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STABILITY OF SOLUTIONS TO COMPLEX
MONGE-AMPÈRE FLOWS

VINCENT GUEDJ, CHINH H. LU, AND AHMED ZERIAHI

Abstract. We establish a stability result for elliptic and parabolic complex Monge-Ampère equations on compact Kähler manifolds, which applies in particular to the Kähler-Ricci flow.

Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday.

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Introduction

Several important problems of Kähler geometry (e.g. finding canonical metrics) necessitate to study the existence and regularity of solutions to certain degenerate complex Monge-Ampère equations. The fundamental work of Yau [Yau78] guarantees the existence of smooth solutions to a large class of equations. It has been generalized in various interesting directions, providing weak solutions to several degenerate situations. We refer the reader to [GZ17] for a recent overview.

In this paper we are interested in the stability properties of solutions to such equations. Let $X$ be a compact Kähler manifold of dimension $n$, $\theta$ be a Kähler form and $dV$ a volume form on $X$. Fix $0 \leq f$ a positive density on $X$ and let $\varphi$ be a $\theta$-plurisubharmonic function solving

\[ \text{MA}_\theta(\varphi) = e^{\varphi} f dV, \]

where $\text{MA}_\theta(\varphi) = (\theta + dd^c \varphi)^n$ denotes the complex Monge-Ampère measure with respect to the form $\theta$. 
If \((f_j dV)\) is a sequence of Borel measures converging in total variation to \(f dV\) then, it follows from [GZ12] that the corresponding sequence of solutions \((\varphi_j)\) converges in \(L^1(X)\) to \(\varphi\). Our aim is to establish a quantitative version of this convergence.

Our first main result gives a satisfactory answer for \(L^p\) densities with respect to Lebesgue measure, \(p > 1\) (see [Kol03, Blo03, GZ12] for related results).

**Theorem A.** Fix \(p > 1\) and \(0 \leq f, g \in L^p(X, dV)\). If \(\varphi, \psi\) are bounded \(\theta\)-plurisubharmonic functions on \(X\) such that
\[
\text{MA}_\theta(\varphi) = e^\varphi f dV; \quad \text{MA}_\theta(\psi) = e^\psi g dV,
\]
then
\[
\|\varphi - \psi\|_\infty \leq C\|f - g\|_p^{1/n},
\]
where \(C > 0\) depends on \(p, n, X, \theta, dV\) and uniform bounds on \(\|f\|_p, \|g\|_p\).

We use some ideas of the proof of Theorem A to establish a stability result for parabolic complex Monge-Ampère flows. We consider the equation
\[(0.2) \quad (\omega_t + dd^c \varphi_t)^n = e^{\dot\varphi + F(t, \cdot, \varphi)} f dV,\]
in \(X_T := ]0, T[ \times X\), where \(0 < T < +\infty\) and
- \((\omega_t)_{t \in [0, T]}\) is a smooth family of Kähler forms on \(X\);
- there exists a fixed Kähler form \(\theta\) such that \(\theta \leq \omega_t\) for all \(t\);
- \(F = F(t, x, r) : \hat{X}_T := [0, T] \times X \times \mathbb{R} \to \mathbb{R}\) is smooth, non-decreasing in the last variable, uniformly Lipschitz in the first and the last variables, i.e. \(\exists L > 0\) s.t. \(\forall((t_1, t_2), x, (r_1, r_2)) \in [0, T]^2 \times X \times \mathbb{R}^2\),
\[
|F(t_1, x, r_1) - F(t_2, x, r_2)| \leq L (|r_1 - r_2| + |t_1 - t_2|),
\]
- \(0 < f \in C^\infty(X, \mathbb{R})\).

Our second main result is the following:

**Theorem B.** Fix \(F, G : \hat{X}_T := [0, T] \times X \times \mathbb{R} \to \mathbb{R}\) satisfying the above conditions and fix \(p > 1\). Assume that \(\varphi : [0, T] \times X \to \mathbb{R}\) is a smooth solution to the parabolic equation \((0.2)\) with data \((F, f)\) and \(\psi : [0, T] \times X \to \mathbb{R}\) is a smooth solution to \((0.2)\) with data \((G, g)\). Then
\[
\sup_{X_T} |\varphi - \psi| \leq \sup_X |\varphi_0 - \psi_0| + T \sup_{X_T} \|F - G\| + A \|g - f\|_p^{1/n},
\]
where \(A > 0\) is a constant depending on \(X, \theta, n, p, L\), a uniform bound on \(\varphi_0, \psi_0, \dot\varphi_0, \dot\psi_0\) on \(X\) and a uniform bound on \(\|f\|_p\) and \(\|g\|_p\).

**Dédiacce.** C’est un plaisir de contribuer à ce volume en l’honneur de Jean-Pierre Demailly, dont nous apprécions l’exigence et la générosité. Ses notes de cours [Dem89, Dem13] ont fortement contribué au développement de la théorie du pluripotentiel en France, nous lui en sommes très reconnaissants.

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1. Preliminaries

1.1. Elliptic complex Monge-Ampère equations. Let $(X, \theta)$ be a compact Kähler manifold of dimension $n$.

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called \emph{quasi-plurisubharmonic} (quasi-psh for short) on $X$ if it can locally be written $u = \rho + \varphi$, where $\varphi$ is a plurisubharmonic function and $\rho$ is a smooth function.

A function $u$ is called $\theta$-\emph{plurisubharmonic} ($\theta$-psh for short) if it is quasi-psh on $X$ and $\theta + dd^c u \geq 0$ in the weak sense of currents on $X$. The set of all $\theta$-psh functions on $X$ is denoted by $\text{PSH}(X, \theta)$.

It follows from the seminal work of Bedford and Taylor [BT76, BT82] that the Monge-Ampère measure $(\theta + dd^c u)^n =: \text{MA}_\theta(u)$ is well-defined when $u$ is a bounded $\theta$-psh function, and it satisfies several continuity properties.

We refer the readers to [GZ17] for a detailed exposition of global pluripotential theory. In this note we will need the following comparison principle.

**Proposition 1.1.** Assume that $u, v \in \text{PSH}(X, \theta) \cap \text{L}^\infty(X)$ are such that $e^{-u} \text{MA}_\theta(u) \geq e^{-v} \text{MA}_\theta(v)$ on $X$. Then $u \leq v$ on $X$.

This result is well known (a proof can be found in [DDNL16, Lemma 2.5]). We shall also need the following version of the domination principle:

**Proposition 1.2.** Fix a non-empty open subset $D \subset X$ and let $u, v$ be bounded $\theta$-psh functions on $X$ such that for all $\zeta \in \partial D$,

$$\limsup_{D \ni z \to \zeta} (u(z) - v(\zeta)) \geq 0.$$ 

If $\text{MA}_\theta(u)\{u < v\} \cap D = 0$ then $u \geq v$ in $D$.

**Proof.** Adding a large constant to both $u$ and $v$, we can assume that $v \geq 0$. For each $\varepsilon > 0$ consider $v_\varepsilon := (1 - \varepsilon)v$. Then $v_\varepsilon \in \text{PSH}(X, \theta)$ and $v_\varepsilon \leq v$, hence

$$\limsup_{D \ni z \to \zeta} (u(z) - v_\varepsilon(z)) \geq 0.$$ 

The comparison principle (see [CKZ11]) yields

$$\int_{\{u < v_\varepsilon\} \cap D} \theta^n_{v_\varepsilon} \leq \int_{\{u < v_\varepsilon\} \cap D} \theta^n_u \leq \int_{\{u < v\} \cap D} \theta^n_u = 0.$$ 

Since $\theta^n_{v_\varepsilon} \geq \varepsilon^n \theta^n$, we deduce that $u \geq v_\varepsilon$ almost everywhere (with respect to Lebesgue measure) in the open set $D$, hence everywhere in $D$. The result follows by letting $\varepsilon \to 0$. \qed

1.2. Complex Monge-Ampère flows. We recall the following definition of (sub/super)solution which will be used in this note.

**Definition 1.3.** Let $\varphi : [0, T] \times X \to \mathbb{R}$ be a function satisfying:

- $\varphi$ is continuous in $X_T := [0, T] \times X$,
- for any $x \in X$, the function $\varphi(\cdot, x)$ is $C^1$ in $[0, T]$ and $\varphi = \partial_t \varphi$ its partial derivative in $t$ is continuous in $[0, T] \times X$;
- for any $t \in [0, T]$ the function $\varphi_t$ is bounded and $\omega_t$-plurisubharmonic.

We say that the function $\varphi$ is a
solution to the equation (0.2) with data \((F, f)\) if for any \(t \in ]0, T[\),
\[
(\omega_t + dd^c \varphi_t)^n = e^{\hat{\varphi}(t, \cdot) + F(t, \cdot; \varphi_t)} f dV.
\]

• subsolution to the equation (0.2) with data \((F, f)\) if for any \(t \in ]0, T[\),
\[
(\omega_t + dd^c \varphi_t)^n \geq e^{\hat{\varphi}(t, \cdot) + F(t, \cdot; \varphi_t)} f dV.
\]

• supersolution to (0.2) with data \((F, f)\) if for any \(t \in ]0, T[\),
\[
(\omega_t + dd^c \varphi_t)^n \leq e^{\hat{\varphi}(t, \cdot) + F(t, \cdot; \varphi_t)} f dV.
\]

All the above inequalities have to be understood in the weak sense of currents: the LHS is a well defined Borel measure since \(\varphi_t\) is a bounded \(\omega_t\)-psh function (see [BT76]), while the RHS is a well defined measure which is absolutely continuous with respect to Lebesgue measure.

2. Stability in the elliptic case

2.1. A general semi-stability result. Theorem A is a consequence of the following more general result which generalizes stability results where the right-hand side does not depend on the unknown function (see [Kol03], [DZ10]).

**Theorem 2.1.** Fix \(p > 1\). Assume that \(0 \leq f, g \in L^p(X, dV)\) and \(\varphi, \psi\) are bounded \(\theta\)-plurisubharmonic functions on \(X\) such that
\[
\text{MA}_{\theta}(\varphi) \geq e^\varphi f dV \quad \text{and} \quad \text{MA}_{\theta}(\psi) \leq e^\psi g dV.
\]

Then there is a constant \(C > 0\) depending on \(p, n, X, \theta\) and a uniform bound on \(\log \|f\|_p\) and \(\log \|g\|_p\) such that
\[
\varphi \leq \psi + (C + 2\text{osc}_X \varphi + 2\text{osc}_X \psi) \exp \left(\frac{\text{osc}_X \varphi}{n}\right) \|g - f\|_p^{1/n}.
\]

**Proof.** We use a perturbation argument inspired by an idea of Kolodziej [Kol96] (see also [Ngu14, proof of Theorem 3.11]) who considered the local case.

For simplicity we normalize \(\theta\) and \(dV\) so that \(\int_X dV = \text{Vol}(\theta) = 1\), and we denote by \(\|f\|_p\) the \(L^p\)-norm of \(f\) with respect to the volume form \(dV\). We assume that \(\|f\|_p, \|g\|_p\) are uniformly bounded away from zero and infinity (i.e. \(\log \|f\|_p, \log \|g\|_p\) are uniformly bounded).

If \(\|(g - f)\|_p = 0\) then \(g \leq f\) almost everywhere in \(X\). In this case, \(\varphi\) is a subsolution and \(\psi\) is a supersolution to the same complex Monge-Ampère equation. Then the comparison principle (Proposition 1.1) yields \(\varphi \leq \psi\) in \(X\), which proves the result.

We assume in the sequel that \(\|(g - f)\|_p > 0\). Integrating the inequality \(\text{MA}_{\theta}(\varphi) \geq e^\varphi f dV\), we see that
\[
\inf_X \varphi \leq -\log \left(\int_X f dV\right) \leq -\log \|f\|_p,
\]
hence \(\sup_X \varphi \leq \text{osc}_X \varphi - \log \|f\|_p\). Similarly \(\sup_X \psi \geq -\log \|g\|_p\), hence \(-\inf_X \psi \leq \text{osc}_X \psi + \log \|g\|_p\).
Set $\varepsilon := e^{\sup_X \varphi/n} \|(g - f)\|_p^{1/n}$. If $\varepsilon \geq 1/2$ then
\[
\sup_X (\varphi - \psi) \leq \sup_X \varphi - \inf_X \psi
\leq 2 \left( \sup_X [\varphi - \inf_X \psi] \exp \left( \frac{\sup_X \varphi}{n} \right) \|(g - f)\|_p^{1/n} \right),
\]
as desired.

So we can assume that $\varepsilon < 1/2$. Hölder inequality yields
\[
\int_X (g - f) + \| (g - f) \|_p dV \leq 1.
\]

It follows from [Ko98] that there exists a bounded $\theta$-psh function $\rho$ such that
\[
\text{MA}_\theta(\rho) = h dV := \left( a + \frac{(g - f)_+}{\|(g - f)\|_p} \right) dV, \quad \sup_X \rho = 0,
\]
where $a \geq 0$ is a normalization constant to insure that $\int_X h dV = \int_X dV$. Since $\|h\|_p \leq 2$ the uniform estimate of Kolodziej [Ko98] guarantees
\[
-C_1 \leq \rho \leq 0,
\]
where $C_1$ only depends on $p, \theta, n$.

Observe that one can use here soft techniques and avoid the use of Yau’s Theorem (see [EGZ11, GZ17]). Consider now
\[
\varphi_{\varepsilon} := (1 - \varepsilon)\varphi + \varepsilon \rho + C_2 \varepsilon + n \log(1 - \varepsilon),
\]
where $C_2$ is a positive constant to be specified hereafter, and $\varepsilon < 1/2$ is defined as above. Then $\varphi_{\varepsilon}$ is a bounded $\theta$-psh function, and a direct computation shows that
\[
\text{MA}_\theta(\varphi_{\varepsilon}) \geq (1 - \varepsilon)^n \text{MA}_\theta(\varphi) + \varepsilon^n \text{MA}_\theta(\rho)
\geq e^{\varphi + n \log(1 - \varepsilon)} f dV + e^{\varphi} (g - f)_+ dV.
\]
We choose $C_2 = -\inf_X \varphi$ so $\rho - \varphi \leq C_2$ and $\varphi_{\varepsilon} \leq \varphi + n \log(1 - \varepsilon) \leq \varphi$. Thus
\[
\text{MA}_\theta(\varphi_{\varepsilon}) \geq e^{\varphi_{\varepsilon}} (f + (g - f)_+) dV \geq e^{\varphi_{\varepsilon}} g dV.
\]
In other words, $\varphi_{\varepsilon}$ is a subsolution and $\psi$ is a supersolution to the equation $\text{MA}_\theta(\varphi) = e^\varphi g dV$. The comparison principle (Proposition 1.1) insures that $\varphi_{\varepsilon} \leq \psi$, hence
\[
\varphi - \psi = \varphi_{\varepsilon} - \psi + \varepsilon (\varphi - \rho) + C_2 \varepsilon - n \log(1 - \varepsilon)
\leq (C_1 - \inf_X \varphi + \sup_X \varphi - \inf_X \psi + 2n) \varepsilon
\leq (C_1 + \text{osc}_X \varphi - \inf_X \psi + 2n) \varepsilon \exp \left( \frac{\sup_X \varphi}{n} \right) \|(g - f)\|_p^{1/n}.
\]
Since $C_1$ only depends on $p, \theta, n$, the result follows.  \qed
2.2. **The proof of Theorem A.** Without loss of generality we normalize $f, g$ and $\theta$ so that $\int_X f dV = \int_X g dV = \int_X \theta^n = \int_X dV$. Let $\phi$ be the unique bounded $\theta$-plurisubharmonic function on $X$ normalized by $\sup_X \phi = 0$ such that $\text{MA}_\theta(\phi) = fdV$. The existence of $\phi$ follows from Kolodziej’s celebrated work [Kol98] and moreover, $-C \leq \phi \leq 0$, where $C$ is a uniform constant depending on $X, \theta, p, \|f\|_p$. Then $\phi$ is a subsolution while $\phi + C$ is a supersolution to the Monge-Ampère equation

$$\text{MA}_\theta(u) = e^u fdV.$$ 

It thus follows from the comparison principle (Proposition 1.1) that $\phi \leq \varphi \leq \phi + C$.

We thus obtain a uniform bound for $\varphi$. The same arguments give a uniform bound for $\psi$ as well. Theorem A follows therefore from Theorem 2.1.

**Remark 2.2.** In Theorem A, one can replace the $L^p$-norm of $f - g$ by the $L^1$-norm, at the cost of decreasing the exponent $1/n$ to $1/(n+\varepsilon)$ ($\varepsilon > 0$ arbitrarily small).

Namely under the assumptions of Theorem A, for any $\varepsilon > 0$, there exists a constant $C > 0$ which depends on $p, n, X, \theta, \varepsilon$ and a uniform bound on $\|f\|_p$ and $\|g\|_p$ such that

$$\|\varphi - \psi\|_\infty \leq C \|f - g\|_1^{1/(n+\varepsilon)}.$$

Indeed repeating the proof with an exponent $1 < r < p$ close to 1, we get a bound in terms of $\|f-g\|_r^{1/n}$. Then observe that if we write $r = (1-t) + tp$, by Hölder-interpolation inequality applied to $h := |f - g| \in L^p(X)$, we have

$$\int_X h^r dV \leq \left( \int_X h dV \right)^{1-t} \left( \int_X h^p \right)^t,$$

which implies that

$$\|h\|_r \leq \|h\|_1^{(1-t)/r} \|h\|_p^{tp/r}.$$ 

Since $t = (r-1)/(p-1)$ and $1-t = (p-r)/(p-1)$, we see that the exponent $(1-t)/r = (p-r)/r(p-1)$ is arbitrary close to 1 as $r \to 1$.

3. **Stability in the parabolic case**

3.1. **A parabolic comparison principle.** We establish in this section a maximum principle which is classical when the data are smooth. It has been obtained for continuous data in [EGZ16], we propose here a different approach which applies to our present setting:

**Theorem 3.1.** Fix $\varphi : [0,T] \times X \to \mathbb{R}$ a subsolution, $\psi : [0,T] \times X \to \mathbb{R}$ a supersolution to the parabolic equation (0.2), where $0 \leq f \in L^p(X)$ with $p > 1$. Then

$$\sup_X (\varphi - \psi) \leq \sup_X (\varphi_0 - \psi_0)_+.$$ 

Recall that $(\varphi_0 - \psi_0)_+ := \sup\{\varphi_0 - \psi_0, 0\}$. 

Proof. Fix $T' < T$, $0 < \varepsilon$. We first assume that $M_0 := \sup_X (\varphi_0 - \psi_0) \leq 0$ and we are going prove that $\varphi \leq \psi + 2\varepsilon t$ in $X_{T'}$.

Consider $w(t, x) := \varphi(t, x) - \psi(t, x) - 2\varepsilon t$. This function is upper semi-continuous on the compact space $[0, T'] \times X$. Hence $w$ attains a maximum at some point $(t_0, x_0) \in [0, T'] \times X$. We claim that $w(t_0, x_0) \leq 0$. Assume by contradiction that $w(t_0, x_0) > 0$, in particular $t_0 > 0$, and set

$$K := \{x \in X; w(t_0, x) = w(t_0, x_0)\}.$$  

The classical maximum principle insures that for all $x \in K$,

$$\partial_t \varphi(t_0, x) \geq \partial_t \psi(t_0, x) + 2\varepsilon.$$

By continuity of the partial derivatives in $(t, x)$, we can find an open neighborhood $D$ of $K$ such that for all $x \in D$

$$\partial_t \varphi(t_0, x) > \partial_t \psi(t_0, x) + \varepsilon.$$

Set $u := \varphi(t_0, \cdot)$ and $v := \psi(t_0, \cdot)$. Since $\varphi$ is a subsolution and $\psi$ is a supersolution to (0.2) we infer

$$(\omega_{t_0} + dd^c u)^n \geq e^{F(t_0, x, u(x)) - F(t_0, x, v(x)) + \varepsilon} (\omega_{t_0} + dd^c v)^n,$$

in the weak sense of measures in $D$. Recall that

- $u$ and $v$ are continuous on $D$,
- $F$ is non-decreasing in $r$,
- $u(x) > v(x) + \varepsilon t_0$ for any $x \in K$.

Shrinking $D$ if necessary, we can assume that the latter inequality is true in $D$. We thus get

$$(\omega_{t_0} + dd^c u)^n \geq e^\varepsilon (\omega_{t_0} + dd^c v)^n.$$

From this we see in particular that $D \neq X$, hence $\partial D \neq \emptyset$.

Consider now $\tilde{u} := u + \min_{\partial D} (v - u)$. Since $v \geq \tilde{u}$ on $\partial D$, Proposition 1.2 yields

$$\int_{\{v < \tilde{u}\} \cap D} e^\varepsilon (\omega_{t_0} + dd^c v)^n \leq \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + dd^c u)^n \leq \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + dd^c v)^n.$$

It then follows that $\tilde{u} \leq v$, almost everywhere in $D$ with respect to the measure $(\omega_{t_0} + dd^c v)^n$, hence everywhere in $D$ by the domination principle (see Proposition 1.2). In particular for all $x \in D$,

$$(3.1) \quad u(x) - v(x) + \min_{\partial D} (v - u) = \tilde{u}(x) - v(x) \leq 0,$$

Since $K \cap \partial D = \emptyset$, we infer $w(t_0, x) < w(t_0, x_0)$, for all $x \in \partial D$, i.e.

$$u(x) - v(x) < u(x_0) - v(x_0) \quad \text{for all } x \in \partial D,$$

contradicting (3.1). Altogether this shows that $t_0 = 0$, thus $\varphi \leq \psi + 2\varepsilon t$ in $X_{T'}$. Letting $\varepsilon \to 0$ and $T' \to T$ we obtain that $\varphi \leq \psi$ in $X_T$.

We finally get rid of the assumption $\varphi_0 \leq \psi_0$. If $M_0 := \sup_X (\varphi_0 - \psi_0) > 0$ then $\varphi - M_0$ is a subsolution of the same equation since $F$ is non decreasing in the last variable. Hence $\varphi - M_0 \leq \psi$ in $X_T$. This proves the required inequality. \hfill \Box
Remark 3.2. We note for later works that the above proof only requires $t \mapsto \varphi(t, \cdot), \psi(t, \cdot)$ to be $C^1$ in $]0, T[ \times X$. This should be useful in analyzing the smoothing properties of complex Monge-Ampère flows at time zero.

3.2. A parabolic semi-stability theorem. We now establish a technical comparison principle which is a key step in the proof of Theorem B. The proof of this result does not require the smoothness assumption on $F, G, f, g$.

We assume in this subsection that
- $F, G : \hat{X}_T := [0, T[ \times X \times \mathbb{R} \to \mathbb{R}$ are continuous;
- $F, G$ are non decreasing in the last variable;
- $F, G$ satisfy condition (0.3) with the same constant $L > 0$.
- $0 \leq f, g \in L^p(X)$ with $p > 1$.

Theorem 3.3. Assume that $\varphi : [0, T[ \times X \to \mathbb{R}$ is a subsolution to the parabolic equation (0.2) with data $(F, f)$ and $\psi : [0, T[ \times X \to \mathbb{R}$ is a supersolution to the parabolic equation (0.2) with data $(G, g)$. Then

$$\sup_{X_T} (\varphi - \psi) \leq \sup_{X_T} (\varphi_0 - \psi_0) + T \sup_{\hat{X}_T} (G - F) + A \|(g - f)\|_p^{1/n},$$

where $A > 0$ depends on $X, \theta, n, p$ and a uniform bound on $\varphi, \varphi, \psi$, and $\sup_{X_T} G(t, x, \sup_{X_T} \varphi)$.

Remark 3.4. In the second term of the estimate above one can replace $\sup_{X_T} (G - F)$ by $\sup_{[0, T[ \times X \times I} (G - F)$, where $I = [\inf_{X_T} \varphi, \sup_{X_T} \varphi]$ is a compact interval in $\mathbb{R}$.

Proof. We first assume that $\|(g - f)\|_p > 0$. Since $\int_X dV = \int_X \theta^n = 1$, it follows from [Kol98] that there exists $\rho \in \text{PSH}(X, \theta) \cap C^0(X)$ such that

$$\left(\theta + dd^c \rho\right)^n = \left(a + \frac{(g - f)\|_p}{(g - f)\|_p} \right) dV$$

normalized by $\max_X \rho = 0$, where $a \geq 0$ is a normalizing constant given by

$$a := 1 - \frac{\|(g - f)\|_p}{(g - f)\|_p} \in [0, 1].$$

We moreover have a uniform bound on $\rho$ which only depends on the $L^p$ norm of the density of $(\theta + dd^c \rho)^n$ which is here bounded from above by 2,

$$\|\rho\|_\infty \leq C_0(a + 1) \leq 2C_0,$$

where $C_0 > 0$ is a uniform constant depending only on $(X, \theta, p)$.

Fix $B, M > 0$. For $0 < \delta < 1$ and $(t, x) \in X_T$ we set

$$\varphi_\delta(t, x) := (1 - \delta) \varphi(t, x) + \delta \rho + n \log(1 - \delta) - B\delta t - Mt.$$

The plan is to choose $B, M > 0$ in such a way that $\varphi_\delta$ be a subsolution for the parabolic equation (0.2) with data $(G, g)$. The conclusion will then follow from the comparison principle (Theorem 3.1).
Observe that for \( t \in [0, T] \) fixed, \( \varphi_\delta(t, \cdot) \) is \( \omega_t \)-plurisubharmonic in \( X \) and
\[
(\omega_t + dd^c \varphi_\delta(t, \cdot))^n \geq (1 - \delta)^n(\omega_t + dd^c \varphi_t)^n + \delta^n(\theta + dd^c \rho)^n.
\]
Using that \( \varphi \) is a subsolution to (0.2) with density \( f \), we infer
\[
(3.4) \quad (\omega_t + dd^c \varphi_\delta(t, \cdot))^n \geq e^{\varphi + F(t, \cdot) + n \log(1 - \delta)} f dV + \delta^n \frac{(g - f)_+}{\|g - f\|_p} dV.
\]
Set \( m_0 := \inf_{X_T} \varphi_\delta \), \( m_1 := \inf_{X_T} \varphi \) and choose \( M := \sup_{X_T} (G - F)_+ \).
Noting that \( \varphi \geq \varphi_\delta + \delta \varphi \) and recalling that \( G \) is non-decreasing in the last variable, we obtain
\[
\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta)
\geq \dot{\varphi}_\delta(t, x) + \delta \dot{\varphi}(t, x) + G(t, x, \varphi_\delta(t, x)) - M + n \log(1 - \delta) + B \delta + M
\geq \dot{\varphi}_\delta(t, x) + \delta \dot{\varphi}(t, x) + G(t, x, \varphi_\delta(t, x) + \delta \varphi(t, x)) + n \log(1 - \delta) + B \delta
\geq \dot{\varphi}_\delta(t, x) + \delta m_1 + G(t, x, \varphi_\delta(t, x) + \delta m_0) + n \log(1 - \delta) + B \delta.
\]
The Lipschitz condition (0.3) yields
\[
\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta)
\geq \dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x)) + B \delta - L \delta m_0 + \delta m_1 + n \log(1 - \delta).
\]
Using the elementary inequality \( \log(1 - \delta) \geq -2(\log 2)\delta \) for \( 0 < \delta \leq 1/2 \), it follows that for \( 0 < \delta \leq 1/2 \),
\[
B \delta - L \delta m_0 + \delta m_1 + n \log(1 - \delta) \geq (B - L m_0 + m_1 - 2n \log 2)\delta.
\]
We now choose \( B := L m_0 - m_1 + 2n \log 2 \) so that
\[
\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta) \geq \dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x)),
\]
which, together with (3.4), yields
\[
(3.5) \quad (\omega_t + dd^c \varphi_\delta(t, \cdot))^n \geq e^{\varphi_\delta(t, \cdot) + G(t, \cdot, \varphi_\delta(t, \cdot))} f dV + \delta^n \frac{(g - f)_+}{\|g - f\|_p} dV.
\]
On the other hand, if we set
\[
M_1 := \sup_{X_T} \dot{\varphi}, \quad M_0 := \sup_{X_T} \varphi \quad \text{and} \quad N := \sup_{X_T} G(t, x, M_0),
\]
then the properties of \( G \) insure
\[
\dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x)) \leq (1 - \delta) \sup_{X_T} \dot{\varphi} + \sup_{X_T} G(t, x, (1 - \delta) \varphi(t, x))
\leq (1 - \delta) M_1 + \sup_{X_T} G(t, x, (1 - \delta) M_0)
\leq (1 - \delta) M_1 + M_0 L \delta + N \leq N + \max\{L M_0, M_1\}
\]
Using (3.5) we conclude that for \( 0 < \delta < 1/2 \),
\[
(3.6) \quad (\omega_t + dd^c \varphi_\delta(t, \cdot))^n \geq e^{\varphi_\delta(t, \cdot) + G(t, \cdot, \varphi_\delta(t, \cdot))} (f + \delta^n e^{-M_2} \frac{(g - f)_+}{\|g - f\|_p}) dV,
\]
where \( M_2 := N + \max\{L M_0, M_1\} \).
To conclude that \( \varphi_\delta \) is a subsolution, we finally set

\[
\delta := \| (g - f)_+ \|_p^{1/n} e^{M_2/n}.
\]

Assume first that \( \| (g - f)_+ \|_p \leq 2^{-n} e^{M_2} \) so that \( \delta \leq 1/2 \). It follows from (3.6) that

\[
(\omega_t + dd^c \varphi_\delta(t,\cdot))^{n/\omega} \geq e^{\psi_\delta(t,\cdot) + G(t,x,\varphi_\delta(t,\cdot))} (f + (g - f)_+)dV
\]

hence \( \varphi_\delta \) is a subsolution to (0.2) for the data \((G, g)\) in \( D \). The comparison principle (Theorem 3.1) insures that for all \((t, x) \in X_T\),

\[
\varphi_\delta(t, x) - \psi(t, x) \leq \max_X (\varphi_\delta(0, \cdot) - \psi(0, \cdot)\).
\]

Taking into account the estimates (3.3) and (3.7), we get

\[
\varphi(t, x) - \psi(t, x) \leq \max_X (\varphi(0, \cdot) - \psi(0, \cdot) + TM + A_1 \| (g - f)_+ \|_p^{1/n},
\]

where

\[
A_1 := (M_0 + 2C_0 + 2n \log 2 + BT)e^{M_2/n}.
\]

When \( \| (g - f)_+ \|_p > 2^{-n} e^{M_2} \), we can choose a constant \( A_2 > 0 \) so that

\[
\varphi(t, x) - \psi(t, x) \leq \max_X (\varphi_0 - \psi_0)_+ + A_2 2^{-n} e^{-M_2}.
\]

We eventually take \( A = \max\{A_1, A_2\} \).

Assume finally that \( \| (g - f)_+ \|_p = 0 \) which means that \( g \leq f \) almost everywhere in \( X \). In this case we solve the equation (3.2) with the right hand side equal to \( dV \) and repeat the same arguments with an arbitrary \( \delta > 0 \). The conclusion follows by letting \( \delta \to 0 \).

\[\square\]

3.3. Proof of Theorem B. The proof of Theorem B goes by symmetrizing the roles of \( \varphi \) and \( \psi \), and establishing uniform bounds on \( \varphi, \psi, \varphi, \psi \) depending on uniform bounds for \( \varphi_0, \psi_0, \varphi_0, \psi_0 \) and \( \|f\|_p, \|g\|_p \).

3.3.1. Bounds on \( \varphi \). By assumption, we can fix \( a, A > 0 \) such that

\[
a \theta \leq \omega_t \leq A \theta, \ \forall (t, x) \in X_T.
\]

Let \( \rho \) be the unique bounded normalized \( \theta \)-psh function \( \sup_X \rho = 0 \) such that \( MA_\theta(\rho) = C_0 fdV \), where \( C_0 > 0 \) is a uniform normalization constant. Then \( \rho \) is bounded by a constant depending only on the \( L^p \) norm of \( f \) and on \( \theta, p \). We set

\[
C_1 := \sup_X (a \rho - \varphi_0) \text{ and } C_2 := \sup_X (\varphi_0 - A \rho).
\]

We next introduce the following uniform constants

\[
C_3 := \sup_{(t, x) \in [0,T] \times X} F(t, x, \varphi_0(t, x)) \text{ and } C_4 := \inf_{(t, x) \in [0,T] \times X} F(t, x, \varphi_0(t, x)).
\]

A direct computation shows that the function defined on \( X_T \) by

\[
u(t, x) := a \rho - C_1 - \max(C_3 - n \log a - \log C_0, 0)t\]
is a subsolution to the parabolic equation (0.2) with data \((f, F)\). Indeed, for fixed \(t \in [0, T]\), the Monge-Ampère measure of \(u_t\) can be estimated as

\[
(\omega_t + dd^c u_t)^n \geq a^n(\theta + dd^c \rho)^n = a^nC_0 fdV.
\]

Since \(F\) is non-decreasing in the last variable and \(\varphi_0 \geq u_t\) we obtain \(C_3 \geq F(t, x, \varphi_0) \geq F(t, x, u_t)\). Thus

\[
e^{\tilde{u} + F(t, ; u_t)} fdV \leq e^{-\max(C_3 - n \log a - \log C_0, 0) + C_3} fdV \leq a^nC_0 fdV \leq (\omega_t + dd^c u_t)^n.
\]

A similar computation shows that the function defined on \(X_T\) by

\[
v(t, x) := A\rho + C_2 + \max(-C_4 + n \log A + \log C_0, 0) t
\]

is a supersolution to the parabolic equation

\[
(A\theta + dd^c v_t)^n = e^{\tilde{u} + F(t, ; v_t)} fdV.
\]

Since \(A\theta \geq \omega_t\) we see that \(\varphi_t\) is a subsolution to the equation (3.8). The parabolic comparison principle (Theorem 3.1) therefore yields

\[u \leq \varphi \leq v,\]

in \(X_T\).

3.3.2. Bounds on \(\varphi\). We now provide a uniform bound on \(\varphi_t\), assuming all data are smooth. We only outline the proof since the arguments are classical. The previous subsection has provided a uniform bound

\[-B_0 \leq \varphi(t, x) \leq B_0, \forall (t, x) \in [0, T] \times X.
\]

Since \(\omega_t\) is smooth in \(t\) and \(\omega_t\) is uniformly Kähler we can fix a positive constant \(B_1\) such that

\[-B_1\omega_t \leq \omega_t \leq B_1\omega_t.
\]

Up to enlarging \(B_1\) we can further assume that

\[-B_1 \leq \partial_r F(t, x, r) \leq B_1, \forall (t, x, r) \in [0, T] \times X \times [-B_0, B_0].
\]

Set

\[
\Delta_t(h) = \Delta_{\omega_t + dd^c \varphi_t}(h) = n \frac{dd^c h \wedge (\omega_t + dd^c \varphi_t)^{n-1}}{(\omega_t + dd^c \varphi_t)^n},
\]

and

\[
\operatorname{Tr}_t(\eta) := n \frac{\eta \wedge (\omega_t + dd^c \varphi_t)^{n-1}}{(\omega_t + dd^c \varphi_t)^n}.
\]

A straightforward computation yields

\[
\varphi_t = \Delta_t(\varphi_t) + \operatorname{Tr}_t(\omega_t) - \partial_t F(t, x, \varphi_t) - \varphi_t \partial_r F(t, x, \varphi_t) \\
\leq \Delta_t(\varphi_t) - B_1 \Delta_t(\varphi_t) + C - \varphi_t \partial_r F(t, x, \varphi_t),
\]

using that \(\varphi_t\), hence \(\partial_r F(t, x, \varphi_t)\), is uniformly bounded on \(X_T\), and that

\[
\operatorname{Tr}_t(\omega_t) \leq B_1 \operatorname{Tr}_t(\omega_t) = B_1 n - B_1 \Delta_t(\varphi_t).
\]

Consider

\[
H(t, x) := \varphi_t(x) - B_1 \varphi_t(x) - (C + 1) t.
\]
It follows from the above computation that
\[
\left( \frac{\partial}{\partial t} - \Delta_t \right) H \leq -1 - [B_1 + \partial_t F(t, x, \varphi_t)]\dot{\varphi}_t.
\]

If \( H \) reaches its maximum at time zero, then
\[
H \leq \sup_X \dot{\varphi}_0 - B_1 \inf_X \varphi_0,
\]
hence \( \dot{\varphi}_t \leq C(T, B_0, B_1) + \sup_X \dot{\varphi}_0 \). If \( H \) reaches its maximum at \((t_0, x_0)\) with \( t_0 > 0 \), we obtain at \((t_0, x_0)\),
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta_t \right) H \leq -1 - [B_1 + \partial_t F(t, x, \varphi_t)]\dot{\varphi}_t,
\]
hence \( \dot{\varphi}_t(t_0, x_0) \leq 0 \) since \( [B_1 + \partial_t F(t, x, \varphi_t)] \geq 0 \). Therefore
\[
H_{\max} = H(t_0, x_0) \leq -B_1 \inf_X \varphi_{t_0} \leq B_0 B_1,
\]
and we obtain an appropriate bound from above for \( \dot{\varphi}_t \).

The proof for the lower bound goes along similar lines, considering
\[
G = \dot{\varphi}_t(x) + B_1 \varphi_t(x) + (C' + 1)t.
\]

4. Concluding remarks

4.1. Varying the reference forms. It is certainly interesting to study the
stability properties when the reference forms \( \theta, \omega_t \) are varying. For simplicity
we address this issue here only in the elliptic case.

**Theorem 4.1.** Fix \( \theta, \omega \) Kähler forms. Fix \( p > 1 \) and \( 0 \leq f, g \in L^p(X, dV) \).
If \( \varphi \) (resp. \( \psi \)) is a bounded \( \theta \)-plurisubharmonic (resp. \( \omega \)-plurisubharmonic)
function on \( X \) such that
\[
\text{MA}_\theta(\varphi) = e^\varphi f dV \quad \text{and} \quad \text{MA}_\omega(\psi) = e^\psi g dV,
\]
then
\[
\|\varphi - \psi\|_\infty \leq C \left\{ \|f - g\|_p^{1/n} + d(\omega, \theta) \right\},
\]
where \( C > 0 \) depends on \( p, n, X, \omega, dV \) and uniform bounds on \( \|f\|_p, \|g\|_p \).

We use here the following distance on positive forms,
\[
d(\omega, \theta) := \inf\{t > 0 ; e^{-t} \omega \leq \theta \leq e^t \omega\}.
\]

**Proof.** Set
\[
c = \inf\{t > 0 ; (1 - t) \omega \leq \theta \leq (1 + t) \omega\}.
\]
Adjusting the constant we can assume that \( c \leq 1/2 \) and \( c \approx d(\omega, \theta) \). Now
\[
\psi_c := (1 - c) \psi + n \log(1 - c) + c \inf_X \psi
\]
is a \( \theta \)-psh function whose Monge-Ampère measure can be estimated as
\[
\text{MA}_\theta(\psi_c) \geq (1 - c)^n e^\varphi g dV \geq e^{\psi_c} g dV.
\]

It thus follows from Theorem 2.1 that
\[
\psi_c \leq \varphi + C \|f - g\|_p^{1/n},
\]
for a uniform constant $C$. Note that the uniform norm of $\psi_c$ is uniformly controlled by $\|\psi\|$ because $c \leq 1/2$. From this and the definition of $\psi_c$ we obtain

$$\psi \leq \varphi + C'(\|f - g\|_p^n + c),$$

where $C'$ is a uniform constant. Exchanging the roles of $\varphi$ and $\psi$ yields the conclusion. \qed

4.2. Big classes. The ideas we have developed so far can also be applied to the more general setting of cohomology classes that are merely big rather than Kähler. We briefly explain the set up for the elliptic stability.

We assume that $\theta$ is a smooth form representing a big cohomology class. A $\theta$-psh function $\varphi$ is no longer bounded on $X$, but it can have minimal singularities. The function $V_{\theta} = \sup\{u \in \mathrm{PSH}(X, \theta) ; u \leq 0\}$ is an example of $\theta$-psh with minimal singularities. Any other $\theta$-psh function $\varphi$ with minimal singularities satisfies $\|\varphi - V_{\theta}\|_{L^\infty(X)} < +\infty$.

The pluripotential theory in big cohomology classes has been developed in [BEGZ10]. In short, the Bedford-Taylor theory can be developed, replacing $X$ by the ample locus of $\theta$, a Zariski open subset in which $V_{\theta}$ is locally bounded.

The a priori estimate of Kołodziej can be extended, as well as Theorem A. It suffices indeed to establish the following:

**Theorem 4.2.** Fix $p > 1$ and assume that $0 \leq f, g \in L^p(X, dV)$, and $\varphi, \psi$ are $\theta$-psh functions on $X$ with minimal singularities such that

$$\operatorname{MA}_\theta(\varphi) \geq e^{\varphi} f dV ; \quad \operatorname{MA}_\theta(\psi) \leq e^{\psi} g dV.$$

Then there is a constant $C > 0$ depending on $p, n, X, \theta$ such that

$$\varphi \leq \psi + \left(C + 2 \sup_X |V_{\theta} - \varphi| + 2 \max_X (V_{\theta} - \psi, 0) \right) \exp \left(\frac{\sup_X \varphi}{n}\right) \|f - g\|_p^{1/n}.$$

**Proof.** Set $\varepsilon := e^{\sup_X \varphi/\|f - g\|_p^{1/n}}$. If $\varepsilon \geq 1/2$ then

$$\sup_X (\varphi - \psi) \leq \left(\sup_X |\varphi - V_{\theta}| + \sup_X (V_{\theta} - \psi, 0) \right) \leq 2 \left(\sup_X |\varphi - V_{\theta}| + \sup_X (V_{\theta} - \psi, 0) \right) \exp \left(\frac{\sup_X \varphi}{n}\right) \|f - g\|_p^{1/n},$$

as desired. So we can assume that $\varepsilon < 1/2$. Hölder inequality yields

$$\int_X \frac{|f - g|}{\|f - g\|_p} dV \leq 1.$$

Let $\rho$ be the unique $\theta$-psh function with minimal singularities such that

$$\operatorname{MA}_\theta(\rho) = h dV := \left(a + \frac{|f - g|}{\|f - g\|_p} \right) dV, \quad \sup_X \rho = 0,$$
where \( a \geq 0 \) is a normalization constant to insure that \( \int_X h dV = \int_X dV \).

Since \( \| h \|_p \leq 2 \), it follows from [BEGZ10, Theorem 4.1] that
\[
-C_1 \leq \rho - V_\theta \leq 0,
\]
where \( C_1 \) only depends on \( p, \theta, n \).

Consider now \( \varphi_\varepsilon := (1 - \varepsilon)\varphi + \varepsilon \rho - C_2 \varepsilon + n \log(1 - \varepsilon) \),
where \( C_2 \) is a positive constant to be specified hereafter. Then \( \varphi_\varepsilon \) is a \( \theta \)-psh function with minimal singularities, and a direct computation shows that
\[
\text{MA}_\theta(\varphi_\varepsilon) \geq (1 - \varepsilon)^n \text{MA}_\theta(\varphi) + \varepsilon^n \text{MA}_\theta(\rho) \geq e^{\varepsilon + n \log(1 - \varepsilon)} f dV + e^\varphi |f - g| dV.
\]
If we choose \( C_2 = \sup_X (V_\theta - \varphi) \) then \( \rho - \varphi \leq C_2 \) and \( \varphi_\varepsilon \leq \varphi + n \log(1 - \varepsilon) \leq \varphi \).

So we can continue the above estimate to arrive at
\[
\text{MA}_\theta(\varphi_\varepsilon) \geq e^{\varphi_\varepsilon}(f + |f - g|) dV \geq e^{\varphi_\varepsilon} g dV.
\]
It follows from (4.1) that \( \varphi_\varepsilon \) is a subsolution hereafter and \( \psi \) is a supersolution of the equation \( \text{MA}_\theta(\varphi) = e^\varphi g dV \). The comparison principle [BEGZ10, Proposition 6.3] insures that \( \varphi_\varepsilon \leq \psi \), hence
\[
\varphi - \psi = \varphi_\varepsilon - \psi + \varepsilon(\varphi - \rho) + C_2 \varepsilon - n \log(1 - \varepsilon) \leq (C_1 + C_2 + \sup_X (\varphi - V_\theta) + 2n) \varepsilon
\]
\[
= (C_1 + \text{osc}_X (\varphi - V_\theta) + 2n) \exp \left( \frac{\sup_X \varphi^n}{n} \right) \| f - g \|_p^{1/n}.
\]

The result follows since \( C_1 \) only depends on \( p, \theta, n \). \( \square \)

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