The Effect of Rotation on Fingering Convection in Stellar Interiors

S. Sengupta© and P. Garaud

Department of Applied Mathematics, Baskin School of Engineering, University of California Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, USA
sutirtha@ucsc.edu

Received 2018 April 11; revised 2018 May 25; accepted 2018 June 7; published 2018 July 31

Abstract

We study the effects of rotation on the growth and saturation of the double-diffusive fingering (thermohaline) instability at low Prandtl number. Using direct numerical simulations, we estimate the compositional transport rates as a function of the relevant nondimensional parameters—the Rossby number, inversely proportional to the rotation rate, and the density ratio that measures the relative thermal and compositional stratifications. Within our explored range of parameters, we generally find rotation to have little effect on vertical transport. However, we also present one exceptional case where a cyclonic large-scale vortex (LSV) is observed at low density ratio and fairly low Rossby number. The LSV leads to significant enhancement in the fingering transport rates by concentrating compositionally dense downflows at its core. We argue that the formation of such LSVs could be relevant to solving the missing-mixing problem in RGB stars.

Key words: hydrodynamics – instabilities – stars: interiors – stars: rotation – stars: abundances

1. Introduction

Over the past decade or so, there has been a resurgence in interest about the role of fingering convection as a mechanism for transport of chemical species in the radiative zones of a variety of objects, ranging from accreting main-sequence stars and white dwarfs in binary systems (Marks et al. 1997; Marks & Sarna 1998; Theado & Vauclair 2010; Vauclair & Théado 2012; Stanclifte et al. 2007; Deal et al. 2013; Denissenkov et al. 2013) to exoplanet host stars (Vauclair 2004; Garaud 2011; Vauclair & Théado 2012), as well as in the interiors of more evolved low-mass red-giant branch (RGB) stars (Charbonnel & Zahn 2007b; Denissenkov & Pinsonneault 2008; Wachlin et al. 2014) and possibly also in planetary atmospheres due to chemical reactions (Tremblin et al. 2015, 2016; although see Leconte 2018). Recent numerical simulations of fingering convection by Denissenkov (2010), Denissenkov & Merryfield (2011), Traxler et al. (2011b), and Brown et al. (2013) (see the review by Garaud 2018) have consistently shown that the typical values of mixing rates are two orders of magnitude below those required to match observed abundance patterns in RGB stars above the so-called “luminosity bump” (Gratton et al. 2000; Charbonnel & Zahn 2007b). The only way to reconcile theory and observations is to invoke the existence of some previously unaccounted for mechanism that could somehow significantly enhance mixing by fingering convection in these stars (see, e.g., Medrano et al. 2014 or Garaud et al. 2015 for some first attempts at cracking the problem).

The obvious candidates for such mechanisms in stars are rotation, shear, and magnetic fields. While the latter two remain to be explored, the effect of rotation on oscillatory double-diffusive convection (ODDC) has recently been studied in Moll & Garaud (2017) using direct numerical simulations (DNSs) with the code PADDI (Traxler et al. 2011b; Stellmach et al. 2011). In this paper, we apply the framework of Moll & Garaud (2017) to the fingering regime and attempt to quantify the effect of rotation on the growth and development of fingering instabilities in parameter regimes relevant for stars. We begin by presenting the model setup (Section 2) followed by a linear stability analysis of the fingering instability in the presence of rotation (Section 3) before quantifying its effect in stellar interiors (Section 4) with the help of DNSs (Section 5). We conclude in Sections 6 by discussing the relevance of our findings for RGB stars.

2. The Model

In this work, we use the Boussinesq approximation (Boussinesq 1903; Spiegel & Veronis 1960) in a Cartesian setup that assumes constant background temperature and composition gradients over the height of the computational domain, and a linearized equation of state given by

\[ \frac{\rho}{\rho_0} = -\alpha \tilde{T} + \beta \tilde{\rho}, \]

where \( \rho, \tilde{T}, \) and \( \tilde{\rho} \) are the perturbations to the background density, temperature, and composition, respectively, and \( \rho_0 \) is the mean density of the fluid in the region considered. The coefficients \( \alpha \) and \( \beta \) are defined as

\[ \alpha = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial T}, \quad \beta = \frac{1}{\rho_0} \frac{\partial \rho}{\partial \mu}, \]

where \( \rho \) denotes pressure. We assume a constant background rotation defined by the angular velocity vector \( \Omega = \Omega e_\theta \), with \( e_\theta \) being the unit vector in the direction of \( \Omega \):

\[ e_\theta = (0, \sin \theta, \cos \theta), \]

where \( \theta \) is the angle between the rotation axis and the \( z \)-axis of our domain, which is aligned with gravity.

Following Traxler et al. (2011a), we use the following units for length \( [l] \), time \( [t] \), temperature \( [T] \), and chemical composition \( [\mu] \):

\[ [l] = d = \left( \frac{\kappa_T}{\alpha g |T_\infty - T_{\infty}^\text{ad}|} \right)^{1/3}, \quad [t] = \frac{d^2}{\kappa_T}, \]
\[ [T] = d |T_\infty - T_{\infty}^\text{ad}|, \quad [\mu] = \frac{\alpha}{\beta} d |T_\infty - T_{\infty}^\text{ad}|, \]
where \( g \) is the local acceleration due to gravity, \( \nu \) is the viscosity of the medium, \( \kappa_T \) is the thermal diffusivity, \( T_0 \) is the background temperature gradient with respect to position \( z \), and \( T_{0ad} = -\frac{\partial}{\partial z} T_{0ad} \) is the corresponding adiabatic temperature gradient, where \( c_p \) is the specific heat at constant pressure. Using this choice of units, we can write the nondimensional form of the Navier–Stokes equations for the velocity field \( \mathbf{u} = (u, v, w) \) as follows:

\[
\frac{1}{\operatorname{Pr}} \left[ \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \sqrt{\operatorname{Ta}^* (\theta_0^* \times \mathbf{u})} \right] \\
= -\nabla \bar{p} + \left( \bar{T} - \bar{p} \right) e_c + \nabla^2 \mathbf{u},
\]

(4)

\[
\frac{\partial \bar{T}}{\partial t} + \mathbf{u} \cdot \nabla \bar{T} + w = \nabla^2 \bar{T},
\]

(5)

\[
\frac{\partial \bar{\mu}}{\partial t} + \mathbf{u} \cdot \nabla \bar{\mu} + \frac{w}{\mathcal{R}_0} = \tau \nabla^2 \bar{\mu},
\]

(6)

\[
\nabla \cdot \mathbf{u} = 0,
\]

(7)

with four relevant nondimensional parameters being the Prandtl number \( \operatorname{Pr} \), the diffusivity ratio \( \tau \), the density ratio \( \mathcal{R}_0 \), and the finger-based Taylor number \( \operatorname{Ta}^* \) defined as (Moll & Garaud 2017):

\[
\operatorname{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_T}{\kappa_T}, \quad \mathcal{R}_0 = \frac{\alpha [T_0 - T_{0ad}]}{\beta [\mu_0]}, \quad \operatorname{Ta}^* = \frac{4 \pi^2 \mu_0}{\kappa_T^2}.
\]

(8)

As shown by Garaud (2018), the density ratio \( \mathcal{R}_0 \) measures the effective stratification of the system, with \( \mathcal{R}_0 = 1 \) corresponding to the limit of overturning convection. In nonrotating stars, a region is unstable to basic fingering when

\[
1 < \mathcal{R}_0 < \frac{1}{\tau}.
\]

(9)

The effect of rotation in turn is described by the finger-based Taylor number \( \operatorname{Ta}^* \) (see Section 4 for more detail on the significance of \( \operatorname{Ta}^* \)). In what follows, we assume that the computational domain is triply periodic, which greatly simplifies both the linear stability analysis (Section 3) and the numerics (Section 5 and beyond).

### 3. Linear Stability Analysis

We linearize the set of governing equations (Equations (4)–(7)) and use the ansatz:

\[
q(x, y, z, t) = \hat{q} e^{i (k_x x + m y + \ell z) + \lambda t},
\]

(10)

for \( q = (u, \bar{T}, \bar{\mu}) \). After some algebra, we obtain a quartic polynomial equation for the growth rate \( \lambda \):

\[
(\lambda + \operatorname{Pr} \kappa_T^2 \lambda + \kappa_T^2 \lambda + \kappa_T^2) + \frac{k_h^2 \operatorname{Pr} (\lambda + \kappa_T^2) [((\lambda + \kappa_T^2) - R_{\mu}^{-1} (\lambda + \kappa_T^2)]}
\]

\[
+ \operatorname{Ta}^* (m \sin \theta + k \cos \theta)^2 (\lambda + \tau \kappa_T^2) (\lambda + \kappa_T^2) = 0,
\]

(11)

where \( K = \sqrt{k_h^2 + k^2} \) is the total wavenumber and \( k_h = \sqrt{l^2 + m^2} \) is the horizontal wavenumber. This is almost identical to the growth rate equation obtained in the ODDC case (Equation (16) in Moll & Garaud 2017) except for the sign in front of the second term (namely, the term proportional to \( \frac{k_h^2}{K^2} \) which is positive in the fingering case, and negative in the ODDC case.

#### 3.1. Regime of Instability

It can be shown that the fastest-growing modes (i.e., modes with largest \( \operatorname{Re}(\lambda) \) satisfying Equation (11)) have \( k = 0 \) and \( m = 0 \) (Moll & Garaud 2017). Thus, these modes remain unaffected by rotation (since the rotation term in Equation (11) drops out for \( k = m = 0 \)), and the range of density ratios for which fingering takes place is unchanged:

\[
1 < \mathcal{R}_0 < \frac{1}{\tau},
\]

(12)

where the lower limit of \( \mathcal{R}_0 = 1 \) corresponds to the system being unstable to overturning convection (Ledoux unstable), while the upper limit \( \mathcal{R}_0 = 1 / \tau \) corresponds to marginal stability to fingering convection.

#### 3.2. Fastest-growing Modes

Equation (11) can be solved numerically for the growth rate \( \lambda \) of the instability. The results for the rotating case are shown in Figure 1 for \( \operatorname{Ta}^* = 0.01 \) and \( \operatorname{Ta}^* = 1 \), for \( \tau = \operatorname{Pr} = 0.1 \), \( \mathcal{R}_0 = 1.25 \). As discussed above, we see that for \( \theta = 0 \), the fastest-growing modes are those with \( k = 0 \), as is found for the nonrotating fingering unstable modes (Brown et al. 2013).

These so-called “elevator” modes are unaffected by rotation as can be seen in Figure 1 and by direct inspection of Equation (11) since the last term (containing \( \operatorname{Ta}^* \)) vanishes for \( k = 0 \) and \( \theta = 0 \). The modes with \( k \neq 0 \) by contrast are suppressed by rotation in the sense that the higher \( k \) modes grow more slowly or become stable with increasing \( \operatorname{Ta}^* \).

While linear theory helps to determine the linearly unstable region of parameter space, quantifying mixing by fingering convection can only be done using nonlinear arguments. In the nonrotating case, Radko & Smith (2012) and Brown et al. (2013) showed that the nonlinear saturation of the fingering instability is due to the shear that inevitably develops between upflowing and downflowing fingers. By matching the growth rate of the fingers to the growth rate of the emerging shear instability, they successfully predicted the amplitude of the vertical velocity at saturation, which they then used to model the turbulent mixing coefficient.

Since rotation has a tendency to stabilize a system against motion perpendicular to the rotation axis, we may expect it to stabilize the fingers against the shear instabilities that cause their nonlinear saturation. In that case, the vertical velocity within the fingers might be permitted to grow to much larger amplitude before the secondary shear instabilities develop, which could in turn lead to an enhancement in the efficiency of vertical transport in rotating fingering convection compared with the nonrotating case. This intuitive picture, and its obvious potential for explaining the “missing mixing” in RGB stars, motivated us to run DNSs of rotating fingering convection. In what follows, we first attempt to estimate when the effects of
rotation may become important, and then present nonlinear DNSs of rotating fingering convection to test these ideas.

4. Estimating When Rotation Is Important in Stellar Interiors

While rotation does not have any effect on the growth rate of the fastest-growing fingering modes, it is very likely to have one on their nonlinear saturation (see our discussion above and the findings of Moll & Garaud 2017 for the effect of rotation on the nonlinear saturation of the ODDC instability). A commonly used measure of the relative strength of inertial forces \((\mathbf{u} \cdot \nabla \mathbf{u})\) to Coriolis forces \((2\Omega \times \mathbf{u})\) is the Rossby number, defined as

\[
Ro = \frac{U}{2\Omega L},
\]

(13)

where \(U\) and \(L\) are typical dimensional velocities and length-scales associated with the fluid motions in consideration. In turbulent flows, the effect of rotation is therefore negligible if \(Ro \gg 1\), but dominant if \(Ro \ll 1\). For moderate and high \(Pr\) fingering convection and ODDC, since \(U \sim \frac{\kappa^*}{\tau} \) and \(L \sim \tau\) (Traxler et al. 2011b; Mirouh et al. 2012; Wood et al. 2013; Moll et al. 2016), one may estimate \(Ro\) as

\[
Ro \sim \frac{\kappa^*}{2\Omega \tau} \sim \frac{1}{\sqrt{Ta^*}},
\]

(14)

which would imply that \(Ta^* \gg 1\) double-diffusive systems should be strongly rotationally constrained, while \(Ta^* \ll 1\) systems should not feel the effect of rotation at all. This was verified to be true for \(Pr \sim 1\) down to \(Pr \sim 0.01\) for ODDC (see Moll & Garaud 2017), for instance.

However, in stellar interiors, the Prandtl number is asymptotically small, taking values ranging from \(10^{-6}\) down to \(10^{-9}\). In this regime, the vertical velocities within individual fingers do not scale as above, but instead are expected to scale with \(Pr\) (Brown et al. 2013, and see below). Hence, the effective Rossby number of rotating fingering convection is predicted to be significantly different from the estimate given in (14).

Indeed, for the parameter regime appropriate for stellar interiors, the dimensional quantities \(U\) and \(L\) can be estimated using the results of Brown et al. (2013) as

\[
U \sim \lambda_{\text{max}} L \frac{\kappa^*}{\tau^2} \quad \text{and} \quad L \sim \frac{2\pi}{\lambda_{\text{max}}} \frac{d}{\tau},
\]

(15)

The quantities \(\lambda_{\text{max}}\) and \(\tau_{\text{max}}\) are the nondimensional growth rate and the associated nondimensional horizontal wavenumber of the fastest-growing linearly unstable mode, found numerically by maximizing the solutions of (11) for \(Ta^* = 0\), \(k = 0\) and \(\theta = 0\) against all possible values of \(l\). We can test this scaling by comparing, for instance, the rms nondimensional vertical velocity \(w_{\text{rms}}\) extracted by reanalyzing nonrotating DNSs at moderately low values of \(Pr\) presented in Traxler et al. (2011b), Brown et al. (2013), and Garaud (2018), against our nondimensional theoretical prediction from (15), namely,

\[
w_{\text{model}} = \frac{U}{\kappa^* \tau^2} = \frac{\lambda_{\text{max}}}{\lambda_{\text{max}}} \frac{L}{\frac{2\pi}{\lambda_{\text{max}}} ^{2}} = \frac{\lambda_{\text{max}}}{l_{\text{max}}} \frac{2\pi}{l_{\text{max}}}.
\]

(16)

This comparison is shown in Figure 2, and clearly demonstrates that \(w_{\text{model}}\) is a remarkably accurate estimate for \(w_{\text{rms}}\) across the entire range of Prandtl numbers and density ratios tested. This would in turn imply that the Rossby number of rotating fingering convection could, at a first
approximation, be given by
\[ \text{Ro} = \frac{\lambda_{\text{max}} \kappa_T}{2\Omega d^2} = \frac{\lambda_{\text{max}}}{\sqrt{\text{Ta}^*}}. \]  

(17)

In the asymptotic regime, where \( \text{Pr}, \tau \ll r \ll 1 \), where \( r = \frac{R_0 - 1}{\tau^2 - 1} \) (which is the regime most appropriate for stellar interiors), Brown et al. (2013) further showed (see equation B 14–15 in the Appendix B of their paper) that
\[ \lambda_{\text{max}} \simeq \left( \frac{1}{1 + \frac{r}{\tau}} \right)^{\frac{1}{2}} \simeq 1 \]
and
\[ \lambda_{\text{max}} \simeq \text{Pr} \sqrt{\frac{\tau}{r \text{Pr}}} \]
resulting in the following predicted scaling for Ro with all the input parameters:
\[ \text{Ro} = \frac{\sqrt{\text{Pr}}}{\sqrt{R_0 - 1}} \frac{1}{\sqrt{\text{Ta}^*}}. \]

(19)

Since \( d \) is related to the buoyancy frequency, \( N \), as
\[ d^4 = \frac{\kappa_T \nu}{N^4}, \]
we can write \( \text{Ta}^* \) (given by (8)) as
\[ \text{Ta}^* = 4\text{Pr} \frac{\Omega^2}{N^4}. \]

Thus, our estimate for the Rossby number associated with fingering convection in stars is simply given by
\[ \text{Ro} = \frac{N^2}{4\Omega^2 R_0 - 1}. \]

(22)

Using a typical value of \( N^2 \sim 10^{-4} \) in the radiative zone of a solar-mass RGB star, one can estimate the Rossby number in the region just above the hydrogen-burning shell. We use observed estimates for red-giant rotation rates inferred from asteroseismic data from Kepler (Deheuvels et al. 2014) that range between \( \sim 0.25 \) and 10 times the solar rotation rate (400 nHz). Assuming a rather extreme estimate for density ratio \( R_0 \sim 10^4 \) (Denissenkov 2010), we find that the Rossby number would be in the range \( 0.1 \lesssim \text{Ro} \lesssim 10 \) for slow rotators and \( 0.016 \lesssim \text{Ro} \lesssim 1.6 \) for fast rotators, at the onset of fingering convection in RGB stars.

This then strongly suggests that rotation must be taken into account in modeling fingering convection in these objects.

In the following section, we therefore present new DNSs of rotating fingering convection, with values of \( \text{Ro} \) spanning the anticipated range (0.05–5) for RGB stars.

5. Numerical Simulations
5.1. Numerical Tool: PADDI

We use a version of the pseudo-spectral, triply periodic PADDI code (Traxler et al. 2011b; Stellmach et al. 2011) modified in Moll & Garaud (2017) to include the effects of rotation. We perform DNSs for \( \text{Ta}^* = 0, 0.01, 0.1, 1, 10, 25, \) and 100. We anticipate fingers to become taller for increasing values of \( R_0 \) or \( \text{Ta}^* \) and hence choose an elongated (rectangular) box with dimensions \( 100d \times 100d \times 200d \) as our default domain size. This is adjusted as and when required for varying \( R_0 \) and \( \text{Ta}^* \) (see Table 1). For simplicity, we only present results for a domain at the poles with the rotation axis aligned with the z-direction, i.e., \( \theta = 0 \) in (3). In all of our simulations, the temperature and composition fields are initialized with small amplitude random noise. Since performing DNSs at realistic values of \( \text{Pr}, \tau \) for stellar interiors is computationally unfeasible as of now, we can only run simulations at parameters down to \( \text{Pr} = \tau = 0.01 \) at best. For this exploratory work, we prefer \( \text{Pr} = \tau = 0.1 \), because it allows us to comprehensively explore the effects of varying \( R_0 \) and \( \text{Ta}^* \). We now look at a few sample simulations.

5.2. Sample Runs at \( \text{Pr} = \tau = 0.1 \)

For this choice of \( \tau = 0.1 \), a system is unstable to fingering provided \( 1 < R_0 < 10 \). We focus our study on two values of \( R_0 = 1.45 \) and 5—the former representing conditions close to being convectively unstable and the latter being half-way through the fingering-unstable range. We summarize the results of our DNSs for different choices of \( \text{Ta}^* \) (which varies the Rossby number \( \text{Ro} \)) and \( R_0 \) in Table 1.

Figure 3 shows snapshots of the vertical velocity field in six different simulations spanning values of \( \text{Ta}^* = 0.01, 0.1, 0.5 \) for two values of \( R_0 = 1.45 \) and 5. As can be readily seen from the snapshots, at \( \text{Ta}^* = 0.01 \) (which is in the “slowly rotating” regime), the fingers become more stable with increasing stratification (i.e., increasing \( R_0 \)), which has also been observed experimentally (Krishnamurti 2003) and in DNSs of nonrotating fingering convection (Traxler et al. 2011b). With increasing values of \( \text{Ta}^* \), we observe a propensity of the flow to become invariant along the axis of rotation. This is in accordance with the Taylor–Proudman theorem (Proudman 1916; Taylor 1917), which becomes relevant when the Rossby number becomes much smaller than 1. The Taylor–Proudman constraint is significantly more pronounced for the \( R_0 = 5 \) case than for the \( R_0 = 1.45 \) case, at fixed \( \text{Ta}^* \). To understand why this is the case, we note that the effective Rossby number, given by (19), is significantly higher at \( R_0 = 1.45 \) (\( \text{Ro} = 0.47 \) for \( \text{Ta}^* = 1 \)) than \( R_0 = 5 \) (\( \text{Ro} = 0.16 \) for \( \text{Ta}^* = 1 \)); hence, achieving a Taylor–Proudman state at smaller \( R_0 \) requires larger values of \( \text{Ta}^* \).

Using the set of rotating DNSs, we can actually compare our theoretical estimate for the Rossby number \( \text{Ro} \) (see (19)), to the effective Rossby number of the simulations, which is given by
\[ \text{Ro}_f \sim \frac{W_{\text{rms}}}{10\sqrt{\text{Ta}^*}}. \]

(23)

where \( W_{\text{rms}} \) is the measured rms vertical velocity in the DNS, and we have used the approximation
\[ \frac{L}{d} \simeq 10, \]

(24)

from (15) and (18). The results are shown in Figure 4 and confirm that the predicted \( \text{Ro} \) derived in Section 4 is a fairly good estimate of the effective Rossby number of the fingers (\( \text{Ro}_f \)) in all of our rotating simulations as long as it is multiplied by the constant 0.15.

Finally, note that as in Traxler et al. (2011b), the elongation of the fingers along the vertical direction (either for high \( R_0 \), or high \( \text{Ta}^* \), or both) poses a numerical challenge. Indeed, we
need to ensure that our domain is tall enough so that the fingers do not “feel” its boundaries, which would lead to artificial enhancements of the transport rates due to the assumption of periodic boundary conditions. This problem is discussed in more detail in the Appendix.

5.3. Effect of Rotation on Compositional Transport by Small-scale Fingering Convection

In what follows, we now only report on the simulations with the largest resolution and domain sizes available at $R_0 = 1.45$ and 5. As usual, the turbulent transport of heat is negligible in fingering convection. We measure the vertical flux of composition in terms of the compositional Nusselt number, $N_u$, defined as:

$$N_u = 1 - \frac{R_0}{\tau} \langle \overline{\nu} \rangle,$$

where $\langle \rangle$ denotes a volume average over the entire domain. $N_u$ can be interpreted as the ratio of the effective diffusivity $D_\nu$ to the microscopic diffusivity $\nu$, i.e.,

$$D_\nu = \nu N_u,$$

The time-evolution of $N_u$ is shown in Figure 5 for different values of $T^*$. As expected, we see the development of the fingering instability at early times, followed by its nonlinear saturation. As anticipated from our naive argument of Section 3, we find that the peak compositional transport increases significantly with $T^*$, suggesting that the shear instability between the fingers is indeed stabilized by rotation. However, we also see that the turbulent transport rates after saturation of the fingering instability, once the system has achieved a statistically steady state, do not depend on $T^*$ nearly as much. To see this more quantitatively, we measure the transport properties of fingering convection in that statistically stationary state. The time-averaged $N_u$ values thus extracted for different values of $T^*$ are presented in Figure 6 for both values of $R_0$, as a function of the corresponding Rossby numbers, $R_o$ (as given by Equation (19)).

This shows that rotation actually tends to lower the vertical transport rates by a factor of up to 2 compared with the nonrotating case, when the Rossby number drops significantly below unity.

This is a rather unexpected finding in light of our discussions in Section 3 where we expected that rotation would act to enhance the rms vertical velocities and therefore also the mixing rates. Instead, we find that both vertical and horizontal rms velocities remains almost unchanged as the rotation rate is increased (see Figure 7).

5.4. Emergence of a Large-scale Vortex

While all the results reported so far were from high-resolution simulations, we ran a few additional simulations at half their resolution for much longer to see if any longer-term dynamics emerge. These runs are not particularly under-resolved, so their dynamics are still reliable, i.e., the fingers and their structure are still well resolved. Interestingly, one such run at a resolution of $64 \times 64 \times 128$ Fourier modes for $R_0 = 1.45$ and $T^* = 10$
shows a significant enhancement in Nu$_\mu$ over a long timescale (~3000 time units), as shown in the left panel of Figure 8. It also shows a steady increase in the rms values of the vertical velocity, chemical field ($\mu_{\text{rms}}$) as well as the vertical component of the vorticity field, $\omega_{\text{rms}}$ (see the right panel of Figure 8). Figure 9 shows horizontal ($x-y$ plane) snapshots of the vertical velocity and the chemical fields at times $t = 217$ and $t = 1300$, and reveals the emergence of a cyclonic large-scale vortex (hereafter, referred to as LSV). The LSV shows a substantial enhancement in the concentration of high-$\mu$ fluid at its core, associated with a strengthening of the downward vertical component of velocity. It is to be noted that LSVs seen in other simulations (see S. Sengupta & P. Garaud 2018, in preparation) are always cyclonic, but can have the reverse situation, with low-$\mu$ material flowing upward. In both cases, this causes the enhancement in chemical transport measured through the increase in Nu$_\mu$ in Figure 8. Figure 10 presents volume rendered snapshots of the vertical vorticity in the flow at the same times ($t = 217$ and $t = 1300$), and clearly shows the emergence of long coherent cyclonic vortices that later merge into a single cyclonic LSV spanning the entire height and width of our domain.

The emergence of this LSV is reminiscent of similar findings reported in simulations of compressible rotating convection by Chan (2007; see also Chan & Mayr 2013) and Käpylä et al. (2011), of rotating Rayleigh–Bénard convection using asymptotically reduced equations by Julien et al. (2012; see also Rubio et al. 2014), of rotating Rayleigh–Bénard convection using the full equations by Guervilly et al. (2014), Favier et al. (2014), and Julien et al. (2018), of forced stratified rotating turbulence by

---

**Figure 3.** Snapshots of vertical velocity fields (after saturation) at $\text{Ta}^* = 0.01$ (left), 1 (middle) and 10 (right) for $R_0 = 1.45$ (top) and 5 (bottom); the values of Ro corresponding to each run are computed from (19).

**Figure 4.** Comparison of the predicted Rossby number, $R_o$ (given by 19), with the effective Rossby number of the fingers, $R_{o_f}$, measured from the DNSs according to (23). The solid line corresponds to $R_{o_f} = 0.15R_o$. 

The Astrophysical Journal, 862:136 (12pp), 2018 August 1
Sengupta & Garaud
Seshasayanan & Alexakis (2018), and of oscillatory double-
diffusive convection by Moll & Garaud (2017). In all cases, the
LSVs are always seen to fill the domain, and have been
interpreted as resulting from a nonlocal inverse cascade in
spectral space (Favier et al. 2014; Julien et al. 2018). The power
spectrum of the vertically invariant structures was found to scale
with $k_h^{-3}$, which is consistent with the assumption that turbulence
is rotationally dominated (Smith & Waleffe 1999).

We come to similar conclusions for our own simulations: the
horizontal energy spectrum (shown in the left panel of
Figure 11) clearly shows the development over time of a
well-defined power law at low horizontal wavenumber $k_h$. The
power law is seen for values of $k_h \lesssim 0.5$; this cutoff corresponds to the energy injection scale given by the typical
wavenumber of the fastest-growing fingering modes. We can
also estimate the rate at which the LSV grows in strength by
fitting an exponential to the rms vorticity, $\omega_{rmss}$ (between
$t \sim 500 - 2000$) as shown in the right panel of Figure 8,
which gives a value of $\sim 0.00039$ per unit time. The
corresponding growth timescale, which would be of the order
of 3000, is much larger than an eddy turnover timescale (which
is of the order of 10), but much smaller than the thermal or
viscous diffusion timescales across the domain (which are of
the order of $10^4$ and $10^5$ respectively).

6. Discussion

6.1. Summary of Our Findings

We have investigated the effect of rotation on the linear growth
of the fingering instability (Section 3) and found that rotation does
not affect the growth rate of the fastest-growing modes of the
basic linear instability. It does, however, influence their nonlinear
evolution and saturation. With the help of DNSs (Section 5) using
the PADDI code, we have measured the compositional transport
rates of rotating fingering convection in a parameter regime that
approaches stellar conditions. In general, we have found that
rotation does not enhance mixing by fingering convection contrary
to our original expectations. In fact, rotation seems to have a mild stabilizing effect on mixing. The compositional transport rates predicted across a wide range of rotation rates are
consistently lower than the corresponding nonrotating values
measured in previous DNSs (Traxler et al. 2011b; Brown et al. 2013). For simplicity, we restricted our present study to the polar configuration only, so these findings need to be verified for the non-polar cases. We suspect, however, that non-polar
configurations will have even weaker vertical mixing rates, simply by virtue of their geometry.

We have observed a possible exception to this general finding for a particularly turbulent (low $R_0$) and rapidly rotating (low Ro)
run in which coherent large-scale structures naturally emerge and
gradually evolve to merge into a single cyclonic large-scale
vortex spanning the entire computation domain. This LSV causes
a significant enhancement in the compositional transport rates by
concentrating high-$\mu$ material at its core that is advected
downward. Inspection of the horizontal kinetic energy spectrum
suggests that the LSV forms through a rotationally driven
nonlocal inverse cascade that draws its energy from the basic
instability at the finger scale. The LSV formation and its spectral
properties are strongly reminiscent of those observed in a variety
of other rapidly rotating turbulent systems, such as convection
(Käpylä et al. 2011; Julien et al. 2012; Chan & Mayr 2013;
Favier et al. 2014; Guervilly et al. 2014; Rubio et al. 2014; Julien
et al. 2018), stratified turbulence (Marino et al. 2013; Oks
et al. 2017; Seshasayanan & Alexakis 2018), and oscillatory
double-diffusive convection (Moll & Garaud 2017).

6.2. Implications for Mixing in Stars

Our findings raise a tantalizing possibility: if these large-
scale vortices also form in the fingering regions of RGB stars,
they could substantially enhance the efficiency of mixing by fingering convection, and thereby provide a self-consistent scenario to explain the observed abundance changes on the upper RGB (Gratton et al. 2000; Charbonnel & Zahn 2007b). This raises the obvious question of whether LSVs would form under more realistic stellar conditions. Studies of rapidly rotating convection and oscillatory double-diffusive convection in the polar configuration (i.e., with rotation aligned with gravity) have generally concluded that LSVs are only observed in a rotationally constrained (low Rossby number) and yet also strongly turbulent (high Reynolds number) regime (Guervilly et al. 2014; Julien et al. 2018; Seshasayanan & Alexakis 2018), which is also what we found here. The first of these conditions can be understood by noting that strong rotation is required to trigger an inverse energy cascade. However, rotation cannot be too strong otherwise the flow becomes vertically invariant (through the Taylor–Proudman constraint) and horizontal motions can only decay in that case. To see this, note that the vertical component of the vorticity equation (obtained by taking the curl of Equation 4) reduces to

\[
\frac{\partial \omega_z}{\partial t} + \mathbf{u} \cdot \nabla \omega_z = \Pr \nabla^2 \omega_z
\]

when motions are independent of \(z\). In that limit, \(\omega_z\) must ultimately decay with time (since this advection-diffusion equation contains no source term), which in turn shows that horizontal motions must necessarily decay as well. In other words, the flow must remain sufficiently three-dimensional to continually feed energy into the vortex and maintain it against viscous decay, hence the need for a sufficiently large Reynolds number.

In Section 4 (combined with the results of Figure 4), we showed that the fingering regions of RGB stars would indeed satisfy the low Rossby number requirement, with estimated values in the range of \(10^{-13}\) for rapid and moderate rotators. To estimate the Reynolds number, \(\text{Re}\), expected in these regions, we use a similar argument as that in Section 4. Since \(\text{Re} = \frac{UL}{\nu}\), where \(U\) and \(L\) are the characteristic velocities and lengthscale of fingering flows (given by Equation (15)) and \(\nu\) is the viscosity,

\[
\text{Re} = \frac{L^2 \lambda_{\text{max}}}{d^2 \Pr}.
\]

Using \(\lambda_{\text{max}}\) from (18) and the typical lengthscale given by (24), we have

\[
\text{Re} \sim \frac{100}{\sqrt{\text{Pr}(R_0 - 1)}}.
\]
According to this estimate, using $Pr \sim 10^{-6}$ and $R_0 \sim 10^3$, as before, we find that $Re \sim 10^4$, which should indeed be sufficiently high for LSVs to form.

We therefore conclude that fingering regions of RGB stars can indeed potentially be the home of large-scale vortices near the poles, which would cause a very substantial enhancement of the compositional fluxes and could in turn explain the observed evolution of the surface abundances after the luminosity bump.

Of course, much remains to be done to confirm this scenario. In particular, recent results on the formation of large-scale vortices in other systems such as rotating convection and oscillatory double-diffusive convection suggest that they may not develop (1) at lower latitudes (Moll & Garaud 2017), and (2) unless the computational domain has a unit aspect ratio (Julien et al. 2018). In these cases, large-scale horizontal jets form instead. Whether these would also be more common in the case of rotating fingering convection remains to be determined, but is likely. Whether compositional transport would similarly be enhanced in the presence of jets or not also remains to be determined, but also seems likely. These questions will be answered in future work, as they require substantial computational resources to fully explore.

The simulations presented here clearly point out the need to understand better the interplay of different physical mechanisms in order to provide robust estimates of mixing to be used in stellar evolutionary calculations. Most modern stellar evolution codes treat mixing processes independently, by computing a simple diffusion coefficient for each one of them and then adding them together (Lagarde et al. 2011; Matrozis & Stancliffe 2017; Paxton et al. 2018). This study reveals that although rotation and fingering convection can indeed be fairly well understood independently in some regimes, other regimes exist in which they strongly reinforce one another. We showed that this regime is precisely the one that is relevant for the RGB “extra-mixing” problem. If indeed LSVs form in the radiative zone above the H-burning shell in the interiors of RGB stars, they could greatly enhance transport and provide a self-consistent scenario explaining the observed abundance changes on the upper RGB which nonrotating model predictions fail to do. This is the only possible scenario in the context of the “missing-mixing” problem of the RGBs, which could help to explain the observed change in abundances of red-giants above the luminosity bump in a self-consistent way without the need to invoke physical mechanisms that are not specific to this particular evolutionary phase (Charbonnel & Zahn 2007a; Denissenkov et al. 2009). We aim to explore the emergence of these LSVs across a wider range of parameter space in a future work (S. Sengupta & P. Garaud 2018, in progress).

![Figure 9. Horizontal snapshots of chemical (left) and vertical velocity (right) fields at $t = 217$ (top) and $t = 1300$ (bottom) for the $64 \times 64 \times 128$ run at $R_0 = 1.45$, $Ta^* = 10$—red shows positive and blue shows negative values of the quantities.](image-url)
preparation) to make more systematic predictions for the conditions in which one can expect them to form.

S.S. and P.G. were funded from NSF AST 1412951. We thank S. Stellmach for the use of the PADDI code. The simulations were performed on the Hyades supercomputer, purchased using an NSF MRI grant. Figures 3, 10, and 13 were rendered using VisIt, a product of the Lawrence Livermore National Laboratory.

Software: PADDI (Traxler et al. 2011b; Stellmach et al. 2011).

Figure 10. Snapshots of vertical vorticity $\omega_z = (\nabla \times \nu_z)$, for the run described in Section 5.4; the left panel shows two cyclonic vortices at $t = 217$ that later merge into a single LSV as shown in the right panel at $t = 1300$.

Figure 11. Evolution of the horizontal energy, $E_h = \frac{1}{2}(u^2 + v^2)$, for the lower resolution run (see the text for details) at $R_0 = 1.45$ and $Ta^* = 10$, showing the gradual growth in energy of low $k_h = \sqrt{k^2 + m^2}$ modes with time, that follows a power law in $k_h^{-3}$; the arrow head shows the position of the energy injection scale ($k_h \sim 0.5$) corresponding to the typical wavenumber of the fastest-growing fingering modes.

Figure 12. Compositional fluxes for simulations using domain heights of 200d and 800d at $R_0 = 9.1$ for $Ta^* = 1, 10$, and 25 corresponding to Rossby numbers $Ro = 0.01, 0.004, and 0.002$ respectively.

Appendix

Effect of Domain Size

From Table 1, we can note that for $R_0 = 5$, the difference in the compositional Nusselt numbers between two simulations for which the height of the domain differs by a factor of 2, is at most a few percent even for our highest $Ta^*$ runs. However, for a more extreme choice of $R_0 = 9.1$, Figure 12 shows that using our default domain size at $Ta^* = 10$ or 25 gives compositional fluxes that differ by up to an order of magnitude from those obtained by using a taller domain ($100d \times 100d \times 800d$). A similar effect was also observed.
by Traxler (2011) even for the nonrotating case for very high values of \(R_0\) close to the marginal stability threshold of \(\frac{1}{7}\).

Figure 13 shows snapshots of the vertical velocity in two simulations for \(R_0 = 9.1\), Ta* = 10—the left panel using our default domain size and the right panel with a \(100d \times 100d \times 800d\) domain. The \(200d\)-tall domain has fingers that are perfectly vertical, whereas the \(800d\)-tall domain, shows fingers that no longer remain perfectly vertical. We conjecture that the fastest-growing wavelength of the shear instability between upflowing and downflowing fingers increases with increasing rotation rate. When the latter exceeds the domain size, the shear instability is suppressed and the transport is vastly enhanced. This effect is artificial, however, and must be avoided by making sure the domain is indeed tall enough to contain the shear-unstable modes.

**ORCID iDs**

S. Sengupta @ https://orcid.org/0000-0003-0191-4157

**References**

- Boussinesq, J. 1903, Théorie Analytique De La Chaleur: Mise en Harmonie Avec La Thermodynamique et Avec La Théorie Mécanique De La Lumière, Vol. 2 (Paris: Gauthier-Villars).
- Brown, J. M., Garaud, P., & Stellmach, S. 2013, *ApJ*, 768, 34
- Chan, K. L., & Mayr, H. G. 2013, *EPJSL*, 371, 212
- Charbonnel, C., & Zahn, J.-P. 2007a, *A&A*, 467, L29
- Deal, M., Deheuvels, S., Vauclair, G., Vauclair, S., & Wachlin, F. C. 2013, *A&A*, 557, L12
- Deheuvels, S., Doğan, G., Goupil, M. J., et al. 2014, *A&A*, 564, A27
- Denissenkov, P. A., & Merryfield, W. J. 2011, *ApJL*, 727, L8
- Denissenkov, P. A., & Pinsonneault, M. 2008, *ApJ*, 684, 626
- Denissenkov, P. A., Pinsonneault, M., & MacGregor, K. B. 2009, *ApJ*, 696, 1823
- Denissenkov, P. A., Herwig, F., Bildsten, L., & Paxton, B. 2013, *ApJ*, 762, 8
- Favieir, B., Silvers, L. J., & Proctor, M. R. E. 2014, *PhFl*, 26, 096605
- Garaud, P., Medrano, M., Brown, J. M., Mankovich, C., & Moore, K. 2015, *ApJ*, 808, 89
- Garaud, P. 2011, *ApJL*, 728, L30
- Garaud, P. 2018, *AnRFM*, 50, 275
- Gratton, R. G., Sneden, C., Carretta, E., & Bragaglia, A. 2000, *A&A*, 354, 169
- Guervilly, C., Hughes, D. W., & Jones, C. A. 2014, *JFM*, 758, 407
- Julien, K., Rubio, A. M., Grooms, I., & Knobloch, E. 2012, *GalFt*, 106, 392
- Julien, K., Knobloch, E., & Plumley, M. 2018, *JFM*, 837, R4
- Käpylä, P. J., Mantere, M. J., & Hackman, T. 2011, *ApJ*, 742, 34
- Krishnamurthi, R. 2003, *JFM*, 483, 287
- Lagarde, N., Charbonnel, C., Decressin, T., & Hagelberg, J. 2011, *A&A*, 536, A28
- Leconte, J. 2018, *ApJL*, 853, L30
- Marino, R., Mininni, P. D., Rosenberg, D., & Pouquet, A. 2013, *El*, 102, 44006
- Marks, P. B., & Sarna, M. J. 1998, *MNRAS*, 301, 699
- Marks, P. B., Sarna, M. J., & Prihaln, D. 1997, *MNRAS*, 290, 283
- Matrozis, E., & Stancliffe, R. J. 2017, *A&A*, 606, A55
- Medrano, M., Garaud, P., & Stellmach, S. 2014, *ApJL*, 792, L30
- Mirosh, G. M., Garaud, P., Stellmach, S., Traxler, A. L., & Wood, T. S. 2012, *ApJ*, 750, 61
- Moll, R., & Garaud, P. 2017, *ApJ*, 834, 44
- Moll, R., Garaud, P., & Stellmach, S. 2016, *ApJ*, 823, 33
- Oks, D., Mininni, P. D., Marino, R., & Pouquet, A. 2017, *PhFl*, 29, 111109
- Paxton, B., Schwab, J., Bauer, E. B., et al. 2018, *ApJS*, 234, 34
- Proudnman, J. 1916, *RSPSA*, 92, 408
- Radko, T., & Smith, D. P. 2012, *JFM*, 692, 5
- Rubio, A. M., Julien, K., Knobloch, E., & Weiss, J. B. 2014, *PhRvL*, 112, 144501
- Seshasayanan, K., & Alexakis, A. 2018, *JFM*, 841, 434
- Smith, L. M., & Waleffe, F. 1999, *PhFl*, 11, 1608
- Spiegel, E. A., & Veronis, G. 1960, *ApJ*, 131, 442
- Stancliffe, R. J., Gubbels, E., Izzard, R. G., & Pols, O. R. 2007, *A&A*, 464, L57
- Stellmach, S., & Hansen, U. 2008, *GGG*, 9, Q5003

**Figure 13.** Vertical velocity fields (after saturation) for \(R_0 = 9.1\) and Ta* = 10 (corresponding to a Rossby number Ro = 0.004) for domain heights of 200d (left) and 800d (right), showing the need for using an elongated domain (in the vertical direction) at high \(R_0\). The vertical scale of the 800d-tall domain was shrunk by a factor of 4, for ease of comparison with the 200d-tall domain.
Stellmach, S., Traxler, A., Garaud, P., Brummell, N., & Radko, T. 2011, JFM, 677, 554
Taylor, G. I. 1917, RSPSA, 93, 99
Theado, S., & Vauclair, S. 2010, Ap&SS, 328, 209
Traxler, A., Garaud, P., & Stellmach, S. 2011a, ApJL, 728, L29
Traxler, A., Stellmach, S., Garaud, P., Radko, T., & Brummell, N. 2011b, JFM, 677, 530

Traxler, A. L. 2011, PhD thesis, Univ. California, Santa Cruz
Tremblin, P., Amundsen, D. S., Mourier, P., et al. 2015, ApJL, 804, L17
Tremblin, P., Amundsen, D. S., Chabrier, G., et al. 2016, ApJL, 817, L19
Vauclair, S., & Théado, S. 2012, ApJ, 753, 49
Vauclair, S. 2004, ApJ, 605, 874
Wachlin, F. C., Vauclair, S., & Althaus, L. G. 2014, A&A, 570, A58
Wood, T. S., Garaud, P., & Stellmach, S. 2013, ApJ, 768, 157