The Hua operators on Homogeneous Line Bundles over Bounded Symmetric Domains of Tube Type.

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Abstract

Let $D = G/K$ be a bounded symmetric domain of tube type. We show that the image of the Poisson transform on the degenerate principal series representation of $G$ attached to the Shilov boundary of $D$ is characterized by a $K$-covariant differential operator on a homogeneous line bundle over $D$. As a consequence of our result we get the eigenvalues of the Casimir operator for Poisson transforms on homogeneous line bundles over $G/K$. This extends a result of Imemura and all [5] on symmetric domains of classical type to all symmetric domains. Also we compute a class of Hua type integrals generalizing an earlier result of Faraut and Koranyi[3].

Key words: Hua Operators; Poisson transforms; Hua-type integrals

1 Introduction and main result

Let $G/K$ be an irreducible Hermitian symmetric space of tube type with Shilov boundary $G/P_{\Xi}$. In [15] Shimeno proved that the Poisson transform maps the space $B(G/P_{\Xi}, L_{\lambda})$ of hyperfunction-valued sections of a degenerate spherical series representation attached to $G/P_{\Xi}$ bijectively onto an eigenspace of the Hua operator $\mathcal{H}$ on $G/K$, under certain conditions on the complex parameter $\lambda$.

The aim of this paper is to generalize the above result to homogeneous line bundles over $G/K$. To state our result in rough form let us fix some notations referring to section 2 for more details. For $\nu \in \mathbb{Z}$ let $E_{\nu}$ denote the homogeneous line bundle on $G/K$ associated to the one dimensional representation $\tau_{\nu}$ of $K$. Let $\mathcal{D}_{\nu}(G/K)$ be the algebra of $G$-invariant differential operators on $E_{\nu}$.

Shimeno [14] proved that the Poisson transform is a $G$-isomorphism from the space $B(G/P, L_{\mu,\nu})$ of hyperfunction-valued sections of principal series representations attached to the Furstenberg boundary $G/P$ onto the solutions space $\mathcal{A}(G/K, \mathcal{M}_{\mu,\nu})$ of the system of differential equations on $E_{\nu}$

$$\mathcal{M}_{\mu,\nu} : \quad (D - \chi_{\mu,\nu}(D))F = 0 \quad \forall D \in \mathcal{D}_{\nu}(G/K),$$

under certain conditions on $\nu$ and $\mu \in a^*_c$. In above $\chi_{\mu,\nu}$ is a certain character of the algebra $\mathcal{D}_{\nu}(G/K)$.

Let $r$ and $m$ denote respectively the rank of $G/K$ and the multiplicity of the short restricted

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roots. Let $P_{\Xi}$ be a maximal standard parabolic subgroup of $G$ with the Langlands decomposition $P_{\Xi} = M_{\Xi}A_{\Xi}N_{\Xi}$ such that $A_{\Xi}$ is of real dimension one. Let $\xi$ be the one dimensional representation of $P_{\Xi}$ defined by

$$\xi(m_{1}a) = \tau_{\nu}(m) a^{\rho_{\Xi} - \lambda_{\rho_{0}}}, \quad m_{1} \in M_{\Xi,s}, m \in M, a \in A_{\Xi}, n \in N_{\Xi}.$$ 

In above $M_{\Xi,s}$ is the semisimple part of $M_{\Xi}$, $\rho_{0}$ and $\rho_{\Xi}$ are linear forms on $a_{\Xi} = \mathbb{R}X_{0}$ (the Lie algebra of $A_{\Xi}$) defined by $\rho_{0}(X_{0}) = r$ and $\rho_{\Xi} = \eta\rho_{0}$ with $\eta = \frac{w_{0}}{2}(r - 1) + 1$.

Let $L_{\xi}$ be the homogeneous line bundle over the Shilov boundary $G/P_{\Xi}$ associated to $\xi$. For $f$ in $B(G/P_{\Xi}, L_{\lambda,\nu})$ the space of hyperfunction-valued sections of the homogeneous line bundle $L_{\xi}$ we define its Poisson transform by

$$[P_{\lambda,\nu}f](g) = \int_{K} f(\xi g) \tau_{\nu}(k) dk,$$

where $dk$ is the normalized Haar measure of $K$.

The degenerate series representation $B(G/P_{\Xi}, L_{\lambda,\nu})$ attached to $G/P_{\Xi}$ is a $G$-sub-module of a principal series representation $B(G/P, L_{\mu_{\lambda},\nu})$ for $\mu_{\lambda} = \lambda\rho_{0} - \rho_{\Xi} + \rho$, and the image $P_{\lambda,\nu}(B(G/P_{\Xi}, L_{\lambda,\nu}))$ is a $G$-submodule of the solution space $A(G/K, M_{\mu_{\lambda},\nu})$.

Therefore it is natural to pose the problem of characterizing this image by differential operators on $E_{\nu}$.

In the trivial cases the origin of this problem goes back to L. H. Hua [4] who showed in the case of the classical Cartan domain of $n \times n$ matrices that for $f$ in $B(G/P_{\Xi})$ $P_{\rho_{\Xi}}f$ is annihilated by $n^{2}$-second order differential operators. Since then many authors considered the problem of constructing differential operators characterizing the image $P_{\rho_{\Xi}}(B(G/P_{\Xi}, L_{\lambda,\nu}))$, see [3], [12] in the case of the Siegel upper half plane. In [9] Johnson and Koranyi constructed second order differential operator $\mathcal{H}$-called then after Hua operator- and showed that $\mathcal{H}$ characterizes the image $P_{\rho_{\Xi}}(B(G/P_{\Xi}, L_{\lambda,\nu}))$. Lassalle [11] reproved their result introducing the operator $\mathcal{H}_{q}$, cutting down the number of equations.

In this paper we will show that the image of $B(G/P_{\Xi}, L_{\lambda,\nu})$ under $P_{\lambda,\nu}$ is characterized by a $K$- covariant differential operator on $E_{\nu}$.

Let $g$ and $k$ be the Lie algebras of $G$ and $K$ respectively and $g = k \oplus p$ be the Cartan decomposition of $G$ with Cartan involution $\theta$. We denote the complexifications of $g, k, p$ by $g_{c}, k_{c}, p_{c}$ respectively.

The center $\mathfrak{z}$ of $k$ is of dimension one and there exists $Z_{0} \in \mathfrak{z}$ such that $adZ_{0}$ define a complex structure on $p_{c}$. Let

$$g_{c} = p_{-} \oplus k_{c} \oplus p_{+},$$

be the corresponding eigenspace decomposition of $p_{c}$.

Let $E_{i}$ be a basis of $p_{+}$ and $E_{j}^{*}$ be the dual basis of $p_{-}$ with respect to the Killing form of $g_{c}$. We consider the element of $\mathcal{U}(g_{c}) \otimes k_{c}$ -called here the Hua operator- defined by

$$\mathcal{H} = \sum_{i,j} E_{i} E_{j}^{*} \otimes [E_{j}, E_{i}^{*}]. \quad (1.1)$$

The operator $\mathcal{H}$ is a homogeneous differential operator from the space of $C^{\infty}$-sections of $E_{\nu}$ to the space of $C^{\infty}$-sections of the homogeneous vector bundle on $G/K$ associated to the
representation $\tau_\nu \otimes Ad_K|_{k_c}$, which does not depend on the choice of basis. (see section 4)
The pair $(K, K \cap M_\Xi)$ is a compact symmetric pair. Let $\bar{\tau}$ be the corresponding involution of $\mathfrak{k}$, and let $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{k}$ into eigenspaces of $\bar{\tau}$. Here $\mathfrak{l}$ is the Lie algebra of $K \cap M_\Xi$.

Let $\mathcal{H}_q$ be the element of $\mathcal{U}(\mathfrak{g}_c) \otimes q_c$ defined by

$$\mathcal{H}_q = \sum_{i,j} E_i E_j^* \otimes p([E_j, E_i^*]),$$

where $p$ denotes the orthogonal projection of $k_c$ onto $q_c$.

The main result of this paper can be stated as follows

**Theorem 1.1** Let $\lambda$ be a complex number and let $\nu \in \mathbb{Z}$ such that

$$- \lambda - \frac{m}{2} (-r + 2 + j) \notin \{1, 2, \ldots\} \text{ for } j = 0, 1,$$

$$- \lambda + \eta - |\nu| \notin 2\mathbb{Z}^+ + 2.$$  \hspace{1cm} (1.3)

Then the Poisson transform $P_{\lambda,\nu}$ is a $G$-isomorphism from the space $B(G/P_\Xi, L_{\lambda,\nu})$ onto the space of $C^\infty$-sections of $E_\nu$ satisfying the system of differential equations

$$\mathcal{H}_q F = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} F(-iZ_0).$$ \hspace{1cm} (1.5)

In the above $p = 2\eta$ is the genus of the bounded symmetric domain $G/K$.

In the case $\tau_\nu$ is the trivial representation, Theorem 1.1 has been established by Lassalle [11] for $\lambda = \rho_\Xi$ and generalizes to generic $\lambda$ by Shimeno [15].

To prove our result we first show that every solution of the Hua system is a joint eigenfunction of the algebra $D_\nu(G/K)$ (Theorem 6.1).

To do so, let $h \subset q$ be a Cartan subalgebra of the symmetric pair $(\mathfrak{k}, \mathfrak{l})$ and let $s$ be the orthogonal complement of $h$ in $q$ with respect to the Killing form $B$. Write

$$\mathcal{H}_q = \mathcal{H}_h + \mathcal{H}_s$$

with $\mathcal{H}_h \in \mathcal{U}(\mathfrak{g}_c) \otimes h_c$ and $\mathcal{H}_s \in \mathcal{U}(\mathfrak{g}_c) \otimes s_c$.

Then we prove (Theorem 6.2) that if $F$ is a $\tau_{-\nu}$-spherical function on $G$ satisfying

$$\mathcal{H}_h F = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} F(-iZ_0),$$

then the function

$$\phi(t_1, \ldots, t_r) = \prod_{j=1}^r (\cosh t_j)^\nu F(t_1, \ldots, t_r),$$

satisfies the following system of differential equations

$$\frac{\partial^2 \phi}{\partial t_k^2} + 2 \coth 2t_k \frac{\partial \phi}{\partial t_k} - 2\nu \tanh t_k \frac{\partial \phi}{\partial t_k} + \frac{m}{2} \sum_{j=1}^r \frac{1}{(\sinh^2 t_j - \sinh^2 t_k)}(\sinh 2t_j \frac{\partial \phi}{\partial t_j} - \sinh 2t_k \frac{\partial \phi}{\partial t_k})$$

\text{for } j \neq k.$$
for all $k = 1, ..., r$.

Then by using a result of Yan [17] on generalized hypergeometric functions in several variables we deduce that $F$ is given (up to a constant) in terms of the generalized Gauss-hypergeometric function $2F_1^{(m)}(b, c; d; x_1, ..., x_r)$ associated to the parameter $m$, see [17]. Namely

$$F(t_1, ..., t_r) = \prod_{j=1}^{r}(1 - \tanh^2 t_j)^{\frac{\lambda + \eta - \nu}{2}} 2F_1^{(m)}\left(\frac{\lambda + \eta - \nu}{2}, \frac{\lambda + \eta + \nu}{2}; \eta, \tanh^2 t_1, ..., \tanh^2 t_r\right).$$

To finish the proof of our main result we show that under the conditions (1.3) and (1.4) the induced equations of the subsystem $H \xi F = 0$ for boundary values on the maximal boundary $G/P$ characterize the space $\mathcal{B}(G/P, L_{\lambda, \xi})$.

### 1.1 Consequences.

For the symmetric domains of classical type, the eigenvalues of the Casimir operator for Poisson transforms on $B(G/P, L_{\lambda, \nu})$ have been computed by Imemura and all [5]. Using our result we can compute them for all symmetric domains without using their classification as in [5]. Thus, of course the exceptional domains are included.

More precisely, let $L$ be the Casimir operator acting on $C^\infty$-sections of $E_{\nu}$.

**Corollary 1.1** Let $\lambda \in \mathbb{C}, \nu \in \mathbb{Z}$ and let $F = P_{\lambda, \nu}f$ with $f \in \mathcal{B}(G/P, L_{\lambda, \xi})$. Then we have

$$LF = \frac{\lambda^2 - (\eta - \nu)^2}{4r} F. \quad (1.6)$$

**Proof.** The Laplacian $L$ is the projection of the Hua operator $\mathcal{H}$ onto the center $\mathfrak{z}_c$ of $\mathfrak{g}_c$. Thus $H_z = iL \otimes \mathfrak{z}_0^*$. Since $\mathfrak{z} \subset \mathfrak{h}$, then

$$\mathcal{H}_z F = \frac{\lambda^2 - (\eta - \nu)^2}{4r} F(-i\mathfrak{z}_0), \quad (1.7)$$

by Theorem 1.1.

Let $B$ be the Killing form of $\mathfrak{g}$. Then we have

$$B(\mathcal{H}_z F, Z_0) = iLF,$$

from which we deduce that $LF = -\frac{\lambda^2 - (\eta - \nu)^2}{4r} BF(Z_0, Z_0)$. Since $B(Z_0, Z_0) = -2\lambda r$, the result follows.

**Remark 1.1** The above corollary agrees with [12], Theorem 5.2 with $q = \eta$ and $s = \eta - \lambda$, where the eigenvalues of the Casimir operator have been computed by Koranyi using a different method.

Let $h(z, w)$ be the canonical Jordan polynomial associated to the bounded symmetric domain $G/K$ and let $S$ denote its Shilov boundary.
Then by using trivialization of the homogeneous line bundles $E_\nu$ and $L_\xi$ we can rewrite the Poisson transform $P_{\lambda,\nu}$ as

$$P_{\lambda,\nu} f(z) = \int_S \left[ \frac{h(z, z)}{h(z, u)} \right]^{\frac{\lambda + n - \nu}{2}} h(z, u)^{-\nu} f(u) du,$$

see Proposition 5.1, for more details.

Letting $f = 1$ in (1.8) we deduce from the method of the proof of Theorem 1.1, an explicit expression of a class of Hua type integrals. Namely, we have

**Corollary 1.2** let $\lambda \in \mathbb{C}$ and let $\nu \in \mathbb{Z}$. Then we have

$$\int_S \left[ \frac{h(z, z)}{h(z, u)} \right]^{\frac{\lambda + n - \nu}{2}} h(z, u)^{-\nu} du = h(z, z)^{\frac{\lambda + n - \nu}{2}}$$

$$2F_1^{(m)} \left( \begin{array}{c} \lambda + \eta - \nu, \lambda + \eta + \nu \\ \frac{2}{\eta}, \tanh^2 t_1, ..., \tanh^2 t_r \end{array} \right),$$

$$z = Ad(k) \sum_{j=1}^r \tanh t_j E_{\gamma_j}.$$ 

The above formula has been established for $\nu = 0$ by Faraut and Koranyi in [3].

In the case of the trivial line bundle on $SU(n, n)/S(U(n) \times U(n))$, we computed a more general class of the following Hua type integrals. More precisely, we have

$$\int_{U(n)} \left[ \frac{h(\tanh t I, \tanh t I)}{h(\tanh t I, u)} \right]^{\frac{\lambda + n}{2}} \phi_m(u) du = \frac{n!}{d_m} \det(\phi_{\lambda,|m|-i+j}(t))_{i,j},$$

where $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ such that $m_1 \geq m_2 \geq ... \geq m_n$ and $d_m = \Pi_{i<j} (1 + \frac{m_i - m_j}{j-i}),$ with

$$\phi_{\lambda,k}(t) = (1 - \tanh^2 t)^{\frac{\lambda+n}{2}} \left( \frac{(\lambda+n)k}{(1)_k} \right) \tanh^{k} t \quad 2F_1 \left( \begin{array}{c} \lambda + n, \lambda + n \\ \frac{2}{2}, \tanh^2 t \end{array} \right)$$

In above $2F_1(a, b; c; x)$ is the classical Gauss hypergeometric function, $(a)_k$ the Pochhammer symbol and $\phi_m$ a zonal spherical function related to Schur functions, see [2] for more details.

Before ending this section, we should mention that Professor Koufany informed me that he and Professor Zhang had also obtained a characterization of Poisson integrals on homogeneous line bundles over tube type symmetric domains see [10]. See also [16]. Namely, they showed that the image $P_{\lambda,\nu}(B(G/P_\xi, L_{\lambda,\nu}))$ can be characterized by the operator $H$ defined in (1.1). The system (1.1) has of course $dim_{\mathbb{R}}K$ differential equations.

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Regarding to the scalar case, this system has too many equations.

Our result shows that, as in the scalar case, a subsystem of $dim_{\mathbb{R}}S = dim_{\mathbb{C}}G/K$ equations is sufficient to characterize Poisson integrals on homogeneous line bundles over $S$.

The organization of this paper is as follows. After a preliminaries on Hermitian symmetric spaces we review in section 3 the results of [15] on the Poisson transform on homogeneous line bundles on the Furstenberg boundary. In section 4 we define the Poisson transform on
degenerate principal series representation attached to the Shilov boundary and introduce the Hua operator $\mathcal{H}$ on $E_\nu$. Using a trivialization of the space of $C^\infty$-sections of $E_\nu$ we give a realization of $\mathcal{H}$ on the Harish-Chandra realization of $G/K$ (Proposition 4.1). Section 5, 6 and 7 are devoted to the proof of our main result.

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## 2 Preliminaries and notations

In this section we recall some structural results on Hermitian symmetric space, see [4] for more details.

For a real Lie algebra $\mathfrak{b}$ we shall denote by $\mathfrak{b}_c$ its complexification.

Let $G/K$ be an irreducible hermitian symmetric space of noncompact type with rank $r$. The group $G$ is the connected component of its isometry group and $K$ is the isotropy subgroup of $G$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$ with respect to a Cartan involution $\theta$. Thus $\mathfrak{k}$ has one dimensional center $\mathfrak{z}$, $\mathfrak{k}_c = [\mathfrak{k}, \mathfrak{k}] \neq \mathfrak{k}$ and $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{k}_c$.

Let $Z_0 \in \mathfrak{z}$ such that $(adZ_0)^2 = -1$ on $\mathfrak{p}_c$. Let $\mathfrak{p}_+$ (respectively $\mathfrak{p}_-$) be the $i$ (respectively $-i$) $\mathfrak{k}$-eigenspace of $adZ_0$ in $\mathfrak{g}_c$.

Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are Abelian subalgebras. Moreover $[\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{t}_c$. We thus have the Harish-Chandra decomposition of $\mathfrak{g}_c$:

$$\mathfrak{g}_c = \mathfrak{p}_+ + \mathfrak{t}_c + \mathfrak{p}_-.$$

Let $G_c$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_c$. Let $P_+, P_-$ and $K_c$ denote the analytic subgroup of $G_c$ corresponding to the Lie subalgebras $\mathfrak{p}_+, \mathfrak{p}_-$ and $\mathfrak{t}_c$ respectively. Then $P_+ K_c P_-$ is an open dense subset of $G_c$ containing $G$. For $z \in \mathfrak{p}_+$ and $g \in G$ we denote by $U(g : z)$ the $K_c$ component of $g \exp z$. That is

$$g \exp z = \exp(g.z)U(g : z)p_-(g). \quad (2.1)$$

Under the above action the $G$-orbit $D = G.0$ of $z = 0 \in \mathfrak{p}_+$ is a bounded domain in $\mathfrak{p}_+$ and $K$ is the isotropy subgroup of 0. This is the Harish-Chandra realization of the Hermitian symmetric space $G/K$.

Recall that the $K_c$-component $U(g : z)$ is the canonical automorphy factor and that it satisfies the multiplier identity

$$U(g_1 g_2 : z) = U(g_1 : g_2.z)U(g_2 : z). \quad (2.2)$$

### 2.1 The Roots.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ (and hence also of $\mathfrak{g}$). Let $\Delta$ denote the root system of $(\mathfrak{g}_c, \mathfrak{t}_c)$.

For $\alpha \in \Delta$ let $\mathfrak{g}_\alpha$ denote the root space for $\alpha$. A root $\alpha$ is said to be compact (resp. noncompact) if the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{t}_c$ (resp. $\mathfrak{p}_c$). Let $B$ denote the Killing form of $\mathfrak{g}_c$. For each root $\alpha$ we can choose root vectors $\vec{H}_\alpha \in \mathfrak{t}_c$, $\vec{E}_\alpha \in \mathfrak{g}_\alpha$, and $\vec{E}_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$\alpha(H) = B(H, \vec{H}_\alpha), \quad \forall H \in \mathfrak{t}_c,$$
\[ [E_\alpha, E_{-\alpha}] = \hat{H}_\alpha, \quad \tau E_\alpha = -E_{-\alpha}, \]
with \( B(E_\alpha, E_{-\alpha}) = 1. \)

In above \( \tau \) denotes the conjugation in \( \mathfrak{g}_C \) with respect to the real form \( \mathfrak{k} + i\mathfrak{p}. \)

For \( \alpha, \beta \) in \( \Delta \), we set \( <\alpha, \beta> = B(H_\alpha, H_\beta). \) Then the length \( |\alpha| \) of a root \( \alpha \) is defined by \( |\alpha| = \sqrt{<\alpha, \alpha>}. \)

Put \( c_\alpha = \frac{\sqrt{2}}{|\alpha|}. \) Let \( H_\alpha = c_\alpha^2 \hat{H}_\alpha \) and \( E_\alpha = c_\alpha \tilde{E}_\alpha. \) Then the root vectors \( H_\alpha, E_\alpha \) and \( E_{-\alpha} \) satisfy

\[ [E_\alpha, E_{-\alpha}] = H_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha, \quad B(E_\alpha, E_{-\alpha}) = \frac{2}{|\alpha|^2}. \]

Moreover \( \alpha(H_\alpha) = 2. \)

We choose an ordering on \( \Delta \) such that a noncompact root is positive if and only if \( g_\alpha \subset p_+ \).
We denote by \( \Phi^+ \) the set of positive noncompact roots. Then we have

\[ p_+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad p_- = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}. \]

For each \( \alpha \in \Phi^+ \) we set

\[ X_\alpha = E_\alpha + E_{-\alpha}, \quad Y_\alpha = i(E_\alpha - E_{-\alpha}). \]

Two roots \( \alpha \) and \( \beta \) are said strongly orthogonal if neither \( \alpha + \beta \) nor \( \alpha - \beta \) are roots.

Let \( \gamma_j \) be a maximal set of strongly orthogonal noncompact roots, such that \( \gamma_j \) is the highest element of \( \Phi^+ \) strongly orthogonal to \( \gamma_{j+1}, \ldots, \gamma_r, \) for \( j = r, \ldots, 1. \)

Then \( \mathfrak{a} = \sum_{j=1}^{r} \mathbb{R} X_{\gamma_j} \) is a maximal Abelian subspace of \( \mathfrak{p}. \)

Let \( \mathfrak{h} \) denote the Abelian subalgebra generated by the elements \( iH_{\gamma_j}, j = 1, \ldots, r \) and let \( \mathfrak{h}^\perp \) be its orthogonal in \( \mathfrak{t} \) with respect to the Killing form \( B. \) Then \( \mathfrak{h}^\perp = \{ H \in \mathfrak{t}; \gamma_j(H) = 0, j = 1, \ldots, r \}. \)

For \( \alpha, \beta \in \Delta \) denote \( \alpha \sim \beta \) if and only if \( \alpha |_{\mathfrak{h}} = \beta |_{\mathfrak{h}}. \) Let

\[ \Phi^+_{ij} = \{ \alpha \in \Phi^+; \alpha \sim \frac{\gamma_i + \gamma_j}{2} \}, \]

for \( 1 \leq i < j \leq r. \)

Let \( \mathfrak{C} \) denote the set of compact roots in \( \phi. \) Define

\[ C_0 = \{ \alpha \in \mathfrak{C}; \alpha \sim 0 \}, \]

and

\[ C_{ij} = \{ \alpha \in \mathfrak{C}; \alpha \sim \frac{\gamma_j - \gamma_i}{2} \}, \quad \text{for} \ 1 \leq i < j \leq r. \]

Then it is known that \( \Delta^+ \) is the disjoint union of the sets \( \Gamma, C_0, C_{ij}, \Phi^+_{ij} \) and \( \Phi^+ \) is the disjoint union of the sets \( \Gamma \) and \( \Phi^+_{ij}. \)

Let \( \alpha \in \Phi^+_{ij}. \) Define \( \tilde{\alpha} \in \mathfrak{t}^* \) by \( \tilde{\alpha} = \gamma_i + \gamma_j - \alpha. \) Then \( \tilde{\alpha} \in \Phi^+_{ij} \) see [11].

Now we recall a result from [1] which will be helpful later.
Proposition 2.1 Let k be a fixed integer, 1 ≤ k ≤ r.

(i) For γj ∈ Γ we have γj(Hγk) = 2δjk.
(ii) If α ∈ Φ+jk, then α(Hγk) = 1.
(iii) In all other cases α(Hγk) = 0.
(iv) Let α ∈ Φ+jk. Then <α, α> = <γk, γk> if α ̸= ̺ and <α, α> = 1/2 <γk, γk> if α = ̺.

Let c be the Cayley transform of g, given by
\[ c = \exp \frac{\pi}{4}(E_0 - E_0), \]
where \( E_0 = \sum_{k=1}^{r} E_{γk} \). Then \( Adc(Hγj) = Xγj \), and \( Adc(\mathfrak{h}) = i\mathfrak{a} \).

It is well known that \( Adc^1 = 1 \) and \( Adc^2 \) is an automorphism of \( \mathfrak{k} \). Let \( l \) (respectively \( q \)) be the +1 (respectively -1) eigenspace of \( Adc^2 \) in \( \mathfrak{k} \). Then we have \( \mathfrak{z} \) is a subset of \( \mathfrak{q} \). Moreover, let \( q_s = q \cap \mathfrak{t} \) then \( \mathfrak{z} = q_s + l \) and \( q = q_s + \mathfrak{z} \).

Put \( β_j = γ_j \circ (Adc^{-1}|a) \), for \( j = 1, ..., r \). Then the set of restricted roots \( Σ \) of the pair \( (g, a) \) is given by
\[ Σ = \{ ±β_j \ (1 ≤ j ≤ r), \pm β_j \pm β_k 2 \ (1 ≤ j ≠ k ≤ r) \}. \]
Let \( Σ^+ \) be the set of positive roots in \( Σ \). Then
\[ Σ^+ = \{ β_j \ (1 ≤ j ≤ r), \frac{β_j ± β_k}{2} \ (1 ≤ j < k ≤ r) \}. \]
The Weyl group \( W \) of \( Σ \) acts as the group of all permutations and sign changes of the set \( \{ β_1, ..., β_r \} \), so it is isomorphic to the semi-direct product of \( (\mathbb{Z}/2\mathbb{Z})^r \) and the symmetric group.

We set
\[ α_j = \frac{β_r-j+1 - β_r-j}{2}, \ (1 ≤ j ≤ r-1) \quad α_r = β_1. \]
Then \( Γ = \{ α_1, ..., α_r \} \) is the set of simple roots in \( Σ^+ \). Let \( \{ H_1, ..., H_r \} \) denote the basis of \( \mathfrak{a} \) which is dual to \( \{ α_1, ..., α_r \} \).

For \( α ∈ Σ \) let \( g^α \) denote the corresponding root space and let \( m_α \) be the multiplicity of \( α \).

The multiplicities of the roots \( ±β_j ± β_k \) are \( m_α \) and \( m_β \) are \( m_α \) and \( m_β \) respectively.

As usual set \( ρ = \frac{1}{2} \sum_{α ∈ Σ^+} m_αα \), \( n^+ = \sum_{α ∈ Σ^+} g^α \) and \( n^- = θ(n) \).

Let \( A, N^+ \) and \( N^- \) be the analytic subgroups of \( G \) corresponding to \( a, n^+ \) and \( n^- \) respectively. The group \( G \) has the Iwasawa decomposition \( G = KAN^+ \). Let \( M \) be the centralizer of \( a \) in \( K \). Then \( P = MAN^+ \) is a minimal parabolic subgroup of \( G \).

Let \( Ξ = Γ \setminus \{ α_r \} \) and let \( P_Ξ \) be the corresponding standard parabolic subgroup of \( G \) with the Langlands decomposition \( P_Ξ = M_ΞA_ΞN_Ξ \) such that \( A_Ξ ⊂ A \). Then \( P_Ξ \) is a maximal standard parabolic subgroup of \( G \) and the space \( G/P_Ξ \) is the Shilov boundary of \( X \).

If \( a_Ξ \) denotes the Lie algebra of \( A_Ξ \), then
\[ a_Ξ = \{ H ∈ a; γ(H) = 0, ∀γ ∈ Ξ \}. \]
Moreover \( a_Ξ = \mathbb{R}X_0 \) where \( X_0 = \sum_{j=1}^{r} X_γj \).

On \( a_Ξ \) we define the linear form \( ρ_0 \) by \( ρ_0(X_0) = r \). Let \( ρ_Ξ \) be the restriction of \( ρ \) to \( a_Ξ \). Then
\[ ρ_Ξ = (m \cdot \frac{r-1}{2} + 1)ρ_0. \]
The algebras \( n^\pm \) decomposes as \( n^\pm = n^\pm_\Xi + n(\Xi)^\pm \), where
\[
n^\pm_\Xi = \sum_{j \leq k} g^{\pm(\beta_j + \beta_k)/2}, \quad n(\Xi)^\pm = \sum_{j > k} g^{\pm(\beta_j - \beta_k)/2},
\]
and we have \( p_\Xi = m + a + n^+ + n(\Xi)^- \), \( p_\Xi \) been the Lie algebra of \( P_\Xi \).

## 3 Eigensections of invariant differential operators

We review the main result of [14] on the image of the Poisson transform on the principal series representation attached to the Furstenberg boundary \( G/P \).

Recall that the genus \( p \) of the bounded domain \( D \), is given by
\[
p = m(r - 1) + 2.
\]
Then the length \( | \gamma_j | \) of the roots \( \gamma_j \) is such that \( | \gamma_j | = \frac{1}{\sqrt{p}} \).

Let \( Z = \frac{2}{n} Z_0 \) where \( n = \text{dim}_\mathbb{C} p_+ \).

Let \( K_s \) be the analytic subgroup of \( K \) with Lie algebra \( \mathfrak{k}_s \). For \( \nu \in \mathbb{Z} \) define \( \tau_\nu : K \to \mathbb{C}^\times \) by
\[
\tau_\nu(k) = 1 \quad \text{if} \quad k \in K_s,
\]
and
\[
\tau_\nu(\exp(tZ)) = e^{-i\nu t} \quad \text{for} \quad t \in \mathbb{R}.
\]
Then \( \tau_\nu \) determines a one dimensional representation of \( K \) and all one dimensional representations of \( K \) have this form, see [13] and the remark below.

**Remark 3.1** Since
\[
B(Z, Z_0) = -2p,
\]
it follows that \( Z \) is the same as the element
\[
\frac{1}{r} \sum_{j=1}^{r} \frac{2i\tilde{H}_{\gamma_j}}{<\gamma_j, \gamma_j>},
\]
in [I3].

Let \( E_\nu \) be the homogeneous line bundle on \( G/K \) associated to \( \tau_\nu \). The space of \( C^\infty \)-sections of \( E_\nu \) can be identified with the space
\[
C^\infty(G/K, \tau_\nu) = \{ f \in C^\infty(G); f(gk) = \tau_\nu(k)^{-1}f(g) \quad \text{for all} \quad g \in G, k \in K \}.
\]
The group \( G \) acts on \( C^\infty(G/K, \tau_\nu) \) by the left regular representation \( \pi(g)f(x) = f(g^{-1}x) \).

Let \( \mathbb{D}_\nu(G/K) \) be the set of all left-invariant differential operators on \( G \) that map \( C^\infty(G/K, \tau_\nu) \) into itself. Then, accordingly to Shimeno result [14] \( \mathbb{D}_\nu(G/K) \) is isomorphic, via the Harish-Chandra isomorphism \( \gamma_\nu \), to \( \mathcal{U}(\mathfrak{a})^W \) the set of Weyl group invariant elements in \( \mathcal{U}(\mathfrak{a}) \).

For \( \mu \in \mathfrak{a}_c^* \) and \( \nu \in \mathbb{Z} \) we define an algebra homomorphism \( \chi_{\mu, \nu} \) of \( \mathbb{D}_\nu(G/K) \) by
\[
\chi_{\mu, \nu}(D) = \gamma_\nu(D)(\mu).
\]
For $\mu \in a_+^*$ we denote by $\mathcal{B}(G/P, L_{\mu,\nu})$ the space of hyperfunction-valued sections of the homogeneous line bundle on $G/P$ associated to the character $\sigma_{\mu,\nu}$ of $P$ given by

$$\sigma_{\mu,\nu}(\text{man}) = a^{\rho - \mu} \tau_\nu(m) \quad m \in M, a \in A, n \in N^+.$$  

For $f \in \mathcal{B}(G/P, L_{\mu,\nu})$ we define the Poisson transform $\mathcal{P}_{\mu,\nu}$ by

$$\mathcal{P}_{\mu,\nu} f(g) = \int_K f(gk) \tau_\nu(k) dk.$$  

A straightforward computation shows that

$$\mathcal{P}_{\mu,\nu} f(g) = \int_K e^{-(\mu + \rho)H(g^{-1}k)} f(k) \tau_\nu(k^{-1}k) dk,$$

where $\kappa : G \to K$ and $H : G \to a$ are the projections defined by $g \in \kappa(g)e^{H(g)N^+}$.

Let $\mathcal{B}(G/K, \tau_\nu)$ be the space of hyperfunction-valued sections of the homogeneous line bundle $E_\nu$ on $G/K$. We denote by $\mathcal{A}(G/K, \mathcal{M}_{\mu,\nu})$ the space of all real analytic functions in $\mathcal{B}(G/K, \tau_\nu)$ which satisfy the system of differential equations

$$\mathcal{M}_{\mu,\nu} : DF = \gamma_\nu(D)(\mu)F, \quad D \in \mathbb{D}_\nu(G/K).$$

Let $e_\nu(\mu)^{-1}$ be the denominator of the $c$-function associated to $\mathcal{P}_{\mu,\lambda}$. That is

$$e_\nu(\mu)^{-1} = \prod_{1 \leq j < k \leq r} \Gamma(\frac{1}{2}(m + \mu_j + \mu_k)) \Gamma(\frac{1}{2}(m + \mu_k - \mu_j)) \times \prod_{1 \leq j \leq r} \Gamma(\frac{1}{2}(1 + \mu_j + \nu)) \Gamma(\frac{1}{2}(1 + \mu_j - \nu)).$$

**Theorem 3.1** [15] Let $\mu \in a_+^*$ and $\nu \in \mathbb{Z}$ satisfying the conditions

$$-2 \frac{\mu, \alpha}{\alpha, \alpha} \notin \{1, 2, \ldots\} \quad \text{for all} \quad \alpha \in \Sigma^+$$

and

$$e_\nu(\mu) \neq 0,$$

then the Poisson transform $\mathcal{P}_{\mu,\nu}$ is a $G$-isomorphism from $\mathcal{B}(G/P, L_{\mu,\nu})$ onto $\mathcal{A}(G/K, \mathcal{M}_{\mu,\nu})$. The inverse of $\mathcal{P}_{\mu,\nu}$ is given by the boundary value map up to a non-zero constant multiple.

Recall that a $\mathbb{C}$-valued function on $G$ is called $\tau_{-\nu}$-spherical function if it satisfies

$$f(k_1gk_2) = \tau_{-\nu}(k_2)f(g)\tau_{-\nu}(k_1) \quad (3.1)$$

A $\tau_{-\nu}$-spherical function $\Phi$ on $G$ will be called an elementary spherical function of type $\tau_{-\nu}$ if it satisfies

$$D\Phi = \chi_{\mu,\nu}\Phi, \quad \forall D \in \mathbb{D}_\nu(G/K),$$

$$\Phi(e) = 1$$

According to [15], for $\mu \in a_+^*$ and $\nu \in \mathbb{Z}$, there exists a unique elementary spherical function of type $\tau_{-\nu}$, given by

$$\Phi_{\mu,\nu}(g) = \int_K e^{-(\mu + \rho)H(g^{-1}k)} \tau_\nu(k^{-1}k) \tau_{-\nu}(k^{-1}k) dk. \quad (3.2)$$
4 The Poisson transform and the Hua operator

4.1 The Poisson transform on a homogeneous line bundle

In this subsection we define the Poisson transform on degenerate principal series representation attached to the Shilov boundary $G/P_\Xi$ of $G/K$.

Let $M_{\Xi,s}$ be the analytic subgroup of $M_\Xi$ with Lie algebra $[m_\Xi,m_\Xi]$. From here on we suppose that $\tau_\nu$ is such that $\tau_\nu|_{K\cap M_{\Xi,s}} = 1$.

For $\lambda \in \mathbb{C}$ and $\nu \in \mathbb{Z}$ let $\xi_{\lambda,\nu}$ denote the one dimensional representation of $P_\Xi$ defined by

$$\xi_{\lambda,\nu}(m_{1\text{man}}) = a^{\rho_{\Xi} - \lambda \rho_0} \tau_\nu(m), \text{ for all } m_{1\text{man}} \in M_{\Xi,s}, m \in M, a \in A_\Xi, n \in N_\Xi.$$

Let $B(G/P_\Xi, L_{\lambda,\nu})$ be the space of hyperfunction-valued sections of the line bundle on $G/P_\Xi$ associated to the character $\xi_{\lambda,\nu}$. The Poisson transform $P_{\lambda,\nu}$ of an element $f \in B(G/P_\Xi, L_{\lambda,\nu})$ is defined by

$$P_{\lambda,\nu}f(g) = \int_K f(k) \tau_\nu(k) dk. \quad (4.1)$$

By the generalized Iwasawa decomposition $G = KP_\Xi$, the restriction map from $G$ to $K$ gives an isomorphism from $B(G/P_\Xi, L_{\lambda,\nu})$ onto the space $B(K/K_\Xi, L_\nu)$ of hyperfunction-valued sections of the homogeneous line bundle on $K/K_\Xi$ associated to the representation $\tau_\nu$. Here $K_\Xi = K \cap M_\Xi$.

For $\lambda \in \mathbb{C}$ define the following $\mathbb{C}$-linear form $\mu_\lambda$ on $a_\Xi^*$ by

$$\mu_\lambda(H) = (\lambda \rho_0 - \rho_\Xi)(H_\Xi) + \rho(H),$$

where $H_\Xi$ is the $a_\Xi$-component of $H$ with respect to the orthogonal decomposition $a = a_\Xi \oplus a(\Xi)$.

We have

$$B(G/P_\Xi, L_{\lambda,\nu}) \subset B(G/P, L_{\mu_\lambda,\nu}). \quad (4.2)$$

From (4.2) we deduce

$$P_{\lambda,\nu}(B(G/P_\Xi, L_{\lambda,\nu})) \subset A(G/K, M_{\mu_\lambda,\nu}). \quad (4.3)$$

A straightforward computation shows that the Poisson transform of $f \in B(K/K_\Xi, L_\nu)$ is given by

$$P_{\lambda,\nu}f(g) = \int_K e^{-(\lambda + \eta)\rho_0(H_\Xi(g^{-1}k))} f(k) \tau_\nu(k(g^{-1}k)) dk.$$

The space $B(K/K_\Xi, L_\nu)$ can be identified to the space of all hyperfunctions $f$ on $K$ such that

$$f(km) = \tau_\nu^{-1}(m)f(k) \quad \forall \ m \in K_\Xi.$$

Let $\Lambda$ be the map from $B(K/K_\Xi, L_\xi)$ into the space $B(K/K_\Xi)$ of all hyperfunctions on the Shilov boundary $K/K_\Xi$ defined by

$$\Lambda f(k) = \tau_\nu(k)f(k).$$

Then $\Lambda$ is a $K$-isomorphism.

Using $\Lambda$ the Poisson transform (denoted again by $P_{\lambda,\nu}$) of an element $f \in B(K/K_\Xi)$ is given by

$$P_{\lambda,\nu}f(g) = \int_K e^{-(\lambda + \eta)\rho_0(H_\Xi(g^{-1}k))} f(k) \tau_\nu(k^{-1}k(g^{-1}k)) dk, \quad (4.4)$$
4.2 The Hua operator.

If $(\theta, V)$ is a finite dimensional representation of the compact group $K$, we denote by $C^\infty(G/K, \theta)$ the space of $C^\infty$-sections of the homogeneous vector bundle on $G/K$ associated to $\theta$.

Let $E_i$ be a basis of $\mathfrak{p}_+$ and $E^*_i$ be the dual basis of $\mathfrak{p}_-$ with respect to the Killing form $B$.

Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. We consider the element of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes \mathfrak{k}_{\mathbb{C}}$ defined by

$$H = \sum_{i,j} E_i E^*_j \otimes [E_j, E^*_i].$$

Then $H$ defines a homogeneous differential operator from the space $C^\infty(G/K, \tau^\nu)$ to the space $C^\infty(G/K, \tau^\nu \otimes \text{Ad}_K |_{\mathfrak{k}_{\mathbb{C}}})$, which does not depend on the choice of the basis.

Let $V$ be a linear subspace of $\mathfrak{k}$ and let $V_c$ be its complexification. We denote by $p$ the orthogonal projection from $\mathfrak{k}_{\mathbb{C}}$ onto $V_c$. We extend $p$ on $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes \mathfrak{k}_{\mathbb{C}}$ by setting

$$p(U \otimes X) = U \otimes p(X) \quad (U \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), X \in \mathfrak{k}_{\mathbb{C}}).$$

We put $H_V = p(H)$.

If $v_j$ is a basis of $V_c$ and $v^*_j$ is the dual basis with respect to $B$. Then

$$H_V = \sum_k U_k \otimes v^*_k,$$

where $U_k$ is the element of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ given by

$$U_k = \sum_j [v_k, E_j] E^*_j,$$

see [11] for more details.

Let $\Lambda^\nu$ be the operator defined on $C^\infty(G/K, \tau^\nu)$ by

$$\Lambda^\nu F(z) = \tau^\nu(U(g : 0)) F(g), \quad z = g.0. \quad (4.5)$$

Then $\Lambda^\nu$ is an isomorphism from $C^\infty(G/K, \tau^\nu)$ onto $C^\infty(D)$. Notice that

$$\tau^\mu(U(g : z)) = [J(g, z)]^\nu_{\mathfrak{p}},$$

where $J(g, z)$ stands for the Jacobian of the transformation $g$.

Now we define an action $T^\nu$ of $G$ on $D$ as follows:

For each $g \in G$ define $T^\nu(g)$ such that the following diagram

$$C^\infty(G/K, \tau^\nu) \xrightarrow{\pi(g)} C^\infty(G/K, \tau^\nu)$$

$$\downarrow \Lambda^\nu \quad \downarrow \Lambda^\nu$$

$$C^\infty(D) \xrightarrow{T^\nu(g)} C^\infty(D)$$

is commutative.

The following result can be proved by direct computations.
Lemma 4.1  i) Let $F \in C^\infty(D)$. For any $g \in G$ we have
\[ T_\nu(g)F(z) = \tau_\nu(U(g^{-1}:z))F(g^{-1}z) \]

ii) The operator $\Lambda_\nu$ is a $G$-intertwining operator from $C^\infty(G/K, \tau_\nu)$ onto $C^\infty(D)$.

Using the above $G$-intertwining operator the Hua operator may be viewed as acting on $C^\infty(D)$. The new operator which we denote by $\mathcal{H}_\nu$ will be given below.

For $f \in C^\infty(G/K, \tau_\nu \otimes \text{Ad})$, we define the function $\Lambda_{\tau_\nu \otimes \text{Ad}}f : G/K \to \mathbb{C} \otimes \mathfrak{t}_\mathbb{C}$ by
\[ \Lambda_{\tau_\nu \otimes \text{Ad}}f(z) = (\tau_\nu \otimes \text{Ad})(U(g:0))f(g), \quad z = g.0. \]

We define the Hua operator $\mathcal{H}_\nu$ on $C^\infty(D)$ such that the following diagram
\[
\begin{array}{ccc}
C^\infty(G/K, \tau_\nu) & \xrightarrow{\Lambda_\nu} & C^\infty(G/K, \tau_\nu \otimes \text{Ad}_K |_{\mathfrak{k}_\nu}) \\
\downarrow & & \downarrow \\
C^\infty(D) & \xrightarrow{\mathcal{H}_\nu} & C^\infty(D, \mathbb{C} \otimes \mathfrak{t}_\mathbb{C})
\end{array}
\]
is commutative.

**Proposition 4.1** Let $F \in C^\infty(D)$. Then we have
\[
\mathcal{H}_\nu F(z) = \tau_\nu(U(g:0)) \sum_{i,j} [\text{Ad}(U(g:0)^{-1})E_i \text{Ad}(U(g:0)^{-1})E_j^*] \Lambda_\nu^{-1} F(g) \otimes [E_j, E_i^*], \quad (4.6)
\]
with $z = g.0$.

Proof. The proof follows by direct computations.

The Hua operator $\mathcal{H}_\nu$ has the following invariance property

**Proposition 4.2** For any $h \in G$ and $F \in C^\infty(D)$ we have
\[
T_\nu(h)(\mathcal{H}_\nu)F(z) = \text{Ad}(U(h^{-1}:z))\mathcal{H}_\nu(T_\nu(h)F)(z).
\]

Proof. Let $g \in G$ such that $g.0 = z$. We have
\[
\mathcal{H}_\nu F(h^{-1}.z) = \tau_\nu(U(h^{-1}g:0))
\times \sum_{i,j} [\text{Ad}(U(h^{-1}g:0)^{-1})E_i \text{Ad}(U(h^{-1}g:0)^{-1})E_j^*] \Lambda_\nu^{-1} F(h^{-1}g) \otimes [E_j, E_i^*],
\]
by Proposition 4.1.

Use the identity (2.2) on the factor of automorphy $U(g:z)$ to write
\[
U(h^{-1}g:0) = U(h^{-1},g.0)U(g:0),
\]
and the identity
\[
\sum_{i,j} \text{Ad}(k^{-1})E_i \text{Ad}(k^{-1})E_j^* f(g) \otimes [E_j, E_i^*] = \sum_{i,j} E_i E_j^* f(g) \otimes [\text{Ad}(k)E_j, \text{Ad}(k)E_i^*], \quad \forall k \in K_c,
\]

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to get
\[
\mathcal{H}_\nu F(h^{-1}, z) = \tau_\nu(U(h^{-1} : z)) \text{Ad}(U(h^{-1} : z)) \tau_\nu(U(g : 0)) \\
\times \sum_{i,j} \text{Ad}(U(g : 0)^{-1}) E_i \text{Ad}(U(g : 0)^{-1}) E_j^* \Lambda_\nu^{-1} F(h^{-1} g) \otimes [E_j, E_i^*].
\] (4.7)

Next by using the fact that \(\Lambda_\nu^{-1} F(h^{-1} g) = \Lambda_\nu^{-1} (T_\nu(h) F)(g)\) as well as the formula (4.6) we obtain
\[
\tau_{-\nu}(U(h^{-1} : z)) \mathcal{H}_\nu F(h^{-1}, z) = \text{Ad}(U(h^{-1} : z)) \mathcal{H}_\nu((T_\nu(h) F)(z),
\]
and the proposition follows.

5 Necessity of the conditions

In this section we will establish that Poisson integrals are eigenfunctions of the Hua operator.

**Lemma 5.1** Let \(F \in C^\infty(G/K, \tau)\). Then
\[
\mathcal{H} F = \left(\frac{\lambda^2 - (\eta - \nu)^2}{4p}\right) F(-IZ_0).
\]

**Proof.** It follows by direct computations.

Let \(du\) be the normalized \(K\)-invariant measure on the Shilov boundary \(S\). By using the operator \(\Lambda_\nu\) we can rewrite the Poisson transform (4.4) from \(B(S)\) to \(C^\infty(D)\)
\[
P_{\lambda,\nu}(z, u) f(u) du,
\]
where
\[
P_{\lambda,\nu}(z, u) = e^{-(\lambda+\eta)\rho_0(H_\Xi(g^{-1} k))} \tau_\nu(U(g : 0)) \tau_\nu(k^{-1} \kappa(g^{-1} k)),
\]
with \(z = g.o \) and \(u = k.E_0\).

Next we introduce a \(K\)-invariant polynomial \(h(z)\) on \(p_+\) whose restriction on \(\sum_{j=1}^r \mathbb{R} E_{\gamma_j}\) is given by
\[
h(\sum_{j=1}^r a_j E_{\gamma_j}) = \Pi_{j=1}^r (1 - a_j^2).
\]

Let \(h(z, w)\) denote its polarization, see [3] for more details.

**Proposition 5.1** i) The Poisson kernels for the homogeneous line bundles are given by
\[
P_{\lambda,\nu}(z, v) = \left[\frac{h(z, z)}{h(z, v)}\right]^{\lambda+\eta-\nu} h(z, v)^{-\nu}.
\]

ii) \(P_{\lambda,\nu}(g, z, g, u) = P_{\lambda,\nu}(z, u) J_g(z) \frac{\nu}{2p} J_g(u) \frac{\lambda+\eta+\nu}{2p} J_g(u)\),
for every \(g \in G\).
Proof. i) Let
\[ \Psi_{\lambda,\nu}(z) = e^{-(\lambda+\eta)\rho_0(H\Xi(g^{-1}))} \tau_\nu(U(g : 0)) \tau_\nu(\kappa(g^{-1})) \quad z = g.0 \quad (5.2) \]
Observe that the right hand side of (5.2) is right \( K \)-invariant, hence \( \Psi_{\lambda,\nu}(z) \) is well defined on \( D \). Define \( \mu_\lambda \in \mathfrak{a}_c^* \) by \( \mu_\lambda = \lambda \rho_0 + \rho \Xi - \rho \). Then
\[ e^{(\lambda \rho_0 + \rho \Xi - \rho)H\Xi(g^{-1})} = e^{(\mu_\lambda + \rho)H(g^{-1})}. \]
Next recall that if \( g = ne^A(g) \kappa_1(g) \) with respect to the decomposition \( G = NAK \), then
\[ A(g) = -H(g^{-1}), \kappa_1(g) = (\kappa(g^{-1}))^{-1}. \]
Henceforth
\[ \Psi_{\lambda,\nu}(g) = e^{(\mu_\lambda + \rho)A(g)} \tau_\nu(U(g : 0)) \tau_\nu(\kappa_1(g)). \quad (5.3) \]
But the right hand-side of (5.3) is nothing but the generalized Harish-Chandra \( c \)-function \( e_{\lambda,\nu} \) on \( G/K \), introduced in [18]. Accordingly to [18], in the Siegel domain realization \( T_\Omega \) of \( G/K \) the function \( e_{\lambda,\nu} \) is given by
\[ \tilde{e}_{\lambda,\nu}(w) = \Delta(\lambda \rho_0 + \rho \Xi - \rho) \Delta(\omega(\gamma(z)))^{-\nu}, \quad w \in T_\Omega. \]
In above \( \Delta \) denotes the Koecher norm function on \( \mathfrak{p}_+ \), and \( \omega(w) = \frac{w + \bar{w}}{2} \), see [3] for more details. Let \( \gamma \) be the Cayley transform from \( D \) onto the Siegel realization \( T_\Omega \) of \( G/K \). Then we have
\[ \Psi_{\lambda,\nu}(z) = \Delta(e - z)^{-\nu} \Delta(\omega(\gamma(z))) \Delta(\omega(\gamma(z)))^{-\nu}, \]
and since
\[ h(z, z) = \Delta(e - z) \Delta(\omega(\gamma(z))) \Delta(e - z), \]
we get
\[ \Psi_{\lambda,\nu}(z) = \left[ \frac{h(z, z)}{h(z, e)} \right]^{-\nu} h(z, e)^{-\nu}. \]
Observing that \( P_{\lambda,\nu}(z, v) = \Psi_{\lambda,\nu}(k^{-1}g) \) and that \( h(z, w) \) is \( K \) bi-invariant we obtain the desired result.

ii) The identity (5.1) is easily derived from the following identity on the Jordan polynomial \( h(z, w) \)
\[ h(g.z, g.w) = J_g(z)^{\frac{1}{2}} h(z, w) J_g(u)^{\frac{1}{2}}, \]
and the proof of Proposition 5.1 is finished.

It follows from above that the Poisson transform can be now rewritten explicitly as
\[ P_{\lambda,\nu}f(z) = \int_S \left( \frac{h(z, z)}{h(z, u)^2} \right)^{\frac{\lambda + \eta - \nu}{2}} h(z, u)^{-\nu} f(u) du, \quad (5.4) \]

Remark 5.1 This result has been proved recently by Koranyi [9] by using a different method. Formula (5.4) agrees with Theorem 4.2 of [9], with \( l = \nu, q = \eta \) and \( s = \eta - \lambda \).

Now we are ready to prove the main result of this section.
**Proposition 5.2** Let \( F = P_{\lambda,\nu}f \) with \( f \in B(G/P, L_{\lambda,\xi}) \). Then

\[
\mathcal{H}F = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} F(-iZ_0).
\] (5.5)

Proof. In view of Lemma 5.1 and the invariance property (5.1) of the Poisson kernel, it suffices to show that \( \mathcal{H}P_{\lambda,\nu}(z,v)|_{z=0} = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} (-iZ_0) \).

To do so let \( E_j \) be an orthonormal basis of \( p_+ \) and \( E^*_j \) be a dual basis of \( p_- \) with respect to \( B \) (for example \( \{\tilde{E}_{\alpha}\}_{\alpha \in \Phi^+} \) and \( \{\tilde{E}_{-\alpha}\}_{\alpha \in \Phi^+} \) are such basis). Let \( z_1, ..., z_n \) be coordinates for \( p_+ \).

Regarding \( K \)-invariant functions on \( G \) as functions on \( D \) and vice versa, we have

\[
E_i E^*_j F(e) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} F(0).
\]

We know that

\[
h(z,v)^{-\nu} = 1 + \nu < z, v > \text{ + higher order homogeneous terms},
\]

where \( < z, v > = -\frac{1}{2p} B(z, \tau v) \). A simple computation gives

\[
\frac{\partial^2}{\partial z_i \partial \bar{z}_j} P_{\lambda,\nu}(z,v)|_{z=0} = \frac{(\lambda + \eta - \nu)(\lambda + \eta + \nu)}{4} v_j \bar{v}_i - \frac{\lambda + \eta - \nu}{2} \delta_{ij}.
\]

Therefore

\[
\mathcal{H}P_{\lambda,\nu}(z,v)|_{z=0} = \sum_{i,j} [E_j, E^*_i] \left( \frac{(\lambda + \eta - \nu)(\lambda + \eta + \nu)}{4} v_j \bar{v}_i - \frac{\lambda + \eta - \nu}{2} \delta_{ij} \right)
\]

\[
= \frac{(\lambda + \eta - \nu)(\lambda + \eta + \nu)}{4} [v, \bar{v}] - \frac{\lambda + \eta - \nu}{2} \sum_{\alpha \in \Phi^+} \tilde{H}_\alpha.
\]

Notice that \( [v, \bar{v}] = [Ad(k) \tilde{E}_0, Ad(k) \tilde{E}_0] = \sum_{j=1}^r \tilde{H}_{\gamma_j} \).

Since

\[
\frac{1}{\eta} \sum_{\alpha \in \Phi^+} \tilde{H}_\alpha = \sum_{j=1}^r \tilde{H}_{\gamma_j},
\]

we get

\[
\mathcal{H}P_{\lambda,\nu}(z,v)|_{z=0} = \frac{(\lambda + \eta - \nu)(\lambda - \eta + \nu)}{4} \tilde{H}_0,
\]

as

\[
\sum_{j=1}^r \tilde{H}_{\gamma_j} = -\frac{i}{p} Z_0,
\]

the result follows.
6 The Hua eigensections

In this section we shall consider the subsystem

\[ \mathcal{H}_q F = \left( \frac{\lambda^2 - (\eta - \nu)^2}{4p} \right) F(-iZ_0), \]

and prove the main result of this section.

**Theorem 6.1** Let \( F \in B(G/K, \tau) \) such that

\[ \mathcal{H}_q F = \left( \frac{\lambda^2 - (\eta - \nu)^2}{4p} \right) F(-iZ_0). \]

Then \( F \in \mathcal{A}(G/K, \mathcal{M}_{\mu, \lambda, \nu}). \)

Most of the Proof of Theorem 6.1 consists in proving the following

**Theorem 6.2** Let \( F \) be a \( \tau_\nu \)-spherical function on \( G \) satisfying

\[ \mathcal{H}_h F = \lambda^2 - (\eta - \nu)^2 \]

(6.1)

Then up to a constant multiple we have

\[ F(a_t) = \prod_{j=1}^r (1 - \tanh^2 t_j)^{\frac{\lambda + \eta}{2}} \, 2F_1^{(m)} \left( \frac{\lambda + \eta - \nu}{2}, \frac{\lambda + \eta + \nu}{2}; \eta; \tanh^2 t_1, \ldots, \tanh^2 t_r \right), \]

where \( a_t = \exp(\sum_{j=1}^r t_j X_{\gamma_j}) \).

In particular the elementary spherical function \( \Phi_{\mu, \lambda, \nu} \) is given by

\[ \Phi_{\mu, \lambda, \nu}(a_t) = \prod_{j=1}^r (1 - \tanh^2 t_j)^{\frac{\lambda + \eta}{2}} \, 2F_1^{(m)} \left( \frac{\lambda + \eta - \nu}{2}, \frac{\lambda + \eta + \nu}{2}; \eta; \tanh^2 t_1, \ldots, \tanh^2 t_r \right). \] (6.2)

In the above \( 2F_1^{(m)}(\alpha, \beta; \gamma; x_1, \ldots, x_r) \) is the generalized Gauss hypergeometric function, see \cite{17} for more details.

**Corollary 6.1** let \( \lambda \in \mathbb{C} \) and let \( \nu \in \mathbb{Z} \). Then we have

\[ \int_S \left[ \frac{h(z, z)}{|h(z, u)|^2} \right]^{\frac{\lambda + \eta - \nu}{2}} h(z, u)^{-\nu} du = h(z, z) \frac{\lambda + \eta - \nu}{2} \, 2F_1^{(m)} \left( \frac{\lambda + \eta - \nu}{2}, \frac{\lambda + \eta + \nu}{2}; \eta; \tanh^2 t_1, \ldots, \tanh^2 t_r \right), \]

z = Ad(k) \sum_{j=1}^r \tanh t_j E_{\gamma_j}.

**Proof.** We first note that

\[ \int_S \left[ \frac{h(z, z)}{|h(z, u)|^2} \right]^{\frac{\lambda + \eta - \nu}{2}} h(z, u)^{-\nu} du = \tau_\nu(U(g : 0)) \Phi_{\mu, \lambda, \nu}(g), \quad z = g.0 \] (6.4)
We have
\[ \tau_{\nu}(U(g : 0))\Phi_{\mu,\lambda}(g) = \tau_{\nu}(U(a_t : 0))\Phi_{\mu,\lambda}(a_t), \]
if \( g = h a_t k \) with respect to the Cartan decomposition \( G = K C l(A^+) K \).

Now recall that the \( P^+ K \_ P^- \) decomposition of \( a_t = \exp(\sum_{j=1}^r t_j X_{\gamma_j}) \)
is
\[ a_t = \exp(\sum_{j=1}^r \tanh t_j E_{\gamma_j}) \exp(\sum_{j=1}^r (\log \cosh t_j) H_{\gamma_j}) \exp(\sum_{j=1}^r (\tanh t_j E_{-\gamma_j}). \]

Hence
\[ \tau_{\nu}(U(a_t : 0)) = (\prod_{j=1}^r \cosh t_j)^{\nu}. \]

We therefore obtain
\[ \int_S h(z, z) \frac{h(z, u)^{\lambda + \eta - \nu}}{|h(z, u)|^2} du = (\prod_{j=1}^r \cosh t_j)^{\nu} \Phi_{\mu,\lambda}(x) F(g) \]
and from (6.2) we obtain the identity (6.3).

**Proof of Theorem 6.1.** Let \( F \in B(G/K, \tau_{\nu}) \) such that
\[ \mathcal{H}_q F = \frac{\chi^2 - (\eta - \nu)^2}{4p} F(-iZ_0). \quad (6.5) \]

Fix \( g \in G \) and put \( F_g(x) = \int_K F(gkx)\tau_{\nu}(k)dk \). Then \( F_g \) satisfies
\[ F_g(k_1 x k_2) = \tau_{\nu}^{-1}(k_1) F(g(x))\tau_{\nu}^{-1}(k_2), \]
and the system (6.1). Therefore \( F_g = F_g(e)\Phi_{\mu,\lambda,\nu} \), by Theorem 6.2.

That is
\[ \int_K F(gkx)\tau_{\nu}(k)dk = \Phi_{\mu,\lambda,\nu}(x) F(g). \]

As in the case \( \nu = 0 \) we can prove that the above functional equation characterizes the joint eigensections of \( D_{\nu}(G/K) \), so \( F \in \mathcal{A}(G/K, \mathcal{M}_{\mu,\lambda,\nu}) \) and the proof of Theorem 6.1 is finished.

To prove Theorem 6.2 we shall need an explicit form of the radial components of the operator \( \mathcal{H}_q \).

**6.1 Radial components.**

Recall that function \( F \) on \( G \) is \( \tau_{-\nu} \)-spherical if it satisfies
\[ F(k_1 g k_2) = \tau_{\nu}(k_2)^{-1} F(g)\tau_{\nu}(k_1)^{-1}. \]

Let \( \overline{F} \) denote the restriction to \( A^+ \) of a \( \tau \)-spherical function \( F \). By the Cartan decomposition \( G = K C l(A^+) K \) a \( \tau_{-\nu} \)-spherical function \( F \) is essentially determined by \( \overline{F} \).
For $U \in U(g_\mathfrak{c})$, we denote by $\Delta_\tau(U)$ its $\tau$-radial component. That is $\Delta_\tau(U) \in U(a_\mathfrak{c})$ and satisfies

$$(UF) = \Delta_\tau(U)F,$$

for every $\tau_{-\nu}$-spherical function $F$ on $G$.

Let $H^*_{\gamma_k}$ ($k = 1, \ldots, r$) be the basis of $\mathfrak{h}$ which is dual to $H_{\gamma_k}$. Then we have

$$\mathcal{H}_\mathfrak{h} = \sum_{k=1}^r U_k \otimes H^*_{\gamma_k},$$

where the components $U_k$ are given by

$$U_k = \sum_{\alpha \in \Phi^+} \alpha(H_{\gamma_k}) E_\alpha E^*_\alpha.$$

We have

$$\Delta_\tau(U_k) = \sum_{\alpha \in \Phi^+} \alpha(H_{\gamma_k}) \Delta_\tau(E_\alpha E^*_\alpha).$$

Let $F$ be a $\tau_{-\nu}$-spherical function on $G$. Then the system $\mathcal{H}_\mathfrak{h}F = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} F, (-i)Z_0$ reads as

$$(\Delta_\tau(U_k)) F(a) = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} F,$$  \hspace{1cm} (6.6)

for $k = 1, 2, \ldots, r$.

Recall that the dual basis $E^*_\alpha$ of $\mathfrak{p}_-$ is given by

$$E^*_\alpha = \frac{\langle \alpha, \alpha \rangle}{2} E_{-\alpha}.$$

In view of Proposition 2.1, $U_k$ can be rewritten as

$$U_k = |\gamma_k|^2 E_{\gamma_k} E_{-\gamma_k} + \frac{1}{2} \sum_j \sum_{\alpha \in \Phi^+_j} <\alpha, \alpha > E_\alpha E_{-\alpha}.$$

To compute the $\tau$-radial parts $\Delta_\tau(U_k)$ explicitly, we should determine $\Delta_\tau(E_\alpha E_{-\alpha})$ for all positive noncompact roots $\alpha$.

The representation $\tau_\nu$ of $K$ induces differentiated representation of the Lie algebra $\mathfrak{k}$. We shall denote this representation by the same latter $\tau_\nu$.

**Lemma 6.1** Let $F$ be a $\tau_\nu$-spherical function on $G$ and let $\alpha \in \Phi_\pm$.

i) If $\alpha \in \Gamma$ or $\alpha \in \Phi^+_i$ with $\alpha \neq \tilde{\alpha}$, then

$$\tau_\nu(-iH_\alpha)F(a) = -i\nu$$

ii) If $\alpha \in \Phi^+_i$ with $\alpha = \tilde{\alpha}$, then

$$\tau_\nu(-iH_\alpha)F(a) = -2i\nu$$
Proof. We have
\[ B(-iH_\alpha, Z_0) = \frac{2}{|\alpha|^2}, \]  
(6.7)
i) If \( \alpha \in \gamma_k \) for some \( k = 1, \ldots, r \) or \( \alpha \in \Phi_{ij}^+ \) with \( \alpha \neq \tilde{\alpha} \) then
\[ < Z - iH_\alpha, Z_0 > = 0, \]
by iv) of Proposition 2.1. Thus \( Z - iH_\alpha \in t_s \). Therefore
\[ \tau_\nu^{-1}(\exp t(-iH_\alpha) = e^{-i\nu t}, \]
and i) follows.

ii) In the case \( \alpha \in \Phi_{ij}^+ \) with \( \alpha = \tilde{\alpha} \) then
\[ < Z - \frac{i}{2}H_\alpha, Z_0 > = 0, \]
from which we deduce that \( \tau_\nu^{-1}(\exp t(\frac{i}{2}H_\alpha) = e^{-i\nu t}. \) Hence ii).

Proposition 6.1
\[ 4\Delta_r(E_{\gamma_k} E_{-\gamma_k}) = X_{\gamma_k}^2 + 2\coth 2t_k X_{\gamma_k} - \nu^2 \tanh^2 t_k + 2\nu. \]

Proof. Write
\[ E_{\gamma_k} E_{-\gamma_k} = \frac{1}{4}(X_{\gamma_k}^2 + Y_{\gamma_k}^2 + i[X_{\gamma_k}, Y_{\gamma_k}]}, \]
and since
\[ [X_{\gamma_k}, Y_{\gamma_k}] = -2iH_{\gamma_k}, \]
we get
\[ E_{\gamma_k} E_{-\gamma_k} = \frac{1}{4}(X_{\gamma_k}^2 + Y_{\gamma_k}^2 + i(-2iH_{\gamma_k})). \]
Hence we need to compute only \( \Delta_r(Y_{\gamma_k}^2) \).

Let \( a = \exp \sum_{j=1}^r t_j X_{\gamma_j} \). Then we have
\[ Ad(a^{-1})iH_{\gamma_k} = (\cosh 2t_k iH_{\gamma_k} + (\sinh 2t_k)Y_{\gamma_k}. \]
Thus
\[ Y_{\gamma_k}^2 = (\coth 2t_{\gamma_k})(iH_{\gamma_k})^2 + \sinh^{-2} 2t_k(Ad(a^{-1})iH_{\gamma_k})^2 \]
\[ - \coth 2t_{\gamma_k} \sinh^{-1} 2t_k(iH_{\gamma_k}Ad(a^{-1})iH_{\gamma_k}) + (Ad(a^{-1})iH_{\gamma_k})iH_{\gamma_k}. \]
Observe that
\[ [Ad(a^{-1})iH_{\gamma_k}, iH_{\gamma_k}] = (2\sinh 2t_k)X_{\gamma_k}, \]
since \( [Y_{\gamma_k}, iH_{\gamma_k}] = 2X_{\gamma_k}. \)
Next, since \( F \) is a \( \tau_\nu \)-spherical function we have
\[ (Ad(a^{-1})iH_{\gamma_k})iH_{\gamma_k} F(a) = -\nu^2 F(a), \]
hence
\[ (iH_{\gamma_k}Ad(a^{-1})iH_{\gamma_k})F(a) = -2\sinh 2t_k X_{\gamma_k} - \nu^2, \]
from which we deduce that
\[ Y_{\gamma_k}^2 = 2\coth 2t_k X_{\gamma_k} - \nu^2 \tanh^2 t_k, \]
and the proof of Proposition 6.1 is finished.
Lemma 6.2 \textit{Let }$\alpha \in \Phi^+$\textit{ such that }$\alpha \sim \frac{2\gamma_i + \gamma_j}{2}$ (i $\neq$ j), with $\alpha \neq \tilde{\alpha}$. Then we have}

\[ \Delta_r( E_\alpha E_{-\alpha} + E_{\tilde{\alpha}} E_{-\tilde{\alpha}}) = \frac{1}{2} [\coth(t_i + t_j)(X_{\gamma_i} + X_{\gamma_j}) + \coth(t_i - t_j)(X_{\gamma_i} - X_{\gamma_j}) + 2 \nu]. \]

We first prepare the following Lemma

Lemma 6.2 \textit{Proposition} 6.2 \textit{[11]. Let}

\[ \text{Ad}(a^{-1})U_1 = \cosh(t_i + t_j)U_1 - \sinh(t_i + t_j)(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}), \]
\[ \text{Ad}(a^{-1})U_2 = \cosh(t_i + t_j)U_2 - \sinh(t_i + t_j)(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}}), \]
\[ \text{Ad}(a^{-1})V_1 = \cosh(t_i - t_j)V_1 - \sinh(t_i - t_j)(X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}}), \]
\[ \text{Ad}(a^{-1})V_2 = \cosh(t_i - t_j)V_2 - \sinh(t_i - t_j)(Y_\alpha + \epsilon_\alpha Y_{\tilde{\alpha}}). \]

To prove the above lemma we need the following result from [11]

Lemma 6.3 \textit{[11]. Let }$\alpha \in \Phi^+_{ij}$. \textit{Then there exists }$U_1, U_2$\textit{ in }$q$\textit{ and }$V_1, V_2$\textit{ in }$l$\textit{ such that}

\[ i) \text{ad}(X_{\gamma_k})U_1 = (\delta_{ik} + \delta_{jk})(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}), \]
\[ ii) \text{ad}(X_{\gamma_k})(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}) = (\delta_{ik} + \delta_{jk})U_1, \]
\[ iii) \text{ad}(X_{\gamma_k})U_2 = (\delta_{ik} + \delta_{jk})(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}}), \]
\[ iv) \text{ad}(X_{\gamma_k})(X_\alpha + \epsilon_\alpha X_{\tilde{\alpha}}) = (\delta_{ik} + \delta_{jk})U_2, \]
\[ v) \text{ad}(X_{\gamma_k})V_1 = (\delta_{ik} - \delta_{jk})(X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}}), \]
\[ vi) \text{ad}(X_{\gamma_k})(X_\alpha - \epsilon_\alpha X_{\tilde{\alpha}}) = (\delta_{ik} - \delta_{jk})V_1, \]
\[ vii) \text{ad}(X_{\gamma_k})V_2 = (\delta_{ik} - \delta_{jk})(Y_\alpha + \epsilon_\alpha Y_{\tilde{\alpha}}), \]
\[ viii) \text{ad}(X_{\gamma_k})(Y_\alpha + \epsilon_\alpha Y_{\tilde{\alpha}}) = (\delta_{ik} - \delta_{jk})V_2. \]

Proof of Lemma 6.2. We shall prove only i), the others assertions can be proved in a similar way.

\[ \text{we have} \]
\[ \text{Ad}(\exp(-\sum_{k=1}^{r} t_k X_{\gamma_k}))U_1 = \exp(\text{ad}(-\sum_{k=1}^{r} X_{\gamma_k}))U_1. \]

By i) of the above lemma, we have

\[ \text{ad}(-\sum_{k=1}^{r} X_{\gamma_k})^{2n}U_1 = (t_i + t_j)^{2n}U_1, \]

and

\[ \text{ad}(-\sum_{k=1}^{r} X_{\gamma_k})^{2n+1}U_1 = (t_i + t_j)^{2n+1}(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}), \]

from which we get

\[ \text{Ad}(a^{-1})U_1 = \cosh(t_i + t_j)U_1 - \sinh(t_i + t_j)(Y_\alpha - \epsilon_\alpha Y_{\tilde{\alpha}}). \]
This finishes the proof.
Now we prove Proposition 6.2 giving the \( \tau \)-radial part of \( E_\alpha E_{-\alpha} + E_{\bar{\alpha}} E_{-\bar{\alpha}} \).

**Proof of Proposition 6.2.** First observe that
\[
4(E_\alpha E_{-\alpha} + E_{\bar{\alpha}} E_{-\bar{\alpha}}) = X_\alpha^2 + X_{\bar{\alpha}}^2 + Y_\alpha^2 + Y_{\bar{\alpha}}^2 + i(2iH_\alpha + 2iH_{\bar{\alpha}}).
\]
Next, since
\[
X_\alpha^2 + X_{\bar{\alpha}}^2 = \frac{1}{2}[(X_\alpha + \epsilon_\alpha X_{\bar{\alpha}})^2 + (X_\alpha - \epsilon_\alpha X_{\bar{\alpha}})^2],
\]
we will compute \( \Delta_\tau((X_\alpha \pm \epsilon_\alpha X_{\bar{\alpha}})^2) \).
To this end, consider the element \( U_2 \) given by Lemma 6.3 and let \( a = \exp(\sum_{k=1}^r t_j X_{\gamma_j}) \). Then we have
\[
Ad(a^{-1})U_2 = \cosh(t_i + t_j)U_2 - \sinh(t_i + t_j)(X_\alpha + \epsilon X_{\bar{\alpha}}),
\]
from which we get
\[
\sinh^2(t_i + t_j)(X_\alpha + \epsilon X_{\bar{\alpha}}) = \cosh^2(t_i + t_j)U_2 - \cosh(t_i + t_j)U_2 Ad(a^{-1})U_2 - \cosh(t_i + t_j)(Ad(a^{-1})U_2)U_2 + (Ad(a^{-1})U_2)^2.
\]
Recall from [11], that \( U_2 = i(Q_\alpha - \bar{Q}_\alpha) \) where
\[
Q_\alpha = N_{\alpha,-\gamma}E_{\alpha,-\gamma} + N_{\alpha,-\gamma}E_{\alpha,-\gamma},
\]
from which we obtain \( <U_2, Z_0> = 0 \), therefore \( U_2 \in \mathfrak{q} \cap \mathfrak{t} \) and \( \tau_\nu(U_2) = 0 \).
By ii) in Lemma 6.2, we have
\[
[Ad(a^{-1})U_2), U_2] = -2\sinh(t_i + t_j)(X_\gamma + X_{\gamma_j}),
\]
which gives us
\[
\Delta_\tau((X_\alpha + \epsilon X_{\bar{\alpha}})^2) = 2\coth(t_i + t_j)(X_\gamma + X_{\gamma_j}).
\]
Similarly by considering \( V_1 \) and noticing that \( V_1 \in \mathfrak{t} \cap \mathfrak{t} \) we get
\[
\Delta_\tau((X_\alpha - \epsilon X_{\bar{\alpha}})^2) = 2\coth(t_i - t_j)(X_\gamma - X_{\gamma_j}).
\]
Finally since \( -2iH_\alpha F(a) = -2i\nu F(a) \), by Lemma 6.1, the result follows.

**Proposition 6.3** Let \( \alpha \in \Phi^+ \) such that \( \alpha = \frac{\gamma_i + \gamma_j}{2} \) (\( i \neq j \)), that is \( \alpha = \bar{\alpha} \). Then we have
\[
\Delta_\tau(E_\alpha E_{-\alpha}) = \frac{1}{2}[\coth(t_i + t_j)(X_\gamma + X_{\gamma_j}) + \coth(t_i - t_j)(X_\gamma - X_{\gamma_j}) + 2\nu].
\]

**Proof.** We first suppose that \( \epsilon_\alpha = 1 \). Accordingly to Proposition 13 and Proposition 14 in [11], \( U_1 = V_1 = 0 \).
We compute first \( \Delta_\tau(X_\alpha^2) \) and \( \Delta_\tau(Y_\alpha^2) \) in the case \( \epsilon_\alpha = 1 \).
We have
\[
Ad(a^{-1})U_2 = \cosh(t_i + t_j)U_2 - 2\sinh(t_i + t_j)X_\alpha,
\]
hence
\[
4\sinh^2(t_i + t_j)X_\alpha^2 = \cosh^2(t_i + t_j)U_2^2 + (Ad(a^{-1})U_2)^2 - \cosh(t_i + t_j)(U_2 Ad(a^{-1})U_2 + (Ad(a^{-1})U_2)U_2).
\]
Noticing that

\[ U_2 Ad(a^{-1}) U_2 = -[Ad(a^{-1}) U_2, U_2] \]
\[ = -8 \sinh(t_i + t_j)(X_{\gamma_i} + X_{\gamma_j}), \]
we get

\[ \Delta_r(X_\alpha^2) = 2 \coth(t_i + t_j)(X_{\gamma_i} + X_{\gamma_j}), \]
and similarly

\[ \Delta_r(Y_\alpha^2) = 2 \coth(t_i - t_j)(X_{\gamma_i} - X_{\gamma_j}). \]

in the case \( \epsilon_\alpha = -1 \), analogous computations give

\[ \Delta_r(Y_\alpha^2) = 2 \coth(t_i + t_j)(X_{\gamma_i} + X_{\gamma_j}), \]
and

\[ \Delta_r(X_\alpha^2) = 2 \coth(t_i - t_j)(X_{\gamma_i} - X_{\gamma_j}). \]

To finish the proof of Proposition 6.3, notice that in the case \( \alpha = \tilde{\alpha} \)

\[ -i2H_\alpha F(a) = -4i\nu. \]

**Proposition 6.4** The \( \tau \)-radial part of the operator \( U_k \) is given by

\[
\frac{4}{|\gamma_k|^2} \Delta_r(U_k) = \frac{\partial^2}{\partial t_k^2} + 2 \coth 2t_k \frac{\partial}{\partial t_k} - \nu^2 \tanh^2 t_k + 2\nu \\
+ \frac{m}{2} \sum_{j \neq k} \left[ \coth(t_j + t_k)(\frac{\partial}{\partial t_j} + \frac{\partial}{\partial t_k}) + \coth(t_j - t_k)(\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_k}) + 2\nu \right].
\]

Proof. Recall that

\[ \Delta_r(U_k) = \sum_{\alpha \in \Phi^+} \alpha(H_{\gamma_k}) \Delta_r(E_\alpha E_\alpha^*). \]

By using (i), (ii) and (iii) of proposition 2.1, we obtain

\[ \Delta_r(U_k) = \langle \gamma_k, \gamma_k \rangle \Delta_r(E_{\gamma_k} E_{-\gamma_k}) + \frac{1}{2} \sum_{j \neq k} \sum_{\alpha \in \Phi_{\gamma_k}^+} < \alpha, \alpha > \Delta_r(E_{\alpha} E_{-\alpha}), \]

from which we deduce

\[
\frac{4}{|\gamma_k|^2} \Delta_r(U_k) = \Delta_r(E_{\gamma_k} E_{-\gamma_k}) + \frac{m}{2} \sum_{j \neq k} \left[ \coth(t_j + t_k)(X_{\gamma_j} + X_{\gamma_k}) + \coth(t_i - t_j)(X_{\gamma_j} - X_{\gamma_k}) + 2\nu \right],
\]

by the results of Proposition 6.1, Proposition 6.2 and Proposition 6.3.

Next consider a coordinate system \( t = (t_1, ..., t_r) \in \mathbb{R}^r \mapsto \exp(\sum_{j=1}^r t_j X_{\gamma_j}) \in A \), such that the Weyl group \( W \) acts as the group of all permutations and sign changes of the coordinates \( (t_1, ..., t_r) \) to get the result. This finishes the proof of Proposition 6.4.
6.2 Proof of Theorem 6.2

In this subsection we give the proof Theorem 6.2 which is the main step in the proof of our main result.

Proof of Theorem 6.2. Let $F$ be a $\tau-\nu$-spherical function satisfying the system (6.1). Recall from subsection 6.1 (equation (6.6)) that $F$ satisfies

$$(\Delta_\tau(U_k))F(a) = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p}F, \quad k = 1, \ldots, r$$

Let

$$\phi(t_1, \ldots, t_r) = \prod_{j=1}^{r} \cosh t_j)^\nu F(t_1, \ldots, t_r).$$

It follows from Proposition 6.4 that the function $\phi(t_1, \ldots, t_r)$ satisfies the system of differential equations

$$\frac{\partial^2 \phi}{\partial t_k^2} + 2 \coth 2t_k \frac{\partial \phi}{\partial t_k} - 2\nu \tanh t_k \frac{\partial \phi}{\partial t_k} + \frac{m}{2} \sum_{j=1}^{r} \frac{1}{(\sinh^2 t_j - \sinh^2 t_k)}(\sinh 2t_j \frac{\partial \phi}{\partial t_j} - \sinh 2t_k \frac{\partial \phi}{\partial t_k})$$

$$= \frac{(\lambda^2 - (\eta - \nu)^2)}{4} \phi,$$

for all $k = 1, \ldots, r$.

Put $x_i = -\sinh^2 t_i$ and $\psi(x_1, \ldots, x_r) = \phi((t_1, \ldots, t_r))$. Then the function $\psi$ satisfies the system

$$x_k(1-x_k) \frac{\partial^2 \psi}{\partial x_k^2} + (1 - (2 - \nu)x_k) \frac{\partial \psi}{\partial x_k} - \frac{m}{2} \sum_{j \neq k} \frac{x_j(1-x_j)}{x_k - x_j} \frac{\partial \psi}{\partial x_j} - \frac{x_k(1-x_k)}{x_k - x_j} \frac{\partial \psi}{\partial x_k}$$

$$= \frac{(\eta - \nu)^2 - \lambda^2}{4} \psi,$$

for all $k = 1, \ldots, r$.

Since $\mathfrak{z} \subset \mathfrak{b}$, $F$ is an eigenfunction of the Laplace-Beltrami operator (see the proof of Theorem 6.1). Hence $F$ is analytic. Being $\tau$-spherical $F$ is $W$-invariant. Since the Weyl group $W$ acts as the group of all permutations and sign changes of the coordinates $(t_1, \ldots, t_r)$, it follows that $\psi$ is a symmetric function of $x_1, \ldots, x_r$ and analytic at $x_1 = \ldots = x_r = 0$.

From Theorem 2.1 [17] we deduce that

$$\psi(x_1, \ldots, x_r) = c \ F_1^{(m)}\left(\frac{\lambda + \eta - \nu}{2}, -\frac{\lambda + \eta - \nu}{2}, \eta; x_1, \ldots, x_r\right),$$

where $c$ is some numerical constant.

Thus

$$F(a_t) = \prod_{j=1}^{r} \cosh t_j)^{-\nu} \ F_1^{(m)}\left(\frac{\lambda + \eta - \nu}{2}, -\frac{\lambda + \eta - \nu}{2}, \eta; -\sinh^2 t_1, \ldots, -\sinh^2 t_r\right).$$
Next using the following formula on the generalized Gauss hypergeometric function \[ \text{II7} \]

\[ 2F_1^{(m)}(\alpha, \beta, \gamma; y_1, \ldots, y_r) = \prod_{j=1}^{r} (1 - y_j)^{a} \quad 2F_1^{(m)}(\alpha, \gamma - \beta, \gamma; y_1, \ldots, y_r - 1). \]

we get

\[ F(a_t) = \prod_{j=1}^{r} (1 - \tanh^2 t_j)^{\frac{\lambda + \eta - \nu}{2}} \quad 2F_1^{(m)}\left(\frac{\lambda + \eta - \nu}{2}, \frac{\lambda + \nu}{2}, \eta; \tanh^2 t_1, \ldots, \tanh^2 t_r, \right) \]

and the proof of Theorem 6.2 is finished.

7 The sufficiency of the conditions

In this section we shall complete the proof of our main result. Let \( F \in B(G/K, \tau_\nu) \) such that \( F \) satisfies the Hua system

\[ \mathcal{H}F = \frac{(\lambda^2 - (\eta - \nu)^2)}{4p} F(-i)Z_0. \]  

(7.1)

Then \( F \in A(G/K, \mathcal{M}_{\mu, \lambda, \nu}) \), by Theorem 6.1.

By Theorem 3.1, it suffices to prove that the boundary value \( \tilde{\beta}_{\mu, \lambda, \nu} F \) is in \( B(G/P_\Lambda, L_{\lambda, \nu}) \). To do so, we will show that the induced equations of the subsystem \( \mathcal{H}_s F = 0 \) of (7.1) for boundary values on \( G/P \) characterize \( B(G/P_\Lambda, L_{\lambda, \nu}) \), see [7] for more details on induced equations that boundary values satisfy. The method of the proof for \( \nu = 0 \) in [15] can be generalized to our situation, see also [11]. Below, we give an outline of the proof. Let \( s \) be the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{q} \) with respect to \( B \). Denote by \( C^+ \) the set of positive compact roots \( \beta \) such that \( \beta \sim \frac{\gamma_j - \gamma_i}{2} \) for some \( j > i \). Let \( \{S_{\beta}^*, \beta \in C^+\} \) be the basis of \( s_c \) that is dual to \( \{S_\beta, \beta \in C^+\} \), with respect to \( B \). We have

\[ \mathcal{H}_s = \sum_{\beta \in C^+} U_\beta \otimes S_{\beta}^*, \]

where \( U_\beta \in U(\mathfrak{g}_C) \) is given by

\[ U_\beta = \sum_{\alpha \in \Phi^+} [S_\beta, E_\alpha] E_\alpha^*. \]

The condition \( \mathcal{H}_s F = 0 \) implies

\[ U_\beta F = 0, \quad \forall \beta \in C^+, \]

Consider the Poincaré-Birkhoff-Witt Theorem decomposition

\[ U(\mathfrak{g}_C) = U(\mathfrak{n}_C^- + \mathfrak{a}_C) + \sum_{X \in \mathfrak{c}} U(\mathfrak{g}_C)(X - \tau_\nu(X)), \]  

(7.2)

and let \( \Pi_1 \) denote the projection of \( U(\mathfrak{g}_C) \) to \( U(\mathfrak{n}_C^- + \mathfrak{a}_C) \) with respect to the decomposition (7.2).
For $U \in U(g_{\mathbb{C}})$, let $\tilde{U}$ be the differential operator on $N^- \times \mathbb{R}^r$ defined by $\tilde{U} = P(\Pi_1(U))$ where $P$ denotes the operator defined by $P(X_\alpha) = t^\alpha X_\alpha$ and $P(H_j) = -t_j \frac{\partial}{\partial t_j}$, under the isomorphism from $\mathbb{R}^r$ onto $A$ given by $t \mapsto a(t) = \exp(-\sum_{t_j \neq 0} \log |t_j| H_j)$.

By the Iwasawa decomposition $G = N^- AK$, the restriction map from $G$ to $N^- A$ gives an isomorphism from $B(G/K, \tau_\nu)$ to $B(N^- A)$.

Now $F$ regarded as an element of $B(N^- A)$ satisfies the differential equation

$$\tilde{U}_\beta F = 0, \quad \beta \in C^+.$$

Fix $\beta \in C^+$ such that $\beta \sim \frac{\gamma_i - \gamma_{i-1}}{2}$ ($2 \leq i \leq r$).

Similar computations as in [15], Proposition 4.4 and Proposition 4.5 show that the operator

$$t^{\frac{1}{2}(\beta_i - \beta_{i-1})}\tilde{U}_\beta$$

is well defined on $N^- \times \mathbb{R}$ and has analytic coefficients near $t = 0$, and that the induced equations for the system $\mathcal{H}_s F = 0$ are

$$Adc(E_{-\beta})\beta_{\mu, \nu}(F) = 0, \quad \forall \beta \in C^+; \beta \sim \frac{\gamma_i - \gamma_{i-1}}{2} \quad (2 \leq i \leq r).$$

To conclude recall that the vectors $\{Adc(E_{-\beta}), \beta \sim \frac{\gamma_i - \gamma_{i-1}}{2}\}$ span the root space $g_{\beta_i - \beta_{i-1}}$ and that $\{\frac{\beta_i - \beta_{i-1}}{2}, 2 \leq i \leq r\}$ are the simple roots of $\{\frac{\beta_i - \beta_j}{2}, 1 \leq j < i \leq r\}$. Thus

$$X_\alpha\beta_{\mu, \nu}(F) = 0 \quad \forall \alpha; \quad \alpha = \frac{\beta_i - \beta_j}{2},$$

with $1 \leq j < i \leq r$.

This shows that $\tilde{\beta}_{\mu, \nu} F \in B(G/P_\Xi, L_{\lambda, \nu})$ and the proof the main result of this paper is finished.

References

[1] N. Berline, M. Vergne, Equations de Hua et noyau de Poisson. Lecture Notes in Mathematics, vol. 880, 1-51, Berlin-Heidelberg-New York. Springer 1981.

[2] A. Boussejra, $L^2$-Poisson integral representations of solutions of the Hua system on the bounded symmetric domain $SU(n, n)/S(U(n) \times U(n))$, J. Funct. Anal. 202 (2003) 25-43.

[3] J. Faraut, A. Koranyi, Analysis on symmetric cones, Oxford University Press, New York, 1993.

[4] S. Helgason, Group and Geometric Analysis, Academic Press, New York, 1984.

[5] E. Imamura, K. Okamoto, M. Tsukamoto, A. Yamamori, Eigenvalues of generalized Laplacians for generalized Poisson-Cauchy transforms on classical domains, Hiroshima Math. Journal 39 (2009), 237-275.

[6] K. D. Johnson, A. Koranyi, The Hua operators on bounded symmetric domains of tube type, Ann. of Math, 111 (1980), 589-608.
[7] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, Ann. of Math (2) 111 (1980), no. 3, 589-608.

[8] A. Koranyi, P. Malliavin, Poisson formula and compound diffusion associated to an over determined elliptic system on the Siegel half-plane of rank two, Acta Math, 134,(1975), 185-209.

[9] A. Koranyi, Poisson transforms for line bundles from the Shilov boundary to bounded symmetric domains, preprint.

[10] K. Koufany, G. Zhang, Hua operators, Poisson Transform and Relative Discrete Series on Line Bundle over Bounded Symmetric Domains, arXiv:1105.3806.

[11] M. Lassalle, Les équations de Hua d’un domaine borné symétrique de type tube, Invent. Math. 77 (1984) 129-161.

[12] K. D. Johnson. Differential equations and the Bergman-Shilov boundary on the Siegel upper half-plane. Arkiv for Mathematic, 16 (1978), 95-108.

[13] H. Schlichtkrull, One-Dimensional $K$-types In Finite Dimensional Representations Of Semisimple Lie Groups: A generalization of Helgason’s Theorem. Math. Scand. 54 (1984) 279-294.

[14] N. Shimeno, The Plancherel formula for spherical functions with a one dimensional $K$-type on a simply connected simple Lie group of hermitian type, J. Funct. Anal. 121 (1994) 330-388.

[15] N. Shimeno, Boundary Value Problems for the Shilov Boundary of a Bounded Symmetric Domain of Tube Type, J. Funct. Anal. 140 (1996) 124-141.

[16] T. Oshima, N. Shimeno, Boundary Value Problems on Riemannian Symmetric Spaces of the Noncompact Type. Preprint.

[17] Z. Yan; A class of Generalized Hypergeometric Functions in several variables, Canad. J. Math. vol 44 (6) (1992), 1317-1338.

[18] G. Zhang, Berezin transform on line bundles over bounded symmetric domains. J. Lie. Theory (10) (2000) 111-126.