HYPERBOLIC COXETER GROUPS AND MINIMAL GROWTH RATE IN DIMENSION FOUR

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Abstract. For small \( n \), the known compact hyperbolic \( n \)-orbifolds of minimal volume are intimately related to Coxeter groups of smallest rank. For \( n = 2 \) and \( n = 3 \), these Coxeter groups are intimately related to the triangle group \([7,3]\) and the tetrahedral group \([3,5,3]\), and they are also distinguished by the fact that they have minimal growth rate among all cocompact hyperbolic Coxeter groups in \( \text{Isom} \mathbb{H}^n \), respectively. In this work, we prove that the Coxeter simplex group \([5,3,3,3]\), which is the fundamental group of the minimal volume arithmetic compact hyperbolic 4-orbifold, has smallest growth rate among all cocompact Coxeter groups in \( \text{Isom} \mathbb{H}^4 \) as well. The proof is based on certain combinatorial properties of compact hyperbolic Coxeter polyhedra and some monotonicity properties of growth rates of the associated Coxeter groups.

In memoriam Ernest B. Vinberg

1. Introduction

Let \( \mathbb{H}^n \) denote the real hyperbolic \( n \)-space with its isometry group \( \text{Isom} \mathbb{H}^n \). A hyperbolic Coxeter group \( G \subset \text{Isom} \mathbb{H}^n \) of rank \( N \) is a cofinite discrete group generated by \( N \) reflections with respect to hyperplanes in \( \mathbb{H}^n \). Such a group is closely related to a finite volume Coxeter polyhedron \( P \subset \mathbb{H}^n \) with \( N \) facets, which in turn is a convex polyhedron all of whose dihedral angles are of the form \( \pi/k \) for an integer \( k \geq 2 \). Hyperbolic Coxeter groups are geometric realisations of abstract Coxeter systems \((W,S)\) consisting of a group \( W \) with a finite set \( S \) of generators satisfying the relations \( s^2 = 1 \) and \((ss')^{m_{ss'}} = 1\) where \( m_{ss'} = m_{s's} \in \{2,3,\ldots,\infty\} \) for \( s \neq s' \). For small rank \( N \), the group \( W \) is characterised most conveniently by its Coxeter symbol or its Coxeter graph.

Hyperbolic Coxeter groups are not only characterised by a simple presentation but they are also distinguished in other ways. For example, for

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small $n$, they appear as fundamental groups of smallest volume orbifolds $O^n = \mathbb{H}^n/\Gamma$ where $\Gamma \subset \text{Isom}\mathbb{H}^n$ is a discrete subgroup (see [12], [15], [8] and [19], for example). In particular, for $n = 2$, $3$ and $4$, the compact hyperbolic $n$-orbifold of minimal volume is the quotient of $\mathbb{H}^n$ by a Coxeter group of smallest rank and given by the triangle group $[7, 3]$, the $\mathbb{Z}_2$-extension of the tetrahedral group $[3, 5, 3]$ and, when restricted to the arithmetic context, by the simplex group $[5, 3, 3, 3]$, respectively.

In parallel to volume we are interested in the spectrum of small growth rates of hyperbolic Coxeter groups $G = (W, S)$. In general, the growth series $f_S(t)$ of a Coxeter system $(W, S)$ is given by

$$f_S(t) = 1 + \sum_{k \geq 1} a_k t^k,$$

where $a_k \in \mathbb{Z}$ is the number of words $w \in W$ with $S$-length $k$. The series $f_S(t)$ is can be computed by Steinberg’s formula

$$\frac{1}{f_S(t^{-1})} = \sum_{\substack{W_T < W \mid |W_T| < \infty \atop |T| \leq \infty}} (-1)^{|T|} f_T(t^{-1}),$$

where $W_T, T \subset S$, is a finite Coxeter subgroup of $W$, and where $W_\emptyset = \{1\}$. In particular, $f_S(t)$ is a rational function that can be expressed as the quotient of coprime monic polynomials $p(t), q(t) \in \mathbb{Z}[t]$ of equal degree. For cocompact hyperbolic Coxeter groups, the series $f_S(t)$ is infinite and has radius of convergence $R < 1$ which can be identified with the real algebraic integer given by the smallest positive root of the denominator polynomial $q(t)$. The growth rate $\tau_G = \tau_{(W,S)}$ is defined by the inverse of the radius of convergence $R$ of $f_S(t)$. In contrast to the finite and affine cases, hyperbolic Coxeter groups are of exponential growth.

In [16] and [20], it is shown that the triangle group $[7,3]$ and the tetrahedral group $[3,5,3]$ have minimal growth rate among all cocompact hyperbolic Coxeter groups in $\text{Isom}\mathbb{H}^n$ for $n = 2$ and $n = 3$, respectively. These results have an interesting number theoretical component since the growth rate $\tau$ of any Coxeter group acting cocompactly on $\mathbb{H}^n$, $2 \leq n \leq 3$, is a Salem number, that is, $\tau$ is a real algebraic integer $> 1$ of degree at least 4 whose inverse $\tau^{-1}$ is a conjugate, and all further conjugates of $\tau$ lie on the unit circle. In particular, the growth rate $\tau_{[7,3]}$ equals the smallest known Salem number, and it is given by Lehmer’s number $\alpha_L \approx 1.17628$ with minimal polynomial $L(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$. The constant $\alpha_L$ plays an important role in the strong version of Lehmer’s problem about a universal lower bound for Mahler measures of nonzero noncyclotomic irreducible integer polynomials (see [28]).

The proof in [20] of the two results above is based on the fact that for $n = 2$ and $n = 3$ the rational function $f_S(t)$ comes with an explicit formula in terms of the exponents of the Coxeter group $G = (W, S) \subset \text{Isom}\mathbb{H}^n$. 

For dimensions $n \geq 4$, however, there are only a few structural results, and closed formulas for growth functions do not exist in general. In this work, we establish the following result for $n = 4$ by developing a new strategy for the proof.

**Theorem.** Among all Coxeter groups acting cocompactly on $\mathbb{H}^4$, the Coxeter simplex group $[5, 3, 3, 3]$ has minimal growth rate, and as such it is unique.

In [4], and by an extension of our methods, an analogous result for growth rates of hyperbolic Coxeter groups acting cocompactly on hyperbolic space of higher dimension is shown. For $n = 5$, the group of reference is the Coxeter prism group $[5, 3, 3, 3, 3]$ found by Makarov whose quotient space, by [8], has minimal volume among all arithmetic compact hyperbolic 5-orbifolds.

The work is organised as follows. In Section 2.1 we provide the necessary background about hyperbolic Coxeter polyhedra, their reflection groups and the characterisation by means of the Coxeter symbol, the Vinberg graph and the Gram matrix. We present the relevant classification results for families of Coxeter polyhedra with few facets due to Esselmann, Kaplinskaja and Tumarkin. In Section 2.2 we consider abstract Coxeter systems and Coxeter diagrams and introduce the notions of growth series and growth rate. Of importance will be the growth monotonicity result of Terragni as given in Lemma 3. The proof of the above Theorem is presented in Section 3. We start with the Lemma 4 about the comparison of growth rates of certain Coxeter groups of rank 5 and consider then compact Coxeter polyhedra $P \subset \mathbb{H}^4$ with $N = 5$, $N = 6$ and $N \geq 7$ facets, respectively. In the most challenging case $N \geq 7$, we look at a node of maximal valency and its neighborhood in the Coxeter graph of $P$ in order to conclude by means of the lemmas 3 and 4.

### 2. Hyperbolic Coxeter polyhedra and growth rates

#### 2.1. Hyperbolic Coxeter polyhedra and their reflection groups.

Denote by $\mathbb{H}^n$ the standard hyperbolic $n$-space realised by the upper sheet of the hyperboloid in $\mathbb{R}^{n+1}$ according to

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} \mid q_{n,1}(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0 \}.$$ 

A hyperbolic hyperplane $H$ is the intersection of a vector subspace of dimension $n$ with $\mathbb{H}^n$ and can be represented as the Lorentz-orthogonal complement $H = e^L$ by means of a vector $e$ of (normalised) Lorentzian norm $q_{n,1}(e) = 1$. The isometry group $\text{Isom}\mathbb{H}^n$ of $\mathbb{H}^n$ is given by the group $O^+(n,1)$ of positive Lorentzian matrices leaving the bilinear form $\langle x, y \rangle_{n,1}$ associated to $q_{n,1}$ and the upper sheet invariant. It is well known that $O^+(n,1)$ is generated by linear reflections $r = r_H : x \mapsto x - 2\langle e, x \rangle_{n,1} e$ with respect to hyperplanes $H = e^L$ (see [3] Section A.2).

A hyperbolic $n$-polyhedron $P \subset \mathbb{H}^n$ is the intersection of a finite number $N \geq n+1$ of half-spaces $H_i^-$ bounded by hyperplanes $H_i$ all of whose normal
unit vectors $e_i$ are directed outwards with respect to $P$, say. A facet of $P$ is the intersection of $P$ with one of the hyperplanes $H_i$, $1 \leq i \leq N$. A polyhedron is a Coxeter polyhedron if all of its dihedral angles are of the form $\frac{\pi}{k}$ for an integer $k \geq 2$.

In this work, we suppose that $P$ is a compact hyperbolic Coxeter polyhedron so that $P$ is the convex hull of finitely many points in $\mathbb{H}^n$. In particular, $P$ is simple since all dihedral angles of $P$ are less than or equal to $\frac{\pi}{2}$. As a consequence, each vertex $p$ of $P$ is the intersection of $n$ hyperplanes bounding $P$ and characterised by a vertex neighborhood which is a cone over a spherical Coxeter $(n-1)$-simplex.

The following structural result of A. Felikson and P. Tumarkin [11, Theorem A] will be of importance later in this work. For its statement, the compact Coxeter polyhedra in $\mathbb{H}^4$ that are products of two simplices will play a certain role. There are seven such polyhedra which were discovered by F. Esselmann [9] (see also [10] and Example 2).

**Theorem 1.** If all facets of a compact Coxeter polyhedron $P \subset \mathbb{H}^4$ are mutually intersecting, then $P$ is either a simplex or one of the seven Esselmann polyhedra.

Fix a compact Coxeter polyhedron $P \subset \mathbb{H}^n$ with its bounding hyperplanes $H_1, \ldots, H_N$ as above. Denote by $G$ the group generated by the reflections $r_i = r_{H_i}$, $1 \leq i \leq N$. Then, $G$ is a cocompact discrete subgroup of $\text{Isom} \mathbb{H}^n$ with $P$ equal to the closure of a fundamental domain for $G$. The group $G$ is called a (cocompact) hyperbolic Coxeter group. It follows that $G$ is finitely presented with natural generating set $S = \{r_1, \ldots, r_N\}$ and relations

\[(1) \quad r_i^2 = 1 \quad \text{and} \quad (r_ir_j)^{m_{ij}} = 1,\]

where $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$ for $i \neq j$. Here, $m_{ij} = \infty$ means that the product $r_ir_j$ is of infinite order which fits into the following geometric picture. Denote by $\text{Gr}(P) = ((e_i, e_j)_{n,1}) \in \text{Mat}(N; \mathbb{R})$ the Gram matrix of $P$. Then, the coefficients of $\text{Gr}(P)$ off its diagonal can be interpreted as follows.

\[(2) \quad (e_i, e_j)_{n,1} = \begin{cases} 
\cos \frac{\pi}{m_{ij}} & \text{if } \angle(H_i, H_j) = \frac{\pi}{m_{ij}}; \\
\cosh l_{ij} & \text{if } d_{\mathbb{H}}(H_i, H_j) = l_{ij} > 0.
\end{cases}\]

The matrix $\text{Gr}(P)$ is of signature $(n, 1)$ and - apart from this - contains important information about $P$. For example, each vertex of $P$ is characterised by a positive definite $n \times n$ principal submatrix of $\text{Gr}(P)$.

Beside the Gram matrix $\text{Gr}(P)$, the Vinberg graph $\Sigma(P)$ is very useful to describe a Coxeter polyhedron $P$ if the number $N$ of its facets is small in comparison with the dimension $n$. For $1 \leq i \leq N$, the Vinberg graph $\Sigma(P)$ consists of nodes $v_i$ which correspond to the hyperplanes $H_i$ or their unit normal vectors $e_i$. The number $N$ of nodes is called the order of $\Sigma(P)$. If the hyperplanes $H_i$ and $H_j$ are not orthogonal, the corresponding nodes $v_i$ and $v_j$ are connected by an edge with weight $m_{ij} \geq 3$ if $\angle(H_i, H_j) = \frac{\pi}{m_{ij}}$;
they are connected by a dotted edge \((l_{ij})\) if \(H_i\) and \(H_j\) are at distance \(l_{ij} > 0\) in \(\mathbb{H}^n\). The weight \(m_{ij} = 3\) is omitted since it occurs very frequently.

Since \(P\) is compact (and hence of finite volume), the Vinberg graph \(\Sigma(P)\) is connected. Furthermore, by deleting a node together with the edges emanating from it so that \(\Sigma(P)\) gives rise to two connected components \(\Sigma_1\) and \(\Sigma_2\), at most one of two subgraphs \(\Sigma_1, \Sigma_2\) can have a dotted edge (since otherwise, the signature condition of \(\text{Gr}(P)\) is violated). If \(p\) is a vertex of \(P\), then the \(n\) hyperplanes in the boundary of \(P\) whose intersection equals \(p\) give rise to an elliptic subgraph \(\Sigma(p)\) of order \(n\) of \(\Sigma(P)\), that is, \(\Sigma(p)\) is the Vinberg graph of the spherical Coxeter \((n - 1)\)-simplex associated to the vertex neighborhood of \(p\) in \(P\), and vice versa. In particular, there is a correspondence between the vertices of \(P\) and the elliptic subgraphs of order \(n\) of \(\Sigma(P)\).

The subsequent examples summarise the classification results for compact Coxeter \(n\)-polyhedra in terms of the number \(n + k, 1 \leq k \leq 3\), of their facets.

**Example 1.** The compact hyperbolic Coxeter simplices were classified by Lannér [23] and exist for \(n \leq 4\), only. In the case \(n = 4\), there are precisely five simplices \(L_i\) whose Vinberg graphs \(\Sigma_i = \Sigma(L_i), 1 \leq i \leq 5\), are given in Figure 1. The simplex \(L = L_1\) described by the top left Vinberg graph (or by its Coxeter symbol \([5,3,3,3]\); see [18]) will be of particular importance.

**Example 2.** The compact Coxeter polyhedra with \(n + 2\) facets in \(\mathbb{H}^n\) have been classified. The list consists of the examples of Esselmann and the (glueings of) straight Coxeter prisms due to I. Kaplinskaja (see [10], [27], for example).
Example 3. The compact hyperbolic Coxeter polyhedra $P \subset \mathbb{H}^n$, $n \geq 4$, with $n+3$ facets exist up to $n=8$ and have been enumerated by Tumarkin [31]. For $n=4$, his list comprises 40 Vinberg graphs. These graphs have at least one and at most three dotted edges, and each node has valency at most 4.

Remark 2. By a result of Felikson and Tumarkin [12, Corollary], the Coxeter polyhedra in Examples 1, 2 and 3 contain all compact Coxeter polyhedra with exactly one pair of non-intersecting facets.

For further details and results about hyperbolic Coxeter polyhedra and Coxeter groups, their geometric-combinatorial and arithmetical characterisation as well as general (non-)existence results, we refer to the foundational work of E. Vinberg [32, 33]. An overview about the diverse partial classification results can be found in [10].

2.2. Coxeter groups and growth rates. A hyperbolic Coxeter group $G=\langle G, S \rangle$ with $S=\{r_1, \ldots, r_N\}$ as above is the geometric realisation of an abstract Coxeter system $(W, S)$ of rank $N$ consisting of a group $W$ generated by a subset $S$ of elements $s_1, \ldots, s_N$ satisfying the relations as given by (1). In the fundamental work [7] of Coxeter, the irreducible finite (or spherical) and affine Coxeter groups are classified. Abstract Coxeter groups are most conveniently described by their Coxeter graphs or by their Coxeter symbols. More precisely, the Coxeter graph $\Sigma=\Sigma(W)$ of a Coxeter system $(W, S)$ has nodes $v_1, \ldots, v_N$ corresponding to the generators $s_1, \ldots, s_N$ of $W$, and two nodes $v_i$ and $v_j$ are joined by an edge with weight $m_{ij} \geq 3$. In particular, there will be no edge if $m_{ij}=2$ and there will be an edge decorated by $\infty$ if the product element $s_is_j$ is of infinite order $m_{ij}=\infty$. In this way, the Vinberg graph of a hyperbolic Coxeter group is a refined version of its Coxeter graph.

Observe that the Coxeter graph $\bullet \overset{\infty}{\longrightarrow} \bullet$ describes the affine group $\tilde{A}_1$ and - simultaneously - is underlying the Vinberg graph $\bullet \cdots \bullet$ of a compact hyperbolic Coxeter $1$-simplex as given by any geodesic segment.

In the case that the rank $N$ of the Coxeter system $(W, S)$ is small, a description by the Coxeter symbol is more convenient. For example, $[p_1, \ldots, p_k]$ with integer labels $p_i \geq 3$ is associated to a linear Coxeter graph with $k+1$ edges marked by the respective weights. The Coxeter symbol $[(q^k, q)]$ describes a cyclic Coxeter graph with $k \geq 1$ consecutive edge weights $p$ followed by the weight $q$; see [18] Appendix, for example.

For a Coxeter system $(W, S)$ with generating set $S=\{s_1, \ldots, s_N\}$, the (spherical) growth series $f_S(t)$ is defined by

$$f_S(t) = 1 + \sum_{k \geq 1} a_k t^k,$$

where $a_k \in \mathbb{Z}$ is the number of words $w \in W$ with $S$-length $k$. For references of the subsequent basic properties of $f_S(t)$, see for example [17, 20, 22]. The
series \( f_S(t) \) can be computed by Steinberg’s formula

\[
\frac{1}{f_S(t^{-1})} = \sum_{W_T < W, |W_T| < \infty} (-1)^{|T|} f_T(t),
\]

where \( W_T, T \subset S \), is a finite Coxeter subgroup of \( W \), and where \( W_\emptyset = \{1\} \). By a result of Solomon, the growth polynomials \( f_T(t) \) in (3) can be expressed by means of their exponents \( m_1, m_2, \ldots, m_p \) according to the formula

\[
f_T(t) = \prod_{s = 1}^{p} [m_i + 1].
\]

Here we use the standard notation \([k] = 1 + t + \cdots + t^{k-1}\) with \([1] = 1\) as well as \([k, l] = [k] \cdot [l]\) and so on. By replacing the variable \( t \) by \( t^{-1} \), the function \([k]\) satisfies the property \([k](t) = t^{k-1}[k](t^{-1})\).

Table 1 lists all irreducible finite Coxeter groups together with their growth polynomials up to the exceptional groups \( E_6, E_7 \) and \( E_8 \) which are irrelevant for this work. Let us add that the growth series of a reducible Coxeter system \((W, S)\) with factor groups \((W_1, S_1)\) and \((W_2, S_2)\) such that \( S = (S_1 \times \{1_{W_2}\}) \cup (\{1_{W_1}\} \times S_2)\), satisfies the product formula \( f_S(t) = f_{S_1}(t) \cdot f_{S_2}(t)\).

By the above, in its disk of convergence, the growth series \( f_S(t) \) is a rational function that can be expressed as the quotient of coprime monic polynomials \( p(t), q(t) \in \mathbb{Z}[t] \) of equal degree. The growth rate \( \tau_W = \tau(W, S) \) is defined by the inverse of the radius of convergence \( R \) of \( f_S(t) \). Since \( \tau_W \) equals the biggest real root of the denominator polynomial \( q(t) \), it is a real algebraic integer.

Consider a cocompact hyperbolic Coxeter group \( G = (G, S) \). Then, the rational function \( f_G(t) \) is reciprocal (resp. anti-reciprocal) for even \( n \) odd \( n \) odd (see [22], for example). In particular, for \( n = 4 \), one has \( f_G(t^{-1}) = f_G(t) \) for all \( t \neq 0 \). Furthermore, a result of Milnor [24] implies that the growth rate \( \tau_G \) is strictly bigger than 1 so that \( G \) is of exponential growth. More specifically, for \( n = 2 \) and \( n = 3 \), \( \tau_G \) is always a Salem number, that is, \( \tau_G \) is a real algebraic integer \( \alpha \) whose inverse is a conjugate of \( \alpha \), and all other conjugates lie on the unit circle. However, by a result of Cannon [5] (see also [22] Theorem 4.1), the growth rates of the five Lannér groups acting on \( \mathbb{H}^4 \) and shown in Figure 1 are not Salem numbers anymore; they are so-called Perron numbers, that is, real algebraic integers \( > 1 \) all of whose conjugates are of strictly smaller absolute values.

**Example 4.** The smallest known Salem number \( \alpha_L \approx 1.176281 \) with minimal polynomial \( L(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1 \) equals the growth rate \( \tau_{[7, 3]} \) of the cocompact Coxeter triangle group \( G = [7, 3] \) with Coxeter graph \( \bullet - - - \bullet \) which in turn is the smallest growth rate among all cocompact planar hyperbolic Coxeter groups (see [16] [20]). The second smallest growth rate is realised by the Coxeter triangle group \([8, 3]\) with Coxeter graph \( \bullet - - - - - \bullet \)
and appears as the seventh smallest known Salem number $\approx 1.23039$ given by the minimal polynomial $t^{10} - t^7 - t^5 - t^3 + 1$ (see [21]).

**Example 5.** Among the cocompact Coxeter tetrahedral groups, the smallest growth rate is about 1.35098 with minimal polynomial $t^{10} - t^9 - t^6 + t^5 - t^4 - t + 1$; it is achieved in a unique way by the group $G = [3, 5, 3]$ with Coxeter graph $\bullet \cdots \bullet$ (see [21]).

**Example 6.** Consider the (arithmetic) Lannéer group $L = [5, 3, 3, 3]$ with Coxeter graph $\bullet \cdots \bullet$ mentioned in Example 1. By means of Steinberg’s formula (3), the growth function $f_S(t)$ equals

\[
1 - t - t^4 + t^8 - t^9 + t^{10} - t^{11} + t^{14} - t^{15} + t^{16} - 2t^{17} + 2t^{18} - t^{19} + t^{20} - t^{21} + t^{22} - t^{23} + 2t^{24} - 2t^{25} + 2t^{26} - 2t^{27} + 2t^{28} - t^{29} + t^{30} - t^{31} + 2t^{32} - 2t^{33} + 2t^{34} - 2t^{35} + 2t^{36} - t^{37} + t^{38} - t^{39} + t^{40} - t^{41} + 2t^{42} - 2t^{43} + t^{44} - t^{45} + t^{46} - t^{49} + t^{50} - t^{51} + t^{52} - t^{53} - t^{59} + t^{60}
\]

The denominator polynomial $q(t)$ of $f_S(t)$ is palindromic and of degree 60.

By means of the software PARI/GP [26], one checks that $q(t)$ is irreducible and has – beside non-real roots some of them being of absolute value one – exactly two inversive pairs $\alpha^{\pm 1}, \beta^{\pm 1}$ of real roots such that $\alpha > \beta > 1$. Indeed, by the results in [5, 6], $\alpha$ is not a Salem number anymore. As a consequence, the growth rate $\tau_G = \alpha \approx 1.19988$ of the Lannéer group $[5, 3, 3, 3]$ is not a Salem number. However, $\tau_{[5,3,3,3]}$ is a Perron number. These properties can be checked by the software CoxIter [13, 14] developed by R. Guglielmetti. In a similar way, one computes the individual growth series and related invariants and properties of any cocompact (or cofinite) hyperbolic Coxeter group with given Vinberg graph.

| Group | Exponents | Growth polynomial $f_S(x)$ |
|-------|-----------|-----------------------------|
| $A_n$ | $1, 2, \ldots, n - 1, n$ | $[2, 3, \ldots, n, n + 1]$ |
| $B_n$ | $1, 3, \ldots, 2n - 3, 2n - 1$ | $[2, 4, \ldots, 2n - 2, 2n]$ |
| $D_n$ | $1, 3, \ldots, 2n - 5, 2n - 3, n - 1$ | $[2, 4, \ldots, 2n - 2 \cdot [n]$ |
| $G_2^{(m)}$ | $1, m - 1$ | $[2, m]$ |
| $F_4$ | $1, 5, 7, 11$ | $[2, 6, 8, 12]$ |
| $H_3$ | $1, 5, 9$ | $[2, 6, 10]$ |
| $H_4$ | $1, 11, 19, 29$ | $[2, 12, 20, 30]$ |

**Table 1.** Exponents and growth polynomials of irreducible finite Coxeter groups
Growth rates satisfy an important monotonicity property on the partially ordered set of Coxeter systems as follows. For two Coxeter systems \((W, S)\) and \((W', S')\), one defines \( (W, S) \leq (W', S') \) if there is an injective map \( \iota : S \to S' \) such that \( m_{st} \leq m'_{\iota(s)(\iota(t))} \) for all \( s, t \in S \). If \( \iota \) extends to an isomorphism between \( W \) and \( W' \), one writes \( (W, S) \simeq (W', S') \), and \( (W, S) < (W', S') \) otherwise. In the latter case, we often say that the Coxeter system \((W', S')\) dominates the system \((W, S)\). This partial order satisfies the descending chain condition since \( m_{st} \in \{2, 3, \ldots, \infty\} \) where \( s \neq t \). In particular, any strictly decreasing sequence of Coxeter systems is finite (see [25]). In this work, the following result of Terragni [29, Section 3] will play an essential role.

**Lemma 3.** If \( (W, S) < (W', S') \), then \( \tau_{(W, S)} < \tau_{(W', S')} \).

In the sequel we sometimes use a graphical notation as follows. Let \( \Sigma \) and \( \Sigma' \) be the Coxeter graphs of \((W, S)\) and \((W', S')\), respectively. Then, we abbreviate and write

\[ \Sigma <_{\tau} \Sigma' \quad \text{if} \quad (W, S) < (W', S'). \]

**Notice.** We will work with the Coxeter graph instead of the Vinberg graph associated to a hyperbolic Coxeter group \((W, S)\) and, hence, we replace each dotted edge between two nodes \( \nu_s \) and \( \nu_{s'} \) by an edge with weight \( \infty \) indicating that the product element \( ss' \in W \) is of infinite order.

**Example 7.** Consider the Kaplinskaja polyhedron \( K \subset \text{Isom} \mathbb{H}^4 \) as described in Figure 2. The polyhedron \( K \) defines a Coxeter system \((W', S')\) with associated Coxeter graph \( \Sigma' \). The graph \( \Sigma' \) has the subgraph \( \Sigma \) given by \( \bullet \cdots \bullet \cdots \bullet \infty \bullet \cdots \bullet \) which in turn defines a Coxeter subgroup \((W, S)\) of \((W', S')\) with \( S \subset S' \). By Lemma 3 it follows first that \( \Sigma <_{\tau} \Sigma' \) and finally that \( \tau_{[5,3,3,3]} < \tau_{[\infty,3,3,3]} < \tau_K \).

Notice that the same reasoning applies for the remaining Coxeter polyhedra found by Kaplinskaja since each of their Coxeter graphs contains a linear Coxeter subgraph of the type \( \bullet \cdots \bullet \cdots \bullet \cdots \bullet \infty \bullet \cdots \bullet \cdots \bullet \) with \( k, l, m \geq 3 \). As a consequence, all the compact Coxeter polyhedra in \( \mathbb{H}^4 \) with 6 facets found by Kaplinskaja have a growth rate which is strictly bigger than the one of the Coxeter simplex \([5,3,3,3]\).

### 3. Growth minimality and the Coxeter group \([5,3,3,3]\)

In this section, we prove the following result as announced in the Introduction (see Section 1).

**Theorem.** Among all Coxeter groups acting cocompactly on \( \mathbb{H}^4 \), the Coxeter simplex group \([5,3,3,3]\) has minimal growth rate, and as such it is unique.

3.1. **An auxiliary result.** Consider the three abstract Coxeter groups \( W_1, W_2 \) and \( W_3 \) with generating subsets \( S_1, S_2 \) and \( S_3 \) of rank 5 as defined by the Coxeter graphs in Figure 3. Each of the three Coxeter groups
Figure 3. The three abstract Coxeter groups $W_1, W_2$ and $W_3$

$W_1, W_2$ and $W_3$ can be represented geometrically as a discrete subgroup of $O^+(4,1)$ generated by reflections in the facets of a Coxeter 4-simplex of infinite volume. Indeed, one easily checks that the associated Tits form is of signature $(4,1)$ and that some of the simplex vertices are not hyperbolic but ultra-ideal points (of positive Lorentzian norm).

The hyperbolic Coxeter groups $G_i$ associated to the Coxeter systems $(W_i, S_i)$ for $1 \leq i \leq 3$ will play an important role when comparing growth rates. By Lemma 3, one has that

$$
\tau^{[5,3,3,3]} < \tau_{G_1}.
$$

In addition, the following result holds.

**Lemma 4.**

(1) $\tau_{G_1} < \tau_{G_2}$;
(2) $\tau_{G_1} < \tau_{G_3}$.

**Proof.** By means of Steinberg’s formula (3), we identify the finite Coxeter subgroups with their growth polynomials of $G_i$ according to Table 1 in order to deduce the following expressions for their growth functions $f_i(t), 1 \leq i \leq 3$.

(a) $$
\frac{1}{f_1(t^{-1})} = h(t) - \frac{1}{[2,2,2]} - \frac{4}{[2,2,3]} - \frac{2}{[2,3,4]} + \frac{1}{[2,2,3,4]} + \frac{1}{[2,3,4,5]}
$$

(b) $$
\frac{1}{f_2(t^{-1})} = h(t) - \frac{1}{[2,2,2]} - \frac{5}{[2,2,3]} - \frac{1}{[2,3,4]} + \frac{1}{[2,2,3,4]} + \frac{1}{[2,3,4,4]}
$$

(c) $$
\frac{1}{f_3(t^{-1})} = h(t) - \frac{2}{[2,2,2]} - \frac{3}{[2,2,3]} - \frac{2}{[2,3,4]} + \frac{1}{[2,2,3,3]} + \frac{1}{[2,3,4,5]}
$$

Here, the help function $h(t)$ is given by

$$
h(t) = 1 - \frac{5}{[2]} + \frac{6}{[2,2]} + \frac{3}{[2,3]}.
$$

By taking the difference of (a) and (b), one easily computes that

$$
\frac{1}{f_1(t^{-1})} - \frac{1}{f_2(t^{-1})} = \frac{t^4}{[3,3,5]} > 0 \quad \forall t > 0.
$$

For $x = t^{-1} \in (0,1)$, we deduce that the smallest zero of $1/f_1(x)$ as given by the radius of convergence of the growth series $f_1(x)$ of $G_1$ is strictly bigger than the one of $1/f_2(x)$. Hence, we get $\tau_{G_1} < \tau_{G_2}$.

The identities (a) and (c) combined with the property

$$
\frac{1}{[m]} > \frac{1}{[l]} \quad \text{for} \quad 1 \leq m < l, \ t > 0,
$$

...
lead to the estimate
\[
\frac{1}{f_1(t^{-1})} - \frac{1}{f_3(t^{-1})} = \frac{1}{[2,2,2]} - \frac{1}{[2,2,2,3]} + \frac{1}{[2,2,3]} + \frac{1}{[2,2,3,4]}
\]
\[
= \frac{1}{[2,2,2]} \left\{ 1 - \frac{1}{[3]} \right\} + \frac{1}{[2,2,3]} \left\{ \frac{1}{[4]} - 1 \right\}
\]
\[
> \frac{1}{[2,2,3]} \left\{ \frac{1}{[3]} + \frac{1}{[4]} \right\} > 0, \quad \forall t > 0
\]
which proves that \( \tau_{G_1} < \tau_{G_3} \).

3.2. The proof. We now turn to the proof of the Theorem above and consider a group \( G \) in Isom\( \mathbb{H}^4 \) generated by the set \( S \) of reflections \( r_1, \ldots, r_N \) in the \( N \) facet hyperplanes bounding a compact Coxeter polyhedron \( P \subset \mathbb{H}^4 \).

The group \( G = (G,S) \) is a cocompact hyperbolic Coxeter group of rank \( N \geq 5 \). Assume that the group \( G \) is not isomorphic to the group \( G_* = [5,3,3,3] \).

We have to show that \( \tau_G > \tau_{[5,3,3,3]} \approx 1.19988 \).

In view of Theorem 1, we distinguish between the cases \( N = 5 \), \( N = 6 \) and \( N \geq 7 \). In fact, for \( N \geq 7 \), the Vinberg graph of \( G \) contains at least one dotted edge since there are at least two facets of \( P \) which are at positive distance in \( \mathbb{H}^4 \). Hence, the associated Coxeter graph \( \Sigma \) of \( G \) has at least one edge of type \( \bullet - \infty - \bullet - \bullet \).

The case \( N = 5 \). A Coxeter group \( G \) of rank \( N = 5 \) as above is related to one of the five compact Coxeter simplices \( L_1, \ldots, L_5 \) given by the Coxeter graphs in Figure 1. By applying the reasoning as in Example 7, it remains to check that \( \tau_{[5,3,3,3]} \) is strictly smaller than the growth rates of \( L_4 = [5,3,3,3,1] \), given by the Coxeter graph with exactly one node of valency bigger than 2, and of \( L_5 = [(3^4,4)] \) given by the cyclic Coxeter graph. By means of CoxIter, one computes the two growth functions and looks for the smallest positive real root of the denominator polynomials. It turns out that their inverses are given by \( \tau_{[5,3,3,3,1]} \approx 1.44970 \) with minimal polynomial of degree 60, and by \( \tau_{[(3^4,4)]} \approx 1.62282 \) with minimal polynomial of degree 28, respectively (see also [5], [27], [30]). Hence, the group \([5,3,3,3]\) has minimal growth rate among all cocompact hyperbolic Coxeter groups of rank \( N = 5 \).

The case \( N = 6 \). The cocompact hyperbolic Coxeter groups of rank \( N = 6 \) are classified and consist of the seven Esselmann polyhedra and the straight Coxeter prisms and their glueings found by Kaplinskaja (see Example 2). The only case which cannot be treated by the argument explained in Example 7 is the group related to the Esselmann polyhedron \( E \) depicted in Figure 2. Indeed, its Coxeter graph does not contain a connected subgraph of order 5 with a weight \( \infty \). By means of CoxIter (see also [27, Appendix A]), one sees that the denominator polynomial of the growth function of \( E \) is irreducible of degree 24 whose biggest positive root yields \( \tau_E \approx 2.43645 \) (which is not a Salem number but a Perron number; see [22, Chapter 4], [4]). In particular, \( \tau_E > \tau_{[5,3,3,3]} \).
The case $N \geq 7$. As mentioned above, the Coxeter graph $\Sigma$ of $G$ has at least one edge marked $\infty$. Our strategy is to detect a connected Coxeter subgraph $\sigma$ of order 5 of $\Sigma$ which describes a Coxeter subgroup $H$ of $G$ generated by a subset $T \subseteq S$ and to verify that the Coxeter system $(H, T)$ satisfies $(W_i, S_i) \leq (H, T)$ where $(W_i, S_i)$ are associated to the Coxeter groups $G_i, 1 \leq i \leq 3$, given in Figure 3. By means of Lemma 4, equation (5) and Lemma 3, we can then conclude that

$$\tau_{[5,3,3,3]} < \tau G_1 \leq \tau H < \tau G.$$ 

To this end, we analyse the graph $\Sigma$ of $G$ in terms of the maximal valency $\mu \geq 2$ of the $N$ nodes of $\Sigma$.

- If $\mu = 2$, then $\Sigma$ is either a linear graph or a cyclic graph of order at least 7 and with at least one edge with weight $\infty$. Hence, $\Sigma$ contains a connected subgraph $\sigma$ of order 5 associated to a Coxeter subgroup $H$ which dominates one of the Coxeter groups $W_1$ and $W_2$ depicted in Figure 3. In particular, by Lemma 3, $[5,3,3,3] < \tau G_1 \leq \tau H < \tau G$, and the claim follows.

Let $v$ be a node of valency $\mu$ of (the metric graph) $\Sigma$ and denote by $v_1, \ldots, v_\mu$ its $\mu$ neighboring nodes at distance 1. The nodes $v, v_1, \ldots, v_\mu$ describe a subset $S_v \subset S$ generating an irreducible Coxeter subgroup $H_v$ of $G$. Denote by $\sigma_v$ the Coxeter graph of $H_v$. Next, we associate to the Coxeter systems $(H_v, S_v)$ and $(G, S)$ the additional Coxeter systems $(\Gamma_v, S_v)$ and $(\Gamma, S)$ defined by the Coxeter graphs $\gamma_v$ and $\gamma$ that arise from the graphs $\sigma_v$ and $\Sigma$ of $H_v$ and $G$, respectively, by replacing all finite weights $m \geq 3$ by the weight $m = 3$. Observe that the graph $\gamma_v$ contains a tree with root $v$ and $\mu$ outgoing edges. It follows that $(\Gamma_v, S_v) \leq (H_v, S_v)$ as well as $(\Gamma, S) \leq (G, S)$. In the following, we study the Coxeter graph $\sigma_v$ of $H_v$ (resp. $\gamma_v$) and its embedding in $\Sigma$ (resp. $\gamma$) by distinguishing between the cases $2 < \mu < 5$ and $\mu \geq 5$.

- Let $3 \leq \mu \leq 4$. Although the two cases are very similar, we treat first the case $\mu = 3$ and then the case $\mu = 4$ for sake of illustration.

For $\mu = 3$, consider the Coxeter graph $\sigma_v$ with associated graph $\gamma_v$. Two examples for $\gamma_v$ are depicted in Figure 4. Notice that the weight $\infty$ can appear several times or not at all. Since $\gamma$ is connected having $N \geq 7$ nodes and at least one edge with weight $\infty$, the Coxeter graph $\gamma_v$ can be embedded.
as a subgraph \( \hat{\gamma}_v \) of order 7 in \( \gamma \) by adding \( N - 4 \) further nodes together with their edges inheriting the weights 3 or \( \infty \). In Figure 5, for \( N = 7 \), two examples of such embeddings \( \hat{\gamma}_v \) for the graphs \( \gamma_v \) in Figure 4 are given, both respecting the condition \( \mu = 3 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Examples of embeddings \( \hat{\gamma}_v \) of \( \gamma_v \) in \( \gamma \)}
\end{figure}

Suppose that the graph \( \hat{\gamma}_v \) contains at least one edge with weight \( \infty \). Then, one easily checks that for all the finitely many realisations, the graph \( \hat{\gamma}_v \) always contains a subgraph of order 5 associated to one of the Coxeter groups \( W_1, W_2 \) or \( W_3 \) of rank 5 as given in Figure 4.

Suppose that the edge weights of the graph \( \hat{\gamma}_v \) are all finite (and equal to 3). It follows that the Coxeter graph \( \gamma \) contains an edge with weight \( \infty \) that does not belong to the subgraph \( \hat{\gamma}_v \). Since \( \gamma \) is connected, one finds a subgraph of order 5 of \( \gamma \) associated to one of the Coxeter groups \( W_1, W_2 \) or \( W_3 \) of rank 5 as above. Indeed, it suffices to look at any partial spanning tree of order \( N - 7 \), starting with an edge of weight \( \infty \), which connects with \( \hat{\gamma}_v \) in \( \gamma \).

Of course, the two step procedure as described above can be abbreviated by extending \( \gamma_v \) directly to a subgraph of order \( N \) of \( \gamma \). In any case, we deduce that \( \tau_{[5,3,3,3]} \leq \tau_{\Gamma} \leq \tau_G \).

For \( \mu = 4 \), the situation is very similar, and we summarise the different steps, only. Consider the Coxeter subgroup \( H_v \subset G \) given by the Coxeter graph \( \sigma_v \) of order 5 with node \( v \) of valency 4 and its neighboring nodes \( v_1, \ldots, v_4 \). Again, construct the Coxeter graphs \( \gamma_v \) and \( \gamma \) from the graphs \( \sigma_v \) and \( \Sigma \) of \( H_v \) and \( G \), respectively, by discarding all finite weights \( \geq 4 \) from the edges. Two simple examples are shown in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Examples of Coxeter graphs \( \gamma_v \)}
\end{figure}
Since $\gamma$ is connected having $N \geq 7$ nodes, the Coxeter graph $\gamma_v$ can be embedded as a subgraph $\hat{\gamma}_v$ of order 7 in $\gamma$ by adding $N - 7$ further nodes together with their edges of weight 3 or $\infty$. There are finitely many possibilities subject to the constraint $\mu = 4$. In Figure 7, two examples of such embeddings $\hat{\gamma}_v$ for the graphs $\gamma_v$ in Figure 6 are given.

Using the same arguments as in the case $\mu = 3$, we can conclude that the Coxeter graph $\gamma$ contains a (proper) Coxeter subgraph associated to one of the Coxeter groups $G_1, G_2$ or $G_3$ from Section 3.1. Hence, by Lemma 3 and Lemma 4, $\tau[5,3,3,3] < \tau G_1 < \tau \Gamma \leq \tau G$.

- Let $\mu \geq 5$. In this case, the Coxeter subgroup $H_v$ of $G$ is of rank at least 6. Consider the Coxeter subgroup of $H_v$, with graph $\theta$, that is generated by the subset $T_v \subset S_v$ of the $\mu$ reflections corresponding to the neighboring nodes $v_1, \ldots, v_\mu$ of $v$. The Coxeter graph $\theta$ cannot be elliptic since otherwise it would give rise to a positive definite principal submatrix of rank $\mu \geq 5$ in the Gram matrix $\text{Gr}(P)$ of signature (4,1) of $P$ (see Section 2.1). In particular, the graph $\theta$ cannot be totally disconnected but has at least one edge of weight $\geq 3$. We analyse this setting in more detail in view of $\mu \geq 5$ and $\theta$.

For $\mu = 5$, we construct the Coxeter graphs $\gamma_v$ and $\gamma$ from the graphs $\sigma_v$ and $\Sigma$ as in the cases $2 < \mu < 5$ above. Denote by $\gamma_\theta$ the subgraph of $\gamma_v$ induced by $\theta$. The graph $\gamma_\theta$ has at least one edge. Since $N \geq 7$, $\gamma_v$ has a non-trivial embedding as a connected subgraph $\hat{\gamma}_v$ of order $N$ of $\gamma$ by adding $N - 6$ nodes together with the corresponding edges of weight 3 or $\infty$. An example of an embedding $\hat{\gamma}_v$ in $\gamma$ for $N = 7$ is depicted in Figure 8. By assumption, $\gamma$ has at least one edge with weight $\infty$. As a consequence, it is not difficult to see that $\gamma$ contains a Coxeter subgraph of order 5 associated to one of the Coxeter groups $W_1, W_2$ of $W_3$ given in Figure 3, which proves the claim $\tau[5,3,3,3] < \tau G$. 

---

**Figure 7.** Examples of embeddings $\hat{\gamma}_v$ of $\gamma_v$ in $\gamma$
Figure 8. An example for $\hat{\gamma}_v$

For $\mu = 6$, the Coxeter graph $\sigma_v$ is of order 7. Hence, if $N = 7$, the graph $\sigma_v$ coincides with the Coxeter graph $\Sigma$ of $G$. However, this case can be excluded since the graph of a compact Coxeter polyhedron with 7 facets in $\mathbb{H}^4$ does not have a node with valency 6 (see Remark 2). Hence, we deduce that in the presence of a node $v$ of valency 6, we have $N \geq 8$. This fact allows us to conclude exactly as in the case of $\mu = 5$.

For $\mu \geq 7$, we have $N \geq \mu + 1 \geq 8$. Consider the Coxeter subgroup of $H_v$ defined by the graph $\theta$ of order $\mu$ as above. The graph $\theta$ has a connected Coxeter subgraph of order at least 3 since otherwise there is again a conflict with the signature $(4,1)$ of $\text{Gr}(P)$. If an edge of $\hat{\gamma}_v$ has the weight $\infty$, we easily find a subgraph of $\sigma_v$ of the type as depicted in Figure 3. In the other case, $N \geq 9$ so that $\gamma_v$ has an embedding as a connected subgraph $\hat{\gamma}_v$ of order $N$ with at least one edge of weight $\infty$ in $\gamma$. We apply the same reasoning as in the previous cases and conclude for the Coxeter group $\hat{G}$ associated to $\hat{\gamma}_v$ that $\tau_{[5,3,3,3]} < \tau_{\hat{G}_1} < \tau_{\hat{G}} \leq \tau_G$.

This finishes the proof of the Theorem.

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