Study of Steady State Heat Conduction in Composite Layer by Linear Triangular Finite Element

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Abstract. Finite element method with linear triangular shape function was used to numerically investigate steady state heat conduction in composite layer. This method divides each layer into a number of small elements. By defining elements, local node, global node, and temperature boundary conditions, an element coefficient matrix can be formed. Element coefficient matrix needs to be assembled to form a global coefficient matrix. The solution of the global coefficient matrix will provide the temperature value at each node. In this research, variations of the thermal conductivity value and temperature boundary conditions are applied. The results of the study will show the effect of different thermal conductivity value against temperature gradient.

1. Introduction
Heat conduction is a form of heat transfer which is very important to be reviewed. Many systems in the world of engineering are related to this process. For example, in the field of electronics will take good care of the heat conduction process so that electric machines can achieve optimum conversion efficiency. The world of architecture needs to study heat conduction because it relates to the thermal stresses experienced by the material. Energy conversion machines produced by nuclear power can be said to have absolute requirements for detailed analysis of the heat transfer process. This is related to the safety factor in the operation of the power plant. There are many more energy engineering systems that are directly related to the heat transfer process. The process of heat transfer can occur not only involving one material but also on the layer of material with different thermal conductivity. Scientific studies can be carried out from simple conditions for heat transfer in one-dimensional direction, or even up to two and three-dimensional analysis. The heat conduction equation can be solved by using numerical methods such as monte carlo method [1], finite volume method [2], and finite block method [3].

Besides these methods, the finite element method is one of the numerical methods that can be used to solve the heat conduction equation. The calculation step with this method is very dependent on the shape function selection. For two-dimensional analysis can be used linear triangular or quadratic
triangular. In this study the heat conduction process in the composite layer will be analyzed using a linear triangular finite element. The results of the calculation will give an overview of the heat distribution process due to the given conduction. And also can be seen the relationship between the value of thermal conductivity of the material to its temperature distribution.

2. Method and Parameter

The heat conduction equation without a heat source for the steady state case is given by [4],

$$\frac{d^2 T}{dx^2} = 0$$

(1)

$k$ is the thermal conductivity and $A$ is the cross-sectional area perpendicular to the direction of heat flow. This equation is solved by the finite element method which has the following calculation steps.

2.1. Discretization of the system into finite elements

System discretization or meshing on the finite element method is done by dividing the total area or region into finite elements. The element or sub-region itself has several forms specified in this method. Different forms will give different governing equations to solve. Discrete elements in this method are formed by dots or called nodes. It should be noted that the node consists of local nodes and global nodes. Local nodes are related to element types while global nodes are related to the overall system. Discretization can be done by selecting the element type, defining each node and its coordinates, then defining elements, local nodes, and global nodes and determining the value of the boundary conditions at each node.

2.2. Determine the governing equation for each element

In this section we will form a matrix in each element called the element coefficient matrix. Suppose for a linear triangle element, it has a element coefficient matrix which is [5],

$$[C^{(e)}] = \begin{bmatrix} C^{(e)}_{11} & C^{(e)}_{12} & C^{(e)}_{13} \\ C^{(e)}_{21} & C^{(e)}_{22} & C^{(e)}_{23} \\ C^{(e)}_{31} & C^{(e)}_{32} & C^{(e)}_{33} \end{bmatrix}$$

(2)

The value of each element is determined by

$$C^{(e)}_{ij} = \frac{1}{4A} [P_i P_j + Q_i Q_j]$$

(3)

P and Q values are determined as follows.

$$P_1 = (y_2 - y_3), \quad P_2 = (y_3 - y_1), \quad P_3 = (y_1 - y_2)$$

$$Q_1 = (x_3 - x_2), \quad Q_2 = (x_1 - x_3), \quad Q_3 = (x_2 - x_1)$$

(4)

In the finite element method, $A$ is the element area obtained from this equation

$$A = \frac{1}{2} (P_2 Q_3 - P_3 Q_2)$$

(5)

$$P_1 + P_2 + P_3 = 0 = Q_1 + Q_2 + Q_3$$

(6)

$$\sum_{j=1}^{3} C^{(e)}_{ij} = 0 = \sum_{j=1}^{3} C^{(e)}_{ij}$$

(7)
2.3. Assemble the element equation

After we have calculated the element coefficient matrix for all elements, then we will assemble it to form a global coefficient matrix. Suppose there are two linear triangular elements as shown below.

![Figure 1. Two linear triangular elements](image)

Each has a element coefficient matrix that is,

$$
K_1 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
$$

$$
K_2 = \begin{bmatrix}
b_{22} & b_{23} & b_{24} \\
b_{32} & b_{33} & b_{34} \\
b_{42} & b_{43} & b_{44}
\end{bmatrix}
$$

(8)

then we can form a global coefficient matrix as follows,

$$
[K] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 \\
a_{21} & a_{22} + b_{22} & a_{23} + b_{23} & b_{24} \\
a_{31} & a_{32} + b_{32} & a_{33} + b_{33} & b_{34} \\
0 & b_{42} & b_{43} & b_{44}
\end{bmatrix}
$$

(9)

2.4. Solve the system of equations

In the discretization step we have defined the value of the boundary conditions at each node. This means that the global coefficient matrix will consist of elements with temperature values that are known and unknown. If the index f and p are defined as free and prescribed, the solution for the temperature value can be written,

$$
[A][T]=[B]
$$

(10)

$$
[T]=[A]^{-1}[B]
$$

(11)

$$
[T]=[T_f], \quad [A]=[C_{ff}]
$$

(12)

$$
[B]=-C_{ff}[T_p]
$$

(13)

In this study, the system consists of two layers arranged vertically as shown in figure 2. The thickness of each material is 0.5 m so the total area of the system is 1 m².
The definition of nodes and elements uses a linear triangular finite element for the above system, as shown in Figure 3. It can be seen that the number of nodes is 441 nodes with 800 elements. This means that in the later calculation the global matrix formed is a $441 \times 441$ matrix. In figure 3, the text that is red is an element and the text is black is a node.

Regarding the purpose of this study is to see the effect of the thermal conductivity value of the material on the temperature gradient, variations in $k$ values and variations in temperature difference ($\Delta T$) are carried out. The relationship between thermal conductivity and temperature gradients is trying to be understood by taking the pairs of values for three groups, namely low, middle, and high thermal conductivity groups. The properties of each group are shown in table 1.
Figure 4. Low thermal conductivity group (G1)

Figure 5. Middle thermal conductivity group (G2)

Figure 6. High thermal conductivity group (G3)

Table 1. Material properties [6]

| Material      | $k$ (W/cm °C) |
|---------------|---------------|
| Fiber Glass   | 0.00035       |
| Gypsum        | 0.0051        |
| Zirconium     | 0.228         |
| Lead          | 0.348         |
| Copper        | 3.98          |
| Aluminum      | 2.04          |

For variations in temperature differences, several values are taken, as shown in the following table.

Table 2. Temperature differences

| $T_{Top}$ (°C) | $T_{Bottom}$ (°C) | $ΔT$ (°C) |
|---------------|------------------|----------|
| 50            | 20               | 30       |
| 60            | 20               | 40       |
| 80            | 20               | 60       |
| 100           | 20               | 80       |
In addition to geometry in the form of vertical composite layers (figure 2), the study of heat conduction is also applied to other geometries in the form of a heat sink used in electronic cooling. The cross-section of heat sink is shown in figure 7.

Heat sink consists of two parts, namely the bottom plate and fins. Both are combined according to the thermal conductivity group (G1, G2, and G3). All the fins are of the same size (a = 0.15 m). The distance between fins is constant for all pairs of thermal conductivity groups with b = 0.15 m, c = 0.10 m, and d = 0.15 m. To find out about the effect of the thickness of the material on heat conduction, the thickness of the bottom plate (f) is varied by 0.25, 0.50 and 0.75 m. The temperature boundary conditions on the bottom plate are 100 °C and the top fins are 20 °C.

3. Results and Discussions
Results of equation (1) using linear triangular finite element for $\Delta T = 30 \, ^\circ\text{C}$ is presented in figure 8. Figure 8a, 8b, and 8c are temperature distributions for low, middle, and high thermal conductivity groups, respectively. The results in this figure show the temperature boundary conditions on the upper and lower sides of the material and their temperature distribution. There is a difference in the rate of heat transfer between each material group. This difference can be seen clearly in the color changes that occur. Because the boundary conditions are defined only on the upper and lower sides of the material, the heat conduction changes significantly in the vertical direction. Changes in the horizontal direction are not too different between adjacent nodes. For example, changes in the vertical direction at points y = 0.5 m are for G1, G2, and G3 at 21 °C, 31 °C, and 39 °C, respectively.
This relationship can also be seen when plotted the temperature distribution curve in figure 8d. The black curve is the temperature distribution in G1, the color is red for G2, and the color is green for G3. Each curve shows the difference in temperature gradient. Small temperature gradients indicate that heat conduction occurs quickly or vice versa. The G1 and G2 curves have the same trend of heat conduction rates because both of all have a thermal conductivity value for the material on the upper side which is smaller than the bottom side. This is different from the G3 curve which has a material on the top side with a greater thermal conductivity value than the bottom side. For temperature difference $\Delta T = 40 \, ^\circ C$ is presented in figure 9.
Figure 9. Temperature distribution with $\Delta T = 40 \, ^oC$, (a) Low thermal conductivity group, (b) Middle thermal conductivity group, (c) High thermal conductivity group, and (d) graph of $G_1$, $G_2$, and $G_3$ at $x = 0.5 \, m$.

In Figure 9, changes in the vertical direction at points $y = 0.5 \, m$ are for $G_1$, $G_2$, and $G_3$ at $22 \, ^oC$, $35 \, ^oC$, and $46 \, ^oC$, respectively. The curve obtained in figure 9d shows a trend similar to the curve in Figure 8d. This is because the composition and pair of materials used are still the same.
Figure 10. Temperature distribution with $\Delta T = 60 \, ^{\circ}C$, (a) Low thermal conductivity group, (b) Middle thermal conductivity group, (c) High thermal conductivity group, and (d) graph of $G_1$, $G_2$, and $G_3$ at $x = 0.5 \, m$

Temperature distribution for each group with $\Delta T = 60 \, ^{\circ}C$ is presented in Figure 10. This result shows a change in the vertical direction at the point $y = 0.5 \, m$ are for $G_1$, $G_2$, and $G_3$ at $23 \, ^{\circ}C$, $43 \, ^{\circ}C$, and $59 \, ^{\circ}C$, respectively. Whereas for $\Delta T = 80 \, ^{\circ}C$ a change in the vertical direction is obtained at the point $y = 0.5 \, m$ are for $G_1$, $G_2$, and $G_3$ at $25 \, ^{\circ}C$, $51 \, ^{\circ}C$, and $72 \, ^{\circ}C$, respectively. As shown in Figure 11.
Figure 11. Temperature distribution with $\Delta T = 80$ °C, (a) Low thermal conductivity group, (b) Middle thermal conductivity group, (c) High thermal conductivity group, and (d) graph of $G_1$, $G_2$, and $G_3$ at $x = 0.5$ m

The results of the calculation of heat conduction on the heat sink are shown in Figures 12, 13, and 14, respectively for $f = 0.25$, $f = 0.50$, and $f = 0.75$ m.
The results in Figure 12 show the temperature distribution on the bottom plate and fins. In accordance with the thermal conductivity group used it can be seen that figure 12c because it is a high thermal conductivity group, the heat transfer rate is faster than the results in figures 12b and 12a. In the boundary area between the fins and the bottom plate at 0.25 m thickness, for the G1 group the average temperature reached 99.69 °C, the G2 group reached 97.16, and the G3 group reached 92.16. The same thing was also seen for temperature distribution with f = 0.50 m. In figure 13, the results are obtained for the boundary area between the fins and the bottom plate for the G1 group reaching 96.80 °C, the G2 group reaching 77.50 °C, and the G3 group reaching 57.47 °C. Similar to the results of the calculation of the temperature distribution for the geometry in Figure 2, changes in the horizontal direction for each group are not too significant because the temperature boundary conditions are given in the vertical direction at the bottom of the plate and at the top of the fins.
For a bottom plate thickness of 0.75 m (figure 14), the calculation results for the boundary area of the two layers with G1 group temperatures reaching 75.24 °C, G2 groups reaching 36.23 °C, and G3 groups reaching 27.08 °C.

Overall these results indicate that there is an influence or relationship between the thermal conductivity of the material on the heat conduction produced. With the same principle, it is also possible to calculate heat conduction for more than two layers of material. Convergent requirements as validation of calculations with this method have also been achieved for all temperature differences. It is shown that for the steady state case the temperature boundary conditions on the upper and lower sides of the material layer can be fulfilled.
Figure 14. Temperature distribution of heat sink with f = 0.75 m , (a) Low thermal conductivity group, (b) Middle thermal conductivity group, (c) High thermal conductivity group

4. Conclusions
The solution of the heat conduction equation in the composite layer by linear triangular finite element shows that there is a relationship between the value of thermal conductivity and temperature gradient. In this calculation, the results show that G1 has a transfer rate smaller than G2 and G3. This is because G1 is a group of materials with a smaller thermal conductivity compared to G2 and G3. A small temperature gradient indicates that heat conduction occurs rapidly.

References
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