Standard decomposition of expansive ergodically supported dynamics

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Abstract

In this work we introduce the notion of weak quasigroups, that are quasigroup operations defined almost everywhere on some set. Then we prove that the topological entropy and the ergodic period of an invertible expansive ergodically-supported dynamical system with the shadowing property \((X, T)\) establishes a sufficient criterion for the existence of quasigroup operations defined almost everywhere outside of universally null sets and for what \(T\) is an automorphism. Furthermore, we find out a decomposition of the dynamics of \(T\) in terms of \(T\)-invariant weak topological subquasigroups.

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1 Introduction

The problem of characterizing the dynamical behavior of maps which are endomorphisms for compact groups have been widely studied in the last years. One of the first works on this subject is due to R.

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Bowen [2], who studied the entropy of such maps and showed that the Haar measure is the maximum entropy measure for certain class of algebraic dynamical systems. Latter, in [3], D. Lind proved that ergodic maps which are automorphisms for compact Abelian groups are always conjugated to some full shift. For the case where \((X,+)\) is any topological group with \(X\) being a zero-dimensional space, B. Kitchens [4] proved that any expansive endomorphism \(T: X \to X\) can be represented as a shift map defined on the cartesian product of a full shift with a finite set. In [5], this result was extended for expansive maps which are endomorphisms for certain class of zero-dimensional quasigroups.

We say \((X,T)\) is a topological dynamical system if \(X\) is a compact metric space and \(T: X \to X\) is a continuous onto map. The topological entropy of \((X,T)\) will be denoted by \(h(T)\). We say \((X,T)\) and \((Y,S)\) are conjugated if there exists an invertible map \(f: X \to Y\) such that \(f \circ T = S \circ f\). In the case when \(f\) is a homeomorphism we say \((X,T)\) and \((Y,S)\) are topologically conjugated.

Given a finite alphabet \(\mathcal{A}\), define \(\mathcal{A}^S := \{(x_i)_{i \in S} : x_i \in \mathcal{A}, \forall i \in S\}\), with \(S = \mathbb{Z}\) or \(S = \mathbb{N}\). We consider in \(\mathcal{A}^S\) the product topology which is generated by the clopen subsets called cylinders. Let \(\sigma_{\mathcal{A}^S}: \mathcal{A}^S \to \mathcal{A}^S\) be the shift map defined by \(\sigma_{\mathcal{A}^S}((x_i)_{i \in S}) = (x_{i+1})_{i \in S}\). Therefore, a symbolic dynamical system is a topological dynamical system \((\Lambda, \sigma_{\Lambda})\) where \(\Lambda \subseteq \mathcal{A}^S\) is a closed subset such that \(\sigma_{\mathcal{A}^S}(\Lambda) = \Lambda\), and \(\sigma_{\Lambda}\) is the restriction of \(\sigma_{\mathcal{A}^S}\) to \(\Lambda\) (in this case we refer \(\Lambda\) as a shift space). A special type of shift spaces are the Markov shifts, which are those symbolic dynamical systems that can be constructed from walks on finite directed graphs (see [6] for more details).

A topological dynamical system \((X,T)\), is said to be expansive if there exists a family \(\{U_i\}_{1 \leq i \leq k}\) of open sets, such that \(\bigcup_{1 \leq i \leq k} U_i = X\) and for \(x, y \in X, x \neq y\), there exists \(1 \leq i \leq k\) and \(n \in S\) (with \(S = \mathbb{Z}\) if \(T\) is invertible and \(S = \mathbb{N}\) otherwise) for that \(T^n(x) \in U_i\) and \(T^n(y) \notin U_i\). When \((X,T)\) is expansive we can define its symbolic representation as the shift space \((\Lambda, \sigma_{\Lambda})\), where \(\Lambda \subseteq \{1, \ldots, k\}^S\) is such that \((q_i)_{i \in S} \in \Lambda\) if and only if there exists \(x_0 \in X\) such that for all \(i \in S\) we have \(T^i(x_0) \in U_{q_i}\).

We say \((X,T)\) has the shadowing property if for any \(\epsilon > 0\) there exists \(\delta > 0\) such that if \((y_n)_{n \in S} \subset X\) is a sequence which verifies
Given \((X, T)\), define \(X^T := \{(x_i)_{i \in \mathbb{Z}} : x_{i+1} = T(x_i), \forall i \in \mathbb{Z}\}\). It is well known that there exists a standard metric on \(X^T\) which makes it compact, and for which the shift map \(\sigma_T : X^T \to X^T\) is continuous (see [3]). The topological dynamical system \((X^T, \sigma_T)\) is called the inverse limit system of \((X, T)\). The projection \(p : X^T \to X\) which takes the sequence \((x_i)_{i \in \mathbb{Z}}\) to \(x_0\) is continuous and commutes with the maps \(\sigma_T\) and \(T\), that is, \(p \circ \sigma_T = T \circ p\). Note that, if \(T\) is invertible, then \(p\) is also invertible and therefore \(p^{-1}\) is also continuous. In fact, if \(T\) is invertible, then for each \(a \in X\) the unique sequence in \(X^T\) with \(x_0 = a\) is \((T^i(a))_{i \in \mathbb{Z}}\), which means \(p\) is invertible and, since \(p\) is a continuous function between compact spaces, it implies that \(p^{-1}\) is also continuous. Therefore, in such a case, \(p\) is a topological conjugation of \((X, T)\) and \((X^T, \sigma_T)\).

A probability measure on \(X\) is said to be an ergodically supported measure if it is an ergodic measure which assigns positive measure for any nonempty open subset of \(X\). Thus, we say \((X, T)\) is ergodically supported if there exists an ergodically supported measure for it. A set \(E \subset X\) which has zero measure for any ergodically supported measure is said a universally null set.

If for all \(n \geq 1\) we have that \((X, T^n)\) is ergodically supported, then we say \((X, T)\) is ergodically aperiodic. On the other hand, we say \((X, T)\) has ergodic period \(B\) if \((X, T)\) is ergodically supported and there exists a finite family \(\{C_i\}_{0 \leq i \leq B-1}\) of closed sets, such that: \(X = \bigcup_{i=0}^{B-1} C_i\); \(C_i \cap C_j\) is universally null set for any \(i \neq j\); \(T(C_i) = C_{i+1}\) (mod \(B\)); and \((C_i, T^B)\) is ergodically aperiodic for all \(i\).

An important concept in dynamical system is the almost topological conjugacy of two dynamical systems. Such a concept was introduced by R. Adler and B. Marcus in [1] to study invariants of Markov shifts and latter extended by W. Sun in [3] to dynamical systems whose symbolic representations are Markov shifts. Due to the central role played by almost topological conjugacies in this work, we present its definition due to Sun:

**Definition 1.1.** Two ergodically supported topological dynamical systems \((X, T)\) and \((Y, S)\) are said almost topologically conjugate if there exist an ergodically supported Markov shift \((\Lambda, \sigma_\Lambda)\) and two continuous onto maps \(f_T : \Lambda \to X^T\) and \(f_S : \Lambda \to Y^S\) such that:
(i) $\sigma_{X^T} \circ f_T = f_T \circ \sigma$ and $\sigma_{Y^S} \circ f_S = f_S \circ \sigma$;

(ii) There exist a $\sigma_{X^T}$-invariant universally null set $M_2 \subset X^T$ and a $\sigma_{Y^S}$-invariant universally null set $P_2 \subset Y^S$, such that $f_T : \Lambda \setminus M_1 \to X^T \setminus M_2$ and $f_S : \Lambda \setminus P_1 \to Y^S \setminus P_2$ are one-to-one, where $M_1 = f_T^{-1}(M_2)$ and $P_1 = f_S^{-1}(P_2)$.

In this work, we prove that the topological entropy and the ergodic period of an invertible expansive ergodically-supported dynamical system with the shadowing property $(X, T)$ establishes a sufficient criterion for the existence of quasigroup operations defined almost everywhere outside of universally null sets and for what $T$ is an automorphism (Theorem 3.4). As consequence of this result, we prove that if $(X, T)$ is aperiodic and has topological entropy $\log(N)$ for an integer $N \in \{2, 8, 18\} \cup \{2p : p \text{ is prime}\}$, then we can find a quasigroup operation defined almost everywhere and decompose the dynamics of $T$ in terms of a finite family of subquasigroups (Theorem 3.5). In this way we obtain for ergodic maps an analogous to the decomposition of linear maps in terms of their eigenspaces.

2 Quasigroups and weak quasigroups

Let $G$ be a set and let $\ast$ be a binary operation on $G$. We say that $\ast$ is a quasigroup operation if $\ast$ is left and right cancelable. If, in addition, $G$ is a topological space and $\ast$ is continuous, we say that $\ast$ is a topological quasigroup operation. For the case when $G$ is finite the quasigroup operation $\ast$ is associated to a Latin square. Furthermore, it is easy to check that if for any $g \in G$ it follows that $g \ast G = G$ (which always occurs if $G$ is finite), then $\ast$ is an associative quasigroup operation if and only if $\ast$ is a group operation.

For a finite set $G$, we will say that $s : G \to G$ is a translation on $G$, if given any $x \in G$ we have that $G = \{s^k(x) : k = 0, \ldots, \#G - 1\}$, where $\#G$ denotes the cardinality of $G$. The following theorem gives a sufficient and necessary condition on the cardinality of a finite set $G$ for the existence of quasigroup operations on $G$ for which a given translation is an automorphism.
Lemma 2.1. Given a finite set $G$ and a translation $s : G \rightarrow G$, there exists a quasigroup operation $*$ on $G$ for which $s$ is an automorphism if and only if $G$ has an odd quantity of elements.

Proof. Let $n := \#G$ and, without loss of generality, we can consider $G := \{0, \ldots, n - 1\}$ and the translation on $G$ in the form

$$s(x) = x \hat{+} 1,$$

where $\hat{+}$ is the sum mod $n$.

If $n$ is odd, we can define $\lambda := (n + 1)/2$ and, since $gcd(\lambda, n) = 1$ we have a quasigroup operation $*$ defined for all $x, y \in G$ by

$$x * y := \lambda(x \hat{+} y),$$

where $\lambda z$ stands for $z$ summed $\lambda$ times with itself (mod $n$). Therefore, for any $x, y \in G$ we have that

$$s(x) * s(y) = (x \hat{+} 1) * (y \hat{+} 1) = \lambda[(x \hat{+} 1) \hat{+} (y \hat{+} 1)] =$$

$$= \lambda(x \hat{+} y \hat{+} 2) = \lambda(x \hat{+} y) \hat{+} \lambda 2 = (x * y) \hat{+} 1 = s(x * y).$$

Now, let us show that if $n$ is even, then it does not exit a quasigroup operation for which $s$ is automorphism. For this, consider that $*$ is some binary operation on $G$ for which $s$ is an automorphism. We can represent the action of $*$ on $G$ by a table where the entry in the row indexed by $x$ and column indexed by $y$ represents the product $x * y$. Note that since $s$ is an automorphism for $*$, then the table shapes as follows:

|   | 0     | 1     | 2     | $\ldots$ | $n-1$ |
|---|-------|-------|-------|----------|-------|
| 0 | $a_0$ | $a_1$ | $a_2$ | $\cdots$ | $a_{n-1}$ |
| 1 | $a_{n-1-1}$ | $a_0+1$ | $a_1+1$ | $\cdots$ | $a_{n-2-1}$ |
| 2 | $a_{n-2-2}$ | $a_{n-1-2}$ | $a_0+2$ | $\cdots$ | $a_{n-3-2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-1$ | $a_{1+}(n-1)$ | $a_{2+}(n-1)$ | $a_{3+}(n-1)$ | $\cdots$ | $a_{0+}(n-1)$ |

where $a_0, a_1, \ldots, a_{n-1} \in \{0, \ldots, n-1\}$. Hence, for all $x, y \in G$, we can write $x * y = a_{y+}(n-x)+x$. We recall that $*$ was a quasigroup operation if and only if the above table was a Latin square, that is, if and only
if there was not repetition of elements in any row and any column of the table. Therefore, the sum mod $n$ of all elements of any row or of all elements of any column should result in the same value. But, if $n$ is even, supposing by contradiction that there is no repetitions in any row of the table (that is, $a_i \neq a_j$, for $i \neq j$), we get that the sum mod $n$ over any row is

$$\sum_{y=0}^{n-1} x \ast y = \sum_{y=0}^{n-1} (a_{y+(n-x)} \ast x) = \frac{1}{2} (n-1)n+x (\text{mod } n) = \frac{n}{2},$$

while the sum (mod $n$) over any column is

$$\sum_{x=0}^{n-1} x \ast y = \sum_{x=0}^{n-1} (a_{y+(n-x)} \ast x) = \frac{1}{2} (n-1)n+\frac{1}{2}(n-1)n (\text{mod } n) = 0,$$

which contradicts that the table is a Latin square.

In the next section we shall look for quasigroup operations, for which a given dynamical system is an automorphism. In general, if $(X, T)$ is ergodic with nonzero entropy, then there does not exist such a quasigroup operation (with exception for dynamical systems on zero-dimensional spaces). In fact, to the general case, we will need some ‘weakness’ in the operation.

**Definition 2.2.** Given a probability measure $\mu$ on the Borelians of $G$, we will say $\ast$ is a weak-quasigroup operation with respect to $\mu$ if $\ast$ is a quasigroup operation which is well defined for $\mu \times \mu$-almost all $(a, b) \in G \times G$. If in addition the operation $\ast$ is continuous on its domain, then we will say it is a topological weak quasigroup operation with respect to $\mu$. When $\ast$ is a weak quasigroup operation with respect to $\mu$ on $G$, we will call $(G, \ast, \mu)$ a (topological) weak quasigroup. Furthermore, if a (topological) weak quasigroup is associative, then we will simply say it is a (topological) weak group.

Note that the definition of a weak quasigroup operation is made on the product space $G \times G$ and not on the space $G$. Thus, it is possible that there exist $x \in G$ and a non-null measure subset of $A \subset G$ such that for all $x \in A$ the products $x \ast y$ and $y \ast x$ are not defined for any $y \in G$. 

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On the other hand, if $*$ is a weak quasigroup operation with respect to some measure $\mu$ on $G$, then given $x, y \in G$, the existence of the product $x * y$ does not imply the existence of $y * x$ (but when $*$ is commutative). Furthermore, the cancelable property of a weak quasigroup operation $*$ means that if $x * y$ (or $y * x$) and $x * z$ (or $z * x$) are defined, then $x * y = x * z$ (or $y * x = z * x$) if and only if $y = z$. In the same way, the associativity of a weak group holds only if both $x * (y * z)$ and $(x * y) * z$ are defined.

3 Weak quasigroups and expansive ergodically supported automorphisms

In order to construct a topological weak quasigroup for which a given topological dynamical system is automorphism, we need the following results.

Lemma 3.1. Let $\alpha : X \to Y$ be a continuous and onto map between topological spaces, and suppose $X$ is compact and $Y$ is Hausdorff. Given $\tilde{Y} \subseteq Y$, define $\tilde{X} := \alpha^{-1}(\tilde{Y})$. If the restriction $\tilde{\alpha} : \tilde{X} \to \tilde{Y}$ is one-to-one, then $\tilde{\alpha}$ is a homeomorphism.

Hence, by using the above lemma we can prove that two invertible dynamical systems that are almost topologically conjugated are topologically conjugated outside of universally null sets:

Theorem 3.2. If two invertible dynamical systems $(X, T)$ and $(Y, S)$ are almost topologically conjugated, then there exists a homeomorphism $\varphi : \tilde{X} \to \tilde{Y}$ between $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$ total-measure subsets with respect to any ergodically supported measure, which is a topological conjugacy between $(\tilde{X}, T)$ and $(\tilde{Y}, S)$.

Proof. Let $(X^T, \sigma_{X^T})$ and $(Y^S, \sigma_{Y^S})$ be the inverse limit systems of $(X, T)$ and $(Y, S)$, respectively. Note that since both $(X, T)$ and $(Y, S)$ are invertible, then the projections $p_T : X^T \to X$ and $p_S : Y^S \to Y$ are homeomorphism.

Let $(\Sigma, \sigma)$, $f_T : \Sigma \to X^T$ and $f_S : \Sigma \to Y^S$, be the Markov shift and the maps given in the definition of almost topological conjugacy. Also denote as $M_2 \subseteq X^T$ and $P_2 \subseteq Y^S$, and as $M_1 := f_T^{-1}(M_2)$
and $P_1 := f^{-1}_S(P_2)$, the universally null sets which make the maps $f_T : \Sigma \setminus M_1 \to X^T \setminus M_2$ and $f_S : \Sigma \setminus P_1 \to Y^S \setminus P_2$ to be bijections. Denote as $\bar{f}_T$ and $\bar{f}_S$ these restrictions of $f_T$ to $\Sigma \setminus M_1$ and of $f_S$ to $\Sigma \setminus P_1$, respectively. From Lemma 3.1, we get that $\bar{f}_T$ and $\bar{f}_S$ are homeomorphisms.

Note that since $p_T$ and $p_S$ are homeomorphisms, the sets $M_3 := p_T(M_2)$ and $P_3 := p_S(P_2)$ are also universally null sets. Denote as $\bar{p}_T$ and as $\bar{p}_S$ the restrictions $p_T : X^T \setminus M_2 \to X \setminus M_3$ and $p_S : Y^S \setminus P_2 \to Y \setminus P_3$.

Thus, the maps $\gamma_T : \Sigma \setminus M_1 \to X \setminus M_3$ and $\gamma_S : \Sigma \setminus P_1 \to Y \setminus P_3$ defined by $\gamma_T := \bar{p}_T \circ \bar{f}_T$ and $\gamma_S := \bar{p}_S \circ \bar{f}_S$ are also homeomorphisms.

Note that $\tilde{\Sigma} := \Sigma \setminus (M_1 \cup P_1)$, is a total-measure subset of $\Sigma \setminus M_1$ and of $\Sigma \setminus P_1$, with respect to any ergodically supported measure. Hence, $\tilde{X} := \gamma_T(\tilde{\Sigma}) \subseteq X$ and $\tilde{Y} := \gamma_S(\tilde{\Sigma}) \subseteq Y$ are total-measure subsets with respect to any ergodically supported measure. Therefore we can consider $\tilde{\gamma}_T : \tilde{\Sigma} \to \tilde{X}$ and $\tilde{\gamma}_S : \tilde{\Sigma} \to \tilde{Y}$ the restrictions of $\gamma_T$ and $\gamma_S$, respectively.

Finally, we define the homeomorphism $\varphi : \tilde{X} \to \tilde{Y}$ given by

$$\varphi := \tilde{\gamma}_S \circ \tilde{\gamma}_T^{-1}.$$

Since, all maps involved in the definition of $\varphi$ commute with the dynamical systems, we get that $\varphi$ is a topological conjugacy between $(\tilde{X}, T)$ and $(\tilde{Y}, S)$.

\[\square\]

**Corollary 3.3.** Let $(X, T)$ and $(Y, S)$ be two topological dynamical systems, and assume they are invertible, expansive, ergodically supported, have the shadowing property, and have the same ergodic period. Then, there exist $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ total-measure subset with respect to any ergodically supported measure such that $(\tilde{X}, T)$ and $(\tilde{Y}, S)$ are topologically conjugated.

**Proof.** It is a consequence of Theorem 1.2 in [8] and the previous theorem.

\[\square\]

Now, we are able to get sufficient conditions on a dynamical system that allow to define a topological weak quasigroup operation for which the map of a dynamical system becomes an automorphism.

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Theorem 3.4. Let \((X, T)\) be a topological dynamical system, and assume it is invertible, expansive, ergodically supported with odd ergodic period, and has the shadowing property. If \(h(T) = \log(N)\) for some positive integer \(N\), then there exists a topological weak quasigroup operation \(\ast\) with respect to any ergodically supported measure of \((X, T)\), for which \(T\) is an automorphism.

Proof. Let \(B\) be the ergodic period of \((X, T)\). Define the dynamical system \((Y, S)\) as \(Y := \{0, 1, \ldots, N-1\} \times \{0, 1, \ldots, B-1\}\) and \(S := \sigma \times s\) the map where \(\sigma : \{0, 1, \ldots, N-1\} \times \{0, 1, \ldots, B-1\} \to \{0, 1, \ldots, N-1\}\) is the shift map and \(s : \{0, 1, \ldots, B-1\} \to \{0, 1, \ldots, B-1\}\) is the permutation defined by \(s(i) := i + 1 (\text{mod } B)\).

Since \((X, T)\) and \((Y, S)\) are invertible, expansive, ergodically supported, have the shadowing property, and have the same topological entropy and the same ergodic period, then from Corollary 3.3 there exist \(\tilde{X} \subset X\) and \(\tilde{Y} \subset Y\) total-measure subsets with respect to any ergodically supported measure and \(\varphi : \tilde{X} \to \tilde{Y}\) which is topological conjugacy between \((\tilde{X}, T)\) and \((\tilde{Y}, S)\).

Define on \(Y\) the quasigroup operation \(\ast\) given by
\[
(\{x_i\}_{i \in \mathbb{Z}}, a) \ast (\{y_i\}_{i \in \mathbb{Z}}, b) := (\{x_i \ast ^{\tilde{x}} y_i\}_{i \in \mathbb{Z}}, \lambda(a \ast ^{\tilde{+}} b)),
\]
where \(^{\tilde{x}}\) is any quasigroup operation on \(\{0, 1, \ldots, N-1\}\), \(\lambda := (B + 1)/2\) and \(^{\tilde{+}}\) is the sum \(\text{mod } B\). It is straightforward that the shift map \(\sigma\) is an automorphism for \(^{\tilde{x}}\) and, from Lemma 2.1, the map \(s\) is an automorphism for \(^{\tilde{+}}\). Thus, \(S\) is an automorphism for \(\ast\). Furthermore, since \(^{\tilde{x}}\) is a 1-block operation (see [7]) and \(^{\tilde{+}}\) is continuous for the power set topology on \(\{0, 1, \ldots, B-1\}\), then \(\ast\) is a topological quasigroup operation.

Denote by \(\Theta : Y \times Y \to Y\) the map given by \(\Theta(x, y) = x \ast y\), for any \(x, y \in Y\). Since \(\Theta\) is continuous, the set \(\Theta^{-1}(\tilde{Y}) \subseteq Y \times Y\) has total measure subset with respect to the product measure on \(Y \times Y\) of any ergodically supported measure on \(Y\).

Thus,
\[
\Lambda := (\tilde{Y} \times \tilde{Y}) \cap \Theta^{-1}(\tilde{Y})
\]
is also a total measure subset with respect to the product measure on \(Y \times Y\) of any ergodically supported measure on \(Y\). Note that, \(\Lambda\) is the set of all pair of points of \(\tilde{Y} \times \tilde{Y}\) for what the product by \(\ast\) is a point
lying in $\tilde{Y}$. Furthermore, since $\Theta$ commutes with the maps $S \times S$ and $S$, and $\tilde{Y}$ is $S$-invariant, we get that $\Lambda$ is $S \times S$-invariant.

Define $\Omega \subseteq \tilde{X} \times \tilde{X}$ by

$$\Omega := (\varphi \times \varphi)^{-1}(\Lambda).$$

Since $\Lambda$ is a total measure subset with respect to the product measure on $Y \times Y$ of any ergodically supported measure on $Y$, and $\varphi \times \varphi : \tilde{X} \times \tilde{X} \rightarrow \tilde{Y} \times \tilde{Y}$ is a homeomorphism, then $\Omega$ is a total measure subset with respect to the product measure on $X \times X$ of any ergodically supported measure on $X$. Hence, for any pair $(x, y) \in \Omega$ we can define the quasigroup operation $\bullet$ given by

$$x \bullet y := \varphi^{-1}(\varphi(x) \ast \varphi(y)).$$

(2)

Note that $\bullet$ is well defined. In fact, since $(x, y) \in \Omega$, then $(\varphi(x), \varphi(y)) \in \Lambda$. Therefore $(\varphi(x) \ast \varphi(y)) \in \tilde{Y}$ and $\varphi^{-1}(\varphi(x) \ast \varphi(y)) \in \tilde{X}$.

Furthermore, for any $(x, y) \in \Omega$ it follows that

$$T(x \bullet y) = T(\varphi^{-1}(\varphi(x) \ast \varphi(y))) = \varphi^{-1}(S(\varphi(x) \ast \varphi(y)))$$

$$= \varphi^{-1}(S(\varphi(x)) \ast \varphi(y)) = \varphi^{-1}(\varphi(T(x)) \ast \varphi(T(y))) = T(x) \bullet T(y).$$

Note that we can get the weak quasigroup operation in the previous theorem as being associative if, and only if, $(X, T)$ is ergodically aperiodic. In fact, if $(X, T)$ is not ergodically aperiodic then $(Y, S)$ is also not ergodically aperiodic. On the other hand, if $S$ would be an expansive and ergodic automorphism for some group operation, then from Theorem 1(iv) in [4] we could get that $(X, T)$ is topologically conjugated to a full shift, which would contradict that $(Y, S)$ was not ergodically aperiodic.

The next theorem give sufficient conditions to decompose the dynamics of $(X, T)$ in terms of $T$-invariant weak subquasigroups.

**Theorem 3.5.** Let $(X, T)$ be an invertible, expansive, ergodically-supported and aperiodic topological dynamical system with the shadowing property. Suppose $h(T) = \log(N)$, with $N \in \mathbb{N}$. If $p_1p_2 \cdots p_q = N$
is a decomposition of $N$ by integers such that $p_i \not\in \{2,6\}$ for all $i = 1,\ldots,q$, then there exists a topological weak quasigroup operation $\bullet$ on $X$ for which $T$ is an automorphism, and $T$-invariant weak subquasigroups $X_k \subseteq X$ for $k = 1,\ldots,q$, such that almost all $x \in X$ with respect to the maximum-entropy measure of $(X,T)$ can be written as $x = x_1 \bullet (x_2 \bullet \cdots (x_{q-2} \bullet (x_{q-1} \bullet x_q)))$, with $x_k \in X_k$.

Proof. Firstly, we need appropriately to define the topological quasi-group shift $Y$ in Theorem 3.4. For this, for each $k = 1,\ldots,q$, consider the finite alphabet $A_k = \{1,\ldots,p_k\}$ and the full shift over $p_k$ symbols $Y_k := \mathbb{Z}^{p_k}$.

Since for any $k$ the alphabet $A_k$ has cardinality $p_k \not\in \{2,6\}$, we can define on $A_k$ an idempotent quasigroup operation, that is, a quasigroup operation $\ast_k$ such that for any $a \in A_k$ it follows $a \ast_k a = a$ (see [3]).

Therefore, define $Y := Y_1 \times \cdots \times Y_q = \{(x^i_1,\ldots,x^i_q)_{i\in\mathbb{Z}} : (x^i_k)_{i\in\mathbb{Z}} \in Y_k, k = 1,\ldots,q\}$, and define on $Y$ the idempotent quasigroup operation $\ast$ given by

$$(x^i_1,\ldots,x^i_q)_{i\in\mathbb{Z}} \ast (y^i_1,\ldots,y^i_q)_{i\in\mathbb{Z}} = (x^i_1 \ast_1 y^i_1,\ldots,x^i_q \ast_q y^i_q)_{i\in\mathbb{Z}}.$$

Given $S \subset Y$ denote as $\langle S \rangle$ the subquasigroup of $(Y,\ast)$ generated by $S$, that is, the smallest subquasigroup which contains $S$. For $m \geq 1$ let $S^m$ be the set obtained multiplying $S$ by itself $m$ times in any possible associative way. For example,

$S^1 := S$

$S^2 := S \ast S$

$S^3 := S \ast (S \ast S) \cup (S \ast S) \ast S$

$\vdots$

It is easy to check that

$$\langle S \rangle = \bigcup_{m \geq 1} S^m.$$

Note that $Y$ is ergodically aperiodic and with topological entropy $\log(N)$. Now, let $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ be the total-measure subsets with
respect to any ergodically supported measure, and \( \varphi : \tilde{X} \to \tilde{Y} \) be the topological conjugacy between \((\tilde{X}, T)\) and \((\tilde{Y}, \sigma)\), given by Theorem 3.4.

Let \( \nu \) be the uniform Bernoulli measure on \( Y \). We recall that \( \nu \) is an ergodically supported measure and it is the maximum entropy measure for \((Y, \sigma)\). For each \( k = 1, \ldots, q \), denote as \( \nu_k \) the projection of \( \nu \) on \( k^{th} \) coordinate, that is, \( \nu_k \) is the uniform Bernoulli measure on \( Y_k \). In particular, \( \nu = \bigotimes_{k=1}^{q} \nu_k = \nu_1 \times \cdots \times \nu_q \).

Given \( k = 1, \ldots, q \), for each \( j = 1, \ldots, q \) and \( j \neq k \), we can take \((z_{k,j}^i)_{i \in \mathbb{Z}} \in Y_j \) and define the section

\[
S_k = \{ (x_1^i, \ldots, x_q^i)_{i \in \mathbb{Z}} \in Y : x_j^i = z_{k,j}^i \ \forall j \neq k, \ \forall i \in \mathbb{Z} \}. \tag{3}
\]

Note that, we can identify \( S_k \) with \( Y_k \) and without loss of generality we can consider the measure \( \nu_k \) on \( S_k \). Furthermore, since \( * \) is idempotent, then for each \( k = 1, \ldots, q \), \( S_k \) is a topological subquasigroup and \( Y = S_1 * (S_2 * (\cdots (S_{q-2} * (S_{q-1} * S_q)))) \).

On the other hand, since \( \tilde{Y} \) has total measure, due to Fubini’s Theorem we can get that for \( \bigotimes_{j \neq k} \nu_j \)-almost all choice of \((z_{k,j}^i)_{i \in \mathbb{Z}} \in Y_j \), \( j \neq k \), we have

\[
\nu_k(S_k \cap \tilde{Y}) = 1.
\]

Now, let \( S_k \), for \( k = 1, \ldots, q \), be sections for which the above equality hold, and define

\[
\hat{S}_k := \left\langle \bigcup_{n \in \mathbb{Z}} \sigma^n(S_k) \right\rangle.
\]

We have that \( \hat{S}_k \) is a \( \sigma \)-invariant subquasigroup of \( Y \) and \( \nu_k(\hat{S}_k \cap \tilde{Y}) = 1 \) for each \( k = 1, \ldots, q \). It means that \( \nu \)-almost all \( x \in \tilde{Y} \) can be written as \( x = x_1 * (x_2 * (\cdots (x_{q-2} * (x_{q-1} * x_q)))) \), with \( x_k \in \hat{S}_k \).

Finally, since the sets \( \hat{S}_k \cap \tilde{Y} \) are \( \sigma \)-invariant and, since \( \mu := \nu \circ \varphi \) is the maximum-entropy measure for \((X, T)\), then the weak subquasigroups \( X_k := \varphi^{-1}(\hat{S}_k \cap \tilde{Y}) \) satisfy the theorem.

\[\square\]

**Corollary 3.6.** Under the same hypotheses of Theorem 3.4, if \( x \in X \) can be decomposed as \( x = x_1 \cdot (x_2 \cdot (\cdots (x_{q-2} \cdot (x_{q-1} \cdot x_q)))) \), with \( x_k \in X_k \), then \( T(x) = T(x_1) \cdot (T(x_2) \cdot (\cdots (T(x_{q-2}) \cdot (T(x_{q-1}) \cdot T(x_{q}))))) \).
The previous result means that whenever dynamical system has topological entropy \( \log(N) \) with \( N \notin \{2, 8, 18\} \cup \{2p : p \text{ is prime}\} \), its behavior can be decomposed (except for a universally null set) as the action of the map on subquasigroups in an analogous way to the decomposition of the action of linear maps in terms of their eigenspaces.

Note that if \( Q_k \) and \( R_k \) are two distinct sections given by 3, then they are disjoint. Furthermore, \( Q_k * R_k \) is also a section on the same coordinates. Observe that any section \( R_k \) is a universally null set. In fact, for any measure \( \gamma \) on \( Y \) it follows that

\[
\gamma \left( \bigcup_{n \in \mathbb{Z}} \sigma^n(R_k) \right) = \sum_{n \in I} \gamma(\sigma^n(R_k)) = \sum_{n \in I} \gamma(R_k),
\]

where \( I \) is finite if the pairwise disjoint family \( \{\sigma^n(R_k) : n \in \mathbb{Z}\} \) is finite and \( I = \mathbb{Z} \) otherwise. Hence, if by contradiction we suppose \( \gamma \) is ergodically supported and \( \gamma(R_k) > 0 \), then it follows that necessarily \( I \) is finite. Therefore, \( \bigcup_{n \in I} \sigma^n(R_k) \subseteq Y \) is closed and \( \sigma \)-invariant and, since \( \gamma \) is ergodic and \( \gamma(R_k) > 0 \), then \( \gamma \left( \bigcup_{n \in I} \sigma^n(R_k) \right) = 1 \). But this implies that \( Y \setminus \left( \bigcup_{n \in I} \sigma^n(R_k) \right) \) is non-empty open set with null measure, which is a contradiction with the hypothesis that \( \gamma \) is ergodically supported.

Now observe that the sets \( \hat{S}_k \) in the proof of Theorem 3.5 can be written as

\[
\hat{S}_k = \left\langle \bigcup_{n \in \mathbb{Z}} \sigma^n(S_k) \right\rangle = \bigcup_{m \geq 1} \left( \bigcup_{n \in \mathbb{Z}} \sigma^n(S_k) \right)^m,
\]

where \( \left( \bigcup_{n \in \mathbb{Z}} \sigma^n(S_k) \right)^m \) is the set obtained making all possible products between \( m \) sections selected from \( \bigcup_{n \in \mathbb{Z}} \sigma^n(S_k) \). Thus, for each \( k \) and \( m \) the set \( \left( \bigcup_{n \in \mathbb{Z}} \sigma^n(S_k) \right)^m \) is a countable union of sections and therefore the subquasigroup \( \hat{S}_k \) is also a countable union of sections and thus it is an universally null set. Consequently each weak subquasigroup \( X_k \) is also an universally null sets.
4 Final discussion

Note that the converse statement of Theorem 3.4 would allow to extend the results of [5] for the more general case where $T : X \to X$ is an expansive ergodically supported map with the shadowing property and an automorphism for some topological quasigroup operation. However, seems to be not direct that the existence of a topological (weak) quasigroup operation on $X$ for which $T$ is an automorphism implies that $h(T) = \log(N)$. Note that, since any topological (weak) quasigroup operation on $X$ induces a topological weak quasigroup operation on the symbolic representation of $X$, then we should be able to assure that the existence of weak quasigroup operation on a shift space product a finite set implies that this shift space has topological entropy $\log(N)$. It would be some kind of extension of the consequences of Theorem 1 in [4] (for groups) and Theorem 4.25 in [7] (for quasigroups).

It would also be interesting to study the case of non-invertible maps. In such a case we cannot apply Theorem 3.3 since the projections $p_T : X^T \to X$ and $p_S : Y^S \to Y$ are not invertible. In fact, we only use the hypothesis that $T$ is invertible just to get $p_T : X^T \to X$ and $p_S : Y^S \to Y$ being homeomorphism and thus to assure that $f_T$ and $f_S$ are bimeasurable functions in Theorem 3.2 which allowed to construct the topological conjugacy $\varphi : \tilde{X} \to \tilde{Y}$. However, there are several examples of non-invertible maps which are topologically conjugated to shift spaces outside of a universally null set and thus we can apply Theorem 3.4 to construct an algebraic operation for which $T$ is automorphism without to use Theorem 3.2 (for instance, the maps on the unit interval with the form $T(x) = Mx \ (\text{mod} \ M)$ are topologically conjugate to $\{0, \ldots, M-1\}^N$ outside of the sets $\{i/M^k : k \geq 1, 0 \leq i \leq M^k-1\} \subset [0,1]$ and $\{(x_i)_{i\in\mathbb{N}} : \sum_{i\in\mathbb{N}} x_i < \infty\} \subset \{0, \ldots, M-1\}^\mathbb{N}$).

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