The Lemniscate of Bernoulli, Without Formulas

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

In this article, we present purely geometrical proofs of the well-known properties of the lemniscate of Bernoulli.

What Is the Lemniscate?

A polynomial lemniscate with foci \( F_1, F_2, \ldots, F_n \) is a locus of points \( X \) such that the product of distances from \( X \) to the foci is constant (\( \prod_{i=1}^{n} |F_iX| = \text{const} \)).

The \( n \)-th root of this value is called the radius of the lemniscate. It is clear that a lemniscate is an algebraic curve of degree (at most) \( 2n \). You can see a family of lemniscates with three foci in Figure 1.

A lemniscate with two foci is called a Cassini oval. It is named after the astronomer Giovanni Domenico Cassini who studied it in 1680. The best-known Cassini oval is the lemniscate of Bernoulli, which was described by Jakob Bernoulli in 1694. For each point of the curve, the product of the distances to the foci equals one quarter of the square of the distance between the foci (Fig. 2). Bernoulli considered it as a modification of an ellipse, which has a similar definition: the locus of points with the sum of distances to the foci being constant. (Bernoulli was not familiar with the work of Cassini.)

It is clear that the lemniscate of Bernoulli passes through the midpoint between the foci. This point is called the juncture or double point of the lemniscate.

The lemniscate of Bernoulli has many very interesting properties. For example, the area bounded by the lemniscate is equal to \( \frac{1}{2} |F_1F_2|^2 \). In this article we prove some other properties, mainly using purely synthetic arguments.

How Do We Construct the Lemniscate of Bernoulli?

There exists a very simple method for constructing the lemniscate of Bernoulli using the following three-bar linkage. James Watt invented this construction: Take two equal rods \( F_1A \) and \( F_2B \) each of length \( \frac{1}{2} |F_1F_2| \) and fixed at the points \( F_1 \) and \( F_2 \), respectively. Let points \( A \) and \( B \) lie on opposite sides of the line \( F_1F_2 \). The third rod connects the points \( A \) and \( B \) and its length equals \( |F_1F_2| \) (Fig. 3). Then, during the motion of the linkage the midpoint \( X \) of the rod \( AB \) traces the lemniscate of Bernoulli with foci at \( F_1 \) and \( F_2 \).

To see this note that the quadrilateral \( F_1AF_2B \) is an isosceles trapezoid (Fig. 4). Moreover, triangles \( \Delta AF_1X \) and \( \Delta ABF_1 \) are similar, because they have the common angle \( A \) and the following relation on their sides holds:

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For the same reason, triangles $\triangle BXF_2$ and $\triangle BF_2$ are similar. They have the common angle $B$, and the ratios of the length of the sides with endpoints at $B$ is $\sqrt{2}$. Therefore, $\frac{AF_1}{AB} = \frac{AF_1}{AF_2} = \frac{XF_2}{BF_2}$.

Let us remark that in the trapezoid $F_1AF_2B$ angles $\angle A$ and $\angle F_2$ are equal. Because angles $\angle XAF_2$ and $\angle XF_2B$ are equal too, we obtain $\angle F_1AX = \angle XF_2A$. This implies that triangles $\triangle F_1AX$ and $\triangle AF_2X$ are similar. Therefore, $\frac{|F_1X|}{|AX|} = \frac{|AX|}{|XF_2|} \Rightarrow |XF_1| \cdot |XF_2| = |AX|^2 = |F_1O|^2$.

Thus, we have shown that point $X$ lies on the lemniscate of Bernoulli.

Since the motion of the point $X$ is continuous and $X$ attains the farthest points of the lemniscate, the trajectory of $X$ is the whole lemniscate of Bernoulli.

Let $O$ be the midpoint of the segment $F_1F_2$ (double point of the lemniscate). Denote by $M$ and $N$ the midpoints of the segments $F_1A$ and $F_1B$, respectively (Fig. 5). Translate the point $O$ by the vector $\overrightarrow{OF_1}$. Denote the new point by $O'$. Observe that triangles $\triangle F_1MO'$ and $\triangle NMO$ are congruent.

Moreover, the following equation holds:

$$|F_1M| = |F_1O| = \frac{1}{\sqrt{2}}|F_1O|.$$

In other words, the points $M$ and $O'$ lie on the circle with center at $F_1$ and radius $\frac{1}{\sqrt{2}}|F_1O|$.

Using this observation, we can obtain another elegant method for constructing the lemniscate of Bernoulli.

Let us construct the circle with center at one of the foci and radius $\frac{1}{\sqrt{2}}|F_1O|$. On each secant $OAB$ (where $A$ and $B$ are the points of intersection of the circle and the secant) choose points $X$ and $X'$ such that $|AB| = |OX| = |OX'|$ (Fig. 6). The union of all points $X$ and $X'$ form the lemniscate of Bernoulli with foci $F_1$ and $F_2$.

Another interesting way to construct a lemniscate is with the linkages given in Figure 7. The lengths of the segments $F_1A$ and $F_1O$ are equal. The point $A$ is the intersection of rods $AX$ and $AY$ each of length $\sqrt{2}|F_1O|$. Denote the midpoints of these rods by $B$ and $C$, and join them with $O$ by another rod of the length $\frac{|AX|}{2}$. In the
process of rotating point $A$ around the circle, each of the points $X$ and $Y$ generates half of the lemniscate of Bernoulli with foci $F_1$ and $F_2$.

**The Lemniscate of Bernoulli and The Equilateral Hyperbola**

The hyperbola is a much better-known curve. An hyperbola with foci $F_1$ and $F_2$ is the set of all points $X$ such that the value $|F_1X| - |F_2X|$ is constant. Points $F_1$ and $F_2$ are called the foci of the hyperbola. Among all hyperbolas, we single out equilateral hyperbolas, i.e., the set of points $X$ such that $|F_1X| - |F_2X| = \frac{|F_1F_2|}{2}$.

The lemniscate of Bernoulli is an inversion image of an equilateral hyperbola. Before proving this claim, let us recall the definition of an inversion.

**Definition** Inversion with respect to the circle with center $O$ and radius $r$ is the transformation that maps every point $X$ in the plane to the point $X'$ lying on the ray $OX$ such that $|OX'| = \frac{r^2}{|OX|}$.

Inversion has many interesting properties; see, for example [2]. Among the properties, is the following: a circle or a line will invert to either a circle or a line, depending on whether it passes through the origin. We will prove here just one simple lemma that will help us later.

**Proposition** Suppose $A$ is an orthogonal projection of the point $O$ on some line $\ell$. Then the inversion image of the line $\ell$ with respect to a circle with center at $O$ is the circle with diameter $OA^*$, where $A^*$ is the inversion image of the point $A$.

**Proof** Let $B$ be any point on the line $\ell$, and let $B^*$ be its image (Fig. 8). Since $|OA^*| = \frac{r^2}{|OA|}$ and $|OB^*| = \frac{r^2}{|OB|}$, we see that triangles $\triangle OAB$ and $\triangle OB^*A^*$ are similar. Therefore the angle $\angle BOB^*$ is a right angle and the point $B^*$ lies on the circle with diameter $OA^*$.

Note that the center $O_1$ of this circle is the inversion of the point $O_2$, where $O_1$ is the point symmetric to $O$ with respect to the line $\ell$.

Now, let us prove that the lemniscate of Bernoulli with foci $F_1$ and $F_2$ is an inversion of the equilateral hyperbola with foci $F_1$ and $F_2$ with respect to the circle with center at $O$ and radius $|OF_1|$. For this proof, we will use the results we obtained in the proof of correctness of the first method for constructing the lemniscate (Fig. 4). Let $P$ be the point of intersection of the lines $F_1A$ and $F_2B$ and let $Q$ be the point symmetric to $P$ with respect to the line $F_1F_2$ (Fig. 9). Note that

$$|F_2Q| = |F_1Q| = |F_2P| = |F_1P| = |PA| = |F_1F_2| = \frac{|F_1F_2|}{\sqrt{2}}.$$ 

Therefore, the points $P$ and $Q$ lie on the equilateral hyperbola with foci $F_1$ and $F_2$. Now it remains to show that points $X$ and $Q$ are the images of each other under the inversion with center at $O$ and radius $|OF_1|$. First, let us show that triangles $\triangle F_1XO$ and $\triangle PF_1O$ are similar.

The quadrilateral $F_1XOB$ is a trapezoid. Therefore $\angle OF_1X + \angle OF_1B = 180^\circ$. Also, we have $\angle AF_1O + \angle OF_1P = 180^\circ$. Since angle $\angle XF_1B$ is equal to angle $\angle AF_1O$, we obtain that $\angle OF_1X$ and $\angle OF_1P$ are equal to each other.

Because angles $\angle XF_2B$ and $\angle XF_1A$ are equal, we have that $\angle XF_1P + \angle PF_2X = 180^\circ$. In other words, the quadrilateral $PF_1XF_2$ is inscribed. Therefore, we have

$$\angle F_2F_1X = \angle F_2PX = \angle F_1PO.$$ 

The last equation proves that points $O$ and $X$ are symmetric to each other with respect to the perpendicular bisector of the segment $F_1B$.

Thus, triangles $\triangle F_1XO$ and $\triangle PF_1O$ are similar because they have two corresponding pairs of equal angles. It follows that $\angle F_1AX$ and $\angle OF_1P$ are equal, and we have that the point $Q$ lies on the ray $OX$. In addition, from similarity of triangles $\triangle F_1XO$ and $\triangle QF_1O$ (it is congruent to the
triangle $\Delta PF_1O$, we obtain

$$\frac{|OX|}{|OF_1|} = \frac{|OF_1|}{|OQ|} \Rightarrow |OX| \cdot |OQ| = |OF_1|^2.$$  

This means that points $Q$ and $X$ are images of each other under the inversion with center at $O$ and radius $|OF_1|$ (Fig. 10).

If we look at Figure 9, we can make another observation: the points $X$ and $O$ lie on the circle centered at $P$.

It is interesting that this circle touches the lemniscate of Bernoulli. For suppose $\ell$ is the tangent to the hyperbola at the point $Q$. From Lemma 1, it follows that the image of the line $\ell$ under the inversion with center at $O$ and radius $|F_1O|$ is a circle $\omega_t$ passing through the point $Q$. Since $X$ is the inverse image of the point $Q$, we see that the circle $\omega_t$ touches the lemniscate at the point $X$. From the same Lemma we conclude that the center of this circle lies on the normal line from the point $O$ to the line $\ell$.

Let us show that lines $OP$ and $OQ$ are symmetric to each other with respect to the line $F_1F_2$. It will follow that the point $P$ is the center of the circle $\omega_t$. Without loss of generality, we can assume that the equation of the hyperbola is $y = \frac{1}{x}$. Suppose line $\ell$ intersects the abscissa and the ordinate in the points $R$ and $S$, respectively (Fig. 11). It is well known that the derivative of the function $\frac{1}{x}$ at the point $x_0$ is equal to $-\frac{1}{x_0^2}$. It follows that the point $Q$ is the midpoint of the segment $RS$, and $OQ$ is the median of the right triangle $\Delta ROS$. Therefore, the angles $\angle QOR$ and $\angle QRO$ are equal. Since angles $\angle POS$ and $\angle QOR$ are also equal, we obtain that the lines $OP$ and $RS$ are perpendicular, as was to be proved.

Let us note the following: Since the circle $\omega_t$ touches the lemniscate at the point $X$, the radius $PX$ of this circle is a normal (perpendicular to the tangent line) to the lemniscate at $X$ (Fig. 12). Note that the triangle $\Delta XPO$ is isosceles, and the lines $XO$ and $PO$ are symmetric with respect to the line $F_1O$. Therefore, we can write these equations:

$$\angle PXO = \angle XOP = 2\angle POF_1.$$  

The following very simple method for constructing the normal to the lemniscate of Bernoulli emerges: For any point $X$ on the lemniscate, take the line forming with the line intersecting $OX$ at $X$ at the angle $2\angle XOF$. This line will be a normal to the lemniscate.

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