ANALYSIS OF AN AGE-STRUCTURED MODEL FOR HIV-TB CO-INFECTION

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ABSTRACT. According to the report of the WHO, there is a strong relationship between AIDS and tuberculosis (TB). Therefore, it is very important to study how to control TB in the context of the global AIDS epidemic. In this paper, we establish an age structured mathematical model of HIV-TB co-infection to study the transmission dynamics of this co-infection, and consider awareness in the modeling. We give the basic reproduction numbers for each of the two diseases and find four equilibria, namely, disease-free equilibrium, TB-free equilibrium, HIV-free equilibrium and endemic disease equilibrium. Then we discuss the local stability of the equilibria according to the range of values of the two basic reproduction numbers, and find the endemic equilibrium is unstable. We also discuss the global stability of the disease-free equilibrium and the TB-free equilibrium. Based on the new HIV-positive cases and TB cases data in China, the best-fit parameter values and initial values of the model are identified by the MCMC algorithm. Then we perform uncertainty and sensitivity analysis to identify the parameters that have significant impact on the basic reproduction number $R_T$. Finally, combined with the established model, we give some measures that may help China achieve the goal of WHO of reducing the incidence of TB by 80% by 2030 compared to 2015.

1. Introduction. Human immunodeficiency virus (HIV) is transmitted through specific body fluids such as blood and semen. Therefore, HIV carriers can transmit the virus to other persons through sexual behavior, sharing syringes, etc. HIV attacks the body’s immune system, specifically the CD4$^+$T cells, which help the immune system fight off infections. Untreated, HIV reduces the number of CD4$^+$T cells in the body, making people more likely to get other infections. Over time, HIV destroy so many of these cells that the body cannot fight off infections and disease, This is the signal that people have AIDS. Unfortunately, so far there is no cure for
HIV infection, but there are currently medical methods that can control the amount of HIV in the human body, so that the HIV viral load can become undetectable. Today, someone diagnosed with HIV and treated can live nearly as long as someone who does not have HIV[1].

TB is one of the top 10 causes of death worldwide, about 1.7 billion people are estimated to have a latent TB infection, about 5 – 10% of them will develop active TB disease during their lifetime, that is, they can infect other people. The worry is that millions of people continue to fall sick with TB each year. In the 1940s, with the discovery of effective drugs to treat TB, the number of infections fell dramatically, at that time, people believed that TB would disappear like smallpox. However, in the 1990s, the number of people infected with TB increased. Subsequently, the WHO and many countries and regions worked together to formulate various measures to control the epidemic of TB, but with little success. Many researchers believe that this is related to the global epidemic of HIV in the same period[2].

According to the WHO report, an estimated 9% of the incident TB cases in 2017 were among people living with HIV. The risk of developing TB in the 37 million people living with HIV was 20 times higher than the risk in the rest of the world population, and latent TB infection in HIV-infected individuals develop active TB disease at a rate of about 5% to 12% per year. This shows that HIV carriers once infected with TB have a very high possibility of developing active TB disease. In addition, TB is the main cause of death among HIV-positive people, about one in three HIV-positive people died of TB (see Figure 1, source [3]). Therefore, it is very important and meaningful to study the dynamics of HIV-TB co-infection.

Mathematical models are very useful and important to investigate the co-infection epidemic dynamics, and to assess the treatment programs. Many articles have studied the dynamics and control of HIV-TB co-infection[7, 5, 16, 13, 19, 20]. Gakkhar and Chavda[7] studied a mathematical model for the spread of HIV and TB infection in the human population. One of the objects of their study was to decrease the fatality of HIV infection by avoiding co-infection. They then studied the stability of the boundary equilibria and showed that the endemic equilibrium was unstable. This achieved the object of their study to eliminate HIV-TB co-infection. However, they did not consider the effect of latent period of TB and treatment of TB patients on the model dynamics. These two factors affect the reliability of the model. In addition, they did not quantitatively analyze the model based on actual data. Agusto et al. [5] formulated a mathematical model of the transmission of TB-HIV/AIDS

![Figure 1](image-url). Estimated number of deaths from HIV/AIDS and TB in 2017. Deaths from TB among HIV-positive people are shown in grey.
co-infection. By using treatment of infected individuals with TB as the system control variable, they investigated optimal strategies for controlling the spread of the disease. However, they did not analyze the global stability of the model. In addition, they did not use the actual data in the study of control strategies, which may affect the application of their proposed control strategies. The researches in reference [7, 5, 16, 13, 19, 20] are based on ODE models. In modeling with ODEs, it is assumed that the time to recovery or death is exponentially distributed. This assumption is unreasonable for slowly progressive diseases that necessarily include a long-term latent or chronic stage, because infectivity for infectious individuals has changed over the time since infection. Kermack and McKendrick argued that age-structured models should be used to model slowly progressive diseases. HIV/AIDS and TB are slowly progressive diseases, it is reasonable to use age-structured model to study HIV-TB co-infection[18].

Based on the above analyses, in this paper, we formulate an age-structured HIV-TB co-infection model to study how to control the spread of TB in the context of the global AIDS epidemic. This paper is organized as follows. In sect. 2, we introduce an age-structured HIV-TB co-infection model and present some basic properties. In sect. 3, we define the basic reproductive numbers for each of the two diseases and prove the local stability of the equilibria. In Sect. 4, we present the uniform persistence result and prove the global stability of the disease-free equilibrium and the TB-free equilibrium. In Sect. 5, we perform data fitting and sensitivity analysis of the basic reproductive number $R_T$, and assess the feasibility of the WHO End TB Strategy by 2030. In Sect. 6, we give a brief discussion.

2. Description of model and basic properties.

2.1. Model formulation. In our model, we will study the transmission of HIV-TB co-infection, the main objective of our model is to study how to control TB in the context of the global AIDS epidemic. Therefore, we will focus on assessing treatment programs for individuals infected with TB. Since HIV-positive people are infectious at all stages, we do not consider different stages of HIV disease. Moreover, since only people with active TB are infectious, people infected with TB can be classified into two groups: infected but not yet infectious and infectious. Mallela et.al [16] believed that the co-infected individuals are considered in the critical stage regardless of their stage of individual disease, then we do not consider the different stages of co-infection, and assume that they can get special treatment or isolation without infecting others. Since HIV is a disease transmitted through specific body fluids, it is possible for people infected with TB to refrain from HIV infection with proper awareness and suitable precautionary measures. So we can assume that people infected with TB is not susceptible to HIV. However, TB is an airborne transmitted disease, HIV-positive people with lowered immunity may get TB infection. Hence, HIV infected class is considered susceptible to TB infection. Based on these analyses and assumptions, our model divides the total population at time $t$ into five mutually-exclusive subgroups: susceptible class to both diseases, TB-susceptible and HIV-infected class, TB-latent and HIV-susceptible class (those who have TB but are not infectious), TB-infectious $[\sigma e]$ and HIV-susceptible class, co-infected with HIV and TB class denoted by $S(t)$, $i(t, a)$, $e(t, \theta)$, $I_T(t)$ and $I_c(t)$ respectively. Here the parameter $a$ denotes the infection age, $\theta$ denotes the latent age. The flow among those subgroups is shown in the following flowchart (Figure 2).
Base on the above notation and the flowchart (Figure 2), we formulate the HIV-TB co-infection model as follows:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \Lambda + \alpha I_T + \int_0^{+\infty} \delta(\theta) c(t,\theta) d\theta - S \int_0^{+\infty} \beta(a) i(t, a) da - \beta T S I_T - \mu S, \\
\frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} &= - (\mu + \delta_i(a) + \delta_T(a) I_T) i(t,a), \\
\frac{\partial e(t,\theta)}{\partial t} + \frac{\partial e(t,\theta)}{\partial \theta} &= - (\mu + \delta(\theta) + \sigma(\theta)) e(t,\theta), \\
\frac{dI_T(t)}{dt} &= \int_0^{+\infty} \sigma(\theta) c(t,\theta) d\theta - (\mu + \mu_T + \alpha) I_T, \\
\frac{dI_c(t)}{dt} &= I_T \int_0^{+\infty} \delta_T(a) i(t,a) da - (\mu + \mu_c) I_c,
\end{align*}
\]

with the following boundary and initial conditions

\[
i(t,0) = S \int_0^{+\infty} \beta(a) i(t,a) da, \quad e(t,0) = \beta_T S I_T, \\
i(0,a) = i_0(a), \quad e(0,\theta) = e_0(\theta), S(0) = s_0, I_T(0) = i_{T0}, \quad I_c(0) = i_{c0}.
\]

where \(i_0(a), e_0(\theta) \in L^1_+(0, +\infty)\), and \(s_0, i_{T0}, i_{c0} \in \mathbb{R}_+\). The meanings of the parameters in (1) are explained in Table 1.

Throughout this paper, we make the following assumptions and notations.

\((A1)\): \(\Lambda, \mu, \beta(a), \beta_T, \delta_i(a), \delta_T(a), \sigma(\theta), \mu_T, \mu_c, \alpha, \delta(\theta), \delta_T(a) > 0\);

\((A2)\): \(\beta(a), \delta_i(a), \sigma(\theta), \delta_T(a) \in L^+\infty_+(0, +\infty)\) with essential upper bounds \(\bar{\beta}, \bar{\delta}_i, \bar{\sigma}, \bar{\delta}_T > 0\), respectively, and \(\beta(a)\) is Lipschitz continuous on \(\mathbb{R}_+\) with Lipschitz coefficients \(M_\beta\);

\((A3)\): \(\beta(a) \in L^1_+ (0, +\infty);\)

\((A4)\): \(i(0,0) = S(0) \int_0^{+\infty} \beta(a) i(0,a) da, \quad e(0,0) = \beta_T I_T(0) S(0).\)

For \(a, \theta \geq 0\), we denote

\[
\begin{align*}
k_1(a) &= e^{- \int_0^a (\mu + \delta_3(s)) \, ds}, \quad k_2(\theta) = e^{- \int_0^\theta (\mu + \delta(s) + \sigma(s)) \, ds}, \\
k(t, a) &= e^{- \int_0^t (\mu + \delta_3(s) + \delta_T(s) I_T(t-a+s)) \, ds} \quad \text{for } t \geq a, \\
k_1(t, a) &= e^{- \int_0^a (\mu + \delta_3(s) + \delta_T(s) I_T(t-a+s)) \, ds} \quad \text{for } a > t,
\end{align*}
\]
| Parameters | Description |
|------------|-------------|
| $\Lambda$ | the recruitment rate of the susceptible class |
| $\mu$ | the natural death rate of the population |
| $\beta(a)$ | the transmission coefficient of HIV class to susceptible class |
| $\beta_T$ | the transmission coefficient of active TB class to susceptible class |
| $\delta_i(a)$ | the death rate due to HIV |
| $\sigma(\theta)$ | the rate at which latent class progress into infectious class |
| $\mu_T$ | the death rate due to TB |
| $\mu_c$ | the death rate due to co-infection |
| $\alpha$ | the rate at which treatment infectious individuals progress into susceptible class |
| $\delta(\theta)$ | the rate at which treatment latent individuals progress into susceptible class |
| $\delta_T(a)$ | the transmission coefficient of TB infectious class to HIV class |

$$\mathcal{K}_1 = \int_0^{+\infty} \beta(a)k_1(a)da, \quad \mathcal{K}_2 = \int_0^{+\infty} \sigma(\theta)k_2(\theta)d\theta, \quad \mathcal{K}_3 = \int_0^{+\infty} \delta(\theta)k_2(\theta)d\theta.$$  

Since the first four equations of system (1) do not contain $I_c(t)$, then we will only discuss the following model:

$$\frac{dS(t)}{dt} = \Lambda + \alpha I_T + \int_0^{+\infty} \delta(\theta)e(t,\theta)d\theta - S \int_0^{+\infty} \beta(a)i(t,a)da - \beta_T SI_T - \mu S, \quad (3)$$

$$i(t,0) = S \int_0^{+\infty} \beta(a)i(t,a)da, \quad e(t,0) = \beta_T SI_T, \quad i(0,a) = i_0(a), \quad e(0,\theta) = e_0(\theta), S(0) = s_0, I_T(0) = i_{T0}.$$  

Let $\mathcal{Y} = \mathbb{R}_+ \times (L^1_+(0,+\infty))^2 \times \mathbb{R}_+$, with norm

$$\| (x_1,x_2,x_3,x_4) \|_{\mathcal{Y}} = | x_1 | + \sum_{i=2}^3 \int_0^{+\infty} | x_i(s) | ds + | x_4 | .$$

2.2. Well-posedness. Base on above assumptions, we can verify the local existence of unique and nonnegative solution of the model (3) with the nonnegative initial conditions (see Webb [24] and Iannelli [12] ), thus we obtain the following proposition.

**Proposition 1.** Let $x_0 \in \mathcal{Y}$, then there exists $\epsilon > 0$ and neighborhood $B_0 \subset \mathcal{Y}$ with $x_0 \in B_0$ such that there exists a unique continuous function, $\Phi : [0,\epsilon] \times B_0 \to \mathcal{Y}$, where $\Phi(t,x_0)$ is the solution to (1) with $\Phi(0,x_0) = x_0$. 

Lemma 2.1. \(\text{of the semiflow.}\) A positively invariant bounded closed set is attracted by a nonempty compact set. According to the comparison principle, we have

\[
\frac{d}{dt} \| \Phi(t, x_0) \|_\mathcal{Y} \leq \Lambda - \mu \| \Phi(t, x_0) \|_\mathcal{Y}.
\]

According to the comparison principle, we have

\[
\| \Phi(t, x_0) \|_\mathcal{Y} \leq \frac{\Lambda}{\mu} - \mu^t \| x_0 \|_\mathcal{Y},
\]

which yields

\[
\| \Phi(t, x_0) \|_\mathcal{Y} \leq \max \{ \frac{\Lambda}{\mu}, \| x_0 \|_\mathcal{Y} \}.
\]

Boundedness is a direct consequence of nonnegativity of solution. Then following proposition is immediate

**Proposition 2.** Let \(x_0 \in \mathcal{Y},\) then there exists a unique continuous semiflow, \(\Phi: \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y},\) where \(\Phi(t, x_0)\) is the solution to (3) with \(\Phi(0, x_0) = x_0,\) and (4), (5) are satisfied for \(t \in \mathbb{R}_+.\) The following set is positively invariant for system (1)

\[\Omega = \{ x = (S(t), i(t, a), e(t, \theta), I_T(t)) : \| x \|_\mathcal{Y} \leq \frac{\Lambda}{\mu} \} .\]

From Proposition 2 and inequality (4), we obtain the following results.

**Proposition 3.** (1) The solution of (3), \(\Phi(t, \cdot),\) is point dissipative, and \(\Omega\) attracts all points in \(\mathcal{Y} ;\)

(2) Let \(B \subset \mathcal{Y}\) be bounded, then \(\Phi(t, B)\) is bounded;

(3) If \(x_0 \in \mathcal{Y}\) and \(\| x_0 \|_\mathcal{Y} \leq A\) for some \(A \geq \frac{\Lambda}{\mu},\) then \(S(t), I(t), T(t),\)
\(e(t, \cdot) \|_{L_1^\infty}, \| r(t, \cdot) \|_{L_1^\infty} \leq A.\)

**Remark.** From now on, we only consider solutions of (3) with initial conditions in \(\Omega,\) since we focus on the limiting behavior of (3).

2.3. Asymptotic smoothness. Integrating the equations for \(i(t, a), e(t, \theta)\) in (3) along the characteristic lines, \(t - a = \text{const.}, t - \theta = \text{cost.},\) respectively, we deduce that

\[
i(t, a) = \begin{cases} i(t - a, 0)k(t, a) & 0 \leq a < t, \\ i_0(a - t)k_1(t, a) & 0 \leq t < a, \end{cases}
\]

\[
e(t, \theta) = \begin{cases} e(t - \theta, 0)k_2(\theta) & 0 \leq \theta \leq t, \\ e(\theta - t)k_2(\theta - t) & 0 \leq t < \theta. \end{cases}
\]

The continuous semiflow \(\{ \Phi(t, \cdot) \}_{t \geq 0}\) is said to be asymptotically smooth, if each positively invariant bounded closed set is attracted by a nonempty compact set.

We will use the following two lemmas [23] to prove the asymptotic smoothness of the semiflow.

**Lemma 2.1.** The semiflow \(\Phi: \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}\) is asymptotically smooth if there are maps \(U_1, U_2: \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}\) such that \(\Phi(t, x) = U_1(t, x) + U_2(t, x),\) and the following hold for any bounded closed set \(\mathcal{A} \subset \mathcal{Y}\) that is forward invariant under \(\Phi:\)

(1) \(\lim_{t \to +\infty} \text{diam}U_2(t, \mathcal{A}) = 0;\)

(2) there exists \(t_\mathcal{A} \geq 0\) such that \(U_1(t, \mathcal{A})\) has compact closure for each \(t \geq t_\mathcal{A}.\)

Since \(\mathcal{Y}\) is an infinite dimensional space, space \(L_1^\infty(0, +\infty)\) is a component of \(\mathcal{Y}\). For infinite dimensional space, we cannot deduce precompactness only from boundedness. Thus, to derive the precompactness of \(\mathcal{Y}\), we apply the following lemma:
Lemma 2.2. Let $\mathcal{B}$ be a bounded subset of $L^1_+(0, +\infty)$. Then $\mathcal{B}$ has compact closure if and only if the following conditions hold:

(i) $\sup_{f \in \mathcal{B}} \int_0^{+\infty} |f(s)| ds < +\infty$;

(ii) $\lim_{h \to +\infty} \int_h^{+\infty} |f(s)| ds = 0$ uniformly in $f \in \mathcal{B}$;

(iii) $\lim_{h \to 0^+} \int_0^{h} |f(s) - f(s+h)| ds = 0$ uniformly in $f \in \mathcal{B}$;

(iv) $\lim_{h \to 0^+} \int_0^{h} |f(s)| ds = 0$ uniformly in $f \in \mathcal{B}$.

We are now ready to prove a result on the semiflow $\Phi$ generated by system (3)

Theorem 2.3. The semiflow $\Phi$ generated by system (3) is asymptotically smooth.

Proof. We first decompose $\Phi$ into the following two parts: $U_1, U_2$ defined respectively by

$$U_1(t, x) = (S(t), \tilde{i}(t, \cdot), \tilde{e}(t, \cdot), I_T(t)),$$

$$U_2(t, x) = (0, \psi_1(t, \cdot), \psi_e(t, \cdot), 0),$$

where

$$\psi_1(t, a) = \begin{cases} 0 & 0 \leq a \leq t, \\ i_0(a-t)k_1(t, a) & 0 \leq t < a, \end{cases}$$

$$\psi_e(t, \theta) = \begin{cases} 0 & 0 \leq \theta \leq t, \\ e_0(\theta-t)\frac{k_2(\theta)}{k_2(\theta-t)} & 0 \leq t < \theta, \end{cases}$$

$$\tilde{i}(t, a) = \begin{cases} i(t-a, 0)k(t, a) & 0 \leq a \leq t, \\ 0 & 0 \leq t < a, \end{cases}$$

$$\tilde{e}(t, \theta) = \begin{cases} e(t-\theta, 0)k_2(\theta) & 0 \leq \theta \leq t, \\ 0 & 0 \leq t < \theta. \end{cases}$$

for $x = (S(0), i_0(a), e_0(\theta), I_T(0))$, clearly, we have $\Phi(t, x) = U_1(t, x) + U_2(t, x)$.

Let $\mathscr{A} \subset \Omega$, $r = \frac{A}{\mu}$, for each $x \in \mathscr{A}$, we know $\|x\|_{\mathscr{A}} \leq r$. So we can derive

$$\|U_2(t, x)\|_{\mathscr{A}} = \int_t^{+\infty} i_0(a-t)k_1(t, a) da + \int_t^{+\infty} e_0(\theta-t)\frac{k_2(\theta)}{k_2(\theta-t)} d\theta$$

$$= \int_0^{+\infty} i_0(s)k_1(t, s+t) ds + \int_0^{+\infty} e_0(s)\frac{k_2(s+t)}{k_2(s)} ds$$

$$= \int_0^{+\infty} i_0(s)e^{-\int_s^{+\infty}(\mu+\delta(l)+\sigma(l)) dl} ds$$

$$+ \int_0^{+\infty} e_0(s)e^{-\int_s^{+\infty}(\mu+\delta(l)+\sigma(l)) dl} ds$$

$$\leq e^{-\mu t} \|x\|_{\mathscr{A}} \leq re^{-\mu t}.$$

Thus, $\lim_{t \to +\infty} \text{diam } U_2(t, \mathscr{A}) = 0$. In the following we will show that $U_1(t, \mathscr{A})$ has compact closure for each $t \geq 0$.

From Proposition 3, we know that $S(t), I_T(t)$ remain in the compact set $[0, r]$ for all $t \geq 0$. Next, we will show that $\tilde{i}(t, a)$ and $\tilde{e}(t, \theta)$ remain in a precompact subset of $L^1_+(0, +\infty)$ which is independent of $x$. According to

$$0 \leq \tilde{i}(t, a) = \begin{cases} i(t-a, 0)k(t, a) & 0 \leq a \leq t, \\ 0 & 0 \leq t < a, \end{cases}$$

and (2), it is easy to show that

$$0 \leq \tilde{i}(t, a) \leq \tilde{\beta}r^2e^{-\mu a}.$$
Therefore, the conditions (i), (ii) and (iv) of Lemma 2.2 are satisfied. Now, we only need to check the condition (iii) of Lemma 2.2.

Notice that \(| \frac{dR(t)}{dt} | \leq (\sigma + \mu + \mu_T + \alpha)r \leq M_1, \quad | \frac{dS(t)}{dt} | \leq \Lambda + (\alpha + \delta r + \beta_T r + \delta + \mu)r \leq M_2, | \frac{d(k(t, a))}{da} | \leq e^{-\mu a}(\mu + \delta + \delta_T r + \delta_T M_1 a) \leq e^{-\mu a}(N_1 + N_2 a)\), then

\[
\int_{0}^{t-h} | \tilde{i}(t, a + h) - \tilde{i}(t, a) | \, da = \int_{0}^{t-h} | \tilde{i}(t, a + h) - \tilde{i}(t, a) | \, da + \int_{t-h}^{t} | \tilde{i}(t, a) | \, da
\]

\[
= \int_{0}^{t-h} \left| i(t - a, h, 0)k(t, a + h) - i(t - a, 0)k(t, a) \right| \, da + \int_{t-h}^{t} \left| i(t, a) \right| \, da
\]

\[
\leq \int_{0}^{t-h} \left| i(t - a, h, 0) \right| \left| k(t, a + h) - k(t, a) \right| \, da + \int_{t-h}^{t} \left| i(t, a) \right| \, da + \beta r^2 h,
\]

where

\[
\int_{0}^{t-h} \left| i(t - a, h, 0) \right| \left| k(t, a + h) - k(t, a) \right| \, da \leq \beta r^2 h \int_{0}^{t-h} e^{-\mu a}(N_1 + N_2 a) \, da = \beta r^2 \left( \frac{N_1}{\mu} + \frac{N_2}{\mu^2} \right)h.
\]

Also,

\[
| i(t - a, h, 0) - i(t - a, 0) | = \left| S(t - a - h) \int_{0}^{t-h} \beta(s)i(t - a - h, s) \, ds - S(t - a) \int_{0}^{t-h} \beta(s)i(t - a, s) \, ds \right|
\]

\[
= \left| S(t - a - h) - S(t - a) \right| \left| \int_{0}^{t-h} \beta(s)i(t - a - h, s) \, ds \right|
\]

\[
+ \left| S(t - a) \right| \left| \int_{0}^{t-h} \beta(s)(i(t - a - h, s) - i(t - a, s)) \, ds \right|
\]

\[
\leq M_2 \beta r h + r \left| \int_{0}^{t-h} \beta(s)(i(t - a - h, s) - i(t - a, s)) \, ds \right|
\]

\[
= M_2 \beta r h + r \left| \int_{0}^{h} \beta(s)i(t - a, s) \, ds + \int_{h}^{t-h} \beta(s)i(t - a, s) \, ds \right|
\]

\[
- \int_{0}^{\theta} \beta(s)i(t - a - h, s) \, ds \right|
\]

\[
\leq M_2 \beta r h + r^3 \beta^2 h
\]

\[
+ r \left| \int_{0}^{t-h} \beta(s + h)i(t - a, s + h) \, ds - \int_{0}^{t-h} \beta(s)i(t - a - h, s) \, ds \right|
\]

\[
= M_2 \beta r h + r^3 \beta^2 h
\]

\[
+ r \left| \int_{0}^{t-h} \beta(s + h)i(t - a - h, s)e^{-\int_{s}^{t-h}(\mu + \delta_T l + \delta_T T r(t - a l)) \, dl} \, ds \right|
\]
- \int_0^{+\infty} \beta(s) i(t - a - h, s) ds | \
\leq M_2 \tilde{\beta} r h + r^3 \tilde{\beta}^2 h \\
+ r \int_0^{+\infty} (1 - e^{-\int_0^{+h} (\mu + \delta_i(t) + \delta_T(t)(t-a+h)dt)} \beta(s + h) i(t - a - h, s) ds \\
+ r \int_0^{+\infty} |\beta(s + h) - \beta(s)| i(t - a - h, s) ds \\
\leq M_2 \tilde{\beta} r h + r^3 \tilde{\beta}^2 h + \tilde{\beta} r^2 (\mu + \delta_i + \delta_T) r h + r^2 M_3 h \equiv \Delta h.

Then
\int_0^{t-h} |i(t - a - h, 0) - i(t - a, 0)| k(t, a) | da \leq \Delta t \int_0^{t-h} e^{-\mu a} ds \leq \frac{\Delta}{\mu} h.

Hence,
\int_0^{+\infty} |i(t, a + h) - i(t, a)| da \leq \tilde{\beta} r^2 \left(\frac{N_1}{\mu} + \frac{N_2}{\mu^2}\right) + \tilde{\beta} r^2 + \frac{\Delta}{\mu} h.

Thus, the condition (iii) of Lemma 2.2 holds, then we can get that \( \tilde{i}(t, a) \) satisfies the conditions of Lemma 2.2. In a similar way, \( \tilde{e}(t, \theta) \) also satisfies the conditions of Lemma 2.2. Therefore, we obtain \( U \) has compact closure for each \( t \geq 0 \).

Using Lemma 2.1, we know semiflow \( \Phi \) is asymptotically smooth. This completes the proof.

Combining Proposition 3, asymptotically smoothness of \( \Phi \) and the existence theory of global attractors, the following result is immediate from Theorem 2.6 in [15] and Theorem 2.4 in [4].

**Theorem 2.4.** The semiflow \( \Phi \) has a global attractor \( A \) in \( Y \), which attracts any bounded subset of \( Y \).

3. **Equilibria and their local stability.**

3.1. **Existence of equilibria.** System (3) always has a disease free equilibrium \( E_0 = (\frac{\Lambda}{\mu}, 0, L_1(0, +\infty), 0, L_1(0, +\infty), 0) \). One can obtain the reproduction number for HIV from system (3)

\[ \mathcal{R}_H = \frac{\Lambda}{\mu} \mathcal{X}_1, \]

and the basic reproduction number for TB is obtained

\[ \mathcal{R}_T = \frac{\Lambda \beta_T \mathcal{X}_2}{\mu (\mu + \mu_T + \alpha)}. \]

Next, we investigate the single-disease equilibria and the endemic equilibria of system (3). A TB-free equilibrium \( E_1 = (\tilde{S}, \tilde{i}(a), 0, 0) \) of system (3) satisfies the following equations:

\[ \Lambda - \tilde{S} \int_0^{+\infty} \beta(a) \tilde{i}(a) da - \mu \tilde{S} = 0, \]

\[ \frac{d\tilde{i}(a)}{da} = - (\mu + \delta_i(a)) \tilde{i}(a), \]

\[ \tilde{i}(0) = \tilde{S} \int_0^{+\infty} \beta(a) \tilde{i}(a) da. \]
From (7) we can easily find that if \( R_H > 1 \), system (3) has a TB-free equilibrium \( E_1 = (\bar{S}, \bar{i}(a), 0, 0) \), where \( \bar{S} = \frac{1}{\mathcal{K}^-} \), \( \bar{i}(0) = \frac{\mu}{\mathcal{K}^-} (R_H - 1) \) and \( \bar{i}(a) = \bar{i}(0)k_1(a) \).

A HIV-free equilibrium \( E_2 = (\bar{S}, 0, \bar{e}(\theta), \bar{I}_T) \) of system (3) satisfies the following equations:

\[
\begin{align*}
\Lambda + \alpha \bar{I}_T - \beta_T \bar{I}_T \bar{S} + \int_{0}^{+\infty} \delta(\theta) \bar{e}(\theta) d\theta - \mu \bar{S} &= 0, \\
\frac{d\bar{e}(\theta)}{d\theta} &= -(\mu + \delta(\theta) + \sigma(\theta)) \bar{e}(\theta), \\
\int_{0}^{+\infty} \sigma(\theta) \bar{e}(\theta) d\theta - (\mu + \mu_T + \alpha) \bar{I}_T &= 0 \\
\bar{e}(0) &= \beta_T \bar{I}_T \bar{S}.
\end{align*}
\]

When solving the HIV-free equilibrium, we notice that \( \mathcal{K}_2 + \mathcal{K}_3 < 1 \). From (8) we can easily find that if \( R_T > 1 \), system (3) has a HIV-free equilibrium \( E_2 = (\bar{S}, 0, \bar{e}(\theta), \bar{I}_T) \), where

\[
\bar{S} = \frac{\mu + \mu_T + \alpha}{\beta_T \mathcal{K}_2}, \quad \bar{I}_T = \frac{\Lambda (1 - \frac{1}{\mathcal{K}_T})}{\beta_T S (1 - \mathcal{K}_3) - \alpha}, \quad \bar{e}(0) = \beta_T \bar{I}_T \bar{S} \text{ and } \bar{e}(\theta) = \bar{e}(0)k_2(\theta).
\]

A endemic equilibrium \((S^*, i^*(a), e^*(\theta), I_T^*)\) of system (3) satisfies the following equations:

\[
\begin{align*}
\Lambda + \alpha S^* + \int_{0}^{+\infty} \delta(\theta) e^*(\theta) d\theta - S^* \int_{0}^{+\infty} \beta(a) i^*(a) da - \beta_T I_T^* S^* - \mu S^* &= 0, \\
\frac{di^*(a)}{da} &= -(\mu + \delta_i(a) + \delta_T(a) I_T^*) i^*(a), \\
\frac{de^*(\theta)}{d\theta} &= -(\mu + \delta(\theta) + \sigma(\theta)) e^*(\theta), \\
\int_{0}^{+\infty} \sigma(\theta) e^*(\theta) d\theta - (\mu + \mu_T + \alpha) I_T^* &= 0 \\
e^*(0) &= \beta_T I_T^* S^*, \quad i^*(0) = S^* \int_{0}^{+\infty} \beta(a) i^*(a) da.
\end{align*}
\]

Solving the third equation of the system (9) gives \( e^*(\theta) = e^*(0)k_2(\theta) \). This substitution in the fourth equation yields \( \mathcal{K}_3^* e^*(0) = (\mu + \mu_T + \alpha) I_T^* \). Then substitution in the fifth equation yields

\[
S^* = \frac{\mu + \mu_T + \alpha}{\beta_T \mathcal{K}_2}.
\]

Note:

\[
k_{I_T^*}(a) = e^{-\int_{0}^{a} (\mu + \delta_i(s) + \delta_T(s) I_T^*) ds}, \quad \mathcal{K}_{I_T^*} = \int_{0}^{+\infty} \beta(a) k_{I_T^*}(a) da.
\]

Solving the second equation of the system (9) gives \( i^*(a) = i^*(0)k_{I_T^*}(a) \). This substitution in the fifth equation yields \( \mathcal{K}_{I_T^*} = \frac{1}{S^*} \). Then we can obtain the following necessary and sufficient conditions

\[
I_T^* > 0 \iff \mathcal{K}_{I_T^*} < \mathcal{K}_1 \iff \frac{1}{S^*} < \mathcal{K}_1 \iff R_T < R_H.
\]

Therefore, we know that a condition for the existence of \( E^* \) is

\[
R_T < R_H.
\]
Using the above conditions, the first equation of (9) can be written as
\[ \Lambda + \alpha I_T - i^*(0) - \beta_T I_T S^* + \beta_T I_T S^* \mathcal{X}_3 - \mu S^* = 0, \]
we can get
\[ i^*(0) = [(1 - \mathcal{X}_3)\beta_T S^* - \alpha]\left(\frac{\Lambda - \mu S^*}{(1 - \mathcal{X}_3)\beta_T S^* - \alpha} - I^*\right). \]
Notice that \((1 - \mathcal{X}_3)\beta_T S^* - \alpha > 0, \tilde{S} = S^*\).
Then we can obtain the following necessary and sufficient conditions
\[ i^*(0) > 0 \iff \Lambda - \mu S^* > 0, \frac{\Lambda - \mu S^*}{(1 - \mathcal{X}_3)\beta_T S^* - \alpha} - I^* > 0 \iff \mathcal{R}_T > 1 \text{ and } \bar{I}_T > I_T^*. \]
Thus if \(\mathcal{R}_T > \alpha > 0, I_T > I_T^*, \) then the endemic equilibrium
\[ E^* = (S^*, I^*, T^*, e^*(a), r^*(\theta)) \]
exists, where
\[ S^* = \frac{\mu + \mu T + \alpha}{\beta_T \mathcal{X}_2}, \]
\(I_T^*\) is the solution of equation \(\mathcal{X}_T = \frac{1}{S^*},\)
\[ i^*(a) = [(1 - \mathcal{X}_3)\beta_T S^* - \alpha]\left(\frac{\Lambda - \mu S^*}{(1 - \mathcal{X}_3)\beta_T S^* - \alpha} - I^*\right)k_{I_T}(a), \]
and
\[ e^*(\theta) = \beta_T I_T^* S^* k_{I_T}(a). \]

Summarizing the discussions above, we have the following theorem.

**Theorem 3.1.** (1) The disease free equilibrium \(E_0\) of system (3) always exists;
(2) if \(\mathcal{R}_H > 1,\) system (3) exists a TB free equilibrium \(E_1;\)
(3) if \(\mathcal{R}_H > 1,\) system (3) exists a HIV free equilibrium \(E_2;\)
(4) if \(\mathcal{R}_T > \mathcal{R}_H > 1\) and \(I_T > I_T^*,\) then the endemic equilibrium \(E^*\) exists.

3.2. **Local stability of the Equilibria.** Now we consider the linearized system of (3) at an equilibrium \(E = (\tilde{S}, \tilde{i}(a), \tilde{e}(\theta), \tilde{I}_T).\) Let \(S(t) = S(t) - \tilde{S}, I(t, a) = i(t, a) - \tilde{i}(a), e(t, \theta) = e(t, \theta) - \tilde{e}(\theta), I_T(t) = I_T(t) - \tilde{I}_T,\) then removing the bar, we obtain the following linearized system:
\[
\frac{dS(t)}{dt} = \alpha I_T + \int_0^{+\infty} \delta(\theta)e(t, \theta)d\theta - \tilde{S} \int_0^{+\infty} \beta(a)i(t, a)da \\
- S \int_0^{+\infty} \beta(a)\tilde{i}(a)da - \beta_T S \tilde{I}_T - \beta_T S \tilde{I}_T - \mu S,
\]
\[
\frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(\mu + \delta_i(a) + \delta_T(a)\tilde{I}_T)i(t, a) - \delta_T(a)\tilde{i}(a)I_T,
\]
\[
\frac{\partial e(t, \theta)}{\partial t} + \frac{\partial e(t, \theta)}{\partial \theta} = -\mu + \delta(\theta) + \sigma(\theta)\sigma(\theta)e(t, \theta),
\]
\[
\frac{dI_T(t)}{dt} = \int_0^{+\infty} \sigma(\theta)e(t, \theta)d\theta - (\mu + \mu_T + \alpha)I_T,
\]
i(t, 0) = S \int_0^{+\infty} \beta(a)\tilde{i}(a)da + \tilde{S} \int_0^{+\infty} \beta(a)i(t, a)da,
\]
e(t, 0) = \beta_T \tilde{S} I_T + \beta_T \tilde{I}_T,
In order to simplify the representation of the characteristic equation, we use the following notations:

\[ \mathcal{K}_1(\lambda) = \int_0^{+\infty} \beta(a)e^{-\int_0^a (\lambda + \mu + \delta_i(s))ds} da, \]

\[ \mathcal{K}_2(\lambda) = \int_0^{+\infty} \sigma(\theta)e^{-\int_0^\infty (\lambda + \mu + \delta_i(s) + \sigma(s))ds} d\theta, \]

\[ \mathcal{K}_3(\lambda) = \int_0^{+\infty} \delta(\theta)e^{-\int_0^\infty (\lambda + \mu + \delta_i(s) + \sigma(s))ds} d\theta, \]

\[ \mathcal{K}(\lambda) = \int_0^{+\infty} \beta(a)e^{-\int_0^a (\lambda + \mu + \delta_i(s) + \delta_T(s)\dot{T}_t)ds} da, \]

\[ \mathcal{K}_5(\lambda) = \int_0^{+\infty} \beta(a)e^{-\int_0^a (\lambda + \mu + \delta_i(s) + \delta_T(s)\dot{T}_t)ds} da, \]

\[ \hat{B} = \int_0^{+\infty} \beta(a)\hat{i}(a)da, \]

\[ B_1 = \mathcal{K}_1(0) = \mu(\mathcal{K}_H - 1), \quad B_2 = \dot{\iota}(0), \mathcal{K}_I, S^0 = \frac{\Lambda}{\mu}, \]

\[ \Omega_2(\lambda) = \int_0^{+\infty} \beta(a)e^{-\int_0^a (\lambda + \mu + \delta_i(s) + \delta_T(s)\dot{T}_t)ds} da, \]

\[ \Omega(\lambda) = \int_0^{+\infty} \beta(a)e^{-\int_0^a (\lambda + \mu + \delta_i(s) + \delta_T(s)\dot{T}_t)ds} da. \]

For (10), let \( S(t) = S_1 e^{\lambda t}, \quad I_T(t) = I_T e^{\lambda t}, \quad i(t, a) = i_1(a)e^{\lambda t}, e(t, \theta) = e_1(\theta)e^{\lambda t} \), we have

\[
(\lambda + \mu)S_1 = \alpha I_T + \int_0^{+\infty} \delta(\theta)e_1(\theta)d\theta - S_1 \int_0^{+\infty} \beta(a)\hat{i}(a)da
- \hat{S} \int_0^{+\infty} \beta(a)i_1(a)da - \beta_T S_1 \dot{I}_T - \beta_T \hat{S}I_T,
\]

\[
\frac{di_1(a)}{da} = -(\lambda + \mu + \delta_i(a) + \delta_T(a)\dot{I}_T)i_1(a) - \delta_T(a)\hat{i}(a)I_T,
\]

\[
\frac{de_1(\theta)}{d\theta} = -(\lambda + \mu + \delta(\theta) + \sigma(\theta))e_1(\theta),
\]

\[
(\lambda + \mu + \mu_T + \alpha)I_T = \int_0^{+\infty} \sigma(\theta)e_1(\theta)d\theta,
\]

\[
i_1(0) = S_1 \int_0^{+\infty} \beta(a)\hat{i}(a)da + \hat{S} \int_0^{+\infty} \beta(a)i_1(a)da,
\]

\[
e_1(0) = \beta_T \hat{S}I_T + \beta_T S_1 \dot{I}_T.
\]

We obtain the following equation from the system (11)

\[
\begin{bmatrix}
\lambda + \mu + \beta_T \dot{I}_T & -\alpha + \beta_T \hat{S} & 1 & -\mathcal{K}_3(\lambda) \\
0 & \lambda + \mu + \mu_T + \alpha & 0 & -\mathcal{K}_2(\lambda) \\
-\hat{B} & \Omega(\lambda) \hat{S} & 1 & -\mathcal{K}(\lambda) \hat{S} \\
-\beta_T \dot{I}_T & -\beta_T \hat{S} & 0 & 1
\end{bmatrix}
= 0. \quad (12)
\]

Further, we obtain the characteristic equation of system (3) at an equilibrium \( \tilde{E} \) as follows: \( f(\lambda) = 0 \), where

\[
f(\lambda) = (\lambda + \mu + \beta_T \dot{I}_T)(1 - \mathcal{K}(\lambda) \hat{S})(\lambda + \mu + \mu_T + \alpha - \beta_T \hat{S} \mathcal{K}_2(\lambda)) + \hat{B}(\lambda + \mu + \mu_T + \alpha - \beta_T \hat{S} \mathcal{K}_2(\lambda)) + \beta_T \dot{I}_T [-\mathcal{K}_2(\lambda) \Omega(\lambda) \hat{S}
\]
Theorem 3.2. (Local stability) (i) The disease-free equilibrium \( E_0 \) of system (3) is locally stable if \( \max\{\mathcal{R}_H, \mathcal{R}_T\} < 1 \) and is unstable if \( \max\{\mathcal{R}_H, \mathcal{R}_T\} > 1 \).

(ii) The TB-free equilibrium \( E_1 \) of system (3) is locally stable if \( \mathcal{R}_H > 1 \) and \( \mathcal{R}_T < \mathcal{R}_H \), and is unstable if \( \mathcal{R}_H > 1 \) and \( \mathcal{R}_T > \mathcal{R}_H \).

(iii) The endemic equilibrium \( E^* \) of system (3) is unstable if it exists.

Proof. We first consider the local stability of the disease-free equilibrium \( E_0 \). The characteristic equation corresponding to \( E_0 \) is

\[
f(\lambda) = (\lambda + \mu)(1 - S^0\mathcal{K}_1(\lambda))(\lambda + \mu + \mu_T + \alpha - \beta_T S^0 \mathcal{K}_2(\lambda)) = 0.
\]

Clearly, \( f(\lambda) = 0 \) has one negative real root \(-\mu\), and other roots satisfy

\[
\mathcal{K}_1(\lambda) = \frac{1}{S^0} \quad \text{or} \quad g_0(\lambda) = \lambda + \mu + \mu_T + \alpha - \beta_T S^0 \mathcal{K}_2(\lambda) = 0.
\]

We have \( \mathcal{K}_1(0) = \frac{\mathcal{R}_H}{S^0} \), \( \mathcal{K}_1(+\infty) = 0 \) and \( \mathcal{K}_1(\lambda) \) monotonically decreasing. Hence, if \( \mathcal{R}_H > 1 \), then \( \mathcal{K}_1(\lambda) = \frac{1}{S^0} \) has a positive real root. That is, if \( \mathcal{R}_H > 1 \), the disease-free equilibrium \( E_0 \) is unstable. If \( \mathcal{R}_H < 1 \), all roots of \( \mathcal{K}_1(\lambda) = \frac{1}{S^0} \) have negative real parts. Otherwise, \( \mathcal{K}_1(\lambda) = \frac{1}{S^0} \) has at least one root \( \lambda_0 = a_0 + ib_0 \) satisfying \( a_0 \geq 0 \). But

\[
|\mathcal{K}_1(\lambda_0)| \leq \mathcal{K}_1(a_0) \leq \frac{\mathcal{R}_H}{S^0} < \frac{1}{S^0}.
\]

In addition, \( g_0(0) = (\mu + \mu_T + \alpha)(1 - \mathcal{R}_T) \), \( g_0(+\infty) = +\infty \) and \( g_0(\lambda) \) monotonically increasing, then we know, if \( \mathcal{R}_T > 1 \), then \( g(\lambda) = 0 \) has a positive real root. That is, if \( \mathcal{R}_T > 1 \), the disease-free equilibrium \( E_0 \) is unstable. If \( \mathcal{R}_T < 1 \), all roots of \( g_0(\lambda) = 0 \) have negative real parts. Otherwise, \( g_0(\lambda) = 0 \) has at least one root \( \lambda_1 = a_1 + ib_1 \) satisfying \( a_1 \geq 0 \). But the real part of \( g_0(\lambda_1) \)

\[
\text{Re}(g_0(\lambda_1)) \geq (a_1 + \mu + \mu_T + \alpha - \beta_T S^0 \mathcal{K}_2(a_1)) \geq (\mu + \mu_T + \alpha)(1 - \mathcal{R}_T) > 0.
\]

Hence, if \( \max\{\mathcal{R}_H, \mathcal{R}_T\} < 1 \), all roots of \( f(\lambda) = 0 \) have negative real parts, then the disease-free equilibrium \( E_0 \) is locally stable, and is unstable if \( \max\{\mathcal{R}_H, \mathcal{R}_T\} > 1 \).

Next, we consider the local stability of the TB-free equilibrium \( E_1 \). We notice that that the TB-free equilibrium exists only when \( \mathcal{R}_H > 1 \), so the following discussion is based on \( \mathcal{R}_H > 1 \). The characteristic equation corresponding to \( E_1 \) is

\[
f(\lambda) = [(\lambda + \mu)(1 - \mathcal{K}_1(\lambda)S^0) + B_1](\lambda + \mu + \mu_T + \alpha - \beta_T S^0 \mathcal{K}_2(\lambda)) = 0.
\]

Clearly, whether \(-\mu\) is the characteristic root of equation \( f(\lambda) = 0 \) does not affect the stability of the TB-free equilibrium \( E_1 \). Therefore, we will discuss the sign of eigenvalues other than \(-\mu\). At this time, the equation \( f(\lambda) = 0 \) can be rewritten as

\[
f(\lambda) = (\lambda + \mu)[(1 - \mathcal{K}_1(\lambda)S^0) + \frac{B_1}{(\lambda + \mu)}](\lambda + \mu + \mu_T + \alpha - \beta_T S^0 \mathcal{K}_2(\lambda)) = 0,
\]

and other roots satisfy

\[
g_1(\lambda) = (1 - \mathcal{K}_1(\lambda)S^0) + \frac{B_1}{(\lambda + \mu)} = 0 \quad \text{or} \quad g_2(\lambda) = \lambda + \mu + \mu_T + \alpha - \beta_T S^0 \mathcal{K}_2(\lambda) = 0.
\]
We have \( g_1(0) = (1 - \mathcal{K}_1 \mathcal{S}) + (\mathcal{R}_H - 1)(\mathcal{R}_H - 1) = (\mathcal{R}_H - 1), g_1(+\infty) = 1 \), we then know if \( \mathcal{R}_H < 1 \), then \( g_1(\lambda) = 0 \) has a positive real root. That is, if \( \mathcal{R}_H < 1 \), the TB-free equilibrium \( E_1 \) is unstable. Similar to the discussion of \( g_0(\lambda) = 0 \), we know that if \( \mathcal{R}_H > 1 \), all roots of \( g_1(\lambda) = 0 \) have negative real parts.

Also, \( g_2(0) = (\mu + \mu_T + \alpha)(1 - \frac{H}{\mathcal{R}_H}), g_2(+\infty) = +\infty \) and \( g_2(\lambda) \) monotonically increasing, we know if \( \mathcal{R}_H < \mathcal{R}_T \), then \( g_2(\lambda) = 0 \) has a positive real root. That is, if \( \mathcal{R}_H < \mathcal{R}_T \), the TB-free equilibrium \( E_1 \) is unstable. Similar to the discussion of \( g_0(\lambda) = 0 \), we know that if \( \mathcal{R}_H > \mathcal{R}_T \), all roots of \( g_2(\lambda) = 0 \) have negative real parts.

Hence, if \( \mathcal{R}_H > 1 \) and \( \mathcal{R}_T < \mathcal{R}_H \), all roots of \( f(\lambda) = 0 \) have negative real parts, then the TB-free equilibrium \( E_1 \) is locally stable, and is unstable if \( \mathcal{R}_H > 1 \) and \( \mathcal{R}_H < \mathcal{R}_T \).

Finally, we consider the local stability of the endemic equilibrium \( E^* \). The characteristic equation corresponding to \( E^* \) is

\[
f(\lambda) = (\lambda + \mu + \beta_T I_1^*(1 - \mathcal{K}_5(\lambda)S^*)(\lambda + \mu + \mu_T + \alpha - \beta_T S^* \mathcal{K}_2(\lambda)) + B_2(\lambda + \mu + \mu_T + \alpha - \beta_T S^* \mathcal{K}_2(\lambda)) + \beta_T I_1^*(1 - \mathcal{K}_2(\lambda)S^* - (1 - \mathcal{K}_5(\lambda)S^*)(\lambda + \mu + \mu_T + \alpha - \beta_T S^*)) = 0.
\]

We have

\[
f(0) = -\beta_T I_1^* \mathcal{K}_2(0)S^* < 0
\]

and

\[
0 \leq \Omega_2(\lambda) = \int_0^{+\infty} \beta(\lambda)e^{-\int_0^\lambda \mu + \delta_1(s) + \delta_T(s)I_2^*ds} \int_0^\lambda i^*(\lambda)\delta_T(s)e^{\int_0^\lambda \mu + \delta_1(t) + \delta_T(t)I_2^*dt}e^{-(\alpha - s)\lambda}dsda.
\]

Apparently \( \Omega_2(\lambda) \) is monotonically decreasing, we then know \( f(+\infty) = +\infty \), so \( f(\lambda) = 0 \) has a positive real root. That is, \( E^* \) is unstable if it exists.  

4. Uniform persistence and global stability.

4.1. Global stability of the disease-free equilibrium \( E_0 \).

**Theorem 4.1.** The disease-free equilibrium \( E_0 \) of system (3) is globally asymptotically stable if \( \max\{\mathcal{R}_H, \mathcal{R}_T\} < 1 \).

**Proof.** Let \( g(x) = x - \ln x - 1 \), note that \( g(x) \) is non-negative and continuous in \((0, +\infty)\) with a unique root at \( x = 1 \). Using similar arguments to the proof of lemma 4.2[6], we can get that any solution in \( A \) satisfies that \( S(t) > 0 \) for \( t \in \mathbb{R} \). Next, we construct the following Lyapunov function \( L = L_0 + L_1 + L_2 + L_3 \) on the global attractor \( A \), by the compactness of \( A \), we can easily deduce \( L \) is bounded on \( A \), where

\[
L_0 = S^0 g \left( \frac{S}{S^0} \right), \quad S^0 = \frac{\Lambda}{\mu}, \quad L_1 = \int_0^{+\infty} F_0(a) i(t, a) da,
\]

\[
F_0(a) = \int_a^{+\infty} \frac{1}{\mathcal{K}_3} \beta(u)e^{-\int_u^\lambda (\mu + \delta_1(s))ds} du, \quad L_2 = \int_0^{+\infty} F_1(\theta) e(t, \theta) d\theta,
\]

\[
F_1(\theta) = \int_0^{+\infty} \beta_T S^0 \sigma(u) e^{-\int_u^\lambda (\mu + \delta_1(s) + \delta_T(s))ds} du, \quad L_3 = \frac{\beta_T S^0}{\mu + \mu_T + \alpha} I_T.
\]
Calculating the derivative of $L_0$, $L_1$, $L_2$, $L_3$ along solutions of system (3), respectively. We can deduce

\[
\dot{L}_0 = -\mu \left( \frac{S - S^0}{S} \right)^2 + (1 - \frac{S^0}{S}) \mu I_T - \beta_T I_T S + (1 - \frac{S^0}{S}) \int_0^{+\infty} \delta(\theta) e(t, \theta) d\theta \\
- S(1 - \frac{S^0}{S}) \int_0^{+\infty} \beta(a) i(t, a) da.
\]

\[
\dot{L}_1 = -\int_0^{+\infty} F_0(a)((\mu + \delta_i(a) + \delta_T(a) I_T)i(t, a) + \frac{\partial i}{\partial a} da)
\]

\[
= F_0(0)i(t, 0) - \int_0^{+\infty} F_0(a) \delta_T(a) I_T i(t, a) da - \int_0^{+\infty} \frac{1}{\mathcal{K}_1} \beta(a) i(t, a) da.
\]

\[
\dot{L}_2 = \left( \frac{\beta_T S^0 \mathcal{K}_2}{\mu + \mu_T + \alpha} + \frac{(1 - \mathcal{R}_T) \mathcal{K}_3}{\mathcal{K}_3} \right) \beta_T I_T S \\
- \int_0^{+\infty} \left( \frac{\beta_T S^0 \sigma(\theta)}{\mu + \mu_T + \alpha} + \frac{(1 - \mathcal{R}_T) \delta(\theta)}{\mathcal{K}_3} \right) e(t, \theta) d\theta.
\]

\[
\dot{L}_3 = \frac{\beta_T S^0}{\mu + \mu_T + \alpha} \int_0^{+\infty} \sigma(\theta) e(t, \theta) d\theta - \beta_T S^0 I_T.
\]

Therefore,

\[
\frac{dL}{dt} = -\mu \left( \frac{S - S^0}{S} \right)^2 + (1 - \frac{S^0}{S}) \mu I_T - \int_0^{+\infty} \frac{(1 - \mathcal{R}_T) \delta(\theta)}{\mathcal{K}_3} e(t, \theta) d\theta \\
+ (1 - \frac{S^0}{S}) \int_0^{+\infty} \delta(\theta) e(t, \theta) d\theta + \frac{(\mathcal{R}_H - 1) \mathcal{K}_3}{\mathcal{K}_1} \int_0^{+\infty} \beta(a) i(t, a) da \\
- \int_0^{+\infty} F_0(a) \delta_T(a) I_T i(t, a) da.
\]

Notice that $S \leq S^0$, if $\mathcal{R}_H < 1$, $\mathcal{R}_T < 1$, then $\frac{dL}{dt} \leq 0$ holds. According to the proof of the theorem 4.3[9], we can easily obtain that $\mathcal{A} = \{E_0\}$. This proves that $E_0$ is globally asymptotically stable. The proof is complete.

Since the main objective of our model is to study how to control the spread of TB in the context of the global HIV epidemic. We have great interest in studying the global stability of the TB-free equilibrium $E_1$ when HIV persists.

4.2. **Uniform persistence of HIV transmission.** In this subsection, our purpose is to show that system (3) is uniformly persistent when $\mathcal{R}_T < 1$, $\mathcal{R}_H > 1$. Define

\[
M_0 = \{(x_1, x_2, x_3, x_4) \in \Omega : \int_0^{+\infty} x_2(a) da > 0\},
\]

let $\partial M_0 = \Omega \setminus M_0$. Then we have $\Omega = M_0 \cup \partial M_0$.

**Theorem 4.2.** The sets $M_0$ and $\partial M_0$ are forward invariant under the semiflow $\Phi(t, \cdot)$. Also, if $\mathcal{R}_T < 1$ the disease-free equilibrium $E_0$ of system (3) is globally asymptotically stable for the semiflow $\Phi(t, \cdot)$ restricted to $\partial M_0$.

**Proof.** First we prove $M_0$ is forward invariant under the semiflow $\Phi(t, \cdot)$. Let $\Phi(0, x_0) \in M_0$, $I(t) = \int_0^{+\infty} i(t, a) da$, we have

\[
\frac{dI(t)}{dt} \geq -(\mu + \delta_i + \delta_T + \frac{\Lambda}{\mu}) I(t), \quad I(t) \geq e^{-(\mu + \delta_i + \delta_T + \frac{\Lambda}{\mu}) t} I(0) > 0.
\]
This implies the fact that $\Phi(t, M_0) \subset M_0$, i.e. $M_0$ is forward invariant under the semiflow $\Phi(t, \cdot)$. Next, we will prove $\partial M_0$ is forward invariant under the semiflow $\Phi(t, \cdot)$.

$$\frac{dl(t)}{dt} \leq (\beta \frac{A}{\mu} - \mu)l(t), \quad l(t) \leq e^{-(\beta \frac{A}{\mu} - \mu)t}l(0) = 0.$$

Finally, we prove the disease-free equilibrium $E_0$ of system (3) is globally asymptotically stable for the semiflow $\Phi(t, \cdot)$ restricted to $\partial M_0$. In $\partial M_0$, system (3) can be written as follows

$$\begin{align*}
\frac{ds(t)}{dt} &= \Lambda + \alpha I_T + \int_{0}^{+\infty} \delta(\theta) e(t, \theta) d\theta - \beta_T S I_T - \mu S, \\
\frac{\partial e(t, \theta)}{\partial t} + \frac{\partial e(t, \theta)}{\partial \theta} &= - (\mu + \delta(\theta) + \sigma(\theta)) e(t, \theta), \\
\frac{di(t)}{dt} &= \int_{0}^{+\infty} \sigma(\theta) e(t, \theta) d\theta - (\mu + \mu_T + \alpha) I_T, \\
e(t, 0) &= \beta_T S I_T, \quad e(0, \theta) = e_0(\theta), \quad S(0) = s_0, \quad I_T(0) = i_{T0}.
\end{align*}$$

(13)

Similar to the discussion of Theorem 2.4, we know system (13) has a global attractor $B$ in $\partial M_0$.

Using similar arguments to the proof of lemma 4.2[6], we can get that any solution in $B$ satisfies that $S(t) > 0$ for $t \in \mathbb{R}$. Next, we construct the following Lyapunov function $V = L_0 + L_2 + L_3$ on the global attractor $B$, by the compactness of $B$, we can easily deduce $L$ is bounded on $B$, where $L_0, L_2$ and $L_3$ have been defined in Theorem 4.1.

Calculating the derivative of $L_0, L_2, L_3$ along solutions of system (3), respectively. We can deduce

$$\begin{align*}
\dot{L}_0 &= -\mu \left(\frac{(S - S^0)^2}{S}\right) + (1 - \frac{S^0}{S})(\mu I_T - \beta_T I_T S) + (1 - \frac{S^0}{S}) \int_{0}^{+\infty} \delta(\theta) e(t, \theta) d\theta, \\
\dot{L}_2 &= \frac{\beta_T S^0 \mathcal{K}_2}{\mu + \mu_T + \alpha} + (1 - \frac{\mathcal{R}_T}{\mathcal{K}_3}) \beta_T I_T S \\
&\quad - \int_{0}^{+\infty} \frac{\beta_T S^0 \sigma(\theta)}{\mu + \mu_T + \alpha} + (1 - \frac{\mathcal{R}_T}{\mathcal{K}_3}) \delta(\theta) e(t, \theta) d\theta, \\
\dot{L}_3 &= \frac{\beta_T S^0}{\mu + \mu_T + \alpha} \int_{0}^{+\infty} \sigma(\theta) e(t, \theta) d\theta - \beta_T S^0 I_T.
\end{align*}$$

Therefore,

$$\frac{dV}{dt} = -\mu \left(\frac{(S - S^0)^2}{S}\right) + (1 - \frac{S^0}{S}) \alpha I_T - \int_{0}^{+\infty} \frac{1 - \mathcal{R}_T}{\mathcal{K}_3} \delta(\theta) e(t, \theta) d\theta$$

$$+ (1 - \frac{S^0}{S}) \int_{0}^{+\infty} \delta(\theta) e(t, \theta) d\theta.$$

Notice that $S \leq S^0, \mathcal{R}_T < 1$, then $\frac{dV}{dt} \leq 0$ holds. According to the proof of the theorem 4.3[9], we can easily obtain that $B = \{(\frac{\Lambda}{\mu}, 0, 0)\}$. This proves that $(\frac{\Lambda}{\mu}, 0, 0)$ is globally asymptotically stable. Furthermore, we are able to deduce that $E_0$ is globally asymptotically stable in $\partial M_0$. The proof is complete.

**Theorem 4.3.** If $\mathcal{R}_T < 1, \mathcal{R}_H > 1$, then semiflow $\{\Phi(t, \cdot)\}_{t \geq 0}$ generated by system (3) is uniformly persistent with respect to the decomposition $(M_0, \partial M_0)$. 

Proof. Since the disease-free equilibrium $E_0$ is globally asymptotically stable restricted to $\partial M_0$ when $\mathcal{R}_T < 1$, applying Theorem 4.2 in [11], we only need to prove
\[
W_s(E_0) \cap M_0 = \emptyset,
\]
where $W_s(E_0) = \{ x \in \mathcal{Y} : \lim_{t \to +\infty} \Phi(t, x) = E_0 \}$. By way of contradiction, we assume that there exists a $x_0 \in M_0$ such that $x_0 \in W_s(E_0)$. Then we can find a list of $\{ x_n \} \subset M_0$ such that
\[
\| \Phi(t, x_n) - E_0 \|_s < \frac{1}{n}, \quad t \geq 0.
\]
Denote $\Phi(t, x_n) = (S_n(t), I_n(t), E_n(t), \mathcal{T}_n(t))$. Then for all $t \geq 0$, we have
\[
\frac{\Lambda}{\mu} - \frac{1}{n} < S_n(t) < \frac{\Lambda}{\mu} + \frac{1}{n}, \quad 0 < \mathcal{T}_n(t) < \frac{1}{n}
\]
and $\Phi(t, x_n) \subset M_0$. From the system (3), we know that $\int_0^{+\infty} i_t(t, a) da > 0$ for $t \geq 0$.

Letting $k_0(a) = e^{-\int_0^a (\mu + \delta_i(s) + \delta_T(s) \frac{1}{n}) ds}$, we obtain
\[
\int_0^{+\infty} \beta(a)e^{-\int_0^a (\mu + \delta_i(s) + \delta_T(s) \frac{1}{n}) ds} da
\geq \int_0^{+\infty} \beta(a)e^{-\int_0^a (\mu + \delta_i(s)) ds} (\frac{1}{n} - \int_0^a \delta_T(s) \frac{1}{n}) ds da
\geq \mathcal{K}_1 - \frac{\beta \delta_T}{\mu n}.
\]

Since $\mathcal{R}_T > 1$, we can choose sufficiently large $n$ such that $\frac{\Lambda}{\mu} > \frac{1}{n}$, $\mathcal{K}_1 - \frac{\beta \delta_T}{\mu n} > 0$ and
\[
f(n) = (\frac{\Lambda}{\mu} - \frac{1}{n}) (\mathcal{K}_1 - \frac{\beta \delta_T}{\mu n}) > 1.
\]

Now we construct the following system
\[
\frac{\partial i_t(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(\mu + \delta_i(a) + \delta_T(a) \frac{1}{n}) i(t, a),
\]
\[
i(t, 0) = \left(\frac{\Lambda}{\mu} - \frac{1}{n}\right) \int_0^{+\infty} \beta(a) i(t, a) da,
\]
\[
i(0, a) = i_0(a).
\]

Using similar analysis as in Sect. 2, we can get existence, uniqueness and nonnegative of solution to system (15). By the comparison principle, we know
\[
i(t, s) \geq \hat{i}(t, s).
\]

Letting $\hat{i}(t) = \hat{i}(t, 0)$, By use of Volterra formulation (6), we have
\[
\hat{i}(t, a) = \begin{cases} i(t - a, 0) k_0(a), & 0 \leq a \leq t, \\ i_0(a - t) \frac{k_0(a)}{k_0(a - t)}, & 0 \leq t < a. \end{cases}
\]

Substituting (17) into the second equation of (15) yields
\[
i(t) \geq \left(\frac{\Lambda}{\mu} - \frac{1}{n}\right) \int_0^t \beta(a) i(t - a) k_0(a) da.
\]
We can claim that \( \hat{i}(t) \) is unbounded. Otherwise, we can use Laplace transform of (18) yield
\[
\mathcal{L}[\hat{i}](\lambda) \geq \mathcal{L}[\hat{i}](\lambda)\mathcal{L}[\beta](\lambda),
\]
where
\[
\mathcal{L}[\hat{i}](\lambda) = \int_0^{+\infty} e^{-\lambda t} \hat{i}(t) dt, \quad \mathcal{L}[\beta](\lambda) = \int_0^{\infty} \beta(a)k_0(a)(\frac{\lambda}{\mu} - \frac{1}{n})e^{-\lambda a} da.
\]
We obtain
\[
(1 - \mathcal{L}[\beta](\lambda))\mathcal{L}[\hat{i}](\lambda) \geq 0.
\]
\[
\mathcal{L}[\beta](\lambda) \to \mathcal{L}[\beta](0), \text{ by the Dominated Convergence Theorem.}
\]
Since
\[
(1 - \mathcal{L}[\beta](\lambda))|_{\lambda=0} = (1 - f(n)) < 0,
\]
then there exists \( \varepsilon > 0 \) such that
\[
(1 - \mathcal{L}[\beta](\lambda)) < 0,
\]
for all \( \lambda \in [0, \varepsilon) \). According to (20), we have \( \mathcal{L}[\hat{i}](\lambda) < 0 \) for all \( \lambda \in (0, \varepsilon) \). But this contradicts the nonnegative of \( \hat{i}(t)(t \geq 0) \). Hence, \( \hat{i}(t) \) is unbounded. Since \( i(t, 0) \geq \hat{i}(t) \), we get that \( i(t, 0) \) is unbounded. This contradicts the Proposition 3. Therefore, \( W_s(E_0) \cap M_0 = \emptyset \). By Theorem 4.2 [11], we get that semiflow \( \{\Phi(t, \cdot)\}_{t \geq 0} \) generated by system (3) is uniformly persistent. This shows that under the conditions of \( R_T < 1 \) and \( R_H > 1 \), HIV will continue to spread.

4.3. Global stability of the TB-free equilibrium \( E_1 \).

**Theorem 4.4.** If \( R_T < 1, R_H > 1 \), then the TB-free equilibrium \( E_1 \) of system (3) is globally asymptotically stable in \( M_0 \).

**Proof.** We define the following system
\[
\frac{\partial \hat{e}(t, \theta)}{\partial t} + \frac{\partial \hat{e}(t, \theta)}{\partial \theta} = -((\mu + \delta(\theta) + \sigma(\theta))\hat{e}(t, \theta),
\]
\[
\frac{dI_T(t)}{dt} = \int_0^{+\infty} \sigma(\theta)\hat{e}(t, \theta)d\theta - (\mu + \mu_T + \alpha)I_T,
\]
\[
\hat{e}(t, 0) = \beta_T S^0 \hat{I}_T, \hat{e}(0, \theta) = e_0(\theta), \hat{I}_T(0) = i_{T0}.
\]
System (21) has an equilibrium \( \hat{E} = (0, 0) \). By the comparison principle, we know \( e(t, \theta) \leq \hat{e}(t, \theta), I_T(t) \leq \hat{I}_T(t) \). Using the method of proving the existence of global attractor in Subsection 2.3, it can be deduced that system (21) has a global attractor \( B_1 \), and we know by Theorem 2.4 in [4] that this attractor is a strong global attractor.

We define Lyapunov function \( V = V_1 + V_2 \) on the global attractor \( B_1 \), by the compactness of \( B_1 \), we can easily deduce \( V \) is bounded on \( B_1 \), where
\[
V_1 = \int_0^{+\infty} F_1(\theta)\hat{e}(t, \theta)d\theta, \quad F_1(\theta) = \int_0^{+\infty} \sigma(u)e^{-\int_0^u(\mu + \delta(s) + \sigma(s))ds}d\theta, \quad V_2 = \hat{I}_T.
\]
Calculating the derivative of \( V \) along a solution.
\[
\frac{dV_1}{dt} = \int_0^{+\infty} F_1(\theta)(-\frac{\partial \hat{e}(t, \theta)}{\partial \theta} - (\mu + \delta(\theta) + \sigma(\theta))\hat{e}(t, \theta))
\]
\[
= \mathcal{X}_2 \hat{e}(t, 0) - \int_0^{+\infty} \sigma(\theta)\hat{e}(t, \theta)d\theta = \mathcal{X}_2 \beta_T S^0 \hat{I}_T - \int_0^{+\infty} \sigma(\theta)\hat{e}(t, \theta)d\theta,
\]
\[
\frac{dV_2}{dt} = \int_0^{+\infty} \sigma(\theta)\hat{e}(t, \theta)d\theta - (\mu + \mu_T + \alpha)\hat{I}_T.
\]
Thus, \( \frac{d(V_1 + V_2)}{dt} = (\mu + \mu_T + \alpha)(R_T - 1)I_T \leq 0. \)

We know that \( \frac{dV}{dt} \leq 0, \) and according to the proof of the theorem 4.3[9], we can easily obtain that \( B_1 = \{E\}. \) This proves that \( E \) is globally asymptotically stable. According to the fact that \( B_1 \) is a strong global attractor, we know that for sufficiently large \( n, \) there exists \( t_n, \) such that when \( t > t_n, \) \( \int_0^{+\infty} e(t, \theta)d\theta \leq \frac{1}{n}, \) \( I_T(t) \leq \frac{1}{n}, \)

for all \( \Phi(0, x_0) \in \Omega, \) and \( \mathcal{R}_H = \frac{\mathcal{X}_1(\frac{1}{n})}{\mu + \beta T \frac{n}{g}} > 1 \) and \( \mathcal{R}_H = \frac{(\Lambda + (\alpha + \delta) \frac{1}{n})}{\mu + \beta T \frac{n}{g}} > 1, \)

where \( \mathcal{X}_1(\frac{1}{n}) = \int_0^{+\infty} \beta(a)e^{-\int_0^{t} (\mu + \delta_i(s) + \frac{1}{n}) ds} da. \)

Next, we define the following system

\begin{align*}
\frac{d\tilde{S}(t)}{dt} &= \Lambda - \tilde{S} \int_0^{+\infty} \beta(a)\tilde{i}(t, a)da - \beta_T \tilde{S} \frac{1}{n} - \mu \tilde{S}, \\
\frac{\partial \tilde{i}(t, a)}{\partial t} + \frac{\partial \tilde{i}(t, a)}{\partial a} &= - (\mu + \delta_i(a) + \bar{\delta}_T \frac{1}{n}) \tilde{i}(t, a), \\
\tilde{i}(t, 0) &= \tilde{S} \int_0^{+\infty} \beta(a)\tilde{i}(t, a)da, \quad \tilde{i}(t, a) = \tilde{i}(t, a), \quad \tilde{S}(t) = S(t_n).
\end{align*}

Similar to the discussion in Subsection 2.2, it can be deduced that the solution of system (22) is non-negative. Using the method of proving the existence of global attractor in Subsection 2.3, it can be deduced that system (22) has a global attractor \( B_2. \) By the comparison principle, we know \( i(t, a) \geq \tilde{i}(t, a), \) \( S(t) \geq \tilde{S}(t) \) for \( t \geq t_n. \)

For any positive equilibrium \((\tilde{S}^*, \tilde{i}^*(a))\) of system (22), it should satisfy the following equations:

\begin{align*}
\Lambda - \tilde{i}^*(0) - (\beta_T \frac{1}{n} + \mu)\tilde{S}^* &= 0, \\
\tilde{i}^*(a) &= \tilde{i}^*(0) \tilde{k}(a), \\
\tilde{i}^*(0) &= \tilde{S}^* \mathcal{X}_1(\frac{1}{n}) \tilde{i}^*(0),
\end{align*}

where \( \tilde{k}(a) = e^{-\int_0^{t} (\mu + \delta_i(s) + \bar{\delta}_T \frac{1}{n}) ds} da. \)

Solving the third equation of the system (23), gives \( \tilde{S}^* = \frac{1}{\mathcal{X}_1(\frac{1}{n})}. \) This substitution in the first equation yields \( \tilde{i}^*(0) = \Lambda (1 - \frac{1}{\mathcal{R}_H}) > 0. \) Therefore, we know that system (23) has a positive equilibrium \( \tilde{E} = (\tilde{S}^*, \tilde{i}^*(a)). \)

Let \((\tilde{S}(t), \tilde{i}(t, a)) \in B_2, \) we will prove that \( \tilde{S}(t), \int_0^{+\infty} \tilde{i}(t, a)da > 0 \) for \( t \in R. \) We assume that there exists \( t_0 \in R \) such that \( \tilde{S}(t_0) = 0. \) From (22) we have \( \frac{d\tilde{S}(t)}{dt} = \Lambda > 0, \) then \( \exists \eta > 0 \) such that \( \tilde{S}(t_0 - \eta) < 0. \) This contradicts the non-negative of the solution. Thus, \( \tilde{S}(t) > 0 \) for \( t \in R. \) Also, we assume that there exists \( t_1 \in R \) such that \( \int_0^{+\infty} \tilde{i}(t_1, a) = 0, \) we have

\begin{align*}
\frac{d \int_0^{+\infty} \tilde{i}(t, a)da}{dt} &= - \int_0^{+\infty} (\mu + \delta_i(a) + \bar{\delta}_T \frac{1}{n}) \tilde{i}(t, a)da + \tilde{S} \int_0^{+\infty} \beta(a)\tilde{i}(t, a)da,
\end{align*}

then \( \int_0^{+\infty} \tilde{i}(t, a) = 0 \) for \( t \in R. \)
Further, we can deduce $\int_0^{+\infty} \tilde{i}(t_n, a) dt = \int_0^{+\infty} i(t_n, a) dt = 0$, this contradicts the $(S(t_n), i(t_n, a), e(t_n, \theta), \tilde{I}_R(t_n)) \in M_0$. Thus, $\tilde{S}(t), \int_0^{+\infty} \tilde{i}(t, a) da > 0$ for $t \in R$. According to the compactness of $B_2$, we know there exist $\varepsilon, M > 0$, such that all solutions in $B_2$ are satisfied $\varepsilon \leq \tilde{S}(t) \leq M$ and $\varepsilon \tilde{k}(a) \leq i(t, a) \leq M \tilde{k}(a)$.

We define Lyapunov function $U = U_1 + U_2$, on the global attractor $B_2$, by the compactness of $B_2$, we can easily deduce $U$ is bounded on $B_2$, where

$$U_1 = \tilde{S}^*(\frac{\tilde{S}}{S}), \quad U_2 = \int_0^{+\infty} F_3(a)g(\tilde{i}(t, a)) \frac{\tilde{i}(t, a)}{\tilde{i}^*(a)} da,$$

$$F_3(a) = \int_a^{+\infty} \beta(u) \frac{1}{K_1(\frac{u}{n})} e^{-\int_a^{u} (\mu + \delta_i(s) + \tilde{\delta}_{\tilde{R}} s) ds} \frac{\tilde{i}(t, a)}{\tilde{i}^*(a)} da.$$

Calculating the derivative of $U$ along a solution.

$$\frac{dU_1}{dt} = (1 - \frac{\tilde{S}^*}{S})(\Lambda - \tilde{S} \int_0^{+\infty} \beta(a) \tilde{i}(t, a) da - \beta_T \tilde{S} \frac{1}{n} - \mu \tilde{S}).$$

Notice that $\Lambda = \tilde{i}^*(0) + (\beta_T \frac{1}{n} + \mu) \tilde{S}^*$, we have

$$\frac{dU_1}{dt} = (1 - \frac{\tilde{S}^*}{S})(\tilde{i}^*(0) - \tilde{i}(t, 0) + (\beta_T \frac{1}{n} + \mu)(\tilde{S}^* - \tilde{S})).$$

Notice that $F_3^*(a) = -\beta(a)\frac{1}{x_1(\frac{a}{x})} + (\mu + \delta_i(a) + \tilde{\delta}_{\tilde{R}} \tilde{\delta}) F_3(a)$ and

$$\frac{\partial}{\partial a}g(\tilde{i}(t, a)) \frac{\tilde{i}(t, a)}{\tilde{i}^*(a)} = (1 - \frac{\tilde{i}^*(a)}{\tilde{i}(t, a)}) \frac{1}{\tilde{i}^*(a)} \frac{\partial}{\partial a} \frac{\tilde{i}(t, a)}{\tilde{i}^*(a)} + (\mu + \delta_i(a) + \tilde{\delta}_{\tilde{R}} \frac{1}{n} \tilde{i}(t, a)),$$

we have

$$\frac{dU_2}{dt} = \int_0^{+\infty} F_3(a)(1 - \frac{\tilde{i}^*(a)}{\tilde{i}(t, a)}) \frac{\tilde{i}(t, a)}{\tilde{i}^*(a)} - (\mu + \delta_i(a) + \tilde{\delta}_{\tilde{R}} \frac{1}{n} \tilde{i}(t, a)) da$$

$$- \int_0^{+\infty} F_3(a) \tilde{i}^*(a) \frac{\partial}{\partial a} \frac{\tilde{i}(t, a)}{\tilde{i}^*(a)} da$$

$$= \tilde{i}^*(0) g(\tilde{i}(t, 0)) - \int_0^{+\infty} \beta(a) \frac{1}{K_1(\frac{1}{a})} \tilde{i}^*(a) g(\frac{\tilde{i}(t, a)}{\tilde{i}^*(a)}) da.$$
Notice that \( K \) that system (25) has a positive equilibrium \( \hat{S}^* \), this proves that \( \hat{\xi} \).

We assume that there exists \( t \) such that system (22) is non-negative. Using the method of proving the existence of global asymptotically stable, it can be deduced that the solution of following equations:

\[
\frac{dU_1 + U_2}{dt} = - (\beta \frac{1}{n} + \mu) \frac{(\hat{S}^* - \hat{S})^2}{\hat{S}} + \int_0^{+\infty} \beta(a) \frac{1}{\mathcal{H}_1 \left( \frac{1}{n} \right)} \hat{i}^*(a) \ln \frac{\hat{i}(t, a)}{\hat{i}^*(a)} da.
\]

\( \mathcal{H}_1 \) is sufficiently large, \( t \rightarrow +\infty \), \( S(t) \geq \hat{S}, i(t, a) \geq \hat{i}(a) \).

Finally, we define the following system

\[
\frac{d\hat{S}(t)}{dt} = (\Lambda + \alpha \frac{1}{n} + \delta - \frac{1}{n}) - \hat{S} \int_0^{+\infty} \beta(a) \hat{i}(t, a) da - \mu \hat{S},
\]

\[
\frac{\partial \hat{i}(t, a)}{\partial t} + \frac{\partial \hat{i}(t, a)}{\partial a} = - (\mu + \delta \hat{i}(a)) \hat{i}(t, a),
\]

\[
\hat{i}(t, 0) = \hat{S} \int_0^{+\infty} \beta(a) \hat{i}(t, a) da, \hat{i}(t_n, a) = \hat{i}(t_n, a), \hat{S}(t_n) = S(t_n).
\]

Similar to the discussion in Subsection 2.2, it can be deduced that the solution of system (22) is non-negative. Using the method of proving the existence of global attractor in Subsection 2.3, it can be deduced that system (24) has a global attractor \( \mathcal{B}_3 \). By the comparison principle, we know \( i(t, a) \leq \hat{i}(t, a), S(t) \leq \hat{S}(t) \) for \( t \geq t_n \).

For any positive equilibrium \( (\hat{S}^*, i^*(a)) \) of system (24), it should satisfy the following equations:

\[
(\Lambda + \alpha \frac{1}{n} + \delta - \frac{1}{n}) - i^*(0) - \mu \hat{S}^* = 0,
\]

\[
i^*(a) = \hat{i}^*(0)k_1(a),
\]

\[
i^*(0) = \hat{S}^* \mathcal{H}_1 \hat{i}^*(0).
\]

We obtain \( \hat{S}^* = \frac{1}{\mathcal{H}_1} \) and \( \hat{i}^*(0) = (\Lambda + \alpha \frac{1}{n} + \delta - \frac{1}{n})(1 - \frac{1}{\mathcal{R}_H}) \) > 0. Therefore, we know that system (25) has a positive equilibrium \( \hat{E} = (\hat{S}^*, \hat{i}^*(a)) \).

Let \( (\hat{S}(t), \hat{i}(t, a)) \in \mathcal{B}_3 \), we will prove that \( \hat{S}(t), \int_0^{+\infty} \hat{i}(t, a) da > 0 \) for \( t \in R \). We assume that there exists \( t_0 \in R \) such that \( \hat{S}(t_0) = 0 \). From (24) we have
we have the non-negative of the solution. Thus, $\hat{S}(t_0 - \eta) < 0$. This contradicts the non-negative of the solution. Thus, $\hat{S}(t) > 0$ for $t \in R$. Also, we assume that there exists $t_1 \in R$ such that $\int_{t_0}^{+\infty} \hat{\gamma}(t_1, a) = 0$, we have

$$\frac{d}{dt} \int_{t_0}^{+\infty} \hat{\gamma}(t_1, a) da = - \int_{t_0}^{+\infty} (\mu + \delta_i(a)) \hat{\gamma}(t_1, a) da + \hat{S} \int_{t_0}^{+\infty} \beta(a) \hat{\gamma}(t_1, a) da,$$

then $\int_{t_0}^{+\infty} \hat{\gamma}(t_1, a) = 0$ for $t \in R$.

Further, we can deduce $\int_{t_0}^{+\infty} \hat{\gamma}(t_n, a) = \int_{t_0}^{+\infty} \hat{\gamma}(t_n, a) = 0$, this contradicts the $(S(t_n), i(t_n, a), e(t_n, \theta), I_T(t_n)) \in M_0$. Thus, $\hat{S}(t), \int_{t_0}^{+\infty} \hat{\gamma}(t_1, a) da > 0$ for $t \in R$. According to the compactness of $B_3$, we know there exist $\varepsilon, M > 0$, such that all solutions in $B_3$ are satisfied $\varepsilon \leq \hat{S}(t) \leq M$ and $\varepsilon k(a) \leq \hat{i}(t, a) \leq M k(a)$.

We define Lyapunov function $U^* = U_3 + U_4$, on the global attractor $B_3$, by the compactness of $B_3$, we can easily deduce $U^*$ is bounded on $B_2$, where

$$U_3 = \hat{S} \hat{g}(\frac{\hat{S}}{\hat{S}}), \quad U_4 = \int_{t_0}^{+\infty} F_4(a) g(\frac{\hat{i}(t, a)}{\hat{i}^*(a)}) da,$$

$$F_4(a) = \int_{a}^{+\infty} \beta(u) \frac{1}{x^1} e^{-\int_{u}^{+\infty} (\mu + \delta_i(s)) ds} da.$$

Calculating the derivative of $U^*$ along a solution.

$$\frac{dU_3}{dt} = (1 - \frac{\hat{S}^*}{\hat{S}})(\Lambda + \alpha \frac{1}{n} + \hat{\delta}_i \frac{1}{n} - \hat{S} \int_{t_0}^{+\infty} \beta(a) \hat{\gamma}(t_1, a) da - \mu \hat{S}).$$

Notice that $\Lambda + \alpha \frac{1}{n} + \hat{\delta}_i \frac{1}{n} = \hat{i}^*(0) + \mu \hat{S}^*$, we know

$$\frac{dU_3}{dt} = (1 - \frac{\hat{S}^*}{\hat{S}})(\hat{i}^*(0) - \hat{i}(t, 0) + \mu (\hat{S}^* - \hat{S})).$$

Notice that $F'_4(a) = -\beta(a) \frac{1}{x^1} (\mu + \delta_i(a))F'_4(a)$,

$$\frac{\partial}{\partial a} \hat{g}(\frac{\hat{i}(t, a)}{\hat{i}^*(a)}) = (1 - \frac{\hat{i}^*(a)}{\hat{i}(t, a)}) \frac{1}{\hat{i}^*(a)} \left( \frac{\partial \hat{i}(t, a)}{\partial a} + (\mu + \delta_i(a)) \hat{i}(t, a) \right),$$

we have

$$\frac{dU_4}{dt} = \int_{t_0}^{+\infty} F_4(a) (1 - \frac{\hat{i}^*(a)}{\hat{i}(t, a)}) \frac{\partial \hat{i}(t, a)}{\partial a} -(\mu + \delta_i(a)) \hat{i}(t, a) da$$

$$= - \int_{t_0}^{+\infty} F_4(a) \hat{i}^*(a) \frac{\partial}{\partial a} \hat{g}(\frac{\hat{i}(t, a)}{\hat{i}^*(a)}) da$$

$$= \hat{i}^*(0) \hat{g}(\frac{\hat{i}(t, 0)}{\hat{i}^*(0)}) - \int_{t_0}^{+\infty} \beta(a) \frac{1}{x^1} \hat{i}^*(a) \hat{g}(\frac{\hat{i}(t, a)}{\hat{i}^*(a)}) da.$$}

Thus,

$$\frac{dU_3 + U_4}{dt} = - \mu \frac{(\hat{S}^* - \hat{S})^2}{\hat{S}} + (1 - \frac{\hat{S}^*}{\hat{S}})(\hat{i}^*(0) - \hat{i}(t, 0))$$

$$+ (\hat{i}(t, 0) - \hat{i}^*(0) - \hat{i}^*(0) \ln \frac{\hat{i}(t, 0)}{\hat{i}^*(0)})$$

$$- \int_{t_0}^{+\infty} \beta(a) \frac{1}{x^1} (\hat{i}(t, a) - \hat{i}^*(a) - \hat{i}^*(a) \ln \frac{\hat{i}(t, a)}{\hat{i}^*(a)}) da.$$
5. HIV-TB co-infection case studies of China.

5.1. Data sources. According to the data from National Bureau of Statistics of China (NBSC)[25], the average number of births and the average natural mortality rate between 2005 and 2017 are \( \Lambda = 16,439,333 \) persons year\(^{-1} \) and \( \mu = 1/74.7 \) year\(^{-1} \), respectively, and the number of the initial susceptible population \( S(0) = 1,307,560,000 \) persons. The number of new TB cases and the number of HIV-positive cases (Table 2) from the Chinese Center for Disease Control and Prevention[26].

| Year | 2005  | 2006  | 2007  | 2008  | 2009  | 2010  | 2011  |
|------|-------|-------|-------|-------|-------|-------|-------|
| TB cases | 1,259,308 | 1,127,541 | 1,161,359 | 1,169,549 | 1,169,538 | 1,397,638 | 391,325 |
| HIV-positive cases | 50,887 | 38,262 | 42,633 | 51,525 | 57,473 | 61,622 | 73,196 |

| Year | 2012  | 2013  | 2014  | 2015  | 2016  | 2017  |
|------|-------|-------|-------|-------|-------|-------|
| TB cases | 101,010 | 102,474 | 105,991 | 106,912 | 830,290 | 830,290 |
| HIV-positive cases | 10,328 | 105,784 | 119,194 | 132,016 | 142,124 | 152,746 |

Note: new TB cases refer to active TB cases, HIV-positive cases include HIV carriers and AIDS patients.
5.2. Data fitting. Reference[18] mentioned that the California Partners’ Study examined 212 females having regular sexual contacts with their HIV-infected male partners. Couples were followed for different durations (duration of exposure) up to 100 months. All partners were already infected before the contact began. Only about 20% of the females were eventually infected. Shiboski and Jewell [22] use the data to estimate a time-since-infection-dependent infectivity. No explicit form of the function is given. They found HIV transmission functions first increase from 0 to 40 months after infection, and then rapidly decrease. In [14], the transmission coefficient, which increases first and then decreases, is described by the following function.

\[
\beta(a) = \begin{cases} 
0.0, & \text{if } 0.0 \leq a \leq 5.0, \\
0.66667(a - 5.0)^2e^{-0.05(a - 5.0)}, & \text{if } a > 5.0,
\end{cases}
\] (26)

where time units are days. Reference[18] also mentioned that the HIV viral load will rise in a short time after infection, the infectivity is also considered to increase, and generally lower during the latent stage of the infection. Accordingly, we use the transmission function \(\beta(a), \delta_T(a)\) similar to (26) as follows:

\[
\beta(a) = m_3ae^{-\beta a}, \delta_T(a) = m_2ae^{-\beta a}.
\]

During the spread of TB, it is worth noting that after becoming infected, some people develop TB disease soon (within weeks), before their immune system can fight the TB bacteria. When their immune system is established, the possibility of developing tuberculosis decreases, and the possibility of recovery increases [2, 3]. The time of a few weeks is negligible compared with the time unit of our research. Thus, we choose the following age-dependent monotonic decreasing function and monotonic increasing function to represent \(\sigma(\theta)\) and \(\delta(\theta)\), respectively.

\[
\sigma(\theta) = m_4e^{-\sigma \theta}, \delta(\theta) = m_5(1 - e^{-\delta \theta}).
\]

In addition, we can make a reasonable assumption that the longer the infection time, the higher the possibility of death. Thus, we choose age-dependent monotonic increasing function \(\delta_i(a)\) as follows:

\[
\delta_i(a) = m_1(1 - e^{-\delta_i a}).
\]

Next, we estimate the unknown parameters and initial values

\[
\hat{\theta} = (\alpha, m_1, m_2, m_3, m_4, m_5, \beta, \beta_T, \delta_i, \delta, \sigma, \mu_T, \mu_c, i(0), e(0), I_T(0), I_c(0), i_c(0), t_s(0))
\]

of system (1), where we assume \(i_0(a) = i(0)ue^{-\mu a}, e_0(\theta) = e(0)ue^{-\mu \theta}\).

Let \(P(t, \hat{\theta})\) be the number of new TB cases from model (1) at the \(t\)th year, then \(P(t, \hat{\theta})\) can be written as

\[
P(t, \hat{\theta}) = X(t) - X(t - 1),
\]

where \(X(t)\) represents the cumulative number of people with active TB from model (1) at the \(t\)th year, \(X(0) = i_c(0)\), and satisfies the following equation

\[
\frac{dX(t)}{dt} = \int_0^{+\infty} \sigma(\theta)e(t, \theta)d\theta + I_T \int_0^{+\infty} \delta_T(a)i(t, a)\,da.
\]

Let \(Q(t, \hat{\theta})\) be the number of new HIV-positive cases from model (1) at the \(t\)th year, then \(Q(t, \hat{\theta})\) can be written as

\[
Q(t, \hat{\theta}) = Y(t) - Y(t - 1),
\]
where $Y(t)$ represents the cumulative number of people infected with HIV from model (1) at the $t^{th}$ year, $Y(0) = t_0(0)$, and satisfies the following equation

$$\frac{dY(t)}{dt} = S \int_0^{+\infty} \beta(a) i(t,a) da.$$ 

In what follows, we will use $P(t, \hat{\theta}), Q(t, \hat{\theta})$ to simulate the number of new cases of TB and HIV-positive in China. We use MATLAB 2018b software to simulate $\hat{\theta}$. In this paper, we use an Adaptive Metropolis (DRAM) algorithm to carry out the Markov chain Monte Carlo (MCMC) procedure [10]. We can estimate the convergence of the Markov chain by the closeness of the Geweke value to 1. The mean, standard deviation and 95% confidence interval of the estimated parameters are shown in Table 3, and the fitting result can be seen in Figure 3. In addition, according to the mean values of parameters in Table 3, we also find that the values of two basic reproduction numbers $R_H$ and $R_T$ are 1.3928 and 0.6834, respectively. This means that under the current situation that HIV transmission is not effectively controlled, TB control has improved in China. However, whether the current TB control measures in China can achieve the WHO End TB Strategy in 2030, that is, the number of new cases of TB will be reduced by 80% by 2030 compared with 2015, which still needs further study.

![Figure 3](image-url)  
**Figure 3.** Data fitting: (a) the fitting results of the number of new TB cases reported from 2005 to 2017; (b) the fitting results of the number of new HIV-positive cases reported from 2005 to 2017. The solid black line represents the fitted data, and the red dots represent the actual data. The areas from the darkest to the lightest correspond to the 50%, 90%, 95% and 99% posterior limits of the model uncertainty.

5.3. **Uncertainty and sensitivity analysis.** The outputs of system (3) is governed by the system input parameters and the initial values, but some of these parameters and initial values are obtained by data fitting, which may exhibit some uncertainty in their selection. The purpose of uncertainty analysis (UA) [21, 8, 17] is to determine the reliability of parameter estimates. In order to realize the UA, the most commonly used sampling method is Latin hypercube sampling (LHS). For each parameter, sampling is guided by the specification of a probability density function, i.e. normal, uniform, etc. Next, we study some parameters that affect $R_T$. For the
Table 3. The parameters values and initial values of the model (3).

| Parameter | Mean     | Std       | 95% CI                  | Source   |
|-----------|----------|-----------|-------------------------|----------|
| Λ         | 1643933  | -         | -                       | [25]     |
| µ         | -        | -         | -                       | -        |
| S(0)      | 13075600 | -         | -                       | [25]     |
| α         | 0.002489 | 0.000239  | [0.002483197,0.00249482]| MCMC     |
| β         | 0.054717 | 0.000305  | [0.05467444,0.05470502]| MCMC     |
| m1        | 3.07 × 10^{-9}  | 7.63 × 10^{-11} | [3.070478 × 10^{-9}, 3.073469 × 10^{-9}] | MCMC |
| m2        | 0.073713 | 0.002395  | [0.073666415,0.073760303]| MCMC     |
| δ         | 0.054719 | 0.002311  | [0.05467444,0.05470502]| MCMC     |
| m3        | 5.07 × 10^{-9} | 7.63 × 10^{-11} | [5.069704 × 10^{-9}, 5.073469 × 10^{-9}] | MCMC |
| m4        | 0.085041 | 0.002759  | [0.084987398,0.085095587]| MCMC     |
| m5        | 1.59175  | 0.024036  | [1.591275856,1.592218155]| MCMC     |
| βT        | 1.495393 | 1.496291  | [1.495393 × 10^{-9}, 1.496291 × 10^{-9}] | MCMC |
| δi        | 0.921802 | 0.036083  | [0.921094653,0.922509267]| MCMC     |
| m1        | 0.921801 | 0.036083  | [0.921094653,0.922509267]| MCMC     |
| m2        | 1.405996 | 9.99339   | [1.404030 × 10^{-9}, 1.407948 × 10^{-9}] | MCMC |
| m3        | 0.010988 | 0.000573  | [0.01097653,0.01099909]| MCMC     |
| m4        | 0.894160 | 0.895137  | [0.89416205,0.895136999]| MCMC     |
| m5        | 0.000943 | 0.000422  | [0.000942785,0.000944442]| MCMC     |
| σ         | 0.010987 | 0.010999  | [0.01097653,0.01099909]| MCMC     |
| µT        | 0.000043 | 0.000043  | [0.000042785,0.000044442]| MCMC     |
| µe        | 0.000043 | 0.000043  | [0.000042785,0.000044442]| MCMC     |
| e(0)      | 15413672 | 566182    | [15412562,15414782]    | MCMC     |
| i(0)      | 2566318  | 597       | [2566006,2566629]      | MCMC     |
| t(0)      | 7433129  | 289907    | [7427446,7438811]     | MCMC     |
| I(t)      | 14052775 | 30076     | [14051685,14052864]   | MCMC     |
| l(0)      | 27800    | 4091      | [27765,27835]         | MCMC     |

In Table 3, we let $m_1, m_2, m_3, \beta, \delta, \mu_T, \mu_e, i(0), e(0), I_T(0), I_c(0), i_c(0), t_s(0)$ take the mean value, and select normal distribution for $\alpha, m_4, m_5, \beta_T, \delta, \sigma$, where the mean and standard deviation are given in Table 3. We draw 1000 samples and obtain distribution histogram of the basic reproduction number $R_T$ (see Figure 4). From Figure 4, we know that the distribution of the basic reproduction number in the range $[0.6257, 0.7389]$, and the mean value is 0.6836. Combined with the analysis in section 4, we can conclude that under current control measures, the possibility of eliminating TB is very great.

![Figure 4](image-url)  

**Figure 4.** The distribution histogram of the basic reproduction number $R_T$.

Sensitivity analysis (SA) is to identify critical parameters that have significant impact on the basic reproduction numbers $R_T$ and to quantify how parameters uncertainty impact $R_T$. Now, a global sensitivity analysis is usually implemented using sampling-based methods. We will use partial rank correlation coefficient (PRCC) method to study SA. We calculate the PRCC between the parameters and the basic reproduction number $R_T$ (see Figure 5). From the PRCC values, we can know that $\beta_T, m_4, m_5$ have the most important impact on $R_T$. 

![Figure 5](image-url)
5.4. Effective Measures to achieve the WHO strategy. China is one of the countries with the largest number of new TB cases. If China can make a breakthrough in TB control, it will be of far-reaching significance to TB control in China and the world. However, based on the prediction of model (1), we find that with the current control measures, China may not reach the WHO’s goal by 2030 (see Figure 6). To do so, China should give more feasible control measures. In the above analysis, we know that $m_4, m_5, \beta_T$ are the most important factors for TB control. Next, we adjust $m_4, m_5$ and $\beta_T$ to predict the number of new TB cases in the next few years.

We use the mean values in table 3 as the baseline to compare the following control effects. First, we only consider changing the value of parameter $m_5$ (see Figure 6), we can find that the value of $m_5$ needs to be increased to 4 times the baseline to reach the WHO 2030 target. If we only consider changing the value of parameter $m_4$, we can find that the value of $m_4$ needs to be reduced by 60% to reach the WHO 2030 target (see Figure 7). Next, we can find that even if $\beta_T$ is reduced by 80%, the WHO 2030 target will not be achieved(see Figure 8). Finally, we consider changing these three parameters at the same time (see Figure 9), we can find that if we reduce the parameters $m_4$ and $\beta_T$ by 20% and 50% respectively, and increase the parameters $m_5$ by 100%, then we will reach the WHO 2030 target.

$\delta(\theta) = m_5(1 - e^{-\delta\theta}), \sigma(\theta) = m_4 e^{-\sigma\theta}$ represent the rate at which latent individuals progress into susceptible class and at which latent class progress into infectious class, respectively. $\beta_T$ represents the transmission coefficient of active TB class to susceptible class. Therefore, China should strengthen the detection and the treatment of latent individuals, so as to not only prevent them from becoming active TB, but also make more people recover. In addition, China should also strengthen the publicity of tuberculosis knowledge so that susceptible people can know how to avoid being infected. These measures are reflected in model (1), which can reduce $m_4$, $\beta_T$ and increase $m_5$. By using these measures, China may reach the WHO End TB Strategy in 2030.

6. Discussion. In this paper, we used an age-structured mathematical model to study the transmission dynamics of HIV-TB co-infection. Sufficient conditions were derived for the global asymptotic stability of the disease-free equilibrium and the TB-free equilibrium. We estimated model parameters by fitting the annual new TB cases and HIV-positive cases data of China, and concluded that, by using current TB control measures, China may not reach the WHO’s goal by 2030. In order
Figure 6. The effect of changes in $m_5$ on the number of new TB cases.

Figure 7. The effect of changes in $m_4$ on the number of new TB cases.

Figure 8. The effect of changes in $\beta_T$ on the number of new TB cases.
The reported TB infected cases

baseline parameter value
Actual data

0.8*m_4,0.5*T,2*m_5

The 2030 targets of WHO

Figure 9. The effect of changes in $m_5$, $\beta_T$ and $m_4$ on the number of new TB cases.

to achieve WHO’s goal, China needs to develop more practical control measures. PRCC values of the basic reproduction number, $R_T$, with respect to some important model parameters show that effective measures to control the spread of TB should include strengthening the detection and treatment of latent TB cases and public education campaigns.

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