The Capacity of the Semi-Deterministic Cognitive Interference Channel with a Common Cognitive Message and Approximate Capacity for the Gaussian Case

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Abstract—In this paper the study of the cognitive interference channel with a common message, a variation of the classical cognitive interference channel in which the cognitive message is decoded at both receivers. We derive the capacity for the semi-deterministic channel in which the output at the cognitive decoder is a deterministic function of the channel inputs. We also show capacity to within a constant gap and a constant factor for the Gaussian channel in which the outputs are linear combinations of the channel inputs plus an additive Gaussian noise term. Most of these results are shown using an interesting transmission scheme in which the cognitive message, decoded at both receivers, is also pre-coded against the interference experienced at the cognitive decoder. The pre-coding of the cognitive message does not allow the primary decoder to reconstruct the interfering signal. The cognitive message acts instead as a side information at the primary receiver when decoding its intended message.

Index Terms—cognitive interference channel, superposition coding, binning, semi-deterministic channel, approximate capacity.

I. INTRODUCTION

Cognitive networks are transmission networks where the message of one user is known at multiple nodes. The study of cognitive network was inspired by newfound abilities of smart radios to overhear the transmission taking place over the channel and gather information about neighboring nodes [3]. The information theoretical study of cognitive networks has so far focused on small networks with a limited number of users and messages. A classical such model is the cognitive interference channel [2]: a channel where two sets of transmitter/receiver pair communicate over a shared channel, thus interfering with each other transmission. One of the encoders in the network has knowledge of only one of the messages to be transmitted –the primary transmitter– while the other node has knowledge of both messages –the cognitive transmitter. The extra knowledge at the cognitive encoder models a smart and adaptable device that is able to acquire the primary message from previous or simultaneous transmissions. This model has been of great interest in the recent years, see [9] and [10] for a summary of the results, and capacity is known for specific regimes. Capacity is known in the “weak interference” regime of [10], a regime where the interference created by the cognitive transmitter at the primary user is negligible and can be treated. In the “very strong interference” regime of [5] instead, capacity is achieved by having the primary receiver decode the interference created by the cognitive user and strip it from the received signal. Capacity for the Gaussian case is also known in the “primary decodes cognitive” regime of [9], where the cognitive message is decoded at both receivers and pre-coded against the interference created by the primary user at the cognitive decoder.

Despite of these and other results, available for the cognitive interference channel, capacity is not known in general. In particular no capacity result is available for the channel where the cognitive output is a degraded version of the primary output. The difficulty in determining capacity for this channel follows from the fact that pre-coding the cognitive message against the interference from the primary user has the effect of also canceling the primary signal at the primary receiver. To gain new insights on this problem, we focus on a variation of the cognitive interference channel where the primary receiver also decodes the cognitive message.

Paper organization and contributions

- Sec. II: we introduce the model of the cognitive interference channel with a common cognitive message, a variation of the cognitive interference channel where the primary decoder decodes both messages.
- Sec. III: we derive inner and outer bounds to the capacity region. Both bounds are inspired by the cognitive interference channel in the “strong interference” regime. In this regime the primary decoder can reconstruct the channel output of the cognitive receiver after having decoded its intended message.
- Sec. IV: we show the capacity for the semi-deterministic case, that is the channel where the channel output at the cognitive decoder is a deterministic function of the channel inputs while the output at the primary receiver is any random function.
- Sec. V: we derive the capacity in the “very strong interference regime” and the “primary decodes cognitive regime”. In the “very strong interference” regime there is...
no loss of optimality in having both decoders decode both messages while in the “primary decodes cognitive” regime there is no rate loss in having the primary receiver decode the cognitive message.

- **Sec. VI** we prove the capacity region of the Gaussian case to within a constant gap and a constant factor. That is we bound the difference between inner and outer bounds as well as the ratio between the two. These two results characterize the capacity region of the Gaussian case at large and small SNR respectively.

- **Sec. VII** concludes the paper.

II. CHANNEL MODEL

The Cognitive Interference Channel with a Common Cognitive Message (CIFC-CCM), as shown in Fig. 1 is obtained from the classical Cognitive Interference Channel (CIFC) by having the primary decoder decode both messages. It consists of two transmitter-receiver pairs that exchange independent messages over a common channel. Transmitter \( i, i \in \{1, 2\} \), has discrete input alphabet \( X_i \) and its receiver has discrete output alphabet \( Y_i \). The channel is assumed to be memoryless with transition probability \( P_{Y_i|X_i|X_j,X_k} \). Encoder 2 wishes to communicate a message \( W_2 \) uniformly distributed on \( W_2 = [1:2^{NR_2}] \) to decoder 2 in \( N \) channel uses at rate \( R_2 \). Similarly, encoder 1, wishes to communicate a message \( W_1 \) uniformly distributed on \( W_1 = [1:2^{NR_1}] \) to both decoder 1 and decoder 2 in \( N \) channel uses at rate \( R_1 \). Encoder 1 (i.e., the cognitive user) knows its own message \( W_1 \) and that of encoder 2 (i.e. the primary user), \( W_2 \). A rate pair \((R_1, R_2)\) is achievable if there exist sequences of encoding functions

\[
X_i^N = f_{X_i}^N(W_i, W_2), \quad f_{X_i}^N : W_i \times W_2 \rightarrow X_i^N;
\]

\[
X_2^N = f_{X_2}^N(W_2), \quad f_{X_2}^N : W_2 \rightarrow X_2^N,
\]

with corresponding sequences of decoding functions

\[
\hat{W}_i^1 = f_{\hat{W}_i}^1(Y_i^{N}), \quad f_{\hat{W}_i}^1 : Y_i^{N} \rightarrow W_1;
\]

\[
\hat{W}_2^1 = f_{\hat{W}_2}^1(Y_1^{N}), \quad f_{\hat{W}_2}^1 : Y_1^{N} \rightarrow W_1;
\]

\[
\hat{W}_2^2 = f_{\hat{W}_2}^2(Y_2^{N}), \quad f_{\hat{W}_2}^2 : Y_2^{N} \rightarrow W_2.
\]

The capacity region is defined as the closure of the region of achievable \((R_1, R_2)\) pairs \([1]\). Standard strong-typicality is assumed; properties may be found in \([4]\).

In the following we focus in particular on Gaussian CIFC-CCM in Fig. 2. For this class of channels the input/output relationship is:

\[
Y_1 = X_1 + aX_2 + Z_1, \tag{1}
\]

\[
Y_2 = X_2 + b|X_1| + Z_2,
\]

for \(a, b \in \mathbb{C}\) and for \(Z_i \sim \mathcal{N}_C(0,1)\), where the \(\mathcal{N}_C\) indicates complex, circularly symmetric jointly Gaussian RV. Moreover, the channel inputs are subject to the power constraints

\[
\mathbb{E}[X_i] \leq P_i, \quad i \in \{1, 2\}. \tag{2}
\]

A channel where the outputs are obtained from a linear combination of the input plus an additional complex Gaussian term can be reduced to the formulation in (1) and (2) without loss of generality [9] App. A].

III. OUTER AND INNER BOUNDS FOR THE COGNITIVE INTERFERENCE CHANNEL WITH A COMMON COGNITIVE MESSAGE

We start by deriving an outer bound for the capacity region of the general CIFC-CCM. This outer bound is based on the results known for the cognitive interference channel in the “strong interference” regime.

**Theorem III.1. An Outer Bound for the CIFC-CCM**

Any achievable region for the CIFC-CCM is contained in the region

\[
R_1 \leq I(Y_1;X_1|X_2), \tag{3a}
\]

\[
R_1 \leq I(Y_2;X_1|X_2), \tag{3b}
\]

\[
R_1 + R_2 \leq I(Y_2;X_1, X_2), \tag{3c}
\]

union over all the joint distributions of the channel inputs \(P_{X_1,X_2}\).

*Proof:*

The outer bound in (3a) was originally devised for the classical CIFC in [10] and is valid for the CIFC-CCM as well, since the cognitive decoder is decoding only the cognitive
Thus it can be dropped.

Remark III.2. The bound (35) is redundant if

\[ I(X_1; Y_1 | X_2) \leq I(X_1; Y_2 | X_2), \]  

for the distributions \( P_{X_1, X_2} \). Condition (34) corresponds to the “strong interference” regime for the CIFC. Thus, when dropping (35) from the outer bound in Th. III.1 one obtains the “strong interference” outer bound for the CIFC [3]. This outer bound is capacity in the “very strong interference” regime for the general CIFC and is capacity in the “primary decodes cognitive” regime for the Gaussian CIFC.

Rem. III.2 formally defines the relationship between the CIFC in “strong interference” and the CIFC-CCM. For a CIFC in the “strong interference” regime there is no loss of optimality in having the primary decoder decodes both messages. Under condition (34), the rate of the cognitive message is not bounded by the decoding capabilities of the primary receiver. For these reasons, the CIFC is equivalent to the CIFC-CCM when condition (34) holds. We avoid referring to condition (34) as “strong interference” condition as one cannot properly define “interference” in the CIFC-CCM since the primary receiver is decoding both the cognitive message and the primary message.

The next theorem specializes the outer bound of Th. III.1 to the Gaussian channel in (1).

Corollary III.3. An Outer Bound for the Gaussian CIFC-CCM

Any achievable region for the Gaussian CIFC-CCM is contained in the region

\[ R_1 \leq \mathcal{C}(\alpha \min\{1, |b|^2\} P_1), \]  

\[ R_1 + R_2 \leq \mathcal{C}(P_2 + b^2 P_1 + 2 \sqrt{\alpha} |b|^2 P_1 P_2), \]  

for \( \mathcal{C}(x) = \log(1 + x) \).

Proof: The outer bound is obtained from Th. III.1 by noting that complex, circularly symmetric channel inputs maximize all the rate bounds simultaneously.

We now develop an inner bound for the CIFC-CCM as follows: we rate-split the primary message in public and private part. The primary private message is superposed to the public primary message; the cognitive message is superposed to the primary public message and pre-coded against the primary private message.

Theorem III.4. Inner Bounds for the CIFC-CCM

The following region is achievable for a general CIFC-CCM

\[ R_1 \leq I(Y_1; U_{1c} | U_{2c}) - I(U_{1c}; X_2 | U_{2c}), \]  

\[ R_1 \leq I(Y_2; X_1 | X_2, U_{2c}), \]  

\[ R_1 + R_2 \leq I(Y_1; U_{1c}, U_{2c}) + I(Y_2; X_2 | U_{1c}, U_{2c}), \]  

\[ R_1 + R_2 \leq I(Y_2; X_1, X_2), \]  

\[ 2R_1 + R_2 \leq I(Y_1; U_{1c}, U_{2c}) + I(Y_2; X_1, X_2 | U_{2c}) - I(U_{1c}; X_2 | U_{2c}), \]  

for any distribution that factors as \( P_{U_{1c}, U_{2c}, X_1, X_2} \).

Proof: The common cognitive message is embedded in the codeword \( U_{1c}^N \), with rate \( R_{1c} \), while the primary common message in the codeword \( U_{2c}^N \), with rate \( R_{2c} \), and the primary private message in the codeword \( X_2^N \) with rate \( R_2p \). The codeword \( U_{1c}^N \) is binned against \( X_2^N \) and the codewords \( U_{1c}^N \) and \( X_2^N \) are both superposed to \( U_{2c}^N \). The channel input \( X_1^N \) is finally obtained as a deterministic function of \( U_{1c}^N, U_{2c}^N \) and \( X_2^N \). From (6) we obtain the achievable region

\[ R_{1c} \geq I(U_{1c}; X_2 | U_{2c}), \]  

\[ R_{1c} + R_{1c} + R_{2c} \leq I(Y_1; U_{1c}, U_{2c}), \]  

\[ R_{1c} + R_{1c} + R_{2c} \leq I(U_{1c}; X_2 | U_{2c}) + I(U_{1c}; X_2 | U_{2c}), \]  

\[ R_{1c} + R_{1c} \leq I(Y_2; U_{1c} | X_2, U_{2c}) + I(U_{1c}; X_2 | U_{2c}), \]  

\[ 2R_2p \leq I(Y_2; X_1 | U_{1c}, U_{2c}) + I(U_{1c}; X_2 | U_{2c}). \]  

By applying the FME with

\[ R_1 = R_{1c}, \]  

\[ R_2 = R_{2c} + R_{2p}, \]  

we obtain the region in (6).
not required in the CIFC and some error events in the scheme of Th. III.4 are not errors in the achievable scheme of [9, Th. 7].

IV. CAPACITY FOR THE SEMI-DETERMINISTIC COGNITIVE INTERFERENCE CHANNEL WITH A COMMON COGNITIVE MESSAGE

The semi-deterministic CIFC-CCM is a general CIFC-CCM where the channel output at the cognitive decoder is a deterministic function of the channel inputs, i.e. 

$$Y_1 = f(Y_1, X_2)$$

(9)

while the primary output is any random function of the inputs. When condition (9) holds, binning at the cognitive transmitter can fully pre-cancel the effect of the interference at the cognitive receiver thus making (3a) achievable.

**Theorem IV.1. Capacity of the Semi-Deterministic CIFC-CCM**

The capacity of the semi-deterministic channel is 

$$R_1 \leq H(Y_1|X_2)$$

(10a)

$$R_1 \leq I(Y_2; X_1|X_2)$$

(10b)

$$R_1 + R_2 \leq I(Y_2; X_1, X_2)$$

(10c)

union over all the distributions $P_{X_1, X_2}$.

Proof: Consider the transmission scheme in Th. III.4 for $U_{2c} = \emptyset$ to obtain the region 

$$R_1 \leq I(Y_1; U_{1c}) - I(U_{1c}; X_2)$$

(11a)

$$R_1 \leq I(Y_2; X_1|X_2)$$

(11b)

$$R_1 + R_2 \leq I(Y_2; X_1, X_2)$$

(11c)

$$R_1 + R_2 \leq I(Y_1; U_{1c}) + I(Y_2; X_2|U_{1c})$$

(11d)

where we have dropped (6c) since, with $U_{2c} = \emptyset$, 

$$\text{RHS-}(11a) + \text{RHS-}(11c) = \text{RHS-}(6c).$$

(12)

For the assignment $U_{1c} = Y_1$, which is possible given (6a), the inner bound in (11) coincides with (10) since 

$$\text{RHS-}(11d) = H(Y_1) + H(Y_2|Y_1) + H(Y_2|X_1, X_2)$$

$$= I(Y_2; X_1, X_2) + H(Y_1|Y_2) \geq \text{RHS-}(11c).$$

(13a)

which is also equivalent to the outer bound.

V. CAPACITY IN THE “VERY STRONG INTERFERENCE” REGIME AND THE “PRIMARY DECODES COGNITIVE” REGIME

Capacity in the “very strong interference” regime for the CIFC is achieved by having both decoders decode both messages and by superposing the cognitive message over the primary message [5]. This strategy achieves capacity also for a class of CIFC-CCM that we also term “very strong interference” regime. This definition is not fully accurate since the primary receiver decodes both messages, but is coherent with the CIFC literature.

**Theorem V.1. Capacity in the Very Strong Interference Regime**

$$I(Y_2; X_1, X_2) \leq I(Y_1; X_1, X_2),$$

(14)

the region in (3) is capacity.

Proof: Consider the scheme in (6). For $X_2 = U_{2c}$ and $U_{1c} = X_1$ the achievable region is 

$$R_1 \leq I(Y_1; X_1|X_2)$$

(15a)

$$R_1 \leq I(Y_2; X_1|X_2)$$

(15b)

$$R_1 + R_2 \leq I(Y_1; X_1, X_2)$$

(15c)

$$R_1 + R_2 \leq I(Y_2; X_1, X_2).$$

(15d)

Under condition (14) the bound in (15c) can be eliminated from the inner bound and the inner bound is then equivalent to the outer bound in (3).

**Remark V.2.** The “very strong interference” regime for the CIFC is defined by condition (14) and (4). However, condition (4) is not required to prove capacity for the CIFC-CCM.

The following corollary states the result of Th. V.1 for the Gaussian case in (1).

**Corollary V.3. Capacity for the Gaussian CIFC-CCM in the Very Strong Interference Regime**

$$((|a|^2 - 1)P_2 - (|b|^2 - 1)P_1 - 2a|b|\sqrt{P_1 P_2}) \geq 0$$

(16)

the capacity of the Gaussian CIFC-CCM is given by [5].

Proof: Condition (16) is derived from (14) for the Gaussian model in (1). Details can be found in [9, App. B].

We can extend the “primary decodes cognitive” regime of (9) to the Gaussian CIFC-CCM:

**Theorem V.4. The Primary Decodes Cognitive Interference Regime for the CIFC-CCM**

$$P_2 |1 - a| |b|^2 \geq (|b|^2 - 1)(1 + P_1 + |a|^2 P_2) - P_1 P_2 |1 - a| |b|^2$$

$$P_2 |1 - a| |b|^2 \geq (|b|^2 - 1)(1 + P_1 + |a|^2 P_2 + 2Re(a)\sqrt{P_1 P_2}),$$

(17)

then (5) is the capacity of the Gaussian CIFC-CCM.

Proof: Consider the scheme in (11) with the assignment 

$$X_1 \sim N(0, P_1) \quad i \in \{1, 2\}$$

(18a)

$$U_{1c} = X_1 + \frac{\alpha P}{\alpha P_1 + a} X_2,$$

(18b)
which yields the achievable region

\[ R_1 \leq C \left( \alpha \min \{ 1, |b|^2 \} P_1 \right) \]

\[ R_1 + R_2 \leq C \left( |b|^2 P_1 + P_2 + 2 \sqrt{\pi |b|^2 P_1 P_2} \right) \]

\[ + f \left( \alpha + \sqrt{\frac{\alpha P_1}{P_2}}, \frac{\alpha P_1}{\alpha P_1 + 1} \right) \]

\[ - f \left( \frac{1}{|b|^2 + \sqrt{\frac{\alpha P_1}{P_2}} \frac{1}{\alpha P_1 + 1} |b|^2 \sigma^2} \right) \]

\[ R_1 + R_2 \leq C \left( |b|^2 P_1 + P_2 + 2 \sqrt{\pi |b|^2 P_1 P_2} \right), \]

for

\[ f(h, \sigma^2, \lambda) = \log \left( \frac{\sigma^2 + \alpha P_1}{\alpha^2 |h|^2 P_2 + \sigma^2} \right) \]

and

\[ \lambda_{\text{Costa}}(h, \sigma^2) = \frac{\alpha P_1}{\alpha P_1 + \sigma^2}; h. \]

This scheme achieves capacity when (19b) is larger than (19c). The conditions were determined in [9] to prove the “primary decodes cognitive regime” for the CIFC.

Remark V.5. The “primary decodes cognitive regime” for the CIFC is defined by condition (17) and condition (4) which is given by |b| ≥ 1 in the Gaussian case. Condition (4) is not required to prove capacity for the CIFC-CCM. The capacity of the Gaussian CIFC-CCM for |b| ≤ 1 is given by Corollary V.4.

VI. CAPACITY TO WITHIN A CONSTANT GAP AND A CONSTANT FACTOR

Theorem VI.1. Capacity to within 1.87 bits For any Gaussian CIFC-CCM, the outer bound region in (5) can be achieved to within 1.87 bits/s/Hz.

Proof: Capacity is known for |b| ≤ 1. The achievability of the outer bound in (5) to within 1.87 bits/s/Hz for |b| > 1 using the scheme in (11) is shown in [7].

Theorem VI.2. Capacity to within a factor 2 For any Gaussian CIFC-CCM, the outer bound region in (5) can be achieved to within a factor 2.

Proof: Capacity is known for |b| ≤ 1. The achievability of the outer bound in (5) to within a factor 2 for |b| > 1 by a simple time division scheme is shown in [9].

A plot of the capacity results available for the Gaussian CIFC-CCM is depicted in Fig. 4 in the α × b plane we plot the “strong interference” regime (green, right-hatched area) and the “primary decodes cognitive regime” (blue, left-hatched area).

VII. CONCLUSION

In this paper we study a variation of the classical cognitive interference channel where the primary receiver decodes both messages. This channel is related to the cognitive interference channel in the “strong interference” regime and many results for this channel apply to the model under consideration. We derive the capacity for the semi-deterministic case, where the cognitive output is a deterministic function of the channel inputs. We also show capacity in the “strong interference” regime, where there is no rate loss in having both receivers decode both messages. For the Gaussian channel, we determine capacity in the “primary decodes cognitive” regime and determine capacity to within a constant gap and to within a constant factor.