Noncommutative Yang-Mills in IIB Matrix Model

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Abstract

We show that twisted reduced models can be interpreted as noncommutative Yang-Mills theory. Based upon this correspondence, we obtain noncommutative Yang-Mills theory with D-brane backgrounds in IIB matrix model. We propose that IIB matrix model with D-brane backgrounds serve as a concrete definition of noncommutative Yang-Mills. We investigate D-instanton solutions as local excitations on D3-branes. When instantons overlap, their interaction can be well described in gauge theory and AdS/CFT correspondence. We show that IIB matrix model gives us the consistent potential with IIB supergravity when they are well separated.

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1 Introduction

A large $N$ reduced model has been proposed as a nonperturbative formulation of type IIB superstring theory[1][2]. It is defined by the following action:

$$S = -\frac{1}{g^2} \text{Tr}(\frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi]).$$  \hspace{1cm} (1.1)

It is a large $N$ reduced model of ten dimensional super Yang-Mills theory. Here $\psi$ is a ten dimensional Majorana-Weyl spinor field, and $A_\mu$ and $\psi$ are $N \times N$ Hermitian matrices. It is formulated in a manifestly covariant way which enables us to study the nonperturbative issues of superstring theory. In fact we can in principle predict the dimensionality of spacetime, the gauge group and the matter contents by solving this model. We have already initiated such investigations in [3][4]. We refer our recent review for more detailed expositions and references[5]. We also note a deep connection between our approach and noncommutative geometry[6][7].

This action can be related to the Green-Schwarz action of superstring[8] by using the semiclassical correspondence in the large $N$ limit:

$$-i[,] \rightarrow \{,\}, \quad \text{Tr} \rightarrow \int d^2 \sigma \sqrt{\hat{g}}.$$  \hspace{1cm} (1.2)

In fact eq.(1.1) reduces to the Green-Schwarz action in the Schild gauge[3][4][15]:

$$S_{\text{Schild}} = \int d^2 \sigma (\sqrt{\hat{g}} \alpha(\frac{1}{4}(X^\mu, X^\nu)^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu \{X^\mu, \psi\}) + \beta \sqrt{\hat{g}}).$$  \hspace{1cm} (1.3)

Through this correspondence, the eigenvalues of $A_\mu$ matrices are identified with the spacetime coordinates $X^\mu(\sigma)$. The $\mathcal{N}=2$ supersymmetry manifests itself in $S_{\text{Schild}}$ as

$$\delta^{(1)} \psi = -\frac{1}{2} \sigma_{\mu\nu} \Gamma^{\mu\nu} \epsilon,$$
$$\delta^{(1)} X^\mu = i\bar{\epsilon} \Gamma^\mu \psi,$$  \hspace{1cm} (1.4)

and

$$\delta^{(2)} \psi = \xi,$$
$$\delta^{(2)} X^\mu = 0.$$  \hspace{1cm} (1.5)

The $\mathcal{N}=2$ supersymmetry (1.4) and (1.3) is directly translated into the symmetry of $S$ as

$$\delta^{(1)} \psi = \frac{i}{2} [A_\mu, A_\nu] \Gamma^{\mu\nu} \epsilon,$$
$$\delta^{(1)} A_\mu = i\bar{\epsilon} \Gamma^\mu \psi.$$  \hspace{1cm} (1.6)
and
\[ \delta^{(2)} \psi = \xi, \]
\[ \delta^{(2)} A_\mu = 0. \]  \hspace{1cm} (1.7)

If we take a linear combination of \( \delta^{(1)} \) and \( \delta^{(2)} \) as
\[ \tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}, \]
\[ \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}), \]  \hspace{1cm} (1.8)
we obtain the \( \mathcal{N}=2 \) supersymmetry algebra.
\[ (\tilde{\delta}^{(i)} \tilde{\delta}^{(j)} - \tilde{\delta}^{(j)} \tilde{\delta}^{(i)}) \psi = 0, \]
\[ (\tilde{\delta}^{(i)} \tilde{\delta}^{(j)} - \tilde{\delta}^{(j)} \tilde{\delta}^{(i)}) A_\mu = 2i\bar{\epsilon} \Gamma^\mu \xi \delta_{ij}. \]  \hspace{1cm} (1.9)

The \( \mathcal{N}=2 \) supersymmetry is a crucial element of superstring theory. It imposes strong constraints on the spectra of particles. Furthermore it determines the structure of the interactions uniquely in the light-cone string field theory\[^2\]. The IIB matrix model is a nonperturbative formulation which possesses such a symmetry. Therefore it has a very good chance to capture the universality class of IIB superstring theory. These symmetry considerations force us to interpret the eigenvalues of \( A_\mu \) as the space-time coordinates. Note that our argument is independent of the D-brane interpretations which are inevitably of semiclassical nature\[^10\].

We recall the typical classical solutions of (1.1) which represent infinitely long static D-strings. When \( \psi = 0 \), the equation of motion of (1.3) is
\[ \{ X^\mu, \{ X^\mu, X^\nu \} \} = 0. \]  \hspace{1cm} (1.10)
Corresponding to this, the equation of motion of (1.4) is
\[ [A_\mu, [A_\mu, A_\nu]] = 0. \]  \hspace{1cm} (1.11)
We can easily construct a solution of (1.10), which represents a static D-string extending straight in the \( X^1 \) direction:
\[ X^0 = T_\tau, \]
\[ X^1 = \frac{L}{2\pi} \sigma, \]
\[ \text{other } X^\mu \text{'s } = 0, \]  \hspace{1cm} (1.12)
where $T$ and $L$ are large enough extensions of a D-string and
\[
0 \leq \tau \leq 1, \\
0 \leq \sigma \leq 2\pi.
\] (1.13)

Considering the relation between the commutator and the Poisson bracket, we obtain a solution of (1.11) corresponding to the above one as follows:
\[
A_0 = \frac{T}{\sqrt{2\pi n}} \hat{q} \equiv \hat{p}_0, \\
A_1 = \frac{L}{\sqrt{2\pi n}} \hat{p} \equiv \hat{p}_1, \\
\text{other } A_\mu \text{'s} = 0,
\] (1.14)

where $T$ and $L$ are large enough extensions of a D-string, and $\hat{q}$ and $\hat{p}$ are $n \times n$ Hermitian matrices having the following commutation relation and the eigenvalue distributions:
\[
[\hat{q}, \hat{p}] = i,
\] (1.15)

and
\[
0 \leq \hat{q} \leq \sqrt{2\pi n}, \\
0 \leq \hat{p} \leq \sqrt{2\pi n}.
\] (1.16)

Strictly speaking such $\hat{p}$ and $\hat{q}$ do not exist for finite values of $n$. For large values of $n$, however, we expect that (1.15) can be approximately satisfied, because it is nothing but the canonical commutation relation. As is well-known in the correspondence between the classical and quantum mechanics, the total area of the $p-q$ phase space is equal to $2\pi$ multiplied by the dimension of the representation. In this sense (1.16) indicates that $\hat{p}$ and $\hat{q}$ are $n \times n$ matrices.

The cases in which $[A_\mu, A_\nu] = c-number \equiv c_{\mu\nu}$ have a special meaning. These correspond to BPS-saturated backgrounds [11]. Indeed, by setting $\xi$ equal to $\pm \frac{1}{2} c_{\mu\nu} \Gamma^{\mu\nu} \epsilon$ in the $\mathcal{N}=2$ supersymmetry (1.6) and (1.7), we obtain the relations
\[
(\delta^{(1)} \mp \delta^{(2)}) \psi = 0, \\
(\delta^{(1)} \mp \delta^{(2)}) A_\mu = 0.
\] (1.17)

Namely, half of the supersymmetry is preserved in these backgrounds. It is possible to construct higher dimensional solutions which preserve half of the supersymmetry in an analogous way. The D-branes in IIB matrix model have been investigated in [16] [17] [18] [19].
The bosonic part of the action vanishes for the commuting matrices \((A_\mu)_{ij} = x_\mu^i \delta_{ij}\) where \(i\) and \(j\) are color indices. These are the generic classical vacuum configurations of the model. We have proposed to interpret \(x_\mu^i\) as the space-time coordinates. If such an interpretation is correct, the distributions of the eigenvalues determine the extent and the dimensionality of spacetime. Hence the structure of spacetime is dynamically determined by the theory. As we have shown in [3], spacetime exits as a single bunch and no single eigenvalue can escape from the rest. However the appearance of a smooth manifold itself is not apparent in this approach since we find four dimensional fractals in a simple approximation. Although it is very plausible that gauge theory and gravitation may appear as low energy effective theory, we are still not sure how matter fields propagate [4].

The situation drastically simplifies if we consider noncommutative backgrounds. These are the D-brane like solutions which preserve a part of SUSY. Although the ultimate relevance of these solutions to the vacuum of IIB matrix model is not clear, we can certainly test our ideas to get a realistic model for space-time and matter with these backgrounds. We can indeed show that gauge theory appears as the low energy effective theory. In the case of \(m\) coincident D-branes, we obtain noncommutative super-Yang Mills theory of 16 supercharges in the gauge group of \(U(m)\).

It is of course well-known that the low energy effective action for D-branes is super Yang-Mills theory. If we mod out the theory with the translation operator, we immediately find the corresponding super Yang-Mills theory [20][21]. Noncommutative Yang-Mills theories have been obtained by the compactification on noncummutative tori[7]. By ‘compactification’, we may modify the theory by throwing away many degrees of freedom. We are essentially left with gauge theory. It is now well perceived through AdS/CFT correspondence that gauge theory can represent gravitation in the vicinity of the D-branes[23]. However gauge theory is not capable to describe gravitation in the flat space-time far from the brane.

We point out that well-known twisted reduced models[13] are equivalent to noncommutative Yang-Mills theory. The expansion around the infinitely extended D-branes in IIB matrix model defines a twisted reduced model. Using the equivalence, we find noncommutative Yang-Mills theory in IIB matrix model. We therefore propose that IIB matrix model with D-brane backgrounds provides us a concrete definition of noncommutative Yang-Mills theory. We point out that IIB matrix model contains nonlocal degrees of freedom which can represent the gravitational interaction in the flat ten dimensional space-time far from the branes. This fact will be demonstrated by calculating the potential between a D-instanton
and an anti-D-instanton on D3-branes.

The organization of this paper is as follows. In section 2, we show that noncommutative Yang-Mills theory is equivalent to large $N$ twisted reduced models. In section 3, we apply the result of section 2 to IIB matrix model with D-brane backgrounds. We study D-instantons in section 4. Section 5 is devoted to conclusions and discussions. There we discuss the relation between the Maldacena conjecture and the IIB matrix model conjecture.

2 Noncommutative Yang-Mills as twisted reduced model

In this section, we show that noncommutative Yang-Mills theory is equivalent to twisted reduced models. Reduced models are defined by the dimensional reduction of $d$ dimensional gauge theory down to zero dimension (a point)\cite{12}. We consider $d$ dimensional $U(n)$ gauge theory coupled to adjoint matter as an example:

\[
S = -\int d^d x \frac{1}{g^2} Tr\left(\frac{1}{4}[D_\mu, D_\nu][D_\mu, D_\nu] + \frac{1}{2}\bar{\psi}\Gamma_\mu[D_\mu, \psi]\right),
\]

(2.1)

where $\psi$ is a Majorana spinor field. The corresponding reduced model is

\[
S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{1}{2}\bar{\psi}\Gamma_\mu[A_\mu, \psi]\right).
\]

(2.2)

Now $A_\mu$ and $\psi$ are $n \times n$ Hermitian matrices and each component of $\psi$ is $d$-dimensional Majorana-spinor. We expand the theory around the following classical solution.

\[
[\hat{p}_\mu, \hat{p}_\nu] = iB_{\mu\nu},
\]

(2.3)

where $B_{\mu\nu}$ are c-numbers. We remark that this commutation relation cannot be satisfied with finite size matrices of dimension $n$. In fact it is spoiled at the boundary. Nevertheless we can still assume it as long as we consider the degrees of freedom which are localized in the region far from the boundary. Although such a background is no longer a stable solution, its life time becomes arbitrary large in the large $n$ limit. We assume the rank of $B_{\mu\nu}$ to be $\tilde{d}$ and define its inverse $C^{\mu\nu}$ in $\tilde{d}$ dimensional subspace. The directions orthogonal to the subspace is called the transverse directions. $\hat{p}_\mu$ satisfy the canonical commutation relations and they span the $\tilde{d}$ dimensional phase space. The semiclassical correspondence shows that the volume of the phase space is $V_p = n(2\pi)^{\tilde{d}/2}\sqrt{detB}$. Since we identify $\hat{p}_\mu$ as momenta, the phase space corresponds to momentum space which is also called by the same name in particle physics.
We consider the product of two matrices.

\[ A_\mu = \hat{p}_\mu + \hat{a}_\mu. \]

We Fourier decompose \( \hat{a}_\mu \) and \( \psi \) fields as

\[
\hat{a} = \sum_k \hat{a}(k) \exp(iC^{\mu\nu}k_\mu\hat{p}_\nu), \\
\psi = \sum_k \psi(k) \exp(iC^{\mu\nu}k_\mu\hat{p}_\nu). \tag{2.4}
\]

The Hermiticity requires that \( \hat{a}^*(k) = \hat{a}(-k) \) and \( \hat{\psi}^*(k) = \hat{\psi}(-k) \). Let us consider the case that \( \hat{p}_\mu \) consist of \( d/2 \) canonical pairs \((\hat{p}_i, \hat{q}_i)\) which satisfy \([\hat{p}_i, \hat{q}_j] = iB\delta_{ij}\). We also assume that the solutions possess the discrete symmetry which exchanges canonical pairs \((\hat{p}_i \leftrightarrow \hat{q}_i)\) in each canonical pair. We then find \( V_p = L^d \) where \( L \) is the extension of each \( \hat{p}_\mu \). The volume of the unit quantum in phase space is \( L^d/n^d = \lambda^d \) where \( \lambda \) is the spacing of the quanta. \( B \) is related to \( \lambda \) as \( B = \lambda^2/(2\pi) \). The eigenvalues of \( \hat{p}_\mu \) are quantized in the unit of \( L/n^2/\lambda = n^{1/d} \). So we restrict the range of \( k_\mu \) as \(-n^{1/d}\lambda/2 < k_\mu < n^{1/d}\lambda/2\).

Since \( |\hat{p}_\mu| < L \), we can assume that \( k_\mu \) is quantized in the unit of \( \lambda/n^{1/d} \). So \( \sum_k \) runs over \( n^2 \) degrees of freedom which coincide with those of \( n \) dimensional Hermitian matrices.

We can construct a map from a matrix to a function as

\[ \hat{a} \rightarrow a(x) = \sum_k \hat{a}(k) \exp(ik_\mu x^\mu). \tag{2.5} \]

We consider the product of two matrices

\[
\hat{a} = \sum_k \hat{a}(k) \exp(iC^{\mu\nu}k_\mu\hat{p}_\nu), \\
\hat{b} = \sum_k \hat{b}(k) \exp(iC^{\mu\nu}k_\mu\hat{p}_\nu), \\
\hat{a}\hat{b} = \sum_{k,l} \hat{a}(k)\hat{b}(l) \exp(iC^{\mu\nu}k_\mu l_\nu) \exp(iC^{\rho\sigma}(k_\rho + l_\rho)\hat{p}_\sigma). \tag{2.6}
\]

By this construction, we obtain the \( * \) product

\[
\hat{a}\hat{b} \rightarrow a(x) \ast b(x), \\
a(x) \ast b(x) \equiv \exp(iC^{\mu\nu}\frac{\partial^2}{\partial\xi^\mu\partial\eta^\nu})a(x + \xi)b(x + \eta)|_{\xi=\eta=0}. \tag{2.7}
\]

The operation \( Tr \) over matrices can be exactly mapped on the integration over functions as

\[ Tr[\hat{a}] = \sqrt{detB}(\frac{1}{2\pi})^\frac{d}{2} \int d^dx a(x). \tag{2.8} \]

It is easy to understand this statement if we consider the two dimensional case.

\[
Tr(\exp(iC^{\mu\nu}k_\mu\hat{p}_\nu)) \\
= \int dq <q|\exp(ik_0\hat{p}/B)\exp(-ik_1\hat{q}/B)\exp(ik_0k_1/B)|q> \\
= 2\pi B\delta(k_0)\delta(k_1). \tag{2.9}
\]
From these considerations, we find the following map from matrices onto functions:

\[
\begin{align*}
\hat{a} & \rightarrow a(x), \\
\hat{a}\hat{b} & \rightarrow a(x) \ast b(x), \\
Tr & \rightarrow \sqrt{\det B} \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \int d^d x.
\end{align*}
\] (2.10)

\[P_\mu\] acts on \(\hat{a}\) as

\[
[\hat{p}_\mu, \hat{a}] = \sum_k k_\mu \tilde{a}(k) \exp(iC^{\rho\sigma} k_\rho \hat{p}_\sigma).
\] (2.11)

This motivates us to find the following correspondence:

\[
[\hat{p}_\mu + \hat{a}_\mu, \hat{o}] \rightarrow \frac{1}{i} \partial_\mu o(x) + a_\mu(x) \ast o(x) - o(x) \ast a_\mu(x),
\] (2.12)

The low energy excitations with \(|k| << \lambda\) are commutative since

\[
[\hat{a}_\mu, \hat{a}_\nu] = \sum_k \sum_l \tilde{a}(k)_{\mu} \tilde{a}(l)_\nu \exp(iC^{\rho\sigma} k_\rho \hat{p}_\sigma), \exp(iC^{\rho'\sigma'} l_\rho' \hat{p}_{\sigma'})]
\] \[
= \sum_k \sum_l \tilde{a}(k)_{\mu} \tilde{a}(l)_\nu \exp(iC^{\rho\sigma}(k_\rho + l_\rho) \hat{p}_\sigma)
\times 2i \sin \left( \frac{1}{2} C^{\rho'\sigma'} k_\rho' l_\sigma' \right).
\] (2.13)

We may interpret the newly emerged coordinate space as the semiclassical limit of \(\hat{x}^\mu = C^{\mu\nu} \hat{p}_\nu\). In such an interpretation

\[a(x) = Tr[\rho_x \hat{a}],\] (2.14)

where \(\rho_x\) denotes a density matrix localized around the eigenvalue \(x\). Semiclassically we indeed find eqs. (2.5) and (2.8) although we emphasize that eq.(2.10) is the exact correspondence.

Applying the rule eq.(2.10), the bosonic action becomes

\[
-\frac{1}{4g^2} Tr[A_\mu, A_\nu][A_\mu, A_\nu] = \frac{d^n B^2}{4g^2} - \sqrt{\det B} \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \int d^d x \frac{1}{g^2} \left( \frac{1}{4} [D_\alpha, D_\beta][D_\alpha, D_\beta] + \frac{1}{2} [D_\alpha, \varphi_\nu][D_\alpha, \varphi_\nu] + \frac{1}{4} [\varphi_\nu, \varphi_\rho][\varphi_\nu, \varphi_\rho] \right).
\] (2.15)

In this expression, the indices \(\alpha, \beta\) run over \(\tilde{d}\) dimensional world volume directions and \(\nu, \rho\) over the transverse directions. We have replaced \(a_\nu \rightarrow \varphi_\nu\) in the transverse directions. Inside
the products should be understood as $\ast$ products and hence commutators do not vanish. The fermionic action becomes

$$
\frac{1}{g^2} Tr \bar{\psi} \Gamma_{\mu}[A_{\mu}, \psi] = \sqrt{\text{det} B} \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \int d^d x \frac{1}{g^2} \left[ (\bar{\psi} \Gamma_{\alpha}[D_{\alpha}, \psi] + \bar{\psi} \Gamma_{\nu}[\varphi_{\nu}, \psi]) \ast \right].
$$

(2.16)

We therefore find noncommutative U(1) gauge theory.

In order to obtain noncommutative Yang-Mills theory with $U(m)$ gauge group, we consider new classical solutions which are obtained by replacing each element of $\hat{p}_\mu$ by the $m \times m$ unit matrix:

$$
\hat{p}_\mu \rightarrow \hat{p}_\mu \otimes \mathbf{1}_m.
$$

(2.17)

The fluctuations around this background $\hat{a}$ and $\hat{\psi}$ can be Fourier decomposed in the analogous way as in eq.(2.4) with $m$ dimensional matrices $\tilde{a}(k)$ and $\tilde{\psi}(k)$ which satisfy $\tilde{a}(-k) = \tilde{a}^\dagger(k)$ and $\tilde{\psi}(-k) = \tilde{\psi}^\dagger(k)$. It is then clear that $[\hat{p}_\mu + \hat{a}_\mu, \hat{\phi}]$ can be mapped onto the nonabelian covariant derivative $D_{\mu}\phi(x)$ once we use $\ast$ product. Applying our rule (2.10) to the action in this case, we obtain

$$
\frac{\tilde{d}nB^2}{4g^2} - \sqrt{\text{det} B} \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \int d^d x \frac{1}{g^2} \text{tr} \left[ \frac{1}{4} [D_{\alpha}, D_{\beta}] [D_{\alpha}, D_{\beta}] + \frac{1}{2} [D_{\alpha}, \varphi_{\nu}] [D_{\alpha}, \varphi_{\nu}] + \frac{1}{4} [\varphi_{\nu}, \varphi_{\rho}] [\varphi_{\nu}, \varphi_{\rho}] + \frac{1}{2} \bar{\psi} \Gamma_{\alpha}[D_{\alpha}, \psi] + \frac{1}{2} \bar{\psi} \Gamma_{\nu}[\varphi_{\nu}, \psi] \right] \ast.
$$

(2.18)

The Yang-Mills coupling is found to be $(2\pi)^{\frac{d}{2}} g^2 / B^{d/2}$. Therefore it will decrease if the density of quanta in phase space decreases with fixed $g^2$.

Although our arguments here has been in the continuum theory, it is straightforward to generalize our arguments to lattice gauge theory by replacing

$$
\exp(iC_{\mu\nu} k_{\mu}^{\min} \hat{p}_\nu) \rightarrow U_\mu,
$$

(2.19)

where the index $\mu$ should not be summed. $U_\mu$ are t'Hooft matrices whose canonical pairs satisfy

$$
U_\mu U_\nu = U_\nu U_\mu \exp \left( \frac{2\pi i}{n^{2/d}} \right),
$$

(2.20)

since $|k_\mu^{\min}| = \lambda / n^{1/d}$ as we have explained in this section. Here we need to remark on the novelty of our interpretation of twisted reduced models since it has been interpreted as large
$N$ limit of $U(N)$ gauge theory. Our innovation is that we have constructed the coordinate space in the matrices by the relation $\hat{x}^\mu = C^\mu_\nu \hat{p}_\nu$. The remarkable feature of our construction is the appearance of the coordinate space out of momentum space. It is thanks to the noncommutativity of momentum space. We may interpret that the noncommutativity effectively introduces the maximum momentum scale $\lambda$. Since $\hat{x}^\mu = C^\mu_\nu \hat{p}_\nu$, the large momentum region corresponds to large length scale in the dual $\hat{x}^\mu$ space. Thus the physics beyond this momentum scale is better understood in the dual coordinate space. Therefore the whole construction reminds us string theory and T duality. In fact we will see in the following section that the identical structure emerges in association with the D-branes in IIB matrix model.

### 3 Noncommutative Yang-Mills and D-branes

In this section we apply the results of the preceding section to D-branes. We notice that the D-string solution in eq.(1.14) is precisely the type of the backgrounds in twisted reduced models we have considered in the preceding section. As we have found in section 2, the both momentum space and coordinate space are embedded in the matrices of twisted reduced models. They are related by $\hat{x}^\mu = C^\mu_\nu \hat{p}_\nu$. This relation should be understood in the following sense. The plane wave with a wave vector $k_\mu$ corresponds to an eigenstate of $\hat{P}_\mu = [\hat{p}_\mu, ]$ with $k_\mu$ as the eigenvalue. They are commutative to each other since $|k_\mu| < \lambda$. The dual coordinate space is embedded in the large eigenvalues of the matrices which are rotated by $C^\mu_\nu$. They are also commutative. We obtain the same physics if we interpret $k_\mu$ as momenta or $C^\mu_\nu \hat{p}_\nu$ as coordinates $\hat{x}^\mu$.

We need to interpret $A_\mu$ as coordinates in IIB matrix model due to $\mathcal{N}=2$ SUSY as we have emphasized in the introduction. For this purpose, we identify the solution of IIB matrix model as $\hat{x}^\mu$ which satisfy:

$$[\hat{x}^\mu, \hat{x}^\nu] = -iC^\mu_\nu.$$

(3.1)

Now the plane waves correspond to the eigenstates of $\hat{P}_\mu$ with small eigenvalue, where $\hat{p}_\mu = B_\mu_\nu \hat{x}^\nu$. $\hat{x}^\mu$ and $\hat{p}_\nu$ satisfy the canonical commutation relation: $[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu$. We expand $A_\mu = \hat{x}_\mu + \hat{a}_\mu$ as before and $\hat{a}_\mu$ and $\hat{\psi}$ can be Fourier decomposed as in

$$\hat{a} = \sum_k \hat{a}(k) \exp(ik_\mu \hat{x}^\mu),$$

$$\hat{\psi} = \sum_k \hat{\psi}(k) \exp(ik_\mu \hat{x}^\mu).$$

(3.2)
The space-time translation is realized by the following Unitary operator:

\[
\exp(i\hat{p}_\nu d^\nu)(\hat{x}^\mu + \hat{a}^\mu)\exp(-i\hat{p}_\nu d^\nu) = \hat{x}^\mu + d^\mu + \sum_k \tilde{a}(k)\exp(ik_\nu d^\nu)\exp(ik_\rho \hat{x}^\rho). \tag{3.3}
\]

We find that \( \tilde{a}(k) \) is multiplied by the phase \( \exp(ik_\nu d^\nu)\tilde{a}(k) \).

Once the eigenvalues of \( \hat{P}_\mu \) are identified with \( k_\mu \), the coordinate space has to be embedded in the rotated matrices as we have seen in section 2. If we identify the large eigenvalues of \( A^\mu \) as the coordinates \( x^\mu \), we have to rotate the covariant derivatives as follows:

\[
[\hat{x}^\mu + \hat{a}^\mu, \hat{\theta}] \rightarrow C^{\mu
u}(\frac{1}{i}\partial_\nu o(x) + b_\nu(x) \ast o(x) - o(x) \ast b_\nu(x)). \tag{3.4}
\]

Note that we have defined a new gauge field \( b_\mu(x) \) by this expression. We can map the matrices onto functions by using the rule eq.(2.10) which is derived in the preceding section.

We consider the gauge invariance on D-string. The IIB matrix model is invariant under the Unitary transformation: \( A_\mu \rightarrow UA_\mu U^\dagger, \psi \rightarrow U\psi U^\dagger \). As we shall see, the gauge symmetry can be embedded in the \( U(n) \) symmetry. We expand \( U = \exp(i\tilde{\lambda}) \) and parameterize

\[
\tilde{\lambda} = \sum_k \tilde{\lambda}(k)\exp(ik_0 \hat{x}^0 + ik_1 \hat{x}^1). \tag{3.5}
\]

Under the gauge transformation, we find the fluctuations around the fixed background transform as

\[
\hat{a}^\mu \rightarrow \hat{a}^\mu + i[\hat{x}^\mu, \hat{\lambda}] - i[\hat{a}^\mu, \hat{\lambda}]
= \sum_k \exp(ik_0 \hat{x}^0 + ik_1 \hat{x}^1)(\tilde{a}(k)^\mu + iC^{\mu\nu}k_\nu \tilde{\lambda}(k))
+ \sum_k \sum_l \tilde{a}(k)^\mu \tilde{\lambda}(l)\exp(i(k_0 + l_0) \hat{x}^0 + i(k_1 + l_1) \hat{x}^1) \frac{i(k_0 l_1 - k_1 l_0)}{B} + \cdots. \tag{3.6}
\]

\[
\hat{\psi} \rightarrow \hat{\psi} - i[\hat{\psi}, \hat{\lambda}]
= \sum_k \exp(ik_0 \hat{x}^0 + ik_1 \hat{x}^1)\tilde{\psi}(k)
+ \sum_k \sum_l \tilde{\psi}(k)\tilde{\lambda}(l)\exp(i(k_0 + l_0) \hat{x}^0 + i(k_1 + l_1) \hat{x}^1) \frac{i(k_0 l_1 - k_1 l_0)}{B} + \cdots. \tag{3.7}
\]

We interpret the above result as

\[
b_\alpha(x) \rightarrow b_\alpha(x) + \frac{\partial}{\partial x^\alpha}\theta(x) + \eta_\beta \frac{\partial b_\alpha(x)}{\partial x^\beta} + \cdots,
\]

\[
a_\nu(x) \rightarrow a_\nu(x) + \eta_\beta \frac{\partial a_\nu(x)}{\partial x^\beta} + \cdots,
\]

\[
\psi(x) \rightarrow \psi(x) + \eta_\beta \frac{\partial \psi(x)}{\partial x^\beta} + \cdots, \tag{3.8}
\]
where $\eta_{\alpha} = \epsilon^{\alpha\beta}\partial_{\beta}\lambda(x)$. We indeed find $U(1)$ gauge group in the commutative limit. The leading corrections in $1/B$ represent the volume preserving diffeomorphism.

Applying the rule eq.(2.10), the bosonic action becomes

$$-\frac{1}{4g^2}Tr[A_\mu, A_\nu][A_\mu, A_\nu]$$

$$= \frac{LT}{B4\pi g^2} - \frac{1}{B^22\pi g^2} \int d^2x \left( \frac{1}{4B}[D_\alpha, D_\beta][D_\alpha, D_\beta] - \frac{B}{4}[\varphi_\alpha, \varphi_\beta][\varphi_\alpha, \varphi_\beta] \right),$$

(3.9)

where $D_\alpha = \partial/(i\partial x^\alpha) + b_\alpha$ and $a_\nu = \sqrt{1/B}\varphi_\nu$. The fermionic action becomes

$$Tr\bar{\psi}\Gamma_\mu[A_\mu, \psi]$$

$$= \int d^2x Tr(\bar{\psi}\Gamma_\alpha[D_\alpha, \psi] + \sqrt{B}\bar{\psi}\Gamma_\nu[\varphi_\nu, \psi] \star),$$

(3.10)

where $\Gamma_\alpha = \epsilon_{\alpha\beta}\tilde{\Gamma}_\beta$. We therefore find noncommutative two dimensional $\mathcal{N}=8 U(1)$ gauge theory.

We next consider $m$ parallel D-strings. We can construct such a solution $A^d_\mu$ by replacing each element of a D-string solution by an $m$ by $m$ unit matrix. As before we decompose $A_\mu = A^d_\mu + \hat{a}_\mu$ where each element of $\hat{a}_\mu$ is now an $m$ by $m$ matrix. We can simply apply the generalized rule which implies $[A_\alpha, \ ] \to C^{\alpha\beta}D_\beta$ where $D_\alpha$ is the nonabelian covariant derivative. The bosonic action becomes

$$-\frac{1}{4g^2}Tr[A_\mu, A_\nu][A_\mu, A_\nu]$$

$$= \frac{mLT}{B4\pi g^2} - \frac{1}{B^22\pi g^2} \int d^2x Tr\left( \frac{1}{4B}[D_\alpha, D_\beta][D_\alpha, D_\beta] - \frac{B}{4}[\varphi_\alpha, \varphi_\beta][\varphi_\alpha, \varphi_\beta] \right),$$

(3.11)

The symbol $tr$ implies taking the trace over $U(m)$ gauge group. The fermionic action becomes

$$Tr\bar{\psi}\Gamma_\mu[A_\mu, \psi]$$

$$= \int d^2x Tr(\bar{\psi}\Gamma_\alpha[D_\alpha, \psi] + \sqrt{B}\bar{\psi}\Gamma_\nu[\varphi_\nu, \psi] \star).$$

(3.12)

We therefore find two dimensional $\mathcal{N}=8$ super Yang-Mills theory with $U(m)$ gauge group.

We move on to consider higher dimensional D-branes. A D3-brane may be constructed as follows:

$$A_0 = \frac{T}{\sqrt{2\pi n_1}}\hat{q} \equiv \hat{x}^0;$$

11
\[ A_1 = \frac{L}{\sqrt{2\pi n_1}} \hat{p} \equiv \hat{x}^1, \]
\[ A_2 = \frac{L}{\sqrt{2\pi n_2}} \hat{q}' \equiv \hat{x}^2, \]
\[ A_3 = \frac{L}{\sqrt{2\pi n_2}} \hat{p}' \equiv \hat{x}^3, \]
\[ \text{other } A_\mu's = 0, \quad (3.13) \]

which may be embedded into \( n_1 n_2 \) dimensional matrices. We can further consider \( m \) parallel D3-branes after replacing each element of the D3-brane solution by \( m \) by \( m \) unit matrix. Under the replacements \( A_\alpha \rightarrow C^{\alpha \beta} D_\beta \) and \( A_\nu \rightarrow (1/B) \varphi_\nu \), the bosonic action becomes
\[
-\frac{1}{4g^2} Tr [A_\mu, A_\nu] [A_\mu, A_\nu]
= \frac{mTL^3}{(2\pi)^2 g^2} - \frac{1}{B^2 (2\pi)^2 g^2} \int d^4x \text{tr} \left( \frac{1}{4} [D_\alpha, D_\beta] [D_\alpha, D_\beta] \right)
+ \frac{1}{2} [D_\alpha, \varphi_\nu] [D_\alpha, \varphi_\nu] + \frac{1}{4} [\varphi_\nu, \varphi_\rho] [\varphi_\nu, \varphi_\rho])_*, \quad (3.14)
\]

where we integrate over the four dimensional world volume of D3-branes. As for the fermionic action, we find
\[
Tr \bar{\psi} \Gamma_\mu [A_\mu, \psi]
= \int d^4x \text{tr} (\bar{\psi} \tilde{\Gamma}_\alpha [D_\alpha, \psi] + \bar{\psi} \Gamma_\nu [\varphi_\nu, \psi])_*. \quad (3.15)
\]

We thus find four dimensional \( \mathcal{N}=4 \) super Yang-Mills theory. The Yang-Mills coupling is found to be \( g^2 B^2 \). Recall that \((2\pi/B)^2 = R^4\) is the unit volume of a quantum and \( R \) is the average spacing. If we consider a background with larger \( R \) for fixed \( g^2 \), we find that the Yang-Mills coupling decreases.

In this section, we have obtained noncommutative Yang-Mills theory with D-brane backgrounds in IIB matrix model to all orders in power series of \( 1/B \). Alternatively we can view the IIB matrix model with D-brane backgrounds as a concrete definition of noncommutative Yang-Mills theory.

### 4 D-instantons in IIB matrix model

As we have shown in section 3, the low energy effective theory with D3-backgrounds are super Yang-Mills theory. In super Yang-Mills theory, there are local nontrivial solutions, namely instantons. The equation of motion in IIB matrix model is
\[
[A_\mu, [A_\mu, A_\nu]] = 0. \quad (4.1)
\]
With our substitution rule, \( A_\alpha \to C^{\alpha \beta} D_\beta \), we obtain,

\[
[D_\alpha, [D_\alpha, D_\beta]] = 0. \tag{4.2}
\]

Since the instantons are nontrivial solutions of gauge theory, they must become those of IIB matrix model after ramifications at short distances. As we have shown, short distance modification of Yang-Mills theory in IIB matrix model is to render it noncommutative. In fact such solutions on noncommutative \( R^4 \) are constructed in [22].

The ’t Hooft solution is

\[
A_\mu(x) = i \Sigma_{\mu \nu} \Phi(x)^{-1} \partial_\nu \Phi(x),
\]

\[
\Phi(x) = 1 + \rho^2 \frac{|x - x_i|^2}{},
\tag{4.3}
\]

where \( \Sigma_{\mu \nu} \) is self-dual in \( \mu \nu \) and takes values in traceless two by two Hermitian matrices. The location of the instanton is denoted by \( x_i \) and \( \rho \) is its size. The prescription is just replace \( \Phi(x) \) by its noncommutative analog. Although it is an interesting problem to study small instantons whose size is comparable to \( R \), it suffices to consider instantons whose size is much larger than \( R \) in this section.

Let us consider an instanton solution first. The classical value of the action is

\[
S = \frac{2TL^3}{g^2(2\pi)^2} + \frac{1}{B^2g^2}. \tag{4.4}
\]

We interpret the first and the second term of eq.(4.4) as the action of two D3-branes and that of a D-instanton respectively. It preserves one fourth of the supersymmetry and hence eq.(4.4) receives no quantum corrections. In order to see whether the solution preserves a part of supersymmetry, we consider

\[
\delta^{(1)} \psi = \frac{i}{2} [A_\mu, A_\nu] \Gamma_{\mu \nu} \epsilon,
\]

\[
= \frac{i}{2} (-iC^{\alpha \beta} + C^{\alpha \gamma}C^{\beta \delta} [D_\gamma, D_\delta] \frac{1 \mp \Gamma^5}{2}) \Gamma_{\alpha \beta} \epsilon,
\]

\[
\delta^{(2)} \psi = \xi \tag{4.5}
\]

where \( \mp \) correspond to self-dual and anti self-dual field strengths respectively. Therefore an (anti)instanton solution preserves one fourth of the supersymmetry which satisfy

\[
\Gamma^5 \epsilon = \pm \epsilon,
\]

\[
\xi = \frac{1}{2} C^{\alpha \beta} \Gamma_{\alpha \beta} \epsilon. \tag{4.6}
\]
This argument is valid for generic (anti)self-dual field configurations.

We next consider a classical solution of IIB matrix model which represents an instanton and an (anti)instanton. We can realize $U(4)$ gauge theory by considering four D3 branes. We embed an instanton into the first $SU(2)$ part and the other (anti)instanton into the remaining $SU(2)$ part. We separate them in the fifth dimension by the distance $b$:

$$
A_0 = \begin{pmatrix}
\hat{x}^0 + a_0 & 0 \\
0 & \hat{x}^0 + a'_0
\end{pmatrix},
$$

$$
A_1 = \begin{pmatrix}
\hat{x}^1 + a_1 & 0 \\
0 & \hat{x}^1 + a'_1
\end{pmatrix},
$$

$$
A_2 = \begin{pmatrix}
\hat{x}^2 + a_2 & 0 \\
0 & \hat{x}^2 + a'_2
\end{pmatrix},
$$

$$
A_3 = \begin{pmatrix}
\hat{x}^3 + a_3 & 0 \\
0 & \hat{x}^3 + a'_3
\end{pmatrix},
$$

$$
A_4 = \begin{pmatrix}
b & 0 \\
0 & -\frac{b}{2}
\end{pmatrix},
$$

$$
A_\rho = 0,
$$

where $\rho = 5, \ldots, 9$.

The classical action is twice of eq.(4.4). While two instanton system receives no quantum corrections, the instanton - anti-instanton system receives quantum corrections since it is no longer BPS. We now evaluate the one loop effective potential due to an instanton and (anti)instanton. Since they are local excitations, they must couple to gravity. These solutions are characterized by the adjoint field strength $F_{\mu\nu}$ which does not vanish at the locations of the instantons. Let us assume that they are separated by a long distance compared to their sizes. We also assume that $b >> R$. Then we can choose two disjoint blocks in each of which a large part of an (anti)instanton is contained. Let the location and the size of instantons $(x_i, \rho_i)$ and $(x_j, \rho_j)$. The ten dimensional distance of them is $r^2 = (x_i - x_j)^2 + b^2$. Here we have assumed that $r >> \rho$. The potential between the $i$-th and the $j$-th blocks due to the off-diagonal $(i, j)$ block has been calculated in [1].

$$
W^{(i,j)} = \frac{1}{r^8}(-Tr^{(i,j)}(F_{\mu\nu}F_{\nu\lambda}F_{\lambda\rho}F_{\rho\mu}) - 2Tr^{(i,j)}(F_{\mu\nu}F_{\lambda\rho}F_{\mu\rho}F_{\lambda\nu}) + \frac{1}{2}Tr^{(i,j)}(F_{\mu\nu}F_{\lambda\rho}F_{\nu\rho}F_{\lambda\mu}) + \frac{1}{4}Tr^{(i,j)}(F_{\mu\nu}F_{\lambda\rho}F_{\mu\rho}F_{\lambda\nu}) + 3 \frac{1}{2r^8}(-n_i\tilde{b}_8(f^{(i)}) - n_i\tilde{b}_8(f^{(j)}) - 8Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\nu\sigma})Tr(f^{(j)}_{\mu\nu}f^{(j)}_{\nu\sigma}) + Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\nu\sigma})Tr(f^{(j)}_{\mu\nu}f^{(j)}_{\nu\sigma}) + Tr(f^{(i)}_{\mu\nu}f^{(i)}_{\mu\nu})Tr(f^{(j)}_{\mu\nu}f^{(j)}_{\mu\nu})),$$

(4.8)
where $\tilde{f}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}/2$ and

$$\tilde{b}_8(f) = \frac{2}{3} Tr(f_{\mu\nu} f_{\nu\lambda} f_{\lambda\rho} f_{\rho\mu}) + 2 Tr(f_{\mu\nu} f_{\lambda\rho} f_{\mu\lambda} f_{\nu\rho}) - \frac{1}{2} Tr(f_{\mu\nu} f_{\mu\nu} f_{\lambda\rho} f_{\lambda\rho}) - \frac{1}{4} Tr(f_{\mu\nu} f_{\mu\nu} f_{\lambda\rho} f_{\lambda\rho}).$$  \hspace{1cm} (4.9)

The novel feature of eq.(4.8) is that we have kept axion type interactions also. Note that the $\tilde{b}_8(f) = 0$ for an (anti)instanton configuration. So the potential between an instanton and an (anti)instanton is

$$\frac{3}{2r^8}(-8 Tr(f^{(i)}_{\mu\nu} f^{(i)}_{\nu\rho}) Tr(f^{(i)}_{\mu\rho} f^{(i)}_{\rho\sigma}) + Tr(f^{(i)}_{\mu\nu} f^{(i)}_{\mu\rho} Tr(f^{(i)}_{\rho\sigma} f^{(i)}_{\rho\sigma}) + Tr(f^{(i)}_{\mu\nu} \tilde{f}^{(i)}_{\mu\nu} Tr(f^{(i)}_{\rho\sigma} \tilde{f}^{(i)}_{\rho\sigma})).$$ \hspace{1cm} (4.10)

Here we can apply the low energy approximation such as

$$Tr(D^i_{\mu}, D^i_{\nu}) = \frac{1}{B^2(2\pi)^2} \int d^4x Tr(D^i_{\mu}, D^i_{\nu}) = -\frac{R^4}{\pi^2},$$

$$Tr(D^i_{\mu}, D^i_{\nu}) = \frac{1}{B^2(2\pi)^2} \int d^4x Tr(\epsilon_{\mu\rho\sigma} (D^i_{\mu}, D^i_{\nu}) (D^i_{\rho}, D^i_{\sigma}) = \pm \frac{R^4}{\pi^2},$$

$$Tr(D^i_{\mu}, D^i_{\nu}) = \frac{1}{B^2(2\pi)^2} \int d^4x Tr([D^i_{\mu}, D^i_{\nu}] [D^i_{\rho}, D^i_{\sigma}) = -\frac{R^4}{4\pi^2} \delta_{\mu\rho},$$  \hspace{1cm} (4.11)

where $D^i_{\mu}$ denotes the covariant derivative of the instanton background which is localized at the $i$-th block. So the interactions eq.(4.11) can be interpreted due to the exchange of dilaton, axion and gravitons. We find that the potential between two instantons vanish due to their BPS nature. On the other hand, the following potential is found between an instanton and an anti-instanton

$$-3 \frac{R^8}{\pi^4 r^8}.$$

(4.12)

We remark that the above approximation is no longer valid when $b < R$. In this case the interactions between an instanton and an anti-instanton is well described by the gauge fields which are low energy modes of IIB matrix model. They are close to diagonal degrees of freedom in IIB matrix model. Their contribution can be estimated by gauge theory. On the other hand when $b >> R$, the standard gauge theory description breaks down since we have to take account of the noncommutativity. In that case, we believe that the block-block interaction gives us a correct result. It is in a sense T dual descript ion to gauge theory. The one loop effective potential can be calculated by gauge theory when $b << R$

$$\Gamma = -\frac{1}{B^4(2\pi)^2 b^4} \int d^4x b_8,$$

(4.13)

where we have assumed $b \rho >> 1/B$. The above expression is estimated as follows:

$$\Gamma = \frac{-144}{B^4 \pi^2 b^4} \int d^4x \frac{\rho^4}{((x-y)^2 + \rho^2)^4} \frac{\rho^4}{((x-z)^2 + \rho^2)^4}$$
where $r = |y - z|$ is assumed to be much larger than $\rho$. We note that eq. (4.14) falls off with the identical power for large $r$ with eq. (4.12).

In $AdS$/CFT correspondence, the instanton size is interpreted as the radial coordinate of a D-instanton in $AdS_5$ [24]. With this interpretation, the D-instanton approaches the boundary of $AdS_5$ when the instanton size vanishes. The gravitational interaction between D-instantons in flat space is known to be of the form $\alpha'^4/r^8$. If we assume that $\alpha' \sim 1/B$ and $g^2 \sim g_s \alpha'^2$, we find that eq. (4.12) indeed of such a type. In $AdS_5$, it is modified as

$$\frac{\alpha'^4}{L^8 ((y - z)^2 + (\rho - \rho')^2)^4},$$

where $L$ is the radius of $AdS_5$. Here we notice a limitation of gauge theory. We cannot describe gravitational interaction between a D-instanton and an anti-D-instanton in flat space far from the branes by gauge theory. On the other hand the IIB matrix model can describe the gravitational interaction between them in both regions near and far from the branes. Hence we find an important advantage of IIB matrix model over gauge theory as a nonperturbative formulation of superstring. It is due to the existence of nonlocal (or noncommutative) degrees of freedom in IIB matrix model.

We have proposed that the fundamental strings are created by the Wilson loop operators. Simple examples are the following vertex operators for a dilaton, an axion and gravitons which are consistent with eq. (4.10):

$$Tr\{[A_\alpha, A_\beta][A_\alpha, A_\beta]\exp(ik\cdot A_\gamma)\} + \text{fermionic terms},$$
$$\epsilon_{\alpha\beta\gamma\delta} Tr\{[A_\alpha, A_\beta][A_\gamma, A_\delta]\exp(ik\cdot A_{\gamma'})\} + \text{fermionic terms},$$
$$Tr\{[A_\alpha, A_\mu][A_\mu, A_\beta]\exp(ik\cdot A_\gamma)\} + \text{fermionic terms}. \quad (4.16)$$

We have the corresponding vertex operators in CFT:

$$\int d^4x\exp(ik \cdot x) + \text{fermionic terms},$$
$$\int d^4x\epsilon_{\alpha\beta\gamma\delta} \exp(ik \cdot x) + \text{fermionic terms},$$
$$\int d^4x\exp(ik \cdot x) + \text{fermionic terms}. \quad (4.17)$$

Under the replacements $A_\alpha \to C^{\alpha\beta} D_\beta, A_\mu \to 1/B\varphi_\nu$, the Wilson loops in IIB matrix model coincide with those in CFT such as

$$Tr\{[A_\alpha, A_\mu][A_\mu, A_\beta]\exp(ik(\hat{x}_\gamma + \hat{a}_\gamma))\}$$
\rightarrow \int d^4x tr \{ [D_\alpha, D_\gamma][D_\gamma, D_\beta] + [D_\alpha, \varphi_\nu][D_\beta, \varphi_\nu] \} \exp(ik \cdot x) + \cdots, \quad (4.18)

where we have replaced $\hat{x} \rightarrow x$ in the argument of the exponential. It is interesting to investigate in more detail the above noticed relationship between the Wilson loops in IIB matrix model and the vertex operators in CFT.

5 Conclusions and discussions

In this paper we have shown that large $N$ twisted reduced models can be interpreted in terms of noncommutative Yang-Mills theory. Such a system appears in IIB matrix model as infinitely extended D-brane solutions which preserve a part of SUSY. We therefore obtain noncommutative Yang-Mills theory with such backgrounds. We propose that IIB matrix model with such a background provides us a precise definition of noncommutative Yang-Mills theory. The novel feature is that the real coordinate space and the conjugate momentum space can be embedded into matrices in the large $n$ limit. We have studied D-instanton interactions. When they overlap, their interaction is well described by gauge theory. When they are well separated in the fifth dimension, the gauge theory description breaks down. In that situation, IIB matrix model provides an accurate description and we find the result is consistent with IIB supergravity. In this sense, IIB matrix model is valid in both the gauge theory region and supergravity regions. Therefore we need not assume the overlap of the two unlike $AdS$/CFT correspondence.

Let us consider the implication of our results on Maldacena conjecture[23]. We have shown that $U(N)$ gauge theory is obtained if we consider $N$ parallel D-branes in IIB matrix model whose distances are much smaller compared to string scale. The basic conjecture of IIB matrix model is that it is a nonperturbative formulation of IIB superstring theory. Its low energy limit is IIB supergravity. The tree level string theory is considered to be obtained by summing planar diagrams and string perturbation theory is identified with the topological expansion of the matrix model. If we consider the large $N$ limit with fixed t’Hooft coupling, we are left with the planar diagrams. Therefore the Maldacena conjecture follows from our IIB matrix model conjecture. We can also point out the limitations of Maldacena conjecture. The gauge theory description is valid only when the distances between the D-branes are much smaller than the string scale. Therefore it is not applicable when the branes are separated by the distance much longer than the string scale. In other words, CFT cannot describe the flat space far from the brane. In this regard, IIB matrix model has the definite advantage
as we have demonstrated by the instanton interactions.

Although we have obtained 4 dimensional gauge theory with D-brane backgrounds, we find ten dimensional gravity. It is presumably not a bad news for our scenario to get realistic models out of IIB matrix model, since D3-brane backgrounds possess enormous vacuum energy density $(1/\alpha')^2$. This is certainly what has been expected from string perturbation theory. Although the eigenvalue distributions of $A_\mu$ are sharply peaked at four dimensional hyperplane, they presumably spread out into the transverse directions due to the massless scalars. It suggests us to consider the solutions without massless scalars (broken SUSY?) to obtain four dimensional gravity. In any case, noncommutative geometry may be a crucial element to obtain gauge theory in IIB matrix model.

We conclude this section with the following comment. Chepelev and Tseytlin argued that our D-brane solutions can be interpreted as not pure D-branes but those with nonvanishing $U(1)$ field strength[17]. In this paper we have shown that our solution can be interpreted in terms of noncommutative geometry with vanishing $U(1)$ field strength. Recently Seiberg and Witten has pointed out that noncommutative Yang-Mills is equivalent to ordinary Yang-Mills theory with a nonvanishing $B$-field or $U(1)$ field strength[25]. Apparently these arguments are consistent with each other and reflect the T duality of string theory.

Acknowledgments

The final part of this work was carried out while we got together at PIMS of UBC. We would like to thank the organizers of the conference PFS ’99:T.Lee, G. Semenoff and especially Y. Makeenko for their warm hospitality. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan.
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