Anisotropic solid dark energy

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In this paper, we study a triad of inhomogeneous scalar fields, known as “solid”, as a source of homogeneous but anisotropic dark energy. By using the dynamical system approach, we find that anisotropic accelerated solutions can be realized as attractor points for suitable choices of the parameters of the model. We complement the dynamical analysis with a numerical solution whose initial conditions are set in the deep radiation epoch. The model predicts a non-negligible spatial shear within the observational bounds nowadays, even when it is set to zero at early times. The anisotropic attractor and the ultra slowly varying equation of state of dark energy very close to $-1$ are key features of this scenario. Following a similar approach, we also analyzed the isotropic version of the model. We find that the solid can be characterized by a nearly constant equation of state and thus being able to simulate the behavior of a cosmological constant.

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I. INTRODUCTION

The current Universe is expanding at an accelerated rate [1, 2] and it is highly homogeneous and isotropic at cosmological scales [3]. The simplest description of the Universe is based on the standard Λ Cold Dark Matter (ΛCDM) model. In this model, the current accelerated expansion of the Universe is due to the repulsive effect of a constant energy density with negative pressure, which is given by the cosmological constant Λ [4]. Despite its success, there are several theoretical and observational problems with this scenario. One of the theoretical difficulties is the so-called cosmological constant problem, which asserts that if Λ is associated with the vacuum energy density of the Universe, the value predicted by the theory and the value obtained from observations differs by several tens of orders of magnitude [5, 6]. On the observational side, the $H_0$ tension states that the current value of the Hubble parameter calculated from the cosmic microwave background (CMB) data does not agree with the value computed from local measurements of type Ia supernovae (SNe Ia) [7, 8]. It seems that this tension could be alleviated if extensions to ΛCDM model are considered [9].

The search for alternatives to the standard model is generally split into two broad categories: modified gravity theories and dynamical dark energy. While the former has been recently under observational pressure [10–14], the latter is usually based on time-dependent fields with vanishing spatial gradients [15, 16]. This choice ensures that the background geometry can be described by a homogeneous and isotropic metric, and thus the evolution of the Universe is also homogeneous and isotropic. On the other hand, some observations seem to imply a violation of the Universe’s isotropy at large scales—the so called CMB anomalies [17, 18]—suggesting that background metrics different to the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric should be considered [19–21]. It has been also pointed out that some of these CMB anomalies could be explained by the introduction of an anisotropic late-time accelerated expansion.

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An anisotropic dark energy can be realized by considering a homogeneous but spatially anisotropic metric, together with a suitable arrangement of the fields driving the expansion of the universe. Among the proposals for anisotropic dark energy, we can find models of vector fields \[ \phi \] p-forms \[ \phi \] or non-Abelian gauge fields \[ \phi \]. All of these models are based on time-dependent fields, so as to comply with the homogeneity of the background metric. Nonetheless, in Refs. \[ 31 \] \[ 32 \], it was shown that a triad of scalar fields with spatially constant and isotropic energy-momentum tensor, i.e. invariant under translations and spatial rotations. This triad, known as “solid”, is given by

$$\phi^I \equiv x^I, \quad I \in \{1, 2, 3\} \tag{1}$$

where \( \phi^I \) is a scalar field and \( x^I \) is a comoving cartesian coordinate. The solid configuration for homogeneous scalar fields is similar to other configurations for different fields. For instance, the cosmic triad for vector and nonabelian gauge vector fields \[ \phi \] \[ \phi \] \[ \phi \] the recent Higgs triad \[ \phi \], among others.

In Refs. \[ 39 \] \[ 40 \], it was shown that the solid configuration in Eq. \[ \phi \] can support prolonged anisotropic inflationary solutions because it is almost insensitive to the expansion. In this work, we show that this characteristic behavior is also present at late-times and the final stage of the Universe is given by an anisotropic accelerated expansion, even if the initial spatial shear is set to zero. We also show that dark energy dominance is possible when the background metric is homogeneous and isotropic.

This paper is organized in the following way. In Sec. \[ \text{II} \] we present the action and the energy-momentum tensor of the model. In Sec. \[ \text{III} \] the equations of motion in an homogeneous but anisotropic background are derived. Section \[ \text{IV} \] is dedicated to the dynamical analysis of the model. A numerical integration of the background equations and the general cosmological evolution is presented in Sec. \[ \text{V} \]. The isotropic version of the model is treated in Sec. \[ \text{VI} \]. Finally, our conclusions are presented in Sec. \[ \text{VII} \].

**II. GENERAL MODEL**

Consider the following action for the solid

$$S = \int d^4x \sqrt{-g} \left[ \frac{m_p^2}{2} R - \sum_I F^I (X^I) + \mathcal{L}_m + \mathcal{L}_r \right] \tag{2}$$

where \( m_p \) is the reduced Planck mass, \( R \) is the Ricci scalar, \( F^I \) is the Lagrangian characterizing the dynamics of the scalar field \( \phi^I \), whose argument is the canonical kinetic-type term

$$X^I = g_{\mu\nu} \nabla^I \phi^I \nabla^I \phi^I \tag{3}$$

and \( \mathcal{L}_m \) and \( \mathcal{L}_r \) are the Lagrangians for matter and radiation fluids, respectively. Varying the action in Eq. \[ \phi \] with respect to the space-time metric \( g^{\mu\nu} \), we get the gravitational field equations

$$T^\mu_\nu = 2 \sum_I F^I \nabla^I \phi^I \nabla^I \phi^I - g^{\mu\nu} \sum_I F^I + T^m_\nu + T^r_\nu \tag{4}$$

where \( T^m_\nu \) and \( T^r_\nu \) are the energy tensors associated to the matter and radiation perfect fluids, respectively, and we have used the shorthand notation \( F_X \equiv \frac{\partial}{\partial x} F_x \). Varying the action with respect to \( \phi^I \), we get the equation of motion of the solid

$$\nabla^\mu (F_X \nabla^\mu \phi^I) = 0 \tag{5}$$

Note that the field \( \phi^I \) itself trivially satisfies the Klein-Gordon equation for a massless field: \( \nabla^\mu \nabla^\mu \phi^I = 0 \).

**III. BACKGROUND EQUATIONS OF MOTION**

In Ref. \[ 31 \], it was shown that the action \[ \phi \] can be compatible with the symmetries of the FLRW metric whenever the three Lagrangians \( F^I \) are identical. Since we are interested in the dynamics of an anisotropic solid, we adopt the geometry of a homogeneous but anisotropic Bianchi-I metric. For simplicity, we assume that there exists a residual isotropy in the \( (y, z) \) plane, such that the background geometry is given by

$$ds^2 = -dt^2 + a(t)^2 \left[ e^{-2\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right] \tag{6}$$

where \( a(t) \) is the average scale factor and \( \sigma(t) \) is the geometrical shear, being both functions of the cosmic time \( t \). In this background, the canonical kinetic-type terms are

$$X^1(t) = \frac{e^{4\sigma(t)}}{a^2(t)} \quad X^2(t) = X^3(t) = \frac{e^{-2\sigma(t)}}{a^2(t)} \tag{7}$$

and the requirement for the Lagrangians \( F^I \) in this case is

$$\sum_I F^I (X^I) = F^1 (X^1) + 2F^2 (X^2) \tag{8}$$
Considering the 00 component of the gravitational field equations, the first “Friedman” equation reads

\[ 3m_p^2 H^2 = F^1 + 2F^2 + 3m_p^2 \sigma^2 + \rho_m + \rho_r, \]  

where \( H = \dot{a}/a \) is the Hubble parameter\(^1\) and we have defined \( \rho_m \) and \( \rho_r \) as the densities for the matter and radiation perfect fluids, respectively. The second Friedman equation follows from \( m_p^2 \text{tr} G_{\mu\nu} = \text{tr} T_{\mu\nu} \), which can be written as

\[ -2m_p^2 \dot{H} = \frac{2}{3} X^1 F_{X^1} + \frac{4}{3} X^2 F_{X^2} + \rho_m + \frac{4}{3} \rho_r + 6m_p^2 \sigma^2. \]  

Finally, the evolution equation for the geometrical shear is obtained from the relation

\[ m_p^2(G^2 - G^1) = T^2 - T^1 \]

as

\[ \dot{\sigma} + 3H \sigma = \frac{2}{3m_p^2} (X^2 F_{X^2} - X^1 F_{X^1}). \]  

Since we are interested in anisotropic late-time accelerated solutions, it is necessary to characterize the dark energy fluid. We define the density and pressure of dark energy by

\[ \rho_{\text{DE}} \equiv F^1 + 2F^2 + 3m_p^2 \sigma^2, \]
\[ p_{\text{DE}} \equiv \frac{1}{3}(2x_1 - 3)F^1 + \frac{2}{3}(2x_2 - 3)F^2 + 3m_p^2 \sigma^2, \]

where we have defined the quantities

\[ x_1 \equiv \frac{X^1 F_{X^1}}{F^1}, \quad x_2 \equiv \frac{X^2 F_{X^2}}{F^2}, \]  

which characterize the form of the Lagrangians \( F^1 \) and \( F^2 \), respectively.

Note that our choice to include the geometrical shear in \( \rho_{\text{DE}} \) and \( p_{\text{DE}} \) (instead of considering only the contribution coming from the solid) allows us to write the continuity equation simply as

\[ \dot{\rho}_{\text{DE}} + 3H (\rho_{\text{DE}} + p_{\text{DE}}) = 0, \]

which greatly simplifies our analysis\(^2\).

The set of Eqs. (9)-(11) describes the cosmological background dynamics. In the next section, we will study the asymptotic behavior of this set of equations through a dynamical system analysis\(^3\).

### IV. DYNAMICAL SYSTEM

#### A. Autonomous System

In order to proceed, we introduce the following dimensionless variables

\[ f_1 \equiv \frac{F^1}{3m_p^2 H^2}, \quad f_2 \equiv \frac{F^2}{3m_p^2 H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3m_p^2 H^2}, \]

\[ \Omega_r \equiv \frac{\rho_r}{3m_p^2 H^2}, \quad \Sigma \equiv \frac{\dot{\sigma}}{H}, \]

such that the first Friedman equation (9) becomes the constraint

\[ \Omega_m = 1 - f_1^2 - 2f_2^2 - \Sigma^2 - \Omega_r. \]

Changing the cosmic time \( t \) for the number \( N \) of e-folds defined as \( dN \equiv H dt \), the background equations (9)-(11) are replaced by the autonomous system

\[ f_1' = f_1 [q + 1 - x_1 (1 - 2\Sigma)], \]
\[ f_2' = f_2 [q + 1 - x_2 (1 + \Sigma)], \]
\[ \Sigma' = \Sigma(q - 2) - 2 (x_1 f_1^2 - x_2 f_2^2), \]
\[ \Omega_r' = 2\Omega_r (q - 1), \]

where the deceleration parameter, \( q \equiv -\dot{a}/a^2 \), is given by

\[ q = \frac{1}{2} [1 + (2x_1 - 3)f_1^2 + 2(2x_2 - 3)f_2^2 + 3\Sigma^2 + \Omega_r]. \]

However, instead of the deceleration parameter, we equivalently characterize the evolution of the average scale factor \( a(t) \) in terms of the effective equation of state \( w_{\text{eff}} \equiv (2q - 1)/3 \). The dark sector is characterized by its equation of state \( w_{\text{DE}} \equiv p_{\text{DE}}/\rho_{\text{DE}} \), which in terms of the dynamical variables reads

\[ w_{\text{DE}} = -1 + \frac{2}{3} \frac{x_1 f_1^2 + 2x_2 f_2^2 + 3\Sigma^2}{f_1^2 + 2f_2^2 + \Sigma^2}, \]

and its density parameter \( \Omega_{\text{DE}} \equiv \rho_{\text{DE}}/3m_p^2 H^2 \).

Since the functions \( x_1 \) and \( x_2 \) cannot themselves be expressed in terms of the dimensionless variables, it is necessary to choose the specific Lagrangians \( F^1 \) and \( F^2 \) in order to get a closed autonomous system. In this case, the simplest model is obtained when \( x_1 \)

\(^1\) Here, an overdot denotes a derivative with respect to the cosmic time \( t \).

\(^2\) Had we chosen to separate the contributions of the solid and the geometry, we would end up with an equation of the form \( \dot{\rho}_{\text{DE}} + 3H (\dot{\rho}_{\text{DE}} + p_{\text{DE}}) \propto g_{ij} \pi^{ij} \), where \( g_{ij} \) is the time derivative of the spatial part of the metric, and \( \pi^{ij} \) is the trace-free part of the energy-momentum tensor.

\(^3\) Here, a prime denotes a derivative with respect to the number of e-folds \( N \).
and $x_2$ are constants, which corresponds to a power law model
\[ F^1 \propto (X^1)^n, \quad F^2 \propto (X^2)^m, \]
such that
\[ x_1 = n, \quad x_2 = m. \tag{22} \]

In the next subsection, we will study the asymptotic behavior of the system by finding the fixed points of the autonomous system.

### B. Fixed Points and Stability

In the following, we discuss the fixed points relevant to the radiation ($\Omega_r \simeq 1, w_{\text{eff}} \simeq 1/3$), matter ($\Omega_m \simeq 1, w_{\text{eff}} \simeq 0$), and dark energy eras ($\Omega_{mDE} \simeq 1, w_{\text{eff}} < -1/3$), which can be obtained by setting $f_1' = 0, f_2' = 0, \Sigma' = 0,$ and $\Omega_r' = 0$ in equations 16-19. The stability of these points can be known by perturbing the autonomous set around them. Up to linear order, the perturbations $\delta X = (\delta f_1, \delta f_2, \delta \Sigma, \delta \Omega_r)$ satisfy the differential equation,

\[ \delta X = \mathbb{M} X, \tag{23} \]

where $\mathbb{M}$ is a $4 \times 4$ Jacobian matrix. The sign of the real part of the eigenvalues $\lambda_{1,2,3,4}$ of $\mathbb{M}$ determines the stability of the point. A fixed point is an attractor, or sink, if the real part of all the eigenvalues are negative. If at least one of the eigenvalues has positive real part it is called a saddle. If the real part of all the eigenvalues are positive the fixed point is called a repeller or source.

In what follows, we refer to each point by its name, which is defined as $R, M$ or $DE$ – depending on whether it corresponds to a radiation, matter or dark energy dominated universe – followed by a number. The points and their eigenvalues are gathered in Tables I and II respectively.

#### 1. Radiation Dominance

- **(R-1) Isotropic radiation**: This point corresponds to a radiation-dominated and isotropic universe, and it trivially satisfies the constraint $15$. One can check that the eigenvector associated with the eigenvalue $\lambda_1 = -1$ points in the $\Sigma$ direction in the phase space $(f_1, f_2, \Sigma, \Omega_r)$, indicating that the trajectories around this point are attracted in this direction. This means that the shear decays from its value around this point. On the other hand, the eigenvector associated with the eigenvalue $\lambda_2 = 1$ points to the $\Omega_r$ direction, meaning that radiation is also decaying. The eigenvalues $\lambda_3$ and $\lambda_4$ are positive for $m$ and $n$ less than 2, respectively. Under this condition, $(R-1)$ is a saddle with three positive eigenvalues, and the dark components $f_1$ and $f_2$ grow during the radiation epoch, since the eigenvectors associated to these eigenvalues $(\lambda_{3,4})$ point to the $f_2$ and $f_1$ directions, respectively.

- **(R-2) Anisotropic radiation scaling with $F^1$**:

  This corresponds to an anisotropic solution where the density parameter and equation of state of dark energy are given by

  \[ \Omega_{DE} = \frac{(n-2)(n-3)}{4n^2}, \quad w_{DE} = \frac{1}{3}, \tag{24} \]

  indicating that dark energy scales as a radiation fluid, or “dark radiation”. This point is a viable solution if the conditions

  \[ f_1^2 \geq 0, \quad 0 \leq \Omega_{DE} \leq 1, \quad \text{and} \quad 0 \leq \Omega_r \leq 1, \]

  are satisfied. Furthermore, the big-bang nucleosynthesis (BBN) gives the bound $\Omega_{DE} < 0.045_{15}$. Imposing all of these conditions, we determine that the physical region of existence of the point $(R-2)$ is

  \[ 1.64237 < n \leq 2, \quad \forall \quad m. \tag{25} \]

  The eigenvalues of $\mathbb{M}$ in this point (see Table I) tell us that this is a saddle point. The eigenvalues $\lambda_3$ and $\lambda_4$ are negative in the region of existence given in Eq. 25. The second eigenvalue is positive in the region of existence of the point for $m < 4n/(3n - 2)$, and negative otherwise. Since the eigenvector associated to this eigenvalue points to the $f_2$ direction when $\lambda_2 > 0$, the dark component $f_2$ grows during the radiation epoch.

- **(R-3) Anisotropic radiation scaling with $F^2$**:

  The density parameter and equation of state of dark energy in this solution are given by

  \[ \Omega_{DE} = \frac{(m-2)(m-3)}{m^2}, \quad w_{DE} = \frac{1}{3}. \tag{26} \]

  Imposing the conditions

  \[ f_2^2 \geq 0, \quad 0 \leq \Omega_{DE} \leq 1, \quad \text{and} \quad 0 \leq \Omega_r \leq 1, \]

  as well as the BBN bound $\Omega_{DE} < 0.045$, the physical region of existence of $(R-3)$ becomes

  \[ 1.86271 < m \leq 2, \quad \forall \quad n. \tag{27} \]
\[
\begin{array}{cccccccc}
\text{Fixed Point} & f_1 & f_2 & \Omega_r & \Omega_m & \Sigma & w_{\text{eff}} & \text{stability} \\
R-1 & 0 & 0 & 1 & 0 & 0 & 1/3 & \text{saddle} \\
R-2 & \frac{\sqrt{2}\pi}{2n} & 0 & \frac{3n^2+5n-6}{4n^2} & 0 & \frac{n-2}{2n} & 1/3 & \text{saddle} \\
R-3 & 0 & \frac{\sqrt{2}}{2m} & \frac{5n^2-6}{2m^2} & 0 & \frac{2-m}{m} & 1/3 & \text{saddle} \\
M-1 & 0 & 0 & 0 & 1 & 0 & 0 & \text{saddle/attractor} \\
M-2 & \frac{3(3-2n)}{4n} & 0 & \frac{3(2n^2+3n-3)}{4n^4} & \frac{n-3}{2n} & 0 & \text{saddle/attractor} \\
M-3 & 0 & \frac{3(n+1)(3-n)}{4n^3} & 0 & \frac{4(n-5)m-9}{2m^2} & \frac{3-2m}{2m} & 0 & \text{saddle/attractor} \\
DE-1 & \frac{3(3+n)(1-n)}{3n-m} & 0 & 0 & 0 & \frac{2n}{3n-m} - 1 + \frac{2(n+1)n}{3-n} & \text{saddle/attractor} \\
DE-2 & 0 & \frac{\sqrt{3(3-2m)}}{3-m} & 0 & 0 & \frac{m}{3} & -1 + \frac{2m}{3m} & \text{saddle/attractor} \\
DE-3 & \frac{3(n+1)+m(n+1)}{m+m-6} & \frac{\sqrt{3(n+1)+m(n+1)}}{m+m-6} & 0 & 0 & \frac{m-n}{m+2n} & -1 + \frac{2m}{m+2n} & \text{saddle/attractor} \\
\end{array}
\]

**TABLE I.** Fixed points for the dynamical system (16)-(19). The points are labelled according to the cosmological regime as $R$- (radiation), $M$- (matter) and $DE$- (dark energy).

\[
\begin{array}{cccc}
\text{Fixed Point} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
R-1 & -1 & 1 & \frac{2-m}{m} & \frac{2-n}{n} \\
R-2 & 1 & 2 \left[ 1 + \frac{m}{2} \left( \frac{3}{n} - \frac{3}{2} \right) \right] - \frac{3}{2} & \frac{1}{4} - \frac{2m}{n} & \frac{1}{4} - \frac{2m}{n} \\
R-3 & 1 & 2 \left[ 1 + \frac{m}{2} \left( \frac{3}{n} - \frac{3}{2} \right) \right] & \frac{1}{4} - \frac{2m}{n} & \frac{1}{4} - \frac{2m}{n} \\
M-1 & -\frac{3}{2} & -1 & \frac{3}{2} & \frac{3}{2} \\
M-2 & -1 & \frac{3}{2} \left[ 1 - m \left( 1 - \frac{1}{4n} \right) \right] - \frac{3}{4} & \frac{1}{4} - \frac{2m}{n} & \frac{1}{4} - \frac{2m}{n} \\
M-3 & -1 & \frac{3}{2} \left[ 1 + 2n \left( \frac{3}{m} - 1 \right) \right] - \frac{3}{4} & \frac{1}{4} - \frac{2m}{n} & \frac{1}{4} - \frac{2m}{n} \\
DE-1 & \frac{3(n+1)+m(n+1)}{m+m-6} & \frac{3(n+1)+m(n+1)}{m+m-6} & \frac{3(n+1)+m(n+1)}{m+m-6} & \frac{3(n+1)+m(n+1)}{m+m-6} \\
DE-2 & \frac{2(n-6)}{3-m} & \frac{3(n+1)}{3-m} & \frac{3(n+1)}{3-m} & \frac{3(n+1)}{3-m} \\
DE-3 & \text{too long to show} & \text{too long to show} & \text{too long to show} & \text{too long to show} \\
\end{array}
\]

**TABLE II.** Eigenvalues for the equilibrium points in Table I. The expressions for the eigenvalues of the point $DE-3$ are too long, and thus omitted.

The eigenvalues of $\mathbb{M}$ in this point show that this is a saddle point. The eigenvalues $\lambda_3$ and $\lambda_4$ are negative in the region of existence given in Eq. (27). The second eigenvalue is positive in the region of existence of the point for $n \leq 2m/(3m-4)$, and negative otherwise. Since the eigenvector associated to this eigenvalue points to the $f_1$ direction, when $\lambda_2 > 0$, the dark component $f_1$ grows during the radiation epoch.

\[ \cdot (M-1) \text{ Isotropic matter:} \] This corresponds to an isotropic matter-dominated universe with $\Omega_{DE} = 0$, and $w_{DE}$ undetermined. The eigenvector associated with the eigenvalue $\lambda_1 = -3/2$ points to the $\Sigma$ direction indicating that the trajectories around this point are attracted in this direction (faster than around the point $(R-I)$). In this case, the eigenvector associated with the eigenvalues $\lambda_2 = -1$ points to the $\Omega_r$ direction, meaning that radiation is decaying. The eigenvalues $\lambda_3$ and $\lambda_4$ are positive for $m$ and $n$ less than $3/2$, respectively. Under this condition, $(M-1)$ is a saddle with two positive eigenvalues, and the dark components $f_1$ and $f_2$ grow during the matter epoch, since the eigenvector associated to these eigenvalues point to the $f_2$ and $f_1$ directions, respectively.

\[ \cdot (M-2) \text{ Anisotropic matter scaling with } F^1: \] The energy-density and equation of state of dark energy in this solution are given by

\[
\Omega_{DE} = \frac{(2n-3)(n-3)}{8n^2}, \quad w_{DE} = 0, \tag{28}
\]

indicating that $\rho_{DE}$ scales as a pressureless fluid. This point is a viable solution if

\[ f_1^2 \geq 0, \quad 0 \leq \Omega_{DE} \leq 1, \quad \text{and} \quad 0 \leq \Omega_m \leq 1, \]
are satisfied. Moreover, CMB anisotropies give the bound $\Omega_{DE} < 0.02$ around the redshift $z = 50$ [3] (which ensures that we are deep in the matter-dominated era). Therefore, the physical region of existence of $(M-2)$ is

$$1.40166 < n \leq 1.5, \quad \forall \ m.$$  \hspace{1cm} (29)

The eigenvalues $\lambda_3$ and $\lambda_4$ are negative in the region of existence given in Eq. (29). In order for this point to be cosmologically viable, it has to be a saddle rather than an attractor. This means that the second eigenvalue has to be positive. We have $\lambda_2 > 0$ in the region of existence of the point for $m < 2n/(2n - 1)$. Since the eigenvector associated to this eigenvalue points towards the $f_2$ direction, when $\lambda_2 > 0$, the dark component $f_2$ grows during the matter epoch.

- $(M-3)$ Anisotropic matter scaling with $F^2$: For this point, the energy-density and equation of dark energy are

$$\Omega_{DE} = \frac{(2m - 3)(n - 3)}{2m^2}, \quad w_{DE} = 0.$$  \hspace{1cm} (30)

Imposing the conditions

$$f_2^2 \geq 0, \quad 0 \leq \Omega_{DE} \leq 1, \quad \text{and} \quad 0 \leq \Omega_r \leq 1,$$

together with the CMB bound $\Omega_{DE} < 0.02$, the physical region of existence of $(M-3)$ is found to be

$$1.47166 < m \leq 1.5, \quad \forall \ n.$$  \hspace{1cm} (31)

The eigenvalues $\lambda_3$ and $\lambda_4$ are negative in the region of existence given in Eq. (31). For this point to be a saddle, the second eigenvalue has to be positive, which is the case in its region of existence if $n \leq m/(2m - 2)$. Since the eigenvector associated to this eigenvalue points to the $f_1$ direction, when $\lambda_2 > 0$, the dark component $f_1$ grows during the matter epoch.

3. Dark Energy Dominance

- $(DE-1)$ Anisotropic dark energy scaling with $F^1$: This is the first solution corresponding to an anisotropic dark energy dominated universe. The energy density and equation of state of dark energy are given by

$$\Omega_{DE} = 1, \quad w_{DE} = w_{eff} = -1 + \frac{2n(1+n)}{3-n}.$$  \hspace{1cm} (32)

If we now impose the conditions $f_1^2 \geq 0$ and $-1 \leq w_{DE} < -1/3$ – the latter being necessary for having accelerated solutions, we find the following regions of existence

$$-1.86852 < n \leq -1 \quad \lor \quad 0 \leq n < 0.535184.$$  \hspace{1cm} (33)

However, since current observations favour an equation of state of dark energy $w_{DE} \approx -1$ nowadays [3], and a small anisotropy of $|\Sigma_0| < O(0.001)$ [14-15], we take the region of existence of this point as

$$0 \leq n < 0.535184, \quad \forall \ m.$$  \hspace{1cm} (34)

The first branch in (33), although leading to a viable equation of state for dark energy, leads to a too large $|\Sigma_0|$, and is thus discarded.

The eigenvalues $\lambda_{2,3,4}$ are negative in the region of existence given in Eq. (34), while $\lambda_1$ is negative in this region when

$$m \geq n \left( \frac{1+n}{1-n} \right).$$  \hspace{1cm} (35)

Therefore, we conclude that $(DE-1)$ is an attractor inside the region of existence in Eq. (34) whenever Eq. (35) is obeyed.

- $(DE-2)$ Anisotropic dark energy with $F^2$: The dark energy parameters in this solution are

$$\Omega_{DE} = 1, \quad w_{DE} = w_{eff} = -1 + \frac{2m}{3-m}.$$  \hspace{1cm} (36)

Imposing the conditions $f_2^2 \geq 0$ and $-1 \leq w_{DE} < -1/3$, we arrive at the following region of existence

$$0 \leq m < 0.75, \quad \forall \ n.$$  \hspace{1cm} (37)

The eigenvalues $\lambda_{1,2,3}$ are negative in the region of existence given by Eq. (37), while $\lambda_4$ is negative in this region when

$$n \geq \frac{m}{1-m}.$$  \hspace{1cm} (38)

Thus, $(DE-2)$ is an attractor inside the region of existence in Eq. (37) when Eq. (38) is satisfied.

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[4] We concentrate in no-ghost solutions; i.e. $w_{DE} \geq -1$, although this possibility has not been discarded by observations yet [3].

[5] The symbol $\lor$ stands for the logic “OR”.

[6] Here, the subscript 0 means that the corresponding quantity is evaluated nowadays.
FIG. 1. Region of existence of (DE-3) when it is an accelerated solution without phantom line crossing. Regions (I) and (II) are given by Eqs. (40) and (41), respectively.

- **(DE-3) Anisotropic dark energy scaling with $F^1$ and $F^2$:** This is the only solution in which dark energy scales with both $F^1$ and $F^2$. The parameters of dark energy in this case are

$$\Omega_{\text{DE}} = 1, \quad w_{\text{DE}} = w_{\text{eff}} = -1 + \frac{2mn}{m + 2n}. \quad (39)$$

By demanding that both $f_2^2$ and $f_3^2$ are positive, and that $-1 \leq w_{\text{DE}} < -1/3$, we arrive at two possible regions for the parameters $m$ and $n$:

- (I): $0 < n < 0.535184$ \& $\frac{n}{1 + n} \leq m < n \left(\frac{1 + n}{1 - n}\right)$,

- (II): $0.535184 \leq n < 3$ \& $\frac{n}{1 + n} \leq m < \frac{2n}{3n - 1}$.

These regions are plotted in Fig. 1. Notice that the cases $m = 0$ or $n = 0$ are not allowed in the region of existence. The eigenvalues of $\mathbb{M}$ in this fixed point are given by too long algebraic expressions involving the parameters $n$ and $m$. We omit them here since only the sign of the real part of the eigenvalues is relevant to the stability analysis. We investigated the parameter window, inside the regions of existence in Fig. 1, where $\lambda_{1,2,3,4}$ are negative, and we found that the four eigenvalues are negative inside the whole region of existence of the point; i.e., when (DE-3) exists, it is an attractor.

From the previous analysis, it is clear that the three dark energy dominated points can be attractors. In Fig. 2 we plot the $(m, n)$ parameter space where each dark energy dominated point is an attractor. We can see that these regions are separated by bifurcation curves (black solid lines), meaning that they are mutually excluded. This ensures that the system has only one dark energy attractor for a particular set of parameters $(m, n)$.

V. COSMOLOGICAL EVOLUTION

Figure 2 summarizes our main results regarding the theoretical viability of the solid as a model of anisotropic dark energy. In this section we want to study the dynamics of the model for the parameters $(m, n)$ inside the coloured regions of Fig. 2. We will implicitly assume that, prior to the radiation epoch, the Universe underwent an inflationary period which perfectly smoothed any initial spatial shear or inhomogeneities. Therefore, we choose $\Sigma_i = 0$ as an ini-
tial condition\footnote{Here, the subscript \( i \) means that the corresponding quantity is evaluated at some time deep in the radiation epoch.} such that the starting point for any cosmological trajectory is from the isotropic radiation point \((R-1)\). Moreover, we will choose parameters \( m \) and \( n \) such that the attractor point is given by \((DE-2)\). This choice allows us to give a simple and concrete example of the cosmological dynamics which can be easily extended to the other attractor points. Since \( \Sigma_i = 0 \), it is natural to assume that the contributions to the energy budget coming from the variables \( f_1 \) and \( f_2 \) are the same at the starting point. Having this in mind, we have chosen

\[
\Omega_{r_i} = 0.99995, \quad f_{1i} = f_{2i} = 10^{-14}, \quad \Sigma_i = 0 \quad (42)
\]
as initial conditions at the redshift \( z = 7.25 \times 10^7 \), and we have integrated the system up to \( z \to -1 \). Since observations favor an equation of state of dark energy close to \(-1\), from Eq. \((36)\) we have to choose \( m \approx 0 \). In particular, we have chosen \( m = 10^{-5} \). From Fig. \( 2 \) we can see that there are less restrictions regarding the choice of the parameter \( n \). For example, we could assume a value for \( n \) allowing the existence of the scaling points \((R-2)\) or \((M-2)\). However, for simplicity, we have chosen \( n = 10^{-2} \), such that the scaling points do not exist.

In Fig. \( 3 \) we plot the dynamical evolution of \( \Omega_r, \Omega_m, \Omega_{DE}, \Sigma \) and \( w_{DE} \) obtained from the numerical integration of Eqs. \((16)-(19)\). In the case where \( \Sigma_i \approx 0 \), there are no appreciable changes in the cosmological behavior presented in Fig. \( 3 \). However, a “stiff matter” epoch driven by the spatial shear appears before the radiation era, which we briefly treat in Appendix A.

In particular, Fig. \( 3 \) shows that the radiation-dominated epoch \((\Omega_r \approx 1 \text{ and } w_{eff} \approx 1/3)\) runs from \( z = 7.25 \times 10^7 \) to \( z \approx 3200 \) where the radiation-matter transition occurs. Moreover, \( \Omega_{DE} \approx 3.52 \times 10^{-11} \) during this transition, obeying the BBN constraint \( \Omega_{DE} < 0.045 \). The length of this radiation phase is in agreement with the constraint given in Ref. \([43]\). From \( z \approx 3200 \), the Universe is dust-dominated \((\Omega_m \approx 1 \text{ and } w_{eff} \approx 0)\) until \( z \approx 0.3 \) at the matter-dark energy transition. The contribution of the dark sector is \( \Omega_{DE} \approx 1.58 \times 10^{-5} \) at \( z = 50 \), value which is within the CMB bound \( \Omega_{DE} < 0.02 \). The dark energy-dominance \((\Omega_{DE} \approx 1 \text{ and } w_{eff} < -1/3)\) starts from \( z = 0.3 \) and on into the future, agreeing with the results obtained by the dynamical system analysis, i.e. \((DE-2)\) is an attractor. This is further supported by the fact that the values of \( f_1, f_2, \Sigma \) and \( w_{DE} \) are those predicted by the dynamical system. Explicitly, \( f_1 \approx 0, f_2 \approx 0.707107, \Sigma \approx 3.33334 \times 10^{-6}, \) and \( w_{DE} \approx -0.999993 \) in the far future \((z \to -1)\), which are consistent with the values computed from the \((DE-2)\) line in Table \( 1 \) and Eq. \((36)\). Although \( w_{DE} \) seems to be constant during the whole expansion history, this is not the case. Indeed, during the radiation-matter transition \((z \approx 3200)\) we have \( w_{DE} \approx -0.998059 \), while during the matter-dark energy transition \((z \approx 0.3)\) we have \( w_{DE} \approx -0.998171 \). The final value is \( w_{DE} \approx -0.999993 \) which corresponds to the value in the attractor point [see Eq. \((36)\)]. Thus, the numerical solution shows an ultra slowly varying equation of state of the dark energy, changing only about \( \approx 0.001\% \) during this particular cosmological trajectory.

From Fig. \( 3 \) we can also notice that dark energy does not behave as radiation or dust, given that \( w_{DE} \not\in \{1/3, 0\} \), confirming that the Universe does not approach the scaling anisotropic points. Indeed, we have confirmed, for several pairs of parameters \((n, m)\), that the expansion history of the Universe is

![FIG. 3. Evolution of the density parameters, the effective equation of state, and the equation of state of dark energy during the whole expansion history. The initial conditions were chosen deep in the radiation era at the redshift \( z = 7.25 \times 10^7 \). The Universe passes through radiation dominance at early times (red dotted line), followed by a matter dominance (light brown dashed line), and ends in the dark energy dominance (black solid line) characterized by \( w_{eff} \approx -1 \) (blue dot-dashed line). The dark sector behaves very similar to a cosmological constant since \( w_{DE} \approx -1 \) (magenta small-dashed line).](image-url)
very similar to that shown in Fig. 3 and thus the cosmological trajectories of the Universe are never close to the scaling anisotropic points. We can thus conclude that the typical cosmological evolution is given by

\[(R-I) \rightarrow (M-I) \rightarrow (DE-i),\]

where \(i\) can be 1, 2 or 3, depending on the \(m\) and \(n\) values we choose (see Fig. 2).

As shown in Fig. 4, the density parameters \(f_1^2 \equiv \Omega_{f_1}\) and \(f_2^2 \equiv \Omega_{f_2}\) associated with each solid Lagrangian \(F^i\) grow during the late matter-dominated epoch around \(z = 10\), while they are subdominant in the whole prior cosmological evolution, as expected from the dynamical analysis.

We have also investigated the evolution of \(\Sigma\) taking the same initial conditions in Eq. (42) and \(n = 10^{-2}\), but this time with different choices for the parameter \(m\) in such a way that \((DE-2)\) is the only attractor of the system. The results are shown in Fig. 5 where we can see that \(|\Sigma|\) starts to grow around \(z = 10\), similarly to \(f_1\) and \(f_2\) as seen in Fig. 4. The values predicted for the present spatial shear are \(|\Sigma_0| \leq 9.4 \times 10^{-4}\), corresponding to the black solid curve in Fig. 5, which are in agreement with the observational bounds \(|\Sigma_0| \leq O(0.001)\) [44-45]. We want to stress that, even for \(\Sigma_0 = 0\), the final state of the Universe is an anisotropic accelerated expansion.

On the other hand, we corroborated that the other two dark energy points, namely \((DE-1)\) and \((DE-3)\), can be attractors in their respective stability regions (see Fig. 2). In order for \((DE-1)\) to be the only attractor of the system, we choose \(n = 10^{-5}\) and \(m = 10^{-3}\) according to the existence and stability conditions in Eqs. (34) and (35). The initial conditions are the same as in Eq. (42) and, although the values of \(f_1, f_2, \Sigma\) and \(w_{DE}\) change, the cosmological evolution is qualitatively the same. In particular, the shear and the equation of state of dark energy today are \(|\Sigma_0| \approx 1.2 \times 10^{-4}\) and \(w_{DE} \approx -0.999559\). We confirm that the values predicted by the dynamical system analysis are in agreement with our numerical solution: \(f_1 = 1, f_2 = 0, \Sigma = -6.66669 \times 10^{-6}\) and \(w_{DE} = -0.999993\) [see Table 1 and Eq. (22)], when evaluated for redshifts \(z \rightarrow -1\). Finally, for the point \((DE-3)\), we choose \(n = 10^{-3}\) and \(m = 10^{-3}\), so that this point is the only attractor of the system. Using the same initial conditions as in Eq. (42), the cosmological dynamics is not significantly changed apart from the important fact that \(\Sigma = 0\) during the whole cosmological evolution, i.e. the expansion history is isotropic. By changing the values of the parameters \(n\) and \(m\) while keeping \((DE-3)\) as the only attractor, we verified that the case \(n = m\) is the unique isotropic solution. The numerical results are also consistent with the analytical findings, namely: \(f_1 = f_2 = 0.57735, w_{DE} = -0.999333\) and \(\Sigma = 0\) for redshifts \(z \lesssim -0.99\) [see Table 1 and Eq. (38)], i.e. the attractor point is reached very quickly.
A. Observational Signatures

From the magnitude-redshift data of SNe Ia, the analysis of Ref. [41] showed that the present value of the spatial shear is constrained to be $|\Sigma_0| \lesssim \mathcal{O}(0.01)$. More recently, the authors of Ref. [45] used a combination of the observational Hubble data $H(z)$ and SNe Ia measurements to put the tighter bound of $|\Sigma_0| \lesssim \mathcal{O}(0.008)$ [45]. For the set of parameters and initial conditions used in Fig. 5, we can see that $|\Sigma_0|$ is within these bounds, although it can take greater values in the future cosmological evolution.

An anisotropic dark energy has the potential for breaking of statistical isotropy and explain of some of the large scale CMB anomalies. In particular, anisotropic pressure can induce peculiar velocity flows, anisotropy in the SNe Ia data, and significant CMB dipole and quadrupole [17] fluctuations. The main observational effect of an anisotropic shear is its contribution to the CMB temperature anisotropies. This contribution is introduced through the redshift at last scattering surface, which becomes anisotropic. Considering only large-scale fluctuations, this can be quantified as [26, 49]:

$$\frac{\delta T}{T}(\hat{n}) \lesssim |\sigma_0 - \sigma_{\text{dec}}|,$$  \hspace{1cm} (43)

where $\delta T/T$ is the CMB temperature anisotropies, $\hat{n}$ is the unit vector along the line-of-sight, and $\sigma_{\text{dec}}$ is the value of the geometrical shear at the time of decoupling ($z \simeq 1090$). Since $\sigma' = \Sigma$, we can obtain the values for the geometrical shear by numerical integration. For the cases with the largest and smallest shear today, as depicted in Fig. 5 and using the same initial conditions of Eq. [42], we obtain

largest $\Sigma_0$ : $|\sigma_0 - \sigma_{\text{dec}}| \approx 5 \times 10^{-4}$, \hspace{1cm} (44)

smallest $\Sigma_0$ : $|\sigma_0 - \sigma_{\text{dec}}| \approx 3 \times 10^{-5}$, \hspace{1cm} (45)

which are in qualitative agreement with the conservative bound $|\sigma_0 - \sigma_{\text{dec}}| < 10^{-4}$ [3], thus, a solid anisotropic dark energy could alleviate the observed CMB quadrupole anomaly.

VI. ISOTROPIC DARK ENERGY

So far, we have been mainly interested in the solid as a viable model for an anisotropic late-time expansion of the Universe. Interestingly, note that the spatial shear vanishes if $n = 0, m = 0$, or $n = m$, when the attractor point is $(DE-1), (DE-2)$, or $(DE-3)$, respectively. However, this does not mean that $\Sigma$ is zero during the whole expansion history, since the dynamical analysis only provides the asymptotic behavior of the system. Nonetheless, the solid is also compatible with the FLRW metric, and it can also be used to describe an isotropic late-time expansion with equation of state close to $-1$. For this we have to consider that the Lagrangians $F^I$ are identical. Thus, the isotropic model requires $\sigma = 0$ in the Bianchi-I metric in Eq. [6], and:

$$X^I = X = 1/a^2, \quad F^I = F, \quad I = 1, 2, 3.$$  \hspace{1cm} (46)

The Friedman equations in Eqs. [9] and [10] are simplified as

$$3m_0^2H^2 = 3F + \rho_m + \rho_r,$$  \hspace{1cm} (47)

$$-2m_0^2\dot{H} = 2XF_X + \rho_m + \frac{4}{3}\rho_r.$$  \hspace{1cm} (48)

Defining $f^2 = F/(m_0^2H^2)$ and using the density parameter $\Omega_r$ in Eq. [14], the autonomous set is reduced to

$$f' = f(q + 1 - x),$$  \hspace{1cm} (49)

$$\Omega_r^* = 2\Omega_r(q - 1),$$  \hspace{1cm} (50)

where $\Omega_m$ is given in terms of $f$ and $\Omega_r$ by the Friedman constraint coming from Eq. [47], and the deceleration parameter is given by

$$q = \frac{1}{2} [1 + (2x - 3)f^2 + \Omega_r],$$  \hspace{1cm} (51)

where $x = XF_X/F$ is a function characterizing the form of the Lagrangian $F$. Assuming a power law model, $F \propto X^n$, we get $x = n$ and the system is closed. Three fixed points can be found: a radiation dominated point which is a source for $n < 2$, a matter dominated point which is a saddle for $n < 3/2$, and a dark energy dominated point which is an attractor for $n < 3/2$.

In Fig. 3 we numerically integrate the set given by Eqs. [49] and [50]. We assume $n = 10^{-3}$ and the initial conditions according with the present observed values for the density parameters: \[2\]

$$\Omega_{r,0} = 10^{-4}, \quad \Omega_{m,0} = 0.3, \quad \Omega_{DE,0} \equiv f_0^2 \approx 0.7.$$  \hspace{1cm} (52)

We can see that the cosmological evolution is quite similar to the one of the anisotropic case, which is shown in Fig. 3 and detailed in Sec. [7].

In contrast to the quintessence model, this model does not have a kination epoch previous to the radiation domination period. This is due to the fact that the kination epoch requires $\dot{\phi}^2 \gg V(\phi)$, where $\phi$ is...
the quintessence scalar field and $V$ its potential. In the quintessence model, the potential is necessary in order to get accelerated expansion since this stage is provided by $\dot{\phi}^2 \ll V(\phi)$. In the present case, such potential is not needed since the accelerated expansion is driven by the proper kinetic terms of the scalar fields through the function $F$. Another important difference between this model and quintessence is that the equation of state of dark energy can be constant or dynamical. The equation of state of dark energy is

$$w_{\text{DE}} = -1 + \frac{2}{3} x_i,$$  \hspace{1cm} \text{(53)}

which is constant for constant $x$ (power law model), thus mimicking a cosmological constant. However, $w_{\text{DE}}$ can also be dynamical for a general time-dependent $x$ function.

![Graph showing evolution of density parameters](image)

**FIG. 6.** Evolution of the density parameters, the effective equation of state, and the equation of state of dark energy for $n = 10^{-3}$. The initial conditions were chosen according with the present observed values for the density parameters, i.e.: $\Omega_{r,0} = 10^{-4}$, $\Omega_{m,0} = 0.3$, $\Omega_{\text{DE}a} \equiv f_a^2 \approx 0.7$. The Universe begins in a radiation dominated epoch (red dotted line), followed by a matter dominance one (light brown dashed line) and ends in the dark energy epoch (black solid line) characterized by $w_{\text{eff}} \simeq -1$ (blue dot-dashed line). We can see that the dark sector behaves as a cosmological constant since $w_{\text{DE}} \simeq -1$ (magenta small-dashed line).

**VII. CONCLUSIONS**

In this work we studied a set of three scalar fields with a constant but nonvanishing spatial gradients as a source of anisotropic dark energy. This particular set of inhomogeneous scalar fields, known as “solid”, has been used in the inflationary context showing interesting features at the perturbative level \cite{31,32}. It was also shown that anisotropic inflationary solutions are possible \cite{39,40}. Here, we have investigated the late-time dynamics of the solid in a Bianchi-I expanding universe. Through a dynamical system analysis, we showed that for a particular power-law model, the solid can generate an anisotropic late-time accelerated expansion, being this epoch an attractor of the system for suitable values of the free parameters of the model (see Fig. 2). In these regions, the cosmological evolution starts in a radiation-dominance epoch, followed by a matter-dominance epoch, and ending in an anisotropic dark energy-dominated period which can be realized by three different points, whose attractor regions are separated by bifurcation curves, i.e. they are mutually excluded.

The dynamical analysis was complemented with a numerical integration of the dynamical system. The parameters were fixed in such a way that (DE-2) was the only attractor point and the equation of state of dark energy was close to $-1$ as expected from observations \cite{3}. The initial conditions were chosen in the deep radiation era (redshift $z = 7.25 \times 10^7$) and assuming a zero spatial shear ($\Sigma_i = 0$). We found that the spatial shear differs sensibly from zero at around $z = 10$, taking nonnegligible values nowadays but within the observational bounds $|\Sigma_0| \lesssim \mathcal{O}(0.001)$ \cite{44,45}. The numerical solution also showed an ultra slowly varying equation of state of dark energy, $w_{\text{DE}}$, that only changed about $\sim 0.001\%$ from its value at $z = 7.25 \times 10^7$ to its final value in the far future ($z \rightarrow -1$). We verified that similar behaviors are obtained if the parameters are chosen to establish (DE-1) or (DE-3) as the attractor points.

The nonvanishing shear after the radiation-dominated epoch leaves imprints on CMB and SNe Ia data. In particular, the shear affects the CMB quadrupole temperature anisotropy through the standard Sachs-Wolfe formula. We showed that the change of the spatial shear from decoupling to today can be compatible with the CMB quadrupole data. In particular, if $|\sigma_0 - \sigma_{\text{dec}}|$ is of order $10^{-4}$, there may be an interesting possibility for addressing the CMB quadrupole anomaly.

We have also investigated the isotropic version of the solid model as a candidate for (isotropic) dark energy. In this case, only three fixed points were found: a source radiation point, a saddle matter point, and
an attractor dark energy point. The evolution of the Universe is very similar to the anisotropic case (see Figs. [3]). However, for a power-law model, the equation of state of dark energy is indeed a constant [see Eq. (53)], and is thus phenomenologically distinguishable from quintessential models.

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Appendix A: Other fixed point

The fixed points presented are the relevant point for the cosmological history of the Universe. Nonetheless, the dynamical system has another fixed points, which we present here.

(S) Stiff matter domination: This fixed point is characterized by

\[ f_1 = 0, \quad f_2 = 0, \quad \Sigma = \pm 1, \quad \Omega_c = 0, \]

with \( \Omega_m = 0, \Omega_{\text{DE}} = 1 \) and \( w_{\text{DE}} = 1 \). Since \( w_{\text{eff}} = 1 \), these points correspond to a “stiff matter” domination driven by the spatial shear \( \Sigma \). This also implies that the energy density of dark energy decays as \( \rho_{\text{DE}} \propto a^{-6} \), and thus this period is prior to the radiation domination. Although these points are the only possible sources of the model (i.e, all the eigenvalues of the Jacobian matrix \( \mathbb{M} \) are positive), they can be arbitrarily pushed back to the past depending on how small \( \Sigma \) is. For example, these points are in the infinite past for \( \Sigma = 0 \). This stiff fluid is common in quintessence models where the kinetic term of the scalar field has the chance to dominate [4], and it has been pointed out in some works that this period can be useful in the study of the reheating process [50–55].

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