Conformal Anomalies and Gravitational Waves

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Work based on:
K. Meissner and H.N.: arXiv:1607.07312
H. Godazgar, K. Meissner and H.N.: in progress
Executive Summary

Is the cancellation of conformal anomalies required

- **Quantum mechanically:** to ensure quantum consistency of perturbative quantum gravity?

... in analogy with cancellation of gauge anomalies for Standard Model (where they are required to maintain renormalizability), and/or

- **already at classical level:** corrections from induced anomalous non-local action to Einstein Field Equations may potentially overwhelm smallness of Planck scale $\ell_{PL} \Rightarrow$ huge corrections to any solution?

If so, cancellation requirement could lead to very strong restrictions on admissible theories!
Conformal Symmetry

Conformal symmetry comes in two versions:

1. Global conformal symmetry = extension of Poincaré group by dilatations $D$ and conformal boosts $K^\mu$

2. Local dilatations = Weyl transformations

$$g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)}g_{\mu\nu}(x)$$

Important consequence: flat space limit of Weyl and diffeomorphism invariant theories exhibits full (global) conformal symmetry (via conformal Killing vectors) → important restrictions on effective actions $\Gamma = \Gamma[g]$. 
Conformal Anomaly $\equiv$ Trace Anomaly

Conformal anomaly ($\equiv$ trace anomaly) \cite{Deser,Duff,Isham(1976)}

\[
T_{\mu}^{\mu}(x) = a E_2(x) \equiv a R(x) \quad (D = 2)
\]

\[
T_{\mu}^{\mu}(x) = A(x) \equiv a E_4(x) + c C_{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}(x) \quad (D = 4)
\]

where $E_4(x) \equiv$ Euler number density

\[
E_4 \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2
\]

\[
C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2
\]

Coefficients $c_s$ and $a_s$ for fields of spin $s$ (with $s \leq 2$) were computed already long ago.

\cite{Duff(1977);Christenses,Duff(1978);Fradkin,Tseytlin(1982);Tseytlin(2013);}

see also: Eguchi,Gilkey,Hanson, Phys.Rep.66(1980)213
Anomalous Effective Action

Anomaly can be obtained by varying anomalous effective action $\Gamma_{\text{anom}} = \Gamma_{\text{anom}}[g]$

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}}g_{\mu\nu}(x)\frac{\delta \Gamma_{\text{anom}}[g]}{\delta g_{\mu\nu}(x)}$$

but this effective action is necessarily *non-local*.

Simplest example: string theory in *non-critical* dimension has a trace anomaly $T^{\mu\mu} \propto R \Rightarrow$ leads to anomalous effective action = Liouville theory. [Polyakov(1981)]

$$\Gamma_{\text{anom}}^{D=2} \propto \int d^2x \sqrt{-g} R \Box_g^{-1} R$$

- new propagating degree of freedom (longitudinal mode of world sheet metric = Liouville field)
  ⇒ changes physics in dramatic ways!
Analog for gravity in $D = 4$: non-local actions that give anomaly \textit{exactly} are known, for instance [Riegert(1984)]

$$\Gamma_{\text{anom}}[g] = \int d^4 x d^4 y \sqrt{-g(x)} \sqrt{-g(y)} \left( E_4 - \frac{2}{3} \Box_g R \right)(x) G^P(x, y) \left( E_4 - \frac{2}{3} \Box_g R \right)(y)$$

with $\triangle^P G^P(x) = \delta^{(4)}(x)$, and the 4th order operator

$$\triangle^P \equiv \Box_g \Box_g + 2 \nabla_\mu \left( R^{\mu \nu} - \frac{1}{3} g^{\mu \nu} R \right) \nabla_\nu$$

However, no closed form actions are known that have the correct conformal properties (as would be obtained from Feynman diagrams), despite many efforts.

[Deser,Schwimmer(1993);Erdmenger,Osborn(1998);Deser(2000);Barvinsky et al.(1998);Mazur,Mottola(2001);...]

In lowest order

$$\Gamma_{\text{anom}}^{D=4} \propto \int d^4 x \sqrt{-g} E_4 \Box_g^{-1} R + \cdots$$

where $\cdots$ stands for \textit{infinitely many} (non-local) terms.
While
\[ A(x) = -\frac{2}{\sqrt{-g(x)}} g_{\mu\nu}(x) \frac{\delta \Gamma_{\text{anom}}[g]}{\delta g_{\mu\nu}(x)} \]
is local, contribution to Einstein equations
\[ \ell_{PL}^{-2} \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] = -\frac{2}{\sqrt{-g(x)}} \frac{\delta \Gamma_{\text{anom}}[g]}{\delta g^{\mu\nu}(x)} + \ldots \]
in general remains non-local for non-scalar modes.

Claim: non-localities from \( \Box_g^{-1} \) in \( \Gamma_{\text{anom}}[g] \) can ‘overwhelm’ smallness of Planck scale and produce observable deviations for Einstein’s equations!

Typical correction is (symmetrized traceless part of)
\[ \propto \nabla_\mu (G_{\text{ret}} \ast E_4) \nabla_\nu (G_{\text{ret}} \ast R) + \ldots \]
with retarded propagator \( G_{\text{ret}} \) in space-time background given by metric \( g_{\mu\nu} \) solving classical Einstein equations.
For order of magnitude estimate, evaluate this integral for a (conformally flat) cosmological background

\[ ds^2 = a(\eta)^2(-d\eta^2 + dx^2) \]

by integrating from end of radiation era (= \( t_{\text{rad}} \)) back to \( t_0 = n_\ast \ell_{\text{PL}} \), with \( a(\eta) = \eta/(2t_{\text{rad}}) \) and \( \eta = 2\sqrt{tt_{\text{rad}}} \) and with retarded Green’s function [Waylen(1978)]

\[ G_{\text{ret}}(\eta, x; \eta', y) = \frac{1}{4\pi|x - y|} \cdot \frac{\delta(\eta - \eta' - |x - y|)}{a(\eta)a(\eta')} \]

Resulting correction on r.h.s. of Einstein’s equations

\[ T_{00}^{\text{anom}} \sim 10^{-5} t_{\text{rad}}^{-1} (n_\ast \ell_{\text{PL}})^{-3} \]

‘beats’ factor \( \sim (t_{\text{rad}}\ell_{\text{PL}})^{-2} \) on l.h.s. even for \( n_\ast \sim 10^8 \)!

Similar results from evaluating contribution of Riegert action → could be a generic phenomenon, and thus affect any solution of Einstein equations. [Godazgar,Meissner,HN]
Cancelling conformal anomalies

|     | massless |     | massive |     |
|-----|----------|-----|---------|-----|
|     |          | c_s | a_s     | c̄_s|
| 0(0*) | \(\frac{3}{2}\) | -\(\frac{1}{2}\)(\(\frac{179}{2}\)) | \(\frac{3}{2}\)(\(\emptyset\)) | -\(\frac{1}{2}\)(\(\emptyset\)) |
| \(\frac{1}{2}\) | \(\frac{9}{2}\) | -\(\frac{11}{4}\) | \(\frac{9}{2}\) | -\(\frac{11}{4}\) |
| 1    | 18       | -31 | \(\frac{39}{2}\) | -\(\frac{63}{2}\) |
| \(\frac{3}{2}\) | -\(\frac{411}{2}\) | \(\frac{589}{4}\) | -201 | \(\frac{289}{2}\) |
| 2    | 783      | -571| \(\frac{1605}{2}\) | -\(\frac{1205}{2}\) |

- \(\bar{c}_s\) and \(\bar{a}_s\) include lower helicities: \(\bar{c}_1 = c_1 + c_0\), *etc.*
- Gravitinos and supergravity needed for cancellation
- No cancellation possible for \(N \leq 4\) supergravities
NB: gravitino contribution may evade positivity properties because there does not exist a gauge invariant traceless energy momentum tensor for \( s = \frac{3}{2} \). [A.Schwimmer]

\[
\begin{align*}
    c_2 + 5c_3 + 10c_1 + 11c_1^{\frac{1}{2}} + 10c_0 &= 0 \quad (N = 5) \\
    c_2 + 6c_3 + 16c_1 + 26c_1^{\frac{1}{2}} + 30c_0 &= 0 \quad (N = 6) \\
    c_2 + 8c_3 + 28c_1 + 56c_1^{\frac{1}{2}} + 70c_0 &= 0 \quad (N = 8)
\end{align*}
\]

Old result: combined contribution \( \sum_s (c_s + a_s) \) vanishes for all \( N \geq 3 \) theories with appropriate choice of field representations for spin zero fields [Townsend,HN(1981)].

Thus: conformal anomalies for \( \sum_s a_s \) and \( \sum_s c_s \) cancel only for \( N \geq 5 \) supergravities! [Meissner,HN]

... as they do for ‘composite’ \( \text{U}(5), \text{U}(6) \) and \( \text{SU}(8) \) \( R \)-symmetry anomalies. [Marcus(1985)]

Implications for finiteness of \( N \geq 5 \) supergravities?

[Cf. Carrasco,Kallosh,Roiban,Tseytlin(2013);Bern,Davies,Dennen(2014)]
Idem for D=11 SUGRA compactified AdS$_4 \times S^7$

|       | $SO(8)$ representations                                      |
|-------|-------------------------------------------------------------|
| 0     | $[n+2\ 0\ 0\ 0]$ , $[n\ 0\ 2\ 0]$ , $[n-2\ 2\ 0\ 0]$ ,   |
|       | $[n-2\ 0\ 0\ 2]$ , $[n-2\ 0\ 0\ 0]$                      |
| $\frac{1}{2}$ | $[n+1\ 0\ 1\ 0]$ , $n-1\ 1\ 1\ 0]$ ,                     |
|       | $[n-2\ 1\ 0\ 1]$ , $[n-2\ 0\ 0\ 1]$                      |
| 1     | $[n\ 1\ 0\ 0]$ , $[n-1\ 0\ 1\ 1]$ , $[n-2\ 1\ 0\ 0]$    |
| $\frac{3}{2}$ | $[n\ 0\ 0\ 1]$ , $[n-1\ 0\ 1\ 0]$                        |
| 2     | $[n\ 0\ 0\ 0]$                                             |

‘Floor-by-floor’ cancellation [Cf.Gibbons,HN(1985)]: for all $n$

$$\bar{c}_2 f_2(n) + \bar{c}_3 f_{\frac{3}{2}}(n) + \bar{c}_1 f_{1}(n) + \bar{c}_{\frac{1}{2}} f_{\frac{1}{2}}(n) + \bar{c}_0 f_{0}(n) = 0$$

where $f_s(n) \equiv \sum$ (dimensions of $SO(8)$ spin-$s$ irreps) at Kaluza-Klein level $n$ (no anomalies for odd $D$).
Conceptual Issues

Why worry about conformal anomalies in theories that are not even classically conformally invariant?

HOWEVER: recall axial anomaly and anomalous conservation of axial current

\[ \partial^\mu J_\mu^A = 2im\bar{\psi}\gamma^5\psi + \frac{\alpha}{8\pi} F^{\mu\nu} \tilde{F}_{\mu\nu} \]

→ anomaly is crucial even in presence of explicitly broken axial symmetry \((m \neq 0)\).

Idem for gauge anomalies in Standard Model: these must cancel even when quarks and leptons acquire masses via spontaneous symmetry breaking.

Is there a hidden conformal structure behind \(N \geq 5\) supergravities (and M Theory)? But cannot be conformal supergravity in any conventional sense...
Outlook

V. Mukhanov: “You cannot figure out the fundamental theory by simply looking at the sky!”

But maybe there is a way...