal{SH}_m(X, \omega) \), \( j \in \mathbb{N} \) then \( \sup_j u_j^* \in \mathcal{SH}_m(X, \omega) \). Here \( (\cdot)^* \) denotes the upper semicontinuous regularization;
5. if \( u \in \mathcal{SH}_m(X, \omega) \), then there exists a decreasing sequence \( u_j \in \mathcal{SH}_m(X, \omega) \cap C^\infty(X) \) such that \( u_j \to u \), \( j \to \infty \).

Following the idea from [14] one can define the complex Hessian operator for \((\omega, m)\)-subharmonic through the following construction. First assume that \( u \in \mathcal{SH}_m(X, \omega) \cap L^\infty(X) \), then

\[
H_m(u) := \omega_u^m \wedge \omega^{n-m},
\]
which is a non-negative (regular) Borel measure on $X$. For an arbitrary (not necessarily bounded) $(\omega, m)$-subharmonic function $u$ let $u_j = \max(u, -j)$ be the canonical approximation of $u$. Then define

$$H_m(u) := \lim_{j \to \infty} \chi_{\{u > -j\}} \; H_m(u_j).$$

The complex Hessian operator is then used to construct the following function class.

**Definition 2.4.** Let $\mathcal{E}_m(X, \omega)$ be the class of all $(\omega, m)$-subharmonic functions defined as

$$\mathcal{E}_m(X, \omega) = \left\{ u \in \mathcal{SH}_m(X, \omega) : \int_X H_m(u) = 1 \right\}.$$

**Remark.** Note that $u \in \mathcal{E}_m(X, \omega)$ if, and only if,

$$H_m(u_j)(\{u_j \leq -j\}) \to 0, \quad \text{as } j \to \infty.$$

Here, $u_j = \max(u, -j)$.

Let us collect some properties of the class $\mathcal{E}_m(X, \omega)$. Proofs can be found in [12].

**Theorem 2.5.** Let $(X, \omega)$ be a compact Kähler manifold.

1. If $u, v \in \mathcal{E}_m(X, \omega)$, $t \in [0, 1]$, then $tu + (1-t)v \in \mathcal{E}_m(X, \omega)$, and $\max(u, v) \in \mathcal{E}_m(X, \omega)$. In particular, if $u, v \in \mathcal{SH}_m(X, \omega)$, $u \in \mathcal{E}_m(X, \omega)$ and $u \leq v$, then $v \in \mathcal{E}_m(X, \omega)$.

2. If $u_j \in \mathcal{E}_m(X, \omega)$ is decreasing sequence converging to $u \in \mathcal{E}_m(X, \omega)$, then $H_m(u_j)$ converges weakly to $H_m(u)$.

3. If $u, v \in \mathcal{E}_m(X, \omega)$, then

$$\chi_{\{u < v\}} H_m(\max(u, v)) = \chi_{\{u < v\}} H_m(v).$$

In particular, if $H_m(u) \geq \mu$, $H_m(v) \geq \mu$ for some Borel measure $\mu$, then $H_m(\max(u, v)) \geq \mu$.

4. (The comparison principle) Let $T$ be a positive current of the type

$$T = \omega_{\psi_1} \wedge \cdots \wedge \omega_{\psi_k} \wedge \omega^{n-m},$$

where $k < m$, and $\psi_1, \ldots, \psi_k \in \mathcal{E}_m(X, \omega)$. Then for $u, v \in \mathcal{E}_m(X, \omega)$ it holds

$$\int_{\{u < v\}} \omega_v^{m-k} \wedge T \leq \int_{\{u < v\}} \omega_u^{m-k} \wedge T.$$

5. (The Dirichlet problem for the complex Hessian operator) Let $\mu$ be a probability measure on $X$ that does not charge $m$-polar sets. Then there exists a unique function $u \in \mathcal{E}_m(X, \omega)$ such that $H_m(u) = \mu$.

We shall be in need of the $m$-capacity defined on a compact Kähler manifold $X$.

**Definition 2.6.** For any Borel set $A \subset X$ define the $m$-capacity of $A$ as

$$\text{cap}_m(A) := \sup \left\{ \int_A H_m(u) : u \in \mathcal{SH}_m(X, \omega), -1 \leq u \leq 0 \right\}.$$

We say that a Borel set $A \subset X$ is $m$-polar if $\text{cap}_m(A) = 0$. 
Proposition 2.7. If $u \in \mathcal{E}_m(X, \omega)$, $u \leq 0$, then for any $t < 0$ it holds
\[
\operatorname{cap}_m(\{u < -t\}) \leq \frac{C}{t},
\]
where the constant $C$ does not depend on $u$.

A central part of the proof of Lemma 4.2 is the following estimate due to Dinew and Kołodziej [11].

Lemma 2.8. For any $1 < \alpha < \frac{n}{n-m}$ there exists a constant $C(\alpha) > 0$ such that for any Borel set $A \subset X$ it holds
\[
V(A) \leq C(\alpha) \cap_m(A)^\alpha,
\]
where $V(A) = \int_A \omega^n$.

3. Functions with bounded $(p,m)$-energy

In this section we focus on $(\omega,m)$-subharmonic functions with bounded $(p,m)$-energy, and prove some necessary properties that is needed for the rest of this paper.

Definition 3.1. Let $n \geq 2$, $p > 0$, and let $1 \leq m \leq n$. Assume that $(X, \omega)$ is a connected and compact Kähler manifold of complex dimension $n$, where $\omega$ is a Kähler form on $X$ such that $\int_X \omega^n = 1$. We define the class of $(\omega,m)$-subharmonic functions with bounded $(p,m)$-energy as
\[
\mathcal{E}^p_m(X, \omega) := \{u \in \mathcal{E}_m(X, \omega) : u \leq 0, e_{p,m}(u) < \infty\},
\]
where
\[
e_{p,m}(u) = \int_X (-u)^p H_m(u).
\]

Remark. It was proved in [14, 16] (see also [7, 9]) that $u \in \mathcal{E}^p_m(X, \omega)$ if, and only if, $\sup_j e_{p,m}(u_j) < \infty$, where $u_j = \max(u, -j)$ is the canonical approximation of $u$. Furthermore, $e_{p,m}(u_j) \to e_{p,m}(u)$ as $j \to \infty$.

Lemma 3.2. Let $(X, \omega)$ be a compact Kähler manifold, and $p > 0$. Furthermore, let $1 \leq j \leq m$, and let $T$ be a positive current of the type
\[
T = \omega_{\psi_1} \wedge \cdots \wedge \omega_{\psi_{m-j}} \wedge \omega^{n-m},
\]
where $\psi_1, \ldots, \psi_{m-j} \in \mathcal{E}_m(X, \omega)$. Then for any $u, v \in \mathcal{E}_m(X, \omega) \cap L^\infty(X)$, $u, v \leq 0$, it holds
\[
\int_X (-u)^p \omega_u^j \wedge T \leq 2^p \int_X (-u)^p \omega_u^j \wedge T + 2^p \int_X (-v)^p \omega_v^j \wedge T.
\]

Proof. Note that we have the following $\{u < -2s\} \subset \{u < v - s\} \cup \{v < -s\}$, and therefore by Theorem 2.5 (4)
\[
\int_{\{u < v - s\}} \omega_u^j \wedge T \leq \int_{\{u < v - s\}} \omega_u^j \wedge T \leq \int_{\{u < -s\}} \omega_u^j \wedge T,
\]
Lemma 3.3. Let \( (X, \omega) \) be a compact Kähler manifold, and \( p > 0 \). Let \( T \) be a positive current of the type
\[
T = \omega \psi_1 \wedge \cdots \wedge \omega \psi_{m-1} \wedge \omega^{n-m},
\]
where \( \psi_1, \ldots, \psi_{m-1} \in \mathcal{E}_m(X, \omega) \). Then for any \( u, v \in \mathcal{E}_m(X, \omega) \cap L^\infty(X) \) such that \( u \leq v \leq 0 \) it holds
\[
\int_X (-u)^p \omega_v \wedge T \leq (p + 1) \int_X (-u)^p \omega_u \wedge T. \tag{3.1}
\]
In particular, if \( u, v \in \mathcal{E}_m^p(X, \omega) \) are such that \( u \leq v \leq 0 \), then
\[
e_{p,m}(v) \leq (p + 1)^m e_{p,m}(u).
\]

Proof. From Proposition 2.3 it follows that we can assume that \( u \) is smooth and \( u \leq v < 0 \).

Case 1: \( p \geq 1 \). We have
\[
\int_X (-u)^p \omega_v \wedge T = \int_X (-u)^p \omega \wedge T + \int_X v dd^c(-u)^p \wedge T = I_1 + I_2. \tag{3.2}
\]
Note that
\[
I_1 = \int_X (-u)^p \omega \wedge T \leq \int_X (-u)^p \omega \wedge T + p \int_X (-u)^{p-1} du \wedge d^c u \wedge T
= \int_X (-u)^p \omega_u \wedge T, \tag{3.3}
\]
and
\[
dd^c(-u)^p = p(p - 1)(-u)^{p-2} du \wedge d^c u - p(-u)^{p-1} dd^c u \geq -p(-u)^{p-1} dd^c u.
\]
The integral \( I_2 \) can be estimated as follows
\[
I_2 = \int_X v dd^c(-u)^p \wedge T \leq p \int_X (-v)(-u)^{p-1} dd^c u \wedge T
\leq p \int_X (-v)(-u)^{p-1} \omega_u \wedge T \leq p \int_X (-u)^p \omega_u \wedge T. \tag{3.4}
\]
Combining the inequalities (3.2), (3.3) and (3.4) we get (3.1).
Case 2: \((0 < p < 1)\). We have by (3.3)
\[
\int_X (-u)^p \omega \wedge T = \int_X (-u)^p \omega \wedge T + \int_X (-v)dd^c (-(-u)^p) \wedge T
\]
\[
\leq \int_X (-u)^p \omega \wedge T + \int_X (-v) [p(-u)^{p-1} \omega + dd^c (-(-u)^p)] \wedge T
\]
\[
\leq \int_X (-u)^p \omega \wedge T + \int_X (-v) [p(-u)^{p-1} \omega + dd^c (-(-u)^p)] \wedge T
\]
\[
= p \int_X (-u)^p \omega \wedge T + \int_X (-u)^p \omega \wedge T \leq (p + 1) \int_X (-u)^p \omega \wedge T.
\]

The last statement of this lemma follows from the canonical approximation, and inequality (3.1) applied \(m\)-times
\[
e_{p,m}(v) = \int_X (-v)^p \omega_v^m \wedge \omega^{n-m} \leq \int_X (-u)^p \omega_v^m \wedge \omega^{n-m}
\]
\[
\leq (p + 1) \int_X (-u)^p \omega \wedge \omega_v^{m-1} \wedge \omega^{n-m} \leq \cdots \leq (p + 1)^m \int_X (-u)^p \omega_v^m \wedge \omega^{n-m} = (p + 1)^m e_{p,m}(u).
\]

\[\square\]

**Corollary 3.4.** Let \((X, \omega)\) be a compact Kähler manifold, and \(p > 0\). The following conditions are then equivalent

1. \(u \in \mathcal{E}_m^p(X, \omega)\);

2. for any decreasing sequence \(u_j \in \mathcal{E}_m^p(X, \omega)\), \(u_j \searrow u\) we have
\[
\sup_j e_{p,m}(u_j) < \infty;
\]

3. there exists a decreasing sequence \(u_j \in \mathcal{E}_m^p(X, \omega)\), \(u_j \searrow u\) such that
\[
\sup_j e_{p,m}(u_j) < \infty.
\]

**Proof.** The equivalence \((1) \Leftrightarrow (3)\) follows from the remark after Definition 3.1 and implication \((2) \Rightarrow (3)\) is immediate. Finally, we prove \((3) \Rightarrow (2)\). Assume that there exists a decreasing sequence \(u_j \in \mathcal{E}_m^p(X, \omega)\), \(u_j \searrow u\) such that
\[
\sup_j e_{p,m}(u_j) < \infty,
\]
and let \(v_j\) be any sequence decreasing to \(u\). Then for any \(j\) there exists \(k_j\) such that \(v_j \geq u_{k_j}\). Therefore by Lemma 3.3 the sequence \(e_{p,m}(v_j)\) is also bounded. \[\square\]

**Lemma 3.5.** Let \((X, \omega)\) be a compact Kähler manifold, and \(p > 0\). Then there exists a constant \(C > 0\) such that for any \(u_0, u_1, \ldots, u_m \in \mathcal{E}_m^p(X, \omega)\) it holds
\[
\int_X (-u_0)^p \omega_1 \wedge \cdots \wedge \omega_m \wedge \omega^{n-m} \leq C \max_{j=1, \ldots, m} e_{p,m}(u_j).
\]

**Proof.** By using the canonical approximation we can assume without loss of generality that all functions \(u_0, \ldots, u_m\) are bounded. For \(T = \omega_1 \wedge \cdots \wedge \omega_m \wedge \omega^{n-m}\), Lemma 3.2 yields
\[
\int_X (-u_0)^p \omega_1 \wedge T \leq 2^p \int_X (-u_0)^p \omega \wedge T + 2^p \int_X (-u_1)^p \omega_1 \wedge T.
\]
Therefore we can assume that \( u_0 = u_1 \). Set \( u = \epsilon \sum_{j=1}^{m} u_j \), where \( \epsilon \) is a small positive constant that will be specified later. It is sufficient to estimate integrals of the type \( \int_X (-u_1)^p \omega_u^m \land \omega^{n-m} \), since
\[
\omega_u^m \land \omega^{n-m} \geq \epsilon^m \omega_{u_1} \land \cdots \land \omega_{u_m} \land \omega^{n-m}.
\] (3.6)

Again by using Lemma 3.2
\[
\int_X (-u_1)^p \omega_u^m \land \omega^{n-m} \leq 2^p e_{p,m}(u_1) + 2^p e_{p,m}(u),
\]
and
\[
e_{p,m}(u) = \int_X \left(-\epsilon \sum_{j=1}^{m} u_j \right)^p \omega_u^m \land \omega^{n-m} \leq \max(\epsilon^p, \epsilon) \sum_{j=1}^{m} \int_X (-u_j)^p \omega_u^m \land \omega^{n-m}
\[
\leq \max(\epsilon^p, \epsilon) 2^p m \left( \max_{j=1,\ldots,m} e_{p,m}(u_j) + e_{p,m}(u) \right).
\] (3.7)

Now take \( \epsilon \) such that \( 1 - 2^p m \max(\epsilon, \epsilon^p) > \frac{1}{2} \), then by (3.6) and (3.7) we get
\[
\int_X (-u_1)^p \omega_u^m \land \omega^{n-m} \leq \epsilon^m \int_X (-u_1)^p \omega_u^m \land \omega^{n-m}
\[
\leq \frac{4^p m}{\epsilon^m (1 - 2^p m \max(\epsilon, \epsilon^p))} \max_{j=1,\ldots,m} e_{p,m}(u_j) + 2^p \epsilon^{-m} e_{p,m}(u_j)
\[
\leq 2^p + 1 \epsilon^{-m} \max_{j=1,\ldots,m} e_{p,m}(u_j).
\]

\[\Box\]

**Remark.** Assume that the functions \( u_j \in E_m^p(X, \omega) \) are such that \( \sup_{X} u_j = -1 \), and \( \sup_{j \in \mathbb{N}} e_{p,m}(u_j) < \infty \). Then
\[
u = \sum_{j=1}^{\infty} \frac{1}{2^j} u_j \in E_m^p(X, \omega).\]
(3.8)

By using Corollary 3.3 it is sufficient to construct a decreasing sequence of functions \( v_j \in E_m^p(X, \omega), v_j \searrow u, j \to \infty \), such that \( \sup_{j \in \mathbb{N}} e_{p,m}(v_j) < \infty \). Let us next define
\[
v_j = \sum_{k=1}^{j} a_k u_k, \text{ where } a_k = \frac{2^j}{2^k (2^j - 1)}.
\]

Then \( v_j \in \mathcal{S}_m(X, \omega), v_j \searrow u \), and by Lemma 3.6 we get
\[
e_{p,m}(v_j) = \int_X \left(-\sum_{k=1}^{j} a_k u_k \right)^p \omega_{v_j}^m \land \omega^{n-m}
\[
\leq \sum_{k=1}^{j} \max(a_k, a_k^p) \sum_{k_1 + \cdots + k_j = m} \left( \frac{m}{k_1 \cdots k_j} \right) a_1^{k_1} \cdots a_j^{k_j} \int_X (-u_k)^p \omega_{u_j}^m \land \omega^{n-m}
\[
\leq \sum_{k=1}^{j} \max(a_k, a_k^p) \sum_{k_1 + \cdots + k_j = m} \left( \frac{m}{k_1 \cdots k_j} \right) a_1^{k_1} \cdots a_j^{k_j} C \max_{k=1,\ldots,j} e_{p,m}(u_k)
\[
\leq C \sup_{k \in \mathbb{N}} e_{p,m}(u_k),
\]
which means that \( v_j \in \mathcal{E}_m^p(X, \omega) \), and \( \sup_{j \in \mathbb{N}} \epsilon_{p,m}(v_j) < \infty \). Thus, (4.8) holds.

4. A Sobolev type inequality

The aim of this section is to prove the Sobolev type inequality in Theorem 1.3.

We shall first need to prove the estimates of the sublevel sets for \((\omega, m)\)-subharmonic functions with bounded \((p, m)\)-energy in Lemma 4.4.

**Lemma 4.1.** If \( u \in \mathcal{E}_m(X, \omega) \), \( t \in [0, 1] \), \( s > 0 \) then

\[ t^m \cap_m \{u < -s - t\} \leq \int_{\{u < -s\}} H_m(u) \leq s^m \cap_m \{u < -s\}. \] (4.1)

Furthermore, if \( u \in \mathcal{E}_m^p(X, \omega) \), \( p > 0 \) and \( s > 1 \), then

\[ \cap_m \{u < -s\} \leq (s - 1)^{-p} e_{p,m}(u). \] (4.2)

**Proof.** Let \( v \in \mathcal{E}_m(X, \omega) \) be such that \(-1 \leq v \leq 0\). Then for \( t \in [0, 1] \) we get that \( tv \in \mathcal{E}_m(X, \omega) \). Note that we have

\[ \{u < -s - t\} \subset \{u < -s + tv\} \subset \{u < -s\}, \]

and therefore by Theorem 2.5 (4)

\[
\int_{\{u < -s-t\}} H_m(v) \leq \int_{\{u < -s+tv\}} H_m(v) \leq t^{-m} \int_{\{u < -s-t\}} H_m(s + tv)
\]

\[
\leq t^{-m} \int_{\{u < -s+tv\}} H_m(u) \leq t^{-m} \int_{\{u < -s\}} H_m(u).
\]

This proves the left inequality in (4.1).

To prove the right inequality in (4.1) we assume for a moment that \( u \) is continuous and let \( 1 \leq s < s_0 \). Then

\[
\int_{\{u < s\}} H_m(\max(u, -s_0)) = -\int_{\{u \geq s\}} H_m(\max(u, -s_0)) + \int_X H_m(\max(u, -s_0))
\]

\[
= -\int_{\{u \geq s\}} H_m(u) + \int_X H_m(u) = \int_{\{u < s\}} H_m(u).
\]

Note that \( \frac{1}{s_0} \max(u, -s_0) \in \mathcal{E}_m(X, \omega) \), and \(-1 \leq \frac{1}{s_0} \max(u, -s_0) \leq 0\), and therefore

\[
\cap_m \{u < -s\} \geq \int_{\{u < s\}} H_m \left( \frac{1}{s_0} \max(u, -s_0) \right)
\]

\[
\geq s_0^{-m} \int_{\{u < s\}} H_m(\max(u, -s_0)) = s_0^{-m} \int_{\{u < s\}} H_m(u).
\]

If \( s_0 \downarrow s \), then we get

\[
\cap_m \{u < -s\} \geq s^{-m} \int_{\{u < s\}} H_m(u).
\]

For the general situation take a smooth decreasing sequence \( u_j \downarrow u \) and observe that

\[
\int_{\{u < s\}} H_m(u) \leq \liminf_{j \to \infty} \int_{\{u_j < s\}} H_m(u_j)
\]

\[
\leq s^m \liminf_{j \to \infty} \cap_m \{u_j < -s\} \leq s^m \cap_m \{u < -s\}.
\]
To prove inequality (4.2) assume that \( u \in \mathcal{E}_m^p(X, \omega) \), and \( s > 1 \). We shall use (4.1) to obtain
\[
t^m \operatorname{cap}_m(\{u < -s - t\}) \leq \int_{\{u < -s\}} H_m(u) \leq s^{-p} \int_{\{u < -s\}} (-u)^p H_m(u) \leq s^{-p} e_{p,m}(u).
\]
For \( t = 1 \) we get the desired conclusion.

**Lemma 4.2.** If \( u \in \mathcal{E}_m^p(X, \omega) \), then \( u \in L^q(X) \) for \( 0 < q < \frac{\max(p,1)n}{n-m} \).

**Proof.** Let \( u \in \mathcal{E}_m^p(X, \omega) \). Assume first that \( p \geq 1 \). From Lemma 2.8 and Lemma 4.1 we have for \( \alpha < \frac{n}{n-m} \)
\[
\int_X (-u)\alpha \omega^n \leq 2\alpha + q \int_2^\infty t^{q-1} V(\{u < -t\}) dt
\leq 2\alpha + qC(\alpha) \int_2^\infty t^{q-1} (t - 1)^{-p} e_{p,m}(u) \alpha dt
= 2\alpha + qC(\alpha)e_{p,m}(u) \int_2^\infty t^{q-1} (t - 1)^{-p \alpha} dt.
\]
The right hand side is finite if, and only if, \( q < p \alpha < \frac{\alpha n}{n-m} \).

For \( p < 1 \) we use Proposition 4.1 to obtain, in a similar way as above, that
\[
\int_X (-u)\alpha \omega^n \leq 2\alpha + q \int_2^\infty t^{q-1} (t - 1)^{-\alpha} dt.
\]
The right hand side is finite if, and only if, \( q > \alpha < \frac{n}{n-m} \). \( \square \)

We shall need the following elementary fact.

**Proposition 4.3.** Let \( \alpha > 0 \) and \( F : [0, \infty) \to [0, \infty) \) a decreasing function such that
\[
\int_0^\infty t^\alpha F(t) dt < \infty.
\]
Then there exists a constant \( C > 0 \) such that for all \( t > 0 \) we have \( F(t) \leq Ct^{-\alpha-1} \).

**Proof.** Using integration by parts we get
\[
C = \int_0^\infty t^\alpha F(t) dt = \int_0^s t^\alpha F(t) dt + \int_s^\infty t^\alpha F(t) dt
= \frac{s^{\alpha+1} F(s)}{\alpha + 1} - \frac{1}{\alpha + 1} \int_0^s t^{\alpha+1} F'(t) dt + \int_s^\infty t^\alpha F(t) dt \geq \frac{s^{\alpha+1} F(s)}{\alpha + 1}.
\]
\( \square \)

**Remark.** From Lemma 4.2 and Proposition 4.3 it follows that for all \( u \in \mathcal{E}_m^p(X, \omega) \) there exists a constant \( C(u, q) \) depending only on \( u \) and \( q \) such that
\[
V(\{u < -t\}) \leq \frac{C(u, q)}{t^q}, \quad \text{for} \quad 0 < q < \frac{\max(p,1)n}{n-m}.
\]

In Theorem 4.4 we prove estimates of the sublevel sets of \((\omega, m)\)-subharmonic functions with bounded \((p, m)\)-energy. For the case \( p = 1 \), Theorem 4.4 gives sharper estimates than those proved in [10].

**Theorem 4.4.** Let \( n > 2 \), and let \( 1 \leq m \leq n \). Assume that \((X, \omega)\) is a connected and compact Kähler manifold of complex dimension \( n \), where \( \omega \) is a Kähler form on \( X \) such that \( \int_X \omega^n = 1 \). If \( u \in \mathcal{E}_m^p(X, \omega) \), then
(1) there exists a constant $C(u)$ depending only on $u$ such that for all $t > 1$
\[ \text{cap}_{m}(\{u < -t\}) \leq \frac{C(u)}{t^{p+1}}; \]

(2) there exists a constant $C(u, q)$ depending only on $u$ and $q$ such that for all $t > 1$, and $0 < q < \frac{(p+1)\alpha}{n-m}$,
\[ V(\{u < -t\}) \leq \frac{C(u, q)}{t^{q}}; \]

(3) for all $0 < q < \frac{(p+1)\alpha}{n-m}$, we have that $u \in L^{q}(X)$.

**Proof.** By Lemma 1.2 we know that if $u \in E_{m}(X, \omega)$, then $u \in L^{q}(X)$ for $0 < q < \frac{\max(p, 1)\alpha}{n-m}$. Fix $u \in E_{m}(X, \omega)$, $u \leq -1$, and $v \in E_{m}(X, \omega)$, $-1 \leq v \leq 0$, and let $t \geq 1$. Then $\frac{u}{t} \in E_{m}(X, \omega)$, and
\[ \{u < -2t\} \subset \left\{ \frac{u}{t} < v - 1 \right\} \subset \{u < -t\}. \]

By Theorem 2.3 (4) we obtain
\[ \int_{\{u < -2t\}} \omega_{v}^{m} \land \omega^{n-m} \leq \int_{\{\frac{u}{t} < v - 1\}} \omega_{v}^{m} \land \omega^{n-m} \leq \int_{\{\frac{u}{t} < v - 1\}} \omega_{v}^{m} \land \omega^{n-m} \]
\[ \leq \int_{\{u < -t\}} \omega_{v}^{m} \land \omega^{n-m} \leq \int_{\{u < -t\}} \omega_{v}^{m} \land \omega^{n-m} \leq \int_{\{u < -t\}} \omega^{n} + t^{-1} \sum_{j=1}^{n} \int_{\{u < -t\}} \omega_{v}^{m} \land \omega^{n-j}. \]

Now observe that by Lemma 3.2
\[ \int_{\{u < -t\}} \omega_{v}^{m} \land \omega^{n-m} \leq t^{-p} \int_{X} (-u)^{p} \omega_{v}^{m} \land \omega^{n-m} \leq \frac{2p}{\Delta} e_{p, m}(u), \]
and by (1.3)
\[ \int_{\{u < -t\}} \omega^{n} = V(\{u < -t\}) \leq \frac{C(u, q)}{t^{q}}, \text{ for } 0 < q < \frac{\max(p, 1)\alpha}{n-m}. \]

Therefore we get
\[ \text{cap}_{m}(\{u < -t\}) \leq \frac{C_{1}(u, q)}{t^{p+1}}, \text{ for } q < \frac{\max(p, 1)\alpha}{n-m}. \]  (4.4)

Let $p_{1} = \min(p + 1, q)$, where $q < \frac{\max(p, 1)\alpha}{n-m}$. Then by (4.4) we have
\[ \text{cap}_{m}(\{u < -t\}) \leq \frac{C_{2}(u, q)}{t^{p_{1}}}. \]

By the proof of Lemma 1.2 we get that $u \in L^{q_{1}}$, where $q_{1} = p_{1}\frac{n}{n-m}$, and then it follows from (1.3) that
\[ V(\{u < -t\}) \leq \frac{C_{3}(u, q_{1})}{t^{q_{1}}}. \]

Now we can once more repeat the argument above and obtain
\[ \text{cap}_{m}(\{u < -t\}) \leq \frac{C_{4}(u, q_{1})}{t^{p_{2}}} e_{p, m}(u), \]
where $p_{2} = \min(p + 1, q_{1})$. It would again imply that $u \in L^{q_{2}}(X)$ and
\[ V(\{u < -t\}) \leq \frac{C_{5}(u, q_{2})}{t^{q_{2}}}, \]
where \( q_2 < p_2 \frac{n}{n-m} \). We can repeat this argument \( l \)-times until \( p_l = p + 1 \) (which is possible since \( \frac{n}{n-m} > 1 \)). Finally, we get

\[
\text{cap}_m(\{u < -t\}) \leq \frac{\bar{C}(u)}{p+1},
\]

\( u \in L^q(X) \), and

\[
V(\{u < -t\}) \leq \frac{\bar{C}(u,q)}{t^q}, \quad \text{for } 0 < q < \frac{(p+1)n}{n-m}.
\]

Now we can prove a Sobolev type inequality for \((\omega, m)\)-subharmonic functions with bounded \((p, m)\)-energy.

**Theorem 4.5.** Let \( X \) be a connected and compact Kähler manifold of complex dimension \( n \), where \( \omega \) is a Kähler form on \( X \) such that \( \int_X \omega^n = 1 \). Also let \( 1 \leq m \leq n \) and \( p \geq 1 \). Then for any \( 1 < q < \frac{pm}{n-m} \), and any \( \epsilon > 0 \), there exists constant \( C(\epsilon) \) such that for any \( u \in \mathcal{E}^m_p(X, \omega) \), \( \sup_X u = -1 \), we have that

\[
\int_X (-u)^q \omega^n \leq C(\epsilon) e_{p,m}(u)^{\frac{(p-1)n}{n-p-n+m} + \epsilon}.
\]

**Proof.** Take \( u \in \mathcal{E}^m_p(X, \omega) \), \( \sup_X u = -1 \), and fix \( q < \frac{pm}{n-m} \). Also, let \( q < Q < p\alpha \), where \( 1 < \alpha < \frac{n}{n-m} \). Then we have by Lemma [14]

\[
\int_X (-u)^q \omega^n = q \int_0^{\infty} t^{q-1} V(\{u < -t\}) \, dt \leq q \left( \int_0^{\infty} t^{Q-1} V(\{u < -t\}) \, dt \right)^{\frac{q-1}{Q-1}} \left( \int_0^{\infty} V(\{u < -t\}) \, dt \right)^{\frac{Q-q}{Q-1}} \leq q \left( \frac{2^Q}{Q} + \int_2^{\infty} t^{Q-1} e_{p,m}(u)(t-1)^{-p\alpha} \, dt \right)^{\frac{q-1}{Q-1}} \left( \int_X (-u) \omega^n \right)^{\frac{Q-q}{Q}} \leq C e_{p,m}(u)^{\frac{(p-1)n}{n-p-n+m}} \left( \int_X (-u) \omega^n \right)^{\frac{Q-q}{Q}}.
\]

It follows from [15] that if \( u \) is \((\omega, m)\)-subharmonic function such that \( \sup_X u = -1 \), then there exists a constant \( C' \) that does not depending on \( u \) such that \( \sup_X \int_X (-u) \omega^n \leq C' \). Note that

\[
\inf_{\alpha,Q} \left\{ \left( \frac{\alpha q - 1}{Q - 1} \right) : q < Q < p\alpha \text{ and } 1 < \alpha < \frac{n}{n-m} \right\} = \frac{n(q-1)}{np-n+m}.
\]

Therefore, for any \( \epsilon > 0 \) there exists constant \( C(\epsilon) \) that does not depending on \( u \) such that

\[
\int_X (-u)^q \omega^n \leq C(\epsilon) e_{p,m}(u)^{\frac{(p-1)n}{n-p-n+m} + \epsilon}.
\]

\( \square \)

At the end of this section we can prove the following partial characterizaton of negative \((\omega, m)\)-subharmonic functions with bounded \((p, m)\)-energy.
Proposition 4.6. Let $X$ be a connected and compact Kähler manifold of complex dimension $n$, where $\omega$ is a Kähler form on $X$ such that $\int_X \omega^n = 1$. Also let $1 \leq m \leq n$ and $p > 0$. Then

$$\left\{ u \in \mathcal{SH}_m(X, \omega) : \int_0^\infty t^{m+p-1} \text{cap}_m(\{u < -t\}) dt < \infty \right\} \subset \mathcal{E}_m^p(X, \omega).$$

In particular, if for $u \in \mathcal{SH}_m^-(X, \omega)$ there exist constants $C(u) > 0$ and $\epsilon > 0$ such that

$$\text{cap}_m(\{u < -t\}) \leq \frac{C(u)}{t^{p+m+\epsilon}},$$

then $u \in \mathcal{E}_m^p(X, \omega)$.

Proof. Let $u \in \mathcal{SH}_m^-(X, \omega)$, and assume that

$$\int_0^\infty t^{m+p-1} \text{cap}_m(\{u < -t\}) dt < \infty.$$ 

Then without lost of generality we can assume that $u \leq -1$. Let us define $u_t = \max(u, -t)$, $t \geq 1$, then $v = \frac{u_t}{t} \in \mathcal{SH}_m(X, \omega)$, $-1 \leq v \leq 0$, and

$$\omega_u^m \wedge \omega^{n-m} \geq t^{-m} \omega_u^m \wedge \omega^{n-m}.$$ 

We then have by Proposition 4.6

$$(\omega_u^m \wedge \omega^{n-m})(\{u < -t\}) \leq t^m (\omega_u^m \wedge \omega^{n-m})(\{u < -t\}) \leq t^m \text{cap}_m(\{u < -t\}) \to 0, \text{ as } t \to \infty.$$ 

Thus, $u \in \mathcal{E}_m(X, \omega)$. Furthermore,

$$\begin{align*}
(\omega_u^m \wedge \omega^{n-m})(\{u \leq -t\}) &= \int_X \omega^n - (\omega_u^m \wedge \omega^{n-m})(\{u > -t\}) \\
&= \int_X \omega^n - (\omega_u^m \wedge \omega^{n-m})(\{u > -t\}) = (\omega_u^m \wedge \omega^{n-m})(\{u \leq -t\}).
\end{align*}$$

Finally,

$$\begin{align*}
\int_X (-u)^p H_m(u) &= p \int_1^\infty t^{p-1} (\omega_u^m \wedge \omega^{n-m})(\{u < -t\}) dt \\
&\leq p \int_1^\infty t^{p-1} (\omega_u^m \wedge \omega^{n-m})(\{u \leq -t\}) dt \leq p \int_1^\infty t^{m+p-1} \text{cap}_m(\{u \leq -t\}) dt < \infty.
\end{align*}$$

□

5. THE COMPLEX HESSIAN EQUATIONS

In this section we consider complex Hessian equations for $\mathcal{E}_m^p(X, \omega)$. We need the following generalization of Theorem 4.3.

Theorem 5.1. Let $n \geq 2$, $p > 0$, and let $1 \leq m \leq n$. Assume that $(X, \omega)$ is a connected and compact Kähler manifold of complex dimension $n$, where $\omega$ is a Kähler form on $X$ such that $\int_X \omega^n = 1$. Furthermore, assume that $\mu$ is a Borel measure defined on $X$. Fix a constant $\beta$ such that $1 > \beta > \max\left(\frac{p}{p+m-n}, \frac{p}{p+1}\right)$, for $p > 1$, and $\beta = \frac{p}{p+1}$ for $p \leq 1$. The following conditions are then equivalent:

1. $\mathcal{E}_m^p(X, \omega) \subset L^q(X, \mu)$;
(2) there exists a constant $C > 0$ such that for all $u \in \mathcal{E}_m(X, \omega) \cap L^\infty(X)$ with $\sup_X u = -1$ it holds
\[ \int_X (-u)^q \, d\mu \leq C e_{p,m}(u)^{\frac{q}{p}}. \]

(3) there exists a constant $C > 0$ such that for all $u \in \mathcal{E}_m^p(X, \omega)$ with $\sup_X u = -1$ it holds
\[ \int_X (-u)^q \, d\mu \leq C e_{p,m}(u)^{\frac{q}{p}}. \]

**Proof.** The implication $(2) \Rightarrow (1)$ is obvious. The equivalence $(2) \Leftrightarrow (3)$ follows by approximation. We shall prove $(1) \Rightarrow (2)$.

Assume first that $p > 1$. To prove this implication assume that condition (2) is not true, i.e., there exists a sequence $u_j \in \mathcal{E}_m(X, \omega) \cap L^\infty(X)$, $\sup_X u_j = -1$, such that
\[ \int_X (-u_j)^q \, d\mu \geq 4^j q e_{p,m}(u_j)^{\frac{q}{p}}. \] (5.1)

**Case 1.** If the sequence $e_{p,m}(u_j)$ is bounded (or it contains a bounded subsequence), then let us define
\[ u = \sum_{j=1}^{\infty} \frac{1}{2^j} u_j. \]

Then $u$ belongs to $\mathcal{SH}_m(X, \omega)$, and by Lemma it follows
\[ \int_X (-u)^p \, H_m(u) \leq C(p, m) \sup_{j \in \mathbb{N}} e_{p,m}(u_j) < \infty. \]

Hence, $u \in \mathcal{E}_m^p(X, \omega)$. On the other hand by (5.1)
\[ \int_X (-u)^q \, d\mu \geq \frac{1}{2^j q} \int_X (-u_j)^q \, d\mu \geq \frac{1}{2^j q} 4^j q e_{p,m}(u_j)^{\frac{q}{p}} \geq 2^j q \to \infty, \quad \text{as } j \to \infty. \]

Thus, $u \notin L^q(X, \mu)$.

**Case 2.** Now assume that $e_{p,m}(u_j) \to \infty$. Let us define $v_j = t_j u_j$, where
\[ t_j = e_{p,m}(u_j)^{\frac{q}{p}}. \] (5.2)

Then we have by Theorem and Lemma
\[ e_{p,m}(v_j) = t_j^p \int_X (-u_j)^p \omega^n + t_j^p \sum_{k=1}^{m} \binom{m}{k} t_j^k (1 - t_j)^{m-k} \int_X (-u_j)^p \omega^k \wedge \omega^{n-k} \]
\[ \leq t_j^p \int_X (-u_j)^p \omega^n + C^p m^p e_{p,m}(u_j) \]
\[ \leq C' t_j^p e_{p,m}(u_j)^{\beta} + C^p m^p e_{p,m}(u_j) < +\infty. \]

Therefore, we can repeat the argument from the first case to show that function
\[ v = \sum_{j=1}^{\infty} \frac{1}{2^j} v_j \]
belongs to $\mathcal{SH}_m^p(X,\omega)$, but $v \notin L^q(X,\mu)$, since
\[
\int_X (-v)^q \, d\mu \geq \frac{1}{2q} t_j^q \int_X (-u_j)^q \, d\mu
\geq \frac{1}{2q} t_j^q e_{p,m}(u_j) \frac{u_j}{\beta} = 2^q \to \infty, \text{ as } j \to \infty.
\]

Next, assume that $p \leq 1$ and $\beta = \frac{p}{p+1}$. By [15] it follows that if $u$ is $(\omega, m)$-subharmonic function such that $\sup_X u = -1$, then there exists a constant $C'$ which does not depending on $u$ such that
\[
\int_X (-u)^p \omega^n \leq \left( \int_X (-u)^n \omega^n \right)^p \leq (C')^p,
\]
and then we repeat the above proof for the case when $p > 1$.

By making the best use of Theorem 5.1 we prove the following theorem. Theorem 5.2 was in the case $p = 1$ proved in [16].

**Theorem 5.2.** Let $n \geq 2$, $p > 0$, and let $1 \leq m \leq n$. Assume that $(X,\omega)$ is a connected and compact Kähler manifold of complex dimension $n$, where $\omega$ is a Kähler form on $X$ such that $\int_X \omega^n = 1$. Furthermore, assume that $\mu$ is a Borel probability measure defined on $X$. The following conditions are then equivalent:

1. $E_m^p(X,\omega) \subset L^p(X,\mu)$;
2. there exists unique $(\omega, m)$-subharmonic function $u$ in $E_m^p(X,\omega)$ such that $\sup_X u = -1$ and $H_m(u) = \mu$.

**Proof.** Implication (2)⇒(1) follows from Lemma 3.5. Next, we shall prove implication (1)⇒(2). To do so let us define the following collection of Borel probability measures
\[
\mathcal{M} = \{ \mu : \mu(X) = 1, \mu(K) \leq \text{cap}_m(K), K \subset X \}.
\]

It was proved in [16] that $\mathcal{M}$ is convex and compact. Furthermore, for any Borel probability measure $\mu$ we have the following decomposition
\[
\mu = f \nu + \sigma, \text{ where } \nu \in \mathcal{M}, \sigma \perp \mathcal{M}, f \in L^1(\nu).
\]
If we assume that $\mu$ vanishes on $m$-polar sets, then $\sigma = 0$. By assumption $\mu$ is a Borel probability measure defined on $X$ such that $E_m^p(X,\omega) \subset L^p(X,\mu)$. Thus, $\mu$ vanishes on $m$-polar sets, so there exist $\nu \in \mathcal{M}$ and $f \in L^1(\nu)$ such that $\mu = f \nu$. Set
\[
\mu_j = c_j \min(f, j) \nu,
\]
where $c_j > 0$ is such that $\mu_j(X) = 1$. It follows from [16] that there exists $u_j \in E_m(X,\omega)$ such that $H_m(u_j) = \mu_j$ and $\sup_X u_j = -1$. Without loss of generality we can assume that $u_j \to u$ in $L^1(X)$. Next, we define
\[
u = \max(u_j, -k) \in L^\infty(X).\]
This construction implies that $u_{j,k} \in E_m^p(X,\omega)$, and $u_{j,k} \searrow u_j$, as $k \to \infty$. Hence, $e_{p,m}(u_{j,k}) \to e_{p,m}(u_j)$, $k \to \infty$. By Theorem 5.1 it follows for some $\beta < 1$ that
\[
\int_X (-u_{j,k})^p H_m(u_j) \leq \int_X (-u_{j,k})^p \, d\mu_j \leq C \left( \int_X (-u_{j,k})^p H_m(u_{j,k}) \right)^\beta,
\]
and since we have \( \int_X (-u_{j,k})^p H_m(u_j) \to e_{p,m}(u_j) \) and \( e_{p,m}(u_{j,k}) \to e_{p,m}(u_j) \), as \( k \to \infty \), we get
\[
\sup_k e_{p,m}(u_{j,k}) < \infty.
\]

Thus, \( u_j \in \mathcal{E}_m^p(X,\omega) \). Theorem 5.1 yields, again for some \( \beta < 1 \), that
\[
\int_X (-u_j)^p H_m(u_j) \leq c_j \int_X (-u_j)^p \, d\mu \leq c_j C \left( \int_X (-u_j)^p H_m(u_j) \right)^\beta.
\]

Thus, \( \sup_j e_{p,m}(u_j) < \infty \). Let us define \( v_j = (\sup_{k \geq j} u_k)^\ast \). Here \((\cdot)^\ast\) denotes the upper semicontinuous regularization. Then \( v_j \) is a decreasing sequence of function from \( \mathcal{E}_m^p(X,\omega) \), \( v_j \searrow u \), \( j \to \infty \). Furthermore, since \( \sup_j e_{p,m}(v_j) < \infty \), then we can conclude that \( u \in \mathcal{E}_m^p(X,\omega) \). Then by [16] we conclude that \( H_m(v_j) \geq \min(f,j)\nu \), after passing to the limit with \( j \) we get \( H_m(u) \geq \mu \), but since both measure \( H_m(u) \) and \( \mu \) have the same total mass we conclude that \( H_m(u) = \mu \). \( \square \)

At the end of this section we shall prove the following proposition.

**Proposition 5.3.** Assume the same conditions as in Theorem 5.2.

a) If there exist constants \( \alpha > \frac{1}{p+1} \) and \( C > 0 \) such that for all Borel sets \( E \) it holds
\[
\mu(E) \leq C \text{cap}_m(E)^\alpha,
\]
then \( \mathcal{E}_m^p(X,\omega) \subset L^p(X,\mu) \).

b) If \( \mathcal{E}_m^p(X,\omega) \subset L^p(X,\mu) \), then for fixed \( \beta > \max \left( \frac{p-n}{m-n+m}, \frac{p}{p+1} \right) \), if \( p > 1 \), and \( \beta = \frac{1}{p+1} \) if \( p \leq 1 \), there exists a constant \( C > 0 \) such that for all Borel sets \( E \) it holds
\[
\mu(E) \leq C \text{cap}_m(E)^\beta.
\]

**Proof.** a) Let \( u \in \mathcal{E}_m^p(X,\omega) \) with \( \sup_X u = -1 \). From Theorem 4.4 it follows that
\[
\int_X (-u)^p \, d\mu = p \int_1^\infty t^{p-1} \mu(\{u < -t\}) \, dt \\
\leq pC \int_1^\infty t^{p-1} \text{cap}_m(\{u < -t\}) \, dt = pC \int_1^\infty t^{p-1} \left( \frac{C'(u)}{tp+1} \right)^\alpha \, dt \\
= pC(C'(u))^\alpha \int_1^\infty t^{p-1-\alpha p-\alpha} \, dt < \infty.
\]

b) Assume that \( \mathcal{E}_m^p(X,\omega) \subset L^p(X,\mu) \). From Theorem 5.1 it follows that there exists a constant \( C > 0 \) such that for all \( v \in \mathcal{E}_m^p(X,\omega) \cap L^\infty(X) \), with \( \sup_X v = -1 \), it holds
\[
\int_X (-v)^p \, d\mu \leq C e_{p,m}(v)^\beta. \tag{5.3}
\]

Let \( E \) be a Borel set, and let \( h_{m,E} \) be the \( m \)-extremal function for the set \( E \), i.e.
\[
h_{m,E} = \left( \sup \{ u \in S\mathcal{H}_m(X,\omega) : u \leq -1 \text{ on } E, u \leq 0 \text{ on } X \} \right)^\ast,
\]
where \((\cdot)^\ast\) denotes the upper semicontinuous regularization (see e.g. [16] Section 4) for further information). Using \( h_{m,E} \) in (5.3) we arrive at
\[
\mu(E) \leq \int_X (-h_{m,E})^p \, d\mu \leq C e_{p,m}(h_{m,E})^\beta = C \text{cap}_m(E)^\beta.
\]

\( \square \)
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Departement of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden
E-mail address: Per.Ahag@math.umu.se

Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: Rafal.Czyz@im.uj.edu.pl