On certain sums of arithmetic functions involving the gcd and lcm of two positive integers

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Abstract

We obtain asymptotic formulas with remainder terms for the hyperbolic summations $\sum_{mn \leq x} f((m,n))$ and $\sum_{mn \leq x} f([m,n])$, where $f$ belongs to certain classes of arithmetic functions, $(m,n)$ and $[m,n]$ denoting the gcd and lcm of the integers $m,n$. In particular, we investigate the functions $f(n) = \tau(n), \log n, \omega(n)$ and $\Omega(n)$. We also define a common generalization of the latter three functions, and prove a corresponding result.

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1 Introduction

Let $F : \mathbb{N}^2 \to \mathbb{C}$ be an arithmetic function of two variables. Several asymptotic results for sums $\sum F(m,n)$ with various bounds of summation are given in the literature. The usual ‘rectangular’ summations are of form $\sum_{m \leq x, n \leq y} F(m,n)$, in particular with $x = y$. The ‘triangular’ summations can be written as $\sum_{n \leq x} \sum_{m \leq n} F(m,n)$. Note that if the function $F$ is symmetric in the variables, then

$$\sum_{m,n \leq x} F(m,n) = 2 \sum_{n \leq x} \sum_{m \leq n} F(m,n) - \sum_{n \leq x} F(n,n).$$

The ‘hyperbolic’ summations have the shape $\sum_{mn \leq x} F(m,n)$, the sum being over the Dirichlet region $\{(m,n) \in \mathbb{N}^2 : mn \leq x\}$. Hyperbolic summations have been less studied than rectangular and triangular summations and it is hyperbolic summations that are estimated in this paper.

We mention a few examples for functions $F$ involving the greatest common divisor (gcd) and the least common multiple (lcm) of integers. If $F(m,n) = (m,n)$, the gcd of $m$ and $n$, then

$$\sum_{m,n \leq x} (m,n) = \frac{x^2}{\zeta(2)} \left( \log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{\zeta(2)} - \frac{\zeta'(2)}{\zeta(2)} \right) + O \left( x^{1+\theta+\varepsilon} \right),$$

holds for every $\varepsilon > 0$, where $\zeta$ is the Riemann zeta function, $\zeta'$ is its derivative, $\gamma$ is Euler’s constant, and $\theta$ denotes the exponent appearing in Dirichlet’s divisor problem. Furthermore,

$$\sum_{mn \leq x} (m,n) = \frac{1}{4\zeta(2)} x(x \log x)^2 + c_1 x \log x + c_2 x + O \left( x^\beta (\log x)^\beta \right),$$

where $\beta$ is a parameter.

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where \(c_1, c_2\) are explicit constants, and \(\beta = \frac{547}{832} \approx 0.657451\), \(\beta' = \frac{26947}{8320} \approx 3.238822\).

Estimate (1.1) (in the form of a triangular summation, involving Pillai’s arithmetic function) was obtained by Chidambaraswamy and Sitaramachandrarao [4, Th. 3.1] using elementary arguments. Formula (1.2) was deduced applying analytic methods by Krätzel, Nowak and Tóth [11, Th. 3.5].

If \(F(m, n) = [m, n]\), the lcm of \(m\) and \(n\), then we have
\[
\sum_{m,n \leq x} [m, n] = \frac{\zeta(3)}{4\zeta(2)} x^4 + O \left( x^3 (\log x)^{2/3} (\log \log x)^{1/3} \right),
\]
established by Bordellès [2, Th. 6.3] with a slightly weaker error term. The error in (1.3) comes from Liu’s improvement for the error term by Walfisz on \(\sum_{n \leq x} \varphi(n)\), where \(\varphi\) is Euler’s function. See [8].

We also have
\[
\sum_{m,n \leq x} \frac{(m,n)}{[m,n]} = 3x + O \left( (\log x)^2 \right),
\]
which follows from the general arithmetic identities for \(G_f(k)\) in Proposition 2.1, see in particular (2.2), and the summation of the divisor function \(\tau(1, 1, 2; k)\) is well known in the literature. See, e.g., Krätzel [10, Ch. 6]. We deduce that
\[
\sum_{mn=k} \tau((m,n)) = \sum_{abc^2=k} 1 =: \tau(1, 1, 2; k),
\]
for every \(k \in \mathbb{N}\),
\[
\sum_{mn \leq x} \tau((m,n)) = \zeta(2)x \left( \log x + (2\gamma - 1) + 2\frac{\zeta'(2)}{\zeta(2)} \right) + \zeta^2(1/2)x^{1/2} + R(x),
\]
where the best known related error term, to our knowledge, is \(R(x) \ll x^{63/178 + \varepsilon}\), with \(63/178 \approx 0.353932\), given by Liu [12] using deep analytic methods.
In an analogous manner to (1.5), let us define
\[ \sum_{mn \leq x} f([m, n]) = \sum_{k \leq x} L_f(k), \] (1.7)
where \( L_f(k) = \sum_{mn = k} f([m, n]). \)

We remark that if \( F \) is an arbitrary arithmetic function of two variables, then the one variable function
\[ \tilde{F}(k) = \sum_{mn = k} F(m, n) \]
is called the convolute of \( F \). The function \( F \) of two variables is said to be multiplicative if \( F(m_1m_2, n_1n_2) = F(m_1, n_1)F(m_2, n_2) \) provided that \( (m_1n_1, m_2n_2) = 1 \). If \( F \) is multiplicative, then \( \tilde{F} \) is also multiplicative. See Vaidyanathaswamy [19], Tóth [16, Sect. 6]. The functions \( G_f \) and \( L_f \) of above are special cases of this general concept. If \( f \) is multiplicative, then \( G_f(k) \) and \( L_f(k) \) are multiplicative as well.

In the present paper we deduce simple arithmetic representations of the functions \( G_f(k) \) and \( L_f(k) \) (Proposition 2.1), and establish new asymptotic estimates for sums of type (1.5) and (1.7). Namely, we give estimates for \( \sum_{mn \leq x} \tilde{f}((m, n)) \) when \( f \) belongs to a wide class of functions (Theorem 2.2), and obtain better error terms in the case of a narrower class of functions (Theorem 2.3). In particular, we consider the functions \( f(n) = \log n, \omega(n) \) and \( \Omega(n) \) (Corollary 2.5). Actually, we define a common generalization of these three functions and prove a corresponding result (Corollary 2.4). We also investigate the function \( f(n) = 1/n \), the related result on \( \sum_{mn \leq x} (m, n)^{-1} \) (Theorem 2.6) being strongly connected with the sum \( \sum_{mn \leq x} [m, n] \) (Theorem 2.7). Furthermore, we deduce estimates for the sums \( \sum_{mn \leq x} f([m, n]) \) in the cases of \( f(n) = \log n, \Omega(n) \) (Theorems 2.8, 2.9) and \( f(n) = \tau(n) \) (Theorem 2.10), respectively. We pose as an open problem to obtain an estimate for the sum \( \sum_{mn \leq x} \omega([m, n]) \). Finally we obtain a formula for \( \sum_{mn \leq x} (m, n)[m, n]^{-1} \) (Theorem 2.11). The proofs are given in Section 3.

Throughout the paper we use the following notation: \( \mathbb{N} = \{1, 2, \ldots\}; \mathbb{P} = \{2, 3, 5, \ldots\} \) is the set of primes; \( n = \prod_p p^{\nu_p(n)} \) is the prime power factorization of \( n \in \mathbb{N} \), the product being over \( p \in \mathbb{P} \), where all but a finite number of the exponents \( \nu_p(n) \) are zero; \( \tau(n) = \sum_{d \mid n} 1 \) is the divisor function; \( \mu(n) = n \) (\( n \in \mathbb{N} \)); \( \mu \) is the Möbius function; \( \omega(n) = \# \{ p : \nu_p(n) \neq 0 \} \); \( \Omega(n) = \sum_p \nu_p(n) \); \( \kappa(n) = \prod_{\nu_p(n) \neq 0} p \) is the squarefree kernel of \( n \); * is the Dirichlet convolution of arithmetic functions; \( \zeta \) is the Riemann zeta function, \( \zeta' \) is its derivative, \( \pi(x) = \sum_{p \leq x} 1 \); \( \gamma \) is Euler’s constant.

## 2 Main results

Useful arithmetic representations of the functions \( G_f(n) = \sum_{ab = n} f([a, b]) \) and \( L_f(n) = \sum_{ab = n} f([a, b]) \), already defined in the Introduction, are given by the next result.
Proposition 2.1. Let $f$ be an arbitrary arithmetic function. Then for every $n \in \mathbb{N}$,

$$G_f(n) = \sum_{a^2b^2c=n} f(a)\mu(b)\tau(c) = \sum_{a^2c=n} (f \ast \mu)(a)\tau(c) = \sum_{a^2c=n} f(a)2^{\omega(c)},$$

and

$$L_f(n) = \sum_{a^2b^2c=n} f(n/a)\mu(b)\tau(c) = \sum_{a^2c=n} f(ac)2^{\omega(c)}.$$  

If $f$ is completely additive, then for every $n \in \mathbb{N}$,

$$L_f(n) = f(n)\tau(n) - G_f(n).$$

In terms of formal Dirichlet series, identities (2.1), (2.2) and (2.3) show that for every arithmetic function $f$,

$$\sum_{n=1}^{\infty} \frac{G_f(n)}{n^z} = \frac{\zeta^2(z)}{\zeta(2z)} \sum_{n=1}^{\infty} \frac{f(n)}{n^{2z}}.$$  

See [11, Prop. 5.1] for a similar formula on the sum $\sum_{d_1\cdots d_k=n} g((d_1, \ldots, d_k))$, where $k \in \mathbb{N}$ and $g$ is an arithmetic function.

Our first asymptotic formula applies to every function $f$ satisfying a condition on its order of magnitude.

Theorem 2.2. Let $f$ be an arithmetic function such that $f(n) \ll n^\beta \log n^\delta$, as $n \to \infty$, for some fixed $\beta, \delta \in \mathbb{R}$ with $\beta < 1$. Then

$$\sum_{mn \leq x} f((m, n)) = x(C_f \log x + D_f) + R_f(x),$$

where the constants $C_f$ and $D_f$ are given by

$$C_f = \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{f(n)}{n^2},$$

$$D_f = \frac{1}{\zeta(2)} \left( C \sum_{n=1}^{\infty} \frac{f(n)}{n^2} - 2 \sum_{n=1}^{\infty} \frac{f(n) \log n}{n^2} \right),$$

with $C$ defined by

$$C = 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)},$$
and the error term is

$$R_f(x) \ll \begin{cases} x^{(\beta+1)/2}(\log x)^{\delta+1}, & \text{if } 0 < \beta < 1 \text{ or } \beta = 0, \delta \neq -1, \\ x^{1/2}\log x, & \text{if } \beta = 0, \delta = -1, \\ x^{1/2}, & \text{if } \beta < 0. \end{cases}$$

The error term $R_f(x)$ can be improved assuming that the Riemann Hypothesis is true. For example, let $q = 221/608 \doteq 0.363486$. If $\beta, \delta \in \mathbb{R}$ and $\beta < 2q - 1 \doteq -0.273026$, then $R_f(x) \ll x^{\nu+\varepsilon}$.

Theorem 2.2 applies, e.g., to the functions $f(n) = (\log n)^\delta$ (with $\beta = 0$, $\delta \in \mathbb{R}$), $f(n) = \tau^k(n)$ ($k \in \mathbb{N}$, with $\beta = k\varepsilon$, $\varepsilon > 0$ arbitrary small), $\delta = 0$), $f(n) = \omega(n)$ or $\Omega(n)$ (with $0 < \beta = \varepsilon$ arbitrary small, $\delta = 0$). However, for some special functions asymptotic formulas with more terms or with better unconditional errors can be obtained. See, e.g. (1.6), namely the case $f(n) = \tau(n)$ and our next theorems.

Let $f$ be a function such that $(\mu * f)(n) = 0$ for all $n \neq p^\nu$ ($n$ is not a prime power), $(\mu * f)(p^\nu) = g(p)$ does not depend on $\nu$ and $g(p)$ is sufficiently small for the primes $p$. More exactly, we have the next result.

**Theorem 2.3.** Let $f$ be an arithmetic function such that there exists a subset $Q$ of the set of primes $\mathbb{P}$ and there exists a subset $S$ of $\mathbb{N}$ with $1 \in S$, satisfying the following properties:

i) $(\mu * f)(n) = 0$ for all $n \neq p^\nu$, where $p \in Q$ and $\nu \in S$,

ii) $(\mu * f)(p^\nu) = g(p)$, depending only on $p$, for all prime powers $p^\nu$ with $p \in Q$, $\nu \in S$.

iii) $g(p) \ll (\log p)^\eta$, as $p \to \infty$, where $\eta \geq 0$ is a fixed real number.

Then for the error term in (2.7) we have $R_f(x) \ll x^{1/2}(\log x)^\eta$. Furthermore, the constants $C_f$ and $D_f$ can be given as

$$C_f = \sum_{p \in Q} g(p) \sum_{\nu \in S} \frac{1}{p^{2\nu}},$$

$$D_f = (2\gamma - 1)C_f - 2 \sum_{p \in Q} g(p) \log p \sum_{\nu \in S} \frac{\nu}{p^{2\nu}}.$$

The prototype of functions $f$ to which Theorem 2.3 applies is the function $f_{S,\eta}$ implicitly defined by

$$h_{S,\eta}(n) := (\mu * f_{S,\eta})(n) = \begin{cases} (\log p)^\eta, & \text{if } n = p^\nu \text{ a prime power with } \nu \in S, \\ 0, & \text{otherwise}, \end{cases}$$

(2.9)

where $1 \in S \subseteq \mathbb{N}$, $\eta \geq 0$ is real and $Q = \mathbb{P}$. It is possible to consider the corresponding generalization with $Q \subseteq \mathbb{P}$, as well. By Möbius inversion we obtain that for $n = \prod_p p^{\nu_p(n)} \in \mathbb{N}$,

$$f_{S,\eta}(n) = \sum_{d|n} h_{S,\eta}(d) = \sum_{p|n} (\log p)^\eta \#\{\nu : 1 \leq \nu \leq \nu_p(n), \nu \in S\},$$

where $f_{S,\eta}(1) = 0$ (empty sum).
Let \( S = \mathbb{N} \). Then

\[
f_{N, \eta}(n) := \sum_{p | n} \nu_p(n)(\log p)^\eta,
\]

which gives for \( \eta = 1 \), \( f_{N,1}(n) = \log n \), while \( h_{N,1}(n) = \Lambda(n) \) is the von Mangoldt function. If \( \eta = 0 \), then \( f_{N,0}(n) = \Omega(n) \).

Now let \( S = \{1\} \). Then

\[
f_{\{1\}, \eta}(n) := \sum_{p | n} (\log p)^\eta,
\]

and if \( \eta = 0 \), then \( f_{\{1\},0}(n) = \omega(n) \). If \( \eta = 1 \), then \( f_{\{1\},1}(n) = \log \kappa(n) \), where \( \kappa(n) = \prod_{p | n} p \).

The functions \( f_{S, \eta}(n) \) and \( h_{S, \eta}(n) \) have not been studied in the literature, as far as we know.

According to (2.9), the conditions of Theorem 2.3 are satisfied and we deduce the next result.

**Corollary 2.4.** If \( 1 \in S \subseteq \mathbb{N} \) and \( \eta \geq 0 \) is real number, then

\[
\sum_{mn \leq x} f_{S, \eta}((m, n)) = x(C_{f_{S, \eta}} \log x + D_{f_{S, \eta}}) + O(x^{1/2}(\log x)^\eta),
\]

where the constants \( C_{f_{S, \eta}} \) and \( D_{f_{S, \eta}} \) are given by

\[
C_{f_{S, \eta}} = \sum_p (\log p)^\eta \sum_{\nu \in S} \frac{1}{p^{2\nu}},
\]

\[
D_{f_{S, \eta}} = (2\gamma - 1)C_{f_{S, \eta}} - 2 \sum_p (\log p)^{\eta+1} \sum_{\nu \in S} \frac{\nu}{p^{2\nu}}.
\]

In the special cases mentioned above we obtain the following results.

**Corollary 2.5.** We have

\[
\sum_{mn \leq x} \log(m, n) = x(C_{\log} \log x + D_{\log}) + O(x^{1/2} \log x),
\]

(2.10)

\[
\sum_{mn \leq x} \log \kappa((m, n)) = x(C_{\log \kappa} \log x + D_{\log \kappa}) + O(x^{1/2} \log x),
\]

(2.11)

\[
\sum_{mn \leq x} \omega((m, n)) = x(C_{\omega} \log x + D_{\omega}) + O(x^{1/2}),
\]

\[
\sum_{mn \leq x} \Omega((m, n)) = x(C_{\Omega} \log x + D_{\Omega}) + O(x^{1/2}),
\]

(2.12)

where

\[
C_{\log} = -\frac{\zeta'(2)}{\zeta(2)} = \sum_p \frac{\log p}{p^2 - 1} \approx 0.569960, \quad D_{\log} = -\frac{\zeta'(2)}{\zeta(2)} \left(2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)} + 2\frac{\zeta''(2)}{\zeta(2)}\right),
\]

(2.13)
\[ C_{\log \kappa} = \sum_p \frac{\log p}{p^2} \doteq 0.493091, \quad D_{\log \kappa} = (2\gamma - 1) \sum_p \frac{\log p}{p^2} - 2 \sum_p \frac{(\log p)^2}{p^2}, \]
\[ C_\omega = \sum_p \frac{1}{p^2} \doteq 0.452247, \quad D_\omega = (2\gamma - 1) \sum_p \frac{1}{p^2} - 2 \sum_p \log p \frac{p}{p^2}, \]
\[ C_\Omega = \sum_p \frac{1}{p^2 - 1} \doteq 0.551693, \quad D_\Omega = (2\gamma - 1) \sum_p \frac{1}{p^2 - 1} - 2 \sum_p \frac{p^2 \log p}{(p^2 - 1)^2}. \]  

(2.14)

We deduce by (2.10) and (2.11) that \( \prod_{mn \leq x} (m, n) \Rightarrow \kappa((m, n)) \sim x^{\log \kappa(x)}, \) as \( x \to \infty. \)

The next result concerns the function \( f(n) = 1/n \) and gives a better error term than the error obtained from Theorem 2.2.

**Theorem 2.6.** We have

\[ \sum_{mn \leq x} \frac{1}{(m, n)} = \zeta(3) \zeta(2) x (\log x + D) + O(x^{\theta + \varepsilon}), \tag{2.15} \]

where

\[ D = 2\gamma - 1 - 2 \frac{\zeta'(2)}{\zeta(2)} + 2 \frac{\zeta'(3)}{\zeta(3)}, \]

and \( \theta \) is the exponent in Dirichlet’s divisor problem.

Now consider the functions given by \( L_f(n) = \sum_{ab = n} f([a, b]). \) If \( f = \text{id}, \) then \( L_{\text{id}}(n) = \sum_{ab = n} [a, b] = n \sum_{ab = n} (a, b)^{-1}. \) The next result follows from Theorem 2.6 by partial summation. It may be compared to estimates (1.1), (1.2) and (1.3).

**Theorem 2.7.** We have

\[ \sum_{mn \leq x} [m, n] = \frac{\zeta(3)}{2 \zeta(2)} x^2 (\log x + E) + O(x^{\theta + 1 + \varepsilon}), \]

where

\[ E = 2\gamma - 1 - \frac{1}{2} - 2 \frac{\zeta'(2)}{\zeta(2)} + 2 \frac{\zeta'(3)}{\zeta(3)}, \]

and \( \theta \) is the exponent in Dirichlet’s divisor problem.

If the function \( f \) is completely additive, then identity (2.6) can be used to deduce asymptotic estimates for \( \sum_{n \leq x} L_f(n). \)

**Theorem 2.8.** We have

\[ \sum_{mn \leq x} \log[m, n] = x (\log x)^2 + (2\gamma - 2 - C_{\log}) x \log x - (2\gamma - 2 + D_{\log}) x + O(x^{1/2} \log x), \]

where \( C_{\log} \) and \( D_{\log} \) are given by (2.13).
As a consequence we deduce that $\prod_{mn\leq x}[m,n] \sim x^{x\log x}$ as $x \to \infty$.

**Theorem 2.9.** We have

$$\sum_{mn\leq x} \Omega([m,n]) = 2x(\log x)(\log\log x) + (C - C_\Omega)x \log x + O(x),$$

where $C_\Omega$ is given by (2.14) and

$$C = 4 \sum_p \left( \log \left(1 - \frac{1}{p}\right) + \frac{1}{p - 1} \right) - 2\Gamma'(2),$$

(2.16)

$\Gamma$ being the Gamma function.

Our treatment cannot be applied to the function $f(n) = \omega(n)$, which is additive, but not completely additive. We formulate as an open problem to deduce an estimate for $\sum_{mn\leq x} \omega([m,n])$.

Next we consider the divisor function $f(n) = \tau(n)$.

**Theorem 2.10.** We have

$$\sum_{mn\leq x} \tau([m,n]) = x(C_1(\log x)^3 + C_2(\log x)^2 + C_3 \log x + C_4) + O(x^{1/2+\varepsilon})$$

(2.17)

for every $\varepsilon > 0$, where

$$C_1 = \frac{\pi^2}{12} \prod_p \left( 1 - \frac{1}{(p+1)^2} \right) = 0.078613,$$

and the constants $C_2, C_3, C_4$ can also be given explicitly.

Estimate (2.17) may be compared to (1.6) and to that for $\sum_{m,n\leq x} \tau([m,n])$. See Tóth and Zhai [17, Th. 3.4].

Finally, we deduce the counterpart of formula (1.4) with hyperbolic summation.

**Theorem 2.11.** We have

$$\sum_{mn\leq x} \frac{(m,n)}{[m,n]} = \frac{\zeta^2(3/2)}{\zeta(3)} x^{1/2} + O((\log x)^3).$$

(2.18)

### 3 Proofs

**Proof of Proposition 2.1.** Group the terms of the sum $G_f(n) = \sum_{ab=n} f((a,b))$ according to the values $(a,b) = d$, where $a = dc, b = de$ with $(c,e) = 1$. We obtain, using the property of the Möbius function,

$$G_f(n) = \sum_{d^2ce=n} f(d) = \sum_{d^2ce=n} f(d) \sum_{\delta|(c,e)} \mu(\delta)$$

$$= \sum_{d^2\delta^2k\ell=t=n} f(d)\mu(\delta) = \sum_{d^2\delta^2t=n} f(d)\mu(\delta) \sum_{k\ell=t} 1$$
\[ \sum_{d^2 t = n} f(d) \mu(\delta) \tau(t), \]
giving (2.1), which can be written as (2.2) and (2.3) by the definition of the Dirichlet convolution and the identity \( \sum_{d^2 t = k} \mu(\delta) \tau(t) = 2^{\varphi(k)}. \)

Alternatively, use the identity \( f(n) = \sum_{d | n} (f * \mu)(d) \) to deduce that
\[
G_f(n) = \sum_{a b = n} (f * \mu)(d) = \sum_{a b = n} \sum_{d | a, d | b} (f * \mu)(d)
= \sum_{d^2 c e = n} (f * \mu)(d) = \sum_{d^2 t = n} (f * \mu)(d) \sum_{c e = t} 1 = \sum_{d^2 t = n} (f * \mu)(d) \tau(t),
\]
giving (2.2).

For \( L_f(n) \) use that \( [a, b] = ab/(a, b) \) and apply the first method above to deduce (2.4) and (2.5).

Finally, to obtain (2.6), use that if \( f \) is completely additive, then \( f([a, b]) = f(ab/(a, b)) = f(ab) - f((a, b)). \)

For the proof of Theorem 2.2 we need the following Lemmas.

**Lemma 3.1.** Let \( s, \delta \in \mathbb{R} \) with \( s > 1 \). Then
\[
\sum_{n > x} (\log n)^\delta n^s \ll (\log x)^\delta x^{s-1}.
\]

**Proof.** The function \( t \mapsto t^{-s}(\log t)^\delta \) \( (t > x) \) is decreasing for large \( x \). By comparing the sum with the corresponding integral we have
\[
\sum_{n > x} (\log n)^\delta n^s \leq \int_x^\infty \frac{(\log t)^\delta}{t^s} dt.
\]

If \( \delta < 0 \), then trivially,
\[
\int_x^\infty \frac{(\log t)^\delta}{t^s} dt \leq (\log x)^\delta \int_x^\infty \frac{1}{t^s} dt = \frac{(\log x)^\delta}{x^{s-1}}.
\]

If \( \delta > 0 \), then integrating by parts gives
\[
\int_x^\infty \frac{(\log t)^\delta}{t^s} dt \ll \frac{(\log x)^\delta}{x^{s-1}} + \int_x^\infty \frac{(\log t)^{\delta-1}}{t^s} dt,
\]
and repeated applications of the latter estimate, until the exponent of \( \log t \) becomes negative, conclude the proof.

**Lemma 3.2.** Let \( s, \delta \in \mathbb{R} \) with \( s > 0 \). Then
\[
\sum_{2 \leq n \leq x} (\log n)^\delta n^s \ll \begin{cases} x^{1-s}(\log x)^\delta, & \text{if } 0 < s < 1, \delta \in \mathbb{R}, \\ (\log x)^\delta + 1, & \text{if } s = 1, \delta \neq -1, \\ \log \log x, & \text{if } s = 1, \delta = -1, \\ 1, & \text{if } s > 1. \end{cases}
\]
Proof. Let $0 < s < 1$. If $\delta \geq 0$, then trivially
\[
\sum_{n \leq x} \frac{(\log n)^\delta}{n^s} \leq (\log x)^\delta \sum_{n \leq x} \frac{1}{n^s}
\]
and by comparison of the sum with the corresponding integral we have
\[
\sum_{n \leq x} \frac{1}{n^s} \ll x^{1-s}.
\] (3.1)

If $\delta < 0$, then write
\[
\sum_{2 \leq n \leq x} \frac{(\log n)^\delta}{n^s} = \sum_{2 \leq n \leq x^{1/2}} \frac{(\log n)^\delta}{n^s} + \sum_{x^{1/2} < n \leq x} \frac{(\log n)^\delta}{n^s}
\]
\[
\ll \sum_{n \leq x^{1/2}} \frac{1}{n^s} + (\log x^{1/2})^\delta \sum_{x^{1/2} < n \leq x} \frac{1}{n^s},
\]
which is, using again (3.1),
\[
\ll x^{(1-s)/2} + (\log x)^\delta x^{1-s} \ll (\log x)^\delta x^{1-s},
\]
where $1 - s > 0$. The case $s = 1$ is well-known. If $s > 1$, then the corresponding series is convergent. \hfill \Box

Proof of Theorem 2.2. We use identity (2.3) and the known estimate (see [6])
\[
\sum_{n \leq x} 2^{\omega(n)} = \frac{x}{\zeta(2)} (\log x + C) + O(x^{1/2}),
\] (3.2)
where $C$ is defined by (2.8). We deduce by standard arguments that
\[
\sum_{mn \leq x} f((m, n)) = \sum_{d^2 c \leq x} f(d) 2^{\omega(c)} = \sum_{d \leq x^{1/2}} f(d) \sum_{c \leq x/d^2} 2^{\omega(c)}
\]
\[
= \sum_{d \leq x^{1/2}} f(d) \left( \frac{1}{\zeta(2)} \frac{x}{d^2} (\log x/d^2 + C) + O((x/d^2)^{1/2}) \right)
\]
\[
= \frac{x}{\zeta(2)} \left( (\log x + C) \sum_{d \leq x^{1/2}} \frac{f(d)}{d^2} - 2 \sum_{d \leq x^{1/2}} \frac{f(d) \log d}{d^2} \right) + O(x^{1/2} \sum_{d \leq x} \frac{|f(d)|}{d})
\]
Here
\[
\sum_{d \leq x^{1/2}} \frac{f(d)}{d^2} = \sum_{d=1}^\infty \frac{f(d)}{d^2} + A_f(x),
\]
where the series converges absolutely by the given assumption on \( f \), and
\[
A_f(x) = \sum_{d>x^{1/2}} \frac{|f(d)|}{d^2} \ll \sum_{d>x^{1/2}} \frac{(\log d)\delta}{d^{2-\beta}} \ll x^{(\beta-1)/2}(\log x)^\delta
\]
by using Lemma 3.1, where \( 2-\beta > 1 \), leading to the error \( x(\log x)x^{(\beta-1)/2}(\log x)^\delta = x^{(\beta+1)/2}(\log x)^{\delta+1} \).

Furthermore,
\[
\sum_{d\leq x^{1/2}} \frac{f(d)\log d}{d^2} = \sum_{d=1}^{\infty} \frac{f(d)\log d}{d^2} + B_f(x),
\]
where the series converges absolutely and
\[
B_f(x) = \sum_{d>x^{1/2}} \frac{|f(d)|\log d}{d^2} \ll \sum_{d>x^{1/2}} \frac{(\log d)\delta+1}{d^{2-\beta}} \ll x^{(\beta+1)/2}(\log x)^{\delta+1}
\]
by using Lemma 3.1 again, giving the same error \( x^{(\beta+1)/2}(\log x)^{\delta+1} \). Finally,
\[
x^{1/2} \sum_{d\leq x} \frac{|f(d)|}{d} \ll x^{1/2} \sum_{d\leq x} \frac{(\log d)\delta}{d^{1-\beta}} \ll \begin{cases} x^{(\beta+1)/2}(\log x)^\delta, & \text{if } 0 < \beta < 1, \\ x^{1/2}(\log x)^{\delta+1}, & \text{if } \beta = 0, \delta \neq -1, \\ x^{1/2}(\log x), & \text{if } \beta = 0, \delta = -1, \\ x^{1/2}, & \text{if } \beta < 0, \end{cases}
\]
using Lemma 3.2.

Baker [1] proved that under the Riemann Hypothesis for the error term \( R(x) \) of estimate (3.2) one has \( R(x) \ll x^{1/4+\varepsilon} \), whilst Kaczorowski and Wiertelak [9] remarked that a slight modification of the treatment in [20] yields \( R(x) \ll x^{2/3+\varepsilon} \). This leads to the desired improvement of the error.

Now we will prove Theorem 2.3. We need the following Lemmas.

**Lemma 3.3.** If \( \eta \geq 0 \) and \( s > 1 \) are real numbers, then
\[
\sum_{p>x} \frac{(\log p)^\eta}{p^s} \ll \frac{(\log x)^{\eta-1}}{x^{s-1}}.
\]

**Proof.** We have, by using Riemann-Stieltjes integration, integration by parts and the Chebyshev estimate \( \pi(x) \ll x/\log x \),
\[
\sum_{p>x} \frac{(\log p)^\eta}{p^s} = \int_x^\infty \frac{(\log t)^\eta}{t^s} d(\pi(t)) = \left[ \frac{(\log t)^\eta}{t^s} \pi(t) \right]_{t=x}^{t=\infty} - \int_x^\infty \left( \frac{(\log t)^\eta}{t^s} \right)' \pi(t) dt
\]
\[
\ll \frac{(\log x)^{\eta-1}}{x^{s-1}} + \int_x^\infty \frac{(\log t)^{\eta-1}}{t^s} dt.
\]

Integration by parts, again, gives
\[
\int_x^\infty \frac{(\log t)^{\eta-1}}{t^s} dt \ll \frac{(\log x)^{\eta-1}}{x^{s-1}} + \int_x^\infty \frac{(\log t)^{\eta-2}}{t^s} dt,
\]
and repeated applications of the latter estimate conclude the result. \( \square \)
Lemma 3.4. If \( \eta, s \geq 0 \) are real numbers, then
\[
\sum_{p \leq x} \frac{(\log p)^\eta}{p^s} \ll \frac{(\log x)^{\eta-1}}{x^{s-1}}.
\]

Proof. Similar to the previous proof, by using Riemann-Stieltjes integration,
\[
\sum_{p \leq x} \frac{(\log p)^\eta}{p^s} = \int_2^x \frac{(\log t)^\eta}{t^s} \pi(t) dt = \left[ \frac{(\log t)^\eta}{t^s} \pi(t) \right]_{t=2}^{t=x} - \int_2^x \left( \frac{(\log t)^\eta}{t^s} \right)' \pi(t) dt
\]
\[
\ll \frac{(\log x)^{\eta-1}}{x^{s-1}} + \frac{x}{\log x} \left[ \frac{(\log t)^\eta}{t^s} \right]_2^x \ll \frac{(\log x)^{\eta-1}}{x^{s-1}}.
\]
\[
\square
\]

Proof of Theorem 2.3. Now we use identity (2.2) and the well-known estimate on \( \tau(n) \),
\[
\sum_{n \leq x} \tau(n) = x(\log x + C_1) + O(x^{\theta + \varepsilon}),
\]
where \( C_1 = 2\gamma - 1 \) and \( 1/4 < \theta < 1/3 \). Put \( \theta_1 = \theta + \varepsilon \). We have
\[
\sum_{mn \leq x} f((m, n)) = \sum_{d^2c \leq x} (\mu * f)(d)\tau(c) = \sum_{d \leq x^{1/2}} (\mu * f)(d) \sum_{c \leq x/d^2} \tau(c)
\]
\[
= \sum_{d \leq x^{1/2}} (\mu * f)(d) \left( \frac{x}{d^2} (\log \frac{x}{d^2} + C_1) + O((\frac{x}{d^2})^{\theta_1}) \right)
\]
\[
= x \left( (\log x + C_1) \sum_{d \leq x^{1/2}} \frac{(\mu * f)(d)}{d^2} - 2 \sum_{d \leq x^{1/2}} \frac{(\mu * f)(d) \log d}{d^2} \right) + O(x^{\theta_1} \sum_{d \leq x^{1/2}} \frac{|(\mu * f)(d)|}{d^{2\theta_1}}).
\]

Here
\[
A := \sum_{d \leq x^{1/2}} \frac{(\mu * f)(d)}{d^2} = \sum_{p^\nu \leq x, p \in Q} \frac{g(p)}{p^{2\nu}} = \sum_{p \leq x^{1/2}, p \in Q} \frac{g(p)}{\nu} \sum_{\nu \in S} \frac{1}{p^{2\nu}}
\]
\[
= \sum_{p \leq x^{1/2}, p \in Q} g(p) \left( H_S(p) - \sum_{\nu \geq m+1} \frac{1}{p^{2\nu}} \right),
\]
\[
(3.4)
\]
where \( m := \lfloor \frac{\log x}{\log p} \rfloor \), and for every prime \( p \),
\[
\frac{1}{p^2} \leq H_S(p) := \sum_{\nu \in S} \frac{1}{p^{2\nu}} \leq \sum_{\nu = 1}^{\infty} \frac{1}{p^{2\nu}} = \frac{1}{p^2 - 1},
\]
\[
(3.5)
\]
using that $1 \in S$. Here
\[
\sum_{p \leq x^{1/2}} \sum_{p \in Q} g(p) H_S(p) = \sum_{p \in Q} g(p) H_S(p) - \sum_{p > x^{1/2}} g(p) H_S(p),
\] (3.6)
where the series is absolutely convergent by the condition $g(p) \ll (\log p)^{\eta}$ and by (3.5), and the last sum is
\[
\ll \sum_{p > x^{1/2}} \frac{(\log p)^{\eta}}{p^2 - 1} \ll \sum_{p > x^{1/2}} \frac{(\log p)^{\eta}}{p^2} \ll \frac{(\log x)^{\eta-1}}{x^{1/2}}
\]
by Lemma 3.3. Also,
\[
A_1 := \sum_{p \leq x^{1/2}} \sum_{p \in Q} g(p) \sum_{\nu \geq m+1} \frac{1}{p^{2\nu}} \ll \sum_{p \leq x^{1/2}} (\log p)^{\eta} \sum_{\nu \geq m+1} \frac{1}{p^{2\nu}}
\] (3.7)
\[
= \sum_{p \leq x^{1/2}} (\log p)^{\eta} p^{2m}(p^2 - 1).
\] (3.8)
By the definition of $m$ we have $m > \frac{\log x}{2 \log p} - 1$, hence $p^{2m} > \frac{x}{p^2}$. Thus the sum in (3.8) is
\[
\leq \frac{1}{x} \sum_{p \leq x^{1/2}} \frac{p^2(\log p)^{\eta}}{p^2} - 1 \ll \frac{1}{x} \sum_{p \leq x^{1/2}} (\log p)^{\eta} \leq \frac{1}{x} (\log x)^{\eta} \pi(x^{1/2}),
\] (3.9)
hence
\[
A_1 \ll \frac{(\log x)^{\eta-1}}{x^{1/2}},
\] (3.10)
using $\eta \geq 0$ and the estimate $\pi(x^{1/2}) \ll \frac{x^{1/2}}{\log x}$.

We deduce by (3.4), (3.6), (3.7) and (3.10) that
\[
A = \sum_{p \in Q} g(p) H_S(p) + O \left( \frac{(\log x)^{\eta-1}}{x^{1/2}} \right),
\]
which leads to the error term $\ll x^{1/2}(\log x)^{\eta}$.

In a similar way,
\[
B := \sum_{d \leq x^{1/2}} \frac{(\mu * f)(d) \log d}{d^2} = \sum_{p \leq x} \sum_{\nu \in S} g(p) \log p^\nu = \sum_{p \leq x^{1/2}} g(p) \log p \sum_{\nu \geq m+1} \frac{\nu}{p^{2\nu}}
\]
\[
= \sum_{p \leq x^{1/2}} \sum_{p \in Q} g(p) \log p \left( K_S(p) - \sum_{\nu \geq m+1} \frac{\nu}{p^{2\nu}} \right),
\]

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where for every prime $p$,

$$
\frac{1}{p^2} \leq K_S(p) := \sum_{\nu=1}^{\infty} \frac{\nu}{p^{2\nu}} \leq \sum_{\nu=1}^{\infty} \frac{\nu}{p^2} = \frac{p^2}{(p^2 - 1)^2} \ll \frac{1}{p^2},
$$

(3.11) since $1 \in S$. We write

$$
\sum_{p \leq x^{1/2}} g(p)(\log p)K_S(p) = \sum_{p \in Q} g(p)(\log p)K_S(p) - \sum_{p > x^{1/2}} g(p)(\log p)K_S(p),
$$

where the series is absolutely convergent by (3.11), and the last sum is

$$
\ll \sum_{p > x^{1/2}} \frac{(\log p)^{\eta+1}}{p^2} \ll \frac{(\log x)^{\eta}}{x^{1/2}}
$$

by Lemma 3.3. Also,

$$
B_1 := \sum_{p \leq x^{1/2}} g(p) \log p \sum_{\nu \geq m+1} \frac{\nu}{p^{2\nu}} \ll \sum_{p \leq x^{1/2}} (\log p)^{\eta+1} \frac{\nu}{p^{2\nu}},
$$

(3.12)

where

$$
\sum_{\nu \geq m+1} \frac{\nu}{p^{2\nu}} = \frac{p^2}{(p^2 - 1)^2} \left( \frac{m+1}{p^{2m+2}} - \frac{m}{p^{2m}} \right) \ll \frac{1}{p^2} \frac{m}{p^{2m}} \ll \frac{\log x}{x \log p},
$$

using that $p^{2m} > \frac{x}{p^m}$ and $m \ll \frac{\log x}{\log p}$. This gives that for the sum $B_1$ in (3.12),

$$
B_1 \leq \frac{\log x}{x} \sum_{p \leq x^{1/2}} (\log p)^{\eta} \ll \frac{(\log x)^{\eta}}{x^{1/2}},
$$

see (3.9). Putting all of this together we obtain that

$$
B = \sum_{p \in Q} g(p)(\log p)K_S(p) + O\left( \frac{(\log x)^{\eta}}{x^{1/2}} \right),
$$

which leads to the error term $\ll x^{1/2}(\log x)^{\eta}$, the same as above.

Finally,

$$
C := x^{\theta_1} \sum_{d \leq x^{1/2}} \frac{|(\mu \ast f)(d)|}{d^{2\theta_1}} = x^{\theta_1} \sum_{d \leq x^{1/2}} \frac{g(p)}{p^{2\nu \theta_1}} \leq x^{\theta_1} \sum_{p \leq x^{1/2}} (\log p)^{\eta} \sum_{\nu=1}^{\infty} \frac{1}{p^{2\nu \theta_1}},
$$

$$
\ll x^{\theta_1} \sum_{p \leq x^{1/2}} \frac{(\log p)^{\eta}}{p^{2\theta_1}} \ll x^{1/2}(\log x)^{\eta-1}
$$

by Lemma 3.4. This finishes the proof.
Proof of Theorem 2.6. According to identity (2.2),
\[
\sum_{ab=n} \frac{1}{(a,b)} = \sum_{d^2c=n} (\mu \ast \text{id}_{-1})(d)\tau(c),
\]
where \(\text{id}_{-1}(n) = n^{-1} (n \in \mathbb{N})\). Using estimate (3.3) we deduce
\[
\sum_{mn \leq x} \frac{1}{(m,n)} = \sum_{d^2c \leq x} (\mu \ast \text{id}_{-1})(d)\tau(c) = \sum_{d \leq x^{1/2}} (\mu \ast \text{id}_{-1})(d) \sum_{c \leq x/d^2} \tau(c)
\]
\[
= \sum_{d \leq x^{1/2}} (\mu \ast \text{id}_{-1})(d) \left( \frac{x}{d^2} (\log \frac{x}{d^2} + 2\gamma - 1) + O((\frac{x}{d^2})^{\theta + \varepsilon}) \right).
\]
Here \((\mu \ast \text{id}_{-1})(n) \leq n^{-1}\) for every \(n \in \mathbb{N}\), and known elementary arguments lead to the given asymptotic formula. \(\square\)

Proof of Theorem 2.7. We have
\[
\sum_{ab \leq x} [a,b] = \sum_{n \leq x} n \sum_{ab=n} \frac{1}{(a,b)},
\]
and partial summation applied to estimate (2.15) gives the result. \(\square\)

Proof of Theorem 2.8. We have, by using identity (2.6),
\[
\sum_{mn \leq x} \log[m,n] = \sum_{n \leq x} \tau(n) \log n - \sum_{mn \leq x} \log(m,n).
\]
We obtain by partial summation on (3.3) that
\[
\sum_{n \leq x} \tau(n) \log n = x((\log x)^2 + (2\gamma - 2) \log x + 2 - 2\gamma) + O(x^\theta \log x),
\]
which can be combined with formula (2.10) on \(\sum_{mn \leq x} \log(m,n)\). \(\square\)

Proof of Theorem 2.9. The function \(\Omega(n)\) is completely additive, hence by (2.6),
\[
\sum_{mn \leq x} \Omega([m,n]) = \sum_{n \leq x} \tau(n)\Omega(n) - \sum_{mn \leq x} \Omega((m,n)).
\]
It follows from the general result by De Koninck and Mercier [5, Th. 8], applied to the function \(\Omega(n)\) that
\[
\sum_{n \leq x} \tau(n)\Omega(n) = 2x(\log x)(\log \log x) + Cx \log x + O(x),
\]
where the constant \(C\) is defined by (2.16). Now using estimate (2.12) on \(\sum_{mn \leq x} \Omega((m,n))\) finishes the proof. \(\square\)
Proof of Theorem 2.10. We show that
\[ h(n) := \sum_{ab=n} \tau([a,b]) = \sum_{dk=n} \psi(d)\tau^2(k), \]
(3.13)
where the function \( \psi \) is multiplicative and \( \psi(p^n) = (-1)^{n-1}(n-1) \) for every prime power \( p^n \) \( (n \geq 0) \).

This can be done by multiplicativity and computing the values of both sides for prime powers. However, we present here a different approach, based on identity (2.5). The Dirichlet series of \( h(n) \) is
\[ H(z) := \sum_{n=1}^{\infty} \frac{h(n)}{n^z} = \sum_{dk=1}^{\infty} \tau(dk)2^{\omega(k)} = \sum_{k=1}^{\infty} \frac{2^{\omega(k)}}{k^z} \sum_{d=1}^{\infty} \tau(dk). \]
(3.14)

If \( f \) is any multiplicative function and \( k = \prod p^{\nu_p(k)} \) a positive integer, then
\[ \sum_{n=1}^{\infty} \frac{f(kn)}{n^z} = \prod_p \sum_{\nu=0}^{\infty} \frac{f(p^{\nu+p(k)})}{p^{\nu z}}. \]

If \( f(n) = \tau(n) \), then this gives (see Titchmarsh [14, Sect. 1.4.2])
\[ \sum_{n=1}^{\infty} \tau(kn) = \zeta^2(2z) \prod_p (\nu_p(k) + 1 - \frac{\nu_p(k)}{p^z}). \]
(3.15)

By inserting (3.15) into (3.14) we deduce
\[ H(z) = \zeta^2(2z) \sum_{k=0}^{\infty} \frac{2^{\omega(k)}h_z(k)}{k^z}, \]
where \( h_z \) is the multiplicative function given by \( h_z(k) = \prod_p \left( \nu_p(k) + 1 - \frac{\nu_p(k)}{p^z} \right) \), depending on \( z \). Therefore, by the Euler product formula,
\[ H(z) = \zeta^2(2z) \prod_p \left( 1 + \sum_{\nu=1}^{\infty} \frac{2}{p^{\nu z}} \left( \nu + 1 - \frac{\nu}{p^{2z}} \right) \right) \]
\[ = \zeta^2(2z) \prod_p \left( 1 + 2(1 - \frac{1}{p^{2z}}) \sum_{\nu=1}^{\infty} \frac{\nu + 1}{p^{\nu z}} + \sum_{\nu=1}^{\infty} \frac{1}{p^{\nu z}} \right) \]
\[ = \zeta^2(2z) \prod_p \left( 1 + 2 \left( 1 - \frac{1}{p^{2z}} \right) \left( 1 - \frac{1}{p^z} \right)^{-2} - 1 \right) \]
\[ = \zeta^2(2z) \prod_p \left( 1 + 2 \left( 1 - \frac{1}{p^z} \right) \left( 1 - \frac{1}{p^z} \right)^{-1} - 2 \left( 1 - \frac{1}{p^{2z}} \right) \right) \]
\[ = \zeta^2(2z) \prod_p \left( 1 + 2 \left( 1 + \frac{1}{p^z} \right) \left( 1 - \frac{1}{p^z} \right)^{-2} - 2 \left( 1 - \frac{1}{p^{2z}} \right) \right) \]
\[ = \zeta(z)\zeta^2(2z) \prod_p \left(1 + \frac{1}{p^z}\right) \left(1 + \frac{2}{p^z}\right) \]

\[ = \zeta^2(z) \zeta(2z) \prod_p \left(1 + \frac{2}{p^z}\right), \]

which can be written as

\[ H(z) = \frac{\zeta^4(z)}{\zeta(2z)} G(z), \]

where

\[ G(z) = \prod_p \left(1 - \frac{1}{(p^z + 1)^2}\right) = \prod_p \sum_{\nu=0}^{\infty} \frac{(-1)^\nu - 1 (\nu - 1)}{p^{\nu z}}. \tag{3.16} \]

Here \( \zeta^4(z) = \sum_{n=1}^{\infty} \frac{\tau^n(n)}{n^z} \), as well known. This proves identity (3.13).

The infinite product in (3.16) is absolutely convergent for \( \Re z > 1/2 \), Using Ramanujan’s formula

\[ \sum_{n\leq x} \tau^2(n) = x(a(\log x)^3 + b(\log x) + c\log x + d) + O(x^{1/2+\epsilon}), \]

where \( a = 1/\pi^2 \), the convolution method leads to asymptotic formula (2.17). The main coefficient is \((1/\pi^2)G(1) = (1/\pi^2) \prod_p \left(1 - \frac{1}{(p+1)^2}\right)\). See the similar proof of [15, Th. 1]. \(\square\)

**Proof of Theorem 2.11.** We have

\[ \sum_{ab \leq x} \frac{\phi(n)}{[a,b]^2} = \sum_{n \leq x} \frac{1}{n} \sum_{ab=n} (a, b)^2. \tag{3.17} \]

Let \( f(n) = n^2 \). Then \((\mu * f)(n) = \phi_2(n) = n^2 \prod_{p|n} (1 - 1/p^2)\) is the Jordan function of order 2. Here Theorems 2.2 and 2.3 cannot be applied. However, the estimate

\[ \sum_{n \leq x} \phi_2(n) = \frac{x^3}{3\zeta(3)} + O(x^2), \]

is well-known, and using identity (2.2) we deduce that

\[ \sum_{ab \leq x} (a, b)^2 = \sum_{d^2k=n} \phi_2(d)\tau(k) = \sum_{k \leq x} \sum_{d \leq (x/k)^{1/2}} \phi_2(d) \]

\[ = \frac{x^{3/2}}{3\zeta(3)} \sum_{k \leq x} \frac{\tau(k)}{k^{3/2}} + O \left( x \sum_{k \leq x} \frac{\tau(k)}{k} \right) = \frac{\zeta^2(3/2)}{3\zeta(3)} x^{3/2} + O \left( x(\log x)^2 \right). \]

Now, taking into account (3.17), partial summation concludes formula (2.18). \(\square\)
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