Volume form as volume of infinitesimal simplices

Anders Kock
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Abstract. In the context of Synthetic Differential Geometry, we describe the square volume of a “second-infinitesimal simplex”, in terms of square-distance between its vertices. The square-volume function thus described is symmetric in the vertices. The square-volume gives rise to a characterization of the volume form in the top dimension.

Introduction

In the context of Synthetic Differential Geometry (SDG), it is possible to express some of the “infinitesimal geometric” notions for a manifold $M$ directly in terms of the points of the manifold, rather than in terms of the tangent bundle $TM$; this was demonstrated for the case of connections, differential forms, and Riemannian metric, in a series of papers, [2], [3], [4], [5], [6], and the present note belongs to the same series: here, we want to pass directly from the metric to the measure of volume.

The crux is that in SDG, one has the notion of when points $x, y$ in a manifold are first, resp. second, . . . order neighbours, (written $x \sim_1 y$, resp. $x \sim_2 y$, . . .).

Recall [4] that an infinitesimal $k$-simplex is a $k + 1$ tuple $(x_0, \ldots, x_k)$ of mutual first order neighbours. In this note, we shall consider also $k + 1$ tuples $(x_0, \ldots, x_k)$ of mutual second order neighbours, so we shall need a more elaborate terminology: call the former first-infinitesimal $k$-simplices, the latter second-infinitesimal $k$-simplices.

Recall also [3] that a Riemannian metric synthetically may be expressed in terms of a function $g(x_0, x_1)$, applicable to any second-infinitesimal 1-simplex
to be thought of as the square length (square of the length) of the simplex. In particular, $g$ should be symmetric in its two arguments, and vanish on any first-infinitesimal 1-simplex. Actually, the second condition follows from the first (provided $g(x, x) = 0$), see Proposition \[1\] below.

We shall in the present note construct, for any second-infinitesimal $k$-simplex in a Riemannian manifold $(M, g)$, its square volume, and prove, Proposition \[2\] that the square-volume is a symmetric function in the $k + 1$ vertices; also, the square volume will vanish if two of the vertices are first order neighbours. For the case where $k = n$, the dimension $n$ of the manifold, it is possible in a certain sense to extract square roots and derive a characterization of the volume form (volume of first-infinitesimal $n$-simplices) of a Riemannian manifold; see Theorem \[3\].

1 Square volume: Heron’s and Gram’s constructions

The set of first-infinitesimal 1-simplices in a manifold $M$ equals what in other contexts is called the first neighbourhood of the diagonal, $M_{(1)}$, and similarly the set of second-infinitesimal simplices is the second neighbourhood of the diagonal, $M_{(2)}$. As observed in \[2\] Proposition 4.1, a function $f(x, y)$, vanishing on the diagonal, $f(x, x) = 0$, is alternating on the first neighbourhood, $f(x, y) = -f(y, x)$. For reference, we state this fact again, together with a slight extension of it:

**Proposition 1** If a function in two variables vanishes on the diagonal, then it is alternating ($f(x, y) = -f(y, x)$) on $M_{(1)}$. If a function is symmetric ($f(x, y) = f(y, x)$) on $M_{(2)}$, and vanishes on the diagonal, then it vanishes on $M_{(1)}$.

For, on $M_{(1)}$, $f$ is alternating as well as symmetric.

For a Riemannian manifold, i.e. a manifold $M$ equipped with a Riemannian metric $g$, we shall construct “square $k$-volumes” for any second-infinitesimal $k$-simplex (where $k \leq n$, $n$ the dimension of the manifold $M$ under consideration).

The basic idea for the construction of a square $k$-volume function goes, for the case $k = 2$, back to Heron of Alexandria (perhaps even to Archimedes);
they knew how to express the square of the area of a triangle $S$ in terms of an expression involving only the lengths $a, b, c$ of the three sides:

$$\text{area}^2(S) = t \cdot (t - a) \cdot (t - b) \cdot (t - c)$$

where $t = \frac{1}{2}(a + b + c)$. Substituting for $t$, and multiplying out, one discovers (cf. [1] §18.4) that all terms involving an odd number of any of the variables $a, b, c$ cancel, and we are left with

$$\text{area}^2(S) = \frac{1}{16}(-a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2), \quad (1)$$

an expression that only involves the squares $a^2, b^2$ and $c^2$ of the lengths of the sides.

For the case where the triangle is a second-infinitesimal 2-simplex, the three first terms in the above parenthesis vanish, and we are left with

$$\text{area}^2(S) = \frac{1}{8}(a^2b^2 + a^2c^2 + b^2c^2),$$

or expressed in terms of $g$ and the vertices $x_0, x_1, x_2$ of the simplex

$$\text{area}^2(x_0, x_1, x_2) = \frac{1}{8}(g(x_0, x_1)g(x_0, x_2) + g(x_1, x_0)g(x_1, x_2) + g(x_2, x_0)g(x_2, x_1)). \quad (2)$$

Since $g$ is symmetric in its two arguments, the expression here is symmetric in the three arguments. Also, if two vertices, say $x_0$ and $x_1$ are equal, the two first terms in (2) vanish, but the third one does too, being $g(x_2, x_0)^2$. (Note that the values of $g$ are always numbers with square zero.) From Proposition [1] then follows that (2) vanishes if $x_0$ and $x_1$ are 1-neighbours; similarly, by symmetry, for $x_0, x_2$ or $x_1, x_2$.

Since the formula (2) only involves $g$, we do have a “square-area” construction for second-infinitesimal 2-simplices, satisfying the conditions mentioned in the introduction: symmetry, and vanishing if two vertices are 1-neighbours.

The expression in (2) has a rather evident generalization to higher $k$, but we shall prefer to generalize a different (less symmetric) expression for square-area. This formulation involves the “Gram determinant”. It is well known from linear algebra books that the square of the volume of the box spanned by $k$ vectors $x_1, \ldots, x_k$ in $\mathbb{R}^n$ is the Gram-determinant of the $n \times k$ matrix $X$ with the $x_i$’s as columns, where the Gram determinant of such
matrix $X$ is the determinant of the $k \times k$ matrix $X^T \cdot X$; or, equivalently, the matrix whose $ij$’th entry is $x_i \cdot x_j$, where $\cdot$ denotes the standard inner product in $\mathbb{R}^n$. Recall also that this inner product may be written in terms of square-norm, by the polarization formula

$$x \cdot y = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2).$$

Thus the volume of the $k$-box spanned by the vectors $x_1, \ldots, x_k$ can be expressed in terms of square-norm. The volume of the $k$-simplex spanned by some vectors (and 0) differs from the volume of the box spanned by the vectors by a factor $1/k!$. This motivates the following construction of a square-$k$-volume of a second-infinitesimal $k$-simplex in a Riemannian manifold $M, g$:

$$\text{vol}^2(x_0, \ldots, x_k) = \frac{1}{(k!)^2} \det((x_i - x_0) \cdot (x_j - x_0)), \quad (3)$$

where in turn

$$(x_i - x_0) \cdot (x_j - x_0) = \frac{1}{2}(-g(x_i, x_j) + g(x_i, x_0) + g(x_j, x_0)), \quad (4)$$

Note that the minus signs used on the left hand side are purely symbolical, and also a question whether $\cdot$ is bilinear makes no sense.

We shall want to make sense to the expression $(3)$ without assuming $x_i \sim_2 x_j$, but assuming, as before, $x_i \sim_2 x_0$ and $x_j \sim_2 x_0$, whence at least $x_i \sim_4 x_j$. This depends on choosing an extension $\overline{g}$ of $g$ to the fourth neighbourhood of the diagonal $M(4)$, so that the first term on the right hand side of $(3)$ makes sense. The value of the $\cdot$ expression will in general depend on the choice of $\overline{g}$, but the dependence will be “controlled”, in a sense that is best made precise by working in a coordinate neighbourhood around $x_0$:

The Riemannian metric $g$ on a neighbourhood of 0 in $\mathbb{R}^n$ takes the form of a matrix product

$$g(x, y) = (y - x)^T \cdot G(x) \cdot (y - x) \quad (5)$$

where $G(x)$ for each $x$ is a symmetric $n \times n$ matrix with positive determinant.

Let $x \sim_2 0$ and $y \sim_2 0$. The value of $x - 0 \cdot y - 0$, as given by $(4)$ with $x_0 = 0, x_i = x,$ and $x_j = y$ will depend on the extension $\overline{g}$ chosen; however, since a different choice will mean a difference for $\overline{g}(x, y)$ which is of order $\geq 3$ in $y - x$, and since $x$ and $y$ are assumed to be $\sim_2 0$, this difference will be of
total order \( \geq 3 \) in \( x, y \), and so may be subsumed in the “error term” \( \epsilon \) in the conclusion in the following Lemma. So in the proof of the Lemma, we may as well assume that the expression for \( \mathcal{f} \) is the expression (3) for all \( x, y \).

**Lemma 1** Assume \( x \sim_2 0 \) and \( y \sim_2 0 \) (hence \( x \sim_4 y \)). Then

\[
x \cdot y = x \cdot G(0) \cdot y + \epsilon(x, y),
\]

where \( \epsilon(x, y) \) is of total degree \( \geq 3 \) in \( x \) and \( y \). In particular, \( x \cdot x = x^T \cdot G(0) \cdot x \).

**Proof.** Let us calculate \( 2 x \cdot y \), i.e. \(-\mathcal{f}(x, y) + g(0, x) + g(0, y) \). We get

\[-(y - x)^T \cdot G(x) \cdot (y - x) + x^T \cdot G(0) \cdot x + y^T \cdot G(0) \cdot y;
\]

we Taylor expand \( G(x) \) and thus rewrite the first term as \(-(y - x)^T \cdot G(0) \cdot (y - x) \) plus a term \( \epsilon \) which is of degree \( \geq 1 \) in \( x \), and well as quadratic in \( y - x \), and hence of total degree \( \geq 3 \) in \( x, y \). The three terms involving \( G(0) \) now are, by the standard polarization formula, 2 times \( x^T \cdot G(0) \cdot y \). This proves the first assertion.

The assertion about \( x \cdot x \) is then clear.

We consider now the determinant on the right hand side of (3) in the coordinatized situation, i.e. with the metric given in coordinates by (5).

**Proposition 2** Consider a second-infinitesimal \( k \)-simplex \( x_0, \ldots, x_k \). Then the determinant \( \text{det}((x_i - x_0) \cdot (x_j - x_0)) \) equals the determinant of the \( k \times k \) matrix \( X^T \cdot G(x_0) \cdot X \), where \( X \) denotes the \( n \times k \) matrix with the \( x_i - x_0 \) as columns.

**Proof.** The \( ij \)’th entry \( (x_i - x_0) \cdot (x_j - x_0) \) may by Lemma 1 be written

\[(x_i - x_0)^T \cdot G(x_0) \cdot (x_j - x_0) + \epsilon_{i,j},\]

where \( \epsilon_{i,j} \) is of total degree \( \geq 3 \) in \( (x_i - x_0), (x_j - x_0) \). We claim that all the “error terms” \( \epsilon_{i,j} \) get killed when expanding the determinant as a sum of products of the entries. Note that the error terms only occur in off-diagonal entries, \( i \neq j \). We may write \( \epsilon_{i,j} \) as \( \epsilon'_{i,j} + \epsilon''_{i,j} \), where \( \epsilon'_{i,j} \) is of degree \( \geq 2 \) in \( x_i - x_0 \) and \( \epsilon''_{i,j} \) is of degree \( \geq 2 \) in \( x_j - x_0 \) (bidegrees \((3, 0)\) or \((0, 3)\) do not occur, since \( x_i \sim_2 x_0 \) for all \( i \)). Now \( \epsilon'_{i,j} \), living in the \( j \)’th column, gets, in
each of the terms of the determinant expansion, multiplied by something from
the \(i\)’th column. But each entry in the \(i\)’th column, say

\[(x_i - x_0) \cdot G(x_0) \cdot (x_i - x_0) + \epsilon_{i,i}\]

is of degree \(\geq 1\) in \(x_i - x_0\) and therefore kills \(\epsilon'_{i,j}\). Similarly, \(\epsilon^*_{i,j}\) gets killed by
being multiplied by anything in the \(j\)’th row.

So the only factors left in the determinant expansion are the \((x_i - x_0)^T \cdot G(x_0) \cdot (x_j - x_0)\), i.e. those of the matrix \(X^T \cdot G(x_0) \cdot X\), as claimed.

Whereas the “Heron” formulas (1) and (2) for square area are evidently
symmetric in the three vertices \(x_0, x_1, x_2\), the Gram determinant formula (3)
gives a preferential status to the vertex \(x_0\). Nevertheless,

**Proposition 3** The expression (3) for square volume of a second-infinitesimal \(k\)-simplex is symmetric in the \(k + 1\) arguments \(x_0, \ldots, x_k\). Also, the expression vanishes if two of the vertices \(x_i\) and \(x_j\) are 1-neighbours.

**Proof.** Symmetry in the arguments \(x_1, \ldots, x_k\) is clear. So it suffices to prove that

\[\text{vol}^2(x_0, x_1, \ldots, x_k) = \text{vol}^2(x_1, x_0, \ldots, x_k).\]  \hfill (6)

By Proposition 2, it suffices to prove that

\[\det(X^T \cdot G(x_0) \cdot X) = \det(Y^T \cdot G(x_1) \cdot Y)\]  \hfill (7)

where \(X\) (as above) is the \(n \times k\) matrix with columns \(x_i - x_0\) \((i = 1, 2, \ldots, k)\) and \(Y\) similarly is the \(n \times k\) matrix with columns \(x_i - x_1\) \((i = 0, 2, \ldots, k)\).

Now each of the \(k!\) terms in the determinant \(\det(X^T \cdot G(x_0) \cdot X)\) is of degree \(\geq 2\) in \(x_1 - x_0\). For, the entries of the first row of the matrix \(X^T \cdot G(x_0) \cdot X\) are linear in \((x_1 - x_0)\), and likewise: each entry in the first column is linear in \((x_1 - x_0)\); and finally the entry in position \((1, 1)\) is quadratic in \((x_1 - x_0)\).

Any of the \(k!\) terms of the determinant expansion contains as well a factor from the first row, as one from the first column, and thus is of degree \(\geq 2\) in \(x_1 - x_0\).

Now since each term in the determinant expansion of \(X^T \cdot G(x_0) \cdot X\) is
of degree \(\geq 2\) in \(x_1 - x_0\), we may replace the \(G(x_0)\) by \(G(x_1)\); for Taylor expanding \(G(x_1)\) from \(x_0\) gives \(G(x_0)\) plus terms of degree \(\geq 1\) in \(x_1 - x_0\), but these terms get killed because our expressions are already of degree \(\geq 2\) in \(x_1 - x_0\).
Thus it suffices to prove that the $k \times k$ matrices $X^T \cdot G \cdot X$ and $Y^T \cdot G \cdot Y$ have the same determinant, where $G$ denotes $G(x_1)$. This is a matter of elementary linear algebra. It is easy to see that one can obtain the matrix $Y$ from the matrix $X$ by elementary column operations: changing sign on a column, and adding a multiple of one column to another one. This can be expressed matrix theoretically by saying $Y = X \cdot Z$ where $Z$ is a $k \times k$ matrix of determinant $\pm 1$. Hence we have

$$
\det(Y^T \cdot G \cdot Y) = \det((X \cdot Z)^T \cdot G \cdot (X \cdot Z))
$$

$$
= \det(Z^T \cdot (X^T \cdot G \cdot X) \cdot Z) = \det(X^T \cdot G \cdot X),
$$

since $\det(Z) = \det(Z^T) = \pm 1$, proving (1), and hence the symmetry assertion of the Proposition.

To see that the expression (3) vanishes if two vertices are 1-neighbours, it suffices, by the symmetry already proved, and by Proposition 1, to prove that we get value zero if two vertices are equal. This is clear from Proposition 2, since a determinant of form $X^T \cdot G \cdot X$ is zero if two columns of $X$ are equal.

Remark. The above exposition does not, even in dimension 2, prove or argue that the Heron expression and the Gram expression agree; I suppose this is rather elaborate, or requires some more penetrating concepts.

## 2 Volume form

We are now going to compare square volumes with differential forms in the top dimension. Differential forms are defined on first-infinitesimal simplices, and any square-volume function applied to such simplex yields value 0. So no immediate comparison can be made. The comparison therefore proceeds via a somewhat ad hoc notion of extended $k$-simplex and extended $k$-form $\Omega$.

**Definition 1** An extended $k$-simplex at $x_0$ is a $k + 1$-tuple $(x_0, \ldots, x_k)$ of points with $x_i \sim_2 x_0$ for all $i = 1, \ldots, k$.

In particular, any second-infinitesimal simplex qualifies as an extended simplex.

Note that we do not assume $x_i \sim_2 x_j$ unless $i$ or $j$ is 0, and so the notion of extended simplex is asymmetric. (There are not $(k + 1)!$ symmetries of an extended $k$-simplex, but only $k!$.)
It is possible to define “square volume” not just for second-infinite simplices, but also for extended \(k\)-simplices. To define \(\text{vol}^2(X)\) of an extended \(k\)-simplex \(X\), we first choose an extension of \(g\) to \(\mathcal{G}\), as above, so (3) and hence (2) make sense, and serve as the definition of \(\text{vol}^2(X)\); we have to argue that the value does not depend on the choice of \(\mathcal{G}\). This follows because a different choice will, when working in coordinates, give a difference in the \(\epsilon\) part of the \((x_i - x_0) \cdot (x_j - x_0)\) only. The same argument as used in the proof of Proposition 2 then shows that all these \(\epsilon\) differences vanish when expanding the determinant.

**Definition 2** An extended \(k\)-form \(\Omega\) is a law which to any extended \(k\)-simplex assigns a number; \(\Omega\) should be alternating (change sign when swapping two entries \(x_i\) and \(x_j\) with \(i\) and \(j\) different, and \(\geq 1\)) and give value 0 if \(x_i = x_j\) for some \(i \neq j\).

It makes sense to say that the extended \(k\)-form \(\Omega\) is an extension of the \(k\)-form \(\omega\). — The notion of extended form will only be used for the case \(k = n\). — We have, for general \(k\),

**Lemma 2** Assume that the extended \(k\)-form \(\Omega\) extends the non-degenerate \(k\)-form \(\omega\), and similarly \(\Omega'\) extends \(\omega'\). If

\[
\Omega(X)^2 = \Omega'(X)^2
\]

for every extended \(k\)-simplex \(X\), then locally \(\omega = \omega'\) or \(\omega = -\omega'\).

**Proof.** We intend to prove \(\omega(x_0, \ldots, x_k) = \omega'(x_0, \ldots, x_k)\) for any \(k\)-simplex \((x_0, \ldots, x_k)\). We introduce coordinates, and assume \(x_0 = 0 \in \mathbb{R}^n\). Then \(\Omega\) as a function of the remaining \(k\) variables \(x_1, \ldots, x_k\) may be seen as a function \(D_2(n) \times \ldots \times D_2(n) \to \mathbb{R}^k\) (\(k\) copies of \(D_2(n)\)). The ring \(B\) of such functions is a tensor product of \(k\) copies of the ring \(A\) of functions on \(D_2(n)\); this latter ring is graded,

\[A = A_0 \oplus A_1 \oplus A_2.\]

So \(B\) is multi-graded; for simplicity of notation, let us do the case where \(k = 2\), so \(B\) is bigraded,

\[B = \bigoplus_{i,j} A_i \otimes A_j.\]
The assumption that $\Omega$ vanishes if one of the $x_i$’s is 0 now implies that $\Omega$ belongs to the summand $\bigoplus_{i>0,j>0} A_i \otimes A_j$, and so may be written

$$\Omega = \omega + H$$

where $\omega \in A_1 \otimes A_1$ and $H \in A_1 \otimes A_2 \oplus A_2 \otimes A_1 \oplus A_2 \otimes A_2$. Then for (bi-)degree reasons ($A_3 = 0$), it follows that $\Omega^2 = \omega^2$ in the bigraded ring $B$. The bilinear function which $\omega$ defines may be identified with the given form $\omega$, and since this form is non-degenerate, we may from $\Omega^2 = \omega^2$ conclude that $\omega = \pm \sqrt{\Omega^2}$, so $\omega$ can, modulo sign, be reconstructed from $\Omega^2$. This gives the desired uniqueness assertion.

A similar argument gives that if $\Omega$ and $\Omega'$ are two extended $k$-forms extending $\omega$, then $\Omega(X)^2 = \Omega'(X)^2$, for any extended $k$-simplex $X$.

Let $Vol^2$ (with capital $V$) denote the square-volume construction in the top dimension $n$, given by the formula (3); as noted above, $vol^2(X)$ makes invariantly sense even for an extended $k$-simplex $X$.

We shall prove

**Theorem 1** Let $M, g$ be a Riemannian manifold of dimension $n$. Then there exists locally, and uniquely up to sign, an $n$-form $\omega$, such that for some, or equivalently for any, extended $n$-form $\Omega$ extending $\omega$,

$$\Omega(x_0, \ldots, x_n)^2 = Vol^2(x_0, \ldots, x_n).$$

The $n$-form (unique up to sign) thus characterized, deserves the name **volume form** for the Riemannian manifold $M, g$.

**Proof.** From the Lemma follows the local uniqueness of volume forms, modulo sign. (So, if $M$ is connected, there are at most two volume forms $\omega$ and $-\omega$.) The existence of a volume form will proceed in coordinates, and will ultimately not be very different from the standard description in terms of square root of the determinant of the symmetric matrix representing $g$ in coordinates (as in the proof of Proposition 3). The $n$ form $\omega$ we construct as a candidate for a volume form is then defined by

$$\omega(x_0, \ldots, x_n) = \frac{1}{n!} \sqrt{\det G(x_0)} \det(x_1 - x_0, \ldots, x_n - x_0); \quad (8)$$
the same expression then defines a value for any \( x_0, \ldots, x_n \), so in particular defines an extended \( n \)-form \( \Omega \). Then by linear algebra,

\[
\Omega(x_1 - x_0, \ldots, x_n - x_0)^2 = \frac{1}{(n!)^2} \det(X^T \cdot G(x_0) \cdot X),
\]

where \( X \) denotes the \( n \times n \) matrix with columns the \( x_i - x_0 \).

We now observe that

\[
\det(X^T \cdot G(x_0) \cdot X) = \det((x_i - x_0) \bullet (x_j - x_0)),
\]

which will then prove the existence of the volume form. This is essentially immediate from Proposition 4, because the proof about the cancellation of error terms depended on \( x_i \sim_2 x_0 \), but not on \( x_i \sim_2 x_j \).

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