WORLD-SHEET OF THE DISCRETE LIGHT FRONT STRING

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Abstract

Some aspects of light-like compactifications of superstring theory and their implications for the matrix model of M-theory are discussed.

1 Light-Like compactification

T-duality is one of the most profound of stringy phenomena. Closed string theory on a space-time which has a compact dimension with radius $R$ has the same spectrum as a string theory on a space with radius $\alpha'/R$. This leads to many interesting properties and is an essential part of the web of dualities which relate the different superstring theories and M theory. One might ask whether it is important that the dimension which is compactified is space-like. In this talk, I will review some recent work[1] which asks what happens when the dimension that is compactified is light-like, rather than space-like.

Of course, we could get a light-like circle by boosting a compactified spatial circle by an infinite amount[2]. Consider a closed string with compactified spatial direction, $X_{D-1} \sim X_{D-1} + 2\pi R$. In terms of light-cone coordinates, $X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X_{D-1})$

$$ (X^+, X^-) \sim \left( X^+ + \sqrt{2}\pi R, X^- - \sqrt{2}\pi R \right)$$

We consider a boosted reference frame, where $\tilde{X}^+ = \Lambda X^+$ and $\tilde{X}^- = \Lambda^{-1} X^-$. We fix $\Lambda = \sqrt{2R^+}/R$ and take the limit $\Lambda \to \infty$ with $R^+$ fixed, we finally get the light-like direction compactified,

$$ (\tilde{X}^+, \tilde{X}^-) \sim (\tilde{X}^+ + 2\pi R^+, X^-) \quad (1) $$

The original spatial circle is vanishingly small, $R = \sqrt{2R^+}/\Lambda$. The momenta transform to

$$ P^+ = \Lambda \frac{\sqrt{(P_{D-1})^2 + \vec{p}^2 + M^2 + P_{D-1}}}{\sqrt{2}} $$

$$ P^- = \frac{\sqrt{(P_{D-1})^2 + \vec{p}^2 + M^2 - P_{D-1}}}{\sqrt{2}\Lambda} $$

with $P_{D-1} = -N/R$ and $\vec{p}^2 = \sum_{D-2} p_i^2$. In the limit of infinite boost this becomes

$$ (P^+, P^-) = \left( \frac{R^+}{2N} \left( \vec{p}^2 + M^2 \right), \frac{N}{R^+} \right) $$

Here, $P^+$ is the infinite momentum frame Hamiltonian which generates translations of $X^-$. Also, since $X^+ \sim X^+ + 2\pi R^+$ the conjugate momentum is quantized, $P^+ = N/R^+$. Of the states with $N = 0$, all but the massless, low momentum ones get infinite energy.

Closed string theory on a space with one dimension compactified to a vanishingly small circle is T-dual to a closed string theory on the un-compactified space.

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As a consequence, the spatial compactification to a vanishingly small circle has no effect on the spectrum of the theory. For any light-like compactification radius $R^+$, the energy spectrum of the rest-frame states is just that of the de-compactified theory.

Of course the states which have finite energy and momentum in the frame with compact light-cone must have infinite momentum in the compact spatial direction and infinite energy in the original rest frame. Under T-duality, states with nonzero momentum in the vanishingly small compact direction are exchanged with states with fundamental strings wrapping the very large dual circle. Since they are very long, they have large energy. Thus, light-like compactification does not alter the spectrum of the string theory. What it does is explores the theory in a kinematical regime where there are long fundamental strings wrapping an almost infinite compact direction. This a high energy state in the rest frame string theory and one could in principle study it there. In the infinite momentum frame it is a generic state with finite energy and momentum. This will be the reason why the zero temperature limit of the partition functions that we shall compute in the following are independent of $R^+$.

The thermodynamic partition function is obtained from the trace over physical states of the Boltzmann factor, \( \exp(-\beta \mu P^\mu) \) \[\begin{align*}
Z &= \sum_{N=0}^{\infty} e^{-\frac{\beta}{\mu N/R^+}} \int \frac{d^{D-2}P}{(2\pi)^{D-2}} \cdot e^{-\frac{\beta}{\mu}P^2/2N} \\
&\quad \cdot e^{-\frac{\beta}{8}R^+P^2/2N} \sum_{M^2} \rho(M^2) e^{-\frac{\beta}{8}R^+M^2/2N}
\end{align*}\] where we sum over states in the mass spectrum. These are conveniently found in the light cone gauge by imposing the constraints $L_0 + \bar{L}_0 = 0$ and the level matching condition $L_0 - \bar{L}_0 = N \cdot \text{integer}$. (Details are explained in \[3\]) The first constraint gives the mass shell condition. The second is the level matching condition and is imposed using an integer-valued Lagrange multiplier. The result for the NSR superstring is elegantly summarized in terms of the Hecke operator \[4\] acting on the partition function of the superconformal field theory with target space $R^8$, \[\begin{align*}
- \frac{2\pi \beta \mu R^\mu}{V} &= \mathcal{H}[e^{-\beta \mu \beta^\nu / 2\beta \mu R^\nu}] \ast F[\tau, \bar{\tau}]
\end{align*}\] where \[\begin{align*}
F &= \left[ \left( \frac{1}{4\pi^2 \alpha' \tau_2} \right)^4 \frac{1}{|\eta(\tau)|^{24}} |\theta_2(0, \tau)|^8 \right]_{\tau = \nu}
\end{align*}\] and $\nu = 2\pi \alpha' / \beta \mu R^\mu$ is a fixed constant ($\beta \mu$ is space-like and $R^\mu$ is light-like). The factor in front contains the ratio of volumes of $R^8$ and $R^9 \times S^1$ with compactified light cone. The action of $\mathcal{H}[p]$ on a modular function $\phi(\tau, \bar{\tau})$ is defined by \[\begin{align*}
\mathcal{H}[p] \ast \phi(\tau, \bar{\tau}) &= \sum_{N=1}^{\infty} \sum_{p} P^N \sum_{s \mod k, r, \nu = \text{odd}} \frac{1}{N} \phi \left( \frac{s + \tau r}{k}, \frac{s + \bar{\tau}r}{k} \right)
\end{align*}\] This is similar to other partition functions for conformal field theories on symmetric orbifolds. For a recent discussion see ref.\[5\].

This result is a discretization of the usual Teichmuller space which occurs in the genus 1 string amplitude. Recall that the genus zero contribution is insensitive to compactifications and for the superstring it vanishes. At genus 1, because of the modification of the GSO projection by the finite temperature boundary conditions, the superstring torus amplitude is non-zero. Here, we see that the usual integration over the Teichmuller space of tori is replaced by a summation over discrete parameters

\[\tau = \frac{s + \nu r}{k}\]

2 A theorem about path integrals

It is interesting to see what happens when we compactify a null direction in the path integral representation of the string free energy. We can do this for the
In order to implement this compactification in the path integral, we assume that the world-sheet is a Riemann surface \( \Sigma_g \) of genus \( g \) whose homology group \( H_1(\Sigma_g) \) is generated by the closed curves, \( a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g \)

\[ a_i \cap a_j = \emptyset, \quad b_i \cap b_j = \emptyset, \quad a_i \cap b_j = \delta_{ij} \tag{10} \]

Furthermore, one may pick a basis of holomorphic differentials \( \omega_i \in H^1(\Sigma_g) \) with the properties

\[ \oint_{a_i} \omega_j = \delta_{ij}, \quad \oint_{b_i} \omega_j = \Omega_{ij} \tag{11} \]

where \( \Omega \) is the period matrix. It is complex, symmetric, \( \Omega_{ij} = \Omega_{ji} \), and has positive definite imaginary part.

Compactification is implemented by including the possible windings of the string world-sheet on the compact dimensions. These form distinct topological sectors in the path integration in \( (6) \). In the winding sectors, the bosonic coordinates of the string should have a multi-valued part which changes by \( \beta \)-integer or \( (i) \sqrt{2\pi R} \)-integer as it is moved along a homology cycle. The derivatives of these coordinates should be single-valued functions. It is convenient to consider their exterior derivatives which can be expressed as linear combinations of the holomorphic and anti-holomorphic 1-forms and exact parts,

\[ dX^0 = \sum_{i=1}^{g} (\lambda_i \omega_i + \bar{\lambda}_i \bar{\omega}_i) + \text{exact} \]

\[ dX^9 = \sum_{i=1}^{g} (\gamma_i \omega_i + \bar{\gamma}_i \bar{\omega}_i) + \text{exact} \tag{12} \]

Then, we require

\[ \oint_{a_i} dX^0 = \beta n_i + \sqrt{2\pi R^+} i p_i \]

\[ \oint_{b_i} dX^0 = \beta m_i + \sqrt{2\pi R^+} i q_i \]

\[ \oint_{a_i} dX^9 = \sqrt{2\pi R^+} p_i \]

\[ \oint_{b_i} dX^9 = \sqrt{2\pi R^+} q_i \tag{13} \]
with \(p_i, q_i, m_i, n_i\) integers. With \([13]\), we use these equations to solve for the constants in \([12]\). With the formula \(\int \omega \bar{\omega} = -2\pi (\Omega)_{ij}\), we compute the part of the string action which contains the winding integers, 
\[
S = \frac{\beta^2}{4\pi \alpha'} (n \Omega^i \Omega^{-1} (\Omega n - m) + 2\pi i \sqrt{2} R^+ \frac{1}{4\pi \alpha'} \left[ (p \Omega^i - q) \Omega^{-1} (\Omega n - m) + (n \Omega^i - m) \Omega^{-1} (\Omega p - q) \right] + \ldots
\]
(14)
Note that the integers \(p_i\) and \(q_i\) appear linearly in a purely imaginary term in the action. This is the only place that they appear in the string path integral (unlike \(m_i\) and \(n_i\) which could appear in the GSO projection). When the action is exponentiated and summed over \(p_i\) and \(q_i\), the result will be periodic Dirac delta functions. These delta functions impose a linear constraint on the period matrix of the world-sheet. Thus, with the appropriate Jacobian factor, the net effect is to insert into the path integral measure the following expression,
\[
\sum_{n=1}^{g} \frac{e^{-\frac{\beta^2}{4\pi \alpha'} (n \Omega^i \Omega^{-1} (\Omega n - m) |\det \Omega_2|}}{\nu^{2g} \prod_{j=1}^{g} \delta ((n_i + i\nu r_i) \Omega_{ij} - (m_j + i\nu s_j))}
\]
(15)
where \(\nu = 4\pi \alpha' / \sqrt{2} R^+\), the same constant as in \([1]\) if we specialize to the temperature D-vector \(\beta_0 \equiv \beta\), all other components vanishing. Consequently, the integration over metrics in the string path integral is restricted to those for which the period matrix obeys the constraint
\[
\sum_{i=1}^{g} (n_i + i\nu r_i) \Omega_{ij} - (m_j + i\nu s_j) = 0
\]
(16)
for all combinations of the 4g integers \(m_i, n_i, r_i, s_i\) such that \(\Omega\) is in a fundamental domain of period matrices for surfaces of genus \(g\).

Since the columns of the period matrix are linearly independent vectors, these are \(g\) independent complex constraints on the moduli space of \(\Sigma_g\). Thus its complex dimension \(3g - 3\) is reduced to \(2g - 3\) and there is further discrete data contained in the integers. One would expect that, when the compactifications are removed, either \(\beta \to \infty\) or \(R^+ \to \infty\), the discrete data assembles itself to a “continuum limit” which restores the complex dimension of moduli space.

It is interesting to ask whether the Riemann surfaces with the constraint \([16]\) can be classified in a sensible way. The answer to this question is yes, a Riemann surface obeys the constraint \([16]\) if and only if it is a branched cover of the torus, \(T^2\), with Teichmuller parameter \(iv\). This is established through the

**Theorem:** \(\Sigma_g\) is a branched cover of \(T^2\) if and only if the period matrix obeys \([16]\), for some choice of integers \(m_i, n_i, r_i\) and \(s_i\).

The proof can be found in ref. \([1]\).

As a concrete example, the constraint \([16]\) can be solved explicitly for genus one. The torus amplitude for the finite temperature type II superstring was given in the NSR formulation by Attick and Witten \([3]\).

The modification of their formula by the null compactification can be found using \([13]\),
\[
\frac{F}{V} = -\sum_{\tau \in F} \frac{\nu^2 e^{-\frac{\beta^2 |\tau + m|}{4\pi \alpha' \tau_2}}}{m^2 + \nu^2 n^2} \left( \frac{1}{4\pi \alpha' \tau_2} \right)^5
\]
\[
\cdot \left[ \frac{1}{4\pi |\eta(\tau)|^{24}} \left( \theta_1^2 \theta_2 \theta_4^4 + \theta_2^2 \theta_3 \theta_4^4 + \theta_3^2 \theta_4^4 \right) (0, \tau) + e^{i\pi(m+n)} \left( \theta_2^2 \theta_3 \theta_4^4 + \theta_3^2 \theta_4^4 \right) (0, \tau) - e^{i\pi m} \left( \theta_3^2 \theta_4^4 + \theta_4^2 \theta_4^4 \right) (0, \tau) \right]
\]
(17)
where the solution of \([16]\) yields the discrete Teichmuller parameter,
\[
\tau = \frac{m + ivs}{n + ivr}
\]
and one should sum over the integers so that \(\tau\) is in the fundamental domain, \(F\),
\[
F \equiv \left\{ \tau_1 + iv \tau_2 \left| -\frac{1}{2} < \tau_1 \leq \frac{1}{2} ; |\tau| \geq 1 ; \tau_2 > 0 \right\}
\]
(18)
Modular transformations and identities for theta functions can be used to rewrite \([17]\) as the Hecke
operator acting on the partition function of a superconformal field theory, with torus world-sheet and target space $R^8$ seen in the formulae (3), (4) and (5).

3 Implications for the matrix model

M-theory is a parameter free quantum mechanical system which has 11-dimensional super-Poincare symmetry, 11-dimensional supergravity as its low energy limit and produces the five known consistent superstring theories at various limits of its moduli space. Details of its dynamics are thus far unknown. The matrix model [7], [8] model is conjectured to describe the full dynamics of M-theory in a particular kinematical context, the infinite-momentum frame.

An important check of the matrix model conjecture would be to use it to reproduce perturbative string theory. Most straightforward is the IIA superstring which is gotten by compactification of a spatial direction of M-theory. With this compactification, the matrix model itself becomes 1+1-dimensional, maximally-supersymmetric Yang-Mills theory. According to Dijkgraaf, Verlinde and Verlinde [9], the string degrees of freedom which emerge in the perturbative string limit are simultaneous eigenvalues of the matrices. At finite temperature, the matrices are defined on a torus [3] and their eigenvalues, since they solve polynomial equations, are functions on branched covers of the torus. If the matrix model is to agree with perturbative string theory, these branched covers must be the full set of Riemann surfaces that contribute to the string path integral. In the work that we have reviewed here, we have indeed seen that this is the case, the moduli spaces of branched covers which occur in the matrix model and the world-sheets that occur in the measure in the string path integral are identical.

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