NUMERICAL SEMIGROUPS WITH UNIQUE APÉRY EXPANSIONS

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ABSTRACT. In this paper, we carry out a fairly comprehensive study of two special classes of numerical semigroups, one generated by the sequence of partial sums of an arithmetic progression and the other one generated by the partial sums of a geometric progression, in embedding dimension 4. Both these classes have the common feature that they have unique expansions of the Apéry set elements.

1. INTRODUCTION

The set $\Gamma$, subset of the set of nonnegative integers $\mathbb{N}$, is called a numerical semigroup if it is closed under addition, contains zero and generates $\mathbb{Z}$ as a group. Every numerical semigroup $\Gamma$ satisfies the following two fundamental properties (see [6]): The complement $\mathbb{N} \setminus \Gamma$ is finite and $\Gamma$ has a unique minimal system of generators $a_1 < \cdots < a_n$. The greatest integer not belonging to $\Gamma$, usually denoted by $F(\Gamma)$ is called the Frobenius number of $\Gamma$. The integers $a_1$ and $n$, denoted by $m(\Gamma)$ and $e(\Gamma)$ respectively are known as the multiplicity and the embedding dimension of the semigroup $\Gamma$. The Apéry set of $\Gamma$ with respect to a non-zero $a \in \Gamma$ is defined to be the set $\text{Ap}(\Gamma, a) = \{ s \in \Gamma \mid s - a \notin \Gamma \}$. In this paper, we study a class of numerical semigroups, which are special in the sense that they have the uniqueness of representation of each element in the Apéry set $\text{Ap}(\Gamma, a)$. Given positive integers $a_1 < \cdots < a_n$, every numerical semigroup ring $k[\Gamma] = k[t^{a_1}, \ldots, t^{a_n}]$ is the coordinate ring of an affine monomial curve.

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given by the monomial parametrization \( \nu : k[x_1, \ldots, x_n] \rightarrow k[t] \), such that \( \nu(x_i) = t^{a_i}, 1 \leq i \leq p \). The ideal \( \ker(\nu) = p \) is the defining ideal of the parametrized monomial curve, which is graded with respect to the weighted gradation.

It is known that uniqueness of representations of the Apéry set elements of a numerical semigroup is actually quite helpful; see [7], [13]. Let us call these as numerical semigroups with unique Apéry expansions. One requires the Apéry table in order to understand the tangent cone, which is quite hard to compute in general. However, uniqueness of expressions of the Apéry set elements makes it easier. In this paper, we will be presenting two classes of numerical semigroups with this property. In fact, we have stumbled upon these classes while looking for a large class of numerical semigroups with this property, especially from the standpoint of computing tangent cones.

Let \( f(x), g(x) \in \mathbb{Q}[x] \) such that \( f(\mathbb{N}) \subset \mathbb{N}, g(\mathbb{N}) \subset \mathbb{N} \) and both are increasing, so called increasing numerical polynomials. In this paper we study numerical semigroups minimally generated by integers of the form \( \{ a, g(i)a + f(i)d \mid \gcd(a, d) = 1, 1 \leq i \leq n \} \). Some of the interesting classes of numerical semigroups that have been studied fall under this general class. For example, \( g(x) = 1, f(x) = x \) gives an arithmetic sequence (see [15], [9]) and \( g(x) = \text{constant}, f(x) = x \) gives a generalized arithmetic sequence (see [12]). We study the following two cases:

1. When \( g(x) = x + 1, f(x) = \frac{x(x + 1)}{2} \) and \( n = 3 \); we denote the semigroup by \( \Gamma_4 \). A complete study has been carried out in sections 2 through 6 in the following order - Apéry set, the pseudo Frobenius numbers, the defining ideal, syzygies and finally the Apéry table and the tangent cone. All these are known to be extremely hard to compute in general. We have used the computer algebra system [10] to form initial guesses for many of the theorems that we have proved here.

2. We take \( g(x) \) to be a constant numerical function, \( f(x) = r^x \) and define the numerical semigroup \( \mathcal{S}_{n+2} \), in section 7. We compute the the Apéry table and the tangent cone for \( \mathcal{S}_{n+2} \).

2. APÉRY SET OF \( \Gamma_4 \)

We now consider the numerical semigroup generated by the positive integers \( s_1, \ldots, s_4 \), where \( d > 0 \) and \( a > 0 \) are integers with \( \gcd(a, d) = 1 \), and \( s_n = \frac{a}{2}[2a + (n - 1)d] \), for \( 1 \leq n \leq 4 \). We denote this numerical semigroup by \( \Gamma_4 \), the semigroup ring by \( k[\Gamma_4] \) and the defining ideal by \( p_4 \). We will
see in the next Theorem that we need to impose some bounds on $a$ so that \{s_1, \ldots, s_4\} is a minimal generating set for the numerical semigroup $\Gamma_4$.

**Theorem 2.1.** Let $d > 0$ and $a \geq 7$ be integers with $\gcd(a, d) = 1$. Let $s_n = \frac{n}{2}[2a + (n - 1)d]$, for $1 \leq n \leq 4$. The set $T_4 = \{s_1, \ldots, s_4\}$ is a minimal generating set for the numerical semigroup $\Gamma_4 = \langle s_1, \ldots, s_4 \rangle$.

**Proof.** Suppose $s_3 = m_1s_1 + m_2s_2$, for some $m_1, m_2 \geq 0$. We get,

\begin{equation}
(1) \quad (m_1 + 2m_2 - 3)a = (3 - m_2)d.
\end{equation}

Since $\gcd(a, d) = 1$, we get $a | (3 - m_2)$.

If $m_2 \leq 3$, then $3 = m_2 + ka$, for some $k \geq 0$, and we get $m_2 = 3$, since $a \geq 7$. Therefore, $m_1 + 3 = 0$ (using equation 2.1) - a contradiction. If $m_2 > 3$, then the L.H.S. of equation 2.1 is positive whereas the R.H.S. is negative - a contradiction.

Suppose $s_4 = m_1s_1 + m_2s_2 + m_3s_3$, for some $m_1, m_2, m_3 \geq 0$. We get,

\begin{equation}
(2) \quad (m_1 + 2m_2 + 3m_3 - 4)a = (6 - m_2 - 3m_3)d.
\end{equation}

Therefore $a | (6 - m_2 - 3m_3)$, since $\gcd(a, d) = 1$.

If $m_2 + 3m_3 \leq 6$, then $6 = m_2 + 3m_3 + ka$, for some $k \geq 0$. Therefore, we get $6 = m_2 + 3m_3$, since $a \geq 7$. Possible solutions for $(m_2, m_3)$ are $(0, 2), (3, 1), (6, 0)$. Substituting these values of $m_2, m_3$ in the equation 2.2 we get $m_1 < 0$ - a contradiction.

If $m_2 + 3m_3 > 6$, then the L.H.S. of the equation 2.2 is positive, whereas the R.H.S. is negative - a contradiction. 

\hfill $\Box$

**Theorem 2.2.** Let $a \geq 7$. For each $1 \leq i \leq a - 1$, let $i = 6\mu_i + q_i$, such that $0 \leq q_i < 6$. For each $1 \leq i \leq a - 1$, we define $\nu_i, \xi_i$ as follows;

(i) $(\nu_i, \xi_i) = (1, q_i - 3)$, if $q_i \geq 3$;
(ii) $(\nu_i, \xi_i) = (0, q_i)$, if $q_i < 3$.

Let $\text{Ap}(\Gamma_4, a)$ denote the Apéry set of $\Gamma_4$, with respect to the element $a$. Then \( \text{Ap}(\Gamma_4, a) = \{(4\mu_i + 3\nu_i + 2\xi_i)a + id \mid 1 \leq i \leq a - 1\} \cup \{0\} \).

**Proof.** Let $T = \{(4\mu_i + 3\nu_i + 2\xi_i)a + id \mid 1 \leq i \leq a - 1\}$. We notice that $i = 6\mu_i + 3\nu_i + \xi_i$, therefore for $1 \leq i \leq a - 1$, we have

\((4\mu_i + 3\nu_i + 2\xi_i)a + id = \mu_i(4a + 6d) + \nu_i(3a + 3d) + \xi_i(2a + d) \in \Gamma_4 \).

Hence $T \subset \Gamma_4$. Let $s \in \text{Ap}(\Gamma_4, a) \setminus \{0\}$, with $s \equiv id(\mod a)$. Suppose

\[
\begin{align*}
s &= c_1(2a + d) + c_2(3a + 3d) + c_3(4a + 6d) \\
  &= (2c_1 + 3c_2 + 4c_3)a + (c_1 + 3c_2 + 6c_3)d,
\end{align*}
\]
then \((c_1 + 3c_2 + 6c_3) \equiv i (\mod a)\), as \(\gcd(a, d) = 1\). Therefore

\[ (2.3) \quad c_1 + 3c_2 + 6c_3 = i + ka = 6\mu_i + q_i + ka, \]

for some \(k \geq 0\). It is enough to show that, \(4c_3 + 3c_2 + 2c_1 \geq 4\mu_i + 3\nu_i + 2\xi_i\). Suppose \(4c_3 + 3c_2 + 2c_1 < 4\mu_i + 3\nu_i + 2\xi_i\), then from \(2.3\) substituting \(\mu_i\), we have

\[ (2.4) \quad 6\xi_i + 9\nu_i - 2q_i > 4c_1 + 3c_2 + 2ka. \]

We consider the following cases:

**Case A.** If \(q_i = 0\), then \((\nu_i, \xi_i) = (0, 0)\), and from \(2.4\) we get \(0 > 4c_1 + 3c_2 + 2ka\), which is impossible.

**Case B.** If \(q_i = 1\), then \((\nu_i, \xi_i) = (0, 1)\) and from \(2.4\) we get \(4 > 4c_1 + 3c_2 + 2ka\). Therefore \(k = 0\) and \((c_1, c_2) \in \{(0, 0), (0, 1)\}\). Putting values of \(c_1\) and \(c_2\) in equation \(2.3\) we get the following:

\[
\begin{align*}
6c_3 &= 6\mu_i + 1 & \text{if } (c_1, c_2) = (0, 0), \\
6c_3 &= 6\mu_i - 2 & \text{if } (c_1, c_2) = (0, 1).
\end{align*}
\]

All lead to contradictions.

**Case C.** If \(q_i = 2\), then \((\nu_i, \xi_i) = (0, 2)\), and from \(2.4\) we get \(8 > 4c_1 + 3c_2 + 2ka\). Therefore \(k = 0\) and \((c_1, c_2) \in \{(0, 0), (1, 0), (1, 1), (0, 1), (0, 2)\}\). Putting values of \(c_1\) and \(c_2\) in equation \(2.3\) we get the following:

\[
\begin{align*}
6c_3 &= 6\mu_i + 2 & \text{if } (c_1, c_2) = (0, 0), \\
6c_3 &= 6\mu_i + 1 & \text{if } (c_1, c_2) = (1, 0), \\
6c_3 &= 6\mu_i - 2 & \text{if } (c_1, c_2) = (1, 1), \\
6c_3 &= 6\mu_i - 1 & \text{if } (c_1, c_2) = (0, 1), \\
6c_3 &= 6\mu_i - 4 & \text{if } (c_1, c_2) = (0, 2).
\end{align*}
\]

All lead to contradictions.

**Case D.** If \(q_i = 3\), then \((\nu_i, \xi_i) = (1, 0)\), and from \(2.4\) we get \(3 > 4c_1 + 3c_2 + 2ka\). Therefore \(k = 0\) and \((c_1, c_2) = (0, 0)\). Then from equation \(2.3\) we get \(6c_3 = 6\mu_i + 3\); which is not possible.

**Case E.** If \(q_i = 4\), then \((\nu_i, \xi_i) = (1, 1)\), and from \(2.4\) we get \(7 > 4c_1 + 3c_2 + 2ka\). Therefore \(k = 0\) and \((c_1, c_2) \in \{(0, 0), (1, 0), (1, 1), (0, 1), (0, 2)\}\). Putting values of \(c_1\) and \(c_2\) in equation \(2.3\) we get the following:

\[
\begin{align*}
6c_3 &= 6\mu_i + 4 & \text{if } (c_1, c_2) = (0, 0), \\
6c_3 &= 6\mu_i + 3 & \text{if } (c_1, c_2) = (1, 0), \\
6c_3 &= 6\mu_i + 1 & \text{if } (c_1, c_2) = (0, 1), \\
6c_3 &= 6\mu_i - 2 & \text{if } (c_1, c_2) = (0, 2).
\end{align*}
\]
All lead to contradictions.

**Case F.** If \( q_1 = 5 \), then \( (\nu_i, \xi_i) = (1, 2) \), from \( [2, 4] \) we get \( 11 > 4c_1 + 3c_2 + 2ka \). Therefore \( k = 0 \) and
\[
(c_1, c_2) \in \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0)\}.
\]

Putting values of \( c_1 \) and \( c_2 \) in equation \( [2, 3] \), we get the following:
\[
\begin{align*}
6c_3 &= 6\mu_i + 5 \quad \text{if} \quad (c_1, c_2) = (0, 0), \\
6c_3 &= 6\mu_i + 2 \quad \text{if} \quad (c_1, c_2) = (0, 1), \\
6c_3 &= 6\mu_i - 1 \quad \text{if} \quad (c_1, c_2) = (0, 2), \\
6c_3 &= 6\mu_i - 4 \quad \text{if} \quad (c_1, c_2) = (0, 3), \\
6c_3 &= 6\mu_i + 4 \quad \text{if} \quad (c_1, c_2) = (1, 0), \\
6c_3 &= 6\mu_i + 1 \quad \text{if} \quad (c_1, c_2) = (1, 1), \\
6c_3 &= 6\mu_i - 2 \quad \text{if} \quad (c_1, c_2) = (1, 2), \\
6c_3 &= 6\mu_i + 3 \quad \text{if} \quad (c_1, c_2) = (2, 0).
\end{align*}
\]

All lead to contradictions. \( \square \)

An example has been discussed in \( [6, 9] \).

### 3. Pseudo Frobenius Numbers and Type of \( \Gamma_4 \)

**Definition 1.** Let \( \Gamma \) be a numerical semigroup, we say that \( x \in \mathbb{Z} \) is a *pseudo-Frobenius number* if \( x \notin \Gamma \) and \( x + s \in \Gamma \) for all \( s \in \Gamma \setminus \{0\} \). We denote by \( \text{PF}(\Gamma) \) the set of pseudo-Frobenius numbers of \( \Gamma \). The cardinality of \( \text{PF}(\Gamma) \) is denoted by \( t(\Gamma) \) and we call it the *type* of \( \Gamma \).

Let \( a, b \in \mathbb{Z} \). We define \( \leq_{\Gamma} \) as \( a \leq_{\Gamma} b \) if \( b - a \in \Gamma \). This order relation defines a poset structure on \( \mathbb{Z} \).

**Theorem 3.1.** Let \( \Gamma \) be a numerical semigroup and \( a \in \Gamma \setminus \{0\} \). Then \( \text{PF}(\Gamma) = \{w - a \mid w \in \text{Maximals}_{\leq_{\Gamma}} \text{Ap}(\Gamma, a)\} \).

**Proof.** See Proposition 8 in \( [11] \). \( \square \)

Let \( \omega(i) = (4\mu_i + 3\nu_i + 2\xi_i)a + id \), for \( 1 \leq i \leq a - 1 \). Therefore, \( \text{Ap}(\Gamma_4, a) = \{\omega(i) \mid 1 \leq i \leq a - 1\} \).

**Theorem 3.2.** Let \( a \geq 7 \) and \( d \) be two integers such that \( \gcd(a, d) = 1 \). Suppose \( \Gamma_4 = \langle s_1, \ldots, s_4 \rangle \), where \( s_n = \frac{n}{2}[2a + (n-1)d] \), for \( 1 \leq n \leq 4 \). Let
\( \mathbb{PF}(\Gamma_4) \) be the set of pseudo Frobenius numbers of the numerical semigroup \( \Gamma_4 \). Write \( a = 6m + q, 0 \leq q \leq 5; \) then

\[
\mathbb{PF}(\Gamma_4) = \begin{cases}
\{\omega(a-1)\}, & \text{if } q = 0; \\
\{\omega(a-1), \omega(a-2)\}, & \text{if } q = 1; \\
\{\omega(a-1), \omega(a-3)\}, & \text{if } q = 2; \\
\{\omega(a-1), \omega(a-4)\}, & \text{if } q = 3; \\
\{\omega(a-1), \omega(a-2), \omega(a-5)\}, & \text{if } q = 4; \\
\{\omega(a-1), \omega(a-3), \omega(a-6)\}, & \text{if } q = 5.
\end{cases}
\]

**Proof.** We first note that \( \omega(i+6) - \omega(i) = 4a + 6d \in \Gamma_4 \), for \( 0 \leq i < a-6 \). Therefore, \( \omega(i) \leq \omega(i+6), 0 \leq i < a-6 \). Hence, by Theorem 3.1 \( \mathbb{PF}(\Gamma_4) \subset \{\omega(a-i) \mid 1 \leq i \leq 6\} \). Proof of the theorem follows easily by checking each case. \( \square \)

**Corollary 3.3.** Let \( \text{Der}_k(\Gamma_4) \) be the set of \( k \)-derivations of \( k[[t^a, t^{2a+d}, t^{3a+3d}, t^{4a+6d}]] \), then

\[
\text{Der}_k(\Gamma_4) = \{t^{\alpha+1} \mid \alpha \in \mathbb{PF}(\Gamma_4)\}.
\]

**Proof.** Follows from [11], page 875. \( \square \)

**Corollary 3.4.** Let \( F(\Gamma_4) \) be the Frobenius number of \( \Gamma_4 \). Then

(i) \( F(\Gamma_4) = \omega(a-1), \) if \( q \in \{0, 3, 5\}; \)

(ii) If \( q = 1 \), then

\[
F(\Gamma_4) = \begin{cases}
\omega(a-2), & \text{if } 3a > d; \\
\omega(a-1), & \text{otherwise}.
\end{cases}
\]

(iii) If \( q = 2 \) then

\[
F(\Gamma_4) = \begin{cases}
\omega(a-3), & \text{if } a > 2; \\
\omega(a-1), & \text{otherwise}.
\end{cases}
\]

(iv) If \( q = 4 \) then

\[
F(\Gamma_4) = \begin{cases}
\omega(a-2), & \text{if } a > d; \\
\omega(a-1), & \text{otherwise}.
\end{cases}
\]

**Proof.** One can easily find the maximum element from 3.2. \( \square \)
4. Minimal Generating Set for the Defining Ideal $p_4$

Let us begin with the following theorem from [8], which helps us compute a minimal generating set for the defining ideal of a monomial curve.

**Theorem 4.1.** Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring, $I \subset A$ the defining ideal of a monomial curve defined by natural numbers $a_1, \ldots, a_n$, whose greatest common divisor is 1. Let $J \subset I$ be a subideal. Then $J = I$ if and only if $\text{dim}_k A/(J + (x_i)) = a_i$ for some $i$. (Note that the above conditions are also equivalent to $\text{dim}_k A/(J + (x_i)) = a_i$ for any $i$.)

**Proof.** See [8]. □

**Lemma 4.2.** Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring. For a monomial ideal $J$ of $A$, we write the unique minimal generating set of $J$ as $G(J)$. Let $I = (f_1, \ldots, f_k)$ and $I_i = (f_1, \ldots, \hat{f}_i, \ldots, f_k)$, $1 \leq i \leq k$. Suppose that, with respect to some monomial order on $A$, $\{\text{LT}(f_1), \ldots, \text{LT}(f_k)\} \subset G(\text{LT}(I))$ and $G(\text{LT}(I)) \subset G(\text{LT}(I)) \setminus \{\text{LT}(f_i)\}$ for all $1 \leq i \leq k$. Then $I$ is minimally generated by $\{f_1, \ldots, f_k\}$.

**Proof.** Suppose $I$ is not minimally generated by $\{f_1, \ldots, f_k\}$. Then there is a polynomial $f_i$ such that $f_i \in I_i$. Therefore there is a monomial $m \in G(\text{LT}(I_i))$, such that $m \mid \text{LT}(f_i)$. But $m$ and $\text{LT}(f_i)$ are distinct elements of $G(\text{LT}(I))$, which gives a contradiction. □

**Notations.** We now introduce some notations specific to the polynomial ring with 4 variables $A = k[x_1, x_2, x_3, x_4]$. Let $m$ and $d$ be fixed positive integers. We define the subsets $H_0, H_1, H_2, H_3, H_4, H_5$ of $A$ as follows:

(i) $H_0 = \{x_2^2 - x_1^2 x_4, x_2^3 - 3 x_1 x_3 x_4, x_1^2 x_4^d - x_4^m\}$.

(ii) (a) $H_1 = \{x_2^2 - x_1^2 x_4, x_2^3 - 3 x_1 x_3 x_4, x_1^7 - x_2 x_4, x_1^2 x_2^2 - x_3 x_4, x_1^2 x_2^2 x_3 - x_2^2\}$, if $m = d = 1$.

(b) $H_1 = \{x_2^2 - x_1^2 x_4, x_2^3 - 3 x_1 x_3 x_4, x_1^{4m+d-6} x_2^5 - x_4^m + 1, x_1^{4m+d+1} x_2 - x_3 x_4^m, x_1^{4m+d+2} - x_2 x_4^m\}$, otherwise.

(iii) $H_2 = \{x_3^2 - x_1^2 x_4, x_3^3 - 3 x_1 x_3 x_4, x_1^{4m+d-4} x_2^4 - x_4^m + 1, x_1^{4m+d+1} x_2 - x_3 x_4^m, x_1^{4m+d+4} - x_2 x_4^m\}.$

(iv) $H_3 = \{x_3^2 - x_1^2 x_4, x_3^3 - 3 x_1 x_3 x_4, x_1^{4m+d-2} x_2^3 - x_4^m + 1, x_1^{4m+d+3} - x_3 x_4^m\}.$

(v) $H_4 = \{x_3^2 - x_1^2 x_4, x_3^3 - 3 x_1 x_3 x_4, x_1^{4m+d} x_2^2 - x_4^m + 1, x_1^{4m+d+5} - x_2 x_3 x_4^m\}.$
Theorem 4.3. Suppose $a = 6m + q$, where $0 \leq q \leq 5$. Then $H_q$ is a minimal generating set for the ideal $P_a$.

Proof. We now use Theorem 4.1 to show that $\dim_k(A/(H_q, x_1)) = a$. Let $B = k[x_2, x_3, x_4]$ and $H_q = \langle H_q, x_1 \rangle / \langle x_1 \rangle \subset B$. Therefore we need to show that $\dim_k(B/(H'_q)) = a$. Let $\kappa_q$ be the dimension of the vector space $B/(H'_q)$. We define

$$\mathfrak{B} = \{x_i x_j x_k^l \mid 0 \leq i \leq 2, 0 \leq j \leq 1, 0 \leq k \leq m\},$$

and show that image of the set $\mathfrak{B} \setminus \mathfrak{B}_q$ forms a basis of the vector space $B/(H'_q)$, through the following cases:

(A) We have $H'_0 = \{x_3^2, x_2 x_4^3\}$ and $\mathfrak{B}_0 = \{x_4^m, x_2 x_4^m, x_3 x_4^m, x_2 x_3 x_4^m, x_2^2 x_3 x_4^m\}$. Hence $\kappa_0 = 6m$.

(B) $H'_1 = \{x_3^2, x_2 x_4^{m+1}, x_3 x_4^m, x_2 x_4^m\}$ and $\mathfrak{B}_1 = \{x_2 x_4^m, x_2 x_4^m, x_3 x_4^m, x_2 x_3 x_4^m, x_2^2 x_3 x_4^m\}$. Hence $\kappa_1 = 6m + 1$.

(C) $H'_2 = \{x_3^2, x_2 x_4^{m+1}, x_3 x_4^m, x_2 x_4^m\}$ and $\mathfrak{B}_2 = \{x_2 x_4^m, x_3 x_4^m, x_2 x_3 x_4^m, x_2^2 x_3 x_4^m\}$. Therefore $\kappa_2 = 6m + 2$.

(D) $H'_3 = \{x_3^2, x_2 x_4^{m+1}, x_3 x_4^m\}$ and $\mathfrak{B}_3 = \{x_3 x_4^m, x_2 x_3 x_4^m, x_2^2 x_3 x_4^m\}$. Hence $\kappa_3 = 6m + 3$.

(E) $H'_4 = \{x_3^2, x_2 x_4^{m+1}, x_2 x_3 x_4^m\}$ and $\mathfrak{B}_4 = \{x_2 x_3 x_4^m, x_2^2 x_3 x_4^m\}$. Hence $\kappa_4 = 6m + 4$.

(F) $H'_5 = \{x_3^2, x_2 x_4^{m+1}, x_2^2 x_3 x_4^m\}$ and $\mathfrak{B}_5 = \{x_2^2 x_3 x_4^m\}$. Hence $\kappa_5 = 6m + 5$.

We now apply Lemma 4.2 to each case to prove that these indeed give us the minimal generating sets for the ideal $P_a$ in various cases.

5. Syzygies of $k[\Gamma_a]$

Lemma 5.1. Suppose $a = 6m$, and $\gcd(a, d) = 1$. Then, the set $\{x_3^2 - x_1^2 x_4, x_2^3 - x_3 x_4^3, x_1^{4m+d} - x_4^m\}$ forms a regular sequence in $A$.

Proof. With respect to the lexicographic monomial order induced by $x_2 > x_3 > x_1$, the leading terms of these polynomials are mutually coprime. Hence the set $\{x_3^2 - x_1^2 x_4, x_2^3 - x_3 x_4^3, x_1^{4m+d} - x_4^m\}$ forms a regular sequence.
**Corollary 5.2.** Suppose $a = 6m$ and $\gcd(a, d) = 1$, then the Koszul complex resolves $A/p_4$ and the Betti numbers are $\beta_0 = 1, \beta_1 = 3, \beta_2 = 3, \beta_3 = 1$. Hence the ring $A/\mathfrak{P}_4$ is complete intersection.

**Proof.** Proof follows from lemma 5.1. \hfill \qed

**Lemma 5.3.** Let $m, d$ be two positive integers; consider the polynomials

\[ g_1 = -x_1^{(4m+d+2)} + x_2^2 x_4^m, \quad g_2 = x_1^5 x_4 - x_2^3 x_3 \quad \text{and} \quad g_3 = x_1^{(4m+d-1)} x_2^2 - x_3 x_4^m. \]

The set $\{g_1, g_2, g_3\}$ forms a regular sequence in $A = k[x_1, x_2, x_3, x_4]$.

**Proof.** Let $x_3 > x_1 > x_2 > x_4$ induce the lexicographic monomial order on $A$. Then $\text{Lt}(g_1) = -x_1^{(4m+d+2)}$, $\text{Lt}(g_2) = -x_2^3 x_3$ and $\text{Lt}(g_3) = -x_3 x_4^m$.

Since $\gcd(\text{Lt}(g_1), \text{Lt}(g_2)) = 1$, the set $\{g_1, g_2\}$ forms Gröbner basis of $\mathfrak{G}$, with respect to the chosen monomial order and forms a regular sequence. Let $\mathfrak{G} = \langle g_1, g_2 \rangle$ and $g_3 h \in \mathfrak{G}$; we have to show that $h \in \mathfrak{G}$. After division we may assume that $\text{Lt}(g_1) \downarrow \text{Lt}(h)$ and $\text{Lt}(g_2) \downarrow \text{Lt}(h)$. Since $g_3 h \in \mathfrak{G}$ and $\text{Lt}(g_1) \downarrow \text{Lt}(h), \text{Lt}(g_2) \downarrow \text{Lt}(h)$, we have $x_2^3 \mid \text{Lt}(h)$ and $x_3 \mid \text{Lt}(h)$. We write $h = m_0 + \cdots + m_r$, where each $m_i$’s are monomials and $m_0 \geq \cdots > m_r$, with respect to the chosen monomial order. Since $x_3 > x_1 > x_2 > x_4$ is the lexicographic monomial order on $A$, $x_3 \nmid m_0$ implies $x_3 \nmid m_i$, for $1 \leq i \leq r$. Suppose $x_1^i \mid m_i$ but $x_1^{i+1} \nmid m_i$, then $i < j$ implies $l_j \leq i < 4m + d + 2$.

Let $m_0 = x_2^3 m_0'$, then

\[ (x_1^{(4m+d-1)} x_2^2 - x_3 x_4^m) (x_2^3 m_0' + m_1 + \cdots + m_r) \in \mathfrak{G}. \]

After dividing by $g_2$ we get

\[ (x_1^{(4m+d-1)} x_2^5 - x_1^5 x_4^{m+1}) m_0' + (x_1^{(4m+d-1)} x_2^2 - x_3 x_4^m) (m_1 + \cdots + m_r) \in \mathfrak{G}. \]

Then leading term of above polynomial is $-x_3 x_4^m m_1$ and we can divide by $g_2$. Continuing this way we get

\[ (x_1^{(4m+d-1)} x_2^5 - x_1^5 x_4^{m+1}) (m_0' + \cdots + m_r) \in \mathfrak{G}, \]

where $m_i = x_2^3 m_i'$, for $0 \leq i \leq r$. Notice that $m_0' \geq \cdots > m_r'$. If $m = d = 1$, then $-x_1^5 x_4^2 m_0'$, otherwise $x_1^{(4m+d-1)} x_2^5 m_0$ is the leading term of the above polynomial.

**Case 1.** Suppose $m = d = 1$, then

\[ (x_1^4 x_2^5 - x_1^5 x_4^2) (m_0' + \cdots + m_r) \in \mathfrak{G}. \]

We have $\text{Lt}(g_1) = -x_1^7 \mid -x_1^5 x_2^3 m_0'$ (since $x_3 \nmid m_0 = \text{Lt}(h)$), hence $x_1^2 \mid m_0'$. Let $m_0 = x_1^2 m_0' = x_1^2 x_2^3 m_0'$. After dividing by $g_1$ we get

\[ (x_1^6 x_2^5 - x_2 x_4^3) m_0'' + (x_1^4 x_2 - x_1^5 x_4^2) (m_1' + \cdots + m_r') \in \mathfrak{G}. \]

We continuously divide the above polynomial by $g_1$ and we get,

\[ (x_1^6 x_2^5 - x_2 x_4^3) (m_0'' + \cdots + m_s'') + (x_1^4 x_2 - x_1^5 x_4^2) (m_{s+1}' + \cdots + m_r') \in \mathfrak{G}, \]
where $0 \leq s \leq r$, and for $0 \leq i \leq s$ we have $m'_i = x^2_i m''$ and the leading term is $x^6 x^5 m_0''$. Therefore $x^2 \nmid m_i'$ for $s + 1 \leq i \leq r$. Hence $\text{Lt}(g_1) = -x^7_1 \mid x^4_1 x^2_2 m_0$, therefore $x_1 \mid m''_s$. Let $m_0 = x^3_2 x^3 m''_s$, then we have

\[(x^6_2 x^4_4 - x^3_1 x^2_2 x^2_2) m''_0 + (x^6_1 x^5_2 - x^3_1 x^2_4)(m''_1 + \cdots + m''_s) + (x^4_1 x^2_2 - x^5_1 x^2_4)(m''_{s+1} + \cdots + m''_r) \in \mathfrak{S}.
\]

Again we continuously divide by $g_1$ and for $0 \leq s \leq r$ we get

\[(x^6_2 x^4_4 - x^3_1 x^2_2 x^2_2)(m''_0 + \cdots + m''_s) + (x^4_1 x^2_2 - x^5_1 x^2_4)(m''_{s+1} + \cdots + m''_r) \in \mathfrak{S}.
\]

If the leading term of above polynomial is $-x^5_1 x^2_4 m''_{s+1}$, then $\text{Lt}(g_1) \nmid -x^5_1 x^2_4 m''_{s+1}$ (as $x^7_1 \nmid m''_{s+1}$). Therefore the leading term of above polynomial is $-x^3_2 x^2_2 m''_0$.

Thus, we have $\text{Lt}(g_1) = -x^7_1 \mid -x^3_2 x^2_2 x^2_2 m''_0$, hence $x^6_1 \mid m''_s$, which implies that $\text{Lt}(g_1) \mid \text{Lt}(h)$ - a contradiction.

**Case 2.** If $m$ or $d$ is greater than 1, then $x_1^{(4m+d-1)} x^5_2 m_0$ is the leading term of $(x_1^{(4m+d-1)} x^5_2 - x^5_4 x^4_{m+1})(m'_1 + \cdots + m'_r)$. Therefore $\text{Lt}(g_1) = -x_1^{4m+d+2} \mid x_1^{(4m+d-1)} x^5_2 m_0$, hence $x_1^3 \mid m'_r$. After dividing by $g_1$ we get

\[(x^6_2 x^4_4 - x^8_1 x^4_{m+1}) m''_0 + (x_1^{(4m+d-1)} x^5_2 - x^5_4 x^4_{m+1})(m'_1 + \cdots + m'_r) \in \mathfrak{S}.
\]

We proceed along the same line of argument as in Case 1. The variable $x_3$ is not present in the polynomial in each step, the leading term is always divisible by $\text{Lt}(g_1)$ and after finite steps we get $\text{Lt}(g_1) \mid \text{Lt}(h)$ - a contradiction.

**Proposition 5.4.** Suppose $a = 6$ and $d = 1$. Then the complex,

\[0 \rightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \rightarrow A/p_4 \rightarrow 0 \]

is a minimal graded free resolution of $A/p_4$, where the maps $\phi_i$ are given by

\[\phi_1 = (f_1, f_2, f_3, f_4, f_5),\]

with $f_1 = -x^3_2 + x^3_1 x_3$, $f_2 = -x^3_2 + x^3_1 x_4$, $f_3 = x^7_1 - x_2 x_4$, $f_4 = x_1^4 x^2_2 - x_3 x_4$, $f_5 = x^4_1 x^2_2 x_3 - x^4_2$,

\[\phi_2 = \begin{pmatrix}
x^4_1 & x_4 & 0 & x^7_1 x_3 & 0 & x_3^2 \\
0 & 0 & x_4 & x^5_1 & x^2_1 x_2 & x^2_1 x_3 - x^2_2 \\
x_2 & x^2_3 - x_3 & 0 & -x_4 & 0 & -x_1^2 x^2_2 \\
x_2 & x^3_1 & -x_3 & 0 & -x_4 & x^2_3 \\
0 & 0 & x^2_1 & x_2 & x_3 & 0
\end{pmatrix}.
\]
and

\[ \phi_3 = \begin{pmatrix} x_4 & x_1^2 x_3 \\ -x_1^4 & -x_3^2 \\ 0 & -x_1^3 x_3 + x_2^3 \\ -x_3 & -x_1^2 x_2 \\ x_2^2 & x_1 \\ x_1 & x_4 \end{pmatrix} \]

**Proof.** We use the Buchsbaum-Eisenbud acyclicity theorem (see in [2]). It is easy to show that \( \text{grade}(I_4(\phi_2), A) \geq 2 \). We take the minors,

\[ [1234 \mid 1235] = (x_3 x_4 - x_1^4 x_2^2)(x_1^2 x_2 x_3 - x_4^2), \quad [2345 \mid 1236] = x_1^2(x_1^3 x_3 - x_2^3)^2, \]

which have distinct irreducible factors, hence they form a regular sequence. We now consider the following minors,

\[ [56 \mid 12] = -x_1^7 + x_2 x_4, [15 \mid 12] = x_1^5 x_4 - x_3^3 x_3, [46 \mid 12] = x_1^4 x_2^3 - x_3 x_4^m, \]

which form a regular sequence by Lemma 5.3.

**Proposition 5.5.** Suppose \( a = 6m + 1 \) and either \( m \) or \( d \) is greater than 1. Then the complex

\[ 0 \rightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \rightarrow A/p_4 \rightarrow 0 \]

is a minimal graded free resolution of \( A/p_4 \), where the maps \( \phi_i \) are given by

\[ \phi_1 = (f_1, f_2, f_3, f_4, f_5), \]

where \( f_1 = -x_2^3 + x_1^3 x_3, f_2 = -x_3^2 + x_2^2 x_4, f_3 = x_1^{(4m+d+2)} - x_2 x_4^m, f_4 = x_1^{(4m+d-1)} x_2^2 - x_3 x_4^m, f_5 = x_1^{(4m+d-6)} x_2^5 - x_4^{m+1} \);

\[ \phi_2 = \begin{pmatrix} x_1^{(4m+d-3)} x_3 + x_2^3 x_4 & 0 & 0 & 0 & 0 \\ -x_3 x_4 & -x_2 & 0 & 0 & 0 \\ 0 & x_2 & x_1^2 & -x_3 & 0 \\ 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_1 & 0 \end{pmatrix} \]

and

\[ \phi_3 = \begin{pmatrix} x_1^{(4m+d-3)} & x_4^m \\ -x_4 & -x_2 x_3 \\ 0 & -x_2^2 x_4 + x_3^2 \\ 0 & x_1^3 x_3 - x_2^3 \\ x_3 & x_1^2 x_2^2 \\ -x_2 & -x_1^2 \end{pmatrix} \]
**Proof.** We use the Buchsbaum-Eisenbud acyclicity theorem. It is easy to show that \( \text{grade}(I_2(\phi_2), A) \geq 2 \). We take the minors \([1345 \mid 1246] = -x_3(x_1^2x_4 - x_3^2)^2 \) and \([2345 \mid 2345] = (x_4^m x_2 - x_1^{4m+d+2})(-x_3^2 x_3 + x_2^2) \). These have distinct irreducible factors. Hence they form a regular sequence. Next we have to show that \( \text{grade}(I_2(\phi_3), A) \geq 3 \). By Lemma 5.6, the minors \([15 \mid 12] = -x_1^{4m+d+2} + 2x_2 x_4^m \), \([26 \mid 12] = x_1^5 x_4 - x_3^3 x_3 \) and \([15 \mid 12] = x_1^{4m+d-1} x_2^2 - x_3 x_4^m \) form a regular sequence. \( \square \)

**Proposition 5.6.** Suppose \( a = 6m + 2 \). Then the complex,

\[
0 \longrightarrow A^2 \xrightarrow{\phi_2} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow A/p_4 \longrightarrow 0
\]

is a minimal graded free resolution of \( A/p_4 \), where the maps \( \phi_i \) are given by

\[
\phi_1 = (f_1, f_2, f_3, f_4, f_5),
\]

with \( f_1 = -x_3^2 + x_1^3 x_4, f_2 = -x_3^2 + x_1^2 x_4, f_3 = x_1^{4m+d+1} x_2 - x_3 x_4^m, f_4 = x_1^{4m+d+4} - x_2^2 x_4, f_5 = x_1^{4m+d+4} - x_4^m + 1 \),

\[
\phi_2 = \begin{pmatrix}
    x_1^2 x_4 - x_3^3 & x_3^4 & x_1^{4m+d-2} x_2 & x_1^{4m+d+1} x_2 & x_1^{4m+d-4} x_2 x_3 & x_1^{4m+d-1} x_3 - x_1^{4m+d-1} x_2^2 \\
    x_3^4 & x_3^4 & 0 & x_1^{4m+d+1} x_2 & x_1^{4m+d-4} x_2 x_3 & x_1^{4m+d-1} x_3 - x_1^{4m+d-1} x_2^2 \\
    -x_3^2 x_3 + x_3^2 & 0 & x_1^3 & 0 & x_1^{4m+d+1} x_2 & x_1^{4m+d-4} x_2 x_3 \\
    0 & x_1^3 & -x_3 x_3 & x_2 & -x_4 & 0 \\
    0 & -x_3 x_3 & 0 & -x_3 x_3 & 0 & -x_4 \\
    0 & 0 & x_1^3 & 0 & x_3 & x_3^2
\end{pmatrix}
\]

and

\[
\phi_3 = \begin{pmatrix}
    x_1^{4m+d-1} x_4 & x_3^m \\
    0 & -x_1^3 x_4 + x_3^3 \\
    0 & x_1^3 x_4 - x_3^3 \\
    -x_4 & -x_2 x_3 \\
    -x_3 & -x_1 x_2 \\
    x_3 & x_1 x_2
\end{pmatrix}.
\]

**Proof.** The proof is similar to that of Proposition 5.6. We note that the following minors \([1345 \mid 1235] = x_2 (x_1^2 x_4 - x_3^2)^2 \), \([2345 \mid 2345] = (x_4^m x_3 - x_1^{4m+d+1}) x_2 (x_1^2 x_3 + x_2^3) \) in \( I_4(\phi_2) \) form a regular sequence. We then show that the minors belonging to \( I_2(\phi_3) \), given by \([15 \mid 12] = -x_1^{4m+d+1} + x_2 x_4^m \), \([45 \mid 12] = x_1^5 x_4 - x_3^2 x_3 \) and \([16 \mid 12] = x_1^{4m+d+1} x_2^2 - x_3 x_4^m \), form a regular sequence. \( \square \)

**Proposition 5.7.** Suppose \( a = 6m + 3 \). Then the complex,

\[
0 \longrightarrow A^2 \xrightarrow{\phi_2} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow A/p_4 \longrightarrow 0
\]

is a minimal graded free resolution of \( A/p_4 \), where the maps \( \phi_i \) are given by

\[
\phi_1 = (f_1, f_2, f_3, f_4),
\]
with \( f_1 = -x_2^3 + x_3^3 x_3, f_2 = -x_2^2 + x_3^2 x_4, f_3 = x_1^{(4m+d+3)} - x_3 x_3^m, f_4 = x_1^{(4m+d-2)} x_2 - x_4^{m+1}; \)

\[
\phi_2 = \begin{pmatrix}
    x_1^{(4m+d)} & x_3 x_3^m & x_1^{(4m+d-2)} x_3 & x_1^{(4m+d-2)} x_2 - x_4^{m+1} \\
    x_1^{(4m+d+2)} & x_3 x_3^m & x_1^{(4m+d+3)} x_3 & x_1^{(4m+d+3)} x_2 - x_4^{m+1} \\
    -x_3^3 & x_4^m & -x_2 & -x_4 & 0 \\
    0 & -x_3 & -x_2 & -x_4 & 0 \\
    0 & x_1^2 & x_2 x_3 & -x_1^2 x_3 + x_2^3 & -x_1^2 x_4 + x_3^2
\end{pmatrix}
\]

and

\[
\phi_3 = \begin{pmatrix}
    x_1^{(4m+d+1)} \\
    -x_2^3 \\
    x_3 \\
    0 \\
    x_1^2
\end{pmatrix}.
\]

**Proof.** The proof is similar to that of Proposition 5.5. We observe that the minors \([134 | 124] = (x_1^2 x_4 - x_2^3)^2\) and \([234 | 123] = x_1^2 (-x_1^3 x_3 + x_3^3)^2\) belonging to \(I_3(\phi_2)\) form a regular sequence. We further observe that the minors \([15 | 12] = -x_1^{(4m+d+3)} + x_3 x_3^m, [13 | 12] = x_1^{m+1} - x_1^{4m+d+1} x_3\) and \([34 | 12] = x_1^3 x_3^2 - x_3^2 x_3\), belonging to \(I_2(\phi_3)\), form a regular sequence. \(\square\)

**Proposition 5.8.** Suppose \(a = 6m + 4\). Then the complex,

\[
0 \rightarrow A^3 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^4 \xrightarrow{\phi_1} A \rightarrow A/p_1 \rightarrow 0
\]

is minimal graded free resolution of \(R/4\), where the maps \(\phi_i\) are given by

\[
\phi_1 = (f_1, f_2, f_3, f_4),
\]

with \( f_1 = -x_2^3 + x_3 x_3^3, f_2 = -x_2^2 + x_3^2 x_4, f_3 = x_1^{(4m+d+5)} - x_2 x_3 x_4^m, f_4 = x_1^{(4m+d)} x_2 - x_4^{m+1}; \)

\[
\phi_2 = \begin{pmatrix}
    x_1^{(4m+d+2)} & x_3 x_3^m & x_1^{(4m+d+3)} x_3 & x_1^{(4m+d+3)} x_2 - x_4^{m+1} \\
    x_1^{(4m+d+2)} & x_3 x_3^m & x_1^{(4m+d+3)} x_3 & x_1^{(4m+d+3)} x_2 - x_4^{m+1} \\
    -x_3^3 & x_4^m & -x_2 & -x_4 & 0 \\
    0 & -x_3 & -x_2 & -x_4 & 0 \\
    0 & x_1^2 & x_2 x_3 & -x_1^2 x_3 + x_2^3 & -x_1^2 x_4 + x_3^2
\end{pmatrix}
\]

and

\[
\phi_3 = \begin{pmatrix}
    x_1^{(4m+d+1)} \\
    -x_2^3 \\
    x_3 \\
    0 \\
    x_1^2
\end{pmatrix}.
\]

**Proof.** The proof is similar to that of Proposition 5.5. We observe that the minors \([134 | 124] = x_2 (x_1^2 x_4 - x_2^3)^2\) and \([234 | 123] = x_1^2 (-x_1^3 x_3 + x_3^3)^2\) in \(I_3(\phi_2)\) form a regular sequence. We also observe that the minors \([124 |
123] = x^{(4m+d)}_1 x_2^4 - x_2^2 x_4^{m+1}, [246 | 123] = x_1^3 x_2^2 x_3 - x_3^2 and [256 | 123] = x_2^3 x_3 - x_3^5 x_4, belonging to $I_3(\phi_3)$, form a regular sequence. □

**Proposition 5.9.** Suppose $a = 6m + 5$. Then the complex,

$$0 \rightarrow A^3 \xrightarrow{\phi_1} A^6 \xrightarrow{\phi_2} A^4 \xrightarrow{\phi_3} A \rightarrow A/\mathcal{P}_4 \rightarrow 0$$

is a minimal graded free resolution of $A/\mathcal{P}_4$, where the maps $\phi_i$ are given by

$$\phi_1 = (f_1, f_2, f_3, f_4),$$

with $f_1 = -x_3^3 + x_2^2 x_3$, $f_2 = -x_3^2 + x_2^3 x_4$, $f_3 = x_1^{(4m+d+2)} x_2 - x_4^{m+1}$, $f_4 = x_1^{(4m+d+}\) - x_2^2 x_3 x_4^m$;

$$\phi_2 = \begin{pmatrix}
              x_1^2 x_4 - x_2^3 & x_3 x_4^m & x_2 x_4^{m+1} & x_1^{(4m+d+4)} x_2 - x_4^{m+1} & 0 & x_1^{(4m+d+2)} x_3 \\
              -x_1^2 x_3 + x_2^3 & x_1^2 x_4 & x_2 x_4^2 & 0 & x_1^{(4m+d+2)} x_2 - x_4^{m+1} & x_1^{(4m+d+5)} x_3 \\
              0 & x_1^2 & x_1 x_4 & -x_3 x_3 + x_2^3 & -x_1^2 x_4 + x_3^4 & 0 \\
              0 & -x_2 & -x_3 & 0 & x_1^3 & -x_2^2 \\
              0 & 0 & 0 & -x_2^2 & x_3 & 0
\end{pmatrix}$$

and

$$\phi_3 = \begin{pmatrix}
              x_4^m & 0 & x_1^{(4m+d+2)} \\
              x_4 & x_1 & 0 \\
              -x_2 & 0 & -x_4 \\
              x_4 & x_3 & 0 \\
              0 & x_1^3 & -x_2^2 \\
              0 & -x_2 & x_3
\end{pmatrix}.$$ 

**Proof.** The proof is similar to that of Proposition 5.5. We observe that the minors $[134 | 125] = x_2(x_1^2 x_4 - x_3^2)^2$ and $[234 | 123] = x_2^2(-x_1^2 x_3 + x_3^2)^2$ belonging to $I_3(\phi_2)$ form a regular sequence. We further observe that the minors $[123 | 123] = x_1^{(4m+d+2)} x_2 x_4 - x_4^{m+2}$, $[345 | 123] = x_3^2 x_3 - x_1^5 x_4$, $[456 | 123] = x_1^5 x_3 - x_1^3 x_2^3$, belonging to $I_3(\phi_3)$, form a regular sequence. □

**Lemma 5.10.** The curve $k[\Gamma_4]$ is a set-theoretic complete intersection if $a \equiv i (\mod a)$, where $i \in \{0, 3, 4, 5\}$.

**Proof.** If $a = 6m$, then $k[\Gamma_4]$ is a set-theoretic complete intersection by Corollary 5.2. For the other cases it follows from theorem 5.3. in [5]. □

6. APÉRY TABLE AND TANGENT CONE OF $k[\Gamma_4]$

Throughout this section we assume that the field $k$ is infinite.
Definition 2. Let \((R, m)\) be a Noetherian local ring and \(I \subset R\) be an ideal of \(R\). The fibre cone of \(I\) is the ring
\[
F(I) = \bigoplus_{n \geq 0} \frac{I^n}{mI^n} \cong R[It] \otimes R/m.
\]

Krull dimension of the ring \(F(I)\) is called the analytic spread of \(I\), denoted by \(\ell(I)\).

An ideal \(J \subset I\) is called a reduction of \(I\) if there exists an integer \(n > 0\) such that \(I^{n+1} = JJ^n\). A reduction \(J\) of \(I\) is a minimal reduction if \(J\) is minimal with respect to inclusion among reductions of \(I\). A minimal reduction always exists by [4]. It is well known that \(J\) is a minimal reduction of \(I\) if and only if \(J\) is minimally generated by \(\ell(I)\) number of elements, i.e, \(\mu(J) = \ell(I)\). If \(J\) is a minimal reduction of \(I\), then the least integer \(r\) such that \(I^{r+1} = JJ^r\), is the reduction number of \(I\) with respect to \(J\), denoted by \(r_J(I)\).

We are particularly interested in the semigroup ring \(k[[\Gamma_4]]\), which is the coordinate ring of the algebroid monomial curve defined by the numerical semigroup \(\Gamma_4\). Let \(a \geq 7\) and \(d > 0\) be two integers, such that \(gcd(a, d) = 1\). Let \(R = k[[t^a, t^{2a+d}, t^{3a+3d}, t^{4a+6d}]]\) and \(m\) is the maximal ideal \((t^a, t^{2a+d}, t^{3a+3d}, t^{4a+6d})\). Consider the principal ideal \(I = (t^a) \subset R\). The fibre cone of \(I\) is the ring
\[
F(I) = \bigoplus_{n \geq 0} \frac{I^n}{mI^n} \cong R[It] \otimes R/m.
\]

We note that here \(\ell(I) = 1\) and the tangent cone \(G_m = \bigoplus_{n \geq 0} \frac{m^n}{m^{n+1}}\) is an \(F(I)\)-algebra. Moreover \(F(I) \hookrightarrow G_m\) is a Noether normalisation (see [3] and [4]).

Suppose \(\Gamma\) be a numerical semigroup minimally generated by \(a_1 < \cdots < a_e\). Let \(M = \Gamma \setminus \{0\}\) and for a positive integer \(n\), we write \(nM := M + \cdots + M\) \((n\)-copies\). Let \(m\) be the maximal ideal of the ring \(k[[t^{a_1}, \ldots t^{a_e}]]\). Then \((n+1)M = a + nM\) for all \(n \geq r\) if and only if \(r = r_{(t^{a_1})}(m)\).

Let \(\text{Ap}(\Gamma, a_1) = \{0, \omega_1, \ldots, \omega_{a_1-1}\}\). Now for each \(n \geq 1\), let us define \(\text{Ap}(nM) = \{\omega_{n,0}, \ldots, \omega_{n,a_1-1}\}\) inductively. We define \(\omega_{1,0} = a_1\) and \(\omega_{1,i} = \omega_i\), for \(1 \leq i \leq a_1 - 1\). Then \(\text{Ap}(M) = \{a_1, \omega_1, \ldots, \omega_{a_1-1}\}\). Now we define \(\omega_{n+1,i} = \omega_{n,i}\), if \(\omega_{n,i} \in (n+1)M\), and \(\omega_{n+1,i} = \omega_{n,i} + a_1\), otherwise.

We note that \(\omega_{n+1,i} = \omega_{n,i} + a_1\) for all \(0 \leq i \leq a_1 - 1\) and \(n \leq r_{(t^{a_1})}(m)\). Then, the Apéry table \(\text{AT}(\Gamma, a_1)\) of \(\Gamma\) is a table of size \((r_{(t^{a_1})}(m) + 1) \times a_1\).
whose \((0, t)\) entry is \(\omega_t\), for \(0 \leq t \leq a_1 - 1\) (we take \(\omega_0 = 0\)), and the \((s, t)\) entry is \(\omega_{st}\), for \(1 \leq s \leq r_{(e)}(m)\) and \(0 \leq t \leq a_1 - 1\).

Next we want to describe Apery table of \(\Gamma_4\) and we need following Lemmas.

**Lemma 6.1.** Elements of the Apery set \(\text{Ap}(\Gamma_4, a)\) have unique expressions.

**Proof.** We have \((4\mu_i + 3\nu_t + 2\xi_i)a + id = \mu_i(4a + 6d) + \nu_t(3a + 3d) + \xi_i(2a + d)\), for \(i \leq i \leq a - 1\). Suppose for some \(i \leq i \leq a - 1\),

\[
(4\mu_i + 3\nu_t + 2\xi_i)a + id = c_1(2a + d) + c_2(3a + 3d) + c_3(4a + 6d).
\]

Then

\[
(6.1) \quad [(4c_3 + 3c_2 + 2c_1) - (4\mu_i + 3\nu_t + 2\xi_i)]a = [(6\mu_i + 3\nu_t + \xi_i) - (6c_3 + 3c_2 + c_1)]d.
\]

We have already shown in Theorem 7.2, that, \(4c_3 + 3c_2 + 2c_1 \geq 4\mu_i + 3\nu_t + 2\xi_i\) and \(6c_3 + 3c_2 + c_1 \geq 6\mu_i + 3\nu_t + \xi_i\). From equation 6.1 we have,

\[
(6.2) \quad 4c_3 + 3c_2 + 2c_1 = 4\mu_i + 3\nu_t + 2\xi_i
\]

\[
(6.3) \quad 6c_3 + 3c_2 + c_1 = 6\mu_i + 3\nu_t + \xi_i.
\]

We eliminate \(\mu_i\) and get,

\[
(6.4) \quad 3(c_2 - \nu_t) = 4(\xi_i - c_1).
\]

Let \(c_2 - \nu_t = 4k\) and \(\xi_i - c_1 = 3k\), for \(k \in \mathbb{Z}\). If \(k > 0\) then \(\xi_i = 3k + c_1\), which is impossible, since \(0 \leq \xi_i \leq 2\). If \(k < 0\) then \(\nu_t = c_2 - 4k\), again a contradiction, since \(0 \leq \nu_t \leq 1\). Therefore \(k = 0\), hence \((\mu_i, \nu_t, \xi_i) = (c_3, c_2, c_1)\).

**Lemma 6.2.** Let \((\mu_i, \nu_t, \xi_i)\) be the same as in Theorem 7.2 for \(1 \leq i \leq a - 1\), and \((\mu_0, \nu_0, \xi_0) = (0, 0, 0)\). Then \(\lfloor \frac{a}{6} \rfloor + 2 = \max\{\mu_i + \nu_t + \xi_i \mid 1 \leq i \leq a - 1\}\), where \(\lfloor \cdot \rfloor\) denotes the greatest integer function.

**Proof.** Let \(a = 6\mu + q\), \(0 \leq q \leq 5\). We note that \(i \leq 6\mu + 4\) for all \(1 \leq i \leq a - 1\). Therefore, \(\mu_i + \nu_t + \xi_i \leq \mu + 2\), for all \(1 \leq i \leq a - 1\). On the other hand, \(\mu_{a-q} + \nu_{a-q-1} + \xi_{a-q-1} = \mu + 2 = \lfloor \frac{a}{6} \rfloor + 2\).

**Corollary 6.3.** Let \(\text{AT}(\Gamma_4)\) denote the Apery table for \(\Gamma_4\). Then \(\text{AT}(\Gamma_4)\) will be of order \(\lfloor \frac{a}{6} \rfloor + 3 \times a\). Let \(\omega_{st}\) be the \((s, t)\) entry of the table \(\text{AT}(\Gamma_4)\). Then, \(\omega_{st} = (4\mu_t + 3\nu_t + 2\xi_i)a + td\), if \(0 \leq s \leq \mu_t + \nu_t + \xi_t\) and \(0 \leq t \leq a - 1\). On the other hand, \(\omega_{st} = (3\mu_t + 2\nu_t + \xi_t + s)a + td\), if \(\mu_t + \nu_t + \xi_t < s \leq \lfloor \frac{a}{6} \rfloor + 2\) and \(0 \leq t \leq a - 1\). Hence the reduction number of \(r_3(m)\) is \(\lfloor \frac{a}{6} \rfloor + 2\).
**Proof.** Proof follows from Lemmas [6.1](#) and [6.2](#). □

**Remark.** Minimal generating set of the defining ideal can be found abstractly in [7], when elements of Apery set has unique representation. But here we have written explicitly.

**Lemma 6.4.** Let \((\mu_i, \nu_i, \xi_i)\) be the same as in Lemma [6.2](#) for \(0 \leq i \leq a - 1\). Let \(a = 6\mu + q\), \(\mu \geq 1\), \(0 \leq q \leq 5\). Let \(t_k\) be the number of solutions of the equation \(\mu_i + \nu_i + \xi_i = k\), for \(0 \leq k \leq \mu + 2\). Then

\[
t_k = \begin{cases} 
1 & \text{if } k = 0, \\
3 & \text{if } k = 1, \\
\left\lfloor \frac{q}{2} \right\rfloor + 2 & \text{if } k = 2 \text{ and } \mu = 1, \\
5 & \text{if } k = 2 \text{ and } \mu \geq 2, \\
6 & \text{if } 3 \leq k \leq \mu, \\
\left\lfloor \frac{q}{2} \right\rfloor + 3 & \text{if } k = \mu + 1 \text{ and } \mu \geq 2, \\
1 & \text{if } k = \mu + 2 \text{ and } q \in \{0, 1, 2\}, \\
2 & \text{if } k = \mu + 2 \text{ and } q \in \{3, 4\}, \\
3 & \text{if } k = \mu + 2 \text{ and } q = 5.
\end{cases}
\]

**Proof.** For each of the following cases we write the set of solutions.

Case 1. If \(k = 0\) then \((\mu_i, \nu_i, \xi_i) = (0, 0, 0)\) is the only solution.

Case 2. If \(k = 1\) then \(\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}\) is the set of solutions.

Case 3. If \(k = 2\) and \(\mu \geq 2\), then \(\{(2, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 2)\}\) is the set of solutions.

Case 4. If \(3 \leq k \leq \mu\), then \(\{(k, 0, 0), (k - 1, 1, 0), (k - 1, 0, 1), (k - 2, 1, 1), (k - 2, 0, 2), (k - 3, 1, 2)\}\) is the set of solutions.

Case 5. If \(k = \mu + 1\) and \(\mu \geq 2\), then,

(i) if \(q \in \{0, 1\}\) then \(\{(\mu - 1, 1, 1), (\mu - 1, 0, 2), (\mu - 2, 1, 2)\}\) is the set of solutions;

(ii) if \(q \in \{2, 3\}\) then \(\{(\mu, 0, 1), (\mu - 1, 1, 1), (\mu - 1, 0, 2), (\mu - 2, 1, 2)\}\) is the set of solutions;

(iii) if \(q \in \{4, 5\}\) then \(\{(\mu, 0, 1), (\mu, 1, 0), (\mu - 1, 1, 1), (\mu - 1, 0, 2), (\mu - 2, 1, 2)\}\) is the set of solutions.

Case 6. If \(k = \mu + 2\), then,

(i) if \(q \in \{0, 1, 2\}\) then \(\{(\mu - 1, 1, 2)\}\) is the set of solutions;

(ii) if \(q \in \{3, 4\}\) then \(\{(\mu, 0, 2), (\mu - 1, 1, 2)\}\) is the set of solutions;

(iii) if \(q = 5\) then \(\{(\mu, 1, 1), (\mu, 0, 2), (\mu - 1, 1, 2)\}\) is the set of solutions.

For the case \(\mu = 1\) and \(k = 2\), it is easy to calculate (see example [6.9](#)).
We take some definitions from [4]. Let \( W = \{a_0, \ldots, a_n\} \) be a set of integers. We call it a ladder if \( a_0 \leq \ldots \leq a_n \). Given a ladder, we say that a subset \( L = \{a_i, \ldots, a_{i+k}\} \), with \( k \geq 1 \), is a landing of length \( k \) if \( a_{i-1} < a_i = \cdots = a_{i+k} < a_{i+k+1} \) (where \( a_{-1} = -\infty \) and \( a_{n+1} = \infty \)).

In this case, \( s(L) = i \) and \( e(L) = i + k \). A landing \( L \) is said to be a true landing if \( s(L) \geq 1 \). Given two landings \( L \) and \( L' \), we set \( L < L' \) if \( s(L) < s(L') \). Let \( p(W) + 1 \) be the number of landings and assume that \( L_0 < \cdots < L_{p(W)} \) are the distinct landings. Then we define the following numbers: \( s_j(W) = s(L_j) \), \( e_j(W) = e(L_j) \), for each \( 0 \leq j \leq p(W) \); \( c_j(W) = s_j(W) - e_{j-1}(W) \), for each \( 0 \leq j \leq p(W) \).

Suppose \( \Gamma \) be a numerical semigroup minimally generated by \( a_1 < \cdots < a_e \) and \( \text{m} \) be the maximal ideal of \( \mathbb{k}[[t^{a_1}, \ldots, t^{a_e}]] \). Let \( r = r_{(t^{a_1})}(\text{m}) \), \( M = \Gamma \setminus \{0\} \) and \( Ap(nM) = \{\omega_{n,0}, \ldots, \omega_{n,a_1-1}\} \) for \( 0 \leq n \leq r \). For every \( 1 \leq i \leq a_1 - 1 \), consider the ladder of the values \( W^i = \{\omega_{n,i}\}_{0 \leq n \leq r} \) and define the following integers:

(i) \( p_i = p(W^i) \)

(ii) \( d_i = e_{p_i}(W^i) \)

(iii) \( b_j^i = e_{j-1}(W^i) \) and \( c_j^i = c_j(W^i) \), for \( 1 \leq j \leq p_i \).

**Theorem 6.5.** (Cortadellas, Zarzuela.) With the above notations,

\[
G_m \cong F \bigoplus_{i=1}^{a_1-1} \left( F(-d_i) \bigoplus_{j=1}^{p_i} \left( \frac{F}{((t^{a_1})^*c_j^i)}F(-b_j^i) \right) \right),
\]

where \( G_m \) is the tangent cone of \( \Gamma \) and \( F = F((t^{a_1})) \) is the fiber cone.

**Proof.** See Theorem 2.3 in [4]. \( \square \)

**Corollary 6.6.** The tangent cone \( G_m \) of \( \Gamma_4 \) is a free \( F(\mathcal{I}) \)-module. Moreover

\[
G_m = \bigoplus_{k=0}^{\lfloor \frac{r+2}{2} \rfloor} (F(\mathcal{I})(-k))^{t_k},
\]

where \( t_k \)'s are given in Lemma 6.4.

**Proof.** Proof follows from corollary [6.1] and [6.5] \( \square \)

**Corollary 6.7.** The following properties hold good for the tangent cone \( G_m \) of \( \Gamma_4 \):

(i) \( G_m \) is Cohen-Macaulay;

(ii) \( G_m \) is not Gorenstein;

(iii) \( G_m \) is Buchsbaum.
Proof. (i) and (ii) easily follow from the fact that $G_m$ is a free $F(I)$-module (see section 4 in [4]). For proving (ii), we use Theorem 20 in [3]. Here we observe that if $G_m$ is Gorenstein then $m^n \cap (m^{n+2} : I) = m^{n+1}$, for $1 \leq n \leq r_3(m)$. Now $m^n \cap (m^{n+2} : I) = m^{n+1}$, for all $1 \leq n \leq r_3(m)$ if and only if $nM_4 \cap (n + 2)M_4 - a = (n + 1)M_4$, for all $1 \leq n \leq r_3(m)$, where $M_4 = \Gamma_4 \setminus \{0\}$. Which is impossible, since $(n + 1)a \notin nM_4$. □

**Corollary 6.8.** Let $HG_m(x)$ be the Hilbert series of $G_m$. Then

$$HG_m(x) = \left(\frac{\binom{n_a}{2} + 2}{1 - x}\right)_k x^k \right)/(1 - x).$$

Where $t_k$’s are given in Lemma 6.4.

*Proof.* Follows from Corollary 6.6. □

**Example 6.9.** Let us consider an example where $a = 11$ and $d = 24$. Hence $\Gamma_4 = \langle 11, 46, 105, 188 \rangle$. Here $d \equiv 2(\text{mod } a)$ and we have $Ap(\Gamma_4, a) = \{(4\mu_i + 3\nu_i + 2\xi_i)a + id \mid 1 \leq i \leq a - 1\} \cup \{0\}$, where $(\mu_i, \nu_i, \xi_i)$ are same as in 6.2. Let $\omega_i = (4\mu_i + 3\nu_i + 2\xi_i)a + id$ for $0 \leq i \leq a - 1$; the values are given in the table below:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| $\xi_i$ | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 |
| $\nu_i$ | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $\mu_i$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\omega_i$ | 0 | 46 | 92 | 105 | 151 | 197 | 188 | 234 | 280 | 293 | 339 |

Let $M_4 = \Gamma_4 \setminus \{0\}$, then, we have,

| $Ap(\Gamma_4)$ | 0 | 46 | 92 | 105 | 151 | 197 | 188 | 234 | 280 | 293 | 339 |
| $Ap(M_4)$ | 11 | 46 | 92 | 105 | 151 | 197 | 188 | 234 | 280 | 293 | 339 |
| $Ap(2M_4)$ | 22 | 57 | 92 | 116 | 151 | 197 | 199 | 234 | 280 | 293 | 339 |
| $Ap(3M_4)$ | 33 | 68 | 103 | 127 | 162 | 197 | 210 | 245 | 280 | 304 | 339 |

From the Apéry table we get $G_m = F \oplus F(-1)^3 \oplus F(-2)^4 \oplus F(-3)^3$, where $F = F(t^a)$, the fiber cone of $(t^a)$. Therefore we have the Hilbert series

$$HG_m(x) = \frac{1 + 3x + 4x^2 + 3x^3}{1 - x}.$$
7. APÉRY SET, APÉRY TABLE AND THE TANGENT CONE OF $k[\mathcal{S}_{n+2}]$

Let $a, d, r, h$ be positive integers with $\gcd(a, d) = \gcd(a, r) = 1$ and $d > hn(r - 1)$. Suppose $a_0 = a$ and $a_{k+1} = ha + r^kd$, for $0 \leq k \leq n$. Let $\mathcal{S}_{n+2} = \langle \{a_0, a_1, \ldots, a_{n+1}\} \rangle$ be the numerical semigroup with embedding dimension $n + 2$, such that $\{a_0, a_1, \ldots, a_{n+1}\}$ form a minimal system of generators for $\mathcal{S}_{n+2}$.

Definition 3. Let $m, r, n$ be positive integers and $m = \sum_{k=0}^{n} \alpha_k r^k$, where $0 \leq \alpha_i \leq r - 1$ for $i \in \{0, \ldots, n - 1\}$. Then the expression $m = \sum_{k=0}^{n} \alpha_k r^k$ is called the $r$-adic representation of $m$ upto order $n$.

Lemma 7.1. Let $m$ and $r$ be two positive integers and $m = \sum_{k=0}^{n} \alpha_k r^k$ be the $r$-adic representation of $m$ upto order $n$. Then for any expression $m = \sum_{k=0}^{n} \beta_k r^k$, we have

$$\sum_{k=0}^{n} \alpha_k \leq \sum_{k=0}^{n} \beta_k.$$ 

Moreover, $\sum_{k=0}^{n} \alpha_k < \sum_{k=0}^{n} \beta_k$, if $\sum_{k=0}^{n} \beta_k r^k$ is not an $r$-adic representation of $m$ upto order $n$.

Proof. We proceed by induction on $n$. If $n = 0$ then it follows trivially. At first we claim that $\beta_n \leq \alpha_n$. If not, then $\alpha_n + 1 < \beta_n$, hence $(\alpha_n + 1)r^n \leq \beta_n r^n$. Now

$$m = \sum_{k=0}^{n} \alpha_k r^k \leq \sum_{k=0}^{n} (r-1)r^k + \alpha_n r^n = (r^n - 1) + \alpha_n r^n < (\alpha_n + 1)r^n \leq \beta_n r^n,$$

which is a contradiction. Let $\alpha_n = t + \beta_n$, where $t \geq 0$. Again $\sum_{k=0}^{n-1} \alpha_k r^k + (t + \beta_n)r^n = \sum_{k=0}^{n} \beta_k r^k$, therefore $\sum_{k=0}^{n-2} \alpha_k r^k + (tr + \alpha_{n-1})r^{n-1} = \sum_{k=0}^{n-1} \beta_k r^k$.

By the induction hypothesis, $\sum_{k=0}^{n-1} \alpha_k + (tr + \alpha_{n-1}) \leq \sum_{k=0}^{n-1} \beta_k$. Hence, we
have,
\[
\sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n-1} \alpha_k + t + \beta_n \\
\leq \sum_{k=0}^{n-2} \alpha_k + (tr + \alpha_{n-1}) + \beta_n \\
\leq \sum_{k=0}^{n} \beta_k.
\]
\[\square\]

**Theorem 7.2.** Let for each \(i \in \{1, \ldots, a-1\}\), \(i = \sum_{k=0}^{n} a_{ki}r^k\) be the \(r\)-adic representation of \(i\) up to order \(n\). Suppose \(\ell_i = \sum_{k=0}^{n} a_{ki}\), for \(1 \leq i \leq a-1\).

Then \(\text{Ap}({\mathcal{G}}_{n+2}, a) = \{\ell_iha + id \mid 1 \leq i \leq a-1\} \cup \{0\}\).

**Proof.** Let \(T = \{\ell_iha + id \mid 1 \leq i \leq a-1\}\). At first we note that
\[\ell_iha + id = \sum_{k=0}^{n} a_{ki}(ha + r^kd), \quad 1 \leq i \leq a-1.\]
Therefore \(T \subset {\mathcal{G}}_{n+2}\). Suppose \(s \in \text{Ap}({\mathcal{G}}_{n+2}, a) \setminus \{0\}\), with \(s \equiv id\, (mod\, a)\).

Let \(s = \sum_{k=0}^{n} c_{k+1}(ha + r^kd)\), then \(\gcd(a, d) = 1\) forces that \(\sum_{k=0}^{n} c_{k+1}r^k \equiv i\, (mod\, a)\). Therefore \(\sum_{k=0}^{n} c_{k+1}r^k = i+pa\), and we have \(s = \left(\sum_{k=0}^{n} c_{k+1}\right)ha + (i + pa)d\).

If \(p > 0\) then
\[s = \left(\sum_{k=0}^{n} c_{k+1}\right)ha + (i + pa)d \\
\geq \left(\sum_{k=0}^{n} c_{k+1} + n(r - 1)\right)ha + id \\
> nh(r - 1)a + id \quad (as\ s > 0\ implies\ \sum_{k=0}^{n} c_{k+1} > 0) \\
\geq \ell_i + id.\]
This gives a contradiction as $s \in \text{Ap}(\mathfrak{S}_{n+2}, a)$ and $s \equiv \ell_i + id (\text{mod} a)$. If $p = 0$, then by Lemma 7.1, we have $\ell_i \leq \sum_{k=0}^{n} c_{k+1}$. Therefore $s \geq \ell_i + id$.

Now $s \in \text{Ap}(\mathfrak{S}_{n+2}, a)$ and $s \equiv \ell_i + id (\text{mod} a)$, therefore we have $s = \ell_i + id$, hence $s \in T$. □

**Lemma 7.3.** Every element of $\text{Ap}(\mathfrak{S}_{n+2})$ has a unique expression.

**Proof.** Let

$$\omega(i) = \ell_i ha + id = \sum_{k=0}^{n} c_{k+1}(ha + r^k d) = \left(\sum_{k=0}^{n} c_{k+1}\right)ha + \left(\sum_{k=0}^{n} c_{k+1}r^k\right)d,$$

for $1 \leq i \leq a - 1$, where $\ell_i$’s are the same as in Theorem 7.2. Therefore, $\sum_{k=0}^{n} c_{k+1}r^k \equiv i (\text{mod} a)$, hence $\sum_{k=0}^{n} c_{k+1}r^k = i + pa$ for some $p \geq 0$. If $p > 0$ then,

$$\omega(i) = \left(\sum_{k=0}^{n} c_{k+1}\right)ha + (i + pa)d$$

$$\geq \left(\sum_{k=0}^{n} c_{k+1} + n(r - 1)\right)ha + id$$

$$> nh(r - 1)a + id \quad \text{(as } \omega(i) > 0 \text{ implies } \sum_{k=0}^{n} c_{k+1} > 0)$$

$$\geq \ell_i + id.$$

This gives a contradiction as $\omega(i) \in \text{Ap}(\mathfrak{S}_{n+2}, a)$. Therefore $\sum_{k=0}^{n} c_{k+1}r^k = i$. If the expression $\sum_{k=0}^{n} c_{k+1}r^k$ is not an $r$-adic representation of $i$ upto order $n$, then $\sum_{k=0}^{n} c_{k+1} > \ell_i$ by lemma 7.1 which is a contradiction. Therefore $\sum_{k=0}^{n} c_{k+1}r^k$ is an $r$-adic representation of $i$ upto order $n$ and by the uniqueness of $r$-adic representation, $\omega(i)$ upto order $n$ has unique expression for $1 \leq i \leq a - 1$. □

**Theorem 7.4.** Let $r = \max\{\ell_i \mid 1 \leq i \leq a - 1\}$, where $\ell_0 = 0$ and $\ell_i$’s are the same as in Theorem 7.2 for $1 \leq i \leq a - 1$. Let $\text{AT}(\mathfrak{S}_{n+2}, a)$ denote the
Apéry table of \( S_{n+2} \). Then \( \text{AT}(S_{n+2}, a) \) will be of order \( r \times a \). Let \( \omega_{st} \) be the \((s,t)\) entry of the table \( \text{AT}(S_{n+2}, a) \). Then

\[
\omega_{st} = \begin{cases} 
\ell_t a + td & \text{if } 0 \leq s \leq \ell_t, \ 0 \leq t \leq a - 1; \\
\ell_t a + td + (s - \ell_t)a & \text{if } \ell_t < s \leq r, \ 0 \leq t \leq a - 1.
\end{cases}
\]

**Proof.** Follows from Lemma 7.3. \( \square \)

**Theorem 7.5.** Let \( k \) be an infinite field. The following properties hold for the tangent cone \( G_m \) of \( S_{n+2} \):

(i) \( G_m \) is Cohen-Macaulay,
(ii) \( G_m \) is not Gorenstein,
(iii) \( G_m \) is Buchsbaum.

**Proof.** Follows from Theorem 7.4 and [4]. \( \square \)

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