Extensivity and nonextensivity of two-parameter entropies

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Abstract

In this paper, we investigate two-parameter entropies and obtain some conditions for their extensivity. By using a generalized \((k,r)\) − product, correlations for subsystems are related to the joint probabilities, so that the entropy remains extensive.

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1 Introduction

A quantity $X(A)$ associated with a system $A$ is said additive with regard to a specific composition of $A$ and $B$ if it satisfies

$$X(A + B) = X(A) + X(B)$$  \hspace{1cm} (1)

where $+$ inside the argument of $X$ precisely indicates that composition. Suppose, instead of two subsystems $A$ and $B$, we have $N$ of them $(A_1, A_2, ..., A_N)$. Then the quantity $X$ is additive if we have

$$X(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} X(A_i)$$  \hspace{1cm} (2)

supposing that all subsystems are equal,

$$X(N) = NX(1)$$  \hspace{1cm} (3)

with the notation $X(N) \equiv X(\sum_{i=1}^{n} A_i)$ and $X(1) \equiv X(A_1)$. Another related concept is extensivity which corresponds to a weaker demand, namely that of,

$$\lim_{N \to \infty} \frac{|X(N)|}{N} < \infty$$  \hspace{1cm} (4)

Clearly, all quantities which are additive, are also extensive, whereas the opposite is not necessarily true. In other words, extensivity is defined as additivity when $N \to \infty$. Of course, there are quantities that are neither additive nor extensive. They are called nonextensive. Boltzmann-Gibbs (BG) statistical mechanics is based on the entropy

$$S_{BG} \equiv -k \sum_{i=1}^{w} p_i \ln p_i$$  \hspace{1cm} (5)

with

$$\sum_{i=1}^{w} p_i = 1$$  \hspace{1cm} (6)

where $p_i$ is the probability associated with the $i^{th}$ microscopic state of the system and $k$ is Boltzmann constant. From now on, and without loss of generality, we shall take $k$ equal to unity.
Nonextensive statistical mechanics, first introduced by C. Tsallis in 1988 [1, 2, 3], is based on the so-called ‘nonextensive’ entropy $S_q$ defined as follows:

$$S_q \equiv \frac{1 - \sum_{i=1}^{w} p_i^q}{q - 1} \quad (7)$$

As we see this entropy depends on parameter $q$. Afterwards, some other entropies were suggested depending on one parameter [4, 5, 6, 7]. Recently, an entropy was introduced [8, 9] that depends on two parameters, and in some special limits recovers other entropies that had been introduced previously. That is

$$S_{k,r} \equiv -\sum_{i=1}^{w} p_i \ln_{k,r} p_i \quad (8)$$

with

$$\ln_{k,r}(x) = x^r x^k - x^{-k} \quad (9)$$

The concept of extensivity has been investigated mostly for systems with no correlation, namely independent systems. In that case, the probabilities belong to the composition system are defined as the product of the probabilities in each subsystem. If the composition law is not explicitly indicated, it is tacitly assumed that systems are statistically independent. In that case, for two systems A and B, it immediately follows that

$$S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B) \quad (10)$$

hence, BG-entropy is additive and also extensive, but for q-entropy we have

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \quad (11)$$

hence, q-entropy is nonextensive for $q \neq 1$. In [10] Tsallis has illustrated the remarkable changes that occur when A and B are specially correlated. Indeed, he has shown that in such case

$$S_q(A + B) = S_q(A) + S_q(B) \quad (12)$$

for the appropriate value of $q$ (hence extensive), whereas

$$S_{BG}(A + B) \neq S_{BG}(A) + S_{BG}(B) \quad (13)$$
hence BG-entropy isn’t extensive in the case of correlated systems. This paper is organized as follows. In sec. 2, the nonextensivity of $S_{k,r}$ is discussed, where the extensivity of BG-entropy is recovered in a special limit. In sec. 3, we investigate how to interpret entropy $S_{k,r}$ extensive and finally in sec. 4 extensivity of entropy with canonic ensemble is discussed when we have correlated subsystems.

2 nonextensivity of $S_{k,r}$ in the case of independent systems

As said, the entropy $S_{k,r}$ Eqs. (8) and (9) is more general than the other entropies introduced previously and in some special limits recovers them. We prove that this entropy is nonextensive in the case of independent subsystems. Supposing two independent subsystems $A$ and $B$, for the probability in the composite system $A + B$ we have

$$p_{ij}^{A+B} = p_i^A p_j^B \quad \forall (i, j)$$

with the definitions

$$S_{k,r}(A) \equiv -\sum_{i=1}^{W_A} p_i^A \ln_{k,r} p_i^A$$

$$S_{k,r}(A + B) \equiv -\sum_{i=1}^{W_A} \sum_{j=1}^{W_B} p_{ij}^{A+B} \ln_{k,r} p_{ij}^{A+B}$$

By adding and subtracting the phrase $p_i^{r+k+1} p_j^{r-k+1}$ and using Eq. (9), we can find

$$S_{k,r}(A + B) = \sum_{j=1}^{W_B} p_j^{r-k+1} S_{k,r}(A) + \sum_{i=1}^{W_A} p_i^{r+k+1} S_{k,r}(B)$$

As we see

$$S_{k,r}(A + B) \neq S_{k,r}(A) + S_{k,r}(B)$$

hence $S_{k,r}$ isn’t extensive in general. However, one may choose some special range of parameters where Eq. (17) is extensive. We study the extensivity of $S_{k,r}$ in some special limits.
2.1 q-entropy (Tsallis entropy)

The q-logarithm that is usually used is

\[ \ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q} \]  (19)

With this logarithm the q-entropy is defined as

\[ S_q(p) \equiv -\sum_{i=1}^{W} p_i q \ln_q p_i \]  (20)

by choosing \( r = k \) and \( q = 1 + 2k \) in Eqs. (8) and (9), one has

\[ S_q(p) \equiv -\sum_{i=1}^{W} p_i \ln_q p_i \]  (21)

where

\[ \ln_q(x) \equiv \frac{x^{q-1} - 1}{q-1} \]  (22)

which gives an equivalent entropy to Eq. (20). For \( \exp_q(x) \) it is obtained

\[ \exp_q(x) \equiv [1 + (q-1)x]^{\frac{1}{q-1}} \]  (23)

In the limit \( r = k \) and \( q = 1 + 2k \) from Eq. (17), one recovers

\[ S_q(A + B) = \sum_{j=1}^{W_B} p_j S_q(A) + \sum_{i=1}^{W_A} p_i q S_q(B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \]  (24)

that is the familiar expression for nonextensivity of q-entropy. In the limit \( q \to 1 \) extensivity of BG-entropy is obtained.

2.2 k-entropy

The k-entropy introduced in [6, 7] is

\[ S_k(p) \equiv -\sum_{i=1}^{W} p_i \ln_k p_i \]  (25)
where
\[ \ln_k(x) \equiv \frac{x^k - x^{-k}}{2k} \] (26)

It is clear that we can recover k-logarithm from Eq. (9) in the limit \( r \to 0 \). For \( \exp_k(x) \) we have
\[ \exp_k(x) \equiv (\sqrt{1 + k^2x^2} + kx)^\frac{1}{k} \] (27)

In that limit Eq. (17) results in
\[ S_k(A + B) = \sum_{j=1}^{W_B} p_j^{-k+1} S_k(A) + \sum_{i=1}^{W_A} p_i^{k+1} S_k(B) \]
\[ \neq S_k(A) + S_k(B) \] (28)

that ensures the nonextensivity of the k-entropy. It is clear that in the limit \( k \to 0 \) the extensivity of BG-entropy is obtained again.

3 How to interpret the entropy \( S_{k,r} \) extensive

Suppose, we have \( N \) subsystems \( (A_1, A_2, ..., A_N) \). We define the probabilities in the composite system \( p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} \) that satisfy the condition
\[ \sum_{i_1i_2...i_N} p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} = 1 \] (29)

and marginal probabilities as follows
\[ p_{i_s}^{A_s} \equiv \sum_{i_1i_2...i_{s-1}i_{s+1}i_N} p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} \] (30)

If \( p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} \) also satisfies the condition
\[ p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} = \exp_{k,r}(\sum_{s=1}^{N} \ln_{k,r} p_{i_s}^{A_s}) \] (31)

then for the entropy of the composite system with the definition
\[ S_{k,r}(\sum_{s=1}^{N} A_s) \equiv - \sum_{i_1i_2...i_N} p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} \ln_{k,r} p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} \] (32)
we have
\[ S_{k,r}(\sum_{s=1}^{N} A_s) = - \sum_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} p_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} \ln k,r[p_{i_s}^{A_s}] \]
\[ = - \sum_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} p_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} \sum_{s=1}^{N} \ln k,r p_{i_s}^{A_s} \]
\[ = - \sum_{s=1}^{N} \sum_{i_s}^{A_s} \ln k,r p_{i_s}^{A_s} = \sum_{s=1}^{N} S_{k,r}(A_s) \quad (33) \]

It is useful at this point to connect the present problem to some generalized algebra which have been discussed by many authors. We use the product introduced in [9]. It is defined as follows:
\[ x \otimes_{k,r} y \equiv \exp_{k,r}(\ln_{k,r}(x) + \ln_{k,r}(y)) \quad (34) \]
hence we can write (31) as
\[ p_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} = \exp_{k,r}(\ln_{k,r}(x) + \ln_{k,r}(y)) \]
\[ = p_{i_1}^{A_1} \otimes_{k,r} p_{i_2}^{A_2} \otimes_{k,r} \ldots \otimes_{k,r} p_{i_N}^{A_N} \quad (35) \]

So extensivity of the entropy is satisfied if we use logarithm, exponential and also the product based on (34). In the limit \( k \to 0 \) and \( r \to 0 \) (BG-limit), the usual product is recovered and (35) describes the probability of composite system in the case of independent subsystems and also extensivity of BG-entropy in that case which is expected. Eq. (31) is a very special correlation for subsystems which leads to extensivity of entropy. However, it is possible to define a general correlation among subsystems so that the entropy remains extensive. Consider the following relation
\[ \tilde{p}_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} \equiv p_{i_1}^{A_1} \otimes_{k,r} p_{i_2}^{A_2} \otimes_{k,r} \ldots \otimes_{k,r} p_{i_N}^{A_N} \quad (36) \]
where \( p_{i_s}^{A_s} \)s are the probabilities of each subsystem, but \( \tilde{p}_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} \)s are not necessarily represent the joint probabilities. Now the sum of subsystem entropies can be written as
\[ \sum_{s=1}^{N} S_{k,r}(A_s) = - \sum_{s=1}^{N} p_{i_s}^{A_s} \ln k,r p_{i_s}^{A_s} = - \sum_{i_1 i_2 \ldots i_N}^{A_1+\ldots+A_N} \ln k,r p_{i_s}^{A_s} \]
we choose which is equivalent to the Tsallis proposal for the joint probabilities $[1, 0]$ if system and each subsystem are shown in the following table

Consider two subsystems A and B where the probabilities of composite and each subsystem are shown in the following table

| $A \setminus B$ | 1 | 2 |
|-----------------|---|---|
| 1               | $p_{11}^{A+B}$ | $p_{12}^{A+B}$ | $p_1^A$ |
| 2               | $p_{21}^{A+B}$ | $p_{22}^{A+B}$ | $1 - p_1^A$ |

So entropy is extensive if

$$S_{k,r} \left( \sum_{s=1}^{N} A_s \right) = - \sum_{i_{12}...i_N} p_{i_{11}...i_N}^{A_1+...+A_N} \ln k_r p_{i_{11}...i_N}^{A_1+...+A_N} \ln k_r p_{i_{11}...i_N}^{A_1+...+A_N}$$

$$= - \sum_{i_{12}...i_N} p_{i_{11}...i_N}^{A_1+...+A_N} \ln k_r p_{i_{11}...i_N}^{A_1+...+A_N}$$

It is clear that $\tilde{p}_{i_{11}...i_N}^{A_1+...+A_N}$ and $p_{i_{11}...i_N}^{A_1+...+A_N}$ can be related to each other by the following relations

$$\ln k_r p_{i_{11}...i_N}^{A_1+...+A_N} - \ln k_r \tilde{p}_{i_{11}...i_N}^{A_1+...+A_N} = \phi_{i_{11}...i_N}$$

$$\sum_{i_{12}...i_N} p_{i_{11}...i_N}^{A_1+...+A_N} \phi_{i_{11}...i_N} = 0 \quad (40)$$

where $\phi_{i_{11}...i_N}$ are arbitrary functions with (10) as a constraint. Eqs. (36) and (39) result in

$$p_{i_{11}...i_N}^{A_1+...+A_N} = \exp_{k,r} \left( \sum_{s=1}^{N} \ln k_r p_{i_s}^{A_s} + \phi_{i_{11}...i_N} \right) \quad (41)$$

In the Tsallis limit Eq. (11) can be written as (by using (22) and (23))

$$p_{i_{11}...i_N}^{A_1+...+A_N} = \left[ 1 - N + (q - 1) \phi_{i_{11}...i_N} + \sum_{s=1}^{N} (p_{i_s}^{A_s})^{q-1} \right]^{1 \over q-1} \quad (42)$$

which is equivalent to the Tsallis proposal for the joint probabilities (10) if we choose

$$\phi_{i_{11}...i_N}^{(q)} = (q - 1) \phi_{i_{11}...i_N} \quad (43)$$

Consider two subsystems A and B where the probabilities of composite system and each subsystem are shown in the following table
with the following relations

\[ p_{11}^{A+B} + p_{12}^{A+B} = p_1^A \]  \hspace{1cm} (44)
\[ p_{21}^{A+B} + p_{22}^{A+B} = p_2^A = 1 - p_1^A \]  \hspace{1cm} (45)
\[ p_{11}^{A+B} + p_{21}^{A+B} = p_1^B \]  \hspace{1cm} (46)
\[ p_{12}^{A+B} + p_{22}^{A+B} = p_2^B = 1 - p_1^B \]  \hspace{1cm} (47)

and also a constraint (40)

\[ p_{11}^{A+B} \phi_{11} + p_{12}^{A+B} \phi_{12} + p_{21}^{A+B} \phi_{21} + p_{22}^{A+B} \phi_{22} = 0 \]  \hspace{1cm} (48)

Using Eq. (41), it is possible to write Eqs. (44) to (47) in terms of \( p_1^A, p_2^A, \phi_{11}, \phi_{12}, \phi_{21} \) and \( \phi_{22} \). So \( \phi_{ij}s \) can be determined. For simplicity we use Tsallis limit and so (42) for the probabilities of the composite system. We also assume that both subsystems \( A \) and \( B \) are equal, namely \( p_1^A = p_1^B = p \). So we have

\[ p_{11}^{A+B} = [2p^{q-1} + (q-1)\phi_{11} - 1]^{\frac{1}{q-1}} \]  \hspace{1cm} (49)
\[ p_{12}^{A+B} = [p^{q-1} + (1-p)^{q-1} + (q-1)\phi_{12} - 1]^{\frac{1}{q-1}} \]  \hspace{1cm} (50)
\[ p_{21}^{A+B} = [(1-p)^{q-1} + p^{q-1} + (q-1)\phi_{21} - 1]^{\frac{1}{q-1}} \]  \hspace{1cm} (51)
\[ p_{22}^{A+B} = [2(1-p)^{q-1} + (q-1)\phi_{22} - 1]^{\frac{1}{q-1}} \]  \hspace{1cm} (52)

By substituting Eqs. (49) to (52) in (44) to (48), we obtain \( \phi_{12} = \phi_{21} \) and so

\[ [2p^{q-1} + (q-1)\phi_{11} - 1]^{\frac{1}{q-1}} + [p^{q-1} + (1-p)^{q-1} + (q-1)\phi_{12} - 1]^{\frac{1}{q-1}} = p \]  \hspace{1cm} (53)
\[ [p^{q-1} + (1-p)^{q-1} + (q-1)\phi_{12} - 1]^{\frac{1}{q-1}} + [2(1-p)^{q-1} + (q-1)\phi_{22} - 1]^{\frac{1}{q-1}} = 1 - p \]  \hspace{1cm} (54)
\[ \phi_{11}[2p^{q-1} + (q-1)\phi_{11} - 1]^{\frac{1}{q-1}} + 2\phi_{12}[p^{q-1} + (1-p)^{q-1} + (q-1)\phi_{12} - 1]^{\frac{1}{q-1}} + \phi_{22}[2(1-p)^{q-1} + (q-1)\phi_{22} - 1]^{\frac{1}{q-1}} = 0 \]  \hspace{1cm} (55)

With a given value of \( q \), above equations can be solved and one may obtain \( \phi_{11}(p), \phi_{12}(p) = \phi_{21}(p) \) and \( \phi_{22}(p) \). A few typical \((q, \phi_{11}(1/2), \phi_{12}(1/2))\) points are:

\((0.4, -7.57873, 0.71845), (0.5, -4.20199, 0.62528), (0.6, -2.5339, 0.53573), \)
where it should be noted that for $p = 1/2$ symmetries of equations ensures that $\phi_{11}(p) = \phi_{22}(p)$. We will investigate more numerical estimates for two-parameter entropies in another paper.

4 Extensive entropy in the case of correlated subsystems, a constraint approach

In this section, a new approach is used where the condition (40) is entered to the entropy as a constraint and then the entropy is maximized. Parallel to what is done in [9], we introduce the entropy in the composite system as

$$S(\sum_{s=1}^{N} A_s) \equiv - \sum_{i_1 i_2 \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} \Lambda(p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N})$$  (56)

where $\Lambda(x)$ is a generalization of the logarithm. We have the constraints

$$\sum_{i_1 i_2 \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} = 1$$  (57)

$$\sum_{i_1 i_2 \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} E_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} = U$$  (58)

$$\sum_{i_1 i_2 \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} \phi_{i_1 i_2 \ldots i_N} = 0$$  (59)

For simplicity we use the notation $\{i\}$ instead of $\{i_1 i_2 \ldots i_N\}$. Then the entropic functional can be introduced as

$$F[p] = S(p) - \beta' \left( \sum_{\{i\}} p_{\{i\}} - 1 \right) - \beta \left( \sum_{\{i\}} p_{\{i\}} E_{\{i\}} - U \right) - \beta'' \left( \sum_{\{i\}} p_{\{i\}} \phi_{\{i\}} \right)$$  (60)

where $\beta$, $\beta'$ and $\beta''$ are Lagrange multipliers and it has been supposed that $\phi_{\{i\}}$ isn’t an explicit function of $p_{\{i\}}$. If $F[p]$ in Eq. (60) is stationary for variations of the probabilities $p_{\{j\}}$,

$$\frac{\delta}{\delta p_{\{j\}}} F[p] = 0$$  (61)

one finds

$$\frac{d}{dp_{\{j\}}} \left[ p_{\{j\}} \Lambda(p_{\{j\}}) \right] = -\beta (E_{\{j\}} - \mu - \mu' \phi_{\{j\}})$$  (62)
where $\mu = -\beta'/\beta$ and $\mu' = -\beta''/\beta$.

Without loss of generality, we can express the probability distribution $p_j$ as

$$p_j = \alpha \varepsilon \left( -\frac{\beta}{\lambda} (E_j - \mu - \mu' \phi_j) \right)$$

(63)

where $\alpha$ and $\lambda$ are two arbitrary, real and positive constants, and $\varepsilon(x)$ an invertible function that can be a generalization of, and in some limit reduce to, the exponential function. If we require that $\varepsilon(x)$ be the inverse of $\Lambda(x)$, Eqs. (62) and (63) result in

$$\frac{d}{dp_j} [p_j \Lambda(p_j)] = \lambda \varepsilon^{-1} \left( \frac{p_j}{\alpha} \right)$$

(64)

that can be rewritten as [9]

$$\frac{d}{dx} \left[ x \Lambda(x) \right] = \lambda \varepsilon^{-1} \left( \frac{x}{\alpha} \right)$$

(65)

So, for $\Lambda(x)$ we have

$$\Lambda(x) = \ln_{k,r}(x) = x^r \frac{x^k - x^{-k}}{2k}$$

(66)

and the constants $\alpha$ and $\lambda$ can be expressed in terms of $k$ and $r$

$$\alpha = \left( \frac{1 + r - k}{1 + r + k} \right)^{1/(2k)}$$

(67)

$$\lambda = \frac{1 + r - k}{1 + r + k} \frac{(r+k)/(2k)}{(r-k)/(2k)}$$

(68)

Eq. (66) indicates that by imposing the condition (59), the definition of logarithm dose not change and the only thing we must change is the definition of probability in the composite system.

It is useful here to interpret each subsystem separately. By imposing the conditions

$$\sum_{i_s} p_{i_s}^{A_s} = 1$$

(69)

$$\sum_{i_s} p_{i_s}^{A_s} E_{i_s}^{A_s} = U_s$$

(70)
For the subsystems, the entropic functional will be

\[ \mathcal{F}_s[p] = S_s(p) - \beta_s' \left( \sum_{i_s} p_{i_s} - 1 \right) - \beta_s \left( \sum_{i_s} p_{i_s} E_{i_s} - U_s \right) \quad (71) \]

and by maximizing the entropic functional in the way similar to the case of composite system, we obtain

\[ p_{i_s} = \alpha \exp_{k,r} \left( -\frac{\beta_s}{\lambda} (E_{i_s} - \mu_s) \right) \quad (72) \]

where \( \mu_s = -\beta_s'/\beta_s \), \( \exp_{k,r}(x) \) is inverse function of \( \ln_{k,r}(x) \) and \( \alpha \) and \( \lambda \) are defined in Eqs. (67) and (68).

Using (63) and (72), Eq. (31) can be written as

\[ \ln_{k,r}[\alpha \exp_{k,r}(-\frac{\beta_s}{\lambda}(E_{(j)} - \mu - \mu'\phi_{(j)}))] = \sum_{s=1}^{N} \ln_{k,r}[\alpha \exp_{k,r}(-\frac{\beta_s}{\lambda}(E_{i_s} - \mu_s))] \quad (73) \]

Where parameters \( \phi_{(j)} \) can be used to ensure extensivity of the two-parameter entropies. From Eq. (73), it is clear that extensivity of entropy dose not necessarily ensures extensivity of energy (For a discussion in the case of q-entropy see [11]). In the Boltzmann-Gibbs limit Eq. (73) becomes

\[ \beta(E_{(j)} - \mu - \mu'\phi_{(j)}) = \sum_{s=1}^{N} \beta_s(E_{i_s} - \mu_s) \quad (74) \]

where only in a special case leads to the extensivity of energy.

### 5 Probabilities and effective number of states

Our motivation for studying such kind of correlations and extensivity of the two-parameter entropies of correlated subsystems was the following argument by Tsallis [10, 12] which defines effective number of states. Suppose that the probability distribution in phase space is uniform within a volume \( W \) and also \( S_q \) is given by

\[ S_q = \ln_q W \quad (75) \]

With the help of q-product [13] defined as

\[ x \otimes_q y \equiv \exp_q(\ln_q x + \ln_q y) \]

\[ = (x^{1-q} + y^{1-q} - 1)^{1/q} \quad (76) \]
it is possible to interpret $S_q$ extensive. Supposing that $W_A$ and $W_B$ be the number of states for subsystems A and B. Equation

$$W_{A+B}^{\text{eff}} \equiv W_A \otimes_q W_B$$

(77)
can be interpreted as a definition for effective number of states for the system $A + B$. Definition (76) ensures that

$$\ln_q W_{A+B}^{\text{eff}} = \ln_q W_A \otimes_q W_B = \ln_q W_A + \ln_q W_B$$

(78)

Eq. (78) shows extensivity of the entropy (75). If we suppose

$$W_A^{A_1} = W_A^{A_2} = ... = W_A^{A_N} = 1/p$$

(79)
the probability in the composite system will be

$$(1/p_{A_1+...+A_N}^{i_1i_2...i_N}) = (1/p) \otimes_q (1/p) \otimes_q ... \otimes_q (1/p)$$

(80)

and hence

$$p_{A_1+...+A_N}^{i_1i_2...i_N} = p \otimes_{2-q} p \otimes_{2-q} ... \otimes_{2-q} p$$

(81)

where (81) is obtained from (80) by the properties of q-product. At this point it is appropriate to use the following q-product which is used in this paper

$$x \otimes_{q'} y \equiv \exp_{q'}(\ln_{q'} x + \ln_{q'} y)$$

$$= (x^{q'-1} + y^{q'-1} - 1)^{1/q'-1}$$

(82)

comparing Eq. (82) with Eq. (76) shows that $q' = 2 - q$. By our q-product Eq. (81) can be written as

$$p_{i_1i_2...i_N}^{A_1+A_2+...+A_N} = p \otimes_q p \otimes_q ... \otimes_q p$$

(83)

This is a hinting point to define the probability of composite system in terms of the probabilities of subsystems by a generalized $(k, r)$-product.

6 conclusion

In this paper, it is shown that two-parameter entropies $S_{k,r}$ are not in general extensive. A formulation is given where by $(k, r)$-products of subsystem probabilities one may obtain joint probabilities involving some functions $\phi_{i_1i_2...i_N}$. Demanding extensivity of the entropy imposes some constraints on $\phi_{i_1i_2...i_N}$ and so joint probabilities are identified. We believe this is the most general representation for obtaining extensive entropies in the case of correlated subsystems.
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