DOUBLE-PARTITION QUANTUM CLUSTER ALGEBRAS

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ABSTRACT. A family of quantum cluster algebras is introduced and studied. In general, these algebras are new, but subclasses have been studied previously by other authors. The algebras are indexed by double partitions or double flag varieties. Equivalently, they are indexed by broken lines $L$. By grouping together neighboring mutations into quantum line mutations we can mutate from the cluster algebra of one broken line to another. Compatible pairs can be written down. The algebras are equal to their upper cluster algebras. The variables of the quantum seeds are given by elements of the dual canonical basis.

1. Introduction

A cluster algebra, as invented by Fomin and Zelevinsky, is a commutative algebra generated by a family of generators called cluster variables. The generators are grouped into clusters and the cluster variables can be computed recursively from the initial cluster.

The theory of cluster algebra is related to a wide range of subjects such as Poisson geometry, integrable systems, higher Teichmüller spaces, combinatorics, commutative and non-commutative algebraic geometry, and the representation theory of quivers and finite-dimensional algebras.

In [28], it is proved that the coordinate rings of $SL(n, \mathbb{C})$ and its maximal double Bruhat cell $SL(n, \mathbb{C})^{w_0,w_0}$ are cluster algebras. This is generalized in the recent work [11] where it is proved that the coordinate ring of any double Bruhat cell $G^{u,v}$ of any semi-simple algebraic group is a cluster algebra.

Quantum cluster algebras were introduced and studied by Berenstein and Zelevinsky [3]. A main motivation was to understand the dual canonical basis. Following Lusztig (23), the dual canonical basis for the coordinate algebra $O_q(M(n))$ of an $n \times n$ quantum matrix was shown to exist in [19]. This construction can be carried over to the algebra $O_q(M(m,n))$ for all $m$ and $n$ verbatim.

A quantum mutation is governed by a pair of matrices, called a compatible pair, with certain favorable properties. To construct a quantum cluster, one of the main difficulties is to construct the compatible pairs. In the present
paper we construct a family of quasi commuting quantum minors of the algebra \( O_q(M(m, n)) \) associated to each so-called broken line \( L \), and construct a corresponding compatible pair \((\Lambda_L, B_L)\).

The set of broken lines has a natural partial ordering with unique biggest and smallest elements.

Let us be more specific: A **broken line** from \((1, n)\) to \((m, 1)\) is a path in \(\mathbb{N} \times \mathbb{N}\) starting at \((1, n)\) and terminating at \((m, 1)\) while alternating between horizontal and vertical segments and passing through smaller column numbers (in the horizontal direction) and bigger row numbers (in vertical direction). To each broken line we construct in Section 7 a family of \(nm q\) commuting s.

With the line fixed, each of these quantum minors is uniquely given by a point \((i, j)\) \(\in \mathbb{N} \times \mathbb{N}\) with \(1 \leq i \leq m\) and \(1 \leq j \leq n\). The quantum cluster algebra \(A_{\Lambda_L}^{-}\) is then determined by the quantum minors corresponding to the points on or below the line \(L\). We prove that monomials in these are members of the dual canonical basis. We introduce the natural ordering on the set of broken lines and introduce some natural subalgebras. One such is \(O_q(T_L \cup L)\) which denotes the subalgebra of \(O_q(M(m, n))\) generated by the standard elements \(Z_{i,j}\) of \(O_q(M(m, n))\) (cf. Section 3) for which \((i, j)\) is on the line, or below it.

We introduce a special class of mutations that are called quantum line mutations. To each triple of broken lines \(L_a, L_b\) with \(L_a, L_b \leq L\) we can mutate by quantum line mutations from \(L_a\) to \(L_b\) by what is denoted by \(\mu^L(L_a, L_b)\).

For a compatible pair \((\Lambda_L, B_L)\) connected with \(A_{\Lambda_L}^{-}\) we show that we can mutate by quantum line mutations to a bigger line \(L_1\) inside \(O_q(M(m, n))\) and, by carefully keeping track, construct a compatible pair \((\Lambda_{L_1}, B_{L_1})\) connected with \(A_{\Lambda_{L_1}}^{-}\) in the process. Starting at a particularly simple broken line, namely the one corresponding to the smallest broken line \(L^{-}\), we can, by repeated quantum line mutations, construct a compatible pair for \(A_{\Lambda_L}^{-}\). Thus, we obtain compatible pairs for all broken lines. At first they are just compatible pairs for the smaller algebras. The algebra \(O_q(M(m, n))\) corresponds to the unique maximal broken line \(L^+\). However, mutating in the opposite direction, we get a compatible pair for this bigger algebra for any line. Or, indeed, mutating backwards from any bigger line algebra to a smaller, we get a quantum seed \(O_{L_1, L}\) for the bigger line algebra \(L\) indexed by the smaller line algebra \(L_1\).

Instances of such algebras have been studied in [21], [22], and [13].

The main technical result is the following: Let \(A\) be an \(n \times n\) matrix whose entries are non-negative integers and let \(b(A)\) be the element of the dual canonical basis of \(O_q(M(m, n))\) corresponding to this. Let \(\det_q\) denote the quantum determinant. If \(I\) denotes the \(n \times n\) identity matrix then

\[
\det_q = b(I)
\]

and (this is (4.2))

\[
b(A)\det_q = b(A + I).
\]

Once this has been established, it can be generalized to several other configurations involving quantum minors.
After this introduction, the article continues in Section 2 with a review of quantum cluster algebras, followed in Section 3 by basic facts and structures relating to the quantized matrix algebra. The matters concerning the technical result (1.2) and its generalizations, take up Sections 4 and 5.

In Section 6, using the results on dual canonical bases, we strengthen a result of Parshall and Wang considerably. In so doing, we obtain a crucial mutation identity: Theorem 6.16. This result then makes it possible to introduce the class of mutations called quantum line mutations. We also include an observation relating this to totally positive matrices.

In Section 7, we construct compatible pairs \((\Lambda^0_L, B^0_L)\) and \((\Lambda_L, B_L)\). At first just for the algebra \(O_q(T_L \cup L)\), but later also for the full algebra.

Finally, in Section 8, we extend slightly a result of Goodearl and Lenagan (15) saying that the \(q\)-determinantal ideal is prime. We then use quantum line mutations to give an inductive proof of the following, where \(C^\pm_L\) are the non-mutable (covariant) elements, and \(U^\pm_L\) is the upper cluster algebra:

**Theorem** Let \(C^+_L = \{Y_1, \ldots, Y_s\}\). Then,

\[
U^+_L = O_q(T_L \cup L)[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}] = A^-_L.
\]

This result is Theorem 8.5. As a consequence, we conclude that in the case of \(O_q(M(m, n))\), the quantum cluster algebra is equal to its upper cluster algebra.

2. **Basics of Quantum Cluster Algebras**

Throughout the paper, the base field is \(K = \mathbb{Q}(q)\), where \(q\) is an indeterminate over the rational numbers. To avoid terms involving \(q^{1/2}\), we work with the square of the \(q\) used by Benstein and Zelevinsky; \(q^2 = q^2_{our} = q_{BZ}\).

Given an integral skew-symmetric matrix \(\Lambda = (\lambda_{ij}) \in M_m(\mathbb{Z})\), the Laurent quasi polynomial algebra \(\mathcal{L}(\Lambda)\) associated to the matrix \(\Lambda\) is an associative algebra generated by \(x_1, x_2, \ldots, x_m; x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}\) with the defining relations

\[
\begin{align*}
x_i x_j &= q^{2 \lambda_{ij}} x_j x_i.
\end{align*}
\]

Conversely, given such relations, the matrix \(\Lambda = (\lambda_{ij}) \in M_m(\mathbb{Z})\) will be called the \(\Lambda\)-matrix of the variables \(x_1 \cdots, x_m\).

The set of ordered monomials

\[
\{x^\mathbf{a} := x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \mid \mathbf{a} = (a_1, a_2, \cdots, a_m) \in \mathbb{Z}^m\}
\]

is a basis of \(\mathcal{L}(\Lambda)\). It is well known that \(\mathcal{L}(\Lambda)\) is a Noetherian domain and one can talk about its skew field of fractions which is denoted by \(\mathcal{F}(\Lambda)\). Using \(\Lambda\), one can define a bilinear form on \(\mathbb{Z}^m\) as follows:

\[
\begin{align*}
\Lambda : \mathbb{Z}^m \times \mathbb{Z}^m &\longrightarrow \mathbb{Z} \\
\Lambda(a, b) &= a\Lambda b^T.
\end{align*}
\]
For any \( a \in \mathbb{Z}^m \), the normalized monomial is defined as
\[
x(a) = q^{\sum_{i<j} \lambda_{ij} a_i a_j} x^a.
\]
The map
\[
(2.3) \quad \forall i = 1, \ldots, m : x_i \mapsto x_i, \quad q \mapsto q^{-1}
\]
extends to a \( \mathbb{Q} \)-algebra anti-automorphism which actually does not depend on the ordering. Then
\[
(2.4) \quad \overline{x(a)} = x(a).
\]
It is easy to check that
\[
x(a)x(b) = q^{\Lambda(a,b)} x(a+b),
\]
which, of course, is equivalent to the commutation relations (2.1).

Denote by \( K^* := \mathbb{Q}(q) - \{0\} \) the multiplicative group of non-zero elements. The group \((K^*)^m \) acts on \( \mathcal{L}(\Lambda) \) as an automorphism group. Explicitly, for any \( h = (h_1, h_2, \ldots, h_m) \in (K^*)^m \), it acts on \( \mathcal{L}(\Lambda) \) according to the formulae
\[
h(x_i) = h_i x_i \text{ for all } i.
\]

**Remark 2.1.** If a subspace \( S \subset \mathcal{A}(\Lambda) \) is invariant under the action of the group \((K^*)^m \), then it is spanned by the monomials that it contains.

Notice that if \( a = (a_1, a_2, \ldots, a_m) \) and \( f = (f_1, f_2, \ldots, f_m) \) are vectors then

**Lemma 2.2.**
\[
(2.5) \quad \Lambda(a)^t = (f)^t \Leftrightarrow \forall i : x_i x^a = q^{2f_i} x^a x_i.
\]
However simple this actually is, it will have a great importance later on.

In [3], the notion of a quantum cluster algebra was introduced. Let us recall the definition.

**Definition 2.3.** Let \( B \) be an \( m \times n \) integer matrix with rows labeled by \([1, m]\) and columns labeled by an \( n \)-element subset \( ex \subset [1, m] \). Let \( \Lambda \) be a skew-symmetric \( m \times m \) integer matrix with rows and columns labeled by \([1, m]\). We say that a pair \((\Lambda, B)\) is compatible if, for every \( j \in ex \) and \( i \in [1, m] \), we have
\[
\sum_{k=1}^{m} b_{kj} \lambda_{ki} = 2\delta_{ij} d_j
\]
for some positive integers \( d_j \) \((j \in ex)\). [The factor 2 is an artifact of our working with \( q^2 \).]

If one arranges the symbols such that \( ex = \{1, 2, \ldots, n\} \), the compatibility condition states that the \( n \times m \) matrix \( \tilde{D} = B^T \Lambda \) consists of the two blocks: the \( n \times n \) diagonal matrix \( D \) with positive integer diagonal entries \( d_j \), and the \( n \times (m - n) \) zero block.

With the above setup, the triple \( (\{x_1, x_2, \ldots, x_m\}, \Lambda, B) \) is an example of a **quantum seed** of \( \mathcal{V}(\Lambda) \). The notion of a quantum seed is more general than
the one presented here, but ours suffice for the purposes below. The variables $x_i$ are called quantum cluster variables. The variables $x_i, i \in \text{ex}$ are called mutable variables and the set of these is called the cluster. The variables $x_j, j \notin \text{ex}$ are called non-mutable variables.

Denote by $e_1, e_2, \cdots, e_m$ the standard basis of $\mathbb{Z}^m$. For a given compatible pair $(\Lambda, B = (b_{ik}))$, one can mutate the cluster in the direction of $i \in \text{ex}$, thereby obtaining a new cluster whose variables are $x_1, \cdots, x_i', x_{i+1}, \cdots, x_m$. The unique new variable is defined by

$$x_i' = x \left( \sum_{b_{ki}>0} b_{ki} e_k - e_i \right) + x \left( \sum_{b_{ki}<0} -b_{ki} e_k - e_i \right).$$

One can check that $x_1, \cdots, x_i', x_{i+1}, \cdots, x_m$ is a q-commuting family.

We will extend matrix mutations to those of compatible pairs. Fix an index $i \in \text{ex}$. The matrix $B'_i = \mu_i(B)$ can be written as

$$B'_i = E_i B F_i,$$

where

$\bullet$ $E_i$ is the $m \times m$ matrix with entries

$$e_{ab} = \begin{cases} \delta_{ab} & \text{if } b \neq i; \\ -1 & \text{if } a = b = i; \\ \max(0, -b_{ai}) & \text{if } a \neq b = i. \end{cases}$$

$\bullet$ $F_i$ is the $n \times n$ matrix with rows and columns labeled by $\text{ex}$, and entries given by

$$f_{ab} = \begin{cases} \delta_{ab} & \text{if } a \neq i; \\ -1 & \text{if } a = b = i; \\ \max(0, b_{ab}) & \text{if } a = i \neq b. \end{cases}$$

The triple $(\{x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_m\}, \Lambda_i = E_i^T \Lambda E_i, B'_i)$ is also a quantum seed. The above process of passing from a quantum seed to another is called a quantum mutation in the direction $i$. We say that two quantum seeds are mutation equivalent if they can be obtained from each other by a sequence of quantum mutations. In the general definition of BZ there is an additional parameter $\varepsilon = \pm 1$ in the definition of the matrices $E_i, F_i$. Throughout this article we restrict to $\varepsilon = 1$ and for this reason we suppress it.

Given a quantum seed, let $S$ be the set of all quantum seeds which are mutation equivalent to the given one. The quantum cluster algebra $A(S)$ associated to the given quantum seed is the $\mathbb{Q}(q)$ subalgebra of $V(\Lambda)$ generated by all quantum cluster variables contained in $S$.

3. The quantum matrices the dual canonical bases

The coordinate algebra $O_q(M(m, n))$ of the quantum $m \times n$ matrix is an associative algebra, generated by elements $Z_{ij}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n,$
subject to the following defining relations:

\((3.1)\) \[ Z_{ij} Z_{ik} = q^2 Z_{ik} Z_{ij} \text{ if } j < k, \]
\((3.2)\) \[ Z_{ij} Z_{kj} = q^2 Z_{kj} Z_{ij} \text{ if } i < k, \]
\((3.3)\) \[ Z_{ij} Z_{st} = Z_{st} Z_{ij} \text{ if } i > s, j < t, \]
\((3.4)\) \[ Z_{ij} Z_{st} = Z_{st} Z_{ij} + (q^2 - q^{-2}) Z_{it} Z_{sj} \text{ if } i < s, j < t. \]

The associated quasi-polynomial algebra \(o_q(M(m, n))\) of the quantum \(m \times n\) matrix is an associative algebra, generated by elements \(z_{ij}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\), subject to the following defining relations:

\((3.5)\) \[ z_{ij} z_{ik} = q^2 z_{ik} z_{ij} \text{ if } j < k, \]
\((3.6)\) \[ z_{ij} z_{kj} = q^2 z_{kj} z_{ij} \text{ if } i < k, \]
\((3.7)\) \[ z_{ij} z_{st} = z_{st} z_{ij} \text{ in all other cases.} \]

For any matrix \(A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \in M_{m,n}(\mathbb{Z}_+), \) where \(\mathbb{Z}_+ = \{0, 1, \cdots\}\), we define a monomial \(Z^A\) by

\[(3.8)\] \[ Z^A = \prod_{i,j=1}^{n} Z_{ij}^{a_{ij}}, \]

where the factors are arranged in the lexicographic order on \(I(m, n) = \{(i, j) \mid i = 1, 2, \cdots, m; j = 1, \cdots, n\}\) given by \((1, 1) > (1, 2) > \cdots > (1, n) > (2, 1) > \cdots\). We define similar elements \(z^A \in o_q\). It is well known that the set \(\{Z^A \mid A \in M_{m,n}(\mathbb{Z}_+)\}\) is a basis of the algebra \(O_q(M(m, n))\).

From the defining relations \((3.1) - (3.7)\) of the algebras \(O_q(M(m, n))\) and \(o_q(M(m, n))\) it is easy to show the following lemma. The last statement in the lemma, though trivial, is included for its usefulness.

**Lemma 3.1.** The mapping

\[(3.9)\] \[ : Z_{ij} \mapsto Z_{ij} \quad q \mapsto q^{-1} \]

extends to an anti-automorphism of the algebra \(O_q(M(m, n))\) as an algebra over \(\mathbb{Q}\). The mapping

\[(3.10)\] \[ : z_{ij} \mapsto z_{ij} \quad q \mapsto q^{-1} \]

extends to an anti-automorphism of the algebra \(o_q(M(m, n))\) as an algebra over \(\mathbb{Q}\). If an element \(P\) in \(O_q(M(m, n))\) satisfies \(\overline{P} = P\) then any rewriting \(P_r\) of this element (such as an ordering of it) will satisfy \(\overline{P_r} = P_r\). A similar statement holds in \(o_q(M(m, n))\).

For any \(A = (a_{ij}) \in M_{m,n}(\mathbb{Z}_+), \)

\[ r \circ (A) := (\sum_j a_{1j}, \cdots, \sum_j a_{mj}) := (r_0, r_1, r_2, \cdots, r_n). \]
This is called the row sum of $A$. 
\[
\co(A) := \left( \sum_j a_{j1}, \cdots, \sum_j a_{jn} \right) := (\co_1, \co_2, \cdots, \co_n).
\]
This is called the column sum of $A$.

The following result follows easily from the defining relations (3.1) - (3.4):

**Lemma 3.2.** Let 
\[
Z^A Z^B = \sum_C a_{C}^{A,B} Z^C,
\]
where $a_{C}^{A,B} \in \mathbb{Q}[q^2, q^{-2}]$. Then $\forall a_{C}^{A,B} \neq 0$:
\[
\begin{align*}
\ro(C) &= \ro(A) + \ro(B), \\
\co(C) &= \co(A) + \co(B).
\end{align*}
\]

From the defining relations we also have
\[
(3.11) \quad \overline{Z}^A = E(A) Z^A + \sum_{B < A} c_B(A) Z^B,
\]
where
\[
E(A) = q^{-2(\sum_i \sum_{j > k} a_{ij} a_{ik} + \sum_i \sum_{j > k} a_{ji} a_{ki})}
\]
and $\forall B < A$: $c_B(A) \neq 0 \Rightarrow \ro(B) = \ro(A)$, and $\co(B) = \co(A)$. Here, 
\[
c_B(A) \in \mathbb{Z}[q, q^{-1}],
\]
and the lexicographic order on $M_{m,n}(\mathbb{Z}_+)$, obtained by augmenting the previous order on $I(m,n)$ by the natural order on $\mathbb{Z}_+$, is denoted $\leq$.

Let
\[
(3.12) \quad N(A) = q^{-\sum_i \sum_{j > k} a_{ij} a_{ik} - \sum_i \sum_{j > k} a_{ji} a_{ki}} \quad \text{and} \quad Z(A) = N(A) Z^A.
\]

It is easy to see that (compare with (2.4))
\[
(3.13) \quad \overline{Z(A)} = Z(A) \mod \text{lower order terms}.
\]

In lack of better words we introduce:

**Definition 3.3.** We call $Z(A)$ the normalized form of $Z^A$. We call $N(A)$ the normalization factor.

Let $i < s$ and $j < t$. Set $E_{i,j,s,t} = E_{i,j} + E_{s,t} - E_{i,t} - E_{s,j}$, where for any of the mentioned pairs $(a, b)$, $E_{a,b}$ is the $(a, b)$th matrix unit. Upon rewriting $\overline{Z}^A$ according to our lexicographic order, one picks up terms $c_{A'} Z^{A'}$, where $A'$ is obtained from $A$ by subtraction of elements of the form $E_{i,j,s,t}$. The next result follows directly from (3.12).

**Lemma 3.4.** If $A' = A - E_{i,j,s,t}$, then
\[
N(A') = N(A) q^{4 - 2(a_{ij} + a_{st} - a_{it} - a_{sj})}.
\]

To facilitate the following proofs, we introduce a notion of a level in $M_{m,n}(\mathbb{Z}_+)$:
Definition 3.5. Let
\[ \mathcal{D} = \{ E_{i,j,s,t} \mid i < s \text{ and } < t \}. \]
The matrix \( A \in M_{m,n}(\mathbb{Z}_+) \) is of level \( L(A) = 0 \) if there are no elements \( D \in \mathcal{D} \) and \( A_1 \in M_{m,n}(\mathbb{Z}_+) \) such that \( A = D + A_1 \). Let \( \mathcal{L}_0 \) denote the set of matrices of level 0. We define the level \( L(A) \) of any \( A \) not of level zero by
\[ L(A) := \max \{ r \in \mathbb{N} \mid \exists D_1, \ldots, D_r \in \mathcal{D}, \exists A_0 \in \mathcal{L}_0 : A = D_1 + \cdots + D_k + A_0 \}. \]
Set
\[ L^* = \bigoplus_{A \in M_{m,n}(\mathbb{Z}_+)} \mathbb{Z}[q] Z(A). \]

Proposition 3.6. There is a unique \( \mathbb{Z}[q] \)-basis \( B^* = \{ b(A) \mid A \in M_{m,n}(\mathbb{Z}_+) \} \) of \( L^* \) in which each element \( b(A) \) is determined uniquely by the following conditions:
1. \( \overline{b(A)} = b(A) \) for all \( A \).
2. \( b(A) = Z(A) + \sum_{B < A} h_B(A) Z(B) \) where \( h_B(A) \in q^2 \mathbb{Z}[q^2] \) and \( \overline{\rho}(B) = \overline{\rho}(A), \overline{\sigma}(B) = \overline{\sigma}(A) \).
The basis \( B^* \) is called the dual canonical basis of \( \mathcal{O}_q(M(m,n)) \).

Corollary 3.7. If we number our basis vectors in the two bases \( B_1 = \{ b(B) \mid B \in M_{m,n}(\mathbb{Z}_+) \} \) and \( B_2 = \{ Z(B) \mid B \in M_{m,n}(\mathbb{Z}_+) \} \) according to the lexicographic ordering then the change of basis matrices are lower triangular with 1’s in the diagonal and elements from \( q^2 \mathbb{Z}[q^2] \) in all other non-zero positions.

Proof of Proposition 3.6 and Corollary 3.7. Noticing the \( q^2 \) factors in Lemma 3.4, this can be proved in analogy with Lusztig ([23, 2. Proposition]), see also [19, Theorem 3.5]. However, we will sketch a proof for clarity: We proceed by induction on the level \( k \) utilizing that if Proposition 3.6 holds up to level \( k \) then so does Corollary 3.7. The case of level 0 is trivial since if \( L(A) = 0 \) then \( b(A) = Z(A) \). Suppose then that the result holds up to, and including level \( k \) and let \( A \) be of level \( k + 1 \). It is easy to see from the defining relations (3.1)-(3.4) together with Corollary 3.7 (up to level \( k \)) that
\[ \overline{Z(A)} - Z(A) = \sum_{B < A; \ L(B) < L(A)} h_B b(B) \]
with elements \( h_B \in \mathbb{Z}[q^2] \). Since the left hand side of (3.15) is skew under the bar operator, each \( h_B \) can be decomposed as \( h_B = h_B^+ + h_B^- \) with \( h_B^+ \in q^2 \mathbb{Z}[q^2] \) and \( h_B^- = -\overline{h_B^+} \). Then
\[ b(A) = Z(A) + \sum_{B < A} h_B^+ b(B) \]
is the unique solution. Invoking Corollary 3.7 (up to \( k \)) once again, the proof is complete.

The following simple principle is very useful:
Proposition 3.8. If \( m_1 \leq m \) and \( n_1 \leq n \) we may view \( \mathcal{O}_q(M(m_1,n_1)) \) as the subalgebra of \( \mathcal{O}_q(M(m,n)) \) generated by the elements of (some) \( m_1 \) rows and \( n_1 \) columns. If, correspondingly, we consider \( M_{m_1,n_1}(\mathbb{Z}_+) \subseteq M_{m,n}(\mathbb{Z}_+) \) then for any \( A \in M_{m_1,n_1}(\mathbb{Z}_+) \), upon these identifications, the basis vector \( b(A) \in \mathcal{O}_q(M(m_1,n_1)) \) is also a basis vector in \( \mathcal{O}_q(M(m,n)) \).

If, under such identifications, \( \mathcal{O}_q(M(m_1,n_1)) \) and \( \mathcal{O}_q(M(m_2,n_2)) \) are two commuting subalgebras of \( \mathcal{O}_q(M(m,n)) \) and if \( b(A_i) \in \mathcal{O}_q(M(m_i,n_i)), \ i = 1,2, \) are members of the respective dual canonical bases, then \( b(A_1 + A_2) = b(A_1)b(A_2) \) is in the dual canonical basis of \( \mathcal{O}_q(M(m,n)) \).

Proof. The relations, the bar operator, and the order on the subalgebras are restrictions of the relations, the bar operator, and the order on the full algebra. The result then follows by the uniqueness. \( \square \)

Remark 3.9. The commutativity condition in Proposition 3.8 is equivalent to having all elements of one subalgebra positioned NE of the other.

If \( m = n \), one may define the quantum determinant \( \text{det}_q \) as follows:

\[
\text{det}_q(n) = \text{det}_q = \sum_{\sigma \in S_n} (-q^2)^{\ell(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}
\]

(3.17)

\[
= \sum_{\sigma \in S_n} (-q^2)^{\ell(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}
\]

(3.18)

We recall some results from [25] regarding the Quantum Laplace Expansion:

Suppose \( I = \{i_1 < i_2 < \cdots < i_r\} \) and \( J = \{j_1 < j_2 < \cdots < j_r\} \) are subsets of \( I = \{1,2,\ldots,n\} \). Define

\[
\xi^I_J = \sum_{\sigma \in S_r} (-q^2)^{\ell(\sigma)} Z_{i_1,j_{\sigma(1)}} Z_{i_2,j_{\sigma(2)}} \cdots Z_{i_r,j_{\sigma(r)}}
\]

(3.19)

\[
= \sum_{\sigma \in S_r} (-q^2)^{\ell(\sigma)} Z_{i_1,j_1} Z_{i_2,j_2} \cdots Z_{i_r,j_r}
\]

(3.20)

These elements are called quantum minors. Notice that they are only defined if \( \#J = \#I \). For two subsets \( I, J \subseteq \{1,2,\ldots,n\} \), the symbol \( sgn_q(I;J) \) is defined by

\[
sgn_q(I;J) = \begin{cases} 
0 & \text{if } I \cap J \neq \emptyset \\
(-q^2)^{\ell(I;J)} & \text{if } I \cap J = \emptyset 
\end{cases}
\]

(3.21)

where \( \ell(I;J) = \#\{ (i,j) | i \in I, j \in J, i > j \} \). Then,

\[
Sgn_q(J_1;J_2) \xi^I_J = \sum_{I_1 \cup I_2 = I} \xi^I_{I_1} \xi^I_{I_2} Sgn_q(I_1;I_2)
\]

(3.22)

\[
Sgn_q(J_1;J_2) \xi^I_J = \sum_{I_1 \cup I_2 = I} \xi^I_{I_1} \xi^I_{I_2} Sgn_q(I_1;I_2)
\]

(3.23)

If \( m = n \) and \( I = \{1,2,\cdots,n\} \setminus \{i\} \), \( J = \{1,2,\cdots,n\} \setminus \{j\} \), \( \xi^I_J \) will (occasionally) be denoted by \( A(i,j) \).

The following was proved by Parshall and Wang in [26]:
Proposition 3.10. $\det_q$ is central. Furthermore, let $i, k \leq n$ be fixed integers. Then

\[
\delta_{i,k}\det_q = \sum_{j=1}^{n} (-q^2)^{i-k} Z_{i,j} A(k, j) = \sum_{j} (-q^2)^{i-j} A(i, j) Z_{k,j} 
\]

\[
= \sum_{j} (-q^2)^{i-k} Z_{j,i} A(j, k) = \sum_{j} (-q^2)^{i-j} A(j, i) Z_{j,k}.
\]

It is of key importance for the rest of the article to note the following which is proved by an easy induction argument using (3.24):

Corollary 3.11.

\[
\overline{\det_q} = \det_q = b(I).
\]

Thus, all quantum minors are members of the dual canonical basis.

Definition 3.12. An element $x \in \mathcal{O}_q(M(m,n))$ is called covariant if for any $Z_{ij}$ there exists an integer $n_{i,j}$ such that

\[
x Z_{i,j} = q^{2n_{i,j}} Z_{i,j} x.
\]

Clearly, $Z_{1,n}$ and $Z_{m,1}$ are covariant. Two elements $x, y \in \mathcal{O}_q(M(m,n))$ are said to $q$-commute if there exists an integer $p$ such that

\[
xy = q^{2p}yx.
\]

Let $\det_q(t) = \xi^{\{1, \ldots, t\}}_{\{n-t+1, \ldots, n\}}$, for $t = 1, 2, \ldots, \min\{m, n\}$. It is easy to extend [17, Theorem 4.3] from the $n \times n$ case to the general rectangular case:

Proposition 3.13. The element $\det_q(t)$ is covariant for all $t$. More precisely, let $M_i^- = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq t \text{ and } 1 \leq j \leq n-t\}$, $M_i^+ = \{(i, j) \in \mathbb{N}^2 \mid t+1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$, $M_t^l = \{(i, j) \in \mathbb{N}^2 \mid t+1 \leq i \leq m \text{ and } 1 \leq j \leq n-t\}$, and $M_t^r = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq t \text{ and } n-t+1 \leq j \leq n\}$.

\[
Z_{i,j}\det_q(t) = \det_q(t) Z_{i,j} \text{ if } (i, j) \in M_t^l \cup M_t^r,
\]

\[
Z_{i,j}\det_q(t) = q^{2t} \det_q(t) Z_{i,j} \text{ if } (i, j) \in M_t^l, \text{ and}
\]

\[
Z_{i,j}\det_q(t) = q^{-2t} \det_q(t) Z_{i,j} \text{ if } (i, j) \in M_t^r.
\]

Recall from [17] the result for quantum $2 \times 2$ matrices:

\[
\forall a \in \mathbb{N} : \ Z_{2,2}^{a} Z_{1,1} = Z_{1,1} Z_{2,2}^a - q^2(1 - q^{-4a}) Z_{2,2}^{a-1} Z_{2,1} Z_{1,2}.
\]
For later purposes, we need the following results for \( n \times n \) matrices regarding \( Z_{n,n}A(n,n) \): Using (3.24) we can define elements \( M_1, M_2 \) by

\[
\det_q = \sum_{j=1}^{n} (-q^2)^{j-n} Z_{n,j} A(n,j) = \sum_{j=1}^{n} (-q^2)^{n-j} A(n,j) Z_{n,j}
\]

\[
= Z_{n,n} A(n,n) + M_1
\]

\[
= A(n,n) Z_{n,n} + M_2
\]

An easy application of Proposition 3.13 gives that for \( j \neq n \), \( Z_{n,n} A(n,j) = q^{-2} A(n,j) Z_{n,n} \) and similarly for \( A(j,n) \), and it then follows that

\[
Z_{n,n} M_i = q^{-4} M_i Z_{n,n} \quad \text{for} \quad i = 1, 2.
\]

In the ring of fractions of \( O_q(M(n)) \) we can write \( A(n,n) = (Z_{n,n})^{-1} (\det_q - M_1) = (\det_q - M_2) (Z_{n,n})^{-1} \), which is useful since, by Proposition 3.13 the terms \( M_1, M_2 \) have simple \( q \)-relations with \( Z_{n,n} \). Thus,

\[
Z_{n,n} A(n,n) = \det_q - M_1 = \det_q - q^{-4} M_2
\]

\[
A(n,n) Z_{n,n} = \det_q - q^4 M_1 = \det_q - M_2, \quad \text{and hence}
\]

\[
[Z_{n,n}, A(n,n)] = (q^4 - 1) M_1
\]

\[
= (1 - q^{-4}) M_2.
\]

Notice that all monomials in \( M_2 \) contain factors of \( q^{2\ell} \) with \( \ell \geq 1 \).

More generally, we get

\[
[Z_{r,n}, A(n,n)] = q^4 (1 - q^{-4r}) M_1 Z_{n,n}^{r-1}
\]

\[
= (1 - q^{-4r}) M_2 Z_{n,n}^{r-1}.
\]

Likewise, it follows by induction that for all \( r \in \mathbb{N} \),

\[
(3.29) \quad [Z_{r,n}^{r-1}, A(1,1)] = -(1 - q^{-4r}) Z_{1,1}^{r-1} N_1 \quad \text{where}
\]

\[
(3.30) \quad N_1 = \sum_{j=2}^{n} (-q^2)^{j-1} Z_{1,j} A(1,j)
\]

\[
(3.31) \quad = \sum_{\sigma \in S_n : \sigma(1) \neq 1} (-q^2)^{\ell(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)},
\]

where, for each \( \sigma \) in the last sum, \( \ell(\sigma) \in \mathbb{N} \). This observation is an important ingredient in the proof of Theorem 4.1 below.

4. \( \det_q \) AND DUAL CANONICAL BASES

In this section, \( n = m \) throughout. The following result is of key importance. However simple to formulate, it is remarkably difficult to prove.
Theorem 4.1. For all $A \in M_n(Z_+)$ there are integers $c_B \in \{-1, 0, 1\}$ and integers $\gamma_B > 0$ such that

$$Z(A) \cdot \det_q = Z(A + I) + \sum_{B < A} q^{2\gamma_B} c_B Z(B).$$

In particular,

$$b(A) \cdot \det_q = b(A + I) = b(A)b(I).$$

Proof: By using Corollary 3.7, the second claim follows easily from the first. We proceed to prove (4.1) by induction on the number $c$ such that there are non-zero elements in at most the columns 1, $\cdots$, $c$ of $A$. For a fixed such $c$ we proceed by induction on the number $r$ such that there are non-zero elements in at most the rows 1, $\cdots$, $r$ in the $c$th column. Notice that the formula (4.1) holds for any $A$ with non-zero entries at most in the first column. Indeed, as follows by an elementary computation,

$$Z(A)Z(E_\sigma)(-q^2)^{f(\sigma)} = (-q^2)^{f(\sigma)} q^{2\gamma_i - (a_{i,1} + a_{i,1} + \cdots + a_{n,1})} Z(A + E_\sigma),$$

where $E_\sigma$ is the matrix of the permutation $\sigma$, $i = \sigma^{-1}(1)$, and $Z(E_\sigma) = Z^{E_\sigma}$.

It is likewise easy to see that if the theorem holds for any $A$ with non-zero entries in at most the first $c$ columns 1, 2, $\cdots$, $c$, then it is also true if we replace $A$ by $A + a_{1,c+1} E_{1,c+1}$ for any $a_{1,c+1} \in \mathbb{N}$.

Now let us assume that the theorem holds up to the $r$th row in the $c$th column. Let $Z^{A_0}$ correspond to a matrix $A_0$ fulfilling the requirements up to, and including, row $r$ and column $c$, and consider $A = A_0 + a_{r+1,c} \cdot E_{r+1,c}$.

Before getting further into the details, let us remark that the two lexicographic orderings $(1, 1) > (1, 2) > \cdots > (1, n) > (2, 1) > \cdots > (n, 1) > (2, 1) > \cdots$ and $(1, 1) > (2, 1) > \cdots > (n, 1) > (2, 1) > \cdots$ have the same monomials. By this we mean that if $Z^A$ is written according to one of the orderings, then rewriting it according to the other will not create auxiliary terms. Indeed, not even a factor different from 1. Let us denote the former ordering by row-column and the latter by column-row.

Consider

$$Z(A) \cdot \sum_{\sigma \in S_n} (-q^2)^{f(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}.$$

Set $\alpha = \left( \sum_{k=1}^{c-1} a_{(r+1),c} a_{(r+1),k} + \sum_{l=1}^c a_{r+1,c} \alpha_{l,c} \right)$. Then,

$$Z(A) = Z(A_0) \cdot q^{-\alpha} Z^{\sigma_{r+1,c}}.$$

The task now is to order each summand in

$$Z(A) \cdot \sum_{\sigma \in S_n} (-q)^{2f(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}$$

lexicographically. To do so, we will group the terms in $\det_q$ together strategically into sums of products of quantum minors.

We can safely assume $1 \leq r \leq n - 1$. Decompose

$$\{(i, j) \in \mathbb{N} \mid 1 \leq i, j \leq n\} = R_1 \cup R_2 \cup R_3 \cup R_4,$$
Notice that it follows from the defining relations that
\[ \det_R \text{ and } \det_J \]
where \( \tilde{\cdot} \) in the positions of \( R \) decomposition \( R \) entries at most in the positions of \( R \) minors having entries from \( \mathbb{Z} \).

For instance, no pair can share a row or a column. All combinations of quantum minors will occur with a non-zero coefficient, of course. For instance, no pair can share a row or a column.

Let \( R_{1,2,3} \) denote the set of matrices over \( \mathbb{Z}_+ \) with non-zero entries at most in the positions of \( R_1 \cup R_2 \cup R_3 \), let \( R_4 \) denote the set of matrices with non-zero entries at most in the positions of \( R_4 \), and let \( M_4 \) denote the set of quantum minors having entries from \( R_4 \).

It follows that
\[ \det_q = \sum_{M_4 \in M_4} \sum_{G \in R_{1,2,3}} P_{G,M_4} Z^G M_4, \]
where each \( P_{G,M_4} \in \mathbb{Z}[q^2] \). Then, because reordering elements from \( R_{1,2,3} \) does not introduce terms from \( R_4 \),
\[ Z(A_0) \det_q = \sum_{M_4 \in M_4} \sum_{H \in R_{1,2,3}} \hat{P}_{H,M_4} Z^H M_4 \]
for some elements \( \hat{P}_{H,M_4} \in \mathbb{Z}[q^2,q^{-2}] \). At the same time, by the induction hypothesis,
\[ Z(A_0) \det_q = \sum_{L \in R_4} \sum_{K \in R_{1,2,3}} \tilde{c}_{L,K} Z(K + L), \]
where \( \tilde{c}_{L,K} = 1 \) for the unique configuration corresponding to \( Z(A_0 + I) \) and in all other cases, if non-zero, \( \tilde{c}_{L,K} = \pm q^{2\gamma_{K,L}} \) where \( \gamma_{K,L} \in \mathbb{N} \). The expression \( Z^H M_4 \) in (4.7) is a sum of monomials \( q^{2p_i} Z^H Z^{S_{4,i}} \) corresponding to \( M_4 = \sum_i q^{2p_i} Z^{S_{4,i}} \) as a quantum minor. Here, \( \forall i : p_i \in \mathbb{Z}_+ \) and \( p_i \in \mathbb{N} \) for all but one \( i \). The normalization factors \( N(H + S_{4,i}) \) are independent of \( i \) and may thus for instance be computed for the unique \( i \) for which \( p_i = 0 \). It follows from this that we have a formula
\[ Z(A_0) \det_q = \sum_{M_4 \in M_4} \sum_{H \in R_{1,2,3}} c_{H,M_4} q^{2p_{H,M_4}} q^{-v(H;M_4) - v(H;M_4)} Z(H) M_4 \]
for some constants \( c_{H,M_4} \in \{-1,0,1\} \) and some elements \( p_{H,M_4} \in \mathbb{Z}_+ \). With the exception of one pair \((H_s,M_{4,s})\) where \( p_{H_s,M_{4,s}} = 0 \), we have furthermore
that $p_{H,M_4} \in \mathbb{N}$ when $c_{H,M_4} \neq 0$. The symbols \( \mathcal{r}_q(H;M_4) \) and \( \mathcal{c}_q(H;M_4) \) denote the row sums, respectively column sums, of \( H \) corresponding to the rows, respectively columns, of \( M_4 \). Each summand in \( q^{-\mathcal{r}_q(H;M_4)}Z(H)M_4 \) is normalized.

Observing that \( \det q \) is central, we can insert it in any position we prefer. Returning to (4.4) we will therefore consider \( Z(A_0) \cdot \det q \cdot q^{-\mathcal{a}}Z_{r+1,c} \). In view of (4.7) we need to focus on the rewriting of expressions of the form \( M_4Z_{r+1,c} \) and, in particular, to keep careful track of the \( q \) factors we pick up. This is the only place where negative exponents might originate.

Four different situations may occur:

- 1) \( M_4 \) has all row numbers greater than \( r+1 \) and all column numbers greater than \( c \).
- 2) \( M_4 \) has all row numbers greater than \( r+1 \) but a column number equal to \( c \).
- 3) \( M_4 \) has a row number equal to \( r+1 \) but all column numbers greater than \( c \).
- 4) \( Z_{r+1,c} \) occurs in \( M_4 \).

In cases 2) and 3), \( M_4Z_{r+1,c} = q^{-2}Z_{r+1,c}M_2 \) and in case 4), \( M_4Z_{r+1,c} = Z_{r+1,c}M_2 \). What they have in common is the essential fact that they obey a quasi-commutation relation. Furthermore, once they are rewritten according to these relations, they are ordered correctly according to the lexicographic ordering. In view of the last statements in Lemma 3.1, this means that the monomials in \( M_4 \) can be treated individually, obeying the same row and column sum relations. Thus, in the cases 2), 3), 4) we obtain the desired.

In case 1) we have a reinterpretation of (3.29):

\[
[Z_{r+1,c}, M_4] = -q^2(1 - q^{-4a})Z_{r+1,c} T,
\]

where \( Z_{r+1,c} T = q^4T Z_{r+1,c} \). The factor \( T \) will be discussed shortly.

Notice that the left hand side of (4.10) evidently is skew under the bar operator. Thus, it follows that \( Z_{r+1,c}^{-1}T = q^{-4a+4}Z_{r+1,c}^{-1}T \). We have \( T = q^2N_1 \) in terms of (3.29) so each monomial \( Z_{Y_i} = Z(Y_i) \) in \( T \) occurs with a factor \( \pm q^{2p_i} \) with \( p_i \in \mathbb{Z}_+ \). However, we must utilize even finer details of \( T \). Specifically, we may assume that the monomial summands of \( T \) each have a contribution \( Z_{x,c} \) with \( x > r + 1 \) and a contribution \( Z_{r+1,y} \) with \( y > c \). Furthermore, \( T \) is ordered according to the lexicographic ordering column-row and a factor of \( q^2 \) is taken out of the original determinantal expression which involves expressions \( (-q^2)\ell \) where \( \ell \geq 2 \). It is clearly the term with \( q^{2-4a} \) we must be able to handle. Before addressing this, we remark that the term \( Z_{r+1,c}^{-1}M_4 \) from the commutator is handled by the same argument as in cases 2), 3), and 4).

We know from the construction that each \( K + L \) in (3.31), appearing with a non-zero coefficient, compared to \( A_0 \) has an additional element in each row and column coming from the various summands in the determinant. With the given \( M_4 \) we then know that the extra element \( W_{r+1,a} \) in the \((r + 1)\)th
row must have $u < c$ and the extra element $W_{v,c}$ in the $c$th column must have $v < r + 1$.

The above observations easily imply that

$$Z(H)Z_{r+1,c}^{a-1}T = q^{-4a+4}Z_{r+1,c}^{a-1}T \overline{Z(H)}$$

$$= q^{-4a+4}q^{-2q^{-2}(H:M_4)}Z(H)Z_{r+1,c}^{a-1}T$$

$$+ \text{lower order terms}$$

Equation (4.11) implies that the term in $X$ coming from the leading term in $T$ is normalized. The other terms are then positive powers of $q^2$ times normalized elements.

This completes the proof. $\square$

5. Covariant Minors and the dual canonical basis

Let us consider an $n \times n$ matrix $X$ decomposed into

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$  

We assume furthermore that $C$ is quadratic of size $s$. We denote the $s \times s$ quantum minor corresponding to the lower left corner by $I_{s,ll}$.

Lemma 5.1. Let $b(X)$ be an element of the dual canonical basis with $X$ given as in (5.1). Then

$$b(X)I_{s,ll} = q^{(S(A)-S(D))}b(\tilde{X}),$$

with $\tilde{X} = \begin{pmatrix} A & B \\ C + I_s & D \end{pmatrix}$. Here, $I_s$ is the $s \times s$ identity matrix, while $S(A)$ and $S(D)$ denote the sum of all entries in $A$ and $D$, respectively.

Proof: It is easy to see that $Z^X = Z^A Z^B Z^C Z^D$. Suppose then, by Proposition 3.6, that

$$b(X) = \sum_{X' \leq X} q^{2p'} Z(X'),$$

Here we set $X' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ and then

$$Z(X') = C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'}$$

where

$$C_{A',B',C',D'} = N(A')N(B')N(C')N(D')N(A',B')N(A',C')N(B',D')N(C',D')$$
The factors $N(A'), N(B'), N(C'),$ and $N(D')$ are given by (3.12) as are the factors $N(A', B'), N(A', C'), N(B', D').$ Likewise $N(C', D')$ (there will be no factors $N(A', D')$ and $N(B', C')$) are factors having their origin in (3.12). For clarity we exemplify this by

$$N(C', D') = q^{-\sum C_0'(C') C_0'(D')}$$

and $N(A', C') = q^{-\sum C_0(A') C_0(A')}.$

It follows, since row and column sums are the same for the matrices in the right hand side, that we have $-S(A') + S(D') = -S(A) + S(D).$ Indeed, one first considers eg. the blocks $A, B$ and then the blocks $B, D.$ Thus,

$$q^{-(S(A)+S(D))} b(X) I_{s,ll} = \sum A',B',C',D' \ q^{-(S(A') + S(D'))} q^{2p'} C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll}.$$

It follows from Proposition 3.13 that

$$I_{s,ll} Z^{A'} Z^{B'} Z^{C'} Z^{D'} = q^{-2S(A)+2S(D)} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll}. $$

Thus, the left hand side, and hence both sides of (5.4) are bar invariant. Now consider a term in (5.4) of the form

$$q^{-(S(A')+S(D'))} C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll} = q^{-(S(A')-S(D'))} C_{A',B',C',D'} Z^{A'} Z^{B'} Z^{C'} Z^{D'} I_{s,ll} Z^{D'}.$$

Here, $Z(C') I_{s,ll} = N(C') Z^{C'} I_{s,ll}.$ By Theorem 4.1, this equals $Z(C' + I_s) + \sum_{C'' < C'} f_{C''} Z(C'')$ and for all $C''$, $f_{C''}$ a polynomial in $q^2$ which vanishes at $q = 0.$ Notice that each $C''$ has the same row and column sums as $C' + I_s$ and that $\forall i : r_{0i} (C' + I_s) = r_{0i} (C') + 1$ and, similarly, $\forall j : c_{0j} (C' + I_s) = c_{0j} (C') + 1.$ But then $q^{-S(A')-S(D')} N(A', C') N(C'', D') = N(A', C'') N(C'', D).$ Thus, the right hand side of (5.4) is a sum of terms $q^{p_i} Z(Y_i).$ Precisely the term with $A' = A, B' = B, C' = C,$ and $D' = D$ has a factor $q^{p_i} = 1.$ Thus the right hand side has the right expansion properties, hence is a member of the dual canonical basis corresponding to the stated element $X.$

Let us instead consider an $n \times n$ matrix $X$ decomposed into

$$X = \left( \begin{array}{cc} 0 & B \\ C & D \end{array} \right),$$

where we now assume that $D$ is quadratic of size $s$. We denote the $s \times s$ quantum minor corresponding to the lower right hand corner (as occupied by $D$) by $I_{s,lr}.$

**Lemma 5.2.** Let $b(X)$ be an element of the dual canonical basis with $X$ given as in (5.3). Then

$$b(X) I_{s,lr} = q^{(S(B)+S(C))} b(\tilde{X}),$$

with $\tilde{X} = \left( \begin{array}{cc} 0 & B \\ C & D + I_s \end{array} \right).$ As before, $I_s$ is the $s \times s$ identity matrix, while $S(B)$ and $S(C)$ denote the sum of all entries in $B$ and $C$, respectively.
The proof follows the same lines as that of Lemma 5.1 and is omitted. By Proposition 3.8 the following case encompasses the two former. The proof is omitted for similar reasons.

**Lemma 5.3.** Let \( b(X) \) be an element of the dual canonical basis with

\[
X = \begin{pmatrix}
  0 & B_1 & B_2 \\
  C_1 & D & C_2 \\
  G_1 & G_2 & 0
\end{pmatrix}.
\]

Then

\[
(5.7) \quad b(X)I_{s,cc} = q^{S(B_1)+S(C_1)-S(C_2)-S(G_2)}b(\tilde{X}),
\]

with \( \tilde{X} = \begin{pmatrix}
  0 & B_1 & B_2 \\
  C_1 & D + I_s & C_2 \\
  G_1 & G_2 & 0
\end{pmatrix} \). As before, \( I_s \) is the \( s \times s \) identity matrix.

**Remark 5.4.** Using Proposition 3.8 it follows that analogous results hold for the configurations

\[
X = \begin{pmatrix}
  C_1 & D & C_2 \\
  G_1 & G_2 & 0
\end{pmatrix}, \quad \tilde{X} = \begin{pmatrix}
  C_1 & D + I_s & C_2 \\
  G_1 & G_2 & 0
\end{pmatrix}
\]

and

\[
X = \begin{pmatrix}
  B_1 & B_2 \\
  D & C_2 \\
  G_2 & 0
\end{pmatrix}, \quad \tilde{X} = \begin{pmatrix}
  B_1 & B_2 \\
  D + I_s & C_2 \\
  G_2 & 0
\end{pmatrix},
\]

in which the matrix \( X \) is not necessarily quadratic.

### 6. Broken line constructions

Consider the \( m \times n \) quantum matrix algebra \( \mathcal{O}_q(M(m, n)) \). In this section, all elements \( Z_{i,j} \) and all quantum minors are elements of this algebra.

**Definition 6.1.** A broken line in \( M_{m,n}(\mathbb{Z}_+) \) is a path in \( \mathbb{N} \times \mathbb{N} \) starting at \((1, n)\) and terminating at \((m, 1)\). We will occasionally also refer to this as a broken line from \((1, n)\) to \((m, 1)\). It must satisfy furthermore that it alternates between horizontal and vertical segments while passing through smaller column numbers (in the horizontal direction) and bigger row numbers (in vertical direction).

Unless we are in the extreme cases \((1, n) \mapsto (1, 1) \mapsto (m, 1)\) or \((1, n) \mapsto (m, n) \mapsto (m, 1)\), this will divide the indices \((i, j)\) into 3 disjoint sets \( S_L, L, \) and \( T_L \). Here, \( S_L \) is the set of points above the line (where there are 3 subsets, we will say that \((1, 1)\) is above the line), \( L \) is the line itself, and \( T_L \) is the set of points below the line.
Remark 6.2. A broken line is determined by a double partition
\[
1 = i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_s = m \quad \text{and} \quad n = j_1 \geq j_2 \geq j_3 \geq \cdots \geq j_s = 1,
\]
such that the corners in the line \( L \) are \((i_t, j_t); t = 1, 2, \ldots, s\). This, naturally, dictates that in the partitions, precisely every second inequality is sharp. Furthermore, if in a given position, one is sharp, then the other is not, and vice versa.

In a similar vein, the broken line is given by a double flag variety.

For a given broken line \( L \), we now construct a family \( \mathcal{V}_L \) with \( mn \) elements consisting of certain quantum minors: (It will be proved below that all members \( q \)-commute.) For points in \((i, j) \in T_L \cup L\) we take the biggest quantum minor having its bottom right corner in \((i, j)\) and completely contained in \( T_L \cup L \). One can also say that it is the biggest quantum minor consisting of adjacent rows and columns (we call such a quantum minor solid) and which contains \((i, j)\) as well as points from \( L \) but no points from \( S_L \). The line \( L \) is thus represented by points, that is, \( 1 \times 1 \) matrices. For the points in \( S_L \) we do something else: For \((i, j) \in S_L\) we take the biggest quantum minor consisting of adjacent rows and columns and which contains \((i, j)\) in the upper left corner (all other rows have numbers bigger than \( i \) and all columns have numbers bigger than \( j \)). Notice that with \( L \) fixed, each such quantum minor corresponds uniquely to a point \((i, j)\). By the quantum minor corresponding to a point we then mean this quantum minor.

In the sequel, we shall consider the following more general family \( \mathcal{V}_{L_1, L_2} \subset \mathcal{V}_L \) where, clearly, \( \mathcal{V}_L = \mathcal{V}_{L^+, L} \):

**Definition 6.3.** Let \( L_1, L_2 \) be broken lines with \( L_1 < L_2 \). The family \( \mathcal{V}_{L_1, L_2} \) is the subfamily of \( \mathcal{V}_{L_1} \) that corresponds to the points in \( T_{L_2} \cup L_2 \).

The first important observation is:

**Proposition 6.4.** Any quantum minor corresponding to a point in \( S_L \) \( q \)-commutes with any \( Z_{i,j} \) for which \((i, j) \notin S_L\).

This follows immediately from Proposition 3.13.

**Proposition 6.5.** Let \( M = M_{a,b}(k) \) be a \( k \times k \) quantum minor with upper left corner in \((a, b)\) and lower right corner given as \((a + k - 1, b + k - 1)\) and such that \( M \) is inside the \( m \times n \) quantum matrices. Refer to the 9 different positions of a pair \((i, j)\) relative to \( M \) as NW, N, NE, \ldots, SE such that NW is \((a > i \text{ and } b > j)\) and SE is \((i > a + k - 1 \text{ and } j > b + k - 1)\). Let \( IM \) denote the indices of \( M \). Then \( Z_{i,j} \) \( q \)-commutes with \( M \) unless \((i, j)\) is NW or SE. For the remaining pairs, in the \( q \)-commutation formulas \( Z_{i,j} M = q^2 p_{i,j} M Z_{i,j} \), \( p_{i,j} \) depends only on the relative positions. Indeed, \( p_{i,j} = 1 \) for \((i, j)\) \( W, N \) of \( M \), \( p_{i,j} = 0 \) for \((i, j)\) \( IM, SW \) and \( NE \), and \( p_{i,j} = -1 \) for \((i, j)\) \( SE \).

**Proof.** With the exception of NW and SE, the \( q \)-commutation relation may be seen as taking place inside a smaller matrix algebra in which \( M \) is a covariant quantum minor. \( \square \)
Proposition 6.6. All members of $\mathcal{V}_L$ $q$-commute.

Proof. Let $A, B \in \mathcal{V}_L$. It follows by inspection from the construction that the possible positions of $A$ relative to $B$ are at most $W$, $SW$, $S$, and $IB$, or that the positions of $B$ relative to $A$ all are at most $W$, $SW$, $S$, and $IA$. It suffices to consider the first of these. We consider then $B$ as fixed and consider the expansion of $A$ into a linear combination of monomials of the form $Z_{i_{1},\sigma(1)} \cdots Z_{i_{1}+r,\sigma(i_{1}+r)}$ for some $\sigma \in S_r$. Using Proposition 6.4 and Proposition 6.5 we obtain the following: If $W_{\sigma}(B)$ and $S_{\sigma}(B)$ denote the number of terms $Z_{i_{1}+i,\sigma(i_{1})+i}$ to the west, respectively to the south, of $B$, the given monomial will $q$-commute with $B$ with a factor $q^{2(W_{\sigma}(B) - S_{\sigma}(B))}$. It is easily seen that $W_{\sigma}(B) - S_{\sigma}(B)$ is independent of $\sigma$, and thus the claim follows. □

Remark 6.7. One gets a similar family by interchanging $S_L$ and $T_L$. Colloquially speaking, if one allows $L$ to vary, this can be accomplished by a reflection mapping $m \times n$ matrices to $n \times m$ matrices while interchanging rows and columns.

Definition 6.8. We introduce a partial ordering of the broken lines:

$$L_1 \leq L_2 \iff S_{L_2} \subseteq S_{L_1}.$$ 

In this ordering, the line $L^+$ corresponding to the empty set: $(1, n) \rightarrow (1, 1) \rightarrow (m, 1)$, is the unique maximal element, and the line $L^-$ corresponding to $T_L = \emptyset$: $(1, n) \rightarrow (m, n) \rightarrow (m, 1)$, is the unique smallest element.

In the extreme case of $L^+$, the $q$-commuting quantum minors in the corresponding family are the following:

1. For $i \geq j$, $\xi_{\{i-j+1,i-j+2,\ldots,i\}}$
2. For $j > i$, $\xi_{\{1,2,\ldots,i\}}$.

Definition 6.9. Given a broken line $L$, let $\mathcal{O}_q(T_L \cup L)$ denote the subalgebra of $\mathcal{O}_q(M(m,n))$ generated by the $Z_{i,j}$ for which $(i,j) \in T_L \cup L$. Let $\mathcal{V}_L^-$ denote the set of variables in $\mathcal{V}_L$ corresponding to the points in $L \cup T_L$, and let $\mathcal{C}_L^-$ denote the set of variables for the points in $L^- \subseteq (T_L \cup L)$). Analogously, let $\mathcal{V}_L^+$ denote the variables corresponding to the points in $S_L$.

In the following we will consider cluster algebra constructions inside an ambient space which is either i) the skew field of fractions $\mathcal{F}_L$ constructed from $\mathcal{O}_q(M(m,n))$ (or, equivalently, $\mathcal{V}_L$) and where $\mathcal{V}_L$ is part of an initial seed and ii) skew field of fractions $\mathcal{F}_L^-$ constructed from $\mathcal{O}_q(T_L \cup L)$ (or, equivalently, $\mathcal{V}_L^-$) and where $\mathcal{V}_L^-$ is part of an initial seed.

Proposition 6.4 can be stated as the fact that any variable in $\mathcal{V}_L^+$ is covariant with respect to the full subalgebra $\mathcal{O}_q(T_L \cup L)$. The algebra $\mathcal{O}_q(T_L \cup L)$ has previously been studied in [22, section 3].
Theorem 6.10. Let $\mathcal{V}_L$ denote the family of $mn$ $q$-commuting quantum minors constructed from a broken line as above. Then up to multiplication by a power of $q$, any monomial in the members of $\mathcal{V}_L$ is a member of the dual canonical basis.

Proof: The main tool is Lemma 5.3, but Proposition 3.8 is also important, c.f. the remark following Lemma 5.3. Consider then a monomial. Rewrite it, if necessary, in such a way that the factors coming from the points on $L$ are furthest to the left. Then place to the right of these the $2 \times 2$-minors corresponding to the points one step below the line. Continue in this way until all the factors corresponding to the points on, or below, the line are positioned. While continuing to add from the right, order the factors coming from $S_L$ in a similar fashion and such that the factor corresponding to the position $(1, 1)$ is furthest to the right. The finer order is not important. We view the monomial as the result of a sequence of multiplications from the right by minors according to this ordering. Inductively, we may at each step $r$ in the sequence assume that what we are multiplying the minor onto is some $q_{2r} b(X_r)$. The start is clearly trivial. Furthermore, at each step we can apply Lemma 5.3, and the result follows. □

Definition 6.11. For a given line $L$, we say that the line $L_1$ is a closest bigger line to $L$ if $L < L_1$ and there is no other line $L_2$ such that $L < L_2 < L_1$. In this case, if $L = (1, n) \rightarrow \cdots \rightarrow (f, d) \rightarrow (c, d) \rightarrow (c, g) \rightarrow \cdots \rightarrow (m, 1)$, then $L_1 = (1, n) \rightarrow \cdots \rightarrow (f, d) \rightarrow (c - 1, d) \rightarrow (c - 1, d - 1) \rightarrow \cdots \rightarrow (m, 1)$ for some such “corner” $(f, d) \rightarrow (c, d) \rightarrow (g, d)$, where we, naturally, also allow $f = c - 1$ and $g = d - 1$.

We will call the given corner of $L$ convex and the resulting corner of $L_1$ concave. We will also write

$$L_1 = L \uparrow (c, d) \text{ or } L = L_1 \downarrow (c - 1, d - 1).$$

6.1. Key technical results. Focus on a position $(i_0, j_0)$ inside the quantum matrix algebra $\mathcal{O}_q(M(n_0, r_0))$. Consider the subalgebra $M = \mathcal{O}_q(M(s))$ generated by the variables $Z_{i_0+a,j_0+b}$ with $0 \leq a, b \leq s - 1$ where $s$ is the biggest positive integer such that $Z_{i_0+s-1,j_0+s-1} \in \mathcal{O}_q(M(n_0, r_0))$. Naturally, this subalgebra is isomorphic to $\mathcal{O}_q(M(s))$ in which we number the rows and columns as $0, 1, \ldots, s - 1$. Assume $s \geq 2$. Inside $M$ are the quantum minors $Y_r = Y_r(s-2) = \xi_{1, \ldots, s-2}^{0, 1, \ldots, s-2}$, $Y_t = Y_t(s-2) = \xi_{0, \ldots, s-2}^{1, \ldots, s-1}$, $X_o = X_o(s-2) = \xi_{1, \ldots, s-2}^{1, \ldots, s-2}$, $X_t = X_t(s-2) = \xi_{0, \ldots, s-2}^{0, 1, \ldots, s-2}$, $X_b = X_b(s-2) = \xi_{1, \ldots, s-2}^{1, \ldots, s-1}$, and $D = D(s-2) = \xi^{0, 1, \ldots, s-1}_{0, 1, \ldots, s-1}$. The last is just the full quantum determinant in $M$. $X_o(0)$ is defined as the the constant 1.

Definition 6.12. We will call a set $\{X_t, X_b, D, X_o, Y_t, Y_r\} \subset \mathcal{O}_q(M(n_0, r_0))$ an $\mathcal{M}$-set if there exists $i_0, j_0, s \in \mathbb{N}$ such that they are given as above.
We have the following facts which follow by direct computation:

**Lemma 6.13.** The element $X_o^{-1}D^{-1}Y_rY_l$ commutes with all elements $Z_{i,j} \in M$ with the exception of $Z_{i_0,j_0}$ and $Z_{i_0+s-1,j_0+s-1}$. In particular, it commutes with the quantum minors $X_{(a)} = \xi_{\{1,\ldots,a\}}$ for $a = 1, \ldots, s - 2$. It $q$-commutes with the quantum minor $X_b$ according to

$$X_bX_o^{-1}D^{-1}Y_rY_l = q^{-4}X_o^{-1}D^{-1}Y_rY_lX_b.$$ \hspace{1cm} (6.1)

In case $(i_0,j_0)$ is a point on a line $L$ then $X_o^{-1}D^{-1}Y_rY_l$ commutes with all other elements in $V_L$.

The following result will allow us to construct the $B$ matrices of the compatible pairs. It follows directly from Lemma 6.13, see Lemma 2.2.

**Corollary 6.14.** Let $\Lambda$ be defined as the $\Lambda$-matrix of the variables $X_b, X_o, D, Y_r, Y_l$. Then,

$$\Lambda \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$ \hspace{1cm} (6.2)

The following was proved by Parshall and Wang in [26, Theorem 5.2.1] but is also a special case of [14] Theorem 6.2.

**Proposition 6.15** (Parshall and Wang).

$$X_tX_b - X_bX_t = (q^2 - q^{-2})Y_rY_l.$$ \hspace{1cm}

We wish to strengthen this result considerably, namely to the following equation which will play an important role later when we consider the quantum mutations in certain directions.

**Theorem 6.16.**

$$X_tX_b = X_oD + q^2Y_rY_l.$$ \hspace{1cm}

**Proof:** We first observe that by Theorem 4.11 and Lemma 5.1 $X_oD$ and $Y_rY_l$ are members of the dual canonical basis; $X_oD = b(A_1)$ and $Y_rY_l = b(A_2)$ for some specific matrices $A_1, A_2$. We consider the expansion of $X_tX_b$ onto the dual canonical basis;

$$X_tX_b = \sum_i c_i(q)b(C_i).$$ \hspace{1cm} (6.2)

The coefficients $c_i$ are Laurent polynomials in $q^2$. The leading term must be $b(A_1)$ with coefficient 1. If we can prove that the other coefficients actually are polynomials in $q^2$ without constant term then the proof follows from
Proposition 6.15. We proceed to prove this: First we expand

\[ X_t = Z_{0,0} X_0 + \sum_j (-q^2)^{\ell_{\sigma_j}} Z(C_{\sigma_j}), \]

where the powers of \( q^2 \) are strictly positive and where \( Z(C_{\sigma_j}) \) is a normalized monomial without contribution from \( Z_{0,0} \). Thus, \( Z(C_{\sigma_j}) = b(C_{\sigma_j}) + \text{t.o.t.s} \), where the lower order terms in their own right will have polynomial coefficients with no constant terms. According to Lemma 5.2,

\[ (\sum_j (-q^2)^{\ell_{\sigma_j}} Z(C_{\sigma_j})) X_b \]

is then alright. It remains to consider \( Z_{0,0} X_0 X_b \). But here we notice that \( X_0 X_b \) again is a member of the dual canonical basis; \( X_0 X_b = b(A_3) \) for some matrix \( A_3 \). This matrix is, and so is then \( b(A_3) \), without contributions from the first row and the first column. Now notice that

\[ b(A_3) = Z(A_3) + q^2 \sum_{G_k < A_3} d_k(q) Z(G_k), \]

where the coefficients are polynomials in \( q^2 \). It then follows from the above remarks that

\[ Z_{0,0} b(A_3) = Z(A_3 + E_{0,0}) + q^2 \sum_{G_k < A_3} d_k(q) Z(G_k + E_{0,0}). \]

Expanding the right hand side in terms of the canonical basis, we get the result since the basis change matrix is lower diagonal with 1’s in the diagonal and all non-diagonal terms have zero constant terms.

The above lemma tells us that \( X_t X_b - q^2 Y_t Y_l \) is bar invariant and therefore coincides with the dual canonical basis elements with the same leading term which is \( X_0 D \).

□

Remark 6.17. The referee has kindly informed us that another, in a way easier, proof may be obtained as follows: Consider the equation \( Z_{1,1} Z_{n,n} = q^2 Z_{1,n} Z_{n,1} + \xi_{1,n}^{1,n} \) in \( \mathcal{O}_q(M(n)) \) and apply the anti-endomorphism \( \Gamma \) given in [26, Corollary 5.2.2] to it. The effect of \( \Gamma \) on quantum minors has been computed in [20, Lemma 4.1], and from this one can see that the formula is obtained.

The following result is an easy variation of [16, Lemma 4.1, Proposition 4.5]:

Lemma 6.18. Let \( \Lambda_{L^+} \) be the \( \Lambda \)-matrix of the family \( \mathcal{V}_{L^+} \) and let \( \mathcal{C} = \mathcal{C}_{L^+}^- \) be the \( n + m - 1 \) covariant quantum minors determined as the variables in \( \mathcal{V}_{L^+} \) corresponding to the points in \( L^- \). Let \( s = \text{corank}(\Lambda_{L^+}) \). The kernel of \( \Lambda_{L^+} \) is then generated by \( s \) monomials in the elements of \( \mathcal{C}^{\pm 1} \).
Theorem 6.19. Consider two broken lines $L_1, L_2$ in $M_{m,n}(\mathbb{Z}_+)$ Let $Q_{L_1} = (V_{L_1}, \Lambda_{L_1}, B_{L_1})$ and $Q_{L_2} = (V_{L_2}, \Lambda_{L_2}, B_{L_2})$ be quantum seeds corresponding to these such that the set of non-mutable elements in both cases is $C$ as above. Then there exists a quantum seed $\tilde{Q}_{L_2} = (V_{L_2}, \Lambda_{L_2}, B_{L_2})$ which is quantum mutation equivalent to $Q_{L_1}$ and such that $B_{L_2}$ and $B_{L_2}$ have the same principal part.

Proof. Let $L$ be an arbitrary broken line and let $(i, j)$ be a concave corner of $L$, specifically, assume that $(i, j + 1), (i, j), (i + 1, j)$ are points on the broken line $L$. If we replace $(i, j)$ by $(i + 1, j + 1)$ while keeping the other points, we get another broken line $L'$. We claim that $Q_L$ and some appropriate quantum seed $\tilde{Q}_{L'}$ are quantum mutation equivalent. We will construct a sequence of interim quantum seeds $Q_a = (V_a, \Lambda_a, B_a)$ and $\tilde{Q}_a = (V_a, \Lambda_a, B_a)$ such that

\begin{equation}
Q_L = Q_0 \Rightarrow \tilde{Q}_0 \Rightarrow Q_1 \Rightarrow \tilde{Q}_1 \Rightarrow \cdots \Rightarrow \tilde{Q}_{m-i} = \tilde{Q}_{L'}.
\end{equation}

The double arrows are quantum mutations while the single arrows indicate some change on the level of the B-matrix.

Without loss of generality, we may assume that $i \geq j$ and to avoid limiting cases, assume $j \geq 3$. By construction, the quantum minor $\xi_{\{i, j+1, \ldots, n\}}^{\{i, j+1, \ldots, n-i\}}$ is both a quantum cluster variable for the quantum seeds associated to $L$ and to $L'$, but labeled by different points, namely, labeled by $(n, j + n - i)$ in the quantum seed associated to $L$ and labeled by $(i, j)$ in the quantum seed associated to $L'$. This quantum minor is not affected by the following manipulations. The quantum minors $\xi_{\{i,j\}}, \xi_{\{i,j+1\}}, \ldots, \xi_{\{i,m-1\}}$ are changed into $\xi_{\{j+1,j\}}, \xi_{\{j+1,j+1\}}, \ldots, \xi_{\{j+1,m-1\}}$, and all other quantum cluster variables stay unchanged.

Specifically, we do the following sequence of replacements:

\begin{align}
\xi_{\{i,j\}} & \rightarrow \xi_{\{i,j+1\}} \\
\xi_{\{i,j+1\}} & \rightarrow \xi_{\{i+1,j+2\}} \\
\ldots & \\
\xi_{\{i,j+a\}} & \rightarrow \xi_{\{i+1,i+1+a\}} \\
\ldots & \\
\xi_{\{i,m-1\}} & \rightarrow \xi_{\{i+1,m\}} \\
\xi_{\{j,j+m-1-i\}} & \rightarrow \xi_{\{j+1,j+m-i\}}
\end{align}

Each replacement is a quantum mutation in the sense of Berenstein, Zelevinsky. The quantum mutations are governed by the Theorem 6.16. At any level $a < m - i$, the quantum minors $\xi_{\{j,j+a\}}^{\{i,i+a\}} = X_{t}^{(a)}$, $\xi_{\{j+1,j+1+a\}}^{\{i,j+1+a\}} = X_{b}^{(a)}$, $\xi_{\{j+1,j+1+a\}}^{\{i+1,i+1+a\}} = D^{(a)}$, $\xi_{\{j,j+1\}}^{\{i+1,i+1+a\}} = X_{a}^{(a)}$, $\xi_{\{j,j+1\}}^{\{i+1,i+1+a\}} = Y_{t}^{(a)}$, $\xi_{\{j+1,j+1\}}^{\{i+1,i+1+a\}} = Y_{a}^{(a)}$, $\xi_{\{j+1,j+1\}}^{\{i+1,i+1+a\}} = Y_{r}^{(a)}$, and $\xi_{\{j+1,j+1\}}^{\{i+1,i+1+a\}} = Y_{r}^{(a)}$ constitute an $M$-set. The elements $X_{t}^{(a)}, D^{(a)}, X_{a}^{(a)}, Y_{t}^{(a)}$, and $Y_{r}^{(a)}$ are all quantum cluster variables in $V_a$. Lemma 6.13 shows that the element $X_{a}^{(a)} D^{(a)} Y_{t}^{(a)}^{-1} Y_{r}^{(a)}^{-1}$ commutes with all the quantum cluster
variables except $X_t^{(a)}$. In fact,

$$X_t^{(a)} X_o^{(a)} D^{(a)}(Y_t^{(a)})^{-1}(Y_r^{(a)})^{-1} = q^{-4} X_o^{(a)} D^{(a)}(Y_t^{(a)})^{-1}(Y_r^{(a)})^{-1} X_t^{(a)}.$$  

This implies that, module the kernel of $\Lambda_a$, the only non-zero entries in the column of $B_a$ corresponding to the variable $X_t^{(a)}$ are at the row positions of the variables $Y_t^{(a)}$ and $Y_r^{(a)}$ where it is 1, and at the row positions of the variables $X_o^{(a)}$ and $D^{(a)}$ where it is $-1$. We then change (if needed) $B_a$ into $\tilde{B}_a$ such that the the column in the latter corresponding to $X_t^{(a)}$ is non-zero precisely at the mentioned 4 places. With this, the changing of $X_t^{(a)}$ to $X_b^{(a)}$ is indeed a quantum mutation, and by Theorem 6.16 $X_b^{(a)}$ is the target of this mutation. We then perform this quantum mutation and obtain a new interim quantum seed $Q_{a+1} = (V_{a+1}, \Lambda_{a+1}, B_{a+1})$. The variables $Y_t^{(a)}, Y_r^{(a)}, a = 0, 1, \ldots$ are both in $V_a$ and in $V_{a+1}$. In the step $a + 1$, $D^{(a)} = X_t^{(a+1)}$ and, most importantly, $X_o^{(a+1)} = X_b^{(a)}$ which is now a variable in $V_{a+1}$. In this way we can carry out the entire transition from $L$ to $L'$. Hence our changing of the set of variables for a broken line at a concave point is obtained through a sequence of quantum mutations in the sense of Berenstein and Zelevinsky. Furthermore, it is elementary to see that any broken line can be obtained as a sequence of such moves from the broken line $(1, n) \rightarrow (1, 1) \rightarrow (m, 1)$. Therefore, the quantum seeds associated to the broken lines $L, L_1$ are quantum mutation equivalent to each other.

\[\square\]

[As an aside, we observe that we, starting at the top, could break off the above replacements at any lower level, but we shall not find it useful to do so.]  

Now, for each broken line $L$, we have a family of $mn q$ commuting quantum minors which by construction is a generating set of the fraction field of the Noetherian domain $O_q(M(m, n))$.

**Corollary 6.20.** Let $L$ be an arbitrary broken line and Let $\xi_I^J$ be any solid quantum minor. Then $\xi_I^J$ can be written as a Laurent polynomial with coefficient in $\mathbb{Z}[q, q^{-1}]$ of the cluster variables associated to $L$.

**Proof.** By our construction using broken lines, one can see that the solid quantum minor $\xi_I^J$ belongs to some quantum seed associated to a broken line $L'$. By the above theorem, $\xi_I^J$ can be obtained through a sequence of quantum mutations from the quantum cluster variables associated to $L$. Now the statement follows from the quantum Laurent phenomenon established in [3]. \[\square\]

**Remark 6.21.** Recall that a real matrix $A$ is totally positive (resp. totally non-negative) if all of its minors are positive (resp. non-negative). In [12], it is shown that a matrix is totally positive if all of its solid minors are positive. Moreover, in [8], it is shown that a matrix is totally positive if some specially
chosen minors (in fact a cluster) are positive. The above result is related to the total positivity of real matrices. Specializing $q$ to 1, we obtain a family of seeds (associated to broken lines) which are mutation equivalent to each other. To test if a matrix is total positive one only need to check if the minors in an arbitrary cluster associated to a broken line is positive.

6.2. Quantum line mutations.

Definition 6.22. In the general setting of $\mathcal{F}_L = \mathcal{F}_{L^+}$, let $L_1$ be a closest bigger line to the line $L$. Assume the configurations are as in Definition 6.14. The restricted quantum line mutation $\mu_R(L_1, L)$ is the map $\mathcal{V}_{L_1} \mapsto \mathcal{V}_L$ given as the composite map $(6.8)$ where $(i, j)$ is replaced by $(c - 1, d - 1)$.

If $Q_{L_1} = (V_{L_1}, \Lambda_{L_1}, B_{L_1})$ and $Q_L = (V_L, \Lambda_L, B_L)$ are quantum seeds, the quantum line mutation $\mu(L_1, L) : Q_{L_1} \mapsto Q_L$ is a map as given by the analogue of (6.7) but where it furthermore is demanded that at each level $i$, $Q_i = Q_i$. For practical purposes, we also consider the trivial quantum mutation as a quantum line mutation and denote it by $\mu(L, L)$.

Definition 6.23. In the general setting of $\mathcal{F}_L^-$, let $L_2 \leq L_3 \leq L$ be broken lines such that $L_3$ is a closest bigger line to the line $L_2$. The quantum line mutation $\mu^L(L_3, L_2)$ is the map $\mathcal{V}_{L_3} \mapsto \mathcal{V}_{L_2, L_1}$ defined in analogy with Definition 6.22. In particular, $\mu(L_1, L) = \mu^L(L_1, L)$. We denote the inverse of $\mu^L(L_3, L_2)$ by $\mu^L(L_2, L_3)$.

We have the following diamond lemma for quantum line mutations, cf. (2):

Lemma 6.24. Let $L_1 \leq L$. Let $\mu^L(L_1, L_1 \downarrow (c_1, d_1))$ and $\mu^L(L_1, L_1 \downarrow (c_2, d_2))$ be quantum line mutations. Then $\mu^L(L_1 \downarrow (c_1, d_1), (L_1 \downarrow (c_1, d_1)) \downarrow (c_2, d_2))$ and $\mu^L(L_1 \downarrow (c_2, d_2), (L_1 \downarrow (c_2, d_2)) \downarrow (c_1, d_1))$ are quantum line mutations. Furthermore,

$$
\mu^L(L_1 \downarrow (c_1, d_1), (L_1 \downarrow (c_1, d_1)) \downarrow (c_2, d_2)) \circ \mu^L(L_1, (L_1 \downarrow (c_1, d_1)) = \\
\mu^L(L_1 \downarrow (c_2, d_2), (L_1 \downarrow (c_2, d_2)) \downarrow (c_1, d_1)) \circ \mu^L(L_1, (L_1 \downarrow (c_2, d_2)).
$$

Proof. The key to this Lemma is Corollary 6.14 as well as the explicit formulae (2.7) and (2.8). The mutation which does the replacement $X^{(a)}_t \mapsto X^{(a)}_b$ makes changes to the rows in $B$ corresponding to the quantum minors $D^{(a)}_t, X^{(a)}_o, Y^{(a)}_t$, and $Y^{(a)}_r$. It follows by direct inspection that if an entry in the row of $X^{(a)}_t$ in some position $v$ is zero then the column of $v$ stays unchanged under the quantum mutation. At this level the entry at the position of $X^{(a)}_t$ of course is zero. Clearly, $X^{(a)}_t$ is not a member of any of the subsequent sets $D^{(a+p)}_t, X^{(a+p)}_o, Y^{(a+p)}_t, Y^{(a+p)}_r$, where $p = 1, \ldots, p_0$ for some specific positive integer $p_0$. It follows that the positions in the row of $X^{(a)}_t$ corresponding to the later values $X^{(a+p)}_t$ must be zero since we know precisely what the column
of $X_i^{(a+p)}$ looks like. These considerations can easily be extended to include
the case of a second quantum line mutation since none of the variables $X_i^{(a)}$
take part in any way in the second quantum line mutation. In the case where $(c_2, d_2) = c_1 - 1, d_1 + 1$ there is an overlap of variables in the sense that the $Y_r$
variables belonging to $(c_1, d_1)$ play the role of $Y_l$ variables belonging to $(c_2, d_2)$ but this is easily taken care of: They are not the sources or targets of
mutations and then the effects of the two different quantum line mutations
on the rows of such elements are independent of each other. The crucial ob-
servation is that neither of the two quantum line mutations affects the rows
of the variables involved in the other. □

The following result concerning independence of paths, follows easily since
one may fill in diamonds as in Lemma 6.24:

**Corollary 6.25.** If $L_1 \leq L_2 \leq \cdots \leq L_{n-1} \leq L_n$ and $L_1 \leq L'_2 \leq \cdots \leq L'_{n-1} \leq L_n$ are broken lines such that at each level the bigger line is a closest bigger line to the neighboring smaller. Then

$$\mu^L(L_2, L_1) \circ \cdots \circ \mu^L(L_n, L_{n-1}) = \mu^L(L'_2, L_1) \circ \cdots \circ \mu^L(L_n, L'_{n-1}).$$

In view of Corollary 6.25 we extend our definition of a quantum line mutation to the following

**Definition 6.26.** Let $L_1 \leq L_n \leq L$ be broken lines. The quantum line mutation $\mu^L(L_n, L_1)$ is the composite of any sequence as in Corollary 6.25 between $L_1$ and $L_n$.

Let $L_a \leq L$ and $L_b \leq L$ be broken lines. The quantum line mutation $\mu^L(L_a, L_b)$ is defined in terms of any broken line $L_c \leq L_a, L_b$ as

$$\mu^L(L_a, L_b) = \mu^L(L_b, L_c)^{-1} \circ \mu^L(L_a, L_c).$$

We shall also need

**Definition 6.27.** A position $(c, d) \in T_L \cup L$ is called attractive with respect to $T_L \cup L$ if either there exist $i > 0$, $j > 0$ such that $(c-i, d) \in T_L \cup L$ and $(a, d+j) \in T_L \cup L$ or if there exist $i > 0$, $j > 0$ such that $(c+i, b) \in T_L \cup L$ and $(c, d-j) \in T_L \cup L$. Clearly, if $(c, d)$ satisfies the first condition of attraction then $(c-i, d+j)$ satisfies the second, and vice versa. If $(c, d)$ is not attractive we call it repulsive.

The following is obvious

**Lemma 6.28.** The concave corners of $L$ are repulsive. The point $(m, n)$ is also repulsive.
6.3. **Covariant elements.** We extend Definition 3.12 in the obvious way to $\mathcal{O}_q(T_L \cup L)$. The next observation we wish to make is that the seeds we construct are minimal in the following sense:

**Proposition 6.29.** The set of covariant elements for $\mathcal{O}_q(T_L \cup L)$ is generated by the $m + n - 1$ elements in $C_L^{-}$.

**Proof.** First of all it is clear that the elements in $C_L^{-}$ are covariant, and hence, so is any monomial in these.

Since there is a unique smallest element in the set of broken lines, this may be proved by induction. For the line $L^-$ it is clear that we have a quasi-polynomial algebra so here, the claim is trivial. Consider then a line $L$ for which the claim is true and let $L_1$ be a closest bigger line. Assume the configurations are as in Definition 6.11. (Thus, $(i, j) = (c - 1, d - 1)$.) It is clear that $\mathcal{O}_q(T_L \cup L_1)$ is obtained by adjoining $Z_{c-1,d-1}$ to $\mathcal{O}_q(T_L \cup L)$. There is a unique element $X_b = X_b^{(m-c)}$ from $C_L^{-}$ having its upper left corner in $(c, d)$. This is the largest solid quantum minor with its upper left corner in this position and completely contained in $\mathcal{O}_q(T_L \cup L)$. It is clear from Proposition 6.5 that this is the only element from $C_L^{-}$ which does not $q$-commute with $Z_{c-1,d-1}$. On the other hand, when $(c - 1, d - 1)$ is viewed as an element in $S_L$, the variable $D = D^{(m-c)}$ does, by Proposition 6.4, $q$-commute with $\mathcal{O}_q(T_L \cup L)$ - and clearly also with $Z_{c-1,d-1}$. Next observe that evidently $D \in C_L^{-}$.

Suppose then that $C \in \mathcal{O}_q(T_L \cup L_1)$ is covariant. It is clear that

$$\mathcal{O}_q(T_L \cup L_1) \subseteq \mathcal{O}_q(T_L \cup L)[D, X_b^{-1}].$$

Both adjoined elements are covariant as far as $\mathcal{O}_q(T_L \cup L)$ is concerned, and it follows easily that $C$ must be a polynomial in the variables from $C_L^{-}$ together with $D$ and $X_b^{-1}$. The element $Z_{c-1,d-1}$ $q$-commutes with all these generators except $X_b$ and this easily implies that $X_b^{\pm 1}$ cannot appear.

□

**Proposition 6.30.** Consider the quadratic algebra $\mathcal{O}_q(T_L \cup L) \subseteq \mathcal{O}_q(M(m, n))$. Then there is a non-trivial center if and only if $m = n$ and $L = L^+$. This center is generated by $\det_q(n)$.

**Proof.** Consider the covariant element $M = M_{m,n}$ corresponding to the position $(m, n)$. If $N$ is any other covariant element and $MN = q^\delta NM$ then $\delta \neq 0 \Rightarrow \delta < 0$. This follows by easy inspection. Hence, since any central element must be a polynomial in the elements in $C_L^{-}$, the central element must be made up of those elements from $C_L^{-}$ that are contained in $M$. If there are points in the algebra not inside $M$, then there will be non-trivial commutation relations between these and the quantum minors from $M$. Thus, there can be no points outside $M$. The remaining details now follow from the classical result of Parshall and Wang ([26]). □
7. On compatible pairs

We now settle the existential questions implicitly raised in Theorem \[6.19\], Definition \[6.22\], and Definition \[6.23\].

**Proposition 7.1.** To a given broken line \(L\) in \(M_{m,n}(\mathbb{Z}_+)\) one can construct the following data \(D_L\):

- An ordering of the set of variables \(V_L = V_L^+ \cup V_L^-\) of the broken line such that the variables in \(V_L^-\) are assigned the numbers from 1 to \(N_L\) and the remaining variables the numbers from \(N_L + 1\) to \(mn\). Let \(c_L^- = \#\{C_L^-\}\) denote the number of covariant elements. (These are the non-mutable elements when \(V_L^-\) is considered in its own right. We have, of course, that \(c_L^- = m + n - 1\) but it is convenient to have the additional notation.) We will even assume that the mutable variables are assigned the numbers from 1 to \(\tilde{N}_L = N_L - c_L^-\).
- The \(mn \times mn\) matrix \(\Lambda_L\) of the full quantized matrix algebra corresponding to the variables \(V_L\) of the broken line.
- A matrix \(B_L\) of size \(mn \times (m - 1)(n - 1)\) such that \((\Lambda_L, B_L)\) is a compatible pair.
- The \(N_L \times N_L\) matrix \(\Lambda_L^0\) of the variables in \(V_L^-\). We view this as a submatrix of \(\Lambda_L\).
- An \(N_L \times \tilde{N}_L\) matrix \(B_L^0\) such that \((\Lambda_L^0, B_L^0)\) is a compatible pair for the set of variables \(V_L^-\). If one defines a \(mn \times \tilde{N}_L\) matrix \(B_L^R\) by adding \(mn - \tilde{N}_L\) rows of zeros to \(B_L^0\) such that \(B_L^0\) occupies the top rows of \(B_L\), the following holds in addition:
  \[\Lambda_L B_L^0 = -4D_L,\] where \(D_L = I_{\tilde{N}_L} \oplus 0\) is the \(mn \times \tilde{N}_L\) matrix consisting of \(I_{\tilde{N}_L}\) in the top \(\tilde{N}_L \times \tilde{N}_L\) corner augmented by an appropriate number of rows of zeros. Here, \(I_{\tilde{N}_L}\) is the \(\tilde{N}_L \times \tilde{N}_L\) identity matrix.
  \[
  \Lambda_L B_L^0 = -4D_L, \quad D_L = I_{\tilde{N}_L} \oplus 0
  \]
  where \(\tilde{D}_L\) is the \(\tilde{N}_L \times \tilde{N}_L\) identity matrix.
  \[
  \Lambda_L B_L^R = -4D_L, \quad D_L = I_{\tilde{N}_L} \oplus 0
  \]
  An ordering of the set non-mutable elements given as \(C_L^-\). If \(L_1\) is a broken line in \(M_{m,n}(\mathbb{Z}_+)\) and \(L_1 < L\), there exists a quantum seed \(O_{L_1, L}^- = (V_{L_1, L}^-, \Lambda_{L_1, L}, B_{L_1, L}^-)\) with the set non-mutable elements given as \(C_{L_1}^-\). If \(L_1\) is a broken line in \(M_{m,n}(\mathbb{Z}_+)\) and \(L_1 < L\), there exists a quantum seed \(O_{L_1, L} = (V_{L_1, L}^-, \Lambda_{L_1, L}, B_{L_1, L})\) which is equivalent by the quantum line mutations \(\mu_L(L_1, L)\) to \(O_L^-\) and where the pairs \((\Lambda_{L_1, L}, B_{L_1, L})\) and \((\Lambda_{L_1}^0, B_{L_1}^0)\) are related in a way that generalizes in an obvious manner the way \((\Lambda_{L_1}, B_{L_1})\) is related to \((\Lambda_{L_1}^0, B_{L_1}^0)\).

**Proof:** A short proof would be to say that this follows by bootstrapping. We give here a more detailed proof using the same principle: First assume that \(n = m + 1\) (or, analogously, \(n = m - 1\)). The existence of \(B_L\) will follow from the first parts of the proof. We prove the claims involving \(B_L^R\) and \(B_L^0\) by induction on the partial order on the set of broken lines. The induction starts with the line \(L^-\). There are no mutable elements in \(V_L^-\) so \(B_L^R\) and \(B_L^0\) are empty. The other structure we start with is \(V_L^-\) and a compatible pair \((\Lambda_L^-, B_L^-)\) connected with this set of variables. The set of non-mutable elements is \(\mathcal{C} = C_L^-\) as before. The matrix \(\Lambda_L^-\) can, naturally,
be explicitly written down. It is known from [16][Proposition 4.11] that $\Lambda_{L^-}$ is invertible. Notice that all $2 \times 2$ block matrices in [16] must be multiplied by 2 to conform with the current assumptions. It follows then from the discussion on p. 85 in [16] that there exists an integer matrix $A$ with $\det A = 1$ such that $A(\Lambda_{L^-})A^t = \tilde{D}$ where $\tilde{D}$ is a block diagonal matrix consisting of $\frac{1}{2}m(m - 1)$ $2 \times 2$ blocks \[
abla \hspace{1cm} \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} \] and $m$ $2 \times 2$ blocks \[
abla \hspace{1cm} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \]. The existence of $B_{L^-}$ follows easily from this simply as a submatrix of $4\Lambda^{-1}$. It is a consequence of the analysis below, that only blocks with off diagonal entry $\pm 4$ appear. At the moment, it is only the existence and uniqueness of $B_{L^-}$ that matters. After we have presented the induction step, we encourage the reader to take it right at the start as a simple exercise.

Suppose now that we have a line $L$ with data $D_L$. Let $L_1 > L$ be a broken line closest to $L$. We must now construct the data $D_{L_1}$ for $L_1$. For this purpose we consider the inverse, $\mu(L, L_1)$, of the quantum line mutation $\mu(L_1, L)$. We view these mutations as taking place inside the full $m \times (m+1)$ matrix algebra, but we shall keep a keen eye on its relation to the ambient spaces $F_L^-$ and $F_{L_1}^-$. Specifically, how $Q_{L_1}^-$ grows from $Q_L^-$. The mutation $\mu(L, L_1)$ begins with a mutation of the form of the reversed of the bottom line in (6.8) and where the element we mutate from, $X_b^{(m-i-1)}$, is a covariant, viz. non-mutable, element of $V_L^-$. This is not represented in the matrices $B^0_L$ and $B^R_L$, so we define a new matrix $B^R_{L_1}$ by joining one new column to $B^R_L$ in the position $\tilde{N}_L + 1$ and labeled by $X_b^{(m-i-1)}$. If this approach is possible, it follows from Corollary 6.14 in combination with Definition 2.3 and Lemma 2.2 what this column must look like in the full algebra: There should be the value -1 at the positions corresponding to $X_b^{(m-i-1)}$ and $D^{(m-i-1)}$ and the value 1 at the positions of $Y_r^{(m-i-1)}$, and $Y_l^{(m-i-1)}$. All other elements must be zero. Of these, $X_o^{(m-i-1)}$, $Y_r^{(m-i-1)}$, and $Y_l^{(m-i-1)}$ are variables of $V_L^-$. The element $D^{(m-i-1)}$ is not a variable of $V_L^-$ but it is a covariant (non-mutable) element of $V_{L_1}^-$. To begin with we add $D^{(m-i-1)}$ to the set of variables and thus obtain a set of variables for $V_{L_1}^-$, and corresponding to this we obtain a $\Lambda$-matrix $\Lambda_{L_1}^0$ with one more column and row than $\Lambda_{L}^0$. Since $-\frac{1}{4}B_L$ is a part of the inverse matrix of $\Lambda_L$ it is clear that the matrix $B_{L_1}^R$ is part of the matrix $B_{L_1}$ corresponding to the given set of variables with $\Lambda$-matrix $\Lambda_{L_1}$. We now perform the desired mutation inside the full matrix algebra, thus obtaining a new compatible pair $(\Lambda_{L_1}, B_{L_1})$. According to Theorem 6.10, with the given column of $B_{L_1}$ this is exactly the mutation

$$X_b^{(m-i-1)} \mapsto X_l^{(m-i-1)}.$$

At the same time we observe that the difference of doing the mutation inside this algebra and doing it in the smaller algebra based on $V_{L_1}^-$ lies entirely in the matrix we use for $F$ in (2.8). Here it makes a difference where we are since $D^{(m-i-1)}$, and one or both of $Y_r^{(m-i-1)}, Y_l^{(m-i-1)}$ are mutable in the full
algebra but non-mutable in the small algebra. At this moment of the proof we are not concerned with the columns of $B_{L_1}$ that are not in $B_{L_1}^R$, though they are known in principle. The two mutations differ only in what happens to those the columns in the way that different choices of $F$ may result in a multiple of the column corresponding to $X_b$ being added or subtracted. Furthermore, if one or both of $Y_i^{(m-i-1)}, Y_i^{(m-i-1)}$ are non-mutable in $V_L^-$, they stay so in $V_{L_1}^-$. So, since we need not concern ourselves with the non-mutable variables of $V_{L_1}^-$, it is clear that the analogous new compatible pair $(\Lambda_{L_1}^0, B_{L_1}^0)$ - obtained by mutation in the ambient space $F_{L_1}$ from the seed whose set of variables is $V_{L_1}^- \cup \{D^{(m-i-1)}\}$ and whose $\Lambda, B$ part is as described - is a subpair of $(\Lambda_{L_1}, B_{L_1})$. Now perform the remaining mutations in the quantum line mutation. These only involve mutable variables and are easily seen to preserve the general form. Finally, one can reshuffle the variables to obtain the wanted ordering. Again, this does not change the general form. Thereby the induction step is completed.

The part of the mutation which involves the matrix $E$ affects the rows corresponding to the variables $X_o, D$. All in all, the variables corresponding to the points in $S_{L_1}$ do not take part in any of the manipulations.

In this way we build up $B$ matrices with more and more columns. In the end we reach $O_L = (V_{L^+}, \Lambda_{L^+}, B_{L^+})$ of the extremal line $L^+$. Once we have that, we can mutate back, by quantum line mutations, to any quantum seed $O_{L^+} = (V_L, \Lambda_L, B_L)$. We can also stop the growing process at an earlier point, where we have obtained a seed $O_L^- = (V_L^-, \Lambda_L^0, B_L^0)$ and use mutations $\mu^L$ inside the ambient space $F_L^-$ to obtain the seeds $O_{L_1,L}$ mentioned in the proposition. Finally recall the independence of path result Corollary 6.25.

Let us now consider the general situation of $O_q(M(m,n))$. Suppose for simplicity that $m = n + r$ with $r \geq 2$. We can view this as the subalgebra of $O_q(M(m,n + r + 1))$ generated by the elements $Z_{i,j}$ with $1 \leq i \leq m$ and $2 + r \leq j \leq n + r + 1$. Any broken line $L : (1, n) \to \cdots \to (m, 1)$ in $M_{m,n}(Z_+)$ is similarly considered as a line $\bar{L} : (1, n + r + 1) \to \cdots \to (m, r+2)$ in $M_{m,m+1}(Z_+)$ which is then extended by the segment $\bar{L} \to (m, 1)$. This corresponds to adding the non-mutable covariant variables $Z_{m,1}, \ldots, Z_{m,r+1}$ to all sets of variables in all quantum seeds. If we stipulate that the mutations and other operations in $O_q(M(m,n + r + 1))$ should never involve these we clearly get the result as a subcase of the full case based on $(m, m + 1)$.

\[\square\]

**Remark 7.2.** Also for the remaining mutations in the quantum line mutation $\mu(L, L_1)$ we can write down explicitly the values in the $B$-column which we mutate from simply by using Corollary 6.14 repeatedly. In this way one can in fact “explicitly” write down the compatible pairs at each step.
8. The quantum (upper) cluster algebra of a broken line

**Definition 8.1.** The cluster algebra $\mathcal{A}^-_L$ connected with a broken line $L$ in $M_{m,n}(\mathbb{Z}_+)$ is the $\mathbb{Z}[q]$-algebra generated in the space $\mathcal{F}^-_L$ by the inverses of the non-mutable elements $C^-_L$ together with the union of the sets of all variables obtainable from the initial seed $Q^-_L$ by composites of quantum line mutations $\mu^L(L, L_1)$ with $L_1 \leq L$.

Observe that we include $C^-_L$ in the set of variables.

**Definition 8.2.** The upper cluster algebra $\mathcal{U}^-_L$ connected with a broken line $L$ in $M_{m,n}(\mathbb{Z}_+)$ is the $\mathbb{Z}[q]$-algebra in $\mathcal{F}^-_L$ given as the intersection of all the Laurent algebras of the sets of variables obtainable from the initial seed $Q^-_L$ by composites of quantum line mutations $\mu^L(L, L_1)$ with $L_1 \leq L$.

**Remark 8.3.** Our terminology may seem a bit unfortunate since the notions of a cluster algebra and an upper cluster algebra already have been introduced by Berenstein and Zelevinsky in terms of all mutations. We only use quantum line mutations which form a proper subset of the set of all quantum mutations. However, it will be a corollary to what follows that the two notions in fact coincide, and for this reason we do not introduce some auxiliary notation.

**Remark 8.4.** The algebras $\mathcal{A}^-_L$ and $\mathcal{U}^-_L$ of a broken line $L$ are defined in terms of some $\mathcal{O}_q(M(m, n))$, but of course, if the line has a segment $(m, 1) \leftarrow (m, u)$, with $(m, u)$ denoting a corner, and $u > 1$, then the elements $Z_{m,1}, \ldots, Z_{m,u-1}$ are all covariant. Thus, $\mathcal{O}_q(T_L \cup L) = \mathcal{O}_q(T_{L_1} \cup L_1)[Z_{m,1}, \ldots, Z_{m,u-1}]$ and $\mathcal{A}^-_L = \mathcal{A}^-_{L_1}[Z_{m,1}^{\pm 1}, \ldots, Z_{m,u-1}^{\pm 1}]$, where $L_1$ is what remains of $L$ after these elements have been removed. Similarly with segments $(1, n) \rightarrow (u, n)$. In the same spirit, covariant elements may be added if it is convenient to view $\mathcal{A}^-_L$ as a part of a quantum cluster algebra based on some other $\mathcal{O}_q(M(m_1, n_1))$ with $m \leq m_1$ and $n \leq n_1$. See also the last part of the proof of Proposition 7.1.

It is clear that $\mathcal{O}_q(T_L \cup L) \subseteq \mathcal{U}^-_L$ and that $Y_{i}^{\pm 1} \in \mathcal{U}^-_L$ for all $Y_i \in C^-_L$. Indeed, by the $q$ Laurent Phenomenon, $\mathcal{O}_q(T_L \cup L) \subseteq \mathcal{A}^-_L \subseteq \mathcal{U}^-_L$.

**Theorem 8.5.** Let $\mathcal{C}^-_L = \{Y_1, \ldots, Y_s\}$. Then,

$$\mathcal{U}^-_L = \mathcal{O}_q(T_L \cup L)[Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}] = \mathcal{A}^-_L.$$

We need only establish the first equality. We will in the proof of that use the following

**Proposition 8.6.** A quantum minor $M \in \mathcal{C}^-_L$ satisfies the following crucial property:

If $p_1 M = p_2 p_3$ in $\mathcal{O}_q(T_L \cup L)$ then

$$p_2 = p_4 M \text{ or } p_3 = p_5 M \text{ for some } p_4, p_5 \text{ in } \mathcal{O}_q(T_L \cup L).$$
Proof. Goodearl and Lenegan proved in ([15]) that the determinantal ideal is prime. We can reduce our case, in which \( M \) is a covariant quantum minor, to theirs by using a PBW basis of the full set of variables in which the variables of the row and columns of \( M \) are written to the right. The elements \( p_2 \) and \( p_3 \) may then be written as sums of polynomials in the variables of \( M \) with coefficients (to the left) that are monomials in the variables outside \( M \). Let us be specific and say that \( M = \xi_{\{i,j+1,\ldots,n\}}^{\{j,j+1,\ldots,j+n-i\}} \). Let us order the monomials so that the points with column number less than \( j \) are biggest, and ordered lexicographically with the biggest being the point with smallest row and column number. The finer details are irrelevant. Next in the ordering we take those points having a column number between \( j \) and \( j + n - i \) with a similar lexicographical ordering. Finally we take those with a column number bigger than \( j + n - i \). Here we chose an opposite ordering. We can then focus on the monomials that are the biggest in \( p_2 \) and \( p_3 \). The point of the chosen ordering is that one does not pick up bigger terms via (3.4) while rewriting a product. Let \( v_0^0p_2^0 \) be the summand in \( p_2 \) corresponding to the biggest monomial \( v_0^0 \). Here \( p_2^0 \) is a polynomial in the variables of \( M \). Let \( v_0^0p_3^0 \) be the analogous summand for \( p_3 \). Regrouping in the product \( p_2p_3 \) according to our total ordering results in a unique highest term (up to a factor of \( q \) to some power) \( v_0^0v_0^0p_2^0p_3^0 \). This must then match a term corresponding to the same monomial in \( p_1M \). By ([15]), \( p_2^0 = p_4^0M \) or \( p_3^0 = p_5^0M \). Say it is \( p_2^0 = p_4^0M \). Since \( M \) is covariant with respect to all the variables of \( \mathcal{O}_q(T_L \cup L) \) we can just drop the expression \( v_0^0p_2^0 \) from \( p_2 \). Indeed, by looking at the biggest elements, we can assume from the beginning that neither \( p_2 \) nor \( p_3 \) contains a summand of the form \( pM \) and then argue by contradiction. 

Proof of Theorem 8.5. We prove this by induction. For the unique smallest line \( L^- \) the algebra \( \mathcal{O}_q(T_L \cup L^-) = \mathcal{O}_q(L^-) \) is generated by the covariant elements in \( \mathcal{C}^-_{L^-} \). The algebra is quasi-polynomial and there are no quantum line mutations except the trivial. Thus the claim is trivially true. [Actually, there is also a unique line \( L_1 \) closest to \( L^- \) and the situation here essentially corresponds to \( \mathcal{O}_q(M(2,2)) \). This case is also true and well-known.] Let us then consider a line \( L \) and let \( L_1 \) be a closest line with \( L < L_1 \). Let the notation be as in the proof of Proposition 6.29. Then \( \mathcal{C}^-_{L_1} \) is obtained from \( \mathcal{C}^-_L \) by replacing \( X_b \) by \( D \). Suppose that \( u \in \mathcal{U}^-_{L_1} \). Since \( \mathcal{U}^-_{L_1} \) is an algebra it is clear that we may assume that when \( u \) is expressed as a \( q \)-Laurent polynomial of some set of variables, all powers of the covariant elements are non-negative. Moreover, \( L \) is obtained from \( L_1 \) by a quantum line mutation and all subsequent quantum line mutations of \( L \) are thus also quantum line mutations of \( L_1 \). In all these mutations \( D \) stays unchanged. It is then clear that \( \mathcal{U}^-_{L_1} \subseteq \mathcal{U}^-_{L}[[D^{\pm 1}]] \). Now, the non-mutable elements of \( L \) are the same as those of \( L_1 \) with the exception of \( X_b \). By the argument about the positivity of the non-mutable elements we can then assume that all elements except the latter occur with a positive power. Thus, we can assume \( u \in \mathcal{U}^-_{L}[[X_b^{\pm 1}, D]] \). By
the induction hypothesis we then have

\[(8.1) \quad u \in \mathcal{O}(T_L \cup L)[X_b^{-1}, D],\]

and to meet our goal, we only need be concerned about the elements with a strictly negative power of \(X_b\).

Naturally, \(\mathcal{O}(T_L \cup L)\) can be viewed as a subalgebra of \(\mathcal{O}(T_{L_1} \cup L_1)\).

Let us denote the initial variables of \(L_1\) by \(D = X_t^{(m-c+1)} \cup X_t^{(m-c)} \cup \ldots \cup X_t^{(0)} \cup W_1 \cup \ldots \cup W_N\). The initial variables of \(L\) are then \(X_b = X_t^{(m-c)} \cup X_t^{(m-c-1)} \cup \ldots \cup X_t^{(0)} \cup W_1 \cup \ldots \cup W_N\). Let us look at the element \(u\). This can be written as a Laurent polynomial in the given initial variables of \(L_1\), one of which is \(D\):

\[
\begin{align*}
\sum_{\alpha, \beta} W_\beta^\alpha &\prod_{i=0}^{m+1-c} \left( X_t^{(m-c+1-i)} \right)^{\alpha_i}.
\end{align*}
\]

We can factor out the biggest negative powers such that

\[(8.2) \quad u = p_{top} \cdot W^{-\beta_0} \prod_{i=0}^{m+1-c} \left( X_t^{(m-c+1-i)} \right)^{-\alpha_i^0},
\]

where \(p_{top} \in \mathcal{O}(T_{L_1} \cup L_1)\) and, in particular, \(p_{top}\) contains no positive powers of \(D\). We wish to argue by contradiction and thus assume that the multi-indices \(\alpha_0, \beta_0\) are non-negative, and at least one \(\alpha_i^0\) or \(\beta_i^0\) is positive.

Set \(Z = Z_{c-1,d-1}\) Then \(D = ZX_b\) modulo \(\mathcal{O}(T_L \cup L)\). By (8.1) we have

\[(8.3) \quad u = \left( \sum_i Z^i p_i X_b^{k_i} \right) X_b^{-\rho},
\]

where \(\forall i : 0 \leq k_i < \rho\). Furthermore the elements \(p_i \in \mathcal{O}(T_L \cup L)\) contain no power of \(Z\) nor \(X_b\). Combining (8.2) and (8.3), we have

\[
\left( \sum_i Z^i p_i X_b^{k_i} \right) X_b^{-\rho} = p_{top} \cdot W^{-\beta_0} \prod_{i=0}^{m+1-c} \left( X_t^{(m-c+1-i)} \right)^{-\alpha_i^0}.
\]

Now, in the \(q\)-Laurent algebra we clearly get

\[(8.4) \quad \left( \sum_i Z^i p_i X_b^{k_i} \right) \prod_{i=0}^{m+1-c} \left( X_t^{(m-c+1-i)} \right)^{\alpha_0^0} W^{-\beta_0} = q^{2\gamma} p_{top} X_b^\rho,
\]

where \(q^{2\gamma}\) is an irrelevant factor stemming from the constant commutativity between \(X_b\) and the elements \(W_i\). We ignore this and similar factors in the following.

The first crucial observation is that by Proposition 8.6, the term \((M_i)^{\alpha_0}\) does not contain a positive power of \(D\) since the right hand side of (8.4) clearly does not.

The next important fact is that the position \((c-1,d-1)\) is repulsive. This implies that it is straightforward to look at the highest order terms of \(Z\) in (8.4). In the right hand side we simply write \(p_{top} = Z^S u_S + l.o.t.s\) where
$u_S$ is a polynomial in the variables of $T_L \cup L$. In the left hand side, let us say that $Z^K p_K X_b^{kK}$ is the term with the highest $Z$ exponent $K$. We then get additional $Z$ terms from $\prod_{i=1}^{m+1-c} \left( X_t^{(m-c+1-i)} \right)^{\alpha_i^0}$, and here the highest $Z$ term is $Z^{\alpha_0^0} \prod_{i=1}^{m-c} \left( X_b^{(m-c-i)} \right)^{\alpha_i^0}$, where $\alpha_S^0 = \sum_{i=1}^{m-c-1} \alpha_i^0$.

Using the repulsiveness again, we get

$$Z^{K+\alpha_0^0} p_K \prod_{i=1}^{m-c} \left( X_b^{(m-c-i)} \right)^{\alpha_i^0} X_b^{kK} = Z^S u_S X_b^\rho.$$  

And thus,

$$p_K \prod_{i=1}^{m-c} \left( X_b^{(m-c-i)} \right)^{\alpha_i^0} = u_S X_b^{\rho-kK}.$$

This identity holds in $\mathcal{O}_q(T_L \cup L)$. Since $\rho - k_\ell > 0$ it follows by Proposition 8.6 that $X_b$ must be a right divisor of one of the terms on the left hand side. The $X_b^{(a)}$ with $a = 0, 1, \ldots, m - c - 1$ terms of course are impossible in this respect. Thus, $p_K = \tilde{p} X_b$ for some $\tilde{p}$. This is a contradiction to the way $p_K$ was defined. Hence, there can be no negative power $X_b^{-\rho}$ in (8.3).

\[\square\]

**Corollary 8.7.** For the case of the quantum $n \times r$ matrix algebra, the quantum cluster algebra is equal to its upper bound.

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