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On robust feedback for systems with multidimensional control

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The paper deals with local robust feedback synthesis for systems with multidimensional control and unknown bounded perturbations. Using V. I. Korobov’s controllability function method, we construct a bounded control which steers an arbitrary initial point to the origin in some finite time; an estimate from above for the time of motion is given. We have found the range of a segment where the perturbations can vary. As an example we consider the problem of stopping the oscillations of the system of two coupled pendulums.

Key words: controllability function method, systems with multidimensional control, robust feedback synthesis, finite-time stabilization, unknown bounded perturbations, uncertain systems.

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1 Introduction and Problem Statement

The paper deals with the synthesis problem, i.e. the problem of constructing a control which depends on phase coordinates and steers an arbitrary initial point from some neighborhood of the origin to the origin in some finite time. Besides the control should satisfy some preassigned constrains. In methods for solving the feedback synthesis problem for a linear system are given. Further we consider the synthesis problem for the linear system with continuous bounded unknown perturbations.

In the present paper we find such constraint for the unknown perturbations that the control which solves the synthesis problem for the system without the perturbation also solves the synthesis problem for the perturbed system.

For the first time, the concept of the feedback synthesis has been introduced and investigated in paper [2] written in Russian. In the English translation of this paper and in other papers of its author, this concept has been literally translated from Russian as ”positional synthesis”. Later the concept has been introduced and studied in [3] wherein it has been called “feedback synthesis”. Now, the term ”feedback synthesis” is generally used for the concept of the synthesis introduced in [2]. The controllability function method is introduced in [2]. In this method the angle between the direction of motion and the direction of decrease of the controllability function is not less than the corresponding angle in the dynamic programming method, and no more than in a method of Lyapunov function [1, p. 10]. The main advance of the controllability function method is finiteness of the motion time. Among other authors developing such approach we would like to mention [5]. Herein the concept of finite time stability involves the bounding of trajectories within specifical domains of the state space during a given finite time interval. A bit later, the problem of steering an arbitrary initial point from some neighborhood of the origin to the origin (or in general case in equilibrium point) in a finite time has
been called ”finite-time stabilization” (see, e.g., [6, 7]). In contrast to this problem the controllability function method is solve the problem of steering an arbitrary initial point to generally non-equilibrium point in a finite time. The paper [8] is devoted to the problem of construction of a constrained control, which transfers a control system from any point to a given non-equilibrium point in a finite time in global sense.

Let us consider the system

\[ \dot{x} = (A_0 + K + R(t, x))x + B_0u, \quad (1) \]

where \( t \geq 0, \ x \in Q \subset \mathbb{R}^n, \ Q \) is a neighborhood of the origin; \( u \in \mathbb{R}^r \) is a control satisfying the constraint \( \|u\| \leq 1; \) \( A_0 \) is \((n \times n)\) matrix of the form \( A_0 = \text{diag}(A_{01}, \ldots, A_{0r}) \), where \( A_{0i} \) are \((n_i \times n_i)\) matrices of the form

\[
A_{0i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, r; \quad n_1 \geq n_2 \geq \cdots \geq n_r \geq 1,
\]

\( n_1 + \ldots + n_r = n; \) \( B_0 \) is a \((n \times r)\) matrix whose elements \((B_0)_{s,i}\) are equal to 1, \( s_i = n_1 + \ldots + n_i, \ i = 1, \ldots, r \) and the others are equal to zero; the elements of matrix \( K \) which are in row \( s_i \) (in other words a row which contains a control) are equal to \( k_{s_1,j} \), and the other elements are equal to zero, \( R(t, x) = \text{diag}(R_1(t, x), \ldots, R_r(t, x)) + \tilde{R}(t, x), \ R_i(t, x) = \)

\[
= \begin{pmatrix} r_{(s_i-1)+1} & r_{(s_i-1)+2} & 0 & 0 & \cdots & 0 & 0 \\ r_{(s_i-2)+1} & r_{(s_i-2)+2} & r_{(s_i-2)+3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{s_i-1} & r_{s_i-1} & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad (2)
\]

the elements of matrix \( \tilde{R}(t, x) \) which are in row \( s_i \) (in other words, a row which contains a control) are equal to \( r_{s_1,j} \), and the other elements are equal to zero, \( r_{mj} = r_{mj}(t, x) \). We assume that functions \( r_{mj}(t, x) \) are unknown, and we call such systems robust systems, see for ex. [9] p. 173]. We assume that the functions \( r_{mj}(t, x) \) satisfy an imposed constraints

\[
\max_{1 \leq j \leq m+1 \leq n, \ i = 1, \ldots, r} |r_{mj}(t, x)| \leq \Delta. \quad (3)
\]

It is necessary to to find \( \Delta \) and to construct a bounded control which steers an arbitrary initial point \( x_0 \in Q \) to the origin in a finite time for any perturbation matrix \( R(t, x) \) under condition [9].

As a classical example of problem of this kind, we can mention the problem of control over the motion of a cart over the surface with an unknown bounded friction. The process of motion of this system is described by the following equations

\[
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = r_{22}(t, x_1, x_2)x_2 + u.
\end{cases}
\]
The term $r_{22}(t,x_1,x_2)x_2$ is sliding frictional force and $r_{22}(t,x_1,x_2)$ is the coefficient of the nonlinear viscous friction which is an unknown function and satisfies the constraint $|r_{22}(t,x_1,x_2)| \leq \Delta$. The constraint under consideration on $r_{22}(t,x_1,x_2)$ allow a "negative" friction.

The general approach to admissible control synthesis problem for an arbitrary nonlinear autonomous control system has been given by V. I. Korobov in [2]. In the same paper an estimate for the time of motion (settling-time function) from an arbitrary initial point to the origin has been given. Recently, the problem of finite-time stabilization has been formulated in several different ways [1], [9]-[15]. The article [16] describes a method for solving the feedback synthesis problem for systems with multidimensional control and without perturbations (i.e. $R(t,x)\equiv 0$). Moreover, in this case the controllability function is the time of motion. In [10], we have solved the robust synthesis problem for a case with one perturbation and a scalar control. In [11], the case when $R(t,x) = p(t,x)R$, $K \equiv 0$ and the control is scalar has been considered.

In [12], an adaptive fuzzy finite-time control scheme has been proposed for a class of nonlinear systems with unknown nonlinearities. The proposed scheme can guarantee that states of the closed-loop system converge to a small neighborhood of the origin in finite time. The book [9, p. 201] deals with the problem of robust stabilization for systems with constant affine perturbations. In [14], the Lyapunov function method has been suggested to analyze the finite-time stabilization of the system $\dot{x}(t) = A_0x + B_0u(t) + d(t,x(t))$, where $u(t)$ is a scalar function and $d(t,x)$ is measurable and uniformly bounded in the variable $t$ function. In [13, 14], the finite-time stabilization conditions have been formulated in the form of linear matrix inequalities. In [15], the problem of finite-time stabilization for the second order system of general form (or double integrator) with a scalar control has been considered.

First of all, we describe the conditions which the perturbations $r_{mj}(t,x)$ must satisfy.

**Definition 1.1.** By set of admissible perturbations $\mathcal{R}$ we denote a set of matrices $R(t,x)$ whose elements are functions $r_{mj}(t,x) : [0, +\infty) \times Q \to \mathbb{R}$ such that the following conditions are satisfied:

i) $r_{mj}(t,x)$ are continuous in variables $t$ and $x$;

ii) $\max_{1 \leq j \leq m+1 \leq n, i=1,\ldots,n} |r_{mj}(t,x)| \leq \Delta$ for all $(t,x) \in [0, +\infty) \times Q$;

iii) in each domain $\mathcal{R}_1(\rho_2) = \{(t,x) : 0 \leq t < +\infty, \|x\| \leq \rho_2\}$, the vector function $R(t,x)x$ satisfies the Lipschitz condition

$$|R(t,x'')x'' - R(t,x')x'| \leq \ell_1(\rho_2)\|x'' - x'\|.$$  

If $R(t,x) \equiv 0$, then (1) is canonical system: $\dot{x} = (A_0 + K)x + B_0u$. This concept has been introduced in [2] for the first time. Also this system has been called "chain of integrators system" (for second order system see for ex. [7]). In the analyzed approach, this system plays the key role because the solution of the synthesis problem for an arbitrary linear system with a multidimensional control can be reduced to the solution of the synthesis problem for the canonical system [1, p. 105]. The canonical system is completely controllable. In [1, Theorem 2.3]; [16] the control $u(x)$ which solves the synthesis problem for the canonical system is given.

**Definition 1.2.** The problem of finding such range of perturbations $r$ that the trajectory $x(t)$ of the
closed-loop system with the control \( u(x) \)
\[
\dot{x} = (A_0 + K + R(t, x))x + B_0u(x),
\]
\[ (4) \]
starting at an arbitrary initial point \( x(0) = x_0 \in Q \), ends at the origin at some finite time \( T(x_0, R) \),
i. e. \( \lim_{t \to T(x_0, R)} x(t) = 0 \), is said to be the local robust feedback synthesis. If \( Q = \mathbb{R}^n \), this problem is called the global robust feedback synthesis.

Obviously, if \( r_{11}(t, x) \equiv 0 \) and \( r_{12}(t, x) \equiv -1 \) then the first coordinate \( x_1 \) in (1) is uncontrollable; therefore, the problem will not be solvable for any value of \( \Delta \).

The paper is organized as follows. In Section 2, some basic concepts of the controllability function method are given. Section 3 represents the main results. In Section 3.3 we consider the problem of stopping the oscillations of the system of two coupled pendulums.

## 2 Background: the Controllability Function Method

In this Section we recall some basic concepts and some results of the controllability function method [1, 2]. Let us consider a nonlinear system of the form
\[
\dot{x} = f(x, u),
\]
\[ (5) \]
where \( x \in Q \subset \mathbb{R}^n \) and \( u \in \Omega \subset \mathbb{R}^r \), moreover, \( \Omega \) is such that \( 0 \in \text{int} \Omega, \ f(0,0)=0 \).

**Definition 2.1.** The problem of constructing a control of the form \( u = u(x), \ x \in Q \) is said to be the local feedback synthesis if:

i) \( u(x) \in \Omega \);

ii) the trajectory \( x(t) \) of the closed-loop system \( \dot{x} = f(x, u(x)) \), starting at an arbitrary initial point \( x_0 \in Q \), ends at the origin at some finite time \( T(x_0) \). If \( Q = \mathbb{R}^n \), the problem is called the global feedback synthesis.

The sufficient conditions for solvability the feedback synthesis for system (1) were formulated in [1, Theorem 1.1].

Let us describe one of possible approaches to the solution of the feedback synthesis for the canonical system [1, Theorem 2.3]; [16]:
\[
\dot{x} = (A_0 + K)x + B_0u,
\]
\[ (6) \]
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^r \) is a control which satisfies the constraint \( ||u|| \leq 1 \). It should be noted that system (1) coincides with the completely controllable system [1] when \( R(t, x) \equiv 0 \). Let us set
\[
F^{-1} = \int_{0}^{1} (1-t)e^{-A_0t}B_0B_0^*e^{-A_0^*t}dt.
\]
\[ (7) \]

Let \( D(\Theta) \) be a diagonal matrix of the form
\[
D(\Theta) = \text{diag}(D_1(\Theta), \ldots, D_r(\Theta)), \quad \text{where} \quad D_i(\Theta) = \text{diag} \left( \Theta^{-\frac{2n_i-2j+1}{2}} \right)_{j=1}^{n_i}.
\]
\[ (8) \]
Theorem 2.1. [1, Theorem 2.3] The controllability function \( \Theta = \Theta(x) \) is defined for \( x \neq 0 \) as a unique positive solution of the equation

\[
2a_0 \Theta = (D(\Theta)FD(\Theta)x, x),
\]

where the constant \( a_0 \) satisfies the inequality

\[
0 < a_0 \leq \frac{2}{\|F^{-1}\| \cdot (\|B_0^*F\| + 2 \max\{c^2, c\}\|B_0^*K\|)^2},
\]

besides the domain of solvability synthesis problem is ellipsoid of the form \( Q = \{ x : \Theta(x) \leq c \} \). At \( x = 0 \) we put \( \Theta(0) = 0 \).

Then at the domain \( Q \) the control

\[
u(x) = -\left( \frac{1}{2} B_0^*D(\Theta(x))FD(\Theta(x)) + B_0^*K \right) x
\]

solves the local feedback synthesis for system (6) and satisfies the constraint \( \|u(x)\| \leq 1 \). Moreover, in this case the equation \( \dot{\Theta}(x) = -1 \) holds, i.e. the controllability function \( \Theta(x) \) equals to the time of motion from any initial point \( x \in Q \) to the origin.

In the case when \( K \equiv 0 \), the synthesis is global.

3 The Solution of the Robust Feedback Synthesis

Let us consider system (1). Eq. (4) with control (11) takes the following form

\[
(A_0 + K + R(t, x))x + B_0u(x) =
\]

\[
= (A_0 + K + R(t, x))x - \left( \frac{1}{2} B_0^*D(\Theta(x))FD(\Theta(x)) + B_0^*K \right) x.
\]

Due to the fact that \( B_0B_0^*K = K \), the last equation takes the form

\[
(A_0 + K + R(t, x))x + B_0u(x) = (A_0 + R(t, x))x - \frac{1}{2} B_0^*D(\Theta(x))FD(\Theta(x))x.
\]

Put \( y(\Theta, x) = D(\Theta)x \). Then Eq. (9) takes the following form

\[
2a_0 \Theta = (F_y(\Theta, x), y(\Theta, x)).
\]

Let us set

\[
H = \text{diag}(H_1, \ldots, H_r), \quad \text{where} \quad H_i = \text{diag} \left( -\frac{2n_i - 2j + 1}{2} \right)_{j=1}^{n_i}
\]

and

\[
F^1 = F - FH - HF = (2n_i - i - j + 2)f_{ij})_{i,j=1}^n.
\]

If the matrix \( F \) is positive defined, then Eq. (12) has a unique positive solution \( \Theta = \Theta(y) \) [1, p. 108]. Since the controllability function is the time of motion, then the matrix \( F^1 \) is positive defined [1 p. 106]. Let the constant \( a_0 \) satisfies inequality (10). Let us investigate the closed-loop system (1) with
control given by relation (11). Let us denote the trajectory of this system by \( x(t) \) and let us find the derivative with respect to the system \( \dot{\Theta} = \frac{d}{dt} \Theta(x(t)) \). From Eq. (12) it follows that

\[
2a_0 \dot{\Theta} = (F \dot{y}(\Theta, x), y(\Theta, x)) + (F y(\Theta, x), \dot{y}(\Theta, x)).
\]  

(14)

Let us find \( \dot{y}(\Theta, x) \). We obtain that \( \frac{d}{d\Theta} D(\Theta) = \frac{1}{\Theta} HD(\Theta) \). Therefore,

\[
\dot{y}(\Theta, x) = \dot{D}(\Theta)x + D(\Theta) \dot{x} = \frac{\dot{\Theta}}{\Theta} H y(\Theta, x) + D(\Theta) A_0 D^{-1}(\Theta) y(\Theta, x) +

+ D(\Theta) R(t, x) D^{-1}(\Theta) y(\Theta, x) - \frac{1}{2} D(\Theta) B_0 B_0^* D(\Theta) F y(\Theta, x).
\]

Let us set

\[
S(\Theta, t, x) = \Theta(F D(\Theta) R(t, x) D^{-1}(\Theta) + D^{-1}(\Theta) R^* (t, x) D(\Theta) F).
\]

(15)

In [1, p. 109] it was proved that

\[
D(\Theta) A_0 D^{-1}(\Theta) = \Theta^{-1} A_0, \quad D(\Theta) b_0 = \Theta^{-1/2} b_0, \quad FA_0 + A_0^* F - FB_0 B_0^* F = -F^1.
\]

From (14) we see that

\[
\dot{\Theta} (2a_0 - \frac{1}{\Theta} ((FH + HF) y(\Theta, x), y(\Theta, x))) = \frac{1}{\Theta} (-F^1 + S(\Theta, t, x) y(\Theta, x), y(\Theta, x)).
\]

Taking into account Eq. (12), we obtain that the derivative of the controllability function with respect to system (4) is of the form:

\[
\dot{\Theta} = -1 + \frac{(S(\Theta, t, x) y(\Theta, x), y(\Theta, x))}{(F^1 y(\Theta, x), y(\Theta, x))}.
\]

(16)

Let us introduce the following notation:

- \( M^* \) is the transpose matrix to the matrix \( M \);
- \( \sigma(M) \) is the spectrum of matrix \( M \);
- \( \lambda_{\text{min}}(M) = \min \{ \lambda : \lambda \in \sigma(M) \} \);
- \( \lambda_{\text{max}}(M) = \max \{ \lambda : \lambda \in \sigma(M) \} \);
- \( \rho(M) = \max \{ |\lambda|, \lambda \in \sigma(M) \} \) is spectral radius of matrix \( M \);
- \( |M| = (|m_{ij}|)_{i,j=1}^n \) is the absolute value of matrix \( M \), i.e. matrix which consists of absolute values of the elements of matrix \( M \);
- \( \widetilde{G} = |(F^1)^{-1}| \cdot (F \widetilde{R} + \widetilde{R}^* F) \), where the matrix \( \widetilde{R} \) coincides with the matrix \( R(t, x) \) at \( r_{m_j}(t, x) = 1 \).

Let us set \( y = y(\Theta, x) \). Let us find the exact estimate for \( \dot{\Theta} \). To this end we find the largest and smallest values of the ratio \( (S(\Theta, t, x) y, y) / (F^1 y, y) \) at \( y \neq 0 \). Let us consider the problem

\[
(S(\Theta, t, x) y, y) \rightarrow \text{extr}, \quad y \in \{ y : (F^1 y, y) = c \}.
\]

We solve this problem using the method of Lagrange multipliers. The Lagrange function takes the form

\[
\mathcal{L}(y, \lambda) = (S(\Theta, t, x) y, y) - \lambda [(F^1 y, y) - c].
\]
From the necessary condition of the extremum we obtain that
\[ S(\Theta, t, x)y - \lambda F^1 y = 0. \]
So at the extremum point the following condition holds: \( S(\Theta, t, x)y, y) = \lambda(F^1 y, y) \), moreover \( \lambda \in \sigma((F^1)^{-1}S(\Theta, t, x)) \). Therefore,
\[ \lambda_{\min}((F^1)^{-1}S(\Theta, t, x)) \leq \frac{(S(\Theta, t, x)y, y)}{(F^1 y, y)} \leq \lambda_{\max}((F^1)^{-1}S(\Theta, t, x)). \]
Thus, from (16) we obtain that
\[ \dot{\Theta} \leq -1 + \lambda_{\max}((F^1)^{-1}S(\Theta, t, x)). \] (17)

### 3.1 Perturbations of the superdiagonal elements

Suppose that the \((n_i \times n_i)\) matrices \(R_i(t, x)\) have nonzero elements only at the main superdiagonal and \(\hat{R}(t, x) \equiv 0\). Then system (11) has the following form:
\[
\begin{align*}
\dot{x}_{s_{i-1}+j} &= (1 + r(s_{i-1}+j)(s_{i-1}+j+1)(t, x))x_{s_{i-1}+j+1}, \quad j = 1, \ldots, n_i - 1, \\
\dot{x}_s &= \sum_{j=1}^{n} k_{s,j}x_j + u, \quad i = 1, \ldots, r. 
\end{align*}
\] (18)

Similarly to [1] p. 109, one can show that \(D(\Theta)R(t, x)D^{-1}(\Theta) = \Theta^{-1}R(t, x)\) (by using the fact that in the case under consideration the matrix \(R(t, x)\) has the same structure as \(A_0\)). So we obtain that
\[ S(\Theta, t, x) = S_0(t(x) = FR(t, x) + R^*(t, x)F. \] (19)

It should be noted that the matrix \(S_0(t, x)\) does not depend on \(\Theta\). This observation is crucial for our method of solving the robust feedback synthesis. Indeed, the explicit form of \(S_0(t(x) is 
\[ S_0(t, x) = \text{diag}(S_1(t, x), \ldots, S_r(t, x)), \]
where \(S_i(t, x) =
\[
\begin{pmatrix}
0 & f_{11}r_{12} & \cdots & f_{1(n_i-1)}r_{(n_i-1)n_i} \\
-f_{11}r_{12} & 2f_{12}r_{12} & \cdots & f_{1n_i}r_{12} + f_{2(n_i-1)}r_{(n_i-1)n_i} \\
-f_{12}r_{23} & f_{13}r_{23} & \cdots & f_{2n_i}r_{23} + f_{3(n_i-1)}r_{(n_i-1)n_i} \\
& \cdots & \cdots & 2f_{(n_i-1)n_i}r_{(n_i-1)n_i}
\end{pmatrix},
\]

and \(r_{mj} = r_{mj}(t, x)\).

**Theorem 3.1.** Let \(\gamma\) be an arbitrary number which satisfies the inequality \(0 < \gamma < 1\). Let
\[ \Delta = \frac{(1 - \gamma)}{\rho(G)}. \] (20)

Let the controllability function \(\Theta = \Theta(x), x \neq 0\), be a unique positive solution of Eq. (14), where the constant \(a_0\) satisfies inequality (10).

Then at the domain \(Q\) specified by the equality \(Q = \{x : \Theta(x) \leq c\}\) the control given by relation (11) solves the local robust feedback synthesis for system (15). Moreover, the trajectory \(x(t)\) of the closed-loop system (14), starting at an arbitrary initial point \(x(0) = x_0 \in Q\), ends at the origin at some finite time \(T(x_0, \mathcal{R})\) satisfying the estimate
\[ T(x_0, \mathcal{R}) \leq \frac{\Theta(x_0)}{\gamma}. \] (21)

In the case when \(K \equiv 0\), the robust feedback synthesis is global.
**Proof.** Since \( B_0 = \text{diag}(B_{01}, \ldots, B_{0r}) \), then the matrices \( A_0 \) and \( B_0 \) have a block structure. So the matrix \( F^{-1} \) given by (7) is of the form
\[
F^{-1} = \text{diag}(F_1^{-1}, \ldots, F_r^{-1}),
\]
where (see [1, p. 98])
\[
F_i^{-1} = \frac{1}{\int_0^1 e^{-A_0 t} B_0 B_0^t e^{-A_0 t} dt} \left( \frac{1}{(ni - m - j + 1)(2ni - m - j + 2)(ni - m - j + 2)} \right)^{ni}_{m,j=1}.
\]

Let us fix value of \( i \) and consider the matrix \( F_i \) which is inverse to the matrix \( F_i^{-1} \). Let us prove that the elements of the matrix \( F_i \) are positive. To this end we analyze the matrix
\[
\tilde{M} = \left( \frac{1}{(2ni - m - j + 1)(2ni - m - j + 2)} \right)^{ni}_{m,j=1}.
\]

Put \( d_m = (-1)^m(n_i - m)! \). The elements of the matrix \( F_i^{-1} \) can be calculated from the elements of the matrix \( \tilde{M} \) by multiplying every element of row \( m \) by \( d_m \) and every element of column \( j \) by \( d_j \).

It is known that if every element of row \( m \) of the matrix is multiplied by \( \varepsilon \neq 0 \), then every element of column \( m \) in the inverse matrix will be divided by \( \varepsilon \). A similar assertion is true for the columns.

Then, in order to determine the elements of the matrix \( F_i \) we should divide every element of column \( m \) of the matrix \( \tilde{M}^{-1} \) by \( d_m \), and every element of row \( j \) of the matrix \( \tilde{M}^{-1} \) by \( d_j \). Therefore, the element with the number \( mj \) will be divided by \( d_m d_j \), \( \text{sign} d_m d_j = (-1)^{m+j} \).

Let us prove that all the minors of the matrix \( \tilde{M} \) are positive. It is known that all the minors of \( ni \times ni \) matrix \( \tilde{M} \) are positive if its \( s \) order minors composed from consecutive \( s \) rows and consecutive \( s \) columns are positive [17, Theorem 3.3]. This theorem was first proved in [18]. So in the matrix \( \tilde{M} \) we consider only submatrices composed from consecutive \( s \) rows \( \bar{r} + 1, \bar{r} + 2, \ldots, \bar{r} + s \) and consecutive \( s \) columns \( \bar{c} + 1, \bar{c} + 2, \ldots, \bar{c} + s \). In addition, any such submatrix is the Schur product of the Cauchy matrices. A Cauchy matrix is a matrix of the form \( \left( \frac{1}{x_m + y_j} \right)^{ni}_{m,j=1} \) [19, Theorem 1.2.12.1]. Each consecutive submatrix of the matrices \( \left( \frac{1}{2ni - m - j + 1} \right)^{ni}_{m,j=1} \) and \( \left( \frac{1}{2ni - m - j + 2} \right)^{ni}_{m,j=1} \) is a Cauchy matrix (put for the first matrix \( x_m = n_i - m \), \( y_j = n_i - j + 1 \)). The determinant of the Cauchy matrix is determined in [19, Theorem 1.2.12.1] by the formula
\[
\prod_{m>j} \frac{(x_m - x_j)(y_m - y_j)}{\prod_{m,j} (x_m + y_j)}.
\]

Each consecutive submatrix of the matrices \( \left( \frac{1}{2ni - m - j + 1} \right)^{ni}_{m,j=1} \) and \( \left( \frac{1}{2ni - m - j + 2} \right)^{ni}_{m,j=1} \) is a positive definite matrix due to the Silvester criteria and formula (23). The Schur product of the matrices \( \left( \frac{1}{2ni - m - j + 1} \right)^{ni}_{m,j=1} \) and \( \left( \frac{1}{2ni - m - j + 2} \right)^{ni}_{m,j=1} \) is the matrix of the form
\[
\left( \frac{1}{(2ni - m - j + 1)(2ni - m - j + 2)} \right)^{ni}_{m,j=1}.
\]
and it is equal to \( \tilde{M} \). The Schur product of positive definite matrices is a positive definite matrix [19 Theorem 6.4.2.1]. Hence, in the matrix \( \tilde{M} \) all the submatrices composed from consecutive \( s \) rows \( \bar{r} + 1, \bar{r} + 2, \ldots, \bar{r} + s \) and consecutive \( s \) columns \( \bar{c} + 1, \bar{c} + 2, \ldots, \bar{c} + s \) are positive. Therefore, all minors of the matrix \( \tilde{M} \) are positive. Then the minors of the order \( n_i - 1 \) and \( n_i \), in particular, are also positive. Hence, the elements of the matrix inverse to the matrix \( \tilde{M} \) have the sign \((-1)^{m+j}\). This implies that all the elements of the matrix \( \tilde{F} \) inverse to matrix \( \tilde{F}^{-1} \) are positive.

It is known that \( \lambda_{\max}((F^1)^{-1}S_0(t,x)) \leq \rho((F^1)^{-1}S_0(t,x)) \). We claim that \( \rho((F^1)^{-1}S_0(t,x)) \leq \rho((F^1)^{-1}S_0(t,x)) \). To prove this inequality we need the following Theorem.

**Theorem 3.2.** [20 Theorem 8.1.18] Let \( M \) and \( N \) be some matrices. Then
1. \( |M \cdot N| \leq |M| \cdot |N| \);
2. If \( |M| \leq N \), then \( \rho(M) \leq \rho(|M|) \leq \rho(N) \).

Therefore \( \rho((F^1)^{-1}S_0(t,x)) \leq \rho((F^1)^{-1}S_0(t,x)) \) \( \leq \Delta \rho(\tilde{G}) \). Here we use the fact that the elements of the matrix \( F \) are positive. Let us substitute the last inequality into inequality (17). We obtain that

\[
\dot{\Theta} \leq -1 + \Delta \rho(\tilde{G}).
\] (24)

If we assume that \(-1 + \Delta \rho(\tilde{G}) \leq -\gamma \), then \( \dot{\Theta} \leq -\gamma \). Similarly to [1 Theorem 1.2], the estimate on the time of motion [21] follows from the last inequality.

To complete the proof of the theorem, boundedness of the control has to be established. Since \( B_0^\ast D(\Theta) = \Theta^{-\frac{1}{2}}B_0^\ast \), the control given by (11) can be rewritten in the form

\[
u(x) = -\left(\frac{\Theta^{-\frac{1}{2}}}{2} B_0^\ast F + B_0^\ast KD^{-1}(\Theta(x))\right) y(\Theta, x)
\]

Since \( \|y(\Theta, x)\|^2 \leq 2a_0 \Theta(x) \|F^{-1}\| \) and

\[
\|D^{-1}(\Theta(x))\| = \begin{cases} \Theta^{\frac{1}{2}} & \text{if } 0 < \Theta < 1, \\ \Theta^{\frac{2n_1-1}{2}} & \text{if } \Theta \geq 1, \end{cases}
\]
at \( \Theta(x) \leq c \) we get

\[
\|u(x)\| \leq \left(\frac{1}{2} \|B_0^\ast F\| + \max\{c^{n_1}, c\} \|B_0^\ast K\|\right) \sqrt{2a_0 \|F^{-1}\|}.
\]

Let the constant \( a_0 \) satisfy inequality (10). Then from the last inequality we obtain that \( \|u(x)\| \leq 1 \) for all \( x \in Q \). Due to [1 Theorem 2.3] the control \( u(x) \) of the form (11) solves the local feedback synthesis for system (18). The proof of theorem is completed.

### 3.2 The general case

Let the matrix \( R(t, x) \) has the form given in (2). Then the elements of \( S(\Theta, t, x) \) defined by relation (15) are polynomials in \( \Theta \) whose degree does not exceed \( n_1 \). Reasoning similarly to the case with the perturbations of the superdiagonal elements, from inequality (17) it follows that

\[
\dot{\Theta} \leq -1 + \rho((F^1)^{-1}S(\Theta, t, x)) \leq -1 + \Delta \max\{c^{n_1}, c\} \cdot \rho(\tilde{G})
\]
at $\Theta(x) \leq c$. If we assume that
\begin{equation}
-1 + \Delta \max \{c^{n_1}, c \} \cdot \rho(\tilde{G}) \leq -\gamma,
\end{equation}
then $\dot{\Theta} \leq -\gamma$. Thus, the following theorem is valid.

**Theorem 3.3.** Let the controllability function $\Theta = \Theta(x)$, $x \neq 0$, be a unique positive solution of Eq. (9), where the constant $a_0$ satisfies inequality (10). Let the solvability domain be defined by $Q = \{x : \Theta(x) \leq c\}$, where $Q$ is ellipsoid. Let $\gamma$ be an arbitrary number which satisfies the inequality $0 < \gamma < 1$. Let
\begin{equation}
\Delta = \frac{(1 - \gamma)}{\max \{c^{n_1}, c \} \cdot \rho(\tilde{G})}.
\end{equation}

Then in the domain $Q$ the control given by relation (11) solves the local robust feedback synthesis for system (1). Moreover, the trajectory $x(t)$ of the closed-loop system (4), starting at an arbitrary initial point $x(0) = x_0 \in Q$, ends at the origin at some finite time $T(x_0, R)$, where the time of motion $T(x_0, R)$ satisfies inequality (21).

**Remark 3.1.** If we solve inequality (25) with respect to $c$ and consider $\Delta$ to be arbitrary, then we obtain the following solvability domain of the synthesis problem: $Q = \{x : \Theta(x) \leq c\}$.

**Remark 3.2.** Value of $\Delta$ is monotonically decreasing in $\gamma$. In addition, the inequality for the time of motion $T(x_0, R)$ given by (21) is also monotonically decreasing in $\gamma$. The value $\Delta \to \max$ at $\gamma \to 0$. Moreover $T(x_0, R) \to +\infty$ at $\Delta \to 0$.

**Remark 3.3.** Let $R(t, x) \in R$. To determine the trajectory starting at a given initial point $x_0 \in Q$ we act as follows. We solve Eq. (9) at $x = x_0$ and find its unique positive root $\Theta(x_0) = \Theta_0$. Put $\theta(t) = \Theta(x(t))$. The trajectory satisfies the following system:
\begin{equation}
\begin{cases}
\dot{x} = (A_0 + R(t,x))x - \frac{1}{2} B^*_0 D(\theta(x))FD(\theta(x))x, \\
\dot{\theta} = \frac{(-F^1 + S(\Theta,t,x))D(\theta)x, D(\theta)x)}{(F^1 D(\theta)x, D(\theta)x)}, \\
x(0) = x_0, \ \theta(0) = \Theta_0.
\end{cases}
\end{equation}

It should be noted that in order to determine $\Theta_0$ it suffices to solve Eq. (9) only once.

### 3.3 Stopping the oscillations of the system of two coupled pendulums

Let us consider a mechanical system which consists of two pendulums coupled by a spring. Pendulums oscillate in the same plane. We denote by $l_1$ and $l_2$ lengths of pendulums and $m_1$ and $m_2$ theirs masses. The lengths from the suspension points of two pendulums to the spring attachment points are considered to be equal to each other and we denote them by $h$. The spring stiffness is equal to $k$. Oscillations of this system without a control were considered in many books (see for ex. [21, Sect. 6.1], [22, Sect. 132]).
Let us consider the controllable motion of this system. Pairs of forces \( u_1 \) and \( u_2 \) act as shown in Fig. 1. The linearized equations of the motion of this pendulums are of the form:

\[
\begin{align*}
\ddot{\varphi}_1 &= -\frac{m_1gl_1 + kh^2}{m_1l_1^2} \varphi_1 + \frac{kh^2}{m_1l_1^2} \varphi_2 + u_1, \\
\ddot{\varphi}_2 &= \frac{kh^2}{m_2l_2^2} \varphi_1 - \frac{m_2gl_2 + kh^2}{m_2l_2^2} \varphi_2 + u_2.
\end{align*}
\]

Pairs of forces \( u_1 \) and \( u_2 \) satisfy the inequality \( \|(u_1, u_2)\| = \sqrt{u_1^2 + u_2^2} \leq 1 \). We assume that positive value of \( u_i \) corresponds the case then moments of the force acts in a clockwise direction. The force act tangentially to trajectory of motion.

The first case. Suppose that the values of \( m_1, m_2, l_1, l_2 \) and \( h \) are known. Suppose that the spring stiffness \( k \) is unknown. Let us set

\[
\frac{kh^2}{m_1l_1^2} = r_{21}, \quad \frac{kh^2}{m_2l_2^2} = r_{41}, \quad \frac{g}{l_1} = k_{21}, \quad \frac{g}{l_2} = k_{43}.
\]

By changing the variables

\[
\begin{align*}
x_1 &= \varphi_1, & x_2 &= \dot{\varphi}_1, & x_3 &= \varphi_2, & x_4 &= \dot{\varphi}_2
\end{align*}
\]

system (28) is reduced to the following form:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -(r_{21} + k_{21})x_1 + r_{21}x_3 + u_1, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= r_{41}x_1 - (r_{41} + k_{43})x_3 + u_2.
\end{align*}
\]

The coefficients \( r_{21} \) and \( r_{41} \) are unknown constants.

System (29) can be written in the matrix form:

\[
\dot{x} = (A_0 + K + R)x + B_0u,
\]

where

\[
A_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad B_0 = \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix},
\]

(31)
where \( \Theta = \Theta(\gamma) \) following equation holds: \( r g(B_0, (A_0 + K + R)B_0) = 4 \), then this system is completely controllable.

The matrices \( F \) and \( D(\Theta) \) given by relations (17) and (3) correspondingly are of the following form:

\[
F = \begin{pmatrix} 36 & 12 & 0 & 0 \\ 12 & 6 & 0 & 0 \\ 0 & 0 & 36 & 12 \\ 0 & 0 & 12 & 6 \end{pmatrix}, \quad D(\Theta) = \begin{pmatrix} \Theta^{-\frac{3}{2}} & 0 & 0 & 0 \\ 0 & \Theta^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \Theta^{-\frac{3}{2}} & 0 \\ 0 & 0 & 0 & \Theta^{-\frac{1}{2}} \end{pmatrix}.
\]

Let \( x = (x_1, x_2, x_3, x_4) \neq 0 \) and determine the controllability function \( \Theta = \Theta(x) \) as a unique positive solution of Eq. (9). In the analyzed case, this equation takes the form:

\[
2a_0\Theta^4 = 36x_1^2 + 24\Theta x_1x_2 + 6\Theta^2x_2^2 + 36x_3^2 + 24\Theta x_3x_4 + 6\Theta^2x_4^2.
\]

At \( x = 0 \) we put \( \Theta(0) = 0 \). We consider the solution of the robust feedback synthesis in the ellipsoid \( Q = \{ x : \Theta(x) \leq c \} \). The constant \( c > 0 \) is defined below. The constant \( a_0 \) satisfies inequality (10) which takes the form:

\[
0 < a_0 \leq \frac{3.58}{(13.42 + 2\max\{c^2, c\} \max\{k_{21}, k_{43}\})^2}.
\]

In order to the solvability domain contains the ellipsoid of the largest size, we choose \( a_0 \) as the largest value which satisfies (34).

The control given by relation (11) which solves the robust feedback synthesis is of the following form:

\[
u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} -\frac{6x_1}{\Theta^2(x)} - \frac{3x_2}{\Theta(x)} + k_{21}x_1 \\ \frac{6x_3}{\Theta^2(x)} - \frac{3x_4}{\Theta(x)} + k_{43}x_3 \end{pmatrix},
\]

where \( \Theta = \Theta(x) \) is a unique positive solution of Eq. (33). For any value of \( k \) this control steers an arbitrary initial point \( x_0 \) to the origin in some finite time \( T(x_0, k) \leq \Theta(x_0)/\gamma \), where \( \gamma \) is an arbitrary number which satisfies the inequality \( 0 < \gamma < 1 \).

The matrix \( S = S(\Theta, t, x) \) given by relation (15) is of the form:

\[
S(\Theta) = \begin{pmatrix} -24r_{21}\Theta^2 & -6r_{21}\Theta^2 & 12(r_{21} + r_{41})\Theta^2 & 6r_{41}\Theta^2 \\ -6r_{21}\Theta^2 & 0 & 6r_{21}\Theta^2 & 0 \\ 12(r_{21} + r_{41})\Theta^2 & 6r_{21}\Theta^2 & -24r_{41}\Theta^2 & -6r_{41}\Theta^2 \\ 6r_{41}\Theta^2 & 0 & -6r_{41}\Theta^2 & 0 \end{pmatrix},
\]

where \( \Theta = \Theta(x) \) is a unique positive solution of Eq. (33).
Let us find an estimate for the solvability domain. To this end we find $c$ from inequality (25), which takes the form

$$-1 + \Delta \max\{c^2, c\} \cdot \rho(\tilde{G}) \leq -\gamma,$$

(35)

where $\tilde{G} = \begin{pmatrix} 7 & 1 & 7 & 1 \\ 6 & 6 & 6 & 6 \\ 4 & 1 & 4 & 1 \\ 2 & 2 & 2 & 2 \\ 7 & 1 & 7 & 1 \\ 6 & 6 & 6 & 6 \\ 4 & 1 & 4 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$, $\rho(\tilde{G}) \approx 8.4$, $\Delta = k \cdot \max\left\{\frac{h^2}{m_1l_1^2}, \frac{h^2}{m_2l_2^2}\right\}$.

From (35) it follows that

$$\max\{c^2, c\} \leq \frac{0.12(1 - \gamma)}{r} = \frac{0.12(1 - \gamma)}{k \max\left\{\frac{h^2}{m_1l_1^2}, \frac{h^2}{m_2l_2^2}\right\}}.$$

Taking into account the inequality (17), let us find a more precise estimate for $c$. At $x \in Q$ from (17) it follows that

$$\dot{\Theta} \leq -1 + \lambda_{\max}(F^1) \leq -1 + \left(\frac{r_{21} + r_{41} + \sqrt{2(r_{21}^2 + r_{41}^2)}}{6}\right)^2 \leq -1 + \left(\frac{r_{21} + r_{41} + 2\sqrt{2(r_{21}^2 + r_{41}^2)}}{6}\right)^2c^2.$$

Let $c > 0$ be such that the following inequality holds:

$$-1 + \left(\frac{r_{21} + r_{41} + 2\sqrt{2(r_{21}^2 + r_{41}^2)}}{6}\right)^2c^2 \leq -\gamma.$$

Then $\dot{\Theta} \leq -\gamma$. From (36) it follows that $c \leq \sqrt{\frac{6(1 - \gamma)}{r_{21} + r_{41} + 2\sqrt{2(r_{21}^2 + r_{41}^2)}}}$. In order to solvability domain contains the ellipsoid of the largest size, we choose $c$ as the largest value which satisfies (36). So, we obtain the following solvability domain:

$$Q = \left\{ x : \Theta(x) \leq \sqrt{\frac{6(1 - \gamma)}{k \left(\frac{h^2}{m_1l_1^2} + \frac{h^2}{m_2l_2^2} + 2\sqrt{2h^2/m_1l_1^2 + 2h^2/m_2l_2^2}\right)}} \right\}.$$

(37)

Let us consider the values of the parameters

$$m_1 = 1, m_2 = 2, l_1 = 60, l_2 = 30, h = 7.5, \gamma = 0.001.$$

Then $h = \frac{1}{8}, l_1 = \frac{1}{4}, \frac{k_{21}}{l_1} \approx 0.16, \frac{k_{43}}{l_2} \approx 0.32, r_{21} = \frac{k}{64}, r_{41} = \frac{k}{32}.$

Let the stiffness $k$ satisfies the constraint $k \leq 4$, but the value of $k$ is unknown. Then the set of points (37) from which we may steer to the origin is the ellipsoid of the form $Q = \{ x : \Theta(x) \leq 3.2 \}$. Besides from (37) it follows that the stiffness $k$ decreases as values of axes of ellipsoid $Q$ increases. At $c = 3.2$ inequality (34) on $a_0$ takes the form: $a_0 \leq 0.0088 \ldots$ Put $a_0 = 0.0088.$
Let the initial point be equal to \(x(0) = (-0.3, 0.3, 0, 0)\), \(x(0) \in Q\). The unique positive solution \(\Theta_0\) of Eq. (33) \(\Theta_0 \approx 3.2\). Let \(x = x(t, k_0)\) be the trajectory of system (27), which is realized at some coefficient of stiffness \(k_0\) which satisfies inequality \(k_0 \leq 4\). Put \(\theta(t) = \Theta(x(t, k_0))\). The trajectory \(x = x(t, k_0)\) satisfies the following system:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{k_0}{64}(-x_1 + x_3) - \frac{6 x_1}{\theta^2} - \frac{3x_2}{\theta}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= \frac{k_0}{32}(x_1 - x_3) - \frac{6 x_1}{\theta^2} - \frac{3x_2}{\theta}, \\
\dot{\theta} &= \phi,
\end{align*}
\]

where

\[
\phi = -((12 + 0.03 k_0 \theta^2) x_1^2 + (6 + 0.02 k_0 \theta^2) x_1 x_2 \theta + x_2^2 \theta^2 + (12 + 0.06 k_0 \theta^2) x_2^2 + (6 + 0.03 k_0 \theta^2) x_3 x_4 \theta + x_4^2 \theta^2 - 0.09 k_0 x_1 x_3 \theta^2 - 0.02 k_0 x_2 x_3 \theta^3 - 0.03 k_0 x_1 x_4 \theta^3) / (12 x_1^2 + 6 x_1 x_2 \theta + x_2^2 \theta^2 + 12 x_3^2 + 6x_3 x_4 \theta + x_4^2 \theta^2).
\]

The two-dimensional projection of domain \(Q\) on the plane \(Ox_1x_2 = O\varphi_1\varphi_1\) or equally \(Q = \{(x_1^0, x_2^0, 0, 0) : \Theta(x_1^0, x_2^0, 0, 0) \leq 3.2\}\) is given in Fig. 2. Let \((x_1^0(t), x_2^0(t), x_3^0(t), x_4^0(t), \theta(t))\) be the solution of system (38) at \(k_0 = 4\). The curve \((x_1^0(t), x_2^0(t))\) is also given in Fig. 2 (the solid line). The curve \((\bar{x}_1^0(t), \bar{x}_2^0(t))\) (the dashed line), which corresponds to the case \(k_0 = 0\) is also given in Fig. 2. All the other trajectories fill up the domain between the trajectories corresponding to \(k_0 = 0\) and \(k_0 = 4\) if stiffness \(k_0\) satisfies the inequality \(0 \leq k_0 \leq 4\) and trajectories begin from \(x(0)\). At \(k_0 = 0\) the trajectory may be found from the following system:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\frac{6 x_1}{\theta^2} - \frac{3x_2}{\theta}, \quad \dot{x}_3 = x_4, \\
\dot{x}_4 &= -\frac{6 x_1}{\theta^2} - \frac{3x_2}{\theta}, \\
\dot{\theta} &= -1, \\
x_1(0) &= -0.3, \quad x_2(0) = 0.3, \quad x_2(0) = 0, \quad x_2(0) = 0, \quad \theta(0) = 3.2.
\end{align*}
\]

Figure 2: The projection of the phase trajectory and the ellipsoid \(Q\) on the plane \(O\varphi_1\varphi_1\)
The plot of the components of the control on the trajectory

\[ u_1 = u_1(x_1^0(t), x_2^0(t), x_3^0(t), x_4^0(t)) = -\frac{6x_1^0(t)}{\theta^2(t)} - \frac{3x_2^0(t)}{\theta(t)} + 0.16x_1^0(t), \]

\[ u_2 = u_2(x_1^0(t), x_2^0(t), x_3^0(t), x_4^0(t)) = -\frac{6x_3^0(t)}{\theta^2(t)} - \frac{3x_4^0(t)}{\theta(t)} + 0.32x_3^0(t) \]

are given in Fig. 3. The norm of the control \( \| (u_1, u_2)^* \| = \sqrt{u_1^2 + u_2^2} \) is given in Fig. 4 and we can see that \( \| (u_1, u_2)^* \| \leq 1 \). The controllability function \( \theta(t) \) shown in in Fig. 5 is close to the linear \( y = 3.2 - t \). The derivative of the controllability function with respect to the system is given in Fig. 6, and we can see that it is negative. The estimate for the time of motion (21) is of the form: \( T \leq 3206 \). It is fulfilled at all \( 0 \leq k_0 \leq 4 \), but at a particular value of \( k_0 \) the value of \( T \) is less than 3206. The results of the numerical calculations demonstrate that the time of motion \( T \) from the point \( x(0) \) at \( k_0 = 4 \) is \( T \approx 3.43 \), besides it can be shown numerically that at \( 0 \leq k_0 \leq 4 \) the following inequality holds: \( 3.2 \leq T \leq 3.43 \). All graphs are given at the trajectory at \( k_0 = 4 \). At the other values of \( k_0 \) graphs are similarly to that for present at Fig. 2-6.

The second case. Let \( l_1 = l_2 = l \). Let us consider that the values \( m_1, m_2 \) and \( k \) are known. Also
we consider that the pendulum length \( l \) is unknown. Besides, the ratio \( \frac{h}{l} \) is known. Let us set

\[
\frac{k h^2}{m_1 l_1^2} = k_{21}, \quad \frac{k h^2}{m_2 l_2^2} = k_{41}, \quad \frac{g}{l} = r_{21}.
\]

By changing the variables

\[
x_1 = \varphi_1, \quad x_2 = \dot{\varphi}_1, \quad x_3 = \varphi_2, \quad x_4 = \dot{\varphi}_2
\]

system (28) is reduced to the following form:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -(r_{21} + k_{21}) x_1 + k_{21} x_3 + u_1, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= k_{41} x_1 - (r_{21} + k_{41}) x_3 + u_2.
\end{align*}
\]

The coefficient \( r_{21} \) is unknown constant.

This system can be written in the matrix form (30) where the matrices \( A_0 \) and \( B_0 \) given by relations (31) and the matrices \( K \) and \( R \) are of the form:

\[
K = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-k_{21} & 0 & k_{21} & 0 \\
0 & 0 & 0 & 0 \\
k_{41} & 0 & -k_{41} & 0
\end{pmatrix} \quad R = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-r_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -r_{21} & 0
\end{pmatrix}.
\]

The matrices \( F \) and \( D(\Theta) \) are given by relations (32). Let us define the controllability function \( \Theta = \Theta(x) \) at \( x \neq 0 \) as a unique positive solution of Eq. (33). At \( x = 0 \) we put \( \Theta(0) = 0 \). Similarly to the first case we consider the solution of the robust feedback synthesis in the ellipsoid \( Q = \{ x : \Theta(x) \leq c \} \). The constant \( c > 0 \) is defined below. The constant \( a_0 \) satisfies inequality (10) that takes the form:

\[
0 < a_0 \leq \frac{3.58}{(13.42 + 2.83 \max\{c^2, c\} \sqrt{k_{21}^2 + k_{41}^2})^2}.
\]

In order to the solvability domain contains the ellipsoid of the largest size, we choose \( a_0 \) as the largest value which satisfies (39).

The control given by relation (11) which solves the robust feedback synthesis is of the following form:

\[
u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix}
\frac{6x_1}{\Theta^2(x)} - \frac{3x_2}{\Theta(x)} + k_{21}(x_1 - x_3) \\
\frac{6x_3}{\Theta^2(x)} - \frac{3x_4}{\Theta(x)} + k_{41}(-x_1 + x_3)
\end{pmatrix},
\]

where \( \Theta = \Theta(x) \) is a unique positive solution of Eq. (33). For any value of \( l \) this control steers an arbitrary initial point \( x_0 \) to the origin in some finite time \( T(x_0, l) \leq \Theta(x_0)/\gamma \), where \( \gamma \) is an arbitrary number which satisfies the inequality \( 0 < \gamma < 1 \).
The matrix $S = S(\Theta, t, x)$ given by relation (15) has the following form:

$$
S = \begin{pmatrix}
-\frac{24g\Theta^2}{l_1} & -\frac{6g\Theta^2}{l_1} & 0 & 0 \\
-\frac{6g\Theta^2}{l_1} & 0 & 0 & 0 \\
0 & 0 & -\frac{24g\Theta^1}{l_1} & -\frac{6g\Theta^2}{l_1} \\
0 & 0 & -\frac{6g\Theta^2}{l_1} & 0
\end{pmatrix},
$$

where $\Theta = \Theta(x)$ is a unique positive solution of Eq. (33).

Taking into account the inequality (17), let us find an exact estimate for $c$. Since $\lambda_{\text{max}}((F^1)^{-1}S(\Theta)) = \frac{g\Theta^2}{2l}$, then at $x \in Q$ from (17) it follows that

$$
\dot{\Theta} \leq -1 + \lambda_{\text{max}}((F^1)^{-1}S(\Theta)) = -1 + \frac{g\Theta^2}{2l} \leq -1 + \frac{gc^2}{2l}.
$$

Let $c > 0$ be such that the following inequality holds:

$$
-1 + \frac{gc^2}{2l} \leq -\gamma. \tag{40}
$$

Then $\dot{\Theta} \leq -\gamma$. From (40) it follows that $c \leq \sqrt{0.2 \, l(1-\gamma)}$. In order to solvability domain contain the ellipsoid of the largest size, we choose $c$ as the largest value which satisfies (40). So, we obtain the following solvability domain:

$$
Q = \{ x : \Theta(x) \leq \sqrt{0.2 \, l(1-\gamma)} \}. \tag{41}
$$

Let

$$
m_1 = 1, \ m_2 = 2, \ k = 1, \ \frac{h}{l} = \frac{1}{4}, \ \gamma = 0.001.
$$

Then $k_{21} = \frac{kh^2}{m_1l^2} = \frac{1}{16}$, $k_{41} = \frac{kh^2}{m_2l^2} = \frac{1}{32}$, $r_{21} = 9.8 \cdot \frac{l}{l}$.

Let the length $l$ satisfies the constraint $l \geq 30$, but the value of $l$ is unknown. Then the set of points (41) from which we may steer to the origin is the ellipsoid of the form $Q = \{ x : \Theta(x) \leq 2.47 \}$. Besides from (41) it follows that the length $l$ decreases as values of axes of ellipsoid $Q$ decrease. At $c = 2.47$ inequality (39) on $a_0$ takes the form: $a_0 \leq 0.016 \ldots$ Put $a_0 = 0.016$.

Similarly to the first case let the initial point be equal to $x(0) = (-0.3, 0.3, 0, 0)$, $x(0) \in Q$. The unique positive solution $\Theta_0$ of Eq. (33) is $\Theta_0 \approx 2.44$. The estimate for the time of motion (21) is of the form: $T \leq 2438$. It is fulfilled at all $l \geq 30$, but at a particular value of $l$ the value of $T$ is less than 2438. The results of the numerical calculations demonstrate that the time of motion $T$ from the point $x(0)$ at $l = 30$ is $T \approx 3$, besides it can be shown numerically that at $l \geq 30$ the following inequality holds: $2.44 \leq T \leq 3$. The further considerations are similar to those in the first case.

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