The $B_c$ mass up to order $\alpha_s^4$

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We evaluate in perturbative QCD, up to order $\alpha_s^4$, the mass of the $B_c$. We use the so-called $1S$-mass in order to improve the convergence of the perturbative series. Our result is $E(B_c)_{\text{pert}} = 6326^{+29}_{-9}$ MeV. Non-perturbative effects are discussed. A comparison with potential models seems to be consistent with non-perturbative contributions of the order $-(40 \div 100)$ MeV.

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I. INTRODUCTION

The discovery of the $B_c$ meson (the lowest pseudoscalar $^1S_0$ state of the $\bar{b}c$ system) has been reported in 1998 by the CDF collaboration in the 1.8 TeV $p\bar{p}$ collisions at the Fermilab Tevatron [1]. The mass has been measured to be $6.40 \pm 0.39 \pm 0.13$ GeV.

The fact that the mass of the quarks of quark–antiquark systems built up by $b$ and $c$ quarks is much larger than the typical binding energy suggests that these systems are non-relativistic, i.e. that the heavy-quark velocity $v$ is small. The typical scales of these systems are the binding energy $\sim mv^2$ and the momentum transfer $\sim mv$; moreover, because of the non-relativistic nature of the system, $m \gg mv \gg mv^2$ (for the purposes of this discussion $m$ and $v$ can be identified with the mass and the velocity of the lightest component of the bound state respectively). Let us call $\Lambda_{\text{QCD}}$ the scale at which non-perturbative effects become important.

If $\Lambda_{\text{QCD}} \lesssim mv^2$, then the scale $mv$ can be integrated out order by order in $\alpha_s$ at a scale $mv \gg \mu' \gg mv^2$. The system is described up to order $\alpha_s^4$ by a potential which is entirely accessible to perturbative QCD and at the leading order is the Coulomb potential. Non-potential effects start at order $\alpha_s^5 \ln \mu'$ [2,3]. This kind of system is called Coulombic.
perturbative effects are of non-potential type. In the particular situation $m v^2 \gg \Lambda_{\text{QCD}}$ they can be encoded into local condensates [4]. This condition seems to be fulfilled by the bottomonium ground state, which has been studied in this way in [5]. Also the charmonium ground state has been analysed as a Coulombic bound state by the same authors. In both cases (but with caveats) the non-perturbative corrections à la Voloshin–Leutwyler, i.e. in terms of local condensates, have been claimed to be under control [6].

For heavy quarkonium states higher than the ground state the condition $\Lambda_{\text{QCD}} \lesssim m v^2$ is not fulfilled and non-perturbative terms affect the potential. The system is no longer Coulombic. Traditionally the energy of these systems has been calculated within QCD-inspired confining potential models. A large variety of them exists in the literature and they have been on the whole quite successful (cf. [7] for some recent reviews). However, the usual criticisms apply. Their connection with the QCD parameters is hidden, the scale at which they are defined is not clear, they cannot be systematically improved and they usually contain a superposition by hand of perturbative and non-perturbative effects. For this reason a lot of effort has been devoted, over the years, to obtaining the relevant potentials from QCD by relating them to some Wilson loops expectation values [8, 7]. Anyway, these have to be eventually computed either via lattice simulations or in QCD vacuum models [9]. In the specific situation $m v \gg \Lambda_{\text{QCD}} \gg m v^2$ the scale $m v$ can still be integrated out perturbatively, giving rise to a Coulomb-type potential. Non-perturbative contributions to the potential will arise when integrating out the scale $\Lambda_{\text{QCD}}$. This situation has been studied in [2]. We will call quasi-Coulombic the systems described by the situation $m v \gg \Lambda_{\text{QCD}} \gg m v^2$, when the non-perturbative piece of the potential can be considered small with respect to the Coulombic one and treated as a perturbation.

The only available theoretical predictions (to our knowledge) of the $B_c$ mass resort to (confining) potential models or to the lattice. In this work we will carry out the calculation
of the perturbative $B_c$ mass up to order $\alpha_s^4$. We will call it $E(B_c)_{\text{pert}}$. This calculation will be relevant to a QCD determination of the $\bar{b}c$ ground state if this system is Coulombic or at least quasi-Coulombic. Moreover, in the way we are doing the calculation, we also assume the $\Upsilon(1S)$ and the $J/\psi$ to be Coulombic or at least quasi-Coulombic systems. The question if these assumptions correspond to the actual systems cannot be settled at this point. On the other hand there is no a priori reason to rule them out. Let us consider, for instance, the argument used in [5] for the $J/\psi$. Lattice data show that the static potential clearly deviates from a $1/r$ behaviour for distances larger than 1 GeV$^{-1}$ (see [7] and references therein). Therefore the $J/\psi$ is Coulombic or quasi-Coulombic if the characteristic scale of the bound state, $\mu \sim m_cv_c$, is bigger than 1 GeV. If we assume $m_c \simeq (1.6 \div 2.0)$ GeV and if we fix that scale on the Bohr radius, $a$, of the $J/\psi$, $\mu = 2/a(\mu)$, then we get $\mu \simeq (1.5 \div 1.6)$ GeV. Since this scale falls into the energy window between the mass scale $m_c$ and 1 GeV, these figures are consistent with a Coulombic or quasi-Coulombic picture of the $J/\psi$.

The main problem of the calculation of the perturbative $B_c$ mass is the well-known bad convergence of the perturbative series when using the pole mass. This is due to a renormalon cancellation occurring between the pole mass and the static Coulomb potential [10]. We handle the problem by expressing the $c$ and the $b$ pole mass in the perturbative

\footnotetext[1]{It is in general somehow ambiguous to separate in a physical quantity perturbative from non-perturbative contributions. From this point of view the following Eq. (2) may be seen as a definition of what we call here perturbative $B_c$ mass. Analogous definitions have to be understood for the perturbative $J/\psi$ and $\Upsilon(1S)$ masses.}

\footnotetext[2]{One may wonder if these figures are consistent with the non-relativistic expansion underlying NRQCD. Only an actual calculation may decide this, since a break-down of the NRQCD expansion, if it occurs, should be manifest in a breakdown of the expansion of the energy levels. In the specific situation of the $B_c$, as we will see later on in this paper, the expansion that we get shows a still convergent behaviour.
expression of the $B_c$ mass as half of the perturbative mass of the $J/\psi$ ($E(J/\psi)_{\text{pert}}$) and the $\Upsilon(1S)$ ($E(\Upsilon(1S))_{\text{pert}}$) respectively. This corresponds to using the quark mass in the so-called 1S scheme introduced in [11]. In this way, by expressing $E(B_c)_{\text{pert}}$ in terms of quantities that are infrared safe at order $\Lambda_{\text{QCD}}$ (the $^3S_1$ perturbative masses), the pathologies of the perturbative series, due to the renormalon ambiguities affecting the pole mass, are cured. We will explicitly show that, in fact, we obtain a better convergence of the perturbative expansion and a stable determination of the perturbative mass of the $B_c$, just in the energy range that the above discussion on the $J/\psi$ suggests to be also the relevant one for the $B_c$. Non-perturbative terms are of potential type in the quasi-Coulombic situation and of non-potential type in the Coulombic situation. They affect the identification of the perturbative masses $E(J/\psi)_{\text{pert}}$, $E(\Upsilon(1S))_{\text{pert}}$ and $E(B_c)_{\text{pert}}$ with the corresponding physical ones. If we aim at obtaining a good estimate of the physical $B_c$ mass, it is not important for each of these contributions to be individually small, as long as the sum of them in the $B_c$ mass is small. As we will discuss at the end, a picture with non-perturbative corrections to the $B_c$ mass of a not too large size ($\lesssim 100\text{ MeV}$) seems to be consistent with the experimental data and with the potential models.

The paper is organized in the following way. In the next section we set up the formalism and perform the calculation of the perturbative $B_c$ mass. In section 3 we briefly discuss the non-perturbative corrections and compare our result with other determinations of the $B_c$ mass available in the literature.

**II. CALCULATION OF $E(B_C)_{\text{PERT}}$**

In order to calculate the $B_c$ mass in perturbation theory up to order $\alpha_s^4$, we need to consider the following contributions to the potential: the perturbative static potential at two loops, the $1/m$ relativistic corrections at one loop, the spin-independent $1/m^2$ relativistic corrections at tree level and the $1/m^3$ correction to the kinetic energy. We will not consider here effects due to a non-zero charm quark mass on the $b$ (and the $b\bar{b}$ system) of the type
discussed in [12]. We will follow the derivation of the heavy quarkonium mass of Ref. [5].

The static potential at two-loops has been calculated in [13]. It is useful, in order to perform an analytic calculation, to split it as

$$V_0(r) = v_0(r) + \delta v_0(r),$$

where $v_0$ is the part that does not contain logarithms,

$$v_0(r) \equiv -C_F \frac{\tilde{\alpha}_s(\mu)}{r},$$

$$\tilde{\alpha}_s(\mu) \equiv \alpha_s(\mu) \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left[ \frac{5}{12} \beta_0 - \frac{2}{3} C_A + \frac{\beta_0}{2} \gamma_E \right] + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \beta_0^2 \left( \frac{\gamma_E}{4} + \frac{\pi^2}{48} \right) + \left( \frac{\beta_1}{8} + \frac{5}{12} \beta_0^2 - \frac{2}{3} C_A^2 \beta_0 \right) \gamma_E + \frac{c}{16} \right] \right\},$$

where $\beta_n$ are the $\beta$-function coefficients $^3, c \equiv \left( \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta_3 \right) C_A^2 - \left( \frac{899}{81} + \frac{28}{3} \zeta_3 \right) C_A N_f - \left( \frac{55}{6} - 8 \zeta_3 \right) C_F N_f + \left( \frac{10}{9} N_f \right)^2, \gamma_E = 0.5772 \ldots$ is the Euler constant, $C_A = 3, C_F = 4/3$ and $N_f$ is the number of flavours (we will take $N_f = 3$ $^4$). $\delta v_0$ is given by

$$\delta v_0(r) \equiv -\frac{C_F \alpha_s(\mu)^2}{\pi r} \ln(\mu r) \left\{ \frac{\beta_0}{2} + \frac{\alpha_s}{\pi} \left[ \frac{\beta_0}{4} \ln(\mu r) + 2 \gamma_E \right] + \frac{\beta_1}{8} + \frac{5}{12} \beta_0^2 - \frac{2}{3} C_A \beta_0 \right\}.$$

The strong coupling constant $\alpha_s$ is understood in the \(\overline{\text{MS}}\) scheme. At the scale $\mu$ we will take the value of $\alpha_s$ from the three-loop expression with $\Lambda_{N_f=3}^{\overline{\text{MS}}} = (300 \pm 50)$ MeV. The $1/m$ relativistic corrections at one loop, the $1/m^2$ tree-level spin-independent terms and the $1/m^3$ correction to the kinetic energy are given by $^4$

$$\delta v_1(r) \equiv \left( 2 C_F^2 \frac{m_{\text{red}}}{m_b m_c} - \frac{C_A C_F}{m_{\text{red}}} \right) \frac{\alpha_s^2}{4 r^2} + \frac{C_F \alpha_s}{m_b m_c} \frac{1}{r} \Delta + \frac{1}{2} \left( \frac{1}{m_b^2} + \frac{1}{m_c^2} - \frac{2}{m_b m_c} \right) C_F \pi \alpha_s \delta^{(3)}(r) - \frac{\Delta^2}{8} \left( \frac{1}{m_b^3} + \frac{1}{m_c^3} \right).$$

$^3 \beta_0 = 11 C_A / 3 - 2/3 N_f, \beta_1 = 34 C_A^2 / 3 - 10 N_f C_A / 3 - 2 N_f C_F, \ldots$

$^4$ The result one obtains by choosing $N_f = 4$ and $\Lambda_{N_f=4}^{\overline{\text{MS}}} = 230$ MeV has been checked to be consistent with the central value and the errors of Eq. (5).
Then, the Hamiltonian relevant in order to get the $B_\alpha$ after an explicit calculation we get from Eq. (2), up to order $m_{\text{red}}$ where at leading order $\alpha^4$ produces relevant at order $\alpha^4$ perturbation theory ($G_c$ stands for the Coulombic intermediate states) and can be read off from the second reference in [3]. The other averages can be easily evaluated by means of the standard formulas

$$\langle \delta v_0 G_c \delta v_0 \rangle = -m_{\text{red}} \frac{C_F^2 \beta_0^2 \alpha^4}{2\pi^2} \left( \frac{3 + 3 \gamma_E^2 - \pi^2 + 6 \zeta(3)}{12} - \frac{\gamma_E^2}{2} \ln(\mu a/2) + \frac{1}{4} \ln^2(\mu a/2) \right),$$

with $a(\mu) \equiv 1/(m_{\text{red}} C_F \tilde{\alpha}_s(\mu))$, the Bohr radius of the system. This corresponds to the only contribution relevant at order $\alpha^4$ produced by the Hamiltonian (1) in second-order perturbation theory ($G_c$ stands for the Coulombic intermediate states) and can be read off from the second reference in [3]. The other averages can be easily evaluated by means of the standard formulas

$$\left\langle \frac{1}{r} \ln^2(\mu r) \right\rangle = \frac{1}{a} \left( \ln^2(\mu a/2) + 2(1 - \gamma_E) \ln(\mu a/2) + (1 - \gamma_E)^2 + \frac{\pi^2}{6} - 1 \right),$$

$$\left\langle \frac{1}{r} \ln(\mu r) \right\rangle = \frac{1}{a} (\ln(\mu a/2) - \gamma_E + 1), \quad \left\langle \frac{1}{r} \right\rangle = \frac{1}{a}, \quad \left\langle \frac{1}{r^2} \right\rangle = \frac{2}{a^2},$$

$$\left\langle \frac{1}{r} \Delta \right\rangle = -\frac{3}{a^2}, \quad \langle \Delta^2 \rangle = \frac{5}{a^4}, \quad \langle \delta^{(3)}(r) \rangle = \frac{1}{\pi a^3}.$$

After an explicit calculation we get from Eq. (2), up to order $\alpha^4$:

$$E(B_\alpha)_{\text{pert}} = m_b + m_c + E_0(\mu) \left\{ 1 - \frac{\alpha_s(\mu)}{\pi} \left[ \beta_0 \ln \left( \frac{2 C_F \alpha_s m_{\text{red}}}{\mu} \right) + \frac{4}{3} C_A - \frac{11}{6} \beta_0 \right] + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left[ \frac{3}{4} \beta_0 \ln^2 \left( \frac{2 C_F \alpha_s m_{\text{red}}}{\mu} \right) + \left( 2 C_A \beta_0 - \frac{9}{4} \beta_0^2 - \frac{\beta_1}{4} \right) \ln \left( \frac{2 C_F \alpha_s m_{\text{red}}}{\mu} \right) \right] - \pi^2 C_F^2 \left( \frac{1}{m_b^2} + \frac{1}{m_c^2} - \frac{6}{m_b m_c} \right) m_{\text{red}}^2 + \frac{5}{4} \pi^2 C_F^2 \left( \frac{1}{m_b^3} + \frac{1}{m_c^3} \right) m_{\text{red}}^3 + \pi^2 C_F C_A - \frac{4}{9} C_A^2 - \frac{17}{9} C_A \beta_0 + \frac{1}{2} \zeta(3) + \frac{\pi^2}{24} \left( \beta_0^2 + \frac{\beta_1}{4} + \frac{c}{8} \right) \right\},$$

(3)
where $E_0(\mu) = -m_{\text{red}} \frac{(C_F \alpha_s(\mu))^2}{2}$.

The main problem connected with the perturbative series (3) is the bad convergence in terms of the heavy-quark pole masses. Let us consider, for instance, $\mu = 1.6$ GeV, $m_b = 5$ GeV and $m_c = 1.8$ GeV. Then we get $E(B_c)_{\text{pert}} \simeq 6149$ MeV $\simeq 6800 - 115 - 183 - 353$ MeV, where the second, third and fourth figures are the corrections of order $\alpha_s^2$, $\alpha_s^3$ and $\alpha_s^4$ respectively. The series turns out to be very badly convergent. This reflects also in a strong dependence on the normalization scale $\mu$: at $\mu = 1.2$ GeV we would get $E(B_c)_{\text{pert}} \simeq 5860$ MeV, while at $\mu = 2.0$ GeV we would get $E(B_c)_{\text{pert}} \simeq 6279$ MeV. The origin of this behaviour can be understood in the renormalon language. The pole mass is affected by an IR renormalon ambiguity that cancels against an IR renormalon ambiguity of order $\Lambda_{\text{QCD}}$ present in the static potential [10]. The non-convergence of the perturbative series (3) signals the fact that large $\beta_0$ contributions (coming from the static potential renormalon) are not summed up and cancelled against the pole masses. A possible solution, in order to avoid large perturbative corrections and large cancellations, or, in other words, in order to obtain a well-behaved perturbative expansion, is to resort to a different definition of the mass. The so-called 1$S$ mass of a heavy quark $Q$ is defined as half of the perturbative contribution of the $^3S_1$ $Q - \bar{Q}$ mass [11]. Unlike the pole mass, the 1$S$ mass, containing, by construction, half of the total static energy $\langle 2m + V_{\text{Coul}} \rangle$, is free of ambiguities of order $\Lambda_{\text{QCD}}$. Our strategy will be the following. First, we consider the perturbative contribution (up to order $\alpha_s^4$) of the $^3S_1$ levels of charmonium and bottomonium:

$$E(J/\psi)_{\text{pert}} = f(m_c), \quad E(\Upsilon(1S))_{\text{pert}} = f(m_b),$$

which are respectively a function of the $c$ and the $b$ pole mass and can be read off from Eq. (3) in the equal-mass case, adding to it the spin-spin interaction energy: $m(C_F \alpha_s)^4/3$. We

$^5$The result also depends on the $c$ and $b$ pole masses, which are poorly known. See the following discussion.
invert these relations in order to obtain the pole masses as a formal perturbative expansion depending on the $1S$ mass. Finally, we insert the expressions $m_c = f^{-1}(E(J/\psi)_{\text{pert}})$ and $m_b = f^{-1}(E(\Upsilon(1S))_{\text{pert}})$ in Eq. (3). At this point we have the perturbative mass of the $B_c$ as a function of the $J/\psi$ and $\Upsilon(1S)$ perturbative masses

$$E(B_c)_{\text{pert}} = f(f^{-1}(E(J/\psi)_{\text{pert}}), f^{-1}(E(\Upsilon(1S))_{\text{pert}})).$$  

(4)

If we identify the perturbative masses $E(J/\psi)_{\text{pert}}$, $E(\Upsilon(1S))_{\text{pert}}$ with the physical ones, i.e. $E(J/\psi)_{\text{phys}} = 3097$ MeV and $E(\Upsilon(1S))_{\text{phys}} = 9460$ MeV [15], then the expansion (4) depends only on the scale $\mu$.

![FIG. 1. $|E_3/E_2|$ and $|E_4/E_2|$ as a function of $\mu$, being $E_n$ the order $\alpha_s^n$ contribution to $E(B_c)_{\text{pert}}$. The continuous lines correspond to $\Lambda_{\text{MS}}^{N_f=3} = 300$ MeV, the dashed lines to $\Lambda_{\text{MS}}^{N_f=3} = 250$ and $\Lambda_{\text{MS}}^{N_f=3} = 350$ respectively. $E_3$ vanishes for values of $\mu$ around 1.4 GeV.](image)

In Fig. 1 we show the dependence on $\mu$ of the order $\alpha_s^3$ and $\alpha_s^4$ contributions to $E(B_c)_{\text{pert}}$ respectively. Taking into account that the order $\alpha_s^3$ contribution vanishes at $\mu \simeq 1.4$ GeV, the perturbative series seems to be reliable for values of $\mu$ bigger than $(1.2 \div 1.3)$ GeV and lower than $(2.6 \div 2.8)$ GeV. For instance, $E(B_c)_{\text{pert}} = 6278.5 + 35 + 6.5 + 5.5$ MeV at the scale $\mu = 1.6$ GeV. This is consistent with: i) the fact that, for values of $\mu$ close to or less than 1 GeV, the perturbative calculation (and the initial assumption that $B_c$ is Coulombic or quasi-Coulombic) is expected to break down; ii) the fact that higher values of $\mu$ do not
correspond to the characteristic scale of the system (this is signalled by the appearance of big logarithms in the perturbative expansion); iii) the estimate of the scale $\mu$ inferred in the introduction from the size of the $J/\psi$. More precisely we will take in our analysis $1.2 \text{ GeV} \leq \mu \leq 2.0 \text{ GeV}$ and $250 \text{ MeV} \leq \Lambda_{\overline{\text{MS}}}^{N_f=3} \leq 350 \text{ MeV}$ (in terms of $\alpha_s$ this corresponds to $0.26 \lesssim \alpha_s(2 \text{ GeV}) \lesssim 0.30$). In this way we entirely cover the energy range used in [5] in order to study the $J/\psi$.\footnote{The inclusion of a somewhat higher energy region, which seems to be allowed by Fig. 1, would not change our final result \footnote{Actually the range considered in \footnote{was 1.36 GeV $\leq \mu \leq 1.76$ GeV.}}. E.g. taking $1.2 \text{ GeV} \leq \mu \leq 2.6 \text{ GeV}$ we would get, by keeping the same central values as above, $E(B_c)_{\text{pert}} = 6326^{+29}_{-10} \text{ MeV}$.}

By varying $\mu$ from 1.2 GeV to 2.0 GeV and $\Lambda_{\overline{\text{MS}}}^{N_f=3}$ from 250 MeV to 350 MeV and by calculating the maximum variation of $E(B_c)_{\text{pert}}$ in the given range of parameters, we get as our final result

$$E(B_c)_{\text{pert}} = 6326^{+29}_{-10} \text{ MeV}. \quad (5)$$
The upper limit corresponds to the choice of parameters \( \Lambda_{\text{MS}}^{N_f=3} = 350 \text{ MeV}, \mu = 1.2 \text{ GeV}, \) while the lower limit to \( \Lambda_{\text{MS}}^{N_f=3} = 250 \text{ MeV} \) and \( \mu = 2.0 \text{ GeV}. \) As a consequence of the now obtained good behaviour of the perturbative series in the considered range of parameters, our result appears stable with respect to variations of \( \mu \) (see Fig. 2). We would like to note that the main source of error in Eq. (5) comes from the border region \( 1.2 \text{ GeV} \lesssim \mu < 1.4 \text{ GeV} \) at \( \Lambda_{\text{MS}}^{N_f=3} \lesssim 350 \text{ MeV}, \) where it may become questionable to treat the \( B_c \) as a Coulombic or quasi-Coulombic system (see Fig. 1).

III. DISCUSSION AND CONCLUSIONS

We have calculated the perturbative \( B_c \) mass as defined by Eq. (2). The problem of the bad behaviour of the perturbative series has been overcome by expressing the perturbative \( B_c \) mass in terms of the perturbative \( J/\psi \) and \( \Upsilon(1S) \) masses. The series we obtain has a good convergent behaviour. This fact is relevant since it shows that the scale hierarchy considered in the introduction \( (m > m_{\nu} > \Lambda_{\text{QCD}}) \), which led to the Hamiltonian (1), correctly applies to the system under consideration. In other words, the result we get is consistent with the assumption made that the \( B_c \) system is Coulombic or quasi-Coulombic. Moreover, the perturbative series turns out to be weakly sensitive to variations of \( \mu \) (the renormalization scale) and \( \Lambda_{\text{MS}}^{N_f=3} \) in the range \( 1.2 \text{ GeV} \leq \mu \leq 2.0 \text{ GeV} \) and \( 250 \text{ MeV} \leq \Lambda_{\text{MS}}^{N_f=3} \leq 350 \text{ MeV}. \) The result appears, therefore, reliable from a perturbative point of view. Non-perturbative contributions have not been taken into account so far. They affect the identification of the perturbative masses \( E(B_c)_{\text{pert}}, E(\Upsilon(1S))_{\text{pert}}, E(J/\psi)_{\text{pert}}, \) with the corresponding physical ones through Eq. (3). Let us call these non-perturbative contributions \( \delta E(B_c), \delta E(\Upsilon(1S)) \) and \( \delta E(J/\psi) \) respectively. As discussed in the introduction, depending on the actual kinematic situation of the system, they can be of potential or non-potential nature. In the last

\[8\] For instance, an analogous analysis carried out on the \( B_s \) system does not show any sign of convergence.
case they can be encoded into non-local condensates or into local condensates. There is no way to discriminate among these situations, since the size of what would be the energy scale $mv^2$ of the system with respect to $\Lambda_{\text{QCD}}$ is unknown. Non-perturbative contributions affect the identification of Eq. (4) with the physical $B_c$ mass roughly by an amount $\simeq -\frac{\delta E(J/\psi)}{2}$.

Assuming $|\delta E(J/\psi)| \lesssim 300$ MeV, $|\delta E(\Upsilon(1S))| \lesssim 100$ MeV and $\delta E(\Upsilon(1S)) \lesssim \delta E(B_c) \lesssim \delta E(J/\psi)$, the identification of our result (5) with the physical $B_c$ mass may, in principle, be affected by uncertainties, due to the unknown non-perturbative contributions, as big as $\pm 200$ MeV. However, the different $\delta E$ are correlated, so that we expect, indeed, smaller uncertainties. If we assume, for instance, $\delta E(\Upsilon(1S))$ and $\delta E(J/\psi)$ to have the same sign, which seems to be quite reasonable, then the above uncertainty reduces to $\pm 100$ MeV. Constraining even more the form of $\delta E$, by evaluating it from the Voloshin–Leutwyler formula (i.e. in terms of local condensates), as given in Ref. [5], we get a negative contribution (since the term coming from the $J/\psi$ is the dominating one) of less than 100 MeV. This feature, if preserved also in the other kinematic situations, would confirm, indeed, that the effect of the non-perturbative contributions is not too big and that its effect is to lower down the perturbative result given in Eq. (5). More quantitative statements are difficult to make, since, differently from the perturbative case discussed in the previous section, they appear to be dependent on the choice of the parameters.

The result we get in Eq. (5) is compatible with the experimental value $E(B_c)_{\text{phys}} = 6.40 \pm 0.39 \pm 0.13$ GeV reported in [1]. We mention that OPAL reports in [21] 2 candidates $B_c$ in hadronic $Z^0$ decays events, with an estimated mass $E(B_c)_{\text{phys}} = 6.32 \pm 0.06$ GeV. Also this value compares favourably with ours. Having more precise and established experimental data will make it possible to make some more definite statements. In particular, it will be possible to give, inside a Coulombic or quasi-Coulombic picture, a precise estimate of the size of the non-perturbative effects in the $B_c$ mass. In the table, we also report, for comparison, some of the other determinations of the $B_c$ mass available in the literature. The results quoted in [16–19] rely on potential models (essentially a Coulomb plus a confining potential)
and are reported without errors. The figure that appears in the table in correspondence of Ref. [19] refers to an average of different models performed by those authors. Finally [20] reports the result of a very recent lattice calculation. We would like to note that, if one assumes that potential models give a $B_c$ mass close to reality, then, comparing the potential model predictions with Eq. (5), non-perturbative contributions seem to be of the order $-(40 \div 100)$ MeV (consistently with expectations, non-perturbative corrections become as smaller as perturbation theory better works, i.e. in correspondence of low values of $\Lambda_{\overline{MS}}^{N_f=3}$ and high values of $\mu$). Finally, it is interesting to notice that these figures are completely consistent with the general discussion on the uncertainties, coming from non-perturbative contributions, done above.

| Obtained from | $B_c(1^1S_0)$ in MeV |
|---------------|----------------------|
| Experiment [1] | $6400 \pm 390 \pm 130$ |
| Eq. (5) (pert. mass) | $6326_{-9}^{+29}$ |
| [16] | $6264$ |
| [17] | $6253$ |
| [18] | $6286$ |
| [19] | $6255$ |
| [20] | $6.386(9)(98)(15)$ |

TABLE I. Mass of the $B_c$ meson. The result labelled with Eq. (5) refers to the present work.
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