Symmetry properties of finite sums involving generalized Fibonacci numbers

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Abstract

We extend a result of I. J. Good and prove more symmetry properties of sums involving generalized Fibonacci numbers.

1 Introduction

The generalized Fibonacci numbers $G_i$, $i \geq 0$, with which we are mainly concerned in this paper, are defined through the second order recurrence relation $G_{i+1} = G_i + G_{i-1}$, where the seeds $G_0$ and $G_1$ need to be specified. As particular cases, when $G_0 = 0$ and $G_1 = 1$, we have the Fibonacci numbers, denoted $F_i$, while when $G_0 = 2$ and $G_1 = 1$, we have the Lucas numbers, $L_i$.

I. J. Good [1] proved the symmetry property:

$$F_q \sum_{k=1}^{n} \frac{(-1)^k}{G_k G_{k+q}} = F_n \sum_{k=1}^{q} \frac{(-1)^k}{G_k G_{k+n}},$$

(1.1)

where $q$ and $n$ are nonnegative integers, and all the numbers $G_1, G_2, \ldots, G_{n+q}$ are nonzero.

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The identity (1.1) is a particular case (corresponding to setting $p = 1$) of the following result, to be proved in this present paper:

$$F_{pq} \sum_{k=1}^{n} \frac{(-1)^{pk}}{G_{pk}G_{pk+pq}} = F_{pn} \sum_{k=1}^{q} \frac{(-1)^{pk}}{G_{pk}G_{pk+pm}},$$

(1.2)

where $q$, $p$ and $n$ are nonnegative integers, and all the numbers $G_p$, $G_{2p}$, $\ldots$, $G_{pn+pq}$ are nonzero.

In the limit as $n$ approaches infinity, and specializing to Fibonacci numbers, the identity (1.2) gives

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{(-1)^{pk}}{F_{pk}F_{pk+pq}} = \frac{1}{F_{pq}} \sum_{k=1}^{q} \left\{ \frac{(-1)^{pk}}{F_{pk}} \lim_{n \to \infty} \left( \frac{F_{pn}}{F_{pk+pq}} \right) \right\}$$

(1.3)

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

The identity (1.3) generalizes Bruckman and Good’s result (identity (19) of [2], which corresponds to setting $q = 1$ in (1.3)).

In sections 3.1 – 3.3 we will prove identity (1.2) and discover more symmetry properties of sums involving generalized Fibonacci numbers. In section 3.4 we shall extend the discussion to Horadam sequences $W_i$ and $U_i$ by proving

$$U_{pq} \sum_{k=1}^{n} \frac{Q^{pk}}{W_{pk}W_{pk+pq}} = U_{pn} \sum_{k=1}^{q} \frac{Q^{pk}}{W_{pk}W_{pk+pm}},$$

(1.4)

and

$$U_{2pq} \sum_{k=1}^{2n} \frac{(-Q^p)^k}{W_{pk}W_{pk+2pq}} = U_{2pn} \sum_{k=1}^{2q} \frac{(-Q^p)^k}{W_{pk}W_{pk+2pm}},$$

(1.5)

for integers $p$, $q$, $Q$ and $n$, thereby extending André-Jeannin’s result (Theorem 1 of [5]) and further generalizing the identity (1.2).
2 Required identities

2.1 Telescoping summation identities

The following telescoping summation identities are special cases of the more general identities proved in [3].

Lemma 2.1. If \( f(k) \) is a real sequence and \( u, v \) and \( w \) are positive integers, then

\[
\sum_{k=1}^{w} [f(uk + uv) - f(uk)] = \sum_{k=1}^{v} [f(uk + uw) - f(uk)].
\]

Lemma 2.2. If \( f(k) \) is a real sequence and \( u, v \) and \( w \) are positive integers such that \( v \) is even and \( w \) is even, then

\[
\sum_{k=1}^{w} (\pm 1)^{k-1} (f(uk + uv) - f(uk)) = \sum_{k=1}^{v} (\pm 1)^{k-1} (f(uk + uw) - f(uk)).
\]

Lemma 2.3. If \( f(k) \) is a real sequence and \( u, v \) and \( w \) are positive integers such that \( vw \) is odd, then

\[
\sum_{k=1}^{w} (-1)^{k-1} (f(uk + uv) + f(uk)) = \sum_{k=1}^{v} (-1)^{k-1} (f(uk + uw) + f(uk)).
\]

2.2 Product of a Fibonacci number and a generalized Fibonacci number

Lemma 2.4 (Howard [5], Corollary 3.5). For integers \( a, b, c \),

\[
F_a G_{2b+a+c} = \begin{cases} 
F_{a+b} G_{a+b+c} - F_b G_{b+c} & \text{if } a \text{ is even}, \\
F_{a+b} G_{a+b+c} + F_b G_{b+c} & \text{if } a \text{ is odd}.
\end{cases}
\]

2.3 Product of a Lucas number and a generalized Fibonacci number

Lemma 2.5 (Vajda [4], Formula 10a). For integers \( a, b \),

\[
L_a G_b = \begin{cases} 
G_{b+a} + G_{b-a} & \text{if } a \text{ is even}, \\
G_{b+a} - G_{b-a} & \text{if } a \text{ is odd}.
\end{cases}
\]
2.4 Difference of products of a Fibonacci number and a generalized Fibonacci number

Lemma 2.6 (Vajda [4], Formula 21). For integers $a$, $b$,

$$F_b G_a - F_a G_b = (-1)^a G_0 F_{b-a}.$$

3 Main Results: Symmetry properties

3.1 Sums of products of reciprocals

Theorem 3.1. If $n$ and $q$ are nonnegative integers and $p$ is a nonzero integer, then

$$F_{pq} \sum_{k=1}^{n} \frac{(-1)^{pk}}{G_{pk}G_{pk+pq}} = F_{pn} \sum_{k=1}^{q} \frac{(-1)^{pk}}{G_{pk}G_{pk+pn}}.$$

Proof. Dividing through the identity in Lemma 2.6 by $G_aG_b$ and setting $b = pk + pq$ and $a = pk$, we have:

$$\frac{F_{pk+pq}}{G_{pk+pq}} - \frac{F_{pk}}{G_{pk}} = (-1)^{pk} \frac{G_0 F_{pq}}{G_{pk}G_{pk+pq}}. \quad (3.1)$$

Similarly,

$$\frac{F_{pk+pn}}{G_{pk+pn}} - \frac{F_{pk}}{G_{pk}} = (-1)^{pk} \frac{G_0 F_{pn}}{G_{pk}G_{pk+pn}}. \quad (3.2)$$

We now use the sequence $f(k) = F_k/G_k$ in Lemma 2.1 with $u = p$, $v = q$ and $w = n$, while taking into consideration identities (3.1) and (3.2).

Theorem 3.2. If $n$ and $q$ are nonnegative even integers and $p$ is a nonzero integer, then

$$F_{pq} \sum_{k=1}^{n} \frac{(\pm 1)^k}{G_{pk}G_{pk+pq}} = F_{pn} \sum_{k=1}^{q} \frac{(\pm 1)^k}{G_{pk}G_{pk+pn}}.$$

Proof. We use the sequence $f(k) = F_k/G_k$ in Lemma 2.2 with $u = p$, $v = q$ and $w = n$. \qed
3.2 First-power sums

**Theorem 3.3.** If \( p, q, n \) and \( t \) are integers such that \( pqn \) is odd, then

\[
L_{pq} \sum_{k=1}^{2n} (\pm 1)^{k-1} G_{pk+pq+t} = L_{pn} \sum_{k=1}^{2q} (\pm 1)^{k-1} G_{pk+pn+t}, \quad (3.3)
\]

\[
L_{pq} \sum_{k=1}^{n} G_{2pk+pq+t} = L_{pn} \sum_{k=1}^{q} G_{2pk+pn+t}. \quad (3.4)
\]

**Proof.** Consider the generalized Fibonacci sequence \( f(k) = G_{k+t} \). If we choose \( u = p, v = 2q \) and \( w = 2n \), then Lemma 2.2 gives

\[
\sum_{k=1}^{2n} (\pm 1)^{k-1} (G_{pk+2pq+t} - G_{pk+t}) = \sum_{k=1}^{2q} (\pm 1)^{k-1} (G_{pk+2pn+t} - G_{pk+t}). \quad (3.5)
\]

But from the second identity of Lemma 2.5 we have

\[
G_{pk+2pq+t} - G_{pk+t} = L_{pq} G_{pk+pq+t}, \quad pq \text{ odd,} \quad (3.6)
\]

and

\[
G_{pk+2pn+t} - G_{pk+t} = L_{pn} G_{pk+pn+t}, \quad pn \text{ odd.} \quad (3.7)
\]

Using (3.6) and (3.7) in (3.5), identity (3.3) is proved.

The proof of identity (3.4) is similar, we use the sequence \( f(k) = G_{2k+t} \) in Lemma 2.1 with \( u = 2p, v = q \) and \( w = n \).

\[\Box\]

**Theorem 3.4.** If \( p, q, n \) and \( t \) are integers such that \( pqn \) is odd or \( q \) and \( n \) are even, then

\[
F_{pq} \sum_{k=1}^{n} (-1)^{k-1} G_{2pk+pq+t} = F_{pn} \sum_{k=1}^{q} (-1)^{k-1} G_{2pk+pn+t}. \quad (3.8)
\]

**Proof.** Consider the sequence \( f(k) = F_{k} G_{k+t} \). If we choose \( u = p, v = q \) and \( w = n \), then Lemma 2.3 gives

\[
\sum_{k=1}^{n} (-1)^{k-1} (F_{pk+pq} G_{pk+pq+t} + F_{pk} G_{pk+t})
\]

\[
= \sum_{k=1}^{q} (-1)^{k-1} (F_{pk+pn} G_{pk+pn+t} + F_{pk} G_{pk+t}). \quad (3.8)
\]

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From the second identity of Lemma 2.4 we have

\[ F_{pk+pq}G_{pk+pq} + F_{pk}G_{pk+t} = F_{pq}G_{2pk+pq+t}, \quad pq \text{ odd}, \quad (3.9) \]

and

\[ F_{pk+pn}G_{pk+pn} + F_{pk}G_{pk+t} = F_{pn}G_{2pk+pn+t}, \quad pn \text{ odd}. \quad (3.10) \]

The theorem then follows from using (3.9) and (3.10) in (3.8). If \( q \) and \( n \) are even then we use \( f(k) = F_kG_{k+t} \) with \( u = p, v = q \) and \( w = n \) in Lemma 2.2 together with the first identity of Lemma 2.4.

**Theorem 3.5.** If \( p, q, n \) and \( t \) are integers such that \( p \) is even or \( q \) and \( n \) are even, then

\[ F_{pq} \sum_{k=1}^{n} G_{2pk+pq} + t = F_{pn} \sum_{k=1}^{q} G_{2pk+pn} + t. \]

**Proof.** Consider the sequence \( f(k) = F_kG_{k+t} \). Lemma 2.4 with \( u = p, v = q \) and \( w = n \) gives

\[ \sum_{k=1}^{n} (F_{pk+pq}G_{pk+pq} + t - F_{pk}G_{pk+t}) = \sum_{k=1}^{q} (F_{pk+pn}G_{pk+pn} + t - F_{pk}G_{pk+t}). \quad (3.11) \]

From the first identity of Lemma 2.4 we have

\[ F_{pk+pq}G_{pk+pq} + t - F_{pk}G_{pk+t} = F_{pq}G_{2pk+pq+t}, \quad pq \text{ even}, \quad (3.12) \]

and

\[ F_{pk+pn}G_{pk+pn} + t - F_{pk}G_{pk+t} = F_{pn}G_{2pk+pn+t}, \quad pn \text{ even}. \quad (3.13) \]

Using (3.12) and (3.13) in (3.11), Theorem 3.5 is proved.

**Theorem 3.6.** If \( p, q, n \) and \( t \) are integers such that \( p \) is even, then

\[ F_{pq} \sum_{k=1}^{2n} (\pm 1)^{k-1}G_{pk+pq} + t = F_{pn} \sum_{k=1}^{2q} (\pm 1)^{k-1}G_{pk+pn} + t. \]
Proof. Consider the sequence \( f(k) = F_k G_{k+t} \). Lemma 2.2 with \( u = p \), \( v = 2q \) and \( w = 2n \) gives

\[
\sum_{k=1}^{2n} (\pm1)^k (F_{pk+2pq}G_{pk+2pq+t} - F_{pk}G_{pk+t})
= \sum_{k=1}^{2q} (\pm1)^k (F_{pk+2pn}G_{pk+2pn+t} - F_{pk}G_{pk+t}) .
\] (3.14)

From identities (3.12) and (3.13) we have

\[
F_{pk+2pq}G_{pk+2pq+t} - F_{pk}G_{pk+t} = F_{2pq}G_{2pk+2pq+t} ,
\] (3.15)

and

\[
F_{pk+2pn}G_{pk+2pn+t} - F_{pk}G_{pk+t} = F_{2pn}G_{2pk+2pn+t} .
\] (3.16)

Using (3.15) and (3.16) in (3.14), Theorem 3.6 is proved. \( \square \)

**Theorem 3.7.** If \( p, q, n \) and \( t \) are integers such that \( p \) is even and \( nq \) is odd, then

\[
L_{pq} \sum_{k=1}^{n} (\pm1)^k G_{2pk+pq+t} = L_{pn} \sum_{k=1}^{q} (\pm1)^k G_{2pk+pn+t} ,
\]

Proof. Consider the sequence \( f(k) = G_{2k+t} \). If we choose \( u = 2p \), \( v = q \) and \( w = n \), then Lemma 2.3 gives

\[
\sum_{k=1}^{n} (\pm1)^k (G_{2pk+2pq+t} + G_{2pk+t})
= \sum_{k=1}^{q} (\pm1)^k (G_{2pk+2pn+t} + G_{2pk+t}) , \quad nq \text{ odd} .
\] (3.17)

From the first identity in Lemma 2.5, we have

\[
G_{2pk+2pq+t} + G_{2pk+t} = L_{pq}G_{2pk+pq+t} , \quad pq \text{ even} ,
\] (3.18)

and

\[
G_{2pk+2pn+t} + G_{2pk+t} = L_{pn}G_{2pk+pn+t} , \quad pn \text{ even} .
\] (3.19)

Using (3.18) and (3.19) in (3.17), Theorem 3.7 is proved. \( \square \)
3.3 More sums involving products of reciprocals

**Theorem 3.8.** If \( p, q, n \) and \( t \) are positive integers such that \( pnq \) is odd, then

\[
L_{pq} \sum_{k=1}^{2n} \frac{(\pm 1)^{k-1}G_{pk+pq+t}}{G_{pk+t}G_{pk+2pq+t}} = L_{pn} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}G_{pk+pn+t}}{G_{pk+t}G_{pk+2pn+t}}, \tag{3.20}
\]

\[
L_{pq} \sum_{k=1}^{n} \frac{G_{2pk+pq+t}}{G_{2pk+t}G_{2pk+2pq+t}} = L_{pn} \sum_{k=1}^{q} \frac{G_{2pk+pn+t}}{G_{2pk+t}G_{2pk+2pn+t}}. \tag{3.21}
\]

**Proof.** Use of \( f(k) = 1/G_{k+t} \) in Lemma 2.2 with \( u = p, v = 2q \) and \( w = 2n \), noting the identities (3.6) and (3.7) proves identity (3.20). To prove identity (3.21), we use \( f(k) = 1/G_{2k+t} \) in Lemma 2.1 with \( u = p, v = q \) and \( w = n \), together with the second identity in Lemma 2.5. \( \square \)

**Theorem 3.9.** If \( p, q, n \) and \( t \) are positive integers such that \( p \) is even and \( nq \) is odd, then

\[
L_{pq} \sum_{k=1}^{n} \frac{(-1)^{k-1}G_{2pk+pq+t}}{G_{2pk+t}G_{2pk+2pq+t}} = L_{pn} \sum_{k=1}^{q} \frac{(-1)^{k-1}G_{2pk+pn+t}}{G_{2pk+t}G_{2pk+2pn+t}}.
\]

**Proof.** Use \( f(k) = 1/G_{2k+t} \) in Lemma 2.3 with \( u = p, v = q \) and \( w = n \), employing the identities (3.18) and (3.19). \( \square \)

**Theorem 3.10.** If \( p, q, n \) and \( t \) are positive integers such that \( p \) is even or \( n \) and \( q \) are even, then

\[
F_{pq} \sum_{k=1}^{n} \frac{G_{2pk+pq+t}}{F_{pk}G_{pk+t}F_{pk+pq}G_{pk+pq+t}} = F_{pn} \sum_{k=1}^{q} \frac{G_{2pk+pn+t}}{F_{pk}G_{pk+t}F_{pk+pn}G_{pk+pn+t}}.
\]

**Proof.** Use \( f(k) = 1/(F_{k}G_{k+t}) \) in Lemma 2.1 with \( u = p, v = q \) and \( w = n \), while taking cognisance of the following identities which follow from identities (3.12) and (3.13):

\[
\frac{1}{F_{pk}G_{pk+t}} - \frac{1}{F_{pk+pq}G_{pk+pq+t}} = \frac{F_{pq}G_{2pk+pq+t}}{F_{pk}G_{pk+t}F_{pk+pq}G_{pk+pq+t}}, \quad pq \text{ even,}
\]

\[
(3.22)
\]
and
\[ \frac{1}{F_{pk}G_{pk+t}} - \frac{1}{F_{pk+pn}G_{pk+pn+t}} = \frac{F_{pn}G_{2pk+pn+t}}{F_{pk}G_{pk+t}F_{pk+pn}G_{pk+pn+t}}, \quad pn \text{ even}. \] (3.23)

**Theorem 3.11.** If \( p, q, n \) and \( t \) are positive integers such that \( p \) is odd or \( n \) and \( q \) are even, then
\[ F_{pq} \sum_{k=1}^{n} \frac{(-1)^{k-1}G_{2pk+pq+t}}{F_{pk}G_{pk+t}F_{pk+pq}G_{pk+pq+t}} = F_{pn} \sum_{k=1}^{q} \frac{(-1)^{k-1}G_{2pk+pn+t}}{F_{pk}G_{pk+t}F_{pk+pn}G_{pk+pn+t}}. \]

### 3.4 Horadam sequence

Some of the above results can be extended to the Horadam sequence [7], \( \{W_i\} = \{W_i(a, b; P, Q)\} \) defined by
\[ W_0 = a, W_1 = b, W_i = PW_{i-1} - QW_{i-2}, (i > 2), \] (3.24)
where \( a, b, P, \) and \( Q \) are integers, with \( PQ \neq 0 \) and \( \Delta = P^2 - 4Q > 0 \).

We define the sequence \( \{U_i\} \) by \( U_i = W_i(0, 1; P, Q) \) and note also that our sequence \( \{G_i\} \) is given by \( G_i = W_i(G_0, G_1; 1, -1) \). It is readily established that [7] [6]:
\[ W_i = \frac{a\alpha^i - b\beta^i}{\alpha - \beta}, \] (3.25)
where \( \alpha = (P + \sqrt{\Delta})/2, \beta = (P - \sqrt{\Delta})/2, A = b - \beta a \) and \( B = b - \alpha a \).

**Theorem 3.12.** If \( n \) and \( q \) are nonnegative integers and \( p \) is a nonzero integer, then
\[ U_{pq} \sum_{k=1}^{n} \frac{Q^{pk}}{W_{pk}W_{pk+pq}} = U_{pn} \sum_{k=1}^{q} \frac{Q^{pk}}{W_{pk}W_{pk+pn}}. \]

Note that when \( p = 1 \), Theorem 3.12 reduces to Theorem 1 of [6].

**Proof.** Since \( n \) and \( k \) in identity (4.1) of [6] are arbitrary nonnegative integers, we substitute \( pk \) for \( n \) and \( pq \) for \( k \) in the identity, obtaining
\[ \frac{\beta^{pk}}{W_{pk}} - \frac{\beta^{pk+pq}}{W_{pk+pq}} = \frac{AQ^{pk}U_{pq}}{W_{pk}W_{pk+pq}}. \] (3.26)
The theorem now follows by choosing \( f(k) = \beta^k/W_k \) in Lemma 2.1 with \( w = n, u = p \) and \( v = q \) while making use of (3.26).

**Theorem 3.13.** If \( n \) and \( q \) are nonnegative even integers and \( p \) is a nonzero integer, then

\[
U_{pq} \sum_{k=1}^{n} \frac{(\pm Q)^k}{W_{pk}W_{pk+pq}} = U_{pn} \sum_{k=1}^{q} \frac{(\pm Q)^k}{W_{pk}W_{pk+pn}}.
\]

**Proof.** The theorem follows by choosing \( f(k) = \beta^k/W_k \) in Lemma 2.2 with \( w = n, u = p \) and \( v = q \), while making use of (3.26).

**References**

[1] I. J. GOOD (1994), A symmetry property of alternating sums of products of reciprocals, *The Fibonacci Quarterly* 32 (3):284–287.

[2] P. S. BRUCKMAN and I. J. GOOD (1976), A generalization of a series of de Morgan, with applications of Fibonacci type, *The Fibonacci Quarterly* 14 (3):193–196.

[3] K. ADEGOKE (2017), Generalizations for reciprocal Fibonacci-Lucas sums of Brousseau, arXiv:1703.06075 https://arxiv.org/abs/1703.06075.

[4] S. VAJDA (2008), Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover Press

[5] F. T. HOWARD (2003), The sum of the squares of two generalized Fibonacci numbers, *The Fibonacci Quarterly* 41 (1):80–84.

[6] R. ANDRÉ-JEANNIN (1997), Summation of reciprocals in certain second-order recurring sequences, *The Fibonacci Quarterly* 35 (1):68–74.

[7] A. F. HORADAM (1965), Basic properties of a certain generalized sequence of numbers, *The Fibonacci Quarterly* 3 (3): 161–176.