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Exact Replication of the Best Rebalancing Rule in Hindsight

Alex Garivaltis*

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Abstract

This paper prices and replicates the financial derivative whose payoff at $T$ is the wealth that would have accrued to a $1$ deposit into the best continuously-rebalanced portfolio (or fixed-fraction betting scheme) determined in hindsight.

For the single-stock Black-Scholes market, Cover and Ordentlich (1998) only priced this derivative at time-$0$, giving $C(S,0) = 1 + \sigma \sqrt{T/(2\pi)}$. Of course, it is not true that $C(S,t) = 1 + \sigma \sqrt{(T-t)/(2\pi)}$.

I complete the Cover-Ordentlich (1998) analysis by deriving the price at any time $t$. By contrast, I also study the more natural case of the best levered rebalancing rule in hindsight. This yields $C(S,t) = \sqrt{T/t} \cdot \exp(rt + \sigma^2 b(S,t)^2 t/2)$, where $b(S,t)$ is the best rebalancing rule in hindsight over the observed history $[0,t]$. I show that the replicating strategy amounts to betting the fraction $b(S,t)$ of wealth on the stock over the interval $[t,t + dt]$. This fact holds for the general market with $n$ correlated stocks in geometric Brownian motion: we get $C(S,t) = (T/t)^{n/2} \exp(rt + t \cdot b'\Sigma b/2)$, where $\Sigma$ is the covariance of instantaneous returns per unit time. This result matches the $O(T^{n/2})$ “cost of universality” derived by Cover in his “universal portfolio theory” (1986, 1991, 1996, 1998), which super-replicates the same derivative in discrete-time. The replicating strategy compounds its money at the same asymptotic rate as the best levered rebalancing rule in hindsight, thereby beating the market asymptotically. Naturally enough, we find that the American-style version of Cover’s Derivative is never exercised early in equilibrium.

**Keywords:** Exotic options, Lookback options, Continuously-rebalanced portfolios, Kelly rule, Universal portfolios, Dynamic replication

**JEL codes:** C44, D53, D81, G11, G13

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1 Introduction

1.1 Literature review

The exotic option literature has several examples (Wilmott 1998) of derivatives with “lookback” or “no-regret” features. For example, a floating-strike lookback call allows its owner to look back at the price history of a given stock, buy a share at the realized minimum $m = \min_{1 \leq t \leq T} S_t$, and sell it at the terminal price $S_T$. Similarly, a fixed-strike lookback call allows its owner to buy one share at a fixed price $K$, and sell it at the historical maximum $M = \max_{1 \leq t \leq T} S_t$.

This paper prices and replicates a markedly different type of lookback option, whose payoff is equal to the final wealth that would have accrued to a $1 deposit into the best continuous rebalancing rule (or fixed-fraction betting scheme) determined in hindsight. This contingent claim has been studied by Cover and his collaborators (1986, 1991, 1996, 1998) who used it as a performance benchmark for discrete-time portfolio algorithms. Cover and Ordentlich’s important (1998) paper (on the “max-min universal portfolio”) super-replicates this derivative in discrete-time.

In the context of one underlying stock, a rebalancing rule is a fixed-fraction betting scheme that continuously maintains some fraction $b \in \mathbb{R}$ of wealth in the stock and keeps the rest in cash. The portfolio is held for the differential time interval $[t, t+dt]$, at which point it is rebalanced to the target allocation. If $b > 1$, the scheme uses margin loans, but continuously maintains a fixed debt-to-assets ratio of $(b-1)/b$. Say, for $b = 2$ the scheme would keep a 50% loan-to-value ratio at all times. Thus, when the stock rises, the trader instantly adjusts by borrowing additional cash against his new wealth. Similarly, on a downtick he will de-lever himself by selling a precise amount of the stock. For example, using $b = 2$ on the S&P 500 index from January 2012 through August 2018 would have, under monthly rebalancing, compounded one’s money at
31.8% annually, as compared to buying and holding the index \((b = 1)\), which would have yielded 15.6% annually. This is illustrated in Figure 1.

In contrast to the leveraged “buy high, sell low” strategy discussed above, rebalancing rules \(b \in (0, 1)\) amount to “volatility harvesting” strategies (Luenberger 1997) that “live off the fluctuations” of the underlying. Such rules are mechanical schemes for buying low and selling high, schemes that profit from mean-reversion in cyclical or “sideways” markets. For example, using \(b = 0.5\) for shares of Advanced Micro Devices (AMD) with monthly rebalancing over the author’s lifetime (April 1986 through August 2018), the trader would have compounded at 7.79% per year, compared to 1.77% for \(b = 1\). This is illustrated in Figure 2.

These examples make it clear that the best rebalancing rule in hindsight will handily outperform the underlying over long periods. For particularly bad underlyings, the best rebalancing rule in hindsight will outperform by holding all cash \((b = 0)\) or by shorting the stock \((b < 0)\). Inevitably, one lives to regret the fact that he did not use the best rebalancing rule in hindsight. In 1986, no one could have reli-
ably predicted that $b = 0.5$ would beat AMD by 6 percent a year. But (at least in the Black-Scholes world) it was possible to delta-hedge the final wealth of the best continuous rebalancing rule in hindsight. Such is the business of this paper.

1.2 Contribution

Cover and Ordentlich (1998) priced this derivative at time-0, for a single underlying with unlevered hindsight optimization. The last result in their paper is the formula $C(S, 0) = 1 + \sigma \sqrt{T/(2\pi)}$, where $T$ is the horizon and $\sigma$ is the volatility. Of course, it is not the case that $C(S, t) = 1 + \sigma \sqrt{(T - t)/(2\pi)}$. Accordingly, this paper completes the Cover-Ordentlich analysis, deriving the eponymous (1998) Cost of Achieving the Best Portfolio in Hindsight at any time $t$, for levered hindsight optimization over any number of correlated stocks in geometric Brownian motion. When leverage is allowed in the hindsight optimization, replication becomes especially simple. At time $t$, we just look back at the observed history $[0, t]$ and compute the best (currently known) rebalancing rule in hindsight, here denoted $b(S, t)$. We then bet the fraction

![Figure 2: $b = 0.5$ for AMD shares under monthly rebalancing, Apr 1986-Aug 2018.](image)
$b(S,t)$ of wealth on the stock over $[t,t+dt]$. This is equivalent to holding $\Delta(S,t) = b(S,t)C(S,t)/S$ shares of the stock in state $(S,t)$. The replicating strategy serves to translate Cover and Ordentlich’s (1998) “max-min universal portfolio” into continuous time. Thus, the present paper does for Cover and Ordentlich (1998) what Jamshidian (1992) did for Cover’s original (1991) performance-weighted universal portfolio.

2 One underlying

2.1 Payoff computation

For simplicity, we start with a single underlying stock whose price $S_t$ follows the geometric Brownian motion

$$dS_t/S_t = \mu dt + \sigma dW_t,$$

and a risk-free bond whose price $B_t = e^{rt}$ follows

$$dB_t/B_t = r dt,$$

where $W_t$ is a standard Brownian motion. We consider constant rebalancing rules, or fixed-fraction betting schemes, that “bet” the fraction $b \in \mathbb{R}$ of wealth on the stock over the interval $[t,t+dt]$. Assume that the gambler starts with $1, and let $V_t = V_t(b)$ be his wealth at $t$. He thus owns $bV_t/S_t$ shares of the stock at $t$, and has the remaining $(1-b)V_t$ dollars invested in bonds. The gambler’s wealth evolves according to

$$dV_t/V_t = b(dS_t/S_t) + (1-b)dB_t/B_t = (b\mu + (1-b)r)dt + b\sigma dW_t.$$
Since $V_t$ is a geometric Brownian motion, we have

$$V_t(b) = \exp\{ (r + (\mu - r)b - \sigma^2 b^2/2)t + b\sigma W_t \}. \quad (4)$$

In the formula

$$S_t = S_0 \cdot \exp((\mu - \sigma^2/2)t + \sigma W_t), \quad (5)$$

we can solve for $\sigma W_t$ in terms of $S_t$, and substitute it back into the equation for $V_t(b)$. This yields

$$V_t(b) = \exp\{ rt + (\log(S_t/S_0) - (r - \sigma^2/2)t)b - \sigma^2 b^2 t/2 \}. \quad (6)$$

Note that the drift $\mu$ has disappeared from the expression for $V_t(b)$. The trader’s wealth is Markovian: it depends only on the current state $(S_t,t)$.

To find the best rebalancing rule in hindsight over $[0,t]$, we maximize $V_t(b)$ with respect to $b$. Since the exponent is quadratic in $b$, the best rebalancing rule in hindsight over $[0,t]$ is

$$b(S,t) = \frac{\log(S_t/S_0) - (r - \sigma^2/2)t}{\sigma^2 t}. \quad (7)$$

If we write $\hat{\mu} = \log(S_t/S_0)/t + \sigma^2/2$, we have

$$b(S,t) = \frac{\hat{\mu}(S,t) - r}{\sigma^2}. \quad (8)$$

Let $V_t^*$ denote the final wealth of the best rebalancing rule in hindsight over $[0,t]$. Then

$$V_t^* = \exp\{ rt + \sigma^2 b(S,t)^2 \cdot t/2 \}. \quad (9)$$

Figure 3 plots this payoff for different volatilities, assuming a risk-free rate of $r = $
Figure 3: The payoff of Cover’s Derivative for $T = 5, S_0 = 100, r = 0$.

0 over a horizon of $T = 5$ years. In Cover and Ordentlich (1998), the hindsight optimization is over unlevered rebalancing rules $0 \leq b \leq 1$, and in that context, the best rebalancing rule in hindsight is $\text{Max}(0, \text{Min}(b(S,t), 1))$. Thus, they use the payoff

$$V_t^* = \begin{cases} 
  e^{rt} & \text{if } b(S,t) \leq 0 \\
  \exp\left(rt + \sigma^2b(S,t)^2 \cdot t/2\right) & \text{if } 0 \leq b(S,t) \leq 1 \\
  S_t/S_0 & \text{if } b(S,t) \geq 1 
\end{cases}$$

(10)

Figure 4 plots the unlevered payoff for different volatilities, assuming a risk-free rate of $r = 0$ over a horizon of $T = 2$ years. Consider a derivative (“hindsight allocation option”) whose payoff at $T$ is $V_T^*(S_T)$. Let $\mathbb{Q}$ denote the equivalent martingale
Figure 4: The payoff of Cover’s Derivative for unlevered hindsight optimization, $T = 2$, $S_0 = 100$, $r = 0$. 
measure. Cover and Ordentlich (1998) computed the expectation

\[ C(S, 0) = E_0^Q[V^*_T] = 1 + \sigma \sqrt{T/(2\pi)} \]  

(11)

with respect to \( Q \) and the information available at \( t = 0 \). If someone buys a dollar’s worth of this derivative at \( t = 0 \) (for some distant expiration date \( T \)), he will compound his money at the same asymptotic rate as the best unlevered rebalancing rule in hindsight. His initial dollar buys him \( 1/C(S, 0) \) units of the derivative, yielding final wealth \( V^*_T/(1 + \sigma \sqrt{T/(2\pi)}) \). After holding the option for \( T \) years, the excess continuously-compounded growth rate of the best rebalancing rule in hindsight (over and above that of the option holder) is

\[ \log\left(\frac{V^*_T}{1 + \sigma \sqrt{T/(2\pi)}}\right)/T - \log\left\{\frac{V^*_T}{1 + \sigma \sqrt{T/(2\pi)}}\right\}/T = \log\left\{1 + \sigma \sqrt{T/(2\pi)}\right\}/T, \]

(12)

which tends to 0 as \( T \to \infty \). Note that the excess growth rate is deterministic. Figure 5 plots the excess growth rate for different volatilities and maturities.

### 2.2 No-arbitrage price

Regardless of whether or not the hindsight optimization permits leverage, the Black-Scholes (delta-hedging) strategy will always use leverage in certain situations. Of course, even if the replicating strategy does not use leverage in a particular state \((S, t)\), its whole definition is predicated on the ability to do so at any time in the future. Thus, we find it more natural to start with levered hindsight optimization. Accordingly, we take up the Black-Scholes (1973) equation

\[ (\sigma^2 S^2/2) \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0 \]

(13)
Figure 5: Excess continuously-compounded annual growth rate of the best (unlevered) rebalancing rule in hindsight over that of the replicating strategy.
subject to the boundary condition \( C(S, T) = \exp(rT + \sigma^2 b(S, T)^2 \cdot T/2) \). For convenience, we define the variable

\[
    z_t = \frac{\log(S_t/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}},
\]

which is a unit normal under the equivalent martingale measure. We have \( b(S, t) = z/(\sigma\sqrt{t}) \). Thus, the final payoff of Cover’s Derivative is

\[
    V_T^* = \exp(rT + z_T^2/2).
\]

The intrinsic value at time \( t \) is

\[
    V_t^* = \exp(rt + z_t^2/2).
\]

We proceed to compute the expected discounted payoff with respect to the equivalent martingale measure. To this end, we write

\[
    z_T = \sqrt{t/T} \cdot z_t + \sqrt{(T-t)/T} \cdot y,
\]

where

\[
    y = \frac{\log(S_T/S_t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.
\]

\( y \) is a unit normal with respect to the equivalent martingale measure and the information available at \( t \). Thus, we have

\[
    E_t[\exp(z_T^2/2)] = \exp\{t z_T^2/(2T)\}/\sqrt{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{t}{2T} y^2 + \frac{\sqrt{t(T-t)}}{T} z_t y \right) dy.
\]
To evaluate the integral, we make note of the general formula (e.g. Rubinstein and Reiner 1992)

\[
\int_{A}^{B} e^{-\alpha y^2 + \beta y} dy = \sqrt{\pi/\alpha} \cdot \exp\{\beta^2/(4\alpha)\} \cdot \left[ N(B\sqrt{2\alpha} - \beta/\sqrt{2\alpha}) - N(A\sqrt{2\alpha} - \beta/\sqrt{2\alpha}) \right],
\]

(20)

where \( N(\cdot) \) is the cumulative normal distribution function. Putting \( \alpha = t/(2T) \), \( \beta = \sqrt{t(T-t)} z_t \), \( A = -\infty \), and \( B = +\infty \), we get

\[
E_t[\exp(z_t^2/2)] = \sqrt{T/t} \cdot \exp(z_t^2/2).
\]

(21)

**Theorem 1.** For levered hindsight optimization (over all \( b \in \mathbb{R} \)), the price of Cover’s Derivative is

\[
C(S,t) = \sqrt{T/t} \cdot \exp(rt + z^2/2) = \sqrt{T/t} \cdot \exp(rt + \sigma^2 b(S,t)^2 t/2) = \sqrt{T/t} \cdot V_t^*.
\]

(22)

where \( z = (\log(S/S_0) - (r - \sigma^2/2)t)/(\sigma\sqrt{t}) \), \( b(S,t) \) is the best rebalancing rule in hindsight over \([0,t]\), and \( V_t^* \) is the intrinsic value at time \( t \).

**Corollary 1.** The American-style version of Cover’s Derivative (that pays \( V_t^* \) if exercised at \( t \)) will never be exercised early in equilibrium.

### 2.3 Replicating strategy and the Greeks

Differentiating the price, we find at once that

\[
\Delta = \frac{\partial C}{\partial S} = C z/(S \sigma \sqrt{t}) = C b(S,t)/S
\]

(23)

or, equivalently, that \( \Delta S/C = b(S,t) \).
Theorem 2. The replicating strategy for Cover’s Derivative bets the fraction \( b(S,t) \) of wealth on the stock in state \((S,t)\). Thus, to replicate Cover’s Derivative, one just uses the best rebalancing rule in hindsight as it is known at time \( t \).

Hence, for the complete market with a single stock in geometric Brownian motion, assuming leveraged hindsight optimization, the following three trading strategies are identical:

1. The strategy that looks back over the known price history \([0,t]\), finds the best continuously-rebalanced portfolio in hindsight, and uses that portfolio over the interval \([t, t + dt]\)

2. The \( \Delta \)-hedging strategy induced by Cover’s Derivative

3. The natural estimator \( (\hat{\mu} - r)/\sigma^2 \) of the continuous-time Kelly rule (Luenberger 1997), which is \( (\mu - r)/\sigma^2 \)

For reference, we catalog the rest of the Greeks below.

\[
\Gamma = \frac{\partial \Delta}{\partial S} = \frac{C(z^2 - 1)}{S^3 \sigma^3 t^{3/2}} \quad (24)
\]

\[
\Theta = \frac{\partial C}{\partial t} = C(r - 1/(2t) - z^2/2) \quad (25)
\]

Thus, there will be significant time decay in the option value for small times \( t \) and for extreme price realizations in either direction.

\[
\nu = \frac{\partial C}{\partial \sigma} = Cz \left( \sqrt{t/2} + \frac{r - \log(S/S_0)}{\sigma^2 t} \right) \quad (26)
\]

Generally, there are two implied volatilities that rationalize a given option price \( C \),
Figure 6: The dual implied volatilities that rationalize any observed price of Cover’s Derivative: \( t = 0.5, \ T = 1, \ r = 0.03, \ S_0 = 100, \ S_t = 105. \)

corresponding to the values

\[
z = \pm \sqrt{2(\log C - rt) + \log(t/T)}. \tag{27}
\]

The lowest possible rational option price, which corresponds to \( z_t = 0 \), is \( \sqrt{T/t} \cdot e^{rt} \).

This happens when \( S_t = S_0 e^{(r - \sigma^2/2)t} \), e.g. when \( W_t = 0 \). Figure 6 plots the option price against \( \sigma \) for the parameters \( t = 0.5, \ T = 1, \ r = 0.03, \ S_0 = 100, \) and \( S_t = 105. \)

Finally, we have the interest rate sensitivity

\[
\rho = \frac{\partial C}{\partial r} = Ct(1 - b(S,t)). \tag{28}
\]

Thus, when the best rebalancing rule in hindsight makes a positive allocation to cash, higher interest rates will make the option more valuable. When the hindsight-
optimized rebalancing rule uses margin debt \((b(S,t) > 1)\), higher interest rates will make the option less valuable.

### 2.4 Unlevered hindsight optimization

In this subsection, we take up the case of unlevered hindsight optimization, obtaining a more direct generalization of Cover and Ordentlich’s (1998) formula \(C(S,0) = 1 + \sigma \sqrt{T/(2\pi)}\). Thus, we consider the payoff

\[
V_t^* = \begin{cases} 
  e^r & \text{if } z_t \leq 0 \\
  \exp(rt + z_t^2/2) & \text{if } 0 \leq z_t \leq \sigma \sqrt{t}, \\
  S_t/S_0 & \text{if } z_t \geq \sigma \sqrt{t}
\end{cases}
\]

where \(z_t = (\log(S_t/S_0) - (r - \sigma^2/2)t)/(\sigma \sqrt{t})\). In this connection, the replicating strategy will no longer coincide with the best (unlevered) rebalancing rule in hindsight over the known history \([0,t]\).

Again, we make the decomposition \(z_T = \sqrt{t/T} \cdot z_t + \sqrt{(T-t)/T} \cdot y\), where \(y = (\log(S_T/S_t) - (r - \sigma^2/2)(T-t))/(\sigma \sqrt{T-t})\). With this terminology, the final payoff becomes

\[
V_T^* = \begin{cases} 
  e^{rT} & \text{if } y \leq -z_t \sqrt{t/(T-t)} \\
  \exp(rt + z_T^2/2) & \text{if } -z_t \sqrt{t/(T-t)} \leq y \leq (\sigma T - \sqrt{t} z_t)/\sqrt{T-t} \\
  S_T/S_0 & \text{if } y \geq (\sigma T - \sqrt{t} z_t)/\sqrt{T-t}
\end{cases}
\]

The expected discounted payoff will be the sum of three terms \(e^{-r(T-t)}(I_1 + I_2 + I_3)\), corresponding to the three cases \(b^* \leq 0, 0 \leq b^* < 1, \text{ and } b^* \geq 1\). Each constitutes a separate solution of the Black-Scholes equation. To further simplify the notation, we
define \( A = -z_t \sqrt{t/(T-t)} \) and \( B = A + \sigma T/\sqrt{T-t} \). We have

\[
e^{-r(T-t)} I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A} \exp(rt - y^2/2) dy = e^{rt} N(A),
\]

(31)

where \( N(\cdot) \) is the cumulative normal distribution function. Next, we get

\[
e^{-r(T-t)} I_2 = \frac{\exp\{rt + t z_t^2/(2T)\}}{\sqrt{2\pi}} \int_{A}^{B} \exp \left( - \frac{t}{2T} y^2 + \frac{\sqrt{t(T-t)}}{T} z_t y \right) dy.
\]

(32)

Evaluating the integral and simplifying, we get

\[
e^{-r(T-t)} I_2 = \sqrt{T/t} \cdot \exp(rt + z_t^2/2) \cdot \left\{ N(A \sqrt{T/t} + \sigma tT/(T-t)) - N(A \sqrt{T/t}) \right\}.
\]

(33)

Finally, we calculate

\[
e^{-r(T-t)} I_3 = e^{-r(T-t)} \frac{S_t}{S_0} \cdot \frac{\exp\{(r - \sigma^2/2)(T-t)\}}{\sqrt{2\pi}} \int_{B}^{\infty} \exp \left( - \frac{y^2}{2} + \sigma \sqrt{T-t} \cdot y \right) dy,
\]

(34)

which simplifies to

\[
e^{-r(T-t)} I_3 = \frac{S_t}{S_0} N(\sigma \sqrt{T-t} - B).
\]

(35)

**Theorem 3.** For the single-stock Black-Scholes market with unlevered hindsight optimization, the price of Cover’s Derivative is

\[
\hat{C}(S,t) = e^{rt} N(A) + C(S,t) \left\{ N(A \sqrt{T/t} + \sigma tT/(T-t)) - N(A \sqrt{T/t}) \right\},
\]

\[
+ \frac{S_t}{S_0} N(\sigma \sqrt{T-t} - B),
\]

(36)
where $z = \log\left(\frac{S_t}{S_0} - (r - \sigma^2/2)t\right) / (\sigma \sqrt{t})$, $A = -z \sqrt{t}/(T-t)$, $B = A + \sigma T / \sqrt{T-t}$, and $C(S, t) = \sqrt{T/t} \cdot \exp(rt + z^2/2)$ is the price of Cover’s Derivative under leveraged hindsight optimization.

2.5 Binomial lattice price

For completeness, we derive the price of Cover’s Derivative on the Cox-Ross-Rubinstein (1979) binomial lattice. By abuse of notation, let $r$ be the per-period interest rate, with $R = 1 + r$ being the gross rate of interest. We subdivide the interval $[0, T]$ into $N$ subintervals of length $\Delta t = T/N$. The stock price evolves according to $S(t + \Delta t) = S(t) \cdot u$ with probability $p$, and $S(t + \Delta t) = S(t) \cdot d$ with probability $1-p$, where $u, d$ are constants such that $d < R < u$. We let $q = (R - d)/(u - d)$ be the risk-neutral probability, with $1 - q = (u - R)/(u - d)$. The payoff-relevant state is the number of ups $j$, where $0 \leq j \leq N$. After $N$ plays, the (possibly leveraged) rebalancing rule $b$ has grown the initial dollar into

$$V_T(b) = R^N \left(1 + b(u - R)/R\right)^j \left(1 + b(d - R)/R\right)^{N-j}. \quad (37)$$

Taking logs, differentiating with respect to $b$, and clearing fractions, we find that

$$0 = j(u - R)(1 + b(d - R)/R) + (n - j)(d - R)(1 + b(u - R)/R) \quad (38)$$

Solving and simplifying, the best rebalancing rule in hindsight (after $j$ ups and $N - j$ downs) is

$$b(j, N) = \frac{R}{N(u - d)} \left(\frac{j}{q} - \frac{N - j}{1 - q}\right) \quad (39)$$
The final payoff of Cover’s Derivative is

\[ V^*_T(j, N) = R^N \left( \frac{j}{qN} \right)^j \left( \frac{N-j}{(1-q)N} \right)^{N-j}. \]  

(40)

For this formula, and those that follow, we are tacitly using the convention that \( 0^0 = 1 \). If the hindsight-optimized rebalancing rules are restricted to \( 0 \leq b \leq 1 \), the payoff becomes

\[ V^*_T = \begin{cases} 
R^N & \text{if } j \leq Nq \\
R^N \left( \frac{j}{qN} \right)^j \left( \frac{N-j}{(1-q)N} \right)^{N-j} & \text{if } Nq < j < Nq + \frac{u-d}{R_q(1-q)} \\
u^j d^{N-j} & \text{if } j \geq Nq + \frac{u-d}{R_q(1-q)}
\end{cases}. \]  

(41)

For unlevered hindsight optimization, Cover and Ordentlich (1998) gave the time-0 price as

\[ P\{j \leq Nq\} + \sum_{Nq < j < Nq + \frac{u-d}{R_q(1-q)}} \binom{N}{j} (j/N)^j ((N-j)/N)^{N-j} + \sum_{j \geq Nq + \frac{u-d}{R_q(1-q)}} \binom{N}{j} (qu)^j ((1-q)d)^{N-j}. \]  

(42)

We compute the price, for levered hindsight optimization, in the general state \((k,n)\), where \(k\) upticks have occurred in the first \(n\) time steps. Letting \(j\) denote the number of upticks in the next \(N-n\) steps, we find the expected discounted payoff to be

\[ C(k, n) = q^{-k} (1-q)^{-\frac{n-k}{n}} \sum_{j=0}^{N-n} \binom{N-n}{j} \left( \frac{k+j}{N} \right)^{k+j} \left( \frac{N-k-j}{N} \right)^{N-k-j}. \]  

(43)

Finally, we are able to replicate Cover’s Derivative on the binomial lattice, using the option price in conjunction with the formula

\[ \Delta = \frac{C_u - C_d}{S(u-d)} = \frac{C(k+1, n+1) - C(k, n+1)}{S(u-d)}, \]  

(44)
where $S$ is the current price, $n$ is the number of time steps to date, and $k$ is the number of upticks that have occurred so far.

We close this section by computing the price of Cover’s (unlevered) Derivative in all possible situations $(k, n)$. The price will consist of three terms $\Sigma_1 + \Sigma_2 + \Sigma_3$, corresponding to the three events $b \leq 0$, $0 < b < 1$, $b \geq 1$. Again, $j$ will denote the number of upticks that occur over the next $N - n$ time steps. To start, we have

$$\Sigma_1 = \sum_{0 \leq j \leq Nq - k} \binom{N - n}{j} q^j (1 - q)^{N - n - j}. \quad (45)$$

Next, we get

$$\Sigma_2 = q^{-k} (1 - q)^{-(n-k)} \sum_{Nq - k < j < Nq - k + \frac{u - d}{Rq(1 - q)}} \binom{N - n}{j} \left( \frac{k + j}{N} \right)^{k+j} \left( \frac{N - k - j}{N} \right)^{N-k-j}. \quad (46)$$

Finally, we have

$$\Sigma_3 = q^{-k} (1 - q)^{-(n-k)} \sum_{j \geq Nq - k + \frac{u - d}{Rq(1 - q)}} \binom{N - n}{j} (qu)^{k+j} ((1 - q)d)^{N-k-j}. \quad (47)$$

### 2.5.1 Simulation: “Shannon’s Demon”

To illustrate the replication of Cover’s Derivative on the binomial lattice, we simulate Shannon’s canonical discrete-time example. This amounts to the parameters $u = 2$, $d = 1/2$, $r = 0$, $R = 1$, and a risk-neutral probability of $q = 1/3$. The gambler buys (replicates) a dollar’s worth of derivative at $n = 0$, and holds it until $n = N$. His wealth after $n$ steps (and $k$ upticks) is $C(k,n)/C(0,0)$. By comparison, the stock price will be $2^{2k-n}$. Figure 7 plots a sample path for $N = 300$ periods.
Figure 7: Replication of Cover’s Derivative in the canonical discrete-time example (“Shannon’s Demon”)
3 Several underlyings

We turn our attention to the general stock market with \( n \) correlated stocks \((i = 1, \ldots, n)\) that follow the geometric Brownian motions

\[
\frac{dS_{it}}{S_{it}} = \mu_i \, dt + \sigma_i dW_{it}, \quad (48)
\]

where \( S_{it} \) is the price of stock \( i \) at \( t \) and \((\mu_i, \sigma_i)\) are the drift and volatility of stock \( i \), respectively. Let \( \rho_{ij} = \text{Corr}(dW_{it}, dW_{jt}) \) be the correlation coefficient of the instantaneous changes of the Wiener processes \( W_{it} \) and \( W_{jt} \). The correlation matrix will be denoted \( R = [\rho_{ij}]_{n \times n} \). Next, we let

\[
\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j = \text{Cov}(dS_{it}/S_{it}, dS_{jt}/S_{jt})/dt. \quad (49)
\]

We let \( \Sigma = [\sigma_{ij}]_{n \times n} \) be the covariance matrix of instantaneous returns per unit time, and we write \( \Sigma = MRM \), where \( M = \text{diag}(\sigma_1, \ldots, \sigma_n) \) is the diagonal matrix of volatilities.

We take up the general rebalancing rules \( b = (b_1, \ldots, b_n)' \in \mathbb{R}^n \), where \( b_i \) is the fraction of wealth bet on stock \( i \) over the interval \([t, t + dt]\). Thus, the trader keeps the fraction \( 1 - \sum_{i=1}^{n} b_i \) of his wealth in bonds over the interval \([t, t + dt]\). As before, we let \( V_t(b) \) denote the gambler’s wealth at \( t \), where \( V_0 = 1 \). The trader’s wealth evolves
according to

\[
dV_t(b)/V_t(b) = \sum_{i=1}^{n} b_i(dS_{it}/S_{it}) + \left(1 - \sum_{i=1}^{n} b_i\right)dB_t/B_t \\
= \left(\sum_{i=1}^{n} b_i\mu_i + \left(1 - \sum_{i=1}^{n} b_i\right)r\right)dt + \sum_{i=1}^{n} b_i\sigma_idW_{it}.
\]

(50)

For brevity, let \( \mu = (\mu_1, ..., \mu_n)' \) be the vector of drifts. We then have

\[
dV_t/V_t = (r + (\mu - r1)'b)dt + \sum_{i=1}^{n} b_i\sigma_i dW_{it},
\]

(51)

where \( 1 = (1, ..., 1)' \) is an \( n \times 1 \) vector of ones. The solution of this stochastic differential equation is given by

\[
V_t(b) = \exp\left\{ (r + (\mu - r1)'b - b'\Sigma b/2)t + \sum_{i=1}^{n} b_i\sigma_iW_{it} \right\}.
\]

(52)

This can be verified directly by applying the multivariate version of Itô’s Lemma (Björk 1998) to the function

\[
F(W_1, ..., W_n, t) = \exp\left\{ (r + (\mu - r1)'b - b'\Sigma b/2)t + \sum_{i=1}^{n} b_i\sigma_iW_{i} \right\}.
\]

(53)

Indeed, we get

\[
dF_t = \frac{\partial F}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial F}{\partial W_i}dW_{it} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 F}{\partial W_i\partial W_j} \rho_{ij}dt.
\]

(54)

Substituting \( \partial F/\partial t = F \cdot (r + (\mu - r1)'b - b'\Sigma b/2), \partial F/\partial W_i = F \cdot b_i\sigma_i, \) and \( \partial^2 F/\partial W_i\partial W_j = \)
$F \cdot b_i b_j \sigma_i \sigma_j$ yields the desired result. Proceeding as before, we take the expression

$$
\sigma_i W_{it} = \log(S_{it}/S_{i0}) - (\mu_i - \sigma_i^2/2)t
$$

and substitute it into the above formula for $V_t(b)$. This yields

$$
V_t(b) = \exp \left\{ rt + \sum_{i=1}^n \left( \log(S_{it}/S_{i0}) - (r - \sigma_i^2/2)t \right) b_i - (t/2)b'\Sigma b \right\}.
$$

(56)

For brevity, let

$$
\hat{z}_i = \frac{\log(S_{it}/S_{i0}) - (r - \sigma_i^2/2)t}{\sigma_i\sqrt{t}}.
$$

(57)

Under the equivalent martingale measure, the variables $z = (z_1, ..., z_n)'$ are all unit normals, with correlation matrix $R = [\rho_{ij}]$. Thus, we can write

$$
V_t(b) = \exp \left\{ rt + \sqrt{t}(z'Mb) - (t/2)b'\Sigma b \right\}.
$$

(58)

Maximizing $V_t(b)$ with respect to $b$, we get the first-order condition

$$
t\Sigma b = \sqrt{t}Mz.
$$

(59)

For simplicity, let $S = (S_1, ..., S_n)'$ be the vector of stock prices, and let $b(S, t)$ denote the best rebalancing rule in hindsight over $[0, t]$. Solving the first-order condition yields

$$
b(S, t) = M^{-1}R^{-1}z/\sqrt{t}.
$$

(60)

The final wealth that accrues to a $1$ deposit into the best rebalancing rule in hindsight over $[0, t]$ is

$$
V_t^* = \exp(rt + z'R^{-1}z/2) = \exp(rt + (t/2)b'\Sigma b).
$$

(61)
Hence, the final payoff of Cover’s Derivative is \( V^*_T = \exp(rT + z'R^{-1}z/2) \). Again, we see that the final wealth of the best (leveraged) rebalancing rule in hindsight is Markovian: it depends only on the current state \((S_1, ..., S_n, t)\).

We pass to the multivariate version of the Black-Scholes equation (Wilmott 2001), which governs the no-arbitrage price of “rainbow,” “basket,” or “correlation” options that depend on several underlyings. As usual \( C(S_1, ..., S_n, t) = C(S, t) \) will denote the price of Cover’s Derivative. We solve the differential equation

\[
\frac{1}{2} \sum_{i,j} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 C}{\partial S_i \partial S_j} + r \sum_{i=1}^{n} S_i \frac{\partial C}{\partial S_i} + \frac{\partial C}{\partial t} - rC = 0
\]  

subject to the boundary condition \( C(S, T) = V^*_T(S) = \exp(rT + z'_T R^{-1}z_T/2) \). As usual, we do this by computing the expected discounted payoff with respect to the equivalent martingale measure.

To this end, we write

\[
z_T = \sqrt{t/T} \cdot z_t + \sqrt{(T-t)/T} \cdot y, \tag{63}
\]

where

\[
y_i = \frac{\log(S_{iT}/S_{it}) - (r - \sigma_i^2/2)(T-t)}{\sigma_i \sqrt{T-t}}. \tag{64}
\]

The \( y_i \) are all unit normals with respect to the equivalent martingale measure \( Q \) and the information available at \( t \). \( R \) is the correlation matrix of the random vector \( y = (y_1, ..., y_n)' \). The conditional density of \( y \) is \( f(y) = (2\pi)^{-n/2} \det(R)^{-1/2} \exp(-y'R^{-1}y/2) \).

Expanding the quadratic form \( z'_T R^{-1}z_T \), we get

\[
z'_T R^{-1}z_T/2 = \frac{t}{2T} z'_t R^{-1}z_t + \frac{\sqrt{t(T-t)}}{T} z'_t R^{-1}y + \frac{T-t}{2T} y'R^{-1}y. \tag{65}
\]
Thus, we find that $E^Q_t[\exp(z'_t R^{-1} z_T/2)] =$

$$(2\pi)^{-n/2} \det(R)^{-1/2} \exp \left( \frac{t}{2T} z'_t R^{-1} z_t \right) \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( -\frac{t}{2T} y'R^{-1} y + \frac{\sqrt{t(T-t)}}{T} z'_t R^{-1} y \right) dy_1 \cdots dy_n. \quad (66)$$

To evaluate the multiple integral, we use the general formula

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-y'A y + \beta' y) dy_1 \cdots dy_n = \pi^{n/2} \det(A)^{-1/2} \exp(\beta'A^{-1} \beta/4), \quad (67)$$

where $A$ is any symmetric positive definite $n \times n$ matrix and $\beta = (\beta_1, ..., \beta_n)'$ is any vector of constants. Putting $A = t/(2T) \cdot R^{-1}$, $\beta = \sqrt{t(T-t)/T} \cdot R^{-1} z_t$, and simplifying, we get

$$E^Q_t[\exp(z'_t R^{-1} z_T/2)] = (T/t)^{n/2} \exp(z'_t R^{-1} z_t/2). \quad (68)$$

**Theorem 4.** For levered hindsight optimization (over all $b \in \mathbb{R}^n$), the price of Cover’s Derivative is

$C(S, t) = (T/t)^{n/2} \exp(r t + z'R^{-1} z/2) = (T/t)^{n/2} \exp(r t + t \cdot b' \Sigma b/2) = (T/t)^{n/2} \cdot V^*_t \quad (69)$

where $z_i = (\log(S_{it}/S_{i0}) - (r - \sigma_i^2/2)t)/(\sigma_i \sqrt{t})$, $b(S, t)$ is the best rebalancing rule in hindsight over $[0, t]$, and $V^*_t$ is the intrinsic value at time $t$.

**Corollary 2.** For the general market with $n$ correlated stocks in geometric Brownian motion, the American-style version of Cover’s Derivative (that pays $V^*_t$ upon exercise
at $t$) will never be exercised early in equilibrium.

To find the replicating strategy, we again differentiate the price, getting

$$
\Delta_i = \frac{\partial C}{\partial S_i} = \frac{C \cdot (R^{-1}z_t)_i}{S_i \sigma_i \sqrt{t}} = \frac{C \cdot b_i(S,t)}{S_i} \tag{70}
$$

where $(R^{-1}z_t)_i$ is the $i^{th}$ coordinate of the vector $R^{-1}z_t$. Thus, we have the relation $\Delta_i S_i / C = b_i(S,t)$.

Theorem 5. The replicating strategy for Cover’s Derivative bets the fraction $b_i(S,t)$ of wealth on stock $i$ in state $(S,t)$. Thus, to replicate Cover’s Derivative, one just uses the best rebalancing rule in hindsight as it is known at time $t$.

For the general stock market, we have again concluded that the following three trading strategies are identical:

1. The strategy that looks back over the known price history $[0,t]$, finds the best continuously-rebalanced portfolio in hindsight, and uses that portfolio over the interval $[t, t+dt]$.

2. The $\Delta$-hedging strategy induced by Cover’s Derivative.

3. The natural estimator $\Sigma^{-1}(\hat{\mu} - r1)$ of the continuous-time Kelly rule (which is $\Sigma^{-1}(\mu - r1)$).

4 Simulations

We close with three simulations that help visualize the behavior of the replicating strategy over $T = 200$ years under a risk-free rate of $r = 0.02$. We let $\nu_i = \mu_i - \sigma_i^2/2$ denote the compound-annual growth rate of stock $i$, and we normalize the initial stock prices to $S_{i0} = 1$. We also normalize the trader’s initial wealth to $\$1$. Simulations 1
Figure 8: OUtcome for $\nu = 0.04, \sigma = 0.7$.

and 2 deal with the univariate case. For the first 5 years of the experiment, the trader holds a single share of the stock. Then at $t = 5$, he puts all his money into Cover’s Derivative. The waiting period is necessary because $C \to +\infty$ as $t \to 0^+$. Thus, for $t \leq 5$ the trader’s wealth is $S_t$, and for $t \geq 5$ his wealth is $S_5C(S_t, t)/C(S_5, 5)$.

Simulation 1

We put $\nu = 0.04$ and $\sigma = 0.7$. The Kelly growth rate (Luenberger 1997) for this market is 9.17% and the Kelly bet is $b^* = 0.54$. The replicating strategy learns to hold significant cash balances and “live off the fluctuations,” which are substantial on account of the 70% annual volatility. Figure 8 gives a sample path.
Simulation 2

Next, we use $\nu = 0.08$ and $\sigma = 0.17$. The Kelly growth rate is 11.6\% and the Kelly bet is $b^* = 2.57$. The replicating strategy uses enormous leverage in an effort to exploit low interest rates and the favorable risk/return profile. This is Figure 9. After 200 years, the stock price has appreciated from $1$ a share to $100$ million a share, but the replicating strategy has grown the initial dollar into $1$ trillion.
Simulation 3

Finally, we simulate the bivariate case. At \( t = 0 \), the trader puts $0.50 into each stock. He holds this portfolio for 5 years, and then he puts all his money into Cover’s Derivative. Thus, for \( t \leq 5 \) his wealth is \( 0.5(S_1(t) + S_2(t)) \), and for \( t \geq 5 \) his wealth is \( 0.5(S_1(5) + S_2(5))C(S_1, t)/C(S_5, 5) \).

We use \( \nu = (0.03, 0.08)' \) and \( \sigma = (0.55, 0.7)' \), with \( \rho = 0.2 \) being the correlation of instantaneous returns. The Kelly growth rate is 13.7% and the Kelly fractions are \( b = (0.39, 0.56)' \). Figure 10 gives the result. On this particular sample path, the replicating strategy uses leverage for decades on end, in spite of the fact that a Kelly gambler would continuously hold 5% of wealth in cash.
5 Conclusion

This paper priced and replicated an exotic option (“Cover’s Derivative”) whose payoff equals the final wealth that would have accrued to a $1 deposit into the best leveraged, continuously-rebalanced portfolio in hindsight. A rebalancing rule is a fixed-fraction betting scheme that trades continuously so as to maintain a target proportion \( b_i \) of wealth in each stock \( i \). For the Black-Scholes market with \( n \) correlated stocks in geometric Brownian motion, the no-arbitrage price of Cover’s Derivative is 

\[
C(S,t) = \left(\frac{T}{t}\right)^{n/2} \cdot \exp(rt + t \cdot b^T \Sigma b/2),
\]

where \( b = b(S,t) \) is the best rebalancing rule in hindsight over \([0,t]\) and \( \Sigma \) is the covariance of instantaneous returns per unit time. Since \( C \) is equal to \( (T/t)^{n/2} \) times intrinsic value, the American-style version of Cover’s Derivative will never be exercised early in equilibrium.

The order of magnitude \( C(S,t;T) = \mathcal{O}(T^{n/2}) \) agrees with the super-replicating price derived by Cover in his discrete-time universal portfolio theory. A sophisticated, long-lived institution that puts all its money into the replicating strategy (a strategy which turns out to be horizon-free) will grow its endowment at the same asymptotic rate as the best levered rebalancing rule in hindsight. In the long-run, with probability approaching 1, it will beat the market averages by an exponential factor.

The replicating strategy amounts to betting the fraction \( b_i(S,t) \) of wealth on stock \( i \) at time \( t \), where \( b(S,t) \) is the best rebalancing rule in hindsight over the currently known price history. If someone knows the covariance \( \Sigma \) of instantaneous returns (but not necessarily the drifts of the various stocks), he can use the formula 

\[
b(S,t) = M^{-1}R^{-1}z/\sqrt{t},
\]

where \( R \) is the correlation matrix of the instantaneous returns, \( M = \text{diag}(\sigma_1, ..., \sigma_n) \) is the (diagonal) matrix of volatilities, and 

\[
z_i = \left(\log\left(\frac{S_{it}}{S_{i0}}\right) - (r - \sigma_i^2/2)t\right)/(\sigma_i \sqrt{t}).
\]

But even if he is ignorant of \( \Sigma \), he can still find \( b(S,t) \) at any given time by hindsight-optimizing over the known price history.
Another expression for the replicating strategy is \( b(S, t) = \Sigma^{-1}(\hat{\mu} - r1) \), where 
\( \hat{\mu}_i = \log(S_{it}/S_{i0}) + \sigma_i^2/2 \) is the natural estimator of the drift of stock \( i \). The replicating strategy converges in mean square to the continuous-time Kelly rule, \( b^* = \Sigma^{-1}(\mu - r1) \), and its realized compound-growth rate converges to the Kelly (1956) optimum asymptotic growth rate, which is \( \gamma^* = r + (1/2)(\mu - r1)'\Sigma^{-1}(\mu - r1) \). This happens because the intrinsic value of Cover’s Derivative grows at an asymptotic rate of \( \gamma^* \) per unit time. A $1 deposit into the replicating strategy at time \( t \) guarantees that the trader will achieve, at \( T \), the deterministic fraction \( V^*_T/C(S, t) \) of the final wealth of the best rebalancing rule in hindsight. The excess continuously-compounded growth rate of \( V^*_T \) (over and above that of the replicating strategy) is at most \( \{rt + z_i^t R^{-1} z_i/2 + n \log(T/t)/2\} / (T - t) \), which tends to 0 as \( T \to \infty \).

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