MODULI SPACES OF PARABOLIC $U(p,q)$-HIGGS BUNDLES

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Abstract. Using the $L^2$-norm of the Higgs field as a Morse function, we study the moduli space of parabolic $U(p,q)$-Higgs bundles over a Riemann surface with a finite number of marked points, under certain genericity conditions on the parabolic structure. When the parabolic degree is zero this space is homeomorphic to the moduli space of representations of the fundamental group of the punctured surface in $U(p,q)$, with fixed compact holonomy classes around the marked points. By means of this homeomorphism we count the number of connected components of this moduli space of representations. Finally, we apply our results to the study of representations of the fundamental group of elliptic surfaces of general type.

1. Introduction

A parabolic vector bundle over a compact Riemann surface with marked points consists of a vector bundle, equipped with a weighted flag structure on the fibre over each marked point. These objects were introduced by Seshadri [Se] in relation to certain desingularisations of the moduli space of semistable vector bundles. It turns out that, similarly to the Narasimhan and Seshadri correspondence [NS, D] between stable vector bundles and representations of the fundamental group of the surface in the unitary group $U(n)$, there is an analogous correspondence, proved by Metha and Seshadri [MS] (see also [Bi]), relating stable parabolic bundles to unitary representations of the fundamental group of the punctured surface with a fixed holonomy class around each marked point.

In order to study representations of the fundamental group of the punctured surface in $GL(n, \mathbb{C})$ one has to consider parabolic Higgs bundles. These are pairs consisting of a parabolic vector bundle and a meromorphic endomorphism valued one-form with a simple pole along each marked point, whose residue is nilpotent with respect to the flag. Moduli spaces of parabolic Higgs bundles provide interesting examples of hyperkähler manifolds. This theory, studied by Simpson in [S2] and others [HoY, K, Na, NS1], generalizes the non-parabolic Higgs bundle theory studied by Hitchin [H], Donaldson [D2], Simpson [S1] and Corlette [C].

In this paper we study parabolic $U(p,q)$-Higgs bundles. These are the objects that correspond to representations of the fundamental group of the punctured surface in $U(p,q)$, with fixed compact holonomy classes around the marked points. Our approach combines the techniques used in [BCG] in the study of $U(p,q)$-Higgs bundles in the non-parabolic case as well as those used in [GGM] to study the topology of moduli spaces of $GL(n, \mathbb{C})$-parabolic Higgs bundles.

Date: 15 March 2006. Revised 21 January 2008.
2000 Mathematics Subject Classification. 14D20, 14H60.
Key words and phrases. Parabolic bundles, Higgs bundles, moduli spaces.
Partially supported by Ministerio de Educación y Ciencia (Spain) through Project MTM2004-07090-C03-01.
For a parabolic $U(p,q)$-Higgs bundle there is an invariant, similar to the Toledo invariant in the non-parabolic case. We show that this parabolic Toledo invariant has a bound provided by a generalization of the Milnor–Wood inequality. Our main result in the paper is to show that if the genus of the surface and the number of marked points are both at least one, then the moduli space of parabolic $U(p,q)$-Higgs bundles with fixed topological type, generic parabolic weights and full flags is non-empty and connected if and only if the parabolic Toledo invariant satisfies a generalized Milnor–Wood inequality (see Theorem 6.13).

As in [BGG] and [GGM], the main strategy is to use the Bott-Morse-theoretic techniques introduced by Hitchin [H]. The connectedness properties of our moduli space reduce to the connectedness of a certain moduli space of parabolic triples introduced in [BiG] in connection to the study of the parabolic vortex equations and instantons of infinite energy. Much of the paper is devoted to a thorough study of these moduli spaces of triples and their connectedness properties.

After spelling out the correspondence between parabolic $U(p,q)$-Higgs bundles and representations of the fundamental group of the punctured surface in $U(p,q)$, we transfer our results on connectedness of the moduli space of parabolic $U(p,q)$-Higgs bundles to the moduli space of representations (see Theorems 13.2 and 13.3). We then apply this to the study of representations of the fundamental group of certain complex elliptic surfaces of general type (see Theorem 14.4). These are complex surfaces whose fundamental group is isomorphic to the orbifold fundamental group of an orbifold Riemann surface.

We should point out that our main results do not apply when the genus of the Riemann surface is zero. This is not surprising if we have in mind that on $\mathbb{P}^1$ the parabolic weights must satisfy certain inequalities in order for parabolic bundles to exist ([Bis, Bel]). Presumably, something similar must be true also in the case of parabolic $U(p,q)$-Higgs bundles. We plan to come back to this problem in a future paper.

In the process of finishing our paper we have come across several papers ([BI, KM, Kr]) that seem to be related to our work in the case of $U(p,1)$. It would be interesting to investigate further the relationship between these different approaches.

Acknowledgments: We thank the referee for a very careful reading of the manuscript and for numerous suggestions.

2. PARABOLIC HIGGS BUNDLES

Let $X$ be a closed, connected, smooth Riemann surface of genus $g \geq 0$ together with a finite set of marked points $x_1, \ldots, x_s$. Denote by $D$ the effective divisor $D = x_1 + \cdots + x_s$ defined by the marked points. A parabolic vector bundle $E$ over $X$ consists of a holomorphic vector bundle together with a parabolic structure at each $x \in D$, that is, a weighted flag on the fibre $E_x$,

$$E_x = E_{x,1} \supset E_{x,2} \supset \cdots \supset E_{x,r(x)+1} = \{0\},$$

$$0 \leq \alpha_1(x) < \cdots < \alpha_{r(x)}(x) < 1.$$

We denote $k_i(x) = \dim(E_{x,i}/E_{x,i+1})$ the multiplicity of the weight $\alpha_i(x)$. It will sometimes be convenient to repeat each weight according to its multiplicity, i.e., we set $\tilde{\alpha}_1(x) = \ldots = \tilde{\alpha}_{r(x)}(x)$, $\tilde{k}_1(x) = \tilde{k}_2(x) = \cdots = \tilde{k}_{r(x)}(x)$.

$$E_x = E_{x,1} \supset E_{x,2} \supset \cdots \supset E_{x,r(x)+1} = \{0\},$$

$$0 \leq \tilde{\alpha}_1(x) < \cdots < \tilde{\alpha}_{r(x)}(x) < 1.$$
\[ \tilde{\alpha}_k(x)(x) = \alpha_1(x), \] etc. We then have weights \( 0 \leq \tilde{\alpha}_1(x) \leq \ldots \leq \tilde{\alpha}_n(x) < 1, \) where \( n = \text{rk} \ E. \) Denote also \( \alpha(x) = (\tilde{\alpha}_1(x), \ldots, \tilde{\alpha}_n(x)) \) the system of weights at \( x \in D \) and by \( \alpha = (\alpha(x))_{x \in D} \) the weight type of \( E. \) We say that the flags are full if \( k_i(x) = 1 \) for all \( i \) and \( x \in D. \) Note that in this case \( \alpha(x) = (\tilde{\alpha}_1(x), \ldots, \tilde{\alpha}_n(x)) = (\alpha_1(x), \ldots, \alpha_n(x)). \) A holomorphic map \( f : E \to E' \) between parabolic bundles is called parabolic if \( \alpha_i(x) > \alpha'_j(x) \) implies \( f(E_{x,i}) \subset E'_{x,j+1} \) for all \( x \in D, \) and \( f \) is strongly parabolic if \( \alpha_i(x) \geq \alpha'_j(x) \) implies \( f(E_{x,i}) \subset E'_{x,j+1} \) for all \( x \in D, \) where we denote by \( \alpha'_j(x) \) the weights on \( E'. \) We denote \( \text{ParHom}(E, E') \) and \( \text{SParHom}(E, E') \) the sheaves of parabolic and strongly parabolic morphisms from \( E \) to \( E', \) respectively. If \( E' = E \) we denote these sheaves by \( \text{ParEnd}(E) \) and \( \text{SParEnd}(E), \) respectively.

We define the parabolic degree and parabolic slope of \( E \) by

\[
(1) \quad \text{pardeg}(E) = \deg(E) + \sum_{x \in D} \sum_{i=1}^{r(x)} k_i(x)\alpha_i(x),
\]

\[
(2) \quad \text{par} \mu(E) = \frac{\text{pardeg}(E)}{\text{rk}(E)}.
\]

A parabolic bundle \( E \) is said to be (semi)-stable if for every non-trivial proper parabolic subbundle \( E' \) of \( E \) we have \( \text{par} \mu(E') < \text{par} \mu(E) \) (resp. \( \text{par} \mu(E') \leq \text{par} \mu(E) \)).

In the following we will use the following construction for parabolic bundles, called parabolic direct sum. Let \( V \) and \( W \) two parabolic bundles with weight types \( \alpha \) and \( \alpha' \) we say that \( E \) is the parabolic direct sum of \( V \) and \( W \) if and only if \( E = V \oplus W \) as holomorphic bundles, the system of weights, \( \tilde{\alpha}, \) on \( E \) consists of the ordered collection of the weights in \( \alpha \) and \( \alpha', \) and the corresponding filtration is such that

\[ E_{x,k} = V_{x,i} \oplus W_{x,j}, \]

where \( i \) (resp. \( j \)) is the smallest integer such that \( \tilde{\alpha}_k(x) \leq \alpha_i(x) \) (resp. \( \tilde{\alpha}_k(x) \leq \alpha'_j(x) \)).

A parabolic Higgs bundle is a pair \( (E, \Phi) \) consisting of a parabolic bundle \( E \) and \( \Phi \in H^0(\text{SParEnd}(E) \otimes K(D)), \) i.e. \( \Phi \) is a meromorphic endomorphism valued one-form with simple poles along \( D \) whose residue at \( x \in D \) is nilpotent with respect to the flag. A parabolic Higgs bundle is called (semi)-stable if for every \( \Phi \)-invariant subbundle \( E' \) of \( E, \) its parabolic slope satisfies \( \text{par} \mu(E') < \text{par} \mu(E) \) (resp. \( \text{par} \mu(E') \leq \text{par} \mu(E) \)), and it is said to be polystable if it is the direct sum of stable parabolic Higgs bundles of the same parabolic slope.

Fixing the topological invariants \( n = \text{rk} \ E \) and \( d = \deg E \) and the weight type \( \alpha, \) the moduli space \( \mathcal{M} = \mathcal{M}(n, d; \alpha) \) is defined as the set of isomorphism classes of polystable parabolic Higgs bundles of type \( (n, d; \alpha). \) Using Geometric Invariant Theory, Yokogawa \[ Y1 \ Y2 \] has showed that \( \mathcal{M} \) is a complex quasi-projective variety, which is smooth at the stable points.

A parabolic \( U(p, q) \)-Higgs bundle on \( X \) is a parabolic Higgs bundle \( (E, \Phi) \) such that \( E = V \oplus W, \) where \( V \) and \( W \) are parabolic vector bundles of ranks \( p \) and \( q \) respectively, and

\[ \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V \oplus W) \to (V \oplus W) \otimes K(D), \]

where \( \beta : W \to V \otimes K(D) \) and \( \gamma : V \to W \otimes K(D) \) are strongly parabolic morphisms. A parabolic \( U(p, q) \)-Higgs bundle \( (E = V \oplus W, \Phi) \) is (semi)-stable if the slope stability condition \( \text{par} \mu(E') < \text{par} \mu(E) \) (resp. \( \text{par} \mu(E') \leq \text{par} \mu(E) \)) is satisfied for all \( \Phi \)-invariant parabolic
subbundles of the form \( E' = V' \oplus W' \), i.e. for all parabolic subbundles \( V' \subset V \) and \( W' \subset W \) such that \( \beta(W') \subseteq V' \otimes K(D) \) and \( \gamma(V') \subseteq W' \otimes K(D) \). Note that, a priori, this definition of stability seems to be weaker than the stability definition for parabolic Higgs bundles (we ask for \( V' \subset V \) and \( W' \subset W \)). But this is not the case, since for any \( \Phi \)-invariant \( E' \subset E \), we apply the \( U(p,q) \)-stability condition to \( V' \oplus W' \) and to \( V'' \oplus W'' \), where \( V'' = V \cap E' \), \( W'' = W \cap E' \), \( V'' = \pi_V(E') \), \( W'' = \pi_W(E') \) (where \( \pi_V \), \( \pi_W \) are the projections of \( V \oplus W \) onto \( V \), \( W \), respectively). Then using the exact sequences \( V' \to E' \to W'' \) and \( W' \to E' \to V'' \), one gets easily that \( \text{par} \mu(E') \leq \text{par} \mu(E) \).

Fix the topological invariants \( a = \text{deg} V \) and \( b = \text{deg} W \) and the weight types \( \alpha \) and \( \alpha' \) for \( V \) and \( W \), respectively. This determines a system of weights \( \tilde{\alpha} \) and a flag structure, given by the parabolic direct sum construction, on \( E = V \oplus W \). Let

\[ \mathcal{U} = \mathcal{U}(p, q, a, b; \alpha, \alpha') \]

be the moduli space of polystable parabolic \( U(p, q) \)-Higgs bundles of degrees \((a, b)\) and weights \((\alpha, \alpha')\).

We say that the weights are generic when every semistable parabolic Higgs bundle is automatically stable, that is, there are no properly semistable parabolic Higgs bundles. We will keep the following assumption on the weights all throughout the paper (although some of the results hold in more general situations):

**Assumption 2.1.** The weights of \((E, \Phi)\) are generic and \((E, \Phi)\) has full flags at each parabolic point. This means that all the weights of \( V \) and \( W \) are different and of multiplicity one.

Note that the set of weights such that, for fixed degree and rank of \( E \), make \((E, \Phi)\) strictly semistable has positive codimension. This justifies the term generic for the weights which do not allow strict semistability.

The construction of \( \mathcal{U} \) follows the same arguments given in the non-parabolic case (see [BGG]).

**Proposition 2.2.** Let \( n = p + q \), \( d = a + b \), and let \( \tilde{\alpha} \) be the system of weights defined by \( \alpha \) and \( \alpha' \) as above. Then \( \mathcal{U}(p, q, a, b; \alpha, \alpha') \) embeds as a closed subvariety in \( \mathcal{M}(n, d; \tilde{\alpha}) \).

**Proof.** The proof is similar to that in the non parabolic case (see Proposition 3.11 in [BGG]). One only notices that in the case \( p = q \), the parabolic bundles \( V \) and \( W \) can not be parabolically isomorphic since they have different weights. \( \square \)

**Remark 2.3.** Sometimes we refer to elements \((E, \Phi) \in \mathcal{M}\) as parabolic \( \text{GL}(n, \mathbb{C}) \)-Higgs bundles, since the structure group of the frame bundle of \( E \) is \( \text{GL}(n, \mathbb{C}) \).

### 3. Deformation theory

The results of Yokogawa [Y1] and [BGG] readily adapt to describe the deformation theory of parabolic \( U(p, q) \)-Higgs bundles.
Let \((E = V \oplus W, \Phi)\) be a parabolic \(U(p,q)\)-Higgs bundle. We introduce the following notation:

\[
U = \text{ParEnd}(E), \quad \hat{U} = \text{SParEnd}(E),
\]
\[
U^+ = \text{ParEnd}(V) \oplus \text{ParEnd}(W), \quad \hat{U}^+ = \text{SParEnd}(V) \oplus \text{SParEnd}(W),
\]
\[
U^- = \text{ParHom}(W, V) \oplus \text{ParHom}(V, W), \quad \hat{U}^- = \text{SParHom}(W, V) \oplus \text{SParHom}(V, W).
\]

With this notation, \(U = U^+ \oplus U^-\), \(\hat{U} = \hat{U}^+ \oplus \hat{U}^-\), \(\Phi \in H^0(\hat{U}^- \otimes K(D))\), and \(\text{ad}(\Phi)\) sends \(U^+\) to \(\hat{U}^−\) and \(U^-\) to \(\hat{U}^+\). We consider the complex of sheaves

\[
C^\bullet : U^+ \xrightarrow{\text{ad}(\Phi)} \hat{U}^- \otimes K(D).
\]

**Lemma 3.1.** Let \((E, \Phi)\) be a stable parabolic \(U(p,q)\)-Higgs bundle. Then

\[
\ker(\text{ad}(\Phi) : H^0(U^+) \to H^0(\hat{U}^- \otimes K(D))) = \mathbb{C},
\]
\[
\ker(\text{ad}(\Phi) : H^0(U^-) \to H^0(\hat{U}^+ \otimes K(D))) = 0.
\]

**Proof.** Since \((E, \Phi)\) is stable as a parabolic \(GL(n, \mathbb{C})\)-Higgs bundle, it is simple, that is, its only endomorphisms are the non-zero scalars. Thus,

\[
\ker(\text{ad}(\Phi) : H^0(U) \to H^0(\hat{U} \otimes K(D))) = \mathbb{C}.
\]

Since \(U = U^+ \oplus U^-\) and \(\text{ad}(\Phi)\) sends \(U^+\) to \(\hat{U}^-\) and \(U^-\) to \(\hat{U}^+\), the statements of the Lemma follow. □

**Proposition 3.2.**

(i) The space of endomorphisms of \((E, \Phi)\) is isomorphic to the zeroth hypercohomology group \(H^0(C^\bullet)\).

(ii) The space of infinitesimal deformations of \((E, \Phi)\) is isomorphic to the first hypercohomology group \(H^1(C^\bullet)\).

(iii) There is a long exact sequence

\[
0 \to H^0(C^\bullet) \to H^0(U^+) \to H^0(\hat{U}^- \otimes K(D)) \to H^1(C^\bullet) \to H^1(U^+) \to H^1(\hat{U}^- \otimes K(D)) \to H^2(C^\bullet) \to 0,
\]

where the maps \(H^i(U^+) \to H^i(\hat{U}^- \otimes K(D))\) are induced by \(\text{ad}(\Phi)\).

□

**Proposition 3.3.** Let \((E, \Phi)\) be a stable parabolic \(U(p,q)\)-Higgs bundle, then

(a) \(H^0(C^\bullet) = \mathbb{C}\) (in other words \((E, \Phi)\) is simple) and

(b) \(H^2(C^\bullet) = 0\).

**Proof.** (a) This follows immediately from Lemma 3.1 and (iii) of Proposition 3.2.

(b) For parabolic bundles \(E\) and \(F\) the sheaves \(\text{ParHom}(E, F)\) and \(\text{SParHom}(F, E) \otimes O(D)\) are naturally dual to each other (see for example [BoY]) and we thus have that

\[
\text{ad}(\Phi) : H^1(U^+) \to H^1(\hat{U}^- \otimes K(D))
\]
is Serre dual to \(\text{ad}(\Phi) : H^0(U^-) \to H^0(\hat{U}^+ \otimes K(D))\). Hence Lemma 3.1 and (iii) of Proposition 3.2 show that \(H^2(C^\bullet) = 0\). □
Proposition 3.4. Assuming Assumption 2.1, the moduli space $\mathcal{U}$ of stable parabolic $U(p, q)$-Higgs bundles is a smooth complex variety of dimension

$$(7) \quad 1 + (g - 1)(p + q)^2 + \frac{s}{2}((p + q)^2 - (p + q)),$$

where $g$ is the genus of $X$, and $s$ is the number of marked points.

Remark 3.5. The formula in (7) is also valid in the case $s = 0$ and genus $g \geq 2$. In such case we recover the formula for the dimension of the moduli space of non parabolic $U(p, q)$-Higgs bundles given in [BGG]. As expected, this dimension is half the dimension of the moduli space $\mathcal{M}$ of parabolic $GL(n, \mathbb{C})$-Higgs bundles of rank $n = p + q$. Observe also that, in order to have a non empty moduli space we need $s \geq 3$ when $g = 0$.

Proof. Our assumption on the genericity of the weights implies that there are no properly semistable parabolic $U(p, q)$-Higgs bundles and hence every point in $\mathcal{U}$ is stable. Smoothness follows from Propositions 3.2 and 3.3. Now, our assumption on having full flags and different weights on $V$ and $W$ imply that

$$\text{SParHom}(V, W) = \text{ParHom}(V, W),$$

and

$$\dim \text{ParHom}(V, W)_x + \dim \text{ParHom}(W, V)_x = pq,$$

$$\dim \text{ParEnd}(V)_x = \frac{p(p + 1)}{2},$$

$$\dim \text{ParEnd}(W)_x = \frac{q(q + 1)}{2}.$$

Also, the short exact sequence

$$0 \to \text{ParHom}(V, W) \to \text{Hom}(V, W) \to \bigoplus_{x \in D} \text{ParHom}(V_x, W_x) \to 0$$

implies that

$$\deg(\text{ParHom}(V, W)) = p \deg(W) - q \deg(V) + \sum_{x \in D} (\dim \text{ParHom}(V_x, W_x) - pq).$$

Using the above information and Proposition 3.2 we have that the dimension of the tangent space of $\mathcal{U}$ at a point $(E, \Phi)$ is

$$\dim \mathbb{H}^1(C^\bullet) = \dim \mathbb{H}^0(C^\bullet) + \dim \mathbb{H}^2(C^\bullet) - \chi(C^\bullet)$$

$$= 1 - \chi(\text{ParEnd}(V) \oplus \text{ParEnd}(W)) + \chi((\text{SParHom}(V, W) \oplus \text{SParHom}(W, V)) \otimes K(D))$$

$$= 1 - (p^2 + q^2)(1 - g) - \deg(\text{ParEnd}(V)) - \deg(\text{ParEnd}(W)) + 2pq(1 - g) + \deg(\text{ParHom}(V, W)) + \deg(\text{ParHom}(W, V)) + 2pq(2g - 2) + 2pq$$

$$= 1 + (g - 1)(p + q)^2 + 2pq + (p^2 + q^2 - 2pq)s + \sum_{x \in D} \left( \dim \text{ParHom}(V, W)_x + \dim \text{ParHom}(W, V)_x - \dim \text{ParEnd}(V)_x - \dim \text{ParEnd}(W)_x \right)$$

$$= 1 + (g - 1)(p + q)^2 + \frac{s}{2}((p + q)^2 - (p + q)).$$

$\square$
4. Parabolic Toledo invariant

In analogy with the non-parabolic case [BGG], one can associate a Toledo invariant to a parabolic \( U(p,q) \)-Higgs bundle.

**Definition 4.1.** The parabolic Toledo invariant corresponding to the parabolic Higgs bundle \((E = V \oplus W, \Phi)\) is

\[
\tau = 2 \frac{pq}{p+q} (\text{par} \mu(V) - \text{par} \mu(W))
\]

The Toledo invariant will give us a way to classify components of the moduli space of parabolic \( U(p,q) \)-Higgs bundles. So we first determine the possible values that it can take.

**Proposition 4.2.** Let \((E = V \oplus W, \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})\) be a semistable parabolic \( U(p,q) \)-Higgs bundle. Then

\[
p(\text{par} \mu(V) - \text{par} \mu(E)) \leq \text{rk} (\gamma) \left(g - 1 + \frac{s}{2}\right),
\]

\[
q(\text{par} \mu(W) - \text{par} \mu(E)) \leq \text{rk} (\beta) \left(g - 1 + \frac{s}{2}\right).
\]

**Proof.** Consider the parabolic bundles \( N = \ker(\gamma) \) and \( I = \text{im} (\gamma) \otimes K(D)^{-1} \). We have an exact sequence of parabolic bundles

\[
0 \to N \to V \to I \otimes K(D) \to 0
\]

and

\[
\text{pardeg}(V) = \text{pardeg}(N) + \text{pardeg}(I \otimes K(D))
\]

\[
= \text{pardeg}(N) + \text{pardeg}(I) + \text{rk}(I)(2g - 2 + s).
\]

Note that \( I \) is a subsheaf of \( W \) and the map \( I \to W \) is a parabolic map. Let \( \tilde{I} \subset W \) be its saturation, which is a subbundle of \( W \), and endow it with the induced parabolic structure. So \( N, V \oplus \tilde{I} \subset E \) are \( \Phi \)-invariant parabolic subbundles of \( E \). The semistability of \((E, \Phi)\) implies that

\[
\text{par} \mu(N) \leq \text{par} \mu(E),
\]

\[
\text{par} \mu(V \oplus I) \leq \text{par} \mu(V \oplus \tilde{I}) \leq \text{par} \mu(E).
\]

This yields

\[
\text{pardeg}(N) \leq \text{rk}(N) \text{par} \mu(E),
\]

\[
\text{pardeg}(V) + \text{pardeg}(I) \leq (p + \text{rk}(I)) \text{par} \mu(E).
\]

Adding both and using (9) we have the inequality

\[
2 \text{pardeg}(V) \leq 2p \text{par} \mu(E) + \text{rk}(I)(2g - 2 + s),
\]

and hence

\[
p(\text{par} \mu(V) - \text{par} \mu(E)) \leq \text{rk} (\gamma) \left(g - 1 + \frac{s}{2}\right).
\]

The other case is analogous. \( \square \)

**Remark 4.3.** The inequalities in Proposition 4.2 are not sharp. This is due to the fact that (10) can be improved by assigning to \( I \) the weights induced by the inclusion \( I \subset W \).
One has the following bound for the Toledo invariant.

**Proposition 4.4.** Let \((E, \Phi)\) be a semistable parabolic \(U(p,q)\)-Higgs subbundle. Then,

\[ |\tau| \leq \tau_M = \min\{p, q\}(2g - 2 + s), \]

**Proof.** Noting that

\[\text{par} \mu (E) = \frac{p}{p + q} \text{par} \mu (V) + \frac{q}{p + q} \text{par} \mu (W),\]

Proposition 4.2 may be rewritten as

\[ q(\text{par} \mu (E) - \text{par} \mu (W)) \leq \text{rk} (\gamma) \left( g - 1 + \frac{s}{2} \right), \]

\[ p(\text{par} \mu (E) - \text{par} \mu (V)) \leq \text{rk} (\beta) \left( g - 1 + \frac{s}{2} \right). \]

By (11) we also have \(\tau = 2p(\text{par} \mu (V) - \text{par} \mu (E)) = 2q(\text{par} \mu (E) - \text{par} \mu (W)).\) The result follows. \(\square\)

5. Hitchin equations and parabolic Higgs bundles

In order to study the topology of \(\mathcal{U}\) we need a gauge-theoretic interpretation of this moduli space in terms of solutions to the Hitchin equations. One can adapt the arguments given by Simpson \([S2]\) for the case of parabolic \(GL(n, \mathbb{C})\)-Higgs bundles to the \(U(p,q)\) situation, along the lines of what is done in \([BGG]\) in the non-parabolic case. Similarly, to construct the moduli space from this point of view, one can adapt the construction given by Konno \([K]\) (see also \([NSI]\)) in the parabolic \(GL(n, \mathbb{C})\) case.

A parabolic structure on a smooth vector bundle is defined in a similar way to what is done in the holomorphic category. Let \(E\) be a smooth parabolic vector bundle of rank \(n\) and fix a hermitian metric \(h\) on \(E\) which is smooth in \(X \setminus D\) and whose (degenerate) behaviour around the marked points is given as follows. We say that a local frame \(\{e_1, \ldots, e_n\}\) for \(E\) around \(x\) respects the flag at \(x\) if \(E_{x,i}\) is spanned by the vectors \(\{e_{M_i+1}(x), \ldots, e_n(x)\}\), where \(M_i = \sum_{j \leq i} k_j(x)\). Let \(z\) be a local coordinate around \(x\) such that \(z(x) = 0\). We require that \(h\) be of the form

\[
h = \begin{pmatrix}
|z|^{2\alpha_1} & 0 \\
& \ddots & 0 \\
0 & & |z|^{2\alpha_n}
\end{pmatrix}
\]

with respect to some local frame around \(x\) which respects the flag at \(x\), where \(\alpha_i = \tilde{\alpha}_i(x)\).

A unitary connection \(d_A\) associated to a smooth \(\bar{\partial}\) operator \(\bar{\partial}_E\) on \(E\) via the hermitian metric \(h\) is singular at the marked points: if we write \(z = \rho \exp(\sqrt{-1}\theta)\) and \(\{e_i\}\) is the local frame used in the definition of \(h\), then with respect to the local frame \(\{\epsilon_i = e_i/|z|^{\tilde{\alpha}_i}\}\), the connection is of the form

\[
d_A = d + \sqrt{-1} \begin{pmatrix}
\tilde{\alpha}_1 & 0 \\
& \ddots & 0 \\
0 & & \tilde{\alpha}_r
\end{pmatrix} d\theta' + A',
\]

where \(A'\) is regular. We denote the space of smooth \(\bar{\partial}\)-operators on \(E\) by \(\mathcal{C}_E\), the space of associated \(h\)-unitary connections by \(\mathcal{C}_E^h\), the group of complex parabolic gauge transformations by \(\mathcal{G}_E^c\) and the subgroup of \(h\)-unitary parabolic gauge transformations by \(\mathcal{G}_E^h\).
Let $V$ and $W$ be smooth parabolic vector bundles equipped with hermitian metrics $h_V$ and $h_W$ adapted to the parabolic structures in the sense explained above. We denote $\mathcal{C} := \mathcal{C}_V \times \mathcal{C}_W$, $\mathcal{C}^C := \mathcal{C}_V^C \times \mathcal{C}_W^C$, $\mathcal{G} := \mathcal{G}_V \times \mathcal{G}_W$. The space of Higgs fields is $\Omega = \Omega^+ \oplus \Omega^-$, where $\Omega^+ = \Omega_{1,0}(\text{ParHom}(W, V) \otimes \mathcal{O}(D))$ and $\Omega^- = \Omega_{1,0}(\text{ParHom}(V, W) \otimes \mathcal{O}(D))$. Here we regard $\text{ParHom}(W, V)$ and $\text{ParHom}(V, W)$ as smooth vector bundles defined like in the holomorphic category.

Following Biquard [Bi] and Konno [K], we introduce certain weighted Sobolev norms and denote the corresponding Sobolev completions of the spaces defined above by $\mathcal{C}_k^1$, $\Omega_1^k$, $(\mathcal{G}^C)_2^k$ and $\mathcal{G}_2^k$. Let

$$\mathcal{H} = \{(\partial E, \Phi) \in \mathcal{C} \times \Omega \mid \partial E \Phi = 0\}$$

and let $\mathcal{H}_1^k$ be the corresponding subspace of $\mathcal{C}^k_1 \times \Omega^k_1$.

Let $\partial E = (\partial V, \partial W)$ where $\partial V \in \mathcal{C}_V$ and $\partial W \in \mathcal{C}_W$, and $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ with $\beta \in \Omega^+$ and $\gamma \in \Omega^-$. Let $F(A_V)$ and $F(A_W)$ be the curvatures of the $h_V$ and $h_W$-unitary connections corresponding to $\partial V$ and $\partial W$, respectively. Let $\beta^*$ and $\gamma^*$ be the adjoints with respect to $h_V$ and $h_W$. Fix a Kähler form $\omega$ on $X$ with volume of $X$ normalized to $2\pi$. We consider the moduli space $S$ defined by the subspace of elements in $\mathcal{H}^k_1$ satisfying Hitchin equations

$$F(A_V) + \beta \beta^* + \gamma^* \gamma = -\sqrt{-1} \mu \text{Id}_V \omega,$$

$$F(A_W) + \gamma \gamma^* + \beta^* \beta = -\sqrt{-1} \mu \text{Id}_W \omega,$$

modulo gauge transformations in $\mathcal{G}_2^k$, where the equations are only defined on $X \setminus D$. Taking the traces of the equations, adding them, integrating over $X \setminus D$, and using the Chern–Weil formula for parabolic bundles, we find that $\mu = \text{par}_1 \mu(V \oplus W)$.

The subspace of smooth points in $\mathcal{H}^k_1$ carries a Kähler metric induced by the complex structure of $X$ and the hermitian metrics $h_V$ and $h_W$. The Hitchin equations are moment map equations for the action of $\mathcal{G}_2^k$ on this subspace. In particular, the smooth part of $\mathcal{S}$, which corresponds to irreducible solutions, is obtained as a Kähler quotient. Under the genericity assumptions on the parabolic weights in Assumption 2.1, all the solutions are irreducible and the moduli space $S$ is a smooth Kähler manifold.

Fix the topological invariants $p = \text{rk } V$, $q = \text{rk } W$, $a = \deg V$, $b = \deg W$ and the weight types $\alpha$ and $\alpha'$ of $V$ and $W$, respectively. Then

$$\mathcal{U}(p, q, a; b, \alpha, \alpha') \cong (\mathcal{H}^s)^k_1/(\mathcal{G}^C)^k_2,$$

where $\mathcal{H}^s$ are the stable elements in $\mathcal{H}$. Moreover, if $\mathcal{S}(p, q, a; b, \alpha, \alpha')$ is the moduli space of solutions for these fixed invariants, we have the following.

**Theorem 5.1.** There is a homeomorphism

$$\mathcal{U}(p, q, a; b, \alpha, \alpha') \cong \mathcal{S}(p, q, a; b, \alpha, \alpha').$$
6. Morse theory on the moduli space of parabolic $U(p,q)$-Higgs bundles.

In this section we recall the Bott-Morse theory used already in the study of parabolic Higgs bundles in [GGM, BoY]. There is an action of $\mathbb{C}^*$ on $U$ given by

$$\psi : \mathbb{C}^* \times U \to U$$

$$\left( \lambda, (E, \Phi) \right) \mapsto (E, \lambda \Phi).$$

This restricts to a Hamiltonian action of the circle on the moduli space $S$ of solutions to the Hitchin equations, which is isomorphic to $U$ (Theorem 5.1), with associated moment map

$$[(E, \Phi)] \mapsto -\frac{1}{2} \|\Phi\|^2 = -\sqrt{-1} \int_X \text{Tr} (\Phi \Phi^*) .$$

We choose to use the positive function, $f : U \to \mathbb{R}$

$$f((E, \Phi)) = \|\Phi\|^2 .$$

(12)

Clearly $f$ is bounded below since it is non-negative. It is also proper, this follows from the properness of the moment map associated to the circle action on $M$ [Bis] (see also [GGM]) and the fact that $U \subset M$ is a closed subset.

To study the connectedness properties of $U$, we use the following basic result: if $Z$ is a Hausdorff space and $f : Z \to \mathbb{R}$ is proper and bounded below then $f$ attains a minimum on each connected component of $Z$. Therefore, if the subspace of local minima of $f$ is connected then so is $Z$. We thus have the following.

**Lemma 6.1.** The function $f : U \to \mathbb{R}$ defined in (12) has a minimum on each connected component of $U$. Moreover, if the subspace of local minima of $f$ is connected then so is $U$. □

Now we will describe the minima of $f$. For this we introduce the subset of $U$ defined by

$$(13) \quad N = N(p, q, a, b; \alpha, \alpha') = \{(E, \Phi) \in U(p, q, a, b; \alpha, \alpha') \text{ such that } \beta = 0 \text{ or } \gamma = 0\}.$$ 

**Proposition 6.2.** For every $(E, \Phi) \in U$

$$f(E, \Phi) \geq \frac{|\tau|}{2} ,$$

with equality if and only if $(E, \Phi) \in N$.

**Proof.** The proof is similar to the one for Proposition 4.5 in [BGG] apart from the fact that we are using adapted metrics on the bundle. □

We will prove that $N$ is the subvariety of local minima of $f$. For this we have to describe the critical points of $f$ and characterize the local minima. By a theorem of Frankel [F], the critical points of $f$ are exactly the fixed points of the circle action.

For a fixed point $(E, \Phi)$ of the circle action, we have an isomorphism $(E, \Phi) \cong (E, e^{\sqrt{-1} \theta} \Phi)$ which yields the following commutative diagram.

$$
\begin{array}{c}
E \xrightarrow{\Phi} E \otimes K(D) \\
\downarrow \psi \quad \downarrow \psi \otimes 1_{K(D)} \\
E \xrightarrow{e^{\sqrt{-1} \theta} \Phi} E \otimes K(D).
\end{array}
$$
Proposition 6.3 ([S2 Thm. 8]). The equivalence class of a stable parabolic Higgs bundle \( (E, \Phi) \) is fixed under the action of \( S^1 \) if and only if it is a parabolic Hodge bundle. This means that \( E \) decomposes as a direct sum

\[
E = E_0 \oplus E_1 \oplus \cdots \oplus E_m
\]

of parabolic bundles, such that \( \Phi_l = \Phi|_{E_l} \) belongs to \( H^0(\text{SParHom}(E_l, E_{l+1}) \otimes K(D)) \). If \( \Phi_l \neq 0 \), then the weight of the isomorphism \( \psi_\theta : E \rightarrow E \) on \( E_{l+1} \) is one plus the weight of \( \psi_\theta \) on \( E_l \).

The decomposition of \( E \) is given by the eigenbundles corresponding to the eigenvalues of the circle action on \( (E, \Phi) \).

Corollary 6.4. In the situation of Proposition 6.3, if \( (E, \Phi) \) is stable, then each \( \Phi_l \) is nonzero and the \( E_l \) are alternately contained in \( V \) and \( W \).

Proof. The proof goes similarly to the non parabolic case (see Proposition 4.10 from [BGG]). □

Now we want to compute the index of a critical point \( (E, \Phi) \). For this we need to write the complex in (3) in terms of the eigenbundle decomposition provided by Proposition 6.3. Hence

\[
\text{ParEnd}(V) \oplus \text{ParEnd}(W) = \bigoplus_{-m \leq 2k \leq m} U_{2k},
\]

\[
\text{SParHom}(V, W) \oplus \text{SParHom}(W, V) = \bigoplus_{-m \leq 2k+1 \leq m} \hat{U}_{2k+1}.
\]

where

\[
U_l = \bigoplus_{i-j=l} \text{ParHom}(E_j, E_i),
\]

\[
\hat{U}_l = \bigoplus_{i-j=l} \text{SParHom}(E_j, E_i).
\]

Therefore the deformation complex (3) for a parabolic \( U(p,q) \)-Higgs bundle \( (E, \Phi) \) can be written as

\[
C^\bullet : \bigoplus_{-m \leq 2k \leq m} U_{2k} \xrightarrow{\text{ad}(\Phi)} \bigoplus_{-m \leq 2k+1 \leq m} \hat{U}_{2k+1} \otimes K(D).
\]

Each piece of this complex gives a subcomplex whose hypercohomology gives an eigenspace of the tangent space \( T_{(E, \Phi)} \mathcal{U} \) for the circle action.

Proposition 6.5. Let \( (E, \Phi) \) be a stable parabolic \( U(p,q) \)-Higgs bundle which represents a fixed point of the circle action on \( \mathcal{U} \). Then the eigenspace of the Hessian of \( f \) corresponding to the eigenvalue \(-2k \) is \( \mathbb{H}^1 \) of the following complex

\[
C^\bullet_{2k} : U_{2k} \xrightarrow{\text{ad}(\Phi)} \hat{U}_{2k+1} \otimes K(D).
\]

Proof. Similar to the non parabolic case (see Proposition 4.11 from [BGG]). □

Corollary 6.6. \( (E, \Phi) \) is a local minimum of \( f \) if and only if \( \mathbb{H}^1(C^\bullet_{2k}) = 0 \) for all \( k \geq 1 \).
Proposition 6.7. Let \((E, \Phi)\) be a stable parabolic \(U(p,q)\)-Higgs bundle which is a fixed point of the \(S^1\)-action on \(\mathcal{U}\). Then \(\chi(C^k_{2k}) \leq 0\) for all \(k \geq 1\), and equality holds if and only if
\[
\text{ad}(\Phi)|_{U_{2k}} : U_{2k} \to \hat{U}_{2k+1} \otimes K(D)
\]
is an isomorphism of bundles.

Proof. We want to get a bound for
\[
\chi(C^k_{2k}) = \chi(U_{2k}) - \chi(\hat{U}_{2k+1} \otimes K(D)).
\]
The dual of each \(U_l\) is
\[
U_l^\vee = \bigoplus_{i-j=l} (\text{ParHom}(E_j, E_i))^\vee = \bigoplus_{i-j=l} \text{SParHom}(E_i, E_j(D)) = \hat{U}_l(D).
\]
The dual of \(\text{ad}(\Phi)|_{U_{2k}}\) is
\[
(\text{ad}(\Phi)|_{U_{2k}})^l = \text{ad}(\Phi)|_{U_{-2k-1}} \otimes 1_{K^{-1}} : U_{-2k-1} \otimes K^{-1} \to \hat{U}_{-2k}(D).
\]
The vector bundle \(\text{ParEnd}(E)\) has a natural parabolic structure induced by the parabolic structure of \(E\). In fact \(\text{ParEnd}(E)\) as a parabolic bundle is the parabolic tensor product of the parabolic bundle \(E\) and the parabolic dual of \(E\) (see [Y1]), and hence its parabolic degree is 0. With respect to this parabolic structure \((\text{ParEnd}(E), \text{ad}(\Phi))\), where \(\text{ad}(\Phi) : \text{ParEnd}(E) \to \text{SParEnd}(E) \otimes K(D)\), is a parabolic Higgs bundle. Now, the stability of \((E, \Phi)\) implies the polystability of \((\text{ParEnd}(E), \text{ad}(\Phi))\). This can be seen by producing a solution to the Hitchin equations on \((\text{ParEnd}(E), \text{ad}(\Phi))\) out of the solution on \((E, \Phi)\), which exists by Theorem 5.1. Since the solution on \((\text{ParEnd}(E), \text{ad}(\Phi))\) may not be irreducible, we only have polystability (in particular, semistability) of \((\text{ParEnd}(E), \text{ad}(\Phi))\). The subbundles \(\ker(\text{ad}(\Phi)|_{U_{2k}})\) and \(\ker(\text{ad}(\Phi)|_{U_{-2k-1}})\) of \(\text{ParEnd}(E)\) are \(\text{ad}(\Phi)\)-invariant and hence we can apply the stability condition on the parabolic slopes. Since the ordinary degree is smaller than the parabolic degree, we have \(\deg(\ker(\text{ad}(\Phi)|_{U_{2k}})) \leq 0\) and \(\deg(\ker(\text{ad}(\Phi)|_{U_{-2k-1}}))) \leq 0\). Therefore we have the following chain of inequalities
\[
\deg(U_{2k}) = \deg(\ker(\text{ad}(\Phi)|_{U_{2k}})) + \deg(\text{im}(\text{ad}(\Phi)|_{U_{2k}})) \\
\leq \deg(\text{im}(\text{ad}(\Phi)|_{U_{2k}})) \\
\leq -\deg(\text{im}(\text{ad}(\Phi)|_{U_{2k}})^l)) \\
= -\deg(\text{im}(\text{ad}(\Phi)|_{U_{-2k-1}} \otimes 1_{K^{-1}})) \\
= -\deg(\text{im}(\text{ad}(\Phi)|_{U_{-2k-1}})) + \text{rk}(\text{im}(\text{ad}(\Phi)|_{U_{-2k-1}}))(2g - 2) \\
= \deg(\ker(\text{ad}(\Phi)|_{U_{-2k-1}})) - \deg(U_{-2k-1}) + \text{rk}(\text{im}(\text{ad}(\Phi)|_{U_{-2k-1}}))(2g - 2) \\
\leq -\deg(U_{-2k-1}) + \text{rk}(\text{im}(\text{ad}(\Phi)|_{U_{-2k-1}}))(2g - 2) \\
= \deg(\hat{U}_{2k+1}(D)) + \text{rk}(\text{im}(\text{ad}(\Phi)|_{U_{-2k-1}}))(2g - 2),
\]
where we have used that \(\text{rk}(\text{im}(h)) = \text{rk}(\text{im}(h^l))\) and that \(\deg(\text{im}(h)) \leq -\deg(\text{im}(h^l))\) for any morphism of sheaves \(h\).
Using this we have that
\[
\chi(C_{2k}^*) = \deg(U_{2k}) + \text{rk}(U_{2k})(1 - g) - \deg(U_{2k+1} \otimes K(D)) - \text{rk}(U_{2k+1})(1 - g)
\]
\[
= \deg(U_{2k}) + \text{rk}(U_{2k})(1 - g) - \deg(U_{2k+1}) - \text{rk}(U_{2k+1})(g - 1 + s)
\]
\[
\leq \deg(U_{2k+1}(D)) + \text{rk}(\im(\text{ad}(\Phi)|_{U_{2k}}))(2g - 2) + \text{rk}(U_{2k})(1 - g) - \deg(U_{2k+1})
\]
\[
- \text{rk}(U_{2k+1})(g - 1 + s)
\]
\[
= (g - 1)(2 \text{rk}(\im(\text{ad}(\Phi)|_{U_{2k}})) - \text{rk}(U_{2k}) - \text{rk}(U_{2k+1})),
\]
where we have used that \(\hat{U}_{2k+1} = U_{2k+1}\) since all the weights are different and of multiplicity 1, and hence for \(i \neq j\) it is \(\text{SParHom}(E_i, E_j) = \text{ParHom}(E_i, E_j)\), since \(E_i\) and \(E_j\) are different pieces in the decomposition of Proposition 6.3. We thus have \(\chi(C_{2k}^*) \leq 0\). If equality holds then \(\text{rk}(\im(\text{ad}(\Phi)|_{U_{2k}})) = \text{rk}(U_{2k}) = \text{rk}(U_{2k+1})\), and also equality holds in (16), showing that \(\text{ad}(\Phi)|_{U_{2k}}\) is an isomorphism as claimed.

\textbf{Corollary 6.8.} Let \((E, \Phi)\) be a stable parabolic \(\text{U}(p, q)\)-Higgs bundle which represents a critical point of the Morse function \(f\). This critical point is a minimum if and only if
\[
\text{ad}(\Phi)|_{U_{2k}} : U_{2k} \rightarrow \hat{U}_{2k+1} \otimes K(D)
\]
is an isomorphism for all \(k \geq 1\).

\textbf{Proof.} By Corollary 6.6 \((E, \Phi)\) is a local minimum if and only if
\[
\mathbb{H}^1(C_{2k}^*) = 0, \quad \forall k \geq 1.
\]
Note that by Proposition 6.3 \(\mathbb{H}^0(C_{2k}^*) = 0\) and \(\mathbb{H}^2(C_{2k}^*) = 0\), for \(k \geq 1\). Hence \((E, \Phi)\) is a local minimum if and only if
\[
\chi(C_{2k}^*) = \sum (-1)^i \dim \mathbb{H}^i(C_{2k}^*) = 0, \quad \forall k \geq 1.
\]
By Proposition 6.7 this is equivalent to requiring that
\[
\text{ad}(\Phi) : U_{2k} \rightarrow \hat{U}_{2k+1} \otimes K(D)
\]
be an isomorphism of sheaves. \(\square\)

Finally, we show that all these minima are in \(\mathcal{N}\).

\textbf{Proposition 6.9.} Let \((E, \Phi) = (E_0 \oplus \cdots \oplus E_m, \Phi)\) be stable and a fixed point of the circle action, with \(m \geq 2\). Then \((E, \Phi)\) is not a local minimum.

\textbf{Proof.} First note that \(U_l = \hat{U}_l = 0\) for \(l > m\), and note also that for \(l = m\), \(U_m = \text{ParHom}(E_0, E_m)\). Now we divide the proof conforming the different possibilities for \(U_l\) and \(\hat{U}_l\) as the number \(m\) of terms in the bundle decomposition of \(E\) is even or odd.

If \(m\) is even then \(2k = m\) and
\[
\text{ad}(\Phi)|_{U_m} : \text{ParHom}(E_0, E_m) \rightarrow 0
\]
does not satisfy Corollary 6.8 hence \((E, \Phi)\) is not a local minimum.

If \(m \geq 2\) is odd, then \(2k = m - 1\) and
\[
\text{ad}(\Phi)|_{U_{m-1}} : \text{ParHom}(E_0, E_{m-1}) \oplus \text{ParHom}(E_1, E_m) \rightarrow \text{SParHom}(E_0, E_m) \otimes K(D).
\]
We will show that this is not an injective map of sheaves, and therefore \((E_0 \oplus \cdots \oplus E_m, \Phi)\) is not a minimum. We prove this in a small open set where all the bundles trivialize. We need to find \(\zeta = (\zeta_1, \zeta_2) \in U_{m-1}, \zeta \neq 0\) such that \(\text{ad}(\Phi)|_{U_{m-1}}(\zeta) = 0\), i.e. we need to find \(\zeta_1\) and \(\zeta_2\) making the following diagram commutative.

\[
\begin{array}{ccc}
E_0 & \xrightarrow{\Phi} & E_1 \otimes K(D) \\
\downarrow \zeta_1 & & \downarrow \zeta_2 \otimes 1_{K(D)} \\
E_{m-1} & \xrightarrow{\Phi} & E_m \otimes K(D)
\end{array}
\]

For this, take \(\zeta_2 \neq 0\) such that \(\zeta_2 \otimes 1_{K(D)}(E_1 \otimes K(D)) \subset \Phi(E_{m-1})\), this is possible by taking \(\zeta_2\) as the composition of \(\Phi_l\) in Proposition 6.3 tensor the appropriate power of \(K(D)\), note that they are nonzero by Corollary 6.4. Now take \(\zeta_1\) making \((E, \Phi)\) an injective map of sheaves.

Corollary 6.10. The subvariety of local minima of \(f: U(p, q, a, b; \alpha, \alpha') \to \mathbb{R}\) coincides with the set \(N(p, q, a, b; \alpha, \alpha')\) defined in (13).

Proof. By Proposition 6.9, for \((E, \Phi)\) to be a minimum it must have a decomposition of the form \(E = E_0 \oplus E_1\) with \(\Phi\) mapping \(E_0\) into \(E_1\). But by definition the only possible decompositions are \(E = V \oplus W\) with \(\Phi = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}\) and \(E = W \oplus V\) with \(\Phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\). So \((E, \Phi) \in \mathcal{N}\).

Conversely, if \((E, \Phi) \in \mathcal{N}\) then \(m = 1\) and \(U_{2k} = U_{2k+1} = 0\), for \(k \geq 1\). So Corollary 6.9 applies and \((E, \Phi)\) is a minimum. \(\square\)

Which of the two components of the Higgs field vanishes is given by the following.

Lemma 6.11. Let \((E, \Phi) \in \mathcal{N}\). Then the Toledo invariant \(\tau \neq 0\) and

(i) \(\gamma = 0\) if and only if \(\tau < 0\).

(ii) \(\beta = 0\) if and only if \(\tau > 0\).

Proof. Observe that \(\tau\) can not be equal to zero because this implies \(\gamma = \beta = 0\) and then \((E, \Phi)\) cannot be stable. The rest follows directly from the definition of the Toledo invariant. \(\square\)

Our main goal in the rest of the paper is to show the following.

Theorem 6.12. Suppose \(g > 0\). Then there is a value

\[
\tau_L = \min\{p, q\}(2g - 2 + s) - \frac{|p - q|}{p + q}\epsilon,
\]

with \(\epsilon > 0\) explicitly computable (see Remark 11.9), such that the subvariety \(\mathcal{N}(p, q, a, b; \alpha, \alpha')\) is non-empty and connected if and only if the parabolic Toledo invariant \(\tau\) satisfies the bound \(|\tau| \leq \tau_L\). The moduli space of parabolic \(U(p, q)\)-Higgs bundles \(\mathcal{U}(p, q, a, b; \alpha, \alpha')\) is empty for \(|\tau| > \tau_L\).
Proof. In the case \( p \neq q \), the result will follow from Proposition 7.3 and Theorem 11.8. In the case \( p = q \), the result will follow from Propositions 7.3 and 7.7. Corollary 12.12 and Remark 12.13. Note that \( \tau_L = \tau_M \) for \( p = q \). \( \Box \)

Combining Theorem 6.12, Corollary 6.10 and Lemma 6.1, we have the main result of our paper.

**Theorem 6.13.** Suppose \( g > 0 \) and \( s > 0 \). The moduli space of parabolic \( U(p,q) \)-Higgs bundles \( U(p,q,a,b;\alpha,\alpha') \) is non-empty and connected if and only if \( |\tau| \leq \tau_L \). The moduli space is empty whenever \( |\tau| \geq \tau_L \).

**Remark 6.14.** It is likely that Theorem 6.13 holds more generally than under Assumption 2.1. It should be enough to assume that \( V \oplus W \) have full flags, but arbitrary (non-generic) weights. The reason is that the assumption of full flags is strong enough to avoid the type of problem that comes up in Theorem 3.32 of [BGG], since all the weights are distinct. One way to prove this would be to show that the moduli spaces for different choices of weights are related by flips as with the moduli spaces of triples (as in [Th]).

**Remark 6.15.** Actually, in both Theorems 6.12 and 6.13, the case \( |\tau| = \tau_L \) does not occur under Assumption 2.1. This is true since \( \sigma = 2g - 2 \) is not a critical value for the appropriate moduli space of triples appearing in Proposition 7.3 (see Remark 7.3). For \( p = q \), it cannot happen that \( |\tau| = \tau_M \), as pointed out in Remark 12.13.

7. Parabolic triples

In the previous section, we have concluded that it is necessary to study the connectedness of the subspace \( \mathcal{N} \) of \( \mathcal{U} \). This subset consists of parabolic \( U(p,q) \)-Higgs bundles with \( \gamma = 0 \) or \( \beta = 0 \), hence giving rise in a natural way to objects called parabolic triples.

We recall the basics of parabolic triples from [BiG, GGM]. A **parabolic triple** is a holomorphic triple \( T = (E_1, E_2, \phi) \) where \( E_1 \) and \( E_2 \) are parabolic bundles and \( \phi : E_2 \to E_1(D) \) is a strongly parabolic homomorphism, i.e. \( \phi \in H^0(\text{SParHom}(E_2, E_1(D))) \). We denote by \( \alpha = (\alpha^1, \alpha^2) \) the parabolic system of weights for the triple \( (E_1, E_2, \phi) \), where \( \alpha^i \) is the system of weights of \( E_i \) with \( i = 1, 2 \).

For \( \sigma \in \mathbb{R} \) the parabolic \( \sigma \)-**degree** and \( \sigma \)-**slope** of \( T \) are defined as

\[
\text{pardeg}_{\sigma}(T) = \text{pardeg}(E_1) + \text{pardeg}(E_2) + \sigma \text{rk}(E_2),
\]

\[
\text{par} \mu_{\sigma}(T) = \frac{\text{pardeg}(E_1 + \text{pardeg}(E_2))}{\text{rk}(E_1) + \text{rk}(E_2)} + \sigma \frac{\text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)}.
\]

A parabolic triple \( T' = (E'_1, E'_2, \phi') \) is a **parabolic subtriple** of \( T = (E_1, E_2, \phi) \) if \( E'_i \subset E_i \) are parabolic subbundles for \( i = 1, 2 \) and \( \phi' = \phi|_{E'_2} \) being \( \phi(E'_2) \subset E'_1(D) \). As usual, \( T \) is called \( \sigma \)-stable (resp. \( \sigma \)-semistable) if for any non-zero proper subtriple \( T' \) we have \( \text{par} \mu_{\sigma}(T') \leq \text{par} \mu_{\sigma}(T) \) (resp. \( \text{par} \mu_{\sigma}(T') < \text{par} \mu_{\sigma}(T) \)). The triple \( T \) is called \( \sigma \)-polystable if it is the direct sum of parabolic triples with the same parabolic \( \sigma \)-slope.

Let

\[
\mathcal{N}_{\sigma} = \mathcal{N}_{\sigma}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)
\]
be the moduli space of isomorphism classes of \( \sigma \)-polystable triples with fixed system of weights \((\alpha^1, \alpha^2)\) and \(r_1 = \text{rk}(E_1), r_2 = \text{rk}(E_2), d_1 = \text{deg}(E_1), d_2 = \text{deg}(E_2)\). Let
\[
\mathcal{N}_\sigma^p \subset \mathcal{N}_\sigma
\]
be the open subset consisting of \( \sigma \)-stable triples.

**Proposition 7.1.** A necessary condition for \( \mathcal{N}_\sigma(r_1, r_2, d_1, d_2, \alpha^1, \alpha^2) \) to be non-empty is
\[
\sigma_m < \sigma < \sigma_M \quad \text{if } r_1 \neq r_2
\]
\[
\sigma_m < \sigma \quad \text{if } r_1 = r_2
\]
where
\[
\sigma_m = \text{par}\mu(E_1) - \text{par}\mu(E_2)
\]
\[
\sigma_M = \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) \left(\text{par}\mu(E_1) - \text{par}\mu(E_2)\right) + s \frac{r_1 + r_2}{|r_1 - r_2|}, \quad \text{if } r_1 \neq r_2.
\]

**Proof.** See Proposition 4.3 from [GGM]. \(\square\)

**Remark 7.2.** We will see later on that there is an effective upper bound \(\sigma_L\) given by (38) which in general is strictly smaller than \(\sigma_M\).

The correspondence between parabolic triples and parabolic \(U(p, q)\)-Higgs bundles goes as follows. Let \((E, \Phi)\) be a parabolic \(U(p, q)\)-Higgs bundle with \(\Phi = \beta : W \to V \otimes K(D)\). This defines a triple \(T = (E_1, E_2, \phi)\) where \(E_1 = V \otimes K, E_2 = W, \phi = \beta\). Conversely, given a parabolic triple \(T = (E_1, E_2, \phi)\) we get a parabolic \(U(p, q)\)-Higgs bundle with \(\Phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\) by defining \((E = V \oplus W, \Phi)\) where \(V = E_1 \otimes K^{-1}, W = E_2\) and \(\beta = \phi\). When \((E, \Phi)\) is a parabolic \(U(p, q)\)-Higgs bundle with \(\Phi = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}: V \to W \otimes K(D)\) we have an analogous correspondence. That is, the corresponding triple to \((E, \Phi)\) is \(T = (W \otimes K, V, \gamma)\).

**Lemma 7.3.** A parabolic \(U(p, q)\)-Higgs bundle \((E, \Phi)\) with \(\beta = 0\) or \(\gamma = 0\) is parabolically (semi)stable if and only if the corresponding parabolic triple is \(\sigma\)-(semi)stable for \(\sigma = 2g - 2\).

**Proof.** Let \(T = (E_1, E_2, \phi)\) be the triple defined by \((E, \Phi)\) (without loss of generality we assume \(\gamma = 0\)). Therefore if we set \(\sigma = 2g - 2\) we have
\[
\text{par}\mu_{\sigma}(T) = \frac{\text{pardeg}(E_1) + \text{pardeg}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)} + \sigma \frac{\text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)}
\]
\[
= \frac{\text{pardeg}(V) + \text{pardeg}(W) + p(2g - 2)}{p + q} + \sigma \frac{q}{p + q}
\]
\[
= \text{par}\mu(E) + 2g - 2.
\]

Note that the correspondence between parabolic triples and \(U(p, q)\) parabolic bundles with \(\beta = 0\) or \(\gamma = 0\) gives also a correspondence between parabolic subtriples and parabolic subbundles. That is, given a subtriple \(T'\) of \(T\) the corresponding parabolic \(U(p, q)\)-Higgs bundle is a \(\Phi\)-invariant subbundle of \((E, \Phi)\), and conversely given \((E', \Phi')\) the corresponding triple gives a parabolic subtriple of \(T\). Hence equation (19) gives that \(\text{par}\mu_{2g - 2}(T') < \text{par}\mu_{2g - 2}(T)\) if and only if \(\text{par}\mu(E') < \text{par}\mu(E)\) (analogously for the semistability condition). \(\square\)
Combining the arguments above and Lemma 6.11 we have the following correspondence.

**Proposition 7.4.** Let \( \mathcal{N}(p, q, a, b; \alpha, \alpha') \) be the submanifold of local minima of \( \mathcal{U}(p, q, a, b; \alpha, \alpha') \) and let \( \tau \) be the Toledo invariant then,

(i) If \( \tau < 0 \) then \( \mathcal{N}(p, q, a, b; \alpha, \alpha') = \mathcal{N}_{g-2}(p, q, a + p(2g - 2), b; \alpha, \alpha'). \)

(ii) If \( \tau > 0 \) then \( \mathcal{N}(p, q, a, b; \alpha, \alpha') = \mathcal{N}_{g-2}(q, p, b + q(2g - 2), a; \alpha', \alpha'). \)

**Proof.** It follows immediately from Lemma 6.11. \( \square \)

**Remark 7.5.** Note that the genericity condition on the weights implies that there are no properly \((2g - 2)\)-semistable triples for \( \sigma = \tau \geq 2 \), that is, \( \mathcal{N}_{g-2}^\sigma = \mathcal{N}_{g-2}. \)

So we state the following assumption that we shall use during the rest of the paper, and which is a translation of Assumption 2.1 via Proposition 7.4.

**Assumption 7.6.** We consider moduli spaces of \( \sigma \)-stable triples \( \mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha_1, \alpha_2) \) satisfying that there are no properly \((2g - 2)\)-semistable triples and such that all the weights are of multiplicity one, and the weights of \( E_1 \) and \( E_2 \) are all different.

It is clear that in order for \( \mathcal{N}(p, q, a, b; \alpha, \alpha') \) to be non-empty, \( 2g - 2 \) must be in the range for \( \sigma \) given by Proposition 7.1, where \( \sigma_m \) and \( \sigma_M \) are determined by the correspondence given in Proposition 7.4. In fact, one has the following comparison of such necessary condition with the Milnor–Wood inequality for the parabolic Toledo invariant \( \tau \) given in Proposition 4.3.

**Proposition 7.7.** Let \( \sigma_m \) and \( \sigma_M \) be the bounds for \( \sigma \) defined in Proposition 7.1 for the moduli space of parabolic triples identified in Proposition 7.4 with the subvariety \( \mathcal{N}(p, q, a, b; \alpha, \alpha') \). Recall \( \tau_M = \min\{p, q\}(2g - 2 + s) \). Then

\[
0 \leq |\tau| \leq \tau_M \iff \begin{cases} 
\sigma_m \leq 2g - 2 \leq \sigma_M, & \text{if } p \neq q, \\
\sigma_m \leq 2g - 2, & \text{if } p = q.
\end{cases}
\]

**Proof.** Write \( \sigma_m \) and \( \sigma_M \) in terms of \( \tau \), that is,

\[
\begin{align*}
\sigma_m &= \frac{(p+q)\tau}{2pq} - 2g + 2, & \text{if } \tau < 0, \\
\sigma_m &= -\frac{(p+q)\tau}{2pq} + 2g - 2, & \text{if } \tau > 0, \\
\sigma_M &= \left(1 + \frac{p+q}{p-q}\right)\left(\frac{(p+q)\tau}{2pq} + 2g - 2\right) + s\frac{p+q}{p-q}, & \text{if } \tau < 0, \\
\sigma_M &= \left(1 + \frac{p+q}{p-q}\right)\left(-\frac{(p+q)\tau}{2pq} + 2g - 2\right) + s\frac{p+q}{p-q}, & \text{if } \tau > 0.
\end{align*}
\]

From these equalities, the result is clear. \( \square \)

**Remark 7.8.** Proposition 7.7 gives a condition for the number of marked points in order for \( \mathcal{N} \) to be non-empty. Namely,

(i) If \( g = 0 \) then \( s \geq 3 \),

(ii) If \( g = 1 \) then \( s \geq 1 \),

and no extra condition when \( g \geq 2 \).
8. Extensions and deformations of parabolic triples

In order to study the differences between the moduli spaces $N_\sigma$ as $\sigma$ changes, we need to study extensions and deformations of parabolic triples. This study is done in [GGM]. We summarize the main results.

Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two parabolic triples. Let $\text{Hom}(T'', T')$ denote the vector space of homomorphisms from $T''$ to $T'$, and $\text{Ext}^1(T'', T')$ be the vector space of extensions of the form

$$0 \to T' \to T \to T'' \to 0,$$

that is, commutative diagrams:

$$
\begin{array}{cccc}
0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0 \\
\downarrow \phi' & & \downarrow \phi & & \downarrow \phi'' & & \\
0 & \longrightarrow & E'_1(D) & \longrightarrow & E_1(D) & \longrightarrow & E''_1(D) & \longrightarrow & 0.
\end{array}
$$

In order to study extensions of parabolic triples, we consider the following complex of sheaves

$$C^k(T'', T') : \text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2) \to \text{SParHom}(E''_2, E'_1(D))$$

and a long exact sequence:

$$
\begin{align*}
0 & \to \mathbb{H}^0(\text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2)) \to \mathbb{H}^0(\text{SParHom}(E''_2, E'_1(D))) \\
& \to \mathbb{H}^1(\text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2)) \to \mathbb{H}^1(\text{SParHom}(E''_2, E'_1(D))) \\
& \to \mathbb{H}^2 \to 0.
\end{align*}
$$

We denote:

$$h^i(T'', T') = \dim \mathbb{H}^i(C^*(T'', T')),$$

$$\chi(T'', T') = h^0(T'', T') - h^1(T'', T') + h^2(T'', T').$$

**Proposition 8.2** (Proposition 4.8 [GGM]). For parabolic triples $T'$ and $T''$

$$\chi(T'', T') = \chi(\text{ParHom}(E''_1, E'_1)) + \chi(\text{ParHom}(E''_2, E'_2)) - \chi(\text{SParHom}(E''_2, E'_1(D))).$$

**Corollary 8.3** (Corollary 4.9 [GGM]). For any extension $0 \to T' \to T \to T'' \to 0$ of parabolic triples we have that

$$\chi(T, T) = \chi(T', T') + \chi(T'', T'') + \chi(T', T'') + \chi(T'', T').$$

Using the same arguments as in Proposition 3.5 of [BGG2] one can prove the following.

**Proposition 8.4.** Suppose that $T'$ and $T''$ are $\sigma$-semistable.

(i) If $\text{par}_\sigma(T') < \text{par}_\sigma(T'')$, then $\mathbb{H}^0(C^*(T'', T')) \cong 0$. 

(ii) If \( \text{par}_\sigma(T') = \text{par}_\sigma(T'') \) and \( T', T'' \) are \( \sigma \)-stable, then

\[
\mathbb{H}^0(C^\bullet(T'', T')) \cong \begin{cases} 
\mathbb{C}, & \text{if } T' \cong T'', \\
0, & \text{if } T' \not\cong T''.
\end{cases}
\]

Theorem 8.5. Let \( T = (E_1, E_2, \phi) \) be a \( \sigma \)-stable parabolic triple.

(i) The Zariski tangent space at the point defined by \( T \) in the moduli space \( N^\sigma_\sigma \) of \( \sigma \)-stable triples is isomorphic to \( H^1(C^\bullet(T, T)) \).

(ii) If \( \mathbb{H}^2(C^\bullet(T, T)) = 0 \), then the moduli space \( N^\sigma_\sigma \) of \( \sigma \)-stable parabolic triples is smooth in a neighbourhood of the point defined by \( T \).

(iii) \( \mathbb{H}^2(C^\bullet(T, T)) = 0 \) if and only if the homomorphism

\[
H^1(\text{ParEnd}(E_1)) \oplus H^1(\text{ParEnd}(E_2)) \to H^1(\text{SParHom}(E_2, E_1(D)))
\]

is surjective.

(iv) At the smooth point in \( N^\sigma_\sigma \) represented by \( T \), the dimension of the moduli space of \( \sigma \)-stable parabolic triples is

\[
\dim N^\sigma_\sigma = h^1(T, T) = 1 - \chi(T, T)
\]

\[
= 1 - \chi(\text{ParEnd}(E_1)) - \chi(\text{ParEnd}(E_2)) + \chi(\text{SParHom}(E_2, E_1(D)))
\]

(v) If \( \phi \) is injective or surjective then \( T \) defines a smooth point in the moduli space.

Proof. The proof runs analogous to the non parabolic situation (see proof of Theorem 3.8 in [BGG2]). \qed

9. Critical values

A parabolic triple \( T = (E_1, E_2, \phi) \) is strictly \( \sigma \)-semistable if and only if there is a proper subtriple \( T' = (E'_1, E'_2, \phi') \) such that \( \text{par}_\sigma(T) = \text{par}_\sigma(T') \), i.e.,

\[
\text{par}_\sigma(T') + \sigma \frac{r'_2}{r'_1 + r'_2} = \text{par}_\sigma(T) + \sigma \frac{r_2}{r_1 + r_2},
\]

where \( r'_1 = \text{rk}(E'_1) \), \( r'_2 = \text{rk}(E'_2) \). There are two ways in which this can happen. One is that there exists a parabolic subtriple such that

\[
\frac{r'_2}{r'_1 + r'_2} = \frac{r_2}{r_1 + r_2}
\]

therefore this implies

\[
\text{par}_\sigma(T') = \text{par}_\sigma(T).
\]

In this case \( T \) is strictly \( \sigma \)-semistable for all \( \sigma \) (or at least for an interval of values of \( \sigma \)) and it is called \( \sigma \)-independent semistable. The other way in which strict \( \sigma \)-semistability can happen is if equality holds for (24) but with

\[
\frac{r'_2}{r'_1 + r'_2} \neq \frac{r_2}{r_1 + r_2}.
\]
Definition 9.1. The values of \( \sigma \) such that there exists a strictly \( \sigma \)-semistable triple \( T \) with a subtriple \( T' \) such that \( \text{par}\mu_\sigma(T') = \text{par}\mu_\sigma(T) \) and
\[
\frac{r'_2}{r'_1 + r'_2} \neq \frac{r_2}{r_1 + r_2}
\]
are called critical values.

Proposition 9.2 (Proposition 5.2 [GGM]).
(i) The critical values of \( \sigma \) form a discrete subset of \([\sigma_m, \sigma_M]\) if \( r_1 \neq r_2 \), and of \([\sigma_m, \infty)\) if \( r_1 = r_2 \).
(ii) The stability criteria for two values of \( \sigma \) between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.
(iii) For generic weights, \( \sigma = 2g - 2 \) is not a critical value.

Let \( \sigma_\epsilon \) be a critical value such that \( \sigma_m < \sigma_\epsilon < \sigma_M \). Here we adopt the convention that \( \sigma_M = \infty \) when \( r_1 = r_2 \). Set
\[
\sigma^+_\epsilon = \sigma_\epsilon + \epsilon, \quad \sigma^-_\epsilon = \sigma_\epsilon - \epsilon,
\]
where \( \epsilon > 0 \) is small enough so that \( \sigma_\epsilon \) is the only critical value in the interval \((\sigma^-_\epsilon, \sigma^+_\epsilon)\).

Lemma 9.3. Let \( \sigma_\epsilon \in (\sigma_m, \sigma_M) \) be a critical value. We define the flip loci \( S_{\sigma^+_\epsilon} \) as the set of triples in \( N^s_{\sigma^+_\epsilon} \) which are \( \sigma^+_\epsilon \)-stable but not \( \sigma^-_\epsilon \)-stable. Then
\[
N^s_{\sigma^+_\epsilon} - S_{\sigma^+_\epsilon} = N^s_{\sigma^-_\epsilon} = N^s_{\sigma^-_\epsilon} - S_{\sigma^-_\epsilon}.
\]

The following result is analogous to [BGG2] Proposition 5.4.

Proposition 9.4. Let \( \sigma_\epsilon \in (\sigma_m, \sigma_M) \) be a critical value. Let \( T = (E_1, E_2, \phi) \) be a triple which is \( \sigma_\epsilon \)-semistable.

1. Suppose that \( T \) represents a point in \( S_{\sigma^+_\epsilon} \), i.e. suppose that \( T \) is \( \sigma^+_\epsilon \)-stable but not \( \sigma^-_\epsilon \)-stable. Then \( T \) has a description as the middle term in an extension
\[
0 \to T' \to T \to T'' \to 0
\]
in which
(a) \( T' \) and \( T'' \) are both \( \sigma^+_\epsilon \)-stable, with \( \text{par}\mu_{\sigma^+_\epsilon}(T') < \text{par}\mu_{\sigma^+_\epsilon}(T) \),
(b) \( T' \) and \( T'' \) are both \( \sigma^-_\epsilon \)-semistable with \( \text{par}\mu_{\sigma^-_\epsilon}(T') = \text{par}\mu_{\sigma^-_\epsilon}(T) \).

2. Similarly, if \( T \) represents a point in \( S_{\sigma^-_\epsilon} \), i.e. if \( T \) is \( \sigma^-_\epsilon \)-stable but not \( \sigma^+_\epsilon \)-stable, then \( T \) has a description as the middle term in an extension \((25)\) in which
(a) \( T' \) and \( T'' \) are both \( \sigma^-_\epsilon \)-stable with \( \text{par}\mu_{\sigma^-_\epsilon}(T') < \text{par}\mu_{\sigma^-_\epsilon}(T) \),
(b) \( T' \) and \( T'' \) are both \( \sigma^-_\epsilon \)-semistable with \( \text{par}\mu_{\sigma^-_\epsilon}(T') = \text{par}\mu_{\sigma^-_\epsilon}(T) \).

The following lemma is proved with analogous arguments as in Proposition 3.6 of [BGG2].

Lemma 9.5. Let \( T' \) and \( T'' \) be triples which are \( \sigma \)-stable and of the same \( \sigma \)-slope, for some \( \sigma \geq 2g - 2 \). Then
\[
\mathbb{H}^2(C^*(T'', T')) = 0.
\]

Corollary 9.6. \( N_\sigma \) is smooth of the expected dimension, for any \( \sigma \geq 2g - 2 \).

Proposition 9.7. If \( \sigma_\epsilon > 2g - 2 \) then the loci \( S_{\sigma^+_\epsilon} \subset N^s_{\sigma^+_\epsilon} \) have codimension bigger than or equal to \(-\chi(T', T'')\).
Proof. Let us do the case of $\sigma_c^\pm$. For simplicity we denote
\[
N^{\prime\prime}_{\sigma_c^\pm} = N^{\prime\prime}_{\sigma_c^\pm}(r_1', r_2', d_1', d_2'; \alpha^\prime, \alpha^\prime'),
N''_{\sigma_c^\pm} = N^{\prime\prime}_{\sigma_c^\pm}(r_1'', r_2'', d_1'', d_2''; \alpha^\prime'', \alpha^{\prime''}').
\]
It is known from \cite{Y2} that $N^{\prime\prime}_{\sigma_c^\pm}$ and $N''_{\sigma_c^\pm}$ are fine moduli spaces. That is, there are universal parabolic triples $T' = (E'_1, E'_2, \Phi')$ and $T'' = (E''_1, E''_2, \Phi)$ over $N^{\prime\prime}_{\sigma_c^\pm} \times X$ and $N''_{\sigma_c^\pm} \times X$ respectively. Thus we consider the complex $C^\bullet(T'', T')$ as defined in (20) and take relative hypercohomology with respect to the projection
\[
\pi : X \times N^{\prime\prime}_{\sigma_c^\pm} \times N''_{\sigma_c^\pm} \rightarrow N''_{\sigma_c^\pm} \times N''_{\sigma_c^\pm}.
\]
We define $W^+ := H^1(C^\bullet(T'', T'))$. By Proposition 9.4, $S_{\sigma_c^\pm}$ is a subset of the projective fibration $\mathbb{P}W^+$ over $N^{\prime\prime}_{\sigma_c^\pm} \times N''_{\sigma_c^\pm}$. The fibres of this fibration are projective spaces of dimension
\[
\dim \mathbb{P}(\text{Ext}^1(T'', T')) = \dim \text{Ext}^1(T'', T') - 1
= h^0(T'', T') + h^2(T'', T') - \chi(T'', T') - 1
= -\chi(T'', T') - 1,
\]
using Lemma 9.4 and Proposition 8.5 to substitute $h^0(T'', T') = h^2(T'', T') = 0$. Therefore
\[
\dim S_{\sigma_c^\pm} \leq -\chi(T'', T') + \dim(N''_{\sigma_c^\pm} \times N''_{\sigma_c^\pm})
= -\chi(T'', T') - 1 + 1 - \chi(T', T') + 1 - \chi(T'', T'')
= \dim N_{\sigma_c^\pm} + \chi(T', T''),
\]
since the moduli spaces $N^{\prime\prime}_{\sigma_c^\pm}$ and $N''_{\sigma_c^\pm}$ are smooth of the expected dimension. Therefore
\[
\dim N_{\sigma_c^\pm} - \dim S_{\sigma_c^\pm} \geq -\chi(T', T'').
\]
Hence, if we prove that this codimension is positive then the moduli spaces $N_{\sigma_c^\pm}$ for different values of $\sigma \geq 2g - 2$ are birational, and in particular have the same number of irreducible components.

10. Codimension of the flip loci

Let $\sigma_c$ be a critical value in the interval $(\sigma_m, \sigma_M)$ such that $\sigma_c \geq 2g - 2$. Let $T'$ and $T''$ be two $\sigma_c^\pm$-stable (and $\sigma_c$-semistable) parabolic triples with $\text{par} \mu_{\sigma_c}(T') = \text{par} \mu_{\sigma_c}(T'')$. Changing the roles of $T'$ and $T''$, we may compute the bound $\chi(T'', T')$ for the codimension of the flip locus (Proposition 8.7) using the complex (20). Under our Assumption 7.6, we have $\text{SParHom}(E'_2, E'_1(D)) = \text{ParHom}(E'_2, E'_1(D))$, and hence the complex (20) is
\[
C^\bullet(T'', T') : C_1 = \text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2) \xrightarrow{\alpha_1} C_0(D) = \text{ParHom}(E''_2, E'_1(D))
(\xi_1, \xi_2) \mapsto \phi' \xi_2 - \xi_1 \phi''.
\]
Our task is to bound the Euler characteristic of the complex $C^\bullet(T'', T')$, that is,
\[
\chi(C^\bullet(T'', T')) = (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + \deg(C_1) - \deg(C_0(D)).
\]
In order to obtain bounds for $\deg(C_1)$ and $\deg(C_0)$, we follow a similar strategy to that used in \cite{BGG} in the non-parabolic case, exploiting the existence theorem for parabolic vortex equations.
Theorem 10.1 ([BG] Thm. 3.4]). Let $T = (E_1, E_2, \phi)$ be a parabolic triple. Let $\tau_1$ and $\tau_2$ satisfy $\tau_1 \text{rk}(E_1) + \tau_2 \text{rk}(E_2) = \text{pardeg}(E_1) + \text{pardeg}(E_2)$, and let $\sigma = \tau_1 - \tau_2$. Then $E_1$ and $E_2$ admit hermitian metrics, adapted to the parabolic structures, satisfying

$$\sqrt{-1}AF(E_1) + \phi\phi^* = \tau_1 \text{Id}_{E_1},$$

$$\sqrt{-1}AF(E_2) - \phi^*\phi = \tau_2 \text{Id}_{E_2},$$

if and only if $T$ is $\sigma$-polystable. Here $F(E_i)$ is the curvature of the hermitian metric of $E_i$ and $\Lambda$ is the contraction with a Kähler form on $X$ with volume normalized to $2\pi$

One can easily show that

$$\tau_1 = \text{par} \mu_\sigma(T),$$

$$\tau_2 = \text{par} \mu_\sigma(T) - \sigma.$$

Moreover, adding up the equations in Theorem 10.1, integrating, and using the Chern–Weil formula for parabolic bundles, we have that

$$r_1 \tau_1 + r_2 \tau_2 = \text{pardeg}(E_1) + \text{pardeg}(E_2).$$

In our situation, the triples $T'$ and $T''$ are $\sigma$-stable for $\sigma = \sigma_\epsilon^\pm$, and hence, by Theorem 10.1 there exist adapted hermitian metrics such that

$$\sqrt{-1}AF''(E_1') + \phi''(\phi')^* = \tau_1' \text{Id}_{E_1'}, \quad \sqrt{-1}AF''(E_2') - (\phi')^*\phi' = \tau_2' \text{Id}_{E_2'},$$

$$\sqrt{-1}AF''(E_1'') + \phi''(\phi')^* = \tau_1'' \text{Id}_{E_1''}, \quad \sqrt{-1}AF''(E_2'') - (\phi'')^*\phi'' = \tau_2'' \text{Id}_{E_2''},$$

where $\sigma = \tau_1' - \tau_2' = \tau_1'' - \tau_2''$. In particular, $\tau_1' - \tau_1'' = \tau_2' - \tau_2''$.

Let us consider the induced adapted hermitian metrics on $C_0$ and $C_1$. The corresponding curvatures are given by

$$F(C_0) = -F(E_2')^t \otimes \text{Id}_{E_1'} + \text{Id} \otimes F(E_1'),$$

$$F(C_1) = \left( -F(E_2')^t \otimes \text{Id}_{E_1'} + \text{Id}_{E_1'} \otimes F(E_1') - F(E_2')^t \otimes \text{Id}_{E_2'} + \text{Id}_{E_2'} \otimes F(E_2') \right).$$

Actually, we have defined $C_0$ and $C_1$ as holomorphic bundles, but they admit parabolic structures in a natural way: given parabolic bundles $E$ and $F$, there are parabolic duals $E^{\otimes p}$ and parabolic tensor products $E \otimes^p F$ (see [Y1][GGM]). Then the parabolic structure on $\text{ParHom}(E, F)$ is given by $E^{\otimes p} \otimes^p F$. In the formulas for $F(C_0)$ and $F(C_1)$ we have to consider the adapted metrics for the parabolic structures on each $(E_j')^{\otimes p} \otimes^p E_i'$, induced by the adapted metrics on the bundles $E_k'$ and $E_k''$, for $k = 1, 2$.

Consider the homomorphism $a_2$ defined by

$$\text{ParHom}(E_1'', E_1')(-D) \xrightarrow{a_2} \text{ParHom}(E_1'', E_1') \oplus \text{ParHom}(E_2'', E_2')$$

$$\xi \rightarrow (\phi' \xi, \xi \phi'').$$

The connections on $C_0$ and $C_1$ satisfy

$$\sqrt{-1}AF(C_0) + a_1 a_1^* = (\tau_1' - \tau_2'') \text{Id}_{C_0}$$

$$\sqrt{-1}AF(C_1) - a_1^* a_1 + a_2 a_2^* = (\tau_1' - \tau_2'') \text{Id}_{C_1}.$$
Lemma 10.2. Let $K$ and $Q(D)$ denote the kernel and the torsion-free part of the cokernel, respectively, of the homomorphism $a_1$. Then
\[
\text{par}\mu(K) \leq \text{par}\mu_\sigma(T') - \text{par}\mu_\sigma(T''), \\
\text{par}\mu(Q) \geq \text{par}\mu_\sigma(T'') - \text{par}\mu_\sigma(T') + \sigma.
\]

Proof. The kernel $K$ is a subbundle of the hermitian bundle $C_1$, so that we may take the $C^\infty$ orthogonal splitting $C_1 = K \oplus S$. Since $K$ is a holomorphic subbundle, the induced connection $D_K$ on $K$ satisfies $D_{C_1}|_K = D_K + A$, where $D_{C_1}$ is the connection on $C_1$ and $A \in \Omega^{1,0}(\text{Hom}(K,S))$ is the second fundamental form of $K \subset C_1$. Therefore the curvature $F(K)$ of the connection on $K$ satisfies $F(C_1)|_K = F(K) + \bar{A}^t \land A$.

We now use the second equation in (26) restricted to $K$, take the trace and integrate on $X \setminus D$, to get
\[
\int_{X \setminus D} \text{Tr} (\sqrt{-1} \Lambda (F(K) + \bar{A}^t \land A) - a_1^* a_1|_K + a_2 a_2^*|_K) = \int_{X \setminus D} \text{Tr} ((\tau'_1 - \tau''_1) \text{Id}_{C_1}|_K).
\]
That is
\[
\text{par}\deg(K) + \|A\|^2_{L^2} + \int_{X \setminus D} \text{Tr} (a_2 a_2^*|_K) = (\tau'_1 - \tau''_1) \text{rk}(K),
\]
obtaining
\[
\text{par}\deg(K) \leq (\tau'_1 - \tau''_1) \text{rk}(K)
\]
as desired, since $\tau'_1 = \text{par}\mu_\sigma(T')$ and $\tau''_1 = \text{par}\mu_\sigma(T'')$.

To get the second inequality, let $S'(D)$ be the saturation of the image of $a_1$, which is holomorphic subbundle of $C_0(D)$. Then there is a $C^\infty$ orthogonal splitting $C_0 = S' \oplus Q$. The curvature of the induced connection on $Q$ satisfies $F(C_0)|_Q = F(Q) + B \land \bar{B}^t$ with $B \in \Omega^{0,1}(\text{Hom}(Q,S'))$. If we consider the first equation in (26) restricted to $Q$, take the trace and integrate, we get
\[
\int_{X \setminus D} \text{Tr} (\sqrt{-1} \Lambda (F(Q) + B \land \bar{B}^t) + a_1 a_1^*)|_Q = \int_{X \setminus D} \text{Tr} ((\tau'_2 - \tau''_2) \text{Id}_{C_0}|_Q).
\]
That is,
\[
\text{par}\deg(Q) - \|B\|^2_{L^2} = (\tau'_2 - \tau''_2)(\text{rk}(C_0) - \text{rk}(\text{im}(a_1))).
\]
Hence,
\[
(27) \quad \text{par}\deg(Q) \geq (\tau'_2 - \tau''_2)(\text{rk}(C_0) - \text{rk}(\text{im}(a_1))),
\]
as stated. $\square$

Theorem 10.3. Let $T'$ and $T''$ be $\sigma_c^\pm$-stable parabolic triples over a punctured Riemann surface of genus $g > 0$ such that $\text{par} \mu_{\sigma_c}(T') = \text{par} \mu_{\sigma_c}(T'')$ for $\sigma_c \geq 2g - 2$. Suppose that the morphism $a_1$ is not an isomorphism of bundles. Then
\[
\chi(C^*(T'', T')) < 0.
\]
Proof. We have
\[
\chi(C^\bullet(T'', T')) = (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + \text{deg}(C_1) - \text{deg}(C_0(D)) \\
\leq (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + \text{deg}(K) - \text{deg}(Q) \\
= (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + \text{deg}(K) - \text{deg}(Q(-D)(D)).
\]
(28)

Observe that for any (non-zero) parabolic bundle \(E, \text{deg}(E(D)) > \text{pardeg}(E) \geq \text{deg}(E)\), where the strict inequality is given by the fact that the weights on \(E\) always satisfy \(0 \leq \alpha_i(x) < 1\), for all \(i\) and all \(x \in D\). Using this, the hypothesis \(\sigma \geq 2g - 2\), and Lemma 10.2 we have
\[
\chi(C^\bullet(T'', T')) \leq (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + \text{pardeg}(K) - \text{pardeg}(Q(-D)) \\
= (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) - \sigma(\text{rk}(C_0(D)) - \text{rk}(\text{im}(a_1))) \\
\leq (1 - g)(\text{rk}(C_1) - \text{rk}(C_0)) + 2(1 - g)(\text{rk}(C_0) - \text{rk}(\text{im}(a_1))) \\
= (1 - g)(\text{rk}(C_1) + \text{rk}(C_0) - 2 \text{rk}(\text{im}(a_1))) \\
\leq 0,
\]
(29)

using that \(g \geq 1\). If either \(K\) or \(Q\) is a non-zero bundle, then the first line of (29) is a strict inequality. If both are zero and \(a_1\) is not an isomorphism, then the third line of (28) is a strict inequality since \(\text{im}(a_1) \neq C_0(D)\). In both cases,
\[
\chi(C^\bullet(T'', T')) < 0.
\]
\(\square\)

Remark 10.4. Note that this theorem does not cover the case \(g = 0\). This is not so surprising if we recall that, in order for parabolic bundles to exist on \(\mathbb{P}^1\), the parabolic weights must satisfy certain inequalities ([Bis] [Bel]). Presumably, something similar must be true also in the case of parabolic \(U(p, q)\)-Higgs bundles.

The following result will be useful in the next sections.

Lemma 10.5. If \(a_1\) is generically an isomorphism of bundles, then either

(a) \(E''_1 = 0\) and \(\phi' : E'_2 \rightarrow E'_1\) is generically an isomorphism. In this case, \(r_2 > r_1\).
(b) \(E''_2 = 0\) and \(\phi'' : E''_2 \rightarrow E''_1\) is generically an isomorphism. In this case, \(r_2 < r_1\).

Proof. One may look at a generic point \(x \in X \setminus D\), i.e., a point where the maps \(\phi'\) and \(\phi''\) are generic. We have
\[
(a_1)_x : \text{ParHom}(C^{r'_1}, C^{r'_1}) \oplus \text{ParHom}(C^{r'_2}, C^{r'_2}) \rightarrow \text{ParHom}(C^{r''_2}, C^{r''_1}) \\
(\alpha, \beta) \mapsto \phi'_x \beta - \alpha \phi''_x.
\]

If \(\phi''_x\) is not surjective, take \(\beta = 0\) and \(\alpha \neq 0\) with \(\alpha|_{\text{im}(\phi''_x)} = 0\). Then \((a_1)_x(\alpha, \beta) = 0\). If \(\phi'_x\) is not injective, take \(\alpha = 0\) and \(\beta \neq 0\) with \(\text{im}(\beta) \subset \ker(\phi'_x)\), to get \((a_1)_x(\alpha, \beta) = 0\). Both possibilities contradict the injectivity of \((a_1)_x\). Therefore \(\phi''_x\) is surjective and \(\phi'_x\) is injective.

If neither of \(\phi'_x\) and \(\phi''_x\) is an isomorphism, then take a map \(C^{r''_2} \rightarrow C^{r'_1}\) which induces a non-zero map \(\ker(\phi''_x) \rightarrow \text{coker}(\phi'_x)\). This cannot be in the image of \((a_1)_x\), contradicting our assumption. So either \(\phi'_x\) or \(\phi''_x\) are isomorphisms. In the first case \(r'_1 r'_1 + r'_2 r'_2 = r''_2 r'_1\) gives \(r'_1 = 0\) and we are in case (a). In the second, we are in case (b). \(\square\)
11. Irreducibility of the moduli space of triples for \( r_1 \neq r_2 \)

This section is devoted to study the irreducibility and non-emptiness of the moduli space of \( \sigma \)-stable parabolic triples for ranks \( r_1 \neq r_2 \).

Given a triple \( T = (E_1, E_2, \phi) \) one has the dual triple \( T^* = (E_2^{\text{op}}, E_1^{\text{op}}, \phi^t) \) where \( E_i^{\text{op}} \) is the parabolic dual of \( E_i \) and \( \phi^t \) is the dual of \( \phi \).

**Proposition 11.1.** The \( \sigma \)-stability of \( T \) is equivalent to the \( \sigma \)-stability of \( T^* \). The map \( T \mapsto T^* \) defines an isomorphism of the corresponding moduli spaces of \( \sigma \)-stable triples.

This allows us to restrict to the case \( r_1 > r_2 \) and appeal to duality for the case \( r_1 < r_2 \). So throughout this section we assume that \( r_1 > r_2 \).

**Lemma 11.2.** Let \( X \) be a Riemann surface with a finite number of marked points and let \( E, F \) be parabolic bundles on \( X \). Let \( p \in X \) be a parabolic point. Then there is a natural exact sequence

\[
0 \to \frac{\text{Hom}(E_p, F_p)}{\text{ParHom}(E_p, F_p)} \otimes \mathcal{O}(-p) \to \text{ParHom}(E, F)_p \to \text{ParHom}(E_p, F_p) \to 0.
\]

The second map is induced by restriction to \( p \). The first map is multiplication by a holomorphic function vanishing once at \( p \).

**Proof.** We have a defining exact sequence for the bundle of parabolic homomorphisms from \( E \) to \( F \) given by

\[
0 \to \text{ParHom}(E, F) \to \text{Hom}(E, F) \to \bigoplus_{x \in D} \text{Hom}(E_x, F_x) \to 0.
\]

Now we tensor with the skyscraper sheaf \( \mathbb{C}(p) \), to get

\[
0 \to \text{Tor} \left( \frac{\text{Hom}(E_p, F_p)}{\text{ParHom}(E_p, F_p)}, \mathbb{C}(p) \right) \to \text{ParHom}(E, F)_p \to \text{Hom}(E, F)_p \to \frac{\text{Hom}(E_p, F_p)}{\text{ParHom}(E_p, F_p)} \to 0.
\]

This is because \( \text{Tor} \left( \frac{\text{Hom}(E_x, F_x)}{\text{ParHom}(E_x, F_x)}, \mathbb{C}(p) \right) = 0 \) for \( p \neq x \), and the fact that if \( \Theta \) is a torsion sheaf supported scheme-theoretically at \( p \) (i.e., supported at \( p \) and with no infinitesimal information), we have that \( \text{Tor}(\Theta, \mathbb{C}(p)) \cong \Theta \otimes \mathcal{O}(-p) \) naturally (to see this, tensor the exact sequence \( \mathcal{O}(-p) \to \mathcal{O} \to \mathbb{C}(p) \) with \( \Theta \)). Hence

\[
0 \to \frac{\text{Hom}(E_p, F_p)}{\text{ParHom}(E_p, F_p)} \otimes \mathcal{O}(-p) \to \text{ParHom}(E, F)_p \to \text{Hom}(E_p, F_p) \to \frac{\text{Hom}(E_p, F_p)}{\text{ParHom}(E_p, F_p)} \to 0,
\]

which yields

\[
0 \to \frac{\text{Hom}(E_p, F_p)}{\text{ParHom}(E_p, F_p)} \otimes \mathcal{O}(-p) \to \text{ParHom}(E, F)_p \to \text{ParHom}(E_p, F_p) \to 0.
\]

Locally, with a local coordinate \( z \) vanishing at \( p \), the second map is given by \((f_0 + f_1 z + \cdots)_p \mapsto f_0 + f_1 z \). The first map is \( f_1 \mapsto (f_1 z)_p \). \( \square \)
To clarify the Lemma, let us see an example, where $E$ has rank 3 and weights $\beta_i$, $F$ has rank 4 and weights $\alpha_j$, and $\beta_1 < \alpha_1 < \alpha_2 < \alpha_3 < \beta_2 < \beta_3 < \alpha_4$. Then a typical parabolic homomorphism from $E$ to $F$ has matrix of the form:

$$
\phi(z) = \begin{pmatrix}
\phi_{11}(z) & \phi_{12}(z) & \phi_{13}(z) \\
\phi_{21}(z) & \phi_{22}(z) & \phi_{23}(z) \\
\phi_{31}(z) & \phi_{32}(z) & \phi_{33}(z) \\
\phi_{41}(z) & \phi_{42}(z) & \phi_{43}(z)
\end{pmatrix}
$$

around $p$. The parabolicity of $\phi$ means that for $z = 0$, the only non-zero entries are those below the broken line. The line in the matrix is easy to construct: starting by the upper-left corner, draw a horizontal line for each $\beta_j$, and a vertical line for each $\alpha_i$, considering the $\alpha$’s and $\beta$’s in increasing order. The sheaf $\text{ParHom}(E, F)$ is actually a bundle (since it is torsion-free) of rank $\text{rk}(E)\text{rk}(F)$. Its stalk at $p$, $\text{ParHom}(E, F)_p$, is formed by the matrices with entries which are complex numbers below the broken line, and which are complex numbers times $z$ above the line.

**Proposition 11.3.** Assume that $g > 0$, $\sigma_e \geq 2g - 2$ and $r_1 > r_2$. Let $T', T''$ be $\sigma_e^\pm$-stable triples with $\mu_{\sigma_e}(T') = \mu_{\sigma_e}(T'')$. Then $\chi(C^\bullet(T'', T')) = 0$ if and only if the following conditions hold:

1. $E_2' = 0$.
2. $\phi'' : E_2'' \to E_1''(D)$ is a fibre bundle isomorphism at $X \setminus D$. In particular, $r_2'' = r_1''$.
3. At any point $p \in D$, write $\phi'' = z^{-1}(\phi_0 + \phi_1 z + \phi_2 z^2 + \cdots)$, where $z$ is a local holomorphic coordinate around $p$ in $X$. Then $\text{ParHom}(E_{1,p}', E_1') \to \text{ParHom}(E_{2,p}', E_1')$, $f \mapsto -f \circ \phi_0$, is surjective.
4. At any $p \in D$, consider the induced homomorphism $\phi_1 : \ker \phi_0 \to \text{coker} \phi_0$. Then $\text{ParHom}(\text{coker} \phi_0, E_{1,p}') \to \ker(\phi_0, E_{1,p}')$, $f \mapsto -f \circ \phi_1$, is surjective.

**Proof.** By Theorem 10.3, $\chi(C^\bullet(T'', T')) = 0$ if and only if $a_1$ is an isomorphism. By Lemma 10.5 if $a_1$ is generically an isomorphism and $r_1 > r_2$ then $E_2' = 0$. This proves (1). Also $\phi' : E_2' \to E_1'(D)$ is generically an isomorphism. Moreover the two bundles involved in the complex $C^\bullet(T'', T')$ must be of the same rank and of the same degree. The complex $C^\bullet(T'', T')$ reduces to

$$
\text{ParHom}(E_{1}', E_1') \xrightarrow{a_1} \text{ParHom}(E_2', E_1'(D)),
$$

where $a_1(f) = -f \circ \phi''$ is an isomorphism of bundles. Restricting $a_1$ to the open subset $U = X \setminus D$, we have that $\text{Hom}(E_1', E_1')|U \to \text{Hom}(E_2', E_1'(D))|U$ is an isomorphism. Hence $E_2''|U \to E_1''(D)|U$ is an isomorphism of bundles, and (2) follows.

Now let $p \in D$, take a neighbourhood $U$ of $p$, and a coordinate $z$ vanishing at $p$. Hence we may write $\phi'' = \phi_0 z^{-1} + \phi_1 + \phi_2 z + \cdots$, where $\phi_i \in \text{Hom}(E_{2,p}', E_{1,p}')$ and $\phi_0 \in \text{ParHom}(E_{2,p}', E_{1,p}')$, on $U$. We want to characterize when

$$
\text{ParHom}(E_{1,p}', E_1') \to \text{ParHom}(E_2, E_1'(D))_p = \text{ParHom}(E_{2,p}', E_{1,p}'),
$$
is an isomorphism of vector spaces. It is enough to analyze when this map is surjective. Using Lemma 11.2, we have a commutative diagram whose rows are short exact sequences:

$$
\begin{array}{c}
\text{Hom}(E''_1, E'_1) \otimes \mathcal{O} \rightarrow \text{ParHom}(E''_1, E'_1) \\
\downarrow b_0 \quad \downarrow b_1 \quad \downarrow b_2 \\
\text{ParHom}(E''_1, E'_1) \rightarrow \text{ParHom}(E''_2, E'_1(p)) \rightarrow \text{ParHom}(E''_2, E'_1(p)) \otimes \mathcal{O}.
\end{array}
$$

The middle vertical arrow is induced by $f \mapsto -f \circ \phi''$. Thus the right vertical arrow is induced by $f_0 \mapsto -(f_0 \circ \phi_0)z^{-1}$. The left vertical arrow is thus given by $f_1 \mapsto -(f_1 \circ \phi_0)z^{-1}$.

We want to characterize the cases where the middle vertical arrow is surjective. Using the long exact sequence produced by the snake lemma, we see that $b_1$ being surjective is equivalent to $b_2$ being surjective and the connecting homomorphism $\ker b_2 \rightarrow \coker b_0$ also being surjective. The condition that $b_2$ is surjective is exactly (3).

For the remaining condition, we need to spell out the connecting homomorphism. Take $f_0 \in \text{ParHom}(E''_1, E'_1)$ lying in

$$
\ker b_2 = \text{ParHom}(E''_1/\phi_0(E''_2), E'_1).
$$

Lift $f_0$ to a local section of $\text{ParHom}(E''_1, E'_1)$ on $U$, e.g. taking $f(z) \equiv f_0$. Compose with $\phi''$ to get $-(f \circ \phi_0 + f \circ \phi_1 z + \cdots )z^{-1}$. Recalling that $f \circ \phi_0 = 0$, the leading term is

$$
-f_0 \circ \phi_1 \in \coker b_0 = \text{Hom}(E''_2, E'_1)/b_0(\text{Hom}(E''_1, E'_1)).
$$

Assuming that (3) holds already, we have that $\text{ParHom}(E''_2, E'_1) \subset b_0(\text{ParHom}(E''_1, E'_1)) \subset b_0(\text{Hom}(E''_1, E'_1))$, since the maps $b_0$ and $b_2$ are both composition with $\phi_0$. Hence the image of $f_0$ under the connecting homomorphism is

$$
-f_0 \circ \phi_1 \in \coker b_0 = \text{Hom}(E''_2, E'_1)/b_0(\text{Hom}(E''_1, E'_1)) = \text{Hom}(\ker \phi_0, E'_1).
$$

Therefore the surjectivity of the connecting homomorphism is equivalent to (4).

**Lemma 11.4.** Condition (4) of Proposition 11.3 holds if and only if all the weights of $E'_1$ are bigger than those of $\coker \phi_0$, and $\phi_1 : \ker \phi_0 \rightarrow \coker \phi_0$ is an isomorphism.

**Proof.** The condition (4) says that

$$
\text{ParHom} \left( \frac{E''_1}{\phi_0(E''_2)}, E'_1 \right) \rightarrow \text{Hom}(\ker \phi_0, E'_1), \quad f \mapsto -f \circ \phi_1,
$$

is surjective. Since $E''_1/\phi_0(E''_2)$ and $\ker \phi_0$ are vector spaces of the same dimension, this is equivalent to the following two conditions:

- $\phi_1 : E''_1 \rightarrow E''_1/\phi_0(E''_2)$ satisfies that $\phi_1 : \ker \phi_0 \rightarrow \coker \phi_0$ is an isomorphism.
- $\text{ParHom}(E''_1/\phi_0(E''_2), E'_1) = \text{Hom}(E''_1/\phi_0(E''_2), E'_1)$. Hence all the weights of $E''_1/\phi_0(E''_2)$ are smaller than those of $E'_1$.

□
Let $\sigma_c \in (\sigma_m, \sigma_M)$ be a critical value with $\sigma_c \geq 2g - 2$. We aim to characterize when $N_s^{\sigma_c}$ and $N_s^{\sigma_c}$ are birational by using Proposition 9.7. Let us deal with either of $S_{\sigma_c}^{\pm}$. Suppose that $T'$ and $T''$ are $\sigma_c$-semistable, $\sigma_c$-stable triples with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T'')$. We consider extensions

\[ 0 \to T'' \to T \to T' \to 0 \tag{30} \]

(note that we have changed the role of $T'$ and $T''$ in the computation of the codimension of the flip loci in Section 10 so that now $T''$ is the subtriple), where $\mu_{\sigma_c}(T'') < \mu_{\sigma_c}(T)$, by Proposition 9.7. The first conclusion to infer from Proposition 11.3 is that, if $\chi(C^*(T'', T')) = 0$ then $r''_2 = 0$ and $r''_2 = r'_1$. So $\mu_{\sigma_c}(T'') > \mu_{\sigma_c}(T)$. Therefore $S_{\sigma_c}$ cannot be of zero codimension. So our study is limited to $S_{\sigma_c}$: the only situation we may encounter when $\chi(C^*(T'', T')) = 0$ is that $N_s^{\sigma_c}$ has more irreducible components than $N_s^{\sigma_c}$.

To analyze when $\chi(C^*(T'', T')) = 0$ we have to check when conditions (3) and (4) of Proposition 11.3 are satisfied. Let $p \in D$ be a parabolic point. We need to understand the parabolic vector spaces $E_{2,p}$ and $E_{1,p}$. These have parabolic weights of multiplicity one and all weights are different, by Assumption 7.8. We shall keep the following notation for the rest of the section: $\alpha_i$ denote the weights of $E_{1,p}$ and $\beta_j$ denote the weights of $E_{2,p}$ (we drop $p$ from the notation in the weights when this causes no confusion).

Since $T$ is a triple which is an extension (31) with $r'_2 = 0$ and $r''_2 = r'_1$, then $\phi : E_2 \to E_1(D)$ comes from a map $\phi'' : E_2 \to E''_1(D)$ as follows

\begin{align*}
E_2 & \xrightarrow{\phi''} E_1(D) \\
\phi'' & \downarrow \phi \downarrow \downarrow \\
E'_1(D) & \to E_1(D) \to E'_1(D).
\end{align*}

Take a neighbourhood $U$ of $p$ where $E_1|_U = E'_1|_U \oplus E''_1|_U$. Then $\phi = (\phi_0 + \phi_1 z + \cdots)z^{-1}$ and $\phi_0 : E_{2,p} \to E''_{1,p}$ is a parabolic map. This gives decompositions of the parabolic vector spaces

\[ E_{1,p} = E'_{1,p} \oplus E''_{1,p}, \]

\[ E''_{1,p} = \text{im} \phi_0 \oplus \text{coker} \phi_0, \tag{31} \]

as direct sums of parabolic vector subspaces (the splitting is non-canonical, but the weights of the different subspaces are well-determined).

Let us see that there is a “canonical” distribution of weights in (31) such that conditions (3) and (4) hold. Note that $\text{ParHom}(E_{2,p}, E_{1,p})$ is a vector space, in particular an irreducible affine variety. We may consider the action of $\text{ParAut}(E_{2,p}) \times \text{ParAut}(E_{1,p})$ on this space (this corresponds to lower triangular changes of bases). Then there is a unique open dense orbit, which is the only orbit of maximal dimension. We shall call an element of such orbit a generic parabolic homomorphism of $E_{2,p}$ to $E_{1,p}$. For instance, if $E_{2,p}$ is 7-dimensional with weights $\beta_j$ and $E_{1,p}$ is 9-dimensional with weights $\alpha_i$, and

\[ \alpha_1 < \beta_1 < \beta_2 < \alpha_2 < \beta_3 < \beta_4 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6 < \beta_5 < \alpha_7 < \alpha_8 < \beta_6 < \beta_7 < \alpha_9, \]
then the generic elements are the orbit of the element

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

(32)

Lemma 11.5. Suppose that \( \phi_0 : E_{2,p} \to E_{1,p} \) is a generic parabolic homomorphism, and let \( E_{1,p} = E'_{1,p} \oplus E''_{1,p} \) be any parabolic splitting with \( \text{im} \phi_0 \subset E''_{1,p} \). Then condition (3) in Proposition 11.3 is satisfied.

Proof. Suppose that \( \phi_0 \) is a generic element in \( \text{ParHom} (E_{2,p}, E_{1,p}) \), and let us see that the map \( \text{ParHom} (E''_{1,p}, E'_{1,p}) \to \text{ParHom} (E_{2,p}, E'_{1,p}) \), \( f \mapsto -f \circ \phi_0 \), is surjective. Take \( g \in \text{ParHom} (E_{2,p}, E'_{1,p}) \). Consider the map \( \phi_c = \phi_0 \oplus cg : E_{2,p} \to E''_{1,p} \oplus E'_{1,p} \). For \( \epsilon \) small we have that \( \phi_c \) also lives in the generic open set, so it is equivalent to \( \phi_0 \) by the action of \( \text{ParAut} (E_{2,p}) \times \text{ParAut} (E_{1,p}) \). This means that

\[
\begin{pmatrix}
a_e & b_e \\
c_e & d_e
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
\phi_0 \\
\epsilon g
\end{pmatrix}.
\]

Both matrices, \( \begin{pmatrix} a_e & b_e \\ c_e & d_e \end{pmatrix} \) and \( M_\epsilon \), are the identity for \( \epsilon = 0 \), so \( a_\epsilon \) is invertible for small \( \epsilon \). Therefore \( \phi_0 M_\epsilon = a_\epsilon^{-1} \phi_0 \) and \( c_\epsilon \phi_0 M_\epsilon = \epsilon g \). This yields

\[
g = \epsilon^{-1} c_\epsilon a_\epsilon^{-1} \phi_0,
\]

as required. \( \square \)

Recall that we have fixed topological data (fixed ranks, degrees and parabolic weights) for the triples \( T \) we are studying. When we write such a triple \( T \) as an extension \( T'' \to T \to T' \), there are different possible topological types for \( T' \) and \( T'' \). By the above discussion, our best chance to obtain \( \chi(C^\bullet(T', T'')) = 0 \) is to arrange the topological types as follows:

- Fix the ranks \( r'_2 = 0 \), \( r''_2 = r_2 \), \( r''_1 = r_2 \), \( r'_1 = r_1 - r_2 \). This is necessary for conditions (1) and (2) to hold. So \( \phi : E_2 \to E_1(D) \) should be induced by \( \phi'' : E''_2 \to E''_1(D) \) by means of the inclusion \( E''_1(D) \to E_1(D) \).
- At each \( p \in D \), consider a generic element \( \phi_p \in \text{ParHom} (E_{2,p}, E_{1,p}) \). This determines the weights of \( \text{im} \phi_p \subset E''_{1,p} \). By Lemma 11.3 condition (3) is satisfied.
- Choose the weights of \( \text{coker} \phi''_p \) in the unique way such that Lemma 11.3 is satisfied.
- This gives the weights of \( E''_{1,p} = \text{im} \phi_p \oplus \text{coker} \phi''_p \) at each \( p \in D \), and hence the weights of \( E'_{1,p} \).
- \( d''_2 = d_2 \). Now condition (2) determines the degree of \( E'_1 \), since the map \( \phi'' : E_2 \to E''_1(D) \) is an isomorphism on \( X \setminus D \) and it is of a specified form at each \( p \in D \). Namely,
introduce the number

\[ r_p = \min \{ \dim \text{coker } \psi_0 \mid \psi_0 \in \text{ParHom} (E_{2,p}, E_{1,p}) \} - (r_1 - r_2). \]

Obviously this minimum is obtained for a generic parabolic morphism. Moreover \( r_p = \dim \text{coker } \phi_0 \), where \( \phi_0 : E_{2,p} \rightarrow E_{1,p} \) is generic, and \( \phi_0 = \phi_0 : E_{2,p} \rightarrow E_{1,p}^\prime \), using that \( E_{1,p}^\prime \subset E_{1,p} \). With this notation, \( E_2 \rightarrow E_1^\prime (D) \rightarrow \oplus_{p \in D} C(p)^{r_p} \) is an exact sequence of sheaves, so \( d_i^p = d_2 - r_2 s + \sum_{p \in D} r_p \).

This does not guarantee the existence or uniqueness of the topological types of \( T' \) and \( T'' \) to have \( \chi(C^* (T', T'')) = 0 \), but helps us in which direction to look for such distributions of topological types.

Let us see this discussion in the particular example \( \text{(32)} \). For a generic \( \phi_p : E_{2,p} \rightarrow E_{1,p} \), the weights of \( \text{im } \phi_0 \) are \( \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_9 \), and the weight of \( \text{coker } \phi_0 \) is \( \alpha_1 \). Thus the weights of \( E_{1,p}^\prime \) are \( \alpha_6, \alpha_8 \). The map \( \phi \) takes the form:

\[
\phi = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & z & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} + O(z^2) \cdot z^{-1},
\]

around \( p \in D \). Note that such \( \phi : E_2 \rightarrow E_1 (D) \) is injective for \( z \neq 0 \), as required by condition (2).

**Remark 11.6.** The definitions of generic parabolic map and of \( r_p \) given in \( \text{(33)} \) are also valid in the case \( r_1 = r_2 \).

**Proposition 11.7.** Assume \( g > 0 \), \( r_1 > r_2 \) and \( \sigma_c \geq 2g - 2 \). Let \( T', T'' \) be \( \sigma_c^- \)-stable triples with \( \mu_{\sigma_c} (T') = \mu_{\sigma_c} (T'') \). If \( \chi(C^* (T'', T')) = 0 \) then the following holds:

(i) \( r_1' = 0, r_2'' = r_2 = r_1', d_2'' = d_2 \).

(ii) For each \( p \in D \), the parabolic map \( \phi''_p : E_{2,p} \rightarrow E_{1,p}'' \) has rank \( r_2 - r_p \), with \( r_p \) defined in \( \text{(33)} \).

(iii) \( d_i'' = d_2 - r_2 s + \sum_{p \in D} r_p \).

(iv) For each \( 1 \leq k \leq r_2 - r_p \), define

\[ i_k = \min \{ j \mid 1 \leq j \leq r_1, \beta_k < \alpha_j, j > i_{k-1} \}. \]

and let \( I = \{ i_1, \ldots, i_{r_2-r_p} \} \). Let \( J \subset \{1, \ldots, r_1\} - I \) be the set of the lowest \( r_p \) elements of \( \{1, \ldots, r_1\} - I \). Then the weights of \( E_{1,p}'' \) are exactly \( \{ \alpha_i \mid i \in I \cup J \} \).

In particular, the ranks, degrees and weights of \( T' \) and \( T'' \) are univocally determined. Thus there is at most one possible value of \( \sigma_c \) for which \( \chi(C^* (T', T'')) = 0 \).

**Proof.** Item (i) follows from Proposition \( \text{[11.3]} \) (1).
Item (iii) follows once we know item (ii) and using Proposition 11.3 (2), since in this case we have an exact sequence of sheaves

\[ E_2 = E''_2 \xrightarrow{\phi''} E''_1(D) \to \bigoplus_{p \in D} C(p)^{\tau_p}. \]

Next, note that the increasing sequence of numbers \( i_1, i_2, \ldots \in \{1, \ldots, r_1\} \) is well-defined for \( 1 \leq k \leq r_2 - r_p \). Actually, looking at a generic parabolic map \( \psi_0 : E_{2,p} \to E_{1,p} \), the weights of \( \text{im} \ \psi_0 \) are \( \alpha_{i_1}, \ldots, \alpha_{i_2 - r_p} \), with \( r_2 - r_p = \dim \psi_0 \) (see [32] for a specific example).

Now we shall prove (ii) and (iv) using Proposition 11.3 (3), i.e., that

\[ \text{ParHom} (E''_{1,p}, E'_{1,p}) \to \text{ParHom} (E_{2,p}, E'_{1,p}) \]

(35)

is surjective, denoting as before, \( \phi_0 = \phi''_{1,p} \). Let \( \{e_1, \ldots, e_{r_1}\} \) be a basis for \( E_{1,p} \) adapted to its parabolic structure (and adapted to the splitting \( E''_{1,p} \oplus E'_{1,p} \), i.e. each \( e_i \) belongs either to \( E''_{1,p} \) or \( E'_{1,p} \)), and let \( \{v_1, \ldots, v_{r_2}\} \) be a basis for \( E_{2,p} \) adapted to its parabolic structure.

Now let \( t_0 \in \{1, \ldots, r_1\} \) such that \( \alpha_{t_0} \) is the lowest weight of \( E'_{1,p} \). Let \( 0 \leq a \leq r_2 - r_p \) such that \( i_a < t_0 \leq i_{a+1} \) (introducing the notation \( i_0 = 0, i_{r_2-r_p+1} = r_1 + 1 \)). Let us see that \( \alpha_{i_{a+1}}, \ldots, \alpha_{i_2 - r_p} \) are weights of \( \text{im} \ \phi_0 \) (if \( a = r_2 - r_p \) then there is nothing to prove). Actually, they cannot be weights of \( \text{coker} \ \phi_0 \), since by Lemma 11.4 all the weights of \( \text{coker} \ \phi_0 \) are smaller than \( \alpha_{t_0} \). So they are weights of \( \text{im} \ \phi_0 \) or of \( E'_{1,p} \) by (31). Suppose that \( \alpha_{i_{a+1}}, \ldots, \alpha_{i_{b-1}} \) are weights of \( \text{im} \ \phi_0 \) but \( \alpha_{t_0} \) is the first weight of \( E'_{1,p} \) in the list. Then take \( V = \langle v_1, \ldots, v_a \rangle \subset E_{2,p} \).

The surjectivity of (35) gives that

\[ \text{ParHom} (E''_{1,p}, \langle e_{i_0} \rangle) \to \text{ParHom} (E_{2,p}, \langle e_{i_0} \rangle) \to \text{ParHom} (V, \langle e_{i_0} \rangle) = \text{Hom} (V, \langle e_{i_0} \rangle) \]

is surjective (the last equality follows from \( \alpha_{t_0} > \beta_0 \)). Therefore \( \phi_0 \mid_V : V \to E''_{1,p} \) must be injective, and all the weights or \( \phi_0 (V) \subset E''_{1,p} \) should be smaller than \( \alpha_{t_0} \). So there are weights \( \alpha_{x_1} < \ldots < \alpha_{x_b} < \alpha_{t_0} \) with \( \beta_j < \alpha_{x_j} \). This implies that \( i_j \leq x_j, j = 1, \ldots, b \), which contradicts that \( x_b < i_t \).

Next step is to see that there are \( y_1 < \ldots < y_a < t_0 \) such that \( i_j \leq y_j, j = 1, \ldots, a \) and \( \alpha_{y_j} \) are weights of \( \text{im} \ \phi_0 \). As before, take \( V = \langle v_1, \ldots, v_a \rangle \subset E_{2,p} \). The surjectivity of (35) gives that

\[ \text{ParHom} (E''_{1,p}, \langle e_{i_0} \rangle) \to \text{ParHom} (E_{2,p}, \langle e_{i_0} \rangle) \to \text{ParHom} (V, \langle e_{i_0} \rangle) = \text{Hom} (V, \langle e_{i_0} \rangle) \]

is surjective. So \( \phi_0 \mid_V : V \to E''_{1,p} \) must be injective, and all the weights or \( \phi_0 (V) \subset E''_{1,p} \) should be smaller than \( \alpha_{t_0} \). So there are weights \( \alpha_{y_1} < \ldots < \alpha_{y_a} < \alpha_{t_0} \) with \( \beta_j < \alpha_{y_j} \). This implies that \( i_j \leq y_j, j = 1, \ldots, a \).

The elements

\[ \{y_1, \ldots, y_a, i_{a+1}, \ldots, i_{r_2 - r_p}\} \]

are weights of \( \text{im} \ \phi_0 \). So \( \dim \text{im} \ \phi_0 \geq r_2 - r_p \). As obviously \( \dim \text{im} \ \phi_0 \leq r_2 - r_p \), it must be \( \dim \text{im} \ \phi_0 = r_2 - r_p \), implying item (ii). Thus the weights of \( \text{im} \ \phi_0 \) are exactly those in (36).

The elements

\[ \{1, \ldots, t_0 - 1\} - \{y_1, \ldots, y_a\} \]
are the sub-indices of the weights of \( \text{coker } \phi_0 \), by Lemma \[11.4\]. So \( t_0 - 1 - a = r_p \), i.e. \( t_0 = r_p + a + 1 \). Finally (the sub-indices of) the weights of \( E_{1.p}' \) are

\[
\{(1, \ldots, t_0 - 1) - \{y_1, \ldots, y_a\} \} \cup \{y_1, \ldots, y_a; i_{a+1}, \ldots, i_{r_2 - r_p}\} = \\
\{1, \ldots, t_0 - 1\} \cup \{i_{a+1}, \ldots, i_{r_2 - r_p}\} = I \cup J,
\]
as required. □

Our final result in this section completes the proof of Theorem \[6.12\]. We have to use Theorem \[12.10\] which will be proven in the next section. First, consider the distribution of weights and degrees given by Proposition \[11.7\], and consider the critical value associated to it, which is \( 12.10 \), which will be proven in the next section. First, consider the distribution of weights and degree of \( E_{1.2}'' \). Consider the triples \( \tau_0 \), and \( \sigma > \sigma \) is non-empty for some value of \( C \) and \( N_2' \) is very close to \( \tau_0 \). Now the \( \tau_0 \) and \( \sigma \) are birational, by Proposition \[9.7\]. So all moduli spaces \( N_s' = N_s'' \) of \( N_2' \) is very close to \( \tau_0 \). Thus, the moduli spaces \( N_s' \) and \( N_s'' \) are birational, by Proposition \[9.7\]. So all moduli spaces \( N_s' \) are birational for \( 2g - 2 \leq \sigma < \sigma_L \).

Moreover there may be different distributions of weights, ranks and degrees giving rise to the critical value \( \sigma_L \), but only the one given by Proposition \[11.7\] gives critical subsets \( S_{\sigma_L} \) of codimension zero. So the number of irreducible components is given by the number of irreducible components of a subset of the space of extensions \( T'' \rightarrow T \rightarrow T' \) with the distribution of weights, ranks and degrees given by Proposition \[11.7\]. Let us see that this space of extensions is non-empty and irreducible: the triples \( T'' \) have \( r_2'' = 0, r_1' = r_1 - r_2, \) so they are parametrized by a moduli space of parabolic bundles \( E_1' \), which is non-empty, irreducible and of the expected dimension by \[BY2\]. The triples \( T'' \) have \( r_1'' = r_2'' = r_2, \) and \( d_1'' + r_2''s - d_2'' - \sum r_p = 0, \) so they are parametrized by a moduli space of \( \sigma_{<}L \)-stable triples which is non-empty, irreducible and of the expected dimension by Theorem \[12.10\]. Now the
dimension of the projective fibres of the space of extensions $T'' \rightarrow T \rightarrow T'$ is
\[-\chi(C^*(T', T'')) - 1 \geq 0,
\]
since $\chi(C^*(T', T'')) < 0$, by Theorem 10.3. Therefore there is a non-empty space of extensions. Moreover, a generic triple $T'$ is $\sigma_L$-stable. In that case, any non-trivial extension $T'' \rightarrow T \rightarrow T'$ is $\sigma_L$-stable (see Proposition 9.4). So the space $S_{\sigma_L}$ is non-empty, and irreducible.

Finally, if $\sigma_L > 2g - 2$, the argument above proves that $N_{\sigma_L}^*$ is non-empty, so there is some non-empty $N_{\sigma}$ with $\sigma > 2g - 2$ and the statement of the theorem follows. Conversely, if some $N_{\sigma}$ with $\sigma > 2g - 2$ is non-empty, then it must be $\sigma_L > 2g - 2$ completing the argument. □

Now Proposition 7.7 transfers the inequalities $\sigma_m \leq 2g - 2 < \sigma_L$ into a Milnor–Wood type inequality $0 \leq |\tau| < \tau_L$, where
\[(40) \quad \tau_L = \min\{p, q\}(2g - 2 + s) - \frac{|p - q|}{p + q} \epsilon,
\]
where $\epsilon$ is given in (39).

Remark 11.9. One can spell out the process for computing $\epsilon$, by using the procedure of Proposition 11.4 and the identification of Proposition 7.4. Let $p = \text{rk}(V)$, $q = \text{rk}(W)$, $\alpha$ the system of weights of $V$ and $\beta$ the system of weights of $W$. Suppose that $q \leq p$ (the other case is similar, interchanging the roles of $V$ and $W$). Define, at each $x \in D$, $\alpha_i + p l(x) = \alpha_i(x) + l$, for any $l \geq 1$. Put $i_0 = 0$ and define, for $1 \leq k \leq q$,
\[i_k = \min\{j \mid j > i_{k-1}, \alpha_j > \beta_k\}.
\]
Then
\[\epsilon = \sum_{x \in D} \sum_{k=1}^{p} (\alpha_{i_k}(x) - \beta_k(x)).
\]

12. THE MODULI SPACE OF TRIPLES FOR $r_1 = r_2$ AND LARGE $\sigma$

In this section, we study the moduli space of triples with equal ranks $r_1 = r_2$. We prove that some of them are irreducible and non-empty for $\sigma \geq 2g - 2$. The results here are enough for the proof of Theorem 11.8 to work, but we also analyze some other cases. It is likely that the result holds in general.

Proposition 12.1. Suppose that $r_1 = r_2$ and $g > 0$. Then all the moduli spaces $N_{\sigma}$, for $\sigma \geq 2g - 2$ are birational to each other.

Proof. This is a consequence of Theorem 10.3 and Proposition 9.7. For $\chi(C^*(T', T''))$ to vanish, it must be $a_1$ an isomorphism. But this is impossible if $r_1 = r_2$ by Lemma 10.3. □

Now let us see that the moduli spaces $N_{\sigma}$ stabilizes for $\sigma$ large.

Proposition 12.2. Suppose that $r_1 = r_2$. Then there is a value $\sigma_1$ such that any $\sigma$-stable parabolic triple $T = (E_1, E_2, \phi)$ with $\sigma > \sigma_1$ satisfies that $\phi$ is injective. Hence
\[(41) \quad 0 \rightarrow E_2 \rightarrow E_1(D) \rightarrow S \rightarrow 0,
\]
where $S$ is a torsion sheaf.
Proof. Denote $N = \ker \phi$ and consider the parabolic subtriple $(0, N, \phi)$. Suppose that $k = \text{rk}(N) > 0$. The $\sigma$-stability of $T$ implies that
\[
\text{pardeg } N + k\sigma < k \left( \frac{\text{pardeg } (E_1 \oplus E_2)}{2r_1} + \frac{1}{2} \sigma \right).
\]
Now consider the subtriple $(I, E_2, \phi)$ where $I(D)$ is the parabolic image sheaf of $\phi$, with rank $\text{rk}(I) = r_1 - k$. The $\sigma$-stability of $T$ gives us
\[
\text{pardeg } (I \oplus E_2) + r_1\sigma < (2r_1 - k) \left( \frac{\text{pardeg } (E_1 \oplus E_2)}{2r_1} + \frac{1}{2} \sigma \right).
\]
Adding up both equations, and noting that $\text{pardeg } E_1 \oplus E_2 = \text{pardeg } (E_1 \oplus E_2)$, we get
\[
2\text{pardeg } E_2 - (r_1 - k)s + (r_1 + k)\sigma < \text{pardeg } (E_1 \oplus E_2) + r_1\sigma,
\]
which is rewritten as
\[
\sigma < \frac{\text{pardeg } E_1 - \text{pardeg } E_2 + (r_1 - k)s}{k}.
\]
So for $\sigma_1 = \text{pardeg } E_1 - \text{pardeg } E_2 + (r_1 - 1)s$ the result follows. \(\square\)

Lemma 12.3. Suppose that $r_1 = r_2$ and $\sigma > \sigma_1$. Let $T$ be a $\sigma$-stable triple and $T'$ a subtriple of $T$ with $r_1' = r_2'$. Write $E_2 \rightarrow E_1(D) \rightarrow S$, $E_2' \rightarrow E_1'(D) \rightarrow S'$, $t = \text{length } S$, $t' = \text{length } S'$. Then
\[
\text{par}\mu(E_1') < \text{par}\mu(E_1) + \left( \frac{t'}{r_1'} - \frac{t}{r_1} \right) + s,
\]
\[
\text{par}\mu(E_2') < \text{par}\mu(E_2) - \left( \frac{t'}{r_1'} - \frac{t}{r_1} \right) + s.
\]
Proof. From Proposition 12.2 as $\sigma > \sigma_1$, $\phi$ is an injective morphism. So $\phi'$ is injective for any subtriple $T'$ of $T$. Hence for a subtriple $T'$ with $r_1' = r_2'$ we have the following commutative diagram
\[
\begin{array}{c}
\begin{array}{ccc}
0 & \longrightarrow & E_2' \\
& & \downarrow \\
0 & \longrightarrow & E_2' \\
\end{array} \\
\begin{array}{ccc}
E_1'(D) & \longrightarrow & S' \\
& & \downarrow \\
E_1(D) & \longrightarrow & S \\
\end{array} \\
\begin{array}{c}
0 \\
0
\end{array}
\end{array}
\]
where $S$ and $S'$ are torsion sheaves. Let $t$ and $t'$ denote the lengths of $S$ and $S'$ respectively, as in the statement. By stability,
\[
0 > \text{par}\mu_{\sigma}(T') - \text{par}\mu_{\sigma}(T)
= \frac{1}{2} \left( \text{par}\mu(E_1') + \text{par}\mu(E_2') - \text{par}\mu(E_1) - \text{par}\mu(E_2) \right)
= \text{par}\mu(E_1') - \text{par}\mu(E_1) - \frac{1}{2} (\text{par}\mu(E_1') - \text{par}\mu(E_2')) + \frac{1}{2} (\text{par}\mu(E_1) - \text{par}\mu(E_2))
= \text{par}\mu(E_2') - \text{par}\mu(E_2) + \frac{1}{2} (\text{par}\mu(E_1') - \text{par}\mu(E_2')) - \frac{1}{2} (\text{par}\mu(E_1) - \text{par}\mu(E_2)).
\]
Now at each point $p \in D$, $|\sum \beta_j(p) - \sum \alpha_i(p)| \leq r_1$, so $t - r_1s \leq \text{pardeg } E_1(D) - \text{pardeg } E_2 \leq t + r_1s$, equivalently $t - 2r_1s \leq \text{pardeg } E_1 - \text{pardeg } E_2 \leq t$ or
\[
\frac{t}{r_1} - 2s \leq \text{par}\mu(E_1) - \text{par}\mu(E_2) \leq \frac{t}{r_1}.
\]
Analogously, for $T'$ we have
\[ \frac{t}{r_1'} - 2s \leq \par\mu(E_1') - \par\mu(E_2') \leq \frac{t}{r_1'} \]
Substituting into the formulae above, we get the result in the statement.

Proposition 12.4. Suppose that $r_1 = r_2$. Then there is a value $\sigma_2 \geq \sigma_1$ such that $N^s_{\sigma_2} = N^s_{\sigma_1}$ for any $\sigma, \sigma' \geq \sigma_2$, i.e. there are no critical values above $\sigma_2$.

Proof. Consider a $\sigma$-stable triple $T = (E_1, E_2, \phi)$ with $\sigma > \sigma_1$. Suppose that $T$ is properly $\sigma_c$-semistable for some $\sigma_c$, and let $T' \subset T$ be a $\sigma_c$-destabilizing subtriple. Clearly $r_2' \leq r_1'$, since $\phi$ being injective implies that $\phi'$ is also injective. On the other hand, if $r_1' = r_2'$ then $T$ is $\sigma$-semistable for generic values of $\sigma$ and could not be $\sigma$-stable for some $\sigma$. Therefore $r_2' < r_1'$.

In the formula
\begin{equation}
(42) \quad \sigma_c = 2\par\mu(E_1') \frac{r_1'}{r_1' - r_2'} + 2\par\mu(E_2') \frac{r_2'}{r_1' - r_2'} - (\par\mu(E_1) + \par\mu(E_2)) \frac{r_1' + r_2'}{r_1' - r_2'}
\end{equation}
we want to bound the values of $\par\mu(E_1')$ and $\par\mu(E_2')$ in order to get a bound for the critical value $\sigma_c$ which is independent of $T$.

Apply Lemma 12.3 to the subtriples $(\phi'(E_1')(-D), E_2', \phi')$ and $(E_1', (\phi')^{-1}(E_1(D)), \phi')$, both of which satisfy the equal rank condition. The first one has no torsion, the second has torsion with $0 \leq t' \leq t$. We get
\[ \par\mu(E_2') < \par\mu(E_2) + \frac{t}{2r_1'} + s, \]
\[ \par\mu(E_1') < \par\mu(E_1) + \frac{1}{2} \left( \frac{t'}{r_1'} - \frac{1}{r_1} \right) + s \leq \par\mu(E_1) + \frac{t(r_1 - r_1')}{2r_1 r_1'} + s. \]
Using that $\frac{1}{r_1} \leq \par\mu(E_1) - \par\mu(E_2) + 2s$, by the exact sequence (11) and $1 \leq r_1' \leq r_1 - 1$, we get bounds on $\par\mu(E_1')$ and $\par\mu(E_2')$. Substituting these bounds into (42) and using that $r_1' - r_2' \geq 1$ and $r_1', r_2' \leq r_1 = r_2$, we get a bound on $\sigma_c$, as required.

With this result, we may introduce the notation $N^s_L$ for the moduli space of $\sigma$-stable triples for any value $\sigma > \sigma_2$. We shall refer to this as the moduli space for large values of $\sigma$. There is an obvious condition for $N^s_L$ to be non-empty. Let $\phi : E_2 \rightarrow E_1(D)$ be a parabolic morphism which is moreover injective. For any $p \in D$, it induces a parabolic map $\phi_p \in \text{ParHom}(E_{2,p}, E_{1,p})$. This satisfies
\[ \dim \phi_p \leq r_1 - r_p, \]
with $r_p$ defined in (33) (cf. Remark 11.6). Therefore for any parabolic map $\phi \in \text{ParHom}(E_2, E_1(D))$, we have that
\begin{equation}
(43) \quad d_1 + r_1 s - d_2 \geq \sum_{p \in D} r_p.
\end{equation}
Let us now see that this is a sufficient condition for non-emptiness and irreducibility of $N^s_L$.

First we need some preliminary results.

Lemma 12.5. If both $E_2$ and $E_1$ are parabolic stable bundles, and $\phi : E_2 \rightarrow E_1(D)$ is an injective parabolic map, then $T = (E_1, E_2, \phi)$ is a $\sigma$-stable triple for large values of $\sigma$. 
Proof. Any subtriple $T' \subset T$ should have $r'_2 \leq r'_1$. The stability of the bundles implies that $\text{par}_\mu(E'_1) < \text{par}_\mu(E_1)$ and $\text{par}_\mu(E'_2) < \text{par}_\mu(E_2)$, from where it follows that $\text{par}_\mu(T') < \text{par}_\mu(T)$, for any $\sigma$, in particular for large values of $\sigma$.

\begin{lemma}
Let $L$ be a fixed parabolic line bundle. Consider the moduli space $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha, \beta)$ of $\sigma$-stable parabolic triples $T = (E_1, E_2, \phi)$ of degrees $(d_1, d_2)$ and weight types $(\alpha, \beta)$. Let $(\tilde{d}_1, \tilde{d}_2)$ and $(\tilde{\alpha}, \tilde{\beta})$ the degrees and weight types of the triples of the form $(E_1 \otimes^p L, E_2 \otimes^p L, \phi)$. Then $(E_1, E_2, \phi) \mapsto (E_1 \otimes^p L, E_2 \otimes^p L, \phi)$ gives an isomorphism $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha, \beta) \cong \mathcal{N}_\sigma(r_1, r_2, \tilde{d}_1, \tilde{d}_2; \tilde{\alpha}, \tilde{\beta})$.
\end{lemma}

Let us see that tensoring with a suitable parabolic line bundle allows us to reduce to the case $r_p = 0$ for all $p \in D$. For this we need an alternative characterization of $r_p$. Fix $p \in D$, and denote by $\alpha_1 < \cdots < \alpha_{r_1}$ the weights of $E_{1,p}$ and by $\beta_1 < \cdots < \beta_{r_1}$ the weights of $E_{2,p}$, since $r_2 = r_1$. Extend the weights to an infinite sequence of real numbers by declaring $\alpha_{k+r_1m} = \alpha_k + m$, $1 \leq k \leq r_1$, $m \in \mathbb{Z}$. This means that we have a sequence

$$
\alpha_{r_1} - 1 < \alpha_1 < \cdots < \alpha_{r_1} < \alpha_1 + 1 < \alpha_2 + 1 < \cdots
$$

In this strictly increasing sequence $\mathbb{Z} \to \mathbb{R}$, 1 is sent to $\alpha_1$ characterized as the smallest non-negative number in the sequence. Similarly consider the infinite sequence $\beta_k$ from the weights of $E_{2,p}$. Define the functions:

\begin{align}
 f : [0, \infty) & \to \mathbb{R}, \\
 g : [0, \infty) & \to \mathbb{R}, \\
 x & \mapsto \# \{ \alpha_k \mid 0 < \alpha_k < x \}, \\
 x & \mapsto \# \{ \beta_k \mid 0 < \beta_k \leq x \}.
\end{align}

Note that $f(x + 1) = f(x) + r_1$ and $g(x + 1) = g(x) + r_1$. Now we have

\begin{lemma}
$r_p = \max\{f(g) = \max_{[0,1]}(f-g)$.\n\end{lemma}

Proof. The way $f$ and $g$ are defined, $f - g$ is a right-continuous step function, with jumps by +1 at the points $\alpha_k$ and −1 at the points $\beta_k$. As $f - g$ is 1-periodic, the existence of maximum and the equality $\max(f-g) = \max_{[0,1]}(f-g)$ are clear. Let $M = \max(f - g)$ and $x_0 \in [0,1)$ be a point which is not a weight and satisfies $(f - g)(x_0) = M$. Then, writing $k = f(x_0)$, we have $\alpha_k < x_0 < \alpha_{k+1}$ and $k - M = g(x_0)$, i.e. $\beta_{k-1} < x_0 < \beta_{k-1}$. The maximality of $f - g$ at $x_0$ implies that we have $\beta_{k-M} < \alpha_k < x_0 < \beta_{k-M+1} < \alpha_{k+1}$. So any parabolic map $\phi_0 : E_{2,p} \to E_{1,p}$ satisfies that $\phi_0(E_{2,p,k-M+1}) \subset E_{1,p,k+1}$ and hence

$$
\dim \ker \phi_0 \geq \dim E_{2,p,k-M+1} - \dim E_{1,p,k+1} = (r_1 - k + M) - (r_1 - k) = M.
$$

Conversely, let $\phi_0 : E_{2,p} \to E_{1,p}$ be a map such that $\phi_0(E_{2,p,k-M+1}) \subset E_{1,p,k+1}$ for each $k$. Then $\phi_0$ is a parabolic map: for if $\beta_i > \alpha_j$, take $\beta_i > x > \alpha_j$. So $g(x) \leq i - 1$ and $f(x) \geq j$. So $j - i + 1 \leq f(x) - g(x) \leq M$ and hence $i \geq j - M - 1$. Thus $\phi_0(E_{2,p,i}) \subset \phi_0(E_{2,p,j-M+1}) \subset E_{1,p,j+1}$. On the other hand, it is clear that there are maps satisfying $\phi_0(E_{2,p,k-M+1}) \subset E_{1,p,k+1}$ for each $k$ with $\dim \ker \phi_0 = M$. Hence there are parabolic maps $\phi_0$ with $\dim \ker \phi_0 = M$, completing the proof that $M = r_p$.

\begin{proposition}
There exists a suitable parabolic line bundle $L$ such that the moduli space of $\sigma$-stable triples of the form $(E_1 \otimes^p L, E_2 \otimes^p L, \phi)$ has associated $\tilde{r}_p = 0$, for all $p \in D$.
\end{proposition}
Proof. We shall assume that there is only one point \( p \in D \) and we shall tensor with a parabolic line bundle of the form \( L = \mathcal{O}_{[x]} \), i.e. the trivial line bundle with weight \( x \in [0,1) \) at \( p \). Take \( x_0 \in (0,1) \) which does not coincide with any weight and gives the maximum value of the function \( f - g \). Let \( L = \mathcal{O}_{[1-x_0]} \). Denoting by \( k_0 = f(x_0) \), the weights of \( E_2 \otimes^p L \) are

\[
0 \leq \alpha_{k_0+1} - x_0 < \cdots < \alpha_{r_1} - x_0 < \alpha_1 - x_0 + 1 < \cdots < \alpha_{k_0} - x_0 + 1 < 1
\]

(see \([GGM]\)). Said otherwise, if \( \tilde{\alpha}_k \) is the infinite sequence associated to the weights of \( \tilde{E}_2 = E_2 \otimes^p L \), then \( \tilde{\alpha}_k = \alpha_{k+k_0} - x_0 \). The function \( \tilde{f} \) associated to \( \tilde{E}_2 \) as in (44) is

\[
\tilde{f}(x) = \#\{\tilde{\alpha}_k \mid 0 < \tilde{\alpha}_k < x\} = \#\{\alpha_k \mid 0 < \alpha_k - x_0 < x\} = \#\{\alpha_k \mid x_0 < \alpha_k < x + x_0\} = f(x + x_0) - f(x_0),
\]

the last equality because \( x_0 \) is not a weight of \( E_2 \). Analogously for \( \tilde{E}_1 = E_1 \otimes^p L \), the function \( \tilde{g} \) associated to it is

\[
\tilde{g}(x) = g(x + x_0) - g(x_0).
\]

Then the number \( r_p \) associated to the moduli spaces of triples \( (\tilde{E}_1, \tilde{E}_2, \phi) \) is

\[
\tilde{r}_p = \max(\tilde{f}(x) - \tilde{g}(x)) = \max(f(x + x_0) - g(x + x_0)) - M = 0.
\]

\[ \square \]

**Proposition 12.9.** Assume that \( r_1 = r_2 \) and \( r_p = 0 \) for all \( p \in D \). Then the moduli space of \( \sigma \)-stable triples for \( \sigma \) large and \( d_2 + r_2 s = d_1 \) is irreducible.

**Proof.** Any triple \( T = (E_1, E_2, \phi) \) in \( \mathcal{N}^s_L \) satisfies that \( \phi : E_2 \rightarrow E_1(D) \) is generically an isomorphism by Proposition 12.2. So the condition on the degrees implies that it is an isomorphism of bundles. Moreover, by Lemma 12.3 the family \( \mathcal{H} \) of bundles \( E_1 \) appearing as part of triples of \( \mathcal{N}^s_L \) is a bounded family which is irreducible and the generic element is a stable bundle (see \([GGM]\)).

Let us study the fibres of \( \mathcal{N}^s_L \rightarrow \mathcal{H} \). Fix \( E_1 \in \mathcal{H} \) and consider the fibre over \( E_1 \). Identifying \( E_2 \) with \( E_1(D) \) (as bundles) via the isomorphism \( \phi \), an element \( (E_1, E_2, \phi) = (E_1, E_1(D), \text{Id}) \) in the fibre consists on giving for each \( p \in D \) a flag for \( V = E_{1,p} \) and a flag for \( V = E_{2,p} \) such that the identity map \( \text{Id} : V \rightarrow V \) is a parabolic map with respect to these flags. For simplicity, assume there is only one point \( p \in D \). Let

\[
\mathcal{F}_1 = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{r_1} = E_{1,p} \mid \dim V_i = i\}
\]

be the space parametrizing (complete) flags at \( E_{1,p} \), with fixed weights \( \alpha_1 < \cdots < \alpha_{r_1} \). This is an irreducible variety. Analogously define the space

\[
\mathcal{F}_2 = \{0 \subset W_1 \subset W_2 \subset \cdots \subset W_{r_1} = E_{2,p} \mid \dim W_i = i\}
\]

of (complete) flags for \( E_{2,p} \), with fixed weights \( \beta_1 < \cdots < \beta_{r_1} \). The condition \( r_p = 0 \) means that \( g(x) \leq f(x) \), for all \( x \), with the notation (44). The identity map is parabolic if \( W_i \subset V_{i+k(i)} \), \( 1 \leq i \leq r_1 \), for some set of integers \( k(i) \geq 0 \) such that \( 0 < 1 + k(1) \leq 2 + k(2) \leq \cdots \leq r_1 + k(r_1) = r_1 \). The set of compatible flags if given by

\[
(45) \quad \mathcal{F} = \{(F_1, F_2) \mid W_i \subset V_{i+k(i)}, 1 \leq i \leq r_1\} \subset \mathcal{F}_1 \times \mathcal{F}_2 .
\]
This is also an irreducible variety, as $F \to F_1$ is a fibration with irreducible base and irreducible fibres. Note that the other projection $F \to F_2$ is also surjective.

A generic stable bundle $E_1$ satisfies that a generic flag $F_1 \in F_1$ gives a parabolic stable bundle. Let $U_1 \subset F_1$ be a (dense) open subset with this property. Analogously consider a dense open subset $U_2 \subset F_2$ such that $E_2 = E_1(D)$ with a flag $F_2 \in F_2$ is parabolically stable. If $F \cap (U_1 \times U_2) = \emptyset$ then $F \subset ((F_1 - U_1) \times F_2) \cup (F_1 \times (F_2 - U_2))$. Being irreducible, $F$ should be contained in either of $((F_1 - U_1) \times F_2)$ or $(F_1 \times (F_2 - U_2))$. This contradicts the surjectivity of both $F \to F_1$ and $F \to F_2$. This proves that $F \cap (U_1 \times U_2) \neq \emptyset$, so the generic element of $F$ gives parabolic stable bundles $E_1$ and $E_2$. By Lemma 12.5 such element is $\sigma$-stable for $\sigma$ large. Therefore the generic stable bundle $E_1$ satisfies that the fibre of $\mathcal{N}_L^\sigma \to \mathcal{H}$ is an open subset of the space of compatible flags $F$. This shows that $\mathcal{N}_L^\sigma$ is irreducible and non-empty.

**Theorem 12.10.** Suppose that $r_1 = r_2$ and that $d_1 + r_1s - d_2 = \sum p \in D r_p$. Then the moduli space $\mathcal{N}_L^\sigma$ is irreducible, of the expected dimension and non-empty.

*Proof.* By Proposition 12.8 there exists a parabolic line bundle $L$ such that $(E_1, E_2, \phi) \mapsto (E_1 = E_1 \otimes^p L, E_2 = E_2 \otimes^p L, \phi)$ gives an isomorphism of moduli spaces of $\sigma$-stable triples $\mathcal{N}_\sigma(r_1, r_1, d_1, d_2; \alpha, \beta) \cong \mathcal{N}_\sigma(r_1, r_1, \tilde{d}_1, \tilde{d}_2; \tilde{\alpha}, \tilde{\beta})$ such that $\tilde{r}_p = 0$ for each $p \in D$. Then

$$\tilde{d}_1 + r_1 s - \tilde{d}_2 = d_1 + r_1 s - d_2 - \sum p \in D r_p.$$

This is easily seen by computing the degrees $\tilde{d}_1$ and $\tilde{d}_2$. For instance, suppose that there is only one point $p \in D$. Then, with the notations of the proof of Proposition 12.8

$$\begin{align*}
\tilde{d}_1 &= \deg \tilde{E}_1 = \operatorname{pardeg} (E_1 \otimes^p L) - \sum \tilde{\alpha}_k \\
&= \operatorname{pardeg} (E_1) + r_1 \operatorname{pardeg} (L) - \left( \sum (\alpha_k - x_0) + k_0 \right) \\
&= d_1 + \sum \alpha_k + r_1 (1 - x_0) - \sum \alpha_k + r_1 x_0 - k_0 \\
&= d_1 + r_1 - k_0 = d_1 + r_1 - f(x_0).
\end{align*}$$

Analogously, $\tilde{d}_2 = d_2 + r_1 - g(x_0)$, so that $\tilde{d}_1 - \tilde{d}_2 = d_1 - d_2 - r_p$.

Now the moduli space $\mathcal{N}_L^\sigma(r_1, r_1, \tilde{d}_1, \tilde{d}_2; \tilde{\alpha}, \tilde{\beta})$ is non-empty and irreducible by Proposition 12.9. So the same is true of our initial moduli space by using Lemma 12.6. The dimension statement follows from Corollary 9.6.

**Theorem 12.11.** Suppose that $r_1 = r_2$ and $d_1 + r_1s - d_2 \geq \sum p \in D r_p$. Then the moduli space $\mathcal{N}_L^\sigma$ is non-empty, of the expected dimension and irreducible.

*Proof.* The dimension statement follows from Corollary 9.6. Arguing as in the proof of Theorem 12.10 we may suppose that $r_p = 0$, for $p \in D$. Now, there exist triples $\phi : E_2 \to E_1(D)$, with $\phi$ injective, $E_1$ and $E_2$ stable bundles, and satisfying that the torsion sheaf quotient of the map $\phi$ is generic (in particular, supported on $X \setminus D$). This follows from [BGC2], where non parabolic $\sigma$-stable triples for $\sigma$ large are found by constructing $\sigma$-stable triples with these properties.

Now the argument of the proof of Proposition 12.9 works here to find parabolic structures on $E_1$ and $E_2$ such that $(E_1, E_2, \phi)$ is a $\sigma$-stable parabolic triple for $\sigma$ large, since the only
necessary fact is that $\phi_p : E_{2,p} \to E_{1,p}$ is an isomorphism for all $p \in D$. This gives the non-emptiness of $N_L^s$.

For proving the irreducibility of $N_L^s$, the main obstacle are the triples with quotient supported at points of $D$. We work as follows. Let $H$ be the family of bundles $E_1$ appearing in triples $T = (E_1, E_2, \phi) \in N_L^s$. This is a bounded and irreducible family whose generic element $E_1 \in H$ is a generic stable bundle. Let $Q = \text{Quot}^t(H)$ be the Quot scheme parametrizing quotients $E_1(D) \to S$, with $E_1 \in H$ and $t = \text{length } S = d_1 + r_1 s - d_2 - \sum_{p \in D} r_p$. The kernel of a generic element in $Q$ is a stable bundle $E_2$. If the support of $S$ is contained in $X \setminus D$, then the fiber of the map $N_L^s \to Q$ over a quotient $E_1(D) \to S$ in $Q$ is a subset of the set of compatible flags $F$ defined in (45). For a generic element in $Q$, this is actually an open subset of $F$, as proved in the proof of Proposition (12.9). This produces an open subset $U \subset N_L^s$, which is of dimension

$$\dim Q + \dim F.$$ 

Let us see the irreducibility of $N_L^s$ by checking that $\dim(N_L^s \setminus U) < \dim U$. Certainly, the only effect that we must take care of is the jumping in the dimension of the fiber of $N_L^s \to Q$ when the torsion sheaf is supported at some points of $D$. Let $p \in D$, and suppose that $p$ is in the support of $S$, say $S_p = C^l$. The set of quotients $E_1, p \to S_p$ is parametrized by the grassmannian $\text{Gr}(l, r_1)$. The codimension of the space $Q^l \subset Q$ parametrizing such quotients is

$$r_1 \text{length } S - (r_1 \text{length } S - l) = r_1 l - lr_1 + l^2 = l^2.$$

Now let us compute the dimension of the fiber of $N_L^s \to Q$ over a point in $Q^l$. With the definition of $k(i)$ given in Proposition (12.9), such fiber is the space

$$F_s = \{ (W_i, V_i) \in F_1 \times F_2 \mid \phi(W_i) \subset V_{i+k(i)} \}. $$

Equivalently, $(W_i, V_i) \in F_s \iff W_i \subset \phi^{-1}(V_{i+k(i)})$. It remains to see that

$$\dim F_s - \dim F < l^2.$$

The fibration $F \to F_1$ is surjective and the dimension of the fiber is

$$\sum_{i=1}^{r_1} k(i).$$

Let us compute the dimension of a fiber of $F_s \to F_1$. Such dimension depends on the flag $\{V_i\} \in F_1$, so we need to stratify $F_1$ as follows. The flag $\{V_i\}$ is determined by a collection of numbers $0 \leq a_1 \leq \ldots \leq a_{r_1} = r_1 - l$ such that

$$0 \subset V_1 \cap \text{Im} (\phi) \subset \ldots \subset V_{r_1} \cap \text{Im} (\phi) = \text{Im} (\phi)$$

$$0 \subset \mathbb{C}^{a_1} \subset \ldots \subset \mathbb{C}^{a_{r_1}} = \mathbb{C}^{r_1-l}$$

Clearly, $a_{i+1} = a_i + \delta_{i+1} (a_0 = 0)$ where there are uniquely defined $1 \leq i_1 < \ldots < i_{r_1-l} \leq r_1$ such that $\delta_k = 1$ and $\delta_j = 0$ for $j \neq i_k$, $k = 1, \ldots, r_1 - l$. The codimension of the stratum $S_{a_1, \ldots, a_{r_1}} \subset F_1$ defined by such $\{V_i\}$ is

$$\sum_{k=1}^{r_1-l} (l - i_k + k).$$
The fiber of $F_s \to F_1$ over $\{V_i\} \in S_{a_1,\ldots,a_r}$ is given by flags $\{W_i\} \in F_2$ such that $W_i \subset \tilde{V}_{l+i+k(i)}$, with $\tilde{V}_i = \phi^{-1}(V_i) \cong \mathbb{C}^{l+\alpha}$. The dimension of such fiber is thus

$$\sum_{i=1}^{r_1} (l + a_i + k(i) - \ell) \leq \sum_{i=1}^{r_1} (l + a_i - \ell) + \sum k(i)$$

$$= \sum_{i=1}^{r_1} (l - \ell) + \sum_{k=1}^{r_1-l} (r_1 - i_k + 1) + \sum k(i)$$

So the dimension of the preimage of $S_{a_1,\ldots,a_r}$ by the map $F_s \to F_1$ is less than or equal to

$$\dim F_1 - \sum_{k=1}^{r_1-l} (l - i_k + k) + \sum_{i=1}^{r_1} (l - \ell) + \sum_{k=1}^{r_1-l} (r_1 - i_k + 1) + \sum k(i)$$

$$= \dim F_1 + \sum k(i) + \frac{l^2 - l}{2} = \dim F + \frac{l^2 - l}{2}.$$ 

Since this is true for any stratum, we have

$$\dim F_s \leq \dim F + \frac{l^2 - l}{2} < \dim F + l^2,$$

as required. \hfill \Box

Combining Theorem 12.11 with Proposition 12.1 we have the following.

**Corollary 12.12.** Let $g > 0$, $r_1 = r_2$ and $d_1 + r_1 s - d_2 \geq \sum_{p \in D} r_p$. Then the moduli spaces $\mathcal{N}_{\sigma}$ are non-empty, irreducible and of the expected dimension for any $\sigma \geq 2g - 2$.

**Remark 12.13.** Corollary 12.12 and the correspondence in Proposition 7.3 gives that the moduli space $\mathcal{U}(p, p; a; b; \alpha, \beta)$ is non-empty and connected if and only if the following is satisfied:

(i) In the case $\tau < 0$. It must be $|\tau| \leq \tau_M$ by Proposition 7.4. Also, defining $r_x = \min\{\dim \text{coker } \phi \mid \phi \in \text{ParHom}(V_x, W_x)\}$, for $x \in D$, we must have $b + (2g - 2 + s)p - a \geq \sum_{x \in D} r_x$, by Corollary 12.12. But this last condition is redundant: $\tau < 0$ is equivalent to $\text{par} \mu(V) < \text{par} \mu(W)$, hence

$$a = \deg(V) < \text{pardeg}(V) < \text{pardeg}(W) < \deg(W) + ps = b + ps + (2g - 2)s,$$

since $g > 0$. Also, we may tensor with a suitable parabolic line bundle $L$ to arrange $r_x = 0$, for all $x \in D$, by Proposition 12.8 (this does not change $\tau$ or the inequality that we need to check). So $b + (2g - 2 + s)p - a \geq 0$, as required.

(ii) The case $\tau > 0$ is worked out similarly, and the only condition we obtain is $|\tau| \leq \tau_M$.

Note that the genericity of the weights (Assumption 2.1) prevents the case $|\tau| = \tau_M$ to happen.

### 13. Representations of fundamental groups in $U(p, q)$

Let $X$ be a compact Riemann surface of genus $g \geq 0$ and let $S = \{x_1, \ldots, x_s\}$ be a set of distinct points of $X$. Let $\Gamma = \pi_1(X \setminus S)$ be the fundamental group of $X \setminus S$. The group $\Gamma$ is generated by the usual generators $a_i, b_i$, $1 \leq i \leq g$, of $\pi_1(X)$, together with additional generators $\gamma_1, \ldots, \gamma_s$ corresponding to loops enclosing each $x_i$ simply, not enclosing any $x_j$. 

j ≠ i, and which are homotopic to zero relatively to the base point on X. There is also the relation

\[ [a_1, b_1] \cdots [a_g, b_g] \gamma_1 \cdots \gamma_s = 1, \]

where \([a_i, b_i]\) is the commutator of \(a_i\) and \(b_i\).

Parabolic Higgs bundles are related to representations of \(\Gamma\). To be precise, let us fix integers \(n = \text{rk} E, d = \text{deg} E\) and the weight type \(\alpha = \{\alpha(x)\}_{x \in S}\), where \(\alpha(x) = (\alpha_1(x), \ldots, \alpha_r(x))\) are weights with multiplicities \(k_i(x)\) for every \(x \in S\). It is convenient to repeat each weight according to its multiplicity, by setting \(\tilde{\alpha}_1(x) = \ldots = \tilde{\alpha}_{k_i(x)}(x) = \alpha_1(x)\), etc., thus having weights \(0 ≤ \tilde{\alpha}_1(x) ≤ \ldots ≤ \tilde{\alpha}_n(x) < 1\) (see Section \([\text{I}]\)).

For every \(x_i \in S\) there is a \(C_i \in \text{U}(n)\) defined by

\[
C_i = \begin{pmatrix}
\exp(2\pi \sqrt{-1}\tilde{\alpha}_1(x_i)) & 0 \\
0 & \ddots
\end{pmatrix},
\]

for every \(x_i \in S\).

Consider the set of representations \(\text{Hom}_+^+(\Gamma, \text{GL}(n, \mathbb{C}))\) defined by semisimple homomorphisms \(\rho: \Gamma \to \text{GL}(n, \mathbb{C})\) such that \(\rho(\gamma_i)\) is conjugated to \(C_i\) by an element in \(\text{GL}(n, \mathbb{C})\) for \(1 ≤ i ≤ s\). Here by semisimple we mean that \(\rho\) is a direct sum of irreducible representations. The moduli space of representations of \(\Gamma\) in \(\text{GL}(n, \mathbb{C})\) with fixed holonomy in the conjugacy class of \(C_i\), is defined by the quotient

\[
\mathcal{R}(n; \alpha) := \frac{\text{Hom}_+^+(\Gamma, \text{GL}(n, \mathbb{C}))}{\text{GL}(n, \mathbb{C})},
\]

where \(\text{GL}(n, \mathbb{C})\) acts by conjugation. The set \(\mathcal{R}(n; \alpha)\) has a natural structure of a complex algebraic variety. The following is proved by Simpson in \([S2]\).

**Theorem 13.1.** Let \((n, d; \alpha)\) be such that

\[
d + \sum_{x \in S} (\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_n(x)) = 0,
\]

i.e., the parabolic degree vanishes. Then there is a homeomorphism

\[
\mathcal{R}(n; \alpha) \cong \mathcal{M}(n, d; \alpha).
\]

This generalizes the theorem of Metha–Seshadri \([MS]\) which identifies the moduli space of parabolic bundles of type \((n, d, \alpha)\) with vanishing parabolic degree with the moduli space of representations of \(\Gamma\) in \(\text{U}(n)\) with fixed holonomy conjugated to \(C_i\) around the marked points.

There is a similar correspondence between representations of \(\Gamma\) in \(\text{U}(p, q)\) and parabolic \(\text{U}(p, q)\)-Higgs bundles. To explain this, let us come back to the notation in Section \([2]\) and fix the types of the parabolic bundles \(V\) and \(W\) to be \((p, a, \alpha)\) and \((q, b, \alpha')\), respectively. For every \(x_i \in S\) there are matrices \(C_i \in \text{U}(p)\) and \(C_i' \in \text{U}(q)\) defined as in \([16]\) by the weight systems \(\alpha\) and \(\alpha'\), respectively.

Consider now the set of representations \(\text{Hom}_{a,\alpha'}^+(\Gamma, \text{U}(p, q))\) defined by semisimple homomorphisms \(\rho: \Gamma \to \text{U}(p, q)\) such that \(\rho(\gamma_i)\) is conjugated to \(C_i \times C_i' \in \text{U}(p) \times \text{U}(q)\) (recall that \(\text{U}(p) \times \text{U}(q)\) is the maximal compact subgroup of \(\text{U}(p, q)\)) by an element in \(\text{U}(p, q)\) for \(1 ≤ i ≤ s\). Define the moduli space of representations of \(\Gamma\) in \(\text{U}(p, q)\) with fixed holonomy \(\text{U}(p, q)\)-conjugated to \(C_i \times C_i'\) by the quotient

\[
\mathcal{R}(p, q; \alpha, \alpha') := \frac{\text{Hom}_{a,\alpha'}^+(\Gamma, \text{U}(p, q))}{\text{U}(p, q)}.
\]
The set $\mathcal{R}(p, q; \alpha, \alpha')$ is a real analytic variety. We can adapt the arguments of Simpson [S2] to prove the following.

**Theorem 13.2.** Let $(p, a, \alpha)$ and $(q, b, \alpha')$ be such that

$$\text{pardeg}(V) + \text{pardeg}(W) = a + b + \sum_{x \in S}(\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_p(x) + \tilde{\alpha}'_1(x) + \ldots + \tilde{\alpha}'_q(x)) = 0.$$  

Then there is a homeomorphism

$$\mathcal{R}(p, q; \alpha, \alpha') \cong \bigsqcup_{a, b} \mathcal{U}(p, q, a, b; \alpha, \alpha').$$

Note that $(p, q, a, b; \alpha, \alpha')$ must also satisfy the Milnor–Wood inequality, which in these cases reduces to

$$|\text{pardeg}(V)| \leq \min\{p, q\}(g - 1 + s/2),$$

since $\text{pardeg}(W) = -\text{pardeg}(V)$.

Combining Theorem 13.2 and Theorem 6.12 we have the following.

**Theorem 13.3.** Under the genericity conditions given by Assumption 2.1, and for $g > 0$, the number of non-empty connected components of $\mathcal{R}(p, q; \alpha, \alpha')$ equals the number of integers $a$ such that

$$|a + \sum_{x \in S}(\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_p(x))| \leq \tau_L/2,$$

where $\tau_L$ is given by (40).

**Remark 13.4.** The condition on the genus $g$ comes from Theorem 13.2.

Like in the proof of Theorem 13.1 (S2), the main ingredients in the proof of Theorem 13.2 are, on the one hand, the correspondence given by Theorem 5.1 between polystable parabolic $U(p, q)$-Higgs bundles and solutions to Hitchin equations, and, on the other, the existence of a harmonic adapted metric on a $U(p, q)$-bundle with a semisimple meromorphic flat connection with simple poles. To see this, let us come back to the framework of Section 5, and consider smooth parabolic vector bundles $V$ and $W$ of types $(p, a; \alpha)$ and $(q, b; \alpha')$, respectively. On the bundle $V \oplus W$ we consider flat $U(p, q)$-connections $D$ on $X \setminus S$, meromorphic at $x_i \in S$ and whose residue at $x_i$ is conjugated to $C_i \times C'_i$. We say that $D$ is semisimple if the corresponding representation is semisimple. These connections are in correspondence with elements in $\text{Hom}_{\tilde{\alpha}, \alpha}(\Gamma, U(p, q))$.

Let $h = (h_V, h_W)$, where $h_V$ and $h_W$ are adapted hermitian metrics on $V$ and $W$, respectively. We decompose $D$ as $D = d_A + \Psi$, where $d_A$ is a $U(p) \times U(q)$ connection and $\Psi$ takes values in $\mathfrak{m}$, where $u(p, q) = u(p) \oplus u(q) + \mathfrak{m}$ is the Cartan decomposition of the Lie algebra of $U(p, q)$. We say that $h$ is harmonic if $d_A^* \Psi = 0$. Then the following can be proved easily adapting the results in [C], [S2].

**Theorem 13.5.** A connection $D$ as above is semisimple if and only if there exists a harmonic hermitian metric $h = (h_V, h_W)$.
Theorem 14.1. Given an orbifold fundamental group for basic facts on orbifold surfaces). Conversely, given an elliptic surface an elliptic surface is a smooth compact complex surface $Y$ with a fibration $f: Y \rightarrow X$ onto a Riemann surface $X$, and an integer $\chi > 0$, there is an elliptic surface $Y$, unique up to diffeomorphism, with

$$\pi_1(Y) = \pi_1^{orb}(X), \quad \text{and} \quad \chi(\mathcal{O}_Y) = \chi.$$ 

Conversely, given an elliptic surface $Y$ with $b_1(Y)$ even, $\chi(\mathcal{O}_Y) > 0$ and $\text{kod}(Y) = 1$ we have

$$\pi_1(Y) = \pi_1^{orb}(X),$$

for some 2-orbifold Riemann surface $X$.

To understand this result and the relation of $Y$ to the 2-orbifold $X$, recall that an elliptic surface is a smooth compact complex surface $Y$ with a fibration $f: Y \rightarrow X$ onto a Riemann surface $X$. The following is proved in [Dol, Ue] (see also [Fri, SS]).
surface $X$ such that the generic fibre is an elliptic curve (the complex structure of the fibre may vary from point to point). In some special points the fibre may degenerate into nodal fibers. This is always the case for the elliptic surfaces we are dealing with. Technically this is the condition $\chi > 0$. The effect of these singularities is that they kill the extra generators of the fundamental group determined by the fibre. In addition to these nodal fibres there are multiple fibres, located over the marked points of $X$. They are defined analogously to orbifold singularities: a neighbourhood $Y_m$ of such a multiple fibre in $X$ is the quotient by a finite cyclic group,

$$f : Y_m \cong (\Delta \times E_{\tau(z)})/\mathbb{Z}_m \longrightarrow \Delta/\mathbb{Z}_m \cong \Delta$$

defined by $[(t, c)] \mapsto t^m = z$, where $\Delta$ is the unit disc in $\mathbb{C}$, $E_{\tau}$ is the torus $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, and the generator of $\mathbb{Z}_m$ acts as $(t, c) \mapsto (t \cdot \exp(2\pi \sqrt{-1}/m), c + 1/m)$. The crucial difference of a multiple fibre of $Y$ and the orbifold point is, however, that this action is free and hence the quotient is smooth. Roughly speaking, the orbifold singularity is now hidden in the map $f$ between two smooth manifolds $Y$ and $X$.

To relate representations $\rho : \pi_1^{\text{orb}}(X) \to \text{GL}(n, \mathbb{C})$ to parabolic Higgs bundles, we observe that $\rho(\gamma_i)$ must be conjugated to a matrix of the form

$$C_i = \begin{pmatrix}
\exp(2\pi \sqrt{-1} \frac{l_1(x_i)}{m_i}) & 0 & \ldots & 0 \\
0 & \exp(2\pi \sqrt{-1} \frac{l_2(x_i)}{m_i}) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \exp(2\pi \sqrt{-1} \frac{l_n(x_i)}{m_i})
\end{pmatrix}$$

for integers $l_j(x_i)$ such that

$$0 \leq l_1(x_i) \leq \ldots \leq l_n(x_i) < m_i.$$

This follows from the fact that $\rho(\gamma_i)^m_i = I$. Such a representation of $\pi_1^{\text{orb}}(X)$ lifts to a representation $\tilde{\rho} : \Gamma \to \text{GL}(n, \mathbb{C})$. Conversely, if $\tilde{\rho} : \Gamma \to \text{GL}(n, \mathbb{C})$ is such that $\rho(\gamma_i)$ is conjugated to a matrix $C_i$ as above then $\tilde{\rho}$ descends to a representation $\rho : \pi_1^{\text{orb}}(X) \to \text{GL}(n, \mathbb{C})$. We thus have proved the following.

**Proposition 14.2.** There is a one-to-one correspondence between representations $\rho : \pi_1^{\text{orb}}(X) \to \text{GL}(n, \mathbb{C})$ and representations $\tilde{\rho} : \Gamma \to \text{GL}(n, \mathbb{C})$ such that $\tilde{\rho}(\gamma_i)$ is conjugated to a matrix of the form (47) for integers $l_j(x_i)$ satisfying (48).

Similarly, we have the following.

**Proposition 14.3.** There is a one-to-one correspondence between representations $\rho : \pi_1^{\text{orb}}(X) \to \text{U}(p, q)$ and representations $\tilde{\rho} : \Gamma \to \text{U}(p, q)$ such that $\tilde{\rho}(\gamma_i)$ is $\text{U}(p, q)$-conjugated to an element of the form $C_i \times C'_i \subset \text{U}(p) \times \text{U}(q)$ with $C_i$ and $C'_i$ like in (47), defined for integers $l_j(x_i)$ and $l'_k(x_i)$ satisfying

$$0 \leq l_1(x_i) \leq \ldots \leq l_p(x_i) < m_i \quad \text{and} \quad 0 \leq l'_1(x_i) \leq \ldots \leq l'_q(x_i) < m_i.$$

Let

$$\lambda = \{\lambda(x_i) = (l_1(x_i), \ldots, l_n(x_i))\}_{x_i \in S},$$
where \( l_j(x_i) \) are integers satisfying (45). Let \( \mathcal{R}_{orb}^Y(n; \lambda) \) and \( \mathcal{R}_Y(n; \lambda) \) be the moduli spaces of semisimple representations of \( \pi_1^{orb}(X) \) and \( \pi_1(Y) \) in \( \text{GL}(n, \mathbb{C}) \) such that \( \rho(\gamma_i) \) is conjugated to the matrix (47). Similarly, let

\[
\lambda = \{ \lambda(x_i) = (l_1(x_i), \ldots, l_p(x_i)) \}_{x_i \in S} \quad \text{and} \quad \lambda' = \{ \lambda'(x_i) = (l'_1(x_i), \ldots, l'_q(x_i)) \}_{x_i \in S}
\]

satisfying (49). Let \( \mathcal{R}_{orb}^X(p, q; \lambda, \lambda') \) and \( \mathcal{R}_Y(p, q; \lambda, \lambda') \) be the moduli spaces of semisimple representations of \( \pi_1^{orb}(X) \) and \( \pi_1(Y) \) in \( \text{U}(p, q) \) such that \( \rho(\gamma_i) \) is conjugated to a matrix \( C_1 \times C'_1 \) like in Proposition 14.3. Of course, since \( \pi_1^{orb}(X) \cong \pi_1(Y) \), \( \mathcal{R}_{orb}^X(p, q; \lambda, \lambda') \cong \mathcal{R}_Y(p, q; \lambda, \lambda') \).

Combining Propositions 14.2 and 14.3 and Theorems 13.1 and 13.2 we have the following.

**Theorem 14.4.** Let \( \lambda \) given by (57) satisfying (48) and let \( \tilde{\alpha}(x_i) = \lambda(x_i)/m_i \). Let \( (n, d) \) be such that

\[
d + \sum_{x_i \in S} (\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_n(x)) = 0.
\]

Then

\[
\mathcal{R}_{orb}^X(n; \lambda) \cong \mathcal{R}_Y(n; \lambda) \cong \mathcal{R}(n, d; \alpha) \cong \mathcal{M}(n, d; \alpha).
\]

Similarly, let \( \lambda \) and \( \lambda' \) given by (57) satisfying (49) and let \( \tilde{\alpha}(x_i) = \lambda(x_i)/m_i \) and \( \tilde{\alpha}'(x_i) = \lambda'(x_i)/m_i \). Let \( (p, q, a, b) \) be such that

\[
a + b + \sum_{x_i \in S} (\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_p(x) + \tilde{\alpha}'_1(x) + \ldots + \tilde{\alpha}'_q(x)) = 0.
\]

Then

\[
\mathcal{R}_{orb}^X(p, q; \lambda, \lambda') \cong \mathcal{R}_Y(p, q; \lambda, \lambda') \cong \mathcal{R}(p, q; \alpha, \alpha') \cong \bigsqcup_{a, b} \mathcal{U}(p, q, a, b; \alpha, \alpha').
\]

As established by Simpson and Corlette, higher dimensional non-abelian Hodge theory (S1, C) gives a correspondence between semisimple flat bundles or representations of the fundamental group of a compact Kähler manifold \( (Y, \omega) \), and polystable Higgs bundles on \( (Y, \omega) \) with vanishing first and second Chern classes (see S1 for the definition of stability). Now, a \( \text{GL}(n, \mathbb{C}) \)-Higgs bundle on \( Y \) is defined as a pair \( (E, \Phi) \) consisting of a holomorphic vector bundle \( E \) over \( Y \) and a homomorphism \( \Phi : E \rightarrow E \otimes \Omega_Y^1 \) such that \( [\Phi, \Phi] = 0 \), where \( \Omega_Y^1 \) is the bundle of holomorphic one-forms on \( Y \). If \( E = V \oplus W \), where \( V \) and \( W \) are holomorphic bundles of ranks \( p \) and \( q \) respectively, and

\[
\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V \oplus W) \rightarrow (V \oplus W) \otimes \Omega_Y^1,
\]

then \( (E, \Phi) \) is said to be a \( \text{U}(p, q) \)-Higgs bundle. Of course, when \( Y \) is a Riemann surface we recover the original definition of Higgs bundle since \( \Omega_Y^1 \) is the canonical bundle and the condition \( [\Phi, \Phi] = 0 \) is trivially satisfied.

If \( Y \) is a complex elliptic surface as above, equipped with a Kähler metric \( \omega \), non-abelian Hodge theory on \( (Y, \omega) \) combined with Theorem 14.4 gives the following.

**Theorem 14.5.** There is a one-to-one correspondence between the moduli space of polystable \( \text{GL}(n, \mathbb{C}) \)-Higgs bundles on \( (Y, \omega) \) with vanishing Chern classes and the moduli space of parabolic \( \text{GL}(n, \mathbb{C}) \)-Higgs bundles on \( X \) with parabolic structure on the orbifold points.
Similarly, there is a one-to-one correspondence between the moduli space of polystable $U(p, q)$-Higgs bundles on $(Y, \omega)$ with vanishing Chern classes and the moduli space of parabolic $U(p, q)$-Higgs bundles on $X$ with parabolic structure on the orbifold points.

Remark 14.6. It would be very interesting to work out this correspondence directly in a similar fashion to what is done by Bauer [Ba] for the case of moduli spaces of vector bundles. We plan to come back to this problem in a future paper.

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