DERIVED EQUIVALENCE AND GROTHENDIECK RING OF VARIETIES: THE CASE OF K3 SURFACES OF DEGREE 12 AND ABELIAN VARIETIES

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ABSTRACT. In this paper, we ask the question of whether the difference $[X] - [Y]$ of the classes of a Fourier–Mukai pair $(X, Y)$ of smooth projective varieties in the Grothendieck ring of varieties is annihilated by some power of the class $L = [A^1]$ of the affine line. We give an affirmative answer for a very general K3 surface of degree 12, and a negative answer for a very general abelian varieties of dimension greater than one.

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1. Introduction

Let $X$ and $Y$ be a pair of smooth projective varieties (or more generally smooth and proper Deligne–Mumford stacks) over a field $k$. We say that $X$ is $D$-equivalent to $Y$ if the bounded derived category of coherent sheaves $D(X) := D^b \text{coh} X$ is equivalent to $D(Y)$ as a $k$-linear triangulated category (we also say that $Y$ is a Fourier–Mukai partner of $X$).

It is shown in the pioneering paper [Muk81] that an abelian variety is D-equivalent to its dual, meaning that non-birational varieties could be D-equivalent. The following natural question arises from this observation.

Question 1.1. Which piece of information of a variety (or more generally Deligne–Mumford stack) does the derived category have? In other words, if $X$ and $Y$ are D-equivalent, which invariants of $X$ and $Y$ do coincide?

It follows from the uniqueness of the Serre functor [BK89] that $X$ and $Y$ have the same dimension and isomorphic (anti-)canonical rings. The Hochschild cohomology ring, whose graded pieces are the group of natural transformations from the identity functor to the shift functor, is also derived invariant. Combined with the Hochschild–Kostant–Rosenberg isomorphism [HKR62, Swa96, Yek02, Cal05, ACT12], this gives the following partial coincidence of the Hodge numbers for any integer $i \in \mathbb{Z}$:

$$
\sum_{q-p=i} h^{p,q}(X) = \sum_{q-p=i} h^{p,q}(Y). \quad (1.1)
$$
Another deep result is the coincidence of Chow motives with rational coefficients up to Tate twists [Orl05, Tab05, Kon09, Tab13].

On the other hand, Popa and Schnell [PS11] proved (in characteristic zero) that the Picard schemes of $X$ and $Y$ are isogenous. As a corollary, it follows that the Hodge numbers of $X$ and $Y$ coincide for all $(p, q)$ under the assumption that $\dim X(= \dim Y) \leq 3$. Honigs [Hon16] also showed that, if $k = \mathbb{F}_q$, the congruence zeta functions coincide again in dimension 3 or less. The aim of this paper, in a word, is to discuss simultaneous refinements of these results.

Recall that smooth projective varieties $X$ and $Y$ are $K$-equivalent if there is a diagram

$$
\begin{array}{ccc}
W & \rightarrow & X \\
\downarrow p & & \downarrow q \\
Y & \rightarrow & \end{array}
$$

(1.2)

where $W$ is a normal variety and $p, q$ are birational projective morphisms satisfying $p^*K_X \simeq q^*K_Y$. It is conjectured and has been shown in some cases that K-equivalent varieties (and stacks) are D-equivalent. Conversely, it is also conjectured that birational and D-equivalent smooth projective varieties of non-negative Kodaira dimension are K-equivalent. For the details and the history of DK hypothesis, see the recent survey article [Kaw17] and references therein.

Combined with the arguments above, it is natural to ask if K-equivalent varieties have the same Hodge numbers when $k = \mathbb{C}$. In fact, this is known to be the case in its full generality. One way to prove this is to utilize the method of motivic integration, and here the Grothendieck ring of varieties comes into the picture.

The Grothendieck ring of varieties over a field $k$, which will be denoted by $K_0(\text{Var}/k)$ in this paper, is the quotient of the free abelian group generated by the set of isomorphism classes of schemes of finite type over $k$ modulo the relations

$$[X] = [X \setminus Z] + [Z] \quad (1.3)$$

for closed embeddings $Z \subset X$. Multiplication in $K_0(\text{Var}/k)$ is defined by the Cartesian product, which is easily seen to be associative, commutative, and unital with $1 = [\text{Spec } k]$.

It is shown by means of the motivic integration of Kontsevich [Bat98, DL99] that classes of smooth projective K-equivalent varieties coincide in the completed Grothendieck ring of varieties. This is a completion of the localized Grothendieck ring $K_0(\text{Var}/k) [L^{-1}]$, where $L = [A^1]$ is the class of the affine line (see Definition 7.1). From this result, by taking (an extension of) the Hodge–Deligne polynomial, one immediately obtains the coincidence of the Hodge numbers for K-equivalent varieties.

In view of this result, it is natural to ask if D-equivalent varieties have the same class in the completed (or localized) Grothendieck ring of varieties. Recently a number of positive results have been obtained by several groups of people.

(1) Borisov [Bor] showed that the Pfaffian-Grassmannian pairs $(X, Y)$ of Calabi–Yau 3-folds [Rød00] satisfy

$$([X] - [Y]) (L^2 - 1) (L - 1) L^7 = 0 \quad (1.4)$$

in $K_0(\text{Var}/k)$, giving a first counter-example to the cancellation problem, which asks the injectivity of the homomorphism $K_0(\text{Var}/k) \rightarrow K_0(\text{Var}/k) [L^{-1}]$. (1.4) is subsequently refined by Martin [Mar16] to

$$([X] - [Y]) \cdot L^6 = 0. \quad (1.5)$$
It is shown in [IMOUa] that for pairs \((X', Y')\) of smooth Calabi–Yau 3-folds obtained as certain degenerations of \(X\) and \(Y\), one has
\[
([X'] - [Y']) \cdot \mathbb{L} = 0. \tag{1.6}
\]
Both the Pfaffian-Grassmannian pairs \((X, Y)\) of Calabi–Yau 3-folds and their degenerations \((X', Y')\) are Fourier–Mukai partners by [BC09, Kuzb] and [Kuza] (see also [Ued]).

In [KSa], Kuznetsov and Shinder studied the Fourier–Mukai partners of K3 surfaces of degree 8 and 2. A K3 surface \(X\) is said to be of degree \(d\) if there is an ample line bundle \(L\) on \(X\) satisfying \(L \cdot L = d\). They proved that there are such pairs \((X'', Y'')\) satisfying
\[
([X''] - [Y'']) \cdot \mathbb{L} = 0. \tag{1.7}
\]

In [BCP17, OR], it is shown that a generic pair of the so-called Kanazawa Calabi–Yau 3-folds \((X''', Y''')\) are \(D\)-equivalent, non-birational, and satisfy the equality
\[
([X'''] - [Y''']) \cdot \mathbb{L}^4 = 0. \tag{1.8}
\]

These results lead to the following problem:

**Problem 1.2** (IMOUb Problem 1.3). Let \((X, Y)\) be a Fourier–Mukai pair. Does the equality
\[
([X] - [Y]) \cdot \mathbb{L}^k = 0 \in K_0(\text{Var}/k) \tag{1.9}
\]
hold for a non-negative integer \(k\)?

For example, one can take \(k = 0\) if \(X\) and \(Y\) are related by an elementary flop appearing in [BO]. The affirmative answer to Problem 1.2 was stated as a conjecture in [KSa, Conjecture 1.6] around the same time as [IMOUb]. Theorem 1.5 below gives an example of a Fourier–Mukai pair where (1.9) does not hold for any \(k\) (i.e., a counter-example to [KSa, Conjecture 1.6]). The same example is also discovered independently in [Efi, Theorem 3.1]. After the discovery of this example, a new conjecture is proposed in [KSb, Conjecture 1.6], stating that the answer to Problem 1.2 is affirmative if \(X\) and \(Y\) are simply connected.

One of the purposes of this paper is, as an application of the geometry of equivariant vector bundles on homogeneous spaces of type \(D\), to prove that a general Fourier–Mukai pair of K3 surfaces of degree 12 gives another example of an affirmative answer to Problem 1.2.

**Theorem 1.3.** Let \(X\) be a general K3 surface of degree 12 over \(\mathbb{C}\). Then there exists a non-isomorphic Fourier–Mukai partner \(Y\) of \(X\) satisfying
\[
([X] - [Y]) \cdot \mathbb{L}^3 = 0. \tag{1.10}
\]

**Remark 1.4.** After the completion of the first draft of this paper, Brendan Hassett and Kuan-Wen Lai proved in [HL, Theorem 4.1] the stronger equality
\[
([X] - [Y]) \cdot \mathbb{L} = 0 \in K_0(\text{Var}/\mathbb{C}), \tag{1.11}
\]
by a completely different method based on Cremona transformations of \(\mathbb{P}^4\).

It follows from [Ogu02, Proposition 1.10] (see also [HLOY04, Corollary 2.7.4]) that the number of isomorphism classes of Fourier–Mukai partners of a K3 surface over \(\mathbb{C}\) with Picard number 1 and degree 12 is 2. Hence Theorem 1.3 and [HL, Theorem 4.1] gives an affirmative answer to Problem 1.2 for very general K3 surfaces of degree 12 over \(\mathbb{C}\).

On the other hand, there are also cases where Problem 1.2 has a negative answer.
Theorem 1.5. For any integer \( g \geq 2 \), there exists a pair of non-isomorphic complex abelian \( g \)-folds \((A, B)\) which are \( D \)-equivalent but \([A] \neq [B]\) in the completed Grothendieck ring of varieties \( \hat{K}_0(V_{\text{ar}}/\mathbb{C}) \).

Another example of a negative answer to Problem 1.2 is given in Example 7.8. It is an interesting problem to modify Problem 1.2 in such a way that it still has meaningful implication(s) such as the coincidence of the Hodge numbers and that of the number of rational points.

Another interesting problem is to extend the whole picture in such a way that not only equivalences but also admissible embeddings of triangulated categories are taken into account. Let \( \Gamma_k \) be the ring defined in [BLL04, Definition 8.1] as the quotient of the free abelian group generated by quasi-equivalence classes of enhanced (=pretriangulated dg) bounded derived categories \( \mathcal{D}(X) \) of smooth complex projective varieties \( X \) by the relations

\[
[\mathcal{D}(X)] = [\mathcal{D}(Y_1)] + \cdots + [\mathcal{D}(Y_n)]
\]

for semiorthogonal decompositions

\[
\mathcal{D}(X) = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle
\]

with \( \mathcal{D}(Y_i) \cong \mathcal{B}_i \) for \( i = 1, \ldots, n \). Multiplication in \( \Gamma_k \) is defined by the tensor product of dg categories. It is shown in [BLL04, Section 8] that there exists a motivic measure

\[
K_0(\text{Var}/k) \to \Gamma_k
\]

sending the class \([X]\) of a smooth projective variety to \([\mathcal{D}(X)]\), which descends to a ring homomorphism

\[
K_0(\text{Var}/k)/(\mathbb{L} - 1) \to \Gamma_k.
\]

Problem 1.2 is closely related to the problem, given implicitly at the end of [BLL04, Section 8], of asking how close to being injective the map (1.15) is. [Orl05, Conjecture 1] also has a flavor similar to Problem 1.2 although there is no apparent implication in either direction.

This paper is organized as follows: In Section 2, we give pairs \((X, Y)\) of Calabi–Yau manifolds defined as zeros of sections of equivariant vector bundles on homogeneous spaces of \( \text{Spin}(2m) \), and prove the equality (2.13) in the Grothendieck ring of varieties. By specializing to \( m = 4 \), we obtain pairs of K3 surfaces of degree 12. In Section 3, we show that \( X \) can also be described as a linear section of a homogeneous space of \( \text{Spin}(2m + 2) \). In Section 4, we use results of Mukai to complete the proof of Theorem 1.3. In Section 5, we give pairs \((X, Y)\) of K3 surfaces in 5-dimensional quadrics satisfying \(([X'] - [Y']) \cdot \mathbb{L}^2 = 0\). This is a side result which is not on the main line of discussion in this paper. In Section 6, we discuss the relation between semiorthogonal decompositions of derived categories and relations in the Grothendieck ring of varieties in an example. In Section 7, we discuss negative answers to Problem 1.2.

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2. K3 SURFACES IN $OG(4,8)$

Let $V_0$ be a vector space of dimension $m$ and $V_{nat} := V_0 \oplus V_0^\vee$ be a vector space of dimension $2m$ equipped with the natural non-degenerate bilinear form

$$\langle v + \tilde{v}, v' + \tilde{v}' \rangle = \tilde{v}(v') + \tilde{v}'(v).$$  \hfill (2.1)

The spin group $G := \text{Spin}(V_{nat})$ is the simply-connected simple algebraic group, which is obtained as the double cover of $\text{SO}(V_{nat})$. The spinor representation is a $2m$-dimensional representation of $G$ on $\bigwedge V_0$, which decomposes as the direct sum of half spinor representations $V_1 := \bigwedge^{\text{even}} V_0$ and $V_2 := \bigwedge^{\text{odd}} V_0$. These half spinor representations are related to each other by an outer automorphism of $\text{Spin}(2m)$. They are self-dual if $m$ is even, and dual to each other if $m$ is odd. We number the simple roots in such a way that the fundamental weights $\omega_1$ and $\omega_2$ correspond to the half spinor representations $V_1$ and $V_2$ respectively. Given a dominant integral weight $\lambda$ of $G$, the irreducible representation of $G$ with highest weight $\lambda$ will be denoted by $V_\lambda$. We also write $(i, j) := i\omega_1 + j\omega_2$, so that $V_1 = V_{(1,0)}$ and $V_2 = V_{(0,1)}$.

Let $F_i$ be the homogeneous space of $G$ associated with the Dynkin diagram with the $i$-th simple root crossed out. We number simple roots in such a way that

$$F_i = OG(m - i + 1, V_{nat})$$

$$:= \{ V \subset V_{nat} \mid \langle -, - \rangle |_V = 0 \text{ and } \dim V = m - i + 1 \}$$  \hfill (2.2)

(2.3)

for $i = 3, \ldots, m$. Let further $F_{12}$ be the homogeneous space of $G$ associated with the crossed Dynkin diagram with the first and the second simple root crossed out. It can naturally be identified with $OG(m - 1, V_{nat}) \cong \mathbb{P}_{F_1}(S_1^\vee)$, where $S_i$ is the tautological bundle on $F_i \cong OG(m, V_{nat})$ for $i = 1, 2$. They fit into the following diagram:

$$\begin{array}{c}
F_{12} \\
| \quad \downarrow p_1 \quad \downarrow p_2 \\
F_1 & F_2
\end{array}$$

\hfill (2.4)

The Picard group of $F_{12}$ is given by

$$\text{Pic } (F_{12}) = p_1^* \text{Pic } (F_1) \oplus p_2^* \text{Pic } (F_2) \cong \mathbb{Z}^2.$$  \hfill (2.5)

The line bundle $\mathcal{O}_{F_{12}}(1, 1) := \mathcal{O}_{F_1}(1) \boxtimes \mathcal{O}_{F_2}(1)$ is very ample, and one has

$$F_{12} \cong \mathbb{P}_{F_1}((p_1^* \mathcal{O}_{F_{12}}(1, 1))^\vee)$$  \hfill (2.6)

for both $i = 1, 2$, where $(p_i^* \mathcal{O}_{F_{12}}(1, 1))$ is the locally free sheaf of rank $m$ on $F_i$ associated with the representation of $P_i \cong S(\text{GL}(m) \times \text{GL}(1))$ with highest weight $(1, 1)$. If $i$ and $j$ are non-negative integers, then one has

$$H^0(\mathcal{O}_{F_{12}}(i, j)) \cong V_{(i,j)}^\vee$$  \hfill (2.7)

by the Borel–Weil theorem.

Let

$$s \in H^0(F_{12}, \mathcal{O}_{F_{12}}(1, 1)) \simeq H^0(F_1, p_1^* \mathcal{O}_{F_{12}}(1, 1)) \simeq H^0(F_2, p_2^* \mathcal{O}_{F_{12}}(1, 1))$$  \hfill (2.8)
be a general section and let
\[ D := Z(s) \subset F_{12}, \]
\[ X := Z(p_{1\ast}s) \subset F_1, \]
\[ Y := Z(p_{2\ast}s) \subset F_2, \]
be its zero loci, which are smooth complete intersections by Bertini [Muk92, Theorem 1.10]. One can easily compute the rank and the degree of \( (p_i)_\ast \mathcal{O}_{F_{12}}(1, 1) \) to show that \( X \) and \( Y \) are Calabi–Yau of dimension \( m(m - 3)/2 \). Set \( \pi_i = p_i|_D : D \to F_i \) for \( i = 1, 2 \).

**Lemma 2.1.** The morphism \( \pi_1 \) is a \( \mathbb{P}^{m-2} \)-bundle over \( F_1 \setminus X \) and a \( \mathbb{P}^{m-1} \)-bundle over \( X \), which are locally trivial in the Zariski topology. The same holds for \( \pi_2 \).

**Proof.** Since the arguments for \( \pi_1 \) and \( \pi_2 \) are the same, we only give it for \( \pi_1 \). Fix a point \( x \in F_1 \). Since \( p_1 \) is a projective bundle and \( \mathcal{O}_{F_{12}}(1, 1) \) is \( p_1 \)-ample, by cohomology-and-base-change, we obtain the standard isomorphism
\[ p_{1\ast}\mathcal{O}_{F_{12}}(1, 1)|_x \iso H^0 \left( p_1^{-1}(x), \mathcal{O}_{F_{12}}(1, 1)|_{p_1^{-1}(x)} \right) \]
sending \( (p_{1\ast}s)(x) \in p_{1\ast}\mathcal{O}_{F_{12}}(1, 1)|_x \) to \( s|_{p_1^{-1}(x)} \). Hence
\[ D \cap p_1^{-1}(x) = \pi_1^{-1}(x) \]
is isomorphic to \( p_1^{-1}(x) \iso \mathbb{P}^{m-1} \) if \( x \in X \), and to a hyperplane therein otherwise. It follows that the short exact sequence
\[ 0 \to \mathcal{O}_{F_1} \to p_{1\ast}\mathcal{O}_{F_{12}}(1, 1) \to p_{1\ast}\mathcal{O}_{F_{12}}(1, 1)/\mathcal{O}_{F_1} =: \mathcal{Q} \to 0 \]
splits Zariski locally on \( F_1 \setminus X \), so that \( \pi_1 \) is the \( \mathbb{P}^{m-2} \)-bundle over \( F_1 \setminus X \) associated to the locally free sheaf \( \mathcal{Q} \). The latter claim follows from \( s|_{p_1^{-1}(X)} = 0 \).

**Corollary 2.2.** One has
\[ ([X] - [Y])\mathbb{L}^{m-1} = 0 \]
in the Grothendieck ring of varieties.

**Proof.** One has
\[ [D] = ([F_1] - [X])[\mathbb{P}^{m-2}] + [X][\mathbb{P}^{m-1}] = ([F_2] - [Y])[\mathbb{P}^{m-2}] + [Y][\mathbb{P}^{m-1}] \]
by Lemma 2.1. Since \( F_1 \) and \( F_2 \) are related by an outer automorphism of \( \text{Spin}(2m) \) and hence isomorphic as algebraic varieties, one has
\[ 0 = ([X] - [Y])([\mathbb{P}^{m-1}] - [\mathbb{P}^{m-2}]) = ([X] - [Y])\mathbb{L}^{m-1}, \]
and Corollary 2.2 is proved.

Let \( G(m, V_0 \oplus V_0^\vee) \) be the Grassmannian of \( m \)-space in \( V_0 \oplus V_0^\vee \), and \( S \) be the universal subbundle on it. The zero of the section \( s' \) of \( \text{Sym}^2 S^\vee \) associated with the natural pairing \( \langle -, - \rangle \) on \( V_0 \oplus V_0^\vee \) is the homogeneous space of the form
\[ \mathcal{O}(V_0 \oplus V_0^\vee)/\text{GL}(V_0). \]
It has two connected components, which can be identified with \( F_1 \) and \( F_2 \) in such a way that \( S^\vee|_{F_i} \) is the equivariant vector bundle associated with the irreducible representation of \( P_i \) with the highest weight \( \omega_{2i} \) for both \( i = 1 \) and 2. Note that the lowest weight of this irreducible representation of \( P_1 \) (resp. \( P_2 \)) is \( \omega_1 + \omega_2 \) (resp. \( -\omega_1 + \omega_2 \)), so that \( S|_{F_1} \) (resp. \( S|_{F_2} \)) is the equivariant vector bundle associated with the irreducible representation
of $P_1$ (resp. $P_2$) with the highest weight $-\omega_1 + \omega_2$ (resp. $\omega_1 - \omega_2$). Since the ample generator of Pic $G(m, V_0 \oplus V_0^\vee)$ restricts to twice the ample generator of Pic $F_1$, one has
\[ S(1)|_{F_1} \cong (p_1)_* \mathcal{O}_{F_{12}}(-1, 1) \otimes (p_1)_* \mathcal{O}_{F_{12}}(2, 0) \cong (p_1)_* \mathcal{O}_{F_{12}}(1, 1). \] (2.17)

The same reasoning for $F_2$ gives
\[ S(1)|_{F_2} = (p_2)_* \mathcal{O}_{F_{12}}(1, 1). \] (2.19)

It follows that the zero of the section
\[ s + s' \in H^0(\mathcal{S}(1) \oplus \text{Sym}^2 \mathcal{S}^\vee) \] (2.20)
is isomorphic to the disjoint union of $X$ and $Y$.

When $m = 4$, the homogeneous spaces $F_1$ and $F_2$ are related to $F_4$ by an outer automorphism of Spin(8) called the triality automorphism. The homogeneous space $F_4 = \text{OG}(1, V_{\text{nat}})$ is a quadric hypersurface in $G(1, V_{\text{nat}}) \cong \mathbb{P}^7$, and $X$ and $Y$ are K3 surfaces of degree 12.

3. K3 SURFACES IN OG(5, 10)

Set $\mathcal{O}_{F_1}(i) := \mathcal{E}_{(i,0)}$ and $\mathcal{E} := \mathcal{E}_{(0,1)}$, so that $H^0(\mathcal{O}_{F_1}(1)) \cong V_1^\vee$ and $H^0(\mathcal{E}) \cong V_2^\vee$. Let
\[ \mathcal{K} := \text{Ker}(H^0(\mathcal{E}) \otimes \mathcal{O}_{F_1} \to \mathcal{E}) \subset H^0(\mathcal{E}) \otimes \mathcal{O}_{F_1} \cong V_2^\vee \otimes \mathcal{O}_{F_1} \] (3.1)
be the kernel of the natural morphism, which is locally free since $\mathcal{E}$ is globally generated. Consider the following diagram:
\[ \begin{array}{ccc}
\mathbb{P} := \mathbb{P}_{F_1}(\mathcal{O}_{F_1}(-1) \otimes \mathcal{K}) & \xrightarrow{\pi} & \mathbb{P}(V_1 \oplus V_2^\vee) \\
F_1 & \xrightarrow{\mu} & \end{array} \] (3.2)

Here $\pi$ is the structure morphism of the $\mathbb{P}^4$-bundle and $\mu$ is the morphism defined by
\[ \mathcal{O}_p(-1) \subset \pi^*(\mathcal{O}_{F_1}(-1) \otimes \mathcal{K}) \subset (V_1 \oplus V_2^\vee) \otimes \mathcal{O}_p. \] (3.3)

Set
\[ \Sigma := \mu(\mathbb{P}) \subset \mathbb{P}(V_1 \oplus V_2^\vee). \] (3.4)

Since $\mathcal{O}_{F_1}(1)$ is very ample on $F_1$,
\[ \mu : \mathbb{P} \setminus \mathbb{P}_{F_1}(\mathcal{K}) \to \Sigma \setminus (\Sigma \cap \mathbb{P}(V_2^\vee)) \] (3.5)
is an isomorphism, where $\mathbb{P}_{F_1}(\mathcal{K})$ is identified with
\[ \mathbb{P}_{F_1}(0 \otimes \mathcal{K}) \subset \mathbb{P} := \mathbb{P}_{F_1}(\mathcal{O}_{F_1}(-1) \otimes \mathcal{K}) \] (3.6)
and similarly for $\mathbb{P}(V_2^\vee) \subset \mathbb{P}(V_1 \oplus V_2^\vee)$. Let
\[ s : V_1 \to V_2^\vee \] (3.7)
be a linear map, and $s \in H^0(\mathcal{E}(1))$ be its image by the map
\[ V_1^\vee \otimes V_2^\vee \cong H^0(\mathcal{O}_{F_1}(1)) \otimes H^0(\mathcal{E}) \to H^0(\mathcal{E}(1)). \] (3.8)

**Lemma 3.1.** The map (3.8) is surjective. In particular, $s$ is general if so is $\tilde{s}$.

**Proof.** This follows from the fact that the map (3.8) is a non-zero $G$-equivariant map and $H^0(\mathcal{E}(1)) \cong V_{(1,1)}^\vee$ is an irreducible representation of $G$. \qed

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The map \( \tilde{s} \) induces a linear embedding
\[
\mathbb{P}(V_1) \hookrightarrow \mathbb{P}(V_1 \oplus V_2^\vee), \quad [p] \mapsto [(p, \tilde{s}(p))],
\] (3.9)
whose image will be denoted by \( L_\tilde{s} \). Conversely, if a linear subspace \( L \subset \mathbb{P}(V_1 \oplus V_2^\vee) \) of codimension \( 2^{m-1} \) satisfies \( L \cap \mathbb{P}(V_1) = \emptyset \), then there exists an element \( \tilde{s} \in V_1^\vee \otimes V_2^\vee \) such that \( L = L_{\tilde{s}} \).

Proposition 3.3 below is an adaptation of [IM] Proposition 4.1 to the present situation.

**Proposition 3.2.** \( Z(s) \subset F_1 \subset \mathbb{P}(V_1) \) and \( \Sigma \cap L_{\tilde{s}} \subset L_{\tilde{s}} \cong \mathbb{P}(V_1) \) are projectively equivalent.

The proof of Proposition 3.2 is identical to that of [IM] Proposition 4.1, and hence omitted.

Let
\[
\begin{align*}
&\xymatrix{ & H_{12} \ar@{-}[dl]_{h_1} \ar@{.}[dr]^{h_2} & \\
H_1 & & H_2}
\end{align*}
\] (3.10)
be the diagram (2.4) with \( m \) replaced with \( m + 1 \). Proposition 3.3 below is an adaptation of [IM] Lemma 4.3:

**Proposition 3.3.** Let \( F_1 \subset H_1 \) be an equivariant embedding corresponding to the unique inclusion of the Dynkin diagram of type \( D_m \) into that of type \( D_{m+1} \). Then we have the following commutative diagram:
\[
\begin{array}{c}
\xymatrix{ h_1^{-1}(F_1) \ar[r]^-{\sim} & \mathbb{P} \\
H_2 \ar[u]_{h_2 h_1^{-1}(F_1)} \ar[r]^-{\sim} & \Sigma \ar[u]_\mu }
\end{array}
\] (3.11)

**Proof.** Since the argument is the same as the proof of [IM] Lemma 4.3], we will be brief. Let \( B \subset P_1 \) be the Borel and a parabolic subgroup of \( G = \text{Spin}(2m+2) \) with \( H_1 = G/P_1 \), and \( W \) the Weyl group of \( G \). By using [CR13] Lemma 2.4 and Lemma 3.4 below, we can see that the Tits transform \( h_2 (h_1^{-1}(F_1)) \) of \( F_1 = \tilde{B}w^{-1}P_1/P_1 \subset H_1 \) in the diagram (3.10) is \( H_2 = G/P_2 \), where \( w \in W \) is also computed by regarding \( F_1 \) as the Tits transform of the Borel fixed point in a \((2m)\)-dimensional quadric hypersurface with the \( G \)-action.

The isomorphism \( h_1^{-1}(F_1) \simeq \mathbb{P} \) and the commutativity of (3.11) follow from the branching rule of the representations of parabolic subgroups \( P_i \cap \text{Spin}(2m) \subset P_1 \).

**Lemma 3.4.** Let \( W \) and \( W_{P_i} \) be the Weyl groups of \( G = \text{Spin}(2m+2) \) and its parabolic subgroups \( P_i \) for \( i = 1, \ldots, m+1 \). Let further \( W_{P_i} \subset W \) be the set of minimal length representatives of \( W_{P_i} \setminus W \) for \( i = 1, \ldots, m+1 \). The longest element of \( W_{P_i} \) and \( W_{P_i} \) will be denoted by \( w_i \) and \( w_i' \), respectively. Let \( \tilde{w}_{m+1} \in W_{P_{m+1}} \) be the minimal length representative of \( W_{P_i}w_{m+1} \in W_{P_i} \setminus W \). Then one has
\[
W_{P_i}w_1 \tilde{w}_{m+1} = W_{P_2}w_2.
\] (3.12)

**Proof.** One of a reduced decomposition of the longest element of the Weyl group of each type is given by [Lit98]. In particular, we have
\[
w_1 = (s_2)(s_3s_2)(s_4s_3s_2)(\ldots)(s_{m+1}\ldots s_4s_3s_2),
\] (3.13)
\[
w_{m+1} = (s_1s_2)(s_3s_1s_2s_3)(s_4s_3s_1s_2s_3s_4)(\ldots)(s_m\ldots s_4s_3s_1s_2s_3s_4\ldots s_m).
\] (3.14)
By using (3.14) and by computing the right action of \( w_{m+1} \in W \) on a weight \( \omega_1 \), we can derive the following formula inductively:

\[
\tilde{w}_{m+1} = \begin{cases} 
(s_1)(s_3s_2)(s_4s_3s_1)(\ldots)(s_m \ldots s_4s_3s_1) & \text{for even } m, \\
(s_1)(s_3s_2)(s_4s_3s_1)(\ldots)(s_m \ldots s_4s_3s_2) & \text{for odd } m.
\end{cases}
\] (3.15)

Similarly, we can also check by an induction that the right action of \( w_1 \tilde{w}_{m+1} \in W \) on a weight \( \omega_2 \) is the same as that of \( (s_2s_3s_4 \ldots s_{m+1}) \tilde{w}_{m+1} \), \( \tilde{w}_{m+1} \), which is nothing but the longest element \( w^2 \in W \). This concludes the proof of Lemma 3.4. □

By setting \( m = 4 \), one concludes that a general linear section of \( \text{OG}(5,10) \) is projectively equivalent (and hence isomorphic) to the zero of a general section of \( E(1) \) on \( \text{OG}(4,8) \).

4. Projective duality and derived equivalence

The orthogonal Grassmannian \( \text{OG}(5,10) \) of dimension 10 can be embedded into the projectivization \( \mathbb{P} \) of 16-dimensional half spinor representation of \( \text{Spin}(10) \). The projective dual variety of \( \text{OG}(5,10) \) in the dual projective space \( \mathcal{P} \) is also isomorphic to \( \text{OG}(5,10) \). It is known in [Muk88, Theorem 0.3] that a generic K3 surface of degree 12 can be described as the intersection \( \text{OG}(5,10) \cap L \) with a linear subspace \( L \subset \mathbb{P}_{15} \) of codimension 8, which is unique up to the action of \( \text{Spin}(10) \). As explained in [Muk99, Example 1.3], the intersection

\[
\check{X} := \text{OG}(5,10) \cap L^\perp \in \check{\mathbb{P}}_{15}
\] (4.1)

of the dual \( \text{OG}(5,10) \) with \( L^\perp \) in the dual projective space is the moduli space

\[
\check{X} \cong \mathcal{M}_S(2, \mathcal{O}_X(1), 3),
\] (4.2)

where \( \mathcal{M}_S(r, \ell, t) \) is the moduli space of stable sheaves \( E \) on \( S \) satisfying

\[
\text{rank } E = r, \quad c_1(E) = \ell, \quad \chi(E) = r + t.
\] (4.3)

\((X, \check{X})\) give Fourier–Mukai partners, which are not isomorphic if \( X \) has Picard number one by [HLOY03, Theorem 2.1] (see also its proof).

**Proposition 4.1.** One has an isomorphism \( \check{X} \cong Y \).

**Proof.** A general element \( \tilde{s} \in V_1^\vee \otimes V_2^\vee \) defines the linear subspaces

\[
L_1 := \text{Im } ((\text{id}_{V_1}, \tilde{s}) : \mathbb{P}(V_1) \to \mathbb{P}(V_1 \oplus V_2^\vee))
\] (4.4)

and

\[
L_2 := \text{Im } ((-\tilde{s}, \text{id}_{V_2}) : \mathbb{P}(V_2) \to \mathbb{P}(V_1^\vee \oplus V_2)),
\] (4.5)

in the dual projective spaces, which are mutually orthogonal;

\[
L_1 = (L_2)^\perp, \quad L_2 = (L_1)^\perp.
\] (4.6)

Proposition 3.2 shows

\[
X = L_1 \cap H_2 \subset F_1, \quad Y = L_2 \cap H_1 \subset F_2,
\] (4.7)

which immediately implies Proposition 4.1. □

Theorem 1.3 follows from Corollary 2.2 for \( m = 4 \), the last line of Section 3, and [Muk99, Example 1.3].
Let $s_2 \in \text{Hom}(\mathcal{O}_{F_1}, \mathcal{E}) \cong V_2^\vee$ be a general element and $\mathcal{E}' := \text{coker}(s_2)$ be its cokernel, which is a globally generated locally free sheaf of rank 3. The exact sequence
\[ 0 \to \mathcal{O}_{F_1}(1) \xrightarrow{s_2} \mathcal{E}(1) \to \mathcal{E}'(1) \to 0 \] (5.1)
shows that the zero locus $X$ of a general section of $\mathcal{E}(1)$ degenerates to the zero $X'$ of a general section of $\mathcal{O}_{F_1}(1) \oplus \mathcal{E}'(1)$, which again is a K3 surface of degree 12. Take a general section $s_1 \in H^0(\mathcal{O}_{F_1}(1)) \cong V_1^\vee$ and set
\[ Q_1 := Z(s_1) \subset F_1. \] (5.2)
Similarly, $Q_2 \subset F_2$ is defined as the zero of $s_2$ considered as an element of $H^0(\mathcal{O}_{F_2}(1)) \cong V_2^\vee$. Both $Q_1$ and $Q_2$ are smooth 5-dimensional quadrics, since they are general hyperplane sections of 6-dimensional quadrics. Since $\mathbb{P}F_1((\mathcal{E}')^\vee) \subset \mathbb{P}F_1(\mathcal{E}^\vee) = F_1$ is equal to $p_2^{-1}(Q_2)$, the diagram (2.4) restricts to the diagram
\[ \begin{array}{ccc}
\mathbb{P}F_1((\mathcal{E}')^\vee) & \xrightarrow{q_1} & Q_1 \\
\downarrow & & \downarrow \\
F_1 & \to & Q_2.
\end{array} \] (5.3)
The bundle $\mathcal{E}'|_{Q_1}$ is an Ottaviani bundle in the sense of [Kan, Definition 2.1]. We set
\[ Q_{12} := \mathbb{P}q_1\left(\left(\mathcal{E}'|_{Q_1}\right)^\vee\right) \subset \mathbb{P}F_1((\mathcal{E}')^\vee). \] (5.4)
By restricting (5.3) to $Q_{12}$, we obtain the diagram
\[ \begin{array}{ccc}
Q_{12} & \xrightarrow{q_1} & Q_1 \\
\downarrow & & \downarrow \\
Q_2 & \xrightarrow{q_2} & Q_2,
\end{array} \] (5.5)
where both $q_1$ and $q_2$ are $\mathbb{P}^2$-bundles. We write elements of $\text{Pic} Q_{12} \cong \mathbb{Z}^2$ as
\[ \mathcal{O}_{Q_{12}}(i, j) := \mathcal{O}_{Q_1}(i) \boxtimes \mathcal{O}_{Q_2}(j). \] (5.6)
Let $\mathcal{O}$ be the complexified Cayley octonion algebra. According to [Kan, Theorem 2.6], the projective space of null-square imaginary octonions
\[ Q := \left\{ \left[ \sum_{i=0}^{7} x_i e_i \right] \in \mathbb{P}(\mathcal{O}) \mid x_0 = \sum_{i=1}^{7} x_i^2 = 0 \right\} \] (5.7)
can naturally be identified with both $Q_1$ and $Q_2$ in such a way that $Q_{12}$ is identified with
\[ \left\{ (x, y) \in Q \times Q \mid x \cdot y = 0 \in \mathcal{O} \right\} \] (5.8)
and $q_1$ and $q_2$ are identified with the first and the second projections respectively. It follows that the diagram (5.5) is symmetric with respect to the exchange of $Q_1$ and $Q_2$.

Let $s'$ be a general section of $H^0(\mathcal{O}_{Q_{12}}(1, 1))$ and set
\[ \begin{array}{c}
D' := Z(s') \subset Q_{12}, \\
X' := Z(q_1 s') \subset Q_1, \\
Y' := Z(q_2 s') \subset Q_2.
\end{array} \] (5.9)
Then both $X'$ and $Y'$ are K3 surfaces of degree 12, and the same reasoning as Corollary 2.2 gives the equality

$$([X'] - [Y']) \mathcal{L}^2 = 0$$  (5.10)

in the Grothendieck ring of varieties. It is an interesting problem to see if $X'$ and $Y'$ are not isomorphic, and characterize K3 surfaces which can be obtained in this way.

6. K3 surfaces in $G(2, 6)$

Let $W$ be a vector space of dimension 6. The Pfaffian hypersurface $\text{Pf}(W) \subset \mathbb{P} \left( \bigwedge^2 W^\vee \right)$ consists of skew bilinear forms on $W$ whose rank is strictly smaller than 6. A general linear subspace $L \subset \mathbb{P} \left( \bigwedge^2 W^\vee \right)$ of dimension 5 determines a cubic 4-fold

$$X := \text{Pf}(W) \cap L$$  (6.1)

and a K3 surface

$$Y := G(2, W) \cap L^\perp.$$  (6.2)

It is known by [Kuzb, Theorem 2] that there is a fully faithful functor $\iota : D(Y) \to D(X)$ and a semiorthogonal decomposition

$$D(X) = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \iota D(Y) \rangle.$$  (6.3)

Fix a linear subspace $U \subset W$ of dimension 5. The rational map

$$\varphi : \text{Pf}(W) \dashrightarrow \mathbb{P}(U)$$  (6.4)

induced by the composition of the wedge square $(-) \wedge^2 : \bigwedge^2 W^\vee \to \bigwedge^4 W^\vee$, the projection $\bigwedge^4 W^\vee \to \bigwedge^4 U^\vee$, and the (canonical up to scalar) isomorphism $\bigwedge^4 U^\vee \cong U$ sends $x \in \bigwedge^2 W^\vee \subset \text{Hom}(W, W^\vee)$ to $\ker x \cap U \subset U$. It is defined by a linear subsystem of $|2H|$, where $H$ is the hyperplane section of $\text{Pf}(W)$. The base locus of this linear subsystem is the union of

$$B_1 := \{ x \in \text{Pf}(W) \mid \ker x \subset U \}$$  (6.5)

and

$$B_2 := \{ x \in \text{Pf}(W) \mid \dim \ker x = 4 \}.$$  (6.6)

The inclusion $U \subset W$ induces a linear projection $\pi_U : \mathbb{P} \left( \bigwedge^2 W^\vee \right) \dashrightarrow \mathbb{P} \left( \bigwedge^2 U^\vee \right)$, and $B_1$ is the closure of the inverse image of

$$G(2, U) = \left\{ x \in \mathbb{P} \left( \bigwedge^2 U^\vee \right) \mid \dim \ker x = 3 \right\} \subset \mathbb{P} \left( \bigwedge^2 U^\vee \right)$$  (6.7)

by this rational map. If $L$ is general, then $B_2 \cap L$ is empty, and $S := B_1 \cap L$ projects isomorphically by $\pi_U$ to the section $G(2, U) \cap \pi_U(L) \subset \mathbb{P} \left( \bigwedge^2 U^\vee \right)$ of $G(2, U)$ by the linear subspace $\pi_U(L)$ of dimension 5, which is a del Pezzo surface of degree 5 embedded anti-canonically into $\pi_U(L) \cong L$. Let $\tilde{X} := \text{Bl}_S X$ be the blow-up of $X$ along $S$, whose exceptional divisor will be denoted by $E$. It is known by [Tre84, Proposition 2] that

- the linear system $|2\tilde{H} - E|$ gives a birational morphism $\tilde{\varphi} : \tilde{X} \to \mathbb{P}(U)$, where $\tilde{H}$ is the total transform of the hyperplane section of $X$, and
- the exceptional locus $F \subset \tilde{X}$ of $\tilde{\varphi}$ is a smooth divisor, whose image $G \subset \mathbb{P}^4$ is isomorphic to the 5-point blow-up of $Y$. 

The proof of [Tre84, Proposition 2], which is based on [Dan80, Theorem 1], actually shows that the morphism \( \tilde{\varphi} \) is the blow-up of \( \mathbb{P}^4 \) along \( G \). Hence one has

\[
[\tilde{X}] = [X] + \mathbb{L}[S] = [X] + \mathbb{L}(1 + \mathbb{L} + \mathbb{L}^2 + 4\mathbb{L})
\]  
(6.8)

and

\[
[\tilde{X}] = [\mathbb{P}^4] + \mathbb{L}[G] = 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}([Y] + 5\mathbb{L}),
\]  
(6.9)

so that

\[
[X] = 1 + \mathbb{L}^2 + \mathbb{L}^4 + \mathbb{L}[Y].
\]  
(6.12)

**Remark 6.1.** We learned from Genki Ouchi that he independently obtained the formula (6.12) multiplied by \( \mathbb{L}^5(L - 1)^2(L + 1) \) using an argument similar to [Bor]. See [KSa, Section 2.6] for similar results for other types of cubic fourfolds.

### 7. Abelian varieties

We prove Theorem 1.5 in this section. First recall the following definition.

**Definition 7.1.** The localized Grothendieck ring of varieties \( K_0(\Var/k)[L^{-1}] \) is the localization of the Grothendieck ring of varieties \( K_0(\Var/k) \) by the class \( L \) of the affine line. For each integer \( i \in \mathbb{Z} \), let \( \Fil_i \subset K_0(\Var/k)[L^{-1}] \) be the abelian subgroup spanned by the elements of the form \( [X] \cdot L^{-m} \), where \( m \in \mathbb{Z} \) is an integer and \( X \) is a variety such that \( \dim X - m \leq i \). The subgroups \( (\Fil_i)_{i \in \mathbb{Z}} \) form an ascending filtration of \( K_0(\Var/k)[L^{-1}] \) satisfying \( \Fil_i \cdot \Fil_j \subset \Fil_{i+j} \). The completed Grothendieck ring of varieties is the completion with respect to this filtration:

\[
K_0(\Var/k)[L^{-1}] \rightarrow \lim_{\substack{\longrightarrow \vphantom{x_i} \\ i \in \mathbb{Z}}} K_0(\Var/k)[L^{-1}] / \Fil_i =: \widehat{K}_0(\Var/k).
\]  
(7.1)

Let \( A_k \) be the group completion of the commutative monoid of isomorphism classes of algebraic group schemes over \( k \) whose connected components are abelian varieties and whose group of geometric connected components is a finitely generated group, where the binary operation + of the monoid is defined by the direct sum; \([A] + [B] := [A \oplus B] \). When \( k = \mathbb{C} \), by using the Bittner presentation of the Grothendieck ring [Bit04] (which in turn is based on resolution of singularities [Hir64] and weak factorization [AKMW02]), Ekedahl proved the following:

**Theorem 7.2** ([Eke09, Theorem 3.4], cf. also [Cau16, Appendix A]). There is a homomorphism of abelian groups

\[
\Pic_{\mathbb{C}} : \widehat{K}_0(\Var/\mathbb{C}) \rightarrow A_{\mathbb{C}}
\]  
(7.2)

sending the class \( [X] \) of a smooth proper variety \( X \) to the class \( \Pic(X) = [\Pic^0(X)] + [\NS(X)] \).

Ekedahl also proved the following:

**Proposition 7.3** ([Eke09, Proposition 3.6]). Assume that \( A \), \( B \) and \( C \) are abelian varieties over \( \mathbb{C} \) satisfying \( A \oplus C \cong B \oplus C \). Then \( \text{Hom}(A,B) \) is a locally free right module of rank 1 over \( R := \text{End}(A) \) and the natural morphism of abelian varieties \( \text{Hom}(A,B) \otimes_R A \rightarrow B \) is an isomorphism.
The proof of \cite{Eke09} Proposition 3.6] is based on the Tate’s isogeny theorem for abelian varieties over a field which is finitely generated over its prime field. This is first shown in \cite{Fal83} over number fields, and is generalized later (see \cite{FW84} Chapter VI §3 Theorem 1 b]). See also \cite{FW84} Chapter IV §1 Corollary 1.2] and its proof).

**Corollary 7.4.** Let $X$ and $Y$ be smooth projective varieties over $\mathbb{C}$ such that $\text{End}(\text{Pic}^0(X)) \cong \mathbb{Z}$. If $[X] = [Y]$ holds in $\widehat{K}_0(\mathcal{V}ar/\mathbb{C})$, then $\text{Pic}^0(X)$ is isomorphic to $\text{Pic}^0(Y)$.

**Proof.** Theorem 7.2 gives the equality $[\text{Pic}(X)] = [\text{Pic}(Y)]$ in the group $A_{\mathbb{C}}$. This is equivalent to saying that there exists a group scheme $G$ whose neutral component is an abelian variety and the group of components is a finitely generated abelian group such that

\[ \text{Pic}(X) \times G \cong \text{Pic}(Y) \times G. \]  

(7.3)

By taking the neutral components, we obtain the isomorphism

\[ \text{Pic}^0(X) \times G^0 = \text{Pic}^0(Y) \times G^0. \]  

(7.4)

Now Proposition 7.3 gives an isomorphism $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$, since a locally free $\mathbb{Z}$-module of rank 1 is unique up to isomorphism. □

**Remark 7.5.** A ring $R$ is said to have the cancellation property if an isomorphism of finitely generated one-sided modules $A \oplus C \cong B \oplus C$ over $R$ implies $A \cong B$ whenever $C$ is a projective module. In Corollary 7.4 it suffices to assume that $\text{End} \left( \text{Pic}^0(X) \right)$ is a hereditary ring with the cancellation property.

**Lemma 7.6.** For any integer $g \geq 2$, there exists an abelian $g$-fold $A$ such that $\text{End} \left( A \right) = \mathbb{Z}$ and $A \not\cong \widehat{A} := \text{Pic}^0 \left( A \right)$.

**Proof.** The authors learned the following construction of $A$ from the answer by Bjorn Poonen to a question in MathOverflow \cite{Poo}. For the sake of completeness, here we give an explanation with details (with a minor change).

Let $J$ be the Jacobian of a very general smooth projective curve of genus $g$. As explained in \cite{Dol16} Lecture 15], by applying \cite{ACGH85} p. 359 Lemma] for the case $d = 2$ and \cite{BL04} Theorem 11.5.1] we obtain $\text{End}(J) = \mathbb{Z}$. Note also that $J$ is isomorphic to its dual, since it is the Jacobian of a curve and hence is a principally polarized abelian variety. Let $G \subset J$ be a finite subgroup whose order $n$ is not a $g$-th power of an integer. Then $A := J/G$ is never isomorphic to its dual, since otherwise the composition

\[ J \xrightarrow{q} A \xrightarrow{\sim} \widehat{A} \xrightarrow{\#} \widehat{J} \sim J \]  

(7.5)

gives an endomorphism of $A$ of degree $n^2$, contradicting $\text{End}(J) = \mathbb{Z}$ and the choice of $n$. The isogeny $q: J \to A$ induces the isomorphism $\text{End}(A) \otimes \mathbb{Q} \cong \text{End}(J) \otimes \mathbb{Q} = \mathbb{Q}$, and hence one has $\text{End}(A) = \mathbb{Z}$. □

**Remark 7.7.** The difference $[A] - \left[ \widehat{A} \right] \in \widehat{K}_0(\mathcal{V}ar/\mathbb{C})[L^{-1}]$ for an abelian variety $A$ as above is a non-trivial element of the kernel of the map

\[ \chi_{\text{mot}}: \widehat{K}_0(\mathcal{V}ar/\mathbb{C})[L^{-1}] \to \widehat{K}_0(\text{Mot}(\mathbb{C})) \]  

(7.6)

to the Grothendieck ring of Chow motives; since $A$ and $\widehat{A}$ are isogenous, they have isomorphic Chow motives by \cite{MNP13} Theorem 2.7.2 (c)].
Theorems 1.5 is an immediate consequence of Corollary 7.4 and Lemma 7.6. Note that the existence of one example in Lemma 7.6 implies that a very general abelian variety also is an example.

We can also construct another class of examples of D-equivalent varieties whose classes are distinct in \( \hat{K}_0(\text{Var}/\mathbb{C}) \).

**Example 7.8.** Let \( A \) be a complex abelian surface such that \( A \not\cong \hat{A} \) and \( \text{End} (A) = \mathbb{Z} \). Choose \( n \geq 1 \) and consider the Hilbert schemes of \( n \)-points \( A^{[n]} \) and \( \hat{A}^{[n]} \). By [Plo07], Proposition 8], the derived equivalence between \( A \) and \( \hat{A} \) induces a derived equivalence between \( A^{[n]} \) and \( \hat{A}^{[n]} \). On the other hand, the summation maps \( s : A^{[n]} \rightarrow A \) and \( \hat{s} : \hat{A}^{[n]} \rightarrow \hat{A} \) are the Albanese maps of \( A^{[n]} \) and \( \hat{A}^{[n]} \), respectively. Hence it follows from Corollary 7.4 that

\[
[A^{[n]}] \neq [\hat{A}^{[n]}] \in \hat{K}_0(\text{Var}/\mathbb{C}) .
\]  

(7.7)

In the study of arithmetic and geometry of abelian varieties, it is often convenient to work with the category of abelian varieties up to isogeny. With this in mind, let \( K'_0(\text{Var}/k) \) be the quotient of \( K_0(\text{Var}/k) \) by the ideal generated by \( [A] - [B] \), where \( A \) and \( B \) are isogenous abelian varieties. The counting measure factors through \( K'_0(\text{Var}/\mathbb{F}_q) [L^{-1}] \), and the Hodge–Deligne polynomial factors through the composition with the completion \( K'_0(\text{Var}/\mathbb{C}) [L^{-1}] \rightarrow \hat{K}'_0(\text{Var}/\mathbb{C}) \) with respect to the filtration induced by that of \( K_0(\text{Var}/\mathbb{C}) \) in Definition 7.1 Combined with the result of Orlov [Orl02] that D-equivalent abelian varieties are isogenous, it is natural to ask if \( [X] = [Y] \) holds either in \( K'_0(\text{Var}/\mathbb{C}) [L^{-1}] \) or \( \hat{K}'_0(\text{Var}/\mathbb{C}) \), when \( X \) and \( Y \) are Fourier–Mukai partners. It is interesting to ask if this tentative modification of Problem 1.2 works for Example 7.8.

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