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Holonomic rank of $A$-hypergeometric differential-difference equations

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Abstract

We introduce $A$-hypergeometric differential-difference equation $H_A$ and prove that its holonomic rank is equal to the normalized volume of $A$ with giving a set of convergent series solutions.

1 Introduction

In this paper, we introduce $A$-hypergeometric differential-difference equation $H_A$ and study its series solutions and holonomic rank.

Let $A = (a_{ij})_{i=1}^d_{j=1}^n$ be a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of $A$ spans $\mathbb{Z}^d$ and there is no zero column vector. Let $a_i$ be the $i$-th column vector of the matrix $A$ and $F(\beta, x)$ the integral

$$F(\beta, x) = \int_C \exp \left( \sum_{i=1}^n x_i t^{a_i} \right) t^{-\beta-1} dt, \quad t = (t_1, \ldots, t_d), \ \beta = (\beta_1, \ldots, \beta_d).$$

The integral $F(\beta, x)$ satisfies the $A$-hypergeometric differential system associated to $A$ and $\beta$ “formally”. We use the word “formally” because, there is no general and rigorous description about the cycle $C$ ([11, p.222]).

We will regard the parameters $\beta$ as variables. Then, the function $F(s, x)$ on the $(s, x)$ space satisfies differential-difference equations “formally”, which will be our $A$-hypergeometric differential-difference system.

Rank theories of $A$-hypergeometric differential system have been developed since Gel’fand, Zelevinsky and Kapranov [4]. In the end of 1980’s,

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under the condition that the points lie on a same hyperplane, they proved
that the rank of \( A \)-hypergeometric differential system \( H_A(\beta) \) agrees with
the normalized volume of \( A \) for any parameter \( \beta \in \mathbb{C}^d \) if the toric ideal
\( I_A \) has the Cohen-Macaulay property. After their result had been gotten,
many people have studied on conditions such that the rank equals the nor-
amalized volume. In particular, Matuschevich, Miller and Walther proved that
\( I_A \) has the Cohen-Macaulay property if the rank of \( H_A(\beta) \) agrees with the
normalized volume of \( A \) for any \( \beta \in \mathbb{C}^d \) ([5]).

In this paper, we will introduce \( A \)-hypergeometric differential-difference
system, which can be regarded as a generalization of difference equation
for the \( \Gamma \)-function, the Beta function, and the Gauss hypergeometric dif-
ference equations. As the first step on this differential-difference system,
we will prove our main Theorem 3 utilizing theorems on \( A \)-hypergeometric
differential equations, construction of convergent series solutions with a ho-
mogenization technique, uniform convergence of series solutions, and Mut-
sumi Saito’s results for contiguity relations [9], [10], [11, Chapter 4]. The
existence theorem 2 on convergent series fundamental set of solutions for
\( A \)-hypergeometric differential equation for generic \( \beta \) is the second main the-
orem of our paper. Finally, we note that, for studying our \( A \)-hypergeometric
differential-difference system, we wrote a program “yang” ([6], [8]) on a com-
puter algebra system Risa/Asir and did several experiments on computers
to conjecture and prove our theorems.

2 Holonomic rank

Let \( D \) be the ring of differential-difference operators
\[
\mathbb{C}\langle x_1, \ldots, x_n, s_1, \ldots, s_d, \partial_1, \ldots, \partial_n, S_1, \ldots, S_d, S_1^{-1}, \ldots, S_d^{-1}\rangle
\]
where the following (non-commutative) product rules are assumed
\[
S_i s_i = (s_i + 1)S_i, \quad S_i^{-1} s_i = (s_i - 1)S_i^{-1}, \quad \partial_i x_i = x_i \partial_i + 1
\]
and the other types of the product of two generators commute.

Holonomic rank of a system of differential-difference equations will be
defined by using the following ring of differential-difference operators with
rational function coefficients
\[
U = \mathbb{C}\langle s_1, \ldots, s_d, x_1, \ldots, x_n \rangle \langle S_1, \ldots, S_d, S_1^{-1}, \ldots, S_d^{-1}, \partial_1, \ldots, \partial_n \rangle
\]
It is a \( \mathbb{C} \)-algebra generated by rational functions in \( s_1, \ldots, s_d, x_1, \ldots, x_n \) and
differential operators \( \partial_1, \ldots, \partial_n \) and difference operators \( S_1, \ldots, S_d, S_1^{-1}, \ldots, S_d^{-1} \).
The commutation relations are defined by \( \partial_i c(s, x) = c(s, x)\partial_i + \frac{\partial c}{\partial x_i}, \) \( S_i c(s, x) = c(s_1, \ldots, s_i + 1, \ldots, s_d, x)S_i, \) \( S_i^{-1} c(s, x) = c(s_1, \ldots, s_i - 1, \ldots, s_d, x)S_i^{-1}. \)

Let \( I \) be a left ideal in \( D. \) The holonomic rank of \( I \) is the number \( \text{rank}(I) = \dim_{\mathbb{C}(s,x)} U / (UI). \)

In case of the ring of differential operators \( (d = 0), \) the definition of the holonomic rank agrees with the standard definition of holonomic rank in the ring of differential operators.

For a given left ideal \( I, \) the holonomic rank can be evaluated by a Gröbner basis computation in \( U. \)

### 3 \( \mathcal{A} \)-hypergeometric differential-difference equations

Let \( A = (a_{ij})_{i=1,\ldots,d, j=1,\ldots,n} \) be an integer \( d \times n \) matrix of rank \( d. \) We assume that the column vectors \( \{a_i\} \) of \( A \) generates \( \mathbb{Z}^d \) and there is no zero vector. The \( \mathcal{A} \)-hypergeometric differential-difference system \( H_A \) is the following system of differential-difference equations

\[
\left( \sum_{j=1}^{n} a_{ij} x_j \partial_j - s_i \right) \bullet f = 0 \quad \text{for } i = 1, \ldots, d \quad \text{and}
\]

\[
\left( \partial_j - \prod_{i=1}^{n} S_i^{-a_{ij}} \right) \bullet f = 0 \quad \text{for } j = 1, \ldots, n.
\]

Note that \( H_A \) contains the toric ideal \( I_A. \) (use [12, Algorithm 4.5] to prove it.)

**Definition 1.** Define the unit volume in \( \mathbb{R}^d \) as the volume of the unit simplex \( \{0, e_1, \ldots, e_d\}. \) For a given set of points \( \mathcal{A} = \{a_1, \ldots, a_n\} \) in \( \mathbb{R}^d, \) the normalized volume \( \text{vol}(\mathcal{A}) \) is the volume of the convex hull of the origin and \( \mathcal{A}. \)

**Theorem 1.** \( \mathcal{A} \)-hypergeometric differential-difference system \( H_A \) has linearly independent \( \text{vol}(A) \) series solutions.

The proof of this theorem is divided into two parts. The matrix \( A \) is called homogeneous when it contains a row of the form \( (1, \ldots, 1). \) If \( A \) is homogeneous, then the associated toric ideal \( I_A \) is homogeneous ideal [12]. The first part is the case that \( A \) is homogeneous. The second part is the case that \( A \) is not homogeneous.
Proof. (A is homogeneous.) We will prove the theorem with the homogeneity assumption of $A$. In other words, we suppose that $A$ is written as follows:

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ * & \end{pmatrix}. $$

Gel’fand, Kapranov, Zelevinski gave a method to construct $m = \text{vol}(A)$ linearly independent solutions of $H_A(\beta)$ with the homogeneity condition of $A$ ([4]). They suppose that $\beta$ is fixed as a generic $C$-vector. Let us denote their series solutions by $f_1(\beta; x), \ldots, f_m(\beta; x)$. It is easy to see that the functions $f_i(s; x)$ are solutions of the differential-difference equations $H_A$. We can show, by carefully checking the estimates of their convergence proof, that there exists an open set in the $(s, x)$ space such that $f_i(s; x)$ is locally uniformly convergent with respect to $s$ and $x$. Let us sketch their proof to see that their series converge as solutions of $H_A$. The discussion is given in [4], but we need to rediscuss it in a suitable form to apply it to the case of inhomogeneous $A$.

Let $B$ be a matrix of which the set of column vectors is a basis of $\text{Ker}(A : \mathbb{Q}^n \to \mathbb{Q}^d)$ and is normalized as follows:

$$B = \begin{pmatrix} 1 & \cdots & 1 \\ * & \end{pmatrix} \in M(n, n - d, \mathbb{Q}).$$

We denote by $b^{(i)}$ the $i$-th column vector of $B$ and by $b_{ij}$ the $j$-th element of $b^{(i)}$. Then the homogeneity of $A$ implies

$$\sum_{j=1}^{n} b_{ij} = 0.$$

Let us fix a regular triangulation $\Delta$ of $\mathcal{A} = \{a_1, \ldots, a_n\}$ following the construction by Gel’fand, Kapranov, Zelevinsky. Take a $d$-simplex $\tau$ in the triangulation $\Delta$. If $\lambda \in \mathbb{C}^n$ is admissible for a $d$-simplex $\tau$ of $\{1, 2, \ldots, n\}$ (admissible $\Leftrightarrow$ for all $j \notin \tau$, $\lambda_j \in \mathbb{Z}$), and $A\lambda = s$ holds, then $H_A$ has a formal series solution

$$\phi_{\tau}(\lambda; x) = \sum_{l \in L} \frac{x^{\lambda+l}}{\Gamma(\lambda+l+1)},$$

where $L = \text{Ker}(A : \mathbb{Z}^n \to \mathbb{Z}^d)$ and $\Gamma(\lambda+l+1) = \prod_{i=1}^{n} \Gamma(\lambda_i+l_i+1)$ and when a factor of the denominator of a term in the sum, we regard the term
is zero. Put \( \# \tau = n' \). Note that there exists an open set \( U \) in the \( s \) space such that \( \lambda_i, \ i \in \tau \) lie in a compact set in \( \mathbb{C}^{n'} \setminus \mathbb{Z}^{n'} \). Moreover, this open set \( U \) can be taken as a common open set for all \( d \)-simplices in the triangulation \( \Delta \) and the associated admissible \( \lambda \)'s when the integral values \( \lambda_j (j \notin \tau) \) are fixed for all \( \tau \in \Delta \).

Put \( L' = \{(k_1, \ldots, k_{n-d}) \in \mathbb{Z}^{n-d} \mid \sum_{i=1}^{n-d} k_ib^{(i)} \in \mathbb{Z}^n\} \). Then, \( L' \) is \( \mathbb{Z} \)-submodule of \( \mathbb{Z}^{n-d} \) and \( L = \{\sum_{i=1}^{n-d} k_ib^{(i)} \mid k \in L'\} \). In other words, \( L \) can be parametrized with \( L' \). Without loss of the generality, we may suppose that \( \tau = \{n-d+1, \ldots, n\} \). Then, we have

\[
\phi_\tau(\lambda; x) = \sum_{l \in L} \frac{x^{\lambda+l}}{\Gamma(\lambda+l+1)} = \sum_{k \in L'} \frac{x^{\lambda+\sum_{i=1}^{n-d} k_ib^{(i)}}}{\Gamma(\lambda+\sum_{i=1}^{n-d} k_ib^{(i)}+1)}
\]

Note that the first \( n-d \) rows of \( B \) are normalized. Then, we have

\[
\lambda_j + \sum_{i=1}^{n-d} k_ib_{ij} + 1 = \lambda_j + k_j + 1 \in \mathbb{Z} \quad (j = 1, \ldots, n-d)
\]

Since \( 1/\Gamma(0) = 1/\Gamma(-1) = 1/\Gamma(-2) = \cdots = 0 \), the sum can be written as

\[
\phi_\tau(\lambda; x) = \sum_{k \in L'} \frac{x^{\lambda+\sum_{i=1}^{n-d} k_ib^{(i)}}}{\Gamma(\lambda+\sum_{i=1}^{n-d} k_ib^{(i)}+1)}
\]

Moreover, when we put

\[
k'_j = \lambda_j + k_j, \quad (j = 1, \ldots, n-d)
\]

\[
\lambda' = \lambda - \sum_{i=1}^{n-d} \lambda_ib^{(i)}
\]

\[
\hat{\lambda} = (\lambda_1, \ldots, \lambda_{n-d})
\]

we have

\[
\sum_{i=1}^{n-d} k_ib^{(i)} = -\sum_{i=1}^{n-d} \lambda_ib^{(i)} + \sum_{i=1}^{n-d} k'_ib^{(i)}
\]
Hence, the sum $\phi_\tau (\lambda; x)$ can be written as

$$
\phi_\tau (\lambda; x) = \sum_{k' \in L'} x^{\lambda - \sum_{i=1}^{n-d} \lambda_i b^{(i)}} \cdot x^{\sum_{i=1}^{n-d} k'_i b^{(i)}} \Gamma (\lambda - \sum_{i=1}^{n-d} \lambda_i b^{(i)} + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)
$$

Note that our series with the coefficients in terms of Gamma functions agree with those in [11, §3.4], which do not contain Gamma functions, by multiplying suitable constants. Hence we will apply some results on series solutions in [11] to our discussions in the sequel.

**Lemma 1.** Let $(k_i) \in (\mathbb{Z}_{\geq 0})^m$ and $(b_{ij}) \in M(m, n, \mathbb{Q})$. Suppose that

$$
\sum_{i=1}^{m} k_i b_{ij} \in \mathbb{Z}, \quad \sum_{j=1}^{n} b_{ij} = 0
$$

and parameters $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to a compact set $K$. Then there exists a positive number $r$, which is independent of $\lambda$, such that the power series

$$
\sum_{k' \in L' + \hat{\lambda}} \frac{x^{b^{(1)}_{k'}} \cdots x^{b^{(n-d)}_{k'}}}{\Gamma (\lambda + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)}
$$

is convergent in $|x^{b^{(1)}_{k'}}|, \ldots, |x^{b^{(n-d)}_{k'}}| < r$.

The proof of this lemma can be done by elementary estimates of Gamma functions. See [7, pp.18–21] if readers are interested in the details. Since

$$
k' \in L' + \hat{\lambda} \iff \sum_{i=1}^{n-d} k'_i b^{(i)} \in \mathbb{Z}^n
$$

it follows from Lemma 1 that there exists a positive constant $r$ such that the series converge in

$$
|x^{b^{(1)}_k}|, \ldots, |x^{b^{(n-d)}_k}| < r
$$

for any $s$ in the open set $U$. We may suppose $r < 1$. Take the log of (3.1). Then we have

$$
b^{(k)} \cdot (\log |x_1|, \ldots, \log |x_n|) < \log |r| < 0 \quad \forall k \in \{1, \ldots, n - d\}
$$
Following [4], for the simplex $\tau$ and $r$, we define the set $C(A, \tau, r)$ as follows.

$$C(A, \tau, r) = \{ \psi \in \mathbb{R}^n \mid \exists \varphi \in \mathbb{R}^d, \psi_i - (\varphi, a_i) \begin{cases} > -\log |r|, & i \notin \tau, \\ = 0, & i \in \tau, \end{cases} \}$$

The condition (3.2) and $(-\log |x_1|, \ldots, -\log |x_n|) \in C(A, \tau, r)$ is equivalent (see [3, section 4] as to the proof).

Since $\Delta$ is a regular triangulation of $A$, $\bigcap_{r \in \Delta} C(A, \tau, r)$ is an open set. Therefore, when $s$ lies in the open set $U$ and $-\log |x|$ lies in the above open set, the $\text{vol}(A)$ linearly independent solutions converge.

Let us proceed on the proof for the inhomogeneous case. We suppose that $A$ is not homogeneous and has only non-zero column vectors. We define the homogenized matrix as

$$\tilde{A} = \begin{pmatrix} 1 & \cdots & 1 & 1 \\ a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{dn} & \cdots & a_{dn} & 0 \end{pmatrix} \in M(d+1, n+1, \mathbb{Z}).$$

For $s = (s_1, \ldots, s_n) \in \mathbb{C}^d$ and a generic complex number $s_0$, we put $\tilde{s} = (s_0, s_1, \ldots, s_d)$. We suppose that $\tau = \{n-d+1, \ldots, d, d+1\}$ is a $(d+1)$-simplex. Let us take an admissible $\lambda$ for $\tau$ such that $\tilde{A}\lambda = \tilde{s}$ and $\hat{\lambda} = (\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ as in the proof of the homogeneous case. Put $\lambda = (\lambda_1, \ldots, \lambda_n)$. Consider the solution of the hypergeometric system for $\tilde{A}$

$$\tilde{\phi}_\tau(\hat{\lambda}; \tilde{x}) = \sum_{k' \in L' \cap \mathbb{Z}} \tilde{x}^{\lambda + \sum_{i=1}^{n-d} k'_i b_i(i)} \Gamma(\lambda + \sum_{i=1}^{n-d} k'_i b_i + 1)$$

and the series

$$\phi_\tau(\lambda; x) = \sum_{k' \in L' \cap \mathbb{Z}} \frac{\prod_{j=1}^n x_j^{\lambda + \sum_{i=1}^{n-d} k'_i b_{ij}}}{\prod_{j=1}^n x_j^{\lambda + \sum_{i=1}^{n-d} k'_i b_{ij} + 1}}$$

($\tilde{x} = (x_1, \ldots, x_{n+1}), x = (x_1, \ldots, x_n)$). Here, the set $S$ is a subset of $L'$ such that an integer in $\mathbb{Z}_{\leq 0}$ does not appear in the arguments of the Gamma functions in the denominator. We note that $L'$ for $\tilde{A}$ and $L'$ for $A$ agree, which can be proved as follows. Let $(k_1, \ldots, k_{n+1})$ be in the kernel of $\tilde{A}$ in $\mathbb{Q}^{n+1}$. Since $\tilde{A}$ contains the row of the form $(1, \ldots, 1)$, then $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ implies that $k_{n+1}$ is an integer. The conclusion follows from the definition of $L'$.
Definition 2. We call $\phi_\tau(\lambda; x)$ the dehomogenization of $\tilde{\phi}_\tau(\lambda; \tilde{x})$.

Intuitively speaking, the dehomogenization is defined by “forgetting” the last variable $x_{n+1}$ associated $\Gamma$ factors. See Example 1.

Formal series solutions for the hypergeometric system for inhomogeneous $A$ do not converge in general. However, we can construct $\text{vol}(A)$ convergent series solutions as the dehomogenization of a set of series solutions for $\tilde{A}$ hypergeometric system associated to a regular triangulation on $\tilde{A}$ induced by a “nice” weight vector $\tilde{w}(\varepsilon)$, which we will define. Put $\tilde{w} = (1, \ldots, 1, 0) \in \mathbb{R}^{n+1}$. Since the Gröbner fan for the toric variety $I_{\tilde{A}}$ is a polyhedral fan, the following fact holds.

Lemma 2. For any $\varepsilon > 0$, there exists $\tilde{v} \in \mathbb{R}^{n+1}$ such that $\tilde{w}(\varepsilon) := \tilde{w} + \varepsilon \tilde{v}$ lies in the interior of a maximal dimensional Gröbner cone of $I_{\tilde{A}}$. We may also suppose $\tilde{v}_{n+1} = 0$.

Proof. Let us prove the lemma. The first part is a consequence of an elementary property of the fan. When $I$ is a homogeneous ideal in the ring of polynomials of $n+1$ variables, we have

$$\text{in}_u(I) = \text{in}_{u+t(1, \ldots, 1)}(I)$$

for any $t$ and any weight vector $u$. In other words, $\tilde{u}$ and $\tilde{u} + t(1, \ldots, 1)$ lie in the interior of the same Gröbner cone.

When the weight vector $\tilde{w}(\varepsilon) = \tilde{w} + \varepsilon \tilde{v}$ lies in the interior of the Gröbner cone, we define a new $\tilde{v}$ by $\tilde{v} - \tilde{v}_{n+1}(1, \ldots, 1)$. Since the initial ideal does not change with this change of weight, we may assume that $\tilde{v}_{n+1} = 0$ for the new $\tilde{v}$.

Since the Gröbner fan is a refinement of the secondary fan and hence $\tilde{w}(\varepsilon)$ is an interior point of a maximal dimensional secondary cone, it induces a regular triangulation ([12] p.71, Proposition 8.15). We denote by $\Delta$ the regular triangulation on $\tilde{A}$ induced by $\tilde{w}(\varepsilon)$. For a $d$-simplex $\tau \in \Delta$, we define $b^{(i)}$ as in the proof of the homogeneous case. Since the weight for $\tilde{a}_{n+1}$ is the lowest, $n+1 \in \tau$ holds. We can change indices of $\tilde{a}_1, \ldots, \tilde{a}_n$ so that $\tau = \{n-d+1, \ldots, n+1\}$ without loss of generality.

Let us prove that the dehomogenized series $\phi_\tau(\lambda; x)$ converge. It follows from a characterization of the support of the series [11, Theorem 3.4.2] that we have

$$\tilde{w}(\varepsilon) \cdot \left(\sum_{i=1}^{n-d} k^i b^{(i)} + \lambda\right) \geq \tilde{w}(\varepsilon) \cdot \lambda, \quad \forall k' \in L' \cap S.$$
Here, \( S \) is a set such that \( Z_{<0} \) does not appear in the denominator of the \( \Gamma \) factors. Take the limit \( \varepsilon \to 0 \) and we have

\[
\tilde{w} \cdot \sum_{i=1}^{n-d} k'_i b^{(i)} \geq 0, \quad \forall k' \in L' \cap S.
\]

From Lemma 2, \( \tilde{w}(\varepsilon) \in C(\tilde{A}, \tau) \) holds and then

\[
\tilde{w}(\varepsilon) \cdot b^{(i)} \geq 0.
\]

Similarly, by taking the limit \( \varepsilon \to 0 \), we have

\[
\tilde{w} \cdot b^{(i)} = \sum_{j=1}^{n} b_{ij} \geq 0.
\]

Therefore, we have \( \sum_{j=1}^{n+1} b_{ij} = 0 \), the inequality \( b_{i,n+1} \leq 0 \) holds for all \( i \).

Since \( k'_1 \geq -\lambda_1, \ldots, k'_{n-d} \geq -\lambda_{n-d} \), we have

\[
\sum_{i=1}^{n-d} k'_i b_{i,n+1} \leq -\sum_{i=1}^{n-d} \lambda_i b_{i,n+1}
\]

Note that the right hand side is a non-negative number. Suppose that \( \lambda_{n+1} \) is negative. In terms of the Pochhammer symbol we have \( \Gamma(\lambda_{n+1} - m) = \Gamma(\lambda_{n+1})(-\lambda_{n+1} + 1; m)^{-1} (-1)^m \), then we can estimate the \((n+1)\)-th gamma factors as

\[
\left| \Gamma(\lambda_{n+1} + \sum_{i=1}^{n-d} k'_i b_{i,n+1} + 1) \right| = \left| \Gamma(\lambda_{n+1} + 1) \right| \left| \left( -\lambda_{n+1} \sum_{i=1}^{n-d} k'_i b_{i,n+1} \right)^{-1} \right| \\
\leq c' \left| \Gamma(\lambda_{n+1} + 1) \right| \left| \left( -\lambda_{n+1} \sum_{i=1}^{n-d} \lambda_i b_{i,n+1} \right)^{-1} \right| \\
= c
\]

(3.4)

Here, \( c' \) and \( c \) are suitable constants.

When \( \lambda_{n+1} \geq 0 \), there exists only finite set of values such that \( \lambda_{n+1} + \sum_{i=1}^{n-d} k'_i b_{i,n+1} \geq 0 \). Then, we can show the inequality (3.4) in an analogous way.

Now, by (3.4), we have

\[
\frac{1}{\prod_{j=1}^{n} \Gamma(\lambda_j + \sum_{i=1}^{n-d} k'_i b_{ij} + 1)} \leq c \frac{1}{\Gamma(\lambda + \sum_{i=1}^{n-d} k'_i b^{(i)} + 1)}
\]
We note that the right hand side is the coefficient of the series solution for the homogeneous system for $\tilde{A}$ and the series converge for $(-\log|x_1|, \ldots, -\log|x_{n+1}|) \in C(\tilde{A}, \tau, r) \ (r < 1)$ uniformly with respect to $\tilde{s}$ in an open set.

Put $x_{n+1} = 1$. Since $-\log|x_{n+1}| = 0$ and $\tilde{w}(\varepsilon) \in \{y \mid y_{n+1} = 0\}$, we can see that

$$\bigcap_{\tau \in \Delta} C(\tilde{A}, \tau, r) \cap \{y \mid y_{n+1} = 0\}$$

is a non-empty open set of $\mathbb{R}^n$. Therefore the dehomogenized series $\phi_r(\lambda; x)$ converge in an open set in the $(s, x)$ space.

**Theorem 2.** The dehomogenized series $\phi_r(\lambda; x)$ satisfies the hypergeometric differential-difference system $H_A$ and they are linearly independent convergent solutions of $H_A$ when $\lambda$ runs over admissible exponents associated to the initial system induced by the weight vector $\tilde{w}(\varepsilon)$.

**Proof.** Since $A\lambda = s$, it is easy to show that they are formal solutions of the differential-difference system $H_A$. We will prove that we can construct $m$ linearly independent solutions. We note that the weight vector $\tilde{w}(\varepsilon) = (1, \ldots, 1, 0) + \varepsilon v \in \mathbb{R}^{n+1}$ is in the neighborhood of $(1, \ldots, 1, 0) \in \mathbb{R}^{n+1}$ and in the interior of a maximal dimensional Gröbner cone of $I_{\tilde{A}}$.

It follows from [11, p.119] that the minimal generating set of in$(1, \ldots, 1, 0) I_{\tilde{A}}$ does not contain $\partial_{n+1}$. Since

$$\text{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}} = \text{in}_v(\text{in}_{(1, \ldots, 1, 0)} I_{\tilde{A}})$$

does not contain $\partial_{n+1}$, we have

$$M = \langle \text{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}} \rangle = \langle \text{in}_w I_A \rangle \quad \text{in } \mathbb{C}[\partial_1, \ldots, \partial_{n+1}].$$

Here, we define $w(\varepsilon)$ with $\tilde{w}(\varepsilon) = (w(\varepsilon), 0)$. Put $\tilde{\theta} = (\theta_1, \ldots, \theta_{n+1})$. From [11, Theorem 3.1.3], for generic $\tilde{\beta} = (\beta_0, \tilde{\beta})$, $\tilde{\beta} \in \mathbb{C}^d$, the initial ideal $\text{in}_{(\tilde{w}(\varepsilon), \tilde{w}(\varepsilon))} H_{\tilde{A}}(\tilde{\beta})$ is generated by $\text{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})$ and $\tilde{A}\tilde{\theta} - \tilde{\beta}$. Let us denote by $T(M)$ the standard pairs of $M$. From [11, Theorem 3.2.10], the initial ideal

$$\langle \text{in}_{\tilde{w}(\varepsilon)} I_{\tilde{A}}, \tilde{A}\tilde{\theta} - \tilde{\beta} \rangle$$

has $\#T(M) = \text{vol}(\tilde{A})$ linearly independent solutions of the form

$$\{\tilde{x}^{\tilde{\lambda}} \mid (\partial^n, T) \in T(M)\}$$

Here, $\tilde{\lambda}$ is defined by $\tilde{\lambda}_i = a_i \in \mathbb{Z}_{\geq 0}, \forall i \notin T$ and $\tilde{A}\tilde{\lambda} = \tilde{\beta}$. Note that $\tilde{\lambda}$ is admissible for the $d$-simplex $T$. 

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Since we have
\[(\text{in}_{\varepsilon}(\varepsilon I_A)) = (\text{in}_{\varepsilon}(\varepsilon I_A))\]
the difference between
\[(\text{in}_{\varepsilon}(\varepsilon I_A), A\theta - \beta)\]
and (3.5) is only
\[\theta_1 + \cdots + \theta_n + \theta_{n+1} - \beta_0\]
and other equations do not contain \(x_{n+1}, \partial_{n+1}\).

For any \((\partial^a, T) \in T(M)\), we have \(n + 1 \in T\). Therefore, the two solution
spaces (3.6) and (3.5) are isomorphic under the correspondence
\[x^\lambda \mapsto \tilde{x}^{\tilde{\lambda}}\]
(3.7)

Here, we put \(\tilde{\lambda} = (\lambda, \lambda_{n+1})\) and \(\lambda_{n+1}\) is defined by
\[\sum_{i=1}^{n} \lambda_i + \lambda_{n+1} - \beta_0 = 0\]
It follows from [11, Theorem 2.3.11 and Theorem 3.2.10] that
\[\{\tilde{x}^{\tilde{\lambda}} \mid (\partial^a, T) \in T(M)\}\]
are \(C\)-linearly independent. Therefore, from the correspondence (3.7), the functions
\[\{x^\lambda \mid (\partial^a, T) \in T(M)\},\]
of which cardinality is \(\text{vol}(A)\), are \(C\)-linearly independent. Hence, series
solutions with the initial terms
\[\left\{\frac{x^\lambda}{\Gamma(\lambda + 1)} \mid (\partial^a, T) \in T(M)\right\}\]
are \(C\) linearly independent, which implies the linear independence of series
solutions with these starting terms [11]. We have completed the proof of the
theorem and also that of Theorem 1. \(\Box\)

**Theorem 3.** The holonomic rank of \(H_A\) is equal to the normalized volume
of \(A\).
Proof. First we will prove $\text{rank}(H_A) \leq \text{vol}(A)$. It follows from the Adolphson’s theorem ([1]) that the holonomic rank of $A$-hypergeometric system $H_A(\beta)$ is equal to the normalized volume of $A$ for generic parameters $\beta$. It implies that the standard monomials for a Gröbner basis of the $A$-hypergeometric system $H_A(s)$ in $C(s, x)(\partial_1, \ldots, \partial_n)$ consists of $\text{vol}(A)$ elements. We note that elements in the Gröbner basis can be regarded as an element in the ring of differential-difference operators with rational function coefficients $U$. We denote by $\partial_j$ and $r_j$ the creation and annihilation operators. The existence of them are proved in [10, Chapter 4]. Then, we have

$$H_j = \partial_j - \prod_{i=1}^{n} S_i^{-a_{ij}} \in H_A$$

and

$$B_j = r_j - \prod_{i=1}^{n} S_i^{a_{ij}} \in H_A, \quad r_j \in C(s, x)(\partial_1, \ldots, \partial_n).$$

Since the column vectors of $A$ generate the lattice $\mathbb{Z}^d$, we obtain from $B_j$’s and $H_j$’s elements of the form $S_i - p(s, x)\partial, \ S_i^{-1} - q(s, x)\partial \in H_A$. It implies the number of standard monomials of a Gröbner basis of $H_A$ with respect to a block order such that $S_1, \ldots, S_n > S_1^{-1}, \ldots, S_n^{-1} > \partial_1, \ldots, \partial_n$ is less than or equal to $\text{vol}(A)$.

Second, we will prove $\text{rank}(H_A) \geq \text{vol}(A)$. We suppose that $\text{rank}(H_A) < \text{vol}(A)$ and will induce a contradiction. For the block order $S_1, \ldots, S_d > S_1^{-1}, \ldots, S_d^{-1} > \partial_1, \ldots, \partial_n$, we can show that the standard monomials $T$ of a Gröbner basis of $H_A$ in $U$ contains only differential terms and $\# T < \text{vol}(A)$ by the assumption. Let $T'$ be the standard monomials of Gröbner basis $G(s)$ of $H_A(s)$ in the ring of differential operators with rational function coefficients $D(s)$. Note that $\# T' = \text{vol}(A)$. Then $T$ is a proper subset of the set $T'$. For $r \in T' \setminus T$, it follows that

$$\partial^r \equiv \sum_{\alpha \in T} c_\alpha(x, s)\partial^\alpha \pmod{H_A}.$$

From Theorem 2, we have convergent series solutions $f_1(s, x), \ldots, f_m(s, x)$ of $H_A$, where $m = \text{vol}(A)$. So,

$$\partial^r \cdot f_i = \sum_{\alpha \in T} c_\alpha(x, s)\partial^\alpha \cdot f_i \quad (3.8)$$

Since $f_1(s, x), \ldots, f_m(s, x)$ are linearly independent, the Wronskian standing
for $T'$

$$W(T'; f)(x, s) = \begin{vmatrix} f_1(s; x) & \cdots & f_m(\beta; x) \\ \partial^{\delta} f_1(s; x) & \cdots & \partial^{\delta} f_m(\beta; x) \\ \vdots & \cdots & \vdots \\ \partial^{\delta} f_1(\beta; x) & \cdots & \partial^{\delta} f_m(\beta; x) \end{vmatrix} (\partial^{\delta} \in T' \setminus \{1\})$$

is non-zero for generic number $s$. However $r \in T'$ and (3.8) induce the Wronskian $W(T'; f)(s, x)$ is equal to zero.

Finally, by $\text{rank}(H_A) \leq \text{vol}(A)$ and $\text{rank}(H_A) \geq \text{vol}(A)$, the theorem is proved.

**Example 1.** Put $A = (1 \ 2 \ 3)$ and $\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$. This is Airy type integral [11, p.223].

The matrix $\tilde{A}$ is homogeneous. For $\tilde{w}(\varepsilon) = (1, 1, 1, 0) + \frac{1}{100} (1, 0, 0, 0)$, the initial ideal $\text{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})$ is generated by $\partial_1^1, \partial_1 \partial_2, \partial_1 \partial_3, \partial_2^2$. Note that the initial ideal does not contain $\partial_4$. We solve the initial system $(\tilde{A} \tilde{\theta} - \tilde{s}) \bullet \ g = 0$, $(\text{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})) \bullet \ g = 0$. The standard pairs $(\partial^a, T)$ for $\text{in}_{\tilde{w}(\varepsilon)}(I_{\tilde{A}})$ are $(\partial_1^1 \partial_2^1, \{3, 4\})$, $(\partial_1^1 \partial_2^2, \{3, 4\})$, $(\partial_1^1 \partial_2^3, \{3, 4\})$. Hence, the solutions for the initial system are

$$x_1^0 x_2^0 x_3^{(s_1-2)/3} x_4^{s_0-1-(s_1-2)/3}, x_1^0 x_2^0 x_3^{s_1/3} x_4^{s_0-s_1/3}, x_1^0 x_2^0 x_3^{(s_1-4)/3} x_4^{s_0-2-(s_1-4)/3}$$

([11]). Therefore, the $\mathcal{A}$-hypergeometric differential-difference system $H_{\tilde{A}}$
has the following series solutions.

\[ \tilde{\phi}_1(\tilde{\lambda}, \tilde{x}) = x_4^{s_0} \left( \frac{x_2}{x_4} \right) \left( \frac{x_3}{x_4} \right)^{s_1-2} \sum_{k_1 \geq 0, \ k_2 \geq -1 \atop (k_1, k_2) \in L'} \frac{\left( x_1 x_3^{-1/3} x_4^{-2/3} \right)^{k_1} \left( x_2 x_3^{-2/3} x_4^{-1/3} \right)^{k_2}}{k_1! (k_2 + 1)! \Gamma \left( \frac{s_1 - k_1 - 2k_2 + 1}{3} \right) \Gamma \left( \frac{s_0 - s_1 - 2k_1 - k_2 + 2}{3} \right)} \]

\[ \tilde{\phi}_2(\tilde{\lambda}, \tilde{x}) = x_4^{s_0} \left( \frac{x_3}{x_4} \right)^{s_1} \sum_{k_1 \geq 0, \ k_2 \geq 0 \atop (k_1, k_2) \in L'} \frac{\left( x_1 x_3^{-1/3} x_4^{-2/3} \right)^{k_1} \left( x_2 x_3^{-2/3} x_4^{-1/3} \right)^{k_2}}{k_1! k_2! \Gamma \left( \frac{s_1 - k_1 - 2k_2 + 1}{3} \right) \Gamma \left( \frac{s_0 - s_1 - 2k_1 - k_2 + 3}{3} \right)} \]

\[ \tilde{\phi}_3(\tilde{\lambda}, \tilde{x}) = x_4^{s_0} \left( \frac{x_2}{x_4} \right)^2 \left( \frac{x_3}{x_4} \right)^{s_1-4} \sum_{k_1 \geq 0, \ k_2 \geq -2 \atop (k_1, k_2) \in L'} \frac{\left( x_1 x_3^{-1/3} x_4^{-2/3} \right)^{k_1} \left( x_2 x_3^{-2/3} x_4^{-1/3} \right)^{k_2}}{k_1! (k_2 + 2)! \Gamma \left( \frac{s_1 - k_1 - 2k_2 - 1}{3} \right) \Gamma \left( \frac{s_0 - s_1 - 2k_1 - k_2 + 2}{3} \right)} \]

Here,

\[ L' = \{(k_1, k_2) \mid k_1 \equiv 0 \mod 3, k_2 \equiv 0 \mod 3\} \cup \{(k_1, k_2) \mid k_1 \equiv 1 \mod 3, k_2 \equiv 1 \mod 3\} \]

The matrix \( A \) is not homogeneous and by dehomogenizing the series solution for \( \tilde{A} \) we obtain the following series solutions for the \( A \)-hypergeometric differential-difference system \( H_A \).

\[ \phi_1(\lambda, x) = x_2 x_3^{s_1-2} \sum_{k_1 \geq 0, \ k_2 \geq -1 \atop (k_1, k_2) \in L'} \frac{\left( x_1 x_3^{-1/3} x_4^{-2/3} \right)^{k_1} \left( x_2 x_3^{-2/3} x_4^{-1/3} \right)^{k_2}}{k_1! (k_2 + 1)! \Gamma \left( \frac{s_1 - k_1 - 2k_2 + 1}{3} \right)} \]

\[ \phi_2(\lambda, x) = x_3^{s_1} \sum_{k_1 \geq 0, \ k_2 \geq 0 \atop (k_1, k_2) \in L'} \frac{\left( x_1 x_3^{-1/3} x_4^{-2/3} \right)^{k_1} \left( x_2 x_3^{-2/3} x_4^{-1/3} \right)^{k_2}}{k_1! k_2! \Gamma \left( \frac{s_1 - k_1 - 2k_2 + 3}{3} \right)} \]

\[ \phi_3(\lambda, x) = x_2 x_3^{s_1-4} \sum_{k_1 \geq 0, \ k_2 \geq -2 \atop (k_1, k_2) \in L'} \frac{\left( x_1 x_3^{-1/3} x_4^{-2/3} \right)^{k_1} \left( x_2 x_3^{-2/3} x_4^{-1/3} \right)^{k_2}}{k_1! (k_2 + 2)! \Gamma \left( \frac{s_1 - k_1 - 2k_2 - 1}{3} \right)} \]
Here $\phi_k(x)$ is the dehomogenization of $\tilde{\phi}_k(x)$.

Finally, let us present a difference Pfaffian system for $A$. It can be derived by using Gröbner bases of $H_A$ and has the following form:

$$S_1 \begin{pmatrix} f \\ x_3 \partial_3 \bullet f \\ S_1 \bullet f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{s_1 x_1}{6 x_2} & -\frac{3 x_1 x_3 - 4 x_2^2}{6 x_2^2} & \frac{2 (s_1 - 1) x_2 + x_1^2}{6 x_2^2} \\ \frac{5}{6 x_2} & -\frac{2 x_1}{3 x_2} & -\frac{x_1^2}{x_2} \end{pmatrix} \begin{pmatrix} f \\ x_3 \partial_3 \bullet f \\ S_1 \bullet f \end{pmatrix}.$$ 

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