UNIQUENESS PROPERTY FOR C*-ALGEBRAS GIVEN BY
RELATIONS WITH CIRCULAR SYMMETRY

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Abstract. A general method of investigation of the uniqueness property for
C*-algebra equipped with a circle gauge action is discussed. It unifies isomor-
phism theorems for various crossed products and Cuntz-Krieger uniqueness
theorem for Cuntz-Krieger algebras.

1. Introduction

The origins of C*-theory and particularly the theory of universal C*-algebras
generated by operators that satisfy prescribed relations go back to the work of
W. Heisenberg, M. Bohr and P. Jordan on matrix formulation of quantum me-
chanics, and among the most stimulating examples are algebras generated by anti-
commutation relations and canonical commutation relations (in the Weyl form).
The great advantage of relations of CAR and CCR type is uniqueness of represen-
tation. Namely, due to the celebrated Slawny’s theorem, see e.g. [1], the
C*-algebras generated by such relations are defined uniquely up to isomorphisms preserving
the generators and relations. This uniqueness property is not only a strong mathemat-
ical tool but also has a significant physical meaning – if we had no such uniqueness,
different representations would yield different physics.

The aim of the present note is to advertise a program of developing a general
approach to investigation of uniqueness property and related problems based on
exploring the symmetries of relations. We focus here, as a first attempt, on circular
symmetries and propose a two-step method of investigation universal C*-algebra
C*(G, R) generated by a set of generators G subject to relations R which could be
schematically presented as follows:

\[(G, R, \{\gamma_\lambda\}_{\lambda \in \mathbb{T}}) \xrightarrow{\text{step 1}} (B_0, B_1) \text{ (Hilbert bimodule, reversible dynamics)} \xrightarrow{\text{step 2}} C^*(G, R) = B_0 \rtimes_{\mathcal{B}_1} \mathbb{Z} \text{ (universal C*-algebra)}\]

- we fix a circle gauge action \(\gamma = \{\gamma\}_{\lambda \in \mathbb{T}}\) on \(C^*(G, R)\) which is induced by a
circular symmetry in \((G, R)\); in the first step we associate to \(\gamma\) a non-commutative
reversible dynamical system which is realized via a Hilbert bimodule \((B_0, B_1)\), and
in the second step we use this system to determine the uniqueness property for
\(C^*(G, R)\).

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2. **Uniqueness property, universal C*-algebras and gauge actions**

Suppose we are given an abstract set of generators $G$ and a set of *-algebraic relations $R$ that we want to impose on $G$. Formally $G$ is a set and $R$ is a set consisting of certain *-algebraic relations in a free non-unital *-$\mathcal{A}$-algebra $\mathcal{A}$ generated by $G$. By a representation of the pair $(G, R)$ we mean a set of bounded operators $\pi = \{\pi(g)\}_{g \in G} \subseteq L(H)$ on a Hilbert space $H$ satisfying the relations $R$, and denote by $C^*(\pi)$ the C*-algebra generated by $\pi(g)$, $g \in G$. At this very beginning one faces the following two fundamental problems:

1. **(non-degeneracy problem)** Do there exists a faithful representation of $(G, R)$, i.e. a representation $\{\pi(g)\}_{g \in G}$ of $(G, R)$ such that $\pi(g) \neq 0$ for all $g \in G$?
2. **(uniqueness problem)** If one has two different faithful representations of $(G, R)$, do they generate isomorphic C*-algebras? More precisely, does for any two faithful representations $\pi_1, \pi_2$ of $(G, R)$ the mapping

$$\pi_1(g) \mapsto \pi_2(g), \quad g \in G,$$

extends to the (necessarily unique) isomorphism $C^*(\pi_1) \cong C^*(\pi_2)$?

The first problem is important and interesting in its own rights, see [2, 3], however here we would like to focus on the second problem and thus throughout we assume that all the pairs $(G, R)$ under consideration are non-degenerate. We say that $(G, R)$ possess uniqueness property if the answer to question 2 is positive.

Any representation $\pi$ of $(G, R)$ extends uniquely to a *-homomorphism, also denoted by $\pi$, from $\mathcal{A}$ into $L(H)$. The pair $(G, R)$ is said to be admissible if the function $|||·||| : \mathcal{A} \to [0, \infty]$ given by

$$|||w||| = \sup\{||\pi(w)|| : \pi \text{ is a representation of } (G, R)\}$$

is finite. Plainly, admissibility is a necessary condition for uniqueness property and therefore we make it our another standing assumption. Then the function $|||·||| : \mathcal{A} \to [0, \infty]$ is a C*-seminorm on $\mathcal{A}$ and its kernel

$$\mathbb{I} := \{w \in \mathcal{A} : |||w||| = 0\}$$

is a self-adjoint ideal in $\mathcal{A}$ – it is the smallest self-adjoint ideal in $\mathcal{A}$ such that the relations $R$ become valid in the quotient $\mathcal{A}/\mathbb{I}$. We put

$$C^*(G, R) := \frac{\mathcal{A}}{\mathbb{I}}$$

and call it a universal C*-algebra generated by $G$ subject to relations $R$, cf. [4]. C*-algebra $C^*(G, R)$ is characterized by the property that any representation of $(G, R)$ extends uniquely to a representation of $C^*(G, R)$ and all representations of $C^*(G, R)$ arise in that manner. In particular, $(G, R)$ possess uniqueness property if and only if any faithful representation of $(G, R)$ extends to a faithful representation of $C^*(G, R)$.

3. **Gauge actions – exploring the symmetries in the relations**

We would like to identify the uniqueness property of $(G, R)$ by looking at the symmetries in $(G, R)$. In order to formalize this we use a natural torus action $\{\gamma_\lambda\}_{\lambda \in \mathbb{T}^d}$ on $\mathcal{A}$ determined by the formula

$$\gamma_\lambda(g) = \lambda g, \quad \text{for } g \in G \quad \text{and} \quad \lambda = \{\lambda_h\}_{h \in \mathbb{G}} \in \mathbb{T}^\mathbb{G}$$

where $\mathbb{G}$ is a group.
where \( T = \{ z \in \mathbb{C} : |z| = 1 \} \) is a unit circle. A closed subgroup \( G \subset T^0 \) may be considered as a group of symmetries in the pair \((G, R)\) if the restricted action \( \gamma = \{ \gamma_\lambda \}_{\lambda \in \mathbb{T}} \) leaves invariant the ideal \( \mathbb{I} \). Any such group gives rise to a pointwisely continuous group action on \( C^*(G, R) \) and actions that arise in that manner are called gauge actions.

Let us from now on consider the case when \( G \cong T \), that is we have a circle gauge action \( \gamma = \{ \gamma_\lambda \}_{\lambda \in T} \) on \( C^*(G, R) \). Then for each \( n \in \mathbb{Z} \) the formula

\[
\mathcal{E}_n(b) := \int_T \gamma_\lambda(b) \lambda^{-n} \, d\lambda
\]

defines a projection \( \mathcal{E}_n : C^*(G, R) \rightarrow C^*(G, R) \), called \( n \)-th spectral projection, onto the subspace

\[
B_n := \{ b \in C^*(G, R) : \gamma_\lambda(b) = \lambda^n b \}
\]
called \( n \)-th spectral subspace for \( \gamma \), cf. e.g. [5]. Spectral subspaces specify a \( \mathbb{Z} \)-gradation on \( C^*(G, R) \). Namely, \( \bigoplus_{n \in \mathbb{Z}} B_n \) is dense in \( C^*(G, R) \), and

\[
B_n B_m \subset B_{n+m}, \quad B_n^* = B_{-n} \quad \text{for all} \quad n, m \in \mathbb{Z}.
\]

In particular, \( B_0 \) is a \( C^* \)-algebra – the fixed point algebra for \( \gamma \), and \( \mathcal{E}_0 : B \rightarrow B_0 \) is a conditional expectation. A circle action on a \( C^* \)-algebra \( B \) is called semi-saturated [4] if \( B \) is generated as a \( C^* \)-algebra by its first and zeroth spectral subspaces. We note that every continuous group endomorphism of \( T \) is of the form \( \lambda \mapsto \lambda^n \), for certain \( n \in \mathbb{Z} \), and hence it follows that \( G \subset \bigcup_{n \in \mathbb{Z}} B_n \). In particular, we have

**Lemma 1.** The circle gauge action \( \gamma = \{ \gamma_\lambda \}_{\lambda \in T} \) on \( C^*(G, R) \) is semi-saturated, that is \( C^*(G, R) = C^*(B_0, B_1) \) if and only if \( G = G_0 \cup G_1 \) for some disjoint sets \( G_0, G_1 \) and \( \gamma_\lambda(g_0) = g_0, \gamma_\lambda(g_1) = \lambda^n g_1 \), for all \( g_0, g_1 \in G_i \).

We introduce an important necessary condition for \((G, R)\) to possess uniqueness property.

**Proposition 2.** The following conditions are equivalent:

i) each faithful representation of \((G, R)\) give rise to a faithful representation of the fixed-point algebra \( B_0 \).

ii) each faithful representation \( \pi \) of \((G, R)\) give rise to a faithful representation of \( C^*(G, R) \) if and only if there is a circle action \( \beta \) on \( C^*(\pi) \) such that

\[
\beta_\pi(\pi(g)) = \pi(\gamma_\pi(g)), \quad g \in G.
\]

**Proof:** i) \( \Rightarrow \) ii). It suffices to apply the gauge invariance uniqueness for circle actions, see e.g. [3] 2.9] or [6] 4.2]. ii) \( \Rightarrow \) i). Assume that \( \pi \) is a faithful representation of \((G, R)\) such that its extension is not faithful on \( B_0 \). The spaces \( \{ \pi(B_n) \}_{n \in \mathbb{Z}} \) form a \( \mathbb{Z} \)-graded \( C^* \)-algebra and thus by [6] 4.2], there is a (unique) \( C^* \)-norm \( \| \cdot \|_\beta \) on \( \bigoplus_{n \in \mathbb{Z}} \pi(B_n) \) such that the circle action \( \beta \) on \( \bigoplus_{n \in \mathbb{Z}} \pi(B_n) \) established by gradation extends onto the \( C^* \)-algebra \( B = \bigoplus_{n \in \mathbb{Z}} \pi(B_n) \). Mapping \( \pi \) with the embedding \( \bigoplus_{n \in \mathbb{Z}} \pi(B_n) \subset B \) one gets a faithful representation \( \pi' \) of \((G, R)\) which is gauge-invariant but not faithful on \( C^*(G, R) \).

In the literature the statements showing that the condition ii) in Proposition 2 holds are often called gauge-invariance uniqueness theorems and therefore we shall say that the triple \((G, R, \gamma)\) has the gauge-invariance uniqueness property if each faithful representation of \((G, R)\) give rise to a faithful representation of the fixed-point algebra \( B_0 \). In particular, this always holds for triples \((G, R, \gamma)\) such that
$C^*(G, \mathcal{R})$ can be modeled as relative Cuntz-Pimsner algebra, see [3] Sect. 9] and sources cited there.

4. From relations to Hilbert bimodules

Let us fix a pair $(G, \mathcal{R})$ with a circle gauge action $\gamma = \{ \gamma_\lambda \}_{\lambda \in \mathbb{T}}$. It follows from (11) that $B_1$ can be naturally viewed as a Hilbert bimodule over $B_0$, in the sense introduced in [7, 1.8]. Namely, $B_1$ is a $B_0$-bimodule with bimodule operations inherited from $C^*(G, \mathcal{R})$ and additionally is equipped with two $B_0$-valued inner products

$$(a, b)_R := a^*b, \quad L(a, b) := ab^*$$

that satisfy the so-called imprimitivity condition: $a \cdot (b, c)_R = L(a, b) \cdot c = ab^*c$, for all $a, b, c \in B_1$. Thus we can consider crossed product $B_1 \rtimes_{B_0} \mathbb{Z}$ of $B_0$ by the Hilbert bimodule $B_1$ constructed in [8], which could be alternatively defined as the universal $C^*$-algebra:

$$B_1 \rtimes_{B_0} \mathbb{Z} = C^*(G_\gamma, \mathcal{R}_\gamma)$$

where $G_\gamma = B_0 \cup B_1$ and $\mathcal{R}_\gamma$ consists of all algebraic relations in the Hilbert bimodule $(B_0, B_1)$.

**Proposition 3.** We have a natural embedding $B_1 \rtimes_{B_0} \mathbb{Z} \rightarrow C^*(G, \mathcal{R})$ which is an isomorphism if and only if $\gamma$ is semi-saturated. Moreover, if $\gamma$ is semi-saturated, then the following conditions are equivalent:

i) $(G, \mathcal{R})$ possess uniqueness property

ii) $(G, \mathcal{R}, \gamma)$ has gauge-invariance uniqueness property and $(G_\gamma, \mathcal{R}_\gamma)$ possess uniqueness property

**Proof.** Since the homomorphism $B_1 \rtimes_{B_0} \mathbb{Z} \rightarrow C^*(G, \mathcal{R})$ is gauge-invariant and injective on $B_0$ it is injective onto the whole $B_1 \rtimes_{B_0} \mathbb{Z}$ by [5] 2.9. The rest, in view of Proposition 2 is clear.

The Hilbert bimodule $(B_0, B_1)$ is an imprimitivity bimodule (called also Morita-Rieffel equivalence bimodule), see [9], if and only if $B_1^*B_0 = B_0$ and $B_1B_0^* = B_0$. In general, $B_1^*B_1$ and $B_1B_0^*$ are non-trivial ideals in $B_0$ and we may treat $B_0$ as a $B_1^*B_0 \rightarrow B_1B_0^*$-imprimitivity bimodule. This means, cf. [9] Cor. 3.33], that the induced representation functor

$$\hat{h} : B_1 \rightarrow \text{Ind}$$

is a homeomorphism $\hat{h} : B_1^*B_1 \rightarrow B_1B_0^*$ between the spectra of $B_1^*B_1$ and $B_1B_0^*$. Treating these spectra as open subsets of the spectrum $\hat{B}_0$ of $B_0$ we may treat $\hat{h}$ as a partial homeomorphism of $\hat{B}_0$. We shall say that $(B, h)$ is a partial dynamical system dual to the bimodule $(B_0, B_1)$. Partial homeomorphism $\hat{h}$ is said to be topologically free if for each $n \in \mathbb{N}$ the set of points in $\hat{B}_0$ for which the equality $\hat{h}^n(x) = x$ (makes sense and) holds has empty interior.

**Theorem 4** (main result). Suppose that the partial homeomorphism $\hat{h} = B_1 \rightarrow \text{Ind}$ is topologically free. Then the pair $(G_\gamma, \mathcal{R}_\gamma)$ possess uniqueness property and in particular

i) if $(G, \mathcal{R}, \gamma)$ possess gauge-invariance uniqueness property, then any faithful representation of $(G, \mathcal{R})$ extends to the faithful representation of $B_1 \rtimes_{B_0} \mathbb{Z} \subset C^*(G, \mathcal{R})$. 
ii) if $\gamma$ is semi-saturated and $(G, R, \gamma)$ possess gauge-invariance uniqueness property, then $(G, R)$ possess uniqueness property.

Proof. Apply the main result of [10] and Proposition [3] \hfill \Box

5. APPLICATIONS TO CROSSED PRODUCTS AND CUNTZ-KRIEGER ALGEBRAS

We show that our main result is a generalization of the so called isomorphisms theorem for crossed products by automorphisms (see, for instance, [11] pp. 225, 226] for a brief survey of such results) by applying it to a crossed product by an endomorphism which is considered to be one of the most successful constructions of this sort, see [12] and sources cited there. In particular, we shall use this crossed product to identify the uniqueness property for Cuntz-Krieger algebras.

5.1. Crossed products by monomorphisms with hereditary range. Let $\alpha : A \to A$ be a monomorphism of a unital $C^*$-algebra $A$. Let $G = A \cup \{S\}$ and let $R$ consists of all $*$-algebraic relations in $A$ plus the covariance relations

(2) \quad Sa^* = \alpha(a), \quad S^*S = 1, \quad a \in A.

Then $C^*(G, R) \cong A \times_\alpha \mathbb{N}$ is the crossed product of $A$ by $\alpha$, which is equipped with a semi-saturated circle gauge action: $\gamma_\lambda(a) = a, \gamma_\lambda(S) = \lambda S, a \in A$. Let us additionally assume that $\alpha(A)$ is a hereditary subalgebra of $A$. This is equivalent to $\alpha(A) = \alpha(1)A\alpha(1)$. Then we have $S^*AS \subset A$ since for any $a \in A$ there is $b \in A$ such that $\alpha(b) = \alpha(1)a\alpha(1)$ and therefore

$$S^*aS = S^*\alpha(1)a\alpha(1)S = S^*\alpha(b)S = S^*bS^*S = b \in A.$$ 

Hence on one hand $A = B_0$ is the fixed point algebra for $\gamma$ and $B_1 = B_0S$ is the first spectral subspace. On the other hand the spectrum of the hereditary subalgebra $\alpha(A)$ may be naturally identified with an open subset of $A$, see e.g. [13] Thm 5.5.5, and then the dual $\alpha : \hat{A} \to \hat{A}$ to the isomorphism $\alpha : A \to \alpha(A)$ becomes a partial homeomorphism of $\hat{A}$. Under this identification one gets

$$\hat{\alpha} = B_1\text{-Ind}$$

and hence if the partial system $(\hat{A}, \hat{\alpha})$ dual to $(A, \alpha)$ is topologically free, then $(G, R)$ possess uniqueness property.

5.2. CUNTZ-KRIEGER ALGEBRAS. Let $G = \{S_i : i = 1, \ldots, n\}$, where $n \geq 2$, and let $R$ consists of the Cuntz-Krieger relations

(3) \quad S_i^*S_j = \sum_{j=1}^n A(i, j)S_jS_i^*, \quad S_i^*S_k = \delta_{i,k}S_i^*S_i, \quad i, k = 1, \ldots, n,$

where $\{A(i, j)\}$ is a given $n \times n$ zero-one matrix such that every row and every column of $A$ is non-zero, and $\delta_{i,j}$ is Kronecker symbol. Then $C^*(G, R)$ is the Cuntz-Krieger algebra $O_A$ and the celebrated Cuntz-Krieger uniqueness theorem, cf. [14] Thm. 2.13, states that the pair $(G, R)$ possess uniqueness property if and only if the so called condition (I) holds:

(I) \quad the space $X_A := \{(x_1, x_2, \ldots) \in \{1, \ldots, n\}^\mathbb{N} : A(x_k, x_{k+1}) = 1\}$ has no isolated points (considered with the product topology)
We may recover this result applying our method to the standard circle gauge action on $O_{\alpha}$ determined by equations $\gamma_{\lambda}(S_i) = \lambda S_i$, $i = 1, ..., n$. Indeed, the fixed point $C^*$-algebra for $\alpha$ coincides with the so called AF-core

$$F_{\alpha} = \{ S_{\mu} S_{\nu}^* : |\mu| = |\nu| = k, k = 1, \ldots \}$$

where for a multiindex $\mu = (i_1, \ldots, i_k)$, with $i_j \in 1, ..., n$, we denote by $|\mu|$ the length $k$ of $\mu$ and write $S_{\mu} = S_{i_1} S_{i_2} \cdots S_{i_k}$. Moreover, any faithful representation of the Cuntz-Krieger relations (3) yields a faithful representation of $F_{\alpha}$, that is $(\mathcal{G}, \mathcal{R}, \gamma)$ possess gauge-invariance uniqueness property. Following [12] we put

$$S := \sum_{i,j} \frac{1}{\sqrt{n_j}} S_i P_j$$

where $n_j = \sum_{i=1}^{n} A(i, j)$ and $P_j = S_j S_j^*$, $j = 1, ..., n$. A routine computation shows that $S F_{\alpha} S^* \subset F_{\alpha}$, $S^* F_{\alpha} S \subset F_{\alpha}$ and $S^* S = 1$ ($S$ is an isometry). Hence the mapping $F_{\alpha} \ni \alpha \mapsto \alpha(a) := S a S^* \in F_{\alpha}$ is a monomorphism with a hereditary range. It is uniquely determined by the formula

$$\alpha(S_{i,j} S_{i,j}^* S_{j,j}^*) = \frac{1}{\sqrt{n_{i,j}}} \sum_{i,j} S_{i,j} S_{i,j}^* S_{j,j}^*$$

From the construction any representation of relations (3) yields a representation of $(F_{\alpha}, \alpha)$ as introduced in the previous subsection. Conversely, if $S$ satisfies (3) where $A = F_{\alpha}$, then one gets representation of (3) by putting $S_i := \sum_{j=1}^{n} A(i, j) \sqrt{n_j} P_i S P_j$. Thus we have a natural isomorphism

$$O_{\alpha} \cong F_{\alpha} \rtimes_{\alpha} \mathbb{N}$$

under which the introduced gauge actions coincide. Hence we may identify the partial dynamical system dual to the Hilbert bimodule $(B_1, B_0)$ where $B_0 = F_{\alpha}$ and $B_1 = F_{\alpha} S$ with the partial dynamical system $(\hat{F}_{\alpha}, \hat{\alpha})$ dual to $(F_{\alpha}, \alpha)$, as introduced in the previous subsection.

In order to identify the the topological freeness of $\hat{\alpha}$ we define $\pi_{\mu} \in \hat{A}$ for any infinite path $\mu = (i_1, i_2, \ldots)$, $A(i_j, i_{j+1}) = 1$, $j \in \mathbb{N}$, to be the the GNS-representation associated to the pure state $\omega_{\mu} : F_{\alpha} \rightarrow \mathbb{C}$ determined by the formula

$$\omega_{\mu}(S_{\mu} S_{\eta}^*) = \begin{cases} 1 & \nu = \eta = (\mu_1, \ldots, \mu_n) \\ 0 & \text{otherwise} \end{cases} \quad \text{for} \quad |\nu| = |\eta| = n.$$

Using description of the ideal structure in $F_{\alpha}$ in terms of Bratteli diagrams [15], similarly as in [10], one can show that representations $\pi_{\mu}$ form a dense subset of $\hat{F}_{\alpha}$ and

$$\hat{\alpha}(\pi_{(\mu_1, \mu_2, \mu_3, \ldots)}) = \pi_{(\mu_2, \mu_3, \ldots)}, \quad \text{for any} \ (\mu_1, \mu_2, \mu_3, \ldots).$$

In particular, it follows that topological freeness of $\hat{\alpha}$ is equivalent to condition (1).

Accordingly

our main result, Theorem 4 when applied to Cuntz-Krieger relations is equivalent to the Cuntz-Krieger uniqueness theorem.
References

[1] D. E. Evans and J. T. Lewis. *Dilations of irreversible evolutions in algebraic quantum theory*. Comm. Dublin Inst. Adv. Studies Ser. A, 24, 1977.

[2] B. K. Kwański, and A. V. Lebedev. Relative Cuntz-Pimsner algebras, partial isometric crossed products and reduction of relations. preprint [arXiv:0704.3811](http://arxiv.org/abs/0704.3811).

[3] B. K. Kwański. *C*-algebras generalizing both relative Cuntz-Pimsner and Doplicher-Roberts algebras*. preprint [arXiv:0906.4382](http://arxiv.org/abs/0906.4382) accepted in Trans. Amer. Math. Soc.

[4] B. Blackadar. *Shape theory for C*-algebras*. Math. Scand., 56(2):249–275, 1985.

[5] R. Exel. *Circle actions on C*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequence*. J. Funct. Analysis, 122:361–401, 1994.

[6] S. Doplicher and J. E. Roberts. A new duality theory for compact groups. *Invent. Math.*, 98:157–218, 1989.

[7] L. Brown, J. Mingo, and N. Shen. Quasi-multipliers and embeddings of Hilbert C*-modules. *Canad. J. Math.*, 71:1150–1174, 1994.

[8] B. Abadie, S. Eilers, and R. Exel. Morita equivalence for crossed products by Hilbert C*-bimodules. *Trans. Amer. Math. Soc.*, 350:3043–3054, 1998.

[9] I. Raeburn and D. P. Williams. *Morita equivalence and continuous-trace C*-algebras*. Amer. Math. Soc., Providence, 1998.

[10] B. K. Kwański. Cuntz-Krieger uniqueness theorem for crossed products by Hilbert bimodules. preprint [arXiv:1010.0446](http://arxiv.org/abs/1010.0446).

[11] A. B. Antonevich and A. V. Lebedev. *Functional differential equations: I. C*-theory*. Longman Scientific & Technical, Harlow, Essex, England, 1994.

[12] A. B. Antonevich, V. I. Bakhtin, and A. V. Lebedev. Crossed product of C*-algebra by an endomorphism, coefficient algebras and transfer operators. *Sb. Math.*, 202(9):1253–1283, 2011.

[13] G. J. Murphy. *C*-Algebras and operator theory*. Academic Press, Boston, 1990.

[14] J. Cuntz and W. Krieger. A class of C*-algebras and topological markov chains. *Inventiones Math.*, 56:251–268, 1980.

[15] O. Bratteli. Inductive limits of finite dimensional C*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.

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