Abstract. The paper is devoted to generalizations of actions of topological groups on manifolds. Instead of a topological group, we consider a local topological group generalizing the notion of a germ or a neighborhood in a topological group. The notion of an action of a local group on a topological space is introduced.

The paper constructs the theory of local sharply $n$-transitive groups and local $n$-pseudofields. Local sharply $n$-transitive groups are reduced to simpler algebraic objects — local $n$-pseudofields, similarly to the way Lie groups are reduced to Lie algebras, and sharply two-transitive groups, are reduced to neardomains. This can be useful, since, opposite to locally compact and connected sharply $n$-transitive groups, which are absent for $n > 3$, local sharply $n$-transitive groups exist for any $n$, for example, the group $GL_n(\mathbb{R})$. Being boundedly sharply $n$-transitive, the groups under consideration are also Lie groups, which gives extra methods for their study.

Keywords: Local topological group, local sharply $n$-transitive group, local $n$-pseudofield.

MSC Classification: 22A99, 22A30, 18F60, 20B22.

1 Introduction

C. Jordan [1] discovered in 1872 that among finite groups, with the exception of the symmetric groups $S_n$, alternating groups $A_{n+2}$ and Mathieu groups $M_{11}$ and $M_{12}$, there are no sharply $n$-transitive groups for $n > 3$.

In 1931 Carmichael [2] came to the conclusion that finite sharply 2-transitive permutation groups are groups of affine transformations $x \mapsto xb + a, b \neq 0$, of a finite nearfield. In 1936 Zassenhaus [3, 4] recovered this result and in addition showed that every finite sharply 3-transitive group is isomorphic to a group of transformations $x \mapsto \frac{xb + c}{xd + e}$ with suitable conditions on $a, b, c, d$ over a field $\mathbb{F} \cup \{\infty\}$ and, in certain cases, over a nearfield.

Tits [5] showed that if a sharply 2-transitive group is locally compact and connected and acts on a topological space, then it is isomorphic to the group of transformations $x \mapsto xb + a, b \neq 0$, of the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$, or the skew-field of quaternions $\mathbb{H}$. In this case a sharply 3-transitive group is isomorphic to the group of transformations $x \mapsto \frac{xb + c}{xd + e}$ with the condition $ad - cb \neq 0$. Such a group can be constructed only over the fields $\mathbb{R}$ or $\mathbb{C}$. In spite of the absence of infinite sharply $k$-transitive groups for $k > 3$, it is known [6] that infinite $k$-transitive but not $(k + 1)$-transitive groups exist for arbitrary $k$.

If we do not require the group to be locally compact and connected, then a sharply 2-transitive group is isomorphic to the affine group of transformations $x \mapsto xb + a$ of some pseudo-field. At the beginning of the 1950s Tits [7] defined a pseudo-field as an algebraic system $\mathbb{B} = \langle B; +, -, \cdot, 0 \rangle$ with two binary operations satisfying the following axioms:

1) $\langle B; +, -, 0 \rangle$ is a magma with neutral element 0;
2) $a + (-a) = 0 \Rightarrow (-a) + a = 0$;
3) \( \langle B_1; ·, -1, 1 \rangle \) is a group with neutral element 1, where \( B_1 = B \setminus \{0\} \);
4) \( x \cdot 0 = 0 \cdot x = 0 \);
5) \( (x + y) \cdot z = x \cdot z + y \cdot z \);
6) \( (\exists r_{ab} \in B_1) \) such that \( (x + a) + b = x \cdot r_{ab} + (a + b) \) for any \( x \in B \).

Isomorphic sharply 2-transitive groups can be constructed over non-isomorphic pseudo-fields. To avoid such a situation, in the mid-1960s Karzel [7, 8] introduced a similar algebraic system, a neardomain, as a system \( \mathbb{B} \) with two binary operations. Here, axiom 1) was strengthened and axiom 4) was changed:

1*) \( \langle B; +, - , 0 \rangle \) is a loop with neutral element 0;
4*) \( x \cdot 0 = 0 \).

Along with the generalization of a nearfield to a neardomain, a KT-field was introduced in [9] for the construction of sharply 3-transitive groups in the infinite case. This is a pair \( (\mathbb{B}, \varepsilon) \), where \( \mathbb{B} \) is a neardomain and \( \varepsilon \) is an automorphism of the group \( B_1 \). This automorphism satisfies the identity

\[
\varepsilon(1 - \varepsilon(x)) = 1 - \varepsilon(1 - x). \tag{1}
\]

P.M. Cohn [10, Lemma 7.5.1.] considered an equivalent definition of a skew-field \( \mathbb{F} = \langle F; ·, +, -1, -1, -1, 1, 0 \rangle \), which he constructed using a unary operation \( \varphi : F_0 \to F_0 \) acting on a multiplicative group \( F_1 = \langle F_1; ·, -1, 1 \rangle \), where \( F_0 = F_1 \setminus \{1\} \). The operation \( \varphi \) satisfies the following axioms:

1. \( \varphi(yxy^{-1}) = y\varphi(x)y^{-1}, \ x, y \in F_0 \);
2. \( \varphi(\varphi(x)) = x, \ x \in F_0 \);
3. \( \varphi(xy^{-1}) = \varphi(\varphi(x)(\varphi(y))^{-1})\varphi(y^{-1}), \ x, y \in F_0, x \neq y \);
4. the element \( b = \varphi(x^{-1})x(\varphi(x))^{-1} \) does not depend on the chosen \( x \in F_0 \).

Here it turns out that \( b = -1 \) and \( \varphi(x) = 1 - x \).

W. Leissner obtained similar results independently (see [11]). He also showed in [12] that when only part of the requirements on the function \( \varphi : B_0 \to B_0 \) were applied, one could obtain a nearfield (using only Axioms 2, 3, and 4) or a neardomain (only by Axioms 2 and 3).

When developing his approach with a view to constructing sharply \((k + 1)\)-transitive groups, instead of a single automorphism \( \varepsilon \subseteq Aut(\mathbb{B}_1) \) with condition (1) for a KT-field, Leissner included a symmetric group of automorphisms \( S_{k-1} \subseteq Aut(\mathbb{B}_1) \) such that \( S_k = \langle \varphi, S_{k-1} \rangle \). By using such an algebraic system Leissner constructed the sharply \( n \)-transitive groups \( S_n, S_{n+1}, A_{n+2} \) and the sharply 4- and 5-transitive Mathieu groups \( M_{11}, M_{12} \). He called the algebra that he introduced a \( \mathbb{B}_1 \)-field of degree \( n \) (see [13]), where \( \mathbb{B}_1 \) is the multiplicative group over which the field of degree \( n \) is constructed. For example, a \( \mathbb{B}_1 \)-field of degree 3 is associated with a KT-field.

In [14], A. A. Simonov constructed a generalization of sharply \( n \)-transitive groups to boundedly sharply \( n \)-transitive groups. Among them, there are local sharply 2-transitive groups that cannot be constructed over local neardomains.

\[\text{1This notation means that the group } S_k \text{ is generated by the subgroup } S_{k-1} \text{ and the element } \varphi.\]
The paper develops the theory of local sharply $n$-transitive groups and local $n$-pseudofields in line with [12, 13] and [14]. Local sharply $n$-transitive groups are reduced to simpler algebraic objects — local $n$-pseudofields. It can be useful because, opposite to locally compact and connected sharply $n$-transitive groups, which are absent for $n > 3$, there are local sharply $n$-transitive groups for arbitrary $n$, for example, the group $\text{GL}_n(\mathbb{R})$.

In Section 2, we give the definitions of a local group, a local group isomorphism, a continuous group of transformations, and a local $n$-pseudofield.

Section 3 contains the main constructions. A local sharply 2-transitive group is constructed in Theorem 1 from a local pseudofield. Then the result is extended, and a local sharply $n$-transitive group is constructed from a local $n$-pseudofield in Theorem 2.

At the next step, Theorem 3 is applied to solve the inverse problem — a local $n$-pseudofield is constructed from a local sharply $n$-transitive group. The section is finished by Theorem 4, which proves the equivalence of the categories of local sharply $n$-transitive groups and local $n$-pseudofields.

We now explain our terminology. To describe sharply 2-transitive groups, in [5], Tits introduced the algebraic system, a pseudofield, as the generalization of the concepts of a field, a skew-field, and a nearfield. But later, a close concept, a near-domain, was applied for describing such groups, and so the term pseudofield got vacant. In [13], Leissner introduced the notion of a G-field of degree $n$ for describing sharply $n$-transitive groups. In this article, following the previously introduced notion of an $n$-pseudofield (see [14]), we define a local $n$-pseudofield for describing the algebraic systems associated with local sharply $n$-transitive groups.

2 Definitions

2.1 A local group

Give the definition of local topological groups and a local isomorphism (see [15, §23]):

**Definition 1** A topological space $G$ is called a local group if the product $ab \in G$ is defined for some pairs $a, b$ of elements of $G$; moreover, the following conditions must be satisfied:

1. If the products $ab, (ab)c, bc, a(bc)$ are defined then the equality $(ab)c = a(bc)$ holds.
2. If the product $ab$ is defined then, for every neighborhood $W$ of $ab$, there are neighborhoods $U$ and $V$ of $a$ and $b$ respectively such that if $x \in U$ and $y \in V$ then the product $xy$ is defined and $xy \in W$.
3. $G$ contains a distinguished element $e$, called the unit, such that if $a \in G$ then the product $ea$ is defined and $ea = a$.
4. If the product $ab$ is determined for a pair $a, b$ and $ab = e$ then $a$ is said to be the left inverse for $b$, $a = b^{-1}$. If $b$ has a left inverse then, for every neighborhood $V$ of $b$, there is a neighborhood $U$ of $b^{-1}$ such that each $y \in V$ has a left inverse $y^{-1} \in U$. 
Let $G$ be a local group. Refer to any neighborhood $U$ of the unit $e$ in $G$ as a part of the local group $G$. Every part $U$ of a local group $G$ is itself a local group with the operations induced from $G$.

**Definition 2** Let $G$ and $G'$ be two local groups and let $U$ and $U'$ be their parts. A mapping $f$ is said to be a local isomorphism from $G$ onto $G'$ if $f$ is a homeomorphism from $U$ onto $U'$ and the following conditions hold:

1. If the product $ab$ is defined in $U$ then the product $f(a)f(b)$ is defined in $U'$ and $f(ab) = f(a)f(b)$.
2. $f$ takes the unit into the the unit.
3. $f$ is invertible, and its inverse $f^{-1}$ satisfies the same conditions as $f$.

If there is a local isomorphism from a local group $G$ onto a local group $G'$ then $G$ and $G'$ are said to be locally isomorphic.

Two local isomorphisms of $f$ and $f'$ of a group $G$ onto a group $G'$ are called equivalent if they coincide on some part of $G$. Below we will analyze local isomorphisms only up to equivalence.

Let us give also the definitions of the groups of transformations [15, §24]:

**Definition 3** A topological group $G$ is called a continuous group of transformations of a topological space $\Gamma$ if for any element $x \in G$ there corresponds a transformation $x^*$ of $\Gamma$ so that $(xy)^* = x^*y^*$ and the function $\sigma$ of two variables $x \in G$ and $\xi \in \Gamma$ defined by the relation $\sigma(x, \xi) = x^*(\xi)$ is continuous, i.e. gives a continuous mapping of the direct product $G \times \Gamma$ of the topological spaces $G$ and $\Gamma$ onto $\Gamma$.

If different elements in the group $G$ give different transformations then $G$ is called an effective group of transformations. In this case, the elements $G$ can be treated as transformations ($x = x^*$).

A continuous group $G$ of transformations of a space $\Gamma$ is called transitive if the abstract group $G$ of transformations of $\Gamma$ is transitive.

Henceforth, by a continuous group of transformations we mean a pair $(\Gamma, G)$, where $G$ is a topological group and $\Gamma$ is a topological space. Let us now consider mappings $\psi : G \to G'$ and $\chi : \Gamma \to \Gamma'$.

**Definition 4** A pair of mappings $(\chi, \psi)$ is called a similarity of the pair $(\Gamma, G)$ onto the pair $(\Gamma', G')$ if $\psi : G \to G'$ is a group isomorphism, $\chi : \Gamma \to \Gamma'$ is a homeomorphism of topological spaces, and

$$\chi[g(x)] = \psi(g)(\chi[x]),$$

where $g \in G$, $x \in \Gamma$.

If there is a pair of mappings $(\chi, \psi)$ that is a similarity of $(\Gamma, G)$ and $(\Gamma', G')$ then the pairs $(\Gamma, G), (\Gamma', G')$ are called similar.

**Definition 5** Call a continuous group of transformations $(\Gamma, G)$ acting on a space $\Gamma$ as locally sharply $n$-transitive if $G$ is a local group acting on some open subspace $M \subset \Gamma^n$ sharply transitively.
2.2 A local pseudofield

Consider the symmetric group $S_n$ and a group of transformations $(G, S_n)$, acting locally $G \times S_n \to G$ in the space $G$. In other words, local homeomorphisms $f_\alpha$ are defined in $G$; they are indexed by elements $\alpha \in S_n$ for which $f_\beta(f_\alpha(x)) = f_{\alpha \beta}(x)$.

It is known that $S_n$ is generated by the transpositions $(1, i)$, where $i = 2, 3, \ldots, n$. Note that

$$(1, i)(1, j)(1, i) = (1, j)(1, i)(1, j) = (i, j) \quad \text{for} \quad i \neq j.$$

Denote the involute local homeomorphisms defined by transpositions as follows:

$$f_{(1,i)} = e_i, \quad \text{for} \quad (i = 2, \ldots, n).$$

The binary operation $(\cdot) : G \times G \to G$ is defined almost everywhere in $G^2$ and its restriction to $G_1$ gives the local structure $(G_1; \cdot, E, e)$ on $G_1 \subset G$. Using the local homeomorphisms $e_i$, from the local group $(G_1; \cdot, E, e)$, construct the locally isomorphic groups

$$\varphi_i : (G_1; \cdot, E, e) \to (G_1; \cdot, E_i, e_i),$$

where $E(x) = x^{-1}$ is the local homeomorphism of taking the inverse in the group $G_1$, and $x \cdot_i y = \varphi_i(\varphi_i(x)\varphi_i(y))$, $E_i(x) = \varphi_i(E(\varphi_i(x)))$ are the multiplication and the inverse taken in $G_i$; $e_i = \varphi_i(e)$ are the local units of the local groups $G_1$ and $G_i$ respectively.

**Definition 6** Say that a group of transformations $(G, S_n)$ defines a local $n$-pseudofield $(G; \cdot, E, \varphi_2, \ldots, \varphi_n, e)$ if the following conditions are fulfilled:

1. if the products $a \varphi_i(b^{-1})$, $\varphi_i(a \varphi_i(b^{-1}))b$, $a \cdot_i b$, are defined then

$$a \cdot_i b = \varphi_i(\varphi_i(a)\varphi_i(b)) = \varphi_i(a \varphi_i(b^{-1}))b;$$

2. if the product $a \cdot_i b$ is defined then, for every neighborhood $W$ of the element $a \cdot_i b$ there exist neighborhoods $U$ and $V$ of $a$ and $b$ such that for $x \in U$, $y \in V$ the products $x \cdot_i y$, $x \varphi_i(y^{-1})$, $\varphi_i(x \varphi_i(y^{-1}))y$ are defined and $x \cdot_i y = \varphi_i(x \varphi_i(y^{-1}))y \in W$.

3. The local homeomorphism $\sigma_{ij} = \varphi_j \varphi_i \varphi_j$ for $i \neq j$ is a local automorphism of the group $G_1$.

4. If $\varphi_i E \varphi_i(a)$ and $E \varphi_i E(a)$ are defined for some $a \in G$ then

$$\varphi_i E \varphi_i(a) = E \varphi_i E(a).$$

5. The elements $e_i = \varphi_i(e) \in G$ are left zeros for the binary operation in $G$, i.e., $e_i \cdot x = e_i$, for $x \in U$ from a neighborhood of the unit $e \in G_1$.

Equation (2) can be written down as a relation between two group operations:

$$(a \cdot_i b)b^{-1} = \varphi_i(a) \cdot_i b^{-1}$$

for $a \in G, b \in V \subset U \cap \varphi_i(U)$.

\footnote{The dimension of the space where the operation is undefined is less than the dimension of $G.$}
3 Basic constructions

3.1 A local sharply 2-transitive group

Theorem 1 From a local 2-pseudofield \( \langle G; \cdot, E, \varphi_2, e \rangle \), one can construct a local sharply 2-transitive group of transformations \((G, G^2)\).

Consider the topological space \( G \) and its square \( G^2 \). Separate neighborhoods \( U, U_2 \subset G \); \( W, W_2 \subset G^2 \) of the local units \( e \in U \), \( e_2 \in U_2 = \varphi_2(U) \), \( W \subset G \times U \), \( W_2 \subset U_2 \times G \), such that the following hold for arbitrary \( x \in G \), \((y_1, y_2) \in W \) and \( x' \in U_2 \), \((y'_1, y'_2) \in W_2 \):

\[
\varphi_2(y_1 y_2^{-1}) \in U, \quad \varphi_2(x \varphi_2(y_1 y_2^{-1})) y_2 \in G
\]

and

\[
\varphi_2(y'_2 \cdot 2 E_2(y'_1)) \in U_2, \quad \varphi_2(x' \cdot 2 \varphi_2(y'_2 \cdot 2 E_2(y'_1))) \cdot 2 y'_1 \in G.
\]

Define the functions \( f_1 : G \times G^2 \to G \),

\[
f_1(x, y_1, y_2) = \varphi_2(x \varphi_2(y_1 y_2^{-1})) y_2, \quad \text{for } x \in G, (y_1, y_2) \in W \quad (3)
\]

and

\[
f_2(x, y_1, y_2) = \varphi_2(x \cdot 2 \varphi_2(y_2 \cdot 2 E_2(y_1))) \cdot 2 y_1, \quad \text{for } x \in G, (y_1, y_2) \in W_2. \quad (4)
\]

For \( x \in G \), \((y_1, y_2) \in W \cap W_2 \), both functions \( f_1(x, y_1, y_2) \) and \( f_2(x, y_1, y_2) \) are defined and coincide with account taken of (2):

\[
f_2(x, y_1, y_2) = \varphi_2(x \cdot 2 \varphi_2(y_2 \cdot 2 E_2(y_1))) \cdot 2 y_1 = (\varphi_2(x) \cdot (y_2 \cdot 2 E_2(y_1))) \cdot 2 y_1
\]

\[
= \varphi_2(\varphi_2(x) \cdot \varphi_2(\varphi_2(y_2) \varphi_2 E_2(y_1))) \cdot \varphi_2(y_1)
\]

\[
= \varphi_2(x \varphi_2(\varphi_2(y_1) E \varphi_2(y_2)) \cdot \varphi_2 E(y_2)) y_2
\]

\[
= \varphi_2(x \varphi_2(y_1 E(y_2)) E \varphi_2 E(y_2) \varphi_2 E(y_2)) y_2
\]

\[
= \varphi_2(x \varphi_2(y_1 E(y_2))) y_2 = f_1(x, y_1, y_2).
\]

For \( x = e \) and \((y_1, y_2) \in W \), we have

\[
f_1(e, y_1, y_2) = \varphi_2(e \varphi_2(y_1 y_2^{-1})) y_2 = \varphi_2(\varphi_2(y_1 y_2^{-1})) y_2 = (y_1 y_2^{-1}) y_2 = y_1. \quad (5)
\]

For \( x = e_2 \) and \((y_1, y_2) \in W_2 \), by analogy, we have the second function:

\[
f_2(e_2, y_1, y_2) = y_2. \quad (6)
\]

Consider the function \( f_2 \) for \( x \in U, y \in U \cap U_2 \):

\[
f_2(x, y, e_2) = \varphi_2(x \cdot 2 \varphi_2(e_2 \cdot 2 E_2(y))) \cdot 2 y
\]

\[
= \varphi_2(x \cdot 2 \varphi_2 E_2(y)) \cdot 2 y = \varphi_2 \left( (x \cdot 2 \varphi_2 E_2(y)) \cdot \varphi_2(y) \right)
\]

\[
= \varphi_2 \left( x \cdot 2 \varphi_2 E_2(y) \right) \cdot \varphi_2(y) = \varphi_2 \left( \varphi_2(x) \cdot 2 \varphi_2(y) \right) = x \cdot y. \quad (7)
\]

Similarly, for the function \( f_1 \) and \( x \in U_2, y \in U \cap U_2 \), we have:

\[
f_1(x, e_1, y) = x \cdot 2 y.
\]
Define a function $f$ as follows:

$$
\begin{cases}
  f_1(x, y, z) \quad &\text{for } x \in G, (y, z) \in W, \\
  x \cdot y \quad &\text{for } x, y \in U, z = e_2, \\
  f_2(x, y, z) \quad &\text{for } x \in G, (y, z) \in W_2, \\
  x \cdot \_z \quad &\text{for } x, z \in U_2, y = e.
\end{cases}
$$

Further define a binary local operation $(\circ_2) : G^2 \times G^2 \to G^2$ as follows:

$$
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\circ_2
\begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix}
= 
\begin{pmatrix}
  f(x_1, y_1, y_2) \\
  f(x_2, y_1, y_2)
\end{pmatrix}.
$$

For convenience, we do not differ the pairs from $G^2$ written as a column
$$
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
$$
and a row $(x_1, x_2)$.

2°. Check condition (1) of Definition 1 of a local group (the associativity of the product of pairs):

$$(x_1, x_2), (y_1, y_2), (z_1, z_2) \in G^2.$$

On the one hand, for the $i$th component of the product

$$(x_1, x_2) \circ_2 ((y_1, y_2) \circ_2 (z_1, z_2)),$$

we can write

$$
\varphi_2(\varphi_2(x_1, \varphi_2(y_1, y_2^{-1})) \varphi_2(z_1, z_2^{-1})) z_2
\begin{equation}
= \varphi_2(\varphi_2(x_1, \varphi_2(y_1, y_2^{-1})) \varphi_2(y_2, \varphi_2(z_1, z_2^{-1}))) z_2
= \varphi_2(x_1, \varphi_2(y_1, y_2^{-1})) \varphi_2(y_2, \varphi_2(z_1, z_2^{-1})) \varphi_2(y_2, \varphi_2(z_1, z_2^{-1})) z_2.
\end{equation}
$$

The transformation of $\varphi_2$ has led to a representation of the $i$th component already of the product $((x_1, x_2) \circ_2 (y_1, y_2)) \circ_2 (z_1, z_2)$ so that the local operation (its local nature will be checked later) $\circ_2$ is associative, and so it one can assert that $(G^2; \circ_2)$ is a local semigroup.

3°. Let us check condition (2) of Definition 1 of a local group.

Suppose that the value $f(a, b, c)$ is defined for some $a, b, c$. Since a local group and a group isomorphic to it with multiplication $(\cdot)$ are defined in $G$, for every neighborhood $W_1$ of the element $cb^{-1}$, there are neighborhoods $U_1$ and $V_1$ of $c$ and $b^{-1}$ such that the product $xy$ is defined for $x \in U_1, y \in V_1$ and $xy \in W_1$. Then, for every neighborhood $W_2$ of $a' \cdot b'$, there are neighborhoods $U_2$ and $V_2$ of $a'$ and $b'$ such that the product $x \cdot y$ is defined for $x \in U_2, y \in V_2$ and $x \cdot y \in W_2$. And finally, for every neighborhood $W_3$ of $a''b''$, there are neighborhoods $U_3$ and $V_3$ of $a''$ and $b''$ such that the product $xy$ is defined for $x \in U_3, y \in V_3$ and $xy \in W_3$. Then, by superposition, for an arbitrary neighborhood $W = W_3 \ni f(a, b, c)$, there exist neighborhoods $U = \varphi_2(U_2), V' \subseteq V_3 \cap E(V_1), V \subseteq U_1$, with $V_2 \subseteq W_1, U_3 \subseteq W_2$ such that $f(x, y, z) \in W$ holds for arbitrary $x \in U, y \in V', z \in V$. 

7
Consider arbitrary pairs \((a_1, a_2), (b_1, b_2)\) for which
\[
(c_1, c_2) = (f(a_1, b_1), f(a_2, b_1, b_2)),
\]
is defined but, in this case, from the previous construction, for any neighborhood \(W \ni (c_1, c_2)\), there exist neighborhoods \(U \ni (a_1, a_2), V \ni (b_1, b_2)\) such that, for arbitrary \((x_1, x_2) \in U, (y_1, y_2) \in V\), we have
\[
(f(x_1, y_1, y_2), f(x_2, y_1, y_2)) = (x_1, x_2) \circ_2 (y_1, y_2) \in W.
\]
Hence, the operation \(\circ_2\) is local.

4°. Consider the pair \((e, e_2)\) as the local unit; then, reckoning with \((5)\) and \((6)\), we have: \((e, e_2) \circ_2 (y_1, y_2) = (y_1, y_2)\). Thus, the pair \((e, e_2) \in G^2\) is the local unit, whereas \(\langle G^2; \circ_2 \rangle\) is the local magma.

Verify that the left inverse to \((x_1, x_2)\) is
\[
\begin{pmatrix}
\varphi_2(x_2^{-1}) E\varphi_2(x_1 x_2^{-1}) \\
E\varphi_2(x_1 x_2^{-1})
\end{pmatrix}
\]
Indeed, in the product,
\[
\begin{pmatrix}
\varphi_2(x_2^{-1}) E\varphi_2(x_1 x_2^{-1}) \\
E\varphi_2(x_1 x_2^{-1})
\end{pmatrix} \circ_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]
for the first component, we have
\[
f_1(\varphi_2(x_2^{-1}) E\varphi_2(x_1 x_2^{-1}), x_1, x_2) = \varphi_2(\varphi_2(x_2^{-1}) E\varphi_2(x_1 x_2^{-1}) \varphi_2(x_1 x_2^{-1})) x_2 = \varphi_2^2(x_2^{-1}) x_2 = x_2^{-1} x_2 = e.
\]
For the second component, we get
\[
f_2(E\varphi_2(x_1 x_2^{-1}), x_1, x_2) = \varphi_2 \left( E\varphi_2(x_1 x_2^{-1}) \cdot_2 \varphi_2 \left( x_2 \cdot_2 E(x_1) \right) \right) \cdot_2 x_1
\]
\[
= \left( \varphi_2 E \varphi_2(x_1 x_2^{-1}) \cdot_2 \varphi_2 \left( x_2 \cdot_2 E(x_1) \right) \right) \cdot_2 x_1
\]
\[
= \left( \varphi_2 E \varphi_2(x_1 x_2^{-1}) \cdot_2 \varphi_2 \left( x_2 E(x_1) \right) \right) \cdot_2 x_1
= \varphi_2 E \varphi_2(x_1) \cdot_2 x_1 = E(x_1) \cdot_2 x_1 = e_2.
\]
Condition (4) of Definition 1 of the local group follows from the superposition of the local group operations, the local nature of the transformations \(\varphi_2\), and taking the inverse in the local group. As a result, \(\langle G^2; \circ_2 \rangle\) is a local group. The theorem is proved.

Note that the group \(G\) is embedded in \(G^2\) as \(G \ni x \mapsto (x, e_2) \in G^2\), and the image of \(G\) under this embedding coincides with the stabilizer \(G \simeq (G^2)_{e_2}\) of \(e_2\) in \(G^2\), as follows from \((7)\) and the definition of function \((8)\).

As an example of a group \(G\), consider the multiplicative group \(\mathbb{R}^+\) and the function \(\varphi_2(x) = -x + 1\). The corresponding group \(G^2\) is constructed with the use of the function \(f(x, a, b) = x(a - b) + b\) and is isomorphic to the affine group of transformations of the set \(\mathbb{R}\).
3.2 Infix–postfix notation

Above, using a homeomorphism \( \varphi_2 \) and a group of transformations \((G, G)\), we constructed a group \((G, G^2)\). Considering \(n\)-pseudofields, as \(n\) grows from 2 to 3 and more, the number of parentheses rises substantially. To avoid their complication, we will use the combined infix and postfix notation of formulas.

The postfix notation for a group can be written as group action on itself \(G \times G' \rightarrow G\). For instance, the binary operation of multiplication in \(G\) can be written as a function (or a unary operation) \(x \cdot y \equiv f_y(x)\), whereas, in the postfix notation, it is done through the right action \(f_y(x) \equiv x \cdot [y]\), where \(x \in G, [y] \in G'\). For multiplying three elements, we have

\[
(x \cdot y) \cdot z = f_z(f_y(x)) = x \cdot [y][z].
\]

Associativity leads to the identity

\[
x \cdot [y][z] = x \cdot (y \cdot z) = f_{y \cdot z}(x) = x \cdot [y \cdot z].
\]

For brevity, we omit the multiplication dot, so that

\[
x \cdot [y][z] = x \cdot [y \cdot z] = x \cdot [yz].
\]

For the inverse operation \(x^{-1} = E(x)\), the identity \(E(E(x)) = x\) in the postfix notation looks as follows:

\[
x \cdot EE = x \quad \text{or in short} \quad EE = id.
\]

The identity \(a b b^{-1} = a\) for the group in the postfix notation looks as

\[
a \cdot [b][b^{-1}] = a \quad \text{or in short} \quad [b][b^{-1}] = id.
\]

Finally, in the postfix notation, when the inverse of an element succeeds multiplication by this element, we reduce the product.

Identity 2 of Definition 6 is written down as follows:

\[
a \cdot \varphi_i[\varphi_i(b)]\varphi_i = a \cdot [\varphi_iE(b)]\varphi_i[b],
\]

where \(\varphi_iE(b) = \varphi_i(b^{-1})\), then, for \(b' = \varphi_i(b)\), it is written down briefly as

\[
\varphi_i[b']\varphi_i = [E_i(b')]\varphi_i[\varphi_i(b')],
\]

where, as we recall, \(E_i = \varphi_iE\varphi_i = E\varphi_iE\). The identity

\[
\sigma_{ij}(\sigma_{ij}(x)y) = x\sigma_{ij}(y),
\]

which holds for the automorphism \(\sigma_{ij}\) in Definition 6(3), for a group \(G\) is rewritten as follows:

\[
\sigma_{ij}[y]\sigma_{ij} = [\sigma_{ij}(y)].
\]

For \(\varphi_i\) and \(\sigma_{jk}\), we have

\[
\varphi_i\sigma_{jk} = \sigma_{jk}\varphi_i,
\]

for \(i \neq j, k\) and

\[
\varphi_i\sigma_{ij} = \varphi_j\varphi_i.
\]

Let us sum up the transition to the mixed infix-postfix notation:
• the formulas partition into functions (unary operations) — $\varphi_1$, $E$, $\sigma_{ij}$, and postmultiplication $[y]$ and are written in the postfix form;

• if an element $y = f(x)$ in the unary operation of the postmultiplication $[y]$ is a function then, in the infix form, it looks as $[f(x)]$.

Write the function (3) obtained in theorem 1 as the couple

$$f_1(x, y_1, y_2) \equiv x \cdot [y_1, y_2] = x \cdot [\varphi_2(y_1y_2^{-1})] \varphi_2[y_2]. \quad (15)$$

(Note that, under no circumstances, the notation in square b rackets that we consider the commutator of the elements $y_1$ and $y_2$; it is just the notation for a pair. Moreover, we do not have to consider such a commutator anywhere, and so this notation should not confuse.) Then, for the function $f_2$, we may write

$$f_2(x, y_1, y_2) = x \cdot \varphi_2[\varphi_2(y_2), \varphi_2(y_1)] \varphi_2,$$

and agree the following for the natural notation of function (8):

$$[x, e_2] = [x], \ [e, y] = \varphi_2[\varphi_2(y)] \varphi_2.$$

Prove the following assertion:

**Lemma 3.1** If, for some $x \in U \subset G$ and $(y_1, y_2) \in W \subset G^2$, for which, $x \cdot [y_1, y_2] \varphi_2$ and $x \cdot [\varphi_2(y_1), \varphi_2(y_2)]$, $x \cdot \varphi_2[y_1, y_2]$, and $[x, y_1, y_2]$ are defined then

$$[y_1, y_2] \varphi_2 = [\varphi_2(y_1), \varphi_2(y_2)] \text{ and } \varphi_2[y_1, y_2] = [y_2, y_1].$$

Indeed, transform the first equality:

$$x \cdot [y_1, y_2] \varphi_2 = \varphi_2(\varphi_2(x \varphi_2(y_1y_2^{-1})) y_2)$$

$$= \varphi_2(x \varphi_2(y_1y_2^{-1}) \varphi_2 E \varphi_2(y_2)) \varphi_2(y_2)$$

$$= \varphi_2(x \varphi_2(y_1) \varphi_2 E \varphi_2(y_2)) \varphi_2(y_2^{-1}) \varphi_2 E \varphi_2(y_2) \varphi_2(y_2)$$

$$= \varphi_2(x \varphi_2(y_1 E \varphi_2(y_2))) \varphi_2(y_2) = x \cdot [\varphi_2(y_1), \varphi_2(y_2)].$$

For the second equality, we have

$$x \cdot \varphi_2[y_1, y_2] = \varphi_2(\varphi_2(x) \varphi_2(y_1y_2^{-1})) y_2$$

$$= \varphi_2(x \varphi_2(y_2y_1^{-1})) y_1y_2^{-1}y_2 = \varphi_2(x \varphi_2(y_2y_1^{-1})) y_1 = x \cdot [y_2, y_1].$$

3.3 A locally sharply $n$–transitive group

For a collection $(x_1, \ldots, x_{n-1}, x_n) \in G^n$, define a tuple function as the superposition of a tuple of a lesser dimension and the function $\varphi_n$:

$$[x_1, \ldots, x_{n-1}, x_n] = [\varphi_n(x_1x_n^{-1}), \ldots, \varphi_n(x_{n-1}x_n^{-1})] \varphi_n[x_n]. \quad (16)$$
Lemma 3.2 For tuple (16), we have the following equalities for $i \leq n$:

$$[x_1, \ldots, x_{n-1}, x_n] \varphi_i = [\varphi_i(x_1), \ldots, \varphi_i(x_{n-1}), \varphi_i(x_n)],$$  \hspace{1cm} (17)

$$[x_1, \ldots, x_{n-1}, x_n] [y] = [x_1y, \ldots, x_{n-1}y, x_ny]$$ \hspace{1cm} (18)

and

$$\varphi_i [x_1, \ldots, x_i, \ldots, x_n] = [x_i, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n].$$ \hspace{1cm} (19)

Prove the lemma by induction. Expression (18) is obtained just from the definition of (16) and the equality

$$x_i^{-1} x_n^{-1} = x_i y (x_n y)^{-1}.$$

For obtaining (17), write

$$[x_1, \ldots, x_{n-1}, x_n] \varphi_i \overset{(16) \text{ and } (11)}{=} \varphi_n(x_1 x_n^{-1}), \ldots, \varphi_n(x_{n-1} x_n^{-1})] \varphi_n E_i(x_n) \varphi_i [\varphi_i(x_n)],$$

which, for $i = n$, with account taken of the induction, transforms into the equality

$$[x_1, \ldots, x_{n-1}, x_n] \varphi_n = [\varphi_n(x_1 x_n^{-1}) E_n(x_n), \ldots, \varphi_n(x_{n-1} x_n^{-1}) E_n(x_n)] \varphi_n [\varphi_n(x_n)]$$

$$= [\varphi_n(\varphi_n(x_1) E \varphi_n(x_n)), \ldots, \varphi_n(\varphi_n(x_{n-1}) E \varphi_n(x_n))] \varphi_n [\varphi_n(x_n)]$$

$$= [\varphi_n(x_1), \ldots, \varphi_n(x_{n-1}), \varphi_n(x_n)].$$

For considering the case $i \in \{2, \ldots, n-1\}$, recall that the identity $\varphi_n \varphi_i = \varphi_i \sigma_{in}$ follows from the definition of the automorphism $\sigma_{ij}$. Hence,

$$\varphi_n(x_1 x_n^{-1}), \ldots, \varphi_n(x_{n-1} x_n^{-1})] \varphi_i \sigma_{in} [E_i(x_n)] \varphi_i [\varphi_i(x_n)]$$

$$\overset{(12)}{=} [\varphi_i \varphi_n(x_1 x_n^{-1}), \ldots, \varphi_i \varphi_n(x_{n-1} x_n^{-1})] [\sigma_{in} E_i(x_n)] \sigma_{in} \varphi_i [\varphi_i(x_n)]$$

$$= [\sigma_{in} \varphi_i(x_1 x_n^{-1}) \sigma_{in} E_i(x_n), \ldots, \sigma_{in} \varphi_i(x_{n-1} x_n^{-1}) \sigma_{in} E_i(x_n)] \varphi_i \varphi_n [\varphi_i(x_n)]$$

$$= [\sigma_{in} (\varphi_i(x_1) E \varphi_i(x_n)), \ldots, \sigma_{in} (\varphi_i(x_{n-1}) E \varphi_i(x_n))] \varphi_i \varphi_n [\varphi_i(x_n)]$$

$$= [\varphi_n(x_1), \ldots, \varphi_n(x_{n-1}), \varphi_n(x_n)].$$

Thus, expression (17) is proved. Let us now check (19). For $n = 2$, it was validated in Lemma 3.1. Let us now consider $n = 3$ for $\varphi_3$:

$$\varphi_3[y_1, y_2, y_3] = \varphi_3[\varphi_3(y_1 y_3^{-1}), \varphi_3(y_2 y_3^{-1})] \varphi_3[y_3]$$

$$= \varphi_3[y_1^{(1)}, y_2^{(1)}] \varphi_3[y_3]$$

$$= \varphi_2 \sigma_{23} \varphi_2[y_1^{(1)}, y_2^{(1)}] \varphi_3[y_3]$$

$$= \varphi_2 \sigma_{23} \varphi_2[y_1^{(1)} E(y_1^{(1)})] \sigma_{23} \varphi_2[y_1^{(1)}] \varphi_3[y_3]$$

$$= \varphi_2 \varphi_3 (y_1^{(1)} E(y_1^{(1)})) \varphi_3[y_3]$$

$$= \varphi_2 \varphi_3 (y_1^{(1)} E(y_1^{(1)})) \varphi_2 \varphi_3 [y_1^{(1)}] \varphi_3[y_3]$$
Let us now show that (19) holds if it is fulfilled for tuples of lesser dimension:

\[
\varphi_n[y_1, \ldots, y_{n-2}, y_{n-1}, y_n] = \varphi_{n-1}\sigma_n, n-1\varphi_{n-1-1}[y_1^{(1)}, \ldots, y_{n-2}^{(1)}, y_{n-1}]{\varphi_n[y_n]}
\]

\[
= \varphi_{n-1}\sigma_n, n-1[y_{n-1}^{(1)}, y_2^{(1)}, \ldots, y_{n-2}^{(1)}, y_{n-1}]{\varphi_n[y_n]}
\]

where

\[
y_i^{(1)} = y_i \cdot [E(y_n)]\varphi_n \text{ and } y_i^{(2)} = y_i^{(1)} \cdot [E(y_1^{(1)})]\varphi_{n-1}.
\]

Apply \(\sigma_{n,n-1}\):

\[
\sigma_{n,n-1}(y_i^{(2)}) = y_i \cdot [E(y_n)]\varphi_n[E\varphi_n(y_1{y_n}^{-1})]\varphi_{n-1}\sigma_{n,n-1}
\]

\[
= y_i \cdot [E(y_n)]\varphi_n[E\varphi_n(y_1{y_n}^{-1})]\varphi_{n-1}
\]

\[
= y_i \cdot [E(y_n)][E\varphi_n(y_1{y_n}^{-1})]\varphi_n[E\varphi_n(y_1{y_n}^{-1})]\varphi_{n-1}
\]

\[
= y_i \cdot [E(y_1)]\varphi_n[E\varphi_n(y_1{y_n}^{-1})]\varphi_{n-1} = (y_i^{(2)}).
\]

Continue expression (20) with account taken of (21) and (22):

\[
= \varphi_{n-1}\sigma_n, n-1[y_{n-1}^{(2)}, y_2^{(2)}, \ldots, y_{n-2}^{(2)}, y_{n-1}]{\varphi_n[y_n]}
\]

\[
= \varphi_{n-1}[(y_{n-1}^{(1)})^{(2)}, (y_2^{(1)})^{(2)}, \ldots, (y_{n-2}^{(1)})^{(2)}]{\sigma_n, n-1\varphi_{n-1-1}[y_1^{(1)}]{\varphi_n[y_n]}
\]

\[
= \varphi_{n-1}[(y_{n-1}^{(1)})^{(2)}, (y_2^{(1)})^{(2)}, \ldots, (y_{n-2}^{(1)})^{(2)}]{\varphi_{n-1}\varphi_n[y_1^{(1)}]{\varphi_n[y_n]}
\]

\[
= \varphi_{n-1}[(y_{n-1}^{(1)})^{(2)}, (y_2^{(1)})^{(2)}, \ldots, (y_{n-2}^{(1)})^{(2)}]{\varphi_{n-1}[E\varphi_n(y_1{y_n}^{-1})]\varphi_n(y_1{y_n}^{-1})]{\varphi_n[y_n]}
\]

\[
= \varphi_{n-1}[(y_{n-1}^{(1)})^{(2)}, (y_2^{(1)})^{(2)}, \ldots, (y_{n-2}^{(1)})^{(2)}]{\varphi_{n-1}[E\varphi_n(y_1{y_n}^{-1})]\varphi_n(y_1{y_n}^{-1})]{\varphi_n[y_1]}
\]

\[
= \varphi_{n-1}[(y_{n-1}^{(1)})^{(1)}, (y_2^{(1)})^{(1)}, \ldots, (y_{n-2}^{(1)})^{(1)}]{\varphi_n[y_1]}
\]

where \((y_i^{(1)})^{(1)} = \varphi_n(y_1{y_n}^{-1})\).

It remains to verify (19) for \(n\) and \(\varphi_i\) for \(i < n:\)

\[
\varphi_n[y_1, \ldots, y_i, \ldots, y_n] = \varphi_i[y_1^{(1)}, \ldots, y_i^{(1)}, \ldots, y_{n-1}^{(1)}]{\varphi_n[y_n]}
\]
\[ \left[ y_1^{(1)}, \ldots, y_1^{(1)}, \ldots, y_{n-1}^{(1)} \right] \varphi_n[y_n] = \left[ y_1, \ldots, y_i, \ldots, y_n \right]. \]

The lemma is proved. \hfill \Box

Define a function \( f : G \times G^n \to G \):

\[
f(x; y_1, \ldots, y_n) = \begin{cases} 
  x \cdot [y_1, \ldots, y_{n-1}] & \text{for } y_n = e_n, \\
  x \cdot [y_1, \ldots, y_n] & \text{for } x \in U, \\
  x \cdot \varphi_i [\varphi_i(y_i), \ldots, \varphi_i(y_1), \ldots, \varphi_i(y_n)] \varphi_i & \text{for } x \in \varphi_i(U),
\end{cases}
\]

where \( U \subset G \) is a neighborhood of the unit \( e \in G \). If \( x \in \varphi_i(U) \cap U \) then, with Lemma 3.2 taken into account:

\[ x \cdot [y_1, \ldots, y_n] = x \cdot \varphi_i [\varphi_i(y_i), \ldots, \varphi_i(y_1), \ldots, \varphi_i(y_n)] \varphi_i. \]

Using function (23), construct multiplication in \( G^n \):

\[
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f(x_1, y_1, \ldots, y_n) \\ \vdots \\ f(x_n, y_1, \ldots, y_n) \end{pmatrix},
\]

for collections \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in G^n \).

**Theorem 2** From a local \( n \)-pseudofield, it is possible to construct a local sharply \( n \)-transitive group of transformations \((G, G^n)\) with multiplication (24).

10. Verify condition (1) of the definition 1 of a local group — the associativity of operation (24).

It follows from the definition of tuples (15) and (16) with account taken of Lemma 3.2 that

\[ [x_1, x_2, \ldots, x_n] = \left[ x_1^{(n-1)} \right] \varphi_2 \left[ x_2^{(n-2)} \right] \ldots \left[ x_{n-1}^{(1)} \right] \varphi_n \left[ x_n \right], \]

where \( x_j^{(k)} = \varphi_{n+1-k} \left( x_j^{(k-1)} E \left( x_{n+1-k}^{(k-1)} \right) \right) \) and \( x_j^{(0)} = x_j \). Then, with account taken of (17) and (18), the multiplication of tuples \( [x_1, \ldots, x_n], [y_1, \ldots, y_n] \) is written down as follows:

\[ [x_1, \ldots, x_n] [y_1, \ldots, y_n] = [x_1 \cdot [y_1, \ldots, y_n], \ldots, x_n \cdot [y_1, \ldots, y_n]]. \]
Therefore,

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n 
\end{pmatrix}
\begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n 
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n 
\end{pmatrix}
= 
\begin{pmatrix}
  x_1 \bullet [y_1, \ldots, y_n] \\
  \vdots \\
  x_n \bullet [y_1, \ldots, y_n] 
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n 
\end{pmatrix}
= 
\begin{pmatrix}
  x_1 \bullet [y_1, \ldots, y_n] \cdot [z_1, \ldots, z_n] \\
  \vdots \\
  x_n \bullet [y_1, \ldots, y_n] \cdot [z_1, \ldots, z_n] 
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n 
\end{pmatrix}
\begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n 
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n 
\end{pmatrix}
= 
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n 
\end{pmatrix}
\begin{pmatrix}
  y_1 \\
  \vdots \\
  y_n 
\end{pmatrix}
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_n 
\end{pmatrix}.
\]

Condition (2) of Definition 1 is fulfilled by superposition.

20. Verify condition (3) of Definition 1 of a local group. Make sure that 
\((e,e_2,\ldots,e_n) \in G^n\) defines the left neutral element. For the unit \(e\) we have:

\[
f(e, x_1, \ldots, x_n) = e \bullet [x_1, \ldots, x_n]
= e \bullet [\varphi_n(x_1 x_n^{-1}), \ldots, \varphi_n(x_n x_n^{-1})] \varphi_n [x_n] = \varphi_n(x_1 x_n^{-1}) \bullet \varphi_n [x_n] = x_1.
\]

Hence, for \(i \geq 2\), we infer

\[
f(e, y_1, \ldots, y_n) = e_i \bullet \varphi_i [\varphi_i(y_1), \ldots, \varphi_i(y_1), \ldots, \varphi_i(y_n)] \varphi_i
= e \bullet [\varphi_i(y_1), \ldots, \varphi_i(y_1), \ldots, \varphi_i(y_n)] \varphi_i = \varphi_i(y_i) \bullet \varphi_i = y_i.
\]

30. Check condition (4) of Definition 1 of a local group.

Suppose that, in the local group \(G^{n-1}\), for \((x_1, \ldots, x_{n-1}) \in G^{n-1}\), there exists an inverse \((x_1, \ldots, x_{n-1})^{-1} \in G^{n-1}\) such that

\[
(x_1, \ldots, x_{n-1})^{-1} (x_1, \ldots, x_{n-1}) = (e_1, \ldots, e_{n-1}),
\]

where \(e_1 = e\). Then the tuple \([x_1, \ldots, x_{n-1}]\) has an inverse tuple \([x_1, \ldots, x_{n-1}]^{-1}\), and, for any \(y \in U\),

\[
y \bullet [x_1, \ldots, x_{n-1}]^{-1} [x_1, \ldots, x_{n-1}] = y.
\]

The inverse to an element \((x_1, \ldots, x_n) \in G^n\) is

\[
\left(\begin{array}{c}
  x_1 \\
  \vdots \\
  x_n 
\end{array}\right)^{-1}
= \left(\begin{array}{c}
  \varphi_n(x_n^{-1}) \bullet [x_1^{(1)}, \ldots, x_{n-1}^{(1)}]^{-1} \\
  \vdots \\
  E \varphi_n(x_1) \bullet \varphi_n([\varphi_1(x_1)]^{(1)}, \ldots, ([\varphi_n(x_n)]^{(1)})^{-1} \varphi_i \\
  \vdots \\
  E \varphi_n(x_1) \bullet \varphi_n([\varphi_n(x_n)]^{(1)}, \ldots, ([\varphi_n(x_{n-1})]^{(1)})^{-1} \varphi_n
\end{array}\right),
\]

(27)
Hence, as a local group of transformations $G$, the corresponding group $T$ of transformations $\varphi$ down with the use of the function $E\varphi = (\varphi_1, \ldots, \varphi_n$).

Multiplication by $(x_1, \ldots, x_n)$ from the right leads to multiplication by a tuple. For the first component, we have
\[
\varphi_1(x_1) = \varphi_1(x_1)^{(1)} = \varphi_n(x_1)E\varphi(x_1), \quad 1 \leq i < n,
\]
\[
\varphi_1(x_1) = \varphi_n(x_1)E\varphi(x_1), \quad i = n.
\]

Thus, $E\varphi(x_1) = (\varphi_1(x_1), \ldots, \varphi_{n-1}(x_1), \varphi_n(x_1))$ defines the inverse in the local group $G^n$.

For the components with the numbers $i = 2, \ldots, n - 1$, we infer
\[
\varphi_i E\varphi_i(x_1) = \varphi_i((\varphi_i(x_1))^{(1)}, \ldots, (\varphi_i(x_n-1))^{(1)}, \varphi_i(x_n))
\]
\[
= \varphi_i E\varphi_i(x_1) \cdot \varphi_i((\varphi_i(x_1))^{(1)}, \ldots, (\varphi_i(x_n-1))^{(1)}, \varphi_i(x_n))
\]
\[
= \varphi_i E\varphi_i(x_1) \cdot \varphi_i E\varphi_i(x_1) = E\varphi_i(x_1) \cdot \varphi_i(x_n) = \varphi_i(e) = e_i.
\]

Finally, for the last component, we have
\[
E\varphi_n(x_1) \cdot \varphi_n((\varphi_n(x_1))^{(1)}, \ldots, (\varphi_n(x_n-1))^{(1)}, \varphi_n(x_n))
\]
\[
= E\varphi_n(x_1) \cdot \varphi_n E\varphi_n(x_1) \cdot (\varphi_n(x_1))^{(1)}, \ldots, (\varphi_n(x_n-1))^{(1)}, \varphi_n(x_n))
\]
\[
= E\varphi_n(x_1) \cdot [\varphi_n(x_1)] \varphi_n = \varphi_n(e) = e_n.
\]

Thus, (27) defines the inverse in the local group $G^n$.

The constructed local group $G^n$ is sharply transitive under the action on itself. Hence, as a local group of transformations $G^n$ of $G$, it is sharply $n$-transitive.

The theorem is proved.

Thus, we have constructed a mapping $F_2 : (G, \varphi_2, \ldots, \varphi_n) \to (G, G^n)$, i.e., a procedure that, given an arbitrary $n$-pseudofield $(G, \varphi_2, \ldots, \varphi_n)$, constructs the corresponding group of transformations $(G, G^n)$.

### 3.4 Examples

As a simplest example of a local sharply 2-transitive group, consider the group of affine transformations of the field of real or complex numbers $x \to xa + b$, for which the corresponding group $T_2$ can be written as
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1(y_1 - y_2) + y_2 \\ x_2(y_1 - y_2) + y_2 \end{pmatrix}.
\]

Here $(1, 0)$ is the neutral element. The corresponding local 2-pseudofield is written down with the use of the function $\varphi_2(x) = -x + 1$ acting on the multiplicative group $G$.
Extending this example to the case $n = 3$, pass to the locally isomorphic group $\psi : G \to G'$ by means of the transformation $\psi(x) = \frac{2x}{1 + x}$ and its inverse $\psi^{-1}(x) = \frac{x}{2x}$ so that the multiplication in $G'$ has the form

$$x \cdot' y = \frac{2xy}{1 + x + y - xy}$$

with the functions

$$\varphi'_2(x) = \psi^{-1} \varphi_2 \psi(x) = \frac{1 - x}{1 + 3x} \text{ and } \varphi'_3(x) = -x$$

acting on this group and $e'_1 = 1, e'_2 = 0, e'_3 = -1$. In this case, the group multiplication in $T_3$ can be written through the tuple function

$$x \bullet [y_1, y_2, y_3] = \frac{x(2y_1y_3 - y_2(y_1 + y_3)) + y_2(y_3 - y_1)}{x(y_1 - 2y_2 + y_3) + y_3 - y_1}.$$ 

Other examples for the groups $T_n$ of transformations of $\mathbb{R}^2$ for $n \leq 4$ can be found in [17].

The group $GL_n(\mathbb{R})$ is an example of a local sharply $n$-transitive group of transformations in $\mathbb{R}^n$ constructed by means of the local $n$-pseudofield with distinguished elements

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1),$$

from the multiplicative group $G$ with the multiplication

$$(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) = (x_1y_1, x_1y_2 + x_2, \ldots, x_1y_n + x_n)$$

and the functions $\varphi_i$ which, under the action at the row $(x_1, \ldots, x_n)$, replace two elements $x_i$ and $x_j$ leaving the remaining coordinates fixed.

Using the same group $G$ and the same functions $\varphi_i$ for $i < n$ but $\varphi_n$, which is replaced by

$$\varphi_n(x_1, \ldots, x_n) = (1 - x_1 - x_2 - \ldots - x_{n-1}, x_2, \ldots, x_n),$$

the Mikhaĭlichenko group $T_n$ is constructed [18]; it is nonisomorphic to $GL_n$ but embeddable in $GL_{n+1}$.

### 3.5 A local $n$-pseudofield

Let us show that it is possible to construct a local $n$-pseudofield from a local sharply $n$-transitive group. Namely, we have the following assertion:

**Theorem 3** Given a local sharply $n$-transitive group $T_n$ of transformations of a set $G$, it is possible to construct a local $n$-pseudofield from $T_n$. 

16
10. Since $T_n$ is a local sharply $n$-transitive group, the stabilizer of arbitrary $n$ of various elements from $G$ is trivial. Consider $n$ different elements $e_1, \ldots, e_n$ from $G$ for which there is a nontrivial stabilizer of $e_2, \ldots, e_n$. Fix this collection $[e_1, \ldots, e_n]$.

The action of the group $T_n$ on $G$ is written as $a \mapsto a \cdot x$, where $x \in T_n$, $a \in G$. Define a structure of a local group on $G^n$. With an element $x \in T_n$, associate a tuple $[x_1, \ldots, x_n]$ from $G^n$ by the rule

$$[x_1, \ldots, x_n] = [e_1 \cdot x, \ldots, e_n \cdot x] = [e_1, \ldots, e_n] \cdot x.$$  

Then the neutral element $e \in T_n$ element determines the tuple $[e_1, \ldots, e_n] \in G^n$. Define the multiplication operation of such sets in accordance with the rule:

$$[x_1, \ldots, x_n][y_1, \ldots, y_n] = [e_1, \ldots, e_n] \cdot (xy) = [x_1, \ldots, x_n] \cdot y$$

$$= [x_1 \cdot y_1, \ldots, x_n \cdot y] = [x_1 \cdot [y_1, \ldots, y_n], \ldots, x_n \cdot [y_1, \ldots, y_n]],$$  

(28)

where we use that, owning to the correspondence $T_n \ni x \mapsto [x_1, \ldots, x_n] \in G^n$, the elements of the group $G^n$ act at elements of $G$ by the rule

$$a \cdot [x_1, \ldots, x_n] \equiv a \cdot x, \quad \text{где} \quad a \in G, \ x \in T_n.$$  

By construction, the local groups $T_n$ and $G^n$ are locally isomorphic.

The action of $G^n$ at the elements $e_i$ follows from the identity

$$[e_1, \ldots, e_n] \cdot [x_1, \ldots, x_n]$$

$$= [e_1 \cdot [x_1, \ldots, x_n], \ldots, e_n \cdot [x_1, \ldots, x_n]] = [x_1, \ldots, x_n].$$  

(29)

Denote the stabilizer of $e_2, \ldots, e_n$ in $G^n$ by $G^1$. If $[y_1, \ldots, y_n] \in G^1$ then, by the definition of the stabilizer,

$$e_i \cdot [y_1, \ldots, y_n] = e_i, \quad \text{for} \quad i \in \{2, \ldots, n\}.$$  

Consequently, reckoning with (29), the stabilizer $G^1$ consists of the elements

$$[y_1, e_2, \ldots, e_n] \in G^n.$$  

Put $[y_1] = [y_1, e_2, \ldots, e_n]$, then

$$a \cdot [y_1, e_2, \ldots, e_n] = a \cdot [y_1], \quad e_i \cdot [y_1] = y_1 \quad \text{and} \quad e_i \cdot [y_1] = e_i, \quad \text{where} \ i > 1.$$  

(30)

The following equality holds in $G^1$:

$$[x_1][y_1] = [x_1, e_2, \ldots, e_n][y_1, e_2, \ldots, e_n] = [x_1 y_1, e_2, \ldots, e_n] = [x_1 y_1].$$  

Basing on it, determine the inverse $x_1^{-1}$ for $x_1$:

$$[x_1^{-1}] = [x_1]^{-1} = [x_1, e_2, \ldots, e_n]^{-1}.$$  

Thus, we have transferred the structure of the group $G^1$ to the set $G$ itself and have obtained a group $G$ in which the multiplication is written without a dot. Then we
can state that (30) implies identity (5) in definition 6, i.e., the elements $e_i$ for $i > 1$
are left zeros for the elements of $G$.

Now, denote by $G_i^2$ the stabilizer of the elements

$e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$

in $G^n$. It is easily to see that every element in $G_i^2$ looks as $[x_1, e_2, \ldots, x_i, \ldots, e_n]$ for
some $x_1, x_i \in G$. Introduce the following notation for elements of $G_i^2$:

$[x_1, x_i] \equiv [x_1, e_2, \ldots, x_i, \ldots, e_n]$. 

In the stabilizer $G_i^2$, the element $[e_1, e_i]$ is neutral, and $[e_i, e_1] \in G_i^2$ is an involution:

$[e_i, e_1][e_i, e_1] = [e_1, e_i]$. 

Then, for any $[x_1, x_i] \in G_i^2$, we have the equalities:

$[e_i, e_1][x_1, x_i] = [x_i, x_1]; \quad [x_1, x_i][e_i, e_1] = [\phi_i(x_1), \phi_i(x_i)]$, \quad (31)

where, by definition, 

$\phi_i(a) = a \cdot [e_i, e_1], \quad a \in G$. 

Note that $\phi_i(e_1) = e_i$. 

For arbitrary $[e_i, x_i]$, we have

$[e_i, x_i] = [x_i^{-1}, e_i][x_i, e_2] = [\phi_i(x_i^{-1}), e_i][e_i, e_1][x_i, e_i] = [\phi_i(x_i^{-1})][e_i, e_1][x_i]$. 

On the other hand, with account taken of (31), we get

$[e_i, x_i] = [e_i, e_1][\phi_i(x_i^{-1}), e_i][e_i, e_1] = [e_i, e_1][\phi_i(x_i^{-1})][e_i, e_1]$. 

Thus,

$\phi_i[\phi_i(x_i)] \phi_i = [\phi_i(x_i^{-1})] \phi_i [x_i]$. \quad (32)

Acting at an element $a \in G$ by both sides of the equality, we obtain expression (2)
from definition 6.

Since this identity is obtained on the local group $G^n$, condition (2) of definition 6
is fulfilled.

20. Consider $x_i \in G$ for which $\phi_i(x_i^{-1}), E\phi_i(x_i^{-1}) \in U$, and (32) can be considered
at the action on $E\phi_i(x_i^{-1})$. Then, taking (30) into account, on the one hand, we have

$E\phi_i(x_i^{-1}) \cdot [\phi_i(x_i^{-1})] \phi_i [x_i] = e_i \cdot \phi_i [x_i] = e_i \cdot [x_i] = e_i$, 

and on the other hand, we obtain

$E\phi_i(x_i^{-1}) \cdot \phi_i [\phi_i(x_i)] \phi_i = \phi_i (\phi_i E\phi_i(x_i^{-1}) \phi_i(x_i))$. 

Consequently,

$\phi_i E\phi_i(x_i^{-1}) \phi_i(x_i) = e_1$, 

from which we get identity (4) of Definition 6.
3. For proving Assertion (3) of Definition 6, given arbitrary $X = [x_1, \ldots, x_n] \in G^n$, construct the element $X_{ij} \in G^n$ obtained from $X$ by interchanging $x_i$ and $x_j$. It follows from (29) that

$$E_{ij}X = X_{ij},$$

where $E_{ij} = [e_1, \ldots, e_n]_{ij}$. On the other hand, $E_{ii}^2 = E = [e_1, \ldots, e_n]$, for $i \in \{2, \ldots, n\}$ and

$$E_{ii}E_{ij}E_{ii} = E_{ij}E_{ii}E_{ij} \quad \text{at} \quad i \neq j.$$

In addition to the above-introduced $\phi_i(x) = x \cdot E_{ii}$, $x \in G$, define $\varepsilon_{ij} : G \to G$ as

$$\varepsilon_{ij}(x) = x \cdot E_{ij} = x \cdot E_{ij}E_{ii}E_{ij} = \varphi_j \varphi_i \varphi_j(x).$$

Then, for arbitrary $x \in U$, $y \in \phi_i(U) \cap U$, we have

$$x \cdot [y, e_2, \ldots, e_n]E_{ij} = x \cdot E_{ij}E_{ij}[y, e_2, \ldots, e_n]E_{ij} = x \cdot E_{ij}[\varepsilon_{ij}(y), e_2, \ldots, e_n],$$

and so we arrive at the equality

$$\varepsilon_{ij}(xy) = \varepsilon_{ij}(x)\varepsilon_{ij}(y).$$

Therefore, $\varepsilon_{ij}$ belongs to the group of automorphisms of the local group $G$, which leads us to the fulfillment of condition (3) of Definition 6. The theorem is proved.

Thus, we have constructed the map

$$F_1 : (G, T_n) \to \langle G, \phi_2, \ldots, \phi_n \rangle,$$

which associates with a local group of transformations $(G, T_n)$ the corresponding local $n$-pseudofield.

### 3.6 Categorical equivalence

**Definition 7** For any class of algebras $K\mathfrak{A}$, denote by $\overline{K}\mathfrak{A}$ the category whose objects are algebras $\mathfrak{A} \in K\mathfrak{A}$ and morphisms are homomorphisms of algebras.

Let us now give the definition of an equivalence of categories (see [16, §4.4]):

**Definition 8** A functor $\overline{F}_2 : \overline{K}\mathfrak{A}_1 \to \overline{K}\mathfrak{A}_2$ is called an equivalence of categories and the categories $\overline{K}\mathfrak{A}_1$ and $\overline{K}\mathfrak{A}_2$ are called equivalent if there is an (opposed) functor $\overline{F}_1 : \overline{K}\mathfrak{A}_2 \to \overline{K}\mathfrak{A}_1$ and natural isomorphisms:

$$\overline{F}_1 \overline{F}_2 \cong I : \overline{K}\mathfrak{A}_1 \to \overline{K}\mathfrak{A}_1 \quad \text{and} \quad \overline{F}_2 \overline{F}_1 \cong I : \overline{K}\mathfrak{A}_2 \to \overline{K}\mathfrak{A}_2.$$

Henceforth we will consider the group of transformations $(G, G^n)$ as a two-sorted algebra $\langle G, G^n; \bullet, g, E \rangle$, where $g : G^n \times G^n \to G^n$ is the group operation, $E$ is the unary operation of taking the inverse in $G^n$. The action of the group $G^n$ on the topological space $G$ is written as the multiplication

$$(\bullet) : G \times G^n \to G.$$
Recall that a homomorphism of two groups of transformations
\[ (G, G^n) = \langle G, G^n; \cdot, g, E \rangle \quad \text{and} \quad (G', G'^n) = \langle G', G'^n; \cdot', g', E' \rangle \]
is a pair of mappings
\[ \mu : G \to G' \quad \text{and} \quad \lambda : G^n \to G'^n \]
such that the diagrams
\[
\begin{array}{ccc}
G^n & \xrightarrow{E} & G^n \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
G'^n & \xrightarrow{E'} & G'^n
\end{array}
\]
\[
\begin{array}{ccc}
G^n \times G^n & \xrightarrow{g} & G^n \\
\downarrow{\lambda \times \lambda} & & \downarrow{\lambda} \\
G'^n \times G'^n & \xrightarrow{g'} & G'^n
\end{array}
\]
\[
\begin{array}{ccc}
G \times G^n & \xrightarrow{(\cdot)} & G \\
\downarrow{\mu \times \lambda} & & \downarrow{\mu} \\
G' \times G'^n & \xrightarrow{(\cdot')} & G'
\end{array}
\]
commute.

Regard a local n-pseudofield \( \langle G, \varphi_2, \ldots, \varphi_n \rangle \) as the algebra \( \langle G; \cdot, -^1, \varphi_2, \ldots, \varphi_n \rangle \).

Let \( \mathcal{K}_2 = K\langle G, G^n; \cdot, g, E \rangle \) and \( \mathcal{K}_1 = K\langle G; \cdot, -^1, \varphi_2, \ldots, \varphi_n \rangle \) be the classes of the algebras of local sharply n-transitive groups and local n-pseudofields.

Consider the categories \( \overline{\mathcal{K}}_2, \overline{\mathcal{K}}_1 \), whose objects are the corresponding algebras and whose morphisms are homomorphisms of algebras that preserve the numbers \( n \) (these numbers are the degree of the pseudofield and the sharp transitivity degree of the local group of transformations).

**Theorem 4** The category \( \overline{\mathcal{K}}_2 \) of local sharply n-transitive groups and and the category \( \overline{\mathcal{K}}_1 \) of local n-pseudofields are equivalent.

In Theorems 3 and 2, we constructed two mappings \( F_1 \) and \( F_2 \), and hence, for the corresponding functors \( \overline{F}_1 \) and \( \overline{F}_2 \), we constructed the mappings of the objects of the categories. It remains to define the mappings of morphisms of these categories.

For an arbitrary morphism \( h \in \text{mor}(\overline{\mathcal{K}}_1) \), from the corresponding algebras,
\[
\text{dom} \ h = \langle G; \cdot, -^1, \varphi_2, \ldots, \varphi_n \rangle \quad \text{and} \quad \text{cod} \ h = \langle G^h; h, -^{1h}, \varphi_2^h, \ldots, \varphi_n^h \rangle,
\]
using \( F_2 \), construct their images
\[
\langle G, G^n; \cdot, g, E \rangle \quad \text{and} \quad \langle G^h, (G^h)^n; \cdot', g', E' \rangle.
\]

With account taken of the construction of the group operation \( g \), using the tuple function, we conclude that, in the category \( \overline{\mathcal{K}}\langle G, G^n; \cdot, g, E \rangle \) of transformation groups, the morphism \( \overline{F}_2(h) \) is defined by the pair of morphisms
\[
h : G \to G^h \quad \text{and} \quad h \times \ldots \times h : G^n \to (G^h)^n
\]
so that \( \overline{F}_2(h) = (h, h \times \ldots \times h) \). Under this mapping, the identity morphism is mapped to the identity morphism
\[
\overline{F}_2 : \text{id}_{\mathcal{A}_1} \mapsto \text{id}_{\mathcal{A}_2}
\]
and for arbitrary \( f, h \in \text{mor}(\overline{\mathcal{K}}_1) \), for which the composition \( f \circ h \) is defined, the composition
\[
\overline{F}_2(f \circ h) = \overline{F}_2(f) \circ \overline{F}_2(h)
\]
is also defined.

In the first part of Theorem 3, choosing an arbitrary collection \( [e_1, \ldots, e_n] = e \in G^n \), we passed to the isomorphic group

\[
(id, f_e) : \langle G, T_n; \cdot, ^{-1} \rangle \mapsto \langle G, G^n; \bullet, g, E \rangle,
\]

For arbitrary homomorphic groups of transformations such that

\[
(\mu, \lambda) : \langle G, T_n; \cdot, ^{-1} \rangle \mapsto \langle G', T'_n; \cdot', ^{-1'} \rangle
\]

fixing collections \( e \in G^n, e' \in G'^n \), construct the isomorphic groups

\[
(id, f_e) : \langle G, T_n; \cdot, ^{-1} \rangle \mapsto \langle G, G^n; \bullet, g, E \rangle,
\]

\[
(id, f_{e'}) : \langle G', T'_n; \cdot', ^{-1'} \rangle \mapsto \langle G', G'^n; \bullet', g', E' \rangle.
\]

Then the mapping \((\mu', \lambda') = (id, f_{e'})(\mu, \lambda)(id, f_e)^{-1}\) is a homomorphism of the groups \( \langle G, G^n; \bullet, g, E \rangle \rightarrow \langle G', G'^n; \bullet', g', E' \rangle \), and the diagram

\[
\begin{array}{c}
\langle G, G^n; \bullet, g, E \rangle \\
\downarrow (id, f_e)(\mu, \lambda)(id, f_e)^{-1} \\
\langle G', G'^n; \bullet', g', E' \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle G, G^n; \bullet, ^{-1} \rangle \\
\downarrow (id, f_e)(\mu, \lambda) \\
\langle G', G'^n; \bullet', ^{-1'} \rangle
\end{array}
\]

commutes.

In Theorem 3, from the group \( \langle G, G^n; \bullet, g, E \rangle \), we constructed an \( n \)-pseudofield \( \langle G; \cdot, ^{-1}, \phi_2, \ldots, \phi_n \rangle \). Denoting this mapping by \( f_1 \), represent \( F_1 = f_1 \circ (id, f_e) \) so that the following diagram holds:

\[
\begin{array}{c}
\langle G, T_n; \cdot, ^{-1} \rangle \\
\downarrow F_1 \\
\langle G; \cdot, ^{-1}, \phi_2, \ldots, \phi_n \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle G, G^n; \bullet, g, E \rangle \\
\downarrow f_1 \\
\langle G'; \cdot', ^{-1}, \phi_2, \ldots, \phi_n \rangle
\end{array}
\]

The mapping \( F_2 \) of Theorem 2 is inverse to \( f_1 \). We have the commutative diagram

\[
\begin{array}{c}
\langle G, G^n; \bullet, g, E \rangle \\
\downarrow (\mu', \lambda') \downarrow f_1 \circ (\mu', \lambda') \circ f_1^{-1} \\
\langle G', G'^n; \bullet', g', E' \rangle
\end{array}
\]

\[
\begin{array}{c}
\langle G; \cdot, ^{-1}, \phi_2, \ldots, \phi_n \rangle \\
\downarrow f_1 \circ (\mu', \lambda') \circ f_1^{-1} \\
\langle G'; \cdot', ^{-1'}, \phi_2, \ldots, \phi_n' \rangle
\end{array}
\]

where the morphism \( f_1 \circ (\mu', \lambda') \circ f_1^{-1} \) defines the morphism of the corresponding algebras. Thus, we have constructed the mapping \( F_1 : (\mu', \lambda') \rightarrow f_1 \circ (\mu', \lambda') \circ f_1^{-1} \). This mapping takes the identity morphism in \( \mathbb{K}_2 \) to the identity morphism in \( \mathbb{K}_1 \). If

\[
(f_1, f_2), (h_1, h_2) \in \text{mor}(\mathbb{K}_2)
\]

21
and the composition \((f_1, f_2) \circ (h_1, h_2)\) is defined then

\[
\overline{F_1}((f_1, f_2) \circ (h_1, h_2)) = \overline{F_1}(f_1, f_2) \circ \overline{F_1}(h_1, h_2)
\]

is also defined. Considering the compositions of the mappings \(F_1\) and \(F_2\) of Theorems 3 and 2:

\[
F_1 \circ F_2 \left( (G; \cdot,^{-1}, \varphi_2, \ldots, \varphi_n) \right) = F_1 \left( (G, G^n; g, E) \right) = (G; \cdot,^{-1}, \phi_2, \ldots, \phi_n)
\]

and

\[
F_2 \circ F_1 \left( (G, T_n; \bullet, -,^{-1}) \right) = F_2 \left( (G; \cdot,^{-1}, \varphi_2, \ldots, \varphi_n) \right) = (G, G^n; g, E),
\]

we come to a natural isomorphism \(\overline{F_1} \circ \overline{F_2} \cong I\) and \(\overline{F_2} \circ \overline{F_1} \cong I\).

The theorem is proved. \(\square\)

4 Conclusion

The paper shows that local sharply \(n\)-transitive groups can be constructed over simpler objects — local \(n\)-pseudofields, which are proved to be categorically equivalent.

In conclusion, we want to formulate some problems:

1. Let \(G\) be a Lie group. Classify functions \(\varphi : G \to G\) (possibly, defined only on an open subset) such that
   
   (a) \(\varphi(\varphi(x)\varphi(y)) = \varphi(x\varphi(E(y)))y\),
   
   (b) \(\varphi(\varphi(x)) = x\),
   
   (c) \(\varphi E\varphi(x) = E\varphi E(x)\).

2. At present, the authors are familiar with a classification\(^3\) of local sharply \(n\)-transitive groups of transformations of the set \(\mathbb{R}^2\) [19, 20]. There arises the problem of: as a minimum, to find possible 2-pseudofields over 3D groups, and as a maximum, to construct a classification of local \(n\)-pseudofields for subsequently constructing the corresponding local \(n\)-transitive groups of transformations of \(\mathbb{R}^3\).

3. A more general task is to find the constraints imposed on the Lie algebras for local sharply \(n\)-transitive groups of transformations when they are associated with the corresponding local \(n\)-pseudofields.

References

[1] C. Jordan, “Recherches sur les substitutions,” J. Math. Pures Appl. (2), 17 (1872), 351–367.

[2] R. D. Carmichael, “Algebras of certain doubly transitive groups”, Amer. J. Math. 53:3 (1931), 631–644.

\(^3\)In the case of the set \(\mathbb{R}\), it coincides with the global classification.
[3] H. Zassenhaus, “Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen”, Abh. Math. Sem. Univ. Hamburg 11:1 (1935/1936), 17–40.

[4] H. Zassenhaus, “Über endliche Fastkörper”, Abh. Math. Sem. Univ. Hamburg 11:1 (1935/1936), 187–220.

[5] J. Tits, “Sur les groupes doublement transitiif continus,” Comment. Math. Helv., 26 (1952), 203–224; “Sur les groupes doublement transitiif continus: correction et complements”, Comment. Math. Helv., 30 (1956), 234–240.

[6] A. Barlotti, K. Strambach, “k-Transitive permutation groups and k-planes,” Math. Z., 185:4 (1984), 465–485.

[7] H. Karzel, Inzidenzgruppen I. Lecture Notes by Pieper, I. and Sorensen, K., University of Hamburg (1965), 123–135.

[8] H. Karzel, “Zusammenhänge zwischen Fastbereichen, scharf zweifach transitiven Permutationsgruppen und 2-Strukturen mit Rechtecksaxiom,” Abh. Math. Sem. Univ. Hamburg, 32:3-4 (1968), 191–206.

[9] W. Kerby, H. Wefelscheid, “Über eine scharf 3-fach transitiven Gruppen zugeordnete algebraische Struktur,” Abh. Math. Sem. Univ. Hamburg, 37:3-4 (1972), 225–235.

[10] P. M. Cohn, Free Rings and Their Relations, London–New York: Academic Press (1971).

[11] W. Leissner, Eine Charakterisierung der multiplikativen Gruppe eines Körpers, Jber. Deutsch. Math.–Verein. 73 (1971/72), 92–100

[12] W. Leissner, Ein Stufenaufbau der Fastereiche, Fastkörper und Körper aus ihrer multiplikativen Gruppe. Abh. Math. Sem. Univ. Hamburg 46 (1977), 55–89.

[13] W. Leissner, On sharply n-transitive groups. The Eighteenth International Symposium on Functional Equations, August 26–September 6, 1980, Waterloo and Scarborough, Ontario, Canada.

[14] A.A. Simonov, “On generalized sharply n-transitive groups,” Izv. Math., vol. 78 , no. 6 , 1207–1231 (2014).

[15] L. S. Pontryagin, Topological Groups, Moscow: Nauka (1973) [in Russian].

[16] S. Mac Lane, Categories for the working mathematician. 2nd ed. Graduate Texts in Mathematics. 5. New York, NY: Springer (1998).

[17] A. A. Simonov, “Correspondence between near-domains and groups,” Algebra Logika 45:2, 239–251 (2006); translation in: Algebra Logic 45:2, 139–146 (2006).
[18] V. G. Bardakov and A. A. Simonov, “Rings and groups of matrices with a non-standard product,” Sib. Mat. Zh. 54:3 (2013), 504–519; translation in: Sib. Math. J. 54, 393–405 (2013).

[19] G. G. Mikha’lichenko, “Dimetric physical structures and complex numbers,” Dokl. Akad. Nauk SSSR 321:4, 677–680 (1991); translation in: Sov. Math., Dokl. 44:3, 775–778 (1992).

[20] G. G. Mikha’lichenko, “Bimetric physical structures of rank $(n + 1, 2)$,” Sib. Mat. Zh. 34:3, 132–143 (1993); translation in: Sib. Math. J. 34:3, 513–522 (1993).
Information about authors

Mikhail V. Neshchadim
Sobolev Institute of mathematics SB RAS
4 Koptyug Ave.,
630090, Novosibirsk, Russia
E-mail: neshch@math.nsc.ru

Andrei A. Simonov
Novosibirsk State university,
2 Pirogova str.
630090, Novosibirsk, Russia
E-mail: a.simonov@g.nsu.ru