Lattice W-algebras and logarithmic CFTs

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Abstract
This paper is part of an effort to gain further understanding of 2D logarithmic conformal field theories (LCFTs) by exploring their lattice regularizations. While all work so far has dealt with the Virasoro algebra (or the product $\text{Vir} \otimes \text{Vir}$), the best known (although maybe not the most relevant physically) LCFTs in the continuum are characterized by a W-algebra symmetry, whose presence is powerful, but whose role as a ‘symmetry’ remains mysterious. We explore here the origin of this symmetry in the underlying lattice models. We consider $\mathbb{U}_q \mathfrak{su}(2)$ XXZ spin chains for $q$ a root of unity, and argue that the centralizer of the ‘small’ quantum group $\mathbb{U}_q \mathfrak{su}(2)$ goes over the W-algebra in the continuum limit. We justify this identification by representation theoretic arguments, and give, in particular, lattice versions of the W-algebra generators. In the case $q = i$, which corresponds to symplectic fermions at central charge $c = -2$, we provide a full analysis of the scaling limit of the lattice Virasoro and W generators, and show in details how the corresponding continuum Virasoro and W-algebras are obtained. Striking similarities between the lattice W algebra and the Onsager algebra are observed in this case.

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1. Introduction
It is not surprising that some crucial algebraic objects should play a role both in lattice models and in conformal field theories. For instance, the Yang–Baxter equation expresses factorizability of lattice interactions and determines integrable Boltzmann weights; it also expresses
factorizability of braiding in the CFT, and is related with monodromy properties of conformal blocks.

Work of the last many years however has unraveled deeper and less expected connections. Among these was the surprising observation that order parameters (local height probabilities) are given by CFT branching functions [1], or that the representation theory of lattice algebras such as the Temperley–Lieb (TL) algebra is equivalent, in a certain categorical sense (see [2–6] for more details), to the representation theory of the Virasoro algebra (see also [7, 8]). This has proven particularly useful in understanding logarithmic conformal field theories (LCFTs), a topic which has gained a lot of attention recently.

Indeed, classifying and solving LCFTs from first principles seems very difficult. This is for two main reasons. On the one hand, the loss of unitarity allows a proliferation of new universality classes. On the other hand, many of the tools that were available in the unitary case—such as the correspondence between bulk and boundary theories, or the full factorization of the operator product expansions—are not relevant any longer, or need deep modifications. It seems therefore most reasonable therefore to try and gain additional understanding of the problem by resorting to other approaches than pure bootstrap, conformal field theory ones. The consideration of lattice regularizations is most useful in that respect, and allows one, in particular, to rely on many advanced results of algebra. The representation theory of nonsemi-simple cellular algebras for instance has allowed in this way great progress in the understanding of the types of indecomposable modules or fusion rules appearing in many classes of LCFTs. Further general discussion of this ‘lattice approach’ can be found in [9].

It is crucial to try to understand the observations in [3, 7] more thoroughly, building on the intuitive notion that the algebra of local Hamiltonians (e.g., nearest coupling for spin chains) should in some sense go over to the algebra of local Hamiltonians (the stress energy tensor) in the continuum limit, and for boundary theories, where only one chiral algebra is expected. However, showing precisely how, say, the Virasoro algebra emerges from the TL algebra in the continuum limit in general remains a very hard exercise. This is fully understood so far in the Ising model case only, where the presence of an underlying Majorana fermion makes things quite easy [11].

It is also worth mentioning that among LCFTs, the rational class with ‘large’ chiral algebras like the triplet W-algebras [12, 13] has a finite number of primary fields (or isomorphism classes of simple modules) and is particularly accessible for rigorous algebraic analysis [14, 15], as well as for an explicit treatment of their Verlinde algebra aspects [16–20] and hopefully mapping class group properties [21]. The simplest such LCFT occurs in the so called symplectic fermion theory at $c = -2$ [12, 22], where the chiral algebra is generated by a triplet of fermion bilinears of conformal weight $h = 3$:

\[
W^+ = \partial \eta^+ \eta^+,
\]
\[
W^0 = \frac{1}{2} (\partial \eta^+ \eta^- + \partial \eta^- \eta^+),
\]
\[
W^- = \partial \eta^- \eta^-.
\]
satisfying the nonlinear relations

\[ W^\alpha(z) W^\beta(w) = g^{\alpha\beta} \left( \frac{1}{(z-w)^3} - \frac{3}{2} \frac{T(w)}{(z-w)^2} + \frac{3}{2} \frac{\partial T(w)}{(z-w)} + \frac{3}{8} \frac{\partial^2 T(w)}{(z-w)^3} \right) 
\]

\[ + \frac{3}{8} \frac{\partial T(w)}{(z-w)^3} + \frac{1}{4} \frac{\partial^2 T(w)}{z-w} + \frac{1}{4} \frac{\partial^2 T(w)}{(z-w)^3} + \frac{1}{2} \frac{(TW)(w)}{z-w} + \frac{1}{25} \frac{(T^2)(w)}{z-w} + \frac{12}{25} \frac{(TW)(w)}{z-w} \]  

(1)

with \( g^{+-} = g^{-+} = 4, g^{00} = 2, f_0^{+-} = f_0^{-+} = 2, f_+^{0+} = f_+^{0-} = f_-^{0+} = f_-^{0-} = 1 \) and other \( g^{\alpha\beta} \) and \( f_{\alpha\beta} \) equal to 0.

Most studied generalizations concern the so-called \((1, p)\) models (symplectic fermions being the \((1, 2)\) model) where the chiral algebra is the triplet W-algebra \( W(p) \) defined first in [23] as an extension of the Virasoro algebra by a triplet of primary fields of conformal dimension \( 2p - 1 \). Identifying lattice models which have such W-algebra \( W(p) \) symmetries remains, in our opinion and except for \( p = 2 \), a challenge, even though some formal progress has been made in this direction in the last few years [24].

The study of the W-algebras is notoriously difficult, not only because of the high nonlinearity of the OPEs, but also because they do lack physical understanding. Motivated by the crucial role played by these algebras in the rational class of LCFTs, we are presenting here a fresh look at this question by identifying the lattice equivalents of the triplets in the XX spin chain. To go further and identify lattice algebras that would converge to the chiral triplet W-algebras, we use the so-called Lusztig duality [27, 28] between these chiral algebras and the small quantum group on a finite spin-chain, which is equivalent to a \( g^f(111) \) spin chain built on alternating representations. Using the earlier proposal in [4, 11], a lattice version of the Virasoro algebra is obtained by considering TL generators—recall that the TL algebra is the centralizer of the full quantum group acting on the spin chain. In section 2, we propose similarly a lattice version of the W-algebra obtained as the centralizer of the small quantum group \( \mathcal{U}_q\mathfrak{sl}(2) \) acting on the spin chain. Fourier modes (particular linear combinations of the functions of conformal fields) \[ \mathcal{U}_q\mathfrak{sl}(2) \] modules were proven first in [28] for the \((1, 2)\) models and in [29] for all \((1, p)\) cases. The functor realizing such an equivalence could be explicitly constructed using a bimodule over the mutual centralizers, the triplet W-algebra and the small quantum group.

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lattice W-algebra generators) which constitute ‘approximations’ of the Virasoro or chiral W-algebra modes are then defined, and their commutators are studied in section 2 as well. Of course, for a finite system, these approximations do not form a closed algebra. Calculating further commutators leads to infinitely many such ‘approximations’, and a complicated algebraic structure, which is studied in detail. The scaling limit is then studied in section 3 where we are able, after the introduction of appropriate normal ordering, to obtain exactly, from the lattice and in the limit of large chains, the defining relations of the Virasoro and W algebras for symplectic fermions. In section 4, we generalize our analysis to other values of q. We define the lattice W algebra for the XXZ spin chain, and study its representation theory in details. While refraining from a full analysis of the continuum limit, we show that our results are fully consistent with what is expected in the \((p - 1, p)\) and \((1, p)\) models of LCFTs. Section 5 contains some conclusions and pointers for future work. Five appendices contain supplementary materials like long definitions, proofs and bulky expressions.

2. The triplet W-algebra from XX spin-chain

In this section, we give an explicit definition of the lattice analogue of Kausch’s triplet W-algebra in the case of XX spin-chains. The use of the well-known fermionic formulation of the model allows to introduce all modes of ‘W-currents’ on the lattice. Their scaling limit and commutation relations are then studied in the next section.

2.1. The XX model

We consider the XX model Hamiltonian of \(N\) one-half spins with an open boundary condition described by the ‘quantum-group symmetric’ boundary term [33],

\[
H = \frac{1}{2} \sum_{j=1}^{N-1} \left( \sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} \right) - \frac{i}{2} \left( \sigma^z_j - \sigma^z_{j+1} \right).
\]  

(2)

This is an operator acting on \(H_N = C^{2 \otimes N}\) and \(\sigma^x, \sigma^y\) and \(\sigma^z\) are usual Pauli matrices,

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(3)

2.2. Lattice fermions

It is useful in what follows to reformulate everything in terms of ordinary lattice fermions \(c_j\) and \(c_j^\dagger\). We use the Jordan–Wigner transformation to get the lattice fermions

\[
c_j = i^{-j+1} \prod_{i=j}^{N-1} \sigma^x_i \otimes \sigma^x_j, \quad c_j^\dagger = i^{j-1} \prod_{i=j}^{N-1} \sigma^x_i \otimes \sigma^x_j
\]  

(4)

with the anticommutation relations

\[
\{ c_j^\dagger, c_j \} = \delta_{j,j'}.
\]  

(5)

This transformation gives the free fermion expression for the XX Hamiltonian from (2):

\[
H = - \sum_{j=1}^{N-1} e_j = \sum_{j=1}^{N-1} \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) - i c_1^\dagger c_1 + i c_N^\dagger c_N.
\]  

(6)
where we introduce the Hamiltonian densities
\[ e_j = c_j c_{j+1}^+ + c_{j+1} c_j^+ + i \left( c_j^+ c_j - c_{j+1}^+ c_{j+1} \right), \quad 1 \leq j \leq N - 1, \] (7)
which satisfy defining relations for the TL algebra with zero fugacity parameter
\[ e_i^2 = 0, \]
\[ e_i e_j = e_j e_i, \quad |i - j| > 1. \]
We denote the algebra generated by these \( e_j \), with \( 1 \leq j \leq N - 1 \), and the identity as \( TL_{i,N} \).

2.3. Quantum group results for \( q = i \) case

For applications to open XX spin-chains, we set in what follows \( q = i \). As a module over the quantum group \( U_q sl(2) \), see general definitions in appendix A, the spin chain \( LL_{i,N} \) is a tensor product of two-dimensional irreducible representations such that \( E \rightarrow \sigma^+, \; F \rightarrow \sigma^-, \; K \rightarrow q \sigma^z \), and \( \sigma = f = 0 \), see precise definitions of quantum groups at roots of unity in appendix A. Using the \((N - 1)\)-folded comultiplications (A.12) together with the Jordan–Wigner transformation (4), we obtain the representation \( \rho_{gf}: U_q sl(2) \rightarrow \text{End}_c(H_N) \) (usual fermionic expressions)
\[ \rho_{gf}(E) \equiv \Delta^{N-1}(E) = \sum_{j=1}^{N} q^{j} c_j \rho_{gf}(K), \quad \rho_{gf}(F) \equiv \Delta^{N-1}(F) = \sum_{j=1}^{N} q^{-j} c_j, \] (8)
and
\[ \rho_{gf}(e) \equiv \Delta^{N-1}(e) = \sum_{1 \leq j < j_2 \leq N} (-1)^{j+j_2} q^{j-j_2} c_{j_2}^+ c_j^+, \]
\[ \rho_{gf}(f) \equiv \Delta^{N-1}(f) = \sum_{1 \leq j < j_2 \leq N} q^{j-j_2} c_j c_{j_2}. \] (9)
We can then easily check that all the generators \( e_j \) of \( TL_{i,N} \) commute with the quantum-group action \([33]\). Moreover, the quantum-group generators give the full symmetry of the XX model, or in more technical terms the full quantum group \( U_q sl(2) \) at \( q = i \) gives the centralizer of \( TL_{i,N} \). This is a particular case of a more general (and, of course, well-known) statement discussed in section 4 in the context of XXZ representations.

2.4. Lattice W-algebra: generators and their relations

As was mentioned in the introduction, our paradigm in definition of lattice algebras (as well as chiral algebras in the scaling limit) is based on their centralizers, which are various quantum groups. We will see in section 3 that a properly defined scaling limit of \( TL_{i,N} \) gives all the Virasoro modes \( L_n \) in the symplectic fermions representation, a logarithmic CFT with the central charge \( c = -2 \). It was discussed in \([3, 25]\) that the generators of the full quantum group \( U_q sl(2) \) give also the centralizer of the Virasoro representation, or more formally, an equivalent ‘staircase’ bimodule structure for the commuting actions of the Virasoro at \( c = -2 \) and \( U_q sl(2) \) is obtained as a semi-infinite version of the finite ones for \( TL_{i,N} \) and \( U_q sl(2) \) extracted from the open XX spin-chains.

In the case of the chiral triplet W-algebra, it was proven in \([28]\) that the centralizer of the W-algebra in the symplectic fermions theory is given by the small (or restricted) quantum group \( U_q sl(2) \) generated by the capital generators \( E \) and \( F \), with the Cartan element \( K \), only.
By an analogy with the TL case, we then define the lattice analog of Kausch’s triplet W-algebra as an extension of the TL algebra for \( q = i \) by all operators commuting with the action of the restricted quantum group. The new generators will break the \( \mathfrak{sl}(2) \) symmetry generated by the divided powers \( e \) and \( f \). Put more formally, we define the lattice W-algebra \( \mathcal{W}_{iN} \) as the centralizer (algebra of all operators intertwining the action) of the restricted quantum group \( \mathcal{U}_q \mathfrak{sl}(2) \) on the tensor-product representation \( \mathcal{H}_N \).

In order to describe the additional generators extending \( \mathcal{W}_{iN} \) to \( \mathcal{W}_{iN} \), we first recall the definition of the walled Brauer algebra \([30]\) in the context of alternating \( g\ell(1|1) \) spin-chains introduced in \([10]\).

### 2.4.1. A relation with \( g\ell(1|1) \) spin-chains and the walled Brauer algebra

Up to some simple transformations, the XX spin chain can be reformulated as a \( g\ell(1|1) \) spin chain with alternating fundamental representation and its dual. This is discussed in details for instance in the introductory part of \([4]\) where it is shown, among other things, that the \( g\ell(1|1) \) symmetry is generated by the quantum-group generators \( E, F \), and the \( \mathfrak{sl}(2) \) Cartan element \( h \) via

\[
\rho_{g\ell}(E) \equiv \Delta^{N-1}(E) = F_{(1)}^+ \rho_{g\ell}(K),
\]

\[
\rho_{g\ell}(F) \equiv \Delta^{N-1}(F) = q^{-1} F_{(1)},
\]

\[
\rho_{g\ell}(K) \equiv \Delta^{N-1}(K) = (-1)^{\rho_\ell(2h)},
\]

where \( F_{(1)}^+, F_{(1)} \) are the fermionic generators and \( 2h \) the fermion number (the last generator of \( g\ell(1|1) \) is trivially zero in the alternating product of fundamental and its dual).

We note now that the \( g\ell(1|1) \) action commutes with more than the nearest-neighbor coupling—TL generators. It commutes in particular with permutations \( WB_j \equiv P_j P_{j+1} P_j \) of next-nearest neighbor \( C^{111} \) tensors and \((P_j \text{ is the usual flip } a \otimes b \mapsto b \otimes a \text{ of states on } j\text{th and } (j+1)\text{th sites})\) which in mathematical terms corresponds to an action of the walled Brauer algebra \([30]\) (see also section 5 of \([31]\)) generated by \( e_j \) and \( WB_j \). The walled Brauer generators are represented in our spin-chain by

\[
WB_j = \left( c_j + c_{j+2} \right) \left( c_j^+ + c_{j+2}^+ \right) - 1, \quad j = 1, 2, ..., N-2. \tag{11}
\]

It was proven in \([35]\) that this representation of the walled Brauer algebra gives the centralizer of the \( g\ell(1|1) \) action in the alternating tensor-space representation. We thus see that \( WB_j \in \mathcal{W}_{iN} \) because of the relations \((10)\) which show the the \( g\ell(1|1) \) representation contains the image \( \rho_{g\ell}(\mathcal{U}_q \mathfrak{sl}(2)) \) of the restricted quantum group \( \mathcal{U}_q \mathfrak{sl}(2) \). Obviously, \( WB_j \)’s do not commute with the \( \mathfrak{sl}(2) \) generators \( e \) and \( f \) but still commute with the commutator \([e, f] = 2h = S^z \). In order to find the full centralizer of the quantum group \( \mathcal{U}_q \mathfrak{sl}(2) \) with the Cartan element \( K \) represented by the operator \((-1)^{2h} \), we then introduce two more generators by

\[
WB_j^+ = \left[ e, WB_j \right], \quad WB_j^- = \left[ f, WB_j \right]. \tag{12}
\]

These operators now break the \( S^z \) symmetry (the \( g\ell(1|1) \) generator \( h \) which is not in \( \mathcal{U}_q \mathfrak{sl}(2) \)) but respect the action of the Cartan generator \( K \) of \( \mathcal{U}_q \mathfrak{sl}(2) \) because \( e \) and \( f \) commute with \( K \), see appendix \( A \).

### 2.4.2. Definition of the lattice W-algebra

We conclude that the lattice W-algebra \( \mathcal{W}_{iN} \) can be defined as an extension of the \( \mathcal{T}\mathcal{L}_{iN} \) algebra by the generators.
\[ WB^+_j = (-1)^j \left( c^+_j c^+_{j+1} + i c^+_j c^+_{j+2} - c^+_j c^+_{j+3} \right), \]
\[ WB^0_j = -\frac{1}{2} \left( 1 + ic^+_j c_{j+1} - c^+_j c_{j+2} + ic^+_j c_{j+3} \right) - 2c^+_{j+1}c_{j+2} - ic^+_{j+1}c_{j+2} - c^+_{j+2}c_{j+3} - ic^+_{j+2}c_{j+3}, \]
\[ WB^-_j = (-1)^{j+1} \left( c_j c_{j+1} + ic_j c_{j+2} - c_j c_{j+3} \right), \]

where \( j = 1, 2, \ldots, N - 2 \). Of course, this statement should be proven. We formulate our result in theorem B.1 and give a computational proof in appendix B where we use more convenient fermions (the Fourier transforms of \( c_j \)'s and \( c_j^\dagger \)'s) introduced below.

We note that the relation of \( WB^0_j \) generators with the walled Brauer generators is

\[ WB^0_j = -\frac{1}{2} WB_j + \frac{i}{2} c_j = -\frac{i}{2} c_{j+1}. \]

2.4.3. \( sl(2) \) structure. The introduction of the \( WB^0_j \) generators is convenient for having a triplet of generators with respect to the action of the \( sl(2) \) part of \( U_q sl(2) \) given in (9). Indeed, we note that the generators \( WB^+_j \), \( WB^0_j \) and \( WB^-_j \) for each site \( j \) form an \( sl(2) \) triplet

\[ \left[ e, WB^+_j \right] = 0, \left[ e, WB^0_j \right] = -WB^+_j, \left[ e, WB^-_j \right] = 2WB^0_j, \]
\[ \left[ f, WB^+_j \right] = -2WB^0_j, \left[ f, WB^0_j \right] = WB^-_j, \left[ f, WB^-_j \right] = 0, \]

and themselves satisfy \( sl(2) \) relations on each site

\[ \left[ WB^+_j, WB^-_j \right] = 2WB^0_j, \left[ WB^0_j, WB^\pm_j \right] = \pm WB^\mp_j. \]

We note that these statements agree with a more general statement given below in 4.6.1 for all roots of unity \( q = e^{i\pi \rho/\nu}. \)

The formulas (17) and (18) mean that we have an action of \( sl(2) \) on the algebra \( \mathcal{W}_N \) by derivatives of the multiplication. Because the generators belong to spin-0 and spin-1 representations of the \( sl(2) \), the algebra \( \mathcal{W}_N \) is therefore decomposed onto \( sl(2) \)-modules of integer spins only. The multiplicity \( m_k \) of a \( k \)-dimensional irreducible \( sl(2) \) submodule in \( \mathcal{W}_N \) is

\[ m_k = \binom{2N-2}{N-k} - \binom{2N-2}{N-k-2}, \]

where \( k \) is odd. We note that \( m_1 \) is the Catalan number which equals the dimension of the TL subalgebra. This agrees with the fact that the span of \( U_q sl(2) \)-invariants is the subalgebra \( T_{\ell} L_{\infty} \). We finally obtain the dimension of the \( \mathcal{W}_N \) algebra

\[ \text{dim}(\mathcal{W}_N) = \sum_{k \text{ odd}} km_k = 2^{2N-3}. \]

The numbers \( m_k \) and the dimension of the algebra can be computed explicitly using the representation theory of \( \mathcal{W}_N \) which will be discussed briefly below. We just note now that the number \( 2^{2N-3} \) obtained as the dimension of the algebra follows from the fact that the dimension of its centralizer—\( \rho_{sl(2)}(U_q sl(2)) \)—is \( 8 = 2^3 \) while the full matrix algebra (or the Clifford algebra) has dimension \( 2^{2N} \).
2.4.4. Relations in the algebra. It would be interesting of course to express our algebra $\mathcal{W}_{iN}$ directly in terms of generators and defining relations, much as can be done for the TL algebra, but we have not managed to do so. Investigation of what the relations involving $WB_{j}^{\pm,0}$ generators might be is quite intricate, and can be organized in terms of the number of sites involved. We give examples of the relations in appendix C. We hope that there exist a bigger algebra with simpler defining relations (like for TL or Hecke algebras) such that its spin-chain representation is not faithful and this algebra factors through $\mathcal{W}_{iN}$.

2.5. Fermionic modes

2.5.1. Hamiltonian spectrum and Jordan blocks. In order to study the behavior of the lattice $W$-algebra in the continuum limit, we have to first discuss technical preliminaries. Fourier transform will play an essential role in what follows, and the corresponding definitions are better understood by first studying here the spectrum, eigenvectors and Jordan blocks structure of the Hamiltonian in (6). We first consider the case of even $N$ and compute eigenvectors of the adjoint action of $H$ in the one-particle sector with the result

$$[H, \theta^+_{k}] = 2 \cos \frac{k}{N} \theta^+_{k}, \quad [H, \theta^-_{k}] = -2 \cos \frac{k}{N} \theta^-_{k},$$

(20)

where we introduce fermions in the momentum space

$$\theta^+_{k} = \sum_{j=1}^{N} A_{k}(j)c^+_j, \quad \theta^-_{k} = -i \sum_{j=1}^{N} A_{k}(j)c_j, \quad k = 1, ..., N - 1$$

(21)

with the amplitudes

$$A_{k}(j) = \frac{1}{2} \left( 1 + ie^{-ik} \right) e^{-ik} - \frac{1}{2} \left( 1 + ie^{ik} \right) e^{ik}, \quad k = 1, ..., N - 1.$$  

(22)

These operators satisfy the anti-commutation relations

$$\{ \theta^+_{k}, \theta^-_{k'} \} = N \cos \frac{k}{N} \delta_{k,k'}.$$  

(23)

There are also two root vectors

$$\gamma^+ = i \frac{\sqrt{\pi}}{N} \sum_{j=1}^{N} \left( \frac{N}{2} - j + \frac{1}{2} \left( 1 - (-1)^j \right) \right) i c^-_j,$$

(24)

$$\gamma^- = -\frac{\sqrt{\pi}}{N} \sum_{j=1}^{N} \left( \frac{N}{2} - j + \frac{1}{2} \left( 1 - (-1)^j \right) \right) i c^-_j,$$

(25)

which transform under the action of $H$ as

$$[H, \gamma^+] = -\frac{2\sqrt{\pi}}{N} \theta^+_{N/2}, \quad [H, \gamma^-] = \frac{2\sqrt{\pi}}{N} \theta^-_{N/2}$$

and satisfy the relations

$$\{ \theta^+_{N/2}, \gamma^- \} = \sqrt{\pi}, \quad \{ \theta^-_{N/2}, \gamma^+ \} = -\sqrt{\pi}.$$  

(26)

Using the spectrum (20) of $H$, we conclude that the vacuum or ground state is

$$|0\rangle = \theta^+_{N/2} \theta^+_{N/2 + 1}... \theta^+_{N - 1} | \downarrow \ldots \downarrow \rangle.$$  

(27)
where $| \downarrow \ldots \downarrow \rangle$ is a reference state with all spins down, and the ‘log-partner’ of the vacuum $| \overline{0} \rangle = \gamma^+ \theta^+_{N/2-1} \ldots \theta^+_{1} | \downarrow \ldots \downarrow \rangle$, (28)
i.e., we have $H | 0 \rangle = E_0 | 0 \rangle$ and $H | \overline{0} \rangle = E_0 | \overline{0} \rangle + \frac{2 \pi}{N} | 0 \rangle$, where the ground-state energy $E_0$ is computed below. We thus have a Jordan cell of rank 2. It is easy to see using the fermions that all excited states involved in the Hamiltonian’s Jordan cells are of maximum rank 2.

### 2.5.2. The spectrum generating algebra.

For later convenience in the scaling limit section, we introduce new fermions

$$
n^+_n = - \frac{\sqrt{n}}{N \sin \frac{\pi n}{N}} \theta^+_{N/2+n}, \quad n = -\frac{N}{2} + 1, \ldots, \frac{N}{2} - 1, \quad n \neq 0, \quad (29)$$

together with the fermionic zero modes

$$
n^+_0 = \frac{1}{\sqrt{\rho}} \theta^+_{N/2}, \quad n^-_0 = \frac{1}{\sqrt{\rho}} \theta^-_{N/2}. \quad (30)$$

We have for these generators the anti-commutation relations

$$\{ n^+_n, n^-_{n'} \} = n \delta_{n+n',0}, \quad \{ n^+_0, \gamma^+ \} = \pm 1. \quad (31)$$

We note that the operators $n^+_n$ become in the scaling limit studied in section 3 the modes of a pair of symplectic fermions fields.

The Hamiltonian (6) takes then the normal ordered form

$$
H = \sum_{n=1}^{N/2-1} \frac{2}{n} \sin \frac{\pi n}{N} \left( n^+_n n^-_{n} - n^-_{n} n^+_n \right) + \frac{2 \pi}{N} n^+_0 n^-_0 + \cot \frac{\pi}{2N} - 1, \quad (32)
$$

with the ground state energy

$$
H | 0 \rangle = \left( \cot \frac{\pi}{2N} - 1 \right) | 0 \rangle, \quad (33)
$$

where the vacuum now satisfies

$$n^+_0 | 0 \rangle = 0. \quad (34)$$

We call the algebra generated by these fermions the spectrum generating algebra. Note that the adjoint action by the Hamiltonian is

$$[ H, n^+_n ] = \pm 2 \sin \frac{\pi n}{N} n^+_n, \quad [ H, \gamma^+ ] = - \frac{2 \pi}{N} n^+_0. \quad (35)$$

### 2.5.3. Quantum-group generators in terms of Fourier modes.

The quantum group generators introduced in (8) and (9) are expressed in terms of the $n^\pm$-fermions as

$$E = n^+_0 K, \quad F = n^-_0. \quad (36)$$
with the $\mathfrak{sl}(2)$ generators
\[
e = \sum_{n=1}^{N/2-1} \frac{1}{n} \eta_n^+ \eta_n^- + \gamma^+ \eta_0^+, \quad f = \sum_{n=1}^{N/2-1} \frac{1}{n} \eta_n^- \eta_n^+ - \gamma^- \eta_0^-,
\]
\[
h = -\frac{1}{2} \left( \sum_{n=1}^{N/2-1} \frac{1}{n} (\eta_n^+ \eta_n^- + \eta_n^- \eta_n^+) + \gamma^+ \eta_0^- + \gamma^- \eta_0^+ \right).
\]

We then have the $\mathfrak{sl}(2)$ action on the fermions
\[
[e, \eta_n^-] = \eta_n^+, \quad [f, \eta_n^+] = \eta_n^-, \quad [h, \eta_n^\pm] = \pm \frac{1}{2} \eta_n^\pm,
\]
\[
[e, \gamma^+] = \gamma^+, \quad [f, \gamma^-] = \gamma^-, \quad [h, \gamma^\pm] = \pm \frac{1}{2} \gamma^\pm.
\]

2.5.4. TL and W generators in terms of Fourier modes. Using the transformation (21), the TL generators (7) in terms of fermions (29) take the form
\[
e_k = \frac{4(-1)^{k+1}}{N} \sum_{j_1, j_2 = 0}^{N-1} \frac{N - 1}{2} \sqrt{\frac{\sin \pi j_1 / N}{j_1}} \sqrt{\frac{\sin \pi j_2 / N}{j_2}} \sin \pi \left( \frac{1}{2} + \frac{j_1}{N} \right) \sin \pi \left( \frac{1}{2} + \frac{j_2}{N} \right) \eta_{j_1}^+ \eta_{j_2}^-,
\]
where (and in what follows) we assume
\[
\sin \pi j/N \bigg|_{j=0} = \frac{\pi}{N}.
\]

Meanwhile, the generators $\mathbb{W}_{\mathfrak{b}, \mathfrak{w}}$ (13)--(15) of the algebra $\mathcal{W}_{\mathfrak{b}, \mathfrak{w}}$ can be written as
\[
\mathbb{W}_{\mathfrak{b}}^+ = \frac{2}{N} \sum_{j_1, j_2 = 0}^{N-1} \sqrt{\frac{\sin \pi j_1 / N}{j_1}} \sqrt{\frac{\sin \pi j_2 / N}{j_2}} V_k(j_1, j_2) \eta_{j_1}^+ \eta_{j_2}^+,
\]
\[
\mathbb{W}_{\mathfrak{b}}^- = \frac{2}{N} \sum_{j_1, j_2 = 0}^{N-1} \sqrt{\frac{\sin \pi j_1 / N}{j_1}} \sqrt{\frac{\sin \pi j_2 / N}{j_2}} V_k(j_1, j_2) \left( \eta_{j_1}^+ \eta_{j_2}^- + \eta_{j_1}^- \eta_{j_2}^+ \right),
\]
\[
\mathbb{W}_{\mathfrak{b}}^- = \frac{2}{N} \sum_{j_1, j_2 = 0}^{N-1} \sqrt{\frac{\sin \pi j_1 / N}{j_1}} \sqrt{\frac{\sin \pi j_2 / N}{j_2}} V_k(j_1, j_2) \eta_{j_1}^- \eta_{j_2}^-,
\]
where we introduce trigonometric functions
\[
V_k(j_1, j_2) = \sin \pi \frac{j_1 - j_2}{2N} \cos \pi \frac{(j_1 + j_2)(k + 1/2)}{N} \]
\[
- (-1)^k \cos \pi \frac{j_1 + j_2}{2N} \sin \pi \frac{(j_1 - j_2)(k + 1/2)}{N}.
\]
We note that these functions have the symmetries
\[ V_k(-j_1, -j_2) = V_k(j_2, j_1) = -V_k(j_1, j_2). \]

### 2.6. Generalized lattice modes

Our ultimate purpose is to establish a connection between the lattice algebras and the ones present in the corresponding conformal field theory. The intuition guiding us is that the TL generators are in some sense a lattice version of the stress energy tensor \( T(z) \), and similarly the \( \mathcal{W}_n \) generators a lattice version of the triplet chiral \( \mathcal{W} \) algebra currents \( \mathcal{W}^\alpha(z) \). The strategy to make this more precise is to define Fourier modes expected to converge in a certain sense (to be discussed below) toward the Virasoro \( L_m \) and \( \mathcal{W} \)-algebra modes \( \mathcal{W}^\alpha_n \). If such a convergence occurs, one might expect that the lattice modes obey, in the limit \( N \to \infty \), the commutation relations of the CFT.

Of course, the algebra of lattice modes will not in general be closed, and calculating commutators will generate more quantities on the right-hand side which should also converge to the Virasoro and \( \mathcal{W} \)-algebra modes. This means that there must be a most likely infinite family of lattice versions of the \( L_m \) and \( \mathcal{W}^\alpha_n \), with convergence to a single Virasoro and \( \mathcal{W} \)-algebra occurring in the limit \( N \to \infty \). In this subsection, we define this family explicitly, postponing the discussion of what happens as \( N \to \infty \) to the next section.

#### 2.6.1. Higher approximation of Virasoro modes

The fact that many lattice expressions can converge to the same continuum limit is well known. For instance, there is an infinite number of choices of natural Hamiltonians for the XXZ spin-chains coming out of the quantum inverse scattering formalism which all go over to the Virasoro dilatation operator \( L_0 \) in the continuum limit. The underlying construction [11] suggests a natural extension to the case of non-zero modes—that is Fourier modes at non-vanishing momentum.

We start by defining the operators
\[ H_n^0 = \sum_{j=1}^{N-1} \cos \frac{n j}{N} \epsilon_j, \quad n \in \mathbb{Z}. \]  

The Hamiltonian (6) in this notation is \( H_n^0 \). The adjoint action by the Hamiltonian \( H_0^0 \) on \( H_n^0 \) generates the family \( H_n^r \) with \( r \in \mathbb{N}_0 \) and \( n \in \mathbb{Z} \) in the following way
\[ [H_0^r, H_n^r] = -4 \sin \frac{m}{2N} H_n^{r+1}, \quad r = 0, 1, 2, \ldots \]  

We note that this recurrence does not determine \( H_0^r \) directly but neatly means that \( H_0^r \) commutes with \( H_n^0 \). To define \( H_0^r \) we calculate \( H_m^r \) with \( m \neq 0 \) using the recurrence and then set \( m = 0 \) in the final formula. Relation (46) bears some similarity with the definition of ladder operators in integrable XXZ spin chains [34], but is different.

We recall then the fermionic expression for the Hamiltonian in (32). By a direct calculation using the formulas (41), and (45), and the recurrence relation (46), we obtain fermionic expressions for all the higher modes.
\[ H_n' = \sum_{j_1,j_2 = \frac{j_1}{2} + 1}^{j_1 + 1} \frac{\sin \theta_1 j / N}{j_1} \frac{\sin \theta_2 j / N}{j_2} \times \left( \cos \pi \frac{j_1 - j_2}{2N} \left( (-1)^r \delta_{j_1 + j_2, n} + \delta_{j_1 - j_2, -n} \right) \right) \]

\[ = - \sin \pi \frac{j_1 + j_2}{2N} \left( \delta_{j_1 - j_2, N+n} + \delta_{j_1 - j_2, -N-n} \right) \]

\[ + (-1)^r \delta_{j_1 - j_2, N-n} + (-1)^r \delta_{j_1 - j_2, -N+n} \right) \eta_{j_1}^+ \eta_{j_2}^- \]

That these operators give in the scaling (or continuum) limit the Virasoro modes and more generally elements in the universal enveloping of the Virasoro algebra at central charge \( c = -2 \) will be discussed below in the next section. Now, we introduce what will turn out later to be lattice analogues of modes of the triplet W-algebra currents.

2.6.2. Higher approximations of W-algebra modes. Similarly, we first define the operators

\[ W_n^{\alpha, 0} = \sum_{j=1}^{N-2} \cos \pi \frac{n(j + \frac{1}{2})}{N} \mathcal{W}_j^\alpha, \quad \alpha = +, -, 0. \]

The adjoint action by the Hamiltonian \( H_0' \) on \( W_n^{\alpha, 0} \) generates the family \( W_n^{\alpha, r} \) with \( r \in \mathbb{N}_0 \) and \( n \in \mathbb{Z} \) in the following way

\[ \left[ H_0', W_n^{\alpha, r} \right] = -4 \sin \pi \frac{m}{2N} W_n^{\alpha, r + 1}, \quad r = 0, 1, 2, ..., \quad \alpha = +, -, 0. \]

We note that \( W_n^{0, r} \) is determined in the same way as it is explained after (46).

Using (32) for \( H_0' = H \) and (42)–(44) with the recurrent relation (49), we obtain

\[ W_n^{+, r} = \sum_{j_1,j_2 = \frac{j_1}{2} + 1}^{j_1 + 1} \frac{\sin \theta_1 j_1 / N}{j_1} \frac{\sin \theta_2 j_2 / N}{j_2} \mathcal{U}_{j_1,j_2}(j_1,j_2) \eta_{j_1}^+ \eta_{j_2}^+, \]

\[ W_n^{0, r} = \sum_{j_1,j_2 = \frac{j_1}{2} + 1}^{j_1 + 1} \frac{\sin \theta_1 j_1 / N}{j_1} \frac{\sin \theta_2 j_2 / N}{j_2} \mathcal{U}_{j_1,j_2}(j_1,j_2) \left( \eta_{j_1}^+ \eta_{j_2}^- + \eta_{j_1}^- \eta_{j_2}^+ \right), \]

\[ W_n^{-, r} = \sum_{j_1,j_2 = \frac{j_1}{2} + 1}^{j_1 + 1} \frac{\sin \theta_1 j_1 / N}{j_1} \frac{\sin \theta_2 j_2 / N}{j_2} \mathcal{U}_{j_1,j_2}(j_1,j_2) \eta_{j_1}^- \eta_{j_2}^-, \]

where we introduce trigonometric functions

\[ \mathcal{U}_{j_1,j_2}(j_1,j_2) = \sin \pi \frac{j_1 - j_2}{2N} \cos \pi \frac{j_1 - j_2}{2N} \left( (-1)^r \delta_{j_1 + j_2, n} + \delta_{j_1 - j_2, -n} \right) \]

\[ + 2 \cos \pi \frac{j_1 + j_2}{2N} \sin \pi \frac{j_1 + j_2}{2N} \left( \delta_{j_1 - j_2, N+n} + (-1)^r \delta_{j_1 - j_2, -N+n} \right). \]

2.7. A note on the odd number of sites

We can also consider the spin-chains with odd number \( N \) of sites. They will give below a different sector of symplectic fermions in the scaling limit. For the odd-\( N \), we define the fermions \( \eta_n^ \pm \) with half-integer \( -\frac{N}{2} + 1 \leq n \leq \frac{N}{2} - 1 \) by formulas (29) where the fermionic
operators $\theta_k$ and $\theta_k^\dagger$, with $0 \leq k \leq N - 1$, are defined by (21) with amplitudes $A_k(j)$ defined by (22), for $k = 1, \ldots, N - 1$, and $A_0(j) = i^j$. We also have the $\mathfrak{su}(2)$ generators (8) represented as $E = \theta_0^\dagger K$ and $F = \theta_0$, satisfying

\[\{E^{kN}, \eta_j^\pm\} = 0, \quad \{F, \eta_j^\pm\} = 0, \quad j = -\frac{N}{2} + 1, -\frac{N}{2} + 2, \ldots, \frac{N}{2} - 1, \quad (54)\]

in addition to the quantum group relations. Then the backward transformation is

\[c_k^\dagger = -i \sum_{j=-N/2+1}^{N/2-1} \frac{1}{j} \sqrt{\frac{N}{\sin \pi j/N}} A_{N/2+j}(k) \eta_j^+ - j^k E^{kN}, \quad (55)\]

\[c_k = -\sum_{j=-N/2+1}^{N/2-1} \frac{1}{j} \sqrt{\frac{N}{\sin \pi j/N}} A_{N/2-j}(k) \eta_j^- + i^k F. \quad (56)\]

The $\mathfrak{sl}(2)$ generators are now represented as

\[e = \sum_{n=1/2}^{N/2-1} \frac{1}{n} \eta_n^+ \eta_n^+, \quad f = -\sum_{n=1/2}^{N/2-1} \frac{1}{n} \eta_n^- \eta_n^-, \quad h = -\frac{1}{2} \sum_{n=1/2}^{N/2-1} \frac{1}{n} (\eta_n^+ \eta_n^- + \eta_n^- \eta_n^+). \]

We note that the Hamiltonian (6) takes a slightly different form

\[H = \sum_{n=1/2}^{N/2-1} \frac{2}{n} \sin \frac{\pi n}{N} \left(\eta_n^- \eta_n^+ - \eta_n^+ \eta_n^-\right) + \left(1 - \frac{1}{\sin \pi/N}\right). \quad (57)\]

The definition of the lattice W-algebra is the same as in the even-$N$ case and theorem B.1 is true for odd-$N$ case as well. Formal expressions of the lattice W-algebra generators in terms of $\eta^\pm$ fermions are the same as in the even-$N$ case. So, the generators $e_k$ are given by (41) and $W_k^a$ are given by (42)–(44). Similarly, their linear combinations $H_n^0$ and $W_n^a, 0$ together with higher ‘currents’ $H_n^r$ and $W_n^{a, r}$ are expressed in the same way. So, the currents $H_n^r$ are given by (47) and currents $W_n^{a, r}$ by (50)–(52). For odd $N$ the operators $H_n^r$ and $W_n^{a, r}$ satisfy the same commutation relations as for even $N$.

Next, we turn to the study of the scaling properties of the whole family of the operators (modes) $H_n^r$ and $W_n^{a, r}$ introduced in 2.6.

### 3. Scaling limit of the XX chain and W-algebra modes

In this section we calculate the limit $N \to \infty$ of the commutation relations between modes (47) and (50)–(52) and show that they coincide with the commutation relations of the triplet W-algebra modes. For the convenience of the reader we give all the necessary conventions about symplectic fermions and the triplet W-algebra in appendix D.

#### 3.1. Scaling limit and $1/N$-decomposition

Here, we discuss how to proceed from the $W_{i,N}$ generators to get all the modes of the triplet W-algebra currents, including the Virasoro generators, in the chiral LCFT of symplectic fermions. The procedure is called the scaling limit $N \to \infty$ of the model. This limit involves the rescaled Hamiltonian $NH$ and its ‘low-lying’ eigenvectors: one should choose
the vacuum state $|0\rangle$) and consider in the limit only finite-energy states for $N (H = E_0)$, with the ground state energy $E_0 = \langle 0|H|0\rangle$, or in our cases the states generated by $\eta^\alpha_\alpha$ and $\gamma^\alpha_\alpha$ from $|0\rangle$ for any finite $n$. This way we obtain the CFT Hilbert space and then we should take the limit of the higher modes $H_n^\alpha$ and $W_n^{\alpha\tau}$ rescaled also by $N$ in the basis of the ‘low-lying’ eigenvectors. For these reasons, we found the decomposition of the Fourier modes $H_n^\alpha$ and $W_n^{\alpha\tau}$ in terms of operators that generate eigenvectors from the vacuum of $H$ —the fermionic bilinears $\eta^\alpha_\alpha \eta^{\beta}_\beta$ (note that coefficients in the decompositions are functions of the spin-chain size $N$.) We will show below that $H_n^\alpha$ and $W_n^{\alpha\tau}$ generate a Lie algebra whose structure constants are trigonometric functions of $N$. The scaling limit of the Fourier modes $H_n^\alpha$ and $W_n^{\alpha\tau}$ and their commutators involves thus formal expansions of the operators and the structure constants, respectively, as Laurent series in $N$. Different terms in such expansions for $H_n^\alpha$ and $W_n^{\alpha\tau}$, are then indeed identified with elements from the enveloping algebra of the chiral triplet $W$-algebra. Similar constructions of limits were discussed in [4, 36] where also more formal construction of direct/inductive limits is introduced.

3.1.1. The scaling limit of the Hamiltonian. We first study the scaling limit of the Hamiltonian $H$. We take the Hamiltonian (32), which is written in the normal-ordered form and keep terms up to the order $1/N$ for the very large $N$ decomposition

$$H = -\frac{2N}{\pi} + 1 + \frac{2\pi}{N} \left( -\frac{2}{24} + \sum_{n=1}^{N/2-1} \left( \eta^{-}_n \eta^{+}_n - \eta^{\alpha}_n \eta^{\beta}_n \right) - \eta^{-}_0 \eta^{\alpha}_0 \right) + O\left(1/N^2\right).$$

We see that the leading part (after subtracting the vacuum expectation of the Hamiltonian) which is an operator in front of $1/N$ has at $N \to \infty$ the form $L_0 = c/24$ on the ‘low-lying’ eigenvectors or scaling states, with $c = -2$. This expression of $L_0$ is well-known and it is the zero mode of the stress-energy tensor $T(z)$ in the symplectic fermions theory (compare it with (D.4)). Thus, properly renormalized Hamiltonian $\frac{N}{2\pi}(H - \langle 0|H|0\rangle)$ converges to $L_0 = c/24$.

3.1.2. Higher Hamiltonians. We could expect new features considering the scaling limit of the whole family of the higher Hamiltonians $H_0^\ell$ of the open XX spin-chain introduced above.

The limit $N \to \infty$ of operators $H_0^\ell$ given by (47) is singular and therefore we should subtract the divergent part, as we did it for $H = H_0^0$. To do this we calculate the eigenvalues $2h_\ell$ of $H_0^0$ on the vacuum vector (27) using (47), which gives

$$h_0 = \frac{1}{2} \left( \cot \frac{\pi}{2N} - 1 \right),$$

$$h_\ell = \frac{1}{2(\ell + 1)} + \frac{1}{2^{\ell+1}} \sum_{n=0}^{\ell/2+1} \left( \ell - 1 \right)_{\ell/2+n-1} - \left( \ell - 1 \right)_{\ell/2+n+1} \cot \frac{\pi}{2N},$$

for even $\ell > 0$. (59)

As we will see from general formulas below, the first leading term (in $1/N$) of the renormalized Hamiltonians $\frac{N}{2\pi}(H_0^0 - 2h_\ell)$ gives $L_0$ again but the next subleading terms, those in front of $1/N^k$, are different from those in the $H_0^0$ decomposition and are particular polynomials in zero modes of $T(z)$ and its descendants.
3.1.3. Virasoro modes in symplectic fermions. We start from calculating the scaling limit of the Fourier modes $H_0^n$. Expanding $\sqrt{\sin \pi j N}$ in (47) into a series in $1/N$ we obtain the very large $N$ decomposition

$$
\frac{N}{\pi} \left( H_0^n - 2\hbar_0 \delta_{n,0} \right) = - \sum_{j=\frac{-N-1}{2}}^{\frac{N-1}{2}} : \eta_j^+ \eta_{-j}^- : - \sum_{j=\frac{-N-1}{2}}^{\frac{N-1}{2}} : \eta_j^+ \eta_{-j}^- : + \sum_{j=\frac{-N-1}{2}+1}^{\frac{N-1}{2}+1} \left( \delta_{j_1-j_2,N+n} + \delta_{j_1-j_2,-N-n} + \delta_{j_1-j_2,N+n} + \delta_{j_1-j_2,-N-n} \right) \eta_{j_1}^+ \eta_{j_2}^- + o\left( \frac{1}{N} \right),
$$

where we use the standard normal ordering

$$
: \eta_j^+ \eta_j^- : = \begin{cases} \eta_j^+ \eta_j^- & j_2 \geq 0, \\
- \eta_j^+ \eta_j^- & j_1 \geq 0. 
\end{cases}
$$

We note that the second line of the previous expression vanishes on the scaling or the low-energy states because for any $n$ we can find $N$ such that $j_1$ or $j_2$ will be larger than any fixed number. Therefore, on the scaling states we have

$$
\frac{N}{\pi} \left( H_0^n - 2\hbar_0 \delta_{n,0} \right) = L_{n,N} + L_{-n,N} + o\left( \frac{1}{N} \right),
$$

where

$$
L_{n,N} = - \sum_{j=\frac{-N-1}{2}+1}^{\frac{N-1}{2}} : \eta_j^+ \eta_{-j}^- :.
$$

This expression coincides with (D.4) in the limit $N \to \infty$. In what follows we suppress the superscript $(N)$ in the expressions whenever it cannot lead to a confusion.

Similarly, after subtracting the vacuum eigenvalues we can write the series decomposition for the whole family of higher modes $H_r^n$ in $1/N$ on the scaling states

$$
H_r^n - (1 + (-1)^r) \hbar_r \delta_{n,0} = \sum_{k=1}^\infty \left( \frac{\pi}{N} \right)^{2k-1} H_r^n[k],
$$

where first several leading and subleading terms in front of $1/N^k$ have at $N \to \infty$ the form

$$
H_r^n[1] = (-1)^r L_n + L_{-n},
$$

$$
H_r^n[2] = \frac{1}{24} \left( 4(3r+1) \left( (-1)^r \left( L^2 \right)_n + \left( L^2 \right)_{-n} \right) - 3r \left( (-1)^r \left( \partial^2 L \right)_n + \left( \partial^2 L \right)_{-n} \right) \right) + 3(r+2) \left( (-1)^r (\partial L)_n + (\partial L)_{-n} \right) + 4 \left( (-1)^r L_n + L_{-n} \right).
$$
where the modes \( (\partial^k T_n^m) \), of the composite fields \( \partial^k T(z)^m \), are computed using the fermionic expression (D.1) of the stress energy tensor \( T(z) \), expressions for \( L_n \) and \( (L^2)_n \) given in (D.4), (D.5). Note that for \( n = 0 \) we obtain in this way an infinite family of commuting Hamiltonians \( \{H_n[1, 0], \{H_n[2, 1]\} \) as particular polynomials in zero modes of \( T(z) \) and its descendants.

In order to obtain the defining relations of the Virasoro algebra in the continuum limit, only the leading \( 1/N \) term will be necessary below, that is the \( H_n[1] \). For the triplet \( W \)-algebra relations, the presence of nonlinearities requires keeping also \( H_n[2] \), and similarly (see below) \( \alpha_{W_n[1]} \) and \( \alpha_{W_n[2]} \).

### 3.1.4. W-algebra modes in symplectic fermions

In a similar way, we obtain the decompositions of (50)–(52)

\[
W_n^{\alpha, r} = \sum_{k=1}^{\infty} \left( \frac{\pi}{N} \right)^{2k} W_n^{\alpha, r}[k],
\]

where the first two coefficients have at \( N \to \infty \) the form

\[
W_n^{\alpha, r}[1] = (-1)^{r} W_n^{\alpha} + W_n^{\alpha, r},
\]

\[
W_n^{\alpha, r}[2] = \frac{1}{48} \left( \frac{48}{5} (3r + 2) \right) (-1)^{r} \left( (LW_n^\alpha) + (LW_n^\alpha)_{-n} \right)
\]

\[
- \frac{2}{5} (9r + 1) \left( (-1)^{r} \left( (\partial^2 W_n^\alpha) + (\partial^2 W_n^\alpha)_{-n} \right) \right)
\]

\[
- (18r + 26) \left( (-1)^{r} \left( (\partial W_n^\alpha) + (\partial W_n^\alpha)_{-n} \right) \right)
\]

\[
- (6r + 22) \left( (-1)^{r} (W_n^\alpha + W_n^\alpha) \right),
\]

with expressions for \( W_n^\alpha \) and \( (LW_n^\alpha) \) given in (D.6)–(D.8). Note that the leading term in \( W_n^{\alpha, r} \) appears at \( 1/N^2 \) and we have a convergence of the rescaled Fourier modes \( \frac{\pi^n}{N^n} W_n^{\alpha, r} \) to the modes \( (-1)^{r} W_n^\alpha + W_n^\alpha \) of the chiral W-algebra currents, for any integer \( n \).
3.2. Scaling limit of the commutators

In this section we calculate commutators of $H_n^r$ and $\alpha_W^r n$ and show that in the limit $N \to \infty$ the commutators give the commutation relations (D.9)–(D.11) of the chiral triplet W-algebra of symplectic fermions. Using the explicit expression (47) we obtain the following commutation relations

\[ [H_n^r, H_m^0] = 2 \sin \frac{n-m}{2N} H^1_{n+m} + 2 \sin \frac{n+m}{2N} H^1_{n-m}, \] (70)

\[ [H_n^0, H_m^1] = 2 \sin \frac{2n-m}{2N} H^2_{n+m} - 2 \sin \frac{2n+m}{2N} H^2_{n-m} - 2 \sin \frac{n}{2N} \left( \cos \frac{n-m}{2N} H^0_{n+m} - \cos \frac{n+m}{2N} H^0_{n-m} \right). \] (71)

\[ [H_n^1, H_m^1] = 2 \sin \frac{n-m}{N} H^3_{n+m} - 2 \sin \frac{n+m}{N} H^3_{n-m} - 2 \sin \frac{n}{2N} \left( \cos \frac{n-m}{2N} \sin \frac{m}{2N} H^1_{n+m} + \sin \frac{n}{2N} \sin \frac{m}{2N} H^1_{n-m} \right). \] (72)

Taking (63) and the sign factor $(-1)^r$ in (64) into account we obtain the commutation relations of $L_n^r$’s as the limit

\[ [L_n^r, L_m^r] = \lim_{N \to \infty} \frac{N^2}{4\pi} \left( [H_n^r, H_m^0] - [H_n^0, H_m^1] + [H_n^0, H_m^0] + [H_n^1, H_m^1] \right). \] (73)

Then, we substitute right-hand sides of (70)–(72), then substitute $H_n^r$ from (63) and calculate the Laurent series in $N$ up to the second order which indeed gives the Virasoro commutation relations (D.9). We note that in this calculation we need to keep only first term in the decomposition (63). We also note that the Virasoro central extension appears from the decomposition of the vacuum eigenvalues (59) in $N$.

The commutation relations $[H_n^r, W_m^{\alpha,s}]$ and $[W_m^{\alpha,s}, W_m^{\beta,s}]$ are given in appendix E. To calculate the commutators

\[ [L_n^r, W_m^\alpha] = \lim_{N \to \infty} \frac{N^3}{4\pi} \left( [H_n^r, W_m^{\alpha,0}] - [H_n^0, W_m^{\alpha,1}] + [H_n^1, W_m^{\alpha,0}] + [H_n^1, W_m^{\alpha,1}] \right), \] (74)

we take the commutators from (E.1)–(E.4) and substitute the Laurent decompositions (63) and (67) into them. This gives the relations (D.10).

The calculation of the commutator

\[ [W_m^{\alpha}, W_m^{\beta}] = \lim_{N \to \infty} \frac{N^4}{4\pi} \left( [W_m^{\alpha,0}, W_m^{\beta,0}] - [W_m^{\alpha,1}, W_m^{\beta,1}] \right) \]

\[- [W_m^{\alpha,1}, W_m^{\beta,0}] + [W_m^{\alpha,1}, W_m^{\beta,1}] \] (75)

differs from the previous two in the following point. Due to the factor $N^4$ we should take the decompositions of the right-hand sides of (E.5)–(E.7) up to the 4th order. The leading order coming from the trigonometric functions is 0 or 1 and therefore we should keep the 3rd and 4th orders in the decompositions (63) and (67) respectively. This calculation results in the contribution from $(L^2)_n$ and $(W^\alpha)_n$ in the chiral W-algebra relations (D.11), which is
obtained after taking the limit. We have thus succeeded in reproducing the algebra structure of the Virasoro $L_n$ and the $W^\alpha_n$ modes from our lattice approximations.

3.3. A note on the odd-$N$ case scaling limit

The calculation of the $N \to \infty$ limit for the odd $N$ differs from the calculation for the even $N$ only by normal ordering constants. Thus for the odd case the calculation repeats the even case but now instead of (59) we take the normal ordering constants

$$h_0 = \frac{1}{2} \left( \frac{1}{\sin \frac{\pi}{2N}} - 1 \right),$$

$$h_\ell = -\frac{1}{2(\ell + 1)} + \frac{1}{2^{\ell+1}} \sum_{n=0}^{\ell/2} \left( \frac{\ell - 1}{\ell/2 + n - 1} \right) - \left( \frac{\ell - 1}{\ell/2 + n + 1} \right) \frac{1}{\sin \frac{\pi}{2n + 1}},$$

for even $\ell > 0$. (76)

4. The XXZ representation and the W-algebra on a lattice

In this section, we give our definition of the lattice W-algebra in the context of XXZ spin-chains at any root of unity $q$. We begin with collecting basic facts about XXZ spin-chains with quantum-group symmetry [3, 5, 33]. Then, we define the lattice W-algebra using its centralizing property with the small quantum group and study its properties (generators, relations, representation theory, etc.) Finally, in the last subsection we discuss two scaling limits of this lattice algebra, describing $(p-1, p)$ and $(1, p)$ models.

4.1. The Hamiltonian

We consider the XXZ Hamiltonian of $N$ one-half spins with an open boundary condition described by the ‘quantum-group symmetric’ boundary term [33],

$$H(q) = \frac{1}{2} \sum_{i=1}^{N-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right) = \frac{q - q^{-1}}{4} (\sigma_i^z - \sigma_{i+1}^z),$$

where $\sigma_i^x$, $\sigma_i^y$ and $\sigma_i^z$ are usual Pauli matrices given in (3). We note that we have an opposite sign in front of the boundary term (77), with respect to the standard choice [3, 33], which corresponds to a slightly different basis (corresponding to $q \to q^{-1}$ or relabeling of sites from $N$ to 1) we use in order to adapt to our quantum-group notations. The Hamiltonian $H(q)$ is not hermitian (with respect to the usual bilinear form on the spin chain) due to the imaginary boundary term $\frac{\sin(\pi/\ell)}{2}$ but its eigenvalues are real while its Jordan form has cells of rank 2. An easiest way to see non-trivial Jordan forms in the spectrum is to study symmetries of the Hamiltonian densities which we discuss below.

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We note finally that the Hamiltonian is massless when $|q| = 1$ and we will consider only this critical case. Moreover, roots of unity cases are most interesting for applications in the scaling limit, and these are the ones we will consider for the definition and study of lattice versions of the triplet W-algebras. We recall some basic information about the quantum group
\[ U_q \mathfrak{sl}(2) \] symmetry at the root of unity cases in appendix A where we also give relations to more usual (in the spin-chain literature) quantum group generators \( S^\pm, S^z \) and \( q S^z \).

4.2. The Casimir operator and TL algebra

As a module over \( U_q \mathfrak{sl}(2) \), the XXZ spin chain \( H_N \) is a tensor product of \( N \) copies of the fundamental two-dimensional simple module \( X_{2,1} \) (see notations below in 4.3.1) such that the generators are represented on \( X_{2,1} \) as

\[
E = \sigma^+, \quad F = \sigma^-, \quad K = \frac{q + q^{-1}}{2} - 1 + \frac{q - q^{-1}}{2} \sigma^z, \quad \text{and} \quad e = f = h = 0.
\]

Using the \((N - 1)\)-folded comultiplications (A.12) for the capital generators together with the general comultiplication (A.13) and (A.14) for the divided powers, which we compute in appendix A, we obtain the XXZ representation of \( U_q \mathfrak{sl}(2) \)

\[
\rho_{q,N} : U_q \mathfrak{sl}(2) \rightarrow \text{End}_C(H_N), \quad H_N = \bigotimes_{j=1}^N X_{2,1}.
\]

It has the usual XXZ expressions for the generators

\[
\rho_{q,N}(E) = \sum_{j=1}^N 1 \otimes \cdots \otimes 1 \otimes \sigma_j^+ \otimes q^{r_j} \otimes \cdots \otimes q^{r_{j+1}} \otimes 1 \otimes \cdots \otimes 1
\]

\[
\rho_{q,N}(F) = \sum_{j=1}^N q^{-r_j} \otimes \cdots \otimes q^{-r_{j-1}} \otimes \sigma_j^- \otimes 1 \otimes \cdots \otimes 1
\]

and for the divided powers the action is

\[
\rho_{q,N}(e) = q^{r_{j-1}} \sum_{1 \leq j_1 < j_2 < \cdots < j_N \leq N} (-1)^{j_1} 1 \otimes \cdots \otimes 1 \otimes \sigma_{j_1}^+ \otimes q^{r_{j_1} - 1 - r_{j_2}} \otimes \cdots \otimes q^{r_{j_2} - 1 - r_{j_3}} \otimes \cdots \otimes q^{r_{j_N - 1} - 1 - r_{j_{j_1}}}
\]

\[
\otimes \sigma_{j_1}^+ q^{r_{j_1} - 1} \otimes \sigma_{j_2}^+ \otimes \cdots \otimes \sigma_{j_{j_1}}^+ \otimes \cdots \otimes \sigma_{j_N}^+
\]

and

\[
\rho_{q,N}(f) = (-1)^{j_1} \sum_{j_1=1}^N \frac{q^{r_{j_1} - 1}}{\otimes \cdots \otimes 1 \otimes \sigma_{j_1}^- \otimes q^{r_{j_1} - 1 - r_{j_2}} \otimes \cdots \otimes q^{r_{j_2} - 1 - r_{j_3}} \otimes \cdots \otimes q^{r_{j_N - 1} - 1 - r_{j_{j_1}}}}
\]

\[
\otimes \sigma_{j_1}^- \otimes \sigma_{j_2}^- \otimes \cdots \otimes \sigma_{j_N}^-
\]

Then, we recall that the quantum-group Casimir element

\[
C = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}
\]
commutes with the full quantum group $U_q\mathfrak{sl}(2)$ and it is represented on $X_{2,1} \otimes X_{2,1}$ as
\[
\Delta(C) = \frac{1}{2} \left( \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \frac{q + q^{-1}}{2} \sigma^z \otimes \sigma^z \right) + \frac{q - q^{-1}}{4} \left( 1 \otimes \sigma^z - \sigma^z \otimes 1 \right) + \frac{3q^3 + q + q^{-1} + 3q^{-3}}{4(q - q^{-1})^2} 1 \otimes 1.
\]
This gives the usual XXZ coupling and the Hamiltonian (77) is expressed as a sum over the densities $\Delta(C)_{i,i+1}$ acting on $i$th and $(i + 1)$th tensorands
\[
H(q) = \sum_{i=1}^{N-1} \Delta(C)_{i,i+1} - (N - 1) \frac{3q^3 + q + q^{-1} + 3q^{-3}}{4(q - q^{-1})^2}.
\]
Introducing the more convenient notation for the Hamiltonian densities
\[
e_i = - \Delta(C)_{i,i+1} + \frac{q^3 + q^{-3}}{(q - q^{-1})^2} 1, \quad 1 \leq i \leq N - 1,
\]
the Hamiltonian takes the usual form [3, 33]
\[
H(q) = - \sum_{i=1}^{N-1} e_i + (N - 1) \frac{q + q^{-1}}{4}.
\]
It is straightforward to check that the operators $e_i$, for $1 \leq i \leq N - 1$, satisfy the defining relations of the TL algebra $\mathcal{TL}_qN$
\[
e_i^2 = (q + q^{-1}) e_i,
\]
\[
e_i e_{i \pm 1} e_i = e_i,
\]
\[
e_i e_j = e_j e_i, \quad |i - j| > 1.
\]
The relation (83) of the TL generators $e_i$ with the Casimir element and the coassociativity\(^5\) of the quantum-group comultiplication give the well-known result [33]
\[
\left[ \rho_q(\mathfrak{u}_q(2)), \mathcal{TL}_qN \right] = 0.
\]
Moreover, it was shown first in [32] that $\mathcal{TL}_qN$ is the centralizer of the representation of $U_q\mathfrak{sl}(2)$ on $\mathcal{H}_N$ and vice versa, i.e., they are mutual centralizers,
\[
\mathcal{TL}_qN \cong \text{End}_{U_q\mathfrak{sl}(2)}(\mathcal{H}_N), \quad \rho_q(\mathfrak{u}_q(2)) \cong \text{End}_{\mathcal{TL}_qN}(\mathcal{H}_N),
\]
for any $q$, including the root of unity cases.

It was also shown [3, 32] that the representation (83) is faithful and projective $\mathcal{TL}_qN$ -modules are direct summands in a decomposition of the spin-chain $\mathcal{H}_N$. Such a decomposition over $\mathcal{TL}_qN$ can be systematically found using the symmetry algebra—the centralizing algebra $U_q\mathfrak{sl}(2)$ and a decomposition of $\mathcal{H}_N$ as a module over $U_q\mathfrak{sl}(2)$, see e.g. [5]. We will analyze these decompositions for explicitly describing generators of our new lattice $\mathcal{W}$-algebra and in studying its modules below. It is thus important to recall basics of

\(^5\) By the coassociativity, we can write the action of $E$ on $C^2 \otimes C^2 \otimes C^2$ in two ways: (i) $\Delta^3(E) = \Delta(1) \otimes E + \Delta(E) \otimes K$ which obviously commutes with $e_1$ and (ii) $\Delta^3(E) = 1 \otimes \Delta(E) + E \otimes \Delta(K)$ which now clearly commutes with $e_2$, and similarly for the other $U_q\mathfrak{sl}(2)$ generators and higher values of $N$. 

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4.3. Modules over the quantum groups

In this section, we describe a few important $U_q \mathfrak{sl}(2)$- and $\mathcal{U}_q \mathfrak{sl}(2)$-modules which appear in decompositions of the spin-chains. We begin with recalling results on the simple modules and then describe their projective covers [25].

4.3.1. Simple $U_q \mathfrak{sl}(2)$-modules. A simple $U_q \mathfrak{sl}(2)$-module $X_{s,r}$ is labeled by the pair $(s, r)$, with $1 \leq s \leq p$ and $r \in \mathbb{N}$, and has the highest weights $(-1)^r q^{s-r} \frac{r-1}{2}$ with respect to $K$ and $h$ generators, respectively. The $sr$-dimensional module $X_{s,r}$ is spanned by elements $a_{n,m}$, $0 \leq n \leq s-1$, $0 \leq m \leq r-1$, where $a_{0,0}$ is the highest-weight vector and the left action of the algebra on $X_{s,r}$ is given by

$$K a_{n,m} = (-1)^r q^{s-r-2n} a_{n,m}, \quad h a_{n,m} = \frac{1}{2} (r - 1 - 2n) a_{n,m}, \quad (87)$$

$$E a_{n,m} = (-1)^r [n] [s-n] a_{n-1,m}, \quad e a_{n,m} = m(r-m) a_{n,m-1}, \quad (88)$$

$$F a_{n,m} = a_{n+1,m}, \quad f a_{n,m} = a_{n,m+1}. \quad (89)$$

where we set $a_{-1,m} = a_{n,-1} = a_{n,m} = a_{n,r} = 0$. We note that the $X_{s,r}$ restricted to the subalgebra $\mathcal{U}_q \mathfrak{sl}(2)$ is isomorphic to the $r$-fold direct sum of the simple $\mathcal{U}_q \mathfrak{sl}(2)$-module $X_{\alpha}$ with $\alpha = (-1)^{r-1}$ in notations of [27] and the action of $\mathcal{U}_q \mathfrak{sl}(2)$-generators is given by the first column of $(87)$–$(89)$.

4.3.2. Projective covers. The simple modules $X_{s,r}$ have projective covers $P_{s,r}$ with the subquotient structure

$$X_{s,1} \quad (90)$$

where $r \geq 2$. We also note that the $P_{s,r}$ as a $\mathcal{U}_q \mathfrak{sl}(2)$-module is isomorphic to the $r$-fold direct sum of $P_{s}^{\alpha}$ with $\alpha = (-1)^{r-1}$. The action of $E$, $F$, and $K$ in $P_{s}^{\alpha}$ can be found in [27].

4.4. XXZ decompositions

We are now ready to show the decomposition of $H_N = \bigotimes^N X_{2,1}$ over $U_q \mathfrak{sl}(2)$. Such a decomposition for any root of unity and any $N$ was explicitly written in [5] with the final result
\[
\mathcal{H}_N \left| U_q(\mathfrak{sl}(2)) \right. 
= \bigoplus_{s, t \equiv 0 \mod 2} \dim \left( \mathcal{X}_{s, t} \right) \mathcal{X}_{s, t} \bigoplus \bigoplus_{r = 1}^{n - 1} \dim \left( \mathcal{X}_{p_r, s, t} \right) \mathcal{X}_{p_r, s, t},
\]

where we defined \( N = m_p + m_s \) for \( m_p \in \mathbb{N} \) and \(-1 \leq m_s \leq p - 2\), and the multiplicity of each \( \mathcal{P}_{p-r, s, t} \) and \( \mathcal{X}_{s, t} \) equals \( d_{0, s, t} \) and \( d_{0, s, t} \), respectively, which are dimensions of simple modules \( \mathcal{X}_j \) over \( T \mathcal{L}_{q, N} \):

\[
\dim(\mathcal{X}_j) \equiv d_j = \sum_{n \geq 0} d_{j + np} - \sum_{n \geq (j + 1)} d_{j + np - 1 - 2(j \mod p)}
= \sum_{j \not\equiv 0 \mod p} (-1)^{(j - j \mod p)} d_j, \quad j \mod p \neq \frac{kp - 1}{2}, \quad k = 0, 1, \quad (92)
\]

where we introduce numbers \( d_j = \left( \frac{N}{N/2 + j} \right) - \left( \frac{N}{N/2 + j + 1} \right) \) and a step function \( t(j) \equiv t \) as

\[
t = \begin{cases} 
1, & \text{for } j \mod p > \frac{p - 1}{2}, \\
0, & \text{for } j \mod p < \frac{p - 1}{2}.
\end{cases} \quad (93)
\]

We assume that \( d_0^0 = 0 \) whenever \( j > N/2 \), as usually. Each simple module \( \mathcal{X}_j \) is defined as the head of the standard \( T \mathcal{L}_{q, N} \)-module characterized by \( 2j \) through lines in the links representation (see details in e.g. \([3, 5, 38]\)).

We next turn to an introduction of a lattice \( \mathcal{W} \)-algebra which uses the decompositions over \( U_q(\mathfrak{sl}(2)) \) but before we give explicit examples.

### 4.4.1. Examples

Using (91), we now give several examples of decompositions of the XXZ spin-chains over \( U_q(\mathfrak{sl}(2)) \) for \( p = 2, 3, 4 \).

- For \( p = 2 \), we have
  \[
  \mathcal{H}_2 = \mathcal{P}_{1, 1}, \quad \mathcal{H}_3 = 2\mathcal{X}_{2, 1} \oplus \mathcal{X}_{2, 2}, \quad \mathcal{H}_4 = 2\mathcal{P}_{1, 1} \oplus \mathcal{P}_{1, 2}, \quad ...
  \]

- For \( p = 3 \), we have
  \[
  \mathcal{H}_2 = \mathcal{X}_{1, 1} \oplus \mathcal{X}_{3, 1}, \quad \mathcal{H}_3 = \mathcal{X}_{2, 1} \oplus \mathcal{P}_{2, 1}, \quad \mathcal{H}_4 = \mathcal{X}_{1, 1} \oplus 3\mathcal{X}_{3, 1} \oplus \mathcal{P}_{1, 1}, \quad \mathcal{H}_5 = \mathcal{X}_{2, 1} \oplus 4\mathcal{P}_{2, 1} \oplus \mathcal{X}_{3, 2}, \quad ...
  \]

We assume that \( d_0^0 = 0 \) whenever \( j > N/2 \), as usually. Each simple module \( \mathcal{X}_j \) is defined as the head of the standard \( T \mathcal{L}_{q, N} \)-module characterized by \( 2j \) through lines in the links representation (see details in e.g. \([3, 5, 38]\)).

We next turn to an introduction of a lattice \( \mathcal{W} \)-algebra which uses the decompositions over \( U_q(\mathfrak{sl}(2)) \) but before we give explicit examples.
For $p = 4$, we have
\[
\begin{align*}
H_2 &= X_{1,1} \oplus X_{3,1}, & H_3 &= 2X_{2,1} \oplus X_{4,1}, \\
H_4 &= 2X_{1,1} \oplus 2X_{3,1} \oplus P_{3,1}, & H_5 &= 4X_{2,1} \oplus 4X_{4,1} \oplus P_{3,1}, \\
H_6 &= 4X_{1,1} \oplus 4X_{3,1} \oplus 5P_{3,1} \oplus P_{3,1}, & H_7 &= 8X_{2,1} \oplus 14X_{4,1} \oplus 6P_{3,1} \oplus X_{4,2}, \\
H_8 &= 8X_{1,1} \oplus 8X_{3,1} \oplus 20P_{3,1} \oplus 6P_{3,1} \oplus P_{3,2},
\end{align*}
\]

where the multiplicities in the front of each direct summand are dimensions of irreducible modules over $\mathcal{U}_q \mathfrak{sl}(2)$.

We see from the decompositions in 4.4.1 a general pattern: the simple projective module $X_{p,2}$ appears for the first time at $N = -\frac{1}{2}$ sites. This corresponds to the first time when the algebra of endomorphisms of the module $H_N$ over the small quantum group $\mathcal{U}_q \mathfrak{sl}(2)$ is larger than the algebra of endomorphisms respecting the full quantum group $\mathcal{U}_q \mathfrak{sl}(2)$ (we discuss how $\mathcal{U}_q \mathfrak{sl}(2)$-modules are restricted to $\mathcal{U}_q \mathfrak{sl}(2)$ after (89) and (90).) Therefore, we have isomorphisms

\[
\text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_N) \cong \text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_N), \quad \text{for } 1 \leq N \leq 2p - 2,
\]

while we have an inclusion

\[
\text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_N) \subset \text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_N), \quad \text{for } N \geq 2p - 1.
\]

We recall that the TL algebra $\mathcal{TL}_{q,N}$ is alternatively defined as $\text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_N)$.

**Definition 4.5.** The lattice $W$-algebra $\mathcal{W}_{q,N}$, for $q = e^{2\pi i p}$, is the algebra of endomorphisms $\text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_N)$, i.e., the algebra of operators commuting with the action of the restricted quantum group $\mathcal{U}_q \mathfrak{sl}(2)$ on the tensor-product representation $H_N = \bigotimes_{j=1}^N X_{2,1}$.

We note that the algebra $\mathcal{W}_{q,2p-1}$ is a non-trivial extension of $\mathcal{TL}_{q,2p-1}$. It has dimension $\dim \mathcal{TL}_{q,2p-1} + 3$. Note that the modules $X_{p,1}$ and $X_{p,2}$ restricted to $\mathcal{U}_q \mathfrak{sl}(2)$ are $X_p^+$ and $2X_p^-$, respectively. This means that $\text{Hom}_{\mathcal{U}_q \mathfrak{sl}(2)}(X_{p,2}, X_{p,1}) = 0$ and therefore the three additional elements/generators of $\text{End}_{\mathcal{U}_q \mathfrak{sl}(2)}(H_{2p-1})$ are endomorphisms on the direct summand $X_{p,2}$ and are given by the two maps $e$ and $f$ mixing two copies of $X_p^+$ in the $\mathfrak{sl}(2)$-doublet $X_{p,2}$ and by the generator $h$ restricted to this direct summand. The corresponding generators in $\mathcal{W}_{q,2p-1}$ can be explicitly constructed using the primitive idempotent $e_0$ given in appendix A.3—the normalized projector $(e_0^2 = e_0)$ onto the modules $X_{p,r}$ for any even $r$—as $WB_i^+ = \rho_{q,2p-1}(ee_0)$, $WB_i^- = \rho_{q,2p-1}(fe_0)$, and $WB_i^0 = \rho_{q,2p-1}(he_0)$.

We now turn to a definition of the lattice $W$-algebra $\mathcal{W}_{q,N}$ using similar generators.

**4.6. Generators via the primitive idempotent and $s'$**

We consider an extension $\mathcal{W}_{q,N}$ of the TL algebra $\mathcal{TL}_{q,N}$ by the generators

\[
\text{WB}_j^+ = \rho_{q,2p-1}((ee_0)_{j}, j+2p-2),
\]

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\[ WB_j^0 = \rho_{q,2p-1}(he_0)_{j,j+2p-2}, \]  
\[ WB_j^- = \rho_{q,2p-1}(fe_0)_{j,j+2p-2}, \]  
where \( j = 1, 2, \ldots, N - 2p + 2 \) and we introduce the notation \( x_{j+n} \), with \( x \in \text{End}_{\mathcal{E}} H_{n+1} \), for the operator \( 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \) applied between the \( j \)th and \((j + n)\)th sites and acted by identity otherwise. The algebra \( \mathcal{W}_{q,N} \) commutes with \( U_q \mathfrak{sl}(2) \) 
\[ \left[ WB_j^\alpha, \rho_{q,N}(U_q \mathfrak{sl}(2)) \right] = 0, \quad 1 \leq j \leq N - 2p + 2, \quad \alpha \in \{0, \pm\}. \]  
by construction: \( WB_j^\alpha \) acts non-trivially on \( 2p - 1 \) tensorands between \( j \)th and \((j + 2p - 2)\)th sites and by the identity on the other tensorands, the \( 2p - 1 \) tensorands consist the module isomorphic to the module \( \mathcal{E} \) and the \( \alpha \) \( WB_j \) are intertwiners respecting the \( U_q \mathfrak{sl}(2) \)-action on this \( H_{2p-1} \). One should use then the coassociativity of the comultiplication in \( U_q \mathfrak{sl}(2) \): we apply \( \Delta^{2p-2}(\cdot) \) on the \( j \)th tensor component of the element \( \Delta^{N-2p+1}(x) \), for any \( x \in U_q \mathfrak{sl}(2) \), with the resulting operator \( \Delta^{N-1}(x) \) acting on \( H_N \) obviously commuting with \( WB_j^\alpha \). This finishes our proof of the statement in (99).

We could write an explicit XXZ expression for \( WB_j^\alpha \) generators using the known expression for the central idempotent \( e_0 \) as a polynomial in the Casimir operator, see A.3 
\[ \Delta^n(e_0) = \frac{1}{4} \prod_{j=1}^{p-1} \left( 2 - q^j - q^{-j} \right)^2 \left( \Delta^n(\mathcal{C}) + 2 \prod_{j=1}^{p-1} \left( \Delta^n(\mathcal{C}) - q^j + q^{-j} \right)^2, \right. \]  
where 
\[ \Delta^n(\mathcal{C}) = \left( q - q^{-1} \right)^2 \Delta^n(F) \Delta^n(E) + q \Delta^n(K) + q^{-1} \Delta^n(K^{-1}). \]  
Then, the action of \( \rho_{q,2p-1}(e_0)_{j,j+2p-2} \) is obtained using the action (79), where we appropriately shift the indices of the Pauli \( \sigma \) matrices and replace \( N \) by \( 2p - 1 \). We also need formulas for \( (2p - 2) \)-folded comultiplication of the divided powers \( e \) and \( f \) applied on the spin-1/2 chain. They are similarly obtained from (80) and (81). Using all these formulas one can write the generators \( WB_j^\alpha \) in terms of Pauli matrices, to obtain expressions similar to TL generators \( ej \), but they are actually very bulky and we do not give them here.

Note that the operators (96)–(98) do not commute with the \( U \mathfrak{sl}(2) \) part of \( U_q \mathfrak{sl}(2) \)–they form an \( \mathfrak{sl}(2) \) triplet. We also see from the definition which involves the \( U \mathfrak{sl}(2) \) generators that for all \( j \) the generators \( WB_j^\alpha \) themselves satisfy \( \mathfrak{sl}(2) \) relations.

**Proposition 4.6.1.** For \( 1 \leq j \leq N - 2p + 2 \), the generators \( WB_j^+, WB_j^0 \) and \( WB_j^- \) satisfy the \( \mathfrak{sl}(2) \) relations 
\[ \left[ WB_j^+, WB_j^- \right] = 2WB_j^0, \quad \left[ WB_j^0, WB_j^\pm \right] = \pm WB_j^\pm. \]  
and they span a basis in the adjoint representation of \( \mathfrak{sl}(2) \): 
\[ \left[ e, WB_j^+ \right] = 0, \quad \left[ e, WB_j^- \right] = -WB_j^+, \quad \left[ e, WB_j^0 \right] = 2WB_j^0, \]  
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We therefore have that there exists an action of \( \mathfrak{sl}(2) \) on \( \mathcal{W}_{q,N} \) by derivatives of the associative multiplication.

To get more relations in \( \mathcal{W}_{q,N} \), we need to express \( q^\rho e_j(2) \) in TL generators with \( j \leq k \leq j + 2p - 3 \). We now turn to this problem.

4.7. Generators via the \( q \)-symmetrizer

We now give an alternative description of the generators using the \( q \)-symmetrizers \( S_r \) projecting onto the sub-representation of the maximum spin in the tensor product of spin-\( \frac{1}{2} \) representations; we assume now that \( q \) is generic. The intertwiners \( S_r \) are elements from \( \mathcal{T}_r q^{N} \) and defined by the recursion relation

\[
S_{r+1} = S_r - \frac{[r]}{[r+1]} S_r e_r S_r,
\]

where \( S_1 = 1 \) and \( e_r \) is the XXZ representation of the TL-generator \( e_r \). Using (105), we easily get first ones

\[
S_2 = 1 - \frac{1}{2} e_1,
\]

\[
S_3 = 1 - \frac{2}{3} (e_1 + e_2) + \frac{1}{3} \{ e_1, e_2 \}.
\]

We see that at the case of a root of unity the \( S_r \) are degenerate for \( r \geq p \) (because \( [p] = 0 \) but not all of them. Indeed, \( S_2 \) is degenerate at \( q = i \) while \( S_3 \) is well-defined and \( S_1 \rightarrow 1 - e_1 e_2 - e_2 e_1 \). In general, the nondegenerate ones are \( S_{np-1} \), for any \( n \geq 1 \). This follows from the fact that the (projective and irreducible) module \( X_{p,n} \) is the limit \( q \rightarrow e^{i\pi/p} \) of the \( np \)-dimensional Weyl module (generically irreducible module of the spin \( \frac{np-1}{2} \)) and the \( X_{p,n} \) is the direct summand in the limit. Therefore, there exists a projector onto this direct summand and this projector for \( N = 2p - 1 \) is simply given by the primitive central idempotent \( e_0 \) of the quantum group. We thus observe that the primitive idempotent \( e_0 \) acted on \( H_{2p-1} \) (and used above in the definition of the generators \( \mathcal{W}_{q,N}^r \)) coincides with \( S_{2p-1} \). For example, at \( q = i \) the operator \( S_1 = 1 - e_1 e_2 - e_2 e_1 \) equals \( q_{3,3}(e_2) \) acted on 3 sites. This observation allows us to give a more useful definition of the lattice W-algebra generators

\[
\mathcal{W}^+ = \rho_{4,2p-1}(e_0)|j,j+2p-2 \rangle  S_{2p-1}(j),
\]

\[
\mathcal{W}^0 = \rho_{4,2p-1}(e)|j,j+2p-2 \rangle  S_{2p-1}(j),
\]

\[
\mathcal{W}^- = \rho_{4,2p-1}(f)|j,j+2p-2 \rangle  S_{2p-1}(j),
\]

where \( S_{2p-1}(j) \) is the Jones–Wenzl projector of the TL subalgebra generated by \( e_k \), with \( j \leq k \leq j + 2p - 2 \), i.e., it is defined by the recursion relation
\[ S_{r+1}(j) = S_r(j) - \frac{r}{r+1} S_r(j) e_{r+j-1} S_r(j). \]  

(109)

So, on the full spin-chain, this operator is applied between the \( j \)th and \((j + 2p - 2)\)th sites and acts by identity on the other tensorands.

We first note that the generators (106)–(108) of the algebra \( \mathcal{W}_{q,N} \) (at \( q = i \)) coincide with the ones introduced in 2.4, where we used a representation of the walled Brauer algebra, up to a sign factor \((-1)^j\). As in the case of free fermions, the algebra \( \mathcal{W}_{q,N} \) has many defining relations in addition to the \( \mathfrak{sl}(2) \) relations. We do not see any reasons to write all of them down because they do not actually help in studying representation theory. Moreover, we believe that most of them are relations coming from a particular representation of a bigger algebra which still needs to be discovered. Among the defining relations we can easily get few simple ones. Recall that we have relations in the TL algebra \( S_r e_j = e_j S_r = 0 \), for any \( 1 \leq j \leq r - 1 \). We therefore have

\[ \mathcal{W}B_e e_j = e_j \mathcal{W}B_e \alpha = 0, \quad k \leq j \leq k + 2p - 3, \quad \alpha = \pm, 0. \]  

(110)

Finally, we believe that \( \mathcal{W}_{q,N} \) generates the full centralizer of \( U_q \mathfrak{sl}(2) \). This is indeed true for \( N \leq 2p - 1 \) and for all \( N \) at \( p = 2 \) case where the definition (13)–(15) of the lattice \( \mathfrak{w} \)-algebra coincides with \( \mathcal{W}_{q=N} \) defined here and we have proven theorem B.1 about the centralizer of \( U_q \mathfrak{sl}(2) \) for this \( q = i \) case. We thus finish this subsection with the following very reasonable conjecture which will be proven elsewhere.

**Conjecture 4.8.** The associative algebra \( \mathcal{W}_{q,N} \) defined by the generators \( \mathcal{W}B_e \alpha \), with \( 1 \leq j \leq N - 2p + 2 \) and \( \alpha \in \{ 0, \pm \} \), and \( e_k \), with \( 1 \leq k \leq N - 1 \), in the XXZ representation is isomorphic to the centralizer \( \mathcal{W}_{q,N} \) of \( U_q \mathfrak{sl}(2) \).

We next give a short discussion of our results on representation theory of the lattice \( \mathfrak{w} \)-algebra \( \mathcal{W}_{q,N} \) and relations with results on the representation theory of the chiral triplet \( \mathfrak{w} \)-algebras in the limit, for \( c = -2 \) and \( c = 0 \) cases, in particular.

### 4.9. Representation theory of \( \mathcal{W}_{q,N} \)

By the definition 4.5, the algebra \( \mathcal{W}_{q,N} \) is the full centralizer of \( U_q \mathfrak{sl}(2) \), i.e., it is defined as \( \mathcal{W}_{q,N} = \text{End}_{U_q \mathfrak{sl}(2)}(H_N) \) the algebra of all operators commuting with the small quantum group \( U_q \mathfrak{sl}(2) \). Its representation theory can be thus studied using the decomposition of the XXZ spin-chain over \( U_q \mathfrak{sl}(2) \). Such a decomposition can be easily obtained by restriction of the \( U_q \mathfrak{sl}(2) \)-module \( H_N \) in (91) on the subalgebra \( U_q \mathfrak{sl}(2) \). The idea is then to describe the multiplicity spaces in front of \( U_q \mathfrak{sl}(2) \) direct summands. These multiplicity spaces are then irreducible modules over \( \mathcal{W}_{q,N} \). The next step is to describe all possible homomorphisms between the \( U_q \mathfrak{sl}(2) \) direct summands. These homomorphisms represent the action of \( \mathcal{W}_{q,N} \) in its reducible and indecomposable modules. We only announce our results about \( \mathcal{W}_{q,N} \)-modules in order to make the paper not too long. All the necessary details will be given in a forthcoming publication [41].

We begin with the description of projective covers for simple \( \mathcal{W}_{q,N} \)-modules and describe then their content with respect to the TL subalgebra. Note that \( \mathcal{W}_{q,N} \) is represented faithfully in the spin chain, by its definition as the subalgebra \( \text{End}_{U_q \mathfrak{sl}(2)}(H_N) \) in the algebra of all operators acting on \( H_N \). Therefore, the projective cover of an irreducible \( \mathcal{W}_{q,N} \)-module can be identified with the indecomposable \( \mathcal{W}_{q,N} \)-submodule in \( H_N \) of maximum dimension and with the property being able to cover the irreducible module.
We have found that the algebra $\mathcal{W}_{q,N}$ has the following projective covers. There are modules $\mathcal{X}_s^\pm$ which are irreducible and projective simultaneously and we denote them also by $\mathcal{Y}_s^\pm$. There are $-p + 1$ cells (taking into account odd and even values of $N$) where reducible but indecomposable modules exist. Each cell contains projective modules $P_1^+$, $P_1^-$, and $P_0^+$, with $1 \leq s \leq p - 1$, and their subquotient structure is given in figure 1 (from the left to the right). These modules are projective covers of irreducible modules denoted by $\mathcal{X}_s^\pm$ and $\mathcal{Y}_s^\pm$, respectively. The decomposition of the irreducible $\mathcal{W}_{q,N}$-modules onto modules over the two commuting algebras, $\mathcal{T}_{q,N}$ and $U\mathfrak{s}\ell(2)$, can be written as

$$\mathcal{Y}_s = \mathcal{X}_{s-1} \otimes \mathbb{C}, \quad 1 \leq s \leq p - 1,$$

$$\mathcal{X}_s^+ = \bigoplus_{r \geq 1} \mathcal{X}_{2p-2r+1}\otimes \mathbb{C}^{2r-1}, \quad 1 \leq s \leq p,$$

$$\mathcal{X}_{p-s}^- = \bigoplus_{r \geq 1} \mathcal{X}_{2p+2r-1}\otimes \mathbb{C}^{2r}, \quad 0 \leq s \leq p - 1,$$

where the direct sums are finite, $\mathcal{X}_j$ are irreducible $\mathcal{T}_{q,N}$-modules defined after (92) and as usually we imply that $\mathcal{X}_j \equiv 0$ whenever $j > N/2$ and $s + N = 1 \mod 2$; the tensorands $\mathbb{C}^r$ denote the $r$-dimensional irreducible modules over $\mathfrak{s}\ell(2)$. We note that the modules $\mathcal{X}_{p-s}^-$ are non-zero only for $N \geq 2p - 1$. For $N < 2p - 1$, we have modules $\mathcal{Y}_s$ and $\mathcal{X}_s^+$ with trivial $U\mathfrak{s}\ell(2)$ content and they are also irreducible modules over $\mathcal{T}_{q,N}$, in accordance with (94).

The dimensions of irreducible $\mathcal{W}_{q,N}$-modules can be expressed in terms of the multiplicities $d^j_0$ introduced in (92). The dimension of $\mathcal{Y}_s$ is equal to $d^j_0$ and the module is the singlet with respect to the $U\mathfrak{s}\ell(2)$ action.

The dimensions of the irreducible $\mathcal{W}_{q,N}$-modules $\mathcal{X}_s^\pm$ are

$$\dim(\mathcal{X}_s^+) = \sum_{r \geq 1} (2r - 1)d^{2p-2r+1}_0, \quad \dim(\mathcal{X}_s^-) = \sum_{r \geq 1} 2r \cdot d^0_{s+2p-2r+1},$$

where $d^{j-1}_0$ is a non-negative integer and $j - 1 = 2p - 2r + 1$.

---

**Figure 1.** Projective covers of irreducible $\mathcal{W}_{q,N}$-modules denoted by $\mathcal{X}_s^+$, $\mathcal{X}_{p-s}^-$, and $\mathcal{Y}_s$. Same diagrams also describe subquotient structure of indecomposable modules over the chiral triplet W-algebra from [13].
Finally, we give the $\mathcal{TL}_{q,N} \otimes U\mathfrak{s}\ell(2)$ content of the projective $\mathcal{W}_{q,N}$-modules

\begin{align}
P_s^+ \big|_{\mathcal{TL}_{q,N} \otimes U\mathfrak{s}\ell(2)} &= \bigoplus_{n \geq 1} \mathbb{C} \otimes \mathbb{C}^{2n-1}, \quad 1 \leq s \leq p, \\
P_{p-s}^- \big|_{\mathcal{TL}_{q,N} \otimes U\mathfrak{s}\ell(2)} &= \bigoplus_{n \geq 1} \mathbb{C} \otimes \mathbb{C}^{2n}, \quad 0 \leq s < p - 1,
\end{align}

where the direct sums are finite of course, similarly to those for the simple modules above, and the subquotient structure of the projective $\mathcal{TL}_{q,N}$-modules $P_j$ can be found, e.g., in [5] and we use the identity $P_{p+1} = \mathcal{X}_{p+1}$ for modules over $\mathcal{TL}_{q,N}$.

We note also that the modules $P_{p-s}^-$ are zero unless $N \geq 2p - 1$. For $N < 2p - 1$, we have projective modules $P_i^+$ and $P_s^-$ with trivial $U\mathfrak{s}\ell(2)$ content—the two subquotients $\mathcal{X}_{p-s}$ in the first diagram on the left in figure 1 are actually absent (because of our notation $\mathcal{X}_{N/2} = 0$ for $\mathcal{TL}_{q,N}$-modules), and the module $P_i^+$ has only three irreducible subquotients. These projectives are the projective covers over $\mathcal{TL}_{q,N}$ and the algebra $\mathcal{W}_{q,N}$ has thus the same representation theory whenever $N < 2p - 1$, in accordance with (94).

Finally, the decomposition of the full spin-chain over $\mathcal{W}_{q,N}$ can be written as

$$H_N \big|_{\mathcal{W}_{q,N}} \cong \bigoplus_{s \geq 0 \mod 2} \left( sP_s^+ \oplus (p-s)P_{p-s}^- \right) \oplus \delta_{(p+N) \mod 2,1} P_p^+ \oplus \delta_{N \mod 2,1} P_{p-s}^-, \quad (115)$$

where $\delta_{a,b}$ is the usual Kronecker symbol.

4.10. Two scaling limits: $(p - 1, p)$ and $(1, p)$ theories

In the free fermions case $(p = 2)$ we already took the scaling limit of our spin-chains, and thus of the lattice $\mathcal{W}$-algebra projective modules. It is easy to see that in this case the modules $\mathcal{Y}_s$ are absent and we have only two reducible and indecomposable projectives $P_s^\pm$ of the ‘diamond’ structure (four instead of five subquotients for $P_N^\pm$) on even number of sites. For odd number $N$, the spin-chain is semisimple and both the projectives $P_N^\pm = \mathcal{X}_N^\pm$ are irreducible. This subquotient structure persists in the limit and give exactly the modules over the triplet $\mathcal{W}$-algebra [12], with the same character of course. This result is not very surprising as long as we consider free theories.

The case $p = 2$ enjoys a special symmetry: the models defined by the two Hamiltonians $H(q)$ and $-H(q)$ are equivalent, and exhibit the same scaling limit. This is not the case for larger values of $p$. While the scaling limit of $H(q)$ corresponds to the $(p - 1, p)$ theory, the Hamiltonian $-H(q)$ corresponds to the $(1, p)$ theory. Physically, this is because two Hamiltonians of opposite sign have in general very different structure of ground state and excited states.

We start by considering the case of $H(q)$: for instance, the choice $p = 3$ or $q = e^{2\pi i/3}$, which gives rise to a $c = 0$ logarithmic CFT [3] in the scaling limit. The generating function of energy levels of the Hamiltonian $H(q)$ from (77) is well known [33] thanks to the Bethe ansatz, and the partition function (or energy levels generating function) for scaling states in the XXZ spin-chain can be written as

$$\lim_{N \to \infty} \sum_{\text{states } i} q^{\frac{N}{2}(E_i(N) - N_{\text{chem}})} = q^{-c/24} \sum_{j \geq 0} (2j + 1) q^{h_{1,1-2j}} \prod_{n=1}^{\infty} \left( 1 - q^n \right), \quad (116)$$

28
where \( v_F = \frac{\sin \gamma}{\gamma} \) (with \( 2 \cos \gamma = q + q^{-1} \)) is the Fermi velocity, with the central charge
\[
c = c_{p-1, p} = 1 - \frac{6}{p(p - 1)},
\]
and \( E_i(N) \) is the eigenvalue of the \( i \)th (counted from the vacuum) eigenstate of \( H = \sum_i e_i \).

Here, we have also subtracted from the Hamiltonian \( H \) the density \( \psi_\infty = \lim_{N \to \infty} E_0(N)/N \), with \( E_0(N) \) the ground-state energy; we also use the standard notation for the conformal weights
\[
h_{r,s} = \frac{(pr - (p - 1)s)^2 - 1}{4p(p - 1)}.
\]

Note that the central charge here is the one from \( (p - 1, p) \) Minimal Models but the field content is actually different.

Let us denote by \( \mathcal{V}_h \) the Virasoro Verma module generated from the highest-weight state of weight \( h \). The expression at the multiplicity \( (2j + 1) \) on the right-hand side of (116) coincides with the Virasoro character \( Tr q^{L_0 - c/24} \) of the so-called Kac module with conformal weight \( h_{1,1+j} \) defined as the quotient \( \mathcal{V}_h/\mathcal{V}_{h_{1,1+j}} \) by the submodule corresponding to the singular vector in \( \mathcal{V}_{h_{1,1+j}} \) at level \( 2j + 1 \). These Kac modules correspond to the scaling limit of the standard \( \mathcal{T}_{q,N} \)-modules characterized by \( 2j \) through lines in the links representation (see details in [3, 5]).

It is also well-established [3] (see also [5]) that the simple TL modules \( \mathcal{X}_j \) go over in the scaling limit to the simple Virasoro modules with the highest weight \( h_{1,2j+1} \), which we will simply denote by \( \mathcal{X}_{1,2j+1} \). Then, using the decomposition (111) of the simple \( \mathcal{V}_j \) modules, we see that they correspond to the Virasoro irreducible representations from the \( (p - 1, p) \) minimal model Kac table. In the case \( p = 3 \) or \( c = 0 \), we have only one module \( \mathcal{X}_{1,1} \) which has dimension 1 for any \( N \) and this module corresponds to the unique operator (identity) in the \((2, 3)\ Kac \) table. For any \( p \geq 2 \), these are also simple modules over the triplet chiral W-algebra first observed in [13].

Using now the decomposition (112) and (113) of the simple \( \mathcal{X}^\pm_\ell \) modules as TL-U\( sl(2) \) bimoules and the identification of the simple TL modules \( \mathcal{X}_j \) with the simple Virasoro modules \( \mathcal{X}_{1,2j+1} \) in the scaling limit, we determine the Virasoro and \( U(sl(2)) \) algebras content for the scaling limit of \( \mathcal{X}^\pm_\ell \) (we use the same notation for the limits of \( \mathcal{W}_{q,N} \)-modules)
\[
\mathcal{X}^+_s = \bigoplus_{\ell \geq 1} \mathcal{X}_{1,2\ell+1} \otimes \mathbb{C}^{2\ell-1}, \quad 1 \leq s \leq p,
\]
\[
\mathcal{X}^-_{p-s} = \bigoplus_{\ell \geq 1} \mathcal{X}_{1,2\ell+1} \otimes \mathbb{C}^{2\ell}, \quad 0 \leq s \leq p - 1,
\]
where now the direct sums are infinite and are taken over all \( sl(2) \)-spins of a fixed parity; we also recall that \( \mathcal{X}_{1,1} \) is the simple Virasoro module of the weight \( h_{1,1} \). The decompositions (119) and (120) allow us to compute the characters \( Tr q^{L_0 - c/24} \) of the limits. The characters of \( \mathcal{X}^\pm_\ell \) are then expressed exactly as in [13]. Having an identification of the scaling limit of simples over \( \mathcal{W}_{q,N} \) with simples over the chiral W-algebra as the Virasoro-\( U(sl(2)) \) bimoules, we believe that the action of our lattice W-algebra on the simples indeed converges in \( N \to \infty \) limit to the action of the chiral W-algebra on the corresponding \( \mathcal{X}^\pm_\ell \) spaces, as was shown for \( p = 2 \) case above.

We also note that the limits \( \mathcal{Y}_j \) and \( \mathcal{X}^\pm_\ell \) we have constructed (in total, \( 3p - 1 \) modules) do not give all the possible irreducible modules (in total, \( 1/(2)(p - 1)(p - 2) + 2p(p - 1) \)) for the chiral W-algebra in the \( (p - 1, p) \) models constructed in [13]. It is obvious to us that in order
to obtain all the irreducible modules in the \((p - 1, p)\) model we have to use a bigger quantum group discovered in [39] and the corresponding lattice model, which we leave for a future work.

We can go further and compare also indecomposable but reducible modules. The structure of projective modules \(P_\sigma^\pm\) does not depend on the number of sites and should persists in the scaling limit. Therefore, we get the same diagrams in the scaling limit as those in figure 1. It is interesting to note that indecomposable modules over the chiral triplet \(W\)-algebra for \(c = 0\) \((p = 3)\) with the same subquotient structure were proposed in [40] (note also that for any \((p - 1, p)\) theory similar modules with 5, 4, and 2 subquotients involving the minimal model content were constructed in [13]).

We now consider the Hamiltonian with the opposite sign, that is \(-H(q)\). Its low-lying spectrum is the same as for the Hamiltonian \(H( - q^{-1})\) which has the Heisenberg coupling with the opposite sign. The spectra are related by the similarity transformation

\[
-H(q) = e^{-i\sum_{j=1}^{N} H \left( - q^{-1} \right) e^{i\sum_{j=1}^{N} H \left( - q^{-1} \right)}.
\]

This observation tells us that the generating function of low-lying states for \(-H(q)\) converges to characters from the \((1, p)\) theories. Indeed note that \(-q^{-1} = q^{p-1}\) and recall that the generating function of low-lying states for \(H\) \((e^{i\omega x - z})\) converges to characters from the \((p', p)\) theory [33]. Further, at this limit we observe that the structure of the projective covers \(P_\sigma^\pm\) will be slightly different—the subquotients \(\mathcal{Y}_i\) corresponding to the minimal model content disappear in the \((1, p)\) scaling limit. It is indeed easy to see for instance that when \(p = 3\) when we have just one \(\mathcal{Y}_{i=1}\), and it is spanned by one state which is now highest-energy state for the Hamiltonian \(-H(q)\) \((H( - q^{-1}))\). This state therefore will go to ‘infinity’ in the limit to \((1, 3)\) or \(c = -7\) theory. We expect that similarly all the subquotients \(\mathcal{Y}_i\) have just one point of condensation of energy levels and they thus disappear in the second, \((1, p)\), scaling limit. So, all the diagrams of the reducible but indecomposable triplet \(W\)-algebra projectives will have only four subquotients in the limit corresponding to the \((1, p)\) theories, as it was expected. The characters, Virasoro-\(U_\text{sl}(2)\) bimodule structure on the second scaling limit of simples \(\mathcal{X}_i^\pm\) and the subquotient structure of their projective covers in the limit are indeed the same ones (compare e.g. with [28]) as for the \((1, p)\) triplet \(W\)-algebra. Contrary to \((p - 1, p)\) models, here we recover all the irreducible modules for the chiral algebra.

5. Conclusions

We believe we have reached our main goal of understanding the lattice origin and analog of \(W\)-algebra symmetry in chiral (boundary) LCFTs. The next step should be the analysis of what happens in the non-chiral case, that is for bulk LCFTs. In this direction, the results in [4, 36, 42] could serve as a good starting point. There is now a lot of evidence that the bulk boundary relationship is extremely complicated in the logarithmic case, and that naive ‘periodicized’ versions of the XXZ spin chain or the underlying loop models will not have \(W\)-symmetry. What has to be done to restore this symmetry then remains an open question of crucial importance, and we think the study of lattice \(W\)-algebras in this context will be the source of further progress.

The case \(c = -2\) has also shed light on the mathematical nature of the relationship between the TL and the Virasoro algebras, or the lattice and continuum \(W\)-algebras. For a finite chain, the Fourier modes \(H_\alpha^r\) and \(W_\alpha^{r, r}\) define a finite dimensional algebra whose
structure constants are function of the size $N$. The scaling limit involves not only taking $N \to \infty$ but also normal ordering with respect to the ground state. In this scaling limit, the generators $H_n^r$ and $W_{n}^{a, r}$ expand on elements from the enveloping algebra of the Virasoro and the triplet $W$-algebra. In infinitely many lattice ‘approximations’ converge to the same limit in leading order—e.g., all the $H_n^r$ go to $(-1)^r L_n + L_{-n}$—but they differ at next to leading order. While our calculations were considerably simplified by the underlying presence of free fermions, we believe it should be possible to extend the analysis to another root of unity. It is important to note that different Hamiltonians will lead, for the same lattice algebra, to different continuum limits, as exemplified above in the $(1, p)$ and $(p - 1, p)$ cases.

To finish, we now would like to get back to the algebra of zero modes in the $q = i$ case. The operators $H_0^r$ and $W_0^{a, r}$ have the following properties

$$[H_0^r, H_0^s] = 0, \quad [H_0^r, W_0^{a, s}] = 0$$

and

$$[W_0^{0, r}, W_0^{0, s}] = -8W_0^{r+s, 0} + 8W_0^{r+s, 0};$$
$$[W_0^{0, r}, W_0^{-s}] = 8W_0^{-r+s, 0} - 8W_0^{-r+s, 0};$$
$$[W_0^{r, s}, W_0^{0}] = 4W_0^{0, r+s} - 4W_0^{0, r+s},$$

for $r, s \in 2\mathbb{N}_0$. For finite $N$ operators $H_0^r$ and $W_0^{a, r}$ are linearly dependent

$$H_0^N = \sum_{r=0}^{N-1} (-1)^{\frac{N-1}{2}} S_{-r} \left( \cos^2 \frac{\pi}{N}, \cos^2 \frac{2\pi}{N}, \ldots, \cos^2 \frac{N-1}{N} \right) H_0^r,$$

$$W_0^{a, N-2} = \sum_{r=0}^{N-2} (-1)^{\frac{N-1}{2}} S_{-r-1} \left( \cos^2 \frac{\pi}{N}, \cos^2 \frac{2\pi}{N}, \ldots, \cos^2 \frac{N/2 - 1}{N} \right) W_0^{a, r},$$

where $S_r(x_1, x_2, \ldots, x_k)$ is the $r$th elementary symmetric polynomial in the variables $x_1, x_2, \ldots, x_k$. We also recall that $H_0^0$ coincides with the Hamiltonian $H$, thus we have $4N - 10$ currents that commute with the Hamiltonian.

Interestingly, the algebra of $W_0^{a, r}$ can be identified through a quotient algebra of the $\mathfrak{sl}(2)$ loop algebra $\mathfrak{sl}(2)[t, t^{-1}]$ in the following way

$$W_0^{+, r} = 4 \left( t^2 - 1 \right) t^r e, \quad W_0^{-, r} = -4 \left( t^2 - 1 \right) t^r f, \quad W_0^{0, r} = -8 \left( t^2 - 1 \right) t^r h,$$

where the parameter $t$ satisfies the equation

$$P(t) \equiv \left( t^2 - 1 \right) \sum_{r=0}^{N-1} (-1)^r S_r \left( \cos^2 \frac{\pi}{N}, \cos^2 \frac{2\pi}{N}, \ldots, \cos^2 \frac{N/2 - 1}{N} \right) t^{N-2r} = 0.$$  

Recalling the well known property of elementary symmetric polynomials

$$\prod_{j=1}^{n} (t - t_j) = \sum_{r=0}^{n} (-1)^r S_r (t_1, \ldots, t_n) t^{n-r},$$

we obtain that the solutions of the equation (129) are $\left\{ \pm \cos \frac{\pi j}{N}, \ 0 \leq j \leq N/2 - 1 \right\}$. We have thus the representation of the loop algebra $\mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$ realized by the quotient by the ideal generated by the polynomial $P(t)$ from (129), i.e., we have relations $P(t)e = 0$, $P(t)f = 0$, etc. It is obvious that this representation (or the quotient) of the $\mathfrak{sl}(2)$ loop algebra
is isomorphic to the following tensor-product representation

$$V[t_1, ..., t_N] \equiv \bigotimes_{j=0}^{N/2-1} \left( \mathbb{C}^2 [\cos (\pi j/N)] \otimes \mathbb{C}^2 [-\cos (\pi j/N)] \right),$$

(130)

where by $\mathbb{C}[t]$ we denote the so-called evaluation representation of the loop algebra—any element $p(t)x$ acts by $p(t)x$, with $p(t) \in \mathbb{C}[t, t^{-1}]$ and $x \in \mathfrak{sl}(2)$. The action on the full spin chain (130) (defined on $N$ sites) is then given by the usual repeated comultiplication of a Lie algebra: $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Therefore, the zero modes of the $W^{\alpha, r}$ generators, with $\alpha \in \{ \pm, 0 \}$, can be written as

$$W^{\alpha, r}_{0} = \sum_{j=1}^{N} (t_j^2 - 1) \alpha_r x_a, \quad x_+ = 4e, \quad x_0 = -8h, \quad x_- = -4f,$$

(131)

where we also assume that all $t_j$’s are distinct and are from the set $\{ \pm \cos \frac{\pi j}{N}, 0 \leq j \leq N/2 - 1 \}$. Note that this expression is given in a different basis than the usual spin-basis of the XX chains.

We note further that the representation (130) of the loop algebra is irreducible because all the evaluation parameters $t_j$ are distinct, see e.g. [43]. For the subalgebra generated by the zero modes $W^{\alpha, r}_{0}$, it is a reducible representation. Indeed the representation (130) is isomorphic to the initial spin-chain representation where the zero modes are represented by (50) – (52) and the former one is reducible even for the full lattice W-algebra $\mathcal{W}_{\mathcal{N}}$—due to presence of the fermionic zero modes and the decomposition onto bosonic and fermionic states. The spin-chain representation of $\mathcal{W}_{\mathcal{N}}$ is a direct sum of two reducible but indecomposable representations, as discussed earlier in the more general context of XXZ spin-chains. It is easy to see that the indecomposability here is due the action of non-zero modes $\alpha_{W}^{r}$, as they contain fermionic zero modes $\eta^{\pm}_{0}$, while the action of $W^{\alpha, r}_{0}$ is semisimple. The module $V[t_1, ..., t_N]$ over the algebra of zero modes is thus semisimple and it is decomposed as

$$V[t_1, ..., t_N] \langle W^{0, r}_0 \rangle = 4 \left( \mathcal{X}^+ \oplus \mathcal{X}^- \right),$$

for even $N$,

where $\mathcal{X}^\pm$ are irreducible $\mathcal{W}_{\mathcal{N}}$-modules and they consist of bosonic and fermionic states, respectively, generated from the vacuum state by negative fermionic modes $\eta^{\pm}_{<0}$, and $\mathcal{X}^\pm$ has dimension $2^{N-3}$. Similarly for odd $N$, but we have the multiplicity 2 instead of 4.

The algebra of the zero modes $W^{\alpha, r}_{0}$ is reminiscent of the Onsager algebra [44] that appears in Onsager’s original solution of the Ising model, or the chiral Potts model. Like our algebra of zero modes, the Onsager algebra is also identified as a subalgebra of the $\mathfrak{sl}(2)$ loop algebra [45], but a different subalgebra from the one we have found here. Now the use of such an identification in the context of the Ising or chiral Potts model is that it leads to the spectrum of the Hamiltonian, which in these cases is a particular element of the loop algebra. Things are a bit different for the XX spin chain we are considering. The boundary terms prevent the usual analysis using the Onsager algebra. Meanwhile, we have another ‘version’ of this algebra, where the Hamiltonians $H_0$ are somehow ‘disconnected’, since they commute with all the zero modes. Of course, we can decide to compute the spectrum of the conserved quantities $W^{0, r}_0$ nonetheless. Indeed using the identification (131) we obtain

$$\text{Spec} \left( W^{0, r}_0 \right) = \left\{ \sum_{j=1}^{N} m_j (t_j^2 - 1) \alpha_r, \quad m_j = \pm 4, \quad t_j^2 = \cos \frac{\pi j}{N}, \quad 0 \leq j \leq \frac{N}{2} - 1 \right\},$$

(132)
where $N$ is the number of sites. Of course, this spectrum (the set of eigenvalues of $W_{0}^{r,s}$) coincides with the eigenvalues extracted from the fermionic expression (51) at $n = 0$.

It is natural to wonder at this stage whether higher spin representations, or different choices of the parameters $t_{j}$, are related with interesting new models exhibiting some sort of $W$-algebra symmetry. Another question is whether the lattice $W$-algebras for other roots of unity have anything to do with the $s(2)$ loop algebra.

A last remark of interest concerns the fact that the zero modes $H_{0}^{r}$ and $W_{0}^{r,s}$ commute, which implies that our XX spin chain has more conserved quantities than the $H_{0}^{r}$, which are the standard quantities obtainable from derivatives of the transfer matrix with respect to the spectral parameter. Of course, since we are dealing with free fermions, the existence of other conserved quantities apart from the $H_{0}^{r}$ is obvious. We have not been able however to extend the commutation of the zero modes to other values of $q$. It is tempting to speculate that there are other possible lattice models for which such additional conserved quantities could be built, and that they would be better candidates to reproduce the bulk $W$-symmetric LCFTs in the continuum limit.

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**Appendix A. Quantum groups**

Here, we collect definitions of different quantum groups at roots of unity and iterated comultiplication formulas. In setting the notation and recalling the basic facts about $U_{q}s\ell(2)$ needed below, we largely follow [25]. We introduce the standard notation $[n] = \frac{q^{n} - q^{-n}}{q - q^{-1}}$ for $q$-numbers and set $[n]! = [1][2]...[n]$.

### A.1. The small or restricted quantum group

The quantum group $\mathcal{U}_{q}s\ell(2)$ is the ‘small’ quantum $s\ell(2)$ with $q = e^{i\pi/n}$ and the generators $E$, $F$, and $K^{\pm 1}$ satisfying the standard relations for the quantum $s\ell(2)$,

\begin{equation}
KEK^{-1} = q^{2}E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},
\end{equation}

with additional relations,

\begin{equation}
E^{p} = F^{p} = 0, \quad K^{2p} = 1,
\end{equation}

and the comultiplication is given by

\begin{equation}
\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K.
\end{equation}

This associative algebra is finite-dimensional, $\dim \mathcal{U}_{q}s\ell(2) = 2p^{3}$. 

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A.2. The full or Lusztig quantum group \( U_{q}\mathfrak{sl}(2) \)

The Lusztig (or full) quantum group \( U_{q}\mathfrak{sl}(2) \) with \( q = e^{i\pi/p} \), for any integer \( p \geq 2 \), is generated by \( E, F, \) and \( K \) satisfying the relations (A.1) and (A.2), and additionally by the divided powers \( f \sim F^j/[j!] \) and \( e \sim E^j/[j!] \) which satisfy the usual \( \mathfrak{sl}(2) \)-relations:

\[
[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h. \tag{A.4}
\]

There are also 'mixed' relations

\[
[h, K] = 0, \quad [E, e] = 0, \quad [K, e] = 0, \quad [F, f] = 0, \quad [K, f] = 0, \tag{A.5}
\]

\[
[F, e] = \frac{1}{[p - 1]!} K^p q^i K - q^{-i} K^{-i} - E^{p-1}, \quad [E, f] = \frac{(-1)^{p+1}}{[p - 1]!} F^{p-1} q^i K - q^{-i} K^{-i}, \tag{A.6}
\]

\[
[h, E] = \frac{1}{2} E A, \quad [h, F] = -\frac{1}{2} A F. \tag{A.7}
\]

where

\[
A = \sum_{s=1}^{p-1} \left( u_s(q^{-i-1}) - u_s(q^{i-1}) \right) K + q^{i-1} u_s(q^{i-1}) - q^{-i-1} u_s(q^{-i-1}) - u_s(K) e_s \tag{A.8}
\]

with the polynomials \( u_s(K) = \prod_{n=1, n \neq s}^{p-1} (K - q^n - 2n) \), and \( e_s \) are central primitive idempotents defined just below following [27]. The relations (A.1)–(A.8) are the defining relations of the quantum group \( U_{q}\mathfrak{sl}(2) \).

A.3. Central idempotents

We recall the primitive central idempotents in \( U_{q}\mathfrak{sl}(2) \) [27]

\[
e_s = \frac{1}{\psi_s(\beta)} \psi_s'(\beta) \left( \bar{C} - \bar{C} \right) - \frac{\psi_s'(\beta)}{\psi_s(\beta)} \left( \bar{C} - \beta \right) \psi_s(\beta), \quad 1 \leq s \leq p - 1,
\]

\[
e_0 = \frac{1}{\psi_0(\beta_0)} \psi_0(\beta), \quad e_p = \frac{1}{\psi_p(\beta_p)} \psi_p(\beta),
\]

with the polynomials

\[
\psi_s(x) = (x - \beta_s)(x - \beta_s) \prod_{j \neq s}^{p-1} (x - \beta_j)^2, \quad 1 \leq s \leq p - 1,
\]

\[
\psi_0(x) = (x - \beta_p) \prod_{j=1}^{p-1} (x - \beta_j)^2, \quad \psi_p(x) = (x - \beta_0) \prod_{j=1}^{p-1} (x - \beta_j)^2,
\]

where \( \beta_j = q^j + q^{-j} \), and the Casimir element is

\[
\bar{C} = (q - q^{-1}) EF + q^{-1} K + q K^{-1}.
\]
A.4. Comultiplication in $U_q\mathfrak{su}(2)$

The quantum group $U_q\mathfrak{su}(2)$ has the Hopf-algebra structure with the comultiplication

$$\Delta(E) = 1 \otimes E + E \otimes \mathcal{K}, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

(A.9)

$$\Delta(e) = e \otimes 1 + K^p \otimes e + \frac{1}{[p - 1]!} \sum_{r=1}^{p-1} q^{(p-r)} K^r E^{p-r} \otimes E' K'^{-r},$$

(A.10)

$$\Delta(f) = f \otimes 1 + K^p \otimes f + \frac{(-1)^p}{[p - 1]!} \sum_{r=1}^{p-1} q^{-r} E^{p-r} \otimes F F' K'^{-r},$$

(A.11)

The antipode and counity are not used in the paper but a reader can find them, for example, in [25].

The $N$-folded comultiplication for the capital generators $E$ and $F$ is

$$\Delta^N(E) = \sum_{j_1=1}^{N+1} \cdots \sum_{j_N=1}^{N+1} E \otimes K \otimes \cdots \otimes K,$$

$$\Delta^N(F) = \sum_{j_1=1}^{N+1} \cdots \sum_{j_N=1}^{N+1} F \otimes 1 \otimes \cdots \otimes 1,$$

(A.12)

and for the divided powers $e$ and $f$ is

$$\Delta^N(e) = \sum_{n=1}^{N+1} \otimes K^p \otimes \cdots \otimes K \otimes 1$$

$$+ \sum_{0 \leq j_1, \ldots, j_N \in \mathbb{N}} q^{\sum j_i j_i} \prod_{m=1}^{N+1} E \otimes K \otimes \sum_{r=1}^{N+1} \frac{\sum j_i j_i}{[j_i]_q [j_i + 1]_q} [j_i]_q [j_i + 1]_q \prod_{m=1}^{N+1} E \otimes K \otimes \sum_{r=1}^{N+1} \frac{\sum j_i j_i}{[j_i]_q [j_i + 1]_q} [j_i]_q [j_i + 1]_q,$$

(A.13)

$$\Delta^N(f) = \sum_{n=1}^{N+1} \otimes K^p \otimes f \otimes \cdots \otimes 1$$

$$+ (-1)^{p-1} \sum_{0 \leq j_1, \ldots, j_N \in \mathbb{N}} q^{\sum j_i j_i} \prod_{m=1}^{N+1} F \otimes K \otimes \sum_{r=1}^{N+1} \frac{\sum j_i j_i}{[j_i]_q [j_i + 1]_q} [j_i]_q [j_i + 1]_q \prod_{m=1}^{N+1} F \otimes K \otimes \sum_{r=1}^{N+1} \frac{\sum j_i j_i}{[j_i]_q [j_i + 1]_q} [j_i]_q [j_i + 1]_q.$$

(A.14)

A.5. Standard spin-chain notations

We note the Hopf-algebra homomorphism

$$E \mapsto S^+ k, \quad F \mapsto k^{-1} S^-,$$

where we introduced the more usual (in the spin-chain literature [33, 37]) quantum group generators.

6 We note that our convention for the spin-chain representation differs from the one in [33] by the change $q \rightarrow q^{-1}$. 
\[ S^\pm = \sum_{1 \leq j \leq N^2} q^{-\sigma^\pm_1/2} \otimes \cdots q^{-\sigma^\pm_{j-1}/2} \otimes \sigma^\pm_j \otimes q^{\sigma^\pm_{j+1}/2} \otimes \cdots \otimes q^{\sigma^\pm_{N^2}/2} \quad (A.15) \]

together with \( k = q^S \) and the relations

\[ kS^\pm k^{-1} = q^2 S^\pm, \quad [S^+, S^-] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \]

\[ \Delta(S^\pm) = k^{-1} \otimes S^\pm + S^\pm \otimes k. \]

A.5.1. The case of XX spin-chains. For \( p = 2 \) or ‘XX spin-chain’ case, the \((N - 1)\)-folded coproduct of the renormalized powers \( e \) and \( f \) reads

\[ \Delta^{N-1}e = \sum_{j=1}^{N} 1 \otimes \cdots \otimes 1 \otimes e \otimes K^2 \otimes \cdots \otimes K^2+ \]

\[ + q \sum_{i=0}^{N-2N-1-t} \sum_{j=1}^{N} 1 \otimes \cdots \otimes 1 \otimes E \otimes K \otimes \cdots \otimes K \otimes EK \otimes K^2 \otimes \cdots \otimes K^2 \quad (A.16) \]

and

\[ \Delta^{N-1}f = \sum_{j=1}^{N} K^2 \otimes \cdots \otimes K^2 \otimes f \otimes 1 \otimes \cdots \otimes 1+ \]

\[ + q^{-1} \sum_{i=0}^{N-2N-1-t} \sum_{j=1}^{N} K^2 \otimes \cdots \otimes K^2 \otimes K^{-1}F \otimes K^{-1} \otimes \cdots \otimes K^{-1} \]

\[ \otimes F \otimes 1 \otimes \cdots \otimes 1. \quad (A.17) \]

These renormalized powers can also be expressed in terms of the more usual spin-chain operators, and one finds at \( p = 2 \)

\[ \Delta^{N-1}(e) = qS^{+}(2)k^2, \quad \Delta^{N-1}(f) = q^{-1}k^{-2}S^{-}(2), \]

where \( q = i \) and

\[ S^{\pm}(2) = \sum_{1 \leq j \leq (N-1)} q^{-\sigma^\pm_1} \otimes \cdots \otimes q^{-\sigma^\pm_{j-1}} \otimes \sigma^\pm_j \otimes 1 \otimes \cdots \]

\[ \otimes 1 \otimes \sigma^\pm_i \otimes q^{\sigma^\pm_{i+1}} \otimes \cdots \otimes q^{\sigma^\pm_{N}}. \quad (A.18) \]

Appendix B. The proof about the centralizer of \( \overline{U}_q sl(2) \) for \( q = i \)

In this section we prove the following theorem.

Theorem B.1. The centralizer of the \( \overline{U}_q sl(2) \) representation at \( q = i \)—the open XX spin-chain—is generated by

\[ \epsilon_k = c_k c_k^\dagger + c_{k+1} c_{k+1}^\dagger + i(c_k^\dagger c_k - c_{k+1}^\dagger c_{k+1}) \]
where \( N \leq n, m \leq N - 1 \) and \( 1 \leq j \leq N - 2 \).

Our proof is computational and consists of few lemmas. Recall first the linear combinations (21) and (29) of the \( c_j \) and \( c^\dagger_j \) operators—the Fourier transforms \( \eta^\pm_n \) of the lattice fermions. We recall also the definition (30) of the zero modes \( \eta^\pm_0 \) and conjugate to them \( \gamma^\pm \) operators defined in (24) and (25). The \( \eta^\pm_n \), with \(-\frac{N}{2} + 1 \leq n \leq \frac{N}{2} - 1\), and \( \gamma^\pm \) generate a Clifford algebra of dimension \( 2^{2N}\), see relations in (31). Note that the spin-chain is an irreducible representation of this Clifford algebra. We also recall that each pair of the modes \( \eta^\pm_n \) and the pair \( \gamma^\pm \), form an \( s \ell (2) \) doublet, with respect to the \( s \ell (2) \) representation given in (37), (38), see its action in (39) and (40). Therefore, the \( \eta^\pm_n \) and \( \gamma^\pm \) change an eigenvalue of the Cartan \( h^\pm = S^z \) by \( \pm 1 \). Now, we can formulate and prove the following technical lemma.

**Lemma B.1.1.** The centralizer of the \( g \ell (1|1) \), i.e., the walled Brauer algebra representation in the XX spin-chain is generated by the fermionic bilinears \( \eta^+_n \eta^-_m \), with \(-\frac{N}{2} + 1 \leq n, m \leq \frac{N}{2} - 1 \) and for any \( N \in \mathbb{N} \).

**Proof.** Define an index set \( I \) as the set of integers \(-\frac{N}{2} + 1 \leq n \leq \frac{N}{2} - 1\). We first note that the centralizer of the algebra generated by \( S^z \)—a commutative subalgebra in the \( g \ell (1|1) \) representation—has the fermionic bilinears

\[
\eta^+_n \eta^-_m, \quad \text{with } n, m \in I, \tag{B.1}
\]

and

\[
\gamma^+_n \gamma^-_m, \quad \gamma^+_n \eta^-_m, \quad \eta^+_n \gamma^-_m, \quad \text{with } n \in I, \tag{B.2}
\]

as its generators. Indeed, since the \( \eta^+_n \) and \( \gamma^\pm \)—the generators of the Clifford algebra—change an eigenvalue of \( S^z \) by \( \pm 1 \) it is clear that only monomials in \( \eta^+_n \) and \( \gamma^\pm \) with equal number of ‘+’ and ‘−’ commute with \( S^z \). All such monomials can obviously be generated by the bilinears in (B.1) and (B.2), with the use of the (anti)commutation relations in the Clifford algebra.

We then check by direct calculations that any linear combination of the bilinears in (B.2) does not commute with the fermionic part of \( g \ell (1|1) \)—the zero modes \( \eta^+_0 \)—while any linear combination of \( \eta^+_n \eta^-_m \) does. It is also straightforward to see that any word from the Clifford algebra involving the bilinears from (B.2) and any linear combination of such words do not commute with the \( \eta^+_0 \). Finally, recall that the \( g \ell (1|1) \) representation in our spin-chain is generated by \( S^z \) and \( \eta^+_0 \). We therefore conclude that the centralizer of the \( g \ell (1|1) \) action is generated by the \( \eta^+_n \eta^-_m \), with \( n, m \in I \). \( \square \)
Consider then a subalgebra $\rho_{g\ell}(\mathfrak{U}_q\mathfrak{sl}(2))$ in the (image of) $g\ell(1|1)$, see the definition of $\rho_{g\ell}$ in (10). This subalgebra is generated by $K = (-1)^{\delta}$ and $\eta^\pm_0$. Our aim is to describe the centralizer of the $\mathfrak{U}_q\mathfrak{sl}(2)$ action. Obviously the centralizer of $K$ is generated by all bilinears in the fermionic modes $\eta^+\eta^-$ and $\gamma^\pm$. Following the same lines as in the proof of lemma B.1.1 we prove our second technical lemma.

**Lemma B.1.2.** The centralizer of the $\mathfrak{U}_q\mathfrak{sl}(2)$ for $q = i$ is generated by the fermionic bilinears $\eta^+\eta^-,\eta^+\eta^+,\eta^-\eta^-$, with $-\frac{N}{2} + 1 \leq n, m \leq \frac{N}{2} - 1$ and for any $N \in \mathbb{N}$.

Then, note that the centralizer of $\mathfrak{U}_q\mathfrak{sl}(2)$ is in the image under the adjoint action of the $\mathfrak{g}\ell(1|1)$ algebra on the centralizer of $g\ell(1|1)$, i.e., on the walled Brauer algebra representation. Indeed, in terms of generators for both the centralizers described in lemmas B.1.1 and B.1.2 we have

$$\left[\varnothing, \eta^+\eta^-\right] = \eta^+\eta^-,$$

$$\left[f, \eta^+\eta^-\right] = \eta^+\eta^-,$$

$$-\frac{N}{2} + 1 \leq n, m \leq \frac{N}{2} - 1,$$

where we use the fermionic expressions (37) of the $\varnothing$ and $f$ generators of the $\mathfrak{g}\ell(2)$ algebra. We also note that the action of $\mathfrak{g}\ell(2)$ differentiates the multiplication in the Clifford algebra, i.e., for $a$ and $b$ from the Clifford algebra, we have $\varnothing(ab) = \varnothing(a)b + a\varnothing(b)$, etc. Therefore, the equation (B.3) define the image of any element of the walled Brauer algebra representation under the $\mathfrak{g}\ell(2)$ action. The centralizer of $\mathfrak{U}_q\mathfrak{sl}(2)$ is thus generated by (actually, coincides with) this image of the walled Brauer algebra under the action of $\mathfrak{U}_q\mathfrak{sl}(2)$. We finally note that by definition (12)–(15) the lattice W-algebra $\mathcal{W}_{i,N}$ is also generated by the image under the action of $\mathfrak{U}_q\mathfrak{sl}(2)$ on the same walled Brauer algebra where just another system of generators, the $e_j$ and $\mathcal{W}_B^j$ instead of $\eta^+\eta^-$, is used. The $\mathcal{W}_{i,N}$ is thus isomorphic to the centralizer of $\mathfrak{U}_q\mathfrak{sl}(2)$ which finishes the proof of theorem B.1.

**Appendix C. Examples of relations in the lattice W-algebra**

Here, we give some examples of relations in the lattice W-algebra $\mathcal{W}_{i,N}$.

- **3 sites:** the defining relations are

$$\mathcal{W}_B^m\mathcal{W}_B^m = 0,$$

$$\mathcal{W}_B^0\mathcal{W}_B^m = \pm \frac{1}{2} \mathcal{W}_B^m,$$

$$\mathcal{W}_B^0\mathcal{W}_B^m = \frac{1}{4} (1 - e_m e_{m+1} - e_{m+1} e_m),$$

$$\mathcal{W}_B^m \mathcal{W}_B^{-m} = \mathcal{W}_B^0 + \frac{1}{2} (1 - e_m e_{m+1} - e_{m+1} e_m),$$

$$e_m \mathcal{W}_B^m = \mathcal{W}_B^0 e_m = e_{m+1} \mathcal{W}_B^m = \mathcal{W}_B^0 e_{m+1} = 0.$$

These relations allow to construct a basis in the lattice W-algebra on 3 sites. The elements

$$\mathcal{W}_B^\alpha, \quad \alpha = \pm, 0,$$

together with 5 basis elements from the TL algebra give 8 basis elements in total.

- **4 sites:** let us introduce notations $e_{1,m} = [e_m, e_{m+1}], e_{2,m} = \left[e_m, [e_{m+1}, e_{m+2}]\right], \text{etc.}$, and $\mathcal{W}_B^{a,m} = [\mathcal{W}_B^m, e_{m+1}]$. Then, the defining relations in $\mathcal{W}_{i,4}$ can be written as (we set $m = 1$ here)
\[ WB_m^0 WB_m^+ + 1 = 0, \]
\[ WB_m^0 WB_m^0 + 1 = \frac{1}{4} \left( 1 - 2e_{2,m}e_{m+1} + 2e_{1,m} \right), \]
\[ WB_m^0 + 1 WB_m^0 = \frac{1}{4} \left( 1 - 2e_{2,m}e_{m+1} - 2e_{1,m+1} \right), \]
\[ WB_m^+ WB_m^- + 1 = WB_m^0 + 1 - e_{m+1}e_m WB_m^0 + 1 + e_{1,m+1} - e_{m+1}e_{2,m} + \frac{1}{2}, \]
\[ WB_m^+ WB_m^- = WB_m^0 + e_{m+1}e_m WB_m^0 + 1 - e_{1,m+1} - e_{m+1}e_{2,m} + \frac{1}{2}, \]
in particular
\[ \left[ WB_m^+, WB_m^- \right] = WB_m^0 + WB_m^0 + 1 - e_{1,m} - e_{1,m+1}, \]
\[ \left[ WB_m^+, WB_m^- + 1 \right] = WB_m^0 + WB_m^0 + 1 + e_{1,m} + e_{1,m+1}, \]
and relations with the TL generators
\[ e_{m-1} WB_m^\alpha e_{m-1} = e_{m+2} WB_m^\alpha e_{m+2} = 0, \]
\[ \left[ e_{m}, WB_{m+1}^\alpha \right] = \left[ WB_m^\alpha, e_{m+2} \right]. \]
\[ e_{1,m} WB_m^\alpha + 1 = e_{1,m+1} WB_m^\alpha, \quad WB_m^\alpha + 1 e_{1,m} = WB_m^\alpha e_{1,m+1}. \quad \alpha = +, -, 0. \]
\[ e_{2,m} WB_m^\alpha = - \frac{1}{2} e_m WB_m^\alpha + 1, \quad WB_m^\alpha e_{2,m} = - \frac{1}{2} WB_m^\alpha e_m, \]
\[ \left[ e_{m+1}, WB_{1,m}^\alpha \right] = - \frac{1}{2} WB_m^\alpha + \frac{1}{2} WB_m^\alpha + 1, \]
and
\[ WB_{m+2}^m WB_m^\pm = WB_m^\pm + 1 e_m WB_m^\pm + 1 = 0, \]
\[ WB_m^0 e_{m+2} WB_m^+ = WB_m^0 + 1 e_m WB_m^+ + 1 = \frac{1}{2} \left( WB_m^+ e_{m+2} + WB_m^+ e_m \right). \]
\[ WB_m^+ e_{m+2} WB_m^0 = WB_m^0 + 1 e_m WB_m^0 + 1 = - \frac{1}{2} \left( e_{m+2} WB_m^0 + e_m WB_m^+ \right), \]
\[ WB_m^0 e_{m+2} WB_m^0 = WB_m^0 + 1 e_m WB_m^0 + 1 = \frac{1}{4} \left( e_m + e_{m+2} - 2e_{2,m} + 2e_{1,m}e_{m+1} + 2e_m e_{m+1} e_{m+2} \right), \]
\[ WB_m^+ e_{m+2} WB_m^0 = WB_m^0 + 1 e_m WB_m^+ + 1 = \frac{1}{2} \left( e_m + e_{m+2} - 2e_{2,m} + 2e_{1,m} e_{m+2} + 2e_m e_{m+1} e_{m+2} \right), \]
\[ WB_m^+ e_{m+2} WB_m^0 = - WB_m^0 + 1 e_m WB_m^+ + 1 = - \frac{1}{2} \left( e_m + e_{m+2} - 2e_{2,m} + 2e_{1,m} e_{m+2} + 2e_m e_{m+1} e_{m+2} \right). \]

These relations allow to construct a basis in the lattice W-algebra \( \mathcal{W}_{l,4} \):
\[ WB_m^\alpha, WB_m^\alpha + 1, \quad 6 \text{ elements}, \]
\[ e_{m+2} WB_m^\alpha, WB_m^\alpha e_{m+2}, \quad e_m WB_m^\alpha, \quad 9 \text{ elements}, \]
\[ e_{m+1} e_m WB_m^\alpha, \quad 3 \text{ elements}, \]
which together with 14 basis elements from the TL algebra give 32 basis elements.
5 sites: for 5 sites the relations are very bulky and we give only simple examples of them:

\[
\begin{align*}
\left[ WB_{m+2}^+, WB_m^+ \right] &= \left[ \epsilon_{m}, \left[ \epsilon_{m+1}, WB_{m+2}^- \right] \right] - \epsilon_{3,m}, \\
\left[ WB_{m+2}^+, WB_m^- \right] &= \left[ \epsilon_{m}, \left[ \epsilon_{m+1}, WB_{m+2}^0 \right] \right] + \epsilon_{3,m}, \\
\left[ WB_{m+2}^0, WB_m^0 \right] &= 2 \epsilon_{3,m}.
\end{align*}
\]

Appendix D. The triplet W-algebra currents in symplectic fermions

We recall that the triplet W-algebra currents in terms of the symplectic fermions \cite{22}

\[
\eta^z(z) = \sum z^{-n-1} \eta^z_n, \quad \text{with} \quad \eta^+(z) \eta^-(w) = \frac{1}{(z-w)^2} + \ldots,
\]

where the sum is assumed to be over \( n \in \mathbb{Z} \) an the dots stand for regular terms, are given as

\[
T(z) = -: \eta^+(z) \eta^-(z): = \sum z^{-n-2} L_n, \tag{D.1}
\]

\[
W^\pm(z) = : \partial \eta^\pm(z) \eta^\mp(z): = \sum z^{-n-3} W_n^\pm, \tag{D.2}
\]

\[
W^0(z) = \frac{1}{2} \left( : \partial \eta^+(z) \eta^-(z): - : \eta^+(z) \partial \eta^-(z): \right) = \sum z^{-n-3} W_n^0. \tag{D.3}
\]

We also introduce modes of some composite currents

\[
T(z) T(z) = \sum z^{-n-4} (L^2)_n, \quad : T(z) W^n(z): = \sum z^{-n-5} (LW^n)_n.
\]

These modes are expressed in the modes of the symplectic fermions \( \eta^\pm(z) \) as

\[
L_n = - \sum_{j_1+j_2=n} : \eta_{j_1}^+ \eta_{j_2}^- : , \tag{D.4}
\]

\[
(L^2)_n = - \frac{1}{2} \sum_{j_1+j_2=n} \left( (j_1+1)(j_1+2) + (j_2+1)(j_2+2) \right) : \eta_{j_1}^+ \eta_{j_2}^- : , \tag{D.5}
\]

\[
W_n^\pm = - \sum_{j_1+j_2=n} (j_1+1) \eta_{j_1}^\pm \eta_{j_2}^\pm, \tag{D.6}
\]

\[
W_n^0 = - \sum_{j_1+j_2=n} (j_1-j_2) \eta_{j_1}^+ \eta_{j_2}^-, \tag{D.7}
\]

\[
(LW^+_n) = \sum_{j_1+j_2=n} \left( - \frac{1}{3} (j_2+1)(j_2+2)(j_2+3)
\right.
\]

\[+ \frac{1}{2} (j_1+1)(j_2+1)(j_2+2) \right) \eta_{j_1}^+ \eta_{j_2}^+. \tag{D.8}
\]
The modes of these currents satisfy the commutation relations [12]

\[ [L_n, L_m] = (n - m)L_{n+m} - \frac{1}{6}n\left(n^2 - 1\right)\delta_{n+m,0}, \quad \text{(D.9)} \]

\[ [L_n, W^\alpha_m] = (2n - m)W^{\alpha}_{n+m}, \quad \text{(D.10)} \]

\[ [W^\alpha_n, W^\beta_m] = g^{\alpha\beta} \left(2(n - m)(L^2)^{n+m}_{n+m} - \frac{1}{4}(n - m)(2n + m + 4)(2m + n + 4)L_{n+m} \right.
\]
\[ \left. - \frac{1}{120}n\left(n^2 - 1\right)(n^2 - 4)\delta_{n+m,0}\right)
\]
\[ + f^{\alpha\beta}_y \left(\frac{12}{5}(LW^y)^{n+m}_{n+m} + \frac{1}{10}(2n^2 + 2m^2 - 21nm - 36n \right.
\]
\[ \left. - 36m - 76)W^y_{n+m}\right), \quad \text{(D.11)} \]

where \(g^{\alpha\beta}\) and \(f^{\alpha\beta}_y\) are defined after (1).

Our aim in the paper is to show that these commutation relations can be obtained from the scaling limit of those in the (finite-dimensional) lattice \(W\)-algebra \(\mathcal{W}_{L,N}\) introduced in sections 2.4 and 2.6 as an extension of the TL algebra.

**Appendix E. Commutation relations in the \(\mathcal{W}_{L,N}\) algebra**

We collect here a list of commutators of the Fourier modes in \(\mathcal{W}_{L,N}\) generators

\[ \left[H^0_n, W^{\alpha,0}_m\right] = 2 \sin \frac{\pi}{2N} \left(\frac{2n - m}{N} W^{\alpha,1}_{n+m} + 2 \sin \frac{\pi}{2N} \left(\frac{2n + m}{N} W^{\alpha,1}_{n-m}\right)\right) \quad \text{(E.1)} \]

\[ \left[H^1_n, W^{\alpha,1}_m\right] = 2 \sin \frac{\pi}{2N} \left(\frac{3n - m}{N} W^{\alpha,2}_{n+m} - 2 \sin \frac{\pi}{2N} \left(\frac{3n + m}{N} W^{\alpha,2}_{n-m}\right)\right.
\]
\[ \left. - \sin \frac{\pi}{2N} \left(\cos \frac{n}{2N} W^{\alpha,0}_{n+m} - \cos \frac{n}{2N} W^{\alpha,0}_{n-m}\right)\right) \quad \text{(E.2)} \]

\[ \left[H^1_n, W^{\alpha,0}_m\right] = 2 \sin \frac{\pi}{2N} \left(\frac{n - m}{N} W^{\alpha,2}_{n+m} + 2 \sin \frac{\pi}{2N} \left(\frac{n + m}{N} W^{\alpha,2}_{n-m}\right)\right.
\]
\[ \left. + 2 \cos \frac{\pi}{2N} \sin \frac{\pi}{2N} \left(\cos \frac{n}{2N} W^{\alpha,0}_{n+m} - \cos \frac{n}{2N} W^{\alpha,0}_{n-m}\right)\right) \quad \text{(E.3)} \]

\[ \left[H^1_n, W^{\alpha,1}_m\right] = 2 \sin \frac{\pi}{2N} \left(\frac{3n - 2m}{2N} W^{\alpha,3}_{n+m} - 2 \sin \frac{\pi}{2N} \left(\frac{3n + 2m}{2N} W^{\alpha,3}_{n-m}\right)\right.
\]
\[ \left. + \left(\cos \frac{\pi}{N} \sin \frac{\pi}{2N} - \sin \frac{3n - 2m}{2N}\right) W^{\alpha,1}_{n+m}\right)
\]
\[ \left. - \left(\cos \frac{\pi}{N} \sin \frac{\pi}{2N} - \sin \frac{3n + 2m}{2N}\right) W^{\alpha,1}_{n-m}\right) \quad \text{(E.4)} \]
\[
\begin{align*}
\left[ W_n^{\alpha, 0}, W_m^{\beta, 0} \right] &= f_{\gamma} \left\{ 2 \cos \pi \frac{n - m}{N} W_{n+m}^{\gamma} + 2 \cos \pi \frac{n + m}{N} W_{n-m}^{\gamma} \\
&\quad - \frac{1}{2} \left( \cos \pi \frac{n}{N} + \cos \pi \frac{m}{N} + \cos \pi \frac{n - m}{N} + 1 \right) W_{n+m}^{\gamma} \\
&\quad - \frac{1}{2} \left( \cos \pi \frac{n}{N} + \cos \pi \frac{m}{N} + \cos \pi \frac{n + m}{N} + 1 \right) W_{n-m}^{\gamma} \right\} \\
&\quad + g^{\gamma} \left\{ 4 \sin \pi \frac{n - m}{N} H_{n+m}^{\gamma} + 4 \sin \pi \frac{n + m}{N} H_{n-m}^{\gamma} \\
&\quad - \left( \sin \pi \frac{n}{N} - \sin \pi \frac{m}{N} + 3 \sin \pi \frac{n - m}{N} \right) H_{n+m}^{\gamma} \\
&\quad - \left( \sin \pi \frac{n}{N} + \sin \pi \frac{m}{N} + 3 \sin \pi \frac{n + m}{N} \right) H_{n-m}^{\gamma} \right\},
\end{align*}
\tag{E.5}
\]

\[
\begin{align*}
\left[ W_n^{\alpha, 0}, W_m^{\beta, 1} \right] &= f_{\gamma} \left\{ 2 \cos \pi \frac{3n - 2m}{2N} W_{n+m}^{\gamma} + 2 \cos \pi \frac{3n + 2m}{2N} W_{n-m}^{\gamma} \\
&\quad - \left( \cos \pi \frac{n}{2N} \cos \pi \frac{n}{N} + \cos \pi \frac{3n - 2m}{2N} \right) W_{n+m}^{\gamma} \\
&\quad + \left( \cos \pi \frac{n}{2N} \cos \pi \frac{n}{N} + \cos \pi \frac{3n + 2m}{2N} \right) W_{n-m}^{\gamma} \right\} \\
&\quad + g^{\gamma} \left\{ 4 \sin \pi \frac{3n - 2m}{2N} H_{n+m}^{\gamma} - 4 \sin \pi \frac{3n + 2m}{2N} H_{n-m}^{\gamma} \\
&\quad - 2 \left( \cos \pi \frac{n}{2N} \sin \pi \frac{n}{N} + 2 \sin \pi \frac{3n - 2m}{2N} \right) H_{n+m}^{\gamma} \\
&\quad + 2 \left( \cos \pi \frac{n}{2N} \sin \pi \frac{n}{N} + 2 \sin \pi \frac{3n + 2m}{2N} \right) H_{n-m}^{\gamma} \\
&\quad + 2 \sin \pi \frac{n}{N} \cos \pi \frac{m}{2N} \cos \pi \frac{n - m}{2N} H_{n+m}^{\gamma} \\
&\quad - 2 \sin \pi \frac{n}{N} \cos \pi \frac{m}{2N} \cos \pi \frac{n + m}{2N} H_{n-m}^{\gamma} \right\},
\end{align*}
\tag{E.6}
\]

\[
\begin{align*}
\left[ W_n^{\alpha, 1}, W_m^{\beta, 1} \right] &= f_{\gamma} \left\{ 2 \cos \pi \frac{3n - 3m}{2N} W_{n+m}^{\gamma} + 2 \cos \pi \frac{3n + 3m}{2N} W_{n-m}^{\gamma} \\
&\quad - \frac{1}{2} \left( \cos \pi \frac{n - m}{2N} + 3 \cos \pi \frac{3n - 3m}{2N} \right) W_{n+m}^{\gamma} \\
&\quad + \frac{1}{2} \left( \cos \pi \frac{n + m}{2N} + 3 \cos \pi \frac{3n + 3m}{2N} \right) W_{n-m}^{\gamma} \right\} \\
&\quad + \frac{1}{2} \cos \pi \frac{n - m}{2N} \sin \pi \frac{n}{N} \sin \pi \frac{m}{N} W_{n+m}^{\gamma} \\
&\quad + \frac{1}{2} \cos \pi \frac{n + m}{2N} \sin \pi \frac{n}{N} \sin \pi \frac{m}{N} W_{n-m}^{\gamma} \right\}
\end{align*}
\]
\[ g^{\alpha \beta} \left( 4 \sin \frac{3n - 3m}{2N} H_{n+m}^5 - 4 \sin \frac{3n + 3m}{2N} H_{n-m}^5 \\ - \left( \sin \frac{n - m}{2N} + 5 \sin \frac{3n - 3m}{2N} \right) H_{n+m}^3 \\ + \left( \sin \frac{n + m}{2N} + 5 \sin \frac{3n + 3m}{2N} \right) H_{n-m}^3 \\ - \frac{1}{4} \left( \cos \frac{n + m}{N} \sin \frac{n - m}{2N} - 3 \sin \frac{n - m}{2N} \\ - 5 \sin \frac{3n - 3m}{2N} \right) H_{n+m}^1 \\ + \frac{1}{4} \left( \cos \frac{n - m}{N} \sin \frac{n + m}{2N} - 3 \sin \frac{n + m}{2N} \\ - 5 \sin \frac{3n + 3m}{2N} \right) H_{n-m}^1 \right). \] (E.7)

We note that the commutator \([W_n^{\alpha,1}, W_m^{\beta,0}]\) can be obtained from \([W_n^{\alpha,0}, W_m^{\beta,1}]\) by replacing \(\alpha\) with \(\beta\) and \(n\) with \(m\) and noting that \(H_{r/n} = (-1)^r H_n^r\) and \(W_{-n} = (-1)^r W_n^r\).

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