Conjugacy classes in discrete Heisenberg groups

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Abstract. We study an extension of a discrete Heisenberg group coming from the theory of loop groups and find invariants of conjugacy classes in this group. In some cases, including the case of the integer Heisenberg group, we make these invariants more explicit.

Keywords: discrete Heisenberg group, conjugacy classes, local fields.

Recall that a Heisenberg group \( \text{Heis}(3, R) \) over a ring \( R \) is the group of matrices
\[
\begin{pmatrix}
1 & n & c \\
0 & 1 & p \\
0 & 0 & 1
\end{pmatrix},
\]
where \( n, p, c \in R \). When \( R = \mathbb{Z} \), we call such a group the integer Heisenberg group. We shall use a more general definition of the discrete Heisenberg group from [1]. The discrete Heisenberg group is the triple of abelian groups \((N, P, C)\) and a pairing \((\cdot, \cdot) : N \times P \to C\). The multiplication is the same as in the case of unipotent matrices:
\[
\begin{pmatrix}
1 & n & c \\
0 & 1 & p \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & n' & c' \\
0 & 1 & p' \\
0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & n + n' & c + c' + (n, p') \\
0 & 1 & p + p' \\
0 & 0 & 1
\end{pmatrix}.
\]

We describe an algebraic geometry construction in which \( \text{Heis}(3, \mathbb{Z}) \) appears (cf. [1]). Let \( X \) be a surface over the field \( k \). We fix a flag \( P \in C \) on \( X \) and assume that the point \( P \) is a smooth point on \( X \) and on the curve \( C \). The 2-dimensional local field \( K_{P,C} \) has the discrete valuation subring \( \widehat{O}_{P,C} \). It is mapped onto the local field \( k(C)_{P} \) on \( C \). This local field contains its own discrete valuation subring \( \widehat{O}_{P} \) and we denote its preimage in \( \widehat{O}_{P,C} \) by \( \widehat{O}'_{P,C} \). We set \( \Gamma_{P,C} := K_{P,C}^{\ast}/(\widehat{O}'_{P,C})^{\ast} \).

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\[ \Gamma_{P,C} \text{ is (noncanonically) isomorphic to } \mathbb{Z} \oplus \mathbb{Z}. \] However, there is a canonical exact sequence of abelian groups

\[ 0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_{P,C} \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (0.1) \]

The map to \( \mathbb{Z} \) in the sequence (0.1) corresponds to the discrete valuation \( \nu_C \) with respect to \( C \) and the subgroup \( \mathbb{Z} \) corresponds to the discrete valuation \( \nu_P \) on \( C \) at \( P \). A choice of local coordinates \( u, t \) in a neighbourhood of \( P \) such that locally \( C = \{ t = 0 \} \) provides a splitting of this exact sequence. The group \( \Gamma_{P,C} \) will then be isomorphic to the subgroup \( \{ t^n u^m, n, m \in \mathbb{Z} \} \) in \( K_{P,C}^* \). A change of local parameter \( t_C \) along the curve \( C \) gives the automorphism of \( \Gamma_{P,C} \) preserving the extension (0.1).

There exists a well-known central extension

\[ 1 \longrightarrow k(C)^*_P \overset{i}{\longrightarrow} \tilde{K}_{P,C}^* \overset{\alpha}{\longrightarrow} K_{P,C}^* \longrightarrow 1, \quad (0.2) \]

for which\(^1\) \( \bigwedge^2 K_{P,C}^* \rightarrow K(C)^*_P \) has tame symbol without sign \( (f, g) \mapsto f^{\nu(g)} g^{-\nu(f)}|_C \). One can find an explicit description of this extension using torsors in [2].

Let us denote by \( s \) a canonical section of extension (0.2) over the subgroup \( \check{O}_{P,C}^* \). One may prove that the subgroup \( s((\check{O}'_{P,C})^*)i(\check{O}_P) \) is normal in \( \tilde{K}_{P,C} \). We put \( \tilde{\Gamma}_{P,C} := \tilde{K}_{P,C}/s((\check{O}'_{P,C})^*)i(\check{O}_P) \). Then we have the central extension

\[ 1 \longrightarrow k(C)^*_P/\check{O}_P \longrightarrow \tilde{\Gamma}_{P,C} \longrightarrow \Gamma_{P,C} \longrightarrow 1. \quad (0.3) \]

The map obtained by the commutator of liftings for the extension (0.3) is the standard symplectic form on \( \mathbb{Z}^2 \) so \( \tilde{\Gamma}_{P,C} \simeq \text{Heis}(3, \mathbb{Z}) \). One can find in [1] the construction of some global Heisenberg-type groups corresponding to \( X \). These groups encode a lot of information about \( X \), for example, the Chow groups can be recovered from them. The representation theory of discrete Heisenberg groups is studied in the series of articles [1], [3], [4]. It turns out to be more convenient to consider some extension of the Heisenberg group using a construction from the theory of loop groups. The point is that characters of irreducible representations in the extended Heisenberg group are not just distributions but functions. An important problem is to investigate central functions on the extended Heisenberg group and to obtain a Plancherel-type theorem. That is an explanation of our interest in conjugacy classes in the extended Heisenberg group. In this article we construct a full system of invariants, that is, a set of invariants such that two elements are conjugate if and only if values of invariants from the set coincide on these elements.

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\section*{§ 1. The construction of the extended Heisenberg group}

For abelian groups \( P \) and \( C \) we can consider the group \( \text{Aut}_{gr}(P \oplus C) \) of automorphisms which preserve the filtration \( 0 < C < P \oplus C \) and act trivially on its

\(^1\)This map is obtained from the commutator of liftings of two elements.
adjoint quotients (we call such automorphisms graded). The Heisenberg group can be represented as a semidirect product \((P \oplus C) \rtimes N\) where \(N\) is a subgroup in \(\text{Aut}_{\text{gr}}(P \oplus C)\). A graded automorphism can be given by the homomorphism of \(P\) to \(P \oplus C\) such that the following diagram is commutative:

\[
P \xrightarrow{id} P \oplus C
\]

So this homomorphism is the pair \((\text{id}, n : P \to C)\). It implies that \(\text{Aut}_{\text{gr}}(P \oplus C)\) is abelian and that there is a pairing

\[
\text{Aut}_{\text{gr}}(P \oplus C) \times P \to C,
\]

which gives us the pairing \(N \times P \to C\) from the definition of the Heisenberg group. \(N\) acts on \(P \oplus C\) in the following way:

\[
n(p, c) = (p, c + n(p)).
\]

Now we examine a more general situation. Let \(G = A \rtimes B\). We are interested in graded automorphisms of \(G\), that is, ones preserving the filtration \((1) \subset A \subset A \rtimes B\) and acting trivially on its adjoint quotients. Any such automorphism is defined by a homomorphism \(\varphi : B \to G\) with some conditions. Let \(\varphi(b) = \alpha(b)b\), where \(b \in B, \alpha(b) \in A\). What conditions on \(\alpha\) should we apply to have a graded automorphism?

**Lemma 1.** The map \(\varphi\) defines a graded automorphism \(f\) of \(G\) if and only if the following two conditions hold:

1. \(\alpha(b)\) belongs to \(Z(A)\), the centre of \(A\);
2. \(\alpha\) is a crossed homomorphism \(B \to Z(A)\), that is, we have \(\alpha(b_1)b_1\alpha(b_2)b_2^{-1} = \alpha(b_1b_2)\).

**Proof.** One can prove that \(f\) is an automorphism if and only if \(\varphi\) is a homomorphism preserving the conjugacy action of \(B\) on \(A\). The second condition is equivalent to \(\varphi\) being a homomorphism:

\[
\alpha(b_1)b_1\alpha(b_2)b_2 = \alpha(b_1b_2)b_1b_2, \quad \alpha(b_1)b_1\alpha(b_2)b_2^{-1} = \alpha(b_1b_2).
\]

The first condition is equivalent to \(\varphi\) preserving the conjugacy action of \(B\) on \(A\):

\[
\varphi(b)(a) = f(b(a)) = b(a).
\]

So \(\alpha(b)\) commutes with every \(a \in A\).

Now we consider a group \((A \rtimes B) \rtimes \text{Aut}_{\text{gr}} G\). The following definition is slightly more general than the analogue from [1].

**Definition.** An *extended Heisenberg group* is the group \(((P \oplus C) \rtimes N) \rtimes K\) for some subgroup \(K\) in \(\text{Aut}_{\text{gr}}((P \oplus C) \rtimes N)\).
Let us find crossed homomorphisms \( k: N \to P \oplus C \) satisfying the conditions of Lemma 1. Let \( k(n) = (k_p(n), k_c(n)) \). As \( P \oplus C \) is abelian the first condition is satisfied. We check the second condition. Let \( k(n) = (k_p(n), k_c(n)) \):

\[
\begin{align*}
(k_p(n_1), k_c(n_1))(k_p(n_2), k_c(n_2) + n_1(k_p(n_2))) &= (k_p(n_1 + n_2), k_c(n_1 + n_2)), \\
(k_p(n_1) + k_p(n_2), k_c(n_1) + k_c(n_2) + n_1(k_p(n_2))) &= (k_p(n_1 + n_2), k_c(n_1 + n_2)).
\end{align*}
\]

So \( k_p: N \to P \) is a homomorphism and for \( k_c \) we have the condition:

\[
k_c(n_1 + n_2) = k_c(n_1) + k_c(n_2) + n_1(k_p(n_2)). \tag{1.1}
\]

Remark 1. From (1.1) it follows that \( n_1(k_p(n_2)) = n_2(k_p(n_1)) \) because \( N \) is abelian.

Remark 2. Condition (1.1) holds when we add some homomorphism \( k^e_c: N \to C \) to \( k_c \).

Example 1. The map \( k_c(n) = n(n-1)/2: \mathbb{Z} \to \mathbb{Z} \) satisfies condition (1.1) for \( k_p(n) = n: \mathbb{Z} \to \mathbb{Z} \). So the following action of \( K = \mathbb{Z} \) on the integer Heisenberg group is graded:

\[
k(p, c, n) = \left( p + kn, c + k \frac{n(n-1)}{2}, n \right).
\]

And the definition gives the extended integer Heisenberg group.

So an element \( k \) of \( \text{Aut}_{gr}((P \oplus C) \ltimes N) \) is given by a pair \((k_p, k_c)\), where \( k_p: N \to P \) is a homomorphism and \( k_c: N \to C \) satisfies (1.1). The action of \( k \) on \((P \oplus C) \ltimes N\) is as follows:

\[
k(p, c, n) = (p + k_p(n), c + k_c(n), n).
\]

Compositions of such actions correspond to the mappings \( k_p = k^1_p + k^2_p \) and \( k_c = k^1_c + k^2_c \), so \( \text{Aut}_{gr} \) is abelian. Let us study the functions \( k_c \) more closely.

Lemma 2. Let \( N = \mathbb{Z}/l_1Z e_1 \oplus \mathbb{Z}/l_2Z e_2 \oplus \cdots \oplus \mathbb{Z}/l_sZ e_s \), where \( l_i \in \mathbb{Z} \), and let us have a homomorphism \( k_p: N \to P \). Then \( k_c \) with condition (1.1) exists if and only if:

1) \( n_1(k_p(n_2)) = n_2(k_p(n_1)) \) for any \( n_1, n_2 \);

2) \( \frac{l_i(l_i - 1)}{2} e_i(k_p(e_i)) \in l_iC \).

If \( k_c \) exists then it can be represented as:

\[
k_c(m_1 e_1 + \cdots + m_s e_s) = \sum_{i<j} m_i m_j e_i(k_p(e_j)) + \sum_i \frac{1}{2} m_i (m_i - 1) e_i(k_p(e_i)) + m_i x_i + k^e_c(m_1 e_1 + \cdots + m_s e_s),
\]

where \( m_i \in \mathbb{Z} \), \( k^e_c \) is a homomorphism from \( N \) to \( C \) and \( x_i \) is any solution of the equation

\[
l_i x_i + \frac{l_i(l_i - 1)}{2} e_i(k_p(e_i)) = 0.
\]
Proof. It is not difficult to prove that \( k_c(0) \) equals zero. Also one can prove using condition (1.1) that

\[
k_c(ne_i) = nk_c(e_i) + \frac{n(n-1)}{2} e_i k_p(e_i).
\]

This formula gives us a map \( k'_c(n) \) from \( \mathbb{Z} \) to \( N \) satisfying condition (1.1) for \( k'_p(n) = k_p(ne_i) \). The map \( k'_c \) comes from some \( k_c \) with condition (1.1) from \( \mathbb{Z}/l_i\mathbb{Z}e_i \) to \( N \) if and only if \( k'_c(l_i) = 0 \). So we need condition 2) to define \( k_c \) on \( \mathbb{Z}/l_i\mathbb{Z}e_i \).

Condition 1) is necessary because \( k_c(n_1 + n_2) = k_c(n_2 + n_1) \).

Now suppose that conditions 1) and 2) hold and let us prove that there exists a function \( k_c \) with the required properties. Define \( x_i \in C \) as any solution of the equation

\[
l_i x_i + \frac{l_i(l_i - 1)}{2} e_i(k_p(e_i)) = 0.
\]

Consider a function \( k''_c : N \to C \) defined by the formula

\[
k''_c(m_1e_1 + \cdots + m_se_s) = \sum_{i<j} m_im_j e_i(k_p(e_j)) + \sum_{i} \frac{m_i(m_i - 1)}{2} e_i(k_p(e_i)) + m_ix_i.
\]

Condition 1) implies that this formula does not depend on the order of the summands. Also we have that

\[
\frac{m_i(m_i - 1)}{2} e_i(k_p(e_i)) + m_ix_i = \frac{(m_i + l_i)(m_i + l_i - 1)}{2} e_i(k_p(e_i)) + (m_i + l_i)x_i
\]

because of condition 2) and since \( l_i k_p(e_i) = 0 \). So the function \( k''_c \) is well defined. One can prove that it satisfies condition (1.1). Finally, if there exists some \( k_c \), then the function \( k_c - k''_c \) is a homomorphism. This proves the last statement.

Remark 3. For any function \( \gamma : N \to C \) we can consider its polarization \( \tilde{\gamma}(n_1, n_2) = \gamma(n_1 + n_2) - \gamma(n_1) - \gamma(n_2) \). It is clear that if polarizations of two functions are equal then the difference of these functions is a homomorphism \( N \to C \). Polarizations of \( 2k_c(n) \) and \( nk_p(n) \) are equal so \( 2k_c(n) = \varphi_k(n) + n(k_p(n)) \), where \( \varphi_k(n) \) is a homomorphism \( N \to C \) depending on \( k \).

§ 2. Conjugacy classes in a Heisenberg group

Consider an extended Heisenberg group \( G = ((P \oplus C) \times N) \times K \). The formula for multiplication in this group is

\[
(p', c', n', k') * (p_1, c_1, n_1, k_1)
= (p' + p_1 + k'_p(n_1), c_1 + c' + k'_c(n_1) + n'(p_1 + k'_p(n_1)), n_1 + n', k_1 + k').
\]

Let \( x_2 = (p_2, n_2, c_2, k_2) \) be conjugate to \( x_1 = (p_1, n_1, c_1, k_1) \) by means of \( (p', n', c', k') \). This is equivalent to

\[
(p' + p_1 + k'_p(n_1), c_1 + c' + k'_c(n_1) + n'(p_1 + k'_p(n_1)), n_1 + n', k_1 + k')
= (p_2 + p', k_2, n', c_2 + c' + k_2, n' + n_2(p' + k_2, n'), n_2 + n', k_2 + k').
\]
So we get that $n_1 = n_2 = n$, $k_1 = k_2 = k$ and
\[
\begin{cases}
p_1 - p_2 = k_p(n') - k'_p(n), \\
c_1 + k'_c(n) + n'(p_1 + k'_p(n)) = c_2 + k_c(n') + n(p' + k_p(n')).
\end{cases}
\tag{2.1}
\]

Using the summand $n(p')$ in the second equation of (2.1) and Remark 1 we can reduce the second equation (mod Im $n$), where $n: P \to C$ is a homeomorphism given for fixed $n \in N$. So we get
\[
\begin{cases}
p_1 - p_2 = k_p(n') - k'_p(n), \\
c_1 - c_2 = k_c(n') - k'_c(n) - n'(p_1) \pmod{\text{Im } n}.
\end{cases}
\tag{2.2}
\]

Now we fix $n$ and $k$ until the end of the next section and obtain a system of equations on $n'$, $k'$ with coefficients depending on $p_1$, $c_1$.

Let us denote by $\Lambda(n', k')$ the homomorphism $k_p(n') - k'_p(n)$ from $N \oplus K$ to $P$. It is obvious that $R((p, c, n, k)) = p \in P/\text{Im } \Lambda$ is an invariant of the conjugacy class. Suppose that
\[\text{all considered elements have the same value } r \text{ of the invariant } R.\tag{2.3}\]

Let us denote by $B_{ij}$ the map $k_c(n') - k'_c(n) - n'(p_1) \text{ from } N \oplus K \text{ to } C/\text{Im } n$. We notice that $B_{12}$ depends only on the first index. Now we consider the sets $V_{12} := c_1 - c_2 - B_{12}(\Lambda^{-1}(p_1 - p_2)) \subset C/\text{Im } n$. The indices 12 mean that $B_{12}$ and $V_{12}$ correspond to the pair $(x_1, x_2)$. In the same way we can define $V_{ij}$ when we have a collection $\{x_i \in G, i \in I\}$.

**Lemma 3.** Under condition (2.3) we have
1) $V_{12} = -V_{21}$;
2) $V_{12} + V_{23} = V_{13}$.

**Proof.** 1) Let $n'$, $k'$ be elements of $\Lambda^{-1}(p_1 - p_2)$. Then $-n'$, $-k'$ are elements of $\Lambda^{-1}(p_2 - p_1)$ and it is sufficient to prove that $B_{12}(n', k') = -B_{21}(-n', -k')$.

We have the following chain of equalities:
\[
B_{12}(n', k') + B_{21}(-n', -k') = (k_c(n') - k'_c(n) - n'(p_1)) + k_c(-n') + k'_c(n) + n'(p_2)
= k_c(n') + k_c(-n') + n'(p_2 - p_1)
= k_c(0) - n'(k_p(-n')) + n'(k'_p(n) - k_p(n')) = 0.
\]

Here we use the definitions of $B_{ij}$ and $\Lambda$, the equality $n'(k'_p(n)) = n(k'_p(n')) = 0$ in $C/\text{Im } n$, the equality $k_c(0) = 0$ and equation (1.1).

2) Let $n'_1$, $k'_1$ and $n'_2$, $k'_2$ be elements of $\Lambda^{-1}(p_1 - p_2)$ and $\Lambda^{-1}(p_2 - p_3)$, respectively. Then we have
\[
B_{12}(n'_1, k'_1) + B_{23}(n'_2, k'_2)
= (k_c(n'_1) - k'_1 c(n)) - n'_1(p_1)) + (k_c(n'_2) - k'_2 c(n) - n'_2(p_2))
= (k_c(n'_1) + k_c(n'_2)) + n'_2(p_1 - p_2) - (k'_1 c + k'_2 c)(n) - (n'_1 + n'_2) + (p_1)
= (k_c(n'_1 + n'_2) - n'_1 k_p(n)) + n'_2 (k_p(n') - k'_1 p(n)) - (k'_1 c + k'_2 c(n) - (n'_1 + n'_2)(p_1)
= -n'_1 k_p(n) + n'_2 (k_p(n) - k'_1 p(n)) + B_{13}(n'_1 + n'_2, k'_1 + k'_2)
= B_{13}(n'_1 + n'_2, k'_1 + k'_2).
\]
In the fourth equality we use the first equation in system (2.2), which holds because of condition (2.3). Also we use that \( n'(k_p'(n')) = n(k_p'(n')) = 0 \) in the fifth equality. So from the chain of equations we get that \( V_{12} + V_{23} \subset V_{13} \). Using the fact that \( V_{23} = -V_{32} \) we get the reverse inclusion.

**Remark 4.** Note that \( V = B_{12}(\ker \Lambda) \) does not depend on a pair of elements. It only depends on the value of \( R \). Indeed, we have \( B_{12}(\ker \Lambda) = V_{11} \) and by Lemma 3,

\[
V_{11} = V_{12} + V_{21} = V_{21} + V_{12} = V_{22} = B_{23}(\ker \Lambda).
\]

So \( V \) is also an invariant of a conjugacy class. We have \( V_{11} + V_{11} = V_{11} \) and \( 0 \in V_{11} \) because \( V_{11} = -V_{11} \). So \( V \) is a group.

**Lemma 4.** \( V_{12} \) is a coset of the group \( C/\im n \) by the subgroup \( B_{12}(\ker \Lambda) \).

**Proof.** By Remark 4 we have that \( V_{11} \) is a group. If we fix some element \( x \) in \( V_{12} \), then by Lemma 3 we have that \( x - V_{12} \subset V_{11} \) and on the other hand \( x + V_{11} \subset V_{12} \), so \( x - V_{12} = V_{11} = B_{12}(\ker \Lambda) \).

The subgroup \( V_{12} \) gives us an element \( v_{12} \) in \( C/V \). And the second equation of the system (2.2) is equivalent to \( v_{12} \) being 0, because in this case \( 0 \in V_{12} \). Let us fix some element \( (p_0, c_0, n, k) \) with \( R((p_0, c_0, n, k)) = r \) and set its index equal to 0; this is a base for the construction of our last invariant. Denote by \( S(x_1) \) the element \( v_{01} \). We have the equality \( v_{01} + v_{12} = v_{02} \). Then \( v_{12} = v_{10} - v_{20} \). So the second equation of the system (2.2) is equivalent to \( S(x_1) = S(x_2) \) and \( S(x) \) is an invariant of a conjugacy class. So we have the set of invariants \( (n, k, R, S) \), where \( n \in N, k \in K, R \in P/\im \Lambda \) and \( S \subset C/V \). This is a full system of invariants because if invariants of two elements coincide then system (2.2) is solvable.

**§ 3.** The case of \( N/\im n \) of odd order and \( \varphi_k = 0 \)

If \( N/\im n \) is finite of an odd order then we can simplify the system (2.2). By Remark 3 in this case we have following equalities

\[
k_c(n') = \frac{1}{2}(2k_c(n')) = \frac{1}{2}(\varphi_k(n') + n'(k_p(n'))) = \frac{1}{2}(\varphi_k(n') + n'(p_1 - p_2 + k_p(n')))
\]

\[
= \frac{1}{2}(\varphi_k(n') + n'(p_1 - p_2)) \pmod{\im n},
\]

where \( \varphi_k \) is the homomorphism from \( N \) to \( C \). We use the first equation from system (2.2) in the third equality. In addition, \( k_p'(n) = \frac{1}{2}\varphi_k'(n) \pmod{\im n} \). Now we suppose that \( \varphi_s = 0 \) for all \( s \in K \). Let us denote by \( \bar{K}(n) \) the group \( \{k_p(n), k \in K\} \) for fixed \( n \). Then we get the new system

\[
\begin{aligned}
&\begin{cases}
p_1 - p_2 = k_p(n') \pmod{K(n)}, \\
2c_1 - 2c_2 = -n'(p_1 + p_2) \pmod{\im n}.
\end{cases}
\end{aligned}
\]

The first invariant here is \( R = p_1 \pmod{K(n) + k_p(N)} \). Now we fix a value \( r \) of \( R \). Let us consider for any \( k \in K \) the homomorphism \( k_p \) which is the composition
of $k_p$ and the natural homomorphism from $P$ to $P/K(n)$. In the notation of the previous section, using divisibility by 2 of elements from $C/\text{Im } n$ we get

$$V'_{12} = 2V_{12} = 2c_1 - 2c_2 - (\tilde{k}_p^{-1}(p_1 - p_2))(p_1 + p_2) \subset C/\text{Im } n.$$ 

The second equation in (3.1) is equivalent to $0 \in V'_{12}$. As before, we have that $V'_{12} + V'_{23} = V'_{13}$ and that $V'_{12}$ is a coset by $V' = \text{Ker } \tilde{k}_p(i_1)$. And as the last invariant we can take $V'_{10} \in C/V'$ where the element with index 0 is some fixed element on which the value of $R$ is equal to $r$.

There is an interesting case when $\tilde{k}_p$ is surjective in $P/K(n)$. In this case the set $\tilde{k}_p^{-1}(p_1)(p_2)$ is a singleton in $C$ because $k_p^{-1}(0)(k_p(n'')) = 0 \pmod{\text{Im } n}$ for any $n'' \in N$ and we have that $\tilde{k}_p^{-1}(p_1)(p_2) = k_p^{-1}(p_2)(p_1)$. Therefore, the second equation is equivalent to $2c_1 - k_p^{-1}(p_1)(p_1) = 2c_2 - k_p^{-1}(p_2)(p_2) \pmod{\text{Im } n}$. So the full system of invariants in this case is

$$\{ n \in N, k \in K, p \pmod{(K(n) + k_p(N))} \in P/(K(n) + k_p(N)), 2c + \tilde{k}_p^{-1}(p)(p) \in C/\text{Im } n \}.$$

§ 4. An extended integer Heisenberg group

Now we consider the case of the Heisenberg group from Example 1. So we assume that $N, P, C, K$ are all isomorphic to $\mathbb{Z}$, the pairings $N \times P \to C$ and $N \times K \to P$ are simply multiplication and $k_c(n) = kn(n - 1)/2$. We need the following lemma.

**Lemma 5.** Suppose that we have a system of integer equations

$$\begin{cases}
ax = b \pmod{n}, \\
cx = d \pmod{n}.
\end{cases} \quad (4.1)$$

Then it has a solution $x \in \mathbb{Z}$ if and only if

$$(a, n)|b \text{ and } d(a, n) - bcw = 0 \pmod{n(a, c, n)},$$

where $(a, n) := \gcd(a, n)$, $(a, c, n) := \gcd(a, c, n)$ and $w$ is any solution of the equation

$$\frac{a}{(a, n)}w = 1 \pmod{\frac{n}{(a, n)}}.$$

**Proof.** It is clear that the condition $(a, n)|b$ is equivalent to the first equation being solvable. If $(a, n)|b$ then the first equation can be rewritten as

$$a'x = b' \pmod{\frac{n}{(a, n)}},$$

where $a', b'$ are $a, b$ divided by $(a, n)$. So we have that $x = wb' + in/(a, n)$, where $i \in \mathbb{Z}$ and $w$ is any solution of the equation

$$\frac{a}{(a, n)}w = 1 \pmod{\frac{n}{(a, n)}}.$$
Substituting this expression into the second equation we get

\[ cb'w - d = -c \frac{n}{(a, n)}i \pmod{n}. \]

This is solvable if and only if \( cb'w - d \) is divisible by \((ca'n/(a, n), n)\). Multiplying by \((a, n)\) we get \( bcw - d(a, n) = 0 \pmod{n(a, c, n)}\) because \((cn, n(a, n)) = n(c, (a, n)) = n(a, c, n)\). Let \((p_1, n_1, c_1, k_1)\) be conjugate to \((p_2, n_2, c_2, k_2)\). Then we have that \(n_1 = n_2 = n\), \(k_1 = k_2 = k\) and there is a system similar to (2.2),

\[
\begin{cases}
  p_1 - p_2 = kn' - k'n, \\
  c_1 - c_2 = k\frac{n'(n' - 1)}{2} - k'n(n - 1) - n'p_1 \pmod{n}.
\end{cases}
\]

(4.2)

Now we consider two cases: \(n\) is odd and \(n\) is even.

First suppose that \(n\) is odd. Then as in the previous section we get the system

\[
\begin{cases}
  p_1 - p_2 = kn' \pmod{n}, \\
  (2c_1 + p_1) - (2c_2 + p_2) = -(p_1 + p_2)n' \pmod{n}.
\end{cases}
\]

(4.3)

Using Lemma 5 we obtain that it is solvable if and only if

\[ p_1 = p_2 \pmod{(k, n)}, \]

\[ -p_1^2w - (n, k)(2c_1 + p_1) = -p_2^2w - (n, k)(2c_2 + p_2) \pmod{n(n, k, p)}, \]

where \(w\) is any fixed solution of the equation

\[ \frac{k}{(k, n)}w = 1 \pmod{n \pmod{(n, k)}}. \]

So in this case

\[ \{n, k, p \pmod{(n, k)}, -p^2w - (n, k)(2c + p) \pmod{n(n, k, p)}\} \]

is a full system of invariants.

Now suppose that \(n\) is even. Multiplying the second equation of (4.2) by 2 and expressing \(kn'\) by the first equation we get the system

\[
\begin{cases}
  p_1 - p_2 + nk' = kn' \pmod{2n}, \\
  (2c_1 + p_1) - (2c_2 + p_2) = -(p_1 + p_2 + nk')n' \pmod{2n}.
\end{cases}
\]

(4.4)

We can consider the first equation above \(\pmod{2n}\) because if the pair \((k', n')\) satisfies the second equation then \((k' + 2l, n')\) also satisfies it, for any \(l \in \mathbb{Z}\). So if we have a solution \((k', n')\) of system (4.4) we can find \(l \in \mathbb{Z}\) such that \((k' + 2l, n')\) satisfies the first equation of (4.2). The equality \(n^2 = 0 \pmod{2n}\) is used above (it holds because \(n\) is even).
We consider two cases $k' \equiv 0 \pmod{2}$ and $k' \equiv 1 \pmod{2}$. Let $w$ be any fixed solution of the equation

$$\frac{k}{(k,2n)}w = 1 \left( \text{mod } \frac{2n}{(2n,k)} \right).$$

In the first case, by Lemma 5, solvability is equivalent to

$$\begin{cases} p_1 = p_2 \pmod{(2n,k)}, \\ -p_1^2w - (2n,k)(2c_1 + p_1) = -p_2^2w - (2n,k)(2c_2 + p_2) \pmod{2n(2n,k,2p_1)}. \end{cases} \tag{4.5}$$

In the second case solvability is equivalent to

$$\begin{cases} p_1 = p_2 + n \pmod{(2n,k)}, \\ -p_1^2w - (2n,k)(2c_1 + p_1) - wn(2p_1 + n) \\ = (-p_2^2w - (2n,k)(2c_2 + p_2)) \pmod{2n(2n,k,2p_1)}. \end{cases} \tag{4.6}$$

We call the first case 0-equivalence and the second case 1-equivalence. Two elements are conjugate if and only if they are 0- or 1-equivalent. The composition of $\delta_1$-equivalence and $\delta_2$-equivalence is $(\delta_1 + \delta_2)$-equivalence, where $\delta_1, \delta_2 \in \mathbb{Z}/2\mathbb{Z}$. Let us introduce the following notations:

$$\begin{align*}
I_1((p,c,n,k)) &:= p \pmod{(2n,k)}, \\
I_2((p,c,n,k)) &:= p + n \pmod{(2n,k)}, \\
J_1((p,c,n,k)) &:= -p_1^2w - (2n,k)(2c_1 + p_1), \\
J_2((p,c,n,k)) &:= -p_2^2w - (2n,k)(2c_1 + p_1) - wn(2p_1 + n).
\end{align*}$$

We proved above that if $x_1$ and $x_2$ are 0-equivalent then $I_1(x_1) = I_1(x_2)$ and $J_1(x_1) = J_1(x_2)$; if $x_1$ and $x_2$ are 1-equivalent then $I_2(x_1) = I_1(x_2)$ and $J_2(x_1) = J_1(x_2)$. It is easy to prove that in the first case we have also that $I_2(x_1) = I_2(x_2)$ and $J_2(x_1) = J_2(x_2)$, in the second case $I_1(x_1) = I_2(x_2)$ and $J_1(x_1) = J_2(x_2)$. So we get that the set $\{J_1(x), J_2(x)\}$ and the set $\{I_1(x), I_2(x)\}$ are invariants of a conjugacy class. So

$$\{n,k,\{I_1(x), I_2(x)\}, \{J_1(x), J_2(x)\}\}$$

is the full system of conjugacy class invariants.

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