Generalized Root Models: Beyond Pairwise Graphical Models for Univariate Exponential Families

David I. Inouye  Pradeep Ravikumar  Inderjit S. Dhillon
Dept. of Computer Science, University of Texas at Austin, Austin, TX
{dinouye,pradeepr,inderjit}@cs.utexas.edu

Abstract
We present a novel $k$-way high-dimensional graphical model called the Generalized Root Model (GRM) that explicitly models dependencies between variable sets of size $k \geq 2$—where $k = 2$ is the standard pairwise graphical model. This model is based on taking the $k$-th root of the original sufficient statistics of any univariate exponential family with positive sufficient statistics, including the Poisson and exponential distributions. As in the recent work with square root graphical (SQR) models [1]—which was restricted to pairwise dependencies—we give the conditions of the parameters that are needed for normalization using the radial conditionals similar to the pairwise case [1]. In particular, we show that the Poisson GRM has no restrictions on the parameters and the exponential GRM only has a restriction akin to negative definiteness. We develop a simple but general learning algorithm based on $\ell_1$-regularized node-wise regressions. We also present a general way of numerically approximating the log partition function and associated derivatives of the GRM univariate node conditionals—in contrast to [1] which only provided algorithm for estimating the exponential SQR. To illustrate GRM, we model word counts with a Poisson GRM and show the associated $k$-sized variable sets. We finish by discussing methods for reducing the parameter space in various situations.

1 Introduction

Most standard graphical models are restricted to pairwise dependencies between variables. For example, the Ising model for binary data and the multivariate Gaussian for real-valued data are popular pairwise graphical models. However, real-world data often exhibits triple-wise, or more generally $k$-wise dependencies. For example, the words deep, neural and network often occur together in recent research papers—note that this triple of words refers to something more specific than any of the two words without the third word, i.e. if a document only contains neural and network but not deep, then this may be a more classical paper about shallow neural networks. In the biological domain, genetic, metabolic and protein pathways play an important role in studying the development of diseases and possible interventions. These pathways are known to be complex and involve many genes or proteins rather than just simple pairwise interactions [1].

Thus, we seek to begin bridging this gap between pairwise models and complex real-world data that contain complex $k$-wise interactions by defining a class of $k$-wise graphical models called Generalized Root Models (GRM), which can be instantiated for any $k \geq 1$ and any univariate exponential family with positive sufficient statistics including the Gaussian (using the $x^2$ sufficient statistic), Poisson and exponential distributions. We estimate the graphical model structure and parameters using $\ell_1$-regularized node-wise regressions similar to previous work [2][3][4]. However, unlike previous work, because the log partition function of the GRM node conditionals is not known in closed-form—even for the previous work considering the pairwise case [1]—we develop a novel numerical approximation method for the GRM log partition function and related derivatives. In addition, we present a Newton-like optimization algorithm similar to [5] to solve the node-regressions—which significantly reduces the number of numerical log partition function approximations needed compared to gradient descent. Finally, we demonstrate the GRM model and parameter estimation algorithm on real-world text data.

[1] https://www.genome.gov/27530687/
2 Related Work

This paper generalizes the square root graphical model (SQR) from [1], which only considers pairwise dependencies. [1] followed the idea of constructing a joint distribution by defining the form of the node-conditional distributions as in [3] but introduced the idea of taking the square root of the sufficient statistics \( T(x) \) to form a pairwise term \( \sqrt{T(x)} \) which is linear \( O(T(x)) \) rather than the pairwise term \( T(x)T(x) \) in [3] which is quadratic \( O(T(x)^2) \). This elegant modification allowed for arbitrary positive and negative dependencies in the Poisson SQR graphical model whereas the Poisson graphical model in [3] only permitted negative dependencies—a crucial limitation of the Poisson models from [3]. While [6] proposed three modifications to the original Poisson models as defined in [3], the modifications lead to distributions with either Gaussian-esque thin tails or truncated distributions which required unintuitive cutoff points where the probability mass may concentrate near the corners of the distribution [6]. Though SQR models have great promise, SQR models are limited to pairwise dependencies, and [1] did not provide an estimation algorithm for the Poisson SQR model because the node conditional log partition function is not known in closed form. Thus, this paper extends the SQR model class to include \( k \)-wise interactions where \( k > 2 \) and, in addition, instantiates a concrete approximation algorithm for the node conditional log partition function and associated derivatives.

In a somewhat different direction, latent variable models provide an implicit and indirect way of modeling complex dependencies. Generally, though the explicit dependencies in latent variable models are only pairwise, many variables can be related implicitly through a latent variable. For example, mixture models associate a discrete latent variable with every instance which implicitly introduces dependencies. Other more complex latent variable models such as topic models [7][8] can introduce even more implicit dependencies in interesting ways. While latent variable models have proven to be practically effective in helping to model complex dependencies, the development of GRM models in this paper is distinctive and somewhat orthogonal to latent variable models. As opposed to implicitly modeling dependencies through latent variables, the GRM model explicitly models dependencies between observed variables. Thus, the discovered dependencies have an intuitive and obvious explanation in terms of the observed data variables. In addition, GRM models can be seen as complementary to latent variable models because GRM models can be used as base distributions for these latent variable models. For example, [9][4] explore using count-valued graphical models in mixtures and topic models. Thus, GRMs can provide new components from which to build more interesting models for real-world situations. Finally, node-conditional models such as GRM can be estimated using convex optimization problems, which often have theoretical guarantees [2][3] whereas latent variable models often require optimizing a non-convex function and struggle with theoretical guarantees.

Notation Let \( p \) and \( n \) be the number of dimensions and data instances respectively. Let \( \mathbb{R}_+ \) denote the set of nonnegative real numbers and \( \mathbb{Z}_+ \) denote the set of nonnegative integers. Unless indicated otherwise, we denote vectors with boldface lower case letters (e.g. \( \mathbf{x}, \theta \)) and their corresponding scalar values as normal lower case letters (e.g. \( x, \theta \)). We denote the standard basis vectors as \( e_s = [0, \cdots, 0, 1, 0, \cdots, 0]^T \) and the ones vector as \( e = [1, 1, \cdots, 1]^T \). Let \( x^p \) and \( \sqrt{x} \) to be the entry-wise power and \( j \)-th root of the vector \( x \). We denote tensors (or multidimensional arrays) with parenthesized superscripts as \( X^{(k)} \) where \( k \) is the order of the tensor. For example, \( A^{(2)} \in \mathbb{R}^{p \times p} \) is a matrix, \( A^{(3)} \in \mathbb{R}^{p \times p \times p} \) is a three dimensional tensor, and \( A^{(k)} \in \mathbb{R}^{p \times \cdots \times p} \) is a \( k \)-th order tensor. We index tensors using brackets and subscripts, e.g. \( [A^{(3)}]_{1,2,3} \) is a scalar value in the multidimensional array at index \((1, 2, 3)\). We define \([A^{(k)}]_s \in \mathbb{R}^{p \times p \times \cdots \times p} \) to be a sub tensor created by fixing the last index to \( s \) and letting the others vary—in MATLAB colon indexing notation, this would be \( A(:,:,\cdots,s) \). For example, if \( A^{(3)} \in \mathbb{R}^{p \times p \times p} \), then \([A^{(3)}]_s \in \mathbb{R}^{p \times p} \) is a matrix corresponding to the \( s \)-th slice of the tensor \( A^{(3)} \). We define \( \circ \) to be the outer product operation. For example, \( \mathbf{x} \circ \mathbf{x} = \mathbf{x} \mathbf{x}^T \in \mathbb{R}^{p \times p} \) and \( \mathbf{x} \circ \mathbf{x} \circ \mathbf{x} \in \mathbb{R}^{p \times 3} \), where \([\mathbf{x} \circ \mathbf{x} \circ \mathbf{x}]_{s_1,s_2,s_3} = x_{s_1}x_{s_2}x_{s_3} \). For more general sizes, we denote a \( k \)-th outer product to be \( \mathbf{x} \circ^k = \mathbf{x} \cdots \circ \mathbf{x} \) such that there are \( k \) copies of \( \mathbf{x} \) and the result is a \( k \)-th order tensor. We define \( \mathbf{x} \circ^0 = \mathbf{e} = [1, 1, \cdots, 1]^T \). We also denote the inner product operation of two tensors as \( \langle A^{(k)}, B^{(k)} \rangle = \sum_{s_1,\cdots,s_k} a_{s_1,\cdots,s_k} b_{s_1,\cdots,s_k} \).
3 Generalized Root Model

With the notation given in the previous section, we will define the GRM model. First, let the sufficient statistic and log base measure of a univariate exponential family be denoted as $T(x)$ and $B(x)$ respectively. We will also define the domain (or support) of the random variable to be $D$ and it’s corresponding measure to be $\mu(x)$, which is either the counting measure or Lebesgue measure depending on whether $x$ is discrete or continuous.

Let us denote a new $j$-th root sufficient statistic $\tilde{T}_j(x) = \sqrt[\ell]{T(x)}$ except in the case when $T(x) = f(x)^c$ where $c$ is an even positive integer. If $T(x) = f(x)^c$, then we simplify $\tilde{T}_j(x) \equiv f(x)^{\ell}$ (rather than the usual $|f(x)|^{\ell}$). For example, if $T(x) = x^2$, then $\tilde{T}_2(x) \equiv x$ (rather than $|x|$). As in [1], this nuanced definition is necessary to recover the multivariate Gaussian distribution. However, for notational simplicity, we will merely write $\sqrt[\ell]{x}$ for $\tilde{T}_j(x)$ throughout the paper. Note that $\tilde{T}_j(x) = \sqrt[\ell]{x}$ for the Poisson and exponential GRM models. Using this simplified notation, we can define the Generalized Root Model for $k \leq p$ as:

$$\Pr(x | \Psi^{(1)}_{(j)}) = \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \Psi^{(j)}_{(j)}, \sqrt[\ell]{x} \right) + \sum_s B(x_s) - A(\Psi^{(1)}_{(j)}) \right)$$

$$A(\Psi^{(1)}_{(j)}) = \ln \int_D \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \Psi^{(j)}_{(j)}, \sqrt[\ell]{x} \right) + \sum_s B(x_s) \right) d\mu(x),$$

where $A(\Psi^{(1)}_{(j)})$ is the joint log partition function, $\Psi^{(j)}_{(j)} = \left\{ \left( \Psi^{(j)}_{(j)} : j \in \{1, \cdots, k\}, \ell \leq j \right) \right\}$, $\Psi^{(j)}_{(j)}$ are super symmetric tensors of order $\ell$ which are zero whenever two indices are the same. More formally, letting $\pi(\cdot)$ be an index permutation:

$$\Psi^{(j)}_{(j)} \in \left\{ A^{(j)} : \left[ A^{(j)} \right]_{s_1, \cdots, s_{\ell}} = \left[ A^{(j)} \right]_{\pi(s_1, \cdots, s_{\ell})} = 0 \right\},$$

Note that the non-zeros of $\Psi^{(j)}_{(j)}$ define $\ell$-sized variable sets (or cliques) of the underlying graphical model.

3.1 Special Cases

We now consider several special cases of this model to build some understanding of the GRMs connection to previous models. The independent model is trivially recovered if $k = 1$: $\Pr(x | \Psi^{(1)}_{(1)}) = \exp \left( \left( \Psi^{(1)}_{(1)}, x \right) + \sum_s B(x_s) - A(\Psi^{(1)}_{(1)}) \right)$. 

Square Root Graphical Model [1] Another special case is the previous SQR models (i.e. $k = 2$) from [1] by taking (using the notation from [1]) $\Psi^{(1)}_{(1)} = \text{diag}(\Phi)$, $\Psi^{(2)}_{(1)} = \Theta$ and $\Psi^{(2)}_{(2)} = \tilde{\Phi}$, where $\text{diag}(\Phi)$ is a column vector of the diagonal entries and $\tilde{\Phi}$ has the same off-diagonal entries as $\Phi$ but is zero along the diagonal. Thus, the SQR model can be written as:

$$\Pr(x | \Psi^{(1)}_{(1)}, \Psi^{(1)}_{(2)}, \Psi^{(2)}_{(2)}) = \exp \left( \left( \Psi^{(1)}_{(1)}, x \right) + \left( \Psi^{(1)}_{(2)}, \sqrt[\ell]{x} \right) + \left( \Psi^{(2)}_{(2)}, \sqrt[\ell]{x} \circ \sqrt[\ell]{x} \right) + \sum_s B(x_s) - A(\Psi^{(1)}_{(1)}) \right).$$

Simplified Model with Only Strongest Interaction Terms We consider another special case such that only the strongest interaction (i.e. when $\ell = j$) terms are non-zero:

$$\Pr(x | \Psi^{(1)}_{(j)}) = \exp \left( \sum_{j=1}^{k} \left( \Psi^{(j)}_{(j)}, \sqrt[\ell]{x} \right) + \sum_s B(x_s) - A(\Psi^{(1)}_{(j)}) \right).$$

This restricted parameter space forces $j$-wise dependencies to only be through the $j$-th root term. For example, pairwise interactions are only available through the sufficient statistic $\sqrt[\ell]{x_1 x_2}$ and ternary interactions are only available through the sufficient statistic $\sqrt[\ell]{x_1 x_2 x_3}$. Without this restriction interactions would be allowed through multiple terms, e.g. pairwise interactions would be allowed through multiple sufficient statistics $\sqrt[\ell]{x_1 x_2}, \sqrt[\ell]{x_2 x_3}, \cdots, \sqrt[\ell]{x_{j-1} x_j}$. Thus, this simplified model is more interpretable and easier to learn while still retaining the strongest $j$-wise interaction terms. For our experiments, we assume this simplified model unless specified otherwise.
3.2 Conditional Distributions

As in [1], we derive both the node conditionals and the radial conditional distributions. An illustration of these two types of univariate conditional distributions can be seen in Fig. 1. This node conditional distribution is critical for the parameter estimation that will be described in later sections; whereas the radial conditional distributions are critical for showing the normalization of GRM models.

Figure 1: Node conditional distributions (left) are univariate probability distributions of one variable assuming the other variables are given while radial conditional distributions are univariate probability distributions of vector scaling assuming the vector direction is given. Both conditional distributions are helpful in understanding SQR graphical models. (Figure reproduced from [1] with permission.)

3.2.1 Node Conditionals

The node conditionals are as follows (see appendix for full derivation):

\[
\Pr(x_s \mid x_{-s}, \Psi_{(j)}^{(i)}) \propto \exp \left( \sum_{j=1}^{k} \eta_{js} x_s^{1/j} + B(x_s) \right),
\]

(5)

where \(x_{-s}\) is all other variables except \(x_s\), \(\eta_{js} = \sum_{\ell=1}^{j} \left[ \Psi_{(j)}^{(\ell)} \right]_s, \ell \sqrt{x_{\ell} o_{\ell-1}} \). This is a univariate exponential family with sufficient statistics \(x_s^{1/j}\), natural parameters \(\eta_{js}\) and base measure \(B(x_s)\). Note that this reduces to the original exponential family if the interaction terms \(\eta_{2s} = \cdots = \eta_{ks} = 0\).

3.2.2 Radial Conditionals

As in [1], we define the radial conditional distribution by fixing the unit direction \(v = \frac{x}{\|x\|}\) of the sufficient statistics but allowing the scaling \(z = \|x\|_1\) to be unknown. Thus, we get the following radial conditional distribution (see appendix for derivation):

\[
\Pr(x = zv \mid v, \Psi_{(i)}^{(j)}) \propto \exp \left( \sum_{r \in \mathcal{R}} \eta_r(v) z^r + B_v(z) \right),
\]

(6)

where \(\mathcal{R} = \{\ell/j : j \in \{1, \cdots, k\}, \ell \leq j\}\) is the set of possible ratios, \(\eta_r(v) = \sum_{\ell/r : \ell/j = r} \left[ \Psi_{(j)}^{(\ell)} \right]_s, \ell \sqrt{v_{\ell} o_{\ell}} \) are the exponential family parameters, \(z^r\) are the corresponding sufficient statistics and \(B_v(z) = \sum_s B(zv_s)\) is the base measure. Thus, the radial conditional distribution is a univariate exponential family (as in [1]).
3.3 Normalization

The previous exponential and Poisson graphical models \([10, 3]\) could only model negative dependencies. However, we generalize the results from the pairwise SQR model in \([1]\) and show that GRM normalization for any \(k \geq 2\) puts little to no restriction on the value of the parameters—thus allowing both positive and negative dependencies. For our derivations, let \(V = \{\mathbf{v} : \|\mathbf{v}\|_1 = 1, \mathbf{v} \in \mathbb{R}^p_+\}\) be the set of unit vectors in the positive orthant. The GRM log partition function \(A(\Psi^{(1)}_k)\) can be decomposed into nested integrals over the unit direction and over the scaling \(z\):

\[
A(\theta, \Phi) = \ln \int \int_{\mathcal{Z}(\mathbf{v})} \exp \left( \sum_{j=1}^k \sum_{\ell=1}^j \left( \Psi^{(\ell)}(j), \sqrt{z} \mathbf{v} \circ \mathbf{v}^\ell \right) + \sum_s B(z \mathbf{v}_s) \right) \, d\mu(z) \, d\mathbf{v}
\]

\[
= \ln \int \int_{\mathcal{Z}(\mathbf{v})} \exp \left( \sum_{r \in \mathcal{R}} \eta_r(\mathbf{v}) z^r + \tilde{B}(z) \right) \, d\mu(z) \, d\mathbf{v}
\]

(7)

where \(\mathcal{Z}(\mathbf{v}) = \{z \in \mathbb{R}_+ : z \mathbf{v} \in \mathcal{D}\}\), and \(\mu\) and \(\mathcal{D}\) are the measure and domain (or support) of the random variable. Because \(V\) is bounded, the joint distribution will be normalizable if the radial conditional distribution is normalizable—generalizing the results from \([1]\) for \(k > 2\). Informally, the radial conditional distribution converges if the asymptotically largest term of \(\{\eta_r(\mathbf{v}) z^r\} \cup \{B(z \mathbf{v}_s)\}\) is monotonically decreasing at least linearly.\(^2\) We give several examples in the following paragraphs.

Gaussian GRM For the Gaussian GRM, we take the Gaussian univariate distribution with sufficient statistic \(T(x) = x^2\) and \(B(x) = 0\). When \(k = 2\) (i.e. the standard multivariate Gaussian), the largest radial conditional term is \(\eta_1 x^2\) where \(\eta_1 = \langle \psi^{(1)}(1), \mathbf{v}^2 \rangle + \langle \psi^{(2)}(2), \mathbf{v} \circ \mathbf{v} \rangle\). Note that the radial conditional (i.e. a univariate Gaussian) is normalizable only if \(\eta_1 < 0\) for all \(\mathbf{v} \in V\), which is equivalent to the positive definite condition on the Gaussian inverse covariance matrix. We can also consider a Gaussian-like model with \(k = 3\). In this case, we have that \(\eta_1 = \langle \psi^{(1)}(1), \mathbf{v}^2 \rangle + \langle \psi^{(2)}(2), \mathbf{v} \circ \mathbf{v} \rangle + \langle \psi^{(3)}(3), \mathbf{v}^2 \circ \mathbf{v} \circ \mathbf{v} \rangle\) and we need \(\eta_1 < 0\) \(\forall \mathbf{v} \in V\). Note that the Gaussian GRM models for \(k > 2\) are novel models to the authors’ best knowledge.

Exponential GRM Because the exponential distribution also has a constant base measure like the Gaussian, the asymptotically largest term is \(\eta_1 x^2\) and thus we must have that \(\eta_1 < 0\) \(\forall \mathbf{v} \in V\). However, unlike the Gaussian, in the case of the exponential distribution \(V\) is only positive \(\ell_1\)-normalized vectors. This is a significantly weaker condition on the parameters than for a Gaussian and allows strong positive and negative dependencies.

Poisson GRM For the Poisson distribution, the base measure is the asymptotically largest term \(O(\exp(-z \ln(z)))\). Thus, as in \([1]\), the parameters can be arbitrarily positive or negative because eventually the base measure will ensure normalizability. Note that this is true for arbitrarily large \(k\).

4 Parameter Estimation

As in \([3,4,1]\), we solve a set of independent \(\ell_1\)-regularized node-wise regressions for each node—based on the node conditional distributions in Sec. 3.2.1—using a Newton-like method for convex optimization with an non-smooth \(\ell_1\) penalty as in \([11,12,4]\). More specifically we take the log likelihood of the node conditionals and add an \(\ell_1\) penalty on all interaction terms:

\[
\arg \min_{\Psi^{(1)}_k} - \frac{1}{n} \sum_{s=1}^n \left( \frac{1}{n} \sum_{i=1}^n \left( \sum_{j=1}^k \eta_{j,s} x_{s}^{1/j} - A(\eta_{s}) \right) + \lambda \sum_{j=2}^k \sum_{\ell=1}^j \|\psi^{(\ell)}_{(j)}\|_1 \right), \tag{8}
\]

\(^2\)For more formal proofs, we refer the reader to \([1]\).
where \( \eta_{jsi} = \sum_{\ell=1}^{\ell} \left( \psi^{(\ell)}_{(j)} \right) \), \( \ell \sqrt{x_i \circ \ell} \) and \( \parallel \cdot \parallel_1 \) is an entry-wise sum of absolute values. Note that this is trivially decomposable into \( p \) subproblems and can thus be trivially parallelized to improve computation speed. We use the Newton-like method as in [5] to greatly reduce computation. The initial innovation from [5] was that the Hessian only needed to be computed over a free set of variables each Newton iteration because of the \( \ell_1 \) regularization which suggested sparsity of the parameters. Yet, the number of Newton iterations was very small compared to gradient descent. In the case of GRM models, whose bottleneck is the computation of the gradient of \( A \) (at least under our current implementation though it might be possible to significantly reduce this bottleneck), this Newton-like method provides even more benefit because the gradient only has to be computed a small number of times (roughly 30) in our case rather than the several thousand times that would be needed for running thousands of proximal gradient descent steps for the same level of convergence.

In the next section, we derive the gradient and Hessian for the smooth part of the optimization as a function of the gradient and Hessian of the node conditional log partition function \( A(\eta) \). Then, we develop a general method for bounding the log partition function \( A(\eta) \) and associated derivatives even though usually no closed-form exists.

### 4.1 Gradient and Hessian of GRMs

**Notation for gradient and Hessian** Let \( \text{vec}(\Psi^{(\ell)}) \in \mathbb{R}^{p^2} \) be the vectorized form of a tensor. For example, the vectorized form of a \( p \times p \) matrix is formed by stacking the matrix columns on top of each other to form one long vector. Also, let \( [x | x \in A] \) be analogous to the normal set notation except that the bracket and vertical line notation creates a vector from all the elements concatenated to together. This is similar to a list comprehension in Python. For our gradient and Hessian calculations, we define the following variable transformations and give them as examples of this notation:

\[
B_s = \left\{ \left[ \text{vec} \left( \left[ \psi^{(\ell)}_{(j)} \right]_s \right) \right] : \ell \leq j \in \{1, 2, \ldots, k\} \right\}
\]

\[
\mathcal{Z}_{si} = \left\{ \left[ \text{vec} \left( \ell \sqrt{x_i \circ \ell} \right) \right] : \ell \leq j \in \{1, 2, \ldots, k\} \right\}
\]

With this notation, we have that \( \eta_{jsi} = \beta^T_j z_{jsi} \). Because each node regression is independent, we focus on solving one of the \( p \) subproblems for a particular \( s \) using the notation from above:

\[
\arg\min_{B_s} \sum_{i=1}^{n} f_s(B_s | x_{si}, Z_{si}),
\]

where \( f_s(B_s | x_{si}, Z_{si}) = -\sum_{j=1}^{k} (\beta^T_j z_{jsi}) \sqrt{x_i} + A(\beta^T_j z_{jsi} | j \in \{1, \ldots, k\}) \). For notational simplicity, we suppress the dependence on \( s \) and \( i \) in the derivations of the gradient and Hessian of \( f(\cdot) \) (the gradient and Hessian are merely the sum over all instances). With this simplified notation, the gradient and Hessian are as follows (as functions of \( A, \nabla A \) and \( \nabla^2 A \)):

\[
\nabla f(B | x, Z) = \left[ \left( -\sqrt{x} + \frac{\partial A}{\partial \eta_j} \right) z_j \mid j \in \{1, 2, \ldots, k\} \right],
\]

\[
\nabla^2 f(B | x, Z) = \left[ \left[ \frac{\partial^2 A}{\partial \eta_j \partial \eta_k} z_j \circ z_k \mid j \in \{1, 2, \ldots, k\} \right] \right].
\]
Thus, we need to compute expectations for at most \( g \) original exponential family. Because we can determine the concavity of each region of subdomains. error, which will be described in a future section. For each subdomain, we will form linear upper and lower bounds for integral over different subdomains of the domain. We will choose the subdomains in appropriate way to minimize to use the machinery of the base exponential family to compute the needed integrals.

4.2 Gradient and Hessian of \( A(\eta) \)

Because the node conditional distributions are not standard distributions, we must either derive the closed-form log partition function as done with the specific case of the exponential SQR model in [11], or we must numerically approximate the log partition function and its first and second derivatives. To the authors’ best knowledge, even for the simplified SQR model with \( k = 2 \), no closed-form solution to log partition function exists for SQR node conditionals except for the discrete, Gaussian and exponential SQR models. Thus, we seek a general way to estimate the log partition function as done with the specific case of the exponential SQR model in [1], or we must numerically approximate the log partition function and its first and second derivatives. To the authors’ best knowledge, even for the discrete, Gaussian and exponential SQR models. Thus, we seek a general way to estimate the log partition function and its first and second derivatives. To the authors’ best knowledge, even for the

Derivatives of \( A(\eta) \) Reformulated as Expectations  
We first note that the gradient and Hessian of \( A(\eta) \) are merely functions of particular expectations—a well-known result of exponential families:

\[
A(\eta) = \ln \int_D \exp \left( \sum_{j=1}^{k} \eta_j x_j^\gamma + B(x) \right) d\mu(x) 
\]

\[
\nabla A(\eta) = \left[ \mathbb{E}(x_j^\gamma) \mid j \in \{1, \cdots, k\} \right] 
\]

\[
\nabla^2 A(\eta) = \begin{bmatrix}
\mathbb{E}(x_j^\gamma) - \mathbb{E}(x_j^\gamma)\mathbb{E}(x_\gamma) & \mathbb{E}(x_j^\gamma) - \mathbb{E}(x_j^\gamma)\mathbb{E}(x_\gamma) & \cdots & \mathbb{E}(x_j^\gamma) - \mathbb{E}(x_j^\gamma)\mathbb{E}(x_\gamma)
\end{bmatrix}
\]

Thus, we need to compute expectations for at most \( \binom{k}{2} + k \) functions of the form \( \mathbb{E}(x^a) \).

Definition of \( M(a) \) to Unify Approximations  
To develop our approximations under a unified framework, let us define the following function \( M(a) \) and its subfunctions denoted \( f(x) \) and \( g(x) \):

\[
M(a) = \ln \int_D x^a \exp \left( \sum_{j=1}^{k} \eta_j x_j^\gamma + B(x) \right) d\mu(x) = \ln \int_D \exp \left( \eta_1 x + B(x) + \underbrace{\sum_{j=2}^{k} \eta_j x_j^\gamma + \ln(x^a)}_{g(x)} \right) d\mu(x).
\]

By simple inspection, we see that \( M(0) = A(\eta_1, \eta_2) \) and \( \mathbb{E}(x^a) = \exp \left( M(a) - M(1) \right) \). Thus, by approximating \( M(a) \), we can approximate all the necessary derivatives. If \( g(x) = 0 \), then this is simply the log partition function of the base exponential family, which is usually known in closed form. If \( g(x) \approx bx + c \) (as we will develop in the next section), then we can create a modified \( f \) and \( g \) such that \( \tilde{f}(x) = (\eta_1 + b)x + c \) and \( \tilde{g}(x) = 0 \)—thus also allowing us to use the machinery of the base exponential family to compute the needed integrals.

Overall Approach to Bounding \( M(a) \)  
Our approach splits the integral into \( d = O(1) \) integrals which bound the integral over different subdomains of the domain. We will choose the subdomains in appropriate way to minimize error, which will be described in a future section. For each subdomain, we will form linear upper and lower bounds for \( g(x) \) so that we can then use the CDF function of the base exponential family to approximate the integrals over these subdomains.

First, we will describe how to compute linear upper and lower bounds to \( g(x) \) so that the integrals reduce to the original exponential family. Because we can determine the concavity of each region of \( g(x) \)\(^3\), we can form linear upper

---

\(^3\)This can be done by solving for the zeros of a polynomial.
and lower bounds using the theory of convexity. The secant line and the first-order Taylor series approximation form upper and lower bounds or vice versa depending on concavity. We can bound the tails of \(g(x)\) with a constant function or Taylor series approximation as appropriate. See appendix for details on linear approximations for \(g(x)\).

If \(g(x)\) is upper and lower bounded by a linear functions, i.e. \( b_l x + c_l = g_l(x) \leq g(x) \leq g_u(x) = b_u x + c_u \), then we can form a modified functions of \( f(x) \) that will be upper and lower bounds of \( f(x) + g(x) \):

\[
(\eta_1 + b_l)x + c_l = \hat{f}_l(x) \leq f(x) + g(x) \leq \hat{f}_u(x) = (\eta_1 + b_u)x + c_u.
\]

(16)

Assuming \( \hat{\eta}_l = \eta_1 + b_l \) and \( \hat{\eta}_u = \eta_1 + b_u \) are valid parameters, we can then use the original exponential family CDF—which is usually known in closed form—to compute the needed integrals.

Now that we have linear upper and lower bounds for \(g(x)\), we can upper and lower bound \(M(a)\) using the CDF of the original exponential family to compute the needed integrals (see appendix for more derivation):

\[
M(a) \approx \ln \sum_{i=1}^{d} \int_{D_i} \exp(\hat{f}_i(x))d\mu(x) \\
= \ln \sum_{i=1}^{d} \exp{c_i + A(\hat{\eta}_i)} + \ln(CDF(\max(D_i) | \hat{\eta}_i)) - CDF(\min(D_i) | \hat{\eta}_i)),
\]

(17)

where the domain is split into disjoint subdomains, i.e. \( \{ D_i : D = \bigcup_{i}^d D_i, \forall i \neq j, D_i \cap D_j = \emptyset \} \). \((\hat{\eta}, b)\) are either \((\hat{\eta}_u, b_u)\) or \((\hat{\eta}_l, b_l)\) depending on whether the upper or lower bound is needed. \(A(\hat{\eta})\) and \(CDF(\cdot)\) are the log partition function and CDF of the original exponential family. Note that assuming \(A(\hat{\eta})\) and \(CDF(\cdot)\) are available in closed form—as is the case for the Poisson distribution—this approximation can be computed in \(O(d) = O(1)\) time.

**Algorithm to Find Appropriate Subdomains \(D_i\)** We need that every subdomain has a constant concavity (i.e. either concave or convex over the subdomain) in order to use Taylor series and secant line bounds (and a constant bound for the tails). Thus, we use the following algorithm to find subdomains to help minimize the difference between the upper and lower bounds. (An illustration of the method can be seen in Fig. 2):

1. Find all real roots of \(g''(x)\), denoted \((x'_1, x'_2, \cdots)\) so we know the inflection points (which will define the regions of constant concavity).
2. Use inflection points and endpoints of domain (e.g. 0 and \(\infty\) for Poisson) to define the initial subdomains.
3. Compute initial bounds for these subdomains using Eqs. (18)
4. Split the subdomain with the largest difference between upper and lower bounds (i.e. the subdomain with the largest error).
5. Recompute bounds for the two new subdomains formed by splitting the largest error subdomain.
6. Repeat previous two steps until \(d\) domains have been obtained.

![Figure 2: Approximation of the \(M(a)\) function with \(a = 0\) and \(\eta = [3.0232, -4.4966]\) for 2 subdomains (left) and for 5 subdomains (right) using the algorithm described in Sec. 4.2. The top is the actual values of the summation in Eqn. (15) and the bottom is the linear approximation \(bx + c\) to the non-linear part \(g(x)\) as in Eqn. (16)](image)
Note that the roots of $g''(x)$ can be solved by expanding to a polynomial and then computing the eigenvalues of the companion matrix. For example if $g(x) = x^{1/2} + x^{1/3} + \alpha n(x)$, then $g''(x) = -\frac{1}{2} (x^{1/2} + \frac{2}{3} x^{1/3} + \alpha)$. Note that the zeros of this function are equal to the zeros of $h(x) = \frac{1}{2} x^{1/2} + \frac{2}{3} x^{1/3} + \alpha$. Thus, we can let $y = x^{1/6}$ and form the polynomial function $h(y) = \frac{1}{2} y^3 + \frac{2}{3} y^2 + \alpha$. We can then solve the zeros of this polynomial by forming the companion matrix and solving for the eigenvalues. However, we only need the real zeros and we do not care about multiplicity so it may be faster to use a direct root finding algorithm instead—though we have not explored this option.

5 Results on Text Documents

We computed the Poisson GRM model on two datasets: Classic3 and Grolier encyclopedia articles. The Classic3 dataset contains 3893 research abstracts from library and information sciences, medical science and aeronautical engineering. The Grolier encyclopedia dataset contains 5000 random articles from the Grolier encyclopedia. We set $k = 3, p = 500$ and $\lambda = 0.01$ for our experiments. We chose 10 interval endpoints (i.e. 9 subdomains) for our approximations. Note that this means there are at least $\binom{500}{3} \approx 2 \times 10^7$ possible parameters. We give the top 10 positive parameters for individual, edge-wise and triple-wise combinations. The top 50 (unless there are less than 50 non-zeros) of both negative and positive dependencies for single, pairwise and triple-wise dependencies can be found in the appendix.

| Single | Pairwise | Triple-wise |
|--------|----------|-------------|
| information | boundary + layer | layer + skin + friction |
| flow | heat + transfer | information + retrieval + storage |
| library | tunnel + wind | pressure + number + mach |
| pressure | edge + leading | layer + plate + flat |
| system | bone + marrow | flow + given + case |
| theory | angle + attack | flow + plate + flat |
| results | skin + friction | number + mach + investigation |
| data | growth + hormone | number + mach + conducted |
| patients | plate + flat | wing + ratio + aspect |
| found | shock + wave | number + based + Reynolds |
| method | mach + numbers | pressure + ratio + jet |
| cells | number + mach | heat + transfer + coefficients |
| analysis | number + Reynolds | system + retrieval + user |

Table 1: Table of Tuples

These results illustrate that our model and algorithm can find interesting pairwise and triple-wise words. The timing for these experiments using prototype code in MATLAB on TACC Maverick cluster (https://portal.tacc.utexas.edu/user-guides/maverick) was 2653 seconds for the Classic3 dataset and 5975 seconds for the Grolier dataset. Given the extremely large number of parameters to be optimized, this gives evidence that GRM models are computationally tractable while still wanting for some improvement.

6 Discussion

While it may seem at first that this model is impractical for even $k = 4$, we suggest some practical ideas for reducing the parameter space. First, if some parameters are known or expected a priori to be non-zero, we could only allow those parameters to be non-zero. For example, known genetic pathways could be encoded as $k$-wise cliques. Thousands of known pathways could be added which would only incur thousands of parameters, which is very small relative to all possible parameters. Second, the optimization could proceed in a stage-wise fashion such that the first a model is fit for $k = 1$, then this model is used to choose which parameters to allow in the next model of $k = 2$, etc. For example, we could first train a model with only pairwise parameters ($k = 2$). Then, we could find all triangles in the discovered graph and only add these parameters for training a model with $k = 3$. This heuristic would significantly reduce the number of possible parameters if the parameters are assumed to be sparse (as is usually the case with $\ell_1$-regularized objectives). Third, the tensors could be constrained to be low-rank and thus only $O(p)$ values for each tensor would be needed. For example, we could assume that the pairwise tensors are low-rank matrices. For higher order tensors, a similar idea could hold, e.g. $\Psi^{(k)}_{(j)} = \sum_{i=1}^{M} \theta_i \phi^k$, where $M$ is $O(1)$. 

9
7 Conclusion

We generalize the previous SQR \[1\] model to include factors of size \( k > 2 \). We study this general distribution by giving the node and radial conditional distributions, which provides simple conditions for normalization of the GRM class of models. We then develop an approximation technique for estimating the node-wise log partition function and associated derivatives for the Poisson case—note that \[1\] only provided an algorithm for approximating the exponential SQR model. Finally, we qualitatively demonstrated our model on two real world datasets.

Acknowledgments

This work was supported by NSF (DGE-1110007, IIS-1149803, IIS-1447574, IIS-1546459, DMS-1264033, CCF-1320746) and ARO (W911NF-12-1-0390).

References

[1] D. I. Inouye, P. Ravikumar, and I. S. Dhillon, “Square root graphical models: Multivariate generalizations of univariate exponential families that permit positive dependencies,” in ICML, 2016.

[2] P. Ravikumar, M. Wainwright, and J. Lafferty, “High-dimensional Ising model selection using \( l1 \)-regularized logistic regression,” The Annals of Statistics, vol. 38, pp. 1287–1319, 6 2010.

[3] E. Yang, P. Ravikumar, G. I. Allen, and Z. Liu, “On graphical models via univariate exponential family distributions,” JMLR, vol. 16, pp. 3813–3847, 2015.

[4] D. I. Inouye, P. Ravikumar, and I. S. Dhillon, “Fixed-length Poisson MRF: Adding dependencies to the multinomial,” in NIPS, pp. 3195–3203, 2015.

[5] C. Hsieh, M. Sustik, I. Dhillon, and P. Ravikumar, “Sparse inverse covariance matrix estimation using quadratic approximation.,” Nips, pp. 1–9, 2011.

[6] E. Yang, P. Ravikumar, G. I. Allen, and Z. Liu, “On Poisson graphical models,” in NIPS, p. 1718–1726, 2013.

[7] D. Blei, L. Carin, and D. Dunson, “Probabilistic topic models,” IEEE Signal Processing Magazine, vol. 27, pp. 55–65, 11 2010.

[8] D. M. B. Lafferty and J. D., “Correlated topic models,” in Advances in Neural Information Processing Systems 18, pp. 147–154, 2006.

[9] D. I. Inouye, P. Ravikumar, and I. S. Dhillon, “Admixture of Poisson MRFs: A topic model with word dependencies,” in ICML, 2014.

[10] J. Besag, “Spatial interaction and the statistical analysis of lattice systems,” Journal of the Royal Statistical Society. Series B (Methodological), vol. 36, no. 2, pp. 192–236, 1974.

[11] C.-J. Hsieh, M. A. Sustik, I. S. Dhillon, and P. Ravikumar, “QUIC: Quadratic approximation for sparse inverse covariance estimation,” JMLR, vol. 15, pp. 2911–2947, 2014.

[12] D. I. Inouye, P. Ravikumar, and I. S. Dhillon, “Capturing semantically meaningful word dependencies with an admixture of {poisson mrf}s,” in NIPS, pp. 3158–3166, 2014.
A Node Conditional Derivation

\[ \Pr(x_a | x_{-a}, \Psi_{-a}^{(j)}) = \Pr(x = x_{a0} + x_a e_a | x_{a0}, \Psi_{-a}^{(j)}) \]
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \Psi_{(j)}^{(\ell)} + \sqrt{x_{a0} + x_a e_a} \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (19)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \Psi_{(j)}^{(\ell)} \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (20)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (21)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (22)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (23)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (24)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (25)
\[ \propto \exp \left( \sum_{j=1}^{k} \left( \phi_j^\ell \right) + B(x_a) \right) \]  \hspace{1cm} (26)

where \( \eta_{js} = \sum_{\ell=1}^{j} \ell \left( \phi_{(j)}^{(\ell)} \right) \). See notation section for definition of \( \phi_{(j)}^{(\ell)} \). This is a univariate exponential family with sufficient statistics \( x_a^{1/\ell} \), natural parameters \( \eta_{js} \), and base measure \( B(x_a) \). This recovers the SQR node conditional from \( \Pi \) with \( k = 2 \).

B Radial Conditional Derivation

As in \( \Pi \), we define the radial conditional distribution by fixing the unit direction \( v = \frac{x}{||x||} \) of the sufficient statistics but allowing the scaling \( z = ||x|| \) to be unknown. Thus, we get the following radial conditional distribution:

\[ \Pr(x = z v | v, \Psi_{(j)}) \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_{(j)}^{(\ell)} \right) + \sum_z B(z v_a) \right) \]  \hspace{1cm} (27)
\[ \propto \exp \left( \sum_{j=1}^{k} \sum_{\ell=1}^{j} \left( \phi_{(j)}^{(\ell)} \right) + \sum_z B(z v_a) \right) \]  \hspace{1cm} (28)
\[ \propto \exp \left( \sum_{\ell \in \mathcal{R}} \eta_{\ell}(v) z^\ell + B_v(z) \right) \]  \hspace{1cm} (29)

where \( \mathcal{R} = \{ \ell/j : j \in \{1, \cdots, k\}, \ell \in \{1, \cdots, j\} \} \) is the set of possible ratios, \( \eta_{\ell}(v) = \sum_{(i,j)} \phi_{(j)}^{(\ell)} \phi_{(i)}^{(i)} \) are the exponential family parameters, \( z^\ell \) are the corresponding sufficient statistics, and \( B_v(z) = \sum_z B(z v_a) \) is the base measure. Thus, the radial conditional distribution is a univariate exponential family.
C Derivation of $M(a)$ Approximation

\[ M(a) \approx \ln \sum_{i=1}^{d} \int_{D_i} \exp(f_i(x))d\mu(x) \]  
\[ = \ln \sum_{i=1}^{d} \exp(c_i) \int_{D_i} \exp(\eta_i x + B) d\mu(x) \]  
\[ = \ln \sum_{i=1}^{d} \exp(c_i) \exp \left( A(\eta_i) \left( \text{CDF} \left( \max(D_i) \mid \eta_i \right) - \text{CDF} \left( \min(D_i) \mid \eta_i \right) \right) \right) \]  
\[ = \ln \sum_{i=1}^{d} \exp(c_i + A(\eta_i)) \left( \text{CDF} \left( \max(D_i) \mid \eta_i \right) - \text{CDF} \left( \min(D_i) \mid \eta_i \right) \right) \]  
\[ = \ln \sum_{i=1}^{d} \exp(c_i + A(\eta_i)) + \ln \left( \text{CDF} \left( \max(D_i) \mid \eta_i \right) - \text{CDF} \left( \min(D_i) \mid \eta_i \right) \right) \]  

D Linear Bounds of $g(x)$

**Taylor series linear bound** Upper bound if concavity = -1 and lower bound if concavity = 1:

\[ q^* = \begin{cases} 
q_1, & \text{if } q_2 = \infty \\
q_2, & \text{if } q_1 = -\infty \\
\arg \max_{q_1,q_2} g(q), & \text{otherwise}
\end{cases} \]

\[ \hat{g}(x) = g(q^*) + g'(q^*)(x - q^*) \]

**Secant linear bound** Upper bound if concavity = 1 and lower bound if concavity = -1:

\[ b = \frac{g(q_2) - g(q_1)}{q_2 - q_1} \]

\[ g(q_1) = bq_1 + c \]

\[ \Rightarrow c = g(q_1) - bq_1 \]

**Tail bounds** We know there are only a finite number of inflection points so let us take the $x$ value for the last inflection point, denoted $x^*$. By simple asymptotic analysis, we know that the largest non-zero term will dominate eventually. Let's assume w.l.o.g. that $\eta_j, x^* > 0$ and $\eta_j > 0$. Then, we know that after the last inflection point, the concavity will be negative. In addition, we know that the $g(x)$ is a continuous function and $g(x) \to \infty$ as $x \to \infty$. The function must be monotonically increasing after the last inflection point. Proof by contradiction: Suppose the monotonicity is negative after the last inflection point. Then, because the $g(x)$ is a continuous function and $g(x) \to \infty$ as $x \to \infty$, the function must eventually have a positive monotonicity. Yet this would switch from negative monotonicity to positive monotonicity after the last inflection point. However, this would be an inflection point that is greater than the assumed last inflection point which leads to a contradiction. The case where $\eta_j < 0$ can be proved similarly. Thus, we can use a constant function for an upper bound if concavity = 1 and we can use a constant function as a lower bound if concavity = -1. A Taylor series approximation forms an upper or lower bound depending on concavity.

E Complete Results for Classic3 Dataset

<<< Largest 1-tuples, $j = 1$, $ell = 1$ >>>

| Information | Flow |
|-------------|------|
| -0.6256     | -0.6906 |

\[ ^4 \text{If for all } j, \eta_j = 0, \text{then we can take } q^* = \infty, \eta_j^* = a. \]
-0.8097 library
-1.1693 pressure
-1.4066 system
-1.4090 theory
-1.4209 results
-1.4248 data
-1.4597 patients
-1.5737 found
-1.5791 method
-1.6292 cells
-1.6566 analysis
-1.7161 given
-1.7198 use
-1.7389 number
-1.7525 used
-1.7550 study
-1.7860 made
-1.7884 effect
-1.8054 time
-1.8089 body
-1.8371 research
-1.8563 cases
-1.8958 normal
-1.9246 effects
-1.9436 present
-1.9690 discussed
-1.9803 shock
-1.9899 presented
-2.0112 wing
-2.0135 surface
-2.0275 large
-2.0295 case
-2.0447 obtained
-2.0603 new
-2.0695 paper
-2.0776 libraries
-2.0834 high
-2.0884 problems
-2.1160 methods
-2.1162 well
-2.1311 development
-2.1414 general
-2.1417 growth
-2.1659 problem
-2.2109 jet
-2.2112 terms
-2.2313 systems
-2.2416 form

<<< Positive 2-tuples, \( j = 2 \), \( \ell_1 = 2 \) >>>

4.9827 boundary + layer
4.2583 heat + transfer
3.9493 tunnel + wind
3.3190 edge + leading
3.1510 bone + marrow
3.0395 angle + attack
2.8711 skin + friction
2.5638 growth + hormone
2.3182 plate + flat
2.2675 shock + wave
2.2548 mach + numbers
2.1277 number + mach
2.1047 number + reynolds
2.0561 agreement + good
2.0306 attack + angles
2.0102 document + documents
1.9377 cells + cell
1.7727 journals + journal
1.5436 library + libraries
1.5291 lift + drag
1.5090 wing + wings
1.4283 shells + cylindrical
1.4192 buckling + shells
1.4126 temperature + thermal
1.4080 free + stream
1.3980 ratio + aspect
1.3892 equations + differential
1.3721 boundary + layers
1.3675 point + stagnation
1.2495 heat + temperature
1.2285 reynolds + transition
1.2000 wings + aspect
1.1793 temperature + temperatures
1.1554 thin + shells
1.1335 science + scientific
1.1279 cells + marrow
1.1188 numbers + reynolds
1.1004 cylinder + circular
1.0910 renal + kidney
1.0907 pressure + pressures
1.0384 high + speed
1.0294 layer + laminar
1.0266 information + retrieval
1.0232 patients + therapy
1.0159 patients + cancer
1.0127 jet + nozzle
0.9848 group + groups
0.9660 experimental + theoretical
0.9528 buckling + stress

<<< Negative 2-tuples, j = 2, ell = 2 >>>
-0.8428 flow - library
-0.6400 information - pressure
-0.5896 flow - cells
-0.5885 pressure - library
-0.5737 library - patients
-0.5570 flow - system
-0.5559 information - cells
-0.5533 information - heat
-0.5185 information - patients
-0.5184 flow - patients
0.491 theory - patients
0.492 information - normal
0.4695 information - found
0.460 information - effect
0.4587 library - theory
0.4489 library - normal
0.4479 library - cells
0.4213 library - effects
0.4097 flow - retrieval
0.4071 flow - growth
0.3925 library - found
0.3857 library - cases
0.3667 information - wing
0.3653 information - case
0.3580 pressure - cells
0.3548 flow - information
0.3383 flow - subject
0.3364 results - library
0.3316 information - effects
0.3242 information - temperature
0.3170 information - surface
0.3143 flow - children
0.3131 library - obtained
0.2957 flow - book
0.2904 flow - research
0.2903 information - mach
0.2863 theory - cells
0.2858 library - effect
0.2824 information - equations
0.2817 flow - literature
0.2780 flow - index
0.2754 flow - buckling
0.2745 analysis - patients
0.2692 information - cases
0.2650 information - shock
0.2626 information - boundary
0.2583 information - method
0.2566 information - high
0.2524 library - body
0.2522 information - ratio

<<< Positive 3-tuples, j = 3, ell = 3 >>>
0.5067 layer + skin + friction
0.3171 information + retrieval + storage
0.3149 pressure + number + mach
0.3118 layer + plate + flat
0.2672 flow + given + case
0.2411 flow + plate + flat
0.1759 number + mach + investigation
0.1390 number + mach + conducted
0.1340 wing + ratio + aspect
0.1317 number + based + Reynolds
0.1100 pressure + ratio + jet
0.1072 heat + transfer + coefficients
0.0973 system + retrieval + user
0.0972 boundary + layer + experiments
F Complete Results for Grolier Encyclopedia Dataset

<<< Largest 1-tuples, j = 1, ell = 1 >>>
-1.6202  american
-1.7936  century
-1.8188  john
-1.8830  called
-1.8866  city
-1.9162  world
-1.9543  life
-2.0359  united
-2.1299  system
-2.1328  university
-2.1390  family
-2.1473  time
-2.1591  war
-2.1858  include
-2.1870  english
-2.2457  water
-2.2485  history
-2.2559  de
-2.2694  form
-2.3326  major
-2.3442  national
-2.3523  french
-2.3537  william
-2.3708  art
-2.3808  found
-2.4045  name
-2.4049  modern
-2.4255  music
-2.4315  power
-2.4433  king
-2.4445  social
-2.4455  british
-2.4596  usually
-2.4718  charles
-2.4784  south
-2.4923  law
-2.4995  north
-2.5030  repr
-2.5165  species
-2.5247  theory
-2.5383  human
-2.5520  ft
-2.5530  black
-2.5566  government
-2.5660  west
-2.5773  york
-2.5777  church
-2.5841  school
-2.5890  development
-2.5899  common

<<< Positive 2-tuples, j = 2, ell = 2 >>>
8.7140  km + mi
3.9800 language + languages
2.9617 china + chinese
2.6237 plants + plant
2.5229 deg + temperatures
2.5152 music + musical
2.4147 spanish + spain
2.1059 art + painting
2.0840 poetry + poet
2.0738 agricultural + agriculture
2.0492 war + civil
2.0024 literature + literary
1.8595 french + france
1.8405 german + germany
1.7779 culture + cultural
1.7453 china + asia
1.7368 india + asia
1.7088 system + systems
1.6836 city + york
1.6818 west + east
1.5932 africa + african
1.5932 deg + mm
1.5137 southern + northern
1.4987 architecture + building
1.4788 style + architecture
1.4431 body + blood
1.4359 role + played
1.4313 sea + ocean
1.4304 cells + blood
1.4118 science + scientific
1.4074 century + centuries
1.3808 population + sq
1.3723 social + society
1.3611 italian + renaissance
1.3611 music + opera
1.3600 ocean + pacific
1.3569 cause + disease
1.3548 cities + urban
1.3419 war + army
1.3308 united + countries
1.3065 animals + animal
1.2998 church + christian
1.2991 art + museum
1.2907 education + schools
1.2809 programs + program
1.2715 deg + temperature
1.2550 world + war
1.2515 party + leader
1.2477 government + federal

<<< Negative 2-tuples, j = 2, ell = 2 >>>
-0.2449 life - languages
-0.2179 century - species
-0.2041 city - species
-0.1575 war - species
-0.1156 city - sq
Positive 3-tuples, j = 3, ell = 3

0.3126 american + city + york
0.2773 city + population + center
0.2549 population + deg + mm
0.1971 major + population + persons
0.1641 ft + sea + level
0.1523 american + south + america
0.1515 deg + sq + consists
0.1351 city + deg + july
0.1330 war + civil + union
0.1173 population + deg + elected
American + United + English
War + Congress + Program
Population + Sq + Persons
American + French + British
Language + Includes + Languages
World + War + Japanese
Major + Time + Changes
Century + World + Laws
Life + Human + Stage
North + South + President
City + Population + University
Century + World + War
Km + Mi + Discovered
War + United + Received
City + River + Historical
Century + History + Short
Century + English + Story
War + South + Union
American + War + Congress
World + United + David
Century + History + Active
Century + History + Wide
War + United + Active
War + Army + June
City + Km + Population
Government + National + Rise
Century + Form + Appeared
War + United + Caused
Century + World + Separate
United + People + Continued
City + United + Urban
Century + Time + Applied
World + United + Building
World + South + Iron
World + Population + Rate
City + University + Center
Century + Time + Studied
American + War + People
North + Km + Fish
Century + Time + Applied
World + William + Series

<<< Negative 3-tuples, j = 3, ell = 3 >>>
-0.2560 City - Population - York
-0.1206 Km - Mi - America
-0.1076 American - Km - Mi
-0.0586 Km - Mi - York
-0.0518 War - North - Example
-0.0429 Km - Mi - Social
-0.0413 Km - Mi - Family
-0.0294 Km - Mi - Own
-0.0293 Km - Mi - Style
-0.0263 City - Population - Style
-0.0262 Km - Mi - Theory
-0.0253 City - Center - Sq
-0.0208 Km - Mi - Law
-0.0207 Km - Mi - Human
