On $S$-matrix, and fusion rules for irreducible $V^G$-modules

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Abstract
Let $V$ be a simple vertex operator algebra, and $G$ a finite automorphism group of $V$ such that $V^G$ is regular. The definition of entries in $S$-matrix on $V^G$ is discussed, and then is extended. The set of $V^G$-modules can be considered as a unitary space. In this paper, we obtain some connections between $V$-modules and $V^G$-modules over that unitary space. As an application, we determine the fusion rules for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules by the fusion rules for irreducible $V$-modules and by the structure of $G$.

1 Introduction

This paper deals with fusion rules for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules. In [21, Theorem 2], a lower bound of those fusion rules is introduced. In the recent study of the orbifold theory, Dong-Ren-Xu [4] prove that, assuming certain conditions, every irreducible $V^G$-module is a $V^G$-submodule of irreducible $g$-twisted $V$-modules for some $g \in G$. Motivated by the decomposition [4, Equation 3.1], we show that the lower bound given by Tanabe is the desired fusion rules.

Using the extended definition of entries in $S$-matrix, we establish a unitary space on the set of $V^G$-modules. The inner product on the unitary space is derived from the entries $S_{M_i, M_j}$. We show that the linear subspace spanned by $g$-twisted $V$-modules, $S_V(G) = \bigcup_{g \in G} S_V(g)$, consists of exactly the annihilators of the linear subspace spanned by irreducible $V^G$-modules which occur as submodules of irreducible $g \neq id$ twisted $V$-modules.

Finding all desired entries $S$-matrix is a practical way to compute the fusion rules of irreducible modules of $V^G$. Using the extended definition of entries in $S$-matrix, we show that the entries $S_{M_\lambda, E}$ are distributed proportionally to the quantum dimension of $M_\lambda$. The concept of quantum dimension in vertex operator algebra is first introduced in the paper [6]. As an application, we obtain the following main theorem.
Theorem 1.1. Let \( M, N, F \in \mathcal{M}_V \). The notations are defined in Remark 6.4. Then, we have

\[
N_{M_{\lambda_1}, N_{\chi_1}} = \dim\mathbb{C} \text{Hom}_{\mathcal{A}_{\alpha(F)}}(G, \phi(F))(\text{Ind}_{S(F)}^D(F)V_{\xi}, I_{\lambda_1, \chi_1}(v^1, v^2)).
\]

The paper is organized as follows. In Section 2, we review some results used in this paper for \( g \)-rational vertex operator algebras, the modular invariance of trace functions, Verlinde formula, and orbifold theory from [8], [9], [4], [17]. In Section 3, we extend the definition of the entries in the \( S \) matrix, and then derive some properties based on the extended definition. In Section 4, we establish a unitary space on the set of \( V^G \)-modules, and then analyze the structure of the unitary space. This structure is useful in discussing the action of a group \( G \) on \( g \)-twisted modules. As an application, we show that \( G_M = G \) under certain assumptions. In Section 5, we show the "evenly distributive property" of certain entries in the \( S \)-matrix, and then give a general formula of the fusion rules for irreducible \( V^G \)-modules which occur as submodules of irreducible \( V \)-modules by the fusion rules for irreducible \( V \)-modules and by the structure of \( G \).

2 Preliminaries

2.1 Basics

The definition of vertex operator algebra \( V = (V, Y, 1, \omega) \) is introduced and is developed in [3], [12]. The definition of module (including weak, admissible and ordinary modules) are given in [7], [8].

According to [8], a vertex operator algebra is called rational if any admissible module is a direct sum of irreducible admissible modules. According to [23], a vertex operator algebra \( V \) is called \( C_2 \)-cofinite if the subspace \( C_2(V) \) is spanned by \( u_{-2v} \) for all \( u, v \in V \) has finite codimension in \( V \). A \( C_2 \)-cofinite vertex operator algebra is finitely generated with a PBW-like spanning set [13].

According to [7], a vertex operator algebra is called regular if any weak module is a direct sum of irreducible ordinary modules. In [1], [20], the authors show that regularity is equivalent to the combination of rationality and \( C_2 \)-cofiniteness if \( V \) is of CFT type (\( V_0 = \mathbb{C}1 \) and \( V_n = 0 \) for \( n < 0 \)). Regular vertex operator algebras include many important vertex operator algebras such as the lattice vertex operator algebras, vertex operator algebras associated to the integrable highest weight modules for affine Kac-Moody algebras, vertex operator algebras associated to the discrete series for the Virasoro algebra, the framed vertex operator algebras.

Let \( V \) be a vertex operator algebra and \( g \) an automorphism of \( V \) of finite order \( T \). Then
$V$ is a direct sum of eigenspaces of $g$:

$$V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r,$$

where $V^r = v \in V | gv = e^{-2\pi ir/T} v$. Use $r$ to denote both an integer between 0 and $T - 1$ and its residue class mod $T$ in this situation. The definitions of $g$-twisted $V$-module (including weak, admissible, ordinary modules) and $g$-rational vertex operator algebra are given in [4, Definitions 2.1, 2.2, 2.3, 2.4].

### 2.2 Modular Invariance

Let $V$ be a vertex operator algebra, $g$ an automorphism of $V$ of order $T$ and $M = \bigoplus_{n \in \mathbb{Z}^+} M_{\lambda+n}$ a $g$-twisted $V$-module.

For any homogeneous element $v \in V$, the trace function associated to $v$ is defined to be

$$Z_M(v, q) = \text{tr}_M o(v) q^{L(0) - \frac{c}{24}} = q^{\lambda - \frac{c}{24}} \sum_{n \in \mathbb{Z}^+} \text{tr}_{M_{\lambda+n}} o(v) q^n,$$

where $o(v) = v(\text{wt} v - 1)$ is the degree zero operator of $v$. According to [23], [9], the trace function $Z_M(v, q)$ converges to a holomorphic function on the domain $|q| < 1$ if $V$ is $C_2$-cofinite. Let $\tau$ be in the complex upper half-plane $\mathbb{H}$ and $q = e^{2\pi ir}$. Then, the holomorphic function $Z_M(v, q)$ becomes $Z_M(v, \tau)$.

Let $v = 1$ be the vacuum vector. Then, the holomorphic function $Z_M(1, q)$ becomes the formal character of $M$. Denote $Z_M(1, q)$ and $Z_M(1, \tau)$ by $\chi_M(q)$ and $\chi_M(\tau)$, respectively. We call $\chi_M(q)$ the character of $M$.

In [9], the authors discuss the action of $\text{Aut}(V)$ on twisted modules. Let $g, h \in \text{Aut}(V)$ with $g$ of finite order. If $M, Y_M$ is a weak $g$-twisted $V$-module, there is a weak $h^{-1}gh$-twisted $V$-module $M \circ h, Y_{M\circ h}$, where $M \circ h \cong M$ as vector spaces and

$$Y_{M\circ h}(v, z) = Y_M(hv, z),$$

for $v \in V$. This defines a left action of $\text{Aut}(V)$ on weak twisted $V$-modules and on isomorphism classes of weak twisted $V$-modules. Symbolically, write

$$(M, Y_M) \circ h = (M \circ h, Y_{M\circ h}) = M \circ h,$$

where we sometimes abuse notation slightly by identifying $M, Y_M$ with the isomorphism class that it defines.

If $g, h$ commute, $h$ acts on the $g$-twisted modules. Denote by $\mathcal{M}_V(g)$ the equivalence classes of irreducible $g$-twisted $V$-modules and set $\mathcal{M}_V(g, h) = \{ M \in \mathcal{M}_V(g) | h \circ M \cong M \}$. 

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It is well known [8], [9] that if $V$ is $g$-rational, both $\mathcal{M}_V(g)$ and $\mathcal{M}_V(g, h)$ are finite sets. For any $M \in \mathcal{M}_V(g, h)$, there is a $g$-twisted $V$-modules isomorphism

$$\varphi(h) : M \circ h \to M.$$

The linear map $\varphi(h)$ is unique up to a nonzero scalar. If $h = 1$, we simply take $\varphi(1) = 1$. For $v \in V$, set

$$Z_M(v, (g, h), \tau) = \text{tr}_{M \circ h} \varphi(h)q^{L(0) - \frac{c}{24}}\sum_{n \in \mathbb{Z}^+} \text{tr}_{M \circ h} o(v) \varphi(h) q^n,$$

which is a holomorphic function on $\mathbb{H}$ (see [9]). Note that $Z_M(v, (g, h), \tau)$ is defined up a nonzero scalar. If $h = 1$, then $Z_M(v, (g, h), \tau) = Z_M(v, \tau)$.

The assumptions made in [4] are

- (V1) $V = \bigoplus_{n \geq 0} V_n$ is a simple vertex operator algebra of CFT type,
- (V2) $G$ is a finite automorphism group of $V$ and $V^G$ is a vertex operator algebra of CFT type,
- (V3) $V^G$ is $C_2$-cofinite and rational,
- (V4) The conformal weight of any irreducible $g$ twisted $V$-module for $g \in G$ except $V$ itself is positive.

In the rest of this paper, we also assume these four conditions.

The following results are obtained in [1], [2], [18].

**Lemma 2.1.** Let $V$ and $G$ be as before. Then, $V$ is $C_2$-cofinite, and $V$ is $g$-rational for all $g \in G$.

In Reference [23], Zhu introduced a second vertex operator algebra $(V, Y[^{\tilde{\omega}}, 1, \tilde{\omega})$ associated to $V$, where $\tilde{\omega} = \omega - \frac{c}{24}$ and

$$Y[v, z] = Y(v, e^z - 1)e^{z \cdot (w)(v)} = \sum_{n \in \mathbb{Z}} v[n] z^{n - 1}$$

for homogeneous $v$. Write

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n - 2}.$$

Carefully distinguish between the notion of conformal weight in the original vertex operator algebra and in the second vertex operator algebra $(V, Y[^{\tilde{\omega}}, 1, \tilde{\omega})$. If $v \in V$ is homogenous in the second vertex operator algebra, denote its weight by $\text{wt}[v]$. For such $v$, define an action of the modular group $\Gamma$ on $T_M$ in a familiar way, namely

$$Z_M\gamma(v, (g, h), \gamma \tau) = (c\tau + d)^{-\text{wt}[v]} Z_M(v, (g, h), \gamma \tau), \quad (2.1)$$
where $\gamma \tau$ is the Mobius transformation; that is,

$$
\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL(2, \mathbb{Z}).
$$

Let $P(G)$ denote the commuting pairs of elements in a group $G$. Let $\gamma \in \Gamma$ act on the right of $P(G)$ via

$$(g, h)\gamma = (g^a h^c, g^b h^d).$$

The following results are from [23], [5].

**Theorem 2.2.** Assume $(g, h) \in P(\text{Aut}(V))$ such that the orders of $g$ and $h$ are finite. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Also assume that $V$ is $g^a h^c$-rational and $C_2$-cofinite. If $M^i$ is an irreducible $h$-stable $g$-twisted $V$-module, then

$$T_{M^i}(v, g, h, \tau) = \sum_{N_j \in \mathcal{M}(g^a h^c, g^b h^d)} \gamma_{i,j}(g, h)T_{N_j}(v, (g, h)\gamma, \tau),$$

where $\gamma_{i,j}(g, h)$ are some complex numbers independent of the choice of $v \in V$.

For convention, use $\mathcal{M}_V$ for $\mathcal{M}_V(1)$, the set of all irreducible $V$-modules.

### 2.3 Fusion rules and Verlinde Formula

Let $V$ be as before and $M, N, W \in \mathcal{M}_V$. The fusion rule $N^W_{M,N} = \dim I_V \begin{pmatrix} W \\ M & N \end{pmatrix}$, where $I_V \begin{pmatrix} W \\ M & N \end{pmatrix}$ is the space of intertwining operators of type $\begin{pmatrix} W \\ M & N \end{pmatrix}$.

Since $V$ is rational, there is a tensor product $\boxtimes$ of two $V$-modules (see [15], [16], [19]) such that if $M, N$ are irreducible then $M \boxtimes N = \sum_{W \in \mathcal{M}} N^W_{M,N}$. The irreducible $V$-module is called a simple current if $M \boxtimes N$ is irreducible again for any irreducible module $N$.

The following Verlinde formula (see [22]) is proved in [17].

**Theorem 2.3.** Let $V$ be a rational and $C_2$-cofinite simple vertex operator algebra of CFT type and assume $V \cong V'$. Let $S = (S_{i,j})$ be the $S$-matrix. Then,

1. $(S^{-1})_{i,j} = S_{i',j'} = S'_{i',j}$, and $S'_{i',j'} = S_{i,j}$.
2. $S$ is symmetric and $S^2 = (\delta_{i,j})$.
3. $N_{i,j}^k = \sum_{s=0}^d \frac{1}{S_{0,s}} S_{j,s} S_{i,s} S_{k',s}$.
4. The $S$-matrix diagonalizes the fusion matrix $N(i) = (N_{i,j}^k)_{j,k=0}^d$ with diagonal entries $S_{i,s}$, for $i, s \in \{0, 1, \cdots, d\}$. More explicitly, $S^{-1}N(i)S = \text{diag}(S_{s,i})_{s=0}^d$. In particular, $S_{0,s} \neq 0$, for $s \in \{0, 1, \cdots, d\}$. 

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We also have [10]:

**Proposition 2.4.** The $S$-matrix is unitary and $S_{V,M} = S_{M,V}$ is positive for any irreducible $V$-module $M$.

### 2.4 On Orbifold Theory

The quantum Galois theory was introduced in [11],[14],[6].

**Theorem 2.5.** (Quantum Galois Theory) Let $V$ be a simple vertex operator algebra, $G$ a compact subgroup of $\text{Aut}(V)$ acting continuously on $V$. Then, as a $G,V$-module,

$$ V = \bigoplus_{\chi \in \text{Irr}(G)} (W_{\chi} \otimes V_{\chi}), \quad (2.2) $$

where

- $V_{\chi} \neq 0$, $\forall \chi \in \text{Irr}(G)$,
- $V_{\chi}$ is an irreducible $V^G$-module, $\chi \in \text{Irr}(G)$,
- $V_{\chi} \cong V_{\lambda}$ as $V^G$-modules if and only if $\chi = \lambda$.

Some conjectures on orbifold theory are proved in [4].

**Theorem 2.6.** (Orbifold theory) Let $V$, $G$ be defined in the reference, and $M$ be an irreducible $g$-twisted $V$-module, $N$ an irreducible $h$-twisted $V$-module. Assume that $M$ and $N$ are not in the same orbit of $S$ under the action of $G$. Then, as a $\mathbb{C}M[G_M],V^GM$-module,

$$ M = \bigoplus_{\lambda \in \Lambda_{G_M,\alpha_M}} W_{\lambda} \otimes M_{\lambda}, \quad (2.3) $$

where

- $W_{\lambda} \otimes M_{\lambda}$ is nonzero for any $\lambda \in \Lambda_{G_M,\alpha_M}$,
- each $M_{\lambda}$ is an irreducible $V^{GM}$-module,
- $M_{\lambda} \cong M_{\gamma}$, as $V^{GM}$-modules, if and only if $\lambda = \gamma$,
- each $M_{\lambda}$ is an irreducible $V^G$-module,
- $M_{\lambda} \ncong N_{\mu}$, as $V^G$-modules,
- any irreducible $V^G$-module is isomorphic to $M_\lambda$ for some irreducible $g$-twisted $V$-module $M$ and some $\lambda \in \Lambda_{G_M,\alpha_M}$.

The following results are from [4].
Theorem 2.7. We have
\[ q\dim_{V^g} M = |G|q\dim_V M. \]
for \( M \in \mathcal{M}_V(g) \), and \( g \in G \).

Theorem 2.8. Use the same notations in Theorem 2.6. We have
\[ S_{V^\lambda, V^g} = \frac{\dim W_\lambda}{|G_M|} S_{M, V} \]
(2.4)
\[ q\dim_{V^g} M_\lambda = |G : G_M|(\dim W_\lambda)(q\dim_V M). \]
(2.5)

Theorem 2.9. we have the following relation,
\[ \text{glob}(V^G) = |G|^2 \text{glob}(V). \]
3 Definition extension of the entries in the $S$-matrix

In this paper, $V^G$ is a rational vertex operator algebra satisfying the four conditions (V1)–(V4) in Section 4, where $G$ is a finite automorphism group of $V$. We first introduce some notations used in this paper.

Remark 3.1. The last statement in Theorem 2.6 shows that there are two types of irreducible $V^G$ modules.

• An irreducible $V^G$ module $M$ is of type one if $M$ occurs in the decomposition of irreducible $V$ modules, as $V^G$ modules.

• An irreducible $V^G$ module $M$ is of type two if $M$ does not occur in the decomposition of irreducible $V$ modules, as $V^G$ modules. That is, $M$ occurs in a $g$ twisted $V^G$ module for some $g \in G$ and $g \neq 1$.

Remark 3.2. Denote by $\mathcal{M}_{V^G,1}$ the set of irreducible $V^G$-modules of type one, by $\mathcal{M}_{V^G,II}$ the set of irreducible $V^G$-modules of type two. Set $\mathcal{S}_V(G) = \bigcup_{g \in G} \mathcal{M}(g)$. Let $\mathcal{O}(W) = \{ M \in \mathcal{M}_V | M \circ g \cong W, \text{ for some } g \in G \}$. Set $\mathcal{S}_V(G) = \mathcal{S}_V(G)/G = \{ \mathcal{O}(M) | M \in \mathcal{S}_V(G) \}$.

The following definition of the $S$-matrix is well known (see [23]).

Definition 3.3. In Equation 2.1 let $g = h = 1$, and $\gamma = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then,

$$Z_{M^i}(v, -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{j=0}^{d} S_{i,j} Z_{M^j}(v, \tau). \tag{3.1}$$

The matrix $S = (S_{i,j})$ is called an $S$-matrix which is independent of the choice of $v$.

Then, we can extend the definition of the entries $S_{U,W}$.

Definition 3.4. Extend the notation of the entry $S_{U,W}$, where $U, W \in \mathcal{M}_V$. Add to $S$ a superscript $V$, where $V$ is the associated vertex operator algebra. That is, $S^V$ represents the $S$-matrix of $V$, whereas $S^{V^G}$ represents the $S$-matrix of $V^G$. The entries $S^{V^G}_{U,W}, S^{V^G}_{U,W}$ are defined in the same way. Denote $U = \bigoplus_{i=1}^{n} U_i$ and $W = \bigoplus_{j=1}^{m} W_j$, where $U_i$ and $W_j$ are irreducible $V^G$-modules, for $i \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, m\}$. Note that $U_{i_1}$ and $U_{i_2}$ might be isomorphic $V^G$-modules for $i_1 \neq i_2$. Then, define

$$S^{V^G}_{U,W} = \sum_{i=1}^{n} \sum_{j=1}^{m} S^{V^G}_{U_i,W_j}. \tag{3.2}$$

This extension is well defined, because the module decomposition $U = \bigoplus_{i=1}^{n} U_i$ is unique. The superscript $V$ can be omitted, if no confusion follows.

The following lemma shows that two $V$-modules can exactly be distinguished by the associated entries in the $S$-matrix.
Lemma 3.5. Let $U, W$ be two $V$-modules. Then, $U \cong W$ if and only if $S_{U,M}^V = S_{W,M}^V$ for every $M \in \mathcal{M}_V$.

Proof. Since $V$ is a rational vertex operator algebra, $U, W$ are completely reducible and the module decomposition is unique. For $v \in V^G$ and $W \in \mathcal{M}_V$, and , $Z_W(v, \tau)$ are linearly independent (see [23]). Thus, $U \cong W$, if and only if $Z_U(v, -\frac{1}{\tau}) = Z_W(v, -\frac{1}{\tau})$, if only if $S_{U,M}^V = S_{W,M}^V$ for every $M \in \mathcal{M}_V$. □

The formula in the next lemma is a variation of the third statement in Theorem 2.3.

Lemma 3.6. Let $U$ and $W$ be two $V$-modules. Let $M^k$ be an irreducible $V$-module, where $k \in \{0, 1, \cdots, d\}$. Then, $S_{U \boxtimes W, M^k} = \frac{1}{S_{V,M^k}}S_{U,M^k}S_{W,M^k}$.

Proof. It is sufficient to show the equation is true when $U$ and $W$ are irreducible $V$-modules. Assume that $U$ and $W$ are irreducible $V$-modules. Let $v \in V$ be a homogeneous vector in the second vertex operator algebra. Recall the fusion product relation, $U \boxtimes W = \sum_{k=0}^{d} N_{U,M}^{M^k} M^k$. Then, by Verlinde formula,

$$Z_{U \boxtimes W}(v, -\frac{1}{\tau}) = Z_{\{\sum_{k=0}^{d} N_{U,M}^{M^k} M^k\}}(v, -\frac{1}{\tau}) = \sum_{k=0}^{d} N_{U,M}^{M^k} Z_{M^k}(v, -\frac{1}{\tau}) = \sum_{k=0}^{d} \left(\sum_{l=0}^{d} \frac{1}{S_{V,M^l}} S_{U,M^l} S_{W,M^l} S_{(M^k, M^l)} Z_{M^k}(v, -\frac{1}{\tau})\right) = \sum_{k=0}^{d} \left(\sum_{l=0}^{d} \frac{1}{S_{V,M^l}} S_{U,M^l} S_{W,M^l} S_{(M^k, M^l)} (\tau^{W[v]} \sum_{r=0}^{d} S_{M^r,M^l} Z_{M^r}(v, \tau))\right) = \tau^{W[v]} \sum_{l=0}^{d} \frac{1}{S_{V,M^l}} S_{U,M^l} S_{W,M^l} (\sum_{r=0}^{d} \delta_{M^r,M^l} Z_{M^r}(v, \tau)) = \tau^{W[v]} \sum_{l=0}^{d} \frac{1}{S_{V,M^l}} S_{U,M^l} S_{W,M^l} Z_{M^l}(v, \tau)

On the other hand, $Z_{U \boxtimes W}(v, -\frac{1}{\tau}) = \tau^{W[v]} \sum_{l=0}^{d} S_{U \boxtimes W,M^l} Z_{M^l}(v, \tau)$. A comparison of the coefficients of $Z_{M^l}(v, \tau)$ yields $S_{U \boxtimes W,M^l} = \frac{1}{S_{V,M^l}} S_{U,M^l} S_{W,M^l}$. □

Next, we need to find some methods to compute the entries $S_{M,W}^{V_G}$ based on entries in the $S$-matrix of $V$.

Lemma 3.7. Let $M,W \in \mathcal{M}_V$. By orbifold theory, denote $W = \bigoplus_{j=1}^n W_j \otimes U_j$, where $W_j \in \mathcal{M}_V$, and $W_{j_1} \not\cong W_{j_2}$, for $j_1 \neq j_2$. Then,

$$S_{M,W_j}^{V_G} = (\dim U_j) \sum_{N \in \Theta(W)} S_{M,N}^V. \quad (3.3)$$
Proof. Denote \( M = \bigoplus_{i=1}^{m} M_i \), where \( M_i \in \mathcal{M}_{V,G} \), for \( i \in \{1, 2, \ldots, m\} \). Note that \( M_{i_1} \) and \( M_{i_2} \) might be isomorphic, for \( i_1 \neq i_2 \). Definition 3.4 shows that
\[
Z_M(v, -\frac{1}{\tau}) = \tau^{\text{Wt}[v]} \sum_{N \in \mathcal{M}} S^V_{M,N} Z_N(v, \tau)
\]
\[
= \tau^{\text{Wt}[v]} \left( \sum_{N \notin \mathcal{O}(W)} S^V_{M,N} Z_N(v, \tau) + \sum_{N \in \mathcal{O}(W)} S^V_{M,N} Z_N(v, \tau) \right)
\]
\[
= \tau^{\text{Wt}[v]} \left( \sum_{N \notin \mathcal{O}(W)} S^V_{M,N} Z_N(v, \tau) + \sum_{N \in \mathcal{O}(W)} S^V_{M,N} \sum_{j=1}^{n} (\dim U_j) Z_{W_j}(v, \tau) \right)
\]
\[
= \tau^{\text{Wt}[v]} \left( \sum_{N \notin \mathcal{O}(W)} S^V_{M,N} Z_N(v, \tau) + \sum_{j=1}^{n} (\dim U_j) \sum_{N \in \mathcal{O}(W)} S^V_{M,N} Z_{W_j}(v, \tau) \right).
\]

On the other hand,
\[
Z_M(v, -\frac{1}{\tau}) = \sum_{i=1}^{m} Z_{M_i}(v, -\frac{1}{\tau})
\]
\[
= \sum_{i=1}^{m} \tau^{\text{Wt}[v]} \left( \sum_{N \in \mathcal{M}_{V,G}, N \neq W_j} S^G_{M_i,N} Z_N(v, \tau) + S^G_{M_i,W_j} Z_{W_j}(v, \tau) \right)
\]
\[
= \tau^{\text{Wt}[v]} \left( \sum_{i=1}^{m} \sum_{N \in \mathcal{M}_{V,G}, N \neq W_j} S^G_{M_i,N} Z_N(v, \tau) + \sum_{i=1}^{m} S^G_{M_i,W_j} Z_{W_j}(v, \tau) \right)
\]
\[
= \tau^{\text{Wt}[v]} \left( \sum_{N \in \mathcal{M}_{V,G}, N \neq W_j} \sum_{i=1}^{m} S^G_{M_i,N} Z_N(v, \tau) + S^G_{M,W_j} Z_{W_j}(v, \tau) \right)
\]
\[
= \tau^{\text{Wt}[v]} \left( \sum_{N \in \mathcal{M}_{V,G}, N \neq W_j} S^G_{M,N} Z_N(v, \tau) + S^G_{M,W_j} Z_{W_j}(v, \tau) \right).
\]

A comparison of the coefficients of \( Z_{W_j}(v, \tau) \) shows that
\[
S^G_{M,W_j} = (\dim U_j) \sum_{N \in \mathcal{O}(W)} S^V_{M,N}.
\]

The next corollary follows directly from the preceding lemma.

**Corollary 3.8.** Use the same notation in the preceding lemma. Then,
\[
S^G_{M,W_j} = (\dim U_j) \sum_{g \in G} S^V_{M,Wg} |G_W|.
\]

Proof. Equation 3.3 shows that
\[
S^G_{M,W_j} = (\dim U_j) \sum_{N \in \mathcal{O}(W)} S^V_{M,N}
\]
\[
= (\dim U_j) \frac{\sum_{g \in G} S^V_{M,Wg}}{|G_W|}.
\]

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The formula in the next lemma is useful in our further discussions.

**Lemma 3.9.** Let $M, W \in \mathcal{M}_V$, and $g \in G$. Then,

$$S_{Mog, Wog}^V = S_{M, W}^V.$$  \hspace{1cm} (3.4)

**Proof.** Definition 3.4 shows that

$$Z_{Mog}(v, -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{Nog \in \mathcal{M}_V} S_{Mog, Nog}^V Z_{Nog}(v, \tau).$$

On the other hand,

$$Z_{Mog}(v, -\frac{1}{\tau}) = Z_M(gv, -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{N \in \mathcal{M}_V} S_{M, N}^V Z_N(gv, \tau) = \tau^{\text{wt}[v]} \sum_{N \in \mathcal{M}_V} S_{M, N}^V Z_{Nog}(v, \tau).$$

A comparison of the coefficients of $Z_{Nog}(v, \tau)$ shows that $S_{Mog, Nog}^V = S_{M, N}^V$. \hfill \square

The next corollary follows directly from the preceding lemma.

**Corollary 3.10.** Let $W \in \mathcal{M}_V$, and $g \in G$. Then,

$$S_{V, Wog}^V = S_{V, W}^V.$$  \hspace{1cm} (3.5)

**Proof.** Since $g$ is an automorphism of $V$, we have $V \circ g \cong V$. Lemma 3.9 shows that

$$S_{V, Wog}^V = S_{Vog, Wog}^V = S_{V, W}^V.$$ \hfill \square

**Corollary 3.11.** Use the same notation in Corollary 3.8. Then,

$$S_{V, Wj}^G = \dim U_j \left| G \right| \frac{S_{V, W}^V}{|G_W|}.$$  

**Proof.** By Corollaries 3.8, 3.10 we have

$$S_{V, Wj}^G = \dim U_j \sum_{g \in G} S_{V, Wog}^V = \dim U_j \frac{\sum_{g \in G} S_{V, W}^V}{|G_W|} = \dim U_j \frac{S_{V, W}^V}{|G_W|}.$$ \hfill \square
Formulas in the next lemma play an important role in our further discussions.

**Lemma 3.12.** Let $M, N, W \in \mathcal{M}_V$, such that $W$ is not an irreducible $V^G$-module. By orbifold theory, denote $W = \bigoplus_{i=1}^n W_i \otimes P_i$, where $W_i \in \mathcal{M}_{V^G}$. Then,

\[
S^{V^G}_{\{M \boxtimes V^G N\}, W_i} = \frac{1}{S^{V^G}_{V^G, W_i}} \left( \frac{(\dim P_i)^2}{|G_W|^2} \left( \sum_{g \in G} S^{V}_{M, W_og} \right) \left( \sum_{g \in G} S^{V}_{N, W_og} \right) \right),
\]

and

\[
S^{V^G}_{\{\bigoplus_{g \in G} M_{V^G(Nog)}\}, W_i} = \frac{1}{S^{V^G}_{V^G, W_i}} \left( \frac{\dim P_i}{|G_W|} \left( \sum_{g \in G} S^{V}_{M, W_og} \right) \left( \sum_{g \in G} S^{V}_{N, W_og} \right) \right).
\]

**Proof.** Lemma 3.6 and Corollary 3.8 show that

\[
S^{V^G}_{\{M \boxtimes V^G N\}, W_i} = \frac{1}{S^{V^G}_{V^G, W_i}} S^{V^G}_{M, W_i} S^{V^G}_{N, W_i}
\]

\[
= \frac{1}{S^{V^G}_{V^G, W_i}} (\dim P_i) \frac{\sum_{g \in G} S^{V}_{M, W_og}}{|G_W|} (\dim P_i) \frac{\sum_{g \in G} S^{V}_{N, W_og}}{|G_W|}
\]

\[
= \frac{1}{S^{V^G}_{V^G, W_i}} (\dim P_i)^2 \left( \sum_{g \in G} S^{V}_{M, W_og} \right) \left( \sum_{g \in G} S^{V}_{N, W_og} \right).
\]

Lemmas 3.6, 3.9, and Corollaries 3.8, 3.10 show that

\[
S^{V^G}_{\{\bigoplus_{g \in G} M_{V^G(Nog)}\}, W_i} = \sum_{g \in G} S^{V^G}_{\{M_{V^G(Nog)}\}, W_i}
\]

\[
= \frac{q \dim_{V^G} W_i}{|G_W|} \sum_{g, h \in G} S^{V}_{\{M_{V^G(Nog)}\}, W_{och}}
\]

\[
= \frac{\dim P_i}{|G_W|} \sum_{g, h \in G} \frac{1}{S^{V}_{V^G, W_{oh}}} S^{V}_{M, W_{oh}} S^{V}_{M_{V^G(Nog)}, W_{oh}}
\]

\[
= \frac{\dim P_i}{|G_W|} \sum_{g, h \in G} \frac{1}{S^{V}_{V^G, W_{oh}}} S^{V}_{M, W_{oh}} S^{V}_{M_{V^G(Nog)}, W_{oh}}
\]

\[
= \frac{1}{S^{V^G}_{V^G, W_i}} (\dim P_i)^2 \left( \sum_{g \in G} S^{V}_{M, W_og} \right) \left( \sum_{g \in G} S^{V}_{N, W_og} \right).
\]

\[\Box\]

**Remark 3.13.** Let $M, N \in \mathcal{M}_V$. Let $W \in \mathcal{M}_{V^G, II}$. Then, $S^{V^G}_{M, W} = S^{V^G}_{N, W} = 0$. This implies

\[
S^{V^G}_{\{M \boxplus V^G N\}, W} = S^{V^G}_{\{\bigoplus_{g \in G} M_{V^G(Nog)}\}, W} = 0.
\]

The formula in the next lemma will give an important property of the unitary space constructed in Section 4.
Lemma 3.14. Let $M \in \mathcal{S}_V(G)$, and $F_i \in \mathcal{M}_{V_G,\Pi}$. Then,

$$S_{M,F_i}^{V_G} = 0.$$ 

Proof. Modular invariance shows that

$$Z_M(v, -\frac{1}{\tau}) = \tau^{Wt[v]} \sum_{N \in \mathcal{M}_V} S_{M,N}^V Z_N(V, (1, g), \tau).$$

Reference [4] indicates that

$$Z_N(v, (1, g), \tau) = \sum_{\lambda \in \Lambda_{G_N, \alpha_N}} \lambda(g) Z_{N,\lambda}(v, \tau)$$

Since $N \in \mathcal{M}_V$, $N_\lambda \in \mathcal{M}_{V_G}$ is of type one. Thus,

$$Z_M(v, -\frac{1}{\tau}) = \tau^{Wt[v]} \sum_{N \in \mathcal{M}_V} S_{M,N}^V Z_N(V, (1, g), \tau)
= \tau^{Wt[v]} \sum_{N \in \mathcal{M}_V} (\sum_{\lambda \in \Lambda_{G_N, \alpha_N}} \lambda(g) Z_{N,\lambda}(\tau))
= \tau^{Wt[v]} \sum_{M_i \in \mathcal{M}_{V_G, I}} a_i Z_{M_i}(v, \tau),$$

where $a_i \in \mathbb{C}$. Since none of the modules in $\mathcal{M}_{V_G,\Pi}$ appears in the right side of the preceding equation, we have $S_{M,F_i}^{V_G} = 0$, where $F_i \in \mathcal{M}_{V_G,\Pi}$. 

The next corollary follows directly from the preceding lemma.

Corollary 3.15. Let $M \in \mathcal{M}_{V_G}$, and $N \in \mathcal{S}_V(G)$, and $F_i \in \mathcal{M}_{V_G,\Pi}$. Then,

$$S_{M \boxtimes V_G N,F_i}^{V_G} = 0.$$ 

Proof. Apply Lemmas 3.6, 3.2 we have

$$S_{M \boxtimes V_G N,F_i}^{V_G} = \frac{1}{S_{V_G,F_i}^{V_G}} S_{M,F_i}^{V_G} S_{N,F_i}^{V_G}
= \frac{1}{S_{V_G,F_i}^{V_G}} S_{M,F_i}^{V_G} \times 0
= 0.$$ 

In the next lemma, we show a result for fusion products involving $V$ over $V_G$. 

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Lemma 3.16. Let $M \in \mathcal{S}_V(G)$. Then,

$$V \boxtimes_{V^G} M = |G|M.$$ 

Proof. Assume $F_i \in \mathcal{M}_{V^G}$. Lemma 3.6 and Equation 5.8 show that

$$S_{V \boxtimes V^G M, E_i} = \frac{1}{S_{V^G, F_i}} S_{V^G} S_{V^G}$$

$$= \frac{1}{S_{V^G, F_i}} |G| S_{V^G, F_i} S_{V^G}$$

$$= |G| S_{M, F_i}$$

$$= S_{V^G}.$$ 

Let $F_i \in \mathcal{M}_{V^G}$. Lemma 3.2 and Corollary 3.15 show that

$$S_{V \boxtimes V^G M, F_i} = 0$$

$$= |G| 0$$

$$= |G| S_{M, F_i}$$

$$= S_{V^G}.$$ 

Thus, $S_{V \boxtimes V^G M, F_i} = S_{V^G}$, for every $F_i \in \mathcal{M}_{V^G}$. Hence, Lemma 3.5 implies $V \boxtimes_{V^G} M = |G|M$ as $V^G$-modules. □
4 The unitary space structure on the set of $V$-modules

Remark 4.1. In the first place, we introduce some notations used in this section. Let $E_V$ be the linear space spanned by $\mathcal{M}_V$ over $C$. In the rest of the paper, write $E_{V,G}$ as $E$ for convenience. Then, $\dim E = |\mathcal{M}_{V,G}|$, and $A(\sum a_i M_i, \sum b_i M_i) = S_{V,G}^{a_i M_i \sum b_i M_i}$ is a bilinear form on $E$. Since the $S$-matrix of $V_G$ is unitary, $A(\cdot, \cdot)$ is nondegenerate. Let $W$ be the linear subspace of $E$ spanned by $\mathcal{M}_{V,G}$. Let $\text{W}^A = \{ x \in E | A(x, y) = 0, \forall y \in W \}$ consisting of the annihilators of $W$. Then,

$$\dim(\text{W}^A) = \dim E - \dim W = |\mathcal{M}_{V,G}|.$$

Let $\mathcal{M}_V$ be the set of all $V$-modules. Define

$$\iota : \mathcal{M}_{V,G} \rightarrow E$$

$$\iota(\bigoplus_i a_i M_i) = \sum_i a_i M_i,$$

where $M_i \in \mathcal{M}_{V,G}$, and $a_i \in C$. Let $U$ be the linear subspace of $E$ spanned by $\{ \iota(M) \mid M \in S_V(G) \}$. Then, $\dim U = |\mathcal{S}_V(G)|$.

- (a) If $M \in S_V(G)$, Lemma 3.2 shows that $\iota(M) \in \text{W}^A$.
- (b) If $M \in \mathcal{M}_{V,G}$, and $N \in S_V(G)$, Corollary 3.15 shows that $\iota(M \boxtimes_{V,G} N) \in \text{W}^A$.
- (c) $U = \text{W}^A$.
- (d) $\iota(M \boxtimes_{V,G} N) \in U$, if and only if $M \in U$, or $N \in U$.

Statements (c), (d) will be proved in the following paragraphs.

Lemma 4.2. Use the notations introduced in Remark 4.1. Then,

$$U = \text{W}^A.$$

Proof. Let $v \in V^G$, $\tau \rightarrow -\frac{1}{\tau}$, $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, and $N \in \mathcal{M}(1, g)$, where $g \in G$. Then,

$$Z_N(v, (1, g), -\frac{1}{\tau}) = \tau^{\text{Wt}[v]} \sum_{M \in \mathcal{M}_{V,G}} a_i Z_{M_i}(v, \tau). \quad (4.1)$$

Modular invariance shows that

$$Z_N(v, (1, g), -\frac{1}{\tau}) = \tau^{\text{Wt}[v]} \sum_{M \in \mathcal{M}(g)} S_N, M Z_M(v, \tau). \quad (4.2)$$
Let $M \in \mathcal{M}_V$. Use generalized Galois theory to denote $M = \sum_{\lambda \in \Lambda_{G_M}} W_\lambda \otimes M_\lambda$. By Reference [4],

$$Z_{M_\lambda}(v, -\frac{1}{\tau}) = \frac{1}{|G_M|} \sum_{g \in G_M} Z_M(v, (1, g), -\frac{1}{\tau})\overline{\lambda(g)}.$$  \hspace{1cm} (4.3)

Since $M_\lambda \in \mathcal{M}_{V,G}$, Equations 4.2, 4.3 show that

$$|\mathcal{M}_{V,G}| \leq |S_V(G)|$$ \hspace{1cm} (4.4)

It follows that $\dim \mathcal{W}_A \leq \dim \mathcal{U}$. Statement (a) in Remark 4.1 implies $\dim \mathcal{U} \leq \dim \mathcal{W}_A$, and $\mathcal{U}$ is a subspace of $\mathcal{W}_A$. Thus, $\dim \mathcal{U} = \dim \mathcal{W}_A$, and hence $\mathcal{U} = \mathcal{W}_A$. \hfill \Box

Remark 4.3. Let $M \in \mathcal{M}_{V,G}$ The preceding lemma implies $S_{V,G}^{V,G} = 0$, for every $F_i \in \mathcal{M}_{V,G,II}$, if and only if $M \in \mathcal{B}$.

Remark 4.4. Let $\mathcal{M}_{V,G} = \{M^0, M^1, \ldots, M^d\}$, where $d \in \mathbb{N}$. Let $e_i = \iota(M^i)$. Let $\langle \cdot, \cdot \rangle$ be the Hermitian inner product on $\mathcal{E}$, where $\{e_0, e_1, \ldots, e_d\}$ is an orthonormal basis of $\mathcal{E}$. By References [22], [17], the $S$ matrix is unitary. This implies $\{e_i\}$, where $e_i = \sum_{k=0}^{d} S_{i,k} e_k$ is also an orthonormal basis of $\mathcal{E}$. Let $M \in \mathcal{M}_V(g)$. The one-to-one map $\iota$ is omitted, if no confusion follows. By orbifold theory, denote $M = \bigoplus_{\lambda \in \Lambda_{G_M}} W_\lambda \otimes M_\lambda$, where $M_\lambda \in \mathcal{M}_{V,G}$. Then,

$$\langle M, M \rangle = \langle \bigoplus_{\lambda \in \Lambda_{G_M}} W_\lambda \otimes M_\lambda, \bigoplus_{\lambda \in \Lambda_{G_M}} W_\lambda \otimes M_\lambda \rangle$$

$$= \langle \sum_{\lambda \in \Lambda_{G_M}} (\dim W_\lambda) M_\lambda, \sum_{\lambda \in \Lambda_{G_M}} (\dim W_\lambda) M_\lambda \rangle$$

$$= \sum_{\lambda \in \Lambda_{G_M}} (\dim W_\lambda)^2$$

$$= |G_M|. $$

By [4] Theorem 3.2], with $N \in \mathcal{M}_V(h)$, we have

$$\langle M, N \rangle = \delta_{\iota(M), \iota(N)}|G_M|. $$  \hspace{1cm} (4.5)

In the next proposition, we show $G_M = G$ under certain assumptions.

**Proposition 4.5.** Let $G$ be an abelian group, and $V$ a vertex operator algebra. Assume $G_{M_k} = G$, or $\{e\}$, for $k = 1, 2, \ldots, n$. Then, $G_M = G$, for each $M \in \mathcal{M}_{V,g}$, where $g \neq e$.

**Proof.** Let $\mathcal{M}_V = \{M_1, M_2, M_3, \ldots, M_n\}$. Let $\mathcal{S}_V(\{e\}) = \mathcal{S}_V/G = \{S_1, S_2, \ldots, S_l\}$. Let $G_k = G_M$, where $M = S_k$. Then, we have

$$\sum_{g \neq e} |\mathcal{M}_V(g)| \geq |(\bigcup_{g \neq e})\mathcal{M}_V(g)/G|$$

$$= |\mathcal{S}_V(G) - \iota|$$

$$= \Box$$

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5 Fusion products of irreducible type-one \(V_G\)-modules

In this section, let \(N\) be in \(\mathcal{M}_{V_G}\). Denote the quantum Galois decomposition of \(V\) by 
\[ V = \bigoplus_{\chi \in \text{Irr}(G)} W_\chi \otimes V_\chi, \]
where \(V_\chi\) is an irreducible \(V_G\)-module, \(W_\chi\) is an irreducible \(\mathbb{C}G\)-module, and \(\text{qdim} V_\chi = \dim W_\chi\). Write \(\mathcal{M}_{V_G}\) as \(\{M^0, M^1, M^2, \ldots, M^d\}\), where \(M^0 \cong V_G\).

**Theorem 5.1.** Let \(M \in S_V(G)\), and \(E \in \mathcal{M}_{V_G}\). By orbifold theory, denote \(M = \bigoplus_{\lambda \in \Lambda_{G_M,\alpha M}} W_\lambda \otimes M_\lambda\), where \(M_\lambda \in \mathcal{M}_{V_G}\). Then,

\[
\frac{S_{M_1, E}}{\text{qdim}_{V_G} M_1} = \frac{S_{M_2, E}}{\text{qdim}_{V_G} M_2}, \tag{5.1}
\]

where \(\lambda_1, \lambda_2 \in \Lambda_{G_M,\alpha M}\).

**Proof.** Standard modular invariance formula shows that

\[
Z_{M_\lambda}(v, -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{W \in \mathcal{M}_{V_G}} S^V_{M_\lambda, W} Z_W(v, \tau). \tag{5.2}
\]

Reference [4] implies the following relation

\[
Z_{M_\lambda}(v, -\frac{1}{\tau}) = \frac{1}{|G_M|} \tau^{\text{wt}[v]} \sum_{h \in G_M} \sum_{N \in \mathcal{M}(h, g^{-1})} S^V_{M,N} Z_N(v, (h, g^{-1}), \tau) \overline{\lambda(h)}. \tag{5.3}
\]

Orbifold theory implies that

\[
Z_N(v, (h, g^{-1}), \tau) = \sum_{\mu \in \Lambda_{G_N,\alpha N}} \mu(h) Z_{N_\mu}(v, \tau). \tag{5.4}
\]

Plug Equation 5.4 in Equation 5.3

\[
Z_{M_\lambda}(v, -\frac{1}{\tau}) = \frac{1}{|G_M|} \tau^{\text{wt}[v]} \sum_{h \in G_M} \sum_{N \in \mathcal{M}(h, g^{-1})} S^V_{M,N} \sum_{\mu \in \Lambda_{G_N,\alpha N}} \mu(h) Z_{N_\mu}(v, \tau) \overline{\lambda(h)}. \tag{5.5}
\]

Then, for \(h = 1\), observe the coefficients of \(Z_{N_\mu}(v, \tau)\) in Equations 5.2, 5.5. A comparison of these two coefficients yields

\[
S^V_{M_\lambda, N_\mu} = \frac{1}{|G_M|} S^V_{M,N}(1) \overline{\lambda(1)}.
\]

Thus, by Equation 2.5, we have

\[
\frac{S^V_{M_\lambda_1, N_\mu}}{S^V_{M_\lambda_2, N_\mu}} = \frac{\frac{\lambda_1(1)}{\lambda_2(1)}}{\frac{\dim W_{\lambda_1}}{\dim W_{\lambda_2}}} = \frac{\text{qdim}_{V_G} M_{\lambda_1}}{\text{qdim}_{V_G} M_{\lambda_2}},
\]

where \(\lambda_1, \lambda_2 \in \Lambda_{G_M,\alpha M}\). The desired result follows. \(\square\)
Corollary 5.2. Let $U$, $N$ be irreducible $V$-modules. Assume that $U$ is not irreducible as a $V^g$-module. Denote $U = \bigoplus_{i=0}^{T-1} U^i$. Then, $S_{U^i,N} = S_{U^i,N}$.

Proof. Notice that $\text{qdim}_{V^g} U^i = \text{qdim}_{V^G} U^i$, because $(g)$ is a cyclic group. So, Theorem 5.1 implies the desired result.

Theorem 5.3. Let $M \in S_V(G)$, and $W \in \mathcal{M}_V$. By orbifold theory, denote $M = \bigoplus_{\lambda \in \Lambda_{G,\alpha M}} Q_{\lambda} \otimes M_{\lambda}$, and $W = \bigoplus_{\mu \in \Lambda_{G,\alpha W}} P_{\mu} \otimes W_{\mu}$. Then,

$$S_{V^G}^{M,W_{\mu}} = \frac{\dim Q_{\lambda}}{|G_M|} (\dim P_{\mu}) \sum_{N \in \Theta(W)} S_{V}^{M,N}. \quad (5.6)$$

Proof. Since $M = \bigoplus_{\lambda \in \Lambda_{G,\alpha M}} Q_{\lambda} \otimes M_{\lambda}$, we have

$$S_{V^G}^{M,W_{\mu}} = \sum_{\lambda \in \Lambda_{G,\alpha M}} (\dim Q_{\lambda}) S_{M_{\lambda},W_{\mu}}^{V^G}.$$

On the other hand, by $W = \bigoplus_{\mu \in \Lambda_{G,\alpha W}} P_{\mu} \otimes W_{\mu}$ and Lemma 3.7, we have

$$S_{V^G}^{M,W_{\mu}} = (\dim P_{\mu}) \sum_{N \in \Theta(W)} S_{V}^{M,N}.$$

It follows that

$$\sum_{\lambda \in \Lambda_{G,\alpha M}} (\dim Q_{\lambda}) S_{M_{\lambda},W_{\mu}}^{V^G} = (\dim P_{\mu}) \sum_{N \in \Theta(W)} S_{V}^{M,N}.$$

Let $\lambda$ run over $\Lambda_{G,\alpha M}$. The preceding equation becomes a system of linear equations. By Theorem 5.1, one could solve this system of linear equations, and obtain

$$S_{M_{\lambda},W_{\mu}}^{V^G} = \frac{\dim Q_{\lambda}}{|G_M|} (\dim P_{\mu}) \sum_{N \in \Theta(W)} S_{V}^{M,N}.$$

Remark 5.4. In Equation 5.6, substitute $V$ for $M$. Recall the fact $G_V = G$. By quantum Galois theory, denote $V = \bigoplus_{\chi \in \text{Irr}(G)} (Q_{\chi} \otimes V_{\chi})$. Equation 5.6 becomes

$$S_{V_x,W_{\mu}}^{V^G} = \frac{(\dim Q_{\chi})}{|G|} (\dim P_{\mu}) \sum_{N \in \Theta(W)} S_{V}^{V,N}. \quad (5.7)$$

Corollary 5.5. Let $W \in \mathcal{M}_{V^G}$.

$$S_{V,W}^{V^G} = |G| S_{V,W}^{V^G}. \quad (5.8)$$
Proof. Notice that \( q\dim(V) = |G|q\dim(V^G) \). Theorem 5.1 implies the desired result. \( \blacksquare \)

**Theorem 5.6.** Let \( M, N \in \mathcal{M}_V \), and \( g \in G \). Then,

\[
M \boxtimes_{V^G} N = \bigoplus_{g \in G} M \boxtimes_V (N \circ g).
\]  
(5.9)

Proof. By Corollary 5.5, we have

\[
|G|S_{V^G, W_i}^{V^G} = S_{V, W_i}^V = (\dim P_i)|G|S_{M, W}^{V^G}/|G_W|.
\]

That is,

\[
S_{V^G, W_i}^{V^G} = \frac{(\dim P_i)}{|G_W|} S_{V, W_i}^V.
\]  
(5.10)

This implies the right side of Equation 5.6 equals the right side of Equation 5.7. It follows that \( S_{((\bigoplus_{g \in G} M^G_{V^G}(N^g)), I)}^{V^G} = 0 \) for \( I \in \mathcal{M}_V \) of type one (untwisted). Remark 3.13 shows that \( S_{((\bigoplus_{g \in G} M^G_{V^G}(N^g)), I)}^{V^G} = 0 \) for \( I \in \mathcal{M}_V \) of type two (twisted). Therefore, \( S_{((\bigoplus_{g \in G} M^G_{V^G}(N^g)), I)}^{V^G} = 0 \) for every \( I \in \mathcal{M}_V \).

By Lemma 3.5, \( M \boxtimes_{V^G} N = \bigoplus_{g \in G} M \boxtimes_V (N \circ g) \).

\[ \blacksquare \]

**Lemma 5.7.**

\[
V \boxtimes_{V^G} V = |G|V.
\]  
(5.11)

Proof. This is a special case of Theorem 5.6. \[ \blacksquare \]

### 6 Fusion rules for \( \mathcal{M}_{V^G} \), and their quantum dimensions

**Theorem 6.1.** Let \( M, N, F \in S_V(G) \). By orbifold theory, write

\[
M = \bigoplus_{\lambda \in \Lambda_{M, \alpha_M}} U_{\lambda} \otimes M_{\lambda}
\]

\[
N = \bigoplus_{\chi \in \Lambda_{N, \alpha_N}} W_{\chi} \otimes N_{\chi}
\]

\[
F = \bigoplus_{\xi \in \Lambda_{F, \alpha_F}} V_{\xi} \otimes F_{\xi}.
\]

Then,

\[
\frac{\langle (M_{\lambda_1} \boxtimes_{V^G} N_{\chi_1}), F \rangle}{\langle (M_{\lambda_2} \boxtimes_{V^G} N_{\chi_2}), F \rangle} = \frac{(q\dim_{V^G}M_{\lambda_1})(q\dim_{V^G}N_{\chi_1})}{(q\dim_{V^G}M_{\lambda_2})(q\dim_{V^G}N_{\chi_2})},
\]

where \( \lambda_1, \lambda_2 \in \Lambda_{G_M, \alpha_M}, \) and \( \chi_1, \chi_2 \in \Lambda_{G_N, \alpha_N} \).
Proof. Note that the $S$-matrix is unitary. By Lemma 3.14 $S_{F,E_i}^{V_G} = 0$, for $E_i \in \mathcal{M}_{V_G,II}$. Thus,

$$\langle M, F \rangle_{V_G} = \sum_{E_i \in \mathcal{M}_{V_G}} S_{M,E_i}^{V_G} S_{F,E_i}^{V_G}$$

Apply Corollary 3.15, Lemma 3.6, and Equation 5.10

$$\langle (M_{\lambda_1} \boxtimes_{V_G} N_{\chi_1}), F \rangle = \sum_{E_i \in \mathcal{M}_{V_G,II}} S_{(M_{\lambda_1} \boxtimes_{V_G} N_{\chi_1}),E_i}^{V_G} S_{F,E_i}^{V_G}$$

Notice that the value, $\frac{1}{|G_M|} S_{V_G, E_i}^{V_G} \frac{1}{|G_N|} S_{V_G}^{V_G}$, is independent of choices of $\lambda_1$ and $\chi_1$. By Equation 2.5, the desired result follows

$$\frac{\langle (M_{\lambda_1} \boxtimes_{V_G} N_{\chi_1}), F \rangle}{\langle (M_{\lambda_2} \boxtimes_{V_G} N_{\chi_2}), F \rangle} = \frac{(\dim U_{\lambda_1})(\dim W_{\chi_1})}{(\dim U_{\lambda_2})(\dim W_{\chi_2})} = \frac{(q \dim_{V_G} M_{\lambda_1})(q \dim_{V_G} N_{\chi_1})}{(q \dim_{V_G} M_{\lambda_2})(q \dim_{V_G} N_{\chi_2})}.$$  

Remark 6.2. Theorem 6.1 shows that

$$\langle (M_{\lambda_1} \boxtimes_{V_G} N_{\chi_1}), F \rangle = \langle \sum_{\xi \in \Lambda_{GP,\alpha F}} N_{M_{\lambda_1},N_{\chi_1}}^{F_{\xi}} \sum_{\xi \in \Lambda_{GP,\alpha F}} (\dim V_{\xi}) F_{\xi} \rangle = \sum_{\xi \in \Lambda_{GP,\alpha F}} N_{M_{\lambda_1},N_{\chi_1}}^{F_{\xi}} (\dim V_{\xi}).$$

By Equation 2.5 and Theorem 6.1 it follows that

$$\frac{\langle (M_{\lambda_1} \boxtimes_{V_G} N_{\chi_1}), F \rangle}{\langle (M_{\lambda_2} \boxtimes_{V_G} N_{\chi_2}), F \rangle} = \frac{\sum_{\xi \in \Lambda_{GP,\alpha F}} N_{M_{\lambda_1},N_{\chi_1}}^{F_{\xi}} (\dim V_{\xi})}{\sum_{\xi \in \Lambda_{GP,\alpha F}} N_{M_{\lambda_2},N_{\chi_2}}^{F_{\xi}} (\dim V_{\xi})} = \frac{(q \dim_{V_G} M_{\lambda_1})(q \dim_{V_G} N_{\chi_1})}{(q \dim_{V_G} M_{\lambda_2})(q \dim_{V_G} N_{\chi_2})}.$$
This means, inside every irreducible \( g \)-twisted \( V \)-module \( F \), fusion rules of irreducible \( V^G \)-modules distribute "proportionally" to the product of their quantum dimensions.

**Corollary 6.3.** Use the same notations in Theorem 6.1. We have

\[
\text{qdim}_{V^G}( \bigoplus_{\xi \in \Lambda_{G^F, \alpha_F}} N_{M_{V^G}}(F_\xi)) = \frac{\text{dim}(W_{d_1}) \text{dim}(U_{d_1}) \text{qdim}_V(F)}{|G_M||G_N| |G_F|} \sum_{g, h, l \in G} N_{M_{V^G}}(F_{\omega h})
\]

**Proof.** Note that

\[
\sum_{\chi \in \Lambda_{G^N, \alpha_N}, \lambda \in \Lambda_{G^M, \alpha_M}} \text{dim}(W_\chi)^2 \text{dim}(U_\lambda)^2 = |G_M||G_N|.
\]

Recall that

\[
M \otimes_{V^G} N = \sum_{\chi \in \Lambda_{G^N, \alpha_N}, \lambda \in \Lambda_{G^M, \alpha_M}} \text{dim}(W_\chi) \text{dim}(U_\lambda) M_{V^G, \alpha} N_\chi.
\]

By Theorem 6.1 it follows that

\[
\sum_{\xi \in \Lambda_{G^F, \alpha_F}} N_{M_{V^G}}(F_\xi) (\text{dim}(V_\xi)) = \frac{\text{dim}(W_{d_1}) \text{dim}(U_{d_1}) \langle M \otimes_{V^G} N, F \rangle}{|G_M||G_N|} \sum_{g, h, l \in G} N_{M_{V^G}}(F_{\omega h})
\]

Note that

\[
\text{qdim}_{V^G}(F_\xi) = \text{dim} V_\xi \text{qdim}_V(F) |G| |G_F|,
\]

and hence

\[
\text{qdim}_{V^G}( \bigoplus_{\xi \in \Lambda_{G^F, \alpha_F}} N_{M_{V^G}}(F_\xi)) = \sum_{\xi \in \Lambda_{G^F, \alpha_F}} N_{M_{V^G}}(F_\xi) (\text{dim}(V_\xi)) \text{qdim}_V(F) |G| |G_F| = \frac{\text{dim}(W_{d_1}) \text{dim}(U_{d_1}) \langle M \otimes_{V^G} N, F \rangle \text{qdim}_V(F)|G|}{|G_M||G_N| |G_F|}.
\]

So, by Theorem 5.6 and Equation (4.5), we have

\[
\langle M \otimes_{V^G} N, F \rangle = \sum_{g \in G} \langle M \otimes_V N \circ g, F \rangle = \sum_{g \in G} \langle \sum_{E \in \mathcal{G}} N_{E_{M,N_\omega} E}, F \rangle = \sum_{g \in G} \langle \sum_{E \in \mathcal{G}(F)} N_{E_{M,N_\omega} E}, F \rangle = \sum_{g, h \in G} \langle N_{E_{M,N_\omega} E \circ h}, F \rangle \frac{|\mathcal{G}(F)|}{|G|}.
\]
Note that \( \iota(F \circ h) = \iota(F) \) as vectors in \( E_{V^0} \). By Remark 6.4 we have
\[
\langle M \boxtimes_{V^0} N, F \rangle = \sum_{g,h \in G} N_{F_{\text{coh}} M, N} G_{\iota(F)} \frac{|G|}{|G|} = \sum_{g,h \in G} N_{F_{\text{coh}} M, N} G_{\iota(F)} = \sum_{g,h \in G} N_{F_{\text{coh}} M, N} G_{\iota(F)}.
\]

Lemma 3.9 and Verlinde formula implies \( N_{F_{\text{coh}} M, N} = N_{F_{\text{coh}} M, N} G_{\iota(F)} \). So, it follows that
\[
|G| \sum_{g,h \in G} N_{F_{\text{coh}} M, N} = \sum_{g,h \in G} N_{F_{\text{coh}} M, N} G_{\iota(F)}.
\]

Then, we have the desired result.

\( \square \)

**Remark 6.4.** Use the same notations in Reference [21], assume that \( M, N, F \in \mathcal{M}_V \). Set
\[
\mathcal{I} = \bigoplus_{(L^1, L^2, L^3) \in \theta(M) \times \theta(N) \times \theta(F)} \mathcal{I}_V \left( \begin{pmatrix} L^3 \\ L^1 \\ L^2 \end{pmatrix} \right) \otimes L^1 \otimes L^2.
\]

Let \( X^1 = \text{Ind}_{S(M)}^{D(N)} U_{\lambda_1} \), and \( X^2 = \text{Ind}_{S(N)}^{D(N)} W_{\chi_1} \). Set
\[
\mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2) = \text{Span}_C \{ f \otimes (w^1 \otimes v^1) \otimes (w^2 \otimes v^2) \in \mathcal{I}| w^1 \in X^1, w^2 \in X^2 \}
\]
with \( v^1 \in M_{\lambda_1}, v^2 \in N_{\chi_1} \). Then by Reference [21] Theorem 2, we have
\[
N_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} \geq \dim_C \text{Hom}_{\mathbb{A}^{\alpha(\iota(F))}(G, \iota(F))}(\text{Ind}_{S(F)}^{D(F)} V_{\xi}, \mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2)). \tag{6.1}
\]

Under certain assumption, this inequality is an equation.

**Theorem 6.5.** Let \( M, N, F \in \mathcal{M}_V \). Use the same notations in Remark 6.4. We have
\[
N_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} = \dim_C \text{Hom}_{\mathbb{A}^{\alpha(\iota(F))}(G, \iota(F))}(\text{Ind}_{S(F)}^{D(F)} V_{\xi}, \mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2)).
\]

**Proof.** Let \( H_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} = \dim_C \text{Hom}_{\mathbb{A}^{\alpha(\iota(F))}(G, \iota(F))}(\text{Ind}_{S(F)}^{D(F)} V_{\xi}, \mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2)) \). Note that
\[
\text{qdim}_{V^0}(F_\xi) = \dim V_{\xi} \text{qdim}_{V}(F) \frac{|G|}{|G_F|}.
\]

We have
\[
\text{qdim}_{V^0} \left( \bigoplus_{\xi \in \Lambda_{G_F, \alpha_\mathcal{F}}} H_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} \right) = \sum_{\xi \in \Lambda_{G_F, \alpha_\mathcal{F}}} H_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} \text{dim}(\text{Ind}_{S(F)}^{D(F)} V_{\xi}) \text{qdim}_{V}(F).
\]
Recall that $\mathcal{A}_{\alpha G(F)}(G, \sigma(F))$ is semisimple, and its simple modules are precisely $\text{Ind}_{S(F)}^{D(F)} V_\xi$.

It follows that

$$\sum_{\xi \in \Lambda_{G,F,\sigma(F)}} H_{M_{\lambda_1}}^{F_\xi} N_{\lambda_1} \dim(\text{Ind}_{S(F)}^{D(F)} V_\xi) q\dim_{V_\xi}(F) = q\dim_{V_\xi}(F) \dim(\mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2)).$$

Note that

$$\mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2) \cong \bigoplus_{(L^1, L^2, L^3) \in \mathcal{O}(M) \times \mathcal{O}(N) \times \mathcal{O}(F)} I_{V_\xi} \left( \begin{array}{ccc} L^3 \\ L^1 \\ L^2 \end{array} \right) \otimes U_{\lambda_1} \otimes W_{\chi_1}.$$

So, we have

$$\dim(\mathcal{I}_{\lambda_1, \chi_1}(v^1, v^2)) = \sum_{(L^1, L^2, L^3) \in \mathcal{O}(M) \times \mathcal{O}(N) \times \mathcal{O}(F)} \dim I_{V_\xi} \left( \begin{array}{ccc} L^3 \\ L^1 \\ L^2 \end{array} \right) \dim U_{\lambda_1} \dim W_{\chi_1}$$

$$= \sum_{(L^1, L^2, L^3) \in \mathcal{O}(M) \times \mathcal{O}(N) \times \mathcal{O}(F)} F_{L^1, L^2}^{L^3} \dim U_{\lambda_1} \dim W_{\chi_1}$$

$$= \frac{\dim(W_{\chi_1}) \dim(U_{\lambda_1}) \dim(V_{\xi})}{|G_M||G_N| |G_F|} \sum_{g,h,l \in G} N_{M_{\lambda_1}, N_{\chi_1}}^{F_{\omega h}}.$$

It follows that

$$\sum_{\xi \in \Lambda_{G,F,\sigma(F)}} H_{M_{\lambda_1}}^{F_\xi} N_{\lambda_1} \dim(\text{Ind}_{S(F)}^{D(F)} V_\xi) q\dim_{V_\xi}(F) = \frac{\dim(W_{\chi_1}) \dim(U_{\lambda_1}) \dim(V_{\xi})}{|G_M||G_N| |G_F|} \sum_{g,h,l \in G} N_{M_{\lambda_1}, N_{\chi_1}}^{F_{\omega h}}.$$

By Corollary 6.3, we have

$$\sum_{\xi \in \Lambda_{G,F,\sigma(F)}} N_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} \dim_{V_\xi}(F_\xi) = \sum_{\xi \in \Lambda_{G,F,\sigma(F)}} H_{M_{\lambda_1}}^{F_\xi} N_{\lambda_1} \dim(V_{\xi}).$$

Since we have $\dim(\text{Ind}_{S(F)}^{D(F)} V_\xi) > 0$ and $N_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi} \geq H_{M_{\lambda_1}, N_{\chi_1}}^{F_\xi}$, the desired result follows. \(\square\)

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