ON CERTAIN COMBINATORIAL EXPANSIONS OF THE LEGENDRE-STIRLING NUMBERS

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Abstract. The Legendre-Stirling numbers of the second kind were introduced by Everitt et al. in the spectral theory of powers of the Legendre differential expressions. In this paper, we provide a combinatorial code for Legendre-Stirling set partitions. As an application, we obtain combinatorial expansions of the Legendre-Stirling numbers of both kinds. Moreover, we present grammatical descriptions of the Jacobi-Stirling numbers of both kinds.

Keywords: Legendre-Stirling numbers; Jacobi-Stirling numbers; Context-free grammars

1. Introduction

Let \( \ell[y](t) = -(1-t^2)y''(t) + 2ty'(t) \) be the Legendre differential operator. Then the Legendre polynomial \( y(t) = P_n(t) \) is an eigenvector for the differential operator \( \ell \) corresponding to \( n(n+1) \), i.e., \( \ell[y](t) = n(n+1)y(t) \). Following Everitt et al. [7], for \( n \in \mathbb{N} \), the Legendre-Stirling numbers \( \text{LS}(n, k) \) of the second kind appeared originally as the coefficients in the expansion of the \( n \)-th composite power of \( \ell \), i.e.,

\[
\ell^n[y](t) = \sum_{k=0}^{n} (-1)^k \text{LS}(n, k)((1-t^2)^k y^{(k)}(t))^{(k)}.
\]

For each \( k \in \mathbb{N} \), Everitt et al. [7, Theorem 4.1] obtained that

\[
\prod_{r=1}^{k} \frac{1}{1-r(r+1)x} = \sum_{n=0}^{\infty} \text{LS}(n, k)x^{n-k}, \quad \left| \frac{x}{k(k+1)} \right| \leq \frac{1}{k(k+1)}, \tag{1}
\]

\[
\text{LS}(n, k) = \sum_{r=0}^{k} (-1)^{r+k} \frac{(2r+1)(r^2+r)^n}{(r+k+1)!(k-r)!}.
\]

According to [2, Theorem 5.4], the numbers \( \text{LS}(n, k) \) have the following horizontal generating function

\[
x^n = \sum_{k=0}^{n} \text{LS}(n, k) \prod_{i=0}^{k-1} (x - i(1+i)). \tag{2}
\]

It follows from (2) that the numbers \( \text{LS}(n, k) \) satisfy the recurrence relation

\[
\text{LS}(n, k) = \text{LS}(n-1, k-1) + k(k+1)\text{LS}(n-1, k),
\]

with the initial conditions \( \text{LS}(n, 0) = \delta_{n,0} \) and \( \text{LS}(0, k) = \delta_{0,k} \), where \( \delta_{i,j} \) is the Kronecker’s symbol.
By using (1), Andrews et al. [2, Theorem 5.2] derived that the numbers \( \text{LS}(n, k) \) satisfy the vertical recurrence relation

\[
\text{LS}(n, j) = \sum_{k=j}^{n} \text{LS}(k-1, j-1)(j(j+1))^{n-k}.
\]

A particular value of \( \text{LS}(n, k) \) is provided at the end of [3]:

\[
\text{LS}(n+1, n) = 2 \binom{n+2}{3}.
\]  (3)

In [6, Eq. (19)], Egge found that

\[
\text{LS}(n+2, n) = 40 \binom{n+2}{6} + 72 \binom{n+2}{5} + 36 \binom{n+2}{4} + 4 \binom{n+2}{3}.
\]

Using the triangular recurrence relation \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \), we get

\[
\text{LS}(n+2, n) = 40 \binom{n+3}{6} + 32 \binom{n+3}{5} + 4 \binom{n+3}{4}.
\]  (4)

Egge [6, Theorem 3.1] showed that for \( k \geq 0 \), the quantity \( \text{LS}(n+k, n) \) is a polynomial of degree \( 3k \) in \( n \) with leading coefficient \( \frac{1}{3^k k!} \).

This paper is a continuation of [6], and it is motivated by the following problem.

**Problem 1.** Let \( k \) be a given nonnegative integer. Could the numbers \( \text{LS}(n+k, n) \) be expanded in the binomial basis?

The paper is organized as follows. In Section 2, by introducing a combinatorial code for Legendre-Stirling set partitions, we give a solution of Problem 1. Moreover, we get a combinatorial expansion of the Legendre-Stirling numbers of the first kind. In Section 3, we present grammatical interpretations of Jacobi-Stirling numbers of both kinds.

2. **Legendre-Stirling set partitions**

The combinatorial interpretation of the Legendre-Stirling numbers \( \text{LS}(n, k) \) of the second kind was first given by Andrews and Littlejohn [3]. For \( n \geq 1 \), let \( M_n \) denote the multiset \( \{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\} \), in which we have one unbarred copy and one barred copy of each integer \( i \), where \( 1 \leq i \leq n \). Throughout this paper, we always assume that the elements of \( M_n \) are ordered by

\[
\overline{1} = 1 < \overline{2} = 2 < \cdots < \overline{n} = n.
\]

A **Legendre-Stirling set partition** of \( M_n \) is a set partition of \( M_n \) with \( k+1 \) blocks \( B_0, B_1, \ldots, B_k \) and with the following rules:

- (\( r_1 \)) The ‘zero box’ \( B_0 \) is the only box that may be empty and it may not contain both copies of any number;
- (\( r_2 \)) The ‘nonzero boxes’ \( B_1, B_2, \ldots, B_k \) are indistinguishable and each is non-empty. For any \( i \in [k] \), the box \( B_i \) contains both copies of its smallest element and does not contain both copies of any other number.
Let $\mathcal{LS}(n, k)$ denote the set of Legendre-Stirling set partitions of $M_n$ with one zero box and $k$ nonzero boxes. The standard form of an element of $\mathcal{LS}(n, k)$ is written as

$$\sigma = B_1 B_2 \cdots B_k B_0,$$

where $B_0$ is the zero box and the minima of $B_i$ is less than that of $B_j$ when $1 \leq i < j \leq k$. Clearly, the minima of $B_1$ are 1 and $\mathbf{T}$. Throughout this paper we always write $\sigma \in \mathcal{LS}(n, k)$ in the standard form. As usual, we let angle bracket symbol $< i, j, \ldots >$ and curly bracket symbol $\{ k, \mathbf{F}, \ldots \}$ denote the zero box and nonzero box, respectively. In particular, let $<>$ denote the empty zero box. For example, $\{1, \mathbf{T}, 3\} \{2, \mathbf{F}\} < \mathbf{F} > \in \mathcal{LS}(3, 2)$. A classical result of Andrews and Littlejohn \[ Theorem 2\] says that

$$\mathcal{LS}(n, k) = \# \mathcal{LS}(n, k).$$

We now provide a combinatorial code for Legendre-Stirling partitions (CLS-sequence for short).

**Definition 2.** We call $Y_n = (y_1, y_2, \ldots, y_n)$ a CLS-sequence of length $n$ if $y_1 = X$ and

$$y_{k+1} \in \{ X, A_{i, j}, B_s, \overline{B}_s, 1 \leq i, j, s \leq n_x(Y_k), i \neq j \} \quad \text{for} \ k = 1, 2, \ldots, n - 1,$$

where $n_x(Y_k)$ is the number of the symbol $X$ in $Y_k = (y_1, y_2, \ldots, y_k)$.

For example, $(X, X, A_{1, 2})$ is a CLS-sequence, while $(X, X, A_{1, 2}, B_3)$ is not since $y_4 = B_3$ and $3 > n_x(Y_3) = 2$. Let $\mathcal{CLS}_n$ denote the set of CLS-sequences of length $n$.

The following lemma is a fundamental result.

**Lemma 3.** For $n \geq 1$, we have $\mathcal{LS}(n, k) = \# \{ Y_n \in \mathcal{CLS}_n \mid n_x(Y_n) = k \}$.

**Proof.** Let

$$\mathcal{CLS}(n, k) = \{ Y_n \in \mathcal{CLS}_n \mid n_x(Y_n) = k \}.$$

Now we start to construct a bijection, denoted by $\Phi$, between $\mathcal{LS}(n, k)$ and $\mathcal{CLS}(n, k)$. When $n = 1$, we have $y_1 = X$. Set $\Phi(Y_1) = \{1, \mathbf{T}\} < >$. This gives a bijection from $\mathcal{CLS}(1, 1)$ to $\mathcal{LS}(1, 1)$. Let $n = m$. Suppose $\Phi$ is a bijection from $\mathcal{CLS}(n, k)$ to $\mathcal{CLS}(n, k)$ for all $k$. Consider the case $n = m + 1$. Let

$$Y_{m+1} = (y_1, y_2, \ldots, y_m, y_{m+1}) \in \mathcal{CLS}_{m+1}.$$

Then $Y_m = (y_1, y_2, \ldots, y_m) \in \mathcal{CLS}(m, k)$ for some $k$. Assume $\Phi(Y_m) \in \mathcal{LS}(m, k)$. Consider the following three cases:

(i) If $y_{m+1} = X$, then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by putting the box $\{m + 1, m + \mathbf{T}\}$ just before the zero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m + 1, k + 1)$.

(ii) If $y_{m+1} = A_{i, j}$, then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by inserting the entry $m + 1$ to the $i$th nonzero box and inserting the entry $m + \mathbf{T}$ to the $j$th nonzero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m + 1, k)$.

(iii) If $y_{m+1} = B_s$ (resp. $y_{m+1} = \overline{B}_s$), then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by inserting the entry $m + 1$ (resp. $m + \mathbf{T}$) to the $s$th nonzero box and inserting the entry $m + \mathbf{T}$ (resp. $m + 1$) to the zero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m + 1, k)$. 


After the above step, it is clear that the obtained \( \Phi(Y_{n+1}) \) is in standard form. By induction, we see that \( \Phi \) is the desired bijection from \( \mathcal{CLS}(n,k) \) to \( \mathcal{CLS}(n,k) \), which also gives a constructive proof of Lemma 3.

Example 4. Let \( Y_5 = (X, X, A_{2,1}, B_2, \overline{B}_1) \). The correspondence between \( Y_5 \) and \( \Phi(Y_5) \) is built up as follows:

\[
X \leftrightarrow \{1, \overline{1}\} < > ; \\
X \leftrightarrow \{1, \overline{1}\} \{2, \overline{2}\} < > ; \\
A_{2,1} \leftrightarrow \{1, \overline{1, 3}\} \{2, \overline{2, 3}\} < > ; \\
B_2 \leftrightarrow \{1, \overline{1, 3}\} \{2, \overline{2, 3}, 4\} < 4 > ; \\
\overline{B}_1 \leftrightarrow \{1, \overline{1, 3, 5}\} \{2, \overline{2, 3, 4}\} < 4, 5 > .
\]

As an application of the CLS-sequences, we present the following result.

Lemma 5. Let \( k \) be a given positive integer. Then for \( n \geq 1 \), we have

\[
\text{LS}(n+k, n) = 2^k \sum_{t_k=1}^{n} \left( \frac{t_k + 1}{n} \right) \sum_{t_{k-1}=1}^{t_k} \left( \frac{t_{k-1} + 1}{2} \right) \cdots \sum_{t_2=1}^{t_3} \left( \frac{t_2 + 1}{2} \right) \sum_{t_1=1}^{t_2} \left( \frac{t_1 + 1}{2} \right).
\]

Proof. It follows from Lemma 3 that

\[
\text{LS}(n+k, n) = \#\{Y_{n+k} \in \mathcal{CLS}_{n+k} \mid n_x(Y_{n+k}) = n\}.
\]

Let \( Y_{n+k} = y_1y_2 \cdots y_{n+k} \) be a given element in \( \mathcal{CLS}_{n+k} \). Since \( n_x(Y_{n+k}) = n \), it is natural to assume that \( y_i = X \) except \( i = \ell_1 + 1, \ell_2 + 2, \ldots, t_k + k \). Let \( \sigma \) be the corresponding Legendre-Stirling partition of \( Y_{n+k} \). For \( 1 \leq \ell \leq k \), consider the value of \( y_{\ell+\ell} \). Note that the number of the symbol \( X \) before \( y_{\ell+\ell} \) is \( t_\ell \). Let \( \hat{\sigma} \) be the corresponding Legendre-Stirling partition of \( y_1y_2 \cdots y_{\ell+\ell-1} \). Now we insert \( y_{\ell+\ell} \). We distinguish two cases:

(i) If \( y_{\ell+\ell} = A_{i,j} \), then we should insert the entry \( t_\ell + \ell \) to the \( i \)th nonzero box of \( \hat{\sigma} \) and insert \( t_\ell + \ell \) to the \( j \)th nonzero box. This gives \( 2t_i \binom{t'_i}{2} \) possibilities, since \( 1 \leq i, j \leq t_\ell \) and \( i \neq j \).

(ii) If \( y_{\ell+\ell} = B_s \) (resp. \( y_{\ell+\ell} = \overline{B}_s \)), then we should insert the entry \( t_\ell + \ell \) (resp. \( t_\ell + \ell \)) to the \( s \)th nonzero box of \( \hat{\sigma} \) and insert \( t_\ell + \ell \) (resp. \( t_\ell + \ell \)) to the zero box. This gives \( 2t'_1 \binom{t'_1}{1} \) possibilities, since \( 1 \leq s \leq t_\ell \).

Therefore, there are exactly \( 2t_i \binom{t'_i}{2} + 2t'_1 \binom{t'_1}{1} = 2t_i t'_{i+1} \) Legendre-Stirling partitions of \( M_{t_i+\ell} \) can be generated from \( \hat{\sigma} \) by inserting the entry \( y_{\ell+\ell} \). Note that \( 1 \leq t_{j-1} \leq t_j \leq n \) for \( 2 \leq j \leq k \). Applying the product rule for counting, we immediately get (5). \( \square \)

The following simple result will be used in our discussion.

Lemma 6. Let \( a \) and \( b \) be given integers. Then

\[
\binom{x-b}{2} \binom{x}{a} = \binom{a+2}{2} \binom{x}{a+2} + (a+1)(a-b) \binom{x}{a+1} + (a-b) \binom{x}{2} \binom{x}{a}.
\]
In particular,\[
\binom{x-1}{2} \binom{x}{a} = \binom{a+2}{2} \binom{x}{a+2} + (a^2-1) \binom{x}{a+1} + \binom{a-1}{2} \binom{x}{a}.
\]

**Proof.** Note that\[
\binom{a+2}{2} \binom{x-a}{2} \binom{x-a-1}{a+1} + (a+1)(a-b) \binom{x-a}{a+1} + \binom{a-b}{2} = \binom{x-b}{2}.
\]
This yields the desired result. \(\square\)

We can now conclude the main result of this paper from the discussion above.

**Theorem 7.** Let \(k\) be a given nonnegative integer. Then for \(n \geq 1\), the numbers \(\text{LS}(n+k,n)\) can be expanded in the binomial basis as

\[
\text{LS}(n+k,n) = 2^k \sum_{i=k+2}^{3k} \gamma(k,i) \binom{n+k+1}{i},
\]

where the coefficients \(\gamma(k,i)\) are all positive integers for \(k+2 \leq i \leq 3k\) and satisfy the recurrence relation

\[
\gamma(k+1,i) = \left(\binom{i-k-1}{2}\right) \gamma(k,i-1) + (i-1)(i-k-2) \gamma(k,i-2) + \binom{i-1}{2} \gamma(k,i-3),
\]

with the initial conditions \(\gamma(0,0) = 1\), \(\gamma(0,i) = \gamma(i,0) = 0\) for \(i \neq 0\). Let \(\gamma_k(x) = \sum_{i=k+2}^{3k} \gamma(k,i)x^i\). Then the polynomials \(\gamma_k(x)\) satisfy the recurrence relation

\[
\gamma_{k+1}(x) = \left(\frac{k(k+1)}{2} - 2kx + x^2\right) x \gamma_k(x) - \left(k + (k-2)x - 2x^2\right) \gamma'_k(x) + \frac{(1+x)^2x^3}{2} \gamma''_k(x),
\]

with the initial conditions \(\gamma_0(x) = 1\), \(\gamma_1(x) = x^3\) and \(\gamma_2(x) = x^4 + 8x^5 + 10x^6\).

**Proof.** We prove (6) by induction on \(k\). It is clear that

\[
\text{LS}(n,n) = 1 = \binom{n+1}{0}.
\]

When \(k = 1\), by using the Chu Shih-Chieh’s identity

\[
\binom{n+1}{k+1} = \sum_{i=k}^{n} \binom{i}{k},
\]

we obtain

\[
\sum_{t_1=1}^{n} \binom{t_1+1}{2} = \binom{n+2}{3},
\]

and so (3) is established. When \(k = 2\), it follows from Lemma 5 that

\[
\text{LS}(n+2,n) = 4 \sum_{t_2=1}^{n} \binom{t_2+1}{2} \sum_{t_1=1}^{t_2} \binom{t_1+1}{2} = 4 \sum_{t_2=1}^{n} \binom{t_2+1}{2} \binom{t_2+2}{3}.
\]
Setting \( x = t_2 + 2 \) and \( a = 3 \) in Lemma 6 we get
\[
LS(n + 2, n) = 4 \sum_{t_2=1}^{n} \left( 10 \left( \frac{t_2 + 2}{5} \right) + 8 \left( \frac{t_2 + 2}{4} \right) + \left( \frac{t_2 + 2}{3} \right) \right)
\]
\[
= 4 \left( 10 \left( \frac{n + 3}{6} \right) + 8 \left( \frac{n + 3}{5} \right) + \left( \frac{n + 3}{4} \right) \right),
\]
which yields (4). Along the same lines, it is not hard to verify that
\[
LS(n + 3, n) = 8 \sum_{t_3=1}^{n} \left( \frac{t_3 + 1}{2} \right) \left( 10 \left( \frac{t_3 + 3}{6} \right) + 8 \left( \frac{t_3 + 3}{5} \right) + \left( \frac{t_3 + 3}{4} \right) \right)
\]
\[
= 8 \left( 280 \left( \frac{n + 4}{9} \right) + 448 \left( \frac{n + 4}{8} \right) + 219 \left( \frac{n + 4}{7} \right) + 34 \left( \frac{n + 4}{6} \right) + \left( \frac{n + 4}{5} \right) \right).
\]
Hence the formula (6) holds for \( k = 0, 1, 2, 3 \), so we proceed to the inductive step. For \( k \geq 3 \), assume that
\[
LS(n + k, n) = 2^k \sum_{i=k+2}^{3k} \gamma(k, i) \binom{n + k + 1}{i}.
\]
It follows from Lemma 5 that
\[
LS(n + k + 1, n) = 2^{k+1} \sum_{t_{k+1}=1}^{n} \left( \frac{t_{k+1} + 1}{2} \right) \sum_{i=k+2}^{3k} \gamma(k, i) \binom{t_{k+1} + k + 1}{i}
\]
By using Lemma 6, it is routine to verify that the coefficients \( \gamma(k, i) \) satisfy the recurrence relation (7), and so (6) is established for general \( k \). Multiplying both sides of (7) by \( x^i \) and summing for all \( i \), we immediately get (8).

In [2], Andrews et al. introduced the \((unsigned) Legendre-Stirling numbers \( Lc(n, k) \) of the first kind, which may be defined by the recurrence relation
\[
Lc(n, k) = Lc(n - 1, k - 1) + n(n - 1)Lc(n - 1, k),
\]
with the initial conditions \( Lc(n, 0) = \delta_{n,0} \) and \( Lc(0, n) = \delta_{0,n} \). Let \( f_k(n) = LS(n + k, n) \). According to Egge [6 Eq. (23)], we have
\[
Lc(n - 1, n - k - 1) = (-1)^k f_k(n)
\]
for \( k \geq 0 \). For \( m, k \in \mathbb{N} \), we define
\[
\binom{-m}{k} = \frac{(-m)(-m - 1) \cdots (-m - k + 1)}{k!}.
\]
Combining (6) and (9), we immediately get the following result.

**Corollary 8.** Let \( k \) be a given nonnegative integer. For \( n \geq 1 \), the numbers \( Lc(n - 1, n - k - 1) \) can be expanded in the binomial basis as
\[
Lc(n - 1, n - k - 1) = (-1)^k 2^k \sum_{i=k+2}^{3k} \gamma(k, i) \binom{-n + k + 1}{i},
\]
where the coefficients \( \gamma(k, i) \) are defined by (7).
It follows from (8) that
\[
\gamma(k + 1, k + 3) = \left(\frac{k(k + 1)}{2} - k(k + 2) + \frac{(k + 2)(k + 1)}{2}\right) \gamma(k, k + 2),
\]
\[
\gamma(k + 1, 3k + 3) = \left(1 + 6k + \frac{3k(3k - 1)}{2}\right) \gamma(k, 3k),
\]
\[
\gamma_{k+1}(-1) = -\left(\frac{k(k + 1)}{2} + k + 1\right) \gamma(-1).
\]
Since \(\gamma(1, 3) = 1\) and \(\gamma_1(-1) = -1\), it is easy to verify that for \(k \geq 1\), we have
\[
\gamma(k, k + 2) = 1, \quad \gamma(k, 3k) = (3k)! k! (3!)^k, \quad \gamma_1(-1) = (-1)^k \frac{(k + 1)! k!}{2^k}.
\]
It should be noted that the number \(\gamma(k, 3k)\) is the number of partitions of \(\{1, 2, \ldots, 3k\}\) into blocks of size 3 (see [18, A025035]), and the number \(\frac{(k+1)! k!}{2^k}\) is the product of first \(k\) positive triangular numbers (see [18, A006472]). Moreover, if the number \(\text{LS}(n + k, n)\) is viewed as a polynomial in \(n\), then its degree is \(3k\), which is implied by the quantity \(\binom{n + k + 1}{3k}\). Furthermore, the leading coefficient of \(\text{LS}(n + k, n)\) is given by
\[
2^k \gamma(k, 3k) \frac{1}{(3k)!} = 2^k \frac{(3k)!}{k!(3!)^k} \frac{1}{(3!)^k} = \frac{1}{k!3^k},
\]
which yields [6, Theorem 3.1].

3. Grammatical interpretations of Jacobi-Stirling numbers of both kinds

In this section, a context-free grammar is in the sense of Chen [4]: for an alphabet \(A\), let \(Q[[A]]\) be the rational commutative ring of formal power series in monomials formed from letters in \(A\). A context-free grammar over \(A\) is a function \(G: A \to Q[[A]]\) that replace a letter in \(A\) by a formal function over \(A\). The formal derivative \(D\) is a linear operator defined with respect to a context-free grammar \(G\). More precisely, the derivative \(D = DG: Q[[A]] \to Q[[A]]\) is defined as follows: for \(x \in A\), we have \(D(x) = G(x)\); for a monomial \(u\) in \(Q[[A]]\), \(D(u)\) is defined so that \(D\) is a derivation, and for a general element \(q \in Q[[A]]\), \(D(q)\) is defined by linearity. The reader is referred to [5, 14] for recent progress on this subject.

Let \([n] = \{1, 2, \ldots, n\}\). The Stirling number \(\binom{n}{k}\) of the second kind is the number of ways to partition \([n]\) into \(k\) blocks. Chen [4, Eq. 4.8] showed that if \(G = \{x \to xy, y \to y\}\), then
\[
D^n(x) = x \sum_{k=0}^{n} \binom{n}{k} y^k.
\]
Let \(\mathcal{S}_n\) be the symmetric group of all permutations of \([n]\). Let \(\text{cyc}(\pi)\) be the number of cycles of \(\pi\). The (unsigned) \textit{Stirling number of the first kind} is defined by
\[
\binom{n}{k} = \#\{\pi \in \mathcal{S}_n \mid \text{cyc}(\pi) = k\}.
\]
From [13, Eq. 4.8], we see that if \(G = \{x \to xy, y \to yz, z \to z^2\}\), then
\[
D^n(x) = x \sum_{k=0}^{n} \binom{n}{k} y^k z^{n-k}.
\]
According to [8, Theorem 4.1], the Jacobi-Stirling number $JS_n^k(z)$ of the second kind is defined by

$$x^n = \sum_{k=0}^{n} JS_n^k(z) \prod_{i=0}^{k-1} (x - i(z + i)).$$

(11)

It follows from (11) that the numbers $JS_n^k(z)$ satisfy the recurrence relation

$$JS_n^k(z) = JS_{n-1}^{k-1}(z) + k(k + z)JS_{n-1}^k(z),$$

with the initial conditions $JS_0^0(z) = \delta_{n,0}$ and $JS_0^k(z) = \delta_{0,k}$. It is clear that $JS_n^k(1) = LS(n,k)$. Following [10, Eq. (1.3), Eq. (1.5)], the (unsigned) Jacobi-Stirling number $Jc_n^k(z)$ of the first kind is defined by

$$\prod_{i=0}^{n-1} (x + i(z + i)) = \sum_{k=0}^{n} Jc_n^k(z)x^k,$$

and the numbers $Jc_n^k(z)$ satisfy the following recurrence relation

$$Jc_n^k(z) = Jc_{n-1}^{k-1}(z) + (n - 1)(n - 1 + z)Jc_{n-1}^k(z),$$

with the initial conditions $Jc_0^0(z) = \delta_{n,0}$ and $Jc_0^k(z) = \delta_{k,0}$. In particular, $Jc_n^k(1) = Lc(n,k)$.

Properties and combinatorial interpretations of the Jacobi-Stirling numbers of both kinds were extensively studied in [1, 10, 11, 12, 15, 16, 17]. The Jacobi-Stirling numbers share many similar properties to those of the Stirling numbers. A question arises immediately: are there grammatical descriptions of the Jacobi-Stirling numbers of both kinds? In this section, we give the answer.

As a variant of the CLS-sequence, we now introduce a marked scheme for Legendre-Stirling partitions. Given a Legendre-Stirling partition $\sigma = B_1B_2\cdots B_kB_0 \in LS(n,k)$, where $B_0$ is the zero box of $\sigma$. We mark the box vector $(B_1, B_2, \ldots, B_k)$ by the label $a_k$. We mark any box pair $(B_i, B_j)$ by a label $b$ and mark any box pair $(B_s, B_0)$ by a label $c$, where $1 \leq i < j \leq k$ and $1 \leq s \leq k$. Let $\sigma'$ denote the Legendre-Stirling partition that generated from $\sigma$ by inserting $n + 1$ and $n + 1$ in the same box, then

$$\sigma' = B_1B_2\cdots B_kB_{k+1}B_0,$$

where $B_{k+1} = \{n + 1, n + 1\}$. This case corresponds to the operator $a_k \rightarrow a_{k+1}b^c$. If $n + 1$ and $n + 1$ are in different boxes, then we distinguish two cases:

(i) Given a box pair $(B_i, B_j)$, where $1 \leq i < j \leq k$. We can put $n + 1$ (resp. $n + 1$) into the box $B_i$ and put $n + 1$ (resp. $n + 1$) into the box $B_j$. This case corresponds to the operator $b \rightarrow 2b$.

(ii) Given a box pair $(B_i, B_0)$, where $1 \leq i \leq k$. We can put $n + 1$ (resp. $n + 1$) into the box $B_i$ and put $n + 1$ (resp. $n + 1$) into the zero box $B_0$. Moreover, we mark any barred entry in the zero box $B_0$ by a label $z$. This case corresponds to the operator $c \rightarrow (1 + z)c$.

Let $A = \{a_0, a_1, a_2, a_3, \ldots, b, c\}$ be a set of alphabet. Using the above marked scheme, it is natural to consider the following grammars:

$$G_k = \{a_0 \rightarrow a_1c, a_1 \rightarrow a_2bc, \ldots, a_{k-1} \rightarrow a_kb^{k-1}c, b \rightarrow 2b, c \rightarrow (1 + z)c\}$$

(12)

where $k \geq 1$. 

Theorem 9. Let $G_k$ be the grammars defined by (12). Then we have

$$D_nD_{n-1} \cdots D_1(a_0) = \sum_{k=1}^{n} JS_n^k(z)a_kb(z)^{c^k}.$$ 

Proof. Note that $D_1(a_0) = a_1c$ and $D_2D_1(a_0) = a_2bc^2 + (1 + z)a_1c$. Thus the result holds for $n = 1, 2$. For $m \geq 2$, we define $P_m^k(z)$ by

$$D_mD_{m-1} \cdots D_1(a_0) = \sum_{k=1}^{n} P_m^k(z)a_kb(z)^{c^k}.$$ 

We proceed by induction. Consider the case $n = m + 1$. Since

$$D_{m+1}D_mD_{m-1} \cdots D_1(a_0) = D_{m+1}(D_mD_{m-1} \cdots D_1(a_0)),$$

it follows that

$$D_{m+1}D_m \cdots D_1(a_0) = D_{m+1} \left( \sum_{k=1}^{n} P_m^k(z)a_kb(z)^{c^k} \right)$$

$$= \sum_{k=1}^{n} P_m^k(z) \left( a_{k+1}b(z)^{k+2} + k(k-1)a_kb(z)^{c^k} + (1 + z)k a_kb(z)^{c^k} \right).$$

Therefore, we obtain $P_{m+1}^k(z) = P_{m}^{k-1}(z) + k(k+1)P_{m}^k(z)$. Since the numbers $P_m^k(z)$ and $JS_n^k(z)$ satisfy the same recurrence relation and initial conditions, so they agree. 

Combining the marked scheme for Legendre-Stirling partitions and Theorem 9, it is clear that for $n \geq k$, the number $JS_n^k(z)$ is a polynomial of degree $n - k$ in $z$, and the coefficient $z^i$ of $JS_n^k(z)$ is the number of Legendre-Stirling partitions in $LS(n, k)$ with exactly $i$ barred entries in the zero box, which gives a proof of [10, Theorem 2].

We end this section by giving the following result.

Theorem 10. Let $A = \{a, b_0, b_1, \ldots\}$ be a set of alphabet. Let $G_k$ be the grammars defined by

$$G_k = \{ a \rightarrow (k-1)(k-1+z)a, b_0 \rightarrow b_1, b_1 \rightarrow b_2, \ldots, b_{k-1} \rightarrow b_k \},$$

where $k \geq 1$. Then we have

$$D_nD_{n-1} \cdots D_1(ab_0) = a \sum_{k=1}^{n} JS_n^k(z)b_k.$$ 

Proof. Note that $D_1(ab_0) = ab_1$ and $D_2D_1(ab_0) = (1 + z)ab_1 + ab_2$. Hence the result holds for $n = 1, 2$. For $m \geq 2$, we define $Q_m^k(z)$ by $D_mD_{m-1} \cdots D_1(ab_0) = a \sum_{k=1}^{m} Q_m^k(z)b_k$. We proceed by induction. Consider the case $n = m + 1$. Since

$$D_{m+1}D_mD_{m-1} \cdots D_1(ab_0) = D_{m+1}(D_mD_{m-1} \cdots D_1(ab_0)),$$

it follows that

$$D_{m+1}D_m \cdots D_1(ab_0) = D_{m+1} \left( a \sum_{k=1}^{m} Q_m^k(z)b_k \right)$$

$$= a \sum_{k=1}^{m} Q_m^k(z)m(m+z)b_k + a \sum_{k=1}^{m} Q_m^k b_{k+1}.$$
Therefore, we obtain $Q_{m+1}^k(z) = Q_{m}^{k-1}(z) + m(m+z)Q_{m}^k(z)$. Since the numbers $Q_n^k(z)$ and $J_n^k(z)$ satisfy the same recurrence relation and initial conditions, so they agree. $\square$

4. Concluding remarks

Note that the Jacobi-Stirling numbers are polynomial refinements of the Legendre-Stirling numbers. It would be interesting to explore combinatorial expansions of Jacobi-Stirling numbers of both kinds.

Let $\gamma_k(x)$ be the polynomials defined by (8). We end our paper by proposing the following.

**Conjecture 11.** For any $k \geq 1$, the polynomial $\gamma_k(x)$ has only real zeros. Set

$$\gamma_k(x) = \gamma(k, 3k)x^{k+2} \prod_{i=1}^{2k-2} (x - r_i), \quad \gamma_{k+1}(x) = \gamma(k + 1, 3k + 3)x^{k+3} \prod_{i=1}^{2k} (x - s_i),$$

where $r_{2k-2} < r_{2k-3} < \cdots < r_2 < r_1$ and $s_{2k} < s_{2k-1} < s_{2k-2} < \cdots < s_2 < s_1$. Then

$$s_{2k} < s_{2k-2} < s_{2k-1} < s_{2k-3} < s_{2k-2} < \cdots < r_k < s_{k+1} < s_k < r_{k-1} < \cdots < s_2 < r_1 < s_1,$$

in which the zeros $s_{k+1}$ and $s_k$ of $\gamma_{k+1}(x)$ are continuous appearance, and the other zeros of $\gamma_{k+1}(x)$ separate the zeros of $\gamma_k(x)$.

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