MultiplesolutionsforsuperlinearKlein–Gordon–Maxwell
equations†

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Abstract
In this paper, we consider the following Klein–Gordon–Maxwell equations
\[
\begin{align*}
-\Delta u + V(x)u - (2\omega + \phi)\phi u &= f(x, u) + h(x) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi + \phi u^2 &= -\omega u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]
where \( \omega > 0 \) is a constant, \( u, \phi : \mathbb{R}^3 \to \mathbb{R} \), \( V : \mathbb{R}^3 \to \mathbb{R} \) is a potential function. By assuming the coercive condition on \( V \) and some new superlinear conditions on \( f \), we obtain two nontrivial solutions when \( h \) is nonzero and infinitely many solutions when \( f \) is odd in \( u \) and \( h \equiv 0 \) for above equations.

KEYWORDS
Klein–Gordon–Maxwell equations, multiple solutions, superlinear conditions, variational methods

1 INTRODUCTION AND MAIN RESULTS

In this paper, we considered the following Klein–Gordon–Maxwell equations
\[
\begin{align*}
-\Delta u + V(x)u - (2\omega + \phi)\phi u &= f(x, u) + h(x) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi + \phi u^2 &= -\omega u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]
where \( \omega > 0 \) is a constant, \( u, \phi, V : \mathbb{R}^3 \to \mathbb{R} \). This type of equation has very interesting physical background which is a model to describe the nonlinear Klein–Gordon field interacting with the electromagnetic field. Along with the development of variational methods, many mathematicians used these methods to investigate the existence and multiplicity of...
solutions for differential equations (see [1, 2, 5–24, 26, 28]). In 2001, V. Benci and D. Fortunato [5] considered the following systems

$$\begin{cases}
-\Delta u + (m^2 - (\omega + \phi)^2)u = |u|^{q-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi + \phi u^2 = -\omega u^2 & \text{in } \mathbb{R}^3.
\end{cases} \quad (1.2)$$

By using the variational methods, they obtained infinitely many solitary wave solutions when $|m| > |\omega|$ and $4 < q < 6$. After this first work, many mathematicians have treated this problem with variant cases with $m, \omega$ and $q$. In 2004, D’Aprile and Mugnai [13] dealt with the case $q \in (2, 4]$ with $\sqrt{\left(\frac{q}{2} - 1\right)m > \omega > 0}$. Some existence and nonexistence results are obtained for the problem (1.1) with variant conditions on $m, \omega$ and $q$ (see [1, 2, 13, 14, 24] for more details).

Although we lose the compactness for the problem (1.1) since the problem lies in a unbounded domain, we can also consider this problem in a radially symmetric space $H^1_{\text{rad}}(\mathbb{R}^3) := \{ u \in H^1(\mathbb{R}^3) : u = u(r), r = |x| \}$. It is known that $H^1_{\text{rad}}(\mathbb{R}^3)$ is compactly embedded into $L^s(\mathbb{R}^3)$ for $2 < s < 2^*$. If $V(\cdot)$ is not radial symmetric, there are still some other ways to retrieve the compactness. One classical way is to assume $V(\cdot)$ to be coercive. In this paper, we mainly consider the coercive case. This case has also been studied in many papers (see [4, 10, 12, 16–19]).

For coercive potentials, we will use the following condition.

(V1) Suppose $V \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{\mathbb{R}^3} V(x) = V_0 > 0$ and that there is a constant $r > 0$ such that

$$\lim_{|y| \to +\infty} \text{meas}\left(\{x \in \mathbb{R}^3 : |x - y| \leq r, V(x) \leq M\}\right) = 0, \quad \text{for all } M > 0,$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure. The condition (V1) was introduced by T. Bartsch et al. [4, Lem. 3.1.] to guarantee the compactness of the embedding. We will use this condition to prove our theorem.

As we know, the growth of the nonlinear terms is important in showing the geometric structure of the corresponding functionals (which is defined in (2.2)) and the boundedness of the Plais–Smale ($PS$) sequence. In [11], the author assumed the following condition.

(AR) there exists a constant $\theta > 4$ such that

$$f(x, t)t \geq \theta F(x, t) \quad \text{for } (x, t) \in \mathbb{R}^3 \times \mathbb{R},$$

where $F(x, t) = \int_0^t f(x, s) \, ds$. This condition is used to show the geometric structure of the corresponding functionals and the boundedness of the ($PS$) sequences. Hence (AR) has been widely used in many papers to obtain the existence and multiplicity of elliptic problem with variational methods. This condition implies that the growth exponent of the nonlinear terms is more than 4 at infinity. In order to deal with other nonlinearities with 4-superlinear growth, many authors (see [16, 17]) considered the following growth conditions.

(SL1) $f(x, t)/t^3 \to \infty$ as $|t| \to +\infty$ uniformly in $x$;

(SL2) $\frac{1}{4}f(x, t)t - F(x, t) \geq -D t^2$ for $D > 0$ and $|t|$ large enough uniformly in $x$.

(SL1) and (SL2) has been directly or indirectly used to show the existence and multiplicity of solutions for the problem (1.1). However, above conditions on $f$ eliminate many nonlinearities such as $F(x, t) = |t|^{5/2}$.

In order to deal with the nonlinearities with growth exponent between 2 and 4, Chen and Song [10] obtained two solutions for the problem (1.1) under (AR) condition just requiring $\theta > 2$ with coercive potentials when $h(x) \neq 0$. In 2018, Chen and Tang [12] introduced the following superlinear condition which is weaker than (AR) and obtained infinitely many solutions for the problem (1.1).

(WAR) there exist constants $\theta > 2$ and $K > 0$ such that

$$f(x, t)t - \theta F(x, t) + K t^2 \geq 0 \quad \text{for } (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$ 

There are still many functions cannot be included in above conditions. In this paper, we introduce some new superlinear conditions and an example is given to show the difference from previous conditions. Now, we state our main results.

**Theorem 1.1.** Suppose that $V$ satisfies (V1) and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfies

(F1) $f(x, t) = o(|t|)$ as $t \to 0$ for all $x \in \mathbb{R}^3$;
(F2) $F(x, 0) = 0$ for all $x \in \mathbb{R}^3$ and there exist $\tau \in (2, 6)$ and $d_1 > 0$ such that

$$|f(x, t)| \leq d_1 (|t| + |t|^\tau) \quad \text{for all } (x, t) \in \mathbb{R}^3 \times \mathbb{R};$$

(F3) let $\tilde{F}(x, t) = f(x, t) - 2F(x, t)$, then there exist $d_2 > 0$ and $r_0 > 0$ such that

$$|\tilde{F}(x, t)| \leq d_2 |t|^\tau \quad \text{for all } |t| \geq r_0 \text{ and } x \in \mathbb{R}^3;$$

(F4) $F(x, t)/t^2 \to +\infty$ as $|t| \to \infty$ uniformly in $x$;

(F5) $f(x, t) t / 2 \to +\infty$ as $|t| \to +\infty$ uniformly in $x$.

Then there is a constant $m > 0$ such that, for any $h \not\equiv 0$ satisfying $\|h\|^2 < m$, the problem (1.1) possesses at least two nontrivial solutions.

**Theorem 1.2.** Suppose that (V1), (F2)–(F5) hold, $h(x) \equiv 0$ and $f(x, -t) = -f(x, t)$, for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$. Then the problem (1.1) possesses infinitely many solutions.

**Remark 1.3.** In 2018, Shi and Chen [19] obtained two solutions for the problem (1.1) with nonzero perturbation by using the combination of a cut-off functional and a Pohozaev type identity. Although $f(x, t)$ was required to satisfy some very weak growth conditions in their paper, they needed some smooth conditions on the gradients of $V(x)$ and $h(x)$, which are not needed in our theorems.

**Remark 1.4.** Setting $4 > p > 2$, $0 < \varepsilon < p - 2$, consider

$$F(x, t) = |t|^p + a(p - 2)|t|^{p - \varepsilon} \sin^2(|t|^\varepsilon / \varepsilon).$$

(1.3)

For any $\theta > 2$ and $K > 0$, let max $\left\{ 0, \frac{\theta - \varepsilon}{p - 2} \right\} < a < 1$ and $t_n = \left( \varepsilon \left( n\pi + \frac{3\pi}{4} \right) \right)^{1/\varepsilon}$, then

$$f(x, t_n) t_n - \theta F(x, t_n) - K t_n^2$$

$$= (p - \theta)|t_n|^p + a(p - 2)(p - \theta - \varepsilon)|t_n|^{p - \varepsilon} \sin^2(|t_n|^\varepsilon / \varepsilon) + a(p - 2)|t_n|^p \sin(2|t_n|^\varepsilon / \varepsilon) - K t_n^2$$

$$= |t_n|^p \left[ (p - \theta) - a(p - 2) + \frac{a(p - 2)(p - \theta - \varepsilon)}{2|t_n|^\varepsilon} \right] - K t_n^2$$

$$\to -\infty \quad \text{as } n \to \infty.$$ 

Hence (1.3) does not satisfy (WAR) or the following condition

(FSL) $F(x, t)/t^4 \to +\infty$ as $|t| \to +\infty$ uniformly in $x$.

However, it is easy to see that, for all $x \in \mathbb{R}^3$,

$$f(x, t) t - 2F(x, t) = (p - 2)|t|^p \left[ (1 + a \sin(2|t|^\varepsilon / \varepsilon)) + \frac{a(p - 2) \sin^2(|t|^\varepsilon / \varepsilon)}{|t|^\varepsilon} \right]$$

$$\geq \frac{1}{2}(1 - a)(p - 2)|t|^p$$

for $|t|$ large enough. Then we can check that (1.3) satisfies conditions (F1)–(F5).

### 2 PRELIMINARIES

Let

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx < \infty \right\}.$$
Then \( E \) is a Hilbert space with the inner product
\[
(u, v)_E = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) \, dx
\]
and the norm \( \| u \|_E = (u, u)^{1/2}_E \). Define the function space
\[
D^{1,2}(\mathbb{R}^3) := \{ u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \}
\]
with the norm
\[
\| u \|_{D^{1,2}} := \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{1/2}.
\]
Under (V1), the embedding \( E \hookrightarrow L^s(\mathbb{R}^3) \) are compact for any \( s \in [2, 6) \) (see [4]). Hence for each \( s \in [2, 6) \), there exists a constant \( C_s > 0 \) such that
\[
\| u \|_{L^s} \leq C_s \| u \|_E, \quad \text{for all } u \in E.
\] (2.1)
Obviously, the problem (1.1) has a variational structure. Consider \( J : E \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R} \) defined by
\[
J(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla \phi|^2 + [V(x) - 2\omega + \phi^2]u^2) \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx - \int_{\mathbb{R}^3} h(x)u \, dx.
\]
Evidently, \( J \) belongs to \( C^1(E \times D^{1,2}(\mathbb{R}^3), \mathbb{R}) \). Thus, the pair \((u, \phi)\) is a weak solution of problem (1.1) if and only if it is a critical point of \( J \) in \( E \times D^{1,2}(\mathbb{R}^3) \). We can also see that \( J \) is strong indefinite. To reduce this functional, we need the following lemmas.

**Lemma 2.1.** (See [13].) For every \( u \in H^1(\mathbb{R}^3) \) there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) which solves the second equation of the problem (1.1). Furthermore,

(i) \(-\omega \leq \phi_u \leq 0 \) in \( \mathbb{R}^3 \);

(ii) if \( u \) is radially symmetric, \( \phi_u \) is radial too.

We can consider the functional \( I : E \to \mathbb{R} \) defined by \( I(u) = J(u, \phi_u) \). Therefore
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega \phi_u u^2) \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx - \int_{\mathbb{R}^3} h(x)u \, dx.
\] (2.2)
and we have, for any \( u, v \in E \),
\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv - f(x, u)v - h(x)v) \, dx.
\] (2.3)

**Lemma 2.2.** The following statements are equivalent:

(1) \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) is a critical point of \( J \) (i.e. \((u, \phi)\) is a solution of system (1.1));

(2) \( u \) is a critical point of \( I \) and \( \phi = \phi_u \).

By Lemmas (2.1) and (2.2), we only need to look for the critical points of \( I \) to show the existence and multiplicity of critical points for \( J \). Some details can be found in [18]. Subsequently, we prove our theorems with mountain pass theorem and a abstract critical point theorem introduced by T. Bartsch.
3 \ | \ PROOF OF THEOREM 1.1

**Lemma 3.1.** Suppose that (V1), (F1) and (F2) hold. Let \( h \in L^2(\mathbb{R}^3) \), then there exist some constants \( \rho, \alpha, m > 0 \) such that \( I(u)\|u\|_{L^2} = \rho \geq \alpha \) for all \( h \) satisfying \( \|h\|_{L^2} < m \).

**Proof.** The proof is similar to Lemma 3.1 in [10].

**Lemma 3.2.** Suppose that (V1), (F4), (F5) hold, then there exists \( v \in E \) with \( \|v\|_E > \rho \) such that \( I(v) < 0 \), where \( \rho \) is given in Lemma 3.1.

**Proof.** Set \( e \in C_0^\infty(B_1(0), \mathbb{R}) \), where \( B_1(0) = \{ x \in \mathbb{R}^3 : |x| \leq 1 \} \), such that \( \|e\|_E = 1 \) and \( A = \frac{2(1 + \int_{B_1(0)} \omega^2 e^2 dx)}{\int_{B_1(0)} e^2 dx} \). It follows from (F4) and (F5) that there exists \( Q > 0 \) such that

\[
F(x, t) \geq A \left( t^2 - Q^2 \right)
\]

for all \((x, t) \in B_1(0) \times \mathbb{R})

Then \( I(\eta e) \rightarrow -\infty \) as \( \eta \rightarrow +\infty \). Therefore, there exists \( \eta_0 > 0 \) such that \( I(\eta_0 e) < 0 \) and \( \|\eta_0 e\|_E > \varphi \). Let \( v = \eta_0 e \), we can see \( I(v) < 0 \), which proves this lemma.

**Lemma 3.3.** Assume that (V1) and (F2)–(F5) hold, then \( I \) satisfies the (PS) condition.

**Proof.** Suppose that \( \{u_n\} \subset E \) is a sequence such that \( \{I(u_n)\} \) is bounded and \( I'(u_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Then there exists a constant \( \overline{M} > 0 \) such that

\[
|I(u_n)| \leq \overline{M}, \quad \|I'(u_n)\| \leq \overline{M}.
\]

Now we prove that \( \{u_n\} \) is bounded in \( E \). Arguing in an indirect way, we assume that \( \|u_n\|_E \rightarrow +\infty \) as \( n \rightarrow \infty \). Set \( w_n = \frac{u_n}{\|u_n\|_E} \). Then \( \|w_n\|_E = 1 \) and there exists a subsequence of \( \{w_n\} \), still denoted by \( \{w_n\} \), such that \( w_n \rightharpoonup w_0 \) in \( E \). Then we have

\( w_n \rightarrow w_0 \) in \( L^s(\mathbb{R}^3) \) for any \( s \in [2, 6) \).

Let \( \Omega = \{ x \in \mathbb{R}^3 : |u_n(x)| > 0 \} \). If \( \text{meas}(\Omega) > 0 \), we have \( |u_n| \rightarrow +\infty \) as \( n \rightarrow \infty \) for a.e \( x \in \Omega \). On one hand, by (F4), (F5) and Fatou's lemma, we obtain

\[
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|_E^2} dx \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|_E^2} dx
\]

\[
= \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, u_n)}{|u_n|} |w_n|^2 dx
\]

\[
= +\infty.
\]
On the other hand, by (3.2) and Lemma 2.1, we get

\[
\left| \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|_E^2} \, dx - \frac{1}{2} \right| = \left| - \frac{I(u_n)}{\|u_n\|_E^2} + \frac{\omega}{2} \int_{\mathbb{R}^3} \frac{\phi_{u_n}^2}{\|u_n\|_E^2} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \frac{h(x)u_n}{\|u_n\|_E^2} \, dx \right| \\
\leq \frac{M}{\|u_n\|_E^2} + \frac{\omega^2C_2}{2} + C_2 \|h\|_2 + \frac{\omega^2C_2}{2} \|u_n\|_E \to \omega^2C_2 \|u_n\|_E \text{ as } n \to \infty,
\]

which is a contradiction. Then we have \(\text{meas}(\Omega) = 0\), which implies that \(w_0 = 0\) a.e. \(x \in \mathbb{R}^3\) and \(w_n \to 0\) in \(L^s(\mathbb{R}^3)\) \((2 \leq s < 6)\). By (2.2), (2.3), (F3) and (F5), we obtain

\[
2I(u_n) - \langle I'(u_n), u_n \rangle = \int_{\mathbb{R}^3} (\omega + \phi_{u_n}) \phi_{u_n} u_n^2 \, dx + \int_{\mathbb{R}^3} \bar{F}(x,u_n) \, dx - \int_{\mathbb{R}^3} h(x)u_n \, dx
\\
\geq \int_{\mathbb{R}^3} (\omega + \phi_{u_n}) \phi_{u_n} u_n^2 \, dx + d_2 \int_{|u_n| \geq r_0} |u_n|^r \, dx - \int_{\mathbb{R}^3} h(x)u_n \, dx,
\]

which implies that

\[
\lim_{n \to \infty} \frac{\int_{|u_n| \geq r_0} |u_n|^r \, dx}{\|u_n\|_E^2} = 0. \tag{3.3}
\]

By (2.3), (3.3), (F2) and (2.1), one sees that

\[
\frac{M + 1}{\|u_n\|_E} \geq \frac{\langle I'(u_n), u_n \rangle}{\|u_n\|_E^2}
\\
\geq 1 - d_1 \int_{\mathbb{R}^3} \left( w_{r_0}^2 + \frac{|u_n|^r}{\|u_n\|_E^2} \right) \, dx - \frac{\|h\|_2 \|u_n\|_2}{\|u_n\|_E^2}
\\
\geq 1 - d_1 \int_{\mathbb{R}^3} w_{r_0}^2 \, dx - d_1 \left( \int_{|u_n| \geq r_0} \frac{|u_n|^r}{\|u_n\|_E^2} \, dx \right) + r_0^{-2} \int_{|u_n| \leq r_0} w_{r_0}^2 \, dx - \frac{C_2 \|h\|_2}{\|u_n\|_E}
\\
\to 1 \text{ as } n \to \infty,
\]

which is a contradiction, then \(\{u_n\}\) is bounded in \(E\). Similar to Lemma 3.3 in [10], we can see that \(\{u_n\}\) has a strong convergent subsequence. Then \(I\) satisfies the (PS) condition. \(\Box\)

**Proof of Theorem 1.1.** In order to obtain two nontrivial solutions for the problem (1.1), we will apply the Ekeland’s variational principle and the mountain pass theorem to the functional \(I\). The rest proof of Theorem 1.1 is similar to the proof of Theorem 1.2 in [10]. \(\square\)

### 4 PROOF OF THEOREM 1.2

In order to obtain infinitely many solutions of (1.1), similar to [18], we shall use the abstract critical point theorem introduced by T. Bartsch in [3]. Let space \(X\) be reflexive and separable, then there exist \(e_i \in X\) and \(f_i \in X^*\) such that \(X = \langle e_i, i \in \mathbb{N} \rangle, X^* = \langle f_i, i \in \mathbb{N} \rangle, \langle e_i, f_j \rangle = \delta_{i,j}\), where \(\delta_{i,j}\) denotes the Kronecker symbol. Subsequently, put

\[
X_k = \text{span}\{e_k\}, \quad Y_k = \bigoplus_{i=1}^{k} X_i, \quad Z_k = \bigoplus_{i=k}^{\infty} X_i.
\]

Now we state the critical points theorem by T. Bartsch.
Lemma 4.1. Assume \( \phi \in C^1(X, \mathbb{R}) \) satisfies the (PS) condition, \( \phi(-u) = \phi(u) \). For every \( k \in \mathbb{N} \), there exists \( \rho_k > r_k > 0 \), such that

(i) \( a_k := \max_{u \in Y_k, \|u\| = r_k} \phi(u) \leq 0 \);
(ii) \( b_k := \inf_{u \in Z_k, \|u\| = r_k} \phi(u) \rightarrow +\infty \) as \( k \rightarrow \infty \).

Then \( \phi \) has a sequence of critical values tending to \(+\infty\).

Lemma 4.2. Assume that (V1), (F2), (F4) and (F5) hold. Then for every \( k \in \mathbb{N} \), there exists \( \rho_k > r_k > 0 \), such that

(i) \( a_k := \inf_{u \in Z_k, \|u\| = r_k} I(u) \rightarrow +\infty \) as \( k \rightarrow \infty \);
(ii) \( b_k := \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0 \).

Proof. It follows from (F2) that

\[
|F(x, t)| \leq d_1 \left( \frac{1}{2} t^2 + \frac{1}{r} |t|^r \right), \quad \text{for all } (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{4.1}
\]

For any \( k \in \mathbb{N} \) and \( p \in [2, 6) \), we set

\[
\beta_k(p) = \sup_{u \in Z_k, \|u\|_E = 1} \|u\|_p.
\]

Similar to Lemma 2.8 in [18], we have \( \beta_k(p) \rightarrow 0 \) as \( k \rightarrow \infty \). Letting \( r_k = \left( \frac{\tau}{8 \delta_k \beta_k^r(\tau)} \right)^{1/r} \), for any \( u \in Z_k \), it follows from (2.2), Lemma 2.1 and (4.1) that

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega \phi_u u^2) \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx
\]

\[
\geq \frac{1}{2} \|u\|_E^2 - d_1 \left( \frac{\beta_k^2(2)}{2} \|u\|_E^2 + \frac{\beta_k^r(\tau)}{r} \|u\|_E^r \right)
\]

\[
\geq \frac{1}{8} \|u\|_E^2
\]

for \( k \) large enough. Thus we obtain \( a_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \geq \frac{1}{8} r_k^2 \rightarrow +\infty \) as \( k \rightarrow \infty \).

Subsequently, for any \( u \in Y_k \) and \( \delta > 0 \), set

\[
\Gamma_\delta(u) = \{ x \in \mathbb{R}^3 : |u| \geq \delta \|u\|_E \}.
\]

Similar to Lemma 2.6 in [28], there exists \( \varepsilon_1 > 0 \) such that

\[
\text{meas}(\Gamma_\varepsilon_1(u)) \geq \varepsilon_1.
\]

From (F4), there exists \( r_\infty > 0 \) such that

\[
F(x, u) \geq \frac{1 + \omega^2 C_2^2}{\varepsilon_1^3} |u|^2 \geq \frac{1 + \omega^2 C_2^2}{\varepsilon_1} \|u\|_E^2 \tag{4.2}
\]

for all \( u \in Y_k \) and \( x \in \Gamma_\varepsilon_1(u) \) with \( \|u\|_E \geq r_\infty \). We can choose \( \rho_k > \max \left\{ \frac{r_\infty}{\varepsilon_1}, r_k \right\} \), then for any \( u \in Y_k \) with \( \|u\|_E = \rho_k \), it follows from (2.2), (F4), Lemma 2.1, (4.2) and (F5) that

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega \phi_u u^2) \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx
\]
\[
\leq \frac{1}{2} \|u\|_E^2 + \frac{\omega^2}{2} \|u\|_2^2 - \int_{\Omega(u)} F(x, u) \, dx \\
\leq - \frac{(1 + \omega^2 C_2^2)}{2} \|u\|_E^2,
\]

which means \(b_k \leq 0\) for \(\rho_k\) large enough. Then we finish the proof of this lemma.

\[\square\]

**Proof of Theorem 1.2.** Similar to the proof of Theorem 1.1, we see that \(I\) satisfies the (PS) condition. Furthermore, \(I(-u) = I(u)\), then we obtain our conclusion by using the Lemma 4.1.

\[\square\]

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