Higher-dimensional models in gravitational theories of quartic Lagrangians

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March 24, 2022

Abstract

Ten-dimensional models, arising from a gravitational action which includes terms up to the fourth order in curvature tensor, are discussed. The spacetime consists of one time direction and two maximally symmetric subspaces, filled with matter in the form of an anisotropic fluid. Numerical integration of the cosmological field equations indicates that exponential, as well as power-law, solutions are possible. We carry out a dynamical study of the results in the $H_{\text{ext}} - H_{\text{int}}$ plane and confirm the existence of attractors in the evolution of the Universe. Those attracting points correspond to "extended" De Sitter space-times in which the external space exhibits inflationary expansion, while the internal one contracts.

PACS Codes: 98.80.Hw , 11.10.Kk

I. Introduction

The mathematical background for a non-linear Lagrangian theory of gravity was first formulated by Lovelock\(^1\), who proposed that the most general gravitational Lagrangian is

$$\mathcal{L} = \sqrt{-g} \sum_{m=0}^{n/2} \lambda_{(m)} \mathcal{L}_{(m)} \quad (1.1)$$

where $\lambda_{(m)}$ are coupling constants, $n$ denotes the manifold’s dimensions, $g$ is the determinant of the metric tensor and $\mathcal{L}_{(m)}$ are functions of the Riemann curvature
tensor, of the form

\[ \mathcal{L}_{(m)} = \frac{1}{2^m} \delta_{\alpha_1 \ldots \alpha_{2m}}^{\beta_1 \ldots \beta_{2m}} R^{\alpha_1 \alpha_2}_{\beta_1 \beta_2} \ldots R^{\alpha_{2m-1} \alpha_{2m}}_{\beta_{2m-1} \beta_{2m}} \]

(1.2)

where \( \delta^\alpha_\beta \) is the Kronecker symbol, \( \mathcal{L}_{(0)} \) is the volume \( n \)-form which gives rise to the cosmological constant, \( \mathcal{L}_{(1)} = \frac{1}{2} R \) is the Einstein-Hilbert (EH) Lagrangian and \( \mathcal{L}_{(2)} \) is the quadratic Gauss-Bonnett (GB) combination. Euler variation of the gravitational action corresponding to Eq.(1.1) yields the most general symmetric and divergenceless tensor, which describes the propagation of the gravitational field and depends only on the metric and its first and second order derivatives.

While quadratic Lagrangians have been widely studied (e.g. see Refs. [3,4] and references therein), cubic and/or quartic Lagrangians only recently have been introduced in the discussion of cosmological models in the framework of superstring theories. The reason is that, it is very hard to derive and (even harder) to solve the corresponding field equations. In this case, solutions may be obtained only through certain numerical techniques, where the idea of ”attractor” plays a central role: If some special spacetime is the attractor for a wide range of initial conditions, such a spacetime is naturally realized asymptotically. Since the ten-dimensional superstring theory is a candidate for a realistic unified theory, it is very important to investigate whether a similar attractor exists in this theory.

In the present paper we integrate numerically the field equations, resulting from a quartic gravitational Lagrangian, to obtain anisotropic, ten-dimensional cosmological models. The spacetime consists of one time direction and two maximally symmetric subspaces, FRW\( \otimes \)FRW: The external space, representing the ordinary Universe and the internal one, constituted by the extra dimensions. The internal space is a compact manifold of very small ”physical size” with respect to that of the ”visible” space at the present epoch. Since, on the other hand, at the origin the two subspaces were of comparable physical size, the internal one must have somehow been contracted towards a static value of the order of Planck length, \( l_{Pl} \sim 10^{-33} \text{ cm} \), to achieve ”spontaneous compactification”. Compactification is a topological process of quantum origin, which leads to the separation of the extra dimensions from the ordinary ones. In what follows we consider models of an already compactified internal space, i.e. we study only its contraction.

In Section II we derive the explicit form of the field equations for a quartic theory in ten dimensions, in which both subspaces are filled with an anisotropic fluid. In Section III we solve numerically the field equations, for a wide range of initial conditions and for several values of the ”free” parameters involved, as regards (1) vacuum models of flat subspaces and (2) perfect fluid models of positively curved subspaces. Next, we carry out a dynamical study in the \( H_{ext} - H_{int} \) plane, where each \( H_j \) represents the Hubble parameter of the corresponding subspace. Accordingly, we confirm the existence of attracting points and investigate their evolution with respect to the variation of the coupling constants \( \lambda_{(m)} \). The explicit time-dependence of the unknown scale functions may be subsequently determined by solving the linearized field equations around these attracting points. The corresponding analysis is presented in
II. The field equations in a quartic gravity theory

We consider a ten-dimensional line element, representing cosmological models which consist of two homogeneous and isotropic factor spaces, of the form

$$ds^2 = -dt^2 + R^2(t) \left( \frac{\sum_{i=1}^{3} (dx^i)^2}{1 + \frac{1}{4}k_{ext} \sum_{i=1}^{3} (x^i)^2} \right) + S^2(t) \left( \frac{\sum_{j=4}^{9} (dx^j)^2}{1 + \frac{1}{4}k_{int} \sum_{j=4}^{9} (x^j)^2} \right)$$

(2.1)

where $\hbar = 1 = c$, $R(t)$ and $S(t)$ are the cosmic scale functions of the external and the internal space respectively, $k_{ext} = -1, 0, +1$ is the curvature parameter of the "ordinary" space and $k_{int} = 0, +1$ is the corresponding parameter of the internal one. Therefore, the extra dimensions may be compactified either in a six-dimensional sphere, for $k_{int} = +1$, or in a six-dimensional torus, for $k_{int} = 0$. The spatial section of the metric (2.1) can be viewed as the direct product of two FRW models with three and six dimensions respectively. These models may be obtained through Hamilton’s principle, from a ten-dimensional action in which the gravitational part is of the form

$$I = \frac{1}{V_{int}} \int \sqrt{-\hat{g}} \left[ \lambda(0) \mathcal{L}_0 + \lambda(1) \mathcal{L}_1 + \lambda(1) \mathcal{L}_2 + \lambda(2) \mathcal{L}_3 + \lambda(3) \mathcal{L}_4 + \lambda(4) \mathcal{L}_5 \right] d^{10}x$$

(2.2)

where each of $\mathcal{L}_{(m)}$ is given by Eq.(1.2), $\lambda_{(m)}$ are the corresponding coupling constants and $V_{int}$ is a normalization constant, corresponding to the "physical size volume" of the internal space, once it may be considered static. The field equations read

$$\mathcal{L}_{\mu\nu} = -8\pi G_{10} T_{\mu\nu}$$

(2.3)

where $\mathcal{L}_{\mu\nu}$ is the Lovelock tensor up to the fourth order in curvature (Greek indices refer to the ten-dimensional spacetime) and $G_{10} = G V_{int}$ is the ten-dimensional gravitational constant. $T_{\mu\nu}$ is the energy-momentum tensor of an anisotropic perfect fluid source, of the form $T_{\mu\nu} = \text{diag} (\rho, -p_{ext}, ..., -p_{int}, ...)$, where $\rho$ is the total mass-energy density, while $p_{ext}$ and $p_{int}$ are the pressures associated to each factor space, separately. For the metric (2.1) Eq.(2.3) is decomposed into three independent equations of the form (cf. Ref. [9])

$$16\pi G_{10} \rho = \lambda(0) + 6\lambda(1) \left[ P + 5Q + 6(\frac{\dot{R}}{R})(\frac{\dot{S}}{S}) \right]$$

$$+ 72\lambda(2) \left[ 5Q^2 + 5PQ + 10(\frac{\dot{R}}{R})^2(\frac{\dot{S}}{S})^2 + 2P(\frac{\dot{R}}{R})(\frac{\dot{S}}{S}) + 20Q(\frac{\dot{R}}{R})(\frac{\dot{S}}{S})^2 \right]$$

$$+ 720\lambda(3) \left[ Q^3 + 8(\frac{\dot{R}}{R})^3(\frac{\dot{S}}{S})^3 + 9PQ^2 + 18Q^2(\frac{\dot{R}}{R})(\frac{\dot{S}}{S}) + 12PQ(\frac{\dot{R}}{R})(\frac{\dot{S}}{S}) \right]$$

$$+ 36Q(\frac{\dot{R}}{R})^2(\frac{\dot{S}}{S})^2 + 17280\lambda(4) \left[ PQ^3 \right]$$

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\[ + 6Q^2 \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 6PQ^2 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 8Q \left( \frac{\dot{R}}{R} \right)^3 \left( \frac{\dot{S}}{S} \right)^3 \]  

\[ (2.4a) \]

\[-16\pi G_{10} \rho_{\text{ext}} = \lambda_{(0)} + 2\lambda_{(1)} \left[ P + 15Q + 12 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 2 \left( \frac{\dot{R}}{R} \right) + 6 \left( \frac{\ddot{S}}{S} \right) \right] \]

\[ + 24\lambda_{(2)} \left[ 15Q^2 + 10 \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 5PQ + 40Q \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 20Q \left( \frac{\dot{S}}{S} \right) \right] \]

\[ + 2P \left( \frac{\ddot{S}}{S} \right) + 10Q \left( \frac{\dot{R}}{R} \right) + 20 \left( \frac{\dot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 4 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \]

\[ + 720\lambda_{(3)} \left[ Q^3 + 3Q^2P + 12Q^2 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 6Q^2 \left( \frac{\ddot{S}}{S} \right) + 6Q^2 \left( \frac{\dot{R}}{R} \right) \right] \]

\[ + 12Q \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right) + 4PQ \left( \frac{\dot{S}}{S} \right) + 8 \left( \frac{\dot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 24Q \left( \frac{\dot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \]

\[ + 8Q \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \right] + 5760\lambda_{(4)} \left[ 2Q^3 \left( \frac{\dot{R}}{R} \right) \right] \]

\[ + 12Q^2 \left( \frac{\dot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 12Q^2 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 6PQ^2 \left( \frac{\ddot{S}}{S} \right) \]

\[ + 24Q \left( \frac{\ddot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 2PQ^2 + 6Q^2 \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 \right]  

\[ (2.4b) \]

\[-16\pi G_{10} \rho_{\text{int}} = \lambda_{(0)} + 2\lambda_{(1)} \left[ 3P + 10Q + 15 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 3 \left( \frac{\dot{R}}{R} \right) + 5 \left( \frac{\ddot{S}}{S} \right) \right] \]

\[ + 24\lambda_{(2)} \left[ 5Q^2 + 20 \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 10PQ + 5P \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + P \left( \frac{\dot{R}}{R} \right) \right] \]

\[ + 30Q \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 10Q \left( \frac{\dot{R}}{R} \right) + 5P \left( \frac{\dot{S}}{S} \right) + 10Q \left( \frac{\dot{S}}{S} \right) + 10 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \]

\[ + 20 \left( \frac{\ddot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \right] \]

\[ + 720\lambda_{(3)} \left[ 4 \left( \frac{\dot{R}}{R} \right)^3 \left( \frac{\dot{S}}{S} \right)^3 + 3QPQ^2 + 3Q^2 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 3Q^2 \left( \frac{\dot{R}}{R} \right) \right] \]

\[ + 2Q^2 \left( \frac{\ddot{S}}{S} \right) + 6PQ \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 12Q \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 2PQ \left( \frac{\dot{R}}{R} \right) \]

\[ + 6PQ \left( \frac{\ddot{S}}{S} \right) + 4 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 12 \left( \frac{\dot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 + 4P \left( \frac{\ddot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \]

\[ + 12Q \left( \frac{\dot{R}}{R} \right) \left( \frac{\ddot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) \right] \]

\[ + 5760\lambda_{(4)} \left[ 6Q^2 \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{R}}{R} \right) \left( \frac{\dot{S}}{S} \right) + 3QPQ^2 \left( \frac{\dot{S}}{S} \right) + 12Q \left( \frac{\dot{S}}{S} \right) \left( \frac{\dot{R}}{R} \right)^2 \left( \frac{\dot{S}}{S} \right)^2 \right] \]
\[ + 12PQ\left(\frac{\dot{S}}{S}\right)\left(\frac{\dot{R}}{R}\right) + 3PQ^2\left(\frac{\dot{R}}{R}\right)\left(\frac{\dot{S}}{S}\right) + 4Q\left(\frac{\dot{R}}{R}\right)^3\left(\frac{\dot{S}}{S}\right)^3 \\
+ 3PQ^2\left(\frac{\ddot{R}}{R}\right) + 12Q\left(\frac{\dot{R}}{R}\right)^2\left(\frac{\dot{S}}{S}\right)^2 + 8\left(\frac{\dot{S}}{S}\right)\left(\frac{\dot{R}}{R}\right)^3\left(\frac{\dot{S}}{S}\right)^3 \right] \tag{2.4c}
\]

where an overdot denotes derivative with respect to time and we have set

\[ P = \left(\frac{\dot{R}}{R}\right)^2 + \frac{k_{\text{ext}}}{R^2}, \quad Q = \left(\frac{\dot{S}}{S}\right)^2 + \frac{k_{\text{int}}}{S^2} \tag{2.5} \]

Since the Lovelock tensor is divergenceless, \( L^{\mu\nu}_{\mu\nu} = 0 \), we obtain the conservation law \( T^{\mu\nu}_{\mu\nu} = 0 \), which gives

\[ \dot{\rho} + 3 (\rho + p_{\text{ext}}) \frac{\dot{R}}{R} + 6 (\rho + p_{\text{int}}) \frac{\dot{S}}{S} = 0 \tag{2.6} \]

Further inspection of the system of Eqs. (2.4) and (2.6) shows that only three of them are truly independent. Thus, the problem is completely determined by those, plus the two equations of state for the matter content, one for each subspace\(^{17}\). In the present article we consider two cases with regard to the energy-momentum tensor:

(a) Vacuum models, \( \rho = 0 \), in connection to flat spatial sections \((k_{\text{ext}} = 0 = k_{\text{int}})\) and

(b) Models of an heterotic superstring gas\(^{20}\), \( p_{\text{ext}} = \frac{1}{3}\rho \) and \( p_{\text{int}} = 0 \), in connection to positively curved spatial sections \((k_{\text{ext}} = 1 = k_{\text{int}})\). In the later case, the conservation law (2.6) gives

\[ \rho = \frac{M}{R^4 S^6} \tag{2.7} \]

where \( M \) is an integration constant. Thus, the external space is radiation dominated\(^{9,11,20}\).

In principle, we may integrate the system of Eqs. (2.4) and (2.6) to obtain the form of the unknown scale functions. However this is not an easy task, even in the most simple and symmetric cases\(^9\). Nevertheless, we may get a good estimation of their dynamic behaviour through numerical integration\(^{11}\).

Once the two equations of state are determined, Eq.(2.6) may be readily solved to give the unknown energy density and pressures, as functions of \( R(t) \) and \( S(t) \). These expressions are subsequently introduced in the r.h.s. of Eqs.(2.4). Now, only two of these equations are truly independent. The third one corresponds to a constraint, to be satisfied by the solutions of the system. As such, we choose Eq.(2.4a). The remaining independent field equations (2.4b) and (2.4c) may be recast in the form of a first order system (see also \([11]\)), as follows

\[ \dot{H}_{\text{ext}} = G_1 \left( H_{\text{ext}}, H_{\text{int}}, X, Y \right) \tag{2.8a} \]

\[ \dot{H}_{\text{int}} = G_2 \left( H_{\text{ext}}, H_{\text{int}}, X, Y \right) \tag{2.8b} \]

\[ \dot{X} = -X H_{\text{ext}} \tag{2.8c} \]

\[ \dot{Y} = -Y H_{\text{int}} \tag{2.8d} \]
where we have set
\[ H_{\text{ext}} = \frac{\dot{R}}{R}, \quad H_{\text{int}} = \frac{\dot{S}}{S}, \quad X^2 = \frac{k_{\text{ext}}}{R^2}, \quad Y^2 = \frac{k_{\text{int}}}{S^2} \] (2.9)
and the explicit forms of the functions \( G_1 \) and \( G_2 \) are given in the Appendix A.

Finally, it is convenient to make a parameter rescaling in the field equations, of the form
\[ \kappa_m = \frac{\lambda(m)}{\lambda(1)}, \quad m = 0, 1, 2, 3, 4 \] (2.10)
where \( \lambda(1) = (16\pi G)^{-1} \) is the coupling constant in the four-dimensional General Relativity (GR). The value of the normalized coupling constants, \( \kappa_m (\kappa_m \leq 1) \), is directly proportional to the contribution in the field equations of the corresponding \( m \)-th order non-linear term, with respect to the results obtained in the EH cosmology. Clearly, \( \kappa_1 = 1 \).

### III. Numerical Results

We integrate numerically the system of Eqs.(2.8). The constraint (2.4a) is checked to be satisfied with an accuracy of \( 10^{-10} \) along integration. The initial conditions \( H_{0\text{ext}}, H_{0\text{int}}, X_0, Y_0 \) are chosen so that: (a) \( X_0 = Y_0 \), i.e. at the origin, the two factor spaces are separated, but of the same "physical size"\(^{16} \). (b) \( H_{0\text{ext}} > 0 \), i.e. initially the ordinary space expands, in accordance to what we observe at the present epoch\(^{12,21} \). (c) \( H_{0\text{int}} < 0 \), i.e. at the origin, the internal space contracts, in correspondence to "spontaneous compactification"\(^{12,16,17,21} \). The cases where either \( H_{0\text{ext}} < 0 \) or \( H_{0\text{int}} > 0 \) are not permitted, since the constraint equation is not satisfied. Nevertheless, the case where both conditions \( H_{0\text{ext}} < 0 \) and \( H_{0\text{int}} > 0 \) are valid is acceptable by numerical analysis. Actually, it corresponds to the time-reversed solution of the system (2.8).

The time coordinate is measured in dimensionless units, being normalized with respect to the Planck time, \( \tau = t/t_P \) (\( t_P = \sqrt{G} \sim 10^{-43} \text{sec} \)). The limits of numerical integration range from \( \tau = 0 \) to \( \tau = 10^5 \). The upper limit coincides with the origin of the GUT epoch\(^{21} \), \( t_{GUT} = 10^5 t_P \), corresponding to the end of the string regime\(^{22} \). However we have to point out that, although the origin of the time coordinate is set at \( \tau = 0 \), the equations (2.8) may not be valid in the region \( 0 < \tau \leq 1 \) since, in the absence of a quantum gravity theory, there is always a region of ambiguity around \( t = 0 \), of the order of Planck time\(^{23-25} \).

The solution of the system (2.8) may be represented as curves in the \( H_{\text{ext}} - H_{\text{int}} \) plane. Any point located on these curves always satisfies the constraint condition (2.4a). Thus, the curves actually represent "orbits" of the dynamical system under study. Each curve, corresponding to a different set of initial conditions, is bounded by fixed points (or infinities) and represents a different type of evolution for the Universe.

In what follows, we focus attention on the existence and the evolution of attracting points in the \( H_{\text{ext}} - H_{\text{int}} \) plane. The reason rests in the physical meaning of the
attractor: No matter what the behaviour of a cosmological model at the origin might be, it will always end up to evolve as indicated by the location of the attracting point in the \( H_{\text{ext}} - H_{\text{int}} \) plane.

(1) Vacuum models with spatially flat subspaces

We study the evolution of vacuum ten-dimensional cosmological models, with metric of the form (2.1), in which both subspaces are spatially flat, i.e. \( k_{\text{ext}} = 0 = k_{\text{int}} \). Thus, \( X = 0 = Y \).

The first case to study are the GB models (see also Refs. [11,12]). In this case, \( \kappa_2 = 1 \) and \( \kappa_0 = 0 = \kappa_3 = \kappa_4 \). The non-linear curvature contributions to the field equations come out from the quadratic terms alone. The time evolution of the Hubble parameters is presented in Fig. 1a. We see that both parameters evolve to approach constant values in the later stages. This situation verifies the existence of attracting points in the \( H_{\text{ext}} - H_{\text{int}} \) plane during the evolution of the Universe. Therefore, for a wide range of initial conditions, both subspaces will end up to evolve as De Sitter spaces, in complete correspondence to the results of Ishihara\textsuperscript{12}.

We also observe that \( H_{\text{ext}} > 0 \) and \( H_{\text{int}} < 0 \). Therefore, while the internal space contracts exponentially to achieve spontaneous compactification, the external one expands, a fact that corresponds to an inflationary phase. This result indicates that the introduction of the non-linear curvature terms into the gravitational action may play an important role as far as the inflation is concerned\textsuperscript{26–30}. The explicit location of the attracting point is shown in Fig. 1b. The attractor corresponds to the fixed point \( D_2 \) recognized by Ishihara\textsuperscript{12} in the evolution of the extended De Sitter models in GB theory.

The next step is to introduce into the problem a "bare" cosmological constant, \( \Lambda \), corresponding to the expectation value of the vacuum energy density\textsuperscript{25}. Now, in addition to \( \kappa_2 \), we also have \( \kappa_0 \neq 0 \), while \( \kappa_3 = 0 = \kappa_4 \). When \( \kappa_0 \in [0, 1] \) the value of the cosmological constant in physical units is \( \Lambda = 2\kappa_0 \times 10^{-48} \text{ cm}^{-2} \), which is quite small.

The behaviour of the model is qualitatively similar to the previous case. Again we verify the existence of an "attractor". Both subspaces correspond to De Sitter models. The external space exhibits inflationary expansion, while the internal one contracts. However, in this case, the location of the attracting point \( D_2 \) has changed to higher absolute values in the evolution of \( H_{\text{ext}} \) and \( H_{\text{int}} \) (Fig. 2a). We may determine explicitly the law of the attractor’s displacement in the \( H_{\text{ext}} - H_{\text{int}} \) plane, caused by variations of the cosmological constant.

In general, to determine the exact location of the attracting points in an \( H_{\text{ext}} - H_{\text{int}} \) plane, requires to set

\[
G_1(H_{\text{ext}}, H_{\text{int}}, X, Y) = 0 \tag{3.1.1a}
\]

\[
G_2(H_{\text{ext}}, H_{\text{int}}, X, Y) = 0 \tag{3.1.1b}
\]
In the case of flat and vacuum subspaces \((X = Y = p_{\text{ext}} = p_{\text{int}} = 0)\), Eqs.(3.1.1) read

\[
\begin{align*}
    f_1(H_{\text{ext}}, H_{\text{int}}, \kappa_m) &= [G_{12}G_{20} - G_{22}G_{10}]_{X=Y=0} = 0 \quad (3.1.2a) \\
    f_2(H_{\text{ext}}, H_{\text{int}}, \kappa_m) &= [G_{21}G_{10} - G_{11}G_{20}]_{X=Y=0} = 0 \quad (3.1.2b)
\end{align*}
\]

where \(m = 0, 2, 3, 4\) and the quantities \(G_{ij}\) are presented in the Appendix A. We differentiate the functions \(f_1\) and \(f_2\) with respect to \(H_{\text{ext}}, H_{\text{int}}\) and \(\kappa_m\), to obtain a system of first order differential equations ("variational equations")

\[
\begin{align*}
    df_1 &= \left(\frac{\partial f_1}{\partial H_{\text{ext}}}\right)_{p}dH_{\text{ext}} + \left(\frac{\partial f_1}{\partial H_{\text{int}}}\right)_{p}dH_{\text{int}} + \sum_j \left(\frac{\partial f_1}{\partial \kappa_m}\right)_{p}d\kappa_m = 0 \quad (3.1.3a) \\
    df_2 &= \left(\frac{\partial f_2}{\partial H_{\text{ext}}}\right)_{p}dH_{\text{ext}} + \left(\frac{\partial f_2}{\partial H_{\text{int}}}\right)_{p}dH_{\text{int}} + \sum_j \left(\frac{\partial f_2}{\partial \kappa_m}\right)_{p}d\kappa_m = 0 \quad (3.1.3b)
\end{align*}
\]

The system (3.1.3) may be used, to determine the evolution of the attracting point \(D_2\) \((H_{\text{ext}}, H_{\text{int}})\), under the variation of the normalized coupling constants \(\kappa_m\). For \(\kappa_2 = 1\), in the case of vanishing \(\kappa_3\) and \(\kappa_4\), the evolution of the attractor \(D_2\) \((H_{\text{ext}}, H_{\text{int}})\) with respect to the variation of the cosmological constant \(\kappa_0\), is given by

\[
\begin{align*}
    \frac{dH_{\text{ext}}}{d\kappa_0} &= \left(\frac{QQ_1}{PP}\right) \quad (3.1.4a) \\
    \frac{dH_{\text{int}}}{d\kappa_0} &= \left(\frac{QQ_2}{PP}\right) \quad (3.1.4b)
\end{align*}
\]

where the functions \(PP, QQ_1\) and \(QQ_2\) are given in the Appendix B. Subsequently, the system (3.1.4) is evaluated by numerical integration. The corresponding results are shown in Fig. 2b. Using least square fitting, we see that the displacement of \(D_2\) takes place along the straight line

\[
H_{\text{int}} = -0.075H_{\text{ext}} - 0.071 \quad (3.1.5)
\]

The investigation of the behaviour of the models under consideration by including a third order curvature term, corresponds to study them at earlier epochs in the history of the Universe. Indeed, if we are interested in the behaviour of the model very close to the initial singularity, the leading terms to consider in the field equations are those with the highest power in \(\left(\frac{1}{t}\right)^{1/2}\), i.e. those obtained from the highest order terms in the gravitational action.\(^9\)

The time-evolution of the model is quite similar to the previous cases. In the later stages it corresponds to an extended De Sitter model, in which both subspaces evolve exponentially. The external space expands, while the internal one contracts (Fig. 3a).

Again, we verify the existence of an attracting point \(P\) in the evolution of the Hubble parameters and we investigate its behaviour as \(\kappa_3\) increases, from 0 to 1, i.e. until it becomes as important as the quadratic term. The evolution of the attractor in the \(L_{(3)}\)-theory, with respect to the variation of \(\kappa_3\), may be obtained in a similar way.
as in the $\kappa_0$ case. We differentiate the functions $f_1$ and $f_2$ with respect to $H_{\text{ext}}$, $H_{\text{int}}$ and $\kappa_3$ to obtain a first order system of differential equations which, for $\kappa_2 = 1$ and for vanishing $\kappa_0$ and $\kappa_4$, will determine the displacement of P in the $H_{\text{ext}} - H_{\text{int}}$ plane, under the variation of $\kappa_3$.

The corresponding results are presented in Fig. 3b. We observe that the attractor moves to higher absolute values of $H_{\text{int}}$ as $\kappa_3$ increases. This result has a clear physical meaning. Since increasing $\kappa_3$ corresponds to study the earlier stages in the evolution of the Universe, we see that at these epochs the internal space contracts at higher rates than those of the GB theory. Then Fig. 3b verifies that at the late stages, where the GB theory holds alone, the value of the internal Hubble parameter decreases in order to achieve stabilization.

Again, the law of displacement of P in the $H_{\text{ext}} - H_{\text{int}}$ plane may be estimated using best-fit methods. In this context, we find that it may be represented by a sixth-order polynomial $H_{\text{ext}} = p_6(H_{\text{int}})$, with coefficients: $a_0 = 0.7373$, $a_1 = -25.594$, $a_2 = -457.324$, $a_3 = -3323.9$, $a_4 = -12156.9$, $a_5 = -22135.2$ and $a_6 = -15905.6$.

Finally, to solve the cosmological field equations when all terms in the action (2.2) are included (i.e. $\kappa_4 \neq 0$), corresponds to study the dynamic behaviour of the model under consideration at even earlier epochs. The results are slightly different from those of the previous case (Fig. 4a). Again, in the later stages, the model consists of two De Sitter subspaces and there exists an attracting point. The attractor’s displacement in the $H_{\text{ext}} - H_{\text{int}}$ plane is obtained in a way similar to the $\kappa_0$ and $\kappa_3$ cases and may be represented by a third-order polynomial, $H_{\text{int}} = p_3(H_{\text{ext}})$, with coefficients: $b_0 = 7.47$, $b_1 = -34.27$, $b_2 = 51.33$ and $b_3 = -25.74$. The corresponding result is shown in Fig. 4b.

Hence, we may conclude that in every case where non linear terms are included, the “extended” De Sitter solution (i.e. an exponentially expanding external space in connection to an exponentially contracting internal one) corresponds to an “attractor” of the dynamical system under consideration. Accordingly, (in our model) no matter how the Universe may originate, there is at least one period during its time-evolution in which it exhibits inflation of the ordinary space, accompanied by spontaneous compactification of the internal one\textsuperscript{12,30}.

(2) Perfect fluid models of curved subspaces

We consider a ten-dimensional metric of the form (2.1), which now represents a class of cosmological models with positively curved subspaces ($k_{\text{ext}} = 1 = k_{\text{int}}$). Then, $X = R^{-1}(t)$ and $Y = S^{-1}(t)$ and we study the time-evolution of the cosmological models as results from the solution of the system (2.8).

The numerical analysis is carried out in the same fashion as in the previous case of vacuum models. We consider that at the origin both subspaces are of the same "physical size", i.e. $X_0 = Y_0$, but they have different expansion rates, $H_{0 \text{ext}}$ and $H_{0 \text{int}}$. As such we choose the corresponding range used in the vacuum case. We normalize both scale functions $R(t)$ and $S(t)$ to unity, with respect to their value at the Planck
epoch. That is
\[ R(t) \to \frac{R(t)}{R_{Pl}}, \quad S(t) \to \frac{S(t)}{S_{Pl}} \]
where \( R_{Pl} = S_{Pl} \). As initial conditions we choose \( R_0 = 100 = S_0 \).

We represent the matter filling the Universe by a closed or heterotic superstring perfect gas, with the following equation of state, deduced by Matsuo
\[ p_{\text{ext}} = \frac{1}{3} \rho, \quad p_{\text{int}} = 0 \]
Thus, the external space is radiation-dominated, while the internal one is pressureless. It has been recently shown that, in this case, the two subspaces are completely disjoint. The time-evolution of the total mass-energy density \( \rho \) is accordingly given by Eq.(2.7).

As regards the GB models \( (\kappa_3 = 0 = \kappa_4) \), we have performed a number of computational runs, varying the initial values of the Hubble parameters and the coupling constant \( \kappa_2 \) as well, from \( \kappa_2 = 0.1 \) to \( \kappa_2 = 1 \). Numerical results in this case indicate that there is a considerable difference with respect to the vacuum-flat models. It rests in the fact that the range of values of the coupling constant \( \kappa_2 \) may be splitted into two parts. Each one of these parts leads to a different time-evolution of both the external and the internal scale functions.

The first part consists of values of \( \kappa_2 \) in the interval \( 0.1 \leq \kappa_2 \leq 0.65 \), i.e. when the contribution of the quadratic curvature terms is relatively small. In this case we expect that the time-evolution of the Universe will be only slightly different from the corresponding EH one. Indeed, the numerical results indicate that the system (2.8) admits solutions with a power law dependence of the scale functions upon time, of the form
\[ R(t) \propto t^{m_1} \]  
\[ S(t) \propto t^{-m_2} \]
where the values of the indices \( m_1 \) and \( m_2 \) are continuously increasing in the ranges \( 0.25 \leq m_1 \leq 0.55 \) and \( 0.01 \leq m_2 \leq 0.11 \), as \( \kappa_2 \) increases from 0.1 to 0.65. In this case, there are no attracting points in the evolution of the Universe. The last values in those ranges (0.55 and 0.11, respectively), both corresponding to the value \( \kappa_2 = 0.65 \), represent a Kasner-type regime of the GB models. Indeed, the analytic approach in this case suggests that the two subspaces evolve as
\[ R(t) \sim t^{p_1}, \quad S(t) \sim t^{-p_2} \]
where both \( p_1 \) and \( p_2 \) are positive and in a ten-dimensional spacetime they satisfy the conditions
\[ 3p_1 - 6p_2 = 1 \]  
\[ 3p_1^2 + 6p_2^2 = 1 \]
The only physically acceptable solution of the system (3.2.5), compatible with the condition \( p_1, p_2 > 0 \), is

\[
p_1 = \frac{5}{9} = 0.555, \quad p_2 = \frac{1}{9} = 0.111
\]

Therefore, when \( \kappa_2 = 0.65 \), although the spatial sections are curved, the time-evolution of the Universe admits a Kasner-type solution. This solution actually lies on the interface between two different types of cosmological behaviour (Figs. 5a and 5b).

The second type of time-evolution arises when \( 0.65 < \kappa_2 \leq 1 \). Then the Universe behaves, again, as an extended De Sitter spacetime (where the external space expands while the internal one contracts, both exponentially). In this case there exists an attracting point as in the vacuum-flat models (Fig. 6a).

In conclusion, for a curved ten-dimensional GB cosmological model, filled with matter in the form of a superstring perfect gas, we may obtain three different types of cosmological behaviour, depending on the exact value of the normalized coupling constant \( \kappa_2 \):

(a) Power-law solutions, with no attracting points, when \( 0.1 \leq \kappa_2 < 0.65 \).

(b) A Kasner-type model, when \( \kappa_2 = 0.65 \).

(c) Extended De Sitter models, with an attracting point, when \( 0.65 < \kappa_2 \leq 1 \).

In all cases, the external space expands, while the internal one contracts. The inclusion of the contribution of the third and/or the fourth order terms in the field equations simply amounts to a modulation of those results (Fig. 6b).

IV. Analytic Results

Analytic expressions, for the time-evolution of the model Universe considered, may be obtained by solving the cosmological field equations (2.8) around the attracting points. Accordingly, we investigate the cosmological behaviour of a vacuum, ten-dimensional model with spatially flat subspaces \( X = 0 = Y \) within the context of the quartic Lagrangian theory under consideration. Clearly, setting some of the coupling constants \( \lambda_{(m)} \) equal to zero corresponds to reducing the general theory to its lower case counterparts (EH-cosmology, GB-theory etc.).

Since we are interested in the behaviour of the model around the attracting points, we consider the linearized equations

\[
H_{ext} = A_1 + H_1(t) \tag{4.1a}
\]

\[
H_{int} = A_2 + H_2(t) \tag{4.1b}
\]
where \(A_1\) and \(A_2\) are the coordinates of the attractor, while \(H_1(t)\) and \(H_2(t)\) represent small perturbations around those values \(|H_1|, |H_2| \ll 1\). Therefore, to obtain the time-evolution of \(H_{\text{ext}}\) and \(H_{\text{int}}\), we only have to solve the system (2.8) linearized with respect to \(H_1\) and \(H_2\).

The system of the cosmological field equations (2.8), linearized with respect to \(H_1\) and \(H_2\), may be written in the form

\[
\dot{H}_1(t) = \frac{\beta_1 H_1 + \beta_2 H_2 + \beta_3}{\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3} \quad (4.2a)
\]
\[
\dot{H}_2(t) = \frac{\gamma_1 H_1 + \gamma_2 H_2 + \gamma_3}{\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3} \quad (4.2b)
\]

where \(\alpha_j, \beta_j\) and \(\gamma_j\) \((j = 1, 2, 3)\) are constants, calculated directly from the linearization of the original equations, which depend on \(A_1, A_2\) and \(\lambda_{(m)}\) \((m = 0, 1, 2, 3, 4)\). From Eqs.(4.2) we obtain

\[
\frac{dH_1}{dH_2} = \frac{\beta_1 H_1 + \beta_2 H_2 + \beta_3}{\gamma_1 H_1 + \gamma_2 H_2 + \gamma_3} \quad (4.3)
\]

The solution of Eq.(4.3), in connection to Eqs.(4.1), will give, in the linear approximation, the analytic expression of \(H_{\text{ext}}\) in terms of \(H_{\text{int}}\). To solve Eq.(4.3), we need to have the solution \((h_1, h_2)\) of the algebraic system

\[
\beta_1 H_1 + \beta_2 H_2 + \beta_3 = 0 \quad (4.4a)
\]
\[
\gamma_1 H_1 + \gamma_2 H_2 + \gamma_3 = 0 \quad (4.4b)
\]

We choose

\[
\gamma_1 \neq 0, \quad \beta_2 \gamma_1 - \beta_1 \gamma_2 \neq 0 \quad (4.5)
\]

and furthermore, we set

\[
w = H_1 - h_1 \quad (4.6a)
\]
\[
z = H_2 - h_2 \quad (4.6b)
\]

We verify that the solution of Eq.(4.3) depends on several algebraic combinations of the constants \(\alpha_j, \beta_j\) and \(\gamma_j\), something that leads to several conditions between the coupling constants \(\lambda_{(m)}\). Therefore, we consider the following cases:

(a) \(\beta_1 + \gamma_2 \neq 0\): This combination corresponds to the most general case. Setting

\[
\Delta = -\left[4\beta_2 \gamma_1 + (\beta_1 - \gamma_2)^2\right] \quad (4.7)
\]

the solution of Eq.(4.3) reads

\[
\ln \frac{1}{c} \left[\beta_2 z^2 + (\beta_1 - \gamma_2)zw - \gamma_1 w^2\right] = \begin{cases} 
\frac{\beta_1 + \gamma_2}{\sqrt{|\Delta|}} \ln \frac{\beta_1 - \gamma_2 - \sqrt{|\Delta| - 2\gamma_1 w}}{\beta_1 - \gamma_2 + \sqrt{|\Delta| - 2\gamma_1 w}} \quad & \text{for } \Delta < 0 \\
-\frac{2(\beta_1 + \gamma_2)}{(\beta_1 - \gamma_2)^2 - 2\gamma_1} \quad & \text{for } \Delta = 0 \\
2\frac{\beta_1 + \gamma_2}{\sqrt{|\Delta|}} \arctan \left(\frac{(\beta_1 - \gamma_2)^2 - 2\gamma_1}{\sqrt{|\Delta|}}\right) \quad & \text{for } \Delta > 0 
\end{cases} \quad (4.8)
\]
where $c$ is an arbitrary integration constant.

(b) $\beta_1 + \gamma_2 = 0$ and $\Delta = 0$ with $\beta_2 \gamma_1 < 0$: In this case we may proceed to derive the explicit time-dependence of the Hubble parameters and the corresponding scale functions for both subspaces. From Eq.(4.8) we obtain

$$H_2 = c_1 H_1 + c_2$$  \hspace{1cm} (4.9)

where

$$c_1 = \sqrt{\frac{\gamma_1}{\beta_2}}, \quad c_2 = 1 + h_2 - c_1 h_1$$  \hspace{1cm} (4.10)

Now, Eq.(4.9) is inserted into Eq.(4.2a) to give

$$\dot{H}_1 = \frac{\delta H_1 + \epsilon}{\zeta H_1 + \eta}$$  \hspace{1cm} (4.11)

where the constants $\delta$, $\epsilon$, $\zeta$ and $\eta$ stand for the combinations

$$\delta = \beta_1 c_1 + \beta_2, \quad \epsilon = \beta_3 + \beta_1 c_2$$

$$\zeta = \alpha_1 c_1 + \alpha_2, \quad \eta = \alpha_3 + \alpha_1 c_2$$  \hspace{1cm} (4.12)

We consider the following cases:

(i) $\delta$, $\zeta \neq 0$: In this case, Eq.(4.11) results in

$$\frac{\zeta}{\delta} H_1 + \frac{\zeta}{\delta} \left( \frac{\eta}{\zeta} - \frac{\epsilon}{\delta} \right) \ln \left( \frac{H_1 + \frac{\epsilon}{\delta}}{\delta} \right) = t - t_0$$  \hspace{1cm} (4.13)

where $t_0$ is an integration constant. Now, Eq.(4.1a) in connection with Eq.(4.13), may be easily integrated to give the form of $R(t)$, when the condition

$$\frac{\eta}{\zeta} - \frac{\epsilon}{\delta} = 0$$  \hspace{1cm} (4.14)

holds. Then, we obtain

$$\ln R(t) \sim A_1 (t - t_0) + \frac{\delta}{2 \zeta} (t - t_0)^2$$  \hspace{1cm} (4.15)

which introduces a quadratic correction to the expected De Sitter solution.

(ii) $\delta$, $\eta \neq 0$ and $\zeta = 0$: In this case we rediscover the solutions of Ishihara$^{12}$, obtained in the GB theory, as a particular case of the general solution. Indeed, from Eq.(4.11) we obtain

$$H_1 = C e^{\frac{\delta}{\epsilon} t} - \frac{\epsilon}{\delta}$$  \hspace{1cm} (4.16)
where $C$ is an arbitrary integration constant. Therefore the corresponding external scale function is of the form

$$\ln R(t) \sim \left( A_1 - \frac{\epsilon}{\delta} \right) (t - t_0) + C \frac{\eta}{\delta} e^{\frac{\eta}{\delta} (t - t_0)}$$  \hfill (4.17)

Since the external space expands, we must have $A_1 > \frac{\epsilon}{\delta}$. For $C = 0$, Eq.(4.17) reads

$$R(t) \sim e^{(A_1 - \frac{\epsilon}{\delta})(t - t_0)}$$ \hfill (4.18a)

corresponding again to a De Sitter phase, while for $\frac{2C}{\delta} \ll 1$ it yields

$$R(t) \sim e^{(A_1 - \frac{\epsilon}{\delta})(t - t_0)} \left( 1 + C \left( \frac{\eta}{\delta} \right)^2 e^{\frac{\eta}{\delta} (t - t_0)} \right)$$ \hfill (4.18b)

For $\epsilon = 0$ Eq.(4.18b) corresponds to the solution of Ishihara (Eq.(15) of Ref. [12]) obtained in the framework of the GB theory.

(iii) $\delta = 0$, $\zeta \neq 0$: Finally, in this case, Eqs. (4.1a) and (4.11) result in

$$\ln R(t) \sim \left( A_1 - \frac{\eta}{\zeta} \right) (t - t_0) \pm \frac{1}{2\epsilon \zeta^2} \left[ \eta^2 + 2\epsilon \zeta (t - t_0) \right]^{3/2}$$ \hfill (4.19)

where, in connection with the numerical results we must have $A_1 > \frac{\eta}{\zeta}$.

In concluding, we see that the coupling constants $\lambda_{(m)}$ may not be arbitrary. In every case, they should satisfy certain algebraic relations, depending on the form of the corresponding solution around the attracting points.

Since both Eqs.(4.2) are almost of the same functional form, in all of the preceding cases, similar functional results may be obtained for the internal space, through the solution of Eq.(4.2b). In this case, however, we must take into account the fact that the numerical results indicate that the extra dimensions contract ($A_2 < 0$). This argument may lead to additional constraints on the coupling constants $\lambda_{(m)}$.

V. Discussion and Conclusions

In the present paper we have studied the time evolution of anisotropic, ten-dimensional cosmological models in the framework of a quartic Lovelock-Lagrangian theory of gravity\textsuperscript{1,9–11}. The cosmological models under consideration consist of one time direction and two homogeneous and isotropic subspaces: A three-dimensional external space, which represents the ordinary Universe, and a compact internal space, which is constituted by the extra dimensions. The evolution of the Universe depends on four free parameters. These are the coefficients $\lambda_{(m)}$ which introduce the extra curvature terms in the gravitational Lagrangian ($m = 0, 2, 3, 4$). They are to be regarded as the coupling constants\textsuperscript{3,9}. Since we have considered models of an already compactified internal space\textsuperscript{17}, we accordingly examine the process of its contraction\textsuperscript{15–17,19}.
The Universe is filled with matter in the form of an anisotropic perfect fluid. Given an equation of state for the matter content of each subspace, the time evolution of the Universe is completely determined by a system of three second-order, non-linear differential equations, consisting of the field equations (2.4b) and (2.4c) together with the conservation law (2.6). The initial value field equation (2.4a) corresponds to a constraint which should be satisfied by the cosmological solutions. As regards the energy momentum tensor, we have considered two cases: (a) Vacuum models, \( \rho = 0 \), in connection with spatially flat subspaces and (b) Models of an heterotic superstring gas, with \( p_{\text{ext}} = \frac{1}{3} \rho \) and \( p_{\text{int}} = 0 \), in connection with positively curved subspaces.

The three independent equations may be subsequently expressed in the form of a first order system, Eqs.(2.8), involving the Hubble parameters \( H_{\text{ext}} \), \( H_{\text{int}} \) and the corresponding scale functions \( R(t) \), \( S(t) \) of the two factor spaces. This system is evaluated numerically, for a wide range of initial conditions of the form \( H_{0}\) \( \text{ext} > 0 \) and \( H_{0}\) \( \text{int} < 0 \). Its solutions may be represented by curves in the \( H_{\text{ext}} - H_{\text{int}} \) plane. Those curves correspond to the ”orbits” of the dynamical system under study and each one of them, associated with a different set of initial conditions, represents a different type of evolution for the Universe.

In the case of vacuum models with flat subspaces \( (k_{\text{ext}} = 0 = k_{\text{int}}) \), the numerical results indicate that for all values of the coupling constants involved and also for a wide range of initial conditions, the Universe will always end up to evolve according to an extended De Sitter solution, i.e. an exponentially expanding external space, accompanied by an exponentially contracting internal one. Indeed, in this case the Hubble parameters of both subspaces approach constant values in the later stages. We have confirmed that those values actually represent the attracting points of the dynamical system under consideration\(^{11-13}\). The appearance of attractors in the solution of the cosmological field equations is very important, since, if a spacetime is an attractor for a wide range of initial conditions, then it may be realized asymptotically in the later stages\(^{11,13}\). Those results indicate that the existence of the non-linear curvature terms in the gravitational action may lead to inflation without the use of any phase transition\(^{19,27-30,36}\).

Furthermore, we have investigated the evolution of the attractors under the variation of the normalized coupling constants \( \kappa_m = \lambda(m) / \lambda(1) \) \((m = 0, 3, 4)\). In all cases, the attracting points are displaced at higher absolute values of \( H_{\text{int}} \) as \( \kappa_m \) increases from 0 to 1. As regards the variation of \( \kappa_3 \) and \( \kappa_4 \), this result has a clear physical meaning.

In determining the cosmological behaviour of the model very close to the initial singularity, the leading terms to consider are those with the highest power in \((\frac{1}{t})\), i.e. those obtained from the highest order terms in the gravitational action\(^{5,6}\). Therefore, the increase of \( \kappa_m \) \((m = 3, 4)\) corresponds to a more accurate study of the earlier stages in the evolution of the Universe\(^9\). Then, from Figs. 3b and 4b, we see that at those epochs the internal space contracts at higher rates than those of the GB theory. This means that in the later stages of the time evolution, where the GB theory holds alone, the absolute value of the internal Hubble parameter decreases, in order for the
extra space to achieve stabilization at a small physical size\textsuperscript{15\hbox{-}17,19,32}.

In the vacuum case it is possible to derive the analytic dependence of the scale functions upon time, by linearizing the field equations around the values of $H_{ext}$ and $H_{int}$ at the attracting points. The corresponding results indicate that the functional form of the analytic solution depends on several algebraic conditions between the coupling constants $\lambda_{(m)}$. Therefore, in a Lovelock-Lagrangian theory of gravity, the coupling constants may play an important role in determining the cosmological behaviour of the model Universe. Nevertheless, the coefficients of each term in the Lagrangian either should be determined experimentaly or they should be given by some underlying foundamental theory\textsuperscript{3}. In this context, we have rediscovered the solutions of Ishihara\textsuperscript{12}, obtained in the framework of the GB theory, as particular solutions of the general quartic case.

The cosmological models with matter in the form of a superstring perfect gas, in which both subspaces are possitively curved ($k_{ext} = 1 = k_{int}$), can be treated only numerically. In this case, the evolution of the GB models depends additionally on the exact value of the normalized coupling constant $\kappa_2$. We have obtained three different types of cosmological behaviour:

(a) Power-law solutions, with no attracting points, when $0.1 \leq \kappa_2 < 0.65$.

(b) A Kasner-type model, when $\kappa_2 = 0.65$.

(c) Extended De Sitter models, with an attracting point, when $0.65 < \kappa_2 \leq 1$.

In all cases, the external space expands, while the internal one contracts. The inclusion of the contributions of the third and/or the fourth order terms in the field equations, simply amounts in a modulation of the above results (e.g. see Fig. 6b).

Acknowledgements: The authors would like to express their gratitude to Professor J. Demaret for his suggestions and his comments on the content of this article. Furthermore, they would like to thank Dr. A. Anastasiadis for several helpful advices on their computer work. One of us (K. K.) would like to thank the Greek State Scholarships Foundation for the financial support during this work.

Appendix A

The cosmological field equations (2.4b) and (2.4c) may be recast in the form of a first order system, as follows

\[\dot{H}_{ext} = G_1 (H_{ext} , H_{int} , X , Y) \quad (A.1a)\]
\[\dot{H}_{int} = G_2 (H_{ext} , H_{int} , X , Y) \quad (A.1b)\]
\[\dot{X} = -X H_{ext} \quad (A.1c)\]
\[ \dot{Y} = -Y \, H_{\text{int}} \quad \text{(A.1d)} \]

where
\[ X^2 = \frac{k_{\text{ext}}}{R^2}, \quad Y^2 = \frac{k_{\text{int}}}{S^2} \quad \text{(A.2)} \]

The functions \( G_1 \) and \( G_2 \) are given by the expressions
\[
G_1 = \frac{[G_{12}(16\pi p_{\text{int}} + G_{20}) - G_{22}(16\pi p_{\text{ext}} + G_{10})]}{[G_{11}G_{22} - G_{12}G_{21}]} \quad \text{(A.3a)}
\]
\[
G_2 = \frac{[G_{21}(16\pi p_{\text{ext}} + G_{10}) - G_{11}(16\pi p_{\text{int}} + G_{20})]}{[G_{11}G_{22} - G_{12}G_{21}]} \quad \text{(A.3b)}
\]

where we have set
\[
G_{10} = \lambda_0 + 2\lambda_1 B_{10} + 24\lambda_2 B_{20} + 720\lambda_3 B_{30} + 5760\lambda_4 B_{40} \quad \text{(A.4a)}
\]
\[
G_{11} = 4\lambda_1 + 24\lambda_2 B_{21} + 720\lambda_3 B_{31} + 5760\lambda_4 B_{41} \quad \text{(A.4b)}
\]
\[
G_{12} = 12\lambda_1 + 24\lambda_2 B_{22} + 720\lambda_3 B_{32} + 5760\lambda_4 B_{42} \quad \text{(A.4c)}
\]
\[
G_{20} = \lambda_0 + 2\lambda_1 C_{10} + 24\lambda_2 C_{20} + 720\lambda_3 C_{30} + 5760\lambda_4 C_{40} \quad \text{(A.4d)}
\]
\[
G_{21} = 6\lambda_1 + 24\lambda_2 C_{21} + 720\lambda_3 C_{31} + 5760\lambda_4 C_{41} \quad \text{(A.4e)}
\]
\[
G_{22} = 10\lambda_1 + 24\lambda_2 C_{22} + 720\lambda_3 C_{32} + 5760\lambda_4 C_{42} \quad \text{(A.4f)}
\]

and the quantities \( B_{ij} \) and \( C_{ij} \) are given by
\[
B_{10} = 3H_{\text{ext}}^2 + 21H_{\text{int}}^2 + 12H_{\text{ext}}H_{\text{int}} + X^2 + 15Y^2 \quad \text{(A.5a)}
\]
\[
B_{20} = 35H_{\text{ext}}^4 + 27H_{\text{ext}}^2H_{\text{int}}^2 + 60H_{\text{ext}}H_{\text{int}}^3 + 4H_{\text{ext}}^3H_{\text{int}} + 7H_{\text{int}}^2X^2 + 5Y^2(3Y^2 + X^2)
\]
\[
+ Y^2(50H_{\text{int}}^2 + 15H_{\text{ext}}^2 + 40H_{\text{ext}}H_{\text{int}}) \quad \text{(A.5b)}
\]
\[
B_{30} = 7H_{\text{int}}^6 + 33H_{\text{ext}}^2H_{\text{int}}^4 + 8H_{\text{ext}}^3H_{\text{int}}^3 + 36H_{\text{ext}}H_{\text{int}}^5 + 9Y^4(H_{\text{ext}}^2 + H_{\text{int}}^2) + 3X^2Y^4 + Y^6
\]
\[
+ Y^2(15H_{\text{int}}^4 + 34H_{\text{ext}}^2H_{\text{int}}^2 + 48H_{\text{ext}}H_{\text{int}}^3 + 8H_{\text{ext}}^3H_{\text{int}}) + 10H_{\text{int}}^2X^2Y^2 + 7H_{\text{int}}^4X^2 + 12H_{\text{ext}}H_{\text{int}}Y^4 \quad \text{(A.5c)}
\]
\[
B_{40} = 39H_{\text{ext}}^2H_{\text{int}}^6 + 12H_{\text{ext}}H_{\text{int}}^7 + 12H_{\text{ext}}^3H_{\text{int}}^5 + 3H_{\text{ext}}Y^6 + X^2Y^6
\]
\[
+ Y^4(21H_{\text{ext}}^2H_{\text{int}}^2 + 12H_{\text{ext}}H_{\text{int}}^3 + 12H_{\text{ext}}H_{\text{int}}) + Y^2(57H_{\text{ext}}^2H_{\text{int}}^4 + 24H_{\text{ext}}H_{\text{int}}^5 + 24H_{\text{ext}}^3H_{\text{int}}^3) \quad \text{(A.5d)}
\]
\[ + 9H_{int}^2 X^2 Y^4 + 15H_{int}^4 X^2 Y^2 + 7H_{int}^6 X^2 \quad (A.5d) \]

\[ B_{21} = 10H_{int}^2 + 4H_{ext} H_{int} + 10Y^2 \quad (A.5e) \]

\[ B_{31} = 6H_{int}^4 + 8H_{ext} H_{int}^3 + 6Y^4 + Y^2 \left(12H_{int}^2 + 8H_{ext} H_{int}\right) \quad (A.5f) \]

\[ B_{41} = 2H_{int}^6 + 12H_{ext} H_{int}^5 + Y^4 \left(6H_{int}^2 + 12H_{ext} H_{int}\right) + Y^2 \left(6H_{int}^4 + 24H_{ext} H_{int}^3\right) + 2Y^6 \quad (A.5g) \]

\[ B_{22} = 20H_{int}^2 + 2H_{ext}^2 + 20H_{ext} H_{int} + 2X^2 + 20Y^2 \quad (A.5h) \]

\[ B_{32} = 6H_{int}^4 + 12H_{ext} H_{int}^2 + 24H_{ext} H_{int}^3 + 4Y^2 \left(3H_{int}^2 + H_{ext}^2 + 6H_{ext} H_{int}\right) + 4H_{int}^2 X^2 + \left(6Y^4 + 4X^2 Y^2\right) \quad (A.5i) \]

\[ B_{42} = 12H_{ext} H_{int}^5 + 30H_{ext}^2 H_{int}^4 + Y^4 \left(12H_{ext} H_{int} + 6H_{ext}^2\right) + 12Y^2 \left(2H_{ext}^2 H_{int}^3 + 3H_{ext}^2 H_{int}\right) + 12H_{int}^2 X^2 Y^2 + 6H_{int}^4 X^2 + 6X^2 Y^4 \quad (A.5j) \]

\[ C_{10} = 6H_{ext}^2 + 15H_{int}^2 + 15H_{ext} H_{int} + 3X^2 + 10Y^2 \quad (A.6a) \]

\[ C_{20} = 15H_{int}^4 + 45H_{ext} H_{int}^2 + 15H_{ext}^3 H_{int} + 50H_{ext} H_{int}^3 + H_{ext}^4 + 10Y^2 \left(2H_{int}^2 + 2H_{ext}^2 + 3H_{ext} H_{int}\right) \]

\[ + X^2 \left(15H_{int}^2 + 5H_{ext} H_{int} + H_{ext}^2\right) + 5Y^4 + 10X^2 Y^2 \quad (A.6b) \]
\[ C_{30} = 26H_{\text{ext}}^3H_{\text{int}}^3 + 36H_{\text{ext}}^2H_{\text{int}}^4 + 15H_{\text{ext}}H_{\text{int}}^5 + 6H_{\text{ext}}^4H_{\text{int}}^2 + Y^4(6H_{\text{ext}}^2 + 3H_{\text{ext}}H_{\text{int}} + 2H_{\text{int}}^2) + 2Y^2(15H_{\text{ext}}^2H_{\text{int}}^2 + 9H_{\text{ext}}H_{\text{int}}^3 + 2H_{\text{int}}^4 + 9H_{\text{ext}}^3H_{\text{int}} + H_{\text{ext}}^4) + 2X^2Y^2(6H_{\text{int}}^2 + 3H_{\text{ext}}H_{\text{int}} + H_{\text{ext}}^2) + X^2(9H_{\text{int}}^4 + 10H_{\text{ext}}H_{\text{int}}^3 + 2H_{\text{ext}}H_{\text{int}}^2) + 3X^2Y^4 \quad (A.6c) \]

\[ C_{40} = 33H_{\text{ext}}^3H_{\text{int}}^5 + 15H_{\text{ext}}^2H_{\text{int}}^6 + 15H_{\text{ext}}^4H_{\text{int}}^4 + 3Y^4(3H_{\text{ext}}^3H_{\text{int}} + H_{\text{ext}}^2H_{\text{int}}^2 + H_{\text{ext}}^4) + Y^2(34H_{\text{ext}}^3H_{\text{int}}^3 + 18H_{\text{ext}}^2H_{\text{int}}^4 + 18H_{\text{ext}}^4H_{\text{int}}^2) + 3X^2Y^4(H_{\text{int}}^2 + H_{\text{ext}}^2 + H_{\text{ext}}H_{\text{int}}) + 6X^2Y^2(H_{\text{ext}}H_{\text{int}}^2 + 3H_{\text{ext}}H_{\text{int}}^3 + H_{\text{ext}}^4) + 3X^2(H_{\text{int}}^6 + 5H_{\text{ext}}H_{\text{int}}^5 + H_{\text{ext}}^2H_{\text{int}}^4) \quad (A.6d) \]

\[ C_{21} = H_{\text{ext}}^2 + 10H_{\text{int}}^2 + 10H_{\text{ext}}H_{\text{int}} + X^2 + 10Y^2 \quad (A.6e) \]

\[ C_{31} = 3H_{\text{int}}^3 + 6H_{\text{ext}}^2H_{\text{int}}^2 + 12H_{\text{ext}}H_{\text{int}}^3 + 2Y^2(3H_{\text{int}}^2 + H_{\text{ext}}^2 + 6H_{\text{ext}}H_{\text{int}}) + 2X^2H_{\text{int}}^2 + (3Y^4 + 2X^2Y^2) \quad (A.6f) \]

\[ C_{41} = 6H_{\text{ext}}H_{\text{int}}^5 + 15H_{\text{ext}}^2H_{\text{int}}^4 + 3Y^4(2H_{\text{ext}}H_{\text{int}} + H_{\text{ext}}^2) + 6Y^2(2H_{\text{ext}}H_{\text{int}}^3 + 3H_{\text{ext}}^2H_{\text{int}}) + 6H_{\text{int}}^2X^2Y^2 + 3H_{\text{int}}^4X^2 + 3X^2Y^4 \quad (A.6g) \]

\[ C_{22} = 5H_{\text{ext}}^2 + 10H_{\text{int}}^2 + 20H_{\text{ext}}H_{\text{int}} + 5X^2 + 10Y^2 \quad (A.6h) \]

\[ C_{32} = 2H_{\text{int}}^4 + 18H_{\text{ext}}^2H_{\text{int}}^2 + 4H_{\text{ext}}^3H_{\text{int}} + 12H_{\text{ext}}H_{\text{int}}^3 + 2Y^2(2H_{\text{int}}^2 + 3H_{\text{ext}}^2 + 6H_{\text{ext}}H_{\text{int}}) + 2X^2(3H_{\text{int}}^2 + 2H_{\text{ext}}H_{\text{int}}) + 2Y^4 + 6X^2Y^2 \quad (A.6i) \]
\[ C_{42} = 15H_{ext}^2H_{int}^4 + 20H_{ext}^3H_{int}^3 + 3H_{ext}^2Y^4 \]
\[ + \ 6Y^2 \left( 3H_{ext}^2H_{int}^2 + 2H_{ext}^3H_{int} \right) \]
\[ + \ 3X^2 \left( H_{int}^4 + 4H_{ext}H_{int}^3 \right) \]
\[ + \ 6H_{int}^2X^2Y^2 \]
\[ + \ 12H_{ext}H_{int}X^2Y^2 + 3X^2Y^4 \quad (A.6j) \]

**Appendix B**

For \( \kappa_2 = 1 \), in the case of vanishing \( \kappa_3 \) and \( \kappa_4 \), the evolution of the attracting point \( D_2 \) with respect to the variation of the cosmological constant \( \kappa_0 \), is given by

\[
\left( \frac{dH_{ext}}{d\kappa_0} \right) = \left( \frac{QQ_1}{PP} \right) 
\]
\[ (B.1a) \]

\[
\left( \frac{dH_{int}}{d\kappa_0} \right) = \left( \frac{QQ_2}{PP} \right) 
\]
\[ (B.1b) \]

The functions \( PP, QQ_1, \) and \( QQ_2 \) are given by the expressions

\[ PP = F_{11}F_{22} - F_{12}F_{21} \]
\[ (B.2a) \]

\[ QQ_1 = F_1F_{22} - F_2F_{12} \]
\[ (B.2b) \]

\[ QQ_2 = F_{11}F_2 - F_1F_{21} \]
\[ (B.2c) \]

where we have set

\[ F_1 = -\left[ 2 + 24(10H_{int}^2 - 3H_{ext}^2) \right] \]
\[ (B.3a) \]

\[ F_2 = -\left[ 2 + 24(H_{ext}^2 + 6H_{ext}H_{int}) \right] \]
\[ (B.3b) \]

\[ F_{11} = G_{121}G_{20} + G_{12}G_{201} - G_{221}G_{10} - G_{22}G_{101} \]
\[ (B.3c) \]

\[ F_{12} = G_{122}G_{20} + G_{12}G_{202} - G_{222}G_{10} - G_{22}G_{102} \]
\[ (B.3d) \]

\[ F_{21} = G_{211}G_{10} + G_{21}G_{101} - G_{111}G_{20} - G_{11}G_{201} \]
\[ (B.3e) \]

\[ F_{22} = G_{212}G_{10} + G_{21}G_{102} - G_{112}G_{20} - G_{11}G_{202} \]
\[ (B.3f) \]

Now, the quantities \( G_{ij} \) are given by

\[ G_{11} = 4 + 24(B_{21})_{X=Y=0} \]
\[ (B.4a) \]

\[ G_{12} = 12 + 24(B_{22})_{X=Y=0} \]
\[ (B.4b) \]

\[ G_{10} = \kappa_0 + 2(B_{10})_{X=Y=0} + 24(B_{20})_{X=Y=0} \]
\[ (B.4c) \]

\[ G_{21} = 6 + 24(C_{21})_{X=Y=0} \]
\[ (B.4d) \]

\[ G_{22} = 10 + 24(C_{22})_{X=Y=0} \]
\[ (B.4e) \]
\[ G_{20} = \kappa_0 + 2(C_{10})_{X=Y=0} + 24(C_{20})_{X=Y=0} \]  

(B.4f)

where \((B_{ij})_{X=Y=0}\) and \((C_{ij})_{X=Y=0}\) denote the form of the corresponding quantities for \(X = 0 = Y\) and the symbols \(G_{ijk}\) stand for

\[ G_{ijk} = \left( \frac{\partial G_{ij}}{\partial H_k} \right)_{X=Y=0} \]  

(B.5)

in which, \(k = 1\) for \(H_{ext}\) and \(k = 2\) for \(H_{int}\).

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Figure Captions

**Fig. 1a:** The evolution of the Hubble parameters in the GB theory ($\kappa_2 = 1$), for three different sets of initial conditions ($H_{0}^{\text{ext}}, H_{0}^{\text{int}}$): (0.75, −0.25) [solid line] (0.85, −0.15) [dashed line] and (0.95, −0.05) [squares]. The time coordinate is measured in units of $10^4 t_{Pl}$.

**Fig. 1b:** The orbits (in the $H_{\text{ext}} - H_{\text{int}}$ plane) of the dynamical system determined by the cosmological field equations for a model with flat subspaces, for three different sets of initial conditions. All orbits end at the attracting point $D_2$ (0.8866, −0.1375).

**Fig. 2a:** The change in the location of the attractor $D_2$ when a non-zero cosmological constant is included, for $\kappa_0 = 0.5$.

**Fig. 2b:** The displacement of the attractor on the $H_{\text{ext}} - H_{\text{int}}$ plane for a wide range of values of the cosmological constant (squares). Notice the very good agreement with the least square fitting result $H_{\text{int}} = -0.075 H_{\text{ext}} - 0.071$.

**Fig. 3a:** The evolution of the Hubble parameters in the $L(3)$-theory, for $\kappa_3 = 0.15$ and for three different sets of initial conditions ($H_{0}^{\text{ext}}, H_{0}^{\text{int}}$): (0.75, −0.25) [solid line] (0.85, −0.15) [dashed line] and (0.95, −0.05) [squares]. The time coordinate is measured in units of $10^4 t_{Pl}$.

**Fig. 3b:** The displacement of the attractor on the $H_{\text{ext}} - H_{\text{int}}$ plane for a wide range of values of the coupling constant $\kappa_3$ (squares). There is a very good agreement with the plot of a sixth-order polynomial of the form $H_{\text{ext}} = p_6(H_{\text{int}})$.

**Fig. 4a:** The evolution of the external Hubble parameter for $H_{0}^{\text{ext}} = 0.85$, when several combinations of the non-linear terms are gradually included in the field equations. The time coordinate is measured in units of $10^4 t_{Pl}$.

**Fig. 4b:** The displacement of the attractor on the $H_{\text{ext}} - H_{\text{int}}$ plane for a wide range of values of the coupling constant $\kappa_4$ (squares). Notice the very good agreement with the plot of a third-order polynomial (solid line) of the form $H_{\text{int}} = p_3(H_{\text{ext}})$.

**Fig. 5a:** The time-evolution of the positively curved external space, for several values of the normalized coupling constant $\kappa_2$.

**Fig. 5b:** The corresponding evolution of the positively curved internal space. In both figures the time coordinate is measured in units of $10^4 t_{Pl}$. Notice the change in the cosmological behaviour of both subspaces when $\kappa_2 < 0.65$ and when $\kappa_2 > 0.65$.

**Fig. 6a:** The orbits (in the $H_{\text{ext}} - H_{\text{int}}$ plane) of the dynamical system determined by the cosmological field equations for a model with positively curved subspaces, for three different sets of initial conditions when $\kappa_2 > 0.65$. All orbits end at the attracting point $P$.

**Fig. 6b:** The time-evolution of the Kasner solution $R(t) \sim t^{0.55}$ for the external space, for several values of the normalized coupling constant $\kappa_3$. Again, the time coordinate is measured in units of $10^4 t_{Pl}$. Notice the slight modulation in the time-evolution when $0 \leq \kappa_3 \leq 0.75$.  

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