Approximate Symmetries in General Relativity

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Abstract

The problem of finding an appropriate geometrical/physical index for measuring a degree of inhomogeneity for a given space-time manifold is posed. Interrelations with the problem of understanding the gravitational/informational entropy are pointed out. An approach based on the notion of approximate symmetry is proposed. A number of related results on definitions of approximate symmetries known from literature are briefly reviewed with emphasis on their geometrical/physical content. A definition of a Killing-like symmetry is given and a classification theorem for all possible averaged space-times acquiring Killing-like symmetries upon averaging out a space-time with a homothetic Killing symmetry is proved.

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1 Entropy, Information, Inhomogeneity

In modern gravitational physics there exist a number of proposed definitions of gravitational entropy. Amongst them one can mention the black hole entropy by Bekenstein \[1\] and Hawking \[2\] which relates the gravitational entropy with the black hole surface area, the space-time entropy by Penrose \[3\] based on the idea of employing the Weyl tensor for ‘measuring’ the pure gravitational content of a given space-time, Hu’s cosmological entropy \[4\] due to particle production in anisotropically expanding Universe, Blanderberger-Mukhanov-Prokopec’s non-equilibrium entropy \[5\] for a classical stochastic field applied for density perturbations in an inflationary Universe, the intrinsic entropy by Smolin \[6\] measuring the irreversibility inherent in conversion any form of matter into gravitational radiation. Despite relative consensus of opinions regarding the usefulness and physical adequacy in applying the known definitions of gravitational entropy for the corresponding domains of gravitational phenomena, there is still much controversy in understanding (formulating) the underlying foundations of classical and/or quantum gravitational physics that are generally expected to bring about a generic definition of the notion of gravitational entropy, it being expected to be of geometrical nature. Two aspects here are of the primary importance: (i) missing links and interrelations between different definitions; (ii) absence, in most cases, of clear geometrical interpretations of the proposed notions of entropy.

The challenging problem of gravitational entropy can be considered in broader context as that of finding relations between the informational and gravitational entropy (see a discussion in \[1\], \[7\], \[8\] and references therein). The manifold’s entropy here is to be understood as a kind of geometrical entropy measuring the information encoded in a space-time manifold and being related with manifold’s inhomogeneity. In such approach the key point (assumption) is that an evolving gravitational system tends to a final symmetric (relatively) homogeneous state with subsequently increasing the system’s entropy, or, its physical homogeneity which is reflected, in general, in the homogeneity of the space-time. Indeed, a ‘structureless’ smooth, highly symmetric homogeneous manifold (i.e. when the system is disordered and requires less information to describe it) may be considered to possess maximum entropy, while a lumpy inhomogeneous, highly structured manifold (i.e. when the system is ordered and requires a great deal of information to describe it) does have a smaller value of entropy. In such context the entropy of a system measures one’s uncertainty or lack of information about the actual internal configuration of the system. Given a set of system’s states determined by probabilities \( p_n \), the system entropy is defined due to Shannon’s formula \[10\] as

\[
S = - \sum_n p_n \ln p_n. \tag{1}
\]

New information \( \Delta I \) about the system imposes some constraints on the probabilities \( p_n \), which results \[11\] in a decrease \( \Delta S \) in one’s uncertainty about the internal state of the system. Due to Brillouin’s identification of information with negative entropy \[11\] the property is formalized by the relation

\[
\Delta I = -\Delta S. \tag{2}
\]

\(^1\)An important class of self-organizing, dissipative physical systems \[9\] is not considered here.
Inhomogeneity and Approximate Symmetry

Modern gravitational physics based on the pseudo-Riemannian geometrical picture of a space-time manifold poses another problem of finding an appropriate geometrical/physical index for measuring a degree of inhomogeneity for a given space-time manifold. The theoretical description of gravitational field based, as for all other known interactions, on the notion of symmetry, i.e. the metric tensor is considered to be known if one knows the corresponding group of the space-time manifold symmetries\(^2\). Intuitively, to conclude which manifold is more inhomogeneous one has to compare their symmetry groups - the more symmetries a manifold has, the more homogeneous it is. It is implicitly assumed here that the measure of inhomogeneity is in the difference of the two symmetry groups. However, such an approach does not have any definite means to ‘measure’ inhomogeneity. Indeed, how to decide which manifold is more inhomogeneous - one having, say, a couple of Killing vectors or another possessing only one covariantly constant vector? In practice, to construct an inhomogeneous manifold one usually takes a perturbed manifold with a metric \(g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu}\) with a presumed background manifold \(g^{(0)}_{\mu\nu}\) and the perturbation functions \(h_{\mu\nu}\) where \(\epsilon \ll 0\) is the smallness parameter. The notion of background here is central for it is taken by definition as a smooth and homogeneous reference manifold. It enables one to determine the above mentioned difference between two manifolds (the perturbed and background ones) through the inhomogeneous perturbations \(h_{\mu\nu}\), the notion of background metric itself assuming a kind of smoothing procedure \([14],[15]\) usually taken as a space, or space-time volume averaging.

An index of inhomogeneity is expected to have non-trivial physical and geometrical content, as well as being closely related with information and entropy. Indeed, let us assume that an index \(\chi\) measures inhomogeneity of a gravitating physical system, i.e. that of its space-time, and let initially at \(t_0\) the system have a symmetry group \(G_0\), an inhomogeneity index \(\chi_0\) and a characteristic inhomogeneity length \(l_0\). During evolution of the system to an equilibrium state for \(\Delta t = t_1 - t_0\), its entropy increases for \(\Delta S\) with a decrease in information \(\Delta I\) about its internal configuration (due to washing out system’s initial conditions) in accordance with Brillouin’s relation (2). At the same time the system becomes more homogeneous, i.e. decreases for \(\Delta \chi = \chi_1 - \chi_0\). Therefore, at \(t_1\) the system has a symmetry group \(G_1\), \(G_1 \subset G_0\), an inhomogeneity index \(\chi_1\), \(\chi_1 < \chi_0\), and a characteristic inhomogeneity length \(l_1\), \(l_1 \gg l_0\) and the following relation between information, entropy and inhomogeneity holds

\[
\Delta I = -\Delta S \sim \Delta \chi.
\]

The evolved state \(\{G_1, \chi_1, l_1\}\) is coarse-grained compared with the initial one \(\{G_0, \chi_0, l_0\}\) and the transition from the latter to the former may be accomplished by means of an

\(^2\text{All gauge freedom is always inside the symmetry group.}\)
appropriate smoothing operator with a smoothing scale $d_{\text{aver}}$ such as $l_1 \gg d_{\text{aver}} \gg l_0$.

Through a sequence of more and more coarse-grained states $\{G_i, \chi_i, l_i\}$ satisfying the relation (3) the system reaches eventually the equilibrium state $\{G_{\text{max}}, 0, L\}$, i.e. the space-time manifold becomes homogeneous with the symmetry group $G_{\text{max}}$ and the vanishing inhomogeneity index, $\chi = 0$, the characteristic manifold length $L$ typically being manifold’s curvature radius.$^3$

The goal of this paper is to propose an approach to define an inhomogeneity index by means of finding out an appropriate description of symmetries of inhomogeneous manifolds - the group $G$ from the above discussion. The question is much finer that it seems at first sight. Indeed, such a ‘standard’ symmetry as Killing’s is likely to be ‘too symmetric and smooth’ to serve for description of inhomogeneities usually thought of as a kind of lumpiness or ripples, i.e. an inhomogeneous manifold can be viewed as possessing symmetry that is not precise. The perturbation approach described above illustrates this idea where the group of an inhomogeneous manifold is a distorted, or, approximate in some sense, group of the background manifold, the former being taken as the direct product of the latter by the 4-parametric gauge group of infinitesimal coordinate transformations. The idea of approximate symmetry as relevant to the geometry of inhomogeneous spaces had been put forward first by Matzner [14], [16] who had proposed a way of defining almost Killing vectors, a generalization of the Killing symmetry (see Section 4) useful for description of spaces with gravitational radiation and related to an invariant definition of background. In a physical setting in the early sixties Komar [17], [18] had introduced the concept of semi-Killing vectors, a kind of relaxed Killing symmetry (see Section 4), proved to be useful for the formulation of asymptotic covariant conservation laws for gravitational radiation. It should be pointed out that the problem of defining approximate symmetries in a rigorous way is not simply academic because of its primary importance in modern cosmology for understanding the structure and evolution of our Universe with a hierarchy of physical scales from stars to clusters of galaxies [19]. The Universe is certainly inhomogeneous on smaller scales becoming smooth (averaged) for its largest scale where its space-time geometry possesses isometries in accordance with the Friedmann-Lemaître-Robertson-Walker cosmological model.

In this paper the definitions of approximate symmetries known from literature are briefly reviewed with emphasis on their geometrical/physical content and with pointing out corresponding candidates for the inhomogeneity indexes. A general definition of a Killing-like symmetry [20], [21] is given and a classification theorem for all possible averaged space-times acquiring Killing-like symmetries upon averaging out a space-time with a homothetic Killing symmetry is proved.

3 Approximate Symmetry Groups

The main idea of the approach by Spero and Baierlein [22], [23] is to construct a 3-parameter simply-transitive approximate symmetry group of an inhomogeneous space-time metric$^4$ by minimizing a specific 3-volume-average of deviations of the orthonormal tetrad

$^3$The homogeneous manifold may have several characteristic lengths like for a torus.

$^4$A space-time is said to be spatially homogeneous if it is invariant under a 3-parameter abstract Lie group $G_3$ acting simply transitively on a family of spacelike hypersurfaces.
rotation coefficients from those in a Bianchi type.

Given a subset $U \subset S$, where $S$ is a spacelike hypersurface with a metric $g_{ab}$ of a space-time with a metric $\gamma_{\mu\nu}$, one wishes to find a triad of orthonormal vectors $\{e_A\}$, $g_{ab}e_A^a e_B^b = \delta^A_B$, in $U$ such that the commutation coefficients

$$\gamma^C_{AB} = g_{ab}[e_A, e_B]e^C = 2e^a_{[A} \nabla e^b_{B]}e^c_B$$

are as close as possible to some set of structure constants $C^C_{AB}$. The simply-transitive group 3-parameter Lie group to which $C^C_{AB}$ corresponds is said to be the approximate symmetry group of $g_{ab}$ and $\{e_A\}$ is said to be the best fit triad. This is done by requiring that $\{e_A\}$ and $C^C_{AB}$ of the following 3-volume average of rotation coefficient deviations:

$$I \equiv \frac{1}{V} \int_U \Delta^C_{AB} \Delta^C_{AB} dV + 8\lambda_An_{AB}a_B + 2\lambda_{[AB]}n^{[AB]}$$

give a global minimum $\delta I = 0$ under variation with respect to the $e^a_A$. Here the deviations are $\Delta^C_{AB} = \gamma^C_{AB} - C^C_{AB}$, the Lagrange multipliers $\lambda_A$ and $\lambda_{[AB]}$ ensure that $C^C_{AB}$ are the structure constant antisymmetric in $A$ and $B$ and satisfying the Jacobi identities, the tensor density $n^{AB} = \frac{1}{2}C^{(A}_{CD}e^{B)CD}$ and the vector $a_B = \frac{1}{2}C_{BD}^B$ appear in the irreducible decomposition of the structure constants $C^C_{AB} = \epsilon_{ABD}n^{DC} + 2a_{[A} \delta^C_{B]}$ and $V = \int_U dV$ is the 3-volume of the subset $U$ with the 3-volume invariant measure $dV$.

The approach has enabled to reveal that the simply transitive group $G_3$ in Bianchi types VI$_h$, VII$_h$, VIII and IX are stable symmetries, i.e. a homogeneous metric with one of these symmetry group types will preserve this group type when perturbed by arbitrary small metric perturbations. That means that all the perturbed types possess the approximate symmetry group. The $G_3$ in all other types are unstable. All two-dimensional metrics and some three-dimensional metrics have been also analyzed. It is shown, in particular, that the approximate symmetry group of the 3-dimensional Kantowski-Sachs space-time is the Bianchi type I and those of the Gowdy $T^3$ space-time are types I or VI$_0$.

In the framework of the approach $I$ and $\lambda^2$ measure the degree of metric’s inhomogeneity. $I = 0$ and $\lambda^2 = 0$ iff the metric is homogeneous. $I > 0$ iff the metric is inhomogeneous, the greater $I$ the more inhomogeneous being the metric, and vice-versa. The $I$ may be thought of as measuring the ‘perpendicular distance’ in the 3-metric superspace from the metric under study to the nearest submanifold of homogeneous metrics. The role of $\lambda_A$ is more subtle. No inhomogeneous solutions to the Einstein equations have been found [23] in which $\lambda_A = 0$.

Thus, the approach allows one to find a group of approximate symmetry and the corresponding best fit tetrad may therefore be used to write down Einstein’s equations explicitly. The technique is global apart possible difficulties with topology, and the approach has a sufficiently clear geometrical picture as far as Bianchi geometries are involved. At the same time, generalization to other geometries meets difficulties, for the technique depends drastically on the slicing algorithm of the space-time under study, and there are no links to other approaches to approximate symmetries. The physical meaning of the candidates for inhomogeneity indexes $I$ and $\lambda^2$ is not clear.

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5Capital Latin letters stand for the triad components and the indexes run from 1 to 3.
4 Semi-Killing Vectors

The concept of the semi-Killing symmetry appeared \[17\], \[18\] in searching conserved quantities for physically important space-times (such as those with gravitational radiation) that do not have isometries. The conservation of energy, momentum and angular momentum in Lorentz-covariant field theories in Minkowski space-time is well known to follow by requiring the invariance of physical laws with respect to infinitesimal transformations $\xi^\alpha(x^\mu)$ of the coordinate surfaces

$$x^\alpha = x^\alpha + \xi^\alpha \quad \text{(6)}$$

where $\xi^\alpha(x^\mu) = a^\alpha + \Omega^\alpha_\beta b^\beta$, with $a^\alpha$, $\Omega^\alpha_\beta$ and $b^\beta$ being rigid shift vector, rotation matrix and rotation parameters. Equivalently, one can start requiring the invariance with respect to (6) simultaneously with the form-invariance of the Minkowski metric tensor $\eta_{\mu\nu}$ with respect to the transformations, i.e. vanishing Lie-derivative of the metric $\mathcal{L}_\xi \eta_{\mu\nu} = 0$ that leads to Killing’s equation for the vector $\xi^\alpha$:

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \quad \text{(7)}$$

The Killing vectors $\xi^\alpha$ are then coordinate-hypersurface orthogonal, to fix rigidly the preferred Minkowski coordinates in a flat space-time and single out the preferred conserved quantities, otherwise there are infinitely many conserved quantities. In general relativity the infinitesimal shift vector $\xi^\alpha$ is always a space-time function, so the question is how to single out a preferred system of curvilinear coordinates to get appropriate conserved quantities. In case of existing non-trivial Killing vector fields \[7\], \[24\] they had been shown \[17\], \[24\] to be appropriate in order to obtain the analogs of the preferred conserved quantities of Lorentz-covariant theories mentioned above.

In general, however, Killing vectors cannot be found in many physically important space-time. From an analysis \[17\], \[18\] of the asymptotic properties of radiative solutions it has been suggested to use a semi-Killing vector field $\xi^\alpha$ defined by

$$\xi^\alpha(\xi_{\alpha;\beta} + \xi_{\beta;\alpha}) = 0, \quad \xi^\alpha_{;\alpha} = 0 \quad \text{(8)}$$

which is precisely half of the Killing vector, for finding a preferred time translation. If the vector field is timelike and spacelike-hypersurface orthogonal, the corresponding generalized energy-momentum flux vector of gravitational field $E^\alpha(\xi)) = -2\xi^\beta R^\alpha_\beta$ is conserved, $E^\alpha(\xi)_{;\alpha} = 0$. The total generalized energy defined as

$$E(\xi) = \frac{1}{2\kappa} \int E^\alpha dS_\alpha = \frac{1}{2\kappa} \int E^4(- \det g_{\mu\nu})^{1/2} d^3x \quad \text{(9)}$$

can be shown \[17\] to be positive-definite (here $\kappa$ is Einstein’s constant). It is important to note that introduction of the semi-Killing vectors has been motivated by the observation of Peres \[25\] that if a coordinate system is chosen so that the hypersurfaces of constant time are minimal, and if the energy is determined by the strength of $1/r$ in the asymptotic behaviour of the $g_{00}$ metric component, then the energy is necessarily positive-definite. Indeed, the coordinate hypersurfaces orthogonal to a timelike semi-Killing vector are minimal, i.e. a solution to the Plateau problem - this is due to the first Eq. (8), as well as harmonic due to the second Eq. (8).

\[^6\text{Note that the coordinate hypersurfaces of the Minkowski coordinates of flat space-time are both minimal and harmonic.}\]
In the framework of the approach the total energy $E(\xi) = 0$ implies that the space-time is locally flat.

A definite advantage of the notion of semi-Killing vectors is their transparent physical meaning and motivation and sufficiently clear geometrical meaning, though it is not well-understood which limitations are posed on a space-time by the conditions $(8)$, especially by the second equation. At the same time, the approach is essentially local, it does not have any generalizations and a symmetry group picture is not available. Links to other approximate symmetry approaches are not known except the almost symmetry (see Section 5).

5 Almost Symmetry

The almost symmetry of Matzner [14], [16] defines a measure $\lambda(\xi)$ of deviation from the Killing symmetry as a minimum of the specific functional

$$0 \leq \lambda[\xi] = \frac{\int \xi^{(\alpha;\beta)}\xi_{(\alpha;\beta)}dV}{\int \xi^\alpha\xi_\alpha dV}, \quad dV \equiv (\det g_{\mu\nu})^{\frac{1}{2}}d^4x \tag{10}$$

where $\xi^\alpha$ is an arbitrary vector field. For the positive-definite metrics, $\lambda$ is zero iff $\xi^\alpha$ is Killing. The differential equation resulting from the variational problem $\delta \lambda = 0$ for (10) reads as an eigenvalue problem

$$\xi^{(\alpha;\beta)}_{;\beta} + m\lambda \xi^\alpha = 0 \tag{11}$$

with smallest eigenvalue $0^\lambda$ for positive-definite spaces from all $m\lambda$. The upper bound for the $0^\lambda$ can be shown [16] to be the averaged curvature $0^\lambda \leq \langle R \rangle = L^{-2}$ where $L$ is a typical length of the problem. It is important to note that estimates of this type hold for the eigenvalue $0^\lambda$ in any space and the idea of almost symmetry itself enters when it turns out that $0^\lambda \ll L^{-2}$. It means geometrically that an almost symmetry can be viewed as a kind of a deformation of a space with isometries - what one may call ‘inhomogenization’ of an initially smooth space.

The physical meaning of the almost symmetry becomes more clear while considering its application to high-frequency gravitational waves [27], [28]. Let $g_{\mu\nu}$ be a vacuum metric which admits a steady coordinate system [29] such that the metric can be written

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu} \tag{12}$$

with a background metric $g^{(0)}_{\mu\nu}$ and the perturbation functions $h_{\mu\nu}$ and the smallness parameter $\epsilon \ll 0$. Here $g^{(0)}_{\mu\nu}$ is a slowly varying function of space-time coordinates, $g^{(0)} = \mathcal{O}(1)$, $\partial g^{(0)} = g^{(0)}/L$, $h_{\mu\nu}$ is a rapidly varying function, $h = \mathcal{O}(l/L)$, $\partial h = h/l$, where $l$ is the short wavelength of radiation, $L$ is a typical background length (curvature radius).
and due to the high-frequency approximation adopted $\epsilon \leq l/L$ [27, 30]. Expansion of the Ricci tensor $R_{\mu\nu}$ for $g_{\mu\nu}$ in a series in powers of $\epsilon$ can be shown [27, 30] to bring about a certain propagation equation for $h_{\mu\nu}$, $R_{\mu\nu}^{(1)} = 0$, in the first order and the equation $R_{\mu\nu}^{(2)} = -\epsilon^2 R_{\mu\nu}^{(2)}$ in the second order. To extract the part of $R_{\mu\nu}^{(2)}$ smooth on the scale $l$, its space-time averaging over several wavelengths must be carried out [31], [28], [30], [32], which gives Isaacson’s equation

$$R_{\alpha\beta}^{(0)}(g^{(0)}) = -\kappa T_{\alpha\beta}^{(GW)},$$

where $T_{\alpha\beta}^{(GW)}$ is Isaacson’s energy-momentum tensor for gravitational waves

$$T_{\alpha\beta}^{(GW)} \equiv \frac{\epsilon^2}{\kappa} \langle R_{\alpha\beta}^{(2)} \rangle = \frac{\epsilon^2}{4\kappa} \langle h^{\mu\nu}_{\alpha\beta} h_{\mu\nu;\beta} \rangle. \quad (14)$$

Upon assuming the almost symmetry vector $\xi^\alpha$ to be only slowly varying, $\xi = O(1)$, $\partial \xi = \xi/L$, calculation of the integrand of the functional [10] gives

$$4 \xi^{(\alpha;\beta)} \xi_{(\alpha;\beta)} = g^{(0)\alpha\mu} g^{(0)\beta\nu} [\mathcal{L}_{\xi} g_{\beta\gamma}^{(0)} \mathcal{L}_{\xi} g_{\mu\nu}^{(0)} + 2 \epsilon h_{\alpha;\beta,\sigma} \xi^{\sigma} h_{\mu;\nu,\rho} \xi^{\rho} + O(l)]. \quad (15)$$

With taking into account the denominator $\int \epsilon^{\alpha} \xi^{\beta} g_{\alpha\beta}^{(0)} (-\det g_{\mu\nu}^{(0)})^{1/2} d^4 x$, the first term in the right-hand side of (15) yields a number $4\lambda_{g^{(0)}}[\xi]$ depending only on $\xi^{\alpha}$ and $g_{\alpha\beta}^{(0)}$. In estimating the functional [10] for other terms in the right-hand side of (15) one can observe that the fast-varying second term does not contribute into the integral, and the only contribution coming from the third terms is due to its slowly varying part, $\int \epsilon^{\alpha} g^{(0)\alpha\mu} h_{\alpha;\beta,\sigma} g^{(0)\beta\nu} h_{\mu;\nu,\rho} \xi^{\sigma} \xi^{\rho} (-\det g_{\mu\nu}^{(0)})^{1/2} d^4 x$, the integrand of which contains the Isaacson’s energy-momentum tensor for gravitational waves (14). As a result, $\lambda[\xi]$ for high-frequency radiation has been shown [16] to be of the form

$$\lambda[\xi] = \lambda_{g^{(0)}}[\xi] + \lambda_{rad}[\xi], \quad (16)$$

where

$$\lambda_{rad}[\xi] \equiv \frac{\kappa \int T_{\alpha\beta}^{(GW)} \xi^{\alpha} \xi^{\beta} (-\det g_{\mu\nu}^{(0)})^{1/2} d^4 x}{\int g_{\alpha\beta}^{(0)} \xi^{\alpha} \xi^{\beta} (-\det g_{\mu\nu}^{(0)})^{1/2} d^4 x}, \quad (17)$$

both terms in (16) being independent of $l$.

In the almost symmetry approach $\lambda[\xi]$ (14) stands for a measure of inhomogeneity of a space with almost symmetry compared with that with isometries. As expected from the integral definition (14) for $\lambda[\xi]$ and from the estimation of its upper bound (see above), it is sampling the large-scale curvature of an almost symmetric space. Indeed, the estimation of $\lambda[\xi]$ of a space-time filled with high-frequency radiation (16) shows this explicitly since $\lambda_{g^{(0)}}[\xi]$ is bounded by the background curvature and $\lambda_{rad}[\xi]$ being a kind of effective energy of radiation (17) is also a curvature in the background metric through the energy-momentum tensor $T_{\alpha\beta}^{(GW)}$ and (13). The $\lambda[\xi]$ given by (13) does therefore measure only the large-scale space-time curvature with a contribution from averaged ripples, yielding a total estimate for inhomogeneity of the space-time with high-frequency radiation compared with a space-time $g_{\mu\nu} = g_{\mu\nu}^{(0)}$ with a Killing vector $\zeta^\alpha$, $\mathcal{L}_\zeta g_{\mu\nu}^{(0)} = 0$. It is important to note that
if an almost symmetry vector $\xi^\alpha$ is timelike then $\lambda_{\text{rad}}[\xi]$ has a definite sign, it is positive (negative) if a timelike vector is positive (negative) definite, for the energy density of high-frequency radiation is always positive-definite \[ T^{(GW)}_{\alpha\beta} \xi^\alpha \xi^\beta \geq 0. \] For the $\lambda[\xi]$ itself (and $\lambda_{\text{gEO}}[\xi]$ in the above example) its sign-definiteness can be only shown for the positive-definite metric spaces by finding a lower bound for $\lambda[\xi]$ so that $\lambda[\xi] \geq 0$.

One of the advantages of almost symmetry approach is its clear geometrical meaning and the quantity $\lambda$ possesses also a specific physical interpretation for physically interesting spaces. It should be pointed out that almost symmetry has relevance to semi-Killing vectors (see Section 4) and almost-Killing vectors (see Section 6). Though the technique is global and well-developed for positive-definite metrics, its meets difficulties in generalization to Lorentzian signature spaces. Also no symmetry group formulation of almost symmetry is known.

### 6 Almost-Killing Vectors

The notion of almost-Killing vectors has been introduced by York \[33] in searching for ‘natural’ vector fields in an asymptotically flat space-time. A general equation of this kind reads \[34\]

\[\xi_{\beta;\alpha}^{\alpha} + \xi_{\alpha;\beta}^{\beta} - c\xi_{\alpha}^{\alpha;\alpha} = 0.\] (18)

When $c = 0$ the equation (18) is called the almost-Killing equation\[8\], and when $c = 1/2$ it is called the conformal almost-Killing one. There are basically two observations regarding (18) which make interesting studying the equation in the Kerr space-time \[34\]. Firstly, for $c = 2$ the equation reduces to Maxwell’s equations for a source-free, test electromagnetic field with the vector potential $\xi^\alpha$ in the Kerr background. The equation is well-known to admit a remarkable decoupling of components and variables when being solved employing the Newman-Penrose formalism \[35\]. Secondly, compared with the Killing equation (7) which has 16 components for 4 unknowns and therefore no solution in a general space-time, the equation (18) has 4 components for 4 unknowns and hence it is in general solvable.

York’s almost-Killing equation is obtained by acting on the Killing equation with an additional covariant derivative. Any solution of Killing’s equation is also a solution of the almost-Killing equation, and, generally, any asymptotic Killing vector\[9\] is asymptotic to a solution of the almost-Killing equation. Thus the almost-Killing symmetry gives a natural way of extending symmetries, whether approximate or exact, from infinity to the entire space-time. The integral lines of a solution of the almost-Killing equation, four mutually commuting, linear independent almost-Killing vectors, may be used as a ‘natural’ Kerr asymptotic coordinates and serve as a coordinate grid throughout the entire Kerr space-time.

The situation is analogous with the conformal almost-Killing equation which is a once covariantly differentiated version of the conformal Killing equation

\[\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = \frac{1}{2}g_{\alpha\beta}\nabla_\gamma \xi^\gamma.\] (19)

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8This is in fact Yano-Bochner’s equation, see Section 5.
9An asymptotic Killing vector in the Kerr space-time is defined as a solution of the equation $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = \mathcal{O}(1/r^2)$ where $r$ is the Boyer-Lindquist radial coordinate \[36\].
Again, any Killing vector is also a solution of (13) and a solution to the equation can be viewed as a natural extension of symmetries from infinity.

Four linear independent classes of vector solutions to the generalized almost-Killing equation (18) in the Kerr space-time have been found [34] in terms of Teukolsky’s radial and angular functions. The vector solutions which are asymptotic to the ten Killing vectors of Minkowski space-time have been also given.

Though the almost-Killing symmetry approach does not have any factor measuring the degree of inhomogeneity, it is the only known approximate symmetry approach where the corresponding equations have been solved explicitly and the corresponding group has been found. The technique is global and possesses a clear physical interpretation. There is an explicit relation between almost-Killing and almost symmetry (see Section 5): the almost symmetry eigenvalue equation (11) reduces to the almost-Killing and conformal almost-Killing ones for \( m\lambda = 0 \) and \( m\lambda = 1/2 \), respectively. On the other hand, no generalization of the notion of the almost-Killing symmetry itself, as well as solutions to other metrics, are known.

### 7 Killing-like Symmetry

The main idea of the Killing-like symmetry [20], [21] is to consider the most general form of deviation from the Killing equations. Let us consider the equation for a Killing-like vector \( \xi^\alpha(x^\mu) \)

\[
\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 2\epsilon_{\alpha\beta}
\]

where a symmetric tensor \( \epsilon_{\alpha\beta}(x^\mu) \) measures deviation from the Killing symmetry. The tensor can be small in order to enable a continuous limit to the case \( \epsilon_{\alpha\beta} \rightarrow 0 \).

The equation (20) covers the cases of semi-Killing, almost-Killing and almost symmetries with additional equations for the tensor \( \epsilon_{\alpha\beta}(x^\mu) \). Also covered are standard generalizations of Killing symmetry such as conformal and homothetic Killing vectors [37]. The algebraic classification of the symmetric tensor \( \epsilon_{\alpha\beta} \) gives an invariant way to introduce a set of scalar indexes measuring the degree of inhomogeneity of the space-time with (20) compared with that with isometries, or even weaker symmetry, for example, conformal Killing’s. For the most general case \( A_1[111,1] \) in Segre’s notation [37] \( \epsilon_{\alpha\beta} \) has the form

\[
\epsilon_{\mu\nu} = \lambda g_{\mu\nu} + \rho x_\mu x_\nu + \sigma y_\mu y_\nu + \tau z_\mu z_\nu
\]

where \( g_{\mu\nu} \) is the space-time metric, \( \lambda(x^\mu), \rho(x^\mu), \sigma(x^\mu), \) and \( \tau(x^\mu) \) are eigenvalues of \( \epsilon_{\alpha\beta} \). If all eigenvalues vanish the space-time has an isometry (20), if \( \rho = \sigma = \tau = 0 \) then there is a conformal Killing vector for \( \lambda(x) \neq 0 \) and a homothetic Killing vector for \( \lambda = \text{const.} \). For other algebraic types of Killing-like symmetry the space-time has the following sets of eigenvalues: two complex conjugated to each other and two real scalars for \( A_2[11,ZZ^*] \), three real scalars for \( A_3[11,2] \) and two real scalars for \( B[1,3] \).

This approach enables one to control the situation under averaging to see how an approximate symmetry gives rise to ‘more’ precise one, or remains the same if a space-10

\[\text{An equivalent form of (21) without explicitly using a metric tensor is } \epsilon_{\mu\nu} = -\lambda t_\mu t_\nu + (\rho + \lambda)x_\mu x_\nu + (\sigma + \lambda)y_\mu y_\nu + (\tau + \lambda)z_\mu z_\nu.\]
time initially was enough symmetric (see Section 2). Under a covariant space-time averaging scheme \[^{18},^{39},^{40}\] generalizing the standard Minkowski space-time averaging scheme \[^{11}\], a Killing vector \[^{7}\] has been shown to remain a Killing vector in the averaged space-time

\[
\bar{\xi}_{\alpha\|\beta} + \bar{\xi}_{\beta\|\alpha} = 0
\]  

\[\text{(22)}\]

where \[\|\) is the covariant derivative with respect to macroscopic metric \[^{38},^{39},^{40}\] \(G_{\alpha\beta}\). In contrast to a smooth and highly symmetric macroscopic space-time, a microscopic space-time having plenty of small-scale inhomogeneities cannot be expected in general to possess isometries. Assuming that approximate microscopic symmetry belongs to the class of Killing-like symmetry \[^{(21)}\], one can show \[^{20},^{21}\] that averaged Killing-like equation reads

\[
\bar{\xi}_{\alpha\|\beta} + \bar{\xi}_{\beta\|\alpha} = 2\bar{\tau}_{\alpha\beta}.
\]  

\[\text{(23)}\]

Given a microscopic Killing-like symmetry, for example, \(A_1[111, 1]\) with \[^{(21)}\], the average tensor \(\bar{\tau}\) can be shown to have the following form:

\[
\bar{\tau}_{\mu\nu} = \langle \lambda \rangle \bar{g}_{\mu\nu} + \langle \rho \rangle X_\mu X_\nu + \langle \sigma \rangle Y_\mu Y_\nu + \langle \tau \rangle Z_\mu Z_\nu
\]  

\[\text{(24)}\]

where \(\langle \lambda \rangle(x^\mu), \langle \rho \rangle(x^\mu), \langle \sigma \rangle(x^\mu)\) and \(\langle \tau \rangle(x^\mu)\) are rational functions of eigenvalues of averaged metric \(\bar{g}_{\mu\nu}\), averaged symmetric tensors \(\bar{t}_\mu t_\nu, \bar{g}_\mu y_\nu, \bar{g}_\mu \bar{y}_\nu\) and \(\bar{z}_\mu \bar{z}_\nu\) and of averaged eigenvalues of \[^{(21)}\], \(\bar{\lambda}, \bar{\rho}, \bar{\sigma}, \bar{\tau}\), with the linear dependence on the last-mentioned, and \(\{T^\mu, X^\mu, Y^\mu, Z^\mu\}\) is the macroscopic eigentetrad in the averaged (macroscopic) space-time. The correlations between eigenvalues and eigenvectors of \[^{(21)}\] are taken into account by algebraic decomposition of symmetric second rank correlation tensors with the correlation eigenvalues entering the quantities \(\langle \lambda \rangle(x^\mu), \langle \rho \rangle(x^\mu), \langle \sigma \rangle(x^\mu)\) and \(\langle \tau \rangle(x^\mu)\) in \[^{(24)}\] as additive terms. Without loss of generality one can consider the correlation terms renormalizing the averaged eigenvalues \(\bar{\lambda}, \bar{\rho}, \bar{\sigma}, \bar{\tau}\). The only assumption made in derivation of \[^{(24)}\] is that averaging out \(\textit{does not change}\) the algebraic type, that is the averaging of the types \(A_1[1, 111] A_2[11, ZZ^*], A_3[11, 2]\) and \(B[1, 3]\) may lead at most to degeneration of the same type, not allowing interchanges between the types \[^{4}\].

If upon averaging \(\bar{\lambda} = 0, \bar{\rho} = 0, \bar{\sigma} = 0\) and \(\bar{\tau} = 0\), i.e. the original space-time had no symmetries (was absolutely disordered), then the microscopic \(\xi_{\alpha\beta} = \epsilon_{\alpha\beta}\) leads to a Killing symmetry \(\bar{\xi}_{\alpha\|\beta} = 0\) in the macroscopic space-time. If \(\bar{\lambda} \neq 0, \bar{\rho} = \bar{\sigma} = \bar{\tau} = 0\) then there is a Killing-like symmetry \(\bar{\xi}_{\alpha\|\beta} = \bar{\lambda}_\alpha \bar{\gamma}_\beta\) which reduces to a conformal Killing symmetry \(\bar{\xi}_{\alpha\|\beta} = \bar{\lambda}(x^\mu) G_{\alpha\beta}\) only when there are no correlations in eigenvectors and all eigenvalues of \(\bar{g}_{\alpha\beta}\) are equal to unity, \(\bar{t}_\mu t_\nu = T_\mu T_\nu, \bar{g}_{\alpha\beta} T^\beta = T_\alpha\) etc.

Let us consider now a particular case of the microscopic Killing-like symmetry taken as a homothetic Killing vector with the homothetic parameter \(\lambda = 1\). Then the microscopic equation \(\xi_{\alpha\beta} = g_{\alpha\beta}\) leads to the macroscopic Killing-like symmetry:

\[
\bar{\xi}_{\alpha\|\beta} = \bar{g}_{\alpha\beta}.
\]  

\[\text{(25)}\]

By means of analysis of the integrability conditions\[^{12}\] of \[^{(23)}\] one can prove a theorem classifying all space-times with the Killing-like symmetry \[^{(23)}\].

\[^{11}\]This is dictated by the local character of both the algebraic decomposition and the averaging procedure.

\[^{12}\]In macroscopic gravity the metric tensor and its inverse remain covariantly constant under averaging, \(\bar{g}_{\alpha\beta\|\gamma} = 0, \bar{g}^\gamma\|\alpha = 0, g_{\alpha\beta\gamma} \neq \delta_\beta^\gamma\), with the covariantly constant macroscopic metric \(G_{\alpha\beta\|\gamma} = 0\) and its inverse \(G^\alpha\beta\|\gamma = 0, G_{\alpha\beta} G^{\alpha\gamma} = \delta_\beta^\gamma\).
Theorem Depending on algebraic types of $\mathbf{g}^{\alpha\beta}$, the following space-times admit the Killing-like symmetry (25):

(A) All algebraic types $A_1[111,1]$, $A_2[11,ZZ^*]$, $A_3[11,2]$ and $B[1,3]$ without degeneration have 4 linear independent covariantly constant vector fields, i.e. the macroscopic space-time is flat.

(B) For $A_3[(11,2)]$ when

$$\bar{g}_{\alpha\beta} = \text{const} \ G_{\alpha\beta} - 2\sigma_\alpha \sigma_\beta$$

(a generalized homothetic Killing vector) the space-time has a covariantly constant null vector $\sigma^\alpha \parallel \beta = 0$, i.e. $G_{\alpha\beta}$ is a pp-wave metric [37].

(C) For $A_3[(111,1)]$ when

$$\bar{g}_{\alpha\beta} = \text{const} \ G_{\alpha\beta}$$

(a homothetic Killing vector) there are no conditions on the curvature tensor and space-time structure.

(D) All other cases are given in Table [I].

An important advantage of the Killing-like symmetry approach is that it incorporates all known geometrical approaches, semi-Killing, almost-Killing and almost symmetries. A link to the approximate symmetry group approach is not clear as yet, and a group picture of the Killing-like symmetry is not available. The technique is essentially local due to the local character of algebraic decomposition, though under certain circumstances, such as the existence of non-singular vector fields, it may be extended on the entire manifold. Another advantage is its applicability to metrics of any signature. On the other hand, the physical interpretation of the Killing-like symmetries and of the eigenvalues (20) as candidates for inhomogeneity indexes is not known so far.

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Table 1: Part (D) of the Theorem in Section \([\square]\) includes the following cases (here all vectors and symmetric idempotent second rank tensors are covariantly constant, vanishing eigenvalues are marked by zero and asterisk stands for complex conjugate):

| tensor \(\mathcal{g}_{\alpha\beta}\) | Ricci tensor | Petrov type | Space-time admits               |
|---------------------------------|--------------|-------------|---------------------------------|
| \([11(1,1)]\)                  | \([11(1,1)]\) | \(D\)       | two spacelike vectors & one tensor |
| \([[(11)1,1]\)                  | \([(11)(1,1)]\) | \(D\)       | spacelike & timelike vectors & one tensor |
| \([[(11)(1,1)]\)               | \([(11)(1,1)]\) | \(D\)       | two tensors                      |
| \([1(11,1)]\)                  | \([11,1]\)    | \(I\)       | one spacelike vector & one tensor |
| \([11(1,1)]\)                  | \([1(1,1)]\)  | \(D\)       |                                 |
| \([1(11,1)]\)                  | \([1(1,1)]\)  | \(O\)       |                                 |
| \([1(11,1)]\)                  | \([1(1,1)]\)  | \(I\)       |                                 |
| \([1(1,2)]\)                   | \([1(1,2)]\)  | \(N\)       |                                 |
| \([1,3]\)                      | \([1,3]\)     | \(III\)     |                                 |
| \([[(111),1]\)                 | \([111,1]\)   | \(I\)       | one timelike vector & one tensor |
| \([[(11)1,1]\)                 | \([(11)(1,1)]\) | \(D\)       |                                 |
| \([[(111),1]\)                 | \([(111),1]\) | \(O\)       |                                 |
| \([([11),ZZ^*]\)               | \([(11)(1,1)]\) | \(D\)       | spacelike & timelike vectors & one tensor |
| \([1(1,2)]\)                   | \([(11)(1,1)]\) | \(N\)       | spacelike & null vectors        |
| \([[(11),2]\)                  | \([(11)(1,1)]\) | \(N\)       | spacelike & timelike vectors & one tensor |
| \([1(1,3)]\)                   | \([(11)(1,1)]\) | \(N\)       | spacelike & null vectors        |

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