Regularized expression for the gravitational energy-momentum in teleparallel gravity and the principle of equivalence

J. W. Maluf*, M. V. O. Veiga
Instituto de Física,
Universidade de Brasília
C. P. 04385
70.919-970 Brasília DF, Brazil
and
J. F. da Rocha-Neto
Universidade Federal do Tocantins
Campus Universitário de Arraias
77.330-000 Arraias TO, Brazil

Abstract

The expression of the gravitational energy-momentum defined in the context of the teleparallel equivalent of general relativity is extended to an arbitrary set of real-valued tetrad fields, by adding a suitable reference space subtraction term. The characterization of tetrad fields as reference frames is addressed in the context of the Kerr space-time. It is also pointed out that Einstein’s version of the principle of equivalence does not preclude the existence of a definition for the gravitational energy-momentum density.

Key words: gravitational energy, teleparallelism, principle of equivalence.

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(*) e-mail: wadih@fis.unb.br
1 Introduction

The notion of gravitational energy-momentum has been addressed recently in the framework of the teleparallel equivalent of general relativity (TEGR) \[1, 2, 3, 4\]. The TEGR is an alternative geometrical description of Einstein’s general relativity in terms of tetrad fields $e^a_\mu$ and of the torsion tensor $T^a_\mu\nu = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu$ ($a, b, \ldots$ and $\mu, \nu, \ldots$ are SO(3,1) and space-time indices, respectively). The field equations for the tetrad field $e^a_\mu$ are precisely equivalent to Einstein’s equations. Therefore it is not a new theory for the gravitational field. The torsion tensor is related to the antisymmetric part of the Weitzenböck connection $\Gamma^\lambda_\mu\nu = e^a_\lambda \partial_\mu e^a_\nu$. The curvature tensor constructed out of this connection vanishes identically. Therefore this connection allows the notion of distant parallelism in space-time. Of course one may construct the Christoffel symbols and consider the physical and geometrical properties of the (nonvanishing) Riemann-Christoffel tensor.

The geometrical framework determined by the tetrad field and torsion tensor has proven to be suitable to investigate the problem of defining the gravitational energy-momentum. A consistent expression developed in the realm of the TEGR shares many features with the expected definition. The gravitational energy-momentum $P^a$ \[1, 2\] obtained in the framework of the TEGR has been investigated in the context of several distinct configurations of the gravitational field. For asymptotically flat space-times $P^{(0)}$ yields the ADM energy \[5\]. In the context of tetrad theories of gravity, asymptotically flat space-times may be characterized by the asymptotic boundary condition,

$$e_{a\mu} \cong \eta_{a\mu} + \frac{1}{2} h_{a\mu}(1/r),$$

and by the condition $\partial_\mu e^a_\nu = O(1/r^2)$ in the asymptotic limit $r \to \infty$. In the asymptotic limit above the quantity $\eta_{a\mu}$ in Eq. (1) coincides with the metric tensor of the Minkowski space-time $\eta_{ab} = (- + + +)$. An important property of tetrad fields that satisfy the condition above is that in the flat space-time limit we have $e^a_\mu(t, x, y, z) = \delta^a_\mu$, and therefore $T^a_\mu\nu = 0$. Hence for the flat space-time we normally consider a set of tetrad fields such that $T^a_\mu\nu = 0$ in any coordinate system. This condition establishes the reference space. However, in general an arbitrary set of tetrad fields that yields the metric tensor for asymptotically flat space-times does not satisfy the asymptotic condition given by Eq. (1). Moreover for such tetrad fields we have in general $T^a_\mu\nu \neq 0$ in the flat space-time. It might be argued, therefore, that
the expression for the gravitational energy-momentum mentioned above is restricted to a particular class of tetrad fields, namely, to the class of frames such that \( T^a_{\mu\nu} = 0 \) if \( e^a_{\mu} \) represents the flat space-time tetrad field.

The definition \( P^a \) is invariant under global SO(3,1) transformations. We have argued elsewhere [2, 3, 4] that it makes sense to have a dependence of \( P^a \) on the frame. The energy-momentum in classical theories of particles and fields does depend on the frame, and we assert that such dependence is a natural property of the gravitational energy-momentum. The total energy of a relativistic body, for instance, depends on the frame. We normally assume that a set of tetrad fields is adapted to an ideal observer in the space-time determined by the metric tensor \( g_{\mu\nu} \). For a given gravitational field configuration (a black hole, for instance), the infinity of possible observers is related to the infinity of tetrad fields (related by a local SO(3,1) transformation) that yields the metric tensor \( g_{\mu\nu} \). Let \( x^\mu(s) \) denote the worldline \( C \) of an observer, and \( u^\mu(s) = dx^\mu/ds \) its velocity along \( C \). We may identify the observer’s velocity with the \( a = (0) \) component of \( e^a_{\mu} \), where \( e^a_{\mu} e^a_{\nu} = \delta^\mu_{\nu} \). Thus, \( u^\mu(s) = e^a_{(0)\mu} \) along \( C \) [6]. The acceleration of the observer is given by

\[
a^\mu = \frac{Du^\mu}{ds} = \frac{De^a_{(0)}\mu}{ds} = u^\alpha \nabla_\alpha e^a_{(0)\mu}.
\] (2)

The covariant derivative is constructed out of the Christoffel symbols. We see that \( e^a_{\mu} \) determines the velocity and acceleration along the worldline of an observer adapted to the frame. From this perspective we conclude that a given set of tetrad fields, for which \( e^a_{(0)\mu} \) describes a congruence of timelike curves, is adapted to a particular class of observers, namely, to observers determined by the velocity field \( u^\mu = e^a_{(0)\mu} \), endowed with acceleration \( a^\mu \). If \( e^a_{\mu} \rightarrow \delta^a_{\mu} \) in the limit \( r \rightarrow \infty \), then \( e^a_{\mu} \) is adapted to stationary observers at spacelike infinity. We may say, therefore, that \( P^{(0)} \) yields the ADM energy for such observers.

In this article we will extend the definition \( P^a \) for the gravitational energy-momentum previously considered for arbitrary tetrad fields, namely, for tetrad fields that satisfy \( T^a_{\mu\nu} \neq 0 \) in the flat space-time. The redefinition is the only possible consistent extension of \( P^a \), valid for tetrad fields that do not satisfy boundary conditions like Eq. (1). We will also argue that an existing version of the principle of equivalence, namely, Einstein’s version of the principle, does not pose any obstacle to the concept of localized gravitational energy. We will show that the usual (textbook) version of the principle was never accepted by Einstein.
2 The principle of equivalence and the localizability of the gravitational energy

The concept of energy in classical electrodynamics is very simple and well known. We consider an arbitrary volume in the three-dimensional space and verify the existence of field lines of the electric and/or magnetic field in this region. The electromagnetic energy density is given by the standard expression that consists of the sum of the square of the electric and magnetic fields, and therefore is nonvanishing in a space-time region where the field lines are present. Charged particles in this region experience the Lorentz force. Therefore the manifestation of the Lorentz force is an indication of the existence of electromagnetic energy density.

Unfortunately there is not a simple picture in general relativity that relates gravitational “field lines” to the existence of gravitational energy density. Nevertheless, it is legitimate to expect that the manifestation of gravitational forces in a three-dimensional region is an indication of the existence of gravitational energy-momentum density in this region. However in general relativity there is a point of view according to which the gravitational energy density cannot be localized (see §20.4). The argument is the following. In any given small region of the space-time manifold we can find a coordinate system such that the Christoffel symbols disappear. In terms of this appropriate coordinate system the small region in question is “free of gravitational fields”. In summary, this is the argument that has been endorsed by many authors, who claim that the nonlocalizability of the gravitational energy-momentum is due to the principle of equivalence.

We do not endorse the conclusion above for various reasons. First, it is well known that the vanishing of the Christoffel symbols does not imply the vanishing of tidal forces in any infinitesimal region of the space-time. Therefore, the assertion that this region is free of gravitational fields is questionable, because we should agree that the existence of gravitational forces (e.g., on the worldline of a particle) is due to gravitational fields. It is not reasonable to accept the idea of having a force in a space-time region without the associated field in this same region.

Second, the principle of equivalence that supports the conclusion above is related to Pauli’s version of the principle, but is different from Einstein’s version. According to Pauli’s formulation, for every infinitely small world region there always exist a coordinate system in which gravitation has no
influence either on the motion of particles or any other physical processes. The distinction between Einstein’s and Pauli’s formulation of the principle of equivalence has been addressed by Norton [9]. From the point of view of Pauli’s formulation, the vanishing of the Christoffel symbols in a space-time region implies that gravitation has no effect in this region. We know, however, that what really vanishes in such region are the first derivatives of the metric tensor. In our opinion the mathematical feature that consists of the vanishing of the first derivatives of any metric tensor - but not of the second and highest derivatives - along any worldline in a Riemannian or pseudo-riemannian manifold, in any dimension, cannot be taken as a physical principle. It is just a feature of differential geometry.

Pauli’s formulation of the principle of equivalence is different from Einstein’s formulation [9]. The latter is unquestionably considered to be the breakthrough that led Einstein to establish the conditions under which a noninertial frame is equivalent to an inertial one, extending in this way the principle of relativity. In view of the practical difficulties related to the description of arbitrary gravitational fields by means of the principle of equivalence, the latter was abandoned in favour of the principle of general covariance. Nevertheless the importance of the principle has always been recognized by Einstein in the years after the formulation of general relativity [9].

Einstein’s version of the principle of equivalence [9] consists of considering a reference frame $K$ (a “Galilean system”), and a reference frame $K'$, which is uniformly accelerated with respect to $K$. Then one asks whether an observer in $K'$ must understand his condition as accelerated, or whether there remains a point of view according to which he can interpret his condition as at “rest”. Einstein concludes that by assuming the existence of a homogeneous gravitational field in $K'$ it is possible to consider the latter as at rest. In his words: \textit{The assumption that one may treat $K'$ as at rest, in all strictness without any laws of nature not being fulfilled with respect to $K'$, I call the \textquote{principle of equivalence} [9].}

Again considering Ref. [9], we observe that an important remark about Einstein’s formulation of the principle of equivalence is not widely considered in the literature: Einstein’s formulation is established in Minkowski space-time. The passage from $K$ to $K'$ amounts to a frame transformation in a finite region of the space-time, not a coordinate transformation. Moreover Einstein never endorsed Pauli’s formulation. Einstein objected that \textit{in the infinitely small every continuous line is a straight line} [9]. He believed that
the restriction to infinitesimal regions makes it impossible to distinguish the geodesic world lines of free point masses from other world lines and thus it is impossible to judge whether - in the words of Pauli’s formulation - “gravitation has no influence on the motion of particles”. Quoting Norton [9]: It has rarely been acknowledged that Einstein never endorsed the principle of equivalence which results, here called the “infinitesimal principle of equivalence”. Moreover, his early correspondence contains a devastating objection to this principle: in infinitesimal regions of the space-time manifold it is impossible to distinguish geodesics from many other curves and therefore impossible to decide whether a point mass is in free fall. ¹

The principle of equivalence follows from the equality of inertial and gravitational masses, and really establishes the equivalence between a noninertial reference frame and an inertial one with the addition of a suitable gravitational field. The nonvanishing of tidal forces in infinitesimal regions of the space-time does not allow us to conclude that such regions can be made free of gravitational fields by means of coordinate transformations. Two nearby particles in free fall undergo geodesic deviation irrespective of whether the metric tensor is reduced to the Minkowski form along their worldlines. We recall that by means of coordinate transformations one cannot reduce the tetrad field or the torsion tensor at a space-time point to their flat space-time values. Any space-time region, infinitesimal or not, is flat and consequently “free of gravitational fields” if and only if the Riemann-Christoffel tensor vanishes in this region. Arguments based on the “infinitesimal principle of equivalence” are not conclusive and cannot be taken to rule out the notion of gravitational energy-momentum density.

3 A regularized expression for the gravitational energy-momentum

Let us briefly recall the Lagrangian formulation of the TEGR. The Lagrangian density for the gravitational field in the TEGR in empty space-time is given by

¹It is worthwhile to point out a compact statement of the principle formulated by Einstein in 1918 [9]: Principle of Equivalence: inertia and gravity are wesensgleich (identical in essence). From this and from the results of the special theory of relativity it necessarily follows that the symmetrical tensor $$g_{\mu\nu}$$ determines the metrical properties of space, the inertial behaviour of bodies in it, as well as the gravitational action.
\[ L(e_{\mu}) = -ke \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) \equiv -k e \Sigma^{abc} T_{abc}, \quad (3) \]

where \( k = 1/(16\pi) \) and \( e = \det(e^\alpha_\mu) \). The tensor \( \Sigma^{abc} \) is defined by

\[ \Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c), \quad (4) \]

and \( T^a = T^b \gamma^a_b \). The quadratic combination \( \Sigma^{abc} T_{abc} \) is proportional to the scalar curvature \( R(e) \), except for a total divergence. The field equations for the tetrad field read

\[ e_{a\lambda} e_{b\mu} \partial_{\nu} (e \Sigma^{b\lambda\nu}) - e (\Sigma^{b\nu} a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcde} \Sigma^{bcd}) = 0, \quad (5) \]

It is possible to prove by explicit calculations that the left hand side of Eq. (5) is exactly given by \( \frac{1}{2} e \left[ R_{\mu
u}(e) - \frac{1}{2} e_{\mu\nu} R(e) \right] \). As usual, tetrad fields convert space-time into Lorentz indices and vice-versa.

The definition for the gravitational energy-momentum has first been obtained in the Hamiltonian formulation of the TEGR [10, 11]. However either the Hamiltonian or Lagrangian field equations may be suitably interpreted as equations that define the gravitational energy-momentum. The momentum canonically conjugated to the tetrad components \( e_{aj} \) is given by \( \Pi_{aj} = -4ke \Sigma^{aj} \). The latter quantity yields the definition of the gravitational energy-momentum \( P^a \) contained within a volume \( V \) of the three-dimensional spacelike hypersurface [11 2 3],

\[ P^a = -\int_V d^3x \partial_k \Pi^{ak}. \quad (6) \]

\( P^a \) transforms as a vector under the global SO(3,1) group. It describes the gravitational energy-momentum with respect to observers adapted to \( e^a_\mu \). These observers are characterized by the velocity field \( u^\mu = e_{(0)}^\mu \), and by the acceleration \( a^\mu \) given by Eq. (2).

Let us assume that the space-time is asymptotically flat. The total gravitational energy-momentum is given by

\[ P^a = -\int_{S \to \infty} dS_k \Pi^{ak}. \quad (7) \]

The field quantities are evaluated on a surface \( S \) in the limit \( r \to \infty \).
In Eqs. (6,7) it is implicitly assumed that the reference space is determined by a set of tetrad fields $e^a_\mu$ for the flat space-time such that the condition $T^a_\mu_\nu = 0$ is satisfied. However in general there exist flat space-time tetrad fields for which $T^a_\mu_\nu \neq 0$. In this case we may generalize Eq. (6) by adding a suitable reference space subtraction term, exactly like in the Brown-York formalism [12]. The Brown-York quasi-local energy expression is regularized by subtracting the energy of a flat slice of the flat space-time.

Let us denote $T^a_\mu_\nu (E) = \partial_\mu E^a_\nu - \partial_\nu E^a_\mu$, and $\Pi^{aj}(E)$ as the expression of $\Pi^{aj}$ constructed out of flat tetrads $E^a_\mu$. The regularized form of the gravitational energy-momentum $P^a$ is defined by

$$P^a = - \int_V d^3x \partial_k [\Pi^{ak}(e) - \Pi^{ak}(E)].$$

This definition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by the tetrad fields $E^a_\mu$, obtained from $e^a_\mu$ by requiring the vanishing of the physical parameters like mass, angular momentum, etc.

The total gravitational energy-momentum is obtained by integrating over the whole three-dimensional spacelike section. Assuming again that the space-time is asymptotically flat, we have

$$P^a = - \oint_{S\to\infty} dS_k [\Pi^{ak}(e) - \Pi^{ak}(E)],$$

where the surface $S$ is established at spacelike infinity. Like Eq. (6), the definition above transforms as a vector under the global SO(3,1) group.

The definition given by Eq. (8) is valid also in the context of space-times with an arbitrary topology. It is legitimate to take the tetrad fields $E^a_\mu$, to represent the pure de Sitter or anti-de Sitter spaces, for instance, in which case Eq. (8) represents the gravitational energy-momentum defined about the latter space-times.

4 Reference frames in the Kerr space-time and the total gravitational energy

In this section we will apply Eq. (9) to a simple set of tetrad fields that describes the Kerr space-time, in order to illustrate the procedure (of course the analysis of the gravitational energy of the Kerr space-time may be carried
out by means of several approaches). For this purpose we will evaluate the total gravitational energy. The asymptotic form of the Kerr metric tensor describes the exterior region of a rotating isolated material system. The set of tetrad fields to be considered allows a straightforward evaluation of connections and curvature (in the context of Riemannian geometry), but it has neither a simple geometrical structure when written in cartesian coordinates, nor an appropriate asymptotic behaviour. Before we carry out this analysis, we will recall the construction of tetrad fields as reference frames. We start by considering the flat Minkowski space-time with cartesian coordinates $x^\mu$.

Besides $x^\mu$, the flat space-time is endowed with cartesian coordinates $q^a$. The coordinate system $q^a$ establishes a global reference frame. The transformation matrix that relates the two coordinate systems defines a set of tetrad fields for the Minkowski space-time, $E^a_\mu = \partial_\mu q^a$. The coordinate transformation $dq^a = E^a_\mu dx^\mu$ can be globally integrated, and therefore it establishes a holonomic transformation between $q^a$ and $x^\mu$. Rotations and boosts between $q^a$ and $x^\mu$ are the two basic SO(3,1) transformations. The condition

$$E_{(i)j}(t, x, y, z) = E_{(j)i}(t, x, y, z),$$  \hspace{1cm} (10)

ensures that $q^a$ is not rotating with respect to $x^\mu$, because a rotation will give rise to antisymmetric components in the sector $E_{(i)j}$ (Latin indices from the middle of the alphabet run from 1 to 3). On the other hand, a boost between $q^a$ and $x^\mu$ implies that $E^{(0)}_k \neq 0$ \footnote{[1]}. Therefore by imposing $E^{(0)}_k = 0$, or, equivalently, the time gauge condition,

$$E_{(i)}^0 = 0,$$  \hspace{1cm} (11)

we ensure that the two coordinate systems have a unique time scale. Both $q^a$ and $x^\mu$ describe the flat space-time. Thus we may say that if conditions (10) and (11) are imposed on $E^a_\mu(t, x, y, z)$ the reference space-time with coordinates $q^a$ is neither rotating nor undergoing a boost with respect to the space-time with coordinates $x^\mu$ \footnote{[1]}. If these conditions are imposed we have $E^a_\mu(t, x, y, z) = \delta^a_\mu$, or

$$E^a_\mu(t, r, \theta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & \cos \theta & -r \sin \theta & 0 \end{pmatrix}. \hspace{1cm} (12)$$
From a different but equivalent point of view we may say that 
$E^a_\mu(t, x, y, z) = \delta^a_\mu$ is adapted to stationary observers in space-time, namely, 
observers that are endowed with the velocity field $u^\mu = E_{(0)}^\mu = \delta^\mu_{(0)}$ and 
acceleration $a^\mu$ given by Eq. (2). In this case, $a^\mu = 0$.

A geometrical interpretation of tetrad fields as an observer’s frame can 
be given as follows. We consider an arbitrary path $x^\mu(s)$ of the observer 
in Minkowski space-time, where $s$ is the proper time of the observer. We 
identify $dx^\mu/ds = u^\mu = E_{(0)}^\mu$, where $E_{(0)}^\mu$ is the timelike component of 
the orthonormal frame (the temporal axis of the observer’s local frame).

According to the hypothesis of locality [13], a noninertial observer at each 
instant along its worldline is equivalent to an otherwise identical momentarily 
comoving inertial observer. It follows from the hypothesis of locality that 
each noninertial observer is endowed with an orthonormal tetrad frame $E_a^\mu$, 
whose derivative along the path is given by [14, 15]

$$\frac{dE_a^\mu}{ds} = \phi_a^{\ b} E_b^\mu,$$

(13)

where $\phi_{ab}$ is the antisymmetric acceleration tensor (not to be confused with $\phi^{ij}$ given by Eq. (28)). According to Refs. [14, 15], in analogy with the 
Faraday tensor we can identify $\phi_{ab} \to (-a, \Omega)$, where $a$ is the translational 
acceleration ($\phi_{(0)(i)} = a_{(i)}$) and $\Omega$ is the frequency of rotation of the local 
spatial frame with respect to a nonrotating (Fermi-Walker transported [6]) 
frame. The invariants constructed out of $\phi_{ab}$ establish the acceleration scales 
and lengths [13]. It follows from Eq. (13) that

$$\phi_a^{\ b} = E_b^\mu \frac{dE_a^\mu}{ds} = E_b^\mu u^\lambda \nabla_\lambda E_a^\mu.$$

(14)

Therefore given any set of tetrad fields for an arbitrary gravitational field 
configuration its geometrical interpretation can be obtained by suitably interpreting the velocity field $u^\mu = e_{(0)}^\mu$ and the acceleration tensor $\phi_{ab}$, in 
case we “switch off” the gravitational field by making $e^a_\mu \to E^a_\mu$. In several 
situations it turns out to be easy to impose conditions (10) and (11) on $e^a_\mu$.

However, the proper interpretation of $\phi_{ab}$ along a typical trajectory determined by the velocity vector $u^\mu$ of a class of observers adapted to a tetrad field seems to be a condition stronger than Eqs. (10) and (11).

Now we consider the Kerr space-time. The line element is given by
\[ ds^2 = -\frac{\psi^2}{\rho^2} dt^2 - \frac{2\chi \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\phi^2, \]  \tag{15}

where

\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \]
\[ \Delta = r^2 + a^2 - 2mr, \]
\[ \chi = 2amr, \]
\[ \Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \]
\[ \psi^2 = \Delta - a^2 \sin^2 \theta. \]

Imposition of conditions (10) and (11) yields the following expression for \( e_{a \mu}, \)

\[ e_{a \mu} = \begin{pmatrix} -\frac{1}{\rho} \sqrt{\psi^2 + \chi^2 \sin^2 \theta} & 0 & 0 & 0 \\ \frac{\rho}{\Sigma \rho} \sin \theta \cos \phi & \rho \cos \theta \cos \phi & -\frac{\Sigma}{\rho} \sin \theta \sin \phi \\ \frac{\rho}{\Sigma \rho} \sin \theta \sin \phi & \rho \cos \theta \sin \phi & \frac{\Sigma}{\rho} \sin \theta \cos \phi \\ 0 & -\rho \sin \theta & 0 \end{pmatrix}. \]  \tag{16}

The transformation \( dq^a = e^a_{\mu} dx^\mu \) determined by the expression above cannot be globally integrated, because in this case \( e^a_{\mu} \neq \partial_\mu q^a. \) Therefore \( dq^a = e^a_{\mu} dx^\mu \) is an anholonomic transformation. An important feature of the equation above is that its expression in the asymptotic limit \( r \to \infty \) is given by Eq. (1). Thus we may say that Eq. (16) is adapted to stationary observers at spacelike infinity. We also note that the flat space-time limit of Eq. (16) yields Eq. (12), and therefore \( T^a_{\mu \nu}(E) = 0. \)

Equation (16) above has proven to describe satisfactorily the energy-momentum properties of the Kerr space-time \[ \text{11} \] (we note that tetrad fields for the Kerr space-time have also been addressed in Refs. \[ 16, 17 \]). However, the line element given by Eq. (15) admits a simple form that is useful for computational purposes, and which reads
\[ e_{a\mu} = \left( \begin{array}{cccc} -A & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ B & 0 & 0 & C \end{array} \right), \]  

(17)

where

\[ A = \left( \frac{\chi^2 \sin^2 \theta + \psi^2 \Sigma^2}{\rho^2 \Sigma^2} \right)^{\frac{1}{2}}, \]

\[ B = -\frac{\chi \sin \theta}{\rho \Sigma}, \]

\[ C = \frac{\Sigma \sin \theta}{\rho}. \] \hspace{1cm} (18)

The flat space-time limit of Eq. (17) is given by

\[ E^a_{\mu} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{array} \right). \] \hspace{1cm} (19)

The expression above yields three nonvanishing torsion components:

\( T_{(2)12}(E) = 1, \) \( T_{(3)13}(E) = \sin \theta, \) \( \) and \( T_{(3)23}(E) = r \cos \theta. \) Inspece of its simplicity, this tetrad field has a rather intricate structure when written in cartesian coordinates. It reads

\[ E^a_{\mu}(t, x, y, z) = \left( \begin{array}{cccc} 1 & 0 & \frac{x}{r} & \frac{y}{r} \\ 0 & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ 0 & \frac{y}{r} & \frac{x}{r} & -\sqrt{x^2+y^2} \\ 0 & -\sqrt{x^2+y^2} & \sqrt{x^2+y^2} & 0 \end{array} \right). \] \hspace{1cm} (20)

In view of the geometrical structure of the equation above, we see that, differently from Eq. (16), Eq. (17) does not display the asymptotic behaviour determined by Eq. (1). Moreover, in general the tetrad field determined by Eq. (20) is adapted to accelerated observers. In order to verify this fact, let us consider a boost in the \( x \) direction, say, of Eq. (20). We find
\[ E^a \mu (t, x, y, z) = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\
0 & \frac{x z}{r \sqrt{x^2 + y^2}} & \frac{y z}{r} & -\frac{\sqrt{x^2 + y^2}}{r} \\
0 & \frac{y}{\sqrt{x^2 + y^2}} & \frac{z}{\sqrt{x^2 + y^2}} & 0
\end{pmatrix}, \quad (21) \]

where \( \beta \) and \( \gamma \) are constants defined by \( \beta = v/c \) and \( \gamma = \sqrt{1 - \beta^2} \). It is easy to see that along an observer’s trajectory whose velocity is determined by \( u^\mu = (\gamma, -\beta \gamma, 0, 0) \) the quantities \( \phi_{(j)}^{(k)} = u^i (E^{(k)}_m \partial_i E_{(j)}^m) \) constructed out of Eq. (21) are nonvanishing. This fact indicates that along the observer’s path the spatial axis \( E_{(j)}^\mu \) rotate. Nevertheless Eq. (17) yields a satisfactory value for the total gravitational energy-momentum, as we will see.

We will integrate Eq. (9) over a surface of constant radius \( x^1 = r \), and then we require \( r \to \infty \). Therefore we make \( k = 1 \) in Eq. (9). Out of Eq. (17) we evaluate all torsion components \( T_{a\mu\nu} \). We need the quantity

\[ \Sigma^{(0)01} = e^{(0)}_0 \Sigma^{001} = \frac{1}{2} e^{(0)}_0 (T^{001} - g^{00} T^1). \]

The calculations are lengthy but straightforward, and therefore they will be omitted here. We find

\[ -\Pi^{(0)1}(e) = 4 ke \Sigma^{(0)01} = -\frac{1}{8\pi} \frac{\Delta}{\rho} (\partial_r \Sigma) \sin \theta. \quad (22) \]

The expression of \( \Pi^{(0)1}(E) \) is obtained from Eq. (19) or, equivalently, by just making \( m = a = 0 \) in the expression above. It is given by

\[ \Pi^{(0)1}(E) = \frac{1}{4\pi} r \sin \theta. \quad (23) \]

Thus the gravitational energy contained within a surface \( S \) of constant radius \( r \) reads

\[ P^{(0)} = -\int_S dS_k \left[ \Pi^{(0)k}(e) - \Pi^{(0)k}(E) \right] = \int_0^\infty d\theta d\phi \frac{1}{4\pi} \sin \theta \left( -\frac{1}{2} \frac{\Delta}{\rho} (\partial_r \Sigma) + r \right). \quad (24) \]

In the limit \( r \to \infty \) we have
\[ 4ke \Sigma^{(0)01} \approx -\frac{1}{4\pi} r \sin \theta (1 - \frac{m}{r}). \]  

(25)

Therefore for the total gravitational energy of the Kerr space-time we obtain

\[ P^{(0)} \approx \int_{r \to \infty} d\theta d\phi \frac{1}{4\pi} \sin \theta \left( -r \left( 1 - \frac{m}{r} \right) + r \right) = m, \]  

(26)

which is the expected result.

We may also integrate Eq. (24) on the surface of constant radius \( r = r_+ \), where \( r_+ \) is the external horizon of the Kerr black hole. On this surface the function \( \Delta = r^2 + a^2 - 2mr \) vanishes. Therefore we find \( P^{(0)} = r_+ = m + \sqrt{m^2 - a^2} \), a result that is quite different from the irreducible mass of the Kerr black hole. The localization of gravitational energy in the Kerr space-time is correctly described by Eq. (16), according to the discussion in Ref. [1]. As discussed above, the frame determined by Eq. (16) is adapted to stationary observers at spacelike infinity.

Before we close this section let us recall that by means of simple algebraic manipulations an expression for the gravitational energy-momentum flux was developed in Ref. [2]. This expression follows directly from the field equations (5). It reads

\[ \frac{d}{dt} \left[ -\int d^3 x \partial_j \Phi^{a j} \right] = -\oint dS_j \phi^{a j}, \]  

(27)

where

\[ \phi^{a j} = k [ e e^{a \mu} (4 \Sigma^{b c j} T_{b c \mu} - \delta^{i j} \Sigma^{b c d} T_{b c d})]. \]  

(28)

The quantity above represents the \( a \) component of the flux density in the \( j \) direction. In Ref. [3] this formalism was applied to the evaluation of energy loss in Bondi’s radiative space-time. In Eqs. (27) and (28) it is assumed that for the flat space-time we have \( T_{a \mu \nu} (E) = 0 \). We may address Eq. (27) in the context of the present analysis. Let us assume that \( T_{a \mu \nu} (E) \neq 0 \). Since \( E^a_{\mu} \) is also a solution of the field equations (5), Eq. (27) is trivially satisfied for \( E^a_{\mu} \). Therefore we may write

\[ \frac{d}{dt} \left[ -\int d^3 x \partial_j [\Pi^{a j} (e) - \Pi^{a j} (E)] \right] = -\oint dS_j [\phi^{a j} (e) - \phi^{a j} (E)], \]  

(29)
where $\phi^{a j}(E)$ is constructed out of $E^a\mu$. We observe that as long as $E^a\mu$ (and consequently $\Pi^{a j}(E)$) is time independent, the left hand side of Eq. (29) is simplified and therefore the energy-momentum loss can be easily calculated out of any set of tetrad fields. The vanishing of $\phi^{a j}(e) - \phi^{a j}(E)$ at spacelike infinity (a feature that is expected to take place for asymptotically flat space-times) ensures the conservation of the total gravitational energy-momentum.

5 Discussion

In this article we have extended the definition for the gravitational energy-momentum previously considered in the framework of the TEGR, which requires $T_{\alpha\mu\nu}(E) = 0$ for the flat space-time, to the case where the flat space-time tetrad fields $E^a\mu$ yield $T_{a\mu\nu}(E) \neq 0$. In the context of the regularized gravitational energy-momentum definition it is not strictly necessary to stipulate asymptotic boundary conditions for tetrad fields that describe asymptotically flat space-times.

We have seen that Eqs. (13) and (14) provide a physical interpretation for a set of tetrad fields in Minkowski space-time, in terms of the linear acceleration and rotation of an observer adapted to the frame $E^a\mu$, endowed with velocity $u^\mu = E\left(0\right)^\mu$. We note that all frames obtained from $E^a\mu = \delta^a_\mu$ by means of a global SO(3,1) transformation (determined by constant transformation matrices $\Lambda^a_{b c}$) yield $\phi_a^b = 0$, according to Eq. (14). Thus the requirement $\phi_{ab} = -\phi_{ba} = 0$ seems to be equivalent to conditions (10) and (11).

The definition given by Eq. (8) can be applied to an arbitrary volume $V$ in any space-time, with an arbitrary topology. We propose that Eq. (8) represents the gravitational energy-momentum relative to the frame determined by the tetrad field $e^a\mu$, with $E^a\mu$ representing the tetrad field when the physical parameters of the metric tensor (mass, angular momentum, etc.) vanish.

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