Variational problems related to some fractional kinetic equations

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Abstract
We establish the existence and symmetry of all minimizers of a constrained variational problem involving the fractional gradient. This problem is closely connected to some fractional kinetic equations.

1 Introduction

Fractional calculus has gained a lot of interest during the last decade due to its numerous applications in many fields. It appears in wave propagation, inhomogenous porous material, geology, hydrology, dynamics of earthquakes, bioengineering, chemical engineering signal processing, medicine, electrochemistry, thermodynamics, neural networks, statistical physics, [2], [6], [8] and references therein.

Fractional equations involving the fractional laplacian have also played a crucial role in some kinetic problems, [5], [10], [11], [12], [13] and [14], in which particular solutions are obtained by solving the following minimization problem:

\[(P_c) : I_c = \inf \left\{ \int_{\mathbb{R}^N} |-\Delta^{s/2}(u)|^2 - \int_{\mathbb{R}^N} F(|x|, u) : u \in S_c \right\}\]

\[E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |-\Delta^{s/2}(u)|^2 - \int_{\mathbb{R}^N} F(|x|, u),\]

\[F(r, t) = \int_0^t f(x, p) dp,\]

\[S_c = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = c^2 \right\},\]

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where \( c \) is a prescribed number, \( 0 < s < 1 \) and \( H^s(\mathbb{R}^N) \) is the usual Besov space, \([8]\).

Note also that the minimization problem \((P_c)\) appears in disperative model equations: The generalized Benjamin-On equation, the Benjamin-Bona-Mahong equation and the fractional nonlinear Schrödinger equation.

In this paper, we address the question of existence, radiality and radial decreasiness of all minimizers of \((P_c)\) for integrands \( F \) satisfying some growth conditions. This result generalizes a recent one obtained by Frank and Lenzmann, \([7]\), in which the authors have considered the basic power nonlinearity
\[
F(r, t) = |t|^{\alpha + 2}.
\]

Moreover, they have proved that the following minimization problems
\[
I_M = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}} |-\Delta^{s/2} u|^2 + \frac{1}{\alpha + 2} \int_{\mathbb{R}} |u|^{\alpha + 2} : u \in S_{\sqrt{M}} \right\}
\]
and
\[
J^{s, \alpha}(Q) = \inf_{u \in H^s(\mathbb{R}) \setminus \{0\}} \left( \int_{\mathbb{R}} |-\Delta^{s/2} u|^2 \right)^{\alpha/4s} \left( \int_{\mathbb{R}} u^2 \right)^{(2s-1)+1} \int |u|^{\alpha + 2}
\]
are equivalent.

They have also added that: \( u \) is a solution of \((1.1)\) if and only if \( u = e^{i\theta} \lambda^{1/\alpha} Q(\lambda^{1/2s}(., + y)) \), for some \( \theta \in \mathbb{R}, y \in \mathbb{R}, \lambda > 0 \); \( Q \) is a solution of \((1.2)\).

Finally they have stated that \((1.2)\) (and therefore \((1.1)\)) only has minimizers when \( 0 < \alpha < \alpha_{\text{max}} \), where \( \alpha_{\text{max}} \) is defined as follows:
\[
\alpha_{\text{max}} = \begin{cases} 
\frac{4s}{1-2s}, & 0 < s < \frac{1}{2} \\
\infty, & \frac{1}{2} \leq s < 1
\end{cases}
\]

This result seems to be erroneous and as we will show in section 2, \((1.1)\) admits minimizers if and only if \( \alpha < 4s \).

Moreover, in this paper, we will study \((P_c)\) for general nonlinearities \( F \) such that: \( |F(r, t)| \leq K(t^2 + |t|^{\ell+2}) \) where \( 0 < \ell < \frac{4s}{N} \).

Our main result, Theorem 2.1, states that:

1. If \( 0 < \ell < \frac{4s}{N} \), \((P_c)\) admits solutions and all minimizers are radial and radially decreasing.

2. If \( \ell = \frac{4s}{N} \), \((P_c)\) admits solutions and all minimizers are radial and radially decreasing if \( c^2 \) is small enough (some estimates will be given below).
3. If \( \liminf_{t \to \infty} F(r, t)/t^{\ell+2} \geq A > 0 \) for some \( \ell > \frac{4s}{N} \), then \( I_c = -\infty \) for all \( c \).

Now before stating our main result, let us first mention that definitions and properties of the Schwarz symmetrization are detailed in [4].

If \( u \in H^s_+(\mathbb{R}^N) = \{ u \in H^s(\mathbb{R}^N) : u \geq 0 \} \), then the fractional Polya-Szeg"{o} inequality holds true:

\[
|\nabla_s u^*|^2_2 = | - \Delta^{s/2} u^* |^2_2 \leq | - \Delta^{s/2} u |^2_2 = |\nabla_s u|^2_2
\] (1.3)

which is a direct consequence of the generalized Riesz inequality, [4], since as it was proven in [1] :

\[
|\Delta^{s/2} u|^2_2 = C_{n,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(-x) - u(y)|^2}{|x-y|^{N+2s}} dx dy.
\]

From now on \( 0 < s < 1 \), \( H^s(\mathbb{R}^N) \) is the standard Besov space. The norm of the Lebesgue space \( L^p(\mathbb{R}^N) \) is denoted by \( \| \cdot \|_p \). \( c \) is a prescribed number and \( N \in \mathbb{N}^* \). In an integral where no domain is given, it is to be understood that it extends on \( \mathbb{R}^N \).

2 Main Result

**Theorem 2.1** Suppose that :

1. \( (F_0) F : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function :
   - \( F(., t) : [0, \infty) \to \mathbb{R} \) is measurable in \( \mathbb{R}_+ \setminus \Gamma \) for all \( t \in \mathbb{R} \), where \( \Gamma \) is a subset of \( \mathbb{R}_+ \) having one dimensional measure zero and ;
   - \( F(r, .) : \mathbb{R} \to \mathbb{R} \) is continuous for every \( r \in [0, \infty) \setminus \Gamma \).
2. \( (F_1) F(r, t) \leq F(r, |t|) \) for a.e \( r \geq 0 \) and every \( t \in \mathbb{R} \),
3. \( (F_2) \) For a.e \( r \geq 0 \) and every \( t \geq 0 \).

\[
0 \leq F(r, t) \leq K(t^2 + t^{\ell+2}), \text{ where } K > 0 \text{ and } 0 < \ell < \frac{4s}{N}.
\]
4. \( (F_3) \) For every \( \varepsilon > 0 \), there exist \( R_0 > 0 \) and \( t_0 > 0 \) such that \( F(r, t) \leq \varepsilon t^2 \) for a.e \( r \geq R_0 \) and \( 0 \leq t \leq t_0 \).
5. \( (F_4) (v, y) \to F(\frac{1}{v}, y) \) is supermodular on \( \mathbb{R}_+ \times \mathbb{R}_+ \), i.e

\[
F(r, a) + F(R, A) \geq F(r, A) + F(R, a)
\]
for every $0 \leq r < R$ and $0 \leq a < A$.

Let $(\tilde{P}_c) : \inf \{ E(u) : u \in H^s_+ (\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 \leq c^2 \} = \tilde{I}_c$

(I) $\tilde{I}_c < \tilde{I}_d$ for $d < c$.

Then

1. $(P_c)$ admits a Schwarz symmetric minimizer for any $c$. Moreover if $(F_4)$ holds with a strict sign, then for any $c$, all minimizers of $(P_c)$ are radial and radially decreasing, i.e, Schwarz symmetric.

2. If $\ell = \frac{4s}{N}$, then (1) holds true if and only if $c$ is small enough.

3. If \( \liminf_{t \to \infty} F(r, t)/t^\ell = A > 0, \) with $\ell > \frac{4s}{N}$ then $I_c = -\infty$.

**Proof of 1** Fix $c$

**Step 1** : $(P_c)$ is well posed ($I_c > -\infty$ and all minimizing sequences are bounded in $H^s(\mathbb{R}^N)$).

By $(F_1)$ and $(F_2)$, we can write:

\[
\int F(|x|, u(x))dx \leq \int F(|x|, |u(x)|)dx \\
\leq Kc^2 + K \int |u(u(x)|^{\ell+2}dx
\]

(2.1)

Now using the fractional Gagliardo-Nirenberg inequality [8], [9], it follows that:

\[
|u|_{\ell+2} \leq K'|u|_2^{1-\theta} |\nabla_s u|_2^\theta \quad |\nabla_s u|_2 = \left( \int | - \Delta^{s/2} u|^2 \right)^{1/2} \quad \text{and} \quad \theta = \frac{N\ell}{2s(\ell+2)}.
\]

Thus

\[
\int |u(x)|^{\ell+2}dx \leq K' \left\{ \int u^2(x)dx \right\}^{(1-\theta)(\ell+2)/2} |\nabla_s u|_2^{\theta(\ell+2)}. \quad (2.2)
\]

Therefore using Young inequality, we have

\[
\left\{ \int u^2(x)dx \right\}^{(1-\theta)(\ell+2)/4} |\nabla_s u|_2^{\theta(\ell+2)} \leq \\
\frac{1}{p} e^p \{|\nabla_s u|_2^2\}^{p(\ell+2)/2} + \frac{1}{q^p} \left\{ \int u^2(x)dx \right\}^{q(1-\\theta)(\ell+2)/2} \quad (2.3)
\]
for any $\varepsilon > 0$ and $p > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Choosing $p = \frac{2}{\theta(\ell + 2)} = \frac{4s}{N\ell'}$ we get:

$$
\int |u(x)|^{\ell+2} dx \leq \frac{K'}{p} \varepsilon^p \{ |\nabla_s u|^2 \} + \frac{K'}{q\varepsilon^q} \{ \int u^2(x) dx \}^{\frac{(1-\theta)(\ell+2)}{2}}
$$

$$
= \frac{K'}{p} \varepsilon^p |\nabla_s u|^2 + \frac{K'}{q\varepsilon^q} c^{\theta(1-\theta)(\ell+2)} \tag{2.5}
$$

for any $u \in S_c$.

Therefore:

$$
E(u) \geq \frac{1}{2} |\nabla_s u|^2 - Kc^2 - K' \varepsilon^p |\nabla_s u|^2 - \frac{KK'}{q\varepsilon^q} c^{\theta(1-\theta)(\ell+2)}
$$

$$
= \left( 1 - \frac{KK'}{p} \varepsilon^p \right) |\nabla_s u|^2 - Kc^2 - \frac{KK'}{q\varepsilon^q} c^{\theta(1-\theta)(\ell+2)}.
$$

Thus $I_c > -\infty$ and all minimizing sequences are bounded in $H^s(\mathbb{R}^N)$.

**Step 2**: Existence of a Schwarz symmetric minimizing sequence.

First note that if $u \in H^s(\mathbb{R}^N)$ then $|u| \in H^s(\mathbb{R}^N)$.

Now by $(F_1)$, we certainly have:

$$
E(|u|) \leq E(u) \quad \forall u \in H^s(\mathbb{R}^N).
$$

Now by $(1.3)$, we know that $|\nabla_s |u|^*|_2 \leq |\nabla_s u|_2$, and using Theorem 1 of [4], we have:

$$
\int F(|x|, |u(x)|) dx \leq \int F(|x|, |u(x)|^*) dx
$$

and

$$
\int u^2 = \int (u^*)^2.
$$

Thus without loss of generality, $(P_c)$ always admits a Schwarz symmetric minimizing sequence.

Let $(u_n)$ be a Schwarz symmetric minimizing sequence of $(P_c)$ for a fixed $c$.

**Step 3**: Let $(u_n) = (u_n^*)$ be a Schwarz symmetric minimizing sequence then if $u_n$ converges weakly to $u \Rightarrow E(u) \leq \liminf E(u_n)$.

**Proof** $|\nabla_s u|_2 \leq \liminf |\nabla_s u_n|_2$ by the weak lower semi-continuity of $\| \|$ of
the fractional gradient in \( H^s(\mathbb{R}^N) \).

Let us prove now that:

\[
\lim_{n \to \infty} \int F(|x|, u_n(x)) \, dx = \int F(|x|, u(x)) \, dx.
\]

Let \( R > 0 \), let us first prove that:

\[
\lim_{n \to +\infty} \int_{|x| \leq R} F(|x|, u_n(x)) \, dx = \int_{|x| \leq R} F(|x|, u(x)) \, dx.
\]

Since \( u_n \) converges weakly to \( u \) in \( H^s(\mathbb{R}^N) \), it converges strongly to \( u \) in \( L^{\ell+2}(|x| \leq R) \). Thus there exists a subsequence \( (u_{n_k}) \) of \( (u_n) \) such that \( u_{n_k} \to u \) a.e in \( L^2(B(0, R)) \) and \( |u_{n_k}| \leq h \) with \( h \in L^{\ell+2}(|x| \leq R) \).

Now by \( (F_2) \) :

\[
F(|x|, u_{n_k}(x)) \leq K(h^2(x) + h^{\ell+2}(x)).
\]

Noticing that \( h^2 + h^{\ell+2} \in L^1(|x| \leq R) \), we get thanks to the dominated convergence theorem:

\[
\lim_{n \to +\infty} \int_{|x| \leq R} F(|x|, u_n(x)) \, dx = \int_{|x| \leq R} F(|x|, u(x)) \, dx.
\]

Let us prove now that \( \lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} F(|x|, u_n(x)) \, dx = 0 \).

Let \( n \in \mathbb{N} \), since \( (u_n) = (u_n^*) \), we have that

\[
V_N |x|^N u_n^2(x) \leq \int_{|y| \leq |x|} u_n^2(y) \, dy \leq c^2.
\]

Thus

\[
u_n(x) \leq \frac{c}{V_N^{1/2} |x|^{N/2}} \leq \frac{c}{V_N^{1/2} R^{N/2}} \quad \forall |x| > R.
\]

Now let \( \varepsilon > 0 \) and \( R \) big enough, we obtain thanks to \( (F_3) \) that:

\[
\int_{|x| > R} F(|x|, u_n(|x|)) \, dx \leq \varepsilon \int_{|x| > R} u_n^2(x) \, dx < \varepsilon c^2,
\]

proving that

\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{|x| > R} F(|x|, u_n(x)) \, dx = 0.
\]
But $u$ inherits all the properties of the sequence $(u_n)$ used to get the above limit, then it follows that:

$$\lim_{R \to \infty} \int_{|x| > R} F(|x|, u(x)) \, dx = 0.$$  

**Step 4 :** $I_c$ is achieved.

Denoting $v$ the weak limit of a Schwarz minimizing sequence of $(\tilde{P}_c)$. We certainly have, using previous steps, that

$$E(v) \leq \liminf E(u_n),$$

where

$$\lim_{n \to \infty} E(u_n) = \tilde{I}_c.$$  

On the other hand $|v|^2 = d^2 \leq c^2$.

It follows then by hypothesis $(I)$, that:

$$\tilde{I}_c < \tilde{I}_d \leq E(v) \leq \tilde{I}_c$$

which is impossible, then $|v|^2 = c^2$. Suppose that $|v|^2 = d^2 < c^2$.

Therefore $I_c \leq E(v) = \tilde{I}_c \leq I_c$, proving that $(P_c)$ is achieved by $v = v^*$ a.e.

Now to show that all minimizers of $(P_c)$ are Schwarz symmetric, it is sufficient to notice that if $(F_4)$ holds with a strict sign then it follows by Theorem 1 of [4] that

$$E(u^*) < E(u) \quad \text{for any} \quad u \in H^s_c(\mathbb{R}^N)$$

and the result follows.

**Proof of 2) If** $\ell = \frac{4}{Ns}$, (2.2) becomes:

$$\int |u(x)|^{\ell+2} \, dx \leq K'c^{A/N}|\nabla_s u|^2 \quad \forall u \in S_c.$$

Hence

$$E(u) \geq \frac{1}{2}|\nabla_s u|^2 - Kc^2 - KK'c^{A/N}|\nabla_s u|^2$$

$$= \left( \frac{1}{2} - KK'c^{A/N} \right)|\nabla_s u|^2 - Kc^2.$$  

Thus $I_c > -\infty$ and all minimizing sequences are bounded in $H^s(\mathbb{R}^N)$ provided that $0 < c < \left( \frac{1}{KK'} \right)^{4/N}$.

Then previous steps (2,3 and 4) apply to $c$ such as in the latter interval.

**Proof of 3) :** It suffices to consider $u \in S_c$ and $u_\lambda(x) = \lambda^{N/2} u(\lambda x) (\in S_c)$, then the results follow when $\lambda$ tends to infinity.
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