Inequalities of extended \((p, q)\)-beta and confluent hypergeometric function

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Abstract

In this present paper, we establish the log convexity and Turán type inequalities of extended \((p, q)\)-beta functions. Also, we present the log-convexity, the monotonicity and Turán type inequalities for extended \((p, q)\)-confluent hypergeometric function by using the inequalities of extended \((p, q)\)-beta functions.

keywords: Extended beta functions, extended hypergeometric functions, log-convexity, Turán-type inequalities.

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1 Introduction

We begin with the classical gamma function

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re(z) > 0.
\]

In another way, it is defined as

\[
\Gamma(z) = \lim_{n \to \infty} \frac{n!n^{z-1}}{(z)_n}
\]

where \((\alpha)_n\) is the Pochhammer symbol defined as

\[
(\alpha)_n = \begin{cases} 
\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1); & \text{for } n \geq 1, \alpha \neq 0 \\
1 & \text{if } n = 0 
\end{cases}
\]

and

\[
\Gamma(z+1) = z\Gamma(z)
\]
The relation between Pochhammer symbol and gamma function is given below

\[(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)}.\]

The beta function is defined by

\[B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad (1.1)\]

\[(\Re(x) > 0, \Re(y) > 0)\]

and

\[B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \Re(x) > 0, \Re(y) > 0. \quad (1.2)\]

Chaudhry and Zubair \[4\] and Chaudhry et al. \[5\] defined the following extended gamma and beta functions

\[\Gamma_p(z) = \int_0^\infty t^{z-1}e^{-t-p^t}dt, \quad (1.3)\]

\[\Re(z) > 0, p \geq 0.\]

When \(p = 0\), then \(\Gamma_p\) tends to the classical gamma function \(\Gamma\), and

\[B(x, y; p) = \int_0^1 t^{x-1}(1 - t)^{y-1}e^{-p^t}dt \quad (1.4)\]

(where \(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0\)) respectively. When \(p = 0\), then \(B(x, y; 0) = B(x, y)\). Recently Choi et al. \[6\] introduced the following extension of extended beta function as

\[B(x, y; p, q) = \int_0^1 t^{x-1}(1 - t)^{y-1}e^{-p^t-\frac{q}{1-t}}dt \quad (1.5)\]

(where \(\Re(p) > 0, \Re(q) > 0, \Re(x) > 0, \Re(y) > 0\)).

It is clear that when \(p = q\), then (1.5) reduces to the well known extended beta function (1.4). Similarly if \(p = q = 0\), then (1.5) reduces to the classical beta function (1.1). In the same paper, they also defined the following extension of extended confluent hypergeometric function by

\[\Phi_{p,q}(\beta; \gamma; z) = \sum_{n=0}^\infty \frac{B(\beta + n; \gamma - \beta; p, q)}{B(\beta, \gamma - \beta)} z^n n! \quad (1.6)\]

\[
\left( p \geq 0, q \geq 0, \Re(\gamma) > \Re(\beta) > 0 \right).
\]

The integral representations of extension of extended confluent hypergeometric function is given by

\[\Phi_{p,q}(\beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1 - t)^{\gamma-\beta-1} \exp\left(zt - \frac{p}{t} - \frac{q}{1-t}\right) dt \quad (1.7)\]

\[
\left( p \geq 0, q \geq 0, \Re(\gamma) > \Re(\beta) > 0 \right).
\]

Note that for \(p = q\), the series (1.6) respectively reduces to the extended confluent hypergeometric series. Similarly for \(p = q = 0\) the series (1.6) respectively reduces to the classical confluent hypergeometric series.
2 Main results: Inequalities of extended \((p, q)\)-beta function

In this section, we establish some inequalities which involve extended \((p, q)\)-beta functions by using some natural inequalities \([10]\). For this continuation of our study, we recall the following well-known Chebychev’s integral inequality and Hölder-Rogers inequality.

**Lemma 2.1.** (see \([7, 13]\)) Let the functions \(f, g : [a, b] \subseteq R \rightarrow R\) be asynchronous for all \(x \in [a, b]\) and \(p(x) : [a, b] \subseteq R \rightarrow R\) be a positive integrable function, then

\[
\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx.
\]  

(2.1)

**Definition 2.1.** In \([3]\), a function \(f : (a, b) \rightarrow R\) is said to be convex if for any \(x_1, x_2 \in (a, b)\) and \(\alpha \in (0, 1)\)

\[
f(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha f(x_1) + (1 - \alpha) f(x_2).
\]  

(2.2)

It shows that when we move from \(x_1\) to \(x_2\), the line joining the points \((x_1, f(x_1))\) and \((x_2, f(x_2))\) lies always above the graph of \(f\).

**Definition 2.2.** A function \(f\) is said to be a log-convex if \(f > 0\) and \(\log f\) is convex i.e., for all \(x_1, x_2 \in I\) (where \(I\) is an interval) and \(\alpha \in (0, 1)\), we have

\[
\log f(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha \log f(x_1) + (1 - \alpha) \log f(x_2) = \log(f^\alpha(x)f^{1 - \alpha}(x_2)).
\]

This implies that

\[
f(\alpha x_1 + (1 - \alpha) x_2) \leq f^\alpha(x_1)f^{1 - \alpha}(x_2).
\]  

(2.3)

**Lemma 2.2.** (Hölder inequality \([13]\)) If \(\theta_1\) and \(\theta_2\) are positive real numbers such that \(\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1\), then the following inequality holds for integrable functions \(f, g : [a, b] \rightarrow R\):

\[
|\int_a^b f(x)g(x)dx| \leq (\int_a^b |f|^{\theta_1}dx)^{\frac{\theta_2}{\theta_1}}(\int_a^b |g|^{\theta_2}dx)^{\frac{\theta_1}{\theta_2}}.
\]  

(2.4)

**Theorem 1.** If \(x, y, x_1, y_1\) are positive real numbers satisfying the condition

\[
(x - x_1)(y - y_1) \geq 0,
\]

(2.5)

then for the extended \((p, q)\)-beta function, we have the inequality

\[
B_{p,q}(x, y_1)B_{p,q}(x_1, y) \leq B_{p,q}(x_1, y_1)B_{p,q}(x, y),
\]  

(2.6)

**Proof.** Consider the mappings \(f, g, h : [0, 1] \rightarrow [0, \infty)\) given by

\[
f(t) = t^{x - x_1}, \quad g(t) = (1 - t)^{y - y_1} \quad \text{and} \quad h(t) = t^{x_1 - 1}(1 - t)^{y_1 - 1}\exp\left(-\frac{t}{1 - t}\right).
\]

Now, differentiation of \(f\) and \(g\) gives

\[
f'(t) = (x - x_1)t^{x - x_1 - 1}, \quad g'(t) = (y_1 - y)(1 - x)^{y - y_1 - 1}.
\]
This shows that $f$ and $g$ have the same monotonicity on $[0, 1]$. Applying the Chebyshev’s integral inequality (2.1), for the above defined functions $f$, $g$ and $h$, we have
\[
\left( \int_a^b t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) dt \right) \left( \int_a^b t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) dt \right)
\leq \left( \int_a^b t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) dt \right) \left( \int_a^b t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) dt \right)
\]
which implies that,
\[
B_{p,q}(x, y_1)B_{p,q}(x_1, y) \leq B_{p,q}(x_1, y_1)B_{p,q}(x, y)
\]
which completes the desired proof. \qed

**Theorem 2.** The function $(p, q) \mapsto B_{p,q}(x, y)$ is log convex on $(0, \infty)$ for each $x, y > 0$. Moreover, the function $B_{p,q}(x, y)$ satisfy the following Turán type inequality
\[
B^2_{p,q}(x, y) - B_{p+a,q+a}(x, y)B_{p-a,q-a}(x, y) \leq 0,
\]
for all real $a$.

**Proof.** From the definition of log-convexity, it will be sufficient to prove that
\[
B_{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2}(x, y) \leq \left( B_{p,q}(x, y) \right)^\alpha \left( B_{p,q}(x, y) \right)^{1-\alpha},
\]
for $\alpha \in [0, 1]$, $p_1, p_2, q_1, q_2 > 0$ and for a fixed $x, y > 0$. Obviously, (2.8) is true for $\alpha = 0$ and $\alpha = 1$. Assume that $\alpha \in (0, 1)$, then it follows from (2.5) that
\[
B_{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2}(x, y)
= \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{\alpha p_1 - (1-\alpha)p_2}{t} + \frac{-\alpha q_1 - (1-\alpha)q_2}{1-t} \right)
= \left( \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p_1}{t} - \frac{q_1}{1-t} \right) \right)^\alpha
\times \left( \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p_1}{t} - \frac{q_1}{1-t} \right) \right)^{1-\alpha}
\]
Let $\theta_1 = \frac{1}{\alpha}$ and $\theta_2 = \frac{1}{1-\alpha}$. Clearly $\theta_1 > 1$ and $\theta_1 + \theta_2 = \theta_1\theta_2$. Thus applying the Hölder-Rogers inequality (2.4) for integrals in (2.9) gives
\[
B_{\alpha p_1 + (1-\alpha)p_2, \alpha q_1 + (1-\alpha)q_2}(x, y) < \left( \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p_1}{t} - \frac{q_1}{1-t} \right) \right)^\alpha
\times \left( \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p_1}{t} - \frac{q_1}{1-t} \right) \right)^{1-\alpha}
= \left( B_{p,q}(x, y) \right)^\alpha \left( B_{p,q}(x, y) \right)^{1-\alpha},
\]
This implies that $(p, q) \mapsto B_{p,q}(x, y)$ is log convex on $(0, \infty)$.

Now, taking $\alpha = \frac{1}{2}$, $p_1 = p - a$, $p_2 = p + a$, and $q_1 = q - a$, $q_2 = q + a$, the inequality (2.10) yields
\[
B^2_{p,q}(x, y) - B_{p+a,q+a}(x, y)B_{p-a,q-a}(x, y) \leq 0.
\]
\qed

\[
4
\]
Theorem 3. The function \((x, y) \mapsto B_{p,q}(x, y)\) is logarithmic convex on \((0, \infty) \times (0, \infty)\), for all \(p, q \geq 0\). In particular

\[
B_{p,q}^2 \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \leq B_{p,q}(x_1, y_1)B_{p,q}(x_2, y_2).
\]

Proof. Let \((x_1, y_1), (x_2, y_2) \in (0, \infty)^2\), and \(c, d \geq 0\) with \(c + d = 1\), then we have

\[
B_{p,q} \left( c(x_1, y_1) + d(x_2, y_2) \right) = B_{p,q}(cx_1 + dx_2, cy_1 + dy_2).
\]

Applying the definition of \((p, q)\)-extended beta function on the right hand side of inequality (2.11), we have

\[
B_{p,q} \left( c(x_1, y_1) + d(x_2, y_2) \right)
= \int_0^1 t^{c(x_1,x_2)-(c+d)(1-t)} t^{y_1+y_2-(c+d)(1-t)} \exp \left( -p \frac{t}{t} - q \frac{c+d}{1-t} \right) dt
= \int_0^1 (t^{x_1} (1-t)^{y_1-1} \exp \left( -p \frac{t}{t} - q \frac{c}{1-t} \right) t^{x_2-1} (1-t)^{y_2-1} \exp \left( -p \frac{t}{t} - q \frac{d}{1-t} \right))^d dt.
\]

Again by considering \(\theta_1 = \frac{1}{c}, \theta_2 = \frac{1}{d}\), we can use the Hölder-Rogers inequality for above integrals and it follows

\[
B_{p,q} \left( c(x_1, y_1) + d(x_2, y_2) \right) \leq \left( \int_0^1 t^{x_1-1} (1-t)^{y_1-1} \exp \left( -p \frac{t}{t} - q \frac{c}{1-t} \right) dt \right)^c \times \left( \int_0^1 t^{x_2-1} (1-t)^{y_2-1} \exp \left( -p \frac{t}{t} - q \frac{d}{1-t} \right) dt \right)^d
= \left( B_{p,q}(x_1, y_1) \right)^c \left( B_{p,q}(x_2, y_2) \right)^d.
\]

This shows the logarithmic convexity of extended \((p, q)\)-beta function \(B_{p,q}(x, y)\) on \((0, \infty)^2\).

For \(c = d = \frac{1}{2}\), the above inequality reduces to

\[
B_{p,q}^2 \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \leq B_{p,q}(x_1, y_1)B_{p,q}(x_2, y_2).
\]

Let \(x, y > 0\) be such that \(\min_{a \in R} (x + a, x - a) > 0\), then by taking \(x_1 = x + a, x_2 = x + a, y_1 = y + b\) and \(y_2 = y - b\) in (2.12), we get

\[
\left[ B_{p,q}(x, y) \right]^2 \leq B_{p,q}(x + a, y + b)B_{p,q}(x - a, y - b),
\]

for all \(p, q \geq 0\). \(\square\)
3 Inequalities for \((p, q)\)-extended confluent hypergeometric function

In this section, we present the log-convexity and Turán type inequality for extended confluent hypergeometric function defined in \([1.6]\). For this continuation, we recall the following well-known lemma.

Lemma 3.1. Consider the power series \(f(x) = \sum_{n \geq 0} a_n x^n\) and \(g(x) = \sum_{n \geq 0} b_n x^n\), where \(a_n \in R\) and \(b_n > 0\) for all \(n\). Further assume that both series converge on \(|x| < \alpha\). If the sequence \(\{a_n/b_n\} \geq 0\) is increasing (or decreasing), then \(x \mapsto f(x)/g(x)\) is also increasing (or decreasing) function on \((0, \alpha)\).

Note that the above lemma is valid only if both \(f\) and \(g\) are both even or both odd functions.

Theorem 4. Let \(\beta \geq 0\) and \(\gamma, \delta > 0\), then the following assertions for extended \((p, q)\)-confluent hypergeometric function are true.

(i) For \(\gamma \geq \delta\), the function \(x \mapsto \Phi_{p,q}(\beta; \gamma; x)/\Phi_{p,q}(\beta; \delta; x)\) is increasing on \((0, \infty)\).

(ii) For \(\gamma \geq \delta\), we have

\[
\delta \Phi_{p,q}(\beta + 1; \gamma + 1; x) \Phi_{p,q}(\beta; \delta; x) \geq \gamma \Phi_{p,q}(\beta; \gamma; x) \Phi_{p,q}(\beta + 1; \delta + 1; x).
\]

(iii) The function \(x \mapsto \Phi_{p,q}(\beta; \gamma; x)\) is log-convex on \(R\).

(iv) The function \((p, q) \mapsto \Phi_{p,q}(\beta; \gamma; x)\) is log convex on \((0, \infty)\) for fixed \(x > 0\).

(v) Let \(\sigma > 0\), then the function

\[
\beta \mapsto \frac{B(\beta, \gamma) \Phi_{p,q}(\beta + \sigma; \gamma; x)}{B(\beta + \sigma, \gamma) \Phi_{p,q}(\beta; \gamma; x)}
\]

is decreasing on \((0, \infty)\) for fixed \(\gamma, x > 0\).

Proof. From the definition of \([1.6]\), it follows that

\[
\frac{\Phi_{p,q}(\beta; \gamma; x)}{\Phi_{p,q}(\beta; \delta; x)} = \sum_{n=0}^{\infty} \frac{a_n(c)x^n}{\sum_{n=0}^{\infty} a_n(d)x^n}, \quad \text{where} \quad a_n(t) = \frac{B_{p,q}(\beta + n, t - \beta)}{B(\beta, t - \beta)n!}.
\]

(3.1)

If we denote \(f_n = a_n(c)/a_n(d)\), then

\[
f_n - f_{n+1} = \frac{a_n(c)}{a_n(d)} - \frac{a_{n+1}(c)}{a_{n+1}(d)} = \frac{B(\beta, \delta - \beta)}{B(\beta, \gamma - \beta)} \left( \frac{B_{p,q}(\beta + n, \gamma - \beta)}{B_{p,q}(\beta + n, \delta - \beta)} - \frac{B_{p,q}(\beta + n + 1, \gamma - \beta)}{B_{p,q}(\beta + n + 1, \delta - \beta)} \right).
\]

Now take \(x = \beta + n, y = \delta - \beta, x_1 = \beta + n + 1, y_1 = \gamma - \beta\) in \([2.0]\). Since \((x - x_1)(y - y_1) = \gamma - \delta \geq 0\), it follows from Theorem [1] that

\[
\frac{B_{p,q}(\beta + n, \gamma - \beta)}{B_{p,q}(\beta + n, \delta - \beta)} \leq \frac{B_{p,q}(\beta + n + 1, \gamma - \beta)}{B_{p,q}(\beta + n + 1, \delta - \beta)}.
\]

this is equivalent to say that \(\{f_n\}\) is an increasing sequence and hence with the aid of Lemma 3.1, we observe that \(x \mapsto \Phi_{p,q}(\beta; \gamma; x)/\Phi_{p,q}(\beta; \delta; x)\) is increasing on \((0, \infty)\).

To prove the assertion (ii), we recall the following well-known identity from \([6]\):

\[
\frac{d^n}{dx^n} \Phi_{p,q}(\beta; \gamma; x) = \frac{(\beta_n)}{(\gamma_n)} \Phi_{p,q}(\beta + n; \gamma + n; x).
\]

(3.2)
Since the increasing properties of \( x \mapsto \Phi_{p,q} \left( \beta; \gamma; x \right) / \Phi_{p,q} \left( \beta, \delta; x \right) \) is equivalent to the following inequality

\[
\frac{d}{dx} \left( \frac{\Phi_{p,q} \left( \beta; \gamma; x \right)}{\Phi_{p,q} \left( \beta, \delta; x \right)} \right) \geq 0.
\]  

(3.3)

This together with (5.2) implies

\[
\Phi'_{p,q} \left( \beta; \gamma; x \right) \Phi_{p,q} \left( \beta, \delta; x \right) - \Phi_{p,q} \left( \beta; \gamma; x \right) \Phi'_{p,q} \left( \beta, \delta; x \right) = \frac{\beta \gamma}{\gamma} \Phi_{p,q} \left( \beta + 1; \gamma + 1; x \right) \Phi_{p,q} \left( \beta, \delta; x \right) - \frac{\beta}{\delta} \Phi_{p,q} \left( \beta; \gamma; x \right) \Phi_{p,q} \left( \beta + 1; \delta + 1; x \right) \geq 0.
\]

This implies that

\[
\delta \Phi_{p,q} \left( \beta + 1; \gamma + 1; x \right) \Phi_{p,q} \left( \beta, \delta; x \right) \geq \gamma \Phi_{p,q} \left( \beta; \gamma; x \right) \Phi_{p,q} \left( \beta + 1; \delta + 1; x \right)
\]

which prove the assertion. The log-convexity of \( x \mapsto \Phi_{p,q} \left( \beta; \gamma; x \right) \) can be prove by using the integral representation of extended \((p, q)\)-confluent hypergeometric function as given in (1.7) and by applying the Hölder-Rogers inequality for integrals as follows:

\[
\Phi_{p,q} \left( \beta; \gamma; ax + (1 - \alpha)y \right) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta - 1} (1 - t)^{\gamma - 1 - 1} \exp \left( axt + (1 - \alpha)y - \frac{p}{t} - \frac{q}{1 - t} \right) dt
\]

\[
= \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 \left[ \left( t^{\beta - 1} (1 - t)^{\gamma - 1 - 1} \exp \left( xt - \frac{p}{t} - \frac{q}{1 - t} \right) \right) \right] \alpha \times \left( t^{\beta - 1} (1 - t)^{\gamma - 1 - 1} \exp \left( yt - \frac{p}{t} - \frac{q}{1 - t} \right) \right)^{1 - \alpha} dt
\]

\[
\leq \left[ \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 (1 - t)^{\gamma - 1 - 1} \exp \left( xt - \frac{p}{t} - \frac{q}{1 - t} \right) dt \right]^{\alpha} \times \left[ \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 (1 - t)^{\gamma - 1 - 1} \exp \left( yt - \frac{p}{t} - \frac{q}{1 - t} \right) dt \right]^{1 - \alpha}
\]

\[
= \left( \Phi_{p,q} \left( \beta; \gamma; x \right) \right)^{\alpha} \left( \Phi_{p,q} \left( \beta; \gamma; y \right) \right)^{1 - \alpha}, (x, y > 0, \alpha \in [0, 1]).
\]

This prove that \( x \mapsto \Phi_{p,q} \left( \beta; \gamma; x \right) \) is log-convex for a fixed \( x > 0 \). For the case when \( x < 0 \), then the assertion immediately follows from the identity (see [9]):

\[
\Phi_{p,q} \left( \beta; \gamma; x \right) = e^x \Phi_{q,p} \left( \gamma - \beta; \gamma; -z \right).
\]

Since, the infinite sum of log-convex functions is log-convex for \( x > 0 \). Thus, the log-convexity of \((p, q) \mapsto \Phi_{p,q} \left( \beta; \gamma; x \right)\) is equivalent to prove that \((p, q) \mapsto B(\beta + n, \gamma - \beta)\) is log-convex on \((0, \infty)\) and for non-negative integer \( n \). From Theorem [2] it is clear that \((p, q) \mapsto B(\beta + n, \gamma - \beta)\) is log-convex for \( \gamma > \beta > 0 \) and hence assertion (iv) is true.

Now, let \( \beta' \geq \beta \) and set \( h(t) = t^{\beta' - 1} (1 - t)^{\gamma - \beta' - 1} \exp \left( xt - \frac{p}{t} - \frac{q}{1 - t} \right), f(t) = \left( \frac{t}{1 - t} \right)^{\beta - \beta'} \) and \( g(t) = \left( \frac{t}{1 - t} \right)^\sigma \). Then using the integral representation (1.7) of extended confluent hypergeometric function, we
have
\[ \frac{B(\beta, \gamma)\Phi_{p,q}(\beta + \sigma; \gamma; x)}{B(\beta + \sigma, c)\Phi_{p,q}(\beta; \gamma; x)} - \frac{B(\beta', \gamma)\Phi_{p,q}(\beta' + \sigma; \gamma; x)}{B(\beta' + \sigma, \gamma)\Phi_{p,q}(\beta'; \gamma; x)} = \int_0^1 f(t)g(t)h(t)dt - \int_0^1 f(t)h(t)dt. \] (3.4)

One can easily determine that for \( \beta' \geq \beta \), the function \( f \) is decreasing when \( \sigma \geq 0 \) and the function \( g \) is increasing. Since \( h \) is non-negative function for \( t \in [0,1] \). Thus, by reverse Chebyshev’s reverse inequality (2.1), it follows that
\[ \int_0^1 f(t)g(t)h(t)dt \leq \int_0^1 h(t)dt \int_0^1 f(t)g(t)h(t)dt. \] (3.5)

This together with (3.4) implies
\[ \frac{B(\beta, \gamma)\Phi_{p,q}(\beta + \sigma; \gamma; x)}{B(\beta + \sigma, \gamma)\Phi_{p,q}(\beta; \gamma; x)} \geq 0, \]
which is equivalent to say that the function
\[ \beta \mapsto \frac{B(\beta, \gamma)\Phi_{p,q}(\beta + \sigma; \gamma; x)}{B(\beta + \sigma, \gamma)\Phi_{p,q}(\beta; \gamma; x)} \]
is decreasing on \((0, \infty)\). \( \square \)

Remark 3.1. In particular, the following decreasing property of extended \((p, q)\)-confluent hypergeometric function
\[ \beta \mapsto \frac{B(\beta, \gamma)\Phi_{p,q}(\beta + \sigma; \gamma; x)}{B(\beta + \sigma, \gamma)\Phi_{p,q}(\beta; \gamma; x)} \]
is equivalent to the following inequality
\[ \Phi_{p,q}^2(\beta + \sigma; \gamma; x) \geq \frac{B^2(\beta + \sigma, \gamma)}{B(\beta + 2\sigma, \gamma)B(\beta, \gamma)} \Phi_{p,q}(\beta + 2\sigma; \gamma; x)\Phi_{p,q}(\beta; \gamma; x). \] (3.6)

When \( p = q \), then the above inequality will reduce to the inequality recently proved by [11]. Similarly, when \( p = q = 0 \), then the above inequality reduces to the inequality of confluent hypergeometric which is an improved version of Theorem 4(b) given in [9].

4 conclusion

In this paper, we introduced inequalities for extended \((p, q)\)-beta and \((p, q)\)-confluent hypergeometric function defined by Choi et al. [6]. Throughout in this paper, if we take \( p = q \) then we get the inequalities of extended beta function and extended confluent hypergeometric function recently introduced by Mondal [11]. Similarly if we take \( p = q = 0 \), then the newly defined inequalities for extended \((p, q)\)-beta function will reduce to the inequalities of classical beta function (see [1, 7]).
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