The integrability of the 2-Toda lattice on a simple Lie algebra

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Abstract

We define the 2-Toda lattice on every simple Lie algebra $\mathfrak{g}$, and we show its Liouville integrability. We show that this lattice is given by a pair of Hamiltonian vector fields, associated with a Poisson bracket which results from an $\mathcal{R}$-matrix of the underlying Lie algebra. We construct a big family of constants of motion which we use to prove the Liouville integrability of the system. We achieve the proof of their integrability by using several results on simple Lie algebras, $\mathcal{R}$-matrices, invariant functions and root systems.

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1 Introduction

The 2-Toda lattice associated with $\mathfrak{sl}_n(\mathbb{C})$ is the pair of differential equations given by the Lax equations

\[
\begin{align*}
\frac{\partial (L, M)}{\partial t} &= [(L_u, L_u), (L, M)], \\
\frac{\partial (L, M)}{\partial s} &= [(M_\ell, M_\ell), (L, M)],
\end{align*}
\]

(1)

where $(L, M)$ are traceless matrices of the form

\[
(L, M) = \begin{pmatrix}
    a_{11} & 1 & \cdots & 0 \\
    a_{21} & a_{22} & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{n,n-1} & a_{nn}
\end{pmatrix},
\]

(2)

and where $L_u$ is the upper triangular part of $L$ and $M_\ell$ is the strictly lower triangular part of $M$.

The 2-Toda lattice was first introduced in the context of infinite-dimensional matrices [11, 13, 14]. This system is Hamiltonian with respect to a Poisson structure, associated with an $\mathcal{R}$-matrix with $\frac{1}{2}\text{Trace } L^2$ and $\frac{1}{2}\text{Trace } M^2$ as Hamiltonians. These Hamiltonians admit the Ad-invariant functions on $\mathfrak{gl}(\mathbb{C}) \times \mathfrak{gl}(\mathbb{C})$ as Poisson commuting constants of motion. In the finite-dimensional setting, i.e., on $\mathfrak{sl}_n(\mathbb{C})$, this system is still Hamiltonian with respect to a Poisson structure associated with an $\mathcal{R}$-matrix similar to that of the infinite-dimensional setting. The family of Ad-invariant functions are again in involution with respect to the Poisson structure associated with the $\mathcal{R}$-matrix but their number of independent Ad-invariant functions, which is $2n - 2$, is much too small compared with the dimension of the phase space of the 2-Toda lattice, which is $n^2 + 2n - 3$.

The main purpose of the present article is to introduce a family of functions which contains the Ad-invariant functions and is large enough to give the Liouville integrability of the 2-Toda lattice. This family is $\mathcal{F} := \{F_{j,i}, 1 \leq i \leq n - 1 \text{ and } 0 \leq j \leq i + 1\}$, where $F_{j,i}$ is the function on $\mathfrak{sl}_n(\mathbb{C})$ constructed from the the relation

\[
\text{Trace}(\lambda L - M)^{i+1} = \sum_{j=0}^{i+1} (-1)^{m_{i+1}-j} \lambda^j F_{j,i}(L, M).
\]
The Toda lattice can be defined for every simple Lie algebra [7]. In the same spirit we introduce the 2-Toda system for every simple Lie algebra and we show its Liouville integrability. We recall that, according to [2, Definition 4.13], in the general case a system \((M, \{\cdot, \cdot\}, \mathcal{F})\) is Liouville integrable if \((M, \{\cdot, \cdot\})\) is a Poisson manifold of rank \(2r\) and \(\mathcal{F} = (F_1, \ldots, F_s)\) is involutive and independent, with \(s = \dim M - r\).

The main result of this paper is given in Section 3: we show the Liouville integrability of the 2-Toda lattice, not only on \(\mathfrak{sl}_n(\mathbb{C})\), but also on an arbitrary simple Lie algebra \(\mathfrak{g}\). We define this lattice in this general context. We choose a simple Lie algebra \(\mathfrak{g}\); then we denote by \(\mathfrak{h}\) a Cartan subalgebra, by \(\alpha_1, \ldots, \alpha_\ell\) the simple roots of \(\mathfrak{g}\) with respect to \(\mathfrak{h}\), by \(e_1, \ldots, e_\ell\) the corresponding eigenvectors, and by \(\mathfrak{g}_i\) the subspace spanned by the eigenvectors associated with roots of length \(i \in \mathbb{Z}\). The 2-Toda lattice, associated with \(\mathfrak{g}\), is the pair of differential equations on \(T^2\) given by the following Lax pair equations

\[
\begin{align*}
\frac{\partial (L, M)}{\partial t} &= [(L_+, L_+), (L, M)], \\
\frac{\partial (L, M)}{\partial s} &= [(M_-, M_-), (L, M)],
\end{align*}
\]

where \((L, M)\) belongs to the phase space \(T^2 := \sum_{i \leq 0} \mathfrak{g}_i \times \sum_{i \geq -1} \mathfrak{g}_i + (\sum_{i=1}^{\ell} e_i, 0)\), and here, \(L_+\) (resp. \(M_-\)) stands for the projection of \(L \in \mathfrak{g}\) (resp. \(M \in \mathfrak{g}\)) on \(\sum_{i \geq 0} \mathfrak{g}_i\) (resp. \(\sum_{i < 0} \mathfrak{g}_i\)).

We construct a linear Poisson structure with respect to which the 2-Toda lattice is Hamiltonian. This linear Poisson structure comes from an \(\mathcal{R}\)-matrix of the type studied in Section 2, namely from the endomorphism of \(\mathfrak{g}^2\) given by \(\mathcal{R}(x, y) = (R(x - y) + y, R(x - y) + x)\), for all \(x, y \in \mathfrak{g}\), where \(R := P_+ - P_-\) is the classical \(\mathcal{R}\)-matrix on a simple Lie algebra, associated with the Lie algebra splitting \(\mathfrak{g} = \sum_{i \geq 0} \mathfrak{g}_i \oplus \sum_{i < 0} \mathfrak{g}_i\). Using the Killing form of \(\mathfrak{g}\), the product Lie algebra \(\mathfrak{g}^2\) is equipped with a bilinear, symmetric, Ad-invariant, non-degenerate form, which allows us to identify \(\mathfrak{g}^2\) with its dual and hence to equip \(\mathfrak{g}^2\) with a linear Poisson structure that we denote by \(\{\cdot, \cdot\}_\mathcal{R}\). We then establish that the phase space \(T^2\) of the 2-Toda lattice is a Poisson submanifold of \((\mathfrak{g}^2, \{\cdot, \cdot\}_\mathcal{R})\) (see Propositions [3]) and the Hamiltonian vector fields of functions \(H(x, y) := \frac{1}{2} \langle x \mid x \rangle\) and \(\tilde{H}(x, y) := \frac{1}{2} \langle y \mid y \rangle\) are tangent to \(T^2\) and describe on \(T^2\) the equations of motion (3) for the 2-Toda lattice (see Proposition [10]).

We construct the integrable system. Let \((P_1, \ldots, P_\ell)\) be a generating family of the algebra of Ad-invariant functions, chosen to be of respective degrees \(m_1 + 1, \ldots, m_\ell + 1\). For all \(1 \leq i \leq \ell\) and \(0 \leq j \leq m_i + 1\), we define \(F_{j,i} \in \mathcal{F}(\mathfrak{g}^2)\) to be the coefficient in \(\lambda^j\) of the polynomial \((x, y) \mapsto P_i(\lambda x - y)\), and we consider the following family of functions on \(\mathfrak{g}^2\):

\[
\mathcal{F} = (F_{j,i}, 1 \leq i \leq \ell \text{ and } 0 \leq j \leq m_i + 1).
\]
Our main result is the following theorem:

**Theorem** The triplet \((T^2, F_{|T^2}, \{\cdot, \cdot\}_R)\) is an integrable system.

To prove this result, we proceed as follows:

- We show in Proposition 14 that \(F\) is involutive for \(\{\cdot, \cdot\}_R\): this result is a particular case of the second point of Theorem 4 that we prove in Section 2.
- We show in Proposition 17 the independence of \(F\) restricted to \(T^2\) by using a theorem of Raïs [10].
- We compute the rank of the restriction of the Poisson \(R\)-bracket to \(T^2\). This is done by establishing a Poisson isomorphism between \((T^2, \{\cdot, \cdot\}_R)\) and the product Poisson manifold \((g^*, \{\cdot, \cdot\}) \times (g_0 \oplus g_1, \{\cdot, \cdot\}_R)\), where \(\{\cdot, \cdot\}\) is the Lie-Poisson bracket on \(g^*\), and \(\{\cdot, \cdot\}_R\) is the above \(R\)-bracket (restricted to \(g_0 \oplus g_1\); see Proposition 18).
- We check that \(\text{card} \ F_{|T^2} = \dim T^2 - \frac{1}{2} \text{Rk}(T^2, \{\cdot, \cdot\}_R)\).

We also define the 2-Toda from on \(gl_n(C)\) and we show its Liouville integrability with respect to the linear and quadratic Poisson \(R\)-bracket. This result is mainly based on the fact that the Hamiltonian vector fields of the linear and quadratic Poisson structure are essentially identical.

In the last subsection, by restricting the 2-Toda lattice on \(g\) to a well chosen affine subspace, we find again the usual Toda lattice.

## 2 \(\mathcal{R}\)-matrices, Poisson structures and functions in involution on the square of a Lie algebra

In this section we fix \((g, [\cdot, \cdot])\) a Lie algebra over \(F\), with \(F = R\) or \(F = C\). The vector space \(g^2 := g \times g\), endowed with the Lie bracket

\[
[(x, y), (z, s)] := ([x, z], [y, s]), \quad \forall x, y, z, s \in g,
\]

is a Lie algebra.

We construct an \(\mathcal{R}\)-matrix of \(g^2\) with the help of an endomorphism \(R\) of \(g\) satisfying some conditions. With this \(\mathcal{R}\)-matrix, when \(g\) is finite dimensional, we construct a linear Poisson bracket on the dual \(g^{2*}\) of \(g^2\), we explain the construction of a large family of functions on \(g^{2*}\), which commute for this linear Poisson structure and we spell out the expressions of their Hamiltonian vector fields. We also give some Casimir functions.
2.1 Construction of $R$-matrices on $\mathfrak{g}^2$

We begin by recalling some properties and definitions of $R$-matrices (see [2, Section 4.4]). Let $\mathfrak{g}$ be a Lie algebra. A (vector space) endomorphism $R$ of $\mathfrak{g}$ is called an $R$-matrix of $\mathfrak{g}$ if the bilinear map $\{\cdot, \cdot\}_R : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, defined for all $x, y \in \mathfrak{g}$ by

$$\{x, y\}_R := \frac{1}{2}([Rx, y] + [x, Ry]),$$

defines a (second) Lie bracket on $\mathfrak{g}$, which is then called a Lie $R$-bracket. Let $B_R : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be defined, for all $x, y \in \mathfrak{g}$, by

$$B_R(x, y) := [Rx, Ry] - R([Rx, y] + [x, Ry]).$$

The bracket $\{\cdot, \cdot\}_R$ satisfies the Jacobi identity if and only if

$$[B_R(x, y), z] + [B_R(y, z), x] + [B_R(z, x), y] = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (7)$$

A sufficient condition for (7) to be satisfied is that there exists a constant $c \in \mathbb{F}$ such that

$$B_R(x, y) = -c^2[x, y], \quad \forall x, y \in \mathfrak{g}. \quad (8)$$

We call (8) the modified classical Yang-Baxter equation (mCYBE) of $\mathfrak{g}$ of constant $c$.

We construct in the following proposition an $R$-matrix of $\mathfrak{g}^2$ by using an endomorphism of $\mathfrak{g}$ which satisfies some condition which generalizes the modified classical Yang-Baxter equation of $\mathfrak{g}^2$.

**Proposition 1** Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{z}(\mathfrak{g})$ its center and $c \in \mathbb{F}$ a constant. Let $R$ be an endomorphism of $\mathfrak{g}$ and $R$ the endomorphism of $\mathfrak{g}^2$ defined, for every $(x, y) \in \mathfrak{g}^2$, by

$$\mathcal{R}(x, y) := (R(x - y) + cy, R(x - y) + cx). \quad (9)$$

The endomorphism $\mathcal{R}$ is an $R$-matrix of $\mathfrak{g}^2$ if and only if, for every $x, y \in \mathfrak{g}$,

$$B_R(x, y) + c^2[x, y] \in \mathfrak{z}(\mathfrak{g}). \quad (10)$$

In the particular case where $\mathfrak{g}$ is a complex semi-simple Lie algebra, this is in turn equivalent to $R$ being a solution of (mCYBE) of constant $c$.

**Proof.** Using Definition (6) of $B_R$ and replacing $\mathcal{R}$ by its Expression (9), we obtain

$$B_\mathcal{R}((x, y), (z, s)) = (B_R(x - y, z - s) + c^2([x - y, z - s] - [x, z]), B_R(x - y, z - s) + c^2([x - y, z - s] - [y, s])). \quad (11)$$

According to Formula (7), $\mathcal{R}$ is an $R$-matrix of $\mathfrak{g}^2$ if and only if, for every $(x, y), (x', y')$ and $(x'', y'')$ in $\mathfrak{g}^2$,

$$[B_\mathcal{R}((x, y), (x', y')); (x'', y'')] + \circ = (0, 0), \quad (12)$$

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where \( \odot = \text{cyl}((x,y),(x',y'),(x'',y'')) \). By Formula (11), the left hand side of Equation (12) is equal to

\[
([B_R(x - y, x' - y') + c^2[x - y, x' - y'], x''], [B_R(x - y, x' - y') + c^2[x - y, x' - y'], y'']) + \odot,
\]

where we used the Jacobi identities \([[[x, x'], x'', y''] + \odot = 0 \) and \([[[y', y''], x'] + \odot = 0 \). Now, if \( \mathcal{R} \) is an \( \mathcal{R} \)-matrix of \( \mathfrak{g}^2 \), then (13) is equal to zero (even without circular permutation), so Formula (10) holds. If Formula (10) holds then the first component of the left hand side of (13) is equal to zero:

\[
[B_R(x - y, x' - y') + c^2[x - y, x' - y'], x''] + \text{cyl}((x, y), (x', y'), (x'', y'')) = 0,
\]

for every \( x, y, x', y', x'', y'' \in \mathfrak{g} \); in particular, for \( x' = x'' = 0 \), we obtain

\[
[B_R(y', y'') + c^2[y', y''], x] = 0, \quad \forall y', y'', x \in \mathfrak{g},
\]

hence the endomorphism \( \mathcal{R} \) is an \( \mathcal{R} \)-matrix of \( \mathfrak{g}^2 \) holds. \( \square \)

For future use, we also state the following corollary, which is a consequence of Formula (11) and already appeared in (11) (with \( c = 1 \)).

**Corollary 2** Let \( \mathfrak{g} \) be a Lie algebra and let \( R \) be an endomorphism of \( \mathfrak{g} \). If \( R \) is a solution of (mCYBE) of \( \mathfrak{g} \) of constant \( c \), then the endomorphism \( \mathcal{R} \) of \( \mathfrak{g}^2 \), defined, for all \( (x, y) \in \mathfrak{g}^2 \), by

\[
\mathcal{R}(x, y) := (R(x - y) + cy, R(x - y) + cx),
\]

is a solution of (mCYBE) of \( \mathfrak{g}^2 \) of constant \( c \).

**Example 3** Let \( \mathfrak{g} := \mathfrak{g}_+ \oplus \mathfrak{g}_- \) be a Lie algebra splitting (i.e., \( \mathfrak{g} \) is, as a vector space, the direct sum of Lie subalgebras \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \)). Let \( P_\pm \) be the projections of \( \mathfrak{g} \) on \( \mathfrak{g}_\pm \) and let \( R := P_+ - P_- \). Since \( R \) is a solution of (mCYBE) of \( \mathfrak{g} \) of \( c = 1 \), Corollary 2 implies that the endomorphism \( \mathcal{R} \), defined for every \( x, y \in \mathfrak{g} \) by \( \mathcal{R}(x, y) = (R(x - y) + y, R(x - y) + x) \) is a solution of (mCYBE) of \( \mathfrak{g}^2 \) of \( c = 1 \). Replacing \( R \) by its expression \( P_+ - P_- \), we verify that

\[
\mathcal{R}(x, y) = (x_+ - x_- + 2y_-, y_- - y_+ + 2x_+) = (x_+ + y_-, x_+ + y_-) - (x_- - y_-, y_+ - x_+),
\]

where \( x = x_+ + x_- \) and \( y = y_+ + y_- \) are the decompositions of \( x \) and \( y \) with respect to the splitting \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \). Since the Lie subalgebras

\[
\mathfrak{g}_2^+ := \{(x, x) \mid x \in \mathfrak{g}_+\} \quad \text{and} \quad \mathfrak{g}_2^- := \{(x, y) \mid x \in \mathfrak{g}_- \text{ and } y \in \mathfrak{g}_+\}
\]

have a trivial intersection, (15) implies that \( \mathfrak{g}^2 = \mathfrak{g}_2^+ \oplus \mathfrak{g}_2^- \) is a Lie algebra splitting and that \( \mathcal{R} \) is the difference of the projections on \( \mathfrak{g}_2^+ \) and \( \mathfrak{g}_2^- \).
Note that independently of Proposition 1 or Corollary 2 the above example shows that from an $R$-matrix of $\mathfrak{g}$, which is the difference of the projections on two supplementary Lie subalgebras of $\mathfrak{g}$, the endomorphism $\mathcal{R}$ given by the Formula (9) is automatically an $\mathcal{R}$-matrix of $\mathfrak{g}^2$, which is the difference of projections on two supplementary Lie subalgebras of $\mathfrak{g}^2$.

2.2 Linear Poisson structures on $\mathfrak{g}^* \times \mathfrak{g}^*$

From now and until the end of the article, we assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra and we denote by $\mathcal{F}(\mathfrak{g})$ the algebra of smooth functions on $\mathfrak{g}$ (if $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$) or holomorphic functions on $\mathfrak{g}$ (if $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$). Let $\mathcal{R}$ be an $\mathcal{R}$-matrix of $\mathfrak{g}^2$. The dual $(\mathfrak{g}^2)^*$ of the Lie algebra $(\mathfrak{g}^2, [\cdot, \cdot])_{\mathcal{R}}$ admits a Lie-Poisson structure, defined for every $F, G \in \mathcal{F}((\mathfrak{g}^2)^*)$ at $\varphi \in (\mathfrak{g}^2)^*$ by

$$\{F, G\}_{\mathcal{R}}(\varphi) := \langle \varphi, [d_{\varphi} F, d_{\varphi} G]_{\mathcal{R}} \rangle$$

(16)

where $d_{\varphi} F \in ((\mathfrak{g}^2)^*)^\ast \simeq \mathfrak{g}^* \times \mathfrak{g}^*$ is the differential of $F$ at the point $\varphi \in (\mathfrak{g}^2)^*$. We identify $(\mathfrak{g}^2)^*$ with $\mathfrak{g}^* \times \mathfrak{g}^*$ as follows:

$$\Phi : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow (\mathfrak{g}^2)^*$$

$$(\xi, \eta) \rightarrow ((x, y) \mapsto \langle (\xi, \eta), (x, y) \rangle) := \langle \xi, x \rangle - \langle \eta, y \rangle.$$

By means of this identification, the Poisson structure (16) can be transported to the vector space $\mathfrak{g}^* \times \mathfrak{g}^*$. This transported Poisson structure is given, for all functions $F, G \in \mathcal{F}(\mathfrak{g}^* \times \mathfrak{g}^*)$ at $(\xi, \eta) \in \mathfrak{g}^* \times \mathfrak{g}^*$, by

$$\{F, G\}_{\mathcal{R}}(\xi, \eta) := \langle (\xi, \eta), [d_{(\xi, \eta)} F, d_{(\xi, \eta)} G]_{\mathcal{R}} \rangle,$$

(17)

where $d_{(\xi, \eta)} F \in ((\mathfrak{g}^* \times \mathfrak{g}^*)^\ast \simeq \mathfrak{g} \times \mathfrak{g}$. We call this Poisson structure the Poisson $\mathcal{R}$-bracket on $\mathfrak{g}^* \times \mathfrak{g}^*$.

2.3 Casimirs and functions in involution on $\mathfrak{g}^* \times \mathfrak{g}^*$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. We denote by $\mathbf{G}$ a connected Lie group whose Lie algebra is $\mathfrak{g}$. We denote by $\mathcal{F}(\mathfrak{g}^*)^\mathbf{G}$ the algebra of $\text{Ad}^*$-invariant functions on $\mathfrak{g}^*$.

According to the Adler-Kostant-Symes theorem [2, Theorem 4.37], the $\text{Ad}^*$-invariant functions on $\mathfrak{g}^* \times \mathfrak{g}^*$ (i.e., functions in $\mathcal{F}(\mathfrak{g}^* \times \mathfrak{g}^*)^\mathbf{G} \times \mathbf{G}$) are in involution for the Poisson structure associated with an $\mathcal{R}$-matrix. But, when $\mathfrak{g}$ is a semi-simple Lie algebra, this family of functions is too small to insure the Liouville integrability. We construct a larger family$^1$ of functions in involution.

$^1$Which contains the $\text{Ad}^*$-invariant functions on $\mathfrak{g}^* \times \mathfrak{g}^*$. 

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Theorem 4 Let $\lambda \in F$ be a constant and let $\psi_\lambda$ be the map
\[
\psi_\lambda : g^* \times g^* \to g^* \\
(\xi, \eta) \mapsto \lambda \xi - \eta.
\] (18)

Let $R$ be an endomorphism$^2$ of $g$ and $c \in F$ a constant. Assume that the endomorphism $R$ of $g^2$ defined for every $(x, y) \in g^2$ by
\[
R(x, y) := (R(x - y) + cy, R(x - y) + cx)
\] (19)
is an $R$-matrix of $g^2$, and denote by $\{\cdot, \cdot\}_R$ the Poisson $R$-bracket on $g^* \times g^*$. Then:

(1) For every $F \in \mathcal{F}(g^*)^G$, the function $F \circ \psi_1$ is a Casimir for $\{\cdot, \cdot\}_R$.

(2) For every $F, G \in \mathcal{F}(g^*)^G$ and every $\lambda, \gamma \in F$, the functions $F \circ \psi_\lambda$ and $G \circ \psi_\gamma$ are in involution for $\{\cdot, \cdot\}_R$. In particular, if $F$ and $G$ are polynomials of degree respectively $l$ and $k$, then the functions $F_0, \ldots, F_l, G_0, \ldots, G_k$, defined, for every $(\xi, \eta) \in g^* \times g^*$, by
\[
F(\psi_\lambda(\xi, \eta)) = \sum_{i=0}^l \lambda^i F_i(\xi, \eta) \quad \text{and} \quad G(\psi_\gamma(\xi, \eta)) = \sum_{j=0}^k \gamma^j G_j(\xi, \eta),
\]
are in involution for $\{\cdot, \cdot\}_R$.

(3) If $c = 1$, the map $\psi_1 : (g^* \times g^*, \{\cdot, \cdot\}_R) \to (g^*, \{\cdot, \cdot\})$ is a Poisson morphism.

(4) For every $H \in \mathcal{F}(g^*)^G$, the Hamiltonian vector field $X_{H \circ \psi_\lambda} := \{\cdot, H \circ \psi_\lambda\}_R$ is given at $(\xi, \mu) \in g^* \times g^*$ by
\[
X_{H \circ \psi_\lambda}(\xi, \eta) = \frac{1}{2}(1 - \lambda) \text{ad}^*_{\{R - cI\}_R(\lambda \xi - \eta)} H(\{R + cI\}_R d_{\lambda \xi - \eta} H)(\xi, \eta).
\] (20)

In particular, if $g := g_+ \oplus g_-$ is a Lie algebra splitting and $R := P_+ - P_-$ is the difference of the projections on $g_+$ and $g_-$, we have
\[
X_{H \circ \phi_\lambda}(\xi, \eta) = (1 - \lambda) \text{ad}^*_{\{(-d_{\lambda \xi - \eta} H)_-, (d_{\lambda \xi - \eta} H)_+\}}(\xi, \eta). \tag{21}
\]

To prove this theorem, we need the following lemmas, the proof of the first of which is left to the reader.

Lemma 5 Let $F \in \mathcal{F}(g^*)$, $(\xi, \eta) \in g^* \times g^*$ and $\lambda \in F$ be a constant.

(1) The differential of $F \circ \psi_\lambda$ at $(\xi, \eta) \in g^* \times g^*$ is given by
\[
d_{(\xi, \eta)}(F \circ \psi_\lambda) = (\lambda d_{\lambda \xi - \eta} F, d_{\lambda \xi - \eta} F).
\] (22)

$^2R$ is not necessarily an $R$-matrix.
(2) Let $R$ be an endomorphism of $\mathfrak{g}$ and let $\mathcal{R}$ be the endomorphism of $\mathfrak{g}^2$ defined in (19). Then
\[
\mathcal{R} d_{(\xi,\eta)}(F \circ \psi) = (\lambda - 1)(R d_{\lambda \xi - \eta} F, R d_{\lambda \xi - \eta} F) + c(d_{\lambda \xi - \eta} F, \lambda d_{\lambda \xi - \eta} F).
\] (23)

**Lemma 6** Let $F, G \in \mathcal{F}(\mathfrak{g}^*)^G$.

(1) The differentials of $F$ and $G$ commute at every point of $\mathfrak{g}^*$, i.e.,
\[
[d_{\xi} F, d_{\xi} G] = 0, \quad \forall \xi \in \mathfrak{g}^*.
\] (24)

(2) For every $\xi, \eta \in \mathfrak{g}^*$ and every $a, b \in F$,
\[
\langle a \xi + b \eta, [d_{\xi} F, d_{\xi} G] \rangle = 0.
\] (25)

**Proof.** Using the $\text{Ad}^*$-invariance of $F$ and $G$, we obtain
\[
\text{ad}_{d_{\xi} F}^* \xi = 0 \quad \text{and} \quad \text{ad}_{d_{\xi} G}^* \xi = 0.
\] (26)

Stated differently, $d_{\xi} F \in \mathfrak{g}^\xi$ and $d_{\xi} G \in \mathfrak{g}^\xi$ where $\mathfrak{g}^\xi$ is the centralizer of $\xi$. For $\xi$ regular, according to [12, Proposition 19.7.5], the centralizer $\mathfrak{g}^\xi$ is abelian, so Formula (24) holds at least for every regular point $\xi$ of $\mathfrak{g}^*$. Since the set of regular elements of $\mathfrak{g}^*$ is dense in $\mathfrak{g}^*$ (see [12, Proposition 19.7.5]), the continuous map $\xi \to [d_{\xi} F, d_{\xi} G]$ is zero at all points. This establishes (24). The second point of the lemma follows directly from (26). $\Box$

Let us now prove Theorem 4

**Proof.** (1) Let $K \in \mathcal{F}(\mathfrak{g}^* \times \mathfrak{g}^*)$ and let $(\xi, \eta) \in \mathfrak{g}^* \times \mathfrak{g}^*$. The Poisson $\mathcal{R}$-bracket between $\psi^*_x F = F \circ \psi_1$ and $K$ at $(\xi, \eta)$ is given by
\[
\{ \psi^*_x F, K \}_{\mathcal{R}}(\xi, \eta) = \frac{1}{2} \langle (\xi, \eta), [\mathcal{R} d_{(\xi,\eta)}(F \circ \psi_1), d_{(\xi,\eta)} K] \rangle
\]
\[
+ \frac{1}{2} \langle (\xi, \eta), [d_{(\xi,\eta)}(F \circ \psi_1), \mathcal{R} d_{(\xi,\eta)} K] \rangle.
\]

According to (22) and (23), we obtain $d_{(\xi,\eta)}(F \circ \psi_1) = (d_{\xi - \eta} F, d_{\xi - \eta} F)$ and $\mathcal{R} d_{(\xi,\eta)}(F \circ \psi_1) = c(d_{\xi - \eta} F, d_{\xi - \eta} F)$, so
\[
\{ \psi^*_x F, K \}_{\mathcal{R}}(\xi, \eta) = \frac{1}{2} \langle (\xi, \eta), [(d_{\xi - \eta} F, d_{\xi - \eta} F), \mathcal{R} d_{(\xi,\eta)} K + c d_{(\xi,\eta)} K] \rangle.
\] (27)

Using Formula (19), we have that $\mathcal{R} d_{(\xi,\eta)} K + c d_{(\xi,\eta)} K$ is of the form $(x, x)$ for some $x \in \mathfrak{g}$, the explicit expression of which is not needed in this proof. Then (27) becomes
\[
\{ \psi^*_x F, K \}_{\mathcal{R}}(\xi, \eta) = \frac{1}{2} \langle (\xi, \eta), [(d_{\xi - \eta} F, d_{\xi - \eta} F), (x, x)] \rangle
\]
\[
= \frac{1}{2} \langle \xi - \eta, [d_{\xi - \eta} F, x] \rangle
\]
\[
= 0.
\]
where we used that $F$ is an Ad*-invariant function on $\mathfrak{g}^*$ in the last line. This shows that $\psi^*_\lambda F$ is a Casimir on $((\mathfrak{g}^*)^2, \{\cdot,\cdot\}_{\mathcal{R}})$.

(2) Let us prove that the Poisson $\mathcal{R}$-bracket between $F \circ \psi_\lambda$ and $G \circ \psi_\gamma$ at an arbitrary point $(\xi, \eta) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is equal to zero. According to (17),

$$\{\psi^*_\lambda F, \psi^*_\gamma G\}_{\mathcal{R}}(\xi, \eta) = \frac{1}{2} \left( \langle (\xi, \eta), [\mathcal{R} d(\xi, \eta), d(\xi, \eta)(G \circ \psi_\gamma)] \rangle \right)$$

Using Formulae (22) and (23), the Poisson $\mathcal{R}$-bracket of (28) becomes

$$\{\psi^*_\lambda F, \psi^*_\gamma G\}_{\mathcal{R}}(\xi, \eta) = \left[ \begin{array}{c} \frac{1}{2} \langle (\xi, \eta), [(\lambda - 1)(R d(\xi, \eta)F, R d(\xi, \eta)F), (\gamma d(\xi, \eta)G, d(\xi, \eta)G)] \rangle \\ + \frac{1}{2} \langle (\xi, \eta), [(c(\gamma d(\xi, \eta)F, c(\lambda d(\xi, \eta)F), (\gamma d(\xi, \eta)G, d(\xi, \eta)G)] \rangle \\ - \langle (F, \lambda) \leftrightarrow (G, \gamma) \rangle \\ = \frac{\lambda - 1}{2} \langle \xi - \eta, [R d(\xi, \eta)F, d(\xi, \eta)G] \rangle \\ + \frac{c\gamma}{2} \langle \xi, [\lambda d(\xi, \eta)F, d(\xi, \eta)G] \rangle - \frac{c\lambda}{2} \langle \eta, [\lambda d(\xi, \eta)F, d(\xi, \eta)G] \rangle \\ - \langle (F, \lambda) \leftrightarrow (G, \gamma) \rangle \\ = \frac{c\gamma}{2} \langle \xi, [\lambda d(\xi, \eta)F, d(\xi, \eta)G] \rangle - \frac{c\lambda}{2} \langle \eta, [\lambda d(\xi, \eta)F, d(\xi, \eta)G] \rangle \\ - \langle (F, \lambda) \leftrightarrow (G, \gamma) \rangle \end{array} \right)$$

where we have used the Ad*-invariance of $F$ and $G$ on $\mathfrak{g}^*$ to simplify the expression. According to Formula (25), all four terms of the previous expression vanish if $\lambda \neq \gamma$. If $\lambda = \gamma$, the bracket $\{\psi^*_\lambda F, \psi^*_\lambda G\}_{\mathcal{R}}$ becomes

$$\{\psi^*_\lambda F, \psi^*_\lambda G\}_{\mathcal{R}}(\xi, \eta) = c\lambda \langle \xi - \eta, [\lambda d(\xi, \eta)F, d(\xi, \eta)G] \rangle$$

which is zero in view of Formula (24).

We now suppose that $F$ and $G$ are polynomials of degrees $l$ and $k$, so for all $(\xi, \eta) \in \mathfrak{g} \times \mathfrak{g}^*$, $F \circ \psi_\lambda(\xi, \eta) = \sum_{i=0}^l \lambda^i F_i(\xi, \eta)$ and $G \circ \psi_\gamma(\xi, \eta) = \sum_{j=0}^k \gamma^j G_j(\xi, \eta)$; hence

$$\{F \circ \psi_\lambda, G \circ \psi_\gamma\}_{\mathcal{R}} = \sum_{i=0}^l \sum_{j=0}^k \lambda^i \gamma^j \{F_i, G_j\}_{\mathcal{R}}.$$ (29)

As we just showed, the term on the left hand side of Equation (29) is zero, for all $\lambda, \gamma \in \mathbb{F}$. Therefore, all the coefficients of $\sum_{i=0}^l \sum_{j=0}^k \lambda^i \gamma^j \{F_i, G_j\}_{\mathcal{R}}$ are zero.
\( \psi \) computation gives

\[
\{ F \circ \psi_1, G \circ \psi_1 \} \mathcal{R}(\xi, \eta) = \langle \xi - \eta, [d_{\xi - \eta} F, d_{\xi - \eta} G] \rangle = \{ F, G \} (\psi_1(\xi, \eta)).
\]

Hence \( \psi^* \) is a Poisson map.

(4) Let \( K \) be a function on \( \mathfrak{g}^* \times \mathfrak{g}^* \) and \( (\xi, \eta) \) be an element of \( \mathfrak{g}^* \times \mathfrak{g}^* \). A direct computation gives

\[
\mathcal{X}_{H \circ \psi} (\xi, \eta)[K] = \{ K, H \circ \psi \} \mathcal{R}(\xi, \eta)
= \frac{1}{2} \langle \xi, \eta \rangle [\mathcal{R} d(\xi, \eta) K, d(\xi, \eta)(H \circ \psi)] + [d(\xi, \eta) K, \mathcal{R} d(\xi, \eta)(H \circ \psi)] \rangle.
\]

In order to rewrite this formula, we temporarily use \( (x, y) := d(\xi, \eta) K \). According to (19), \( \mathcal{R} d(\xi, \eta) K = \mathcal{R}(x, y) \) is given by

\[
\mathcal{R} d(\xi, \eta) K = (R(x - y), R(x - y)) + c(y, x).
\]

Formulae (31) and (22) allow one to rewrite the first term of (30) in the following manner:

\[
\langle \xi, \eta \rangle [\mathcal{R} d(\xi, \eta) K, d(\xi, \eta)(H \circ \psi)]
= \langle \xi, \eta \rangle [(R(x - y), R(x - y)), (\lambda d_{\lambda \xi - \eta} H, d_{\lambda \xi - \eta} H)]
+ c \langle \xi, \eta \rangle [(y, x), (\lambda d_{\lambda \xi - \eta} H, d_{\lambda \xi - \eta} H)]
= \lambda \xi - \eta, [R(x - y), d_{\lambda \xi - \eta} H] + c \lambda \xi, [y, d_{\lambda \xi - \eta} H] - c \eta, [x, d_{\lambda \xi - \eta} H]
= c \eta, [y, d_{\lambda \xi - \eta} H] - c \lambda \xi, [x, d_{\lambda \xi - \eta} H],
\]

where, in the second line, we used, three times, the Ad*-invariance of \( H \). Thus

\[
\langle \xi, \eta \rangle [\mathcal{R} d(\xi, \eta) K, d(\xi, \eta)(H \circ \psi)] = \langle \xi, \eta \rangle [(c \lambda d_{\lambda \xi - \eta} H, c d_{\lambda \xi - \eta} H), (x, y)] .
\]

By replacing the first term of (30) by its expression given in (33) and by using Formula (23) to express the second term of (30), we rewrite \( \mathcal{X}_{H \circ \psi} (\xi, \eta)[F] \) as follows:

\[
\mathcal{X}_{H \circ \psi} (\xi, \eta)[K] = \frac{1}{2} \langle \xi, \eta \rangle [(c \lambda d_{\lambda \xi - \eta} H, c d_{\lambda \xi - \eta} H), (x, y)]
- \frac{(\lambda - 1)}{2} \langle \xi, \eta \rangle [(R d_{\lambda \xi - \eta} H, R d_{\lambda \xi - \eta} H), (x, y)]
- \frac{1}{2} \langle \xi, \eta \rangle [(c d_{\lambda \xi - \eta} H, c \lambda d_{\lambda \xi - \eta} H), (x, y)]
= \frac{(\lambda - 1)}{2} \langle \xi, \eta \rangle [(c I - R) d_{\lambda \xi - \eta} H, -(R + c I) d_{\lambda \xi - \eta} H, d(\xi, \eta) K] .
\]

We then deduce Formula (20). When \( \mathfrak{g} \) is a Lie algebra splitting and \( R = P_+ - P_- \), we have \( c = 1 \), \( R - c I = -2 P_- \) and \( R + c I = 2 P_+ \). This implies that (20) gives (21). \( \square \)
3 The Liouville integrability of the 2-Toda lattice

In this section we define the 2-Toda lattice for every complex simple Lie algebra and we prove its Liouville integrability.

In order to define the 2-Toda lattice for every simple Lie algebra, we need some notation. Let $g$ be a simple Lie algebra of rank $\ell$, with Killing form $\langle \cdot | \cdot \rangle$. We choose $h$, a Cartan subalgebra with roots system $\Phi$, and $\Pi = (\alpha_1, \ldots, \alpha_\ell)$, a system of simple roots with respect to $h$. For every $\alpha$ in $\Phi \setminus \{-\Pi, \Pi\}$, we denote by $e_\alpha$ a non-zero eigenvector associated with eigenvalue $\alpha$, and, for every $1 \leq i \leq \ell$, we denote by $e_i$ and $e_{-i}$ non-zero eigenvectors associated respectively with $\alpha_i$ and $-\alpha_i$. The Lie algebra $g = \sum_{k \in \mathbb{Z}} g_k$ is endowed with the natural grading (i.e., for every $k, l \in \mathbb{Z}$, $[g_k, g_l] \subset g_{k+l}$) defined by $g_0 := h$ and, for every $k \in \mathbb{Z}$, $g_k := \{e_\alpha | \alpha \in \Phi, |\alpha| = k\}$, for $|\alpha|$ the length of the root $\alpha$, i.e., $|\alpha| = \sum_{i=1}^\ell a_i$ for $\alpha = \sum_{i=1}^\ell a_i \alpha_i$. In the sequel, we shall use the following property: $\langle g_k | g_l \rangle = 0$ if $k + l \neq 0$.

We introduce the following notation:

$$g_{<k} := \sum_{i<k} g_i, \quad g_{\leq k} := \sum_{i \leq k} g_i, \quad g_{>k} := \sum_{i>k} g_i, \quad g_{\geq k} := \sum_{i \geq k} g_i.$$ 

Also, $g_+ := g_{\geq 0}$ and $g_- := g_{<0}$.

3.1 Definition of the 2-Toda lattice

The next definition gives again the definition given in (1) when specialized to the case $g = \mathfrak{sl}_n(\mathbb{C})$, taking for $h$ the Lie subalgebra of diagonal matrices.

**Definition 7** The 2-Toda lattice associated with a simple Lie algebra $g$ is the system of differential equations given by the following Lax equations:

$$\frac{\partial (L, M)}{\partial t} = [(L_+, L_+), (L, M)],$$

$$\frac{\partial (L, M)}{\partial s} = [(M_-, M_-), (L, M)],$$

where $(L, M)$ is an element of the phase space of the 2-Toda lattice $T^2 := g_{<0} \times g_{>\ell} + (\sum_{i=1}^\ell e_i, 0)$, where $L_+ := P_+(L)$, $M_- := P_-(M)$, and where $P_{\pm}$ is the projection of $g$ on $g_{\pm}$.

3.2 The 2-Toda lattice is a Hamiltonian system

We recall from Example 3 that when $g = g_+ \oplus g_-$ is a Lie algebra splitting then $g^2 = g^2_+ \oplus g^2_-$ is also a Lie algebra splitting, where

$$g^2_+ := \{(x, x) | x \in g\} \quad \text{et} \quad g^2_- := \{(x, y) | x \in g_-, y \in g_+\}.$$
Also, for every \((x, y) \in \mathfrak{g}^2\) we have \((x, y) = (x, y)_+ + (x, y)_-\), where
\[
\begin{align*}
(x, y)_+ &= (x_+ + y_-, x_+ + y_-) \in \mathfrak{g}_2^+, \\
(x, y)_- &= (x_- - y_-, y_+ - x_+) \in \mathfrak{g}_2^-.
\end{align*}
\] (37)

Let \(\mathcal{R}\) be the difference of the projections of \(\mathfrak{g}^2\) on the Lie subalgebras \(\mathfrak{g}_2^+\) and \(\mathfrak{g}_2^-\). According to Example 3, we have
\[
\mathcal{R}(x, y) = (x_+ - x_- + 2y_-, y_- - y_+ + 2x_+) \quad \text{(38)}
\]
where \(R\) is the difference of the projections of \(\mathfrak{g}\) on \(\mathfrak{g}_+\) and on \(\mathfrak{g}_-\). We provide \(\mathfrak{g}^2\) with the following \(\text{Ad}\)-invariant, non-degenerate symmetric bilinear form:
\[
\langle \cdot | \cdot \rangle_2 : \mathfrak{g}^2 \times \mathfrak{g}^2 \to \mathbb{C}, \quad ((x_1, y_1), (x_2, y_2)) \mapsto \langle x_1 | x_2 \rangle - \langle y_1 | y_2 \rangle. \quad \text{(40)}
\]

We use it to identify \(\mathfrak{g}^2\) with its dual and we obtain according to (16) a linear Poisson structure on \(\mathfrak{g}^2\), defined for every \(F, G \in \mathcal{F}(\mathfrak{g}^2)\) at \((x, y) \in \mathfrak{g}^2\) by
\[
\{F, G\}_\mathcal{R}(x, y) = \frac{1}{2} \langle (x, y) | [\mathcal{R}\nabla_{(x,y)}F, \nabla_{(x,y)}G] + [\nabla_{(x,y)}F, \mathcal{R}\nabla_{(x,y)}G] \rangle_2, \quad \text{(41)}
\]
where \(\nabla_{(x,y)}F\) is the gradient of \(F\) at \((x, y) \in \mathfrak{g}^2\) (with respect to \(\langle \cdot | \cdot \rangle_2\)), i.e.,
\[
\langle \nabla_{(x,y)}F | (z, s) \rangle_2 = \langle d_{(x,y)}F; (z, s) \rangle, \quad \forall (z, s) \in \mathfrak{g}^2.
\]

We show that the phase space \(\mathcal{T}^2\) is equipped with a Poisson structure and the equations of motion of the 2-Toda lattice are Hamiltonian.

**Proposition 8** \(\mathcal{T}^2\) is a Poisson submanifold of \((\mathfrak{g}^2, \{\cdot, \cdot\}_\mathcal{R})\).

We use the following lemma to show the above proposition.

**Lemma 9** Let \(\mathfrak{g}\) be a Lie algebra equipped with a non-degenerate, symmetric, bilinear form \(\langle \cdot | \cdot \rangle\). Let \(a \in \mathfrak{g}\) and let \(E\) be a subspace of \(\mathfrak{g}\). We suppose that:
1. The orthogonal \(E^\perp\) of \(E\) is a Lie ideal of \((\mathfrak{g}, [\cdot, \cdot])\).
2. For every \(x, y \in \mathfrak{g}\), we have \(\langle a | [x, y] \rangle = 0\).

Then \(a + E\) and \(E\) are Poisson submanifolds of \((\mathfrak{g}, \{\cdot, \cdot\})\), equipped with its linear Poisson structure \(\{\cdot, \cdot\}\). The map \(x \to a + x\) is a Poisson isomorphism between them. Moreover \(E\) is Poisson isomorphic to the Lie-Poisson manifold \((\mathfrak{g}/E^\perp)^*\), also equipped with its linear Poisson structure.

\(^3\)Here the orthogonality is with respect to the form \(\langle \cdot | \cdot \rangle\).
Proof. We start to show that $E$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot, \cdot\})$. Let $J := \langle F \in \mathcal{F}(\mathfrak{g}) \mid F \equiv 0 \text{ on } E \rangle$. To show that $E$ is a Poisson submanifold is equivalent to show that $J$ is Poisson ideal. Let $F \in J$ and let $x \in E$, we notice that $\nabla_x F \in E^\perp$. According to the first condition of the proposition, for every $G \in \mathcal{F}(\mathfrak{g})$, we have $[\nabla_x f, \nabla_x G] \in E^\perp$. This implies that $\{F, G\} (x) = \langle x \mid [\nabla_x f, \nabla_x G] \rangle = 0$. Then $J$ is a Poisson ideal.

We now show that $E + a$ is a Poisson submanifold. Let $I := \langle F \in \mathcal{F}(\mathfrak{g}^2) \mid F \equiv 0 \text{ on } E + a \rangle$, $F \in I$ and let $x + a \in E + a$. Notice that $\nabla_{x+a} F$ is an element of $E^\perp$, which is according to first condition of these proposition is a Lie ideal of $(\mathfrak{g}, [\cdot, \cdot])$. Then, for every $G \in \mathcal{F}(\mathfrak{g})$, $[\nabla_{x+a} F, \nabla_{x+a} G] \in E^\perp$. Let now compute the Poisson bracket between $F$ and $G \in \mathcal{F}(\mathfrak{g})$ at a point $x + a$.

\[
\{F, G\} (x + a) = \langle x + a \mid [\nabla_{x+a} F, \nabla_{x+a} G] \rangle = \langle x \mid [\nabla_{x+a} F, \nabla_{x+a} G] \rangle = 0,
\]

where we have used the second condition of proposition to justify the transition from first to second line and we have zero in the last line because $x \in E$ and $[\nabla_{x+a} F, \nabla_{x+a} G] \in E^\perp$.

Let now show the translation by $a$ is a Poisson isomorphism between $\mathfrak{g}$ and $\mathfrak{g}$. We denote by $T_a$ the translation by $a$, defined for every $x \in \mathfrak{g}$, by $T_a(x) = x + a$. For every $F, G \in \mathcal{F}(\mathfrak{g})$ and every $x \in \mathfrak{g}$, we have

\[
\{F \circ T_a, G \circ T_a\} (x) = \langle x \mid [\nabla_x (F \circ T_a), \nabla_x (G \circ T_a)] \rangle = \langle x \mid [\nabla_{T_a(x)} F, \nabla_{T_a(x)} G] \rangle = \langle x + a \mid [\nabla_{T_a(x)} F, \nabla_{T_a(x)} G] \rangle = \{F, G\} (T_a(x))\]

where we have used Condition (2) to justify the transition from second to third line. Since $E$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot, \cdot\})$ the restriction of the Poisson isomorphism $T_a$ to $E$ is also a Poisson isomorphism on its image. \hfill $\square$

We now prove Proposition \[.\]
Proof. Its easy to verify that $\mathcal{T}^2$ admits the following description as an affine subspace of $\mathfrak{g}^2$:

\[
\mathcal{T}^2 = (e, e) + \Delta(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \oplus \mathfrak{g}^2_-, \quad (42)
\]

where $\Delta(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) := \{(x, x) \mid x \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}\}$ and $e = \sum_{i=1}^\ell e_i$. We check that the two assumptions of Lemma \[.\] are satisfied, with $a = (e, e)$, $E = \Delta(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \oplus \mathfrak{g}^2_-$, $\mathfrak{g} = \mathfrak{g}^2$ and $[\cdot, \cdot] = [\cdot, \cdot]_R$.

1. It is clear that $E = \mathfrak{g}_{\leq 0} \times \mathfrak{g}_{\geq 1}$. The orthogonal of $E$ is $E^\perp = \mathfrak{g}_{< 0} \times \mathfrak{g}_{> 1}$, which is a subspace of $\mathfrak{g}^2_-$ and it is an ideal of $(\mathfrak{g}^2, [\cdot, \cdot]_R)$, because, for every $(x, y) \in \mathfrak{g}_{< 0} \times \mathfrak{g}_{> 1}$ and $(z, s) \in \mathfrak{g}^2$,

\[
[(x, y), (z, s)]_R \in [\mathfrak{g}_{< 0} \times \mathfrak{g}_{> 1}, \mathfrak{g}_{< 0} \times \mathfrak{g}_{> 0}] \subset \mathfrak{g}_{< -1} \times \mathfrak{g}_{> 1} \subset \mathfrak{g}_{< 0} \times \mathfrak{g}_{> 1}.
\]
(2) For every \((x, y), (x', y') \in \mathfrak{g}^2\), we have
\[
\langle (e, e) \mid [(x, y), (x', y')] \rangle = \langle (e, e) \mid [(x, y)_+, (x', y')_+] - [(x, y)_-, (x', y')_-] \rangle_2.
\] (43)
Since \([(x, y)_+, (x', y')_+] = (u, u)\) for some \(u \in \mathfrak{g}\) it follows that the first term on the right side of Equation (43) is zero. Also \([(x, y)_-, (x', y')_-) \in \mathfrak{g}_{<1} \times \mathfrak{g}_{>0}\), which is orthogonal to \((e, e) \in \mathfrak{g}_1 \times \mathfrak{g}_1\); hence the second term on the right hand side of Equation (43) is zero.

\[\square\]

**Proposition 10** Let \(H, \tilde{H} \in \mathcal{F}(\mathfrak{g}^2)\) be defined at every point \((x, y)\) of \(\mathfrak{g}^2\) by
\[
H(x, y) := \frac{1}{2} \langle x \mid x \rangle \quad \text{and} \quad \tilde{H}(x, y) := \frac{1}{2} \langle y \mid y \rangle.
\] (44)
The Hamiltonian vector field \(X_H := \{\cdot, H\}_\mathcal{R}\) (resp. \(X_{\tilde{H}} := \{\cdot, \tilde{H}\}_\mathcal{R}\)) is tangent to \(\mathcal{T}^2\) and describes on \(\mathcal{T}^2\) the equation of motion (34) (resp. (35)) for the 2-Toda lattice.

**Proof.** Let \(F \in \mathcal{F}(\mathfrak{g}^2)\) and let \((L, M) \in \mathcal{T}^2\). We have
\[
X_H[F](L, M) = \{F, H\}_\mathcal{R}(L, M)
\]
\[
= \frac{1}{2} \langle (L, M) \mid \mathcal{R}\nabla_{(L,M)}F, \nabla_{(L,M)}H \rangle \rangle_2
\]
\[
+ \frac{1}{2} \langle (L, M) \mid \nabla_{(L,M)}F, \mathcal{R}\nabla_{(L,M)}H \rangle \rangle_2
\]
\[
= -\frac{1}{2} \langle [(L, M), \mathcal{R}\nabla_{(L,M)}H] \mid \nabla_{(L,M)}F \rangle_2,
\] (45)
where we have used the Ad-invariance on \(\mathfrak{g}^2\) of \(H\) to justify the transition from second to third equality. We deduce from Equation (45) that \(X_H(L, M) = \frac{1}{2} [\mathcal{R}\nabla_{(L,M)}H, (L, M)]\).
Hence according to Formula (34), we have
\[
X_H(L, M) = [(\nabla_{(L,M)}H)_+, (L, M)] = [(L, 0)_+, (L, M)] = [(L_+, L_+), (L, M)].
\]
To show that \(X_{\tilde{H}}\) describes Equation (35) it suffices to repeat the same reasoning of \(X_H\).
Since \(\mathcal{T}^2\) is a Poisson submanifold of \((\mathfrak{g}^2, \{\cdot, \cdot\})\), the Hamiltonian vector fields \(X_H\) and \(X_{\tilde{H}}\) are tangent to \(\mathcal{T}^2\). \(\square\)

### 3.3 The integrability of the 2-Toda lattice

According to [5, Theorem 7.3.8], for every simple Lie algebra \(\mathfrak{g}\) of rank \(\ell\), there exist \(\ell\) homogeneous, independent, Ad-invariant polynomials \(P_1, \ldots, P_\ell\) which generate the algebra of Ad-invariant polynomial functions on \(\mathfrak{g}\) and which are of degree, respectively, \(m_1 + 1, \ldots, m_\ell + 1\), where \(m_1, \ldots, m_\ell\) are the exponents of \(\mathfrak{g}\) (we note that \(m_1 \leq \ldots \leq m_\ell\)). Each \(P_i\) induces \(m_i + 2\) functions \(F_{j,i} \in \mathcal{F}(\mathfrak{g}^2)\), as follows:
\[ P_i(\lambda x - y) = \sum_{0 \leq j \leq m_i + 1} (-1)^{m_i + 1 - j} \lambda^j F_{j,i}(x,y). \]  

(46)

Every function \( F_{j,i} \) for \( 1 \leq i \leq \ell \) and \( 0 \leq j \leq m_i + 1 \) is homogeneous of degree \( j \) with respect to its first variable and of degree \( m_i + 1 - j \) with respect to its second variable.

**Notation 11** We denote by \( \mathcal{F} \) the family of functions on \( \mathfrak{g}^2 \) given by

\[ \mathcal{F} := (F_{j,i}, 1 \leq i \leq \ell \text{ and } 0 \leq j \leq m_i + 1). \]

(47)

**Remark 12** (1) The functions \( F_{0,1} \) and \( F_{2,1} \) are the Hamiltonians of the 2-Toda lattice introduced in Proposition [14].

(2) The functions \( F_{0,i} \) and \( F_{m_i+1,i} \) for \( 1 \leq i \leq \ell \) are Ad-invariant functions on \( \mathfrak{g}^2 \). According to the Adler-Kostant-Symes theorem [2, Theorem 4.37] they are in involution with respect to the bracket \( \{\cdot, \cdot\}_\mathcal{R} \). Also they are independent on \( \mathfrak{g}^2 \). They are therefore a good candidates for giving the Liouville integrability of the 2-Toda lattice. However their cardinal \( 2\ell \) is very small compared to \( \dim T^2 - \frac{1}{2} \text{Rk}(T^2, \{\cdot, \cdot\}_\mathcal{R}) \).

Since, as we noticed in Remark [12] the Hamiltonians of the 2-Toda lattice appear among the functions composing \( \mathcal{F} \), the next theorem gives the Liouville integrability of the 2-Toda lattice.

**Theorem 13** The triplet \( (T^2, \mathcal{F}|_{T^2}, \{\cdot, \cdot\}_\mathcal{R}) \) is an integrable system.

**Proof.** According to the definition of integrability in the sense of Liouville (see [2, Definition 4.13]), to prove Theorem [3] we must show that

(1) \( \mathcal{F}|_{T^2} \) is involutive for the Poisson \( \mathcal{R} \)-bracket \( \{\cdot, \cdot\}_\mathcal{R} \).

(2) \( \mathcal{F}|_{T^2} \) is independent.

(3) The cardinal of the restriction of \( \mathcal{F} \) to \( T^2 \) satisfies

\[ \text{card} \mathcal{F}|_{T^2} = \dim T^2 - \frac{1}{2} \text{Rk}(T^2, \{\cdot, \cdot\}_\mathcal{R}). \]

(48)

The proofs of these three points are given in respectively Proposition [14], Proposition [17] and Proposition [19] which are given in the next three subsections. \( \square \)

### 3.3.1 The restriction of \( \mathcal{F} \) to \( T^2 \) is involutive for \( \{\cdot, \cdot\}_\mathcal{R} \)

Unlike in the Toda lattice case, it is not the Adler-Kostant-Symes theorem that gives us the commutativity of the considered family of functions \( \mathcal{F} \), which is not formed only by the Ad-invariant functions. This commutativity arises from the fact our phase space is a Poisson submanifold of \( (\mathfrak{g}^2, \{\cdot, \cdot\}_\mathcal{R}) \), and from the second item of Theorem [4].
**Proposition 14** The family of functions $\mathcal{F}|_{T^2}$ is involutive for the Poisson structure $\{\cdot,\cdot\}_\mathbb{R}$.

**Proof.** Since the polynomials $P_1, \ldots, P_\ell$ are $\text{Ad}$-invariant on $\mathfrak{g}$ (and hence $\text{Ad}^*$-invariant upon identifying $\mathfrak{g}$ with $\mathfrak{g}^*$), according to the second point of Theorem 4 the family $F$ is involutive on $(\mathfrak{g}^2,\{\cdot,\cdot\}_\mathbb{R})$. Furthermore, $T^2$ being a Poisson submanifold of $(\mathfrak{g}^2,\{\cdot,\cdot\}_\mathbb{R})$ (see Proposition 3), the restriction of $\mathcal{F}$ to $T^2$ is involutive. \hfill $\Box$

### 3.3.2 The restriction of $\mathcal{F}$ to $T^2$ is independent

We use an unpublished result of Raïs [10], which establishes the independence of a large family of functions on $\mathfrak{g}^2$. We state this result below and the proof is in [4, Section 1].

**Theorem 15** Let $P_1, \ldots, P_\ell$ be a generating family of homogeneous polynomials of the algebra of $\text{Ad}$-invariant polynomial functions on $\mathfrak{g}$. Let $e$ and $h$ be two elements of $\mathfrak{g}$ such that $e$ is regular and $[h,e] = 2e$.

For every $F \in \mathcal{F}(\mathfrak{g})$, and every $y \in \mathfrak{g}$, we denote by $d_y^k F$ the differential of order $k$ of $F$ at $y$. Denote by $V_{k,i}$, for every $1 \leq i \leq \ell$ and $0 \leq k \leq m_i$, the element of $\mathfrak{g}$ defined by

$$\langle V_{k,i} | z \rangle = \left( d_h^{k+1} P_i, (e^k, z) \right), \quad \forall z \in \mathfrak{g},$$

(49)

where, for every $x \in \mathfrak{g}$ and $k \in \mathbb{N}$, $x^k$ is shorthand for $(x, \ldots, x)$ ($k$ times).

(1) The family $\mathcal{F}_1 := (V_{k,i}, 1 \leq i \leq \ell$ and $0 \leq k \leq m_i)$ is linearly independent.

(2) The subspace generated by $\mathcal{F}_1$ is the Lie subalgebra formed by the sum of the all eigenspaces of $\text{ad}_h$ associated with positive or zero eigenvalues.

We first prove, using the first point of Theorem 15, the independence of $\mathcal{F}$ (which is a family $\mathcal{F}(\mathfrak{g}^2)$) at a well chosen point $(e,h) \in T^2 \cap \mathfrak{g}^2$. Afterwards, using the second point of Theorem 15 we show that the restriction of $\mathcal{F}$ to the phase space $T^2$ of the 2-Toda lattice is also an independent family of functions.

**Proposition 16** Let $h \in \mathfrak{h}$ be such that $[h,e] = 2e$.

(1) The polynomial functions $F_{0,0}(x,y), \ldots, F_{0,\ell}(x,y)$ only depend on the second variable $y$. Their differentials at the point $(e,h)$ are independent;

(2) We denote by $\pd{}{x}$ the partial differential with respect to the variable $x$. The $\frac{1}{2} (\dim \mathfrak{g} + \ell)$ partial derivatives $\pd{F_{j,i}}{x}$, $1 \leq i \leq \ell$ and $1 \leq j \leq m_i$ are independent at the point $(e,h)$;

(3) The family $(d_{(e,h)} F_{j,i}, 1 \leq i \leq \ell, 0 \leq j \leq m_i + 1)$ is independent.

**Proof.** (1) For every $1 \leq i \leq \ell$ and every $(x,y) \in \mathfrak{g}^2$, $F_{0,i}(x,y)$ is the term of degree 0 in $\lambda$ of $P_i(\lambda x - y)$, that is $P_i(y)$. Hence, the function $F_{0,i}$, for $1 \leq i \leq \ell$, is a homogeneous polynomial of degree $m_i + 1$ which only depends on the second variable $y$. Moreover, according to two theorems of Kostant [6, Theorem 9] and [7, Theorem 5.2], the differentials of the polynomials $P_1, \ldots, P_\ell$ are independent at every regular point of $\mathfrak{g}$. In particular,
they are independent at the regular point \( h \). In conclusion the differentials of the functions \( F_{0,1}, \ldots, F_{0,\ell} \) are independent at \((e, h)\).

(2) For \( 1 \leq i \leq \ell \), the Taylor Formula applied to the polynomial \( P_i, 1 \leq i \leq \ell \), at \( \lambda x - y \), yields

\[
P_i(\lambda x - y) = \sum_{k=0}^{m_i+1} (-1)^{m_i+1-k} \frac{\lambda^k}{k!} \left\langle d_y^k P_i, x^k \right\rangle.
\]  

(50)

By identifying the coefficients of Equations (50) and (46), we obtain, for every \( 1 \leq i \leq \ell \) and \( 1 \leq k \leq m_i + 1 \), the equality

\[
F_{k,i}(x, y) = \frac{1}{k!} \left\langle d_y^k P_i, x^k \right\rangle.
\]  

(51)

(3) Denote by \( M \) the matrix

\[
M = (d_{(e,h)} F_{k,i}, \ 0 \leq k \leq m_i + 1 \text{ and } 1 \leq i \leq \ell)
\]

\[
= \begin{pmatrix}
\frac{\partial F_{0,1}}{\partial x}(e, h) & \frac{\partial F_{0,1}}{\partial y}(e, h) \\
\vdots & \vdots \\
\frac{\partial F_{0,\ell}}{\partial x}(e, h) & \frac{\partial F_{0,\ell}}{\partial y}(e, h) \\
\frac{\partial F_{1,1}}{\partial x}(e, h) & \frac{\partial F_{1,1}}{\partial y}(e, h) \\
\vdots & \vdots \\
\frac{\partial F_{m_i+1,1}}{\partial x}(e, h) & \frac{\partial F_{m_i+1,1}}{\partial y}(e, h)
\end{pmatrix}.
\]

According to item (1), the form of the matrix \( M \) is

\[
M = \begin{pmatrix}
0 & A \\
B & \ast
\end{pmatrix}
\]

\footnote{The element \( h \) of \( g \) is regular. Indeed, since it verifies \([h, e] = 2e\) with \( e \) regular, it belongs to a principal \( \mathfrak{sl}(2) \) triple (see \cite[Theorem 32.1.5]{12}).}
where $A = \left( \frac{\partial F_{0,i}}{\partial y}(e, h), 1 \leq i \leq \ell \right)$ and $B = \left( \frac{\partial F_{k,i}}{\partial x}(e, h), 1 \leq k \leq m_i + 1 \text{ and } 1 \leq i \leq \ell \right)$.

According to (1) and (2), the $\ell$ rows of the matrix $A$ and the $\frac{1}{2}(\dim g + \ell)$ rows of the matrix $B$ are independent. The independence of the rows of $M$ yields the independence of the differentials of the family of functions $F$ at $(e, h) \in T^2$.

\[\text{□}\]

**Proposition 17** The restriction of the family of functions $F$ to the phase space $T^2$ of the 2-Toda lattice is an independent family.

**Proof.** It suffices to show that the restrictions on the tangent space $T_{(e,h)}(e,h) T^2$ of the differentials

\[(d_{(e,h)} F_{j,i}), \ 1 \leq i \leq \ell \text{ and } 0 \leq j \leq m_i + 1\]

are independent. Since the tangent space of $T^2$ is $g_{\leq 0} \times g_{\geq -1}$ and since $(d_{(e,h)} F_{0,i})\ 1 \leq i \leq \ell$ vanish on $g_{\leq 0} \times \{0\}$, it suffices to show that:

(a) The restriction to the space $\{0\} \times \mathfrak{h}$ of the linear forms

\[(d_{(e,h)} F_{0,i}), \ 1 \leq i \leq \ell \]

is an independent family.

(b) The restriction to the space $g_{\leq 0} \times \{0\}$ of the linear independent forms

\[(d_{(e,h)} F_{j,i}), \ 1 \leq i \leq \ell \text{ and } 1 \leq j \leq m_i + 1\]

is an independent family.

According to Definition (46), $F_{0,i}(x, y) = P_i(y)$ for every $1 \leq i \leq \ell$ and every $(x, y) \in g^2$. Then $\frac{\partial F_{0,i}}{\partial y}(e, h) = 0$ and

\[\left\langle \frac{\partial F_{0,i}}{\partial y}(e, h), z \right\rangle = \left\langle \nabla_h P_i \mid z \right\rangle, \quad \forall z \in g. \quad (54)\]

Since $\nabla_h P_i \in \mathfrak{h}$ (because $[h, \nabla_h P_i] = 0$ and $h \in \mathfrak{h}$ is regular), the restriction to $\mathfrak{h}$ of the family $(\frac{\partial F_{0,1}}{\partial y}(e, h), \ldots, \frac{\partial F_{0,\ell}}{\partial y}(e, h))$ is an independent family of linear forms. This implies the independence of the restriction to $\{0\} \times \mathfrak{h}$ of $d_{(e,h)} F_{0,1}, \ldots, d_{(e,h)} F_{0,\ell}$.

According to equation (53), for every $1 \leq i \leq \ell$ and $0 \leq k \leq \ell$,

\[\left\langle V_{k,i} \mid z \right\rangle = k! \left\langle \frac{\partial F_{k+1,i}}{\partial x}(e, h), z \right\rangle. \quad (55)\]

The second point of Theorem (15) shows that the subspace generated by the family $(V_{k,i})\ 1 \leq i \leq \ell$ and $0 \leq k \leq m_i$ is contained in the Lie subalgebra obtained by summing the eigenspaces of $ad_h$, associated with the nonnegative eigenvalues, which is in our case $g_{\geq 0}$. This proves, using Equation (53), that the restriction to $g_{\leq 0}$ of the family of linear forms $(\frac{\partial F_{k,i}}{\partial x}(e, h), 1 \leq i \leq \ell \text{ and } 1 \leq k \leq m_i + 1)$ is an independent family. Therefore the restriction to $g_{\leq 0} \times \{0\}$ of $(d_{(e,h)} F_{k,i}, 1 \leq i \leq \ell \text{ and } 1 \leq i \leq m_i + 1)$ is an independent family.

\[\text{□}\]
3.3.3 The exact number of functions

According to Equation (47), the cardinal of $\mathcal{F}$ is related to the exponents $m_i$, $1 \leq i \leq \ell$, as follows

$$\text{card } \mathcal{F} = \sum_{i=1}^{\ell} (m_i + 2) = \sum_{i=1}^{\ell} m_i + 2\ell.$$ 

Since $\sum_{i=1}^{\ell} m_i = \frac{1}{2}(\dim \mathfrak{g} - \ell)$ (see [5, Theorem 7.3.8]) and $\mathcal{F}|_{T^2}$ is an independent family, we have $\text{card } \mathcal{F}|_{T^2} = \frac{1}{2}(\dim \mathfrak{g} + 3\ell)$. The dimension of $T^2$ is equal to $\dim \mathfrak{g} + 2\ell$. In conclusion, the relation below is satisfied:

$$\text{card } \mathcal{F}|_{T^2} = \dim T^2 - \frac{1}{2} \text{Rk}(T^2, \{\cdot, \cdot\}_R)$$

if and only if $\text{Rk}(T^2, \{\cdot, \cdot\}_R) = \dim \mathfrak{g} + \ell$. We need therefore to prove this last result, which will be done in Proposition 19 below.

The rank of the restriction of $\{\cdot, \cdot\}_R$ to $T^2$: The purpose of this part is to compute the rank of the Poisson manifold $(T^2, \{\cdot, \cdot\}_R)$, i.e., the maximum of the rank at $x$ of the Poisson $\mathcal{R}$-bracket for every $x \in T^2$. We begin by establishing an isomorphism between the Poisson manifold $(T^2, \{\cdot, \cdot\}_R)$ and a product Poisson manifold.

Before doing this, notice that it follows from Lemma 9 applied to $a = 0$ and $E = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g} = \mathfrak{g}$ and $[\cdot, \cdot] = [\cdot, \cdot]_R$ that (the subspace) $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot, \cdot\}_R)$, since the orthogonal of $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie ideal of $(\mathfrak{g}, [\cdot, \cdot]_R)$.

**Proposition 18** The Poisson manifold $(T^2, \{\cdot, \cdot\}_R)$ is isomorphic to the product Poisson manifold $(\mathfrak{g}^*, \{\cdot, \cdot\}_R) \times (\mathfrak{g}_0 \oplus \mathfrak{g}_1, \{\cdot, \cdot\}_R)$, where $\{\cdot, \cdot\}_R$ is the Lie-Poisson bracket on $\mathfrak{g}^*$, $\mathcal{R}$ is the difference of projections on $\mathfrak{g}_+$ and $\mathfrak{g}_-$, and $\{\cdot, \cdot\}_R$ is the Poisson $\mathcal{R}$-bracket on $\mathfrak{g}$ (restricted to $\mathfrak{g}_0 \oplus \mathfrak{g}_1$).

**Proof.** As we saw in the proof of Proposition 8 a := $(e,e)$ and $E := \mathfrak{g}_2^* \Delta (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$ satisfy Conditions (1) and (2) of Lemma 9 hence $T^2 = (e,e) + E$ and $E$ are isomorphic Poisson submanifolds of $(\mathfrak{g}^2, \{\cdot, \cdot\}_R)$. According to Lemma 9 again, $E$ is Poisson isomorphic to $(\mathfrak{g}^2/E^\perp)^*$, where $\mathfrak{g}^2/E^\perp$ is equipped with the quotient of the Lie bracket $[\cdot, \cdot]_R$ with respect to the Lie ideal $E^\perp$. In conclusion, $(T^2, \{\cdot, \cdot\}_R)$ is Poisson isomorphic to $((\mathfrak{g}^2/E^\perp)^*, \{\cdot, \cdot\}'_R)$, where $\{\cdot, \cdot\}'_R$ is the Lie Poisson bracket of $(\mathfrak{g}^2/E^\perp)^*$. Let us determine the quotient Lie algebra $\mathfrak{g}^2/E^\perp$. The Lie algebra $\mathfrak{g}^2$ (endowed with the bracket $[\cdot, \cdot]_R$) is isomorphic to the direct sum of the Lie algebras $\mathfrak{g}$, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{0}$, the isomorphism being given, for every $x \in \mathfrak{g}, y_- \in \mathfrak{g}_-, y_+ \in \mathfrak{g}_+$, by

$$\mathfrak{g} \times \mathfrak{g}_- \times \mathfrak{g}_+ \simeq \mathfrak{g}^2$$

$$(x, y_-, y_+) \to (x, x) + (y_-, y_+). \quad (56)$$
The Lie algebra isomorphism \((53)\) identifies the subspace \(E^\perp = g_- \times g_{\geq 1} \subset g^2\) with \((0, g_- \times g_{\geq 2})\). Hence the quotient \(g^2 / E^\perp\) is the direct sum of the Lie algebra \(g\) (endowed with the usual bracket) and the quotient Lie algebra \(g_+ / g_{\geq 2}\).

We conclude that \(T^2\) is Poisson isomorphic to the dual of the direct sum Lie algebra \(g \oplus (g_+ / g_{\geq 2})\), endowed with the Lie-Poisson structure, and then is isomorphic to the product of the Poisson manifold \(g^*\) (endowed with the Lie-Poisson structure) with \((g_+ / g_{\geq 2})^*\) (endowed with the Lie-Poisson structure). To complete the proof, it suffices to recall (see Lemma 9) that \(g_0 \oplus g_1\) is a Poisson submanifold of \((g, \{\cdot, \cdot\}_R)\), isomorphic to the Lie-Poisson structure on the dual of the quotient Lie algebra \(g / (g_0 \oplus g_1)^\perp = g_+ / g_{\geq 2}\).

**Proposition 19** The rank of the restriction of the Poisson \(\mathcal{R}\)-bracket \(\{\cdot, \cdot\}_\mathcal{R}\) to the manifold \(T^2\) is \(\dim g + \ell\). As a consequence, the following relation is satisfied:

\[
\text{card } \mathcal{F}_{T^2} = \dim T^2 - \frac{1}{2} \text{Rk}(T^2, \{\cdot, \cdot\}_\mathcal{R}).
\]

**Proof.** According to Proposition 18 the Poisson submanifold \((T^2, \{\cdot, \cdot\}_\mathcal{R})\) is isomorphic to the product manifold \((g^*, \{\cdot, \cdot\}_R) \times (g_0 \oplus g_1, \{\cdot, \cdot\}_R)\). This result proves that the restriction of the rank of Poisson \(\mathcal{R}\)-bracket \(\{\cdot, \cdot\}_\mathcal{R}\) to the manifold \(T^2\) is the sum of the rank of the Lie-Poisson structure on \(g^*\), which is \(\dim g - \ell\) (see [12, Proposition 29.3.2]), and the rank of the Poisson \(R\)-bracket on \(g_0 \oplus g_1\).

We calculate the latter rank and show that it is \(2\ell\), which finishes the proof. Let \((z_1, \ldots, z_{2\ell})\) be the coordinate system on \(g_0 \oplus g_1\), defined by

\[
\begin{align*}
z_i(x) &= \langle h_i | x \rangle, \\
z_{i+\ell}(x) &= \langle e_{-i} | x \rangle,
\end{align*}
\]

where as before, for \(1 \leq i \leq \ell\), the element \(e_{-i}\) is a non-zero eigenvector associated with the root \(-\alpha_i\). The Lie-Poisson brackets between these coordinate functions are given by the following formulae:

\[
\begin{align*}
\{z_i, z_j\}_R &= \{z_{i+\ell}, z_{j+\ell}\}_R = 0, \\
\{z_{i+\ell}, z_j\}_R &= c_{ij} z_{i+\ell},
\end{align*}
\]

where \(C := (c_{ij})_{1 \leq i,j \leq \ell}\) is the Cartan matrix of \(g\). Then the Poisson matrix \(M = (\{z_i, z_j\}_R)_{1 \leq i,j \leq 2\ell}\) is equal to

\[
M = \begin{pmatrix}
0 & -A^T \\
A & 0
\end{pmatrix},
\]

where \(A := v C\) and \(v := \text{diag}(z_{1+\ell}, \ldots, z_{2\ell})\). Since \(C\) is invertible, the rank of \(A\) at any point for which \((z_{1+\ell}, \ldots, z_{2\ell}) = (1, \ldots, 1)\) is \(\ell\); therefore the rank of \(M\) is \(2\ell\), which implies that the rank of the Poisson \(R\)-structure on \(g_0 \oplus g_1\) is \(2\ell\). \(\square\)
3.4 The integrability for the quadratic Poisson $\mathcal{R}$-bracket

In this subsection we study the integrability of the 2-Toda lattice on some Lie algebra $\mathfrak{g}$ with respect to a quadratic Poisson $\mathcal{R}$-bracket. To construct a quadratic Poisson $\mathcal{R}$-bracket on $\mathfrak{g}^2$ it is necessary that $\mathfrak{g}$ is an associative algebra of finite dimensional and $\mathcal{R}$ and its antisymmetric part $\mathcal{R}_-$ are solutions of (mCYBE). For this it suffices to choose $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathcal{R}(x, y) = (R(x - y) + y, R(x - y) + x)$, where where $R = P_+ - P_-$, $P_+$ (resp. $P_-$) being the projection of $\mathfrak{gl}_n(\mathbb{C})$ on the subalgebra of upper (resp. strictly lower) triangular matrices (we will show again later that $\mathcal{R}_-$ is a solution of (mCYBE) of $\mathfrak{g}^2$).

In this subsection we show the integrability of a system of equations which is exactly the system \((\ref{1})\), up to the fact that we do no longer assume the matrices $L, M$ to be traceless. We denote by $\mathcal{T}^{2'}$ the phase space of the 2-Toda lattice, i.e.,

\[
\mathcal{T}^{2'} := \{ (L, M) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \mid (L, M) = \begin{pmatrix}
\begin{pmatrix} a_{11} & 1 & 0 \\
 a_{21} & a_{22} & \ddots \\
 \vdots & \ddots & \ddots & 1 \\
 a_{n1} & \ldots & a_{n,n-1} & a_{nn}
\end{pmatrix}, \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1n} \\
 b_{21} & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \vdots \\
 0 & b_{n,n-1} & b_{nn}
\end{pmatrix}\end{pmatrix}\}
\]

and we name as the 2-Toda lattice on $\mathfrak{gl}_n(\mathbb{C})$ the system of differential equations \((\ref{1})\) with the constraint $(L, M) \in \mathcal{T}^{2'}$.

Before studying the integrability of the 2-Toda lattice with respect to the quadratic Poisson $\mathcal{R}$-bracket we will study the integrability of the latter system with respect to the linear Poisson $\mathcal{R}$-bracket.

3.4.1 The Liouville integrability of the 2-Toda lattice on $\mathfrak{gl}_n(\mathbb{C})$ for the linear Poisson $\mathcal{R}$-bracket

We equip $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ with the Ad-invariant, non-degenerate, symmetric, bilinear form $\langle \cdot | \cdot \rangle_2$, defined for every $(x, y), (x', y') \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ by

\[
\langle (x, y) | (x', y') \rangle_2 := \langle x | x' \rangle - \langle y | y' \rangle = \text{Trace}(xx') - \text{Trace}(yy').
\]  \hfill (57)

As in Subsection 3.2 we consider $\mathcal{R}(x, y) = (R(x - y) + y, R(x - y) + x)$ and we consider the linear Poisson $\mathcal{R}$-bracket\footnote{According to Corollary 2 the endomorphism $\mathcal{R}$ is an $\mathcal{R}$-matrix on $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$.} defined for every $F, G \in \mathcal{F}(\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}))$ at $(x, y) \in \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$, by

\[
\{F, G\}_\mathcal{R}(x, y) = \frac{1}{2}(\langle (x, y) | [\mathcal{R}\nabla_{(x,y)}F, \nabla_{(x,y)}G] \rangle_2 + \langle (x, y) | [\nabla_{(x,y)}F, \mathcal{R}\nabla_{(x,y)}G] \rangle_2).
\]
By using the proofs of Propositions [8] and [10] we show that the phase space $T^{2'}$ is a Poisson submanifold of $(\mathfrak{gl}_n(C) \times \mathfrak{gl}_n(C), \{\cdot,\cdot\}_R)$ and $X_H := \{\cdot,H\}_R$ and $X_{H'} := \{\cdot,H'\}_R$ where $H(x,y) = \frac{1}{2} \text{Trace} x^2$ and $H'(x,y) = \frac{1}{2} \text{Trace} y^2$ describes on $T^{2'}$ the equations of motion of the 2-Toda lattice.

Let us now study the integrability of the 2-Toda lattice on $\mathfrak{gl}_n(C)$. Let $P_i$, for every $i \in \mathbb{N}$ be the Ad-invariant function of $\mathfrak{gl}_n(C)$ defined for all $x \in \mathfrak{gl}_n(C)$ by $P_i(x) = \frac{1}{i+1} \text{Trace}(x^{i+1})$. We define $F_{j,i} \in F(\mathfrak{gl}_n(C) \times \mathfrak{gl}_n(C))$ by $P_i(\lambda x - y) = \sum_{j=0}^{i+1} (-1)^{m+1-j} \lambda^j F_{j,i}(x,y)$ and we define

$$F' = (F_{j,i}, 0 \leq i \leq n - 1 \text{ and } 0 \leq j \leq i + 1).$$

Note that the functions that make up $F'$ are the functions that make up $F$ together with the functions $F_{1,0}(x,y) := \text{Trace}(x)$ and $F_{0,0}(x,y) := \text{Trace}(y)$. We have the following proposition.

**Proposition 20** The triplet $(T^{2'}, F', \{\cdot,\cdot\}_R)$ is an integrable system.

**Proof.** According to the item (2) of Theorem [4] the family $F'$ is involutive for the Poisson bracket $\{\cdot,\cdot\}_R$. Since $F'$ is an independent family on $T^{2'}$ (this follows from the independence of the differentials $dF_{0,0}$ and $dF_{1,0}$ at all points of $\mathfrak{gl}_n(C) \times \mathfrak{gl}_n(C)$ and the independence of $F$ on the submanifold $T^2$, which is the submanifold of $T^{2'}$ defined by $F_{1,0} = F_{0,0} = 0$), and since the cardinal of $F'$ is $\frac{n(n+3)}{2}$ we have, according to [2] Proposition 4.12], the inequality

$$\text{Rk}(T^{2'}, \{\cdot,\cdot\}_R) \leq 2(\text{dim}(T^{2'}) - \text{Card}(F')) = n^2 + n - 2. \quad (58)$$

Furthermore $\text{Rk}(T^{2'}, \{\cdot,\cdot\}_R) = \text{Rk}(T^2, \{\cdot,\cdot\}_R) = n^2 + n - 2$. This implies that the rank of the restriction to $T^{2'}$ of $\{\cdot,\cdot\}_R$ is exactly $n^2 + n - 2$. Then $\text{card} F'|_{T^{2'}} = \text{dim} T^{2'} - \frac{1}{2} \text{Rk}(T^{2'}, \{\cdot,\cdot\}_R)$. This completes the proof. \qed

### 3.4.2 The integrability of the 2-Toda lattice on $\mathfrak{gl}_n(C)$ for the quadratic Poisson $R$-bracket

Let the quadratic $R$-bracket be defined for every $F,G \in F(\mathfrak{gl}_n(C) \times \mathfrak{gl}_n(C))$ at $(x,y) \in \mathfrak{gl}_n(C) \times \mathfrak{gl}_n(C)$, by

$$\{F,G\}_R(x,y) := \frac{1}{2} \langle [(x,y), \nabla_{(x,y)} F] | \mathcal{R}(y) \nabla_{(x,y)} G + \nabla_{(x,y)} G(x,y) \rangle - \langle F \leftrightarrow G \rangle. \quad (59)$$

Since $\mathcal{R}_-(x,y) = \frac{1}{2}(\mathcal{R} - \mathcal{R}^*)(x,y) = (R_-(x-y) + y, R_-(x-y) + x)$ and since both the endomorphism $R = P_+ - P_-$ and its antisymmetric part $R_- = \frac{1}{2}(R - R^*)$ are solutions of (mCYBE) of $\mathfrak{gl}_n(C)$ of constant $c = 1$, it follows from Corollary [2] that $\mathcal{R}$ and $\mathcal{R}_-$ are solutions of (mCYBE) of $\mathfrak{gl}_n(C) \times \mathfrak{gl}_n(C)$ of constant $c = 1$. According to [3 Section 4] the
The \( R \)-bracket ([29]) is indeed a Poisson bracket, that we call the quadratic Poisson \( R \)-bracket (on \( \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \)).

By a direct computation in coordinates we can prove the following proposition.

**Proposition 21** The phase space \( T^{2'} \) of the 2-Toda lattice on \( \mathfrak{gl}_n(\mathbb{C}) \) is a Poisson submanifold of \( (\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}), \{ \cdot, \cdot \}_R^Q) \).

We show the following proposition.

**Proposition 22** Let \( P_i \), for every \( i \in \mathbb{N} \) be the Ad-invariant functions of \( \mathfrak{gl}_n(\mathbb{C}) \) defined for all \( x \in \mathfrak{gl}_n(\mathbb{C}) \) by \( P_i(x) = \frac{1}{i+1} \text{Trace}(x^{i+1}) \) and let \( \phi_\lambda : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \rightarrow \mathfrak{gl}_n(\mathbb{C}) \), for every \( \lambda \in \mathbb{F} \), be defined by \( \phi_\lambda : (x,y) \rightarrow \lambda x - y \).

1. For every \( i, j \in \mathbb{N} \) and every \( \lambda, \gamma \in \mathbb{C} \), the functions \( P_i \circ \phi_\lambda \) and \( P_j \circ \phi_\gamma \) are in involution for \( \{ \cdot, \cdot \}_R^Q \).

2. The Hamiltonian vector field \( \mathcal{X}_{P_i \circ \phi_\lambda}^Q := \{ \cdot, P_i \circ \phi_\lambda \}_R^Q \) is given by

\[
\mathcal{X}_{P_i \circ \phi_\lambda}^Q = -\{(x,y), ((R-I)(\lambda x - y)^{i+1}, (R+I)(\lambda x - y)^{j+1})\}. \tag{60}
\]

**Proof.** (1) According to ([30]) and ([22]), for every \( x, y \in \mathfrak{g} \),

\[
\{ P_i \circ \phi_\lambda, P_j \circ \phi_\mu \}_R^Q(x,y) = \frac{1}{2} \langle (x,y), (\lambda(\lambda x - y)^i, (\lambda x - y)^j) \rangle \mid \mathcal{R}(\mu(\mu x - y)^j, y(\mu x - y)^j) \rangle_2 + \frac{1}{2} \langle (x,y), (\lambda(\lambda x - y)^i, (\lambda x - y)^j) \rangle \mid \mathcal{R}(\mu(\mu x - y)^j x, (\mu x - y)^j y) \rangle_2 - (j, \mu) \leftrightarrow (i, \lambda).
\]

By replacing \( \mathcal{R} \) by its expression \( \mathcal{R}(x,y) = (R(x-y) + y, R(x-y) + x) \), we obtain

\[
\{ P_i \circ \phi_\lambda, P_j \circ \phi_\mu \}_R^Q(x,y) = \langle (x,y), (\lambda(\lambda x - y)^i, (\lambda x - y)^j) \rangle \mid (R((\mu x - y)^{j+1}), R((\mu x - y)^{j+1})) \rangle_2 + \frac{1}{2} \langle (x,y), (\lambda(\lambda x - y)^i, (\lambda x - \mu)^j) \rangle \mid (y(\mu x - y)^j \mu(\mu x - y)^j) \rangle_2 + \frac{1}{2} \langle (x,y), (\lambda(\lambda x - y)^i, (\lambda x - \mu)^j) \rangle \mid ((\mu x - y)^j y, \mu(\mu x - y)^j x) \rangle_2 - (j, \mu) \leftrightarrow (i, \lambda).
\]

By using Formula ([57]) we obtain

\[
\{ P_i \circ \phi_\lambda, P_j \circ \phi_\mu \}_R^Q(x,y) = \langle [\lambda x - y, (\lambda x - y)^i] \mid R((\mu x - y)^{j+1}) \rangle + \frac{1}{2} \langle [\lambda x, (\lambda x - y)^i] \mid y(\mu x - y)^j + (\mu x - y)^j y \rangle - \frac{1}{2} \langle [y, (\lambda x - y)^i] \mid \mu x(\mu x - y)^j + \mu(\mu x - y)^j x \rangle - (j, \mu) \leftrightarrow (i, \lambda).
\]
Since \([\lambda x, (\lambda x - y)^i] = [y, (\lambda x - y)^i]\),

\[
\{P_i \circ \phi_{\lambda}, P_j \circ \phi_{\mu}\}_R^Q(x, y) = -\frac{1}{2} \langle [y, (\lambda x - y)^i] \mid 2(\mu x - y)^{j+1} \rangle - (j, \mu) \longleftrightarrow (i, \lambda)
\]

\[
= \langle y \mid [(\lambda x - y)^i, (\mu x - y)^{j+1}] \rangle - (j, \mu) \longleftrightarrow (i, \lambda)
\]

\[
= 0,
\]

where we used Lemma 6 to provide the last line.

(2) Let \(K\) be a function of \(g^2\); we denote by \((a, b) = \nabla_{(x, y)} K\); according to (59) and (22),

\[
\mathcal{X}_{P_i \circ \phi_{\lambda}}^Q (x, y)[K]
\]

\[
= \frac{1}{2} \langle [(x, y), (a, b)] \mid R(\lambda x) \rangle \lambda (\lambda x - y)^i + \lambda (\lambda x - y)^i x, y(\lambda x - y)^i + (\lambda x - y)^i y \rangle \rangle_2
\]

\[
- \frac{1}{2} \langle [(x, y), R(\lambda x - y)^i] \rangle \rangle_2
\]

\[
- \frac{1}{2} \langle [(x, y), R(\lambda x - y)^i] \rangle \rangle_2
\]

\[
= -\langle [(x, y), (R((\lambda x - y)^{j+1}), R((\lambda x - y)^{j+1}))] \rangle_2
\]

\[
- \frac{1}{2} \langle [(x, y), (\lambda x - y)^i] \rangle \rangle_2
\]

\[
- \frac{1}{2} \langle [(x, y), (\lambda x - y)^i] \rangle \rangle_2
\]

\[
= \langle [(x, y), (\lambda x - y)^i] \rangle \rangle_2
\]

Moreover according to the Ad-invariance of \(P_i\) and the property \(\langle x \mid yz \rangle = \langle xy \mid z \rangle\), \(\forall x, y, z \in g\), we obtain

\[
\mathcal{X}_{P_i \circ \phi_{\lambda}}^Q (x, y)[K] = -\langle [(x, y), (R((\lambda x - y)^{j+1}), R((\lambda x - y)^{j+1}))] \rangle_2
\]

\[
- \frac{1}{2} \langle [(x, y), (\lambda x - y)^i] \rangle \rangle_2
\]

\[
- \frac{1}{2} \langle [(x, y), \lambda x \rangle \rangle_2
\]

\[
+ \frac{1}{2} \langle x \rangle \rangle_2
\]

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We then deduce that
\[
\mathcal{X}^Q_{P_{i_0} \circ \phi_\lambda}(x, y) = -[(x, y), (R((\lambda x - y)^{i+1}), R((\lambda x - y)^{i+1}))]
\]
\[
-\frac{1}{2}[(x, y), (y(\lambda x - y)^{i} + (\lambda x - y)^{i}y, \lambda x(\lambda x - y)^{i} + \lambda(\lambda x - y)^{i}x)]
\]
\[
+ \frac{1}{2}(x[y, (\lambda x - y)^{i}] + [y, (\lambda x - y)^{i}]x, y[x, \lambda(\lambda x - y)^{i}] + [x, \lambda(\lambda x - y)^{i}]y).
\]

Let us compute separately the two last lines:
\[
-\frac{1}{2}[(x, y), (y(\lambda x - y)^{i} + (\lambda x - y)^{i}y, \lambda x(\lambda x - y)^{i} + \lambda(\lambda x - y)^{i}x)]
\]
\[
+ \frac{1}{2}(x[y, (\lambda x - y)^{i}] + [y, (\lambda x - y)^{i}]x, y[x, \lambda(\lambda x - y)^{i}] + [x, \lambda(\lambda x - y)^{i}]y)
\]
\[
= -(x(\lambda x - y)^{i}y - y(\lambda x - y)^{i}x, \lambda y(\lambda x - y)^{i}x - \lambda x(\lambda x - y)^{i}y)
\]
\[
= ([x, (\lambda x - y)^{i+1}], -\lambda[x, (\lambda x - y)^{i+1}])
\]
\[
= [(x, y), ((\lambda x - y)^{i+1}, -(\lambda x - y)^{i+1})].
\]

Then we have Formula (60). □

Let us compare the Hamiltonian vector fields for the quadratic \(\mathcal{R}\)-Poisson bracket and the linear \(\mathcal{R}\)-Poisson bracket. Formula (20) gives, in our case, the following expression for the Hamiltonian vector field of \(P_{i+1} \circ \phi_\lambda\) with respect to the linear \(\mathcal{R}\)-Poisson structure:
\[
\mathcal{X}_{P_{i+1} \circ \phi_\lambda} = \frac{1}{2}(\lambda - 1)\left((x, y), ((R - I)(\lambda x - y)^{i+1}, (R + I)(\lambda x - y)^{i+1})\right).
\]  
(62)

Comparing Formulae (62) and (60), we obtain, for \(\lambda \neq 1\),
\[
\mathcal{X}^Q_{F_{i_0} \circ \phi_\lambda}(x, y) = \frac{2}{1 - \lambda}\mathcal{X}_{P_{i+1} \circ \phi_\lambda}(x, y).
\]  
(63)

The relation (63) implies that for every \(0 \leq i \leq n - 1\), we have
\[
\begin{align*}
\mathcal{X}^Q_{F_0, i} &= 2\mathcal{X}_{F_{0, i+1}}, \\
\mathcal{X}^Q_{F_j, i} - \mathcal{X}^Q_{F_{j-1}, i} &= 2\mathcal{X}_{F_{j, i+1}}, \quad 1 \leq j \leq i + 1, \\
\mathcal{X}^Q_{F_{i+1}, i} &= -2\mathcal{X}_{F_{i+2, i+1}}.
\end{align*}
\]  
(64)

Notice also that the Hamiltonian vector fields \(\mathcal{X}^Q_{F_{0, 0}}\) and \(\mathcal{X}^Q_{F_{1, 0}}\) are precisely the equations of the 2-Toda lattice on \(\mathfrak{gl}_n(\mathbb{C})\) as can be shown by specializing (64) to \(i = 0\) and the first item of Remark 12.
Theorem 23. The triplet \((\mathcal{T}^{2'}, \mathcal{F}'|_{\mathcal{T}^{2'}}, \{\cdot, \cdot\}_{Q,R})\) is an integrable system.

Proof. The involutivity of the family \(\mathcal{F}'\) on \((g^2, \{\cdot, \cdot\}_R)\) follows from item (1) in Proposition \[22\] Since \(\mathcal{F}'\) is an independent family on \(\mathcal{T}^{2'}\) (see the proof of Proposition \[20\]), and since the cardinal of \(\mathcal{F}'\) is \(\frac{n(n+3)}{2}\), we have, according to \[2\] Proposition 4.12, the inequality

\[
\text{Rk}(\mathcal{T}^{2'}, \{\cdot, \cdot\}_{Q,R}) \leq 2(\text{dim}(\mathcal{T}^{2'}) - \text{Card}(\mathcal{F}')) = n^2 + n - 2.
\]

Moreover, according to Formula \[64\], the family of vector fields \(\mathcal{X}_\mathcal{F'} := (\mathcal{X}_{F,j,i}^Q, 0 \leq i \leq n - 1 \text{ and } 0 \leq j \leq i)\) and the family of vector fields \(\mathcal{X}_\mathcal{F} := (\mathcal{X}_{F,j,i}, 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq i+1)\) have the same rank at all points. By choosing a point in \(\mathcal{T}^2\), we can deduce from the fact that \(\mathcal{F}\) is an integrable system on \(\mathcal{T}^2\) (Theorem \[13\]) that this rank is at least the cardinal of \(\mathcal{F'} (= \frac{n(n+3)}{2})\) minus the number of independent Casimir functions on \(\mathcal{T}\) for the linear bracket \(\{\cdot, \cdot\}_R\) \((= n - 1)\). Since \(\mathcal{F}'\) is involutive, we have therefore the inequality

\[
\text{Rk} \{\cdot, \cdot\}_{Q,R} \geq 2(\text{Rk}(\mathcal{X}_{\mathcal{F}'})) = 2(\text{Rk}(\mathcal{X}_{\mathcal{F}})) \geq 2\left(\frac{n(n+3)}{2} - 2 - n + 1\right) = n^2 + n - 2.
\]

Together with \[65\], this implies that the rank of the restriction to \(\mathcal{T}^{2'}\) of \(\{\cdot, \cdot\}_{Q,R}\) is exactly \(n^2 + n - 2\). The identity \(\text{card} \mathcal{F}'|_{\mathcal{T}^{2'}} = \dim \mathcal{T}^{2'} - \frac{1}{2} \text{Rk}(\mathcal{T}^{2'}, \{\cdot, \cdot\}_{Q,R})\) follows and completes the proof. \[\square\]

3.5 The relation between the 2-Toda lattice and the Toda lattice

In this section we show that the Toda lattice is a restriction of the 2-Toda lattice. We begin by recalling the Liouville integrable system of the Toda lattice.

3.5.1 The Toda lattice

We give some definitions and properties of the Toda lattice, which will be useful afterwards in this section.

Definition 24. (1) The phase space \(\mathcal{T}\) of the Toda lattice is the affine subspace of \(g\) given by

\[
\mathcal{T} := g_{-1} \oplus g_0 + e,
\]

where \(e = \sum_{i=1}^\ell e_i\).

(2) The Toda lattice is the system of differential equations on \(\mathcal{T}\) given by the Lax equation

\[
\dot{A} = [A_+, A],
\]

where \(A_+\) is the projection of \(A\) on \(g_+\).
We consider the endomorphism \( R := P_+ - P_- \) of \( g \), the difference of the projections on \( g_+ \) and \( g_- \). The Poisson \( R \)-bracket on \( g \) is defined, for every \( F, G \in \mathcal{F}(g) \) and every \( x \in g \), by
\[
\{F, G\}_R := \frac{1}{2} \langle x | [\nabla_x F, \nabla_x G] + [\nabla_x F, R \nabla_x G] \rangle.
\] (68)

**Theorem 25** [9, Section 4.1] (1) The affine subspace \( \mathcal{T} \) of \( g \) is a Poisson submanifold of \((g, \{\cdot, \cdot\}_R)\).

(2) Let \( H \in \mathcal{F}(g) \), defined for every \( x \in g \) by \( H(x) = \frac{1}{2} \langle x | x \rangle \). The equation of the Hamiltonian field \( \mathcal{X}_H := \{\cdot, H\}_R \) is the equation of motion (67) of the Toda lattice.

(3) Let \( P_1, \ldots, P_\ell \) be a generating family of homogeneous polynomials of the algebra of \( \text{Ad} \)-invariant functions on \( g \) of degree respectively, \( m_1 + 1, \ldots, m_\ell + 1 \). We define \( F_0 := (P_1, \ldots, P_\ell) \). The triplet \( \mathcal{F}_0, \{\cdot, \cdot\}_R, \mathcal{T} \) is Liouville integrable system and the equation of motion of the Toda lattice is
\[
\dot{A} := \{\cdot, P_1\}_R(A) = [(\nabla_A P_1)_+, A].
\] (69)

### 3.5.2 Restriction of the 2-Toda lattice and construction of the Toda lattice

The phase space \( \mathcal{T}^2 \) of the 2-Toda lattice decomposes as
\[
\mathcal{T}^2 := g_{\leq -1} \times g_{> 0} \oplus \Delta(g_{-1} \oplus g_0) + (e, e),
\] (70)

where \( \Delta(g_{-1} \oplus g_0) := \{(x, x) | x \in g_{-1} \oplus g_0\} \).

**Theorem 26** Let \( g^2 \) be the Lie algebra \( g \times g \) with \( \{\cdot, \cdot\}_R \) the \( R \)-Poisson bracket, \( \mathcal{T}^2 \) the phase space of the 2-Toda lattice and \( \mathcal{T}' := \Delta(g_{-1} \oplus g_0) + (e, e) \).

(1) The submanifold \( \mathcal{T}' \) is a Poisson submanifold of \((g^2, \{\cdot, \cdot\}_R)\).

(2) Let \( (\mathcal{T}, \{\cdot, \cdot\}_R) \) be the phase space of the Toda lattice, equipped with the Poisson \( R \)-bracket given as in (68). The map
\[
\varphi : (\mathcal{T}, \{\cdot, \cdot\}_R) \rightarrow (\mathcal{T}', \{\cdot, \cdot\}_R)
\] (71)

is an isomorphism of integrable systems.

**Proof.** (1) It is clear that \( \mathcal{T}' = \Delta(g_{-1} \oplus g_0 + e) = \mathcal{T}^2 \cap g^2_+ \). According to Proposition, \( \mathcal{T}^2 \) is a Poisson submanifold of \((g^2, \{\cdot, \cdot\}_R)\) while \( g^2_+ \) is a Poisson submanifold of \((g^2, \{\cdot, \cdot\}_R)\) because the orthogonal of \( g^0 \), which is \( g^2_+ \), is a Lie ideal for the bracket \( \{\cdot, \cdot\}_R \); their intersection \( \mathcal{T}' \) is then a Poisson submanifold of \((g^2, \{\cdot, \cdot\}_R)\).

(2) By using the coordinate functions on \( \mathcal{T} \) and on \( \mathcal{T}' \) we show that \( \varphi \) is a Poisson isomorphism. Furthermore, the functions of the integrable system \( \mathcal{F} = (F_{k,i}, 1 \leq i \leq \ell \text{ and } 0 \leq k \leq \ell) \)
\[ k \leq m_i + 1 \), restricted to \( T' \), and pulled back on \( T \) by \( \varphi \), give again the functions of the family \( F_0 \) of the Toda lattice. Specifically, for every \( i \) in \( 1, \ldots, \ell \) and every \( 0 \leq k \leq m_i + 1 \), the functions \( F_{k,i} \) are all equal (up to multiplicative constants) to the function \( P_i \). In fact, for every \( x \in g_0 \oplus g_{-1} \),

\[
P_i \circ \phi_\lambda(\varphi(x)) = P_i(\phi_\lambda(x, -x)) = P_i(\lambda x - x) = (\lambda - 1)^{m_i + 1} P_i(x),
\]

and so

\[
\sum_{k=0}^{m_i + 1} (-1)^{m_i + 1 - k} \lambda^k F_{k,i}(\varphi(x)) = \sum_{k=0}^{m_i + 1} (-1)^{m_i + 1 - k} \lambda^k C_{m_i + 1}^k P_i(x)
\]

and \( F_{k,i}(\varphi(x)) = C_{m_i + 1}^k P_i(x) \). In conclusion, \( \varphi : T \to T' \) is Poisson isomorphic and \( \varphi^* F = F_0 \) (see item (3) of Theorem 25 for the definition of \( F_0 \)). This proves the claim. \( \square \)

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