Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, and $\varphi_t(x) = t^{-n} \varphi(x/t)$. Define the maximal operator associated to $\varphi$ by $M\varphi f(x) = \sup_{t > 0} \varphi_t * |f|(x)$. When $\varphi = \chi_{|x| \leq 1}$, this is just the Hardy-Littlewood maximal function $Mf(x)$. J. Kinnunen [Isr. J. Math. 100, 117–124 (1997; Zbl 0882.43003)] showed that $\|Mf\|_{W^{1,p}(\mathbb{R}^n)} \leq C_p \|f\|_{W^{1,p}(\mathbb{R}^n)}$ for $1 < p < \infty$. The authors extend this result to the case $0 < p \leq 1$ in the following way. Let $\varphi \in S(\mathbb{R}^n)$ be nonnegative, $\int \varphi > 0$ and $p > n/(n + 1)$. If $f \in H^{1,p}(\mathbb{R}^n)$, then $M\varphi f \in H^{1,p}(\mathbb{R}^n)$ and there exists $C > 0$ such that $\|M\varphi f\|_{H^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{H^{1,p}(\mathbb{R}^n)}$, where $H^{1,p}(\mathbb{R}^n)$ is the homogeneous Hardy-Sobolev space, i.e. it is the set of all $f \in S'(\mathbb{R}^n)$ which has distributional derivatives $\partial_j f$ in the Hardy space $H^p(\mathbb{R}^n)$ with quasi-norm $\|f\|_{H^{1,p}(\mathbb{R}^n)} = \sum_{j=1}^n \|\partial_j f\|_{H^p(\mathbb{R}^n)}$.

To prove their result they use a characterization of Hardy-Sobolev spaces by A. Miyachi [J. Math. Soc. Japan 42, No. 1, 73–90 (1990; Zbl 0677.42017)] and a self-improving lemma by A. K. Lerner and C. Pérez [Math. Scand. 97, No. 2, 217–234 (2005; Zbl 1101.42010)].

The sharpness of exponent $n/(n + 1)$ is checked. Related results are discussed for $f \in W^{1,1}(\mathbb{R}^n)$ satisfying some additional assumption. As a by-product of the proof, they obtain similar result for the local Hardy-Sobolev spaces $H^{1,p}(\mathbb{R}^n)$ in the same range of exponents.

Reviewer: Kôzô Yabuta (Nishinomiya)

MSC:

42B25 Maximal functions, Littlewood-Paley theory
42B30 $H^p$-spaces
42B35 Function spaces arising in harmonic analysis
46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

Keywords:

maximal function of convolution type; regularity; Hardy-Sobolev spaces

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