First passage in discrete-time absorbing Markov chains under stochastic resetting

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First passage of stochastic processes under resetting has recently been an active research topic in the field of statistical physics. However, most of previous studies mainly focused on the systems with continuous time and space. In this paper, we study the effect of stochastic resetting on first passage properties of discrete-time absorbing Markov chains, described by a transition matrix $Q$ between transient states and a transition matrix $R$ from transient states to absorbing states. Using a renewal approach, we exactly derive the unconditional mean first passage time (MFPT) to either of absorbing states, the splitting probability the and conditional MFPT to each absorbing state. All the quantities can be expressed in terms of a deformed fundamental matrix $Z_\gamma = [I - (1 - \gamma)Q]^{-1}$ and $R$, where $I$ is the identity matrix, and $\gamma$ is the resetting probability at each time step. We further show a sufficient condition under which the unconditional MFPT can be optimized by stochastic resetting. Finally, we apply our results to two concrete examples: symmetric random walks on one-dimensional lattices with absorbing boundaries and voter model on complete graphs.

I. INTRODUCTION

First passage underlies a wide variety of stochastic phenomena that have broad applications in phase transitions, neural firing, searching processes, epidemic extinction, and consensus formation, and so on \cite{1 3}. Recently, first passages under resetting has been an active topic in the field of statistical physics (see \cite{3} for a recent review), due to its numerous applications spanning across interdisciplinary fields ranging from search problems \cite{7,8}, the optimization of randomized computer algorithms \cite{9}, and to chemical and biological processes \cite{10,11}. Resetting refers to a sudden interruption of a stochastic process followed by its starting anew.

A canonical diffusion model subject to stochastic resetting was studied by Evans and Majumdar \cite{12,13}. Resetting can produce a counterintuitive effect: it renders an infinite mean first passage time (MFPT) finite, which can be also minimized at a specific resetting rate. Some extensions have been made in the field, such as temporally or spatially dependent resetting rate \cite{13,14}, higher dimensions \cite{16,17}, complex geometries \cite{18,19}, noninstantaneous resetting \cite{21,22}, in the presence of external potential \cite{23,24}, or in the presence of multiple targets \cite{25,26}, other types of Brownian motion, like run-to-tumble particles \cite{27,28}, active particles \cite{29,30}, and so on \cite{31}. These nontrivial findings have triggered enormous recent activities in the field, including statistical physics \cite{32,33}, stochastic thermodynamics \cite{34,35}, and single-particle experiments \cite{36,37}.

An impressive advantage of resetting is its ability to accelerate the completion of a stochastic process. However, in many cases the resetting can also slow down the completion of a stochastic process. Pal and Reuveni \cite{37} derived a criterion for restart to be beneficial. They showed restart has the ability to expedite the completion of the underlying stochastic process if the relative standard deviation associated with the first passage time (FPT) without resetting is larger than one. As a corollary, if a non-zero optimal resetting rate exists, the relative standard deviation is always unity at optimality \cite{38}. The resetting criterion was also interpreted by so-called “inspection paradox” \cite{19}. The usefulness of the criterion was demonstrated in systems of a Brownian walker in a one-dimensional domain with and without force field \cite{28,50,54}. There has been realizations that so-called “resetting transition” occurs at some parameter of the underlying model, which distinguishes that resetting can either hinder or facilitate in the completion of a stochastic process. Resetting transition can be both first \cite{12,13,14} and second order \cite{18,52} like in the classical phase transition. A Landau-like theory was also used to characterize phase transitions in resetting systems \cite{52}.

Most of previous works have focused on the systems with continuous time and space. However, the impact of resetting on the systems with discrete time and space has only received less attention. Montero and Villarreal \cite{53} studies a discrete time unidirectional random walk on an infinite one-dimensional lattice subject to resetting with a random or site-dependent probability. They analyzed the FPT and survival probabilities for the walker to reach a certain threshold in the lattice. Boyer and Solis-Salas \cite{54} proposed a preferential visit model in order to incorporate the memory effect into the resetting processes. The walker either performs a random move locally or relocates to a previously visited site with a probability proportional to the number of past visits to that site. It was shown that the model generates slow sub-diffusion due to the dynamics of memory-driven resetting. The preferential visit model was further studied in the presence of a single defect site, in which an Anderson-like localization transition was observed \cite{55,56}. Majumdar et al. \cite{61}, studied analytically a simple random walk model on a one-dimensional lattice, where at each time
step the walker either resets to the maximum of the already visited positions or undergoes symmetric random walks. They found that for any nonzero resetting probability both the average maximum and the average position grow ballistically with a common velocity. Bonomo and Pal \cite{52} derived a criterion that dictates when restart remains beneficial in discrete space and time restarted processes, and then applied the result to a symmetric and a biased random walker in one-dimensional lattice confined within two absorbing boundaries. Riascos et al. \cite{62} studied random walks on arbitrary networks subject to resetting with a constant probability. They derived the exact expressions of the stationary probability distribution and the MFPT by the spectral representation of the transition matrix without resetting. Subsequently, the results are generalized to the case of multiple resetting nodes \cite{63, 64}. Wald and Böttcher \cite{65} introduced a framework for studying classical, quantum, and hybrid random walks with stochastic resetting on arbitrary networks, in which they derived analytical solutions of the occupation probability for a classical random walk. They found that for any nonzero resetting probability both the average maximum and the average position grow ballistically with a common velocity. Bonomo and Pal \cite{54} derived a criterion that dictates when restart remains beneficial in discrete space and time restarted processes.

In Sec.V we demonstrate our results by two concrete examples: symmetric random walks on one-dimensional lattices with two absorbing boundaries and voter model on trees. With these examples, we apply our results into two concrete examples: symmetric random walks on one-dimensional lattices with two absorbing boundaries and voter model on trees. With these examples, we apply our results into two concrete examples: symmetric random walks on one-dimensional lattices with two absorbing boundaries and voter model on trees.

Let us consider a discrete-time Markovian process between \( N \) different states, described by a stochastic matrix \( \mathbf{W} \) whose element \( W_{ij} \) gives the transition probability from state \( i \) to state \( j \). Among \( N \) states, there are \( m \) (\( m < N \)) different states that are the searching targets, denoted by \( \{ o_1, \ldots, o_m \} \). Once the system enters into either of targets, the searching process is terminated. For convenience, the \( m \) targets are numbered as the last \( m \) states. The transition matrix \( \mathbf{W} \) can be written in the block form,

\[
\mathbf{W} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{O} & \mathbf{I} \end{pmatrix},
\]

where \( \mathbf{Q} \) is the \( n \times n \) transition matrix between non-target (transient) states (here \( n = N - m \) is the number of transient states), and \( \mathbf{R} \) is the \( n \times m \) transition matrix from transient states to targets (absorbing states). \( \mathbf{O} \) is the null matrix and \( \mathbf{I} \) is the identity matrix.

In the present work, we aim to study the effect of stochastic resetting on first-passage properties of general absorbing Markovian networks. The underlying network is composed of absorbing nodes and transient nodes. The system starts from a transient node, and either performs random walks on the Markovian network or is reset to a given transient node with a constant probability. Once the system enters into either of absorbing nodes, the process is terminated. Using a renewal approach, we derive the exact expressions of the unconditional MFPT (uMFPT), splitting probabilities, and conditional MFPT (cMFPT). We also deduce a sufficient condition under which the uMFPT is expedited via stochastic resetting. Finally, we apply our results into two concrete examples: symmetric random walks on one-dimensional lattices with two absorbing endpoints and voter model on complete graphs.

The paper is structured as follows. In Sec.III and Sec.IV we present first passage properties of an absorbing Markovian network without and with stochastic resetting, respectively. In Sec.V we give a sufficient condition for accelerating uMPFT by stochastic resetting. In Sec.VI we demonstrate our results by two concrete examples. Finally, in Sec.VII we provide the conclusions.
\( j \)th absorbing state at time \( t \) for the first time. The first-passage probability \( f_{ij}^0(t) \) is given by
\[
f_{ij}^0(t) = (Q^{-1}R)_{ij}. \tag{8}
\]
Taking the sum for Eq. (8) over \( t \), we get the splitting (exit) probability to the \( j \)th absorbing state having started from the \( i \)th transient state,
\[
\pi_{ij}^0 = \sum_{t=1}^{\infty} f_{ij}^0(t) = \left[(I + Q + Q^2 + \cdots)R\right]_{ij} = (Z_0R)_{ij}. \tag{9}
\]
Eq. (9) states that the splitting probability \( \pi_{ij}^0 \) is equal to the \((i,j)\)-entry of the matrix \( Z_0R \).

The conditional MFPT started from the \( i \)th transient state and ending in the \( j \)th absorbing state is given by
\[
\langle \tau_{ij}^0 \rangle = \frac{1}{\pi_{ij}} \sum_{t=1}^{\infty} tf_{ij}^0(t) = \frac{1}{\pi_{ij}} \left[(I + 2Q + 3Q^2 + \cdots)R\right]_{ij} = \frac{1}{\pi_{ij}} (Z_0^2R)_{ij}. \tag{10}
\]

### III. SEARCH ON A MARKOVIAN NETWORK UNDER STOCHASTIC RESETTING

We now consider that the system may undergo a resetting process at each time step. With a constant probability \( \gamma \), the system is reset to a transient state \( r \) (different from any target). With the complementary probability \( 1 - \gamma \), the system goes from one state to another in terms of the transition matrix \( W \). As long as the system enters into any absorbing state, the process will be terminated.

Let us denote by \( S_i(t) \) the survival probability of the system starting from the \( i \)th transient state until time \( t \) in the presence of resetting, which satisfies a first renewal equation [14 41],
\[
S_i(t) = (1 - \gamma)^t S_i^0(t) + \sum_{t'=1}^{t} (1 - \gamma)^{t'-1} \gamma S_i^0(t' - 1) S_r(t - t'). \tag{11}
\]
The first term in Eq. (11) corresponds to the case where there is no resetting event up to all time \( t \), which occurs with probability \( (1 - \gamma)^t \). The second term in Eq. (11) accounts for the event where the first resetting takes place at time \( t' \), which occurs with probability \( (1 - \gamma)^{t'-1}\gamma \). Before the first resetting, the particle survives with probability \( S_i^0(t' - 1) \), after which the particle survives with probability \( S_r(t - t') \).

Taking the Laplace transform for Eq. (11), we obtain
\[
\tilde{f}_{ij}(s) = \frac{\tilde{f}_{ij}(s') + e^{-s} \tilde{S}_i(s') \tilde{S}_r(s)}{1 - e^{-s} \tilde{S}_i(s')}. \tag{12}
\]
where \( s' = s - \ln (1 - \gamma) \).

Letting \( i = r \) in Eq. (12), we have
\[
\tilde{S}_r(s) = \frac{\tilde{S}_i(s') + e^{-s} \tilde{S}_i(s') \tilde{S}_r(s)}{1 - e^{-s} \tilde{S}_i(s')}. \tag{13}
\]
Substituting Eq. (13) into Eq. (12), we obtain
\[
\tilde{S}_i(s) = \frac{\tilde{S}_i(s') + e^{-s} \tilde{S}_i(s') \tilde{S}_r(s)}{1 - e^{-s} \tilde{S}_i(s')} \tag{14}
\]
Letting \( s = 0 \) in Eq. (14), we obtain the uMFPT in the presence of resetting
\[
\langle \tau_i \rangle = \tilde{S}_i(0) = \frac{\tilde{S}_i(0) (- \ln (1 - \gamma))}{1 - \gamma \tilde{S}_i(0) (- \ln (1 - \gamma))}. \tag{15}
\]
In Eq. (15), \( \tilde{S}_i(0) (- \ln (1 - \gamma)) \) can be obtained by Eq. (4).

Substituting Eq. (16) into Eq. (15), we obtain,
\[
\langle \tau_i \rangle = \frac{\sum_{j=1}^{n} (Z_\gamma)_{ij}}{1 - \gamma \sum_{j=1}^{n} (Z_\gamma)_{rij}}. \tag{18}
\]
Let us denote by \( f_{ij}(t) \) the first-passage probability of the system in the presence of resetting, which can establish the connection with \( f_{ij}^0(t) \) by the first renewal equation [14 41],
\[
f_{ij}(t) = (1 - \gamma)^t f_{ij}^0(t) + \sum_{t'=1}^{t} (1 - \gamma)^{t'-1} \gamma f_{ij}^0(t' - 1) f_{rj}(t - t'). \tag{19}
\]
Taking the Laplace transform for Eq. (19), we obtain
\[
\tilde{f}_{ij}(s) = \tilde{f}_{ij}(s') + e^{-s} \tilde{S}_i(s') \tilde{f}_{rj}(s), \tag{20}
\]
where \( s' = s - \ln (1 - \gamma) \) as before.

Letting \( i = r \) in Eq. (20), we have
\[
\tilde{f}_{ij}(s) = \frac{\tilde{f}_{ij}(s') + e^{-s} \tilde{S}_i(s') \tilde{f}_{rj}(s)}{1 - e^{-s} \tilde{S}_i(s')} \tag{21}
\]
Substituting Eq. (21) into Eq. (20), we obtain
\[
\tilde{f}_{ij}(s) = \frac{\tilde{f}_{ij}(s') + e^{-s} \tilde{S}_i(s') \tilde{f}_{rj}(s)}{1 - e^{-s} \tilde{S}_i(s')} \tag{22}
\]
The splitting probabilities in the presence of resetting can be deduced by
\[
\pi_{ij} = \sum_{t=1}^{\infty} f_{ij}(t) = \tilde{f}_{ij}(0) = \frac{\tilde{S}_i(0) (- \ln (1 - \gamma))}{1 - \gamma \tilde{S}_i(0) (- \ln (1 - \gamma))} \tag{23}
\]
\[
+ \frac{\gamma \tilde{S}_i(0) (- \ln (1 - \gamma))}{1 - \gamma \tilde{S}_i(0) (- \ln (1 - \gamma))} \tilde{f}_{ij}(0) \]


where \( \tilde{f}_ij^0 (\ln(1-\gamma)) \) can be calculated by Eq. [8],

\[
\tilde{f}_ij^0 (\ln(1-\gamma)) = (1-\gamma)(\mathbf{I} - (1-\gamma)\mathbf{Q})^{-1}\mathbf{R})_{ij} = (1-\gamma)(\mathbf{Z}_\gamma \mathbf{R})_{ij}.
\]

Substituting Eq. [16] and Eq. [24] into Eq. [23], we obtain

\[
\frac{\partial}{\partial \gamma} \left( \mathbb{E}[\tau_{ij}] \right) = \frac{\gamma \sum_{j=1}^{n} (Z_{\gamma})_{ij}}{1 - \gamma \sum_{j=1}^{n} (Z_{\gamma})_{rj}} - (1-\gamma)(\mathbf{Z}_\gamma \mathbf{R})_{ij} - (1-\gamma)(\mathbf{Z}_\gamma \mathbf{R})_{rj}.
\]

During the derivation of Eq. [27], we have used

\[
\tilde{S}_i^0 (\ln(1-\gamma)) = -(1-\gamma) \sum_{j=1}^{n} (\mathbf{QZ}_\gamma^2)_{ij},
\]

and

\[
\tilde{f}_ij^0 (-\ln(1-\gamma)) = -(1-\gamma)(\mathbf{Z}_\gamma^2 \mathbf{R})_{ij}.
\]

IV. CONDITION FOR OPTIMIZING cMFPT BY STOCHASTIC resetting

To search the condition for optimizing \( \langle \tau_{ij} \rangle \) by stochastic resetting, we take the derivative for Eq. [18] with respect to \( \gamma \), which yields

\[
\frac{\partial}{\partial \gamma} \left( \langle \tau_{ij} \rangle \right) = \frac{\sum_{j=1}^{n} (Z_{\gamma})_{ij} \sum_{j=1}^{n} (Z_{\gamma})_{rj} - \gamma \sum_{j=1}^{n} (QZ_{\gamma}^2)_{ij} }{1 - \gamma \sum_{j=1}^{n} (Z_{\gamma})_{rj}^2} - \frac{\sum_{j=1}^{n} (QZ_{\gamma}^2)_{ij}}{1 - \gamma \sum_{j=1}^{n} (Z_{\gamma})_{rj}}.
\]

where we have used \( \partial Z_{\gamma} / \partial \gamma = -QZ_{\gamma}^2 \) in terms of Eq. [17]. At \( \gamma = 0 \), Eq. [30] becomes

\[
\Delta = \frac{\partial}{\partial \gamma} \left( \langle \tau_{ij} \rangle \right) \bigg|_{\gamma=0} = \sum_{j=1}^{n} (Z_0)_{ij} \sum_{j=1}^{n} (Z_0)_{rj} - \sum_{j=1}^{n} (QZ_0^2)_{ij} = \langle \tau_{ij}^0 \rangle \langle \tau_{ij}^0 \rangle - \sum_{j=1}^{n} (QZ_0^2)_{ij}.
\]

the splitting probabilities in the presence of resetting,

\[
\pi_{ij} = (1-\gamma)(Z_\gamma \mathbf{R})_{ij} + \frac{\gamma \sum_{j=1}^{n} (Z_{\gamma})_{ij}}{1 - \gamma \sum_{j=1}^{n} (Z_{\gamma})_{rj}} (1-\gamma)(Z_\gamma \mathbf{R})_{rj}.
\]

V. APPLICATIONS

A. Symmetric random walks on one-dimensional lattices with absorbing boundaries

We consider a symmetric random walk on a one-dimensional lattice of size \( N = n+2 \), where both ends are set to be absorbing boundaries. At each time step, the walker hops to either left or to right with equal probability \( \frac{1}{2} \). The \( n \times n \) transition matrix \( \mathbf{Q} \) between transient states and the \( n \times 2 \) transition matrix \( \mathbf{R} \) from transient states to absorbing states can be written as

\[
\mathbf{Q} = \begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
& \ddots & \ddots \\
& & \frac{1}{2} & 0
\end{pmatrix},
\]

\[
\mathbf{R} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
and

\[ R = \begin{pmatrix} 1/2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/2 \end{pmatrix}. \] (33)

respectively. We can see that \( Q \) is a symmetric tridiagonal matrix. \( I - Q \) is also a symmetric tridiagonal matrix, and its inverse \( Z_0 \) can be obtained explicitly [69].

\[ (Z_0)_{ij} = \frac{2 \min \{i, j\} \left( n + 1 - \max \{i, j\} \right)}{n + 1}. \] (34)

Substituting Eq. (34) into Eq. (3), we obtain

\[ \langle r_i^0 \rangle = \sum_{j=1}^{n} (Z_0)_{ij} = (n + 1 - i)i. \] (35)

\[ \sum_{j=1}^{n} (Z_0^2)_{ij} = \frac{i}{6} \left[ 2 - i + i^3 + 4n + 3n^2 + n^3 - 2i^2 (1 + n) \right]. \] (36)

Substituting Eq. (35) and Eq. (37) into Eq. (31), we obtain

\[ \Delta = \frac{i(n + 1 - i)}{6} \left[ 4 + i(i - n - 1) - n(n + 2) + 6r(n + 1 - r) \right] \] (38)

If the resetting node is the same as the original one, \( r = i \), Eq. (38) leads to the sufficient condition for optimization by stochastic resetting,

\[ i < i_{c_1} \quad \text{or} \quad i > i_{c_2}, \] (39)

with

\[ i_{c_{1,2}} = \frac{n + 1}{2} \pm \frac{1}{2} \sqrt{\frac{n^2 + 2n + 21}{5}}. \] (40)

In the limit of \( n \to \infty \), Eq. (40) simplifies to

\[ i_{c_{1,2}} = \frac{5 + \sqrt{5}}{10} (n + 1). \] (41)

If the length between neighboring site is equal to one, \( n + 1 \) is the total length of the lattice, and then Eq. (41)

recover to the result on the continuous case [28, 51, 56].

When the optimization condition in Eq. (39) holds, (37) there exists an optimal resetting probability, \( \gamma = \gamma_{\text{opt}}, \)
Fig. 2. Results on one-dimensional symmetric random walks. (a) The minimum of uMFPT in the presence of resetting, \(\langle \tau_i \rangle_{\text{min}}\), and the uMFPT without resetting, \(\langle \tau_i^a \rangle\), as a function of the starting position \(i\) of the walker. (b) The optimal resetting probability \(\gamma_{\text{opt}}\) as a function of the starting position \(i\) of the walker. The vertical dashed lines indicate the locations of \(i_{\text{c}(2)}\).

at which the uMFPT is a minimum. For example, for a one-dimensional lattice with \(N = 100\) sites, \(\langle \tau_i \rangle\) can be optimized by stochastic resetting when \(i < 28\) or \(i > 72\). In Fig. 2(a) we show the uMFPT, splitting probabilities, and cMFPT as a function of resetting probability \(\gamma\) for two different starting position: \(i = 10\) (left panel) and \(i = 30\) (right panel). Simulation results (symbols) finds excellent agreement with the theoretical ones (lines). In Fig. 2(b), we compare the minimum of the uMFPT in the presence of resetting with the uMFPT without resetting.

As expected, when the starting node \(i\) of the walker is close to either end the uMFPT can be optimized by resetting and attains a minimum at \(\gamma = \gamma_{\text{opt}}\). Otherwise, the resetting is not beneficial for accelerating the uMFPT. In Fig. 2(b), we show that the value of \(\gamma_{\text{opt}}\) decreases monotonically and becomes zero until the condition in Eq. 39 is no longer satisfied.

For the system, the uMFPT can be also obtained explicitly by calculating the elements of \(Z_\gamma\), given by

\[
(Z_\gamma)_{ij} = \begin{cases} 
(1-\gamma)\delta_{ij} - \theta_i \phi_{i+1}, & i < j \\
(1-\gamma)\delta_{ij} - \theta_i \phi_{i+1}, & i = j \\
(1-\gamma)\delta_{ij} - \theta_i \phi_{i+1}, & i > j 
\end{cases}
\]

where

\[
\begin{align*}
\theta_i &= \frac{\mu_{i-1} - \mu_2 \mu_{i-1}}{\nu_{i-1} - \nu_2 \nu_{i-1}} \\
\phi_i &= \frac{\mu_2 \mu_{i-1}}{\nu_{i-1} - \nu_2 \nu_{i-1}} + \frac{\mu_{i-1} - \mu_2 \mu_{i-1}}{\nu_{i-1} - \nu_2 \nu_{i-1}} 
\end{align*}
\]

with

\[
\begin{align*}
\mu_{1,2} &= \frac{1}{2} \left( 1 \pm \sqrt{2 \gamma - \gamma^2} \right) \\
\nu_{1,2} &= \frac{1}{2} \left( 1 \pm \sqrt{2 \gamma - \gamma^2} \right)
\end{align*}
\]

B. Voter model on complete graphs

We consider voter model on a complete graph of size \(N\) \([70, 71]\). Each node can be in one of two discrete states: 0 and 1. In each time step, a node is randomly chosen, and it adopts the state of a random neighbor. Let us denote \(m\) the number of nodes with state 1, with \(0 \leq m \leq N\). \(m = 0\) and \(m = N\) are two absorbing states, corresponding to all nodes achieving consensus. The model can be viewed as random walks in the \(m\)-space. The element of transition matrix is

\[
W_{m,m'} = \frac{(N-m)m}{N(N-1)} \delta_{m',m+1} + \frac{m(N-m)}{N(N-1)} \delta_{m',m-1} + \left[ 1 - 2\frac{(N-m)m}{N(N-1)} \right] \delta_{m',m}.
\]

From Eq. (41), one can extract the \((N-1) \times (N-1)\) transition matrix \(Q\) between transient states, given by

\[
Q = \begin{pmatrix}
1-2g_1 & g_1 & 0 & \cdots & \cdots & g_{N-2} \\
g_2 & 1-2g_2 & g_3 & \cdots & \cdots & g_{N-1} \\
g_2 & \cdots & \cdots & \cdots & \cdots & g_{N-2} \\
g_2 & \cdots & \cdots & \cdots & \cdots & g_{N-1} \\
go_2 & \cdots & \cdots & \cdots & \cdots & g_{N-2} \\
go_2 & \cdots & \cdots & \cdots & \cdots & g_{N-1}
\end{pmatrix}
\]

and the \((N-1) \times 2\) transition matrix \(R\) from transient states to absorbing states

\[
R = \begin{pmatrix}
g_1 & 0 \\
\vdots & \vdots \\
0 & g_{N-1}
\end{pmatrix}.
\]

where \(g_m = \frac{m(N-m)}{N(N-1)}\) with \(m = 1, \cdots, N-1\).

The model is initialized with \(m_0\) voters with state 1, and the remaining voters with state 0. When the system has not reached the consensus state, it evolves in the following way. At each time step, the model is updated according to the usual rule as mentioned before with the probability \(1-\gamma\). With the complementary probability \(\gamma\), the model is reset to the initial configuration and then the process starts anew. Once the model enters into either of two absorbing states, the process is terminated.

In Fig. 3 we show the uMFPT, splitting probabilities and cMFPT to two absorbing states as a function of resetting probability \(\gamma\) for two different initial conditions on a complete graph with \(N = 100\) nodes. When the initial number \(m_0\) of voters with state 1 is close to zero or the total number \(N\) of voters, the uMFPT attains a minimum at \(\gamma = \gamma_{\text{opt}}\) (see left panel of Fig. 3), implying that the resetting can optimize the uMFPT. Otherwise, the uMFPT shows a monotonic increase with \(\gamma\), and thus the resetting is against the acceleration of the uMFPT (see right panel of Fig. 3). By Eq. (47), we find that when \(m_0 \leq m_{c1} = 18\) or \(m_0 \geq m_{c2} = 82\), the uMFPT, i.e. the mean time to consensus, can be optimized by the stochastic resetting. That is to say, when \(m_0 \leq m_{c1}\) or
m_0 \geq m_{c_2}, there exists a nonzero resetting probability \( \gamma_{\text{opt}} \) at which the uMFPT is a minimum. The is clearly realized from Fig.3(a), in which we show the minimum of the uMFPT in the presence of resetting, \( \langle \tau_{m_0} \rangle_{\text{min}} \), and the uMFPT \( \langle \tau_{m_0} \rangle \) in the resetting-free process, as a function of \( m_0 \). \( \gamma_{\text{opt}} \) decreases as \( m_0 \) approaches \( m_{c_1} \) from below or \( m_{c_2} \) from above, as shown in Fig.3(b). In the limit of \( N \to \infty \), the voter model can be described a diffusion-like equation, from which we can solve the survival probability that the system has not reached the fully ordered state up to time \( t \), and then the uMFPT and the unconditional mean squared FPT. According to the condition for optimization by resetting in the continuous version [32], we can obtain the values of \( m_{c_{1(2)}}/N \) in the limit of \( N \to \infty \) (see Appendix A for details),

\[
m_{c_1}/N = 1 - m_{c_2}/N = 0.1846905.
\] (48)

In Fig.4 we show the values of \( m_{c_{1(2)}}/N \) as a function of \( N \), from which we can see that \( m_{c_{1(2)}}/N \) converge to the limiting values given in Eq.48 in an oscillatory way.

FIG. 3. Results of voter model on a complete graph of size \( N = 100 \) with two different initial number \( m_0 \) of voters with state 1: \( m_0 = 5 \) (left panel) or \( m_0 = 20 \) (right panel). (a) and (d): The uMFPT as a function of resetting probability \( \gamma \). (b) and (e): the splitting probability to \( m = 0 \) (all voters with state 0) or to \( m = N \) (all voters with state 1) as a function of \( \gamma \). (c) and (f): the cMFPT to \( m = 0 \) (all voters with state 0) or to \( m = N \) (all voters with state 1) as a function \( \gamma \). Lines and symbols respectively represent the theoretical and simulation results, and they are excellent agreement.

FIG. 4. Results of voter model on a complete graph of size \( N = 100 \). (a) The minimum of uMFPT in the presence of resetting, \( \langle \tau_i \rangle_{\text{min}} \), and the conditional MFPT without resetting, \( \langle \tau_i^0 \rangle \), as a function of the initial number \( m_0 \) of voters with state 1. (b) The optimal resetting probability \( \gamma_{\text{opt}} \) as a function of \( m_0 \). The vertical dashed lines indicate the locations of \( m_{c_{1(2)}} \).
FIG. 5. \(m_{c_1(2)}/N\) as a function of network size \(N\). The horizontal dotted lines indicate the values of \(m_{c_1(2)}/N\) in the limit of \(N \to \infty\) given in Eq. (18).

VI. CONCLUSIONS

In conclusion, we have studied the impact of stochastic resetting on the first passage of a general absorbing Markovian network. Thanks to the renewal structures, we have established the connection between first passage properties with and without resetting. Based on the connection, we have derived exact expressions of the uMFPT, splitting probabilities, and cMPFT as a function of resetting probability. Furthermore, we present a sufficient condition under which the resetting can expedite the uMFPT. Finally, we apply our results to two typical examples: symmetric random walks on one-dimensional lattices with two absorbing ends, and the voter model on complete graphs. In the two examples, we have found that when the initial condition is prepared such that it is close to either of absorbing states, the resetting is able to accelerate the completion of the underlying stochastic process. In the two examples, we have also shown explicitly the conditions for acceleration via resetting. When the size of system tends to infinity, the conditions recover to their counterparts in the continuous case. In the future, we hope that our results can be applied to more complex situations, which may inspire practical implications in stochastic processes by taking advantage of restart.

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Appendix A: Derivation of optimization condition for the voter model in the continuous limit

In the continuous limit \((N \to \infty)\), the voter model can be described by the Fokker-Planck equation \([70, 71]\),

\[
\frac{\partial c(x,t)}{\partial t} = \frac{1}{N} \frac{\partial^2}{\partial x^2} \left[ (1 - x^2) c(x,t) \right],
\]

where \(x = 2m/N - 1 \in [-1, 1]\) and \(c(x,t)\) is the probability density of the system having \(x\) at time \(t\). The general solution of Eq. (A1) is given by the series expansion \([72, 74]\)

\[
c(x,t) = \sum_{l=0}^{\infty} A_l C_l^{3/2}(x)e^{-\frac{(l+1)(l+2)}{8}t},
\]

where \(A_l\) are coefficients determined by the initial condition and \(C_l^{3/2}(x)\) are the Gegenbauer polynomials. For the initial condition \(c(x,0) = \delta(x-x_0)\), the coefficients \(A_l\) are given by

\[
A_l = \frac{(2l+3)(1-x_0^2)C_l^{3/2}(x_0)}{2(l+1)(l+2)}.
\]

The survival probability is given by

\[
S^0(t|x_0) = \int_{-1}^{1} c(x,t) dx = \sum_{l=0}^{\infty} A_l B_l e^{-\frac{(l+1)(l+2)}{8}t}.
\]

where

\[
B_l = \int_{-1}^{1} C_l^{3/2}(x) dx = 1 - (-1)^{l+1}.
\]

The Laplace transform of \(S^0(t|x_0)\) is

\[
\tilde{S}^0(s|x_0) = \sum_{l=0}^{\infty} \frac{N A_l B_l}{2 + l(l+3) + Ns}.
\]

The MFPT is

\[
\langle \tau^0(x_0) \rangle = \tilde{S}^0(0) = \sum_{l=0}^{\infty} \frac{N A_l B_l}{2 + l(l+3)},
\]

and the mean squared FPT is

\[
\langle (\tau^0(x_0))^2 \rangle = -\frac{d\tilde{S}^0(s)}{ds}|_{s=0} = \sum_{l=0}^{\infty} \frac{2N^2 A_l B_l}{[2 + l(l+3)]^2}.
\]

If we only consider \(l = 0\) (corresponding to the slowest eigenmode), we find that the condition for optimization becomes \([57]\)

\[
\Delta = 2\langle \tau^0(x_0) \rangle^2 - \langle (\tau^0(x_0))^2 \rangle
\]

\[
= \frac{3}{8} (1 - 4x_0^2 + 3x_0^4) < 0,
\]

leading to

\[
x_0 \in (-1, -0.57735) \cup (0.57735, 1),
\]
or equivalently,
\begin{equation}
   m_0/N \in (0, 0.211325) \cup (0.788675, 1). \quad (A11)
\end{equation}

Furthermore, we consider much more eigenmodes, \( l = 0, 2, \ldots, 100 \) (in terms of Eq. (A5), only the even values of \( l \) contribute to the sum in Eq. (A6)), we obtain
\begin{equation}
   x_0 \in (-1, -0.630619) \cup (0.630619, 1) \quad (A12)
\end{equation}

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