Abstract. Let $R$ be a semi-local regular domain containing an infinite perfect field $k$, and let $K$ be the field of fractions of $R$. Let $G$ be a reductive semi-simple simply connected $R$-group scheme such that each of its $R$-indecomposable factors is isotropic. We prove that for any Noetherian affine scheme $A =$ Spec $A$ over $k$, the kernel of the map
\[ H^1_{\text{ét}}(A \times_{\text{Spec } k} \text{Spec } R, G) \to H^1_{\text{ét}}(A \times_{\text{Spec } k} \text{Spec } K, G) \]
induced by the inclusion of $R$ into $K$, is trivial. If $R$ is the semi-local ring of several points on a $k$-smooth scheme, then it suffices to require that $k$ is infinite and keep the same assumption concerning $G$. The results extend the Serre—Grothendieck conjecture for such $R$ and $G$, proved in [PaStV].

1. Introduction

Recall that an $R$-group scheme $G$ is called reductive (respectively, semi-simple; respectively, simple), if it is affine and smooth as an $R$-scheme and if, moreover, for each ring homomorphism $s : R \to \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a reductive (respectively, semi-simple; respectively, simple) algebraic group over $\Omega(s)$. This notion of a simple $R$-group scheme coincides with the notion of a simple semi-simple $R$-group scheme of [SGA3, Exp. XIX, Def. 2.7 and Exp. XXIV, 5.3].

Such an $R$-group scheme $G$ is called simply-connected (respectively, adjoint), if for any homomorphism $s : R \to \Omega(s)$ of $R$ to an algebraically closed field $\Omega(s)$, the group $G_{\Omega(s)}$ is a simply-connected (respectively, adjoint) $\Omega(s)$-group scheme (see [SGA3, Exp. XXII, Def. 4.3.3]). A simple group scheme $G$ is called isotropic, if it contains a split torus $G_{m,R}$.

We prove the following theorem, which is an extension of the results on the Serre—Grothendieck conjecture obtained in [PaStV].

**Theorem 1.1.** Let $R$ be a regular semi-local domain containing an infinite perfect field $k$. Let $K$ be the field of fractions of $R$. Let $G$ be an isotropic simple simply connected $R$-group scheme.

For any Noetherian affine scheme $A =$ Spec $A$ over $k$, the map
\[ H^1_{\text{ét}}(A \times_{\text{Spec } k} \text{Spec } R, G) \to H^1_{\text{ét}}(A \times_{\text{Spec } k} \text{Spec } K, G) \]
induced by the inclusion of $R$ into $K$, has trivial kernel.

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This theorem is deduced, via a theorem of D. Popescu, from its following “geometric” version.

**Theorem 1.2.** Let \( R \) be a semi-local ring of several points on a \( k \)-smooth scheme over an infinite field \( k \). Let \( K \) be the field of fractions of \( R \). Let \( G \) be an isotropic simple simply connected \( R \)-group scheme.

For any Noetherian affine scheme \( A = \text{Spec} \, A \) over \( k \), the map

\[
H^1_{\text{ét}}(A \times_{\text{Spec} \, k} \text{Spec} \, R, G) \to H^1_{\text{ét}}(A \times_{\text{Spec} \, k} \text{Spec} \, K, G)
\]

induced by the inclusion of \( R \) into \( K \), has trivial kernel.

The proof of this theorem is given in Section 3. It uses, on one hand, the constructions and results of \([\text{PaStV}]\). On the other hand, it requires the following extension of Quillen’s local-global principle for projective modules \([\text{Q}, \text{Theorem } 1]\), due to L.-F. Moser (previously announced without proof by Raghunathan \([\text{R1, Theorem } 2]\), and hinted in \([\text{BCW}]\)).

**Theorem.** \([\text{Mo, Korollar } 3.5.2]\) Let \( A \) be a Noetherian commutative ring, \( G \) a group scheme over \( A \) admitting a closed embedding \( G \to \text{GL}_n,A \) for some \( n \geq 1 \). Let \( E \) be a \( G \)-torsor over \( A \), such that \( E \) is trivial on \( A_{\{0\}} \times \text{Spec} \, A \) for all elements \( U_i \) of a Zariski covering \( \text{Spec} \, A = \bigcup U_i \), and on the zero-section \( \{0\} \times \text{Spec} \, A \). Then \( E \) is trivial.

Using this local-global principle one more time, we obtain the following corollary of Theorem 1.1.

**Corollary 1.3.** Let \( S \) be a Noetherian ring such that for any maximal ideal \( m \) of \( S \), the local ring \( S_m \) satisfies the conditions imposed on \( R \) in Theorem 1.1 or in Theorem 1.2. Let \( G \) be a simple simply connected \( S \)-group scheme admitting a closed embedding \( G \to \text{GL}_n,S \) for some \( n \geq 1 \), and such that for any maximal ideal \( m \) of \( S \), the group \( G_{S_m} \) is isotropic. Let \( K \) be the field of fractions of \( S \). Then the natural map

\[
H^1_{\text{ét}}(S[t], G) \to H^1_{\text{ét}}(K(t), G)
\]

has trivial kernel.

**Proof.** Consider the composition

\[
H^1_{\text{ét}}(S[t], G) \to H^1_{\text{ét}}(K[t], G) \to H^1_{\text{ét}}(K(t), G).
\]

By \([? , \text{Prop. } 2.2]\) the map \( H^1_{\text{ét}}(K[t], G) \to H^1_{\text{ét}}(K(t), G) \) has trivial kernel. It remains to prove that \( H^1_{\text{ét}}(S[t], G) \to H^1_{\text{ét}}(K[t], G) \) has trivial kernel. By the local-global principle, we can substitute \( S \) by its localization at a maximal ideal, and then apply Theorem 1.1 for \( A = A_k \).

**Remark 1.** The conditions of Corollary 1.3 on \( S \) are satisfied, in particular, if \( S \) is a (not necessarily semilocal) regular ring containing an infinite perfect field, or if \( \text{Spec} \, S \) is a smooth affine scheme over an infinite field.

**Corollary 1.4.** Let \( S, G \) be as in Corollary 1.3. Assume moreover that the field of fractions \( K \) of \( S \) is perfect. Then the map

\[
H^1_{\text{ét}}(S[t], G) \to H^1_{\text{ét}}(S, G)
\]

...
induced by evaluation at $t = 0$, has trivial kernel.

Proof. We have a commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{ét}}(S[t], G) & \xrightarrow{t=0} & H^1_{\text{ét}}(S, G) \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(K[t], G) & \xrightarrow{t=0} & H^1_{\text{ét}}(K, G).
\end{array}
\]

Since $K$ is perfect, the bottom line is an isomorphism by the main result of [RR]. The left vertical line has trivial kernel by Corollary 1.3.

Remark 2. The conditions of Corollary 1.4 on $S$ are satisfied, in particular, if $S$ is a (not necessarily semilocal) regular ring containing $\mathbb{Q}$, or if $\text{Spec } S$ is a smooth affine scheme over a field of characteristic 0.

Remark 3. All the above results can be easily extended to the case where $G$ is not simple but semisimple, and satisfies the following isotropy condition: every semisimple normal subgroup of $G$ is isotropic. This follows from Faddeev—Shapiro lemma [SGA3, Exp. XXIV Prop. 8.4] (see also [PaStV, Section 12]).

2. Construction of a bundle over an affine line

Let $k$, $R$, $K$, $G$ be as in Theorem 1.2. Let $\mathcal{A}$ be any scheme over $k$. In this section we show that any (étale) principal $G$-bundle over $\mathcal{A} \times_{\text{Spec } k} \text{Spec } R$ which becomes trivial over $\mathcal{A} \times_{\text{Spec } k} \text{Spec } K$ can be substituted by a principal $G$-bundle $P_t$ over $\mathcal{A} \times \mathbb{A}^1_R \times \mathcal{A}$, for some monic polynomial $f \in R[t]$, in such a way that the triviality of this new bundle implies the triviality of $P$. The argument is an extension of the argument of [PaStV, §6].

For compatibility with [PaStV], in this section we denote $R$ by $\mathcal{O}$. We set

$Y := \mathcal{A} \times_{\text{Spec } k} \text{Spec } \mathcal{O} = \mathcal{A} \times_{\text{Spec } k} \text{Spec } R$

for shortness.

Fix a smooth affine $k$-scheme $X$ and a finite family of points $x_1, x_2, \ldots, x_n$ on $X$, such that $\mathcal{O} = O_X,\{x_1, x_2, \ldots, x_n\}$. Set $U := \text{Spec}(\mathcal{O})$. Let

$\text{can} : U \to X$

be the canonical map. Further, consider a simple simply-connected $U$-group scheme $G$.

Let $P$ be a principal $G$-bundle over the scheme $Y$ which is trivial over $Y \times_{\text{Spec } \mathcal{O}} \text{Spec } K$. We may and will assume that for certain $f \in \mathcal{O}$ the principal $G$-bundle $P$ is trivial over $Y \times_{\text{Spec } \mathcal{O}} \text{Spec } \mathcal{O}_f$. Shrinking $X$ if necessary, we may secure the following properties

(i) The points $x_1, x_2, \ldots, x_n$ are still in $X$.

(ii) The group scheme $G$ is defined over $X$ and it is a simple group scheme. We will often denote this $X$-group scheme by $G_X$ and write $G_U$ for the original $G$.

(iii) The principal $G$-bundle $P$ is defined over $Y \times_{\text{Spec } \mathcal{O}} X$ and the function $f \in \mathcal{O}$ belongs to $k[X]$. 

(iv) The restriction $P_f$ of the bundle $P$ to the open subset $Y \times_{\text{Spec} \mathcal{O}} X_f$ is trivial and $f$ vanishes at each $x_i$’s.

In particular, now we are given the smooth irreducible affine $k$-scheme $X$, the finite family of points $x_1, x_2, \ldots, x_n$ on $X$, and the non-zero function $f \in k[X]$ vanishing at each point $x_i$. It was shown in [PaStV, Section 5] that, starting with these data, one can construct what is called there a nice triple [PaStV, Def. 4.1], of the form $(q_U : X \to U, f, \Delta)$. This triple fits into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{q_X} & X \\
\downarrow q_U \quad \quad \quad \downarrow \Delta_{\text{can}} \\
U & \xrightarrow{\Delta} & X
\end{array}
$$

where

$$q_X \circ \Delta = \text{can}$$

and

$$q_U \circ \Delta = \text{id}_U.$$ 

Moreover, $f := q_X^*(f)$. We did that shrinking $X$ along the way, but all properties (i) to (iv) were preserved.

In particular, the restriction $P_f$ of the bundle $P$ to the open subscheme $Y \times_{\text{Spec} \mathcal{O}} X_f$ is trivial by Item (iv) above.

Set $G_X := (q_X)^*(G)$, and let $G_{\text{const}}$ be the pull-back of $G_U$ to $\mathcal{X}$ via $q_U$. By [PaStV, Theorem 4.3] there exists a morphism of nice triples [PaStV, Def. 4.2]

$$\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$$

and an isomorphism

$$\Phi : \theta^*(G_{\text{const}}) \to \theta^*(G_X) =: G_{\mathcal{X}'}$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \text{id}_{G_{\mathcal{X}'}}$.

Set

$$q_X' = q_X \circ \theta : \mathcal{X}' \to X.$$ 

Recall that

$$q_U' = q_U \circ \theta : \mathcal{X}' \to U,$$

since $\theta$ is a morphism of nice triples.

Consider the pullback $(q_X')^*(P)$ of $P$ from $Y \times_U X$ to $Y \times_U \mathcal{X}'$ as a principal $(q_U')^*(G_U) = \theta^*(G_{\text{const}})$-bundle via the isomorphism $\Phi$.

Recall that $P$ is trivial as a $G$-bundle over $Y \times_U X_f$. Therefore, $(q_X')^*(P)$ is trivial as a principal $G_{\mathcal{X}'}$-bundle over $Y \times_U \mathcal{X}'_{q_U(f')}$. Since $\theta$ is a nice triple morphism one has $f' = \theta^*(f) \cdot g'$, and thus the principal $G_{\mathcal{X}'}$-bundle $(q_X')^*(P)$ is trivial over $Y \times_U \mathcal{X}_{f'}$. 
We conclude that \((q'_X)^*(P)\) is trivial over \(Y \times_U \mathcal{X}'_f\), when regarded as a principal \(G_U\)-bundle (more precisely, \((q'_U)^*(G_U)\)-bundle; we omit this base change from the notation) via the isomorphism \(\Phi\).

By [PaStV, Theorem 4.5] there exists a finite surjective morphism \(\sigma : \mathcal{X}' \to \mathbb{A}^1 \times U\) of \(U\)-schemes satisfying

1. \(\sigma\) is étale along the closed subset \(\{f' = 0\} \cup \Delta'(U)\).
2. For a certain element \(g'_{f',\sigma} \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}\) and a unitary polynomial \(N(f') \in \mathcal{O}[t] \to \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})\), defined by the distinguished \(\sigma\) as in [PaStV, Section 4], one has
   \[
   \sigma^{-1}\left(\{f' = 0\}\right) = \{N(f') = 0\} = \{f' = 0\} \cup \{g'_{f',\sigma} = 0\}.
   \]
3. Denote by \((\mathcal{X}')^0 \to \mathcal{X}'\) the largest open sub-scheme, where the morphism \(\sigma\) is étale. Write \(g'\) for \(g'_{f',\sigma}\) from now on. Then the square
   \[
   (\mathcal{X}')^0_0 = (\mathcal{X}')^0_{g'} \xrightarrow{\text{inc}} (\mathcal{X}')^0_{g'}
   \]

is an elementary Nisnevich square. (Here \(\sigma_{g'}^0\) and \(\sigma_{g'}^0\) stand for the corresponding restrictions of \(\sigma\).)

4. One has \(\Delta'(U) \subset (\mathcal{X}')^0_g\).

Regarded as a principal \(G_U\)-bundle via the isomorphism \(\Phi\), the bundle \((q'_X)^*(P)\) over \(Y \times_U \mathcal{X}'\) becomes trivial over \(Y \times_U \mathcal{X}'_f\), and a fortiori over \(Y \times_U (\mathcal{X}')^0_{g'}\). Now, glueing the trivial \(G_U\)-bundle over \(Y \times_U (\mathbb{A}^1 \times U)_{N(f')}\), to the bundle \((q'_X)^*(P)\) along the isomorphism

\[
\psi : Y \times_U (\mathcal{X}')^0_{N(f')} \times U G_U \to (q'_X)^*(P)|_{Y \times_U (\mathcal{X}')^0_{N(f')}}
\]

of principal \(G_U\)-bundles, we get a principal \(G_U\)-bundle \(P_\mathcal{X}\) over \(Y \times_U (\mathbb{A}^1 \times U)\) such that

1. it is trivial over \(Y \times_U (\mathbb{A}^1 \times U)_{N(f')}\),
2. \((\sigma)^*(\mathcal{P}_1)\) and \((q'_X)^*(\mathcal{P})\) are isomorphic as principal \(G_U\)-bundles over \(Y \times_U (\mathcal{X}')^0_{g'}\).

Here \((q'_X)^*(\mathcal{P})\) is regarded as a principal \(G_U\)-bundle via the \(\mathcal{X}'\)-group scheme isomorphism \(\Phi\) from \((3)\);

3. over \(Y \times_U (\mathcal{X}')^0_{N(f')}\) the two \(G_U\)-bundles are identified via the isomorphism \(\psi\) from \((9)\).

Finally, form the following diagram

\[
\begin{array}{ccc}
\mathbb{A}^1_U & \xrightarrow{\sigma_{g'}^0 = \sigma_{\mathcal{X}'}^0} & (\mathcal{X}')^0_g' \\
pr & \downarrow{\mathcal{q}_U} & \downarrow{\Delta'} \\
U & \xrightarrow{\mathcal{q}_U^*} & X
\end{array}
\]

This diagram is well-defined since by Item (4) above the image of the morphism \(\Delta'\) lands in \((X')^0_{g'}\).
Lemma 2.1. The unitary polynomial $h = N(f') \in \Theta [t]$, the principal $G_U$-bundle $P_t$ over $Y \times U \mathbb{A}^1_U$, the diagram [(10)] and the isomorphism [(5)] constructed above has the following properties:

$(1^*) \; q'_U = \text{pr} \circ \sigma$,

$(2^*) \; \sigma$ is étale,

$(3^*) \; q'_U \circ \Delta' = \text{id}_U$,

$(4^*) \; q'_X \circ \Delta' = \text{can}$,

$(5^*)$ the restriction of $P_t$ to $Y \times_U (\mathbb{A}^1_U)_h$ is a trivial $G_U$-bundle,

$(6^*) \; (\sigma)^*(P_t)$ and $(q'_X)^*(P)$ are isomorphic as principal $G_U$-bundles over $Y \times_U (\mathcal{X})^0_f$. Here $(q'_X)^*(P)$ is regarded as a principal $G_U$-bundle via the group scheme isomorphism $\Phi$.

Proof. By the very choice of $\sigma$ it is an $U$-scheme morphism, which proves $(1^*)$. Since $(\mathcal{X})^0 \hookrightarrow \mathcal{X}'$ is the largest open sub-scheme where the morphism $\sigma$ is étale, one gets $(2^*)$. Property $(3^*)$ holds for $\Delta'$ since $(\mathcal{X}', f', \Delta')$ is a nice triple and, in particular, $\Delta'$ is a section of $q'_U$. Property $(4^*)$ can be established as follows:

$$q'_X \circ \Delta' = (q_X \circ \theta) \circ \Delta' = q_X \circ \Delta = \text{can}.$$ 

The first equality here holds by the definition of $q'_X$, see [(3)]; the second one holds, since $\theta$ is a morphism of nice triples; the third one follows from [(3)]. Property $(5^*)$ is just Property 1 in the above construction of $P_t$. Property $(6^*)$ is precisely Property 2 in our construction of $P_t$. \hfill \square

One readily sees that the properties in Lemma 2.1 imply that if the $G$-bundle $P_t$ is trivial on $Y \times_U \mathbb{A}^1_U$, then the original bundle $P$ is trivial on $Y$.

Indeed, if $P_t$ is trivial, then by Property $(6^*)$ in Lemma 2.1, the $G_U$-bundle $(q'_X)^*(P)$ over $Y \times_U (\mathcal{X})^0_f$ is trivial as well. Hence, using Property $(4^*)$, we deduce that the bundle $(\Delta')^*((q'_X)^*(P)) = \text{can}^*(P)$ is a trivial $(\Delta')^*((q'_X)^*(G)) = \text{can}^*(G)$-bundle over $Y \times_U U = Y$.

3. Proofs of Theorems 1.1 and 1.2

The following easy lemma was essentially proved inside the proof of [PaStV, Theorem 8.6]. Here we provide a more detailed proof in a slightly more general situation.

Lemma 3.1. Let $R$ be a semilocal ring, $G$ a simply connected semisimple group scheme over $R$. There exists a closed embedding $G \to \text{GL}_{n,R}$ for some $n \geq 1$.

Proof. We can assume without loss of generality that $R$ is connected. Let $U = \text{Spec} R$. The $U$-group scheme $G$ is given by a 1-cocycle $\xi \in Z^1(U, \text{Aut}(G_0))$, where $G_0$ is the split simply connected simple group scheme over $U$ of the same type as $G$, and $\text{Aut}(G_0)$ is the automorphism group scheme of $G_0$. Recall that $\text{Aut}(G_0) \cong G_0^{\text{ad}} \rtimes N$, where $N$ is the finite group of automorphisms of the Dynkin diagram of $G_0$, and $G_0^{\text{ad}}$ is the adjoint group corresponding to $G_0$. Since $\text{Aut}(G_0) \cong G_0^{\text{ad}} \rtimes N$, we have an exact sequence of pointed sets

$$\{1\} \to H^1(U, G_0^{\text{ad}}) \to H^1(U, G_0^{\text{ad}} \rtimes N) \to H^1(U, N).$$
Thus there is a finite étale morphism \( \pi : V \to U \) such that \( G_V := G \times_U V \) is given by a 1-cocycle \( \xi_V \in Z^1(U, G^{ad}_0) \). We can choose \( V \) so that \( V/U \) is moreover a Galois extension.

For each fundamental weight \( \lambda \) of \( G_0 \), there is a central (also called center preserving, see \cite{PeSt}) representation \( \rho_\lambda : G_0 \to GL_{V_\lambda \otimes \mathbb{Z} U} \), where \( V_\lambda \) is the Weyl module over \( \mathbb{Z} \) corresponding to \( \lambda \). This gives a commutative diagram of \( U \)-group morphisms

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\rho_\lambda} & GL_{V_\lambda \otimes \mathbb{Z} U} \\
\downarrow & & \downarrow \\
G^{ad}_0 & \xrightarrow{\bar{\rho}_\lambda} & PGL_{V_\lambda \otimes \mathbb{Z} U}.
\end{array}
\]

Considering the product of \( \rho_\lambda \)'s with \( \lambda \) running over the set \( \Lambda \) of all fundamental weights, we obtain the following commutative diagram of algebraic \( k \)-group homomorphisms:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\rho} & \prod_{\lambda \in \Lambda} GL_{V_\lambda \otimes \mathbb{Z} U} \\
\downarrow & & \downarrow \\
G^{ad}_0 & \xrightarrow{\bar{\rho}} & \prod_{\lambda \in \Lambda} PGL_{V_\lambda \otimes \mathbb{Z} U}.
\end{array}
\]

By the definition of Weyl modules, \( \rho \) is a closed embedding (cf. \cite{PeSt} Lemma 2).

Twisting the \( V \)-group morphism \( \rho \) with the 1-cocycle \( \xi_V \) we get an \( V \)-group scheme morphism \( \rho_V : G_V \to \prod_{\lambda \in \Lambda} GL_1(A_\lambda) \), where the product is a product of group schemes over \( V \), and each \( A_\lambda \) is an Azumaya algebra over \( V \) obtained from \( \text{End}(V_\lambda \otimes \mathbb{Z} U) \) via the 1-cocycle \( \theta_\lambda = (\bar{\rho}_\lambda)_*(\xi_V) \in Z^1(V, PGL_{V_\lambda \otimes \mathbb{Z} U}) \). Composing \( \rho_V \) with the natural closed embedding \( \prod_{\lambda \in \Lambda} GL_1(A_\lambda) \hookrightarrow GL_{\oplus A_\lambda} \), we obtain a closed embedding

\[
G_V \hookrightarrow GL_{m,V},
\]

for a large enough integer \( m \).

One has

\[
\text{Hom}_V(G_V, GL_{m,V}) = \text{Hom}_U(G, R_{V/U}(GL_{m,V})),
\]

where \( R_{V/U} \) is the Weil restriction functor. Thus \( \rho_V \) determines an \( U \)-morphism

\[
\rho_U : G \hookrightarrow R_{V/U}(GL_{m,V}).
\]

Here \( \rho_U \) is a \( U \)-group scheme morphism, and, since \( \rho \) is a closed embedding, \( \rho_U \) is a closed embedding as well (étale descent).

Let \( d \) be the degree of the Galois extension \( V = \text{Spec} S \) over \( U = \text{Spec} R \). The \( U \)-group scheme \( R_{V/U}(GL_{m,V}) \) admits a natural closed embedding into \( GL_{md,U} \), such that, for any \( R \)-algebra \( X \), the image of \( g \in R_{V/U}(GL_{m,V})(X) = GL_m(X \otimes_R S) \) is the corresponding element of \( GL_{md}(X) \), the \( X \)-linear automorphism of \( X^{\otimes md} \cong (X \otimes_R S)^{\otimes m} \). Now, composing this embedding with \( \rho_U \), we obtain a closed embedding \( G \hookrightarrow R_{V/U}(GL_{m,V}) \hookrightarrow GL_{m,U} \), for \( n = md \).

\[\square\]
Theorem 3.2. Let $B$ be a semi-local Noetherian ring containing an infinite field. Let $G$ be an isotropic simply connected simple group scheme over $B$. Let $P$ be a principal $G$-bundle over $\mathbb{A}^1_B$ trivial over $(\mathbb{A}^1_B)_f$ for a monic polynomial $f \in B[t]$. Then $P$ is trivial.

Proof. This theorem was proved in [PaStV]. Indeed, this is precisely [PaStV, Theorem 2.1], except that in that theorem the base ring $B$ was required to be “of geometric type”, i.e. a semilocal ring of finitely many points on a smooth variety over an infinite field. However, tracing the proof of this statement, one readily sees that the only properties of $B$ that are used are that $B$ is semi-local, Noetherian, and contains an infinite field. (The “geometric type” assumption was an umbrella assumption in the most part of [PaStV], since it is crucial for the validity of the main theorem [PaStV, Theorem 1.2].)

Proof of Theorem 1.2. Consider the case where $R$ is a semi-local ring of several points on a $k$-smooth scheme over an infinite field $k$ (the “geometric case”). Let $P$ be a principal $G$-bundle which is in the kernel of the map $H^1_{\text{ét}}(\mathcal{A} \times \text{Spec } k \text{Spec } R, G) \to H^1_{\text{ét}}(\mathcal{A} \times \text{Spec } k \text{Spec } K, G)$. By considerations in § 2 there is a principal $G$-bundle $P_t$ over $\mathcal{A} \times_k \mathbb{A}^1_R$ trivial over $\mathcal{A} \times_k (\mathbb{A}^1_R)_f$ for a monic polynomial $f \in R[t]$, and such that if $P_t$ is trivial on the whole $\mathcal{A} \times_k \mathbb{A}^1_R$, then the original $G$-bundle $P$ over $\mathcal{A} \times_k \text{Spec } R$ is trivial as well. Thus, it is enough to show that $P_t$ is trivial.

Since $R$ is a semilocal ring containing an infinite field, and $f$ is monic, the Chinese remainder theorem implies that there is $a \in R$ with $f(a) \in R^\times$; changing the variable, we can assume that $f(0) \in R^\times$.

Set $B = \mathcal{A} \otimes_k R$. Note that $B$ is a Noetherian commutative ring containing an infinite field $k$. By Theorem 3.2 for any localization $B_m$ of $B$ at a maximal ideal $m \subseteq B$, the bundle $P_t \times_{\text{Spec } B} \text{Spec } B_m$ is trivial. By Lemma 3.1 the group scheme $G$ admits a closed embedding into some $\text{GL}_n$ over $R$, and hence, by base change, over $B$. Thus, we are given a principal $G$-bundle $P_t$ over $\mathbb{A}^1_B = \mathbb{A}^1_k \times_k \text{Spec } B$, which is trivial Zariski-locally in $\text{Spec } B$, as well as on $\{0\} \times \text{Spec } B$; and $G$ is a linear group. Then by Moser’s local-global principle [Mo, Korollar 3.5.2] $P_t$ is trivial on $\mathbb{A}^1_B = \mathcal{A} \times_k \mathbb{A}^1_R$.

Proof of Theorem 1.1. The claim follows from Theorem 1.2 via the well-known result of D. Popescu [Po, Sw]. Since the field $k$ is perfect, the morphism $k \to R$ is geometrically regular. Therefore, by Popescu’s theory $R$ is a filtered direct limit of smooth $k$-algebras. One readily sees that, since $R$ is semilocal, these smooth $k$-algebras can also be chosen to be semilocal rings of several points on a smooth $k$-variety. Since the functor $H^1_{\text{ét}}(\mathcal{A}, G)$ commutes with filtered direct limits, the result follows.

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