ON SYMMETRIC DEGENERACY LOCI, SPACES OF SYMMETRIC MATRICES OF CONSTANT RANK AND DUAL VARIETIES

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Abstract. Let $X$ be a nonsingular simply connected projective variety of dimension $m$, $E$ a rank $n$ vector bundle on $X$, and $L$ a line bundle on $X$. Suppose that $S^2(E^*) \otimes L$ is an ample vector bundle and that there is a constant even rank $r \geq 2$ symmetric bundle map $E \to E^* \otimes L$. We prove that $m \leq n - r$. We use this result to solve the constant rank problem for symmetric matrices, proving that the maximal dimension of a linear subspace of the space of $m \times m$ symmetric matrices such that each nonzero element has even rank $r \geq 2$ is $m - r + 1$. We explain how this result relates to the study of dual varieties in projective geometry and give some applications and examples.

§0. Introduction

Let $V$ be a complex vector space of dimension $m$. Consider a linear subspace $A \subset S^2V^*$. We can think of $A$ as a vector space of symmetric $m \times m$ matrices or equivalently, as a symmetric $m \times m$ matrix whose entries are linear forms on $A$. We say that $A$ has constant rank $r$ if every nonzero element of $A$ has rank $r$. We ask two basic questions about constant rank subspaces $A$: how large can such subspaces be and what invariants can we attach to them? These questions come up in a number of contexts: linear algebra, vector bundles on projective space, symmetric degeneracy loci and dual varieties.

In linear algebra, an important motivation for this work is the classical theory, due to Kronecker and Weierstrass, giving a normal form for singular pencils of matrices or quadratic forms. That is, if $A \subset \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ or $A \subset S^2V^*$ is a linear subspace such that each element of $A$ has rank $\leq r$ (i.e. $A$ is of bounded
rank $r$) and $\dim(A) = 2$, then a convenient normal form can be given from which geometric properties of the pencil are easy to read off. An excellent exposition of both cases is given by F. R. Gantmacher [G, Chp. XII] or, for the case of quadratic forms one can consult Hodge and Pedoe [HP, Chp. VIII, Sec. 10]. If $A \subset S^2V^*$ is a pencil of constant rank $r$, the normal form shows that in fact $r$ must be even. R. Meshulam proved a generalization of this result without using the normal form for the pencil [M]. We also prove it in two different ways in Theorem 2.15.

The classification problem when $\dim(A) \geq 3$ is more difficult although several interesting results are known. R. Westwick, building on work of J. Sylvester, showed that if a linear subspace $A \subset \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ has constant rank $r$, then $\dim(A) \leq m + n - 2r + 1$ [W], [S]. H. Flanders proved that if $r \leq m \leq n$, and $A \subset \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ is a linear space of bounded rank $r$ then $\dim(A) \leq rn$ [F] (see also [M2]). There is also a classification of such subspaces of near maximal dimension with the most recent results due to L.B. Beasley [B]. R. Meshulam found bounds for symmetric and skew symmetric linear families analogous to Flanders’ bound for general linear families mentioned above [M3]. Another approach is to try to classify subspaces $A \subset \text{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ of bounded rank $r$ for small values of $r$. This was accomplished first by M.D. Atkinson for $r \leq 3$ and then reproved by D. Eisenbud and J. Harris [A], [EH].

Our initial interest in linear spaces $A \subset S^2V^*$ of constant rank $r$ arose because such spaces can be constructed from smooth varieties having degenerate dual varieties. Recall that if $X^n \subset \mathbb{P}^N$ is a nonsingular projective variety of dimension $n$, then the dual variety $X^* \subset \mathbb{P}^N^*$ is the union of all hyperplanes $H$ such such that $H \cap X$ is singular, i.e. such that $H$ is tangent to $X$. A dimension count shows that one expects $X^*$ to be a hypersurface and it is interesting to study when this fails to occur. To that end, define the defect of $X$, $\delta$ by $\delta := N - 1 - \dim(X^*)$ and assume that $X^*$ is degenerate, i.e. $\delta \geq 1$. Let $H$ be a smooth point of $X^*$ and let $|II_{X^*,H}| \subset \mathbb{P}(S^2T^*_HX^*)$ denote the linear system of quadrics generated by the second fundamental form of $X^*$ at $H$ (see e.g. [Lan]). We prove:

Theorem 3.4. $|II_{X^*,H}|$ is a linear system of quadrics of projective dimension $\delta$ and constant rank $n - \delta$.

Thus, given a smooth variety with degenerate dual variety, Theorem 3.4 constructs a $\delta + 1$ dimensional linear subspace $A \subset S^2\mathbb{C}^{N-1-\delta}$ of constant rank $n - \delta$. This constant rank space of quadrics can also be viewed as arising from a result of L. Ein [E, Theorem 2.2]. We also prove an explicit inversion formula that allows one to recover $II_{X^*,H}$ from the second fundamental form and cubic form of any smooth point on $X$ that $H$ is tangent to (Theorem 3.9). Also, Griffiths and Harris proved that linear spaces $A \subset S^2V^*$ of bounded rank $r$ can be constructed from not necessarily smooth varieties which have degenerate duals [GH2, 3.5]. We give details and complete references in §3.

Our main linear algebraic result is:
Theorem 2.16. If \( r \) is even and \( \geq 2 \) then

\[
\max \{ \dim(A) \mid A \subset S^2V^* \text{ is of constant rank } r \} = m - r + 1.
\]

If \( X_r \subset \mathbb{P}(S^2V^*) \) is the projective variety of rank \( \leq r \) matrices, then an elementary dimension count shows that \( \text{codim}(X_{r-1}, X_r) = m - r + 1 \) which explains why one expects this bound. As noted above, the result when \( r \) is odd is classical; see Theorem 2.15. As an application of Theorem 2.16 and Theorem 3.4 we give a new proof of F. Zak’s result that for a smooth variety \( X \) with dual variety \( X^* \), \( \dim(X^*) \geq \dim(X) \). Moreover, Theorem 2.15 and Theorem 3.4 give a proof of the Landman parity theorem: if \( X \) is a nonsingular projective variety with degenerate dual variety then \( n - \delta \) is even (notation as above).

We prove Theorem 2.16 by first proving a more general step-wise result on symmetric degeneracy loci:

**Theorem 1.2.** Let \( X \) be a nonsingular simply connected projective variety of dimension \( m \), \( E \) a rank \( n \) vector bundle on \( X \), and \( L \) a line bundle on \( X \). Suppose that \( S^2(E^*) \otimes L \) is an ample vector bundle and that there is a constant even rank \( r \geq 2 \) symmetric bundle map \( E \to E^* \otimes L \). Then \( m \leq n - r \).

To \( A \subset S^2V^* \) we associate a short exact sequence of vector bundles on \( \mathbb{P}(A) \):

\[
0 \to K \to \mathcal{O}_m^m \to E \to 0
\]

where \( E \) has rank \( r \) and \( E \cong E^*(1) \) and then investigate restrictions on \( E \) and its chern classes in 2.4-2.6. We give examples of symmetric (and skew symmetric) linear spaces \( A \) and compute the associated exact sequences in 2.9-2.14.

We also survey some of the results of the linear algebraists, most recently due to Westwick, sometimes providing alternate proofs and references and tying them in with the algebraic geometry literature.

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§1. Even rank symmetric degeneracy loci

For the basic properties of ample vector bundles and the Lefschetz theorem used in this section we refer to [Laz]. If \( W \) is a variety, we use the notation \( b^i(W) := \dim(H^i(W, \mathbb{C})) \). The following is a well known proposition that will be used in the proof of the next theorem:

**Proposition 1.1.** Let \( Q^{r-2} \subset \mathbb{P}^{r-1} \) be a nonsingular quadric hypersurface. Then.

1. \( b^i(Q) = 1 \) if \( 0 \leq i \leq 2(r-2) \), \( i \) is even and \( \neq r-2 \).
2. \( b^{r-2}(Q) = 2 \) if \( r \) is even.
3. \( b^i(Q) = 0 \) for all other cases.
Proof. Except for $b^{r-2}(Q)$, all the cohomology of $Q$ can be computed by using the Lefschetz theorem, Poincaré duality and the cohomology of projective space. But $e(Q)$, the topological Euler characteristic of $Q$ is just the degree of the top chern class $\int_Q c_{r-2}(T_Q)$ which is $r$ if $r$ is even and $r-1$ if $r$ is odd. (The chern class computation is done for nonsingular surfaces in $\mathbb{P}^3$ in [GH, pg 601]. The computation for nonsingular hypersurfaces is analogous). This determines $b^{r-2}(Q)$. □

**Theorem 1.2.** Let $X$ be a nonsingular simply connected projective variety of dimension $m$, $E$ a rank $n$ vector bundle on $X$, and $L$ a line bundle on $X$. Suppose that $S^2(E^*) \otimes L$ is an ample vector bundle and that there is a constant even rank $r \geq 2$ symmetric bundle map $E \to E^* \otimes L$. Then $m \leq n - r$.

**Proof.** Consider the projective bundle $\pi : \mathbb{P}(E) \to X$. We first show that there is a subvariety $Y \subset \mathbb{P}(E)$ given as the zero locus of a section $t \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L)$ such that the fiber of $\pi|_Y : Y \to X$ over each $x \in X$ is a rank $r$ quadric hypersurface in $\mathbb{P}(E(x))$.

Indeed, there is a natural map $\pi^* \pi_*, \mathcal{O}_{\mathbb{P}(E)}(2) \to \mathcal{O}_{\mathbb{P}(E)}(2)$. Since $\pi_*, \mathcal{O}_{\mathbb{P}(E)}(2) \cong S^2(E^*)$ this gives after tensoring with $\pi^*L$ a map $u : \pi^*(S^2(E^*) \otimes L) \to \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L$. On the other hand, the symmetric bundle map $E \to E^* \otimes L$ defines a section $s \in H^0(X, S^2(E^*) \otimes L)$ which pulls back to give a section $\pi^*s \in H^0(\mathbb{P}(E), \pi^*(S^2(E^*) \otimes L))$. Let $t = u \circ \pi^*s \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L)$.

The section $t$ at $[v] \in \pi(E(x))$ is the linear map $\lambda \to s(x)(v, v)\lambda$. and so vanishes iff $s(x)(v, v) = 0$. But by hypothesis, this defines a rank $r$ quadric hypersurface in $\mathbb{P}(E(x))$.

**Claim 1.3.** $\mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L$ is ample.

**Proof.** First, recall that if $E$ is any vector bundle and $L$ any line bundle then $\mathbb{P}(E \otimes L) \cong \mathbb{P}(E)$ and $\mathcal{O}_{\mathbb{P}(E \otimes L^r)}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*L$.

Then since $S^2(E^*) \otimes L$ is ample on $X$, $\mathcal{O}_{\mathbb{P}(S^2(E) \otimes L^r)}(1) \cong \mathcal{O}_{\mathbb{P}(S^2(E))}(1) \otimes \sigma^*L$ is ample on $\mathbb{P}(S^2(E))$ where $\sigma : \mathbb{P}(S^2(E)) \to X$ is the natural projection. Now the second Veronese gives an inclusion $i : \mathbb{P}(E) \subset \mathbb{P}(S^2(E))$ such that $\pi = i \circ \sigma$. Then $i^* \mathcal{O}_{\mathbb{P}(S^2(E))}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(2)$. So $i^* \mathcal{O}_{\mathbb{P}(S^2(E))}(1) \otimes \sigma^*L \cong \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L$ is ample on $\mathbb{P}(E)$ as required. □

Thus, $\mathbb{P}(E) \setminus Y$ is an affine variety and has the homotopy type of a CW complex of (real) dimension $\leq m + n - 1$. Thus $H_i(\mathbb{P}(E) \setminus Y) = 0$ for $i \geq m + n$.

Let $K$ denote the kernel and $F$ denote the image of the map $E \to E^* \otimes L$. Then since the map has constant rank $r$, $K$ and $F$ are vector bundles on $X$ of ranks $n - r$ and $r$ respectively. Since the map is symmetric, there is a symmetric isomorphism $F \cong F^* \otimes L$. We have a natural map $p : \mathbb{P}(E) \setminus \mathbb{P}(K) \to \mathbb{P}(F)$ given on the fibers by linear projection centered at $\mathbb{P}(K(x)) \subset \mathbb{P}(E(x))$. $p$ is a $\mathbb{C}^{n-r}$-fiber bundle.

The isomorphism $F \cong F^* \otimes L$ determines a hypersurface $Z \subset \mathbb{P}(F)$ such that if $p : \mathbb{P}(F) \to X$ is the natural projection then the fiber of $p|_Z : Z \to X$ over each $x \in X$ is a smooth quadric in $\mathbb{P}(F(x))$. 


Now \( \mathbb{P}(K) \subset Y \) and so \( p \) restricts to a \( \mathbb{C}^{n-r} \)-fiber bundle map \( \mathbb{P}(E) \setminus Y \to \mathbb{P}(F) \setminus Z \). Thus, \( H_i(\mathbb{P}(E) \setminus Y) \cong H_i(\mathbb{P}(F) \setminus Z) \) and by Lefschetz duality this is isomorphic to \( H^{2(m+r-1)-i}(\mathbb{P}(F), Z) \). Hence \( H^i(\mathbb{P}(F), Z) = 0 \) for \( i \leq 2r+m-n-2 \) and then using the long exact sequence of the pair \( (\mathbb{P}(F), Z) \) we conclude that: \( H^i(Z) \cong H^i(\mathbb{P}(F)) \) for \( i \leq 2r + m - n - 3 \).

Now \( b^i(\mathbb{P}(F)) = b^i(X \times \mathbb{P}^{r-1}) \). So if \( f^{p,q} = b^p(X)b^q(\mathbb{P}^{r-1}) \) then by the Kunneth formula, \( b^i(\mathbb{P}(F)) = \sum_{p+q=i} f^{p,q} \).

By Deligne’s theorem [GH, pg 466], the Leray spectral sequence for \( \rho|_Z : Z \to X \) degenerates at the \( E_2 \) term. Since \( X \) is simply connected there is no monodromy so \( (R^i\rho|_Z)(\mathbb{C}) = \) the constant sheaf \( H^i(Q, \mathbb{C}) \) where \( Q \subset \mathbb{P}^{r-1} \) is a smooth quadric. Thus, \( E_2^{p,q} = H^p(X) \otimes H^q(Q) \). Let \( e_2^{p,q} = \dim(E_2^{p,q}) \). Then \( b^i(Z) = \sum_{p+q=i} e_2^{p,q} \).

Let \( g^{p,q} = e_2^{p,q} - f^{p,q} \). Thus by the above, \( \sum_{p+q=i} g^{p,q} = 0 \) for \( i \leq 2r + m + n - 3 \). On the other hand, since \( b^{-2}(Q) = 2 \) (this is where we use the fact that \( r \) is even), \( \sum_{p+q=-2} g^{p,q} = b^0(X) = 1 \). Thus \( 2r + m - n - 3 < r - 2 \) and so \( m \leq n - r \) as required. \( \Box \)

The previous theorem can be viewed as a version for symmetric maps of the following theorem of R. Lazarsfeld:

**Theorem 1.4.** [Laz, Theorem 2.2, pg. 41] Let \( X \) be a projective variety of dimension \( m \). Let \( E \) and \( F \) be vector bundles on \( X \) of ranks \( e \) and \( f \) respectively. Suppose that \( E^* \otimes F \) is ample and that there is a constant rank \( r \) vector bundle map \( E \to F \). Then \( m \leq e + f - 2r \).

Lazarsfeld used this theorem to give a step-wise proof of the non-emptiness of degeneracy loci which was originally proved by Fulton and Lazarsfeld:

**Theorem 1.5.** [Laz, Theorem 2.1, pg. 40] Let \( X \) be a projective variety of dimension \( m \). Let \( E \) and \( F \) be vector bundles on \( X \) of ranks \( e \) and \( f \) respectively. Suppose that \( E^* \otimes F \) is ample and let \( \phi : E \to F \) be a vector bundle map. If \( m \geq (e-r)(f-r) \) then \( X_r(\phi) = \{ x \in X \mid \text{rank}(\phi(x)) \leq r \} \) is non-empty.

There are analogues of the above theorem for symmetric maps and skew symmetric maps due to Harris and Tu [T], [HT]. However, in the case that \( r \) is odd, the proven result is not as sharp as the result conjectured.

**Question 1.6.** Is Theorem 1.2 true without the assumptions that \( X \) is nonsingular and simply connected and that the rank \( r \) is even? An affirmative answer would give a step-wise proof of the non-emptiness of symmetric degeneracy loci.

§2. Linear spaces of matrices satisfying rank conditions

Let \( V = \mathbb{C}^m, W = \mathbb{C}^n \) and let \( A \subset \text{Hom}(V, W) \) (or \( A \subset S^2V^* \) or \( A \subset \Lambda^2V^* \)) be a linear subspace. \( A \) is said to be of bounded rank \( r \) if for all \( q \in A \), \( \text{rank}(q) \leq r \), of rank bounded below by \( r \) if for all nonzero \( q \in A \), \( \text{rank}(q) \geq r \), and of constant rank \( r \) if for all nonzero \( q \in A \), \( \text{rank}(q) = r \).
Definition 2.1.

\[ l(r, m, n) = \max \{ \dim(A) \mid A \subset V \otimes W \text{ is of constant rank } r \} \]

\[ l'(r, m, n) = \max \{ \dim(A) \mid A \subset V \otimes W \text{ is of bounded rank } r \} \]

\[ \overline{\lambda}(r, m) = \max \{ \dim(A) \mid A \subset V \otimes W \text{ is of rank bounded below by } r \} \]

Similarly, define \( c(r, m), \underline{c}(r, m), \) and \( \underline{\lambda}(r, m) \) in the symmetric case and \( \lambda(r, m), \underline{\lambda}(r, m) \) in the skew symmetric case.

2.2 Vector bundles and maps associated to linear spaces of matrices.

For the basic concepts and definitions of vector bundles on projective space used in this part (e.g. chern classes, the splitting principle and uniform bundles) we refer to [OSS]. We say that a vector bundle \( E \) is free if it splits as a direct sum of line bundles. Given a linear subspace \( A \subset \Hom(V, W) \) with \( \dim(A) = l + 1 \) we can associate a vector bundle map \( \psi : V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}(A)} \) on \( \mathbb{P}(A) \) as follows: at \([q] \in \mathbb{P}(A)\) the fiber of \( \mathcal{O}_{\mathbb{P}(A)}(-1) \) is \( \lambda q, \lambda \in \mathbb{C} \) and \( \psi(v \otimes \lambda q) = \lambda \cdot q(v) \). Tensoring by \( \mathcal{O}_{\mathbb{P}(A)}(1) \) we get a map \( \phi : \mathcal{O}_{\mathbb{P}(A)}^m \to \mathcal{O}_{\mathbb{P}(A)}^n(1) \). Alternatively, choosing a basis for \( V \) and \( W \) and a basis \( x_0, \ldots, x_l \) for \( A^* \), \( A \) can be viewed as an \( n \times m \) matrix of linear forms in the \( x_i \) and such a matrix gives \( \phi \).

If \( A \) has constant rank \( r \) then the kernel \( K \), cokernel \( N \) and image \( E \) are vector bundles of rank \( m - r, n - r \) and \( r \) respectively and determine short exact sequences: \( 0 \to K \to \mathcal{O}_{\mathbb{P}(1)}^m \to E \to 0 \) and \( 0 \to E \to \mathcal{O}_{\mathbb{P}(1)}^n(1) \to N \to 0 \).

Let \( h = c_1(\mathcal{O}_{\mathbb{P}(1)}) \). If \( F \) is any vector bundle on \( \mathbb{P}(1) \), we can write \( c_i(F) = f_i h^i \) for some \( f_i \in \mathbb{Z} \). Thus \( c(F) = 1 + c_1(F) + \cdots + c_l(F) = 1 + f_1 h + \cdots + f_l h^l \).

The following theorem is due to R. Westwick [W]. The above construction and the use of chern classes to obtain a weaker results along the lines of the next theorem are due to J. Sylvester [S].

Theorem 2.3. [W] Suppose \( 2 \leq r \leq m \leq n \). Then

1. \( l(r, m, n) \leq m + n - 2r + 1 \)
2. \( l(r, m, n) = n - r + 1 \) if \( n - r + 1 \) does not divide \( (m - 1)!/(r - 1)! \)
3. \( l(r, r + 1, 2r - 1) = r + 1 \)

Proof. We’ll sketch Westwick’s proof of (1) and the \( \leq \) direction of (2). (The \( \geq \) direction is classical; see Proposition 2.10 below). From the above exact sequences, \( c(K)c(E) = 1 \) and \( c(E)c(N) = (1 + h)^n \). Thus, \( c(K)(1 + h)^n = c(N) \).

If \( n - r + 1 \leq i \leq l \) then \( c_i(N) = 0 \) and looking at the coefficient of \( h^i \) we get \( \sum_{j=0}^{m-r} \binom{n-j}{i-j} k_j = 0 \) where \( k_j h^j = c_j(K) \) and we use the convention that \( \binom{n}{j} = 0 \) if \( j < 0 \) or \( j > n \). The coefficient matrix of this collection of linear equations is \( M = \binom{n-j}{i-j} |_{0 \leq j \leq m-r, \ n-r+1 \leq i \leq l} \). If \( l = m + n - 2r + 1 \) then this is a square invertible matrix with determinant \( \prod_{j=0}^{m-r} j! \). Thus \( k_0 = 0 \) which is a contradiction since \( k_0 = 1 \). This proves (1). Westwick refers to a privately published manuscript of Muir and Metzler as a reference for evaluating this determinant however one
can also refer to e.g. [ACGH, pg. 93-95]. (2) follows directly from considering the linear equation with \( i = n - r + 1 \).

We remark that (1) is also a direct consequence of the vector bundle construction above and Lazarsfeld’s result, Theorem 1.4. (This was also previously observed by R. Meshulam). □

If \( A \subseteq S^2 V^* \) (resp. \( A \subseteq \Lambda^2 V^* \)) we similarly get a symmetric (resp. skew symmetric) map \( \phi : O_{\mathbb{P}(A)}^{m} \cong V \otimes O_{\mathbb{P}(A)} \rightarrow V^* \otimes O_{\mathbb{P}(A)}(1) \cong O_{\mathbb{P}(1)}^{m}(1) \) which can be viewed as given by an \( m \times m \) symmetric (resp. skew symmetric) matrix of linear forms in the \( x_i \).

If \( A \) has constant rank \( r \), the symmetry or skew symmetry implies that \( N \cong K^*(1) \) and \( E \cong E^*(1) \). This is because dualizing and then twisting the map \( \phi : O_{\mathbb{P}(1)}^{m} \rightarrow O_{\mathbb{P}(1)}(1) \) by \( O_{\mathbb{P}(1)}(1) \) we get the same map \( \phi \) in the symmetric case and \( -\phi \) in the skew symmetric case. The two short exact sequences above thus reduce to the single sequence \( 0 \rightarrow K \rightarrow O_{\mathbb{P}(1)}^{m} \rightarrow E \rightarrow 0 \) with \( E \cong E^*(1) \).

Conversely, given a surjection \( O_{\mathbb{P}(1)}^{m} \rightarrow E \rightarrow 0 \) where \( E \) is a vector bundle of rank \( r \) satisfying \( E \cong E^*(1) \) we can dualize and twist the surjection by \( O_{\mathbb{P}(1)}(1) \) to obtain: \( O_{\mathbb{P}(1)}^{m} \rightarrow E \cong E^*(1) \rightarrow O_{\mathbb{P}(1)}^{m}(1) \). The above composition then is given by an \( m \times m \) matrix of linear forms and leads to a dimension \( l + 1 \) linear space \( A \subseteq \text{Hom}(\mathbb{C}^m, \mathbb{C}^m) \) of constant rank \( r \). Note that the subspace \( A \) need not be symmetric or skew symmetric. For example, it could be the direct sum of symmetric and skew symmetric subspaces.

**Proposition 2.4.** Let \( A \subseteq S^2 V^* \) or \( A \subseteq \Lambda^2 V^* \) be a constant rank \( r \) subspace of dimension \( \geq 2 \). Then \( E \) is a uniform vector bundle of splitting type \( O_{\mathbb{P}(1)}^{\frac{r}{2}} \oplus O_{\mathbb{P}(1)}^{\frac{r}{2}}(1) \).

In particular, \( r \) is even.

*Proof.* Since \( E \cong E^*(1) \), \( E \) must be uniform of splitting type \( (a_1, \ldots, a_{\frac{r}{2}}, b_1, \ldots, b_{\frac{r}{2}}) \) where \( b_i = 1 - a_i \) (here one may need to reorder). But since \( E \) is globally generated, all the factors in its splitting type must be nonnegative. □

Using the isomorphism \( E \cong E^*(1) \) we get linear relations on the chern classes of \( E \). Specifically, suppose that \( r \geq l \), which is the only interesting case by Proposition 2.6. If \( c_i(E) = e_i h^i \) then for \( 0 \leq i \leq l \),

\[
e_i = \sum_{j=0}^{i} \binom{r-j}{i-j} (-1)^j e_j.
\]

For \( i \) odd, this expresses \( e_i \) as a linear combination of \( e_j \)'s with \( j < i \). Thus there are \( \geq \lceil l/2 \rceil \) linearly independent relations and it is not too hard to see that there are in fact exactly \( \lceil l/2 \rceil \). If \( i = 1 \) we get \( e_1 = r/2 \) (which gives another proof that \( r \) is even) and if \( i = 3 \), using \( e_1 = r/2 \) we get \( (r-2)e_2 - 2e_3 = r(r-1)(r-2)/12 \).

**Proposition 2.5.** For \( 1 \leq i \leq n \), \( 0 \leq e_i \leq e_i^1 = (r/2)^i \). Thus \( \{(c_1(E), \ldots, c_l(E)) \mid E \text{ is a globally generated vector bundle of rank } r \text{ on } \mathbb{P}^l \text{ and } E \cong E^*(1) \} \) is a finite set.
Proof. This follows by the discussion in [DPS, pg 317]. □

**Proposition 2.6.** If $r \leq l$ then $E \cong O_{pl}^{r/2} \oplus O_{pl}^{r/2}(1)$ or $r = l = 2$ and $E \cong T_{p2}(-1)$.

Proof. If $E$ is free then $E \cong O_{pl}^{r/2} \oplus O_{pl}^{r/2}(1)$ by Proposition 2.4. If $r < l$ then since $E$ is uniform by [OSS, Theorem 3.2.3, pg 55] $E$ is free. If $r = l$ and $E$ is not free then by the discussion on [OSS, pg 71], $E \cong \Omega_{p2}(a)$ or $\Omega_{p2}(2)$; but these last two bundles are isomorphic. □

### 2.7 Linear spaces with rank bounded below by $r$.

Let $X_r = \{ [q] \in P(\text{Hom}(V,W)) \mid \text{rank}(q) \leq r \}$ and similarly for $X_r \subset P(S^2V^*)$ and $X_r \subset P(\Lambda^2V^*)$. Then an elementary dimension count (see e.g. [ACGH, pg. 67 & pg. 101]) shows that:

1. $\text{codim}(X_r \subset P(\text{Hom}(V,W))) = (m-r)(n-r)$
2. $\text{codim}(X_r \subset P(S^2V^*)) = \binom{m-r+1}{2}$
3. $\text{codim}(X_r \subset P(\Lambda^2V^*)) = \binom{m-r}{2}$ (r even)

By Bezout’s theorem, this implies the essentially classical:

**Proposition 2.8.**

1. $\mathcal{l}(r,m,n) = (m-r)(n-r)$.
2. $\mathcal{c}(r,m) = \binom{m-r+1}{2}$.
3. $\mathcal{\lambda}(r,m) = \binom{m-r}{2}$ (r even).

### 2.9 Some examples of symmetric and skew-symmetric constant rank linear systems.

Given a linear subspace $A \subset \text{Hom}(V,W)$, we can consider $A$ as a linear subspace of $S^2(V^* \oplus W)$ via the natural inclusion $\text{Hom}(V,W) = V^* \otimes W \subset S^2(V^* \oplus W)$. To distinguish these two different linear embeddings of $A$, we’ll denote $A \subset S^2(V^* \oplus W)$ by $B$. If $A$ has constant rank (resp. bounded rank, resp. rank bounded below by) $r$ then $B$ is a linear space of quadrics of constant rank (resp. bounded rank, resp. rank bounded below by) $2r$. We say that $B$ is a *doubling* of $A$. In matrices, systems formed by doubling look like:

\[
\left\{ \begin{pmatrix} 0 & a \\ t_a & 0 \end{pmatrix} \mid a \in A \right\}.
\]

Similarly, we can double $A \subset \text{Hom}(V,W)$ to get $B \subset \Lambda^2(V^* \oplus W)$.

The following proposition is essentially classical:

**Proposition 2.10.**

1. If $0 < r \leq m \leq n$ then $l(r,m,n) \geq n-r+1$.
2. If $r$ is even and $\geq 2$ then $c(r,m) \geq m-r+1$ and $\lambda(r,m) \geq m-r+1$. 

8
Proof. $X_{r-1} \subset \mathbb{P}(\text{Hom}(\mathbb{C}^r, \mathbb{C}^n))$ has codimension $n-r+1$. Thus we can find an $n-r+1$ dimensional subspace $A \subset \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ of constant rank $r$ from which (1) follows. For (2), note that doubling $A$ produces an $n-r+1$ dimensional subspace $B \subset S^2\mathbb{C}^{r+n}$ (or $\Lambda^2\mathbb{C}^{r+n}$) of constant rank $2r$. □

2.11 Example of linear systems with $E$ free. For the $m-r+1$ dimensional subspace $B$ of $S^2\mathbb{C}^m$ (or $\Lambda^2\mathbb{C}^m$) of constant even rank $r$ produced in the above proof the associated vector bundle $E \cong \mathcal{O}_{\mathbb{P}^m}^{r/2} \oplus \mathcal{O}_{\mathbb{P}^m}^{r/2}(1)$. This follows for the symmetric case since in matrices

$$B = \begin{pmatrix} 0 & A \\ tA & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{r/2} \\ tA & 0 \end{pmatrix} \begin{pmatrix} I_{r/2} & 0 \\ 0 & A \end{pmatrix}$$

and so the image of the induced map $\phi : \mathcal{O}_{\mathbb{P}^m}^m \to \mathcal{O}_{\mathbb{P}^m}^m(1)$ is clearly $\mathcal{O}_{\mathbb{P}^m}^{r/2} \oplus \mathcal{O}_{\mathbb{P}^m}^{r/2}(1)$.

2.12 Examples of linear systems with $E$ not free arising from the Euler sequence.

Given any vector space $V$ of dimension $m$, for $1 \leq k \leq m-1$ there is a natural inclusion

$$V \to \text{Hom}(\Lambda^k V, \Lambda^{k+1} V)$$

$$v \to (\phi_v : \alpha \to v \wedge \alpha)$$

Now if $v \neq 0$, $\ker(\phi_v) = v \wedge \Lambda^{k-1} V$ and so this gives an $m$-dimensional linear space of linear maps of constant rank $(m\choose k) - (m-1\choose k-1) = (m-1\choose k)$. This observation is due to Atkinson and Westwick [AW, pg 233].

Now suppose that $\dim(V) = 2a+1$ and fix a volume form on $V$ i.e. an isomorphism $\Lambda^{2a+1} V \cong \mathbb{C}$ so that we can identify $\Lambda^a V^* \cong \Lambda^{a+1} V$. Then $V \subset \text{Hom}(\Lambda^aV, \Lambda^a V^*)$ and we can think of $\phi_v$ as a bilinear form on $\Lambda^a V$. Then $\phi_v(\alpha, \beta) = v \wedge \alpha \wedge \beta = (-1)^a v \wedge \beta \wedge \alpha = (-1)^a \phi_v(\beta, \alpha)$. Thus, if $a$ is even then in fact $V \subset S^2(\Lambda^a V^*)$ and if $a$ is odd then $V \subset \Lambda^2(\Lambda^a V^*)$. So, we have constructed a $2a + 1$ dimensional linear space of $(2a+1\choose a) \times (2a+1\choose a)$ matrices (symmetric for $a$ even and skew symmetric for $a$ odd) of rank exactly $(2a\choose a)$.

From the vector bundle point of view, at $[v] \in \mathbb{P}(V)$, the map $\Lambda^a V \to \Lambda^a V^*$ factors: $\Lambda^a V \to \Lambda^a (V/v) \to \Lambda^a V^*$. Recall that the fiber of $T_{\mathbb{P}(V)}(-1)$ at $[v]$ is $V/v$ so we conclude that $E$ is just $\Lambda^a(T_{\mathbb{P}(V)}(-1))$. Thus the associated short exact sequence is just the $a$-th exterior power of the Euler sequence i.e.

$$0 \to \mathcal{O}_{\mathbb{P}^{2a}}(-1) \otimes \Lambda^{a-1}(T_{\mathbb{P}^{2a}}(-1)) \to \Lambda^a \mathcal{O}_{\mathbb{P}^{2a}}^{2a+1} \to \Lambda^a(T_{\mathbb{P}^{2a}}(-1)) \to 0$$

In particular, $E$ is not free.

The $a = 1$ case yields the $3 \times 3$ skew symmetric matrices; doubled it is the linear system of quadrics corresponding to the second fundamental form of the
Grassmanian $G(2, 5)$ (which is self-dual) (see §3), which is also the space of quadrics vanishing on the Segre, $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The $a = 2$ case (which is already symmetric) is the linear system of quadrics corresponding to the second fundamental form of the ten dimensional spinor variety (which is also self-dual), which is also the space of quadrics vanishing on the Grassmanian, $G(2, 5)$. (One can prove these statements via a direct coordinate computation).

2.13 Westwick’s example. What follows is an intrinsic construction of a 3 dimensional linear space of $2a + 1 \times 2a + 1$ skew symmetric matrices of constant rank $2a$ where $a \geq 1$. For $a = 1$ this is the $3 \times 3$ skew symmetric matrices. This example is given in coordinates in Sylvester’s paper [S, pg 4] where it is attributed to Westwick. Let $V$ be a $2a + 1$ dimensional vector space. Then $\mathbb{P} := \mathbb{P}(\Lambda^2 V^*) = X_{2a}$ and $X_{2a-1} = X_{2a-2}$. Since a skew symmetric matrix can not have odd rank. Now $\text{codim}(X_{2a-2}, \mathbb{P}) = 3$ so any $\mathbb{P}^2 \subset \mathbb{P} \setminus X_{2a-2}$ will work. An easy chern class computation gives that the associated short exact sequence is:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{O}_{\mathbb{P}^2}^{2a+1} \to E \to 0.$$

Example 2.14. Consider the special case when $l = 3$ and $m - r = 2$. Such an example can’t arise from a linear space of symmetric matrices of constant rank by Theorem 2.16. Then $0 = c_3(K) = -e_3 + 2e_1e_2 - e_1^3$. Using the linear relations in the $e_i$ from the discussion following Proposition 2.4 it follows that $c(K) = 1 + (-r/2)h + (r(r+1)/12)h^2$ and $c(E) = 1 + (r/2)h + (r(2r-1)/12)h^2 + (r^2(r-2)/24)h^3$ and so $r = 8$ gives the smallest rank possibility: $c(E) = 1 + 4h + 10h^2 + 16h^3$. In fact, such an example, given by a skew symmetric family $A$ was given by Westwick [W2, pg 168].

Theorem 2.15. If $r$ is odd then $c(r, m) = 1$.

This theorem follows immediately from Proposition 2.4 or alternatively, from the first chern class computation in the discussion after Proposition 2.4. However, as mentioned in the introduction, it was known classically as a consequence of the Kronecker-Weierstrass theory giving a normal form for pencils of symmetric matrices of bounded rank $r$ [G, Chp. XII], [HP, Chp. VIII, Sec. 10]. There is also a generalization by Roy Meshulam [M].

Theorem 2.16. If $r$ is even and $\geq 2$ then $c(r, m) = m - r + 1$.

Proof. By Proposition 2.10 we only need to prove the $\leq$ inequality. Given $A \subset S^2V^*$ of constant rank $r$ with $\dim(A) = l + 1$ and $\dim(V) = m$ as above we obtain a symmetric constant even rank $r$ map $\mathcal{O}_p^m \to \mathcal{O}_p^m(1)$. Now $\mathbb{P}^l$ is simply connected and $(S^2\mathcal{O}_p^m) \otimes \mathcal{O}_p^l(1)$ is ample and so by Theorem 1.2, $l \leq m - r$ as required.

§3. Applications to dual varieties

For the basic background on dual varieties for this section see [E] and the references there. Let $X^n \subset \mathbb{P}^N$ be a nonsingular projective variety and let $X^* \subset
\( \mathbb{P}^N^* \) be the dual variety of \( X \). Define the defect of \( X \), \( \delta \), by \( \delta := N - 1 - \dim(X^*) \).

Assume that \( \delta \geq 1 \) and let \( H \) be a smooth point of \( X^* \). Then \( H \) can be considered as a hyperplane in \( \mathbb{P}^N \) and the contact locus \( L = L_H = \{ x \in X \mid \tilde{T}_x X \subset H \} \) is a \( \delta \) dimensional (projective) linear space. Let \( N_{L/X} \) be the normal bundle to \( L \) in \( X \).

In his work on dual varieties Lawrence Ein proved that there is an isomorphism \( N_{L/X} \cong N_{L/X}^*(1) \) [E, Theorem 2.2]. In fact, it follows from Ein’s proof that this isomorphism is symmetric.

It is convenient to reformulate Ein’s result as follows in order to construct vector spaces of symmetric matrices of constant rank:

**Lemma 3.1.** A nonsingular projective variety \( X^n \subset \mathbb{P}^N \) with defect \( \delta > 0 \) determines a \( \delta + 1 \) dimensional linear subspace \( A \subset S^2 \mathcal{C}^{N-1-\delta} \) of constant rank \( n - \delta \) with associated short exact sequence:

\[
0 \rightarrow N^*_X/H(1)|_L \rightarrow \mathcal{O}_{\mathbb{P}^N}^{N-1-\delta} \rightarrow N_{L/X} \rightarrow 0.
\]

**Proof.** Start with the exact sequence

\[
0 \rightarrow N_{L/X} \rightarrow N_{L/H} \rightarrow N_{X/H}|_L \rightarrow 0
\]

which holds since \( H \) is tangent to \( X \) along \( L \). Then dualize, twist by \( \mathcal{O}_{\mathbb{P}^N}(1) \) and use the fact that \( N_{L/H} \cong \mathcal{O}_{\mathbb{P}^N}^{N-1-\delta}(1) \) and \( N_{L/X} \cong N^*_L/X(1) \) to obtain the short exact sequence in the statement of the lemma. Thus since \( N_{L/X} \cong N^*_L/X(1) \) is a symmetric isomorphism, we can obtain the constant rank linear subspace of symmetric matrices as described in 2.2. \( \Box \)

The above lemma, along with Theorem 2.16 gives a new proof of a theorem of F. Zak:

**Theorem 3.2.** [Z, Cor. 10, pg. 39] Let \( X^n \subset \mathbb{P}^N \) be a nonsingular projective variety. Then \( \dim(X^*) \geq \dim(X) \).

Let \( x \in X \) be a smooth point and let \( II_{X,x} \in S^2T^*_x X \otimes N_x X \) denote the projective second fundamental form of \( X \) at \( x \) (see e.g. [Lan]). Let \( |II_{X,x}| = \mathbb{P}(I^*I^*(N^*_x X)) \subset \mathbb{P}(S^2T^*_x X) \) denote the linear system of quadrics it generates.

The following theorem is a result proved (but not stated explicitly) by Griffiths and Harris [GH2, 3.5]. It shows that linear spaces of symmetric matrices of bounded rank arise from varieties \( X \) with degenerate duals. Note that the result does not assume that \( X \) is nonsingular.

**Theorem 3.3.** [Lan, 5.2] Let \( X^n \subset \mathbb{P}^N \) be a variety with defect \( \delta \geq 1 \) and let \( x \in X \) be a general point. Then \( |II_{X,x}| \) is a linear system of quadrics of projective dimension \( \delta \) and bounded rank \( n - \delta \).

Another interpretation of Lemma 3.1 is:
Theorem 3.4. Let \( X^n \subset \mathbb{P}^N \) be a smooth variety with defect \( \delta \geq 1 \) and let \( H \in X^* \) be any smooth point. Then \( |I_x^* \subset H| \) is a linear system of quadrics of projective dimension \( \delta \) and constant rank \( n - \delta \).

We remark that Theorem 3.4 provides a nice geometric way of seeing the linear system of quadrics of constant rank whose study is the basis of this paper. In fact, this observation was the beginning of our investigation of this problem and only later did we see the connection with the prior work of Ein.

We present our original proof of (3.4) as we derive a slightly more refined formula which shows how a singular quadric in \( |I_x^* \subset H| \) exactly corresponds to a singular point of \( X \).

Write \( \mathbb{P}^N = \mathbb{P}V \) and let \( \mathcal{I} \subset X \times X^* \) denote the standard incidence correspondence. Denote the smooth points of a variety \( Y \) by \( Y_{sm} \). Let \( H \in X_{sm}^* \) and let \( x \in L_H \cap X_{sm} \). Let \( :, \rangle : V \times V^* \to \mathbb{C} \) denote the canonical pairing. We let \( T_x X \subset V \) denote the cone over the embedded tangent space and in general a hat on an object in projective space denotes the corresponding cone. Note that \( :, \rangle \) descends to a nondegenerate pairing between \( r \) dimensional vector spaces:

\[
(\hat{T}_x X/\hat{T}_x L_H) \times (\hat{T}_H X^*/\hat{T}_H L_x) \to \mathbb{C}
\]

which we continue to denote by \( :, \rangle \). (To see why, recall that \( \hat{T}_x L_H = N^*_H X^*(1) \).)

Let \( (x_t, H_t) \) be the lifting of a curve in \( \mathcal{I} \) to \( V \times V^* \). Note that \( < x_0', H_0 > = 0 \), and \( < x_0, H'_0 > = 0 \), where prime denotes derivative. Differentiating \( < x_0, H'_0 > \), we have

\[
< x_0', H'_0 > = < x_0, H''_0 > .
\]

Now \( x'_0 \in \hat{T}_x X, H'_0 \in \hat{T}_H X^* \) and up to twists, \( < x_0, H''_0 > = n_x \langle I_x^* \subset H(W, W) \rangle \) where \( W \in T_H X^* \) corresponds to \( H'_0 \) and \( n_x \in N^*_H X^* \) corresponds to \( x_0 \). We polarize (3.6) to conclude that the quadratic form \( n_x \langle I_x^* \subset H \rangle \) is of rank \( r \). If we assume that \( L_H \cap X_{sing} = \emptyset \), then \( x \) corresponds to an arbitrary point of \( \mathbb{P}N^*_H X^* \) and we conclude:

Theorem 3.7. Let \( X^n \subset \mathbb{P}^N \) be a variety with dual variety \( X^* \) having defect \( \delta = n - r \). Say there exists an \( H \in X_{sm}^* \) such that \( H \) is only tangent to smooth points of \( X \). Then \( |I_x^* \subset H| \) is a \( \delta \) dimensional system of quadrics of constant rank \( r \) on \( T_H X^* \cong \mathbb{C}^{N-\delta-1} \).

Let \( x \in X_{sm} \) and let \( H \in X_{sm}^* \) be tangent to \( x \). Then

\[
q_H = Q_x
\]

where \( q_H \subset \mathbb{P}(T_x X/T_x L_H) \) is the smooth quadric hypersurface corresponding to evaluating \( I_x^* \subset H \) at a vector representing \( H \) and projecting to the quotient, \( Q_x \subset \mathbb{P}^{N-\delta-1} \).
\( \mathbb{P}(T_H X^*/T_H L_x) \) is the corresponding dual object, and \( T_x X/T_x L_H \) is canonically identified with \( (T_H X^*/T_H L_x)^* \) up to twists by (3.5) (and both spaces are identified with their duals by virtue of the quadrics on them).

To see the connection with ([Ein], 2.2), note that \( T_x X/T_x L_H = N_{L/X,x} \) and the symmetric isomorphism is given by \( q_H \), where one must put in the twist to account for the scaling of \( q_H \).

Differentiating again, we obtain

\[
< x'_0, H''_0 > = - < x'''_0, H_0 > - 2 < x''_0, H'_0 > .
\]

The first term may be interpreted as \( w II_{X^*, H}(W, W) \) where \( W \in T_H X^* \) corresponds to \( H'_0 \) and \( w \in N^*_H X^* \) corresponds to \( x'_0 \). The second term can be interpreted as \( -n_H \partial II_{X,x}(w, w, w) \) where \( \partial II_{X,x} \) is the cubic form (see [Lan]).

The third term can be interpreted as \( -2W \partial II_{X,x}(w, w) \) where here we consider \( W \in N^*_x X \) and \( w \in T_x X \) via the identifications described above. Polarizing, we have our complete inversion formula:

**Theorem 3.9.** Let \( H \in X_{sm}^* \) be such that \( L_H \cap X_{sing} = \emptyset \). Let \( x \in L_H \) be any point. Then \( II_{X^*, H} \) is determined by \( II_{X,x} \) and \( \partial II_{X,x} \). More precisely, given \( y \in N^*_H X^*, v, w \in T_H X^* \), one has the following formula defined up to twists and scales:

\[
(3.10) \quad y \partial II_{X^*, H}(v, w) = n_H \partial II_{X,x}(y, v_1, w_1) + n_H \partial II_{X,x}(v_1, w_1) + v_2 \partial II_{X,x}(v_1, w_1) + w_2 \partial II_{X,x}(y, v_1)
\]

All vector bundles in (3.10) are pulled back to \( I \). \( n_H \in N^*_x X \) denotes a vector representing \( H \). \( v_1, w_1 \in T_H X^*/T_H L_x \) denotes the projection of \( v, w \) to the quotient, and \( v_2, w_2 \in T_H L_x \) denotes their projection to the subspace. All vectors appearing on the right hand side of (3.10) are well defined elements up to twists and scales of the appropriate spaces via repeated use of (3.5).

The ambiguity usually occurring with the cubic form is eliminated by lifting to \( I \) and evaluating it on \( n_H \) paired with at least one element of the singular locus of \( q_H \).

One can get a completely well defined formula using the twists, but the resulting system of quadrics of course will be the same.

**3.11 Some questions.** Do the examples of 2.12 for \( a \geq 3 \) arise from interesting dual varieties? Of course, for odd \( a \) we must first double to get a symmetric system. \( a = 3 \) would yield a 46 dimensional variety in \( \mathbb{P}^{77} \) with defect 6 and \( a = 4 \) would be a 78 dimensional variety in \( \mathbb{P}^{135} \) with defect 8.

In the boundary case of Theorem 2.16, i.e. \( l = m - r \) (with \( l \geq 1 \)), the only examples we know have \( E = T_{\mathbb{P}^2}(-1)^{\otimes 2}, \Lambda^2(T_{\mathbb{P}^4}(-1)) \) or \( O_{\mathbb{P}^1}^{r/2} \oplus O_{\mathbb{P}^1}^{r/2}(1) \). Are there any others? Smooth varieties with \( \text{dim}(X) = \text{dim}(X^*) \) provide examples,
but assuming Hartshorne’s conjecture, Ein’s classification theorem [E, Theorem 4.5] along with the remarks at the end of 2.13 show that no further examples arise in this way. On the other hand, without assuming Hartshorne’s conjecture, Ein’s result shows that if $l \geq r \geq 4$ then $\mathcal{O}_{P^l}^{r/2} \oplus \mathcal{O}_{P^l}^{r/2}(1)$ does not arise from dual varieties.

Suppose $\mathcal{O}_{P^l}^m \to E \to 0$ and $E \cong E^*(1)$. If $r \leq 2l - 4$ is $E$ free? (One can also ask the same question with the more restrictive hypotheses that the isomorphism is symmetric or skew symmetric). The evidence for this is that this is the line below $(l, n) = (2, 2)$ and $(4, 6)$ corresponding to the non-free examples $T_{P^2}(-1)$ and $\Lambda^2(T_{P^4}(-1))$.

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