INTERSECTION OF DUALITY AND DERIVATION RELATIONS FOR MULTIPLE ZETA VALUES

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Abstract. The duality relation is a basic family of linear relations for multiple zeta values. The extended double shuffle relation (EDSR) is one of the families of relations expected to generate all linear relations among multiple zeta values, but it remains unclear as to whether all duality relations can be deduced from the EDSR. In the present paper, regarding the family generated by the duality relation and the family generated by the derivation relation, an explicit characterization of their intersection is obtained. Here, the derivation relation is a specialization of the EDSR.

1. Introduction

Multiple zeta values (MZVs) are defined by the convergent series

$$\zeta(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

for positive integers $k_1, \ldots, k_r$ with $k_1 \geq 2$. For an index $(k_1, \ldots, k_r)$, we call $k_1 + \cdots + k_r$ weight, $r$ depth, and $\# \{ k_i | k_i \geq 2 \}$ height. Regarding the $\mathbb{Q}$-linear space spanned by all MZVs, there exist many $\mathbb{Q}$-linear relations among MZVs. Moreover, this $\mathbb{Q}$-linear space has an algebraic structure, but its structure remains unexplained. The duality relation is a basic and important family of $\mathbb{Q}$-linear relations for MZVs. In [2], the extended double shuffle relation (EDSR) is also one of the families of relations expected to generate all $\mathbb{Q}$-linear relations among MZVs. However, it remains unclear as to whether all duality relations can be deduced from the EDSR.

We consider the family generated by the duality relation and the family generated by the derivation relation. Here, the derivation relation is a specialization of the EDSR. Our first goal is to characterize the intersection of these families explicitly. Based on this characterization, we can generate all relations of the intersection. In particular, we present the four identities as special cases. From two of these cases, we obtain new proofs of the results of Kajikawa [3] and Li [4], and from the other two identities, it is newly pointed out that the two families of the duality relations can be deduced from the derivation relation. Note that all of the duality relations cannot be deduced from the intersection. This fact is checked by numerical calculation (see [4] for details).

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The remainder of the present article is organized as follows. In Section 2, we review the basic terminology and well-known results regarding the duality and the derivation relations. In Section 3, we first state the key equation on the intersection of the duality and the derivation relations (Theorem 3.2). Next, we see that the key equation characterizes the entire intersection explicitly (Theorem 3.5). In Section 4, we present the four identities as special cases of Theorem 3.2 (Corollary 4.1). In Section 5, we prove Theorems 3.2 and 3.5. In Appendix A, we show that Corollary 4.1 (i) and (iii) are equivalent to the results of Kajikawa [3] and Li [5], respectively. We also mention the results obtained by Kawasaki and Tanaka in Remark A.5. In Appendix B, we also show a property related to the characterization (Corollary B.2).

2. Preparation

To state our results, we use the algebraic setup introduced by Hoffman [1]. Let \( S = \mathbb{Q}(x, y) \) be the non-commutative polynomial algebra over the rationals in two indeterminates \( x \) and \( y \), and let \( S^0 \) be its subalgebra \( \mathbb{Q} + xS y \). We define the \( \mathbb{Q} \)-linear map \( Z : S^0 \rightarrow \mathbb{R} \) by

\[
Z(x^{k_1-1}y \cdots x^{k_r-1}y) = \zeta(k_1, \ldots, k_r) \quad (k_1 \geq 2).
\]

For an index \((k_1, \ldots, k_r)\), the weight \( k \) and the depth \( r \) correspond to the total degree and the degree in \( y \) of the monomial \( x^{k_1-1}y \cdots x^{k_r-1}y \), respectively. The height also corresponds to the number of \( xy \) in the monomial. To obtain a \( \mathbb{Q} \)-linear relation among MZVs is precisely to obtain an element in the kernel of the map \( Z \). Let \( \tau : S \rightarrow S \) be the anti-automorphism of the algebra \( S \) defined by

\[
\tau(x) = y, \quad \tau(y) = x.
\]

The map \( \tau \) is an involution and preserves \( S^0 \). Then, the duality relation is stated as

\[
(\tau - \text{id}_S)(w) \in \ker(Z)
\]

for any \( w \in S^0 \) (see, e.g., [6]). For a positive integer \( n \), we define the \( \mathbb{Q} \)-linear map \( \partial_n : S \rightarrow S \) by

\[
\partial_n(x) = -\partial_n(y) = x(x + y)^{n-1}y
\]

and the Leibniz rule

\[
\partial_n(w_1w_2) = \partial_n(w_1)w_2 + w_1\partial_n(w_2),
\]

where \( w_1, w_2 \in S \). We find in [2] that \( \partial_n(S^0) \subset S^0 \) and that \( \deg(\partial_n(w)) = \deg(w) + n \) for any monomial \( w \in S \). We also find that \( \partial_n \) and \( \partial_m \) commute for any \( n, m \geq 1 \). The derivation relation, which is obtained by Ihara, Kaneko, and Zagier [2, Theorem 3], is stated as

\[
\partial_n(w) \in \ker(Z)
\]

for any positive integer \( n \) and \( w \in S^0 \). Let \( S[[u]] \) be the formal power series ring generated by the indeterminate \( u \) over \( S \). By extending \( \tau \) and \( \partial_n \) to \( \mathbb{Q}[[u]] \)-linear maps, \( \tau \) and \( \partial_n \)
both become maps on $\mathcal{H}[u]$. Let $\Delta_u$ be the map on $\mathcal{H}[u]$ defined by

$$\Delta_u = \exp\left(\sum_{n \geq 1} \frac{\partial_u}{n} u^n\right).$$

Then, $\Delta_u$ is an automorphism of $\mathcal{H}[u]$ and satisfies

$$\Delta_u(x) = x \frac{1}{1 - yu}, \quad \Delta_u(y) = (1 - xu - yu) \frac{y}{1 - yu}, \quad \Delta_u(x + y) = x + y,$$

where $1/(1 - wu) := \sum_{i \geq 0} w^i u^i$ ($w \in \mathcal{H}[u]$). (See [2, Corollary 3 and Theorem 4] for details.) The inverse map $\Delta_u^{-1}$ also satisfies

$$\Delta_u^{-1}(x) = \frac{x}{1 - xu}(1 - xu - yu), \quad \Delta_u^{-1}(y) = \frac{1}{1 - xu}y, \quad \Delta_u^{-1}(x + y) = x + y.$$

We denote by $\Delta_u^e := \Delta_u \circ \cdots \circ \Delta_u$, $\Delta_u^{-e} := \Delta_u^{-1} \circ \cdots \circ \Delta_u^{-1}$, and $\Delta_u^0 := \text{id} = \text{id}_{\mathcal{H}[u]}$.

For a positive integer $i$, let $\theta_i$ be the coefficient of $u^i$ in the expansion of $\Delta_u$, that is, $\Delta_u = \text{id} + \sum_{i \geq 1} \theta_i u^i$. Then, it is easy to see that each $\theta_i$ is composed of the maps $\partial_n$.

**Theorem 2.1** (Ihara-Kaneko-Zagier [2, Theorem 3]). The following properties are equivalent:

1. $\partial_n(w) \in \text{Ker}(Z)$ for all $w \in \mathcal{H}^0$ and $n \in \mathbb{Z}_{>0}$;
2. $\theta_i(w) \in \text{Ker}(Z)$ for all $w \in \mathcal{H}^0$ and $i \in \mathbb{Z}_{>0}$.

3. **Characterization**

In this section, considering the family generated by the duality relation and the family generated by the derivation relation, we state the main results for their intersection. First, we see the key equation on the intersection of the duality and derivation relations (Theorem 3.2). Second, we explicitly characterize the entire intersection (Theorem 3.5). The proofs of both theorems are given in Section 5.

Fix a positive integer $s$. For the indeterminates $u_1, \ldots, u_s$, let $\mathcal{H}[u_1, \ldots, u_s]$ be the commutative formal power series ring generated by $u_1, \ldots, u_s$ over $\mathcal{H}$. By setting $\Delta_{u_i}(u_j) = u_j$ for $i, j$ ($1 \leq i, j \leq s$), each $\Delta_{u_1}, \ldots, \Delta_{u_s}$ is extended to an automorphism of $\mathcal{H}[u_1, \ldots, u_s]$.

**Proposition 3.1.** On $\mathcal{H}[u_1, \ldots, u_s]$, the maps $\Delta_{u_i}$ and $\Delta_{u_j}$ are commutative for $i$ and $j$ ($1 \leq i, j \leq s$).

**Proof.** The maps $\partial_n$ and $\partial_m$ are commutative for any positive integers $n$ and $m$. Therefore, this assertion follows from the definitions of $\Delta_{u_i}$ and $\Delta_{u_j}$. □

By setting $\tau(u_i) = u_i$ ($1 \leq i \leq s$), the map $\tau$ is also extended to an anti-automorphism of $\mathcal{H}[u_1, \ldots, u_s]$. For an automorphism $\Delta = \Delta_{u_1}^{e_1} \circ \cdots \circ \Delta_{u_s}^{e_s}$, we define the $\mathbb{Q}$-linear space $\mathcal{D}_\Delta$ by

$$\mathcal{D}_\Delta := \text{span}_\mathbb{Q} \{ ab\tau(\Delta(a)) \mid a \in \mathcal{H}[u_1, \ldots, u_s], b \in (\mathbb{Q}[z])[u_1, \ldots, u_s] \},$$
where $z = x + y$. By considering the case $a = 1$, it is easy to see that $\mathcal{D}_\Delta$ includes $(\mathbb{Q}[z])[u_1, \ldots, u_s]$, so $\mathcal{D}_\Delta$ is not an empty set. Let $\mathcal{D}_\Delta^0$ denote $\mathcal{D}_\Delta \cap \mathcal{H}^0[u_1, \ldots, u_s]$.

**Theorem 3.2.** A certain element $w \in \mathcal{H}[u_1, \ldots, u_s]$ satisfies the following equation if and only if $w \in \mathcal{D}_\Delta$:

\[(\tau - \text{id})(w) = (\Delta - \text{id})(w).\]

We give the proof of Theorem 3.2 in Section 5.

**Remark 3.3.** In Equation (1), if $w \in \mathcal{H}^0[u_1, \ldots, u_s]$, i.e., $w \in \mathcal{D}_\Delta^0$, then the left- and right-hand sides coincide with the duality and derivation relations, respectively. In other words, the duality relation of the left-hand side is generated by the derivation relation of the right-hand side, explicitly.

**Proposition 3.4.** For any $\Delta$, if $w \in \mathcal{D}_\Delta$, then $w^d \in \mathcal{D}_\Delta$ ($d \in \mathbb{Z}_{\geq 0}$).

**Proof.** For any $w \in \mathcal{D}_\Delta$, it is easy to see $\tau(w) = \Delta(w)$ from Theorem 3.2. Then, we have

\[\tau(w^d) = \tau(w)^d = \Delta(w)^d = \Delta(w^d).\]

Therefore, we also obtain $(\tau - \text{id})(w^d) = (\Delta - \text{id})(w^d)$, and so this proof is completed using Theorem 3.2.

For a subset $\mathcal{A}$ of $\mathcal{H}[u_1, \ldots, u_s]$, let Coef$_{u_1, \ldots, u_s}(\mathcal{A})$ denote the set that is composed of all of the coefficients $w_{i_1, \ldots, i_s}$ appearing in each element $\sum_{i_1, \ldots, i_s \geq 0} w_{i_1, \ldots, i_s} u_1^{i_1} \cdots u_s^{i_s} \in \mathcal{A}$. If the subset $\mathcal{A}$ is a $\mathbb{Q}$-linear space, then Coef$_{u_1, \ldots, u_s}(\mathcal{A})$ is a $\mathbb{Q}$-linear space, too. Let $\partial(\mathcal{A})$ be the $\mathbb{Q}$-linear space span$_{\mathbb{Q}} \{\partial_n(w) \mid n \in \mathbb{Z}_{\geq 0}, w \in \mathcal{A}\}$.

**Theorem 3.5.** For integers $e_1, \ldots, e_s$ with $(e_1, \ldots, e_s) \neq (0, \ldots, 0)$, let $\Delta = \Delta_{u_1}^{e_1} \circ \cdots \circ \Delta_{u_s}^{e_s}$. Then, the following sets are equal:

(i) $(\tau - \text{id})(\mathcal{H}) \cap \partial(\mathcal{H})$;
(ii) Coef$_{u_1, \ldots, u_s}((\tau - \text{id})(\mathcal{D}_\Delta))$;
(iii) Coef$_{u_1, \ldots, u_s}((\Delta - \text{id})(\mathcal{D}_\Delta))$.

We give the proof of Theorem 3.5 in Section 5. This theorem still remains true even if we restrict to $\mathcal{H}^0$.

**Corollary 3.6.** For integers $e_1, \ldots, e_s$ with $(e_1, \ldots, e_s) \neq (0, \ldots, 0)$, let $\Delta = \Delta_{u_1}^{e_1} \circ \cdots \circ \Delta_{u_s}^{e_s}$. Then, the following sets are equal:

(i) $(\tau - \text{id})(\mathcal{H}^0) \cap \partial(\mathcal{H}^0)$;
(ii) Coef$_{u_1, \ldots, u_s}((\tau - \text{id})(\mathcal{D}_\Delta^0))$;
(iii) Coef$_{u_1, \ldots, u_s}((\Delta - \text{id})(\mathcal{D}_\Delta^0))$.

**Remark 3.7.** Corollary 3.6 (i) coincides exactly with the entire intersection of the duality and derivation relations. Since both Corollary 3.6 (ii) and (iii) are stated by the explicit set $\mathcal{D}_\Delta^0$, each of these explicitly characterizes the intersection. Furthermore, all relations of the intersection can be generated by substituting elements of $\mathcal{D}_\Delta^0$ into Equation (1).
Note that Theorem 3.5 and Corollary 3.6 hold for any positive integer \( s \). It suffices to consider only the case in which \( s = 1 \). However, in Section 4, we explain that the multiplex indeterminates \( u_1, \ldots, u_s \) are useful to fix the parameters.

4. Explicit Identities

As special cases of Theorem 3.2, we show the four explicit identities on the intersection of the duality and derivation relations. From two of the four identities, we reobtain the results of Kajikawa [3] and Li [5], respectively. From the other two identities, it is newly pointed out that the derivation relation can generate two families of the duality relations for the special indices. Note that these results are obtained due to the extension of the multiple indeterminates \( u_1, \ldots, u_s \).

**Corollary 4.1.** Assume \( s \geq 3 \) and \( d \) is a positive integer. Then, we have the following:

(i) \((\tau - \text{id})\left(\frac{x}{1 - xu_1} \frac{y}{1 - yu_2}\right)^d = (\Delta_{u_1} \circ \Delta_{u_2}^{-1} - \text{id})\left(\frac{x}{1 - xu_1} \frac{y}{1 - yu_2}\right)^d\).

(ii) \((\tau - \text{id})\left(\frac{1}{1 - yu_1} \frac{1}{1 - xu_2} \frac{1}{y}\right)^d = (\Delta_{u_1}^{-1} \circ \Delta_{u_2} - \text{id})\left(\frac{1}{1 - yu_1} \frac{1}{1 - xu_2} \frac{1}{y}\right)^d\).

(iii) \((\tau - \text{id})\left(\frac{xu_1}{1 - xu_1} \frac{1}{1 - yu_2} \frac{1}{1 - xu_3} \frac{1}{y}\right) = -(\Delta_{u_1} \circ \Delta_{u_2}^{-1} \circ \Delta_{u_3} - \text{id})\left(\frac{x}{1 - xu_1} (1 - xu_1 - yu_1) \frac{1}{1 - yu_2} \frac{1}{1 - xu_3} \frac{1}{y}\right) + (\Delta_{u_2}^{-1} \circ \Delta_{u_3} - \text{id})\left(\frac{x}{1 - xu_1} \frac{1}{1 - yu_2} \frac{1}{1 - xu_3} \frac{1}{y}\right)\).

(iv) \((\tau - \text{id})\left(\frac{1}{1 - yu_1} \frac{1}{1 - xu_2} \frac{1}{y} \frac{1}{1 - xu_3} \frac{1}{y}\right) = -(\Delta_{u_1}^{-1} \circ \Delta_{u_2} \circ \Delta_{u_3}^{-1} - \text{id})\left(\frac{1}{1 - yu_1} \frac{1}{1 - xu_2} \frac{1}{1 - yu_1} \frac{1}{1 - xu_3} \frac{1}{y}\right) + (\Delta_{u_1}^{-1} \circ \Delta_{u_2} - \text{id})\left(\frac{1}{1 - yu_1} \frac{1}{1 - xu_2} \frac{1}{1 - yu_1} \frac{1}{y}\right)\).

**Proof of Corollary 4.1.** Substituting one or two special elements of \( \mathcal{D}_\Delta^0 \) into Equation (1), we obtain each identity. For the identity (i), we take the special element \( ab\tau(\Delta(a)) \), where \( \Delta = \Delta_{u_1} \circ \Delta_{u_2}^{-1} \), \( a = \Delta_{u_1}^{-1}(x) \) and \( b = (1 - xu_1 - yu_1)^{-1} (1 - xu_2 - yu_2)^{-1} \). From Proposition 3.4, we see that \{\( ab\tau(\Delta(a)) \)^d \} \( \in \mathcal{D}_\Delta^0 \), i.e.,

\[
\left(\frac{x}{1 - xu_1} \frac{1}{1 - yu_2} \right)^d \in \mathcal{D}_\Delta^0.
\]
By substituting the above element into Equation (11), the identity (i) is obtained. Similarly, for the identity (ii), we also take a special element \( ab\tau(\Delta(a)) \), where \( \Delta = \Delta_{u_1}^{-1} \circ \Delta_{u_2} \), \( a = \Delta_{u_1}(x) \), and \( b = 1 \). By substituting \( \{ab\tau(\Delta(a))\}^d \) into Equation (11), the identity (ii) is obtained, too. For the identity (iii), we take two special elements \( ab\tau(\Delta(a)) \) and \( a'b'\tau(\Delta'(a')) \), where \( \Delta = \Delta_{u_1} \circ \Delta_{u_2}^{-1} \circ \Delta_{u_3} \), \( a = \Delta_{u_1}^{-1}(x) \), and \( b = (1 - xu_2 - yu_2)^{-1} \); \( \Delta' = \Delta_{u_2}^{-1} \circ \Delta_{u_3} \), \( a' = x \), and \( b' = (1 - xu_2 - yu_2)^{-1} \). Then, we see

\[
\frac{x}{1 - xu_1(1 - xu_1 - yu_1)}\frac{1}{1 - \frac{x}{1 - xu_3}yu_2}y = ab\tau(\Delta(a)) \in \mathcal{D}_0^\Delta,
\]

\[
\frac{x}{1 - \frac{x}{1 - xu_3}yu_2}1 - xu_3 y = a'b'\tau(\Delta'(a')) \in \mathcal{D}_0^\Delta'.
\]

By substituting each of the above two elements into Equation (11) and subtracting these equations, the identity (iii) is exactly obtained. Similarly, for the identity (iv), we also take two special elements \( ab\tau(\Delta(a)) \) and \( a'b'\tau(\Delta'(a')) \), where \( \Delta = \Delta_{u_1}^{-1} \circ \Delta_{u_2} \circ \Delta_{u_3} \), \( a = \Delta_{u_1} \circ \Delta_{u_2}^{-1}(x) \), and \( b = (1 - xu_2 - yu_2)^{-1}(1 - xu_3 - yu_3) \); \( \Delta' = \Delta_{u_1}^{-1} \circ \Delta_{u_2} \), \( a' = \Delta_{u_1} \circ \Delta_{u_2}^{-1}(x) \), and \( b' = (1 - xu_2 - yu_2)^{-1} \). By substituting each element into Equation (11) and subtracting these equations, the identity (iv) is exactly obtained, too.

Considering the left-hand sides of identities (ii) and (iv), we newly point out that the two families of the duality relations can be generated by the derivation relation.

**Corollary 4.2.**

(i) For positive integers \( d, m, \) and \( n \), we have

\[
(\tau - \text{id})\left(\sum_{m_1 + \ldots + m_d = m} \sum_{n_1 + \ldots + n_d = n} xy^{m_1}x^{m_1}y \cdots xy^{m_d}x^{n_d}y\right) \in \partial(\mathcal{F}^0).
\]

(ii) For positive integers \( k, r, \) and \( m \) \((k > r + m)\), we have

\[
(\tau - \text{id})\left(\sum_{m_1 + \ldots + m_r = k - r - m - 1} x(x^{m_1}y \cdots x^{m_r}y)x^{m-1}y\right) \in \partial(\mathcal{F}^0).
\]

Corollary 4.2(i) follows by considering the part of Corollary 4.1(ii) that satisfies degree \( m \) in \( u_1 \) and degree \( n \) in \( u_2 \). Similarly, Corollary 4.2(ii) also follows by considering the part of Corollary 4.1(iv) that satisfies total degree \( k - 2 \) in \( u_1, u_2, \) and \( u_3 \), degree \( r - 1 \) in \( u_1 \), and degree \( m \) in \( u_3 \). Applying the map \( Z \) to Corollary 4.2(i) and (ii), we find that the derivation relations can deduce these duality relations for the sum of MZVs with special indices. From Corollary 4.2(i), we obtain the duality relation for the sum of MZVs with indices \( \{1 + \{1, \}^{m_1}n_1, 1, \ldots, 1 + \{1, \}^{m_d}n_d + 1\} \), where \( \{a\}^m \) denotes \( a \cdots a \). Similarly, from Corollary 4.2(ii), we also obtain the duality relation for the sum of MZVs with
weight \( k \), depth \( r + 1 \), and \( k_{r+1} = m \). In particular, taking \( d = 1 \) in Corollary 4.2 (i), we find that the simple duality relation \( \zeta(2, \{1,\}^{n-1}n + 1) - \zeta(2, \{1,\}^{n-1}m + 1) \) can be generated by the derivation relation. This result is not reported in Kajikawa [3], Kawasaki and Tanaka [4], or Li [5], so it is newly pointed out here.

Of course, similar results also follow from each of Corollary 4.1 (i) and (iii). The left-hand side of Corollary 4.1 (i) coincides with the duality relation for the sum of MZVs with fixed weight, depth, and height. Here, the weight, depth, and height are fixed by the total degree in \( u_1 \) and \( u_2 \), the degree in \( u_2 \), and \( d \), respectively. Similarly, the left-hand side of Corollary 4.1 (iii) expresses the duality relation for the sum of MZVs with fixed weight, depth, and the first component \( k_1 \) of the index. Here, the weight, depth, and \( k_1 \) are also fixed by the total degree in \( u_1 \), \( u_2 \), and \( u_3 \), the degree in \( u_2 \), and the degree in \( u_3 \), respectively. In fact, Corollary 4.1 (i) is equivalent to the identity of Kajikawa [3] Main Theorem] (Proposition A.2), and Corollary 4.1 (iii) is equivalent to the identity of Li [5] Theorem 1] (Proposition A.4). The proofs are given in Appendix A. Note that Corollary 4.1 (iii) contains both results given by Kawasaki and Tanaka [4, Theorem (i) and (ii)] as well as the result of Li [5, Theorem 1]. We mention this in Remark A.5.

5. Proof of Theorems

In this section, we prove Theorems 3.2 and 3.5. First, to prove Theorem 3.2, we show the key lemma as follows.

**Lemma 5.1.** On \( \mathcal{H}[u_1, \ldots, u_s] \), we have
\[
\tau \circ \Delta_{u_i}^{-1} \circ \tau \equiv \Delta_{u_i} \quad (1 \leq i \leq s).
\]

**Proof.** It is easy to check that the map \( \tau \circ \Delta_{u_i}^{-1} \circ \tau \) is a ring homomorphism. For each generator \( x, y, \) and \( u_j \) \((1 \leq j \leq s)\), we see that
\[
\tau \circ \Delta_{u_i}^{-1} \circ \tau(x) = \Delta_{u_i}(x), \quad \tau \circ \Delta_{u_i}^{-1} \circ \tau(y) = \Delta_{u_i}(y), \quad \tau \circ \Delta_{u_i}^{-1} \circ \tau(u_j) = \Delta_{u_i}(u_j).
\]

By using Lemma 5.1, we prove Theorem 3.2.

**Proof of Theorem 3.2.** It is straightforward that \( w \in \text{Ker}(\Delta - \tau) \) if and only if \( w \) satisfies Equation (1). Hence, we prove the following equation:
\[
(2) \quad \text{Ker}(\Delta - \tau) = \mathcal{D}_\Delta.
\]

Since \( \text{Ker}(\Delta - \tau) \) is a \( \mathbb{Q} \)-linear space, we first check that each generator \( ab \tau(\Delta(a)) \in \mathcal{D}_\Delta \) belongs to \( \text{Ker}(\Delta - \tau) \), where \( a \in \mathcal{H}[u_1, \ldots, u_s] \) and \( b \in (\mathbb{Q}[z])[u_1, \ldots, u_s] \) \((z = x + y)\).
By using Lemma 5.1 and recalling $\Delta(z) = \tau(z) = z$, we obtain the following:

$$\begin{align*}
(\Delta - \tau)(ab\tau(\Delta(a))) &= \Delta(ab\tau(\Delta(a))) - \tau(ab\tau(\Delta(a))) \\
&= \Delta(a)\Delta(b)\Delta(\tau(\Delta(a))) - \tau(\Delta(a))\tau(b)\tau(a) \\
&= \Delta(a)\Delta(b)\Delta(\Delta^{-1}(\tau(a))) - \Delta(a)\tau(b)\tau(a) \\
&= \Delta(a)b\tau(a) - \Delta(a)b\tau(a) \\
&= 0.
\end{align*}$$

Thus, $\text{Ker}(\Delta - \tau) \supset \mathcal{D}_\Delta$ is obtained. Next, we check that $\text{Ker}(\Delta - \tau) \subset \mathcal{D}_\Delta$. For $w \in \text{Ker}(\Delta - \tau)$, it is easy to see that $w = \tau(\Delta(w))$, so we have the following:

$$
\begin{align*}
w &= \frac{1}{2}w + \frac{1}{2}\tau(\Delta(w)) \\
&= \frac{1}{2}w - \frac{1}{2}(1 - x - y)\tau(\Delta(w)) \\
&= \frac{1}{2}(w + 1 - x - y)\frac{1}{1 - x - y}\{\tau(\Delta(w)) + 1 - x - y\} \\
&\quad - \frac{1}{2}w\frac{1}{1 - x - y}\tau(\Delta(w)) - \frac{1}{2}(1 - x - y)\frac{1}{1 - x - y}(1 - x - y) \\
&= \frac{1}{2}(w + 1 - x - y)\frac{1}{1 - x - y}\tau(\Delta(w + 1 - x - y)) \\
&\quad - \frac{1}{2}w\frac{1}{1 - x - y}\tau(\Delta(w)) - \frac{1}{2}(1 - x - y)\frac{1}{1 - x - y}\tau(\Delta(1 - x - y)).
\end{align*}
$$

Since each of the three terms is a generator of $\mathcal{D}_\Delta$, we find that $w \in \mathcal{D}_\Delta$. In other words, $\text{Ker}(\Delta - \tau) \subset \mathcal{D}_\Delta$, and so Equation (2) holds. \qed

Next, we show the following lemma needed for the proof of Theorem 3.5.

**Lemma 5.2.** On $\mathfrak{H}[u_1, \ldots, u_s]$, we have the following identities:

(i) $\text{Ker}(\Delta - \tau) = \text{Im}(\Delta^{-1} + \tau)$.

(ii) $\text{Ker}(\Delta + \tau) = \text{Im}(\Delta^{-1} - \tau)$.

Note that we use only Lemma 5.2 (i) in the proof of Theorem 3.5. However, Lemma 5.2 (ii) is used in the proof of Proposition B.1.

**Proof.** We prove Lemma 5.2 (i). By using Lemma 5.1 we obtain the following:

$$
(\Delta - \tau)((\Delta^{-1} + \tau)(w)) = (\text{id} + \Delta \circ \tau - \tau \circ \Delta^{-1} - \text{id})(w) = 0
$$

Next, we show the following lemma needed for the proof of Theorem 3.5.

On $\mathfrak{H}[u_1, \ldots, u_s]$, we have the following identities:

(i) $\text{Ker}(\Delta - \tau) = \text{Im}(\Delta^{-1} + \tau)$.

(ii) $\text{Ker}(\Delta + \tau) = \text{Im}(\Delta^{-1} - \tau)$.

Note that we use only Lemma 5.2 (i) in the proof of Theorem 3.5. However, Lemma 5.2 (ii) is used in the proof of Proposition B.1.

**Proof.** We prove Lemma 5.2 (i). By using Lemma 5.1 we obtain the following:

$$
(\Delta - \tau)((\Delta^{-1} + \tau)(w)) = (\text{id} + \Delta \circ \tau - \tau \circ \Delta^{-1} - \text{id})(w) = 0
$$

Thus, $\text{Ker}(\Delta - \tau) \supset \mathcal{D}_\Delta$ is obtained. Next, we check that $\text{Ker}(\Delta - \tau) \subset \mathcal{D}_\Delta$. For $w \in \text{Ker}(\Delta - \tau)$, it is easy to see that $w = \tau(\Delta(w))$, so we have the following:

$$
\begin{align*}
w &= \frac{1}{2}w + \frac{1}{2}\tau(\Delta(w)) \\
&= \frac{1}{2}w - \frac{1}{2}(1 - x - y)\tau(\Delta(w)) \\
&= \frac{1}{2}(w + 1 - x - y)\frac{1}{1 - x - y}\{\tau(\Delta(w)) + 1 - x - y\} \\
&\quad - \frac{1}{2}w\frac{1}{1 - x - y}\tau(\Delta(w)) - \frac{1}{2}(1 - x - y)\frac{1}{1 - x - y}(1 - x - y) \\
&= \frac{1}{2}(w + 1 - x - y)\frac{1}{1 - x - y}\tau(\Delta(w + 1 - x - y)) \\
&\quad - \frac{1}{2}w\frac{1}{1 - x - y}\tau(\Delta(w)) - \frac{1}{2}(1 - x - y)\frac{1}{1 - x - y}\tau(\Delta(1 - x - y)).
\end{align*}
$$

Since each of the three terms is a generator of $\mathcal{D}_\Delta$, we find that $w \in \mathcal{D}_\Delta$. In other words, $\text{Ker}(\Delta - \tau) \subset \mathcal{D}_\Delta$, and so Equation (2) holds. \qed
for any \( w \in \mathfrak{H}[u_1, \ldots, u_s] \). Thus, we have \( \ker(\Delta - \tau) \supset \text{Im}(\Delta^{-1} + \tau) \). Next, reviewing \( w = \Delta^{-1}(\tau(w)) \) for any \( w \in \ker(\Delta - \tau) \), we have
\[
\begin{align*}
w &= \frac{1}{2} \left\{ \Delta^{-1}(\tau(w)) + w \right\} \\
&= \frac{1}{2} \left\{ \Delta^{-1}(\tau(w)) + \tau(\tau(w)) \right\} \\
&= (\Delta^{-1} + \tau) \left( \frac{1}{2} \tau(w) \right) \in \text{Im}(\Delta^{-1} + \tau).
\end{align*}
\]
Therefore, we have \( \ker(\Delta - \tau) \subset \text{Im}(\Delta^{-1} + \tau) \), and so \( \ker(\Delta - \tau) \) and \( \text{Im}(\Delta^{-1} + \tau) \) are equal. The proof of Lemma 5.2 (ii) is similar and so will be omitted. \( \square \)

Using these properties, we prove Theorem 3.5.

\textbf{Proof of Theorem 3.5.} From Theorem 3.2, it is easy to see that the sets (ii) and (iii) are equal and that both of these sets are contained in the set (i), i.e., \((i) \supset (ii) = (iii)\). To prove \((i) \subset (ii)\), it suffices to check the following:
\[(3) \quad (\tau - \text{id})(\mathfrak{H}) \cap \partial(\mathfrak{H}) \subset (\tau - \text{id})(\partial(\mathfrak{H})) ,
\]
\[(4) \quad (\tau - \text{id})(\partial(\mathfrak{H})) \subset \text{Coeff}_{u_1, \ldots, u_s}((\tau - \text{id})(\mathcal{D}_\Delta)) .
\]
First, we check Equation (3). Each element of \((\tau - \text{id})(\mathfrak{H}) \cap \partial(\mathfrak{H})\) can be expressed as \((\tau - \text{id})(w)\), where \( w \in \mathfrak{H} \). Then, there exists a certain positive integer \( N \) such that
\[
\begin{align*}
(\tau - \text{id})(w) &= \sum_{n=1}^{N} \partial_n(v_n),
\end{align*}
\]
where \( v_n \in \mathfrak{H} \). Applying the map \( \tau - \text{id} \) to both sides, we have
\[
\begin{align*}
\sum_{n=1}^{N} (\tau - \text{id}) (\partial_n(v_n)) &= (\tau - \text{id})^2(w) \\
&= -2(\tau - \text{id})(w).
\end{align*}
\]
Recalling that \((\tau - \text{id})(\partial(\mathfrak{H}))\) is a \( \mathbb{Q} \)-linear space, we obtain \((\tau - \text{id})(w) \in (\tau - \text{id})(\partial(\mathfrak{H}))\). Therefore, Equation (3) holds. Next, we prove Equation (4). Since the set \((\tau - \text{id})(\mathcal{D}_\Delta)\) is a \( \mathbb{Q} \)-linear space, the set \( \text{Coeff}_{u_1, \ldots, u_s}((\tau - \text{id})(\mathcal{D}_\Delta))\) is a \( \mathbb{Q} \)-linear space, too. Reviewing that all \( \theta_n(w) \) \( (n \in \mathbb{Z}_{>0}, w \in \mathfrak{H}) \) are generators of \( \partial(\mathfrak{H}) \) from Theorem 2.1, it suffices to check that
\[(5) \quad (\tau - \text{id}) (\theta_n(w)) \in \text{Coeff}_{u_1, \ldots, u_s}((\tau - \text{id})(\mathcal{D}_\Delta))
\]
for any integer \( n \) and \( w \in \mathfrak{H} \). We may assume \( e_1 \neq 0 \) without loss of generality. First, in the case of \( \Delta = \Delta_{u_1, e_1} \) \( (e_1 > 0) \), we prove Equation (5) by induction on \( n \). We will take the element such that one of its coefficients is \((\tau - \text{id})(\theta_n(w))\). By using Equation (2) and Lemma 5.2 (i), we obtain
\[(6) \quad \mathcal{D}_\Delta = \ker(\Delta - \tau) = \text{Im}(\Delta^{-1} + \tau) .
\]
Thus, for any element \( w \in \mathcal{F} \), we have
\[
(\Delta_{u_1}^{-e_1} + \tau)(w) \in \mathcal{D}_{\Delta_{u_1}^e}.
\]
Since the map \( \tau \) is an anti-automorphism of \( \mathcal{F} \), let us replace \( w \) with \( \tau(w) \). By using Lemma 5.11 we obtain
\[
(\Delta_{u_1}^{-e_1} + \tau)(\tau(w)) = (\tau \circ \Delta_{u_1}^{e_1} + \tau \circ \tau)(w) = -\Delta_{u_1}^{e_1} + \tau(w).
\]
The point is that \( (\tau - \text{id})(\Delta_{u_1}^{e_1} + \tau)(w) \) is exactly the element we wanted. Now, in the expansion of \( \Delta_{u_1}^{e_1} \), we consider the coefficients of \( u_1^i \):
\[
\Delta_{u_1}^{e_1} = \left( \sum_{i \geq 0} \theta_i u_1^i \right)^{e_1} = \text{id} + \left( \frac{e_1}{1!} \right) \theta_1 u_1 + \left\{ \left( \frac{e_1}{2!} \right) \theta_1^2 + \left( \frac{e_1}{1!} \right) \theta_2 \right\} u_1^2 + \cdots
\]
\[
= \sum_{i \geq 0} \sum_{j_0 + j_1 + \cdots + j_i = e_1} \frac{e_1^i}{j_0! j_1! \cdots j_i!} \theta_1^j \circ \cdots \circ \theta_i^j u_1^i,
\]
where \( \binom{i}{j} \) is the binomial coefficient. From the above, we find that the coefficient of \( u_1 \) is just \( e_1(\tau - \text{id})(\theta_1(w)) \), and so the set membership holds for \( n = 1 \) in the case of \( \Delta = \Delta_{u_1}^{e_1} \).
Suppose \( n \geq 2 \) and assume the following set membership holds for any \( i < n \) in the case of \( \Delta = \Delta_{u_1}^{e_1} \) \((e_1 > 0)\):
\[
(\tau - \text{id})(\theta_i(w)) \in \text{Coef}_{u_1}((\tau - \text{id})(\mathcal{D}_{\Delta_{u_1}^{e_1}})).
\]
Similarly, for an element \((\tau - \text{id})(\Delta_{u_1}^{e_1} + \tau)(w))\), the coefficient of \( u_1^n \) is
\[
e_1(\tau - \text{id})(\theta_n(w)) + \sum_{j_0 + j_1 + \cdots + j_{n-1} = e_1} \frac{e_1^n}{j_0! j_1! \cdots j_{n-1}!} (\tau - \text{id})\left( (\theta_1 \circ \cdots \circ \theta_{n-1} \circ \theta_{n-1}^{j_{n-1}}(w) \right).
\]
By the induction hypothesis, we see that the second term belongs to \( \text{Coef}_{u_1}((\tau - \text{id})(\mathcal{D}_{\Delta_{u_1}^{e_1}})) \).
Since \( \text{Coef}_{u_1}((\tau - \text{id})(\mathcal{D}_{\Delta_{u_1}^{e_1}})) \) is a \( \mathbb{Q} \)-linear space, \( e_1(\tau - \text{id})(\theta_n(w)) \in \text{Coef}_{u_1}((\tau - \text{id})(\mathcal{D}_{\Delta_{u_1}^{e_1}})) \).
That is, Equation (5) is valid in the case of \( \Delta = \Delta_{u_1}^{e_1} \) \((e_1 > 0)\). Second, in the case of \( \Delta = \Delta_{u_1}^{e_1} \) \((e_1 < 0)\), we can apply the same argument to coefficients of \((\tau - \text{id})(\Delta_{u_1}^{e_1} + \tau)(w)\) taken from \((\tau - \text{id})(\mathcal{D}_{\Delta_{u_1}^{e_1}})) \). Since \(-e_1\) is also a positive integer, Equation (5) is valid in the case of \( \Delta = \Delta_{u_1}^{e_1} \) \((e_1 \neq 0)\). Finally, in the case of \( \Delta = \Delta_{u_1}^{e_1} \circ \cdots \circ \Delta_{u_s}^{e_s} \) \((e_1, \ldots, e_s \in \mathbb{Z}, e_1 > 0)\), let \( v = (\Delta^{-1} + \tau)(\Delta_{u_2}^{e_2} \circ \cdots \circ \Delta_{u_s}^{e_s}(\tau(w))) \). Then, the element \( v \) belongs to \( \mathcal{D}_{\Delta} \) because of Equation (6), and the element \( v \) is easily transformed into \( \Delta_{u_1}^{e_1}(\tau(w)) + \tau(\Delta_{u_2}^{e_2} \circ \cdots \cdots \circ \Delta_{u_s}^{e_s}(\tau(w)))) \). Now, we consider the coefficient of \( u_1^i \) in the element \((\tau - \text{id})(v)\). By using Lemma 5.11 we obtain
\[
(\tau - \text{id})(\Delta_{u_1}^{e_1}(\tau(w))) = (\tau - \text{id})(\tau(\Delta_{u_1}^{e_1}(w))) = -(\tau - \text{id})(\Delta_{u_1}^{e_1}(w))\].
That is, the first term of \((\tau - \text{id})(v)\) is \(- (\tau - \text{id})(\Delta u_1^{e_1}(w))\). Since the second term of \(v\) has no \(u_1\), it suffices to consider the coefficient of \(u_1^i\) in \(-(\tau - \text{id})(\Delta u_1^{e_1}(w))\). By the same argument, Equation (5) is valid in this case. Similarly, in the case of \(e_1 < 0\), this follows by replacing \(\tau(w)\) with \(w\). Therefore, Equation (5) holds, and Equation (4) is obtained.

**Appendix A. Equivalence of Previous Works and Corollary 4.1**

In this appendix, we show that Corollary 4.1 (i) is equivalent to the result of Kajikawa [3] and that Corollary 4.1 (iii) is equivalent to the result of Li [5]. We also mention the result obtained by Kawasaki and Tanaka [4] in Remark A.5. To state the result of Kajikawa, let \(\hat{H}\) be the formal power series ring \(\mathbb{Q}[x, y]\) generated by the indeterminates \(x\) and \(y\) over \(\mathbb{Q}\).

**Theorem A.1 (Kajikawa [3]).** For positive integers \(r\) and \(d\) with \(r \geq d\), we have the following identity on \(\hat{H}^l\):

\[
(\tau - \text{id}) \left( \sum_{i_1 + \cdots + i_d = r, i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \frac{x}{1 - x y_j} \right) = \sum_{m \geq 0} \theta_m \left( \sum_{i_1 + \cdots + i_d = r, i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \left( \frac{x - x}{1 - x y_j} \right)^{i_j - 1} \frac{x}{1 - x} \right)

- \sum_{n=0}^{r-d} \theta_n \left( \sum_{i_1 + \cdots + i_d = r-n, i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \left( \frac{x - x}{1 - x y_j} \right)^{i_j - 1} \frac{x}{1 - x} \right),
\]

where \(\prod_{j=1}^d w_j = w_1 \cdots w_d\).

**Proposition A.2.** Corollary 4.1 (i) and Theorem A.1 are equivalent.

**Proof.** We prove that Corollary 4.1 (i) is exactly the generating function of Theorem A.1 with respect to degree. First, for the left-hand side of Corollary 4.1 (i), we have

\[
(\tau - \text{id}) \left( \frac{x}{1 - x u_1} \frac{y}{1 - y u_2} \right)^d = (\tau - \text{id}) \left( \sum_{r_1 \geq 0} \frac{x}{1 - x u_1} y(y u_2)^{r_1} \cdots \sum_{r_d \geq 0} \frac{x}{1 - x u_1} y(y u_2)^{r_d} \right)

= \sum_{r \geq 0} (\tau - \text{id}) \left( \sum_{r_1, \ldots, r_d \geq 0} \prod_{j=1}^d \frac{x}{1 - x u_1} y^{r_j + 1} \right) u_2^r

= \sum_{r \geq 0} (\tau - \text{id}) \left( \sum_{r_1, \ldots, r_d \geq r+1} \prod_{j=1}^d \frac{x}{1 - x u_1} y^{r_j} \right) u_2^r

= \sum_{r \geq d} (\tau - \text{id}) \left( \sum_{r_1, \ldots, r_d \geq r} \prod_{j=1}^d \frac{x}{1 - x u_1} y^{r_j} \right) u_2^{r-d}.
\]

(7)
Second, the part of $\Delta_{u_1} \circ \Delta_{u_2}^{-1}$ on the right-hand side is transformed as follows:

$$
\Delta_{u_1} \circ \Delta_{u_2}^{-1}\left(\left(\frac{x}{1-xu_1} \frac{y}{1-yu_2}\right)^d\right)
= \left(\frac{x}{1-xu_2} \frac{y}{1-yu_1}\right)^d
= \sum_{r \geq 0} \sum_{i_1, \ldots, i_d \geq 0} \prod_{j=1}^d \left(\frac{x^{i_j+1}}{1-xu_1}\frac{y}{1-yu_1}\right) u_2^r
= \sum_{r \geq 0} \sum_{i_1, \ldots, i_d \geq 0} \prod_{j=1}^d \Delta_{u_1} \left(\left(\Delta_{u_1}^{-1}(x)\right)^{i_j} x \Delta_{u_1}^{-1}(y)\right) u_2^r
= \sum_{r \geq d} \Delta_{u_1} \left(\sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \left\{\left(\frac{x}{1-xu_1} y^{i_j-1} \frac{x}{1-xu_1} y\right)\right\} u_2^r\right)
= \sum_{r \geq d} \sum_{n \geq 0} \theta_n \left(\sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \{\Delta_{u_1}^{-1}(x)\Delta_{u_1}^{-1}(y)\}\right) u_2^{r+n}
= \sum_{r \geq d} \sum_{n \geq 0} \theta_n \left(\prod_{j=1}^d \{\Delta_{u_1}^{-1}(x^{i_j})\Delta_{u_1}^{-1}(y)\}\right) u_2^{r-d}
= \sum_{r \geq d} \sum_{n \geq 0} \theta_n \left(\sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \left\{\left(\frac{x}{1-xu_1} y^{i_j-1} \frac{x}{1-xu_1} y\right)\right\} u_2^{r-d}\right).
$$

Finally, reviewing $\Delta_{u_2} = \sum_{n \geq 0} \theta_n u_2^n$, we transform the identity map part on the right-hand side:

$$
\left(\frac{x}{1-xu_1} \frac{y}{1-yu_2}\right)^d
= \Delta_{u_2} \circ \Delta_{u_1}^{-1}\left(\left(\frac{x}{1-xu_2} \frac{y}{1-yu_1}\right)^d\right)
= \Delta_{u_2} \left(\left(\frac{1}{1-\Delta_{u_1}^{-1}(x)u_2} x \Delta_{u_1}^{-1}(y)\right)^d\right)
= \Delta_{u_2} \left(\sum_{r \geq 0} \sum_{i_1, \ldots, i_d \geq 0} \prod_{j=1}^d \{\Delta_{u_1}^{-1}(x)\Delta_{u_1}^{-1}(y)\}\right) u_2^r
= \sum_{r \geq 0} \sum_{n \geq 0} \theta_n \left(\sum_{i_1, \ldots, i_d \geq 0} \prod_{j=1}^d \{\Delta_{u_1}^{-1}(x^{i_j})\Delta_{u_1}^{-1}(y)\}\right) u_2^{r+n}
= \sum_{r \geq d} \sum_{n \geq 0} \theta_n \left(\prod_{j=1}^d \{\Delta_{u_1}^{-1}(x^{i_j})\Delta_{u_1}^{-1}(y)\}\right) u_2^{r-d}
= \sum_{r \geq d} \sum_{n \geq 0} \theta_n \left(\sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^d \left\{\left(\frac{x}{1-xu_1} y^{i_j-1} \frac{x}{1-xu_1} y\right)\right\} u_2^{r-d}\right).
$$
Substituting (7), (8), and (9) into Corollary 4.1 (i), we obtain the identity composed of these generating functions. Then, we focus on the coefficient of $u^2_{r-d}$:

$$(\tau - \text{id}) \left( \sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^{d} \frac{x}{1-x u_1 y_i^j} \right)$$

$$= \sum_{m \geq 0} \theta_m u^m \left( \sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^{d} \left( x - \frac{x}{1-x u_1 y} y_i^{j-1} \frac{x}{1-x u_1 y} \right) \right)$$

$$- \sum_{n=0}^{r-d} \theta_n \left( \sum_{i_1, \ldots, i_d \geq 1} \prod_{j=1}^{d} \left( x - \frac{x}{1-x u_1 y} y^{i_j-1} \frac{x}{1-x u_1 y} \right) \right).$$

This identity is exactly the generating function of Theorem A.1 with respect to degree. □

**Theorem A.3** (Li [5]). On $\mathcal{S}^0[[u_1, u_2, u_3]]$, we have

$$(\tau - \text{id}) \left( \frac{x}{1-x u_1 y} \frac{1}{1-x u_3 - y u_2} \right)$$

$$= - \frac{1}{u_2 - u_3} (\Delta_{u_2} - \Delta_{u_3}) \left( \frac{x}{1-x u_1 - x u_2 + (x^2 + y x) u_1 u_2} \frac{1}{1-x u_3} (1-x u_3 - y u_3) \right)$$

$$- (1 - \Delta_{u_1}) \left( \frac{x}{1-x u_1 - x u_2 + (x^2 + y x) u_1 u_2 - y u_3} (1-x u_1 - y u_1) \frac{x}{1-x u_1 y} \right).$$

**Proposition A.4.** Corollary 4.1 (iii) and Theorem A.3 are equivalent.

**Proof.** We check that Corollary 4.1 (iii) is exactly Theorem A.3 multiplied by $u_1$ on both sides. To prove this proposition, we show the following:

$$(10) \quad \frac{1}{1-x u_3 - y u_2} \frac{1}{1-x u_3 - y u_2} = \frac{1}{1-x u_3}. \quad \text{It is easy to see that } (1-x u_3) \left( 1 - \frac{1}{1-x u_3} y u_2 \right) = 1 - x u_3 - y u_2. \text{ Thus, we obtain}$$

$$\frac{1}{1-x u_3 - y u_2} = \frac{1}{1-x u_3} y u_2 \frac{1}{1-x u_3} (1-x u_3) \left( 1 - \frac{1}{1-x u_3} y u_2 \right) \frac{1}{1-x u_3 - y u_2}$$

$$= \frac{1}{1-x u_3} y u_2 \frac{1}{1-x u_3} (1-x u_3 - y u_2) \frac{1}{1-x u_3 - y u_2}$$

$$= \frac{1}{1-x u_3} y u_2 \frac{1}{1-x u_3}. $$
Therefore, the left-hand side of Corollary 4.1 (iii) is transformed as follows:

\[
(\tau - \text{id}) \left( \frac{xu_1 y}{1 - xu_1} - \frac{1}{1 - xu_3} yu_2 - \frac{1}{1 - xu_3 - yu_2} y \right) = (\tau - \text{id}) \left( \frac{x}{1 - xu_1} - \frac{1}{1 - xu_3} yu_2 \right) u_1.
\]

The above is exactly the left-hand side of Theorem A.3 multiplied by \( u_1 \).

\[\square\]

**Remark A.5.** Similar to the method mentioned in Li [5], formally multiplying Corollary 4.1 (iii) by \( 1/u_1 \) and setting \( u_1 = 0 \) (or \( u_2 = 0 \)), we obtain the generating function of the result given by Kawasaki and Tanaka [4, Theorem (ii) (or (i))].

**Appendix B. Property Related to the Characterization**

In this appendix, we show that there is no inclusion relation between \( \mathfrak{D}_\Delta \) and \( \mathfrak{D}_{\Delta'} \) (Corollary B.2).

**Proposition B.1.** Let \((e_1, \ldots, e_s), (f_1, \ldots, f_s) \in \mathbb{Z}^s\) with \((e_1, \ldots, e_s) \neq (f_1, \ldots, f_s)\), \( \Delta = \Delta_{e_1} \circ \cdots \circ \Delta_{e_s} \) and \( \Delta' = \Delta_{f_1} \circ \cdots \circ \Delta_{f_s} \). Then, we have the following on \( \mathbb{F}[u_1, \ldots, u_s] \):

1. \( \text{Im}(\Delta - \tau) \cap \text{Im}(\Delta' - \tau) = \{0\} \).
2. \( \text{Ker}(\Delta - \tau) \cap \text{Ker}(\Delta' - \tau) = \left( \mathbb{Q}[z] \right)[u_1, \ldots, u_s] \),

where \( z = x + y \).

Reviewing Equation (2), we obtain the following property from Proposition B.1 (ii).

**Corollary B.2.**

\[ \mathfrak{D}_\Delta \cap \mathfrak{D}_{\Delta'} = \left( \mathbb{Q}[z] \right)[u_1, \ldots, u_s]. \]

In particular,

\[ \mathfrak{D}^0_\Delta \cap \mathfrak{D}^0_{\Delta'} = \mathbb{Q}[u_1, \ldots, u_s]. \]

**Remark B.3.** Recalling that \( \mathfrak{D}_\Delta \) obviously includes \( \left( \mathbb{Q}[z] \right)[u_1, \ldots, u_s] \), we find that \( \mathfrak{D}_\Delta \) and \( \mathfrak{D}_{\Delta'} \) have no common element, except for trivial elements. That is, there is no inclusion relation between \( \mathfrak{D}_\Delta \) and \( \mathfrak{D}_{\Delta'} \).

In order to prove Proposition B.1, we show three lemmas. Let \( \mathbb{Q}(x, z) \) be the non-commutative polynomial algebra over the rational in indeterminates \( x \) and \( z \), where \( z = x + y \). First, we introduce the lexicographic order \( 1 < x < z \) for words of \( \mathbb{Q}(x, z) \). Here, the empty word is 1. For a nonzero polynomial \( p(x, z) \in \mathbb{Q}(x, z) \), let \( \text{LW}(p(x, z)) \) denote the largest word for which the coefficient in \( p(x, z) \) is nonzero. Note that we are not concerned with the values of the coefficients.

**Lemma B.4.** Let \( n \) be a positive integer.

1. For a word \( w \in \mathbb{Q}(x, z) \), we have

\[ \text{LW}(\partial_n(xw)) = xz^n w. \]

2. For words \( w_1, w_2 \in \mathbb{Q}(x, z) - \mathbb{Q}[z] \) with \( w_1 < w_2 \), we have

\[ \text{LW}(\partial_n(w_1)) < \text{LW}(\partial_n(w_2)). \]
Proof. From the definition of $\partial_n$, it is obvious that $\partial_n(x) = xz^{n-1}(z - x)$ and $\partial_n(z) = 0$. In the case of $w \in \mathbb{Q}[z]$, (i) is valid. In the other case, it suffices to check $LW(x\partial_n(w)) < LW(\partial_n(x)w)$. We see
\begin{equation}
LW(\partial_n(x)w) = LW(xz^{n-1}(z - x)w) = xz^nw.
\end{equation}
For the term $x\partial_n(w)$, since there exists a non-negative integer $m$ such that $z^mxw' = w$ ($w' \in \mathbb{Q}[x, z]$), we have
\[
\begin{align*}
x\partial_n(w) &= x\partial_n(z^mxw') \\
&= xz^m\partial_n(xw') \\
&= xz^m\partial_n(x)w' + xz^m\partial_n(w') \\
&= xz^mxz^{n-1}(z - x)w' + xz^m\partial_n(w').
\end{align*}
\]
Note that the above equation holds even if $m = 0$. By comparing the $m + 2$th letter from the left, $LW(x\partial_n(w)) < LW(\partial_n(x)w)$ follows. Therefore, we obtain
\[
LW(\partial_n(xw)) = LW(\partial_n(x)w + x\partial_n(w)) = LW(\partial_n(x)w) = xz^nw.
\]
Next, we prove (ii) in three cases. First, in the case of $w_1 = xw'_1$ and $w_2 = xw'_2$ with $w'_1 < w'_2$, we obtain the following using (i):
\[
LW(\partial_n(xw'_1)) = xz^nw'_1 < xz^nw'_2 = LW(\partial_n(xw'_2)).
\]
Second, in the case of $w_1 = xw'_1$ and $w_2 = zw'_2$, we obtain
\[
LW(\partial_n(xw'_1)) = xz^nw'_1 < zLW(\partial_n(w'_2)) = LW(\partial_n(zw'_2)).
\]
Finally, in the case of $w_1 = z^mxw'_1$ and $w_2 = z^mw'_2$, where $m$ is a positive integer and $xw'_1 < w'_2$, then
\[
LW(\partial_n(z^mxw'_1)) = z^mLW(\partial_n(xw'_1))
\]
and
\[
LW(\partial_n(z^mw'_2)) = z^mLW(\partial_n(w'_2)).
\]
Now, applying other cases, we obtain $LW(\partial_n(xw'_1)) < LW(\partial_n(w'_2))$, and so (ii) is proved. \qed

Lemma B.5. For any positive integer $n$, we have the following on $\mathbb{F}[u_1, \ldots, u_s]$: 
\[
\text{Ker}(\partial_n) = (\mathbb{Q}[z])[u_1, \ldots, u_s].
\]
Proof. It is trivial that $\text{Ker}(\partial_n) \supset (\mathbb{Q}[z])[u_1, \ldots, u_s]$. We check that the inclusion relation $\text{Ker}(\partial_n) \subset (\mathbb{Q}[z])[u_1, \ldots, u_s]$. By using $\mathbb{F} \simeq \mathbb{Q}(x, z)$, the elements of $\mathbb{F}[u_1, \ldots, u_s]$ can be discussed as elements of $(\mathbb{Q}(x, z))[u_1, \ldots, u_s]$. Any element can be expressed by $\sum_{i=0}^\infty w_i$ satisfying $\deg_{x,z}(w_i) = i$ unless $w_i = 0$, and so can each element of $\text{Ker}(\partial_n)$. Given $\sum_{i=0}^\infty w_i \in \text{Ker}(\partial_n)$, we prove $w_i \in (\mathbb{Q}[z])[u_1, \ldots, u_s]$ by induction on $i$. It is obvious that
$w_0 \in (\mathbb{Q}[z])[u_1, \ldots, u_s]$, and that $\partial_i(w_i) = 0$ for any positive integer $i$. Assume this holds in the case of $i - 1$ for some $i \geq 1$. There are $c_{a_1, \ldots, a_i} \in \mathbb{Q}[u_1, \ldots, u_s]$ such that

$$w_i = \sum_{a_1, \ldots, a_i \in \{x, z\}} c_{a_1, \ldots, a_i} a_1 \cdots a_i.$$ 

By applying $\partial_n$ to the above expression and using $\partial_n(zw) = z\partial(w)$ ($w \in \mathfrak{N}$), we have

$$\partial_n(w_i) = \partial_n \left( \sum_{a_1, \ldots, a_i \in \{x, z\}} c_{a_1, \ldots, a_i} a_1 \cdots a_i \right)$$

$$= \partial_n \left( \sum_{a_2, \ldots, a_i \in \{x, z\}} c_{x, a_2, \ldots, a_i} x a_2 \cdots a_i \right) + z\partial_n \left( \sum_{a_2, \ldots, a_i \in \{x, z\}} c_{z, a_2, \ldots, a_i} a_2 \cdots a_i \right).$$

The first letters of the first and second terms are $x$ and $z$, respectively. Since the first letters of the two terms are different, we find that $\partial_n(w_i) = 0$ if and only if

$$\begin{cases} 
\partial_n \left( \sum_{a_2, \ldots, a_i \in \{x, z\}} c_{x, a_2, \ldots, a_i} x a_2 \cdots a_i \right) = 0, \\
z\partial_n \left( \sum_{a_2, \ldots, a_i \in \{x, z\}} c_{z, a_2, \ldots, a_i} a_2 \cdots a_i \right) = 0.
\end{cases}$$

By the induction hypothesis applied to the second equation, we obtain

$$\sum_{a_2, \ldots, a_i \in \{x, z\}} c_{z, a_2, \ldots, a_i} a_2 \cdots a_i \in (\mathbb{Q}[z])[u_1, \ldots, u_s].$$

This means that $c_{z, a_2, \ldots, a_i} = 0$ unless $a_2 = \ldots = a_i = z$. On the other hand, we derive a contradiction from the first equation to prove $c_{x, a_2, \ldots, a_i} = 0$. Suppose that at least one of $c_{x, a_2, \ldots, a_i}$ is nonzero. Let words $v_1 < \ldots < v_l$ with degree $i - 1$ for a positive number $l$, then $\text{LW}(\partial_n(xv_1)) < \ldots < \text{LW}(\partial_n(xv_l))$ follows from Lemma B.4. Now, there exists the largest word of $x a_2 \cdots a_i$ such that $c_{x, a_2, \ldots, a_i}$ is nonzero. Let $x b_2 \cdots b_l$ be this word. Since $\text{LW}(\partial_n(xb_2 \cdots b_l))$ is larger than any other $\text{LW}(\partial_n(xa_2 \cdots a_i))$, we have that

$$\text{LW} \left( \partial_n \left( \sum_{a_2, \ldots, a_i \in \{x, z\}} c_{x, a_2, \ldots, a_i} x a_2 \cdots a_i \right) \right) = \text{LW}(\partial_n(xb_2 \cdots b_l)).$$

However, it is not zero, which is a contradiction. Therefore, we obtain that $c_{a_1, \ldots, a_i} = 0$, unless $a_1 = \cdots = a_i = z$, i.e., $w_i \in (\mathbb{Q}[z])[u_1, \ldots, u_s].$ \[ \square \]

**Lemma B.6.** For integers $e_1, \ldots, e_s$, let $\Delta = \Delta_{u_1}^{e_1} \circ \cdots \circ \Delta_{u_s}^{e_s}$ on $\mathfrak{N}[u_1, \ldots, u_s]$. If $(e_1, \ldots, e_s) \neq (0, \ldots, 0)$, then we have

$$\text{Ker}(\Delta - \text{id}) = (\mathbb{Q}[z])[u_1, \ldots, u_s].$$

**Proof.** It is trivial that $\text{Ker}(\Delta - \text{id}) \supset (\mathbb{Q}[z])[u_1, \ldots, u_s]$, and so we check $\text{Ker}(\Delta - \text{id}) \subset (\mathbb{Q}[z])[u_1, \ldots, u_s]$. We may assume $\Delta = \Delta_{u_1}^{e_1} \circ \cdots \circ \Delta_{u_p}^{e_p} \circ \Delta_{u_{p+1}}^{-e_{p+1}} \circ \cdots \circ \Delta_{u_{p+q}}^{-e_{p+q}}$.
\((e_1, \ldots, e_{p+q} > 0, 1 \leq p + q \leq s)\) without loss of generality. Then, by replacing \(w \in \mathfrak{S}[u_1, \ldots, u_s]\) with \(\Delta_{u_{p+1}} \cdot \Delta_{u_{p+2}} \cdots \Delta_{u_{p+q}}(w)\), it suffices to prove the following:

\[
\text{Ker } (\Delta_{u_1} \cdot \Delta_{u_2} \cdots \Delta_{u_{p+1}} \cdot \Delta_{u_{p+2}} \cdots \Delta_{u_{p+q}}) \subset (\mathbb{Q}[z])[u_1, \ldots, u_s].
\]

Now, by focusing on the part of total degree 1 in \(u_1, \ldots, u_s\), each \(\Delta_{u_i} e_i\) is expanded as follows:

\[
\begin{align*}
\Delta_{u_1} e_1 \cdots \Delta_{u_p} e_p - \Delta_{u_{p+1}} e_{p+1} \cdots \Delta_{u_{p+q}} e_{p+q} &= \prod_{i=1}^{p} \left( \text{id} + \left( \frac{e_i}{1} \right) \theta_1 u_i + \left( \frac{e_i}{2} \right) \theta_1^2 u_i^2 + \cdots \right) \\
&\quad - \prod_{j=1}^{q} \left( \text{id} + \left( \frac{e_{p+j}}{1} \right) \theta_1 u_{p+j} + \left( \frac{e_{p+j}}{2} \right) \theta_1^2 u_{p+j}^2 + \cdots \right) \\
&= \text{id} + \sum_{i=1}^{p} e_i \theta_1 u_i + \text{(the total degree 2 or higher part)} \\
&\quad - \sum_{j=1}^{q} e_{p+j} \theta_1 u_{p+j} + \text{(the total degree 2 or higher part)} \\
&= \sum_{i=1}^{p} e_i \theta_1 u_i - \sum_{j=1}^{q} e_{p+j} \theta_1 u_{p+j} + \text{(the total degree 2 or higher part)},
\end{align*}
\]

where \(\prod\) implies the composition of maps. For an element \(w\) in the kernel, let \(m_0\) be the lowest degree of all words with respect to \(x\) and \(y\). Setting \(w = \sum_{m \geq m_0} w_m (w_m \in \mathfrak{S}[u_1, \ldots, u_s])\) with \(\text{deg}_{x,y}(w_m) = m\) unless \(w_m = 0\), we have

\[
\sum_{i=1}^{p} e_i \sum_{m \geq m_0} \theta_1(w_m) u_i - \sum_{j=1}^{q} e_{p+j} \sum_{m \geq m_0} \theta_1(w_m) u_{p+j} + \text{(the total degree 2 or higher part)} = 0.
\]

Focusing the lowest-degree part in \(x\) and \(y\), we obtain

\[
\sum_{i=1}^{p} e_i \theta_1(w_{m_0}) u_i - \sum_{j=1}^{q} e_{p+j} \theta_1(w_{m_0}) u_{p+j} = 0.
\]

The above equation is transformed as follows:

\[
\theta_1(w_{m_0}) \left( \sum_{i=1}^{p} e_i u_i - \sum_{j=1}^{q} e_{p+j} u_{p+j} \right) = 0.
\]

Since \((e_1, \ldots, e_s) \neq (0, \ldots, 0)\) and \(\mathfrak{S}[u_1, \ldots, u_s]\) is an integral domain, we see \(\theta_1(w_{m_0}) = 0\). By using Lemma B.5 and \(\theta_1 = \partial_1\), we obtain that \(w_{m_0} \in (\mathbb{Q}[z])[u_1, \ldots, u_s]\) holds. Now, it is easy to see that

\[
(\Delta - \text{id})(w - w_{m_0}) = 0.
\]
The lowest degree of all words appearing in \( w - w_{m_0} \) is higher than \( m_0 \), so we obtain \( w \in (\mathbb{Q}[z])[u_1, \ldots, u_s] \) by applying the same argument to \( w - w_{m_0} \) recursively. \( \square \)

Proposition \[B.1\] is proved by using Lemma \[B.6\].

**Proof of Proposition \[B.1\]** We prove Proposition \[B.1\] (i). By applying Lemma 5.2 (i) to the left-hand side, it suffices to check

\[
\ker(\Delta^{-1} + \tau) \cap \ker(\Delta'^{-1} + \tau) = \{0\}.
\]

Suppose \( w \in \ker(\Delta^{-1} + \tau) \cap \ker(\Delta'^{-1} + \tau) \), then we see

\[
(\Delta^{-1} + \tau)(w) = (\Delta'^{-1} + \tau)(w) = 0.
\]

Therefore, we have

\[
\Delta(w) = -\tau(w) = \Delta'(w).
\]

Since it is easy to see that \( \Delta'^{-1} \circ \Delta(w) = w \), we obtain \( w \in (\mathbb{Q}[z])[u_1, \ldots, u_s] \) from Lemma \[B.6\]. Reviewing \( \Delta(z) = \tau(z) = z \), we have

\[
(\Delta^{-1} + \tau)(w) = 2w.
\]

Then, \( w = 0 \) follows from \( w \in \ker(\Delta^{-1} + \tau) \). The proof of Proposition \[B.1\] (ii) is similar and so will be omitted. \( \square \)

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