Nonlocal quadratic Poisson algebras, monodromy map, and Bogoyavlensky lattices

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Abstract. A new Lax representation for the Bogoyavlensky lattice is found, its $r$–matrix interpretation is elaborated. The $r$–matrix structure turns out to be related to a highly nonlocal quadratic Poisson structure on a direct sum of associative algebras. The theory of such nonlocal structures is developed, the Poisson property of the monodromy map is worked out in the most general situation. Some problems concerning the duality of Lax representations are raised.
1 Introduction

In this work we find some new results on the well known integrable system, the Bogoyavlensky lattice. However, these results, dealing with one particular system, allow to touch two general problems of the Hamiltonian theory of integrable systems.

The first of these problems is the nature of the most general quadratic Poisson brackets on associative algebras (or, in a different setting, on Lie groups). These brackets were invented and used for various purposes in [1], [17], [20], [21], [22], [23], [15]. They serve as a wide generalization of the so called Sklyanin bracket [13], [18]. The characteristic property of the Sklyanin bracket in the Lie groups setting is that it defines a Lie–Poisson structure, i.e. the group multiplication is a Poisson map with respect to this bracket. In the framework of associative algebras the corresponding property may be formulated in the following manner. Let the algebra \( g \) carry the Sklyanin bracket, consider the direct sum \( g = g^{(1)} \oplus \ldots \oplus g^{(n)} \) (\( g^{(k)} \) are \( n \) copies of \( g \)), and supply \( g \) with the direct sum of the Sklyanin brackets. Then the monodromy map \( g \mapsto g \), defined as \((u_1, \ldots, u_n) \mapsto u_n \cdot \ldots \cdot u_1\), has the Poisson property. Although several generalizations of this statement are available in the literature, the most general version still has not appeared, presumably because of the lack of interesting examples.

The most examples where the monodomy map is of interest, are connected with the lattice models, where each \( g^{(k)} \) in the direct sum \( g \) is attached to the \( k \)th lattice site and plays the role of the phase space of the \( k \)th particle. In all known examples the Poisson bracket on \( g \) is either ultralocal (the functions depending only on the \( g^{(k)} \) are in involution for different \( k \)), or non–ultralocal, but still with some locality properties (usually the functions on \( g^{(k)} \) are in involution with the functions on \( g^{(j)} \), unless \(|j − k| ≤ 1|\).

In this paper we establish the Poisson property of the monodromy map in the most general situation (see Proposition 5 and Theorem 1). Namely, we allow an arbitrary nonlocality, when the functions on \( g^{(k)} \) do not commute with the functions on \( g^{(j)} \), irrespectively of the value of \(|j − k| \). We give also a rather spectacular example of the situation where our construction works (and is natural and necessary) – the well known Bogoyavlensky lattice. It is interesting to note that in this example different \( g^{(k)} \)'s are attached not to single lattice sites, but to sets of equidistant sites (or, in other terminology, to different sorts of particles).
Not very surprisingly, to obtain these new results for an old system, we have to deal with a novel Lax representation for it. Here we touch the second general problem: the duality. It is well known that often one and the same integrable system admits two (or more) Lax representations, living in different Lie algebras. In particular, the Lax matrices of these representations have different dimensions.

If one knows a Lax representation for a system at hand, then it is often possible to find an alternative Lax representation by the following trick. Take a characteristic polynomial of the known Lax matrix. It serves as a generating function for the integrals of motion. Using some determinantal identities, one can often represent this function as a characteristic polynomial of some other matrix. Then there is a good chance that this matrix is indeed a (new) Lax matrix, i.e. that it takes part in a Lax representation for our system.

One of the determinantal identities often used by such transformations, is the so called Weinstein–Aronszajn formula [12], [1]. A novel feature of the present work is that instead of the latter formula we use the Laplace expansion in order to find new Lax matrices.

What is however very important and not provided by determinantal identities, is establishing a correspondence (if possible, possessing certain Poisson properties) between different Lax matrices, rather then between their characteristic polynomials. To my knowledge, it was done only in the framework of linear Poisson brackets [1]. It would be highly desirable to work out such correspondences in various different contexts, e.g., if one of the Lax matrices belongs to a linear and another to a quadratic Poisson algebra (the most prominent example being the usual Toda lattice, cf. [18], [6]), or if the both matrices belong to quadratic Poisson algebras (e.g., the relativistic Toda lattice, cf. [24], [25], [26]).

The example elaborated in the present paper belongs to the latter class (both algebras quadratic), and has two additional interesting features. First, the Poisson structure of the first (old) Lax formulation includes a Dirac reduction, and, second, the dimensions of the both Lax matrices (old and new) grow with the number of particles.
2 Quadratic brackets on associative algebras

Let $g$ be an associative algebra equipped with a nondegenerate scalar product $\langle \cdot, \cdot \rangle$ which is invariant in the following sense:

$$\langle u \cdot v, w \rangle = \langle u, v \cdot w \rangle. \quad (2.1)$$

The gradient $\nabla \varphi \in g$ of a smooth function $\varphi$ on $g$ is defined by the following relation:

$$\langle \nabla \varphi(u), X \rangle = \left. \frac{d}{d\varepsilon} \varphi(u + \varepsilon X) \right|_{\varepsilon = 0} \quad \forall X \in g. \quad (2.2)$$

Denote also

$$d \varphi(u) = u \cdot \nabla \varphi(u), \quad d' \varphi(u) = \nabla \varphi(u) \cdot u. \quad (2.3)$$

The following ”hierarchy” of quadratic brackets on $g$ is known.

1) The Sklyanin bracket [19, 20]:

$$\{ \varphi, \psi \}(u) = \langle R(d' \varphi), d' \psi \rangle - \langle R(d \varphi), d \psi \rangle.$$

The linear operator $R$ on $g$ has to be skew–symmetric:

$$R^* = -R$$

in order to assure the skew–symmetry of the bracket; here $^\ast$ denotes the adjoint operator with respect to the scalar product $\langle \cdot, \cdot \rangle$. A sufficient condition for this bracket to satisfy the Jacobi identity is the so called modified Yang–Baxter equation for the operator $R$:

$$[R(X), R(Y)] = R([R(X), Y] + [X, R(Y)]) - \alpha [X, Y] \quad \forall X, Y \in g \quad (2.4)$$

with some constant $\alpha$. We shall denote this condition by mYB($R; \alpha$).

2) The Li–Parmentier–Oevel–Ragnisco bracket [10, 16]:

$$\{ \varphi, \psi \}(u) = \langle A(d' \varphi), d' \psi \rangle - \langle A(d \varphi), d \psi \rangle + \langle S(d \varphi), d' \psi \rangle - \langle S(d' \varphi), d \psi \rangle.$$

For the skew–symmetry of this bracket one needs

$$A^* = -A, \quad S^* = S.$$
A sufficient condition for the Jacobi identity is given by \( m_{YB}(A; \alpha) \) and an additional property of the linear operators \( A, S \):
\[
[S(X), S(Y)] = S\left([A(X), Y] + [X, A(Y)]\right) \quad \forall X, Y \in g. \tag{2.5}
\]
We shall denote this condition by \( \text{Hom}(S, A) \). It turns out that the set of two conditions \( m_{YB}(A; \alpha), \text{Hom}(S, A) \) is equivalent also to the set of two conditions \( m_{YB}(A; \alpha), m_{YB}(A + S; \alpha) \).

3) The most general quadratic bracket on \( g \) [7], [17], [26]:
\[
\{\varphi, \psi\}(u) = \langle A(d'\varphi), d'\psi \rangle - \langle D(d\varphi), d\psi \rangle + \langle B(d\varphi), d'\psi \rangle - \langle C(d'\varphi), d\psi \rangle. \tag{2.6}
\]
We shall denote this expression by \( \text{PB}(A, B, C, D) \). For the skew–symmetry one needs the following conditions on the linear operators \( A, B, C, D \):
\[
A^* = -A, \quad D^* = -D, \quad B^* = C. \tag{2.7}
\]

**Proposition 1** Sufficient conditions for (2.6) to be a Poisson bracket are given by the conditions
\[
m_{YB}(A; \alpha), \quad m_{YB}(D; \alpha), \quad \text{Hom}(B, D), \quad \text{Hom}(C, A). \tag{2.8}
\]
If
\[
A + B = C + D = R, \tag{2.9}
\]
then the above sufficient conditions are equivalent to
\[
m_{YB}(A; \alpha), \quad m_{YB}(D; \alpha), \quad m_{YB}(R; \alpha). \tag{2.10}
\]
Under conditions given in the Proposition 1, following Hamiltonian system on \( g \) is defined for every smooth function \( \varphi \):
\[
\dot{u} = \{\varphi, u\} = u \cdot A(d'\varphi(u)) - D(d\varphi(u)) \cdot u \\
+ u \cdot B(d\varphi(u)) - C(d'\varphi(u)) \cdot u. \tag{2.11}
\]
If (2.9) is fulfilled, then this equation takes the Lax form for \( Ad \)–invariant Hamiltonian functions \( \varphi \): for such functions
\[
d\varphi(u) = d'\varphi(u) \tag{2.12}
\]
and we get the equation in the Lax form:
\[
\dot{u} = \{\varphi, u\} = [u, R(d\varphi(u))]. \tag{2.13}
\]
Proposition 2 Under the condition (2.9) Ad–invariant functions are in involution with respect to the bracket \{\cdot, \cdot\}, hence each Ad–invariant function is an integral of motion for (2.13).

3 Poisson brackets on direct sums

Consider a "big" algebra \( g = \bigoplus_{k=1}^{n} g \), a direct sum of \( n \) copies of the algebra \( g \). So, the multiplication in \( g \) is componentwise: if \( u = (u_1, \ldots, u_n) \in g \), \( v = (v_1, \ldots, v_n) \in g \), then
\[ u \cdot v = (u_1 \cdot v_1, \ldots, u_n \cdot v_n). \]

Define the (nondegenerate, invariant) scalar product \( \langle\langle \cdot, \cdot \rangle\rangle \) on \( g \) as
\[ \langle\langle u, v \rangle\rangle = \sum_{k=1}^{n} (u_k, v_k). \]

Now let \( A, B, C, D \) be linear operators on \( g \) satisfying conditions analogous to (2.7) and to (2.8) (or (2.10)). Then one can define the bracket \( \text{PB}(A, B, C, D) \) on \( g \):
\[ \{ \Phi, \Psi \}(u) = \langle\langle A(d'\Phi), d'\Psi \rangle\rangle - \langle\langle D(d\Phi), d\Psi \rangle\rangle + \langle\langle B(d\Phi), d'\Psi \rangle\rangle - \langle\langle C(d'\Phi), d\Psi \rangle\rangle. \] (3.1)

To be able to deal with these objects, let us introduce following (natural) notations. Let \( A_i(u) \) be the \( i \)th component of \( A(u) \); then we set
\[ A_i(u) = \sum_{j=1}^{n} A_{ij}(u_j). \] (3.2)

Let also \( \nabla_j \Phi, d_j \Phi, d'_j \Phi \) denote the \( j \)th components of the corresponding objects for a smooth function \( \Phi(u) \) on \( g \).

Then the conditions (2.7) for the operators \( A, B, C, D \) read:
\[ A^*_i = -A_{ji}, \quad D^*_i = -D_{ji}, \quad B^*_i = C_{ji}. \]

The bracket (3.1) takes the form
\[ \{ \Phi, \Psi \}(u) = \sum_{i,j=1}^{n} \langle A_{ij}(d'_j \Phi), d'_i \Psi \rangle - \sum_{i,j=1}^{n} \langle D_{ij}(d_j \Phi), d_i \Psi \rangle + \sum_{i,j=1}^{n} \langle B_{ij}(d_j \Phi), d'_i \Psi \rangle - \sum_{i,j=1}^{n} \langle C_{ij}(d'_j \Phi), d_i \Psi \rangle \] (3.3)
It is interesting to reformulate the conditions of the Proposition 1 for the operators $A, B, C, D$ acting on $g$ in terms of the operators $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ acting on $g$.

**Proposition 3** The condition $mYB(A; \alpha)$ for a skew symmetric operator $A$ is equivalent to the following set of conditions:

1) $n$ equations $mYB(A_{jj}; \alpha)$;
2) $n(n-1)$ equations $\text{Hom}(A_{ij}, A_{jj})$ (for all $i \neq j$);
3) $n(n-1)(n-2)/6$ conditions (for all $i < j < k$):

$$[A_{ij}(X), A_{ik}(Y)] = A_{ik}([A_{kj}(X), Y]) + A_{ij}([X, A_{jk}(Y)]) \quad \forall X, Y \in g.$$

We shall denote the last condition by $\text{Aux}(A_{ij}, A_{ik}; A_{kj}, A_{jk})$.

**Proposition 4** The condition $\text{Hom}(C, A)$ is equivalent to the set of

1) $n^2$ conditions $\text{Hom}(C_{ij}, A_{jj})$ (for all $1 \leq i, j \leq n$);
2) $n^2(n-1)/2$ conditions $\text{Aux}(C_{ij}, C_{ik}; A_{kj}, A_{jk})$ (for all $1 \leq i, j, k \leq n$, $j < k$).

**Remark 1.** Let us comment, why in the Proposition 3 it is enough to require $\text{Aux}(A_{ij}, A_{ik}; A_{kj}, A_{jk})$ only for $i < j < k$. First, this condition is obviously (skew)symmetric with respect to the interchange $j \leftrightarrow k$. Further, it turns out to be symmetric also with respect to cyclic shifts of the triple $(i, j, k)$ of distinct indices. To demonstrate this, notice that due to nondegeneracy of the scalar product the above condition may be expressed as

$$\langle [A_{ij}(X), A_{ik}(Y)], Z \rangle = \langle A_{ik}([A_{kj}(X), Y]), Z \rangle + \langle A_{ij}([X, A_{jk}(Y)]), Z \rangle$$

for arbitrary $X, Y, Z \in g$. Using the skew–symmetry of the operator $A$ ($A_{ij} = -A_{ji}$) and invariance of the scalar product, we can transform this into

$$-\langle X, A_{ji}([A_{ik}(Y), Z]) \rangle = \langle X, A_{jk}([Y, A_{ki}(Z)]) \rangle - \langle X, [A_{jk}(Y), A_{ji}(Z)] \rangle$$

for arbitrary $X, Y, Z \in g$. Again, due to nondegeneracy of the scalar product this is equivalent to

$$-A_{ji}([A_{ik}(Y), Z]) = A_{jk}([Y, A_{ki}(Z)]) - [A_{jk}(Y), A_{ji}(Z)]$$

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for arbitrary $Y, Z \in g$. This is exactly $\text{Aux}(A_{jk}, A_{ji}; A_{ik}, A_{ki})$, i.e. the above condition for the triple $(j, k, i)$.

**Remark 2.** For $n = 2$ we see that the $mYB(A; \alpha)$ for the skew symmetric operator

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

on $g \oplus g$ is equivalent to the set of four conditions

$$mYB(A_{11}; \alpha), \ mYB(A_{22}; \alpha), \ \text{Hom}(A_{12}, A_{22}), \ \text{Hom}(A_{21}, A_{11})$$

for the operators on $g$. Comparing with the Proposition 1, we see that it is exactly the sufficient conditions for $\text{PB}(A_{11}, A_{12}, A_{21}, A_{22})$ to be a Poisson bracket. This coincidence was first pointed out in [21].

### 4 Poisson properties of the monodromy map

An interesting and important question arising in connection with the Poisson brackets on the direct sums is the question on the Poisson properties of the monodromy maps.

**Proposition 5** Let $g$ be equipped with the Poisson bracket $\text{PB}(A, B, C, D)$. Suppose that the following relations hold:

$$A_{i+1,j+1} - D_{i,j} + B_{i+1,j} - C_{i,j+1} = 0 \quad \text{for} \quad 1 \leq i, j \leq n - 1;$$

$$A_{1,j+1} + B_{1,j} = 0 \iff A_{j+1,1} - C_{j,1} = 0 \quad \text{for} \quad 1 \leq j \leq n - 1;$$

$$D_{n,j} + C_{n,j+1} = 0 \iff D_{j,n} - B_{j+1,n} = 0 \quad \text{for} \quad 1 \leq j \leq n - 1.$$

Then the map

$$\mathcal{M} : g \mapsto g, \quad \mathcal{M}(u) = \mathcal{M}(u_1, \ldots, u_n) = u_n \cdot \ldots \cdot u_1$$

(4.1)

is Poisson, if $g$ is equipped with $\text{PB}(A_{11}, B_{1n}, C_{n1}, D_{nn})$.

**Proof.** Take two smooth functions $\varphi, \psi$ on $g$, and form two functions $\Phi, \Psi$ on $g$ according to

$$\Phi(u) = \varphi(T), \quad \Psi(u) = \psi(T),$$

(4.2)
where
\[ T = \mathcal{M}(u) = u_n \cdot \ldots \cdot u_1. \]

It is easy to see that
\[ \nabla_j \Phi(u) = u_{j-1} \cdot \ldots \cdot u_1 \cdot \nabla \varphi(T) \cdot u_n \cdot \ldots \cdot u_{j+1}. \]

Consequently,
\begin{align*}
  d_j \Phi(u) &= u_j \cdot \nabla_j \Phi(u) = u_j \cdot \ldots \cdot u_1 \cdot \nabla \varphi(T) \cdot u_n \cdot \ldots \cdot u_{j+1}, \quad \text{(4.2)} \\
  d_j' \Phi(u) &= \nabla_j \Phi(u) \cdot u_j = u_{j-1} \cdot \ldots \cdot u_1 \cdot \nabla \varphi(T) \cdot u_n \cdot \ldots \cdot u_j. \quad \text{(4.3)}
\end{align*}

In particular,
\[ d_n \Phi(u) = d\varphi(T), \quad d_1' \Phi(u) = d'\varphi(T), \]

and for \( 1 \leq j \leq n - 1 \) we have
\[ d_j \Phi(u) = d_{j+1}' \Phi(u). \]

Substituting the last two formulas into (3.3), we see that under the conditions of the Proposition, almost all the terms cancel, leaving us with
\[ \{\Phi, \Psi\}(u) = \langle A_{11}(d'\varphi(T)), d'\psi(T) \rangle - \langle D_{nn}(d\varphi(T)), d\psi(T) \rangle \]
\[ + \langle B_{1n}(d\varphi(T)), d'\psi(T) \rangle - \langle C_{n1}(d'\varphi(T)), d\psi(T) \rangle, \]

which proves the Proposition. \( \blacksquare \)

A further important observation is related to the form of the Lax equations on \( \mathfrak{g} \) in the case when the Hamiltonian function has the form
\[ \Phi(u) = \varphi(\mathcal{M}(u)) = \varphi(u_n \cdot \ldots \cdot u_1), \quad \text{(4.4)} \]

and \( \varphi \) is an \( \text{Ad} \)-invariant function on \( \mathfrak{g} \). Under this condition we can represent the formulas (4.2), (4.3) as
\begin{align*}
  d_j \Phi(u) &= d\varphi(T_j), \quad d_j' \Phi(u) = d'\varphi(T_{j-1}), \quad \text{(4.5)}
\end{align*}

where
\[ T_j = u_j \cdot \ldots \cdot u_1 \cdot u_n \cdot \ldots \cdot u_{j+1} \]

(so that, in particular, \( T_0 = T_n = T = \mathcal{M}(u) \)).
Proposition 6 Let $g$ be equipped with the Poisson bracket $\text{PB}(A, B, C, D)$. Then the Hamiltonian equations of motion generated by the Hamiltonian function (4.4) with an $\text{Ad}$–invariant function $\varphi$, may be presented in the form
\[ \dot{u}_i = u_i R_i - L_i u_i, \] (4.6)
where
\[ R_i = \sum_{j=1}^{n} (A_{i,j+1} + B_{i,j}) (d\varphi(T_j)), \]
\[ L_i = \sum_{j=1}^{n} (D_{i,j} + C_{i,j+1}) (d\varphi(T_j)), \]
(in these formulas the subscripts should be taken (mod $n$), so that $A_{i,n+1} = A_{i,1}$ and $C_{i,n+1} = C_{i,1}$).

Proof. This follows immediately from the equations of motion analogous to (2.11), the notation (3.2), and the formulas (4.5). \[ \blacksquare \]

The most important and interesting for applications is the situation, when not only the monodromy map $\mathcal{M}$ is Poisson, but also its compositions with the different powers of the shift
\[ \sigma : g \mapsto g, \quad \sigma(u_1, \ldots, u_{n-1}, u_n) = (u_2, \ldots, u_n, u_1). \] (4.7)

Shifting subscripts in the conditions of the Proposition 5, we arrive at the following fundamental statement.

Theorem 1 Let $g$ be equipped with the Poisson bracket $\text{PB}(A, B, C, D)$. Suppose that the following relations hold:
\[ A_{i+1,j+1} = -B_{i+1,j} = C_{i,j+1} = -D_{i,j} \quad \text{for} \quad i \neq j; \]
\[ A_{j+1,j+1} - D_{j,j} + B_{j+1,j} - C_{j,j+1} = 0 \quad \text{for all} \quad 1 \leq j \leq n. \]
Then each map
\[ \mathcal{M} \circ \sigma^j : g \mapsto g, \quad \mathcal{M} \circ \sigma^j(u) = T_j \] (4.8)
is Poisson, if $g$ is equipped with the Poisson bracket
\[ \text{PB}(A_{j+1,j+1}, B_{j+1,j}, C_{j,j+1}, D_{j,j}). \]
The Hamiltonian equations of motion generated by the Hamiltonian function (4.4) with an Ad–invariant function $\varphi$, are of the form

$$\dot{u}_j = u_j \mathcal{L}_{j-1} - \mathcal{L}_j u_j, \quad \mathcal{L}_j = R_j(d\varphi(T_j)),$$

where

$$R_j = A_{j+1,j+1} + B_{j+1,j} = D_{j,j} + C_{j,j+1}.$$ (4.9)

(In all the formulas the subscripts should be taken (mod $n$)).

This Theorem may be considered as a final link in the chain of generalizations [19], [20], [10], [11], [9], [14], [22], [23]. However, for the non–ultralocal quadratic Poisson structures on direct sums discussed in these papers the operators $A, B, C, D$ always had only few nonvanishing ”operator entries”, namely $A_{j,j}, D_{j,j}, B_{j+1,j}$, and $C_{j,j+1}$. The ”most nonlocal” example (though only with $n = 2$) appeared in [26] in connection with the relativistic Toda lattice.

5 Basic algebras and operators

Two sorts of algebras will play the basic role in our presentation. They are well suited to describe various lattice systems with the so called open–end and periodic boundary conditions, respectively.

1) For the open–end case we set $g = gl(N)$, the algebra of $N \times N$ matrices

$$u = \sum_{i,j=1}^{N} u_{ij} E_{ij}.$$ (5.1)

The (nondegenerate, invariant) scalar product is choosen as

$$\langle u, v \rangle = \text{tr}(u \cdot v).$$ (5.2)

2) For the periodic case $g \subset gl(N)[\lambda, \lambda^{-1}]$ is a twisted subalgebra of the loop algebra $gl(N)[\lambda, \lambda^{-1}]$, consisting of formal semiinfinite Laurent series over $gl(N)$ satisfying

$$u(\lambda) = \sum_{p \in \mathbb{Z}} \sum_{i-j \neq p (\text{mod } N)} \lambda^p u_{ij}^{(p)} E_{ij}.$$ (5.3)
The scalar product is chosen as
\[
\langle u(\lambda), v(\lambda) \rangle = \text{tr}(u(\lambda) \cdot v(\lambda))_0,
\]
the subscript 0 denoting the free term of the formal Laurent series.

In these cases of matrix algebras the Poisson bracket \( \text{PB}(A, B, C, D) \) may be written in a fine tensor form.

1) In the open–end case functions \( u_{ij} \) form the functional basis of the set of functions on \( g \). It is easy to see that \( \nabla u_{ij} = E_{ji} \). For a linear operator \( R \) on \( g \) define the corresponding \( N^2 \times N^2 \) matrix \( r \):
\[
r = \sum_{i,j,k,l} \langle R(\nabla u_{ij}), \nabla u_{kl} \rangle E_{ij} \otimes E_{kl}.
\]
Then it is easy to check that the pairwise Poisson brackets of the coordinate functions may be cast into the formula
\[
\{ u \otimes u \} = (u \otimes u) a - d (u \otimes u) + (I \otimes u) b (u \otimes I) - (u \otimes I) c (I \otimes u).
\]
Here the matrices \( a, b, c, d \) correspond to the operators \( A, B, C, D \) in the same manner, as \( r \) corresponds to \( R \).

2) Analogously, in the periodic case the functional basis of the set of functions on \( g \) is formed by the coordinates \( u_{ij}^{(p)} \), for which \( \nabla u_{ij}^{(p)} = \lambda^{-p} E_{ji} \). The \( n^2 \times N^2 \) matrix \( r(\lambda, \mu) \) depending on two parameters \( \lambda, \mu \), corresponding to a linear operator \( R \), is now defined as
\[
r(\lambda, \mu) = \sum_{p,q} \sum_{i-j \equiv p \ (\text{mod} N)} \sum_{k-l \equiv q \ (\text{mod} N)} \langle R(\nabla u_{ij}^{(p)}), \nabla u_{kl}^{(q)} \rangle \lambda^p \mu^q E_{ij} \otimes E_{kl}.
\]
The pairwise Poisson brackets of the coordinate functions may be presented in the form analogous to (5.6):
\[
\{ u(\lambda) \otimes u(\mu) \}
= (u(\lambda) \otimes u(\mu)) a(\lambda, \mu) - d(\lambda, \mu) (u(\lambda) \otimes u(\mu))
+ (I \otimes u(\mu)) b(\lambda, \mu) (u(\lambda) \otimes I) - (u(\lambda) \otimes I) c(\lambda, \mu) (I \otimes u(\mu)).
\]
representation of the bracket \((3.3)\) reads:

\[
\{u_j(\lambda) \otimes u_i(\mu)\} = (u_j(\lambda) \otimes u_i(\mu)) a_{ij}(\lambda,\mu) (u_j(\lambda) \otimes u_i(\mu)) \\
+ (I \otimes u_i(\mu)) b_{ij}(\lambda,\mu) (u_j(\lambda) \otimes I) - (u_j(\lambda) \otimes I) c_{ij}(\lambda,\mu) (I \otimes u_i(\mu)).
\]

(In the open-end case one has simply to omit everywhere the spectral parameters \(\lambda, \mu\).)

The usefulness of the tensor notation lies in that it provides us with an efficient method of finding Poisson submanifolds for the bracket \(PB(A, B, C, D)\). Suppose that there is a Poisson space \(S\) with a Poisson bracket \(\{ , \}_0\) and a local coordinates \(s\), and a map \(T(s) : S \mapsto g\). Suppose also that the matrix \(\{T(s) \otimes T(s)\}_0\) composed of pairwise Poisson brackets of entries of the matrix \(T(s)\), may be presented in the form \((5.6)\) or \((5.8)\), respectively (naturally, in terms of \(T(s)\) instead of \(u\)). Then the set \(T(S) \subset g\) is a Poisson submanifold for the bracket \(PB(A, B, C, D)\). For a simple proof see \([26]\).

Now we introduce several operators which will be widely used in the following presentation.

1) In the open-end case \(g\) as a linear space may be presented as a direct sum

\[
g = \bigoplus_{k=-(N+1)}^{N-1} g_k,
\]

where \(g_k\) is the set of matrices with nonzero entries only on the \(k\)th diagonal, i.e. in the positions \((i,j)\) with \(i-j=k\). In particular, \(g_0\) is a set of diagonal matrices, which serves as a commutative subalgebra of \(g\). The other two obvious subalgebras are:

\[
g_+ = \bigoplus_{k=1}^{M-1} g_k, \quad g_- = \bigoplus_{k=-M+1}^{-1} g_k,
\]

i.e. the sets of strictly lower and upper triangular matrices.

2) In the periodic case \(g\) as a linear space is a direct sum

\[
g = \bigoplus_{k \in \mathbb{Z}} g_k,
\]

where \(g_k\) is the set of monomial matrices from \(g\) containing only the power \(\lambda^k\). Again \(g_0\) is a commutative subalgebra of \(g\). The two other subalgebras
are:
\[ g_+ = \bigoplus_{0<k<\infty} g_k, \quad g_- = \bigoplus_{k<0} g_k, \]
i.e. the sets of Laurent series with positive and with negative powers of \( \lambda \), respectively.

In the both cases, \( g \) as a linear space is a direct sum of three subspaces \( g_+ \), \( g_- \), \( g_0 \), and we denote the corresponding projections by \( P_+ \), \( P_- \), \( P_0 \), respectively. (Obviously, as a set \( g_0 \) is identical in both the open–end and the periodic case, but, of course, \( P_0 \) has different meaning in these two cases).

Define a linear operator \( R_0 \) on \( g \) by
\[ R_0 = P_+ - P_- . \quad (5.9) \]
Define also a skew–symmetric linear operator \( \Pi \) on \( g_0 \) by
\[ \Pi(E_{jj}) = \sum_{k=1}^{N} \pi_{jk} E_{kk} , \quad \pi_{jk} = \begin{cases} 1, & j > k \\ -1, & j < k \\ 0, & j = k \end{cases} , \quad (5.10) \]
and continue \( \Pi \) on the whole \( g \) according to \( \Pi = \Pi \circ P_0 \).

Then the following operators satisfy all conditions of the Proposition 1:
\[ A = \frac{1}{2}(R_0 + \Pi) , \quad D = \frac{1}{2}(R_0 - \Pi) , \]
\[ B = \frac{1}{2}(P_0 - \Pi) , \quad C = \frac{1}{2}(P_0 + \Pi) , \quad (5.11) \]
so that the Poisson bracket \( \text{PB}(A, B, C, D) \) is defined. Note that these operators have the property \( (2.9) \):
\[ A + B = C + D = R = \frac{1}{2}(R_0 + P_0) . \quad (5.12) \]

An important property of these operators was established in [27].

Proposition 7 Fix an element \( \mathcal{E} \in g_1 \):
\[ \mathcal{E} = \sum_{k=1}^{N-1} E_{k+1,k} \quad \text{or} \quad \mathcal{E} = \lambda \sum_{k=1}^{N} E_{k+1,k} \]
in the open–end and periodic case, respectively. Then for an arbitrary natural number \( m \geq 2 \) the set
\[ \mathcal{P}_{m-1} = \mathcal{E} \bigoplus g_{-j} \quad (5.13) \]
is a Poisson submanifold for the bracket \( \text{PB}(A, B, C, D) \).
To conclude this section we give also the expressions for the matrices $r_0$, $p_0$, $\pi$ corresponding to the operators $R_0$, $P_0$, $\Pi$, respectively. The expression for the $r_0$ depends on the case. In the open–end case:

$$r_0 = \sum_{i<j} E_{ij} \otimes E_{ji} - \sum_{i>j} E_{ij} \otimes E_{ji}. \quad (5.14)$$

In the periodic case:

$$r_0(\lambda, \mu) = \frac{\lambda^N + \mu^N}{\lambda^N - \mu^N} \sum_{i=1}^N E_{ii} \otimes E_{ii} + \sum_{p=1}^{N-1} \frac{2\lambda^p\mu^{N-p}}{\lambda^N - \mu^N} \sum_{i-j \equiv p \mod N} E_{ij} \otimes E_{ji}. \quad (5.15)$$

The expressions for $p_0$, $\pi$ are identical in both cases:

$$p_0 = \sum_{i=1}^N E_{ii} \otimes E_{ii}, \quad (5.16)$$

$$\pi = \sum_{i,j=1}^N \pi_{ij} E_{ii} \otimes E_{jj}. \quad (5.17)$$

The matrices $a, b, c, d$ corresponding to the operators $A, B, C, D$ are given in the periodic case by

$$a(\lambda, \mu) = \frac{1}{2} \left( r_0(\lambda, \mu) + \pi \right), \quad d(\lambda, \mu) = \frac{1}{2} \left( r_0(\lambda, \mu) - \pi \right),$$

$$b = \frac{1}{2} (p_0 - \pi), \quad c = \frac{1}{2} (p_0 + \pi), \quad (5.18)$$

and in the open–end case one has to omit the spectral parameters $\lambda, \mu$.

6 Bogoyavlensky lattice

We will be studying the following integrable lattice system, known as Bogoyavlensky lattice [2] (although it was introduced earlier in [13], and its certain special case appeared also in [3]):

$$\dot{z}_k = z_k \left( \sum_{j=1}^{m-1} z_{k+j} - \sum_{j=1}^{m-1} z_{k-j} \right). \quad (6.1)$$
It may be considered on an infinite lattice (all the subscripts belong to \( \mathbb{Z} \)), and admits also finite-dimensional reductions of two types: open-end and periodic. We shall denote the number of lattice cites (particles) in the finite reductions by \( M - m + 1 \) in the open-end case, and by \( M \) in the periodic case, for the reasons which will become clear soon. The boundary conditions in the open-end case are:

\[
z_k = 0 \quad \text{for} \quad k \leq 0 \quad \text{and} \quad k \geq M - m + 2;
\]

in the periodic case all the subscripts belong to \( \mathbb{Z}/M\mathbb{Z} \).

A Hamiltonian interpretation of this system was found in [2]: the system (6.1) is Hamiltonian with the Hamiltonian function

\[
H(z) = \sum z_k
\]

with respect to the Poisson bracket given by:

\[
\{z_j, z_k\} = \begin{cases} 
z_j z_k, & j - k = 1, \ldots, m - 1 \\
-z_j z_k, & j - k = -m + 1, \ldots, -1 \\
0, & \text{else}
\end{cases}
\]

(in the periodic case with \( M > 2m - 2 \) the conditions of the type \( j - k = 1, \ldots, m - 1 \) have to be understood (mod \( M \)).

Bogoyavlensky has found also the Lax representations for this system:

\[
\dot{T} = [T, P],
\]

where

\[
T(z, \lambda) = \lambda^{-m+1} \sum z_k E_{k, k+m-1} + \lambda \sum E_{k+1, k},
\]

\[
P(z, \lambda) = \sum (z_k + z_{k-1} + \ldots + z_{k-m+1}) E_{kk} + \lambda^m \sum E_{k+m, k}.
\]

Here for the infinite lattices all the subscripts belong to \( \mathbb{Z} \), for the periodic case all the subscripts belong to \( \mathbb{Z}/M\mathbb{Z} \), and for the open-end case all the subscripts belong to \( 1, \ldots, M \) (so that for the both types of finite reductions the matrices involved in the Lax representations are \( M \times M \)). Moreover, in the infinite-dimensional and open-end cases the dependence on the spectral parameter \( \lambda \) becomes inessential and may be suppressed by setting \( \lambda = 1 \). Below we consider only finite lattices. It can be said that in these cases the Lax matrix \( T \) belongs to the algebra \( g \), which is defined exactly as \( g \) in the
previous section with the only difference: $N$ should be everywhere replaced by $M$.

In [27] we gave an $r$–matrix interpretation of the Bogoyavlensky lattices. The main results of [27] concerning the lattice (6.1) may be summarized as follows. Let the operators $A, B, C, D$ on $g$ be defined as in the previous section, with $N$ replaced by $M$. Define also a Poisson bracket $\text{PB}(A, B, C, D)$ on $g$. Then, according to the Proposition 7, the set $\mathcal{P}_{m-1} \subset g$ defined as in (5.13), is a Poisson submanifold. Obviously, the set $\mathcal{T}_{m-1}$ consisting of the Lax matrices (6.5), is a subset of $\mathcal{P}_{m-1}$.

**Proposition 8** The Dirac reduction of the bracket $\text{PB}(A, B, C, D)$ to the subset $\mathcal{T}_{m-1} \subset \mathcal{P}_{m-1}$ coincides with the bracket (6.3). The Dirac correction to the Hamiltonian vector field $\{\varphi, u\}$, when reducing it to the set $\mathcal{T}_{m-1}$, vanishes, if $\varphi(u) = \psi(u^m)$ with some $\text{Ad}$–invariant function $\psi$.

This Proposition explains the Lax equation (6.4), because the Hamiltonian function (6.2) can obviously be seen as

$$H = \frac{1}{m} \text{tr}(T^m), \quad \text{so that } dH = d'H = T^m,$$

and it is not difficult to check that for the matrix (6.3) there holds:

$$P = (P_+ + P_0)(T^m) = \left(R + \frac{1}{2} I\right)(T^m).$$

**7 Illustrative example: Volterra lattice**

We intend now to establish another Lax representation for the Bogoyavlensky lattices, alternative to (6.4). To illustrate our approach by the simple particular case, we start with the lattice (6.1) corresponding to $m = 2$, and known also as the Volterra (or Lotka–Volterra, or Langmuir) lattice:

$$\dot{z}_k = z_k(z_{k+1} - z_{k-1}), \quad (7.1)$$

The invariant Poisson bracket (6.3) for this system is:

$$\{z_{k+1}, z_k\} = z_{k+1} z_k, \quad (7.2)$$
and the corresponding Hamiltonian function

\[ H(z) = \sum_k z_k. \]  

(7.3)

We want to separate all the particles into two "sorts":

\[ u_k = z_{2k-1}, \quad v_k = z_{2k}. \]  

(7.4)

In terms of these two sorts of particles the equations of motion (7.1) may be put down as

\[ \dot{u}_k = u_k(v_k - v_{k-1}), \quad \dot{v}_k = v_k(u_{k+1} - u_k). \]  

(7.5)

The underlying Poisson bracket (7.2) is now

\[ \{v_k, u_k\} = v_k u_k, \quad \{u_{k+1}, v_k\} = u_{k+1} v_k, \]  

(7.6)

and the Hamiltonian function (7.3) is equal to

\[ H(u, v) = \sum_k (u_k + v_k). \]  

(7.7)

### 7.1 Periodic case

The notation (7.4) is consistent with the \(M\)-periodic boundary conditions, only if the total number of particles is even. Hence we consider in this subsection only the case

\[ M = 2N. \]  

(7.8)

The usual \(M \times M = 2N \times 2N\) Lax matrix for the Volterra lattice is:

\[ T(z, \lambda) = \lambda \sum_{k=1}^{2N} E_{k+1,k} + \lambda^{-1} \sum_{k=1}^{2N} z_k E_{k,k+1}. \]  

(7.9)

Its natural ambient space is the algebra \(g\) defined in the previous Section.

Our aim in this Section is to elaborate an alternative \(N \times N\) Lax representation for this system, naturally living in the algebra \(g\) whose definition was given in the Section 5.

Define two Lax matrices from \(g\):

\[ L_1(u, v, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_{k-1}) E_{kk} + \lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1}. \]  

(7.10)
\[ L_2(u, v, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_k) E_{kk} + \lambda^{-1} \sum_{k=1}^{N} u_{k+1} v_k E_{k,k+1}, \quad (7.11) \]

**Proposition 9**

1) The set of matrices \( L_1(u, v, \lambda) \) (or \( L_2(u, v, \lambda) \)) forms a Poisson submanifold in \( g \), if the latter is equipped with the Poisson bracket \( \text{PB}(A, B, C, D) \).

2) The Volterra lattice (7.5) admits two (equivalent) Lax representations

\[ \dot{L}_1 = [L_1, R(L_1)], \quad \dot{L}_2 = [L_2, R(L_2)]. \quad (7.12) \]

3) The function

\[ \det \left( L_1(u, v, \lambda) - \mu I_N \right) = \det \left( L_2(u, v, \lambda) - \mu I_N \right) \quad (7.13) \]

serves as a generating function of integrals of motion of the Volterra lattice. Moreover,

\[ \det \left( T^2(z, \lambda) - \mu I_M \right) = \det \left( L_1(u, v, \lambda^2) - \mu I_N \right) \det \left( L_2(u, v, \lambda^2) - \mu I_N \right). \quad (7.14) \]

**Proof.** We prove first of all the third statement, which will also give a motivation for considering the matrices (7.10), (7.11).

The generating function of integrals of motion for the periodic Volterra lattice, following from the \( M \times M \) Lax representation, can be chosen as

\[ \det(T^2(z, \lambda) - \mu I). \quad (7.15) \]

Here the matrix \( T^2(z, \lambda) \) has the following structure:

\[ T^2(z, \lambda) = \lambda^2 \sum_{k=1}^{2N} E_{k+2,k} + \sum_{k=1}^{2N} (z_k + z_{k-1}) E_{kk} + \lambda^{-2} \sum_{k=1}^{2N} z_k z_{k+1} E_{k,k+2}. \quad (7.16) \]

We use the Laplace formula to represent the determinant (7.15) as

\[
\det(T^2 - \mu I) = \sum_{i_1 < \ldots < i_N} \det(T^2 - \mu I) \begin{pmatrix} 1 & 3 & \ldots & 2N - 1 \\ i_1 & i_2 & \ldots & i_N \end{pmatrix} \\
\times \det(T^2 - \mu I) \begin{pmatrix} 2 & 4 & \ldots & 2N \\ j_1 & j_2 & \ldots & j_N \end{pmatrix} \quad (7.17)
\]
Here the sum is taken over all possible ordered \( N \)-tuples \((i_1, \ldots, i_N)\) of the natural numbers from \((1, \ldots, 2N)\), and the numbers \( j_1 < \ldots < j_N \) form the complement of \((i_1, \ldots, i_N)\) to \((1, \ldots, 2N)\). We use the notation

\[
\mathcal{A} \left( \begin{array}{cccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_N \\
\beta_1 & \beta_2 & \cdots & \beta_N 
\end{array} \right)
\]

for the submatrix of \(\mathcal{A}\) formed by the elements standing on the intersection of the rows \((\alpha_1, \ldots, \alpha_N)\) with the columns \((\beta_1, \ldots, \beta_N)\).

Now the crucial observation based on the explicit formula (7.16) and on the hypothesis (7.8) is the following: in fact in the huge sum (7.17) all but one terms vanish identically, so that we are left with

\[
\det(T^2 - \mu I) = \det(T^2 - \mu I) \begin{pmatrix} 1 & 3 & \cdots & 2N-1 \\ 1 & 3 & \cdots & 2N-1 \end{pmatrix} \times \det(T^2 - \mu I) \begin{pmatrix} 2 & 4 & \cdots & 2N \\ 2 & 4 & \cdots & 2N \end{pmatrix}.
\]

(7.18)

But it is easy to calculate that

\[
(T^2(z, \lambda) - \mu I_{2N}) \begin{pmatrix} 1 & 3 & \cdots & 2N-1 \\ 1 & 3 & \cdots & 2N-1 \end{pmatrix} = L_1(u, v, \lambda^2) - \mu I_N,
\]

(7.19)

\[
(T^2(z, \lambda) - \mu I_{2N}) \begin{pmatrix} 2 & 4 & \cdots & 2N \\ 2 & 4 & \cdots & 2N \end{pmatrix} = L_2(u, v, \lambda^2) - \mu I_N.
\]

(7.20)

This proves the formula (7.14). To prove the formula (7.13), notice that the matrices \(L_1, L_2\) are in fact connected by means of a similarity transformation, due to the following important observation:

\[
L_1(u, v, \lambda) = U(u, \lambda)V(v, \lambda), \quad L_2(u, v, \lambda) = V(v, \lambda)U(u, \lambda),
\]

(7.21)

where

\[
U(u, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} u_k E_{kk},
\]

(7.22)

\[
V(v, \lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1}.
\]

(7.23)

The third statement of the Proposition is herewith completely proved.
Turning to the first statement, we notice that the both matrices $L_1, L_2$ have the same structure as the Lax matrix of the Toda lattice (cf. [26]):

$$L(a, b, \lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} b_k E_{kk} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1}. \quad (7.24)$$

The expressions for the "Toda coordinates" $(a, b)$ are given for the matrix $L_1$ by

$$b_k = u_k + v_{k-1}, \quad a_k = u_k v_k, \quad (7.25)$$

and for the matrix $L_2$ by

$$b_k = u_k + v_k, \quad a_k = u_{k+1} v_k. \quad (7.26)$$

Now a very remarkable circumstance is that in the both parametrizations $(7.25), (7.26)$ the equations of motion $(7.5)$ imply the equations of motion of the Toda lattice:

$$\dot{b}_k = a_k - a_{k-1}, \quad \dot{a}_k = a_k (b_{k+1} - b_k), \quad (7.27)$$

while the Poisson brackets $(7.6)$ imply the quadratic Poisson brackets for the Toda lattice:

$$\{b_{k+1}, b_k\} = a_k, \quad \{a_{k+1}, a_k\} = a_{k+1} a_k, \quad \{b_k, a_k\} = -b_k a_k, \quad \{b_{k+1}, a_k\} = b_{k+1} a_k. \quad (7.28)$$

The corresponding Hamiltonian function $(7.7)$ in the both parametrizations is equal to

$$H = \sum_{k=1}^{N} b_k. \quad (7.29)$$

An $r$–matrix interpretation of the quadratic Poisson bracket $(7.28)$ for the Toda lattice was found in [26]. It was demonstrated there that the matrices $L(a, b, \lambda)$ form a Poisson submanifold in the algebra $g$ equipped with PB$(A, B, C, D)$. The proof in [26] consisted of a verification of the tensor form of this result:

$$\{L(\lambda) \circ L(\mu)\} = \left(\left[L(\lambda) \otimes L(\mu)\right] a(\lambda, \mu) - d(\lambda, \mu) \left[L(\lambda) \otimes L(\mu)\right]\right)$$

$$+ \left[I \otimes L(\mu)\right] b \left[L(\lambda) \otimes I\right] - \left[L(\lambda) \otimes I\right] c \left[I \otimes L(\mu)\right]. \quad (7.30)$$
In the formula (7.30) $L$ stands for the Lax matrix (7.27) of the Toda lattice, but it might as well stand for either of the Lax matrices (7.10), (7.11) of the Volterra lattice. This already proves the first statement of the Proposition. Taking into account that the Hamiltonian function (7.7) is equal to
\[ H = \varphi(L_1) = \varphi(L_2), \]
where
\[ \varphi(L) = \text{tr}(L_0) \quad \text{so that} \quad d\varphi(L) = L, \]
we see that the second statement of the Proposition is also proved. □

However, we want to re-derive the basic result (7.30) here from the viewpoint of the monodromy maps, which will simplify its proof and provide an additional information.

**Proposition 10** The pairs of matrices $(U, V)$ form a Poisson submanifold in the algebra $g = g \oplus g$ equipped with the Poisson bracket $PB(A, B, C, D)$, where
\[
A = \begin{pmatrix} A & -B \\ C & A \end{pmatrix}, \quad B = \begin{pmatrix} B & B \\ B & -C \end{pmatrix}, \\
C = \begin{pmatrix} C & C \\ C & -B \end{pmatrix}, \quad D = \begin{pmatrix} D & -C \\ B & D \end{pmatrix}.
\]

**Proof.** The proof of this statement consists of the direct verification of the following identities (each of them is simpler then (7.30), because they deal only with bidiagonal matrices, while (7.30) – with tridiagonal ones):

\[
\{U(\lambda) \triangledown U(\mu)\} = \left( U(\lambda) \otimes U(\mu) \right) a(\lambda, \mu) - d(\lambda, \mu) \left( U(\lambda) \otimes U(\mu) \right) \\
\left. + \left( I \otimes U(\mu) \right) b \left( U(\lambda) \otimes I \right) - \left( U(\lambda) \otimes I \right) c \left( I \otimes U(\mu) \right), \right.
\]

\[
\{V(\lambda) \triangledown V(\mu)\} = \left( V(\lambda) \otimes V(\mu) \right) a(\lambda, \mu) - d(\lambda, \mu) \left( V(\lambda) \otimes V(\mu) \right) \\
\left. - \left( I \otimes V(\mu) \right) c \left( V(\lambda) \otimes I \right) + \left( V(\lambda) \otimes I \right) b \left( I \otimes V(\mu) \right), \right.
\]

\[
\{U(\lambda) \triangledown V(\mu)\}
\]

21
\[
(U(\lambda) \otimes V(\mu)) c - b \left( U(\lambda) \otimes V(\mu) \right) \\
+ \left( I \otimes V(\mu) \right) b \left( U(\lambda) \otimes I \right) - \left( U(\lambda) \otimes I \right) c \left( I \otimes V(\mu) \right),
\]

\[
\{ V(\lambda) \diamond U(\mu) \}
= - \left( V(\lambda) \otimes U(\mu) \right) b + c \left( V(\lambda) \otimes U(\mu) \right) \\
+ \left( I \otimes U(\mu) \right) b \left( V(\lambda) \otimes I \right) - \left( V(\lambda) \otimes I \right) c \left( I \otimes U(\mu) \right).
\]

(Of course, the last two identities are equivalent, so that it is enough to verify one of them). □

Now the conditions of the Theorem 1 with \( n = 2 \) read:

\[
A_{12} = - B_{11} = C_{22} = -D_{21}, \quad A_{21} = -B_{22} = C_{11} = -D_{12},
\]

\[
A_{22} - D_{11} + B_{21} - C_{12} = 0,
\]

\[
A_{11} - D_{22} + B_{12} - C_{21} = 0,
\]

and are, obviously, fulfilled. This Theorem assures that the monodromy map \( g \mapsto g \) defined as \( (U, V) \mapsto V \cdot U = L_2 \) is Poisson, if the target space \( g \) is equipped with the Poisson bracket

\[
\text{PB}(A_{11}, B_{12}, C_{21}, D_{22}) = \text{PB}(A, B, C, D),
\]

and that the map \( (U, V) \mapsto U \cdot V = L_1 \) is also Poisson, if the target space \( g \) is equipped with the Poisson bracket

\[
\text{PB}(A_{22}, B_{21}, C_{12}, D_{11}) = \text{PB}(A, B, C, D),
\]

which coincides with the previous one. It follows that the manifold consisting of the matrices \( L_2(u, v, \lambda) \) (or of the matrices \( L_1(u, v, \lambda) \)) is Poisson in this bracket, as an image of a Poisson manifold under a Poisson map.

The Theorem 1 implies also that the Hamiltonian equations of motion on the pairs \( (U, V) \) with the Hamiltonian function

\[
\varphi(UV) = \varphi(VU) \quad (\varphi \ Ad - invariant)
\]

may be presented in the form of the "Lax triads"

\[
\dot{U} = U \cdot R(d\varphi(VU)) - R(d\varphi(UV)) \cdot U,
\]

22
\[ \dot{V} = V \cdot R(d\varphi(UV)) - R(d\varphi(VU)) \cdot V, \]

where

\[ R = A + B = C + D = A_{11} + B_{12} = A_{22} + B_{21} = D_{11} + C_{12} = D_{22} + C_{21}. \]

These Lax triads imply also the Lax equations for the matrices \( L_1 = VU, \)
\( L_2 = UV: \)

\[ \dot{L}_1 = [L_1, R(d\varphi(L_1))], \quad \dot{L}_2 = [L_2, R(d\varphi(L_2))]. \]

We would like to notice that these results are parallel to those obtained in [26] for the relativistic Toda lattice.

### 7.2 Open–end case

We outline briefly the features of the open–end case different from that of the periodic one.

First of all, the condition (7.8) is no more necessary for the applicability of our construction. However, it remains more elegant in this case being completely parallel to periodic case. We consider this variant first, i.e. the Volterra lattice with open ends and odd number of particles:

\[ u_k, \quad k = 1, \ldots, N, \quad \text{and} \quad v_k, \quad k = 1, \ldots, N - 1. \]

The both Lax matrices \( L_1, L_2 \) belong to one and the same algebra \( g \), the \( N \times N \) open–end version of the algebra \( g \):

\[ L_1(u, v) = \sum_{k=1}^{N-1} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_{k-1})E_{kk} + \sum_{k=1}^{N-1} u_k v_k E_{k,k+1}. \quad (7.31) \]

\[ L_2(u, v) = \sum_{k=1}^{N-1} E_{k+1,k} + \sum_{k=1}^{N} (u_k + v_k)E_{kk} + \sum_{k=1}^{N-1} u_{k+1} v_k E_{k,k+1}, \quad (7.32) \]

(In the first of these formulas \( v_0 = 0 \), in the second one \( v_N = 0 \)). All the statements of the Proposition 10 remain valid, if one drops out the dependence on the spectral parameter \( \lambda \). Also the factorization

\[ L_1(u, v) = U(u)V(v), \quad L_2(u, v) = V(v)U(u) \quad (7.33) \]
holds, with the matrices

\[ U(u) = \sum_{k=1}^{N-1} E_{k+1,k} + \sum_{k=1}^{N} u_k E_{kk}, \quad V(v) = I + \sum_{k=1}^{N-1} v_k E_{k,k+1}. \]

As in the periodic case, the pairs \((U, V)\) form a Poisson submanifold in \(g \oplus g\) carrying the correspondent Poisson bracket.

Consider now the variant with

\[ M = 2N - 1, \]

when the open-end Volterra lattice consists of an even number of particles:

\[ u_k, \quad k = 1, \ldots, N - 1, \quad \text{and} \quad v_k, \quad k = 1, \ldots, N - 1, \]

and the usual Lax matrix is \(2N - 1 \times 2N - 1\). In this case the matrices \(L_1, L_2\) have different dimensions, namely \(L_1\) is still \(N \times N\) and is given by (7.31) (this time with vanishing \(u_N\)), while \(L_2\) is \(N - 1 \times N - 1\):

\[ L_2(u, v) = \sum_{k=1}^{N-2} E_{k+1,k} + \sum_{k=1}^{N-1} (u_k + v_k) E_{kk} + \sum_{k=1}^{N-2} u_{k+1} v_k E_{k,k+1}. \quad (7.34) \]

Correspondingly, the two alternative Lax representations of the Volterra lattice, analogous to (7.12), are still valid, but live in two different algebras; the Lax matrices still form Poisson submanifolds, but also in two different algebras; their characteristic polynomials do not coincide any more, but satisfy instead the identity

\[ \det \left( L_1(u, v) - \mu I_N \right) = (-\mu) \det \left( L_2(u, v) - \mu I_{N-1} \right). \]

An interpretation in terms of the Poisson bracket on \(g \oplus g\) fails in this formulation.

All these inconveniences, however, can be repaired if we include the Volterra lattice with \(2N - 2\) particles into the one with \(2N - 1\) particles, i.e. add a dummy particle \(u_N\) with a trivial evolution: \(u_N = 0\). Then the both matrices \(L_1, L_2\) become \(N \times N\) and coincide with (7.31), (7.32) \((L_2\) having vanishing dummy entries in the positions \((N, N)\) and \((N - 1, N)\)), and the whole construction (including the factorization (7.33) and the Poisson property of the manifold \((U, V)\)) remains valid. Indeed, the condition \(u_N = 0\) is compatible not only with the equations of motion (7.5), but also with the Poisson brackets (7.4).
8 Alternative Lax representation for the general Bogoyavlensky lattice

8.1 Periodic case

For the general Bogoyavlensky lattice (6.1) we separate all the particles into \( m \) sorts, according to:

\[
v^{(j)}_k = z_{m(k-1)+j}, \quad j = 1, 2, \ldots, m.
\]

This is consistent with the \( M \)-periodic boundary conditions, only if

\[
M = mN.
\]

In these new notations the equations of motion (6.1) take the form:

\[
\dot{v}^{(j)}_k = v^{(j)}_k \left( \sum_{i=j+1}^{m} (v^{(i)}_k - v^{(i)}_{k-1}) + \sum_{i=1}^{j-1} (v^{(i)}_{k+1} - v^{(i)}_k) \right).
\]

The quadratic Poisson bracket (6.3) invariant under this flow, may be presented in the following nice form:

\[
\{ v^{(i)}_k, v^{(j)}_k \} = v^{(i)}_k v^{(j)}_k, \quad \{ v^{(j)}_{k+1}, v^{(i)}_k \} = v^{(j)}_{k+1} v^{(i)}_k \quad \text{for} \quad 1 \leq j < i \leq m.
\]

(As usual, all other brackets vanish). The corresponding Hamiltonian function (6.2) is equal to

\[
H = \sum_{k=1}^{N} \sum_{j=1}^{m} v^{(j)}_k.
\]

Introduce the following \( N \times N \) matrices, depending on the variables corresponding to the particles of only one sort, and on the spectral parameter \( \lambda \):

\[
V_j(\lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v^{(j)}_k E_{k,k+1}, \quad j = 1, 2, \ldots, m.
\]

We suppress in this notation the argument \( v^{(j)} \) of the matrix \( V_j \), because it can be restored unambiguously from the subscript. We shall need also the matrix

\[
U_1(\lambda) = \lambda \sum_{k=1}^{N} E_{k+1,k} + \sum_{k=1}^{N} v^{(1)}_k E_{kk}.
\]
Now we can define the Lax matrix as a “monodromy matrix”:

\[ L_m(v^{(1)}, \ldots, v^{(m)}, \lambda) = V_m(\lambda) \cdot \ldots \cdot V_2(\lambda) \cdot U_1(\lambda) \]  

and its shifted versions as

\[ L_j(v^{(1)}, \ldots, v^{(m)}, \lambda) = V_j(\lambda) \cdot \ldots \cdot V_2(\lambda) \cdot U_1(\lambda) \cdot V_m(\lambda) \cdot \ldots \cdot V_{j+1}(\lambda). \]

**Theorem 2**

1) The set of matrices \( L_j(v^{(1)}, \ldots, v^{(m)}, \lambda) \) for each \( 1 \leq j \leq m \) forms a Poisson submanifold in the algebra \( g \) equipped with the Poisson bracket \( PB(A, B, C, D) \).

2) The Bogoyavlensky lattice (8.3) admits a set of \( m \) (equivalent) Lax representations:

\[ \dot{L}_j = [L_j, R(L_j)]. \]

3) The generating function of integrals of motion in the \( N \times N \) representation is connected with an analogous object in the \( M \times M \) representation by

\[ \det(T^m(z, \lambda) - \mu I_M) = \prod_{j=1}^m \det(L_j(v^{(1)}, \ldots, v^{(m)}, \lambda^m) - \mu I_N), \]

all factors in this product being mutually equal.

**Proof.** As by the proof of the Proposition 8, we start with the third statement. Apply the Laplace formula to expand the determinant

\[ \det(T^m(z, \lambda) - \mu I_M), \]

according to the decomposition of the rows into \( m \) complementary sets, the \( j \)th of them \((1 \leq j \leq m)\) having the numbers \((j, m+j, \ldots, m(N-1)+j)\).

The matrix \( T^m \) has a very special structure, namely

\[ T^m(z, \lambda) \in \mathcal{E}^m + \bigoplus_{i=0}^{m-1} g_{-mi} \]

(cf. (7.16) for \( m = 2 \)). Taking this structure into account, we see that only one from the huge number of terms in the Laplace formula does not vanish:

\[ \det(T^m - \mu I_M) = \prod_{j=1}^m \det(T^m - \mu I_M) \left( \begin{array}{cccc} j & m+j & \ldots & m(N-1)+j \\ j & m+j & \ldots & m(N-1)+j \end{array} \right). \]
Now the submatrices of $T^m$ can be directly calculated, which gives:

$$\left( T^m(z, \lambda) - \mu I_M \right) \left( \begin{array}{cccc} j & m+j & \ldots & m(N-1)+j \\ j & m+j & \ldots & m(N-1)+j \end{array} \right) =$$

$$= L_j(v^{(1)}, \ldots, v^{(m)}, \lambda^m) - \mu I_N$$

(8.13)

(cf. for $m = 2$ the calculation of the submatrices (7.19), (7.20) resulting in the expressions (7.10), (7.11)).

Since all matrices $L_j(v^{(1)}, \ldots, v^{(m)}, \lambda^m)$ are connected by means of a similarity transformation, their determinants are mutually equal, which proves the third statement of the Proposition.

The first statement follows from the Proposition 7. Indeed, it is easy to see from (8.13), (8.12) that the set of matrices $L_j(v^{(1)}, \ldots, v^{(m)}, \lambda)$ for each $j$ is exactly the

$$\mathcal{P}_{m-1} = \mathcal{E} \oplus \bigoplus_{i=0}^{m-1} g_{-i}.$$  

The second statement follows now from the general $r$–matrix theory, because the function (8.3) may be presented as $H = \varphi(L_j)$, where $\varphi(L) = \text{tr}(L)_0$, so that $d\varphi(L) = L$. □

We want, however, just as in the previous Section, to give another proof of the first statement of the Theorem 2, based upon the interpretation of the matrices $L_j(v^{(1)}, \ldots, v^{(m)}, \lambda)$ as monodromy matrices.

**Theorem 3** The ordered $m$–tuples $(U_1, V_2, \ldots, V_m)$ form a Poisson submanifold in the algebra $g = \bigoplus_{j=1}^{m} g$, if the latter is equipped with the Poisson bracket $\text{PB}(A, B, C, D)$, where the operators $A, B, C, D$ are defined according to the formulas:

$$A_{ij} = \begin{cases} A, & \text{if } i = j \\ -B, & \text{if } i = 1, j > 1 \text{ or } i > j > 1 \\ C, & \text{if } i > 1, j = 1 \text{ or } j > i > 1 \end{cases}$$

$$B_{ij} = \begin{cases} B, & \text{if } i = 1 \text{ or } i > j \\ -C, & \text{if } j \geq i > 1 \end{cases}$$

$$C_{ij} = \begin{cases} -B, & \text{if } i \geq j > 1 \\ -C, & \text{if } j > i \text{ or } j = 1 \end{cases}$$

27
\[
D_{ij} = \begin{cases} 
D, & \text{if } i = j \\
B, & \text{if } i > j \\
-C, & \text{if } j > i 
\end{cases}
\]

**Proof.** To prove this statement, one may straightforwardly verify the following identities:

\[
\{U_1(\lambda) \otimes U_1(\mu)\} \\
= \left( U_1(\lambda) \otimes U_1(\mu) \right) a(\lambda, \mu) - d(\lambda, \mu) \left( U_1(\lambda) \otimes U_1(\mu) \right) \\
+ \left( I \otimes U_1(\mu) \right) b \left( U_1(\lambda) \otimes I \right) - \left( U_1(\lambda) \otimes I \right) c \left( I \otimes U_1(\mu) \right),
\]

\[
\{V_j(\lambda) \otimes V_j(\mu)\} \\
= \left( V_j(\lambda) \otimes V_j(\mu) \right) a(\lambda, \mu) - d(\lambda, \mu) \left( V_j(\lambda) \otimes V_j(\mu) \right) \\
- \left( I \otimes V_j(\mu) \right) c \left( V_j(\lambda) \otimes I \right) + \left( V_j(\lambda) \otimes I \right) b \left( I \otimes V_j(\mu) \right),
\]

\[
\{U_1(\lambda) \otimes V_j(\mu)\} \\
= \left( U_1(\lambda) \otimes V_j(\mu) \right) c - b \left( U_1(\lambda) \otimes V_j(\mu) \right) \\
+ \left( I \otimes V_j(\mu) \right) b \left( U_1(\lambda) \otimes I \right) - \left( U_1(\lambda) \otimes I \right) c \left( I \otimes V_j(\mu) \right) \quad (j > 1),
\]

\[
\{V_j(\lambda) \otimes U_1(\mu)\} \\
= - \left( V_j(\lambda) \otimes U_1(\mu) \right) b + c \left( V_j(\lambda) \otimes U_1(\mu) \right) \\
+ \left( I \otimes U_1(\mu) \right) b \left( V_j(\lambda) \otimes I \right) - \left( V_j(\lambda) \otimes I \right) c \left( I \otimes U_1(\mu) \right) \quad (j > 1),
\]

\[
\{V_i(\lambda) \otimes V_j(\mu)\} \\
= - \left( V_i(\lambda) \otimes V_j(\mu) \right) b - b \left( V_i(\lambda) \otimes V_j(\mu) \right) \\
+ \left( I \otimes V_j(\mu) \right) b \left( V_i(\lambda) \otimes I \right) + \left( V_i(\lambda) \otimes I \right) b \left( I \otimes V_j(\mu) \right) \quad (i > j),
\]

\[
\{V_i(\lambda) \otimes V_j(\mu)\} \\
= \left( V_i(\lambda) \otimes V_j(\mu) \right) c + c \left( V_i(\lambda) \otimes V_j(\mu) \right) \\
- \left( I \otimes V_j(\mu) \right) c \left( V_i(\lambda) \otimes I \right) - \left( V_i(\lambda) \otimes I \right) c \left( I \otimes V_j(\mu) \right) \quad (i < j).
\]
(The third of these identities is equivalent to the fourth one, and the fifth is equivalent to the sixth. Therefore in fact one needs to verify only four identities). Propositions 3, 4 allow to check easily that the operators $A, B, C, D$ satisfy the conditions of the Proposition 1 and thus indeed define a Poisson bracket $\text{PB}(A, B, C, D)$. ■

Now a careful inspection convinces that all the conditions of the Theorem 1 with $n = m$ are fulfilled. This Theorem states that the maps

$$(U_1, V_2, \ldots, V_m) \mapsto L_j$$

are Poisson, if the target spaces carry the Poisson brackets

$$\text{PB}(A_{j+1}, B_{j+1}, C_{j+1}, D_{j+1}) = \text{PB}(A, B, C, D).$$

Together with the Theorem 3 this implies the Poisson property of the manifold formed by the matrices $L_j$ with respect to the latter bracket.

Further, it follows from the Theorem 1 that the equations of motion of an arbitrary flow of the Bogoyavlensky lattice hierarchy with the Hamiltonian function $\varphi(L)$ ($L + L_j, \varphi \text{ Ad–invariant}$) admits a Lax representation:

$$
\begin{align*}
\dot{U}_1 &= U_1 \cdot R(d\varphi(L_m)) - R(d\varphi(L_1)) \cdot U_1, \\
\dot{V}_j &= V_j \cdot R(d\varphi(L_{j-1})) - R(d\varphi(L_j)) \cdot V_j, \quad 2 \leq j \leq m.
\end{align*}
$$

As a consequence, the following Lax equations hold:

$$
\dot{L}_j = [L_j, R(d\varphi(L_j))], \quad 1 \leq j \leq m.
$$

The Bogoyavlensky lattice proper corresponds here to $\varphi(L) = L, d\varphi(L) = L$.

### 8.2 Open–end case

As for the Volterra lattice ($m = 2$), the Theorem 2 holds almost literally in the open–end case, if

$$M = mN,$$

i.e. for the lattice consisting of $m(N - 1) + 1$ particles

$$v^{(1)}_k, \quad 1 \leq k \leq N; \quad \text{and} \quad v^{(j)}_k, \quad 1 \leq k \leq N - 1 \quad \text{for} \quad j = 2, \ldots, m.$$
All that has to be changed, is to omit the spectral parameter $\lambda$ in all formulas and to define the matrices $U_1$, $V_j$ as

$$U_1 = \sum_{k=1}^{N-1} E_{k+1,k} + \sum_{k=1}^{N} v_k^{(1)} E_{kk},$$

$$V_j = I + \sum_{k=1}^{N-1} v_k^{(j)} E_{k,k+1}.$$

For the lattices with the number of particles different from $m(N - 1) + 1$ we can add one or several dummy particles with the trivial dynamics, so that all the results remain valid.

There is, however, one more problem concerning the open–end case that has to be mentioned, namely the problem of lacking integrals of motion. Indeed, the generating function

$$\det(L_j - \mu I_N)$$

gives now only $N$ integrals of motion, which for $m > 2$ is much less than necessary for complete integrability. The lacking integrals in a closely related problem (full Toda lattice) were constructed in [4], [5]. Analogous construction can be performed also for the both Lax representations ($M \times M$ and $N \times N$) for the Bogoyavlensky lattice, leading presumably to two sets of additional integrals. However, our argument based on the Laplace formula, does not imply the coincidence or even some relations between these two sets. It would be very important to study this problem in detail.

9 Concluding remarks

In the present paper a well known integrable lattice system was further studied, which gave an opportunity to touch two general problems of the theory of integrable systems, more precisely, of the Hamiltonian aspects of this theory. The first of these problems was completely solved here, namely it was found the most general conditions for monodromy map on a direct sum of associative algebras to be Poisson with respect to some general quadratic brackets. We were able to get rid of all sorts of locality conditions, and to provide an example where such nonlocal structures naturally arise.
Another general problem, a problem of *duality* between different Lax representations for one and the same system, remains almost completely open. Our present results merely add one new example of this situation, and we hope that its careful analysis will bring us more close to the solution of the general problem.

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