From differential crossed modules to tensor hierarchies

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Abstract

The present paper, though inspired by the use of tensor hierarchies in theoretical physics, establishes their mathematical credentials, especially as genetically related to differential crossed modules. Gauging procedures in supergravity rely on a pairing – the embedding tensor – between a Leibniz algebra and a Lie algebra. Two such algebras, together with their embedding tensor, form a triple called a Lie-Leibniz triple, of which differential crossed modules are particular cases. This paper is devoted to showing that any Lie-Leibniz triple induces a differential graded Lie algebra – its associated tensor hierarchy – whose restriction to the category of differential crossed modules is the canonical assignment associating to any differential crossed module its corresponding unique 2-term differential graded Lie algebra. This shows that Lie-Leibniz triples form natural generalizations of differential crossed modules and that their associated tensor hierarchies can be considered as some kind of ‘lie-ization’ of the former. We deem the present construction of such tensor hierarchies clearer and more straightforward than previous derivations. We stress that such a construction suggests the existence of further well-defined Leibniz gauge theories.

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1 Introduction

Gauged models of supergravity usually involve a Lie algebra of symmetries \( \mathfrak{g} \), of which only a sub-algebra \( \mathfrak{h} \) is promoted to become a gauge algebra. To preserve the symmetries manifest when performing their computations, physicists stick to a formulation of the theory that involves the bigger Lie algebra \( \mathfrak{g} \). One of the consequences is that the gauge theory needs the addition of several fields of higher form degree—and their associated field strengths, leading to what physicists call a tensor hierarchy \[8, 9, 10\]. Mathematically, it is

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a (possibly infinite) tower of \( \mathfrak{g} \)-modules – that by convention physicists take to be positively graded\( ^1 \) – such that the space at degree \( k > 0 \) depends exclusively on the spaces of lower degree. These spaces contain the redundant information carried by the fields of higher degree; they are necessary to preserve the covariance of the theory. The need for a unified perspective on gauging procedures in supergravity, as well as in Double and Exceptional field theories, has been salient in theoretical physics for some years now [3, 4, 6, 15].

On the mathematical side, another approach to tensor hierarchies has been recently explored: that of Leibniz algebras [2, 18, 19]. A Leibniz algebra is a vector space \( V \) equipped with a bilinear product \( \circ \) which is a derivation of itself, i.e. it satisfies the Leibniz rule, hence the denomination [21]:

\[
x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z)
\]

The construction of tensor hierarchies relies on the existence of embedding tensors. If \( \mathfrak{g} \) is a Lie algebra and if a Leibniz algebra \( V \) is a \( \mathfrak{g} \)-module, an embedding tensor is a map \( \Theta : V \to \mathfrak{g} \) satisfying some consistency conditions – called the linear and quadratic constraints, respectively. The quadratic constraint implies that \( \mathfrak{h} := \text{Im}(\Theta) \) is a sub-Lie algebra of \( \mathfrak{g} \), whereas what we call the linear constraint requires that the Leibniz product on \( V \) is generated by the action of \( \mathfrak{h} \) on \( V \):

\[
x \circ y = \Theta(x) \cdot y
\]

In other words, the embedding tensor lifts the adjoint action to \( \mathfrak{g} \):

\[\begin{array}{ccc}
V & \xrightarrow{\text{ad}} & \text{End}(V) \\
\Theta & \downarrow & \rho \\
\mathfrak{g} & \downarrow & \end{array}\]

Some authors do not put much emphasis on this latter constraint, or consider that the Leibniz algebra structure is a mere by-product of the definition of \( \Theta \). On the contrary, we believe that Equation (3) is an important aspect of the embedding tensor, because it draws a parallel with the notion of differential crossed modules. We call Lie-Leibniz triples such compatible triples \((\mathfrak{g}, V, \Theta)\), and we indeed notice that differential crossed modules are precisely those Lie-Leibniz triples such that \( V \) is a Lie algebra and \( \Theta \) is \( \mathfrak{g} \)-equivariant.

It has been proved that any Lie-Leibniz triple \((\mathfrak{g}, V, \Theta)\) gives rise canonically to a non-positively graded differential graded Lie algebra [19], which might coincide with the one defined from Borcherds algebras in [23]. Unfortunately, the construction presented in [19] is not very satisfying, mathematically speaking, because the role of \( \mathfrak{g} \) – although central in the Lie-Leibniz triple – is not salient in the resulting differential graded Lie algebra. In the present paper we provide an alternative, simpler construction, that generalizes the results in [19] and that additionally reinstates the prominent role of \( \mathfrak{g} \). The result is a differential graded Lie algebra \( \mathcal{T} = \bigoplus_{i=-1}^{\infty} T_{-i} \) concentrated in degrees lesser than or equal to +1:

\[
\ldots \rightarrow T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V[1] \xrightarrow{\partial_{0}=\Theta} \mathfrak{g} \xrightarrow{\partial_{1}} R_{\Theta}[-1]
\]

\(^1\)We have decided, in the present paper, to take the opposite convention: the tensor hierarchies will be negatively graded. The justification of this choice is that when one goes from tensor hierarchies to \( L_{\infty} \)-algebras, the degree of the \( k \)-brackets is indeed \( 2-k \) (and not \( k-2 \), as it would be if the algebra had been positively graded).

\(^2\)Notice that what we call 'linear constraint' actually slightly differs from what physicists call 'linear constraint' in this context (see footnote 6).

\(^3\)In the present paper, to represent the action of an element \( a \in \mathfrak{g} \) on a vector \( x \) of any \( \mathfrak{g} \)-module \( R \), we use a map \( \rho : \mathfrak{g} \to \text{End}(R) \) or we use a dot, and we write:

\[
a \cdot x = \rho(a; x)
\]
where \( R_\Theta[-1] \) is the cyclic \( \mathfrak{g} \)-submodule of \( \text{Hom}(V, \mathfrak{g}) \) generated by \( \Theta \), the degree having been shifted by +1. We call tensor hierarchy associated to the Lie-Leibniz triple \((g, V, \Theta)\) this differential graded Lie algebra \( T \). The negatively graded part \( T_\bullet = \bigoplus_{i=1}^\infty T_{-i} \) is obtained as a quotient of the free graded Lie algebra of \( V[1] \) by a particular graded ideal, as was proposed by earlier work in theoretical physics \([5, 12]\). A priori, it is not a resolution, although we raise the question that under some circumstances it might be (see Conjecture \([1]\)).

Although it is injective on objects, the function that assigns to each Lie-Leibniz triple its associated tensor hierarchy is not functorial on the category \( \text{Lie-Leib} \) of Lie-Leibniz triples. However, we notice that its restriction to the full subcategory \( \text{diff} \times \text{Mod} \) of differential crossed modules has a very particular image. More precisely, any differential crossed module \( g_{-1} \xrightarrow{\Theta} g_0 \) is sent by this function to its corresponding canonical 2-term differential graded Lie algebra \([1]\) or, more precisely, to the following 3-term differential graded Lie algebra:

\[
g_{-1} \xrightarrow{\Theta} g_0 \xrightarrow{0} \mathbb{R}[-1] \tag{4}\]

Here, \( \mathbb{R}[-1] \) is the irreducible trivial representation of \( g_0 \), to which the \( g_0 \)-submodule \( R_\Theta \subset \text{Hom}(g_{-1}, g_0) \) is isomorphic, since \( \Theta \) is \( g_0 \)-equivariant by definition of differential crossed modules. We observe that this assignment defines a faithful functor on \( \text{diff} \times \text{Mod} \). It is thus reasonable to look for the biggest subcategory of \( \text{Lie-Leib} \) on which the restriction of the function assigning to each Lie-Leibniz triple its associated tensor hierarchy is functorial.

By construction, the tensor hierarchy associated to a Lie-Leibniz triple encodes the symmetric part of the Leibniz product:

\[
\{x, y\} = \frac{1}{2}(x \circ y + y \circ x) \tag{5}\]

This bracket vanishes when \( V \) is a Lie algebra, so in particular when \((g, V, \Theta)\) is a differential crossed module, resulting in the chain complex \([4]\). The prominence of the symmetric bracket \([5]\) and the \( g \)-action in the construction of the tensor hierarchy associated to a given Lie-Leibniz triple \((g, V, \Theta)\) can be mostly seen in the fact that first space of the hierarchy, namely \( T_{-2} \), is the quotient of \( S^2(V) \) by the biggest \( g \)-submodule of \( \text{Ker} \{\ldots, \} \subset S^2(V) \), denoted \( K \). We say that a morphism of Lie-Leibniz triples \((\varphi, \chi) : (g, V, \Theta) \to (g', V', \Theta')\) is compatible is it sends \( K \subset \text{Ker} \{\ldots, \} \subset S^2(V) \) to \( K' \subset \text{Ker} \{\ldots, \}' \subset S^2(V') \). Then, Lie-Leibniz triples together with compatible morphisms form a wide subcategory \( \text{compLie-Leib} \) of the category \( \text{Lie-Leib} \), and \( \text{diff} \times \text{Mod} \) is a full subcategory of both: \( \text{diff} \times \text{Mod} \subset \text{compLie-Leib} \subset \text{Lie-Leib} \). We can now state the main result of this paper, which can be found as Theorem \([13]\).

**Theorem.** The functor

\[
\overline{G} : \text{diff} \times \text{Mod} \longrightarrow \text{DGLie} \leq_1 \]

\[
(g_{-1} \xrightarrow{\Theta} g_0) \longmapsto (g_{-1} \xrightarrow{\Theta} g_0 \xrightarrow{0} \mathbb{R}[-1])
\]

can be canonically extended to an injective-on-objects function \( G : \text{Lie-Leib} \to \text{DGLie} \leq_1 \) such that the restriction of this function to the wide sub-category \( \text{compLie-Leib} \) is a faithful functor. Moreover, \( \text{compLie-Leib} \) is the biggest wide subcategory of \( \text{Lie-Leib} \) (with respect to inclusion) such that this functorial property holds.

This theorem shows that, while Lie-Leibniz triples form a natural generalization of differential crossed modules, tensor hierarchies materialize as a generalization of the differential graded Lie algebra formulation of such differential crossed modules, as given in Equation \([1]\). Section \([2]\) of the paper introduces the mathematical background necessary to state the theorem (subsection \([2.1]\)), as well as an explanation of the relationship between differential crossed modules and Lie-Leibniz triples (subsection \([2.2]\)). Section \([2]\) contains the proof of Theorem \([13]\) and is subdivided into four parts: building the graded vector space \( T_\bullet \) and equipping it with a graded Lie bracket in subsection \([3.1]\) showing that it moreover admits a non-trivial differential in subsection
proving that the differential and the graded Lie bracket are compatible in subsection 3.3 and finally proving Theorem 13 in the final subsection 3.4. Though the construction(s) and proofs are technical, they proceed by a sequence of steps, each of which is meaningful and fairly transparent. Clarifying the relationship between differential crossed modules and Lie-Leibniz triples opens the possibility to better understand higher gauge theories. Indeed, the content of this paper suggests that a unified perspective on standard gauge theories and Leibniz gauge theories [2, 26] can emerge through the use of Lie-Leibniz triples and tensor hierarchies.

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2 Lie-Leibniz triples and differential crossed modules

2.1 Leibniz algebras and Lie-Leibniz triples

The beginning of the section involves well-known facts about Leibniz algebras [21] but then we introduce novel notions that will be thoroughly used in the paper.

Definition 1. A (left) Leibniz algebra is a vector space $V$ together with a bilinear operation $\circ : V \otimes V \to V$ satisfying the relation

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z).$$

(6)

A Leibniz subalgebra of $V$ is a subspace $U$ which is stable under the Leibniz product.

A morphism of Leibniz algebras, or Leibniz algebra morphism, between $(V, \circ)$ and $(V', \circ')$ is a linear map $\chi : V \to V'$ that commutes with the respective Leibniz products:

$$\chi(x \circ y) = \chi(x) \circ' \chi(y)$$

(7)

We call $\text{Leib}$ the category of Leibniz algebras, together with Leibniz algebra morphisms.

Remark 1. Given a Leibniz algebra $V$, the vector space of Leibniz algebra morphisms from $V$ to itself should not be confused with the space $\text{End}(V)$ of vector spaces endomorphisms.

Let $(V, \circ)$ be a Leibniz algebra. The Leibniz product $\circ$ can be split into its skew-symmetric part, denoted $[\,,\,] : \wedge^2 V \to V$, and its symmetric part, denoted $\{\,,\,\} : S^2(V) \to V$, defined by:

$$[x, y] = \frac{1}{2}(x \circ y - y \circ x)$$

and

$$\{x, y\} = \frac{1}{2}(x \circ y + y \circ x)$$

for any $x, y \in V$, so that:

$$x \circ y = [x, y] + \{x, y\}$$

(8)
Example 1. A Lie algebra is a Leibniz algebra whose product is skew-symmetric, i.e. such that \( \{ \ldots \} = 0 \). The category \( \text{Lie} \) of Lie algebras with Lie algebra morphisms is a full subcategory of \( \text{Leib} \).

Definition 2. An ideal of \( V \) is a subspace \( K \) of \( V \) that satisfies the following condition:

\[ V \circ K \subseteq K \]

An ideal \( K \) of \( V \) whose left \( \circ \)-product with \( V \) is trivial, i.e. such that \( K \circ V = 0 \), is called central.

Example 2. The sub-space \( I \) of \( V \) generated by elements of the form \( \{ x, x \} \) is an ideal called the ideal of squares of \( V \). A direct application of Equation (6) shows that it is central. The set \( Z \) of all central elements of \( V \), defined by:

\[ Z = \left\{ x \in V \mid x \circ y = 0 \text{ for all } y \in V \right\} \] (9)

is a central ideal of \( V \), and is the biggest such. It is called the center of \( V \).

In a general Leibniz algebra, the symmetric part of the bracket is not associative nor does the skew-symmetric part of the bracket satisfy the Jacobi identity (otherwise it would be a Lie bracket), but using Equation (6), we have:

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = -\frac{1}{3} \left( \{ x, [y, z] \} + \{ y, [z, x] \} + \{ z, [x, y] \} \right) \] (10)

for every \( x, y, z \in V \). Since the right-hand side lies in the ideal of squares \( I \), we deduce that for any central ideal \( K \) satisfying the following inclusions:

\[ I \subseteq K \subseteq Z, \]

the Leibniz product canonically induces a Lie algebra structure on the quotient \( V/K \). The Lie algebra \( V/I \) is the biggest such, whereas \( V/Z \) is the smallest. The quotient map \( V \to V/I \) is universal with respect to morphisms of Leibniz algebras from \( V \) to any Lie algebra \( [21] \).

A (symmetric) representation of a (left) Leibniz algebra is the natural generalization of the notion of a Lie algebra representation:

Definition 3. A (symmetric) representation of a Leibniz algebra \( V \) is the data of a vector space \( E \) and a morphism of Leibniz algebras \( \rho : V \to \text{End}(E) \).

In particular, since \( \text{End}(E) \) is a Lie algebra, the linear map \( \rho \) satisfies the following identity:

\[
\rho(x \circ y) = [\rho(x), \rho(y)] \] (11)

for every \( x, y \in V \). Since the right-hand side is fully skew-symmetric, so should be the left-hand side. Then, the ideal of squares \( I \) is necessarily in the kernel of the map \( \rho \). This is true for every symmetric representation of \( V \). In particular, this implies that such representations correspond to representations of the Lie algebra \( V/I \) \( [21] \).

As an example, one may consider the adjoint action of \( V \) on itself. Indeed, the Leibniz algebra \( V \) is a representation of \( V \) through the adjoint map – denoted \( \text{ad} \) – and defined by:

\[
\text{ad} : V \longrightarrow \text{End}(V) \quad x \longmapsto \text{ad}_x : y \longmapsto x \circ y
\]

The correct notion of representation in the category of Leibniz algebras is the data of a left and a right actions that satisfy some natural consistency conditions \( [21] \). It is only when the right action equals minus the left action that one says that the representation is symmetric.
One can rewrite the Leibniz identity (6) as:

\[ \text{ad}_{\text{ad}_x(y)}(z) = \text{ad}_x(\text{ad}_y(z)) - \text{ad}_y(\text{ad}_x(z)) \]  

which, in turn, can be compactly summarized as:

\[ \text{ad}_{\text{ad}_x(y)} = [\text{ad}_x, \text{ad}_y] \]

where the bracket on the right hand side is the Lie bracket on End(V). Since \( \text{ad}_x(y) = x \circ y \) on the left hand side, this equation proves that the adjoint action indeed induces a representation of \( V \) on itself.

Moreover, the Leibniz identity (6) shows that the map \( \text{ad} \) lands in the derivations of \( V \). A derivation of a Leibniz algebra \((V, \circ)\) is a linear map \( \delta : V \to V \) such that

\[ \delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y). \]

The space of derivations of \( V \), denoted \( \text{Der}(V) \), is a Lie subalgebra of \( \text{End}(V) \). If \( \delta \) is a derivation of \( V \), then it is a derivation of the symmetric and the skew-symmetric parts of the Leibniz product. These results imply that both the ideal of squares \( \mathcal{I} \) and the center \( \mathcal{Z} \) are stable by every derivation of \( V \). The image of the adjoint action \( \text{ad} : V \to \text{Der}(V) \) is a Lie subalgebra of \( \text{Der}(V) \) called the inner derivations of \( V \), and noted \( \text{inn} \). The kernel of this map is the center \( \mathcal{Z} \) of the Leibniz algebra. Hence, the adjoint map defines a Lie algebra isomorphism between \( V / \mathcal{Z} \) and \( \text{inn}(V) \).

We are now interested in the relationship between the adjoint action of a Leibniz algebra and the embedding tensor of gauging procedures in supergravity. In these theories, the (super)-symmetries are controlled by a Lie algebra \( \mathfrak{g} \) and the field content of the physical model lies in various \( \mathfrak{g} \)-modules. It turns out that the gauge fields \( A_\mu \) take values in the fundamental representation of \( \mathfrak{g} \). This is rather unusual, compared to classical Yang-Mills gauge theories where such fields take values in the adjoint representation of \( \mathfrak{g} \). Since gauge transformations are local symmetries generated by the action of vector fields on the Lagrangian, physicists have developed a natural set-up to couple the fundamental representation in which such fields take values with the Lie algebra \( \mathfrak{g} \) of global symmetries, see e.g. [15].

More precisely, the basic data consist of a Lie algebra \( \mathfrak{g} \) and a \( \mathfrak{g} \)-module \( V \) (usually the fundamental representation of \( \mathfrak{g} \)), in which gauge fields take values. Then physicists observe that under some conditions the Lie algebra structure on \( \mathfrak{g} \) can be lifted to a Leibniz algebra structure on \( V \) [15]. Although the elements of \( V \) would play the role of gauge fields, their associated field strength – as given by the usual definition \( F = dA + A \wedge A \) – are not covariant under gauge transformations, due precisely to the absence of skew-symmetry of the Leibniz algebra structure. This observation forces physicists to add higher fields to the model, taking values in additional \( \mathfrak{g} \)-modules, hence ending up with a tower of spaces that physicists call tensor hierarchy [8, 10]. This is a new manifestation of higher gauge theories in theoretical physics. Physicists lift the Lie algebra structure of \( \mathfrak{g} \) to a Leibniz algebra structure on \( V \) by using what they call an embedding tensor. It is a linear mapping \( \Theta : V \to \mathfrak{g} \) satisfying some conditions which, if satisfied, ensure the compatibility between the Lie algebra structure on \( \mathfrak{g} \) and the Leibniz algebra structure on \( V \) in a sense that resembles what occurs in a differential crossed module.

Due to their importance in supergravity theories and in the present paper, triples of elements \((\mathfrak{g}, V, \Theta)\) satisfying these conditions deserve their own name:

**Definition 4.** A Lie-Leibniz triple is a triple \((\mathfrak{g}, V, \Theta)\) where:

- \( \mathfrak{g} \) is a Lie algebra,
- \( V \) is a \( \mathfrak{g} \)-module equipped with a Leibniz algebra structure \( \circ : V \otimes V \to V \),
- \( \Theta \) is an embedding tensor.

The fundamental representation of a Lie algebra is defined as its smallest-dimensional faithful representation.
• Θ : V → g is a linear mapping called the embedding tensor, that satisfies two compatibility conditions. The first one is the linear constraint:

\[ x \circ y = \Theta(x) \cdot y \]  \hspace{1cm} (15)

The second one is called the quadratic constraint:

\[ \Theta(x \circ y) = [\Theta(x), \Theta(y)]_g \]  \hspace{1cm} (16)

where \([\ldots]_g\) is the Lie bracket on g.

A sub-Lie-Leibniz triple of \((g, V, \Theta)\) is a Lie-Leibniz triple \((t, U, \Theta|_U)\) such that t (resp. U) is a Lie subalgebra of g (resp. Leibniz subalgebra of V).

Remark 2. The Leibniz algebra structure on V is uniquely defined from the action of the Lie algebra g on V and Equation (15). This linear constraint shows that the embedding tensor is a lift of the adjoint action along the representation \(\rho\):

\[ \begin{array}{ccc}
V & \xrightarrow{\text{ad}} & \text{End}(V) \\
\downarrow \Theta & & \downarrow \rho \\
g & & 
\end{array} \]

It is thus redundant to assume in Definition 4 that V is a Leibniz algebra beforehand. The same redundancy occurs in the definition of differential crossed modules. Moreover, Equations (15) and (16) are similar to those defining differential crossed modules. We will investigate in subsection 2.2 the extent of the precise relationship between the former and the latter notions.

Usually, in supergravity models, g is often taken to be the split real form of a semi-simple Lie algebra and V is the smallest-dimensional faithful representation of g. Physicists usually call V the fundamental – or defining – representation of g \([27]\). The term ‘embedding tensor’ has been introduced in the 2000s and refers to the fact that the image of the map Θ defines a Lie sub-algebra of g: this can be straightforwardly read on Equation (16). This Lie sub-algebra \(\text{Im}(\Theta) \subset g\) plays the role of the gauge algebra in supergravity theories. For the particular role that this Lie subalgebra plays in this paper, we denote it by h. Throughout this section and the next one, major emphasis will be on the respective roles of g and of h in a general Lie-Leibniz triple, as well as on the relationship between \(\text{ad}\) and Θ. The quadratic constraint (16) additionally implies two things: that \(\text{Ker}(\Theta)\) is an ideal of V and that \(\mathcal{I} \subset \text{Ker}(\Theta)\). The linear constraint then implies that this ideal is central. Thus we have the following successive inclusions:

\[ \mathcal{I} \subset \text{Ker}(\Theta) \subset Z \]  \hspace{1cm} (17)

The embedding tensor Θ then induces a Lie algebra isomorphism between \(V/\text{Ker}(\Theta)\) and h = \(\text{Im}(\Theta)\). The equality \(\text{Ker}(\Theta) = Z\) is satisfied only when V is a faithful representation of g.

Example 3. Let \((V, \circ)\) be a Leibniz algebra and take g = \(\text{End}(V)\). Let \(\Theta = \text{ad}\) be the map that sends any element \(x \in V\) to the adjoint endomorphism \(\text{ad}_x : y \mapsto x \circ y\). The gauge algebra h = \(\text{Im}(\Theta)\) consists of the inner derivations of V. Then \((\text{End}(V), V, \text{ad})\) is a Lie-Leibniz triple.

\[ 6\text{What physicists call linear constraint, or representation constraint, is a condition on } \Theta \text{ that singles out the } g \text{ sub-module of } \text{Hom}(V, g) \text{ to which } \Theta \text{ belongs. In the physics literature, the linear constraint sometimes appears under the form that the symmetric bracket } \{\ldots\} \text{ factors through a } g \text{ sub-module of } S^2(V), \text{ see e.g. Equation (6) in [13]. This version of the linear constraint can also be interpreted as a weakening of one of the axioms of the notion of differential crossed module; see Lemma 10 and the subsequent discussion.} \]
Another way of seeing the quadratic constraint is to notice that it implies that \( \Theta \) is \( \text{Im}(\Theta) \)-equivariant (but not necessarily \( \mathfrak{g} \)-equivariant). This can be shown as follows: \( \text{Hom}(V, \mathfrak{g}) \) is equipped with a \( \mathfrak{g} \)-module structure through the induced action of \( \mathfrak{g} \) on \( V^* \) on the one hand and through the adjoint action of \( \mathfrak{g} \) on itself on the other hand. This defines a map:

\[
\eta : \mathfrak{g} \longrightarrow \text{End}\left(\text{Hom}(V, \mathfrak{g})\right)
\]

\[
a \longmapsto \eta(a; -) : \Xi \mapsto \left( x \mapsto [a, \Xi(x)]_{\mathfrak{g}} - \Xi(a \cdot x) \right)
\]

Then, for any Lie subalgebra \( \mathfrak{t} \) of the Lie algebra \( \mathfrak{g} \), we say that \( \Theta \) is \( \mathfrak{t} \)-equivariant if

\[
\forall a \in \mathfrak{t} \quad a \cdot \Theta = 0 \quad (18)
\]

Then, considering how \( \mathfrak{g} \) acts on \( V \) and on itself, Equation (16) can be rewritten:

\[
\forall x \in V \quad \Theta(x) \cdot \Theta = 0 \quad (19)
\]

In other words, the embedding tensor is \( \text{Im}(\Theta) \)-equivariant. We emphasize that \( \Theta \) needs not be \( \mathfrak{g} \)-equivariant. The difference between the \( \mathfrak{h} \)-equivariance – recall that we set \( \mathfrak{h} = \text{Im}(\Theta) \) – versus the \( \mathfrak{g} \)-equivariance of the embedding tensor \( \Theta \) plays a central role in gauging procedures in supergravity.

Let us now investigate the relationship between the adjoint map and the embedding tensor \( \Theta \), as we have already seen in Remark 2 that the latter is the lift of the former. Given a Lie-Leibniz triple \( (\mathfrak{g}, V, \Theta) \), the adjoint map induced by the Leibniz product is an element of \( \text{Hom}(V, \text{End}(V)) \). Since \( V \) is a representation of \( \mathfrak{g} \), the spaces \( \text{End}(V) \) and \( \text{Hom}(V, \text{End}(V)) \) canonically inherit a structure of \( \mathfrak{g} \)-module. Then, Equation (12) can be rewritten as:

\[
\Theta(x) \cdot \text{ad}_y - \text{ad}_{\Theta(x)}y = 0 \quad (20)
\]

for every \( x, y \in V \). In turn, this is equivalent to:

\[
a \cdot \text{ad} = 0 \quad (21)
\]

for every \( a \in \mathfrak{h} \). That is to say: the map \( \text{ad} \) is \( \mathfrak{h} \)-equivariant, as is the embedding tensor. However, it is not necessarily \( \mathfrak{g} \)-equivariant, i.e. the action of \( \mathfrak{g} \) on \( \text{ad} \) may not be trivial. This discussion justifies the following definition:

**Definition 5.** We say that a Lie-Leibniz triple \( (\mathfrak{g}, V, \Theta) \) is semi-strict if the adjoint map \( \text{ad} : V \to \text{End}(V) \) is \( \mathfrak{g} \)-equivariant, and we say that it is strict if the embedding tensor \( \Theta \) is \( \mathfrak{g} \)-equivariant.

The semi-strictness and the strictness conditions are equivariance properties satisfied by, respectively, the adjoint action and its lift, the embedding tensor. The relationship between these two conditions relies on the relationship between these two maps. As the name indicates, a strict Lie-Leibniz triple is semi-strict but the converse is not necessarily true, as we will now show using another equivalent characterization of semi-strictness:

**Lemma 6.** The Lie-Leibniz triple \( (\mathfrak{g}, V, \Theta) \) is semi-strict if and only if the Lie algebra morphism \( \rho : \mathfrak{g} \to \text{End}(V) \) defining the \( \mathfrak{g} \)-module structure on \( V \) takes values in \( \text{Der}(V) \).

**Proof.** The fact that the map \( \text{ad} \in \text{Hom}(V, \text{End}(V)) \) is \( \mathfrak{g} \)-equivariant translates as Equation (21) for every \( a \in \mathfrak{g} \). Then, following the reverse reasoning than led to Equation (21), one finds that this equation can be equivalently written as:

\[
a \cdot (x \circ y) - x \circ (a \cdot y) - (a \cdot x) \circ y = 0 \quad (22)
\]

for every \( a \in \mathfrak{g} \) and \( x, y \in V \). This equation characterizes the fact that \( \rho : \mathfrak{g} \to \text{End}(V) \) takes values in \( \text{Der}(V) \), hence the result.
Going back to our discussion: using the linear constraint \( (15) \), one can rewrite Equation \((22)\) as:

\[
a \cdot (\Theta(x) \cdot y) - \Theta(a \cdot x) \cdot y - \Theta(x) \cdot (a \cdot y) = 0
\]

(23)

The left-hand side can be rewritten as \((a \cdot \Theta)(x) \cdot y\), hence the property that \((g, V, \Theta)\) is semi-strict is given by the following condition:

\[
\forall a \in g, \quad \forall x, y \in V \quad (a \cdot \Theta)(x) \cdot y = 0
\]

(24)

Thus, if the Lie-Leibniz triple \((g, V, \Theta)\) is strict, i.e. if \(a \cdot \Theta = 0\) for every \(a \in g\), then it is semi-strict, but the converse is not true, for Equation \((24)\) does not necessarily imply that \(\Theta\) is \(g\)-equivariant, unless \(\rho : g \rightarrow \text{End}(V)\) is injective, i.e. unless the representation \(V\) is faithful.\(^7\)

**Remark 3.** The notion of strict Lie-Leibniz triples had already been implicitly introduced in \([16]\), in Definition 2.3., although it had not been given a name at the time because it was not central in the paper.

**Example 4.** Let \((g, V, \Theta)\) be a Lie-Leibniz triple, and set \(h = \text{Im}(\Theta)\). Then, by Equation \((16)\), the Lie-Leibniz triple \((h, V, \Theta)\) is strict.

**Example 5.** By Lemma \([6]\), the Lie-Leibniz triple \((\text{Der}(V), V, \text{ad})\) is semi-strict. Since the following equality \(f(\text{ad}_x(-)) = \text{ad}_f(x)(-)+\text{ad}_x(f(-))\) holds for every derivation \(f\) of \(V\) and \(x \in V\), we deduce that the embedding tensor \(\text{ad}\) is \(\text{Der}(V)\)-equivariant. This means that \((\text{Der}(V), V, \text{ad})\) is actually a strict Lie-Leibniz triple.

**Example 6.** We take the canonical example of \([21]\) generalizing the construction of a Lie algebra associated to any associative algebra. Let \((A, \cdot)\) be a \(\mathbb{R}\) associative algebra equipped with an endomorphism \(D : A \rightarrow A\) which is not necessarily an algebra homomorphism but which satisfies the following identities:

\[
D(x \cdot Dy) = Dx \cdot Dy = D(Dx \cdot y)
\]

(25)

Then the bilinear product \(\circ : A \times A \rightarrow A\) defined as:

\[
x \circ y = Dx \cdot y - y \cdot Dx
\]

(26)

is a Leibniz product on \(A\). If \(D = \text{id}_A\), then we obtain the canonical Lie algebra structure associated to \(A\). Let us denote the corresponding Lie bracket by \(\{,\}A\) - not to be confused with the skew-symmetric part of the Leibniz product \((26)\) - so that Equation \((26)\) then reads:

\[
x \circ y = [Dx, y]_A
\]

(27)

and we say that \(\circ\) is a derived bracket.

With these data we can define a canonical Lie-Leibniz triple associated to any associative algebra \(A\): \(g\) is the Lie algebra \((A, \{,\}A)\), the Leibniz algebra \(V\) is \((A, \circ)\) and the embedding tensor \(\Theta\) is the endomorphism \(D\). The action of the Lie algebra \(g\) on \(V\) is mediated through the adjoint action of the Lie bracket \([,\]A\): \(\rho_x(y) = [x, y]_A\). The linear constraint is precisely Equation \((27)\), while the quadratic constraint is a consequence of properties \((25)\):

\[
D(x \circ y) = D(Dx \cdot y) - D(y \cdot Dx) = Dx \cdot Dy - Dy \cdot Dx = [Dx, Dy]_A
\]

(28)

Moreover, the gauge algebra \(h\) is precisely the image of the endomorphism \(D\), itself a subspace of \(A\). Now, since the Leibniz product is a derived bracket, the Jacobi identity on \(g\) together with properties \((25)\) imply that

\(^7\)In supergravity theories, Lie-Leibniz triples are not necessarily semi-strict, let alone strict, but the representation \(V\) is often taken to be the defining representation of \(g\), which is faithful.
we have, for any \( x, y, z \in A \):

\[
[x, y \circ z]_A = [x, [Dy, z]_A]_A \\
= [[x, Dy]_A, z]_A + [Dy, [x, z]_A]_A \\
= -[y \circ x, z]_A + y \circ [x, z]_A \\
= [x, y]_A \circ z + y \circ [x, z]_A - [y \circ x - D([y, x]_A), z]_A
\]

So, the (semi)-strictness of the Lie-Leibniz triple associated to the associative algebra \( A \) depends on the endomorphism \( D \). The Lie-Leibniz triple is semi-strict if the bracket \([y \circ x - D([y, x]_A), z]_A\) vanishes for every \( x, y, z \) while the strictness condition – the \( g \)-equivariance of the embedding tensor – here reads \([x, Dy]_A = D[x, y]_A\) and is precisely equivalent to the vanishing of \( y \circ x - D([y, x]_A) \).

Let us apply these observations to the following two cases, that were introduced in [21]:

1. \( D \) is an algebra homomorphism and it is idempotent: \( D^2 = D \);
2. \( D \) is a differential: \( D(x \cdot y) = Dx \cdot y + x \cdot Dy \) and \( D^2 = 0 \).

In the first case, we first deduce from the homomorphism property that \( D([x, y]_A) = [Dx, Dy]_A \). By Equation (29) we then conclude that \( D([x, y]_A) = D(x \circ y) \) so the Lie-Leibniz triple associated to the associative algebra \( A \) and the endomorphism \( D \) is strict if and only if \( x \circ y = D(x \circ y) \). In the second case, we deduce from the derivation property that \( [Dx, y]_A = D([x, y]_A) + [Dy, x]_A \) so we obtain the skew-symmetric bracket of the Leibniz product:

\[
[x, y] = \frac{1}{2}D([x, y]_A)
\]

(29)

This has the following consequences: first, \( D(x \circ y) = D([x, y]) = 0 \) because \( D^2 = 0 \), and second: \( y \circ x - D([y, x]_A) = x \circ y \) so the corresponding Lie-Leibniz triple is strict if and only if \( x \circ y = 0 \), i.e. the Leibniz algebra structure is zero.

Let us now observe some consequence of semi-strictness on the symmetric bracket \( \{ \ldots \} \) entailed by the Leibniz algebra structure \( \circ \) on \( V \). This bracket defines a map from \( S^2(V) \) to \( V \), also denoted \( \{ \ldots \} \). This map, as an element of \( \operatorname{Hom}(S^2(V), V) \) is subject to the action of \( g \) induced from the \( g \)-module structure on \( V \):

\[
a \cdot (\{ \ldots \})(x, y) = a \cdot \{x, y\} - \{a \cdot x, y\} - \{x, a \cdot y\}
\]

(30)

for every \( a \in g \) and \( x, y \in V \). The \( g \)-equivariance of the symmetric bracket is equivalent to the vanishing of the right hand side, which is a priori not granted. However, thanks to the linear constraint (15), the left-hand side of Equation (30) can be written as the following:

\[
a \cdot (2\{ \ldots \})(x, y) = ((a \cdot \Theta)(x)) \cdot y + ((a \cdot \Theta)(y)) \cdot x
\]

(31)

Notice that, due to Equation (19), this term is always 0 when \( a \in \mathfrak{h} \), thus implying that the symmetric bracket is always \( \mathfrak{h} \)-equivariant, but there is no guarantee that it is \( g \)-equivariant. However we deduce from Equation (24) that a sufficient condition for the symmetric bracket to be \( g \)-equivariant is that the Lie-Leibniz triple \( (g, V, \Theta) \) is semi-strict, so that the right-hand side of Equation (31) vanishes. These observations imply that, for any Lie-Leibniz triple, the space \( \operatorname{Ker}(\{ \ldots \}) \subseteq S^2(V) \) is always a \( \mathfrak{h} \)-module but not necessarily a \( g \)-module. This latter fact is central in the construction of the tensor hierarchy associated to a Lie-Leibniz triple.
2.2 A relationship between Lie-Leibniz triples and differential crossed modules

In order to study in a more consistent way the notion of Lie-Leibniz triples with respect to what is already known for differential crossed modules, we make use of categories. That is why we now introduce the correct notion for morphisms of Lie-Leibniz triples:

**Definition 7.** A morphism between two Lie-Leibniz triples \((g, V, \Theta)\) and \((g', V', \Theta')\) is a pair \((\varphi, \chi)\) consisting of a Lie algebra morphism \(\varphi : g \to g'\) and a Leibniz algebra morphism \(\chi : V \to V'\), satisfying the following compatibility conditions:

\[
\Theta'(\chi(x)) = \varphi(\Theta(x)) \tag{32}
\]
\[
\varphi(a) \cdot \chi(x) = \chi(a \cdot x) \tag{33}
\]

for every \(a \in g\) and \(x \in V\). That is to say, the following prism is commutative:

\[
\begin{array}{ccc}
g \otimes V & \xrightarrow{\varphi \otimes \chi} & g' \otimes V' \\
\Theta \otimes \text{id} & & \Theta' \otimes \text{id} \\
V \otimes V & \xrightarrow{\chi \otimes \chi} & V' \otimes V' \\
\rho & & \rho' \\
\text{V} & & \text{V'}
\end{array}
\]

where \(\rho\) (resp. \(\rho'\)) denotes the action of \(g\) (resp. \(g'\)) on \(V\) (resp. \(V'\)). We say that \((\varphi, \chi)\) is an isomorphism of Lie-Leibniz triples when both \(\varphi\) and \(\chi\) are isomorphisms in their respective categories.

**Remark 4.** Let \((V, \circ)\) be a Leibniz algebra, then any Lie-Leibniz triple \((g, V, \Theta)\) whose embedding tensor generates the Leibniz product \(\circ\) as given in Equation (15) induces a canonical Lie-Leibniz triple morphism \((\rho, \text{id}_V)\) from \((g, V, \Theta)\) to \((\text{End}(V), V, \text{ad})\). Moreover, if \((g, V, \Theta)\) is semi-strict, the image of the Lie-Leibniz triple morphism \((\rho, \text{id}_V)\) is a sub-Lie-Leibniz triple of \((\text{Der}(V), V, \text{ad})\).

A particular case of Lie-Leibniz triple morphisms are those for which the linear map \(\chi \otimes \chi : S^2V \to S^2V'\) – here, \(\otimes\) symbolizes the symmetric product – preserves a particular submodule of \(\text{Ker}\{\ldots\}\) \(\subset S^2(V)\). Assume that we have a morphism \((\varphi, \chi)\) between two Lie-Leibniz triples \((g, V, \Theta)\) and \((g', V', \Theta')\). The kernel of the symmetric bracket on \(V\) is a subspace \(\text{Ker}\{\ldots\}\) of \(S^2(V)\) and a \(\mathfrak{h}\)-module but not necessarily a \(g\)-module. However, it admits \(g\)-submodules, and among them one is the biggest, that we denote by \(K\). The same occurs for the symmetric bracket on \(V'\) and the biggest \(g'\)-submodule of \(\text{Ker}\{\ldots\}'\) \(\subset S^2(V')\) is denoted \(K'\). Since \(\chi\) is a morphism of Leibniz algebras, we have that:

\[
\chi \otimes \chi(K) \subset \text{Ker}\{\ldots\}' \tag{34}
\]

By Equation (33), the vector space \(\chi \otimes \chi(K)\) is only a \(\varphi(g)\)-module, and not a \(g'\)-module. So a priori, there is no reason that \(\chi \otimes \chi(K)\) would be a subspace of \(K'\). We decide to give a name to morphisms that precisely have this property:

**Definition 8.** We say that a morphism of Lie-Leibniz triples \((\varphi, \chi) : (g, V, \Theta) \to (g', V', \Theta')\) is compatible if

\[
\chi \otimes \chi(K) \subset K' \tag{35}
\]
This notion of compatible morphisms allows us to define several categories: we call \textbf{Lie-Leib} the category of Lie-Leibniz triples with their associated morphisms given in Definition 7. We call \textbf{compLie-Leib} the wide subcategory \textbf{compLie-Leib} of \textbf{Lie-Leib} whose morphisms are the compatible morphisms of Lie-Leibniz triples. In other words, \text{Ob(compLie-Leib)} = \text{Ob(Lie-Leib)}, while \text{Mor(compLie-Leib)} is contained into \text{Mor(Lie-Leib)}. The semi-strict (resp. strict) Lie-Leibniz triples form a full subcategory \textbf{strLie-Leib} (resp. \textbf{strLie-Leib}), denoted \textbf{compLie-Leib} (resp. \textbf{strLie-Leib}). This can be seen from the fact that in semi-strict Lie-Leibniz triples, the biggest \g-module of \text{Ker}\left(\{\ldots\}\right) \subset S^2(V) is the kernel itself. So, for \( K = \text{Ker}\left(\{\ldots\}\right) \) and \( K' = \text{Ker}\left(\{\ldots\}'\right) \), inclusion \( (34) \), which is valid for any morphism of Lie-Leibniz triples, becomes \( \chi \circ \chi(K) \subset K' \). Hence any morphism of semi-strict Lie-Leibniz triples is compatible.

These inclusions of subcategories will allow us to be more precise when studying the relationship between Lie-Leibniz triples and differential crossed modules. Recall the definition of the latter notion:

\textbf{Definition 9.} A differential crossed module consists of the following data: two Lie algebras \((\g_0, \ldots)_0\) and \((\g_{-1}, \ldots)_{-1}\), a linear map \( \Theta : \g_{-1} \rightarrow \g_0 \) and a Lie algebra morphism \( \rho : \g_0 \rightarrow \text{Der}(\g_{-1}) \), satisfying the following axioms:

\[
\rho(\Theta(x); y) = [x, y]_{-1} \tag{36}
\]
\[
\Theta(\rho(a; x)) = [a, \Theta(x)]_0 \tag{37}
\]
for every \( x, y \in \g_{-1} \) and \( a \in \g_0 \). We denote by \textbf{diff} \times \textbf{Mod} the category of differential crossed modules.\footnote{A \textit{wide} subcategory of a category \( \mathcal{C} \) is a category \( \mathcal{B} \) whose collection of objects is the same as that of \( \mathcal{C} - \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{C}) \) – but whose collection of morphisms does not necessarily coincides with that of \( \mathcal{C} \).}

Notice that the definition of \( \rho \) implies that \( \g_{-1} \) is a \( \g_0 \)-module, equipped with a Lie algebra structure entirely defined by Equation \( (36) \). Moreover, Equation \( (36) \) for \( a = \Theta(z) \) implies that \( \Theta \) is a Lie algebra morphism as well: \( \Theta([z, x]_{-1}) = [\Theta(z), \Theta(x)]_0 \). These observation imply that we have an alternative, equivalent definition of differential crossed modules that relies on the following data:

\textbf{Lemma 10.} A differential crossed module consists of the following data:

\begin{enumerate}
  \item a Lie algebra \( \g_0 \),
  \item a \( \g_0 \)-module \( \g_{-1} \), and
  \item a \( \g_0 \)-equivariant linear map \( \Theta : \g_{-1} \rightarrow \g_0 \) satisfying:
    \[
    \Theta(x) \cdot y = -\Theta(y) \cdot x \tag{38}
    \]
\end{enumerate}

\textit{Proof.} Because of Equation \( (38) \), the vector space \( \g_{-1} \) can be equipped with a skew-symmetric bracket \([x, y]_{-1} = \Theta(x) \cdot y \). This definition, together with the fact that \( \g_{-1} \) is a \( \g_0 \)-module has two consequences: on the one hand, the bracket \([\ldots]_{-1}\) satisfies the Jacobi identity and on the other hand, the representation map \( \rho : \g_0 \rightarrow \text{End}(\g_{-1}) \) takes values in \( \text{Der}(\g_{-1}) \). Eventually Equation \( (36) \) is the definition of the bracket on \( \g_{-1} \) while Equation \( (37) \) corresponds to the \( \g_0 \)-equivariance of \( \Theta \). \( \square \)

The criteria established in Lemma 10 are reminiscent of the definition of a Lie-Leibniz triples. Items 1. and 2. establish that one can equip \( \g_{-1} \) with a Leibniz product, as in Equation \( (15) \), but Equation \( (38) \) implies that this product is skew-symmetric, hence a Lie bracket. Moreover, the \( \g_0 \)-equivariance of \( \Theta \) that is constitutive of differential crossed modules straightforwardly implies the quadratic constraint \( (16) \). We then have the following relationship between differential crossed modules and Lie-Leibniz triples:

\footnote{A \textit{full} subcategory of a category \( \mathcal{C} \) is a category \( \mathcal{B} \) whose collection of objects is included in that of \( \mathcal{C} - \text{Ob}(\mathcal{B}) \subset \text{Ob}(\mathcal{C}) \) – but whose morphisms between elements are the same as that of \( \mathcal{C} \), in the following sense: for every pair of objects \( X, Y \in \text{Ob}(\mathcal{B}) \), we have the identity \( \text{Hom}_B(X, Y) = \text{Hom}_C(X, Y) \).}

\footnote{Morphisms in the category of differential crossed modules are pairs of Lie algebra morphisms \( (\varphi : \g_0 \rightarrow \g'_0, \chi : \g_{-1} \rightarrow \g'_{-1}) \) satisfying Equations \( (32) \) and \( (33) \).}
Lemma 11. The category $\text{diff} \times \text{Mod}$ forms a full subcategory of $\text{strLie-Leib}$. 

Proof. Let $g_{-1} \xrightarrow{\Theta} g_0$ be a differential crossed module. Then, Equations (36) and (37) imply Equations (15) and (16), under the following assumptions:

1. $g = g_0$ and $V = g_{-1}$;

2. the Leibniz product $\circ$ on $g_{-1}$ corresponds to the Lie algebra structure $[\cdot, \cdot]_{-1}$.

This makes $(g_0, g_{-1}, \Theta)$ a Lie-Leibniz triple. Moreover, Equation (37) tells us that the embedding tensor $\Theta$ is $g_0$-equivariant, i.e. that the Lie-Leibniz triple is strict. Morphisms of differential crossed modules are precisely the morphisms of the corresponding Lie-Leibniz triples, hence the result.

This lemma characterizes differential crossed modules as the strict Lie-Leibniz triples $(g, V, \Theta)$ for which $V$ is a Lie algebra. It moreover implies that we have the following inclusions of categories:

$$\text{diff} \times \text{Mod} \subset \text{strLie-Leib} \subset \text{semLie-Leib} \subset \text{compLie-Leib} \subset \text{Lie-Leib}$$

where each inclusion is full in the sense of categories, except on the right-most inclusion where $\text{compLie-Leib}$ is a wide subcategory of $\text{Lie-Leib}$. The similarity between differential crossed modules and Lie-Leibniz triples is striking: one can see the latter as natural generalizations of the former, in which some stringent conditions (a Lie algebra structure on $g_{-1}$ and $g_0$-equivariance of $\Theta$) have been relaxed. Let us now push the analogy further by analyzing the differential graded Lie algebra structure of differential crossed modules and its natural extension to Lie-Leibniz triples. Indeed, an important property of differential crossed modules is that they can be considered as 2-term differential graded Lie algebras concentrated in degrees 0 and 1 [1] or, equivalently, in degrees 0 and $-1$. In the following we will chose the latter convention. We recall here how the identification works, since it will be a cornerstone of the present section.

Given a differential crossed module $g_{-1} \xrightarrow{\Theta} g_0$, assign degree $-1$ to $g_{-1}$ and degree 0 to $g_0$. Then, define the graded Lie bracket on $g_{-1} \oplus g_0$ as follows:

$$[a, b] = [a, b]_0$$
$$[x, y] = 0$$
$$[a, x] = a \cdot x \quad \text{and} \quad [x, a] = -a \cdot x$$

for every $a, b \in g_0$ and $x, y \in g_{-1}$. This bracket is indeed compatible with the differential $\Theta$, in the sense that it forms a differential graded Lie algebra structure. The first non-trivial compatibility condition between the graded Lie bracket $[\cdot, \cdot]$ and the differential $\Theta$ is:

$$\Theta([a, x]) = [a, \Theta(x)]$$

This is nothing but Equation (37). The other non-trivial compatibility condition is:

$$\Theta([x, y]) = [\Theta(x), y] - [x, \Theta(y)]$$

Notice that the left hand side vanishes identically because of Equation (40), and then the right hand side just tells us that $\Theta(x) \cdot y = -\Theta(y) \cdot x$. However this is a tautology because by Equation (36), $\Theta(x) \cdot y = [x, y]_{-1}$ and we know that the Lie bracket on $g_{-1}$ is skew-symmetric. Eventually, the bracket defined in Equations (39)-(41) satisfies the Jacobi identity as the only non-obvious identity to satisfy is the following one:

$$[a, [b, x]] - [b, [a, x]] = [[a, b], x]$$

This is an alternative, equivalent rewriting of the fact that $\rho : g_0 \to \text{Der}(g_{-1})$ is a Lie algebra morphism. Thus, a differential crossed module is equivalent to a 2-term differential graded Lie algebra concentrated in degrees 0 and $-1$. 


Let us now present an alternative, equivalent, formulation of this observation by using the cyclic representation generated by $\Theta$. Let $(g, V, \Theta)$ be a Lie-Leibniz triple and let $\eta : g \to \text{End}(\text{Hom}(V, g))$ denotes the induced action of $g$ on $\text{Hom}(V, g)$. As an element of $\text{Hom}(V, g)$, the embedding tensor $\Theta$ generates a cyclic $g$-submodule of the $g$-module $\text{Hom}(V, g)$; we call it $R_\Theta$. More precisely, noting
\[
\Theta_{a_1a_2\ldots a_n} = a_1 \cdot (a_2 \cdot \ldots (a_n \cdot \Theta) \ldots)) = \eta(a_1; \eta(a_2; \ldots (\eta(a_n; \Theta)) \ldots))
\]
for any $a_1, a_2, \ldots, a_n \in g$, we have:
\[
R_\Theta = \text{Span}(\Theta, \Theta_{a_1a_2\ldots a_m} | a_1, a_2, \ldots, a_m \in g) \subset \text{Hom}(V, g)
\]
In the case where the Lie-Leibniz triple $(g, V, \Theta)$ is a differential crossed module – i.e. when $g = g_0$ and $V = g_{-1}$ – then $\Theta$ being $g_0$-equivariant means that $R_\Theta$ is the one-dimensional irreducible trivial representation of $g_0$ and can be identified with $\mathbb{R}$. Then, the differential graded Lie algebra structure on $g_{-1} \oplus g_0$ presented earlier can be straightforwardly extended to a 3-term differential graded Lie algebra structure on the following chain complex:
\[
g_{-1} \xrightarrow{\Theta} g_0 \xrightarrow{0} \mathbb{R}
\]
One usually consider that elements of $\mathbb{R}$ carry a degree $+1$, so that the grading on the complex is increasing from left to right. Actually, the above sequence is a particular case of the following observation:

**Proposition 12.** To every Lie-Leibniz triple $(g, V, \Theta)$ in which $V$ is a Lie algebra, corresponds a unique 3-term differential graded Lie algebra concentrated in degrees $-1$, $0$ and $1$:

\[
V[1] \xrightarrow{\Theta} g \xrightarrow{-\eta(-; \Theta)} R_\Theta[-1]
\]
which canonically extends the graded Lie algebra structure on $V[1] \oplus g$ defined by Equations (39)-(41).

**Remark 5.** Recall that, for any graded vector space $U_\bullet = \bigoplus_{i \in \mathbb{Z}} U_i$, the operator $[\cdot, \cdot] : U \to U[1]$ is the suspension operator that shifts the degree of every elements in $U$ by $-1$, i.e. it is such that $(U[1])_i = U_{i+1}$. It admits an inverse desuspension operator $[-1] : U \to U[-1]$, that has the reverse effect. See [20] for more details.

**Proof.** Given a Lie-Leibniz triple $(g, V, \Theta)$ in which $V$ is assumed to be a Lie algebra (i.e. the Leibniz product $\circ$ is skew-symmetric), then one defines the following bracket (the first four being that of Equation (39)-(41)):
\[
[a, b] = [a, b]_g, \quad [x, y] = 0, \quad [a, x] = a \cdot x, \quad [x, a] = -a \cdot x, \quad [a, \Theta] = a \cdot \Theta, \quad [\Theta, a] = -a \cdot \Theta, \quad [\Theta, x] = \Theta(x) \quad \text{and} \quad [x, \Theta] = \Theta(x)
\]
for every $a, b \in g$ and $x, y \in V[1]$. Since $R_\Theta$ is generated by the successive actions of $g$ on $\Theta$, one extends the above bracket to the whole of $R_\Theta[-1]$ by the following formulas:
\[
[\Theta_{a_1a_2\ldots a_m}, u] = a_1 \cdot [\Theta_{a_2\ldots a_m}, u] - [\Theta_{a_2\ldots a_m}, a_1 \cdot u] \quad \text{(51)}
\]
\[
[u, \Theta_{a_1a_2\ldots a_m}] = -(-1)^{|u|}[\Theta_{a_1a_2\ldots a_m}, u] \quad \text{(52)}
\]
\[
0 = [[R_\Theta[-1], R_\Theta[-1]]] \quad \text{(53)}
\]
for any homogeneous element $u \in g \oplus V[1]$ of degree $|u|$, and where $\Theta_{a_1a_2\ldots a_m} = a_1 \cdot (a_2 \cdot (\ldots (a_m \cdot \Theta) \ldots))$ for any $a_1, a_2, \ldots, a_m \in g$. Moreover setting $\partial_1 = -\eta(-, \Theta)$ and $\partial_0 = \Theta$, one has $\partial_1(\partial_0(x)) = 0$ by Equation (19). Actually, Lines (19) and (50) imply that the induced differential $\partial$ coincides with the adjoint action of $\Theta$:
\[
\partial = [\Theta, -]
\]
(54)
Using the properties of Lie-Leibniz triples, one may check that the above brackets and differentials satisfy the graded Jacobi identities and Leibniz rule. We have already shown several of them: the Jacobi identity on $g \oplus V[1]$ is Equation (41) and the Leibniz rule for any pair of element $x \in V[1]$ and $a \in g$, see Equations (12) and (13). We will now show that their validity extend to $R_\Theta[-1]$. In the following, we assume that $a, b, a_1, \ldots, a_m \in g$, and $x, y \in V$. We first turn to the Leibniz rules.

Using Equation (51) for $[[\Theta, a], b]$ and reorganizing the terms, one has:

\[
\partial \left( [a, b] \right) = \left[ [\Theta, [a, b]], g \right] = a \cdot [\Theta, b] - [\Theta a, b] = [a, [\Theta, b]] + [[\Theta, a], b] = [a, \partial(b)] + [\partial(a), b]
\]

Hence proving the Leibniz rule on the bracket $[a, b]$. Next, because of the degree, one knows that any Leibniz rule involving an element of $R_\Theta[-1]$ and an element of $g$ is automatically zero. This leaves us with only one Leibniz rule to check: the one involving one element $x \in V[1]$ and every generators of $R_\Theta[-1]$. The Leibniz rule between $x$ and $\Theta$ should read:

\[
\partial([\Theta, x]) = -[[\Theta, \partial(x)]
\]

because $\partial(R_\Theta[-1]) = 0$ by construction. But Equation (54) implies that the left-hand side is zero, while the right-hand side is $\eta(\partial(x), \Theta) = \Theta(x) \cdot \Theta$ which also vanishes by the quadratic constraint (19). This proves that the Leibniz rules is satisfied on the pair of elements $x$ and $\Theta$.

It is now sufficient to prove it on any other choice of generator $\Theta_{a_1 \ldots a_m}$ of $R_\Theta$. While one can deduce from Equation (51) that $\partial([[\Theta, x]], \partial(x)] = -[[\Theta_a, \partial(x)]$, the same does not hold for generators of $R_\Theta[-1]$ of the form $\Theta_{a_1 \ldots a_m}$ for $m \geq 2$ (see the discussion following Equation (137)). Then, as a further assumption, one is allowed to additionally require that:

\[
\partial([\Theta_{a_1 \ldots a_m}, x]) = -[[\Theta_{a_1 \ldots a_m}, \partial(x)]
\]

This is precisely the Leibniz rule for $x$ and $\Theta_{a_1 \ldots a_m}$. One concludes that the Leibniz rule holds for every pair of elements taken from $V[1]$ and $R_\Theta[-1]$, and thus on the whole of $V[1] \oplus g \oplus R_\Theta[-1]$.

We can now turn to proving the remaining Jacobi identities. We already know by Equation (41) and because $g$ is a Lie algebra that the graded Jacobi identity is satisfied on $V[1] \oplus g$. Moreover it is trivially satisfied on $R_\Theta[-1]$ because the bracket is zero on this space. Then we need only check the cases for which one or two terms belong to $R_\Theta[-1]$ and the other one or two belong to $V[1] \oplus g$. The proof is made by induction and, in order to avoid unnecessary technical computation at this stage of the paper, we postpone it to the second part of the proof of Proposition 23, see in particular the discussion below Equation (152). Hence, this concludes the proof of Proposition 12.

This result shows that the assignment between differential crossed modules and differential graded Lie algebras can be extended to specific cases of Lie-Leibniz triples: those for which the Leibniz algebra is a Lie algebra. The objective of the paper is to show that one can generalize Proposition 12 to every Lie-Leibniz triples. However, since Lie-Leibniz triples involve Leibniz algebras and not just Lie algebras, we expect new intricate structures to emerge. In particular, one would expect to associate a bigger differential graded Lie algebra to every Lie-Leibniz triple $(g, V, \Theta)$ whose Leibniz algebra $V$ is not a Lie algebra:

\[
\ldots \longrightarrow V[1] \xrightarrow{\Theta} g \xrightarrow{-\eta(-, \Theta)} R_\Theta[-1]
\]

More precisely, setting

\[
\mathcal{G} : \text{diff} \times \text{Mod} \longrightarrow \text{DGLie}_{\leq 1}
\]

the functor that associates to each differential crossed module the corresponding unique 3-term differential graded Lie algebra (where $\mathbb{R}$ is considered to be in degree $+1$), then the rest of the paper is devoted to prove the following result:
We will show that, in the presence of a symmetric component in the Leibniz product of a Lie-Leibniz triple \((g, V, \Theta)\), the chain complex of Proposition 12 actually extends to the left:

\[
\cdots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V[1] \xrightarrow{\Theta} g \xrightarrow{-\eta(-; \Theta)} R_\Theta[-1]
\]

(59)

The existence of a tower of spaces \(T_i = \bigoplus_{i \geq 1} T_{-i}\) beyond degree \(-1\) is thus the sign that the Leibniz algebra \(V\) is not a Lie algebra. The \(g\)-module structure of \(V\) is key in defining \(T_{-2}\), as it is used in its very definition. The other \(T_{-i}\) for \(i \geq 3\) are built by induction as quotients of sums of products of other \(T_{-j}\) for \(1 \leq j \leq i - 1\) so that the property of being a \(g\)-module is inherited by induction. The action of \(g\) on \(T_{-i}\) is moreover necessary to define the differential on \(T_i\), showing the centrality of the Lie algebra \(g\) in the construction. As proven in Proposition 23, the total space \(T = T_\bullet \oplus g \oplus R_\Theta[-1]\) then canonically inherits a differential graded Lie algebra structure, which restricts to that of Proposition 12 when \(V\) is a Lie algebra.

This construction improves and simplifies the one presented in [19]. In that paper, the author associated a differential graded Lie algebra \(((T_{-i})_{i \geq 0}, \partial, [\ldots])\) to any Lie-Leibniz triple such that \(T_{-1} = V[1]\), \(T_0 = h = \text{Im}(\Theta)\) and \(\partial_0 = \Theta\):

\[
\cdots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V[1] \xrightarrow{\Theta} h
\]

(60)

Here, the spaces \(T_{-i}\) and the linear maps \(\partial_{-i}\) are the same as in the sequence (59). That construction had the serious drawback that the resulting differential graded Lie algebra lost important informations about the Lie algebra \(g\) (for example it did not appear in the resulting dgLa). It used a dual point of view which implied rather cumbersome computations, whereas in the present paper, we propose to offer a novel and direct construction of the chain complex. Moreover, we shall prove that each map \(\partial_{-i}\), as an element of \(\text{Hom}(T_{-i-1}, T_{-i})\) on which \(g\) acts, generates a cyclic \(g\)-submodule of \(\text{Hom}(T_{-i-1}, T_{-i})\) which is isomorphic to a (sub)-module of \(R_\Theta\). This result, although having been conjectured and used for a long time by physicists, is new and is needed precisely in order to extend the chain complex (60) to the right as in diagram (59).

### 3 The tensor hierarchy associated to a Lie-Leibniz triple

Subsection 3.1 is dedicated to define the particular graded vector space \(T_\bullet\) and equip it with a graded Lie bracket. Subsection 3.2 is dedicated to define the differential, while subsection 3.3 proves the compatibility of the bracket with the differential, turning the total space \(T = T_\bullet \oplus g \oplus R_\Theta[-1]\) into a differential graded Lie algebra. This algebra is the one defining the function \(G\) in Theorem 13. Finally, relying on this result, subsection 3.4 is devoted to showing that the restriction of \(G\) to the subcategory \(\text{compLie-Leib}\) is a faithful functor. The present discussion draws on theoretical work done by physicists, among others [5, 12].
3.1 Associating a graded Lie algebra to a Lie-Leibniz triple

The main result of the current subsection shows that any Lie-Leibniz triple gives rise to a graded Lie algebra generalizing the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ that appears in any differential crossed module $\mathfrak{g}_{-1} \xrightarrow{\Theta} \mathfrak{g}_0$.

**Proposition 14.** To any Lie-Leibniz triple $(\mathfrak{g}, V, \Theta)$ one can associate a negatively graded Lie algebra $(T_\bullet = (T_{-i})_{i \geq 1}, \ldots )$, which satisfies the following three conditions:

1. $T_\bullet$ is a graded $\mathfrak{g}$-module, and the corresponding representation $\rho : \mathfrak{g} \to \text{End}(T_\bullet)$ takes values in $\text{Der}(T_\bullet)$, the space of derivations of $[\ldots]$;
2. $T_\bullet = V[1] \oplus [T_1, T_\bullet]$;
3. $T_\bullet = T_{-1} = V[1]$ if and only if $V$ is a Lie algebra.

The first item of the proposition means that for every $i \geq 1$, $T_{-i}$ is a $\mathfrak{g}$-module under the action of a Lie algebra morphism $\rho_{-i} : \mathfrak{g} \to \text{End}(T_{-i})$, so that $\rho$ is the unique Lie algebra morphism restricting to $\rho_{-i}$ on $\text{End}(T_{-i})$. Moreover for every $a \in \mathfrak{g}, x \in T_{-i}$ and $y \in T_{-j}$, $\rho(a ; -)$ is a derivation of the graded Lie bracket $[x, y]$:

$$\rho_{-i,j}(a; [x, y]) = [[\rho_{-i}(a; x), y] + [x, \rho_{-j}(a; y)]$$

(61)

The second item can be reformulated as follows: $T_{-1} = V[1]$ and for every $i \geq 2$, $T_{-i} = \sum_{j=1}^{i-1} [T_{-j}, T_{-i+j}]$. Eventually, the last item is the condition that $T_{-i} = 0$ for every $i \geq 2$ if and only if $V$ is a Lie algebra. In other words, when $V$ is a Lie algebra, the tower of vector spaces reduces to $T_\bullet = T_{-1} = V[1]$ and the graded Lie bracket $[\ldots]$ is identically zero, whereas when $V$ is a Leibniz algebra whose Leibniz product is not fully skew-symmetric, then the spaces of lower degrees exist and satisfy item 2. of Proposition 14.

The proof is split in two parts. First, we will define the $T_{-i}$ recursively, starting by setting $T_{-1} = V[1]$, and then by quotienting ideals out of the free graded Lie algebra generated by $V[1]$. The induction will then give a (possibly infinite) graded vector space $T_\bullet = \bigoplus_{i=1}^{\infty} T_{-i}$ which has the following properties:

1. Every vector space $T_{-i}$ is a $\mathfrak{g}$-module;
2. $T_{-1} = V[1]$;
3. $T_{-i} = 0$ for every $i \geq 2$ if and only if $V$ is a Lie algebra.

This represents the biggest part of the proof, and it crucially relies on key Lemmas [15] and [16]. Then, we will define a graded Lie bracket on $T_\bullet$, so that items 1. and 2. of Proposition 14 are satisfied by construction.

**Remark 7.** The idea behind the proof (quotienting a graded ideal out of a free graded Lie algebra) in the context of building a tensor hierarchy has first been given in [13], where the authors use a partition function in order to compute the $\mathfrak{g}$-modules that form the building blocks of the hierarchy. On the other hand, our proof relies on an inductive argument that is algebraic and that enables us to make explicit the definition of the graded Lie bracket. The relationship between the graded Lie algebra structure presented in [13] and ours has still to be investigated, but we expect it to be very close if not identical.

Let us first give a brief account of our conventions and a technical reminder of graded algebra. Given a finite dimensional vector space $X$, one can define the free Lie algebra $\text{Free}(X)$ as the Lie algebra generated by $X$, imposing only the defining relations of skew-symmetry and the Jacobi identity. It can be thought of as the vector space generated by the successive commutators of a basis of $X$. The vector space $W = \text{Free}(X)$ is graded by the length of the commutators, e.g. $[\cdots [x_1, x_2], x_3], \cdots x_i]$, for any $x_1, \ldots, x_i \in X$, has length $i$. There exist various formulas for a basis of $W = \bigoplus_{k \geq 1} W_k$ [14, 22, 24]. The construction also works for graded
vector spaces $X$, hence defining free graded Lie algebras, also called free Lie superalgebras when the emphasis is on the parity \([7, 25]\).

First, let us recall that for a graded vector space $X$, the free graded skew-commutative algebra generated by $X$ is denoted by $\Lambda^*(X)$. That is, the algebra spanned by elements $x_1 \wedge x_2 \wedge \cdots \wedge x_n$. With the degree of $x$ denoted $|x|$, we have

$$x \wedge y = -(-1)^{|x||y|} y \wedge x.$$  \hfill (62)

For example, pick a vector space $V$ and define $X = V[1]$ to be its suspension, that is to say: $X$ is isomorphic to $V$ as a vector space, but its elements are considered to carry a grading defined to be $-1$ (see Remark \([5]\)). Then, due to the shifted degree, the vector space $\Lambda^2(X)$ is isomorphic to $S^2(V)$ (understood in the category of vector spaces). Then, the free graded Lie algebra $F_\bullet = \text{Free}(X)$ generated by the degree $-1$ vector space $F_{-1} = X = V[1]$ admits $\Lambda^2(X) \simeq S^2(V)$ as the desired space $F_{-2}$. Notice that for $x, y \in F_{-1}$, we have $[x, y] = [y, x]$, where here the bracket is the free graded Lie bracket on $F_\bullet$. The next vector space $F_{-3}$ has a basis of some elements of the form $[[x, y], z]$ for $x, y, z \in F_{-1}$, subject to the Jacobi identity. Thus $F_{-3}$ is isomorphic to the quotient of $F_{-1} \otimes \Lambda^2(F_{-1}) \simeq V \otimes S^2(V)$ by $\Lambda^3(F_{-1}) \simeq S^3(V)$. More generally, the negative grading on $F_\bullet = \bigoplus_{k \geq 1} F_{-k}$ is defined as follows: the number $k \geq 1$ labelling $F_{-k}$ indicates the number of elements $x_1, \ldots, x_k \in X = V[1]$ used in the brackets $[\cdots [x_1, x_2], x_3], \ldots, x_k]$ defining the generators of $F_{-k}$.

In this section, we will consider the total exterior algebra $\Lambda(\bullet) = \bigoplus_{p \geq 1} \Lambda^p \bullet$ of a free graded Lie algebra of the form $F_\bullet = \text{Free}(V[1])$. The exterior algebra inherits a grading from $F_\bullet$ which is such that the component in total degree $-i$ is denoted $\Lambda(F_\bullet)_{-i} = \bigoplus_{p \geq 1} \Lambda^p(F_\bullet)|_{-i}$. This exterior algebra is obtained as the quotient of the tensor algebra $T(F_\bullet)$ by the (graded) ideal generated by elements of the form:

$$x \otimes y + (-1)^{|x||y|} y \otimes x$$  \hfill (63)

for every two homogeneous elements $x, y \in F_\bullet$. The resulting quotient is isomorphic to a graded sub-vector space of the tensor algebra, and is a graded algebra on its own, when equipped with the wedge product. Let us discuss the first few spaces appearing in the decomposition of $\Lambda^2(F_\bullet)$. As a vector space, $\Lambda^2(F_\bullet)|_{-2}$ is canonically isomorphic to $F_{-1} \wedge F_{-1}$, while $\Lambda^2(F_\bullet)|_{-3}$ is canonically isomorphic to $F_{-1} \otimes F_{-2}$, whereas $\Lambda^2(F_\bullet)|_{-4}$ is canonically isomorphic to $F_{-1} \otimes F_{-3} \oplus F_{-2} \wedge F_{-2}$. More generally, $\Lambda^2(F_\bullet)|_{-i}$ (for $i \geq 3$) is canonically isomorphic to the following vector spaces depending on the parity of $i$:

$$\text{if } i \text{ is even: } \quad F_{-\frac{i}{2}} \wedge F_{-\frac{i}{2}} \oplus \bigoplus_{1 \leq j \leq \frac{i}{2}} F_{-j} \otimes F_{-i+j}$$  \hfill (64)

$$\text{if } i \text{ is odd: } \quad \bigoplus_{1 \leq j \leq \frac{i+1}{2}} F_{-j} \otimes F_{-i+j}$$  \hfill (65)

When $j \neq \frac{i}{2}$, what we denote by $F_{-j} \wedge F_{-i+j}$ is the vector space isomorphic to $F_{-j} \otimes F_{-i+j}$, understood as a subspace of $\Lambda^2(F_\bullet)|_{-i}$. In particular, if $x \in F_{-j}$ and $y \in F_{-i+j}$ then we will write $x \wedge y$ instead of $x \otimes y$ when we want to emphasize that we work in $F_{-j} \wedge F_{-i+j}$. Regarding $F_{-j} \wedge F_{-i+j}$ as isomorphic to both $F_{-j} \otimes F_{-i+j}$ and $F_{-i+j} \otimes F_{-j}$, we then have $x \wedge y$ represented by $x \otimes y \in F_{-j} \otimes F_{-i+j}$ or equally by $(-1)^{j-i} y \otimes x \in F_{-i+j} \otimes F_{-j}$.

The Chevalley-Eilenberg homology of a graded Lie algebra $\mathfrak{g}$ is the homology of their chain complex $CE^*(\mathfrak{g})$ with $CE_n(\mathfrak{g}) = \Lambda^n \mathfrak{g}$ and differential $d = \{d_n : CE^n(\mathfrak{g}) \to CE_{n-1}(\mathfrak{g})\}$ given by (111) p. 471:

$$d_n(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j+|x_j||x_j|+|x_i||x_i|-|x_j||x_j|} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_n$$  \hfill (66)

where $\eta_m = \sum_{1 \leq k \leq n-1} |x_k|$. Degrees of the elements appear in the formula because one has to permute $x_i$ and $x_j$ to the first and second positions in order to eventually take the Lie bracket of $x_i$ and $x_j$. There exists another, identical but more compact formula, which uses unshuffles:

$$d_n(x_1 \wedge \cdots \wedge x_n) = - \sum_{\sigma \in \text{Un}(2, n-2)} \epsilon^\sigma [x_{\sigma(1)}, x_{\sigma(2)}] \wedge x_{\sigma(3)} \wedge \cdots \wedge x_{\sigma(n)}$$  \hfill (67)

where $\text{Un}(2, n-2)$ are the $(2, n-2)$-unshuffles\footnote{A permutation $\sigma \in S_n$ is called a $(p, n-p)$-unshuffle if $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$.}, and where $\epsilon^\sigma$ is the sign that is induced by the permutation $\sigma$.\footnote{A permutation $\sigma \in S_n$ is called a $(p, n-p)$-unshuffle if $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$.}
One may check that Equation (66) is the same as Equation (67), and that they correspond to the common definition of the Chevalley-Eilenberg differential (in homology) when $g$ is concentrated in degree 0 (see [17] p. 284). Then the following result holds:

**Lemma 15.** For a free graded Lie algebra $F_\bullet = (F_i)_{i \geq 1}$, the following sequences are exact:

$$0 \longrightarrow \Lambda^2(F_{-1}) \xrightarrow{d_2} F_{-2} \longrightarrow 0$$

(68)

$$\Lambda^3(F_{-1}) \xrightarrow{d_3} F_{-1} \wedge F_{-2} \xrightarrow{d_2} F_{-3} \longrightarrow 0$$

(69)

and more generally, for every $i \geq 4$:

$$\Lambda^3\left( \bigoplus_{j=1}^{i-2} F_{-j} \right)_{-i} \xrightarrow{d_3} \Lambda^2\left( \bigoplus_{j=1}^{i-2} F_{-j} \right)_{-i} \xrightarrow{d_2} F_{-i} \longrightarrow 0$$

(70)

**Proof.** For a free graded Lie algebra $F_\bullet$, one has $H_1(F_\bullet) = 0$ and $H_2(F_\bullet) = 0$. The first equality is justified because on the one hand, the map $d_1 : F_\bullet \rightarrow \mathbb{K}$ is the zero map and, on the other hand, $d_2 : \Lambda^2(F_\bullet) \rightarrow F_{\leq -2}$ is by definition surjective: $F_\bullet$ is a free graded Lie algebra so each $F_{-i}$ (for $i \geq 2$) is spanned by (iterated) brackets. The second equation, $H_2(F_\bullet) = 0$, is deduced from Corollary 6.5 in [11] by keeping track of the natural graduation of $F_\bullet$. Then it suffices to write the equations $H_1(F_\bullet) = 0$ and $H_2(F_\bullet) = 0$ degree-wise, starting at $n = 2$, to obtain the exact sequences (68), (69) and (70).

The proof of Proposition 14 then relies of the use of Lemma 15 in conjunction with the following Lemma, whose statement and proof has been brought to our knowledge by Jonathan Wise:

**Lemma 16.** Let $\mathcal{A}$ be an abelian category and assume that the following diagram in $\mathcal{A}$ is commutative:

$$
\begin{array}{ccc}
0 & \longrightarrow & A \\
& \downarrow & \uparrow \text{id} \\
W & \xrightarrow{a} & X & \xrightarrow{b} & Y & \xrightarrow{c} & Z \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
W & \xrightarrow{\pi} & X/\text{im}(a) & \xrightarrow{\tilde{\eta}} & Y/\text{im}(b) & \xrightarrow{\tilde{\nu}} & Z \\
\end{array}
$$

where $\pi$, $\tilde{\eta}$ and $\tilde{\nu}$ are the linear maps canonically induced from $a$, $b$ and $c$. Then exactness of the second row implies exactness of the third.

**Remark 8.** Here, given a morphism $\chi \in \text{Hom}_\mathcal{A}(P, Q)$, the quotient notation $P/\chi(Q)$ denotes the cokernel of the map $\chi$, that is: $P/\chi(Q) := \text{coker}(\chi : P \rightarrow Q)$. In the category of $R$-modules, this reduces to the usual quotient of $R$-modules.

**Proof.** Exactness of the sequence:

$$W \xrightarrow{a} X \xrightarrow{b} Y \xrightarrow{c} Z$$

12More precisely, $x_1 \wedge \ldots \wedge x_n = e^x x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)}$, so, for example $x \wedge y = (-1)^{|x||y|} y \wedge x$, where $|x|$ denotes the degree of the (homogeneous) element $x$. In full rigor, $e^x$ should be labelled by the data of the elements with respect to which it is defined.
is equivalent to the existence of two isomorphisms:

\[ \text{Coker}(a) \simeq \text{Im}(b) \quad \text{and} \quad \text{Coker}(b) \simeq \text{Im}(c) \]

Now we have the following isomorphisms:

\[ \text{Im}(\pi) \simeq \text{Im}(c) \simeq \text{Coker}(b) \simeq \text{Coker}(\overline{b}) \quad (71) \]

which implies that the sequence:

\[ W \xrightarrow{\pi} X/\text{u}(A) \xrightarrow{\overline{\pi}} Y/\text{k}(A) \xrightarrow{\pi} Z \]

is exact at \( Y/\text{k}(A) \). Exactness at \( X/\text{u}(A) \) is implied by the following sequences of isomorphisms:

\[ \text{Coker}(a) \simeq \text{Coker}(A \to \text{Coker}(a)) \simeq \text{Coker}(A \to \text{Im}(b)) \simeq \text{Im}(\overline{b}) \quad (72) \]

where the first and last equalities are obtained by definition of the maps \( \pi \) and \( \overline{b} \). This concludes the proof. \( \Box \)

**Proof of Proposition 17.** The first part of the proof relies on constructing the graded vector space \( T_\bullet \) by quotienting a graded ideal \( K_\bullet = \bigoplus_{i=2}^{\infty} K_{-i} \) out of the free graded Lie algebra \( F_\bullet = \text{Free}(V[1]) \). More precisely, for every \( i \geq 2 \):

\[ T_{-i} = F_{-i}/K_{-i} \]

and \( T_{-1} = V[1] \) of course. This will be done by using Lemma 16, where we justify that the lines are exact by invoking Lemma 15.

Given this strategy, let us first define \( T_{-2} \) as a quotient of \( \Lambda^2(F_{-1}) \simeq S^2(V) \). The kernel \( \text{Ker}(\{\ldots\}) \) of the symmetric bracket on \( V \) is a subspace of \( S^2(V) \); as seen on Equation (31), it is a \( \mathfrak{g} \)-module but not necessarily a \( \mathfrak{g} \)-module. However, it admits \( \mathfrak{g} \)-submodules, and among them one is the biggest, that we denote by \( K \). We then set \( T_{-2} \):

\[ T_{-2} := \left( S^2(V)/K \right)[2] \]

Here, the suspension operator \([1]\) is applied twice so that elements of \( T_{-2} \) are considered to carry degree \(-2\). Identifying \( \Lambda^2(F_{-1}) \) and \( S^2(V)[2] \), and setting \( K_{-2} = K[2] \), the vector space \( T_{-2} \) can moreover be identified with:

\[ T_{-2} = F_{-2}/K_{-2} \]

since \( \Lambda^2(F_{-1}) \) is by construction canonically isomorphic to \( F_{-2} \).

This vector space is a \( \mathfrak{g} \)-module, inheriting this structure from the canonical quotient map \( p : F_{-2} \to T_{-2} \). This choice of \( T_{-2} \) satisfies all required conditions: it is a \( \mathfrak{g} \)-module and it is zero if and only if \( V \) is a Lie algebra, since then \( \{\ldots\} = 0 \) and \( K = S^2(V) \).\(^{13}\) Recalling that \( F_{-1} = T_{-1} = V[1] \), let us define \( q_{-2} : \Lambda^2(T_{-1}) \to T_{-2} \)

\[ \Lambda^2(T_{-1}) \xrightarrow{q_{-2}} T_{-2} \]

which is by construction \( \mathfrak{g} \)-equivariant:

\[ \begin{array}{ccc}
-2p & & F_{-2} \\
\Lambda^2(T_{-1}) & \xrightarrow{q_{-2}} & T_{-2}
\end{array} \]

\[^{13}\text{In particular, it is precisely for this last reason that we did not pick up any } \mathfrak{g} \text{-submodule of Ker}(\{\ldots\}) \text{ but the biggest such.} \]

\[^{14}\text{The presence of the factor 2 is a convention that makes certain calculations much simpler and consistent with the definition of maps in subsection 3.2.} \]
Let us now use Lemmas \[15\] and \[16\] to define $T_{-3}$. Since $K_{-2}$ is a submodule of $F_{-2}$, there is a canonical embedding $u_{-3} : F_{-1} \otimes K_{-2} \to F_{-1} \wedge F_{-2}$. This makes the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_{-1} \otimes K_{-2} & \longrightarrow & F_{-1} \otimes K_{-2} & \longrightarrow & 0 \\
& & \downarrow^{u_{-3}} & \downarrow & \downarrow^{k_{-3}} & \downarrow & \\
\Lambda^3(F_{-1}) & \longrightarrow & F_{-1} \wedge F_{-2} & \longrightarrow & F_{-3} & \longrightarrow & 0 \\
\end{array}
$$

commutative with exact rows (the bottom row is exact by Lemma \[15\]). We appended a minus sign in front of the linear maps $d_2$ and $d_3$ because the definition of the Chevalley-Eilenberg differential $d_2$ is minus the Lie bracket defining $F_{-3}$ (see Equation \[67\]). The map $k_{-3} : F_{-1} \otimes K_{-2} \to F_{-3}$ is the composite $-d_2 \circ u_{-3}$, making this diagram commutative. As a composition of $\mathfrak{g}$-equivariant morphisms, the map $k_{-3}$ is itself $\mathfrak{g}$-equivariant, so that $\text{Im}(k_{-3})$ defines a $\mathfrak{g}$-submodule $K_{-3}$ of $F_{-3}$.

Since $T_{-1} = F_{-1}$ the quotient of $F_{-1} \wedge F_{-2}$ by $F_{-1} \wedge K_{-2}$ is $T_{-1} \wedge T_{-2}$, so by Lemma \[16\] we deduce that:

$$
\Lambda^3(F_{-1}) \longrightarrow T_{-1} \wedge T_{-2} \longrightarrow F_{-3} / K_{-3} \longrightarrow 0
$$

is an exact sequence. Here, we adopted the same convention for $T_{-1} \wedge T_{-2}$ as we did for $F_{-1} \wedge F_{-2}$, so that it can be identified with $T_{-1} \otimes T_{-2}$. We set:

$$
T_{-3} = F_{-3} / K_{-3}
$$

and we denote by $q_{-3} : T_{-1} \wedge T_{-2} \to T_{-3}$ the map induced from $-d_2|_{F_{-1} \wedge F_{-2}}$ by Lemma \[16\] i.e. the unique map such that the following diagram is commutative, with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_{-1} \otimes K_{-2} & \longrightarrow & F_{-1} \otimes K_{-2} & \longrightarrow & 0 \\
& & \downarrow^{u_{-3}} & \downarrow & \downarrow^{k_{-3}} & \downarrow & \\
\Lambda^3(F_{-1}) & \longrightarrow & F_{-1} \wedge F_{-2} & \longrightarrow & F_{-3} & \longrightarrow & 0 \\
& & \downarrow & \downarrow & \downarrow & \\
\Lambda^3(F_{-1}) & \longrightarrow & T_{-1} \wedge T_{-2} & \longrightarrow & T_{-3} & \longrightarrow & 0 \\
\end{array}
$$

The bottom vertical rows are the quotient maps induced by $u_{-3}$ and $k_{-3}$. Elements of $T_{-3}$ are considered to carry homogeneous degree $-3$. By construction, $T_{-3}$ is a $\mathfrak{g}$-module and $q_{-3}$ is $\mathfrak{g}$-equivariant. Finally, if $V$ is a Lie algebra, then $T_{-2} = 0$ which implies in turn that $T_{-3} = 0$ as well, as required by item 3. of Proposition \[14\].

The next step is not as straightforward and illustrates what happens for higher degrees; for this reason, it is worth its own treatment. Since $K_{-2}$ (resp. $K_{-3}$) is a well-defined subspace of $F_{-2}$ (resp. $F_{-3}$), one can define a linear map $u_{-4} : F_{-1} \otimes K_{-3} \oplus F_{-2} \otimes K_{-2} \to \Lambda^2(F_{\bullet})|_{-4}$ as the composition of the following two maps:

$$(F_{-1} \otimes K_{-3}) \oplus (F_{-2} \otimes K_{-2}) \longrightarrow (F_{-1} \otimes F_{-3}) \oplus (F_{-2} \otimes F_{-2}) \oplus (F_{-3} \otimes F_{-1}) \longrightarrow \Lambda^2(F_{\bullet})|_{-4}$$

The first arrow is the canonical inclusion, whereas the second arrow is the quotient map defining the exterior algebra. The action of $u_{-4}$ can be explicitly given by the following definition:

$$
u_{-4}(x \otimes y) = x \wedge y \quad \text{for every} \quad x \otimes y \in (F_{-1} \otimes K_{-3}) \oplus (F_{-2} \otimes K_{-2}) \quad (73)$$

Then, by construction, the map $u_{-4}$ is injective. This implies in turn that it is $\mathfrak{g}$-equivariant because $K_{-2}$ and $K_{-3}$ are $\mathfrak{g}$-modules. We have the following commutative diagram:
Notice that the bottom row is exact by Lemma 15. The dashed vertical arrow $k_{-4}$ is the composite $-d_2 \circ u_{-4}$ making this diagram commutative. As a composition of $g$-equivariant morphisms, this map is $g$-equivariant. Its image is thus a $g$-submodule of $F_{-4}$, denoted $K_{-4}$. By construction, this space is uniquely defined. Then, recalling that $T_{-4} = F_{-4}$, one can quotient the bottom row by the respective images of $u_{-4}$ and $k_{-4}$ and obtain:

$$\Lambda^3(F_{-4})_{-4} \longrightarrow T_{-4} \wedge T_{-3} \oplus \Lambda^2(T_{-2}) \longrightarrow F_{-4}/K_{-4} \longrightarrow 0 \quad (74)$$

We set:

$$T_{-4} = F_{-4}/K_{-4}$$

so that elements of $T_{-4}$ are considered to carry homogeneous degree $-4$. We call $q_{-4} : T_{-4} \wedge T_{-3} \oplus \Lambda^2(T_{-2}) \rightarrow T_{-4}$ the map induced from $-d_2|_{\Lambda^2(F_{\ast})_{-4}}$ by the quotient. $T_{-4}$ inherits the quotient $g$-module structure and $q_{-4}$ is by construction $g$-equivariant. By Lemma 16, the sequence (74) is exact. It implies in particular that $q_{-4}$ is surjective. In particular, if $V$ is a Lie algebra, then $T_{-2} = 0$ and $T_{-3} = 0$, which implies that $T_{-4} = 0$ as desired.

Now, let $n \geq 4$ and assume that the spaces $T_{-i}$, for $4 \leq i \leq n$, have been defined as quotients:

$$T_{-i} = F_{-i}/K_{-i}$$

where $K_{-i}$ is the $g$-submodule of $F_{-i}$ (of degree $-i$) corresponding to the canonical image of $\bigoplus_{j=1}^{i-2} F_{-j} \otimes K_{-i+j}$ in $F_{-i}$, as the following commutative diagram (with exact rows) shows:

$$\begin{array}{cccccc}
0 & \longrightarrow & \bigoplus_{j=1}^{i-2} F_{-j} \otimes K_{-i+j} & \overset{id}{\longrightarrow} & \bigoplus_{j=1}^{i-2} F_{-j} \otimes K_{-i+j} & \longrightarrow 0 \\
& \downarrow{u_{-i}} & \downarrow{k_{-i}} & \downarrow{d_2} & \downarrow{d_3} & \\
\Lambda^3\left(\bigoplus_{j=1}^{i-2} F_{-j}\right)_{-i} & \longrightarrow & \Lambda^2\left(\bigoplus_{j=1}^{i-1} F_{-j}\right)_{-i} & \longrightarrow & F_{-i} & \longrightarrow 0
\end{array}$$

where the (injective) map $u_{-i}$ is the composition of the two following maps:

$$\bigoplus_{j=1}^{i-2} F_{-j} \otimes K_{-i+j} \longrightarrow \bigoplus_{j=1}^{i-1} F_{-j} \otimes F_{-i+j} \longrightarrow \Lambda^2(F_{\ast})_{-i}$$

and where $k_{-j}$ is the unique linear making the diagram commutative. We can assume that both $u_{-i}$ and $k_{-i}$ are $g$-equivariant. Moreover, if $V$ is a Lie algebra, assume that every $T_{-i} = 0$, for $2 \leq i \leq n$.

Let us show that these data uniquely define a vector space $T_{-n-1}$ of degree $-(n+1)$ satisfying the same conditions, one level higher. We will then use this induction to conclude the proof of Proposition (14). The finite direct sum $\bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j}$ is a subspace of $\bigoplus_{j=1}^{n-1} F_{-j} \otimes F_{-(n+1)+j}$ and then, this induces a unique linear map $u_{-n-1} : \bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j} \rightarrow \Lambda^2(F_{\ast})_{-n-1}$ through the composition:

$$\bigoplus_{j=1}^{n-1} F_{-j} \otimes K_{-(n+1)+j} \longrightarrow \bigoplus_{j=1}^{n-1} F_{-j} \otimes F_{-(n+1)+j} \longrightarrow \Lambda^2\left(\bigoplus_{i=1}^{n} F_{-i}\right)_{-n-1}$$

The map $u_{-n-1}$ is injective and a morphism of $g$-modules. One can then define a commutative diagram:
The dashed vertical line on the right is defined as the unique linear map — denoted $k_{-n-1}$ — making the diagram commutative. As a composition of two $g$-equivariant morphisms, it is $g$-equivariant. The image of this map uniquely defines a $g$-submodule of $F_{-n-1}$ that we call $K_{-n-1}$. Then, since by Lemma 15 the bottom row of the above diagram is exact, it passes to the quotient:

$$
\Lambda^3\left(\bigoplus_{i=1}^{n-1} F_{-i}\right)\bigg|_{-n-1} \xrightarrow{-d_3} \Lambda^2\left(\bigoplus_{i=1}^{n} T_{-i}\right)\bigg|_{-n-1} \xrightarrow{-d_2} F_{n-1} \xrightarrow{q_{=n+1}} F_{-n-1}/K_{-n-1} \rightarrow 0
$$

and by Lemma 16 this sequence is exact. We set:

$$T_{-n-1} = F_{n-1}/K_{-n-1}$$

and elements of $T_{-n-1}$ are considered to carry homogeneous degree $-(n+1)$. We call $q_{-n-1} : \Lambda^2\left(\bigoplus_{i=1}^{n} T_{-i}\right)\bigg|_{-n-1} \rightarrow T_{-n-1}$ the map induced from $-d_2|\Lambda^2(F\cdot)|_{-n-1}$ by taking the quotient. By construction, $T_{-n-1}$ is a $g$-module and $q_{-n-1}$ is $g$-equivariant. Exactness of the sequence (75) implies that $q_{-n-1}$ is surjective. Thus, if $V$ is a Lie algebra, since by induction $T_{-i} = 0$ for every $2 \leq i \leq n$, we obtain that $T_{-n-1} = 0$ as well. We then have proved that the induction hypothesis is true at level $n+1$.

**Remark 9.** Notice that exactness of the sequence (75) is preserved if one changes each $F_{-i}$ by $T_{-i}$ in the term at the extreme left, so that the following sequence is also exact:

$$
\Lambda^3\left(\bigoplus_{i=1}^{n-1} T_{-i}\right)\bigg|_{-n-1} \xrightarrow{-d_3} \Lambda^2\left(\bigoplus_{i=1}^{n} T_{-i}\right)\bigg|_{-n-1} \xrightarrow{-d_2} T_{n-1} \rightarrow 0
$$

This fact will be used in the proof of Lemma 19.

Continuing the induction provides us with a (possibly infinite) graded vector space $T\cdot = \bigoplus_{i=1}^\infty T_{-i}$, which has the following properties:

1. Every vector space $T_{-i}$ is a $g$-module;
2. $T_{-1} = V[1]$;
3. $T_{-i} = 0$ for every $i \geq 2$ if and only if $V$ is a Lie algebra.

So, items 1. and 2. of Proposition 14 are partially shown, whereas item 3. is proved. Let us now define a graded Lie bracket on $T\cdot$ so that $T\cdot = V[1] \oplus \llbracket T\cdot, T\cdot \rrbracket$. The graded vector space $T\cdot$ can be seen as a quotient of the free graded Lie algebra $F\cdot = \text{Free}(V[1])$ by the graded $g$-submodule $K\cdot = \bigoplus_{i=2}^\infty K_{-i}$. Additionally, by construction, this graded module is a Lie ideal in the free graded Lie algebra $F\cdot$. This implies in particular that the graded vector space $T\cdot$ inherits a graded Lie algebra structure that descends from the free graded Lie algebra structure on $F\cdot$.

The corresponding graded Lie bracket $\llbracket \ldots \rrbracket$ can be made explicit: let $q : \Lambda^2(T\cdot) \rightarrow T\cdot$ be the degree 0 map whose restriction to $\Lambda^2(T\cdot)_{-i}$ coincides with $q_{-i}$ for every $i \geq 2$. This condition uniquely defines the map $q$. This bilinear map, being induced from the Chevalley-Eilenberg differential, satisfies the cohomological condition:

$$q^2 = 0$$  (77)
Additionally, it is surjective and $g$-equivariant, by construction. Then, for every two homogeneous elements $x, y \in T_\bullet$, one has:

$$[x, y] = q(x, y)$$  \hfill (78)

This bracket is a graded bracket – i.e. $[x, y] \in F_{|x|+|y|}$, and it is graded skew symmetric:

$$[x, y] = -(-1)^{|x||y|}[y, x]$$  \hfill (79)

Moreover, it satisfies the Jacobi identity – which is equivalent to $q$ satisfying Equation (77). Since $q$ is surjective on $T_{-i}$ for every $i \geq 2$, the condition $T_\bullet = V[1] \oplus [T_\bullet, T_\bullet]$ is automatically satisfied. Item 2. of Proposition 14 is proved, remains the last part of item 1.

Let us call $\rho_{-i} : g \otimes T_{-i} \rightarrow T_{-i}$ the map representing the inherited action of $g$ on $T_{-i}$, and let $\rho : g \otimes T_\bullet \rightarrow T_\bullet$ be the degree 0 map restricting to $\rho_{-i}$ on $g \otimes T_{-i}$. Alternatively, $\rho$ can of course be seen as a degree 0 Lie algebra morphism from $g$ to $\text{End}(T_\bullet)$. Then, since $q$ is $g$-equivariant, it means that the action of $g$ on $T_\bullet$ is a derivation of the Lie bracket, i.e. the representation $\rho : g \rightarrow \text{End}(T_\bullet)$ actually takes values in $\text{Der}(T_\bullet)$. This concludes the proof of Proposition 14.

**Remark 10.** The graded Lie algebra structure defined on $T_\bullet$ is not independent of the Leibniz algebra structure on $V$ since we defined $T_{-2}$ as a quotient of $S^2(V)$ by the biggest $g$-submodule $K$ of $\text{Ker} \{ \ldots \} \subset S^2(V)$. By induction, the dependence on the Leibniz structure has thus been propagated implicitly along the whole structure of $T_\bullet$, in particular though the maps $q_{-i}$. In particular, semi-strict Lie-Leibniz triples satisfy

$$T_{-2} := \left( S^2(V) \big/ \text{Ker}(\{ \ldots \}) \right) [2]$$

**Remark 11.** The graded Lie algebra obtained through Proposition 14 coincides with the one defined in Lemma 3.13 in [19]. This comes from the two following facts: 1. the choice for $T_{-1}$ and $T_{-2}$ are identical in both constructions and 2. the map $q$ in the present section is dual to the map $\pi$ defined in [19]. The surjectivity of $q$ hence corresponds to the injectivity of $\pi$, then the rest of both hierarchies coincide. Notice that a typo has unfortunately been left in the final, published version of the afore mentioned paper [19]: in Definition 2.13, one should read $H^2_{\text{CE}}(g) = \wedge^2(g_{-1})^* / d_{\text{CE}}(g_{-2})^*$.

### 3.2 Existence and uniqueness of the differential

From now on, in order to avoid any misunderstanding, we will denote by $T_{\leq -1}$ the graded vector space $T_\bullet$ constructed in Proposition 14, and more generally we will write $T_{\leq -i} = \bigoplus_{k \geq 1} T_{-k}$. The direct sum $T_\bullet \oplus g$ is thus denoted $T_{\leq 0}$ while $T_\bullet \oplus g \oplus R_\Theta[1]$ is denoted $T_{\leq +1}$, where the space $T_{+1} = R_\Theta[-1]$ is understood as the space $R_\Theta$, shifted by degree +1. Additionally, for brevity sometimes we will write $\mathbb{T}$ in order to symbolize $T_{\leq +1}$.

This subsection is dedicated to building a (non-trivial) differential $\partial = (\partial_{-i})_{i \geq 1}$ on the graded vector space $T_{\leq -1}$ so that it eventually extends to the right as a chain complex:

$$\ldots \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} V[1] \xrightarrow{\Theta} g \xrightarrow{-e(-\Theta)} R_\Theta[-1]$$

The differential will be constructed precisely so that it is compatible with the graded Lie algebra structure on $T_{\leq -1}$ defined in Proposition 14. However we postpone to subsection 3.3 the proof that such objects equip the graded vector space $\mathbb{T}$ with a differential graded Lie algebra structure. The following proposition then provides existence and unicity of the desired differential:
Proposition 17. Let \((g, V, \Theta)\) be a Lie-Leibniz triple, and let \((T_{\leq -1}, [\ldots])\) be the negatively graded Lie algebra associated to \((g, V, \Theta)\) by Proposition \([14]\). Then there exists a unique family of degree +1 \(\mathfrak{h}\)-equivariant linear maps \(\partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{i \geq 1}\) and satisfying the following conditions:

\[
\begin{align*}
\partial([u, v]) &= 2\{u, v\} \\
\partial([u, x]) &= \rho(\Theta(u); (x)) - [u, \partial(x)] \\
\partial([x, y]) &= [[\partial(x), y] + (-1)^{|x|}[x, \partial(y)]
\end{align*}
\]

(80)

(81)

(82)

for every \(u, v \in T_{-1}\) and \(x, y \in T_{\leq -2}\). Moreover, the degree +1 linear map \(\partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{i \geq 1}\) is a differential, i.e. for every \(i \geq 1\),

\[
\partial_{-i} \circ \partial_{-i-1} = 0.
\]

(83)

Remark 12. The differential defined in Proposition \([17]\) does not turn \(T_{\leq -1}\) into a differential graded Lie algebra, but when one appends \(T_0 = g\) and \(T_{+1} = R_\Theta[-1]\) to \(T_{\leq -1}\) with adequate graded Lie brackets, the total graded vector space \(T = T_{\leq +1}\) becomes a differential graded Lie algebra, as is shown in Proposition \([23]\).

Proof. The proof is build on three lemmas \([13]\) [19] and \([20]\) involving a bilinear map \(m : T_{-1} \wedge T_{\leq -1} \to T_{\leq -1}\) that is a convenient rewriting of the bilinear map \(\rho(\Theta(-); -)\) appearing in the right-hand side of Equation \([81]\). We define it component-wise, so that for every \(i \geq 2\), let \(m_{-i} : T_{-1} \wedge T_{-i} \to T_{-i}\) be the linear map defined by composing \(\rho_{-i}\) with \(\Theta\):

\[
\begin{array}{ccc}
T_{-1} \wedge T_{-i} & \xrightarrow{m_{-i}} & T_{-i} \\
\downarrow \quad \Theta \otimes \text{id} & & \downarrow \rho_{-i} \\
g \otimes T_{-i} & & \\
\end{array}
\]

As for \(i = 1\), we take \(m_{-1} = 2\{\ldots\} : \Lambda^2(T_{-1}) \to T_{-1}\), as we have the following identity:

\[
m_{-1}(x, y) = \rho_{-1}(\Theta(x); y) + \rho_{-1}(\Theta(y); x) = 2\{x, y\}
\]

(84)

where \(\rho_{-1}\) denotes the action of \(g\) on \(T_{-1} = V[1]\). Then, the map \(m_{-i}\) lifts the action of \(g\) on \(T_{-i}\) to an action of \(V\) on \(T_{-i}\) and the quadratic constraint \([19]\) implies that each \(m_{-i}\) is \(\mathfrak{h}\)-equivariant. Let \(m : \Lambda^2(T_{\leq -1}) \to T_{\leq -1}\) be the unique morphism whose restriction to \(\Lambda^2(T_{\leq -1})|_{-i-1}\) is \(m_{-i}\) (for every \(i \geq 1\)). In particular, on any subspace of \(\Lambda^2(T_{\leq -1})|_{-i-1}\) other than \(T_{-1} \wedge T_{-i}\) (for \(i \geq 2\)), the morphism \(m\) is the zero morphism.

Both the bilinear map \(q\) defined at the end of the proof of Proposition \([14]\) and the bilinear map \(m\) can be extended to multilinear maps on \(\Lambda^p(T_{\leq -1})\) (for every \(p \geq 3\)) as follows:

\[
q(x_1 \wedge \ldots \wedge x_p) = \sum_{\sigma \in \text{Un}(2,p-2)} e^\sigma q(x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge x_{\sigma(3)} \wedge \ldots \wedge x_{\sigma(p)})
\]

(85)

\[
m(x_1 \wedge \ldots \wedge x_p) = \sum_{\sigma \in \text{Un}(2,p-2)} e^\sigma m(x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge x_{\sigma(3)} \wedge \ldots \wedge x_{\sigma(p)})
\]

(86)

where \(\text{Un}(2,p-2)\) and \(e^\sigma\) have been defined in Equation \([67]\). Then, the first Lemma is the following:

Lemma 18. The following identity holds:

\[
m \circ q + q \circ m = 0
\]

(87)

That is: the following diagram is commutative, for every \(i \geq 1\):

\[
\begin{array}{ccc}
\Lambda^p(T_{\leq -1}) & \xrightarrow{\partial_{-i}} & T_{-i} \\
& \xrightarrow{m \circ q + q \circ m} & \\
\end{array}
\]
Proof. Let \( x \wedge y \wedge z \in \Lambda^3(T_{-1})_{-i-2} \). If \( x, y, \) and \( z \) have degree strictly lower than \(-1\), then (87) is trivially satisfied (because \( m \) and \(-m\) both vanish). Then, one can assume that one element is in \( T_{-1} = V[1] \), say \( x \), which is then of degree \(-1\). In particular \(|y| + |z| = -i - 1\). The action of \( q \) on \( x \wedge y \wedge z \) is defined through formula (85):

\[
q(x \wedge y \wedge z) = q(x \wedge y) \wedge z - (-1)^{|y||z|}q(x \wedge z) \wedge y + (-1)^{|y|-|z|}q(y \wedge z) \wedge x
\]  

In the following we will define \( \Theta \) to be 0 on any element whose degree is lower than \(-1\). This will allow us to keep track of those terms even if they formally vanish. Keeping in mind that \(|x| = -1\), than applying \( m \) on both sides of Equation (88) implies that:

\[
m(q(x \wedge y \wedge z))
= m\left(-(-1)^{|z|(|y|-1)}z \wedge q(x \wedge y) + (-1)^{|y|-|z|}y \wedge q(x \wedge z) - (-1)^{|y|-|z|+1}x \wedge q(y \wedge z)\right)
\]

\[
= -(-1)^{|z|(|y|-1)}\Theta(z) \cdot q(x \wedge y) + (-1)^{|y|}\Theta(y) \cdot q(x \wedge z) - \Theta(x) \cdot q(y \wedge z)
\]

\[
= -(-1)^{|z|(|y|-1)}q\left((\Theta(z) \cdot x) \wedge y + x \wedge (\Theta(z) \cdot y)\right)
\]

\[
+ (-1)^{|y|}q\left((\Theta(y) \cdot x) \wedge z + x \wedge (\Theta(y) \cdot z)\right)
\]

\[
- q\left((\Theta(x) \cdot y) \wedge z + y \wedge (\Theta(x) \cdot z)\right)
\]

\[
= -q\left(m(x \wedge y) \wedge z - (-1)^{|y||z|}m(x \wedge z) \wedge y + (-1)^{|y|+|z|}m(y \wedge z) \wedge x\right)
\]

\[
= -q(m(x \wedge y \wedge z))
\]

as desired (where we used the fact that \( q \) is \( \Phi \)-equivariant, thus in particular \( \Psi \)-equivariant). To pass from the antepenultimate line to the penultimate one, we used the fact that \( m(y, z) \neq 0 \) only if neither \( y \) nor \( z \) have simultaneously their degrees strictly lower than \(-1\). Obviously Equation (87) extends to the whole of \( \Lambda(F_\bullet) \). \( \square \)

Going back to the proof of Proposition 17, the construction of the degree +1 linear endomorphism \( \partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{i \geq 1} \) is made by induction. First, for \( i = 1 \), the definition of \( \partial_{-1} \) is a by-product of the observation that \( m_{-1} \) factors through \( T_{-2} \):

\[
\Lambda^2(T_{-1}) \xrightarrow{q_{-2}} T_{-2}
\]

\[
\Lambda^3(T_{-1})_{-i-2} \xrightarrow{q} \wedge^2(T_{-1})_{-i-2}
\]

\[
\xymatrix{\Lambda^2(T_{-1})_{-i-1} \ar[r]^q \ar[dr]_{m} & T_{-i-1} \ar[d]_{\partial_{-1}} \\
& T_{-1} \ar[u]}
\]

The map \( \partial_{-1} \) is defined as the unique linear map making the above triangle commutative, i.e. such that

\[
\partial_{-1} \circ q(u, v) = m(u, v)
\]
for every \( u, v \in T_{-1} \). In particular, it is surjective on the ideal of squares \( I \) and it is \( \mathfrak{h} \)-equivariant because \( \{ \ldots \} \) has both of these properties. A priori \( \partial_{-1} \) is not \( \mathfrak{g} \)-equivariant because the symmetric bracket \( \{ \ldots \} \) – equivalently \( m_{-1} \) – is not. By construction, Equation (97) is equivalent to Equation (98).

Now assume that we have constructed a family of \( \mathfrak{h} \)-equivariant linear maps \( \partial_{(-n)} = (\partial_{-i} : T_{-i-1} \to T_{-i})_{1 \leq i \leq n} \) up to order \( n \geq 1 \) satisfying the following conditions: \( \partial_{(-n)} \) is the unique linear degree +1 endomorphism of \( T_{-1} \) restricting to \( \partial_{-i} \) on \( T_{-i-1} \) for every \( 1 \leq i \leq n \), extending as a graded derivation on \( \Lambda^* (\bigoplus_{i=1}^{n+1} T_{-i}) \), and satisfying the following equation on \( \Lambda^* (\bigoplus_{i=1}^{n+1} T_{-i}) \):

\[
\partial_{(-n)} \circ q = m + q \circ \partial_{(-n)}
\tag{98}
\]

To define the map \( \partial_{-n-1} : T_{-n-2} \to T_{-n-1} \) such that it extends Equation (98) one step further, we first define a degree +1 linear map \( j_{-n-1} : \Lambda^2 (T_{-1}){|}_{-n-2} \to T_{-n-1} \) as the sum:

\[
j_{-n-1} = m_{-n-1} + q_{-n-1} \circ \partial_{(-n)}.
\tag{99}
\]

By construction the map \( j_{-n-1} \) is \( \mathfrak{h} \)-equivariant, and allows us to prove the existence and uniqueness of the linear map \( \partial_{-n-1} \):

**Lemma 19.** The map \( j_{-n-1} \) factors through \( T_{-n-2} \); in fact, there exists a unique \( \mathfrak{h} \)-equivariant linear map \( \partial_{-n-1} : T_{-n-2} \to T_{-n-1} \) such that the following triangle is commutative:

\[
\begin{array}{c}
\Lambda^2 (T_{-1}){|}_{-n-2} \\
\downarrow q_{-n-2} \\
T_{-n-1} \\
\downarrow \partial_{-n-1} \\
T_{-n-2}
\end{array}
\]

**Proof.** Let \( x \in T_{-n-2} \). By surjectivity of \( q_{-n-2} \), there exists \( y \in \Lambda^2 (T_{-1}){|}_{-n-2} \) such that \( x = q_{-n-2} (y) \). Then one would define:

\[
\partial_{-n-1}(x) = j_{-n-1}(y) \tag{100}
\]

if guaranteed that any other choice of pre-image of \( x \) does not change the result, i.e. that \( j_{-n-1} (\text{Ker}(q_{-n-2})) = 0 \). Thus, let \( w \in \text{Ker}(q_{-n-2}) \). Exactness of the sequence (76):

\[
\Lambda^3 (T_{-1}){|}_{-n-2} \xrightarrow{q} \Lambda^2 (T_{-1}){|}_{-n-2} \xrightarrow{q_{-n-2}} T_{-n-2} \to 0 \tag{101}
\]

implies the equality \( \text{Ker}(q_{-n-2}) = q (\Lambda^3 (T_{-1}){|}_{-n-2}) \). Then there exists \( \alpha \in \Lambda^3 (T_{-1}){|}_{-n-2} \) such that \( q(\alpha) = w \). Then, by Lemma 13 we have:

\[
m_{-n-1}(w) = -q \circ m(\alpha) \tag{102}
\]

But by the induction hypothesis, \( m \) satisfies Equation (98) on \( \Lambda^3 (T_{-1}){|}_{-n-2} \), so we obtain that:

\[
m_{-n-1}(w) = -q_{-n-1} \circ \partial_{(-n)} \circ q(\alpha) + q_{-n-1} \circ q \circ \partial_{(n)}(\alpha) = -q_{-n-1} \circ \partial_{(-n)}(w) \tag{103}
\]

where we used Equation (77) to get rid of the second term on the right hand side. Hence the result: \( j_{-n-1}(w) = 0 \). \( \mathfrak{h} \)-equivariance of \( \partial_{-n-1} \) is straightforward because \( q_{-n-2} \) is \( \mathfrak{g} \)-equivariant and \( j_{-n-1} \) is \( \mathfrak{h} \)-equivariant, so for any \( a \in \mathfrak{h} \) one has:

\[
\partial_{-n-1}(a \cdot x) = \partial_{-n-1}(a \cdot q_{-n-2}(y)) = \partial_{-n-1}(q_{-n-2}(a \cdot y)) = j_{-n-1}(a \cdot y) = a \cdot (j_{-n-1}(y)) = a \cdot \partial_{-n-1}(x) \tag{104}
\]

Notice that a priori \( \partial_{-n-1} \) is not \( \mathfrak{g} \)-equivariant.

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13In particular it vanishes on \( T_{-1} \).
The linear map $\partial_{-(n+1)}$ defined in Lemma 19 is unique and has the advantage of extending Equation (98) to the level $-(n+1)$, by construction. That is to say, setting $\partial_{-(n+1)}$ to be the unique linear degree $+1$ endomorphism of $T_{\leq -1}$ that restricts to $\partial_{-i}$ on $T_{-i-1}$ for every $1 \leq i \leq n+1$, it extends as a graded derivation on $\Lambda^*(\bigoplus_{i=1}^{n+2} T_{-i})$, and satisfies the following equation on $\Lambda^*(\bigoplus_{i=1}^{n+2} T_{-i})$:

$$\partial_{-(n+1)} \circ q = m + q \circ \partial_{-(n+1)}$$  \hspace{1cm} (105)

By induction, this eventually demonstrates the existence and the uniqueness of a degree $+1$ map $\partial$ satisfying the following identity on $\Lambda(T_{\leq -1})$:

$$\partial \circ q = m + q \circ \partial$$  \hspace{1cm} (106)

Notice that this equation, when restricted to $\Lambda^2(T_{\leq -1})$, is equivalent to the three Equations (80), (81) and (82). Indeed, if two elements of $T_{-1}$ are involved in Equation (106), then the last term disappears and $m = m_{-1}$ so that we are left with Equation (97), which is equivalent to Equation (80); if only one element of $T_{-1}$ is involved in Equation (106), then we obtain (81), while if no element of $T_{-1}$ is involved, $m = 0$ and we obtain Equation (82). This argument proves the first part of Proposition 17.

Now we will show that this degree $+1$ linear map $\partial$ is a differential. The proof is made by induction and relies on the following lemma:

**Lemma 20.** The following identity holds:

$$m \circ \partial + \partial \circ m = 0$$  \hspace{1cm} (107)

That is: the following diagram is commutative for every $i \geq 1$:

```
            -m_{-i}   
    \Lambda^2(T_{\leq -1})_{-i-2} \arrow[rr] \arrow[dr] \arrow[u] & & T_{-i-1} \arrow[dl] \arrow[u] \arrow[d] \partial_{-i} \arrow[dr] \\
      \Lambda^2(T_{\leq -1})_{-i-1} \arrow[u] & & T_{-i} \arrow[u]
```

*Proof.* Let $i \geq 1$ and let $x \wedge y \in \Lambda^2(T_{\leq -1})_{-i-2}$. If neither $x$ nor $y$ belong to $T_{-1}$, then $m(x \wedge y) = 0$ and $m(\partial(x) \wedge y + (-1)^{|x|} x \wedge \partial(y)) = 0$ (because either $\Theta$ vanishes on $x, y$ or $\Theta \circ \partial = 0$), so that commutativity of the diagram is trivially satisfied. Thus, we can assume that at least one element, say $x$, belongs to $T_{-1}$. In that case $|y| = -i - 1 < -1$. As usual we assume that $\Theta$ vanishes on elements of degree lower than $-1$, and that $\partial(T_{-1}) = 0$. Then, one has:

$$-m \circ \partial(x \wedge y) = m(x \wedge \partial(y)) = \Theta(x) \cdot \partial(y) - (-1)^{-i} \Theta(\partial(y)) \cdot x = \partial(\Theta(x) \cdot y) = \partial \circ m(x \wedge y)$$

as desired. Notice that we used the $h$-equivariance of $\partial$ to pass from the antepenultimate line to the penultimate one. \qed
Let $\partial_{-1} \circ j_{-2}(y) = \partial_{-1} \circ m_{-2}(y) + \partial_{-1} \circ q_{-2} \circ \partial_{(-1)}(y)\ (108)

= \partial_{-1} \circ m_{-2}(y) + m_{-1} \circ \partial_{(-1)}(y)\ (109)

= 0\ (110)

where we used Lemma 20 to make the penultimate line vanish.

Let $n \geq 1$ and assume that Equation (83) has been shown for every $1 \leq i \leq n$. In particular the following equation is valid:

$$\partial_{(-i)} \circ \partial_{(-i-1)} = 0.\ (111)$$

for every $1 \leq i \leq n$, where $\partial_{(-i)}$ is the unique linear degree +1 endomorphism of $T_{\leq -1}$ that restricts to $\partial_{-j}$ on $T_{-j-1}$ for every $1 \leq j \leq i$. We want to show that:

$$\partial_{-n-1} \circ \partial_{-n-2} = 0.\ (112)$$

Let $x \in T_{-n-3}$. By construction, we have $\partial_{-n-2}(x) = j_{-n-2}(y)$ for some $y \in \Lambda^2(T_{\leq -1})_{-n-3}$. Then, using Lemma 20 we have:

$$\partial_{-n-1} \circ \partial_{-n-2}(x) = \partial_{-n-1} \circ j_{-n-2}(y)$$

= $\partial_{-n-1} \circ (m_{-n-2} + q_{-n-2} \circ \partial_{(-n-1)})(y)$

= $(-m_{-n-1} \circ \partial_{(-n-1)} + \partial_{-n-1} \circ q_{-n-1} \circ \partial_{(-n-1)})(y)$

= $(-m_{-n-1} + \partial_{-n-1} \circ q_{-n-2}) \circ \partial_{(-n-1)}(y)$

= $q_{-n-1} \circ \partial_{(-n)} \circ \partial_{(-n-1)}(y)$

= 0,

where we used Equation (98) between the fourth and fifth line, and the induction hypothesis Equation (111) for $i = n$ at the penultimate line. This concludes the proof of Proposition 17.

Before going further, we need to investigate the relationship between the linear maps $(\partial_{-i})_{i \geq 1}$, and the embedding tensor $\Theta$. As $\Theta$, each of the linear map $\partial_{-i}$ is $\mathfrak{h}$-equivariant, but certainly not $\mathfrak{g}$-equivariant. The cyclic module generated by each linear map $\partial_{-i}$ is a $\mathfrak{g}$-submodule of $\text{Hom}(T_{i-1}, T_{-i})$, that we denote $R_{-i}$:

$$R_{-i} = \text{Span}\left(\partial_{-i}, a_1 \cdot (a_2 \cdot (\ldots (a_m \cdot \partial_{-i}) \ldots)) \middle| a_1, a_2, \ldots, a_m \in \mathfrak{g}\right) \subset \text{Hom}(T_{i-1}, T_{-i})\ (113)$$

Recall that $R_{\Theta}$ is itself a cyclic $\mathfrak{g}$-submodule of $\text{Hom}(V, \mathfrak{g})$ (see Equation (10)). Then the $R_{-i}$ are characterized by the following observation:

**Proposition 21.** For every $i \geq 1$, there exists a surjective morphism of $\mathfrak{g}$-modules $\mu_{-i}: R_{\Theta} \rightarrow R_{-i}$, which is such that:

$$\partial_{-i} = \mu_{-i}(\Theta)\ (114)$$

**Proof.** The proof is made by induction. We will show that the $R_{-i}$ are quotients of the cyclic module $R_{\Theta}$, so that the $\mu_{-i}$ are the quotient maps. First let us observe that, for every $i \geq 2$, as an element of $\text{Hom}(T_{i-1} \wedge T_{-i}, T_{-i})$, the map $m_{-i}$ generates a $\mathfrak{g}$-submodule $M_{-i}$ of $\text{Hom}(T_{i-1} \wedge T_{-i}, T_{-i})$ through the successive actions of $\mathfrak{g}$ on $m_{-i}$:

$$M_{-i} = \text{Span}\left(m_{-i}, a_1 \cdot (a_2 \cdot (\ldots (a_m \cdot m_{-i}) \ldots)) \middle| a_1, a_2, \ldots, a_m \in \mathfrak{g}\right) \subset \text{Hom}(T_{i-1} \wedge T_{-i}, T_{-i})\ (115)$$
The representation \( \rho_{-i} : \mathfrak{g} \otimes T_{-i} \to T_{-i} \) induces a morphism of \( \mathfrak{g} \)-modules \( \overline{\rho}_{-i} \) from \( \text{Hom}(T_{-1} \land T_{-i}, \mathfrak{g} \otimes T_{-i}) \) to \( \text{Hom}(T_{-1} \land T_{-i}, T_{-i}) \). By construction, this homomorphism satisfies:

\[
\mu_{-i} = \overline{\rho}_{-i}(\Theta \land \text{id}_{T_{-i}})
\]  

(116)

One notices that the right hand side is the composition of \( \overline{\rho}_{-i} \) with the following injective linear map:

\[
\begin{array}{ccc}
\text{Hom}(T_{-1}, \mathfrak{g}) & \longrightarrow & \text{Hom}(T_{-1} \land T_{-i}, \mathfrak{g} \otimes T_{-i})
\end{array}
\]

\[
\alpha \longmapsto \alpha \land \text{id}_{T_{-i}}
\]

This composition of maps yields a morphism of \( \mathfrak{g} \)-modules \( \nu_{-i} = \overline{\rho}_{-i} \circ \alpha : \text{Hom}(T_{-1}, \mathfrak{g}) \to \text{Hom}(T_{-1} \land T_{-i}, T_{-i}) \) that sends \( \Theta \) onto \( m_{-i} \):

\[
m_{-i} = \nu_{-i}(\Theta)
\]  

(117)

In particular, since \( \nu_{-i} \) is \( \mathfrak{g} \)-equivariant, the action of \( \mathfrak{g} \) on \( m_{-i} \) depends only on the action of \( \mathfrak{g} \) on \( \Theta \), through \( \nu_{-i} \). In particular, it sends \( R_\Theta \) onto \( M_{-i} \), which is then isomorphic to a quotient of \( R_\Theta \). This line of argument is still valid for \( i = 1 \), where \( m_{-1} = 2 \{ \ldots \} \).

**Initialization:** Let us now turn to the maps \( \partial_{-i} \): assume that we have built the differential \( \partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{i \geq 1} \) as in Proposition [17]. Given that the map \( q_{-2} : T_{-1} \land T_{-1} \to T_{-2} \) is surjective and \( \mathfrak{g} \)-equivariant, and that \( m_{-1} = \partial_{-1} \circ q_{-2} \), it follows that \( q_{-2} \) induces a morphism of \( \mathfrak{g} \)-modules \( \overline{q}_{-2} : M_{-1} \to R_{-1} \), sending \( m_{-1} \) onto \( \partial_{-1} \). Pre-composing it with \( \nu_{-1} \) induces a homomorphism of \( \mathfrak{g} \)-modules \( \mu_{-1} : R_\Theta \to R_{-1} \):

\[
\begin{array}{ccc}
R_\Theta & \xrightarrow{\nu_{-1}} & M_{-1} & \xrightarrow{\overline{q}_{-2}} & R_{-1}
\end{array}
\]

\[
\begin{array}{ccc}
\mu_{-1} \downarrow & & \\
\alpha & \longmapsto & \overline{q}_{-2}(\alpha)
\end{array}
\]

In particular the composition of maps implies that:

\[
\partial_{-1} = \mu_{-1}(\Theta)
\]  

(118)

which, by definition of \( R_{-1} \), implies that \( \mu_{-1} \) is onto. Moreover, since \( \text{Ker}(\mu_{-1}) \) is a \( \mathfrak{g} \)-submodule of \( R_\Theta \) then \( R_{-1} \) is isomorphic, as a \( \mathfrak{g} \)-module, to the quotient \( R_\Theta / \text{Ker}(\mu_{-1}) \).

**Inductive step:** Let \( n \geq 1 \) and assume that for every \( 1 \leq i \leq n \) there exists a surjective homomorphism of \( \mathfrak{g} \)-modules \( \mu_{-i} : R_\Theta \to R_{-i} \), such that:

\[
\partial_{-i} = \mu_{-i}(\Theta)
\]  

(119)

Let us prove that this is still the case one level higher.

The bilinear map \( j_{-n-1} \), as an element of \( \text{Hom}(\Lambda^2(T_{\leq -1})|_{-n-2}, T_{-n-1}) \), generates a \( \mathfrak{g} \)-submodule that we call \( S_{-n-1} \):

\[
S_{-n-1} = \text{Span}(j_{-n-1}, a_1 \cdot (a_2 \cdot (\ldots (a_m \cdot j_{-n-1}) \ldots)) \mid a_1, a_2, \ldots, a_m \in \mathfrak{g}) \subset \text{Hom}(\Lambda^2(T_{\leq -1})|_{-n-2}, T_{-n-1})
\]  

(120)

By the induction hypothesis and Equation (117) at level \( -n-1 \), the right hand side of Equation (99) consists of a set of terms, each of which is the image of \( \Theta \) through a \( \mathfrak{g} \)-equivariant map. Then it means that there exists a morphism of \( \mathfrak{g} \)-modules \( s_{-n-1} : R_\Theta \to S_{-n-1} \) such that:

\[
j_{-n-1} = s_{-n-1}(\Theta)
\]  

(121)

By definition of \( S_{-n-1} \), this morphism is surjective.

Now given that the map \( q_{-n-2} : \Lambda^2(T_{\leq -1})|_{-n-2} \to T_{-n-2} \) is surjective and \( \mathfrak{g} \)-equivariant, and that \( \partial_{-n-1} \circ q_{-n-2} = j_{-n-1} \) by Lemma [19] it induces a morphism of \( \mathfrak{g} \)-modules \( \overline{q}_{-n-2} : S_{-n-1} \to R_{-n-1} \), which is such that:

\[
\overline{q}_{-n-2}(j_{-n-1}) = \partial_{-n-1}
\]

Pre-composing it with \( s_{-n-1} \) induces in turn a homomorphism of \( \mathfrak{g} \)-modules \( \mu_{-n-1} : R_\Theta \to R_{-n-1} \):
The composition of maps implies that:

$$\partial_{-n-1} = \mu_{-n-1}(\Theta) \quad (122)$$

which, by definition of $R_{-n-1}$, implies that $\mu_{-n-1}$ is onto. Moreover, since $\text{Ker}(\mu_{-n-1})$ is a $g$-submodule of $R_\Theta$ then $R_{-n-1}$ is isomorphic, as a $g$-module, to the quotient $R_\Theta/\text{Ker}(\mu_{-n-1})$. This concludes the proof of Proposition 21. \hfill \square

Recall what is it we know so far. Let $(g, V, \Theta)$ be a Lie-Leibniz triple. Proposition 14 uniquely associates to this Lie-Leibniz triple a negatively graded vector space $T_{\leq -1}$ equipped with a graded Lie algebra structure. Then, Proposition 17 has shown that the complex $T_{\leq -1}$ can be equipped with a differential $\partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{i \geq 1}$ which is almost compatible with the graded Lie bracket, see Equations (80)-(82). We will now extend this differential and the Lie bracket to the graded vector space $T = T_{\leq -1} \oplus g \oplus R_\Theta[-1]$, making it a differential graded Lie algebra. The construction is precisely made so that when the Lie-Leibniz triple is a differential crossed module or more generally when $V$ is a Lie algebra, one finds the differential graded Lie algebra associated to it as in Proposition 12.

### 3.3 The differential graded Lie algebra structure on $T$

The present subsection is dedicated to showing that the graded Lie bracket on $T_{\leq -1}$ defined in Proposition 14 and the differential defined in Proposition 17 can be extended to the whole of $T = T_{\leq -1} \oplus g \oplus R_\Theta[-1]$ in such a way that it forms a differential graded Lie algebra.

Let $(g, V, \Theta)$ be a Lie-Leibniz triple, and let $T_{\leq -1}$ be the negatively graded Lie algebra associated to $(g, V, \Theta)$ by Proposition 14. Recall that $T_0 = g$ and $T_{+1} = R_\Theta[-1]$. The differential defined on $T_{\leq -1}$ by Proposition 17 straightforwardly extends to $T_-$ and $T_0$ by setting:

$$\partial_0 = \Theta : T_- \to T_0 \quad \text{and} \quad \partial_{+1} = -\eta(-, \Theta) : T_0 \to T_{+1} \quad (123)$$

We indeed have $\partial_0 \circ \partial_{-1} = 0$ because the subspace $\text{Im}(\partial_{-1}) = I$ is by construction included in $\text{Ker}(\Theta)$, while $\partial_{+1} \circ \partial_0 = 0$ because $\Theta$ is $\text{Im}(\Theta)$-equivariant, by the quadratic constraint (19). For convenience, we denote the extended differential $\partial$ as well.

We can also straightforwardly extend the result of Proposition 21 to $\partial_0$ and $\partial_{+1}$. Since $\partial_0 = \Theta$, we know that $R_0 = R_\Theta$ so that we set $\mu_0 = \text{id}_{R_\Theta}$. Next, we define $R_{+1}$ to be the cyclic $g$-submodule of $\text{Hom}(g, R_\Theta)$ generated by $\partial_{+1}$, so that the $g$-equivariant linear map $-\eta$ sending $\Theta$ to $\partial_{+1}$ induces a morphism of $g$-modules $\mu_{+1} : R_\Theta \to R_{+1}$ such that $\mu_{+1}(\Theta) = \partial_{+1}$. Then, one can set $\mu = (\mu_{-i} : R_\Theta \to R_{-i})_{i \geq -1}$ to be the family of degree 0 $g$-equivariant linear maps sending $\Theta$ to $\partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{i \geq -1}$:

$$\partial_{-i} = \mu_{-i}(\Theta) \quad \text{for every } i \geq -1. \quad (124)$$

In other words, it can by formally understood as a $g$-equivariant linear map

$$\mu : R_\Theta[-1] \longrightarrow \text{Hom}(T, T)|_{i+1} = \bigoplus_{i \geq -1} \text{Hom}(T_{-i-1}, T_{-i}) \quad (125)$$

whose co-restriction to $\text{Hom}(T_{-i-1}, T_{-i})$ coincides with $\mu_{-i}$.
Now, let us extend the graded Lie algebra structure defined in Proposition 14 to the higher levels \( T_0 = \mathfrak{g} \) and \( T_{+1} = R_\Theta[-1] \). For every \( a, b \in \mathfrak{g} \) and \( x \in T_{\leq -1} \), let us set:
\[
[a, x] = a \cdot x \quad \text{and} \quad [x, a] = -a \cdot x, \tag{126}
\]

\[
\lbrack a, b \rbrack = \lbrack a, b \rbrack
\]

where \([\ldots]\) is the Lie bracket on \( \mathfrak{g} \). The first result is the following:

**Lemma 22.** \((T_{\leq 0}, [\ldots])\) is a graded Lie algebra.

*Proof.* The Jacobi identity for the Lie bracket on \( T_0 \) is automatically satisfied because \( \mathfrak{g} \) is a Lie algebra. Then, one need only check that the bracket defined in Equation (126) is compatible with the graded Lie bracket defined in Equation (128), in the sense that it satisfies the two Jacobi identities:
\[
\lbrack a, \lbrack b, x \rbrack \rbrack = \lbrack \lbrack a, b \rbrack, x \rbrack + \lbrack b, \lbrack a, x \rbrack \rbrack \tag{128}
\]

\[
\lbrack a, \lbrack x, y \rbrack \rbrack = \lbrack \lbrack a, x \rbrack, y \rbrack + \lbrack x, \lbrack a, y \rbrack \rbrack \tag{129}
\]

for any \( a, b \in \mathfrak{g} \) and \( x, y \in T_{\leq -1} \). The first equation encodes the fact that \( T_{[x]} \) is a \( \mathfrak{g} \)-module:
\[
a \cdot (b \cdot x) = [a, b] \cdot x + b \cdot (a \cdot x) \tag{130}
\]

so it is automatically satisfied. Recalling that on \( T_{\leq -1} \) the bracket corresponds to the family of maps \( q = (\mathbf{g}_{-i} : \Lambda^2(T_{\leq -1})_{-i} \to T_{-i})_{i \leq -2} \), Equation (129) reads:
\[
a \cdot q(x, y) = q(a \cdot x, y) + q(x, a \cdot y) \tag{131}
\]

which corresponds to the \( \mathfrak{g} \)-equivariance of the map \( q \), which is satisfied by construction. \( \square \)

We will now extend the graded Lie algebra structure to \( T_{+1} = R_\Theta[-1] \) and check at the same time that it forms a graded Lie algebra structure on \( \mathbb{T} \) which is compatible with the differential \( \partial \). Recalling that the generators of \( R_\Theta \) are of the form \( \Theta_{a_1 a_2 \ldots a_m} = a_1 \cdot (a_2 \cdot (\ldots (a_m \cdot \Theta) \ldots )) \) for some \( a_1, a_2, \ldots, a_m \in \mathfrak{g} \), we set the following definitions for the bracket involving \( T_{+1} = R_\Theta[-1] \):
\[
\lbrack \Theta, x \rbrack = \partial(x) \quad \text{and} \quad [x, \Theta] = -(-1)^{|x|} \partial(x) \tag{132}
\]

\[
\lbrack \Theta_{a_1 a_2 \ldots a_m}, x \rbrack = a_1 \cdot \lbrack \Theta_{a_2 \ldots a_m}, x \rbrack - \lbrack \Theta_{a_2 \ldots a_m}, a_1 \cdot x \rbrack \tag{133}
\]

\[
[x, \Theta_{a_1 a_2 \ldots a_m}] = -(-1)^{|x|}[\Theta_{a_1 a_2 \ldots a_m}, x] \tag{134}
\]

\[
[T_{+1}, T_{+1}] = 0 \tag{135}
\]

for every homogeneous element \( x \in T_{\leq 0} \) and every \( a_1, \ldots, a_m \in \mathfrak{g} \). As a side remark, be aware that physicists do not necessarily impose that \( \lbrack T_{+1, T_{+1}} \rbrack = 0 \), see Section 3.4. of [3].

Notice that Equations (132) and (133) need a bit of explanation. Indeed, \( \Theta \) is a linear map of vector spaces whereas \( \partial \) is a family of linear maps, which do not necessarily even belong to the same \( \mathfrak{g} \)-modules. However, we have seen in the proof of Proposition 21 that each cyclic \( \mathfrak{g} \)-module \( R_{-i} \) is a quotient of \( R_\Theta \), through a quotient map \( \mu_{-i} \) sending \( \Theta \) on \( \partial_{-i} \), as in Equation (124). Altogether, the latter form a family of linear maps \( \mu = (\mu_{-i} : R_\Theta \to R_{-i})_{i \geq -1} \) so that one can make sense of Equations (132), by understanding that \([\Theta, -] \), evaluated on a homogeneous element \( x \), actually corresponds to \( \mu_{|x|+1}(\partial)(x) = \partial_{|x|+1}(x) \). Moreover, \( \mathfrak{g} \)-equivariance of \( \mu \) allows to make sense of Equation (133) as:
\[
\lbrack \Theta_{a_1 a_2 \ldots a_m}, x \rbrack = a_1 \cdot (a_2 \cdot (\ldots (a_m \cdot \partial) \ldots ))(x) \tag{136}
\]

The right-hand side of Equation (135) could have been alternatively found by iterating Equation (133) \( m \) times on the left-hand side of Equation (136).
From Equations (132)-(135), we can deduce the following identities:

\[ \partial \left( [\Theta_a, x] \right) = -[\Theta_a, \partial(x)] \quad \text{and} \quad a \cdot (\partial) \left( [\Theta, x] \right) = -[\Theta, a \cdot (\partial)(x)] \]  

(137)

Then, Equations (137) imply that the following identities are equivalent:

\[ \partial \left( [\Theta_{ab}, x] \right) = -[\Theta_{ab}, \partial(x)] \quad \iff \quad a \cdot (\partial) \left( [\Theta_b, x] \right) = -[\Theta_b, a \cdot (\partial)(x)] \]  

(138)

\[ \iff \quad b \cdot (a \cdot (\partial)) \left( [\Theta, x] \right) = -[\Theta, b \cdot (a \cdot (\partial))(x)] \]  

(139)

However, none of them is deducible from existing equations. Then, one is free to choose these identities to hold, implying that at the next level the following are equivalent:

\[ \partial \left( [\Theta_{abc}, x] \right) = -[\Theta_{abc}, \partial(x)] \quad \iff \quad a \cdot (\partial) \left( [\Theta_{bc}, x] \right) = -[\Theta_{bc}, a \cdot (\partial)(x)] \]  

(140)

\[ \iff \quad b \cdot (a \cdot (\partial)) \left( [\Theta_c, x] \right) = -[\Theta_c, b \cdot (a \cdot (\partial))(x)] \]  

(141)

\[ \iff \quad c \cdot (b \cdot (a \cdot (\partial))) \left( [\Theta, x] \right) = -[\Theta, c \cdot (b \cdot (a \cdot (\partial)))(x)] \]  

(142)

being understood that none of them is deducible from existing equations. Assuming that any – equivalently, all – of them hold, we then find another set of equivalent identities at the next level, etc. By induction, we are thus free (and led) to assume the following compatibility condition between the bracket and the differential, generalizing the left-hand sides of Equations (137), (138) and (140) to any level:

\[ \forall u \in T_{+1}, v \in T_{-1} \quad \partial([u, v]) = -[u, \partial(v)] \]  

(143)

It is now time to check the compatibility of the extended bracket and the extended differential on \( T \). Recall that the graded bracket \([\ldots, \ldots]\) is defined by Proposition 14 and Equations (126), (127), (132), (133), (134) and (135), while the differential is defined in Proposition 17 and extended to \( T_0 \) and \( T_{+1} \) by Equations (128). Additionally, we have required that these two objects satisfy Equation (143). Then, under these assumptions, we have the following important result, which generalizes Proposition 12 to any Lie-Leibniz triple:

**Proposition 23.** \((T, [\ldots, \ldots], \partial)\) is a differential graded Lie algebra.

**Proof.** The proof is not much different than that of Proposition 12 because most problems only arise in the interaction between elements at level +1 and 0. We already know that \( \partial^2 = 0 \) by construction, and that the Jacobi identity is satisfied on \( T_{<0} \) by Lemma 22. First let us prove that the bracket and the differential are compatible, i.e. that they satisfy the Leibniz rule:

\[ \partial([u, v]) = [\partial(u), v] + (-1)^{|u|}[u, \partial(v)] \]  

(144)

for every \( u, v \in T \). We have several cases to handle:

1. If \( u, v \in T_{+1} \), or if \( u \in T_{+1} \) and \( v \in T_0 \) (or conversely) the identity is trivial as a consequence of Equation (133) and because \( \partial|_{T_{+1}} \) is supposed to be the zero map.

2. Let \( u \in T_{+1} \) and \( v \in T_{-1} \). Then, the Leibniz rule for \( u \) and \( v \) is nothing but Equation (143).

3. If \( u, v \in T_0 = g \) then, recalling that \( \partial|_{T_0} = -\eta(-; \Theta) \), Equation (144) is equivalent to the following:

\[ -[u, v] \cdot \Theta = v \cdot (u \cdot \Theta) - u \cdot (v \cdot \Theta) \]  

(145)

which corresponds to the fact that \( R_\Theta \) is a \( g \)-module.

4. If \( u \in T_0 \) and \( v \in T_{-1} \), then Equation (144) becomes:

\[ \partial(u \cdot v) = \left[ -u \cdot \Theta, v \right] + u \cdot \partial(v) \]  

(146)
This equation is equivalent to:
\[ \left[u \cdot \Theta, v \right] = u \cdot (\partial(v)) - \partial(u \cdot v) \]  
(147)

which is nothing but Equation (133) for \( \Theta_u = u \cdot \Theta \).

5. If \( u, v \in T_{\leq -1} \), then we have several subcases (we will implicitly use Equation (78) in this part):

5.a. If \( u, v \in T_{-1} \), recalling that \( \partial_0 = \Theta \), Equation (144) reads:
\[ \partial_{-1}(q_{-2}(u, v)) = \Theta(u) \cdot v + \Theta(v) \cdot u \]  
(148)

This equation is nothing but Equation (80).

5.b. If \( u \in T_{-1} \) and \( v \in T_{\leq -2} \), then Equation (144) becomes:
\[ \partial\left(q(u, v)\right) = \Theta(u) \cdot v - q\left(u, \partial(v)\right) \]  
(149)

This equation can be rewritten as:
\[ \Theta(u) \cdot v = \partial\left(q(u, v)\right) + q\left(u, \partial(v)\right) \]  
(150)

which, given the definition of \( m_v \), is nothing but the identity (82).

5.c. If \( u, v \in T_{\leq -2} \), then Equation (144) can be rewritten as:
\[ \partial\left(q(u, v)\right) = q \circ \partial(u \wedge v) \]  
(151)

which is, once the term on the right hand side is transported to the left hand side, nothing but identity (82). Hence the three subcases of item 5., gathered together, are equivalent to the identities (80), (81), and (82). This result was expected since by construction the bilinear map \( m : \Lambda^2(T_{\leq -1}) \rightarrow T_{\leq -1} \) is the composition of \( \partial_0 \) with the graded Lie bracket. To conclude, the Leibniz rule is thus satisfied on the whole of \( \mathbb{T} \).

Now let us check the Jacobi identity for the graded Lie bracket. For notational purposes, recall that for \( u, v, w \in \mathbb{T} \), the Jacobi identity reads:
\[ \left[u, [v, w]\right] = [\left[u, v\right], w] + (-1)^{|u||w|} [v, [u, w]] \]  
(152)

We already know by Lemma 22 that the graded Jacobi identity is satisfied on \( T_{\leq 0} \). Moreover it is trivially satisfied on \( T_{+1} \) because the bracket is zero on this space. Then we need only check the cases for which one or two terms belong to \( T_{+1} \), that is to say: \( \mathbf{I} \), \( u \in T_{+1} \) and \( v, w \in T_{\leq 0} \), and \( \mathbf{II} \), \( u, v \in T_{+1} \) and \( w \in T_{\leq 0} \). It is sufficient to check both of these Jacobi identities on the generators of \( T_{+1} \).

\( \mathbf{I} \). First notice that if \( v \) or \( w \) belongs to \( T_0 = g \) then Equation (152) is Equation (133). Thus one can restrict oneself to the situation where \( v, w \in T_{\leq -1} \). The proof is made by induction. First assume that \( u = \Theta \). Then, since \( [[\Theta, -]] = \partial \) (see Equation (132)), Equation (152) becomes:
\[ \partial\left([v, w]\right) = \left[\partial(v), w\right] + (-1)^{|v|} [v, \partial(w)] \]  
(153)

which is Equation (144) for \( v, w \in T_{\leq -1} \), and has been proved to be satisfied in item 5. above. Then assume that \( u = \Theta_a \) for some \( a \in g \). Then, using Equation (133), one obtains:
\[ [[\Theta_a, [v, w]] = a \cdot [\Theta, [v, w]] - [\Theta, a \cdot [v, w]] \]  
(154)

\[ = a \cdot \left( [\Theta, v], w \right) + (-1)^{|v|} [v, \left[ \Theta, w \right]] \right) - \left[ \Theta, a \cdot [v, w] - [v, a \cdot w] \right] \]  
(155)

\[ = [a \cdot [\Theta, v], w] + [[\Theta, v], a \cdot w] + (-1)^{|v|} \left( [a \cdot v, \left[ \Theta, w \right]] + [v, a \cdot [\Theta, w]] \right) \]  
(156)

\[ - [\left[ \Theta, a \cdot v \right], w] + (-1)^{|v|} [a \cdot v, \left[ \Theta, w \right]] - [\left[ \Theta, v \right], a \cdot w] + (-1)^{|v|} [v, \left[ \Theta, a \cdot w \right]] \]  
(157)
where, from the first line to the second line we used Equation (153) on the term on the left, and Lemma 22 on the term on the right; from the second to the third line, we used Lemma 22 on the first term, and Equation (153) on the second one; the transition from the third line to the fourth and last one results from cancellation of four terms, and the use of Equation (153).

Now let $n \geq 1$ and assume that Equation (152) has been shown for every term $v \in T_{+1}$ of the form $\Theta_{a_{1}...a_{n}}$, and every $v, w \in T_{\leq -1}$. Chose $n + 1$ elements $a_{1}, ..., a_{n+1} \in \mathfrak{g}$, then the proof that:

$$[\Theta_{a_{1}...a_{n+1}}, [v, w]] = [[\Theta_{a_{1}...a_{n+1}}, v], w] + (-1)^{|v|}[v, [\Theta_{a_{1}...a_{n+1}}, w]]$$

rely on the same step as in Equations (154)–(157), except that one uses the induction hypothesis in place of Equation (153). This proves the first case.

II. Now assume that $u, v \in T_{+1}$. Then, by Equation (159), the graded Jacobi identity reduces to:

$$[u, [v, w]] = -[v, [u, w]]$$

Notice that if $w \in T_{0} = \mathfrak{g}$, then it is automatically satisfied because both $[v, w]$ and $[u, w]$ belong to $T_{+1}$. Hence one can assume that $w \in T_{\leq -1}$. The proof is made by induction as well. It is sufficient to check Equation (159) on the generators of $T_{+1}$. Let $v$ be a fixed element of $T_{+1}$. If $u = \Theta$ then Equation (159) corresponds to Equation (143). Next, let $u = \Theta_{a}$ for some $a \in \mathfrak{g}$. Then, following the same lines of argument as in Equations (154)–(156) and recalling that $[T_{+1}, T_{+1}] = 0$, one obtains the following identity:

$$[\Theta_{a}, [v, w]] = -\left( [a \cdot v, [\Theta, w]] + [v, a \cdot [\Theta, w]] + [a \cdot v, [\Theta, w]] + [v, [\Theta, a \cdot w]] \right)$$

The right hand side can be recasted as $[v, [\Theta, w]]$, which is the desired result. Notice that in the present context, where both $u$ and $v$ belong to $T_{+1}$, the use of Equation (143) was crucial to pass from the left-hand side of Equation (160) to the right-hand side.

Now let $n \geq 1$ and assume that Equation (159) has been proved for every $u$ of the form $\Theta_{a_{1}...a_{n}}$. Then pick some $a_{1}, ..., a_{n+1} \in \mathfrak{g}$ so, using Equation (153), one has:

$$[\Theta_{a_{1}...a_{n+1}}, [v, w]] = a_{1} \cdot [\Theta_{a_{2}...a_{n+1}}, [v, w]] - [\Theta_{a_{2}...a_{n+1}}, a_{1} \cdot [v, w]]$$

$$= a_{1} \cdot \left( [v, [\Theta_{a_{2}...a_{n+1}}, w]] - [\Theta_{a_{2}...a_{n+1}}, [a_{1} \cdot v, w]] - [v, a_{1} \cdot w] \right)$$

$$= -[a_{1} \cdot v, [\Theta_{a_{2}...a_{n+1}}, w]] - [v, a_{1} \cdot [\Theta_{a_{2}...a_{n+1}}, w]]$$

$$+ [a_{1} \cdot v, [\Theta_{a_{2}...a_{n+1}}, w]] + [v, [\Theta_{a_{2}...a_{n+1}}, a_{1} \cdot w]]$$

$$= -[v, [\Theta_{a_{1}...a_{n+1}}, w]]$$

where we used three times the induction hypothesis, applied to $\Theta_{a_{2}...a_{n+1}}$. This proves that Equation (159) is satisfied at level $n+1$. Notice that this result could have been straightforwardly obtained from the generalization of equivalences (138)–(142) at every level, justified by the fact that we assumed Equation (143) to hold. By induction this proves that Equation (159) is satisfied over the generators of $T_{+1}$, and thus for every $u, v \in T_{+1}$.

This proves that $[\ldots]$ is a graded Lie bracket on $T$.

The fact that the maps $(\partial_{-i})_{i \geq -1}$ generate $\mathfrak{g}$-modules $(R_{-i})_{i \geq -1}$ satisfying Proposition 21 is of crucial importance in the proof. Indeed it allows us to identify the action of the differential $\partial$ and the adjoint action of the embedding tensor $\Theta$, thus proving the Jacobi identities that the bracket has to satisfy. Without this identification, one cannot simply extend the graded Lie algebra structure defined on $T_{\leq -1}$ in Proposition 14 to the whole complex $T = T_{\leq -1} \oplus \mathfrak{g} \oplus R_{0}[-1]$, and should rather limit oneself to extend it to $T_{\leq -1} \oplus \mathfrak{h}$ only. However, this latter partial result, although more straightforward [19], is quite unsatisfying since one loses most data from $\mathfrak{g}$. Proposition 23 is thus a great improvement to this situation, since we do not lose any data in the process. This justifies the following characterization:
Definition 24. We call tensor hierarchy associated to a Lie-Leibniz triple \((\mathfrak{g}, V, \Theta)\) the (differential) graded Lie algebra \((\mathfrak{t}, \ldots, \delta)\) defined in Proposition 23.

Remark 13. The tensor hierarchy here defined should not be confused with the `tensor hierarchy algebra’ defined in [23], which a priori is not a differential graded Lie algebra, but a \(\mathbb{Z}\)-graded Lie superalgebra with a subspace at degree +1 (in our convention) accommodating all possible embedding tensors satisfying the representation constraint.

When applying this construction to specific cases of Lie-Leibniz triples we obtain the following result, whose proof is immediate:

Proposition 25. The tensor hierarchy associated to a Lie-Leibniz triple \((\mathfrak{g}, V, \Theta)\) for which \(V\) is a Lie algebra is precisely the 3-term differential graded Lie algebra obtained in Proposition 12. In particular the tensor hierarchy associated to a differential crossed module \(\mathfrak{g}_1 \xrightarrow{\Theta} \mathfrak{g}_0\) is the 3-term differential graded Lie algebra:

\[
\mathfrak{g}_1 \xrightarrow{\Theta} \mathfrak{g} \xrightarrow{0} \mathbb{R}[-1]
\]

Example 7. We take a particular case of the general Example 6. Let \(A\) be the associative algebra of 2-by-2 real matrices, i.e. \(A = \mathcal{M}_{2 \times 2}(\mathbb{R})\) and the associative product \(\cdot\) is the matrix product \(\times\). The associative algebra structure on \(A\) induces a Lie algebra structure on \(A\) where the Lie bracket \([\ldots]_A\) is the commutator of matrices; this is \(\mathfrak{gl}_2(\mathbb{R})\). Let \(D : A \to A\) be the linear projection on the diagonal matrices:

\[
D : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}
\] (165)

This linear map satisfies Equation (25). Then, we can define a Leibniz product on \(A\) via Equation (27):

\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0 & (a_{11} - a_{22})b_{12} \\ -(a_{11} - a_{22})b_{21} & 0 \end{pmatrix}
\] (166)

From now on, when we want to emphasize the algebraic structures equipping \(A\), we denote by \(\mathfrak{g}\) the Lie algebra structure \((A, [\ldots]_A)\) and by \(V\) the Leibniz algebra structure \((A, \circ)\). The action of \(\mathfrak{g}\) on \(V\) (equivalently, on \(A\)) is materialized via the adjoint action of the Lie bracket. As the image of \(D\) consists of the Lie subalgebra \(\mathfrak{h}\) of diagonal 2-by-2 matrices, Equation (166) shows that Equation (28) is automatically satisfied. Together with Equation (27), it implies that the triple \((\mathfrak{g}, V, D)\) is a Lie-Leibniz triple. Since \([a \circ b - D([a, b]_A), c]_A \neq 0\), by the discussion at the end of Example 6, this Lie-Leibniz triple is not semi-strict. Let us compute how \(\mathfrak{g}\) acts on the embedding tensor \(D\). Let \(a, b \in A\), if we denote by \(\eta : \mathfrak{g} \to \text{End}(\text{End}(A))\) the representation of \(\mathfrak{g}\) on \(\text{End}(A)\) induced by the adjoint action of \(\mathfrak{g}\) on \(A\), then we have:

\[
\eta(a; D)(b) = [a, Db]_A - D([a, b]_A) = -b \circ a - D([b, a]_A) = \begin{pmatrix} b_{12}a_{21} - a_{12}b_{21} & -(b_{11} - b_{22})a_{12} \\ (b_{11} - b_{22})a_{12} & b_{21}a_{12} - a_{21}b_{12} \end{pmatrix}
\] (167)

In particular it confirms that the embedding tensor is not \(\mathfrak{g}\)-equivariant, but that it is \(\mathfrak{h}\)-equivariant (the right-hand side of Equation (167) vanishes when \(a\) is a diagonal matrix).

As a consequence, the symmetric bracket \([\ldots] : S^2 A \to A\) is not \(\mathfrak{g}\)-equivariant and its kernel is not a \(\mathfrak{g}\)-module. Let us decompose \(S^2 A\) into \(\mathfrak{g}\)-modules to find the biggest \(\mathfrak{g}\)-submodule \(K\) of \(\text{Ker}(\{\ldots\})\). A basis of \(A\) is given by the four matrices:

\[
E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\] (168)
The \( \mathfrak{g} \)-action of a matrix \( a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) on each of them is given by the adjoint action \([a, \cdot]_A\):

\[
\begin{align*}
[a, E_{11}]_A &= -a_{12}E_{12} + a_{21}E_{21}, & [a, E_{12}]_A &= -a_{21}E_{11} + (a_{11} - a_{22})E_{12} + a_{21}E_{22} \\
[a, E_{22}]_A &= a_{12}E_{12} - a_{21}E_{21}, & [a, E_{21}]_A &= a_{12}E_{11} - (a_{11} - a_{22})E_{21} - a_{12}E_{22}
\end{align*}
(169)
(170)

The matrices \( E_{ij} \) induce a basis of the 10-dimensional space \( S^2A \) via the symmetric product \( E_{ij} \odot E_{kl} \). We can alternatively describe this space with degree 2 polynomials of four variables by assigning the variable \( X \) to \( E_{11} \), \( Y \) to \( E_{12} \), \( Z \) to \( E_{21} \) and \( T \) to \( E_{22} \). Then the action (169), (170) of \( \mathfrak{g} \) on the matrices \( E_{ij} \) induces an action on the monomials \( X, Y, Z, T \):

\[
\begin{align*}
a \cdot X &= -a_{12}Y + a_{21}Z, & a \cdot Y &= -a_{21}X + (a_{11} - a_{22})Y + a_{21}T \\
a \cdot T &= a_{12}Y - a_{21}Z, & a \cdot Z &= a_{12}X - (a_{11} - a_{22})Z - a_{12}T
\end{align*}
(171)
(172)

The action of \( \mathfrak{g} \) on \( S^2A \) is induced from Equations (171) and (172) by derivation. Then one can check that the symmetric space \( S^2A \) is completely reducible into four irreducible \( \mathfrak{g} \)-submodules:\(^{16}\)

\[
U_1 = \langle XT - YZ \rangle
\]
\[
U_2 = \langle X^2 + T^2 + XT + YZ \rangle
\]
\[
U_3 = \langle X^2 - T^2, XY + YT, XZ + ZT \rangle
\]
\[
U_4 = \langle Y^2, Z^2, X^2 + T^2 - 2(XT + YZ), XY - YT, XZ - ZT \rangle
\]

The generators of the modules form an alternative basis for \( S^2A \) because one can pass from the standard basis of \( S^2A \) to the generators of \( U_1, U_2, U_3 \) and \( U_4 \) by the following formula:

\[
\begin{align*}
a_{X^2}X^2 + a_{XY}XY + a_{XZ}XZ + a_{XT}XT + a_{Y^2}Y^2 \\
+ a_{YZ}YZ + a_{YT}YT + a_{Z^2}Z^2 + a_{ZT}ZT + a_{T^2}T^2 &= a_{XT}a_{YZ}(XT - YZ) + \frac{2(a_{X^2} + a_{T^2}) + a_{XT} + a_{Y^2}}{6} (X^2 + T^2 + XT + YZ) \\
&+ \frac{a_{X^2} + a_{T^2}}{2} (X^2 - T^2) + a_{XY}a_{YT}(XY + YT) + \frac{a_{XZ} + a_{T^2}}{2} (XZ + ZT) \\
&+ a_{Y^2}a_{Z^2} + a_{YT}a_{ZT}a_{XT} + a_{Y^2} + a_{Z^2} - a_{XT} - a_{Y^2}a_{YT} + \frac{a_{XZ} + a_{T^2}}{2} (XZ + ZT)
\end{align*}
(173)

We then have two equivalent ways of parametrizing \( S^2A \), whose respective usefulness is relative to the given computation.

The symmetric product \( a \odot b \in S^2A \) of two matrices can be decomposed on the polynomials in \( X, Y, Z, T \) as the following:

\[
a \odot b = a_{11}b_{11}X^2 + \frac{1}{2}(a_{11}b_{12} + b_{11}a_{12})XY + \frac{1}{2}(a_{11}b_{21} + b_{11}a_{21})XZ + \frac{1}{2}(a_{11}b_{22} + b_{11}a_{22})XT + a_{12}b_{12}Y^2
\]
\[
+ \frac{1}{2}(a_{21}b_{21} + b_{21}a_{21})YZ + \frac{1}{2}(a_{21}b_{22} + b_{21}a_{22})YT + a_{21}b_{21}Z^2 + \frac{1}{2}(a_{21}b_{22} + b_{21}a_{22})ZT + a_{22}b_{22}T^2
\]
(174)

By comparing the anti-diagonal matrix elements of the matrix on the right-hand side of the symmetrized version of Equation (169) to the decomposition (174), we deduce that the symmetric bracket \( \{ \ldots \} : S^2A \to A \) is defined on an element of \( S^2A \) as:

\[
\begin{pmatrix}
0 & a_{XY} - a_{YT} \\
-a_{XZ} + a_{ZT} & 0
\end{pmatrix}
(175)
\]

\(^{16}\)If one understands \( \mathfrak{g} \) as the Lie algebra \( \mathfrak{gl}_2(\mathbb{R}) \), which is reductive as it can be written as \( \mathfrak{gl}_2(\mathbb{R}) = \mathfrak{s} \mathfrak{l}_2(\mathbb{R}) \oplus \langle I_2 \rangle \), then the four modules \( U_i \) correspond to the decomposition of \( S^2(\mathfrak{gl}_2(\mathbb{R})) \) into irreducible \( \mathfrak{gl}_2(\mathbb{R}) \)-modules (equivalently: \( \mathfrak{s} \mathfrak{l}_2(\mathbb{R}) \)-modules because \( [\mathfrak{gl}_2(\mathbb{R}), \mathfrak{gl}_2(\mathbb{R})] = \mathfrak{s} \mathfrak{l}_2(\mathbb{R}) \)).
The elements of the left-hand side of Equation (175) that do not appear on the right-hand side live in the kernel of the symmetric bracket, which is then the 8-dimensional subspace of $S^2A$ generated by the following vectors:

$$\text{Ker}\{\ldots\} = \langle X^2, Y^2, Z^2, T^2, XT, YZ, XY + YT, XZ + ZT \rangle$$

The three first $g$-submodules $U_1, U_2, U_3$ are contained in $\text{Ker}\{\ldots\}$ while for the last module, only the first three generators belong to the kernel:

$$\text{Ker}\{\ldots\} = U_1 \oplus U_2 \oplus U_3 \oplus \langle Y^2, Z^2, X^2 + T^2 - 2(XT + YZ) \rangle$$

As the action of $g$ on the remaining generators $Y^2, Z^2, X^2 + T^2 - 2(XT + YZ)$ leaves the kernel, the biggest $g$-submodule of the kernel is the 5-dimensional subspace $K = U_1 \oplus U_2 \oplus U_3$. We can then identify $T_{-2}$, the degree $-2$ space of the tensor hierarchy, with $U_4[2]$ ($U_4$ shifted by $-2$). The projection $p : S^2A \rightarrow U_4$ then induces the graded Lie bracket on $T_{-1} = A[1]$ taking values in $T_{-2}$ by Equation (178), that is to say $[a, b] = q_{-2}(a \odot b) = 2p(a \odot b)$. We can give an explicit formula of this bracket, thanks to the decomposition (174) and the correspondence (173):

$$\left[\begin{array}{cc}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{array}\right], \left[\begin{array}{cc}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{array}\right] = \frac{1}{6} \begin{pmatrix}
  2a_{11}b_{11} - (a_{11}b_{22} + b_{11}a_{21}) \\
  + 2a_{22}b_{22} - (a_{12}b_{21} + b_{12}a_{21})
\end{pmatrix}(X^2 + T^2 - 2(XT + YZ))$$

$$+ \frac{1}{2} \begin{pmatrix}
  (a_{11}b_{12} + b_{11}a_{12}) - (a_{12}b_{22} + b_{12}a_{22}) \\
  + ((a_{11}b_{21} + b_{11}a_{21}) - (a_{21}b_{22} + b_{21}a_{22}))
\end{pmatrix}(XY - YT) + 2a_{12}b_{12}Y^2 + 2a_{22}b_{22}Z^2$$

The linear map $\partial_{-1} : T_{-2} \rightarrow T_{-1}$ is then defined as:

$$\begin{array}{c}
pY^2 + qZ^2 \\
+r(X^2 + T^2 - 2(XT + YZ)) \\
+s(XY - YT) + t(XZ - ZT)
\end{array} \xrightarrow{\partial_{-1}} \begin{pmatrix}
  0 & 2s \\
-2t & 0
\end{pmatrix}$$

so that we indeed have $\partial_{-1}([a, b]) = 2\{a, b\}$. The map $\partial_{-1}$ has a 3-dimensional kernel. The tensor hierarchy associated to the Lie-Leibniz triple $(g, V, D)$ then begins with the following sequence:

$$\ldots \rightarrow U_4[2] \xrightarrow{\partial_{-1}} A[1] \xrightarrow{D} A \xrightarrow{-\eta(-D)} R_D[-1]$$

where $R_D \subset \text{End}(A)$ is the $g$-submodule generated by $D$.

In order to define $T_{-3}$, one needs to decompose the following diagram in terms of irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules (because they are the irreducible $\mathfrak{gl}_2(\mathbb{R})$-modules):

$$\begin{array}{c}
0 \xrightarrow{0} A \otimes (U_1 \oplus U_2 \oplus U_3) \xrightarrow{id} A \otimes (U_1 \oplus U_2 \oplus U_3) \xrightarrow{0}
\end{array}$$

$$\begin{array}{c}
S^3(A) \xrightarrow{-d_3} A \otimes S^2(A) \xrightarrow{-d_2} F_{-3} \xrightarrow{0}
\end{array}$$

Here, recall that lines and columns are exact sequences. For any $k \geq 0$, let us denote by $V_k$ the irreducible $\mathfrak{sl}_2(\mathbb{R})$-module of highest weight $k$, so in particular it is a $k + 1$-dimensional vector space so that $V_2$ denotes
the 3-dimensional adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$, and $U_4 = V_4$. Given the discussion above and since
$\mathfrak{gl}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R}) \oplus \langle I_2 \rangle = V_2 \oplus V_6$, the decomposition of the second symmetric power of $A = \mathfrak{gl}_2(\mathbb{R})$ in terms of the $\mathfrak{sl}_2(\mathbb{R})$-modules $U_1, U_2, U_3, U_4$ can be rewritten as:

$$S^2(\mathfrak{gl}_2(\mathbb{R})) = S^2(V_2 \oplus V_6) = V_0 \oplus V_0 \oplus V_2 \oplus V_4 \quad (182)$$

Then, using the classical formulas for reducibility of (symmetric) powers of $\mathfrak{sl}_2(\mathbb{R})$-modules, we have:

$$S^3(\mathfrak{gl}_2(\mathbb{R})) = 2V_0 \oplus 2V_2 \oplus V_4 \oplus V_6 \quad (183)$$

$$\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{gl}_2(\mathbb{R})) = 3V_0 \oplus 5V_2 \oplus 3V_4 \oplus V_6 \quad (184)$$

where the integers in front of each module symbolize their multiplicities. By exactness of the lines of Diagram (181), the difference between Lines (181) and (183) gives:

$$F_{-3} = V_0 \oplus 3V_2 \oplus 2V_4 \quad (185)$$

As every space in the present example is completely reducible, the module $T_{-3}$ can be understood as a submodule of both $F_{-3}$ and $A \otimes U_4 = \mathfrak{gl}_2(\mathbb{R}) \otimes V_4 = V_2 \oplus 2V_4 \oplus V_6$. From this observation we deduce that:

$$T_{-3} \subset F_{-3} \cap (A \otimes U_4) = V_2 \oplus 2V_4 \quad (186)$$

Since $T_{-3} \simeq A \otimes U_4 / S^3 A$, by carefully analyzing the intersection of $A \otimes U_4$ with $S^3(\mathfrak{gl}_2(\mathbb{R}))$, we can deduce the explicit decomposition of $T_{-3}$. For example, as we have the inclusion

$$A \otimes U_4 \subset \mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{sl}_2)$$

where the latter tensor product contains $S^3(\mathfrak{sl}_2) = V_2 \oplus V_6$, we deduce that the term $V_2 \oplus V_6$ in $A \otimes U_4$ corresponds to $S^3(\mathfrak{sl}_2)$ so, by exactness of the bottom line in Diagram (181), $V_2$ cannot appear in $T_{-3}$. Now, there is only one module $V_4$ in $S^3(\mathfrak{gl}_2(\mathbb{R}))$, while there are two in $A \otimes U_4$ so $T_{-3}$ contains at least one module $V_4$, if not two. The question is then if either one of the two modules $V_4$ in the decomposition of $A \otimes U_4$ lies in the image of $S^3(\mathfrak{gl}_2(\mathbb{R}))$. Since the two modules $V_4$ in $A \otimes U_4$ are precisely those of $\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{sl}_2(\mathbb{R}))$, it is equivalent to asking if either one of the two submodules $V_4$ of $\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{sl}_2(\mathbb{R}))$ lies in the image of $S^3(\mathfrak{gl}_2(\mathbb{R}))$.

The following argument suggests that it is not the case. First, notice that we have:

$$S^3(\mathfrak{gl}_2(\mathbb{R})) = S^3(\mathfrak{sl}_2(\mathbb{R})) \oplus S^2(\mathfrak{gl}_2(\mathbb{R})) \quad (187)$$

$$\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{gl}_2(\mathbb{R})) = (\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{sl}_2(\mathbb{R}))) \oplus (\mathfrak{gl}_2(\mathbb{R}) \otimes \mathfrak{gl}_2(\mathbb{R})) \quad (188)$$

Then, comparing the decompositions (187) and (188), we deduce that the submodule $S^2(\mathfrak{gl}_2(\mathbb{R}))$ of $S^3(\mathfrak{gl}_2(\mathbb{R}))$ sits in the submodule $\mathfrak{gl}_2(\mathbb{R}) \otimes \mathfrak{gl}_2(\mathbb{R})$ of $\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{gl}_2(\mathbb{R}))$. Then, since $S^3(\mathfrak{sl}_2) = V_2 \oplus V_6$, the submodule $V_4$ of $S^3(\mathfrak{gl}_2(\mathbb{R}))$ is necessarily in $S^2(\mathfrak{gl}_2(\mathbb{R}))$, so it seems to be sent to the submodule $V_4$ of $\mathfrak{gl}_2(\mathbb{R}) \otimes \mathfrak{gl}_2(\mathbb{R})$ in Line (188). From this, we deduce that none of the two submodules $V_4$ of $\mathfrak{gl}_2(\mathbb{R}) \otimes S^2(\mathfrak{sl}_2(\mathbb{R}))$ - and thus, of $A \otimes U_4$ - lies in the image of $S^3(\mathfrak{gl}_2(\mathbb{R}))$. Then, Diagram (181) can be recasted in the following form, where the colors mark the correspondence between the domain and the range of each map (and where the purple color means that it is both red and blue):

$$
\begin{align*}
0 & \longrightarrow 3V_0 \oplus 4V_2 \oplus V_4 \quad \text{id} \quad 3V_0 \oplus 4V_2 \oplus V_4 \quad \longrightarrow 0 \\
2V_0 \oplus 2V_2 & \oplus V_4 \oplus V_6 \quad \oplus V_2 \oplus V_2 \oplus V_3 V_2 \quad \oplus V_4 \oplus V_6 \quad \oplus V_2 \oplus V_4 \oplus V_6 \quad \longrightarrow V_0 \oplus 3V_2 \oplus 2V_4 \quad \longrightarrow 0 \\
2V_0 \oplus 2V_2 & \oplus V_4 \oplus V_6 \quad \longrightarrow V_2 \oplus 2V_4 \oplus V_6 \quad \longrightarrow 2V_4 \quad \longrightarrow 0
\end{align*}
$$
From this we deduce that \( T_{-3} = (V_4 \oplus V_4)[3] \). The Lie bracket between \( T_{-2} = V_4[2] \) and \( T_{-1} = A[1] \) is uniquely defined as the bottom horizontal line, while the linear map \( \partial_{-2} : T_{-3} \to T_{-2} \) is also uniquely defined by Lemma 19. This gives the next level of the tensor hierarchy associated to the Lie-Leibniz triple \((\mathfrak{g}, V, D)\):

\[
\cdots \to (V_4 \oplus V_4)[3] \xrightarrow{\partial_{-2}} V_4[2] \xrightarrow{\partial_{-1}} A[1] \xrightarrow{D} A \xrightarrow{-\eta(-;D)} R_D[-1]
\]

We will not provide the next space of the hierarchy as the computations become more cumbersome, but in principle there is no mathematical obstruction to this task.

### 3.4 Proof of theorem \([13]\)

Since the tensor hierarchy \( \mathcal{T} \) associated to a given Lie-Leibniz triple \((\mathfrak{g}, V, \Theta)\) includes \( \mathfrak{g} \) and \( V \) as \( T_0 \) and \( T_{-1} \), we immediately have the following result, which is the first part of Theorem \([13]\).

**Proposition 26.** The function \( G : \text{Lie-Leib} \to \text{DGLie}_{\leq 1} \) associating a tensor hierarchy to a given Lie-Leibniz triple is injective-on-objects.

Thus, tensor hierarchies can be seen as Lie-ifications of Lie-Leibniz triples, where no information has been lost, generalizing what occurs for differential crossed modules. We now show that, when restricted to a particular subcategory of \( \text{Lie-Leib} \), this function is a functor. The following Proposition is the second part of Theorem \([13]\).

**Proposition 27.** The restriction of the function \( G : \text{Lie-Leib} \to \text{DGLie}_{\leq 1} \) to \( \text{compLie-Leib} \) is a faithful functor.

**Proof.** Let us first prove that the assignment \( G \) is functorial on \( \text{compLie-Leib} \), i.e. that a compatible morphism of Lie-Leibniz triples induces a dgLa morphism between their tensor hierarchies. Faithfulness will follow from the definition of the action of \( G \) on morphisms.

Let \((\mathfrak{g}, V, \Theta)\) and \((\mathfrak{g}', V', \Theta')\) be Lie-Leibniz triples, and let \((\mathcal{T}, \lbrack \ldots \rbrack, \partial)\) and \((\mathcal{T}', \lbrack \ldots \rbrack', \partial')\) be their associated tensor hierarchies. Let us introduce some notation: for every \( i \geq 1 \), let \( T_{(-i)} \) be the direct sum \( \oplus_{0 \leq j \leq i} T_{-j} \), the same convention applying to \( \mathcal{T}' \). In the proof, \( \phi_{-i} \) will always denote a degree 0 linear map from \( T_{-i} \) to \( T'_{-i} \), and \( \phi_{(-i)} \) will denote the unique degree 0 linear map from \( T_{(-i)} \) to \( T'_{(-i)} \) that restricts to \( \phi_{-j} \) on \( T_{-j} \). It can be straightforwardly extended to \( \Lambda^*(T_{(-i)}) \) as a morphism of graded algebras.

**Definition of \( \phi_{+1} \):** Let \((\varphi, \chi) : (\mathfrak{g}, V, \Theta) \to (\mathfrak{g}', V', \Theta')\) be any (compatible) morphism of Lie-Leibniz triples, and let us denote \( \phi_0 = \varphi \) and \( \phi_{-1} = \chi : V[1] \to V'[1] \). The compatibility of the respective graded Lie brackets on \( \mathcal{T} \) and \( \mathcal{T}' \) with \( \phi_0 \) and \( \phi_{-1} \) is deduced from the fact that \( \phi_0 \) is a Lie algebra morphism and by Equation \((183)\):

\[
\lbrack \phi_0(a), \phi_0(b) \rbrack = \phi_0(\lbrack a, b \rbrack) \quad (189)
\]

\[
\lbrack \phi_0(a), \phi_{-1}(x) \rbrack = \phi_{-1}(\lbrack a, x \rbrack) \quad (190)
\]

for every \( a, b \in T_0 \) and \( x \in T_{-1} \). Now, let \( R_\Theta \subset \text{Hom}(V, \mathfrak{g}) \) and \( R_{\Theta'} \subset \text{Hom}(V', \mathfrak{g}') \) the cyclic \( \mathfrak{g} \)-modules generated by \( \Theta \) and \( \Theta' \), respectively (see Equation \((146)\)). Then we define the linear map \( \phi_{+1} : R_\Theta[-1] \to R_{\Theta'}[-1] \) to be the unique morphism of cyclic \( \mathfrak{g} \)-modules sending \( \Theta \) to \( \Theta' \) and compatible with \( \phi_0 \). That is to say, since \( \Theta \) (resp. \( \Theta' \)) generates \( R_\Theta \) (resp. \( R_{\Theta'} \)), we have:

\[
\phi_{+1}(\Theta) = \Theta' \quad (191)
\]

where \( \Theta = a_1 \cdots a_n = a_1 \cdot (a_2 \cdot (\ldots (a_n \cdot \Theta) \ldots )) \), for every \( a_1, \ldots, a_n \in \mathfrak{g} \). Moreover, since \( R_{-i} \) (resp. \( R'_{-i} \)) is generated by \( \partial_{-i} \) (resp. \( \partial'_{-i} \)) – see Equation \((113)\) – the fact that \( \phi_{+1}(\Theta) = \Theta' \) implies that the map \( \phi_{+1} \) induces a morphism of cyclic \( \mathfrak{g} \)-modules from \( R_{-i} \) to \( R'_{-i} \) – also denoted \( \phi_{+1} \) for convenience – which is such that:

\[
\phi_{+1}(\partial_{-i}) = \partial'_{-i} \quad (192)
\]
So far, there are two compatibility equations involving \( \phi_{+1} \) and the graded Lie bracket. The first one is of the form:

\[
\left[ \phi_{+1}(\Theta), \phi_{-1}(x) \right] = \phi_0 \left( \left[ \Theta, x \right] \right)
\]  

(193)

for any \( x \in T_{-1} \). It is automatically satisfied because it is a rewriting of Equation \([32]\), under the convention that \( \partial = \left[ \Theta, - \right] \). The second equation is:

\[
\left[ \phi_{+1}(\Theta), \phi_0(a) \right] = \phi_{+1} \left( \left[ \Theta, a \right] \right)
\]  

(194)

for every \( a \in T_0 \). It is also satisfied because it can be equivalently rewritten as

\[-\varphi(a) \cdot \Theta' = \phi_{+1}(-a \cdot \Theta) \]

(195)

which is a particular case of Equation \([191]\). Using successively Equations \([191], [133] \) and \([33] \), together with the fact that \( \varphi \) is a morphism of Lie algebras, one can show that Equations \([193] \) and \([194] \) imply that the following two equations hold for any generator \( \Theta_{a_1, \ldots, a_m} \) of \( T_{+1} = R \Theta [-1] \):

\[
\left[ \phi_{+1}(\Theta_{a_1, \ldots, a_m}), \phi_{-1}(x) \right] = \phi_0 \left( \left[ \Theta_{a_1, \ldots, a_m}, x \right] \right)
\]  

(196)

\[
\left[ \phi_{+1}(\Theta_{a_1, \ldots, a_m}), \phi_0(a) \right] = \phi_{+1} \left( \left[ \Theta_{a_1, \ldots, a_m}, a \right] \right)
\]  

(197)

Hence, the first three maps \( \phi_{+1}, \phi_0 \) and \( \phi_{-1} \) are so far compatible with the graded Lie brackets on \( T \) and \( T' \), respectively. Now, because of Equations \([132] \) and \([192] \), one can check that the identity \( \partial'_0 \circ \phi_{-1} = \phi_0 \circ \partial_0 \) corresponds to Equation \([193] \), and that \( \partial'_+ \circ \phi_0 = \phi_{+1} \circ \partial_+ \) corresponds to Equation \([194] \).

**Definition of the \( \phi_{-i}, i \geq 2 \):** The proof is made by induction. Let \( T_{-2} = S^2(V)/K[2] \) and \( T'_{-2} = S^2(V')/K'[2] \) (meaning shifted by \(-2\)). Since the morphism \( (\varphi, \chi) : (g, V, \Theta) \rightarrow (g', V', \Theta') \) is a compatible morphism, Equation \([35] \) implies that \( \chi \circ \chi \) passes to the quotient, i.e. that it induces a well-defined map \( \phi_{-2} : T_{-2} \rightarrow T'_{-2} \) that renders the following prism commutative:\[17\]

\[
\begin{array}{ccc}
T_{-2} & \overset{\phi_{-2}}{\longrightarrow} & T'_{-2} \\
\downarrow \phi_{-1} \wedge \phi_{-1} & & \downarrow \phi_{-1} \\
\Lambda^2(T_{-1}) & \overset{\partial_{-1}}{\longrightarrow} & \Lambda^2(T'_{-1}) \\
\downarrow m_{-1} & & \downarrow m'_{-1} \\
T_{-1} & \overset{\partial_{-1}}{\longrightarrow} & T'_{-1} \\
\end{array}
\]

where we have written \( \chi \) under its current denomination \( \phi_{-1} : T_{-1} \rightarrow T'_{-1} \). The map \( q_{-2} \) has been defined in subsection \([3.1] \) and the map \( m_{-1} \) has been defined in subsection \([3.2] \).

The commutative diagram proves the following two equations:

\[
\partial'_+ \circ \phi_{-2} = \phi_{-1} \circ \partial_{-1}
\]  

(198)

\[
\phi_{-2} \left( \left[ x, y \right] \right) = \left[ \phi_{-1}(x), \phi_{-1}(y) \right]
\]  

(199)

for every \( x, y \in T_{-1} \). The fact that \( \chi \) satisfies Equation \([33] \) implies that \( \phi_{-2} \) in turn satisfies:

\[
\phi_{-2} \left( \left[ a, x \right] \right) = \left[ \phi_0(a), \phi_{-2}(x) \right]
\]  

(200)

\[17\]This is precisely here that we use the compatibility property of the morphism. See and of the proof for a discussion of what happens when the morphism is not compatible.
for every $a \in \mathfrak{g}$ and $x \in T_{-2}$. The last compatibility equations to check are:

\[
\left[\phi_{+1}(\Theta), \phi_{-2}(x)\right] = \phi_{-1}\left(\left[\Theta, x\right]\right) \quad (201)
\]
\[
\left[\phi_{+1}(\Theta_{a_1 \ldots a_m}), \phi_{-2}(x)\right] = \phi_{-1}\left(\left[\Theta_{a_1 \ldots a_m}, x\right]\right) \quad (202)
\]

for every $x \in T_{-2}$. By Equations (132) and (192), the first equation is an alternative form of Equation (198). Equation (202) can be proven by using Equations (83), (133), (191), (200), as well as Equation (201). For example, for $m = 1$, one has:

\[
\left[\phi_{+1}(\Theta), \phi_{-2}(x)\right] = \left[\Theta'_{\varphi(a)}', \phi_{-2}(x)\right] \quad (203)
\]
\[
= \varphi(a) \cdot \left[\Theta', \phi_{-2}(x)\right] - \left[\Theta', \varphi(a) \cdot \phi_{-2}(x)\right] \quad (204)
\]
\[
= \varphi(a) \cdot \phi_{-1}\left(\left[\Theta, x\right]\right) - \left[\Theta', \phi_{-2}(a \cdot x)\right] \quad (205)
\]
\[
= \phi_{-1}\left(a \cdot \left[\Theta, x\right]\right) - \phi_{-1}\left(\left[\Theta, a \cdot x\right]\right) \quad (206)
\]
\[
= \phi_{-1}\left(\left[\Theta_{a}, x\right]\right) \quad (207)
\]

Let us now turn to the inductive step. Assume that the linear map $\phi_{(-i)} : T_{(-i)} \rightarrow T'_{(-i)}$ is constructed up to order $i \geq 2$, and that is satisfies:

\[
\partial' \circ \phi_{(-i)}(x) = \phi_{(-i)} \circ \partial(x) \quad (208)
\]
\[
\phi_{(-i)}\left(\left[ x, y \right]\right) = \left[ \phi_{(-i)}(x), \phi_{(-i)}(y) \right] \quad (209)
\]

for every $x, y \in T_{\leq 0}$ such that $-i \leq |x| + |y| \leq 0$. Moreover, notice that for $a \in T_0 = \mathfrak{g}$, and $u \in T_{\leq 0}$, Equation (209) implies the following result:

\[
\phi_{(-i)}(a \cdot u) = \phi_{0}(a) \cdot \phi_{(-i)}(u) \quad (210)
\]

This equation extends to every element $y \in \Lambda(T_{\leq-1})$. The only remaining non-trivial equations are:

\[
\left[\phi_{+1}(\Theta), \phi_{-i}(x)\right] = \phi_{-i+1}\left(\left[\Theta, x\right]\right) \quad (211)
\]
\[
\left[\phi_{+1}(\Theta_{a_1 \ldots a_m}), \phi_{-i}(x)\right] = \phi_{-i+1}\left(\left[\Theta_{a_1 \ldots a_m}, x\right]\right) \quad (212)
\]

for every $x \in T_{-i}$. By Equations (132) and (192), Equation (211) is automatically satisfied because it is equivalent to Equation (208), while Equation (212) is a consequence of the former as well as (210), as can be shown by using the same kind of arguments as in Equations (203)–(207).

Let us construct $\phi_{-i-1} : T_{-i-1} \rightarrow T'_{-i-1}$ such that Equations (208) and (209) are valid one step higher, i.e. such that $i$ is replaced by $i + 1$. Then, as explained above, Equations (211) and (212) would be obtained as a mere consequence of the former. Since, by Equation (78), the graded Lie bracket $[\ldots]$ (resp. $[\ldots]'$) coincides with $q$ (resp. $q'$), Equation (209) can be rewritten as:

\[
\phi_{(-i)} \circ q = q' \circ \phi_{(-i)} \quad (213)
\]

and straightforwardly extended to $\Lambda^3(T_{\leq-1})|_{-i-1}$ (in particular, see Equation (85)), making the following diagram commutative:

\[
\begin{array}{c}
\Lambda^3(T_{\leq-1})|_{-i-1} \xrightarrow{\phi_{(-i)}} \Lambda^3(T'_{\leq-1})|_{-i-1} \\
\downarrow q \quad \downarrow q' \\
\Lambda^2(T_{\leq-1})|_{-i-1} \xrightarrow{\phi_{(-i)}} \Lambda^2(T'_{\leq-1})|_{-i-1}
\end{array}
\]
where it is understood that $\phi_{(-i)}$ acts as an algebra morphism. Commutativity of the above diagram, together with the fact that $\text{Im} (q|_{\Lambda^2(T_{\leq -1})}) = \text{Ker} (q|_{\Lambda^2(T_{\leq -1})})$ (and a similar equality for $q'$), implies in turn that:

$$\phi_{(-i)}(\text{Ker} (q|_{\Lambda^2(T_{\leq -1})}|_{-i-1})) \subset \text{Ker} (q'|_{\Lambda^2(T'_{\leq -1})}|_{-i-1})$$

Since $\Lambda^2(T_{\leq -1})|_{-i-1} / \text{Ker} (q|_{\Lambda^2(T_{\leq -1})}|_{-i-1}) \simeq T_{-i-1}$ (and a similar identity for $T'_{-i-1}$), this implies that $\phi_{(-i)}$ passes to the quotient, and induces a linear map $\phi_{-i-1} : T_{-i-1} \to T'_{-i-1}$, which makes the following diagram commutative:

$$\Lambda^2(T_{\leq -1})|_{-i-1} \xrightarrow{\phi_{(-i)}} \Lambda^2(T'_{\leq -1})|_{-i-1} \\
q \downarrow {\phi_{-i-1}} \quad q' \downarrow \\
T_{-i-1} \xrightarrow{\phi_{-i-1}} T'_{-i-1}$$

Replacing back $q$ by $\llbracket \ldots \rrbracket$, commutativity of the diagram is equivalent to the following equation:

$$\phi_{-i-1}(\llbracket x, y \rrbracket) = \llbracket \phi_{(-i)}(x), \phi_{(-i)}(y) \rrbracket$$  \hspace{1cm} (214)

for every $x, y \in T_{\leq -1}$ such that $|x| + |y| = -i - 1$.

The similar condition involving an element of $g$ and an element of $T_{-i-1}$ comes from the fact that $q : \Lambda^2(T_{\leq -1}) \to T_{\leq -2}$ (resp. $q'$) is a surjective $g$-equivariant (resp. $g'$-equivariant) map. More precisely, let $a \in g$ and $x \in T_{-i-1}$, then there exists $u \in \Lambda^2(T_{\leq -1})|_{-i-1}$ such that $x = q(u)$. But then, one obtains:

$$\phi_{-i-1}(\llbracket a, x \rrbracket) = \phi_{-i-1}(a \cdot q(u))$$

$$= \phi_{-i-1}(q(a \cdot u)) = q'(\phi_{(-i)}(a \cdot u))$$

$$= q'(\phi_0(a) \cdot \phi_{(-i)}(u)) = \phi_0(a) \cdot q'(\phi_{(-i)}(u))$$

$$= \phi_0(a) \cdot \phi_{-i-1}(q(u)) = \llbracket \phi_0(a), \phi_{-i-1}(x) \rrbracket$$

where we used successively Equation (126), the $g$-equivariance of $q$, Equation (214), Equation (210), $g'$-equivariance of $q'$, Equation (214) again, and finally Equation (126). The proof does not depend on the choice of preimage $u$ we make because $\text{Ker} (q|_{\Lambda^2(T_{\leq -1})}|_{-i-1})$ is stable under the action of $g$. Thus, we have extended Equation (209) one level higher:

$$\phi_{(-i)}(\llbracket x, y \rrbracket) = \llbracket \phi_{(-i-1)}(x), \phi_{(-i-1)}(y) \rrbracket$$  \hspace{1cm} (219)

for every $x, y \in T_{\leq 0}$ such that $-i - 1 \leq |x| + |y| \leq 0$.

There are two more compatibility equations between $\phi_{-i-1}$ and the graded Lie bracket to check, that is:

$$\phi_{-i}(\llbracket \Theta, x \rrbracket) = \llbracket \phi_{+1}(\Theta), \phi_{-i-1}(x) \rrbracket$$  \hspace{1cm} (220)

$$\phi_{-i}(\llbracket \Theta_{a_1\ldots a_m}, x \rrbracket) = \llbracket \phi_{+1}(\Theta_{a_1\ldots a_m}), \phi_{-i-1}(x) \rrbracket$$  \hspace{1cm} (221)

for every $x \in T_{-i-1}$ and every $a_1, \ldots, a_m \in g$. Equation (220) can be rewritten in the form of Equation (208), at level $-i - 1$:

$$\phi_{-i}(\partial(x)) = \partial' (\phi_{-i-1}(x))$$  \hspace{1cm} (222)
This is indeed satisfied, by choosing a pre-image \( u \in \Lambda^2(T_{\leq -1})_{-i-1} \) of \( x \), one has the following result:

\[
\partial' \left( \phi_{-i-1}(x) \right) = \partial' \left( \phi_{-i-1}(q(u)) \right) = \partial' \left( q' \left( \phi_{-i}(u) \right) \right) = q' \left( \partial' \left( \phi_{-i}(u) \right) \right) + m' \left( \phi_{-i}(u) \right) \\
= q' \left( \phi_{-i}(\partial(u)) \right) + \phi_{-i}(m(u)) = \phi_{-i} \left( q(\partial(u)) + m(u) \right) = \phi_{-i} \left( \partial(x) \right)
\]  

(223)

where we have successively used Equations (214), (81) (or (82)) (208), (209), and Equation (81) (or (82)) again. Notice that we passed from \( m' \left( \phi_{-i}(u) \right) \) to \( \phi_{-i}(m(u)) \) by using Equation (209) and the definition of \( m \) (resp. \( m' \)). Proving Equation (221) then relies on using Equations (219) (on \( T_0 \cap T_{-1} \)) and (220), as well as the same kind of arguments as in Equations (203)-(207). This concludes the proof of the inductive step.

The induction provides us with a linear application \( \phi : T \to T' \) satisfying the following conditions:

\[
\partial' \circ \phi(a) = \phi \circ \partial(a) \\
\phi \left( [a, b] \right) = [\phi(a), \phi(b)]
\]  

(225)

(226)

Moreover, this differential graded Lie algebra morphism between \( T \) and \( T' \) is such that \( \phi|_{T_{-1}} = \chi, \phi|_{T_0} = \varphi, \) and \( \phi_{+1}(\Theta) = \Theta' \). The morphism \( \phi \) is the image through the function \( G : \text{Lie-Leib} \to \text{DGLie}_{\leq 1} \) of the compatible morphism of Lie-Leibniz triples \((\varphi, \chi)\):

\[
G(\varphi, \chi) = \phi
\]

Then, when restricted to the wide subcategory \( \text{compLie-Leib} \), the injective-on-objects function \( G \) is a functor. Moreover, since by construction \( G(\varphi, \chi) = G(\varphi', \chi') \) if only if \( \varphi = \varphi' \) and \( \chi = \chi' \), \( G \) is faithful.

Finally, let us explain why \( \text{compLie-Leib} \) is the biggest wide subcategory of \( \text{Lie-Leib} \) (with respect to inclusion) such that \( G \) is a (faithful) functor. The idea is that in the general case where the morphism of Lie-Leibniz triples \((\varphi, \chi)\) is not compatible in the sense of Definition 8, an obstruction arises. Indeed, let \((\varphi, \chi)\) be any morphism of Lie-Leibniz triples between \((g, V, \Theta)\) and \((g', V', \Theta')\). Then, as seen in the first step of the induction, the couple \((\varphi, \chi)\) would induce a morphism of differential graded Lie algebras between the associated tensor hierarchies \( T \) and \( T' \) if and only if there exists a map \( \tau : S^2(V)/K \to S^2(V')/K' \) making the following diagram commutative:

\[
\begin{array}{ccc}
S^2(V) & \xrightarrow{\chi \odot \chi} & S^2(V') \\
\downarrow p & & \downarrow p' \\
S^2(V)/K & \xrightarrow{\tau} & S^2(V')/K'
\end{array}
\]

However, the existence of such a map \( \tau \) is conditioned to the fact that \( \chi \odot \chi(K) \subset K' \), namely that \((\varphi, \chi)\) is a compatible morphism. \( \square \)

We conclude this section and the paper on the following observation: for every Lie-Leibniz triple \((g, V, \Theta)\) for which:

\[
K = \text{Ker}\{\ldots\}
\]  

(227)

the map \( \partial_{-1} : T_{-2} \to T_{-1} \) is injective because \( T_{-2} = S^2(V)/\text{Ker}\{\ldots\}[2] \) is isomorphic to the ideal of squares \( \mathcal{I} \), the image of \( \partial_{-1} \). Moreover, one can then show that Condition (227) implies that \( \partial_{-2} = 0 \), as would be expected from \( T_{\leq -1} \) being a chain complex. The question arises: what about the higher homology spaces? We have some reason to think that the following conjecture holds:
Conjecture 1. For a Lie-Leibniz triple \((\mathfrak{g}, V, \Theta)\) satisfying condition (227) (so in particular for semi-strict Lie-Leibniz triples), the associated tensor hierarchy induces a resolution of the ideal of squares \(I\):

\[
\ldots \xrightarrow{\partial_{-4}} T_{-4} \xrightarrow{\partial_{-3}} T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} I[1] \longrightarrow 0
\]

This would be plausible since in this case, the role of \(\mathfrak{g}\) in the construction of the tensor hierarchy is not as salient as compared to the general case (e.g. in defining \(T_{-2}\)).

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