Experimental simulation of monogamy relation between contextuality and nonlocality in classical light

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Abstract: The Clauser-Horne-Shimony-Holt (CHSH) inequality and the Klyachko-Can-Binicioglu-Shumovski (KCBS) inequality present a tradeoff on the no-disturbance (ND) principle. Recently, the fundamental monogamy relation between contextuality and nonlocality in quantum theory has been demonstrated experimentally. Here we show that such a relation and tradeoff can also be simulated in classical optical systems. Using polarization, path and orbital angular momentum of the classical optical beam, in classical optical experiment we have observed the stringent monogamy relation between the two inequalities by implementing the projection measurement. Our results show the application prospect of the concepts developed recently in quantum information science to classical optical system and optical information processing.

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1. Introduction

The nonlocality and contextuality are two important aspects in quantum theory. Many investigations have been done about them in the past few decades. The nonlocality was usually discussed by the violation of the Bell inequality [1, 2] and the contextuality was analyzed by the violation of the noncontextuality inequality [3–7]. The violations of the two inequalities have been observed by a large number of experimental tests in quantum systems [8–27]. For a long time, the studies on the nonlocality and contextuality have been carried out independently, the people do not pay attention to their relation. Recently, the relation between nonlocal correlation and contextual correlation has been disclosed [28–33]. It has been proven that the no-disturbance (ND) principle imposes the monogamy relation between the contextuality and the nonlocality [34–38], and the quantum version of this monogamy relation is even more stringent [39, 40].

On the other hand, recent investigations have also shown that the violation of the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality can be simulated in the classical wave systems [41–53]. At the same time, the Klyachko-Can-Binciglu-Shumovski (KCBS) contextuality has also been simulated in the classical wave systems [54–56]. However, these mentioned studies on the CHSH Bell inequality and the KCBS contextuality have also been done independently. The monogamy relation between the nonlocality and the contextuality can exhibit simultaneously two aspects of important characteristics in quantum theory, especially more stringent monogamy relation. The question is whether it can be simulated in classical wave systems?

In this work, we show that the monogamy relation between the nonlocality and the contextuality, even more stringent relation, can be simulated in the classical optical systems. Using polarization, path and orbital angular momentum (OAM) of the classical optical beam, in classical optical experiments we can observe the tradeoff between the two inequalities by implementing the projection measurement.
2. Theoretical description on the monogamy relation in classical light

Based on the ND principle, Kurzyński et al proposed a fundamental monogamy relation between the violations of the KCBS inequality and the CHSH inequality [39]. For the KCBS inequality, there are five observables (\(A_1, \ldots, A_5\), which are pairwise compatible. Each observable has two possible measurement outcomes \(\pm 1\). In the noncontextuality case, for the any choice of the outcomes \(\pm 1\) for the observables, the KCBS inequality satisfies

\[
\kappa_i = A_i A_i + A_{i+1} A_{i+1} + A_{i+2} A_{i+2} + A_{i-1} A_{i-1} + A_{i-2} A_{i-2} \geq -3 ,
\]

where \(\kappa_i\) is the average value of KCBS operator. As the description in Ref [39], the KCBS scenario needs only a system (such as Alice’s system). Because these observables are pairwise compatible, every two observables exist a joint probability distribution. There is not a joint probability for the all five observables. According to this assumption to measure the five observables, the classical probabilities measured cannot be described by noncontextual theory, and the KCBS inequality can be violated. For the CHSH inequality, the observables \(A_{i-1}\) and \(A_{i+1}\) are any two incompatible observables. Similarly, \(B_1\) and \(B_2\) have possible outcomes \(\pm 1\). The CHSH Bell inequality is expressed as [39]

\[
\beta_{AB} = A_{i-1} B_1 + A_{i+1} B_2 + A_{i-1} B_1 - A_{i+1} B_2 \geq -2 ,
\]

where \(\beta_{AB}\) is the average value of CHSH operator. In order to demonstrate the violation of the CHSH inequality, two spatial separated systems (Alice’s and Bob’s systems) are needed, and the measurements in the two systems can impact mutually. In the Alice’s system the observables are \(A_{i-1}\) and \(A_{i+1}\), and in the Bob’s system the observables are \(B_1\) and \(B_2\). \(B_1\) and \(B_2\) are incompatible, but \(A_{i-1}\) or \(A_{i+1}\) is compatible with \(B_1\) or \(B_2\). The spatial separation and the mutual impact show the nonlocality, and the compatible relations between the observables show the contextuality. According to the two assumptions to measure the four observables \(A_{i-1}, A_{i+1}, B_1\) and \(B_2\), the classical probabilities, which cannot be described by local realistic and noncontextual theories, can be obtained. The CHSH inequality can be violated. The ND principle describes that, for any three observables \(W, X\) and \(Y\), such that \(W\) and \(X\) are compatible, \(W\) and \(Y\) are compatible. Under the ND principle, the sum of the KCBS inequality and the CHSH inequality is bounded.

Because the Bell inequality is also the noncontextuality inequality, according to Ref [39], we can sum the above two inequalities to produce a new noncontextuality inequality, which is expressed as

\[
G_i + G_2 \geq -5 ,
\]

where

\[
G_1 = A_{i+1} B_1 + A_{i+1} A_{i+2} + A_{i+2} A_{i+2} + A_{i-1} A_{i-1} + A_{i-2} B_1 ,
\]

\[
G_2 = A_{i+1} A_i + A_{i+1} A_i + A_{i+2} B_2 - A_{i-1} B_2 .
\]

We can see that the \(G_i\) and \(G_2\) have the form of the KCBS and CHSH expression, respectively. Because all \(A_i\) are compatible with the observables \(B_i\) and \(B_2\) (\(A_i, A_{i+1}\) and \(B_i\) or \(B_2\) can be jointly measured), under the ND principle \(G_i\) and \(G_2\) can be recovered by the joint probability distribution. Because of the existence of a joint probability distribution
for the $G_1$ and $G_2$, the lower bounds, $G_1 \geq -3$ and $G_2 \geq -2$, are satisfied [39]. In any ND theory, the sum for $G_1$ and $G_2$ is always bounded from below by $-5$ [39]. That is

$$\kappa_\lambda + \beta_{\lambda\beta} \geq -5.$$  

(6)

Thus there exists a monogamy relation between the two inequalities in the ND theory. The violation of one inequality forbids the violation of the other, and only one of them can be violated at one time.

The quantum theory satisfies the ND principle, but there is a more stringent monogamy relation. The regions spanned by $\kappa_\lambda$ and $\beta_{\lambda\beta}$ are two overlap regions [39]. In the point with $\kappa_\lambda = -2.92$ and $\beta_{\lambda\beta} = -2.08$, the quantum boundary touches the ND boundary, and the inequality becomes an equality. Such a stringent relation has been demonstrated experimentally using quantum states [40]. The KCBS inequality and the CHSH inequality are the bases of the monogamy relation, so we briefly describe the test of the two inequalities in photon experiment.

For the KCBS inequality, there are two measurement methods in the quantum case, joint measurement and sequence measurement. Because our classical light scheme corresponds to the joint measurement method in the quantum case, here we take the joint measurement as an example to describe the implementation process. The experiment process contains the input state preparation and the projection measurement. The correlated photon pairs are generated from the spontaneous parametric down conversion setup, one photon as the trigger, and the other produces the desired qutrit state [40]. In order to measure the compatible observables, their mutual eigenstates are established. When the input state projects onto the mutual eigenstates of the compatible observables, the probabilities of eigenvalues are obtained by the photon coincidence count. Possessing the probabilities of eigenvalues of the compatible observables, each correlation pair can be calculated and the value of KCBS inequality can be gotten further.

For the CHSH scenario, in quantum case it is that the two-qubit input state projects onto the operators corresponding to the four observables, which are two-two spatial separated observables. When the input state projects onto the eigenstates of the observables, these probabilities of eigenvalues are tested by the photon coincidence count. The eigenvalue multiplies by its probability, and we can obtain the observable when the products are summed. Possessing these observables, the result of CHSH operator can also be calculated [40].
Now we describe the above-mentioned phenomenon in classical optical system. For the KCBS inequality, the observables \((A_i')\) described in the classical system are shown in Fig. 1. These observables \((A_i')\) can be expanded by the direction unit vectors \(|\psi_i\rangle = |\tilde{e}_0\rangle + \beta_i |\tilde{e}_1\rangle + \gamma_i |\tilde{e}_2\rangle\), that is \(A_i = 2|\psi_i\rangle\langle\psi_i| - I\) \((i = 1, \ldots, 5)\), where \(I\) is the identity matrix. Here \(|\tilde{e}_0\rangle\), \(|\tilde{e}_1\rangle\) and \(|\tilde{e}_2\rangle\) are the classical trit’s (cetrit’s) bases corresponding to the quantum bases \(|0\rangle\), \(|1\rangle\) and \(|2\rangle\) in three space dimensions, the vectors in the classical optical system are expressed by a slightly modified version of bra-ket notation of quantum mechanics [42]. The direction unit vectors \(|\psi_i\rangle\) are at the five vertexes of pentagram [34], thus the coefficients can be denoted as \(i \alpha_i N = 1/\sqrt{1+\cos(\pi/5)}\) is a normalized constant. \(|\psi_i\rangle\) and \(|\psi_{i+1}\rangle\) are orthogonal, thus, \(A_i'\) and \(A_{i+1}'\) are compatible.

The input state can be also expanded by the basis vectors \(|\tilde{e}_0\rangle\), \(|\tilde{e}_1\rangle\) and \(|\tilde{e}_2\rangle\), which is written as \(|\chi\rangle = E_0|\tilde{e}_0\rangle + E_1|\tilde{e}_1\rangle + E_2|\tilde{e}_2\rangle\), where \(E_0\), \(E_1\) and \(E_2\) represent amplitudes of the classical optical field. In this case, we can calculate the correlations \(A_i'A_{i+1}'\) in the classical light, \(\overline{A_i'A_{i+1}'}\) denotes the average value of the correlation \(A_i'A_{i+1}'\) in the classical optical system. For instance, if \(|\chi\rangle = E_1|\tilde{e}_1\rangle\) is taken as the input state for the KCBS inequality (at the symmetry axis of the pentagram [34]), the correlation \(A_i'A_{i+1}' = 2|\psi_i\rangle\langle\psi_i| - 1\) \([2|\psi_{i+1}\rangle\langle\psi_{i+1}| - 1]\) \(= I - 2|\psi_i\rangle\langle\psi_i| - 2|\psi_{i+1}\rangle\langle\psi_{i+1}|\) , for the input state \(|\chi\rangle\)

\[
\overline{A_i'A_{i+1}'}(\chi) = \langle \chi | A_i'A_{i+1}' | \chi \rangle = E_1^2 E_2 (|\tilde{e}_1\rangle \langle \psi_{i+1}' | - E_2^2 (1 - 2 \gamma_i - 2 \gamma_{i+1}) = |\tilde{e}_2\rangle \langle \psi_i | (1 - \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{5}}) = |\tilde{e}_2\rangle \langle \psi_i | (1 - \frac{4}{\sqrt{5}}) .
\]

Because the KCBS inequality requires the specific input state, the amplitude \(E_2\) need to be normalized. In such a case, the value \(\overline{A_i'A_{i+1}'} = 1 - \frac{4}{\sqrt{5}}\) , thus

\[
\kappa_i' = \overline{A_i'A_{i+1}'}, \quad \kappa_i' = -3.944 < -3,
\]

where \(\kappa_i'\) is the corresponding form of \(\kappa_i\) in Eq. (1), and Eq. (7) is the corresponding expression of the violation of KCBS inequality in the classical light. Its maximum violation is \(5 - 4\sqrt{5} = -3.944\).

In Fig. 1(b), two incompatible observables \(B_i'\) and \(B_{i+1}'\) were chosen. They are compatible with the \(A_i'\) in Fig. 1(a). The compatible ratios are represented in Fig. 1. Meanwhile, \(A_i'\) and \(A_i'\) were chosen, and they are incompatible. Thus \(A_i'\) and \(A_i'\) \((B_i'\) and \(B_i'\)) are impossible to be jointly measured. However, it is possible to perform joint measurement for \(A_i'B_{i+1}'\), \(A_{i+1}'B_i'\), \(A_i'B_{i+1}\) and \(A_{i+1}'B_i\). For the \(B_i'\) and \(B_i'\), their outcomes are \(\pm 1\) similarly. The CHSH Bell inequality can be expressed as

\[
\beta_{12}' = \overline{A_i'B_{i+1}'} + \overline{A_{i+1}'B_i'} - \overline{A_i'B_i'} - \overline{A_{i+1}'B_{i+1}'} \geq -2,
\]

where \(\beta_{12}'\) is the corresponding form of \(\beta_{12}\) in Eq. (2), and Eq. (8) corresponds to Eq. (2). Its maximum violation is \(-2\sqrt{2}\).
Mapping into the classical light system, the compatible relation and the measurement relation are similar, so the ND principle exists similarly. It is similar to the quantum case above-mentioned, and the two inequalities satisfy the monogamy relation

\[ \kappa_A^c + \beta_{ab}^{ND} \geq -5. \]  

Here Eq. (9) corresponds to Eq. (6).

The more stringent monogamy relation between the KCBS inequality and the CHSH inequality also can be represented in classical light. Here the observables \( B'_1 \) and \( B'_2 \) in Fig. 1(b) are chosen as two Pauli operators \( Z' \) and \( X' \), respectively. Two overlap regions and the boundaries can be obtained similarly. The detailed descriptions are given in Appendix A. For the boundaries, the two classes of states (un-normalized) in classical light are

\[ |\chi^+_\varphi\rangle = E_1(\varphi) |\tilde{e}_1\rangle |\tilde{e}_1\rangle + E_2(\varphi) |\tilde{e}_2\rangle + |\tilde{e}_2\rangle, \]

\[ |\chi^-\varphi\rangle = E_1(\varphi) |\tilde{e}_1\rangle |\tilde{e}_1\rangle + E_2(\varphi) |\tilde{e}_2\rangle + |\tilde{e}_2\rangle, \]

where

\[ E_1(\varphi) = -0.05 + 0.15 \cot \varphi - 0.57 \tan \varphi, \]

\[ E_2(\varphi) = 0.72 + 0.32 \cot \varphi + 0.26 \tan \varphi. \]

Here \( \kappa'_A \) and \( \beta'_{ab} \) are the amplitudes of classical optical field, and Eq. (10) and Eq. (11) are the classical correlation states. Using the input state \( |\chi^+_\varphi\rangle \) and \( |\chi^-\varphi\rangle \), the two boundaries can be calculated after the states are normalized. These results are showed in Fig. 2. The boundaries for the two classes of states \( |\chi^+_\varphi\rangle \) and \( |\chi^-\varphi\rangle \) correspond to the blue solid curve and red dashed curve, respectively. At the two classes of states, the violations of the KCBS inequality and CHSH inequality show tradeoff in the classical optical system, one inequality is violated and the other is not violated. In the point \( \varphi = -0.28 \), \( \kappa'_A = -2.995 \) and \( \beta'_{ab} = -2.004 \), the boundary touches the ND boundary. That is, the more stringent monogamy relation can be simulated in the classical optical system, which is attributable to the correlation characteristics of classical optical fields [50]. In the following, we test experimentally the above monogamy relation in the classical light systems.

![Fig. 2. The regions spanned by the values of the KCBS and CHSH operator in classical optical system. The blue solid curve and the red dashed curve correspond to the input state |\chi^+_\varphi\rangle \) and |\chi^-\varphi\rangle, respectively. Every point in the regions is produced by an input state. The orange line corresponds to the ND bound. The wine dots denote the experimental results satisfied with the stringent relation, and they are compared with the theoretical results (black squares).](image-url)
3. Experimental demonstration of the monogamy relation in classical light

The experimental monogamy scenario in the classical optical system is designed by analogy of the photon experiment [40]. The experimental scheme is shown in Fig. 3. It consists of three parts: the state preparation, Alice’s measurement, and Bob’s measurement. Based on such a scheme, the cetrit-cebit system can be established, and the required input state can be obtained. In the Alice’s part, we use the joint measurement method to test the KCBS inequality, and then the CHSH inequality is tested using the cetrit-cebit system. The detailed process is described as follow.

In the state preparation stage, two laser beams from He-Ne lasers transmit through Grin lenses (GL) and then the laser beams with horizontal polarization appear. Here the center wavelengths of the laser beams are 633 nm. After they transmit through the vortex phase plate (VPP), the beams carry the OAM with an azimuthal phase structure of \( \exp(i\sigma) \), where \( \sigma \) is the azimuthal angle of polar coordinates, and \( l \) is a integer. Adjusting the VPPs, let the first beam carry the OAM with \( l = 1 \) and the second OAM beam with \( l = 2 \). After two half wave plates (HWPs) and two polarizing beam splitters (PBSs), the beams with different polarizations are combined in a beam-splitter (BS), then two new beams are obtained. The optical field state at the up output port is \( \frac{1}{\sqrt{2}}(|v_{1}\rangle\langle v_{1}| - |v_{2}\rangle\langle v_{2}|) \). After a HWP2 at 90°, the optical field state at the down output port is expressed as \( \frac{1}{\sqrt{2}}(|v_{1}\rangle\langle v_{1}| - |v_{2}\rangle\langle v_{2}|) \), where \( |A_{1}\rangle \) and \( |B_{1}\rangle \) denote the polarized classical states for fields, The superscripts 1, 2 denote the OAM number \( l \), and the meanings of superscripts are same below.

If the polarization information for one beam is measured, the polarization for the other beam can be determined, and vice versa. Such a feature can be described by the classical correlation state: \( \sin 2\theta |v_{1}\rangle\langle v_{2}| - \cos 2\theta |h_{1}\rangle\langle h_{2}| \) [50]. Then one of the correlated beams is transmitted to Alice to produce the cetrit, and the other is transmitted to Bob to produce the classical bit (cebit). After the beam transmitting to Alice passes through a PBS3, then the outputting vertical component passes through a HWP3 and a PBS4, we can obtain three...
beams of light. The cetrit can be constructed by the fields of the three beams, then the cetrit-cebit system can be established using this cetrit and the cebit in Bob’s part. Thus, the desired input state can be obtained: 
\[ 22 11 22 13 1 1 3 \sin 2\theta_1 \cos 2\theta_1 |\psi\rangle |\varphi\rangle. \]

As showed in Fig. 3, the horizontal polarization of path0 and path1, and the vertical polarization of path2 are encoded as \(|\hat{e}_h\rangle\), \(|\hat{e}_h\rangle\), and \(|\hat{e}_v\rangle\), respectively. Meanwhile, the path0 and path2 carry the OAM of \(l = 2\), the path1 carries the OAM of \(l = 1\). For the beam to Bob, the horizontal and vertical components are encoded as \(|\hat{e}_h\rangle\), \(|\hat{e}_v\rangle\), respectively. The horizontal component carries the OAM of \(l = 1\) and the vertical component carries the OAM of \(l = 2\). After they are encoded, this is
\[ \sin 2\theta_1 \sin 2\theta_1 |\hat{e}_h\rangle |\hat{e}_v\rangle - \cos 2\theta_1 |\hat{e}_h\rangle |\hat{e}_h\rangle - \sin 2\theta_1 \cos 2\theta_1 |\hat{e}_v\rangle |\hat{e}_v\rangle \] (12)
The Eq. (12) corresponds to the state \(|\chi\rangle\) in Eq. (10), where
\[ \theta_1 = \frac{1}{2} \arctan \frac{E_1(\varphi) + 1}{E_2(\varphi)(\cos 2\theta_1 - \sin 2\theta_1)}. \] (13)
\[ \theta_3 = \frac{1}{2} \arctan E_1(\varphi), \] (14)
where the \(E_1(\varphi)\) and \(E_2(\varphi)\) are converted into the setting angles \(\theta_1\) and \(\theta_3\) of HWPs for the experimental implementation. Through tuning the angles \(\theta_1\), \(\theta_1'\) and \(\theta_3\) of HWP1, HWP1’ and HWP3, the different input states can be gotten. In the case of \(|\chi\rangle\), the setting angle \(\theta_{11}\) of the HWP11 is \(0°\) in the Bob’s part. For the state \(|\chi\rangle\), we need to set up the angle \(\theta_{11}\) of HWP11 for \(45°\). The setting angles of HWPs for all input states are listed in Appendix B.

For the measurement of KCBS inequality, we adopt the method of joint measurement \([18, 35]\), which is analogy of the photon experiment. It is that the input state projects onto the mutual eigenstate that is common for the compatible observables \(A'_i\) and \(A'_{i+1}\). Here the mutual eigenstates of the compatible observables are established skillfully. We measure the light intensities to get the probabilities of the eigenvalues. With possessing the probabilities of the eigenvalues corresponding to the different eigenstates, each eigenvalue multiplies by its probability, thus the correlation \(A'_iA'_{i+1}\) is obtained when we sum the products. We establish the eigenstates at the three output port PD1, PD2 and PD3 in Alice’s part. The expressions of the optical fields at the output ports and the setting angles of HWPs are summarized in Appendix C. Taking \(A'_iA'_{i+1}\) as example, the output state at the port PD1 is the eigenstate with the eigenvalues \(A'_i=+1\) and \(A'_{i+1}=-1\); the output state at the port PD2 corresponds to the eigenstate with the eigenvalues \(A'_i=-1\) and \(A'_{i+1}=+1\); the output state at the port PD3 describes the eigenstate with the eigenvalues \(A'_i=-1\) and \(A'_{i+1}=-1\). Here \(A'_i=+1\) denotes that the eigenvalue of \(A'_i\) is \(+1\). The input state maps to these eigenstates, and the joint probabilities of the correlation \(A'_iA'_{i+1}\) can be obtained. Based on the classical correlation, the OAM components of the eigenstate in the Alice’s part correspond to the OAM components in the Bob’s part. Thus, by measuring the intensity of optical field of the OAM beam in the Alice’s part corresponding to the OAM in the Bob’s part, the probability of eigenstate can be tested.

There the eigenstates for KCBS operator in the Alice’s part are irrelevant with the measurement in the Bob’s part. Thereupon we set up the angle of HWP10 at \(0°\), which corresponds to the measurement for \(Z'_1\) (or \(B'_1\)) in the Bob’s part.
The OAM beams at the output ports D1 and D2 in the Bob’s part are the OAM beams of \( l = 1 \) and \( l = 2 \), respectively. They correspond to the OAM beams in the Alice’s part, so the probability is obtained when we test the optical intensity sum of the OAM beams of \( l = 1 \) and \( l = 2 \) of the eigenstate in the Alice’s part. Because the OAM beams of \( l = 1 \) and \( l = 2 \) are incoherent, the optical intensity sum for the sorting OAM beams equals to the intensity of beam propagating in one beam. Thus we just need to measure the entire optical intensity at one output port rather than to sort the different OAM beams to obtain the probabilities. The optical intensities are normalized, and the probabilities of these eigenvalues are obtained finally. Then the correlations \( \lambda'_+ \lambda_{-1} \) can be calculated. The calculation method is showed in Appendix D.

For the measurement of CHSH inequality, the case becomes complex. The measurement method is also adopted by analogy of the photon experiment [40]. We need to measure the correlation observables \( \lambda'_i \lambda'_j \) \((i = 1 \text{ or } 4, j = 1 \text{ or } 2)\), thus the observables \( \lambda'_i \) and \( \lambda'_j \) in the Alice’s part, \( \lambda'_i \) and \( \lambda'_j \) in the Bob’s part are considered. The observables \( \lambda'_i \) and \( \lambda'_j \) in the Bob’s part are the Pauli operators \( Z' \) and \( X' \), respectively. Thereupon, the eigenstates and eigenvalues of \( \lambda'_i \) and \( \lambda'_j \) can be obtained easily. Similarly, we also need to establish the eigenstates of \( \lambda'_i \) and \( \lambda'_j \) in the Alice’s part. The input state projects onto the eigenstates of these observables, and the probabilities of eigenvalues can be gotten. Because of the particularity of the observables \( \lambda'_i \) and \( \lambda'_j \), we follow the eigenstates of \( \lambda'_i \) or \( \lambda'_j \) in the Bob’s part to implement the corresponding measurement in the Alice’s part. The field intensities of these eigenstates are test and normalized, and the probabilities of the corresponding eigenvalues can be gotten. Next, the observables \( \lambda'_i \lambda'_j \) can be calculated by summing the products of eigenvalues and probabilities, and then the result of CHSH operator can be obtained. This process is analogy of those in the photon experiment. We measure concretely these observables as follow. The measurement of observables \( Z' \), \( X' \) can be implemented by a HWP10 and a PBS9 in Bob’s part. Similarly, adjusting the angle of the HWP10, the eigenstates corresponding to eigenvalues +1 and −1 of the two operators can be obtained at the two output ports D1 and D2. The setting angles of HWP10 are 0° and 22.5° for \( Z' \) and \( X' \), respectively. In order to measure \( \lambda'_i \lambda'_j \), the \( \lambda'_i \) in the Alice’s part is also considered. There the \( \lambda'_i \) and \( \lambda'_j \) are needed to test \( \lambda'_i \lambda'_j \) and \( \lambda'_i \lambda'_j \), and they can be test in the correlation pairs \( \lambda'_i \lambda'_j \) and \( \lambda'_i \lambda'_j \), respectively. But the \( \lambda'_i \) and \( \lambda'_j \) are not considered and only \( \lambda'_i \) and \( \lambda'_j \) are used. For the observable \( \lambda'_i \), the eigenstates at the three output ports PD1, PD2 and PD3 correspond to the eigenvalues +1, −1 and −1, respectively. For the observable \( \lambda'_j \), the eigenstates at the three output ports PD1, PD2 and PD3 correspond to the eigenvalues −1, +1 and −1, respectively. The input state maps to these eigenstates in the Alice’s part and the Bob’s part, the probabilities of eigenvalues can be measured. Because the input state is the classical correlation (entanglement) state that is constituted by the different polarizations and OAMs of the classical beams, the probability is obtained by measuring the intensity of the corresponding optical field. The detailed measurement methods are provided in Appendix D. In fact, if we use field correlation measurement method described in Ref [50], instead of the above described method, the same results can be obtained.

Under the adjustments of HWPs and the projection measurement, we obtain the experimental results for 11 different input states. The average values of the KCBS and CHSH operators for the different input states are shown in Fig. 2 and Table 1. The tradeoff of the two inequalities for different input states is observed clearly in the Table 1 and Fig. 2. The violation of one inequality forbids the violation of the other and they satisfy the bound imposed by the ND principle. The more stringent boundary between CHSH inequality and KCBS inequality is also observed distinctly in the classical optical experiment. At this point
\( \phi = -0.28 \), the measurement results \( \overline{\text{KCBS}} = -2.898 \pm 0.40 \) and \( \overline{\text{CHSH}} = -1.887 \pm 0.73 \), the stringent boundary touches the ND boundary except the experimental errors.

Table 1. The experimental results for the average values of KCBS and CHSH operator for the eleven input states. The numbers in the brackets indicate the statistical error.

| Input state | KCBS       | CHSH       |
|-------------|------------|------------|
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.70 \) | -3.445(35) | 1.101(50) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.62 \) | -3.806(83) | 0.795(39) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.52 \) | -3.830(10) | 0.409(44) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.40 \) | -3.734(64) | -0.712(62) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.28 \) | -2.898(40) | -1.887(73) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.24 \) | -2.481(58) | -2.150(34) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.17 \) | -1.653(57) | -2.543(58) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.14 \) | -1.341(73) | -2.593(26) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.10 \) | -0.841(56) | -2.494(55) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.06 \) | -0.683(55) | -2.318(41) |
| \( |\chi^+_{\phi}\rangle \) \( \phi = -0.70 \) | -3.475(24) | -1.249(41) |

4. Conclusion and discussion

Mapping the quantum theory to the classical optical system, the classical eigenstates and correlation states have been constructed by using the polarization, path and OAM of the classical optical beams. The classical input states project onto these eigenstates, and the monogamy relation between the contextuality and nonlocality has been simulated experimentally. Such a relation and tradeoff in the classical optical system are in agreement with the presentation in the quantum experiment.

The phenomenon originates from the correspondence between the Maxwell equations describing classical light fields and the Schrodinger equation. Such a correspondence leads to the similarity between coherent processes in quantum mechanics and classical optics. A typical example is the correspondence between the photonic band structure in photonic crystals and the electron band structure in solid state systems [57]. The photonic band structure describes the solutions of the Maxwell equations in periodic dielectric media, and the electron band structure exhibits the solutions of the Schrodinger equation in periodic potentials. Similarly, quantum nonlocality, contextuality and their monogamy relation have been discussed in the quantum scenario. In the present work, we have demonstrated that these phenomena can also be simulated in the classical optical systems. It is generally regarded as an important difference between the classical system and the quantum system. The former admits local realistic and noncontextual description, whereas the latter does not. However, our results have shown that the mathematical machinery developed in quantum information theory is of direct relevance to the discipline of classical optical coherence theory, which can enrich the coherence theory and information optics.

Appendix A: The more stringent monogamy relation in classical light

In this Appendix, we provide the theoretical demonstration about the more stringent monogamy relation in the classical light. In Fig. 1(a) of main body, it includes observables \( A'_i \) (i=1,…,5) and a classical trit (cetr) in the Alice’s part, which tries to violate the KCBS
inequality. In Fig. 1(b) of main body, it includes the observables $Z'$, $X'$ and a classical bit (cebit). The cebit-cebit system is formed, which tries to violate the CHSH inequality in Alice’s part and Bob’s part. Corresponding to the quantum theory [39], the KCBS operator is diagonal in the classical case, which can be expressed as

$$
\begin{pmatrix}
-5 + 2\sqrt{5} & 0 & 0 & 0 & 0 & 0 \\
0 & -5 + 2\sqrt{5} & 0 & 0 & 0 & 0 \\
0 & 0 & -5 + 2\sqrt{5} & 0 & 0 & 0 \\
0 & 0 & 0 & -5 + 2\sqrt{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 5 - 4\sqrt{5} & 0 \\
0 & 0 & 0 & 0 & 0 & 5 - 4\sqrt{5}
\end{pmatrix}.
$$

(15)

So the eigenvalues of the KCBS operator are degenerate

$$
\omega_1 = \omega_2 = \omega_3 = -5 + 2\sqrt{5}, \quad \omega_4 = \omega_5 = 5 - 4\sqrt{5}.
$$

(16)

Their corresponding eigenvectors are $|\omega_1\rangle = |\tilde{e}_0\rangle$, $|\omega_2\rangle = |\tilde{e}_1\rangle$, $|\omega_3\rangle = |\tilde{e}_2\rangle$, $|\omega_4\rangle = |\tilde{e}_3\rangle$, $|\omega_5\rangle = |\tilde{e}_4\rangle$ and $|\omega_6\rangle = |\tilde{e}_5\rangle$. For the CHSH operator, it can be written as the direct sum $M \oplus -M$, where

$$
M = \begin{pmatrix}
1 - \frac{\sqrt{2}}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & 1 - \sqrt{2} & 2 - \sqrt{1 + \frac{2}{\sqrt{2}}} & 2 - \sqrt{1 + \frac{2}{\sqrt{2}}} & 2 - \sqrt{1 + \frac{2}{\sqrt{2}}} & 2 - \sqrt{1 + \frac{2}{\sqrt{2}}}
\end{pmatrix}.
$$

(17)

However the $M$ is showed in the basis $\{|\tilde{e}_0\rangle, |\tilde{e}_1\rangle, |\tilde{e}_2\rangle\}$, and $-M$ is showed in the basis $\{|\tilde{e}_0\rangle, |\tilde{e}_1\rangle, |\tilde{e}_2\rangle\}$. The eigenvalues of $M$ are

$$
\omega_1 = -2.808, \quad \omega_2 = 2, \quad \omega_3 = 0.336.
$$

(18)

Their corresponding eigenvectors are

$$
|\omega_1\rangle = \frac{1}{\sqrt{2}} |\tilde{e}_0\rangle + \frac{1}{\sqrt{2}} |\tilde{e}_1\rangle - \frac{1}{\sqrt{2}} |\tilde{e}_2\rangle, \quad |\omega_2\rangle = \sqrt{1 - \frac{2}{\sqrt{2}}} |\tilde{e}_0\rangle + \sqrt{\frac{2}{\sqrt{2}}} |\tilde{e}_1\rangle
$$

and

$$
|\omega_3\rangle = \frac{1}{\sqrt{2}} |\tilde{e}_0\rangle - \frac{1}{\sqrt{2}} |\tilde{e}_1\rangle - \frac{1}{\sqrt{2}} |\tilde{e}_2\rangle. \quad \text{The KCBS operator is diagonal form and the CHSH operator can be separated to two symmetric parts, so the monogamy problem may be reduced to three real dimensions. Every input state can produce a point in the region, and the region can be expressed by two real parameters in the three-dimensional space. The KCBS operator can be also written as } N \oplus N, \text{ where}
$$

$$
N = \begin{pmatrix}
-5 + 2\sqrt{5} & 0 & 0 \\
0 & -5 + 2\sqrt{5} & 0 \\
0 & 0 & 5 - 4\sqrt{5}
\end{pmatrix}.
$$

(19)

The two $N$ can be showed in basis $\{|\tilde{e}_0\rangle, |\tilde{e}_1\rangle, |\tilde{e}_2\rangle\}$ and $\{|\tilde{e}_0\rangle, |\tilde{e}_1\rangle, |\tilde{e}_2\rangle\}$, respectively. The region corresponding to the operators $M$ and $N$ can be represented by using the parametrization

$$
|\eta, \varphi\rangle = \cos \eta |a\rangle + \sin \eta \cos \varphi |b\rangle + \sin \eta \sin \varphi |c\rangle,
$$

(20)
where these bases \(|a\rangle\), \(|b\rangle\) and \(|c\rangle\) are

\[
|a\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} 0.42 \\ 0.91 \\ 0 \end{pmatrix}, \quad |c\rangle = \begin{pmatrix} -0.91 \\ 0.42 \\ 0 \end{pmatrix}.
\] (21)

The basis vector \(|a\rangle\) is chosen to maximize \(N\). The basis vectors \(|b\rangle\) and \(|c\rangle\) are chosen to minimize and maximize \(M\), respectively, meanwhile to minimize \(N\). There \(|b\rangle\) and \(|c\rangle\) are also the eigenvectors of the upper-left submatrix

\[
\begin{pmatrix}
1 - \frac{1}{\sqrt{5}} & -\sqrt{2 - \frac{1}{\sqrt{5}}} \\
-\sqrt{2 - \frac{1}{\sqrt{5}}} & 1 - \sqrt{5}
\end{pmatrix}
\]

of \(M\). Thus we can obtain

\[
\overline{M_{\eta,\phi}} = 0.21\cos^2 \eta + (-0.34 - 1.38\cos 2\phi)\sin^2 \eta + \cos \eta \sin \eta (3.47\cos \phi - 1.94\sin \phi),
\] (22)

\[
\overline{N_{\eta}} = -\sqrt{5} + (5 - 3\sqrt{5})\cos 2\eta.
\] (23)

The \(\overline{M_{\eta,\phi}}\) and the other \(\overline{N_{\eta}}\) can be calculated by the same method. But their corresponding computational basis has to change to \(|\tilde{e}_0\tilde{e}_1\rangle, |\tilde{e}_1\tilde{e}_0\rangle, |\tilde{e}_0\tilde{e}_2\rangle\). Therefore the entire regions of the operators are obtained. We can obtain the boundaries of the two regions through minimization and maximization of \(\overline{M_{\eta,\phi}}\) (or \(\overline{M_{\eta,\phi}}\)). In this procedure the two parameters reduce to one, namely

\[
\tan \eta = \frac{\csc 2\phi(-1.94\cos \phi + 3.47\sin \phi)}{2.76}.
\] (24)

The boundaries and two regions are showed in the Fig. 2. In fact, the boundaries can be obtained directly by the two classes of normalized states, that is

\[
\overline{A\;'\;A_{i1}}'(\chi_{\phi}^{\prime})\left[A\;'\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\left[A_{i1}'\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]|\chi_{\phi}^{\prime}\rangle,
\] (25)

\[
\overline{A\;'B'}_{i1} = (\chi_{\phi}^{\prime})\left[A\;'\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\left[1 0 0 \right]_{i1}\otimes B_{i1}'|\chi_{\phi}^{\prime}\rangle.
\] (26)

For the term \(A\;'B'_{ij}\), \(i = 1 \text{ or } 4, j = 1 \text{ or } 2\). Using \(\kappa'_{i} = \overline{A\;'\;A_{i1}} + A_{i1}'\overline{A\;'\;} + A_{i1}'\overline{A\;'\;} + \overline{A\;'\;}A_{i1}' + \overline{A\;'\;}A_{i1}'\) and \(\beta'_{i1} = \overline{A\;'\;B'_{i1}} + A_{i1}'\overline{B'_{i1}} + A_{i1}'\overline{B'_{i1}} - \overline{A\;'\;}B'_{i1} - \overline{A\;'\;}B'_{i1}\), the values of \(\kappa'_{i}\) and \(\beta'_{i1}\) for the input state \(|\chi_{\phi}^{\prime}\rangle\) can be calculated. For instance, in the point \(\phi = -0.28\), \(\kappa'_{i} = -2.995\) and \(\beta'_{i1} = -2.004\), the boundary touches the ND boundary. For the other input state \(|\chi_{\phi}^{\prime}\rangle\), the method is similar. The results of \(\kappa'_{i}\) and \(\beta'_{i1}\) can be drawn as the curves, thus the boundaries can be obtained. The two regions bounded by the boundaries are the regions corresponding to the bases \(|\tilde{e}_0\tilde{e}_1\rangle, |\tilde{e}_1\tilde{e}_0\rangle, |\tilde{e}_0\tilde{e}_2\rangle\) and \(|\tilde{e}_0\tilde{e}_1\rangle, |\tilde{e}_1\tilde{e}_0\rangle, |\tilde{e}_0\tilde{e}_2\rangle\).

Appendix B: The state preparation and the setting angles of the HWPs for different input states

The classical beams possess correlation property, and they can be used to establish the classical correlation (entanglement) state. Two orthometric (horizontal and vertical)
polarization beams transmit through a beam splitter (BS), the classical correlation state can be obtained. There the Jones matrix of BS is \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). For the rotation angle \( \theta \), the Jones matrix of HWP is
\[
\begin{pmatrix}
\cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta
\end{pmatrix}.
\]
These formulas are used for the experimental design. Appropriately setting the angles of HWP1, HWP1' and HWP3, the required input states can be prepared. The angles are list in the follow Table 2. The setting angle \( \theta_1' \) of HWP1' are same with the angle \( \theta_1 \) of HWP1, they all are \( \theta_1 \). The HWP11 is for the preparation of the state \( |\chi_{\phi}^+\rangle \), and for the state \( |\chi_{\phi}^-\rangle \) it is not need (or setting up as 0°).

Table 2. The setting angles of HWPs for the eleven different input states in the state preparation stage.

| \( |\psi\rangle \) \( |\chi\rangle \) | \( \theta_1 \) | \( \theta_1' \) | \( \theta_3 \) | \( \theta_{11} \) |
|---|---|---|---|---|
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.70 \) | 41.65° | -7.07° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.62 \) | 42.57° | -4.18° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.52 \) | 44.65° | -0.41° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.40 \) | -40.88° | 4.65° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.28 \) | -33.29° | 11.09° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.24 \) | -30.01° | 13.81° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.17 \) | -23.75° | 19.78° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.14 \) | -21.06° | 22.98° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.10 \) | -17.85° | 28.05° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.06 \) | -15.17° | 34.15° | - |
| \( |\chi_{\phi}^-\rangle \) \( \varphi = -0.70 \) | 41.65° | -7.07° | 45° |

Appendix C: The expressions of the eigenstates in Alice’s part and Bob’s part

In order to test the KCBS inequality and the CHSH inequality, we need to measure these observables \( A_{\phi}^i, A'_{\phi}^i \). The input state projects onto the eigenstates of these observables, we measure the probabilities of these eigenvalues, and these observables can be obtained. According to the request described in the main text, the establishments of the eigenstates are an important assignment. Firstly, we describe the eigenstates for the test of the KCBS inequality in Alice’s part. Because we adopt the method of joint measurement to test the observable \( A_{\phi}^i, A'_{\phi}^i \), their eigenstates need to be established skillfully [18]. We assume that the input field amplitudes are all unit amplitudes, and at the three output ports PD1, PD2 and PD3, the expressions of the eigenstates are

\[
\sin 2\theta_i \sin 2\theta_i |\bar{e}_1^i\rangle - \cos 2\theta_i \sin 2\theta_i |\bar{e}_2^i\rangle - \cos 2\theta_i |\bar{e}_3^i\rangle - \cos 2\theta_i \cos 2\theta_i |\bar{e}_4^i\rangle,
\]

(27)

\[
(\cos 2\theta_i \cos 2\theta_i - \sin 2\theta_i \cos 2\theta_i |\bar{e}_1^i\rangle |\bar{e}_2^i\rangle + (\sin 2\theta_i \cos 2\theta_i + \cos 2\theta_i \sin 2\theta_i |\bar{e}_1^i\rangle |\bar{e}_3^i\rangle + \sin 2\theta_i \cos 2\theta_i |\bar{e}_1^i\rangle |\bar{e}_4^i\rangle + \sin 2\theta_i |\bar{e}_2^i\rangle |\bar{e}_3^i\rangle)
\]

(28)

\[
(\cos 2\theta_i \sin 2\theta_i + \sin 2\theta_i \cos 2\theta_i |\bar{e}_1^i\rangle |\bar{e}_2^i\rangle + (\sin 2\theta_i \sin 2\theta_i - \cos 2\theta_i \cos 2\theta_i |\bar{e}_1^i\rangle |\bar{e}_3^i\rangle) - \sin 2\theta_i \cos 2\theta_i |\bar{e}_4^i\rangle).
\]

(29)
Here the OAM number $l$ is marked as superscript. Properly setting the angles of the HWP6, 7 and 8, the requisite eigenstates can be established at the output ports PD1, PD2 and PD3. For example, for the observable $A_3 A_4$, the angles of the HWP6, 7 and 8 are set up as 81°, 24° and 122°, respectively. Thus the eigenstates at the output ports PD1, PD2 and PD3 are 0.2296 $|\phi_{e}^{+}\rangle + 0.7068 |\phi_{e}^{-}\rangle + 0.6691 |\phi_{0}\rangle$, 0.6028 $|\phi_{0}\rangle + 0.4365 |\phi_{1}\rangle - 0.6679 |\phi_{2}\rangle$ and 0.7642 $|\phi_{0}\rangle - 0.5567 |\phi_{e}^{+}\rangle + 0.3258 |\phi_{2}\rangle$, respectively. The eigenvalues at the output port PD1 are $A_3 = +1$ and $A_4 = -1$, they are $A_3 = -1$ and $A_4 = +1$ at the output port PD2, and $A_3 = -1$ and $A_4 = -1$ at the output port PD3. The input state projects onto these eigenstates, and we measure the corresponding optical intensity to obtain the probabilities. After the input state projects on the eigenstates, the optical fields of the three output ports PD1, PD2 and PD3 are 

$$0.2296 \sin 2\theta_1 \sin 2\theta_1 |\phi_{e}^{+}\rangle - 0.7068 \cos 2\theta_1 |\phi_{e}^{-}\rangle - 0.6691 \sin 2\theta_1 \cos 2\theta_1 |\phi_{0}\rangle,$$  

$$0.6028 \sin 2\theta_1 \sin 2\theta_1 |\phi_{0}\rangle - 0.4365 \cos 2\theta_1 |\phi_{1}\rangle + 0.6679 \sin 2\theta_1 \cos 2\theta_1 |\phi_{2}\rangle,$$  

$$0.7642 \sin 2\theta_1 \sin 2\theta_1 |\phi_{0}\rangle + 0.5567 \cos 2\theta_1 |\phi_{e}^{+}\rangle - 0.3258 \sin 2\theta_1 \cos 2\theta_1 |\phi_{2}\rangle.$$  

The expressions of eigenstates for the other terms $A_i A'_j$ are similar, and the eigenstates all are used for the measurement of probabilities. The setting angles of HWP6, 7 and 8 for the five pairs of observables are listed in the Table 3. The angles of HWP4 and HWP5 are all set as 45° to transform polarization, and the angle of HWP9 is set as 45° for path-length compensation. The HWP4, HWP5 and HWP9 all are same in the entire experiment process.

Table 3. The setting angles of HWPs for the measurement of KCBS inequality.

| $A_i A'_j$ | HWP6 | HWP7 | HWP8 |
|------------|------|------|------|
| $A_3 A_4$  | 117° | 24°  | 32°  |
| $A_3 A'_4$ | 81°  | 24°  | 58°  |
| $A_3 A'_4$ | 45°  | 24°  | 122° |
| $A_3 A'_4$ | 45°  | 24°  | 148° |

In the Bob’s part, the observables $B'_1$ and $B'_2$ correspond to the Pauli operators $Z'$ and $X'$, respectively. In order to obtain the eigenstates of $Z'$ and $X'$, the angles of HWP10 are set as 0°, 22.5°, respectively. For the case $|\chi_{e}^{+}\rangle$, at the two output ports D1 and D2, the eigenstates of the operator $Z'$ are $|\phi_{e}^{+}\rangle$ and $-|\phi_{e}^{-}\rangle$, respectively, and the eigenstates of the operator $X'$ are $\frac{\sqrt{2}}{2}[|\phi_{e}^{+}\rangle + |\phi_{0}\rangle]$ and $\frac{\sqrt{2}}{2}[|\phi_{e}^{-}\rangle + |\phi_{0}\rangle]$, respectively. After projection, for the Pauli operator $Z'$, the expressions of the optical fields at the output ports D1 and D2 are $-\cos 2\theta_1 |\phi_{e}^{+}\rangle$ and $-\sin 2\theta_1 |\phi_{e}^{-}\rangle$, respectively. For the Pauli operator $X'$, the expressions of the optical fields at the output ports D1 and D2 are $\frac{\sqrt{2}}{2}[\sin 2\theta_1 |\phi_{e}^{+}\rangle - \cos 2\theta_1 |\phi_{0}\rangle]$ and $\frac{\sqrt{2}}{2}[\sin 2\theta_1 |\phi_{e}^{-}\rangle + \cos 2\theta_1 |\phi_{0}\rangle]$, respectively. The results of output ports D1 and D2 indicate the measurement of probabilities in the Alice’s part. For the case $|\chi_{e}^{-}\rangle$, the slight change is taken, and the eigenstates correspond to the measurement of probability.

Appendix D: The calculation method for the KCBS inequality and the CHSH inequality

For the KCBS inequality, the probabilities of eigenvalues can be obtained when we measure the intensities of the classical optical fields at the three output ports PD1, PD2 and PD3 in the Alice’s part. Of course, these intensities need to be normalized. Take $A_i A'_j$ as an example, the correlation pair possess eigenvalues $\pm 1$, the probabilities of eigenvalues are
\[ P(A'_1=+1, A'_2=-1), P(A'_1=-1, A'_2=+1) \] and \[ P(A'_1=-1, A'_2=-1) \] at the three output ports PD1, PD2 and PD3, respectively. Thus the result of \( A'_1A'_2 \) is
\[
\overline{A'_1A'_2} = -P(A'_1=+1, A'_2=-1) - P(A'_1=-1, A'_2=+1) + P(A'_1=-1, A'_2=-1).
\]
(33)

For the other correlation pairs \( A'_iA'_i, \) the method is similar. Every pairs \( A'_iA'_i' \) are calculated, thus the value \( \kappa'_i = A'_iA'_i + A'_iA'_i + A'_iA'_i + A'_iA'_i \) can be obtained.

For the CHSH inequality, \( A'B'_i \) need to be measured. There is \( A'_i \) in the Alice’s part and \( B'_i \) in the Bob’s part. The \( A'_i \) possesses the eigenstates corresponding to eigenvalue ±1, and the \( B'_i \) is also same. For the measurement of \( A'_iB'_i' \) and \( A'_iB'_i' \), in the Bob’s part the operator \( B'_i \) is the Pauli operator \( Z' \), so the HWP10 is set up as 0°. Thus the beam transmits through the PBS9, it possess the OAM of \( l=1 \) at the horizontal polarization output port D1, meanwhile it is the eigenstate with \( Z'=-1 \). The beam possesses the OAM of \( l=2 \) and the eigenstate corresponds to \( Z'=+1 \) at the vertical polarization output port D2. The eigenstate of \( A'_i \) or \( A'_i' \) at every output port in the Alice’s part carries the OAM of \( l=1 \) and \( l=2 \). When the OAMs in Alice’s and Bob’s part are correspondence, we measure the intensity of the corresponding OAM beam in Alice’s part to obtain the probability \( P(A'_1=\pm1, B'_1=\pm1) \). For instance, the eigenstate of \( A'_i \) at the output port PD1 in the Alice’s part possess the eigenstate of the OAM of \( l=1 \) and \( l=2 \), and the eigenstate in the Bob’s part possesses the OAM of \( l=1 \) or \( l=2 \) that correspond to the eigenvalue +1 or -1. The intensity of field with the OAM of \( l=1 \) at the output port PD1 in the Alice’s part corresponding the OAM of \( l=1 \) (eigenvalue \( Z'=+1 \)) in the Bob’s part denotes the probability \( P_i(A'_1=+1, B'_1=+1) \). Using an interferometer with two Dove prisms whose relative rotating angle is 45°, the OAM beam of \( l=1 \) and \( l=2 \) can be sorted at the two output ports of interferometer. The probability \( P_i(A'_1=+1, B'_1=-1) \) are obtained when we measure the intensity of field with the OAM \( l=1 \) in the Alice’s part. The other probabilities \( P_i(A'_1=-1, B'_1=+1) \), \( P_i(A'_1=-1, B'_1=-1) \), \( P_i(A'_1=+1, B'_1=-1) \) and \( P_i(A'_1=-1, B'_1=+1) \) are similar. Of course, the intensities of optical fields need to be normalized, and the probabilities can be obtained finally. The measurement method for the other term \( A'_iB'_i' \) is similar.

For the measurement of \( A'_iB'_i' \), the method is slightly different. Because it is the Pauli operator \( X' \) in the Bob’s part and the setting angle of HWP10 is 22.5°. Corresponding to the eigenstate of \( X'=+1 \) at the horizontal polarization output port D1 in the Bob’s part, the half of the field intensity of the eigenstate with the OAM of \( l=1 \) and \( l=2 \) in the Alice’s part is the probability \( P(A'_1=\pm1, B'_1=\pm1) \), where the observable \( A'_i \) possesses similarly different eigenstate with \( A'_1=+1 \), -1 and -1. Thus the probabilities \( P_i(A'_1=+1, B'_1=+1) \), \( P_i(A'_1=-1, B'_1=+1) \) and \( P_i(A'_1=-1, B'_1=-1) \) can be obtained. For the eigenstate of \( X'=+1 \) at the vertical polarization output port D2 in the Bob’s part, the half of the field intensity, which the component \( |\vec{\hat{e}}_0\rangle \) (OAM \( l=2 \)), \( |\vec{\hat{e}}_1\rangle \) (OAM \( l=1 \)) and \( |\vec{\hat{e}}_2\rangle \) (OAM \( l=2 \)) interfere completely in the Alice’s part, is the probability \( P(A'_1=\pm1, B'_1=\pm1) \). Similarly the observable \( A'_i \) has eigenstates with \( A'_1=+1 \), -1 and -1. Here the VPPs are removed to allow the beam interfere completely. Through the aforementioned measurement, the field intensities of these eigenstates are gotten. The field intensity is normalized and the probability of eigenvalues is obtained. The probabilities are \( P_i(A'_1=+1, B'_1=+1) \), \( P_i(A'_1=-1, B'_1=-1) \), and \( P_i(A'_1=-1, B'_1=+1) \).
Possessing the probabilities \( P_i(A'_i=\pm 1, B'_i=\pm 1) \), \( P_j(A'_j=\pm 1, B'_j=\pm 1) \), \( P_i(A'_i=\pm 1, B'_i=\mp 1) \), \( P_j(A'_j=\pm 1, B'_j=\mp 1) \), and \( P_j(A'_j=\mp 1, B'_j=\pm 1) \), \( \overline{A'B'_j} \) can be calculated.

The \( \overline{A'B'_j} \) can be expressed as

\[
\overline{A'B'_j} = P_i(A'_i=\pm 1, B'_i=\mp 1) - P_j(A'_j=\pm 1, B'_j=\mp 1) - P_i(A'_i=\mp 1, B'_i=\pm 1) - P_j(A'_j=\mp 1, B'_j=\pm 1).
\]

For all terms \( A'B' \), \( A'B' \), \( A'B' \), and \( A'B' \), the calculations are the same. Just the eigenvalues of \( A_4 \) are \(-1\), \(+1\) and \(-1\) at the three output port PD1, PD2 and PD3, and there is slight change for the expression. For the input state \( |\chi_\varphi\rangle \), the output ports are turnover to obtain the measurement results of \( \overline{A'B'_j} \). Using the equation \( \beta'_{ia} = \overline{A'B'_i} + \overline{A'B'_j} + A_i A_j - A'_i A'_j \), the \( \beta'_{ia} \) can be calculated.

Taking the input state \( |\chi_\varphi\rangle \) (\( \varphi=-0.7 \)) as an example, the results of \( \kappa'_4 \) and \( \beta'_{ia} \) are list in the Table 4. The detailed data for the others input states are similar.

**Table 4. The experimental results and the theory values of the KCBS and CHSH operator for the input state \( |\chi_\varphi\rangle \) (\( \varphi=-0.7 \)).**

| Terms | \( \overline{A'_1A'_1} \) | \( \overline{A'_1A'_2} \) | \( \overline{A'_4A'_4} \) | \( \overline{A'_2B'_1} \) | \( \overline{A'_2B'_2} \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Theory | -0.494 | -0.986 | -0.494 | 0.482 | 0.102 |
| Experiment | -0.488(16) | -0.884(1) | -0.502(4) | 0.474(11) | 0.089(8) |

| Terms | \( A'_3A'_4 \) | \( A'_3A'_j \) | \( A'_iB'_j \) | \( A'_iB'_j \) |
|-------|-----------------|-----------------|-----------------|-----------------|
| Theory | -0.861 | -0.861 | 0.482 | -0.102 |
| Experiment | -0.787(11) | -0.784(3) | 0.488(23) | -0.050(8) |

| Terms | \( KCBS \) | \( CHSH \) |
|-------|-----------------|-----------------|
| Theory | -3.696 | 1.168 |
| Experiment | -3.445(35) | 1.101(50) |

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