Exact Tail Asymptotics of Dirichlet Distributions

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Abstract: Let \( X \) be a generalised symmetrised Dirichlet random vector in \( \mathbb{R}^k, k \geq 2 \), and let \( u_n, n \geq 1 \) be such that \( \lim_{n \to \infty} P\{X > u_n\} = 0 \). In this paper we derive an exact asymptotic expansion of \( P\{X > u_n\} \) as \( n \to \infty \), assuming that the associated random radius of \( X \) has distribution function in the Gumbel max-domain of attraction.

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1 Introduction

Let \( X = (X_1, \ldots, X_k) \top \) be a random vector in \( \mathbb{R}^k, k \geq 2 \), and let \( u_n, n \geq 1 \) be a positive sequence converging to infinity as \( n \to \infty \). In our notation \( \top \) stands for the transpose sign, and \( \overset{d}{=} \) below for equality of the distribution functions. For any vector \( a = (a_1, \ldots, a_k) \top \in \mathbb{R}^k \setminus (-\infty, 0]^k \) the events

\[ \{X > u_n a\} := \{X_1 > u_n a_1, \ldots, X_k > u_n a_k\}, \quad n \geq 1 \]

are absorbing, i.e., \( \lim_{n \to \infty} P\{X > u_n a\} = 0 \). For such instances it is of interest to determine the rate of convergence to 0 of \( P\{X > u_n a\} \) as \( n \to \infty \). If \( X \) is a standard Gaussian random vector in \( \mathbb{R}^k \) with non-singular covariance matrix \( \Sigma \), then we have the stochastic representation (see e.g. Cambanis et al. (1981))

\[ X \overset{d}{=} R A \top U, \quad (1.1) \]

with \( R > 0 \) such that \( R^2 \) is Chi-squared distributed with \( k \) degrees of freedom, \( A \) a square matrix satisfying \( A \top A = \Sigma \), and \( U = (U_1, \ldots, U_k) \top \) a random vector uniformly distributed on the unit sphere of \( \mathbb{R}^k \) being independent of \( R \).

Results on the tail asymptotics of Gaussian random vectors are well-known, see e.g., Berman (1962), Dai and Mukherjea (2001), Hashorva (2005), Lu and Li (2009) and the references therein.

Indeed, the radial decomposition in (1.1) is quite crucial also in an asymptotic context; it allows us to consider a general random variable \( R \) with some unspecified distribution function \( F \). For such instances \( X \) is an elliptical random vector. If \( F \) is in the Gumbel max-domain of attraction (see (3.13) below), then Theorem 3.1 in Hashorva (2007) implies

\[ P\{X > u_n a\} = (1 + o(1)) \Psi(u_n) P\{R > \mu u_n\}, \quad n \to \infty \quad (1.2) \]

for any \( a \in \mathbb{R}^k \setminus (-\infty, 0]^k \), with \( \Psi \) a known function and \( \mu = (\overline{a} \top \Sigma^{-1} \overline{a})^{1/2} \) with \( \overline{a} \) the unique solution of the quadratic programming problem

\[ P(Sigma^{-1}, a) : \text{minimise } x \top Sigma^{-1} x \text{ under the constraint } x \in [a_1, \infty) \times \cdots \times [a_k, \infty). \quad (1.3) \]

In this paper we are interested in extending (1.2) for \( X \) with stochastic representation (1.1) where \( U \) is a symmetrised Dirichlet random vector with parameter \( \alpha \in (0, \infty)^k \). In the literature such \( X \) is referred to as a generalised symmetrised Dirichlet random vector, introduced and discussed in detail in Fang and Fang (1990).
Further, we write
\[ e \{ \text{up to some constant} \} \] if the index set \( I \) the columns of \( \Sigma \) with indices in
\[ \mathbb{R}^k \] can have the same tail asymptotic behaviour (up to some constant) if the index set \( \{ i : (\overline{\alpha x})_i = 0, \alpha_i \neq 1/2, 1 \leq i \leq k \} \) is empty.

To this end we mention that some possible results and applications related \( [12] \) are: a) density approximation for Dirichlet random vectors, b) asymptotic expansions of conditional distributions, c) identification of the distribution of minima of Dirichlet random vectors, d) asymptotics of concomitants of order statistics, e) estimation of conditional distribution and conditional quantile function, and f) asymptotic independence of sample maxima. In this paper we present few details regarding the last two applications in Example 2.

Organisation of the paper: In Section 2 we give some preliminaries. The main result is presented in Section 3. We provide two illustrating examples in Section 4. Proofs are relegated to Section 5 followed by an Appendix.

## 2 Preliminaries

We shall introduce first some notation. Let in the following \( I, J \) be two non-empty disjoint index sets such that \( I \cup J = \{1, \ldots, k\}, k \geq 2 \), and define for \( x = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k \) the subvector of \( x \) with respect to the index set \( I \) by
\[ x_I := (x_i, i \in I)^\top \in \mathbb{R}^k. \]
If \( \Sigma \in \mathbb{R}^{k \times k} \) is a square matrix, then the matrix \( \Sigma_{IJ} \) is obtained by retaining both the rows and the columns of \( \Sigma \) with indices in \( I \) and \( J \), respectively. Similarly we define \( \Sigma_{JL}, \Sigma_{JJ}, \Sigma_{II} \). For notational simplicity we write \( x_I^\top, \Sigma_{JJ}^{-1} \) instead of \( (x_I)^\top, (\Sigma_{JJ})^{-1} \), respectively. Given \( x, y \in \mathbb{R}^k \) we define
\[
\begin{align*}
x &> y, \text{ if } x_i > y_i, \quad \forall i = 1, \ldots, k, \\
x &\geq y, \text{ if } x_i \geq y_i, \quad \forall i = 1, \ldots, k, \\
x + y &:= (x_1 + y_1, \ldots, x_k + y_k)^\top, \\
x + y &:= (x_1y_1, \ldots, x_ky_k)^\top, \\
\|x_I\|^2 &:= x_I^\top \Sigma_{JJ}^{-1} x_I. 
\end{align*}
\]

Further, we write \( e_i \) (and not \( e_{i,I} \)) for the \( i \)th unit vector in \( \mathbb{R}^{|I|} \), where \( |I| \geq 1 \) denotes the number of elements of \( I \).

We shall be denoting by \( B_{a,b} \) a Beta random variable with positive parameters \( a \) and \( b \) and density function
\[
\frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} x^{a-1} (1-x)^{b-1}, \quad x \in (0,1),
\]
with \( \Gamma(\cdot) \) the Gamma function. Further we write \( Y \sim H \) if the random vector \( Y \in \mathbb{R}^k, k \geq 1 \) has distribution function \( H \), and set \( \overline{H} := 1 - H \).

Throughout this paper \( \alpha := (\alpha_1, \ldots, \alpha_k)^\top \) stands for a vector in \( \mathbb{R}^k \) with positive components, and
\[
\overline{\alpha} := \sum_{i=1}^k \alpha_i, \quad \overline{\alpha}_K := \sum_{i \in K} \alpha_i, \quad K \subset \{1, \ldots, k\}.
\]
When \( K \) is empty, then \( \overline{\alpha}_K \) equals 1.

**Definition 2.1.** A random vector \( U = (U_1, \ldots, U_b)^\top \) in \( \mathbb{R}^k, k \geq 2 \) is said to have symmetrised Dirichlet distribution with parameter \( \alpha \) (henceforth \( U \sim SD(k, \alpha) \)) if \( U^\top U = 1 \) almost surely, and \((U_1, \ldots, U_{b-1})^\top \) possesses the density function \( h \)
where $\Gamma$ is the Gamma distribution with parameters $\alpha, \beta$.

In the following we focus on distribution functions $F$ with an infinite upper endpoint. Referring to Fang and Fang (1990) the density function (when it exists) of a $GSD$ random vector can be defined via a density generator $g$ and the parameter vector $\alpha \in \mathbb{R}^k$. Specifically, let $g$ be a positive measurable function such that

\[ h(x) := \frac{1}{\Gamma(\alpha)} \prod_{i=1}^{k} \Gamma(\alpha_i) \int_0^\infty g(x) x^{\alpha_i - 1} \, dx, \quad x > 0, \quad \forall \alpha_i \in \mathbb{R}^k \]  

(2.4)

Note that if $\alpha = 1/2$ with $1 := (1, \ldots, 1)^\top \in \mathbb{R}^k$, then $U \sim SD(k, \alpha)$ is uniformly distributed on the unit sphere of $\mathbb{R}^k$.

**Definition 2.2.** A random vector $X$ in $\mathbb{R}^k$, $k \geq 2$, is said to possess a generalised symmetrised Dirichlet distribution ($GSD$) if $X \sim U$, where $R > 0$ almost surely being independent of $U$, and $R \sim SD(k, \alpha)$. Write next $X \sim GSD(k, \alpha, F)$.

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(2.5)

holds. In view of the aforementioned paper, see also Theorem 1 in Hashorva et al. (2007), if $X \sim GSD(k, \alpha, F)$, then $X$ possesses the density function $h$ with density generator $g$ defined by

\[ h(x) := g(\sum_{i=1}^{k} |x_i|^2) \prod_{i=1}^{k} |x_i|^{2\alpha_i - 1}, \quad \forall x \in \mathbb{R}, 1 \leq i \leq k, \]  

(2.6)

if and only if $F$ possesses a density function $f$ given in terms of the density generator $g$ by

\[ f(r) = 2 \prod_{i=1}^{k} \Gamma(\alpha_i) g(r^2)^{2\alpha_i - 1}, \quad \forall r \in (0, \infty). \]  

(2.7)

Any subvector of a $GSD$ random vector is again a $GSD$ random vector (see Lemma 6.3 in Appendix).

A canonical example of a $GSD$ random vector is a Kotz Type I $GSD$ random vector in $\mathbb{R}^k$ with density function

\[ h(x) = \frac{\Gamma((N + s) / 2)}{\Gamma((N + s) / 2)} \prod_{i=1}^{k} \Gamma(\alpha_i) (x^\top x)^N \exp(-r(x^\top x)) \prod_{i=1}^{k} |x_i|^{2\alpha_i - 1}, \quad \forall x \in \mathbb{R}^k, \]  

(2.8)

with $r > 0, s > 0, N > -\alpha$, and density generator $g$ given by

\[ g(x) = c x^N \exp(-r x^\top x), \quad x > 0, c > 0. \]

In the standardised case $N = 0$ and $2r = s = 1$ the random vector $X$ possesses independent components with

\[ |X_i|^2 \sim Gamma(\alpha_i, 1/2), \quad \forall i = 1, \ldots, k, \]  

(2.9)

where $Gamma(\alpha_i, 1/2)$ is the Gamma distribution with parameters $\alpha_i, 1/2$.

If $A \in \mathbb{R}^{k \times k}$ is a non-singular square matrix, then $A^\top X$ possesses the density function

\[ h_A(x) := \left( \frac{1}{2} \right)^\pi \prod_{i=1}^{k} \Gamma(\alpha_i) \exp(-||x||^2 / 2) \prod_{i=1}^{k} ||(C x)_i||^{2\alpha_i - 1}, \quad \forall x \in \mathbb{R}^k, \]  

(2.10)

with $C := (A^{-1})^\top, \Sigma := A^\top A$. In particular, if $\alpha_i = 1/2, 1 \leq i \leq k$, then $h_A$ is the density function of a centered Gaussian random vector with covariance matrix $\Sigma$. 

\[ \alpha \]
3 Main Result

Consider a $\mathcal{GSD}$ random vector $X$ in $\mathbb{R}^k, k \geq 2$ with stochastic representation

$$X \overset{d}{=} A^\top RU,$$  \hspace{1cm} (3.11)

where $R \sim F, k \geq 2$ is independent of $U \sim SD(k, \alpha), \alpha \in (0, \infty)^k$, and $A$ is a non-singular $k$-dimensional square matrix. Without loss of generality we assume in the sequel that $\Sigma := A^\top A$ is a correlation matrix, i.e., all the entries of the main diagonal of $\Sigma$ are equal 1.

For a given sequence $u_n, n \geq 1$ of thresholds in $\mathbb{R}^k$ we are interested in the asymptotic behaviour of the joint survivor probability $P\{X > u_n a\}$. As in the elliptical setup (see Hashorva (2007)) it turns out that the tail asymptotics under consideration is closely related to the solution of the quadratic programming problem (1.3). If

$$\Sigma^{-1} a \geq 0, \quad 0 := (0,\ldots,0)^\top \in \mathbb{R}^k$$  \hspace{1cm} (3.12)

is satisfied, then the minimum of the quadratic programming problem (1.3) is attained at $a$, otherwise there exists a unique non-empty index set $I \subset \{1,\ldots,k\}$ which defines the unique solution $\overline{a}$ of $\mathcal{P}(\Sigma^{-1}, a)$, see Proposition 6.2 in Appendix. In the following we refer to the index set $I$ as the minimal index set.

The only asymptotic assumption imposed below is that $F$ is in the Gumbel max-domain of attraction with some scaling function $w$ (for short $F \in GMDA(w)$), i.e.,

$$\lim_{u \to \infty} \frac{F(u + x/w(u))}{F(u)} = \exp(-x), \quad \forall x \in \mathbb{R},$$  \hspace{1cm} (3.13)

with $F := 1 - F$. See Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), De Haan and Ferreira (2006), or Resnick (2008) for more details on the max-domain of attractions.

In order to avoid repetition we formulate the following assumption on the distribution function $F$.

**Assumption A1.** $F$ is a univariate distribution with infinite upper endpoint such that $F(0) = 0$ satisfying (3.13).

In the sequel $u_n, n \geq 1$ is a sequence of constants converging to infinity, and $a \in \mathbb{R}^k \setminus \{0\}$ is a given vector. If (3.13) holds, then we set

$$\delta_n := u_n \|a_i\|, \quad \zeta_n := w(\delta_n), \quad \lambda_n := \delta_n \zeta_n, \quad n \geq 1,$$  \hspace{1cm} (3.14)

with $I$ the minimal index set of $\mathcal{P}(\Sigma^{-1}, a)$. We note in passing that in view of Proposition 6.2 $\|a_i\| \in (0, \infty)$.

Below we show that also the parameter $\alpha$ of $X$ plays a crucial role in the tail asymptotics of interests since the following two index sets

$$L := \{1 \leq i \leq k : \alpha_i \neq 1/2, \quad (C \alpha)_i = 0\}, \quad M := \{1,\ldots,k\} \setminus L$$  \hspace{1cm} (3.15)

appear explicitly in our asymptotic expansion. It is surprising that when the index set $L$ is empty, then the random vector $X$ and the associated elliptical random vector $X^* \overset{d}{=} ARV$, with $V \sim SD(k,1/2)$ independent of $R$ have the same tail asymptotics (up to some constant), i.e., $\lim_{n \to \infty} P\{X > u_n a\}/P\{X^* > u_n a\} = \mu \in (0, \infty)$. We state next our main result.

**Theorem 3.1.** Let $X$ be a Dirichlet random vector in $\mathbb{R}^k, k \geq 2$ defined in (3.11), and set $C := (A^\top)^{-1}, \Sigma := A^\top A$. For a given vector $a \in \mathbb{R}^k \setminus (-\infty,0]^k$, let $I$ with $m$ elements be the minimal index set corresponding to $\mathcal{P}(\Sigma^{-1}, a)$ with the unique solution $\overline{a} \geq a$. Suppose that Assumption A1 holds, and define $\delta_n, \zeta_n, \lambda_n$ as in (3.13). Let $u_n, n \geq 1$ be a sequence of thresholds in $\mathbb{R}^k$ such that

$$\lim_{n \to \infty} \zeta_n (u_n - u_n \overline{a})_I = q_I \in \mathbb{R}^m,$$  \hspace{1cm} (3.16)
and if $m < k$

$$
\lim_{n \to \infty} \left( \frac{\bar{c}_n}{\delta_n} \right)^{1/2} (u_n - u_n \bar{a})_J = q_J \in [-\infty, \infty)^k, \quad J := \{1, \ldots, k\} \setminus I.
$$

(3.17)

(a) If $J := \{1, \ldots, k\} \setminus I$ is non-empty and $L \subset J$ with $L, M$ defined in (3.15), then (set $u := a/\|a_I\|$)

$$
P\{X > u_n\} = (1 + o(1)) \tau_{J, L} \left( \frac{\Gamma(\pi)}{2 \prod_{i=1}^{|J|} \Gamma(\alpha_i)} \right) \frac{\Gamma(\pi)}{\prod_{i=1}^{|M|} \Gamma(\alpha_i)} \frac{\lambda_n^{1 - \frac{|L|}{2} + \frac{1}{2} |\Sigma|^{-1} u_I}}{\prod_{i \in I} \gamma_i^{|\Sigma|^{-1} u_I}} F(\delta_n),
$$

(3.18)

where

$$
\tau_{J, L} := \int_{y_j > q} \prod_{i \in L} (C_{JJ} y_j)_i^{2\alpha_i - 1} \exp(-y_j^T (\Sigma^{-1})_{JJ} y_j / 2) \, dy_j \in (0, \infty),
$$

$$
\tau^*_M := \prod_{i \in M} (\|a_I\|^{1 - 2\alpha_i} |(C \bar{a})_i|^{2\alpha_i - 1}).
$$

(b) If $J$ is empty, then as $n \to \infty$

$$
P\{X > u_n\} = (1 + o(1)) \tau_L \left( \frac{\Gamma(\pi)}{2 \prod_{i=1}^{|L|} \Gamma(\alpha_i)} \right) \frac{\lambda_n^{1 - \frac{|L|}{2} + \frac{1}{2} |\Sigma|^{-1} u_I}}{\prod_{i \in L} \gamma_i^{|\Sigma|^{-1} u_I}} F(\delta_n),
$$

(3.19)

where $\tau_L := \int_{y_j > q} \prod_{i \in L} (C y)_i^{2\alpha_i - 1} \exp(-u^T (\Sigma^{-1} y) \, dy \in (0, \infty)$.

In our notation $\prod_{i \in K} r_i := 1, r_i \in R, i \leq k$ when $K$ is empty. Next, given $\alpha \in (0, \infty)^k$ we define another vector $\tilde{\alpha} \in (0, \infty)^k$ such that $\tilde{\alpha}_M := 1_{M}/2$, and if $L$ is non-empty set $\tilde{\alpha}_L := \alpha_L$.

Corollary 3.2. Under the assumptions of Theorem 3.1 (and the same notation), if further $J$ is non-empty and either $L$ is empty or

$$
C_{JJ}^{-1}(C_{JJ}^{-1})^T = (\Sigma^{-1})_{JJ},
$$

(3.20)

then we have

$$
P\{X > u_n\} = (1 + o(1)) \left( \frac{\Gamma(\pi)}{2 \prod_{i=1}^{|M|} \Gamma(\alpha_i)} \right) \frac{\Gamma(\pi)}{\prod_{i \in L} \Gamma(\alpha_i)} \frac{\prod_{i \in L} \gamma_i^{1 - \frac{|L|}{2} + \frac{1}{2} |\Sigma|^{-1} u_I}}{\prod_{i \in J \setminus L} \gamma_i^{|\Sigma|^{-1} u_I}} F(\delta_n), \quad n \to \infty,
$$

(3.21)

where $Y_J \overset{d}{=} A_{JJ}^T Y_j$ with $Y_j$ a standard Kotz Type I GSD random vector with parameter $\tilde{\alpha}_J$. Set $P\{Y_J > q_J\}$ to 1 if $|J| = 0$.

Remarks: (a) By the properties of the scaling function $w$ we have (see e.g., Resnick (2008))

$$
\lim_{u \to \infty} uw(u) = \infty.
$$

(3.22)

(b) If $F$ is a univariate distribution function with upper endpoint $\infty$, and further $F \in GMDA(w)$, then for any $r \in (1, \infty), \eta \in R$

$$
\lim_{x \to \infty} \frac{(xw(x))^\eta F(rx)}{F(x)} = 0
$$

(3.23)

holds, see Appendix A1.

(c) Condition (3.20) is satisfied in the special case $J$ has only one element and $\Sigma$ is a correlation matrix.

(d) In the 2-dimensional setup if $A = (a_{ij})_{i,j=1,2}$ is given by

$$
a_{11} = a_{22} = 1, \quad a_{12} = \sigma \neq 0, \quad a_{21} = \rho \neq \sigma,
$$

where $\rho = \cos(\alpha)/\sin(\alpha)$ and $\sigma = \sin(\alpha)/\sin(\alpha), \alpha \neq 0, \pi$.
then clearly $A$ is non-singular and $C := (A^\top)^{-1}$ has elements

$$c_{11} = a_{22} = 1/(1 - \sigma \rho), \quad c_{12} = -\rho/(1 - \sigma \rho), \quad c_{21} = -\sigma/(1 - \sigma \rho).$$

If $a := (1,a)^\top, a \in (-\infty, 1]$, then the index set $L$ in Theorem 3.1 is non-empty for $\sigma = 1/a, a \neq 0$ and $\alpha_1 \neq 1/2$. With this choice of the constant $a$ it follows that also the index set $J$ is non-empty. Consequently, for the bivariate setup it is not possibly to have $L$ non-empty and $J$ empty.

### 4 Examples

We illustrate our result with two examples. First we consider the multivariate setup choosing the parameter vector $\alpha$ to have identical components, and then we deal with the bivariate setup.

**Example 4.1.** Let $X$ be a $k$-dimensional random vector as in Theorem 3.1, where $\alpha = p1 \in \mathbb{R}^k, p \in (0, \infty)$. We suppose that the matrix $A$ is such that $\Sigma = A^\top A$ is given by

$$\Sigma = \rho 11^\top + (1 - \rho)E, \quad \rho \in (-1/(k - 1), 1),$$

with $E \in \mathbb{R}^{k \times k}$ the identity matrix.

Let $u_n := u_n1, n \geq 1$ with $u_n, n \geq 1$ given constants converging to infinity. In view of our asymptotic result we are able to derive the asymptotic of $P\{X > u_n1\}$ as $n \to \infty$. We consider first the quadratic programming problem $P(\Sigma, \mathbf{1})$.

Since the inverse matrix of $\Sigma$ is

$$B := \Sigma^{-1} = \frac{E}{1 - \rho} - \frac{\rho 11^\top}{(1 - \rho)(1 + (k - 1)\rho)},$$

we obtain

$$B1 = \left(\rho 11^\top + (1 - \rho)E1\right)^{-1}1 = \frac{1}{1 + (k - 1)\rho} > 0.$$

Consequently, condition (3.22) holds implying that the unique solution of $P(\Sigma, \mathbf{1})$ is $\mathbf{1}$ with the minimal index set $I = \{1, \ldots, k\}$. If $p = 1/2$, then the index set $L$ is empty, and $X$ is an elliptical random vector. We consider next the case $p \neq 1/2$ implying that $L$ is defined by

$$L := \{i : (C1)_i = 0, \ 1 \leq i \leq k\}, \text{ with } C := (A^{-1})^\top.$$

Suppose that $C$ is a lower triangular matrix. Since $B = C^\top C$ is positive definite, the matrix $C$ can be explicitly determined by the well-known Cholesky decomposition. It follows easily that $c_{kk} = \sqrt{b_{kk}}$ and

$$c_{k1} = c_{k2} = \cdots = c_{k,k-1} = b_{k1}/\sqrt{b_{kk}}.$$

Proceeding analogously we obtain

$$c_{i1} = c_{i2} = \cdots = c_{i,i-1}, \quad c_{ii} + (i - 1)c_{i1} > 0, \quad i = 1, \ldots, k - 1.$$

Hence the index set $L$ is empty, and

$$|(C1)_i| = c_{ii} + (k - i)c_{i1}, \quad i = 1, \ldots, k.$$

Since further

$$\|1\|^2 = 1^\top \Sigma^{-1} 1 = \frac{k}{1 + (k - 1)\rho}, \quad \sqrt{|\Sigma|} = (1 - \rho)^{(k-1)/2} \sqrt{1 + (k - 1)\rho},$$
\textbf{Case (3.23):} it follows easily that \( a \leq \rho \).

Considering the sequence of constants converging to infinity, we have:

\[
P \{ X > u_n 1 \} = (1 + o(1)) \tau_L \frac{\Gamma(k/2) \sqrt{k/(1 + (k - 1)\rho)}}{2^{k/2} \sqrt{\left(1 - \rho^2\right)}} \left( \sum_{i=1}^{k} c_{i} \right) \left( u_n (u_n 1) \right)^{k - 1 - k}, \quad n \to \infty.
\]

Where

\[
\tau_L := \int_{y > 0} \exp(-1 \sum_{i=1}^{k} y_i / ||1||) dy = \left( \frac{1}{1 - \rho^2} \right)^{k}, \quad v_n := (n ||1||^{k / (1 - \rho^2)})^{1 - k}, \quad n \geq 1.
\]

When \( p = 1/2 \) (\( X \) being thus elliptically distributed) we obtain

\[
P \{ X > u_n 1 \} = (1 + o(1)) \frac{\Gamma(k/2)}{2^{k/2} \sqrt{\left(1 - \rho^2\right)}} \left( \sum_{i=1}^{k} c_{i} \right) \left( u_n (u_n 1) \right)^{k - 1 - k} \left( \sum_{i=1}^{k} c_{i} \right)^{1 - k}, \quad n \to \infty.
\]

Note that in the Gaussian case (i.e., \( R^2 \) is Chi-squared distributed with \( k \)-degrees of freedom) we have

\[
\mathcal{F}(u_n) = (1 + o(1)) \frac{u_n^{k - 2}}{2^{k/2} \Gamma(k/2)} \exp(-u_n^2 / 2), \quad n \to \infty.
\]

\textbf{Example 4.2.} Consider \( X = RA^T U \) a bivariate Dirichlet random vector with \( R \sim F \) and \( U \sim SD((\alpha_1, \alpha_2), 2) \). Assume that \( F \) satisfies the Assumption A1, and define the matrix \( A \) by

\[
a_{11} = 1, \quad a_{12} = \rho, \quad a_{21} = 0, \quad a_{22} = \sqrt{1 - \rho^2}, \quad \rho \in (-1, 1).
\]

Consequently, \( \Sigma := A^T A \in \mathbb{R}^{2 \times 2} \) has elements

\[
\sigma_{11} = \sigma_{22} = 1, \quad \sigma_{12} = \sigma_{21} = \rho
\]

And \( C := (A^T)^{-1} \) has elements

\[
c_{11} = 1, \quad c_{12} = 0, \quad c_{21} = -\rho / \sqrt{1 - \rho^2}, \quad c_{22} = 1 / \sqrt{1 - \rho^2}.
\]

We focus next on the asymptotics of \( P \{ X_1 > u_n, X_2 > au_n \}, n \to \infty \) where \( a \in (-\infty, 1] \) and \( u_n, n \geq 1 \) is a positive sequence of constants converging to infinity. Depending on the constant \( a \) we need to consider three cases:

\textbf{Case} \( \rho < a \): It follows easily that \( \overline{\alpha} = (1, a) \) is the solution of \( P(\Sigma^{-1}, \alpha) \) with minimal index set \( I = \{1, 2\} \). Further, both \( J \) and \( L \) are empty and

\[
\delta_n = u_n ((1 - 2 \rho a + a^2) / (1 - \rho^2))^{1/2} = c \| u_n \|, \quad c := \| \overline{\alpha} \| > 0, \quad n \geq 1,
\]

\[
\tau_L = \frac{c^2 (1 - \rho^2)^2}{(1 + \rho)(1 - \rho)} = \tau_M^* := c^{2 - 2 \alpha_1 - 2 \alpha_2} \left( \frac{a - \rho}{\sqrt{1 - \rho^2}} \right)^{2 \alpha_1 - 1}.
\]

Consequently, in view of Theorem 3.1 we obtain

\[
P \{ X_1 > u_n, X_2 > au_n \} = (1 + o(1)) \frac{\Gamma(\alpha_1 + \alpha_2)}{2 \Gamma(\alpha_1) \Gamma(\alpha_2)} \left( 1 - \rho^2 \right)^{2 \alpha_1 - 1} \left( 1 - \alpha \right)^{2 \alpha_2 - 1}
\]

\[
\times \left( u_n w(cu_n) \right)^{-1} \mathcal{F}(cu_n), \quad n \to \infty.
\]

In the special case \( a = 1 \) we have \( c = \sqrt{2/(1 + \rho)} \), and we may further write

\[
P \{ X_1 > u_n, X_2 > u_n \} = (1 + o(1)) \frac{\Gamma(\alpha_1 + \alpha_2)}{2 \Gamma(\alpha_1) \Gamma(\alpha_2)} \left( 1 - \rho^2 \right)^{2 \alpha_1 - 1} \left( 1 + \rho \right)^{2 \alpha_2 - 1}
\]

\[
\times \left( u_n w(cu_n) \right)^{-1} \mathcal{F}(cu_n), \quad n \to \infty.
\]

Let \( b_{ni} := H_i^{-1}(1 - 1/n), n > 1 \) with \( H_i^{-1} \), \( i = 1, 2 \) the generalised inverse of the distribution function of \( X_i \). It follows further that both \( X_1 \) and \( X_2 \) are in the Gumbel max-domain of attraction with the same scaling function \( w \). In view of (3.23) and the above asymptotics

\[
\lim_{n \to \infty} n P \{ X_1 > b_{n1}, X_2 > b_{n2} \} = 0,
\]

(4.25)
hence $X_1, X_2$ are asymptotically independent.

We note in passing that the asymptotic independence is a crucial property of sample extremes with certain consequences for statistical applications. See Reiss and Thomas (2007), Hülsler and Li (2009), Das and Resnick (2009) and Peng (2010) for recent developments and applications.

Case $\rho = a \in (-1, 1)$: The minimal index set is $I := \{1\}$ and if $\alpha_2 \neq 0.5$, then

$$J = L = \{2\}, \quad \text{and} \quad \|\bar{a}\| = 1, \quad (C\bar{a})_1 = 1, \quad (C\bar{a})_2 = 0.$$  

In view of Theorem 5.1 for any $q_2 \in (-\infty, \infty)$ we obtain thus (set $r_n := u_n w(u_n)$)

$$P\{X_1 > u_n, X_2 > \rho u_n + q u_n/r_n\} = (1 + o(1)) \frac{2^{\alpha_2-1} \Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)} P\{\sqrt{1 - \rho^2} Y > q \} r_n^{-\alpha_2} F(u_n), \quad n \to \infty,$$

where $Y$ is symmetric about 0 such that $Y^2 \sim \text{Gamma}(\alpha_2, 1/2)$. If $\alpha_2 = 1/2$, then $L$ is empty. Also in this case the above asymptotics holds. Since $X_1^2 \sim R^2 Z$, with $Z \overset{d}{=} B_{\alpha_1, \alpha_2}$ being independent of $R$, in view of Theorem 12.3.1 of Berman (1992) we obtain the following conditional limit result

$$\lim_{n \to \infty} P\{X_2 > \rho u_n + x \sqrt{w(u_n)/w(u)}\, X_1 > u_n\} = P\{\sqrt{1 - \rho^2} Y > x\}, \quad \forall x \in \mathbb{R}. \quad (4.26)$$

Case $a \in (-\infty, 1), \rho \in (a, 1)$: The only difference to the above case is that for the choice of the threshold $u_n = u_n(1, a)^\top$ the vector $q$ as defined in Theorem 3.1 has components $q_1 = 0$ and $q_2 = -\infty$. Hence we have

$$P\{X_1 > u_n, X_2 > a u_n\} = (1 + o(1)) \frac{2^{\alpha_2-1} \Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)} r_n^{-\alpha_2} F(u_n), \quad n \to \infty.$$

5 Proofs

Proof of Theorem 3.1: The idea of the proof is the same as that for the elliptical setup (Theorem 3.1 in Hashorva (2007)), therefore we give the sketch of the proof omitting several details.

Let $Y^* \sim GSD(k+1, \alpha^*, G)$ be a $k+1$ dimensional random vector where $\alpha^* = (\alpha_1, \ldots, \alpha_k, 1)^\top$ and distribution function $G$ has the asymptotic behaviour

$$G(u) = (1 + o(1)) u w(u) F(u) \frac{\Gamma(\bar{a})}{2 \Gamma(\bar{a} + 1)}, \quad u \to \infty. \quad (5.27)$$

By Lemma 7.4 in Hashorva (2007) such $G$ exists. In view of (5.40) the density function of $Y = A^\top Y^*, N := \{1, \ldots, k\}$ is given by

$$h(x) = \prod_{i=1}^k |(C_i x)|^{2\alpha_i-1} \frac{\Gamma(\bar{a})}{\prod_{i=1}^k \Gamma(\alpha_i) |\Sigma|^{1/2}} \int_{\mathbb{R}^k} \frac{1}{2\pi^k} \frac{1}{\Sigma} d\nu(z), \quad \forall x \in \mathbb{R}^k,$$

where $C := (A^\top)^{-1}, \Sigma := A^\top A$, and $\|x\| := x^\top \Sigma^{-1} x, x \in \mathbb{R}^k$. By the properties of $GSD$ random vectors and Lemma 6.1 it follows that

$$P\{X > u_n\} = (1 + o(1)) P\{Y > u_n\}, \quad n \to \infty. \quad (5.28)$$

Let $I$ be the minimal index set of the quadratic programming problem $P(\Sigma^{-1}, \alpha)$ with the unique solution $\bar{a}$. Applying Proposition 6.2 we obtain

$$\bar{a}_I = a_I, \quad \|\bar{a}_I\| > 0, \quad \bar{a}_J = \Sigma_{J I} \Sigma_I^{-1} a_I \geq a_J.$$  

Set for $n \geq 1$

$$\delta_n := u_n \|a_I\|, \quad \zeta_n := w(u_n \|a_I\|), \quad \xi_n := u_n \|a_I\| w(u_n \|a_I\|)$$
and define \( v_n \in \mathbb{R}^k \) by
\[
(v_n)_I := v_{n,I} = \zeta_n 1_I, \quad (v_n)_J := v_{n,J} = \left( \frac{\zeta_n}{\delta_n} \right)^{1/2} 1_J = \sqrt{\frac{\zeta_n}{\delta_n}} 1_J.
\]
By Proposition 6.2 for any \( y \in \mathbb{R}^k \) (set \( u := a/\|a_I\|) \)
\[
|u_n + y - v_n| = \delta + (1 + o(1))|u_I^\top \Sigma^{-1}_{II} y_I + y_I^\top (\Sigma^{-1})_{JJ} y_J/2|, \quad n \to \infty.
\]
Further, \( \|v_n\| \leq \zeta_n \) and \( J \subseteq J \) with \( |J| \) non-empty imply
\[
\prod_{i=1}^k |(u_n C\alpha + C y/v_n)|^{2\alpha_i - 1} = (1 + o(1))|\zeta_n^{\|L\|/2} \prod_{i \in M} |u_n (C\alpha_i)|^{2\alpha_i - 1} \prod_{i \in L} |(C y_i)|^{2\alpha_i - 1}, \quad n \to \infty. \tag{5.29}
\]
We consider first the case \( J \) is non-empty. For notational simplicity define
\[
\tau_M^\ast := \prod_{i \in M} \|a_I\|^{-2\alpha_i} |(C\alpha_i)|^{2\alpha_i - 1}, \quad \eta_n := \zeta_n^{\|L\|/2 + \|\alpha\| - |M| - |L|/2 + \|\alpha\| - |J|/2 + |L| - |M|/2 - \|\alpha\|}, \quad n \geq 1.
\]
Next, for all \( n \) large we obtain
\[
\frac{P\{Y > u_n\}}{G(\delta_n)} = \frac{\Gamma(\tau_M + 1)}{\prod_{i=1}^k \Gamma(\alpha_i) |\Sigma|^{1/2}} \int_0^\infty \left[ \prod_{i=1}^k |(C x_I)|^{2\alpha_i - 1} \int_0^\infty z^{-\tau_M^\ast} dG(z)/G(\delta_n) \right] dx
\]
\[
= \frac{\Gamma(\tau_M + 1)}{\prod_{i=1}^k \Gamma(\alpha_i) |\Sigma|^{1/2}} \int_{y > u_n} \left[ \prod_{i=1}^k |(C (u_n \alpha + y/v_n))_i|^{2\alpha_i - 1} \int_0^\infty z^{-\tau_M^\ast} dG(z)/G(\delta_n) \right] dy
\]
\[
= \frac{\Gamma(\tau_M + 1)}{\prod_{i=1}^k \Gamma(\alpha_i) |\Sigma|^{1/2}} \tau_M^\ast \eta_n \int_{y > u_n} \left[ \prod_{i=1}^k |(C J J y_J)_i|^{2\alpha_i - 1} \int_0^\infty (\delta_n + s/\delta_n)^{-\tau_M^\ast} dG(\delta_n + s/\delta_n)/G(\delta_n) \right] dy,
\]
with \( q(y) := (1 + o(1))|u_I^\top \Sigma^{-1}_{II} y_I + y_J^\top (\Sigma^{-1})_{JJ} y_J/2| \). By the assumptions \( u_I^\top \Sigma^{-1}_{II} y_I + y_J^\top (\Sigma^{-1})_{JJ} y_J/2 \) is positive for any \( y \in \mathbb{R}^k \) such that \( y_I > 0_I \) and further the components of \( \Sigma_{JJ} u_I \) are all positive. The tail asymptotics of \( G \) and the fact that \( w \) is self-neglecting i.e., (see e.g., Resnick (2008))
\[
w(u + x/w(u))/w(u) \to 1, \quad u \to \infty
\]
locally uniformly in \( \mathbb{R} \) imply \( G \in GMDA(w) \), hence
\[
\lim_{n \to \infty} \frac{G(\delta_n + t/\zeta_n) - G(\delta_n + s/\zeta_n)}{G(\delta_n)} = \exp(-s) - \exp(-t), \quad t > s, t, s \in \mathbb{R}.
\]
Since further
\[
\lim_{n \to \infty} \delta_n \zeta_n = \infty, \quad \lim_{n \to \infty} v_n(u_n - u_n, \alpha) = q
\]
Fatou’s Lemma and \(\mathbb{P}\) yield
\[
\begin{align*}
\liminf_{n \to \infty} \frac{P\{X > u_n\}}{G(\delta_n)} & \geq \frac{\Gamma(\pi + 1)}{\prod_{i=1}^k \Gamma(\alpha_i)\Sigma_i^{1/2}} \tau_M^\alpha \eta_n \\
& \times \liminf_{n \to \infty} \left[ \int_{y > \nu_n(u_n - u_n\alpha)} \prod_{i \in L} \left( (C_{iJ}y_j)_i \right)^{2\alpha_i - 1} \left( \delta_n + s/\zeta_n \right)^{2\sigma_i} dG(\delta_n + s/\zeta_n)/G(\delta_n) \right] dy
\end{align*}
\]
with
\[
\begin{align*}
\tau_M^\alpha & = \left( \frac{\Gamma(\pi + 1)}{\prod_{i=1}^k \Gamma(\alpha_i)\Sigma_i^{1/2}} \prod_{i \in L} e_i^\top \Sigma_i^{-1} u_I \right) \left( \delta_n \zeta_n \right)^{1 - |J|/2 - \pi_L + |L|/2} \prod_{i \in M} \left( \|a_I\|^{1 - 2\alpha_i} \right) \left( C(\pi) \right)^{2\alpha_i - 1} F(\delta_n), \quad n \to \infty,
\end{align*}
\]
Thus the result follows.

**Proof of Corollary 3.4** If \(Y_J\) is a standard Kotz Type I \(GSD\) random vector in \(\mathbb{R}^{k-m}\) with parameters \(\tilde{\alpha}_J\), then (2.10) and (3.20) imply that the random vector \(B^\top Y_J, B := (C_{iJ}^\top)^{-1}\) possesses the density function
\[
h_B(x) := \frac{2 - \gamma_J}{\prod_{i \in J} \Gamma(\alpha_i)(\Sigma^{-1})_{iJ}^{-1/2}} \exp(-x_J^\top(\Sigma^{-1})_{iJ} x_J/2) \prod_{i \in J} \left( (C_{iJ}x_J)_i \right)^{2\alpha_i - 1}, \quad \forall x_J \in \mathbb{R}^{k-m}.
\]
Consequently, since \(\tilde{\alpha}_i = 1/2, i \leq k, i \notin L\) the constant \(\tau_{J,L}\) in Theorem 3.1 can be re-written as
\[
\tau_{J,L} = \int_{y_J > q_J} \prod_{i \in L} \left( (C_{iJ}y_J)_i \right)^{2\alpha_i - 1} \exp(-y_J^\top(\Sigma^{-1})_{iJ}y_J/2) dy_J
\]
\[
= \int_{y_J > q_J} \prod_{i \in J} \left( (C_{iJ}y_J)_i \right)^{2\alpha_i - 1} \exp(-y_J^\top(\Sigma^{-1})_{iJ}y_J/2) dy_J = \frac{\tau_M^\alpha \prod_{i \in J} \Gamma(\tilde{\alpha}_i)}{(|\Sigma^{-1})_{iJ}^{-1/2}|} P\{Y_J > q_J\}.
\]

Hence we may further write
\[
P\{X > u_n\} = (1 + o(1)) \frac{\Gamma(\pi + 1)}{\prod_{i=1}^k \Gamma(\alpha_i)\Sigma_i^{1/2}} \tau_M^\alpha \prod_{i \in L} e_i^\top \Sigma_i^{-1} u_I \left( \delta_n \zeta_n \right)^{1 - |J|/2 - \pi_L + |L|/2} F(\delta_n)
\]
\[
\times \left( \frac{\Gamma(\pi + 1)}{\prod_{i=1}^k \Gamma(\alpha_i)\Sigma_i^{1/2}} \prod_{i \in L} e_i^\top \Sigma_i^{-1} u_I \right) \left( \delta_n \zeta_n \right)^{1 - |J|/2 - \pi_L + |L|/2} F(\delta_n), \quad n \to \infty,
\]

consequently the result follows. \(\square\)
6 Appendix

A1. Proof of (3.23). A direct proof is given in Lemma 4.2 of Hashorva and Pakes (2009). Another (simpler) proof can be obtained by borrowing the arguments of Lemma 2.2(b) in Resnick (2008). With Resnick’s idea it follows that for some \( \varepsilon \in (0, 1) \) small enough

\[
\frac{F(x + s/w(x))}{F(x)} \leq \frac{c}{(1 + \varepsilon s)^{c-1}}
\]

is valid for all \( s \geq 0 \) and any \( x \) large with \( c \) some positive constant. (3.23) follows now easily.

A2. The next lemma is a simple generalisation of Lemma 7.3 in Hashorva (2007).

Lemma 6.1. Let \( R, R^* \) be two independent positive random variables such that

\[
\lim_{u \to \infty} \frac{P\{R > u\}}{P\{R^* > u\}} = c \in (0, \infty).
\]

Suppose that \( R \) has the distribution function \( F \) satisfying the Assumption A1, and let \( U \sim SD(k, \alpha) \) being independent of \( R \) and \( R^* \). Then we have

\[
\lim_{u \to \infty} \frac{P\{R A U > u\}}{P\{R^* A U > u\}} = c \in (0, \infty)
\]

for any \( a \in \mathbb{R}^k \setminus (-\infty, 0]^k \) and \( A \in \mathbb{R}^{k \times k} \) a non-singular matrix.

Proof of Lemma 6.1

Since \( R \) has distribution function in the Gumbel max-domain of attraction, then by (3.23) for any \( K > 1 \) we obtain

\[
\lim_{u \to \infty} \frac{P\{R > Ku\}}{P\{R^* > u\}} = 0.
\]

This fact together with \( A \) being non-singular implies the proof. \( \square \)

A3. The next proposition can be found in Hashorva (2005).

Proposition 6.2. Let \( \Sigma \in \mathbb{R}^{k \times k}, k \geq 2 \), be a positive definite correlation matrix and let \( a \in \mathbb{R}^k \setminus (-\infty, 0]^k \) be a given vector. Then the quadratic programming problem

\[
P(\Sigma^{-1}, a) : \text{minimise } \|x\|^2 \text{ under the linear constraint } x \geq a
\]

has a unique solution \( \overline{a} \) defined by a unique non-empty index set \( I \subset \{1, \ldots, k\} \) such that

\[
\overline{a}_I = a_I > 0_I, \quad \Sigma^{-1}_{I I} a_I > 0_I,
\]

\[
\min_{x \geq a} \|x\|^2 = \min_{x \geq a} x^\top \Sigma^{-1} x = \|\overline{a}\|^2 = \|a_I\|^2 = a_I^\top \Sigma^{-1}_{I I} a_I > 0
\]

and if \( |I| < k \), then with \( J := \{1, \ldots, d\} \setminus I \)

\[
\overline{a}_J = -((\Sigma^{-1})_{J J})^{-1}(\Sigma^{-1})_{J I} a_I = \Sigma_{J I} \Sigma^{-1}_{I I} a_I \geq a_J.
\]

Furthermore, for any \( x \in \mathbb{R}^k \) we have

\[
x^\top \Sigma^{-1} a = x_I^\top \Sigma^{-1}_{I I} a_I = x_I^\top \Sigma^{-1}_{I I} \overline{a}_I
\]

and \( 2 \leq |I| \leq k \), provided that \( a = c1, c \in (0, \infty) \).

A4. The next result follows easily, see Fang and Fang (1990).
Lemma 6.3. Let $X \sim GSD(k, \alpha, F)$ be a random vector in $\mathbb{R}^k, k \geq 2$ where $F$ is a distribution function satisfying $F(0) = 0$. For any index $I \subset \{1, \ldots, k\}$ with $m$ elements we have

$$X_I \overset{d}{=} R_I U_I,$$

(6.38)

with $R_I$ being independent of $U_I \sim SD(k, \alpha_I)$, and $R_I$ possesses the density function $f_I$ defined by

$$f_I(z) = 2z^{2\alpha_I - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha_I)\Gamma(\alpha - \alpha_I)} \int_{z}^{\infty} (r^2 - z^2)^{\alpha_I - 1} r^{2(\alpha_I - 1)} dF(r), \quad \forall z \in (0, \infty).$$

(6.39)

Furthermore for any non-singular matrix $A \in \mathbb{R}^{m \times m}$ the random vector $A^T X_I$ possesses the density function $h_{A,I}$ given by (set $C := (A^T)^{-1}, \Sigma := A^T A$ and write $|\Sigma|$ for the determinant of $\Sigma$)

$$h_{A,I}(x_I) = \frac{\Gamma(\alpha) \prod_{i \in I} |C x_i|^{2\alpha_i - 1}}{\prod_{i \in I} \Gamma(\alpha_i) \Gamma(\alpha - \alpha_I)|\Sigma|^{1/2}} \int_{\|x_I\|}^{\infty} (r^2 - \|x_I\|^2)^{\alpha_I - 1} r^{2(\alpha_I - 1)} dF(r), \quad \forall x \in \mathbb{R}^k.$$

(6.40)

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