Abstract. In this study we consider canal surfaces according to parallel transport frame in Euclidean space $\mathbb{E}^4$. The curvature properties of these surfaces are investigated with respect to $k_1$, $k_2$, and $k_3$ which are principal curvature functions according to parallel transport frame. Finally, we point out that if spine curve $\gamma$ is a straight line, then $M$ is a Weingarten canal surface and also $M$ is linear Weingarten pipe surface.

1. Introduction

A canal surface is defined as envelope of a non-parameter set of spheres, centered at a spine curve $\gamma(t)$ with radius $r(t)$: When $r(t)$ is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe (tube) surface. Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipe surfaces (see, [5], [12], [14]). Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with rational spine curve and rational radius function is rational, and it is therefore natural to ask for methods which allow one to construct a rational parametrization of canal surfaces from its spine curve and radius function. The developable surface plays an important role in CAGD. A natural question is when the canal surface is developable. It is well known that, at regular points, the Gaussian curvature of a developable surface is identically zero. In [15] it has been proved that developable canal surface is either a cylinder or a cone.

The canal surface can be considered as a generalization of the classical notion of an offset of a plane curve. In [6] and [7], the analysis and algebraic properties of offset curves are discussed in detail. In [4], do Carmo discussed some geometrical features of pipe surfaces. Moreover, by using pipe surfaces, do Carmo proved two very important theorems in differential geometry concerning the total curvature of space curves, namely Fenchel’s theorem and the Fary-Milnor theorem.

The well-known surfaces such as tori, Dupin cyclides (ring, spindle and horned) in [13] and pipe surfaces in [10] are the special cases of the canal surfaces.

A surface $M$ in Euclidean space $\mathbb{E}^4$ is called a Weingarten surface if there is a smooth relation $U(k_1, k_2) = 0$ between its two principal curvatures $k_1$ and $k_2$. If $K$ and $H$ denote respectively the Gauss curvature and the mean curvature of $M$, $U(k_1, k_2) = 0$ implies a relation $\Phi(K, H) = 0$. The existence of a non-trivial functional relation $\Phi(K, H) = 0$ on a surface $M$ parameterized by a patch $X(u, v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely

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\[ \frac{\partial (K, H)}{\partial (u, v)} = 0. \]  Also, if a surface satisfies a linear equation with respect to \( K \) and \( H \), that is, \( aK + bH = c \), \( a, b, c \in \mathbb{R} \), \( ((a, b, c) \neq (0, 0, 0)) \), then it is said to be a linear Weingarten surface in \( \mathbb{E}^4 \).

The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. But, curvature may vanish at some points on the curve. That is, the second derivative of the curve may be zero. In this situation, we need an alternative frame in \( \mathbb{E}^3 \). Therefore in \( \mathbb{E}^3 \), Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing the second derivative in 3-dimensional Euclidean space. In \( \mathbb{E}^3 \) the advantages of the Bishop frame and the comparison of Bishop frame with the Frenet frame in Euclidean 3-space were given. In Euclidean 4-space \( \mathbb{E}^4 \), we have the same problem for a curve like being in Euclidean 3-space. That is, one of the \( i-th \) \((1 < i < 4)\) derivative of the curve may be zero. In this situation, we need an alternative frame.

In \( \mathbb{E}^4 \) using the similar idea authors considered such curves and construct an alternative frame. They gave parallel transport frame of a curve and they introduced the relations between the frame and Frenet frame of the curve \( \mathbb{E}^4 \). They generalized the relation which is well known in Euclidean 3-space for 4-dimensional Euclidean space \( \mathbb{E}^4 \).

In \( \mathbb{E}^4 \) authors considered canal surfaces imbedded in an Euclidean space of four dimensions. They investigated the curvature properties of these surface with respect to the variation of the normal vectors and curvature ellipse. They also gave some special examples of canal surfaces in \( \mathbb{E}^4 \). Further, they gave necessary and sufficient condition for canal surfaces in \( \mathbb{E}^4 \) to become superconformal.

In the present study, we consider canal surfaces imbedded in Euclidean 4-space \( \mathbb{E}^4 \) with the spine curve \( \gamma \) given with parallel transport frame in \( \mathbb{E}^4 \).

This paper is organized as follows: Section 2 gives some basic concepts of the Frenet frame and Parallel Transport frame of a curve in \( \mathbb{E}^4 \). Also this section provides some basic properties of canal surfaces in \( \mathbb{E}^4 \) and the structure of their curvatures. Section 3 tells about the canal surfaces in \( \mathbb{E}^4 \) according to parallel transport frame. We also obtain some curvature conditions of these type of canal surface.

2. Basic Concepts

Let \( \gamma = \gamma(s) : I \rightarrow \mathbb{E}^4 \) be a unit speed curve in the Euclidean space \( \mathbb{E}^4 \), where \( I \) is interval in \( \mathbb{R} \). Then the derivatives of the Frenet frame of \( \gamma \) (Frenet-Serret formula);

\[
\begin{bmatrix}
T' \\
N' \\
B'_1 \\
B'_2
\end{bmatrix} = \begin{bmatrix}
0 & \kappa & 0 & 0 \\
-\kappa & 0 & \tau & 0 \\
0 & -\tau & 0 & \sigma \\
0 & 0 & -\sigma & 0
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix},
\]

where \( \{T, N, B_1, B_2\} \) is the Frenet frame of \( \gamma \) and \( \kappa, \tau \) and \( \sigma \) are principal curvature functions according to Frenet frame of the curve \( \gamma \), respectively.

In \( \mathbb{E}^4 \), authors used the tangent vector \( T(s) \) and three relatively parallel vector fields \( M_1(s), M_2(s), \) and \( M_3(s) \) to construct an alternative frame. They called this frame a parallel transport frame along the curve \( \gamma \). Then they gave the following theorem for a parallel transport frame.
Theorem 2.1. Let \( \{ T, N, B_1, B_2 \} \) be a Frenet frame along a unit speed curve \( \gamma = \gamma(s) : I \rightarrow \mathbb{E}^4 \) and \( \{ T, M_1, M_2, M_3 \} \) denotes the parallel transport frame of the curve \( \gamma \). The relation may be expressed as

\[
\begin{align*}
T &= T(s) \\
N &= \cos \theta(s) \cos \psi(s) M_1 + (-\cos \phi(s) \sin \psi(s) + \sin \phi(s) \sin \theta(s) \cos \psi(s)) M_2 \\
&\quad + (\sin \phi(s) \sin \psi(s) + \cos \phi(s) \sin \theta(s) \cos \psi(s)) M_3 \\
B_1 &= \cos \theta(s) \sin \psi(s) M_1 + (\cos \phi(s) \cos \psi(s) + \sin \phi(s) \sin \theta(s) \sin \psi(s)) M_2 \\
&\quad + (-\sin \phi(s) \cos \psi(s) + \cos \phi(s) \sin \theta(s) \sin \psi(s)) M_3 \\
B_2 &= -\sin \theta(s) M_1 + \sin \phi(s) \cos \theta(s) M_2 + \cos \phi(s) \cos \theta(s) M_3,
\end{align*}
\]

where \( \theta, \psi \) and \( \phi \) are the Euler angles. Then the alternative parallel frame equations are

\[
\begin{bmatrix}
T' \\
M_1' \\
M_2' \\
M_3'
\end{bmatrix} =
\begin{bmatrix}
0 & k_1 & k_2 & k_3 \\
-k_1 & 0 & 0 & 0 \\
-k_2 & 0 & 0 & 0 \\
-k_3 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
M_1 \\
M_2 \\
M_3
\end{bmatrix},
\]

where \( k_1, k_2 \) and \( k_3 \) are principal curvature functions according to parallel transport frame of the curve \( \gamma \) and their expression as follows:

\[
\begin{align*}
    k_1 &= \kappa \cos \theta \cos \psi \\
    k_2 &= \kappa (-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi) \\
    k_3 &= \kappa (\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi),
\end{align*}
\]

where \( \theta' = \frac{\sigma}{\sqrt{\kappa^2 + \tau^2}} \), \( \psi' = -\tau - \sigma \frac{\sqrt{\kappa^2 - \tau^2}}{\sqrt{\kappa^2 + \tau^2}} \), \( \phi' = -\frac{\sqrt{\kappa^2 - \tau^2}}{\cos \theta} \) and the following equalities

\[
\begin{align*}
    \kappa &= \sqrt{k_1^2 + k_2^2 + k_3^2}, \\
    \tau &= -\psi' + \phi' \sin \theta, \\
    \sigma &= \frac{\phi'}{\sin \psi}, \\
    \phi' \cos \theta + \theta' \cot \psi &= 0
\end{align*}
\]

hold.

Let \( M \) be a smooth surface in \( \mathbb{E}^4 \) given with the patch \( X(u, v) : (u, v) \in D \subset \mathbb{E}^2 \). The tangent space to \( M \) at an arbitrary point \( p = X(u, v) \) of \( M \) span \( \{ X_u, X_v \} \). In the chart \( (u, v) \) the coefficients of the first fundamental form of \( M \) are given by

\[
\begin{align*}
    E &= \langle X_u, X_u \rangle, \\
    F &= \langle X_u, X_v \rangle, \\
    G &= \langle X_v, X_v \rangle,
\end{align*}
\]

where \( \langle, \rangle \) is the Euclidean inner product. We assume that \( W^2 = EG - F^2 \neq 0 \), i.e. the surface patch \( X(u, v) \) is regular.

For each \( p \) in \( M \), consider the decomposition \( T_p \mathbb{E}^4 = T_p M \oplus T_p^\perp M \) where \( T_p^\perp M \) is the orthogonal component of \( T_p M \) in \( \mathbb{E}^4 \). Let \( \nabla \) be the Riemannian connection of \( \mathbb{E}^4 \).

Given any local vector fields \( X_1, X_2 \) tangent to \( M \), the induced Riemannian connection on \( M \) is defined by

\[
\nabla_{X_1} X_2 = (\nabla_{X_1} X_2)^T,
\]
where $T$ meaning the tangent component.

Let $\chi(M)$ and $\chi^\perp(M)$ be the space of the smooth vector fields tangent to $M$ and the space of the smooth vector fields normal to $M$, respectively. Consider the second fundamental map

$$h : \chi(M) \times \chi(M) \to \chi^\perp(M)$$

(4)

$$h(X_i, X_j) = \tilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i, j \leq 2.$$ 

This map is well-defined, symmetric and bilinear.

**Proposition 2.2.** [2] Let $M \subset E^4$ be a surface in $E^4$ given with the parametrization $X(u, v)$. If the coefficient of the first fundamental form $F = 0$, the second fundamental forms of $M$ becomes

$$h(X_u, X_u) = X_{uu} - \frac{1}{E} (X_{uu}, X_u) X_u + \frac{1}{G} (X_{uv}, X_u) X_v,$$

(5)

$$h(X_u, X_v) = X_{uv} - \frac{1}{E} (X_{uv}, X_u) X_u - \frac{1}{G} (X_{uv}, X_v) X_v,$$

$$h(X_v, X_v) = X_{vv} + \frac{1}{E} (X_{uv}, X_u) X_u - \frac{1}{G} (X_{vv}, X_v) X_v$$

**Proposition 2.3.** [2] Let $M \subset E^4$ be a surface in $E^4$ given with the parametrization $X(u, v)$. Then for the basis $\{X_u, X_v\}$ of $T_pM$, the Gaussian curvature and the mean curvature vector of $M$ is defined as follows respectively,

$$K = \frac{1}{W^2} (\langle h(X_u, X_u), h(X_v, X_v) \rangle - \langle h(X_u, X_v), h(X_u, X_v) \rangle)$$

(6)

$$\vec{H} = \frac{1}{2W^2} \left( Eh(X_v, X_v) - 2Fh(X_u, X_v) + Gh(X_u, X_u) \right),$$

(7)

where $W^2 = EG - F^2$.

3. Canal Surfaces According to Parallel Transport Frame in $E^4$

Let $\gamma(u) = (\gamma_1(u), \gamma_2(u), \gamma_3(u), \gamma_4(u)) \subset E^4$ be a curve parametrized by arclength. The corresponding parallel transport frame formulas have the following form:

$$\gamma'(u) = T$$

$$T'(u) = k_1 M_1 + k_2 M_2 + k_3 M_3$$

$$M_1'(u) = -k_1 T$$

$$M_2'(u) = -k_2 T$$

$$M_3'(u) = -k_3 T,$$

where $\{T, M_1, M_2, M_3\}$ is parallel transport frame and $k_1, k_2$ and $k_3$ are principal curvature functions according to parallel transport frame of the curve $\gamma$. The canal surface in $E^4$ according to Parallel Transport frame has the following parametrization:

$$M : X(u, v) = \gamma(u) + r(u) (M_2(u) \cos v + M_3(u) \sin v).$$

(8)
Proposition 3.1. Let $M$ be a canal surface in $\mathbb{E}^4$ according to Parallel transport frame given with the parametrization in \cite{3}. Then the Gaussian curvature of $M$ at point $p$ is

$$K = -\frac{1}{r^2 (f^2 + r^2)} (f^3 - f^4 + A f r + f^2 r^2)$$

where $A = (f r'' - g r')$.

Proof. The tangent space to $M$ at an arbitrary point $p = X(u, v)$ of $M$ is spanned by

$$X_u = fT + r' \cos vM_2 + r' \sin vM_3$$
$$X_v = -r \sin vM_2 + r \cos vM_3$$

where $f = f(u, v) = 1 - k_2 r \cos v - k_3 r \sin v$. Hence the coefficients of the first fundamental form become

$$E = \langle X_u, X_u \rangle = f^2 + r^2$$
$$F = \langle X_u, X_v \rangle = 0$$
$$G = \langle X_v, X_v \rangle = r^2.$$

The second partial derivatives of $X(u, v)$ are expressed as follows:

$$X_{uu} = gT + f k_1 M_1 + (f k_2 + r'' \cos v)M_2 + (f k_3 + r'' \sin v)M_3$$
$$X_{uv} = f_v T - r' \sin vM_2 + r' \cos vM_3$$
$$X_{vv} = -r \sin vM_2 - r \cos vM_1$$

where $g = g(u, v) = f_u - k_2 r' \cos v - k_3 r' \sin v$. Thus from the equations \cite{10} we get

$$\langle X_{uu}, X_u \rangle = fg + f k_2 r' \cos v + f k_3 r' \sin v + r' r''$$
$$\langle X_{uv}, X_u \rangle = f f_v$$
$$\langle X_{uv}, X_v \rangle = -r r'$$
$$\langle X_{vv}, X_v \rangle = 0.$$

Further, by the use of equations \cite{10}, \cite{12} and \cite{14}, the second fundamental forms of $M$ becomes

$$h(X_u, X_u) = \frac{1}{f^2 + r^2} \left( gr'^2 - f^2 f_u + f^2 g - fr' r'' \right) T + f k_1 M_1$$
$$+ \frac{1}{r (f^2 + r^2)} \left( f^3 \cos v - f^4 \cos v + A f r \cos v \right) M_2$$
$$+ \frac{1}{r (f^2 + r^2)} \left( f^3 \sin v - f^4 \sin v + A f r \sin v \right) M_3$$

$$h(X_u, X_v) = \frac{r^2 f_v}{f^2 + r^2} T - \frac{f f_v r' \cos v}{f^2 + r^2} M_2 - \frac{f f_v r' \sin v}{f^2 + r^2} M_3$$

$$h(X_v, X_v) = \frac{r f r'}{f^2 + r^2} T - \frac{r^2 \cos v}{f^2 + r^2} M_2 - \frac{r^2 \sin v}{f^2 + r^2} M_3,$$

where $W^2 = EG - F^2 = r^2 \left( f^2 + r^2 \right) \neq 0$. From the equations \cite{15} we get the result. \qed
As a consequence of (9) we obtain the following result:

**Corollary 3.1.1.** Let $M$ be a pipe (tube) surface with constant $r = r(u)$. Then the Gaussian curvature of $M$ becomes

\[
K = -\frac{k_2 \cos v + k_3 \sin v}{fr} = \frac{f - 1}{fr^2}.
\]

**Proposition 3.2.** Let $M$ be a canal surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (8). If $\gamma$ is a straight line, then the Gaussian curvature of $M$ at point $p$ is

\[
K = -\frac{r''}{r (1 + r')^2}.
\]

**Proof.** Let $\gamma$ be a straight line, then the equations of parallel transport frame of $\gamma$ become

\[
\begin{align*}
\gamma'(u) &= T(u) \\
T'(u) &= 0 \\
M'_1(u) &= 0 \\
M'_2(u) &= 0 \\
M'_3(u) &= 0.
\end{align*}
\]

Further, the tangent space of $M$ at an arbitrary point $p = X(u, v)$ of $M$ is spanned by

\[
\begin{align*}
X_u &= T + r' \cos v M_2 + r' \sin v M_3 \\
X_v &= -r \sin v M_2 + r \cos v M_3
\end{align*}
\]

Hence the coefficients of first fundamental form become

\[
\begin{align*}
E &= \langle X_u, X_u \rangle = 1 + r'^2 \\
F &= \langle X_u, X_v \rangle = 0 \\
G &= \langle X_v, X_v \rangle = r^2.
\end{align*}
\]

The second partial derivatives of $X(u, v)$ are expressed as follows:

\[
\begin{align*}
X_{uu} &= r'' \cos v M_2 + r'' \sin v M_3 \\
X_{uv} &= -r' \sin v M_2 + r' \cos v M_3 \\
X_{vv} &= -r \cos v M_2 - r \sin v M_3
\end{align*}
\]

Thus from the equations (20) and (23) we get

\[
\begin{align*}
\langle X_{uu}, X_u \rangle &= r''r'' \\
\langle X_{uv}, X_u \rangle &= 0 \\
\langle X_{uv}, X_v \rangle &= rr' \\
\langle X_{vv}, X_v \rangle &= 0.
\end{align*}
\]
Considering the equations (24), (25) and (26), we obtain the second fundamental forms of \( M \) as follows:

\[
(24) \quad h(X_u, X_u) = -\frac{r'v''}{1 + r'^2} + \frac{r''\cos v}{1 + r'^2} M_2 + \frac{r''\sin v}{1 + r'^2} M_3 \\
(25) \quad h(X_u, X_v) = 0 \\
(26) \quad h(X_v, X_v) = \frac{rv'}{1 + r'^2} - \frac{r\cos v}{1 + r'^2} M_2 - \frac{r\sin v}{1 + r'^2} M_3.
\]

where \( W^2 = EG - F^2 = r^2 \left( 1 + r'^2 \right) \neq 0 \). Hence from the equations (24)-(26), we get the result.

**Proposition 3.3.** Let \( M \) be a canal surface in \( \mathbb{E}^4 \) according to parallel transport frame given with the parametrization in (3). If \( \gamma \) is a straight line, the surface \( M \) is flat if and only if \( r \) is a linear function of the form \( r(u) = au + b \) for some real constants \( a, b \).

**Proposition 3.4.** Let \( M \) be a canal surface in \( \mathbb{E}^4 \) according to parallel transport frame given with the parametrization in (3). Then the mean curvature vector of \( M \) at point \( p \) is

\[
\vec{H} = \frac{1}{2r(f^2 + r'^2)} \left\{ \left( fr' (f^2 + r'^2) - Arr' - f^2 r' (1 - f) \right) T \\
+ frk_1 \left( f^2 + r'^2 \right) M_1 \\
+ \left( -f^2 \cos v (f^2 + r'^2) + f^3 \cos v (1 - f) + Afr \cos v \right) M_2 \\
+ \left( -f^2 \sin v (f^2 + r'^2) + f^3 \sin v (1 - f) + Afr \sin v \right) M_3 \right\}
\]

**Proof.** Substituting the equations (15)-(17) into (7), we obtain (27). \( \square \)

As a consequence of (27), we obtain the following results;

**Corollary 3.4.1.** Let \( M \) be a canal surface in \( \mathbb{E}^4 \) according to parallel transport frame given with the parametrization (3). Then the mean curvature of \( M \) at point \( p \) is

\[
H = \frac{1}{2r(f^2 + r'^2)} \left( f^2 (f^2 + r'^2) - 2Afrr'^2 - 2f^3 r'^2 (1 - f) \\
+ A^2 r^2 + 2Af^2 r (1 - f) + f^4 (1 - f)^2 \\
+ f^2 r^2 k_1^2 (f^2 + r'^2) - 2f^5 (1 - f) - 2Af^3 r \right)^{\frac{1}{2}}.
\]

**Corollary 3.4.2.** Let \( M \) be a pipe (tube) surface with constant \( r = r(u) \). Then the mean curvature vector of \( M \) becomes

\[
\vec{H} = \frac{1}{2fr} \left( \frac{rk_1 M_1}{r(2fr) \cos u + \sin v M_2} \right)
\]

**Corollary 3.4.3.** Let \( M \) be a pipe (tube) surface with constant \( r = r(u) \). Then the mean curvature of \( M \) at point \( p \) is

\[
H = \frac{1}{2fr} \left( 4f^2 - 4f + r^2 k_1^2 + 1 \right)^{\frac{1}{2}}.
\]
Proposition 3.5. Let $M$ be a canal surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (8). If $\gamma$ is a straight line then the mean curvature vector of $M$ at point $p$ is

\begin{equation}
\overrightarrow{H} = \frac{1}{2r(1 + r^2)^2} \left\{ \begin{array}{c}
\left( r' + r^2 - rr'' \right)T \\
+ \left( -\cos v - r^2 \cos v + rr'' \cos v \right)M_2 \\
+ \left( -\sin v - r^2 \sin v + rr'' \sin v \right)M_3
\end{array} \right\}.
\end{equation}

Proof. Considering the equations (22) and (24), we obtain the solution. \hfill \Box

Proposition 3.6. Let $M$ be a canal surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (8). If $\gamma$ is a straight line then the mean curvature of $M$ at point $p$ is

\begin{equation}
H = \left| \frac{r'^2 - r''r + 1}{2r(1 + r^2)^{\frac{3}{2}}} \right|.
\end{equation}

Proposition 3.7. Let $M$ be a canal surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (8). If $\gamma$ is a straight line, the surface $M$ is minimal if and only if

\begin{equation}
2r + 2\sqrt{r^2 - c_1^2} = e^{\frac{p^2}{r^2} + c_2}.
\end{equation}

Proof. Let $M$ is minimal. Then from the equation (30), $r'^2 - r''r + 1 = 0$. If we take $r' = p(u)$, the last equation becomes

\begin{equation}
\frac{dr}{r} = \frac{pdv}{p^2 + 1}.
\end{equation}

The solution of the equation (31) is as follows:

\begin{equation}
r^2 = \left( p^2 + 1 \right)^2 c_1.
\end{equation}

Again taking $p(u) = r'$, we obtain the following ordinary differential equation:

\begin{equation}
\frac{dr}{r^2 - c_1^2} = \frac{du}{c_1}.
\end{equation}

If we integrate both sides of the last equation, we get the solution. \hfill \Box

As a consequence of (29), we obtain the following result;

Corollary 3.7.1. Let $M$ be a tube surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (8). If $\gamma$ is a straight line, the mean curvature vector of $M$ at point $p$ is

\begin{equation}
\overrightarrow{H} = \frac{1}{2r} \left( -\cos vM_2 - \sin vM_3 \right).
\end{equation}

Corollary 3.7.2. Let $M$ be a tube surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (8). If $\gamma$ is a straight line, $M$ has constant mean curvature of the form

\begin{equation}
H = \frac{1}{2r} = \text{constant}.
\end{equation}
Proposition 3.8. Let $M$ be a canal surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (3). If $\gamma$ is a straight line, then $M$ is a Weingarten surface.

Proof. Considering the equations (19) and (30), we see that $K$ and $H$ are functions of the variable $u$. So

$$K_v = 0 = H_v,$$

that means $K_u H_v - K_v H_u = 0$. \qed

Proposition 3.9. Let $M$ be a pipe surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (3). If $\gamma$ is a straight line, then $M$ is a linear Weingarten surface.

Proof. Let $M$ be a tube surface in $\mathbb{E}^4$ according to parallel transport frame given with the parametrization in (3) and $\gamma$ is a straight line, then we know that $K = 0$ and $H = \frac{1}{2r}$. Then for $a, b, c \in \mathbb{R}$, we get

$$a.0 + b. \frac{1}{2r} = c,$$

which has the solution $(a, b, c) = (t, 2rk, k)$; $t, k \in \mathbb{R}$. \qed

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