Primordial non-Gaussianity and the inflationary Universe

Tomo Takahashi*
Department of Physics, Saga University, Saga 840-8502, Japan
*E-mail: tomot@cc.saga-u.ac.jp

Received February 08, 2014; Revised April 13, 2014; Accepted April 15, 2014; Published June 11, 2014

Non-Gaussianity of primordial fluctuations is one of the most important probes of the very early Universe and is now severely constrained by cosmological observations. In this review, we describe the formalism to investigate primordial non-Gaussianities such as the bispectrum and trispectrum, and summarize the current observational constraints on the so-called non-linearity parameters. We also discuss models of the origin of primordial fluctuations, paying particular attention to their non-Gaussian nature. The scale-dependence of non-linearity parameters and non-Gaussianities from isocurvature fluctuations are also discussed.

1. Introduction

The origin of primordial density fluctuations is one of the most important issues in understanding the phenomena in the very early Universe. Quantum fluctuations of the inflaton field generated during inflation are usually considered as the origin of density fluctuations [1–5]. However, the precise physical mechanism by which these fluctuations are created remains unknown. Outstanding questions include the shape of the inflaton potential, the structure of the kinetic term of the inflaton, the number of fields other than the inflaton and their roles in generating fluctuations, and so on. In models beyond the standard model of particles physics, such as supersymmetry and superstring theory, scalar fields are ubiquitous and, if some scalar field other than the inflaton acquires fluctuations, such a scalar field can also generate a primordial perturbation. Examples of this include the curvaton model [6–8], the modulated reheating scenario [9,10], and so on.

On the other hand, the current cosmological observations (e.g., Planck [11]) are so precise that we can obtain various pieces of information on the properties of primordial fluctuations. The primary observables are the power spectra of (scalar) density perturbations and gravitational waves, which are well measured or constrained by the current data [12]. However, it has been argued that non-Gaussianities of primordial fluctuations can also give invaluable information. In particular, the degrees of non-Gaussianities are quite different between primordial fluctuations generated from a standard inflation and other mechanisms, and thus it is very important to investigate the non-Gaussian nature of primordial fluctuations. In fact, recent Planck data put a tight constraint on the size of non-Gaussianities [13], which gives many implications for the origin of primordial fluctuations and the early Universe.

In this review, we discuss models of primordial fluctuations, in particular, in light of the Planck result. We first summarize the probes of non-Gaussianities such as the bi- and trispectra, describing...
several types of their shape (the wave-number-dependence of the polyspectra) and non-linearity parameters $f_{\text{NL}}, \tau_{\text{NL}},$ and $g_{\text{NL}},$ which are usually used to investigate primordial non-Gaussianities. Then we discuss some explicit models of the origin of primordial fluctuations, including a general single-field inflation and multi-field models. Furthermore, non-Gaussianities can be highly scale-dependent in some cases; hence we also discuss such models. In addition to the above issues, we also discuss non-Gaussianities from isocurvature fluctuations. When multiple sources of density fluctuations exist, the so-called isocurvature fluctuation can be generated. Although the current cosmological observations put stringent constraints on the contribution of the isocurvature mode, the non-Gaussian nature of such fluctuations may reveal different aspects beyond the power spectrum. Also, such fluctuations can be associated with cold dark matter (CDM) and the baryon; hence they may also have interesting implications for dark matter and scenarios of baryogenesis.

This review is organized as follows. In the next section, we summarize the formalism to probe non-Gaussianities such as bi- and trispectra. Observational constraints on the quantities characterizing non-Gaussianities are also given there. Then, in Sect. 3, models of primordial fluctuations are discussed. The scale-dependence of non-linearity parameters and non-Gaussianities from isocurvature fluctuations are discussed in Sects. 4 and 5, respectively. The final section is devoted to the conclusion.

2. Probes of non-Gaussianities and observational constraints

The properties of the curvature perturbation $\zeta$ can be probed by its statistical measures such as the power spectrum $P_\zeta$, bispectrum $B_\zeta$, and trispectrum $T_\zeta$, which correspond to the 2-, 3-, and 4-point correlation functions of $\zeta$:

\[
\langle \zeta_{k_1} \zeta_{k_2} \rangle = (2\pi)^3 P_\zeta(k) \delta(k_1 + k_2),
\]

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta(k_1 + k_2 + k_3),
\]

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle = (2\pi)^3 T_\zeta(k_1, k_2, k_3, k_4) \delta(k_1 + k_2 + k_3 + k_4),
\]

where the subscript $c$ represents the fact that we take the connected part for the correlation functions.

The power spectrum is well measured by cosmological observations such as the cosmic microwave background (CMB) and is usually parametrized as

\[
P_\zeta(k) = \frac{k^3}{2\pi^2} P_\zeta(k) = A_s(k_{\text{ref}}) \left( \frac{k}{k_{\text{ref}}} \right)^{n_s-1} = \tilde{A}_s k^{n_s-1},
\]

where $A_s(k_{\text{ref}})(= \tilde{A}_s)$ and $n_s$ are the amplitude and the spectral index at the reference scale $k_{\text{ref}}$, respectively. We also introduce $\tilde{k} = k/k_{\text{ref}}$ for shorthand notation. The Planck team has precisely measured these quantities as $\ln(10^{10} \tilde{A}_s) = 3.089^{+0.024}_{-0.027}$ and $n_s = 0.9603 \pm 0.0073$ (1σ CL) with $k_{\text{ref}} = 0.05$ Mpc$^{-1}$ in the framework of the standard flat $\Lambda$CDM model [12,14] \(^1\), using the Planck temperature + WMAP polarization data.

\(^1\) In the usual 6-parameter analysis, the spectral index $n_s$ is assumed to be constant and the values mentioned here are from such an analysis. However, one can also probe the scale-dependence of $n_s$, the so-called runnings, which are defined as

\[
\alpha_s \equiv \frac{dn_s}{d\ln k}, \quad \beta_s \equiv \frac{d\alpha_s}{d\ln k}.
\]

The Planck temperature + WMAP polarization data give the constraints on these variables as $\alpha_s = 0.001^{+0.016}_{-0.014}$ and $\beta_s = 0.020^{+0.016}_{-0.015}$ (1σ CL) [12].
2.1. Bispectrum

The so-called non-linearity parameter $f_{\text{NL}}$ is often used to characterize the amplitude of the bispectrum. Unlike the power spectrum, which depends only on a single wave number, the bispectrum depends on three wave numbers.

Depending on the mechanism of how primordial fluctuations are generated, the shape, or the wave-number-dependence, of the bispectrum changes. This property can be used to differentiate models of the origin of primordial fluctuations. Thus, $f_{\text{NL}}$ are defined for some specific “shapes” of the bispectrum. Popular examples include local, equilateral, and orthogonal types. In the following, we describe these types (see also Ref. [13]). Observational constraints on these $f_{\text{NL}}$ are summarized in Sect. 2.3.

2.1.1. Local type. This type of bispectrum can arise when non-linearities in the primordial adiabatic fluctuations are generated in superhorizon scale, and the non-linearities are local in real space. In this type of model, the curvature perturbation $\zeta$ can be expanded as [15]

$$\zeta = \zeta_g + \frac{3}{5} f_{\text{NL}}^{\text{(local)}} \zeta_g^2 + \cdots,$$

which leads to a bispectrum of the following form:

$$B_{\zeta}^{(\text{local})} (k_1, k_2, k_3) = \frac{6}{5} f_{\text{NL}}^{\text{(local)}} \left[ P_{\zeta} (k_1) P_{\zeta} (k_2) + P_{\zeta} (k_2) P_{\zeta} (k_3) + P_{\zeta} (k_3) P_{\zeta} (k_1) \right]$$

$$= \frac{6}{5} f_{\text{NL}}^{\text{(local)}} \tilde{A}_s^2 \left[ \frac{1}{k_1^{4-n_s} k_2^{4-n_s}} + \frac{1}{k_2^{4-n_s} k_3^{4-n_s}} + \frac{1}{k_3^{4-n_s} k_1^{4-n_s}} \right],$$

where $P_{\zeta} (k)$ is defined in Eq. (4). As shown in Fig. 1, the bispectrum of this form peaks around the squeezed limit ($k_3 \ll k_1 \simeq k_2$).

Primordial fluctuations of this type are typically produced during inflation from an isocurvature field whose fluctuations are converted into an adiabatic one after inflation. Examples of models generating local-type non-Gaussianity include: the curvaton model [6–8], modulated reheating [9,10], inhomogeneous end of hybrid inflation [16–20], mixed inflaton and curvaton model [21–26], mixed inflaton and modulated reheating [27], the ungaussiton [28,29], multi-brid inflation [30,31], multi-field inflation [32–34], multi-curvaton [35–37], inhomogeneous cosmological phase transition [38], inhomogeneous end of thermal inflation [39], modulated trapping [40,41], modulated curvaton [42,43], velocity modulation [44], modulated decay of the curvaton [45–47], hybrid curvaton–modulaton model [45], the ekpyrotic scenario [48–50], some single-field inflation models with an early non-attractor phase [108,109], and so on. Some representative models will be described in Sect. 3.

Local-type non-Gaussianity can also be generated in models with isocurvature fluctuations. However, in such a case, we have to introduce another non-linearity parameter for isocurvature fluctuations. We discuss this issue in Sect. 5.

2.1.2. Equilateral type. When one considers a general single-field inflation with a non-canonical kinetic term, or interactions with general higher-derivative operators, the shape of the bispectrum becomes the so-called (nearly) equilateral type. Models generating this type include: $k$-inflation [51, 52], Dirac-Born-Infeld (DBI) inflation [53,54], ghost inflation [55–57], trapped inflation [58], the Lifshitz scalar [59], and so on. To obtain an observational constraint on this type, a template with the
Fig. 1. Shapes of the bispectrum: local (top), equilateral (bottom left), and orthogonal (bottom right) types.

following form is adopted [60]:

\[
B^{(\text{equil})}_\zeta(k_1, k_2, k_3) = \frac{3}{5} f^{(\text{equil})}_{\text{NL}} \left[ -3 P_\zeta(k_1) P_\zeta(k_2) - 2 P_\zeta^{2/3}(k_1) P_\zeta^{2/3}(k_2) P_\zeta^{2/3}(k_3) 
+ 6 P_\zeta^{1/3}(k_1) P_\zeta^{2/3}(k_2) P_\zeta(k_3) + (5 \text{ permutations}) \right]
\]

\[
= \frac{3}{5} f^{(\text{equil})}_{\text{NL}} A_s^2 \left[ -\frac{3}{k_1^{4-n_s} k_2^{4-n_s} k_3^{4-n_s}} - \frac{2}{(k_1 k_2 k_3)^2(4-n_s)/3} 
+ \frac{1}{k_1^{(4-n_s)/3} k_2^{(4-n_s)/3} k_3^{(4-n_s)/3}} + (5 \text{ permutations}) \right]. 
\]

In this form, the amplitude peaks around \( k_1 \simeq k_2 \simeq k_3 \), which can be seen from Fig. 1.

2.1.3. Orthogonal type. In some models, another type of bispectrum can arise, which is orthogonal to the local and equilateral forms and is called the orthogonal type. Its bispectrum form is [61]:

\[
B^{(\text{ortho})}_\zeta(k_1, k_2, k_3) = \frac{3}{5} f^{(\text{ortho})}_{\text{NL}} \left[ -9 P_\zeta(k_1) P_\zeta(k_2) - 8 P_\zeta^{2/3}(k_1) P_\zeta^{2/3}(k_2) P_\zeta^{2/3}(k_3) 
+ 18 P_\zeta^{1/3}(k_1) P_\zeta^{2/3}(k_2) P_\zeta(k_3) + (5 \text{ permutations}) \right]
\]
In fact, this form can be written in terms of \( F \). For the local-type model, we can further expand the curvature perturbations as

\[
\tau = \zeta + \frac{3}{5} f_{\text{NL}}^{(\text{local})} \zeta^2 + \left( \frac{3}{5} \right) f_{\text{NL}}^{(\text{local})} \zeta^2 = \zeta + 3 \frac{f_{\text{NL}}^{(\text{local})}}{5} \zeta^2 + \ldots ,
\]

from which we can compute the trispectrum:

\[
T_\zeta(k_1, k_2, k_3, k_4) = \tau_{\text{NL}}^{(\text{local})} \left[ P_\zeta(k_1) P_\zeta(k_2) P_\zeta(k_3) + 11 \text{ perms.} \right] + \frac{54}{25} g_{\text{NL}}^{(\text{local})} \left[ P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + 3 \text{ perms.} \right],
\]

where \( k_{13} = |k_1 + k_3| \) and \( \tau_{\text{NL}}^{(\text{local})} \) and \( g_{\text{NL}}^{(\text{local})} \) are non-linearity parameters characterizing the size of the trispectrum (later in this subsection, we omit “(local)” from the non-linearity parameters). When a single field (source) is responsible for the curvature perturbation, \( \tau_{\text{NL}} \) is given by \( f_{\text{NL}} \) as

\[
\tau_{\text{NL}} = \left( \frac{6}{5} f_{\text{NL}} \right)^2.
\]

On the other hand, when multiple sources contribute to the curvature perturbation, \( \tau_{\text{NL}} \) is larger than \( (6 f_{\text{NL}}/5)^2 \). Thus, in some models, even if \( f_{\text{NL}} \sim O(1) \), \( \tau_{\text{NL}} \) can be large. Such examples include

\footnote{In some works, the non-linearity parameters for other shapes have been investigated \cite{13,62}.}

\footnote{Strictly speaking, this equation holds when the so-called loop corrections are negligible. See Sect. 3.2.1 for further discussion.}
mixed inflaton–curvaton (or spectator field) model [21–26], the ungaussiton model [28,29], and so on. Thus, \( \tau_{NL} \) may serve as a first signature of non-Gaussianity in such a model.

Regarding \( g_{NL} \), most models have a particular relation between \( f_{NL} \) and \( g_{NL} \). In Ref. [42], local-type models have been classified into three categories by using the \( f_{NL}-g_{NL} \) relation:

(i) “linear \( g_{NL} \)” type: \( |g_{NL}| \sim |f_{NL}| \).
(ii) “suppressed \( g_{NL} \)” type: \( |g_{NL}| \sim (\text{suppression factor}) \times |f_{NL}| \).
(iii) “enhanced \( g_{NL} \)” type: \( |g_{NL}| \sim |f_{NL}|^{n} \) (with \( n > 1 \)).

Although, precisely speaking, the relation between \( f_{NL} \) and \( g_{NL} \) actually depends on the model parameters, the above classification is useful to give an idea of how large \( g_{NL} \) can be. In fact, even for the “enhanced \( g_{NL} \)” type, the power \( n \) in \( |f_{NL}|^{n} \) is \( n = 2 \) for most models [42]. Thus, \( g_{NL} \) is not so enhanced compared to \( f_{NL} \) in most cases.

However, we remark that, if we allow some level of fine-tuning, large values of \( g_{NL} \) are possible even when \( f_{NL} \simeq O(1) \) [68]. Such examples are the self-interacting curvaton [141–147], the modulated curvaton [42,43], and modulated decay of the curvaton [45–47], in which some cancellation occurs to suppress the amplitude of the bispectrum, while cancellation does not happen in the trispectrum. For details of this issue, see Ref. [68].

2.2.2. Equilateral type. The trispectrum of general single-field inflation models including a non-trivial kinetic term can be schematically written as [69]

\[
T_{\zeta}(k_1, k_2, k_3, k_4) = (2\pi)^{6} P^{3} \prod_{i=1}^{4} \frac{1}{k_{i}^{3}} \left[ T_{s_{1}} + T_{s_{2}} + T_{s_{3}} + T_{c_{1}} + T_{c_{2}} + T_{c_{3}} \right]. \tag{14}
\]

Here, \( T_{s_{1,2,3}} \) are the trispectra coming from the scalar-exchange diagram and \( T_{c_{1,2,3}} \) are those from the contact-interaction diagram (explicit forms of each trispectrum can be found in Ref. [69]). We will also discuss the trispectrum in models with general single-field inflation models in Sect. 3.1. Among the trispectra, \( T_{c_{1}} \) has a diagonal-free form and is strongly correlated with the other trispectra, which motivates us to adopt \( T_{c_{1}} \) as a template to probe the size of the equilateral-type trispectrum. The shape correlation between \( T_{c_{1}} \) and the other functions, and also the trispectra produced in various models such as DBI inflation, ghost inflation, and Lifshitz scalar models are analyzed in Refs. [70–72].

To define the non-linearity parameter of the equilateral trispectrum, first we define the estimator \( t_{NL} \) [69,70], by taking the RT (regular tetrahedron) limit in the trispectrum as

\[
\left\langle \zeta^{4} \right\rangle_{\text{RT limit}} \rightarrow (2\pi)^{9} P^{3} \delta(k_{1} + k_{2} + k_{3} + k_{4}) \frac{1}{k^{5}} t_{NL}^{\text{RT limit}}. \tag{15}
\]

where the RT limit corresponds to \( k_{1} = k_{2} = k_{3} = k_{4} = k_{12} = k_{14} \equiv k \). By putting the \( c1 \)-type for the trispectrum whose explicit form can be given as

\[
T_{c_{1}} = \frac{t_{NL}^{\text{equil}}}{12} \prod_{i=1}^{4} \frac{k_{i}^{2}}{k_{1} + k_{2} + k_{3} + k_{4}/4}^{5}, \tag{16}
\]

\[\text{We call a scalar field whose energy density is negligible during the inflation and scarcely affects the inflationary dynamics a “spectator field.”}\]
we define the non-linearity parameter for the equilateral trispectrum $t_{\text{NL}}^\text{equil}$ with Eq. (15). Observational constraints on the non-linearity parameter will be given in the next subsection.

2.3. Observational constraints

Here we summarize observational constraints on the non-linearity parameters $f_{\text{NL}}$, $\tau_{\text{NL}}$, and $g_{\text{NL}}$ from CMB and large-scale structure (LSS) data.

2.3.1. Bispectrum: $f_{\text{NL}}$

Below we list the constraints on $f_{\text{NL}}$ from (relatively) recent analyses.

- **Planck** (Ade et al., 2013 [13])
  \[
  f_{\text{NL}}^{(\text{local})} = 2.8 \pm 5.7, \quad f_{\text{NL}}^{(\text{equil})} = -42 \pm 75, \quad f_{\text{NL}}^{(\text{ortho})} = -25 \pm 39 \quad (68\% \text{ CL}).
  \] (17)

- **WMAP9** (Bennett et al., 2013 [74])
  \[
  f_{\text{NL}}^{(\text{local})} = 37.2 \pm 19.9, \quad f_{\text{NL}}^{(\text{equil})} = 51 \pm 136, \quad f_{\text{NL}}^{(\text{ortho})} = -245 \pm 100 \quad (68\% \text{ CL}).
  \] (18)

- **WMAP9** (Regan et al., 2013, Needlets [73])
  \[
  f_{\text{NL}}^{(\text{local})} = 38.6 \pm 23.1, \quad f_{\text{NL}}^{(\text{equil})} = 64.5 \pm 117.3, \quad f_{\text{NL}}^{(\text{ortho})} = -175.0 \pm 101.8 \quad (68\% \text{ CL}).
  \] (19)

- **LSS** (Xia et al., 2010 [75])
  \[
  f_{\text{NL}}^{(\text{local})} = 62 \pm 27 \quad (68\% \text{ CL}).
  \] (20)

- **LSS** (Ross et al., 2013 [76])
  \[
  -82 < f_{\text{NL}}^{(\text{local})} < 178 \quad (95\% \text{ CL}).
  \] (21)

- **LSS** (Giannantonio et al., 2014 [77])
  \[
  -36 < f_{\text{NL}}^{(\text{local})} < 45 \quad (95\% \text{ CL}) \left[ f_{\text{NL}}^{(\text{local})} = 5 \pm 21 \quad (68\% \text{ CL}) \right].
  \] (22)

- **LSS** (Mana et al., 2013 [78])
  \[
  f_{\text{NL}}^{(\text{local})} = 12 \pm 157 \quad (68\% \text{ CL}).
  \] (23)

- **LSS** (Shandera et al., 2013 [79])
  \[
  f_{\text{NL}}^{(\text{local})} = -3^{+78}_{-91}, \quad f_{\text{NL}}^{(\text{equil})} = -52^{+85}_{-79}, \quad f_{\text{NL}}^{(\text{ortho})} = 63^{+97}_{-104} \quad (68\% \text{ CL}).
  \] (24)

- **LSS** (Karagiannis et al., 2013 [80])
  \[
  46 < f_{\text{NL}}^{(\text{local})} < 158 \quad (95\% \text{ CL}).
  \] (25)

- **LSS** (Ho et al., 2013 [81])
  \[
  f_{\text{NL}}^{(\text{local})} = 2^{+65}_{-66} \quad (68\% \text{ CL}).
  \] (26)

---

5 This parameter is denoted as $g_{\text{NL}}^{\text{1}}$ in Ref. [73]. $g_{\text{NL}}^{\text{equil}}$, defined in Ref. [72], is related to this parameter as $g_{\text{NL}}^{\text{equil}} = 8_{\text{NL}}^{\text{equil}}/32.$
2.3.2. Trispectrum: $\tau_{NL}$ and $g_{NL}$. Observational constraints on the trispectrum, more specifically, on the non-linearity parameters $\tau_{NL}$ and $g_{NL}$, are less explored compared to $f_{NL}$. However, there are several constraints available from the actual data, which we list below.

- **Planck** (Ade et al., 2013 [13])
  \[
  \tau^{(\text{local})}_{NL} < 2800 \quad (95\% \text{ CL}).
  \] (27)

- **WMAP5** (Smidt et al., 2010, Kurtosis power spectrum [82])
  \[
  \tau^{(\text{local})}_{NL} = (1.35 \pm 0.98) \times 10^4, \quad g^{(\text{local})}_{NL} = (0.42 \pm 3.9) \times 10^5 \quad (68\% \text{ CL}).
  \] (28)

- **WMAP5** (Vielva et al., 2010, N-point probability distribution [83])
  \[
  -5.6 \times 10^5 < g_{NL}^{(\text{local})} < 6.4 \times 10^5 \quad (95\% \text{ CL}).
  \] (29)

- **WMAP5** (Fergusson et al., 2010, Modal expansion [70])
  \[
  g_{NL}^{(\text{local})} = (1.6 \pm 7.0) \times 10^5, \quad \tau^{(\text{equil})}_{NL} = (-3.11 \pm 7.5) \times 10^5 \quad (68\% \text{ CL}).
  \] (30)

- **WMAP7** (Hikage and Matsubara 2012, Minkowski functionals [84])
  \[
  \tau^{(\text{local})}_{NL} = (-7.6 \pm 8.7) \times 10^4, \quad g^{(\text{local})}_{NL} = (-1.9 \pm 6.4) \times 10^5 \quad (68\% \text{ CL}).
  \] (31)

- **WMAP9** (Sekiguchi and Sugiyama, 2013, Real-space quartic statistics [85])
  \[
  g_{NL}^{(\text{local})} = (3.3 \pm 2.2) \times 10^5 \quad (68\% \text{ CL}).
  \] (32)

- **WMAP9** (Regan et al., 2013, Needlets [73])
  \[
  \tau^{(\text{local})}_{NL} < 22000 \quad (95\% \text{ CL}), \quad g_{NL}^{(\text{local})} = (-4.9 \pm 2.3) \times 10^5 \quad (68\% \text{ CL}),
  \]
  \[
  g_{NL}^{c1} = \tau^{(\text{equil})}_{NL} = (-3.2 \pm 2.5) \times 10^6 \quad (68\% \text{ CL}).
  \] (33)

- **LSS** (Desjacques and Seljak, 2010 [86])
  \[
  -3.5 \times 10^5 < g_{NL}^{(\text{local})} < 8.2 \times 10^5 \quad (95\% \text{ CL}).
  \] (34)

- **LSS** (Giannantonio et al., 2014 [77])
  \[
  -5.6 \times 10^5 < g_{NL}^{(\text{local})} < 5.1 \times 10^5 \quad (95\% \text{ CL}).
  \] (35)

3. Models of primordial non-Gaussianities

In this section, we discuss models of the origin of primordial fluctuations, paying particular attention to their non-Gaussian nature. In fact, in most models, the predicted value of $f_{NL}$ can vary from $f_{NL} \sim \mathcal{O}(1)$ to $f_{NL} \gg \mathcal{O}(1)$, depending on the model parameters. Thus, even with a severe constraint on $f_{NL}$ from Planck, most models are still viable and the model parameters are just tightly constrained.

Although many models have been proposed in the literature, here we discuss some representative ones.
3.1. General single-field inflation models

Non-Gaussianity has been extensively investigated by many authors in various extensions of a standard single-field inflation model. In fact, by using the most general scalar–tensor theory Lagrangian with second-order equation of motion proposed by Horndeski [87], one can treat single-field inflation models in a unified way, and the bispectrum has been calculated in this setup in Refs. [88–92]. Another approach to studying a wide class of general single-field inflation models is with the effective field theory of inflation [93,94].

However, in the following, we consider the Lagrangian of the form [52]

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M_{pl}^2 R + P(X, \phi) \right],
\]

where \( R \) is the Ricci scalar and \( X \equiv -(1/2)g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \), with \( \phi \) being the inflaton. Below, we briefly discuss the power, bi-, and trispectra with this Lagrangian. For a discussion of non-Gaussianity in this type of general single-field inflation model, see Refs. [69,95–101]. For general multi-field inflation, see Refs. [102,103].

With this Lagrangian, the power spectrum is given by

\[
P_\zeta = \frac{1}{8 \pi^2 M_{pl}^2 c_s \epsilon},
\]

where \( H \) is the Hubble parameter and \( c_s \) is the sound speed, which is

\[
c_s^2 = \frac{P_X}{P_{XX}}.
\]

Here we define the slow-roll parameters, including those other than \( \epsilon \):

\[
\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H}, \quad s = \frac{c_s}{c_s H}.
\]

The bispectrum can be written as [95]

\[
B_\zeta(k_1, k_2, k_3) = (2\pi)^4 P_\zeta^2 \frac{1}{\prod_{i=1}^3 k_i^3} \left[ \left( \frac{1}{c_s^2} - 1 \right) \frac{3 k_i^2 k_j^2 k_k^2}{2 K_i^2} + \frac{1}{(c_s^2 - 1)} \right]
\]

\[
\left( -\frac{1}{K_i} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2 K_i^2} \sum_{i\neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right) + A_o + A_e + A_\eta + A_s \right],
\]

where we have explicitly shown only the leading terms in the slow-roll parameters \( \epsilon, \eta, \) and \( s \). \( A_o, e, \eta, s \) are the terms with \( \epsilon, \eta, \) and \( s \) whose explicit forms can be found in Ref. [95]. In the above expression, \( K_i = k_1 + k_2 + k_3 \) and

\[
\Sigma = XP_X + 2X^2 P_{XX} = \frac{H^2 \epsilon}{c_s^2}, \quad \lambda = X^2 P_{XX} + \frac{2}{3} X^3 P_{XXX}.
\]

To describe the amplitude of the bispectrum, one can generalize the definition of the non-linearity parameter \( f_{\text{NL}} \) as

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (2\pi)^7 \delta(k_1 + k_2 + k_3) \frac{\sum_{i=1}^3 k_i^3 P_\zeta^2}{\prod_{i=1}^3 k_i^3} \left( 3 \frac{f_{\text{NL}}}{10} \right).
\]
Taking the equilateral limit \((k_1 = k_2 = k_3 = k)\), one can derive \(f_{\text{NL}}^{(\text{equil})}\) for expression (40), with only the leading-order terms, as [92]

\[
f_{\text{NL}}^{(\text{equil})} = \frac{85}{324} \left(1 - \frac{1}{c_s^2}\right) - \frac{10 \lambda}{81 \Sigma}.
\]  

The trispectrum can also be computed as [69]

\[
T_\zeta(k_1, k_2, k_3, k_4) = (2\pi)^9 \rho_\zeta^3 \prod_{i=1}^4 \frac{k_i^3}{\mu} \left[ \left( \frac{\mu}{\Sigma} - \frac{9\lambda^2}{\Sigma^2} \right) T_{c1} + \left( \frac{3\lambda}{\Sigma} - \frac{1}{c_s^2} + 1 \right) T_{c2} + \left( \frac{1}{c_s^2} - 1 \right) T_{c3} + \left( \frac{\lambda}{\Sigma} \right)^2 T_{s1} + \left( \frac{1}{c_s^2} - 1 \right) T_{s2} + \left( \frac{1}{c_s^2} - 1 \right)^2 T_{c3} \right],
\]

where

\[
\mu = \frac{1}{2} X^2 P_{XX} + 2X^3 P_{XXX} + \frac{2}{3} X^4 P_{XXXX}.
\]  

\(T_{c1,c2,c3}\) are the trispectra from the contact-interaction diagrams and \(T_{s1,s2,s3}\) are those from the scalar-exchange diagrams (for explicit forms of these trispectra, see Ref. [69]). As discussed in the previous section, \(T_{c1}\) is adopted as a template to define the non-linearity parameter for the equilateral trispectrum since it has a diagonal-free form.

Up to now, we have given general formulas for the model described with the Lagrangian (36). Below we discuss some concrete models of single-field inflation.

### 3.1.1. Slow-roll single-field inflation with a canonical kinetic term

For a standard slow-roll single-field inflation with a canonical kinetic term, one has \(P = X - V(\phi)\). The bispectrum for this model has been calculated as [104]

\[
B_\zeta(k_1, k_2, k_3) = \frac{(2\pi)^4}{16} \rho_\zeta^2 \prod_{i=1}^3 \frac{k_i^3}{k_i} \left[ (3\epsilon - 2\eta_\phi) \sum_i k_i^3 + \epsilon \sum_{i \neq j} k_i k_j^2 + 8\epsilon \sum_{i>j} \frac{k_i^2 k_j^2}{K_i} \right],
\]

where \(\eta_\phi = M_{\text{pl}}^2 \left( V_{\phi\phi} / V \right)\) is the slow-roll parameter defined with the inflaton potential \(V\). The above expression can be derived from \(A_\epsilon\) and \(A_\eta\) in Eq. (40). We note that \(\eta_\phi\) here and \(\eta_H\) defined in Eq. (39) are different and, for the case of \(P = X - V(\phi)\), they are related as \(\eta_H = -2\eta_\phi + 4\epsilon\). In the squeezed limit where \(k_3 \rightarrow 0\), the above bispectrum reduces to

\[
B_\zeta(k_1, k_2, k_3 \rightarrow 0) = 2(3\epsilon - \eta_\phi) P_\zeta(k_1) P_\zeta(k_3) = (1 - n_s) P_\zeta(k_1) P_\zeta(k_3).
\]  

By taking the same limit in \(B_\zeta^{(\text{local})}\) given in Eq. (7), and comparing it with the above expression, we obtain the consistency relation between \(f_{\text{NL}}^{(\text{local})}\) and \(n_s\) as [104]

\[
f_{\text{NL}}^{(\text{local})} = \frac{5}{12} (1 - n_s).
\]  

In fact, this relation has been shown to hold for any single-field inflation model regardless of its potential and its kinetic term [105], as long as a Bunch–Davies vacuum and a negligible decaying mode are assumed. Counterexamples of the violation of this consistency relation have been discussed in Refs. [106–109], where a nonvacuum initial state is considered or the decaying mode is non-negligible. Since current constraints on the spectral index indicate that \(1 - n_s = O(0.01)\), single-field inflation models are consistent with the Planck constraint shown in Eq. (17). However, if \(f_{\text{NL}}^{(\text{local})}\) is confirmed to be \(O(1)\), single-field inflation models would be critically tested as the origin of primordial fluctuations.
3.1.2. $k$-inflation. $k$-inflation is a model where the inflationary expansion is driven by the (non-standard) kinetic term. A power law $k$-inflation model is described with

$$P(X, \phi) = K(\phi)(-X + X^2), \quad (49)$$

where $K(\phi)$ is a function of $\phi$. When one considers the case where the scale factor is given as $a(t) \propto t^{2/3}$ with $\gamma$ being a constant, the form of $K(\phi)$ becomes

$$K(\phi) = \frac{4(4 - 3\gamma)}{9\gamma^2 \phi^2}. \quad (50)$$

Then the sound speed $c_s$ and the other parameters such as $\Sigma$, $\lambda$, and $\mu$ are written as

$$c_s^2 = \frac{2X - 1}{6X - 1} = \frac{\gamma}{8 - 3\gamma},$$

$$\frac{\lambda}{\Sigma} = \frac{2X}{6X - 1} = \frac{1 - c_s^2}{2}, \quad \frac{\mu}{\Sigma} = \frac{X}{6X - 1} = \frac{1 - c_s^2}{4},$$

where we have used $X = (2 - \gamma)/(4 - 3\gamma)$ in the second equality in $c_s^2$. This equation can be obtained from the equation of motion. By using Eq. (43), one can find the equilateral $f_{NL}$ for $c_s \ll 1$ as

$$f_{NL}^{(\text{equil})} \simeq -\frac{85}{324} \frac{1}{c_s^2} = -\frac{170}{81\gamma},$$

where Eq. (51) is used in the second equality with $c_s \ll 1$. From the Planck constraint given in (17), one can derive the bound on $\gamma$. By adopting the prior $0 < \gamma < 2/3$, the bound $\gamma \geq 0.05$ (95% CL) has been obtained [13]. In fact, this constraint is in conflict with the one derived from the constraint on the spectral index $n_s - 1 = -3\gamma$, which gives the limit as $0.01 \leq \gamma \leq 0.02$ [12]. These arguments are a severe test of this model.

The trispectrum in the $k$-inflation model is given as [69]

$$T_\zeta(k_1, k_2, k_3, k_4) \simeq \frac{1}{4} T_{s1} + \frac{1}{2c_s^2} T_{s2} + \frac{1}{c_s^2} T_{s3} - 2T_{c1} - \frac{1}{c_s^2} T_{c2} + \frac{1}{c_s^2} T_{c3},$$

where $c_s^2 \ll 1$ is assumed. Thus, the contribution from $T_{s3}$ is dominant compared to the other ones.

3.1.3. DBI inflation. Non-Gaussianity in the DBI inflation model has been extensively studied in the literature. Single-field DBI inflation has been investigated in Refs. [54,110,111] and the multi-field DBI case in Refs. [112–117].

For the DBI inflation model, the form of $P(X, \phi)$ is

$$P(X, \phi) = -f^{-1}(\phi) \sqrt{1 - 2Xf(\phi)} - V(\phi).$$

Then we have

$$c_s^2 = 1 - 2Xf,$$ \hspace{1cm} (56)

$$\frac{\lambda}{\Sigma} = \frac{1}{2} \left( \frac{1}{c_s^2} - 1 \right), \quad \frac{\mu}{\Sigma} = \frac{1}{4} \left( \frac{1}{c_s^2} - 1 \right) \left( \frac{5}{c_s^2} - 4 \right).$$ \hspace{1cm} (57)

For the bispectrum, one can find that the term with $(1/c_s^2 - 1 - 2\lambda/\Sigma)$ vanishes in Eq. (40) and the equilateral non-linearity parameter $f_{NL}^{(\text{equil})}$ for the limit of $c_s \ll 1$ is

$$f_{NL}^{(\text{equil})} \simeq -\frac{35}{108} \frac{1}{c_s^2},$$ \hspace{1cm} (58)
In fact, the shape of the bispectrum in DBI inflation is not exactly the equilateral one given in Eq. (8). However, the shape correlation between the DBI inflation and the equilateral ones is quite high [118]. For the shape correlation between various models, see Ref. [118]. Nevertheless, an analysis using the bispectrum shape of the DBI inflation has also been done in Ref. [13], in which the non-linearity parameter for DBI inflation is found to be

$$ f_{DBI}^{NL} = 11 \pm 69 \quad (68\% \, CL). $$

This limit can be translated into the one for the sound speed in the DBI case as

$$ c_{s}^{DBI} > 0.07 \quad (95\% \, CL) \quad [13]. $$

For the trispectrum, we obtain, for the limit $c_{s}^{2} \ll 1$, [69]

$$ T_{\zeta}(k_{1}, k_{2}, k_{3}) \simeq \frac{1}{4c_{s}^{4}}T_{s1} + \frac{1}{2c_{s}^{4}}T_{s2} + \frac{1}{c_{s}^{4}}T_{s3} - \frac{1}{c_{s}^{4}}T_{c1} + \frac{1}{2c_{s}^{4}}T_{c2} + \frac{1}{c_{s}^{4}}T_{c3}. $$

Thus the contributions from $T_{c2,c3}$ are small compared to the other ones.

### 3.2. Multi-field (local-type) models

When the curvature perturbations are generated on superhorizon scales, the shapes of non-Gaussianities become the local type. To describe the curvature perturbation of this type, the $\delta N$ formalism [119–123] is useful and is usually adopted. Thus, before looking at some concrete models of the local type, first we briefly summarize the formulas in this formalism.

#### 3.2.1. $\delta N$ formalism

In the $\delta N$ formalism, the superhorizon curvature perturbation $\zeta$ on the uniform (total) energy density hypersurface at some final time $t_{f}$ is given by fluctuations in the $e$-folding number as

$$ \zeta = N(t_{s}, t_{f}, x) - \bar{N}, $$

where

$$ N(t_{s}, t_{f}, x) = \int_{t_{s}}^{t_{f}} H(t, x)dt, $$

and $\bar{N}$ is calculated for the background expansion. $t_{s}$ is the initial time and is usually taken to be just after the cosmological scale crosses the horizon during inflation. The initial hypersurface is taken to be a flat slice. Then the curvature perturbation $\zeta$ can be expressed as

$$ \zeta \simeq N_{a}^{a} + \frac{1}{2}N_{ab}\delta\phi_{a}^{b} + \frac{1}{6}N_{abc}\delta\phi_{a}^{b}\delta\phi_{b}^{c} + \cdots, $$

where $\delta\phi_{a}^{b}$ is a fluctuation of a scalar field $\phi_{a}$ at the time of horizon crossing. Here we label the scalar field by $a$ and $N_{a} = dN/d\phi_{a}$ and so on. Then the power spectrum is given as

$$ P_{\zeta} = N_{a}N_{a}^{a}P_{\delta\phi} = N_{a}N_{a}^{a}\frac{2\pi^{2}}{k^{3}}P_{\delta\phi}(k) = N_{a}N_{a}^{a}\frac{2\pi^{2}}{k^{3}}\left(\frac{H_{s}}{2\pi}\right)^{2}, $$

where $H_{s}$ is the Hubble parameter at the time of horizon crossing and $P_{\delta\phi}$ is the power spectrum of the scalar field fluctuation $\delta\phi$:

$$ \langle\delta\phi_{k_{1}}^{a}\delta\phi_{k_{2}}^{b}\rangle = (2\pi)^{3}\delta(k_{1} + k_{2})P_{\delta\phi}(k)\delta^{ab}. $$

---

6 This is generally the case for all the equilateral models.
Non-linearity parameters for the bi- and trispectra are calculated as \([124–126]\)

\[
\begin{align*}
\frac{6}{5} f^{\text{(local)}}_{\text{NL}} &= \frac{N_a N_b N^{ab}}{(N_c N^c)^2}, \\
\tau^{\text{(local)}}_{\text{NL}} &= \frac{N_{ab} N^{ac} N^b N^c}{(N_d N^d)^3}, \\
\frac{54}{25} g^{\text{(local)}}_{\text{NL}} &= \frac{N_{abc} N_a N_b N^c}{(N_d N^d)^3}.
\end{align*}
\]  

(66)  

(67)  

(68)

With these formulas, we can explicitly write down the expressions for the non-linearity parameters given a concrete model.

From Eqs. (66) and (67), one can derive an important relation between \(f_{\text{NL}}\) and \(\tau_{\text{NL}}\). With the use of the Cauchy–Schwartz inequality, one finds

\[
\tau_{\text{NL}} \geq \left(\frac{6}{5} f_{\text{NL}}\right)^2,
\]  

(69)

which is called the Suyama–Yamaguchi inequality \([127]\). Although this inequality was originally derived at the tree level, some papers have investigated it including loop corrections \([42,128]\). In particular, even if we include all the loop contributions, the above formula holds \([129]\). At the tree level, the equality holds only when a single field is responsible for primordial fluctuations. However, when a loop contribution becomes sizable, the equality is not necessarily satisfied even for a single-field model \([129,130]\). Nevertheless, as long as the loop contribution is not significant, the above inequality can still be used to probe the multi-field nature of the model.

Now, in the following, we discuss some representative models generating local-type non-Gaussianity. For a more exhaustive discussion of local-type models, see Ref. \([42]\).  

3.2.2. Curvaton model. When there exists another light scalar field other than the inflaton during inflation, such a scalar field also acquires quantum fluctuations. If the energy density of such a scalar field is negligible during inflation, its fluctuations act as isocurvature ones. However, they can be converted into adiabatic ones after inflation when the scalar field decays into radiation. Such a scenario is called the curvaton mechanism (the scalar field that is responsible for the fluctuations is called the curvaton) \([6–8]\).

In this model, since the curvature perturbation \(\zeta\) is generated on superhorizon scales, local-type non-Gaussianity is generated, which has been much investigated in the literature \([124,131–151]\). By adopting the \(\delta N\) formalism, one can calculate \(\zeta\) in the curvaton model as

\[
\begin{align*}
\zeta_{\text{curvaton}} &= \frac{2}{3} r_{\text{dec}} \frac{\sigma'_{\text{osc}}}{\sigma'_{\text{osc}}} \delta\sigma^* + \frac{1}{9} \left[ 3 r_{\text{dec}} \left( 1 + \frac{\sigma''_{\text{osc}}}{\sigma'_{\text{osc}}} \right) - 4 r^2_{\text{dec}} - 2 r^3_{\text{dec}} \right] \left( \frac{\sigma'_{\text{osc}}}{\sigma'_{\text{osc}}} \right)^2 (\delta\sigma^*)^2 \\
&\quad + \frac{4}{81} \left[ 9 r_{\text{dec}} \left( \frac{\sigma'_{\text{osc}}}{\sigma'_{\text{osc}}} + \frac{3}{2} \frac{\sigma''_{\text{osc}}}{\sigma'_{\text{osc}}} \right) + 9 r^2_{\text{dec}} \left( 1 + \frac{\sigma''_{\text{osc}}}{\sigma'_{\text{osc}}} \right) \right] \frac{\sigma'_{\text{osc}}}{\sigma'_{\text{osc}}} \\
&\quad + \frac{r^3_{\text{dec}}}{2} \left( 1 - 9 \frac{\sigma''_{\text{osc}}}{\sigma'_{\text{osc}}} \right) + 10 r^4_{\text{dec}} + 3 r^5_{\text{dec}} \left( \frac{\sigma'_{\text{osc}}}{\sigma'_{\text{osc}}} \right)^3 (\delta\sigma^*)^3.
\end{align*}
\]  

(70)
and the non-linearity parameters $f_{NL}$ and $g_{NL}$ are given by [124,133,135]

$$\frac{6}{5} f_{NL} = \frac{3}{2r_{\text{dec}}} \left( 1 + \frac{\sigma_{\text{osc}}''}{\sigma_{\text{osc}}'^2} \right) - 2 - r_{\text{dec}}, \tag{71}$$

$$\frac{54}{25} g_{NL} = \frac{9}{4r_{\text{dec}}^2} \left( \frac{\sigma_{\text{osc}}'^2}{\sigma_{\text{osc}}'^3} + 3 \frac{\sigma_{\text{osc}}''}{\sigma_{\text{osc}}'^2} \right) - \frac{9}{r_{\text{dec}}} \left( 1 + \frac{\sigma_{\text{osc}}''}{\sigma_{\text{osc}}'^2} \right) + \frac{1}{2} \left( 1 - 9 \frac{\sigma_{\text{osc}}''}{\sigma_{\text{osc}}'^2} \right) + 10r_{\text{dec}} + 3r_{\text{dec}}^2, \tag{72}$$

where $r_{\text{dec}}$ roughly corresponds to the energy density of the curvaton at the time of its decay and is defined as

$$r_{\text{dec}} = \frac{3 \rho_{\sigma}}{4 \rho_{r} + 3 \rho_{\sigma}} \bigg|_{t=t_{\text{dec}}}, \tag{73}$$

with $\rho_{\sigma}$ and $\rho_{r}$ being the energy densities of the curvaton and radiation, respectively. $\sigma_{\text{osc}}$ is the value of $\sigma$ at the beginning of its oscillation and $\sigma_{\text{osc}}' = d\sigma_{\text{osc}}/d\sigma_{*}$ with $\sigma_{*}$ being the value of $\sigma$ at the horizon exit.

When we assume a quadratic potential for the curvaton, $\sigma_{\text{osc}}''$ and $\sigma_{\text{osc}}'''$ vanish and the terms with $\sigma_{\text{osc}}$ in Eqs. (71) and (72) become irrelevant. However, when one considers a curvaton model with a self-interaction [141–147] and axion (pseudo-Nambu–Goldstone) type [148–151], the derivatives of $\sigma_{\text{osc}}$ can give a significant contribution, which may drastically affect the prediction for the non-linearity parameters, depending on the model parameters in the potential.

However, for the case with a quadratic potential, $f_{NL}$ and $g_{NL}$ are determined by a single parameter $r_{\text{dec}}$, and, in turn, the Planck constraint on $f_{NL}$ can give a severe bound on $r_{\text{dec}}$ as [13]

$$r_{\text{dec}} \geq 0.15 \quad (95\% \text{ CL}). \tag{76}$$

Thus, the curvaton should almost dominate at its decay to satisfy the constraint.

Furthermore, sizable isocurvature fluctuations can also be generated in the curvaton model, depending on when and how CDM or the baryon is created, which also gives a severe constraint on the model [131,138–140,154,155].

- **Multi-curvaton model**

A model with multiple curvatons has also been investigated in Refs. [35–37,42]. Even with two curvatons, its predictions for the curvature perturbation generally become very complicated compared to those for a (standard) single curvaton case (for full expressions of $\zeta$ up to third order, see Ref. [42]). Unlike the single curvaton case, the value of $f_{NL}$ from the multi-curvaton case can be large as $f_{NL} \gg O(1)$ or of order unity even when both curvatons are dominant or subdominant at

$$n_s - 1 = -2\epsilon + 2\eta_\sigma, \tag{74}$$

where $\epsilon$ is defined in Eq. (39) and

$$\eta_\sigma = M_{\text{Pl}}^2 \frac{U_\sigma(\sigma)}{3 H^2}, \tag{75}$$

with $U(\sigma)$ being the potential for the spectator field. To be consistent with the observed value ($n_s \simeq 0.96$), one needs to assume (i) a large field inflation model (large $\epsilon$) or (ii) a large negative mass squared for the spectator field. For further discussion, see Ref. [152].
their decays. Assuming a simple quadratic potential for the curvatons, the non-linearity parameters can be given by the ratio of their energy densities to the total one at the first and second curvaton decays, which are the counterparts of the $r_{\text{dec}}$ parameter in the single curvaton case, as well as by the initial amplitudes for the curvatons, which can in turn be constrained by the Planck $f_{\text{NL}}$ limit.

Interestingly, it has been argued in Ref. [37] that the curvature perturbation $\zeta$ can be temporarily enhanced if multiple curvatons dominate the Universe at different epochs. Furthermore, if both curvatons contribute to the final curvature perturbation, it has also been discussed that non-Gaussian signature may come from the trispectrum rather than from the bispectrum. The gravitational waves generated by the temporal enhancement of $\zeta$ from the second-order perturbation have also been investigated in Ref. [37].

### 3.2.3. Modulated reheating scenario

When the decay rate of the inflaton $\phi$ depends on a scalar field $\sigma$ and if $\sigma$ is light enough to acquire quantum fluctuations during inflation, the timing of the decay varies from place to place, which generates fluctuations in the reheating temperature and hence the curvature perturbation $\zeta$. This kind of model is called modulated reheating or inhomogeneous reheating [9,10] and the field $\sigma$ responsible for density fluctuations in this model is called the “modulus” or “modulaton.” A sizable local-type non-Gaussianity can be generated in this model and has been studied in Refs. [27,127,156,157].

By adopting the $\delta N$ formalism, one can write down the curvature perturbation in this model as [127]

$$\zeta_{\text{modulated}} = -\frac{1}{6} \frac{\Gamma_\sigma}{\Gamma} \delta \sigma_* + \frac{1}{12} \left( \frac{\Gamma_\sigma \sigma}{\Gamma} + \frac{\Gamma_\sigma^2}{\Gamma^2} \right) \delta \sigma_*^2 + \frac{1}{36} \left( -\frac{\Gamma_\sigma \sigma \sigma}{\Gamma} + 3 \frac{\Gamma_\sigma \sigma \sigma}{\Gamma^2} - 2 \frac{\Gamma_\sigma^3}{\Gamma^3} \right) \delta \sigma_*^3,$$

(77)

where $\Gamma$ is the decay rate of the inflaton and $\Gamma_\sigma = d\Gamma/d\sigma_*$. Here we assume that the inflaton oscillates under a quadratic potential (hence its energy density decreases as $\rho_\phi \propto a^{-3}$)$^8$ and the decay rate of the inflaton is much smaller than the Hubble parameter at the end of inflation. We note that the second assumption does not necessarily hold [158]. We also assume that the scalar field $\sigma$ does not move significantly during the reheating stage and its dynamics hardly affects the density perturbations. However, in some cases, the dynamics of $\sigma$ does affect the prediction of $\zeta$; such cases are discussed in Ref. [157].

From Eq. (77), one can easily derive the non-linearity parameters as

$$\frac{6}{5} f_{\text{NL}} = 6 - 6 \frac{\Gamma_\sigma \sigma}{\Gamma_\sigma^2},$$

(78)

$$\frac{54}{25} g_{\text{NL}} = 36 \left( 2 - 3 \frac{\Gamma_\sigma \sigma}{\Gamma_\sigma^2} + \frac{\Gamma_\sigma^2 \sigma \sigma \sigma}{\Gamma_\sigma^3} \right).$$

(79)

From these equations, one finds that the predictions for $f_{\text{NL}}$ and $g_{\text{NL}}$ strongly depend on the functional form of $\Gamma$. To discuss this in a quantitative way, here we consider the following form for $\Gamma$:

$$\Gamma = \Gamma_0 \left( 1 + \alpha \frac{\sigma}{M} + \beta \frac{\sigma^2}{M^2} \right),$$

(80)

$^8$ See Ref. [27] for the case in which the inflaton oscillates under other types of potential.
where we Taylor-expand $\Gamma$ in $\sigma/M$ and truncate at the second order with $M$ corresponding to some energy scale. The coefficients $\alpha$ and $\beta$ are the parameters of order unity, which depend on some concrete models. Then the non-linearity parameters are given as

\[
\frac{6}{5} f_{\text{NL}} \simeq 6 \left(1 - \frac{2\beta}{\alpha^2}\right), \quad \frac{54}{25} g_{\text{NL}} \simeq 36 \left(2 - \frac{6\beta}{\alpha^2}\right). \tag{81}
\]

Then, the Planck constraint on $f_{\text{NL}}$ can be translated into that for the combination of $\alpha$ and $\beta$ as $-0.9 < \beta/\alpha^2 < 1.4$ (95% CL)\(^9\) [68]. In this model, the constraint from $f_{\text{NL}}$ restricts the functional form of $\Gamma$ and the parameters in $\Gamma$.

Similarly to the case with the curvaton, isocurvature fluctuations can also arise in some cases [140,159,160], which can also give a severe constraint on the model. Furthermore, we note that the modulaton can be produced from inflaton decay because the modulaton has to couple to the inflaton in this model, which may give a dark radiation component [161].

In the above, we have assumed that the decay rate of the inflaton depends on another scalar field. However, some variants of this model have also been discussed in the literature. We briefly describe some of them below.

- **Modulated decay of the curvaton**
  It is also possible that the decay rate of the curvaton $\sigma$ depends on another scalar field $\chi$. Then the decay rate of the curvaton fluctuates to generate the curvature perturbation [45–47]. The non-linearity parameters in this model can be written as [45]

\[
\frac{5}{6} f_{\text{NL}} = \frac{3}{r_{\text{dec}}} \left(1 - \frac{2\Gamma_{\chi\chi}}{\Gamma_{\chi}^2}\right) - 2 - r_{\text{dec}},
\]

\[
\frac{54}{25} g_{\text{NL}} = \frac{1}{r_{\text{dec}}^2} \left[36 \frac{\Gamma_{\chi\chi}^2}{\Gamma_{\chi}^3} + 18(r_{\text{dec}}^2 + 2r_{\text{dec}} - 9) \frac{\Gamma_{\chi\chi}}{\Gamma_{\chi}^2} + 3r_{\text{dec}}^4 + 10r_{\text{dec}}^3 - 22r_{\text{dec}}^2 - 54r_{\text{dec}} + 135\right].
\tag{83}
\]

where $r_{\text{dec}}$ is the ratio defined for the curvaton energy density given in Eq. (73) and $\Gamma_{\chi} = d\Gamma/d\chi_*$ and so on. As seen from the expressions above, this model shares some similarities with the curvaton and modulated reheating models.

- **Modulated curvaton**
  A modulaton in the modulated reheating model can also act as a curvaton after it generates a curvature perturbation via the modulated reheating mechanism. This kind of model is called the “modulated curvaton” [42,43]. In this model, a single scalar field acts as both the curvaton and modulaton; thus, this is different from the “modulated decay of the curvaton” discussed above, where the modulaton is different from the curvaton. Thus the curvature perturbation in this model is obtained from the sum

\(^9\)Here we have adopted the 95% CL constraint from Planck: $-8.9 < f_{\text{NL}}^{(\text{local})} < 14.3$ [13].
of $\zeta_{\text{curvaton}}$ and $\zeta_{\text{modulated}}$ given in Eqs. (70) and (77), respectively, as

$$\zeta = \zeta_{\text{curvaton}} + \zeta_{\text{modulated}}$$

$$= \left( \frac{2r_{\text{dec}}}{3\sigma_*} - \frac{\Gamma_\sigma}{6\Gamma} \right) \delta \sigma_* + \left[ \frac{1}{9\sigma_*^2} \left( 3r_{\text{dec}} - 4r_{\text{dec}}^2 - 2r_{\text{dec}}^3 \right) - \frac{1}{12} \left( \frac{\Gamma_\sigma}{\Gamma} - \frac{\Gamma_{\sigma\sigma}}{\Gamma_{\sigma}} \right) \right] \delta \sigma_*^2$$

$$+ \left[ \frac{4}{81\sigma_*^5} \left( -9r_{\text{dec}}^2 + \frac{r_{\text{dec}}^3}{2} + 10r_{\text{dec}}^4 + 3r_{\text{dec}}^5 \right) - \frac{1}{36} \left( \frac{2\Gamma_{\sigma\sigma}}{\Gamma^3} - \frac{3\Gamma_\sigma\Gamma_{\sigma\sigma}}{\Gamma^2} + \frac{\Gamma_{\sigma\sigma}}{\Gamma} \right) \right] \delta \sigma_*^3,$$

(84)

from which one can calculate $f_{\text{NL}}$ and $g_{\text{NL}}$.

3.2.4. Inhomogeneous end of hybrid inflation. The curvature perturbation can also be generated at the end of inflation if its end is controlled by some light scalar field that fluctuates. This kind of situation can arise in a hybrid inflation model. Here let us consider the potential for hybrid inflation at the end of inflation if its end is controlled by some light scalar field that fluctuates. This kind of situation can arise in a hybrid inflation model. Here let us consider the potential for hybrid inflation of the form [18]:

$$V = \frac{\lambda}{4} \left( \frac{v^2}{\lambda} - \chi^2 \right)^2 + \frac{1}{2} g^2 \phi^2 \chi^2 + \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} f^2 \sigma^2 \chi^2 + \frac{1}{2} m_\sigma^2 \sigma^2,$$

(85)

where $\phi$ and $\chi$ are an inflaton and a waterfall field, respectively. $\sigma$ is a light field that acquires fluctuations and couples to $\chi$. $m_\phi$ and $m_\sigma$ are the masses for $\phi$ and $\sigma$. $\lambda$, $g$, and $f$ are coupling constants and $v$ corresponds to the vacuum expectation value.

The effective mass squared of the waterfall field $\chi$ is written as

$$m_\chi^2 = -v^2 + g^2 \phi^2 + f^2 \sigma^2,$$

(86)

and when $m_\chi^2$ becomes negative, inflation ends due to the tachyonic instability, which occurs when $\phi$ reaches a critical value:

$$\phi_{\text{cr}} = \sqrt{\frac{v^2 - f^2 \sigma^2}{g}}.$$

(87)

Here it should be noticed that the critical value depends on a fluctuating scalar field $\sigma$, thus $\phi_{\text{cr}}$ also fluctuates, which generates the adiabatic fluctuations [16–19].

The non-linearity parameters in this model can be calculated as [18,42]

$$\frac{6}{5} f_{\text{NL}} \simeq -M_{\text{Pl}} \sqrt{2\epsilon_{\text{cr}} \phi_{\text{cr}}'' / \phi_{\text{cr}}'} = \eta_{\phi|_{\text{cr}}} v^2 / f^2 \sigma^2,$$

(88)

$$\frac{54}{25} g_{\text{NL}} \simeq -(2\epsilon_{\text{cr}} - \eta_{\phi|_{\text{cr}}}) \frac{18}{5} f_{\text{NL}} + 2M_{\text{Pl}}^2 \epsilon_{\text{cr}} \phi_{\text{cr}}'' / \phi_{\text{cr}}'^3 = 6\eta_{\phi|_{\text{cr}}} v^2 / f^2 \sigma^2,$$

(89)

where $\epsilon_{\text{cr}}$ and $\eta_{\phi|_{\text{cr}}}$ are the slow-roll parameters evaluated at the time when $\phi$ reaches $\phi_{\text{cr}}$ and a prime represents the derivative with respect to $\sigma$. In the above expressions, we have neglected slow-roll suppressed terms. The Planck constraint on $f_{\text{NL}}^{(\text{local})}$ can be translated into that for a particular combination of parameters as $-10.7 < \eta_{\text{cr}} v^2 / (f^2 \sigma^2) < 17.2$ (95% CL) [68].

3.2.5. Mixed inflaton and spectator model. Even if we consider models with the curvaton, modulated reheating, and so on, where a spectator field is assumed to be responsible for the generation of the curvature perturbation, we still need the inflaton to drive the inflationary expansion. In general, the inflaton would also acquire quantum fluctuations during inflation and it can contribute to the curvature perturbation $\zeta$ as well. In such a case, multiple sources can be responsible for the final $\zeta$. This
kind of model is called the mixed inflaton and spectator field model [165]; some explicit examples are discussed in the literature, such as the mixed inflaton and curvaton model [21–25], the mixed inflaton and modulated reheating model [27], and so on.

Here we briefly discuss a general mixed inflaton and spectator field model [165], where fluctuations from the inflaton \( \phi \) and the spectator field \( \sigma \) such as the curvaton and the modulaton both contribute to the curvature perturbation. By using the formulas given in Eqs. (66), (67), and (68), the non-linearity parameters in this model are given by

\[
\frac{6}{5} f_{\text{NL}} = \left( \frac{1}{1 + R} \right)^2 \left[ \frac{N_{\phi \phi}}{N_\phi^2} + R^2 \frac{N_{\sigma \sigma}}{N_\sigma^2} \right] = \left( \frac{1}{1 + R} \right)^2 \left[ \frac{6}{5} f_{\text{NL}}^{(\phi)} + R^2 \left( \frac{6}{5} f_{\text{NL}}^{(\sigma)} \right) \right],
\]

\[
\tau_{\text{NL}} = \left( \frac{1}{1 + R} \right)^3 \left[ \left( \frac{N_{\phi \phi}}{N_\phi^2} \right)^2 + R^3 \left( \frac{N_{\sigma \sigma}}{N_\sigma^2} \right)^2 \right] = \left( \frac{1}{1 + R} \right)^3 \left[ \left( \frac{6}{5} f_{\text{NL}}^{(\phi)} \right)^2 + R^3 \left( \frac{6}{5} f_{\text{NL}}^{(\sigma)} \right)^2 \right],
\]

\[
\frac{54}{25} g_{\text{NL}} = \left( \frac{1}{1 + R} \right)^3 \left[ \frac{N_{\phi \phi \phi}}{N_\phi^3} + R^3 \frac{N_{\sigma \sigma \sigma}}{N_\sigma^3} \right] = \left( \frac{1}{1 + R} \right)^3 \left[ \frac{54}{25} g_{\text{NL}}^{(\phi)} + R^3 \left( \frac{54}{25} g_{\text{NL}}^{(\sigma)} \right) \right],
\]

where we have defined the ratio of the power spectra from the inflaton \( P^{(\phi)}_\zeta \) and the spectator \( P^{(\sigma)}_\zeta \) as

\[
R \equiv \frac{P^{(\sigma)}_\zeta}{P^{(\phi)}_\zeta}.
\]

In addition, \( f_{\text{NL}}^{(\phi)} \) (\( f_{\text{NL}}^{(\sigma)} \)) corresponds to \( f_{\text{NL}} \) for the case when only the inflaton (the spectator field) is responsible for the curvature perturbation. In general, non-Gaussianities from the inflaton are small compared to those from the spectator field, and thus, in practice, we can neglect the contributions from the inflaton \( f_{\text{NL}}^{(\phi)} \) and \( g_{\text{NL}}^{(\phi)} \) in the above formulas. In such a case, from Eqs. (90) and (91), one finds that \( \tau_{\text{NL}} \) and \( f_{\text{NL}} \) are related as

\[
\tau_{\text{NL}} \approx \frac{1 + R}{R} \left( \frac{6}{5} f_{\text{NL}} \right)^2.
\]

Thus, even if \( f_{\text{NL}} \) is small, \( \tau_{\text{NL}} \) can be large when the fractional contribution of the spectator field to the power spectrum is small.

Here we also remark that, in the mixed model, the non-linearity parameter for the spectator sector \( f_{\text{NL}}^{(\sigma)} \) can be large by having a small value of \( R \) to satisfy the Planck constraint on \( f_{\text{NL}} \). However, \( \tau_{\text{NL}} \) becomes large instead in this case, which can be seen from Eq. (94). This shows that the trispectrum would still be useful to test this kind of scenario\(^{11}\).

\(^{10}\)Here we neglect the contribution from cross terms like \( N_{\phi \sigma} \), which can be justified when we consider a quadratic potential for the curvaton and a slow-rolling inflaton [166].

\(^{11}\)Constraints on the spectral index and the tensor-to-scalar ratio are also important to test the model. In the mixed model, they are given by

\[
n_s - 1 = -2 \epsilon + 2 \eta_\sigma + \frac{-4 \epsilon + 2 \eta_\phi - 2 \eta_\sigma}{1 + R}, \quad r = \frac{16 \epsilon}{1 + R}.
\]

The implications of the constraints on \( n_s \) and \( r \) from the Planck results for the mixed inflaton and spectator field model are discussed in Ref. [165].
3.2.6. Ungaussiton model. When there are two scalar fields that both contribute to the curvature perturbation, one can consider a situation where one field only contributes to the linear order, but the other only gives the second-order contribution, in which \( \zeta \) can be written as

\[
\zeta = N_\phi \delta \phi + \frac{1}{2} N_\sigma (\delta \sigma)^2 + \cdots,
\]

where \( \phi \) can be regarded as the inflaton and \( \sigma \) is some light scalar field. This kind of model has been discussed in Refs. [28,29] and dubbed the “ungaussiton” model [29].

In this model, the non-linearity parameters are dominated by the so-called loop contributions and can be written as [29]

\[
\frac{6}{5} f_{NL} = \frac{N_\sigma^3 \mathcal{P}_\delta \log (kL)}{N_\phi^4},
\]

\[
\tau_{NL} = \frac{N_\sigma^4 \mathcal{P}_\delta \log (kL)}{N_\phi^6},
\]

\[
\frac{54}{25} g_{NL} = \frac{3N_\sigma^2 N_\sigma N_{\sigma \sigma \sigma \sigma} \mathcal{P}_{\delta \delta} \log (kL) + 3N_\sigma^2 N_{\sigma \sigma} N_{\sigma \sigma \sigma \sigma} \mathcal{P}_{\delta \delta} \log (kL)}{N_\phi^3},
\]

where \( L \) corresponds to the box size in which the Fourier modes are taken and \( \log (kL) \) is usually assumed to be \( O(1) \). From the above expressions, one can derive the relation between \( f_{NL} \) and \( \tau_{NL} \) as

\[
\tau_{NL} = \left( \frac{6}{5} \right)^{4/3} \mathcal{P}_{\zeta}^{-1/3} f_{NL}^{4/3} \sim 10^3 f_{NL}^{4/3},
\]

where we set \( \log (kL) \) to be unity and \( \mathcal{P}_{\zeta} \sim 10^{-9} \) is used in the last equality. Hence, in this model, \( \tau_{NL} \) can be large even when \( f_{NL} \sim O(1) \).

4. Scale-dependence of non-Gaussianity

Usually, the non-linearity parameters such as \( f_{NL} \), \( \tau_{NL} \), and \( g_{NL} \) are assumed to be constant. However, they can be (strongly) scale-dependent in some cases and may serve as a useful probe of models of primordial fluctuations. Furthermore, when multiple observations are made on different scales, and if those measure different values for the non-linearity parameters, it may be explained by the scale-dependence of non-Gaussianity, which we discuss in this section. A general discussion on the scale-dependence of non-linearity parameters is given in Refs. [168–170].

The scale-dependence of \( f_{NL} \) is defined in the same manner as the spectral index for the power spectrum as

\[
n_{f_{NL}} \equiv \frac{d \ln |f_{NL}|}{d \ln k}.
\]

Explicit examples of models producing strong scale-dependence have been discussed in the literature, for the local type, such as a self-interacting curvaton [166,171,172], a pseudo-Nambu–Goldstone curvaton [166,173], the mixed inflaton and curvaton model [166,171], the multi-curvaton model [166,171], a model with a DBI isocurvature field [174], the axion N-flation model [175], and, for the equilateral type, the DBI inflation model [110].

Here we focus on the scale-dependence of the local-type \( f_{NL} \) to discuss some explicit models. For the local type, \( f_{NL} \) can be strongly scale-dependent when: (i) multiple sources simultaneously contribute to density perturbation, or (ii) the third derivative of the potential of a scalar field is non-vanishing. Examples of the former case (i) include the mixed inflaton and curvaton and the multi-curvaton models. First, let us consider the mixed inflaton and curvaton model. When the inflaton
\( \phi \) and the curvaton \( \sigma \) are simultaneously responsible for the density perturbation, we can explicitly write down the expression for \( n_{fNL} \) as [166,168]

\[
n_{fNL} \simeq \frac{5}{6f_{NL}} \left[ \left( \frac{1}{1+R} \right)^2 (8\epsilon^2 - 6\epsilon \eta_\phi + \xi_\phi) + 4 \frac{R}{(1+R)^2} \left( 2\epsilon - \eta_\phi + \eta_\sigma \right) \right] \left\{ \left( \frac{1}{1+R} \right) (2\epsilon - \eta_\phi) + \left( \frac{R}{1+R} \right) \frac{N_{2\sigma}^2}{N_{2\sigma}^2} \right\} ,
\]

where \( R \) is the ratio defined in Eq. (93) and \( \xi_\phi = M_4^4 (V_{\phi} V_{\phi\phi\phi} / V^2) \) is the slow-roll parameter defined with the inflaton potential \( V(\phi) \). Here we have assumed a quadratic potential for the curvaton. In fact, this formula can be applied to general inflaton and spectator field models. A concrete analysis of a mixed inflaton and curvaton model has been done in Ref. [166] and it has been shown that \( n_{fNL} \) can be sizable for some parameter ranges.

When the scalar fields responsible for density fluctuations are both subdominant during inflation (i.e., act as isocurvature fields), such as in the case of the multi-curvaton model, \( n_{fNL} \) is given by

\[
n_{fNL} \simeq \frac{10}{5f_{NL}} K_a K_b (\eta_a - \eta_b) \left[ K_a N_{aa}^2 - K_b N_{bb}^2 - (K_a - K_b) \frac{N_{ab}^2}{N_{aa} N_{bb}} \right],
\]

where “\( a \)” and “\( b \)” denote curvaton fields and \( K_a \) and \( K_b \) are defined as

\[
K_a = \frac{N_a^2}{N_a^2 + N_b^2}, \quad K_b = \frac{N_b^2}{N_a^2 + N_b^2}.
\]

\( \eta_a \) and \( \eta_b \) are the slow-roll \( \eta_\phi \) parameters defined for the curvatons “\( a \)” and “\( b \)”.

Next we consider case (ii). When one assumes a potential for the curvaton that deviates from a quadratic form, the scale-dependence of \( f_{NL} \) appears and \( n_{fNL} \) is given by

\[
n_{fNL} \simeq \frac{5}{6f_{NL} N_\sigma} \frac{U_{\sigma\sigma\sigma}(\sigma^*)}{3H_\sigma^2}.
\]

Thus a self-interacting curvaton and pseudo-Nambu–Goldstone curvaton can produce (strongly) scale-dependent \( f_{NL} \) and some explicit analysis of this case has been done in Refs. [166,171,172].

In Refs. [176–178], the expected constraints on \( n_{fNL} \) have been investigated for CMB and several LSS surveys. Future observations of CMB \( \mu \)-distortion may also be able to probe the scale-dependence [179]. The current limit on \( n_{fNL} \) from the WMAP 7-year data is [180]

\[
n_{fNL} = 0.3^{+1.9}_{-1.2} \quad (95\% \text{ CL}).
\]

We should note here that the observability of \( n_{fNL} \) depends on the value of \( f_{NL} \).

From the Planck results, \( f_{NL}^{(local)} \) on CMB scales is now tightly constrained. However, \( f_{NL} \) can also be probed with observations of small scales such as void and halo abundances [181]. If \( f_{NL} \) is strongly scale-dependent, such observations might see a signature of non-Gaussianity even if one does not see any on large scales.

5. Non-Gaussianity from isocurvature fluctuations

5.1. Formalism and observational constraints

When there are multiple sources of density fluctuations, an isocurvature mode can be generated in addition to the adiabatic one. Although current cosmological observations such as CMB have
already put severe constraints on the size of isocurvature fluctuations, some fractional contribution is still allowed. Furthermore, even when the isocurvature contribution is small at linear level, it does not necessarily mean that non-Gaussianity in isocurvature fluctuations is also small compared to the adiabatic one. Even if we do not find any non-Gaussian signatures from an isocurvature mode, the constraints from non-Gaussianity may give information different from that of the power spectrum. Therefore, isocurvature non-Gaussianity would be worth investigating in light of the above considerations.

First, we briefly describe the formalism to discuss isocurvature non-Gaussianity. The non-linear isocurvature perturbations are defined for a component $i$, with respect to radiation, as

$$S_i \equiv 3(\xi_i - \zeta),$$

where $\xi_i$ is the curvature perturbation on the uniform energy density hypersurface of a component $i$ and is given as

$$\xi_i = \delta N + \frac{1}{3(1 + w_i)} \ln \left( \frac{\rho_i(t, \vec{x})}{\bar{\rho}_i(t)} \right).$$

Here $w_i$ is an equation of state for a component $i$ and is assumed to be constant. $\bar{\rho}_i(t)$ is the background energy density of the component $i$. Then, in the same manner as the curvature perturbation for the local type, we can define the non-linearity parameter for isocurvature fluctuations $f_{NL}^{(iso)}$ as

$$S_i = S_{ig} + f_{NL}^{(iso)} S_{ig}^2 + \cdots,$$

where $S_{ig}$ is the linear part for $S_i$. Since, in general, adiabatic and isocurvature fluctuations coexist, a 3-point correlation function can be written as

$$\langle X_i^{k_1} X_i^{k_2} X_i^{k_3} \rangle = (2\pi)^3 \delta(k_1 + k_2 + k_3) B^{IKK}(k_1, k_2, k_3),$$

where $X^I = \xi$ or $S_i$ and the bispectrum $B^{IKK}(k_1, k_2, k_3)$ is given by

$$B^{IKK}(k_1, k_2, k_3) = f_{NL}^{(iso)} P_{IK}(k_2) P_{IK}(k_3) + f_{NL}^{(iso)} P_{IK}(k_3) P_{IK}(k_1) + f_{NL}^{(iso)} P_{IK}(k_1) P_{IK}(k_2).$$

Here $f_{NL}^{(iso)}$ is a general non-linearity parameter and, because of the symmetric structure of the latter two indices, one has the relation $f_{NL}^{(iso)} = f_{NL}^{(iso)}$. For instance, when all indices are $S$, it matches the one defined in Eq. (109), i.e., $f_{NL}^{SS,SS} = f_{NL}^{SS}$. For $I, J, K = \xi$, it corresponds to the usual non-linearity parameter for the local type, i.e., $f_{NL}^{\xi,\xi,\xi} = \frac{3}{5} f_{NL}^{\xi}$. $P_{IJ}(k)$ is the power spectrum defined as

$$\langle X_i^{k_1} X_i^{k_2} \rangle = P_{II}(k_1) (2\pi)^3 \delta(k_1 + k_2).$$

For the case with $X^I = X^J = \xi$, the definition of the power spectrum matches that of Eq. (1). When isocurvature fluctuations exist, we can also define the power spectra for the isocurvature mode and its cross correlation with the adiabatic one as $P_S(= P_{Si,S})$ and $P_{\xi S}$, which correspond to $\langle S_i S_i \rangle$ and $\langle \xi S \rangle$, respectively. The correlated part $P_{\xi S}$ may vanish, depending on the model. To parametrize the correlation between the adiabatic and isocurvature modes, it is conventional to define the correlation angle $\Delta$ as follows:

$$\cos \Delta \equiv \frac{P_{\xi S}(k_{\text{ref}})}{P_{\xi}^{1/2}(k_{\text{ref}}) P_{S}^{1/2}(k_{\text{ref}})}.$$

When $\cos \Delta = +1$ and $-1$, we denote them as positively and negatively correlated modes, respectively. The case of $\cos \Delta = 0$ corresponds to the uncorrelated mode.
As mentioned above, current cosmological observations already give a severe constraint on the size of the isocurvature power spectrum. To investigate the constraint, it is customary to use the ratio of the power spectrum of the isocurvature mode to the adiabatic one or to the total one, which are defined as

\[
\alpha \equiv \frac{P_S(k_{\text{ref}})}{P_{\zeta}(k_{\text{ref}})}, \quad \beta \equiv \frac{P_S(k_{\text{ref}})}{P_{\zeta}(k_{\text{ref}}) + P_S(k_{\text{ref}})},
\]

(114)

where \(k_{\text{ref}}\) is the scale at which the ratio is evaluated. In the literature, either one is used to characterize the size of the isocurvature contributions. We note that, as far as \(\alpha, \beta \ll 1\), these ratios are almost equivalent.

The Planck team has reported the constraints on \(\beta\) [12]. Depending on the degree of correlation, the constraints are different. Although, in general, the spectral index for the isocurvature power spectrum \(n_{\text{iso}} - 1 \equiv d \log P_S / d \log k\) is regarded as a free parameter, for some models, such as the axion (uncorrelated mode) and the curvaton (correlated mode), \(n_{\text{iso}}\) would be almost scale invariant for the former, and the same as the adiabatic mode for the latter. For these particular cases, the constraints are [12]

\[
\beta_0(n_{\text{iso}} = 1) < 0.036, \quad \beta_1(n_{\text{iso}} = n_s) < 0.0025, \quad \beta_{-1}(n_{\text{iso}} = n_s) < 0.0087 \quad (95\% \text{ CL}),
\]

(115)

where the ratio is defined at \(k_{\text{ref}} = 0.002 \text{ Mpc}^{-1}\) and the subscripts 0, +1, and −1 indicate that the quantities are for uncorrelated, positively (fully) correlated, and negatively (fully) correlated, respectively.

Constraints on non-Gaussianities from isocurvature fluctuations have also been investigated using the WMAP data [184–186] and also for expected future data [187, 188]. In general, there are 6 non-linearity parameters defined as in Eq. (111): \(J_{\text{NL}}^{\zeta, \zeta}, J_{\text{NL}}^{\zeta, S}, J_{\text{NL}}^{S, \zeta}, J_{\text{NL}}^{S, S}, f_{\text{NL}}^{\zeta, \zeta}, f_{\text{NL}}^{S, S}\). However, if we consider an uncorrelated isocurvature mode, they are reduced to two as

\[
f_{\text{NL}}^{\zeta, \zeta} = \frac{3}{5} f_{\text{NL}}^{(\text{local})}, \quad f_{\text{NL}}^{S, S} = \alpha^2 f_{\text{NL}}^{(\text{iso})}, \quad f_{\text{NL}}^{I, JK} = 0 \quad (\text{others}).
\]

(116)

When one considers a correlated isocurvature case, one can write \(f_{\text{NL}}^{I, JK}\) as

\[
f_{\text{NL}}^{\zeta, \zeta} = \frac{3}{5} f_{\text{NL}}^{(\text{local})}, \quad f_{\text{NL}}^{\zeta, S} = \alpha^{1/2} \cos \left( \frac{3}{5} f_{\text{NL}}^{(\text{local})} \right), \quad f_{\text{NL}}^{S, \zeta} = \alpha \cos^2 \left( \frac{3}{5} f_{\text{NL}}^{(\text{local})} \right), \quad f_{\text{NL}}^{S, S} = \alpha^{3/2} \cos \Delta f_{\text{NL}}^{(\text{iso})}, \quad f_{\text{NL}}^{S, I, JK} = \alpha^2 f_{\text{NL}}^{(\text{iso})},
\]

(117)

where we have used the ratio \(\alpha\) here to represent the isocurvature contribution. Observational constraints from the WMAP data obtained in Ref. [186] are summarized in Tables 1 and 2 for the cases with uncorrelated/correlated CDM and neutrino density isocurvature modes. For the uncorrelated case, we show the constraint on \(\alpha^2 f_{\text{NL}}^{(\text{iso})}\). For the correlated case, we show the constraint on \(\alpha f_{\text{NL}}^{(\text{iso})}\), which corresponds to the \(f_{\text{NL}}^{S, \zeta}\) part. The other parts can also contribute to the bispectrum; however, their contribution is subdominant as long as \(\alpha \ll 1\). We thus show the constraint on \(\alpha f_{\text{NL}}^{(\text{iso})}\) alone.

\[\text{In Ref. [12], constraints on the ratio } \beta \text{ are reported for those at three different scales: } k_{\text{low}} = 0.002 \text{ Mpc}^{-1}, \]

\(k_{\text{mid}} = 0.05 \text{ Mpc}^{-1}, \text{ and } k_{\text{high}} = 0.10 \text{ Mpc}^{-1}. \text{ Here we show the values for } k_{\text{low}}. \text{ We note that, when the spectral index for the isocurvature mode is fixed as stated in the text, the constraints on different scales are almost the same. However, when one varies } n_{\text{iso}} \text{ freely in the analysis, the ratio on small scales (or blue-tilted } n_{\text{iso}} \text{) is less constrained by the data, in particular for the case with uncorrelated modes [182,183].} \]
Table 1. Constraints on $f_{NL}$ and $f_{NL}^{\text{iso}}$ for uncorrelated isocurvature fluctuations [186]. Results with template marginalization of the Galactic foregrounds are given. The values without parentheses are the constraints obtained by fixing the other non-linearity parameter to zero. For the values with parentheses, they are obtained by marginalizing over the other non-linearity parameter. In the table, constraints on two different spectral indices $n_{\text{iso}}$ are given. For details of the analysis, see Ref. [186].

| spectral index | $f_{NL}$ | $\alpha f_{NL}^{\text{iso}}$ |
|----------------|---------|------------------------|
| uncorrelated   | $n_{\text{iso}} = 0.963$ | $37 \pm 21$ | $22 \pm 64$ |
| CDM            | $n_{\text{iso}} = 1$ | $(41 \pm 23)$ | $(-28 \pm 71)$ |
|                |         | $(33 \pm 21)$ | $30 \pm 66$ |
|                |         | $(35 \pm 23)$ | $(-15 \pm 72)$ |
| uncorrelated   | $n_{\text{iso}} = 0.963$ | $34 \pm 21$ | $164 \pm 143$ |
| neutrino density | $n_{\text{iso}} = 1$ | $(48 \pm 39)$ | $(-116 \pm 266)$ |
|                |         | $(48 \pm 40)$ | $(-87 \pm 257)$ |

Table 2. Constraints on $f_{NL}$ and $f_{NL}^{\text{iso}}$ for correlated isocurvature fluctuations [186]. Results with template marginalization of the Galactic foregrounds are given. The values without parentheses are the constraints obtained by fixing the other non-linearity parameter to zero. For the values with parentheses, they are obtained by marginalizing over the other non-linearity parameter. In the table, constraints on two different spectral indices $n_{\text{iso}}$ are given. For details of the analysis, see Ref. [186].

| spectral index | $f_{NL}$ | $\alpha f_{NL}^{\text{iso}}$ |
|----------------|---------|------------------------|
| correlated     | $n_{\text{iso}} = 0.963$ | $34 \pm 21$ | $90 \pm 120$ |
| CDM            | $n_{\text{iso}} = 1$ | $(37 \pm 25)$ | $(-26 \pm 144)$ |
|                |         | $(37 \pm 21)$ | $99 \pm 117$ |
|                |         | $(49 \pm 25)$ | $(-25 \pm 141)$ |
| correlated     | $n_{\text{iso}} = 0.963$ | $35 \pm 21$ | $82 \pm 54$ |
| neutrino density | $n_{\text{iso}} = 1$ | $(72 \pm 75)$ | $(-78 \pm 191)$ |
|                |         | $(67 \pm 80)$ | $86 \pm 53$ |
|                |         | $(67 \pm 80)$ | $(-80 \pm 204)$ |

Furthermore, as given in Eq. (117), $f_{NL}^{S,\xi}$ in general corresponds to $\alpha \cos^2 \Delta f_{NL}^{\text{iso}}$. Actually, as far as $\cos \Delta \gg \sqrt{\alpha}$ is satisfied, the constraints on $\alpha f_{NL}^{\text{iso}}$ can be regarded as that on $\alpha \cos^2 \Delta f_{NL}^{\text{iso}}$, which includes the correlation angle.

We do not find any non-Gaussianities from isocurvature fluctuations. However, we can use these bounds to obtain constraints on some explicit models such as the axion and the curvaton.

In addition to the bispectrum, the trispectrum from isocurvature fluctuations has also been investigated [189,190].

5.2. Implications for concrete models

As previously mentioned, the amplitude of isocurvature fluctuations has already been severely constrained by cosmological observations. However, some fractional contribution to the total density fluctuations is still allowed. Isocurvature fluctuations arise in association with CDM, baryons, neutrinos, and also for dark radiation. Therefore the study of isocurvature mode may give useful information on these sectors.

Explicit examples are the axion [191–198] for the CDM mode, and Affleck–Dine baryogenesis [199–201] for the baryonic mode, respectively. Non-Gaussianities in these isocurvature scenarios
have been studied in Refs. [184,185,189,202] for the axion model and in Ref. [203] for Affleck–Dine baryogenesis.

Furthermore, in the curvaton model, correlated CDM or baryon isocurvature fluctuations can be generated depending on when and how CDM and the baryon are produced [131,138–140,154,155], and its non-Gaussianity has been studied in Refs. [186,190,204–206]. When there are non-zero neutrino chemical potentials, neutrino density isocurvature fluctuations can be generated in the curvaton model [155,207,208]. Correlated isocurvature fluctuations can also arise in a double inflation model [209,210], the non-Gaussian nature of which has been investigated in Ref. [204]. The issues of isocurvature fluctuations in dark radiation have been investigated in Refs. [211,212].

Constraints on non-Gaussianity in isocurvature fluctuations described in the previous subsection may probe some aspects of models that cannot be obtained from the power spectrum. For the implications of observational constraints given in Tables 1 and 2, we refer the reader to Refs. [185,186].

6. Conclusion

In this review, we have discussed primordial non-Gaussianities as a probe of the inflationary universe. Now Planck has provided severe constraints on the non-linearity parameters, especially for $f_{NL}$, given in Sect. 2. These constraints have many implications for models of primordial fluctuations. The simplest single-field inflation models are consistent with the current observational bounds on non-Gaussianities. However, many other models are also consistent with the current bounds, since $f_{NL} \sim \mathcal{O}(1)$ can be achieved by choosing suitable model parameters. Nevertheless, the current bounds restrict the parameter space of these more complicated models.

Compared to the bispectrum, the trispectrum is currently less constrained by the data. Although the non-linearity parameters in the trispectrum, $\tau_{NL}$ and $g_{NL}$, in most models are of the same order of magnitude as $f_{NL}$, in some cases the trispectrum parameters can be much larger than $f_{NL}$ (such examples are discussed in Ref. [68]). Furthermore, the non-linearity parameters may be strongly scale-dependent in some models; thus, even if the size of non-Gaussianities is small on CMB scales, they may be large, in particular, on small scales that can be probed with small-scale observations, e.g., void and halo abundances [181]. In addition, as discussed in this review, isocurvature non-Gaussianity may still give constraints on dark matter, baryogenesis, and so on.

In the future, CMB experiments could give constraints on $f_{NL}$ at the level of $\mathcal{O}(1)$ (see, e.g., Ref. [213]). Observations of the 21 cm line of neutral hydrogen may potentially be able to reach down to the level of $f_{NL} \sim \mathcal{O}(0.01)$ [214] (see also Ref. [215]). This level of precision can differentiate whether the origin of primordial fluctuations is a standard slow-roll single-field inflation or not, since the single-field inflation predicts $f_{NL} \sim 1 - n_s \sim \mathcal{O}(0.01)$, which is a firm prediction because the current observations precisely measure $n_s$ as $n_s - 1 \simeq \mathcal{O}(0.01)$. On the other hand, alternative models to inflation generally give $f_{NL} \sim \mathcal{O}(1)$ (unless some fine-tuning is assumed).

In addition to the bispectrum and the trispectrum, which have been well investigated so far, even higher-order statistics such as 5- or 6-point correlation functions might have the potential to give insights into the origin of primordial fluctuations [216–219]. To conclude, further study of primordial non-Gaussianity would pin down the origin of density fluctuations, which in turn will give us a precise picture of the early universe.

Acknowledgements

The author is grateful to Toyokazu Sekiguchi, Teruaki Suyama, Masahide Yamaguchi, and Shuichiro Yokoyama for valuable comments on the manuscript. The author would also like to thank Kazuya Koyama and Shuntaro.
Mizuno for the useful discussion on the equilateral-type trispectrum and clarification of the definition of $g_{NL}^{\text{equil}}$.

This work is partially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports, and Culture, Japan, No. 23740195.

References

[1] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. 33, 549 (1981)].
[2] S. W. Hawking, Phys. Lett. B 115, 295 (1982).
[3] A. A. Starobinsky, Phys. Lett. B 117, 175 (1982).
[4] A. H. Guth and S. Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).
[5] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D 28, 679 (1983).
[6] K. Enqvist and M. S. Sloth, Nucl. Phys. B 626, 395 (2002) [arXiv:0109214 [hep-ph]].
[7] D. H. Lyth and D. Wands, Phys. Lett. B 524, 5 (2002) [arXiv:0110002 [hep-ph]].
[8] T. Moroi and T. Takahashi, Phys. Lett. B 522, 215 (2001); 539, 303 (2002) [erratum] [arXiv:0110096 [hep-ph]].
[9] G. Dvali, A. Gruzinov, and M. Zaldarriaga, Phys. Rev. D 69, 023505 (2004) [arXiv:0303591 [astro-ph]].
[10] L. Kofman, arXiv:0303614 [astro-ph].
[11] P. A. R. Ade et al. [Planck Collaboration], arXiv:1303.5062 [astro-ph.CO].
[12] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D 28, 679 (1983).
[13] M. P. Salem, Phys. Rev. D 72, 023516 (2005) [arXiv:0501096 [hep-ph]].
[14] T. Moroi, T. Takahashi, and Y. Toyoda, Phys. Rev. D 72, 023502 (2005) [arXiv:0501007 [hep-ph]].
[15] K. Ichikawa, T. Suyama, T. Takahashi, and M. Yamaguchi, Phys. Rev. D 78, 023513 (2008) [arXiv:0802.4138 [astro-ph]].
[16] J. Fonseca and D. Wands, J. Cosmol. Astropart. Phys. 1206, 028 (2012) [arXiv:1204.3443 [astro-ph.CO]].
[39] M. Kawasaki, T. Takahashi, and S. Yokoyama, J. Cosmol. Astropart. Phys. 0912, 012 (2009) [arXiv:0910.3053 [hep-th]].
[40] D. Langlois and L. Sorbo, J. Cosmol. Astropart. Phys. 0908, 014 (2009) [arXiv:0906.1813 [astro-ph.CO]].
[41] D. Battefeld, T. Battefeld, C. Byrnes, and D. Langlois, J. Cosmol. Astropart. Phys. 1108, 025 (2011) [arXiv:1106.1891 [astro-ph.CO]].
[42] T. Suyama, T. Takahashi, M. Yamaguchi, and S. Yokoyama, J. Cosmol. Astropart. Phys. 1012, 030 (2010) [arXiv:1009.1979 [astro-ph.CO]].
[43] K.-Y. Choi and O. Seto, Phys. Rev. D 85, 123528 (2012); 87, 029902 (2013) [erratum] [arXiv:1204.1419 [astro-ph.CO]].
[44] K. Nakayama and T. Suyama, Phys. Rev. D 84, 063520 (2011) [arXiv:1107.3003 [astro-ph.CO]].
[45] D. Langlois and T. Takahashi, J. Cosmol. Astropart. Phys. 1304, 014 (2013) [arXiv:1301.3319 [astro-ph.CO]].
[46] H. Assadullahi, H. Firouzjahi, M. H. Namjoo, and D. Wands, J. Cosmol. Astropart. Phys. 1303, 041 (2013) [arXiv:1301.3787 [hep-th]].
[47] S. Enomoto, K. Kohri, and T. Matsuda, J. Cosmol. Astropart. Phys. 1308, 047 (2013) [arXiv:1301.3787 [hep-ph]].
[48] K. Koyama, S. Mizuno, F. Vernizzi, and D. Wands, J. Cosmol. Astropart. Phys. 0711, 024 (2007) [arXiv:0708.4321 [hep-th]].
[49] E. I. Buchbinder, J. Khoury, and B. A. Ovrut, Phys. Rev. D 85, 123528 (2012) [arXiv:1204.1419 [hep-th]].
[70] J. R. Fergusson, D. M. Regan, and E. P. S. Shellard, arXiv:1012.6039 [astro-ph.CO].
[71] S. Mizuno and K. Koyama, J. Cosmol. Astropart. Phys. 1010, 002 (2010) [arXiv:1007.1462 [hep-th]].
[72] K. Izumi, S. Mizuno, and K. Koyama, Phys. Rev. D 85, 023521 (2012) [arXiv:1109.3746 [astro-ph.CO]].
[73] D. Regan, M. Gosenca, and D. Seery, arXiv:1310.8617 [astro-ph.CO].
[74] C. L. Bennett et al. [WMAP Collaboration], Astrophys. J. Suppl. 208, 20 (2013) [arXiv:1212.5225 [astro-ph.CO]].
[75] J.-Q. Xia, M. Viel, C. Baccigalupi, G. De Zotti, S. Matarrese, and L. Verde, Astrophys. J. 717, L17 (2010) [arXiv:1003.3451 [astro-ph.CO]].
[76] A. J. Ross et al., Mon. Not. R. Astron. Soc. 428, 1116 (2013) [arXiv:1208.1491 [astro-ph.CO]].
[77] T. Giannantonio, A. J. Ross, W. J. Percival, R. Crittenden, D. Bacher, M. Kilbinger, R. Nichol, and J. Weller, Phys. Rev. D 89, 023511 (2014) [arXiv:1303.1349 [astro-ph.CO]].
[78] A. Mana, T. Giannantonio, J. Weller, B. Hoyle, G. Huetsi, and B. Sartoris, Mon. Not. R. Astron. Soc. 434, 684 (2013) [arXiv:1303.0287 [astro-ph.CO]].
[79] S. Shandera, A. Mantz, D. Rapetti, and S. W. Allen, J. Cosmol. Astropart. Phys. 1308, 004 (2013) [arXiv:1304.1216 [astro-ph.CO]].
[80] D. Karagiannis, T. Shanks, and N. P. Ross, arXiv:1310.6716 [astro-ph.CO].
[81] S. Ho et al., arXiv:1311.2597 [astro-ph.CO].
[82] J. Smidt, A. Amblard, C. T. Byrnes, A. Cooray, A. Heavens, and D. Munshi, Phys. Rev. D 81, 123007 (2010) [arXiv:1004.1409 [astro-ph.CO]].
[83] P. Creminelli and M. A. Luty, J. Cosmol. Astropart. Phys. 1202, 020 (2012) [arXiv:1109.3746 [astro-ph.CO]].
[84] C. Hikage and T. Matsubara, Mon. Not. R. Astron. Soc. 425, 2187 (2012) [arXiv:1207.1183 [astro-ph.CO]].
[85] T. Sekiguchi and N. Sugiyama, J. Cosmol. Astropart. Phys. 1309, 002 (2013) [arXiv:1303.4626 [astro-ph.CO]].
[86] V. Desjacques and U. Seljak, Phys. Rev. D 81, 023006 (2010) [arXiv:0907.2257 [astro-ph.CO]].
[87] G. W. Horndeski, Int. J. Theor. Phys. 10, 363 (1974).
[88] X. Gao and D. A. Steer, J. Cosmol. Astropart. Phys. 1112, 019 (2011) [arXiv:1107.2642 [astro-ph.CO]].
[89] A. De Felice and S. Tsujikawa, Phys. Rev. D 84, 083504 (2011) [arXiv:1107.3917 [gr-qc]].
[90] S. Renaux-Petel, J. Cosmol. Astropart. Phys. 1202, 020 (2012) [arXiv:1109.3746 [astro-ph.CO]].
[91] X. Gao, T. Kobayashi, M. Shiraishi, M. Yamaguchi, J. i. Yokoyama, and S. Yokoyama, Phys. Rev. D 89, 023511 (2014) [arXiv:1303.0287 [astro-ph.CO]].
[92] A. De Felice and S. Tsujikawa, J. Cosmol. Astropart. Phys. 1303, 030 (2013) [arXiv:1301.5721 [hep-th]].
[93] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, J. High Energy Phys. 0612, 080 (2006) [arXiv:0606090 [hep-th]].
[94] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, J. High Energy Phys. 0803, 014 (2008) [arXiv:0709.0293 [hep-th]].
[95] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, J. Cosmol. Astropart. Phys. 0701, 002 (2007) [arXiv:0605045 [hep-th]].
[96] X. Chen, Adv. Astron. 2010, 638979 (2010) [arXiv:1002.1416 [astro-ph.CO]].
[97] D. Seery and J. E. Lidsey, J. Cosmol. Astropart. Phys. 0506, 003 (2005) [arXiv:0503692 [astro-ph]].
[98] M. Li, T. Wang, and Y. Wang, J. Cosmol. Astropart. Phys. 0803, 028 (2008) [arXiv:0801.0040 [astro-ph]].
[99] F. Arroja and K. Koyama, Phys. Rev. D 77, 083517 (2008) [arXiv:0802.1167 [hep-th]].
[100] Y.-i. Takamizu and S. Mukohyama, J. Cosmol. Astropart. Phys. 0901, 013 (2009) [arXiv:0810.0746 [gr-qc]].
[101] K. T. Engel, K. S. M. Lee, and M. B. Wise, Phys. Rev. D 79, 103530 (2009) [arXiv:0811.3964 [hep-ph]].
[102] X. Gao, J. Cosmol. Astropart. Phys. 0806, 029 (2008) [arXiv:0804.1055 [astro-ph]].
[103] F. Arroja, S. Mizuno, and K. Koyama, J. Cosmol. Astropart. Phys. 0808, 015 (2008) [arXiv:0806.0619 [astro-ph]].
[104] J. M. Maldacena, J. High Energy Phys. 0305, 013 (2003) [arXiv:0210603 [astro-ph]].
[105] P. Creminelli and M. Zaldarriaga, J. Cosmol. Astropart. Phys. 0410, 006 (2004) [arXiv:0407059 [astro-ph]].
[106] I. Agullo and L. Parker, Phys. Rev. D 83, 063526 (2011) [arXiv:1010.5766 [astro-ph.CO]].
[107] J. Ganc, Phys. Rev. D 84, 063514 (2011) [arXiv:1104.0244 [astro-ph.CO]].
[108] M. H. Namjoo, H. Firouzjahi, and M. Sasaki, Europhys. Lett. 101, 39001 (2013) [arXiv:1210.3692 [astro-ph.CO]].
[109] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 102, 59001 (2013) [arXiv:1301.5699 [hep-th]].
[110] J. Ganc, Phys. Rev. D 84, 063514 (2011) [arXiv:1104.0244 [astro-ph.CO]].
[111] M. H. Namjoo, H. Firouzjahi, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[112] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[113] X. Chen, Phys. Rev. D 72, 103508 (2005) [arXiv:0502053 [astro-ph]].
[114] J. Ganc, Phys. Rev. D 84, 063514 (2011) [arXiv:1104.0244 [astro-ph.CO]].
[115] M. H. Namjoo, H. Firouzjahi, and M. Sasaki, Europhys. Lett. 102, 59001 (2013) [arXiv:1301.5699 [hep-th]].
[116] J. Ganc, Phys. Rev. D 84, 063514 (2011) [arXiv:1104.0244 [astro-ph.CO]].
[117] M. H. Namjoo, H. Firouzjahi, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[118] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[119] X. Chen, Phys. Rev. D 72, 103508 (2005) [arXiv:0502053 [astro-ph]].
[120] J. Ganc, Phys. Rev. D 84, 063514 (2011) [arXiv:1104.0244 [astro-ph.CO]].
[121] M. H. Namjoo, H. Firouzjahi, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[122] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[123] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[124] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[125] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[126] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[127] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[128] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[129] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[130] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[131] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[132] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[133] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[134] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[135] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[136] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[137] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[138] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[139] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[140] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[141] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[142] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[143] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[144] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[145] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[146] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, Europhys. Lett. 101, 49004 (2013) [arXiv:1303.2436 [astro-ph.CO]].
[147] J. Fonseca and D. Wands, Phys. Rev. D 83, 064025 (2011) [arXiv:1101.1254 [astro-ph.CO]].
[148] K. Dimopoulos, D. H. Lyth, A. Notari, and A. Riotto, J. High Energy Phys. 0307, 053 (2003) [arXiv:0304050 [hep-ph]].
[149] M. Kawasaki, K. Nakayama, and F. Takahashi, J. Cosmol. Astropart. Phys. 0901, 026 (2009) [arXiv:0810.1585 [hep-ph]].
[150] P. Chingangbam and Q. G. Huang, J. Cosmol. Astropart. Phys. 0904, 031 (2009) [arXiv:0902.2619 [astro-ph.CO]].
[151] M. Kawasaki, T. Kobayashi, and F. Takahashi, Phys. Rev. D 84, 123506 (2011) [arXiv:1107.6011 [astro-ph.CO]].
[152] T. Kobayashi, F. Takahashi, T. Takahashi, and M. Yamaguchi, J. Cosmol. Astropart. Phys. 1310, 042 (2013) [arXiv:1303.6255 [astro-ph.CO]].
[153] C. T. Byrnes and D. Wands, Phys. Rev. D 74, 043529 (2006) [arXiv:0605679 [astro-ph]].
[154] T. Moroi and T. Takahashi, Phys. Rev. D 66, 063501 (2002) [arXiv:0206026 [hep-ph]].
[155] D. H. Lyth and D. Wands, Phys. Rev. D 68, 103516 (2003) [arXiv:0306500 [astro-ph]].
[156] M. Zaldarriaga, Phys. Rev. D 69, 043508 (2004) [arXiv:0306006 [astro-ph]].
[157] N. Kobayashi, T. Kobayashi, and A. L. Erickcek, J. Cosmol. Astropart. Phys. 1301, 036 (2014) [arXiv:1308.4154 [astro-ph.CO]].
[158] Y. Watanabe and J. I. Yokoyama, Phys. Rev. D 87, 103524 (2013) [arXiv:1303.5191 [hep-th]].
[159] T. Takahashi, M. Yamaguchi, J. I. Yokoyama, and S. Yokoyama, Phys. Lett. B 678, 15 (2009) [arXiv:0905.0240 [astro-ph.CO]].
[160] K. Kamada, K. Kohri, and S. Yokoyama, J. Cosmol. Astropart. Phys. 1101, 027 (2011) [arXiv:1008.1450 [astro-ph.CO]].
[161] T. Kobayashi, F. Takahashi, T. Takahashi, and M. Yamaguchi, J. Cosmol. Astropart. Phys. 1203, 036 (2012) [arXiv:1111.1336 [astro-ph.CO]].
[162] D. H. Lyth and E. D. Stewart, Phys. Rev. D 53, 1784 (1996) [arXiv:9510204 [hep-ph]].
[163] O. Elgaroy, S. Hannestad, and T. Haugboelle, J. Cosmol. Astropart. Phys. 0309, 008 (2003) [arXiv:0306229 [astro-ph]].
[164] A. E. Romano and M. Sasaki, Phys. Rev. D 78, 103522 (2008) [arXiv:0809.5142 [gr-qc]].
[165] K. Enqvist and T. Takahashi, J. Cosmol. Astropart. Phys. 1310, 034 (2013) [arXiv:1306.5958 [astro-ph.CO]].
[166] T. Kobayashi and T. Takahashi, J. Cosmol. Astropart. Phys. 1206, 004 (2012) [arXiv:1203.3011 [astro-ph.CO]].
[167] D. H. Lyth and E. D. Stewart, Phys. Rev. D 53, 1784 (1996) [arXiv:9510204 [hep-ph]].
[168] C. T. Byrnes, S. Nurmi, G. Tasinato, and D. Wands, J. Cosmol. Astropart. Phys. 1002, 034 (2010) [arXiv:0911.2780 [astro-ph.CO]].
[169] C. T. Byrnes, M. Gerstenlauer, S. Nurmi, G. Tasinato, and D. Wands, J. Cosmol. Astropart. Phys. 1010, 004 (2010) [arXiv:1007.4277 [astro-ph.CO]].
[170] C. T. Byrnes and J.-O. Gong, Phys. Lett. B 718, 718 (2013) [arXiv:1210.1851 [astro-ph.CO]].
[171] C. T. Byrnes, K. Enqvist, and T. Takahashi, J. Cosmol. Astropart. Phys. 1009, 026 (2010) [arXiv:1007.5148 [astro-ph.CO]].
[172] C. T. Byrnes, K. Enqvist, S. Nurmi, and T. Takahashi, J. Cosmol. Astropart. Phys. 1111, 011 (2011) [arXiv:1108.2708 [astro-ph.CO]].
[173] Q.-G. Huang, J. Cosmol. Astropart. Phys. 1011, 026 (2010); 1102, E01 (2011) [erratum] [arXiv:1008.2641 [astro-ph.CO]].
[174] Q.-G. Huang and C. Lin, J. Cosmol. Astropart. Phys. 1110, 005 (2011) [arXiv:1108.4474 [astro-ph.CO]].
[175] Q.-G. Huang, J. Cosmol. Astropart. Phys. 1012, 017 (2010) [arXiv:1009.3326 [astro-ph.CO]].
[176] M. LoVerde, A. Miller, S. Shandera, and L. Verde, J. Cosmol. Astropart. Phys. 0804, 014 (2008) [arXiv:0711.4126 [astro-ph]].
[177] E. Sefusatti, M. Liguori, A. P. S. Yadav, M. G. Jackson, and E. Pajer, J. Cosmol. Astropart. Phys. 0912, 022 (2009) [arXiv:0906.0232 [astro-ph.CO]].
[178] A. Becker, D. Huterer, and K. Kadota, J. Cosmol. Astropart. Phys. 1212, 034 (2012) [arXiv:1206.6165 [astro-ph.CO]].
[179] M. Biagetti, H. Perrier, A. Riotto, and V. Desjacques, Phys. Rev. D 87, 063521 (2013) [arXiv:1301.2771 [astro-ph.CO]].
[180] A. Becker and D. Huterer, Phys. Rev. Lett. 109, 121302 (2012) [arXiv:1207.5788 [astro-ph.CO]].
[181] M. Kamionkowski, L. Verde, and R. Jimenez, J. Cosmol. Astropart. Phys. 0901, 010 (2009) [arXiv:0809.0506 [astro-ph]].
[182] H. Li, J. Liu, J.-Q. Xia, and Y.-F. Cai, Phys. Rev. D 83, 123517 (2011) [arXiv:1012.2511 [astro-ph.CO]].
[183] M. Savelainen, J. Valiviita, P. Walia, S. Rusak, and H. Kurki-Suonio, Phys. Rev. D 88, 063010 (2013) [arXiv:1307.4398 [astro-ph.CO]].
[184] C. Hikage, K. Koyama, T. Matsubara, T. Takahashi, and M. Yamaguchi, Mon. Not. R. Astron. Soc. 398, 2188 (2009) [arXiv:0812.3500 [astro-ph]].
[185] C. Hikage, M. Kawasaki, T. Sekiguchi, and T. Takahashi, J. Cosmol. Astropart. Phys. 1307, 007 (2013) [arXiv:1211.1095 [astro-ph.CO]].
[186] C. Hikage, M. Kawasaki, T. Sekiguchi, and T. Takahashi, J. Cosmol. Astropart. Phys. 1303, 020 (2013) [arXiv:1212.6001 [astro-ph.CO]].
[187] D. Langlois and B. van Tent, Classical Quantum Gravity 28, 222001 (2011) [arXiv:1104.2567 [astro-ph.CO]].
[188] D. Langlois and B. van Tent, J. Cosmol. Astropart. Phys. 1207, 040 (2012) [arXiv:1204.5042 [astro-ph.CO]].
[189] E. Kawakami, M. Kawasaki, K. Nakayama, and F. Takahashi, J. Cosmol. Astropart. Phys. 0909, 002 (2009) [arXiv:0905.1552 [astro-ph]].
[190] D. Langlois and T. Takahashi, J. Cosmol. Astropart. Phys. 1102, 020 (2011) [arXiv:1012.4885 [astro-ph.CO]].
[191] M. Axenides, R. H. Brandenberger, and M. S. Turner, Phys. Lett. B 126, 178 (1983).
[192] D. Seckel and M. S. Turner, Phys. Rev. D 32, 3178 (1985).
[193] A. D. Linde, Phys. Lett. B 158, 375 (1985).
[194] A. D. Linde and D. H. Lyth, Phys. Lett. B 246, 353 (1990).
[195] M. S. Turner and F. Wilczek, Phys. Rev. Lett. 66, 5 (1991).
[196] A. D. Linde, Phys. Lett. B 259, 38 (1991).
[197] D. H. Lyth, Phys. Rev. D 45, 3394 (1992).
[198] S. Kasuya and M. Kawasaki, Phys. Rev. D 80, 023516 (2009) [arXiv:0904.3800 [astro-ph.CO]].
[199] K. Enqvist and J. McDonald, Phys. Rev. Lett. 83, 2510 (1999) [arXiv:9811412 [hep-ph]].
[200] K. Enqvist and J. McDonald, Phys. Rev. D 62, 043502 (2000) [arXiv:9912478 [hep-ph]].
[201] M. Kawasaki and F. Takahashi, Phys. Lett. B 516, 388 (2001) [arXiv:0105134 [hep-ph]].
[202] M. Kawasaki, K. Nakayama, T. Sekiguchi, T. Suyama, and F. Takahashi, J. Cosmol. Astropart. Phys. 0811, 019 (2008) [arXiv:0808.0009 [astro-ph]].
[203] D. Langlois and F. Vernizzi, J. Cosmol. Astropart. Phys. 1102, 004 (2008) [arXiv:0809.4646 [astro-ph]].
[204] D. Langlois, F. Vernizzi, and D. Wands, J. Cosmol. Astropart. Phys. 0812, 004 (2008) [arXiv:0809.2242 [hep-ph]].
[205] M. Kawasaki, K. Nakayama, T. Sekiguchi, T. Suyama, and F. Takahashi, J. Cosmol. Astropart. Phys. 0901, 042 (2009) [arXiv:0810.0208 [astro-ph]].
[206] D. Langlois and A. Lepidi, J. Cosmol. Astropart. Phys. 1101, 008 (2011) [arXiv:1007.5498 [astro-ph.CO]].
[207] C. Gordon and K. A. Malik, Phys. Rev. D 69, 063508 (2004) [arXiv:0311102 [astro-ph]].
[208] E. Di Valentino, M. Lattanzi, G. Mangano, A. Melchiorri, and P. Serpico, Phys. Rev. D 85, 043511 (2012) [arXiv:1111.3810 [astro-ph.CO]].
[209] D. Polarski and A. A. Starobinsky, Nucl. Phys. B 385, 623 (1992).
[210] D. Langlois, Phys. Rev. D 59, 123512 (1999) [arXiv:9906080 [astro-ph]].
[211] M. Kawasaki, K. Miyamoto, K. Nakayama, and T. Sekiguchi, J. Cosmol. Astropart. Phys. 1202, 022 (2012) [arXiv:1107.4962 [astro-ph.CO]].
[212] E. Kawakami, M. Kawasaki, K. Nakayama, and T. Sekiguchi, J. Cosmol. Astropart. Phys. 1207, 037 (2012) [arXiv:1202.4890 [astro-ph.CO]].
[213] D. Baumann et al. [CMBPol Study Team Collaboration], AIP Conf. Proc. 1141, 10 (2009) [arXiv:0811.3919 [astro-ph]].
[214] A. Cooray, Phys. Rev. Lett. 97, 261301 (2006) [arXiv:0610257 [astro-ph]].
[215] S. Chongchitnan and J. Silk, arXiv:1205.6799 [astro-ph.CO].
[216] S. Yokoyama, T. Suyama, and T. Tanaka, J. Cosmol. Astropart. Phys. 0902, 012 (2009) [arXiv:0810.3053 [astro-ph]].

[217] C. Lin and Y. Wang, J. Cosmol. Astropart. Phys. 1007, 011 (2010) [arXiv:1004.0461 [astro-ph.CO]].

[218] J. Meyers and N. Sivanandam, Phys. Rev. D 84, 063522 (2011) [arXiv:1104.5238 [astro-ph.CO]].

[219] T. Suyama and S. Yokoyama, J. Cosmol. Astropart. Phys. 1107, 033 (2011) [arXiv:1105.5851 [astro-ph.CO]].