Effective Wadge Hierarchy in Computable Quasi-Polish Spaces

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Abstract

We define and study an effective version of the Wadge hierarchy in computable quasi-Polish spaces which include most spaces of interest for computable analysis. Along with hierarchies of sets we study hierarchies of \( k \)-partitions which are interesting on their own. We show that levels of such hierarchies are preserved by the computable effectively open surjections, that if the effective Hausdorff-Kuratowski theorem holds in the Baire space then it holds in every computable quasi-Polish space, and we extend the effective Hausdorff theorem to \( k \)-partitions.

Key words: Computable quasi-Polish space, effective Wadge hierarchy, fine hierarchy, \( k \)-partition, preservation property, effective Hausdorff theorem.

1 Introduction

Classical descriptive set theory (DST) \[3\] deals with hierarchies of sets, functions and equivalence relations in Polish spaces. Recently, classical DST was extended to quasi-Polish spaces which contain many important non-Hausdorff spaces \[2\]. Theoretical Computer Science and Computable Analysis especially need an effective DST for some effective versions of the mentioned spaces. A lot of useful work in this direction was done in Computability Theory but mostly for the discrete space \( \mathbb{N} \), the Baire space \( \mathcal{N} \), and some of their relatives \[9, 8\]. Effective versions of classical Borel, Hausdorff and Luzin hierarchies are naturally defined for every effective space (see e.g.\[11\]) but, as also in the classical case, they behave well only for spaces of special kinds. Recently, a convincing version of a computable quasi-Polish space (CQP-space for short) was suggested in \[3, 4\].

In this paper we continue to develop effective DST in CQP-spaces where effective analogues of some important properties of the classical hierarchies hold. Namely, we develop an effective Wadge hierarchy (including the hierarchy of \( k \)-partitions) in such spaces which subsumes the effective Borel and Hausdorff hierarchies (as well as many others) and is in a sense the finest possible hierarchy of effective Borel sets. In particular, we show that levels of such hierarchies are preserved by the computable effectively open surjections, that if the effective Hausdorff-Kuratowski theorem holds in the Baire space then it holds in every CQP-space, and we extend the effective Hausdorff theorem for CQP-spaces \[15\] to \( k \)-partitions. We hope that these results (together with those already known) show that effective DST reached the state of maturity.

We start in the next section with recalling definitions of relevant notions of the effective
DST. In Section 3 we carefully define rather technical notions related to the effective Wadge hierarchy and establish some of its properties in effective spaces. In Section 4 we establish the preservation property. In Section 5 we discuss transfinite extensions of the effective hierarchies which are used in Section 6 to prove the results about the effective Hausdorff-Kuratowski theorem for \( k \)-partitions.

2 Preliminaries

In this section we briefly recall some notation, notions and facts used throughout the paper. Some more special information is recalled in the corresponding sections below.

2.1 Spaces and trees

We use standard set-theoretical notation. In particular, \( Y^X \) is the set of functions from \( X \) to \( Y \), \( P(X) \) is the class of subsets of a set \( X \), \( \mathcal{C} \) is the class of complements \( X \setminus C \) of sets \( C \) in \( \mathcal{C} \subseteq P(X) \), and \( BC(C) \) is the Boolean closure of \( \mathcal{C} \).

All considered spaces are assumed to be countably based \( T_0 \) (sometimes we call such spaces cb\(_0\)-spaces). By effectivization of a cb\(_0\)-space \( X \) we mean a numbering \( \beta : \omega \to P(X) \) of a base in \( X \) such that there is a uniform sequence \( \{ A_{ij} \} \) of c.e. sets with \( \beta_i \cap \beta_j = \bigcup \beta(A_{ij}) \), where \( \beta(A) \) is the image of \( A \) under \( \beta \). The numbering \( \beta \) is called an effective base of \( X \) while the pair \( (X, \beta) \) is called an effective space. We simplify \( (X, \beta) \) to \( X \) if \( \beta \) is clear from the context. The effectively open sets in \( X \) are the sets \( \bigcup_{i \in W} \beta(i) \), for some c.e. set \( W \subseteq \omega \). The standard numbering \( \{ W_n \} \) of c.e. sets \( [9] \) induces a numbering of the effectively open sets. The notion of effective space allows to define e.g. computable and effectively open functions between such spaces \( [18] \).

Many popular spaces (e.g., the discrete space \( \mathbb{N} \) of natural numbers, the space \( \mathbb{R} \) of reals, the Scott domain \( P\omega \), the Baire space \( \mathcal{N} = \mathbb{N}^\mathbb{N} \), the Cantor space \( A^\omega \) of infinite words in a finite alphabet \( A \)) are effective spaces (with natural numberings of bases). The effective space \( \mathbb{N} \) is trivial topologically but very interesting for Computability Theory.

We use standard notation related to the Baire space. In particular, \( \omega^* \) is the set of finite strings of natural numbers including the empty string \( \varepsilon \), \( |\sigma| \) is the length of a string \( \sigma \), \( [\sigma] \) is the basic open set induced by \( \sigma \in \omega^* \) consisting of all \( p \in \mathcal{N} \) having \( \sigma \) as a prefix. By a tree we mean a nonempty initial segment of \( (\omega^*; \sqsubseteq) \) where \( \sqsubseteq \) is the prefix relation. An infinite path through a tree \( T \) is an element \( p \in \mathcal{N} \) such that \( p[n] \in T \) for each \( n \) where \( p[n] \) is the prefix of \( p \) of length \( n \). A tree \( T \) is well founded if there is no infinite path through \( T \).

2.2 Effective versions of classical hierarchies

Let \( \{ \Sigma^0_n(X) \}_{n<\omega} \) be the effective Borel hierarchy and \( \{ D_n(\Sigma^0_n(X)) \}_{n} \) be the effective Hausdorff difference hierarchy over \( \Sigma^0_n(X) \) in arbitrary effective space \( X \). Another popular notation for levels of the difference hierarchy is \( \Sigma^{-1,m} = D_n(\Sigma^0_m(X)) \), with \( \Sigma^{-1,1} \) usually simplified to \( \Sigma^{-1} \). Let also \( \{ \Sigma^1_{i+n}(X) \} \) be the effective Luzin hierarchy. We do not repeat
the standard definitions (which may be found e.g. in [11, 15]) but mention that the
definitions yield also standard numberings of all levels of the hierarchies, so we can speak
e.g. about uniform sequences of sets in a given level. E.g., $\Sigma^0_1(X)$ is the class of effectively
open sets in $X$, $\Sigma^{-1}_2(X)$ is the class of differences of $\Sigma^0_1(X)$-sets, and $\Sigma^0_2(X)$ is the class
of effective countable unions of $\Sigma^{-1}_2(X)$-sets.

Levels of the effective hierarchies are denoted in the same manner as levels of the correspon-
ding classical hierarchies, using the lightface letters $\Sigma, \Pi, \Delta$ instead of the boldface
$\Sigma, \Pi, \Delta$ used for the classical hierarchies. The boldface classes may be considered as
“limits” of the corresponding lightface levels (where the limit is obtained by taking
the union of the corresponding relativised lightface levels, for all oracles). Thus the effective
hierarchies not only refine but also generalise the classical ones.

2.3 Effective versions of the Wadge hierarchy

By effective Wadge hierarchy in a given effective space we mean a special case of the so
called fine hierarchy (FH) introduced and studied in a series of my publications (see e.g. [12][a]
for a survey). We briefly recall some relevant notions. By a base in a set $X$ we mean
a sequence $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ of subsets of $P(X)$ such that any $\mathcal{L}_n$ is closed under union and
intersection, contains $\emptyset, X$ and satisfies $\mathcal{L}_n \cup \mathcal{L}_n \subseteq \mathcal{L}_{n+1}$. For this paper, the effective
Borel bases $\mathcal{L}(X) = \{\Sigma^0_1(X)\}$ in effective spaces $X$ are especially relevant.

The FH over the base $\mathcal{L}$ is a sequence $\{S_\alpha\}_{\alpha<\varepsilon_0}$, $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$, of subsets of
$P(X)$ constructed from the sets in levels of the base by induction on $\alpha$ using suitable
set-theoretical operations. We recall the precise definition in Section 3 below. The con-
struction is designed in such a way that $S_\alpha \cup \tilde{S}_\alpha \subseteq S_\beta$ for all $\alpha < \beta < \varepsilon_0$, and the FH
subsumes many refinements of the base $\mathcal{L}$. In particular $\mathcal{L}_0 = S_0$, $\mathcal{L}_1 = S_\omega$, $\mathcal{L}_2 = S_{\omega^\omega}$,
$\mathcal{L}_3 = S_{\omega^\omega^\omega}$, $\ldots$, $\{S_n\}_{n<\omega}$ is the difference hierarchy over $\mathcal{L}_0$, $\{S_{\omega^{n+1}}\}_{n<\omega}$ is the difference
hierarchy over $\mathcal{L}_1$, $\{S_{\omega^{n+1}}\}_{n<\omega}$ is the difference hierarchy over $\mathcal{L}_2$, and so on.

The FH over the effective Borel base will be denoted by $\{\Sigma_\alpha(X)\}_{\alpha<\varepsilon_0}$ and called the
effective Wadge hierarchy in $X$. The corresponding boldface sequence $\{\Sigma_\alpha(\mathcal{N})\}_{\alpha<\varepsilon_0}$ forms
a small but important fragment of the classical Wadge hierarchy in the Baire space studied in [10]. Some fragments of the classical Wadge hierarchy in quasi-Polish spaces were
first defined and studied in [16] and recently extended by me to all levels providing the
boldface versions of results of this paper for all levels of the Wadge hierarchy in quasi-
Polish spaces.

Some special cases of the effective Wadge hierarchy were considered before (see [12][b] and
references therein). E.g., for the case $X = \mathbb{N}$ the hierarchy $\{\Sigma_\alpha(\mathbb{N})\}_{\alpha<\varepsilon_0}$ was introduced by me in 1983 and used to classify many natural index sets. The FH over the base
$(\mathcal{R} \cap \Sigma^0_1(A^\omega), \mathcal{R} \cap \Sigma^0_2(A^\omega), BC(\mathcal{R} \cap \Sigma^0_2(A^\omega)), BC(\mathcal{R} \cap \Sigma^0_3(A^\omega)), \ldots)$, where $\mathcal{R}$ is the class
of regular omega-languages, coincides with the famous Wagner hierarchy.

2.4 Computable quasi-Polish spaces

Though the effective hierarchies are naturally defined in arbitrary effective space, some
important properties only hold for special classes of spaces, identification of which was
itself a non-trivial task. Recall a similar situation in classical DST where the spaces with “good” DST (namely, the quasi-Polish spaces) were identified relatively recently [2]. Quasi-Polish spaces [2] have several characterisations. Effectivising one of them we obtain the following notion identified implicitly in [15] and explicitly in [3, 4].

**Definition 1.** By a computable quasi-Polish space we mean an effective space \((X, \beta)\) such that there exists a computable effective open surjection \(\xi: \mathcal{N} \to X\) from the Baire space onto \((X, \beta)\).

As shown in [15] [3] [4], CQP-spaces do satisfy effective versions of several important properties of quasi-Polish spaces. E.g. they subsume computable Polish spaces and computable domains and satisfy the effective Hausdorff and Suslin theorems. The class of CQP-spaces includes most of \(cb\)-spaces considered in the literature. In particular, all spaces mentioned in Section [2,1] are CQP-spaces.

In this paper we establish some good properties of the effective Wadge hierarchy in CQP-spaces. For this we use, in particular, the following corollary of Theorem 3.1 [1] (extending Lemma 3.1 in [4]) which effectivises the corresponding classical fact [17] [2] and shows that Baire category technique is consistent with effectivity. For any continuous function \(f: X \to Y\) and \(S \subseteq X\), let \(f[S]\) consist of all \(y \in Y\) such that \(S \cap f^{-1}(y)\) is not meager in \(f^{-1}(y)\). Please be careful in distinguishing \(f[S]\) and the image \(f(A)\).

**Proposition 1.** [7] Let \(f: X \to Y\) be a computable effectively open surjection between effective spaces and let \(A \in \Sigma^0_n(Y)\). Then \(f^{-1}(A) \in \Sigma^0_n(X)\) iff \(A \in \Sigma^0_n(Y)\), and \(f[S] \in \Sigma^0_n(Y)\) for every \(S \in \Sigma^0_n(X)\).

3 Effective Wadge hierarchy of \(k\)-partitions

Here we discuss FHs not only of subsets of \(X\) but also of \(k\)-partitions \(A: X \to \{0, \ldots, k - 1\} = \mathbb{k}\) which may also be written as \(k\)-tuples \((A_0, \ldots, A_{k-1})\) where \(A_i = A^{-1}(i), i < k\). Note that 2-partitions of \(X\) are essentially subsets of \(X\). Surprisingly, the extension from sets to \(k\)-partitions simplifies many proofs related to the FH. The reason is that defining FHs of sets by induction on ordinals leads to tedious inductive proofs while the FHs of \(k\)-partitions may be defined using suitable labeled trees which are technically easier (see Section 8 of [13] for additional details).

Let \((Q; \leq)\) be a preorder. A \(Q\)-tree is a pair \((T, t)\) consisting of a finite tree \(T \subseteq \omega^*\) and a labeling \(t: T \to Q\). Let \(\mathcal{T}_Q\) denote the set of all finite \(Q\)-trees. The \(h\)-preorder \(\leq_h\) on \(\mathcal{T}_Q\) is defined as follows: \((T, t) \leq_h (S, s)\) if there is a monotone function \(f: (T; \sqsubseteq) \to (S; \sqsubseteq)\) satisfying \(\forall x \in T(t(x)) \leq s(f(x))\).

The preorder \(Q\) is called WQO if it has neither infinite descending chains nor infinite antichains. An example of WQO is the antichain \(\mathbb{k}\) with \(k\) elements. A famous Kruskall’s theorem implies that if \(Q\) is WQO then \((\mathcal{T}_Q; \leq_h)\) is WQO. Define the sequence \(\{\mathcal{T}_k(n)\}_{n < \omega}\) of preorders by induction on \(n\) as follows: \(\mathcal{T}_k(0) = \mathbb{k}\) and \(\mathcal{T}_k(n + 1) = \mathcal{T}_{\mathcal{T}_k(n)}\). The sets \(\mathcal{T}_k(n), n < \omega\), are pairwise disjoint but, identifying the elements \(i\) of \(\mathbb{k}\) with the corresponding singleton trees \(s(i)\) labeled by \(i\) (which are precisely the minimal elements of \(\mathcal{T}_k(1)\)), we may think that \(\mathcal{T}_k(0) \sqsubseteq \mathcal{T}_k(1)\), i.e. the quotient-poset of the first preorder.
is an initial segment of the quotient-poset of the other. This also induces an embedding of \( T_k(n) \) into \( T_k(n + 1) \) as an initial segment, so (abusing notation) we may think that \( T_k(0) \subseteq T_k(1) \subseteq \cdots \), hence \( T_k(\omega) = \bigcup_{n<\omega} T_k(n) \) is WQO w.r.t. the induced preorder which we also denote \( \leq_h \). The embedding \( s \) is extended to \( T_k(\omega) \) by defining \( s(T) \) as the singleton tree labeled by \( T \).

With any base \( \mathcal{L} = \{ \mathcal{L}_n \}_{n<\omega} \) in \( X \) we then associate the fine hierarchy of \( k \)-partitions over \( \mathcal{L} \) which is a family \( \{ \mathcal{L}(X, T) \}_{T \in T_k(\omega)} \) of subsets of \( k^X \) defined below. As shown in \cite{13}, \( T \leq_h S \) implies \( \mathcal{L}(X, T) \subseteq \mathcal{L}(X, S) \), hence \( \{ \mathcal{L}(X, T) \mid T \in T_k(\omega) \} \) is WQO.

The FH of sets from Subsection 2.3 is obtained from this construction for \( k = 2 \) since the quotient-poset of \( (T_2(\omega); \leq_h) \) has order type \( 2 \cdot \varepsilon_0 \), i.e. for some \( T_\alpha, T'_\alpha \in T_2(\omega) \) we have: \( T_\alpha, T'_\alpha <_h T_\beta, T'_\beta \) for \( \alpha < \beta < \varepsilon_0 \), and any element of \( T_2(\omega) \) is \( h \)-equivalent to precisely one of \( T_\alpha, T'_\alpha \). We then set \( \mathcal{S}_\alpha = \mathcal{L}(X, T_\alpha) \) for all \( \alpha < \varepsilon_0 \). For details see Definition 8.27 and Proposition 8.28 in \cite{13}.

With any base \( \mathcal{L}(X) \) we associate some other bases as follows. For any \( m < \omega \), let \( \mathcal{L}^m(X) = \{ \mathcal{L}_{m+n}(X) \}_{n} \), we call this base the \( m \)-shift of \( \mathcal{L}(X) \). For any \( U \in \mathcal{L}_n \), let \( \mathcal{L}(U) = \{ \mathcal{L}_n(U) \}_{n<\omega} \). Then set \( \mathcal{S}_\alpha = \mathcal{L}(X, U) \) for all \( \alpha < \varepsilon_0 \). For details see Definition 8.27 and Proposition 8.28 in \cite{13}.

For any finite tree \( T \subseteq \omega^* \) and a \( T \)-family \( \{ U_\tau \} \) of \( \omega^* \)-sets, we define the \( T \)-family \( \{ U_\tau' \} \) of \( \omega^* \)-sets by \( U_\tau' = U_\tau \cup \{ U_{\tau'} \mid \tau \subseteq \tau' \in T \} \). The \( T \)-family \( \{ U_\tau \} \) is monotone if \( U_\tau \supseteq U_{\tau'} \) for all \( \tau \subseteq \tau' \in T \). We associate with any \( T \)-family \( \{ U_\tau \} \) the monotone \( T \)-family \( \{ U_\tau' \} \) by \( U_\tau' = \bigcup_{\tau \subseteq \tau' \in T} U_{\tau'} \). A \( T \)-family \( \{ U_\tau \} \) is reduced if it is monotone and satisfies \( V_{\tau_i} \cap V_{\tau_j} = \emptyset \) for all \( \tau_i, \tau_j \in T \). Obviously, for any reduced \( T \)-family \( \{ U_\tau \} \) the components \( V_\tau \) are pairwise disjoint.

Recall that a class of sets \( \mathcal{C} \subseteq P(X) \) has the reduction property (resp. \( \sigma \)-reduction property, effective \( \sigma \)-reduction property) if for any \( C_0, C_1 \in \mathcal{C} \) there are disjoint \( R_0, R_1 \in \mathcal{C} \) such that \( R_0 \subseteq C_0, R_1 \subseteq C_1 \) and \( R_0 \cup R_1 = C_0 \cup C_1 \). The \( \sigma \)-reduction property is defined similarly but for infinite sequences \( C_0, C_1, \ldots \). The effective \( \sigma \)-reduction property is the effectivisation of the latter property w.r.t. a given numbering of \( \mathcal{C} \).

Item (1) of the next lemma is straightforward while item (2) is checked by a top-down (assuming that trees grow downwards) application of the reduction property.

**Lemma 1.** (1) Let \( \{ U_\tau \} \) be a \( T \)-family of \( \mathcal{L}_n \)-sets. Then \( \bigcup_\tau U_\tau = U_\tau \cup \bigcup_\tau \bigcup_\tau' U_{\tau'} = \bigcup_\tau \bigcup_\tau' U_{\tau'} \in \mathcal{L}_{m+1} \cap \mathcal{L}_{n+1} \), and \( U_\tau \cap \bigcup_\tau U_\tau' = \emptyset \) for \( \tau \subseteq \tau' \in T \).

(2) Let \( \{ U_\tau \} \) be a monotone \( T \)-family of \( \mathcal{L}_n \)-sets, and \( \mathcal{L}_n \) has the reduction property. Then there is a reduced \( T \)-family \( \{ V_\tau \} \) of \( \mathcal{L}_n \)-sets such that \( V_\tau \subseteq U_\tau \) and \( \bigcup_\tau \{ V_{\tau_i} \mid \tau_i \in T \} = \bigcup_\tau \{ U_{\tau_i} \mid \tau_i \in T \} \), and \( V_\tau \subseteq U_\tau \) for each \( \tau \in T \).

We need two more technical notions. The first one is the notion “\( F \) is a \( T \)-family in \( \mathcal{L}(X) \)” which is defined by induction as follows.

**Definition 2.** (1) If \( T \in T_k(0) \) then \( F = \{ X \} \).

(2) If \( (T, t) \in T_k(n + 1) \) then \( F = \{ (U_\tau), (F_\tau) \} \) where \( \{ U_\tau \} \) is a monotone \( T \)-family of \( \mathcal{L}_0 \)-sets with \( T_\tau = X \) and, for each \( \tau \in T \), \( F_\tau \) is a \( t(\tau) \)-family in \( \mathcal{L}_1(U_\tau) \).
The notion of a reduced family is obtained from this definition by requiring \( \{U_\tau\} \) and \( F_\tau \) in item (2) to be reduced.

The second one is the notion “a \( T \)-family \( F \) in \( \mathcal{L}(X) \) determines a partition \( A : X \to \bar{k} \)” which is defined by induction as follows. (In general, not every family determines a \( k \)-partition but every reduced family does.)

**Definition 3.**

1. If \( T \in \mathcal{T}_k(0) \), \( T = i < k \) (so \( F = \{X\} \)), then \( A = \lambda x. i \).
2. If \( (T, t) \in \mathcal{T}_k(n+1) \) (so \( F \) is of the form \( \{\{U_\tau\}, \{F_\tau\}\} \)) then, for each \( \tau \in T \), \( A|_{\bar{U}_\tau} = B_\tau \) where \( B_\tau : \bar{U}_\tau \to \bar{k} \) is determined by \( F_\tau \).

Since inductive proofs according the given definitions sometimes hide the ideas, let us give examples of explicit descriptions of the introduced notions. For \( T = i \in \mathcal{T}_k(0) \), there is only one \( T \)-family \( \{X\} \) in \( \mathcal{L}(X) \) which determines the constant partition \( \lambda x. i \).

For \( T \in \mathcal{T}_k(1) \), a \( T \)-family \( F \) in \( \mathcal{L}(X) \) is a monotone family \( \{U_\tau\} \) of \( \mathcal{L}_0(X) \)-sets whose components \( U_\tau \) cover \( X \). Such a family determines \( A \) if \( A(x) = t(\tau) \), for any \( \tau \in T \) with \( x \in U_\tau \). Note that \( t : T \to \bar{k} \) and \( x \) may belong to different components \( \bar{U}_\tau, \bar{U}_\sigma \) with incomparable \( \tau, \sigma \).

For \( T \in \mathcal{T}_k(2) \), a \( T \)-family \( F \) in \( \mathcal{L}(X) \) consists of a family \( \{U_\tau\} \) as above, and, for each \( \tau_0 \in T \), a family \( \{U_{\tau_0 \tau_1}\}_{\tau_1 \in t_0(\tau_0)} \) of \( \mathcal{L}_1(X) \)-sets whose components (which we call second-level components) \( \bar{U}_{\tau_0 \tau_1} \) cover \( \bar{U}_{\tau_0} \) (called first-level components). Such an \( F \) determines \( A \) if \( A(x) = t_1(\tau_1) \), for all \( \tau_0 \in T, \tau_1 \in t_0(\tau_0) \) with \( x \in \bar{U}_{\tau_0 \tau_1} \). Note that \( t_0 : T \to \mathcal{T}_k(1), t_1 : t_0(\tau_0) \to \bar{k} \).

For \( T \in \mathcal{T}_k(3) \), a \( T \)-family \( F \) in \( \mathcal{L}(X) \) consists of families \( \{U_\tau\}, \{\bar{U}_{\tau_0 \tau_1}\} \) as above and, for all \( \tau_0 \in T, \tau_1 \in t_0(\tau_0), \tau_2 \in t_1(\tau_1) \) of \( \mathcal{L}_2(X) \)-sets whose components \( \bar{U}_{\tau_0 \tau_1 \tau_2} \) of the third level cover \( \bar{U}_{\tau_0 \tau_1} \). Such \( F \) determines \( A \) if \( A(x) = t_2(\tau_2) \), for all \( \tau_0 \in T, \tau_1 \in t_0(\tau_0), \tau_2 \in t_1(\tau_1) \) with \( x \in \bar{U}_{\tau_0 \tau_1 \tau_2} \). Note that \( t_0 : T \to \mathcal{T}_k(2), t_1 : t_0(\tau_0) \to \mathcal{T}_k(1), t_2 : t_1(\tau_1) \to \bar{k} \).

Intuitively, the \( T \)-family \( F \) (say, in an effective Borel base) that determines \( A \) provides a mind-change algorithm for computing \( A(x) \) for a given \( x \in X \) as follows. First, we search for a component \( \bar{U}_{\tau_0} \) containing \( x \); this is the usual mind-change procedure working with differences of \( \Sigma^0_1 \)-sets. While \( x \) sits in \( \bar{U}_{\tau_0} \), we search for a component \( \bar{U}_{\tau_0 \tau_1} \) containing \( x \); this is a harder mind-change procedure working with differences of \( \Sigma^0_2 \)-sets, and so on. Note that if \( F \) is reduced then the computation is “linear” since the components of each level are pairwise disjoint, otherwise the algorithm is “parallel” since already at the first level \( x \) may belong to several components \( \bar{U}_{\tau_0} \). With this interpretation in mind, we consider the (effective) Wadge hierarchy as an “iterated difference hierarchy”, one should only make precise how to “iterate” them.

The next lemma is immediate by induction (or by looking at the examples above).

**Lemma 2.** *Every \( T \)-family in \( \mathcal{L}(X) \) determines at most one \( k \)-partition of \( X \). Every reduced \( T \)-family in \( \mathcal{L}(X) \) determines precisely one \( k \)-partition of \( X \).*

Finally, we define the level \( \mathcal{L}(X, T) \) of the FH of \( k \)-partitions over \( \mathcal{L}(X) \) as the set of
\( A : X \rightarrow \bar{k} \) determined by some \( T \)-family in \( \mathcal{L}(X) \). The FH over the effective Borel bases \( \mathcal{L}(X) = \{ \Sigma^0_{1+n}(X) \} \) is called the \textit{effective Wadge hierarchy in X} and denoted \( \{ \Sigma(X, T) \}_{T \in \mathcal{T}_k(\omega)} \).

From now on we consider only the effective Borel bases. They have some specific features, e.g. any level \( \Sigma^0_{1+n}(X) \) is closed under effective countable unions. We formulate some more properties. A base \( \mathcal{L}(X) \) is reducible (resp. \( \sigma \)-reducible, effectively \( \sigma \)-reducible) is every its level has the corresponding property. By a \textit{morphism} \( g : \mathcal{L}(X) \rightarrow \mathcal{L}(Y) \) between effective Borel bases we mean a function \( g : P(X) \rightarrow P(Y) \) such that any restriction \( g|_{\Sigma^0_{1+n}(X)} \) is a computable function from \( \Sigma^0_{1+n}(X) \) to \( \Sigma^0_{1+n}(Y) \) which preserves effective countable unions and satisfies \( g(\emptyset) = \emptyset, g(X) = Y \). Obviously, the identity function on \( P(X) \) is a morphism of \( \mathcal{L}(X) \) to itself, and if \( g : \mathcal{L}(X) \rightarrow \mathcal{L}(Y) \) and \( h : \mathcal{L}(Y) \rightarrow \mathcal{L}(Z) \) are morphisms then \( h \circ g : \mathcal{L}(X) \rightarrow \mathcal{L}(Z) \) is also a morphism.

\textbf{Proposition 2.} \hspace{1mm} (1) \hspace{1mm} The 1-shift of every effective Borel base is effectively \( \sigma \)-reducible. 

(2) For any \( X \in \{ \mathbb{N}, \mathcal{N}, A^\omega \} \), the effective Borel base \( \mathcal{L}(X) \) is effectively \( \sigma \)-reducible.

(3) Let \( f : X \rightarrow Y \) be a computable function between effective spaces. Then \( f^{-1} : \mathcal{L}(Y) \rightarrow \mathcal{L}(X) \) is a morphism of effective Borel bases, and \( A \in \Sigma(Y, T) \) implies \( A \circ f \in \Sigma(X, T) \).

\textbf{Proof.} \hspace{1mm} (1), \hspace{1mm} (2) and the first assertion in (3) are easy. If \( A \) is determined by a \( T \)-family \( F \) in \( \mathcal{L}(Y) \) then \( A \circ f \) is determined by a \( T \)-family \( f^{-1}(F) \) in \( \mathcal{L}(X) \) defined by induction as follows: if \( T \in \mathcal{T}_k(0) \) and \( F = \{ Y \} \), then set \( f^{-1}(F) = \{ X \} \); if \( (T, t) \in \mathcal{T}_k(n + 1) \) and \( F = (\{ U_r \}, \{ F_r \}) \) then set \( f^{-1}(F) = (\{ f^{-1}(U_r) \}, \{ f^{-1}(F_r) \}) \). 

The reduction property is crucial for understanding which levels of the effective Wadge hierarchies have "good" numberings and complete sets. By the \textit{effective Wadge reducibility} in an effective space \( X \) we mean the many-one reducibility \( \leq^X_W \) by computable functions on \( X \); this reducibility applies not only to subsets of \( X \) but also to any functions defined on \( X \) (in particular, to \( k \)-partitions).

To define the "good" numberings, we need effective versions of some notions from \[14\] \[16\]. By \textit{effective family of pointclasses} we mean a family \( \{ \Gamma(X) \} \) parametrised by the effective spaces \( X \) such that \( \Gamma(X) \subseteq P(X) \) and \( f^{-1} : \Gamma(Y) \rightarrow \Gamma(X) \) for any computable \( f : X \rightarrow Y \). A numbering \( \nu : \mathbb{N} \rightarrow \Gamma(X) \) is \( \Gamma \)-\textit{computable} if \( \{ (n, x) \mid x \in \nu(n) \} \in \Gamma(\mathbb{N} \times X) \). Such a numbering is \textit{principal} if any \( \Gamma \)-computable numbering \( \mu \) is \( \leq^X_W \)-reducible to \( \nu \), i.e. \( \mu = \nu \circ f \) for a computable function \( f \) on \( X \). The corresponding notions for \( k \)-partitions are defined in a similar way.

\( \Sigma \)-Levels of the effective hierarchies in Section \[2.2\] form effective pointclasses, and their standard numberings are principal computable. The next result partially extends this to the effective Wadge hierarchy.

\textbf{Proposition 3.} \hspace{1mm} (1) \hspace{1mm} For any \( T \in \mathcal{T}_k(\omega) \) and any effective space \( X \), the level \( \Sigma(X, s(T)) \) has a principal computable numbering. In particular, this holds for any level \( \Sigma^\omega_{\omega^\alpha(X)} \) of the effective Wadge hierarchy of sets.

(2) If \( X \in \{ \mathbb{N}, \mathcal{N} \} \) then the assertion (1) holds for all levels of the effective Wadge hierarchies, and any level has a \( \leq^X_W \)-complete set.
Proof Sketch. Consider e.g. the level $\Sigma(\mathcal{N}, T)$ in (2), the other cases are similar. Let $F_0, F_1, \ldots$ be the computable numbering of $T$-families in $\mathcal{L}(\mathcal{N})$ induced by the numberings of levels of the effective Borel hierarchy. For any $i$, let $F'_i$ be a reduced family obtained from $F_i$ by the procedure of top-down application of the reduction property. Observe that $F'_i = F_i$ if $F_i$ was already a reduced family, and that if $F_i$ determines $A$ then so does $F'_i$. By Lemma 1 and Proposition 2, $F'_0, F'_1, \ldots$ is a computable numbering of the reduced $T$-families in $\mathcal{L}(\mathcal{N})$. Then $A_0, A_1, \ldots$, where $A_i$ is determined by $F'_i$, is a desired numbering of $\Sigma(\mathcal{N}, T)$. The $\Sigma(\mathcal{N}, T)$-complete $k$-partition is obtained by essentially taking the disjoint union of $A_0, A_1, \ldots$ (using a computable homeomorphism between $\mathbb{N} \times \mathcal{N}$ and $\mathcal{N}$).

4 Preservation of levels

A main result of this paper is the following preservation property for levels the effective Wadge hierarchy.

Theorem 1. Let $f : X \to Y$ be a computable effectively open surjection between effective spaces and $A : Y \to \check{k}$. Then for any $T \in \mathcal{T}_k(\omega)$ we have: $A \in \Sigma(Y, T)$ iff $A \circ f \in \Sigma(X, T)$.

In particular, for all $A \subseteq Y$ and $\alpha < \varepsilon_0$ we have: $A \in \Sigma_\alpha(Y)$ iff $f^{-1}(A) \in \Sigma_\alpha(X)$.

We fix $f$ as in the formulation above and first prove two lemmas about the function $A \mapsto f[A]$ from Section 2.4 which was used e.g. in [17, 2, 16]. The first lemma follows straightforwardly from Definitions 2, 3 and Proposition 1.

Lemma 3. (1) The function $A \mapsto f[A]$ is a morphism from $\mathcal{L}(X)$ to $\mathcal{L}(Y)$, and $f[A] \subseteq f(A)$ for each $A \subseteq X$.

(2) For all $V, W \subseteq X$, $f[V \setminus f[W] \subseteq f[V \setminus W]$.

(3) If $T$ is a c.e. well founded tree and $\{U_\tau\}$ is an effective $T$-family of $\Sigma^0_{\alpha+n}(X)$-sets then $\{f[U_\tau]\}$ is an effective $T$-family of $\Sigma^0_{\alpha+n}(Y)$-sets, and $f[U_\tau] \subseteq f[\check{U}_\tau]$ for each $\tau \in T$.

We associate with any $T$-family $F$ in $\mathcal{L}(X)$ the $T$-family $f[F]$ in $\mathcal{L}(Y)$ by induction as follows: if $T \in \mathcal{T}_k(0)$ (hence $F = \{X\}$) then we set $f[F] = \{Y\}$; if $T \in \mathcal{T}_k(n+1)$ (hence $F = (\{U_\tau\}_{\tau \in T}, \{F_\tau\}$) and $t(\tau) \in \mathcal{T}_k(n)$ for each $\tau \in T$) then we set $f[F] = (\{f[U_\tau]\}, \{f[F_\tau]\})$. That $f[F]$ is really a $T$-family in $\mathcal{L}(Y)$, follows from Proposition 1 and Lemma 3.

Lemma 4. Let $A : Y \to \check{k}$ and let $A \circ f$ be determined by a $T$-family $F$ in $\mathcal{L}(X)$. Then $A$ is determined by the $T$-family $f[F]$.

Proof. To simplify notation and to illustrate the ideas by typical example, we give a proof only for $T \in \mathcal{T}_k(3)$ (see examples before Lemma 2). Since $A \circ f$ is determined by $F$, $(A \circ f)(x) = t_2(\tau_2)$, for all $\tau_0 \in T, \tau_1 \in t_0(\tau_0), \tau_2 \in t_1(\tau_1)$ with $x \in U_{\tau_0\tau_1\tau_2}$.

We have to show that $A$ is determined by $f[F]$, i.e. $A(y) = t_2(\tau_2)$, for all $y \in Y$ and $\tau_0 \in T, \tau_1 \in t_0(\tau_0), \tau_2 \in t_1(\tau_1)$ with $y \in f[U_{\tau_0\tau_1\tau_2}]$. 

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For any given \( y \in Y \), there exist \( \tau_0, \tau_1, \tau_2 \) with \( y \in f[\bigcup_{\tau_0, \tau_1, \tau_2}] \). By Lemma 3(3), these conditions imply \( y \in f[\bigcup_{\tau_0, \tau_1, \tau_2}] \), so \( y = f(x) \) for some \( x \in \bigcup_{\tau_0, \tau_1, \tau_2} \). Thus, \( A(y) = (A \circ f)(x) = t_2(\tau_2) \).

\[ \square \]

**Proof of Theorem 1**.

Let \( A \in \Sigma(Y, T) \), then \( A \) is determined by a \( T \)-family \( F \) in \( \mathcal{L}(Y) \). By Proposition 2(3), \( A \circ f \in \Sigma(X, T) \). Conversely, let \( A \circ f \in \Sigma(X, T) \), then \( A \circ f \) is determined by a \( T \)-family \( F \) in \( \mathcal{L}(X) \). By Lemma 4 \( A \) is determined by the \( T \)-family \( f[F] \) in \( \mathcal{L}(Y) \), hence \( A \in \Sigma(Y, T) \).

\[ \square \]

## 5 Transfinite extensions

Here we briefly discuss transfinite extensions of the effective hierarchies. The transfinite extension of \( \{\Sigma_n^0(X)\}_{n<\omega} \) is defined in a natural way as in classical DST, only in place of \( \omega_1 \) one has to take the first non-computable ordinal \( \omega_1^{CK} \). In fact, to obtain reasonable effectivity properties one should denote levels \( \Sigma^0_{\alpha}(X) \) of the transfinite hierarchy not by computable ordinals \( \alpha < \omega_1^{CK} \) but rather by their names \( a, |a|_O = \alpha \), in the Kleene notation system \( (O; <_O) \) \( (a \mapsto |a|_O) \) is a surjection from \( O \subseteq \omega \) onto \( \omega_1^{CK} \), see chapter 16 of \[ 8 \]. The resulting transfinite effective Borel hierarchy \( \{\Sigma^0_{\alpha}(X)\}_{a \in O} \) is extensional, i.e. \( \Sigma^0_{\alpha}(X) = \Sigma^0_{\beta}(X) \) whenever \( |a|_O = |b|_O \). The transfinite effective Hausdorff hierarchy \( \{\Sigma^{-1,n}_{\alpha}(X)\}_{a \in O} \) over \( \Sigma^0_n(X) \) is also defined in a natural way. For \( n = 1 \), we abbreviate \( \Sigma^{-1,n}_{\alpha}(X) \) to \( \Sigma^{-1}_{\alpha}(X) \). The effective Hausdorff hierarchy is not extensional.

These transfinite hierarchies were used to prove effective versions of classical results from DST for arbitrary CQP-space \( X \): the effective Suslin theorem \( \bigcup \{\Sigma^0_{\alpha}(X) \mid a \in O\} = \Delta^1_1(X) \) (follows from Theorem 4 in \[ 6 \], see also Theorem 4 in \[ 15 \]) and the effective Hausdorff theorem \( \Delta^0_1(X) = \bigcup \{\Sigma^{-1}_{\alpha}(X) \mid a \in O\} \) (Theorem 5 in \[ 15 \]). This provides wide generalizations of the corresponding classical facts of computability theory: the Suslin-Kleene theorem is the first result for \( X \in \{\mathbb{N}, \mathcal{N}\} \) and the Ershov theorem is the second result for \( X = \mathbb{N} \).

Since the effective Wadge hierarchy was defined above using iterated labeled trees, it is natural to define transfinite extensions of this hierarchy also in terms of trees. The natural choice is to consider the computable well founded trees in place of finite trees. To slightly simplify the proof of Theorem 3 in the next section, we consider c.e. trees instead of computable trees, but the results also hold for computable trees.

Let \( T_k^*(\omega) \) be defined just as \( T_k(\omega) \) but using c.e. well founded trees (and computable labeling functions) in place of finite trees. Definition 2 also makes sense under this change (only we have always require that the corresponding \( T \)-families of \( \Sigma^0_{1+n} \)-sets are uniform w.r.t. the principal computable numbering of \( \Sigma^0_{1+n} \)). In this way we also obtain the definition of levels \( \Sigma(X, T) \) for arbitrary \( T \in T_k^*(\omega) \). Repeating the proof of Theorem 4 we obtain the following.

**Corollary 1.** Theorem 4 remains true for \( T \in T_k^*(\omega) \).

Due to well known relation of computable and c.e. well founded trees to \( \omega_1^{CK} \) and the Kleene notation system, the family \( \{\Sigma(X, T)\}_{T \in T_k^*(1)} \) for \( k = 2 \) is essentially the same.
object as \( \{\Sigma_{(\omega)}^{-1}(X)\}_{n \in \mathbb{O}} \). For \( k > 2 \), the family \( \{\Sigma(X, T)\}_{T \in T_k^* (1)} \) is a natural transfinite extension of the effective Hausdorff hierarchy of sets to \( k \)-partitions.

### 6 Effective Hausdorff-Kuratowski theorem

Here we discuss effective versions of the Hausdorff-Kuratowski (HK) theorem for \( k \)-partitions. We say that an effective space \( X \) satisfies \( n \)-HK theorem if \( \Delta_0^0 (X) = \bigcup \{ \Sigma(X, s^n(T)) \mid T \in T_{k^*} (1) \} \) where \( \Delta_0^0 (X) \) is the set of \( A \in k^X \) with \( A_0, \ldots, A_{k-1} \in \Delta_0^0 (X) \) and \( s^n \) is the \( n \)th iteration of the function \( s \) forming the singleton trees. For \( n = 0 \) the equality simplifies to \( \Delta_0^0 (X) = \bigcup \{ \Sigma(X, T) \mid T \in T_k^* (1) \} \) which we call the effective Hausdorff theorem for \( k \)-partitions. A simple calculation shows that \( n \)-HK theorem is equivalent to \( \Delta_0^0 (X) \subseteq \bigcup \{ \Sigma(X, s^n(T)) \mid T \in T_k^* (1) \} \) because the opposite inclusion holds in every effective space.

We define the preorder \( \leq_{ceo} \) on effective spaces by: \( Y \leq_{ceo} X \) if there is a computable effectively open surjection \( f : X \to Y \). Obviously, \( X \) is a CQP-space iff \( X \leq_{ceo} \mathcal{N} \).

**Theorem 2.** If an effective space \( X \) satisfies \( n \)-HK theorem and \( Y \leq_{ceo} X \) then so does \( Y \). Thus, if \( \mathcal{N} \) satisfies \( n \)-HK theorem then so does every CQP-space.

**Proof.** Let \( f : X \to Y \) be a computable effectively open surjection and \( A \in \Delta_0^0 (X^Y) \), \( A = (A_0, \ldots, A_{k-1}) \). Then \( A \circ f = (f^{-1}(A_0), \ldots, f^{-1}(A_{k-1})) \in \Delta_0^0 (X^k) \), hence \( A \circ f \in \Sigma(X, s^n(T)) \) for some \( T \in T_k^* (1) \). By Corollary 1, \( A \in \Sigma(Y, s^n(T)) \).

Thus, to prove the effective \( n \)-HK theorem for all CQP-spaces it suffices to prove it for the Baire space. Though we do not currently have a proof for \( n > 0 \) we do have one for \( n = 0 \). The next result extends Theorem 5 in [15] to \( k \)-partitions.

**Theorem 3.** Every CQP-space satisfies the effective Hausdorff theorem for \( k \)-partitions.

**Proof.** We have to show that \( \Delta_0^0 (k^Y) \subseteq \bigcup \{ \Sigma(\mathcal{N}, T) \mid T \in T_k^* (1) \} \). Let \( A \in \Delta_0^0 (k^Y) \), then \( A \) is limit-computable, i.e. for some computable function \( \Phi : \mathcal{N} \times \mathbb{N} \to k \) we have \( A(x) = \lim_{n} \Phi(x, n) \) (see e.g. Proposition 5.1 in [10] for \( k = 2 \); the proof works for any \( k \)).

Let \( M \) be an oracle Turing machine with \( \Phi(x, n) = M^x(n) \). Define a uniformly c.e. sequence \( R_0, R_1, \ldots \) of subsets of \( \omega^* \) as follows. Let \( R_0 \) consist of the \( \subseteq \)-minimal strings \( \sigma \) such that the computation \( M^\sigma(0) \) stops within \( |\sigma| \) steps. Note that \( R_0 \) is a non-empty computable set whose elements are pairwise \( \subseteq \)-incomparable. With any \( \sigma \in R_0 \) we associate the number \( i_\sigma = 0 \). Suppose by induction that we already have \( R_n \) and with any \( \sigma \in R_n \) some \( i_\sigma \) is associated such that \( M^\sigma(i_\sigma) \) stops within \( |\sigma| \) steps. Let \( R_{n+1} \) consist of the \( \subseteq \)-minimal strings \( \tau \) such that \( \sigma \subseteq \tau \) for some \( \sigma \in R_n \), for some \( i_\sigma < i < \tau \) the computation \( M^\tau(i) \) stops within \( |\tau| \) steps, and \( M^\tau(i) \neq M^\tau(i_\sigma) \). Let \( i_\tau \) be the smallest such \( i \). Note that \( R_{n+1} \) is c.e. and already \( R_1 \) might be empty.

Let \( T \) be the tree generated by \( \bigcup_n R_n \), then \( T \) is c.e. It is well founded because otherwise we would have an infinite sequence \( \sigma_0 \subseteq \sigma_1 \subseteq \cdots \) such that \( \sigma_i \in T_i \) for all \( i \), hence for \( x = \sup \{ \sigma_0, \sigma_1, \ldots \} \) the sequence \( \{ \Phi(x, n) \} \) changes infinitely often; a contradiction. We define the labeling \( t : T \to k \) as follows: if there is no \( \sigma \in R_0 \) with \( \sigma \subseteq \tau \) then \( t(\tau) = 0 \), otherwise \( t(\tau) = M^\sigma(i_\sigma) \) where \( \sigma \subseteq \tau \) and \( \sigma \in R_n \) for the largest possible \( n \). The function
$t$ is computable. Define also the $T$-family $\{U_\tau\}$ of $\Sigma^0_1(N)$-sets by $U_\tau = [\tau]$. Then this family determines $A$, hence $A \in \Sigma(N, T)$.

7 Future work

To simplify notation, we concentrated in this paper on the finitary version of the effective Wadge hierarchy. Transfinite versions based on objects like the iterated labeled c.e. well founded trees seem adequate to develop transfinite versions of the effective Wadge hierarchy. We expect interesting extensions of the effective Hausdorff-Kuratowski theorem along these lines.

If we consider arbitrary well founded labeled trees and their suitable iterations we obtain a broad extension of the classical Wadge hierarchy (see [7] and references therein). The methods of classical Wadge theory (including those in [7]) work only for zero-dimensional spaces. In [16] we suggested an approach to define and develop the Wadge hierarchy in arbitrary cb$_0$-spaces and demonstrated them for some initial segments of the Wadge hierarchy. Using methods of this paper, we recently extended these partial results to the whole Wadge hierarchy, including the hierarchy of $k$-partitions. In particular, classical analogues of the results of this paper hold for arbitrary quasi-Polish spaces.

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