Intersecting Orbifold Planes and Local Anomaly Cancellation in $M$-Theory

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Abstract

A systematic program is developed for analyzing and cancelling local anomalies on networks of intersecting orbifold planes in the context of $M$-theory. Through a delicate balance of factors, it is discovered that local anomaly matching on the lower-dimensional intersection of two orbifold planes may require twisted matter on those planes which do not conventionally support an anomaly (such as odd-dimensional planes). In this way, gravitational anomalies can, in principle, tell us about (twisted) gauge groups on subspaces which are not necessarily ten-, six- or two-dimensional. An example is worked out for the case of an $S^1/Z_2 \times T^4/Z_2$ orbifold and possible implications for four-dimensional physics are speculated on.
1 Introduction

Anomalies have been curiously adept at providing insight into fundamental concepts and indicating new phenomena. The role of local gauge and gravitational anomaly cancellation in fomenting the so-called first superstring revolution is well known. But the ongoing development of nonperturbative “string” dynamics has also relied strongly on constraints imposed by anomaly cancellation in effective field theories. Since the nonperturbative picture is now understood to involve an eleven-dimensional description, eleven-dimensional supergravity has become a central tool in the exploration of the yet-mysterious underpinnings of whatever dynamics comprise M-theory. Anomalies have an important place in this story.

In this paper, we develop a systematic method for analyzing M-theory orbifold anomalies in situations where there are various orbifold planes of different dimensionalities which intersect. Particularly, we indicate a way in which the requirement of local anomaly cancellation on subspaces corresponding to the intersection of two orbifold planes can require twisted matter propagating on the entirety of one of the planes regardless of whether that plane actually supports a separate local anomaly. This works because fermions propagating on an intersecting plane can couple to currents localized on the intersection via projections which are chiral from the point of view of the (lower-dimensional) intersection. The associated contributions to the anomaly are distinguishable by virtue of divisors which properly modify standard index theory results in these situations.

Eleven-dimensional supergravity has a solitonic superfivebrane with chiral zero-modes. Hence, fivebrane worldvolume gravitational anomalies posed an early puzzle to the consistent realization of M-theory effective actions. As anticipated in [3] and realized in [4], the cancellation of these anomalies requires an extension of the minimally coupled supergravity action in the form of a coupling $G X_7$, where $G$ is the four-form field strength and $X_7$ is a seven-form involving the eleven-dimensional Riemann tensor. The chiral modes on the fivebrane worldvolume also couple to the normal bundle involving $SO(5)$ spacetime symmetries not broken by the presence of the fivebrane. The cancellation of associated $SO(5)$ anomalies posed a further puzzle which was analyzed and partially solved in [5] and more recently resolved in [6] by a subtle mechanism involving the $CGG$ interaction present in the minimally coupled theory.

Chiral anomalies also arise in orbifold compactifications of M-theory. Such constructions provided some of the initial impetus for the current faith in a unified description of the five erstwhile separate ten-dimensional string theories, and thereby have played a role in the so-called second superstring revolution. Generally, orbifolding eleven-dimensional
supergravity removes from the bulk gravitino and the bulk three-form potential all but a chiral projection on any even-dimensional hyperplane fixed by the action of the discrete group that defines the orbifold. In cases where there are ten- or six-dimensional fixed planes, the cancellation of gravitational anomalies induced by couplings involving these projections poses yet more puzzles to the consistent realization of M-theory orbifolds, only some of which have been completely resolved.

The prototype M-theory orbifold was described by Hořava and Witten in [8, 9], where the connection between eleven-dimensional supergravity and the $E_8 \times E_8$ heterotic string was first indicated. In this case, one of the spatial dimensions was compactified on $S^1/\mathbb{Z}_2$, which is an orbifold with two ten-dimensional fixed hyperplanes. In this case cancellation of induced gravitational anomalies necessitates the presence of ten-dimensional $E_8$ gauge matter propagating on each of the two ten-planes. These, of course, provide the two $E_8$ factors known previously from the perturbative heterotic string perspective, in which the eleventh dimension is re-interpreted as the (small) string coupling constant.

A central aspect of the Hořava-Witten analysis was the requirement of anomaly cancellation independently at each point in eleven-dimensional spacetime. Since the one-loop anomalies in that case derive from from the coupling of the eleven-dimensional gravitino to ten-dimensional currents, the anomaly is characterized by expressions which differ from those obtained from usual index theorems. This difference is a specific factor of $1/2$ derived from the fact that there are two orbifold planes.

One would like to have a guiding principle for constructing realistic models of nature from M-theory. In perturbative string theory, the requirement of modular invariance was eventually understood to imply anomaly freedom. In M-theory we have no such principle which we can point to which offers an “explanation” for anomaly freedom. But we do have the requirement of anomaly freedom itself. This turns out to be a powerful tool in its own right. It is conceivable that anomaly cancellation in effective theories are somehow equivalent to the microscopic consistency requirements related to a fundamental description of M-theory.

In this paper, we offer a concise and self-contained account of the anomaly cancellation issues associated with three “basic” M-theory constructions: the $M$-fivebrane, the $S^1/\mathbb{Z}_2$ orbifold, and the $T^5/\mathbb{Z}_2$ orbifold. These are reviewed using the specific tools used afterward to analyze the more interesting $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$ orbifold, and are presented in detail for the reason that the same calculations involved constitute necessary sub-analyses in the latter case. The $M$-fivebrane in particular is central to all phenomenological constructions based on M-theory. Furthermore, the independent presentations of these situations using our specific tools enable us to focus on individual generic aspects such as the con-
cept of anomaly inflow and the issue of “wandering fivebranes” which can mediate phase transitions.

In section 2 we review the cancellation of the worldvolume gravitational anomaly for the $M$-fivebrane. This allows us to motivate and introduce the basic tools common to anomaly analyses for all $M$-theory orbifolds, and also to set our notation. The basic tools are the anomaly polynomials mentioned above, the $CGG$ and $GX_7$ Chern-Simon’s interactions and special objects called brane-currents important for describing magnetic and electric sources for $G$ concentrated on sub-manifolds.

In section 3 we review the cancellation of gravitational and gauge anomalies in the context of the Hořava-Witten $S^1/Z_2$ orbifold. This entire analysis carries over as a sub-analysis in the more complicated case analyzed in section 6. Numerical details derived in section 3 are essential to the analysis of section 6.

In section 4 we review the cancellation of the gravitational anomaly in the context of the $T^5/Z_2$ orbifold [10, 11]. This scenario comprises another sub-analysis for the case studied in section 6, but an orthogonal one. The presentation in this section also allows useful parallels to be drawn in section 6 relating to paradoxes which are discussed in that section.

In section 5 we explain an effect not present in the case of the $Z_2$ orbifold anomalies, but which is essential to anomalies in orbifolds with intersecting fixed-planes. This “I-brane” effect mirrors a synonymous effect in the context of intersecting D-branes, and involves an interplay between electric and magnetic sources for $G$ supported on separate but intersecting orbifold planes. The effect forms a crucial ingredient to the analysis of section 6.

Section 6 involves a detailed local anomaly analysis in the case of a $S^1/Z_2 \times T^4/Z_2$ orbifold. This orbifold is essentially an amalgamation of the two $Z_2$ orbifolds reviewed in sections 3 and 4, but there are important new features. The first one is that, due to the larger discrete group, the bulk supersymmetry is halved again on the fixed-plane intersections. This allows for a richer twisted sector, enables a local Green-Schwarz mechanism precluded in the former cases and involves the interesting I-brane effect described in section 5. Most importantly, the analysis indicates how one can infer seven-dimensional twisted states based on their relationship to anomalies on the six-dimensional subspaces corresponding to the intersections with the ten-planes. There is a paradox, however, which leaves an unanswered question. As far as we are aware, ours is the first analysis into the local aspects of anomaly cancellation in $M$-theory orbifolds other than the two $Z_2$ orbifolds described above.

In section 7 we conclude and speculate on implications our results may have for even
more complicated orbifolds related to realistic four-dimensional physics.

We also include three appendices.

Appendix A includes tables describing some results from group theory required by the analysis described in section 6.

Appendices B and C serve as brief encyclopedias to the relevant anomaly polynomials used in the main part of the paper. These appendices are included for reference purposes and are meant to be practical and concise. For this reason the relationship of these polynomials to index theorems is de-emphasised. Appendix B includes the polynomials relevant to ten-dimensional anomalies. This also includes two well-known “classic” examples from ten-dimensional supergravity to illustrate how these same polynomials are used in more straightforward analyses. These are included for the benefit of readers without much familiarity with anomaly polynomials. Appendix C includes a similar encyclopedia involving the analogous six-dimensional anomaly polynomials and also includes examples.

2 The $M$-Fivebrane and Brane-Currents

The $M$-fivebrane describes a prototypical spacetime defect supporting a potential anomaly. This six-dimensional object has worldvolume fields transforming as a $D=6$ $N=2$ tensor multiplet $1$. This includes five scalars parameterizing the position of the fivebrane in eleven-dimensional spacetime as well as an anti self-dual two-form (ie: the three-form field strength satisfies $H = - \ast H$) and a pair of antichiral spin 1/2 fields. The tensor and the fermions each contribute to a gravitational anomaly at one loop. This anomaly is characterized by an eight-form which can be computed via the relation

$$I_8(1 \text{ loop}) = -I_{\text{GRAV}}^{(3-\text{form})}(R) - 2 I_{\text{GRAV}}^{(1/2)}(R),\quad (2.1)$$

where the minus signs indicate anti self-duality and negative chirality respectively, and the absolute values of the two coefficients reflect the multiplicities of the indicated fields. The polynomials $I_{\text{GRAV}}^{(3-\text{form})}(R)$ and $I_{\text{GRAV}}^{(1/2)}(R)$ encode the contributions to a six-dimensional gravitational anomaly due to a single self-dual tensor field and a single chiral spin 1/2 fermion. These are determined by index theorems and given explicitly in appendix C as equation (C.1). Substituting the polynomials given in (C.1) into the expression (2.1) we easily determine

$$I_8(1 \text{ loop}) = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{8} \text{tr} R^4 - \frac{1}{32} (\text{tr} R^2)^2 \right), \quad (2.2)$$

1In our conventions, $N = 2$ describes 16 supercharges, and is therefore twice the minimum in six-dimensions. The same superalgebra is sometimes called $N = 4$ in the literature because it corresponds to $N = 4$ in four dimensions.
where \( R \) is the six-dimensional Riemann tensor expressed as an \( SO(5,1) \)-valued two-form.

The worldvolume fermions also couple to \( SO(5) \) currents associated with the normal bundle. This gauge group is inherited from the “bulk” \( SO(10,1) \) diffeomorphism group, which is broken by the presence of the fivebrane to \( SO(5,1) \times SO(5) \). Whereas the \( SO(5,1) \) transformations include the fivebrane worldvolume diffeomorphisms, with one-loop anomaly described by (2.2), the \( SO(5) \) anomaly is more subtle. Its cancellation was resolved in [6], and involves mathematics which we will not need or describe in this paper. The normal bundle anomaly can be considered independently. We therefore suppress this issue in the balance of this paper.

The one-loop anomaly (2.2) is canceled via “inflow” from classical variation of the eleven-dimensional action. The classical variation can include an anomalous contribution localized on the fivebrane worldvolume provided the four-form \( G \) of eleven-dimensional supergravity couples magnetically to the fivebrane via modifications to the \( dG \) Bianchi identity. The anomaly inflow arises specifically due to the variation of the following terms in the action \( S \):

\[
S = \cdots - \frac{\pi}{3} \int C \wedge G \wedge G + \int G \wedge X_7. 
\]

(2.3)

The \( CGG \) interaction is required by the minimally-coupled supergravity while the \( GX_7 \) term is an additional higher-derivative interaction required by the fivebrane anomaly cancellation. This also requires that \( X_7 \) transform into a total derivative under local \( SO(10,1) \) Lorentz transformations as \( \delta X_7 = dX_6^1 \). The precise form of \( X_7 \) is dictated by the anomaly cancellation.

**A magnetic source for \( G \):**

In the presence of a single fivebrane, the four-form \( G \) satisfies the Bianchi identity

\[
dG = \delta^{(5)}_{W^6}, 
\]

where the five-form \( \delta^{(5)}_{W^6} \) is, in the terminology of [7], a brane-current with support localized on the fivebrane worldvolume, \( W^6 \). Such objects have received critical attention in various papers, notably [3, 4, 5], and have an interesting and somewhat involved mathematical description. The essential features necessary for our purposes are summarized as follows.

An \((11-d)\)-form \( \delta^{(11-d)}_{M^d} \) is a brane-current if it has localized support on the \( d \)-dimensional defect \( M^d \) and if it satisfies properties of being closed, integrating to one

\[\text{The normalizations are chosen as follows. We start with the conventions employed in [6], and apply a further scaling } C \rightarrow 2\pi C \text{ (so the object we call } C \text{ is the one called } C' \text{ in [6]). This determines the coefficient of the first term to be } -\pi/3. \text{ With this scaling, quanta of } G\text{-flux are measured in units of } \int G \text{ rather than } (2\pi)^{-1} \int G, \text{ which also explains why there is no factor of } 2\pi \text{ in equation (2.4).}\]
over any normal fiber of the embedding space, and performing as a generalized Dirac delta function by collapsing integrals to the support manifold, \( \int \delta_M \wedge \Phi = \int_{M^d} \Phi \). Furthermore, provided that manifolds \( M^{d_1} \) and \( M^{d_2} \) intersect transversally, the product of two associated brane-currents is itself a brane-current. Thus, if \( M^{d_1} \) and \( M^{d_2} \) intersect at right angles in eleven-dimensions,

\[
\delta_{M^{d_1}}^{(11-d_1)} \wedge \delta_{M^{d_2}}^{(11-d_2)} \equiv \delta_{M^{d_1} \cap M^{d_2}}^{(11-I)},
\]

where \( I = d_1 + d_2 - 11 \) is the dimensionality of the transversal intersection \( M^I = M^{d_1} \cap M^{d_2} \). Subtleties involving non-transversal intersections are discussed and resolved in \([7]\).

A fivebrane coupled as in (2.4) will contribute one unit of \( G \)-flux. This is seen by integrating (2.4) over the five dimensions transverse to the fivebrane worldvolume using a region bounded by a four-cycle \( \omega \) which encompasses the fivebrane. Stokes theorem then allows us to express the left-hand side of the integrated Bianchi identity as an integral over \( \omega \), while the right-hand side is unity by virtue of a defining property of the brane-current \( \delta^{(5)}_{W^6} \). Thus, \( \int_\omega G = 1 \).

**Anomaly inflow:**

It is straightforward to determine the variation of the \( GX^7 \) term. Using an integration by parts and applying the Bianchi identity (2.4) we determine

\[
\delta \left( \int G \wedge X^7 \right) = - \int_{W^6} X^1_6.
\]

Note that the brane-current \( \delta^{(5)}_{W^6} \) included in the Bianchi identity collapses the eleven-dimensional integral to a six-dimensional integral over \( W^6 \). The \( GX^7 \) inflow contribution is characterized by the closed gauge-invariant eight-form \( X_8 \) which gives rise to (2.6) upon descent. Thus we can write

\[
I_8(GX^7) = -X_8,
\]

where \( X_8 \equiv dX^7 \) and \( \delta X^7 = dX^1_6 \). This inflow contribution must cancel against the one-loop anomaly given in (2.2).

**The total anomaly:**

The total worldvolume gravitational anomaly is given by the sum of the one-loop anomaly (2.2) and the inflow contribution (2.7). Thus, \( I_8(\text{total}) = I_8(\text{1 loop}) + I_8(GX^7) \). We require that this total anomaly vanish. Using equation (2.7), this indicates that \( X_8 \) must be equal
to $I_8(1 \text{ loop})$, which is given as (2.2). Thus,

$$X_8 = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{8} \text{tr} R^4 - \frac{1}{32} (\text{tr} R^2)^2 \right). \quad (2.8)$$

Note that the CGG term does not contribute to the fivebrane gravitational anomaly. It does, however, contribute importantly to the normal bundle anomaly (which we are suppressing). This is described in [1]. In more general contexts, such as the orbifolds analyzed below, the CGG term provides a crucial ingredient and cannot be neglected.

Comments:

The above analysis is easily generalized to the presence of any number of unit-charge fivebranes. If there are $N_5$ fivebranes, then (2.4) is replaced with

$$dG = \sum_{i=1}^{N_5} \delta^{(5)}_{W_i^6} \quad (2.9)$$

where $W_i^6$ is the worldvolume of the $i$th fivebrane. In this case, each fivebrane will support an independent set of worldvolume fields (comprising a D=6 N=2 tensor multiplet) which contributes to a one-loop anomaly localized on $W_i^6$. At the same time there will be an inflow contribution to the total anomaly concentrated on each fivebrane due to the sum of terms in (2.9), and arising due to the variation of the same $G X_7$ term derived above. The one-loop anomaly and the inflow anomaly will cancel each other independently on each fivebrane.

Since eleven-dimensional supergravity has unit-charge fivebranes as a solitons, then, by virtue of the above discussion, any quantum theory which has eleven-dimensional supergravity as its low-energy description should involve as well the particular $G X_7$ interaction. This term also proves crucial to the unraveling of further puzzles presented upon orbifold compactification.

3 \textbf{M-theory on } $R^{10} \times S^1/Z_2$

The simplest orbifold of eleven-dimensional supergravity has had a significant impact. This construction [8, 9] provided simple answers to longstanding puzzles associated with the role of eleven-dimensions in the scheme of string-theory, provided a satisfying picture of the strongly-coupled dynamics of the $E_8 \times E_8$ heterotic string and has opened the door to a wealth of new ideas pertaining to nonperturbative fundamentals in physics. One thing that makes this construction so powerful is the ease with which some of the essential calculations can be done, a common feature of orbifold models.
Orbifolds are defined by discrete projections which act both on the spacetime manifold and on the field content of the theory. These define a set of invariant hypersurfaces, known as orbifold planes, which provide the main focus for analytical attention. In the case of the $S^1/\mathbb{Z}_2$ orbifold, there are two parallel ten-dimensional orbifold planes within eleven dimensional spacetime. Our primary interest in this paper is in more complicated orbifolds which involve intersecting networks of fixed-planes. Nevertheless, much of the analysis in those cases is identical to that associated with simpler orbifolds without fixed-plane intersections. So in this section we briefly but completely review the anomaly analysis for the case of the $S^1/\mathbb{Z}_2$ orbifold. Since this analysis uses the same tools which we will need later on, this enables a necessary appreciation for the some of the computational mechanics used in more general situations. It also supplies results which carry over to other orbifolds whose orbifold group includes a similar projection as a subgroup.

The structure of the orbifold:

Start with eleven-dimensional supergravity on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$. The $\mathbb{R}^{10}$ factor is parameterized by $x^A \equiv \{x^1,\ldots,x^{10}\}$, while the one compact dimension is parameterized by $x^{11}$, which takes values on the interval $[-\pi,\pi]$ with endpoints identified. Truncate the theory via a $\mathbb{Z}_2$ projection which acts on the compact dimension as $x^{11} \rightarrow -x^{11}$. There are then two ten-dimensional hyperplanes fixed by this projection, namely the surfaces defined by $x^{11} = 0$ and $x^{11} = \pi$.

Demanding $\mathbb{Z}_2$-invariance of the eleven-dimensional interaction term $CGG$ implies that the three-form $C$ is odd under the $\mathbb{Z}_2$ projection described above. Thus, $C_{ABC} \rightarrow -C_{ABC}$, where $A,B,C \in \{1,\ldots,10\}$. The components $C_{ABC}$ therefore vanish on the fixed hyperplanes. However, $C_{(11)AB} \rightarrow C_{(11)AB}$. Therefore, from the point of view of the two ten-dimensional fixed-planes, the three-form $C$ contributes one nonvanishing tensor $C_{(11)AB}$.

Half the supersymmetries of the eleven-dimensional theory are broken on the two ten-planes by the $\mathbb{Z}_2$ projection. Thus, on the fixed-planes the one tensor $C_{(11)AB}$ organizes along with the other fields surviving the $\mathbb{Z}_2$ projection into a D=10 N=1 supergravity multiplet. This constitutes the “untwisted sector” of the orbifold.

We allow for a “twisted sector” involving ten-dimensional Yang-Mills multiplets in the adjoint representation of some gauge group $G_i$ propagating on the $i$th ten-plane. As reviewed below, anomaly cancellation uniquely selects this group.

The one-loop anomaly:

A gravitational anomaly arises due to the coupling of chiral projections of the bulk grav-
itino to currents localized on the two fixed-planes. Since the two ten-planes are indistin-
guishable aside from their position, this anomaly is similar on each of the two, and
can be computed using standard formulae if proper care is used. The reason why ex-
tra care is needed is that the anomaly in question actually derives from the coupling of
eleven-dimensional fermions to ten-dimensional currents, whereby standard index theorem
results (such as those described in appendix B) can be applied directly only in the small-
radius limit when the two fixed-planes coincide. This is because it is only in this limit
that we describe ten-dimensional fermions coupled to ten-dimensional currents. Thus, the
gravitino-induced anomaly on a given ten-plane is one-half of that described by the index
theorem results using the (ten-dimensional) untwisted spectrum described above.

The gaugino fields living in “twisted” Yang-Mills multiplets also contribute a gravita-
tional anomaly, as well as mixed and a pure-gauge anomalies. However, since the twisted
fields are ten-dimensional these can be computed directly using the standard formulae
(without multiplying by 1/2).

It is then straightforward to compute the one-loop anomaly using the formulae de-
scribed in appendix B. The local anomaly on the $i$th fixed-plane is characterized by the
following twelve-form,

$$I_{12}(1\text{ loop})_i = \frac{1}{4} \left( I^{(3/2)}_{\text{GRAV}}(R) - I^{(1/2)}_{\text{GRAV}}(R) \right)$$

$$+ \frac{1}{2} \left( n_i I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{\text{MIXED}}(R, F_i) + I^{(1/2)}_{\text{GAUGE}}(F_i) \right),$$

(3.10)

where the various constituent polynomials are given explicitly in (B.1) and (B.2), and $n_i$
is the dimension of the adjoint representation of $G_i$. As explained in appendix B, each
term includes a factor of 1/2 because the relevant fermions are Majorana-Weyl (having
half the degrees of freedom of a Weyl spinor) while the first two terms obtain an additional
factor of 1/2 (accounting for an overall coefficient of 1/4) for the reasons described above.

Using (B.1) and (B.2) we easily compute the polynomial $I_{12}(1\text{ loop})_i$. The result has a
$\text{tr} R^6$ term, with coefficient proportional to $(n_i - 248)$. However, the anomaly can only
be cancelled if the twelve-form $I_{12}(1\text{ loop})_i$ factorizes as the product of a two-form and
an eight-form. This is because the one-loop anomaly can only be canceled through an
additional anomalous variation of the classical action which is necessarily so-factorized.
The reason why the classical contribution is so-factorized is because this contribution
arises only through the noninvariance of independent factors in the “Chern-Simon’s”
interactions $CGG$ and $GX_7$. But $SO(9,1)$ does not enable factorization of $\text{tr} R^6$ (which
is therefore said to be the irreducible part of the anomaly), so this term must vanish
identically. Therefore the dimension of the group $G_i$ must be $n_i = 248$. Without yet
specifying which 248-dimensional group is permitted, we substitute 248 for \( n_i \), obtaining

\[
I_{12}(1 \text{ loop})_i = \frac{1}{2 (2\pi)^6 6!} \left( -\frac{15}{16} \text{tr} R^4 \text{tr} R^2 - \frac{15}{64} (\text{tr} R^2)^3 + \frac{1}{16} \text{tr} R^4 \text{Tr} F_i^2 \\
+ \frac{5}{64} (\text{tr} R^2)^2 \text{Tr} F_i^2 - \frac{5}{8} \text{tr} R^2 \text{Tr} F_i^4 + \text{Tr} F_i^6 \right),
\]

(3.11)

where factors \( \text{Tr} F_i^2 \) involve a trace over the adjoint representation of \( G_i \). As explained above, it is necessary that this polynomial factorize as the product of a two-form and an eight-form. It is straightforward to algebraically impose this restriction, from which the following is found to be a necessary requirement,

\[
\text{Tr} F_i^6 = \frac{1}{24} \text{Tr} F_i^4 \text{Tr} F_i^2 - \frac{1}{3600} (\text{Tr} F_i^2)^3.
\]

(3.12)

There is exactly one nonabelian Lie group with this property, \( E_8 \). Given the property (3.12), as well as the conventional \( E_8 \) definition of a “tr” operation, \( \text{Tr} F^2 \equiv 30 \text{tr} F^2 \), and the two other \( E_8 \) identities listed in appendix A, after a small amount of straightforward algebra the anomaly polynomial (3.11) can be reexpressed as follows,

\[
I_{12}(1 \text{ loop})_i = \frac{1}{3} \pi I_{4(i)} + X_8 \wedge I_{4(i)},
\]

(3.13)

where \( X_8 \) is the eight-form given in (2.8) and \( I_{4(i)} \) is a four-form given by

\[
I_{4(i)} = \frac{1}{16\pi^2} (\text{tr} F_i^2 - \frac{1}{2} \text{tr} R^2).
\]

(3.14)

The factorization (3.13) was first presented in [8], and provides the key to anomaly cancellation. The first term of (3.13) is canceled by inflow mediated by the \( CGG \) interaction while the second term is canceled by inflow mediated by the \( GX_7 \) interaction.

As in the case of the fivebrane described in the previous section, the one-loop anomaly (3.11) is cancelled via inflow from classical variation of the eleven-dimensional action. This can include an anomalous contribution localized on the two ten-planes provided the Bianchi identity \( dG \) is appropriately modified. Anomaly inflow then arises due to the variation of the same two terms (2.3) which were instrumental to the fivebrane anomaly cancellation. A remarkable feature of \( M \)-theory is that for this orbifold, as well as the one presented in the next section, anomaly cancellation does not require any additional counterterms.

**Magnetic couplings:**

The modifications to the \( dG \) Bianchi identity which enables the necessary inflow mechanism can be derived in a systematic way. Since \( dG \) is a five-form, the most general modification with local support on the ten-plane \( M_1^{10} \) would have to be a gauge-invariant
four-form wedged with the one-form brane current $\delta^{(1)}_M$. The most general gauge-invariant four-form which is available for this purpose would be some linear combination of $\text{tr } R^2$ and $\text{tr } F^2_i$. In a fully general analysis, we would leave the coefficients of these two terms unspecified, finding later that they are fixed by anomaly cancellation. Not surprisingly it is precisely the combination $I_4(i)$ given in (3.14) which is required. For the sake of economy we sacrifice a very small amount of ultimately irrelevant generality by specializing to this case. So the appropriately modified Bianchi identity is given by

$$dG = \sum_{i=1}^{2} I_4(i) \wedge \delta^{(1)}_M,$$

where $I_4(i)$ is the four-form given in (3.14) and $\delta^{(1)}_M$ is a one-form brane-current with support on the $i$th ten-plane.

**Anomaly inflow:**

To determine how the $CGG$ term transforms, we need to determine how the three-form potential $C$ transforms. To determine this, we need an explicit form for $G$. This is determined as the object whose exterior derivative reproduces the right-hand side of (3.15). This implies

$$G = dC + \sum_{i=1}^{2} \left( (b - 1) \delta^{(1)}_M \wedge \omega^0_{3(i)} + \frac{1}{2} b \theta(i) I_4(i) \right),$$

where $\omega^0_{3(i)}$ is the Chern-Simons three-form determined by $d\omega^0_{3(i)} = I_4(i)$, while $\theta(i)$ is a zero-form with the two properties $d\theta(i) = 2\delta^{(1)}_M$ and $\theta^2(i) = 1$, and $b$ is a real parameter unspecified by the Bianchi identity. This parameter is, however, fixed by anomaly cancellation, as described below.

Since the field strength $G$ must be gauge invariant, this requires that $C$ have the following transformation property under gauge transformations and local Lorentz transformations,

$$\delta C = \sum_{i=1}^{2} (b - 1) \omega^1_2(i) \wedge \delta^{(1)}_M.$$

Unless $b = 1$, equation (3.17) implies that $C$ has a nontrivial transformation rule. In fact, as will be shown, anomaly cancellation requires $b = 2$, so this allows for anomaly inflow through the resulting noninvariance of the $CGG$ interaction.

Using the properties of the brane-currents described in section 2, it is then straightforward to determine the transformation of the two interactions $CGG$ and $GX_7$. For the
case of the $CGG$ interaction we determine

$$\delta(-\frac{\pi}{3} \int C \wedge G \wedge G) = -\frac{\pi}{3} \sum_{i=1}^{2} \frac{1}{4} (b-1) b^2 \int_{M_{10}^{i}} \omega_{2}^{4(i)} \wedge I_{4(i)} \wedge I_{4(i)}.$$  \hspace{1cm} (3.18)

To obtain this result, we note that, since $G$ is gauge invariant, only the variation of the factor $C$ on the left-hand side of (3.18) contributes. Using the explicit result (3.17), this tells us that $\delta \int C G G = (b-1) \sum_{i=1}^{2} \int_{M_{10}^{i}} \omega_{2}^{1} (G^2)$, where the bar indicates that $G^2$ is evaluated on the $i$th fixed ten-plane. Since $C_{ABC} | = 0$, only the terms $\partial [A C B]_{(11)}$ contribute to $dC$, so that $dC$ necessarily includes a $dx^{11}$ factor. Since both $\delta C$ and $\delta_{M_{10}}^{(1)}$ also contain $dx^{11}$ factors, we can therefore neglect the first two terms on the right hand side of (3.16) when evaluating $G^2$ | (because $dx^{11} \wedge dx^{11} = 0$). As a result, $G^2$ | is proportional to $\theta_{(i)}^{2} = 1$ so that the product $G^2$ is well-defined on $M_{10}^{i}$. Equation (3.18) describes anomaly inflow to the two fixed ten-planes due to the $CGG$ interaction.

Similarly, we determine

$$\delta( \int G \wedge X_{7}) = -\sum_{i=1}^{2} \int_{M_{10}^{i}} I_{4(i)} \wedge X_{6}^{1}.$$  \hspace{1cm} (3.19)

To obtain (3.19), we have integrated by parts and used the Bianchi identity (3.15). Equation (3.19) describes inflow to the two ten-planes due to the $GX_{7}$ interaction.

The anomaly inflow can be described by a pair of twelve-forms $I_{12}$(inflow)$_{i}$ which give rise to (3.18) and (3.19) upon descent. Thus,

$$I_{12}(CGG)_{i} = -\frac{\pi}{12} (b-1) b^2 I_{4(i)}^{3}$$

$$I_{12}(GX_{7})_{i} = -I_{4(i)} \wedge X_{8}.$$  \hspace{1cm} (3.20)

These two contributions must conspire to cancel against the quantum anomaly given in (3.13).

The total anomaly:

The total anomaly is given by the sum of the one-loop anomaly (3.13) and the inflow contributions (3.20). Thus, $I_{12}$(total) = $I_{12}$(1 loop) + $I_{12}(CGG) + I_{12}(GX_{7})$. We require that this total anomaly vanish. Nicely, the second term of (3.13) is canceled by $I_{12}(GX_{7})_{i}$.

The first term of (3.13) is canceled by $I_{12}(CGG)_{i}$ provided $b$ satisfies the cubic equation $b^3 - b^2 - 4 = 0$. This equation has one real root, so anomaly cancellation uniquely selects

$$b = 2.$$  \hspace{1cm} (3.21)

This value of $b$ is fixed by the consistency requirements. It is gratifying that this requirement is satisfied by a rational (indeed, integer) value for $b$. In contrast to the fivebrane
worldvolume anomaly, the \( CGG \) term provides a crucial ingredient to the removal of the gravitational anomaly.

Comment 1:

The above analysis is easily generalized to the case where fivebranes propagate in the \( \mathbf{R}^{10} \times S^1 / \mathbb{Z}_2 \) background. If there are \( N_5 \) fivebranes, then (3.15) is replaced with

\[
dG = \sum_{i=1}^{2} I_{4(i)} \wedge \delta_{M_i^{10}}^{(1)} + \sum_{i=1}^{N_5} \delta_{W_i^6}^{(5)},
\]

where \( W_i^6 \) is the worldvolume of the \( i \)th fivebrane. In this case, consistency is automatically assured because any additional anomalies associated with the fivebranes are removed by the mechanism described in section 2.

Comment 2:

An essential point is that the factorization criterion (3.12) differs from the analogous criterion encountered in the effective field theory describing the perturbative heterotic string, obtained as the limit that the eleventh dimension shrinks to zero size. In that limit, the theory becomes ten-dimensional and the two orbifold ten-planes coincide. The quantum anomaly is then replaced by the sum of the two previously independent polynomials \( I_{12}(1 \text{ loop})_1 \) and \( I_{12}(1 \text{ loop})_2 \) since these are now evaluated on the same manifold. Furthermore, the D=11 supergravity theory collapses to D=10 N=1 supergravity and the inflow mechanism involving the \( CGG \) and the \( GX_7 \) terms mutates into the ordinary Green-Schwarz mechanism mediated by the surviving two-form, which also requires factorization of the quantum anomaly polynomial. However, since the quantum anomaly in the small-radius limit is given by the sum of the two polynomials described above, the factorization criterion is not the same. (The ten-dimensional requirement is derived in appendix B and given as (3.7), which should be compared with (3.12)). The ten-dimensional condition does allow \( E_8 \times E_8 \) as an allowed gauge group, as one would expect, but it also has another solution, \( SO(32) \), which is not relevant to the full \( M \)-theory construction. The fact that \( SO(32) \) is found in the small-radius limit but not on the expanded orbifold can be phrased as an inability to “pull apart” the gauge group \( SO(32) \) to enable anomaly cancellation on independent orbifold planes. The gauge group \( E_8 \times E_8 \), on the other hand, does have the ability to be “pulled apart” so that each \( E_8 \) factor can be naturally associated with one of the two fixed ten-planes in the \( \mathbb{Z}_2 \) orbifold. The concept of “pulled apart” twisted matter illustrates one difference between what we refer to as “collective” anomaly constraints compared to “local” anomaly constraints, the former referring to weaker conditions which apply in the collapsed limit when orbifold planes coalesce.
4 \( M\)-theory on \( \mathbb{R}^6 \times T^5/\mathbb{Z}_2 \)

There is another \( M\)-theory orbifold which gave rise, through a detailed analysis of local anomaly cancellation, to yet another important insight into a subtle mechanism of \( M\)-theory. In this case \( D=11 \) supergravity on \( \mathbb{R}^6 \times T^5/\mathbb{Z}_2 \), which was first analyzed in [10, 11], a curious degeneracy first implicated the role of “wandering fivebranes” in mediating phase transitions.

As in the case of the \( S^1/\mathbb{Z}_2 \) orbifold described in the previous section, the \( T^5/\mathbb{Z}_2 \) orbifold has a set of parallel fixed-planes which are indistinguishable aside from their position. But in the \( T^5/\mathbb{Z}_2 \) case, there are thirty-two fixed-planes rather than two, and their dimensionality is six rather than ten. Due to important differences from the previous case, and also because much of the analysis in the more complicated orbifolds with intersecting fixed planes again parallels this discussion, it is worthwhile to briefly but completely review the anomaly analysis for the case of the \( T^5/\mathbb{Z}_2 \) orbifold.

\textit{Structure of the orbifold:}

Start with eleven-dimensional supergravity on \( \mathbb{R}^6 \times T^5 \). The \( \mathbb{R}^6 \) factor is parameterized by \( x^\mu \equiv \{x^1, ..., x^6\} \), while the five compact dimensions are parameterized by \( x^i \equiv \{x^7, ..., x^{11}\} \), which each take values on the interval \([-\pi, \pi]\) with endpoints identified. Truncate the theory via a \( \mathbb{Z}_2 \) projection which acts on each of the five compact coordinates as \( x^i \to -x^i \). There are \( 2^5 = 32 \) six-dimensional hyperplanes fixed by this projection, namely the surfaces defined when each of the five \( x^i \) independently assumes the value 0 or \( \pi \).

Demanding \( \mathbb{Z}_2 \) invariance of the eleven-dimensional \( CGG \) interaction term implies that the three-form \( C \) is odd under the \( \mathbb{Z}_2 \) projection. Thus, \( C_{\mu\nu\rho} \to -C_{\mu\nu\rho} \) and \( C_{\mu ij} \to -C_{\mu ij} \). The components \( C_{\mu\nu\rho} \) and \( C_{\mu ij} \) therefore vanish on the fixed six-planes. However, \( C_{\mu ri} \to C_{\mu ri} \). Therefore, from the point of view of the six-dimensional fixed-planes, the three-form \( C \) contributes five nonvanishing tensors \( C_{i\mu\nu} \).

Half the supersymmetries of the eleven-dimensional theory are broken on the fixed-planes by the \( \mathbb{Z}_2 \) projection. Thus, on the fixed-planes the five six-dimensional two-forms \( C_{i\mu\nu} \) organize along with the other fields surviving the \( \mathbb{Z}_2 \) projection into \( D=6 \) \( N=2 \) supermultiplets. Since there are no six-dimensional vector fields surviving the \( \mathbb{Z}_2 \) projection, it follows that the relevant supersymmetry is the chiral \( N=2b \) theory. This is because the alternative, the nonchiral \( N=2a \) theory, necessarily involves vector fields. The \( N=2b \) supergravity multiplet includes five self-dual two-forms, so the remaining five anti self-dual components must organize into matter supermultiplets. In the \( N=2b \) theory, the
only matter multiplet is the tensor multiplet (which includes one anti self-dual two-form). Thus, in addition to the N=2b supergravity, we have five D=6 N=2 tensor multiplets in the “untwisted sector”. The anomaly due to the self-dual and anti self-dual two-forms cancel each other. So the anomalous “untwisted” couplings involve two chiral spin 3/2 fields coming from the supergravity multiplet and ten antichiral spin 1/2 fields coming two each from the five tensor multiplets.

We also allow for a “twisted sector” involving some number \( n_i \) of D=6 N=2 tensor multiplets to propagate on the \( i \)th six-plane. The anomalous “twisted” couplings therefore involve \( n_i \) anti self-dual tensor fields and \( 2n_i \) antichiral spin 1/2 fields.

The one-loop anomaly:

A gravitational anomaly arises due to the coupling of chiral projections of the bulk gravitino to currents localized on the thirty-two fixed-planes. Since the six-planes are indistinguishable aside from their position, this anomaly is similar on each of the thirty-two, and can be computed using standard formulae if proper care is used. The reason why extra care is needed is that the anomaly in question actually derives from the coupling of eleven-dimensional fields to six-dimensional currents, whereby standard index theorem results (such as those described in appendix C) can be applied directly only in the small-radius limit when the thirty-two fixed-planes coincide. This is because it is only in this limit that we actually describe six-dimensional fields coupled to six-dimensional currents. Thus, we can compute the gravitino-induced anomaly on a given six-plane as 1/32 of that described by the index theorem results using the (six-dimensional) untwisted spectrum described above.

The “twisted” tensor multiplets also contribute a gravitational anomaly. However, since the twisted fields are six-dimensional these can be computed directly using the standard formulae (without multiplying by 1/32).

It is then straightforward to compute the one-loop anomaly using the formulae described in appendix C. The local anomaly on the \( i \)th six-plane is characterized by the following eight-form,

\[
I_8(1 \text{ loop})_i = \frac{1}{32} \left( 2 I_{\text{GRAV}}^{(3/2)}(R) - 10 I_{\text{GRAV}}^{(1/2)}(R) \right) - n_i \left( 2 I_{\text{GRAV}}^{(1/2)}(R) + I_{\text{GRAV}}^{(3-\text{form})}(R) \right),
\]

where the various constituent polynomials are given explicitly in (C.1), and \( n_i \) is the number of twisted tensor multiplets. The first line describes the untwisted anomaly and includes an additional factor of 1/32 for the reasons described in the previous paragraph. The second line describes the twisted anomaly coming from \( n_i \) tensor multiplets.
We compute the anomaly by substituting the explicit polynomials given in (C.1) into the expression (4.23). After a small amount of algebra the result organizes as

\[ I_8(1 \text{ loop})_i = (n_i - \frac{1}{2}) X_8, \]  

(4.24)

where \( X_8 \) is given in \( \text{(2.8)} \). Since the number of tensor multiplets should be integer, it is apparent that another mechanism is required to cancel this anomaly. In fact, as in the case of the fivebrane described in section 2 and also in the \( S^1/Z_2 \) orbifold described in section 3, the one-loop anomaly (4.24) is canceled via inflow from classical variation of the eleven-dimensional action. This can include an anomalous contribution localized on the thirty-two fixed planes provided the Bianchi identity \( dG \) is appropriately modified. Anomaly inflow then arises due to the variation of the \( GX_7 \) term.

**Magnetic couplings:**

Since \( dG \) is a five-form, the most general modification with local support on the six-plane \( M^6_i \) would have to be a zero-form (ie: a number) multiplying the five-form brane current \( \delta^{(5)}_{M^6} \). So the appropriately modified Bianchi identity is given by

\[ dG = \sum_{i=1}^{32} g_i \delta^{(5)}_{M^6_i}, \]  

(4.25)

where \( \delta^{(5)}_{M^6} \) is a five-form brane-current with support on the \( i \)th six-plane \( M^6_i \) and \( g_i \) are yet-unspecified rational magnetic charges assigned independently to each of the \( M^6_i \).

**Anomaly inflow:**

It is straightforward to determine the transformation of the \( GX_7 \) interaction. This is found to be

\[ \delta( \int G \wedge X_7 ) = - \sum_{i=1}^{32} g_i \int_{M^6_i} X_1^6. \]  

(4.26)

To obtain (4.26) we have integrated by parts and used the Bianchi identity (4.25). Equation (4.26) describes inflow to the six-dimensional fixed planes due to the \( GX_7 \) interaction. Note that in this case the \( CGG \) interaction does not contribute inflow to the local gravitational anomaly.

The anomaly inflow is characterized by the eight-form \( I_8(GX_7)_i \) which gives rise to (4.26) upon descent. Thus,

\[ I_8(GX_7)_i = - g_i X_8, \]  

(4.27)

This contribution must conspire to cancel against the quantum anomaly given in (4.24).
The total anomaly:

The total anomaly is given by the sum of the one-loop contribution (1.24) and the inflow contribution (1.27). Thus, \( I_8(\text{total})_i = I_8(1 \text{ loop})_i + I_8(GX_7)_i \), which tells us

\[
I_8(\text{total})_i = (n_i - g_i - \frac{1}{2}) X_8 .
\]  

(4.28)

We require that this total anomaly vanish. This requires that

\[
n_i - g_i = \frac{1}{2} .
\]  

(4.29)

Note that whereas \( n_i \) is necessarily an integer, the magnetic charge \( g_i \) can be half-integer. The anomaly vanishes for a given choice of magnetic charge \( g_i \) provided there are \( n_i = g_i + 1/2 \) tensor multiplets. Since \( n_i \) should be a nonnegative integer this tells us that there is a minimum magnetic charge equal to \(-1/2\) with permissible values at successively greater integer increments. Thus, the allowed values of \( g_i \) are given by \(-1/2, +1/2, +3/2, \ldots\).  

**Comment 1:**

The above analysis is easily generalized to the case where fivebranes propagate in the \( T^5/\mathbb{Z}_2 \) background. If there are \( N_5 \) fivebranes, then (4.25) is replaced with

\[
dG = \sum_{i=1}^{32} g_i \delta^{(5)}_{M^6_i} + \sum_{i=1}^{N_5} \delta^{(5)}_{W^6_i},
\]  

(4.30)

where \( W^6_i \) is the worldvolume of the \( i \)th fivebrane. In this case, consistency is automatically assured because any additional anomalies associated with the fivebranes are removed by the mechanism described in section 2.

**Global constraints and wandering fivebranes:**

If we integrate the Bianchi identity (4.30) over the compact \( T^5 \), the left-hand side vanishes due to Stokes theorem (since there is no boundary) and the right-hand becomes \( N_5 + \sum_i g_i \) (since the brane-currents integrate to unity). Therefore

\[
N_5 + \sum_i g_i = 0 .
\]  

(4.31)

Now, if we sum equation (4.29) over the 32 fixed points using this constraint, we determine a second global constraint given by \( N_5 + \sum_i n_i = 16 \). Thus, the total number of twisted tensor multiplets plus the number of fivebranes must be 16. Since each fivebrane also supports a tensor multiplet, this tells us that we have a total of 16 tensor multiplets. There are various ways to realize all of these constraints. For instance, if there are no
fivebranes (so that \( N_5 = 0 \)), one could place tensor multiplets on 16 of the 32 fixed points, and assign magnetic charge \(+1/2\) to these same 16 fixed points and magnetic charge \(-1/2\) to the remaining 16 fixed points which do not support tensor multiplets. There are numerous other possibilities. However, there is no way to solve all of the constraints in a way which treats all 32 fixed-planes identically unless we allow \( N_5 \neq 0 \).

The most symmetrical individual solution has \( N_5 = 16 \) and identical magnetic charge \( g_i = -1/2 \) for each of the thirty-two fixed planes. These assignments satisfy the global constraint (4.31). Since \( g_i = -1/2 \), the local constraint (4.29) requires that \( n_i = 0 \), so that there are no twisted tensor multiplets in this solution.

The presence of fivebranes also allows for a unified description (first presented in [11]) which symmetrically incorporates all of the distinct vacua. This is obtained if all twisted tensors are associated with fivebranes wrapping the fixed six-plane in question. A picture emerges by which fivebranes can detach from a given six-plane, taking one tensor and one unit of charge with it, and “wander” throughout the bulk. Similarly, a “wandering fivebrane” can move to and wrap a particular six-plane, thereby adding one tensor to the twisted spectrum of that plane and simultaneously increasing the magnetic charge by one. The wandering branes will have their anomalies canceled by the mechanism explained in section 2, whereas the six-planes will have any local anomaly canceled by the similar mechanism explained above in this section. In this way all of the unique non-symmetrical configurations are linked by phase transitions mediated by the fivebranes!

Comment:

One may ponder another mechanism whereby, on a fixed six-plane, a twisted tensor field mediates a Green-Schwarz mechanism locally via counterterms in the action describing the tensor dynamics. For the case of the \( T^5/Z_2 \) orbifold, however, this isn’t possible. This is because the anomaly eight-form is proportional to \( X_8 \), which is not factorizable due to the presence of \( \text{tr} \, R^4 \). In the case of the orbifold presented in section 6, however, we find a scenario where we have six-dimensional orbifold planes which can support tensors, but where the anomaly can indeed factorize. In that case, such a local Green-Schwarz mechanism is not only possible, but necessary.

5 The “I-brane effect”

The orbifolds analyzed in sections 3 and 4 are the simplest possible examples. In those cases, all of the orbifold planes are of the same dimensionality (ten in the \( S^1/Z_2 \), and six in the \( T^5/Z_2 \) case) and do not intersect each other. More generally orbifolds will involve
fixed planes of mixed dimensionalities which can intersect. In this case, the analysis of local anomaly cancellation is considerably more involved. The presence of intersections allows for a type of inflow mechanism not encountered in the simpler examples. In this section, we describe this effect in generality in a self-contained manner. In the next section, we analyze a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold which involves this effect as a necessary ingredient.

Brane physics has spawned several cousin-effects to the the Green-Schwarz mechanism. These are necessary to explain anomaly cancellation in various scenarios involving spacetime defects. Whereby the conventional Green-Schwarz mechanism involves non-trivial magnetic and also nontrivial electric couplings to the two-form potential in ten-dimensional N=1 supergravity, neither of which is concentrated on spacetime defects, the cousin-effects include couplings which are localized either on the worldvolume of branes or on orbifold points. In the conventional mechanism, a ten-dimensional quantum anomaly is cancelled by a nonvanishing variation of the classical action involving an interplay between ten-dimensional magnetic couplings, which appear in the $dH$ Bianchi identity, and ten-dimensional electric couplings, which appear as Chern-Simons interactions. By way of comparison, the gravitational anomaly of $M$-fivebranes is removed by an interplay between a six-dimensional magnetic coupling to the three-form potential of eleven-dimensional supergravity which is localized on the worldvolume of the fivebrane, and an eleven-dimensional electric coupling appearing as a Chern-Simons interaction. In this section, we describe a related effect inspired by considerations of intersecting D-branes, dubbed “I-branes” and first presented in [12], and generalized and nicely explained in [7].

The idea behind the “I-brane effect” which we consider involves magnetic and electric couplings to the three-form potential which are variously supported on submanifolds. For instance, a magnetic coupling (appearing in the $dG$ Bianchi identity) can be concentrated on a $d1$-dimensional defect. Additionally, electric couplings can appear as Chern-Simons’s interactions in the worldvolume Lagrangian describing matter propagating on a $d2$-dimensional defect. This could be “twisted” matter concentrated on an orbifold plane. The electric couplings would then appear also in the classical field equation for $G$ obtained by the variational principle. Thus, the magnetic and electric couplings appear in the Bianchi identity and in the classical field equations, respectively, as follows

\[
dG = \delta_{M^d}^{(11-d)} \wedge \tilde{Y}_{d1-6},
\]
\[
d \star G \propto \delta_{M^{d2}}^{(11-d2)} \wedge Y_{d2-3},
\]

where $\tilde{Y}_{d1-6}$ is a closed, gauge-invariant $(d1-6)$-form coupling magnetically to $G$, while $Y_{d2-3}$ is a closed, gauge-invariant $(d2-3)$-form coupling electrically to $G$. The forms $\delta_{M^d}^{(11-d)}$ are $(11-d)$-form brane-currents.
The second equation in (5.1) derives from a Chern-Simon’s interaction localized on the submanifold $M^{d_2}$, given by

$$S_{CS}(M^{d_2}) = \int \delta^{(11-d_2)} \wedge G \wedge Y_{d_2-4}^0,$$

(5.2)

where $Y_{d_2-3} = dY_{d_2-4}^0$. This is easily verified, as the classical variation $\delta/\delta G$ applied to (5.2) gives the the right-hand side of the second equation of (5.1), whereas variation of the $G \star G$ kinetic term supplies the left-hand side.

Varying (5.2), integrating by parts, and using the first line of (5.1), it is straightforward to show that

$$\delta S_{CS}(M^{d_2}) = -\int \delta^{(11-\mathcal{I})} \wedge \left( \tilde{Y}_{d_1-6} \wedge Y_{d_2-3} \right)^1_{\mathcal{I}},$$

(5.3)

where $\mathcal{I} = d_1 + d_2 - 11$ is the dimensionality of the intersection $M^{\mathcal{I}} = M^{d_1} \cap M^{d_2}$. Thus, there is an anomalous classical variation localized on the $\mathcal{I}$-dimensional intersection of $M^{d_1}$ and $M^{d_2}$. This anomaly is characterized by the $(\mathcal{I}+2)$-form which gives rise to (5.3) upon descent. Thus,

$$Y(IB)_{\mathcal{I}+2} = \tilde{Y}_{d_1-6} \wedge Y_{d_2-3}.$$

(5.4)

The intersection anomaly (5.4) involves the product of a magnetically-coupled form localized on one defect with an electrically-coupled form localized on another defect, while the contribution to the anomaly itself is localized on the intersection.

6 M-Theory on $S^1/Z_2 \times T^4/Z_2$

In this section, we consider the simplest nontrivial $M$-theory orbifold which involves multiple intersecting fixed-planes. This example has fixed planes of ten-, seven- and six-dimensions, the six-planes lying at the intersections of the ten-planes with the seven-planes. It is actually a second $Z_2$ orbifolding of the “Hořava-Witten” $Z_2$ orbifold described in section 3, and represents a singular limit of $M$-theory on $S^1/Z_2 \times K3$.

The greater complexity of this orbifold compared to the $Z_2$ orbifolds necessitates a greater systematics. We first define more precisely the structure of the orbifold, then develop the needed machinery, and then use this to determine the twisted states. It\footnote{We use a standard notation to describe forms linked by descent, such that a closed gauge invariant form $Z_q$ is written locally as $Z_q = dZ_{q-1}^0$, where $Z_{q-1}^0$ has gauge variation $\delta Z^0 = dZ_{q-2}^1$.}
Table 1: The action of the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the five compact coordinates of the orbifold discussed in section 6. A plus sign indicates no action on the indicated coordinate and a minus sign indicates a parity reversal, $x^i \rightarrow -x^i$.

|     | $x^7$ | $x^8$ | $x^9$ | $x^{10}$ | $x^{11}$ |
|-----|-------|-------|-------|---------|---------|
| $\alpha$ | +     | +     | +     | +       | -       |
| $\beta$  | -     | -     | -     | -       | +       |
| $\alpha\beta$ | -     | -     | -     | -       | -       |

The action of the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the five compact coordinates of the orbifold discussed in section 6. A plus sign indicates no action on the indicated coordinate and a minus sign indicates a parity reversal, $x^i \rightarrow -x^i$.

It turns out that cancelation of the anomaly at six-dimensional orbifold-plane intersections requires particular twisted states on the entirety of one of the intersecting planes, which is seven-dimensional. This analysis illustrates how gravitational anomalies can be used to determine states in extended regions without a continuous local anomaly.

The structure of the orbifold:

We consider the specific $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$ orbifold defined as follows. Start with eleven-dimensional supergravity on a spacetime with topology $\mathbb{R}^6 \times T^5$. The five compact coordinates, $\{x^7, x^8, x^9, x^{10}, x^{11}\}$ each takes values on the interval $[-\pi, \pi]$ with endpoints identified. In addition to the unit element, the orbifold group includes an element $\alpha$ which reverses the orientation of the eleventh coordinate, $x^{11} \rightarrow -x^{11}$, an element $\beta$ which reverses the orientation on each of the four coordinates $x^i \equiv \{x^7, x^8, x^9, x^{10}\}$, and the product $\alpha\beta$ which reverses the orientation of all five compact coordinates. The action of the three nontrivial elements are displayed in table 1.

The global structure of this orbifold is determined as follows. The element $\alpha$ leaves invariant the two ten-planes defined by $x^{11} = 0$ and $x^{11} = \pi$, while $\beta$ leaves invariant the sixteen seven-planes defined when the four coordinates $x^i$ individually assume the value 0 or $\pi$. Finally, $\alpha\beta$ leaves invariant the thirty-two six-planes defined when all five compact coordinates individually assume the value 0 or $\pi$. The $\alpha\beta$ six-planes coincide with intersections of the $\alpha$ ten-planes with the $\beta$ seven-planes. The global structure is nicely visualized by the diagram in figure 1.

Thus, this orbifold describes a network six-, seven-, and ten-dimensional fixed-planes
Figure 1: The global structure of orbifold planes in the $S^1/Z_2 \times T^4/Z_2$ orbifold. The two horizontal lines represent the two ten-dimensional (“Hořava-Witten”) fixed planes associated with the $Z_2$ factor denoted $\alpha$, while the sixteen vertical lines represent the seven-dimensional fixed-planes associated with the $Z_2$ action denoted $\beta$. The thirty-two six-dimensional fixed planes associated with $\alpha \beta$ are represented by the solid dots. These coincide with the intersection of the $\alpha$ planes and the $\beta$ planes. The $X$ in the figure indicates the presence of a “wandering” fivebrane as described in the text.

which intersect. As described above, chiral projections of bulk objects localized on the fixed-planes induce anomalies at one loop. Additional contributions appear via “inflow” from classical variation of the bulk theory which may include anomalous pieces localized on defects. These occur when $G$ couples magnetically to the defects via modifications to the $dG$ Bianchi identity, and follow specifically due to the variation of the terms shown in \eqref{eq:terms}.

*Magnetic and Electric sources for $G$*

The most general modified Bianchi identity will include terms supported locally on orbifold planes and on fivebrane worldvolumes. Since $dG$ is a five-form, a term with local support on one of the seven-planes would have to be proportional to a closed, gauge-invariant one-form constructed from the available fields, wedged with the four-form brane-current $\delta^{(4)}_{M_i^7}$. But there is no way to construct a closed, gauge-invariant one-form from the available fields, so this kind of coupling is disallowed. Therefore, the most general Bianchi identity is given by

$$
\sum_{i=1}^{2} I_{4(i)} \delta^{(1)}_{M_i^{10}} + \sum_{i=1}^{32} g_i \delta^{(5)}_{M_i^6} + \sum_{i=1}^{N_5} \delta^{(5)}_{W_i^6}, \quad (6.1)
$$

where $\delta^{(1)}_{M_i^{10}}$ has support on the $i$th ten-plane $M_i^{10}$ while $\delta^{(5)}_{M_i^6}$ and $\delta^{(5)}_{W_i^6}$ have support on the
six-planes $M_i^6$ and on the fivebrane worldvolumes $W_i^6$, respectively. The four-form $I_{4(i)}$ is determined by anomaly cancellation on $M_i^{10}$ precisely as described in section 3, and is defined in equation (3.14). Finally, $g_i$ are magnetic charges assigned independently to each of the thirty-two fixed six-planes.

The seven-planes do not couple magnetically to $G$ due to their odd-dimensionality. Alternatively, they can couple electrically in the manner described in section 5. This electric coupling can provide an anomaly at seven-plane/ten-plane intersections due to an interplay with the ten-dimensional magnetic coupling. We discuss this further below.

A significant part of our analysis relates to determining the magnetic charges $g_i$. We will present an argument supporting quarter-integer values for $g_i$ in the context of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. In addition, there is a (global) restriction which correlates the total orbifold charge $\sum_i g_i$ with other topological data pertaining to gauge and gravitational instantons and the number of fivebranes, and is obtained by integrating (6.1) over the five-cycle spanned by the five compact dimensions.

The quarter-integer charge quantization can be motivated by the following arguments. The orbifold in question represents a singular deformed limit of $K3 \times S^1 / \mathbb{Z}_2$, where all of the $K3$ curvature is “pinched” to be concentrated and symmetrically distributed over the sixteen fixed seven-planes associated with the element $\beta$ of the orbifold group (these seven-planes are represented by the vertical lines in figure 1). If we start with the smooth $K3$, we can represent the singular orbifold limit by the following deformation of the eleven-dimensional curvature,

\[ \text{tr} \, R^2 \to \text{tr} \, \mathcal{R}^2 + \pi^2 \chi \sum_{i=1}^{16} \delta_{M_i^7}^{(4)}, \]

where $\delta_{M_i^7}^{(4)}$ has support on the $i$th fixed seven-plane $M_i^7$, $\chi = 24$ is the Euler number of $K3$, and $\int_T \text{tr} \, \mathcal{R}^2 = 0$, where the integration is over a four-cycle $T$. Note that this is consistent with the requirement that $\int_T \text{tr} \, R^2 = 16 \pi^2 \chi$. If we make the replacement (6.2) in the first term on the right-hand side of the Bianchi identity (6.1), we generate new terms given by

\[ \frac{1}{16} \left( -\frac{1}{2} \right) \chi \sum_{i=1}^{16} \sum_{j=1}^{16} \delta_{M_i^7 \Omega}^{(1)} \wedge \delta_{M_j^7}^{(4)} = \sum_{i=1}^{32} \left( -\frac{\chi}{32} \right) \delta_{M_i^6}^{(5)} \]

where we have used equation (2.3). These new terms, however, are absorbed by a shift $g_i \rightarrow g_i - \chi/32$. In this way we see that, in the orbifold limit, the Euler character of the smooth $K3$ is equally distributed as a magnetic charge $-\chi/32 = -3/4$ at each of the 32 orbifold fixed six-planes. This represents the gravitational contribution to the magnetic
charge. Similar reasoning motivates that gauge instantons can yield only positive integer or positive half-integer contributions to $g_i$, whereas a resident fivebrane should contribute only positive integer values. Thus, a given six-plane $M^6_i$ should have a minimum charge equal to $-3/4$ with permissible values at successively greater half-integer increments. Thus, the allowed values of $g_i$ would be

$$g_i = -3/4, -1/4, +1/4, ...$$

(6.4)

As described below, the same restriction on permissible values of $g_i$ is found in an independent manner by factorization requirements on the anomaly polynomial.

An additional (global) constraint follows from integrating the Bianchi identity (6.1) over all five compact dimensions. Since this region has no boundary, the left-hand side of the integrated version of (6.1) vanishes due to Stokes theorem, since the integrand is a total derivative. Due to the properties of the brane-currents described in section 2, the integrated right-hand side of (6.1) reduces to the sum $n_1 + n_2 - \chi + N_5 + \sum_i g_i$, where $n_i$ are the instanton numbers associated with the bundle $E_8(i) \rightarrow M^{10}_i$ and $\chi$ is the Euler number defined by $16\pi^2 \chi = \int_T \text{tr} R^2$. The instanton numbers and the Euler number occur since the brane-current in the first term of (6.1) collapses the integral to a four-cyle integral over $T$. So the generic constraint is

$$n_1 + n_2 - \chi + N_5 + \sum_i g_i = 0.$$  

(6.5)

However, we can simplify this expression considerably in the orbifold case where, as we have explained, all of the local curvature which contributes to the Euler number is concentrated on the orbifold-planes, and is more properly absorbed into the magnetic charges $g_i$. So an explicit $\chi$ term should not be included in (6.5). Similar concerns apply to any “zero-size” gauge instanton which is trapped on one of the fixed six-planes. Furthermore, any gauge instanton on one of the ten-planes which is not trapped at an orbifold singularity should be continuously deformable into a fivebrane by shrinking its size to zero. Such a fivebrane can then detach and move into the bulk. Since the two situations are smoothly related on moduli space, we expect an anomaly-free solution for a fivebrane to imply an anomaly-free solution for the associated instanton (and vice-versa). It is simpler to work with fivebranes and, hence, we omit the instanton terms $n_1$ and $n_2$ in equation (6.3). In the analysis to follow, we therefore replace (6.3) with the minimal

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4The half-integer instanton contributions are associated with the ALE instantons classified by the Stieffel-Whitney class.
constraint

\[ N_5 + \sum_{i=1}^{32} g_i = 0, \quad (6.6) \]

noting that \( N_5 \) should be a positive integer. We also keep in mind the quarter-integer quantization of \( g_i \).

**Anomaly Inflow:**

As in the cases of the \( \mathbb{Z}_2 \) orbifolds described in sections 3 and 4, the cancellation of the local anomaly involves one-loop contributions and also inflow contributions describing transformations of the \( CGG \) and \( GX_7 \) terms. There is also another kind of inflow at work in this case which can arise because of the fixed-plane intersections. This is an “I-brane” contribution of the sort described in section 5. The \( CGG \) and \( GX_7 \) contributions are easiest to analyze, so we begin by working these out. Following this, we independently analyze the “I-brane” contribution before moving on to the one-loop anomaly.

To determine how the \( CGG \) term transforms, we need to determine how the three-form potential \( C \) transforms. To determine this, we need an explicit form for \( G \). This is determined as the object whose exterior derivative reproduces the right-hand side of \((6.1)\). Suppressing the fivebrane contributions\(^5\), this implies the following definition,

\[ G = dC + \sum_{i=1}^{32} \left( (b - 1) \delta_{M_{10}}^{(1)} \wedge \omega_{\beta(i)}^0 + \frac{1}{2} b \theta_{\beta(i)} I_{4(i)} \right) + \sum_{i=1}^{32} \frac{1}{2} g_i \theta_{\beta(i)} \delta_{M_{10}}^{(4)}, \quad (6.7) \]

where \( \omega_{\beta(i)}^0 \) is the Chern-Simons three-form determined by \( d\omega_{\beta(i)}^0 = I_{4(i)} \), while \( \theta_{\beta(i)} \) is a zero-form with the two properties \( d\theta_{\beta(i)} = 2\delta_{M_{10}}^{(1)} \) and \( \theta_{\beta(i)}^2 = 1 \), and \( b \) is a real parameter unspecified by the Bianchi identity. This parameter is, however, fixed by anomaly cancellation, as described below.

Since the field strength \( G \) must be gauge invariant, this requires that \( C \) have the following transformation property,

\[ \delta C = \sum_{i=1}^{2} (b - 1) \omega_{\beta(i)}^1 \wedge \delta_{M_{10}}^{(1)}. \quad (6.8) \]

Unless \( b = 1 \), equation \((6.8)\) implies that \( C \) has a nontrivial transformation rule. In fact, as verified below, anomaly cancellation requires \( b = 2 \). This enables anomaly inflow through the resulting noninvariance of the \( CGG \) interaction.

\(^5\) We suppress the fivebrane contribution to \( G \) because this involves unnecessary complexity; the contributions localized on the fivebranes do not affect the orbifold-plane anomalies and furthermore, as described in section 2, the fivebrane anomalies are independently resolved.
Using the properties of the brane-currents described in section 2, it is then straightforward to determine the transformation of the two interactions $CGG$ and $GX_7$. For the case of the $CGG$ interaction we determine

$$
\delta \left( -\frac{\pi}{3} \int C \wedge G \wedge G \right) = -\frac{\pi}{3} \sum_{i=1}^{2} \frac{1}{4} (b - 1) b^2 \int_{M^{10}_i} \omega_{2(i)}^1 \wedge I_{4(i)} \wedge I_{4(i)}
-\frac{\pi}{3} \sum_{i=1}^{32} \frac{1}{2} (b - 1) b g_i \int_{M^6_i} \omega_{2(i)}^1 \wedge I_{4(i)} .
$$

(6.9)

To obtain this result, we note that since $G$ is gauge invariant, only the variation of the factor $C$ on the left-hand side of (6.9) contributes. Using the explicit result (6.8), this tells us that

$$
\delta \int C G^2 = (b - 1) \int_{M^{10}_i} \omega_{2(i)}^1 (G^2) ,
$$

where the bar indicates that $G^2$ is evaluated on $M^{10}_i$. Since $C_{ABC} = 0$, only the terms $\partial_i A C_{BC}(i)$ contribute to $dC$, so that $dC$ necessarily includes a $dx^{11}$ factor. Since both $\delta C$ and $\delta^{(1)}_{M^{10}_i}$ also contain $dx^{11}$ factors, we can therefore neglect the first two terms on the right-hand side of (6.7) when evaluating $G^2$ (because $dx^{11} \wedge dx^{11} = 0$). As a result, $G^2$ is proportional to $\theta^2_{(i)} = 1$ so that the product $G^2$ is well-defined on $M^{10}_i$.

It is interesting to compare equation (6.9) with the analogous expression (3.18) from the case of the $S^1/Z_2$ orbifold. In the present case, we find the same expression for the inflow to the two ten-planes, but we also find an additional contribution localized on the six-planes $M^6_i$ invariant under the full $Z_2 \times Z_2$ group.

Similarly, for the case of the $GX_7$ interaction we determine

$$
\delta \left( \int G \wedge X_7 \right) = -\sum_{i=1}^{2} \int_{M^{10}_i} I_{4(i)} \wedge X_{6(i)}^1 - \sum_{i=1}^{32} g_i \int_{M^6_i} X_{6(i)}^1 .
$$

(6.10)

To obtain (6.10) we have integrated by parts and used the Bianchi identity (6.4). As in the case of the $CGG$ inflow, we find the same contribution as found in the case of the $S^1/Z_2$ orbifold, (3.19), but we also find an additional contribution localized on $M^6_i$.

The anomaly inflow can be described by a pair of twelve-forms $I_{12}(\text{inflow})$, describing the anomaly on the ten-planes $M^{10}_i$, and by a set of thirty-two eight-forms $I_8(\text{inflow})$, describing the anomaly on the six-planes $M^6_i$. These are the objects which give rise to the sum of (6.9) and (6.10) upon descent. Thus,

$$
I_{12}(\text{inflow})_i = -\frac{\pi}{12} (b - 1) b^2 I_{4(i)}^3 - I_{4(i)} \wedge X_8
$$

$$
I_8(\text{inflow})_i = -\frac{\pi}{6} (b - 1) b g_i I_{4(i)}^2 - g_i X_8 .
$$

(6.11)

One might assume that these inflow terms should cancel against the one-loop anomaly similarly to the $Z_2$ orbifold anomalies described in sections 3 and 4. This turns out to
be only partially true. Such cancellation does occur on the fixed ten-planes, but the six-planes are more subtle. In fact, there is another inflow contribution, anticipated in section 5 which contributes to the six-dimensional anomaly.

An “I-brane” anomaly:

The fixed seven-planes can support the special kind of Chern-Simons interaction described by equation (5.2),

\[
S_{CS}(M^7) = \sum_{i=1}^{16} \int_{M^{11}} \delta_{M^7_i}^{(4)} \wedge G \wedge Y^0_{3(i)},
\]

where \(Y^0_{3(i)}\) is a Chern-Simon’s three-form which can include Lorentz as well as a gauge pieces arising from twisted seven-dimensional Yang-Mills matter. Equation (6.12) describes an electric coupling of \(G\) to each of the sixteen fixed seven-planes. If we define \(Y_{3(i)} \equiv dY^0_{3(i)}\), then the most general \(Y_{3(i)}\) is given by

\[
Y_{3(i)} = \frac{1}{(2\pi)^3 4! 32} \left( \eta \, \text{tr} \, R^2 + \rho \, \text{tr} \, F_i^2 \right),
\]

where \(R\) is the eleven-dimensional curvature and \(F_i\) is a seven-dimensional field strength for vector fields propagating on \(M^7_i\) with values in the adjoint of a group \(\mathcal{G}_i\), to be determined. The parameters \(\eta\) and \(\rho\) are arbitrary rational coefficients. The separate numerical prefactor in (6.13) has been chosen to simplify expressions later on. The interaction (6.12) allows for an anomaly contribution of the sort described by equation (5.4). This would arise at a six-dimensional intersection between a ten-plane and a seven-plane as interplay between the ten-dimensional magnetic coupling in the Bianchi identity (6.1) and the seven-dimensional electric coupling implied by (6.12). The anomaly contribution on the six-plane intersection of the \(i\)th ten-plane and the \(j\)th seven-plane is then given by

\[
I_8(IB)_{ij} = I_{4(i)} \wedge Y_{4(j)},
\]

where \(i\) takes either the value 1 or 2, while \(j\) can assume any value from 1 to 16. An alternate labeling scheme involving only a single index facilitates analysis later on. So we adopt the convention of labeling the thirty-two six-planes with a single index, so that (6.15) is rewritten as

\[
I_8(IB)_i = I_{4(i)} \wedge Y_{4(i)},
\]

where \(i\) now assumes any value from 1 to 32. In this case, \(I_{4(i)}\) is taken on the particular ten-plane \(M^{10}_i\) which intersects \(M^6_i\) while \(Y_{4(i)}\) is taken on the particular seven-plane which also intersects \(M^6_i\).
The combination of the inflow contributions shown in (6.11) and the “I-brane” contribution shown in (6.15) must properly conspire with the one-loop anomalies in order that the theory be consistent. We proceed to analyze first the cancellation of the ten-dimensional anomaly and then the six-dimensional anomaly.

6.1 The Ten-Dimensional Anomaly

The anomaly inflow to the fixed ten-planes was computed above, and expressed in (6.11) as $I_{12}^{\text{(inflow)}}$. The two terms in $I_{12}^{\text{(inflow)}}$ arise due to the classical variation of the $CGG$ and the $GX_7$ terms respectively. So we can write $I_{12}^{\text{(inflow)}} = I_{12}^{(CGG)} + I_{12}^{(GX_7)}$. Another contribution arises from one-loop diagrams. In fact, since the ten-planes are only invariant under the element $\alpha$, and since (along with the unit operator) $\alpha$ describes precisely the same $Z_2$ group which defines the orbifold analyzed in section 3, it follows that the computation of the one-loop anomaly in that section applies here as well. So we do not need to perform a separate computation of the ten-dimensional one-loop anomaly; it is given by equation (3.13). Thus, the three contributions to the ten-dimensional anomaly are given by the following polynomials,

$$
I_{12}^{(CGG)} = -\frac{\pi}{12} (b - 1) b^2 I_4^{(i)} \\
I_{12}^{(GX_7)} = -I_4^{(i)} \wedge X_8 \\
I_{12}^{(1 \text{ loop})} = \frac{\pi}{3} I_4^{(i)} + I_4^{(i)} \wedge X_8,
$$

(6.16)

where the inflow contributions were computed above, and the one-loop contribution was computed in section 3 and given as equation (3.13). Not suprisingly, the inflow terms $I_{12}^{(CGG)}$ and $I_{12}^{(GX_7)}$ are also the same as those derived in section 3, given in equation (3.20).

The total gravitational anomaly is given by the sum of all three contributions in (6.16),

$$
I_{12}^{\text{(total)}} = I_{12}^{(1 \text{ loop})} + I_{12}^{(CGG)} + I_{12}^{(GX_7)}.
$$

We require that this total anomaly vanish. Nicely, the second term of $I_{12}^{(1 \text{ loop})}$ is exactly canceled by $I_{12}^{(GX_7)}$. The first term of $I_{12}^{(1 \text{ loop})}$ is exactly canceled by $I_{12}^{(CGG)}$ provided $b$ satisfies the cubic equation $b^3 - b^2 - 4 = 0$. This equation has one real root, so anomaly cancellation uniquely selects

$$
b = 2.
$$

(6.17)

This value of $b$ is fixed by the consistency requirements. It is gratifying that this requirement is satisfied by a rational (indeed, integer) value for $b$. The sub-analysis of the
ten-dimensional anomaly in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case is precisely the same as that given in section 3 for the simpler $\mathbb{Z}_2$ orbifold.

### 6.2 The Six-Dimensional Anomaly

We are mostly concerned with the additional anomaly localized on the six-dimensional fixed-planes $M^6_i$ invariant under the element $\beta$. Since (along with the unit element) $\beta$ describes precisely the same $\mathbb{Z}_2$ group which defines the orbifold analyzed in section 4, one might believe that we can rely on the computation in section 4 in the same manner that we relied on the computation of section 3 to obtain the ten-dimensional anomaly. But things are not so simple in this case because $M^6_i$ are not just invariant under $\beta$ but are, in fact, invariant under $\alpha$ and $\alpha \beta$ as well! So the six-planes are invariant under the entire $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. As a result, there are crucial differences from the $\mathbb{Z}_2$ orbifold analyzed in section 4. For one thing, the additional $\mathbb{Z}_2$ projects out yet another half of the bulk supercharges, so that the local supersymmetry on these planes is $D=6$ $N=1$ rather than $D=6$ $N=2$ as in section 4. This not only changes the untwisted spectrum, and therefore the one-loop anomaly from that of the $\mathbb{Z}_2$ case, but it also means that we have the freedom to include a richer twisted spectrum of additional local states due to the richer structure of $N=1$ supersymmetry.

**Anomaly inflow:**

Anomaly inflow to the fixed six-planes was computed above, and expressed as $I_8$(inflow)$_i$ in (6.11), where the two terms arise due to the classical variation of the $CGG$ and the $GX_7$ terms respectively. So we can write $I_8$(inflow)$_i = I_8(CGG)_i + I_8(GX_7)_i$. Another contribution $I_8(IB)_i$ arises from the “I-brane” mechanism also described above and given as equation (6.15). We set $b = 2$ as required by the removal of the anomaly on the ten-planes. Thus, the three inflow contributions to the six-dimensional anomaly are given by the following polynomials,

\[
I_8(GX_7)_i = -g_i X_8
\]

\[
I_8(CGG)_i = -\frac{1}{5} \pi I^2_{4(i)}
\]

\[
I_8(IB)_i = I_{4(i)} \wedge Y_{4(i)},
\]

(6.18)

where $X_8$ is given in (2.8), while $I_{4(i)}$ is given in (3.14) and $Y_{4(i)}$ is given in (6.13). As explained earlier, the notation is such that $i$ assumes any value from 1 to 32.

The combination of the three contributions shown in (6.18) must conspire with the
one-loop anomalies in order that the theory be consistent. We proceed to analyze the one-loop contributions, and then discuss the conspiracy which renders the theory consistent.

*The one-loop anomaly:*

The quantum contributions consist of three separate pieces:

A contribution $I_8(SG)_i$ arises from the bulk supergravity due to the coupling of chiral projections of the gravitino to $SO(5,1)$ currents associated with diffeomorphisms of the fixed six-planes. On a given six-plane, only the components $C^{(11)\mu\nu}$ and $C^{(11)ij}$ (where $i,j = 1,\ldots,4$) survive the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection from the three-form $C_{IJK}$. These contribute one two-form and six scalars, while the eleven-dimensional metric supplies a six-dimensional metric and eleven more scalars corresponding to $g^{(11)(11)}$ and $g_{ij}$. So the bosonic untwisted spectrum consists of a metric tensor, one two-form and seventeen scalars. These organize along with the surviving fermions into a $D=6$ $N=1$ supergravity multiplet coupled to four $N=1$ hypermultiplets and one $N=1$ tensor multiplet.

Since the total anomaly is distributed equally over the thirty-two six-planes, and since we can only apply the index theorem results in the (small-radius) limit when all six-planes coincide, we conclude that the anomaly on a given hyperplane is $1/32$ of that described by the index theorem results using the untwisted spectrum associated with the bulk fermions. Collectively, these involve one chiral spin $3/2$ field, five antichiral spin $1/2$ fields and one each of self-dual and anti-self-dual tensors. The anomalies due to the tensors cancel each other, so that

$$I_8(SG)_i = \frac{1}{32} \left( I^{(3/2)}_{GRAV} (R) - 5 I^{(1/2)}_{GRAV} (R) \right).$$  \quad (6.19)

Another contribution, $I_8(E_8)_i$, arises from the $E_8$ matter propagating on the fixed ten-planes. To begin with, we assume that $\beta$ acts trivially on the ten-dimensional vector multiplets, so that the $E_8$ gauge group is not broken by the orbifold action. A given ten-dimensional $E_8$ vector supermultiplet decomposes into a $D=6$ $N=2$ $E_8$ vector multiplet, which further decomposes into an $N=1$ vector multiplet and an $N=1$ hypermultiplet. The first of these involves chiral gauginos while the second involves antichiral hyperinos. The anomalies due to these two factors would cancel against each other. However, the six-planes are fixed under both $\mathbb{Z}_2$ factors $\alpha$ and $\beta$. The second $\mathbb{Z}_2$, denoted $\beta$, acts on the $E_8$ supermatter to project out the $N=1$ hypermultiplet, leaving a contribution only from the $N=1$ $E_8$ vector multiplet, which is anomalous. Since there are sixteen fixed six-planes within a given ten-plane, the contribution $I_8(E_8)_i$ localized on a given six-plane is $1/16$ of that described by the index theorem results pertaining to (ten-dimensional) chiral $E_8$.
gauginos, so that

\[ I_8(E_8)_i = \frac{1}{16} \left( 248 I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{\text{MIXED}}(R, F_i)_{\text{ADJ}} + I^{(1/2)}_{\text{GAUGE}}(F_i)_{\text{ADJ}} \right), \]  

(6.20)

where \( F_i \) takes values in the adjoint 248 representation of the \( E_8 \).

The element \( \beta \) can also act nontrivially on the \( E_8 \) vectors, breaking the group to a maximal subgroup. For instance, the 248 decomposes into \( E_7 \times SU(2) \) representations as \((133, 1) \oplus (1, 3) \oplus (56, 2)\). We could realize the \( Z_2 \) on the \( E_8 \) fields to project out the six-dimensional hypermultiplets from the \((133, 1) \oplus (1, 3)\) fields and project out the six-dimensional vector multiplets from the \((56, 2)\) fields. In this case, we would be left with \( E_7 \times SU(2) \) adjoint vectors and 112 hypermultiplets transforming as \((56, 2)\). Another possibility would break \( E_8 \) to \( \text{Spin}(16) \), leaving us with the adjoint 120 coupled to a hypermultiplet in the 128 spinor representation. The two possibilities described by the \( E_7 \times SU(2) \) and \( \text{Spin}(16) \) cases correspond to the only \( Z_2 \) subgroups of \( E_8 \) [13]. Therefore there are only three possibilities, \( E_8 \rightarrow E_8, E_8 \rightarrow E_7 \times SU(2) \) and \( E_8 \rightarrow \text{Spin}(16) \). We will analyze only the first possibility in detail, and make comments about the other two afterwards.

The third contribution \( I_8(G_i) \) arises from twisted matter which we are free to add to the fixed planes. In fact, we are free to add twisted matter of two significantly different sorts. On the one hand, we can include six-dimensional fields propagating on any or all of the thirty-two six-planes, consisting of some number of D=6 N=1 vector, hyper and/or tensor multiplets. On the other hand, we are also free to add seven-dimensional fields propagating on any or all of the sixteen seven-planes, consisting of seven-dimensional vector multiplets. Each of these possibilities will contribute to the six-dimensional anomaly. In the first case, the chiral fields living in the six-dimensional multiplets will couple anomalously to the six-dimensional Lorentz and gauge currents. In the second case, even though the seven-dimensional gauginos will not contribute to an anomaly on the seven-planes (since they are non-chiral), they will couple anomalously to six-dimensional currents on the subplanes fixed by the entire \( Z_2 \times Z_2 \) group. These are, of course, the same six-planes where six-dimensional twisted fields can propagate, represented by the solid dots in figure 1.

The reason why the seven-dimensional fields can contribute to the six-dimensional anomaly mirrors the way in which ten-dimensional \( E_8 \) gauginos contribute to the six-dimensional anomaly. As described above, each ten-dimensional chiral gaugino decomposes into one six-dimensional chiral gaugino and one six-dimensional antichiral hyperino. In that case, the extra \( Z_2 \) projection which leaves the six-planes fixed serves to remove the hyperinos, so that there is a net (six-dimensional) chirality to the projected fermions. Similarly, a seven-dimensional gaugino also decomposes into a six-dimensional gaugino.
and a six-dimensional hyperino $\mathbf{6}$. Once again the extra $\mathbb{Z}_2$ projection will remove the hyperinos, so that the projected fields have a net (six-dimensional) chirality.

We first consider the case where we add six-dimensional twisted matter. If, on the $i$th six-plane we add $n_{V_i}$ vector multiplets, $n_{H_i}$ hypermultiplets, and $n_{T_i}$ gauge-singlet tensor multiplets, the relevant anomaly is given by

$$ I_8(G_i) = (n_V - n_H - n_T)_i I^{(1/2)}_{\text{GRAV}}(R) - n_{T_i} I^{(3\text{-form})}_{\text{GRAV}}(R) $$

$$ + I^{(1/2)}_{MIXED}(R, F_i)_{\text{ADJ},R} + I^{(1/2)}_{\text{GAUGE}}(F_i)_{\text{ADJ},R}, \quad (6.21) $$

where the mixed and pure gauge anomalies involve field strength tensors $F_i$ taking values in both the adjoint (in the case of vector multiplets), and in the $R$ representation (for the case of hypermultiplets). To make sense of equation (6.21), we should use the polynomials given in (C.1) and (C.2), and replace the trace $F_i^2$ contribution in the mixed anomaly and the trace $F_i^4$ contribution in the gauge anomaly as

$$ \text{trace } F_i^n \equiv \text{Tr} F_i^n - \sum_\alpha h_\alpha \text{tr}_\alpha F_i^n, \quad (6.22) $$

where $\text{Tr}$ is an adjoint trace, $h_\alpha$ is the number of hypermultiplets transforming in the $R_\alpha$ representation, and $\text{tr}_\alpha$ is a trace over the $R_\alpha$ representation. Note that the total number of vector multiplets is $n_{V_i} = \dim (G_i)$ while the total number of hypermultiplets is $n_{H_i} = \sum_\alpha h_\alpha \times \dim (R_\alpha)$. The relative minus sign in (6.22) reflects the antichirality of the hyperinos.

Now consider the case where we add seven-dimensional twisted matter. If on a given $\beta$-invariant seven-plane $M_7^i$ we add $\tilde{n}_{V_i}$ vector multiplets in the adjoint of $G_i$, these will contribute to anomalies on the embedded $\alpha$- and $\beta$-invariant six-planes due to the extra $\alpha$ projection. In the simplest case, $\alpha$ will remove all but a six-dimensional hypermultiplet from the seven-dimensional fields. There are other possibilities where $\alpha$ breaks $G_i$ to a maximal subgroup analogous to the situation involving the $\beta$ projection on the $E_8$ fields discussed above. Once again, we will consider first the simplest case, where $\alpha$ does not break $G_i$ and comment on the other possibilities later.

Since there are two fixed six-planes within a given seven-plane, it follows by reasoning described above that the contribution to the six-dimensional anomaly on each of the two six-planes due to the seven-dimensional gauginos will be given by

$$ I_8(G_i) = \frac{1}{2} \left( \tilde{n}_{V_i} I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{MIXED}(R, F_i)_{\text{ADJ},R} + I^{(1/2)}_{\text{GAUGE}}(F_i)_{\text{ADJ},R} \right), \quad (6.23) $$

This is easy to see from the bosonic components of the seven-dimensional Yang-Mills multiplet, which comprises three scalars and one vector; upon toroidal compactification of one dimension, the seven-dimensional vector will give a six-dimensional vector and a fourth scalar, giving the bosonic components of one D=6 N=1 hyper and one D=6 N=1 vector multiplet.
where the factor of 1/2 arises because the anomaly is equally distributed over the two fixed six-planes. To make sense of equation (6.23), we should use the explicit polynomials given in (C.1) and (C.2), making the substitution indicated in (6.22). If $G_i$ is unbroken by $\alpha$, then only the adjoint vectors will contribute, so the $h_\alpha$ would be zero. If $G_i$ is broken to a maximal subgroup by $\alpha$, then vectors in the adjoint of the subgroup will contribute along with some number of hypermultiplets.

It is also possible to include seven-dimensional adjoint $G_i$ vectors and to include additional six-dimensional hypermultiplets which also transform under $G_i$. In this case, the anomaly due to the vectors would be given by (6.23), while that due to the hypers would be given by (6.21).

In all cases, the only distinguishing qualification is a division by two, as seen in equation (6.23), for the anomaly due to any seven-dimensional field. Any purely six-dimensional field contributes to the anomaly without such a division, as in (6.21). It turns out that further considerations concerning the factorizability of the anomaly polynomial require factors of two in such a way that implies the existence of seven-dimensional twisted matter. This is one of the essential points of this paper.

Our strategy is to include unspecified twisted states in a sufficiently powerful way that anomaly cancellation will select both gauge groups and also the dimensionality of the appropriate fields for us. To do this, we need a certain economy which is had by incorporating the various possibilities involving six- and seven-dimensional states in a unified package. This is facilitated in an obvious way by writing the complete twisted anomaly precisely as in equation (6.21), but with two important distinctions. The first distinction is that, since seven-dimensional fields contribute one-half the anomaly on a given six-plane as six-dimensional fields, we consider $n_{V_i}$ in (6.21) to include this potential divisor. Thus, $n_{V_i} \equiv \dim(G_i)/\mu$ where $\mu$ is 1 or 2 depending on the dimensionality of the fields in question,

$$\mu = \begin{cases} 
1 & \text{six-dimensional fields} \\
2 & \text{seven-dimensional fields} 
\end{cases}$$

The second distinction is that when we substitute the anomaly polynomials (C.1) and (C.2) we should replace the trace $F_i^2$ and trace $F_i^4$ contributions not with (6.22) but rather with the obvious extension

$$\text{trace } F_i^n \equiv \frac{1}{\mu} \text{Tr } F_i^n - \sum_\alpha h_\alpha \text{tr}_\alpha F_i^n,$$

where Tr is an adjoint trace, $h_\alpha$ is the number of six-dimensional hypermultiplets transforming in the $R_\alpha$ representation, and $\text{tr}_\alpha$ is a trace over the $R_\alpha$ representation. In this
generalized formulation, the fact that seven-dimensional fields contribute one-half of the mixed and gauge anomalies as do six-dimensional fields is incorporated in the parameter $\mu$.

The complete quantum anomaly on a given six-plane is given by the sum of (6.19), (6.20) and (6.21). To ease our analysis we repeat the three quantum contributions here,

\[
I_8(\text{SG})_i = \frac{1}{32} \left( I^{(3/2)}_{\text{GRAV}}(R) - 5 I^{(1/2)}_{\text{GRAV}}(R) \right)
\]

\[
I_8(E_8)_i = \frac{1}{16} \left( 248 I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{\text{MIXED}}(R, F_i)_{\text{ADJ}} + I^{(1/2)}_{\text{GAUGE}}(F_i)_{\text{ADJ}} \right)
\]

\[
I_8(G_i) = (n_V - n_H - n_T) I^{(1/2)}_{\text{GRAV}}(R) - n_T I^{(3-\text{form})}_{\text{GRAV}}(R)
\]

\[
+ I^{(1/2)}_{\text{MIXED}}(R, F_i)_{\text{ADJ,R}} + I^{(1/2)}_{\text{GAUGE}}(F_i)_{\text{ADJ,R}}.
\]  

(6.26)

The $I_8(\text{SG})_i$ contribution arises from chiral projections of the bulk supergravity fields; the division by 32 reflects the fact that this anomaly is equally distributed over the thirty-two $M_6^i$. The $I_8(E_8)_i$ contribution arises from chiral projections of the ten-dimensional $E_8$ gauginos; in this case the division by 16 reflects the fact that each ten-plane has sixteen embedded fixed six-planes over which this anomaly is equally distributed. Finally, the $I_8(G_i)$ contribution arises from the (yet-undetermined) twisted fields which can propagate either on the six-planes or on the seven-planes, the distinction being encoded in the parameter $\mu$ as described above. The parameter $\mu$ is an important algebraic tool for ascertaining the dimensionality of twisted states indicated by anomaly cancellation.

Henceforth, since it is clear that we are working on a particular six-plane $M_6^i$ we will suppress the $i$ index on objects like $g_i$ and $n_{H,i}$.

The total anomaly:

The total anomaly on a given six-plane is given by the sum of the three inflow contributions given in (6.18) and the three quantum contributions presented in (6.26). Using the anomaly polynomials given in (C.1) and (C.2), we can work out the full anomaly polynomial. This will have many terms. However, only the term proportional to $\text{tr} R^4$ will not be factorizable. This term has a coefficient proportional to $(n_H - n_V + 29n_T - 30g - 23)$. Because the $\text{tr} R^4$ term cannot factorize, it cannot be removed by a Green-Schwarz mechanism using a local tensor field and therefore must vanish identically. This poses an important constraint on the allowed twisted states, given by

\[
n_H - n_V = 30g + 23 - 29n_T.
\]  

(6.27)

If we sum this equation over the thirty-two six-dimensional fixed-planes, impose the constraint (6.6), and use the following expressions for the total number of hyper, vector and
tensor multiplets,

\[ N_H = 4 + N_5 + \sum n_H \]
\[ N_V = 496 + \sum n_V \]
\[ N_T = 1 + N_5 + \sum n_T, \]

we arrive at the “collective” constraint \( N_H - N_V + 29N_T = 273 \), which is a more familiar anomaly requirement. The numbers 4, 496 and 1 which appear in (6.28) describe the untwisted contributions, since there are 4 untwisted hypers and one untwisted tensor coming from the bulk supergravity and there are 496 untwisted vectors coming from the two \( E_8 \) factors. Also, each fivebrane contributes one hypermultiplet and one tensor multiplet.

In the anomaly polynomial, terms corresponding to mixed and gauge anomalies include traces over powers of the field strength \( \mathcal{F} \) in precisely the combinations shown in equation (6.25). We should represent all traces in terms a fundamental representation. There are identities which relate traces over any representation of a given simple gauge factor in terms of traces over other representations. The dimensions of the most useful representations for the simple gauge groups is given in table 2, and a useful tabulation of the trace relations is given in table 3 for the non-exceptional groups and in table 4 for the exceptional groups.

The anomaly polynomial contains a term proportional to \( \text{tr} \mathcal{F}^4 \), which must factorize as \( (\text{tr} \mathcal{F}^2)^2 \). This is done automatically for the exceptional groups and for \( SU(2) \) and \( SU(3) \), since these groups do not have an independent fourth-order Casimir operator. But for other simple groups this factorization requires a conspiracy involving the multiplicities of the hypermultiplet representations. If we assume that the proper factorization of the \( \text{tr} \mathcal{F}^4 \) term occurs, then in all cases the mixed and gauge anomalies will involve the following two factors

\[ \frac{1}{6} \left( \frac{1}{\mu} \text{Tr} \mathcal{F}^2 - \sum h_\alpha \text{tr} \mathcal{F}^2 \right) \equiv -X \text{tr} \mathcal{F}^2 \]
\[ \frac{2}{3} \left( \frac{1}{\mu} \text{Tr} \mathcal{F}^4 - \sum h_\alpha \text{tr} \mathcal{F}^4 \right) \equiv -Y \left( \text{tr} \mathcal{F} \right)^2, \]

(6.29)

where we have defined \( X \) and \( Y \) as the generic coefficients of the \( (\text{tr} \mathcal{F}^2)^2 \) term (which multiplies \( \text{tr} R^2 \) in the anomaly polynomial to form a mixed anomaly) and the \( \text{tr} \mathcal{F}^4 \) term (which constitutes the pure gauge anomaly). It is possible to determine the coefficients \( X \) and \( Y \) for each possible simple gauge factor by applying the trace relations given in
tables 3 and 4 to the expressions in (6.29). The numbers $X$ and $Y$ for generic choice of simple gauge factor and arbitrary hypermultiplet representation are given in table 5.

We now substitute (6.27) into the anomaly polynomial to remove the $\text{tr} R^4$ term. We also use (6.27) to replace the combination $n_V - n_H$ in terms of $n_T$ and $g_i$. It is then a matter of straightforward algebra to determine the final form of the anomaly polynomial. For convenience, we define a hatted polynomial by removing a common prefactor,

$$I_8(\text{total}) \equiv \frac{1}{(2\pi)^3} \frac{3}{4!} \frac{3}{2} \hat{I}_8(\text{total}). \quad (6.30)$$

Now if we compute the total anomaly by summing up the three inflow contributions (6.18) and the three quantum contributions (6.26), then remove the $\text{tr} R^4$ term by imposing (6.27), and finally rewrite all traces over twisted gauge factors using (6.29) we determine the complete anomaly to be given by the following expression,

$$\hat{I}_8(\text{total}) = \frac{1}{32} (1 + \eta + \frac{8}{3} g - 4n_T) (\text{tr} R^2)^2$$
$$+ \frac{1}{16} (5 - \eta + \frac{8}{3} g) \text{tr} R^2 \wedge \text{tr} F^2 - \frac{1}{24} (9 + 4g) (\text{tr} F^2)^2$$
$$- \frac{1}{16} \rho \text{tr} F^2 \wedge \text{tr} F^2 + \left( \frac{1}{32} \rho - X \right) \text{tr} R^2 \wedge \text{tr} F^2 + Y (\text{tr} F^2)^2, \quad (6.31)$$

where $F$ is the $E_8$ gauge field and $\mathcal{F}$ is the twisted gauge field. The expression (6.31) includes unspecified local magnetic charge $g$, an arbitrary number of local tensor fields $n_T$, an arbitrary “I-brane” contribution parameterized by $(\eta, \rho)$, and an arbitrary twisted spectrum parameterized by $(X,Y)$.

The anomaly (6.31) cannot vanish identically. This can be seen by the following simple observations. Cancelation of the $(\text{tr} F^2)^2$ term would require $g = -9/4$. Using this result, cancellation of the $\text{tr} R^2 \wedge \text{tr} F^2$ term would require $\eta = -1$. Cancellation of the $(\text{tr} R^2)^2$ term would then require that $n_T = -3/2$, which is not positive, and therefore not realizable. So we must resort to a local Green-Schwarz mechanism to cancel the anomaly. In the following subsection we attempt to implement this.

### 6.3 Factorizing the anomaly

Since the anomaly cannot vanish identically, it can only be removed via a Green-Schwarz mechanism realized locally (ie: on the particular six-plane $M_6$) through the local coupling of at least one tensor field. Since the fixed six-planes have $N = 1$ supersymmetry, any such tensor would have an anti self-dual field strength. Therefore, the mechanism necessitates that the anomaly polynomial (6.31) factorize into a sum of perfect squares [1], one term for

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[1] We thank to Stephan Theisen for discussions on this point.
each tensor field. The minimal case would involve one twisted tensor field. In that case, we would impose that (6.31) factorizes as

$$\hat{I}_s(\text{total}) = \frac{1}{32} r (\text{tr} R^2 - u \text{tr} F^2 - t \text{tr} F^2)^2 ,$$

(6.32)

where $r$, $u$ and $t$ are rational coefficients to be determined. We discuss the possibility of more than one tensor field below. The one case not covered by the parameterization (6.32) is where the $(\text{tr} R^2)^2$ term vanishes but other terms do not. This possibility is considered separately and found not to be relevant. The reason why we include $r$ as an overall coefficient, rather than inside the brackets multiplying the $\text{tr} R^2$ term, is because the analysis is more tractable this way. Equating equation (6.32) with (6.31) generates six relations encapsulating the factorization requirement,

$$r = 1 + \eta + \frac{8}{3} g - 4 n_T$$

$$r u = -5 + \eta - \frac{8}{3} g$$

$$r u^2 = -12 - \frac{16}{3} g$$

$$r t = 16 X - \frac{1}{2} \rho$$

$$r u t = -\rho$$

$$r t^2 = 32 Y .$$

(6.33)

Three of these (those in the left-hand column) relate to terms which include $\text{tr} F^2$ and, therefore, concern the $E_8$ anomaly. The other three (those in the right-hand column) relate to terms which include $\text{tr} F^2$ and, therefore, concern the twisted anomaly.

The goal is to find rational values for the magnetic charge $g$, the “electric” parameters $\eta$ and $\rho$, and the group-dependent parameters $X$ and $Y$ (which are defined in (6.29)) in such a way that (6.33) determines rational values for $r$, $u$ and $t$.

The parameters $X$ and $Y$ have a very restricted and group-specialized dependence on the multiplicities of the hypermultiplets. These relationships are exhibited in table 5. Furthermore, $X$ and $Y$ also depend on $\mu$, the parameter defined in (6.24), which tells us the dimensionality of the hyperplane on which the twisted states live. Therefore, even if we manage to find a solution to (6.33), this is not enough, since the values of $X$ and $Y$ must be realizable for some choice of simple gauge factor, hypermultiplet multiplicity and value 1 or 2 for the parameter $\mu$. The reason why we can specialize to simple groups is explained in the following paragraph. “Realizable”, in this instance, means that there is a choice of nonnegative integer multiplicity $h_\alpha$ for hypermultiplets transforming in some representation $R_\alpha$ for some choice of simple gauge factor $G$. This choice must describe precisely the given values of both $X$ and $Y$ using the relationships listed in table 5, for

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8The parameters $\eta$ and $\rho$ have an “electric” nature since they determine the electric couplings described by (6.12) and (6.13).
either of the cases $\mu = 1$ or $\mu = 2$. There is a systematic approach that allows one to sift sequentially through all possibilities.

Even if the ultimate solution involves a semisimple group, there are necessary constraints pertaining to each simple factor of that group which coincide with those which we are analyzing. Therefore, we do not lose generality by specializing to the case of simple gauge groups. Nevertheless, once we find a solution to the factorization problem for a given simple factor, we should explore the possibility that this group can be extended to include additional simple factors. Other constraints discussed below may require such an extension in generic cases.

If one can find a solution to (6.33) which includes “realizable” values for $X$ and $Y$, this is still not enough. There remains the extra constraint (6.27) related to the vanishing of the irreducible part of the anomaly. So, once we have determined a set of rational parameters which satisfy (6.33) and a gauge group $G$ and hypermultiplet multiplicity $h_{\alpha}$ which give rise to the given $X$ and $Y$, this will determine the total number of twisted vector multiplets $n_V$ and the total number of twisted hypermultiplets $n_H$. But these must satisfy (6.27). Finding a solution which meets all of these criteria is nontrivial.

Furthermore, there is yet another constraint. This is the global constraint (6.6) which tells us that the sum of all 32 magnetic charges plus the number of fivebranes must vanish. This requires that at least one of the six-planes has nonpositive magnetic charge. It is, therefore, essential that we find a solution with nonpositive $g$.

A “solution”:

It turns out that there is a unique solution to all of the above constraints, for the case of nonpositive $g$. We emphasize that the equations which we have developed are sensitive to each of a large number of coefficients involved in a lengthy multifaceted analysis. If any one of these coefficients would change slightly, there would be no solution at all. Given the complexity of the analysis, which involves a delicate array of effects, we find the existence of one and only one solution to all of these constraints to be significant.

Nevertheless there remains a paradox associated with the solution. Specifically, the solution requires $n_T = 1/2$, which is not an integer value. However, the mere existence of such a “formal” solution is remarkable for the reasons expressed above. Furthermore, there are significant insights which derive from this solution, and indications that the paradox itself may not be inscrutable. In fact, a similar paradox attended the discoveries outlined in section 4 in the context of the $T^5/Z_2$ orbifold. In that case, a “half-tensor” multiplet seemed to be required by anomaly cancellation before a missing inflow mechanism involving the magnetic charge $g_i$ was included. So, despite the described shortcoming, we
will explain the solution in more detail and speculate on possible resolutions to the above paradox in due course.

There is another notable aspect to the solution. This is that the necessary magnetic charge assignment for the six-plane $M^6_i$ in question turns out to be $g_i = -3/4$. As discussed earlier, there are reasons to expect such a quarter-integer value for $g_i$ in the context of this orbifold because the construction represents a singularly deformed limit of $K3 \times S^1 / \mathbb{Z}_2$. One can then interpret part of the local magnetic charge as being the a remnant of the Euler number of the smooth $K3$ manifold. In fact, this “Euler” contribution was earlier shown to give precisely the value $g_i = -3/4$.

The one valid solution with nonpositive $g$ to the factorization problem described by the six conditions (6.33) is given by the following rational assignments for the parameters of the problem,

\[
(n_T, g) = (1/2, -3/4) \\
(\eta, \rho) = (-5, 16) \\
(X, Y) = (-1/2, -1) \\
(r, u, t) = (-8, 1, 2).
\]

(6.34)

As described above, this solution is “validated” only if we can reproduce the indicated values for $X$ and $Y$ using the relationships in table 5 for some choice of multiplicities $h_\alpha$ for hypermultiplets transforming in some representation $R_\alpha$ for some choice of simple gauge factor $G$ for either of the cases $\mu = 1$ or $\mu = 2$. We must also do this in such a way that (6.27) is satisfied. When we substitute the indicated values $g = -3/4$ and $n_T = 1/2$ into (6.27), we find that this extra constraint becomes

\[
n_H - n_V = -14.
\]

(6.35)

So we must find a twisted gauge group which satisfies both $(X, Y) = (-1/2, -1)$ and also (6.33).

By carefully sifting through the possibilities in table 5, one finds a unique simple factor which solves this problem. This is to put twisted $SO(8)$ gauge matter on the seven-dimensional plane $M^7_i$ (so that $\mu = 2$) and to set the number of hypermultiplets to zero (so that $h = h_s = 0$ in table 5). In that case, we would have $n_H = 0$ while $n_V = \dim G / \mu = 14$, the division by $\mu = 2$ indicating that the $SO(8)$ gauge fields are seven-dimensional, as explained above.

It is now evident how the anomaly analysis has required seven-dimensional twisted matter despite the fact that there is no seven-dimensional anomaly as such. It is also
clear that this kind of mechanism generalizes to other orbifolds involving intersecting orbifold planes.

Given the values of \( r, u \) and \( t \) which we have found in (6.34), we can write down the form of the anomaly. Substituting these values into (6.32) we determine

\[
\hat{I}_8(\text{total}) = -\frac{1}{4} \left( \text{tr} R^2 - \text{tr} F^2 - 2 \text{tr} \mathcal{F}^2 \right)^2.
\]

(6.36)

This is the anomaly which should be canceled using a Green-Schwarz mechanism mediated by a local tensor field. If our solution had had \( n_T = 1 \), it would be clear how to realize this; the three-form field strength would satisfy the Bianchi identity \( dH \propto \text{tr} R^2 - \text{tr} F^2 - 2 \text{tr} \mathcal{F}^2 \), and the tensor dynamics would include a Chern-Simon’s interaction proportional to \( \int_{M_6^i} B \wedge (\text{tr} R^2 - \text{tr} F^2 - 2 \text{tr} \mathcal{F}^2) \). The self-duality of \( H \) would be maintained because the magnetic and electric couplings implied by these are the same. Due to the modified \( dH \) Bianchi identity, the tensor \( B \) would transform in just such a way that the resulting transformation of the Chern-Simon’s interaction would cancel the anomaly (6.36). However, since our “formal” solution has \( n_T = 1/2 \), it is unclear whether we actually have a tensor whose dynamics could include these modifications.

The global constraint (6.6) is satisfied for our “solution” by assuming that each of the 32 fixed points exhibits identical behavior. Thus, we would have a charge of \( g_i = -3/4 \) for each of the thirty-two six-planes \( M_6^i \), implying \( \sum_i g_i = -24 \). Equation (6.6) would then be balanced by including 24 independent fivebranes “wandering” in the bulk. We remark that in this case the magnetic charge associated with the fixed-planes is identical to that ascribable to the “pinched” curvature of the singularly-deformed \( K3 \) manifold, as explained early in this section.

Additional tensors:

In principle, we can involve more than one twisted tensor field. As explained above, this would imply a weaker factorization constraint than the one indicated by (6.33). With more tensor fields, we would generalize that constraint to impose that \( \hat{I}_8(\text{total})_i \) factorize as a sum of perfect squares, one for each tensor field. But the possibility of reasonable solutions for the case \( n_T > 1 \) is hampered by the independent constraint (6.27) necessary to remove the irreducible part of the local anomaly. For the case \( n_T = 1 \) this relation becomes \( n_H - n_V = 30g - 6 \) which implies twisted matter living in relatively small gauge groups for reasonable (ie: relatively small) values of the magnetic charge because, in that case, \( 30g - 6 \) is not a large number. For example, we already described one (formal) solution with \( n_T = 1/2 \) and seven-dimensional twisted gauge group \( \mathcal{G} = SO(8) \). With an arbitrary number of twisted tensors, the constraint (6.27) becomes \( n_H - n_V = 30g + 23 - 29n_T \), which quickly
becomes a large negative number as \( n_T \) increases. The problem is compounded because the global constraint (5.6) requires that we find at least one solution with nonpositive magnetic charge. These considerations indicate that smaller values of \( n_T \), such as \( n_T = 2 \), where we would have \( n_H - n_V = 30g - 35 \), are more likely to lead to reasonable solutions than larger values of \( n_T \). But even for the case \( n_T = 2 \), the problem of systematizing the factorization criterion analogous to (6.33) becomes comparably unwieldy since there are many more variables. We have been unable to find a solution to the factorization problem for the case \( n_T = 2 \) which also satisfies \( n_H - n_V = 30g - 35 \). It is also apparent that seeking solutions for \( n_T \geq 3 \) would be a computational morass unlikely to yield interesting solutions.

**Hidden instantons:**

One aspect which we have not emphasized in our analysis involves the possible scenarios described in the paragraph following equation (6.20) whereby the \( \alpha \) projection can be realized in nontrivial ways to break the \( E_8 \) gauge group to \( E_7 \times SU(2) \) or \( Spin(16) \), thereby describing the effect of “hidden instantons” on the fixed points. These possibilities can be analyzed in a manner very similar to that which we have presented. The essential difference is that the contribution \( I_8(E_8) \), given in (6.20) is replaced with an analogous contribution \( I_8(E_7 \times SU(2)) \), or \( I_8(Spin_{16}) \), which are straightforward to define and to compute. When we repeat the above analysis for these cases we do not find interesting solutions involving nonpositive \( g \) for the \( Spin(16) \) case. However, the \( E_7 \times SU(2) \) case is very intriguing, but involves a different set of puzzles which, at this time, are sufficiently muddy that we should avoid further expansion on the subject. This involves work-in-progress.

### 7 Conclusions

The precise characterization of anomalies in general situations involving \( M \)-theory orbifolds involves an interesting array of effects. On the one hand are the quantum anomaly and also the various inflow contributions enabled by modifications to the \( dG \) Bianchi identity and the standard Chern-Simon’s terms in eleven-dimensional supergravity. These two contributions alone are sufficient to understand anomaly cancellation in the simplest cases, such as the \( M \)-fivebrane and orbifolds which only break half of the bulk supersymmetry. In situations involving more supersymmetry breaking, things are more involved. When there are intersecting orbifold planes, an “I-brane” effect occurs which involves an interplay between electric sources of \( G \) localized on one hyperplane and magnetic sources
of $G$ localized on the intersecting plane. Finally, local (twisted) tensor fields are generally needed to supply a local Green-Schwarz mechanism to cancel against the quantum anomaly, the inflow anomaly and also the I-brane anomaly.

We have presented, in detail, a particular example corresponding to an $S^1/Z_2 \times T^4/Z_2$ orbifold which has seven- and ten-dimensional orbifold planes which intersect at additional six-dimensional planes. In this case, it is shown how local anomaly cancellation on the six-planes requires $SO(8)$ gauge matter propagating on each of the sixteen seven-planes. But there remains an unresolved paradox associated with this situation, which is that it requires $n_T = 1/2$. A related problem had been previously noted by other authors [14, 15] in the context of the heterotic string. The authors of [14] have independently indicated a need for sixteen $28$s of $SO(8)$. The analysis in this paper complements the results of that paper by offering an alternative $M$-theoretic and local explanation for these same factors, associating them with seven-dimensional submanifolds.

The $S^1/Z_2 \times T^4/Z_2$ orbifold represents a particular degeneration of $S^1/Z_2 \times K3$ corresponding to the singular $Z_2$ orbifold limit of the $K3$ factor. In [14] the authors also analyzed a separate case corresponding to the singular $Z_3$ orbifold limit of the $K3$ factor. In that case, they did not find the same peculiarities present in the $Z_2$ case. Since that $Z_2 \times Z_3$ construction also has intersecting orbifold planes of precisely the same dimensionality as the one featured in this paper, it would be most interesting to repeat our analysis in that context. This is also work-in-progress.

A possibility suggested by this paper is that that gravitational anomaly cancellation on ten-, six- and two-dimensional orbifold planes within complicated $T^7$ orbifolds, for example, involving four-dimensional fixed planes and/or intersections would require gauge groups and particle spectra which would have relevance to realistic models. This would bring gravitational anomalies into four-dimensional physics in a novel way.
Appendix A: Tables

In this appendix we compile some results and identities from group theory which are necessary to undertake the detailed analysis described in section 6.

| Group   | Tr   | tr   | tr_\ast | tr_S   |
|---------|------|------|----------|--------|
| SU(n)   | $\frac{1}{2}n^2 - 1$ | $n$  |          |        |
| SO(n)   | $\frac{1}{2}n(n - 1)$ | $n$  |          | $2^{(N-2)/2}$ |
| Sp(n)   | $n(2n + 1)$  | $2n$ | $n(2n - 1)$ |        |
| G_2     | 14    | 7    |          |        |
| F_4     | 52    | 26   |          |        |
| E_6     | 78    | 27   |          |        |
| E_7     | 133   | 56   |          |        |
| E_8     | 248   |      |          |        |

Table 2: Representation dimensions used in the definitions of the various trace operations. The trace Tr refers to the adjoint representation, tr refers to the fundamental representation, tr_\ast refers to the antisymmetric tensor representation and tr_S refers to the spinor representation.
SU(n)  \[ \text{Tr} F^2 = 2n \text{ tr} F^2 \]
\[ \text{Tr} F^4 = 2n \text{ tr} F^4 + 6 (\text{tr} F^2)^2 \]
\[ \text{tr} F^4 = \frac{1}{2} (\text{tr} F^2)^2 \text{ for } SU(2) \text{ or } SU(3) \]

SO(n)  \[ \text{Tr} F^2 = (n - 2) \text{ tr} F^2 \]
\[ \text{Tr} F^4 = (n - 8) \text{ tr} F^4 + 3 (\text{tr} F^2)^2 \]
\[ \text{Tr} F^6 = (n - 32) \text{ tr} F^6 + 15 \text{ tr} F^2 \text{ tr} F^4 \]
\[ \text{tr}_s F^2 = 2^{(N-8)/2} \text{ tr} F^2 \]
\[ \text{tr}_s F^4 = -2^{(N-10)/2} \text{ tr} F^4 + 3 \cdot 2^{(N-14)/2} (\text{tr} F^2)^2 \]

Sp(n)  \[ \text{Tr} F^2 = (2n + 2) \text{ tr} F^2 \]
\[ \text{Tr} F^4 = (2n + 8) \text{ tr} F^4 + 3 (\text{tr} F^2)^2 \]
\[ \text{tr}_s F^2 = (2n - 2) \text{ tr} F^2 \]
\[ \text{tr}_s F^4 = (2n - 8) \text{ tr} F^4 + 3 (\text{tr} F^2)^2 \]

Table 3: Trace relations for the non-exceptional classical groups.
\[
\begin{align*}
G_2 & \quad \text{Tr } F^2 = 4 \text{ tr } F^2 \\
& \quad \text{Tr } F^4 = \frac{5}{2} (\text{tr } F^2)^2 \\
& \quad \text{tr } F^4 = \frac{1}{4} (\text{tr } F^2)^2 \\
F_4 & \quad \text{Tr } F^2 = 3 \text{ tr } F^2 \\
& \quad \text{Tr } F^4 = \frac{5}{12} (\text{tr } F^2)^2 \\
& \quad \text{tr } F^4 = \frac{1}{12} (\text{tr } F^2)^2 \\
E_6 & \quad \text{Tr } F^2 = 4 \text{ tr } F^2 \\
& \quad \text{Tr } F^4 = \frac{1}{2} (\text{tr } F^2)^2 \\
& \quad \text{tr } F^4 = \frac{1}{12} (\text{tr } F^2)^2 \\
E_7 & \quad \text{Tr } F^2 = 3 \text{ tr } F^2 \\
& \quad \text{Tr } F^4 = \frac{1}{6} (\text{tr } F^2)^2 \\
& \quad \text{tr } F^4 = \frac{1}{24} (\text{tr } F^2)^2 \\
E_8 & \quad \text{Tr } F^2 \equiv 30 \text{ tr } F^2 \\
& \quad \text{Tr } F^4 = \frac{1}{100} (\text{Tr } F^2)^2 \\
& \quad \text{Tr } F^6 = \frac{1}{7200} (\text{Tr } F^2)^3
\end{align*}
\]

Table 4: Trace relations for the exceptional groups.
\begin{table}
|       | X       | Y       | factorization criteria |
|-------|---------|---------|------------------------|
| $SU(N)$ | 0       | $-4/\mu$ | $h = 2N/\mu$          |
| $SU(2)$ | $\frac{1}{6}(h - 4/\mu)$ | $\frac{1}{3}(h - 16/\mu)$ |                     |
| $SU(3)$ | $\frac{1}{6}(h - 6/\mu)$ | $\frac{1}{3}(h - 18/\mu)$ |                     |
| $SO(N)$ | $q - 1/\mu$ | $q - 2/\mu$ | $h = (N - 8)/\mu + 2q$ |
|        |         |         | $q \equiv 2^{(N-12)/2} h_S$ |
| $Sp(N)$ | $h_\ast + 1/\mu$ | $2h_\ast - 2/\mu$ | $h = 2N(1/\mu - h_\ast) + 8(1/\mu + h_\ast)$ |
| $G_2$  | $\frac{1}{6}(h - 4/\mu)$ | $\frac{1}{6}(h - 10/\mu)$ |                     |
| $F_4$  | $\frac{1}{6}(h - 3/\mu)$ | $\frac{1}{18}(h - 5/\mu)$ |                     |
| $E_6$  | $\frac{1}{6}(h - 4/\mu)$ | $\frac{1}{18}(h - 6/\mu)$ |                     |
| $E_7$  | $\frac{1}{6}(h - 3/\mu)$ | $\frac{1}{36}(h - 4/\mu)$ |                     |
| $E_8$  | $-5/\mu$ | $-6/\mu$ |                           |

Table 5: Values of the numbers $X$ and $Y$, which are defined in equation (6.29), for the cases of each individual simple gauge factor, as functions of the numbers of hypermultiplets transforming in various representations and as functions of the parameter $\mu$, defined in (6.24). The multiplicity $h$ refers to the fundamental representation whereas $h_\ast$ and $h_S$ refer to the antisymmetric tensor representation and spinor representation, respectively. The right-hand column lists necessary criteria for the factorization described by the second equation of (6.29) as well as the definition of the parameter $q$. 

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| $G$          | $n_H - n_V$                  |
|-------------|-----------------------------|
| $SU(N)$     | $(N^2 + 1)/\mu + s$         |
| $SU(2)$     | $2h - 3/\mu + s$            |
| $SU(3)$     | $3h - 8/\mu + s$            |
| $SO(N)$     | $\frac{1}{2}N(N - 15)/\mu + (2N + 32)q + s$ |
| $Sp(N)$     | $N(2N + 15)/\mu - h_s N(2N - 15) + s$ |
| $G_2$       | $7h + s - 14/\mu$           |
| $F_4$       | $26h + s - 52/\mu$          |
| $E_6$       | $27h + s - 78/\mu$          |
| $E_7$       | $56h + s - 133/\mu$         |
| $E_8$       | $-248/\mu$                 |

Table 6: Expressions for $n_H - n_V$ for the simple gauge groups expressed as functions of the representation multiplicities and as functions of the parameter $\mu$, defined in (6.24). The multiplicities $h$, $h_s$ and $q$ are defined in the caption for table 5, while $s$ is the number of gauge singlet hypermultiplets. Note that the factorization criteria listed in the right-hand column of table 5 have been used.
Appendix B: Anomalies in Ten Dimensions

The essential tools used to study orbifold anomalies are specific polynomials used to describe the anomalies themselves. For a given theory, the relevant polynomial can be readily assembled given the zero-mass spectrum using results from index theory. But since each chiral fermion or self-dual tensor contributes to an anomaly in a linearly independent way, it is practical to have “ready-made” building-block polynomials associated with each type of field. In this way, one can readily determine the anomaly polynomial in a given situation and use this to explore the consistency of the theory and as a guide to additional structure. In this paper, an emphasis is put on the nuts-and-bolts aspects of assembling these polynomials and using them to ones advantage. For this reason, and also for the reason of making this paper reasonably self-contained, we include an encyclopedic review of these polynomials in this and the following appendix. The exact relationship to index theorems is relatively unimportant for our purposes. So these have been de-emphasized.

The presentation in this appendix paraphrases the more comprehensive presentation found in [16].

Ten-dimensional field theories involve three types of fields which contribute to anomalies at one loop. These are chiral spin $3/2$ fermions, chiral spin $1/2$ fermions, and self-dual (or anti self-dual) five-forms. The total anomaly can be deduced via descent equations from a formal twelve-form. Following are master formulae for deducing the twelve form from a given theory. The first of these gives the contribution to purely gravitational anomalies due to a chiral spin $3/2$, chiral spin $1/2$, and self-dual five-form field, respectively,

\[
I^{\text{(3/2)}}_{\text{GRAV}}(R) = \frac{1}{(2\pi)^5 6!} \left( \frac{55}{56} \tr R^6 - \frac{75}{128} \tr R^4 \tr R^2 + \frac{35}{512} (\tr R^2)^3 \right)
\]

\[
I^{\text{(1/2)}}_{\text{GRAV}}(R) = \frac{1}{(2\pi)^5 6!} \left( -\frac{1}{504} \tr R^6 - \frac{1}{384} \tr R^4 \tr R^2 - \frac{5}{4608} (\tr R^2)^3, \right)
\]

\[
I^{\text{(5-form)}}_{\text{GRAV}}(R) = \frac{1}{(2\pi)^5 6!} \left( -\frac{496}{504} \tr R^6 + \frac{7}{12} \tr R^4 \tr R^2 - \frac{5}{72} (\tr R^2)^3 \right). \quad (B.1)
\]

In these expressions, the Riemann tensor is regarded as an $SO(9,1)$-valued two-form, $(R_{\mu \nu})_a^b$. The trace is over the $SO(9,1)$ indices $a, b$, and the coordinate indices are suppressed. Wedge products are assumed. We note that these formulae are additive for each field of a given type. For instance, relevant theories contain a number of chiral spin $1/2$ fields living in vector multiplets. The anomaly due to chiral gauginos would be the second equation of (B.1) times the total number of these gauginos.

Next are the master formulae for mixed and pure gauge anomalies, which are due only
to chiral spin 1/2 fermions,

\[ I^{(1/2)}_{\text{MIXED}}(R, F) = \frac{1}{(2\pi)^5 6!} \left( \frac{1}{16} \text{tr} R^4 \text{Tr} F^2 + \frac{5}{64} (\text{tr} R^2)^2 \text{Tr} F^2 - \frac{5}{8} \text{tr} R^2 \text{trace} F^4 \right) \]
\[ I^{(1/2)}_{\text{GAUGE}}(F) = \frac{1}{(2\pi)^5 6!} \text{Tr} F^6 \]  

(B.2)

In these expressions, the Yang-Mills field strengths are two-forms which take values in the adjoint representation of the gauge group, and \( \text{Tr} \) denotes an adjoint trace.

The above master formulae for \( I^{(1/2)} \) and \( I^{(3/2)} \) are for chiral \((\Gamma_{11} \psi = \psi)\) Weyl spinors. If the fermions in question are Majorana-Weyl, which is possible in ten-dimensions, and have therefore half of the degrees of freedom of a Weyl spinor, then the formula should be multiplied by 1/2. For antichiral spinors \((\Gamma_{11} \psi = -\psi)\) the formula should be multiplied by -1.

### B.1 IIB Supergravity

The ten-dimensional type \( \mathcal{IIB} \) supergravity theory has a single self-dual five-form field strength (with analogous four-form potential), a pair of chiral spin-3/2 Majorana-Weyl gravitinos, and a pair of antichiral spin-1/2 fermions. Thus, the total anomaly is described by

\[ I_{12} = I^{(3/2)}_{\text{GRAV}}(R) - I^{(1/2)}_{\text{GRAV}}(R) + I^{(5-\text{form})}_{\text{GRAV}}(R). \]  

(B.3)

The coefficients of the \( I^{(1/2)} \) and \( I^{(3/2)} \) terms each include a factor of two, since there are two each of the relevant field types, and also a factor of one-half since the relevant fields are Majorana-Weyl spinors and therefore have half the degree of freedom of a Weyl spinor, as described above. Thus, the overall coefficients for these terms have absolute value one. Adding up the various contributions supplied by equation (B.1), we find the result \( I_{12} = 0 \! \! \!. \) Thus, the IIB supergravity theory is anomaly-free.

### B.2 N=1 Supergravity Coupled to Yang-Mills Matter

The fermionic fields of the D=10 N=1 supergravity multiplet comprise a chiral Majorana-Weyl spin-3/2 gravitino and an antichiral Majorana-Weyl spin-1/2 dilatino. There are no (anti) self-dual 5-forms. This multiplet couples to Yang-Mills supermultiplets which contains chiral Majorana-Weyl spin-1/2 gauginos living in the adjoint representation of some gauge group \( \mathcal{G} \). Thus, the total anomaly is described by

\[ I_{12} = \frac{1}{2} \left( I^{(3/2)}_{\text{GRAV}}(R) - I^{(1/2)}_{\text{GRAV}}(R) \right) + \frac{1}{2} \left( n I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{\text{MIXED}}(R, F) + I^{(1/2)}_{\text{GAUGE}}(F) \right). \]  

(B.4)
where \( n = \text{dim}(\mathcal{G}) \). Adding up the various contributions, we then arrive at the total anomaly polynomial for a generic super Yang-Mills theory coupled to \( D = 10 \) \( N = 1 \) supergravity,

\[
I_{12} = \frac{1}{2(2\pi)^5} \frac{496 - n}{504} \left( \frac{224 + n}{384} \text{tr} R^6 - \frac{5}{4608} (64 - n) (\text{tr} R^2)^3 \right)
+ \frac{1}{16} \text{tr} R^4 \text{Tr} F^2 + \frac{5}{64} (\text{tr} R^2)^2 \text{Tr} F^2 - \frac{5}{8} \text{tr} R^2 \text{Tr} F^4 + \text{Tr} F^6 \right) \quad (B.5)
\]

To cancel this anomaly via a Green-Schwarz mechanism, it is necessary that the twelve-form factorize into the product of a four-form and an eight-form. For a judicious choice of gauge group, it is possible that \( \text{Tr} F^6 \) factorizes into a linear combination of \( \text{Tr} F^2 \text{Tr} F^4 \) and \( (\text{Tr} F^2)^3 \). But \( SO(9,1) \) does not enable such a factorization of \( \text{tr} R^6 \); this piece must vanish identically. Therefore \( n = 496 \). In this case, (B.3) becomes

\[
I_{12} = \frac{1}{2(2\pi)^5} \frac{15}{8} \text{tr} R^4 \text{tr} R^2 - \frac{15}{32} (\text{tr} R^2)^3 + \frac{1}{16} \text{tr} R^4 \text{Tr} F^2
+ \frac{5}{64} (\text{tr} R^2)^2 \text{Tr} F^2 - \frac{5}{8} \text{tr} R^2 \text{Tr} F^4 + \text{Tr} F^6 \right) . \quad (B.6)
\]

There is only one possibility to factorize this result into the product of a two-form and an eight form, which requires the following property to be satisfied by \( \mathcal{G} \),

\[
\text{Tr} F^6 = \frac{1}{48} \text{Tr} F^4 \text{Tr} F^2 - \frac{1}{14400} (\text{Tr} F^2)^3 . \quad (B.7)
\]

There are exactly two 496-dimensional nonabelian Lie groups with this property, \( SO(32) \) and \( E_8 \times E_8 \). Given the property (B.7), the anomaly polynomial (B.6) may be expressed as

\[
I_{12} = -\frac{15}{2(2\pi)^5} \frac{1}{6!} (\text{tr} R^2 - \frac{1}{30} \text{Tr} F^2) \wedge X_{(8)} . \quad (B.8)
\]

where \( X_{(8)} \) is an eight-form given by the following expression,

\[
X_{(8)} = \frac{1}{8} \text{tr} R^4 + \frac{3}{32} (\text{tr} R^2)^2 - \frac{1}{240} \text{tr} R^2 \text{Tr} F^2 + \frac{3}{24} \text{Tr} F^4 - \frac{1}{7200} (\text{Tr} F^2)^2 . \quad (B.9)
\]

For \( SO(32) \) there is an identity \( \text{Tr} = 30 \text{tr} \), and for \( E_8 \) a similar identity defines the operation \( \text{tr} \). Therefore, in both cases we can rewrite (B.9) as

\[
X_{(8)} = \frac{1}{8} \text{tr} R^4 + \frac{3}{32} (\text{tr} R^2)^2 - \frac{1}{8} \text{tr} R^2 \text{Tr} F^2 + \frac{5}{4} \text{tr} F^4 - \frac{1}{8} (\text{tr} F^2)^2 . \quad (B.10)
\]

**Appendix C: Anomalies in Six Dimensions**

Six-dimensional field theories also involve three types of fields which contribute to anomalies at one loop. These are chiral spin 3/2 fermions, chiral spin 1/2 fermions, and self-dual...
(or anti self-dual) three-forms. The total anomaly can be deduced via descent equations from a formal eight-form. Following are master formulae for deducing the eight form from a given theory. The first of these gives the contribution to purely gravitational anomalies due to a chiral spin $3/2$, chiral spin $1/2$, and a self-dual three-form field, respectively

$$I_{\text{GRAV}}^{(3/2)}(R) = \frac{1}{(2\pi)^3 4!} \left( -\frac{49}{48} \text{tr} R^4 + \frac{43}{192} (\text{tr} R^2)^2, \right)$$

$$I_{\text{GRAV}}^{(1/2)}(R) = \frac{1}{(2\pi)^3 4!} \left( -\frac{1}{240} \text{tr} R^4 - \frac{1}{192} (\text{tr} R^2)^2 \right)$$

$$I_{\text{GRAV}}^{(3-\text{form})}(R) = \frac{1}{(2\pi)^3 4!} \left( -\frac{7}{60} \text{tr} R^4 + \frac{1}{24} (\text{tr} R^2)^2 \right).$$

In these expressions the Riemann tensor is regarded as an $SO(5,1)$-valued two-form, $(R_{\mu\nu})^a_b$. The trace is over the $SO(5,1)$ indices $a, b$, and the coordinate indices are suppressed. Wedge products are assumed. We note that these formulae are additive for each field of a given type. For instance, relevant theories contain a number of chiral spin $1/2$ fields living in vector multiplets. The contribution to the total anomaly due to chiral gauginos would be the second equation of (C.1) times the total number of these gauginos.

Next are the master formulae for mixed and pure gauge anomalies, which are due only to chiral spin $1/2$ fermions,

$$I_{\text{MIXED}}^{(1/2)}(R, F) = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{4} \text{tr} R^2 \text{trace} F^2 \right)$$

$$I_{\text{GAUGE}}^{(1/2)}(F) = \frac{1}{(2\pi)^3 4!} \left( -\text{trace} F^4 \right)$$

In these expressions, the Yang-Mills field strengths are two-forms which take values according to whichever group representation the gauge fields transform in.

### C.1 $D = 6$, $N = 2$

In six dimensions, there are two distinct $N = 2$ supergravity multiplets, one chiral and the other non-chiral. The chiral supergravity multiplet is denoted $N = 2b$ and comprises a sechsbein, five self-dual two-forms $B_{IJ}^{(+)}$ (ie: the three-form field strengths satisfy $H = *H$), and two chiral spin $3/2$ gravitinos. This multiplet can couple only to $N = 2$ tensor
multiplets, which each comprise five real scalars, a single anti self-dual two-form, and two antichiral spin 1/2 gauginos.

For the $N = 2b$ supergravity multiplet coupled to $n N = 2$ tensor multiplets, the quantum anomaly is characterised by the following eight-form,

$$I_8 = \left( 2iG^{(3/2)}(R) + 5iG^{(3-\text{form})}(R) \right) - n \left( 2iG^{(1/2)}(R) + iG^{(3-\text{form})}(R) \right).$$  \hfill (C.3)

There are no mixed or pure gauge contributions since there are no spin-1 gauge fields in the theory. The coefficients of the various contributions in (C.3) follow from the field content of the multiplets specified in the preceding paragraph. The first two terms in (C.3) are the contribution from the $N = 2b$ supergravity multiplet while the second two terms are the contributions from the tensor multiplets. Using the formulae in (C.1) we determine that

$$I_8(R) = \frac{1}{(2\pi)^3 4!} \frac{n - 21}{8} \left( \text{tr}R^4 - \frac{1}{4}(\text{tr}R^2)^2 \right).$$  \hfill (C.4)

Since tr$R^4$ cannot factorize, the first term in this expression must vanish if the theory is to be anomaly-free. This then requires that $n = 21$.

C.2 $D = 6$, $N = 1$

In six dimensions there is only one supergravity multiplet with $N = 1$ supersymmetry. This multiplet is chiral and comprises a sechsbein, a single self-dual two-form and a chiral spin-3/2 gravitino. There are three distinct matter multiplets to which this multiplet can couple. These are the vector multiplet which includes a spin-1 gauge field and a chiral spin-1/2 gaugino, the hypermultiplet which includes four real scalars and an antichiral spin-1/2 fermion, and the tensor multiplet which includes a single real scalar, a single anti self-dual two-form and an antichiral spin-1/2 fermion. Each of these multiplets contributes to a gravitational anomaly. To evaluate a potential gauge anomaly, we have to specify the group representation relevent to each multiplet. We restrict to the case where vector multiplets transform in the adjoint and tensor multiplets are gauge singlets. The representation of hypermultiplets can be chosen freely.

To begin, we restrict to the case where $G$ is simple. We will generalize this to the case where $G$ is semi-simple below. Thus, given a gauge group, the only freedom we allow is in the choice of representation for the hyper multiplets, some of which can be gauge singlets, and the number of gauge-singlet tensor multiplets. For the case of perturbative string effective theories, $n_T = 1$. Including nonperturbative effects can change this, however. Similarly, $M$-Theory also gives rise to $n_T \neq 1$ effective theories.
If we include $n_V$ vector multiplets, $n_H = \sum \alpha n_\alpha$ hyper multiplets in the representation $R_\alpha$, and $n_T$ tensor multiplets, then the total anomaly is described by the following eight-form,

$$I_8 = I_{\text{GRAV}}^{(3/2)}(R) + (n_V - n_H - n_T) I_{\text{GRAV}}^{(1/2)}(R) + (1 - n_T) I_{\text{GRAV}}^{(3\text{-form})}(R)$$

$$+ \left( I_{\text{MIXED}}^{(1/2)}(R, F) + I_{\text{GAUGE}}^{(1/2)}(F) \right)_{\text{ADJ}}$$

$$- \sum \alpha n_\alpha \left( I_{\text{MIXED}}^{(1/2)}(R, F) + I_{\text{GAUGE}}^{(1/2)}(F) \right)_{R_\alpha}$$

which follows directly from the discussion above, given the field content of the various multiplets. Note that the subscripts ADJ and $R_\alpha$ refer to the representations being traced over in the respective anomaly polynomial, and that $n_\alpha$ is the number of hypermultiplets in the representation $R_\alpha$. Using the formulae in (C.4), we then compute

$$I_8 = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{240} (n_H - n_V + 29n_T - 273) \text{tr} R^4 + \frac{1}{192} (n_H - n_V - 7n_T + 51) (\text{tr} R^2)^2 + \frac{1}{4} \text{tr} R^2 \wedge (\text{Tr} F^2 - \sum \alpha n_\alpha \text{tr} F^2_\alpha) - (\text{Tr} F^4 - \sum \alpha n_\alpha \text{tr} F^4_\alpha) \right).$$

(A more precise description of the traces over the gauge group representations is given below. We require that the anomaly (C.6) factorize so that the anomaly can be canceled locally by a Green-Schwarz mechanism. This requires that the coefficient of the first term in (C.6) vanishes, as this term is irreducible (i.e., it is impossible to factorize $\text{Tr} R^4$). We thus determine the following requirement

$$n_H - n_V + 29n_T = 273.$$  

For the case of perturbative heterotic string compactifications, one finds generically that $n_T = 1$, since there is only one tensor in the relevant effective theory. In that case, equation (C.7) reduces to $n_H - n_V = 244$, a commonly cited string requirement. Note that in $M$-theory we expect more than a single tensor field since the eleven-dimensional three-form $C$ can provide us with several two-forms upon dimensional reduction. In addition, fivebranes, which are important ingredients in $M$-theory, provide additional two-forms since their dynamics involve six-dimensional tensor multiplets.
We impose (C.7). Thus we can reexpress the anomaly (C.6) as follows,

\[ I_8 = \frac{1}{(2\pi)^3} \frac{3}{4!} \left( \frac{9 - n_T}{8} (\text{tr} R^2)^2 + \frac{1}{6} \text{tr} R^2 \wedge (\text{Tr} F^2 - \sum_\alpha n_\alpha \text{tr} F_\alpha^2) \right. \]

\[ \left. - \frac{2}{3} (\text{Tr} F^4 - \sum_\alpha n_\alpha \text{tr} F_\alpha^4) \right). \]  

(C.8)

This expression needs some care to be evaluated properly, especially if semisimple groups are allowed. If the gauge group involves \(N\) simple factors, \(G_1 \times G_2 \times \cdots G_N\) and if the hypermultiplets trasform as \((R_1, R_2, \ldots, R_N)\) then it turns out that \(\text{tr} F^2 = \sum_\alpha n_\alpha \text{tr} F_\alpha^2\) and \(\text{tr} F^4 = \sum_\alpha \text{tr} F_\alpha^4 + 6 \sum_{\alpha < \beta} n_{\alpha\beta} \text{tr} F_\alpha^2 \wedge \text{tr} F_\beta^2\) where \(n_\alpha\) is the number of multiplets transforming as \(R_\alpha\), and \(n_{\alpha\beta}\) are the number of multiplets transforming as \((R_\alpha, R_\beta)\) under the \(G_\alpha \times G_\beta\) subgroup. For example, in the case of two gauge factors \(G_1 \times G_2\), we would find \(n_1 = \dim R_2, n_2 = \dim R_1\) and \(n_{12} = 1\). For vector multiplets transforming in the adjoint, we have the relation \(\text{Tr} F^n = \sum_\alpha \text{Tr} F_\alpha^n\).

Using the relationships discussed above, we generalize (C.8) to the case of semisimple gauge group \(G_1 \times G_2 \times \cdots G_N\) with the representation structure described above, and find the following anomaly polynomial,

\[ I_8 = \frac{1}{(2\pi)^3} \frac{3}{4!} \left( \frac{9 - n_T}{8} (\text{tr} R^2)^2 + \frac{1}{6} \text{tr} R^2 \wedge \sum_\alpha X_\alpha^{(2)} - \frac{2}{3} \sum_\alpha X_\alpha^{(4)} + 4 \sum_{\alpha < \beta} Y_{\alpha\beta} \right). \]  

(C.9)

where the following abbreviations have been used,

\[ X_\alpha^{(2)} = \text{Tr} F_\alpha^2 - n_\alpha \text{tr} F_\alpha^2 \]

\[ X_\alpha^{(4)} = \text{Tr} F_\alpha^4 - n_\alpha \text{tr} F_\alpha^4 \]

\[ Y_{\alpha\beta} = n_{\alpha\beta} \text{tr} F_\alpha^2 \wedge \text{tr} F_\beta^2. \]  

(C.10)

The form of the anomaly polynomial (C.9) was presented in [17], with the same conventions used here, but for the special case \(n_T = 1\). Note that the terms \(n_\alpha \text{tr} F_\alpha^2, n_\alpha \text{tr} F_\alpha^4\) have an implicit sum over the different representations which might be included, and that \(n_\alpha\) therefore represents a set of multiplicities, one for each such representation. A similar statement applies to the definition of \(Y_{\alpha\beta}\).

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