Some Results on Betti Series of Universal Modules of Differential Operators

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Abstract

In this article, we discuss the rationality of the Betti series of $\Omega_n(R_m)$ where $\Omega_n(R_m)$ denotes the universal module of $n$th order derivations of $R_m$. We proved that if $R$ is a coordinate ring of an affine irreducible curve represented by $\frac{k[x_1,x_2,...,x_l]}{(f)}$ and if it has at most one singularity point, then the Betti series of $\Omega_n(R_m)$ is rational where $m$ is a maximal ideal of $R$.

1 Introduction and Preliminaries

The following notations will be fixed throughout the paper: ring means commutative with identity and $R$ is a commutative $k$-algebra where $k$ is an algebraically closed field of characteristic zero.

An $n$th order $k$-derivation $D$ of $R$ into an $R$-module $F$ is an element of $Hom_k(R,F)$ such that for any $n+1$ elements $r_0, r_1, \ldots, r_n$ of $R$, the following identity holds:

$$D(r_0r_1 \ldots r_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{j_1 < j_2 < \ldots < j_i} r_{j_1} \ldots r_{j_i} D(r_0 \ldots \hat{r}_{j_1} \ldots \hat{r}_{j_i} \ldots r_n)$$

where the hat over $r_i$’s means that it is missed. It can be easily seen that a first order derivation is the ordinary derivation of $R$ into an $R$-module $F$.

In [5], a universal object for $n$th order derivations constructed in the following way: Consider the exact sequence

$$0 \rightarrow I \rightarrow R \otimes_k R \xrightarrow{\varphi} R \rightarrow 0$$

where $\varphi$ is defined as $\varphi(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} a_i b_i$ for $a_i, b_i \in R$ and $I$ is the kernel of $\varphi$. It is known that $\ker \varphi$ is generated by the set

$$\{1 \otimes r - r \otimes 1 : r \in R\}$$

as an $R$-module. Then the mapping $d_n$ from $R$ into $I/I^{n+1}$ given by

$$d_n(r) = 1 \otimes r - r \otimes 1 + I^{n+1}$$

is called the universal derivation of order $n$ that is, any $n$th order derivation $D$ from $R$ into $F$ can be factored through $I/I^{n+1}$ where $F$ is an $R$-module. Here, the $R$-module $I/I^{n+1}$ is called the universal module of $n$th order derivations and is denoted by $\Omega_n(R)$.

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Note that if $R$ is a finitely generated $k$-algebra, then $\Omega_n(R)$ is a finitely generated $R$-module. It is proved in [5] Prop. 2] that if $R = k[x_1,\ldots,x_s]$ is a polynomial algebra over $k$ with $s$ variables, then $\Omega_n(R)$ is a free $R$-module of rank $\binom{n+s}{s} - 1$ with basis
$$\{d_n(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_s^{\alpha_s}) : 1 \leq \alpha_1 + \alpha_2 + \ldots + \alpha_s \leq n\}$$
and in [5] Theo. 9] that
$$\Omega_n(R) \otimes_R R_S \cong \Omega_n(R_S)$$
where $S$ is a multiplicatively closed subset of $R$.

A free resolution of $\Omega_n(R)$ where $R$ is a local $k$-algebra with maximal ideal $m$ is called a minimal resolution if the followings are satisfied:

$$\ldots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \Omega_n(R) \to 0$$

$F_i$’s are free $R$-modules of finite rank for all $i$ and $\partial_n(F_n) \subseteq m F_{n-1}$ for all $n \geq 1$ (see [4] for definition).

Let $(R, m)$ be a local ring. The Betti series of $\Omega_n(R)$ is defined to be the series
$$B(\Omega_n(R), t) = \sum_{i \geq 0} \dim_{R/m} \text{Ext}^i(\Omega_n(R), \frac{R}{m}) t^i$$
for all $n \geq 1$.

**Lemma 1** Let $R$ be a local ring with maximal ideal $m$ and $M$ be a finitely generated $R$-module. Suppose that
$$0 \to F_1 \xrightarrow{\partial} F_0 \to M \to 0$$
is a minimal resolution of $M$. Then $\text{Ext}^1(M, R/m)$ is not zero.

Dealing with $n$th order case presents extra difficulties. Let $R = k[x, y, z]$ be a $k$-algebra with $z^2 = x^3$ and $y^2 = xz$. It is shown in [1] ex. 3.1.6 and 3.4.7] that $\text{pd}(\Omega_1(R)) \leq 1$, but $\text{pd}(\Omega_2(R))$ is not finite.

In [6], the following proposition is proved:

**Proposition 2** Let $R = k[x_1,\ldots,x_s]$ and $S = k[y_1,\ldots,y_t]$ be polynomial algebras and let $I$ be an ideal of $R$ generated by elements $\{f_1,\ldots,f_m\}$. Assume that $R/I = k[x_1,\ldots,x_s]/(f_1,\ldots,f_m)$ is an affine $k$-algebra of dimension $s - m$ and $\text{pd}(\Omega_2(R/I \otimes_k S)) \leq 1$.

But, unfortunately, this result is not true even for $n = 3$. So, there are two natural questions arise from these results. Can we generalize the dimension of the ring $R$? Can we generalize the dimension of the universal module $\Omega_n$? In [2] and [3], it is studied the following question:

When is the Betti series of a universal module of second order derivations rational?

Our goal is to establish a result analogue of this question for $n$th order universal differential operator modules.
2 Main Results

Proposition 3 Let $k[x_1, x_2, \ldots, x_s]$ be a polynomial algebra and $m$ be a maximal ideal of $k[x_1, x_2, \ldots, x_s]$ containing an irreducible element $f$. If the elements $d_n(x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_s^{\alpha_s}f)$ belong to $m\Omega_n(k[x_1, x_2, \ldots, x_s])$ whenever $0 \leq \alpha_1 + \alpha_2 + \ldots + \alpha_s \leq n - 1$, then $\Omega_n(k[x_1, x_2, \ldots, x_s]/(f))$ admits a minimal resolution of $(k[x_1, x_2, \ldots, x_s]/(f))_m$-modules where $\bar{m} = m/(f)$ is a maximal ideal of $(k[x_1, x_2, \ldots, x_s]/(f))_m$.

Proof. Let $R = S/I = k[x_1, x_2, \ldots, x_s]$ and $\bar{m}$ be a maximal ideal of $R$. Then by [5, Theo. 14 pg. 24] we have the following short exact sequence of $R$-modules

\[ 0 \longrightarrow \frac{N + f\Omega_n(S)}{f\Omega_n(S)} \longrightarrow \frac{\Omega_n(S)}{f\Omega_n(S)} \longrightarrow \Omega_n(R) \longrightarrow 0 \]  

where $N$ is a submodule of $\Omega_n(S)$ generated by the elements of the form

\[ \{d_n(g) : g \in f\Omega_n(k[x_1, x_2, \ldots, x_s])\}. \]

By localizing (1) at $\bar{m}$, we get the following exact sequence of $R_\bar{m}$-modules:

\[ 0 \longrightarrow \left(\frac{N + f\Omega_n(S)}{f\Omega_n(S)}\right)_\bar{m} \longrightarrow \frac{\Omega_n(S)}{f\Omega_n(S)}_\bar{m} \longrightarrow \Omega_n(R)_\bar{m} \longrightarrow 0. \]

Step 1. A module generated by the set $\{d_n(g) : g \in f\Omega_n(k[x_1, x_2, \ldots, x_s])\}$ is a submodule of $m\Omega_n(k[x_1, x_2, \ldots, x_s])$.

Proof of Step 1. Since $d_n$ is $k$-linear, it suffices to show

\[ d_n(x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_s^{\alpha_s}f) \in m\Omega_n(k[x_1, x_2, \ldots, x_s]). \]

By using the properties of $d_n$, we get

\[ d_n(x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_s^{\alpha_s}f) = \sum_{\gamma} a_{\gamma}(x_1, x_2, \ldots, x_s) d_n(x_1^{\gamma_1}x_2^{\gamma_2} \ldots x_s^{\gamma_s}f) + f(\sum_{\beta} a_{\beta}(x_1, x_2, \ldots, x_s) d_n(x_1^{\beta_1}x_2^{\beta_2} \ldots x_s^{\beta_s})) \]

where $a_{\gamma}(x_1, x_2, \ldots, x_s), a_{\beta}(x_1, x_2, \ldots, x_s) \in k[x_1, x_2, \ldots, x_s], 0 \leq \gamma_1 + \gamma_2 + \ldots + \gamma_s \leq n - 1, 0 < \beta_1 + \beta_2 + \ldots + \beta_s \leq n$. By assumption, we know

\[ d_n(x_1^{\gamma_1}x_2^{\gamma_2} \ldots x_s^{\gamma_s}f) \in m\Omega_n(k[x_1, x_2, \ldots, x_s]) \]

whenever $0 \leq \gamma_1 + \gamma_2 + \ldots + \gamma_s \leq n - 1$ and $f \in m$, then the result follows.

Step 2. $(\frac{N + f\Omega_n(S)}{f\Omega_n(S)})_\bar{m} \subseteq \bar{m}(\frac{\Omega_n(S)}{f\Omega_n(S)})_\bar{m}$.

Proof of Step 2. By step 1, we know $N \subseteq m\Omega_n(S)$ and the rest is clear.

Step 3. $(\frac{N + f\Omega_n(S)}{f\Omega_n(S)})_\bar{m}$ is generated by $\binom{n+s-1}{s}$ elements.

Proof of Step 3. It is known that $(\frac{N + f\Omega_n(S)}{f\Omega_n(S)})_\bar{m}$ is generated by the set

\[ \{d_n(x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_s^{\alpha_s}f) + f\Omega_n(S) : 0 \leq \alpha_1 + \alpha_2 + \ldots + \alpha_s \leq n - 1\}. \]

And, it has $\binom{n+s-1}{s}$ elements.

Step 4. $(\frac{N + f\Omega_n(S)}{f\Omega_n(S)})_\bar{m}$ is a free $R_\bar{m}$-module.

Proof of Step 4. The Krull dimension of $R_\bar{m}$ is $s - 1$ and let $K$ be the field of fractions of $R_\bar{m}$. Then by tensoring the exact sequence in (2) by $K$, we get
\[
0 \longrightarrow K \otimes_{R_m} \left( \frac{N + \Omega_n(S)}{f \Omega_n(S)} \right)_m \longrightarrow K \otimes_{R_m} \left( \frac{\Omega_n(S)}{f \Omega_n(S)} \right)_m \longrightarrow K \otimes_{R_m} \Omega_n(R)_m \longrightarrow 0. \quad (3)
\]

We know that, \( \left( \frac{\Omega_n(S)}{f \Omega_n(S)} \right)_m \) is a free \( R_m \)-module of rank \( \binom{n+s}{s} - 1 \). By using the following isomorphism,

\[
K \otimes_{R_m} \Omega_n(R)_m \cong \Omega_n(K)
\]

we have

\[
dim K \otimes_{R_m} \left( \frac{N + \Omega_n(S)}{f \Omega_n(S)} \right)_m = dim K \otimes_{R_m} \left( \frac{\Omega_n(S)}{f \Omega_n(S)} \right)_m - dim \Omega_n(K) = \binom{n+s}{s} - \binom{n+s-1}{s-1} = \binom{n+s-1}{s-1}.
\]

Since the rank of \( \left( \frac{\Omega_n(S)}{f \Omega_n(S)} \right)_m \) is equal to its number of minimal generators, it is a free \( R_m \)-module. Therefore, the short exact sequence given in (2) is a minimal resolution for \( \Omega_n(R_m) \).

Let \( R \) be a finitely generated regular \( k \)-algebra and \( m \) be a maximal ideal of \( R \) then \( \Omega_n(R_m) \) is a free \( R_m \)-module. Then it is clear that \( B(\Omega_n(R_m, t)) \) is rational.

**Theorem 4** Let \( k[x_1, x_2, \ldots, x_s] \) be a polynomial algebra and \( m \) be a maximal ideal of \( k[x_1, x_2, \ldots, x_s] \) containing an irreducible element \( f \). Let \( d_n(x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_s^{\alpha_s}, f) \in m \Omega_n(k[x_1, x_2, \ldots, x_s]) \) for \( 0 \leq \alpha_1 + \alpha_2 + \ldots + \alpha_s \leq n - 1 \). Assume that \( R = \overline{k[x_1, x_2, \ldots, x_s]}(f) \) is not a regular ring at \( \overline{m} = m/(f) \). Then \( B(\Omega_n(R_m, t)) \) is a rational function.

**Proof.** By the previous proposition, the exact sequence of \( R_m \)-modules in (2) is a minimal resolution of \( \Omega_n(R_m) \). And we get the result.

**Question:** It should be interesting to know whether the Betti Series is rational for the algebra \( R = k[x, y, z] \) with \( z^2 = x^3 \) and \( y^2 = xz \).

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