Graph of \(uv\)-paths in 2-connected graphs \(^*\)

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Abstract

For a 2-connected graph \(G\) and vertices \(u, v\) of \(G\) we define an abstract graph \(P(G_{uv})\) whose vertices are the paths joining \(u\) and \(v\) in \(G\), where paths \(S\) and \(T\) are adjacent if \(T\) is obtained from \(S\) by replacing a subpath \(S_{xy}\) of \(S\) with an internally disjoint subpath \(T_{xy}\) of \(T\). We prove that \(P(G_{uv})\) is always connected and give a necessary and a sufficient condition for connectedness in cases where the cycles formed by the replacing subpaths are restricted to a specific family of cycles of \(G\).

1 Introduction

For any vertices \(x, y\) of a path \(L\), we denote by \(L_{xy}\) the subpath of \(L\) that joins \(x\) and \(y\). Let \(G\) be a 2-connected graph and \(u\) and \(v\) be vertices of \(G\). The \(uv\) path graph of \(G\) is the graph \(P(G_{uv})\) whose vertices are the paths joining \(u\) and \(v\) in \(G\), where two paths \(S\) and \(T\) are adjacent if \(T\) is obtained from \(S\) by replacing a subpath \(S_{xy}\) of \(S\) with an internally disjoint subpath \(T_{xy}\) of \(T\). The \(uv\) path graph \(P(G_{uv})\) is closely related to the graph \(G(P, f)\) of \(f\)-monotone paths on a polytope \(P\) (see C. A. Athanasiadis et al \(\Pi[1, 2]\)), whose vertices are the \(f\)-monotone paths on \(P\) and where two paths \(S\) and \(T\) are adjacent if there is a 2-dimensional face \(F\) of \(P\) such that \(T\) is obtained

\(^*\)Partially supported by Conacyt, México.
from $S$ by replacing an $f$-monotone subpath of $S$ contained in $F$ with the complementary $f$-monotone subpath of $T$ contained in $F$.

In Section 2 we show that the graphs $\mathcal{P}(G_{uv})$ are always connected as is the case for the graphs $G(P, f)$.

If $S$ and $T$ are adjacent paths in a $uv$-path graph $\mathcal{P}(G_{uv})$, then $S \cup T$ is a subgraph of $G$ consisting of a unique cycle $\sigma$ joined to $u$ and $v$ by disjoint paths $P_u$ and $P_v$. See Figure 1.

![Figure 1: $S \cup T$](image)

Let $\mathcal{C}$ be a set of of cycles of $G$; the $uv$-path graph of $G$ defined by $\mathcal{C}$ is the spanning subgraph $\mathcal{P}_C(G_{uv})$ of $\mathcal{P}(G_{uv})$ where two paths $S$ and $T$ are adjacent if and only if the unique cycle $\sigma$ which is contained in $S \cup T$ lies in $\mathcal{C}$. A graph $\mathcal{P}_C(G_{uv})$ may be disconnected.

The $uv$ path graph $\mathcal{P}(G_{uv})$ is also related to the well-known tree graph $\mathcal{T}(G)$ of a connected graph $G$, studied by R. L. Cummins [3], in which the vertices are the spanning trees of $G$ and the edges correspond to pairs of trees $S$ and $R$ which are obtained from each other by a single edge exchange. As in the $uv$ path graph, if two trees $S$ and $R$ are adjacent in $\mathcal{T}(G)$, then $S \cup R$ is a subgraph of $G$ containing a unique cycle. X. Li $et$ al [5] define, in an analogous way, a subgraph $\mathcal{T}_C(G)$ of $\mathcal{T}(G)$ for a set of cycles $\mathcal{C}$ of $G$ and give a necessary condition and a sufficient condition for $\mathcal{T}_C(G)$ to be connected. In sections 3 and 4 we show that the same conditions apply to $uv$ path graphs $\mathcal{P}_C(G_{uv})$.

Similar results are obtained by A. P. Figueroa $et$ al [4] with respect to the perfect matching graph $\mathcal{M}(G)$ of a graph $G$ where the vertices are the perfect matchings of $G$ and in which two matchings $L$ and $N$ are adjacent if their symmetric difference is a cycle of $G$. Again, if $L$ and $N$ are adjacent matchings in $\mathcal{M}(G)$, then $L \cup M$ contains a unique cycle of $G$.

For any subgraphs $F$ and $H$ of a graph $G$, we denote by $F \Delta H$ the subgraph of $G$ induced by the set of edges $(E(F) \setminus E(H)) \cup (E(H) \setminus E(F))$. 
2 Preliminary results

In this section we prove that the $uv$ path graph is connected for any 2-connected graph $G$ and give an upper bound for the diameter of a graph $\mathcal{P}(G_{uv})$.

**Theorem 1.** Let $G$ be a 2-connected graph. The $uv$-path graph $\mathcal{P}(G_{uv})$ is connected for every pair of vertices $u, v$ of $G$.

**Proof.** For any different $uv$ paths $Q$ and $R$ in $G$ denote by $n(Q, R)$ the number of consecutive initial edges $Q$ and $R$ have in common. Assume the result is false and choose two $uv$ paths $S : u = x_0, x_1, \ldots, x_s = v$ and $T : u = y_0, y_1, \ldots, y_t = v$ in different components of $\mathcal{P}(G_{uv})$ for which $n^* = n(S, T)$ is maximum.

Since edges $x_{n^*}x_{n^*+1}$ and $y_{n^*}y_{n^*+1}$ are not equal, $x_{n^*+1} \neq y_{n^*+1}$. Let $j = \min\{i : x_{n^*+i} \in V(T)\}$ and $k = \min\{i : y_{n^*+i} \in V(S)\}$ and let $l$ and $m$ be integers such that $y_l = x_{n^*+j}$, $x_m = y_{n^*+k}$. Consider the path:

$$S' : u = x_0, x_1, \ldots, x_{n^*}, y_{n^*+1}, y_{n^*+2}, \ldots, y_{n^*+k}, x_{m+1}, x_{m+2}, \ldots, x_s = v$$

Paths $S$ and $S'$ are adjacent in $\mathcal{P}(G_{uv})$ since $S'$ is obtained from $S$ by replacing the subpath $x_{n^*}, x_{n^*+1}, \ldots, x_m$ of $S$ with the subpath $y_{n^*}, y_{n^*+1}, \ldots, y_{n^*+k}$ of $S'$. Notice that $n(S', T) \geq n(S, T) + 1$ since $x_0x_1x_2, \ldots, x_{n^*}x_{n^*}$, $x_{n^*}y_{n^*+1} \in E(S') \cap E(T)$. By the choice of $S$, and $T$, paths $S'$ and $T$ are connected in $\mathcal{P}(G_{uv})$. This implies that $S$ and $T$ are also connected in $\mathcal{P}(G_{uv})$ which is a contradiction.

\[ \square \]

For any two vertices $u$ and $v$ of a connected graph $G$ we denote by $d_G(u, v)$ the distance between $u$ and $v$ in $G$, that is the length of a shortest $uv$ path in $G$. The diameter of a connected graph $G$ is the maximum distance among pairs of vertices of $G$. For a path $P$, we denote by $l(P)$ the length of $P$.

**Theorem 2.** Let $u$ and $v$ be vertices of a 2-connected graph $G$. The diameter of the graph $\mathcal{P}(G_{uv})$ is at most $2d_G(u, v)$.

**Proof.** Let $S$ and $T$ be $uv$ paths in $G$ and let $P$ be a shortest $uv$ path in $G$. From the proof of Theorem 1 one can see that there are two paths $Q_S$ and $Q_T$ in $\mathcal{P}(G_{uv})$ each with length at most $l(P)$, joining $S$ to $P$ and $T$ to $P$, respectively. Clearly $Q_S \cup Q_T$ contains a path joining $S$ and $T$ in $\mathcal{P}(G_{uv})$ with length at most $2l(P) = 2d_G(u, v)$.

\[ \square \]
In Figure 2 we show a 2-connected graph \( G^2 \) and paths \( S \) and \( T \) joining vertices \( u \) and \( v \) of \( G^2 \) such that \( d_{G^2}(u, v) = 2 \) and \( d_{P(G^2)}(S, T) = 4 \). For any positive integer \( k > 2 \) the graph \( G^2 \) can be extended to a graph \( G^k \) such that \( d_{G^k}(u, v) = k \) and that the diameter of \( P(G^k_{uv}) \) is \( 2k \). This shows that Theorem 2 is tight.

![Graph G^2 and paths S and T.](image)

**3 Necessory condition**

Let \( u \) and \( v \) be vertices of a 2-connected graph \( G \) and \( S \) and \( T \) be two \( uv \) paths adjacent in \( P(G_{uv}) \). Since \( T \) is obtained from \( S \) by replacing a subpath \( S_{xy} \) of \( S \) with an internally disjoint subpath \( T_{xy} \) of \( T \), the graph \( S \Delta T \) is the cycle \( S_{xy} \cup T_{xy} \).

**Theorem 3.** Let \( G \) be a 2-connected graph, \( u \) and \( v \) be vertices of \( G \) and \( \mathcal{C} \) be a set of cycles of \( G \). If the graph \( \mathcal{P}_C(G_{uv}) \) is connected, then \( \mathcal{C} \) spans the cycle space of \( G \).

**Proof.** Let \( \sigma \) be a cycle of \( G \). Since \( G \) is 2-connected, there are two disjoint paths \( P_u \) and \( P_v \) joining, respectively, \( u \) and \( v \) to \( \sigma \). Denote by \( u' \) and \( v' \) the unique vertices of \( P_u \) and \( P_v \), respectively, that lie in \( \sigma \). Vertices \( u' \) and \( v' \) partition cycle \( \sigma \) into two internally disjoint paths \( Q \) and \( R \). Let \( S = P_u \cup Q \cup P_v \) and \( T = P_u \cup R \cup P_v \). Clearly \( S \) and \( T \) are two different \( uv \) paths in \( G \) such that \( S \Delta T = \sigma \).

Since \( \mathcal{P}_C(G_{uv}) \) is connected, there are \( uv \) paths \( S = W_0, W_1, \ldots, W_k = T \) such that for \( i = 1, 2, \ldots k \), paths \( W_{i-1} \) and \( W_i \) are adjacent in \( \mathcal{P}_C(G_{uv}) \). For
$i = 1, 2, \ldots, k$ let $\alpha_i = W_{i-1} \Delta W_i$. Then $\alpha_1, \alpha_2, \ldots, \alpha_k$ are cycles in $C$ such that:

$$\alpha_1 \Delta \alpha_2 \Delta \cdots \Delta \alpha_k = (W_0 \Delta W_1) \Delta (W_1 \Delta W_2) \Delta \cdots \Delta (W_{k-1} \Delta W_k) = W_0 \Delta W_k = \sigma$$

Therefore $C$ spans $\sigma$.

Let $G$ be a complete graph with four vertices $u, x, y, v$ and let $C = \{\alpha, \beta, \delta\}$, where $\alpha = uxv$, $\beta = uyv$ and $\delta = uxyv$. Set $C$ spans the cycle space of $G$ but the graph $P_C(G_{uv})$ is not connected since the $uv$ path $uyxv$ is an isolated vertex of $P_C(G_{uv})$, see Fig 3. This shows that the condition in Theorem 3 is not sufficient for $P_C(G_{uv})$ to be connected.

Figure 3: Graph $G$, set $C = \{\alpha, \beta, \delta\}$ and graph $P_C(G_{uv})$.

4 Sufficient condition

A unicycle of a connected graph $G$ is a spanning subgraph $U$ of $G$ that contains a unique cycle. Let $u$ and $v$ be vertices of a 2-connected graph $G$. A $uv$-monocle of $G$ is a subgraph of $G$ that consists of a cycle $\sigma$ and two disjoint paths $P_u$ and $P_v$ that join, respectively $u$ and $v$ to $\sigma$, see Fig. 1. Clearly for each $uv$-monocle $M$ of a 2-connected graph $G$, there is a unicycle $U$ of $G$ that contains $M$.

Let $C$ be a set of cycles of $G$. A cycle $\sigma$ of $G$ has Property $\Delta^*$ with respect to $C$ if for every unicycle $U$ containing $\sigma$ there is an edge $e$ of $G$, not in $U$ and two cycles $\alpha, \beta \in C$, contained in $U + e$, such that $\sigma = \alpha \Delta \beta$. 


Lemma 1. Let $G$ be a 2-connected graph and $u$ and $v$ be vertices of $G$. Also let $C$ be a set of cycles of $G$ and $\sigma$ be a cycle having Property $\Delta^*$ with respect to $\mathcal{C}$. The graph $\mathcal{P}_{\mathcal{C} \cup \{\sigma\}}(G_{uv})$ is connected if and only if $\mathcal{P}_C(G_{uv})$ is connected.

Proof. If $\mathcal{P}_C(G_{uv})$ is connected, then $\mathcal{P}_{\mathcal{C} \cup \{\sigma\}}(G_{uv})$ is connected since the former is a subgraph of the latter.

Assume now $\mathcal{P}_{\mathcal{C} \cup \{\sigma\}}(G_{uv})$ is connected and let $S$ and $T$ be $uv$ paths in $G$ which are adjacent in $\mathcal{P}_{\mathcal{C} \cup \{\sigma\}}(G_{uv})$. We show next that $S$ and $T$ are connected in $\mathcal{P}_C(G_{uv})$ by a path of length at most 2.

If $\omega = S \Delta T \in \mathcal{C}$, then $S$ and $T$ are adjacent in $\mathcal{P}_C(G_{uv})$. For the case $\omega = \sigma$ denote by $\mathcal{M}$ the $uv$-monocle given by $S \cup T$.

Let $U$ be a unicycle of $G$ containing $\mathcal{M}$. Since $\sigma$ has Property $\Delta^*$ with respect to $\mathcal{C}$, there exists an edge $e = xy$ of $G$, not in $U$, and two cycles $\alpha, \beta \in \mathcal{C}$ contained in $U + e$ such that $\sigma = \alpha \Delta \beta$.

Let $x'$ and $y'$ denote the vertices in $\mathcal{M}$ which are closest in $U$ to $x$ and $y$, respectively. Then there exists a path $R_{x'y'}$ in $G$, with edges in $E(U + e) \setminus E(\mathcal{M})$ joining $x'$ and $y'$ and such that cycles $\alpha$ and $\beta$ are contained in $\mathcal{M} \cup R_{x'y'}$. We analyze several cases according to the location of $x'$ and $y'$ in $\mathcal{M}$.

Denote by $P_u$ and $P_v$ the unique paths, contained in $\mathcal{M}$, that join $u$ and $v$ to $\sigma$ and by $u'$ and $v'$ the vertices where $P_u$ and $P_v$, respectively, meet $\sigma$.

Case 1.- $x' \in V(P_u)$, $y' \in V(P_v)$. Without loss of generality we assume $\alpha = S_{x'y'} \cup R_{y'x'}$ and $\beta = T_{x'y'} \cup R_{y'x'}$, see Fig. 4.

![Figure 4: Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles $\alpha$ and $\beta$.](image)

Let $Q$ be the $uv$-path obtained from $S$ by replacing $S_{x'y'}$ with $R_{x'y'}$. Notice that $Q$ can also be obtained from $T$ by replacing $T_{x'y'}$ with $R_{x'y'}$. 


Case 2.- $x' \in V(P_u)$, $y' \in S \cap \sigma$. Without loss of generality we assume
$\alpha = S_{x'y'} \cup R_{y'x'}$ and $\beta = T_{x'v'} \cup S_{v'y'} \cup R_{y'x'}$, see Fig. 5.

Figure 5: Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles $\alpha$ and $\beta$.

Again let $Q$ be the $uv$-path obtained from $S$ by replacing $S_{x'y'}$ with $R_{x'y'}$. In this case, $Q$ can also be obtained from $T$ by replacing $T_{x'v'}$ with $R_{x'y'} \cup S_{y'v'}$.

Case 3.- $x', y' \in S \cap \sigma$. Without loss of generality we assume $\alpha = S_{x'y'} \cup R_{y'x'}$ and $\beta = S_{u'x'} \cup R_{x'y'} \cup S_{v'y'} \cup T_{v'w'}$, see Fig. 6.

Figure 6: Left: $\mathcal{M} \cup R_{y'x'}$. Right: Cycles $\alpha$ and $\beta$.

Let $Q$ be the $uv$-path obtained from $S$ by replacing $S_{x'y'}$ with $R_{x'y'}$. Path $Q$ is also obtained from $T$ by replacing $T_{u'v'}$ with $S_{u'x'} \cup R_{x'y'} \cup S_{y'v'}$.

Case 4.- $x' \in S \cap \sigma$ and $y' \in T \cap \sigma$. Without loss of generality we assume $\alpha = S_{u'x'} \cup R_{x'y'} \cup T_{y'u'}$ and $\beta = T_{y'v'} \cup S_{v'x'} \cup R_{x'y'}$, see Fig. 7.

Let $Q$ be the $uv$-path obtained from $S$ by replacing $S_{u'x'}$ with $T_{u'y'} \cup R_{y'x'}$. Now $Q$ can also be obtained from $T$ by replacing $T_{y'v'}$ with $R_{y'x'} \cup S_{x'v'}$.
In each case $S\Delta Q = \alpha$ and $Q\Delta T = \beta$. Since $\alpha, \beta \in C$, path $S$ is adjacent to $Q$ and path $Q$ is adjacent to $T$ in $P_C(G_{uv})$. Therefore $S$ and $T$ are connected in $P_C(G_{uv})$ by a path with length at most 2.

All remaining cases are analogous to either Case 2 or to Case 3. \qed

Consider a 2-connected graph $G$ with two specified vertices $u$ and $v$ and let $C$ be a set of cycles of $G$. Construct a sequence of sets of cycles $C = C_0, C_1, \ldots, C_k$ as follows: If there is a cycle $\sigma_1$ not in $C_0$ that has Property $\Delta^*$ with respect to $C_0$ add $\sigma_1$ to $C_0$ to obtain $C_1$. At step $t$ add to $C_t$ a new cycle $\sigma_{t+1}$ (if it exists) that has Property $\Delta^*$ with respect to $C_t$ to obtain $C_{t+1}$. Stop at a step $k$ where there are no cycles, not in $C_k$, having Property $\Delta^*$ with respect to $C_k$. We denote by $Cl(C)$ the final set obtained with this process. Li et al. [5] proved that the final set of cycles obtained is independent of which cycle $\sigma_t$ is added at each step in the case of multiple possibilities.

A set of cycles of $G$ is $\Delta^*$-dense if $Cl(C)$ is the whole set of cycles of $G$.

**Theorem 4.** If $C$ is $\Delta^*$-dense, then $P_C(G_{uv})$ is connected.

**Proof.** Since $C$ is $\Delta^*$-dense, $Cl(C)$ is the set of cycles of $G$ and therefore $P_{Cl(C)}(G_{uv}) = \mathcal{P}(G_{uv})$ which is connected by Theorem [1].

Let $C = C_0, C_1, \ldots, C_k = Cl(C)$ be a sequence of sets of cycles obtained from $C$ as above. By Lemma [1] all graphs $P_{Cl(C)}(G_{uv}) = P_{C_k}(G_{uv}), P_{C_{k-1}}(G_{uv}), \ldots, P_{C_0}(G_{uv}) = P_C(G_{uv})$ are connected. \qed

Li et al. [5] proved the following:
Theorem 5. If $G$ is a plane 2-connected graph and $C$ is the set of internal faces of $G$, then $C$ is $\Delta^*$-dense.

Theorem 6. If $G$ is a 2-connected graph and $C$ is the set of cycles that contain a given edge $e$ of $G$, then $C$ is $\Delta^*$-dense.

We end this section with the following immediate corollaries.

Corollary 1. Let $u$ and $v$ be vertices of a 2-connected plane graph $G$. If $C$ is the set of internal faces of $G$, then $P_{C}(G_{uv})$ is connected.

Proof. By Theorem 5, $C$ is $\Delta^*$-dense and by Theorem 4, $P_{C}(G_{uv})$ is connected. 

Corollary 2. Let $u$ and $v$ be vertices of a 2-connected graph $G$. If $C$ is the set of cycles of $G$ that contain a given edge $e$, then $P_{C}(G_{uv})$ is connected.

Proof. By Theorem 6, $C$ is $\Delta^*$-dense and by Theorem 4, $P_{C}(G_{uv})$ is connected.

Corollary 3. Let $u$ and $v$ be vertices of a 2-connected graph $G$. If $C_u$ is the set of cycles of $G$ that contain vertex $u$, then $P_{C_u}(G_{uv})$ is connected.

Proof. Let $e$ be an edge of $G$ incident with vertex $u$. Clearly the set $C(e)$ of cycles that contain edge $e$ is a subset of the set $C_u$. Therefore $P_{C(e)}(G_{uv})$ is a subgraph of $P_{C_u}(G_{uv})$. By Corollary 2, the graph $P_{C(e)}(G_{uv})$ is connected.

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