Intuitionistic $L$-fuzzy $\beta$-covering Rough Set

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ABSTRACT

By introducing the concepts of intuitionistic $L$-fuzzy $\beta$-covering and intuitionistic $L$-fuzzy $\beta$-neighborhood, we define three kinds of intuitionistic $L$-fuzzy $\beta$-covering rough set models. The basic properties of those intuitionistic $L$-fuzzy $\beta$-covering rough set models are investigated. Moreover, we define the other three kinds of intuitionistic $L$-fuzzy $\beta$-covering rough set models by using the former three models. Finally, we present the matrix representations of the newly defined lower and upper approximation operators so that the calculation of lower and upper approximations of subsets can be converted into operations on matrices.

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1. Introduction

Rough set (RS) theory was firstly introduced by Pawlak [1]. It may be seen as an extension of set theory and has been found to be a new mathematical tool to deal with insufficient and incomplete information systems. In Pawlak’s rough set model, the equivalence relation is an important concept that is used to construct the lower and upper approximations of an arbitrary subset of the universe of discourse. However, the condition of the equivalence relation is highly restrictive and may limit the application of rough sets in many practical problems. Hence, numerous extensions of Pawlak’s rough set were proposed by replacing the equivalence relation with a few mathematical concepts that are more general in nature, for example, arbitrary binary relations [2–4], neighborhood systems, covering-based and Boolean algebras [5–8].

Covering rough set theory is an important generalization of the classical rough set theory. It was firstly introduced by Zakowski [9]. After then, many kinds of different covering rough sets are introduced [10–13]. Meanwhile, fuzzy covering rough set and fuzzy $\beta$-covering rough set are important generalizations of the classical covering rough set theory [14].

The concept of an intuitionistic fuzzy (IF for short) set, initiated by Atanassov [15–17], is another important tool for dealing with imperfect and imprecise information. Compared with Zadeh’s fuzzy set, an IF set is more objective than a fuzzy set to describe the vagueness
of data, because IF set gives both a membership and a non-membership degree of which an element belongs to a set [18–21]. As an important generalization of IF set, Atanassov and Stoeva defined intuitionistic L-fuzzy (ILF for short) set in 1984, which actually is an IF set based on residuated lattice L.

ILF set and covering rough set are the important generalizations of the classical fuzzy set and rough set, but related work is rarely done to combine the ILF set theory with the covering rough set theory. Motivated by this, in this paper, by introducing the concepts of intuitionistic L-fuzzy β-covering and intuitionistic L-fuzzy β-neighborhood, we define several differentiable rough set models, obtain some basic properties of those models, and present the matrix representations of the newly defined lower and upper approximation operators.

2. Preliminaries

In this section, we introduce some basic concepts of residuated lattice and intuitionistic L-fuzzy set used in this paper. We refer to [22–26] for residuated lattice theory and to [18] for intuitionistic L-fuzzy set.

2.1. Complete Regular Residuated Lattice

Definition 2.1 ([22]): A residuated lattice is an algebraic structure $L = (L; \land, \lor, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(i) $(L; \land, \lor, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1;

(ii) $(L; *, 1)$ is a commutative monoid with the identity 1;

(iii) $L$ satisfies the adjointness property, i.e. $\forall x, y, z \in L, x \ast y \leq z$ iff $x \leq y \rightarrow z$.

A residuated lattice is said to be complete if the underlying lattice is complete.

Definition 2.2 ([26]): The negator on $L$ is the mapping $\neg : L \rightarrow L$ defined by $\neg a = a \rightarrow 0$ for every $a \in L$. If $\neg \neg a = a$ for all $a \in L$, then $L$ is called a regular residuated lattice.

In this paper, if there is no further statement, $L$ always denotes a complete regular residuated lattice.

Theorem 2.1 ([18]): Let $L$ be a complete regular residuated lattice. Then

\begin{align*}
(R1) & \quad a \ast b \leq a \land b; \\
(R2) & \quad a = 1 \rightarrow a; \\
(R3) & \quad a \leq b \iff a \rightarrow b = 1; \\
(R4) & \quad a \rightarrow (b \rightarrow c) = (a \ast b) \rightarrow c; \\
(R5) & \quad a \rightarrow b \geq b; \\
(R6) & \quad a \ast (\lor_i b_i) = \lor_i (a \ast b_i); \\
(R7) & \quad a \rightarrow (\land_i b_i) = \land_i (a \rightarrow b_i);
\end{align*}
(R8) \((\bigvee_i b_i) \rightarrow a = \bigwedge_i (b_i \rightarrow a)\);
(R9) \(\neg(\bigvee_{i \in \Gamma} a_i) = \bigwedge_{i \in \Gamma} \neg a_i, \neg(\bigwedge_{i \in \Gamma} a_i) = \bigvee_{i \in \Gamma} \neg a_i\).

2.2. Intuitionistic L-fuzzy Set

In this subsection, we introduce the notion of ILF set that can be regarded as the generalization of IF set induced by Atansssov [1, 2]. At first, we introduce a special lattice \(\tilde{L}\). This can be seen as a generalization of the set \(\{ (x, y) \in [0, 1] \times [0, 1] | x + y \leq 1 \}\).

**Definition 2.3 ([18]):** \(\tilde{L} = \{ (x_1, x_2) | x_1, x_2 \in L, x_1 \leq \neg x_2 \}\). The partial ordering \(\leq_{\tilde{L}}\) is defined as follows:

\[ x \leq_{\tilde{L}} y \iff x_1 \leq y_1, \quad x_2 \geq y_2, \quad \forall x = (x_1, x_2), \quad y = (y_1, y_2) \in \tilde{L}. \]

The pair \((\tilde{L}, \leq_{\tilde{L}})\) is a complete lattice, with the smallest element \(0_{\tilde{L}} = (0, 1)\) and the greatest element \(1_{\tilde{L}} = (1, 0)\).

The meet operator \(\wedge\) and the join operator \(\vee\) on \(\tilde{L}\) which are connected to the partial ordering \(\leq_{\tilde{L}}\) are defined as follows:

\[ x \wedge_{\tilde{L}} y = (x_1 \wedge y_1, x_2 \vee y_2), \quad x \vee_{\tilde{L}} y = (x_1 \vee y_1, x_2 \wedge y_2). \]

**Definition 2.4 ([18]):** Let \(U\) be a non-empty universe of discourse. An ILF set \(A\) in \(U\) is an object having the form

\[ A = \{ < x, \mu_A(x), \nu_A(x) > | x \in U \}, \]

where \(\mu_A : U \rightarrow L\) and \(\nu_A : U \rightarrow L\) satisfy \(\mu_A(x) \leq \neg \nu_A(x)\) for all \(x \in U\).

They are called the degree of \(L\) membership and the degree of \(L\) non-membership of the element \(x\) to \(A\), respectively. The family of all ILF subsets in \(U\) is denoted by \(ILF(U)\).

We denote \(A(x) = (\mu_A(x), \nu_A(x))\), then \(A \in ILF(U)\) if and only if \(A \in \tilde{L}\) for all \(x \in U\). \((\alpha, \beta)\) denotes the constant ILF sets for all \((\alpha, \beta) \in \tilde{L}\). If \(U\) is a non-empty finite set, then ILF set \(A\) can be denoted by a matrix \(A = ((\mu_A(x_1), \nu_A(x_1)), \ldots, (\mu_A(x_n), \nu_A(x_n)))\).

For any \(A, B, A_i (i \in \Gamma) \in ILF(U)\), some operations are introduced as follows:

1. \(A \subseteq B\) iff \(A(x) \leq_{\tilde{L}} B(x)\).
2. \(A = B\) iff \(A \subseteq B\) and \(B \subseteq A\).
3. \(A \cap B = \{ < x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) > | x \in U \}\) means \((A \cap B)(x) = A(x) \wedge_{\tilde{L}} B(x)\).
4. \(A \cup B = \{ < x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) > | x \in U \}\) means \((A \cup B)(x) = A(x) \vee_{\tilde{L}} B(x)\).
5. \(\bigcap_{i \in \Gamma} A_i = \{ < x, \bigwedge_{i \in \Gamma} \mu_A_i(x), \bigvee_{i \in \Gamma} \nu_A_i(x) > | x \in U \}\) means \((\bigcap_{i \in \Gamma} A_i)(x) = (\bigwedge_{i \in \Gamma} \mu_A_i)(x)\).
6. \(\bigcup_{i \in \Gamma} A_i = \{ < x, \bigvee_{i \in \Gamma} \mu_A_i(x), \bigwedge_{i \in \Gamma} \nu_A_i(x) > | x \in U \}\) means \((\bigcup_{i \in \Gamma} A_i)(x) = (\bigvee_{i \in \Gamma} \mu_A_i)(x)\).
7. \(\sim A = \{ < x, \nu_A(x), \mu_A(x) > | x \in U \} \).
3. Intuitionistic L-fuzzy Rough Set

3.1. Intuitionistic L-fuzzy Covering

In this subsection, we introduce the concepts of intuitionistic L-fuzzy \( \beta \)-covering and intuitionistic L-fuzzy \( \beta \)-neighborhood of a non-empty universe of discourse \( U \), and investigate the basic properties of it.

**Definition 3.1:** Let \( U \) be an arbitrary universal set, and \( L^U \) be the intuitionistic L-fuzzy power set of \( U \). For each \( \beta \in \bar{L} \), we call \( C = \{C_1, C_2, \ldots, C_m\} \), with \( C_i \in ILF(U)(i = 1, 2, \ldots, m) \), an intuitionistic L-fuzzy \( \beta \)-covering of \( U \), if for each \( x \in U \), \( C_i \in C \) exists, such that \( C_i(x) \geq \beta \). We also call \((U, C)\) an intuitionistic L-fuzzy \( \beta \)-covering approximation space.

**Definition 3.2:** Let \( C = \{C_1, C_2, \ldots, C_m\} \) be an intuitionistic L-fuzzy \( \beta \)-covering of \( U \) for some \( \beta \in L \). For each \( x \in U \), we define the intuitionistic L-fuzzy \( \beta \)-neighborhood \( N_x^\beta \) of \( x \) as

\[
N_x^\beta = \cap \{C_i \in C | C_i(x) \geq \beta \}.
\]

**Proposition 3.1:** Let \((U, C)\) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space. \( C = \{C_1, C_2, \ldots, C_m\} \) being a fuzzy \( \beta \)-covering of \( U \) for some \( \beta \in L \), \( N_x^\beta = \{y \in U | N_x^\beta (y) \geq \beta \} \), then

(i) \( x \in \overline{N_x^\beta} \); 
(ii) \( \overline{N_x^\beta} = \bigcup_{y \in \overline{N_x^\beta}} \overline{N_y^\beta} \).

**Proof:**

(i) For all \( x \in U \), \( N_x^\beta (x) = \cap \{C_i \in C | C_i(x) \geq \beta \}(x) = \bigwedge_{C_i(x) \geq \beta} C_i(x) \geq \beta \), hence \( x \in \overline{N_x^\beta} \).

(ii) On the one hand, since \( x \in \overline{N_x^\beta} \), we have \( \overline{N_x^\beta} \subseteq \overline{N_x^\beta} \bigcup \bigcup_{y \neq x, y \in \overline{N_x^\beta}} \overline{N_y^\beta} = \bigcup_{y \in \overline{N_x^\beta}} \overline{N_y^\beta} \).

On the other hand, let \( y \in \overline{N_x^\beta} \), then \( N_x^\beta (y) \geq \beta \), we have \( \overline{N_x^\beta} \subseteq \{C_i | C_i(y) \geq \beta \} \). Therefore, \( \overline{N_x^\beta} \supseteq \overline{N_y^\beta} \). Let \( z \in \overline{N_y^\beta} \), then \( N_y^\beta (z) \geq \beta \), we have \( N_x^\beta (z) \geq \beta \) and \( z \in \overline{N_x^\beta} \). Furthermore, we have \( \overline{N_y^\beta} \subseteq \overline{N_x^\beta} \bigcup_{y \in \overline{N_x^\beta}} \bigcup_{y \in \overline{N_x^\beta}} \overline{N_y^\beta} \subseteq \overline{N_x^\beta} \).

**Proposition 3.2:** Let \((U, C)\) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space. If \( \beta_1 \preceq \beta_2 \), then \( N_x^{\beta_2} \supseteq N_x^{\beta_1} \).

**Proof:** Since \( \beta_1 \preceq \beta_2 \), then for every \( c_i \in \{c_i | C_i(x) \geq \beta_2\} \), \( c_i(x) \geq \beta_2 \geq \beta_1 \), this means \( c_i \in \{c_i | C_i(x) \geq \beta_1\} \). Hence, \( \{c_i | C_i(x) \geq \beta_2\} \subseteq \{c_i | C_i(x) \geq \beta_1\} \). Furthermore, \( N_x^{\beta_2} = \cap \{c_i | C_i(x) \geq \beta_2\} \subseteq \cap \{c_i | C_i(x) \geq \beta_1\} = N_x^{\beta_1} \).

**Proposition 3.3:** Let \((U, C)\) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space. If \( y \in \overline{N_x^\beta} \) and \( z \in \overline{N_y^\beta} \), then \( z \in \overline{N_x^\beta} \).
Proof: Assume that \( y \in \overline{N}_x^\beta \) and \( z \in \overline{N}_y^\beta \), then \( N_x^\beta(y) \geq \beta \) and \( N_y^\beta(z) \geq \beta \). This means \( N_x^\beta \in \{c_i|c_i(y) \geq \beta\} \) and \( N_x^\beta \geq N_y^\beta = \bigcap\{c_i|c_i(y) \geq \beta\} \). Furthermore, we have \( N_x^\beta(z) \geq N_y^\beta(z) \geq \beta \) and \( z \in \overline{N}_x^\beta \).

Example 3.1: Let \( L = \{0, a, b, c, d, e, 1\} \), \( \rightarrow \) and \( \otimes \) be defined as follows:

| \( \rightarrow \) | 0 | a | b | c | d | e | 1 |
|-----------------|---|---|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| a               | e | 1 | 1 | 1 | 1 | 1 | 1 |
| b               | b | e | 1 | e | e | 1 | 1 |
| c               | c | c | e | e | 1 | e | 1 |
| d               | d | e | e | e | e | 1 | 1 |
| e               | a | e | e | e | e | 1 | 1 |
| 1               | 0 | a | b | c | d | e | 1 |

| \( \otimes \) | 0 | a | b | c | d | e | 1 |
|-----------------|---|---|---|---|---|---|---|
| 0               | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a               | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| b               | 0 | 0 | 0 | a | a | a | b |
| c               | 0 | 0 | a | 0 | a | a | c |
| d               | 0 | 0 | a | 0 | a | 0 | d |
| e               | 0 | 0 | a | a | a | a | e |
| 1               | 0 | a | b | c | d | e | 1 |

The partial order \( \leq \) on \( L \) is defined by Figure 1.
It is easy to check that $L$ is a complete regular residuated lattice, and
\[ \tilde{L} = \{(0,0), (0,a), (0,b), (0,c), (0,d), (0,e), (0,1), (a,0), (a,a), (a,b), (a,c), (a,d), (a,e), (b,0), (b,a), (b,b), (c,0), (c,a), (c,c), (d,0), (d,a), (d,d), (e,0), (e,a), (1,0)\}. \]

Let
\[ U = \{x_1, x_2, x_3, x_4\}, \quad \beta = (a,e) \in \tilde{L}, \]
\[ C_1 = \frac{(a,b)}{x_1} + \frac{(0,e)}{x_2} + \frac{(d,a)}{x_3} + \frac{(0,b)}{x_4}, \]
\[ C_2 = \frac{(b,0)}{x_1} + \frac{(b,a)}{x_2} + \frac{(a,b)}{x_3} + \frac{(0,e)}{x_4}, \]
\[ C_3 = \frac{(0,c)}{x_1} + \frac{(a,c)}{x_2} + \frac{(a,d)}{x_3} + \frac{(b,b)}{x_4}, \]
then \{C_1, C_2, C_3\} is a fuzzy $\beta$-covering of $U$.

\[ N^\beta_{x_1} = C_1 \cap C_2 = \frac{(a,b)}{x_1} + \frac{(0,e)}{x_2} + \frac{(a,b)}{x_3} + \frac{(0,e)}{x_4}, \]
\[ N^\beta_{x_2} = C_2 \cap C_3 = \frac{(0,c)}{x_1} + \frac{(a,c)}{x_2} + \frac{(a,e)}{x_3} + \frac{(0,e)}{x_4}, \]
\[ N^\beta_{x_3} = C_1 \cap C_2 \cap C_3 = \frac{(0,e)}{x_1} + \frac{(0,e)}{x_2} + \frac{(a,e)}{x_3} + \frac{(0,e)}{x_4}, \]
\[ N^\beta_{x_4} = C_3 = \frac{(0,c)}{x_1} + \frac{(a,c)}{x_2} + \frac{(a,d)}{x_3} + \frac{(b,b)}{x_4}. \]

Furthermore, we have $N^\beta_{x_1} = \{x_1, x_3\}$, $N^\beta_{x_2} = \{x_2, x_3\}$, $N^\beta_{x_3} = \{x_3\}$, $N^\beta_{x_4} = \{x_2, x_3, x_4\}$.

### 3.2. Intuitionistic L-fuzzy Covering Rough Set

In this subsection, we will define three kinds of basic intuitionistic $L$-fuzzy $\beta$-covering rough set models and investigate the properties of it. At first, we define the I-type $L$-fuzzy $\beta$-covering rough set model by using the $L$-fuzzy $\beta$-covering of $U$.

**Definition 3.3:** Let $(U, C)$ be an intuitionistic $L$-fuzzy $\beta$-covering approximation space with $C = \{C_1, C_2, \ldots, C_m\}$ being a fuzzy $\beta$-covering of $U$ for some $\beta \in L$. For each $A \in ILF(U)$, we define the I-type lower approximation $R^\beta_{\bot}(A)$ and the I-type upper approximation $R^\beta_{\top}(A)$ of $x$ as

\[ R^\beta_{\bot}(A)(x) = \bigwedge_{y \in N^\beta_x} A(y), \]
\[ R^\beta_{\top}(A)(x) = \bigvee_{y \in N^\beta_x} A(y). \]

In fact, the I-type $L$-fuzzy $\beta$-covering model above is a generalization of the classical fuzzy rough set model, which simply uses $\overline{N}^\beta_x$ induced by intuitionistic $L$-fuzzy $\beta$-neighborhood $N^\beta_x$ to replace the equivalence class in the fuzzy rough set model. Therefore, it is one of the most basic covering rough set models.
Definition 3.4: Let \((U, C)\) be an intuitionistic \(L\)-fuzzy \(\beta\)-covering approximation space with \(C = \{C_1, C_2, \ldots, C_m\}\) being a fuzzy \(\beta\)-covering of \(U\) for some \(\beta \in \mathbb{L}\). For each \(A \in \text{ILF}(U)\), we define the III-type lower approximation \(\overline{R}_\beta^L(A)\) and the III-type upper approximation \(\overline{R}_\beta^U(A)\) of \(x\) as

\[
\begin{align*}
\overline{R}_\beta^L(A)(x) &= \bigwedge_{y \in \bigcap N_x^\beta \in U} \left( \neg N_x^\beta(y) \lor A(y) \right), \\
\overline{R}_\beta^U(A)(x) &= \bigvee_{y \in \bigcap N_x^\beta \in U} \left( N_x^\beta(y) \land A(y) \right).
\end{align*}
\]

Definition 3.5: Let \((U, C)\) be an intuitionistic \(L\)-fuzzy \(\beta\)-covering approximation space with \(C = \{C_1, C_2, \ldots, C_m\}\) being a fuzzy \(\beta\)-covering of \(U\) for some \(\beta \in \mathbb{L}\). For each \(A \in \text{ILF}(U)\), we define the III-type lower approximation \(\overline{R}_\beta^L(A)\) and the III-type upper approximation \(\overline{R}_\beta^U(A)\) of \(x\) as

\[
\begin{align*}
\overline{R}_\beta^L(A)(x) &= \bigwedge_{y \in U} \left( \neg N_x^\beta(y) \lor A(y) \right), \\
\overline{R}_\beta^U(A)(x) &= \bigvee_{y \in U} \left( N_x^\beta(y) \land A(y) \right).
\end{align*}
\]

In the definition of III-type lower (upper) approximation operator, if we consider \(N_x^\beta(y)\) as an intuitionistic \(L\)-fuzzy binary relation on \(U\), \(\neg N_x^\beta(y) \lor A(y)\) is understood as a classical fuzzy implication operator \(I(a, b) = (1 - a) \lor b\), then III-type lower (upper) approximation operator is a generalization of lower (upper) approximation operator of definition 4.4 in literature [23]. In fact, if we take arbitrary fuzzy implication \(I\) based on ILF sets instead of classic fuzzy implication \(I\), then we can define some other types of intuitionistic \(L\)-fuzzy covering rough set.

Example 3.2: In Example 3.1, let \(A = (0, d)/x_1 + (a, c)/x_2 + (b, a)/x_3 + (e, a)/x_4\), \(\beta = (a, b)\), then \(C = \{C_1, C_2, C_3\}\) being a fuzzy \(\beta\)-covering of \(U\), and

\[
\begin{align*}
N_{x_1}^\beta &= C_1 \cap C_2 = \frac{(a, b)}{x_1} + \frac{(0, e)}{x_2} + \frac{(a, b)}{x_3} + \frac{(0, e)}{x_4}, \quad \overline{N}_{x_1}^\beta = \{x_1, x_3\}. \\
N_{x_2}^\beta &= C_2 = \frac{(b, 0)}{x_1} + \frac{(b, a)}{x_2} + \frac{(a, b)}{x_3} + \frac{(0, e)}{x_4}, \quad \overline{N}_{x_2}^\beta = \{x_1, x_2, x_3\}. \\
N_{x_3}^\beta &= C_1 \cap C_2 = \frac{(a, b)}{x_1} + \frac{(0, e)}{x_2} + \frac{(a, b)}{x_3} + \frac{(0, e)}{x_4}, \quad \overline{N}_{x_3}^\beta = \{x_1, x_3\}. \\
N_{x_4}^\beta &= C_3 = \frac{(0, c)}{x_1} + \frac{(a, c)}{x_2} + \frac{(a, d)}{x_3} + \frac{(b, b)}{x_4}, \quad \overline{N}_{x_4}^\beta = \{x_4\}.
\end{align*}
\]

Furthermore, we have

\[
\begin{align*}
\overline{R}_\beta^L(A)(x_1) &= A(x_1) \land A(x_3) = (0, d) \land (b, a) = (0, d), \\
\overline{R}_\beta^L(A)(x_2) &= A(x_1) \land A(x_2) \land A(x_3) = (0, d) \land (a, c) \land (b, a) = (0, e), \\
\overline{R}_\beta^L(A)(x_3) &= A(x_1) \land A(x_3) = (0, d) \land (b, a) = (0, d), \\
\overline{R}_\beta^L(A)(x_4) &= A(x_4) = (e, a).
\end{align*}
\]
Hence,
\[
R_{ij}^\beta(A) = \frac{(0, d)}{x_1} + \frac{(0, e)}{x_2} + \frac{(0, d)}{x_3} + \frac{(e, a)}{x_4},
\]
\[
R_{ii}^\beta(A)(x_1) = (\neg N_{x_1}^\beta(x_1) \vee A(x_1)) \land (\neg N_{x_1}^\beta(x_3) \vee A(x_3))
\]
\[
= (\neg (a, b) \vee (0, d)) \land (\neg (a, b) \vee (b, a))
\]
\[
= (b, a),
\]
\[
R_{ii}^\beta(A)(x_2) = (\neg N_{x_2}^\beta(x_1) \vee A(x_1)) \land (\neg N_{x_2}^\beta(x_2) \vee A(x_2)) \land (\neg N_{x_2}^\beta(x_3) \vee A(x_3))
\]
\[
= (\neg (b, 0) \vee (0, d)) \land (\neg (b, a) \vee (a, c)) \land (\neg (a, b) \vee (a, b))
\]
\[
= (0, a),
\]
\[
R_{ii}^\beta(A)(x_3) = (\neg N_{x_3}^\beta(x_1) \vee A(x_1)) \land (\neg N_{x_3}^\beta(x_3) \vee A(x_3))
\]
\[
= (\neg (a, b) \vee (0, d)) \land (\neg (a, b) \vee (b, a))
\]
\[
= (b, a),
\]
\[
R_{ii}^\beta(A)(x_4) = (\neg N_{x_4}^\beta(x_4) \vee A(x_4))
\]
\[
= (\neg (b, b) \vee (e, a))
\]
\[
= (b, b),
\]
\[
R_{il}^\beta(A) = \frac{(b, a)}{x_1} + \frac{(0, a)}{x_2} + \frac{(b, a)}{x_3} + \frac{(e, a)}{x_4}.
\]
\[
R_{il}^\beta(A)(x_1) = (\neg N_{x_1}^\beta(x_1) \vee A(x_1)) \land (\neg N_{x_1}^\beta(x_2) \vee A(x_2)) \land (\neg N_{x_1}^\beta(x_3) \vee A(x_3))
\]
\[
\land (\neg N_{x_1}^\beta(x_4) \vee A(x_4))
\]
\[
= (\neg (a, b) \vee (0, d)) \land (\neg (0, e) \vee (a, c)) \land (\neg (a, b) \vee (b, a)) \land (\neg (0, e) \vee (e, a))
\]
\[
= (b, a),
\]
\[
R_{il}^\beta(A)(x_2) = (\neg N_{x_2}^\beta(x_1) \vee A(x_1)) \land (\neg N_{x_2}^\beta(x_2) \vee A(x_2)) \land (\neg N_{x_2}^\beta(x_3) \vee A(x_3))
\]
\[
\land (\neg N_{x_2}^\beta(x_4) \vee A(x_4))
\]
\[
= (\neg (b, 0) \vee (0, d)) \land (\neg (b, a) \vee (a, c)) \land (\neg (a, b) \vee (b, a)) \land (\neg (0, e) \vee (e, a))
\]
\[
= (0, a),
\]
\[
R_{il}^\beta(A)(x_3) = R_{ii}^\beta(A)(x_1) = (b, a),
\]
\[
R_{il}^\beta(A)(x_4) = (\neg N_{x_4}^\beta(x_1) \vee A(x_1)) \land (\neg N_{x_4}^\beta(x_2) \vee A(x_2)) \land (\neg N_{x_4}^\beta(x_3) \vee A(x_3))
\]
\[
\land (\neg N_{x_4}^\beta(x_4) \vee A(x_4))
\]
\[
= (\neg (0, c) \vee (0, d)) \land (\neg (a, c) \vee (a, c)) \land (\neg (a, d) \vee (b, a)) \land (\neg (b, b) \vee (e, a))
\]
\[
= (c, a),
\]
\[
R_{il}^\beta(A) = \frac{(b, a)}{x_1} + \frac{(0, a)}{x_2} + \frac{(b, a)}{x_3} + \frac{(c, a)}{x_4}.
\]
Similarly, we have
\[
\overline{R}_1^\beta(A) = \frac{(b, a)}{x_1} + \frac{(b, a)}{x_2} + \frac{(b, a)}{x_3} + \frac{(b, c)}{x_4},
\]
\[
\overline{R}_II^\beta(A) = \frac{(a, b)}{x_1} + \frac{(a, a)}{x_2} + \frac{(a, b)}{x_3} + \frac{(b, b)}{x_4},
\]
\[
\overline{R}_{III}^\beta(A) = \frac{(a, a)}{x_1} + \frac{(a, a)}{x_2} + \frac{(a, a)}{x_3} + \frac{(b, a)}{x_4}.
\]

**Remark 3.1:** In Example 3.2, we can see that
\[
\overline{R}_1^\beta(A)(x_1) = (0, d) \neq (b, a) = \overline{R}_II^\beta(A)(x_1),
\]
\[
\overline{R}_II^\beta(A)(x_4) = (b, b) \neq (c, a) = \overline{R}_{III}^\beta(A)(x_4),
\]
\[
\overline{R}_I^\beta(A)(x_1) = (0, d) \neq (b, a) = \overline{R}_{II}^\beta(A)(x_1),
\]
\[
\overline{R}_I^\beta(A)(x_1) = (b, a) \neq (a, b) = \overline{R}_II^\beta(A)(x_1),
\]
\[
\overline{R}_I^\beta(A)(x_1) = (a, b) \neq (a, a) = \overline{R}_{II}^\beta(A)(x_1),
\]
\[
\overline{R}_I^\beta(A)(x_1) = (b, a) \neq (a, a) = \overline{R}_{III}^\beta(A)(x_1),
\]
we have \(\overline{R}_I^\beta(A), \overline{R}_II^\beta(A), \overline{R}_{III}^\beta(A)\) as three kinds of different lower approximation operators and \(\overline{R}_I^\beta(A), \overline{R}_II^\beta(A), \overline{R}_{III}^\beta(A)\) as three kinds of different lower approximation operators. But they have the following relationships:

**Theorem 3.1:** Let \((U, C)\) be an intuitionistic L-fuzzy \(\beta\)-covering approximation space, then

(i) \(R_I^\beta(A) \leq R_{II}^\beta(A), \overline{R}_{II}^\beta(A) \leq \overline{R}_I^\beta(A)\);

(ii) \(R_{III}^\beta(A) \leq R_{II}^\beta(A), \overline{R}_{II}^\beta(A) \leq \overline{R}_{III}^\beta(A)\).

**Proof:**

(i) Since \(A(y) \leq \sim N_\chi^\beta(y) \lor A(y)\),
\[
R_I^\beta(A)(x) = \bigwedge_{y \in N_\chi^\beta} A(y) \leq \bigwedge_{y \in N_\chi^\beta} (\sim N_\chi^\beta(y) \lor A(y)) = \overline{R}_I^\beta(A)(x).
\]
Similarly, we have \(\overline{R}_I^\beta(A) \leq \overline{R}_I^\beta(A)\).

(ii) Since \(N_\chi^\beta \subseteq U\), we have
\[
R_{II}^\beta(A)(x) = \bigwedge_{y \in U} (\sim N_\chi^\beta(y) \lor A(y)) \leq \bigwedge_{y \in N_\chi^\beta} (\sim N_\chi^\beta(y) \lor A(y)) = \overline{R}_{II}^\beta(A)(x).
\]
Similarly, we have \(\overline{R}_{II}^\beta(A) \leq \overline{R}_{II}^\beta(A)\).
**Theorem 3.2:** Let \((U, C)\) be an intuitionistic L-fuzzy \(\beta\)-covering approximation space.

(i) if \(\beta = 1_L\), then \(R^\beta_U(A) = R^\beta_1(A)\);

(ii) if \(\beta = 0_L\), then \(R^\beta_U(A) = R^\beta_{\emptyset}(A)\).

**Proof:**

(i) If \(\beta = 1_L\), then \(\forall y \in \overline{N}_x^\beta\); we have \(\overline{N}_x^\beta(y) = 1_L\). Hence \(R^\beta_U(A)(x) = \bigwedge_{y \in \overline{N}_x^\beta}(\neg\overline{N}_x^\beta(y) \lor A(y)) = \bigwedge_{y \in \overline{N}_x^\beta}A(y) = R^\beta_1(A)(x)\).

(ii) If \(\beta = 0_L\), then \(\overline{N}_x^\beta = U\;\text{or};\) we have \(R^\beta_U(A)(x) = \bigwedge_{y \in \overline{N}_x^\beta}(\neg\overline{N}_x^\beta(y) \lor A(y)) = \bigwedge_{y \in U}(\neg\overline{N}_x^\beta(y) \lor A(y)) = R^\beta_{\emptyset}(A)(x)\).

\[\blacksquare\]

**Example 3.3:** In Example 3.1, let

\[
C_1 = \frac{(1,0)}{x_1} + \frac{(0,e)}{x_2} + \frac{(d,a)}{x_3} + \frac{(0,b)}{x_4},
\]

\[
C_2 = \frac{(b,0)}{x_1} + \frac{(1,0)}{x_2} + \frac{(1,0)}{x_3} + \frac{(0,e)}{x_4},
\]

\[
C_3 = \frac{(0,c)}{x_1} + \frac{(1,0)}{x_2} + \frac{(a,d)}{x_3} + \frac{(1,0)}{x_4},
\]

\[
C_4 = \frac{(0,c)}{x_1} + \frac{(1,0)}{x_2} + \frac{(a,e)}{x_3} + \frac{(1,0)}{x_4}.
\]

\(\beta = (1,0) \in \mathbb{L}\), then \(C_1, C_2, C_3, C_4\) being an \(L\)-fuzzy \(\beta\)-covering of \(U\). Suppose that \(A = (e,a)/x_1 + (b,a)/x_2 + (a,c)/x_3 + (0,e)/x_4\), we have \(R^\beta_U(A)(x_1) = (e,a) \neq (a,a) = R^\beta_1(A)(x_1)\). If \(\beta = (0,1) \in \mathbb{L}\), we have \(R^\beta_U(A)(x_1) = (0,e) \neq (e,a) = R^\beta_{\emptyset}(A)(x_1)\).

**Theorem 3.3:** For the operators \(R^\beta_U(A)\) and \(R^\beta_1(A)\), the following properties hold:

(i) \(R^\beta_U(0_L) = 0_L, R^\beta_1(0_L) = 0_L\);

(ii) \(R^\beta_U(1_L) = 1_L, R^\beta_1(1_L) = 1_L\);

(iii) \(R^\beta_U(A) = \sim R^\beta_1(\sim A)\) and \(R^\beta_1(A) = \sim R^\beta_U(\sim A)\);

(iv) \(R^\beta_U(A \cap B) = R^\beta_U(A) \cap R^\beta_U(B), R^\beta_1(A \cup B) = R^\beta_1(A) \cup R^\beta_1(B)\);

(v) \(R^\beta_U(A \cup B) \supseteq R^\beta_U(A) \cup R^\beta_U(B), R^\beta_1(A \cap B) \subseteq R^\beta_1(A) \cap R^\beta_1(B)\);

(vi) \(\forall A \in ILF(U), R^\beta_U(A) \subseteq A \subseteq R^\beta_1(A)\);

(vii) If \(A \subseteq B\), then \(R^\beta_U(A) \subseteq R^\beta_U(B), R^\beta_1(A) \subseteq R^\beta_1(B)\);

(viii) \(R^\beta_U(R^\beta_1(A)) = R^\beta_U(A), R^\beta_1(R^\beta_U(A)) = R^\beta_1(A)\).

**Proof:** Straightforward. \[\blacksquare\]
Theorem 3.4: For the operators \( R^\beta_\beta(A) \) and \( R^\beta_\beta(A) \), the following properties hold:

(i) \( R^\beta_\beta(1_L) = 1_L, R^\beta_\beta(0_L) = 0_L \);
(ii) \( R^\beta_\beta(A) \Leftrightarrow R^\beta_\beta(\sim A) \) and \( R^\beta_\beta(A) \Leftrightarrow R^\beta_\beta(\sim A) \);
(iii) \( R^\beta_\beta(A \cap B) = R^\beta_\beta(A) \cap R^\beta_\beta(B), R^\beta_\beta(A \cup B) = R^\beta_\beta(A) \cup R^\beta_\beta(B) \);
(iv) \( R^\beta_\beta(A \cup B) \geq R^\beta_\beta(A) \cup R^\beta_\beta(B), R^\beta_\beta(A \cap B) \leq R^\beta_\beta(A) \cap R^\beta_\beta(B) \);
(v) If \( A \subseteq B \), then \( R^\beta_\beta(A) \subseteq R^\beta_\beta(B), R^\beta_\beta(A) \subseteq R^\beta_\beta(B) \);
(vi) \( R^\beta_\beta(A) \leq \beta \lor A \) and \( R^\beta_\beta(A) \geq \beta \land A \). Particularly, when \( \beta = 1_L \), we have that \( R^\beta_\beta(A) \subseteq A \subseteq R^\beta_\beta(A) \).

Proof:

(i) \( R^\beta_\beta(1_L) = \bigwedge_{y \in \mathcal{N}_x} (\sim N^\beta_x(y) \lor (1, 0)) = 1_L, R^\beta_\beta(0_L) = \bigvee_{y \in \mathcal{N}_x} (N^\beta_x(y) \land (0, 1)) = 0_L \).

(ii) \( R^\beta_\beta(\sim A)(x) = \bigwedge_{y \in \mathcal{N}_x} (\sim N^\beta_x(y) \lor \sim A(y)) \) 
\( = \bigwedge_{y \in \mathcal{N}_x} (N^\beta_x(y) \land A(y)) \) 
\( = \bigvee_{y \in \mathcal{N}_x} (N^\beta_x(y) \land A(y)) \) 
\( = R^\beta_\beta(A)(x) \).

Similarly, we have \( R^\beta_\beta(A) \equiv R^\beta_\beta(\sim A) \).

(iii) For each \( x \in U \),
\( R^\beta_\beta(A \cap B)(x) = \bigwedge_{y \in \mathcal{N}_x} (\sim N^\beta_x(y) \lor (A \cap B)(y)) \)
\( = \bigwedge_{y \in \mathcal{N}_x} ((\sim N^\beta_x(y) \lor A(y)) \land (\sim N^\beta_x(y) \land B(y))) \)
\( = \bigwedge_{y \in \mathcal{N}_x} (\sim N^\beta_x(y) \lor (A(y)) \land (\sim N^\beta_x(y) \lor B(y))) \)
\( = R^\beta_\beta(A)(x) \cap R^\beta_\beta(B)(x) \).

Similarly, we have \( R^\beta_\beta(A \cup B) = R^\beta_\beta(A) \cup R^\beta_\beta(B) \).

(iv) For each \( x \in U \),
\( R^\beta_\beta(A \cup B)(x) = \bigwedge_{y \in \mathcal{N}_x} (N^\beta_x(y) \land (A \cup B)(y)) \)
\( = \bigwedge_{y \in \mathcal{N}_x} ((N^\beta_x(y) \land A(y)) \lor (N^\beta_x(y) \land B(y))) \)
\( \geq \bigwedge_{y \in \mathcal{N}_x} (N^\beta_x(y) \land A(y)) \lor (\bigwedge_{y \in \mathcal{N}_x} (N^\beta_x(y) \land B(y))) \)
\( = R^\beta_\beta(A)(x) \cup R^\beta_\beta(B)(x) \).
Similarly, we have \( R^\beta_{III}(A \cap B) \subseteq R^\beta_{III}(A) \cap R^\beta_{III}(B) \).

(v) Since \( A \subseteq B \), we have \( A(x) \leq B(x) \) for every \( x \in U \). Furthermore,

\[
R^\beta_{III}(A)(x) = \bigwedge_{y \in N^\beta_x} \left( \sim N^\beta_x(y) \lor A(y) \right)
\]

\[
\leq \bigwedge_{y \in N^\beta_x} \left( \sim N^\beta_x(y) \lor B(y) \right)
\]

\[
= R^\beta_{III}(B)(x).
\]

(vi) For each \( x \in U \), \( R^\beta_{III}(A)(x) = \bigwedge_{y \in N^\beta_x} \left( \sim N^\beta_x(y) \lor A(y) \right) \leq \sim N^\beta_x(x) \lor A(x) \leq \sim \beta \lor A(x) \), then \( R^\beta_{III}(A) \subseteq \sim \beta \lor A(x) \). Similarly, we have \( R^\beta_{III}(A) \geq \beta \lor A \). Particularly, when \( \beta = 1_L \), we have \( R^\beta_{III}(A) \subseteq A \subseteq R^\beta_{III}(A) \).

\[\Box\]

**Remark 3.2:** In Example 3.2, \( R^\beta_{III}(R^\beta_{III}(A))\) \((x_2) = (a, a) \neq (0, a) = R^\beta_{III}(A)(x_2), R^\beta_{III}(R^\beta_{III}(A))\) \((x_2) = (a, b) \neq (a, a) = R^\beta_{III}(A)(x_2)\). We have \( R^\beta_{III}(R^\beta_{III}(A)) \neq R^\beta_{III}(A), R^\beta_{III}(R^\beta_{III}(A)) \neq R^\beta_{III}(A) \), that is to say the idempotent of lower and upper approximate operators \( R^\beta_{III}, R^\beta_{III} \) does not hold.

**Theorem 3.5:** For the III-type approximate operators \( R^\beta_{III} \) and \( R^\beta_{III} \) the following properties hold:

(i) \( R^\beta_{III}(0_L) = 0_L, R^\beta_{III}(1_L) = 1_L \);

(ii) \( R^\beta_{III}(A) = \sim R^\beta_{III}(\sim A) \) and \( R^\beta_{III}(A) = \sim R^\beta_{III}(\sim A) \);

(iii) \( R^\beta_{III}(A \cap B) = R^\beta_{III}(A) \cap R^\beta_{III}(B), R^\beta_{III}(A \cup B) = R^\beta_{III}(A) \cup R^\beta_{III}(B) \);

(iv) \( R^\beta_{III}(A \cup B) \supseteq R^\beta_{III}(A) \cup R^\beta_{III}(B), R^\beta_{III}(A \cap B) \subseteq R^\beta_{III}(A) \cap R^\beta_{III}(B) \);

(v) If \( A \subseteq B \), then \( R^\beta_{III}(A) \subseteq R^\beta_{III}(B), R^\beta_{III}(A) \subseteq R^\beta_{III}(B) \).

**Definition 3.6:** A subset \( \tau \subseteq ILF(U) \) is called an ILF topology if it satisfies:

(i) \( 1_L, 0_L \in \tau \);

(ii) \( A, B \in \tau \) implies \( A \cap B \in \tau \);

(iii) \( \{ A_i | i \in \Gamma \} \in \tau \) implies \( \bigcup_{i \in \Gamma} A_i \in \tau \).

**Theorem 3.6:** Let \((U, C)\) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space. Then

\[ \tau_C = \{ A \in ILF(U) | R^\beta_{III}(A) = A \} \]

is an ALexandrov topology.

**Proof:**
(i) By (ii) of Theorem 3.3, we have \( R_1^\beta(1_L) = 1_L \). By (i) of Theorem 3.3, we have 
\[ R_1^\beta(0_L) \subseteq 0_L, R_1^\beta(0_L) = 0_L, 0_L \in \tau_C. \]

(ii) Let \( A, B \in \tau_C \), by (iii) of Theorem 3.3, we have 
\[ R_1^\beta(A \cap B) = R_1^\beta(A) \cap R_1^\beta(B) = A \cap B, A \cap B \in \tau_C. \]

(iii) Let \( \{ A_i | i \in \Gamma \} \) be \( \tau \), by (iv) of Theorem 3.3, we have \( A_i = R_1^\beta(A_i) \subseteq R_1^\beta(\bigcup_{i \in \Gamma} A_i) \). Hence, 
\[ \bigcup_{i \in \Gamma} A_i \subseteq R_1^\beta(\bigcup_{i \in \Gamma} A_i). \]
At the same time, by (vi) of Theorem 3.3, we have 
\[ R_1^\beta(\bigcup_{i \in \Gamma} A_i) \subseteq \bigcup_{i \in \Gamma} A_i \].
Hence, \( \bigcup_{i \in \Gamma} A_i = R_1^\beta(\bigcup_{i \in \Gamma} A_i) \). \( \tau \) is an ALEXANDROV topology.

\[ \blacksquare \]

**Corollary 3.1:** Let \( (U, C) \) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space. If \( \beta = 1_L \), then \( \tau_C = \{ A \in ILF(U) | R_1^\beta(A) = A \} \) is an ALEXANDROV topology.

**Proof:** By Theorem 3.2(i), when \( \beta = 1_L \), we have \( R_1^\beta = R_1^\beta \), hence \( \tau_C \) is an ALEXANDROV topology. \[ \blacksquare \]

### 3.3. Some Other Rough Approximate Operators Based on I,II,III-type Rough Approximate Operators

The Pawlak approximation operators satisfy many properties \([4],[5]\). When generalizing Pawlak approximations, one task is to specify a subset of these properties that new approximation operators are required to preserve. Another task is to search for possible generalizations of the various notions used in the three definitions. Furthermore, one must make sure that the generalized approximation operators satisfy the required properties. In this paper, we consider the following properties suggested by Pomykala \([10]\): for all \( A \in ILF \n\)

(i) \( R_1^\beta(\sim A) \equiv \sim R_1^\beta(A) \),

(ii) \( R_1^\beta(A) \subseteq A \subseteq R_1^\beta(A) \).

**Definition 3.7:** Let \((U, C)\) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space with \( C = \{C_1, C_2, \ldots, C_m\} \) being a fuzzy \( \beta \)-covering of \( U \) for some \( \beta \in L \). For each \( A \in ILFS \), we define other three kinds of new lower approximation operators and upper approximation operators of \( A \) as

\[
R_1^\beta(A)(x) = R_1^\beta(A)(x) \land R_1^\beta(A)(x), \quad R_1^\beta(A)(x) = R_1^\beta(A)(x) \lor R_1^\beta(A)(x),
\]

\[
R_1^\beta(A)(x) = R_1^\beta(A)(x) \land R_1^\beta(A)(x), \quad R_1^\beta(A)(x) = R_1^\beta(A)(x) \lor R_1^\beta(A)(x),
\]

\[
R_1^\beta(A)(x) = R_1^\beta(A)(x) \land R_1^\beta(A)(x), \quad R_1^\beta(A)(x) = R_1^\beta(A)(x) \lor R_1^\beta(A)(x).
\]

Then \((R_1^\beta, R_1^\beta, R_1^\beta, R_1^\beta)\), \((R_1^\beta, R_1^\beta, R_1^\beta, R_1^\beta)\), \((R_1^\beta, R_1^\beta, R_1^\beta, R_1^\beta)\) are called IV, V, VI-type rough operators of \( A \), respectively.

**Theorem 3.7:** Let \((U, C)\) be an intuitionistic L-fuzzy \( \beta \)-covering approximation space. Then
(i) $\overline{R}_I^\beta(\sim A) = \sim \overline{R}_I^\beta(A)$;
(ii) $\overline{R}_I^\beta(\sim A) = \sim \overline{R}_I^\beta(A)$;
(iii) $\overline{R}_I^\beta(\sim A) = \sim \overline{R}_I^\beta(A)$.

**Proof:** We only prove (i), (ii) and (iii) can be proved similarly.

$$\overline{R}_I^\beta(\sim A) = \overline{R}_I^\beta(\sim A) \wedge \overline{R}_I^\beta(A)$$
$$= \sim \overline{R}_I^\beta(A) \wedge \sim \overline{R}_I^\beta(A)$$
$$= \sim (\overline{R}_I^\beta(A) \vee \overline{R}_I^\beta(A))$$
$$= \sim \overline{R}_I^\beta(A).$$

**Theorem 3.8:** Let $(U, C)$ be an intuitionistic L-fuzzy $\beta$-covering approximation space. Then

(i) $R_I^\beta(A) \subseteq A \subseteq \overline{R}_I^\beta(A)$;
(ii) $R_I^\beta(A) \subseteq A \subseteq \overline{R}_I^\beta(A)$;
(iii) $R_I^\beta(A) \subseteq A \subseteq \overline{R}_I^\beta(A)$.

**Proof:** (i) By (vi) of Theorem 3.3, for every $x \in U$, we have $\overline{R}_I^\beta(A)(x) \leq A(x) \leq \overline{R}_I^\beta(A)(x)$. Then

$$\overline{R}_I^\beta(A)(x) = \overline{R}_I^\beta(A)(x) \wedge \overline{R}_I^\beta(A)(x) \leq A(x) \wedge \overline{R}_I^\beta(A)(x) \leq A(x)$$

and

$$\overline{R}_V^\beta(A)(x) = \overline{R}_V^\beta(A)(x) \vee \overline{R}_V^\beta(A)(x) \geq A(x) \vee \overline{R}_V^\beta(A)(x) \geq A(x).$$

Hence, $\overline{R}_V^\beta(A)(x) \subseteq A \subseteq \overline{R}_V^\beta(A).$

The relationships between $\overline{R}_I^\beta, \overline{R}_V^\beta, \overline{R}_I^\beta, \overline{R}_V^\beta, \overline{R}_I^\beta, \overline{R}_V^\beta$ can be described as follows (Figure 2):

**Theorem 3.9:** Let $(U, C)$ be an intuitionistic L-fuzzy $\beta$-covering approximation space. Then

(i) $R_I^\beta(\sim A) \leq \overline{R}_I^\beta(\sim A)$;
(ii) $R_I^\beta(\sim A) \leq \overline{R}_I^\beta(\sim A)$;
(iii) $R_I^\beta(\sim A) \leq \overline{R}_I^\beta(\sim A)$;
(iv) $R_I^\beta(\sim A) \leq \overline{R}_I^\beta(\sim A)$.

**Remark 3.3:** In Example 3.2, $\overline{R}_I^\beta(A) = (0, d)/x_1 + (0, e)/x_2 + (0, d)/x_3 + (c, a)/x_4, R_I^\beta(\sim A) = (0, e)/x_1 + (0, e)/x_2 + (0, e)/x_3 + (b, b)/x_4$. $\overline{R}_I^\beta(A)(x_4)$ and $R_I^\beta(\sim A)(x_4)$ are not comparable,
hence $R^β_{IV}(A)$ and $R^β_{IV}(A)$ are incomparable. At the same time, in Example 3.2, $R^β_{IV}(A)(x_3) = (b, a) \geq (b, c) = R^β_{IV}(A)(x_3)$ and $R^β_{IV}(A)(x_4) = (b, e) \leq (b, a) = R^β_{IV}(A)(x_4)$. $R^β_{IV}(A)$ and $R^β_{IV}(A)$ are also incomparable.

3.4. Matrix Representations of Lower and Upper Approximation Operators

In this subsection, we will present matrix representations of the lower and upper approximation operators defined in Definitions 3.3–3.5. The matrix representations of the approximation operators make it possible to calculate the lower and upper approximations of subsets through the operations on matrices, which is algorithmic, and can easily be implemented through the computer.
Definition 3.8: Let \( A = (a_{ij})_{n \times m} \) and \( B = (b_{kj})_{m \times l} \) be two matrices. We define \( C = A \ominus B = (c_{ij})_{n \times l} \) and \( D = A \oplus B = (d_{ij})_{n \times l} \) as follows:

\[
c_{ij} = \bigwedge_{k=1}^{m} (a_{ik} \lor b_{kj}), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, l,
\]

\[
d_{ij} = \bigvee_{k=1}^{m} (a_{ik} \land b_{kj}), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, l.
\]

Obviously, for arbitrary \( A, B, C \), if \( B \leq C \), then \( A \ominus B \leq A \ominus C \), \( B \ominus A \geq C \ominus A \), \( A \oplus B \leq A \oplus C \) and \( B \oplus A \leq C \oplus A \). Meanwhile, \( A \ominus B \neq B \ominus A \) and \( A \oplus B \neq B \oplus A \).

Definition 3.9: Let \( (U, C) \) be an intuitionistic \( L \)-fuzzy \( \beta \)-covering approximation space. We define matrix \( M^\beta_I, M^\beta_{II}, M^\beta_{III} \) as follows:

\[
M^\beta_I(x, y) = \begin{cases} 1_L, & y \in \mathcal{N}_x^\beta, \\ 0_L, & \text{otherwise}, \end{cases}
\]

\[
M^\beta_{II}(x, y) = \begin{cases} \mathcal{N}_x^\beta(y), & y \in \mathcal{N}_x^\beta, \\ 0_L, & \text{otherwise}, \end{cases}
\]

\[
M^\beta_{III}(x, y) = \mathcal{N}_x^\beta(y).
\]

Here \( \mathcal{N}_x^\beta \) is defined as in Definition 3.2.

From Definition 3.9, we can see that \( M^\beta_{II} \leq M^\beta_I \) and \( M^\beta_{III} \leq M^\beta_{II} \).

Theorem 3.10: Let \( (U, C) \) be an intuitionistic \( L \)-fuzzy \( \beta \)-covering approximation space, \( A \in \text{ILF}(U) \). Then \( \overline{R}_I^\beta(A) = M^\beta_I \ominus A^T, \overline{R}_{II}^\beta(A) = M^\beta_{II} \oplus A^T (i \in \{I, II, III\}) \).

Example 3.4: In Example 3.2, take \( \beta = (a, b) \), then

\[
M^\beta_I = \begin{pmatrix} 1_L & 0_L & 1_L & 0_L \\ 1_L & 1_L & 1_L & 0_L \\ 1_L & 0_L & 1_L & 0_L \\ 0_L & 0_L & 0_L & 1_L \end{pmatrix}, \quad M^\beta_{II} = \begin{pmatrix} (a, b) & 0_L & (a, b) & 0_L \\ (b, 0) & (b, a) & (a, b) & 0_L \\ (a, b) & 0_L & (a, b) & 0_L \\ 0_L & 0_L & 0_L & (b, b) \end{pmatrix},
\]

\[
M^\beta_{III} = \begin{pmatrix} (a, b) & (0, e) & (a, b) & (0, e) \\ (b, 0) & (b, a) & (a, b) & (0, e) \\ (a, b) & (0, e) & (a, b) & (0, e) \\ (0, c) & (a, c) & (a, d) & (b, b) \end{pmatrix}.
\]

If \( A = \frac{(0, d)}{x_1} + \frac{(a, c)}{x_2} + \frac{(b, a)}{x_3} + \frac{(e, a)}{x_4} \), we have

\[
\overline{R}_I^\beta(A) = M^\beta_I \ominus A^T = \begin{pmatrix} 1_L & 0_L & 1_L & 0_L \\ 1_L & 1_L & 1_L & 0_L \\ 1_L & 0_L & 1_L & 0_L \\ 0_L & 0_L & 0_L & 1_L \end{pmatrix} \ominus \begin{pmatrix} (0, d) \\ (a, c) \\ (b, a) \\ (e, e) \end{pmatrix} = \begin{pmatrix} (0, d) \\ (0, e) \\ (0, d) \\ (e, e) \end{pmatrix}.
\]
\[
\begin{align*}
R^*_I(A) &= M^*_I \odot A^T = \begin{pmatrix}
(a, b) & 0_L & (a, b) & 0_L \\
(b, 0) & (b, a) & (a, b) & 0_L \\
(a, b) & 0_L & (a, b) & 0_L \\
0_L & 0_L & 0_L & (b, b)
\end{pmatrix} \odot \begin{pmatrix}
(0, d) \\
(a, c) \\
(b, a) \\
(e, a)
\end{pmatrix} = \begin{pmatrix}
(b, a) \\
(0, a) \\
(b, a) \\
(e, a)
\end{pmatrix}, \\
R^*_II(A) &= M^*_II \odot A^T = \begin{pmatrix}
(a, b) & (0, e) & (a, b) & (0, e) \\
(b, 0) & (b, a) & (a, b) & (0, e) \\
(a, b) & (0, e) & (a, b) & (0, e) \\
(0, c) & (a, c) & (a, d) & (b, b)
\end{pmatrix} \odot \begin{pmatrix}
(0, d) \\
(a, c) \\
(b, a) \\
(e, a)
\end{pmatrix} = \begin{pmatrix}
(b, a) \\
(0, a) \\
(b, a) \\
(c, a)
\end{pmatrix}, \\
\overline{R}^*_I(A) &= M^*_I \oplus A^T = \begin{pmatrix}
1_L & 0_L & 1_L & 0_L \\
1_L & 1_L & 1_L & 0_L \\
1_L & 0_L & 1_L & 0_L \\
0_L & 0_L & 0_L & 1_L
\end{pmatrix} \oplus \begin{pmatrix}
(0, d) \\
(a, c) \\
(b, a) \\
(e, a)
\end{pmatrix} = \begin{pmatrix}
(b, a) \\
(b, a) \\
(b, a) \\
(e, a)
\end{pmatrix}, \\
\overline{R}^*_II(A) &= M^*_II \oplus A^T = \begin{pmatrix}
(a, b) & 0_L & (a, b) & 0_L \\
(b, 0) & (b, a) & (a, b) & 0_L \\
(a, b) & 0_L & (a, b) & 0_L \\
0_L & 0_L & 0_L & (b, b)
\end{pmatrix} \oplus \begin{pmatrix}
(0, d) \\
(a, c) \\
(b, a) \\
(e, a)
\end{pmatrix} = \begin{pmatrix}
(a, b) \\
(a, a) \\
(a, b) \\
(b, b)
\end{pmatrix}, \\
\overline{R}^*_III(A) &= M^*_III \oplus A^T = \begin{pmatrix}
(a, b) & (0, e) & (a, b) & (0, e) \\
(b, 0) & (b, a) & (a, b) & (0, e) \\
(a, b) & (0, e) & (a, b) & (0, e) \\
(0, c) & (a, c) & (a, d) & (b, b)
\end{pmatrix} \oplus \begin{pmatrix}
(0, d) \\
(a, c) \\
(b, a) \\
(e, a)
\end{pmatrix} = \begin{pmatrix}
(a, b) \\
(a, a) \\
(a, b) \\
(b, a)
\end{pmatrix}.
\end{align*}
\]

4. Conclusions

The theories of covering rough sets, intuitionistic \(L\)-fuzzy sets, are important for dealing with uncertainty and inaccuracy problems. In order to handle these uncertainty and inaccuracy problems more effectively, the combination of the covering and intuitionistic \(L\)-fuzzy sets is further researched in this paper. The concepts of intuitionistic \(L\)-fuzzy \(\beta\)-covering and intuitionistic \(L\)-fuzzy \(\beta\)-neighborhood are proposed. Several intuitionistic \(L\)-fuzzy \(\beta\)-covering rough set models are discussed. Some properties of these models are proved and demonstrated by some examples. The research results fill in the blanks of the study of the intuitionistic \(L\)-fuzzy rough set with the method of covering. We will investigate the applications of the presented models and construct multi-granulation intuitionistic \(L\)-fuzzy covering rough sets, which will be part of the future research directions considered by our group.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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