Theory of Rashba coupling mediated superconductivity in incipient ferroelectrics

Maria N. Gastiasoro,1, ∗ Maria Eleonora Temperini,2 Paolo Barone,3 and Jose Lorenzana1
1 ISC-CNR and Department of Physics, Sapienza University of Rome, Piazzale Aldo Moro 2, 00185, Rome, Italy
2 ISC-CNR Institute for Complex Systems, via dei Taurini 19, 00185 Rome, Italy
3 SPIN-CNR Institute for superconductors, oxides and other innovative materials, Area della Ricerca di Tor Vergata, Via del Fosso del Cavaliere 106, 00133 Rome, Italy
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Experimental evidence suggest that superconductivity in SrTiO3 is mediated by a soft transverse ferroelectric mode which, according to conventional theories, has negligible coupling with electrons. A phenomenological Rashba type coupling has been proposed on symmetry arguments but a microscopic derivation is lacking. Here we fill this gap and obtain a linear coupling directly from a minimal model of the electronic structure. We find that the effective electron-electron pairing interaction has a strong momentum dependence. This yields an unusual situation in which the leading s-wave channel is followed by a sub-leading triplet channel which shows a stronger pairing instability than the singlet d-wave state. The bare Rashba coupling constant is estimated for the lowest band of doped SrTiO3 with the aid of first-principles computations and found to be much larger than previously thought. We argue that although for a uniform system the BCS coupling λ is small, it can produce the right order of magnitude for Tc in the presence of structural inhomogeneities.

The role of polar fluctuations in metallic systems close to a transition where a polar axis develops has recently attracted considerable attention. Across this structural transition, inversion symmetry is broken through polar displacements of atoms and the system becomes a ferroelectric (FE) if there are no free carriers, or a polar metal if the transition happens in the presence of a Fermi sea. In both cases, the development of the polar axis can often be described by the softening of an infrared-active phonon associated to the relevant polar displacements. The coupling of the electronic Fermi sea to this soft mode and its feedback has been central to many recent theoretical and experimental works. These include superconductivity mediated by FE fluctuations [1–5], the possibility of attraction in the odd-parity Cooper channel [6–9], unusual transport properties [10–12], and novel quantum critical phenomena [13].

SrTiO3 (STO) is a model incipient ferroelectric where polar quantum fluctuations are believed to be strong. It becomes superconducting (SC) upon carrier doping, and experimentally it has been widely reported that tuning the system even closer to the polar transition notably enhances the SC critical temperature Tc [14–20]. This has led to a substantial amount of theoretical works to consider the exchange of polar fluctuations as a potential candidate for the Cooper pairing mechanism [1 3 4 9 21–20].

There are two important issues that remain to be addressed regarding this novel pairing mechanism. The first is the influence of the dynamics of the polar fluctuations on the superconducting instability, known to be important in the well studied cases of antiferromagnetic and ferromagnetic critical fluctuations. The second challenge, which we address in this work, is providing a mechanism that allows for a significant coupling between the soft polar mode and the electronic degrees of freedom. Because the long-range dipolar interaction pushes the longitudinal optical mode to higher energies in the long-wavelength limit, the soft mode has a predominantly transverse polarization, which implies no gradient-like linear coupling, such as Fröhlich coupling, the standard electron-phonon interaction considered in polar crystals. Even including longitudinal components from crystal field corrections to the polarization of the soft mode leads to a negligible coupling [27]. Different alternative mechanisms have been proposed to couple to the polar mode including a revival of an old idea involving the quadratic coupling to the soft phonon [3 25 26 28].

Here we focus on an alternative linear coupling mechanism to the odd-parity FE mode which was proposed using symmetry arguments in the presence of spin-orbit coupling (SOC) [2 6 29] or in multi-band systems [13]. These theories are semi-phenomenological in that they do not completely determine the momentum dependence of the coupling nor its magnitude. We present a minimal microscopic model for the linear coupling to a polar transverse mode in a system with SOC and derive the corresponding Rashba-like electron-phonon coupling Hamiltonian.

At first sight one may expect the electron-phonon interaction to scale with λSO, the atomic spin-orbit parameter as found in some tight-binding models of Rashba splitting at surfaces [30]. We will show that this is not the case but rather the coupling scales with dλ/dμa, a conventional electron-phonon matrix element involving the derivative of a hopping matrix element with respect to the atom separation to be defined more precisely below.

With the aid of DFT frozen-phonon computations we estimate the bare Rashba-like coupling for the model system of doped STO and find it to be substantial. Projecting the interaction into the Cooper channel, we study the pairing weak-coupling solutions for both even and odd-
Eq. (2) becomes

\[ t_{\mu\nu}(k, q) = 2it'_{\mu\nu} \sum_{\Delta} (\hat{\mu} \times \hat{\nu} \cdot \hat{\xi}) \cdot \{ u^O_j(q) \sin \kappa_3 - u^{TM}(q) \left[ (\hat{\mu} \cdot \delta) \sin \left( k_3 - \frac{q_3}{2} \right) + (\hat{\nu} \cdot \delta) \sin \left( k_3 + \frac{q_3}{2} \right) \right] \} \]  

where \( \hat{\mu}, \hat{\nu} \) are unitary vector normal to the plane spanned by the \( t_{2g} \) orbital \( \mu, \nu \). We also used the simplified notation \( t'_{\mu\nu} = \frac{\partial t_{\mu\nu}}{\partial q_{\mu\nu}} = -\frac{\partial t_{\mu\nu}}{\partial q_{\mu\nu}} \) for the derivatives. In the \( k, q \to 0 \) limit, and under a rigid oxygen cage condition \( u^O_j(q) = u^O_j(q) = u^{O}(q) \equiv u^{O}(q) \), Eq. (2) becomes

\[ t_{\mu\nu}(k, q) \approx 2it'_{\mu\nu} (\hat{\mu} \times \hat{\nu} \times k) \cdot (u^O_j(q) - u^{TM}(q)) + it'_{\mu\nu} (q_i u^O_{\nu}(q) + q_\nu u^{TM}(q) + O(k^3)) \]  

The first term is proportional to the polarization vector \( P(q) = Z [u^O(q) - u^{TM}(q)] \) with an effective charge \( Z \) per unit cell volume, and thus describes inter-orbital coupling induced by polarization waves. Fig. 1(a) illustrates the induced hopping between orbitals \( \mu = x \) and \( \nu = y \) for a finite polar displacement along \( \hat{\gamma} \), which changes sign with hopping direction. As we explicitly show below, this process is responsible for the Rashba coupling. The second term in Eq. (3), instead, corresponds to the coupling to strain gradients as produced by acoustic modes and vanishes at the zone center.

In order to keep the interaction with the lattice as simple as possible, we will focus on the \( |3/2, \pm 3/2\rangle \) SOC band at the zone center,

\[ c_{+3/2} = (c_{x\uparrow} + ic_{y\downarrow})/\sqrt{2} \equiv c_{\uparrow} \]  
\[ c_{-3/2} = (c_{x\downarrow} - ic_{y\uparrow})/\sqrt{2} \equiv c_{\downarrow} \]  

formed by degenerate states \( j_z = \pm 3/2 \) with opposite angular momentum and spin [see Fig. 1(b)] and involving only two of the three \( t_{2g} \) orbitals. This situation is relevant for example in perovskite oxides with tetragonal symmetry, like STO below 105 K. The derivation of the coupling to other SOC bands \( |j, m_j\rangle \) is analogous. Projecting Eqs. (1) and (3) to the spinor of the doubly-degenerate SOC band Eqs. (4)-(5), \( \psi = (c_{\uparrow}, c_{\downarrow}) \), we arrive to the deformation potential expression \( \mathcal{H}_u = \sum_{kq} \psi^\dagger (k + \frac{q}{2}) \Lambda(k, q) \psi(k - \frac{q}{2}) \) with

\[ \Lambda(k, q) = 2it'_{xy} \hat{z} \times k \sigma_z \cdot (u^O_j(q) - u^{TM}(q)) \]  

Here \( \Lambda(k, q) \) is a \( 2 \times 2 \) matrix in pseudospin space, with the Pauli matrix \( \sigma_z \). Hence we obtained a coupling between a non-magnetic polar displacement \( u \) with the pseudospin of the electrons. Note that only the polarization-wave coupling term, the first term in Eq. (3), contributes to this Rashba deformation potential.

Interestingly, SOC matrix elements do not appear in Eq. (6). This is because near the zone center the angular momentum is not quenched and states with well defined \( m_j \) [Eqs. (4)-(5)] are split already at zero order in \( \lambda_{SO} \) from the rest of the states. Because of their chiral nature, these states propagate with different velocity in the presence of the polar mode induced inter-orbital hopping without further intervention of the SOC as schematized in Fig. 1(b) and (d).

In order to better describe this effect, we can rewrite the deformation potential matrix Eq. (6) in a more conventional form, i.e. in terms of the Rashba field \( u(k) \) experienced by the SOC electrons, that couples to their pseudospin \( \sigma \):

\[ \Lambda(k, q = 0) = 2it'_{xy} u_0 (k_\perp \times \hat{u}_\perp) \cdot \sigma \equiv u(k) \cdot \sigma. \]  

Here \( k_\perp = (k_x, k_y, 0) \) and \( \hat{u}_\perp = (\hat{u}_x, \hat{u}_y, 0) \) are vectors in the \( xy \) plane, and for simplicity we have chosen a finite uniform polar displacement \( u^{O}(0) - u^{TM}(0) = u_0 \hat{a} \). The
when the gray dashed line represents the doubly-degenerate SOC bands in the presence of the polar displacement shown in (a). The process in the SOC basis $|w\rangle$ Rashba field $w(k)$ [Eq. (7)] (top) and $c_{\alpha}^\dagger c_\beta$ [Eq. (3)] (bottom). The orange and blue arrows show the polarization of the spin along $z$, parallel to the angular momentum. The color legend represents the finite polar displacement along $\hat{z}$, parallel to the bond $\mathbf{r} = 0$.

Finally, in order to obtain the electron-FE-phonon Hamiltonian we quantize the displacements $u(q)$ in Eq. (6) and decompose them in a set of normal modes $\lambda$:

$$\mathcal{H}_u = \frac{1}{\sqrt{N}} \sum_{kq\lambda} g_{\lambda}(k, q) \psi^\dagger(k + \frac{q}{2}) \sigma_z \psi(k - \frac{q}{2}) A_{q\lambda}$$

Eq. (8)

Here $A_{q\lambda} = (a_{q\lambda} + a_{-q\lambda}^\dagger)$ is the phonon operator, $\hat{\mathbf{n}}_\lambda(q)$ the unit polarization vector of mode $\lambda$, and $\zeta = \frac{3M_O}{M_{TO}}$. Following several experiments and ab-initio computations [33-41], we assumed that the soft FE mode can be approximated by the Slater mode where the TM atom moves against a rigid oxygen cage, $\mathbf{u}^{TM}(q) = -\zeta \mathbf{u}^{O}(q)$, with a mass $M_S = M_{TM} + 3M_O$ (see SM for details).

Unlike the conventional gradient deformation potential, the Rashba coupling Eq. (9) stays finite at $q = 0$. Furthermore, it selects the $E_u$ irreducible representation components of the phonon polarization vector $(\hat{n}_x(q), \hat{n}_y(q))$ and thus it is finite for any longitudinal or transverse mode with finite polarization in the $xy$ plane, even in the absence of a crystal field. In general, additional contributions to the Rashba coupling Eq. (9) allowed by symmetry [9] are obtained for the electronic SOC states Eq. (3)-(5) by considering spin-flip hopping processes $t_{xy}^\dagger$ in Eq. (7): a $E_u$ term $k_z(\hat{z} \times \sigma)$ and a $A_{2u}$ term $(\mathbf{k} \times \sigma) \cdot \hat{z}$. They each also carry, accordingly, a different induced hopping parameter $t_{xy}$. For simplicity, we ignore these terms but note that they should be considered for a complete description of the Rashba coupling function of a SOC band.

Neglecting the crystal field on the phonon sector and assuming dispersion-less modes (i.e. staying in the small $q$ limit) we obtain one of our main results which are simple expressions for the coupling Eq. (9) to the transverse optical modes:

$$g_{TO1}(\mathbf{k}, \mathbf{q}) = g_{TO} \frac{i \left( \hat{k}_x \hat{q}_z + \hat{k}_y \hat{q}_y \right)}{\sqrt{1 - q_z^2}}$$

Eq. (10)

$$g_{TO2}(\mathbf{k}, \mathbf{q}) = g_{TO} \frac{\hat{q}_x \left( \hat{k}_y \hat{q}_y - \hat{k}_x \hat{q}_x \right)}{\sqrt{1 - q_z^2}}$$

Eq. (11)

$$g_{TO} = 2t_{xy} \left( \frac{1}{\sqrt{1 + \varsigma}} \right) \sqrt{\frac{\hbar}{2M_S\omega_{TO}}} k_F a = \alpha_{TO} k_F k$$

Eq. (12)

We notice again that the Rashba parameter $\alpha_{TO}$ and Rashba electron-phonon matrix element $g_{TO}$ are independent of the strength of $\lambda_{SO}$. On the other hand $g_{TO}$ is proportional to the electronic Fermi wavevector $k_F$, and thus its magnitude will be accordingly reduced in very low-density systems with a small Fermi energy. The coupling to the gapped longitudinal mode has a similar form, but with a smaller coupling $g_{LO} = \sqrt{\frac{2\pi}{\omega_{LO}}} g_{TO}$ instead, since $\omega_{LO} > \omega_{TO}$.

Superconductivity. We are now ready to study the superconducting solutions mediated by the Rashba coupling between the SOC electrons and the FE soft mode.
We restrict the effective electron-electron interaction into the Cooper channel, \([g_{\text{TO}}(\mathbf{k} + \mathbf{k}', \mathbf{k} - \mathbf{k}')]^2 D_{\text{TO}}\), with the matrix elements of the transverse optical phonon sector given by Eqs. (10)–(11) and the corresponding static phonon propagator \(D_{\text{TO}} = \frac{-i\omega_{\text{TO}}}{\omega_{\text{TO}} + \omega_c}\).

The resulting linearized gap equations at the critical temperature \(T_c\) can be written in terms of a four component vector \(d_a\), for the even-parity \(\Delta_{\text{even}}(\mathbf{k}) = d_0(\mathbf{k})i\sigma_y\) and odd-parity \(\Delta_{\text{odd}}(\mathbf{k}) = d_\mathbf{k}(\mathbf{k}) \cdot \sigma_\mathbf{i} \sigma_y\) channels,

\[
d_a(\mathbf{k}) = N_F \frac{g_{\text{TO}}^2 \omega_{\text{TO}}}{\omega_{\text{TO}} + \omega_c} \ln \left( \frac{1.13 \omega_c}{k_BT_c} \right) \int d\mathbf{k'} \frac{L_0(\mathbf{k}, \mathbf{k'}) d_a(\mathbf{k'})}{4\pi} \tag{13}\]

with \(a = 0\) for the singlet and \(a = x, y, z\) for the triplet channel. \(N_F\) is the density of states (DOS) per spin at the Fermi energy and \(\omega_c\) is a characteristic cut-off frequency (of the order of \(\omega_{\text{TO}}\)). The interaction kernels are the following:

\[
L_0(\mathbf{k}, \mathbf{k'}) = \frac{1}{2} \left[ \ell(\mathbf{k}, \mathbf{k'}) + \ell(\mathbf{k}, -\mathbf{k'}) \right] \tag{14}
\]

\[
L_x(\mathbf{k}, \mathbf{k'}) = \frac{1}{2} \left[ \ell(\mathbf{k}, \mathbf{k'}) - \ell(\mathbf{k}, -\mathbf{k'}) \right] \tag{15}
\]

\[
L_{xy}(\mathbf{k}, \mathbf{k'}) = -L_z(\mathbf{k}, \mathbf{k'}) \tag{16}
\]

with the effective angular dependent FE-mediated interaction

\[
\ell(\mathbf{k}, \mathbf{k'}) = \frac{1}{1 - \mathbf{k} \cdot \mathbf{k'}} \left( \mathbf{k} - \mathbf{k}' \right)^2 \tag{17}
\]

Due to the \(x, y\) orbital character of the electronic SOC and because we are considering isotropic transverse phonon modes \(\bar{n}_{\text{TO}}(\mathbf{q})\) [Eq. (9)], the Rashba induced interaction Eq. (17) has cylindrical symmetry. As can be seen from Figs. 2(a)-(c), the momentum dependence on the Fermi surface is very strong. It has a large matrix element when the momentum of the quasiparticles on the Fermi surface, \(\mathbf{k}, \mathbf{k}'\), is near the equator and the exchanged momentum is small [Fig. 2(c)], while it is much weaker for quasiparticles moving along \(z\) [Fig. 2(b)]. The intermediate case in Fig. 2(a) interpolates between the two previous cases.

Because of cylindrical symmetry, the effective interaction only depends on the polar angles and the difference of the azimuthal angles, \(L_a(\mathbf{k}, \mathbf{k'}) = L_a(\theta, \theta', \varphi - \varphi')\), and it can be decomposed as an expansion of real spherical harmonics of the same index \(m\),

\[
L_a(\mathbf{k}, \mathbf{k'}) = \sum_m \sum_{l,l'} f_a^{(m)}(\mathbf{r}) y_{lm}(\mathbf{k}) y_{l'm}(\mathbf{k'}) \tag{18}
\]

where we have used the normalized harmonics \(y_{lm}(\mathbf{k}) = \sqrt{\frac{4\pi}{2l+1}} y_{lm}(\mathbf{k})\) and the Rashba coupling yields an additional term in the BCS coupling constant, i.e.

\[
\lambda_a^{(m)} = N_F \frac{g_{\text{TO}}^2 \omega_{\text{TO}}}{\omega_{\text{TO}} + \omega_c} \theta_{a}^{(m)} \tag{19}
\]

with \(k_BT_c^{(m)} = 1.13 \omega_c \exp \left(-1/\lambda_{a}^{(m)}\right)\) and the Rashba coupling constant \(g_{\text{TO}} = \alpha\pi k_F\) [Eq. (12)].

One can expect that the strong momentum dependence of the interaction would result in a momentum dependent gap function. Remarkably, the interaction potential is such that upon angular integration on \(\mathbf{k}'\) the result is constant, i.e.

\[
\int \frac{d\mathbf{k}}{2\pi} \int \frac{d\varphi}{2\pi} L_{a}(\mathbf{k}, \mathbf{k'}) = \frac{2}{3}. \tag{20}
\]

Consequently the isotropic state is a solution of the gap equation (13). Furthermore, the leading pairing instability, i.e. the state with the highest eigenvalue, is precisely the isotropic s-wave state \(d_0(\mathbf{k}) = y_{00}\) with eigenvalue \(\theta_{0}^{(m=0)} = \frac{2}{3}\).

Notwithstanding the above trivial solution, the strong momentum dependence has important manifestations. Traditionally, the only attractive channel of electron-phonon coupling is the singlet s-wave (or the trivial \(A_{1g}\) channel). Here instead, the Rashba coupling yields attractive channels also in the triplet odd-parity sector.
The relevance of unconventional odd-parity channels mediated by odd-parity fluctuations in systems with SOC has been recently pointed out in various works [7, 9, 31, 32].

The inset of Fig. 2(d) shows the eigenvalue ranking of the sub-leading solutions mediated by the FE Rashba-like interaction. Since the gap equation for the odd-parity vector component $d_{z}^{m}(\hat{k})$ [Eq. 15], which describes unequal pseudospin pairing, is different from that of the equal pseudospin pairing odd-parity components $d_{x}^{m}(\hat{k})$ and $d_{y}^{m}(\hat{k})$ [Eq. 16], these states have different critical temperatures $T_{c}$. Interestingly, the first two sub-leading solutions belong to the odd-parity (triplet) channel, $d_{z}^{(m=1)}$ and $d_{x,y}^{(m=0)}$, with a stronger pairing instability than the next attractive solution $d_{0}^{(m=2)}$, a d-wave-like singlet state.

The SC order parameters for these three sub-leading solutions, $\Delta_{odd}(\hat{k}) = d_{z}^{m}(\hat{k}) \cdot i\sigma_y$ for odd-parity states and $\Delta_{even}(\hat{k}) = d_{0}(\hat{k})i\sigma_y$ for even-parity states take the form,

$$d_{z}^{(m=1)} : \quad d_{z}^{m} = v_1(0,0,\hat{k}_z) + v_2(0,\hat{k}_y)$$

$$d_{x,y}^{(m=0)} : \quad d_{x,y}^{m} = v_3(f(\hat{k}_z),0) + v_4(0,f(\hat{k}_z),0)$$

$$d_{0}^{(m=2)} : \quad d_{0}^{m} = v_5(\hat{k}_z^2 - \hat{k}_y^2) + v_6\hat{k}_x\hat{k}_y$$

to leading order in the multipole expansion [Eq. 18]. Note that in the case of $d_{x,y}^{(m=0)}$, the truncation had to be done at higher order. The coefficients $v_i$ multiply the degenerate gap functions at $T_c$ in our minimal model. Interestingly, nematic ($v_3 \neq v_4$) and chiral solutions are allowed. A consistent study for which particular combination to expect below $T_c$ requires going beyond the linearized Eq. 13. These will generally depend on details on different compounds and it should be examined on a case-by-case basis. In addition, the role of Coulomb interaction, neglected in this work, should also be considered when studying sub-leading SC channels. Because the strongest pair-breaking effect happens in the leading s-wave channel, Coulomb interaction can stabilize odd-parity solutions [7].

Numerical estimates for STO

In the following, we present estimates for the very low doping region in which only one band of STO is partially filled. We focus on the density $n \approx 2 \times 10^{18}$ cm$^{-3}$ corresponding to the maximum $T_{c}$ of this regime (0.2K).

To obtain the induced hopping parameter $t'_{xy}$ we have performed DFT frozen phonon computations (details will be presented elsewhere). We obtained $t'_{xy} = 108$ meV/Å for the lowest band of doped STO [43], which is described by Eqs. (4)-(5). This value is three times larger than the rough estimate in Ref. [43], believed to be unrealistically high at the time. Experimentally, in this compound the optical gap of the soft FE mode is around 1 meV below 4 K [43, 46] and remains soft in the doping range we consider [44]. This results in a characteristic zero-point motion length of the Slater mode $l_S = \sqrt{\frac{E}{2M_{SO}}}$ = 0.28a$_{0}$ = 0.15Å. Thus we get the following Rashba coupling for the lowest band of STO [Eq. 12].

$$g_{T}^{(STO)} \approx 65 \text{meV} \cdot k_{F}a = (254 \text{meV} \cdot \text{Å}) \cdot k_{F} \approx \alpha_{T}^{(STO)} k_{F}$$

Remarkably, the estimated Rashba parameter $\alpha_{T}^{(STO)}$ is two orders of magnitude larger than previous estimates in Refs.[34, 35] for the lowest $d_{xy}$ band of the two-dimensional electron gas in LAO/STO interfaces. For the quoted density $k_{F}a \approx 0.3$, and thus the Rashba coupling turns out to be moderately large, $g_{T}^{(STO)} \approx 20$ meV.

At this point it is important to highlight the main challenge that arises as a consequence of Eq. 19 for any weak-coupling pairing mechanism in STO and related materials. Often estimates of $T_c$ are done at optimum doping but here we focus on the more dilute systems known, which are more challenging to explain. From specific heat measurements [48], the DOS per spin is $N_{F} \approx 0.4$ eV$^{-1}$ for $n \approx 2 \times 10^{18}$. In order to have a strong enough SC coupling $\lambda = N_{F}V_{eff} \approx 0.25$ that sustains a $T_{c} \approx 0.2$ K, the effective pairing interaction should be in the range $V_{eff} \approx 6 \text{eV}$, which is extremely large for an electron-phonon mechanism.

By using the DFT estimated Rashba coupling Eq. 24, the resulting pairing interaction is $V_{eff} \approx 250 \text{meV}$ in the s-wave channel, and together with the $N_{F}$ quoted above, the SC coupling we obtain is $\lambda_{0}^{(m=0)} \approx 0.01$. Therefore, the coupling estimation coming from our minimal model suggests that the mechanism involving the direct Rashba coupling to the FE fluctuations cannot support superconductivity on its own in the very low doping regime of STO, and assuming a spatial homogeneous scenario. On the other hand, as illustrated in Fig. 2(d), since $\lambda_{0}^{(m=0)} \propto \omega_{TO}^{-2}$ [see Eq. 19] the SC coupling constant quickly grows as the system approaches the polar phase [41] ($\omega_{TO} \rightarrow 0$) and the FE mode mediated mechanism remains an obvious candidate to explain the experimentally observed $T_c$ enhancement in this case [14, 20]. Lastly, the proliferation of tetragonal domains in the samples [49] opens the possibility of having local regions with a softer mode and/or variations of carrier density at grain boundaries and defects [44, 50], all of which would locally increase the SC $\lambda$ and may lead to filamentary superconductivity as in other complex oxides [51, 52].

Conclusions. We developed a minimal microscopic model for a linear Rashba-like coupling between a soft polar mode and conduction electrons, relevant for systems close to a polar instability. The explicit form of the electron-phonon Hamiltonian was derived for a $|\frac{3}{2}, \pm \frac{3}{2} \rangle$ SOC band with a Slater polar phonon, believed to be relevant in a model incipient FE like STO. Indeed, the estimate of the bare linear Rashba-like coupling we obtained with the aid of ab-initio calculations for the lowest band of STO is found to be substantial, unlike previous estimates of the more standard gradient-like electron-phonon coupling. Extending this minimal Rashba-like coupling
model to other systems with strong polar fluctuations should be straightforward, and will result in different electron-phonon coupling functions to the corresponding soft mode.

Following the standard weak-coupling approach, we studied the even and odd-parity Cooper channels by considering the effective interaction arising from the Rashba-like deformation potential. In agreement with what was found in previous works which considered odd-parity fluctuations as a pairing mechanism, we find that the leading s-wave SC channel is followed by an attractive subleading odd-parity channel.

Using the estimated bare Rashba coupling $g_{\text{RTDO}}$ for the lowest band of STO together with experimental parameters, we obtained a BCS coupling $\lambda$ for the leading s-wave solution. While its value cannot explain $T_c$ in a spatially uniform scenario at the lowest densities, it may explain it in the presence of inhomogeneities. Our approach can be easily extended to other incipient ferroelectrics or to interfaces.

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Appendix A: Inter-orbital hopping

In this section we provide more details on the derivation of Eq. (2). In the presence of a finite polar displacement of the lattice new hopping channels are induced between different orbitals $\mu$ and $\nu$,

$$H_u = \sum_{r, \delta; \mu, \nu; \sigma} \xi_{\mu \nu} \xi_{r, \mu, \sigma} c_{r, \mu, \sigma, \nu, \sigma} + \text{h.c.}, \quad \text{(A1)}$$

where $\delta = \hat{x}, \hat{y}, \hat{z}$ is the vector parallel to the bond joining nearest neighbor TM atoms. Assuming that the hopping only depends on the displacement between the two TM atoms and the bridging O atom involved in the process [see Fig. 1(b)], in the form $u_r^{O_{\delta}} - u_r^{TM} - u_r^{TM}$, derivatives in real space are simply related

$$\frac{\partial H_{\mu \nu}}{\partial u_{\nu, r + \delta}} = -\frac{\partial H_{\delta \mu}}{\partial u_{\nu, r}} = -\frac{\partial H_{\mu \delta}}{\partial u_{\nu, r + \delta}}. \quad \text{(A2)}$$

The possible spin-conserving hopping processes between a pair of orbitals $\mu$ and $\nu$, allowed only for displacements perpendicular to the bond $\delta$, are encoded in the following:

$$t^{(\mu \nu)}_{r \delta} = t^{(\mu \nu)}_{\delta} [\delta \times \hat{\mu} \times \hat{\nu}] \left[ u_r^{O_{\delta}} - (\hat{\mu} \cdot \delta) u_r^{TM} - (\hat{\nu} \cdot \delta) u_{r + \delta}^{TM} \right] \quad \text{(A3)}$$

where $\hat{\mu}, \hat{\nu}, \hat{\delta}$ is a unitary vector normal to the plane spanned by the $t_{2g}$ orbital $\mu, \nu$. In addition, we have taken the derivatives Eq. (A2) to be uniform for the pair of orbitals for simplicity, i.e. $t^{(\mu \nu)}_{\delta} \equiv \frac{\partial u^{(\mu \nu)}}{\partial u_{\nu, r + \delta}}$. One can also keep the directional dependence of the derivatives if needed. Finally by Fourier expanding the ionic displacement

$$u_r^\kappa = \sum_q u^\kappa(q) e^{i q \cdot (r + \delta_k)} \quad \text{(A4)}$$

of atom $\kappa$ at position $r + \delta_k$, we arrive at Eq. (2).

Appendix B: Electron-phonon Hamiltonian for a Slater mode

In order to get the electron-phonon Hamiltonian in Eq. (8), we follow the usual procedure and quantize the displacements in Eq. (6) by introducing the polarization vectors $\hat{e}_\lambda$ of the normal mode $\lambda$ for each atom $\kappa$,

$$u^\kappa(q) = \sum_\lambda \sqrt{\frac{\hbar}{2NM\kappa\omega_{\kappa\lambda}}} \hat{e}_\lambda \left( a_{q\lambda} + a_{-q\lambda}^\dagger \right). \quad \text{(B1)}$$

Assuming that the polar mode is a Slater mode [39, 53], i.e. a mode which satisfies the rigid cage condition between the TM atom and the oxygen cage $u^\text{TM}(q) = -u^O(q)$, we have the following relations for the polarization vectors

$$\hat{e}_\lambda^O = \frac{1}{\sqrt{M_O}} \hat{n}_\lambda(q) \quad \text{(B2)}$$
$$\hat{e}_\lambda^\text{TM} = \sqrt{\frac{\varsigma}{M_T}} \hat{n}_\lambda(q). \quad \text{(B3)}$$

Here $\hat{n}_\lambda(q)$ is the unit polarization vector of the mode $\lambda$, the mass of the Slater mode $M_T = M + 3M_O$, and $\varsigma = \frac{3M_O}{M_T}$. Substituting these two expressions one arrives to Eq. (8). Following an equivalent procedure one can derive the Hamiltonian for a different polar mode, with a different set of relations between the atomic displacements $u^\kappa(q)$.

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