ERMAKOV-LEWIS DYNAMIC INVARIANTS
WITH SOME APPLICATIONS

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Master Thesis

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31 January 2000
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1. Introduction.

In this work we present a study of the Ermakov-Lewis invariants that are related to some linear differential equations of second order and one variable which are of much interest in many areas of physics. In particular we shall study in some detail the application of the Ermakov-Lewis formalism to several simple Hamiltonian models of “quantum” cosmology. There is also a formal application to the physical optics of waveguides.

In 1880, Ermakov [1] published excerpts of his course on mathematical analysis where he described a relationship between linear differential equations of second order and a particular type of nonlinear equation. At the beginning of the thirties, Milne [2] developed a method quite similar to the WKB technique where the same nonlinear equation found by Ermakov occurred, and applied it successfully to several model problems in quantum mechanics. Further, in 1950, the solution to this nonlinear differential equation has been given by Pinney [3].

On the other hand, within the study of the adiabatic invariants at the end of the fifties, a number of powerful perturbative methods in the phase space have been developed. In particular, Kruskal [4] introduced a certain type of canonical variables which had the merit of considerably simplifying the mathematical approach and of clarifying some quasi-invariant structures of the phase space. Kruskal’s results have been used by Lewis [5, 6] to prove that the adiabatic invariant found by Kruskal is in fact a true invariant. Lewis applied it to the well-known problem of the harmonic oscillator of time-dependent frequency. Moreover, Lewis and Riesenfeld [7] proceeded to quantize the invariant, although the physical interpretation was still not clear even at the classical level. In other words, a constant of motion without meaning was available.

In a subsequent work of Eliezer and Gray [8], an elementary physical interpretation was achieved in terms of the angular momentum of an auxiliary two-dimensional motion. Even though this interpretation is not fully satisfactory in the general case, it is the clearest at the moment.

Presently, the Ermakov-Lewis dynamical invariants are more and more in use for many different time-dependent problems whose Hamiltonian is a quadratic form in the canonical coordinates.
2. The method of Ermakov.

The Ukrainian mathematician V. Ermakov was the first to notice that some nonlinear differential equations are related in a simple and definite way with the second order linear differential equations. Ermakov gave as an example the so-called Ermakov system for which he formulated the following theorem.

**Theorem 1E.** If an integral of the equation

\[
\frac{d^2y}{dx^2} = My
\]  

is known, we can find an integral of the equation

\[
\frac{d^2z}{dx^2} = Mz + \frac{\alpha}{z^3},
\]  

where \(\alpha\) is some constant.

Eliminating \(M\) from these equations one gets

\[
\frac{d}{dx} \left( y \frac{dz}{dx} - z \frac{dy}{dx} \right) = \frac{\alpha y}{z^3}.
\]

Multiplying both sides by

\[
2 \left( y \frac{dz}{dx} - z \frac{dy}{dx} \right),
\]

the last equation turns into

\[
\frac{d}{dx} \left( y \frac{dz}{dx} - z \frac{dy}{dx} \right)^2 = -\frac{2\alpha y}{z} \frac{d}{dx} \left( \frac{y}{z} \right).
\]

Multiplying now by \(dx\) and integrating both sides we get

\[
\left( y \frac{dz}{dx} - z \frac{dy}{dx} \right)^2 = C - \frac{\alpha y^2}{z^2}.
\]  

If \(y_1\) and \(y_2\) are two particular solutions of the equation (1), substituting them by \(y\) in the latter equation we get two integrals of of the equation (2)

\[
\left( y_1 \frac{dz}{dx} - z \frac{dy_1}{dx} \right)^2 = C_1 - \frac{\alpha y_1^2}{z^2}.
\]
Eliminating \( dz/dx \) from these equations we get a general first integral of (2). One should note that the Ermakov system coincide with the problem of the two-dimensional parametric oscillator (as we shall see in chapter 6). Moreover, the proof of the theorem gives an exact method to solve this important dynamical problem.

The general first integral of equation (2) can be also obtained as follows. Getting \( dx \) from equation (3):

\[
\frac{dx}{y^2} = \frac{ydz - zdy}{\sqrt{C - \alpha y^2 / z^2}}.
\]

Dividing both sides by \( y^2 \) we get the form

\[
\frac{dx}{y^2} = \frac{\frac{dz}{y}}{\sqrt{\frac{Cz^2}{y^2} - \alpha}}.
\]

Multiplying by \( C \) and integrating both sides we get:

\[
C \int \frac{dx}{y^2} + C_3 = \sqrt{\frac{Cz^2}{y^2} - \alpha}.
\]

This is the general first integral of equation (2), where \( C_3 \) is the constant of the last integration. For \( y \) is enough to take any particular integral of equation (1).

As a corollary of the previous theorem we can say that

**Corollary 1Ec.** If a particular solution of (2) is known, we can find the general solution of equation (1).

Since it is sufficient to find particular solutions of (1), we can take \( C = 0 \) in equation (3). Thus we get:

\[
\frac{y}{z} \frac{dz}{dx} - \frac{z}{y} \frac{dy}{dx} = \mp \frac{y}{z} \sqrt{-\alpha}.
\]

and therefore

\[
\frac{dy}{y} = \frac{dz}{z} \pm \frac{dx}{z^2} \sqrt{-\alpha}.
\]
Integrating both sides

\[
\log y = \log z \pm \sqrt{-\alpha} \int \frac{dx}{z^2},
\]

which results in

\[
y = z \exp(\pm \sqrt{-\alpha} \int \frac{dx}{z^2}).
\]

Taking the plus sign first and the minus sign next we get two particular solutions of equation (1).

A generalization of the theorem has been given by the same Ermakov.

**Theorem 2E.** If \( p \) is some known function of \( x \) and \( f \) is any other arbitrary given function, then the general solution of the equation

\[
p \frac{d^2 y}{dx^2} - y \frac{d^2 p}{dx^2} = \frac{1}{p^2} f \left( \frac{y}{p} \right)
\]

can be found by quadratures.

Multiplying the equation by

\[
2 \left( p \frac{dy}{dx} - y \frac{dp}{dx} \right) dx
\]

one gets the following form

\[
d \left( p \frac{dy}{dx} - y \frac{dp}{dx} \right)^2 = 2 f \left( \frac{y}{p} \right) d \left( \frac{y}{p} \right).
\]

Integrating both sides and defining for simplicity reasons

\[
2 \int f(z) dz = \varphi(z),
\]

we get

\[
\left( p \frac{dy}{dx} - y \frac{dp}{dx} \right)^2 = \varphi \left( \frac{y}{p} \right) + C.
\]

This is the expression for a first integral of the equation. Thus, for \( dx \) we have:

\[
dx = \frac{pdy - ydp}{\sqrt{\varphi \left( \frac{y}{p} \right) + C}}.
\]
Dividing by $p^2$ and integrating both sides we find:

$$\int \frac{dx}{p^2} + C_4 = \int \frac{d\left(\frac{y}{p}\right)}{\sqrt{\varphi \left(\frac{y}{p}\right)} + C}.$$ 

This is the general integral of the equation.

A particular case is when $p = x$. Then, the differential equation can be written as

$$x^3 \frac{d^2y}{dx^2} = f\left(\frac{y}{x}\right).$$

### 3. The method of Milne.

In 1930, Milne \[2\] introduced a method to solve the Schrödinger equation taking into account the basic oscillatory structure of the wave function. This method has been one of the first in the class of the so-called phase-amplitude procedures, which allow to get sufficiently exact solutions for the one-dimensional Schrödinger equation at any energy and are used to locate resonances.

Let us consider the one-dimensional Schrödinger equation

$$\frac{d^2\psi}{dx^2} + k^2(x)\psi = 0$$ \hspace{1cm} (1)

where $k^2(x)$ is the local wave vector

$$k^2(x) = 2\mu[E - V(x)].$$ \hspace{1cm} (2)

Milne proposed to write the wave function as a variable amplitude multiplied by the sinus of a variable phase, i.e.,

$$\psi(x) = \left(\frac{2\mu}{\pi}\right)^{1/2} \beta(x) \sin(\phi(x) + \gamma)$$ \hspace{1cm} (3)

where $\mu$ is the mass parameter of the problem at hand, $E$ is the total energy of the system, $\gamma$ is a constant phase, and $V(x)$ is the potential energy. In the original method, $\beta$ and $\phi$ are real and $\beta > 0$. Substituting the previous expression for $\psi$ in the wave equation and solving for $d\phi/dx$ one gets

$$\frac{d^2\beta}{dx^2} + k^2(x)\beta = \frac{1}{\beta^3},$$ \hspace{1cm} (4)
\[ \frac{d\phi}{dx} = \frac{1}{\beta^2}. \]  

As one can see, the equation for Milne’s amplitude coincides with the non-linear equation found by Ermakov, now known as Pinney’s equation.

4. Pinney’s result.

In a brief note Pinney was the first to give without proof (claimed to be trivial) the connection between the solutions of the equation for the parametric oscillator and the nonlinear equation (known as Pinney’s equation or Pinney-Milne equation).

\[ y''(x) + p(x)y(x) + \frac{C}{y^3(x)} = 0 \]  

for \( C = \) constant and \( p(x) \) given. The general solution for which \( y(x_0) = y_0, \ y'(x_0) = y'_0 \) is

\[ y_P(x) = \left[ U^2(x) - CW^{-2}V^2(x) \right]^{1/2}, \]  

where \( U \) and \( V \) are solutions of the linear equation

\[ y''(x) + p(x)y(x) = 0, \]  

for which \( U(x_0) = y_0, U'(x_0) = y'_0; V(x_0) = 0, V'(x_0) \neq y'_0 \) and \( W \) is the Wronskian \( W = UV' - U'V = \) constant \( \neq 0 \) and one takes the square root in (2) for that solution which at \( x_0 \) has the value \( y_0 \).

The proof is very simple as follows. From eq. (2) we get \( y_P = \frac{1}{y_P^3}(U\ddot{U} - CW^{-2}V\ddot{V}) \) and \( \ddot{y}_P = -y_P^3(U\dddot{U} - CW^{-2}V\dddot{V}) + y_P^{-1}(U^2 - CW^{-2}V^2)p(x)p_P. \) From here, explicitly calculating \( \ddot{y}_P + p(x)y_P \) one gets \( -Cy_P^{-3} \) and therefore Pinney’s equation.

5. Lewis’ results.

In 1967 Lewis considered parametric Hamiltonians of the standard form:

\[ H_L = \frac{1}{2\epsilon}[p^2 + \Omega^2(t)q^2], \]  

8
If $\Omega$ is real, the motion of the classical system whose Hamiltonian is given by eq. (1) is oscillatory with an arbitrary high frequency when $\epsilon$ goes to zero. Corresponding to this, there are asymptotic series in positive powers of $\epsilon$, whose partial sums are the adiabatic invariants of the system; the leading term of the series is $\epsilon H/\Omega$. In the problem of the charged particle the adiabatic invariant is the series of the magnetic moment. Lewis’s results came out from a direct application of the asymptotic theory of Kruskal (1962) to the classical system described by $H_L$ with real $\Omega$. Lewis found that Kruskal’s theory could be applied in exact form. As a consequence, an exact invariant, which is precisely the Ermakov-Lewis invariant, has been found as a special case of Kruskal’s adiabatic invariant. Although $\Omega$ was originally supposed to be real, the final results hold for complex $\Omega$ as well. Moreover, the exact invariant is a constant of motion of the quantum system whose Hamiltonian is given by the quantum version of eq. (1).

**The classical case.**

Let us take a real $\Omega$. In order to correctly apply Kruskal’s theory, it is necessary to write the equations of motion as for an autonomous system of first order so that all solutions be periodic in the independent variables for $\epsilon \to 0$. This can be achieved by means of a new independent variable $s$ defined as $s = t/\epsilon$ and formally considering $t$ as a dependent variable. The resulting system of equations is

$$
dq/ds = p,
$$

$$
dp/ds = -\Omega^2(t)q,
$$

$$
dt/ds = \epsilon
$$

Since $t$ is now a dependent variable, this system is autonomous. In the limit $\epsilon \to 0$, the solution of the last equation is $t = \text{constant}$, and therefore the other two equations correspond to a harmonic oscillator of constant frequency. Since $\Omega$ is real, the dependent variables are periodic in $s$ of period $2\pi/\Omega(t)$ in the limit $\epsilon \to 0$, and the system of equations has the form required by Kruskal’s asymptotic theory. A central characteristic of the latter theory is a transformation from the variables $(q,p,t)$ to the so-called “nice variables” $(z_1, z_2, \varphi)$. The latter are chosen in such a way that a two-parameter
family of closed curves in the space \((q, p, t)\) can be defined by the conditions \(z_1 = \text{constant}\) and \(z_2 = \text{constant}\). These closed curves are called rings. The variable \(\phi\) is a variable angle which is defined in such a way as to change by \(2\pi\) if any of the rings is covered once. The rings have the important feature that all the family can be mapped to itself if on each ring \(s\) is changed according to eqs. (2). In the general theory, the transformation from the variables \((q, p, t)\) to the variables \((z_1, z_2, \phi)\) is defined as an asymptotic series in positive powers of \(\epsilon\), and a general prescription is given to determine the transformation order by order. As a matter of fact, Lewis has shown one possible explicit form for this transformation in terms of the variables \(q, p\) and Pinney’s function \(\rho(t)\). Moreover, the inverse transformation can also be obtained in explicit form.

For the parametric oscillator problem, the rings are to be found in the \(t=\text{constant}\) planes. It is this property that allows the usage of the rings for defining the exact invariant \(I\) as the action integral

\[
I = \int_{\text{ring}} pdq .
\]  

By doing explicitly the integral of \(I\) as an integral from 0 to \(2\pi\) over the variable \(\phi\) (see the Appendix), one gets

\[
I = \frac{1}{2}[(q^2/\rho^2) + (\rho \dot{\phi} - \epsilon \dot{\rho} q)^2] ,
\]

where \(\rho\) satisfies Pinney’s equation

\[
\epsilon^2 \ddot{\rho} + \Omega^2(t) \rho - 1/\rho^3 = 0 ,
\]

and the point denotes differentiation with respect to \(t\). The function \(\rho\) can be taken as any particular solution of eq. (5). Although \(\Omega\) was supposed to be real, \(I\) is an invariant even for complex \(\Omega\). It is easy to check that \(dI/dt = 0\) for the general case of complex \(\Omega\) by performing the derivation of eq. (4), using eqs. (2) to eliminate \(dq/dt\) and \(dp/dt\), and eq. (5) to eliminate \(\ddot{\rho}\).

It might appear that the problem of solving the system of linear equations given by eqs. (2) has been merely replaced by the problem of solving the nonlinear eq. (5). This is however not so. First, any particular solution of eq. (5) can be used in the formula of \(I\) with all the initial conditions for the eqs. (2). For the numerical work it is sufficient to find a particular solution
for $\rho$. Second, the exact invariant has a simple and explicit dependence on
the dynamical variables $\rho$ and $q$. Third, taking into account the fact that
$\epsilon^2$ is a factor for $\dot{\rho}$ in eq. (5), one can obtain directly a particular solution
for $\rho$ as a series of positive powers of $\epsilon^2$. If $\Omega$ is real and the leading term
of the series is taken as $\Omega^{-1/2}$, then the corresponding series solution is just
the adiabatic invariant expressed as a series. It is interesting to speculate if
in practice it is more useful to calculate $I$ by means of the solution written
as a truncated series of $\rho$ or by the corresponding expression in series for $I$
truncated at the same power of $\epsilon$. Forth, one can also solve eq. (5) to get $\rho$
as a power series in $1/\epsilon^2$ in terms of integrales. Finally, with the result of
eqs. (4) and (5), it is possible to get a better understanding of the nature of
Kruskal’s adiabatic invariant. Some progress in this regard can be found in
the following general discussion on $I$ and $\rho$.

By adding a constant factor, the invariant $I$ of eq. (4) is the most general
quadratic invariant of the system whose Hamiltonian given by eq. (1) is also
a homogeneous quadratic form in $p$ and $q$. This can be seen by writing the
invariant in terms of two linear independent solutions, $f(t)$ and $g(t)$ of the
parametric equation. If we write the generalized form of $I$

$$I = \delta^2[\rho^{-2}q^2 + (\rho p - \epsilon q)^2], \quad (6)$$

where $\delta$ is an arbitrary constant, and compare this form with that in terms
of $f(t)$ and $g(t)$, then we can infer that the two invariants are identical if $\rho$ is
given by

$$\rho = \gamma_1(\epsilon\alpha)^{-1}\left[\frac{A^2}{\delta^2}g^2 + \frac{B^2}{\delta^2}f^2 + 2\gamma_2\left[\frac{A^2B^2}{\delta^4} - (\epsilon^w)^2\right]^{1/2}fg\right]^{1/2}, \quad (7)$$

where $A$ and $B$ are arbitrary constants, while the constants $\alpha$, $\gamma_1$ and $\gamma_2$
are defined by

$$w = fg' - gf', \quad \gamma_1 = \pm 1, \quad \gamma_2 = \pm 1. \quad (8)$$

Since there are two arbitrary constants, this formula for $\rho$ gives the general
solution of eq. (4) expressed in terms of $f$ and $g$. Using this formula we can
build $\rho$ explicitly for any $\Omega$ for which the eqs. (4) can be solved in an exact
manner. By constructing $\rho$ in this way for special cases, we can infer that
the expansion of $\rho$ in a series of positive powers of $\epsilon^2$ is at least sometimes
convergent. For example, if $\Omega = bt^{-2n/(2n+1)}$, where $b$ is a constant and $n$ is
any integer, the series expansion is a polynomial in \( \epsilon \), and consequently it is convergent with an infinite radius of convergence.

Once we have the explicit form of Kruskal’s invariant, it is possible to find a canonical transformation for which the new momentum is the invariant itself. If we denote the new coordinate by \( q_1 \), the conjugated momentum by \( p_1 \), and the generating function by \( F \), then the results can be written as

\[
q_1 = -\tan^{-1}\left[ \rho^2 \frac{p}{q} - \epsilon \rho \dot{\rho} \right],
\]

\[
p_1 = \frac{1}{2} \left[ \rho^{-2} q^2 + (\rho p - \epsilon \rho q)^2 \right],
\]

\[
F = \frac{1}{2} \epsilon \rho^{-1} \rho q^2 \pm \rho^{-1} q(2p_1 - \rho^{-2} q^2)^2 \pm p_1 \sin^{-1}\left[ \rho^{-1} q/(2p_1)^{1/2} \right] + (n + \frac{1}{2}) \pi p_1
\]

\[
\left( -\frac{\pi}{2} \leq \sin^{-1}\left[ \rho^{-1} q/(2p_1)^{1/2} \right] \leq \frac{\pi}{2} , n = \text{integer} \right),
\]

\[
p = \frac{\partial F}{\partial q}, \quad q_1 = \frac{\partial F}{\partial p_1},
\]

\[
H_{\text{new}} = H + \frac{\partial F}{\partial t} = \frac{1}{\epsilon} \rho^{-2} p_1 . \tag{9}
\]

In the expression for \( F \) the upper or lower signs are taken according to \( p - \epsilon \rho^{-1} \dot{\rho} q \) is greater or less than 0. One can see that \( q_1 \) is a cyclic variable in the new Hamiltonian, as it should be if \( p_1 = I \) can be an exact invariant.

Moreover, Lewis noticed that the second order differential equation for \( q \), namely \( \epsilon^2 d^2 q/dt^2 + \Omega^2(t) q = 0 \), is of the same form as the 1D Schrödinger equation, if \( t \) is considered as the spatial coordinate and \( q \) is taken as the wave function. For bound states, \( \Omega \) is imaginary whereas for the continuous spectrum \( \Omega \) is real. Thus, the \( I \) invariant is a relationship between the wave function and its first derivative \([9]\).

The quantum case.

Let us consider the quantum system with the same Hamiltonian \( H_L \), where \( \hat{q} \) and \( \hat{p} \) should fulfill now the commutation relations

\[
[\hat{q}, \hat{p}] = i\hbar . \tag{10}
\]
We shall take $\rho$ as real, which is possible if $\Omega^2$ is real. Using the commutation relationships and the equation for $\rho$ it is easy to show that $\hat{I}$ is a quantum constant of motion, i.e., it can be an observable since it satisfies

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}] = 0 . \quad (11)$$

It follows that $I$ has eigenfunctions whose eigenvalues are time-dependent. The eigenfunctions and eigenvalues of $\hat{I}$ can be found by a method which is similar to that used by Dirac to find the eigenfunctions and eigenvalues of the harmonic oscillator Hamiltonian. First, we introduce the raising and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$, defined by

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}[\rho^{-1}\hat{q} - i(\rho\hat{p} - \epsilon\dot{\rho}\hat{q})] ,$$

$$\hat{a} = \frac{1}{\sqrt{2}}[\rho^{-1}\hat{q} + i(\rho\hat{p} - \epsilon\dot{\rho}\hat{q})] . \quad (12)$$

These operators fulfill the relationships

$$[\hat{a}, \hat{a}^\dagger] = \hbar ,$$

$$\hat{a}\hat{a}^\dagger = \hat{I} + \frac{1}{2}\hbar . \quad (13)$$

The operator $\hat{a}$ acting on an eigenfunction of $\hat{I}$ gives rise to an eigenfunction of $\hat{I}$ whose eigenvalue is less by $\hbar$ with respect to the initial eigenvalue. Similarly, $\hat{a}^\dagger$ acting on an eigenfunction of $\hat{I}$ raises the eigenvalue by $\hbar$. Once these properties are settled, the normalization of the eigenfunctions of $\hat{I}$ can be used to prove that the eigenvalues of $\hat{I}$ are $(n + \frac{1}{2})\hbar$, where $n$ is 0 or a positive integer. If $|n\rangle$ denotes the normalized eigenfunction of $\hat{I}$ whose eigenvalue is $(n + \frac{1}{2})\hbar$, we can express the relationship between $|n+1\rangle$ and $|n\rangle$ as follows

$$|n+1\rangle = [(n+1)\hbar]^{-1/2}\hat{a}^\dagger|n\rangle . \quad (14)$$

The condition that determines the eigenstate whose eigenvalue is $\frac{1}{2}\hbar$ is given by

$$\hat{a}|0\rangle = 0 . \quad (15)$$

Using these results one can calculate the expectation value of the Hamiltonian in an eigenstate $|n\rangle$. The result is

$$\langle n|\hat{H}|n\rangle = (1/2\epsilon)(\rho^{-2} + \Omega^2\rho^2 + \epsilon^2\dot{\rho}^2)(n + \frac{1}{2})\hbar . \quad (16)$$
It is interesting to note that the expectation values of $\hat{H}$ are equally spaced at each moment and that the lowest value is obtained for $n = 0$, i.e., we have an exact counterpart of the harmonic oscillator. As a matter of fact, we can obtain the harmonic oscillator results if $\Omega$ is taken real and constant with $\rho = \Omega^{-1/2}$, which gives $I = \epsilon H/\Omega$.

6. The interpretation of Eliezer and Gray.

The harmonic linear motion corresponding to the 1D parametric oscillator equation can be seen as the projection of a 2D motion of a particle driven by the same law of force. Thus, the 2D auxiliary motion is described by the equation

$$\frac{d^2 \vec{r}}{dt^2} + \Omega^2(t) \vec{r} = 0 \tag{1}$$

where $\vec{r}$ is expressed in Cartesian coordinates $(x, y)$. Using polar coordinates $(\rho, \theta)$ where $\rho = |\vec{r}|$, $x = \rho \cos \theta$, $y = \rho \sin \theta$. The radial and transversal motions are described now by the equations

$$\ddot{\rho} - \rho \dot{\theta}^2 + \Omega^2 \rho = 0 \tag{2}$$

$$\frac{1}{\rho} \frac{d}{dt} \left( \rho^2 \dot{\theta} \right) = 0 \tag{3}$$

Integrating eq. (3)

$$\rho^2 \dot{\theta} = h \tag{4}$$

where $h$ is the angular momentum, which is constant. Substituting in eq. (2) one gets a Pinney equation of the form:

$$\ddot{\rho} + \Omega^2 \rho = \frac{h^2}{\rho^3} \tag{5}$$

The invariant $I$ corresponding to the eq. (3) is:

$$I = \frac{1}{2} \left[ \frac{h^2 x^2}{\rho^2} + (p \rho - x \dot{\rho})^2 \right] \tag{6}$$

and with the substitutions $x = \rho \cos \theta$ and $p = \dot{x}$ one gets:

$$I = \frac{1}{2} \left[ h^2 \cos^2 \theta + h^2 \sin^2 \theta \right] = \frac{1}{2} h^2 \tag{7}$$
Thus, the constancy of $I$ is equivalent to the constancy of the auxiliary angular momentum.

In the elementary classical mechanics, the study of the simple 1D harmonic oscillator is often made as the projection of the uniform circular motion on one of its diameters. The auxiliary motion introduced by Eliezer and Gray is just a generalization of this elementary procedure to more general laws of force.

The connection between the solutions of the parametric oscillator linear equation and Pinney’s solution is given by the following theorem.

**Theorem 1EG.** If $y_1$ and $y_2$ are linear independent solutions of the equation

$$\frac{d^2 y}{dx^2} + Q(x) y = 0 \quad (8)$$

and $W$ is the Wronskian $y_1 y_2' - y_2 y_1'$ (which, according to Abel’s theorem is constant), then the general solution of

$$\frac{d^2 y}{dx^2} + Q(x) y = \frac{\lambda}{y^4} \quad (9)$$

where $\lambda$ is a constant, can be written as follows

$$y_P = (A y_1^2 + B y_2^2 + 2C y_1 y_2)^{1/2} \quad (10)$$

where $A$, $B$ and $C$ are constants such that

$$A B - C^2 = \frac{\lambda}{W^2} \quad (11)$$

However, it is necessary that these constants be consistent with the initial conditions of the motion. If $x_1 (t)$ and $x_2 (t)$ are linear independent parametric solutions of initial conditions $x_1 (0) = 1$, $\dot{x}_1 (0) = 0$, $x_2 (0) = 0$, $\dot{x}_2 (0) = 1$, the general parametric solution can be written as

$$x (t) = \alpha x_1 (t) + \beta x_2 (t) \quad (12)$$

where $\alpha$ and $\beta$ are arbitrary constants that are related to the initial conditions of the motion by $x (0) = \alpha$ and $\dot{x} (0) = \beta$. The corresponding initial conditions for $\rho$ and $\dot{\rho}$ are obtained from $x = \rho \cos \theta$, $\dot{x} = \dot{\rho} \cos \theta - \rho \dot{\theta} \sin \theta$, where $\theta (0) = 0$ gives $\rho (0) = \alpha$ and $\dot{\rho} (0) = 0$. Using (10) we get

$$\rho (t) = \left[ (\alpha x_1 + \beta x_2)^2 + \left( \frac{h^2}{\alpha^2 x_2^2} \right)^2 \right]^{1/2} \quad (13)$$
as the solution of (3) corresponding to the general parametric solution (12). Moreover, we have

\[
\rho \cos \theta = \alpha x_1 + \beta x_2 \tag{14}
\]

\[
\rho \sin \theta = \frac{hx_2}{\alpha} \tag{15}
\]

The previous considerations can be extended to systems whose equations of motion are of the form

\[
\frac{d^2x}{dt^2} + P(t) \frac{dx}{dt} + Q(t) x = 0 \tag{16}
\]

The I invariant is now

\[
I = \frac{h^2 x^2}{\rho^2} + (\dot{\rho} x - \rho \rho')^2 \exp \left(2 \int_0^t P(t) \, dt \right) \tag{17}
\]

where \( \rho \) is any solution of

\[
\frac{d^2 \rho}{dt^2} + P(t) \frac{d\rho}{dt} + Q(t) \rho = \frac{h^2}{\rho^3} \exp \left(-2 \int_0^t P(t) \, dt \right) \tag{18}
\]

The theorem that connects the solutions of (16) with those of (18) (with a change of notation) can be formulated in the following way.

**Theorem 2EG.** If \( y_1(x) \) and \( y_2(x) \) are two linear independent solutions of

\[
\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0 \tag{19}
\]

the general solution of

\[
\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = \frac{\lambda}{y^3} \exp \left(-2 \int P(t) \, dt \right) \tag{20}
\]

can be written down as

\[
y = (Ay_1^2 + By_2^2 + 2Cy_1y_2)^{1/2} \tag{21}
\]

where \( A \) and \( B \) are arbitrary constants, and

\[
AB - C^2 = \frac{\lambda}{W^2} \exp \left(-2 \int P(t) \, dt \right) \tag{22}
\]
7. The connection between the Ermakov invariant and Nöther’s theorem.

In 1978, Leach [10] found the Ermakov-Lewis invariant for the aforementioned parametric equation with first derivative

$$\ddot{x} + g(t)x + \omega^2(t)x = 0,$$

by making use of a time-dependent canonical transformation leading to a constant new Hamiltonian. That transformation belonged to a symplectic group and has been put forth without details. In the same 1978 year, Lutzky [11] proved that the invariant could be obtained starting from a direct application of Noether’s theorem (1918). This famous theorem makes a connection between the conserved quantities of a Lagrangian system with the group of symmetries that preserves the action as an invariant functional. Moreover, Lutzky discussed the relationships between the solutions of the parametric equation of motion and Pinney’s solution and commented on the great potential of the method for solving the nonlinear equations.

For the parametric equation without first derivative Lutzky used the following formulation of Noether’s theorem.

**Theorem NL.** Let $G$ be the one-parameter Lie group generated by

$$G = \xi(x,t)\frac{\partial}{\partial t} + n(x,t)\frac{\partial}{\partial x}$$

such that the action functional $\int L(x,\dot{x},t)dt$ is left invariant under $G$. Then

$$\xi\frac{\partial L}{\partial t} + n\frac{\partial L}{\partial x} + (\dot{\xi} - \dot{x}\xi)\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L = \dot{f}.\quad (2)$$

where $f = f(x,t)$, and

$$\dot{\xi} = \frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x}, \quad \dot{n} = \frac{\partial n}{\partial t} + \dot{x}\frac{\partial n}{\partial x}, \quad \dot{f} = \frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x}.$$  

Moreover, a constant of motion of the system is given by

$$\Phi = (\xi \dot{x} - n)\frac{\partial L}{\partial \dot{x}} - \xi L + f.\quad (3)$$
The Lagrangian \( L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2) \) gives the equations of motion of the parametric oscillating type; substituting this Lagrangian in (2) and equating to zero the coefficients of the corresponding powers of \( \dot{x} \), one gets a set of equations for \( \xi, n, f \). Next, it is easy to prove that they imply that \( \xi \) is a function only of \( t \) and fulfills

\[
\ddot{\xi} + 4\omega \dot{\omega} + 4\omega^2 \dot{\xi} = 0. \tag{4}
\]

The following results are easy to get

\[
n(x, t) = \frac{1}{2} \dot{\xi} x + \psi(t),
\]

\[
f(x, t) = \frac{1}{4} \ddot{\xi} x^2 + \dot{\psi} x + C, \quad \ddot{\psi} + \omega^2 \psi = 0.
\]

Choosing \( C = 0, \psi = 0 \), and substituting these values in (3), one can find that

\[
\Phi = \frac{1}{2}[\xi \dot{x}^2 + (\xi \omega^2 + \frac{1}{2} \ddot{\xi}) x^2 - \dot{\xi} x \dot{x}] \tag{5}
\]

is a conserved quantity for the parametric undamped oscillatory motion if \( \xi \) satisfies (4). Notice that (4) has the first integral

\[
\xi \dddot{\xi} - \frac{1}{2} \dot{\xi}^2 + 2\xi \omega^2 = C_1. \tag{6}
\]

If we choose \( \xi = \rho^2 \) in (5) and (6), with \( C_1 = 1 \), we get that \( \Phi \) is the Ermakov-Lewis invariant. If the formula for the latter is considered as a differential equation for \( x \), then it is easy to solve in the variable \( x/\rho \); the result can be written in the form

\[
x = \rho[A \cos \phi + B \sin \phi], \quad \phi = \phi(t), \tag{7}
\]

where \( \dot{\phi} = 1/\rho^2 \) and \( A \) and \( B \) are arbitrary constants. Thus, the general parametric solution can be found if a particular solution of Pinney’s equation is known.

Consider now the Ermakov-Lewis invariant as a conserved quantity for Pinney’s equation; this is possible if \( x \) fulfills the parametric equation of motion. This standpoint is interesting because it provides an example of how to use Noether’s theorem to change a problem of solving nonlinear equations.
into an equivalent problem of solving linear equations. Thus, if we take as
our initial task to solve Pinney’s equation, we can use Noether’s theorem
with
\[ L(\rho, \dot{\rho}, t) = \frac{1}{2}(\dot{\rho}^2 - \omega^2 \rho^2 - \frac{1}{\rho^2}) , \]
to prove that
\[ \Phi = \frac{1}{2}\left[ \frac{x^2}{\rho^2} + C_2 \frac{\rho^2}{x^2} + (\rho \dot{x} - \dot{\rho} x)^2 \right] \]
is a conserved quantity for Pinney’s equation leading to
\[ \ddot{x} + \omega^2 x = C_2 / x^3 . \]
(8)
The quantity \( C_2 \) is an arbitrary constant; choosing \( C_2 = 0 \), we reduce Pin-
ney’s solution to the parametric linear solution, while (8) turns into the
Ermakov-Lewis invariant.

If we write the invariant for two different solutions of the linear para-
metric equation, \( x_1 \) and \( x_2 \), while keeping the same \( \rho \), and eliminate \( \dot{\rho} \) in the resulting
equations we get
\[ \rho = \frac{1}{W} \sqrt{I_1 x_2^2 + I_2 x_1^2 + 2 x_1 x_2[I_1 I_2 - W^2]^{1/2}} , \]
(10)
where \( W = \dot{x}_1 x_2 - x_1 \dot{x}_2 \), and \( I_1 \) and \( I_2 \) are constants. Thus, a general
solution of Pinney’s equation can be obtained if two solutions of the linear
parametric equation can be found. (Since the Wronskian \( W \) is constant for
two independent linear solutions, we can find that \( I_1 = 1, I_2 = W^2 \), and
therefore (10) turns into \( \rho = \sqrt{x_1^2 + (1/W^2)x_2^2} \), which is the result given
by Pinney in 1950). Moreover, one can see from (7) that two independent
parametric solutions are \( x_1 = \tilde{\rho} \cos \phi, x_2 = \tilde{\rho} \sin \phi \), where \( \tilde{\rho} \) is any solution of
Pinney’s equation. Then \( W = 1 \), and (10) turns into
\[ \rho = \tilde{\rho} \sqrt{I_1 \sin^2 \phi + I_2 \cos^2 \phi + [I_1 I_2 - 1]^{1/2} \sin 2\phi} , \quad \dot{\phi} = 1/\tilde{\rho}^2 . \]
(11)
This beautiful result obtained by Lutzky by means of Noether’s theorem
gives the general solution of Pinney’s equation in terms of an arbitrary par-
ticular solution of the same equation. Moreover, Lutzky suggested that this
approach can be used to solve certain nonlinear dynamical systems once a
conserved quantity containing an auxiliary function of a corresponding non-
linear differential equation can be found. Even if the auxiliary equation is
nonlinear, sometimes it is simpler to solve than the original linear equation. In any case, one can establish useful relationships between the solutions of the two types of equations.

In conclusion, we mention that Noether’s method can be applied to the equation of parametric motion with first derivative (1); in this way one can reproduce the results of Eliezer and Gray of the previous chapter. The effective Lagrangian for (1) is given by

\[ L = \frac{1}{2} e^{F(t)} [\dot{x}^2 - \omega^2(t)x^2], \]

where \( dF/dt = g(t) \).

8. Possible generalizations of the Ermakov method.

We have seen that there is a simple relationship between the solutions of the parametric oscillator

\[ \ddot{x} + \omega^2(t)x = 0, \tag{1} \]

and the solution of nonlinear differential equations of the Pinney type that differ from eq. (1) only in the nonlinear term. The equation of motion of a charged particle in some types of time-dependent magnetic fields can be written in the above form. Many time-dependent oscillating systems are governed by the same eq. (1). A conserved quantity for eq. (1) is

\[ I_{EL} = \frac{1}{2} [(x^2/\rho^2) + (\rho \dot{x} - \dot{\rho} x)^2], \tag{2} \]

where \( x(t) \) satisfies eq. (1) and \( \rho(t) \) satisfies the auxiliary equation

\[ \ddot{\rho} + \omega^2(t)\rho = 1/\rho^3. \tag{3} \]

Using eq. (1) to eliminate \( \omega^2(t) \) in eq. (3) we find

\[ \ddot{\rho} + (\rho/x)\dot{x} = 1/\rho^3, \tag{4} \]

or

\[ x\ddot{\rho} - \rho \dot{x} = (d/dt)(x\dot{\rho} - \rho \dot{x}) = x/\rho^3. \tag{5} \]

Now, multiplying this equation by \( x\dot{\rho} - \rho \dot{x} \) we can write

\[ (x\dot{\rho} - \rho \dot{x})(d/dt)(x\dot{\rho} - \rho \dot{x}) = (x\dot{\rho} - \rho \dot{x})x/\rho^3, \tag{6} \]
or
\[
\frac{1}{2}(d/dt)(\dot{\rho} - \rho \dot{x})^2 = -\frac{1}{2}(d/dt)(x/\rho^2)
\]
and therefore we have the invariant
\[
I_{EL} = \frac{1}{2}[(x^2/\rho^2) + (\rho \dot{x} - \dot{\rho} x)^2]
\]
where \(x\) is any solution of eq. (1) and \(\rho\) is any solution of eq. (3).

A simple generalization of this result has been proposed by Ray and Reid in 1979 \[12\]. Instead of (3) they considered the following equation
\[
\ddot{\rho} + \omega^2(t)\rho = (1/x\rho^2)f(x/\rho)
\]
where \(x\) is a solution of eq. (1) and \(f(x/\rho)\) is an arbitrary function of \(x/\rho\). If again we eliminate \(\omega^2\) and we employ as a factor \(x\dot{\rho} - \rho \dot{x}\) as a factor to obtain
\[
\frac{1}{2}(d/dt)(x\dot{\rho} - \rho \dot{x})^2 = -(d/dt)\phi(x/\rho)
\]
where
\[
\phi(x/\rho) = 2\int^{x/\rho} f(u)du
\]
From eq. (10) we have the invariant
\[
I_f = \frac{1}{2}[\phi(x/\rho) + (\rho \dot{x} - \dot{\rho} x)^2]
\]
where \(x\) is a solution of eq. (1) and \(\rho\) is a solution of eq. (9). For \(f = x/\rho\) we reobtain the invariant \(I_{EL}\). The result (12) provides a connection between the solutions of the linear equation (1) with the solutions of an infinite number of nonlinear equations (9) by means of the invariant \(I_f\).

As an additional generalization, one can consider the following two equations
\[
\ddot{x} + \omega^2(t)x = (1/\rho x^2)g(\rho/x)
\]
\[
\ddot{\rho} + \omega^2(t)\rho = (1/x\rho^2)f(x/\rho)
\]
where \(g\) and \(f\) are arbitrary functions of their arguments. Applying the same procedure to these equations one can find the invariant
\[
I_{f,g} = \frac{1}{2}[\phi(x/\rho) + \theta(\rho/x) + (x\dot{\rho} - \rho \dot{x})^2]
\]
where
\[ \phi(x/\rho) = 2 \int_{x/\rho}^{x/\rho} f(u)du \tag{16} \]
\[ \theta(\rho/x) = 2 \int_{\rho/x}^{\rho/x} g(u)du \tag{17} \]

The expression (15) is an invariant whenever \( x \) is a solution of eq. (13) and \( \rho \) is a solution of eq. (14). One should notice that the functions \( f \) and \( g \) are arbitrary, and therefore the invariant \( I_{f,g} \) gives the connection between the solutions of many different differential equations. We can see that the Ermakov-Lewis invariant is merely a particular case of \( I_{f,g} \) with \( g = 0, f = x/\rho \).

In the cases \( g = 0, f = 0 \); \( g = 0, f = x/\rho \); \( g = \rho/x, f = 0 \); and \( f = x/\rho, g = \rho/x \) the equations (13) and (14) respectively are not coupled. In general, if we have found a solution for \( x \), then the invariant \( I_{f,g} \) provides some information about the solution \( \rho \).

On the other hand, it is not known if the simple mechanical interpretation of Eliezer and Gray is also available for different choices of \( f \) and \( g \). The simple proof of the existence of \( I_{f,g} \) clarifies how such invariants can occur from pairs of differential equations.

9. Geometrical angles and phases in the Ermakov problem.

The quantum mechanical holonomic effect known as Berry's phase (BP) (1984) has been of much interest in the last fifteen years. In the simplest cases, it shows up when the time-dependent parameters of a system change adiabatically in time in the course of a closed trajectory in the parameter space. The wave function of the system gets, in addition to the common dynamical phase \( \exp(-i\hbar \int_0^T E_n(t)dt) \), a geometrical phase factor given by

\[ \gamma_n(c) = i \int_0^T dt \langle \Psi_n(X(t)) | \frac{d}{dt} \Psi_n(X(t)) \rangle , \tag{1} \]

because the parameters are slowly changing along the closed path \( c \) of the spatial parameter \( X(t) \) during the period \( T \). \( |\Psi_n(X(t))\rangle \) are the eigenfunctions of the instantaneous Hamiltonian \( H(X(t)) \). BP has a classical analogue.
as an angular shift accumulated by the system when its dynamical variables are expressed in angle-action variables. This angular shift is known in the literature as Hannay’s angle (Hannay 1985, Berry 1985). Various model systems have been employed to calculate the BP and its classical analog. One of these systems is the generalized harmonic oscillator whose $H$ is given by (Berry 1985, Hannay 1985)

$$H_{XYZ}(p, q, t) = \frac{1}{2}[X(t)q^2 + 2Y(t)qp + Z(t)p^2] ,$$

where the slow time-varying parameters are $X(t)$, $Y(t)$ and $Z(t)$.

Since $H_{XYZ}$ can be transformed into the $H$ of a parametric oscillator, it follows that there should be a connection between the BP of the system with $H_{XYZ}$ and the Lewis phase for the parametric oscillator [7]. This problem has been first studied by Morales [13]. Interestingly, the results appear to be exact although the system does not evolve adiabatically in time and goes to Berry’s result in the adiabatic limit.

Lewis and Riesenfeld [4] have shown that for a quantum nonstationary system which is characterized by a Hamiltonian $\hat{H}(t)$ and a Hermitian invariant $\hat{I}(t)$, the general solution of the Schroedinger equation

$$i\hbar \frac{\partial \Psi(q, t)}{\partial t} = \hat{H}(t)\Psi(q, t) ,$$

is given by

$$\Psi(q, t) = \sum_n C_n \exp(i\alpha_n(t))\Psi_n(q, t) .$$

$\Psi_n(q, t)$ are the eigenfunctions of the invariant

$$\hat{I}\Psi_n(q, t) = \lambda_n \Psi_n(q, t) ,$$

where the eigenvalues are time-dependent, the coefficients $C_n$ are constants and the phases $\alpha_n(t)$ are obtained from the equation

$$\hbar d\alpha_n(t)/dt = \langle \Psi_n|i\hbar \partial/\partial t - \hat{H}(t)|\Psi_n \rangle .$$

Using this result, Lewis and Riesenfeld obtained solutions for a quantum harmonic oscillator of parametric frequency characterized by the classical Hamiltonian

$$H(t) = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2(t)q^2$$

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and the classical equation of motion
\[\ddot{q} + \Omega^2(t)q^2 = 0,\]  
where the points denote differentiation with respect to time. The matrix elements that are required to evaluate the BP are given by

\[\langle \Psi_n | \frac{\partial}{\partial t} | \Psi_n \rangle = \frac{1}{2}i(\rho \ddot{\rho} - \dot{\rho}^2)(n + \frac{1}{2}).\]  
\[\langle \Psi_n | \dot{H}(t) | \Psi_n \rangle = \frac{1}{2}(\dot{\rho}^2 + \Omega^2(t)\rho^2 + 1/\rho^2)(n + \frac{1}{2}),\]

where \(\rho(t)\) is a real number, satisfying the equation

\[\ddot{\rho} + \Omega^2(t)\rho = 1/\rho^3.\]

Substituting (9) and (10) in (8) and integrating one gets

\[\alpha_n(t) = -(n + \frac{1}{2}) \int_0^t dt' / \rho^2(t').\]

One can show this either by using (9) or (12) and one can get the BP and Hannay’s angle for the system of Hamiltonian \(H_{XYZ}\). For this system, the frequency that can be obtained from the Hamiltonian expressed in the action-angle variables, is given by

\[\omega = \frac{\partial H(I, X(t), Y(t), Z(t))}{\partial I} = (XZ - Y^2)^{1/2}.\]

From (2) one can get the equations of motion for \(q\) and \(p\) and by eliminating \(p\) one can get the Newtonian equation of motion for \(q\) as follows

\[\ddot{q} - (\dot{Z}/Z)\dot{q} + [XZ - Y^2 + (\dot{Z}Y - \dot{Y}Z)/Z]q = 0.\]

The term in \(\dot{q}\) can be eliminated by introducing a new coordinate \(Q(t)\) given by (Berry 1985)

\[q(t) = [Z(t)]^{1/2}Q(t).\]

Substituting (13) in (14) one gets

\[\ddot{Q} + [XZ - Y^2 + (\dot{Z}Y - \dot{Y}Z)/Z + 1/2(\dot{Z}/Z - \dot{Z}^2/Z^2) - 1/4(\dot{Z}/Z)^2]Q = 0.\]
which corresponds to the equation of motion of an oscillator with parametrically forced frequency. Berry found Hannay’s angle $\Delta \theta$ by the WKB method in quantum mechanics, but it can also be obtained by means of (8) or (12).

Comparing (8) with (16) we see that we can define $\Omega^2(t)$ as

$$
\Omega^2(t) = XZ - Y^2 + (\dot{Z}Y - \dot{Y}Z)/Z + [1/4(Z/Z - \dot{Z}^2/Z^2) - 1/4(\dot{Z}/Z)^2].
$$

(17)

With this connection and employing (1) and (9) we get

$$
\gamma_n(C) = -\frac{1}{2}(n + \frac{1}{2}) \int_0^T (\dot{\rho} \rho - \rho^2) dt,
$$

(18)

where $\rho(t)$ is the solution of (11) with $\Omega^2(t)$ given by (17). It is important to notice that (18) is exact even when the system does not evolve slowly in time.

To compare with Berry’s result one should take the adiabatic limit. For that we define an adiabaticity parameter $\epsilon$ and a ‘slow time’ variable $\tau$

$$
\xi \equiv \xi(\tau) \quad \tau = \epsilon t,
$$

(19)

in terms of which $\Omega^2(\tau)$ turns into

$$
\Omega^2(\tau) = \epsilon^{-2}[XZ - Y^2 + \epsilon(Z'Y - Y'Z)/Z + \epsilon^2[1/2(Z'/Z') - 1/4(Z'/Z)^2]],
$$

(20)

where the primes indicate differentiation with respect to $\tau$. It has been shown by Lewis that in the adiabatic limit eq. (20) can be solved as a power series in $\epsilon$ with the leading term given by

$$
\rho_0 = \Omega^{-1/2}(\tau).
$$

(21)

If we plug this expression for $\rho$ and its time derivatives in (18) we could obtain the BP in the adiabatic limit. However, it is easy to calculate the Lewis phase first and then resting it from the dynamical phase $-\hbar^{-1} \langle \Psi_n | H(t) | \Psi_n \rangle$.

Substituting (20) and (21) in (12) we get

$$
\alpha_n(\tau) = -(n + \frac{1}{2}) \left( \frac{1}{\epsilon} \int_0^\tau (XZ - Y^2)^{1/2} d\tau' + \frac{1}{2} \int_0^\tau \frac{(Z'Y - Y'Z)}{Z(XZ - Y^2)^{1/2}} d\tau' + O(\epsilon) \right).
$$

(22)
The first term in the right hand side is the dynamical phase and the second 
and higher order terms are associated with Berry’s phase. Therefore we can 
write the BP as follows

$$\gamma_n(C) = -\frac{1}{2}(n + \frac{1}{2}) \int_0^T \frac{\dot{Z}Y - \dot{Y}Z}{Z(XZ - Y^2)^{1/2}} dt.$$  \hspace{1cm} (23)

Hannay’s angle is obtained by using the correspondence principle in the form 
(Berry 1985, Hannay 1985)

$$\Delta \theta = -\frac{\partial \gamma_n}{\partial n},$$ \hspace{1cm} (24)
as

$$\Delta \theta = \frac{1}{2} \int_0^T \frac{(\dot{Z}Y - \dot{Y}Z)}{Z(XZ - Y^2)^{1/2}} dt,$$ \hspace{1cm} (25)

which is the same result as that obtained by Berry (1985).

In this way, it has been proved that if a time-dependent quadratic $H$ 
can be transformed to the parametric form given by (7), then the Lewis phase 
can be used to calculate the BP and the Hannay’s angle. Although we presented 
the particular case discussed by Morales, it is known that Lewis’ approach 
for time-dependent systems is general. As a matter of fact, one can find more 
general cases in the literature.

10. Application to minisuperspace Hamiltonian cosmology.

The formalism of Ermakov invariants can be a useful alternative to study 
the evolutionary and chaoticity problems of “quantum” canonical universes 
since these invariants are closely related to the Hamiltonian formulation. 
Moreover, as we have seen in the previous chapter, Ermakov’s method is 
imtimately related to geometrical angles and phases [27]. Therefore, it seems 
natural to speak of Hannay’s angle as well as of various types of topological 
phases at the cosmological level.

The Hamiltonian formulation of the general relativity has been developed 
in the classical works of Dirac [14] and Arnowitt, Deser and Misner (ADM) 
[15]. When it was applied to the Bianchi homogeneous cosmological models 
it led to the so-called Hamiltonian cosmology [16]. Its quantum counterpart,
the canonical quantum cosmology \cite{17}, is based on the canonical quantization methods and/or path integral procedures. These cosmologies are often used in heuristic studies of the very early universe, close to the Planck scale epoch $t_P \approx 10^{-43}$ s.

The most general models for homogeneous cosmologies are the Bianchi ones. In particular, those of class A of diagonal metric are at the same time the simplest from the point of view of quantizing them.

Briefly, we can say that in the ADM formalism the metric of these models is of the form

$$ds^2 = -dt^2 + e^{2\alpha(t)}(e^{2\beta(t)})_{ij}\omega^i \omega^j,$$

where $\alpha(t)$ is a scalar function and $\beta_{ij}(t)$ is a diagonal matrix of dimension 3, $\beta_{ij} = \text{diag}(x + \sqrt{3}y, x - \sqrt{3}y, -2x)$, $\omega^i$ are 1-forms characterizing each of the Bianchi models and fulfilling the algebra $d\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k$, where $C^i_{jk}$ are structure constants.

The ADM action has the form

$$I_{\text{ADM}} = \int (P_x dx + P_y dy + P_\alpha d\alpha - N\mathcal{H}_\perp)dt,$$

where the Ps are the canonical moments, $N$ is the lapse function and

$$\mathcal{H}_\perp = e^{-3\alpha} \left(-P_\alpha^2 + P_x^2 + P_y^2 + e^{4\alpha}V(x, y)\right).$$

$e^{4\alpha}V(x, y) = U(q\mu)$ is the potential of the cosmological model under consideration. The Wheeler-DeWitt (WDW) equation can be obtained by canonical quantization, i.e., substituting $P_{q^\mu}$ by $\hat{P}_{q^\mu} = -i\partial_{q^\mu}$ in eq. (3), where $q^\mu = (\alpha, x, y)$. The factor ordering of $e^{-3\alpha}$ with the operator $\hat{P}_\alpha$ is not unique. Hartle and Hawking \cite{18} suggested an almost general ordering of the following type

$$-e^{-(3-Q)\alpha}\partial_\alpha e^{-Q\alpha}\partial_\alpha = -e^{-3\alpha}\partial_\alpha^2 + Q e^{-3\alpha}\partial_\alpha,$$

where $Q$ is any real constant. If $Q = 0$ the WDW equation is

$$\Box \Psi - U(q^\mu) \Psi = 0,$$

Using the ansatz $\Psi(q^\mu) = Ae^{\pm\Phi}$ one gets

$$\pm A\Box \Phi + A[\left(\nabla \Phi\right)^2 - U] = 0,$$

where $\Box = G^{\mu\nu}\frac{\partial^2}{\partial q^\mu \partial q^\nu}$, $(\nabla)^2 = -\left(\frac{\partial}{\partial \alpha}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$, and $G^{\mu\nu} = \text{diag}(-1, 1, 1)$. 

[27]
Employing the change of variable \((\alpha, x, y) \rightarrow (\beta_1, \beta_2, \beta_3)\), where
\[
\begin{align*}
\beta_1 &= \alpha + x + \sqrt{3}y, \\
\beta_2 &= \alpha + x - \sqrt{3}y, \\
\beta_3 &= \alpha - 2x,
\end{align*}
\] (7)
the 1D character of some of the Bianchi models can be studied in a more direct way.

**Empty FRW (EFRW) universes for \(Q = 0\).**

We now apply the Ermakov method to the simplest cosmological oscillators which are the empty quantum universes of Friedmann-Robertson-Walker (EFRW) type. The results included in this chapter have been published recently [19]. When the Hartle-Hawking parameter is equal to zero \((Q = 0)\), the WDW equation for the EFRW universe is
\[
\frac{d^2\Psi}{d\Omega^2} - \kappa e^{-4\Omega}\Psi(\Omega) = 0,
\] (8)
where \(\Omega\) is the Misner time which is related to the volume of the universe \(V\) at a given cosmological epoch as \(\Omega = -\ln(V^{1/3})\) [20], \(\kappa\) is the curvature parameter of the universe (1,0,-1 for closed, plane, open universes, respectively) and \(\Psi\) is the wave function of the universe. The general solution is obtained as a linear superposition of modified Bessel functions of zero order in the case for which \(\kappa = 1\), \(\Psi(\Omega) = C_1 I_0(\frac{1}{2}e^{-2\Omega}) + C_2 K_0(\frac{1}{2}e^{-2\Omega})\). If \(\kappa = -1\) the solution will be a superposition of ordinary Bessel functions of zero order \(\Psi(\Omega) = C_1 J_0(\frac{1}{2}e^{-2\Omega}) + C_2 Y_0(\frac{1}{2}e^{-2\Omega})\). \(C_1\) and \(C_2\) are arbitrary superposition constants that we shall take for simplicity reasons equal \(C_1 = C_2 = 1\).

Eq. (8) can be transformed into the canonical equations of motion for a classical point particle of mass \(M = 1\), generalized coordinate \(q = \Psi\) and moment \(p = \Psi'\), and by considering Misner’s time as a Hamiltonian time for which we shall keep the same notation. Thus, we can write
\[
\begin{align*}
\frac{dq}{d\Omega} &= p, \\
\frac{dp}{d\Omega} &= \kappa e^{-4\Omega}q.
\end{align*}
\] (9) (10)
These equations describe the canonical motion of an inverted oscillator \((\kappa = 1)\) and of a normal one \((\kappa = -1)\), respectively [21], of Hamiltonian
\[
H(\Omega) = \frac{p^2}{2} - \kappa e^{-4\Omega}\frac{q^2}{2}.
\] (11)
For the EFRW Hamiltonian the phase space functions
\[ T_1 = \frac{p^2}{2}, \quad T_2 = pq, \quad T_3 = \frac{q^2}{2} \]
form a dynamical Lie algebra, i.e.,
\[ H = \sum \n h_n(\Omega)T_n(p,q), \quad (12) \]
which is closed with respect to the Poisson brackets \( \{T_1, T_2\} = -2T_1, \quad \{T_2, T_3\} = -2T_3, \quad \{T_1, T_3\} = -T_2 \). The Hamiltonian EFRW Hamiltonian can be written now as
\[ H = T_1 - \kappa e^{4\Omega}T_3. \quad (13) \]
The Ermakov invariant \( I \) is a function in the dynamical algebra
\[ I = \sum_{r} \epsilon_r(\Omega)T_r, \quad (14) \]
and through the time invariance condition
\[ \frac{\partial I}{\partial \Omega} = -\{I,H\}, \quad (15) \]
one is led to the following equations for the unknown functions \( \epsilon_r(\Omega) \)
\[ \dot{\epsilon}_r + \sum_n \left[ \sum_m C_{nm}^r h_m(\Omega) \right] \epsilon_n = 0, \quad (16) \]
where \( C_{nm}^r \) are the structure constants of the Lie algebra given above. Thus, we obtain
\[ \begin{align*}
\dot{\epsilon}_1 &= -2\epsilon_2 \\
\dot{\epsilon}_2 &= -\kappa e^{-4\Omega}\epsilon_1 - \epsilon_3 \\
\dot{\epsilon}_3 &= -2\kappa e^{-4\Omega}\epsilon_2.
\end{align*} \quad (17) \]
The solution of this system of equations can be easily obtained by choosing \( \epsilon_1 = \rho^2 \), which gives \( \epsilon_2 = -\rho \dot{\rho} \) and \( \epsilon_3 = \dot{\rho}^2 + \frac{1}{\rho^2} \), where \( \rho \) is the solution of Pinney’s equation
\[ \ddot{\rho} - \kappa e^{-4\Omega}\rho = \frac{1}{\rho^3}. \quad (18) \]
In terms of \( \rho(\Omega) \) and using (6), the Ermakov invariant can be written as follows
\[ I = I_{\text{kin}} + I_{\text{pot}} = \frac{(\rho \dot{\rho} - \dot{\rho}q)^2}{2} + \frac{q^2}{2\rho^2} = \frac{\rho^4}{2} \left[ \frac{d}{d\Omega} \left( \frac{\Psi}{\rho} \right) \right]^2 + \frac{1}{2} \left( \frac{\Psi}{\rho} \right)^2. \quad (19) \]
We have followed the calculations of Pinney and of Eliezer and Gray for $\rho(\Omega)$ in terms of linear combinations of the aforementioned Bessel functions that satisfy the initial conditions as given by these authors. We have worked with the values $A = 1$, $B = -1/W^2$ and $C = 0$ for Pinney’s constants, where $W$ is the Wronskian of the Bessel functions. We have also chosen an auxiliary angular moment of unit value ($h = 1$). Since $I = h^2/2$, we have to obtain a constant value of one-half for the Ermakov invariant. We have checked this by plotting $I(\Omega)$ for $\kappa = \pm 1$ in fig. 1.

Now we pass to the calculation of the angular variables. We first calculate the time-dependent generating function that allows us to go to the new canonical variables for which $I$ is chosen as the new “moment” [5]

$$S(q, I, \vec{\epsilon}(\Omega)) = \int dq' p(q', I, \vec{\epsilon}(\Omega)),$$  \hspace{1cm} (20)

leading to

$$S(q, I, \vec{\epsilon}(\Omega)) = \frac{q^2 \dot{\rho}}{2 \rho} + I \arcsin \left( \frac{q}{\sqrt{2 I \rho^2}} \right) + \frac{q \sqrt{2 I \rho^2} - q^2}{2 \rho^2},$$  \hspace{1cm} (21)

where we have put to zero the constant of integration. Then

$$\theta = \frac{\partial S}{\partial I} = \arcsin \left( \frac{q}{\sqrt{2 I \rho^2}} \right).$$  \hspace{1cm} (22)

Now, the canonical variables are

$$q_1 = \rho \sqrt{2 I} \sin \theta, \quad p_1 = \frac{\sqrt{2 I}}{\rho} \left( \cos \theta + \dot{\rho} \rho \sin \theta \right).$$  \hspace{1cm} (23)

The dynamical angle is

$$\Delta \theta^d = \int_{\Omega_0}^{\Omega} \left( \frac{\partial H_{\text{new}}}{\partial I} \right) d\Omega' = \int_{\Omega_0}^{\Omega} \left[ \frac{1}{\rho^2} - \frac{\rho^2}{2} \frac{d}{d\Omega} \left( \frac{\dot{\rho}}{\rho} \right) \right] d\Omega',$$  \hspace{1cm} (24)

while the geometrical angle (generalized Hannay angle) is

$$\Delta \theta^g = \frac{1}{2} \int_{\Omega_0}^{\Omega} \left[ \frac{d}{d\Omega}(\dot{\rho} \rho) - 2 \rho^2 \right] d\Omega'.$$  \hspace{1cm} (25)
The sum of $\Delta \theta^d$ and $\Delta \theta^g$ is the total change of angle (Lewis’ angle):

$$\Delta \theta^t = \int_{\Omega_0}^{\Omega} \frac{1}{\rho^2} d\Omega' .$$  \hspace{1cm} (26)

Plots of the angular quantities (24-26) for $\kappa = 1$ are displayed in figs. 2,3, and 4, respectively. For $\kappa = -1$ we’ve got similar plots.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The Ermakov-Lewis invariant for $Q = 0$, $\kappa = \pm 1$, $h = 1$.}
\end{figure}
fig. 2: The dynamical angle as a function of $\Omega$.

fig. 3: The geometrical angle as a function of $\Omega$. 
EFRW universes for $Q \neq 0$.

We now apply the Ermakov procedure to the EFRW oscillators when $Q \neq 0$. These results have been reported at the Third Workshop of the Mexican Division of Gravitation and Mathematical Physics of the Mexican Physical Society [22].

It can be shown that the WDW equation for EFRW universes with $Q$ taken as a free parameter is

$$\frac{d^2\Psi}{d\Omega^2} + Q \frac{d\Psi}{d\Omega} - \kappa e^{-4\Omega} \Psi(\Omega) = 0 ,$$

(27)

where, as previously, $\Omega$ is Misner’s time and $\kappa$ is the curvature index of the FRW universe; $\kappa = 1, 0, -1$ for closed, flat, and open universes, respectively. For $\kappa = \pm 1$ the general solution can be expressed in terms of Bessel functions

$$\Psi_+^\alpha(\Omega) = e^{-2\alpha\Omega} \left( C_1 I_\alpha \left( \frac{1}{2} e^{-2\Omega} \right) + C_2 K_\alpha \left( \frac{1}{2} e^{-2\Omega} \right) \right)$$

(28)
and
\[ \Psi^{-\alpha}_{\alpha}(\Omega) = e^{-2\alpha\Omega} \left( C_1 J_\alpha \left( \frac{1}{2} e^{-2\Omega} \right) + C_2 Y_\alpha \left( \frac{1}{2} e^{-2\Omega} \right) \right), \]  
respectively, where \( \alpha = Q/4 \). The case \( \kappa = 0 \) does not correspond to a parametric oscillator and will not be considered here. The eq. (29) can be turned into the canonical equations for a classical point particle of mass \( M = e^{Q\Omega} \), generalized coordinate \( q = \Psi \) and moment \( p = e^{Q\Omega} \dot{\Psi} \) (i.e., of velocity \( v = \dot{\Psi} \)). Again, identifying Misner’s time \( \Omega \) with the classical Hamiltonian time we obtain the equations of motion
\[ \dot{q} \equiv \frac{dq}{d\Omega} = e^{-Q\Omega} p \]  
(30)

\[ \dot{p} \equiv \frac{dp}{d\Omega} = \kappa e^{(Q-4)\Omega} q. \]  
(31)
as derived from the time-dependent Hamiltonian:
\[ H_{cl}(\Omega) = e^{-Q\Omega} \frac{p^2}{2} - \kappa e^{(Q-4)\Omega} \frac{q^2}{2}. \]  
(32)
The Ermakov invariant \( I(\Omega) \) can be built algebraically to be a constant of motion. The result is
\[ I(\Omega) = (p^2/2) - (e^{Q\Omega} \rho \dot{\rho}) \cdot pq + (e^{2Q\Omega} \rho^2 + 1/\rho^2) \frac{q^2}{2}, \]  
(33)
where \( \rho \) is the solution of Pinney’s equation \( \ddot{\rho} + Q \dot{\rho} - \kappa e^{-4\Omega} \rho = \frac{e^{-2Q\Omega}}{\rho^3} \). In terms of \( \rho_\pm(\Omega) \) the Ermakov invariant for this class of EFRW universes reads
\[ I_{\pm EFRW} = \left( \frac{\rho_\pm^2 - e^{Q\Omega} \rho_\pm q}{2} \right)^2 + \frac{q^2}{2\rho_\pm^2} = \frac{e^{2Q\Omega}}{2} \left( \rho_\pm \dot{\Psi}_\alpha^\pm - \dot{\rho}_\pm \Psi_\alpha^\pm \right)^2 + \frac{1}{2} \left( \frac{\Psi_\alpha^\pm}{\rho_\pm} \right)^2. \]  
(34)
In the calculation of \( I_{\pm EFRW} \) we have used linear combinations of Bessel functions that fulfill the initial conditions for \( \rho \) as explained in chapter 6 and which will be presented in some detail in a separate section of the present chapter.

Calculating again the generating function \( S(q, I, \Omega) \) of the canonical transformations leading to the new momentum \( I \), we obtain
\[ S(q, I, \Omega) = e^{Q\Omega} \frac{q^2}{2} \frac{\dot{\rho}}{\rho} + I \arcsin \left[ \frac{q}{\sqrt{2I\rho^2}} \right] + q \sqrt{2I\rho^2 - q^2} \left[ \frac{q}{2\rho^2} \right], \]  
(35)
where the integration constant is again chosen to be zero. The new canonical variables are 
\[ q_1 = \rho \sqrt{2I} \sin \theta \quad \text{and} \quad p_1 = \frac{\sqrt{2I}}{\rho} \left( \cos \theta + e^{Q\Omega} \dot{\rho} \sin \theta \right). \]

The angular quantities are:
\[ \Delta \theta^d = \int_{\Omega_0}^{\Omega} \frac{\partial H_{\text{new}}}{\partial \Omega} d\Omega' = \int_{\Omega_0}^{\Omega} \left[ e^{Q\Omega'} \frac{\partial \rho}{\partial \Omega'} + e^{Q\Omega'} \dot{\rho}^2 \right] d\Omega', \]
\[ \Delta \theta^g = \frac{1}{2} \int_{\Omega_0}^{\Omega} \left[ \frac{d}{d\Omega'} \left( e^{Q\Omega'} \dot{\rho} \right) - 2 e^{Q\Omega'} \dot{\rho}^2 \right] d\Omega', \]
for the dynamical and geometrical angles, respectively. Thus, the total angle will be
\[ \Delta \theta = \int_{\Omega_0}^{\Omega} e^{Q\Omega'} \dot{\rho}^2 d\Omega'. \]

On the Misner time axis, going to \(-\infty\) means going to the origin of the universe, whereas \(\Omega_0 = 0\) means the present era. With these temporal limits for the cosmological evolution, one finds that the variation of the total angle \(\Delta \theta\) is basically the same as the Laplace transformation of \(1/\rho^2\):

\[ \Delta \theta = -L_{1/\rho^2}(Q). \]

The plots of the invariant and of the variations of the angular quantities are shown next, both for the closed EFRW universes as for the open ones.

---

fig. 5: \( I_{\text{EFRW}}^+(\Omega) \) for \( Q = 3 \) and an initial singularity of unit auxiliary angular momentum.
fig. 6: The dynamical angle as a function of $\Omega$ for a closed EFRW universe and $Q = 1$.

fig. 7: The geometrical angle for the same case.
fig. 8: The total angle as a function of $\Omega$ for the same case.

fig. 9: $I_{EFRW}(\Omega)$ with $Q = 1$ for an initial singularity of auxiliary angular momentum excitation $h = 2$. 
fig. 10: The dynamical angle as a function of $\Omega$ for an open EFRW universe of $Q = 1$.

fig. 11: The geometrical angle as a function of $\Omega$ for the same case.
fig. 12: The total angle as a function of $\Omega$ for the same open case.

Somewhat more complicated cosmological models

We sketch now the Taub pure gravity model whose WDW equation reads

$$\frac{\partial^2 \Psi}{\partial \Omega^2} - \frac{\partial^2 \Psi}{\partial \beta^2} + Q \frac{\partial \Psi}{\partial \Omega} + e^{-4\Omega} V(\beta) \Psi = 0 ,$$  \hspace{1em} (1)

where $V(\beta) = \frac{1}{3}(e^{-8\beta} - 4e^{-2\beta})$. This equation can be separated in the variables $x_1 = -4\Omega - 8\beta$ and $x_2 = -4\Omega - 2\beta$. Thus one is led to the following pair of 1D differential equations for which the Ermakov procedure is similar to the EFRW case

$$\frac{d^2 \Psi_{T1}}{dx_1^2} + \frac{Q}{12} \frac{d\Psi_{T1}}{dx_1} + \left(\frac{\omega^2}{4} - \frac{1}{144} e^{x_1}\right) \Psi_{T1} = 0$$  \hspace{1em} (2)

and

$$\frac{d^2 \Psi_{T2}}{dx_2^2} - \frac{Q}{3} \frac{d\Psi_{T2}}{dx_2} + \left(\omega^2 - \frac{1}{9} e^{x_2}\right) \Psi_{T2} = 0 .$$  \hspace{1em} (3)

where $\omega/2$ is a separation constant. The solutions are $\Psi_{T1} \equiv \Psi_{T\alpha_1} = e^{(-Q/24)x_1} Z_{i\alpha_1}(i e^{x_1/2}/6)$ and $\Psi_{T2} \equiv \Psi_{T\alpha_2} = e^{(Q/6)x_2} Z_{i\alpha_2}(i 2 e^{x_2/2}/3)$, respectively, where $\alpha_1 = \sqrt{\omega^2 - (Q/12)^2}$ and $\alpha_2 = \sqrt{4\omega^2 - (Q/3)^2}$. 

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A more realistic case is that in which a scalar field of minimal coupling to the FRW minisuperspace metric is included. The WDW equation is

\[ \partial_\Omega^2 + Q \partial_\Omega - \partial_\phi^2 - \kappa e^{-4\Omega} + m^2 e^{-6\Omega} \phi^2 |\Psi(\Omega, \phi) = 0, \]  

and can be written as a Schroedinger equation for a two-component wave function (see [23]). This allows to think of squeezed cosmological states in the Ermakov framework [24]. For this, we shall use the following factorization of the invariant \( I = \hbar (bb^\dagger + \frac{1}{2}) \), where \( b = (2\hbar)^{-1/2} [\frac{\rho}{\rho} + i(\rho p - e^{Qc}\Omega \dot{\rho}q)] \) and \( b^\dagger = (2\hbar)^{-1/2} [\rho - i(\rho p - e^{Qc}\Omega \dot{\rho}q)] \). \( Q_c \) is a fixed ordering parameter.

Consider now a Misner reference oscillator of frequency \( \omega_0 \) corresponding to a given cosmological epoch \( \Omega_0 \) for which one can introduce the standard factorization operators \( a = (2\hbar \omega_0)^{-1/2} \rho \) and \( a^\dagger = (2\hbar \omega_0)^{-1/2} \rho \). The connection between the two pairs \( a \) and \( b \) is

\[ b(\Omega) = \mu(\Omega) a + \nu(\Omega) a^\dagger \]
\[ b^\dagger(\Omega) = \mu^*(\Omega) a^\dagger + \nu^*(\Omega) a, \]

where \( \mu(\Omega) = (4\omega_0)^{-1/2} [\rho^{-1} - ie^{Qc}\Omega \dot{\rho} + \omega_0 \rho] \) and \( \nu(\Omega) = (4\omega_0)^{-1/2} [\rho^{-1} - ie^{Qc}\Omega \dot{\rho} - \omega_0 \rho] \) satisfy the relationship \( |\mu(\Omega)|^2 - |\nu(\Omega)|^2 = 1 \). The uncertainties can be calculated \( (\Delta q)^2 = \frac{\hbar}{2\omega_0} |\mu - \nu|^2 \), \( (\Delta p)^2 = \frac{\hbar}{2\omega_0^2} |\mu + \nu|^2 \), and \( (\Delta q)(\Delta p) = \frac{\hbar}{2} |\mu + \nu||\mu - \nu| \) showing that in general these Ermakov states are not of minimum uncertainty [24].

The way one should do the linear combinations for the solutions of the linear differential equations.

As it has been shown, in order to solve Pinney’s equation one should first find the solutions to the equations of motion. Since these equations are linear, we have chosen those combinations which satisfy the initial conditions of motion. According to the interpretation of Eliezer and Gray, the solution of Pinney’s equation is just the amplitude of the 2D auxiliary motion. Therefore, two of the three quadratic terms of the solution can be seen as the amplitudes along each of the axes, respectively. The third one is a mixed term (one can also eliminate it by diagonalizing the quadratic form in the square root).

Let \( q(0) = a \) and \( \dot{q}(0) = b \) be the initial conditions for the equation of motion. The solution can be written as \( x(t) = ax_1(t) + bx_2(t) \), and therefore the functions \( x_1 \) and \( x_2 \) must satisfy the conditions \( x_1(0) = 1, \dot{x}_1(0) = 0, x_2(0) = 0, \dot{x}_2(0) = 1 \). If we take \( \psi_1 \) and \( \psi_2 \) as a pair of linear independent solutions of the parametric equation, then we can build the functions \( x_i \)
as linear combinations of \( \psi_i \): 
\[
x_i = a_i \psi_1 + b_i \psi_2.
\]
It is clear that the linear superpositions that satisfy the initial conditions will be:
\[
x_1 = \frac{1}{W(0)} \left[ \psi_2'(0) \psi_1(t) - \psi_1'(0) \psi_2(t) \right]
\]
\[
x_2 = \frac{1}{W(0)} \left[ -\psi_2(0) \psi_1(t) + \psi_1(0) \psi_2(t) \right]
\]
where \( W(0) \) is the Wronskian of the functions \( \psi_1 \) and \( \psi_2 \) evaluated at zero time parameter. The functions \( x_i \) are the correct ones that should enter the solution of Pinney’s equation written in the form given by Eliezer and Gray.

In this way, we have in the case of the cosmological models that have been discussed:
\[
x_1 = \frac{(2z)^\frac{Q}{2}}{2} \left[ \psi_1'(1/2) K_{\frac{Q}{2}}(z) - \psi_2'(1/2) I_{\frac{Q}{2}}(z) \right]
\]
\[
x_2 = \frac{(2z)^\frac{Q}{2}}{2} \left[ K_{\frac{Q}{2}}(1/2) I_{\frac{Q}{2}}(z) - I_{\frac{Q}{2}}(1/2) K_{\frac{Q}{2}}(z) \right]
\]
where
\[
z = \frac{1}{2} e^{-2\Omega},
\]
\[
\psi_1'(1/2) = -\left[ \frac{Q}{2} I_{\frac{Q}{2}}(1/2) + I_{\frac{Q}{2}}'(1/2) \right],
\]
\[
\psi_2'(1/2) = -\left[ \frac{Q}{2} K_{\frac{Q}{2}}(1/2) + K_{\frac{Q}{2}}'(1/2) \right],
\]
for closed EFRW, and similarly for the open EFRW models.

The superposition coefficients we worked with are of the form 
\[
a_+ = N_K(1/2)/D_+(1/2),
\]
\[
b_+ = N_I(1/2)/D_+(1/2),
\]
\[
c_+ = -K(1/2)/D_+(1/2),
\]
\[
d_+ = I(1/2)/D_+(1/2),
\]
where 
\[
N_K(1/2) = K_{\frac{Q}{2}+1}(1/2) - QK_{\frac{Q}{2}}(1/2),
\]
\[
N_I(1/2) = I_{\frac{Q}{2}+1}(1/2) + QK_{\frac{Q}{2}}(1/2),
\]
\[
D_+(1/2) = I_{\frac{Q}{2}+1}(1/2)K_{\frac{Q}{2}}(1/2) + K_{\frac{Q}{2}+1}(1/2)I_{\frac{Q}{2}}(1/2)
\]
for the closed EFRW case; 
\[
a_- = -N_Y(1/2)/D_- (1/2),
\]
\[
b_- = N_J(1/2)/D_- (1/2),
\]
\[
c_- = Y(1/2)/D_- (1/2),
\]
\[
d_- = -J(1/2)/D_- (1/2),
\]
where 
\[
N_Y(1/2) = Y_{\frac{Q}{2}+1}(1/2) - QY_{\frac{Q}{2}}(1/2),
\]
\[
N_J(1/2) = J_{\frac{Q}{2}+1}(1/2) + QJ_{\frac{Q}{2}}(1/2),
\]
\[
D_- (1/2) = J_{\frac{Q}{2}+1}(1/2)Y_{\frac{Q}{2}}(1/2) - Y_{\frac{Q}{2}+1}(1/2)J_{\frac{Q}{2}}(1/2)
\]
for the open EFRW case.
11. Application to physical optics.

In order to study the Ermakov procedure within physical optics, our starting point will be the 1D Helmholtz equation in the form given by Goyal et al [25] and Delgado et al [26]

\[
\frac{d^2 \psi}{dx^2} + \lambda \phi(x) \psi(x) = 0 ,
\]

that is, as a Sturm-Liouville equation for the set of eigenvalues \( \lambda \in \mathbb{R} \) defining the Helmholtz spectrum within a closed given interval \([a,b]\) on the real line, where the nontrivial function \( \psi \) turns to zero at the end points (Dirichlet boundary conditions). Eq. (1) occurs, for example, in the case of the transversal electric modes (TE) propagating in waveguides that have a continuously varying refractive index in the \( x \) direction but are independent of \( y \) and \( z \). Similar problems in acoustics can be treated along the same lines. The transformation of eq. (1) into the canonical equations of motion of a classical point particle is performed as follows. Let \( \psi(x) \) by any real solution of eq. (1). Define \( x = t, \psi = q, \) and \( \psi' = p; \) then, eq. (1) turns into

\[
\frac{dq}{dt} = p \tag{2}
\]

\[
\frac{dp}{dt} = -\lambda \phi(t)q , \tag{3}
\]

with the boundary conditions \( q(a) = q(b) = 0. \) The corresponding classical Hamiltonian

\[
H(t) = \frac{p^2}{2} + \lambda \phi(t)\frac{q^2}{2} . \tag{4}
\]

is similar to the previous cosmological case of \( Q = 0, \) if one identifies \( \lambda = -\kappa \) and \( \phi = e^{-4\Omega}. \) The procedure to find the Ermakov invariant follows step by step the cosmological case. In the phase space algebra we can write the invariant as

\[
I = \sum_r \mu_r(t)T_r , \tag{5}
\]

and applying

\[
\frac{\partial I}{\partial t} = -\{I, H\} , \tag{6}
\]

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we get the system of equations for the coefficients $\mu_r(t)$

$$
\begin{align*}
\dot{\mu}_1 &= -2\mu_2 \\
\dot{\mu}_2 &= \lambda \phi(t)\mu_1 - \mu_3 \\
\dot{\mu}_3 &= 2\lambda \phi(t)\mu_2.
\end{align*}
$$

The solutions can be written in the conventional form by choosing $\mu_1 = \rho^2$, that gives $\mu_2 = -\rho \dot{\rho}$ and $\mu_3 = \dot{\rho}^2 + \frac{1}{\rho^2}$, where $\rho$ is a solution of the Pinney’s equation of the form: $\ddot{\rho} + \lambda \phi(t)\rho = \frac{1}{\rho^3}$, with the Ermakov invariant of the well-known form $I = \frac{(\rho \dot{\rho} - \rho \phi)^2}{2} + \frac{\dot{\rho}^2}{2\rho^2}$. Next, we calculate the generating function of the canonical transformation for which $I$ is the new momentum

$$
S(q, I, \bar{\mu}(t)) = \int dq' p(q', I, \bar{\mu}(t)).
$$

Thus,

$$
S(q, I, \bar{\mu}(t)) = \frac{q^2}{2} \frac{\dot{\rho}}{\rho} + I \arcsin \left[ \frac{q}{\sqrt{2I \rho^2 - q^2}} \right] + \frac{q \sqrt{2I \rho^2 - q^2}}{2 \rho^2},
$$

where we have put to zero the integration constant. In this way we get

$$
\theta = \frac{\partial S}{\partial I} = \arcsin \left( \frac{q}{\sqrt{2I \rho^2 - q^2}} \right).
$$

The new canonical variables are $q_1 = \rho \sqrt{2I} \sin \theta$ and $p_1 = \frac{\sqrt{2I}}{\rho} \left( \cos \theta + \dot{\rho} \rho \sin \theta \right)$. The dynamical angle is given by

$$
\Delta \theta^d = \int_{t_0}^{t} \left[ \frac{1}{\rho^2} - \frac{\rho^2}{2} \frac{d}{dt} \left( \frac{\dot{\rho}}{\rho} \right) \right] dt',
$$

whereas the geometrical angle is

$$
\Delta \theta^g = \frac{1}{2} \int_{t_0}^{t} \left( \ddot{\rho} \rho - \dot{\rho}^2 \right) dt'.
$$

For periodic parameters $\bar{\mu}(t)$, with all the components of the same period $T$, the geometric angle is known as the nonadiabatic Hannay angle [27] that can be written as a function of $\rho$:

$$
\Delta \theta^g_H = - \oint_C \dot{\rho} d\rho.
$$
Now, in order to proceed with the quantization of the Ermakov problem, we turn \( q \) and \( p \) into operators, \( \hat{q} \) and \( \hat{p} \), but keeping the auxiliary function \( \rho \) as a real number. The Ermakov invariant is now a Hermitian constant operator
\[
\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar} [\hat{I}, \hat{H}] = 0
\]
(14)
and the time-dependent Schrödinger equation for the Helmholtz Hamiltonian is
\[
i\hbar \frac{\partial}{\partial t} |\psi(\hat{q}, t)\rangle = \frac{1}{2}(\hat{p}^2 + \lambda \phi(t) \hat{q}^2)|\psi(\hat{q}, t)\rangle.
\]
(15)
The problem now is to find the eigenvalues of \( \hat{I} \)
\[
\hat{I}|\psi_n(\hat{q}, t)\rangle = \kappa_n|\psi_n(\hat{q}, t)\rangle
\]
(16)
and also to write the explicit form of the general solution of eq. (15)
\[
|\psi(\hat{q}, t)\rangle = \sum_n C_n e^{i\alpha_n(t)} |\psi_n(\hat{q}, t)\rangle
\]
(17)
where \( C_n \) are superposition constants, \( \psi_n \) are (orthonormalized) eigenfunctions of \( \hat{I} \), and the phases \( \alpha_n(t) \) are the Lewis phases \([13, 28]\) that can be found from the equation
\[
\hbar \frac{d\alpha_n(t)}{dt} = \langle \psi_n | i\hbar \frac{\partial}{\partial t} - \hat{H} | \psi_n \rangle.
\]
(18)
The crucial point in the Ermakov quantum problem is to perform a unitary transformation in such a way as to get time-independent eigenvalues for the new Ermakov invariant \( \hat{I}' = \hat{U} \hat{I} \hat{U}^\dagger \). It is easy to obtain the required unitary transformation: \( \hat{U} = \exp[-\frac{i}{\hbar} \hat{\phi} (\hat{q}^2 / 2)] \). The new invariant will be \( \hat{I}' = \frac{\dot{\rho}^2}{2\rho^2} + \frac{\hat{q}^2}{2\rho^2} \). The eigenfunctions are \( \propto e^{-\frac{\hat{q}^2}{2\rho^2}} H_n(\theta/\sqrt{\hbar}) \), where \( H_n \) are the Hermite polynomials, \( \theta = \frac{\hat{q}}{\rho} \), and the eigenvalues are \( \kappa_n = \hbar (n + \frac{1}{2}) \). Thus, one can write the eigenfunctions \( \psi_n \) as follows
\[
\psi_n \propto \frac{1}{\rho^{\frac{\phi}{2}}} \exp \left( \frac{1}{2} \frac{i}{\hbar} \frac{\hat{p}}{\rho} \hat{q}^2 \right) \exp \left( - \frac{q^2}{2\hbar \rho^2} \right) H_n \left( \frac{1}{\sqrt{\hbar}} \frac{q}{\rho} \right).
\]
(19)
The factor \( 1/\rho^{1/2} \) has been introduced for normalization reasons. Using these functions and doing simple calculations one can find the geometrical phase
\[
\alpha_n^g = -\frac{1}{2} \left( n + \frac{1}{2} \right) \int_{t_0}^{t} \left[ (\ddot{\rho} \rho) - \rho^2 \right] dt'.
\]
(20)
The cyclic (nonadiabatic) Berry’s phase \[\alpha_{B,n}^g = \left(n + \frac{1}{2}\right) \oint_C \dot{\rho} d\rho . \] (21)

The results obviously depend on the explicit form of \(\rho\) which in turn depends on the explicit form of \(\phi\).

One can find that a good adiabatic parameter is the inverse of the square root of the Helmoltz eigenvalues, \(1/\sqrt{\lambda}\), with a slow “time” variable \(\tau = 1/\sqrt{\lambda}t\). The adiabatic approximation has been studied in detail by Lewis \[6\]. If the Helmholtz Hamiltonian is written down as

\[ H(t) = \frac{\sqrt{\lambda}}{2} [p^2 + \phi(t)q^2] , \] (22)

then Pinney’s equation is

\[ \frac{1}{\lambda} \ddot{\rho} + \phi(t)\rho = \frac{1}{\rho^3} , \] (23)

while the Ermakov invariant becomes a \(1/\sqrt{\lambda}\)-dependent function

\[ I(1/\sqrt{\lambda}) = \frac{(pp - \dot{\rho}q/\sqrt{\lambda})^2}{2} + \frac{q^2}{2\rho^2} . \] (24)

In the adiabatic approximation, Lewis \[6\] obtained the general Pinney solution in terms of the linear independent solutions \(f\) and \(g\) of the equation of motion \(\frac{1}{\lambda} \ddot{q} + \Omega^2(t)q = 0\) for the classical oscillator (see eq. (45) in \[6\]). Among the examples given by Lewis, it is \(\Omega(t) = b t^{m/2}, m \neq -2, b = \text{constant}\) which is directly related to a realistic dielectric of a waveguide since it corresponds to a power-law index profile \(n(x) \propto x^{m/2}\). For this case, Lewis obtained a simple formula for \(\rho\) of \(O(1)\) order in \(1/\sqrt{\lambda}\)

\[ \rho_m = \gamma_1 \left[ \frac{\gamma_2 \pi \sqrt{\lambda}}{(m+2)} \right] \frac{1}{t^{\frac{1}{2}}} [H_{\beta}^{(1)}(y)H_{\beta}^{(2)}(y)]^{\frac{1}{2}} , \] (25)

where \(H_{\beta}^{(1)}\) and \(H_{\beta}^{(2)}\) are Hankel functions of order \(\beta = 1/(m+2)\), \(y = \sqrt[4]{\frac{b^2}{(m+2)}} x^{m+1}\), and \(\gamma_1 = \pm 1, \gamma_2 = \pm 1\). An even more useful technological
application might be the following proposal of Lewis: $m = -\frac{4n}{2n+1}$, $n = \pm 1, \pm 2, ...$, leading to

$$\rho_n = \gamma_1^{\frac{1}{2}} b^{-\frac{1}{2}} t^{\frac{n}{2n+1}} |G(t, 1/\sqrt{\lambda})|^2,$$  \hspace{1cm} (26)

where

$$G(t, 1/\sqrt{\lambda}) = \left[ \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{k!(n-k)!} \left( \frac{1}{2ib(2n+1)} \right)^k t^{-\frac{k}{2n+1}} \right]^{\frac{1}{2}}.$$

(27)

One gets $\rho$ as a polynomial in the square of the adiabatic parameter, i.e., $\lambda^{-1}$, of infinite radius of convergence. The topological quantities (angles and phases) can be calculated by substituting the explicit form of Pinney’s function in the corresponding formulas. Lewis [6] found a recursive formula in $1/\lambda$ of order $1/\lambda^3$ that can be used for any type of index profile. The recurrence relationship is

$$\rho = \rho_0 + \rho_1/\lambda + \rho_2/\lambda^2 + \rho_3/\lambda^3 + ...,$$

(28)

where $\rho_0 = \Omega^{-1/2} = \phi^{-1/4}(x)$; for the other coefficients $\rho_i$ see the appendix in [6]. The main contribution to the topological quantities are given by $\rho_0$. In the case of a power-law index profile, the geometric angle is

$$\Delta \theta_g = -\frac{m}{4b(m+2)} \left[ t^{-\frac{(m+1)}{2}} - t^{-\frac{(m+1)}{2}} \right],$$

(29)

and a similar formula can be written for the geometric quantum phase. For periodic indices, one can write the Hannay angle and Berry’s phase according to their cyclic integral expressions. Finally, we notice that the choice $\phi(x) = \Phi(x) + \text{Const} \psi^3(x)$, which corresponds to nonlinear waveguides, leads to more general time-dependent Hamiltonians that have been discussed in the Ermakov perspective by Maamache [28].

We have presented in a formal way the application of the Ermakov approach to 1D Helmholtz problems. For more details one can look in a recent work by Rosu and Romero [29].
12. Conclusions.

As one could see from the examples we discussed in this work, the Ermakov-Lewis quadratic invariants are an important method of research for parametric oscillator problems. They are helpful for better understanding this widespread class of phenomena with applications in many areas of physics. One can also say that the Ermakov approach gives a connection between the linear physics of parametric oscillators and the corresponding nonlinear physics.

The cosmological applications of the classical Ermakov procedure we presented herein are based on a classical particle representation of the WDW equation for the EFRW models. We also notice that the Ermakov invariant is equivalent to the Courant-Snyder invariant of use in the accelerator physics [30], allowing an analogy between the physics of beams and the cosmological evolution as suggested by Rosu and Socorro [31].

We end up with a possible interpretation of the Ermakov invariant within the empty minisuperspace cosmology. If one performs an expansion of the invariant in a power series in the adiabatic parameter, the principal term which defines the adiabatic regime gives the number of adiabatic “quanta” and there were authors who gave classical descriptions of the cosmological particle production in such terms [32]. On the other hand, the Eliezer-Gray interpretation as an angular momentum of the 2D auxiliary motion allows one to say that for EFRW minisuperspace models, the Ermakov invariant gives the number of adiabatic excitations of the auxiliary angular momentum with which the universe is created at the initial singularity.
Appendix A: Calculation of the integral of $I$.

The phase space integral of $I$ in chapter 5 can be calculated from the formula (15a) in the paper of Lewis [6]

$$I = -\frac{1}{2\pi} \int_0^{2\pi} X_2 \frac{\partial X_1}{\partial \varphi} d\varphi$$

where $X_1$ and $X_2$ represent the functional dependences of $q$ and $p$, respectively, in terms of the nice variables $z_1$ and $\varphi$, which have been given by Lewis in the formulas (38) of the same paper as follows

$$X_1 = \pm \frac{z_1}{F_1 \Omega [1 + \tan^2(\varphi - F_2)]^{1/2}}$$

and

$$X_2 = \pm \frac{z_1 [\epsilon \frac{d\ln \rho}{dt} + \frac{1}{\rho} \tan(\varphi - F_2)]}{F_1 \Omega [1 + \tan^2(\varphi - F_2)]^{1/2}},$$

where $F_1$ and $F_2$ are two arbitrary functions of time. Thus,

$$\frac{\partial X_1}{\partial \varphi} = \pm \frac{z_1}{F_1 \Omega} [1 + \tan^2(\varphi - F_2)]^{-3/2} (-1/2) 2 \tan(\varphi - F_2) \sec^2(\varphi - F_2).$$

We have the following integral

$$I = \frac{z_1^2}{2\pi F_1^2 \Omega^2} \int_0^{2\pi} \frac{[\epsilon \frac{d\ln \rho}{dt} + \frac{1}{\rho} \tan(\varphi - F_2)]}{[1 + \tan^2(\varphi - F_2)]^2} \tan(\varphi - F_2) \sec^2(\varphi - F_2) d\varphi.$$

Now, employing

$$s = \tan^2(\varphi - F_2), \quad ds = 2\tan(\varphi - F_2) \sec^2(\varphi - F_2) d\varphi$$

one gets

$$I \propto \int \frac{\epsilon d\ln \rho}{(1 + s)^2} + \frac{1}{\rho^2} \int \frac{s^{1/2} ds}{(1 + s)^2}.$$

Therefore

$$I = \frac{z_1^2}{2\pi F_1^2 \Omega^2} \left[ -\epsilon \frac{d\ln \rho}{dt} \frac{1}{(1 + s)} + \frac{1}{\rho^2} \left(-s^{-1/2} + \tan^{-1}\sqrt{s}\right) \right].$$

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Going back to the $\varphi$ variable and taking into account the corresponding 0 and $2\pi$ limits one gets

$$I = \frac{z_1^2}{2F_i^2\Omega^2\rho^2},$$

(9)

which is the result obtained by Lewis. The common form of $I$ can be obtained by going back to the $(q,p)$ variables.

**Appendix B: Calculation of the expectation value of $\hat{H}$ in eigenstates of $\hat{I}$.

From the formulas (12) in chapter 5 for the raising and lowering operators one gets

$$\hat{q} = \frac{\rho}{\sqrt{2}}(\hat{a}^+ + \hat{a}),$$

(1)

$$\hat{p} = \frac{1}{\sqrt{2}}\left[(\dot{\rho} + i/\rho)\hat{a}^+ + (\dot{\rho} - i/\rho)\hat{a}\right].$$

(2)

Performing simple calculations, one gets

$$\hat{H} = f(\rho)\hat{a}^+\hat{a} + f^*(\rho)\hat{a}^2 + \frac{1}{4}\left[\dot{\rho}^2 + \frac{1}{\rho^2} + \omega^2\rho^2\right]2\hat{I},$$

(3)

where $f(\rho) = \dot{\rho}^2 + 2i\dot{\rho}/\rho - 1/\rho^2 + \omega^2\rho^2$. Thus,

$$\langle n|\hat{H}|n\rangle = \langle n|\frac{1}{2}\left[\dot{\rho}^2 + \frac{1}{\rho^2} + \omega^2\rho^2\right]I|n\rangle$$

(4)

from which eq. (16) in chapter 5 is obvious.
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