Research Article

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Regularity results for \( p \)-Laplacians in pre-fractal domains

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Abstract: We study obstacle problems involving \( p \)-Laplace-type operators in non-convex polygons. We establish regularity results in terms of weighted Sobolev spaces. As applications, we obtain estimates for the FEM approximation for obstacle problems in pre-fractal Koch Islands.

Keywords: Degenerate elliptic equations, smoothness and regularity of solutions, FEM, fractals

MSC 2010: 35J70, 35B65, 65N30, 28A80

1 Introduction

In this paper, we deal with obstacle problems involving \( p \)-Laplace-type operators in bad domains in \( \mathbb{R}^2 \). This kind of problems occurs in many mathematical models of physical processes: nonlinear diffusion and filtration, power-law materials and quasi-Newtonian flows (see, for example, [17] and references therein).

Let \( \Omega_\omega \) denote a conical domain (see Section 2 for definitions and properties) and let us consider the two obstacle problem:

\[
\text{find } u \in K \text{ such that } a_p(u, v - u) - \int_{\Omega_\omega} f(v - u) \, dx \, dy \geq 0 \text{ for all } v \in K, \quad (1.1)
\]

where

\[
a_p(u, v) = \int_{\Omega_\omega} \left( k^2 + |\nabla u|^2 \right)^{\frac{p-2}{2}} \nabla u \nabla v \, dx \, dy
\]

and

\[
K = \{ v \in W^{1,p}_0(\Omega_\omega) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\omega \}.
\]

Then, under natural assumptions (see (2.2)), there exists a unique function \( u \) that solves problem (1.1). Properties of first-order derivatives have been established by Li and Martio in [25] and by Lieberman in [27] (see also the references quoted there). In this paper, we face the study of the regularity of the second-order derivatives. To our knowledge, for \( p > 2 \) there are no second-order \( L^2 \) regularity results concerning obstacle problems even if the differentiability of the data and the smoothness of the boundary are assumed; in particular, recent results by Brasco, Santambrogio [5] and by Mercuri, Riey, Sciunzi [29] do not seem to work for obstacle problems. Global regularity results in terms of Sobolev (or Besov) spaces with smoothness index greater than 1 are up to now only established for solutions of obstacle problems for \( p = 2 \) (see [11]).

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In this paper, we establish a regularity result for the solution of obstacle problem (1.1) in terms of the
weighted Sobolev spaces, where the weight is the distance from the conical point (see Theorem 3.1). In our
approach, the Lewy–Stampacchia inequality (see Proposition 2.2) plays a crucial role. We note that this result
is new not only for obstacle problems but also in the case of Dirichlet problems. In fact, there is a huge litera-
ture about the regularity in the Hölder classes for both the solution \( u \) and the gradient \( \nabla u \) (see [22] and
the references quoted there), while the smoothness of the second derivatives is little investigated in such type
of irregular domains. Actually, on this topic we have only the contributions by Borsuk and Kondratiev [4]
and by Cianchi and Maz’ya [16]. More precisely, Borsuk and Kondratiev (see [4, Theorems 8.43, 8.44, 8.46])
deal with Dirichlet problems in conical domains, but they require a stronger assumption of the datum and
prove a weaker regularity. In particular, the exponent of the weight in [4] is greater than the one in our paper
(see (3.2)). On the other side, Cianchi and Maz’ya (see [16, Theorem 2.4]) deal with Dirichlet problems in
domains that either satisfy [16, condition (2.12)] or are convex: here the domain \( \Omega_\omega \) is not convex and \( \partial \Omega_\omega \)
does not satisfy [16, condition (2.12)]. Actually, we use some ideas from [16] in order to obtain local estimates
and estimates far away from the conical point. We note that in this part the boundedness of the data \( f \) and
\( A_p(\varphi_i), i = 1, 2 \) is not required, but only the belonging to \( L^2(\Omega_\omega) \) (see Theorems 3.6 and 3.7). To establish
estimates near the conical point we follow the approach of Tolksdorf [34] and Dobrowolski [19].

In the present paper, we prove also the boundedness of the gradient far away from the conical point (see
Theorem 4.3). Essential tools are some results by Tolksdorf [34], Cianchi and Maz’ya [15] and Barret and
Liu [2] (for \( k = 0 \)).

We think that the established results are interesting in themselves and also from the point of view of
numerical analysis. In fact, as is well known, the regularity results are crucial tools to establish error estimates
for the FEM approximation (see, for instance, [6, 8]). To face the numerical approach of the solutions of
obstacle problems in fractal domains, it is natural to consider the solutions of obstacle problems in pre-
fractal approximating domains and the corresponding FEM-solutions and to evaluate the approximation
error. In this spirit, we apply Theorems 3.1 and 4.3 in the study of the obstacle problems in pre-fractal Koch
Islands. More precisely, in Theorem 5.5 we prove a sharp error estimate for the FEM approximations using
the sharp approach of Grisvard [20]. We remark that for \( p = 2 \) Theorem 5.5 gives the sharp result of Grisvard
(see [20, Corollary 8.4.1.7]). Moreover, Theorem 5.5 improves the results of [12]: in particular, estimate (5.11)
gives a faster convergence than the convergence in [12, estimate (5.63)].

The plan of the paper is the following. In Section 2, we describe the geometry of our domain, we intro-
duce the obstacle problems and we state existence, uniqueness, energy estimates, the Lewy–Stampacchia
inequality and a first regularity result for the solutions in terms of the Besov spaces. In Section 3, we estab-
lish our main result in terms of the weighted Sobolev spaces. In Section 4, we establish some further results
concerning the boundedness of the gradient. In the last section, we show an application of these estimates.

2 Preliminary

Let \( \Omega_\omega \) denote a plane domain with a polygonal boundary \( \partial \Omega_\omega \) union of a finite number \( N \) of linear seg-
ments \( \Gamma_i \) numbered according to the positive orientation. We denote by \( \omega_j \) the angle between \( \Gamma_j \) and
\( \Gamma_{j+1} \) and we assume that \( \omega_j < \pi \) for any \( j < N \) and \( \omega_N = \omega > \pi \). For simplicity, we assume that the corner point
between \( \Gamma_N \) and \( \Gamma_1 \) is the origin and that \( \Gamma_1 \) is included in the positive abscissa axis.

We consider the two obstacle problem:

\[
\text{find } u \in \mathcal{K} \text{ such that } a_p(u, v - u) - \int_{\Omega_\omega} f(v - u) \, dx \, dy \geq 0 \quad \text{for all } v \in \mathcal{K},
\]  

(2.1)

where

\[
a_p(u, v) = \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v \, dx \, dy, \quad k \in \mathbb{R},
\]

and

\[
\mathcal{K} = \{ v \in W_0^{1,p}(\Omega_\omega) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\omega \}.
\]
By using the Poincaré inequality (see, e.g., [28]), the monotonicity properties of the $p$-Laplacian and choosing $\nu = \varphi_2 \wedge (\varphi_1 \vee 0)$ as test function in (2.1), we can prove the following result.

**Proposition 2.1.** Let

$$\begin{aligned}
  f &\in W^{-1,p'}(\Omega_\omega), & \frac{1}{p} + \frac{1}{p'} = 1, & \varphi_i \in W^{1,p}(\Omega_\omega), & i = 1, 2, \\
  \varphi_1 &\leq \varphi_2 \text{ in } \Omega_\omega, & \varphi_1 &\geq 0 \leq \varphi_2 \text{ in } \partial_\Omega_\omega.
\end{aligned}$$

(2.2)

Then there exists a unique function $u$ that solves problem (2.1). Moreover,

$$||u||_{W^{1,p}(\Omega_\omega)} \leq C||f||_{W^{-1,p'}(\Omega_\omega)} + ||\varphi_1||_{W^{1,p}(\Omega_\omega)} + ||\varphi_2||_{W^{1,p}(\Omega_\omega)} + |k|.$$

(2.3)

From now on, we denote by $C$ possibly different constants.

We recall that the solution $u$ to problem (2.1) realizes the minimum on the convex $\mathcal{K}$ of the functional

$$J_p(u) = \min_{v \in \mathcal{K}} J_p(v), \quad \text{where} \quad J_p(v) = \frac{1}{p} \int\limits_{\Omega_\omega} (k^2 + |\nabla v|^2)^{\frac{p}{2}} \, dx \, dy - \int\limits_{\Omega_\omega} fv \, dx \, dy.$$

Now we introduce the Lewy–Stampacchia inequality that plays an important role in our approach to the regularity of the solution. We set

$$A_p(u) = -\text{div}( (k^2 + |\nabla u|^2)^{\frac{p}{2}} \nabla u ).$$

**Proposition 2.2.** We assume hypothesis (2.2) and

$$f, A_p(\varphi_i) \in L^{p'}(\Omega_\omega), \quad i = 1, 2, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

(2.4)

Let $u$ be the solution of (2.1). Then

$$A_p(\varphi_2) \wedge f \leq A_p(u) \leq A_p(\varphi_1) \vee f \quad \text{in } \Omega_\omega.$$  

(2.5)

The Lewy–Stampacchia inequality was first proved in [24] for superharmonic functions which solve a minimum problem, the proof being deeply based on the properties of the Green function. This result has been extended to more general (linear) operators and more general obstacles by Mosco and Troianiello in [31], and for $T$-monotone operators like the $p$-Laplacian in [30]. Actually, inequalities (2.5) hold under assumptions weaker than (2.4) according to [36, Remark 1 in Chapter 4.5].

**Proposition 2.3.** We assume hypotheses (2.2) and (2.4). Then the solution $u$ of problem (2.1) is the solution of the Dirichlet problem

$$\begin{aligned}
  A_p(u) &= f^* \quad \text{in } \Omega_\omega, \\
  u &= 0 \quad \text{in } \partial_\Omega_\omega, \\
\end{aligned}$$

(2.6)

where $f^*$ belongs to the space $L^{p'}(\Omega_\omega)$ and

$$||f^*||_{L^{p'}(\Omega_\omega)} \leq C||f||_{L^{p'}(\Omega_\omega)} + ||A_p(\varphi_1)||_{L^{p'}(\Omega_\omega)} + ||A_p(\varphi_2)||_{L^{p'}(\Omega_\omega)}.$$

By using the Lewy–Stampacchia inequality and [32, Theorem 2], we stated in [12] for $k = 0$ the following regularity result in terms of Besov spaces; the case $k \neq 0$ can be treated analogously. We recall a characterization of Besov spaces

$$B^{1,1}_{p,q}(-\lambda)(\Omega_\omega) := (W^{1,p}(\Omega_\omega), L^p(\Omega_\omega))_{\lambda,q},$$

$$B^{0,1}_{p,q}(-\lambda)(\Omega_\omega) := (W^{1,2}(\Omega_\omega), W^{1,p}(\Omega_\omega))_{\lambda,q} = \{ u \in W^{1,p}(\Omega_\omega) : \nabla u \in B^{1,1}_{p,q}(-\lambda)(\Omega_\omega; \mathbb{R}^2) \},$$

where $\lambda \in [0, 1], p, q \in [1, +\infty]$ and $(\cdot, \cdot)_{\lambda,q}$ is the real interpolation functor (see [3]).

**Theorem 2.4.** We assume hypotheses (2.2) and (2.4). Let $u$ be the solution of (2.1). Then $u$ belongs to the Besov space $B^{1+1/p}_{p,\infty}(\Omega_\omega)$. Moreover,

$$||u||_{B^{1+1/p}_{p,\infty}(\Omega_\omega)} \leq C[1 + ||f||_{L^{p'}(\Omega_\omega)} + \| A_p(\varphi_1) \|_{L^{p'}(\Omega_\omega)} + \| A_p(\varphi_2) \|_{L^{p'}(\Omega_\omega)}].$$
Note that, putting $p = 2$ in the previous theorem, we get $u \in H^{3/2-\epsilon}(\Omega_\omega)$ in the Sobolev scale. We point out that the previous result is, in some sense, the best possible as it holds for any value of $\omega \in (\pi, 2\pi)$, and as $\omega \to 2\pi$, the domain becomes very bad.

A natural question is then if we can expect sharper regularity results if we consider a fixed value of $\omega$. Having in mind the by now classical results of Kondratiev (see [21]), we think that the natural spaces to consider are those of weighted Sobolev spaces.

In this section, we state our regularity result in terms of weighted Sobolev spaces.

## 3 Main result

In this section, we state our regularity result in terms of weighted Sobolev spaces.

**Theorem 3.1.** Assume hypotheses (2.2) and

\[
\begin{align*}
  k & \neq 0, \\
  f, A_p(\varphi_i) & \in L^\infty(\Omega_\omega), \quad i = 1, 2, \\
  A_p(\varphi_2) \land f & \geq 0.
\end{align*}
\]

Then the solution $u$ of obstacle problem (2.1) in $\Omega_\omega$ belongs to the weighted Sobolev space

\[ H^{2,\mu}(\Omega_\omega), \quad \mu > 1 - \gamma, \]

where

\[ \gamma = \gamma(p, \chi) = 1 + \frac{p(1 - \chi)^2 + (1 - \chi)\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2}}{2\chi(2 - \chi)(p - 1)} \]

with $\chi = \frac{\omega}{\pi}$.

Moreover,

\[ \| u \|_{H^{2,\mu}(\Omega_\omega)} \leq C \left[ 1 + \| u \|_{L^\infty(\Omega_\omega)} + \| A_p(\varphi_1) \|_{L^\infty(\Omega_\omega)} + \| A_p(\varphi_2) \|_{L^\infty(\Omega_\omega)} \right]. \]

We note that $\gamma$ is the least positive eigenvalue and $\phi(\theta)$ is the corresponding eigenfunction of the problem (see [34] and [4, Theorem 8.12 and Remark 8.13])

\[
\begin{align*}
  \partial_\theta (\lambda^2 \phi^2 + |\partial_\theta \phi|^2)^{\nu-2} \partial_\theta \phi + \lambda (\lambda(p - 1) + 2 - p)(\lambda^2 \phi^2 + |\partial_\theta \phi|^2)^{\nu-2} \phi &= 0 \quad \text{in } 0 < \theta < \omega, \\
  \phi(0) = \phi(\omega) &= 0.
\end{align*}
\]

**Remark 3.2.** To our knowledge, for $p > 2$ there are no second-order $L^2$ regularity results concerning obstacle problems even if the differentiability of the data and the smoothness of the boundary are assumed; in particular, recent results of Brasco, Santambrogio [5] and Mercuri, Riey, Scianz [29] do not seem to work for
obstacle problems. For properties of first-order derivatives we refer to [25, 27] and to the references quoted there. Global regularity results in terms of Sobolev (or Besov) spaces with smoothness index greater than 1 for solutions of obstacle problems are up to now only established for $p = 2$ (see [11]).

**Remark 3.3.** We note that for any fixed value of $p > 2$ the function $\gamma(p, \cdot)$ decreases as the variable $\chi$ increases, and it tends to the value $p^{\frac{1}{2}}$ as $\chi \to 2$. Similarly, for any fixed value of $\chi < 2$ the function $\gamma(\cdot, \chi)$ increases as the variable $p$ increases, and it tends to the value 1 as $p \to +\infty$. If we choose $\omega = \frac{4\pi}{p}$, then the expression for $\gamma$ becomes

$$\gamma(p, \frac{4}{3}) = 1 + \frac{p - \sqrt{p^2 + 32p - 32}}{16(p - 1)}.$$ 

Note that, putting $p = 2$ in the previous formula, we get $\gamma = \frac{3}{4}$ according to the by now classical results of Kondratiev for equations (see, e.g., [4]).

The behavior of $\gamma(p, \frac{2}{3})$ is shown in Figure 1 for $2 < p < 10$, and in Figure 2 for $2 < p < 10.000$.

**Remark 3.4.** We point out that the regularity result of Theorem 3.1, also in the case of Dirichlet problems with datum $F \in L^\infty$, cannot be deduced from [4, Theorems 8.43, 8.44, 8.46] since we do not assume the differentiability of $F$, and, for any $p > 2$, the exponent of the weight in [4] is greater than the one in (3.2). In fact, the exponent of the weight in formula [4, (8.4.35)] is required to be greater than $\frac{p}{2}(1 - \gamma)$ (in our notation), it is increasing in $p$ and its limit is equal to $\frac{1}{2}$ as $p \to \infty$, while $\mu$ in (3.2) is required to be greater than $(1 - \gamma)$, it is decreasing in $p$ and tends to 0 as $p \to \infty$.

**Remark 3.5.** We point out that this regularity result cannot be deduced from [16, Theorem 2.4] as our boundaries do not satisfy [16, condition (2.12)]. Actually, we use some ideas from [16] in order to obtain local
estimates and estimates far away from the origin. We note that in this part the boundedness of the data \(f\) and \(A_p(\phi_i), i = 1, 2\) is not required, but only the belonging to \(L^2(\Omega_\omega)\) (see Theorems 3.6 and 3.7).

The proof is obtained by combining some preliminary results that actually require weaker conditions than (3.1).

**Theorem 3.6.** We assume hypothesis (2.2) and

\[
\begin{align*}
&k \neq 0, \\
&f, A_p(\phi_i) \in L^2_{\text{loc}}(\Omega_\omega) & i = 1, 2.
\end{align*}
\]  

Then the solution \(u\) of obstacle problem (2.1) in \(\Omega_\omega\) belongs to \(H^2_{\text{loc}}(\Omega_\omega)\).

**Proof.** From the Lewy–Stampacchia inequality (2.5) and assumption (3.8) we derive that the solution \(u\) of problem (2.1) is the solution of the equation \(A_p(u) = f^*\), where \(f^*\) belongs to the space \(L^2_{\text{loc}}(\Omega_\omega)\) and

\[
\|f^*\|_{L^2_{\text{loc}}(\Omega_\omega)} \leq C \{\|f\|_{L^2_{\text{loc}}(\Omega_\omega)} + \|A_p(\phi_1)\|_{L^2_{\text{loc}}(\Omega_\omega)} + \|A_p(\phi_2)\|_{L^2_{\text{loc}}(\Omega_\omega)}\}
\]

Moreover,

\[
\sup_{t > 0} \frac{(p - 2)t^2(k^2 + t^2)^{\frac{p-4}{2}}}{(k^2 + t^2)^{\frac{p-2}{2}}} = p - 2
\]

and, as \(k \neq 0\),

\[
\inf_{t > 0} \frac{(p - 2)t^2(k^2 + t^2)^{\frac{p-4}{2}}}{(k^2 + t^2)^{\frac{p-2}{2}}} = 0.
\]

Then we use [16, (5.11) in the proof of Theorem 2.1] and we obtain

\[
\int_{B_R} (k^2 + |\nabla u|^2)^{p-2} \sum_{|\beta| = 2} |D^\beta u|^2 \, dx 
\]

\[
\leq C \left( \|f\|_{L^2_{\text{loc}}(B_{2R})}^2 + \|A_p(\phi_1)\|_{L^2_{\text{loc}}(B_{2R})}^2 + \|A_p(\phi_2)\|_{L^2_{\text{loc}}(B_{2R})}^2 + \frac{1}{R^2} \int_{B_{2R}} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 \, dx \right)  
\]

for any ball \(B_{2R} \subseteq \Omega_\omega\) with \(C\) independent of \(k\). Then we repeat [16, steps 2 and 3 of the proof of Theorem 2.1], and by (2.3) we obtain that \(u \in H^2_{\text{loc}}(\Omega_\omega)\). \(\square\)

Now we derive estimates far away from the origin. Let \(x \in \partial \Omega_\omega \setminus O\) and \(\Omega_s(x) := B_s(x) \cap \hat{\Omega}_\omega\) for \(s > 0\). Let \(0 < R < \frac{\text{dist}(x, O)}{4}\) be such that \(\Omega_{2R}(x) := B_{2R}(x) \cap \hat{\Omega}_\omega\) is convex.

**Theorem 3.7.** We assume hypothesis (2.2) and

\[
f, A_p(\phi_i) \in L^2(\Omega_\omega), & i = 1, 2.
\]

Then the solution \(u\) of obstacle problem (2.1) in \(\Omega_\omega\) satisfies

\[
\int_{\Omega_\omega(x)} (k^2 + |\nabla u|^2)^{p-2} \sum_{|\beta| = 2} |D^\beta u|^2 \, dx 
\]

\[
\leq C \left( \|f\|_{L^2(\Omega_{2R}(x))}^2 + \|A_p(\phi_1)\|_{L^2(\Omega_{2R}(x))}^2 + \|A_p(\phi_2)\|_{L^2(\Omega_{2R}(x))}^2 + \frac{1}{R^2} \int_{\Omega_{2R}(x)} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u|^2 \, dx \right) 
\]

for any \(x \in \partial \Omega_\omega \setminus O\) and \(R \in (0, \frac{\text{dist}(x, O)}{4})\) such that \(\Omega_{2R}(x) = B_{2R}(x) \cap \hat{\Omega}_\omega\) is convex.

**Proof.** From the Lewy–Stampacchia inequality (2.5) and assumption (3.8) we derive that the solution \(u\) of the problem is the solution of the Dirichlet problem (2.6), where \(f^*\) belongs to the space \(L^2(\Omega_\omega)\) and

\[
\|f^*\|_{L^2(\Omega_\omega)} \leq C \{\|f\|_{L^2(\Omega_\omega)} + \|A_p(\phi_1)\|_{L^2(\Omega_\omega)} + \|A_p(\phi_2)\|_{L^2(\Omega_\omega)}\}
\]

We point out that far away from the origin, according to the terminology of [16], the weak second fundamental form on \(\partial \Omega_\omega\) is non-positive. We choose the cut function \(\xi \in C^0_0(B_{2R}(x))\) with \(\xi = 1\) in \(B_R(x)\).

We proceed as in [16, step 1 of the proof of Theorem 2.4]. We observe that on \(\partial \Omega_\omega \cap \partial B_{2R}(x)\) we have \(\xi = 0\) and on \(\partial \Omega_\omega \cap B_{2R}(x)\) the Dirichlet condition holds, so the boundary integrals (see [16, (4.18)]) can be
neglected. By using estimate (3.10), we obtain (see [16, (4.74)])
\[\int_{\Omega} \xi^2 (k^2 + |\nabla u|^2)^{p-2} \sum_{|\beta|=2} |D^\beta u|^2 \, dx\]
\[\leq C \left( \|\xi^2 \|_{L^2(\Omega_{2\rho})}^2 + \|\xi^2 A_p(\phi_1)\|_{L^2(\Omega_{2\rho})}^2 + \|\xi^2 A_p(\phi_2)\|_{L^2(\Omega_{2\rho})}^2 + \int_{\Omega_{2\rho}} |\nabla \xi|^2 (k^2 + |\nabla u|^2)^{p-2} |\nabla u|^2 \, dx \right).\]

Then we repeat [16, steps 2, 3 and 4 of the proof of Theorem 2.3] and we achieve estimate (3.9), where the constant \(C\) is independent of \(k\).

The next theorem concerns estimates near the origin and it holds true for any \(k \in \mathbb{R}\).

**Theorem 3.8.** Assume hypotheses (2.2), (2.4) and
\[A_p(\phi_2) \wedge f \geq 0, \quad A_p(\phi_1) \vee f \leq C_1 r^{lb_0} \quad \text{with} \quad \lambda_0 > \gamma(p-1) - p \quad \text{in} \quad \Omega_{2\rho},\]
(3.11)
where \(\gamma\) is defined in (3.3). Then the following estimates hold for the solution \(u\) of obstacle problem (2.1):
\[|u(x)| \leq C r^\gamma, \quad |\nabla u(x)| \leq C r^{\gamma-1}, \quad |D^\beta u| \leq C r^{\gamma-2}, \quad |\beta| = 2.\]
(3.12)

**Proof.** From the Lewy–Stampacchia inequality (2.5) and assumption (3.11), we derive that the solution \(u\) of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum \(f^*\) having the property
\[0 \leq f^* \leq C_1 r^{lb_0} \quad \text{with} \quad \lambda_0 > \gamma(p-1) - p.\]
Moreover, we can suppose that \(f^* \neq 0\). In fact, if \(f^* = 0\), then the unique solution \(u\) of problem (2.1) is identically zero and estimates (3.12) are trivial.

If \(f^* \neq 0\), we use [19, Theorem 3 and the subsequent remarks] and we deduce that \(u\) admits the singular expansion
\[u(r, \theta) = C_2 r^\gamma \phi(\theta) + v(x)\]
(3.13)
with \(C_2 > 0\), and
\[|v(x)| \leq C_3 r^{\gamma+\delta}, \quad |\nabla v(x)| \leq C_3 r^{\gamma+\delta-1}, \quad |D^\beta u| \leq C_3 r^{\gamma+\delta-2}, \quad |\beta| = 2.\]
(3.14)
Here \(\gamma\) is defined in (3.3), \(\phi(\theta)\) is the corresponding eigenfunction in problem (3.5) and the maximum \(\delta > 0\) depends on \(\gamma\) and \(\lambda_0\). We deduce estimates (3.12) from (3.13) and (3.14).

We are now in a position to prove our main result.

**Proof of Theorem 3.1.** Since assumptions (2.2) and (3.1) imply the assumptions of Theorem 3.6, Theorem 3.7 and Theorem 3.8 (with \(\lambda_0 = 0\)), we combine all the results and we deduce that the solution \(u\) of problem (2.1) belongs to the weighted Sobolev space \(H^{2, \mu}(\Omega_{2\rho})\) for any \(\mu > 1 - \gamma\) as
\[r^\mu |D^\beta u| \in L^2(\Omega_{2\rho}), \quad |\beta| = 2.\]
Finally, estimate (3.4) follows from (2.3), (3.7), (3.9) and (3.12).

### 4 Boundedness of the gradient far away from the origin

We now investigate boundedness of the gradient in \(L^\infty\) far away from the origin. We stress the fact that the results of Theorems 4.1 and 4.2 hold for any \(k \in \mathbb{R}\).

**Theorem 4.1.** We assume hypotheses (2.2) and
\[f, A_p(\phi_i) \in L^\infty(\Omega_{2\rho}), \quad i = 1, 2.\]
(4.1)
Then the solution \(u\) of obstacle problem (2.1) belongs to the Sobolev space \(W^{1, \infty}_{\text{loc}}(\Omega_{2\rho})\).

**Proof.** From the Lewy–Stampacchia inequality (2.5) and assumption (4.1) we derive that the solution \(u\) of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum \(f^* \in L^\infty(\Omega_{2\rho})\). Then the thesis follows from [35, Theorem 1] (see also [18, 26, 37]).
Theorem 4.2. We assume hypotheses (2.2) and (4.1). Then the solution \( u \) of obstacle problem (2.1) belongs to the Sobolev space \( W^{1,\infty}(\Omega_R(x)) \) for any \( x \in \partial \Omega_\omega \setminus O \) and \( R \in (0, \frac{\text{dist}(x, O)}{4}) \) such that \( \Omega_{2R}(x) = B_{2R}(x) \cap \Omega_\omega \) is convex.

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (4.1) we derive that the solution \( u \) of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum \( f^* \in L^{\infty}(\Omega_\omega) \).

Then we can proceed as in [15, Theorem 2.2 and Remark 2.7]: more precisely, we replace [15, Lemma 5.4] by a localized version involving a cut-off function \( \xi \in C^\infty_0(B_{2R}(x)) \) with \( \xi = 1 \) in \( B_R(x) \) and we obtain, for a smooth function \( v \) such that \( v = 0 \) on \( \partial \Omega_\omega \),

\[
C(k^2 + t^2)^{\frac{\beta}{2}} \int_{\{|\nabla v| = t\}} \xi^2 |\nabla v| \, d\lambda^1(x) \leq t \int_{\{|\nabla v| = t\}} \xi^2 \text{div}(k^2 + |\nabla v|^2)^{\frac{\beta}{2}} \nabla v \, d\lambda^1(x) + \int_{\{|\nabla v| > t\}} \xi^2 \frac{1}{(k^2 + |\nabla v|^2)^{\frac{\beta}{2}}} \text{div}(k^2 + |\nabla v|^2)^{\frac{\beta}{2}} \nabla v)^2 \, dx + C \int_{\{|\nabla v| > t\}} \xi^2 |\nabla v|^p \, dx.
\]

We have exploited the fact that the weak second fundamental form on \( \partial \Omega_\omega \cap B_{2R}(x) \) is non-positive. \( \square \)

We now state a further property for the gradient, useful for the application we have in mind when \( k = 0 \) (see [2, Lemma 4.2]). Here, as before, for any \( x \in \partial \Omega_\omega \setminus O \) we set \( \Omega_{2R}(x) = B_{2R}(x) \cap \Omega_\omega \) and \( R \in (0, \frac{\text{dist}(x, O)}{4}) \) is chosen in such a way that \( \Omega_{2R}(x) = B_{2R}(x) \cap \Omega_\omega \) is convex.

Theorem 4.3. We assume (2.2), (4.1) and

\[
k = 0, \quad A_p(\varphi_2) \wedge f \geq c^* > 0. \tag{4.2}
\]

We suppose that the solution \( u \) of obstacle problem (2.1) belongs to the space \( W^{2,s}_{\text{loc}}(\Omega_\omega) \), and for any \( x \in \partial \Omega_\omega \setminus O \) the restriction of \( u \) to the set \( \Omega_{2R}(x) \) belongs to \( W^{2,\lambda}(\Omega_R(x)) \), \( s \in [1, 2] \). Then, for any \( q \geq 1, p > 2 \), we obtain

\[
|\nabla u| \leq t \in L^1(\Omega_\omega)
\]

with

\[
t \geq \frac{q(p + (p - 2)s)}{q + (p - 2)s}.
\]

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (4.1) we derive that the solution \( u \) of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum \( f^* \in L^{\infty}(\Omega_\omega) \), and by (4.2) also \( f^* \geq c^* > 0 \). In particular, assumption (3.11) of Theorem 3.8 is satisfied with \( \lambda_0 = 0 \). We deduce from (3.13) that \( |\nabla u| \) behaves like \( r^{\gamma - 1} \) in a neighborhood of \( O \), and hence \( |\nabla u|^{-1} \in L^\infty \) near \( O \). Far away from the origin, we apply Theorem 4.1 to obtain that \( u \in W^{1,\infty}_{\text{loc}}(\Omega_\omega) \).

Let \( G \) be a domain with \( G \subseteq \Omega_\omega \). Then \( (v_1, v_2) \equiv u \in (W^1,2(G))^2 \) and \( v \equiv |\nabla u| \in L^\infty(G) \). It follows that \( v \in W^{1,\gamma}(G) \) and \( \nabla v = (v_1 \nabla v_1 + v_2 \nabla v_2)/v \). Moreover, we have that

\[
f^* = -\text{div}(\nu^{p-2} \nabla u) = -\left\{ v^{p-2}(v_{1x_1} + v_{2x_2}) + (p - 2)v^{p-2} \frac{v_1 v_{1x_1} + v_2 v_{2x_2}}{v} \right\}.
\]

Then

\[
c^* \leq f^* \leq M(x)|\nabla u|^{p-2} \quad \text{a.e. in } G,
\]

where \( M(x) \in L^\infty(G) \).

We obtain

\[
\int_G |\nabla u|^{\frac{p-q}{q-p}} \, dx \leq C \int_G (M(x))^{\frac{p-q}{q-p}} \, dx,
\]

and if \( t \geq \frac{q(p + (p - 2)s)}{q + (p - 2)s} \), then \( \frac{p-q}{q-p} \frac{p-q}{q-p} \leq s \).

We repeat the previous proof by replacing \( G \) by \( \Omega_R(x) \) and Theorem 4.1 by 4.2 to complete the proof. \( \square \)
5 Error estimates

Obstacle problems in fractal domains have been studied in [10] in the framework of reinforcement problems. To face the numerical approach to the solutions of obstacle problems in fractal domains, it is natural to consider the solutions of obstacle problems in pre-fractal approximating domains and the corresponding FEM-solutions and to evaluate the approximation error. We consider the pre-fractal Koch Islands $\Omega_\alpha^n$ that are polygonal domains having as sides pre-fractal Koch curves. We start by a regular polygon and we replace each side by a pre-fractal Koch curve (see Figures 3 and 4); we refer to [12, Section 2] for the definition and details.

In [12, Section 3], we showed that, assuming some natural conditions, the solutions $u_n$ of the obstacle problem in $\Omega_\alpha^n$ converge to the fractal solution of the obstacle problem in the Koch Island $\Omega_\alpha$.

For any (fixed) $n$, the number of reentrant angles is fixed and hence we can prove, for the solution $u_n$ of the obstacle problem in $\Omega_\alpha^n$, all the results of previous sections with $\chi = \frac{\omega}{\pi}$, where

$$\omega = \begin{cases} \pi + \theta(\alpha) & \text{if the sides of the polygons are obtained by outward curves}, \\ \pi + 2\theta(\alpha) & \text{if the sides of the polygons are obtained by inward curves}. \end{cases} (5.1)$$

We recall that by $\theta(\alpha)$ we denote the opening of the rotation angle of the similarities involved in the construction of the Koch curve, that is,

$$\theta(\alpha) = \arcsin\left(\frac{\sqrt{\alpha}(4 - \alpha)}{2}\right).$$

Then $\chi \in (1, \frac{3}{2})$ in the case of outward curves or $\chi \in (1, 2)$ in the case of inward curves.

In this framework, the involved weighted Sobolev space is

$$H^{2,\mu}(\Omega_\alpha^n) = \{v \in W^{1,2}(\Omega_\alpha^n) : D^\beta v \in L_{2,\mu}(\Omega_\alpha^n) \text{ for all } |\beta| = 2\}, \quad \beta = (\beta_1, \beta_2), \beta_1, \beta_2 \in \mathbb{N},$$

which is a Hilbert space with the norm

$$\|v\|_{H^{2,\mu}(\Omega_\alpha^n)} = \left\{ \sum_{|\beta|=2} \|D^\beta v\|^2_{L_{2,\mu}(\Omega_\alpha^n)} + \|v\|^2_{W^{1,2}(\Omega_\alpha^n)} \right\}^{1/2}.$$ 

Here $L_{2,\mu}(\Omega_\alpha^n)$ is the completion of the space $C(\bar{\Omega}_\alpha^n)$ with respect to the norm

$$\|v\|_{L_{2,\mu}(\Omega_\alpha^n)} = \left\{ \int_{\Omega_\alpha^n} |v|^2 p^{2\mu} \, dx \right\}^{1/2}.$$
and ρ = ρn(x) denotes the distance function from the set of vertices of the reentrant corners of ΩnR. In this setting, we state the following theorems.

**Theorem 5.1.** We assume
\[
\begin{aligned}
\varphi_i &\in W^{1, p}(\Omega_n^R), & i &= 1, 2, \\
\varphi_1 &\leq \varphi_2 \text{ in } \Omega_n^R, & \varphi_1 \leq 0 &\leq \varphi_2 \text{ in } \partial \Omega_n^R,
\end{aligned}
\]  
(5.2)
and
\[
\begin{aligned}
k &\neq 0 \\
f_i, A_p(\varphi_i) &\in L^\infty(\Omega_n^R), & i &= 1, 2, \\
A_p(\varphi_2) \wedge & f \geq 0.
\end{aligned}
\]  
(5.3)
Then the solution u_n of obstacle problem (2.1) in ΩnR belongs to the weighted Sobolev space
\[
H^{2, \mu}(\Omega_n^R), \quad \mu > 1 - y,
\]  
(5.4)
where
\[
y = y(p, \chi) = 1 + \frac{p(1-\chi)^2 + (1-\chi) \sqrt{p^2 - (2-\chi)(p-2)^2}}{2\chi(2-\chi)(p-1)}
\]  
(5.5)
with χ = \frac{\omega}{n} and ω in (5.1).

Moreover,\[
\|u_n\|_{H^{2, \mu}(\Omega_n^R)} \leq C \{1 + \|f\|_{L^\infty(\Omega_n^R)} + \|A_p(\varphi_1)\|_{L^\infty(\Omega_n^R)} + \|A_p(\varphi_2)\|_{L^\infty(\Omega_n^R)}\}.
\]  
(5.6)
If k = 0, then an analog of Theorem 4.3 holds.

**Theorem 5.2.** We assume (5.2) and
\[
\begin{aligned}
k &= 0 \\
f_i, A_p(\varphi_i) &\in L^\infty(\Omega_n^R), & i &= 1, 2, \\
A_p(\varphi_2) \wedge & f \geq c^* > 0.
\end{aligned}
\]  
(5.7)
If the solution u_n of obstacle problem (2.1) in ΩnR belongs to the space H^{2, \mu}(\Omega_n^R), then for any q ≥ 1 and p > 2 we obtain
\[
|\nabla u_n|^{\frac{q-1}{q}} \in L^1(\Omega_n^R),
\]  
(5.8)
with
\[
t \geq \frac{q(p + (p-2)^2)}{q + (p-2)^2}.
\]

We introduce the triangulation of the domain ΩnR in order to define the approximate solutions u_h according to the Galerkin method. Let T_h be a partitioning of the domain ΩnR into disjoint, open regular triangles τ, each side being bounded by h so that ΩnR = \bigcup_{\tau \in T_h} τ. Associated with T_h, we consider the finite-dimensional spaces
\[
S_h = \{v \in C(\bar{\Omega}_n^R) : v|_\tau \text{ is affine for all } \tau \in T_h\} \quad \text{and} \quad S_{h, 0} = \{v \in S_h : v = 0 \text{ on } \partial \Omega_n^R\}.
\]
By π_h we denote the interpolation operator \(\pi_h : C(\bar{\Omega}_n^R) \rightarrow S_h\) such that \(\pi_h v(P_i) = v(P_i)\) for any vertex \(P_i\) of the partitioning \(T_h\).

**Definition 5.3.** The family of triangulations \(T_h\) is adapted to the H^{2, \mu}(\Omega_n^R)-regularity if the following conditions hold:

- The vertices of the polygonal curves \(\partial \Omega_n^R\) are nodes of the triangulations.
- The meshes are conformal and regular.
- There exists \(\sigma^* > 0\) such that, as \(h \rightarrow 0\),
\[
\begin{aligned}
h_\tau &\leq \sigma^* h^{\frac{1}{p'}} \quad \text{for all } \tau \in T_h \text{ such that one of the vertices of } \tau \text{ belongs to } \mathbb{R}^n, \\
h_\tau &\leq \sigma^* h \cdot \inf_i \rho^\mu \quad \text{for all } \tau \in T_h \text{ with no vertex in } \mathbb{R}^n.
\end{aligned}
\]

Here \(h = \sup \{h_\tau = \text{diam}(\tau) : \tau \in T_h\}\) is the size of the triangulation and \(\rho = \rho_n(x)\) denotes the distance of the point \(x\) from the set \(\mathbb{R}^n\) of the vertices of the reentrant corners of \(\Omega_n^R\).
The construction of triangulations $T_h$ adapted to the $H^{2,\mu}$-regularity was introduced by Grisvard in [20]. This tool has been fruitfully used for the FEM approximation of linear problems in pre-fractal domains by [1, 13, 14, 23, 38, 39].

Consider the two obstacle problem in the finite-dimensional space $S_{h,0}$:

$$
\text{find } u \in \mathcal{X}_h \text{ such that } a_p(u, v - u) - \int_{\Omega^p_0} f(v - u) \, dx \, dy \geq 0 \quad \text{for all } v \in \mathcal{X}_h,
$$

where

$$
a_p(u, v) = \int_{\Omega^p_0} \left( (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v \, dx \, dy \right) \quad \text{and} \quad \mathcal{X}_h = \{v \in S_{h,0} : \varphi_{1,h} \leq v \leq \varphi_{2,h} \text{ in } \Omega^p_0\},
$$

with $\varphi_{1,h} = \pi_h \varphi_1$ and $\varphi_{2,h} = \pi_h \varphi_2$.

**Proposition 5.4.** Let us assume hypothesis (5.2). Then, for any $f \in L^p(\Omega^p_0)$, there exists a unique function $u_h$ that solves problem (5.9). Moreover,

$$
\|u_h\|_{W^{1,p}(\Omega^p_0)} \leq C[|k| + \|f\|_{L^p(\Omega^p_0)} + \|\varphi_1\|_{W^{1,p}(\Omega^p_0)} + \|\varphi_2\|_{W^{1,p}(\Omega^p_0)}].
$$

As previously, the solution $u_h$ to problem (5.9) realizes the minimum on the convex $\mathcal{X}_h$ of the functional $J_p(\cdot)$, i.e.,

$$
J_p(u) = \min_{v \in \mathcal{X}_h} J_p(v), \quad \text{where} \quad J_p(v) = \frac{1}{p} \int_{\Omega^p_0} \left( (k^2 + |\nabla v|^2)^{\frac{p}{2}} \, dx \, dy - \int_{\Omega^p_0} fv \, dx \, dy.\right.
$$

**Theorem 5.5.** Let us denote by $u_n$ and $u_h$ the solutions of problems (2.1) in $\Omega^n_0$ and (5.9), respectively. Let us assume hypotheses (5.2), (5.3) and

$$
\varphi_i \in H^{2,\mu}(\Omega^n_0), \quad i = 1, 2.
$$

Let $T_h$ be a triangulation of $\Omega^n_0$ adapted to the $H^{2,\mu}(\Omega^n_0)$-regularity of the solution $u_n$. Then

$$
\|u_n - u_h\|_{W^{1,p}(\Omega^n_0)} \leq C h^\frac{\mu}{2} \|u_n\|_{H^{2,\mu}(\Omega^n_0)}
$$

for any

$$
r \in \left[1, \frac{2\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2}}{\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2} + (\chi - 1)(p - 2)}\right], \quad t \in [2, p].
$$

**Proof.** For any $\sigma \in (0, p]$ we put

$$
|v|_{(p,\sigma)} = \left( \int_{\Omega^p_0} \left( |k| + |\nabla u_n| + |\nabla v|^{p-\sigma} |\nabla v|^\sigma \, dx \, dy \right)^{\frac{1}{\sigma}} \right)\frac{1}{\sigma}.
$$

Repeating the proof of [12, Lemma 5.2] (given for $k = 0$), we prove for any $v_h \in \mathcal{X}_h$ and

$$
v \in \mathcal{X}_n := \{v \in W^{1,p}_0(\Omega^n_0) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega^n_0\}
$$

that

$$
|u_n - u_h|^p_{(p,t)} \leq C \|u_n - v_h\|^p_{(p,t)} + \|f + A_p(u_n)\|_{L^2(\Omega^n_0)}(\|u_n - v_h\|_{L^2(\Omega^n_0)} + \|v - u_h\|_{L^2(\Omega^n_0)}),
$$

where $r \in [1, 2]$, $t \in [2, p]$ and the constant $C$ does not depend on $h$. Now we evaluate the terms on the right-hand side in estimate (5.13) by choosing the test functions $v_h \in \mathcal{X}_h$ and $v \in \mathcal{X}_n$ in an appropriate way. According to Theorem 5.1, the function $u_n$ belongs to the weighted Sobolev space $H^{2,\mu}(\Omega^n_0)$ for any $\mu > 1 - \gamma$ (see (5.4) and (5.5)).

We choose $v_h = \pi_h u_n$, and by using approximation estimates of Grisvard (see [20, Section 8.4.1]), we derive

$$
\|u_n - \pi_h u_n\|_{L^2(\Omega^n_0)} \leq C h^\mu \|u_n\|_{H^{2,\mu}(\Omega^n_0)}.
$$
Then we choose \( v = \varphi_2 \land (u_h \lor \varphi_1) \) and, as in [12, Lemma 4.4], we have
\[
\| v - u_h \|^2_{L^2(\Omega_h)} \leq \| \pi_h \varphi_2 - \varphi_2 \|^2_{L^2(\Omega_h)} + \| \pi_h \varphi_1 - \varphi_1 \|^2_{L^2(\Omega_h)}.
\]

Again using Grisvard estimates and assumption (5.10), we derive
\[
\| v - u_h \|_{L^2(\Omega_h)} \leq C h^2.
\]  

We compare the seminorm \( |u_n - u_h|_{W^{1,1}(\Omega_n^h)} \) with \( |u_n - u|^p_{(p,t)} \) (defined in (5.12)) and we obtain
\[
|u_n - u|^p_{(p,t)} \leq C \frac{|u_n - u|_{(p,t)}^p}{|k|^{p-t}}.
\]  

We now evaluate the term \( |u_n - v_h|^p_{(p,t)} \), where \( v_h = \pi_h u_n \). By the embedding of weighted Sobolev spaces in the fractional Sobolev spaces (see, for instance, [33]), \( u_n \) belongs to the space \( W^{\alpha,2}(\Omega_n^h) \) for any \( \alpha < 1 + y \).

Taking into account the Sobolev embedding (see, for instance, [7]), we have
\[
|\nabla u_n| \in L^r(\Omega_n^h) \quad \text{with} \quad r^* = \frac{2}{2 - \alpha}.
\]  

By the Hölder inequality, we obtain
\[
|u_n - v_h|_{(p,t)} \leq C(r)|u_n - \pi_h u_n|_{W^{1,2}(\Omega_n^h)},
\]  

where we have used estimate (5.6) with \( r = \frac{2(\sigma - p)}{p - 2} \). Hence, as \( \alpha < 1 + y \), \( r^* \) is given in (5.17) and \( y \) in (5.5), we have to choose \( r < p + \frac{2p}{r} \) and we obtain by calculations that
\[
r < \frac{2\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2}}{\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2 + (\chi - 1)(p - 2)}}.
\]  

Now we use [20, Theorem 8.4.1.6] and we obtain
\[
|u_n - \pi_h u_n|_{W^{1,2}(\Omega_n^h)} \leq C h.
\]  

By taking into account estimates (5.13)–(5.16), (5.18) and (5.19), we conclude the proof using once again the Poincaré inequality.

We note that in Theorem 5.5 we assume \( k \neq 0 \); if \( k = 0 \) the following result holds.

**Theorem 5.6.** Let us denote by \( u_n \) and \( u_h \) the solutions of problems (2.1) in \( \Omega_n^h \) and (5.9), respectively. Let us assume hypotheses (5.2), (5.7), (5.10) and that the solution \( u_n \) belongs to the space \( H^{2,\mu}(\Omega_n^h) \). Let \( T_h \) be a triangulation of \( \Omega_n^h \) adapted to the \( H^{2,\mu}(\Omega_n^h) \)-regularity of the solution \( u_n \). Then
\[
|u_n - u_h|_{W^{1,1}(\Omega_n^h)} \leq C h^\gamma |u_n|_{H^{2,\mu}(\Omega_n^h)}
\]
for any
\[
r \in \left[ 1, \frac{2\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2}}{\sqrt{p^2 - \chi(2 - \chi)(p - 2)^2 + (\chi - 1)(p - 2)}} \right),
\]
t \in [2, p], \( q \in [1, t] \), and for \( q < p \) we require \( t \geq \frac{q(p + (p - 2)/q)}{q(p - 2)/2} \).

**Proof.** We proceed as in the proof of Theorem 5.5: we replace estimate (5.16) by
\[
|u_n - u_h|_{W^{1,1}(\Omega_n^h)} \leq \| |\nabla u_n|^{\frac{(p-t)q}{p-t}} \|_{L^2(\Omega_n^h)} \cdot \| |\nabla u_n - u_h|^{\frac{(p-t)q}{p-t}} \|_{L^2(\Omega_n^h)} \cdot \| \nabla u_n - u_h \|^{p-t} d\Omega_n \leq C |u_n - u_h|^p_{(p,t)}.
\]

Here we have used the Hölder inequality and estimate (5.8).
Remark 5.7. From the previous proofs we deduce that, for the linear case $p = 2$, Theorem 5.5 gives the sharp result of Grisvard (see [20, Corollary 8.4.1.7]): in fact, we have $p = t = 2$ and, in particular, formula (5.18) holds true for $r = 2 = p$.

Remark 5.8. We note that Theorem 5.5 improves the results of [12]: in particular, estimate (5.11) gives a faster convergence than the convergence in [12, estimate (5.63)]. In fact, the solution $u_n$ belongs to the weighted Sobolev space $H^{2,p}(\Omega_n)$ for any $\mu = \mu(p) > 1 - \gamma$. This space is continuously embedded in the fractional Sobolev space $W^{\sigma,2}(\Omega_n)$ for any $\sigma < 2 - \mu$ (see, e.g., [33]). Hence, by the Sobolev embedding (see, e.g., [7]), for any $p \geq 2$, the fractional Sobolev space $W^{\sigma,p}(\Omega_n)$ properly contains the weighted Sobolev space $H^{2,\mu}(\Omega_n)$ for some $\mu = \mu(p) > 1 - \gamma$. Actually, for every $p \geq 2$ the exponent $r$ in (5.11) is strictly greater than $\gamma + \frac{2}{p}$. Namely by writing the expression of $\gamma$ in (5.5) in terms of the parameters $p \in [2, +\infty)$ and $\chi \in (1, 2)$, we obtain that $\gamma + \frac{2}{p} < r$ if and only if

$$\chi(2 - \chi)(p - 1)(p - 2) + (\chi - 1)\sqrt{p^2(\chi - 1)^2 + 4\chi(2 - \chi)(p - 1)} > 0. \quad (5.20)$$

Of course, inequality (5.20) holds for any choice of the parameters.

Remark 5.9. We note that the constant $C$ in estimate (5.11) does not depend on $n$. However, to deduce from (5.11) error estimates for the fractal solution we have to bound the norms $\|u_n\|_{H^{2,\mu}(\Omega)}$ uniformly in $n$. Up to now, this type of results is only established for $p = 2$ (see [9, 11]).

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