Laplace-Runge-Lenz vector for arbitrary spin

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Abstract

A countable set of superintegrable quantum mechanical systems is presented which admit the dynamical symmetry with respect to algebra so(4). This algebra is generated by the Laplace-Runge-Lenz vector generalized to the case of arbitrary spin. The presented systems describe neutral particles with non-trivial multipole momenta. Their spectra can be found algebraically like in the case of Hydrogen atom. Solutions for the systems with spins 1/2 and 1 are presented explicitly, solutions for spin 3/2 are expressed via solutions of an ordinary differential equation of first order.

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I. INTRODUCTION

The Laplace-Runge-Lenz (LRL) vector is a fundamental constant of motion admitted by the classical Kepler problem. That is a cornerstone of celestial mechanics. It also plays a very important role in quantum mechanics, starting with its impressive application by Pauli [1] who had found the spectrum of the Hydrogen atom before the Schrödinger equation was discovered.

However, Pauli did not identify the related symmetry group. It was done later by Fock [2] who discovered the symmetry of the Schrödinger equation for the Hydrogen atom w.r.t. group SO(4) for bound states and group SO(1,3) for states with continuous spectrum. Then Bargman [3] showed that Fock’s group is generated by the integral of motion which is a quantum analogue of LRL vector.

Since the Hydrogen atom admits six integrals of motion including LRL and the orbital momentum vector, it is a maximally superintegrable system. The number of its algebraically independent integrals of motion is equal to 5, and three of them (including the Hamiltonian) commute each other. Such systems have many attractive properties starting with multi separability (which is guaranteed if integrals of motion are second order differential operators) and ending with simple solutions which are polynomials multiplied by some overall factor. Some of that systems, e.g., Hydrogen atom and harmonic oscillator are exceptionally important in physical applications.

Systematic search for superintegrable systems is a very interesting and important business started some time ago in papers [4] and [5]. A relatively new field is a classification of superintegrable systems with spin, which include matrix potentials. The systems with spin-orbit coupling where discussed in recent papers [6]-[8] while the systems with Pauli interaction where investigated in [9]-[11]. Let me mention also earlier papers [12] and [13] where higher order integrals of motion of quantum mechanical systems with $2 \times 2$ and $3 \times 3$ matrix potentials where investigated.

A specific sector of the superintegrability phenomena is formed by QM systems which admit a (generalized) LRL vector. There are various generalizations of the LRL vector in classical mechanics, see, e.g. [14]-[17]. However, we can find only few examples of 3d quantum mechanical systems with spin which admit this symmetry. Namely, they are the MICZ-Kepler system which includes a magnetic monopole with an arbitrary electric charge and a fixed inverse-square term in the potential [18], [19], a charged particle with spin $\frac{1}{2}$ and gyromagnetic ratio $g = 4$, etc.
interacting with a superposition of a point magnetic monopole field and a Coulomb plus a fine-tuned inverse-square potential [20], [21], a system with spin-orbit interaction and a special inverse-square potential with fixed coupling constant [21], and neutral particle with spin $\frac{1}{2}$ and a non-trivial dipole momentum interacting with the external field inverse in radius [10]. The latter system is shape invariant and was solved using tools of SUSY quantum mechanics.

Since $s = \frac{1}{2}$ is only a particular (although very important) possible spin value, a natural desire is to generalize the results of [10] to the case of arbitrary spin $s$. This problem is seemed to be especially provocative since in the 2d case the systems admitting the generalized LRL vector where found for both spin $s = \frac{1}{2}$ [22] and $s$ arbitrary [23], [24].

In the present paper the 3d LRL vectors for arbitrary spin are introduced. Namely, a countable set of exactly solvable QM problems is presented which admit generalized LRL vectors with spin. The geometric symmetry of these problems is exhausted by their invariance w.r.t. the rotation group generated by the total orbital momentum including a spin vector with arbitrary $s$. However, any of the discussed problems admits three additional constants of motion which are components of the LRL vector and generate the dynamical symmetry w.r.t. group SO(4).

This extended symmetry makes it possible to find algebraically Hamiltonian eigenvalues and essentially simplifies the procedure of construction of the corresponding eigenvectors. The Hamiltonian eigenvectors for spins $\frac{1}{2}$ and 1 are presented in explicit form, those ones for spin $\frac{3}{2}$ are expressed via solutions of an ordinary differential equation of the fourth order.

The presented systems describe neutral particles with (arbitrary) spin, which have non-trivial dipole or multipole momenta and are affected by dipole or multipole interactions with the external fields.

II. RUNGE-LENZ VECTOR FOR HYDROGEN ATOM

Let us start with the well known example of the dynamical symmetry w.r.t. group SO(4), which is admitted by the hydrogen atom. This QM system is specified by the following hamiltonian:

$$H = \frac{p^2}{2m} + V(x)$$

where

$$p^2 = p_1^2 + p_2^2 + p_3^2, \quad p_1 = -i \frac{\partial}{\partial x_1}, \quad V = -\frac{\alpha}{x}, \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \alpha = e^2 > 0.$$
We will discuss the eigenvalue problem for hamiltonian \((1)\)

\[ H\psi = E\psi \] (2)

and recall an elegant way to find eigenvalues \(E\) algebraically, without solving the stationary Schrödinger equation \((2)\).

Hamiltonian \((1)\) commutes with generators of group \(O(3)\) which are components of the orbital momentum \(L = x \times p\). We can indicate three additional constants of motion which are components of the LRL vector:

\[ \mathbf{K} = \frac{1}{2m}(p \times L - L \times p) + xV. \] (3)

The components of \(L\) and \(K\) satisfy the following commutation relations:

\[ [L_a, H] = [K_a, H] = 0, \]

\[ [L_a, L_b] = i\varepsilon_{abc}L_c, \quad [K_a, L_b] = i\varepsilon_{abc}K_c, \]

\[ [K_a, K_b] = -\frac{2i}{m}\varepsilon_{abc}L_cH. \] (5)

Changing \(H\) in \((4)\) by its eigenvalue \(E\) we obtain the Lie algebra isomorphic to \(so(4)\) provided \(E < 0\) or to \(o(1,3)\) if \(E\) is positive.

It is the hidden symmetry w.r.t. group \(so(4)\) which causes the degeneration of the spectrum of hamiltonian \((1)\) w.r.t. the orbital quantum number \(l\). This symmetry also causes the maximal superintegrability of the Hydrogen atom. Moreover, we can find the eigenvalues \(E\) of hamiltonian \((1)\) algebraically. Let us present this well known procedure which will be used in the following for another models.

First we rescale \(\mathbf{K}\) and consider the following vectors:

\[ \mathbf{K}' = \sqrt{-\frac{m}{2E}}\mathbf{K}, \quad \mathbf{g} = \frac{1}{2}(\mathbf{L} + \mathbf{K}'), \quad \mathbf{q} = \frac{1}{2}(\mathbf{L} - \mathbf{K}'). \] (6)

Their components satisfy the following commutation relations

\[ [g_a, g_b] = i\varepsilon_{abc}g_c, \quad [q_a, q_b] = i\varepsilon_{abc}q_c, \quad [g_a, q_b] = 0 \] (7)

and so form a basis of the Lie algebra \(so(4)\cong su(2)\oplus su(2)\). The related Casimir operators are

\[ C_- = 4q^2, \quad C_+ = 4g^2. \] (8)
On the other hand, in accordance with (6),

\[ C_\pm = L^2 + \nu^2 K^2 \pm 2\nu L \cdot K \]  

(9)

where \( \nu = \sqrt{-\frac{m}{2E}} \). Since \( L \cdot K \equiv 0 \), in our case \( C_+ = C_- = 4q^2 = 4g^2 \).

In accordance with (7) components \( q_a \) of vector \( q \) form a basis of algebra \( su(2) \). Its irreducible representations are labeled by integers of half integers \( q \). The same is true for vector \( g \). The corresponding eigenvalues \( c_\pm \) of the Casimir operator \( C_\pm \) are:

\[ c_- = 4q(q + 1), \quad c_+ = 4g(g + 1) \]  

(10)

where \( q \) and \( g \) are non-negative integers or half integers.

As it follows from definition (3)

\[ K^2 = \alpha^2 + (L^2 + 1) \frac{2H}{m} \]  

(11)

and so equation (9) gives: \( C_1 = -1 - \frac{a^2 m}{2E} \). Comparing that with (10) one obtains

\[ E = -\frac{ma^2}{2n^2}, \quad n = 2q + 1 = 1, 2, ... \]  

(12)

Thus, using the commutativity of hamiltonian (1) with LRL vector, it is possible to obtain the energy levels of Hydrogen atom algebraically.

**III. SUPERINTEGRABLE SYSTEMS WITH ARBITRARY SPIN**

The model hamiltonian (1) ignores the spin of the orbital electron which in fact is treated as a scalar particle. To introduce a spin it is necessary to change the angular momentum \( L \) by the total angular momentum

\[ L \rightarrow J = L + S \]  

(13)

where \( S \) is the spin vector whose components are matrices satisfying the following relations:

\[ [S_a, S_b] = i\varepsilon_{abc}S_c, \quad S_1^2 + S_2^2 + S_3^2 = s(s + 1)I \]  

(14)

where \( I \) is the unit matrix.

To obtain the corresponding Runge-Lenz vector we change in (3) \( L \rightarrow J \) and \( V \rightarrow \hat{V} \) where \( \hat{V} \) is an unknown potential. As a result we obtain:

\[ \hat{K} = \frac{1}{2m}(p \times J - J \times p) + x\hat{V}. \]  

(15)
By definition vector (15) should commute with the hamiltonian

$$\hat{H} = \frac{p^2}{2m} + \hat{V}. \quad (16)$$

It is the case iff potential $\hat{V}$ satisfies the following conditions:

$$[\hat{V}, \mathbf{J}] = 0, \quad (17)$$

$$\mathbf{x} \cdot \nabla \hat{V} + \hat{V} = 0, \quad (18)$$

$$\mathbf{S} \times (\nabla \hat{V}) - (\nabla \hat{V}) \times \mathbf{S} = 0 \quad (19)$$

where $\cdot$ and $\times$ are the symbols of the scalar and vector products and $\nabla$ is the gradient vector.

If conditions (17) – (19) are satisfied, then operators $\mathbf{J}$, $\hat{K}$ and $\hat{H}$ satisfy the same relations as the orbital momentum, LRL vector and hamiltonian of the Hydrogen atom, i.e.,

$$[J_a, H] = [\hat{K}_a, H] = 0, \quad (20)$$

$$[J_a, J_b] = i\varepsilon_{abc}J_c, \quad [\hat{K}_a, J_b] = i\varepsilon_{abc}\hat{K}_c, \quad (21)$$

$$[\hat{K}_a, \hat{K}_b] = -\frac{2i}{m}\varepsilon_{abc}J_c H.$$

Let us calculate potential $\hat{V}$ for arbitrary spin. In accordance with (17) $\hat{V}$ should be a scalar and so it can be expanded via the complete set of projection operators:

$$\hat{V} = \sum_{\nu=-s}^{s} f_\nu \Lambda_\nu \quad (22)$$

where $f_\nu$ are functions of $r$ and $\Lambda_\nu$ are projectors onto the eigenvector space of matrix $\mathbf{S} \cdot \mathbf{n}$, $\mathbf{n} = \frac{\mathbf{x}}{r}$, corresponding to the eigenvalue $\nu$ ($\nu, \nu' = s, s-1, \ldots, -s$):

$$\Lambda_\nu = \prod_{\nu' \neq \nu} \frac{\mathbf{S} \cdot \mathbf{n} - \nu'}{\nu - \nu'} \quad (23)$$

Matrices $\Lambda_\nu$ satisfy the standard projector properties

$$\Lambda_\nu \Lambda_{\nu'} = \delta_{\nu\nu'}\Lambda_\nu, \quad \sum_{\nu=-s}^{s} \Lambda_\nu = \mathbf{I}, \quad \mathbf{S} \cdot \mathbf{n} = \sum_{\nu=-s}^{s} \nu \Lambda_\nu$$

and form a very convenient basis for expansion of other scalar matrices.

Operators (22) satisfy condition (18) iff functions $f_\nu(x)$ are proportional to the inverse radius:

$$f_\nu(x) = \frac{c_\nu}{x} \quad (24)$$
where $c_{\nu}$ are constants.

Let us substitute (22) and (24) into the remaining equation (18). To calculate the gradients of the projection operators we use the following relations [25], [26]

$$\nabla \Lambda_{\nu} = \frac{i}{2x} n \times S (2\Lambda_{\nu} - \Lambda_{\nu+1} - \Lambda_{\nu-1}) - \frac{1}{2x} (S - n(S \cdot n)) (\Lambda_{\nu+1} - \Lambda_{\nu-1}) .$$

(25)

As a result, equating coefficients for linearly independent terms, we obtain the following conditions for coefficients $c_{\nu}$:

$$\nu c_{\nu} = (\nu + 1) c_{\nu+1}, \quad \nu = -s, -s + 1, ..., s .$$

(26)

If spin is integer then $\nu$ can take zero value, while for half-integer spins all $\nu$ are nonzero. Thus the general solutions of equations (26) are:

$$c_{0} = \alpha, \quad c_{\nu} = 0, \quad \text{if} \quad \nu \neq 0$$

(27)

if spin is integer, and

$$c_{\nu} = \frac{\alpha}{\nu}$$

(28)

for half integer spin. Here $\alpha$ is an arbitrary real parameter.

Substituting (24), (27), (28) into (22) we obtain explicit forms of potentials for arbitrary spins:

$$\hat{V} = \frac{\alpha}{x} \hat{\Lambda} = \frac{\alpha}{x} \Lambda_{0} \quad \text{for integer spins,}$$

(29)

$$\hat{V} = \frac{\alpha}{x} \hat{\Lambda} = \frac{\alpha}{x} \sum_{\nu = -s}^{s} \frac{1}{\nu} \Lambda_{\nu} \quad \text{for half integer spins.}$$

(30)

Formulae (16), (29) and (30) give the general form of hamiltonians of neutral particles with arbitrary spin $s$, which commute with the generalized LRL vector (15). Notice that matrices $\hat{\Lambda}$ solve the following equations:

$$\hat{\Lambda} S \cdot n = 0$$

for integer $s$ and

$$\hat{\Lambda} S \cdot n = I$$

for $s$ half integer, where $I$ is the unit matrix of dimension $(2s + 1) \times (2s + 1)$. 

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IV. IRREDUCIBILITY CONDITIONS

By construction, operators (15) where $\hat{V}$ are potentials (29) or (30) satisfy commutation relations (20) and (21). Thus all obtained hamiltonians admit a hidden symmetry w.r.t. algebra so(4), which make it possible to find their spectra without solving the Schrödinger equation.

We will consider eigenvalue problems for hamiltonians (16) which include potentials (29) and (30):

$$\hat{H}\psi \equiv \left(\frac{p^2}{2m} + \frac{\alpha}{x}\Lambda\right)\psi = E\psi.$$  \hspace{1cm} (31)

The eigenvectors $\psi = \psi(x)$ are supposed to be normalizable and vanishing at $x = 0$.

It is convenient to rescale independent variables making the change

$$x \to r = \sqrt{-2mEx}.$$  \hspace{1cm} (32)

As a result we transform (55) to the following form:

$$(1 - \Delta)\psi = \frac{2k}{r}\Lambda\psi$$ \hspace{1cm} (33)

where $\Delta = \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + \frac{\partial^2}{\partial r_3^2}$, $\hat{n} = n = \frac{x}{r}$, $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$, and

$$k = \alpha \nu = \alpha \sqrt{\frac{m}{-2E}}.$$ \hspace{1cm} (34)

In analogy with (6) we introduce two commuting vectors $\hat{q}$ and $\hat{g}$ such that

$$J = \hat{q} + \hat{g}, \quad \nu\hat{K} = \hat{q} - \hat{g}.$$  \hspace{1cm} (35)

Consider now additional conditions which can be imposed on $\psi$ thanks to the hidden symmetry of equation (33) w.r.t. algebra so(4). First we notice that the Casimir operators (8) for arbitrary spin take the following form (compare with (9)):

$$C_\pm = \mathbf{J}^2 + \nu^2\hat{K}^2 \pm 2\nu\mathbf{J} \cdot \hat{K},$$ \hspace{1cm} (36)

moreover,

$$\mathbf{J} \cdot \hat{K} = \begin{cases} 0 & \text{for integer } s, \\ \alpha sI & \text{for half integer } s. \end{cases}$$ \hspace{1cm} (37)
For irreducible representations of algebra \( \text{so}(4) \) the Casimir operators (36) should be proportional to the unit matrix, and their eigenvalues are given by equation (10). Thus, in addition to the Schrödinger equation (31) with Hamiltonian (16), we can impose two additional constrains on the wave function \( \psi \):

\[
C_- \psi = 4q(g+1)\psi, \quad C_+ \psi = 4g(g+1)\psi
\]

(38)

where \( C_- \) and \( C_+ \) are given by equations (36), \( q \) and \( g \) are arbitrary integers. Substituting (36) and (13), (15) into (38) and using equation (33) we obtain the following systems:

\[
\left( (\mathbf{S} \cdot \tilde{\nabla})^2 + \frac{k}{r}(\mathbf{S} \cdot \mathbf{J} + \Lambda \mathbf{S} \cdot \mathbf{J}) + k^2 \Lambda^2 \right) \psi = \omega_{\pm} \psi
\]

(39)

where \( \tilde{\nabla} \) is the gradient vector for rescaled variables \( \tilde{r} \),

\[
\omega_+ = \omega_- = (2g+1)^2
\]

(40)

for integer \( s \), and

\[
\omega_+ = (2g+1)^2 - 2sk, \quad \omega_- = (2q+1)^2 + 2sk
\]

(41)

for \( s \) half integer.

In the following sections particular cases of Hamiltonians \( \hat{H} \) corresponding to \( s = \frac{1}{2}, 1 \) and \( \frac{3}{2} \) are considered in more detail. We will see that the compatibility condition for equations (33) and (39) fixes the Hamiltonian spectrum.

V. LAPLACE-RUNGE-LENZ VECTOR FOR SPIN \( \frac{1}{2} \)

For the case \( s = \frac{1}{2} \) we have \( \mathbf{S} = \frac{1}{2}\sigma \), components of vector \( \sigma \) are Pauli matrices. The corresponding potential (30) is reduced to the following form:

\[
\hat{V} = \alpha \frac{\sigma \cdot \mathbf{x}}{x^2}
\]

(42)

while the related total orbital momentum (13) and LRL vector (15) can be written as:

\[
\mathbf{J} = \mathbf{L} + \frac{1}{2}\sigma,
\]

\[
\hat{\mathbf{K}} = \frac{1}{2m}(\mathbf{p} \times \mathbf{J} - \mathbf{J} \times \mathbf{p}) + \alpha \frac{\sigma \cdot \mathbf{x}}{x^2}.
\]

(43)
In analogy with section 2 it is possible to find eigenvalues of hamiltonian (16), (42) algebraically, without solving the corresponding equation (31). In accordance with (43), the square of the generalized LRL vector takes the following form:

\[ \hat{K}^2 = \left( 2J^2 + \frac{3}{2} \right) \frac{H}{m} + \alpha^2. \]

Substituting this expression into equation (36) and setting \( H = E \) we obtain:

\[ C_\pm = \left( \nu^2 \alpha^2 \pm \nu \alpha - \frac{3}{4} \right) I \]  

(44)

where \( \nu = \sqrt{-\frac{m}{2E}}. \)

It follows from (44) that for \( s = \frac{1}{2} \) equation (39) is reduced to an algebraic constraint. Indeed, in irreducible representations of algebra \( \text{su}(2) \) \( C_\pm = c_\pm I \) where \( c_\pm \) are given by equation (10). Thus we have algebraic conditions

\[ (\nu \alpha - 1/2)^2 = (2q + 1)^2, \quad (\nu \alpha + 1/2)^2 = (2g + 1)^2 \]

for spectral parameter \( \nu \), and so the eigenvalues \( E \) for coupled states can be represented in the following form:

\[ E = -\frac{ma^2}{2N^2} \]  

(45)

where

\[ N = n + 1/2, \quad n = 2q + 1 = 2g = 1, 2, \ldots \]  

(46)

Thus the energy spectrum for the system with spin \( s = \frac{1}{2} \) is given by relation (45) which is rather similar to the Balmer formula. However, in contrast to the Hydrogen atom, the main quantum number \( N \) should be half integer.

For a fixed \( N \) the integrals of motion (43) generate an irreducible representation \( D(l_0, l_1) \) of algebra \( \text{so}(4) \), where

\[ l_0 = \frac{1}{2}, \quad l_1 = N \]

are the Gelfand-Tsetlin numbers which can expressed via \( q \) and \( g \) as \( l_0 = g - q, \ l_1 = g + q + 1 \). This statement is a direct consequence of equations (46).

The spectrum (45) and the corresponding eigenvectors were found in paper [10] using shape invariance of hamiltonian (16), (42). Let us present a more straightforward way to obtain them.
Consider the eigenvalue problem for the Hamiltonian including potential (42):
\[
\left( \frac{p^2}{2m} + \alpha \frac{\sigma \cdot x}{x^2} \right) \psi = E \psi.
\] (47)

Rescaling independent variables in accordance with (32) we transform (47) to the following form:
\[
(1 + \tilde{p}^2) \psi = \frac{2k}{r} \Lambda \psi
\] (48)

where \( \Lambda = \sigma \cdot \tilde{n}, \tilde{n} = n = \frac{\xi}{r} \) and \( \tilde{p} = \sqrt{-2mE} p. \)

Multiplying (48) by \( \sigma \cdot r \) we obtain:
\[
\sigma \cdot r (1 + \tilde{p}^2) \psi = 2k \psi.
\] (49)

Considering (49) in the momentum representation we come to the system of the first order equations:
\[
i \sigma \cdot \nabla_\rho \Phi = \frac{2k}{1 + \tilde{p}^2} \Phi
\] (50)

where \( \phi = (1 + \tilde{p}^2) \psi \) and \( \nabla_\rho \) is the gradient operator.

Introducing the spherical variables and expanding solutions of equation (50) via spherical harmonics:
\[
\Phi = \frac{1}{p} \sum_{j, \lambda, \kappa} \phi_{j, \lambda, \kappa}(p) \Omega_{j, j-\frac{1}{2}, \kappa}(\varphi, \theta), \quad \lambda = \pm \frac{1}{2}, \quad \kappa = -j, -j + 1, \ldots, j,
\]
\[
\Omega_{j, j-\frac{1}{2}, \kappa} = \left( \sqrt{\frac{j+\kappa}{2j}} Y_{j-\frac{1}{2}, \kappa+\frac{1}{2}} \right), \quad \Omega_{j, j+\frac{1}{2}, \kappa} = \left( -\sqrt{\frac{j-\kappa+1}{2j+2}} Y_{j+\frac{1}{2}, \kappa+\frac{1}{2}} \right)
\] (51)

where \( Y_{j, \pm \frac{1}{2}, k, \pm \frac{1}{2}} \) are spherical functions, we come to the following system of radial equations:
\[
i \left( \frac{\partial}{\partial p} - \frac{2j + 1}{2p} \right) \phi_{-\frac{1}{2} j, \kappa} = \frac{2k}{1 + p^2} \phi_{-\frac{1}{2} j, \kappa},
\]
\[
i \left( \frac{\partial}{\partial p} + \frac{2j + 1}{2p} \right) \phi_{\frac{1}{2} j, \kappa} = \frac{2k}{1 + p^2} \phi_{\frac{1}{2} j, \kappa}.
\]

This system is solved by the following functions:
\[
\phi_{-\frac{1}{2} j, \kappa} = 2k (j + 1) p^{j+\frac{3}{2}} (1 + p^2)^{-k} F(-k + 1, -k + j + 1, j + 2, -p^2),
\]
\[
\phi_{\frac{1}{2} j, \kappa} = i (p^2 + 1)^{-k} \left( \left( p^{j+\frac{3}{2}} + p^{j+\frac{5}{2}} \right) (k - 1) (j + 1 - k) F(2 - k, +2 + j - k, 3 + j, -p^2) + (j + 2) (j + 1 - k) p^{j+\frac{3}{2}} (j + 1) p^{j+\frac{5}{2}} F(-k + 1, -k + j + 1, j + 2, -p^2) \right)
\]

where \( F(\cdot, \cdot, \cdot, -p^2) \) are hypergeometric functions. These solutions are square integrable provided \( k = j + 1 + n, n = 0, 1, 2, \ldots \), which is in a good accordance with equations (45) and (34).
VI. THE SYSTEM WITH SPIN 1

A. Hamiltonian and other integrals of motion

Let us consider the LRL vector for spin 1. It is defined by equations (15) and (13) where \( S = (S_1, S_2, S_3) \) is a matrix vector whose components can be chosen in the following form:

\[
S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\] (52)

In accordance with (22), (23) and (29) the corresponding potential in (15) can be given by the following formula:

\[ \hat{V} = \frac{\alpha}{x}(1 - (S \cdot n)^2). \] (53)

Thus the system with spin \( s = 1 \) which admits the LRL vector is specified by the following hamiltonian

\[ H = \frac{p^2}{2m} + \frac{\alpha}{x}(1 - (S \cdot n)^2). \] (54)

Let us consider the eigenvalue problem for hamiltonian (54):

\[ \left( \frac{p^2}{2m} + \frac{\alpha}{x}(1 - (S \cdot n)^2) \right) \psi = E \psi. \] (55)

Making the change (32) we can transform (55) to the standard form (33) where

\[ \hat{\Lambda} = 1 - (S \cdot \tilde{n})^2. \] (56)

The corresponding system (33) includes three coupled equations and so is rather complicated. Nevertheless it admits a sufficient number of integrals of motion to be integrated explicitly and in closed form. These integrals of motion are the LRL vector and total angular momentum given by equations (15), (13) where \( S \) and \( \hat{V} \) are matrices (52) and potential (53).

Thus we can add to equation (55) the condition (39). This system includes two arbitrary parameters, i.e., \( k \) and \( q \); the latter can take integer or half integer values. We will see that the compatibility condition for equations (55) and (39) is nothing but an algebraic relation for these parameters, which defines possible values of \( E \) in equation (55).
B. Separation of variables and energy spectrum

Let us start with equation (33). Exploiting its invariance w.r.t. the rotation group whose generators are given by equations (13) and (52) it is possible to separate variables. To do it we introduce the spherical variables

\[ x_1 = x \cos \varphi \sin \theta, \quad x_2 = x \sin \varphi \sin \theta, \quad x_3 = x \cos \theta \]  

and expand \( \psi = \psi(x) \) via spherical harmonics \( \Omega_{j,\kappa,\lambda}^s \) which are eigenvectors of the commuting integrals of motion \( J_3, J^2 \) and \( L^2 \) (see Appendix 1):

\[ \psi = \frac{1}{x} \sum_{j,\kappa,\lambda} \psi_{j,\kappa,\lambda}(x) \Omega_{j,\kappa,\lambda}^s(\varphi, \theta). \]

(58)

Here \( s = 1, j = 0, 1, \ldots, \kappa = -j, -j + 1, \ldots j, \lambda = 1, 0, -1. \)

In the spherical harmonics basis, operator \( L^2 \) and matrix \( S \cdot n \) are reduced to the following matrices (see Appendix): \( L^2 \to \hat{L}^2, \quad S \cdot n \to \hat{S}_1 \) where

\[ L^2 = \begin{pmatrix} j(j-1) & 0 & 0 \\ 0 & j(j+1) & 0 \\ 0 & 0 & (j+1)(j+2) \end{pmatrix}, \quad \hat{S}_1 = -\frac{1}{\sqrt{2j+1}} \begin{pmatrix} 0 & \sqrt{j+1} & 0 \\ \sqrt{j+1} & 0 & \sqrt{j} \\ 0 & \sqrt{j} & 0 \end{pmatrix}. \]

(59)

Substituting (58) and (59) into (55) and using variables (32) we obtain the following equation for radial functions:

\[ \frac{\partial^2}{\partial r^2} \Phi = \left( 1 + \frac{L^2}{r^2} - \frac{2k}{r} \tilde{\Lambda} \right) \Phi \]

(60)

where \( \tilde{\Lambda} = 1 - \hat{S}_1^2 \) and

\[ \Phi = \text{column} (\Phi_-, \Phi_0, \Phi_+), \quad \Phi_- = \psi_{j,\kappa,-1}, \quad \Phi_0 = \psi_{j,\kappa,0}, \quad \Phi_+ = \psi_{j,\kappa,1}. \]

(61)

Consider now equations (39). It follows from (36) and (37) that in our case \( g = q \) and so we have the only system. It is invariant w.r.t. the rotation group and admits separation of variables in the spherical coordinates. Making changes (57), (58), using identities (A5), (A6) and applying expressions (60) for the second order derivatives we obtain a system of first order radial equations in the following form:

\[ R \frac{\partial \Phi}{\partial x} = \left( \frac{M}{r} - kN + Kr \right) \Phi \]

(62)
where
\[
R = i\{\hat{S}_1, \hat{S}_2\} = i[\hat{S}_1^2, \mathbf{S} \cdot \mathbf{J}], \quad M = \frac{1}{2} \hat{S}_1^2 \mathbf{L}^2 - \hat{S}_2^2 + i\hat{S}_1 \hat{S}_2, \\
N = \{\mathbf{S} \cdot \mathbf{J}, \tilde{\Lambda}\}, \quad K = (2q + 1)^2 - k^2 \tilde{\Lambda} - \hat{S}_1^2
\]
and \(\hat{S}_2 = i[\hat{S}_1, \mathbf{S} \cdot \mathbf{J}]\). Here [..] and {..} denote commutator and anticommutator respectively.

Using their definitions, all matrices (63) can be found in the explicit forms:

\[
R = \mu \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M = -\mu \begin{pmatrix} 0 & 0 & j + 1 \\ 0 & 0 & 0 \\ j & 0 & 0 \end{pmatrix}, \quad N = \frac{\mu}{2j + 1} \begin{pmatrix} 2\mu & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -2\mu \end{pmatrix}, \\
K = ((2q + 1)^2 - 1)I + (1 - k^2)\tilde{\Lambda}, \quad \tilde{\Lambda} = \frac{1}{2j + 1} \begin{pmatrix} j & 0 & -\mu \\ 0 & 0 & 0 \\ -\mu & 0 & j + 1 \end{pmatrix}
\]

where \(I\) is the unit matrix, and \(\mu = \sqrt{j(j + 1)}\).

It follows from (62), (64) that if \(q \neq 0\) then \(\Phi_0 = 0\), and equation (62) is reduced to the following form:

\[
\left( \frac{\partial}{\partial r} - W \right) \tilde{\Phi} = 0
\]

where \(\tilde{\Phi} = \text{column}(\Phi_- , \Phi_+)\) and \(W\) is a matrix superpotential

\[
W = \tilde{M} \frac{1}{r} - k \tilde{N} + \tilde{K}r
\]

with

\[
\tilde{M} = \begin{pmatrix} j & 0 \\ 0 & -(j + 1) \end{pmatrix}, \quad \tilde{N} = \frac{1}{2j + 1} \begin{pmatrix} 1 & 2\mu \\ 2\mu & -1 \end{pmatrix},
\]

\[
\tilde{K} = 4q(q + 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{k^2 - 1}{\mu(2j + 1)} \begin{pmatrix} -\mu & j + 1 \\ -j & \mu \end{pmatrix}.
\]

A differential consequence of (65) is the following second-order equation:

\[
\left( -\frac{\partial^2}{\partial r^2} + W^2 - W' \right) \tilde{\Phi} = 0
\]
where we denote for short $W' = \frac{\partial W}{\partial r}$.

In order equations (60) and (67) to be compatible the matrix $\hat{K}$ should be nilpotent, i.e., $\hat{K}^2 = 0$. This condition generates the following algebraic constrains on parameters $q$ and $k$:

$$k^2 = (2q + 1)^2$$  \hspace{1cm} (68)

or, alternatively, $q = 0$.

Let condition (68) is satisfied, then, in accordance with (34), the admissible energy values for the coupled states are given by formula (12). The total angular momentum $J$ is a sum of two commuting angular momenta $\hat{q}$ and $\hat{g}$ (see (35)), both of which realize irreducible representation $D(q)$ of algebra $so(3)$. Thus, the admissible values of $j$ are $2q, 2q - 1, 2q - 2, ..., 0$, and it is possible to represent the related energy values as

$$E = -\frac{m\alpha^2}{2(n + j + 1)^2}, \hspace{1cm} n = 0, 1, 2, ...$$  \hspace{1cm} (69)

Moreover, the corresponding LRL vector (15), (53) and total orbital momentum (13) realize representations $D(0, N)$ of algebra $so(4)$, where $N = n + j + 1$.

Since $g = q$, the condition $q = 0$ corresponds to the trivial realization $D(1, 0)$ with $N = 1$, $j = 0$ and $E = -\frac{m\alpha^2}{2}$.

C. Hamiltonian eigenvectors

Thus we obtain the spectrum of hamiltonian (55) in the form (12) without solving the corresponding Schrödinger equation. To find this spectrum it was sufficient to ask for the compatibility of equations (33) and (39).

Let us find the corresponding eigenvectors of hamiltonian (55). Instead of the system of second order equations (33) it is sufficient to solve the first order equations (65) which are reduced to the following system:

$$\Phi'_- = \frac{j}{r} \Phi_- - \frac{k}{2j + 1} (\Phi_- + 2\mu \Phi_+) - \frac{(k^2 - 1)r}{2j + 1} (j \Phi_+ + \mu \Phi_-),$$

$$\Phi'_+ = -\frac{j + 1}{r} \Phi_+ + \frac{k}{2j + 1} (\Phi_+ - 2\mu \Phi_-) + \frac{(k^2 - 1)r}{\mu(2j + 1)} ((j + 1) \Phi_- + \mu \Phi_+).$$  \hspace{1cm} (70)

This system can be simplified by the following change of dependent variables:

$$\Phi_- = \frac{1}{\sqrt{2j + 1}} \left( \sqrt{j + 1} \tilde{\Phi}_- + \sqrt{j} \tilde{\Phi}_+ \right), \hspace{1cm} \Phi_+ = \frac{1}{\sqrt{2j + 1}} \left( \sqrt{j + 1} \tilde{\Phi}_+ - \sqrt{j} \tilde{\Phi}_- \right)$$  \hspace{1cm} (71)
As a result we obtain:

\[ \tilde{\Phi}_+ = -\frac{r}{\mu} (\tilde{\Phi}'_+ + k \tilde{\Phi}_-), \]
\[ \tilde{\Phi}'_+ = k \tilde{\Phi}_+ - \frac{1}{r} \tilde{\Phi}'_+ - \frac{\mu}{r} \tilde{\Phi}_- + \frac{k^2 - 1}{\mu} \tilde{\Phi}_-. \]  

(72)  

(73)

Substituting (72) into (73) we obtain the following equation:

\[ -r^2 \tilde{\Phi}'_+ - 2r \tilde{\Phi}'_+ + (j(j+1) - 2kr + r^2) \tilde{\Phi}_- = 0 \]

This equation is solved by the following function

\[ \tilde{\Phi}_- = C_{kj} r^j \exp(-r) F(j+1-k, 2j+2, 2r) \]  

(74)

where \( F \) is the confluent hypergeometric function and \( C_{kj} \) is an integration constant. The corresponding function \( \tilde{\Phi}_+ \) is defined by equation (72) and is easy calculated:

\[ \tilde{\Phi}_+ = -\frac{1}{\mu} C_{kj} r^j \exp(-r) \left( ((k-1)r+j+1)F(j+1-k, 2j+2, 2r) + \frac{j+1-k}{j+1} rF(j+2-k, 2j+3, 2r) \right). \]  

(75)

Functions (74) and (75) are square integrable provided the first argument of the hypergeometric functions is a negative integer. Moreover, thanks to the multiplier \( j+1-k \) in the last line of equation (75) the admissible values of \( k \) and \( j \) are: \( k = j+1+n, \quad n = 0, 1, \ldots \). Then formula (34) gives the energy values (69) for coupled states, which is in a good accordance with the discussion presented in the previous section. The corresponding eigenvectors of hamiltonian (54) are given by equations (58) and (61) where, in accordance with (71), (74) and (75),

\[ \psi_{j\kappa-1} = C_{jn} \sqrt{j+r^j+1} \exp(-r) \left( (j+n)(j+1)F(-n, 2j+2, 2r) - nF(1-n, 2j+3, 2r) \right), \]
\[ \psi_{j\kappa+1} = C_{jn} \sqrt{j+1+r^j} \exp(-r) \left( nrF(1-n, 2j+3, 2r) - (j+1)(2j+1+(n+j)r)F(-n, 2j+2, 2r) \right), \quad \psi_{j\kappa0} = 0. \]

VII. THE SYSTEM WITH SPIN $\frac{3}{2}$

Consider the LRL vector for spin $\frac{3}{2}$. The corresponding spin matrices are:

\[ S_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad S_2 = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \]
and

\[ S_3 = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \]

The related hamiltonian (16) takes the following form:

\[ H = \frac{p^2}{2m} + \frac{\alpha}{x} \hat{\Lambda}, \quad \hat{\Lambda} = \frac{4}{3} \left( 5S \cdot n - 2(S \cdot n)^3 \right). \]  

(76)

The eigenvalue problem for this hamiltonian in rescaled variables (32) is reduced to the standard form (33) where \( \hat{\Lambda} \) is the matrix defined in (76). By construction, this equation admits six integrals of motion which are the components of total orbital momentum \( J \) and LRL vector \( \mathbf{K} \). Thus, like in sections 5 and 6.1, we can search for values of the spectral parameter \( k \) algebraically.

The eigenvalue problems for the Casimir operator \( C_\pm \) admit the uniform representation (39), (41) where \( s = \frac{3}{2} \). Introducing the spherical variables and expanding solutions of the related equations (33) and (39) via spherical spinors like in (58), we obtain the following systems of equations for radial functions:

\[ \Phi''_1 = \Phi_1 + \frac{(2j-1)(2j-3)}{4r^2} \Phi_1 - \frac{2k}{\mu r} (\sqrt{3}j \Phi_3 - \delta \Phi_4), \]

\[ \Phi''_2 = \Phi_2 + \frac{(2j+1)(2j+3)}{4r^2} - \frac{2k(j+1)\sqrt{3}}{\mu r} \Phi_4, \]

\[ \Phi''_3 = \Phi_3 + \frac{(2j+1)(2j-1)}{4r^2} \Phi_3 - \frac{2kj\sqrt{3}}{\mu r} \Phi_1, \]

\[ \Phi''_4 = \Phi_4 + \frac{(2j+3)(2j+5)}{4\mu r} \Phi_4 + \frac{2k}{\mu r} (\delta \Phi_1 - (j+1)\sqrt{3} \Phi_2) \]

and

\[ \Phi'_1 = \frac{(2j-1)}{2r} \Phi_1 - \frac{k}{\mu} \left( \delta \Phi_4 + \sqrt{3} \Phi_3 \right) + r \sum_c R_{1c} \Phi_c, \]

\[ \Phi'_2 = -\frac{(2j+1)}{2r} \Phi_2 - \frac{k}{\mu} \left( \delta \Phi_3 - \sqrt{3} \Phi_4 \right) + r \sum_c R_{2c} \Phi_c, \]

\[ \Phi'_3 = \frac{(2j+1)}{2r} \Phi_3 - \frac{k}{\mu} \left( \delta \Phi_2 + \sqrt{3} \Phi_1 \right) + r \sum_c R_{3c} \Phi_c, \]

\[ \Phi'_4 = -\frac{(2j+3)}{2r} \Phi_4 - \frac{k}{\mu} \left( \delta \Phi_1 - \sqrt{3} \Phi_2 \right) + r \sum_c R_{4c} \Phi_c \]  

(78)
where we denote for short $\psi_{j,\kappa,\lambda}(r) = \text{column}(\Phi_1, \Phi_3, \Phi_2, \Phi_4)$, and

$$R_{11} = -R_{22} = \frac{1 - 4k^2}{2j\delta}, \quad R_{33} = -R_{44} = \frac{1 - 4k^2}{2(j+1)\delta},$$

$$R_{12} = \frac{1}{j} (12k^2(j+1) - 4j\omega_\pm - 3 + 7j),$$

$$R_{21} = \frac{1}{j} (4\omega_\pm j - 4k^2(7j - 3) - 3j - 3),$$

$$R_{34} = \frac{1}{j+1} (4k^2(7j + 10) + 3j - 4 - 4\omega_\pm(j + 1)),$$

$$R_{43} = \frac{1}{j+1} (4\omega_\pm(j + 1) - 12jk^2 - 7j - 10).$$  

(79)

The remaining matrix entries $R_{nc}$ ($n, c = 1, 2, 3, 4$) in (78) are equal to zero.

A necessary condition of compatibility of equations (77)--(78) is $R^2 = 0$ where $R$ is a matrix whose non-zero entries are given in (79). This condition (which appears to be also sufficient) is reduced to the following constraints for the spectral parameter $k$:

$$k^2 = \omega_\pm - \frac{9}{4}$$  

(80)

or, alternatively,

$$(3k)^2 = \omega_\pm - \frac{1}{4}.$$  

(81)

Using definitions (41) for $\omega_\pm$ we rewrite conditions (80) and (81) as:

$$k = n + \frac{3}{2}, \quad n = 2q + 1 = 1, 2, \ldots, g = q + \frac{3}{2}$$  

(82)

and

$$k = \frac{1}{3} \left( n + \frac{1}{2} \right), \quad n = 2q + 1 = 1, 2, \ldots, g = q + \frac{1}{2}$$  

(83)

respectively.

In accordance with (34), the corresponding energy levels have the form (45) with

$$N = n + \frac{3}{2} = \tilde{n} + j, \quad j = \frac{3}{2}, \frac{5}{2}, \ldots, \tilde{n} = 1, 2, \ldots$$  

(84)

for the case (82) or, alternatively,

$$E = -\frac{9m^2\alpha^2}{2N^2}, \quad N = n + \frac{1}{2} = \tilde{n} + j, \quad \tilde{n} = 1, 2, \ldots$$  

(85)

for the case (83). In the latter case $j$ can take any half integer value including $j = \frac{1}{2}$. The eigenvectors corresponding to the presented eigenvalues can be expressed via solutions of the
fourth order scalar ordinary equations, which we obtain by excluding three out of four dependent
variables from the first order system (78), see Appendix B for detailed calculations.

Thus there are two possible spectrum branches which are generated by the compatibility
condition for the systems (77) and (78). They can be found algebraically using rather simple
calculations. However, to calculate the corresponding eigenvectors is not an easy job, see
Appendix B.

Using (84) and (85) it is possible to specify the irreducible representations of algebra so(4)
which are generated by the total momentum and LRL vectors. Namely, they are representations
$D(\frac{1}{2}, N)$ for the case (84) and $D(\frac{3}{2}, N)$ for the case (85).

VIII. A PHYSICAL INTERPRETATION

The systems discussed above describe particles with spin interacting with an external field.
Since the systems hamiltonians do not include minimal interaction terms, these particles are
neutral but can have nontrivial dipole or multipole momenta. Let us consider some representa-
tions of the considered systems and discuss their possible interpretations.

For $s = \frac{1}{2}$ the corresponding potential (42) represents the dipole interaction with a vector
external field $F$ whose components are:

$$F_a = \frac{x_a}{x^2}.$$  \hspace{1cm} (86)

Vector (86) can be interpreted as the electric field strength. Such a field can be realized
experimentally at least on the finite interval $a < x < b, a > 0$, see, e.g., problem 1018 in [27].

The potential (53) for $s = 1$ can be rewritten as:

$$\hat{V} = \hat{Q}_{ab} \frac{\partial \hat{F}_a}{\partial x_b} \equiv \frac{1}{2} Q_{ab} \frac{\partial \hat{F}_a}{\partial x_b} + \frac{1}{6} \frac{\partial F_a}{\partial x_a}$$  \hspace{1cm} (87)

where $\hat{Q}_{ab} = \frac{1}{2} (S_a S_b + S_b S_a - \delta_{ab})$, $Q_{ab} = S_a S_b + S_b S_a - \frac{4}{3} \delta_{ab}$ is the traceless quadruple inter-
action tensor and

$$\hat{F}_a = -\frac{\alpha x_a}{x}$$  \hspace{1cm} (88)

is a vector of the external field strength. In other words, this potentials represents a superpo-
sition of quadrupole and Darwin interactions of spin one particle with an external field.
Alternatively, it is possible to represent potential (53) as a nonlinear function of field (86):

\[ \hat{V} = \alpha \frac{F^2 - (S \cdot F)^2}{|F|}. \]

Let us recall that a nonlinear generalization of the Pauli interaction is also required to obtain a consistent relativistic description of spin-1 particle interacting with an external field [28].

The potential for spin \( \frac{3}{2} \) presented in equation (76) can be represented as

\[ \hat{V} = \frac{2\alpha}{9} Q_{abc} \frac{\partial^2 \tilde{F}_a}{\partial x_b \partial x_c} \]  

(89)

where

\[ \tilde{F}_a = x_a \ln x, \]  

(90)

and

\[ Q_{abc} = \sum_{P(a,b,c)} \left( S_a S_b S_c - \frac{7}{4} S_a \delta_{bc} \right) \]  

(91)

is the octupole interaction tensor. In equation (91) the summation is imposed over all possible permutations of indices \( a, b \) and \( c \).

Alternatively, this potential can be rewritten as:

\[ \hat{V} = \alpha \frac{4}{3} \left( 5S \cdot F - \frac{2(S \cdot F)^3}{|F|^2} \right) \]

where \( F \) is a vector whose components are given by equation (86).

A more difficult task is a physical interpretation of vectors (88) and (90) since the corresponding classical electric fields needs charge densities which are hardly realized experimentally. However, all vectors (86), (88) and (90) solve equations of the axion electrodynamics [29], [30], and this fact presents additional interesting possibilities for their interpretation.

IX. DISCUSSION

Thus we generalize the Fock symmetry of the nonrelativistic Hydrogen atom to the case of arbitrary spin. More exactly, for any spin \( s \) we specify a QM system which admits two vector integrals of motion which are the total orbital momentum and the generalized LRL vector with arbitrary spin. Moreover, they generate the hidden symmetry of coupled states of the specified QM systems w.r.t. group SO(4).
To construct the LRL vector for arbitrary spin we postulate its generic form (15) and find potentials \( \hat{V} \) for which such vector commutes with the corresponding hamiltonian (16). The determining equations (17)–(19) for the potential can be solved exactly for any \( s \). Moreover, their solution is unique up to a coupling constant.

The representation of the generalized LRL vectors in form (15) was chosen by analogy with the LRL for the Hydrogen atom. However, at least for \( s = 0 \) and \( s = \frac{1}{2} \) this representation is unique for all generic hamiltonians (16) with a (matrix) potential \( \hat{V} \) depending on \( x \). For a scalar hamiltonian this fact was proven, e.g., in [31] where all second order integrals of motion for 3d QM systems had been classified. For spin \( \frac{1}{2} \) the uniqueness of the generalized LRL vector was proven in paper [24] were all the related superintegrable systems which are invariant w.r.t. rotation group and admit second order integrals of motion where specified.

Like in the case of the Hydrogen atom, the generalized LRL vector can be used to find the spectrum of the corresponding hamiltonian algebraically. For \( s = \frac{1}{2}, 1 \) and \( \frac{3}{2} \) it has been done in sections 5, 6.2 and 7 correspondingly. However, except the case \( s = \frac{1}{2} \), the considered superintegrable systems with spin do not succeed one more hidden symmetry of the Hydrogen atom, i.e., the shape invariance. Indeed, the lists of shape invariant matrix potentials presented in [32] and [33] does not include effective potentials of equations (60) and (77) [41]. As a result the construction of eigenvectors of the corresponding Hamiltonians is a rather sophisticated problem. However the extended number of integrals of motion admitted by these systems makes it possible effectively simplify the calculation of eigenvectors and solve the system of the first order equations of the form (65) instead of of coupled Schrödinger equations, like it was done in sections 6.3 and Appendix 2.

It is interesting to compare the results of the present paper with ones obtained in [23] and [24] where 2d superintegrable systems with arbitrary spin which admit two dimensional analogues of the LRL vector were discussed. As it was shown in [24], the 2d systems are both superintegrable and supersymmetric. Namely, they admit 2d LRL vectors and are shape invariant. The latter circumstance makes it possible to find their exact solutions in exact form for any spin \( s \) [24].

In the 3d case considered in the present paper the situation is more complicated. The systems with spins 0 and \( \frac{1}{2} \) are shape invariant, but starting with spin 1 this property is loosed, and the procedure of constructing of exact solutions is rather sophisticated.

Comparing the system with spin 1, whose potential and hamiltonian are given by equations (53) and (54), with the corresponding 2d system discussed in [24] (see equation (4.6) there) we
conclude, that their potentials are qualitatively different. Indeed, potential \(53\) is a product of the Coulomb potential and matrix \(\hat{\Lambda} \quad (56)\) which is a projector satisfying \(\hat{\Lambda}^2 = \hat{\Lambda}\). The corresponding matrix in the 2d model is not a projector and its square is proportional to the unit matrix. Nevertheless, up to notations of the coupling constants and possible values of the summation indices, representation \((87)\) is valid for both the 3d and 2d cases.

The same is true for systems with spin \(\frac{3}{2}\). Again the 3d and 2d matrix potentials are qualitatively different since \(\hat{\Lambda}\) in \((76)\) is a matrix inverse to \(S \cdot n\) while the corresponding matrix in \([24]\) is proportional to the square root of the unit matrix. However, representation \((89)\) is valid for both the 3d and 2d systems. In addition, in both cases there are two branches of energy spectrum given by equations \((47)\), \((84)\) and \((85)\) (but in 2d case \(j\) is replaced by eigenvalues of \(J_3\)).

Thus we extend the results of papers \([23]\), \([24]\) and \([10]\) to the case of 3d systems with arbitrary spin. The next task is to find relativistic counterparts of the discussed systems like it was done in \([10]\) and \([35]\) for \(s = \frac{1}{2}\). We plane to do it using the approach developed in \([36]\) and \([37]\).

There are also other challenges connected with the presented results. Among them is a search for LRL vectors with spin in multi dimensional and curved spaces, discussion of superintegrable superpartners of the presented systems and various generalizations with quadratic algebras of higher symmetries. For a generalized MICZ-Kepler system which loses the symmetry w.r.t. algebra so(4) but is integrable thanks to the quadratic symmetry algebra see \([38]\) and \([39]\).

**Appendix A: Spherical harmonics and invariant matrices**

Here a definition of spherical harmonics and explicit forms of rotationally invariant matrices, used in the main text, are presented. They can be found, e.g., in \([40]\), but we use the notations presented in \([26]\).

Spherical harmonics are eigenvectors of the commuting operators \(J_3\), \(J^2\), \(S^2\) and operator
By definition they satisfy the following equations:

\begin{align}
J^2\Omega^s_{j,\kappa,\lambda} &= j(j+1)\Omega^s_{j,\kappa,\lambda}, \\
J_3\Omega^s_{j,\kappa,\lambda} &= \kappa\Omega^s_{j,\kappa,\lambda}, \\
L^2\Omega^s_{j,\kappa,\lambda} &= (j-\lambda)(j-\lambda+1)\Omega^s_{j,\kappa,\lambda} \\
S^2\Omega^s_{j,\kappa,\lambda} &= s(s+1)\Omega^s_{j,\kappa,\lambda}.
\end{align}
\tag{A1}

Here \( s \) are integers or half integers labeling irreducible representations of the rotation group,

\[ j = 0, 1, \ldots, \kappa = -j, -j+1, \ldots j, \lambda = -s, -s+1, \ldots, -s+2 \min(s,j) \]. \tag{A2}

The spherical harmonic can be represented as a column whose \( \mu \)-th component is given by the following equation:

\[ (\Omega^s_{j,\kappa,\lambda})_\mu = C^j_{\lambda,\mu,s,\mu,\mu}Y^j_{-\lambda,\mu,\mu,\mu}, \quad \mu = 1, 2, \ldots, 2\lambda+1 \] \tag{A3}

where \( Y \) are spherical functions and \( C \) are Wigner coefficients. In addition, the following normalization condition is satisfied:

\[ \int \Omega^s_{j,\kappa,\lambda}^\dagger \Omega^s_{j',\kappa',\lambda'}d\omega = \delta_{jj'}\delta_{\kappa\kappa'}\delta_{\lambda\lambda'} \]

where \( \delta_{jj'} \) is the Kronecker symbol and \( d\omega = \sin \theta d\theta d\phi \).

In the spherical harmonics basis the matrix \( S \cdot n \) is reduced to the following form:

\[ S \cdot n \Omega^s_{j,\kappa,\lambda} = \frac{1}{2} \left( A^s_{j,\lambda-1,\kappa,\lambda-1} + A^s_{j,\lambda+1,\kappa,\lambda+1} \right) \tag{A4} \]

where

\[ A^s_{\mu} = \left( \frac{\mu(2j+1-\mu)(2s+1-\mu)(2j+2s-2-\mu)}{(2j+2s-2\mu+1)(2j+2s-2\mu+3)} \right)^{\frac{1}{2}}. \]

The matrix \( J^2 \) is diagonal. Its entries are given by the first equation \((A1)\). The matrix \( S \cdot J \) is diagonal also, its entries are easy calculated using the identity

\[ S \cdot J = \frac{1}{2} \left( j(j+1) + s(s+1) - L^2 \right). \tag{A5} \]

We used one more rotationally invariant matrix, i.e., \( S \cdot p \), whose entries are differential operators. When acting on functions expanded via spherical harmonics basis like \((58)\) it takes the following form:

\[ S \cdot p = -iS \cdot n \frac{\partial}{\partial x} + \frac{i}{x}[S \cdot n, S \cdot L]. \tag{A6} \]

Just definition \((A3)\) for \( s = \frac{1}{2} \) and the momentum representation was used in equation \((51)\).
Appendix B: Some calculation details for spin $\frac{3}{2}$

In the case of spin $\frac{3}{2}$ we are supposed to solve three systems of equations, i.e., (77) and (78).

If condition (82) is satisfied, system (78) takes the following form:

\[
\begin{align*}
\Phi_1' &= \frac{(2j-1)}{2r} \phi_1 - \frac{k}{\mu} \left(\delta \Phi_4 + \sqrt{3} \Phi_3\right) - \frac{(4k^2-1)r}{2j} \left(\phi_1 + \frac{(2j-1)\sqrt{3}}{\delta} \Phi_2\right), \\
\Phi_2' &= -\frac{(2j+1)}{2r} \phi_2 - \frac{k}{\mu} \left(\delta \Phi_3 - \sqrt{3} \Phi_4\right) + \frac{(4k^2-1)r}{2j} \left(\phi_2 + \frac{2j+3}{\sqrt{3} \delta} \Phi_1 + \Phi_2\right), \\
\Phi_3' &= \frac{(2j+1)}{2r} \phi_3 - \frac{k}{\mu} \left(\delta \Phi_2 + \sqrt{3} \Phi_1\right) - \frac{(4k^2-1)r}{2(j+1)} \left(\phi_3 + \frac{2j-1}{\delta} \Phi_4\right), \\
\Phi_4' &= -\frac{(2j+3)}{2r} \phi_4 - \frac{k}{\mu} \left(\delta \Phi_1 - \sqrt{3} \Phi_2\right) + \frac{(4k^2-1)r}{2(j+1)} \left(\phi_4 + \frac{2j+3}{\sqrt{3} \delta} \Phi_3 + \Phi_4\right). 
\end{align*}
\]

(B1)

On the other hand, for the case (83) we obtain:

\[
\begin{align*}
\Phi_1' &= \frac{(2j-1)}{2r} \phi_1 - \frac{k}{\mu} \left(\delta \Phi_4 + \sqrt{3} \Phi_3\right) - \frac{(4k^2-1)r}{2j} \left(\phi_1 - \frac{2j+3}{\sqrt{3} \delta} \Phi_2\right), \\
\Phi_2' &= -\frac{(2j+1)}{2r} \phi_2 - \frac{k}{\mu} \left(\delta \Phi_3 - \sqrt{3} \Phi_4\right) - \frac{(4k^2-1)r}{2j} \left(\phi_2 - \frac{2j-1}{\sqrt{3} \delta} \Phi_1 - \Phi_2\right), \\
\Phi_3' &= \frac{(2j+1)}{2r} \phi_3 - \frac{k}{\mu} \left(\delta \Phi_2 + \sqrt{3} \Phi_1\right) - \frac{(4k^2-1)r}{2(j+1)} \left(\phi_3 - \frac{(2j+3)\sqrt{3}}{\delta} \Phi_4\right), \\
\Phi_4' &= -\frac{(2j+3)}{2r} \phi_4 - \frac{k}{\mu} \left(\delta \Phi_1 - \sqrt{3} \Phi_2\right) - \frac{(4k^2-1)r}{2(j+1)} \left(\phi_4 - \frac{2j-1}{\sqrt{3} \delta} \Phi_3 - \Phi_4\right). 
\end{align*}
\]

(B2)

(B3)

(B4)

Substituting (B1) and (B2) or (B3) and (B4) into (77) we turn the latter equation to identity. It means that instead of the second order system (77) with an unknown spectral parameter $k$ we can solve the first order system (B1), (B2) or (B3), (B4) with the specified $k$ given by equation (82) or (83). Let us consider these systems consequently.

Solving (B1) for $\Phi_3$ and $\Phi_4$ we obtain

\[
\begin{align*}
\Phi_3 &= \frac{1}{4k\mu} \left((4k^2-1) r \left(a^{-1} \phi_2 + \frac{1}{\sqrt{3}} \phi_1\right) - \sqrt{3} \left(\phi_1' - \frac{2j-1}{2r} \phi_1\right)\right) - \left(\phi_2 + \frac{2j+1}{2r} \phi_2\right), \\
\Phi_4 &= -\frac{1}{4k\mu} \left((4k^2-1) r \left(a \phi_1 + \sqrt{3} \phi_2\right) + \sqrt{3} \left(\phi_2' + \frac{2j+1}{2r} \phi_2\right)\right) - \left(\phi_1' - \frac{2j-1}{2r} \phi_1\right) 
\end{align*}
\]

(B5)

where $a = \sqrt{\frac{2j+3}{2j-1}}$.  

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Substituting these expressions into (B2), we obtain a system of second order equations for two unknowns \( \Phi_1 \) and \( \Phi_2 \):

\[
\Phi_1'' = \frac{1}{2jr} \left( (3 - 4j)\Phi_1' + \delta \sqrt{3}\Phi_2' \right) + \frac{1}{4jr^2} \left( \delta^2(j - 1)\Phi_1 + \sqrt{3}\delta(2j + 1)\Phi_2 \right) \\
+ \frac{1}{2j} \left( (2j - 12k^2 + 3)\Phi_1 - 3a^{-1}\sqrt{3}(1 - 4k^2)\Phi_2 \right),
\]

\[
\Phi_2'' = \frac{1}{2jr} \left( \sqrt{3}\delta\Phi_1' - 3\Phi_2' \right) + \frac{1}{4jr^2} \left( (2j + 1)(2j^2 + 3j - 3)\Phi_2 - (2j - 1)\sqrt{3}\delta\Phi_1 \right) \\
+ \frac{1}{2j} \left( (2j - 3 + 12k^2)\Phi_2 + a\sqrt{3}(4k^2 - 1)\Phi_1 \right).
\]

The change

\[
\Phi_1 = \sqrt{r} \left( a\hat{\Phi}_1 + \frac{1}{r^2}\hat{\Phi}_2 \right), \quad \Phi_2 = \sqrt{r} \left( \sqrt{3}\hat{\Phi}_1 - \frac{a}{\sqrt{3}r^2}\hat{\Phi}_2 \right)
\]

reduces (B6) to a more simple form which does not contain the first order derivatives:

\[
\hat{\Phi}_1'' = \frac{1}{4r^2} \left( (4j^2 + 6j - 1)\hat{\Phi}_1 + 3\delta\hat{\Phi}_2 \right) + \hat{\Phi}_1,
\]

\[
\hat{\Phi}_2'' = \frac{1}{4r^2} \left( (4j^2 + 2j - 3)\hat{\Phi}_2 - 3\delta\hat{\Phi}_1 \right) + \frac{12(4k^2 - 1)}{\delta}\hat{\Phi}_1 + \hat{\Phi}_2.
\]

Solving (B8) for \( \hat{\Phi}_2 \) and substituting the solution into (B9) we obtain:

\[
\hat{\Phi}_2 = \frac{1}{3\delta} \left( 4r^2(\hat{\Phi}_1'' - \hat{\Phi}_1) - (4j^2 + 6j - 1)\hat{\Phi}_1 \right)
\]

and

\[
r^4\hat{\Phi}_1^{IV} + 4r^3\hat{\Phi}_1'' - (2r^2 + 2\mu^2 - 3)r^2\hat{\Phi}_1'' - 4r^3\hat{\Phi}_1'
\]

\[
+ \left( r^4 - 2(18k^2 - \mu^2 - 3)r^2 + \mu^4 + \mu^2 + \frac{3}{2} \right) \hat{\Phi}_1 = 0.
\]

Here \( \mu \) and \( k \) are parameters given by equations (??) and (??).

Thus to find solutions of equation (??) with the additional condition (39) it is sufficient to solve (B11) which is an ordinary differential equation of fourth order. Then using definitions (B5), (B5) and (B10) it is possible to reconstruct all components of eigenvectors of hamiltonian (54). Moreover, we are interested in such solutions of (B11) which generate square integrable components given by (B5), (B5) and (B10).

Equation (B11) is much more simple than the initial system of four second order equations (77) where both the parameter \( k \) and eigenvectors \( \Psi \) are unknowns. It seems to be impossible to find its solutions explicitly. However, it is possible to evaluate them as formal series whose coefficients satisfy clear recurrence relations.
The qualitatively analysis of \((B11)\) makes it possible to prove the existence of good solutions at neighborhood of the critical point \(r = 0\) and for \(r \to \infty\). Indeed, for infinitely small and infinitely large \(r\) this equation can be reduced to the following forms:

\[
r^4 \dot{\Phi}_1^{IV} + 4r^3 \dot{\Phi}_1^{'''} - (2\mu^2 - 3)r^2 \dot{\Phi}_1^{''} - 4r^3 \dot{\Phi}_1' + \left(\mu^4 + \mu^2 - \frac{3}{2}\right) \dot{\Phi}_1 = 0. \tag{B12}
\]

and

\[
r \dot{\Phi}_1^{IV} + 4 \dot{\Phi}_1^{'''} - 4r \dot{\Phi}_1'' - 4 \dot{\Phi}_1' + r \dot{\Phi}_1 = 0. \tag{B13}
\]

correspondingly. Regular (and vanishing) at \(x = 0\) solutions of \((B12)\) are:

\[
\dot{\Phi}_1 = C_1 r^{1 + \kappa_-} \mathcal{F} \left( \frac{1}{4} + \frac{\kappa_-}{2}, 1 + \frac{\kappa_-}{2}, 1 + \frac{\kappa_- - \kappa_+}{4}, 1 + \frac{\kappa_- + \kappa_+}{4}, \frac{r^2}{2} \right) + C_2 r^{1 + \kappa_+} \mathcal{F} \left( \frac{1}{4} + \frac{\kappa_+}{2}, 1 + \frac{\kappa_+}{2}, 1 + \frac{\kappa_+ + \kappa_-}{4}, 1 + \frac{\kappa_+ - \kappa_-}{4}, \frac{r^2}{2} \right) \tag{B14}
\]

where \(\mathcal{F}(., ., .)\) are hypergeometric functions, \(C_1\) and \(C_2\) are arbitrary constants, and

\[
\kappa_{\pm} = \sqrt{4\mu^2 - 1 \pm \sqrt{7 - 8\mu^2}}.
\]

Equation \((B13)\) also has ”good” (i.e., square integrable and vanishing at \(r \to \infty\)) solutions given by the following formulae:

\[
\dot{\Phi}_1 = C_1 \exp(r) + C_2 \frac{\exp(r)}{r}.
\]

In other words, equation \((B11)\) has good asymptotic solutions in critical points \(r = 0\) and \(r \to \infty\).

Analogously, starting with \((B3)\) and \((B4)\), we can solve the first pair of equations for \(\Phi_3\) and \(\Phi_4\) and obtain:

\[
\Phi_3 = \frac{1}{4k\mu} \left( (4k^2 - 1)r \left( a\Phi_2 - \sqrt{3}\Phi_1 \right) - \sqrt{3} \left( \Phi_1' - \frac{2j - 1}{2r} \Phi_1 \right) \right.
\]

\[
- \delta \left( \Phi_2' + \frac{2j + 1}{2r} \Phi_2 \right),
\]

\[
\Phi_4 = -\frac{1}{4k\mu} \left( (4k^2 - 1)r \left( a^{-1}\Phi_1 - \frac{1}{\sqrt{3}}\Phi_2 \right) + \sqrt{3} \left( \Phi_2' + \frac{2j + 1}{2r} \Phi_2 \right) \right.
\]

\[
- \delta \left( \Phi_1' - \frac{2j - 1}{2r} \Phi_1 \right). \tag{B15}
\]
Substituting that into (B4) one obtains the following second order system:

\[
\Phi''_1 = \frac{1}{2jr} \left( (3 - 4j)\Phi'_1 + \delta \sqrt{3}\Phi'_2 \right) + \frac{1}{4jr^2} \left( \delta^2 (j - 1)\Phi_1 + \sqrt{3}\delta (2j + 1)\Phi_2 \right) + \frac{1}{2j} \left( (2j + 4k^2 - 1)\Phi_1 + \frac{a}{\sqrt{3}} (1 - 4k^2)\Phi_2 \right),
\]

\[
\Phi''_2 = \frac{1}{2jr} \left( \sqrt{3}\delta\Phi'_1 - 3\Phi'_2 \right) + \frac{1}{4jr^2} \left( (2j + 1)(2j^2 + 3j - 3)\Phi_2 - (2j - 1)\sqrt{3}\delta\Phi_1 \right) + \frac{1}{2j} \left( (2j + 1 - 4k^2)\Phi_2 + a^{-1}\sqrt{3}(4k^2 - 1)\Phi_1 \right).
\]

The change (B7) reduces this system to the following form:

\[
\hat{\Phi}''_1 = \frac{1}{4r^2} \left( (4j^2 + 6j - 1)\hat{\Phi}_1 + 3\delta\hat{\Phi}_2 \right) + \hat{\Phi}_1 - \frac{4(4k^2 - 1)}{\delta}\hat{\Phi}_1, \quad (B17)
\]

\[
\hat{\Phi}''_2 = \frac{1}{4r^2} \left( (4j^2 + 2j - 3)\hat{\Phi}_2 - 3\delta\hat{\Phi}_1 \right) + \hat{\Phi}_2. \quad (B18)
\]

Solving (B18) for \(\hat{\Phi}_1\) we obtain:

\[
\hat{\Phi}_1 = \frac{1}{3\delta} \left( 4r^2 (\hat{\Phi}_2 - \hat{\Phi}''_2) + (4j^2 + 2j - 3)\hat{\Phi}_2 \right). \quad (B19)
\]

Then substituting (B19) into (B17) we obtain the fourth order equation for \(\hat{\Phi}_2\):

\[
r^4\hat{\Phi}''_2 + 4r^3\hat{\Phi}'''_2 - (2r^2 + 2\mu^2 - 3)r^2\hat{\Phi}''_2 - 4r^3\hat{\Phi}'_2
\]

\[
+ \left( r^4 - 2(2k^2 - \mu^2 + 1)r^2 + \mu^4 + \mu^2 - \frac{3}{2} \right) \hat{\Phi}_2 = 0. \quad (B20)
\]

Equation (B20) differs from (B11) only by the coefficient for the term \(r^2\hat{\Phi}\). In the critical point \(r = 0\) and for \(r \to \infty\) this coefficient is not essential.

Thus the general solution of system (B11), (B2) can be expressed via solutions of the fourth order equation (B20) using ansatzs (B19), (B2) and (B15).

Consider separately the special case \(j = \frac{1}{2}\). In this case there are only two possible values of \(\lambda\) in (A2) and we come to the following \(2 \times 2\) matrices in the spherical harmonics basis:

\[
S \cdot n = \frac{1}{2} \sigma_1, \quad \hat{\Lambda} = 3\sigma_1, \quad \hat{L}^2 = 2(2 + \sigma_3), \quad (B21)
\]

\[
S \cdot J = \frac{1}{4} - \sigma_3, \quad S \cdot \nabla = -\frac{1}{2} \left( \sigma_1 \frac{\partial}{\partial r} + i\sigma_2 \frac{1}{r} \right).
\]

Substituting all that into equations (39) and (60), (76) we obtain the algebraic condition (83) and the following system of second-order radial equations:

\[
\Phi''_1 - \Phi_1 - \frac{6}{r^2} \Phi_1 + \frac{2\tilde{k}}{r} \Phi_2 = 0,
\]

\[
\Phi''_2 - \Phi_2 - \frac{2}{r^2} \Phi_1 + \frac{2\tilde{k}}{r} \Phi_1 = 0. \quad (B22)
\]

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where $\tilde{k} = 3k = n + \frac{1}{2}, n = 1, 2, \ldots$

Up to scaling of independent variables the system (B22) coincides with equation (13) for $j = \frac{3}{2}$ of paper [10], where its solutions can be found. The corresponding eigenvalues are given by equation (85).

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