POISSON–LIE GROUP STRUCTURES ON SEMIDIRECT PRODUCTS

FLORIS ELZINGA AND MAKOTO YAMASHITA

Abstract. We look at the Poisson structure on the total space of the dual bundle to the Lie algebroid arising from a matched pair of Lie groups. This dual bundle, with the natural semidirect product group structure, becomes a Poisson–Lie group as suggested by a recent work of Stachura. Moreover, when we start from matched pairs given by the Iwasawa decomposition of simple Lie groups, the associated Lie bialgebra is coboundary.

1. Introduction

Matched pairs of subgroups have been used to produce interesting examples of Hopf algebras \cite{Tak81} in the 80’s, and were further elaborated upon by Majid \cite{Maj90} and collaborators in the theory of quantum groups, where the primary motivating example is the Drinfeld double.

More recently in \cite{Sta17, Sta19}, Stachura gave a groupoid quantization of the $\kappa$-Poincaré group \cite{Zak94}. His model is based on a Lie groupoid arising from a matched pair of subgroups in $SO(N + 1, 1)$, and its associated Lie algebroid. The total space of the dual bundle of a Lie algebroid can be given the structure of a Poisson manifold \cite{Cou90}, and the main result of \cite{Sta17} amounts to identifying this Poisson structure with that of the $\kappa$-Poincaré group.

Motivated by this work, we look at the general case in the framework of matched pairs of Lie groups, and a particular case arising from the Iwasawa decomposition for real simple Lie groups.

In the general setting, suppose that $B, C \subset G$ is a matched pair of Lie groups, with their Lie algebras $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{g}$. The induced action of $C$ on $B$ defines a Lie groupoid $G_B$ with base $B$ (and vice versa). The dual bundle $E$ of the associated Lie algebroid is trivializable, and the fibres can be identified with the orthogonal complement $\mathfrak{b}^0 \subset \mathfrak{g}^*$ of $\mathfrak{b}$.

The starting observation is that $B$ naturally acts on $\mathfrak{b}^0$, whence $E$ has a semidirect group structure by combining this action with the linear group structure on $\mathfrak{b}^0$. Our main result (Theorem 3.2) is that the Poisson structure on $E$ is multiplicative with respect to this group structure. Hence $E$ becomes a Poisson–Lie group.

This can be further motivated by the fact that, when $BC \subset G$ is dense, the operator algebra associated with $G_B$ represents a locally compact quantum group, namely the bicrossed product of $B$ and $C$ \cite{BSV03}. In this scheme, the crossed product of $C$ with the function algebra of $B$ corresponds to the quantized algebra of functions on the Poisson–Lie group $E$. However, as the action of $B$ on $C$ is not by group automorphisms, we ‘break’ the group structure on $C$ and look at the semidirect product of $\mathfrak{b}^0 \simeq \mathfrak{c}^*$ by $B$.

As an example, when we start from the matched pair of $U(1)$ and the $(ax + b)$-group in $SU(1, 1)$, we get a double cover of the $E(2)$-group as $E$. The corresponding Poisson structure is essentially the one studied by Maślanka in \cite{Maś94}.

One interesting feature of this Poisson–Lie group is that the associated cobracket on its Lie algebra is coboundary. Motivated by this, we show that the cobracket is always coboundary for the matched pairs arising from the Iwasawa decomposition of real simple Lie groups (Theorem 4.1).

The paper is organized as follows: in Section 2 we collect some preliminary material and fix our conventions. In Section 3 we prove our first main result. We also include a small discussion
on the deformation quantization picture. In Section 4, we turn to matched pairs in real simple Lie groups and prove our second main result. Acknowledgement. We thank Piotr Stachura for illuminating comments on an early draft of this work.

2. Preliminaries

2.1. Conventions. Given a (real) vector space $V$, we denote its linear dual by $V^*$, and the complement of a subspace $W \subset V$ by

$$W^0 = \{ \phi \in V^* | \forall w \in W : \phi(w) = 0 \}.$$ We identify the exterior power $\wedge^*V$ with a subspace of the tensor power $T^*V$ in such a way that the duality pairing satisfies

$$\langle x \wedge y, \phi \otimes \psi \rangle = \phi(x)\psi(y) - \psi(x)\phi(y)$$

for $x, y \in V$ and $\phi, \psi \in V^*$.

For (real) Lie groups $G$, $A$, etc., we denote their Lie algebras by $\mathfrak{g}$, $\mathfrak{a}$, etc. The adjoint action of $G$ on $\mathfrak{g}$ is denoted by $\text{Ad}$, and the coadjoint action of $G$ on $\mathfrak{g}^*$ is by $\text{Ad}^*$. The corresponding Lie algebra actions of $\mathfrak{g}$ are denoted by $\text{ad}$ and $\text{ad}^*$, so we have $\text{ad}(x)(y) = [x, y]$ and

$$\langle \text{ad}^*(x)(\phi), y \rangle = -\langle \phi, \text{ad}(x)(y) \rangle = \phi([x, y]).$$

When $B, C$ are subgroups of $G$, $BC$ denotes the set of elements of the form $bc \in G$ for $b \in B$ and $c \in C$.

2.2. Matched pairs and Associated Structures. Let $G$ be a second countable locally compact Hausdorff topological group. By a matched pair of subgroups of $G$, we mean a pair of closed subgroups $B, C$ such that $B \cap C = \{ e \}$ and that $BC$ is open in $G$. Thus, any element $g$ in the open set $BC \cap CB$ has unique factorizations $g = bc = c'b'$ for $b, b' \in B$ and $c, c' \in C$.

Given such a matched pair, we have a groupoid $G_{G,B,C}$ (denoted by $\Gamma_B$ in [Sta17]) defined as follows:

- base space: $G^{(0)}_{G,B,C} = B$
- arrow space: $G^{(1)}_{G,B,C} = BC \cap CB$
- range and source maps: $r(g) = b$, $s(g) = b'$ for $g = bc = c'b'$ as above
- composition: $g \circ g' = bcc'$ (product in $G$) when $g = bc = c'b'$ and $g' = b'c''$

The composition is well-defined as we have

$$bcc'' = gb'^{-1}g' = c'^{b''}.$$ When $g' = b'^{-1}g' = c'^{b''}$. We will write $G_B = G_{G,B,C}$.

If in addition $G$ is a Lie group, $G_B$ becomes a Lie groupoid. The Lie algebras form a matched pair $[\text{Ma}]$ Section 4]: $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$ as vector space, with $\mathfrak{b}$ and $\mathfrak{c}$ sitting inside as subalgebras. There is a Haar system on the groupoid $G_B$ [LR01], hence its groupoid $C^*$-algebras make sense. If $G$ is a double Lie group, i.e., $G = BC$, these algebras are nothing but the crossed products for the corresponding action of $C$ on $C_0(B)$.

In general, when $BC$ is dense in $G$ the partial action of $C$ on $B$, whose graph is $G_B$, is densely defined so that $C$ acts on $L^\infty(B)$, and the associated crossed product von Neumann algebra $M = C \ltimes L^\infty(B)$ admits the structure of a semi-regular locally compact quantum group, which is regular when $G = BC$ [BSV03]. Its dual algebra is given by $\hat{M} = L^\infty(C) \rtimes B$. In particular, the associated reduced $C^*$-algebras are given by the reduced groupoid $C^*$-algebras $A = C^*_r(G_B)$ and $\hat{A} = C^*_r(G_{G,C,B})$. 

2
2.3. Lie Groupoids and Lie Algebroids. Let $\mathcal{G}$ be a Lie groupoid with base $M = \mathcal{G}^{(0)}$. Then we get a Lie algebroid $\mathcal{L}(\mathcal{G})$ on $M$ in the standard way, as follows. As a vector bundle, it is given by $\mathcal{L} = \ker(d\gamma : \iota^*T\mathcal{G} \to TM)$, where $\iota : M \to \mathcal{G}$ is the embedding as identity morphisms in $\mathcal{G}$. The bracket on $\Gamma(\mathcal{L})$ is given by identifying it with the space of left invariant vector fields on $\mathcal{G}$, and restricting the usual bracket on $\mathfrak{X}(\mathcal{G}) = \Gamma(T\mathcal{G})$. The anchor map $\alpha : \mathcal{L} \to TM$ is the restriction of $\text{ds}$.

If $G$ is a Lie group and its subgroups $B, C$ form a matched pair in $G$, then the Lie algebroid $\mathcal{L}(\mathcal{G}_B)$ can be modeled on $B \times \mathfrak{c}$. Namely, at $b \in B$, elements of the fiber $\mathcal{L}(\mathcal{G}_B)_b$ correspond to the tangent vectors at $b$ with integral curve $\exp(ty)$ for $y \in \mathfrak{c}$. Equivalently, we use the left translation map $L_b : g \mapsto bg$ to identify $\mathcal{L}(\mathcal{G}_B)_b$ with $\mathfrak{c}$. This gives the trivialization

$$\mathcal{L}(\mathcal{G}_B) \cong B \times \mathfrak{c}.$$ 

3. Poisson–Lie Groups from Matched Pairs

3.1. Poisson Structures from Lie Algebroids. Let $G$ be a Lie group, and $B, C$ be its subgroups forming a matched pair. Let us look in detail at the groupoid $\mathcal{G}_B$ over $B$ and the associated Lie algebroid $\mathcal{L}(\mathcal{G}_B)$.

Consider $E = (TB)^0$, the (total space of the) subbundle of $T^*G|_B = \iota^*T\mathcal{G}_B$ orthogonal to $TB \subset \iota^*T\mathcal{G}_B$. We identify $E$ with $b^0 \times B$ by right translations. Using this presentation, we interpret it as the semidirect product for the natural action of $B$ on $b^0$. That is, given $g = (v, a)$ and $h = (w, b)$ in $b^0 \times B$, we put

$$gh = (v + \text{Ad}_a^w, ab), \quad g^{-1} = (-\text{Ad}_a^{-1}w, a^{-1}).$$

This is consistent with viewing $E$ as a subgroup of $T^*G$, which has a semidirect product structure $g^* \times G$ coming from $\text{Ad}^*$. Let us write the Lie algebra of $E$ as $\mathfrak{e} = b^0 \oplus b$.

**Lemma 3.1.** The action of $g = (v, a)$ for $\text{Ad}_E^*$ is as follows,

$$\text{Ad}_E^g(\psi) = \text{Ad}_a^\psi \quad (\psi \in b^0),$$

$$\text{Ad}_E^g(y) = \text{Ad}_a y - \text{ad}^*(\text{Ad}_a y)(v) \quad (y \in \mathfrak{e}).$$

**Proof.** First let us consider $\psi$. The vector $\text{Ad}_E^g(\psi)$ is the differential at $t = 0$ of the integral curve $g(t\psi, e)g^{-1}$. By the commutativity of $b^0$, this is equal to

$$\left. \frac{d}{dt} \right|_{t=0} (\text{Ad}_a^t\psi, e) = \text{Ad}_a^\psi.$$ 

As for $y$, the vector $\text{Ad}_E^g(y)$ is the differential of the integral curve $g(\psi, \exp(ty))g^{-1}$. Computing the adjoint by $g$, we get

$$\left( v - \text{Ad}_a^\psi \exp(ty) a^{-1} v, a \exp(ty) a^{-1} \right).$$

Its differential is indeed $\text{Ad}_a y - \text{ad}^*(\text{Ad}_a y)(v)$. \qed

Note that $E$ is isomorphic to the dual vector bundle of $\mathcal{L}(\mathcal{G}_B)$ by the duality between $\mathfrak{e}$ and $b^0$. Let us describe the induced Poisson structure on $E$ [Con90]. The bracket is defined on fiberwise linear functions $\tilde{\mathcal{X}}$ on $E$ coming from the sections $X$ of $\mathcal{L}(\mathcal{G}_B)$, and pullbacks $\pi^*(f)$ of the smooth functions on $B$, as

$$\left\{ \tilde{X}_1, \tilde{X}_2 \right\} = [\tilde{X}_1, \tilde{X}_2], \quad \left\{ \tilde{X}, \pi^*(f_1) \right\} = \pi^*(\alpha(X)f), \quad \pi^*(f_1), \pi^*(f_2) = 0.$$ 

To obtain a more concrete formula for the second relation, let us denote the projections from $\mathfrak{g} \simeq b \oplus \mathfrak{c}$ to $b$ and $\mathfrak{c}$ by $P_b$ and $P_c$ respectively. Then, given $y \in \mathfrak{c}$ and $a \in B$, the corresponding section (up to the trivialization (1)) $X^L_y$ of $\mathcal{L}(\mathcal{G}_B)$ satisfies

$$\alpha(X^L_y)(a) = (\text{id}_{R_a})_c P_b \text{Ad}_a y$$

with respect to the right translation map $R_a : b \to ba$.

Our first goal is to prove the following.
Theorem 3.2. The Poisson bracket on $E$ characterized by (11), together with the semidirect product group structure, defines a Poisson–Lie group structure.

Let us first start with an observation: there is a nondegenerate linear duality pairing between $c$ and $b^0$ given as the restriction of the canonical paring between $g$ and $g^*$, while $b^0$ is invariant under the transformations $Ad_a^*$ for $a \in B$. We then obtain an action of $B$ on $c$ by duality. If $a \in B$ and $y \in c$, the vector $P_c Ad_a y \in c$ satisfies

$$\langle P_c Ad_a y, \phi \rangle = \langle y, Ad_a^* \phi \rangle$$

for $\phi \in b^0$. Thus, the operators $(P_c Ad_a)_{a \in B}$ give this action of $B$ on $c$.

Let us fix a basis $(y_i)_{i \in I}$ of $c$, and take its dual basis $(\psi^j)_{j \in I}$ in $b^0$ with respect to the duality pairing. Then the element $t = \sum_{i \in I} \psi^i \otimes y_i$ is invariantly defined, and we have

$$\sum_i Ad_a^* \psi^i \otimes P_c Ad_a y_i = \sum_i \psi^i \otimes y_i.$$  

(6)

The candidate group 1-cocycle $E \to \wedge^2 c$ for our bracket is the function $\eta = \eta_0 + \eta_b$, with the factors

$$\eta_0(g) = \frac{1}{2} \sum_{i,j} \langle v, Ad_a[y_i, y_j] \rangle Ad_a^* \psi^j \wedge Ad_a^* \psi^j, \quad \eta_b(g) = \sum_i Ad_a^* \psi^i \wedge P_b Ad_a y_i,$$

where we write $g = (v, a)$ as above.

Lemma 3.3. The Poisson bivector $\Pi \in \Gamma(E, \wedge^2 TE)$ for (11) is given by $\Pi_g = (dR_g)_c^\otimes \eta(g)$.

Proof. We have to check

$$\langle \Pi_g, df \otimes df' \rangle = \{f, f'\}(g)$$

for functions $f, f'$ of the form either $\tilde{X}$ or $\pi^*(f^\prime)$. The third relation in (11) is obviously satisfied as $\eta(g)$ does not have components in $\wedge^2 b$.

Note that we have

$$\bigg\langle (d\tilde{y}_i)_{\gamma}, (dR_{\gamma})_{\epsilon} \big(Ad_a^* \psi^j\big) \bigg\rangle = \frac{d}{dt} \bigg|_{t=0} \tilde{g}_i \bigg( v + t Ad_a^* \psi^j, a \bigg)$$

(7)

$$= \frac{d}{dt} \bigg|_{t=0} \bigg\langle v + t Ad_a^* \psi^j, Ad_a y_i \bigg\rangle$$

$$= \frac{d}{dt} \bigg|_{t=0} t \bigg\langle \psi^j, y_i \bigg\rangle = \delta_i j.$$  

Here, the adjoint action of $a$ on $y_i$ comes from the fact that we are comparing left translates from the Lie algebroid with right translates in $E$. This, combined with (11), implies the second relation in (11) for $\Pi$.

Finally, we have

$$\bigg\langle (d\tilde{y}_i)_{\gamma}, (dR_{\gamma})_{\epsilon} \big(P_b Ad_a y_j\big) \bigg\rangle = \frac{d}{dt} \bigg|_{t=0} \bigg\langle Ad_a^* \exp(t P_b Ad_a y_i) (v), Ad_a^* \exp(t P_b Ad_a y_i) a (y_j) \bigg\rangle$$

$$= \frac{d}{dt} \bigg|_{t=0} \{v, Ad_a y_j\} = 0.$$

Combining this with (7), we obtain the first relation in (11) for $\Pi$. $\Box$

Lemma 3.4. With $g \in E$ presented by $(v, a) \in b^0 \times B$, we have

$$\eta_0(g) = \frac{1}{2} \langle v, [y_i, y_j] \rangle \psi^j \wedge \psi^j - Ad_a^* \psi^j \wedge \text{ad}^*(P_b Ad_a y_i)(v).$$

(8)
Proof. The coefficient of $\text{Ad}_a^* \psi^i \wedge \text{Ad}_a^* \psi^j$ in $\eta_0(g)$ can be written as

$$\langle v, \text{Ad}_a[y_i, y_j] \rangle = \langle v, P \text{Ad}_a[y_i, y_j] \rangle = \langle v, P \eta[y_i, \text{Ad}_a y_j] \rangle$$

using the assumption $v \in \mathfrak{b}^0$. The second term on the right hand side can be expanded as

$$P \eta [(P_b + P_c) \text{Ad}_a y_i, (P_b + P_c) \text{Ad}_a y_j] \quad \Rightarrow \quad P \eta [(P_b + P_c) \text{Ad}_a y_i, (P_c + P_c) \text{Ad}_a y_j]$$

which we pair with $v$.

Then, using the invariance (6), we can write

$$\langle v, [P \text{Ad}_a y_i, P \text{Ad}_a y_j] \rangle \text{Ad}_a^* \psi^i \wedge \text{Ad}_a^* \psi^j = \langle v, [y_i, y_j] \rangle \psi^i \wedge \psi^j,$$

$$\langle v, [P_b \text{Ad}_a y_i, P \text{Ad}_a y_j] \rangle \text{Ad}_a^* \psi^i \wedge \text{Ad}_a^* \psi^j = \langle v, [P_b \text{Ad}_a y_i, y_j] \rangle \text{Ad}_a^* \psi^i \wedge \psi^j,$$

$$\langle v, [P_c \text{Ad}_a y_i, P_b \text{Ad}_a y_j] \rangle \text{Ad}_a^* \psi^i \wedge \text{Ad}_a^* \psi^j = \langle v, [y_i, P_b \text{Ad}_a y_j] \rangle \psi^i \wedge \text{Ad}_a^* \psi^j.$$

Notice that the last two terms are the same after swapping the order of the bracket and the wedge product and renaming dummy indices.

Finally, by definition

$$\langle v, [P \text{Ad}_a y_i, y_j] \rangle = -\langle \text{ad}^* (P_b \text{Ad}_a y_i)(v), y_j \rangle,$$

so that we can now complete one of the summations to get that

$$\langle v, [P_b \text{Ad}_a y_i, y_j] \rangle \text{Ad}_a^* \psi^i \wedge \psi^j = -\text{Ad}_a^* \psi^i \wedge \text{ad}^* (P_b \text{Ad}_a y_i)(v).$$

Combining these we obtain the claim. \qed

Now we are ready for the proof of our first main result.

Proof of Theorem 5.2. By Lemma 5.3 it is enough to show that $\eta$ satisfies the cocycle condition

$$\eta(gh) = \eta(g) + (\text{Ad}_g^E \otimes \text{Ad}_g^E) \eta(h),$$

or equivalently,

$$\left( \text{Ad}_g^E \otimes \text{Ad}_g^E \right) \eta(h) = \eta(gh) - \eta(g).$$

We proceed by computing the left-hand side for the two pieces $\eta_0$ and $\eta_b$ separately, and then compare. In the following we write $g = (v, a)$ and $h = (w, b)$.

First we expand $\left( \text{Ad}_g^E \otimes \text{Ad}_g^E \right) \eta_0(h)$ as

$$\frac{1}{2} \left( \langle w, \text{Ad}_b[y_i, y_j] \rangle \text{Ad}_{ab}^* \psi^i \wedge \text{Ad}_{ab}^* \psi^j = \frac{1}{2} \left( \langle \text{Ad}_a^* w, \text{Ad}_{ab}[y_i, y_j] \rangle \text{Ad}_{ab}^* \psi^i \wedge \text{Ad}_{ab}^* \psi^j \right) \right.$$

$$= \frac{1}{2} \left( \langle v + \text{Ad}_a^* w, \text{Ad}_{ab}[y_i, y_j] \rangle \text{Ad}_{ab}^* \psi^i \wedge \text{Ad}_{ab}^* \psi^j - \frac{1}{2} \langle v, \text{Ad}_{ab}[y_i, y_j] \rangle \text{Ad}_{ab}^* \psi^i \wedge \text{Ad}_{ab}^* \psi^j \right)$$

We recognize the first term as $\eta_0(gh)$, so we focus on the second term. If we apply our formula (8) to it, we find

$$\frac{1}{2} \langle v, \text{Ad}_{ab}[y_i, y_j] \rangle \text{Ad}_{ab}^* \psi^i \wedge \text{Ad}_{ab}^* \psi^j = \frac{1}{2} \langle v, [y_i, y_j] \rangle \psi^i \wedge \psi^j - \text{Ad}_{ab}^* \psi^i \wedge \text{ad}^* (P_b \text{Ad}_{ab} y_i)(v).$$

The relation (8) also implies that

$$\frac{1}{2} \langle v, [y_i, y_j] \rangle \psi^i \wedge \psi^j = \frac{1}{2} \langle v, \text{Ad}_a[y_i, y_j] \rangle \text{Ad}_a^* \psi^i \wedge \text{Ad}_a^* \psi^j + \text{Ad}_a^* \psi^i \wedge \text{ad}^* (P_b \text{Ad}_a y_i)(v).$$

Combining the two yields

$$\frac{1}{2} \langle v, \text{Ad}_{ab}[y_i, y_j] \rangle \text{Ad}_{ab}^* \psi^i \wedge \text{Ad}_{ab}^* \psi^j =$$

$$\frac{1}{2} \langle v, \text{Ad}_a[y_i, y_j] \rangle \text{Ad}_a^* \psi^i \wedge \text{Ad}_a^* \psi^j + \text{Ad}_a^* \psi^i \wedge \text{ad}^* (P_b \text{Ad}_a y_i)(v)$$

$$- \text{Ad}_{ab}^* \psi^i \wedge \text{ad}^* (P_b \text{Ad}_{ab} y_i)(v).$$
We proceed by exploiting the duality again, the resulting groupoid will freely use the terminology and notation of [Sta00]. Now we give a sketch of the argument. The first term on the right-hand side is precisely \( \eta_0(g) \), so we now have

\[
(9) \quad \left( \operatorname{Ad}_g^E \otimes \operatorname{Ad}_g^E \right) \eta_0(h) = \eta_0(gh) - \eta_0(g) - \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v) + \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v).
\]

Second, we see that \( \left( \operatorname{Ad}_g^E \otimes \operatorname{Ad}_g^E \right) \eta_0(h) \) is equal to

\[
\operatorname{Ad}^*_{ab} \psi^i \wedge [\operatorname{Ad}_a P_b \operatorname{Ad}_b y_i - \operatorname{ad}^*(\operatorname{Ad}_a P_b \operatorname{Ad}_b y_i)(v)]
\]

by (2) and (3). To obtain projections in the right places, write

\[
\operatorname{Ad}_a P_b \operatorname{Ad}_b y_i = P_b \operatorname{Ad}_a P_b \operatorname{Ad}_b y_i = P_b \operatorname{Ad}_a (1 - \psi) \operatorname{Ad}_b y_i.
\]

We proceed by exploiting the duality again,

\[
\operatorname{Ad}^*_{ab} \psi^i \wedge [P_b \operatorname{Ad}_a y_i - P_b \operatorname{Ad}_a P_c \operatorname{Ad}_b y_i] = \operatorname{Ad}^*_{ab} \psi^i \wedge P_b \operatorname{Ad}_a y_i - \operatorname{Ad}^*_{ab} \psi^i \wedge P_b \operatorname{Ad}_a y_i = \eta_0(gh) - \eta_0(g).
\]

This implies

\[
\left( \operatorname{Ad}_g^E \otimes \operatorname{Ad}_g^E \right) \eta_0(h) = \eta_0(gh) - \eta_0(g) - \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i - P_b \operatorname{Ad}_a P_c \operatorname{Ad}_b y_i)(v),
\]

and by invariance (6) we have

\[
(10) \quad \left( \operatorname{Ad}_g^E \otimes \operatorname{Ad}_g^E \right) \eta_0(h) = \eta_0(gh) - \eta_0(g) - \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v) + \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v).
\]

Combining (9) and (10), we find

\[
\left( \operatorname{Ad}_g^E \otimes \operatorname{Ad}_g^E \right) (\eta_0(h) + \eta_0(h)) = \eta_0(gh) - \eta_0(g) + \eta_0(gh) - \eta_0(g) + \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v) - \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v) - \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v) + \operatorname{Ad}^*_{ab} \psi^i \wedge \operatorname{ad}^*(P_b \operatorname{Ad}_a y_i)(v)
\]

\[
= \eta_0(gh) - \eta_0(g),
\]

as desired. \( \square \)

3.2. Alternative Proof of Theorem 3.2 for Double Lie Groups. It was pointed out to us by P. Stachura that Theorem 3.2 can also be derived from work of Zakrzewski [Zak90] when the matched pair decomposition is global. This situation is referred to as a double Lie group. We will freely use the terminology and notation of [Sta00]. Now we give a sketch of the argument.

Let \( G = BC \) be a double Lie group and consider the groupoids \( \mathcal{G}_B \) and \( \mathcal{G}_C = \mathcal{G}_{G,C,B} \). Write \( m_C \) for the multiplication relation of \( \mathcal{G}_C \). We begin by taking its transpose, which gives the relation

\[
m_C^T(g) = \{ (b_1, c, b_2) | b_1, b_2 \in B, c \in C, b_1 c b_2 = g \}.
\]

This can be viewed as a relation \( m_C^T : \mathcal{G}_B \to \mathcal{G}_B \times \mathcal{G}_B \) and turns out to be a (Zakrzewski) morphism of Lie groupoids.

We now apply the phase lift functor to \( \mathcal{G}_B \) and the morphism \( m_C^T \). The phase lift of a differentiable relation \( r : X \to Y \) is a new relation \( \operatorname{Pr} : T^*X \to T^*Y \) with graph

\[
(\alpha, \beta) \in \operatorname{Gr}(\operatorname{Pr}) \iff \forall (u, v) \in T(\pi_X(\alpha), \pi_Y(\beta)) \operatorname{Gr}(r) : \langle \alpha, u \rangle = \langle \beta, v \rangle
\]

The resulting groupoid \( PG_B \) has as arrow space the total space of the cotangent bundle \( T^*\mathcal{G}_B \). However, the base space is not \( T^*B \), as one might expect. The unit relation \( e : \{ 1 \} \to \mathcal{G}_B \) becomes a relation \( Pe : \{ 1 \} \times \{ 0 \} \to T^*\mathcal{G}_B \). Because the original base space was \( B \), it follows that the conditions for \( (y, X^*) \in T^*\mathcal{G}_B \) to lie in \( \operatorname{Gr}(Pe) \) are that \( y \in B \) and that \( X^* \in \mathfrak{b}^0 \), as the fibre of the tangent and cotangent bundle of \( \{ 1 \} \) is the zero vector space.
We now claim that the base map of the phase lift $Pm^T_C$ is precisely the group operation of $(TB)^0$. The graph of the base map is the transpose of the intersection
\[
Gr(Pm^T_C) \cap (TB)^0 \times (TB)^0 \subset (T^*G_B \times T^*G_B) \times T^*G_B.
\]
Let $((b_1, \psi_1), (b_2, \psi_2), (b, \psi)) \in (TB)^0 \times (TB)^0 \times (TB)^0$. Note that $(b_1, b_2, b)$ lies in the graph of $m^T_C$ if and only if $b = b_1 b_2$. We describe the tangent space to $Gr(m^T_C)$ at $(b_1, b_2, b_1 b_2)$. Any tangent vector in the direction of $B$ is not important, because $\psi_1, \psi_2$ are from $b^0$. Take $X \in \mathfrak{c}$ instead, then we get a curve
\[
\left( b_1 e^{tX}, e^{tX} b_2, b_1 e^{tX} b_2 \right) = \left( \left( b_1 e^{tX} b_1^{-1} \right) b_1, e^{tX} b_2, \left( b_1 e^{tX} b_1^{-1} \right) b_1 b_2 \right).
\]
Hence we get the tangent vectors $(\text{Ad}_{b_1}(X), X, \text{Ad}_{b_1}(X))$, or equivalently $(Y, \text{Ad}_{b_1^{-1}}(Y), Y)$. Plugging this into the equation defining the graph of $Pm^T_C$, we find the condition that
\[
\psi(Y) = \psi_1(Y) + \psi_2(\text{Ad}_{b_1^{-1}}(Y)) = (\psi_1 + \text{Ad}_{b_1}^* \psi_2)(Y).
\]
Therefore the base map of $Pm^T_C$ is
\[
((b_1, \psi_1), (b_2, \psi_2)) \mapsto (b_1 b_2, \psi_1 + \text{Ad}_{b_1}^* \psi_2),
\]
as claimed.

Then the phase lift of a Zakrzewski morphism produces a morphism of symplectic groupoids and that the base map of such a morphism is always a Poisson map, see [Zak90, Section 5].

3.3. Example: the $E(2)$ Group from $SU(1, 1)$. Let us take $G = SU(1, 1)$, and its Iwasawa decomposition $G = KAN$ with subgroups
\[
K = U(1) = \left\{ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \mid \varphi \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in \mathbb{R} \right\},
\]
\[
N = \left\{ \begin{pmatrix} 1 + is & is \\ is & 1 - is \end{pmatrix} \mid s \in \mathbb{R} \right\}.
\]
Take $B = K$ and $C = AN$ as matched pair in $G$. In this case we get $b^0 \simeq \mathbb{C}$, and the group $E$ is the semidirect product $\mathbb{C} \rtimes U(1)$ with the product
\[
(z, e^{i\varphi})(w, e^{i\psi}) = (z + e^{2i\varphi}w, e^{i(\varphi + \psi)}).
\]
Let us denote the semidirect product $\mathbb{R}^2 \rtimes SO(2)$ for the natural rotation action of $SO(2)$ on $\mathbb{R}^2$ by $E^+(2)$, which is the ‘positive’ part of the 2-dimensional Euclidean group. In his construction of $E_q(2)$, Woronowicz [Wor91, Section 3] started with the two-fold cover of $E^+(2)$ given by the matrix group
\[
E(2) = \left\{ \begin{pmatrix} v & n \\ 0 & v^{-1} \end{pmatrix} \mid v \in \mathbb{T}, \ n \in \mathbb{C} \right\},
\]
which acts on $\zeta \in \mathbb{C}$ as $\zeta \mapsto v^2 \zeta + vn$. We have an isomorphism $E \simeq E(2)$ through the identifications $v = e^{i\varphi}$ and $n = e^{-i\varphi} z$. This isomorphism also matches up the actions of the groups on $\mathbb{C}$.

As we will see below, the Poisson–Lie group structure on $E(2)$ obtained by our scheme agrees (up to double covering) with the one considered in [Maś94].

Write the generators of $K$, $A$, $N$ above as
\[
\begin{align*}
    i\hbar &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\
    y^{(a)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
    y^{(a)} &= \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}.
\end{align*}
\]
We realize \( g^* \) as a subspace of \( \mathfrak{sl}_2(\mathbb{C}) \) by
\[
g^* = \left\{ \begin{pmatrix} a & z \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{R}, z \in \mathbb{C} \right\}
\]
compatible with the natural duality pairing
\[
(x, y) = \text{Im} \text{Tr}(xy).
\]

Then \( \mathfrak{b}^0 \) is spanned by
\[
y^{(a)*} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad y^{(2)*} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

To help comparison with [Maś94], let us write our basis of \( \mathfrak{c} \) as
\[
P_1 = y^{(a)*}, \quad P_2 = y^{(2)*}, \quad J = \text{ih}.
\]

Their relations are given by
\[
[P_1, P_2] = 0, \quad [J, P_1] = 2P_2, \quad [J, P_2] = -2P_1.
\]

Let us compute the bracket on \( \mathfrak{c}^* \) induced by the Poisson bracket coming from the Lie groupoid. As before we denote by \( X^L_y \) the sections of \( L(\mathcal{G}_B) \) corresponding to \( y \in \mathfrak{c} \). We will also denote the sections coming from \( y^{(a)} \) and \( y^{(2)} \) by \( X^L_{(a)} \) and \( X^L_{(2)} \). Furthermore, we denote the corresponding fibre-wise linear functions on \( E \) by \( \bar{X}^L_y \), etc. Concretely, we have
\[
\bar{X}^L_{(a)}(P_1, e^{i\varphi}) = (P_1, \text{Ad}_{e^{i\varphi}}y^{(a)}) = \cos(2\varphi), \quad \bar{X}^L_{(2)}(P_2, e^{i\varphi}) = \sin(2\varphi),
\]
\[
\bar{X}^L_{(a)}(P_1, e^{i\varphi}) = -\sin(2\varphi), \quad \bar{X}^L_{(2)}(P_2, e^{i\varphi}) = \cos(2\varphi).
\]

So these two functions can and will be used as coordinate functions \( p_1 \) and \( p_2 \) on the ‘linear part’ of \( E \). On the \( U(1) \) part of \( E \), we have the function
\[
U(1) \to \mathbb{C}, \quad \begin{pmatrix} e^{i\varphi} \\ 0 \\ 0 \end{pmatrix} \mapsto e^{i\varphi},
\]
which we denote by \( e^{i\varphi} \) again.

The anchor map \( a: L(\mathcal{G}_K) \to TK = \mathfrak{k} \times K \) becomes
\[
a(X^L_y)(e^{i\varphi}) = (P_1(\text{Ad}_{e^{i\varphi}}y), e^{i\varphi}),
\]
where \( P_k \) is the projection \( \mathfrak{g} \to \mathfrak{k} \) corresponding to the decomposition \( \mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{a} \oplus \mathfrak{n}) \). For our basis above, we get
\[
a(X^L_{(a)})(e^{i\varphi}) = (\sin(2\varphi)J, e^{i\varphi}), \quad a(X^L_{(2)})(e^{i\varphi}) = ((1 - \cos(2\varphi))J, e^{i\varphi}).
\]

We can now compute the Poisson structure. The bracket on the linear part is given by
\[
\{ \bar{X}^L_{(a)}, \bar{X}^L_{(2)} \} = \bar{X}^L_{[y^{(a)}, y^{(2)}]} = 2\bar{X}^L_{(2)}.
\]

Between the linear part and the base part, we have
\[
\{ \bar{X}^L_{(a)}, e^{i\varphi} \} = a(X^L_{(a)})(e^{i\varphi}) = \sin(2\varphi)J e^{i\varphi} = i \sin(2\varphi) e^{i\varphi}, \quad \{ \bar{X}^L_{(2)}, e^{i\varphi} \} = i(1 - \cos(2\varphi)) e^{i\varphi}.
\]

In other words, if we take a branch of \( \varphi \) as a real function on an interval in \( U(1) \), we get
\[
\{ \bar{X}^L_{(a)}, \varphi \} = \sin(2\varphi), \quad \{ \bar{X}^L_{(2)}, \varphi \} = 1 - \cos(2\varphi).
\]

To compare with [Maś94], we pass to the quotient \( E/\mathbb{Z}_2 \simeq E^+(2) \). This identification is given by \((z, e^{i\varphi}) \mapsto (z, e^{i2\varphi})\), so that the linear functions are unchanged, but the function \( e^{i\varphi} \) on \( E^+(2) \) corresponds to \( e^{i2\varphi} \) on \( E \). As
\[
\{ \bar{X}^L_{(a)}, e^{i2\varphi} \} = 2e^{i\varphi} \{ \bar{X}^L_{(a)}, e^{i\varphi} \} = 2i \sin(2\varphi) e^{i2\varphi}, \quad \{ \bar{X}^L_{(2)}, e^{i2\varphi} \} = 2i (1 - \cos(2\varphi)) e^{i2\varphi},
\]
we obtain the following Poisson bracket on $E^+(2)$:

$$\{X^L(a), X^L(b)\} = 2X^L(a), \quad \{X^L(a), \epsilon^{i\theta}\} = 2i \sin(\theta) \epsilon^{i\theta}, \quad \{X^L(2), \epsilon^{i\theta}\} = 2i (1 - \cos(\theta)) \epsilon^{i\theta}. $$

Then, putting

$$V^1 = -\bar{X^L(a)}, \quad V^2 = X^L(2),$$

we obtain the bracket [Mas94 Equation (10)] for $\omega = -2$.

Note that this structure is different from the one behind Woronowicz’s example. In the notation of Sob90, what we obtained is a scaled version of the cobracket $\delta_3$ on $\epsilon(2)$, while Woronowicz’s $E_q(2)$ corresponds to the family $\delta_1$. Nevertheless, there is still some similarity between the two. If we pass to the dual $\epsilon(2)^*$, with basis $\{J^*, P^*_1, P^*_2\}$, then the Lie brackets dualizing $\delta_1$ and $\delta_3$ yield isomorphic Lie algebras. Indeed, $\delta_1$ gives

$$(11) \quad [P^*_1, P^*_2]_1 = 0, \quad [P^*_1, J^*]_1 = sP^*_1, \quad [P^*_2, J^*]_1 = sP^*_2,$$

where $s$ is an auxiliary parameter, while $\delta_3$ gives

$$[P^*_1, P^*_2]_3 = P^*_2, \quad [P^*_1, J^*]_3 = J^*, \quad [P^*_2, J^*]_3 = 0.$$

So the linear isomorphism $\rho: \epsilon(2)^* \rightarrow \epsilon(2)^*$ given by

$$\rho(J^*) = -sP^*_1, \quad \rho(P^*_1) = J^*, \quad \rho(P^*_2) = P^*_2$$

satisfies $\rho \circ [\cdot, \cdot] = [\rho(\cdot), \rho(\cdot)]_3$.

Let us also note that this Lie algebra structure on $\epsilon(2)^*$ is isomorphic to the standard Lie algebra structure on $\mathfrak{su}(1,1)^* = \mathfrak{su}(2)^*$. The latter is spanned by $h, e, i\epsilon$ as a real Lie subalgebra of $\mathfrak{sl}_2(C)$, which is indeed isomorphic to $\epsilon(2)^*$ with bracket (11). In particular, dualizing the Lie bialgebras $(\epsilon(2), \delta_1)$, $(\epsilon(2), \delta_3)$, and $\mathfrak{su}(1,1)$, we get different Lie bialgebra structures on $\epsilon(2)^*$. The ones from $\delta_1$ and $\mathfrak{su}(1,1)$ are cohomologous as we will see in Section 4 while the one from $\delta_3$ has a different class in cohomology.

### 3.4. Deformation Quantization

Let us explain an analogue of strict deformation quantization, in the framework of unbounded multipliers. The bounded picture, that is, a $\mathbb{C}^*$-algebraic strict deformation quantization in the sense of Rieffel [Rie94], is already provided in [Lan99].

Let $U(\mathfrak{c})$ be the complexified universal enveloping algebra of $\mathfrak{c}$, i.e., the universal associative unital $\mathbb{C}$-algebra generated by a copy of $\mathfrak{c}$ as its real subspace, with relations $xy - yx = [x, y]_\mathfrak{c}$ for $x, y \in \mathfrak{c}$. The Hopf algebra $U(\mathfrak{c})$ acts on $\mathbb{C}_b^\infty(B)$, and the elements of the crossed product $U(\mathfrak{c}) \ltimes \mathbb{C}_b^\infty(B)$ can be regarded as unbounded multipliers of $\mathbb{C}^*(G_B)$, see Appendix A.

Since $\text{Sym}(\mathfrak{c})$ can be identified with the space of complex polynomial functions on $\mathfrak{c}^0$, the elements of $\text{Sym}(\mathfrak{c}) \otimes \mathbb{C}_b^\infty(B)$ can be regarded as unbounded multipliers of $\mathbb{C}_0(E)$. Let us consider the corresponding bracket on $\text{Sym}(\mathfrak{c}) \otimes \mathbb{C}_b^\infty(B)$.

By choosing an ordered basis of $\mathfrak{c}$, by the same argument as the Poincaré–Birkhoff–Witt Theorem, we get a linear isomorphism

$$Q: \text{Sym}(\mathfrak{c}) \otimes \mathbb{C}_b^\infty(B) \rightarrow U(\mathfrak{c}) \ltimes \mathbb{C}_b^\infty(B)$$

which is compatible with the filtrations by degree in $\mathfrak{c}$. For an auxiliary parameter $h$, define $Q_h: \text{Sym}(\mathfrak{c}) \otimes \mathbb{C}_b^\infty(B) \rightarrow U(\mathfrak{c}) \ltimes \mathbb{C}_b^\infty(B)$ by

$$Q_h(y_1 \ldots y_k \otimes f) = h^k Q(y_1 \ldots y_k \otimes f).$$

**Proposition 3.5.** We have

$$\frac{[Q_h(T \otimes f), Q_h(T' \otimes f')]_h}{h} = Q_h\{T \otimes f, T' \otimes f'\} + O(h)$$

for $T, T' \in \text{Sym}(\mathfrak{c})$ and $f, f' \in \mathbb{C}_b^\infty(B)$. 


Proof. For linear polynomials, we have
\[ \frac{[Q_h(y \otimes 1), Q_h(y' \otimes 1)]}{h} = Q_h(\{y \otimes 1, y' \otimes 1\}) \quad (y, y' \in \mathfrak{c}) \]
with exact equality, from the structure of $U(\mathfrak{c})$. ($y \otimes 1$ corresponds to the function $X^f_b$.) For general elements $T, T' \in \text{Sym}(\mathfrak{c})$, induction on degree gives
\[ \frac{[Q_h(T \otimes 1), Q_h(T' \otimes 1)]}{h} = Q_h(\{T \otimes 1, T' \otimes 1\}) + O(h^2). \]
Between $\mathfrak{c}$ and $C^\infty_b(B)$, we again have
\[ \frac{[Q_h(y \otimes 1), Q_h(1 \otimes f)]}{h} = Q_h(\{y \otimes 1, 1 \otimes f\}) \quad (y \in \mathfrak{c}, f \in C^\infty_b(B)) \]
with exact equality, from the way the anchor map is defined. ($1 \otimes f$ corresponds to the function $\pi^* f$.) For general $T \in \text{Sym}(\mathfrak{c})$, again by induction on degree we get
\[ \frac{[Q_h(T \otimes 1), Q_h(1 \otimes f)]}{h} = Q_h(\{T \otimes 1, 1 \otimes f\}) + O(h). \]
Finally, we also have $[Q_h(1 \otimes f), Q_h(1 \otimes f')] = 0$. \qed

There is a structure of a Hopf algebra (up to completion) on $U(\mathfrak{c}) \ltimes C^\infty_b(B)$ corresponding to the bicrossed product structure, as follows. On $C^\infty_b(B)$ we consider the one coming from the group structure of $B$, implemented as
\[ \Delta : C^\infty_b(B) \to C^\infty_b(B \times B) \subset M(C_0(B) \otimes C_0(B)). \]
(We can also take $C^\infty_b(B)$ as the domain.) We mix this with the usual cocommutative coproduct $\Delta : U(\mathfrak{c}) \to U(\mathfrak{c}) \otimes U(\mathfrak{c})$, using the action of $B$ on $\mathfrak{c}$, as follows.

Note that $U(\mathfrak{c}) \otimes C^\infty(B \times B)$ is identified with the space of smooth functions $B \times B \to U(\mathfrak{c})$ with finite dimensional images. For $T \otimes f$ with $T \in U(\mathfrak{c})$ and $f \in C^\infty(B \times B)$, denote by $\theta(T \otimes f)$ the function $(g, h) \mapsto T^g f(g, h)$, where $T^g$ is the right action of $B$ on $U(\mathfrak{c})$ induced by the right action on $\mathfrak{c}$. Since $U(\mathfrak{c})$ is the increasing union of finite dimensional representations of $B$, $\theta$ is well-defined as a transform on $U(\mathfrak{c}) \otimes C^\infty(B \times B)$. We then put
\[ \Delta(T \otimes f) = (T_{(1)} \otimes \theta(T_{(2)} \otimes \Delta(f)))_{1,3,24} \quad (T \in U(\mathfrak{c}), f \in C^\infty_b(B)), \]
where $T_{(1)} \otimes T_{(2)}$ is the Sweedler notation for $\Delta(T)$.

If $B$ is compact, we can have a model of a genuine Hopf algebra by taking the algebra $\mathcal{O}(B)$ of matrix coefficients of finite dimensional complex linear representations instead of $C^\infty(B)$. Indeed, since $U(\mathfrak{c})$ is a union of finite dimensional $B$-modules, the above $\Delta$ restricts to a coproduct map from $\mathcal{H} = U(\mathfrak{c}) \otimes \mathcal{O}(B)$ to the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$.

4. Coboundary Lie Bialgebras from Real Simple Lie Groups

Throughout this section, let $G$ denote a connected real simple Lie group with finite center, and $K$ its maximal compact subgroup. We further assume that $K$ has non-discrete center $Z(\mathfrak{k})$.

For example, we could take $G = SU(p, q)$ and $K = S(U(p) \times U(q))$.

4.1. Finding the r-Matrix. Denote the real Lie algebras of $G$ and $K$ by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Let us take the Cartan decomposition for $K \subset G$, i.e., an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the invariant symmetric bilinear form. We note
\[ \mathfrak{[k, k]} \subset \mathfrak{k}, \quad \mathfrak{[k, p]} \subset \mathfrak{p}, \quad \mathfrak{[p, p]} \subset \mathfrak{k}. \]

Under our assumptions, $Z(\mathfrak{k})$ must be 1-dimensional, and we can pick a spanning element $z$ of $Z(\mathfrak{k})$ satisfying
\[ \text{ad}(z)^2 \bigg|_\mathfrak{p} = -1, \]
see for example [Hel01, Chapter VIII].
The Iwasawa decomposition $G = KAN$ gives a matched pair of subgroups $K$ and $S = AN$ in $G$. By the recipe of our main result, we get a Poisson–Lie group structure on $E = \mathfrak{k}^0 \times K$.

**Theorem 4.1.** Under the above setting, the Lie bialgebra structure on the Lie algebra $\mathfrak{e} = \mathfrak{k}^0 \times \mathfrak{k}$ is coboundary. More precisely, $r = z.\delta(z)$ satisfies $\delta(x) = [r, \Delta(x)]$ for all $x \in \mathfrak{e}$.

**Proof.** The Iwasawa decomposition induces the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ as a vector space. We denote the two projections associated to this decomposition by $P^I_\mathfrak{k}$ and $P^I_\mathfrak{s}$ respectively.

Pick a basis $\{y_i\}$ of $\mathfrak{s}$ and let $\{\psi^j\}$ be the dual basis inside $\mathfrak{k}^0$. Their brackets in $\mathfrak{e}$ are given by

$$\left[\psi^i, \psi^j\right] = 0, \quad \left[y_i, \psi^j\right] = \text{ad}^* (y_i)(\psi^j),$$

while the bracket between $y_i$ and $y_j$ is just the one from $\mathfrak{s}$. Recall that the cobracket $\delta$ is defined according to

$$\delta(x) = P^I_\mathfrak{k}[y_i, x] \wedge \psi^i, \quad \langle \delta(\psi), y \otimes y' \rangle = \langle \psi, [y, y'] \rangle$$

for all $x \in \mathfrak{k}, y, y' \in \mathfrak{s}$ and $\psi \in \mathfrak{s}$.

By the centrality of $z$ and the cocycle condition for $\delta$, we have

$$0 = \delta([x, z]) = x.\delta(z) - z.\delta(x) \quad (x \in \mathfrak{k}),$$

which implies

$$\delta(x) = -z.z.\delta(x) = -x.z.\delta(z) = [z.\delta(z), \Delta x].$$

Thus our candidate for the $r$-matrix,

$$r = z.\delta(z) = P^I_\mathfrak{k}[y_i, z] \wedge \text{ad}^*(z)(\psi^j),$$

works for commutators with elements $x \in \mathfrak{k}$.

It remains to show that

$$\delta(\psi) = [r, \Delta \psi]$$

for all $\psi \in \mathfrak{k}^0$.

Let us look at the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and write the corresponding projections as $P^C_\mathfrak{k}$ and $P^C_\mathfrak{p}$. The two sets of projections are related in the following way.

**Lemma 4.2.** For $y \in \mathfrak{e}$, we have

$$P^C_\mathfrak{k} y = -P^I_\mathfrak{p} P^C_\mathfrak{p} y.$$

**Proof.** Let $\sigma$ be the Cartan involution for $\mathfrak{k} \subset \mathfrak{g}$. Then we have

$$P^C_\mathfrak{k} y = \frac{1}{2}(y + \sigma y), \quad P^C_\mathfrak{p} y = \frac{1}{2}(y - \sigma y).$$

Thus,

$$-P^C_\mathfrak{p} y = \frac{1}{2}(\sigma y + y) - y,$$

and we obtain the claim.  

**Proof of Theorem 4.1 continued.** We claim that our candidate for the $r$-matrix can be also written as

$$(14) \quad r = \sum_i P^C_\mathfrak{k} y_i \wedge \psi^i.$$  

By the invariance of the canonical tensor $\sum_i y_i \otimes \psi^i$ and the centrality of $z$, we have

$$r = -\sum_i P^I_\mathfrak{k}[P^I_\mathfrak{s} \text{ad}(z)(y_i), z] \wedge \psi_i = \sum_i P^I_\mathfrak{k} \text{ad}(z)^2(y_i) \wedge \psi_i.$$

Using (13), we have $\text{ad}(z)^2(y) = -P^C_\mathfrak{p} y$ for $y \in \mathfrak{s}$. Then Lemma 4.2 implies the claim.

Finally, notice that (12) implies

$$\psi([y, y']) = \psi\left(\left[P^C_\mathfrak{k} y, y'\right] - \left[P^C_\mathfrak{s} y', y\right]\right)$$

for all $y, y' \in \mathfrak{s}$. This completes the proof of Theorem 4.1.
for \( \psi \in \mathfrak{k}^0 \). Therefore,
\[
\langle \psi, [y, y'] \rangle = \langle \psi, [P^C_{k} y, y'] - [P^C_{k} y', y] \rangle = \sum_i \langle \psi, [\psi^i(y)P^C_{k} y, y'] - [\psi^i(y')P^C_{k} y, y] \rangle = -\sum_i \langle \psi^i \circ \text{ad}^* (P^C_{k} y_i) (\psi), y \wedge y' \rangle,
\]
but by (12) this is nothing but \( \langle [r, \Delta \psi], y \wedge y' \rangle \).

**Remark 4.3.** We note that the above element \( r \) is the only one satisfying \( \delta(x) = [r, x] \) for all \( x \in \mathfrak{c} \). Indeed, take any other potential \( r \)-matrix \( r' \), then the difference \( r - r' \) would be an invariant element of \( \mathfrak{k} \otimes \mathfrak{k}^0 \oplus \mathfrak{k}^0 \otimes \mathfrak{k} \). Testing against \( \Delta(z) \) shows that no such elements exist except for 0.

**Example 4.4.** Starting from \( \mathfrak{g} = \mathfrak{su}(1,1) \) and \( \mathfrak{k} = \mathfrak{so}(2) \), we get \( r = J \wedge P_2 \) for the cobracket on \( \mathfrak{e}(2) \), cf. [Maś94].

By our assumption on \( K \), there is commutative subgroup \( T \subset K \) such that \( t_\mathfrak{c} \) is a Cartan subalgebra of \( \mathfrak{g}_\mathfrak{c} \). Let us choose an order of roots for this Cartan subalgebra, and take the associated Poisson–Lie structure on \( G \), as in [Kor94]. In particular, the Manin triple is \((\mathfrak{g}_\mathfrak{c}, \mathfrak{g}, \mathfrak{g}^*)\), where \( \mathfrak{g}^* \) is spanned by \( \mathfrak{it} \) and the positive root spaces, and the real invariant bilinear form on \( \mathfrak{g}_\mathfrak{c} \) is given by the imaginary part of the Killing form.

The following shows that the structure of \( E \), in particular the commutativity of \( \mathfrak{k}^0 \), naturally appears from that of \( \mathfrak{g}^* \).

**Proposition 4.5.** The subspace \( \mathfrak{k}^0 \subset \mathfrak{g}^* \) is commutative.

**Proof.** Pick a compact form \( \mathfrak{g}_\mathfrak{c} \) of \( \mathfrak{g}_\mathfrak{C} \) that contains \( \mathfrak{k} \). Then it follows from the construction of \( \mathfrak{g}^* \) in the triple above that \( \mathfrak{g}^*_\mathfrak{c} = \mathfrak{g}^* \). Moreover, \( \mathfrak{k} = \mathfrak{g}^*_\mathfrak{c} \) for some invariant automorphism \( \nu \) of \( \mathfrak{g}_\mathfrak{c} \). The automorphism \( \nu \) must be of the form \( \text{Ad}_{\exp(x)} \) for some \( x \in Z(\mathfrak{k}) \), and hence \( \nu^I \) is also a Lie algebra automorphism of \( \mathfrak{g}^*_\mathfrak{c} \). It is clear that \( \mathfrak{k}^0 \) must be the \((-1\text{-})\text{-eigenspace of } \nu^I \text{ acting on } \mathfrak{g}^*_\mathfrak{c} \), and it is also an ideal of \( \mathfrak{g}^*_\mathfrak{c} \) as \( K \) is a Poisson–Lie subgroup of \( G \). Finally then,
\[
[\mathfrak{k}^0, \mathfrak{k}^0] \subset \mathfrak{k}^0 \cap (\mathfrak{g}^*_\mathfrak{c})^{\nu^I} = \{0\}.
\]
This shows the claim. □

**Corollary 4.6.** The group \( E = \mathfrak{k}^0 \ltimes K \) is a subgroup of \( G^* \).

4.2. **Example:** \( G = \text{SU}(p,1) \). The Lie algebra of \( \text{SU}(p,1) \) for \( p \geq 2 \) is
\[
\mathfrak{su}(p,1) = \left\{ \left( \begin{array}{cc} M & b \\ b^* & -\text{Tr}(M) \end{array} \right) \mid M \in M_p(\mathbb{C}), \ M^* = -M, \ b \in \mathbb{C}^p \right\}.
\]
We pick the maximal compact subalgebra
\[
\mathfrak{k} = \left\{ \left( \begin{array}{cc} M & 0 \\ 0 & -\text{Tr}(M) \end{array} \right) \mid M \in M_p(\mathbb{C}), \ M^* = -M \right\},
\]
which corresponds to the maximal compact subgroup \( \text{S(U}(p) \times \text{U}(1)) \) of \( \text{SU}(p,1) \). We remark that the center of \( \mathfrak{k} \) is one-dimensional and spanned by
\[
z = \frac{1}{p+1} \begin{pmatrix} iI_p & 0 \\ 0 & -pi \end{pmatrix},
\]
where we have normalized such that \( \text{ad}^* (z)^2 |_{\mathfrak{k}^0} = -z \). The associated Cartan decomposition is give by
\[
\mathfrak{su}(p,1) = \mathfrak{k} \oplus \mathfrak{p}, \ \mathfrak{p} = \left\{ \left( \begin{array}{cc} 0 & b \\ b^* & 0 \end{array} \right) \mid b \in \mathbb{C}^p \right\}.
\]
The other decomposition we will use is the Iwasawa decomposition. This begins with a choice of maximal commutative subalgebra $a$ of $p$. We will work with

$$a = \left\{ t \begin{pmatrix} 0 & e_p \\ e_p^* & 0 \end{pmatrix} \bigg| t \in \mathbb{R} \right\},$$

where $e_p$ is the column vector of size $p$ with entries $0, \ldots, 0, 1$. In addition we have the positive restricted root spaces

$$\mathfrak{g}_{f_1} = \left\{ \begin{pmatrix} 0 & v & -v \\ -v^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix} \bigg| v \in \mathbb{C}^{p-1} \right\}, \quad \mathfrak{g}_{2f_1} = \left\{ t \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & i & -i \end{pmatrix} \bigg| t \in \mathbb{R} \right\}.$$

These combine into the nilpotent part $n = \mathfrak{g}_{f_1} \oplus \mathfrak{g}_{2f_1}$, which is normalized by $a$. Finally, this gives the solvable part of the Iwasawa decomposition,

$$s = a \oplus n.$$

The complexification of $\mathfrak{su}(p,1)$ is $\mathfrak{sl}(p+1, \mathbb{C})$, and its dual $\mathfrak{su}(p,1)^*$ (with respect to the imaginary part of the Killing form) is given by the Lie algebra of upper triangular $(p+1) \times (p+1)$-matrices with real entries on the diagonal. Then $\mathfrak{g}^0 \subset \mathfrak{su}(p,1)^*$ is given by

$$\mathfrak{g}^0 = \left\{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \bigg| w \in \mathbb{C}^p \right\},$$

which is indeed a commutative subspace.

Let us take a basis of $s$ as follows:

$$y^{(a)} = \begin{pmatrix} 0 \\ e_p^* \\ 0 \end{pmatrix} \in a, \quad y^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & -i \\ 0 & i & -i \end{pmatrix} \in \mathfrak{g}_{2f_1},$$

$$y^{(R)}_k = \begin{pmatrix} 0 & -e_k & e_k \\ e_k^* & 0 & 0 \\ e_k^* & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{f_1}, \quad y^{(I)}_k = \begin{pmatrix} 0 & i e_k & -i e_k \\ i e_k^* & 0 & 0 \\ i e_k^* & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{f_1} \quad (k = 1, \ldots, p-1).$$

Their brackets are given by

$$[y^{(a)}, y^{(2)}] = 2y^{(2)}, \quad [y^{(a)}, y^{(R)}_k] = y^{(R)}_k, \quad [y^{(a)}, y^{(I)}_k] = y^{(I)}_k, \quad [y^{(R)}_k, y^{(R)}_\ell] = 0, \quad [y^{(I)}_k, y^{(I)}_\ell] = 0, \quad [y^{(R)}_k, y^{(I)}_\ell] = 2 \delta_{k \ell} y^{(2)},$$

while all other brackets involving $y^{(2)}$ vanish.

The dual basis in $\mathfrak{g}^0$ is given by

$$y^{(a)*} = \begin{pmatrix} 0 & i e_p \\ 0 & 0 \end{pmatrix}, \quad y^{(2)*} = \begin{pmatrix} 0 & e_p \\ 0 & 0 \end{pmatrix},$$

$$y^{(R)*}_k = \begin{pmatrix} 0 & i e_k \\ 0 & 0 \end{pmatrix}, \quad y^{(I)*}_k = \begin{pmatrix} 0 & e_k \\ 0 & 0 \end{pmatrix} \quad (k = 1, \ldots, p-1).$$

From above computation, we see that the induced cobracket $\delta$ on $\mathfrak{g}^0$ is given by

$$\delta(y^{(a)*}) = 0, \quad \delta(y^{(R)*}_k) = y^{(a)*} \wedge y^{(R)*}_k, \quad \delta(y^{(2)*}) = y^{(a)*} \wedge y^{(2)*} + 2 \sum_{k=1}^{p-1} y^{(R)*}_k \wedge y^{(I)*}_k,$$

where $\epsilon$ denotes $R$ or $I$ in the second relation. From [14], we see that the $r$-matrix is given by

$$r = P_t^C y^{(2)} \wedge y^{(2)*} + \sum_{k=1}^{p-1} \left[ P_t^C y^{(R)*}_k \wedge y^{(R)*}_k + P_t^C y^{(I)*}_k \wedge y^{(I)*}_k \right]$$

$$= (i e_{pp} - i e_{p+1,p+1}) \wedge y^{(2)*} + \sum_{k=1}^{p-1} \left[ (e_{pk} - e_{kp}) \wedge y^{(R)*}_k + (i e_{kp} + i e_{pk}) \wedge y^{(I)*}_k \right].$$
Let us take a look at the structure one the Lie group level. (The formulas apply to the case \( p = 1 \) as well.) The maximal compact subgroup \( K \) is
\[
S(U(p) \times U(1)) = \left\{ \begin{pmatrix} U & 0 \\ 0 & \det(U) \end{pmatrix} \right\} \cong U(p).
\]
Write \( E(p+1) \) for the Poisson–Lie group \( \mathfrak{t}^0 \rtimes K \) constructed out of \( SU(p,1) \) as in the previous section. Then \( E^C(p+1) \cong C^p \rtimes U(p) \) and a brief computation shows that
\[
\text{Ad}_U^*(z_1, \ldots, z_p) = \det(U) \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}.
\]
Thus, the groups \( E^C(p+1) \) can be regarded as subgroups of the even-dimensional Euclidean groups compatible with the complex structure.

4.3. Another Deformation Scheme. Beyond the case of \( g = su(1,1) \), the Lie algebra \( e^* \) could be non-isomorphic from \( g^* \), which can be directly seen for the above examples \( su(p,1) \).

 Nonetheless, let us mention another similar construction that does relate \( g \) and \( e \). Take the Cartan decomposition \( g = \mathfrak{t} \oplus \mathfrak{p} \). Then we have \( c \simeq \mathfrak{p} \rtimes \mathfrak{t} \), where we treat \( \mathfrak{p} \) as a vector space with an action of \( \mathfrak{t} \). To recover \( g \) from \( c \), we just need to use the restriction of the bracket map \( \Lambda^2 \mathfrak{p} \to \mathfrak{t} \) as a cocycle (say \( c \)), and deform the bracket of \( c \). Moreover, as the standard maximal compact form \( g_c \subset g_C \) is \( \mathfrak{t} \oplus i\mathfrak{p} \), the deformation of \( c \) by the inverted cocycle \( -c \) gives \( g_c \).

 On the other hand, \( \sigma(\mathfrak{t}^0) \) is a subalgebra of \( g_C \) which is stable under \( \text{ad}(\mathfrak{t}) \) and is isomorphic to \( \mathfrak{p} \) as a \( \mathfrak{t} \)-module. Thus, the subalgebra \( g' = \sigma(\mathfrak{t}^0) \rtimes \mathfrak{t} \subset g_C \) is isomorphic to \( g' \simeq c \). Moreover, \( (g_C, g', g^*) \) is a Manin triple.

 Thus, we have three Lie bialgebra structures on \( g^* \) corresponding to \( g, g_c, \) and \( g' \). By the above discussion, the corresponding cobrackets \( \delta_g, \delta_{g_c}, \) and \( \delta_{g'} \) are different by one map \( c' : g^* \to \Lambda^2 g^* \) dualizing \( c \).

**Proposition 4.7.** Let \( s \in \Lambda^2 \mathfrak{p} \) be the element representing the complex structure of \( \mathfrak{p} \). Up to the identification \( \mathfrak{p} \simeq \mathfrak{p}^* \) by the inner product and \( \mathfrak{p}^* \subset g^* \) from the Cartan decomposition, \( s \) satisfies
\[
\frac{1}{2} [s, s] + ds = 0,
\]
where \( d \) is the differential of the Gerstenhaber algebra corresponding to \( \delta_{g'} \). The twist of \( \delta_{g'} \) by \( s \) is \( \delta_g \).

 That is, if \( (y_j)_j \) is an orthonormal basis of \( \mathfrak{p} \) as a real Euclidean space, we have
\[
s = \sum_j iy_j \otimes y_j,
\]
where \( iy_j \) is computed using the complex structure of \( \mathfrak{p} \).

 By the main result of [Hal06], quantized universal algebras \( U_h(g^*) \) arising from these Lie bialgebra structures on \( g^* \) are related by 2-cocycles. On the other hand, by the quantum duality principle, these can be interpreted as function algebras on the quantum groups \( G_h \), \( (G_c)_h \), and \( E_h \). This gives a formal analogue of the functional analytic construction of 2-cocycles by De Commer [DCT11] for the case \( g = su(1,1) \) that connects Woronowicz’s SU\(_q\)(2) [Wor87] to Koelink–Kusterman’s SÜ\(_q\)(1,1) [KK03].

**Appendix A. Groupoid C*-Algebras of Matched Pairs of Lie Groups**

Let \( (B, C) \) be a matched pair of subgroups of a Lie group \( G \). Let us be precise about our convention of the groupoid C*-algebra of the groupoid \( G = GB \). We are going to make sense of unbounded multipliers coming from elements of \( e \).

 First, let us fix our convention of Haar system on \( G \). We mostly follow the convention of [Ren80].
Let $\lambda$ be a left Haar measure on the Lie group $C$, and $\Delta$ be the modular function of $C$. We thus have
\[
\int_C f(cc')d\lambda(c') = \int_C f(c')d\lambda(c'),
\]
for all $c \in C$ and $f \in C_c(C)$. Let us also write
\[
\Delta(y) = \frac{d\Delta(e^y)}{dt}\bigg|_{t=0} (y \in \mathfrak{c}).
\]

On the groupoid $\mathcal{G}$, to avoid the confusion with the group structure of $G$, we write $g \cdot g'$ for the product of composable arrows and $\tilde{g}$ for the groupoid inverse. Thus, given $g = r(g)c = e''s(g)$ and $g' = s(g)c'$ with $s(g), r(g) \in B$, we have $g \cdot g' = r(g)cc'$ and $\tilde{g} = c''^{-1}r(g) = s(g)c^{-1}$.

For a fixed $b \in B$, the set $\mathcal{G}^b$ can be identified with
\[
C^{(b)} = \{ c \in C \mid bc \in CB \},
\]
which is an open subset of $C$ by our assumption. We define the measure $\lambda^b$ on $\mathcal{G}^b$ to be the restriction of $\lambda$ on $C^{(b)}$ up to this identification. Then the invariance condition
\[
\int f(g \cdot g')d\lambda^{s(g)}(g') = \int f(g')d\lambda^{r(g)}(g') \quad (g \in \mathcal{G}_B, f \in C_c(\mathcal{G}^{r(g)}))
\]
follows from the left invariance of $\lambda$. By $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$, we mean the full and reduced groupoid $C^*$-algebras associated with this Haar system.

Next let us make sense of the associated the left derivation in direction of $y \in \mathfrak{c}$ as an unbounded multiplier on $C^*(\mathcal{G})$ (or on $C^*_r(\mathcal{G})$). We follow the approach of [2] Chapters 9 and 10. We consider the $C_c(\mathcal{G})$-valued inner product on $C_c(\mathcal{G})$, defined as
\[
\langle f, f' \rangle(g) = f^*g'\int \langle \tilde{f} \bar{g}', g \rangle d\lambda^{s(y)}(g') = \int \tilde{f}(\tilde{g} \cdot \bar{g})f'(g)\lambda^{s(y)}(g').
\]

Let us fix $y \in \mathfrak{c}$, and formally write
\[
(u^s \cdot f)(g) = \Delta(e^{sy})^{1/2}f(e^{-sy}c''b') \quad (g = e''b').
\]
When $f$ is compactly supported, this is well-defined for small $s$ by our openness assumption of $\mathcal{G} \subset G$. Moreover, we have
\[
(u^s \cdot f) \cdot f' = u^s \cdot (f \cdot f')
\]
from left invariance of $\lambda$, and
\[
(u^s \cdot f)^* \cdot f' = u^s \cdot (u^{-s} \cdot f')
\]
from the defining property of $\Delta$, whenever both sides are well-defined. In particular the transform $f \mapsto u^s \cdot f$ preserves the inner product $\langle \cdot, \cdot \rangle$.

Now, we define the operator $t^0_y$ on $C_c^\infty(\mathcal{G})$ by
\[
t^0_yf = i \frac{d}{ds}u^s \cdot f \bigg|_{s=0} = \frac{i}{2}\Delta(y)f + iX_yf,
\]
where $X_y$ is the smooth vector field on $\mathcal{G}$ whose integral curve is $e''b' \to e^{-sy}c''b'$. By (15), this is a map of right $C_c(\mathcal{G})$-modules. By (16), we also have
\[
\langle t^0_yf, f' \rangle - \langle f, t^0_yf' \rangle = 0.
\]
Polynomials of $t^0_{y_1}, \ldots, t^0_{y_k}$ for $y_1, \ldots, y_k \in \mathfrak{c}$ make sense as operators on $C_c^\infty(\mathcal{G})$. Looking at the commutators we have
\[
[t^0_{y_1}, t^0_{y_2}] = it^0_{[y_1, y_2]}
\]
from the standard property of $X_y$ (note that $\Delta([y_1, y_2])$ always vanishes as $\Delta$ is a homomorphism).

Next, let $f$ be a bounded continuous function on $B$. For $f' \in C_c(\mathcal{G})$, we define the left action of $f$ on $f'$, $f \cdot f' \in C_c(\mathcal{G})$ by
\[
(f \cdot f')(g) = f(r(g))f'(g).
\]
We then have \((f \ast f', f'') = (f', f \ast f'')\) for \(f', f'' \in C_c(\mathcal{G})\). Thus, the left action of \(f\) extends to a bounded adjointable endomorphism of \(A\). When \(f \in C^*_b(B)\) and \(f' \in C^*_c(\mathcal{G})\), we have \(f \ast f' \in C^*_c(\mathcal{G})\). Given \(y \in c\), the differentiation of the action of \(e^{\gamma y}\) on \(B\) defines a vector field \(X'\) on \(B\). We then have the commutation relation

\[
t_0'(f \ast f') - f \ast (t_0' f') = iX'_y(f) \ast f' \quad (f' \in C^*_c(\mathcal{G})),
\]

giving a realization of the algebra \(U(c) \rtimes C^*_b(B)\) inside the space of unbounded multipliers of \(C^*_c(\mathcal{G})\).

In the case of double Lie group, \(G = BC\), we can interpret \(C^*_c(\mathcal{G})\) as the full crossed product \(C^*_b(B) \rtimes C^*_c(\mathcal{G})\) for the induced action of \(C^*_c(\mathcal{G})\) on \(B\). In this case there is a unital \(*\)-homomorphism \(\mathcal{M}(C^*_c(\mathcal{G})) \to \mathcal{M}(C^*_c(\mathcal{G}))\), and \(t'_0\) has an extension to an unbounded self-adjoint element affiliated to \(C^*_c(\mathcal{G})\), see \([WN92]\).

\section*{References}

[BSV03] S. Baaj, G. Skandalis, and S. Vaes, \textit{Non-semi-regular quantum groups coming from number theory}, Comm. Math. Phys. \textbf{235} (2003), no. 1, 139–167, DOI:10.1007/s00220-002-0780-6 MR1960723 (2004g:46083)

[BC90] T. J. Courant, \textit{Dirac manifolds}, Trans. Amer. Math. Soc. \textbf{319} (1990), no. 2, 631–661, DOI:10.2307/2001258 MR981241 [3.1]

[DC11] K. De Commer, \textit{On a correspondence between SU_q(2), \(\hat{E}_6(2)\) and \(SU_q(1,1)\)}, Comm. Math. Phys. \textbf{304} (2011), no. 1, 187–229, DOI:10.1007/s00220-011-1208-y MR2793344 [4.1]

[Hal06] G. Halbout, \textit{Formality theorem for Lie bialgebras and quantization of twist and coboundary r-matrices}, Adv. Math. \textbf{207} (2006), no. 2, 617–633, DOI:10.1016/j.aim.2005.12.006 MR2271019 [4.1]

[Hel01] S. Helgason. (2001). \textit{Differential geometry, Lie groups, and symmetric spaces}, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, DOI:10.1090/gsm/034 ISBN 0-8218-2848-7. Corrected reprint of the 1978 original. MR1834454 [4.1]

[KK03] E. Koelink and J. Kustermans, \textit{A locally compact quantum group analogue of the normalizer of SU(1,1) in SL(2,\mathbb{C})}, Comm. Math. Phys. \textbf{233} (2003), no. 2, 231–296, DOI:10.1007/s00220-002-0736-x MR1962042 [4.1]

[Kor94] L. I. Korogodsky, \textit{Quantum group SU(1,1) \rtimes \mathbb{Z}_2 and “super-tensor” products}, Comm. Math. Phys. \textbf{163} (1994), no. 3, 433–460, DOI:10.12874/91.13 [4.1]

[Lan95] E. C. Lance. (1995). \textit{Hilbert C*-modules}, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, DOI:10.1017/CBO9780511552626 ISBN 0-521-47910-X. A toolkit for operator algebraists. MR1325694 (96k:46010)

[Lan99] N. P. Landsman, \textit{Lie groupoid \(\mathbb{C}^\ast\)-algebras and Weyl quantization}, Comm. Math. Phys. \textbf{206} (1999), no. 2, 367–381, DOI:10.1007/s002200050709 MR1722129 [4.1]

[LR01] N. P. Landsman and B. Ramazan, \textit{Quantization of Poisson algebras associated to Lie algebroids}, Groupoids in analysis, geometry, and physics (Boulder, CO, 1999), 2001, pp. 159–192, Amer. Math. Soc., Providence, RI, DOI:10.1090/conm/282/04685 ISBN 0-8218-2848-7. Corrected reprint of the 1978 original. MR1834454 [4.1]

[MaJ90] S. Majid, \textit{Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bi-crossproduct construction}, J. Algebra \textbf{130} (1990), no. 1, 17–64, DOI:10.1016/0021-8693(90)90099-A MR1045735 [4.1]

[Mat94] F. Maslanka, \textit{The \(E_6(2)\) group via direct quantization of the Lie-Poisson structure and its Lie algebra}, J. Math. Phys. \textbf{35} (1994), no. 4, 1976–1983, DOI:10.1063/1.530582 MR1267935 [4.1]

[Rie94] M. A. Rieffel, \textit{Quantization and \(C^\ast\)-algebras: 1943–1993}, (San Antonio, TX, 1993), 1994, pp. 66–97, Amer. Math. Soc., Providence, RI, DOI:10.1090/conm/167/1292010 ISBN 0-8218-40105 [4.1]

[Ren80] J. Renault. (1980). \textit{A groupoid approach to \(\mathbb{C}^\ast\)-algebras}, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, ISBN 3-540-09977-8. MR584266 [4.1]

[Sob96] J. Sobczyk, \textit{Quantum \(E(2)\) groups and Lie bialgebra structures}, J. Phys. A \textbf{29} (1996), no. 11, 2887–2903, DOI:10.1088/0305-4470/29/11/022 MR1399267 [4.1]

[Sta00] P. Stachura, \textit{\(C^\ast\)-algebra of a differential groupoid}, Poisson geometry (Warsaw, 1998), Banach Center Publ., vol. 51, Polish Acad. Sci. Inst. Math., Warsaw, 2000, pp. 263–281. With an appendix by S. Zakrzewski. MR1764452 [4.2]

[Sta17] P. Stachura, \textit{On Poisson structures related to \(\kappa\)-Poincaré group}, Int. J. Geom. Methods Mod. Phys. \textbf{14} (2017), no. 9, 1750133, 14, DOI:10.1142/S021988781750133X MR3681683 [4.2]

[Sta19] P. Stachura, \textit{The \(\kappa\)-Poincaré group on a \(C^\ast\)-level}, Internat. J. Math. \textbf{30} (2019), no. 4, 1950022, 43, DOI:10.1142/S0129167X19500228 MR3950818 [4.4]

[Tak81] M. Takeuchi, \textit{Matched pairs of groups and bismash products of Hopf algebras}, Comm. Algebra \textbf{9} (1981), no. 8, 841–882, DOI:10.1080/00927878108822621 MR615561 (83f:16013) [4.4]
[Wor87] S. L. Woronowicz, Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci. 23 (1987), no. 1, 117–181, DOI:10.2977/prims/1195176848 MR890482 (88h:46130)

[WH91] S. L. Woronowicz, Unbounded elements affiliated with C*-algebras and noncompact quantum groups, Comm. Math. Phys. 136 (1991), no. 2, 399–432. MR1096123 (92b:46117)

[WN92] S. L. Woronowicz and K. Napiórkowski, Operator theory in the C*-algebra framework, Rep. Math. Phys. 31 (1992), no. 3, 353–371. DOI:10.1016/0034-4877(92)90025-V MR1232646

[Zak90] S. Zakrzewski, Quantum and classical pseudogroups. II. Differential and symplectic pseudogroups, Comm. Math. Phys. 134 (1990), no. 2, 371–395.

[Zak94] S. Zakrzewski, Quantum Poincaré group related to the κ-Poincaré algebra, J. Phys. A 27 (1994), no. 6, 2075–2082. MR1280372

Department of Mathematics, University of Oslo, P.O. box 1053, Blindern, 0316 Oslo, Norway
Email address: florise@math.uio.no

Email address: makotoy@math.uio.no