About the equivalence between monads and monadic functors

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Abstract

Given a $\infty$-category $X$ we exhibit the $\infty$-category of right adjoint functors with target $X$ as a localization of the opposite of the $\infty$-category of monads on $X$. This localization restricts to an equivalence between the $\infty$-category of monadic functors with target $X$ and the opposite of the $\infty$-category of monads on $X$.

We refine this result in the case that the monads carry "algebraic" structure, e.g. are monads on a (symmetric) monoidal $\infty$-category compatible with the (symmetric) monoidal structure, monads on a $\infty$-operad compatible with the $\infty$-operad structure or monads on a double $\infty$-category compatible with this structure.

This says that structure on a monad corresponds to structure on its $\infty$-category of algebras.

Moreover we prove dual results about left adjoint functors, comonads and comonadic functors.

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1 Introduction

Let $X$ be a category. The composition of endofunctors of $X$ defines a monoidal structure on the category of endofunctors of $X$, whose associative algebras we call monads on $X$.

Given a monad $T$ on $X$ we can form the category of $T$-algebras $\text{LMod}_T(X)$, which comes equipped with a forgetful functor $\text{LMod}_T(X) \to X$ and a free functor $X \to \text{LMod}_T(X)$ left adjoint to the forgetful functor.

On the other hand every functor $g : Y \to X$ with left adjoint $f : X \to Y$ gives rise to a monad $T = g \circ f$ on $X$, where $T$ arises as the monad associated to the forgetful functor $\text{LMod}_T(X) \to X$.

Moreover we have a canonical functor $Y \to \text{LMod}_T(X)$ over $X$ that sometimes happens to be an equivalence, in which case we call $g$ a monadic functor.

This way we can turn right adjoint functors to monads and vice versa and obtain a correspondence between monads and monadic functors.

This correspondence between monads and monadic functors is especially desirable when working with $\infty$-categories:

The coherence data that exhibit an endofunctor of $X$ as a monad are quite complicated, where the property of a functor of being monadic is quite easy to check by the famous theorem of Barr-Beck.

Given a $\infty$-category $X$ we have an analogous monoidal $\infty$-category $\text{Fun}(X, X)$ of endofunctors of $X$ and define monads as in the 1-categorical definition as objects of the $\infty$-category $\text{Alg}(\text{Fun}(X, X))$ of associative algebras in $\text{Fun}(X, X)$.

Denote $\text{Cat}_{\infty}$ the $\infty$-category of small $\infty$-categories.

We form the $\infty$-category $(\text{Cat}_{\infty})/X$ of functors with target $X$ and its full subcategory $((\text{Cat}_{\infty})/X)^R$ of right adjoint functors with target $X$ and construct a localization

$((\text{Cat}_{\infty})/X)^R \leftrightarrow \text{Alg}(\text{Fun}(X, X))^{\text{op}}$

with local objects the monadic functors with target $X$ (theorem 5.1).
The left adjoint sends a functor \( g : Y \to X \) with left adjoint \( f : X \to Y \) to its associated monad \( g \circ f \) and the right adjoint sends a monad on \( X \) to its \( \infty \)-category of algebras.

So the localization restricts to an equivalence

\[
((\text{Cat}_\infty)/X)_{\text{mon}} \cong \text{Alg}(\text{Fun}(X, X))^{\text{op}},
\]

where \((\text{Cat}_\infty)/X)_{\text{mon}} \subset (\text{Cat}_\infty)/X\) denotes the full subcategory spanned by the monadic functors with target \( X \).

This result is expected by Lurie [1] remark 4.7.4.8.

Moreover we prove a global version:

We form the arrow-category \( \text{Fun}(\Delta^1, \text{Cat}_\infty) \) and its full subcategories

\[
\text{Fun}(\Delta^1, \text{Cat}_\infty)_{\text{mon}} \subset \text{Fun}(\Delta^1, \text{Cat}_\infty)\]

of monadic functors respectively right adjoint functors and show that the full subcategory

\[
\text{Fun}(\Delta^1, \text{Cat}_\infty)_{\text{mon}} \subset \text{Fun}(\Delta^1, \text{Cat}_\infty)\]

is a localization.

Having this correspondence between monads and monadic functors we study how structure on a monad corresponds to structure on its \( \infty \)-category of algebras.

To make this precise we use the notion of categorical pattern (def. in [1], Def. B.0.19):

Structure on a \( \infty \)-category \( X \) is typically given by a \( \infty \)-category \( S \) equipped with some object \( s \in S \) and a \( \infty \)-category over \( S \) satisfying some properties such that \( X \) is the fiber at \( s \).

For example is a symmetric monoidal structure on a \( \infty \)-category \( X \) encoded by a cocartesian fibration \( X^\otimes \to \text{Fin}_* \) satisfying a segal condition and such that \( X \) is the fiber at \( (1) \in \text{Fin}_* \), where \( \text{Fin}_* \) denotes the category of pointed finite sets. Such a datum can elegantly be described using categorical pattern:

Slightly simplified a categorical pattern \( \mathcal{P} \) on a \( \infty \)-category \( S \) consists of a subcategory \( \mathcal{E} \) of \( S \) and a family \( (K^c_j \to S)_{j \in J} \) of functors that send every morphism to one of \( \mathcal{E} \).

We call a functor \( \phi : \mathcal{B} \to S \) fibered with respect to \( \mathcal{P} \) if the following conditions hold:

- For every morphism \( f : s \to t \) of \( \mathcal{E} \) and every \( b \in \mathcal{B} \) lying over \( s \) there is a \( \phi \)-cocartesian lift of \( f \). This implies that for every \( j \in J \) the pullback \( K^p_j \times_S \mathcal{B} \to K^c_j \) is a cocartesian fibration classifying a functor \( H : K^p_j \to \text{Cat}_\infty \).

- The functor \( H \) is a limit diagram.

- For every cocartesian section \( \alpha \) of \( K^p_j \times_S \mathcal{B} \to K^c_j \) the composition \( K^p_j \xrightarrow{\alpha} K^c_j \) is a \( \phi \)-limit diagram.

Given \( \mathcal{P} \)-fibered objects \( \phi : \mathcal{B} \to S, \phi' : \mathcal{B}' \to S \) we call a functor \( \mathcal{B} \to \mathcal{B}' \) over \( S \) a map of \( \mathcal{P} \)-fibered objects if it sends \( \phi \)-cocartesian morphisms lying over morphisms of \( \mathcal{E} \) to \( \phi' \)-cocartesian morphisms.
Categorial pattern turn out to be very useful as virtually all "algebraic" structures on $\infty$-categories are $\mathfrak{P}$-fibered objects for some categorial pattern $\mathfrak{P}$:

For example symmetric monoidal $\infty$-categories and $\infty$-operads are $\mathfrak{P}$-fibered objects for categorial pattern on $\text{Fin}_*$ or monoidal $\infty$-categories, planar operads and double $\infty$-categories are $\mathfrak{P}$-fibered objects for categorial pattern on $\Delta^\text{op}$.

Let $\mathfrak{P}$ be a categorial pattern on a $\infty$-category $S$ denote $\text{Cat}_\infty^{\mathfrak{P}} \subset \text{Cat}_\infty/S$ the subcategory of $\mathfrak{P}$-fibered objects and maps of those. Given $\mathfrak{P}$-fibered objects $X \to S, Y \to S$ denote $\text{Fun}(X, Y)^{\mathfrak{P}} \subset \text{Fun}_S(X, Y)$ the full subcategory spanned by the functors $X \to Y$ over $S$ that are maps of $\mathfrak{P}$-fibered objects and $(\text{Cat}_\infty^{\mathfrak{P}})^{\mathfrak{P}} \subset \text{Cat}_\infty^{\mathfrak{P}/X}$ the full subcategory spanned by the maps of $\mathfrak{P}$-fibered objects $Y \to X$ that admit a left adjoint relative to $S$ that is a map of $\mathfrak{P}$-fibered objects.

We construct a localization

$$(\text{Cat}_\infty^{\mathfrak{P}/X})^{\mathfrak{P}} \rightleftarrows \text{Alg}(\text{Fun}(X, X)^{\mathfrak{P}})^{\mathfrak{P}}_{\text{op}}$$

with local objects the maps of $\mathfrak{P}$-fibered objects $Y \to X$ that induce for every $\mathfrak{P}$-fibered object $Z \to S$ a monadic functor $\text{Fun}(Z, Y)^{\mathfrak{P}} \to \text{Fun}(Z, X)^{\mathfrak{P}}$ (theorem 5.1).

We interprete this by saying that structure on a monad corresponds to structure on its $\infty$-category of algebras.

Combined with theorem 6.8 this refined localization result specializes to the following situations and generalizes theorems about hopf monads on tensor categories like theorem 7.1. of [3] from 1-categories to $\infty$-categories:

Let $T$ be a monad on a $\infty$-category $\mathcal{C}$.

1. Assume that $\mathcal{C}$ carries the structure of an $\infty$-operad.

Then $T$ lifts to a map of $\infty$-operads such that the unit and multiplication of $T$ are natural transformations of $\infty$-operads if and only if the forgetful functor $\text{LMod}_T(\mathcal{C}) \to \mathcal{C}$ and its left adjoint lift to maps of $\infty$-operads.

2. Assume that $\mathcal{C}$ carries the structure of a symmetric monoidal $\infty$-category.

Then $T$ lifts to an oplax symmetric monoidal functor such that the unit and multiplication of $T$ are oplax symmetric monoidal natural transformations (such a $T$ is called a commutative Hopf-monad or symmetric opmonoidal monad) if and only if the forgetful functor $\text{LMod}_T(\mathcal{C}) \to \mathcal{C}$ lifts to a symmetric monoidal functor.

This result generalizes theorem 7.1. of [3] from 1-categories to $\infty$-categories.

3. Assume that $\mathcal{C}$ carries the structure of a left module over a monoidal $\infty$-category $\mathcal{V}$.

Then $T$ lifts to an oplax $\mathcal{V}$-linear functor such that the unit and multiplication of $T$ are oplax $\mathcal{V}$-linear natural transformations if and only if the forgetful functor $\text{LMod}_T(\mathcal{C}) \to \mathcal{C}$ lifts to a $\mathcal{V}$-linear functor.

4. Assume that $\mathcal{C}$ carries the structure of a symmetric monoidal $\infty$-category and admits geometric realizations that are preserved by the tensor product of $\mathcal{C}$ in each component.
Then $T$ preserves geometric realizations and lifts to a lax symmetric monoidal functor such that the unit and multiplication of $T$ are lax symmetric monoidal natural transformations if and only if the free functor $\mathcal{C} \to \text{LMod}_T(\mathcal{C})$ admits geometric realizations that are preserved by the forgetful functor and the tensor product in each component.

Moreover if $\mathcal{C}$ admits small colimits that are preserved by the tensor product in each component, then the same holds for $\text{LMod}_T(\mathcal{C})$.

5. Assume that $\mathcal{C}$ is a presentable symmetric monoidal closed $\infty$-category and $T$ preserves geometric realizations, is an accessible functor and lifts to a lax symmetric monoidal functor such that the unit and multiplication of $T$ are lax symmetric monoidal natural transformations.

Then $\text{LMod}_T(\mathcal{C})$ is a presentable symmetric monoidal closed $\infty$-category and the free functor $\mathcal{C} \to \text{LMod}_T(\mathcal{C})$ lifts to a symmetric monoidal functor.

6. Assume that $\mathcal{C}$ is a presentable closed left module over a presentable monoidal closed category $\mathcal{D}$ such that $T$ lifts to a lax $\mathcal{D}$-linear functor and the unit and multiplication of $T$ are lax $\mathcal{D}$-linear natural transformations and $T$ is an accessible functor and preserves geometric realizations, then $\text{LMod}_T(\mathcal{C})$ is a presentable closed left module over $\mathcal{D}$ and the free functor $\mathcal{C} \to \text{LMod}_T(\mathcal{C})$ lifts to a $\mathcal{D}$-linear functor.

Let $\mathcal{D}$ be a monoidal $\infty$-category and $A$ an associative algebra of $\mathcal{D}$.

Then the functor $T := A \otimes - : \mathcal{D} \to \mathcal{D}$ that tensors with $A$ is naturally a monad on $\mathcal{D}$ and we have a canonical equivalence $\text{LMod}_A(\mathcal{D}) \simeq \text{LMod}_T(\mathcal{D})$ between the $\infty$-category of left modules in $\mathcal{D}$ over $A$ and the $\infty$-category of $T$-algebras.

This implies the following results:

1. Let $\mathcal{D}$ be an $E_{k+1}$-monoidal $\infty$-category for some natural $k$ and $A$ a bialgebra, i.e. an associative algebra in the monoidal $\infty$-category of $E_k$-coalgebras of $\mathcal{D}$. Then the $\infty$-category $\text{LMod}_A(\mathcal{D})$ is an $E_k$-monoidal $\infty$-category and the forgetful functor $\text{LMod}_A(\mathcal{D}) \to \mathcal{D}$ is an $E_k$-monoidal functor.

2. Let $\mathcal{D}$ be an associative monoid in the $\infty$-category of $E_k$-operads for some natural $k$ and $A$ an $E_k$-algebra of $\mathcal{D}$.

Then the $\infty$-category $\text{LMod}_A(\mathcal{D})$ carries the structure of a $E_k$-operad and the forgetful functor $\text{LMod}_A(\mathcal{D}) \to \mathcal{D}$ and its left adjoint are maps of $E_k$-operads.

If $\mathcal{D}$ is additionally an $E_k$-monoidal $\infty$-category that admits geometric realizations that are preserved by the $E_k$-monoidal structure and the functor $A \otimes - : \mathcal{D} \to \mathcal{D}$ induced by the associative monoid structure on the $E_k$-monoidal $\infty$-category $\mathcal{D}$, then the $\infty$-category $\text{LMod}_A(\mathcal{D})$ is an $E_k$-monoidal $\infty$-category and the free functor $\mathcal{D} \to \text{LMod}_A(\mathcal{D})$ is an $E_k$-monoidal functor.

Moreover if $\mathcal{D}$ admits small colimits that are preserved by the $E_k$-monoidal structure, then the same holds for $\text{LMod}_A(\mathcal{D})$. 

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The ∞-category \( \text{Cat}_\infty \) of small ∞-categories is naturally a \((\infty, 2)\)-category, i.e. a ∞-category enriched over small ∞-categories.

Given two ∞-categories \( \mathcal{C}, \mathcal{D} \) the enrichment is given by the ∞-category of functors \( \text{Fun}(\mathcal{C}, \mathcal{D}) \).

Other examples of \((\infty, 2)\)-categories are \( \text{Cat}_{\infty/\mathcal{S}} \) for every small ∞-category \( \mathcal{S} \) or \( \text{Cat}_P^\mathcal{P} \) for every categorial pattern \( \mathcal{P} \) on \( \mathcal{S} \).

To treat all localization results the same time we work with general \((\infty, 2)\)-categories.

\((\infty, 2)\)-categories are the natural setting, where one can define monads, adjunctions and monadicity:

Given a \((\infty, 2)\)-category \( \mathcal{C} \) and an object \( X \) of \( \mathcal{C} \) one can form the ∞-category \( \mathcal{C}[X, X] \) of endomorphisms in \( \mathcal{C} \), which is equipped with a natural monoidal structure given by composition of endomorphisms.

Similar to the case of categories we define a monad on \( X \) to be an associative algebra in \( \mathcal{C}[X, X] \).

We say that a morphism \( g : Y \to X \) in \( \mathcal{C} \) is left adjoint to a morphism \( f : X \to Y \) in \( \mathcal{C} \) if the pair \((f, g)\) satisfies the triangular identities in \( \mathcal{C} \) or equivalently in the homotopy bicategory of \( \mathcal{C} \) (that arises by taking the homotopy category of each morphism category).

Moreover we call \( g : Y \to X \) monadic if it admits a left adjoint morphism and yields for every \( Z \in \mathcal{C} \) a monadic functor \( \mathcal{C}[Z, Y] \to \mathcal{C}[Z, X] \) on morphism categories.

We show that every right adjoint morphism \( g : Y \to X \) in \( \mathcal{C} \) with left adjoint \( f : X \to Y \) gives rise to a monad \( g \circ f \) on \( X \) (proposition 3.6).

On the other hand given a monad \( T \) on \( X \) which is the associated monad of a morphism \( \psi : Y \to X \) we call \( \psi \) an Eilenberg-Moore object of \( T \).

Eilenberg-Moore objects don’t have to exist in general but if they exist, they are unique. We show that Eilenberg-Moore objects exist in many \((\infty, 2)\)-categories (theorem 4.16):

Given a small ∞-category \( \mathcal{S} \) and a categorial pattern \( \mathcal{P} \) on \( \mathcal{S} \) we show that every monad in the \((\infty, 2)\)-categories \( \text{Cat}_{\infty/\mathcal{S}} \) and \( \text{Cat}_P^\mathcal{P} \) admits an Eilenberg-Moore object.

For \( \text{Cat}_\infty \) the Eilenberg-Moore object of a monad \( T \) on a ∞-category \( X \) is given by the usual ∞-category \( \text{LMod}_T(X) \) of \( T \)-algebras also known as Eilenberg-Moore category, from which the name comes.

For \( \text{Cat}_{\infty/\mathcal{S}} \) and \( \text{Cat}_P^\mathcal{P} \) the Eilenberg-Moore object of a monad \( T \) induces on the fiber over every \( s \in \mathcal{S} \) the ∞-category \( \text{LMod}_{T_s}(X_s) \) of algebras in the fiber \( X_s \) over the induced monad \( T_s \) on the fiber over \( s \).

With this terminology we can formulate our main theorem 5.1 from which we deduce all localization results:

Let \( \mathcal{C} \) be a \((\infty, 2)\)-category and \( X \) an object of \( \mathcal{C} \).

Assume that every monad on \( X \) admits an Eilenberg-Moore object in \( \mathcal{C} \).

There is a localization

\[ \text{End} : (\mathcal{C}/X)^\mathbb{R} \to \text{Alg}([X, X])^{\text{op}} : \text{Alg} \]

between the ∞-category of right adjoint morphisms with target \( X \) and the ∞-category of monads on \( X \).

The functor \( \text{End} \) sends a morphism \( g : Y \to X \) with left adjoint \( f : X \to Y \) to its associated monad \( g \circ f \) and the right adjoint \( \text{Alg} \) associates the Eilenberg-Moore object to a monad.
Moreover the local objects of this localization are exactly the monadic morphisms in $\mathcal{C}$ with target $X$.

So restricting the functor $\text{End}$ to the full subcategory $(\mathcal{C}_X)^{\text{mon}} \subset (\mathcal{C}_X)^{\text{R}}$ spanned by the monadic morphisms we obtain an equivalence $\text{End} : (\mathcal{C}_X)^{\text{mon}} \simeq \text{Alg}([X, X])^{\text{op}}$.

For $\mathcal{C} = \text{Cat}_\infty$ we obtain the localization

$$(\text{Cat}_\infty_X)^{\text{R}} \simeq \text{Alg}(\text{Fun}(X, X))^{\text{op}}$$

and for $\mathcal{C} = \text{Cat}_\infty^p$ we obtain the refined localization

$$(\text{Cat}_\infty^p_X)^{\text{R}} \simeq \text{Alg}(\text{Fun}(X, X)^{\text{p}})^{\text{op}}$$.

Moreover we have the dual notions of comonadic morphism and co-Eilenberg-Moore object, for which we get dual statements applying our statements to the $(\infty,2)$-category $\mathcal{C}^{\text{op}}$ that arises from a $(\infty,2)$-category $\mathcal{C}$ by reversing all 2-morphisms.

1.1 Notation and Terminologie

Fix your preferred model of $\infty$-categories.

By category we always mean $\infty$-category, by 2-category we mean $(\infty,2)$-category and by operad we mean $\infty$-operad.

We describe $\infty$-operads and $(\infty,2)$-categories purely in terms of $\infty$-categories, where we take Lurie’s definitions found in [1] 2.1.1.10. and 4.2.1.28. but interpret them derived or homotopy-invariant (see for example the notion of (locally) cocartesian fibration in the next subsection).

Given a strongly inaccessible cardinal $\kappa$ by saying $\kappa$-small we mean essentially $\kappa$-small.

Given a $\kappa$-small category $\mathcal{C}$ denote $\text{Ho}(\mathcal{C})$ its homotopy category.

Given a strongly inaccessible cardinal $\kappa$ denote $\text{Cat}_\infty(\kappa)$ the category of $\kappa$-small categories and $\mathcal{S}(\kappa)$ the full subcategory of $\text{Cat}_\infty(\kappa)$ spanned by the $\kappa$-small spaces.

$\mathcal{S}(\kappa)$ and $\text{Cat}_\infty(\kappa)$ admit all $\kappa$-small limits and colimits.

Given two $\kappa$-small categories $\mathcal{C}, \mathcal{D}$ denote $\text{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors $\mathcal{C} \rightarrow \mathcal{D}$ being the internal hom of $\text{Ho}(\text{Cat}_\infty(\kappa))$.

Given a $\kappa$-small category and objects $X, Y \in \mathcal{C}$ we write $\mathcal{C}(X, Y)$ for the space of morphisms $X \rightarrow Y$ in $\mathcal{C}$ that can be defined as $\mathcal{C}(X, Y) := \{ (X, Y) \times_{\mathcal{C} \times \mathcal{C}} \text{Fun}(\Delta^1, \mathcal{C})\}.$

Moreover we have a natural equivalence

$$\text{Cat}_\infty(\kappa)(\mathcal{B} \times \mathcal{C}, \mathcal{D}) \simeq \text{Cat}_\infty(\kappa)(\mathcal{B}, \text{Fun}(\mathcal{C}, \mathcal{D}))$$

for $\mathcal{B}, \mathcal{C}, \mathcal{D} \in \text{Cat}_\infty(\kappa)$.

Given a $\kappa$-small category containing a morphism $\iota : X \rightarrow Y$, we call $X$ a subobject of $Y$ if $\iota : X \rightarrow Y$ is a monomorphism, i.e. for every $Z \in \mathcal{C}$ induces a fully faithful map $\mathcal{C}(Z, X) \rightarrow \mathcal{C}(Z, Y)$. 7
If $i$ is clear from the context, we also write $X \subset Y$ to indicate that $X$ is a subobject of $Y$ via $i$.

We often use this notion in the cases of a morphism of $\kappa$-small categories and $\kappa$-small operads, where we also use the term subcategory and suboperad.

Remark that monomorphisms are stable under pullback and thus are preserved by pullback preserving functors.

(laterally) (co)cartesian morphisms and fibrations
Let $\phi: \mathcal{C} \to \mathcal{D}$ be a functor. We call a morphism $f: X \to Y$ in $\mathcal{C}$ $\phi$-cocartesian if the commutative square

\[
\begin{array}{ccc}
\mathcal{C}(Y,Z) & \to & \mathcal{C}(X,Z) \\
\downarrow & & \downarrow \\
\mathcal{D}(\phi(Y),\phi(Z)) & \to & \mathcal{D}(\phi(X),\phi(Z))
\end{array}
\]

is a pullback square of spaces.

By the pasting law for pullbacks the following statements follow immediately from the definition:

1. Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of $\mathcal{C}$.
   Assume that $f$ is $\phi$-cocartesian.
   Then $g$ is $\phi$-cocartesian if and only if $g \circ f$ is $\phi$-cocartesian.

2. Let $\psi: \mathcal{D}' \to \mathcal{D}$ be a functor and $\phi': \mathcal{C}' \to \mathcal{D}'$ the pullback of $\phi: \mathcal{C} \to \mathcal{D}$ along $\psi$.
   Let $f: X \to Y$ be a morphism of $\mathcal{C}'$, whose image in $\mathcal{C}$ is $\phi$-cocartesian.
   Then $f: X \to Y$ is $\phi'$-cocartesian.

3. Let $\varphi: \mathcal{D} \to \mathcal{E}$ be a functor and $f: X \to Y$ a morphism of $\mathcal{C}$ such that $\phi(f)$ is $\varphi$-cocartesian.
   Then $f$ is $\phi$-cocartesian if and only if $f$ is $\varphi \circ \phi$-cocartesian.

We call a morphism $f: X \to Y$ in $\mathcal{C}$ locally $\phi$-cocartesian if one of the following equivalent conditions holds:

1. $f: X \to Y$ is $\phi'$-cocartesian, where $\phi'$ denotes the pullback $\Delta^1 \times_\mathcal{D} \mathcal{C} \to \Delta^1$ of $\phi$ along $\phi(f)$.
2. $f: X \to Y$ is a final object of the category $\{\phi(f)\} \times_{\mathcal{D}_{\mathcal{D}(X,Y)}} \mathcal{C}/f$.
3. For every $Z \in \mathcal{C}$ lying over the object $\phi(Y)$ composition with $f: X \to Y$
   \[
   \{\text{id}\} \times_{\mathcal{D}(\phi(Y),\phi(Y))} \mathcal{C}(Y,Z) \to \{\phi(f)\} \times_{\mathcal{D}(\phi(X),\phi(Y))} \mathcal{C}(X,Z)
   \]
   is an equivalence.
The following statements follow immediately from the definition:

Every $\phi$-cocartesian morphism is locally $\phi$-cocartesian.

Let $\psi : \mathcal{D}' \to \mathcal{D}$ be a functor and $\phi' : \mathcal{C}' \to \mathcal{D}'$ the pullback of $\phi : \mathcal{C} \to \mathcal{D}$ along $\psi$. Let $f : X \to Y$ be a morphism of $\mathcal{C}'$.

Then $f : X \to Y$ is locally $\phi'$-cocartesian if and only if the image of $f$ in $\mathcal{C}$ is locally $\phi$-cocartesian.

We call a functor $\phi : \mathcal{C} \to \Delta^1$ a cocartesian fibration if for every object $X$ of $\mathcal{C}$ lying over 0 there is a $\phi$-cocartesian morphism $X \to Y$ in $\mathcal{C}$ such that $Y$ lies over 1.

We call a functor $\phi : \mathcal{C} \to \mathcal{D}$ a locally cocartesian fibration if the pullback $\Delta^1 \times_\mathcal{D} \mathcal{C} \to \Delta^1$ along every morphism of $\mathcal{D}$ is a cocartesian fibration.

We call a functor $\phi : \mathcal{C} \to \mathcal{D}$ a cocartesian fibration if it is a locally cocartesian fibration and every locally $\phi$-cocartesian morphism is $\phi$-cocartesian.

We call a functor $\mathcal{C} \to \mathcal{D}$ a left fibration if it is a cocartesian fibration and all its fibers over objects of $\mathcal{D}$ are spaces.

Dually, we define (locally) cartesian morphisms, (locally) cartesian fibrations and right fibrations.

Given a strongly inaccessible cardinal $\kappa$ denote

- $\text{Cat}_{\infty}^L(\kappa)$ and $\text{Cat}_{\infty}^R(\kappa)$ the wide subcategories of $\text{Cat}_{\infty}(\kappa)$ with morphisms the left adjoint respectively right adjoint functors
- $\text{Op}_{\infty}(\kappa)$ the category of $\kappa$-small operads
- $\mathcal{L}(\kappa)$ and $\mathcal{R}(\kappa)$ the full subcategories of $\text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa))$ spanned by the left respectively right fibrations
- $\text{Cocart}(\kappa), \text{Cart}(\kappa)$ and $\text{Bicart}(\kappa)$ the subcategories of $\text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa))$ with objects the cocartesian fibrations, cartesian fibrations respectively bicartesian fibrations and morphisms the squares of $\kappa$-small categories, whose top functor preserves cocartesian, cartesian, respectively both cocartesian and cartesian morphisms
- $\mathcal{U}(\kappa)$ the full subcategory of $\mathcal{R}(\kappa)$ spanned by the representable right fibrations.

**Remark 1.1.** The evaluation at the target functor $\text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa)) \to \text{Cat}_{\infty}(\kappa)$ is a cartesian fibration as $\text{Cat}_{\infty}(\kappa)$ admits pullbacks.

As left, right, cocartesian, cartesian and bicartesian fibrations and their morphisms (over a fixed category) are stable under pullback, the restrictions $\mathcal{L}(\kappa) \to \text{Cat}_{\infty}(\kappa)$, $\mathcal{R}(\kappa) \to \text{Cat}_{\infty}(\kappa)$, $\text{Cocart}(\kappa) \to \text{Cat}_{\infty}(\kappa)$, $\text{Cart}(\kappa) \to \text{Cat}_{\infty}(\kappa)$ and $\text{Bicart}(\kappa) \to \text{Cat}_{\infty}(\kappa)$ of the evaluation at the target functor are cartesian fibrations.

Given a $\kappa$-small category $\mathcal{C}$ we usually denote the corresponding fibers by $\mathcal{L}_C(\kappa), \mathcal{R}_C(\kappa), \text{Cat}_{\infty}^{\text{cocart}}(\kappa), \text{Cat}_{\infty}^{\text{cart}}(\kappa)$ respectively $\text{Cat}_{\infty}^{\text{bicart}}(\kappa)$. 

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By proposition 7.3 the restriction \( \mathcal{U}(\kappa) \to \mathbf{Cat}_\infty(\kappa) \) of the evaluation at the target functor \( \text{Fun}(\Delta^1, \mathbf{Cat}_\infty(\kappa)) \to \mathbf{Cat}_\infty(\kappa) \) to \( \mathcal{U}(\kappa) \) is a cocartesian fibration and classifies the identity of \( \mathbf{Cat}_\infty(\kappa) \).

Given a \( \kappa \)-small category \( \mathcal{C} \) denote \( \mathcal{P}_\kappa(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}(\kappa)) \) the category of presheaves on \( \mathcal{C} \).

1.2 Acknowledgements

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1.3 Overview

In section 2. we study parametrized categories of sections what are a basic tool for fibered categorical constructions.

In higher category theory functor-categories and more generally categories of section play a prominent role as (homotopy coherent) structure and relations are usually encoded by diagrams.

When working with higher categories it can be very tricky to make assignments functorial or maps natural as one has to introduce structure that exhibits assignments functorial or maps natural.

To avoid this problem we always work with categories relative to a base category when we do category theory.

Doing this consequently we always get fibered or parametrized constructions from which we get the desired structure that makes assignments functorial and maps natural.

For these reasons parametrized categories of sections are very important for us.

Based on parametrized categories of sections we define parametrized categories of algebras over a family of operads and parametrized categories of left modules over a parametrized associative algebra in the evident way.

As in the absolute case we define parametrized categories of algebras over a parametrized monad as parametrized left modules over it.

In section 3. we give the basic definitions and constructions of 2-categories, where we define 2-categories as categories enriched over categories.

So our theory of 2-categories is a special case of the theory of categories enriched over a monoidal category, where we take Lurie’s model of enriched categories found in [1.4.2.1.28]. (but interprete it derived).

Slightly more general we work with categories (lax) enriched over a representable planar operad \( \mathcal{V} \) and families of such.

In section 3. and the appendix we show that many familiar constructions of the theory of enriched categories generalize to our setting.

Especially we endow the object of endomorphisms \([X,X]\) of an object \( X \) in a \( \mathcal{V} \)-enriched category \( \mathcal{M} \) with the structure of an associative algebra in \( \mathcal{V} \) (using Lurie’s theory of endomorphism objects based on enriched strings in
combination with the enveloping LM*-monoidal category) and show that for every \( Y \in M \) the associative algebra \([X, X]\) acts on \([Y, X]\) from the left.

Applying this result to 2-categories, i.e. \( \mathcal{V} := \text{Cat}_\infty(\kappa) \) we can define left actions of a monad on \( X \), i.e. an associative algebra \([X, X]\), on an object \( \psi \) in \([Y, X]\).

If \( \psi \) is endowed with a left-module structure over \( T \) that exhibits \( T \) as the endomorphism object of \( \psi \), we call \( T \) the associated monad to \( \psi \) and call \( \psi \) the Eilenberg-Moore object of \( T \), which we think of as an internalization of the Eilenberg-Moore category of \( T \)-algebras.

We show in proposition 3.6 that every right adjoint morphism of a 2-category admits an associated monad, whereas not every monad of a general 2-category admits an Eilenberg-Moore object. But if the Eilenberg-Moore object exists, it is unique.

In section 4. we study Eilenberg-Moore objects of a monad in a 2-category systematically and show that for every categorical pattern \( P \) on a category \( S \) (def. in [1], Def. B.0.19.) the subcategory of \( \text{Cat}_\infty/\text{sl}\otimes\text{sh.l}\) of \( P \)-fibered objects admits Eilenberg-Moore objects and co-Eilenberg-Moore objects (for every (co)monad) that are preserved by the subcategory inclusion to \( \text{Cat}_\infty/\text{sl}\otimes\text{sh.l} \) (theorem 4.16).

In section 5. we prove the main localization results:

We write \( \text{Alg}([X, X])^{\text{rep}} \) for the full subcategory of \( \text{Alg}([X, X]) \) spanned by the monads that admit an Eilenberg-Moore object and \( (\mathcal{C}/X)^{\text{rep}} \subset (\mathcal{C}/X)^{R} \) for the full subcategory spanned by the morphisms over \( X \), whose associated monad admits an Eilenberg-Moore object.

So \( \text{End} : (\mathcal{C}/X)^{R} \to \text{Alg}([X, X])^{\text{rep}} \) restricts to a functor \( \text{End} : (\mathcal{C}/X)^{\text{rep}} \to (\text{Alg}([X, X])^{\text{rep}})^{\text{op}} \).

We show that \( \text{End} : (\mathcal{C}/X)^{\text{rep}} \to (\text{Alg}([X, X])^{\text{rep}})^{\text{op}} \) admits a fully faithful right adjoint \( \text{Alg} \), whose essential image are the monadic morphisms over \( X \) (theorem 5.1).

Especially every monad admits at most one Eilenberg-Moore object and the functor \( \text{End} \) restricts to an equivalence \( (\mathcal{C}/X)^{\text{mon}} \to (\text{Alg}([X, X])^{\text{rep}})^{\text{op}} \) inverse to the functor \( \text{Alg} \) that associates the Eilenberg-Moore object to a monad if it exists.

Especially the full subcategory \( (\mathcal{C}/X)^{\text{mon}} \subset (\mathcal{C}/X)^{\text{rep}} \) is a localization.

So if all monads on \( X \) admit an Eilenberg-Moore object, we get a localization

\[
\text{End} : (\mathcal{C}/X)^{R} \rightleftarrows \text{Alg}([X, X])^{\text{op}} \rightleftarrows (\mathcal{C}/X)^{\text{mon}} : \text{Alg},
\]

More coherently we construct for every cocartesian \( S \)-family \( \mathcal{C} \to S \) of \((\infty, 2)\)-categories for some \( \infty \)-category \( S \) and every cocartesian section \( X \) of \( \mathcal{C} \to S \) a localization \( \text{End} : (\mathcal{C}^{S}/X)^{\text{rep}} \rightleftarrows (\text{Alg}^{S}([X, X])^{S})^{\text{rev}} \) relative to \( S \) with essential image \( (\mathcal{C}^{S}/X)^{\text{mon}} \) (theorem 5.1) that induces on every fiber the former localization.

Denote \( \text{Cat}_\infty \) the \( \infty \)-category of small \( \infty \)-categories.

We are especially interested in cocartesian \( S \)-families of \((\infty, 2)\)-categories that are given by a subcategory \( \mathcal{C} \) of the pullback \( S \times_{\text{Cat}_\infty} \text{Fun}(\Delta^1, \text{Cat}_\infty) \) of the
evaluation at the target functor $\text{Fun}(\Delta^1, \text{Cat}_\infty) \to \text{Cat}_\infty$ along a functor $S \to \text{Cat}_\infty$, i.e. for $S$ contractible by a subcategory of $\text{Cat}_{\infty/\mathcal{B}}$ for some $\infty$-category $\mathcal{B}$.

In this case we construct another localization

$$\text{End} : (\mathcal{C}/_\mathcal{X})^\text{rep} \rightleftarrows \text{Alg}^{/S^{\text{op}}} ([X, X]^{/S^{\text{rep}}})^{\text{op}}$$

relative to $S$ that induces on every fiber the first localization of theorem 5.4, where we don’t need to assume the section $X$ to be cocartesian.

In this case the localization relative to $S$ is not between cocartesian fibrations over $S$ anymore, which makes it hard to generalize it to arbitrary $S$-families of $(\infty,2)$-categories.

Having this more flexible localization relative to $S$ we are able to show that for every subcategory $\mathcal{C}$ of $\text{Cat}_{\infty/\mathcal{B}}$ for some $\infty$-category $\mathcal{B}$ the full subcategory

$$\text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^\text{rep}$$

is a localization relative to $\mathcal{C}$ (corollary 5.9) by taking the constant $\mathcal{C}$-family with value $\mathcal{C}$ and its diagonal section, where $\text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^\text{rep} \subset \text{Fun}(\Delta^1, \mathcal{C})$ denote the full subcategories spanned by the monadic morphisms respectively those morphism $Y \to X$, whose associated monad on $X$ admits an Eilenberg-Moore object in $\mathcal{C}$ that is preserved by the subcategory inclusion $\mathcal{C}/_X \subset (\text{Cat}_{\infty/\mathcal{B}})/_X$.

From this result applied to the $\infty$-category $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty)$ and theorem 5.1 we deduce that for every $(\infty,2)$-category $\mathcal{C}$ the full subcategory $\text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^\text{rep}$ is a localization relative to $\mathcal{C}$ (theorem 5.3) generalizing corollary 5.9.

To derive the general result from the special result for $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty)$ we use that the functor $\theta : \mathcal{C} \to \text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty)$ that sends an object $X$ of $\mathcal{C}$ to the functor $[-,X] : \mathcal{C}^{\text{op}} \to \text{Cat}_\infty$ induces a right inverse functor $\mathcal{C} \to \theta(\mathcal{C})$ to its essential image (remark 5.5).

Moreover we show in theorem 5.9 that the localization $\text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^\text{rep}$ relative to $\mathcal{C}$ can be enhanced to a localization of $(\infty,2)$-categories relative to $\mathcal{C}$ if $\mathcal{C}$ is cotensored over $\text{Cat}_\infty$.

So if $\mathcal{C}$ is a $(\infty,2)$-category that admits Eilenberg-Moore objects, we obtain a localization $\text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^R$ relative to $\mathcal{C}$ that can be enhanced to a localization of $(\infty,2)$-categories relative to $\mathcal{C}$ if $\mathcal{C}$ is cotensored over $\text{Cat}_\infty$.

If $\mathcal{C}$ is a subcategory of $\text{Cat}_{\infty/\mathcal{B}}$ for some $\infty$-category $\mathcal{B}$ and $X \in \mathcal{C}$, we give a more explicite description of the adjunction $\text{End} : (\mathcal{C}/_X)^\text{rep} \rightleftarrows \text{Alg}([X, X]^{/\text{rep}})^\text{op} : \text{Alg}$.

We show in theorem 5.3 that $\text{Alg}$ is the restriction of the functor

$$\text{Alg}(\text{Fun}_{\mathcal{B}}(X,X))^\text{op} \to ((\text{Cat}_{\infty/\mathcal{B}})/_X)^R \subset \text{Cat}_{\infty/\mathcal{X}}$$

classified by the map $\text{LMod}_{/\mathcal{B}}(X) \to X \times \text{Alg}(\text{Fun}_{\mathcal{B}}(X,X))$ of cartesian fibrations over $\text{Alg}(\text{Fun}_{\mathcal{B}}(X,X))$, where we use the theory of $\infty$-categories of left modules relative to $\mathcal{B}$ developed in section 2.3.

Let $\mathcal{C}$ be a $(\infty,2)$-category and $\psi : Y \to X$ an Eilenberg-Moore object for some monad $T$ on some object $X$ of $\mathcal{C}$.
In section 6, we study when the existence of kan-extensions for a given object \(X\) of a 2-category \(\mathcal{C}\) implies the existence of kan-extensions for the Eilenberg-Moore object of a given monad \(T\) on \(X\) if it exists (proposition 6.3 and corollary 6.6).

Here we work in a purely 2-categorical setting and derive all our results only from the universal property of the Eilenberg-Moore object.

We abstract the following definitions from category theory that make sense in every 2-category:

Let \(\mathcal{C}\) be a 2-category, \(X\) an object of \(\mathcal{C}\) and \(\varphi : A \to B\) a morphism of \(\mathcal{C}\).

Let \(H : A \to X\) and \(H' : B \to X\) be morphisms of \(\mathcal{C}\) and \(\alpha : H \to H' \circ \varphi\) a 2-morphism of \(\mathcal{C}\).

We say that \(\alpha\) exhibits \(H'\) as the left kan-extension of \(H\) along \(\varphi\) and write \(\text{lan}_\varphi(H)\) for \(H'\) if the canonical map \([B, X](H', G) \to [A, X](H' \circ \varphi, G \circ \varphi) \to [A, X](H, G \circ \varphi)\) is an equivalence. Dually we define right kan-extensions.

Let \(\varphi : A \to B\) be a functor.

We say that \(X\) is compatible with left kan-extensions along \(\varphi\) if for every object \(Y\) of \(\mathcal{C}\) the category \([Y, X]\) admits left kan-extensions along \(\varphi\) and for every morphism \(\beta : Z \to Y\) of \(\mathcal{C}\) the functor \([\beta, X] : [Y, X] \to [Z, X]\) preserves left kan-extensions along \(\varphi\).

Dually we say that \(X\) is compatible with right kan-extensions along \(\varphi\) if for every object \(Y\) of \(\mathcal{C}\) the category \([Y, X]\) admits right kan-extensions along \(\varphi\) and for every morphism \(\beta : Z \to Y\) of \(\mathcal{C}\) the functor \([\beta, X] : [Y, X] \to [Z, X]\) preserves right kan-extensions along \(\varphi\).

Let \(X, X'\) be objects of \(\mathcal{C}\) that are compatible with left kan-extensions along \(\varphi : A \to B\).

We say that a morphism \(\theta : X \to X'\) of \(\mathcal{C}\) is compatible with left kan-extensions along \(\varphi\) if for every object \(Y\) of \(\mathcal{C}\) the functor \([Y, \theta] : [Y, X] \to [Y, X']\) preserves left kan-extensions along \(\varphi\) and dually for right kan-extensions.

With these notions we prove the following:

Let \(\varphi : A \to B\) be a morphism of \(\mathcal{C}\).

If \(X\) admits left kan-extensions along \(\varphi\) and \(T : X \to X\) preserves left kan-extensions along \(\varphi\), then \(Y\) admits left kan-extensions along \(\varphi\) that are preserved and reflected by \(\psi : Y \to X\).

If \(X\) admits right kan-extensions along \(\varphi\), then \(Y\) admits right kan-extensions along \(\varphi\) that are preserved and reflected by \(\psi : Y \to X\).

If \(X\) is compatible with left kan-extensions along \(\varphi\) and \(T : X \to X\) is compatible with left kan-extensions along \(\varphi\), then \(Y\) is compatible with left kan-extensions along \(\varphi\).

If \(X\) is compatible with right kan-extensions along \(\varphi\), then \(Y\) is compatible with right kan-extensions along \(\varphi\).

Let \(\mathcal{C}\) be a \(E_k\)-monoidal category for some natural \(k\) and \(T\) a monad on \(\mathcal{C}\) such that \(T\) is a lax \(E_k\)-monoidal functor and the unit and multiplication of \(T\) are lax \(E_k\)-monoidal natural transformations.
From the results about kan-extensions combined with theorem 4.16 and proposition 6.8, we are able to construct a $E_k$-monoidal structure on $\text{LMod}_T(\mathcal{C})$ with the property that not the forgetful functor $\text{LMod}_T(\mathcal{C}) \to \mathcal{C}$ but the free functor $\mathcal{C} \to \text{LMod}_T(\mathcal{C})$ lifts to a $E_k$-monoidal functor provided that $\mathcal{C}$ admits geometric realizations that are preserved by $T$ and the tensor product in each component.

This allows us to construct a $E_k$-monoidal structure on the category $\text{LMod}_A(\mathcal{D})$ of left modules in an $E_{k+1}$-monoidal category $\mathcal{D}$ over an $E_{k+1}$-algebra $A$ provided that $\mathcal{D}$ admits geometric realizations that are preserved by the monad $T := \Lambda \otimes - : \mathcal{D} \to \mathcal{D}$ and the tensor product of the underlying $E_k$-monoidal category in each component.

The $E_{k+1}$-algebra $A$ corresponds to an associative algebra in the category of $E_k$-algebras and thus the functor $T := \Lambda \otimes - : \mathcal{D} \to \mathcal{D}$ does not only define a monad but a monad such that $T$ is a lax $E_k$-monoidal functor and the unit and multiplication of $T$ are lax $E_k$-monoidal natural transformations.

So by our result we obtain a symmetric monoidal structure on $\text{LMod}_A(\mathcal{D}) \simeq \text{LMod}_T(\mathcal{D})$ such that the free functor $\mathcal{D} \to \text{LMod}_A(\mathcal{D})$ is symmetric monoidal, which is the expected symmetric monoidal structure.

Similar results have been proven by Lurie in 4.8.5.20., 5.1.2.6. [1] but with very different methods.
2 Parametrized categories of sections

2.1 Parametrized categories of sections

In the first section we study parametrized versions of categories of sections, from which we define parametrized versions of categories of algebras and left modules.

Those will serve us as a tool to make constructions involving categories of algebras and modules natural or functorial.

A functor $\psi : T \to S$ between $\kappa$-small categories gives rise to an adjunction

$$\psi^* : \text{Cat}_\infty(\kappa)_T \rightleftarrows \text{Cat}_\infty(\kappa)_S : \psi^* = T \times_S \cdot$$

Being a right adjoint functor $T \times_S \cdot : \text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)_T$ preserves finite products and so endows $\text{Cat}_\infty(\kappa)_T$ with a canonical left module structure over $\text{Cat}_\infty(\kappa)_S$.

Let $\phi : C \to T$ be a $\kappa$-small category over $T$.

The functor $\xi : \text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)_C$ is equivalent to the composition

$$\text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)_T \to \text{Cat}_\infty(\kappa)_C$$

Hence $\xi$ admits a right adjoint if (and only if) the functor $C \times_S \cdot : \text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)_C$ admits a right adjoint or equivalently by the adjoint functor theorem applied to the presentable category $\text{Cat}_\infty(\kappa)_S$ if $C \times_S \cdot$ preserves $\kappa$-small colimits.

Remark 2.1. It follows immediately from the definition that flat functors are closed under composition.

Moreover the opposite functor and the pullback of a flat functor $\xi : S \to S'$ along any functor $\alpha : S' \to S$ are flat as we have commutative diagrams

$$\begin{array}{ccc}
\text{Cat}_\infty(\kappa)_S & \xrightarrow{\text{op}} & \text{Cat}_\infty(\kappa)_{\xi S} \\
\downarrow{\text{op}} & & \downarrow{\text{op}} \\
\text{Cat}_\infty(\kappa)_T & \xrightarrow{\text{op}} & \text{Cat}_\infty(\kappa)_{\xi T}
\end{array}$$

and

$$\begin{array}{ccc}
\text{Cat}_\infty(\kappa)_S & \xrightarrow{\alpha_S} & \text{Cat}_\infty(\kappa)_{\xi'} \\
\downarrow{\alpha'_*} & & \downarrow{\alpha'_*} \\
\text{Cat}_\infty(\kappa)_T & \xrightarrow{\alpha'_*} & \text{Cat}_\infty(\kappa)_{\xi T}
\end{array}$$

with $\alpha' : \xi' := S' \times_S \xi \to S'$ the canonical functor, where $\alpha_*, \alpha'_*$ preserve and reflect $\kappa$-small colimits.

Moreover a functor is a cocartesian fibration if and only if it is a locally cocartesian fibration and flat functor and dually a functor is a cartesian fibration if and only if it is a locally cartesian fibration and flat functor.
adjoint to a functor $\text{Cat}_{\infty}(\kappa)/_T \to \text{Cat}_{\infty}(\kappa)/_S$.

Denote $\text{Cat}_{\infty}(\kappa)_{/\text{flat}} \subset \text{Cat}_{\infty}(\kappa)/_T$ the full subcategory spanned by the categories over $T$ that are flat over $S$.

The left action functor $\text{Cat}_{\infty}(\kappa)/_S \times \text{Cat}_{\infty}(\kappa)/_T \to \text{Cat}_{\infty}(\kappa)/_T$ yields a functor

$$(\text{Cat}_{\infty}(\kappa)/_S)^{\text{op}} \times (\text{Cat}_{\infty}(\kappa)/_T)^{\text{op}} \to (\text{Cat}_{\infty}(\kappa)/_T)^{\text{op}} \subset \text{Fun}(\text{Cat}_{\infty}(\kappa)/_T, \mathcal{S}(\kappa))$$

adjoint to a functor $(\text{Cat}_{\infty}(\kappa)/_S)^{\text{op}} \times (\text{Cat}_{\infty}(\kappa)/_T)^{\text{op}} \times \text{Cat}_{\infty}(\kappa)/_T \to \mathcal{S}(\kappa)$ adjoint to a functor $(\text{Cat}_{\infty}(\kappa)/_T)^{\text{op}} \times \text{Cat}_{\infty}(\kappa)/_T \to \text{Fun}(\text{Cat}_{\infty}(\kappa)/_S)^{\text{op}}, \mathcal{S}(\kappa))$ that restricts to a functor $\text{Fun}^{\text{flat}}_T (\cdot, \cdot) : (\text{Cat}_{\infty}(\kappa)/_T)^{\text{op}} \times \text{Cat}_{\infty}(\kappa)/_T \to \text{Cat}_{\infty}(\kappa)/_S$.

So we get a canonical equivalence

$$\text{Cat}_{\infty}(\kappa)/_S(\mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D})) \cong \text{Cat}_{\infty}(\kappa)/_T (\mathcal{B} \times_\mathcal{S} \mathcal{C}, \mathcal{D})$$

natural in $\kappa$-small categories $\mathcal{C}, \mathcal{D}$ over $T$ with $\mathcal{C}$ flat over $\mathcal{S}$ and a $\kappa$-small category $\mathcal{B}$ over $\mathcal{S}$.

**Remark 2.2.**

1. We have a canonical equivalence

$$\text{Fun}_\mathcal{S}(\mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D})) \cong \text{Fun}_T (\mathcal{B} \times_\mathcal{S} \mathcal{C}, \mathcal{D})$$

natural in $\kappa$-small categories $\mathcal{C}, \mathcal{D}$ over $T$ with $\mathcal{C}$ flat over $\mathcal{S}$ and a $\kappa$-small category $\mathcal{B}$ over $\mathcal{S}$ represented by the canonical equivalence

$$\text{Cat}_{\infty}(\kappa)(\mathcal{K}, \text{Fun}_\mathcal{S}(\mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D}))) \cong \text{Cat}_{\infty}(\kappa)/_S(\mathcal{K} \times \mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D})) \cong \text{Cat}_{\infty}(\kappa)/_T(\mathcal{K} \times (\mathcal{B} \times_\mathcal{S} \mathcal{C}), \mathcal{D}) \cong \text{Cat}_{\infty}(\kappa)(\mathcal{K}, \text{Fun}_T (\mathcal{B} \times_\mathcal{S} \mathcal{C}, \mathcal{D}))$$

natural in $\kappa$-small categories $\mathcal{C}, \mathcal{D}$ over $T$ with $\mathcal{C}$ flat over $\mathcal{S}$, a $\kappa$-small category $\mathcal{B}$ over $\mathcal{S}$ and a $\kappa$-small category $\mathcal{K}$.

2. Let $T \to \mathcal{S}, \mathcal{S} \to \mathcal{R}, \mathcal{B} \to \mathcal{S}, \mathcal{C} \to \mathcal{T}, \mathcal{D} \to \mathcal{T}$ be functors.

Generalizing 1. we have a canonical equivalence

$$\text{Fun}^\mathcal{R}_\mathcal{S}(\mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D})) \cong \text{Fun}^\mathcal{R}_T (\mathcal{B} \times_\mathcal{S} \mathcal{C}, \mathcal{D})$$

over $\mathcal{R}$ represented by the canonical equivalence

$$\text{Fun}_\mathcal{S}(\mathcal{K}, \text{Fun}^\mathcal{R}_\mathcal{S}(\mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D}))) \cong \text{Fun}_\mathcal{S}(\mathcal{K} \times_\mathcal{R} \mathcal{B}, \text{Fun}^{\text{flat}}_T (\mathcal{C}, \mathcal{D})) \cong \text{Fun}_T (\mathcal{K} \times_\mathcal{R} \mathcal{B} \times_\mathcal{S} \mathcal{C}, \mathcal{D}) \cong \text{Fun}_T (\mathcal{K}, \text{Fun}^\mathcal{R}_T (\mathcal{B} \times_\mathcal{S} \mathcal{C}, \mathcal{D}))$$

natural in a $\kappa$-small category $\mathcal{K}$ over $\mathcal{R}$.
3. We have a canonical equivalence

$$\text{Fun}^S_T(\mathcal{E}, \mathcal{D})^{\text{op}} \simeq \text{Fun}^{S^\text{op}}_{T^{\text{op}}}(\mathcal{E}^{\text{op}}, \mathcal{D}^{\text{op}})$$

over $S^\text{op}$ natural in $\kappa$-small categories $\mathcal{E}, \mathcal{D}$ over $T$ with $\mathcal{E}$ flat over $S$ represented by the canonical equivalence

$$\text{Cat}_\infty(\kappa)_{T^{\text{op}}}(\mathcal{B}, \text{Fun}^S_T(\mathcal{E}, \mathcal{D})^{\text{op}}) \simeq \text{Cat}_\infty(\kappa)_{T^{\text{op}}}(\mathcal{B}^{\text{op}}, \text{Fun}^S_T(\mathcal{E}, \mathcal{D})) \simeq$$

$$\text{Cat}_\infty(\kappa)_{T^{\text{op}}}(\mathcal{B}^{\text{op}} \times_S \mathcal{E}, \mathcal{D}) \simeq \text{Cat}_\infty(\kappa)_{T^{\text{op}}}(\mathcal{B} \times_S \mathcal{E}^{\text{op}}, \mathcal{D}^{\text{op}}) \simeq$$

$$\text{Cat}_\infty(\kappa)_{S^{\text{op}}}(\mathcal{B}, \text{Fun}^{S^\text{op}}_{T^{\text{op}}}(\mathcal{E}^{\text{op}}, \mathcal{D}^{\text{op}}))$$

natural in $\kappa$-small categories $\mathcal{E}, \mathcal{D}$ over $T$ with $\mathcal{E}$ flat over $S$ and a $\kappa$-small category $\mathcal{B}$ over $S^\text{op}$.

4. Let $S' \rightarrow S$ be a functor of $\kappa$-small categories. Set $T' := S' \times_S T$.

There is a canonical equivalence

$$S' \times_S \text{Fun}^S_T(\mathcal{E}, \mathcal{D}) \simeq \text{Fun}^{S'}_{T'}(S' \times_S \mathcal{E}, S' \times_S \mathcal{D})$$

of categories over $S'$ represented by the canonical equivalence

$$\text{Fun}_S(K, S' \times_S \text{Fun}^S_T(\mathcal{E}, \mathcal{D})) \simeq \text{Fun}_S(K, \text{Fun}^{S'}_{T'}(\mathcal{E}, \mathcal{D})) \simeq \text{Fun}_T(\mathcal{E} \times_S K, \mathcal{D})$$

$$\simeq \text{Fun}_T(\mathcal{E} \times_S K, T' \times_T \mathcal{D}) = \text{Fun}_T((S' \times_S \mathcal{E}) \times_S S' \times_S \mathcal{D})$$

natural in a $\kappa$-small category $K$ over $S'$.

Especially for every object $s$ of $S$ we have a canonical equivalence

$$\text{Fun}^S_T(\mathcal{E}, \mathcal{D})_s \simeq \text{Fun}_{T_s}(\mathcal{E}_s, \mathcal{D}_s).$$

So if $S$ is contractible, we have $\text{Fun}^S_T(\mathcal{E}, \mathcal{D}) \simeq \text{Fun}_T(\mathcal{E}, \mathcal{D})$.

5. If $\psi : T \rightarrow S$ is the identity, we write $\text{Maps}_S(\mathcal{E}, \mathcal{D})$ for $\text{Fun}^S_T(\mathcal{E}, \mathcal{D})$ and have a canonical equivalence

$$S' \times_S \text{Maps}_S(\mathcal{E}, \mathcal{D}) \simeq \text{Maps}_{S'}(S' \times_S \mathcal{E}, S' \times_S \mathcal{D})$$

of categories over $S'$ and for every object $s$ of $S$ a canonical equivalence

$$\text{Maps}_S(\mathcal{E}, \mathcal{D})_s \simeq \text{Fun}(\mathcal{E}_s, \mathcal{D}_s).$$

6. We have a canonical equivalence

$$\text{Fun}^S_T(\mathcal{E}, \mathcal{D}) \simeq S \times_{\text{Maps}_S(\mathcal{E}, T)} \text{Maps}_S(\mathcal{E}, \mathcal{D})$$

over $S$ represented by the canonical equivalence

$$\text{Fun}_S(K, \text{Fun}^S_T(\mathcal{E}, \mathcal{D})) \simeq \text{Fun}_T(K \times_S \mathcal{E}, \mathcal{D}) \simeq$$
Remark 2.3. Let \( \kappa \) be a strongly inaccessible cardinal, \( R,S,T \) be \( \kappa \)-small categories and \( T \to S \), \( R \to S \) and \( X \to T \times S \) \( R \) be functors.

Let \( \mathcal{B} \) be a \( \kappa \)-small category over \( T \) and \( \mathcal{D} \) be a \( \kappa \)-small category over \( R \).
1. Assume that the functors $\mathcal{B} \to S$ and $\mathcal{D} \to R$ are flat.

There is a canonical equivalence

$$\text{Map}_R(\mathcal{D}, \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, X)) \simeq \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, \text{Map}_{R \times S T}(T \times_S \mathcal{D}, X))$$

of categories over $R$ represented by the following canonical equivalence natural in a $\kappa$-small category $K$ over $R$, where we set $Z := \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, X)$ and $W := \text{Map}_{R \times S T}(T \times_S \mathcal{D}, X)$:

$$\text{Fun}_R(K, \text{Map}_R(\mathcal{D}, Z)) \simeq \text{Fun}_R((K \times_R \mathcal{D}) \times_R (R \times_S \mathcal{B}), X) \simeq \text{Fun}_{R \times S T}((K \times_R (R \times_S \mathcal{B})) \times_R \mathcal{D}, X) \simeq \text{Fun}_{R \times S T}((K \times_S \mathcal{B}) \times_{(T \times_S R)} (T \times_S \mathcal{D}), X) \simeq \text{Fun}_{R \times S T}(K \times_S \mathcal{B}, W) \simeq \text{Fun}_R(K, \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, W)).$$

2. Assume that the functors $\mathcal{B} \to S$ and $\mathcal{D} \to S$ are flat.

There is a canonical equivalence

$$\text{Fun}^S_T(\mathcal{B}, \text{Fun}^T_{T \times S R}(T \times_S \mathcal{D}, X)) \simeq \text{Fun}^S_R(\mathcal{D}, \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, X))$$

over $S$.

We have a canonical equivalence natural in a $\kappa$-small category $L$ over $S$:

$$\text{Fun}_S(L, \text{Fun}^S_T(\mathcal{B}, \text{Fun}^T_{T \times S R}(T \times_S \mathcal{D}, X))) \simeq \text{Fun}_T(L \times_S \mathcal{B}, \text{Fun}^T_{T \times S R}(T \times_S \mathcal{D}, X)) \simeq \text{Fun}_{T \times S R}((L \times_S \mathcal{B}) \times_T (T \times_S \mathcal{D}), X)$$

Changing the roles of $R$ and $T$ and $\mathcal{D}$ and $\mathcal{B}$ we get a canonical equivalence natural in a $\kappa$-small category $L$ over $S$:

$$\text{Fun}_S(L, \text{Fun}^S_R(\mathcal{D}, \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, X))) \simeq \text{Fun}_{R \times S T}((L \times_S \mathcal{D}) \times_S \mathcal{B}, X).$$

So we get a canonical equivalence

$$\text{Fun}_S(L, \text{Fun}^S_T(\mathcal{B}, \text{Fun}^T_{T \times S R}(T \times_S \mathcal{D}, X))) \simeq \text{Fun}_{T \times S R}((L \times_S \mathcal{B}) \times_S \mathcal{D}, X)$$

$$\text{Fun}_{R \times S T}((L \times_S \mathcal{D}) \times_S \mathcal{B}, X) \simeq \text{Fun}_S(L, \text{Fun}^S_R(\mathcal{D}, \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, X)))$$

natural in a $\kappa$-small category $L$ over $S$ that represents a canonical equivalence

$$\text{Fun}^S_T(\mathcal{B}, \text{Fun}^T_{T \times S R}(T \times_S \mathcal{D}, X)) \simeq \text{Fun}^S_R(\mathcal{D}, \text{Fun}^R_{R \times S T}(R \times_S \mathcal{B}, X))$$

over $S$. 

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3. Set \( Y := \text{Fun}^T_{T \times S R}(T \times S R, X) \).

For \( B \to T \) and \( D \to R \) the identities the canonical equivalence

\[
\text{Fun}^S_T(T, \text{Fun}^T_{T \times S R}(T \times S R, X)) \cong \text{Fun}^S_R(R, \text{Fun}^R_{R \times S T}(R \times S T, X))
\]

over \( S \) of 2. is adjoint to the functor

\[
R \times S \text{Fun}^S_T(T, Y) \cong \text{Fun}^R_{R \times S T}(R \times S T, R \times S Y) \to \text{Fun}^R_{R \times S T}(R \times S T, X)
\]

over \( R \) induced by the functor

\[
R \times S Y \cong \text{Fun}^T_{T \times S R}(T \times S R, X) \times_T (T \times S R) \to X
\]

over \( T \times S R \).

**Remark 2.4.** Let \( T \to S \) and \( \phi : C \to T \) be functors such that the composition \( C \to T \to S \) is flat and let \( \varphi : D \to \mathcal{E} \) a functor over \( T \).

If the functor \( \varphi : D \to \mathcal{E} \) is a subcategory inclusion, then the induced functor \( \text{Fun}^S_T(C, D) \to \text{Fun}^S_T(\mathcal{C}, \mathcal{E}) \) also is.

If the functor \( \varphi : D \to \mathcal{E} \) is fully faithful, then the induced functor \( \text{Fun}^S_T(C, D) \to \text{Fun}^S_T(\mathcal{C}, \mathcal{E}) \) also is.

Being right adjoint to the functor \( \text{Cat}_\infty(\kappa)_S \xrightarrow{\kappa \times_S -} \text{Cat}_\infty(\kappa)_T \xrightarrow{\phi} \text{Cat}_\infty(\kappa)_T \), the functor \( \text{Fun}^S_T(\mathcal{C}, -) : \text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)_T \) preserves pullbacks and so monomorphisms.

The forgetful functor \( \text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)_T \) preserves and reflects pullbacks and so preserves and reflects monomorphisms, where the morphisms in \( \text{Cat}_\infty(\kappa)_T \) are by definition the subcategory inclusions.

Assume that \( \varphi : D \to \mathcal{E} \) is fully faithful and so a subcategory inclusion.

Then the induced functor \( \text{Fun}^S_T(\mathcal{C}, D) \to \text{Fun}^S_T(\mathcal{C}, \mathcal{E}) \) is a subcategory inclusion.

Let \( \alpha : \Delta^1 \to \text{Fun}^S_T(\mathcal{C}, \mathcal{E}) \) be a morphism of \( \text{Fun}^S_T(\mathcal{C}, \mathcal{E}) \), whose source and target belong to \( \text{Fun}^S_T(\mathcal{C}, D) \subset \text{Fun}^S_T(\mathcal{C}, \mathcal{E}) \) and that lies over a morphism \( f : s \to t \) of \( S \).

\( \alpha \) corresponds to a functor \( F : \Delta^1 \times_S \mathcal{E} \to \Delta^1 \times_S \mathcal{E} \) over \( \Delta^1 \times_S T \) such that the induced functors \( F_1 : \mathcal{E}_s \to \mathcal{E}_s \) over \( T_s \) and \( F_2 : \mathcal{E}_t \to \mathcal{E}_t \) over \( T_t \) factor through \( D_s \) respectively \( D_t \).

As \( D \) is a full subcategory of \( \mathcal{E} \), the functor \( F : \Delta^1 \times_S \mathcal{E} \to \Delta^1 \times_S \mathcal{E} \) over \( \Delta^1 \times_S T \) induces a functor \( \Delta^1 \times_S \mathcal{E} \to \Delta^1 \times_S \mathcal{D} \) over \( \Delta^1 \times_S T \) corresponding to a morphism \( \Delta^1 \to \text{Fun}^S_T(\mathcal{C}, D) \) of \( \text{Fun}^S_T(\mathcal{C}, D) \) that is sent to \( \alpha \).

Let \( S \) be a category and \( \mathcal{E}_S \subset \text{Fun}(\Delta^1, S), \mathcal{W}_S \subset \text{Fun}(\Delta^2, S) \) full subcategories.

We call the triple \((S, \mathcal{E}_S, \mathcal{W}_S)\) a categorical pre-pattern on \( S \) if \( \mathcal{E}_S \) contains all equivalences of \( S \) and \( \mathcal{W}_S \) contains all functors \( \Delta^2 \to S \) that factor through \( \Delta^1 \).
We call a categorical pre-pattern \((S, \mathcal{E}_S, W_S)\) good if the morphisms of \(S\) corresponding to objects of \(\mathcal{E}_S\) are closed under composition, in other words if \(\mathcal{E}_S\) defines a subcategory of \(S\).

We call a functor \(\mathcal{C} \to S\) fibered with respect to the categorical pre-pattern \((S, \mathcal{E}_S, W_S)\) if the pullback \(\mathcal{E}_S \times_S \mathcal{C} \to \mathcal{E}_S\) is a locally cocartesian fibration and for every functor \(\Delta^2 \to S\) that belongs to \(W_S\) a morphism of \(\Delta^2 \times_S \mathcal{C}\) lying over \(0 \to 1\) is cocartesian with respect to the functor \(\psi : \Delta^2 \times_S \mathcal{C} \to \Delta^2\) if its image in \(S\) belongs to \(\mathcal{E}_S\).

Theorem B.4.2 implies the following properties:

Remark 2.5.

Let \(\alpha : T \to S, \mathcal{C} \to T, \gamma : D \to T\) be functors such that the composition \(\phi : \mathcal{C} \to T \to S\) is flat. Let \((S, \mathcal{E}_S, W_S), (T, \mathcal{E}_T, W_T)\) be a good categorical pre-pattern.

Assume that the pullback \(\mathcal{E}_S \times_S \mathcal{C} \to \mathcal{E}_S\) is a cartesian fibration, whose cartesian morphisms are sent to morphisms of \(\mathcal{E}_T\) by the functor \(\mathcal{C} \to T\).

Assume that every functor \(\Delta^2 \to \mathcal{C}\) that lies over an object of \(W_S\) is sent to \(W_T\).

If \(D \to T\) is fibered with respect to the categorical pattern \((T, \mathcal{E}_T, W_T)\), then \(\psi : \text{Fun}^{|S|}_T(\mathcal{C}, D) \to S\) is fibered with respect to the categorical pattern \((S, \mathcal{E}_S, W_S)\).

A morphism of \(\text{Fun}^{|S|}_T(\mathcal{C}, D)\) lying over a morphism of \(\mathcal{E}_S\) is locally \(\phi\)-cocartesian if and only if the corresponding functor \(\Delta^1 \times_S \mathcal{C} \to \Delta^1 \times_S D\) over \(\Delta^1 \times_S T\) sends locally \(\phi\)-cartesian morphisms to morphisms that are locally cocartesian with respect to the functor \(\mathcal{D} \to T\).

Especially if \(T \to S\) is the identity, \(\psi : \text{Map}_S(\mathcal{C}, D) \to S\) is fibered with respect to the categorical pattern \((S, \mathcal{E}_S, W_S)\) if the pullback \(\mathcal{E}_S \times_S \mathcal{C} \to \mathcal{E}_S\) is a cartesian fibration and \(D \to T\) fibered with respect to the categorical pattern \((S, \mathcal{E}_S, W_S)\).

In the following we will consider the most important cases of the last statement:

1. Let \(W_S = \text{Fun}(\Delta^2, S), W_T = \text{Fun}(\Delta^2, T)\):
   - If the functor \(\mathcal{C} \to S\) is a cartesian fibration relative to \(\mathcal{E}_S\), whose cartesian morphisms are sent to morphisms of \(\mathcal{E}_T\) by the functor \(\mathcal{C} \to T\) and \(D \to T\) is a cocartesian fibration relative to \(\mathcal{E}_T\), then the functor \(\text{Fun}^{|S|}_T(\mathcal{C}, D) \to S\) is a cartesian fibration relative to \(\mathcal{E}_S\).
   - Especially if \(\mathcal{E}_S = S\) and \(\mathcal{E}_T = T\): If the functor \(\mathcal{C} \to S\) is a cartesian fibration and \(D \to T\) is a cocartesian fibration, then the functor \(\text{Fun}^{|S|}_T(\mathcal{C}, D) \to S\) is a cocartesian fibration.

2. By Remark 3, for every functor \(\Delta^1 \to S\) we have a canonical equivalence \(\Delta^1 \times_S \text{Fun}^{|S|}_T(\mathcal{C}, D) \simeq \text{Fun}^{|\Delta^1 \times_S T|}(\Delta^1 \times_S \mathcal{C}, \Delta^1 \times_S D)\) over \(\Delta^1\).

   Consequently if the functor \(\mathcal{E}_S \times_S \mathcal{C} \to \mathcal{E}_S\) is a cartesian fibration and for every functor \(\Delta^1 \to S\) corresponding to a morphism of \(\mathcal{E}_S\) the pullback \(\Delta^1 \times_S D \to \Delta^1 \times_S T\) is a cocartesian fibration, then the pullback \(\mathcal{E}_S \times_S \text{Fun}^{|S|}_T(\mathcal{C}, D) \to \mathcal{E}_S\) is a locally cocartesian fibration.
2. can also deduced directly from the choices $\mathcal{E}_T = \mathcal{E}_S \times_S T \subset T$, $\mathcal{W}_S \subset \text{Fun}(\Delta^2, S)$ the full subcategory spanned by the functors $\Delta^2 \to S$ that factor through $\Delta^1$ and $\mathcal{W}_T \subset \text{Fun}(\Delta^2, T)$ the full subcategory spanned by the functors $\Delta^2 \to T$ that lie over objects of $\mathcal{W}_S$, i.e. that factor through $\Delta^1 \times_S T$ for some functor $\Delta^1 \to S$.

3. Assume that $\alpha : T \to S$ is a cocartesian fibration.

Let $\mathcal{E}_S = S, \mathcal{E}_T \subset T$ the subcategory with the same objects and with morphisms the $\alpha$-cocartesian morphisms. Let $\mathcal{W}_S = \text{Fun}(\Delta^2, S)$ and $\mathcal{W}_T = \text{Fun}(\Delta^2, T)$.

If the functor $\mathcal{E} \to S$ is a cartesian fibration, whose cartesian morphisms are sent to $\alpha$-cocartesian morphisms by the functor $\mathcal{E} \to T$ and $\mathcal{D} \to T$ is a map of cocartesian fibrations over $S$, then $\psi : \text{Fun}_T^S (\mathcal{E}, \mathcal{D}) \to S$ is a cartesian fibration.

By [22], we have a canonical equivalence $\mathcal{E}_S \times_S \text{Fun}_T^S (\mathcal{E}, \mathcal{D}) \cong \text{Fun}_{\mathcal{E}_S \times_S T} (\mathcal{E}_S \times_S \mathcal{E}, \mathcal{E}_S \times_S \mathcal{D})$ over $\mathcal{E}_S$.

So pulling back to $\mathcal{E}_S$ we get the following statement:

If the functor $\mathcal{E} \to S$ is a cartesian fibration relative to $\mathcal{E}_S$, whose cartesian morphisms lying over morphisms of $\mathcal{E}_S$ are sent to $\alpha$-cocartesian morphisms by the functor $\mathcal{E} \to T$ and $\mathcal{D} \to T$ is a map of cocartesian fibrations relative to $\mathcal{E}_S$, then $\psi : \text{Fun}_T^S (\mathcal{E}, \mathcal{D}) \to S$ is a cocartesian fibration relative to $\mathcal{E}_S$.

3. can also be deduced from the choices $\mathcal{E}_T \subset \mathcal{E}_S \times_S T$ the subcategory with the same objects and with morphisms the $\alpha$-cocartesian morphisms, $\mathcal{W}_S = \text{Fun}(\Delta^2, S)$ and $\mathcal{W}_T \subset \text{Fun}(\Delta^2, T)$ the full subcategory spanned by the functors $\Delta^2 \to T$ such that the composition $\Delta^2 \to T \to S$ factors through $\mathcal{E}_S$.

4. Let $\theta : S \to R$ be a cocartesian fibration.

Let $\mathcal{E}_S \subset S$ be the subcategory with the same objects and with morphisms the $\theta$-cocartesian morphisms and $\mathcal{E}_T \subset T$ the subcategory with the same objects and with morphisms the morphisms that are cocartesian with respect to the composition $\alpha : T \to S \to R$.

Assume that the pullback $\Delta^1 \times_S \mathcal{E} \to \Delta^1$ along every $\theta$-cocartesian morphism is a cartesian fibration, whose cartesian morphisms are sent to $\alpha$-cocartesian morphisms by the functor $\mathcal{E} \to T$.

If the functor $\mathcal{D} \to T$ is a map of cocartesian fibrations over $R$, then the functor $\text{Fun}_T^S (\mathcal{E}, \mathcal{D}) \to S$ is a map of cocartesian fibrations over $R$.

5. Let $\mathcal{E}_S = S$ and $\mathcal{E}_T \subset T$ the subcategory with the same objects and with morphisms the morphisms that are cocartesian with respect to the functor $\alpha : T \to S$. Let $\mathcal{W}_S = \text{Fun}(\Delta^2, S), W_T = \text{Fun}(\Delta^2, T)$.

Assume that $\mathcal{E} \to S$ is a cartesian fibration, whose cartesian morphisms are sent to $\alpha$-cocartesian morphisms.
Assume that the pullback $\Delta^1 \times_T D \to \Delta^1$ along every $\alpha$-cartesian morphism is a cocartesian fibration and that every locally $\gamma$-cocartesian morphism lying over a $\alpha$-cartesian morphism is $\gamma$-cocartesian.

Then the functor $\text{Fun}^{\mathcal{E}}_T(\mathcal{C}, D) \to S$ is a cocartesian fibration.

Let $\mathcal{E} \to T, \gamma : D \to T$ be maps of cartesian fibrations over $S$.
Assume that for every morphism $f : s \to t$ of $S$ the induced functor $\mathcal{E}_s \to T_s \times_{T_t} \mathcal{C}_t$ is an equivalence.

By lemma 7.41 the pullback $\Delta^1 \times_T D \to \Delta^1$ along every $\alpha$-cartesian morphism is a cocartesian fibration and every locally $\gamma$-cocartesian morphism lying over a $\alpha$-cartesian morphism is $\gamma$-cocartesian.

So the functor $\text{Fun}^{\mathcal{E}}(\mathcal{C}, D) \to S$ is a cocartesian fibration.

6. Let $\alpha : X \to S$ be a cartesian fibration and $\beta : Y \to S$ a locally cocartesian fibration.

(a) Then $\gamma : \text{Map}_S(X, Y) \to S$ is a locally cocartesian fibration, where a morphism is locally $\gamma$-cocartesian if the corresponding functor $\Delta^1 \times_S X \to \Delta^1 \times_S Y$ over $\Delta^1$ sends locally $\alpha$-cartesian morphisms to locally $\beta$-cocartesian morphisms.

(b) Let $\beta' : T \to S$ be a locally cocartesian fibration, $F : Y \to T$ a map of locally cocartesian fibrations over $S$ and $G : X \to T$ a functor over $S$ that sends locally $\alpha$-cartesian morphisms to locally $\beta'$-cocartesian morphisms.

Then the induced functors $\text{Map}_S(X, Y) \to \text{Map}_S(X, T)$ and $S \to \text{Map}_S(X, T)$ are maps of locally cocartesian fibrations over $S$ and so the pullback

$$\psi : \text{Fun}^{\mathcal{E}}_T(X, Y) \simeq S \times_{\text{Map}_S(X, T)} \text{Map}_S(X, Y) \to S$$

is a locally cocartesian fibration, where a morphism is locally $\psi$-cocartesian if the corresponding functor $\Delta^1 \times_S X \to \Delta^1 \times_S Y$ over $\Delta^1 \times_T S$ sends locally $\alpha$-cartesian morphisms to locally $\beta$-cocartesian morphisms.

7. Let $\mathcal{E} \subset S$ be a subcategory. Let $\alpha : X \to S$ be a flat functor such that the pullback $\mathcal{E} \times_S X \to \mathcal{E}$ is a cartesian fibration and $\beta : Y \to S, \beta' : T \to S$ functors such that the pullbacks $\mathcal{E} \times_S Y \to \mathcal{E}, \mathcal{E} \times_S T \to \mathcal{E}$ are locally cocartesian fibrations.

Let $F : Y \to T$ be a functor over $S$ that sends locally $\beta$-cocartesian morphisms lying over morphisms of $\mathcal{E}$ to locally $\beta'$-cocartesian morphisms and $G : X \to T$ a functor over $S$ that sends locally $\alpha$-cartesian morphisms lying over morphisms of $\mathcal{E}$ to locally $\beta'$-cocartesian morphisms.

(a) By 1. $\mathcal{E} \times_S \psi : \mathcal{E} \times_S \text{Fun}^{\mathcal{E}}(X, Y) \simeq \text{Fun}^{\mathcal{E}}_{\mathcal{E} \times_S T}(\mathcal{E} \times_S X, \mathcal{E} \times_S Y) \to \mathcal{E}$ and $\mathcal{E} \times_S \gamma : \text{Map}_\mathcal{E}(\mathcal{E} \times_S X, \mathcal{E} \times_S Y) \simeq \mathcal{E} \times_S \text{Map}_S(X, Y) \to \mathcal{E}$
are locally cocartesian fibrations, where a morphism is locally $\psi$-cocartesian respectively locally $\gamma$-cocartesian if the corresponding functor $\Delta^1 \times_S X \rightarrow \Delta^1 \times_S Y$ over $\Delta^1 \times_S T$ respectively over $\Delta^1$ sends locally $\alpha$-cartesian morphisms to locally $\beta$-cocartesian morphisms.

(b) By 2.5 we have the following:

If every locally $\beta$-cocartesian morphism lying over a morphism of $E$ is $\beta$-cocartesian, then every locally $\gamma$-cocartesian morphism lying over a morphism of $E$ is $\gamma$-cocartesian.

Consequently if every locally $\beta$-cocartesian morphism lying over a morphism of $E$ is $\beta$-cocartesian and every locally $\beta'$-cocartesian morphism lying over a morphism of $E$ is $\beta'$-cocartesian, then every locally $\psi$-cocartesian morphism lying over a morphism of $E$ is $\psi$-cocartesian.

(c) Especially for $E = S$ we see:

The functor $\gamma : \text{Map}_S(X, Y) \rightarrow S$ is a cocartesian fibration if $\alpha : X \rightarrow S$ is a cartesian fibration and $\beta : Y \rightarrow S$ is a cocartesian fibration.

The functor $\psi : \text{Fun}^S_{T}(X, Y) \rightarrow S$ is a cocartesian fibration if $\alpha : X \rightarrow S$ is a cartesian fibration and $\beta : Y \rightarrow S, \beta' : Y' \rightarrow S$ are cocartesian fibrations.

(d) Let $\theta : S \rightarrow R$ be a cocartesian fibration.

For $E \subset S$ the subcategory with the same objects and with morphisms the $\theta$-cocartesian morphisms we get the following:

Let $\alpha : X \rightarrow S$ be a flat functor such that the pullback $\Delta^1 \times_S X \rightarrow \Delta^1$ along every $\theta$-cocartesian morphism of $S$ is a cartesian fibration.

Let $\beta : Y \rightarrow S, \gamma : T \rightarrow S$ be maps of cartesian fibrations over $R$ and $F : X \rightarrow T, G : Y \rightarrow T$ functors over $S$ such that $F$ sends locally $\alpha$-cartesian morphisms lying over $\theta$-cocartesian morphisms to $\gamma$-cocartesian morphisms and $G$ is a map of cocartesian fibrations over $R$.

Then the functors $\psi : \text{Fun}^S_{T}(X, Y) \rightarrow S$ and $\gamma : \text{Map}_S(X, Y) \rightarrow S$ are maps of cocartesian fibrations over $R$.

8. Let $\alpha : X \rightarrow S, \beta : T \rightarrow S$ be cartesian fibrations and $\rho : X \rightarrow T$ a map of cartesian fibrations over $S$.

Let $\gamma : Y \rightarrow T$ be a functor such that the pullback $\Delta^1 \times_T Y \rightarrow \Delta^1$ along every $\beta$-cartesian morphism is a cocartesian fibration.

Assume that every locally $\gamma$-cocartesian morphism lying over a $\beta$-cartesian morphism is $\gamma$-cocartesian.

The functor $\psi : \text{Fun}^S_{T}(X, Y) \rightarrow S$ is a cocartesian fibration.

9. Let $E \subset S$ be a subcategory. Pulling back along $E$ we get the following:
(a) Let \( \alpha : X \to S \) be a flat functor and \( \beta : T \to S \) a functor such that the pullbacks \( E \times_S X \to E \) and \( E \times_S T \to E \) are cartesian fibrations and let \( \rho : X \to T \) be a functor over \( S \) that sends locally \( \alpha \)-cartesian morphisms lying over morphisms of \( E \) to locally \( \beta \)-cartesian morphisms.

Let \( \gamma : Y \to T \) be a functor such that the pullback \( \Delta^1 \times_T Y \to \Delta^1 \) along every locally \( \beta \)-cartesian morphism lying over a morphism of \( E \) is a cocartesian fibration.

Assume that every locally \( \gamma \)-cocartesian morphism lying over a \( \beta \)-cartesian morphism that lies over a morphism of \( E \) is \( \gamma \)-cocartesian. The functor \( \psi : \text{Fun}^S_{\mathcal{E}}(X, Y) \to S \) is a map of cocartesian fibrations over \( R \).

(b) By 2.5 we have the following:

If every locally \( \alpha \)-cartesian morphism lying over a morphism of \( E \) is \( \alpha \)-cartesian and every locally \( \beta \)-cartesian morphism lying over a morphism of \( E \) is \( \beta \)-cartesian and every locally \( \gamma \)-cocartesian morphism lying over a \( \beta \)-cartesian morphism that lies over a morphism of \( E \) is \( \gamma \)-cocartesian, then every locally \( \psi \)-cocartesian morphism lying over a morphism of \( E \) is \( \psi \)-cocartesian.

(c) Let \( \theta : S \to R \) be a cartesian fibration.

For \( \mathcal{E} \subset S \) the subcategory with the same objects and with morphisms the \( \theta \)-cartesian morphisms we get the following:

Let \( \alpha : X \to S, \beta : T \to S \) and \( \rho : X \to T \) be maps of cartesian fibrations over \( R \). Assume that \( \alpha \) is a flat functor.

Let \( \gamma : Y \to T \) be a functor such that the pullback \( \Delta^1 \times_T Y \to \Delta^1 \) along every \( \theta \circ \beta \)-cartesian morphism is a cocartesian fibration.

Assume that every locally \( \gamma \)-cocartesian morphism lying over a \( \theta \circ \beta \)-cartesian morphism is \( \gamma \)-cocartesian. The functor \( \psi : \text{Fun}^S_{\mathcal{E}}(X, Y) \to S \) is a map of cocartesian fibrations over \( R \).
2.2 Parametrized categories of algebras

Based on parametrized categories of sections we define parametrized categories of algebras in the evident way:

Let $S$ be a $\kappa$-small category, $O^\otimes \to C^\otimes$ and $C^\otimes \to O^\otimes$ be maps of $\kappa$-small $S$-families of operads such that the functor $O^\otimes \to S$ is flat.

We define $\operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})$ to be the full subcategory of $\operatorname{Fun}^S_{O^\otimes/C^\otimes}(O^\otimes, C^\otimes)$ spanned by the functors $O^\otimes_s \to C^\otimes_s$ over $O^\otimes_s$ that preserve inert morphisms for some $s \in S$.

So for every $s \in S$ the canonical equivalence $\operatorname{Fun}^S_{O^\otimes}(O^\otimes, C^\otimes)$ restricts to an equivalence $\operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})_s \simeq \operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})_s$.

More generally given a functor $S' \to S$ the canonical equivalence

$$S' \times_S \operatorname{Fun}^S_{O^\otimes/C^\otimes}(O^\otimes, C^\otimes) \simeq \operatorname{Fun}^S_{S' \times_S O^\otimes}(S' \times_S O^\otimes, S' \times_S C^\otimes)$$

over $S'$ of remark 2.6 restricts to an equivalence

$$S' \times_S \operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E}) \simeq \operatorname{Alg}^S_{S' \times_S O^\otimes/S' \times_S O^\otimes}(S' \times_S \mathcal{E})$$

over $S'$.

For every section $S \to O^\otimes$ of the functor $O^\otimes \to S$ lying over some section $\alpha : S \to O^\otimes$ of the functor $O^\otimes \to S$ we have a forgetful functor

$$\operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})_s \simeq \operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})_s \simeq \alpha(s) \times \mathcal{E}_s,$$

over $S$, which induces on the fiber over every $s \in S$ the forgetful functor

$$\operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})_s \to \mathcal{E}_s.$$  

Given a map of operads $O^\otimes \to C^\otimes$ we often write $\operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})$ for $\operatorname{Alg}^S_{O^\otimes/S \times C^\otimes}(\mathcal{E})$ and $\operatorname{Alg}^S(\mathcal{E})$ for $\operatorname{Alg}^S_{O^\otimes/C^\otimes}(\mathcal{E})$.

Remark 2.6. Given maps $O^\otimes \to O^\otimes$, $O^\otimes \to \tilde{O}^\otimes$ and $C^\otimes \to \tilde{O}^\otimes$ of $\kappa$-small $S$-families of operads we have a canonical equivalence

$$\operatorname{Alg}^S_{O^\otimes/\tilde{O}^\otimes}(\mathcal{E}) \simeq \operatorname{Alg}^S_{O^\otimes/C^\otimes}(O \times \tilde{O}^\otimes, \mathcal{E})$$

over $S$ that is the restriction of the canonical equivalence

$$\operatorname{Fun}^S_{O^\otimes/C^\otimes}(O^\otimes, C^\otimes) \simeq \operatorname{Fun}^S_{O^\otimes/\tilde{O}^\otimes/C^\otimes}(O^\otimes \times \tilde{O}^\otimes, C^\otimes)$$

over $S$ of remark 2.6.4.

Remark 2.7. Let $(S, \mathcal{E}_S, W_S), (O^\otimes, \mathcal{E}_{O^\otimes}, W_{O^\otimes})$ be good categorical pre-pattern.

Assume that the pullback $\phi : \mathcal{E}_S \times S O^\otimes \to \mathcal{E}_S$ is a cartesian fibration, whose cartesian morphisms are sent to morphisms of $\mathcal{E}_{O^\otimes}$ by the functor $O^\otimes \to O^\otimes$.

Assume that every functor $\Delta^2 \to O^\otimes$ that lies over an object of $W_S$ is sent to $W_{O^\otimes}$.

If $\beta : \mathcal{E}_S \to C^\otimes$ is fibered with respect to the categorical pattern $(O^\otimes, \mathcal{E}_{O^\otimes}, W_{O^\otimes})$, then by remark 2.5 the functor $\operatorname{Fun}^S_{O^\otimes/C^\otimes}(O^\otimes, C^\otimes) \to S$ is fibered with respect to the categorical pattern $(S, \mathcal{E}_S, W_S)$.  

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Thus also \( \varphi : \text{Alg}^{/S}_{O^{\text{op}}/S}(\mathcal{C}) \to S \) is fibered with respect to the categorical pattern \((S, \mathcal{E}_S, W_S)\).

A morphism of \( \text{Alg}^{/S}_{O^{\text{op}}/S}(\mathcal{C}) \) lying over a morphism of \( \mathcal{E}_S \) is locally \( \varphi \)-cartesian if and only if the corresponding functor \( \Delta^1 \times_S O^{\text{op}} \to \Delta^1 \times_S \mathcal{C}^{\text{op}} \) over \( \Delta^1 \times_S O^{\text{op}} \) sends locally \( \varphi \)-cartesian morphisms to locally \( \beta \)-cocartesian morphisms.

We are especially interested in the following cases:

**Remark 2.8.**

1. If \( O^{\text{op}} \to S \times O^{\text{op}} \) is a cocartesian \( S \)-family of operads over \( O^{\text{op}} \) for some \( \kappa \)-small operad \( O^{\text{op}} \) classifying a functor \( S \to \text{Op}_{\infty}(\kappa)_{/O^{\text{op}}} \) and \( \mathcal{C}^{\text{op}} \to S \times O^{\text{op}} \) is a cartesian \( S \)-family of operads over \( O^{\text{op}} \) classifying a functor \( S^{\text{op}} \to \text{Op}_{\infty}(\kappa)_{/O^{\text{op}}} \) such that the functor \( O^{\text{op}} \to S \) is flat, the functor \( \text{Alg}^{/S}_{O^{\text{op}}/S \times O^{\text{op}}}(\mathcal{C}) \to S \) is a cartesian fibration classifying the functor \( S^{\text{op}} \to (\text{Op}_{\infty}(\kappa)_{/O^{\text{op}}})^{\text{op}} \times \text{Op}_{\infty}(\kappa)_{/O^{\text{op}}} \to \text{Cat}_{\infty}(\kappa) \) by theorem 7.7.

Let \( \mathcal{C}^{\text{op}} \to S \times O^{\text{op}} \) be a \( O^{\text{op}} \)-monoidal category over \( S \) and \( \mathcal{E} \subset \text{Fun}(\Delta^1, S) \) a full subcategory.

Assume that for all \( X \in \mathcal{O} \) the functor \( \mathcal{C}_X \to S \) is a cartesian fibration relative to \( \mathcal{E} \), then by corollary 7.39 the functor \( \mathcal{C}^{\text{op}} \to S \times O^{\text{op}} \) is a map of cartesian fibrations relative to \( \mathcal{E} \).

This implies the following:

Let \( O^{\text{op}} \to O^{\text{op}} \) be a map of operads and \( \mathcal{C}^{\text{op}} \to S \times O^{\text{op}} \) a \( O^{\text{op}} \)-monoidal category over \( S \) such that for all \( X \in \mathcal{O} \) the functor \( \mathcal{C}_X \to S \) is a cartesian fibration relative to \( \mathcal{E} \), then the functor \( \text{Alg}^{/S}_{O^{\text{op}}/S \times O^{\text{op}}}(\mathcal{C}) \to S \) is a cartesian fibration relative to \( \mathcal{E} \).

2. Assume that the functor \( \theta : S \to R \) is a cocartesian fibration, the pullback \( \Delta^1 \times_S O^{\text{op}} \to \Delta^1 \) along every \( \theta \)-cocartesian morphism is a cartesian fibration, the functors \( \mathcal{C}^{\text{op}} \to O^{\text{op}}, \gamma : O^{\text{op}} \to S \) are maps of cocartesian fibrations over \( R \) and \( \phi : O^{\text{op}} \to O^{\text{op}} \) sends locally \( \gamma \circ \phi \)-cartesian morphisms lying over \( \theta \)-cocartesian morphisms to \( \gamma \)-cocartesian morphisms. Then the functor \( \text{Fun}^{/S}_{O^{\text{op}}}(O^{\text{op}}, \mathcal{C}^{\text{op}}) \to S \) is a map of cocartesian fibration over \( R \) and the cocartesian fibration \( \text{Fun}^{/S}_{O^{\text{op}}}(O^{\text{op}}, \mathcal{C}^{\text{op}}) \to R \) restricts to a cocartesian fibrations \( \varphi : \text{Alg}^{/S}_{O^{\text{op}}/O^{\text{op}}}(\mathcal{C}) \to R \) with the same cocartesian morphisms.

A morphism of \( \text{Alg}^{/S}_{O^{\text{op}}/O^{\text{op}}}(\mathcal{C}) \) is \( \varphi \)-cocartesian if and only if the induced functor \( \Delta^1 \times_R O^{\text{op}} \to \Delta^1 \times_R \mathcal{C}^{\text{op}} \) over \( \Delta^1 \times_R O^{\text{op}} \) sends locally \( \gamma \circ \phi \)-cartesian morphisms lying over \( \theta \)-cocartesian morphisms to \( \gamma \)-cocartesian morphisms.

2. specializes to the following case:

Given a map of operads \( O^{\text{op}} \to O^{\text{op}} \) the induced functor \( \phi : S \times O^{\text{op}} \to S \times O^{\text{op}} \) over \( S \) sends locally \( \gamma \circ \phi \)-cartesian morphisms to \( \gamma \)-cocartesian morphisms,
where $\gamma$ denotes the projection $S \times \mathcal{O} \to S$ and the pullback $\Delta^1 \times \mathcal{O} \to \Delta^1 \times (S \times \mathcal{O}) \to \Delta^1$ along every morphism of $S$ is a cartesian fibration.

Thus by 2. given cocartesian fibrations $\theta : S \to R, \rho : \mathcal{C} \to R$, a map $\mathcal{C} \to S$ of cocartesian fibrations over $R$ and a functor $\mathcal{C} \to \mathcal{O}$ that sends $\rho$-cocartesian morphisms to equivalences such that for every $r \in R$ the induced functor $\mathcal{C}^r \to S_r \times \mathcal{O}$ exhibits $\mathcal{C}^r$ as a $S_r$-family of operads over $\mathcal{O}$, the functor $\text{Alg}^S_{\mathcal{C}, S_r \times \mathcal{O}, \mathcal{C}}(\mathcal{C}) \to S$ is a map of cocartesian fibrations over $R$. 
2.3 \( O^\otimes \)-monoidal categories of sections

For every functor \( \mathcal{C} \to T \) over \( S \) such that \( \mathcal{C} \to S \) is flat we have an adjunction
\[
\mathcal{C} \times_S \dashv : \text{Cat}_\infty(\kappa)/S \cong \text{Cat}_\infty(\kappa)/T : \text{Fun}^S_T(\mathcal{C}, -).
\]

Being a right adjoint functor \( \text{Fun}^S_T(\mathcal{C}, -) : \text{Cat}_\infty(\kappa)/T \to \text{Cat}_\infty(\kappa)/S \) preserves finite products and thus monoid objects.

Let \( O^\otimes \) be a \( \kappa \)-small operad and \( D \to T \times O^\otimes \) a \( O^\otimes \)-monoidal category over \( T \) classified by a \( O^\otimes \)-monoid \( \phi \) of \( \text{Cat}_\infty(\kappa)/T \).

Theorem \[ \text{B.4.2.} \] implies the following: functor \( D \) and \( C \)

Let \( \mathcal{T} \) be a category and \( \mathcal{B} \) a categorical pattern on some category \( \mathcal{B} \).

2.3.2 \( \mathcal{B} \)-fibered objects, the functor \( \text{Fun}^\mathcal{B}_T(T \times \mathcal{B}, \mathcal{D}) \to \mathcal{B} \)

Remark 2.9. Theorem \[ \text{B.4.2.} \] implies the following:

\[
\text{Fun}^\mathcal{B}_T(T \times \mathcal{B}, \mathcal{D}) \to \mathcal{B}
\]
is \( \mathcal{P} \)-fibered.

Especially given an operad \( \mathcal{O} \) and a \( T \)-family \( \mathcal{D} \to T \times \mathcal{O} \) of operads over \( \mathcal{O} \) the functor

\[
\text{Fun}_T(T, \mathcal{D}) := \text{Fun}_T^{\mathcal{O}}(T \times \mathcal{O}, \mathcal{D}) \to \mathcal{O}
\]

is a map of operads that is a (locally) cocartesian fibration if \( \mathcal{D} \to T \times \mathcal{O} \) is a \( T \)-family of representable \( \mathcal{O} \)-operads respectively \( \mathcal{O} \)-monoidal categories.

**Remark 2.10.** Given a map \( \mathcal{O} \to \mathcal{O} \) of \( R \)-families of operads such that the functor \( \mathcal{O} \to R \) is flat, remark 2.4 provides a canonical equivalence

\[
\text{Fun}_T^S(\mathcal{C}, \text{Fun}_T^{T \times R}((T \times R) \mathcal{O}, \mathcal{D})) \simeq \text{Fun}_S^{\mathcal{O}}(S \times R \mathcal{O}, \text{Fun}_T^{S \times R}(C \times R \mathcal{O}, \mathcal{D}))
\]

over \( S \) that restricts to an equivalence

\[
\text{Fun}_T^S(\mathcal{C}, \text{Alg}_T^{T \times R}((T \times R) \mathcal{O}, \mathcal{D})) \simeq \text{Alg}_S^{\mathcal{O}}(S \times R \mathcal{O}, \text{Alg}_T^{S \times R}(C \times R \mathcal{O}, \mathcal{D}))
\]

over \( S \) that induces on the fiber over every object \( r \) of \( R \) the restriction

\[
\text{Fun}_{T,r}^S(\mathcal{C}, \text{Alg}_{T \times r}^{T \times r}((T \times r) \mathcal{O}, \mathcal{D})) \simeq \text{Alg}_S^{\mathcal{O}}(S \times r \mathcal{O}, \text{Alg}_T^{S \times r}(C \times r \mathcal{O}, \mathcal{D}))
\]

of the canonical equivalence

\[
\text{Fun}_T^S(\mathcal{C}, \text{Fun}_{T \times r}^{T \times r}((T \times r) \mathcal{O}, \mathcal{D})) \simeq \text{Fun}_S^{\mathcal{O}}(S \times r \mathcal{O}, \text{Fun}_{T,r}^{S \times r}(C \times r \mathcal{O}, \mathcal{D}))
\]

over \( S \).

To see this, we can reduce to the case that \( R \) and \( S \) are contractible according to remark 2.4.

In this case we have to show that the canonical equivalence

\[
\text{Fun}_T(\mathcal{C}, \text{Fun}_T^{T \times r}((T \times r) \mathcal{O}, \mathcal{D})) \simeq \text{Fun}_{r \times S}^{\mathcal{O}}(O, \text{Fun}_T(\mathcal{C}, \mathcal{D}))
\]

restricts to an equivalence

\[
\text{Fun}_T(\mathcal{C}, \text{Alg}_T^{T \times r}((T \times r) \mathcal{O}, \mathcal{D})) \simeq \text{Alg}_{r \times S}^{\mathcal{O}}(\text{Fun}_T(\mathcal{C}, \mathcal{D})).
\]

By remark 2.5 a functor \( \mathcal{O} \to \text{Fun}_T(\mathcal{C}, \mathcal{D}) \) over \( \mathcal{O} \) belongs to \( \text{Alg}_{r \times S}^{\mathcal{O}}(\text{Fun}_T(\mathcal{C}, \mathcal{D})) \) if and only if their corresponding functor \( \mathcal{C} \times \mathcal{O} \to \mathcal{D} \) over \( T \times \mathcal{O} \) factors through \( \text{Alg}_T^{T \times S \times r \times S}(\mathcal{D}) \) if and only if their corresponding functor \( \mathcal{C} \times \mathcal{O} \to \mathcal{D} \) over \( T \times \mathcal{O} \) sends a morphism \((f, g)\) of \( \mathcal{C} \times \mathcal{O} \) with \( f \) an equivalence of \( \mathcal{C} \) and \( g \) an inert morphism of \( \mathcal{O} \) to an inert morphism of \( \mathcal{D} \).

Moreover by remark 2.4 we have the following compatibility:

Denote \( \varphi \) the evaluation functor

\[
T \times_S \text{Fun}_T^S(T, \mathcal{D}) = T \times_S \text{Fun}_{T \times r}^{S \times r \mathcal{O}}(T \times r \mathcal{O}, \mathcal{D}) =
\]
The composition
\[
T \times S \times \text{Fun}_{T \times R \otimes}^S (T, \text{Alg}_{\otimes/\otimes}^T (D^\otimes)) \cong T \times S \text{Alg}_{\otimes/\otimes}^T (\text{Fun}_{T}^S (T, D^\otimes)) \cong
\]
\[
\text{Alg}_{\otimes/\otimes}^T (T \times S \text{Fun}_{T}^S (T, D^\otimes)) \xrightarrow{\text{Alg}_{\otimes/\otimes}^T (\phi)} \text{Alg}_{\otimes/\otimes}^T (D^\otimes)
\]
is equivalent to the evaluation functor over $T$. 

over $T \times R \otimes$. 

Especially we are interested in the case that for every \( s \in S \) the induced functor \( M^s_\mathcal{O} \to T_s \times LM^s \) on the fiber over \( s \) exhibits \( M^s_\mathcal{O} \) as a \( LM^s \)-monoid of \( \text{Cat}_\infty(\kappa)/T_s \).

## 2.4 Parametrized categories of modules

In this subsection we specialize from parametrized categories of \( \mathcal{O}^\circ \)-algebras and \( \mathcal{O}^\circ \)-monoidal categories of sections to parametrized categories of left modules and \( LM^s \)-monoidal categories of sections by taking \( \mathcal{O}^\circ := LM^s \).

We remark that all results given here work for right modules in a similar way:

Let \( T \) be a category and \( M \to T \times LM \) a \( T \)-family of operads over \( LM \).

Set \( \mathcal{C}^\circ := \text{Ass}^\circ \times_{LM^s} M^\circ \) and \( \mathcal{B} := \{m\} \times_{LM^s} M^\circ \).

We write \( \text{LMod}^T(\mathcal{B}) \) for \( \text{Alg}_{LM^s/\text{LM}^s}^T(M^\circ) \).

For every functor \( T' \to T \) we have a canonical equivalence

\[
T' \times_T \text{LMod}^T(\mathcal{B}) \simeq \text{LMod}^{T'}(T' \times_T \mathcal{B}).
\]

We have forgetful functors

\[
\text{LMod}^T(\mathcal{B}) \cong \text{Alg}_{LM^s/\text{LM}^s}^T(M^\circ) \cong \text{Alg}_{\text{Ass}^\circ/\text{LM}^s}^T(M^\circ) \cong \text{Alg}^T(\mathcal{C})
\]

and

\[
\text{LMod}^T(\mathcal{B}) \cong \text{Fun}^T_{T \times \text{LM}^s}(T \times LM, M^\circ) \to \text{Fun}^T_{T \times \text{LM}^s}(T \times \{m\}, M^\circ) \cong \mathcal{B}
\]

over \( T \).

Given a section \( A \) of \( \text{Alg}^T(\mathcal{C}) \to T \) we set \( \text{LMod}_{\text{Ass}^\circ/\text{LM}^s}^T(A) := T \times_{\text{Alg}^T(\mathcal{C})} \text{LMod}^T(\mathcal{B}) \).

If \( \mathcal{C}^\circ = \mathcal{D}^\circ \times T \) for an operad \( \mathcal{D}^\circ \) over \( \text{Ass}^\circ \) and \( A \) is an associative algebra of \( \mathcal{D} \), we write \( \text{LMod}^T_A(\mathcal{B}) \) for \( \text{LMod}^T_{\mathcal{A}^\circ}^T(\mathcal{B}) \cong \{A\} \times_{\text{Alg}(\mathcal{D}^\circ)} \text{LMod}^T(\mathcal{B}) \), where \( \mathcal{A}^\circ \) denotes the functor \( T \to \text{Alg}^T(\mathcal{C}) \cong T \times \text{Alg}(\mathcal{D}) \) over \( T \) corresponding to the constant functor \( T \to \text{Alg}(\mathcal{D}) \) with image \( A \).

**Remark 2.11.**

Let \( \mathcal{E} \subset \text{Fun}(\Delta^1, T) \) be a full subcategory.

If the functors \( \mathcal{B} \to T, \mathcal{C} \to T \) are cartesian fibrations relative to \( \mathcal{E} \) and the functors \( \text{LMod}^T_{\mathcal{A}^\circ}(\mathcal{B}) \to T \) and \( \text{Alg}^T(\mathcal{C}) \to T \) are cartesian fibrations relative to \( \mathcal{E} \) and the functor \( \text{LMod}^T_{\mathcal{A}^\circ}(\mathcal{B}) \to \text{Alg}^T(\mathcal{C}) \) over \( T \) is a map of cartesian fibrations relative to \( \mathcal{E} \).

Moreover if the functors \( \mathcal{B} \to T, \mathcal{C} \to T \) are cartesian fibrations, the functor \( \Phi : \text{LMod}^T(\mathcal{B}) \to \text{Alg}^T(\mathcal{C}) \) over \( T \) is a cartesian fibration whose cartesian morphisms are those that are sent to cartesian morphisms of \( \mathcal{B} \to T \).

This follows from the fact that \( \Phi \) induces on the fiber over every \( t \in T \) the cartesian fibration \( \text{LMod}(\mathcal{B}_t) \to \text{Alg}(\mathcal{C}_t) \), whose cartesian morphisms are those that get equivalences in \( \mathcal{B}_t \) and are thus preserved by the induced functors on the fibers of the cartesian fibration \( \text{LMod}^T(\mathcal{B}) \to T \).
Thus for every section $A$ of $\text{Alg}^{\mathcal{T}}(\mathcal{C}^\otimes) \to T$ the functor $\text{LMod}^{\mathcal{T}}_A(\mathcal{B}) \to T$ is a cartesian fibration and the composition $\text{LMod}^{\mathcal{T}}_A(\mathcal{B}) \to \text{LMod}^{\mathcal{T}}(\mathcal{B}) \to \mathcal{B}$ is a map of cartesian fibrations over $T$.

Lemma 2.12. Let $S$ be a $\kappa$-small category, $\mathcal{C}^\otimes$ be a $\kappa$-small monoidal category over $S$ and $\varphi : \mathcal{D} \to T$ a map of $\kappa$-small cartesian fibrations over $S$.

Let $\mathcal{M}^\otimes$ be a $\mathcal{LMod}^\otimes$-monoidal category over $T$ that exhibits the functor $\mathcal{D} \to T$ as a left module over the pullback of the monoidal category $\mathcal{C}^\otimes \to \text{Ass}^\otimes \times_S \mathcal{S}$ over $S$ along the functor $T \to S$.

The forgetful functor $\text{LMod}^{\mathcal{T}}(\mathcal{D}) \to \text{Alg}^\mathcal{S}(\mathcal{C}) \times_S \mathcal{D}$ is a map of cartesian fibrations over $\text{Alg}^\mathcal{S}(\mathcal{C})$.

A morphism of $\text{LMod}^{\mathcal{T}}(\mathcal{D})$ is cartesian with respect to the cartesian fibration $\text{LMod}^{\mathcal{T}}(\mathcal{D}) \to \text{Alg}^\mathcal{S}(\mathcal{C})$ if and only if its image in $\mathcal{D}$ is cartesian with respect to the cartesian fibration $\mathcal{D} \to \mathcal{S}$.

Proof. Assume first that $S$ is contractible and $\varphi : \mathcal{D} \to T$ is a cartesian fibration.

In this case remark 2.11 implies that the canonical functor $\Psi : \text{LMod}^{\mathcal{T}}(\mathcal{D}) \to \text{Alg}(\mathcal{C}) \times T$ is a cartesian fibration, where a morphism is $\Psi$-cartesian if and only if its image in $\mathcal{D}$ is $\varphi$-cartesian.

Therefore the composition $\Phi : \text{LMod}^{\mathcal{T}}(\mathcal{D}) \to \text{Alg}(\mathcal{C}) \times T \to \text{Alg}(\mathcal{C})$ is a cartesian fibration, where a morphism is $\Phi$-cartesian if and only if it is $\Psi$-cartesian and its image in $T$ is an equivalence, i.e. if and only if its image in $\mathcal{D}$ is an equivalence.

Now let $\varphi : \mathcal{D} \to T$ be an arbitrary functor but $S$ still be contractible.

In this case we embed the functor $\mathcal{D} \to T$ into a cartesian fibration:

The subcategory inclusion $\text{Cat}^{\text{cart}}_{\infty/T}(\kappa) \subset \text{Cat}_{\infty}(\kappa)_{/T}$ admits a left adjoint $\mathcal{E} : \text{Cat}_{\infty}(\kappa)_{/T} \to \text{Cat}^{\text{cart}}_{\infty/T}(\kappa)$ with the following properties:

1. For every functor $X \to T$ the cartesian fibration $\mathcal{E}(X) \to T$ is equivalent over $T$ to the functor $X \times_{\text{Fun}(\{1\},T)} \text{Fun}(\Delta^1,T) \to \text{Fun}(\Delta^1,T) \to \text{Fun}(\{0\},T)$.

2. The unit $X \to \mathcal{E}(X) = X \times_{\text{Fun}(\{1\},T)} \text{Fun}(\Delta^1,T)$ is the pullback of the fully faithful diagonal embedding $T \to \text{Fun}(\Delta^1,T)$ over $\text{Fun}(\{1\},T)$ along $X \to T$ and is thus itself fully faithful.

3. For every category $K$ and every functor $\mathcal{C} \to T$ the map $\mathcal{E}(K \times \mathcal{C}) \to K \times \mathcal{E}(\mathcal{C})$ of cartesian fibrations over $T$ adjoint to the functor $K \times \mathcal{C} \to K \times \mathcal{E}(\mathcal{C})$ over $T$ is an equivalence.

This follows from the following considerations: Taking the opposite category $\text{Cat}_{\infty/T}(\kappa) = \text{Cat}_{\infty}(\kappa)_{/T}^{\text{op}}$ restricts to an equivalence $\text{Cat}^{\text{cart}}_{\infty/T}(\kappa) \simeq \text{Cat}^{\text{cocart}}_{\infty/T}(\kappa)$.

So it is enough to see that the subcategory inclusion $\text{Cat}^{\text{cart}}_{\infty/T}(\kappa) \subset \text{Cat}_{\infty}(\kappa)_{/T}$ admits a left adjoint $\mathcal{E}$ with properties 1., 2., 3., where we have to change $\{1\}$ with $\{0\}$.
We have a colocalization $ι : \text{Cat}_\infty(κ) ≃ \text{Op}_\infty(κ) : γ$ that induces an equivalence $ι : \text{Cat}_\infty(κ)_{/T} ≃ \text{Op}_\infty(κ)_{/\iota(T)} : γ$ that restricts to an equivalence $\text{Cat}_{\text{cocart}}^\infty(κ) ≃ \text{Op}^\infty_{\text{cocart}}(κ)$. 

But the subcategory inclusion $\text{Op}^\infty_{\text{cocart}}(κ)_{/\iota(T)} ⊂ \text{Op}_\infty(κ)_{/\iota(T)}$ admits a left adjoint given by the enveloping $(\iota(T))$-monoidal category that induces on underlying categories the properties 1., 2., 3., when we change $\{1\}$ with $\{0\}$.

The $\text{Cat}_\infty(κ)$-left module structure on $\text{Cat}_\infty(κ)_{/T}$ induced by the symmetric monoidal functor $- × T : \text{Cat}_\infty(κ)^{\text{op}} → (\text{Cat}_\infty(κ)_{/T})^{\text{op}}$ restricts to a $\text{Cat}_\infty(κ)$-left module structure on $\text{Cat}_{\text{cart}}^\infty(κ)$ as the functor $- × T : \text{Cat}_\infty(κ) → \text{Cat}_\infty(κ)_{/T}$ factors through the subcategory $\text{Cat}_{\text{cart}}^\infty(κ) ⊂ \text{Cat}_\infty(κ)_{/T}$.

So the subcategory inclusion $\text{Cat}_{\text{cart}}^\infty(κ) ⊂ \text{Cat}_\infty(κ)_{/T}$ is a $\text{Cat}_\infty(κ)$-linear functor and so by 3. the adjunction $E : \text{Cat}_\infty(κ)_{/T} ≃ \text{Cat}_{\text{cart}}^\infty(κ)$ is a $\text{Cat}_\infty(κ)$-linear adjunction.

Thus we get an induced adjunction $\text{LMod}_C(\text{Cat}_\infty(κ)_{/T}) ≃ \text{LMod}_C(\text{Cat}_{\text{cart}}^\infty(κ))$ over the adjunction $E : \text{Cat}_\infty(κ)_{/T} ≃ \text{Cat}_{\text{cart}}^\infty(κ)$ so that the unit $D → E(D)$ lifts to a $C$-linear functor over $T$.

So the fully faithful unit $D → E(D)$ induces a full subcategory inclusion $\text{LMod}^T(D) ⊂ \text{LMod}^T(E(D))$ over $T$ such that the functor $\text{LMod}^T(D) → \text{Alg}(C)$ is the restriction of the functor $\psi : \text{LMod}^T(E(D)) → \text{Alg}(C)$.

As the lemma holds for the case that $φ : D → T$ is a cartesian fibration and $S$ is contractible, the functor $ψ : \text{LMod}^T(E(D)) → \text{Alg}(C)$ is a cartesian fibration, where a morphism is $ψ$-cartesian if and only if its image in $E(D)$ is an equivalence.

Consequently every $ψ$-cartesian morphism has with its target also its source in $\text{LMod}^T(D) ≃ D ×_{E(D)} (\text{LMod}^T(E(D)))$ so that the cartesian fibration $ψ$ restricts to a cartesian fibration $\text{LMod}^T(D) → \text{Alg}(C)$ with the same cartesian morphisms.

Now let $S$ be arbitrary.

Let $X → Y$ be a map of cartesian fibrations over $S$ over a cartesian fibration $Z → S$.

Then the functor $X → Y$ is a map of cartesian fibrations over $Z$ if and only if the following two conditions are satisfied:

1. For every object $s$ of $S$ the induced functor $X_s → Y_s$ on the fiber over $s$ is a map of cartesian fibrations over $Z_s$.

2. For every morphism $s' → s$ of $S$ the induced functors $X_s → X_{s'}$ and $Y_s → Y_{s'}$ on the fiber send $X_s → Z_s$-cartesian morphisms to $X_{s'} → Z_{s'}$ -cartesian morphisms respectively $Y_s → Z_s$-cartesian morphisms to $Y_{s'} → Z_{s'}$-cartesian morphisms.

Moreover the functor $X → Y$ is a map of cartesian fibrations over $Z$ that reflects cartesian morphisms over $Z$ if and only if 1. and 2. holds for every object $s$ of $S$ the induced functor $X_s → Y_s$ on the fiber over $s$ reflects cartesian morphisms over $Z_s$.

By remark 2.11 the functor $φ(\text{M}^S, Ω_S) : \text{LMod}^T(D) → \text{Alg}^S(C) ×_S D$ is a map of cartesian fibrations over $S$ over the cartesian fibration $\text{Alg}^S(C) → S$.

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For every object $s$ of $S$ the induced functor $\phi(M^\otimes, C^\otimes)_s : LMod^{/T}(\mathcal{D}) \to LMod^{/T}(\mathcal{D})_s \to \text{Alg}(C^\otimes)_s$ on the fiber over $s$ is equivalent to the functor $\phi(M^\otimes, C^\otimes) : LMod^{/T}(\mathcal{D}) \to \text{Alg}(C^\otimes)$. 

As the lemma holds for the case that $S$ is contractible, the functor $\phi(M^\otimes, C^\otimes) : LMod^{/T}(\mathcal{D}) \to \text{Alg}(C^\otimes)$ is cartesian with respect to the cartesian fibration $LMod^{/T}(\mathcal{D}) \to \text{Alg}(C^\otimes)$ if and only if its image in $\mathcal{D}_s$ is an equivalence.

This implies condition 1. and 2., where we use for condition 2. that the canonical functor $LMod^{/T}(\mathcal{D}) \to \mathcal{D}$ is a map of cartesian fibrations over $S$.

\[ \square \]

**Remark 2.13.** Let $\kappa$ be a strongly inaccessible cardinal, $T \to S$ a functor between $\kappa$-small categories and $M^\otimes \to \text{LM}^\otimes \times T$ a $\text{LM}^\otimes$-monoidal category over $T$ that exhibits a functor $\mathcal{D} \to T$ as a left module over the pullback of a monoidal category $C^\otimes \to \text{Ass}^\otimes \times S$ over $S$ along the functor $T \to S$.

Assume that the functor $\mathcal{D} \to T$ is a map of cartesian fibrations over $S$ classifying a natural transformation $H \to G$ of functors $S^{op} \to \text{Cat}_\infty(\kappa)$.

Then by lemma 2.12 the forgetful functor

\[ LMod^{/T}(\mathcal{D}) \to \mathcal{D} \times_T \text{Alg}_{/T}^S(T \times_S C^\otimes) \simeq \mathcal{D} \times_T (T \times_S \text{Alg}_{/S}^S(C^\otimes)) \simeq T \times_S \text{Alg}_{/S}^S(C^\otimes) \]

is a map of cartesian fibrations over $\text{Alg}_{/S}^S(C^\otimes)$ and so classifies a functor

\[ \text{Alg}_{/S}^S(C^\otimes)^{op} \to \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \]

over $\text{Cat}_\infty(\kappa)$ adjoint to a functor

\[ \phi(M, C^\otimes) : \text{Alg}_{/S}^S(C^\otimes)^{op} \to \text{H}^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

over $S^{op}$.

Denote $X : S^{op} \to \text{G}^* (\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))$ the functor over $S^{op}$ corresponding to the natural transformation $H \to G$ of functors $S^{op} \to \text{Cat}_\infty(\kappa)$.

By lemma ... we have a canonical equivalence

\[ \text{H}^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \simeq S^{op} \times_{\text{G}^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))))} \text{G}^* (\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))))^{\Delta^1} \]

\[ = G^* (\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))/S^{op} \]

over $S^{op}$.

So we obtain a functor

\[ \phi(M, C) : \text{Alg}_{/S}^S(C)^{op} \to \text{H}^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \simeq G^* (\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))/S^{op} \]

over $S^{op}$.

\[ \square \]

**Remark 2.14.**
1. Let \( \varphi : S' \to S \) be a functor. Set \( T' := S' \times_S T \).

Then the pullback of the map \( \text{LMod}^{\text{op}}_{/T}(\calD) \to \calD \times_S \text{Alg}^{/S}(\calC) \) of cartesian fibrations over \( \text{Alg}^{/S}(\calC) \) along the functor

\[
\text{Alg}^{/S}(S' \times_S \calC) \cong S' \times_S \text{Alg}^{/S}(\calC) \to \text{Alg}^{/S}(\calC)
\]

is equivalent to the canonical map \( \text{LMod}^{\text{op}}_{/T}(T' \times_T \calD) \to (T' \times_T \calD) \times_S \text{Alg}^{/S}(S' \times_S \calC) \) of cartesian fibrations over \( \text{Alg}^{/S}(S' \times_S \calC) \).

So the functor \( \phi(T' \times_T M, S' \times_S \calC) : \)

\[
\text{Alg}^{/S}(S' \times_S \calC)^{\text{op}} \to \varphi^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))
\]

over \( S^{\text{op}} \) is equivalent to the pullback of the functor \( \phi(M, \calC) \) over \( S^{\text{op}} \) along the functor \( \varphi^{\text{op}} : S^{\text{op}} \to S^{\text{op}} \).

2. Let \( \beta : \calC^\otimes \to \calC^\otimes \) be a monoidal functor over \( S \) and \( M^\otimes \) the pullback of \( M^\otimes \) along \( T \times_S \beta : T \times_S \calC^\otimes \to T \times_S \calC^\otimes \).

Then the functor \( \text{LMod}^{\text{op}}_{/T}(\calD) \to \calD \times_T \text{Alg}^{/T}(T \times_S \calC^\otimes) \cong \calD \times_S \text{Alg}^{/S}(\calC^\otimes) \) of cartesian fibrations over \( \text{Alg}^{/S}(\calC^\otimes) \) along the functor \( \text{Alg}^{/S}(\beta) : \text{Alg}^{/S}(\calC^\otimes) \to \text{Alg}^{/S}(\calC^\otimes) \).

Thus \( \phi(M', \calC^\otimes) \) is the composition

\[
\text{Alg}^{/S}(\calC^\otimes)^{\text{op}} \xrightarrow{\text{Alg}^{/S}(\beta)^{\text{op}}} \text{Alg}^{/S}(\calC^\otimes)^{\text{op}} \xrightarrow{\phi(M, \calC)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))
\]

of functors over \( S^{\text{op}} \).

3. Let \( \gamma : S \to \calR \) be a cartesian fibration.

Denote \( \gamma' : S^{\text{op}} \to \text{Cat}_\infty(\kappa) \) the functor classified by the composition \( \calD \to S \to \calR \).

Then the functor \( \text{LMod}^{\text{op}}_{/T}(\calD) \to \calD \times_S \text{Alg}^{/S}(S' \times_S \calC) \cong \calD \times_S \text{Alg}^{/S}(\calC) \) considered as a functor over \( \text{Alg}^{/S}(\calC) \) is equivalent to the pullback of the monoidal counit \( \text{Fun}^\otimes_{/S}(S, \calC^\otimes) \to \calC^\otimes \) over \( S^{\text{op}} \).

Denote \( \rho : T \times_S \text{Fun}^\otimes_{/S}(S, \calC^\otimes) \to T \times_S \calC^\otimes \) the pullback of the monoidal counit \( S \times_S \text{Fun}^\otimes_{/S}(S, \calC^\otimes) \to \calC^\otimes \) over \( S \) along the functor \( T \to S \) and \( \rho^*(\calC^\otimes) \) the pullback of the \( T \times_S \calC^\otimes \)-left module structure on \( \calD \to T \) along \( \rho \).

By 2. the functor

\[
\theta : (S \times_S \text{Fun}^\otimes_{/S}(S, \text{Alg}^{/S}(\calC)))^{\text{op}} \cong \text{Alg}^{/S}(S \times_S \text{Fun}^\otimes_{/S}(S, \calC))^{\text{op}}
\]

\[
\phi(\rho^*(M), S \times_S \text{Fun}^\otimes_{/S}(S, \calC)) \to \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))
\]

over \( S^{\text{op}} \) factors as

\[
(S \times_S \text{Fun}^\otimes_{/S}(S, \text{Alg}^{/S}(\calC)))^{\text{op}} \cong \text{Alg}^{/S}(S \times_S \text{Fun}^\otimes_{/S}(S, \calC))^{\text{op}}
\]

\[
(S \times_S \text{Fun}^\otimes_{/S}(S, \text{Alg}^{/S}(\calC)))^{\text{op}} \to \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))
\]
Lemma 2.15. Let 

\[ \text{Alg}^S(\mathcal{C})^{\text{op}} \xrightarrow{\phi(M,C)} H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

in other words \( \theta \) is adjoint to the functor

\[ \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{\text{op}}, \text{Alg}^S(\mathcal{C})^{\text{op}}) \xrightarrow{\phi(M,C)} \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{\text{op}}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \]

over \( R^{\text{op}} \).

So by lemma 2.14, the composition

\[ \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{\text{op}}, \text{Alg}^S(\mathcal{C})^{\text{op}}) \simeq \text{Fun}^R_S(S, \text{Alg}^S(\mathcal{C}))^{\text{op}} \simeq \text{Alg}^R(\text{Fun}^R_S(S, \mathcal{C}))^{\text{op}} \]

\[ \phi(\varphi^*(M), \text{Fun}^R_S(S, \mathcal{C})) \xrightarrow{\phi} H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

is equivalent to the composition

\[ \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{\text{op}}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \subset H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

Lemma 2.15. Let \( Y \to R \) be a functor and \( \gamma : S \to R \) and \( D \to S \) be cartesian fibrations between \( \kappa \)-small categories.

Denote \( H : S^{\text{op}} \to \text{Cat}_\infty(\kappa) \) the functor classified by \( D \to S \) and \( H' : R^{\text{op}} \to \text{Cat}_\infty(\kappa) \) the functor classified by the composition \( D \to S \to R \).

Let \( \varphi : X \to D \times_S (S \times_R Y) \) and \( \varphi' : Y \to Y \) be a map of cartesian fibrations over \( S \times_R Y \) that gives rise to a map of cartesian fibrations \( \varphi' \) over \( Y \) via forgetting along the canonical functor \( S \times_R Y \to Y \).

\( \varphi \) classifies a functor \( H_*(S^{op} \times_{R^{op}} Y^{op}) \to \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \) over \( \text{Cat}_\infty(\kappa) \) adjoint to a functor \( \alpha : S^{op} \times_{R^{op}} Y^{op} \to H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \) over \( S^{op} \) and \( \varphi' \) classifies a functor \( H'_*(Y^{op}) \to \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \) over \( \text{Cat}_\infty(\kappa) \) adjoint to a functor \( \beta : Y^{op} \to H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \) over \( R^{op} \).

Then \( \beta \) factors as the functor

\[ Y^{op} \to \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \]

over \( R^{op} \) adjoint to \( \alpha \) followed by the canonical subcategory inclusion

\[ \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \subset H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

over \( R^{op} \), which is represented by the subcategory inclusion

\[ \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(K, \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))))) \subset H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

over \( R^{op} \), which is represented by the subcategory inclusion

\[ \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(K, \text{Fun}^{R^{\text{op}}}_{S^{\text{op}}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))))) \subset H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \]

natural in a \( \kappa \)-small category \( K \) over \( R^{op} \).
Remark 2.16. If $R$ is contractible, the canonical subcategory inclusion

$$\text{Fun}_{S^{op}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \subset H^\sim(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \simeq \{D\} \times_{\text{Fun}(\{1\}, \text{Cat}_\infty(\kappa))} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \simeq \text{Cat}_\infty(\kappa)/D$$

is the composition

$$\text{Fun}_{S^{op}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \simeq \text{Fun}_{\text{Cat}_\infty(\kappa)}(H_*(S^{op}), \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))$$

$$\simeq \text{Cat}_{\infty/S}(\kappa)^D \subset (\text{Cat}_{\infty/S}(\kappa)^D/\sim = \text{Cat}_\infty(\kappa)^D)$$

Proof. The assertion of the lemma follows tautologically from the definition of the canonical functor $\text{Fun}_{S^{op}}(S^{op}, H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))) \to H^\sim(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))$ over $R^{op}$. □
2.5 \textit{LM\textsuperscript{\textbullet}}-monoidal categories of sections

Let \( \mathcal{D} \to T, T \to S \) be functors such that the composition \( \mathcal{D} \to T \to S \) is flat. Let \( M^\text{\textbullet} \to T \times LM^\text{\textbullet} \) be a T-family of operads over \( LM^\text{\textbullet} \).

We set
\[
\text{Fun}^S_T(\mathcal{D}, M)^\text{\textbullet} := \text{Fun}^S_{T \times LM^\text{\textbullet}}(\mathcal{D} \times LM^\text{\textbullet}, M^\text{\textbullet})
\]
and
\[
\text{Fun}^S_T(\mathcal{D}, C)^\text{\textbullet} := \text{Fun}^S_{T \times Ass^\text{\textbullet}}(\mathcal{D} \times Ass^\text{\textbullet}, C^\text{\textbullet}).
\]

We have canonical equivalences
\[
\text{Ass}^\text{\textbullet} \times_{LM^\text{\textbullet}} \text{Fun}^S_T(\mathcal{D}, M)^\text{\textbullet} \simeq \text{Fun}^S_T(\mathcal{D}, C)^\text{\textbullet}
\]
over \( S \times Ass^\text{\textbullet} \) and
\[
\{m\} \times_{LM^\text{\textbullet}} \text{Fun}^S_T(\mathcal{D}, M)^\text{\textbullet} \simeq \text{Fun}^S_T(\mathcal{D}, B)
\]
over \( S \) and for every functor \( S' \to S \) a canonical equivalence
\[
S' \times_S \text{Fun}^S_T(\mathcal{D}, M)^\text{\textbullet} \simeq \text{Fun}^S_{S' \times_SS}(S' \times_S \mathcal{D}, S' \times_S M)^\text{\textbullet}
\]
over \( S' \times LM^\text{\textbullet} \).

\textbf{Remark 2.17.}

Let \( M^\text{\textbullet} \to T \times LM^\text{\textbullet} \) be a \( LM^\text{\textbullet} \)-monoidal category over \( T \) classifying a \( LM^\text{\textbullet} \)-monoid \( \phi \) of \( \text{Cat}_{\infty}(\kappa)/T \).

Theorem 7.4 implies that the image of \( \phi \) under the finite products preserving functor \( \text{Fun}^S_T(\mathcal{C}, -) : \text{Cat}_{\infty}(\kappa)/T \to \text{Cat}_{\infty}(\kappa)/S \) is classified by the \( LM^\text{\textbullet} \)-monoidal category
\[
\text{Fun}^S_T(T, M)^\text{\textbullet} = \text{Fun}^S_{T \times LM^\text{\textbullet}}(T \times LM^\text{\textbullet}, M^\text{\textbullet}) \to LM^\text{\textbullet}
\]
over \( S \).

\textbf{Remark 2.19.}

Given a T-family \( M^\text{\textbullet} \to T \times LM^\text{\textbullet} \) of operads over \( LM^\text{\textbullet} \) the functor
\[
\text{Fun}_T(T, M)^\text{\textbullet} = \text{Fun}^S_{T \times LM^\text{\textbullet}}(T \times LM^\text{\textbullet}, M^\text{\textbullet}) \to LM^\text{\textbullet}
\]
is a map of operads that is a (locally) cocartesian fibration if \( M^\text{\textbullet} \to T \times LM^\text{\textbullet} \) is a T-family of representable \( LM^\text{\textbullet} \)-operads respectively \( LM^\text{\textbullet} \)-monoidal categories.

\textbf{Remark 2.18.}

1. By \textbf{remark 2.17} we have a canonical equivalence
\[
\text{Fun}^S_T(\mathcal{D}, LMod^T(\mathcal{B})) = \text{Fun}^S_T(\mathcal{D}, \text{Alg}^T_{LM^\text{\textbullet}/LM^\text{\textbullet}}(M)) = \\
\text{Alg}^S_{LM^\text{\textbullet}/LM^\text{\textbullet}}(\text{Fun}^S_T(\mathcal{D}, M)) = \text{LMod}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{B}))
\]
over \( \text{Fun}^S_T(\mathcal{D}, \mathcal{B}) \), whose pullback along the canonical functor \( \text{Ass}^\text{\textbullet} \to LM^\text{\textbullet} \) is the canonical equivalence
\[
\text{Fun}^S_T(\mathcal{D}, \text{Alg}^T(\mathcal{C})) = \text{Alg}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{C}))
\]
2. Especially we are interested in the following situation:

Let \( \mathcal{D} \to \mathcal{T} \to \mathcal{B} \to T \) and such that we have a commutative square

\[
\begin{array}{ccc}
\text{Fun}^S_T(\mathcal{D}, \text{LM}^T(\mathcal{B})) & \overset{\alpha}{\longrightarrow} & \text{LM}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{B})) \\
\downarrow & & \downarrow \\
\text{Fun}^S_T(\mathcal{D}, \text{Alg}^S(\mathcal{C})) & \overset{\alpha}{\longrightarrow} & \text{Alg}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{C}))
\end{array}
\] (1)

of categories over \( S \).

Let \( \mathcal{A} \) be a section of \( \text{Alg}^S(\mathcal{C}) \to \mathcal{T} \) and \( \mathcal{A}' \) the section of \( \text{Alg}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{C})) \) over \( S \) corresponding to the composition \( \mathcal{D} \to \mathcal{T} \overset{\delta}{\to} \text{Alg}^S(\mathcal{C}) \) of functors over \( T \).

Square \( \square \) induces an equivalence

\[
\text{Fun}^S_T(\mathcal{D}, \text{LM}^T(\mathcal{B})) \simeq \text{LM}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{B})).
\]

2. Especially we are interested in the following situation:

Let \( T \to S \) be a functor, \( \mathcal{A}^\otimes \to \text{Ass}^\otimes \times S \) a monoidal category over \( S \) and \( M^\otimes \to \text{LM}^\otimes \times T \) a \( \text{LM}^\otimes \)-monoidal category over \( T \) that exhibits a category \( \mathcal{B} \to T \) over \( T \) as a left module over the category \( T \times_S \mathcal{A} \) over \( T \).

We have a canonical diagonal monoidal functor

\[
\delta : \mathcal{A}^\otimes = \text{Map}_S(S, \mathcal{A})^\otimes \to \text{Map}_S(\mathcal{D}, \mathcal{A})^\otimes = \text{Fun}^S_T(\mathcal{D}, T \times_S \mathcal{A})^\otimes
\]

over \( S \) that induces a functor

\[
\text{Alg}^S(\delta) : \text{Alg}^S(\mathcal{A}) \to \text{Alg}^S(\text{Map}_S(\mathcal{D}, \mathcal{A})) = \text{Alg}^S(\text{Fun}^S_T(\mathcal{D}, T \times_S \mathcal{A})) \simeq \text{Fun}^S_T(\mathcal{D}, T \times_S \text{Alg}^S(\mathcal{A}))
\]

over \( S \) that is equivalent over \( S \) to the diagonal functor

\[
\text{Alg}^S(\mathcal{A}) = \text{Map}_S(S, \text{Alg}^S(\mathcal{A})) \to \text{Map}_S(\mathcal{D}, \text{Alg}^S(\mathcal{A})) \simeq \text{Fun}^S_T(\mathcal{D}, T \times_S \text{Alg}^S(\mathcal{A}))
\]

over \( S \).

Pulling back the \( \text{LM}^\otimes \)-monoidal category \( \text{Fun}^S_T(\mathcal{D}, M)^\otimes \) over \( S \) along \( \delta \) we obtain a \( \text{LM}^\otimes \)-monoidal category \( \delta^*(\text{Fun}^S_T(\mathcal{D}, M)^\otimes) \) over \( S \) that exhibits \( \text{Fun}^S_T(\mathcal{D}, \mathcal{B}) \) as a left module over \( \mathcal{A} \).

Square \( \square \) specializes to the commutative square

\[
\begin{array}{ccc}
\text{Fun}^S_T(\mathcal{D}, \text{LM}^T(\mathcal{B})) & \overset{\alpha}{\longrightarrow} & \text{LM}^S(\text{Fun}^S_T(\mathcal{D}, \mathcal{B})) \\
\downarrow & & \downarrow \\
\text{Maps}(\mathcal{D}, \text{Alg}^S(\mathcal{A})) & \overset{\alpha}{\longrightarrow} & \text{Alg}^S(\text{Maps}(\mathcal{D}, \mathcal{A}))
\end{array}
\] (2)
of categories over $S$.

Pulling back square $\square$ along the functor $\text{Alg}^{S}(\delta) : \text{Alg}^{S}(A) \to \text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))$ over $S$ we obtain a canonical equivalence

$$\text{Alg}^{S}(\delta)^{*}(\text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B}))) \simeq \text{LMod}^{S}(\delta^{*}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{B})))$$

over $\text{Alg}^{S}(A) \times_{S} \text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{B})$.

3. We have a canonical equivalence

$$\text{Alg}^{S}(\delta)^{*}(\text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B})))$$

over $\text{Alg}^{S}(A)$ represented by the following canonical equivalence natural in every functor $\alpha : K \to \text{Alg}^{S}(A)$:

$$\text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B}))) \simeq \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B})))$$

$$\{\delta \circ \alpha\} \times \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B}))) \simeq \{\alpha \circ p\} \times \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B})))$$

$$\text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B}))) \simeq \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B})))$$

where $p : \mathcal{D} \times_{S} K \to K$ denotes the canonical functor.

So we get a canonical equivalence

$$\Psi : \text{LMod}^{S}(\delta^{*}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{B}))) \simeq \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B})))$$

over $\text{Alg}^{S}(A)$ such that we have a commutative square

$$\begin{array}{ccc}
\text{LMod}^{S}(\delta^{*}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{B}))) & \longrightarrow & \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B}))) \\
\downarrow & & \downarrow \\
\text{Alg}^{S}(A) \times_{S} \text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{B}) & \longrightarrow & \text{Fun}_{\text{Maps}_{S}(\mathcal{D}, \text{Alg}^{S}(A))}(\alpha \times \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B})))
\end{array}$$

of categories over $S$.

The pullback of $\Psi$ along a section $\alpha$ of $\text{Alg}^{S}(A) \to S$ is a canonical equivalence

$$\text{LMod}^{S}_{\alpha}(\delta^{*}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{B}))) \simeq \text{Fun}_{T}^{S}(\mathcal{D}, \text{LMod}^{T}(\mathcal{B}))$$

over $S$. 

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3 Endomorphism objects

3.1 Basic notions and constructions of enriched category theory

3.1.1 Basic notions of enriched category theory

We take Lurie’s model of enriched categories with some slight modifications:

Let $M^\otimes \to LM^\otimes$ be an operad over $LM^\otimes$. Set $\mathcal{D} := \{m\} \times_{LM^\otimes} M^\otimes$ and $\mathcal{C}^\otimes := Ass^\otimes \times_{LM^\otimes} M^\otimes$.

Let $X, Y$ be objects of $\mathcal{D}$ and $\alpha$ an object of $\mathcal{C}$ and let $\alpha \in Mul_{M^\otimes}(A, X; Y)$.

If $(A, \alpha)$ represents the presheaf $Mul_{M^\otimes}(-, X; Y) : \mathcal{C}^\op \to \mathcal{S}(\kappa)$, i.e. if evaluation at $\alpha$ induces an equivalence

$$\mathcal{C}(B, A) \to \mathcal{S}(\kappa)(Mul_{M^\otimes}(A, X; Y), Mul_{M^\otimes}(B, X; Y)) \to Mul_{M^\otimes}(B, X; Y),$$

we say that $\alpha \in Mul_{M^\otimes}(A, X; Y)$ exhibits $A$ as the morphism object of $X$ and $Y$ and write $[X, Y]$ for $A$.

If $X = Y$, we say that $\alpha \in Mul_{M^\otimes}(A, X; X)$ exhibits $A$ as the endomorphism object of $X$ and write $[X, X]$ for $A$.

For every $n \in \mathbb{N}$ we set $Ass_n := Mul_{Ass^\otimes}(\underbrace{a, ..., a}_{n}; a)$.

Denote $\sigma \in Mul_{LM^\otimes}(a, m; m)$ the unique object and for every $\alpha \in Ass_n$ for some $n \in \mathbb{N}$ denote $\alpha'$ the identity of $\alpha$, the identity of $m$ and $\sigma$ under the operadic composition $Mul_{LM^\otimes}(a, m; m) \times (Mul_{LM^\otimes}(a', ..., \alpha; a) \times Mul_{LM^\otimes}(m; m)) \to Mul_{LM^\otimes}(a', ..., a, m; m)$.

We say that $M^\otimes \to LM^\otimes$ exhibits $\mathcal{D} = \{m\} \times_{LM^\otimes} M^\otimes$ as pseudo enriched over $\mathcal{C}^\otimes = Ass^\otimes \times_{LM^\otimes} M^\otimes$ if the functor $\mathcal{C}^\otimes \to Ass^\otimes$ is a locally cocartesian fibration and the following condition holds:

For every objects $A_1, ..., A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and $X, Y \in \mathcal{D}$ and every $\alpha \in Ass_n$ the canonical map

$$\zeta : Mul_{M^\otimes}(\otimes_\alpha(A_1, ..., A_n), X; Y) \\
\{\sigma\} \times_{Mul_{LM^\otimes}(a, m; m)} Mul_{M^\otimes}(\otimes_\alpha(A_1, ..., A_n), X; Y) \\
\{\alpha'\} \times_{Mul_{LM^\otimes}(a', ..., a, m; m)} Mul_{M^\otimes}(A_1, ..., A_n, X; Y)$$

is an equivalence.

We say that an operad $M^\otimes \to LM^\otimes$ over $LM^\otimes$ exhibits $\mathcal{D}$ as enriched over $\mathcal{C}^\otimes$ if it exhibits $\mathcal{D}$ as pseudo enriched over $\mathcal{C}^\otimes$ and for every objects $X, Y \in \mathcal{D}$ there exists a morphism object $[X, Y] \in \mathcal{C}$.

Let $M^\otimes, N^\otimes$ be operads over $LM^\otimes$ that exhibit categories $M, N$ as enriched over the same locally cocartesian fibration of operads $\mathcal{C}^\otimes \to Ass^\otimes$.

We call a map of operads $M^\otimes \to N^\otimes$ over $LM^\otimes$, whose pullback to $Ass^\otimes$ is the identity, a $\mathcal{C}$-enriched functor.

Convention 1. We make the following convention for the next sections except the appendix.
Let $\mathcal{M}^\otimes \to \mathcal{L}^\otimes$ be an operad over $\mathcal{L}^\otimes$. Set $\mathcal{D} := \{m\} \times_{\mathcal{L}^\otimes} \mathcal{M}^\otimes$ and $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\mathcal{L}^\otimes} \mathcal{M}^\otimes$. Let $X$ be an object of $\mathcal{D}$.

When we say that $X$ admits an endomorphism object or that an object $Y \in \mathcal{C}$ is the endomorphism object of $X$ or that a morphism $\alpha \in \text{Mul}_{\mathcal{M}^\otimes}(Y, X; X)$ exhibits $Y$ as the endomorphism object of $X$, we implicitly assume that for every objects $A_1, \ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and every $\alpha \in \text{Ass}$, the canonical map

$$
\zeta : \text{Mul}_{\mathcal{M}^\otimes}(\Theta_\alpha(A_1, \ldots, A_n), X; X) \cong \{\sigma\} \times \text{Mul}_{\mathcal{M}^\otimes}(\Theta_\alpha(A_1, \ldots, A_n), X; X) \\
\{\alpha'\} \times \text{Mul}_{\mathcal{M}^\otimes}(A_1, \ldots, A_n, X; X)
$$

is an equivalence.

As usual we give parametrized notions of enrichment:

Let $\mathcal{M}^\otimes \to \mathcal{N}^\otimes \times S$ a locally cocartesian $S$-family of operads over $\mathcal{L}^\otimes$.

Set $\mathcal{D} := \{m\} \times_{\mathcal{L}^\otimes} \mathcal{M}^\otimes$ and $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\mathcal{L}^\otimes} \mathcal{M}^\otimes$.

We call a locally cocartesian $S$-family $\mathcal{M}^\otimes \to \mathcal{N}^\otimes \times S$ of operads over $\mathcal{L}^\otimes$ a locally cocartesian $S$-family of categories pseudo-enriched respectively enriched in $\mathcal{C}^\otimes$ if for all $s \in S$ the induced functor $\mathcal{C}^\otimes_s \to \text{Ass}^\otimes$ is a locally cocartesian fibration (equivalently if the functor $\mathcal{C}^\otimes \to \text{Ass}^\otimes \times S$ is a locally cocartesian fibration) and $\mathcal{C}^\otimes_s$ exhibits $D_s$ as pseudo-enriched respectively enriched in $\mathcal{C}_s$.

Let $\mathcal{M}^\otimes, \mathcal{N}^\otimes$ be locally cocartesian $S$-families of categories enriched in $\mathcal{C}^\otimes$.

We call a map $\mathcal{M}^\otimes \to \mathcal{N}^\otimes$ of locally cocartesian $S$-families of operads over $\mathcal{L}^\otimes$, whose pullback to $\text{Ass}^\otimes$ is the identity, a map of locally cocartesian $S$-families of $\mathcal{C}^\otimes$-enriched categories.

**Example 3.1.** Let $\mathcal{M}^\otimes \to \mathcal{N}^\otimes$ be a $\mathcal{L}^\otimes$-monoidal category that exhibits a category $\mathcal{D}$ as left module over a monoidal category $\mathcal{C}$.

Denote $1_\mathcal{C} : \text{Ass}^\otimes \to \mathcal{C}^\otimes$ the initial object of $\text{Alg}(\mathcal{C})$ and $\mathcal{C}^\otimes_{/1_\mathcal{C}}$ the pullback of the cocartesian fibration $(\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{1}$ of operads along the map $1_\mathcal{C} : \text{Ass}^\otimes \to \mathcal{C}^\otimes$ of operads over $\text{Ass}^\otimes$.

By remark [24] the functor $\text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}((1), \mathcal{D})$ is a left module over $\mathcal{D} \times \mathcal{C}^\otimes_{/1_\mathcal{C}}$ in $\text{Cat}_{\text{cocart}}(\mathcal{C}_s)$.

Given a morphism $f : K \to 1_\mathcal{C}$ in $\mathcal{C}$ and $g : Y \to X$ in $\mathcal{D}$ we have $f \otimes g = K \otimes Y \to 1_\mathcal{C} \otimes X = X$.

Especially the functor $\text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}((1), \mathcal{D})$ can be promoted to a cocartesian $\mathcal{D}$-family of categories pseudo-enriched over $\mathcal{C}^\otimes_{/1_\mathcal{C}}$.

For every category $S \in \text{Cat}_\infty(\kappa)$ we have a finite products preserving functor $- \times S : \text{Cat}_\infty(\kappa) \to \text{Cat}_\infty(\kappa)_S$ that makes $\text{Cat}_\infty(\kappa)_S$ to a left module over $\text{Cat}_\infty(\kappa)$.

For $\mathcal{D} = \text{Cat}_\infty(\kappa)$ with the left $\mathcal{C} = \text{Cat}_\infty(\kappa)$-module structure coming from the cartesian structure we see that the left modules $\text{Cat}_\infty(\kappa)_S$ over $\text{Cat}_\infty(\kappa)$ for $S \in \text{Cat}_\infty(\kappa)$ organize to a left module structure on $\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \to \text{Fun}((1), \text{Cat}_\infty(\kappa))$ over $\text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa) \to \text{Cat}_\infty(\kappa)$.

So the functor $\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \to \text{Fun}((1), \text{Cat}_\infty(\kappa))$ lifts to a cocartesian $\text{Cat}_\infty(\kappa)$-family of categories pseudo-enriched over $\text{Cat}_\infty(\kappa)$.
Given a functor $H : S \to \mathbf{Cat}_\infty(\kappa)$ this left module structure gives rise to a left module structure on $\text{Fun}_{\mathbf{Cat}_\infty(\kappa)}(S, \text{Fun}(\Delta^1, \mathbf{Cat}_\infty(\kappa))) \cong \text{Fun}(S, \mathbf{Cat}_\infty(\kappa))/H$ over $\text{Fun}_{\mathbf{Cat}_\infty(\kappa)}(S, \mathbf{Cat}_\infty(\kappa) \times \mathbf{Cat}_\infty(\kappa)) \cong \text{Fun}(S, \mathbf{Cat}_\infty(\kappa))$, which is the diagonal action.

3.1.2 Functoriality of morphism objects

Let $\mathcal{M}^\otimes$ be a cocartesian $S$-family of categories enriched in $\mathcal{C}^\otimes := \mathbf{Ass}^\otimes \times_{\mathbf{LM}^\otimes} \mathcal{M}^\otimes$.

By $\ldots \ldots$ we have a multi-mapping space functor $\text{Mul}_{\mathcal{M}^\otimes}(\cdot, -; -) : \mathcal{C}^{\text{rev}} \times_S \mathcal{M}^{\text{ev}} \times_S \mathcal{M} \to \mathcal{S}(\kappa)$ relative to $S$ that is adjoint to a functor $\mathcal{M}^{\text{ev}} \times_S \mathcal{M} \to \text{Map}_S(\mathcal{C}^{\text{rev}}, \mathcal{S}(\kappa) \times S)$ over $S$.

As $\mathcal{M}^{\otimes} \to \mathbf{LM}^{\otimes}$ is a cocartesian $S$-family of categories enriched over $\mathcal{C}^{\otimes}$, this functor over $S$ induces a functor $\mathcal{M}^{\text{ev}} \times_S \mathcal{M} \to \mathcal{C}$ over $S$ adjoint to a functor $\theta : \mathcal{M} \to \text{Map}_S(\mathcal{M}^{\text{ev}}, \mathcal{C})$ that sends an object $X$ of $\mathcal{M}$ lying over some $s \in S$ to the functor $[\cdot, X] : \mathcal{M}^{\text{op}} \to \mathcal{C}_s$.

In proposition 7.14 we construct a map $\mathcal{M}^{\otimes} \to \text{Map}_S(\mathcal{M}^{\text{ev}}, \mathcal{C})^\otimes$ of $S$-families of operads over $\mathbf{LM}^{\otimes}$, whose underlying functor is the functor $\theta : \mathcal{M} \to \text{Map}_S(\mathcal{M}^{\text{ev}}, \mathcal{C})$ over $S$ and whose pullback to $\mathbf{Ass}^{\otimes}$ is the diagonal map $\delta : \mathcal{C}^{\otimes} \to \text{Map}_S(\mathcal{M}^{\text{ev}}, \mathcal{C})^\otimes$ of $S$-families of operads over $\mathbf{Ass}^{\otimes}$.

For $S$ contractible this guarantees the following:

Let $X$ be an object of $\mathcal{M}$ and $\beta \in \text{Mul}_{\mathcal{M}^\otimes}(B, X, X)$ an operation that exhibits $B = [X, X]$ as the endomorphism object of $X$ in $\mathcal{C}$.

Being a map of operads over $\mathbf{LM}^{\otimes}$ the functor $\theta$ sends the endomorphism $[X, X]$-left module structure on $X$ to a $\delta([X, X])$-left module structure on $[\cdot, X] : \mathcal{M}^{\text{op}} \to \mathcal{C}$ corresponding to a lift $\mathcal{M}^{\text{op}} \to \text{LMod}_{[X, X]}(\mathcal{C})$ of $[\cdot, X] : \mathcal{M}^{\text{op}} \to \mathcal{C}$.

So for every object $Y$ of $\mathcal{M}$ the morphism object $[Y, X]$ is a left-module over the endomorphism object $[X, X]$ in $\mathcal{C}$ and for every morphism $Y \to Z$ in $\mathcal{M}$ the induced morphism $[Z, X] \to [Y, X]$ is $[X, X]$-linear.

3.1.3 Change of enriching category

Let $\mathcal{M}^\otimes$ be an operad over $\mathbf{LM}^{\otimes}$ that exhibits a category $\mathcal{M}$ as enriched over a locally cocartesian fibration of operads $\mathcal{D}^\otimes \to \mathbf{Ass}^{\otimes}$.

Let $\mathcal{C}^\otimes \to \mathbf{Ass}^{\otimes}$ be a locally cocartesian fibration of operads and $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a map of operads over $\mathbf{Ass}^{\otimes}$ that admits a right adjoint $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ relative to $\mathbf{Ass}^{\otimes}$.

Then by proposition 7.21 combined with lemma 7.22 one can pullback $\mathcal{M}^\otimes$ along $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ to obtain an operad $F^\ast(\mathcal{M})^\otimes$ over $\mathbf{LM}^{\otimes}$ that exhibits $\mathcal{M}$ as enriched in $\mathcal{C}^{\otimes}$.

$F^\ast(\mathcal{M})^\otimes$ is determined by the condition that for every operad $\Omega^\otimes$ over $\mathbf{LM}^{\otimes}$, where we set $\mathcal{B}^\otimes := \mathbf{Ass}^\otimes \times_{\mathbf{LM}^\otimes} \Omega^\otimes$, the commutative square

$$
\begin{array}{ccc}
\text{Alg}_{\mathcal{Q}/\mathbf{LM}^\otimes}(F^\ast(\mathcal{M})^\otimes) & \longrightarrow & \text{Alg}_{\mathcal{Q}/\mathbf{LM}^\otimes}(\mathcal{M}^\otimes) \\
\downarrow & & \downarrow \\
\text{Alg}_{\mathcal{B}^\otimes/\mathbf{Ass}^{\otimes}}(\mathcal{C}^\otimes) & \longrightarrow & \text{Alg}_{\mathcal{B}^\otimes/\mathbf{Ass}^{\otimes}}(\mathcal{D}^\otimes).
\end{array}
$$

(4)
Remark 3.2.

Let \( M^\circ \to LM^\circ, N^\circ \to LM^\circ \) be \( \kappa \)-small operads over \( LM^\circ \) that exhibit categories \( M \) respectively \( N \) as pseudo-enriched over locally cartesian fibrations of operads \( \mathcal{C}^\circ \to \text{Ass}^\circ \) respectively \( \mathcal{D}^\circ \to \text{Ass}^\circ \) and let \( F : M^\circ \to N^\circ \) be a map of operads over \( LM^\circ \).

Let \( X, Y \) be objects of \( M \) that admit a morphism object \([X, Y]\).

The canonical morphisms

\[
F([Y, X]) \to [F(Y), F(X)], \quad F([X, X]) \to [F(X), F(X)]
\]

in \( \mathcal{D} \) organize to a morphism of \( LM^\circ \)-algebras, where \( F([Y, X]) \) carries the \( F([X, X]) \)-left module structure that is the image under \( F : \mathcal{C} \to \mathcal{D} \) of the canonical \([X, X]\)-left module structure on \([Y, X] \) and \([F(Y), F(X)] \) carries the canonical \([F(X), F(X)] \)-left module structure.

Assume that the pullback of \( F \) to \( \text{Ass}^\circ \) is the identity.

Let \( T \in \text{Alg}(\mathcal{C}, \mathcal{X}) \) be a monad on \( \mathcal{X} \) and \( \phi : Y \to X \) a left-module over \( T \) in \([Y, X] \) with respect to the canonical \([X, X]\)-left module structure on \([Y, X] \).

Then the morphism \( F(\phi) : F(Y) \to F(X) \) in \( \mathcal{D} \) is a left-module over the monad \( F(T) \in \text{Alg}(\mathcal{F}(X), \mathcal{F}(X)) \) on \( F(X) \) with respect to the canonical \([F(X), F(X)] \)-left module structure on \([F(Y), F(X)] \).

If \( T \) is the endomorphism object of \( \phi \), the monad \( F(T) \) is the endomorphism object of \( F(\phi) \) by proposition 3.6.

Now we specify to the case \( \mathcal{C}^\circ = \text{Cat}^\kappa(\kappa)^x \): .

We call a category enriched over \( \text{Cat}^\kappa(\kappa)^x \) a 2-category and a \( \text{Cat}^\kappa(\kappa)^x \) enriched functor a 2-functor.

We call a (locally) cocartesian \( S \)-family of categories enriched in \( \text{Cat}^\kappa(\kappa)^x \) a (locally) cocartesian \( S \)-family of 2-categories and a map of (locally) cocartesian \( S \)-families of \( \text{Cat}^\kappa(\kappa)^x \)-enriched categories a map of (locally) cocartesian \( S \)-families of 2-categories.

We denote the pullback of a 2-category \( \mathcal{C} \) along the opposite category involution \( (-)^{\text{op}} : \text{Cat}^\kappa(\kappa) \to \text{Cat}^{\kappa}(\kappa) \) by \( \mathcal{C}_{\text{op}} \) so that in \( \mathcal{C}_{\text{op}} \) the 2-morphisms are reversed.

We have a notion of adjunction in any 2-category \( \mathcal{C} \):

Let \( f : X \to Y \) and \( g : Y \to X \) be morphisms of \( \mathcal{C} \).

We say that \( f \) is left adjoint to \( g \) or \( g \) is right adjoint to \( f \) or that \( (f, g) \) is an adjoint pair if there are 2-morphisms \( \eta : \text{id}_X \to g \circ f \) and \( \epsilon : f \circ g \to \text{id}_Y \) such that the triangular identities \((\epsilon \circ f) \circ (f \circ \eta) = \text{id}_f \) and \((g \circ \epsilon) \circ (\eta \circ g) = \text{id}_g\) hold.
3.2 Endomorphism objects

Let $\mathcal{M}^\otimes \to \text{LM}^\otimes$ be an operad over $\text{LM}^\otimes$. Set $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\text{LM}^\otimes} \mathcal{M}^\otimes$ and $\mathcal{M} := \{m\} \times_{\text{LM}^\otimes} \mathcal{M}^\otimes$. Let $X$ be an object of $\mathcal{M}$.

Denote $\varphi : \Delta^1 \to \text{LM}^\otimes$ the morphism of $\text{LM}^\otimes$ corresponding to the unique object of $\text{Mul}_{\text{LM}^\otimes}(a, m; m)$.

Using $\varphi$ we form the category $\text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes)$ and have canonical functors $\text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \mathcal{M}^\otimes_{\{a, m\}} \cong \mathcal{C} \times \mathcal{M}$ and $\text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \mathcal{M}^\otimes = \mathcal{M}$ evaluating at 0 respectively $\mathcal{I}$.

We set $\mathcal{C}[X] := \{(X, X)\} \times_{\mathcal{M} \times \mathcal{M}} \text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes)$.

We have a forgetful functor $\mathcal{C}[X] \to \text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \mathcal{C}$ that is a right fibration classifying the functor $\text{Mul}_{\text{LM}^\otimes}(-, X; X) : \mathcal{C}^{\text{op}} \to \mathcal{S}(\mathcal{I})$ according to lemma 7.20.

So an object of $\mathcal{C}[X]$ corresponding to a pair $(A, \alpha)$ consisting of an object $A$ of $\mathcal{C}$ and an object $\alpha$ of $\text{Mul}_{\text{LM}^\otimes}(A, X; X)$ is a final object of $\mathcal{C}[X]$ if and only if for all objects $B$ of $\mathcal{C}$ evaluation at $\alpha$ induces an equivalence

$$
\mathcal{C}(B, A) \to \mathcal{S}(\mathcal{I})(\text{Mul}_{\text{LM}^\otimes}(A, X; X), \text{Mul}_{\text{LM}^\otimes}(B, X; X)) \to \text{Mul}_{\text{LM}^\otimes}(B, X; X),
$$

i.e. if and only if $\alpha$ exhibits $A$ as the endomorphism object of $X$.

We have a forgetful functor

$$
\text{LMod}(\mathcal{M}) = \text{Alg}_{\text{LM}^\otimes/}(\mathcal{M}^\otimes) \subset \text{Fun}_{\text{LM}^\otimes}(\text{LM}^\otimes, \mathcal{M}^\otimes) \to \mathcal{M} \times_{\mathcal{M} \times \mathcal{M}} \text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes)
$$

over $\mathcal{M}$, where the functor $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is the diagonal functor, that induces a forgetful functor

$$
\{X\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M}) = \{X\} \times_{\mathcal{M}} \text{Alg}_{\text{LM}^\otimes/}(\mathcal{M}^\otimes) \subset \{X\} \times_{\mathcal{M}} \text{Fun}_{\text{LM}^\otimes}(\text{LM}^\otimes, \mathcal{M}^\otimes) \to \mathcal{C}[X] = \{(X, X)\} \times_{\mathcal{M} \times \mathcal{M}} \text{Fun}_{\text{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes).
$$

By proposition 7.12 and convention 4 if $\mathcal{C}[X]$ admits a final object, the final object lifts to a final object of $\{X\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M})$.

As the forgetful functor $\{X\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M}) \to \mathcal{C}[X]$ is conservative, in this case an object of $\{X\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M})$ is a final object if and only if its image in $\mathcal{C}[X]$ is.

So by abuse of notation we will identify the final object of $\{X\} \times_{\mathcal{M}} \text{LMod}(\mathcal{M})$ with the final object of $\mathcal{C}[X]$ if both exist.

Endomorphism objects are functorial in the following way:

Let $F : \mathcal{M}^\otimes \to \mathcal{M'}^\otimes$ be a map of operads over $\text{LM}^\otimes$. Set $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\text{LM}^\otimes} \mathcal{M}^\otimes$, $\mathcal{D}^\otimes := \text{Ass}^\otimes \times_{\text{LM}^\otimes} \mathcal{M'}^\otimes$, $\mathcal{M} := \{m\} \times_{\text{LM}^\otimes} \mathcal{M}^\otimes$, $\mathcal{M'} := \{m\} \times_{\text{LM}^\otimes} \mathcal{M'}^\otimes$.

Let $X$ be an object of $\mathcal{M}$ such that $X$ and $F(X)$ admit endomorphism objects $[X, X]$ respectively $[F(X), F(X)]$.

The map $F : \mathcal{M}^\otimes \to \mathcal{M'}^\otimes$ of operads over $\text{LM}^\otimes$ gives rise to a commutative square

$$
\text{Alg}(\mathcal{C}) \longrightarrow \{F(X)\} \times_{\mathcal{M'}} \text{LMod}(\mathcal{M'})
$$

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Let \( \mathcal{M}^\otimes \to S \times \mathcal{LMod}^\otimes \) be a cocartesian S-family of categories enriched over the cocartesian S-family \( \mathcal{C}^\otimes \to S \times \text{Ass}^\otimes \) of representable planar operads.

Let \( \mathcal{M}^\otimes \to S \times \mathcal{LMod}^\otimes \) be a S-family of operads over \( \mathcal{LMod}^\otimes \) and \( X \) a section of \( \varphi : \Delta^1 \to \mathcal{LMod}^\otimes \).

Denote \( \varphi : \Delta^1 \to \mathcal{LMod}^\otimes \) the morphism of \( \mathcal{LMod}^\otimes \) corresponding to the unique object of \( \text{Mul}_{\mathcal{LMod}}(a,m;m) \).

Using \( \varphi \) we form the category \( \text{Fun}^S_{S \times \mathcal{LMod}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes) \) and have canonical functors \( \text{Fun}^S_{S \times \mathcal{LMod}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes) \to \mathcal{M}^\otimes \). More coherently we study endomorphism objects relative to \( S \):

We have a forgetful functor
\[
\mathcal{C}[X]^S : \text{Fun}^S_{S \times \mathcal{LMod}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes) \to \mathcal{C}
\]
over \( S \) that induces on the fiber over \( s \in S \) the right fibration
\[
\mathcal{C}_s[X(s)] = \{(X(s), X(s)) \} \times_{\mathcal{M} \times \mathcal{M}} \text{Fun}_{\mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \text{Fun}_{\mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \mathcal{C}_s
\]
classifying the functor \( \text{Mul}_{\mathcal{LMod}}(-, X(s); X(s)) : \mathcal{C}^{op} \to \mathcal{S}(\kappa) \) according to lemma \ref{2.3.2} and on sections the right fibration
\[
\text{Fun}_{\mathcal{S} \times \mathcal{LMod}^\otimes}(\Delta^1, \text{Fun}_{\mathcal{S} \times \mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \text{Fun}_{\mathcal{S} \times \mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \text{Fun}_{\mathcal{S} \times \mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes)
\]
classifying the functor \( \text{Fun}_{\mathcal{S} \times \mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \text{Fun}_{\mathcal{S} \times \mathcal{LMod}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \)

We have a forgetful functor
\[
\text{LMod}^S(\mathcal{M}) = \text{Alg}_{\mathcal{LMod}^\otimes}^S(\mathcal{M}^\otimes) \subset \text{Fun}^S_{S \times \mathcal{LMod}^\otimes}(S \times \mathcal{LMod}^\otimes, \mathcal{M}^\otimes) \to \mathcal{M} \times \text{Fun}^S_{S \times \mathcal{LMod}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes)
\]
over $M$, where the functor $M \to M \times_S M$ is the diagonal functor over $S$, that induces a forgetful functor

$$S \times_M \text{LMod}^S(M) = S \times_M \text{Alg}^S_{S \times_M \text{LMod}^S(M)^*} \subset S \times_M \text{Fun}^S_{S \times_M \text{LMod}^S(M)^*}(S \times \Delta^1, M^*)$$

$$\to \mathcal{C}[X]^S = S \times_{(M \times_S M)} \text{Fun}^S_{S \times_M \text{LMod}^S(M^*)(S \times \Delta^1, M^*)}$$

over $S$ that induces on the fiber over $s \in S$ the forgetful functor

$$(X(s)) \times_M \text{LMod}(M_s) = \{X(s)\} \times_M \text{Alg}^S_{S \times_M \text{LMod}^S(M)^*} \subset \{X(s)\} \times_M \text{Fun}^S_{S \times_M \text{LMod}^S(M)^*}(LM^S, M^*)$$

$$\to \mathcal{C}_s[X(s)] = \{(X(s), X(s))\} \times_{M \times M} \text{Fun}^S_{S \times_M \text{LMod}^S(M^*)}(\Delta^1, M^*_s)$$

and on sections the forgetful functor

$$(X) \times_{\text{Fun}_S(S, M)} \text{LMod}(\text{Fun}_S(S, M)) = \{X\} \times_{\text{Fun}_S(S, M)} \text{Alg}^S_{S \times_M \text{LMod}^S(M)^*}(\text{Fun}_S(S, M)^*) \subset$$

$$(X \times \text{Fun}_S(S, M))^S \text{Fun}^S_{S \times_M \text{LMod}^S(M)^*}(\text{Fun}_S(S, M)^*), \to \text{Fun}_S(S, \mathcal{C})^S[X] =$$

$$\{((X, X)) \times_{(\text{Fun}_S(S, M) \times \text{Fun}_S(S, M))} \text{Fun}^S_{S \times_M \text{LMod}^S(M^*)}(\Delta^1, \text{Fun}_S(S, M)^*)\}.$$

Assume that $M^* \to S \times LM^S$ is a locally cocartesian $S$-family of operads over $LM^S$ and $X$ is a locally cocartesian section of $\varphi : M \to S$ such that for all $s \in S$ the image $X(s) \in \mathcal{C}_s$ admits an endomorphism object, in other words the category $\mathcal{C}[X]^S = \mathcal{C}_s[X(s)]$ admits a final object.

Then by proposition [3.12] for every $s \in S$ the final object of $\mathcal{C}[X]^S$ lifts to a final object of the category $S \times_M \text{LMod}^S(M)^* = \{X(s)\} \times_M \text{LMod}(M_s)$.

So by lemma [3.32] the category

$$\text{Fun}_S(S, S \times_M \text{LMod}^S(M))^* = \{X\} \times_{\text{Fun}_S(S, M)} \text{Fun}_S(S, \text{LMod}^S(M)^*) \subset$$

$$(X) \times_{\text{Fun}_S(S, M)} \text{LMod}(\text{Fun}_S(S, M))$$

admits a final object $Y$ such that for every object $s \in S$ the image $Y(s)$ is the final object of the category $\{X(s)\} \times_M \text{LMod}(M_s)$.

The functor $S \times_M \text{LMod}^S(M) \to \text{Alg}^S(\mathcal{C})$ over $S$ sends $Y$ to an object $[X, X]^S$ of the category $\text{Alg}(\text{Fun}_S(S, \mathcal{C})) \equiv \text{Fun}_S(S, \text{Alg}^S(\mathcal{C}))$.

So $Y$ exhibits $[X, X]^S$ as the endomorphism object of $X$ with respect to $\text{Fun}_S(S, M)^* \to LM^S$.

**Remark 3.3.**

Let $f : S \to t$ be a morphism of $S$ such that the induced map $f_* : M^S \to M^S$ of operads over $LM^S$ preserves the endomorphism object of $X(s)$, in other words such that the functor $(S \times_M \text{LMod}^S(M))^* \to (S \times_M \text{LMod}^S(M))^*$, induced by $f$ preserves the final object.

Then the functor $Y : S \to S \times_M \text{LMod}^S(M)$ over $S$ sends $f$ to a locally cocartesian morphism of the locally cocartesian fibration $S \times_M \text{LMod}^S(M) \to S$ and thus the composition $[X, X]^S : S \to S \times_M \text{LMod}^S(M) \to \text{Alg}^S(\mathcal{C})$ sends $f$ to a locally cocartesian morphism of the locally cocartesian fibration $\text{Alg}^S(\mathcal{C}) \to S$.
Let $\psi : T \to S$ be a $\kappa$-small category over $S$ and $X : T \to M$ a functor over $S$ that sends every morphism of $T$ to a locally $\varphi$-cocartesian morphism corresponding to a cocartesian section of the pullback $T \times_S M \to T$.

Assume that for every object $t \in T$ the image $X(t) \in M_{\psi(t)}$ admits an endomorphism object.

Then the category
\[
\{X\} \times_{\Fun_T(T, M)} \LMod(\Fun_S(T, M)) \cong \Fun_T(T, T \times_{(T \times_{S} M)} LMod^T(T \times_{S} M)) \cong \Fun_T(T, T \times_{M} LMod^S(M))
\]
amits a final object $Y$ such that for every object $t \in T$ the image $Y(t)$ is the final object of the category $(X(t)) \times_{M_{\psi(t)}} \LMod(M_{\psi(t)})$ and that lies over an object $[X, X]^T$ of the category Alg$(\Fun_S(T, \mathfrak{C})) \cong \Fun_S(T, \Alg^S(\mathfrak{C}))$.

In other words $Y$ exhibits $[X, X]^T$ as the endomorphism object of $X$ with respect to $T \times_{S} M \to \LM^S \times T$.

Let $f : s \to t$ be a morphism of $T$ such that the induced map $f_* : M_{\psi(t)}^S \to M_{\psi(s)}^S$ of operads over $\LM^S$ preserves the endomorphism object of $X(s)$, in other words such that the functor $(T \times_{M} LMod^S(M))_s \to (T \times_{M} LMod^S(M))_t$ induced by $f$ preserves the final object.

Then the functor $Y : T \to T \times_{M} LMod^S(M) \to T$ sends $f$ to a locally cartesian morphism of $T \times_{M} LMod^S(M) \to T$ and so the composition $[X, X]^T : T \to T \times_{M} LMod^S(M) \to T \times_{S} \Alg^S(\mathfrak{C})$ sends $f$ to a locally cartesian morphism of the locally cocartesian fibration $T \times_{S} \Alg^S(\mathfrak{C}) \to T$.

Construction 1.
Denote $M^2$ the subcategory of $M$ spanned by the morphisms that are cocartesian with respect to the locally cocartesian fibration $M \to S$.

Let $\mathfrak{E} \subset S$ be a subcategory.

Denote $M^\univ_{\End} \subset M_{\End} \subset M^2$ the full subcategories spanned by the objects of $M$ lying over some object $s$ of $S$ that admit an endomorphism object respectively that admit an endomorphism object that is preserved by the functors on the fibers of the locally cocartesian fibration $M \to S$ induced by morphisms of $\mathfrak{E}$.

The left fibration $M^2 \to S$ restricts to a left fibration $M^\univ_{\End} \to S$.

For $T = M_{\End} \to S$ and $X$ the canonical inclusion $M_{\End} \subset M$ the endomorphism object of $X$ is a functor $\End : M_{\End} \to \Alg^S(\mathfrak{E})$ over $S$.

For $T = M^\univ_{\End} \to S$ and $X$ the canonical inclusion $M^\univ_{\End} \subset M$ the endomorphism object of $X$ is a map $\End : M^\univ_{\End} \to \Alg^S(\mathfrak{E})$ of locally cartesian fibrations relative to $\mathfrak{E}$.

Remark 3.4.

1. By lemma [6.2] we have a canonical equivalence over $\Alg^S(\mathfrak{E})$ between the map $S \times_{M^2} \LMod^S(M) \to \Alg^S(\mathfrak{E})$ of locally cocartesian fibrations over $S$ and the map $\Alg^S(\mathfrak{E})_{\{[X, X] \to S \times_{\Alg^S(\mathfrak{E})} \Alg^S(\mathfrak{E})\}} \to \Alg^S(\mathfrak{E})_{\{1\}}$ of
cocommutative fibrations over $S$ that induces on the fiber over $s \in S$ the canonical equivalence $\{X(s)\} \times_{M_1} \text{LMod}(M_2) \simeq \text{Alg}(\mathcal{C}_2)/\{X(s), X(s)\}$ over $\text{Alg}(\mathcal{C}_2)$.

Pulling back this equivalence over $\text{Alg}^S(\mathcal{C})$ along a section of $\text{Alg}^S(\mathcal{C}) \to S$ we obtain a canonical equivalence

$$S \times_{(\text{Alg}^S(\mathcal{C}) \times_{M_2} \text{LMod}(M_1))} \text{LMod}^S(M) \simeq S \times_{(\text{Alg}^S(\mathcal{C}) \times_{M_2} \text{LMod}(M_1))} \text{Alg}^S(\mathcal{C})^A$$

over $S$.

2. Let $\psi : T \to S$ be a $\kappa$-small category over $S$ and $X : T \to M_{\text{End}}$, $Y : T \to \text{Alg}^S(\mathcal{C})$ functors over $S$.

Applying 1. to the pullback $T \times_S M^\otimes \to M^\otimes \times T$ we obtain a canonical equivalence $T \times_{(\text{Alg}^S(\mathcal{C}) \times_{M_2} \text{LMod}(M_1))} \text{LMod}^S(M) \simeq T \times_{(\text{Alg}^S(\mathcal{C}) \times_{M_2} \text{LMod}(M_1))} \text{Alg}^S(\mathcal{C})^A$ over $T$ that induces on the fiber over $t \in T$ the canonical equivalence

$$\{X(t)\} \times_{M_1} \text{LMod}_{Y(t)}(M_2) \simeq \text{Alg}(\mathcal{C}_2)(Y(t), [X(t), X(t)])$$

Especially we obtain a canonical equivalence

$$(\text{Alg}^S(\mathcal{C}) \times_S M_{\text{End}}) \times_{(\text{Alg}^S(\mathcal{C}) \times_S M_2)} \text{LMod}^S(M) \simeq (\text{Alg}^S(\mathcal{C}) \times_S M_{\text{End}}) \times_{(\text{Alg}^S(\mathcal{C}) \times_S M_2)} \text{Alg}^S(\mathcal{C})^A$$

over $\text{Alg}^S(\mathcal{C}) \times_S M_{\text{End}}$.

For later use we add the following lemma:

**Lemma 3.5.** Let $\mathcal{B}, \mathcal{C}$ be $\kappa$-small monoidal categories and $F : \mathcal{B} \to \mathcal{C}$ a monoidal functor that admits a right adjoint $G : \mathcal{C} \to \mathcal{B}$.

Let $\mathcal{D}$ be a left module over $\mathcal{C}$.

The monoidal functor $F : \mathcal{B} \to \mathcal{C}$ induces a functor on left modules $F^* : \text{LMod}_\mathcal{C}(\text{Cat}_\mathcal{C}(\kappa)) \to \text{LMod}_\mathcal{B}(\text{Cat}_\mathcal{B}(\kappa))$ via the cartesian fibration $\text{LMod}(\text{Cat}_\mathcal{C}(\kappa)) \to \text{Alg}(\text{Cat}_\mathcal{C}(\kappa))$.

Let $X$ be an object of $\mathcal{D}$ that admits an endomorphism object $[X, X] \in \text{Alg}(\mathcal{C})$ with respect to the given $\mathcal{C}$-left module structure on $\mathcal{D}$.

The adjunction $F : \mathcal{B} \rightleftarrows \mathcal{C} : G$ induces an adjunction $\text{Alg}(F) : \text{Alg}(\mathcal{B}) \rightleftarrows \text{Alg}(\mathcal{C}) : \text{Alg}(G)$ on associative algebras.

Denote $\varepsilon : \text{Alg}(F) \circ \text{Alg}(G) \to \text{id}_{\text{Alg}(\mathcal{C})}$ its counit. The counit

$$\varepsilon([X, X]) : \text{Alg}(F)(\text{Alg}(G)([X, X])) \to [X, X]$$

in $\text{Alg}(\mathcal{C})$ induces a functor on left modules $\varepsilon([X, X])^* : \text{LMod}_{[X, X]}(\mathcal{D}) \to \text{LMod}_{\text{Alg}(G)([X, X])}(\mathcal{D}) \simeq \text{LMod}_{\text{Alg}(G)([X, X])}(\mathcal{D})$ via the cartesian fibration $\text{LMod}(\mathcal{D}) \to \text{Alg}(\mathcal{C})$.

The left module structure on $X$ over $\text{Alg}(G)([X, X])$ given by $\varepsilon([X, X])^*(X)$ exhibits $\text{Alg}(G)([X, X])$ as the endomorphism object of $X$ with respect to the $\mathcal{B}$-left module structure on $\mathcal{D}$ given by $F^*(\mathcal{D})$. 50
Proof. Choose a cartesian lift $\varphi : F^*(\mathcal{D}) \to \mathcal{D}$ in $\text{LMod}(\text{Cat}_\infty(\kappa))$ of the monoidal functor $F : B \to \mathcal{C}$ with respect to the cartesian fibration $\text{LMod}(\text{Cat}_\infty(\kappa)) \to \text{Alg}(\text{Cat}_\infty(\kappa))$.

Then $\varphi$ (considered as a $\text{LM}^{\otimes}$-monoidal functor) induces a pullback square

$$
\begin{array}{c}
\text{LMod}(F^*(\mathcal{D}))
\downarrow
\downarrow
\text{Alg}(\mathcal{B})
\downarrow
\downarrow
\text{Alg}(\mathcal{C})
\end{array}
$$

and thus a pullback square

$$
\begin{array}{c}
\{X\} \times_\mathcal{D} \text{LMod}(F^*(\mathcal{D}))
\downarrow
\downarrow
\{X\} \times_\mathcal{D} \text{LMod}(\mathcal{D})
\end{array}
$$

By definition $[X,X]$ represents the right vertical right fibration of this square.

As $\text{Alg}(F) : \text{Alg}(\mathcal{B}) \to \text{Alg}(\mathcal{C})$ is left adjoint to $\text{Alg}(G) : \text{Alg}(\mathcal{C}) \to \text{Alg}(\mathcal{B})$, the left vertical right fibration of the square is represented by $\text{Alg}(G)([X,X])$ and is thus the endomorphism object of $X$ with respect to the $\mathcal{B}$-left module structure on $[Y,X]$ given by $F^*(\mathcal{D})$.

The $\text{Alg}(G)([X,X])$-left module structure on $X$ that exhibits $\text{Alg}(G)([X,X])$ as the endomorphism object of $X$ with respect to $F^*(\mathcal{D})$, i.e. the final object of $\{X\} \times_\mathcal{D} \text{LMod}(F^*(\mathcal{D}))$ is given by the image of the identity of $\text{Alg}(G)([X,X])$ under the equivalence

$$
\text{Alg}(\mathcal{B})_{\text{Alg}(G)([X,X])} \cong \text{Alg}(\mathcal{B}) \times_{\text{Alg}(\mathcal{C})} \text{Alg}(\mathcal{C})_{[X,X]} \cong \\
\text{Alg}(\mathcal{B}) \times_{\text{Alg}(\mathcal{C})} (\{X\} \times_\mathcal{D} \text{LMod}(\mathcal{D})) \cong \{X\} \times_\mathcal{D} \text{LMod}(F^*(\mathcal{D})),
$$

which is $\varepsilon([X,X])^*(X)$.

Now we use the theory of endomorphism objects to associate a monad to a given right adjoint morphism in a 2-category.

Let $\mathcal{C}$ be a $\kappa$-small 2-category and let $g : Y \to X$ be a morphism of $\mathcal{C}$ that admits a left adjoint.

Let $T \in \text{Alg}([X,X])$ be a monad equipped with a left-action on $g : Y \to X$ with respect to the canonical $[X,X]$-left module structure on $[Y,X]$.

We say that the left action map $\mu : T \circ g \to g$ in $[Y,X]$ exhibits $T$ as the monad associated to $g$ or the endomorphism monad of $g$ if $\mu$ exhibits $T$ as the endomorphism object of $g$ with respect to the canonical $[X,X]$-left module structure on $[Y,X]$.

The next proposition tells us that every right adjoint morphism in a 2-category admits an associated monad.
**Proposition 3.6.** Let $\mathcal{C}$ be a $\kappa$-small 2-category for a strongly inaccessible cardinal $\kappa$.

Let $X,Y$ be objects of $\mathcal{C}$ and $g : Y \to X$ a morphism of $\mathcal{C}$ that admits a left adjoint $f : X \to Y$ in $\mathcal{C}$.

Denote $\eta : \text{id}_X \to g \circ f$ the unit and $\varepsilon : f \circ g \to \text{id}_Y$ the counit of this adjunction.

1. For every morphism $h : X \to X$ of $\mathcal{C}$ the map

$$\alpha : [X,X](h,g \circ f) \to [Y,X](h \circ g, g \circ f) \xrightarrow{[Y,X](\text{h\circ g, g\circ f})} [Y,X](h \circ g, g)$$

is an equivalence.

So $g \circ \varepsilon : g \circ f \to g$ exhibits $g \circ f$ as the endomorphism object of $g : Y \to X$ with respect to the canonical $[X,X]$-left module structure on $[Y,X]$.

2. Let $T : X \to X$ be a morphism of $\mathcal{C}$ and $\varphi : T \circ g \to g$ a morphism in $[Y,X]$.

Denote $\psi$ the composition $T \xrightarrow{\varepsilon} T \circ g \circ f \xrightarrow{\varphi} g \circ f$ in $[X,X]$ and $\gamma$ the composition

$$[X,X](h,T) \to [Y,X](h \circ g,T \circ g) \xrightarrow{[Y,X](\text{h\circ g, T\circ f)})} [Y,X](h \circ g, g).$$

The morphism $\psi$ is an equivalence if and only if for every morphism $h : X \to X$ of $\mathcal{C}$ the map $\gamma$ is an equivalence.

So $\varphi : T \circ g \to g$ exhibits $T$ as the endomorphism object of $g : Y \to X$ with respect to the canonical $[X,X]$-left module structure on $[Y,X]$ if and only if $\psi$ is an equivalence.

**Proof.** Statement 1. and 2. follow from the following lemma 3.7.

---

**Lemma 3.7.** Let $\mathcal{C}$ be a $\kappa$-small 2-category for a strongly inaccessible cardinal $\kappa$.

Let $X,Y$ be objects of $\mathcal{C}$ and $g : Y \to X$ a morphism of $\mathcal{C}$ that admits a left adjoint $f : X \to Y$ in $\mathcal{C}$.

Denote $\eta : \text{id}_X \to g \circ f$ the unit and $\varepsilon : f \circ g \to \text{id}_Y$ the counit of this adjunction.

1. For every morphism $h : X \to X$ of $\mathcal{C}$ the following two maps are inverse to each other:

$$\alpha : [X,X](h,g \circ f) \to [Y,X](h \circ g, g \circ f) \xrightarrow{[Y,X](\text{h\circ g, g\circ f})} [Y,X](h \circ g, g)$$

$$\beta : [X,X](h \circ g, g) \to [X,X](h \circ g, g \circ f) \xrightarrow{[X,X](\text{h\circ g, f\circ g})} [X,X](h \circ g, f).$$

2. Let $T : X \to X$ be a morphism of $\mathcal{C}$ and $\varphi : T \circ g \to g$ a morphism in $[Y,X]$.

Denote $\psi$ the composition $T \xrightarrow{\varepsilon} T \circ g \circ f \xrightarrow{\varphi} g \circ f$ in $[X,X]$.

Then $\varphi$ factors as $T \circ g \xrightarrow{\psi} g \circ f \xrightarrow{\varphi} g$.

Consequently for every morphism $h : X \to X$ of $\mathcal{C}$ the map

$$\gamma : [X,X](h,T) \to [Y,X](h \circ g, T \circ g) \xrightarrow{[Y,X](\text{h\circ g, T\circ f)})} [Y,X](h \circ g, g).$$
factors as

\[ [X, X](h, T) \xrightarrow{[X, X](h, \psi)} [X, X](h \circ g \circ f) \xrightarrow{\alpha} [Y, X](h \circ g, g). \]

Thus \( \psi \) is an equivalence if and only if for every morphism \( h : X \to X \) of \( \mathcal{C} \) the map \( \gamma \) is an equivalence.

3. Let \( g : Y \to X, h : Z \to X \) be morphisms of \( \mathcal{C} \) that admit left adjoints \( f : X \to Y \) respectively \( k : X \to Z \) and let \( \phi : Y \to Z \) be a morphism in \( \mathcal{C} \) over \( X \).

Denote \( \omega \) the morphism

\[ h \circ k \to h \circ k \circ g \circ f = h \circ k \circ h \circ \phi \circ f \to h \circ \phi \circ f \circ g \circ f \]

in \([X, X]\).

Then \( h \circ k \circ g \circ f \circ g \to g \) is equivalent to the composition

\[ h \circ k \circ g = h \circ k \circ h \circ \phi \to h \circ \phi \circ g. \]

**Proof.** The composition

\[ [Y, X](h \circ g, g \circ f \circ g) \xrightarrow{[Y, X](h \circ g, g \circ f \circ g)} [Y, X](h \circ g, g) \]

\[ \to [X, X](h \circ g \circ f, g \circ f) \]

is equivalent to the composition

\[ [Y, X](h \circ g, g \circ f \circ g) \to [X, X](h \circ g \circ f, g \circ f \circ g \circ f) \xrightarrow{[X, X](h \circ g \circ f, g \circ f \circ g)} [X, X](h \circ g \circ f, g \circ f) \]

and the composition

\[ [X, X](h \circ g \circ f, g \circ f) \xrightarrow{[X, X](h \circ g \circ f, g \circ f)} [X, X](h, g \circ f) \]

\[ \to [Y, X](h \circ g, g \circ f \circ g) \]

is equivalent to the composition

\[ [X, X](h \circ g \circ f, g \circ f) \to [Y, X](h \circ g \circ f \circ g, g \circ f \circ g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g \circ f \circ g)} [Y, X](h, g \circ g \circ f \circ g). \]

So \( \beta \circ \alpha \) is equivalent to

\[ [X, X](h, g \circ f) \to [X, X](h \circ g \circ f, g \circ f \circ g \circ f) \xrightarrow{[X, X](h \circ g \circ f, g \circ f \circ g)} [X, X](h, g \circ f) \]

and \( \alpha \circ \beta \) is equivalent to

\[ [Y, X](h \circ g, g) \to [Y, X](h \circ g \circ f \circ g, g \circ f \circ g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g \circ f \circ g)} \]

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[Y, X](h \circ g \circ f \circ g, g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g)} [Y, X](h \circ g, g).

As

[X, X](h, g \circ f) \rightarrow [X, X](h \circ g \circ f, g \circ f \circ g \circ f) \xrightarrow{[X, X](h \circ g \circ f \circ g, g)} [X, X](h, g \circ f \circ g \circ f)

is equivalent to

[X, X](h, g \circ f) \xrightarrow{[X, X](h \circ g \circ f \circ g, g)} [X, X](h, g \circ f \circ g \circ f)

and

[Y, X](h \circ g, g) \rightarrow [Y, X](h \circ g \circ f \circ g, g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g)} [Y, X](h \circ g \circ f \circ g, g)

is equivalent to

[Y, X](h \circ g, g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g)} [Y, X](h \circ g \circ f \circ g, g),

\beta \circ \alpha \text{ is equivalent to}

[X, X](h, g \circ f) \xrightarrow{[X, X](h \circ g \circ f \circ g, g)} [X, X](h, g \circ f \circ g \circ f) \xrightarrow{[X, X](h \circ g \circ f \circ g, g)} [X, X](h, g \circ f)

and \alpha \circ \beta \text{ is equivalent to}

[Y, X](h \circ g, g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g)} [Y, X](h \circ g \circ f \circ g, g) \xrightarrow{[Y, X](h \circ g \circ f \circ g, g)} [Y, X](h \circ g, g).

Therefore statement 1 follows from the triangular identities:

The compositions \( f \rightarrow f \circ g \circ f \rightarrow f \) and \( g \rightarrow g \circ f \circ g \rightarrow g \) of morphisms of the category \([X, Y]\) respectively \([Y, X]\) are the identities.

It remains to show 2:

The composition \( \psi : T \circ g \xrightarrow{T \circ g} T \circ g \circ f \circ g \xrightarrow{\phi \circ g \circ f} g \) is equivalent to \( T \circ g \xrightarrow{\phi \circ f \circ g} g \) and is thus equivalent to \( \phi \) due to the triangular identities.

It remains to show 3:

The composition

\( h \circ k \circ g \rightarrow h \circ k \circ g \circ f \circ g \rightarrow h \circ k \circ h \circ f \circ g \rightarrow h \circ f \circ g \rightarrow g \)

is equivalent to the composition

\( h \circ k \circ g \rightarrow h \circ k \circ g \circ f \circ g \rightarrow h \circ k \circ h \circ f \circ g \rightarrow h \circ f \circ g \rightarrow g \)

and thus equivalent to the composition

\( h \circ k \circ g \rightarrow h \circ k \circ g \circ f \circ g \rightarrow h \circ k \circ g \rightarrow h \circ k \circ h \circ f \rightarrow h \circ f \circ g \rightarrow g \),

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which is equivalent to

\[ h \circ k \circ g = h \circ k \circ h \circ \phi \rightarrow h \circ \phi \simeq g \]

by the triangular identities.
4 Eilenberg-Moore objects

In this section we develop the theory of Eilenberg-Moore objects and Co-Eilenberg-Moore objects in a given 2-category that abstract the category of monadic algebras and coalgebras from the 2-category \( \text{Cat}_\infty(\kappa) \) to an arbitrary 2-category.

To do so we abstract in definition 4.3 the notions monadic functor and monad from \( \text{Cat}_\infty(\kappa) \) to an arbitrary 2-category.

We show in example 4.5 that for every \( \kappa \)-small category \( S \) the 2-category \( \text{Cat}_\infty(\kappa)/s_{\lambda} \rtimes s_{\lambda} \) admits Eilenberg-Moore objects and co-Eilenberg-Moore objects.

From this we deduce in theorem 4.16 that for every categorical pattern \( P \) on \( S \) the subcategory of \( P \)-fibered objects of \( \text{Cat}_\infty(\kappa)/s_{\lambda} \rtimes s_{\lambda} \) is closed in \( \text{Cat}_\infty(\kappa)/s_{\lambda} \rtimes s_{\lambda} \) under Eilenberg-Moore objects and Co-Eilenberg-Moore objects.

4.1 Eilenberg-Moore objects

Let \( S \) be a category and \( G : D \to C \) a functor over \( S \) that admits a left adjoint relative to \( S \).

By proposition 7.43 the functor \( G \) over \( S \) admits an endomorphism object \( T \in \text{Fun}_S(C, C) \) with respect to the canonical left module structure on \( \text{Fun}_S(D, C) \) over \( \text{Fun}_S(C, C) \).

By remark 2.18 for every category \( B \) over \( S \) we have a canonical equivalence
\[
\theta : \text{LMod}_T(\text{Fun}_S(B, C)) \simeq \text{Fun}_S(B, \text{LMod}^{s_{\lambda} \rtimes s_{\lambda}}_{T}(C))
\]
over \( \text{Fun}_S(B, C) \).

For \( B = D \) the endomorphism left module structure on \( G \) over \( T \) corresponds to lift \( \alpha : D \to \text{LMod}^{s_{\lambda} \rtimes s_{\lambda}}_{T}(C) \) of \( G \).

We will say that \( G \) is a monadic functor over \( S \) or that \( G \) exhibits \( D \) as monadic over \( C \) relative to \( S \) if \( \alpha \) is an equivalence.

If \( S \) is contractible, we will drop \( S \). In this case our definition coincides with the usual one.

**Remark 4.1.** Let \( G : D \to C \) be a functor over \( S \) that admits a left adjoint relative to \( S \).

Then \( G \) is a monadic functor over \( S \) if and only if for every category \( B \) over \( S \) the induced functor \( \text{Funs}_S(B, G) : \text{Funs}_S(B, D) \to \text{Funs}_S(B, C) \) is a monadic functor.

Let \( \alpha : D \to \text{LMod}^{s_{\lambda} \rtimes s_{\lambda}}_{T}(C) \) be the canonical lift of \( G \) from above.

By Yoneda \( \alpha \) is an equivalence if and only if for every functor \( B \to S \) the induced functor \( \beta : \text{Funs}_S(B, D) \to \text{Funs}_S(B, \text{LMod}^{s_{\lambda} \rtimes s_{\lambda}}_{T}(C)) \simeq \text{LMod}_T(\text{Funs}_S(B, C)) \) over \( \text{Funs}_S(B, C) \) is an equivalence.

The canonical \( \text{Funs}_S(C, C) \)-left module structure on \( \text{Funs}_S(D, C) \) is the pullback of the endomorphism left module structure on \( \text{Funs}_S(D, C) \) over \( \text{Fun}(\text{Funs}_S(B, C), \text{Funs}_S(B, C)) \) along a unique monoidal functor \( \text{Funs}_S(C, C) \to \text{Fun}(\text{Funs}_S(B, C), \text{Funs}_S(B, C)) \) that sends \( T \) to some monad \( T' \) on \( \text{Funs}_S(B, C) \).
By remark 3.2 the 2-functor $\text{Funs}(\mathcal{B}, -) : \text{Cat}_\infty(\kappa)_S \to \text{Cat}_\infty(\kappa)$ sends the endomorphism left module structure on $G$ over $T$ to an endomorphism left module structure on $\text{Funs}_S(\mathcal{B}, G)$ over $T'$ with respect to the canonical left module structure on $\text{Fun}(\text{Funs}_S(\mathcal{B}, \mathcal{D}), \text{Funs}_S(\mathcal{B}, \mathcal{E}))$ over $\text{Fun}(\text{Funs}_S(\mathcal{B}, \mathcal{E}), \text{Funs}_S(\mathcal{B}, \mathcal{E}))$.

By ... we have a commutative square

$$
\begin{array}{c}
\text{LMod}_T(\text{Funs}(\mathcal{D}, \mathcal{E})) \\
\text{LMod}_T(\text{Fun}(\text{Funs}(\mathcal{B}, \mathcal{D}), \text{Funs}(\mathcal{B}, \mathcal{E}))) \to' \text{Fun}(\text{Funs}(\mathcal{B}, \mathcal{D}), \text{LMod}_T(\text{Funs}(\mathcal{B}, \mathcal{E}))).
\end{array}
$$

Hence the endomorphism $T'$-left module structure on $\text{Funs}_S(\mathcal{B}, G)$ corresponds to $\beta$ under the canonical equivalence $\theta'$.

So $\beta$ is an equivalence if and only if $\text{Funs}_S(\mathcal{B}, G)$ is monadic.

**Remark 4.2.** Let $G : \mathcal{D} \to \mathcal{E}$ be a functor over $S$ that admits a left adjoint relative to $S$.

Then $G$ is a monadic functor over $S$ if and only if $G$ is equivalent over $\mathcal{E}$ to the forgetful functor $\text{LMod}_S^G(\mathcal{E}) \to \mathcal{E}$ for some left module structure on $\mathcal{E}$ over $S$.

Over some monoidal category $\mathcal{E}$ over $S$ and some functor $A : S \to \text{Alg}_S^G(\mathcal{E})$ over $S$.

The only if direction is evident.

By proposition 7.43 the forgetful functor $\text{LMod}_S^G(\mathcal{E}) \to \mathcal{E}$ admits a left adjoint relative to $S$. Denote $\delta : \text{Funs}_S(S, \mathcal{E}) \to \text{Funs}_S(\mathcal{B}, \mathcal{E})$ the monoidal diagonal functor.

By remark 2.18 we have a canonical equivalence

$$
\text{LMod}_A(\delta^*(\text{Funs}(\mathcal{B}, \mathcal{E}))) \simeq \text{Funs}(\mathcal{B}, \text{LMod}_A^G(\mathcal{E}))
$$

over $\text{Funs}_S(\mathcal{B}, \mathcal{E})$.

By remark 3.4 the forgetful functor $\text{LMod}_S^G(\mathcal{E}) \to \mathcal{E}$ is a monadic functor over $S$ if and only if for all functors $\mathcal{B} \to S$ the induced functor $\text{LMod}_A(\delta^*(\text{Funs}(\mathcal{B}, \mathcal{E}))) \simeq \text{Funs}(\mathcal{B}, \text{LMod}_A^G(\mathcal{E})) \to \text{Funs}(\mathcal{B}, \mathcal{E})$ is monadic.

So we can reduce to the case that $S$ is contractible.

The left module structure on $\mathcal{E}$ over $\mathcal{E}$ is the pullback of the endomorphism left module structure on $\mathcal{E}$ over $\text{Fun}(\mathcal{E}, \mathcal{E})$ along a unique monoidal functor $\rho : \mathcal{E} \to \text{Fun}(\mathcal{E}, \mathcal{E})$ that sends $A$ to some monad $T$ on $\mathcal{E}$.

Thus by corollary 7.18 we have a canonical equivalence $\text{LMod}_A(\mathcal{E}) \simeq \text{LMod}_T(\mathcal{E})$ over $\mathcal{E}$.

By proposition 7.43 the forgetful functor $\psi : \text{LMod}_T(\mathcal{E}) \to \mathcal{E}$ admits a left adjoint.

For $\mathcal{B} = \text{LMod}_T(\mathcal{E})$ the identity corresponds under $\theta$ to a left $T$-module structure on $\psi : \text{LMod}_T(\mathcal{E}) \to \mathcal{E}$ that exhibits $T$ as the endomorphism object of $\psi$ by lemma 4.17.

So tautologically $\psi$ is a monadic functor.

By remark 4.1 the following definition generalizes the notion of monadic functor over $S$. 

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Definition 4.3. (monadic morphism, Eilenberg-Moore object, representable monad)

Let \( C \) be a 2-category and \( \psi : Z \to X \) a morphism of \( C \).

We say that \( \psi \) exhibits \( Z \) as monadic over \( X \) or call \( \psi : Z \to X \) a monadic morphism if \( \psi \) admits a left adjoint in \( C \) and for every object \( Y \) of \( C \) the induced functor \([Y, Z] \to [Y, X]\) is monadic.

Let \( T \in \text{Alg}([X, X]) \) be a monad.

We say that a morphism \( \phi : Z \to X \) of \( C \) exhibits \( Z \) as an Eilenberg-Moore object of \( T \) or that \( \phi : Z \to X \) is an Eilenberg-Moore object of \( T \) if \( \phi \) is monadic and there is a left \( T \)-module structure on \( \phi \) with respect to the canonical \([X, X]\)-left module structure on \([Z, X]\) that exhibits \( T \) as the endomorphism object of \( \phi \).

In this case we say that \( \phi : Z \to X \) represents the monad \( T \).

We call the monad \( T \) representable if there is an Eilenberg-Moore object.

We say that a morphism \( \phi : Z \to X \) of \( C \) is representable if its associated monad is representable.

If \( \phi : Z \to X \) is a morphism of \( C \) that admits a left adjoint \( f : X \to Z \) and exhibits \( Z \) as an Eilenberg-Moore object of \( T \), by proposition 3.6 we have a canonical equivalence \( T \cong \phi \circ f \) in \( \text{Fun}(X, X) \).

If every monad \( T \in \text{Alg}([X, X]) \) admits an Eilenberg-Moore object, we say that \( X \) admits Eilenberg-Moore objects.

If all objects \( X \) of \( C \) admit Eilenberg-Moore objects, we say that \( C \) admits Eilenberg-Moore objects.

By proposition 3.6 every right adjoint morphism \( Y \to X \) admits an endomorphism object with respect to the canonical \([X, X]\)-left module structure on \([Y, X]\). So every monadic morphism \( Y \to X \) is an Eilenberg-Moore object of some monad \( T \) on \( X \).

Tautologically every monadic morphism is representable.

We have the dual notion of comonadic morphism and coEilenberg-Moore object.

Given a 2-category \( C \) we call a morphism \( \psi : Z \to X \) of \( C \) comonadic if \( \psi \) is a monadic morphism in \( C_{\text{op}} \), i.e. if for every object \( Y \) of \( C \) the induced functor \([Y, Z] \to [Y, X]\) is comonadic.

Let \( T \in \text{Alg}([X, X]_{\text{op}}) \) be a comonad.

We say that a morphism \( \phi : Z \to X \) of \( C \) exhibits \( Z \) as a coEilenberg-Moore object of \( T \) or that \( \phi : Z \to X \) is a coEilenberg-Moore object of \( T \) if \( \phi \) is an Eilenberg-Moore object of \( T \) in \( C_{\text{op}} \).

Remark 4.4. Let \( S \) be a category.

The opposite category involution lifts to a canonical equivalence

\[
(C\text{at}_\infty (\kappa)_S)_{\text{op}} \cong C\text{at}_\infty (\kappa)_S^{\text{op}}
\]
of 2-categories, as the opposite category involution \( \text{Cat}_{\infty}(\kappa)_{/S} \cong \text{Cat}_{\infty}(\kappa)_{/S} \) induces for every operad \( \Delta^\circ \) over \( LM^\circ \), where we set \( B^\circ := \text{Ass}^\circ \times_{LM^\circ} \Delta^\circ \), a pullback square

\[
\begin{array}{ccc}
\text{Alg}_{\Delta^\circ/LM^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}) & \xrightarrow{\sim} & \text{Alg}_{\Delta^\circ/LM^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}) \\
\downarrow & & \downarrow \\
\text{Alg}_{B^\circ/\text{Ass}^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}) & \xrightarrow{\sim} & \text{Alg}_{B^\circ/\text{Ass}^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}).
\end{array}
\]

**Example 4.5.** Let \( \mathcal{C} \to S \) be a functor and \( T \in \text{Alg}(\text{Fun}_S(\mathcal{C}, \mathcal{C})) \) a monad.

By proposition [7.4] the functor \( \psi : \text{LMod}^S_T(\mathcal{C}) \to \mathcal{C} \) over \( S \) admits a left adjoint relative to \( S \).

By remark [2.18] for every category \( B \) over \( S \) we have a canonical equivalence

\[ \theta : \text{LMod}^T_T(\text{Fun}_S(B, \mathcal{C})) \cong \text{Fun}_S(B, \text{LMod}^S_T(\mathcal{C})) \]

over \( \text{Fun}_S(B, \mathcal{C}) \).

For \( B = \text{LMod}^S_T(\mathcal{C}) \) the identity corresponds under \( \theta \) to a left \( T \)-module structure on \( \psi : \text{LMod}^S_T(\mathcal{C}) \to \mathcal{C} \) that exhibits \( T \) as the endomorphism object of \( \psi \) by proposition [4.13].

So \( \psi \) is a monadic functor over \( S \) with associated monad \( T \), in other words \( \psi \) is an Eilenberg-Moore object of \( T \) in \( \text{Cat}_{\infty}(\kappa)_{/S} \).

So for every \( \kappa \)-small category \( S \) the 2-category \( \text{Cat}_{\infty}(\kappa)_{/S} \) admits Eilenberg-Moore objects.

By remark [7.4] the duality involution lifts to a canonical equivalence

\[ (\text{Cat}_{\infty}(\kappa)_{/S})^\text{op} \cong \text{Cat}_{\infty}(\kappa)_{/S} \]

of 2-categories, as the opposite category involution \( \text{Cat}_{\infty}(\kappa)_{/S} \cong \text{Cat}_{\infty}(\kappa)_{/S} \) induces for every operad \( \Delta^\circ \) over \( LM^\circ \), where we set \( B^\circ := \text{Ass}^\circ \times_{LM^\circ} \Delta^\circ \), a pullback square

\[ \begin{array}{ccc}
\text{Alg}_{\Delta^\circ/LM^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}) & \xrightarrow{\sim} & \text{Alg}_{\Delta^\circ/LM^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}) \\
\downarrow & & \downarrow \\
\text{Alg}_{B^\circ/\text{Ass}^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}) & \xrightarrow{\sim} & \text{Alg}_{B^\circ/\text{Ass}^\circ}(\text{Cat}_{\infty}(\kappa)_{/S}).
\end{array} \]

Given a comonad \( L \in \text{coAlg}(\text{Fun}_S(\mathcal{C}, \mathcal{C})) \) on a category \( \mathcal{C} \) over \( S \) its co-Eilenberg-Moore object is given by the forgetful functor

\[ \text{coLMod}_L^S(\mathcal{C}) := \text{LMod}^{S^\text{op}}_L(\mathcal{C})^\text{op} \to \mathcal{C} \]

over \( S \).

Let \( \phi : \mathcal{D} \to \mathcal{C} \) be an Eilenberg-Moore object of \( T \) in \( \text{Cat}_{\infty}(\kappa)_{/S} \).

For \( \mathcal{B} = \mathcal{D} \) the endomorphism \( T \)-left module structure on \( \phi \) corresponds under \( \theta \) to a functor \( \mathcal{D} \to \text{LMod}_T^S(\mathcal{C}) \) over \( \mathcal{C} \).

As \( \phi : \mathcal{D} \to \mathcal{C} \) is a monadic functor over \( S \), the functor \( \mathcal{D} \to \text{LMod}_T^S(\mathcal{C}) \) over \( \mathcal{C} \) is an equivalence.

Let \( F : \mathcal{C} \to \mathcal{D} \) be a 2-functor and \( \psi : Z \to X \) an Eilenberg-Moore object of some monad \( T \in \text{Alg}([X, X]) \) on some object \( X \) of \( \mathcal{C} \).

By remark [3.2] the \( T \)-left module structure on \( \psi \) gives rise to a \( F(T) \)-left module structure on \( F(\psi) : F(Z) \to F(X) \) that exhibits \( F(T) \) as the endomorphism object of \( F(\psi) \).

So if \( F(\psi) \) is a monadic morphism, \( F(\psi) \) is an Eilenberg-Moore object of \( F(T) \).

In this case we say that \( F \) preserves the Eilenberg-Moore object of \( T \).
Remark 4.6. Suppose we have given a commutative square

\[
\begin{array}{ccc}
D' & \rightarrow & D \\
\psi' & \downarrow & \psi \\
C' & \rightarrow & C
\end{array}
\]

of $\kappa$-small categories, where the horizontal functors are fully faithful and the right vertical functor $\psi$ is monadic and its left adjoint $F : C \rightarrow D$ restricts to a functor $C' \rightarrow D'$.

Then the functor $\psi' : D' \rightarrow C'$ is monadic if and only if square (5) is a pullback square.

Denote $\text{Fun}(C, C)' \subset \text{Fun}(C, C)$ the full subcategory spanned by the functors $C \rightarrow C$ that send objects of $C'$ to objects of $C'$. Then the endomorphism left modul structure on $C$ over $\text{Fun}(C, C) \rightarrow C'$ restricts to a left modul structure on $C'$ over $\text{Fun}(C', C')$ along a unique monoidal functor $\text{Fun}(C, C) \rightarrow \text{Fun}(C', C')$.

As $\psi : D \rightarrow C$ is monadic, we have a canonical equivalence $D \simeq \text{LMod}_T(C)$ over $C$ for some monad $T$ on $C$ with $T \simeq \psi \circ F$ in $\text{Fun}(C, C)$.

As $F$ restricts to a functor $C \rightarrow D'$, the monad $T$ is an associative algebra of $\text{Fun}(C, C)'$ and so gives rise to a monad $T'$ on $C'$.

We have a canonical equivalence $\text{LMod}_{T'}(C') \simeq C' \times_C \text{LMod}_T(C) \simeq C' \times_C D$ over $C'$. So the functor $C' \times_C D \rightarrow C'$ is monadic.

Moreover $F : C \rightarrow D$ restricts to a functor $F' : C' \rightarrow C' \times_C D$ that restricts to a functor $C' \rightarrow D'$.

So by theorem 5.1 the canonical functor $D' \rightarrow C' \times_C D$ over $C'$ is an equivalence if and only if $\psi'$ is monadic.

**Remark 4.7.**

Let $C$ be a 2-category, $\mathcal{B}$ a subcategory of $\mathcal{C}$, $\psi : Z \rightarrow X$ a morphism of $\mathcal{B}$ and $T \in \text{Alg}([X, X]_\mathcal{B})$ a monad.

If $\mathcal{B}$ is a full subcategory of $\mathcal{C}$, $\psi : Z \rightarrow X$ is a monadic morphism of $\mathcal{B}$ if and only if $\psi$ is a monadic morphism of $\mathcal{C}$ and $\psi : Z \rightarrow X$ is an Eilenberg-Moore object of $T$ in $\mathcal{B}$ if and only if $\psi$ is an Eilenberg-Moore object of $T$ in $\mathcal{C}$.

If $\mathcal{B}$ is an arbitrary subcategory of $\mathcal{C}$ we have the following:

Assume that $\psi : Z \rightarrow X$ is a monadic morphism of $\mathcal{C}$ with left adjoint $F : X \rightarrow Z$.

Remark 4.6 implies the following:

$\psi$ is a monadic morphism in $\mathcal{B}$ if and only if for all $Y \in \mathcal{B}$ the commutative square

\[
\begin{array}{ccc}
[Y, Z]_\mathcal{B} & \rightarrow & [Y, Z]_\mathcal{C} \\
\downarrow & & \downarrow \\
[Y, X]_\mathcal{B} & \rightarrow & [Y, X]_\mathcal{C}
\end{array}
\]
is a pullback square and $\mathcal{F}: X \to Z$ is a morphism of $\mathcal{B}$ or equivalently if and only if for all morphisms $\alpha: Y \to Z$ of $\mathcal{C}$ with $\psi \circ \alpha: Y \to Z \to X$ also $\alpha$ belongs to $\mathcal{B}$ and the composition $\psi \circ \mathcal{F}: X \to Z \to X$ is a morphism of $\mathcal{B}$.

Assume that $\psi: Z \to X$ is an Eilenberg-Moore object of $T$ in $\mathcal{C}$.

Then $\psi: Z \to X$ is an Eilenberg-Moore object of $T$ in $\mathcal{B}$ if and only if $\psi$ is a monadic morphism of $\mathcal{B}$ or equivalently if and only if for all morphisms $\alpha: Y \to Z$ of $\mathcal{C}$ with $\psi \circ \alpha: Y \to Z \to X$ also $\alpha$ belongs to $\mathcal{B}$.

Let $\mathcal{C}$ be a 2-category, $\mathcal{B}$ a subcategory of $\mathcal{C}$ and $X$ an object of $\mathcal{B}$.

Assume that $X$ admits Eilenberg-Moore objects in $\mathcal{C}$.

We say that $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects of $X$ if $X$ admits Eilenberg-Moore objects in $\mathcal{B}$ that are preserved by the subcategory inclusion $\mathcal{B} \subseteq \mathcal{C}$.

By remark 4.7 $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects of $X$ if and only if for every monad $T \in \mathrm{Alg}(\{X, X\}, \mathcal{B})$ the Eilenberg-Moore object $\psi: Z \to X$ in $\mathcal{C}$ is a monadic morphism of $\mathcal{B}$ or equivalently if and only if for all morphisms $\alpha: Y \to Z$ of $\mathcal{C}$ with $\psi \circ \alpha: Y \to Z \to X$ also $\alpha$ belongs to $\mathcal{B}$.

This has the following consequence:

Let $A \subseteq \mathcal{C}$, $B \subseteq \mathcal{C}$ be subcategories and $X \in A \times \mathcal{C} B \subseteq \mathcal{C}$.

If $A, B$ are closed in $\mathcal{C}$ under Eilenberg-Moore objects of $X$, the pullback $A \times \mathcal{C} B$ is also closed in $\mathcal{C}$ under Eilenberg-Moore objects of $X$.

Assume that $\mathcal{C}$ admits Eilenberg-Moore objects.

We say that $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects if $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects of $X$ for all objects $X$ of $\mathcal{B}$.

So $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects if and only if $\mathcal{B}$ admits Eilenberg-Moore objects that are preserved by the subcategory inclusion $\mathcal{B} \subseteq \mathcal{C}$.

**Remark 4.8.** Let $G: \mathcal{D} \to \mathcal{C}$ be a 2-functor that admits a left adjoint $F$.

Then for all $X \in \mathcal{C}, Y \in \mathcal{D}$ the induced functor

$$[F(X), Y] \to [G(F(X)), G(Y)] \to [X, G(Y)]$$

is an equivalence.

So $G: \mathcal{D} \to \mathcal{C}$ preserves monadic morphisms and Eilenberg-Moore objects:

Being a 2-functor $G: \mathcal{D} \to \mathcal{C}$ preserves right adjoint morphisms.

Let $\psi: Z \to X$ be a monad of $\mathcal{D}$. Then for every object $Y$ of $\mathcal{C}$ the functor $[F(Y), Z] \to [F(Y), X]$ is equivalent to the functor $[Y, G(Z)] \to [Y, G(X)]$ so that with $\psi$ also $G(\psi)$ is monadic.

So by remark 4.7 if $Z \to X$ is an Eilenberg-Moore object for a monad $T$ on $X$ in $\mathcal{D}$, the image $G(Z) \to G(X)$ is an Eilenberg-Moore object for the monad $G(T)$ in $\mathcal{C}$.

Let $\psi: Z \to X$ be a morphism of $\mathcal{D}$ that admits a left adjoint.

The full subcategory of $\mathrm{Cat}_{\kappa}(\kappa)$ spanned by those categories $K$ with the property that $[K, Z] \to [K, X]$ is monadic, is closed under $\kappa$-small colimits as the full
subcategory of $\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))$ spanned by the monadic functors is closed under $\kappa$-small limits in the full subcategory of $\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))^R$ spanned by the right adjoint functors.

So if $\mathcal{D}$ is generated by the essential image of $F$ under $\kappa$-small colimits, $\psi : Z \to X$ is monadic if and only if $G(\psi)$ is monadic.

**Remark 4.9.** Let $\mathcal{C}$ be a 2-category and $\mathcal{B}$ a 2-localization of $\mathcal{C}$.

Let $\phi : Z \to X$ be an Eilenberg-Moore object in $\mathcal{C}$ of a monad $T \in \text{Alg}([X, X])$ on some object $X$ of $\mathcal{B}$.

Denote $\Phi : Z \to X$ the map of cocartesian fibrations over $\text{C}^{\text{op}}$ classifying the natural transformation $[-, \phi] : [-, Z] \to [-, X]$ of functors $\text{C}^{\text{op}} \to \text{Cat}_\infty(\kappa)$.

By corollary 4.11 the canonical map $Z \to \text{LMod}_T(X)$ of cocartesian fibrations over $\text{C}^{\text{op}}$ over the cocartesian fibration $X \to \text{C}^{\text{op}}$ is an equivalence.

Thus for every morphism $f : A \to B$ of $\mathcal{C}$ the commutative square

\[
\begin{array}{ccc}
[B, Z] & \longrightarrow & [A, Z] \\
\downarrow & & \downarrow \\
[B, X] & \longrightarrow & [A, X]
\end{array}
\]

is equivalent to the commutative square

\[
\begin{array}{ccc}
\text{LMod}_T([B, X]) & \longrightarrow & \text{LMod}_T([A, X]) \\
\downarrow & & \downarrow \\
[B, X] & \longrightarrow & [A, X]
\end{array}
\]

So if $f : A \to B$ is a local equivalence of $\mathcal{C}$, the $[X, X]$-linear functor $[B, X] \to [A, X]$ is an equivalence and thus induces an equivalence $\text{LMod}_T([B, X]) \to \text{LMod}_T([A, X])$.

Hence the functor $[B, Z] \to [A, Z]$ is an equivalence so that $Z$ belongs to $\mathcal{B}$. So $\phi : Z \to X$ is an Eilenberg-Moore object of $T$ in $\mathcal{B}$.

Consequently every Eilenberg-Moore object in $\mathcal{C}$ of a monad $T \in \text{Alg}([X, X])$ on some object $X$ of $\mathcal{B}$ is an Eilenberg-Moore object of $T$ in $\mathcal{B}$.

Especially if an object $X$ of $\mathcal{B}$ admits Eilenberg-Moore objects in $\mathcal{C}$, it admits Eilenberg-Moore objects in $\mathcal{B}$ and thus with $\mathcal{C}$ also $\mathcal{B}$ admits Eilenberg-Moore objects.

Let $\mathcal{C}$ be a 2-category and $\mathcal{B}$ a 2-localization of $\mathcal{C}$, then $\mathcal{B}^{\text{op}}$ is a 2-localization of $\mathcal{C}^{\text{op}}$.

So every co-Eilenberg-Moore object in $\mathcal{C}$ of a comonad $T \in \text{coAlg}([X, X])$ on some object $X$ of $\mathcal{B}$ is a co-Eilenberg-Moore object of $T$ in $\mathcal{B}$.

**Proposition 4.10.** Let $\mathcal{C}$ be a $\kappa$-small 2-category for a strongly inaccessible cardinal $\kappa$ and $T \in \text{Alg}([X, X])$ a monad on some object $X$ of $\mathcal{C}$.

Let $\phi : Z \to X$ be a morphism of $\mathcal{C}$.

Denote $T' \in \{T\} \times_{[X, X]} \text{LMod}_T([X, X])$ the left $T$-module structure on $T$ coming from the associative algebra structure on $T$.  

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1. Assume that \( \phi \) is endowed with a left T-module structure such that for every \( Y \in C \) the induced left T-module structure on \([Y, \phi] : [Y, Z] \to [Y, X]\) corresponds to an equivalence \([Y, Z] \to \text{LMod}_T([Y, X])\) over \([Y, X]\).

Denote \( T_\phi \) the induced left T-module structure on \([Y, \phi] : [Y, Z] \to [Y, X]\) such that the induced \( T \) lifts the functor \( T \) along \( \phi : Z \to X \) and denote \( \eta : \text{id}_X \to T \) the unit of the monad \( T \).

Then \( \eta : \text{id}_X \to T = \phi \circ T \) exhibits \( \phi \) as the left adjoint of \( \phi : Z \to X \).

2. Let \([X, X], [Z, X] \) be endowed with the canonical left-module structures over \([X, X] \).

Assume that the functor \( [T, X] : [Z, X] \to [X, X] \) is a \([X, X]\)-linear functor such that the induced \( \mathcal{C}(X, X)\)-linear functor

\[\mathcal{C}(T, X) : \mathcal{C}(Z, X) \to \mathcal{C}(X, X)\]

on maximal subspaces is the canonical \( \mathcal{C}(X, X)\)-linear functor.

Suppose that the commutative square

\[
\begin{array}{ccc}
[Z, Z] & \xrightarrow{[T, Z]} & [X, Z] \\
\downarrow & & \downarrow \\
[Z, X] & \xrightarrow{[T, X]} & [X, X]
\end{array}
\]

is equivalent over \([T, X] : [Z, X] \to [X, X] \) to the commutative square

\[
\begin{array}{ccc}
\text{LMod}_T([Z, X]) & \xrightarrow{\text{LMod}_T([T, X])} & \text{LMod}_T([X, X]) \\
\downarrow & & \downarrow \\
[Z, X] & \xrightarrow{[T, X]} & [X, X].
\end{array}
\]

Denote \( \phi' \) the left T-module structure on \( \phi : Z \to X \) that corresponds to the identity of \( Z \).

Then the action map \( \beta : T \circ \phi \to \phi' \) exhibits \( T \) as the endomorphism object of \( \phi \) with respect to the canonical \([X, X]\)-left module structure on \([Z, X] \).

\[\text{Proof.} \ 1.\]

We first assume that \( \phi : Z \to X \) admits a left adjoint \( \mathcal{F} \).

Denote \( \tilde{\eta} \) the unit of the adjunction \( \mathcal{F} : X \nrightarrow Z : \phi \) in \( C \).

The functor \([X, -] : \mathcal{C} \to \text{Cat}_{\omega}(\omega)\) is a 2-functor.

Thus the natural transformation \([X, \tilde{\eta}] : \text{id}_{[X, X]} \to [X, \phi] \circ [X, \mathcal{F}] \) exhibits \([X, \mathcal{F}] : [X, X] \to [X, Z] \) as the left adjoint of the forgetful functor \( \text{LMod}_T([X, X]) \)

\[\simeq [X, Z]^{[X, \phi]} [X, X],\]

i.e. as the free left T-module functor.

As \( T \xrightarrow{T \circ \eta} T \circ T \xrightarrow{\mu} T \) is the identity, the unit \( \eta : \text{id}_X \to T \) of the monad \( T \) exhibits \( T' \) as the free T-module generated by the tensor unit \( \text{id}_X \) of \([X, X] \).
Thus there is a unique equivalence $\mathcal{F} \to \mathcal{F} \simeq [X,\mathcal{F}](\text{id}_X)$ such that $\eta : \text{id}_X \to \phi \circ \mathcal{F} \simeq \phi \circ \mathcal{F}$ is homotopic to $\overline{\eta} = [\overline{X},\overline{\mathcal{F}}](\text{id}_X) : \text{id}_X \to \phi \circ \mathcal{F}$.

So $\eta : \text{id}_X \to T \simeq \phi \circ \mathcal{F}$ exhibits $\mathcal{F} : X \to Z$ as the left adjoint of $\phi : Z \to X$.

Now let $\phi$ be arbitrary. We will show that $\phi$ admits a left adjoint.

By lemma ... it is enough to see that for every $Y \in \mathcal{C}$ the induced natural transformation $[Y,\eta] : \text{id} \to [Y,\phi] \circ [Y,\mathcal{F}]$ of functors $[Y,X] \to [Y,X]$ exhibits $[Y,\mathcal{F}]$ as left adjoint to $[Y,\phi]$.

The 2-functor $[Y,-] : \mathcal{C} \to \text{Cat}_\infty(\kappa)$ sends the monad $T$ on $X$ to a monad $[Y,T]$ on $[Y,X]$.

The left $T$-module structure on $\phi$ gives rise to a left $[Y,T]$-module structure on $\text{Fun}([Y,X],[Y,\phi]) : \text{Fun}([Y,X],[Y,Z]) \to \text{Fun}([Y,X],[Y,X])$.

We have a canonical equivalence

$$\text{Fun}([Y,X],[Y,Z]) \simeq \text{Fun}([Y,X],\text{LMod}_T([Y,X])) \simeq \text{LMod}_{[Y,T]}(\text{Fun}([Y,X],[Y,X]))$$

over $\text{Fun}([Y,X],[Y,X])$.

We have a commutative square

$$\begin{array}{ccc}
[X,Z] & \longrightarrow & \text{LMod}_T([X,X]) \\
\downarrow & & \downarrow \\
\text{Fun}([Y,X],[Y,Z]) & \longrightarrow & \text{LMod}_{[Y,T]}(\text{Fun}([Y,X],[Y,X])).
\end{array}$$

The functor $[Y,\phi] : [Y,Z] \simeq \text{LMod}_T([Y,X]) \to [Y,X]$ admits a left adjoint.

So by what we have proved so far, $[Y,\eta] : \text{id} \to [Y,T] \simeq [Y,\phi] \circ [Y,\mathcal{F}]$ exhibits $[Y,\mathcal{F}] : [Y,X] \to [Y,Z]$ as the left adjoint of $[Y,\phi] : [Y,Z] \to [Y,X]$.

As next we prove 2:
By (a) the unit $\eta : \text{id}_X \to T \simeq \phi \circ \mathcal{F}$ of the monad $T$ exhibits $\mathcal{F}$ as the left adjoint of $\phi : Z \to X$.

Thus by proposition 3.15 we have to see that the composition $T \xrightarrow{T\eta} T \circ \phi \circ \mathcal{F} \xrightarrow{\beta \circ \mathcal{F}} \phi \circ \mathcal{F}$ is an equivalence.

The $[X,X]$-linear functor $\mathcal{F}^* := [\mathcal{F},X] : [Z,X] \to [X,X]$ yields a functor

$$\text{LMod}_T([Z,X]) \to \text{LMod}_T([X,X])$$

that sends the left $T$-module structure $\phi'$ on $\phi$ to the left $T$-module $T$.

So $T \circ \mathcal{F} \simeq T \circ \mathcal{F}^*(\phi) \simeq \mathcal{F}^*(T \circ \phi) \xrightarrow{\beta \circ \mathcal{F}} \mathcal{F}^*(\phi) \simeq T$ is the multiplication map of the monad $T$ as the canonical equivalence $T \circ \mathcal{F}^*(\phi) \simeq \mathcal{F}^*(T \circ \phi)$ is the associativity equivalence of $\mathcal{C}$.

So $\beta \circ \mathcal{F} : T \circ \phi \circ \mathcal{F} \simeq T \circ T \rightarrow \phi \circ \mathcal{F} \simeq T$ is the multiplication map of the monad $T$.

\begin{flushright}
$\square$
\end{flushright}

**Corollary 4.11.** Let $\mathcal{C}$ be a $\kappa$-small 2-category for a strongly inaccessible cardinal $\kappa$ and $\phi : Z \to X$ an Eilenberg-Moore object of some monad $T \in \text{Alg}([X,X])$ on some object $X$ of $\mathcal{C}$.

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Denote $\Phi : \mathcal{Z} \to \mathcal{X}$ the map of cocartesian fibrations over $\mathcal{C}^{op}$ classifying the natural transformation $[-, \phi] : [-, Z] \to [-, X]$ of functors $\mathcal{C}^{op} \to \mathcal{Cat}_{\infty}(\kappa)$.

Then $\phi : Z \to X$ is an Eilenberg-Moore object of $T$ if and only if $\mathcal{Z}$ is equivalent over $\mathcal{X}$ to $\mathcal{LMod}_T(\mathcal{X})$.

By proposition 7.15 we have a 2-functor $\theta : \mathcal{C} \to \mathcal{Cat}_{\infty}^{cocart}(\kappa)$ that sends the morphism $\phi : Z \to X$ to $\Phi : \mathcal{Z} \to \mathcal{X}$. By definition of the notion of monadic morphism $\theta$ preserves monadic morphisms and thus Eilenberg-Moore objects.

So if $\phi : Z \to X$ is an Eilenberg-Moore object of $T$ in $\mathcal{C}$, the map $\Phi : \mathcal{Z} \to \mathcal{X}$ of cocartesian fibrations over $\mathcal{C}^{op}$ is an Eilenberg-Moore object of $\theta(T)$ in $\mathcal{Cat}_{\infty}^{cocart}(\kappa) \subset \mathcal{Cat}_{\infty}(\kappa)_{/\mathcal{C}}^{lmod}$.

Thus by 4.11 we have a canonical equivalence $\mathcal{Z} \simeq \mathcal{LMod}_T(\mathcal{X})$ over $\mathcal{X}$.

On the other hand if there is an equivalence $\mathcal{LMod}_T^{op}(\mathcal{X}) \simeq \mathcal{Z}$ of cocartesian fibrations over $\mathcal{C}^{op}$ over the cocartesian fibration $\mathcal{X} \to \mathcal{C}^{op}$, the assumptions of proposition 4.10 2. are satisfied so that the morphism $\phi : Z \to X$ is an Eilenberg-Moore object of $T$.

Remark 4.12. In the following we want to apply corollary 4.13 to the 2-category $\mathcal{C} = \mathcal{Cat}_{\infty}(\kappa)_{/S}$ for some $\kappa$-small category $S$.

Let $T \in \text{Alg}(\mathcal{Fun}_S(X, X))$ be a monad on some category $X$ over $S$ and $\phi : Z \to X$ a functor over $S$ that admits a left adjoint relative to $S$.

$\phi : Z \to X$ yields a map $\Phi : \mathcal{Z} \to \mathcal{X} := \mathcal{Fun}([\mathcal{Cat}_\infty(\kappa)_{/S}]^{op} \times \mathcal{Fun}(\mathcal{U}'(\kappa)_S \times (\mathcal{Cat}_\infty(\kappa)_{/S})^{op} \times X), \mathcal{Cat}_\infty(\kappa)_{/S})^{op}$ of cocartesian fibrations over $(\mathcal{Cat}_\infty(\kappa)_{/S})^{op}$ that classifies the natural transformation $\mathcal{Fun}_S(-, \phi) : \mathcal{Fun}_S(-, Z) \to \mathcal{Fun}_S(-, X)$ of functors $(\mathcal{Cat}_\infty(\kappa)_{/S})^{op} \to \mathcal{Cat}_\infty(\kappa)$ due to theorem 7.7.

Denote $\mathcal{M}^\otimes \to \mathcal{LMod}^\otimes \times S$ the $\mathcal{LMod}^\otimes$-monoidal category over $S$ that encodes the endomorphism left module structure on $\mathcal{X} \to S$ over $\mathcal{Fun}_S(X, X)$ and denote $\mathcal{U}'(\kappa)_S \to (\mathcal{Cat}_\infty(\kappa)_{/S})^{op} \times S$ the map of cartesian fibrations over $(\mathcal{Cat}_\infty(\kappa)_{/S})^{op}$ classifying the identity of $\mathcal{Cat}_\infty(\kappa)_{/S}$.

By remark 2.13 we have a $\mathcal{LMod}^\otimes$-monoidal category $\mathcal{Fun}([\mathcal{Cat}_\infty(\kappa)_{/S}]^{op} \times \mathcal{Fun}(\mathcal{U}'(\kappa)_S \times (\mathcal{Cat}_\infty(\kappa)_{/S})^{op} \times S \times \mathcal{Fun}_S(X, X))^\otimes$ over $(\mathcal{Cat}_\infty(\kappa)_{/S})^{op}$, whose pullback along the monoidal diagonal functor $\delta : (\mathcal{Cat}_\infty(\kappa)_{/S})^{op} \times \mathcal{Fun}_S(X, X)^\otimes \to \mathcal{Map}([\mathcal{Cat}_\infty(\kappa)_{/S}]^{op} \times \mathcal{Fun}_S(X, X))^\otimes$ over $(\mathcal{Cat}_\infty(\kappa)_{/S})^{op}$ endows $\mathcal{X}$ with the structure of a left module over $\mathcal{Fun}_S(X, X)$ that is the image of the endomorphism left module structure on $\mathcal{X} \to S$ over $\mathcal{Fun}_S(X, X)$ under the 2-functor $\theta : \mathcal{C} \to \mathcal{Cat}_{\infty}^{cocart}(\kappa)$ due to remark 7.8.

So by 4.17 $\phi : Z \to X$ is an Eilenberg-Moore object of $T$ if and only if there is an equivalence $\mathcal{LMod}_T^{/\mathcal{Cat}_\infty(\kappa)_{/S}}(\mathcal{X}) \simeq \mathcal{Z}$ of cocartesian fibrations over $(\mathcal{Cat}_\infty(\kappa)_{/S})^{op}$ over the cocartesian fibration $\mathcal{X} \to (\mathcal{Cat}_\infty(\kappa)_{/S})^{op}$.

Proposition 4.13. Let $\mathcal{C} \to S$ be a functor and $T \in \text{Alg}(\mathcal{Fun}_S(\mathcal{C}, \mathcal{C}))$ a monad. By remark 2.13 for every category $\mathcal{B}$ over $S$ we have a canonical equivalence $\theta : \mathcal{LMod}_T(\mathcal{Fun}_S(\mathcal{B}, \mathcal{C})) \simeq \mathcal{Fun}_S(\mathcal{B}, \mathcal{LMod}_T^{/S}(\mathcal{C}))$. 

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over \text{Funs}(\mathcal{B}, \mathcal{C})$.

For \( \mathcal{B} = \text{LMod}^T(\mathcal{C}) \) the identity corresponds under \( \theta \) to a left \( T \)-module structure on the forgetful functor \( \psi : \text{LMod}^T(\mathcal{C}) \to \mathcal{C} \) with respect to the canonical \( \text{Funs}(\mathcal{C}, \mathcal{C}) \)-left module structure on \( \text{Funs}(\text{LMod}^T(\mathcal{C}), \mathcal{C}) \).

This left \( T \)-module structure exhibits \( T \) as the endomorphism object of \( \psi \).

**Proof.** Denote \( \mathcal{M}^\theta \to \text{LM}^\theta \times S \) the \( \text{LM}^\theta \)-monoidal category over \( S \) that encodes the endomorphism left module structure on \( X \to S \) over \( \text{Funs}(X, X) \).

Denote \( \mathcal{U}'(\kappa)_S \to (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times S \) the map of cartesian fibrations over \( (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \) classifying the identity of \( \text{Cat}_{\infty}(\kappa)_S \).

The forgetful functor \( \psi : \text{LMod}^T(\mathcal{C}) \to \mathcal{C} \) yields a map
\[
\Psi : \mathcal{Y} := \text{Fun}((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})^T(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \text{LMod}^T(\mathcal{X})) \to \mathcal{X} := \text{Fun}((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \mathcal{X})
\]

of cocartesian fibrations over \( (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \) over \( \text{Cat}_{\infty}(\kappa)_S \).

By remark 2.17 we have a \( \text{LM}^\theta \)-monoidal category
\[
\text{Fun}((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})^T(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \mathcal{M})^\theta \) over \( (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \), whose pullback along the monoidal diagonal functor
\[
\delta : (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \text{Funs}(X, X)^\theta \to \text{Map}(\text{Cat}_{\infty}(\kappa)_S)^{\text{op}}(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \text{Funs}(X, X))^\theta
\]

over \( (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \) endows \( \mathcal{X} \) with a left module structure over \( \text{Funs}(X, X) \).

By remark 4.12 it is enough to show that there is an equivalence
\[
\text{LMod}^T((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})(\mathcal{X}) \simeq \mathcal{Y}
\]
of cocartesian fibrations over \( (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \) over the cocartesian fibration
\[
\mathcal{X} \to (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}}.
\]

This desired equivalence is the composition of canonical equivalences
\[
\text{LMod}^T((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})(\mathcal{X}) = \text{LMod}^T((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})(\text{Fun}((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})^T(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \mathcal{X})) \simeq \text{Fun}((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})^T(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \text{LMod}^T(\mathcal{X})) \simeq \text{Fun}((\text{Cat}_{\infty}(\kappa)_S)^{\text{op}})^T(\mathcal{U}'(\kappa)_S, (\text{Cat}_{\infty}(\kappa)_S)^{\text{op}} \times \text{LMod}^T(\mathcal{X})) = \mathcal{Y}
\]
over \( \mathcal{X} \) provided by remark 2.13.1.

\( \square \)
4.2 An existence result for 2-categories with Eilenberg-Moore objects

Let \( S \) be a \( \kappa \)-small category. By example 4.5 the 2-category \( \text{Cat}_\infty(\kappa)_S \) admits Eilenberg-Moore objects and co-Eilenberg-Moore objects.

Goal of this subsection is to show that many subcategories of \( \text{Cat}_\infty(\kappa)_S \) are closed in \( \text{Cat}_\infty(\kappa)_S \) under Eilenberg-Moore objects and co-Eilenberg-Moore objects.

We will show that for every categorical pattern \( \mathcal{P} \) on \( S \) the subcategory \( \text{Cat}_\infty(\kappa)_S^{\mathcal{P}} \subset \text{Cat}_\infty(\kappa)_S \) with objects the \( \mathcal{P} \)-fibered objects and with morphisms the maps of those admits Eilenberg-Moore objects and co-Eilenberg-Moore objects which are preserved by the subcategory inclusion \( \text{Cat}_\infty(\kappa)_S^{\mathcal{P}} \subset \text{Cat}_\infty(\kappa)_S \) (theorem 4.16).

Example 4.14. Theorem 4.16 will imply that structure on a monad is reflected in structure on its category of algebras and dually structure on a comonad is reflected in structure on its category of coalgebras:

Let \( T \) be a monad on a category \( \mathcal{C} \) and denote \( \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C} \) its category of algebras.

Let \( L \) be a comonad on \( \mathcal{C} \) and denote \( \text{coLMod}_L(\mathcal{C}) = \text{LMod}_T(\mathcal{C}^{\text{op}})^{\text{op}} \rightarrow \mathcal{C} \) its category of coalgebras.

1. If \( \mathcal{C} \) carries the structure of an operad and \( T \) lifts to a map of operads such that the unit and multiplication of \( T \) are natural transformations of operads, then the forgetful functor \( \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C} \) and its left adjoint lift to maps of operads.

2. If \( \mathcal{C} \) carries the structure of an operad and \( L \) lifts to a map of operads such that the counit and comultiplication of \( L \) are natural transformations of operads, then the forgetful functor \( \text{coLMod}_L(\mathcal{C}) \rightarrow \mathcal{C} \) and its right adjoint lift to maps of operads.

Let \( \mathcal{V}^{\otimes} \) be a monoidal category.

3. If \( \mathcal{C} \) carries the structure of a left module over \( \mathcal{V} \) and \( T \) lifts to a \( \mathcal{V} \)-linear functor such that the unit and multiplication of \( T \) are \( \mathcal{V} \)-linear natural transformations, then the forgetful functor \( \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C} \) and its left adjoint lift to \( \mathcal{V} \)-linear functors.

4. If \( \mathcal{C} \) carries the structure of a left module over \( \mathcal{V} \) and \( L \) lifts to a \( \mathcal{V} \)-linear functor such that the counit and comultiplication of \( L \) are \( \mathcal{V} \)-linear natural transformations, then the forgetful functor \( \text{coLMod}_L(\mathcal{C}) \rightarrow \mathcal{C} \) and its right adjoint lift to \( \mathcal{V} \)-linear functors.

5. If \( \mathcal{C} \) carries the structure of a symmetric monoidal category and \( T \) lifts to an oplax symmetric monoidal functor such that the unit and multiplication of \( T \) are oplax symmetric monoidal natural transformations, then the forgetful functor \( \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C} \) lifts to a symmetric monoidal functor.

6. If \( \mathcal{C} \) carries the structure of a symmetric monoidal category and \( L \) lifts to a lax symmetric monoidal functor such that the unit and multiplication of
L are lax symmetric monoidal natural transformations, then the forgetful functor \( \text{coLMod}_L(\mathcal{C}) \to \mathcal{C} \) lifts to a symmetric monoidal functor.

We start with the following observation:

**Proposition 4.15.** Let \( S \) be a \( \kappa \)-small category.

1. The full subcategory of \( \text{Cat}_\infty(\kappa)_S \) spanned by the (locally) cartesian fibrations over \( S \) admits Eilenberg-Moore objects, which are preserved by the full subcategory inclusion to \( \text{Cat}_\infty(\kappa)_S \).

Dually, the full subcategory of \( \text{Cat}_\infty(\kappa)_S \) spanned by the (locally) cocartesian fibrations over \( S \) admits co-Eilenberg-Moore objects, which are preserved by the full subcategory inclusion to \( \text{Cat}_\infty(\kappa)_S \).

Moreover for every (locally) cartesian fibration \( \mathcal{C} \to S \) and every monad \( T \in \text{Alg}(\text{Funs}_S(\mathcal{C}, \mathcal{C})) \) the functor \( \text{LMod}_T^{[\mathcal{C}]_S} \to \mathcal{C} \) is a map of (locally) cartesian fibrations over \( S \).

Dually for every (locally) cocartesian fibration \( \mathcal{C} \to S \) and every comonad \( T \in \text{coAlg}(\text{Funs}_S(\mathcal{C}, \mathcal{C})) \) the functor \( \text{coLMod}_T^{[\mathcal{C}]_S} \to S \) is a map of (locally) cocartesian fibrations over \( S \).

2. The subcategory of \( \text{Cat}_\infty(\kappa)_S \) with objects the (locally) cartesian fibrations over \( S \) and with morphisms the functors over \( S \) that preserve (locally) cartesian morphisms admits Eilenberg-Moore objects and co-Eilenberg-Moore objects, which are preserved by the subcategory inclusion to \( \text{Cat}_\infty(\kappa)_S \).

Dually, the subcategory of \( \text{Cat}_\infty(\kappa)_S \) with objects the (locally) cocartesian fibrations over \( S \) and with morphisms the functors over \( S \) that preserve (locally) cocartesian morphisms admits Eilenberg-Moore objects and co-Eilenberg-Moore objects, which are preserved by the subcategory inclusion to \( \text{Cat}_\infty(\kappa)_S \).

**Proof.** We have a canonical equivalence \( (\text{Cat}_\infty(\kappa)_S)_{\text{op}} \simeq \text{Cat}_\infty(\kappa)_{S^\text{op}} \) of 2-categories that restricts to an equivalence of 2-categories between the full subcategory of \( (\text{Cat}_\infty(\kappa)_S)_{\text{op}} \) spanned by the (locally) cocartesian fibrations over \( S \) and the full subcategory of \( \text{Cat}_\infty(\kappa)_{S^\text{op}} \) spanned by the (locally) cartesian fibrations over \( S^\text{op} \) and restricts to an equivalence of 2-categories between the subcategory of \( (\text{Cat}_\infty(\kappa)_S)_{\text{op}} \) with objects the (locally) cocartesian fibrations over \( S \) and with morphisms the functors over \( S \) that preserve (locally) cocartesian morphisms and the subcategory of \( \text{Cat}_\infty(\kappa)_{S^\text{op}} \) with objects the (locally) cartesian fibrations over \( S^\text{op} \) and with morphisms the functors over \( S^\text{op} \) that preserve (locally) cartesian morphisms.

Thus it is enough to show that the full subcategory of \( \text{Cat}_\infty(\kappa)_S \) spanned by the (locally) cartesian fibrations over \( S \) and the subcategory of \( \text{Cat}_\infty(\kappa)_S \) with objects the (locally) (co)cartesian fibrations over \( S \) and with morphisms the functors over \( S \) that preserve (locally) (co)cartesian morphisms admit Eilenberg-Moore objects, which are preserved by the subcategory inclusions to \( \text{Cat}_\infty(\kappa)_S \).
By remark 4.12, it is enough to see the following:
For every (locally) cartesian fibration \( \mathcal{C} \to S \) and every monad \( T \in \text{Alg}(\text{Func}(\mathcal{C}, \mathcal{C})) \) the functor \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is a (locally) cartesian fibration and the functor \( \text{LMod}_{T}^{S}(\mathcal{C}) \to \mathcal{C} \) over S preserves and reflects (locally) cartesian morphisms.

For every (locally) cocartesian fibration \( \mathcal{C} \to S \) and every monad \( T \in \text{Alg}(\text{Func}(\mathcal{C}, \mathcal{C})) \), whose underlying endofunctor \( \mathcal{C} \to \mathcal{C} \) over S is a map of (locally) cocartesian fibrations over S, the functor \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is a (locally) cocartesian fibration and the functor \( \text{LMod}_{T}^{S}(\mathcal{C}) \to \mathcal{C} \) over S preserves and reflects (locally) cocartesian morphisms.

This follows from remark 2.11 and the fact that the functor \( \text{LMod}_{/S}^{/S} \) preserves and reflects (locally) cocartesian morphisms over S.

From proposition 4.15 we deduce the following theorem:

**Theorem 4.16.** Let \( S \) be a \( k \)-small category and \( \mathfrak{P} \) a categorical pattern on \( S \).

The subcategory \( \text{Cat}_{\infty}(\mathfrak{P})_{/S}^{\mathfrak{P}} \subset \text{Cat}_{\infty}(\mathfrak{P})_{/S} \) with objects the \( \mathfrak{P} \)-fibered objects and with morphisms the maps of those admits Eilenberg-Moore objects and coEilenberg-Moore objects which are preserved by the subcategory inclusion \( \text{Cat}_{\infty}(\mathfrak{P})_{/S}^{\mathfrak{P}} \subset \text{Cat}_{\infty}(\mathfrak{P})_{/S} \).

**Proof.** Denote \( \mathcal{E} := \mathcal{E}_{\mathfrak{P}} \subset \text{Func}(\Delta^{1}, S) \) and \( \mathcal{F} := \mathcal{F}_{\mathfrak{P}} \subset \text{Func}(\Delta^{2}, S) \) the specified morphisms and triangles of \( \mathfrak{P} \).

Let \( \mathfrak{P}' \) be the categorial pattern on \( S \) with \( \mathcal{E}_{\mathfrak{P}'} = \mathcal{E}, \mathcal{F}_{\mathfrak{P}'} \subset \text{Func}(\Delta^{2}, S) \) the functors \( \Delta^{2} \to S \) that factor through \( \Delta^{1} \) and with no diagrams.

The identity of \( S \) defines a map of categorial pattern \( \mathfrak{P}' \to \mathfrak{P} \) that gives rise to a \( \text{Cat}_{\infty}(\mathfrak{P}) \)-linear functor \( \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}'} \to \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}} \) with a lax \( \text{Cat}_{\infty}(\mathfrak{P}) \)-linear fully faithful right adjoint compatible with the subcategory inclusions \( \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}'} \subset \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}} \).

Especially \( \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}'} \) is a 2-localization of \( \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}} \).

So by remark 4.13 it is enough to show that \( \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}'} \) admits Eilenberg-Moore objects and coEilenberg-Moore objects which are preserved by the subcategory inclusion \( \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}'} \subset \text{Cat}_{\infty}(\mathcal{E})_{/S}^{\mathfrak{P}} \).

So we can assume that \( \mathfrak{P} \) has no diagrams and \( \mathcal{F} \) consists of the functors that factor through \( \Delta^{1} \).

A category \( \mathcal{C} \) over \( S \) is \( \mathfrak{P} \)-fibered if and only if for every morphism of \( \mathcal{C} \) corresponding to a functor \( \Delta^{1} \to S \) the pullback \( \Delta^{1} \times_{S} \mathcal{C} \to \Delta^{1} \) is a cocartesian fibration, a functor \( F : \mathcal{C} \to D \) over \( S \) is a map of \( \mathfrak{P} \)-fibered objects if and only if for every morphism of \( \mathcal{E} \) corresponding to a functor \( \Delta^{1} \to S \) the pullback \( \Delta^{1} \times_{S} \mathcal{C} \to \Delta^{1} \times_{S} D \) is a map of cocartesian fibrations over \( \Delta^{1} \).

Let \( \mathcal{C} \to S \) be a functor, \( T \in \text{Alg}(\text{Func}(\mathcal{C}, \mathcal{C})) \) a monad on \( \mathcal{C} \) and \( L \in \text{Alg}(\text{Func}(\mathcal{C}, \mathcal{C})) \text{op}) \simeq \text{Alg}(\text{Func}(\mathcal{C}, \mathcal{C}) \text{op}) \) a comonad on \( \mathcal{C} \).

A functor \( \Delta^{1} \to S \) yields a canonical monoidal functor \( \text{Fun}_{S}(\mathcal{E}, \mathcal{C}) \to \text{Fun}_{S}(\mathcal{E}, \mathcal{C}) \text{op}) \).
\[
\text{Fun}_\Delta(\Delta^1 \times \mathcal{C}, \Delta^1 \times \mathcal{C}) \rightarrow \text{Alg}(\text{Fun}_\mathcal{S}(\mathcal{C}, \mathcal{C})) \rightarrow \text{Alg}(\text{Fun}_\Delta(\Delta^1 \times \mathcal{C}, \Delta^1 \times \mathcal{C}))
\]

and

\[
\text{Alg}(\text{Fun}_\mathcal{S}(\mathcal{C}, \mathcal{C})^{\text{op}}) \rightarrow \text{Alg}(\text{Fun}_\Delta(\Delta^1 \times \mathcal{C}, \Delta^1 \times \mathcal{C})^{\text{op}})
\]

that send \( T \) to a monad structure on \( \Delta^1 \times \mathcal{S} T \) respectively \( L \) to a comonad structure on \( \Delta^1 \times \mathcal{S} L \).

By remark 2.2, we have canonical equivalences

\[
\Delta^1 \times \mathcal{S} \text{LMod}^T_{\Delta^1}(\mathcal{C}) \simeq \text{LMod}^{\Delta^1}_{\Delta^1 \times \mathcal{S} T}(\Delta^1 \times \mathcal{S} \mathcal{C})
\]

and

\[
\Delta^1 \times \mathcal{S} \text{coLMod}^T_{\Delta^1}(\mathcal{C}) \simeq \text{coLMod}^{\Delta^1}_{\Delta^1 \times \mathcal{S} L}(\Delta^1 \times \mathcal{S} \mathcal{C})
\]

over \( \Delta^1 \times \mathcal{S} \mathcal{C} \).

\[
\Delta^1 \times \mathcal{S} \text{LMod}_{\mathcal{S}}^{(\Delta^1)^{\text{op}}}(\mathcal{C})^{\text{op}} \simeq \text{LMod}^{(\Delta^1)^{\text{op}}}_{(\Delta^1 \times \mathcal{S} \mathcal{L})^{\text{op}}}(\Delta^1 \times \mathcal{S} \mathcal{C})^{\text{op}}
\]

By remark 4.7, and example 4.5 we have to check the following:

Let \( \mathcal{C} \to \mathcal{S} \) be a functor, \( T \in \text{Alg}(\text{Fun}_\mathcal{S}(\mathcal{C}, \mathcal{C})) \) a monad on \( \mathcal{C} \) and \( L \in \text{Alg}(\text{Fun}_\mathcal{S}^{\text{op}}(\mathcal{C}, \mathcal{C})^{\text{op}}) = \text{Alg}(\text{Fun}_{\mathcal{S}^{\text{op}}}(\mathcal{C}^{\text{op}}, \mathcal{C}^{\text{op}})) \) a comonad on \( \mathcal{C} \).

If for every morphism of \( \mathcal{C} \) corresponding to a functor \( \Delta^1 \to \mathcal{S} \) the pullback \( \Delta^1 \times \mathcal{S} \mathcal{C} \to \Delta^1 \) is a cocartesian fibration and the functors \( \Delta^1 \times \mathcal{S} T, \Delta^1 \times \mathcal{S} L : \Delta^1 \times \mathcal{S} \mathcal{C} \to \Delta^1 \times \mathcal{S} \mathcal{C} \) over \( \Delta^1 \) are maps of cocartesian fibrations over \( \Delta^1 \), then for every morphism of \( \mathcal{C} \) corresponding to a functor \( \Delta^1 \to \mathcal{S} \) the pullbacks

\[
\Delta^1 \times \mathcal{S} \text{LMod}^T_{\mathcal{S}}(\mathcal{C}) \rightarrow \Delta^1 \text{ and } \Delta^1 \times \mathcal{S} \text{coLMod}^T_{\mathcal{S}}(\mathcal{C}) \rightarrow \Delta^1
\]

are cocartesian fibrations and the functors

\[
\Delta^1 \times \mathcal{S} \text{LMod}^T_{\mathcal{S}}(\mathcal{C}) \rightarrow \Delta^1 \times \mathcal{S} \mathcal{C} \text{ and } \Delta^1 \times \mathcal{S} \text{coLMod}^T_{\mathcal{S}}(\mathcal{C}) \rightarrow \Delta^1 \times \mathcal{S} \mathcal{C}
\]

are cocartesian fibrations over \( \Delta^1 \) preserve and reflect cocartesian morphisms.

Hence we can reduce to the case that \( S = \Delta^1, \mathcal{C} = \text{Fun}(\Delta^1, \Delta^1) \) and \( \mathcal{F} = \text{Fun}(\Delta^2, \Delta^1) \).

But in this case \( \text{Cat}_\infty(\kappa)^{\mathcal{P}} = \text{Cat}_\infty(\kappa)^{\text{cocart}}_{\Delta^1} \) so that the result follows from proposition 1.13.

\[\Box\]

\textbf{Example 4.17.} Let \( \mathcal{O}^\otimes \) be an operad and \( \mathcal{P} \) the categorial pattern for operads over \( \mathcal{O}^\otimes \).

Then \( \text{Op}_\infty(\kappa)^{\mathcal{O}^\otimes} \) is closed in \( \text{Cat}_\infty(\kappa)^{\mathcal{O}^\otimes} \) under Eilenberg-Moore objects and co-Eilenberg-Moore objects.

This implies 1.-4. of example 3.12.

Denote \( \mathcal{W} \) the full subcategory of \( \text{Cat}_\infty(\kappa)^{\mathcal{O}^\otimes} \) spanned by the (locally) cocartesian fibrations over \( \mathcal{O}^\otimes \).

Then \( \text{Op}_\infty(\kappa)^{\mathcal{O}^\otimes} \times_{\text{Cat}_\infty(\kappa)^{\mathcal{O}^\otimes}} \mathcal{W} \) is the category of \( \mathcal{O}^\otimes \)-monoidal categories (respectively representable \( \mathcal{O}^\otimes \)-operads) and lax \( \mathcal{O}^\otimes \)-monoidal functors.

By proposition 4.15 \( \mathcal{W} \) is closed in \( \text{Cat}_\infty(\kappa)^{\mathcal{O}^\otimes} \) under co-Eilenberg-Moore objects.

Thus by remark 4.2 the 2-category \( \text{Op}_\infty(\kappa)^{\mathcal{O}^\otimes} \times_{\text{Cat}_\infty(\kappa)^{\mathcal{O}^\otimes}} \mathcal{W} \) is closed in \( \text{Cat}_\infty(\kappa)^{\mathcal{O}^\otimes} \) under co-Eilenberg-Moore objects.
This together with proposition 4.15 implies 6. of example 4.14.

Let $\mathcal{C} \otimes \to O \otimes$ be an $O \otimes$-monoidal category classifying a $O \otimes$-monoid $\phi$ of $\text{Cat}_\infty(\kappa)$.

Denote $(\mathcal{C} \otimes)_{\text{rev}} \to O \otimes$ the fiberwise dual relative to $O \otimes$ of the cocartesian fibration $\mathcal{C} \otimes \to O \otimes$ and $(\mathcal{C} \otimes)^{\vee} = ((\mathcal{C} \otimes)_{\text{rev}})^{\text{op}}$ the cartesian fibration classifying $\phi$.

Let $T \in \text{Coalg}(\text{Alg}_{O \otimes}((\mathcal{C} \otimes)_{\text{rev}}, (\mathcal{C} \otimes)_{\text{rev}}))$ be a comonad in $\text{Op}_\infty(\kappa)/_{O \otimes} \times_{\text{Cat}_\infty(\kappa)/_{O \otimes}} \mathcal{W}$, which we think of as a opmonoidal monad on $\mathcal{C}$.

So the co-Eilenberg-Moore object

$$\text{coLMod}^{(O \otimes)}_T = \text{LMod}^{((O \otimes)^{\text{rev}})^{\text{op}}}_{(O \otimes)^{\text{rev}}} \to O \otimes$$

of $T$ in $\text{Cat}_\infty(\kappa)/_{O \otimes}$ is a $O \otimes$-monoidal category and the forgetful functor $V : \text{coLMod}^{(O \otimes)}_T \to (O \otimes)_{\text{rev}}$ is a $O \otimes$-monoidal functor.

Thus $\text{LMod}^{((O \otimes)^{\text{rev}})^{\text{op}}}_{(O \otimes)^{\text{rev}}} \to O \otimes$ is a $O \otimes$-monoidal category and the forgetful functor $V^{\text{rev}} : \text{LMod}^{((O \otimes)^{\text{rev}})^{\text{op}}}_{(O \otimes)^{\text{rev}}} \to (O \otimes)^{\text{rev}} \to O \otimes$ is a $O \otimes$-monoidal functor.

For every $X \in O$ the $O \otimes$-monoidal functor $V^{\text{rev}} : \text{LMod}^{((O \otimes)^{\text{rev}})^{\text{op}}}_{(O \otimes)^{\text{rev}}} \to (O \otimes)^{\text{rev}}$ induces the forgetful functor $\text{LMod}^{(T_X)^{\text{op}}}_{(T_X)^{\text{op}}}(\mathcal{C}_X) \to \mathcal{C}_X$. 

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5 A localization between monads and right adjoint morphisms

Let S be a \( \kappa \)-small category, \( \mathcal{C}^\circ \to S \times \text{LM}^\circ \) a cocartesian \( S \)-family of categories enriched in \( \text{Cat}_{\infty}(\kappa) \) and \( X \) a section of \( \mathcal{C} \to S \).

We construct a map \( \text{End} : (\mathcal{E}_{/X}^S)^R \to \text{Alg}^{\text{rev}}(\mathcal{X}, \mathcal{X})^S \) of cocartesian fibrations over \( S \) that sends a morphism \( g : Y \to X(s) \) for some \( s \in S \) with left adjoint \( f : X(s) \to Y \) to its associated monad on \( X(s) \).

Denote \( (\mathcal{E}_{/X}^S)^{\text{mon}} \subset (\mathcal{E}_{/X}^S)^{\text{rep}} \subset (\mathcal{E}_{/X}^S)^R \) the full subcategories spanned by the monadic morphisms respectively morphisms whose associated monad is representable and \( \text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}} \subset \text{Alg}(\mathcal{X}, \mathcal{X})^S \) the full subcategory spanned by the representable monads.

We show that \( \text{End} : (\mathcal{E}_{/X}^S)^{\text{rep}} \to (\text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}})^{\text{rev}} \) admits a fully faithful right adjoint \( \text{Alg} \) relative to \( \mathcal{C} \) with essential image \( (\mathcal{E}_{/X}^S)^{\text{mon}} \) (theorem 5.1).

Thus the functor \( \text{End} \) restricts to an equivalence \( (\mathcal{E}_{/X}^S)^{\text{mon}} \to (\text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}})^{\text{rev}} \) inverse to the functor \( \text{Alg} \) and the full subcategory \( (\mathcal{E}_{/X}^S)^{\text{mon}} \subset (\mathcal{E}_{/X}^S)^{\text{rep}} \) is a localization relative to \( S \).

If \( \mathcal{C} \) is a subcategory of \( \text{Cat}_{\infty}^{\text{rev}} \) for some \( \kappa \)-small category \( S \) and \( X \in \mathcal{C} \), we give a more explicite description of the adjunction \( \text{End} : (\mathcal{E}_{/X}^S)^{\text{rep}} \to (\text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}})^{\text{op}} \).

We show in theorem 5.4 that \( \text{Alg} \) is the restriction of the functor

\[
\text{Alg}(\text{Fun}_{S}(\mathcal{X}, \mathcal{X}))^{\text{op}} \to ((\text{Cat}_{\infty}(\kappa))^{S})^R \subset \text{Cat}_{\infty}(\kappa)_{/X}
\]

classified by the map \( \text{LMMod}^{\text{rev}}(\mathcal{X}) \to X \times \text{Alg}(\text{Fun}_{S}(\mathcal{X}, \mathcal{X})) \) of cartesian fibrations over \( \text{Alg}(\text{Fun}_{S}(\mathcal{X}, \mathcal{X})) \).

Having this more explicite description we are able to give a more coherent version of the adjunction of theorem 5.1 for the case that \( \mathcal{C} \) is a subcategory of \( \text{Cat}_{\infty}^{\text{rev}} \).

We define a category \( \text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}} \) over \( \mathcal{C}^{\text{rep}} \), whose fiber over an object \( \mathcal{X} \) of \( \mathcal{C} \) is the category \( \text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}} \) of monads on \( \mathcal{X} \) that admit an Eilenberg-Moore object in \( \mathcal{C} \) that is preserved by the subcategory inclusion \( \mathcal{E}_{/X} \subset (\text{Cat}_{\infty}^{\text{rev}})_{/X} \).

Denote \( \text{Fun}(\Delta^1, \mathcal{C})^{\text{rep}} \subset \text{Fun}(\Delta^1, \mathcal{C})^{\text{rev}} \) the full subcategory spanned by the morphisms \( Y \to X \), whose associated monad on \( X \) admits an Eilenberg-Moore object in \( \mathcal{C} \) that is preserved by the subcategory inclusion \( \mathcal{E}_{/X} \subset (\text{Cat}_{\infty}^{\text{rev}})_{/X} \).

We construct a localization \( \text{End} : \text{Fun}(\Delta^1, \mathcal{C})^{\text{rev}} \to (\text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}})^{\text{op}} : \text{Alg} \) relative to \( \mathcal{C} \) that induces on the fiber over an object \( \mathcal{X} \) of \( \mathcal{C} \) the localization \( \text{End} : (\mathcal{E}_{/X})^{\text{rev}} \to (\text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}})^{\text{op}} : \text{Alg} \) (theorem 5.5), where we use the explicite description of the functor \( \text{Alg} \) given by theorem 5.4.

So the functor \( \text{End} \) restricts to an equivalence

\[
\text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \to (\text{Alg}(\mathcal{X}, \mathcal{X})^{\text{rep}})^{\text{op}}
\]

relative to \( \mathcal{C} \) and the full subcategory \( \text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^{\text{rep}} \) is a localization relative to \( \mathcal{C} \).
From this we deduce the statement that for every 2-category \( \mathcal{C} \) the full subcategory \( \text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^{\text{rep}} \) is a localization relative to \( \mathcal{C} \) (theorem 5.9).

We deduce this from the following facts:

By proposition 7.13, we have a functor \( \theta : \mathcal{C} \to \text{Cat}_{\infty/\mathcal{C}^{\text{rev}}}^{\text{cocart}}(\kappa) \) that sends an object \( X \) of \( \mathcal{C} \) to the cocartesian fibration classifying the functor \( [-, X] : \mathcal{C}^{\text{rev}} \to \text{Cat}_{\infty}(\kappa) \). The induced functor \( \mathcal{C} \to \theta(\mathcal{C}) \) admits a left inverse due to remark 5.8.

By theorem 5.1, for every \( X \in \mathcal{C} \) the full subcategory \( (\mathcal{C}/X)^{\text{mon}} \subset (\mathcal{C}/X)^{\text{rep}} \) of representable morphisms over \( X \) spanned by the monadic morphisms over \( X \) is a localization due to theorem 5.9.

Moreover, we show in theorem 5.9 that this localization can be enhanced to a localization of 2-categories relative to \( \mathcal{C} \) if \( \mathcal{C} \) is cotensored over \( \text{Cat}_{\infty}(\kappa) \).

So if \( \mathcal{C} \) is a 2-category that admits Eilenberg-Moore objects, we obtain a localization \( \text{Fun}(\Delta^1, \mathcal{C})^{\text{mon}} \subset \text{Fun}(\Delta^1, \mathcal{C})^{R} \) from monadic morphisms into right adjoint morphisms.

### 5.1 A localization between monads and right adjoint morphisms

Let \( S \) be a \( \kappa \)-small category, \( \mathcal{C}^{\text{op}} \to S \times \text{LM}^{\text{op}} \) a cocartesian \( S \) family of 2-categories and \( X \) a cocartesian section of \( \mathcal{C} \to S \).

Denote \( (\mathcal{C}/S)^{\text{mon}} \subset (\mathcal{C}/S)^{\text{rep}} \subset (\mathcal{C}/S)^{R} \) the full subcategories spanned by the morphisms \( Y \to X(s) \) in \( \mathcal{C} \) for some \( s \in S \) that are monadic respectively whose associated monad on \( X(s) \) is representable, i.e., admits an Eilenberg-Moore object, respectively is a right adjoint morphism.

**Construction 2.**

Let \( S \) be a \( \kappa \)-small category and \( \mathcal{C}^{\text{op}} \to S \times \text{LM}^{\text{op}} \) a cocartesian \( S \) family of 2-categories.

By proposition 7.13, we have a map

\[
\theta : \mathcal{C}^{\text{op}} \to \text{Map}_S(\mathcal{C}^{\text{rev}}, S \times \text{Cat}_{\infty}(\kappa))^{\text{op}}
\]

of \( S \) families of operads over \( \text{LM}^{\text{op}} \), whose pullback to \( \text{Ass}^{\text{op}} \) is the diagonal map

\[
S \times \text{Cat}_{\infty}(\kappa)^{\times} \simeq \text{Map}_S(\mathcal{C}^{\text{rev}}, S \times \text{Cat}_{\infty}(\kappa))^{\text{op}} \to \text{Map}_S(\mathcal{C}^{\text{rev}}, S \times \text{Cat}_{\infty}(\kappa))^{\text{op}}
\]

of \( S \) families of operads over \( \text{Ass}^{\text{op}} \).

\( \theta \) induces a \( \text{Funs}(S, S \times \text{Cat}_{\infty}(\kappa))^{\times} \simeq \text{Fun}(S, \text{Cat}_{\infty}(\kappa))^{\times} \simeq (\text{Cat}_{\infty}(\kappa))^{\text{cocart}}^{\times} \)-linear map

\[
\chi : \text{Funs}(S, \mathcal{C})^{\text{op}} \to \text{Funs}(S, \text{Map}_S(\mathcal{C}^{\text{rev}}, S \times \text{Cat}_{\infty}(\kappa)))^{\text{op}} \simeq \text{Funs}(\mathcal{C}^{\text{rev}}, S \times \text{Cat}_{\infty}(\kappa))^{\times}
\]

\[
\simeq \text{Fun}(\mathcal{C}^{\text{rev}}, \text{Cat}_{\infty}(\kappa))^{\times} \simeq (\text{Cat}_{\infty}(\kappa))^{\mathcal{C}^{\text{rev}}}^{\times}
\]

of operads over \( \text{LM}^{\text{op}} \).

The composition

\[
\mathcal{C} \to \text{Map}_S(\mathcal{C}^{\text{rev}}, S \times \text{Cat}_{\infty}(\kappa)) \simeq S \times \text{Cat}_{\infty}(\kappa) \text{Cocart}(\kappa) \to S \times \text{Cat}_{\infty}(\kappa) \mathcal{L}(\kappa)
\]

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of functors over $S$ is equivalent to the Yoneda-embedding

$$\mathcal{E} ≃ S \times \text{Cat}_∞(\kappa) \cup(\kappa) → S \times \text{Cat}_∞(\kappa) \mathcal{R}(\kappa) ≃ S \times \text{Cat}_∞(\kappa) \mathcal{L}(\kappa)$$

over $S$.

Hence the composition

$$\text{Fun}_S(\mathcal{E}, \mathcal{E}) → \text{Fun}_S(\mathcal{M}, (\mathcal{E}^{\text{rev}}, S \times \text{Cat}_∞(\kappa))) \simeq \text{Fun}_S(\mathcal{E}^{\text{rev}}, S \times \text{Cat}_∞(\kappa))$$

$$\simeq \text{Fun}(\mathcal{E}^{\text{rev}}, \text{Cat}_∞(\kappa)) ≃ \text{Cat}_∞(\kappa)^{\text{co-cart}} → \mathcal{L}(\kappa)^{\text{rev}}$$

is equivalent to the Yoneda-embedding

$$\text{Fun}_S(\mathcal{E}, \mathcal{E}) ≃ \text{Fun}_{\text{Cat}_∞(\kappa)}(\mathcal{E}, \mathcal{E}) \subset \text{Fun}_{\text{Cat}_∞(\kappa)}(\mathcal{R}(\kappa)) ≃ \text{Fun}_{\text{Cat}_∞(\kappa)}(\mathcal{L}(\kappa))$$

$$≃ \mathcal{L}(\kappa)^{\text{rev}}.$$

Let $X$ be a cocartesian section of $\mathcal{E} → S$ and $\rho : X → \mathcal{E}^{\text{rev}}$ the cocartesian fibration that classifies the functor $\mathcal{E}^{\text{rev}} → \text{Cat}_∞(\kappa)$ adjoint to the functor $\theta \circ X : \mathcal{E}^{\text{rev}} → \text{Cat}_∞(\kappa) × S$ over $S$.

Let $X$ be a cocartesian section of $\mathcal{E} → S$. We have a canonical endomorphism left module structure on $X$ over the cocartesian fibration $[X, X]^S → S$ with respect to the LM$^i$-operad structure on $\text{Fun}_S(\mathcal{E}, \mathcal{E})$ over $\text{Fun}_S(S × \text{Cat}_∞(\kappa)) ≃ \text{Fun}(S, \text{Cat}_∞(\kappa)) ≃ \text{Cat}_∞(\kappa)^{\text{co-cart}}$, which is sent by $\chi$ to a left module structure on $\rho : X → \mathcal{E}^{\text{rev}}$.

By proposition \ref{prop:canonical_endomorphism_left_module_structure} we have a canonical equivalence $(\mathcal{E}^{S})^{\text{re}} ≃ X^c$ of left fibrations over $\mathcal{E}^{\text{rev}}$, where $X^c$ is the subcategory with the same objects and with morphisms the $\rho$-cocartesian morphisms.

Denote $\mathcal{X}^{\text{univ}}_\text{End} ⊂ X^c = (\mathcal{E}^{S})^{\text{re}}$ the full subcategory spanned by the morphisms $f : Y → X(s)$ in $\mathcal{E}_s$ for some $s ∈ S$ that admit an endomorphism object with respect to the canonical $[X(s), X(s)]$-left module structure on $[Y, X(s)]$ that is sent by any morphism $ϕ : s → t$ of $S$ to an endomorphism object of $φ_*(f)$ with respect to the canonical $[X(t), X(t)]$-left module structure on $[φ_*(Y), X(t)]$.

By proposition \ref{prop:canonical_endomorphism_left_module_structure} we have a full inclusion $((\mathcal{E}^{S})^{\text{re}})^r ⊂ \mathcal{X}^{\text{univ}}_\text{End}$.

So by construction \ref{def:canonical_endomorphism_left_module_structure} we have a map

$$((\mathcal{E}^{S})^{\text{re}})^r → \text{Alg}^S([X, X]^S)$$

of cocartesian fibrations over $S$ that is the endomorphism object of the canonical inclusion $((\mathcal{E}^{S})^{\text{re}})^r ⊂ X$ with respect to the left module structure on

$$\text{Fun}_{\text{re}}(((\mathcal{E}^{S})^{\text{re}})^r, X) \text{ over } \text{Fun}_{\text{re}}(((\mathcal{E}^{S})^{\text{re}})^r, \mathcal{E}^{\text{rev}} × S [X, X]^S) ≃ \text{Fun}(((\mathcal{E}^{S})^{\text{re}})^r, [X, X]^S).$$

Passing to fiberwise duals over $S$ we get a map $\text{End} : (\mathcal{E}^{S})^{\text{re}} → \text{Alg}^S([X, X]^S)^{\text{re}}$ of cocartesian fibrations over $S$.

End sends a morphism $g : Y → X(s)$ for some $s ∈ S$ with left adjoint $f : X(s) → Y$ to its endomorphism object with respect to the canonical $[X(s), X(s)]$-left module structure on $[Y, X(s)]$, which is given by $gφ$ according to proposition \ref{prop:canonical_endomorphism_left_module_structure}.

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Now we are ready to state the main theorem:

**Theorem 5.1.** Let $S$ be a $\kappa$-small category, $\mathcal{C}^\circ \rightarrow S \times \text{LM}^\circ$ a cocartesian $S$-family of 2-categories and $X$ a cocartesian section of $\mathcal{C} \rightarrow S$.

We have a localization $\text{End} : (\mathcal{C}^S_{/X})^{\text{rep}} \rightarrow (\text{Alg}^S([X, X]^S)^{\text{rep}})^{\text{rev}} : \text{Alg}$ relative to $S$.

For every object $s \in S$ the local objects of $((\mathcal{C}_s)_{/(X(s))})^{\text{rep}}$ are the monadic morphisms over $X(s)$ so that the restriction

$$(\mathcal{C}^S_{/X})^{\text{mon}} \subset (\mathcal{C}^S_{/X})^{\text{rep}} \text{End} \rightarrow (\text{Alg}^S([X, X]^S)^{\text{rep}})^{\text{rev}}$$

is an equivalence and $(\mathcal{C}^S_{/X})^{\text{mon}} \subset (\mathcal{C}^S_{/X})^{\text{rep}}$ is a localization relative to $S$.

Let $s \in S$ and let $g : Y \rightarrow X, h : Z \rightarrow X$ be morphisms in $\mathcal{C}_s$ that admit left adjoints $f : X \rightarrow Y$ respectively $k : X \rightarrow Z$.

A morphism $\phi : Y \rightarrow Z$ is a local equivalence if and only if the morphism $h \circ k \dashv h \circ k \circ g \circ f \sim h \circ k \circ h \circ \phi \circ f \rightarrow h \circ \phi \circ f \sim g \circ f$ in $[X, Z]$ is an equivalence.

Especially a morphism $\phi : Y \rightarrow Z$ in $((\mathcal{C}_s)_{/(X(s))})^{\text{rep}}$ with $Z$ a local object is a local equivalence if and only if the morphism $k \dashv k \circ g \circ f \sim k \circ h \circ \phi \circ f \rightarrow \phi \circ f$ in $[X, Z]$ is an equivalence.

Let $s \in S$ and let $g : Y \rightarrow X$ be a right adjoint morphism in $\mathcal{C}_s$ with associated monad $T$ that admits an Eilenberg-Moore object $\psi : Z \rightarrow X$ in $\mathcal{C}_s$.

We have a canonical equivalence $[Y, Z] \simeq \text{LMod}_T([Y, X])$ over $[Y, X]$ under which the endomorphism left module structure on $g : Y \rightarrow X$ over $T$ corresponds to lift $g' : Y \rightarrow Z$ of $g : Y \rightarrow X$.

$g' : Y \rightarrow Z$ is a local equivalence in $((\mathcal{C}_s)_{/(X(s))})^{\text{rep}}$ with target a local object.

**Proof.** Being a map of cocartesian fibrations over $S$ the functor $\text{End} : (\mathcal{C}^S_{/X})^{\text{rep}} \rightarrow (\text{Alg}^S([X, X]^S)^{\text{rep}})^{\text{rev}}$ over $S$ admits a fully faithful right adjoint relative to $S$ if and only if for every $s \in S$ the induced functor $\text{End}_s : ((\mathcal{C}_s)_{/(X(s))})^{\text{rep}} \rightarrow (\text{Alg}([X(s), X(s)])^{\text{rep}})^{\text{op}}$ on the fiber over $s$ admits a fully faithful right adjoint.

So we can reduce to the case that $S$ is contractible.

Let $\phi : Z \rightarrow X$ be a monadic morphism of $\mathcal{C}$ and $T \simeq \text{End}(Z)$ its endomorphism object with respect to the canonical $[X, X]$-left module structure on $[Y, X]$.

It is enough to find an equivalence

$$\alpha : \mathcal{C}_{/X}(-, Z) \simeq \text{Alg}([X, X])^{\text{op}}(\text{End}(-), \text{End}(Z))$$

of functors $((\mathcal{C}_{/X})^{\text{rep}})^{\text{op}} \rightarrow S(\kappa)$ such that under the induced equivalence

$$\mathcal{C}_{/X}(Z, Z) \simeq \text{Alg}([X, X])^{\text{op}}(\text{End}(Z), \text{End}(Z))$$

of spaces the identity of $Z$ corresponds to an autoequivalence of $\text{End}(Z)$.

The morphism $\phi : Z \rightarrow X$ induces a natural transformation $[-, \phi] : [-, Z] \rightarrow [-, X]$ of functors $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}(\kappa)$ classified by a map $\mathcal{Z} \rightarrow \mathfrak{X}$ of cocartesian fibrations over $S$. 

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fibrations over $\mathcal{C}^{\text{op}}$. By [13.1] there is a canonical equivalence $\mathcal{Z} \to \text{LMod}^{/\mathcal{C}^{\text{op}}} (\mathcal{X})$ over $\mathcal{X}$.

By remark 3.4.2, we have a canonical equivalence

$$(\text{Alg}([X, X]) \times ((\mathcal{E}/X)^R)^{\text{op}}) \times_{\text{Alg}([X, X]) \times X} \text{LMod}^{/\mathcal{C}^{\text{op}}} (\mathcal{X}) \simeq$$

$$(\text{Alg}([X, X]) \times ((\mathcal{E}/X)^R)^{\text{op}}) \times_{\text{Alg}([X, X]) \times_{\mathcal{C}^{\text{op}}} \text{Alg}([X, X]) \times_{\mathcal{C}^{\text{op}}} \text{Alg}([X, X]) \Delta^I} \text{Alg}([X, X])^{\Delta^I}$$

er over $\text{Alg}([X, X]) \times ((\mathcal{E}/X)^R)^{\text{op}}$ that gives rise to an equivalence

$$(\mathcal{E}/X)^{\text{op}} \times_{\mathcal{X}} \text{LMod}^{/\mathcal{C}^{\text{op}}} (\mathcal{X}) \simeq ((\mathcal{E}/X)^R)^{\text{op}} \times_{\text{Alg}([X, X]) \times_{\mathcal{C}^{\text{op}}} \text{Alg}([X, X]) \Delta^I} \text{Alg}([X, X])^{\Delta^I}$$

er over $((\mathcal{E}/X)^R)^{\text{op}}$.

As $\phi : Z \to X$ is monadic, for every $Y \in \mathcal{C}$ the functor $[Y, \phi] : [Y, Z] \to [Y, X]$ is monadic and thus conservative.

Hence the commutative square

$$\begin{array}{ccc}
\mathcal{E}/Z & \xrightarrow{\mathcal{Z}^{\text{op}}} & \mathcal{Z}^{\text{op}} \\
\downarrow & & \downarrow \\
\mathcal{E}/X & \to & \mathcal{X}^{\text{op}}
\end{array}$$

of cartesian fibrations over $\mathcal{C}$ is a pullback square as it induces on the fiber over every $Y \in \mathcal{C}$ the pullback square

$$\begin{array}{ccc}
\mathcal{E}(Y, Z) & \xrightarrow{[Y, Z]^{\text{op}}} & [Y, Z]^{\text{op}} \\
\downarrow & & \downarrow \\
\mathcal{E}(Y, X) & \to & [Y, X]^{\text{op}}
\end{array}$$

So we get a canonical equivalence

$$(\mathcal{E}/X)^R \times_{\mathcal{E}/X} (\mathcal{E}/X)_{Z} = (\mathcal{E}/X)^R \times_{\mathcal{E}/X} \mathcal{E}/Z \simeq (\mathcal{E}/X)^R \times_{\mathcal{X}^{\text{op}}} \mathcal{Z}^{\text{op}} \simeq$$

$$(\mathcal{E}/X)^R \times_{\mathcal{X}^{\text{op}}} \text{LMod}^{/\mathcal{C}^{\text{op}}} (\mathcal{X})^{\text{op}} \simeq ((\mathcal{E}/X)^R)^{\text{op}} \times_{\text{Alg}([X, X])^{\text{op}}} \text{Alg}([X, X])^{\Delta^I}$$

of right fibrations over $(\mathcal{E}/X)^R$ that classifies an equivalence

$$\mathcal{E}/X (-, Z) \simeq \text{Alg}([X, X])^{\text{op}} (\text{End} (-), \text{End}(Z))$$

of functors $((\mathcal{E}/X)^R)^{\text{op}} \to \mathcal{S}(\kappa)$, whose restriction to $((\mathcal{E}/X)^{\text{rep}})^{\text{op}} \subset ((\mathcal{E}/X)^R)^{\text{op}}$ is the desired equivalence.

Proposition B.6 guarantees that under the induced equivalence

$$\mathcal{E}/X (Z, Z) \simeq \text{Alg}([X, X])^{\text{op}} (\text{End}(Z), \text{End}(Z))$$

the identity of $Z$ corresponds to an autoequivalence of $\text{End}(Z)$.

So the functor $\text{End} : (\mathcal{E}/X)^{\text{rep}} \to (\text{Alg}([X, X])^{\text{rep}})^{\text{op}}$ admits a fully faithful right adjoint that sends a representable monad $T$ on $X$ to the monadic morphism $Z \to X$ representing the functor $\text{Alg}([X, X])^{\text{op}} (\text{End} (-), T) : ((\mathcal{E}/X)^{\text{op}})^{\text{op}} \to \mathcal{S}(\kappa)$.
So given a monadic morphism \( Z \to X \) the right adjoint sends the representable monad \( \text{End}(Z) \) on \( X \) to \( Z \to X \).

Hence the local objects of \((\mathcal{E}/X)^{\text{rep}}\) are exactly the monadic morphisms over \( X \).

The statements about local equivalences follow from lemma 5.3.7.

Let \( s \in S \) and let \( g : Y \to X \) be a right adjoint morphism in \( \mathcal{E}_s \) with associated monad \( T \) that admits an Eilenberg-Moore object \( \psi : Z \to X \) in \( \mathcal{E}_s \).

By definition of \( \alpha \) under the equivalence

\[
\alpha(Y) : (\mathcal{E}_s)/X(s)(Y, Z) = \text{Alg}([X(s), X(s)])^{\text{op}}(\text{End}(Y), T)
\]

the lift \( g' : Y \to Z \) of \( g : Y \to X \) corresponds to the identity of \( T = \text{End}(Y) \) and is thus the unit and so a local equivalence.

\[ \square \]

**Corollary 5.2.** Let \( S \) be a \( \kappa \)-small category, \( \mathcal{C}^\circ \to S \times LM^\circ \) a cocartesian \( S \)-family of 2-categories and \( X \) a cocartesian section of \( \mathcal{C} \to S \). Assume that for every \( s \in S \) the image \( X(s) \) admits Eilenberg-Moore objects in \( \mathcal{C}_s \).

The map \( \text{End} : (\mathcal{C}^S)^R \to \text{Alg}^S([X, X]^S)^{\text{rev}} \) of cocartesian fibrations over \( S \) admits a fully faithful right adjoint \( \text{Alg} \) relative to \( S \) that sends a monad on \( X(s) \) for some \( s \in S \) to its Eilenberg-Moore object.

For every object \( s \in S \) the local objects of \((\mathcal{E}_s)/X(s)\) are the monadic morphisms over \( X(s) \) so that the restriction

\[
(\mathcal{C}^S)^{\text{mon}} \subset \mathcal{C}^S \xrightarrow{\text{End}} \text{Alg}^S([X, X]^S)^{\text{rev}}
\]

is an equivalence and the full subcategory \((\mathcal{C}^S)^{\text{mon}} \subset (\mathcal{C}^S)^R\) is a localization relative to \( S \).

Let \( s \in S \) and let \( g : Y \to X, h : Z \to X \) be morphisms in \( \mathcal{C}_s \) that admit left adjoints \( f : X \to Y \) respectively \( k : X \to Z \).

A morphism \( \phi : Y \to Z \) in \((\mathcal{C}_s)/X(s)\) is a local equivalence if and only if the morphism \( h \circ k \to h \circ k \circ g \circ f \simeq h \circ k \circ h \circ f \circ f \to h \circ f \circ f \simeq g \circ f \) in \([X, X]\) is an equivalence.

Especially a morphism \( \phi : Y \to Z \) in \((\mathcal{C}_s)/X(s)\) with \( Z \) a local object is a local equivalence if and only if the morphism \( k \circ g \circ f \simeq k \circ f \circ f \to f \circ f \) in \([X, Z]\) is an equivalence.

**Remark 5.3.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a 2-functor.

We have a commutative square

\[
\begin{array}{ccc}
(\mathcal{C}/X)^{\text{rep}} & \xrightarrow{\text{End}} & (\text{Alg}([X, X])^{\text{rep}})^{\text{op}} \\
\downarrow & & \downarrow \\
(\mathcal{D}/F(X))^{\text{rep}} & \xrightarrow{\text{End}} & (\text{Alg}([F(X), F(X)])^{\text{rep}})^{\text{op}}.
\end{array}
\]

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If \( F \) preserves monadic morphisms with target \( X \), by remark 3.2 \( F \) preserves the Eilenberg-Moore object of every monad on \( X \). In this case the last square induces a commutative square

\[
\begin{array}{ccc}
(\mathcal{E}/_X)^{\text{rep}} & \xrightarrow{\text{Alg}} & (\text{Alg}([X,X])^{\text{rep}})^{\text{op}} \\
\downarrow & & \downarrow \\
(\mathcal{D}/F(X))^\text{rep} & \xrightarrow{\text{Alg}} & (\text{Alg}([F(X),F(X)])^{\text{rep}})^{\text{op}}.
\end{array}
\]

Applying remark 5.3 to the 2-functor \( \theta : \mathcal{C} \rightarrow \text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa) \) that preserves monadic morphisms we obtain a commutative square

\[
\begin{array}{ccc}
(\text{Alg}([X,X])^{\text{rep}})^{\text{op}} & \xrightarrow{\text{Alg}} & (\mathcal{E}/_X)^{\text{rep}} \\
\downarrow & & \downarrow \\
\text{Alg}([X,X])^{\text{rep}} & \xrightarrow{\text{Alg}} & (\text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa)/_X)^R
\end{array}
\]

with \( X := \theta(X) \).

As the composition \( \mathcal{C} \xrightarrow{\theta} \text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa) \xrightarrow{(-)^{\text{op}}} \mathcal{L}(\kappa)_{\mathcal{C}^{\text{op}}} \) is the Yoneda-embedding, the composition

\[
(\text{Alg}([X,X])^{\text{rep}})^{\text{op}} \xrightarrow{\text{Alg}} (\mathcal{E}/_X)^{\text{rep}} \subset \mathcal{E}/_X \subset \mathcal{L}(\kappa)_{\mathcal{C}^{\text{op}}/(\mathcal{E}/_X)^{\text{op}}}
\]

is equivalent to the functor

\[
(\text{Alg}([X,X])^{\text{rep}})^{\text{op}} \rightarrow (\text{Alg}([X,X])^{\text{rep}})^{\text{op}} \xrightarrow{\text{Alg}} (\text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa)/_X)^R \xrightarrow{(-)^{\text{op}}} \mathcal{L}(\kappa)_{\mathcal{C}^{\text{op}}/(\mathcal{E}/_X)^{\text{op}}}.
\]

Thus the functor \( \text{Alg} : (\text{Alg}([X,X])^{\text{rep}})^{\text{op}} \rightarrow (\mathcal{E}/_X)^{\text{rep}} \) is induced by the functor \( \text{Alg} : \text{Alg}([X,X])^{\text{op}} \rightarrow (\text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa)/_X)^R \).

In the following we will give a more explicit description of the localization

\[
\text{End} : (\text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa)/_X)^R \cong \text{Alg}([X,X])^{\text{op}} : \text{Alg},
\]

i.e. the localization \( \text{End} : (\mathcal{D}/_X)^{\text{rep}} \cong \text{Alg}([X,X])^{\text{op}} : \text{Alg} \) of theorem 5.1 for \( \mathcal{D} = \text{Cat}^{\text{cocart}}_{\infty / \mathcal{C}^{\text{op}}}(\kappa) \) and \( X = X \in \mathcal{D} \).

More generally we will give a more explicit description of the localization

\[
\text{End} : (\mathcal{D}/_X)^{\text{rep}} \cong \text{Alg}([X,X])^{\text{op}} : \text{Alg}
\]

of theorem 5.1 for \( \mathcal{D} \) a subcategory of \( \text{Cat}_{\infty}(\kappa)/S \) for some \( \kappa \)-small category \( S \) and \( X \in \mathcal{D} \).
Construction 3.
Let $G : S^{\text{op}} \rightarrow \text{Cat}_\infty(\kappa)$ be a functor and $\mathcal{C} \subseteq S^{\text{op}} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))$ a subcategory.

Passing to cotensors over $S^{\text{op}}$ we obtain a subcategory inclusion $\mathcal{C}_s \subseteq G^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \Delta^1 \simeq G^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))^{\Delta^1}$ over $S^{\text{op}}$.

Denote $(\mathcal{C}_s)^{\text{mon}} \subseteq (\mathcal{C}_s)^{\text{rep}} \subseteq (\mathcal{C}_s)_{\text{finite}}^R \subseteq (\mathcal{C}_s)^{\text{finite}}$ the full subcategories spanned by the objects of $\text{Fun}(\Delta^1, \mathcal{C})_s$ for some $s \in S$ corresponding to morphisms in $\mathcal{C}_s$ that are monadic, whose associated monad admits an Eilenberg-Moore object that is preserved by the subcategory inclusion $\mathcal{C}_s \subseteq \text{Cat}_\infty(\kappa)$ respectively that admit a left adjoint.

Given a section $X$ of the functor $\mathcal{C} \subseteq G^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \simeq S^{\text{op}}$ corresponding to a natural transformation $H \rightarrow G$ of functors $S^{\text{op}} \rightarrow \text{Cat}_\infty(\kappa)$ we set

$$(\mathcal{C}^{\text{mon}}_S)^{\text{mon}} := S^{\text{op}} \times_{\text{Cat}_\infty(\kappa)} (\mathcal{C}_s)^{\text{mon}}, (\mathcal{C}^S)^{\text{rep}} := S^{\text{op}} \times_{\text{Cat}_\infty(\kappa)} (\mathcal{C}_s)^{\text{rep}} \text{ and } (\mathcal{C}^S)^{\text{finite}} := S^{\text{op}} \times_{\text{Cat}_\infty(\kappa)} (\mathcal{C}_s)^{\text{finite}}$.

Let $D \rightarrow T$ be a map of $\kappa$-small cartesian fibrations over $S$ classifying the natural transformation $H \rightarrow G$ of functors $S^{\text{op}} \rightarrow \text{Cat}_\infty(\kappa)$.

Denote $[X, X]^S \subseteq \text{Fun}^S(T, D)$ the full subcategory spanned by the objects that belong to $[X(s), X(s)]_c \subseteq \text{Fun}_T(D_c, D_c)$ for some $s \in S$.

As for every $s \in S$ the monoidal structure on $\text{Fun}^S(T, D_c, D_c)$ restricts to a monoidal structure on $[X(s), X(s)]_c$, the monoidal structure on $\text{Fun}^S(T, D, D)$ over $S$ restricts to a monoidal structure on $[X, X]^S$ over $S$.

Denote $\text{Alg}^S([X, X]^S) \subseteq \text{Alg}^S([X, X]^S)$ the full subcategory spanned by the monads on $X(s)$ for some $s \in S$ that admit an Eilenberg-Moore object that is preserved by the subcategory inclusion $\mathcal{C}_s \subseteq \text{Cat}_\infty(\kappa)$.

1. The endomorphism $\text{Fun}^S(T, D, D)$-left module structure on $D \rightarrow T$ corresponds to a left module structure on the functor $D \rightarrow T$ over the pullback $T \times_S \text{Fun}^S(T, D, D)$ of the monoidal category $\text{Fun}^S(T, D, D)$ over $S$ along a functor $T \rightarrow S$.

By lemma 2.12 the forgetful functor

$$\zeta : \text{LMod}^T(D) \rightarrow \text{Alg}^S(\text{Fun}^S(T, D, D)) \times_S D$$

is a map of cartesian fibrations over $\text{Alg}^S(\text{Fun}^S(T, D, D))$, where a morphism of $\text{LMod}^T(D)$ is cartesian with respect to the cartesian fibration $\text{LMod}^T(D) \rightarrow \text{Alg}^S(\text{Fun}^S(T, D, D))$ if and only if its image in $D$ is cartesian with respect to the cartesian fibration $D \rightarrow S$.

So $\zeta$ classifies a functor

$$\xi : \text{Alg}^S(\text{Fun}^S(T, D, D))^{\text{op}} \rightarrow H^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))$$

$S^{\text{op}} \times_{G^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))))^{\Delta^1} \simeq G^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))^{\Delta^1}$

$= G^*(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))^{\Delta^1}$
over $S^{op}$ that induces on the fiber over $s \in S$ the functor
\[
\text{Alg}(\text{Fun}_{T_s}(\mathcal{D}_s, \mathcal{D}_s))^{op} \to (\text{Cat}_\infty(\kappa)/_{H(s)})^{\text{mon}}
\]
classified by the map $\text{LMod}^{T_s}(\mathcal{D}_s) \to \text{Alg}(\text{Fun}_{T_s}(\mathcal{D}_s, \mathcal{D}_s))$ of cartesian fibrations over $\text{Alg}(\text{Fun}_{T_s}(\mathcal{D}_s, \mathcal{D}_s))$.

By ... for every monad $T \in \text{Alg}(\text{Fun}_{T_s}(\mathcal{D}_s, \mathcal{D}_s))$ the functor $\text{LMod}^{T_s}(\mathcal{D}_s) \to \mathcal{D}_s$ is the Eilenberg-Moore object in $(\text{Cat}_\infty(\kappa)/_{G(s)})^{op}$.

As every monad $T \in \text{Alg}([X(s), X(s)])$ admits an Eilenberg-Moore object in $\mathcal{C}_s$ that is preserved by the subcategory inclusion $\mathcal{C}_s \subset (\text{Cat}_\infty(\kappa)/_{G(s)})^{op}$, the function $\text{LMod}^{T_s}(\mathcal{D}_s) \to \mathcal{D}_s$ over $T_s = G(s)$ belongs to $((\mathcal{C}_s)/_{H(s)})^{\text{mon}} \subset (\text{Cat}_\infty(\kappa)/_{G(s)})^{op}$.

Thus $\xi$ induces a functor
\[
\text{Alg} : (\text{Alg}^{JS}([X, X]^{JS})^{op})^{op} \to (\mathcal{C}/_{J_X})^{\text{mon}}^{op}
\]
over $S^{op}$ that induces on the fiber over $s \in S$ the functor
\[
\text{Alg} : (\text{Alg}([X(s), X(s)])^{op})^{op} \to (\mathcal{C}_{s/X(s)})^{\text{mon}}^{op}
\]
that is equivalent to the functor of theorem 3.4 with the same name.

More generally by remark 2.14, for every functor $S' \to S$ the pullback $S^{op} \times_{S^{op}} \text{Alg} : S^{op} \times_{S^{op}} (\text{Alg}^{JS}((\text{Fun}_{T_s}(\mathcal{D}, \mathcal{D}))^{op})^{op})^{op} \to$
\[
S^{op} \times_{S^{op}} (\mathcal{C}/_{J_X})^{\text{mon}}^{op} \subset S^{op} \times_{S^{op}} (\mathcal{C}/_{J_X})^{R}
\]
is equivalent over $S^{op}$ to the functor $S^{op} \times_{S^{op}} (\text{Alg}^{JS}((\text{Fun}_{T'_s}(\mathcal{D}, \mathcal{D}))^{op})^{op} = (\text{Alg}^{JS}((\text{Fun}_{T'_s \times_{T_s} T}((S'_s \times_{S} D, S'_s \times_{S} D))^{op})^{op}$
\[
\text{Alg} : ((S^{op} \times_{S^{op}} \mathcal{C}/_{J_X})^{op})^{op} \to S^{op} \times_{S^{op}} (\mathcal{C}/_{J_X})^{R}
\]
over $S^{op}$.

We have a commutative square
\[
\begin{array}{ccc}
\text{Alg}^{JS}([X, X]^{JS})^{op} & \xrightarrow{\text{Alg}} & (\mathcal{C}/_{J_X})^{op} \\
\text{Alg}^{JS}((\text{Fun}_{T_s}(\mathcal{D}, \mathcal{D}))^{op} & \xrightarrow{\text{Alg}} & ((S^{op} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))^{op})^{op}
\end{array}
\]
of categories over $S^{op}$.

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2. Let

\[ \alpha : R^{op} \to (c_{/X}^{S^{op}})^{R} \subset (S^{op} \times \text{Cat}_{\infty}(\kappa))_{/X}^{S^{op}} = S^{op} \times \text{Cat}_{\infty}(\kappa) \]

be a functor over \( S^{op} \) corresponding to a map of cartesian fibrations \( \varphi : E \to R \times_{S} D \) over \( R \).

For every \( r \in R \) lying over some \( s \in S \) the induced functor \( \varphi_{r} : \mathcal{E}_{r} \to \mathcal{D}_{r} \) over \( T_{r} \) admits a left adjoint relative to \( T_{r} \) that is a morphism of \( \mathcal{C}_{s} \). So \( \varphi \) admits a left adjoint \( F \) relative to \( R \times S T \) being a map of cartesian fibrations over \( R \).

So by proposition \( \text{prop} \text{.} \text{3.6} \), \( \varphi \) admits an endomorphism object \( T \) with respect to the canonical left module structure on \( \text{Fun}_{R \times S T}(E, R \times_{S} D) \) over \( \text{Fun}_{R \times S T}(R \times_{S} D, R \times_{S} D) \), which is given by \( \varphi \circ F \).

We have a canonical equivalence

\[ \text{Fun}_{R \times S T}(R \times_{S} D, R \times_{S} D) = \text{Fun}_{R}(R, \text{Fun}_{R \times S T}^{R}(R \times_{S} D, R \times_{S} D)) \cong \]

\[ \text{Fun}_{R}(R, \text{Fun}_{R \times S T}^{S}(\mathcal{D}, \mathcal{D})) \]

of monoidal categories, under which \( T \) corresponds to an associative algebra of \( \text{Fun}_{S}(R, \text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{D})) \) corresponding to a functor \( \phi : R \to \text{Alg}^{S}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{D})) \) over \( S \) that sends \( r \in R \) lying over some \( s \in S \) to the morphism \( \varphi_{r} \circ F_{r} : \mathcal{D}_{r} \to \mathcal{D}_{r} \) of \( \mathcal{C}_{s} \) that is the endomorphism object of \( \varphi_{r} \).

So \( \phi \) induces a functor \( R \to \text{Alg}^{S}([X, X]^{S}) \subset \text{Alg}^{S}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{D})) \) over \( S \) corresponding to a functor \( R^{op} \to \text{Alg}^{S}([X, X]^{S})^{op} \subset \text{Alg}^{S}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{D}))^{op} \) over \( S^{op} \).

For \( R = (c_{/X}^{S^{op}})^{R} \) and \( \alpha : R^{op} \to (c_{/X}^{S^{op}})^{R} \) the canonical equivalence we obtain a functor \( \text{End} : (c_{/X}^{S^{op}})^{R} \to \text{Alg}^{S}([X, X]^{S})^{op} \) over \( S^{op} \). We have a commutative square

\[
\begin{array}{ccc}
(c_{/X}^{S^{op}})^{R} & \xrightarrow{\text{End}} & \text{Alg}^{S}([X, X]^{S})^{op} \\
\downarrow & & \downarrow \\
((S^{op} \times \text{Cat}_{\infty}(\kappa))_{/X}^{S^{op}})^{R} & \xrightarrow{\text{End}} & \text{Alg}^{S}(\text{Fun}_{T}^{S}(\mathcal{D}, \mathcal{D}))^{op},
\end{array}
\]

where the bottom functor over \( S^{op} \) is the functor \( \text{End} \) for \( C = S^{op} \times \text{Cat}_{\infty}(\kappa) \)

\[ \text{Fun}(\Delta^{1}, \text{Cat}_{\infty}(\kappa)) \].

End restricts to a functor \( (c_{/X}^{S^{op}})^{R} \to (\text{Alg}^{S}([X, X]^{S})^{op})^{op} \) with the same name.

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3. By Theorem 3. we have a canonical equivalence

\[
\text{LMod}_\mathcal{T}(\text{Fun}_{R \times_S T}(\mathcal{E}, R \times_S \mathcal{D})) \simeq \text{Fun}_{R \times S T}(\mathcal{E}, \text{LMod}_{R \times S T}(R \times_S \mathcal{D}))
\]

under which the endomorphism \( \mathcal{T} \)-left module on \( \mathcal{E} \) corresponds to a functor \( \beta : \mathcal{E} \to \text{LMod}_{R \times S T}(R \times_S \mathcal{D}) \) over \( R \times S \mathcal{T} \).

As \( R \times_S \mathcal{D} \to R \times_S \mathcal{T} \) is a map of cartesian fibrations over \( R \), by remark 2.18 the functor \( \text{LMod}_{R \times S T}(R \times_S \mathcal{D}) \to R \) is a cartesian fibration, whose cartesian morphisms are those that get cartesian morphisms of \( R \times_S \mathcal{D} \to R \).

So with \( \varphi \) also \( \beta \) is a map of cartesian fibrations over \( R \).

We have a canonical equivalence

\[
\text{LMod}_{R \times \mathcal{D} \leftarrow \mathcal{T}}(\text{Fun}_{R \times_S T}(\mathcal{E}, \mathcal{D})) \simeq \\
\text{LMod}_{R \times S T}(\mathcal{D} \leftarrow \mathcal{E})
\]

of cartesian fibrations over \( R \) over the cartesian fibration \( \mathcal{R} \times_S \mathcal{D} \to \mathcal{R} \) that sends \( r \in \mathcal{R} \) lying over some \( s \in \mathcal{S} \) to the functor \( \gamma(r) : \mathcal{E}_r \to \text{LMod}_{\mathcal{T}_{r}}(\mathcal{D}_s) \) over \( \mathcal{T}_{r} \) that corresponds to the endomorphism \( \mathcal{T}_r \)-left module structure on the functor \( \varphi_r : \mathcal{E}_r \to \mathcal{D}_s \) over \( \mathcal{T}_r \).

So with \( \varphi_s \) also \( \gamma(r) \) belongs to \( \mathcal{C}_s \).

Thus \( \gamma \) induces a natural transformation \( \alpha : \text{Alg} \circ \phi^{\mathcal{D}} \) of functors \( R^{\mathcal{D}} \to (\mathcal{E}_{/\mathcal{S}}^{\mathcal{D}})^R \) over \( \mathcal{S}^{\mathcal{D}} \) that sends \( r \in \mathcal{R} \) lying over some \( s \in \mathcal{S} \) to the functor \( \gamma(r) : \mathcal{E}_r \to \text{LMod}_{\mathcal{T}_r}(\mathcal{D}_s) \) over \( \mathcal{T}_r \) that corresponds to the endomorphism \( \mathcal{T}_r \)-left module structure on the functor \( \varphi_r : \mathcal{E}_r \to \mathcal{D}_s \) over \( \mathcal{T}_r \).

For \( R = ((\mathcal{E}_{/\mathcal{S}}^{\mathcal{D}})^R)^{\mathcal{R}} \) and \( \alpha : \text{id} \to \text{Alg} \circ \text{End} \) of functors \( (\mathcal{E}_{/\mathcal{S}}^{\mathcal{D}})^R \to (\mathcal{E}_{/\mathcal{S}})^R \) over \( \mathcal{S}^{\mathcal{D}} \) that sends an object \( Y \) \( (\mathcal{E}_s)_{/\mathcal{S}(s)}^R \subset ((\text{Cat}_{\infty}(\mathcal{S}))_{/\mathcal{T}_s})_R \) for some \( s \in \mathcal{S} \) to the the functor \( Y \to \text{LMod}_{\mathcal{T}_s}(\mathcal{D}_s) \) over \( \mathcal{T}_s \) that corresponds to the endomorphism \( \mathcal{T}_s \)-left module structure on the right adjoint functor \( Y \to \mathcal{D}_s \) over \( \mathcal{T}_s \).

**Theorem 5.4.** Let \( \mathcal{C} \) be a \( \kappa \)-small category, \( \mathcal{C} \subset \text{Cat}_{\infty}(\mathcal{C}) \) a subcategory and \( X \) an object of \( \mathcal{C} \).

The functor \( \text{Alg} : (\text{Alg}([X,X])^{\mathcal{C}})^{\text{rep}} \to (\mathcal{C}_{/\mathcal{X}}^{\mathcal{D}})^{\mathcal{D}} \subset (\mathcal{C}_{/\mathcal{X}})^{\mathcal{D}} \) of construction 2.1 is right adjoint to the functor \( \text{End} : (\mathcal{C}_{/\mathcal{X}})^{\mathcal{D}} \to (\text{Alg}([X,X])^{\mathcal{C}})^{\text{rep}} \) of theorem 5.4.
Especially the functor $\text{Alg}$ is fully faithful.

So if every monad on $X$ admits an Eilenberg-Moore object that is preserved by the subcategory inclusion $C \subset \text{Cat}_\kappa(S)$, the functor $\text{Alg}([X, X])^{op} \to (E_{/X})^{mon} \subset (E_{/X})^{R}$ of construction [2.1], is a fully faithful right adjoint of the functor $\text{End} : (E_{/X})^{R} \to \text{Alg}([X, X])^{op}$ of theorem [5.7].

**Proof.** We first observe that we can reduce to the case $C = \text{Cat}_\kappa(S)$:

We have commutative squares

\[
\begin{array}{ccc}
(\text{Alg}([X, X])^{rep})^{op} & \xrightarrow{\text{Alg}} & (E_{/X})^{rep} \\
\downarrow & & \downarrow \\
\text{Alg}(\text{Fun}_S(X, X))^{op} & \xrightarrow{\text{Alg}} & ((\text{Cat}_\kappa(S))_{/X})^{R}.
\end{array}
\]

and

\[
\begin{array}{ccc}
(E_{/X})^{rep} & \xrightarrow{\text{End}} & (\text{Alg}([X, X])^{rep})^{op} \\
\downarrow & & \downarrow \\
((\text{Cat}_\kappa(S))_{/X})^{R} & \xrightarrow{\text{End}} & \text{Alg}(\text{Fun}_S(X, X))^{op},
\end{array}
\]

where the vertical functors are subcategory inclusions.

For every representable monad $T \in \text{Alg}([X, X]) \subset \text{Alg}(\text{Fun}_S(X, X))$ and morphism $\psi : Y \to X$ of $C \subset \text{Cat}_\kappa(S)$ that admits a left adjoint in $C$ the canonical map

\[
\text{Alg}(\text{Fun}_S(X, X))^{op}(\text{End}(\psi), T) \cong (\text{Cat}_\kappa(S))_{/X}(\psi, \text{Alg}(T))
\]

is canonically equivalent to the map

\[
\text{Alg}([X, X])(\text{End}(\psi), T)^{op} \cong E_{/X}(\psi, \text{Alg}(T)),
\]

where by remark [2.1] the full subcategory inclusion $E_{/X}(\psi, \text{Alg}(T)) \subset (\text{Cat}_\kappa(S))_{/X}(\psi, \text{Alg}(T))$ is an equivalence as $\text{Alg}(T)$ is an Eilenberg-Moore object for $T$ that is preserved by the subcategory inclusion $C \subset \text{Cat}_\kappa(S)$.

As $\text{Cat}_\kappa(S)$ admits Eilenberg-Moore objects, for $C = \text{Cat}_\kappa(S)$ we have to show that the functor $\text{Alg} : \text{Alg}(\text{Fun}_S(X, X))^{op} \to ((\text{Cat}_\kappa(S))_{/X})^{mon} \subset ((\text{Cat}_\kappa(S))_{/X})^{R}$ is right adjoint to the functor $((\text{Cat}_\kappa(S))_{/X})^{R} \to \text{Alg}(\text{Fun}_S(X, X))^{op}$.

To show this, we will construct an equivalence

\[
\text{Alg}(\text{Fun}_S(X, X))^{op}(\text{End}(\psi), T) \cong (\text{Cat}_\kappa(S))_{/X}(\psi, \text{Alg}(T))
\]

natural in every monad $T \in \text{Alg}(\text{Fun}_S(X, X))$ and functor $\psi : Y \to X$ over $S$ that admits a left adjoint relative to $S$.

Let $X \to S$ be endowed with the canonical endomorphism left module structure over $\text{Fun}_S(X, X)$.

Denote $U'(\kappa)_S \to (\text{Cat}_\kappa(S))^{op} \times S$ the map of cartesian fibrations over $(\text{Cat}_\kappa(S))^{op}$ classifying the identity of $\text{Cat}_\kappa(S)$. 83
As the functor \( U'_{\mathbf{k}}(\mathbf{k})_X \to (\mathbf{Cat}_\infty(\mathbf{k})/X)^{op} \times X \) is a map of cartesian fibrations over \((\mathbf{Cat}_\infty(\mathbf{k})/X)^{op}\), the functor

\[
\Psi : \text{Fun}_{\text{Alg}(\mathbf{Func}(X,X))}((\mathcal{E}_{/X})^{op}) \to \text{Alg}(\mathbf{Func}(X,X)) \times U'_{\mathbf{k}}(\mathbf{k})_X \times ((\mathcal{E}_{/X})^{op} \times \text{LMod}^S(X)) \to \text{Alg}(\mathbf{Func}(X,X)) \times ((\mathcal{E}_{/X})^{op} \times \text{Alg}(\mathbf{Func}(X,X)))^{op}
\]

is a map of cocartesian fibrations over \(((\mathcal{E}_{/X})^{op})^{op}\).

\(\Psi\) induces on the fiber over a functor \(\psi : Y \to X\) over \(S\) that admits a left adjoint \((\Psi')_f\) relative to \(S\) the functor

\[
\text{Fun}_{\text{Alg}(\mathbf{Func}(X,X))}((\mathcal{E}_{/X})^{op}) \times \text{Alg}(\mathbf{Func}(X,X)) \times \text{LMod}^S(X) \to \text{Alg}(\mathbf{Func}(X,X)) \times \text{Alg}(\mathbf{Func}(X,X))^\mathbf{op}
\]

that is a cartesian fibration by remark 2.3, and the fact that the functor \(\text{LMod}^S(X) \to \text{Alg}(\mathbf{Func}(X,X)) \times X\) is a map of cartesian fibrations over \(\text{Alg}(\mathbf{Func}(X,X))\) due to remark 2.11.

By proposition 7.7 \(\Psi\) classifies the functor \(((\mathcal{E}_{/X})^{op})^{op} \to \text{cat}_{\infty/\text{Alg}(\mathbf{Func}(X,X))}(\mathbf{k}) = \text{Fun}(\text{Alg}(\mathbf{Func}(X,X))^\mathbf{op}, \text{Cat}_{\infty}(\mathbf{k}))\) adjoint to the functor

\[
((\mathcal{E}_{/X})^{op})^{op} \times \text{Alg}(\mathbf{Func}(X,X))^\mathbf{op} \subset (\text{Cat}_{\infty}(\mathbf{k})/X)^{op} \times \text{Alg}(\mathbf{Func}(X,X))^\mathbf{op}
\]

\[
\xrightarrow{\text{id} \times \text{Alg}} (\mathbf{Cat}_\infty(\mathbf{k})/X)^{op} \times \mathbf{Cat}_\infty(\mathbf{k}) \xrightarrow{\text{Fun}_X((-,-))} \mathbf{Cat}_\infty(\mathbf{k})
\]

The functor

\[
\Phi : ((\mathcal{E}_{/X})^{op} \times \text{Alg}(\mathbf{Func}(X,X)))^{(1)} \to \text{Alg}(\mathbf{Func}(X,X)) \times ((\mathcal{E}_{/X})^{op})^{op}
\]

is a map of cocartesian fibrations over \(((\mathcal{E}_{/X})^{op})^{op}\) that induces on the fiber over a functor \(\psi : Y \to X\) over \(S\) that admits a left adjoint \((\Phi')_f\) relative to \(S\) the right fibration \(\text{Alg}(\mathbf{Func}(X,X))/\text{End}(\psi) \to \text{Alg}(\mathbf{Func}(X,X))\).

By ... \(\Phi\) classifies the functor \(((\mathcal{E}_{/X})^{op})^{op} \to \text{R}(\mathbf{k} \text{Alg}(\mathbf{Func}(X,X))) \subset \text{cat}_{\infty/\text{Alg}(\mathbf{Func}(X,X))}(\mathbf{k})\) adjoint to the functor

\[
((\mathcal{E}_{/X})^{op})^{op} \times \text{Alg}(\mathbf{Func}(X,X))^\mathbf{op} \xrightarrow{\text{End} \otimes \text{id}} \text{Alg}(\mathbf{Func}(X,X)) \times \text{Alg}(\mathbf{Func}(X,X))^\mathbf{op}
\]

\[
\xrightarrow{\text{Alg}(\mathbf{Func}(X,X))^\mathbf{op}((-,-))} \text{S}(\mathbf{k}) \subset \text{Cat}_{\infty}(\mathbf{k})
\]

So we have to construct an equivalence

\[
\text{Fun}_{\text{Alg}(\mathbf{Func}(X,X))}((\mathcal{E}_{/X})^{op} \times \text{Alg}(\mathbf{Func}(X,X)))^{(1)} \to \text{Alg}(\mathbf{Func}(X,X)) \times ((\mathcal{E}_{/X})^{op})^{op}
\]

\[
\xrightarrow{\text{Alg}(\mathbf{Func}(X,X))^\mathbf{op}((-,-))} \text{S}(\mathbf{k}) \subset \text{Cat}_{\infty}(\mathbf{k})
\]

Denote \(\mathcal{M}^\otimes \to \text{LMod}^\otimes \times S\) the \(\text{LMod}^\otimes\)-monoidal category over \(S\) that encodes the endomorphism left module structure on \(X \to S\) over \(\text{Func}(X,X)\).

By remark 4.17 \(\mathcal{M}^\otimes\) gives rise to a \(\text{LMod}^\otimes\)-monoidal category

\[
\mathcal{M}^\otimes := \text{Fun}_{(\mathbf{Cat}_\infty(\mathbf{k})/S)^{op} \times \text{S}}(\mathcal{U}'(\mathbf{k})_S, (\mathbf{Cat}_\infty(\mathbf{k})/S)^{op} \times \mathcal{M})^{op} \text{ over } (\mathbf{Cat}_\infty(\mathbf{k})/S)^{op},
\]

whose pullback along the monoidal diagonal functor

\[
\delta : (\mathbf{Cat}_\infty(\mathbf{k})/S)^{op} \times \text{S} \times \text{Func}(X,X)^\otimes \to \text{Map}(\mathbf{Cat}_\infty(\mathbf{k})/S)^{op} \times \text{S} \times \text{Func}(X,X)^\otimes
\]

\[
\xrightarrow{\text{Fun}(-,-)} \text{Func}(\mathbf{Cat}_\infty(\mathbf{k})/S)^{op} \times \text{S} \times \text{Func}(X,X)^\otimes
\]

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over \((\mathbf{Cat}_\infty(\kappa)/S)^{\op}\) exhibits \(\mathcal{X}' := \text{Fun}^{(\mathbf{Cat}_\infty(\kappa)/S)^{\op}}(\mathbf{U}'(\kappa)/S, (\mathbf{Cat}_\infty(\kappa)/S)^{\op} \times X)\) as a left module over \(\text{Fun}_S(X, X)\).

The endomorphism left module structure on \(X \to S\) over \(\text{Fun}_S(X, X)\) gives rise to a canonical left module structure over \(\text{Fun}_S(X, X)\) on the cocartesian fibration \(\mathcal{X} \to (\mathbf{Cat}_\infty(\kappa)/S)^{\op}\) classifying the functor \([- , X] : (\mathbf{Cat}_\infty(\kappa)/S)^{\op} \to \mathbf{Cat}_\infty(\kappa)\).

By remark 2.12 we have a canonical \(\text{Fun}_S(X, X)\)-linear equivalence \(\mathcal{X} \simeq \delta^*(\mathcal{X}')\) of cocartesian fibrations over \((\mathbf{Cat}_\infty(\kappa)/S)^{\op}\).

By remark 2.13, we have a canonical equivalence

\[
\text{LMod}^{(\mathcal{C}/\mathcal{X})^{\op}}(X) \simeq \text{LMod}^{(\mathcal{C}/\mathcal{X})^{\op}}(\delta^*(\mathcal{X}')) \simeq \\
N := \text{Fun}_{\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathcal{C}}(\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathbf{U}'(\kappa)_S, \mathcal{C}^{\op} \times \text{LMod}^S(X))
\]

over \(\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathcal{X} \simeq \text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathcal{X}' \simeq \)

\[\text{Fun}_{\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathcal{C}}(\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathbf{U}'(\kappa)_S, \mathcal{C}^{\op} \times \text{Alg}(\mathbf{Fun}_S(X, X)) \times X).
\]

By remark 2.27, we have a canonical equivalence

\[
((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op} \times \mathcal{X} \simeq (\text{Alg}(\mathbf{Fun}_S(X, X)) \times ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op}) \times \text{Alg}(\mathbf{Fun}_S(X, X), \mathcal{X}') \simeq \\
N \simeq \\
\times \text{Fun}_{\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathcal{C}}(\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathbf{U}'(\kappa)_X, ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op} \times \text{LMod}^S(X))
\]

over \(\text{Alg}(\mathbf{Fun}_S(X, X)) \times ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op}.
\]

By 2.24 we have a canonical equivalence

\[
((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op} \times \mathcal{X} \simeq ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op} \times \text{Alg}(\mathbf{Fun}_S(X, X))^{(\mathcal{X})} \text{Alg}(\mathbf{Fun}_S(X, X))^{\Delta^1}
\]

over \(\text{Alg}(\mathbf{Fun}_S(X, X)) \times ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op}.
\]

So we obtain the desired equivalence

\[
\text{Fun}_{\text{Alg}(\mathbf{Fun}_S(X, X)) \times ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op}}(\text{Alg}(\mathbf{Fun}_S(X, X)) \times \mathbf{U}'(\kappa)_X, ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op} \times \text{LMod}^S(X))
\]

\[
(\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op} \times \text{Alg}(\mathbf{Fun}_S(X, X))^{(\mathcal{X})} \text{Alg}(\mathbf{Fun}_S(X, X))^{\Delta^1}
\]

over \(\text{Alg}(\mathbf{Fun}_S(X, X)) \times ((\mathcal{C}/\mathcal{X})^{\mathcal{R}})^{\op}.
\]

\[
\square
\]

Let \(S\) be a \(\kappa\)-small category, \(G : S^{\op} \to \mathbf{Cat}_\infty(\kappa)\) a functor,
\(\mathcal{C} \subset G^*(\mathbf{Fun}(\Delta^1, \mathbf{Cat}_\infty(\kappa)))\) a subcategory and \(X\) a section of the functor \(\mathcal{C} \subset G^*(\mathbf{Fun}(\Delta^1, \mathbf{Cat}_\infty(\kappa))) \to S^{\op}.
\)
In the following we will see that the functor \( \text{Alg} : (\text{Alg}^{\mathcal{S}}([X, X]^{\mathcal{S}})^{\op})^{\op} \to (\mathcal{E}^{\mathcal{S}}_{/X})^{\op} \) of construction \( \mathbb{3} \) is a fully faithful right adjoint relative to \( \mathcal{S}^{\op} \) of the functor \( \text{End} : (\mathcal{E}^{\mathcal{S}}_{/X})^{\op} \to (\text{Alg}^{\mathcal{S}}([X, X]^{\mathcal{S}})^{\op})^{\op} \) constructed in \( \mathbb{3} \).

By theorem \( \mathbb{5.4} \) this localization

\[
\text{End} : (\mathcal{E}^{\mathcal{S}}_{/X})^{\op} \simeq (\text{Alg}^{\mathcal{S}}([X, X]^{\mathcal{S}})^{\op})^{\op} : \text{Alg}
\]

relative to \( \mathcal{S}^{\op} \) induces on the fiber over every object \( s \in \mathcal{S} \) the localization of theorem \( \mathbb{5.1} \) applied to \( \mathcal{C}_s \) and \( X(s) \).

But different to the situation of theorem \( \mathbb{5.1} \) we don’t need to assume \( X \) to be a cocartesian section.

This flexibility is essential to prove corollary \( \mathbb{5.6} \).

**Theorem 5.5.** Let \( \mathcal{S} \) be a \( \kappa \)-small category, \( G : \mathcal{S}^{\op} \to \text{Cat}_{\infty}(\kappa) \) a functor, \( \mathcal{C} \in G^*(\text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa))) \) a subcategory and \( X \) a section of the functor \( \mathcal{C} \in G^*(\text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa))) \to \mathcal{S}^{\op} \).

We have a localization

\[
\text{End} : (G^{\mathcal{S}}_{/X})^{\op} \simeq (\text{Alg}^{\mathcal{S}}([X, X]^{\mathcal{S}})^{\op})^{\op} : \text{Alg}
\]

relative to \( \mathcal{S}^{\op} \) constructed in \( \mathbb{6} \).

**Proof.** Let \( D \to T \) be the map of cartesian fibrations over \( S \) classifying the natural transformation \( H \to G \) of functors \( S^{\op} \to \text{Cat}_{\infty}(\kappa) \) corresponding to the functor \( X : \mathcal{S}^{\op} \to \mathcal{C} \in G^*(\text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa))) \) over \( S^{\op} \).

In view of the commutative squares \( \mathbb{6} \) and \( \mathbb{7} \) we can reduce to the case that \( \mathcal{C} = \mathcal{S}^{\op} \times_{\text{Cat}_{\infty}(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa)) \).

We first show that the functor

\[
\text{Alg} : \text{Alg}^{\mathcal{S}}(\text{Fun}^{\mathcal{S}}_{T}(\mathcal{D}, \mathcal{D}))^{\op} \to ((\mathcal{S}^{\op} \times_{\text{Cat}_{\infty}(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa)))^{\mathcal{S}^{\op}}_{/X})^{\mon}
\]

is an equivalence.

This is equivalent to the condition that for every functor \( \alpha : S' \to S \) the induced functor

\[
\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, \text{Alg}) : \text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, \text{Alg}^{\mathcal{S}}(\text{Fun}^{\mathcal{S}}_{T}(\mathcal{D}, \mathcal{D}))^{\op}) \to
\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, ((\mathcal{S}^{\op} \times_{\text{Cat}_{\infty}(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa)))^{\mathcal{S}^{\op}}_{/X})^{\mon})
\]

is an equivalence.

By remark \( \mathbb{2.14} \), this functor \( \text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, \text{Alg}) \) is equivalent to the functor

\[
\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, \text{Alg}^{\mathcal{S}}(\text{Fun}^{\mathcal{S}}_{T}(\mathcal{D}, \mathcal{D}))^{\op}) \simeq
\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, \text{Alg}^{\mathcal{S}}(\text{Fun}^{\mathcal{S}^{\op}}_{S \times_{S} T}(S', S' \times_{S} D))^{\op}) \xrightarrow{\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, \text{Alg})}
\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, ((\mathcal{S}^{\op} \times_{\text{Cat}_{\infty}(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa)))^{\mathcal{S}^{\op}}_{/X})^{\mon}) \simeq
\text{Fun}_{S^{\op}}(\mathcal{S}^{\op}, ((\mathcal{S}^{\op} \times_{\text{Cat}_{\infty}(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_{\infty}(\kappa)))^{\mathcal{S}^{\op}}_{/X})^{\mon}).
\]

So we can reduce to the case that \( \alpha : S' \to S \) is the identity.

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By remark 2.13, the functor
\[ \text{Alg}(\text{Fun}_T(D, D))^{op} \cong \text{Fun}_{S^{op}}(S^{op}, \text{Alg}(\text{Fun}_T(D, D))^{op}) \]

is equivalent to the functor
\[ \text{Alg} : \text{Alg}(\text{Fun}_T(D, D))^{op} \to (\text{Cat}_\infty(\kappa)/T)_{/D} \]

By theorem 5.4 this functor induces an equivalence
\[ \text{Alg} : (\text{Alg}(\text{Fun}_T(D, D))^{op}) \to ((\text{Cat}_\infty(\kappa)/T)_{/D})^{\text{mon}}. \]

Consequently it is enough to see that the subcategory inclusion
\[ \text{Fun}_{S^{op}}(S^{op}, (S^{op} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))/_{S^{op}}) \]

restricts to a full subcategory inclusion
\[ \text{Fun}_{S^{op}}(S^{op}, ((S^{op} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))/_{S^{op}})^{\text{mon}}) \subset ((\text{Cat}_\infty(\kappa)/T)_{/D})^{\text{mon}}. \]

Let \( \varphi : E \to D \) be a map of cartesian fibrations over \( S \) over the cartesian fibration \( T \to S \) that induces on the fiber over every \( s \in S \) a functor \( \varphi_s : E_s \to D_s \) over \( T_s \) that admits a left adjoint relative to \( T_s \).

Being a map of cartesian fibrations over \( S \) the functor \( \varphi : E \to D \) admits a left adjoint relative to \( T \) and so admits an associated monad \( \mathcal{T} \) in \( \text{Cat}_\infty(\kappa)/T \).

The \( \mathcal{T} \)-left module structure on \( \varphi \) corresponds to a functor \( \beta : E \to \text{LMod}^{\mathcal{T}}_{/T}(D) \) over \( D \) that induces on the fiber over every \( s \in S \) the functor \( \beta_s : E_s \to \text{LMod}^{\mathcal{T}_s}_{/D_s}(D_s) \) over \( D_s \) for every \( s \in S \).

As \( D \to T \) is a map of cartesian fibrations over \( S \), by remark 2.15 the functor \( \text{LMod}^{\mathcal{T}}_{/T}(D) \to S \) is a cartesian fibration, whose cartesian morphisms are those that get cartesian morphisms of \( D \to S \).

So with \( \varphi \) also \( \beta \) is a map of cartesian fibrations over \( S \).

Hence \( \varphi \) is monadic in \( \text{Cat}_\infty(\kappa)/T \) if and only if for every \( s \in S \) the functor \( \varphi_s : E_s \to D_s \) is monadic in \( \text{Cat}_\infty(\kappa)/T_s \).

In this case a morphism of \( E \) is cartesian with respect to \( E \to S \) if and only if its image in \( D \) is cartesian with respect to \( D \to S \).

So we have seen that
\[ \text{Alg} : \text{Alg}(\text{Fun}_T(D, D))^{op} \to ((S^{op} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))/_{S^{op}})^{R} \]

is fully faithful.

By construction 3.3, we have a natural transformation \( \lambda : \text{id} \to \text{Alg} \circ \text{End} \) of endofunctors of \( ((S^{op} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)))/_{S^{op}})^{R} \) over \( S^{op} \) that sends
an object \( Y \in ((\text{Cat}_\infty(\kappa)/\mathcal{T}_{_m})/\mathcal{D}_m))^R \) for some \( s \in S \) to the the functor \( \lambda(Y) : Y \to \text{LMod}_{\mathcal{T}_m}^T(\mathcal{D}_m) \) over \( \mathcal{T}_m \) that corresponds to the endomorphism \( \mathcal{T}_m \)-left module structure on the right adjoint functor \( Y \to \mathcal{D}_m \) over \( \mathcal{T}_m \) with associated monad \( \mathcal{T}_m \).

We will show that \( \lambda : \text{id} \to \text{Alg} \circ \text{End} \) exhibits \( \text{End} \) as left adjoint to \( \text{Alg} \) relative to \( S^\text{op} \).

As \( \text{Alg} \) is fully faithful, it is enough to see that \( \text{End} \circ \lambda : \text{End} \to \text{End} \circ \text{Alg} \circ \text{End} \) and \( \lambda \circ \text{Alg} : \text{Alg} \to \text{Alg} \circ \text{End} \circ \text{Alg} \) are equivalences or equivalently that for every \( s \in S \) the induced natural transformations \( \text{End}_{s} \circ \lambda_s : \text{End}_{s} \to \text{End}_{s} \circ \text{Alg}_{s} \circ \text{End}_{s} \) and \( \lambda_s \circ \text{Alg}_{s} : \text{Alg}_{s} \to \text{Alg}_{s} \circ \text{End}_{s} \circ \text{Alg}_{s} \) on the fiber over \( s \) are equivalences.

So it is enough to see that for every \( s \in S \) the natural transformation \( \lambda_s : \text{id} \to \text{Alg}_{s} \circ \text{End}_{s} \) exhibits \( \text{End}_{s} \) as left adjoint to the fully faithful functor \( \text{Alg}_{s} \), in other words that for every \( Y \in ((\text{Cat}_\infty(\kappa)/\mathcal{T}_m)/\mathcal{D}_m))^R \) the functor \( \lambda(Y) : Y \to \text{LMod}_{\mathcal{T}_m}^T(\mathcal{D}_m) \) over \( \mathcal{D}_m \) induces for every \( \mathcal{E} \in ((\text{Cat}_\infty(\kappa)/\mathcal{T}_m)/\mathcal{D}_m)^{\text{mon}} \) an equivalence \( \text{Cat}_\infty(\kappa)/\mathcal{D}_m(\text{LMod}_{\mathcal{T}_m}^T(\mathcal{D}_m), \mathcal{E}) \to \text{Cat}_\infty(\kappa)/\mathcal{D}_m(\mathcal{Y}, \mathcal{E}) \).

By theorem 5.1 the full subcategory \( ((\text{Cat}_\infty(\kappa)/\mathcal{T}_m)/\mathcal{D}_m)^{\text{mon}} \subset ((\text{Cat}_\infty(\kappa)/\mathcal{T}_m)/\mathcal{D}_m)^R \) is a localization and \( \lambda(Y) : Y \to \text{LMod}_{\mathcal{T}_m}^T(\mathcal{D}_m) \) is the unit.

\[ \square \]

Let \( S \) be a \( \kappa \)-small category, \( G : S^\text{op} \to \text{Cat}_\infty(\kappa) \) a functor and \( \mathcal{C} \subset G^\star(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \) a subcategory.

Set \( \mathcal{C}' := \mathcal{C} \times_{S^\text{op}} \mathcal{C} \subset \mathcal{C} \times_{\text{Cat}_\infty(\kappa)} \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa)) \) and denote \( U \) the section of \( \mathcal{C}' \to \mathcal{C} \) adjoint to the identity of \( \mathcal{C} \).

Then we have a canonical equivalence \( \mathcal{C}'_{/\mathcal{C}} \simeq \mathcal{C} \Delta^1 \) over \( \mathcal{C}^{(1)} \).

So we obtain the following corollary:

**Corollary 5.6.** Let \( S \) be a \( \kappa \)-small category, \( G : S^\text{op} \to \text{Cat}_\infty(\kappa) \) a functor and \( \mathcal{C} \subset G^\star(\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))) \) a subcategory.

We have a localization

\[ \text{End} : (\mathcal{C} \Delta^1)_{\text{rep}} \Rightarrow (\text{Alg}^{\text{rep}}([U, U]^{\text{rep}})_{\text{rep}})^{\text{op}} : \text{Alg} \]

relative to \( \mathcal{C} \).

So the restriction

\[ (\mathcal{C} \Delta^1)_{\text{mon}} \subset (\mathcal{C} \Delta^1)_{\text{rep}} \xrightarrow{\text{End}} (\text{Alg}^{\text{rep}}([U, U]^{\text{rep}})_{\text{rep}})^{\text{op}} \]

is an equivalence and the full subcategory \( (\mathcal{C} \Delta^1)_{\text{mon}} \subset (\mathcal{C} \Delta^1)_{\text{rep}} \) is a localization.

**Lemma 5.7.** Suppose we have given a commutative square

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{C}' \\
\downarrow & & \downarrow \\
\mathcal{D} & \xrightarrow{\psi} & \mathcal{D}'
\end{array}
\]

(8)
of $\kappa$-small categories and let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}, \mathcal{A}' \subset \mathcal{B}' \subset \mathcal{C}'$ full subcategories with $\varphi(\mathcal{A}) \subset \mathcal{A}', \varphi(\mathcal{B}) \subset \mathcal{B}'$.

1. Assume that the functor $\mathcal{C} \to \varphi(\mathcal{C})$ induced by $\varphi$ admits a left inverse.
   Assume that the full subcategory inclusion $\mathcal{A}' \subset \mathcal{B}'$ admits a left adjoint and that for every object $X$ of $\mathcal{D}$ the full subcategory inclusion $\mathcal{A}_X \subset \mathcal{B}_X$ admits a left adjoint.
   Assume that the induced functor $\mathcal{C}_X \to \mathcal{C}'_{\varphi(X)}$ preserves local equivalences.
   Then the full subcategory inclusion $\mathcal{A} \subset \mathcal{B}$ admits a left adjoint relative to $\mathcal{D}$.

2. Let $\mathcal{V}$ be a $\kappa$-small monoidal category and assume that square $[\mathcal{X}]$ is a commutative square of $\mathcal{V}$-enriched categories.

Assume that one of the following conditions holds:

(a) The $\mathcal{V}$-enriched functor $\mathcal{C} \to \varphi(\mathcal{C})$ induced by $\varphi$ admits a left inverse.
(b) The functor $\mathcal{C} \to \varphi(\mathcal{C})$ induced by the underlying functor of $\varphi$ admits a left inverse and the $\mathcal{V}$-enriched categories $\mathcal{C}, \mathcal{C}'$ are cotensored over $\mathcal{V}$ and $\varphi : \mathcal{C} \to \mathcal{C}'$ commutes with cotensors.

Assume that the full subcategory inclusion $\mathcal{A}' \subset \mathcal{B}'$ admits a $\mathcal{V}$-enriched left adjoint, that for every object $X$ of $\mathcal{D}$ the full subcategory inclusion $\mathcal{A}_X \subset \mathcal{B}_X$ admits a left adjoint and that the induced functor $\mathcal{C}_X \to \mathcal{C}'_{\varphi(X)}$ preserves local equivalences.

Then the full subcategory inclusion $\mathcal{A} \subset \mathcal{B}$ admits a $\mathcal{V}$-enriched left adjoint relative to $\mathcal{D}$.

**Proof.** Let $Y$ be an object of $\mathcal{B}$ lying over some object $X$ of $\mathcal{D}$.

For 1. it is enough to find a morphism $Y \to Z$ of $\mathcal{B}_X$ with $Z \in \mathcal{A}$ such that for every object $A$ of $\mathcal{A}$ the induced map $\mathcal{C}(Z, A) \to \mathcal{C}(Y, A)$ is an equivalence, for 2. it is enough to find a morphism $Y \to Z$ of $\mathcal{B}_X$ with $Z \in \mathcal{A}$ such that for every object $A$ of $\mathcal{A}$ the induced map $[Z, A] \to [Y, A]$ is an equivalence.

As the full subcategory inclusion $\mathcal{A}_X \subset \mathcal{B}_X$ admits a left adjoint, we find a local equivalence $f : Y \to Z$ of $\mathcal{B}_X$ with $Z \in \mathcal{A}$.

Denote $X'$ the image of $X$ in $\mathcal{D}'$.

By assumption the image $\varphi(f) : \varphi(Y) \to \varphi(Z)$ is a local equivalence with respect to the localization $\mathcal{A}' \subset \mathcal{B}'$ for 1. respectively with respect to the $\mathcal{V}$-enriched localization $\mathcal{A}' \subset \mathcal{B}'$ for 2.

So for every object $A$ of $\mathcal{A}$ the induced map $\mathcal{C}'(\varphi(Z), \varphi(A)) \to \mathcal{C}'(\varphi(Y), \varphi(A))$ is an equivalence for 1. respectively the induced map $[\varphi(Z), \varphi(A)] \to [\varphi(Y), \varphi(A)]$ is an equivalence for 2.

As the functor $\mathcal{C} \to \varphi(\mathcal{C})$ induced by $\varphi$ admits a left inverse, we have a commutative square

\[
\begin{array}{ccc}
\mathcal{C}(Z, A) & \to & \mathcal{C}'(\varphi(Z), \varphi(A)) \\
\downarrow & & \downarrow \\
\mathcal{C}(Y, A) & \to & \mathcal{C}'(\varphi(Y), \varphi(A))
\end{array}
\]
of spaces, where the compositions \( \mathcal{C}(Z, A) \to \mathcal{C}'(\varphi(Z), \varphi(A)) \to \mathcal{C}(Z, A) \) and \( \mathcal{C}(Y, A) \to \mathcal{C}'(\varphi(Y), \varphi(A)) \to \mathcal{C}(Y, A) \) are the identity.

So with the map \( \mathcal{C}'(\varphi(Z), \varphi(A)) \to \mathcal{C}'(\varphi(Y), \varphi(A)) \) also the map \( \mathcal{C}(Z, A) \to \mathcal{C}(Y, A) \) is an equivalence. This shows 1.

Similarly for 2. a) we have a commutative square

\[
\begin{array}{ccc}
[Z, A] & \longrightarrow & [\varphi(Z), \varphi(A)] \\
\downarrow & & \downarrow \\
[Y, A] & \longrightarrow & [\varphi(Y), \varphi(A)]
\end{array}
\]

(9)

in \( \mathcal{V} \), where the compositions \( [Z, A] \to [\varphi(Z), \varphi(A)] \to [Z, A] \) and \( [Y, A] \to [\varphi(Y), \varphi(A)] \to [Y, A] \) are the identity.

So with the morphism \( [\varphi(Z), \varphi(A)] \to [\varphi(Y), \varphi(A)] \) also the morphism \( [Z, A] \to [Y, A] \) is an equivalence.

For 2. b) we use that \( \mathcal{C}, \mathcal{C}' \) are cotensored over \( \mathcal{V} \) and that \( \varphi : \mathcal{C} \to \mathcal{C}' \) commutes with cotensors to produce a commutative square like square (9).

We have a commutative square

\[
\begin{array}{ccc}
\mathcal{C}(Z, A^{(-)}) & \longrightarrow & \mathcal{C}'(\varphi(Z), \varphi(A^{(-)})) \\
\downarrow & & \downarrow \\
\mathcal{C}(Y, A^{(-)}) & \longrightarrow & \mathcal{C}'(\varphi(Y), \varphi(A^{(-)}))
\end{array}
\]

of functors \( \mathcal{V}^{op} \to \mathcal{S}(\kappa) \), where the compositions \( \mathcal{C}(Z, A^{(-)}) \to \mathcal{C}'(\varphi(Z), \varphi(A^{(-)})) \to \mathcal{C}(Z, A^{(-)}) \) and \( \mathcal{C}(Y, A^{(-)}) \to \mathcal{C}'(\varphi(Y), \varphi(A^{(-)})) \to \mathcal{C}(Y, A^{(-)}) \) are the identity.

This square is equivalent to a commutative square

\[
\begin{array}{ccc}
\mathcal{V}(-, [Z, A]) & \longrightarrow & \mathcal{V}(-, [\varphi(Z), \varphi(A)]) \\
\downarrow & & \downarrow \\
\mathcal{V}(-, [Y, A]) & \longrightarrow & \mathcal{V}(-, [\varphi(Y), \varphi(A)])
\end{array}
\]

of functors \( \mathcal{V}^{op} \to \mathcal{S}(\kappa) \) that represents a commutative square

\[
\begin{array}{ccc}
[Z, A] & \longrightarrow & [\varphi(Z), \varphi(A)] \\
\downarrow & & \downarrow \\
[Y, A] & \longrightarrow & [\varphi(Y), \varphi(A)]
\end{array}
\]

in \( \mathcal{V} \), where the compositions \( [Z, A] \to [\varphi(Z), \varphi(A)] \to [Z, A] \) and \( [Y, A] \to [\varphi(Y), \varphi(A)] \to [Y, A] \) are the identity.

\[\square\]

**Remark 5.8.** Let \( \mathcal{S} \) be a \( \kappa \)-small category and \( \mathcal{C} \to \mathcal{S} \) a cocartesian \( \mathcal{S} \)-family of 2-categories.
By proposition 5.7 we have a canonical functor \(\theta : \mathcal{C} \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)\)
over S that sends an object X of \(\mathcal{C}\) lying over some \(s \in S\) to the cocartesian
fibration over \(\mathcal{E}_s^{\text{op}}\) classifying the functor \([-,X] : \mathcal{E}_s^{\text{op}} \rightarrow \text{Cat}_{\infty}(\kappa)\).

By ... the composition \(\theta : \mathcal{C} \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)\)
\(\mathcal{C}\) over \(\mathcal{E}\) is the Yoneda-embedding relative to \(S\).

Thus the functor \((-)^{\ast} : S \times \text{Cat}_{\infty}(\kappa) \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)\)
\(\mathcal{C}\) over \(\mathcal{E}\) restricts to a functor \(\theta(\mathcal{C}) \rightarrow \mathcal{C} \in \text{Cocart}(\kappa)\) over \(S\) and the composition \(\mathcal{C} \rightarrow \theta(\mathcal{C}) \rightarrow \mathcal{C}\)
so the functor \(\theta' : \mathcal{C} \rightarrow \theta(\mathcal{C})\) over \(S\) induced by \(\theta\) admits a left inverse.

Especially \(\theta : \mathcal{C} \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)\) is conservative.

**Theorem 5.9.** Let \(S\) be a \(\kappa\)-small category and \(\mathcal{C} \rightarrow S\) a cocartesian \(S\)-family
of 2-categories.

The full inclusion \((\mathcal{E}^{\Delta^1})^{\text{mon}} \subset (\mathcal{E}^{\Delta^1})^{\text{rep}}\) of categories admits a left adjoint relative to \(\mathcal{E}\).

If \(\mathcal{C}\) is cotensored over \(\text{Cat}_{\infty}(\kappa)\), the full inclusion \((\mathcal{E}^{\Delta^1})^{\text{mon}} \subset (\mathcal{E}^{\Delta^1})^{\text{rep}}\) of 2-categories admits a left adjoint relative to \(\mathcal{C}\).

So if \(\mathcal{C} \rightarrow S\) admits Eilenberg-Moore objects, the full inclusion \((\mathcal{E}^{\Delta^1})^{\text{mon}} \subset (\mathcal{E}^{\Delta^1})^{\text{R}}\) of categories admits a left adjoint relative to \(\mathcal{C}\) and if \(\mathcal{C} \rightarrow S\) is
additionally cotensored over \(\text{Cat}_{\infty}(\kappa)\), the full inclusion \((\mathcal{E}^{\Delta^1})^{\text{mon}} \subset (\mathcal{E}^{\Delta^1})^{\text{R}}\) of 2-categories admits a left adjoint relative to \(\mathcal{C}\).

**Proof.** We apply lemma 5.7.

The canonical map of \(S\)-families of 2-categories \(\theta : \mathcal{C} \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)\)
induces a commutative square

\[
\begin{array}{ccc}
\mathcal{E}^{\Delta^1} & \xrightarrow{\theta^{\Delta^1}} & S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)^{\Delta^1} \\
\mathcal{E}^{(1)} & \xrightarrow{\theta} & S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)^{(1)}
\end{array}
\]

of \(\kappa\)-small categories over \(S\).

Being a map of \(S\)-families of 2-categories \(\theta\) induces functors \((\mathcal{E}^{\Delta^1})^{\text{R}} \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)^{\Delta^1}\)
and \((\mathcal{E}^{\Delta^1})^{\text{mon}} \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)^{\Delta^1}\) mon over \(S\).

By remark 5.7 the functor \(\theta' : \mathcal{C} \rightarrow \theta(\mathcal{C})\) over \(S\) induced by \(\theta\) admits a left inverse in \(\text{Cat}_{\infty}(\kappa')_{/S}\).

Thus the functor \(\theta^{\Delta^1} : \mathcal{E}^{\Delta^1} \rightarrow \theta(\mathcal{C})^{\Delta^1}\) over \(S\) also does and so, as we have a full subcategory inclusion \(\theta^{\Delta^1} : \mathcal{E}^{\Delta^1} \subset \theta(\mathcal{C})^{\Delta^1}\), the functor \(\theta^{\Delta^1} : \mathcal{E}^{\Delta^1} \rightarrow \text{Cocart}(\kappa)^{\Delta^1}\)
over \(S\) induced by \(\theta^{\Delta^1} : \mathcal{E}^{\Delta^1} \rightarrow S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)^{\Delta^1}\) admits a left inverse in \(\text{Cat}_{\infty}(\kappa')_{/S}\).

By theorem 5.1 for every object X of \(\mathcal{C}\) lying over some object \(s \in S\) the full subcategory inclusions \((\mathcal{E}_s)_{/X}^{\text{mon}} \subset (\mathcal{E}_s)_{/X}^{\text{R}}\) and \((\text{Cat}_{\infty}^{\text{cocart}}(\kappa)_{/\theta(s)}^{\text{R}})^{\text{mon}} \subset (\text{Cat}_{\infty}^{\text{cocart}}(\kappa)_{/\theta(s)}^{\text{R}})^{\text{R}}\) admit left adjoints and the canonical 2-functor
\(\mathcal{E}_s)_{/X} \rightarrow \text{Cat}_{\infty}^{\text{cocart}}(\kappa)_{/\theta(s)}^{\text{R}} \text{ preserves local equivalences being a 2-functor.}\)

By corollary 5.7 the full subcategory inclusion \(S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)^{\Delta^1}\)
mon admits a left adjoint relative to \(S \times \text{Cat}_{\infty}(\kappa) \rightarrow \text{Cocart}(\kappa)\).
So all requirements are satisfied to apply lemma 5.7.
6 Kan-extensions in Eilenberg-Moore objects

Let $T$ be a monad on some symmetric monoidal category $\mathcal{C}$ such that $T$ lifts to an oplax symmetric monoidal functor and the unit and multiplication of $T$ are oplax symmetric monoidal natural transformations.

Then by theorem 4.16 the forgetful functor $\text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C}$ lifts to a symmetric monoidal functor.

In this section we will construct another symmetric monoidal structure on $\text{LMod}_T(\mathcal{C})$ with the property that not the forgetful functor $\text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C}$ but the free functor $\mathcal{C} \rightarrow \text{LMod}_T(\mathcal{C})$ lifts to a symmetric monoidal functor:

Let $T$ be a monad on some symmetric monoidal category $\mathcal{C}$ such that $T$ lifts to a lax symmetric monoidal functor and the unit and multiplication of $T$ are lax symmetric monoidal natural transformations.

Assume that $\mathcal{C}$ admits geometric realizations that are preserved by $T$ and the tensor product of $\mathcal{C}$ in each component.

Then the free functor $\mathcal{C} \rightarrow \text{LMod}_T(\mathcal{C})$ lifts to a symmetric monoidal functor and the tensor product of $\text{LMod}_T(\mathcal{C})$ preserves geometric realizations in each component.

If the tensor product of $\mathcal{C}$ preserves $\kappa$-small colimits in each component, then the tensor product of $\text{LMod}_T(\mathcal{C})$ preserves $\kappa$-small colimits in each component.

Example 6.1.

1. Let $\mathcal{C}$ be a presentable symmetric monoidal category and $T$ a monad on $\mathcal{C}$ such that $T$ lifts to a lax symmetric monoidal functor and the unit and multiplication of $T$ are lax symmetric monoidal natural transformations.

Assume that $T$ is an accessible functor and preserves geometric realizations (e.g. preserves sifted colimits).

Then $\text{LMod}_T(\mathcal{C})$ is a presentable symmetric monoidal category and the free functor $\mathcal{C} \rightarrow \text{LMod}_T(\mathcal{C})$ lifts to a symmetric monoidal functor.

2. Let $\mathcal{C}$ be a presentable monoidal category and $\mathcal{M}$ a presentable left-modul over $\mathcal{C}$.

Let $T$ be a monad on $\mathcal{M}$ such that $T$ lifts to a lax $\mathcal{C}$-linear functor and the unit and multiplication of $T$ are lax $\mathcal{C}$-linear natural transformations.

Assume that $T$ is an accessible functor and preserves geometric realizations (e.g. preserves sifted colimits).

Then $\text{LMod}_T(\mathcal{M})$ is a presentable left modul over $\mathcal{C}$ and the free functor $\mathcal{M} \rightarrow \text{LMod}_T(\mathcal{M})$ is $\mathcal{C}$-linear.

This follows from theorem 4.16 and proposition 6.8.

Let $\mathcal{C} \rightarrow S$ be a cocartesian fibration between $\kappa$-small categories that is compatible with geometric realizations and let $T \in \text{Alg}(\text{Fun}_S(\mathcal{C}, \mathcal{C}))$ be a monad
such that for every object \( s \) of \( S \) the induced functor \( T_s : \mathcal{C}_s \to \mathcal{C}_s \) on the fiber over \( s \) preserves geometric realizations.

Then \( \text{LMod}^S_T(\mathcal{C}) \to S \) is a cocartesian fibration compatible with geometric realizations and the free functor \( \mathcal{C} \to \text{LMod}^S_T(\mathcal{C}) \) over \( S \) is a map of cocartesian fibrations over \( S \).

Moreover if \( \mathcal{C} \to S \) is compatible with \( \kappa \)-small colimits, then \( \text{LMod}^S_T(\mathcal{C}) \to S \) is compatible with \( \kappa \)-small colimits.

Moreover we can derive the following example:

**Example 6.2.**

Let \( X^\otimes \to E_k^\otimes \) be an associative monoid in the category of \( E_k \)-operads for some natural \( k \) and \( A \) an \( E_{k+1} \)-algebra of \( X \).

Then by theorem 6.8 the category \( \text{LMod}_A(X) \) carries the structure of an \( E_k \)-monoid \( \text{LMod}_A(X) \to X \) and its left adjoint are maps of \( E_k \)-operads.

If \( X^\otimes \to E_k^\otimes \) is additionally an \( E_k \)-monoidal category that admits geometric realizations that are preserved by the tensor product of \( X^\otimes \to E_k^\otimes \) and the functor \( A \otimes - : X \to X \) induced by the associative monoid structure on \( X^\otimes \to E_k^\otimes \), then by proposition 6.2. the category \( \text{LMod}_A(X) \) is an \( E_k \)-monoidal category and the free functor \( X \to \text{LMod}_A(X) \) is an \( E_k \)-monoidal functor.

Moreover if \( X \) admits \( \kappa \)-small colimits that are preserved by the tensor product of \( X^\otimes \to E_k^\otimes \), then the same holds for \( \text{LMod}_A(X) \).

We start with the following definitions:

Let \( \mathcal{C} \) be a 2-category, \( X \) an object of \( \mathcal{C} \) and \( \varphi : A \to B \) a morphism of \( \mathcal{C} \).

Let \( H : A \to X \) and \( H' : B \to X \) be morphisms of \( \mathcal{C} \) and \( \alpha : H \to H' \circ \varphi \) a 2-morphism of \( \mathcal{C} \).

We say that \( \alpha \) exhibits \( H' \) as the left kan-extension of \( H \) along \( \varphi \) and write \( \text{lan}_\varphi(H) \) for \( H' \) if the canonical map \([B,X][H',G) \to [A,X][(H' \circ \varphi,G \circ \varphi) \to [A,X][H,G \circ \varphi]) \) is an equivalence.

We say that \( X \) admits left kan-extensions along \( \varphi \) if every morphisms \( H : A \to X \) admits a left kan-extension \( B \to X \) along \( \varphi \).

\( X \) admits left kan-extensions along \( \varphi \) if and only if the functor \( [\varphi,X] : [B,X] \to [A,X] \) admits a left adjoint \( \text{lan}_\varphi : [A,X] \to [B,X] \).

Let \( \phi : X \to Y \) be a morphism of \( \mathcal{C} \).

Let \( H : A \to X \) and \( H' : B \to X \) be morphisms of \( \mathcal{C} \) and \( \alpha : H \to H' \circ \varphi \) a 2-morphism of \( \mathcal{C} \) that exhibits \( H' \) as the left kan-extension of \( H \) along \( \varphi \).

We say that \( \phi : X \to Y \) preserves the left kan-extensions of \( H \) along \( \varphi \) if \( \phi \circ \alpha : \phi \circ H \to \phi \circ H' \circ \varphi \) exhibits \( \phi \circ H' \) as the left kan-extension of \( \phi \circ H \) along \( \varphi \).

We say that \( \phi : X \to Y \) preserves left kan-extensions along \( \varphi \) if \( \phi : X \to Y \) preserves the left kan-extension of every morphism \( H : A \to X \) of \( \mathcal{C} \) along \( \varphi \).

Let \( \phi' : Y \to Z \) a morphism of \( \mathcal{C} \). If \( \phi : X \to Y \) preserves the left kan-extension of \( H \) along \( \varphi \) and \( \phi' : Y \to Z \) preserves the left kan-extension of \( \phi \circ H \) along \( \varphi \),
then $\phi' \circ \phi : X \to Z$ preserves the left kan-extension of $H$ along $\varphi$.

Thus with $\phi : X \to Y$ and $\phi' : Y \to Z$ also the composition $\phi' \circ \phi : X \to Z$ preserves left kan-extensions along $\varphi$.

Let $C$ be a 2-category, $X$ an object of $C$ and $\varphi : A \to B$ a morphism of $C$.

Let $H : A \to X$ and $H' : B \to X$ be morphisms of $C$ and $\alpha : H' \circ \varphi \to H$ a 2-morphism of $C$.

We say that $\alpha$ exhibits $H'$ as the right kan-extension of $H$ along $\varphi$ and write $\text{ran}_\varphi(H)$ for $H'$ if $\alpha$ exhibits $H'$ as the left kan-extension of $H$ along $\varphi$ in $C_{op}$.

**Proposition 6.3.** Let $C$ be a 2-category and $\psi : Y \to X$ an Eilenberg-Moore object for some monad $T$ on some object $X$ of $C$.

Let $\varphi : A \to B$ be a morphism of $C$.

1. If $X$ admits left kan-extensions along $\varphi$ and $T : X \to X$ preserves left kan-extensions along $\varphi$, then $Y$ admits left kan-extensions along $\varphi$ that are preserved and reflected by $\psi : Y \to X$.

2. If $X$ admits right kan-extensions along $\varphi$, then $Y$ admits right kan-extensions along $\varphi$ that are preserved and reflected by $\psi : Y \to X$.

So the subcategory of $C$ with objects the objects of $C$ that admit left (right) kan-extensions along $\varphi$ and with morphisms the morphisms of $C$ that preserve left (right) kan-extensions along $\varphi$ admits Eilenberg-Moore objects and coEilenberg-Moore objects.

The full subcategory of $C$ spanned by the objects of $C$ that admit left (right) kan-extensions along $\varphi$ admits coEilenberg-Moore objects (Eilenberg-Moore objects).

**Proof.**

1.: Denote $[X,X]'$ the full subcategory of $[X,X]$ spanned by those morphisms $X \to X$ that preserve left kan-extensions along $\varphi : A \to B$.

As $[X,X]'$ is closed under composition in $[X,X]$, the monoidal structure on $[X,X]$ restricts to a monoidal structure on $[X,X]'$.

The functor $[\varphi,X] : [B,X] \to [A,X]$ is $[X,X]$-linear and thus also $[X,X]'$-linear after pulling back along the monoidal full subcategory inclusion $[X,X]' \subset [X,X]$.

If $X$ admits left kan-extensions along $\varphi$, the functor $[\varphi,X] : [B,X] \to [A,X]$ admits a left adjoint $\text{lan}_\varphi : [A,X] \to [B,X]$. Denote $\eta$ the unit of this adjunction and let $\phi : X \to X$ a morphism of $C$ that preserves left kan-extensions along $\varphi$.

Then for every morphisms $H : A \to X$ of $C$ the morphism $\text{lan}_\varphi(\phi \circ H) \to \phi \circ \text{lan}_\varphi(H)$ in $[B,X]$ adjoint to the morphism $\phi \circ \eta : \phi \circ H \to \phi \circ \text{lan}_\varphi(H) \circ \varphi$ in $[A,X]$ is an equivalence.

Hence we obtain a $[X,X]'$-linear adjunction $\text{lan}_\varphi : [A,X] \nrightarrow [B,X] : [\varphi,X]$.

So given a monad $T$ on $X$ that preserves left kan-extensions along $\varphi$, i.e. an associative algebra of $[X,X]'$ we obtain an adjunction $\text{LMod}_T([A,X]) \nrightarrow \text{LMod}_T([B,X])$ and a map of adjunctions from the adjunction $\text{LMod}_T([A,X]) \nrightarrow \text{LMod}_T([B,X])$ to the adjunction $\text{lan}_\varphi : [A,X] \nrightarrow [B,X] : [\varphi,X]$.

Let $\psi : Y \to X$ be an Eilenberg-Moore object for $T$.

Then by corollary 6.11 the induced functor $[B,Y] \to [A,Y]$ is equivalent to the functor $\text{LMod}_T([B,X]) \to \text{LMod}_T([A,X])$ over the functor $[B,X] \to [A,X]$. 

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So the morphism $Y \to X$ yields a map of adjunctions from the adjunction $[A, Y] \rightleftarrows [B, Y]$ to the adjunction $[A, X] \rightleftarrows [B, X]$.

As the forgetful functors $\mathbf{LMod}_T([B, X]) \to [B, X]$ and $\mathbf{LMod}_T([A, X]) \to [A, X]$ are conservative, we see that $Y$ admits left kan-extensions along $\varphi$ that are preserved and reflected by $\psi : Y \to X$.

2.: The proof of 2. is similar but easier than 1.

If $X$ admits right kan-extensions along $\varphi$, the functor $[\varphi, X] : [B, X] \to [A, X]$ admits a right adjoint $\text{ran}_\varphi : [A, X] \to [B, X]$.

Hence we obtain a $[X, X]$-linear adjunction $[\varphi, X] : [B, X] \rightleftarrows [A, X] : \text{ran}_\varphi$.

So given a monad $T$ on $X$ we obtain an adjunction $\mathbf{LMod}_T([B, X]) \rightleftarrows \mathbf{LMod}_T([A, X])$ and a map of adjunctions from the adjunction $\mathbf{LMod}_T([B, X]) \rightleftarrows \mathbf{LMod}_T([A, X])$ to the adjunction $[\varphi, X] : [B, X] \rightleftarrows [A, X] : \text{ran}_\varphi$.

Let $\psi : Y \to X$ be an Eilenberg-Moore object for $T$.

Then by corollary 4.11 the induced functor $[B, Y] \to [A, Y]$ is equivalent to the functor $\mathbf{LMod}_T([B, X]) \to \mathbf{LMod}_T([A, X])$ over the functor $[B, X] \to [A, X]$.

So the morphism $Y \to X$ yields a map of adjunctions from the adjunction $[B, Y] \rightleftarrows [A, Y]$ to the adjunction $[B, X] \rightleftarrows [A, X]$.

As the forgetful functors $\mathbf{LMod}_T([B, X]) \to [B, X]$ and $\mathbf{LMod}_T([A, X]) \to [A, X]$ are conservative, we see that $Y$ admits right kan-extensions along $\varphi$ that are preserved and reflected by $\psi : Y \to X$.

\[\square\]

Especially for $\mathcal{C} = \mathbf{Cat}_\infty(\kappa)$ proposition 6.3 implies the following:

Let $T$ be a monad on a category $X$ and $\varphi : A \to B$ a functor.

If $X$ admits left kan-extensions along $\varphi$ that are preserved by $T$, then $\mathbf{LMod}_T(X)$ admits left kan-extensions along $\varphi$ that are preserved and reflected by the forgetful functor $\mathbf{LMod}_T(X) \to X$.

If $X$ admits right kan-extensions along $\varphi$, then $\mathbf{LMod}_T(X)$ admits right kan-extensions along $\varphi$ that are preserved and reflected by the forgetful functor $\mathbf{LMod}_T(X) \to X$.

In the following we will study some consequences of proposition 6.3.

We begin by giving some further notions:

Let $\mathcal{C}$ be a 2-category, $X$ an object of $\mathcal{C}$ and $\varphi : A \to B$ a functor.

We say that $X$ is compatible with left kan-extensions along $\varphi$ if for every object $Y$ of $\mathcal{C}$ the category $[Y, X]$ admits left kan-extensions along $\varphi$ and for every morphism $\beta : Z \to Y$ of $\mathcal{C}$ the functor $[\beta, X] : [Y, X] \to [Z, X]$ preserves left kan-extensions along $\varphi$.

Dually we say that $X$ is compatible with right kan-extensions along $\varphi$ if for every object $Y$ of $\mathcal{C}$ the category $[Y, X]$ admits right kan-extensions along $\varphi$ and for every morphism $\beta : Z \to Y$ of $\mathcal{C}$ the functor $[\beta, X] : [Y, X] \to [Z, X]$ preserves right kan-extensions along $\varphi$.

If $\varphi$ is the full subcategory inclusion $K \subset K^\circ$ for some category $K$, we say that $X$ is compatible with colimits indexed by $K$ for saying that $X$ is compatible with left kan-extensions along $\varphi$. 

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Dually if \( \varphi \) is the full subcategory inclusion \( \mathcal{K} \subset \mathcal{K}^{\circ} \) for some category \( \mathcal{K} \), we say that \( \mathcal{X} \) is compatible with limits indexed by \( \mathcal{K} \) for saying that \( \mathcal{X} \) is compatible with right kan-extensions along \( \varphi \).

Let \( \mathcal{X}, \mathcal{X}' \) be objects of \( \mathcal{C} \) that are compatible with left kan-extensions along \( \varphi : \mathcal{A} \to \mathcal{B} \)

We say that a morphism \( \theta : \mathcal{X} \to \mathcal{X}' \) of \( \mathcal{C} \) is compatible with left kan-extensions along \( \varphi \) if for every object \( \mathcal{Y} \) of \( \mathcal{C} \) the functor \( \{ \mathcal{Y}, \theta \} : \{ \mathcal{Y}, \mathcal{X} \} \to \{ \mathcal{Y}, \mathcal{X}' \} \)

preserves left kan-extensions along \( \varphi \) and dually for right kan-extensions.

**Remark 6.4.** Let \( \mathcal{C} \) be a cotensored module over \( \mathcal{C}_{\infty}^{\kappa} \).

Then \( \mathcal{X} \) is compatible with left kan-extensions along \( \varphi \) if and only if the morphism \( \mathcal{X}^\varphi : \mathcal{X}^\mathcal{B} \to \mathcal{X}^\mathcal{A} \) of \( \mathcal{C} \) admits a left adjoint:

By proposition \([6.4]\) the morphism \( \mathcal{X}^\varphi : \mathcal{X}^\mathcal{B} \to \mathcal{X}^\mathcal{A} \) of \( \mathcal{C} \) admits a left adjoint if and only if for every object \( \mathcal{Y} \) of \( \mathcal{C} \) the induced functor \( \text{Fun}(\varphi, \mathcal{Y}, \mathcal{X}) : \text{Fun}(\mathcal{B}, \mathcal{Y}, \mathcal{X}) \to \{ \mathcal{Y}, \mathcal{X}^\mathcal{A} \} \equiv \text{Fun}(\mathcal{A}, \mathcal{Y}, \mathcal{X}) \) admits a left adjoint \( \text{lan}^\varphi_{\mathcal{X}} \mathcal{Y} \) and for every morphism \( \beta : \mathcal{Z} \to \mathcal{Y} \) of \( \mathcal{C} \) the natural transformation

\[
\text{lan}^\varphi_{\mathcal{X}} \mathcal{Y} \circ \text{Fun}(\mathcal{A}, [\beta, \mathcal{X}]) \to \text{Fun}(\mathcal{B}, [\beta, \mathcal{X}]) \circ \text{lan}^\varphi_{\mathcal{X}} \mathcal{Y} \]

adjoint to \( \text{Fun}(\mathcal{A}, [\beta, \mathcal{X}]) \to \mathcal{Y} \) is an equivalence.

Dually \( \mathcal{X} \) is compatible with right kan-extensions along \( \varphi \) if and only if the morphism \( \mathcal{X}^\varphi : \mathcal{X}^\mathcal{B} \to \mathcal{X}^\mathcal{A} \) of \( \mathcal{C} \) admits a right adjoint.

If \( \mathcal{X}, \mathcal{X}' \) are objects of \( \mathcal{C} \) that are compatible with left kan-extensions along \( \varphi : \mathcal{A} \to \mathcal{B} \), then a morphism \( \theta : \mathcal{X} \to \mathcal{X}' \) of \( \mathcal{C} \) is compatible with left kan-extensions along \( \varphi \) if and only if \( \theta \) induces a map of adjunctions from the adjunction \( \mathcal{X}^\mathcal{A} \equiv \mathcal{X}^\mathcal{B} \) to the adjunction \( \mathcal{X}'^\mathcal{A} \equiv \mathcal{X}'^\mathcal{B} \).

**Example 6.5.** Let \( \mathcal{C} = \mathcal{C}_{\infty}^{\kappa}_{/\mathcal{S} \text{cart}} \) for some category \( \mathcal{S} \).

Let \( \varphi : \mathcal{A} \to \mathcal{B} \) be a functor and \( \mathcal{X} \to \mathcal{S} \) a cocartesian fibration.

The map \( \mathcal{X}^\varphi : \mathcal{X}^\mathcal{B} \to \mathcal{X}^\mathcal{A} \) of cocartesian fibrations over \( \mathcal{S} \) admits a left adjoint in \( \mathcal{C}_{\infty}^{\kappa}_{/\mathcal{S} \text{cart}} \), i.e. a left adjoint relative to \( \mathcal{S} \), if and only if for every \( \mathcal{s} \in \mathcal{S} \) the induced functor \( \text{Fun}(\varphi, \mathcal{X}_s) : \text{Fun}(\mathcal{B}, \mathcal{X}_s) \to \text{Fun}(\mathcal{A}, \mathcal{X}_s) \) on the fiber over \( \mathcal{s} \) admits a left adjoint \( \text{lan}^\varphi_{\mathcal{X}}_s \) and for every morphism \( \mathcal{f} : \mathcal{s} \to \mathcal{t} \) of \( \mathcal{S} \) the natural transformation

\[
\text{lan}^\varphi_{\mathcal{X}}_s \circ \text{Fun}(\mathcal{A}, \mathcal{f}_s) \to \text{Fun}(\mathcal{B}, \mathcal{f}_s) \circ \text{lan}^\varphi_{\mathcal{X}}_s
\]

adjoint to \( \text{Fun}(\mathcal{A}, \mathcal{f}_s) \to \mathcal{X}_s \) is an equivalence.

Consequently \( \mathcal{X} \) is compatible with left kan-extensions along \( \varphi \) if and only if for every \( \mathcal{s} \in \mathcal{S} \) the fiber \( \mathcal{X}_s \) admits left kan-extensions along \( \varphi \) and for every morphism \( \mathcal{f} : \mathcal{s} \to \mathcal{t} \) of \( \mathcal{S} \) the induced functor \( \mathcal{X}_s \to \mathcal{X}_t \) preserves left kan-extensions along \( \varphi \).

Similarly the map \( \mathcal{X}^\varphi : \mathcal{X}^\mathcal{B} \to \mathcal{X}^\mathcal{A} \) of cocartesian fibrations over \( \mathcal{S} \) admits a right adjoint in \( \mathcal{C}_{\infty}^{\kappa}_{/\mathcal{S} \text{cart}} \), i.e. a right adjoint relative to \( \mathcal{S} \) that is a map of cocartesian fibrations over \( \mathcal{S} \), if and only if for every \( \mathcal{s} \in \mathcal{S} \) the induced functor
\[
\text{Fun}(\varphi, X_s) : \text{Fun}(B, X_s) \to \text{Fun}(A, X_s) \text{ on the fiber over } s \text{ admits a right adjoint } \text{ran}_{\varphi}^X \text{ and for every morphism } f : s \to t \text{ of } S \text{ the natural transformation }
\]
\[
\text{Fun}(B, f_s) \circ \text{ran}_{\varphi}^X \to \text{ran}_{\varphi}^X \circ \text{Fun}(A, f_s)
\]
adjoint to \( \text{Fun}(\varphi, X_s) \circ \text{Fun}(B, f_s) \circ \text{ran}_{\varphi}^X \cong \text{Fun}(A, f_s) \circ \text{Fun}(\varphi, X_s) \circ \text{ran}_{\varphi}^X \to \text{Fun}(A, f_s) \) is an equivalence.

Hence \( X \) is compatible with right kan-extensions along \( \varphi \) if and only if for every \( s \in S \) the fiber \( X_s \) admits right kan-extensions along \( \varphi \) and for every morphism \( f : s \to t \text{ of } S \) the induced functor \( X_s \to X_t \) preserves right kan-extensions along \( \varphi \).

Let \( X \to S, X' \to S \) be cocartesian fibrations that are compatible with left kan-extensions along \( \varphi \).

A map \( \theta : X \to X' \) of cocartesian fibrations over \( S \) is compatible with left kan-extensions along \( \varphi \) if and only if for every \( s \in S \) the induced functor \( X_s \to X'_s \) on the fiber over \( s \) preserves left kan-extensions along \( \varphi \).

By \( \ldots \) \( \theta : X \to X' \) is compatible with left kan-extensions along \( \varphi \) if and only if \( \theta \) induces a map of adjunctions from the adjunction \( X^A \rightleftarrows X^B \) to the adjunction \( X'^A \rightleftarrows X'^B \), which is equivalent to the condition that for every \( s \in S \) the induced functor \( X_s \to X'_s \) on the fiber over \( s \) induces a map of adjunctions from the adjunction \( \text{Fun}(A, X_s) \rightleftarrows \text{Fun}(B, X_s) \) to the adjunction \( \text{Fun}(A, X'_s) \rightleftarrows \text{Fun}(B, X'_s) \).

Dually if \( X \to S, X' \to S \) be cocartesian fibrations that are compatible with right kan-extensions along \( \varphi \), a map \( \theta : X \to X' \) of cocartesian fibrations over \( S \) is compatible with right kan-extensions along \( \varphi \) if and only if for every \( s \in S \) the induced functor \( X_s \to X'_s \) on the fiber over \( s \) preserves right kan-extensions along \( \varphi \).

**Corollary 6.6.** Let \( \mathcal{C} \) be a 2-category and \( \psi : Y \to X \) an Eilenberg-Moore object for some monad \( T \) on some object \( X \) of \( \mathcal{C} \).

Let \( \varphi : A \to B \) be a morphism of \( \mathcal{C} \).

1. If \( X \) is compatible with left kan-extensions along \( \varphi \) and \( T : X \to X \) is compatible with left kan-extensions along \( \varphi \), then \( Y \) is compatible with left kan-extensions along \( \varphi \) and \( \psi : Y \to X \) is compatible with left kan-extensions along \( \varphi \).

2. If \( X \) is compatible with right kan-extensions along \( \varphi \), then \( Y \) is compatible with right kan-extensions along \( \varphi \) and \( \psi : Y \to X \) is compatible with right kan-extensions along \( \varphi \).

Thus the subcategory of \( \mathcal{C} \) with objects the objects of \( \mathcal{C} \) that are compatible with left (right) kan-extensions along \( \varphi \) and with morphisms the morphisms of \( \mathcal{C} \) that are compatible with left (right) kan-extensions along \( \varphi \) admits Eilenberg-Moore objects and coEilenberg-Moore objects.

The full subcategory of \( \mathcal{C} \) spanned by the objects of \( \mathcal{C} \) that are compatible with left (right) kan-extensions along \( \varphi \) admits coEilenberg-Moore objects (Eilenberg-Moore objects).
Lemma 6.7. Let $\mathcal{C}$ be a category, $T$ a monad on $\mathcal{C}$ and $I$ a set. Assume that $\text{LMod}_T(\mathcal{C})$ admits geometric realizations.

1. With $\mathcal{C}$ also $\text{LMod}_T(\mathcal{C})$ admits coproducts indexed by $I$.

2. Let $H : \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{D}$ be a functor that preserves geometric realizations.

If $\text{LMod}_T(\mathcal{C})$ admits coproducts indexed by $I$ and the composition $H \circ T : \mathcal{C} \rightarrow \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{D}$ preserves coproducts indexed by $I$, then $H$ preserves coproducts indexed by $I$.

Proof. 1.

Denote $\gamma : \mathcal{C} \rightarrow \text{Fun}(I, \mathcal{C})$ and $\delta : \text{LMod}_T(\mathcal{C}) \rightarrow \text{Fun}(I, \text{LMod}_T(\mathcal{C}))$ the diagonal functors.

Denote $W \subset \text{Fun}(I, \text{LMod}_T(\mathcal{C}))$ the full subcategory spanned by the families $A = (A_i)_{i \in I}$ in $\text{LMod}_T(\mathcal{C})$ that admit a coproduct indexed by $I$, i.e. that the functor $\text{Fun}(I, \text{LMod}_T(\mathcal{C}))(A, -) \circ \delta : \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{S}(\kappa)$ is corepresentable.

$W$ is closed under geometric realizations as $\text{LMod}_T(\mathcal{C})$ admits geometric realizations.

By ... every object of $\text{LMod}_T(\mathcal{C})$ is the geometric realization of a simplicial object of $\text{LMod}_T(\mathcal{C})$ that takes values in the full subcategory of $\text{LMod}_T(\mathcal{C})$ spanned by the free $T$-algebras of $\mathcal{C}$.

Hence it is enough to see that for every family $B = (B_i)_{i \in I}$ in $\mathcal{C}$ the family $A = (T(B_i))_{i \in I}$ in $\text{LMod}_T(\mathcal{C})$ belongs to $W$.

The functor $\text{Fun}(I, \text{LMod}_T(\mathcal{C}))(A, -) \circ \delta : \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{S}(\kappa)$ factors as the forgetful functor $\text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C}$ followed by the functor $\text{Fun}(I, \mathcal{C})(B, -) \circ \gamma : \mathcal{C} \rightarrow \mathcal{S}(\kappa)$.

As $\mathcal{C}$ admits coproducts indexed by $I$, the functor $\text{Fun}(I, \mathcal{C})(B, -) \circ \gamma : \mathcal{C} \rightarrow \mathcal{S}(\kappa)$ is corepresentable and thus also its composition with the right adjoint forgetful functor $\text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{C}$ is corepresentable.

2:

Replacing $H : \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{D}$ by the functor $\text{LMod}_T(\mathcal{C}) \xrightarrow{H'} \mathcal{D} \subset \text{Fun}(\mathcal{D}, \mathcal{S}(\kappa))^{op}$ we can assume that $\mathcal{D}$ admits coproducts indexed by $I$.

As $\text{LMod}_T(\mathcal{C})$ and $\mathcal{D}$ admit coproducts indexed by $I$, the diagonal functors $\delta : \text{LMod}_T(\mathcal{C}) \rightarrow \text{Fun}(I, \text{LMod}_T(\mathcal{C}))$ and $\delta' : \mathcal{D} \rightarrow \text{Fun}(I, \mathcal{D})$ admit left adjoints $\coprod$ respectively $\coprod'$. Denote

$$\alpha : \coprod' \circ \text{Fun}(I, H) \rightarrow H \circ \coprod$$

the natural transformation of functors $\text{Fun}(I, \text{LMod}_T(\mathcal{C})) \rightarrow \mathcal{D}$ adjoint to the natural transformation $\text{Fun}(I, H) \rightarrow \text{Fun}(I, H) \circ \delta \circ \coprod' \circ H' \circ \coprod$ of functors $\text{Fun}(I, \text{LMod}_T(\mathcal{C})) \rightarrow \text{Fun}(I, \mathcal{D})$.

As $\text{LMod}_T(\mathcal{C})$ admits geometric realizations, with $H : \text{LMod}_T(\mathcal{C}) \rightarrow \mathcal{D}$ also $\text{Fun}(I, H) : \text{Fun}(I, \text{LMod}_T(\mathcal{C})) \rightarrow \text{Fun}(I, \mathcal{D})$ preserves geometric realizations so that source and target of $\alpha$ are geometric realizations preserving functors.

Thus the full subcategory $\mathcal{D} \subset \text{Fun}(I, \text{LMod}_T(\mathcal{C}))$ spanned by the objects $X$ such that $\alpha(X)$ is an equivalence is closed under geometric realizations.

Consequently it is enough to see that for every family $B = (B_i)_{i \in I}$ in $\mathcal{C}$ the morphism $\alpha(T(B_1), ..., T(B_n)) : \coprod_{i=1}^n H'(T(B_i)) \rightarrow H'((\coprod_{i=1}^n T(B_i))$ is an equivalence.
But the composition \( \prod_{i=1}^n H'(T(B_i)) \to H'(\prod_{i=1}^n T(B_i)) \cong H'(\prod_{i=1}^n B_i) \) is the canonical morphism and thus an equivalence as \( H \circ T \) preserves coproducts.

**Proposition 6.8.** Let \( \mathcal{C} \to S \) be a cocartesian fibration between \( \kappa \)-small categories that is compatible with geometric realizations and let \( T \in \text{Alg}(\text{Fun}_S(\mathcal{C}, \mathcal{C})) \) be a monad such that for every object \( s \) of \( S \) the induced functor \( \mathcal{C}_s \to \mathcal{C}_s \) on the fiber over \( s \) preserves geometric realizations.

Then \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is a cocartesian fibration compatible with geometric realizations and the free functor \( \mathcal{C} \to \text{LMod}_{T}^{S}(\mathcal{C}) \) over \( S \) is a map of cocartesian fibrations over \( S \).

Moreover if \( \mathcal{C} \to S \) is compatible with \( \kappa \)-small colimits, then \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is compatible with \( \kappa \)-small colimits.

**Proof.** By corollary 6.6 \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is compatible with geometric realizations.

So by remark 6.4 the functor \( \text{LMod}_{T}^{S}(\mathcal{C})(\Delta^op)^p \to \text{LMod}_{T}^{S}(\mathcal{C})^{\Delta^op} \) over \( S \) admits a left adjoint relative to \( S \).

Being a relative left adjoint the free functor \( \mathcal{C} \to \text{LMod}_{T}^{S}(\mathcal{C}) \) over \( S \) preserves cocartesian morphisms.

For every object \( s \) of \( S \) the fiber \( \mathcal{C}_s \) is the only full subcategory of \( \text{LMod}_{T}^{S}(\mathcal{C})_s = \text{LMod}_{T}^{S}(\mathcal{C})_s \) that contains the free \( T \)-algebras and is closed in \( \text{LMod}_{T}^{S}(\mathcal{C})_s \) under geometric realizations.

Hence by remark 7.34 \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is a cocartesian fibration and the free functor \( \mathcal{C} \to \text{LMod}_{T}^{S}(\mathcal{C}) \) over \( S \) is a map of cocartesian fibrations over \( S \).

By lemma 6.7 \( \text{LMod}_{T}^{S}(\mathcal{C}) \to S \) is compatible with coproducts and is thus compatible with \( \kappa \)-small colimits.

**Example 6.9.**

Let \( S \) be a category and \( M^\otimes \to \text{LM}^\otimes \) be an operad over \( \text{LM}^\otimes \) such that we have a monoidal equivalence \( \text{Ass}^\otimes \times_{\text{Comm}^\otimes} (\text{Cat}_{\infty}(\kappa)/S)^* \cong \text{Ass}^\otimes \times_{\text{LM}^\otimes} M^\otimes \). Set \( M := (m) \times_{\text{LM}^\otimes} M^\otimes \).

Let \( B \) be a monoidal category over \( S \) and \( X \) a left modal over \( B \) in \( M \).

Assume that \( X \) admits an endomorphism object.

Then the left modal structure on \( X \) over \( B \) corresponds to a monoidal functor \( B \to [X, X] \) over \( S \) that gives rise to a functor \( \xi : \text{Alg}_{\otimes}^{S}(B) \to \text{Alg}_{\otimes}^{S}([X, X]) \) over \( S \).

This way every section \( A \) of \( \text{Alg}_{\otimes}^{S}(B) \to S \) gives rise to a monad \( \xi(A) \) on \( X \).

If \( \xi(A) \) admits an Eilenberg-Moore object \( Y \to X \) in \( M \), i.e. in the associated 2-category by applying \( \text{Fun}_S(S, -) \) to all morphism categories of \( M \), then we think of \( Y \to X \) as the object of left \( A \)-modules in \( M \).

We specialize to the following situation:

Let \( T \to S \) be a functor. Then the canonical functor \( \text{Cat}_{\infty}(\kappa)/S \to \text{Cat}_{\infty}(\kappa)/T \) makes \( \text{Cat}_{\infty}(\kappa)/T \) to a left module over \( \text{Cat}_{\infty}(\kappa)/S \).
Let $X \to T$ be a left modul in $\text{Cat}_\infty(\kappa)_T$ over a monoidal category $\mathcal{B} \to S$ over $S$.

Assume that the composition $X \to T \to S$ is flat so that $X$ admits an endomorphism object $\text{Fun}_T^S(X, X)$.

So we get a monoidal functor $\mathcal{B} \to \text{Fun}_T^S(X, X)$ over $S$ that gives rise to a functor $\xi: \text{Alg}^S(\mathcal{B}) \to \text{Alg}^S(\text{Fun}_T^S(X, X))$ over $S$.

Given a section $A$ of $\text{Alg}^S(\text{Fun}_T^S(X, X)) \to S$ we have a canonical equivalence

$$\text{LMod}^T_A(X) \simeq \text{LMod}^{\xi(A)}(X)$$

over $X$.

We further specialize to the following situation:

Let $\mathcal{O}^\circ$ be an operad and $X^\circ \to \mathcal{O}^\circ$ be a left module over an associative monoid $\mathcal{B}^\circ \to \mathcal{O}^\circ$ in $\text{Op}_{\infty}(\kappa)_{/\mathcal{O}^\circ} \subset \text{Cat}_{\infty}(\kappa)_{/\mathcal{O}^\circ}$.

Assume that the functor $X^\circ \to \mathcal{O}^\circ$ is flat.

Then we get a monoidal functor $\mathcal{B}^\circ \to \text{Map}_{\mathcal{O}^\circ}(X^\circ, X^\circ)$ over $\mathcal{O}^\circ$ that gives rise to a monoidal functor $\text{Fun}_{\mathcal{O}^\circ}(\mathcal{O}^\circ, \mathcal{B}^\circ) \to \text{Fun}_{\mathcal{O}^\circ}(\mathcal{O}^\circ, \text{Map}_{\mathcal{O}^\circ}(X^\circ, X^\circ)) \simeq \text{Fun}_{\mathcal{O}^\circ}(X^\circ, X^\circ)$ that restricts to a monoidal functor $\beta: \text{Alg}_{\mathcal{O}^\circ}(\mathcal{B}) \to \text{Alg}_{X^\circ/\mathcal{O}}(X)$ that yields a functor $\text{Alg}_{\mathcal{O}^\circ}(\mathcal{B}) \simeq \text{Alg}(\text{Alg}_{\mathcal{O}^\circ}(\mathcal{B})) \to \text{Alg}(\text{Alg}_{X^\circ/\mathcal{O}}(X))$.

Given an $\text{Ass}^\circ \otimes \mathcal{O}^\circ$-algebra $A$ of $\mathcal{B}$ we have a canonical equivalence

$$\text{LMod}_{A}^{\mathcal{O}^\circ}(X^\circ) \simeq \text{LMod}_{\beta(A)}^{\mathcal{O}^\circ}(X^\circ)$$

over $X^\circ$.

By theorem [1.10] the functor $\text{LMod}_{A}^{\mathcal{O}^\circ}(X^\circ) \to \mathcal{O}^\circ$ is a map of operads and the forgetful functor $\text{LMod}_{A}^{\mathcal{O}^\circ}(X^\circ) \to X^\circ$ and its left adjoint over $\mathcal{O}^\circ$ are maps of operads over $\mathcal{O}^\circ$.

By proposition [0.3] the functor $\text{LMod}_{A}^{\mathcal{O}^\circ}(X^\circ) \to \mathcal{O}^\circ$ is a $\mathcal{O}^\circ$-monoidal category and the free functor $X^\circ \to \text{LMod}_{A}^{\mathcal{O}^\circ}(X^\circ)$ is a $\mathcal{O}^\circ$-monoidal functor if $X^\circ \to \mathcal{O}^\circ$ is a $\mathcal{O}^\circ$-monoidal category compatible with geometric realizations and for every $X \in \mathcal{O}$ the functor $A(X)\otimes: \mathcal{B}_X \to \mathcal{B}_X$ preserves geometric realizations.

Moreover if $X^\circ \to \mathcal{O}^\circ$ is compatible with $\kappa$-small colimits, then $\text{LMod}_{A}^{\mathcal{O}^\circ}(X^\circ)$ is compatible with $\kappa$-small colimits.

Let $X$ be a $\mathcal{O}^\circ \otimes \text{Ass}^\circ$-monoidal category corresponding to an associative algebra in the category of $\mathcal{O}^\circ$-monoidal categories.

Then the fiberwise dual $(X^\circ)^{\text{rev}} \to \mathcal{O}^\circ$ of the $\mathcal{O}^\circ$-monoidal category $X^\circ \to \mathcal{O}^\circ$ is an associative algebra in the category of $\mathcal{O}^\circ$-monoidal categories.

So we have a monoidal functor $\beta: \text{Coalg}_{\mathcal{O}^\circ}(X)^{\text{op}} \to \text{Alg}_{(X^\circ)^{\text{rev}}/\mathcal{O}^\circ}((X^\circ)^{\text{rev}})$ that yields a functor $\text{Alg}(\text{Coalg}_{\mathcal{O}^\circ}(X))^{\text{op}} \to \text{Coalg}(\text{Alg}_{X^\circ/\mathcal{O}}((X^\circ)^{\text{rev}}))$.

So every associative algebra $A$ in the monoidal category of $\mathcal{O}^\circ$-coalgebras of $X$ gives rise to a comonad $T$ on $(X^\circ)^{\text{rev}} \to \mathcal{O}^\circ$ in $\text{Op}_{\infty}(\kappa)_{/\mathcal{O}^\circ}$, i.e. a $\mathcal{O}^\circ$-opmonoidal monad.
So by theorem 4.16 the category $\text{LMod}_A(X) \simeq \text{LMod}_T(X)$ lifts to a $\mathcal{O}^\otimes$-monoidal category and the forgetful functor $\text{LMod}_A(X) \simeq \text{LMod}_T(X) \to X$ lifts to a $\mathcal{O}^\otimes$-monoidal functor.
7 Appendix

7.1 Appendix A: Parametrized categories of sections

In this subsection we prove the following:

Denote $\mathcal{R}(\kappa) \subset \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))$ the full subcategory spanned by the right fibrations and $\mathcal{U}(\kappa) \subset \mathcal{R}(\kappa)$ the full subcategory spanned by the representable right fibrations.

The restriction $\mathcal{U}(\kappa) \subset \mathcal{R}(\kappa) \to \text{Fun}(\{1\}, \text{Cat}_\infty(\kappa))$ is a cocartesian fibration and classifies the identity of $\text{Cat}_\infty(\kappa)$.

We start with the following basic lemma:

**Lemma 7.1.** Let $\kappa$ be a strongly inaccessible cardinal and $\rho : X \to S$ a cocartesian fibration between $\kappa$-small categories.

Denote $\varrho : X^{\Delta^1} \to X^{\{1\}} \cong X$ the induced map of cocartesian fibrations over $S$.

Then $\varrho : X^{\Delta^1} \to X^{\{1\}} \cong X$ is a cocartesian fibration, where a morphism of $X^{\Delta^1}$ is $\varrho$-cocartesian if and only if its image under the map $\sigma : X^{\Delta^1} \to X^{\{0\}} \cong X$ of cocartesian fibrations over $S$ is $\rho$-cocartesian.

**Proof.** 2) follows from 1) by the commutativity of the square

\[
\begin{array}{ccc}
  X^{\Delta^1} & \xrightarrow{\xi^{\Delta^1}} & Y^{\Delta^1} \\
  \downarrow & & \downarrow \\
  X^{\{0\}} & \xrightarrow{\xi^{\{0\}}} & Y^{\{0\}}
\end{array}
\]

of cocartesian fibrations over $S$.

As $\varrho : X^{\Delta^1} \to X^{\{1\}}$ is a map of cocartesian fibrations over $S$, it is enough to see that for every object $Z \in S$ the induced functor on the fiber $\varrho_Z : X_Z^{\Delta^1} \to X_Z^{\{1\}}$ is a cocartesian fibration and for every morphism $Z \to Z'$ in $S$ the induced functor on the fiber $X_Z^{\Delta^1} \to X_Z^{\Delta^1}$ sends $\varrho_Z$-cocartesian morphisms to $\varrho_{Z'}$-cocartesian morphisms.

The induced functor on the fiber $\varrho_Z : X_Z^{\Delta^1} \to X_Z^{\{1\}}$ is equivalent to the evaluation at the target functor $\text{Fun}(\Delta^1, X_Z) \to \text{Fun}(\{1\}, X_Z)$ and thus a cocartesian fibration.

The $\varrho_Z$-cocartesian morphisms are those morphisms of $X_Z^{\Delta^1}$ that are sent to equivalences by the induced functor $X_Z^{\Delta^1} \to X_Z^{\{0\}}$ and similar for the $\varrho_{Z'}$-cocartesian morphisms.

Therefore the commutativity of

\[
\begin{array}{ccc}
  X_Z^{\Delta^1} & \xrightarrow{X_Z^{\{0\}}} & X_Z^{\{0\}} \\
  \downarrow & & \downarrow \\
  X_Z^{\Delta^1} & \xrightarrow{X_Z^{\{0\}}} & X_Z^{\{0\}}
\end{array}
\]

implies that the induced functor on the fiber $X_Z^{\Delta^1} \to X_Z^{\Delta^1}$ sends $\varrho_Z$-cocartesian morphisms to $\varrho_{Z'}$-cocartesian morphisms.
This shows that $\varrho: X^{\Delta^1} \to X^{\{1\}} \simeq X$ is a cocartesian fibration.

It remains to characterize the $\varrho$-cocartesian morphisms.

Let $f: A \to B$ be a morphism of $X^{\Delta^1}$ lying over a morphism $g$ in $S$.

Then we can factor $f$ as a $\rho^{\Delta^1}$-cocartesian morphism $\alpha: A \to g_*(A)$ followed by a morphism $\beta: g_*(A) \to B$ in the fiber over $\rho^{\Delta^1}(B)$.

As $\sigma: X^{\Delta^1} \to X^{\{0\}}$ is a map of cocartesian fibrations over $S$, the morphism $\sigma(\alpha): \sigma(A) \to g_*(\sigma(A))$ is $\varrho$-cocartesian and thus $\alpha: A \to g_*(A)$ is $\varrho$-cocartesian.

Therefore $f: A \to B$ is $\varrho$-cocartesian if and only if $\beta: g_*(A) \to B$ is $\varrho$-cocartesian which is equivalent to the condition that $\beta$ is $g^\varrho_\Delta(B)$-cocartesian because $\varrho$ is a cocartesian fibration.

$\beta$ is $g^\varrho_\Delta(B)$-cocartesian if and only if $\sigma(\beta): g_*(\sigma(A)) \to \sigma(B)$ is an equivalence which is equivalent to the condition that $\sigma(f)$ is $\varrho$-cocartesian.

\[
\text{Lemma 7.2.} \quad \text{Let } S \text{ be a } \kappa \text{-small category and } \phi: \mathcal{C} \to \mathcal{D} \text{ a map of } \kappa \text{-small (locally) cocartesian fibrations over } S \text{ that induces on the fiber over every object } s \text{ of } S \text{ a right fibration.}

\text{Let } X \text{ be a section of the (locally) cocartesian fibration } \mathcal{E} \to S \text{ such that for all } s \in S \text{ the image } X(s) \in \mathcal{C}_s \text{ is a final object of } \mathcal{C}_s.

\text{Then } \mathcal{E} \to \mathcal{D} \text{ is canonically equivalent over } \mathcal{D} \text{ to the pullback } S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{0\}} \text{ of the map } \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{1\}} \text{ of (locally) cocartesian fibrations over } S \
\text{along } \phi \times X: S \to \mathcal{D}.

\text{Proof.} \quad \text{Let } \mathcal{E} \to S \text{ be a } \kappa \text{-small (locally) cocartesian fibration and } Z \text{ a section of } \mathcal{E} \to S.

\text{By lemma 7.7} \text{ the map } \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\{1\}} \text{ of (locally) cocartesian fibrations over } S \text{ is a (locally) cocartesian fibration, whose (locally) cocartesian morphisms are those that are sent to (locally) cocartesian morphisms of } \mathcal{E} \to S \text{ under the map } \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\{0\}} \text{ of (locally) cocartesian fibrations over } S.

\text{Thus also the pullback } S \times_{\mathcal{E}^{\{1\}}} \mathcal{E}^{\Delta^1} \to S \text{ of } \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\{1\}} \text{ along } Z: S \to \mathcal{E} \text{ is a (locally) cocartesian fibration, whose (locally) cocartesian morphisms are those that are sent to (locally) cocartesian morphisms of } \mathcal{E} \to S \text{ under the map } S \times_{\mathcal{E}^{\{1\}}} \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\{0\}}.

\text{So the functor } \xi: S \times_{\mathcal{E}^{\{1\}}} \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\Delta^1} \to \mathcal{E}^{\{0\}} \text{ over } S \text{ is a map of (locally) cocartesian fibrations over } S.

\text{So we get a commutative square}

\[
\begin{array}{ccc}
S \times_{\mathcal{E}^{\{1\}}} \mathcal{E}^{\Delta^1} & \longrightarrow & S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \\
\downarrow & & \downarrow \\
\mathcal{E}^{\{0\}} & \longrightarrow & \mathcal{D}^{\{0\}}
\end{array}
\]

(10)

of (locally) cocartesian fibrations over $S$ that induces on the fiber over every object $s$ of $S$ the commutative square

\[\text{Diagram}(10)\]
\[(\mathcal{C}_s)_{/\mathcal{X}_s} \longrightarrow (\mathcal{D}_s)_{/\phi(\mathcal{X}_s)} \]

\[\downarrow \quad \downarrow \]

\[\mathcal{C}_s \longrightarrow \mathcal{D}_s.\]

(11)

As \(\mathcal{C}_s \to \mathcal{D}_s\) is a right fibration, the top horizontal morphism of square is an equivalence.

As \(\mathcal{X}(s)\) is a final object of \(\mathcal{C}_s\), the left vertical morphism of square is an equivalence.

Hence the left vertical and top horizontal morphism of square is an equivalence.

Proposition 7.3. Denote \(\mathcal{U}(\kappa) \subset \mathcal{R}(\kappa)\) the full subcategory spanned by the representable right fibrations.

The restriction \(\mathcal{U}(\kappa) \subset \mathcal{R}(\kappa) \to \text{Cat}_\infty(\kappa)\) is a cocartesian fibration and classifies the identity of \(\text{Cat}_\infty(\kappa)\).

Remark 7.4. The cartesian fibration \(\mathcal{R}(\kappa) \to \text{Cat}_\infty(\kappa)\) is a bicartesian fibration as for every functor \(\mathcal{C} \to \mathcal{D}\) the induced functor \(\mathcal{R}(\kappa)_{/\mathcal{D}} \to \mathcal{R}(\kappa)_{/\mathcal{C}}\) admits a left adjoint.

The left adjoint \(\mathcal{R}(\kappa)_{/\mathcal{C}} \to \mathcal{R}(\kappa)_{/\mathcal{D}}\) preserves representable right fibrations.

Hence the cocartesian fibration \(\mathcal{R}(\kappa) \to \text{Cat}_\infty(\kappa)\) restricts to a cocartesian fibration \(\mathcal{U}(\kappa) \to \text{Cat}_\infty(\kappa)\) with the same cocartesian morphisms.

Proof. Let \(\mathcal{U}'(\kappa) \to \text{Cat}_\infty(\kappa)\) be the cocartesian fibration classifying the identity of \(\text{Cat}_\infty(\kappa)\).

We will show that there is a canonical equivalence \(\mathcal{U}'(\kappa) \simeq \mathcal{U}(\kappa)\) over \(\text{Cat}_\infty(\kappa)\).

By Yoneda it is enough to find for every functor \(\mathcal{H} : \mathcal{S} \to \text{Cat}_\infty(\kappa)\) a bijection between equivalence classes of objects of the categories \(\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{S}, \mathcal{U}'(\kappa))\) and \(\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{S}, \mathcal{U}(\kappa))\) such that for every functor \(\mathcal{T} \to \mathcal{S}\) over \(\text{Cat}_\infty(\kappa)\) the square

\[
\begin{array}{ccc}
\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{S}, \mathcal{U}'(\kappa)) & \longrightarrow & \text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{S}, \mathcal{U}(\kappa)) \\
\downarrow & & \downarrow \\
\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{T}, \mathcal{U}'(\kappa)) & \longrightarrow & \text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{T}, \mathcal{U}(\kappa))
\end{array}
\]

commutes on equivalence classes.

Denote \(\mathcal{D} \to \mathcal{S}\) the cocartesian fibration classifying \(\mathcal{H} : \mathcal{S} \to \text{Cat}_\infty(\kappa)\) so that we have a canonical equivalence \(\mathcal{D} \simeq \mathcal{S} \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}'(\kappa)\) over \(\mathcal{S}\).

We have a canonical equivalence \(\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{S}, \mathcal{U}'(\kappa)) \simeq \text{Funs}(\mathcal{S}, \mathcal{D})\) such that the square

\[
\begin{array}{ccc}
\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{S}, \mathcal{U}'(\kappa)) & \longrightarrow & \text{Funs}(\mathcal{S}, \mathcal{D}) \\
\downarrow & & \downarrow \\
\text{Fun}_{\text{Cat}_\infty(\kappa)}(\mathcal{T}, \mathcal{U}'(\kappa)) & \longrightarrow & \text{Fun}_{\mathcal{T}}(\mathcal{T}, \mathcal{T} \times_{\mathcal{S}} \mathcal{D})
\end{array}
\]

commutes on equivalence classes.
We have a fully faithful functor

\[ \text{Fun}_{\text{Cat} (\kappa)} (S, \mathcal{U} (\kappa)) \subset \text{Fun}_{\text{Cat} (\kappa)} (S, \mathcal{R} (\kappa)) \subset \text{Fun}_{\text{Cat} (\kappa)} (S, \text{Fun} (\Delta^1, \text{Cat}_\infty (\kappa))) \]

\[ \simeq \text{Fun} (S, \text{Cat}_\infty (\kappa)) / \mathcal{U} \simeq \text{Cat}_{\infty / S}^{\text{cocart}} (\kappa) / \mathcal{D}, \]

whose essential image \( W(S, \mathcal{D}) \) consists of those maps \( \mathcal{E} \to \mathcal{D} \) of cocartesian fibrations over \( S \) that induce on the fiber over every object of \( S \) a representable right fibration and the square

\[
\begin{array}{ccc}
\text{Fun}_{\text{Cat} (\kappa)} (S, \mathcal{U} (\kappa)) & \longrightarrow & \text{Cat}_{\infty / S}^{\text{cocart}} (\kappa) / \mathcal{D} \\
\downarrow & & \downarrow \\
\text{Fun}_{\text{Cat} (\kappa)} (T, \mathcal{U} (\kappa)) & \longrightarrow & \text{Cat}_{\infty / T}^{\text{cocart}}(\kappa) / T \times_S \mathcal{D}
\end{array}
\]

commutes on equivalence classes.

Consequently it is enough to find for every cocartesian fibration \( \mathcal{D} \to S \) a bijection between equivalence classes of objects of the categories \( \text{Fun}_S (\mathcal{S}, \mathcal{D}) \) and \( W(S, \mathcal{D}) \) such that the square

\[
\begin{array}{ccc}
\text{Fun}_S (\mathcal{S}, \mathcal{D}) & \longrightarrow & W(S, \mathcal{D}) \\
\downarrow & & \downarrow \\
\text{Fun}_T (T, T \times_S \mathcal{D}) & \longrightarrow & W(T, T \times_S \mathcal{D})
\end{array}
\]

commutes on equivalence classes.

Let \( X \) be a section of \( \mathcal{D} \to S \).

By lemma 7.31 the map \( \mathcal{D}^{\Delta^1} \to \mathcal{D}^{(1)} \) of cocartesian fibrations over \( S \) is a cocartesian fibration, whose cocartesian morphisms are those that are sent by the map \( \mathcal{D}^{\Delta^1} \to \mathcal{D}^{(0)} \) of cocartesian fibrations over \( S \) to a cocartesian morphism of \( \mathcal{D} \to S \).

So the pullback \( \mathcal{D}^{\Delta^1} / X := S \times_{\mathcal{D}^{(1)}} \mathcal{D}^{\Delta^1} \to S \) along \( X \) is a cocartesian fibration and \( \alpha : S \times_{\mathcal{D}^{(1)}} \mathcal{D}^{\Delta^1} \to \mathcal{D} \) is a map of cocartesian fibrations over \( S \).

The map \( \alpha \) of cocartesian fibrations over \( S \) induces on the fiber over every object \( s \) of \( S \) the representable right fibration \( (\mathcal{D}_s) / \mathcal{X}(s) = \{ X(s) \} \times_{\text{Fun} (\{1\}, \mathcal{D})} \text{Fun} (\Delta^1, \mathcal{D}_s) \to \text{Fun} (\Delta^1, \mathcal{D}_s) \to \text{Fun} (\{0\}, \mathcal{D}_s) \). So \( \alpha \) belongs to \( W(S, \mathcal{D}) \).

Pulling back \( \alpha \) along the functor \( T \to S \) we get the map

\[
\text{Fun}_T (T \times_S \mathcal{D})^{\Delta^1} \to (T \times_S \mathcal{D})^{\Delta^1} \to (T \times_S \mathcal{D})^{(0)}
\]

of cocartesian fibrations over \( T \), where the pullback \( T \times_{(T \times_S \mathcal{D})^{(1)}} (T \times_S \mathcal{D})^{\Delta^1} \) is taken over the functor \( T \to T \times_S \mathcal{D} \) over \( T \) corresponding to the functor \( T \to S \to \mathcal{D} \) over \( S \).

This shows the commutativity of square 12.

On the other hand let \( \mathcal{E} \to \mathcal{D} \) be a map of cocartesian fibrations over \( S \) such that for every object \( s \) of \( S \) the induced functor \( \mathcal{E}_s \to \mathcal{D}_s \) is a representable right fibration.

As for every object \( s \) of \( S \) the category \( \mathcal{E}_s \) admits a final object, by lemma 7.32 the category \( \text{Fun}_S (S, \mathcal{E}) \) admits a final object \( Z \) such that for every object \( s \) of \( S \) the image \( Z(s) \) is the final object of \( \mathcal{E}_s \).
The functor \( \text{Fun}_S(S, \mathcal{C}) \rightarrow \text{Fun}_S(S, \mathcal{D}) \) sends \( Z \) to the desired object \( Y \) of \( \text{Fun}_S(S, \mathcal{D}) \).

We have a canonical equivalence \( \text{Fun}_S(S, S \times_{\mathcal{D}(1)} \mathcal{D}^{\Delta^1}) \simeq \text{Fun}_S(S, \mathcal{D}) \) over \( \text{Fun}_S(S, \mathcal{D}) \) so that the image of the final object of the category \( \text{Fun}_S(S, S \times_{\mathcal{D}(1)} \mathcal{D}^{\Delta^1}) \) under the functor \( \text{Fun}_S(S, S \times_{\mathcal{D}(1)} \mathcal{D}^{\Delta^1}) \rightarrow \text{Fun}_S(S, \mathcal{D}) \rightarrow \text{Fun}_S(S, \mathcal{D}[0]) \) is \( X \).

So the functor \( \text{Fun}_S(S, \mathcal{D}) \rightarrow \mathcal{W}(S, \mathcal{D}) \) induces a retract on equivalence classes.

Lemma 7.2 states that we have a canonical equivalence \( \mathcal{C} \simeq \mathcal{D}/_{\mathcal{W}} \) over \( \mathcal{D} \).

**Proposition 7.5.** Let \( \kappa \) be a strongly inaccessible cardinal.

There is a canonical equivalence

\[
\text{Cocart}(\kappa) \simeq \text{Map}_{\text{Cat}_\infty(\kappa)}(\text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa))
\]

of cartesian fibrations over \( \text{Cat}_\infty(\kappa) \) that induces on the fiber over every \( \kappa \)-small category \( \mathcal{C} \) the canonical equivalence \( \text{Cat}_\infty^{\text{Cocart}}(\kappa) \simeq \text{Fun}(\mathcal{C}, \text{Cat}_\infty(\kappa)) \).

Consequently this equivalence restricts to an equivalence

\[
\mathcal{L}(\kappa) \simeq \text{Map}_{\text{Cat}_\infty(\kappa)}(\text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times S(\kappa))
\]

of cartesian fibrations over \( \text{Cat}_\infty(\kappa) \).

Proof. Let \( \kappa' > \kappa \) be a strongly inaccessible cardinal.

By Yoneda it is enough to show that for every \( \kappa' \)-small category \( S \) over \( \text{Cat}_\infty(\kappa) \) there is a bijection between equivalence classes of functors \( S \rightarrow \text{Map}_{\text{Cat}_\infty(\kappa)}(\text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa)) \) over \( \text{Cat}_\infty(\kappa) \) and equivalence classes of functors \( S \rightarrow \text{Cocart}(\kappa) \) over \( \text{Cat}_\infty(\kappa) \) such that for every functor \( \phi : T \rightarrow S \) of \( \kappa' \)-small categories over \( \text{Cat}_\infty(\kappa) \) the square

\[
\begin{array}{ccc}
\text{Func}_{\text{Cat}_\infty(\kappa)}(S, \text{Map}_{\text{Cat}_\infty(\kappa)}(\text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa))) & \longrightarrow & \text{Func}_{\text{Cat}_\infty(\kappa)}(S, \text{Cocart}(\kappa)) \\
\downarrow & & \downarrow \\
\text{Func}_{\text{Cat}_\infty(\kappa)}(T, \text{Map}_{\text{Cat}_\infty(\kappa)}(\text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa))) & \longrightarrow & \text{Func}_{\text{Cat}_\infty(\kappa)}(T, \text{Cocart}(\kappa))
\end{array}
\]

commutes on equivalence classes.

We have a canonical equivalence

\[
\text{Cat}_\infty(\kappa')/_{\text{Cat}_\infty(\kappa)}(-, \text{Map}_{\text{Cat}_\infty(\kappa)}(\text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa))) \simeq \\
\text{Cat}_\infty(\kappa')/_{\text{Cat}_\infty(\kappa)}(- \times_{\text{Cat}_\infty(\kappa)} \text{ll}(\kappa), \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa)) \simeq \\
\text{Cat}_\infty(\kappa')(\text{ll}(\kappa), \text{Cat}_\infty(\kappa))
\]

of functors \( \text{Cat}_\infty(\kappa')/_{\text{Cat}_\infty(\kappa)} \rightarrow \text{S}(\kappa') \).

Consequently it is enough to see that for every \( \kappa' \)-small category \( \varphi : S \rightarrow \text{Cat}_\infty(\kappa) \) over \( \text{Cat}_\infty(\kappa) \) there is a bijection between equivalence classes of functors \( S \times_{\text{Cat}_\infty(\kappa)} \text{ll}(\kappa) \rightarrow \text{Cat}_\infty(\kappa) \) and equivalence classes of functors \( S \rightarrow \text{Cocart}(\kappa) \).
over $\text{Cat}_\infty(\kappa)$ such that for every functor $\phi : T \to S$ of $\kappa'$-small categories over $\text{Cat}_\infty(\kappa)$ the square

$$
\begin{array}{ccc}
\text{Fun}(S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa), \text{Cat}_\infty(\kappa)) & \longrightarrow & \text{Fun}_{\text{Cat}_\infty(\kappa)}(S, \text{Cocart}(\kappa)) \\
\downarrow & & \downarrow \\
\text{Fun}(T \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa), \text{Cat}_\infty(\kappa)) & \longrightarrow & \text{Fun}_{\text{Cat}_\infty(\kappa)}(T, \text{Cocart}(\kappa))
\end{array}
$$

commutes on equivalence classes.

A functor $S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to \text{Cat}_\infty(\kappa)$ is classified by a cocartesian fibration $S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ with $\kappa$-small fibers.

Being a right fibration the forgetful functor $\text{Cat}_\infty(\kappa')/S \to \text{Cat}_\infty(\kappa')$ induces an equivalence $(\text{Cat}_\infty(\kappa')/S)_{/\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to (\text{Cat}_\infty(\kappa')/S)_{/\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$, under which a functor $\psi : X \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ corresponds to a functor $\beta : X \to S$ of cocartesian fibrations over $S$.

A map $Y \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ of cocartesian fibrations over $S$ is itself a cocartesian fibration if and only if it induces on the fiber over every $s \in S$ a cocartesian fibration, whose cocartesian morphisms are preserved by the induced functors on the fibers of $Y \to S$, in other words if and only if it classifies a natural transformation $S \to \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))$ of functors $S \to \text{Cat}_\infty(\kappa)$ with target $\varphi$ that factors through the subcategory $\text{Cocart}(\kappa) \subset \text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))$.

Consequently $\psi : X \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ is a cocartesian fibration with $\kappa$-small fibers if and only if $\beta : X \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ is a map of cocartesian fibrations over $S$ classifying a functor $S \to \text{Cocart}(\kappa)$ over $\text{Cat}_\infty(\kappa)$.

Given a functor $\phi : T \to S$ of $\kappa'$-small categories over $\text{Cat}_\infty(\kappa)$ and a cocartesian fibration $X \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ classifying a functor $S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$, the composition $T \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to \text{Cat}_\infty(\kappa)$ is classified by the pullback of the cocartesian fibration $X \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ along the functor $T \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$.

Therefore if $\alpha$ denotes the natural transformation of functors $S \to \text{Cat}_\infty(\kappa)$ with target $\varphi$ corresponding to the functor $S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa)$ then $\alpha \circ \phi$ is the natural transformation of functors $T \to \text{Cat}_\infty(\kappa)$ with target $\varphi \circ \phi$ corresponding to the composition $T \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to \text{Cat}_\infty(\kappa)$.

So the functor $T \to \text{Cocart}(\kappa)$ over $\text{Cat}_\infty(\kappa)$ corresponding to the composition $T \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to \text{Cat}_\infty(\kappa)$ is the composition $T \to S \to \text{Cocart}(\kappa)$ of $\phi : T \to S$ and the functor $S \to \text{Cocart}(\kappa)$ over $\text{Cat}_\infty(\kappa)$ corresponding to the functor $S \times_{\text{Cat}_\infty(\kappa)} \mathcal{U}(\kappa) \to \text{Cat}_\infty(\kappa)$.

2. follows from the fact that a cocartesian fibration is a left fibration if and only if all its fibers are spaces.

\[\square\]

Remark 7.6. By proposition 7.3 we have a canonical fully faithful map $\mathcal{U}(\kappa) \subset \mathcal{R}(\kappa)$ of cocartesian fibrations over $\text{Cat}_\infty(\kappa)$ and by proposition 7.3 we have a canonical equivalence $\mathcal{L}(\kappa) \cong \text{Map}_{\text{Cat}_\infty(\kappa)}(\mathcal{U}(\kappa), \text{Cat}_\infty(\kappa) \times \mathcal{S}(\kappa))$ over $\text{Cat}_\infty(\kappa)$, whose pullback along $(\cdot)^{\text{rev}} : \text{Cat}_\infty(\kappa) \to \text{Cat}_\infty(\kappa)$ is a canonical equivalence

$$
\mathcal{R}(\kappa) \cong \text{Map}_{\text{Cat}_\infty(\kappa)}(\mathcal{U}(\kappa)^{\text{rev}}, \text{Cat}_\infty(\kappa) \times \mathcal{S}(\kappa))
$$
over $\text{Cat}_\infty(\kappa)$. So we obtain a canonical fully faithful map
$$\chi : \mathcal{U}(\kappa) \subset \text{Map}_{\text{Cat}_\infty(\kappa)}(\mathcal{U}(\kappa)^{\text{rev}}, \text{Cat}_\infty(\kappa) \times S(\kappa))$$
of cocartesian fibrations over $\text{Cat}_\infty(\kappa)$.

Let $E \to S$ be a cocartesian fibration classifying a functor $\phi : S \to \text{Cat}_\infty(\kappa)$ and $\mathcal{E}^{\text{rev}} \to S$ its fiberwise dual.

Pulling back $\chi$ along $\phi : S \to \text{Cat}_\infty(\kappa)$ we get a fully faithful map $\mathcal{E} \subset \mathcal{P}_\kappa(\mathcal{E}) := \text{Map}_S(\mathcal{E}^{\text{rev}}, S \times S(\kappa))$ of cocartesian fibrations over $S$.

This map of cocartesian fibrations over $S$ is adjoint to a functor
$$\alpha : \mathcal{E}^{\text{rev}} \times_S \mathcal{E} \to S(\kappa)$$
such that for every $s \in S$ the composition $(\mathcal{E}_s)^{\text{op}} \times \mathcal{E}_s \to \mathcal{E}^{\text{rev}} \times_S \mathcal{E} \to S(\kappa)$ is the mapping space functor of $\mathcal{E}_s$.

We call $\alpha$ the mapping space functor of $\mathcal{E} \to S$ relative to $S$.

**Theorem 7.7.** (Fundamental theorem of parametrized categories of sections)

Let $\kappa$ be a strongly inaccessible cardinal.

Let $R, S, T$ be $\kappa$-small categories, $T \to R$ a functor, $\alpha : X \to S \times T$ a map of cartesian fibrations over $S$ and $\beta : Y \to S \times T$ a map of cocartesian fibrations over $S$ corresponding to functors $F : S^{\text{op}} \to \text{Cat}_\infty(\kappa)_{/T}$ respectively $G : S \to \text{Cat}_\infty(\kappa)_{/T}$.

Assume that the composition $X \to S \times T \to S \times R$ is a flat functor so that $F$ induces a functor $S^{\text{op}} \to \text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}$.

The map $\text{Fun}_{S \times T}^{S \times R}(X, Y) \to S \times R$ of cocartesian fibrations over $S$ classifies the functor $S \xrightarrow{\left(F^{\text{op}}, G\right)} \left(\text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}\right)^{\text{op}} \times \text{Cat}_\infty(\kappa)_{/T} \xrightarrow{\text{Fun}_{T}^{\text{op}}(\alpha, \cdot)} \text{Cat}_\infty(\kappa)_{/R}$.

This implies the dual statement:

Assume that the composition $Y \to S \times T \to S \times R$ is a flat functor so that $G$ induces a functor $S^{\text{op}} \to \text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}$.

The map $\text{Fun}_{S \times T}^{S \times R}(Y, X) \to S \times R$ of cartesian fibrations over $S$ classifies the functor $S^{\text{op}} \xrightarrow{\left(G^{\text{op}}, F\right)} \left(\text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}\right)^{\text{op}} \times \text{Cat}_\infty(\kappa)_{/T} \xrightarrow{\text{Fun}_{T}^{\text{op}}(\cdot, \beta)} \text{Cat}_\infty(\kappa)_{/R}$.

**Proof.** We only prove the first statement, the second is dual to the first:

By the first part the map
$$\text{Fun}_{S \times T}^{S \times R}(X, Y)^{\text{op}} \xrightarrow{\text{Fun}_{S \times R}^{S^{\text{op}} \times T^{\text{op}}}(Y^{\text{op}}, X^{\text{op}})} S^{\text{op}} \times R^{\text{op}}$$
of cocartesian fibrations over $S^{\text{op}}$ classifies the functor $S^{\text{op}} \xrightarrow{\left(G^{\text{op}}, F\right)} \left(\text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}\right)^{\text{op}} \times \text{Cat}_\infty(\kappa)_{/T} \xrightarrow{\text{Fun}_{T}^{\text{op}}(\cdot, \beta)} \text{Cat}_\infty(\kappa)_{/R}$

being equivalent to the functor $S^{\text{op}} \xrightarrow{\left(G^{\text{op}}, F\right)} \left(\text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}\right)^{\text{op}} \times \text{Cat}_\infty(\kappa)_{/T} \xrightarrow{\text{Fun}_{T}^{\text{op}}(\cdot, \beta)} \text{Cat}_\infty(\kappa)_{/R}$.

Hence the map $\text{Fun}_{S \times T}^{S \times R}(Y, X) \to S \times R$ of cartesian fibrations over $S$ classifies the functor $S^{\text{op}} \xrightarrow{\left(G^{\text{op}}, F\right)} \left(\text{Cat}_\infty(\kappa)_{/T}^{\text{flat/R}}\right)^{\text{op}} \times \text{Cat}_\infty(\kappa)_{/T} \xrightarrow{\text{Fun}_{T}^{\text{op}}(\cdot, \beta)} \text{Cat}_\infty(\kappa)_{/R}$.

The proof will take place in 4 reduction steps:
1. We reduce to the case that \( R \) is contractible.

2. We reduce to the case that \( R \) and \( T \) are contractible.

3. We reduce to the case that \( R, T \) are contractible and \( Y \to S \) is equivalent over \( S \) to \( \Map_S(Y', S \times S(\kappa)) \) for some bicartesian fibration \( Y' \to S \) classify a functor \( G'' : S^{op} \to \Cat_{\infty}^R(\kappa) \).

4. We reduce to the case that \( R, T \) are contractible and \( Y \to S \) is equivalent over \( S \) to \( S \times S(\kappa) \), in which case we deduce the statement from proposition 7.3.

1. Denote \( \Psi \) the functor \( S \to \Cat_{\infty}(\kappa)/T \) classified by the map \( \Fun_{S \times T}^{S(\kappa)}(X, Y) \to S \times R \) of cocartesian fibrations over \( S \).

We want to see that there is an equivalence \( \Psi \simeq \Fun_{T}(\cdot, \cdot) \circ (F^{op}, G) \) of functors \( S \to \Cat_{\infty}(\kappa)/R \). Such an equivalence is represented by an equivalence

\[
\Cat_{\infty}(\kappa)/R(\B, \Psi(s)) \simeq \Cat_{\infty}(\kappa)/R(\B, \Fun_{T}(F(s), G(s)))
\]

natural in \( \B \in \Cat_{\infty}(\kappa)/R \) and \( s \in S \).

Therefore it is enough to find an equivalence

\[
\rho : \Fun_{R}(\B, \Psi(s)) \simeq \Fun_{T}(\B \times R F(s), G(s))
\]

natural in \( \B \in \Cat_{\infty}(\kappa)/R \) and \( s \in S \).

Assume that we have already proved the statement for \( R \) contractible.

Then \( \rho \) is classified by an equivalence

\[
\Fun_{\Cat_{\infty}(\kappa)/R}^{S \times R}(\Upsilon_{R} \times S, (\Cat_{\infty}(\kappa)/R)^{op} \times \Fun_{S \times T}^{\cdot}(X, Y)) \simeq \Fun_{\Cat_{\infty}(\kappa)/R}^{S \times R}(\Upsilon_{R} \times S, (\Cat_{\infty}(\kappa)/R)^{op} \times \Fun_{S \times T}(X, Y))
\]

over \( S' := (\Cat_{\infty}(\kappa)/R)^{op} \times S \).

But we have a canonical equivalence

\[
\Fun_{\Cat_{\infty}(\kappa)/R}^{S \times R}(\Upsilon_{R} \times S, (\Cat_{\infty}(\kappa)/R)^{op} \times \Fun_{S \times T}(X, Y)) \simeq \Fun_{S \times T}(\Upsilon_{R} \times S, \Fun_{S \times T}^{S \times R}( (\Cat_{\infty}(\kappa)/R)^{op} \times X, (\Cat_{\infty}(\kappa)/R)^{op} \times Y)) \simeq \Fun_{S \times T}(\Upsilon_{R} \times S, (\Cat_{\infty}(\kappa)/R)^{op} \times X, (\Cat_{\infty}(\kappa)/R)^{op} \times Y)
\]

over \( S' \).

We continue with 2:

We want to show that the cocartesian fibration \( \Fun_{S \times T}^{S}(X, Y) \to S \) classifies the functor \( S \xrightarrow{(F^{op}, G)} (\Cat_{\infty}(\kappa)/T)^{op} \times \Cat_{\infty}(\kappa)/T \xrightarrow{\Fun_{T}(\cdot, \cdot)} \Cat_{\infty}(\kappa) \).

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We have a pullback square

\[
\begin{array}{ccc}
\text{Fun}_T(-, -) \circ (F^{\text{op}}, G) & \longrightarrow & \text{Fun}(-, -) \circ (F^{\text{op}}, G) \\
\downarrow & & \downarrow \\
\text{Fun}_T(-, -) \circ (F^{\text{op}}, T) & \longrightarrow & \text{Fun}(-, -) \circ (F^{\text{op}}, T) 
\end{array}
\]

of functors $S \to \text{Cat}_\infty(\kappa)$, where the natural transformation $G \to T$ to the constant functor $S \to \text{Cat}_\infty(\kappa)_T$ with image $T$ is the unique one.

If we assume that the statement holds for $R$ and $T$ contractible, this square is classified by the canonical pullback square

\[
\begin{array}{ccc}
\text{Fun}_{S \times T}^{\text{op}}(X, Y) & \longrightarrow & \text{Map}_S(X, Y) \\
\downarrow & & \downarrow \\
S & \longrightarrow & \text{Map}_S(X, S \times T) 
\end{array}
\]

of cocartesian fibrations over $S$.

3.: The statement we want to prove is equivalent to the following one:

For every cartesian fibration $X \to S$ and every cocartesian fibration $Y \to S$ the cartesian fibration $\text{Map}_S(X, Y^{\text{rev}}) \to S$ classifies the functor $S \xrightarrow{(F^{\text{op}}, (-)^{\text{op}} \circ G)} \text{Cat}_\infty(\kappa)^{\text{op}} \times \text{Cat}_\infty(\kappa)$, where $Y^{\text{rev}} \to S$ denotes the fiberwise dual of the cocartesian fibration $Y \to S$.

To show this, we are free to enlarge $Y \to S$ in the following way:

Let $Z \to S$ be a cocartesian fibration classifying a functor $H : S \to \text{Cat}_\infty(\kappa)$ and let $Y \to Z$ be a fully faithful map of cocartesian fibrations over $S$ classifying a component-wise fully faithful natural transformation $G \to H$ of functors $S \to \text{Cat}_\infty(\kappa)$.

If the cocartesian fibration $\text{Map}_S(X, Z) \to S$ classifies the functor $S \xrightarrow{(F^{\text{op}}, H)} \text{Cat}_\infty(\kappa)^{\text{op}} \times \text{Cat}_\infty(\kappa)$, the cocartesian fibration $\text{Map}_S(X, Y) \to S$ classifies the functor $S \xrightarrow{(F^{\text{op}}, G)} \text{Cat}_\infty(\kappa)^{\text{op}} \times \text{Cat}_\infty(\kappa)$.

This follows from the fact that the induced fully faithful map $\text{Map}_S(X, Y) \subset \text{Map}_S(X, Z)$ of cocartesian fibrations over $S$ and the fully faithful map of cocartesian fibrations over $S$ classified by the component-wise fully faithful natural transformation $\text{Fun}(-, -) \circ (F^{\text{op}}, G) \to \text{Fun}(-, -) \circ (F^{\text{op}}, H)$ have the same essential image.

By remark 7.6 we have fully faithful maps $Y \subset \mathcal{P}_\kappa^{\text{FS}}(Y)$ and $\mathcal{P}_\kappa^{\text{FS}}(Y)^{\text{rev}} \subset \mathcal{P}_\kappa^{\text{FS}}(\mathcal{P}_\kappa^{\text{FS}}(Y)^{\text{rev}}) = \text{Map}_S(\mathcal{P}_\kappa^{\text{FS}}(Y), S \times \delta(\kappa'))$ of cocartesian fibrations over $S$ and so a fully faithful map $Y^{\text{rev}} \subset \mathcal{P}_\kappa^{\text{FS}}(Y)^{\text{rev}} \subset \text{Map}_S(\mathcal{P}_\kappa^{\text{FS}}(Y), S \times \delta(\kappa'))$ of cocartesian fibrations over $S$.

Consequently we can reduce to the case that $Y \to S$ is equivalent over $S$ to $\text{Map}_S(Y', S \times \delta(\kappa'))$ for some bicartesian fibration $Y' \to S$ classifying a functor $G' : S \to \text{Cat}_\infty^L(\kappa)$ respectively a functor $G'' : S^{\text{op}} \xrightarrow{\text{G}} \text{Cat}_\infty^L(\kappa)^{\text{op}} \simeq \text{Cat}_\infty^R(\kappa)$.

4.: Assume we have shown the statement for $Y = S \times \delta(\kappa)$. 

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Then the cartesian fibration $\text{Map}_S(Y', S \times S(\kappa)) \to S$ classifies the functor $S \stackrel{G^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, S(\kappa))}{\longrightarrow} \text{Cat}_\infty(\kappa)$ and the cartesian fibration $\text{Map}_S(X, \text{Map}_S(Y', S \times S(\kappa))) \simeq \text{Map}_S(X \times S Y', S \times S(\kappa)) \to S$ classifies the functor $S \stackrel{(F^{\text{op}}, G^{\text{op}})}{\longrightarrow} \text{Cat}_\infty(\kappa) \times \text{Cat}_\infty(\kappa) \stackrel{\text{Fun}(-, S(\kappa)) \times \text{Fun}(-, S(\kappa))}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \times \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, -)}{\longrightarrow} \text{Cat}_\infty(\kappa)$. It remains to show that the cartesian fibration $\text{Map}_S(X, S \times S(\kappa)) \to S$ classifies the functor $S \stackrel{F^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, S(\kappa))}{\longrightarrow} \text{Cat}_\infty(\kappa)$.

To do so, we show that for every category $B$ the cartesian fibration $\text{Map}_S(X, S \times B) \to S$ classifies the functor $S \stackrel{F^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, B)}{\longrightarrow} \text{Cat}_\infty(\kappa)$. This is equivalent to the dual statement:

For every category $B$ the cartesian fibration $\text{Map}_S(Y, S \times B) \to S$ classifies the functor $S \stackrel{G^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, B)}{\longrightarrow} \text{Cat}_\infty(\kappa)$:

If this holds, the cartesian fibration $\text{Map}_{S^{\text{op}}}(X^{\text{op}}, S^{\text{op}} \times B^{\text{op}}) \to S^{\text{op}}$ classifies the functor $S \stackrel{(\text{Fop})^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, B^{\text{op}})}{\longrightarrow} \text{Cat}_\infty(\kappa)$ so that the cartesian fibration $\text{Map}_S(X, S \times B) \simeq \text{Map}_{S^{\text{op}}}(X^{\text{op}}, S^{\text{op}} \times B^{\text{op}})^{\text{op}} \to S$ classifies the functor $S \stackrel{(\text{Fop})^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, B^{\text{op}})}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \longrightarrow \text{Cat}_\infty(\kappa)$, i.e. the functor $S \stackrel{F^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, -)}{\longrightarrow} \text{Cat}_\infty(\kappa)$.

By the remark from above it is enough to show that for every category $B$ the cartesian fibration $\text{Map}_S(Y, S \times B) \to S$ classifies the functor $S \stackrel{G^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, B)}{\longrightarrow} \text{Cat}_\infty(\kappa)$.

Assume that we have already shown the statement for $B = S(\kappa)$. Then the cartesian fibration $\text{Map}_S(X, S \times S(\kappa)) = \text{Map}_S((S \times S(\kappa)) \times S X, S \times S(\kappa)) \to S$ classifies the functor $S \stackrel{G^{\text{op}}}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, S(\kappa))}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, S(\kappa))}{\longrightarrow} \text{Cat}_\infty(\kappa)^{\text{op}} \longrightarrow \text{Cat}_\infty(\kappa)$.

So it finally remains to show the statement for $B = S(\kappa)$. By proposition 7.6 we have a canonical equivalence $\mathcal{L}(\kappa) \simeq \text{Map}_{\text{Cat}_\kappa(\kappa)}(1 \text{II}(\kappa), \text{Cat}_\kappa(\kappa) \times S(\kappa))$ of cartesian fibrations over $\text{Cat}_\kappa(\kappa)$, whose pullback along $G : S \to \text{Cat}_\kappa(\kappa)$ is a canonical equivalence $G^*(\mathcal{L}(\kappa)) \simeq \text{Map}_{\text{Cat}_\kappa(\kappa)}(Y, S \times S(\kappa))$ of cartesian fibrations over $S$.

By ... the cartesian fibration $\mathcal{L}(\kappa) \to \text{Cat}_\kappa(\kappa)$ classifies the functor $\text{Fun}(-, S(\kappa)) : S^{\text{op}} \to \text{Cat}_\kappa(\kappa)$ so that the cartesian fibration $G^*(\mathcal{L}(\kappa)) \simeq \text{Map}_S(Y, S \times S(\kappa)) \to S$ classifies the functor $S^{\text{op}} \stackrel{G^{\text{op}}}{\longrightarrow} \text{Cat}_\kappa(\kappa)^{\text{op}} \stackrel{\text{Fun}(-, S(\kappa))}{\longrightarrow} \text{Cat}_\kappa(\kappa)$. 

\[ \square \]

Remark 7.8. Let $\phi : C \to T$ be a flat functor between $\kappa$-small categories.

Then by definition of flatness the functor $\xi : \text{Cat}_\kappa(\kappa)_{/S} \stackrel{T_{\text{Eq}-}}{\longrightarrow} \text{Cat}_\kappa(\kappa)_{/T} \stackrel{G_{\text{Eq}-}}{\longrightarrow} \text{Cat}_\kappa(\kappa)_{/T}$ admits a right adjoint $\text{Fun}^S_{/T}(\xi, -) : \text{Cat}_\kappa(\kappa)_{/T} \to \text{Cat}_\kappa(\kappa)_{/S}$.  

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Denote $\text{Cat}_\infty(\kappa)^{\text{flat}/S}_{/T} \subset \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$ the full subcategory spanned by the categories over $T$ that are flat over $S$.

Then we have a functor $\text{Fun}^T_{/S}(-,-) : (\text{Cat}_\infty(\kappa)^{\text{flat}/S}_{/T})^{\text{op}} \times \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T} \to \text{Cat}_\infty(\kappa)^{\text{flat}}_{/S}$.

determined by a canonical equivalence

$$\text{Fun}_S(\mathcal{B}, \text{Fun}^T_{/S}(\mathcal{C}, \mathcal{D})) \cong \text{Fun}_{T}(\mathcal{B} \times_S \mathcal{C}, \mathcal{D})$$

natural in $\kappa$-small categories $\mathcal{C}, \mathcal{D}$ over $T$ with $\mathcal{C}$ flat over $S$ and a $\kappa$-small category $\mathcal{B}$ over $S$.

Let $R, S, T$ be $\kappa$-small categories, $T \to R$ a functor, $\alpha : X \to S \times T$ a map of cartesian fibrations over $S$ corresponding to a functor $F : S^{\text{op}} \to \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$.

Assume that the composition $X : S \times T \to S \times R$ is a flat functor so that $F$ induces a functor $S^{\text{op}} \to \text{Cat}_\infty(\kappa)^{\text{flat}/R}_{/T}$.

Let $\mathcal{O}^\otimes$ be a $\kappa$-small operad and $\mathcal{D}^\otimes \to T \times \mathcal{O}^\otimes$ a $\mathcal{O}^\otimes$-monoidal category over $T$ classified by a $\mathcal{O}^\otimes$-monoid $\phi$ of $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$.

The composition $\times : \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T} \to \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$ is adjoint to a functor $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T} \to \text{Fun}(S, \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T})$

$$\text{Cat}_\infty(\kappa)^{\text{flat}}_{/R}$$

is adjoint to a functor $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T} \to \text{Fun}(S, \text{Cat}_\infty(\kappa)^{\text{flat}}_{/T})$

$$\text{Cat}_\infty(\kappa)^{\text{flat}}_{/R}$$

is a flat functor so that $F$ induces a functor $S^{\text{op}} \to \text{Cat}_\infty(\kappa)^{\text{flat}/R}_{/T}$.

Then $\phi'$ is classified by $\mathcal{O}^\otimes$-monoidal category

$$\text{Fun}^{S \times R}_{S \times T} (X, \mathcal{D}^\otimes)_\otimes := \text{Fun}^{S \times R \times \mathcal{O}^\otimes}_{S \times T \times \mathcal{O}^\otimes} (X \times \mathcal{O}^\otimes, S \times \mathcal{D}^\otimes) \to S \times R \times \mathcal{O}^\otimes$$

over $S \times R$.

Now we study the situation for $\mathcal{O}^\otimes = \mathcal{LM}^\otimes$.

Let $\mathcal{M}^\otimes \to T \times \mathcal{LM}^\otimes$ be a $\mathcal{LM}^\otimes$-monoidal category over $T$ classifying a $\mathcal{LM}^\otimes$-monoid $\phi$ of $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$ that exhibits a category $\mathcal{B}$ over $T$ as a left module over a monoidal category $\mathcal{C}$ over $R$ with respect to the canonical left module structure on $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$ over $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/R}$.

Then $\phi'$ is classified by the $\mathcal{LM}^\otimes$-monoidal category

$$\text{Fun}^{S \times R}_{S \times T} (X, \mathcal{M}^\otimes)_\otimes := \text{Fun}^{S \times R \times \mathcal{LM}^\otimes}_{S \times T \times \mathcal{LM}^\otimes} (X \times \mathcal{LM}^\otimes, S \times \mathcal{M}^\otimes) \to S \times R \times \mathcal{LM}^\otimes$$

over $S \times R$ that exhibits the category $\text{Fun}^{S \times R}_{S \times T} (X, \mathcal{B})$ over $S \times R$ as a left module over the monoidal category

$$\text{Map}_{S \times R}(X, \mathcal{O}^\otimes)_\otimes := \text{Map}_{S \times R \times \mathcal{Ass}^\otimes}(X \times \mathcal{Ass}^\otimes, S \times \mathcal{O}^\otimes) \cong$$

$$\text{Fun}^{S \times R}_{S \times T}(X \times T \times R \mathcal{O}^\otimes)_\otimes := \text{Fun}^{S \times R \times \mathcal{Ass}^\otimes}_{S \times T \times \mathcal{Ass}^\otimes}(X \times \mathcal{Ass}^\otimes, S \times (T \times R \mathcal{O}^\otimes)) \
\to S \times R \times \mathcal{Ass}^\otimes$$

over $S \times R$.

By $\ldots$ the functor $\text{Cat}_\infty(\kappa)^{\text{flat}/R}_{/T} \to \text{Fun}((\text{Cat}_\infty(\kappa)^{\text{flat}/R}_{/T})^{\text{op}}, \text{Cat}_\infty(\kappa)^{\text{flat}}_{/R})$ is lax $\text{Cat}_\infty(\kappa)^{\text{flat}/R}_{/T}$-linear and thus also the functor $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T} \to \text{Cat}_\infty(\kappa)^{\text{flat}}_{/S \times R}$ is lax $\text{Cat}_\infty(\kappa)^{\text{flat}}_{/T}$-linear and so sends $\phi$ to a canonical left module structure on $\mathcal{B}' :=$
Fun_{S\times T}^S(X, S \times B) \to S \times R over the monoidal category $S \times \mathcal{C}^\otimes \to S \times R \times \mathcal{Ass}^\otimes$
over $S \times R$ that is classified by the pullback of the $L\mathcal{M}^\otimes$-monoidal category
Fun_{S\times T}^S(X, \mathcal{M})^\otimes \over S \times R along the monoidal diagonal functor

$$
\delta : S \times \mathcal{C}^\otimes = \text{Map}_{S \times R}(S \times R, \mathcal{C})^\otimes = \text{Map}_{S \times R \times \mathcal{Ass}^\otimes}(S \times R \times \mathcal{Ass}^\otimes, S \times \mathcal{C}^\otimes) \\
\to \text{Map}_{S \times R}(X, \mathcal{C})^\otimes = \text{Map}_{S \times R \times \mathcal{Ass}^\otimes}(X \times \mathcal{Ass}^\otimes, S \times \mathcal{C}^\otimes)
$$
over $S \times R$.

Moreover the induced functor $\text{Alg}^S(\delta)$ over $S$ is canonically equivalent over $S \times \mathcal{C}$

$$
\delta' : S \times \text{Alg}^{R}(\mathcal{C}) \approx \text{Map}_{S \times R}(S \times R, S \times \text{Alg}^{R}(\mathcal{C})) \to \text{Map}_{S \times R}(X, S \times \text{Alg}^{R}(\mathcal{C}))
$$
over $S \times R$.

So by remark 2.18 we have a canonical equivalence

$$
\text{LMod}^{S\times R}(\delta'^*(\mathcal{B}')) \approx \delta'^*(\text{Fun}^{S\times T}_S(X, S \times \text{LMod}^{T}(\mathcal{B})))
$$
over $S \times \text{Alg}^{R}(\mathcal{C})$. 

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7.2 Appendix B: Lurie-enriched category theory

In this subsection we will show that every operad $\mathcal{M}^\circ \to \mathcal{L}^\circ$ over $\mathcal{L}^\circ$ embeds into a $\mathcal{L}^\circ$-monoidal category $\mathcal{M}^\circ \to \mathcal{L}^\circ$ that exhibits a category $\mathcal{M}'$ as enriched over a monoidal category $\mathcal{C}' := \mathcal{Ass}^{\bowtie} \times_{\mathcal{L}^{\bowtie}} \mathcal{M}^{\bowtie}$.

Moreover the operad $\mathcal{M}^\circ \to \mathcal{L}^\circ$ over $\mathcal{L}^\circ$ exhibits a category $\mathcal{M}$ as enriched over a locally cocartesian fibration of operads $\mathcal{C}^\circ := \mathcal{Ass}^{\bowtie} \times_{\mathcal{L}^{\bowtie}} \mathcal{M}^{\bowtie}$ if and only if $\mathcal{M}$ is closed in $\mathcal{M}^{\bowtie}$ under morphism objects, i.e. the morphism object in $\mathcal{C}'$ of every objects $X, Y$ of $\mathcal{M} \subset \mathcal{M}'$ belongs to $\mathcal{C}$.

More generally, we will show that every cocartesian $\mathcal{S}$-family $\mathcal{M}^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ of operads over $\mathcal{L}^\circ$ for a $\kappa$-small category $\mathcal{S}$ embeds into a cocartesian $\mathcal{S}$-family $\mathcal{M}^{\bowtie} \to \mathcal{L}^\circ \times \mathcal{S}$ of $\mathcal{L}^\circ$-monoidal categories that is a cocartesian $\mathcal{S}$-family of categories enriched over $\mathcal{C}^{\bowtie} := \mathcal{Ass}^{\bowtie} \times_{\mathcal{L}^{\bowtie}} \mathcal{M}^{\bowtie}$.

The cocartesian $\mathcal{S}$-family $\mathcal{M}^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ of operads over $\mathcal{L}^\circ$ is a cocartesian $\mathcal{S}$-family of categories enriched over $\mathcal{C}^{\bowtie} := \mathcal{Ass}^{\bowtie} \times_{\mathcal{L}^{\bowtie}} \mathcal{M}^{\bowtie}$ if and only if $\mathcal{C}^\circ \to \mathcal{Ass}^{\bowtie} \times \mathcal{S}$ is a locally cocartesian fibration and $\mathcal{M}^{\bowtie}$ is closed in $\mathcal{M}^{\bowtie}$ under morphism objects, i.e. for every $s \in \mathcal{S}$ the morphism object in $\mathcal{C}'$ of every objects $X, Y$ of $\mathcal{M} \subset \mathcal{M}'$ belongs to $\mathcal{C}$.

Let $\mathcal{M}^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ be a cocartesian $\mathcal{S}$-family of operads over $\mathcal{L}^\circ$. Set $\mathcal{M} := \{m\} \times_{\mathcal{L}^{\bowtie}} \mathcal{M}^{\bowtie}$ and $\mathcal{C}^\circ := \mathcal{Ass}^{\bowtie} \times_{\mathcal{L}^{\bowtie}} \mathcal{M}^{\bowtie}$.

Denote $\text{Env}_{\mathcal{L}^{\bowtie}}(\mathcal{M})^\circ : \mathcal{L}^\circ \times \mathcal{S}$ the enveloping cocartesian $\mathcal{S}$-family of $\mathcal{L}^\circ$-monoidal categories (see remark [7.7] for more details) that exhibits the functor $\text{Env}_{\mathcal{L}^{\bowtie}}(\mathcal{M}) := \{m\} \times_{\mathcal{L}^{\bowtie}} \text{Env}_{\mathcal{L}^{\bowtie}}(\mathcal{M})^\circ \to \mathcal{S}$ as left module in $\text{Cat}_{\infty}(\kappa)_{/\mathcal{S}}$ over the monoidal category $\text{Env}_{\mathcal{Ass}^{\bowtie}}(\mathcal{C})^\circ : \mathcal{Ass}^{\bowtie} \times \mathcal{S}$ over $\mathcal{S}$ according to lemma [7.10].

We first show in lemma [7.11] that $\mathcal{M}^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ is a cocartesian $\mathcal{S}$-family of categories enriched over $\mathcal{C}^\circ$ if and only if $\mathcal{C}^\circ \to \mathcal{Ass}^{\bowtie} \times \mathcal{S}$ is a locally cocartesian fibration and for every $s \in \mathcal{S}$ two objects $X, Y$ of $\mathcal{M}_s$ admit a morphism object that is preserved by the full inclusion $\mathcal{M}^\circ_s \subset \text{Env}_{\mathcal{L}^{\bowtie}}(\mathcal{M})^\circ_s$ of operads over $\mathcal{L}^\circ$.

Given a cocartesian $\mathcal{S}$-family $\mathcal{N}^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ of $\mathcal{L}^\circ$-monoidal categories that exhibits a category $\mathcal{D}$ over $\mathcal{S}$ as a left module in $\text{Cat}_{\text{cocart}}(\kappa)_{/\mathcal{S}}$ over some monoidal category $\mathcal{B} = \mathcal{Ass}^{\bowtie} \times \mathcal{S}$ over $\mathcal{S}$ denote $\mathcal{P}_x(\mathcal{N})^\circ : \mathcal{L}^\circ \times \mathcal{S}$ the Day-convolution cocartesian $\mathcal{S}$-family of $\mathcal{L}^\circ$-monoidal categories defined by applying fiberwise the lax symmetric monoidal functor $\mathcal{P}_x : \text{Cat}_{\infty}(\kappa) \to \text{Cat}_{\infty}(\kappa')$ that takes presheaves.

So $\mathcal{P}_x(\mathcal{N})^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ exhibits $\mathcal{P}_x(\mathcal{D}) \to \mathcal{S}$ as a left module in $\text{Cat}_{\text{cocart}}(\kappa)_{/\mathcal{S}}$ over the monoidal category $\mathcal{P}_x(\mathcal{B})^\circ : \mathcal{Ass}^{\bowtie} \times \mathcal{S}$ over $\mathcal{S}$.

Moreover $\mathcal{P}_x(\mathcal{N})^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ is a cocartesian $\mathcal{S}$-family of categories enriched over $\mathcal{P}_x(\mathcal{B})^\circ$.

We have a Yoneda-embedding map $\mathcal{N}^\circ \subset \mathcal{P}_x(\mathcal{N})^\circ$ of cocartesian $\mathcal{S}$-families of $\mathcal{L}^\circ$-monoidal categories that preserves and reflects (fiberwise) morphism objects by lemma [7.12] as for every $s \in \mathcal{S}$ the category $\mathcal{P}_x(\mathcal{B}_s)$ is generated by $\mathcal{B}_s$ under $\kappa$-small colimits.

Thus $\mathcal{M}^\circ \to \mathcal{L}^\circ \times \mathcal{S}$ is a cocartesian $\mathcal{S}$-family of categories enriched over $\mathcal{C}^\circ$ if and only if $\mathcal{C}^\circ \to \mathcal{Ass}^{\bowtie} \times \mathcal{S}$ is a locally cocartesian fibration and for every $s \in \mathcal{S}$ every two objects $X, Y$ of $\mathcal{M}_s$ admit a morphism object that is preserved by the embedding $\mathcal{M}^\circ_s \subset \text{Env}_{\mathcal{L}^{\bowtie}}(\mathcal{M})^\circ_s \subset \mathcal{P}_x(\text{Env}_{\mathcal{L}^{\bowtie}}(\mathcal{M}))^\circ_s$ of operads over $\mathcal{L}^\circ$.
So every cocartesian S-family $M^\otimes \to LM^\otimes \times S$ of operads over $LM^\otimes$ embeds into the cocartesian S-family $M^\otimes := \mathcal{P}_\kappa(Env_{LM^\otimes}(M))^\otimes \to LM^\otimes \times S$ of $LM^\otimes$-monoidal categories that is a cocartesian S-family of categories enriched over $\mathcal{P}_\kappa(Env_{Ass}(C))^\otimes$.

The cocartesian S-family $M^\otimes \to LM^\otimes \times S$ of operads over $LM^\otimes$ is a cocartesian S-family of categories enriched over $\mathcal{C}^\otimes$ if and only if $\mathcal{C}^\otimes \to Ass^\otimes \times S$ is a locally cocartesian fibration and $M^\otimes$ is closed in $M^\otimes$ under morphism objects.

This follows from the following lemmata:

**Remark 7.9.** In the following we will use the enveloping cocartesian S-family of $\mathcal{O}^\otimes$-monoidal categories of a cocartesian $S$-family of operads over $\mathcal{O}^\otimes$ for a given operad $\mathcal{O}^\otimes$ and a category $S$, defined in the following way:

By ... for every operad $\mathcal{O}^\otimes$ the subcategory inclusion $Op_{\infty/Ass}(\kappa) \subset Op_{\infty}(\kappa)/_{\mathcal{O}^\otimes}$ admits a left adjoint $Env_{\mathcal{O}^\otimes}: Op_{\infty}(\kappa)/_{\mathcal{O}^\otimes} \to Op_{\infty/Ass}(\kappa)$.

By ... for every operad $\mathcal{C}^\otimes$ over $\mathcal{O}^\otimes$ we have a canonical equivalence

$$Env_{\mathcal{O}^\otimes}(\mathcal{C})^\otimes \simeq \text{Act}(\mathcal{O}^\otimes) \times \text{Fun}(\{0\},_{\mathcal{O}^\otimes}) \mathcal{C}^\otimes$$

over $\text{Fun}(\{1\},_{\mathcal{O}^\otimes})$ and the unit $\mathcal{C}^\otimes \to Env_{\mathcal{O}^\otimes}(\mathcal{C})^\otimes \simeq \text{Act}(\mathcal{O}^\otimes) \times \text{Fun}(\{0\},_{\mathcal{O}^\otimes}) \mathcal{C}^\otimes$ is the pullback of the diagonal embedding $\mathcal{O}^\otimes \to \text{Act}(\mathcal{O}^\otimes) \subset \text{Fun}(\Delta^1, \mathcal{O}^\otimes)$ along the functor $\text{Act}(\mathcal{O}^\otimes) \times \text{Fun}(\{0\},_{\mathcal{O}^\otimes}) \mathcal{C}^\otimes \to \text{Act}(\mathcal{O}^\otimes)$ and is thus fully faithful.

For every $\kappa$-small category $S$ we get an induced adjunction $Fun(S, Env_{\mathcal{O}^\otimes}) \dashv Fun(S, Op_{\infty/Ass}(\kappa))$

The image of the subcategory $Fun(S, Op_{\infty}(\kappa)/_{\mathcal{O}^\otimes}) \subset Fun(S, \text{Cat}_{\infty}(\kappa)/_{\mathcal{O}^\otimes})$ respectively $Fun(S, Op_{\infty/Ass}(\kappa)) \subset Fun(S, \text{Cat}_{\infty}(\kappa)/_{\mathcal{O}^\otimes})$ under the canonical equivalence $Fun(S, \text{Cat}_{\infty}(\kappa)/_{\mathcal{O}^\otimes}) \simeq \text{Cat}_{\infty/Ass}(\kappa)/_{S, \mathcal{O}^\otimes}$ is the category of cocartesian $S$-families of operads over $\mathcal{O}^\otimes$ respectively the category of cocartesian $S$-families of $\mathcal{O}^\otimes$-monoidal categories.

Thus the subcategory inclusion from the category of cocartesian $S$-families of $\mathcal{O}^\otimes$-monoidal categories into the category of cocartesian $S$-families of operads over $\mathcal{O}^\otimes$ admits a left adjoint, also denoted by $Env_{\mathcal{O}^\otimes}$.

For every cocartesian $S$-family $\mathcal{C}^\otimes \to S \times \mathcal{O}^\otimes$ of operads over $\mathcal{O}^\otimes$ the unit $\mathcal{C}^\otimes \to Env_{\mathcal{O}^\otimes}(\mathcal{C})^\otimes$ is fully faithful as it induces on the fiber over every $s \in S$ the fully faithful unit $\mathcal{C}_s^\otimes \to Env_{\mathcal{O}^\otimes}(\mathcal{C}_s^\otimes)$.

**Lemma 7.10.** Let $S$ be a $\kappa$-small category and $M^\otimes \to LM^\otimes \times S$ a cocartesian $S$-family of operads over $LM^\otimes$.

Set $M := \{m\} \times_{LM^\otimes} M^\otimes$ and $\mathcal{C}^\otimes := Ass^\otimes \times_{LM^\otimes} M^\otimes$.

Denote $Env_{LM^\otimes}(M)^\otimes \to LM^\otimes \times S$ the enveloping cocartesian $S$-family of $LM^\otimes$-monoidal categories and $Env_{Ass^\otimes}(\mathcal{C})^\otimes \to Ass^\otimes \times S$ the enveloping cocartesian $S$-family of monoidal categories.

Denote $\zeta$ the canonical map

$$Env_{Ass^\otimes}(\mathcal{C})^\otimes \to Ass^\otimes \times_{LM^\otimes} Env_{LM^\otimes}(M)^\otimes$$

of cocartesian $S$-families of monoidal categories adjoint to the map $\mathcal{C}^\otimes = Ass^\otimes \times_{LM^\otimes} M^\otimes \subset Ass^\otimes \times_{LM^\otimes} Env_{LM^\otimes}(M)^\otimes$ of cocartesian $S$-families of operads over $Ass^\otimes$.

$\zeta$ is an equivalence.
Proof. As $\zeta$ is a map of cocartesian $S$-families of operads over $\text{Ass}^\otimes$, it is an equivalence if it induces on the fiber over every $s \in S$ an equivalence.

$\zeta$ induces on the fiber over every $s \in S$ the monoidal functor

$$\text{Env}_{\text{Ass}^\otimes}(\mathcal{C}_s^\otimes) \to \text{Ass}^\otimes \times_{\text{Lm}^\otimes} \text{Env}_{\text{Lm}^\otimes}(M_s^\otimes)^\otimes$$

adjoint to the map $\mathcal{C}_s^\otimes = \text{Ass}^\otimes \times_{\text{Lm}^\otimes} M_s^\otimes \subset \text{Ass}^\otimes \times_{\text{Lm}^\otimes} \text{Env}_{\text{Lm}^\otimes}(M_s^\otimes)^\otimes$ of operads over $\text{Ass}^\otimes$.

Consequently we can reduce to the case that $S$ is contractible. In this case by ... we have canonical equivalences

$$\text{Env}_{\text{Lm}^\otimes}(M)^\otimes \simeq \text{Act}(\text{Lm}^\otimes) \times_{\text{Fun}(\{0\},\text{Lm}^\otimes)} M^\otimes$$

over $\text{Fun}(\{1\},\text{Lm}^\otimes)$ and

$$\text{Env}_{\text{Ass}^\otimes}(\mathcal{C})^\otimes \simeq \text{Act}(\text{Ass}^\otimes) \times_{\text{Fun}(\{0\},\text{Ass}^\otimes)} \mathcal{C}^\otimes$$

over $\text{Fun}(\{1\},\text{Ass}^\otimes)$.

We have a canonical equivalence

$$\text{Act}(\text{Ass}^\otimes) \times_{\text{Fun}(\{0\},\text{Ass}^\otimes)} \mathcal{C}^\otimes \simeq \text{Act}(\text{Ass}^\otimes) \times_{\text{Fun}(\{0\},\text{Ass}^\otimes)} \text{Ass}^\otimes \times_{\text{Lm}^\otimes} M^\otimes \simeq \text{Act}(\text{Ass}^\otimes) \times_{\text{Act}(\text{Lm}^\otimes)} \text{Act}(\text{Lm}^\otimes) \times_{\text{Fun}(\{0\},\text{Lm}^\otimes)} M^\otimes$$

over $\text{Act}(\text{Ass}^\otimes)$.

Consequently it is enough to see that the commutative square

$$\begin{array}{ccc}
\text{Act}(\text{Ass}^\otimes) & \longrightarrow & \text{Act}(\text{Lm}^\otimes) \\
\downarrow & & \downarrow \\
\text{Fun}(\{1\},\text{Ass}^\otimes) & \longrightarrow & \text{Fun}(\{1\},\text{Lm}^\otimes)
\end{array}$$

is a pullback square.

To do so we have to show that for every active morphism $h : Y \to X$ of $\text{Lm}^\otimes$ with $X$ also $Y$ belongs to $\text{Ass}^\otimes$.

But if $h : Y \to X$ lies over the active morphism $f : (m) \to (n)$ we have a canonical equivalence $(f) \times_{\text{In}^\otimes((m)_{(0)})} \text{Lm}^\otimes(Y, X) \simeq \prod_{i=1}^{n} \text{Mul}_{\text{Lm}^\otimes}((Y)_{(f^{-1}(i))}, X_i)$.

Containing $h$ the space $(f) \times_{\text{In}^\otimes((m)_{(0)})} \text{Lm}^\otimes(Y, X)$ is not empty so that for all $i \in \{1, ..., n\}$ the space $\text{Mul}_{\text{Lm}^\otimes}((Y)_{(f^{-1}(i))}, X_i)$ is not empty.

So for every $j \in \{1, ..., m\}$ the object $Y_j$ is the unique color of $\text{Ass}^\otimes$.

\begin{lemma}
Let $M^\otimes \to \text{Lm}^\otimes$ be a map of operads such that the map $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\text{Lm}^\otimes} M^\otimes \to \text{Ass}^\otimes$ is a locally cocartesian fibration. Set $M := \{m\} \times_{\text{Lm}^\otimes} M^\otimes$.

Let $X, Y$ be objects of $M$ and $\beta \in \text{Mul}_{M^\otimes}(B, X; Y)$ an operation that exhibits $B$ as the morphism object of $X$ and $Y$ in $\mathcal{C}$.

Denote $\sigma \in \text{Mul}_{M^\otimes}(a, m; m)$ the unique object and for every $\alpha \in \text{Ass}^\otimes_n$ for some $n \in \mathbb{N}$ denote $\alpha'$ the image of $\alpha$, the identity of $m$ and $\sigma$ under the operadic composition $\text{Mul}_{M^\otimes}(a, m; m) \times (\text{Mul}_{Lm^\otimes}(a, ..., a; a) \times \text{Mul}_{Lm^\otimes}(m; m)) \to \text{Mul}_{M^\otimes}(a, ..., a; m; m)$.

The following conditions are equivalent:

\end{lemma}
1. The full inclusion $\mathcal{M}^\circ \subset \text{Env}_{LM^\circ}(\mathcal{M})^\circ$ of operads over $LM^\circ$ preserves the morphism object of $X$ and $Y$, i.e. $\beta \in \text{Mul}_{LM^\circ}(B,X;Y) \cong \text{Env}_{LM^\circ}(\mathcal{M})(B \otimes X,Y)$ exhibits $B$ as the morphism object of $X$ and $Y$ in $\text{Env}_{Ass^\circ}(\mathcal{C})$.

2. For every objects $A_1,\ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and every $\alpha \in \text{Ass}_n$ the canonical map

$$\text{Mul}_{\mathcal{M}^\circ}(\otimes_\alpha(A_1,\ldots,A_n),X;Y) \simeq \{\sigma\} \times_{\text{Mul}_{LM^\circ}(a,m,m)} \text{Mul}_{\mathcal{M}^\circ}(\otimes_\alpha(A_1,\ldots,A_n),X;Y) \rightarrow$$

$$\{\alpha'\} \times_{\text{Mul}_{LM^\circ}(a,\ldots,m,m)} \text{Mul}_{\mathcal{M}^\circ}(A_1,\ldots,A_n,X;Y)$$

is an equivalence.

Proof. Write $[X,Y]$ for $B$ and let $A$ be an object of $\text{Env}_{Ass^\circ}(\mathcal{C})$ corresponding to objects $A_1,\ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and an operation $\alpha \in \text{Ass}_n$.

The canonical map

$$\text{Env}_{Ass^\circ}(\mathcal{C})(A,[X,Y]) \rightarrow \text{Env}_{LM^\circ}(\mathcal{M})(A \otimes X,[X,Y] \otimes X) \rightarrow \text{Env}_{LM^\circ}(\mathcal{M})(A \otimes X,Y)$$

induced by $\beta \in \text{Mul}_{LM^\circ}(B,X;Y) \simeq \text{Env}_{LM^\circ}(\mathcal{M})(B \otimes X,Y)$ factors as the composition of canonical maps

$$\text{Env}_{Ass^\circ}(\mathcal{C})(A,[X,Y]) \simeq \{\alpha\} \times_{\text{Ass}(n)} \text{Mul}_{\mathcal{C}^\circ}((A_1,\ldots,A_n),[X,Y]) \simeq$$

$$\mathcal{C}(\otimes_\alpha(A_1,\ldots,A_n),[X,Y]) \cong \text{Mul}_{\mathcal{M}^\circ}(\otimes_\alpha(A_1,\ldots,A_n),X;Y) \simeq$$

$$\{\sigma\} \times_{\text{Mul}_{LM^\circ}(a,m,m)} \text{Mul}_{\mathcal{M}^\circ}(\otimes_\alpha(A_1,\ldots,A_n),X;Y) \rightarrow$$

$$\{\alpha'\} \times_{\text{Mul}_{LM^\circ}(a,\ldots,m,m)} \text{Mul}_{\mathcal{M}^\circ}(A_1,\ldots,A_n,X;Y) \simeq \text{Env}_{LM^\circ}(\mathcal{M})(A \otimes X,Y)$$

as for $A = [X,Y]$ both maps send the identity to $\beta$.

\[ \square \]

**Proposition 7.12.** Let $\mathcal{M}^\circ \rightarrow LM^\circ$ be a map of operads such that the map $\mathcal{C}^\circ := \text{Ass}^\circ \times_{LM^\circ} \mathcal{M}^\circ \rightarrow \text{Ass}^\circ$ is a locally coCartesian fibration. Set $\mathcal{M} := \{m\} \times_{LM^\circ} \mathcal{M}^\circ$.

Let $X$ be an object of $\mathcal{M}$ that admits an endomorphism object corresponding to a final object of the category $\mathcal{C}[X]$.

Set $\mathcal{M}^\circ := \mathcal{P}_n(\text{Env}_{LM^\circ}(\mathcal{M}))^\circ$, $\mathcal{C}^\circ := \mathcal{P}_n(\text{Env}_{Ass^\circ}(\mathcal{C}))^\circ$ and $\mathcal{M}^\circ := \{m\} \times_{LM^\circ} \mathcal{M}^\circ$.

Denote $\sigma \in \text{Mul}_{LM^\circ}(a,m;m)$ the unique object and for every $\alpha \in \text{Ass}_n$ for some $n \in \mathbb{N}$ denote $\alpha'$ the image of $\alpha$, the identity of $m$ and $\sigma$ under the operadic composition $\text{Mul}_{LM^\circ}(a,m;m) \times (\text{Mul}_{LM^\circ}(a,\ldots,a) \times \text{Mul}_{LM^\circ}(m;m)) \rightarrow \text{Mul}_{LM^\circ}(a,\ldots,a,m;m)$.

The following conditions are equivalent:

1. For every objects $A_1,\ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and every $\alpha \in \text{Ass}_n$ the canonical map

$$\text{Mul}_{\mathcal{M}^\circ}(\otimes_\alpha(A_1,\ldots,A_n),X;X) \simeq$$

$$\{\sigma\} \times_{\text{Mul}_{LM^\circ}(a,m,m)} \text{Mul}_{\mathcal{M}^\circ}(\otimes_\alpha(A_1,\ldots,A_n),X;X) \rightarrow$$

$$\{\alpha'\} \times_{\text{Mul}_{LM^\circ}(a,\ldots,m,m)} \text{Mul}_{\mathcal{M}^\circ}(A_1,\ldots,A_n,X;X)$$

is an equivalence.
2. The full inclusion $\mathcal{M}^\circ \subset \text{Env}_{\text{LM}(\mathcal{M})}^\circ$ of operads over $\text{LM}^\circ$ preserves the endomorphism object, in other words the full subcategory inclusion $\mathcal{C}[X] \subset \text{Env}_{\text{Ass}^\circ}(\mathcal{C})[X]$ preserves the final object.

3. The final object of $\mathcal{C}[X]$ lifts to a final object of $\{X\} \times_\mathcal{M} \text{LM}(\mathcal{M})$, which is preserved by the canonical fully faithful functor $\{X\} \times_\mathcal{M} \text{LM}(\mathcal{M}) \subset \{X\} \times_\bar{\mathcal{M}} \text{LM}(\bar{\mathcal{M}})$.

**Proof.** Lemma 7.11 implies that 1. and 2. are equivalent.

Let $N^\circ \to \text{LM}^\circ$ be a $\text{LM}^\circ$-monoidal category and $Z \in N := \{m\} \times_{\text{LM}^\circ} N^\circ$. Set $\mathcal{B}^\circ := \text{Ass}^\circ \times_{\text{LM}^\circ} \mathcal{N}^\circ$.

By corollary 4.7.2.40. we know that if $\mathcal{B}[X]$ admits a final object, this final object lifts to a final object of $\{X\} \times_N \text{LM}(\mathcal{N})$.

As the forgetful functor $\{X\} \times_N \text{LM}(\mathcal{N}) \to \mathcal{B}[X]$ is conservative, in this case an object of $\{X\} \times_N \text{LM}(\mathcal{N})$ is final if and only if its image in $\mathcal{B}[X]$ is.

As $\bar{\mathcal{M}}^\circ$ exhibits $\bar{\mathcal{M}}$ as closed left module over $\mathcal{C}^\circ$, the category $\bar{\mathcal{C}}[X]$ admits a final object that lifts to a final object of $\{X\} \times_\bar{\mathcal{M}} \text{LM}(\bar{\mathcal{M}})$.

We have a pullback square

$$
\begin{array}{ccc}
\{X\} \times_\mathcal{M} \text{LM}(\mathcal{M}) & \longrightarrow & \{X\} \times_\bar{\mathcal{M}} \text{LM}(\bar{\mathcal{M}}) \\
\downarrow & & \downarrow \\
\bar{\mathcal{C}}[X] & \longrightarrow & \bar{\mathcal{C}}[X].
\end{array}
$$

The functor $\bar{\mathcal{C}}[X] \subset \bar{\mathcal{C}}[X]$ factors as $\bar{\mathcal{C}}[X] \subset \bar{\mathcal{C}}'[X] \subset \bar{\mathcal{C}}[X]$.

By lemma 7.13 the functor $\text{Env}_{\text{Ass}^\circ}(\bar{\mathcal{C}})[X] \subset \bar{\mathcal{C}}[X]$ preserves the final object.

So 2. is equivalent to the condition that the functor $\bar{\mathcal{C}}[X] \to \bar{\mathcal{C}}[X]$ preserves the final object.

Hence 2. and 3. are equivalent.

\[\square\]

**Lemma 7.13.** Let $\iota : \mathcal{M}^\circ \subset \mathcal{M}^\circ'$ be a full inclusion of operads over $\text{LM}^\circ$.

Set $\mathcal{M} = \{m\} \times_{\text{LM}^\circ} \mathcal{M}, \mathcal{M}' = \{m\} \times_{\text{LM}^\circ} \mathcal{M}^\circ'$ and $\mathcal{C}^\circ = \text{Ass}^\circ \times_{\text{LM}^\circ} \mathcal{M}^\circ, \mathcal{C}^\circ = \text{Ass}^\circ \times_{\text{LM}^\circ} \mathcal{M}^\circ'$.

Let $X, Y$ be objects of $\mathcal{M}$ and $\beta \in \text{Mul}_{\mathcal{M}}(B, X; Y)$ an operation that exhibits $B$ as the morphism object of $X$ and $Y$ in $\mathcal{C}$.

Assume that $\mathcal{C}'$ is the only full subcategory of $\mathcal{C}'$ containing $\mathcal{C}$ and closed under $\kappa$-small colimits and that the functor $\text{Mul}_{\mathcal{M}^\circ}(\iota_X : \iota_Y) : \mathcal{C}^\circ \to \mathcal{S}(\kappa)$ preserves $\kappa$-small limits.

Then $\iota(\beta) \in \text{Mul}_{\mathcal{M}^\circ}(\iota(B), \iota(X); \iota(Y))$ exhibits $\iota(B)$ as the morphism object of $\iota(X)$ and $\iota(Y)$ in $\mathcal{C}'$.

**Proof.** For every object $A$ of $\mathcal{C}'$ denote $\xi_A$ the canonical map

$$
\mathcal{C}'(A, \iota([X, Y])) \to \text{Mul}_{\mathcal{M}^\circ}(\iota([X, Y]), \iota(X); \iota(Y)) \times (\mathcal{C}'(A, \iota([X, Y])) \times \mathcal{M}'(\iota(X), \iota(X))) \\
\to \text{Mul}_{\mathcal{M}^\circ}(A, \iota(X); \iota(Y))
$$

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induced by $\iota(\beta)$.

If $A$ belongs to $\mathcal{C}$, the map $\xi_A$ is canonically equivalent to the canonical map

$$\mathcal{C}(A, [X, Y]) \to \text{Mul}_{LM}([X, Y], X; Y) \times (\mathcal{C}(A, [X, Y]) \times \mathcal{M}(X, X))$$

$$\to \text{Mul}_{LM}(A, X; Y)$$

induced by $\beta$ and is thus an equivalence.

Thus $\mathcal{C}$ is contained in the full subcategory $\mathcal{W}$ of $\mathcal{C}'$ spanned by the objects $A$ such that $\xi_A$ is an equivalence.

But $\mathcal{W}$ is closed under $\kappa$-small colimits as the functor $\text{Mul}_{LM}(\cdot, \iota(X); \iota(Y)) : \mathcal{C}'^{\text{op}} \to \mathcal{S}(\kappa)$ preserves $\kappa$-small limits. So by assumption $\mathcal{W} = \mathcal{C}'$.

So far we have seen that every cocartesian $S$-family $\mathcal{M}^S \to \mathcal{LM}^S \times S$ of categories enriched over $\mathcal{C}^S := \text{Ass}^S \times_{\mathcal{LM}^S} \mathcal{M}^S$ embeds into a cocartesian $S$-family $\mathcal{M}'^S \to \mathcal{LM}^S \times S$ of $\mathcal{LM}^S$-monoidal categories that is a cocartesian $S$-family of categories enriched over $\mathcal{C}'^S := \text{Ass}^S \times_{\mathcal{LM}^S} \mathcal{M}'^S$ such that $\mathcal{M}^S$ is closed in $\mathcal{M}'^S$ under morphism objects.

We call $\mathcal{M}'^S := \mathcal{P}_S(\text{Env}_{\mathcal{LM}^S}(\mathcal{M}))^S \to \mathcal{LM}^S \times S$ the enveloping cocartesian $S$-family of $\mathcal{C}'^S := \text{Ass}^S \times_{\mathcal{LM}^S} \mathcal{M}'^S$-enriched categories of $\mathcal{M}^S$.

Typically we use $\mathcal{M}'^S$ to reduce constructions for enriched categories to the case of enriched left modules over a monoidal category as we do in the following.

Let $\mathcal{M}^S \to \mathcal{LM}^S$ be an operad over $\mathcal{LM}^S$ that exhibits $M := \{m\} \times_{\mathcal{LM}^S} \mathcal{M}^S$ as category enriched over $\mathcal{C}^S := \text{Ass}^S \times_{\mathcal{LM}^S} \mathcal{M}^S$ and let $X, Y$ be objects of $\mathcal{M}$.

We will construct a canonical left modul structure on $[Y, X]$ over $[X, X]$.

As $\mathcal{M}^S$ is closed in $\mathcal{M}'^S$ under morphism objects, we can reduce to the case that $\mathcal{M}^S$ is an enriched left module over a monoidal category $\mathcal{C}^S := \text{Ass}^S \times_{\mathcal{LM}^S} \mathcal{M}^S$.

In this case the functor $- \otimes Y : \mathcal{C} \to \mathcal{M}$ is $\mathcal{C}$-linear and admits a right adjoint $[Y, -] : \mathcal{M} \to \mathcal{C}$ that is lax $\mathcal{C}$-linear in a canonical way.

Being lax $\mathcal{C}$-linear the functor $[Y, -] : \mathcal{M} \to \mathcal{C}$ sends the endomorphism left modul structure on $X$ over $[X, X]$ to a left modul structure on $[Y, X]$ over $[X, X]$.

More coherently we will show the following:

Let $\mathcal{M}^S \to \mathcal{LM}^S$ be a cocartesian $S$-family of categories enriched over $\mathcal{C}^S := \text{Ass}^S \times_{\mathcal{LM}^S} \mathcal{M}^S$.

By ....... we have a multi-mapping space functor $\text{Mul}_{LM}(\cdot, \cdot; \cdot) : \mathcal{C}^{\text{rev}} \times_S \mathcal{M}^{\text{rev}} \times_S \mathcal{M} \to \mathcal{S}(\kappa)$ relative to $S$ that is adjoint to a functor $\mathcal{M}^{\text{rev}} \times_S \mathcal{M} \to \text{Map}_{\mathcal{S}}(\mathcal{C}^{\text{rev}}, \mathcal{S}(\kappa) \times S)$ over $S$.

As $\mathcal{M}^S \to \mathcal{LM}^S$ is a cocartesian $S$-family of categories enriched over $\mathcal{C}^S$, this functor over $S$ induces a functor $\mathcal{M}^{\text{rev}} \times_S \mathcal{M} \to \mathcal{C}$ over $S$ adjoint to a functor $\theta : \mathcal{M} \to \text{Map}_{\mathcal{S}}(\mathcal{M}^{\text{rev}}, \mathcal{C})$ that sends an object $X$ of $\mathcal{M}$ lying over some $s \in S$ to the functor $[\cdot, X] : \mathcal{M}_{s}^{\text{op}} \to \mathcal{C}_s$.

We will show in the following that $\theta$ lifts to a map $\gamma : \mathcal{M}^S \to \text{Map}_{\mathcal{S}}(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ$ of $S$-families of operads over $\mathcal{LM}^S$, whose pullback to $\mathcal{Ass}^S$ is the diagonal map $\delta : \mathcal{C}^S \to \text{Map}_{\mathcal{S}}(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ$ of $S$-families of operads over $\mathcal{Ass}^S$.

For $S$ contractible this especially guarantees the following:
Let $X$ be an object of $\mathcal{M}$ and $\beta \in \text{Mul}_{\mathcal{M}^\otimes}(B,X,X)$ an operation that exhibits $B = [X,X]$ as the endomorphism object of $X$ in $\mathcal{C}$.

As $\gamma$ is a map of operads over $\mathcal{L}^\otimes$, it sends the endomorphism $[X,X]$-left module structure on $X$ to a $\delta([X,X])$-left module structure on $[-,X] : \mathcal{M}^\op \to \mathcal{C}$ corresponding to a lift $\mathcal{M}^\op \to \text{LMod}_{[X,X]}(\mathcal{C})$ of $[-,X] : \mathcal{M}^\op \to \mathcal{C}$.

So for every object $Y$ of $\mathcal{M}$ the morphism object $[Y,X]$ is a left-module over the endomorphism object $[X,X]$ in $\mathcal{C}$ and for every morphism $Y \to Z$ in $\mathcal{M}$ the induced morphism $[Z,X] \to [Y,X]$ is a morphism of $[X,X]$-left modules in $\mathcal{C}$.

**Remark 7.14.** Let $\mathcal{M}^\otimes \to \mathcal{L}^\otimes$ be a cocartesian $S$-family of categories enriched over $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\mathcal{L}^\otimes} \mathcal{M}^\otimes$.

We have a natural transformation over $S$ from the morphism object functor $\mathcal{M}^\text{rev} \times_S \mathcal{M} \to \mathcal{C}' \subset \mathcal{C}'$ of $\mathcal{M}$ to the restricted morphism object functor $\mathcal{M}^\text{rev} \times_S \mathcal{M}' \to \mathcal{C}'$ of $\mathcal{M}'$ that is an equivalence as the embedding $\mathcal{M}^\otimes \subset \mathcal{M}^\text{rev}$ preserves morphism objects.

Hence $\theta : \mathcal{M} \to \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}) \subset \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}')$ factors as $\mathcal{M} \subset \mathcal{M}' \xrightarrow{\theta'} \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}') \xrightarrow{\text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}'\text{rev})} \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}'\text{rev})$, where $\theta'$ is defined similarly.

$\mathcal{M}^\otimes$ is a cocartesian $S$-family of $\mathcal{L}^\otimes$-monoidal categories corresponding to a $\mathcal{L}^\otimes$-monoidal category over $S$ classifying a $\mathcal{L}^\otimes$-monoid of $\text{Cat}_{\text{cocart}}^\text{cocom}(\kappa)$.

Thus the multi-mapping space functor $\text{Mul}_{\mathcal{M}^\text{rev}}(\mathcal{C}^\otimes, \mathcal{C}') \times_S \mathcal{M}^\text{rev} \times_S \mathcal{M}' \to \mathcal{S}(\kappa)$ of $\mathcal{M}^\otimes$ relative to $S$ is the composition of the functor $\mathcal{C}^\text{rev} \times_S \mathcal{M}^\text{rev} \times_S \mathcal{M}' \to \mathcal{M}^\text{rev} \times_S \mathcal{M}'$ over $S$ induced by the action map $\mathcal{C} \times_S \mathcal{M}' \to \mathcal{M}'$ of the $\mathcal{L}^\otimes$-monoid of $\text{Cat}_{\text{cocart}}^\text{cocom}(\kappa)$ classified by $\mathcal{M}^\otimes \to S \times \mathcal{L}^\otimes$ followed by the mapping space functor $\mathcal{M}^\text{rev} \times_S \mathcal{M}' \to \mathcal{S}(\kappa)$ of $\mathcal{M}'$ relative to $S$.

Denote $\alpha : S \to \mathcal{C}'$ the unit of the associative monoid of $\text{Cat}_{\text{cocart}}^\text{cocom}(\kappa)$ classified by $\mathcal{C}^\otimes \to S \times \text{Ass}^\otimes$.

Then the composition $\mathcal{M}^\text{rev} \times_S \mathcal{M}' \xrightarrow{\alpha^\text{rev} \times_S \mathcal{M}^\text{rev} \times_S \mathcal{M}'} \mathcal{C}^\text{rev} \times_S \mathcal{M}^\text{rev} \times_S \mathcal{M}' \xrightarrow{\text{Map}_S(\alpha^\text{rev}, \mathcal{S}(\kappa) \times S)} \text{Map}_S(\mathcal{S}(\kappa) \times S, \mathcal{S}(\kappa) \times S)$ is equivalent to the mapping space functor $\mathcal{M}^\text{rev} \times_S \mathcal{M}' \to \mathcal{S}(\kappa)$ of $\mathcal{M}'$ relative to $S$.

Denote $\beta$ the composition $\mathcal{C}' \subset \text{Map}_S(\mathcal{C}^\text{rev}, \mathcal{S}(\kappa) \times S) \xrightarrow{\text{Map}_S(\gamma^\text{rev}, \mathcal{S}(\kappa) \times S)} \text{Map}_S(\mathcal{S}(\kappa) \times S, \mathcal{S}(\kappa) \times S)$.

Then the composition $\mathcal{M}' \xrightarrow{\gamma'} \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}') \xrightarrow{\text{Map}_S(\mathcal{M}^\text{rev}, \beta)} \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{S}(\kappa) \times S)$ is the Yoneda-embedding relative to $S$.

By proposition 7.25, the full subcategory inclusion $\mathcal{C} \subset \mathcal{C}'$ of categories over $S$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}$ of maps of cocartesian fibrations over $S$.

Then the restriction $\mathcal{C} \subset \mathcal{C} \xrightarrow{\beta} \mathcal{S}(\kappa) \times S$ is equivalent to the functor $\mathcal{C} \subset \text{Map}_S(\mathcal{C}^\text{rev}, \mathcal{S}(\kappa) \times S) \xrightarrow{\text{Map}_S(\gamma^\text{rev}, \mathcal{S}(\kappa) \times S)} \text{Map}_S(\mathcal{S}(\kappa) \times S, \mathcal{S}(\kappa) \times S)$ that is equivalent to the functor $\mathcal{C} \xrightarrow{\mathcal{S}(\kappa) \times S} \mathcal{C} \xrightarrow{\mathcal{S}(\kappa) \times S} \mathcal{S}(\kappa) \times S$ that induces on the fiber over every $s \in S$ the functor $\mathcal{C}_s(1_{\mathcal{C}_s}, -) : \mathcal{C}_s \to \mathcal{S}(\kappa)$.

So $\mathcal{M} \xrightarrow{\delta} \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{C}) \xrightarrow{\text{Map}_S(\mathcal{M}^\text{rev}, \lambda)} \text{Map}_S(\mathcal{M}^\text{rev}, \mathcal{S}(\kappa) \times S)$ is the Yoneda-embedding relative to $S$.
Proposition 7.15. Let $S$ be a $\kappa$-small category and $\mathcal{M}^\circ \rightarrow \text{LM}^\circ \times S$ a cocartesian $S$-family of $\kappa$-small categories enriched in $\mathcal{C}^\circ := \text{Ass}^\circ \times _{\text{LM}^\circ} \mathcal{M}^\circ$.

Set $\mathcal{M} := \{ m \} \times _{\text{LM}^\circ} \mathcal{M}^\circ$.

There is a map $\gamma : \mathcal{M}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ$ of $S$-families of operads over $\text{LM}^\circ$, whose underlying functor is the functor $\theta : \mathcal{M} \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})$ over $S$ and whose pullback to $\text{Ass}^\circ$ is the diagonal map $\delta : \mathcal{C}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ$ of $S$-families of operads over $\text{Ass}^\circ$.

$\gamma$ corresponds to a $\mathcal{C}$-linear map $\mathcal{M}^\circ \rightarrow \delta^* (\text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ)$ of $S$-families, i.e. a map of $S$-families of operads over $\text{LM}^\circ$, whose pullback to $\text{Ass}^\circ$ is the identity of $\mathcal{C}^\circ$.

For $S = \Delta^1$ we obtain the following:

Let $\mathcal{M}^\circ \rightarrow \text{LM}^\circ, \mathcal{N}^\circ \rightarrow \text{LM}^\circ$ be $\kappa$-small operads over $\text{LM}^\circ$ that exhibit categories $\mathcal{M}, \mathcal{N}$ as enriched over locally cocartesian fibrations of operads $\mathcal{C}^\circ \rightarrow \text{Ass}^\circ, \mathcal{D}^\circ \rightarrow \text{Ass}^\circ$ and let $F : \mathcal{M}^\circ \rightarrow \mathcal{N}^\circ$ be a map of operads over $\text{LM}^\circ$.

The natural transformation $\text{Fun}(\mathcal{M}^{\text{op}}, F) \circ \theta \rightarrow \text{Fun}(\mathcal{F}^{\text{op}}, D) \circ \theta \circ F$ of functors $\mathcal{M} \rightarrow \text{Fun}(\mathcal{M}^{\text{op}}, D)$ adjoint to the canonical natural transformation $F \circ [-,-] \rightarrow [-,-] \circ (\mathcal{F}^{\text{op}} \times F)$ of functors $\mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{D}$ lifts to a natural transformation $F^{\text{op}} \circ \gamma \rightarrow (\mathcal{D}^\circ)^{\mathcal{F}^{\text{op}}} \circ \gamma \circ F$ over $\text{LM}^\circ$ of maps of operads $\mathcal{M}^\circ \rightarrow (\mathcal{D}^\circ)^{\mathcal{F}^{\text{op}}}$ over $\text{LM}^\circ$.

Proof. We first show that we can reduce to the case that $\mathcal{M}^\circ \rightarrow \text{LM}^\circ \times S$ is a cocartesian $S$-family of $\kappa$-small $\text{LM}^\circ$-monoidal categories.

Let $\mathcal{M}^\circ := \text{P}_{\kappa}(\text{Env}_{\text{LM}^\circ}(\mathcal{M}))^\circ \rightarrow \text{LM}^\circ \times S$ be the enveloping cocartesian $S$-family of $\kappa$-small $\text{LM}^\circ$-enriched categories of $\mathcal{M}^\circ$.

Assume that there is a map $\mathcal{M}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ$ of $S$-families of operads over $\text{LM}^\circ$, whose underlying functor is the functor $\theta' : \mathcal{M}' \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')$ over $S$ and whose pullback to $\text{Ass}^\circ$ is the diagonal map $\delta' : \mathcal{C}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ$ of $S$-families of operads over $\text{Ass}^\circ$.

Then the underlying functor over $S$ of the map $\mathcal{M}^\circ \subset \mathcal{M}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C'})^\circ$ of $S$-families of operads over $\text{LM}^\circ$ is equivalent to $\mathcal{M} \subset \mathcal{M}' \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}') \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')$ being equivalent to

$\mathcal{M} \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}) \subset \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')$ by remark [4.13] and whose pullback to $\text{Ass}^\circ$ is the map $\mathcal{C}^\circ \subset \mathcal{C}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}') \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ$ of $S$-families of operads over $\text{Ass}^\circ$ being equivalent to $\mathcal{C}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}) \subset \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ$.

Hence the map $\mathcal{M}^\circ \subset \mathcal{M}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C}')^\circ$ of $S$-families of operads over $\text{LM}^\circ$ induces a map $\mathcal{M}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ$ of $S$-families of operads over $\text{LM}^\circ$, whose underlying functor over $S$ is the functor $\theta : \mathcal{M} \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})$ and whose pullback to $\text{Ass}^\circ$ is the diagonal functor $\delta : \mathcal{C}^\circ \rightarrow \text{Map}_S(\mathcal{M}^{\text{rev}}, \mathcal{C})^\circ$.

So we can assume that $\mathcal{M}^\circ \rightarrow \text{LM}^\circ \times S$ is a cocartesian $S$-family of $\kappa$-small $\text{LM}^\circ$-monoidal categories.
Given a $\kappa$-small operad $O^\otimes$ and cocartesian $S$-families of $\kappa$-small $O^\otimes$-monoidal categories $D^\otimes \to O^\otimes$, $E^\otimes \to O^\otimes$ we write

- $\text{Fun}^{S,\otimes,lax}_O(D, E) := \text{Alg}^{S}_{O^\otimes/O^\otimes \times O^\otimes}(E^\otimes)$,

- $\text{Fun}^{S,\otimes}_O(D, E) \subset \text{Fun}^{S,\otimes,lax}_O(D, E)$ the full subcategory spanned by the functors over $S$ that induce on the fiber over every object of $S$ a $O^\otimes$-monoidal functor,

- $\text{Fun}^{S,\otimes,lax,R}_O(D, E) \subset \text{Fun}^{S,\otimes,lax}_O(D, E)$ the full subcategory spanned by the functors over $S \times O^\otimes$ that induce on the fiber over every object of $S \times O$ a right adjoint functor.

By ... we have a canonical equivalence

$$\text{Fun}^{S,\otimes,lax}_O(E, P_{\kappa}(D)) \simeq \text{Fun}^{S,\otimes,lax}_O(E \times_O D^{\text{rev}}, S(\kappa) \times S)$$

over $S$.

Especially for $O^\otimes = \text{Triv}^\otimes$ we get a canonical equivalence

$$P_{\kappa}(D) \simeq \text{Maps}_S(D^{\text{rev}}, S(\kappa) \times S)$$

over $S$.

So we get a canonical equivalence

$$\text{Fun}^{S,\otimes,lax}_O(E, P_{\kappa}(D)) \simeq \text{Fun}^{S,\otimes,lax}_O(E \times_O D^{\text{rev}}, S(\kappa) \times S)$$

$$\simeq \text{Fun}^{S,\otimes,lax}_O(D^{\text{rev}} \times_O E, S(\kappa) \times S) \simeq \text{Fun}^{S,\otimes,lax}_O(D^{\text{rev}}, P_{\kappa}(E^{\text{rev}}))$$

over $S$ that restricts to an equivalence

$$\text{Fun}^{S,\otimes,lax,R}_O(E, D) \simeq \text{Fun}^{S,\otimes,lax,R}_O(D^{\text{rev}}, E^{\text{rev}})$$

over $S$.

Specializing to our situation we make the following definitions:

Given cocartesian $S$-families of $\kappa$-small LM$^\otimes$-monoidal categories $N^\otimes \to LM^\otimes \times S$, $N'^\otimes \to LM^\otimes \times S$ we write

- $\text{LinFun}^{S,lax}_{\text{Ass}}(N, N') := \{id\} \times_{\text{Fun}^{S,\otimes,lax}_{\text{Ass}}(\mathfrak{c}, \mathfrak{c})} \text{Fun}^{S,\otimes,lax}_{\text{LM}}(N, N')$,

- $\text{LinFun}^{S,\otimes}_{\text{Ass}}(N, N') := \{id\} \times_{\text{Fun}^{S,\otimes}_{\text{Ass}}(\mathfrak{c}, \mathfrak{c})} \text{Fun}^{S,\otimes}_{\text{LM}}(N, N')$,

- $\text{LinFun}^{S,lax,R}_{\text{Ass}}(N, N') := \{id\} \times_{\text{Fun}^{S,\otimes,lax,R}_{\text{Ass}}(\mathfrak{c}, \mathfrak{c})} \text{Fun}^{S,\otimes,lax,R}_{\text{LM}}(N, N')$.

So we get canonical equivalences

$$\text{Fun}^{S,\otimes,lax,R}_{\text{LM}}(N'^{\text{rev}}, N'^{\text{rev}}) \simeq \text{Fun}^{S,\otimes,lax,R}_{\text{LM}}(N', N)$$

and

$$\text{Fun}^{S,\otimes,lax,R}_{\text{Ass}}(\mathfrak{c}^{\text{rev}}, \mathfrak{c}^{\text{rev}}) \simeq \text{Fun}^{S,\otimes,lax,R}_{\text{Ass}}(\mathfrak{c}, \mathfrak{c})$$

over $S$ and so a canonical equivalence

$$\text{LinFun}^{S,lax,R}_{\text{Ass}}(N'^{\text{rev}}, N'^{\text{rev}}) = \{id\} \times_{\text{Fun}^{S,\otimes,lax,R}_{\text{Ass}}(\mathfrak{c}^{\text{rev}}, \mathfrak{c}^{\text{rev}})} \text{Fun}^{S,\otimes,lax,R}_{\text{LM}}(N', N) \simeq$$

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over $S$.

Especially we get a canonical equivalence
\[
\text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^\text{rev}, \mathcal{M}^{\text{rev}}) \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{F} (\mathcal{F}^\text{rev}, \mathcal{N}^{\text{rev}})
\]
over $S$.

By lemma 7.16 we have a canonical equivalence
\[
\text{Map}_S (\mathcal{M}^{\text{rev}}, \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}, \mathcal{E})) \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}, \delta^* (\text{Map}_S (\mathcal{M}^{\text{rev}}, \mathcal{E})))
\]
over $S$ that induces on sections a canonical equivalence
\[
\text{Fun}_S (\mathcal{M}^{\text{rev}}, \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}, \mathcal{E})) \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}, \delta^* (\text{Map}_S (\mathcal{M}^{\text{rev}}, \mathcal{E})))
\]
over Fun$_S (\mathcal{M}^{\text{rev}}, \text{Map}_S (\mathcal{M}^{\text{rev}}, \mathcal{E})) \cong \text{Fun}_S (\mathcal{M}, \text{Map}_S (\mathcal{M}^{\text{rev}}, \mathcal{E}))$.

Consequently it is enough to find a canonical functor $\mathcal{M}^{\text{rev}} \to \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}, \mathcal{E})$ over $S$ such that the composition $\mathcal{M}^{\text{rev}} \to \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}, \mathcal{E}) \to \text{Map}_S (\mathcal{M}, \mathcal{E})$ corresponds to $\theta : \mathcal{M} \to \text{Map}_S (\mathcal{M}^{\text{rev}}, \mathcal{E})$.

By lemma 7.19 we have a canonical equivalence
\[
\alpha : \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}) \cong \mathcal{M}^{\text{rev}}
\]
over $S$.

The composition $\mathcal{M}^{\text{rev}} \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}) \to \text{Map}_S (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}})$ is adjoint to the left action functor $\mathcal{C}^{\text{rev}} \times_S \mathcal{M}^{\text{rev}} \to \mathcal{M}^{\text{rev}}$ over $S$ of the $\mathcal{C}^{\text{rev}}$-left modul $\mathcal{M}^{\text{rev}}$ in Cat$_\infty (\mathcal{K})_S$.

$\alpha$ induces on the fiber over every $s \in S$ the canonical equivalence
\[
\text{LinFun}^{\mathcal{E}^{\text{rev}}}_s (\mathcal{E}^{\text{rev}}, \mathcal{M}_s^{\text{rev}}) \cong \mathcal{M}_s^{\text{rev}}.
\]

So every $\mathcal{C}_s$-linear functor $\mathcal{E}_s \to \mathcal{M}_s$ is of the form $- \otimes X$ for some $X \in \mathcal{M}_s$ and so admits a right adjoint as $\mathcal{M}_s$ is enriched in $\mathcal{C}_s$.

So every $\mathcal{C}_s^{\text{op}}$-linear functor $\mathcal{E}_s^{\text{op}} \to \mathcal{M}_s^{\text{op}}$ admits a left adjoint.

Thus the full subcategory inclusion
\[
\mathcal{M}^{\text{rev}} \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}) \subset \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}})
\]
induces a full subcategory inclusion
\[
\mathcal{M}^{\text{rev}} \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}) \subset \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}).
\]

So we get a full subcategory inclusion
\[
\varphi : \mathcal{M}^{\text{rev}} \cong \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}) \subset \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}})
\]
\[
= \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}})
\]
over $S$.

The functor
\[
\mathcal{M}^{\text{rev}} \varphi : \text{LinFun}^{S,\text{rev}, \mathcal{L}}_\mathcal{C} (\mathcal{C}^{\text{rev}}, \mathcal{M}^{\text{rev}}) \to \text{Map}_S (\mathcal{M}, \mathcal{E})
\]

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over $S$ is equivalent to the composition

$$\beta : M^{\text{rev}} \to \text{Map}^R_S(\mathcal{C}^{\text{rev}}, M^{\text{rev}}) \cong \text{Map}^R_S(M, \mathcal{C}) \subset \text{Map}_S(M, \mathcal{C})$$

of functors over $S$, where the functor $M^{\text{rev}} \to \text{Map}^R_S(\mathcal{C}^{\text{rev}}, M^{\text{rev}})$ is adjoint to the left action functor $\mathcal{C}^{\text{rev}} \times_S M^{\text{rev}} \to M^{\text{rev}}$ over $S$ of the $\mathcal{C}^{\text{rev}}$-left module $M^{\text{rev}}$ in $\text{Cat}_\infty(\kappa)_S$.

So $M^{\text{rev}} \overset{\delta}{\to} \text{Map}_S(M, \mathcal{C}) \subset \text{Map}_S(M, \mathcal{P}_e(\mathcal{C})) \cong \text{Map}_S(\mathcal{C}^{\text{rev}} \times_S M, S(\kappa) \times S)$ is adjoint to the functor $M^{\text{rev}} \times_S \mathcal{C}^{\text{rev}} \times_S M \cong \mathcal{C}^{\text{rev}} \times_S M^{\text{rev}} \times_S M \to M^{\text{rev}} \times_S M \to S(\kappa)$.

Thus the functor

$$M^{\text{rev}} \overset{\varphi}{\to} \text{LinFun}^{S, \text{lax}}_{\mathcal{C}}(M, \mathcal{C}) \to \text{Map}_S(M, \mathcal{C})$$

over $S$ is adjoint to $\theta$.

The following three lemmata are used for the proof of proposition 7.15.

Given $\kappa$-small operads $M^\otimes : \text{LM}^\otimes, N^\otimes : \text{LM}^\otimes$ over $\text{LM}^\otimes$ we set

$$\text{LinFun}_{\mathcal{C}}^{\text{lax}}(M, N) = \{\text{id}\} \times_{\text{Alg}^\otimes_{/\text{Ass}^\otimes}(\mathcal{C})} \text{Alg}^\otimes_{/\text{LMS}}(N^\otimes).$$

So we have $\text{LinFun}_{\mathcal{C}}^{\text{lax}}(M, N)^\otimes \cong \{\text{id}\} \times_{\text{Op}^\otimes_{/\text{Ass}^\otimes}(\mathcal{C})} \text{Op}^\otimes_{/\text{LMS}}(M^\otimes, N^\otimes) \cong (\{\mathcal{C}\} \times_{\text{Op}^\otimes_{/\text{Ass}^\otimes}} \text{Op}^\otimes_{/\text{LMS}}(M^\otimes, N^\otimes))$.

**Lemma 7.16.** Let $S$ be a $\kappa$-small category and $N^\otimes : \text{LM}^\otimes \times S$ a $\text{LM}^\otimes$-monoid of $\text{Cat}_\infty(\kappa)_S$ that exhibits a cartesian fibration $N \to S$ as a left-module over some cartesian fibration $\mathcal{C} \to S$. Set $\mathcal{C}^\otimes := \text{Ass}^\otimes \times_{\text{LM}^\otimes} N^\otimes$.

Let $M^\otimes : \text{LM}^\otimes \times S$ be a cartesian $S$-family of operads over $\text{LM}^\otimes$ such that we have an equivalence $\text{Ass}^\otimes \times_{\text{LM}^\otimes} M^\otimes \cong \mathcal{C}^\otimes$ over $S$.

Let $\psi : K \to S$ be a cartesian fibration and denote $\delta : \mathcal{C}^\otimes \cong \text{Map}_S(S, \mathcal{C}) \to \text{Map}_S(K, \mathcal{C})$ the functor over $S$ induced by $\psi$.

**Remark 7.17.** By remark 6.12, $\text{Map}_S(K, N^\otimes) \to S \times \text{LM}^\otimes$ is a $\text{LM}^\otimes$-monoid of $\text{Cat}_\infty(\kappa)_S$ that exhibits the cartesian fibration $\text{Map}_S(K, N) \to S$ as a left module over the cartesian fibration $\text{Map}_S(K, \mathcal{C}) \to S$.

$$\delta : \mathcal{C}^\otimes \cong \text{Map}_S(S, \mathcal{C}) \to \text{Map}_S(K, \mathcal{C})^\otimes$$

is a map of associative monoids in $\text{Cat}_\infty(\kappa)_S$, whose underlying functor $\mathcal{C} \cong \text{Map}_S(S, \mathcal{C}) \to \text{Map}_S(K, \mathcal{C})$ is a map of cartesian fibrations over $S$ being induced by the unique map $K \to S$ of cocartesian fibrations over $S$.

Denote $\delta^* (\text{Map}_S(K, N^\otimes)) \to \text{Map}_S(K, N^\otimes)$ a cartesian lift of $\delta$ with respect to the cartesian fibration $\text{Alg}_{/\text{LMS}}(\text{Cat}_\infty(\kappa)_S) \to \text{Alg}^\otimes_{/\text{Ass}^\otimes}(\text{Cat}_\infty(\kappa)_S)$ induced by composition with the map of operads $\text{Ass}^\otimes \to \text{LM}^\otimes$.

So $\delta^* (\text{Map}_S(K, N^\otimes))$ is a $\text{LM}^\otimes$-monoid of $\text{Cat}_\infty(\kappa)_S$ that exhibits the cartesian fibration $\text{Map}_S(K, N) \to S$ as a left module over the cartesian fibration $\mathcal{C} \to S$.

**There is a canonical equivalence**

$$\text{Map}_S(K, \text{LinFun}_{\mathcal{C}}^{S, \text{lax}}(M, N)) \cong \text{LinFun}_{\mathcal{C}}^{S, \text{lax}}(M, \delta^* (\text{Map}_S(K, N)))$$
over $S$ that induces on the fiber over $s \in S$ the canonical equivalence
\[
\text{Fun}(K_s, \text{LinFun}_C^{lax}(M_s, N_s)) \simeq \text{LinFun}_C^{lax}(M_s, \delta_s^*(\mathcal{C}^S)).
\]

**Proof.** Remark 7.17 implies that we can apply lemma 7.18 to deduce that the commutative square
\[
\begin{array}{ccc}
\text{Alg}_{S/M/\mathcal{L}M}^S(\delta^*(\text{Map}_S(K, N))) & \longrightarrow & \text{Alg}_{S/M/\mathcal{L}M}^S(\text{Map}_S(K, N)) \\
\text{Alg}_{C/\text{Ass}}^S(\mathcal{C}) & \longrightarrow & \text{Alg}_{C/\text{Ass}}^S(\text{Map}_S(K, \mathcal{C}))
\end{array}
\]
over $S$ is a pullback square.

Pulling back this square along the section of $\text{Alg}_{C/\text{Ass}}^S(\mathcal{C}) \to S$ corresponding to the identity of $\mathcal{C}^S$ we get a canonical equivalence
\[
S \times \text{Alg}_{C/\text{Ass}}^S(\mathcal{C}) \simeq \text{Alg}_{S/M/\mathcal{L}M}^S(\delta^*(\text{Map}_S(K, N)))
\]
\[
S \times \text{Alg}_{C/\text{Ass}}^S(\text{Map}_S(K, \mathcal{C})) \simeq \text{Alg}_{S/M/\mathcal{L}M}^S(\text{Map}_S(K, N)).
\]

The desired equivalence over $S$ is the composition of canonical equivalences
\[
\text{Map}_S(K, \text{LinFun}_C^{S,lax}(M, N)) \simeq
\]
\[
S \times (\text{Alg}_{C/\text{Ass}}^S(\mathcal{C}) \simeq \text{Alg}_{S/M/\mathcal{L}M}^S(\text{Map}_S(K, N)) \simeq
\]
\[
S \times (\text{Alg}_{S/M/\mathcal{L}M}^S(\delta^*(\text{Map}_S(K, N))) \simeq
\]
\[
\text{LinFun}_C^{S,lax}(M, \delta^*(\text{Map}_S(K, N)))
\]
over $S$, where the first equivalence exists as the functor $\text{Map}_S(K, -) : \text{Cat}_K^\infty(K)_S \to \text{Cat}_K^\infty(K)_S$ preserves pullbacks being the right adjoint of the functor $K \times_S - : \text{Cat}_K^\infty(K)_S \to \text{Cat}_K^\infty(K)_S$, the second equivalence is due to remark 2.31 and the third equivalence is those from above.

\[\square\]

**Corollary 7.18.** Let $S$ be a $\kappa$-small category and $\mathcal{N}^S \to \mathcal{L}M^S \times S$ a cartesian $S$-family of operads over $\mathcal{L}M^S$.

Set $\mathcal{C}^S := \text{Ass}^S \times_{\mathcal{L}M^S} \mathcal{N}^S$.

Let $\overline{\phi} : \mathcal{B}^S \to \mathcal{C}^S$ be a map of cartesian $S$-families of operads over $\text{Ass}^S$.

Let $\chi : \phi^*(\mathcal{N}^S) \to \mathcal{N}^S$ be a map of cartesian $S$-families of operads over $\mathcal{L}M^S$ that is a cartesian lift of $\phi : \mathcal{B}^S \to \mathcal{C}^S$ with respect to the cartesian fibration $\text{Fun}(\mathcal{S}^\mathcal{L}M^S, \mathcal{O} \mathcal{P}_\kappa(K)_{\mathcal{L}M^S}) \to \text{Fun}(\mathcal{S}^\mathcal{L}M^S, \mathcal{O} \mathcal{P}_\kappa(K)_{\text{Ass}^S})$ induced by taking pullback along the map of operads $\text{Ass}^S \to \mathcal{L}M^S$.
For every cocartesian $S$-family $M^\otimes \to LM^\otimes \times S$ of operads over $LM^\otimes$, where we set $D^\otimes := Ass^\otimes \times_{LM^\otimes} M^\otimes$, the commutative square

\[
\begin{array}{ccc}
\Alg_{M^\otimes/\LM^\otimes}^{/\Sl}(\phi^*(N^\otimes)) & \longrightarrow & \Alg_{M^\otimes/\LM^\otimes}^{/\Sl}(N^\otimes) \\
\downarrow & & \downarrow \\
\Alg_{D^\otimes/\Ass^\otimes}^{/\Sl}(B^\otimes) & \longrightarrow & \Alg_{D^\otimes/\Ass^\otimes}^{/\Sl}(C^\otimes)
\end{array}
\] (13)

of cartesian fibrations over $S$ is a pullback square.

**Proof.** By remark 2.8, square (13) is a square of cartesian fibrations over $S$. Consequently it is enough to see that square (13) induces a pullback square on the fiber over every object $s$ of $S$.

Square (13) induces on the fiber over every object $s$ of $S$ the commutative square

\[
\begin{array}{ccc}
\Alg_{M^\otimes/\LM^\otimes}^{/\Sl}(\phi^*(N^\otimes))_s & \longrightarrow & \Alg_{M^\otimes/\LM^\otimes}^{/\Sl}(N^\otimes)_s \\
\downarrow & & \downarrow \\
\Alg_{D^\otimes/\Ass^\otimes}^{/\Sl}(B^\otimes)_s & \longrightarrow & \Alg_{D^\otimes/\Ass^\otimes}^{/\Sl}(C^\otimes)_s
\end{array}
\]

of categories.

Consequently we can reduce to the case that $S$ is contractible.

But then the statement follows from proposition 7.21.

\[\square\]

**Lemma 7.19.** Let $S$ be a $\kappa$-small category and $N^\otimes \to LM^\otimes \times S$ a $LM^\otimes$-monoid of $\Cat_{\cocoscat}(\kappa)$ that exhibits a cocartesian fibration $N \to S$ as a left-modal over some cocartesian fibration $C \to S$ (equivalently a cocartesian $S$-family $N^\otimes \to LM^\otimes \times S$ of $LM^\otimes$-monoidal categories.)

Set $\mathcal{O}^\otimes := Ass^\otimes \times_{LM^\otimes} N^\otimes$.

Denote

\[
\psi_N : \LinFun^/\Sl^/\Sl(C, N) \to \Map^/\Sl^/\Sl(C, N) \to \Map^/\Sl^/\Sl(S, N) \simeq N
\]

the composition of the forgetful functor over $S$ and the functor over $S$ induced by the unit $S \to C$ of the associative monoid $C$ of $\Cat_{\cocoscat}(\kappa)^/\Sl$.

$\psi_N$ is an equivalence.

**Proof.** By Yoneda it is enough to show that for every category $K$ over $S$ the induced map

\[
\Cat_{\cocoscat}(\kappa)^/\Sl(K, \psi_N) : \Cat_{\cocoscat}(\kappa)^/\Sl(K, \LinFun^/\Sl^/\Sl(C, N)) \to \Cat_{\cocoscat}(\kappa)^/\Sl(K, N)
\]

is an equivalence.

The map $\Cat_{\cocoscat}(\kappa)^/\Sl(K, \psi_N)$ is equivalent to the map

\[
\Cat_{\cocoscat}(\kappa)^/\Sl(K, K \times_S \psi_N) : \Cat_{\cocoscat}(\kappa)^/\Sl(K, K \times_S \LinFun^/\Sl^/\Sl(C, N)) \simeq \Cat_{\cocoscat}(\kappa)^/\Sl(K, K \times_S N).
\]
The functor $K \times_{\mathcal{S}} \psi_{\mathcal{N}} : K \times_{\mathcal{S}} \text{LinFun}_{\mathcal{C}}^{/S}(\mathcal{E}, \mathcal{N}) \to K \times_{\mathcal{S}} \mathcal{N}$ over $K$ is equivalent to

the functor $K \times_{\mathcal{S}} \text{LinFun}_{\mathcal{C}}^{/S}(\mathcal{E}, \mathcal{N}) \simeq \text{LinFun}_{K \times_{\mathcal{C}} \mathcal{E}}^{/K}(K \times_{\mathcal{S}} \mathcal{E}, K \times_{\mathcal{S}} \mathcal{N}) \xrightarrow{\psi_{K \times_{\mathcal{S}} \mathcal{N}}} K \times_{\mathcal{S}} \mathcal{N}$

over $K$.

Consequently it is enough to show that the map

$$\text{Cat}_{\infty}(\kappa)_{/S}(S, \psi_{\mathcal{N}}) : \text{Cat}_{\infty}(\kappa)_{/S}(S, \text{LinFun}_{\mathcal{C}}^{/S}(\mathcal{E}, \mathcal{N})) \to \text{Cat}_{\infty}(\kappa)_{/S}(S, \mathcal{N})$$

is an equivalence.

Given $\text{LM}^{\text{op}}$-monoids $M, M'$ of $\text{Cat}_{\infty}^{\text{cart}}(\kappa)$ we have a canonical equivalence

$$\text{Cat}_{\infty}(\kappa)_{/S}(S, \text{LinFun}_{\mathcal{C}}^{/S}(M, M')) \simeq \text{Funs}(S, \text{LinFun}_{\mathcal{C}}^{/S}(M, M'))$$

Moreover the forgetful functor $\text{LinFun}_{\mathcal{C}}^{/S}(M, M') \to \text{Map}^{/S}(M, M')$ over $S$ induces the forgetful map

$$\text{LM}_{\mathcal{C}}(\text{Cat}_{\infty}(\kappa)_{/S})(M, M') \simeq \text{Cat}_{\infty}(\kappa)_{/S}(S, \text{LinFun}_{\mathcal{C}}^{/S}(M, M'))$$

$$\to \text{Cat}_{\infty}(\kappa)_{/S}(S, \text{Map}^{/S}(M, M')) \simeq \text{Cat}_{\infty}(\kappa)_{/S}(M, M').$$

So $\text{Cat}_{\infty}(\kappa)_{/S}(S, \psi_{\mathcal{N}})$ factors as

$$\text{Cat}_{\infty}(\kappa)_{/S}(S, \text{LinFun}_{\mathcal{C}}^{/S}(\mathcal{E}, \mathcal{N})) \simeq \text{LM}_{\mathcal{C}}(\text{Cat}_{\infty}(\kappa)_{/S})(\mathcal{E}, \mathcal{N}) \to \text{Cat}_{\infty}(\kappa)_{/S}(\mathcal{E}, \mathcal{N})$$

$$\to \text{Cat}_{\infty}(\kappa)_{/S}(S, \mathcal{N}),$$

where the last map is induced by the unit of $\mathcal{E}$.

But the map $\text{LM}_{\mathcal{C}}(\text{Cat}_{\infty}(\kappa)_{/S})(\mathcal{E}, \mathcal{N}) \to \text{Cat}_{\infty}(\kappa)_{/S}(\mathcal{E}, \mathcal{N}) \to \text{Cat}_{\infty}(\kappa)_{/S}(S, \mathcal{N})$ is an equivalence as the unit $S \to \mathcal{E}$ of $\mathcal{E}$ exhibits $\mathcal{E}$ as the free left $\mathcal{E}$-module on the tensorunit $S$ of $\text{Cat}_{\infty}(\kappa)_{/S}$. \qed

**Proposition 7.20.** Let $M^{\text{op}} \to \text{LM}^{\text{op}}, N^{\text{op}} \to \text{LM}^{\text{op}}$ be $\kappa$-small operads over $\text{LM}^{\text{op}}$ that exhibit categories $\mathcal{M}$ respectively $\mathcal{N}$ as pseudo-enriched over locally co-cartesian fibrations of operads $\mathcal{C}^{\text{op}} \to \text{Ass}^{\text{op}}$ respectively $\mathcal{D}^{\text{op}} \to \text{Ass}^{\text{op}}$ and let $F : M^{\text{op}} \to N^{\text{op}}$ be a map of operads over $\text{LM}^{\text{op}}$.

Let $X, Y$ be objects of $\mathcal{M}$ that admit a morphism object $[X, Y]$.

The canonical morphisms

$$F([Y, X]) \to [F(Y), F(X)], \quad F([X, X]) \to [F(X), F(X)]$$

in $\mathcal{D}$ organize to a morphism of $\text{LM}^{\text{op}}$-algebras, where $F([Y, X])$ carries the $F([X, X])$-left modul structure that is the image under $F : \mathcal{E} \to \mathcal{D}$ of the canonical $[X, X]$-left modul structure on $[Y, X]$ and $[F(Y), F(X)]$ carries the canonical $[F(X), F(X)]$-left modul structure.
Proof. Denote \( F' : M^\otimes \to N^\otimes \) the \( LM^\otimes \)-monoidal functor \( P_\alpha(Env_{LM^\otimes}(M))^\otimes \to P_\alpha(Env_{LM^\otimes}(N))^\otimes \) and \( \iota : M^\otimes \subset M^\otimes \), \( \iota' : N^\otimes \subset N^\otimes \) the canonical full embeddings of operads over \( LM^\otimes \). Set \( \mathcal{C}^\otimes := P_\alpha(Env_{A^\otimes}(\mathcal{C}))^\otimes , \mathcal{D}^\otimes := P_\alpha(Env_{A^\otimes}(\mathcal{D}))^\otimes \).

We have a canonical equivalence \( F' \circ \iota \simeq \iota' \circ F \) of maps of operads \( M^\otimes \to N^\otimes \) over \( LM^\otimes \).

By lemma \( \{11 \} \) and \( \{13 \} \), \( \iota : M^\otimes \subset M^\otimes \) and \( \iota' : N^\otimes \subset N^\otimes \) preserve morphism objects.

It is enough to see that for every objects \( X, Y \) of \( M \) the canonical morphisms
\[
F'(\iota(\iota(Y),\iota(X))) \simeq F'(\iota(X)) \simeq \iota'(\iota((Y),\iota(X))) \simeq \iota'(\iota(X))
\]

and
\[
F'(\iota(\iota(Y),\iota(X))) \simeq F'(\iota(X)) \simeq \iota'(\iota((Y),\iota(X))) \simeq \iota'(\iota(X))
\]
in \( \mathcal{D}' \) lift to a morphism of \( LM^\otimes \)-algebras, where \( \iota'(\iota((Y),\iota(X))) \simeq F'(\iota(X)) \simeq \iota'(\iota(X)) \simeq \iota'(\iota(X)) \) carries the \( \iota'(\iota((Y),\iota(X))) \simeq F'(\iota(X)) \simeq \iota'(\iota(X)) \)-left module structure that is the image under \( F' : \mathcal{C}' \to \mathcal{D}' \) of the canonical \( \iota(X),\iota(X) \)-left modul structure on \( \iota(X) \) and \( \iota'(\iota(X)) \).

Consequently we can assume that \( M^\otimes \to LM^\otimes , N^\otimes \to LM^\otimes \) are \( LM^\otimes \)-monoidal categories that exhibit categories \( M \) respectively \( N \) as enriched over monoidal categories \( \mathcal{C} \) respectively \( \mathcal{D} \) and that \( F : M^\otimes \to N^\otimes \) is a \( LM^\otimes \)-monoidal functor.

We will show that for every object \( Y \) of \( M \) the canonical natural transformation
\[
F \circ [Y,-] \to [F(Y),-] \circ F
\]
of functors \( M \to \mathcal{D} \) lifts to a natural transformation over \( LM^\otimes \) of maps of operads \( M^\otimes \to LM^\otimes \times_{A^\otimes} \mathcal{D}^\otimes \) over \( LM^\otimes \), whose pullback to \( A^\otimes \) is the identity of \( F \).

If this is shown, the endomorphism \([X,X]\)-left modul structure on \( X \) yields the desired map of \( LM^\otimes \)-algebras in \( \mathcal{D} \).

Evaluation at the tensorunit of \( \mathcal{C} \) induces a functor \( LinFun_{\mathcal{C}}(\mathcal{C}, M) \to Fun(\mathcal{C}, M) \to M \). The canonical functor
\[
LinFun_{\mathcal{D}}(\mathcal{D}, N) \to LinFun_{\mathcal{C}}(Fun(\mathcal{D}), Fun(\mathcal{C}, F^*(N))) \to LinFun_{\mathcal{C}}(Fun(\mathcal{C}, F^*(N))) \simeq N
\]
is equivalent to evaluation at the tensorunit of \( \mathcal{D} \) and is thus an equivalence.

Moreover \( F : M \to N \) is equivalent to the composition
\[
M = LinFun_{\mathcal{C}}(\mathcal{C}, M) \to LinFun_{\mathcal{C}}(\mathcal{C}, F^*(N)) \simeq N.
\]

Thus we obtain a canonical equivalence \((\otimes Y) \circ F \to F \circ (\otimes Y) \) in \( LinFun_{\mathcal{C}}(\mathcal{C}, F^*(N)) \) corresponding to an equivalence of \( LM^\otimes \)-monoidal functors \( LM^\otimes \times_{A^\otimes} \mathcal{C}^\otimes \to N^\otimes \), whose pullback to \( A^\otimes \) is the identity of \( F \).
The desired natural transformation \([Y, -] \to [F(Y), -] \circ F\) over \(LM^\otimes\) is the composition

\[
F \circ [Y, -] = [F(Y), -] \circ (\_ \otimes F(Y)) \circ F \circ [Y, -] = [F(Y), -] \circ F \circ (\_ \otimes Y) \circ [Y, -] = [F(Y), -] \circ F
\]
of natural transformation over \(LM^\otimes\) of maps of operads \(M^\otimes \to LM^\otimes \times_{Ass^\otimes} D^\otimes\) over \(LM^\otimes\).

\[
\square
\]

Let \(M^\otimes\) be an operad over \(LM^\otimes\) that exhibits a category \(M\) as enriched over a locally cocartesian fibration of operads \(D^\otimes \to Ass^\otimes\).

Let \(C^\otimes \to Ass^\otimes\) be a locally cocartesian fibration of operads and \(F : C^\otimes \to D^\otimes\) a map of operads over \(Ass^\otimes\) that admits a right adjoint \(G : D^\otimes \to C^\otimes\) relative to \(Ass^\otimes\).

We will show in the following that one can pullback \(M^\otimes\) along \(F : C^\otimes \to D^\otimes\) to obtain an operad \(F^*(M)^\otimes\) over \(LM^\otimes\) that exhibits \(M\) as enriched over the locally cocartesian fibration of operads \(C^\otimes \to Ass^\otimes\).

We start with the following construction:

**Construction 4.** Let \(M^\otimes\) be an operad over \(LM^\otimes\). Set \(D^\otimes := Ass^\otimes \times_{LM^\otimes} M^\otimes\).

Let \(F : C^\otimes \to D^\otimes\) be a map of operads over \(Ass^\otimes\).

Pulling back the \(LM^\otimes\)-monoidal category \(\hat{M}^\otimes := Env_{LM^\otimes}(M^\otimes) \to LM^\otimes\) along the monoidal functor \(\hat{F} := Env_{Ass^\otimes}(F) : C^\otimes := Env_{Ass^\otimes}(C^\otimes) \to Env_{Ass^\otimes}(D^\otimes) := D^\otimes\) we get a \(LM^\otimes\)-monoidal category \(F^*(M)^\otimes \to LM^\otimes\) that exhibits \(\hat{M} := \{m\} \times_{LM^\otimes} \hat{M}^\otimes\) as left module over the monoidal category \(C^\otimes = Env_{Ass^\otimes}(C^\otimes)\).

Denote \(F^*(M)^\otimes \subset F^*(\hat{M}^\otimes)\) the full suboperad spanned by the objects that belong to \(C\) or \(M\).

Then we have a canonical equivalence \(C^\otimes \simeq Ass^\otimes \times_{LM^\otimes} F^*(M)^\otimes\) of operads over \(Ass^\otimes\) and a canonical equivalence \(M \simeq \{m\} \times_{LM^\otimes} F^*(M)^\otimes\).

The map \(F^*(M)^\otimes \subset F^*(\hat{M}^\otimes)\) of operads over \(LM^\otimes\) induces a map \(F^*(M)^\otimes \to M^\otimes\) of operads over \(LM^\otimes\), whose fiber over \(\{m\} \in LM\) is the identity of \(M\) and whose pullback to \(Ass^\otimes\) is \(F : C^\otimes \to D^\otimes\).

As next we show that the canonical map \(F^*(M)^\otimes \to M^\otimes\) of operads over \(LM^\otimes\) is cartesian with respect to the forgetful functor \(Op^\infty(\kappa)_{LM^\otimes} \to Op^\infty(\kappa)_{Ass^\otimes}\) so that the forgetful functor \(Op^\infty(\kappa)_{LM^\otimes} \to Op^\infty(\kappa)_{Ass^\otimes}\) is a cartesian fibration.

**Lemma 4.22** states that if \(M^\otimes\) exhibits a category \(M\) as enriched over a locally cocartesian fibration of operads \(D^\otimes \to Ass^\otimes\), then \(F^*(M)^\otimes\) exhibits \(M\) as enriched over \(C^\otimes \to Ass^\otimes\).

Denote \((Op^\infty(\kappa)_{Ass^\otimes})^{rep} \subset Op^\infty(\kappa)_{Ass^\otimes}\) and \((Op^\infty(\kappa)_{Ass^\otimes})^{rep,L} \subset (Op^\infty(\kappa)_{Ass^\otimes})^{L}\) the full subcategories spanned by the locally cocartesian fibrations of operads over \(Ass^\otimes\) and \((Op^\infty(\kappa)_{LM^\otimes})^{en} \subset Op^\infty(\kappa)_{LM^\otimes}\) the full subcategory spanned by the operads \(M^\otimes\) over \(LM^\otimes\) that exhibit a category \(M\) as enriched over a locally cocartesian fibration of operads \(D^\otimes \to Ass^\otimes\).

The forgetful functor \((Op^\infty(\kappa)_{LM^\otimes})^{en} \to Op^\infty(\kappa)_{Ass^\otimes})^{rep}\) restricts to a functor \((Op^\infty(\kappa)_{LM^\otimes})^{en} \to (Op^\infty(\kappa)_{Ass^\otimes})^{rep,L}\) over \(LM^\otimes\).
By lemma [7.22] the cartesian fibration

\[(\text{Op}_\infty(\kappa)/\text{Ass}^\circ)^{\text{rep},L} \times_{\text{Op}_\infty(\kappa)/\text{Ass}^\circ} \text{Op}_\infty(\kappa)/\text{LM}^\circ \to \text{Op}_\infty(\kappa)/\text{Ass}^\circ)^{\text{rep},L}\]

restricts to a cartesian fibration

\[(\text{Op}_\infty(\kappa)/\text{Ass}^\circ)^{\text{rep},L} \times (\text{Op}_\infty(\kappa)/\text{Ass}^\circ)^{\text{rep},c} \to (\text{Op}_\infty(\kappa)/\text{Ass}^\circ)^{\text{rep},L}\]

with the same cartesian morphisms.

**Proposition 7.21.** Let $\mathcal{M}^\circ$ be an operad over $\text{LM}^\circ$. Set $\mathcal{D}^\circ := \text{Ass}^\circ \times_{\text{LM}^\circ} \mathcal{M}^\circ$.

Let $F : \mathcal{C}^\circ \to \mathcal{D}^\circ$ be a map of operads over $\text{Ass}^\circ$.

For every operad $\mathcal{Q}^\circ$ over $\text{LM}^\circ$, where we set $\mathcal{B}^\circ := \text{Ass}^\circ \times_{\text{LM}^\circ} \mathcal{Q}^\circ$, the canonical map $F^*(\mathcal{M}^\circ) \to \mathcal{M}^\circ$ of operads over $\text{LM}^\circ$ induces a pullback square

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(F^*(\mathcal{M}^\circ)) & \to & \text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(\mathcal{M}^\circ) \\
\downarrow & & \downarrow \\
\text{Alg}_{\mathcal{B}^\circ/\text{Ass}^\circ}(\mathcal{Q}^\circ) & \to & \text{Alg}_{\mathcal{B}^\circ/\text{Ass}^\circ}(\mathcal{D}^\circ).
\end{array}
\]

Especially the canonical map $F^*(\mathcal{M}^\circ) \to \mathcal{M}^\circ$ of operads over $\text{LM}^\circ$ is cartesian with respect to the forgetful functor $\text{Op}_\infty(\kappa)/\text{LM}^\circ \to \text{Op}_\infty(\kappa)/\text{Ass}^\circ$.

Thus the forgetful functor $\text{Op}_\infty(\kappa)/\text{LM}^\circ \to \text{Op}_\infty(\kappa)/\text{Ass}^\circ$ is a cartesian fibration.

**Proof.** Square (14) embeds into the commutative square

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(F^*(\mathcal{M}^\circ)) & \to & \text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(\mathcal{M}^\circ) \\
\downarrow & & \downarrow \\
\text{Alg}_{\mathcal{B}^\circ/\text{Ass}^\circ}(\mathcal{Q}^\circ) & \to & \text{Alg}_{\mathcal{B}^\circ/\text{Ass}^\circ}(\mathcal{D}^\circ).
\end{array}
\]

Assume that we have already shown that square (14) is a pullback square. Then the full subcategory inclusion

\[
\text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(F^*(\mathcal{M}^\circ)) \subset \text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(\mathcal{M}^\circ)
\]

factors as

\[
\text{Alg}_{\mathcal{Q}^\circ/\text{LM}^\circ}(F^*(\mathcal{M}^\circ)) \subset \text{Alg}_{\mathcal{B}^\circ/\text{Ass}^\circ}(\mathcal{Q}^\circ) \times_{\text{Alg}_{\mathcal{B}^\circ/\text{Ass}^\circ}(\mathcal{D}^\circ)} \text{Alg}_{\mathcal{B}^\circ/\mathcal{Q}^\circ/\text{LM}^\circ}(\mathcal{M}^\circ)
\]

Hence the canonical functor $\chi$ is fully faithful.

Let $\psi : \mathcal{Q}^\circ \to F^*(\mathcal{M}^\circ)$ be a map of operads over $\text{LM}^\circ$, whose pullback to $\text{Ass}^\circ$ induces a map $\mathcal{B}^\circ \to \mathcal{C}^\circ$ of operads over $\text{Ass}^\circ$ and such that the composition $\psi' : \mathcal{Q}^\circ \to F^*(\mathcal{M}^\circ)$ factors through $\mathcal{M}^\circ$.

Thus $\chi$ is essentially surjective and so an equivalence.

So it remains to show that square (14) is a pullback square.

Set $\mathcal{Q}^\circ := \text{Env}_{\text{LM}^\circ}(\mathcal{Q})$ and $\mathcal{B}^\circ := \text{Env}_{\text{LM}^\circ}(\mathcal{B})$.

Using lemma [7.21] square (14) is equivalent to the commutative square

\[
\begin{array}{ccc}
\text{Fun}_{\text{LM}^\circ}(\mathcal{Q}^\circ, F^*(\mathcal{M}^\circ)) & \to & \text{Fun}_{\text{LM}^\circ}(\mathcal{Q}^\circ, \mathcal{M}^\circ) \\
\downarrow & & \downarrow \\
\text{Fun}_{\mathcal{B}^\circ/\mathcal{C}^\circ}(\mathcal{B}^\circ, \mathcal{D}^\circ) & \to & \text{Fun}_{\mathcal{B}^\circ/\mathcal{C}^\circ}(\mathcal{B}^\circ, \mathcal{D}^\circ).
\end{array}
\]
So it is enough to see the following:

Let \( \mathcal{M}^\oplus \) and \( \mathcal{Q}^\oplus \) be \( \text{LM}^\oplus \)-monoidal categories and \( F: \mathcal{C}^\oplus \to \mathcal{D}^\oplus \) a monoidal functor.

Denote \( \gamma \) the cartesian fibration \( \text{Op}_\infty(\kappa)^{\text{cocart}} \to \text{LM}_\infty(\kappa)^{\text{cocart}} \) (being equivalent to the cartesian fibration \( \text{Alg}_{\text{LM}_\infty(\kappa)^{\ast}}(\mathbb{C}) \to \text{Alg}_{\text{Ass}_\infty(\kappa)^{\ast}}(\mathbb{C}) \))

Let \( F^\ast (\mathcal{M})^\oplus \to \mathcal{M}^\oplus \) be a \( \gamma \)-cartesian lift of \( F \).

Then the commutative square

\[
\begin{array}{ccc}
\text{Fun}_{\text{LM}_\infty}^\oplus(\mathcal{Q}^\oplus, F^\ast (\mathcal{M})^\oplus) & \to & \text{Fun}_{\text{LM}_\infty}^\oplus(\mathcal{Q}^\oplus, \mathcal{M}^\oplus) \\
\downarrow & & \downarrow \\
\text{Fun}_{\text{Ass}_\infty}^\oplus(\mathcal{B}^\oplus, \mathcal{C}^\oplus) & \to & \text{Fun}_{\text{Ass}_\infty}^\oplus(\mathcal{B}^\oplus, \mathcal{D}^\oplus)
\end{array}
\]

is a pullback square.

This square is a pullback square if and only if for every \( \kappa \)-small category \( T \) the commutative square

\[
\begin{array}{ccc}
\text{Cat}_\infty(\kappa)(T, \text{Fun}_{\text{LM}_\infty}^\oplus(\mathcal{Q}^\oplus, F^\ast (\mathcal{M})^\oplus)) & \to & \text{Cat}_\infty(\kappa)(T, \text{Fun}_{\text{LM}_\infty}^\oplus(\mathcal{Q}^\oplus, \mathcal{M}^\oplus)) \\
\downarrow & & \downarrow \\
\text{Cat}_\infty(\kappa)(T, \text{Fun}_{\text{Ass}_\infty}^\oplus(\mathcal{B}^\oplus, \mathcal{C}^\oplus)) & \to & \text{Cat}_\infty(\kappa)(T, \text{Fun}_{\text{Ass}_\infty}^\oplus(\mathcal{B}^\oplus, \mathcal{D}^\oplus))
\end{array}
\]

is a pullback square.

This square is equivalent to the commutative square

\[
\begin{array}{ccc}
\text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{Q}^\oplus, (F^\ast (\mathcal{M})^\oplus)^T) & \to & \text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{Q}^\oplus, (\mathcal{M}^\oplus)^T) \\
\downarrow & & \downarrow \\
\text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{B}^\oplus, (\mathcal{C}^\oplus)^T) & \to & \text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{B}^\oplus, (\mathcal{D}^\oplus)^T).
\end{array}
\] (16)

Being right adjoint to the functor \( T \times - : \text{Cat}_\infty(\kappa) \to \text{Cat}_\infty(\kappa) \) the functor \( \text{Fun}(T, -) : \text{Cat}_\infty(\kappa) \to \text{Cat}_\infty(\kappa) \) preserves finite products and so lifts to a symmetric monoidal functor that induces functors

\[
\beta : \text{Alg}_{\text{LM}_\infty(\kappa)^{\ast}}(\mathbb{C}) \to \text{Alg}_{\text{Ass}_\infty(\kappa)^{\ast}}(\mathbb{C})
\]

and

\[
\text{Alg}_{\text{Ass}_\infty(\kappa)^{\ast}}(\mathbb{C}) \to \text{Alg}_{\text{Ass}_\infty(\kappa)^{\ast}}(\mathbb{C})
\]

that are equivalent to the functors \( (-)^T : \text{Alg}_{\text{LM}_\infty(\kappa)^{\ast}}(\mathbb{C}) \to \text{Alg}_{\text{LM}_\infty(\kappa)^{\ast}}(\mathbb{C}) \) respectively \( (-)^T : \text{Alg}_{\text{Ass}_\infty(\kappa)^{\ast}}(\mathbb{C}) \to \text{Alg}_{\text{Ass}_\infty(\kappa)^{\ast}}(\mathbb{C}) \).

The \( \gamma \)-cartesian morphisms are those that get equivalences in \( \text{Cat}_\infty(\kappa) \).

Thus \( \beta \) sends \( \gamma \)-cartesian morphisms to \( \gamma \)-cartesian morphisms so that \( F^\ast (\mathcal{M}^\oplus)^T \to (\mathcal{M}^\oplus)^T \) factors as \( F^\ast (\mathcal{M}^\oplus)^T = (F^T)^\ast ((\mathcal{M}^\oplus)^T) \to (\mathcal{M}^\oplus)^T \) in \( \text{Alg}_{\text{LM}_\infty(\kappa)^{\ast}}(\mathbb{C}) \).

Thus square (16) is equivalent to the commutative square

\[
\begin{array}{ccc}
\text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{Q}^\oplus, (F^T)^\ast ((\mathcal{M}^\oplus)^T)) & \to & \text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{Q}^\oplus, (\mathcal{M}^\oplus)^T) \\
\downarrow & & \downarrow \\
\text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{B}^\oplus, (\mathcal{C}^\oplus)^T) & \to & \text{Op}_\infty(\kappa)^{\text{cocart}}(\mathcal{B}^\oplus, (\mathcal{D}^\oplus)^T).
\end{array}
\]
Consequently it is enough to see that the commutative square

$$\begin{array}{ccc}
O_{\infty}(\kappa)_{/\operatorname{Ass}^\otimes}(\Omega^\otimes,\mathcal{F}^*(\mathcal{M})^\otimes) & \longrightarrow & O_{\infty}(\kappa)_{/\operatorname{Ass}^\otimes}(\Omega^\otimes,\mathcal{M}^\otimes) \\
\downarrow & & \downarrow \\
O_{\infty}(\kappa)_{/\operatorname{Ass}^\otimes}(\mathcal{B}^\otimes,\mathcal{C}^\otimes) & \longrightarrow & O_{\infty}(\kappa)_{/\operatorname{Ass}^\otimes}(\mathcal{B}^\otimes,\mathcal{D}^\otimes)
\end{array}$$

is a pullback square, which follows from the fact that $\mathcal{F}^*(\mathcal{M})^\otimes \rightarrow \mathcal{M}^\otimes$ is $\gamma$-cartesian.

Lemma 7.22. Let $\mathcal{M}^\otimes$ be an operad over $\operatorname{LM}^\otimes$ that exhibits a category $\mathcal{M}$ as enriched over a locally cocartesian fibration of operads $\mathcal{D}^\otimes \rightarrow \operatorname{Ass}^\otimes$.

Let $\mathcal{C}^\otimes \rightarrow \operatorname{Ass}^\otimes$ be a locally cocartesian fibration of operads and $\mathcal{F} : \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ a map of operads over $\operatorname{Ass}^\otimes$ that admits a right adjoint $\mathcal{G} : \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ relative to $\operatorname{Ass}^\otimes$.

The operad $\mathcal{F}^*(\mathcal{M})^\otimes$ over $\operatorname{LM}^\otimes$ exhibits $\mathcal{M}$ as enriched over $\mathcal{C}^\otimes \rightarrow \operatorname{Ass}^\otimes$.

The morphism object of $\mathcal{F}^*(\mathcal{M})^\otimes$ of two objects $X, Y$ of $\mathcal{M}$ is given by $G([X, Y]) \in \mathcal{C}$, where $[X, Y]$ denotes the morphism object of $X$ and $Y$ of $\mathcal{M}^\otimes$.

Proof. Pulling back the $\operatorname{LM}^\otimes$-monoidal category $\bar{\mathcal{M}}^\otimes := \operatorname{Env}_{\operatorname{LM}^\otimes}(\mathcal{M})^\otimes \rightarrow \operatorname{LM}^\otimes$ along the monoidal functor $\bar{\mathcal{F}} : \bar{\mathcal{C}}^\otimes \rightarrow \bar{\mathcal{D}}^\otimes$ we get a $\operatorname{LM}^\otimes$-monoidal category $\mathcal{F}^*(\mathcal{M})^\otimes \rightarrow \mathcal{L}^\otimes$ that exhibits $\bar{\mathcal{M}}^\otimes := \{m\} \times_{\operatorname{LM}^\otimes} \mathcal{M}^\otimes$ as left module over the monoidal category $\bar{\mathcal{C}}^\otimes = \operatorname{Env}_{\operatorname{Ass}^\otimes}(\bar{\mathcal{C}}^\otimes)$ and $\mathcal{F}^*(\mathcal{M})^\otimes \subset \mathcal{F}^*(\mathcal{M})^\otimes$ is defined to be the full suboperad spanned by the objects that belong to $\mathcal{M}$ or $\mathcal{C}$.

Being a 2-functor $\operatorname{Env}_{\operatorname{Ass}^\otimes} : O_{\infty}(\kappa)_{/\operatorname{Ass}^\otimes} \rightarrow O_{\infty}(\kappa)_{/\operatorname{Ass}^\otimes}$ sends the adjunction $\mathcal{F} : \mathcal{C}^\otimes \rightleftarrows \mathcal{D}^\otimes : \mathcal{G}$ relative to $\operatorname{Ass}^\otimes$ to an adjunction $\bar{\mathcal{F}} := \operatorname{Env}_{\operatorname{Ass}^\otimes}(\mathcal{F}) : \bar{\mathcal{C}}^\otimes \rightleftarrows \bar{\mathcal{D}}^\otimes := \operatorname{Env}_{\operatorname{Ass}^\otimes}(\mathcal{G}) := : G$ relative to $\operatorname{Ass}^\otimes$.

Given two objects $X, Y$ of $\mathcal{M}$ by lemma 7.11 the morphism object $[X, Y]$ of $\mathcal{M}^\otimes$ is a morphism object of $\bar{\mathcal{M}}^\otimes$.

So given an object $A \in \operatorname{Env}_{\operatorname{Ass}^\otimes}(\mathcal{C})$ we have a canonical equivalence $\bar{\mathcal{C}}(A, G([X, Y])) \cong \bar{\mathcal{D}}(\bar{\mathcal{F}}(A), [X, Y]) \cong \bar{\mathcal{M}}(\bar{\mathcal{F}}(A) \otimes X, Y) \cong \bar{\mathcal{F}}^*(\bar{\mathcal{M}})(A \otimes X, Y)$.

Thus the statement follows from lemma 7.21.

Lemma 7.23. Let $\varphi : \mathcal{C}^\otimes \rightarrow \operatorname{Ass}^\otimes$ be a locally cocartesian fibration of operads and $\mathcal{M}^\otimes \rightarrow \operatorname{LM}^\otimes$ an operad over $\operatorname{LM}^\otimes$ that exhibits a category $\mathcal{M}$ as pseudo-enriched over the monoidal category $\varphi^! : \operatorname{Env}_{\operatorname{Ass}^\otimes}(\mathcal{C}^\otimes) \rightarrow \operatorname{Ass}^\otimes$.

Let $\mathcal{N} \subset \mathcal{M}$ be a full subcategory.

Assume that every objects $X, Y \in \mathcal{N}$ admit a morphism object $[X, Y]$ in $\operatorname{Env}_{\operatorname{Ass}^\otimes}(\mathcal{C})$ that belongs to $\mathcal{C}$.

Denote $\mathcal{N}^\otimes \subset \mathcal{M}^\otimes$ the full suboperad spanned by the objects that belong to $\mathcal{N}$ or $\mathcal{C}$.

Then $\mathcal{N}^\otimes \rightarrow \operatorname{LM}^\otimes$ exhibits the category $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\varphi : \mathcal{C}^\otimes \rightarrow \operatorname{Ass}^\otimes$. 

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Proof. Let $A_1, \ldots, A_n \in \mathcal{C}$ be objects of $\mathcal{C}$ for some $n \in \mathbb{N}$ and $\alpha \in \text{Ass}_n$ an operation.

By ... we have a canonical equivalence $\text{Env}_{\text{Ass}}^\otimes(\mathcal{C})^\otimes \simeq \text{Act}(\text{Ass})^\otimes \times \text{Fun}(\{1\}, \text{Ass})^\otimes$ $\mathcal{C}^\otimes$ over $\text{Fun}(\{1\}, \text{Ass})^\otimes$ and the full suboperad inclusion $\mathcal{C}^\otimes \subset \text{Env}_{\text{Ass}}^\otimes(\mathcal{C})^\otimes \simeq \text{Act}(\text{Ass})^\otimes \times \text{Fun}(\{0\}, \text{Ass})^\otimes$ $\mathcal{C}^\otimes$ over $\text{Ass}^\otimes$ is the pullback of the diagonal embedding $\text{Ass}^\otimes \subset \text{Act}(\text{Ass})^\otimes$ along the functor $\text{Act}(\text{Ass})^\otimes \times \text{Fun}(\{0\}, \text{Ass})^\otimes$ $\mathcal{C}^\otimes \to \text{Act}(\text{Ass})^\otimes$.

Thus $(A_1, \ldots, A_n, \alpha)$ corresponds to an object $A$ of $\text{Env}_{\text{Ass}}^\otimes(\mathcal{C})^\otimes$, which can be obtained as $A \simeq \otimes_a (A_1, \ldots, A_n)$, where we consider $A_1, \ldots, A_n$ as objects of $\text{Env}_{\text{Ass}}^\otimes(\mathcal{C})$ via the natural embedding $\mathcal{C} \subset \text{Env}_{\text{Ass}}^\otimes(\mathcal{C})$ and form the tensor-product of the monoidal category $\text{Env}_{\text{Ass}}^\otimes(\mathcal{C})^\otimes$.

Denote $\beta : (A_1, \ldots, A_n) \to \otimes_a (A_1, \ldots, A_n) \simeq A$ a $\varphi'$-cocartesian lift of $\alpha$.

Denote $\sigma \in \text{Mul}_{LM^\otimes}(a, m; m)$ the unique object and $\alpha'$ the image of $\alpha$, the identity of $m$ and $\sigma$ under the operadic composition $\text{Mul}_{LM^\otimes}(a, m; m) \times (\text{Mul}_{LM^\otimes}(a, a; a) \times \text{Mul}_{LM^\otimes}(m; m)) \to \text{Mul}_{LM^\otimes}(a, a; m)$.

Let $X, Y$ be objects of $N$.

As $M^\otimes \to LM^\otimes$ exhibits $M$ as pseudo-enriched over the monoidal category $\mathcal{C}^\otimes$ under the operadic composition with $\beta$

$$\text{Mul}_{LM^\otimes}(A, X; Y) \to \{\alpha'\} \times \text{Mul}_{LM^\otimes}(a, a; m) \text{Mul}_{LM^\otimes}(A_1, \ldots, A_n, X; Y)$$

is an equivalence.

Denote $\gamma : (A_1, \ldots, A_n) \to \otimes_a (A_1, \ldots, A_n)$ a locally $\varphi$-cocartesian lift of $\alpha$.

We have to see that composition with $\gamma$

$$\text{Mul}_{LM^\otimes}(\otimes_a (A_1, \ldots, A_n), X; Y) \to \{\alpha'\} \times \text{Mul}_{LM^\otimes}(a, a; m) \text{Mul}_{LM^\otimes}(A_1, \ldots, A_n, X; Y)$$

is an equivalence.

If this is shown, $N^\otimes \to LM^\otimes$ exhibits the category $N$ as pseudo-enriched over the locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \text{Ass}^\otimes$.

As every objects $X, Y \in N$ admit a morphism object $[X, Y]$ in $\text{Env}_{\text{Ass}}^\otimes(\mathcal{C})$ that belongs to $\mathcal{C}$, then $N^\otimes \to LM^\otimes$ exhibits the category $N$ as enriched over the locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \text{Ass}^\otimes$.

By [122, 1], the full suboperad inclusion $\mathcal{C} \subset \text{Env}_{\text{Ass}}^\otimes(\mathcal{C})$ admits a left adjoint $L_{\gamma}$, where the unit $\eta : A \to L(A)$ corresponds to $\gamma : (A_1, \ldots, A_n) \to \otimes_a (A_1, \ldots, A_n)$ together with the commutative square

$$\begin{array}{ccc}
\langle 1 \rangle & \xrightarrow{\alpha} & \langle 1 \rangle \\
\alpha \downarrow & & \downarrow \text{id} \\
\langle 1 \rangle & \xrightarrow{\text{id}} & \langle 1 \rangle
\end{array}$$

in $\text{Ass}^\otimes$.

$\beta$ corresponds to the identity of $(A_1, \ldots, A_n)$ in $\mathcal{C}^\otimes$ together with the commutative square

$$\begin{array}{ccc}
\langle 1 \rangle & \xrightarrow{\text{id}} & \langle 1 \rangle \\
\text{id} \downarrow & & \downarrow \alpha \\
\langle n \rangle & \xrightarrow{\alpha} & \langle 1 \rangle
\end{array}$$

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in \( \text{Ass}^\circ \).

So the image of \( \gamma \) in \( \text{Env}_{\text{Ass}^\circ}(\mathcal{C}) \) factors as \((A_1, \ldots, A_n) \rightarrow A \rightarrow L(A)\) so that composition with \( \gamma \) factors as

\[
\text{Mul}_{M^\#}(L(A), X; Y) \rightarrow \text{Mul}_{M^\#}(A, X; Y) = \{\alpha'\} \times_{\text{Mul}_{LM^\#}(a, \ldots, a, m, m)} \text{Mul}_{M^\#}(A_1, \ldots, A_n, X; Y).
\]

Consequently it is enough to see that composition with \( \eta : A \rightarrow L(A) \)

\[
\text{Mul}_{M^\#}(L(A), X; Y) \rightarrow \text{Mul}_{M^\#}(A, X; Y)
\]

is an equivalence.

As we assumed that \( X, Y \) admit a morphism object \([X, Y] \) in \( \text{Env}_{\text{Ass}^\circ}(\mathcal{C}) \) this map factors as

\[
\text{Mul}_{M^\#}(L(A), X; Y) = \text{Env}_{\text{Ass}^\circ}(\mathcal{C})(L(A), [X, Y]) \rightarrow \text{Env}_{\text{Ass}^\circ}(\mathcal{C})(A, [X, Y])
\]

\[
\rightarrow \text{Mul}_{M^\#}(A, X; Y).
\]

As we assumed that \([X, Y] \) belongs to \( \mathcal{C} \), composition with \( \eta : A \rightarrow L(A) \)

\[
\text{Env}_{\text{Ass}^\circ}(\mathcal{C})(L(A), [X, Y]) \rightarrow \text{Env}_{\text{Ass}^\circ}(\mathcal{C})(A, [X, Y])
\]

is an equivalence.

\[\square\]

7.2.1 Enriched adjunctions

**Lemma 7.24.** Let \( \mathcal{O}^\circ \) be a \( \kappa \)-small operad.

Let \( G : D^\circ \rightarrow \mathcal{C}^\circ \) be a map of \( \kappa \)-small operads over \( \mathcal{O}^\circ \).

The following conditions are equivalent:

1. \( G \) admits a left adjoint relative to \( \mathcal{O}^\circ \).

2. For every object \( X \) of \( \mathcal{O} \) the induced functor \( G_X : D_X \rightarrow \mathcal{C}_X \) on the fiber over \( X \) admits a left adjoint \( F_X : \mathcal{C}_X \rightarrow D_X \) and for all \( n \in \mathbb{N} \) and objects \( X_1, \ldots, X_n, W \) of \( \mathcal{O} \) and objects \( Y_1 \in \mathcal{C}_{X_1}, \ldots, Y_n \in \mathcal{C}_{X_n}, Z \in D_W \) the canonical map

\[
\text{Mul}_{D}(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), Z) \rightarrow \text{Mul}_{C}(G_{X_1}(F_{X_1}(Y_1)), \ldots, G_{X_n}(F_{X_n}(Y_n)), G_W(Z)) \rightarrow \text{Mul}_{C}(Y_1, \ldots, Y_n, G_W(Z))
\]

is an equivalence.

Let \( F : \mathcal{C}^\circ \rightarrow D^\circ \) be a map of \( \kappa \)-small operads over \( \mathcal{O}^\circ \).

The following conditions are equivalent:

1. \( F \) admits a left adjoint relative to \( \mathcal{O}^\circ \).
2. For every object \( X \) of \( \mathcal{O} \) the induced functor \( F_X : \mathcal{E}_X \to \mathcal{D}_X \) on the fiber over \( X \) admits a right adjoint \( G_X : \mathcal{D}_X \to \mathcal{E}_X \) and for all \( n \in \mathbb{N} \) and objects \( X_1, \ldots, X_n, W \) of \( \mathcal{O} \) and objects \( Y_1 \in \mathcal{E}_{X_1}, \ldots, Y_n \in \mathcal{E}_{X_n}, Z \in \mathcal{D}_W \) the canonical map

\[
\text{Mul}_C(Y_1, \ldots, Y_n, G_W(Z)) \to \\
\text{Mul}_D(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), F_W(G_W(Z))) \to \text{Mul}_D(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), Z)
\]

is an equivalence.

Proof. If \( G \) admits a left adjoint relative to \( \mathcal{O}^\circ \), for every object \( X \) of \( \mathcal{O} \) the induced functor \( G_X : \mathcal{D}_X \to \mathcal{E}_X \) on the fiber over \( X \) admits a left adjoint \( F_X : \mathcal{E}_X \to \mathcal{D}_X \).

Moreover for every objects \( Y \in \mathcal{O}^\circ \) lying over some object \( X \) of \( \mathcal{O}^\circ \) and all objects \( Z \in \mathcal{D} \) lying over some object \( W \) of \( \mathcal{O} \) the canonical map \( \Phi : \mathcal{D}^\circ(F_X(Y), Z) \to \mathcal{O}^\circ(G_X(F_X(Y)), G_W(Z)) \to \mathcal{O}^\circ(Y, G_W(Z)) \) over \( \mathcal{O}^\circ(X, W) \) is an equivalence.

The pullback of \( \Phi \) along the full subspace inclusion \( \text{Mul}_O(X_1, \ldots, X_n; W) \subset \mathcal{O}^\circ(X, W) \) is equivalent to the canonical map

\[
\text{Mul}_D(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), Z) \to \\
\text{Mul}_C(G_{X_1}(F_{X_1}(Y_1)), \ldots, G_{X_n}(F_{X_n}(Y_n)), G_W(Z)) \to \text{Mul}_C(Y_1, \ldots, Y_n, G_W(Z)),
\]

where \( X_1, \ldots, X_n \) denote the components of \( X \) and \( Y_1, \ldots, Y_n \) denote the components of \( Y \) for some \( n \in \mathbb{N} \).

So 1. implies 2.

Condition 1. is equivalent to the condition that for all \( Y \in \mathcal{O}^\circ \) lying over some object \( X \) of \( \mathcal{O}^\circ \) there is an object \( T \in \mathcal{D}^\circ \) and a morphism \( \alpha : Y \to G_X(T) \) in \( \mathcal{O}^\circ \) such that for all objects \( Z \in \mathcal{D} \) lying over some object \( W \) of \( \mathcal{O} \) the canonical map \( \Psi : \mathcal{D}^\circ(T, Z) \to \mathcal{O}^\circ(G_X(T), G_W(Z)) \to \mathcal{O}^\circ(Y, G_W(Z)) \) is an equivalence.

The map \( \Psi \) is a map over \( \mathcal{O}^\circ(X, W) \) and is thus an equivalence if and only if it induces on the fiber over every morphism \( \varphi : X \to W \) of \( \mathcal{O}^\circ \) an equivalence.

Using that \( \mathcal{O}^\circ, \mathcal{E}^\circ, \mathcal{D}^\circ \) are operads and \( G : \mathcal{D}^\circ \to \mathcal{E}^\circ \) is a map of operads over \( \mathcal{O}^\circ \) this is equivalent to the condition that \( \Psi \) induces an equivalence on the fiber over every active morphism \( \varphi : X \to W \) of \( \mathcal{O}^\circ \) with \( W \in \mathcal{O} \).

Hence \( \Psi \) is an equivalence if and only if the pullback \( \Psi' \) of \( \Psi \) along the full subspace inclusion \( \text{Mul}_O(X_1, \ldots, X_n; W) \subset \mathcal{O}^\circ(X, W) \) is an equivalence, where \( X_1, \ldots, X_n \) denote the components of \( X \) for some \( n \in \mathbb{N} \).

But \( \Psi' \) is equivalent to the canonical map

\[
\text{Mul}_D(T_1, \ldots, T_n, Z) \to \\
\text{Mul}_C(G_{X_1}(T_1), \ldots, G_{X_n}(T_n), G_W(Z)) \to \text{Mul}_C(Y_1, \ldots, Y_n, G_W(Z))
\]

induced by the components \( \alpha_i : Y_i \to G_{X_i}(T_i) \) of \( \alpha \) in \( \mathcal{D}_{X_i} \) for \( i \in \{1, \ldots, n\} \), where \( Y_1, \ldots, Y_n \) and \( T_1, \ldots, T_n \) denote the components of \( Y \) respectively \( T \).

Hence 2. implies 1.

If \( G \) admits a left adjoint relative to \( \mathcal{O}^\circ \), for every object \( X \) of \( \mathcal{O} \) the induced functor \( G_X : \mathcal{D}_X \to \mathcal{E}_X \) on the fiber over \( X \) admits a left adjoint \( F_X : \mathcal{E}_X \to \mathcal{D}_X \).
Moreover for every objects $Y \in \mathcal{O}^\circ$ lying over some object $X$ of $\mathcal{O}^\circ$ and all objects $Z \in \mathcal{D}$ lying over some object $W$ of $\mathcal{O}$ the canonical map $\Phi : \mathcal{D}^\circ(F_X(Y), Z) \to \mathcal{O}^\circ(G_X(F_X(Y)), G_W(Z))$ over $\mathcal{O}^\circ(X, W)$ is an equivalence.

The pullback of $\Phi$ along the full subspace inclusion $\text{Mul}_\mathcal{O}(X_1, \ldots, X_n; W) \subset \mathcal{O}^\circ(X, W)$ is equivalent to the canonical map

$$\text{Mul}_\mathcal{D}(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), Z) \to \text{Mul}_\mathcal{O}(G_{X_1}(F_{X_1}(Y_1)), \ldots, G_{X_n}(F_{X_n}(Y_n)), G_W(Z)) \to \text{Mul}_\mathcal{O}(Y_1, \ldots, Y_n, G_W(Z))$$

where $X_1, \ldots, X_n$ denote the components of $X$ and $Y_1, \ldots, Y_n$ denote the components of $Y$ for some $n \in \mathbb{N}$.

So 1. implies 2.

The second part is similar.

If $F$ admits a right adjoint relative to $\mathcal{O}^\circ$, for every object $X$ of $\mathcal{O}$ the induced functor $F_X : \mathcal{C}_X \to \mathcal{D}_X$ on the fiber over $X$ admits a right adjoint $G_X : \mathcal{D}_X \to \mathcal{C}_X$.

Moreover for every objects $Y \in \mathcal{C}^\circ$ lying over some object $X$ of $\mathcal{O}^\circ$ and all objects $Z \in \mathcal{D}$ lying over some object $W$ of $\mathcal{O}$ the canonical map $\phi : \mathcal{C}^\circ(Y, G_W(Z)) \to \mathcal{D}^\circ(F_X(Y), F_W(G_W(Z))) \to \mathcal{D}^\circ(F_X(Y), Z)$ over $\mathcal{O}^\circ(X, W)$ is an equivalence.

The pullback of $\phi$ along the full subspace inclusion $\text{Mul}_\mathcal{O}(X_1, \ldots, X_n; W) \subset \mathcal{O}^\circ(X, W)$ is equivalent to the canonical map

$$\text{Mul}_\mathcal{C}(Y_1, \ldots, Y_n, G_W(Z)) \to \text{Mul}_\mathcal{D}(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), G_W(Z)) \to \text{Mul}_\mathcal{O}(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), Z)$$

where $X_1, \ldots, X_n$ denote the components of $X$ and $Y_1, \ldots, Y_n$ denote the components of $Y$ for some $n \in \mathbb{N}$.

So 1. implies 2.

Condition 1. is equivalent to the condition that for all $Z \in \mathcal{D}$ lying over some object $W$ of $\mathcal{O}$ there is an object $T \in \mathcal{C}_W$ and a morphism $\alpha : F_W(T) \to Z$ in $\mathcal{D}_W$ such that for all objects $Y \in \mathcal{C}^\circ$ lying over some object $X$ of $\mathcal{O}^\circ$ the canonical map $\psi : \mathcal{C}^\circ(Y, T) \to \mathcal{D}^\circ(F_X(Y), F_W(T)) \to \mathcal{D}^\circ(F_X(Y), Z)$ is an equivalence.

The map $\psi$ is a map over $\mathcal{O}^\circ(X, W)$ and is thus an equivalence if and only if it induces on the fiber over every morphism $\varphi : X \to W$ of $\mathcal{O}^\circ$ an equivalence.

Using that $\mathcal{O}^\circ, \mathcal{C}^\circ, \mathcal{D}^\circ$ are operads and $G : \mathcal{O}^\circ \to \mathcal{C}^\circ$ is a map of operads over $\mathcal{O}^\circ$ this is equivalent to the condition that $\psi$ induces an equivalence on the fiber over every active morphism $\varphi : X \to W$ of $\mathcal{O}^\circ$ with $W \in \mathcal{O}$.

Hence $\psi$ is an equivalence if and only if the pullback $\psi'$ of $\psi$ along the full subspace inclusion $\text{Mul}_\mathcal{O}(X_1, \ldots, X_n; W) \subset \mathcal{O}^\circ(X, W)$ is an equivalence, where $X_1, \ldots, X_n$ denote the components of $X$ for some $n \in \mathbb{N}$.

But $\psi'$ is equivalent to the canonical map

$$\text{Mul}_\mathcal{C}(Y_1, \ldots, Y_n, T) \to \text{Mul}_\mathcal{D}(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), F_W(T)) \to \text{Mul}_\mathcal{O}(F_{X_1}(Y_1), \ldots, F_{X_n}(Y_n), Z)$$

induced by $\alpha : F_W(T) \to Z$, where $Y_1, \ldots, Y_n$ denote the components of $Y$.

Hence 2. implies 1.
Corollary 7.25. Let $\mathcal{M}^\circ \to \mathcal{L}^\circ, \mathcal{N}^\circ \to \mathcal{L}^\circ$ be $\kappa$-small operads over $\mathcal{L}^\circ$ that exhibit categories $\mathcal{M}$ respectively $\mathcal{N}$ as pseudo-enriched over a locally cocartesian fibration of operads $\mathcal{C}^\circ \to \text{Ass}^\circ$.

Let $G : \mathcal{N}^\circ \to \mathcal{M}^\circ$ be a lax $\mathcal{C}^\circ$-linear functor.

Then $G$ admits a left adjoint relative to $\mathcal{L}^\circ$ if and only if the underlying functor $\mathcal{N} \to \mathcal{M}$ admits a left adjoint $F : \mathcal{M} \to \mathcal{N}$ and for all objects $A \in \mathcal{C}, M \in \mathcal{M}, N \in \mathcal{N}$ the canonical map

$$\text{Mul}_{\mathcal{N}}(A, F(M); N) \to \text{Mul}_{\mathcal{M}}(A, G(F(M)); G(N)) \to \text{Mul}_{\mathcal{M}}(A, M; G(N))$$

is an equivalence.

$$\mathcal{M}^\circ \to \mathcal{L}^\circ, \mathcal{N}^\circ \to \mathcal{L}^\circ$$

exhibit $\mathcal{M}$ respectively $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\mathcal{C}^\circ \to \text{Ass}^\circ$.

Assume that $\mathcal{M}^\circ \to \mathcal{L}^\circ, \mathcal{N}^\circ \to \mathcal{L}^\circ$ exhibit $\mathcal{M}$ respectively $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\mathcal{C}^\circ \to \text{Ass}^\circ$.

Then $G$ admits a left adjoint relative to $\mathcal{L}^\circ$ if and only if the underlying functor $\mathcal{N} \to \mathcal{M}$ admits a left adjoint $F : \mathcal{M} \to \mathcal{N}$ and for all objects $M \in \mathcal{M}, N \in \mathcal{N}$ the canonical morphism

$$[F(M), N] \to [G(F(M)), G(N)] \to [M, G(N)]$$

is an equivalence.

Let $F : \mathcal{M}^\circ \to \mathcal{N}^\circ$ be a lax $\mathcal{C}^\circ$-linear functor.

Then $F$ admits a left adjoint relative to $\mathcal{L}^\circ$ if and only if the underlying functor $\mathcal{M} \to \mathcal{N}$ admits a right adjoint $G : \mathcal{N} \to \mathcal{M}$ and for all objects $A \in \mathcal{C}, M \in \mathcal{M}, N \in \mathcal{N}$ the canonical map

$$\text{Mul}_{\mathcal{M}}(A, M; G(N)) \to \text{Mul}_{\mathcal{N}}(A, F(M), F(G(N))) \to \text{Mul}_{\mathcal{N}}(A, F(M), N)$$

is an equivalence.

Assume that $\mathcal{M}^\circ \to \mathcal{L}^\circ, \mathcal{N}^\circ \to \mathcal{L}^\circ$ exhibit $\mathcal{M}$ respectively $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\mathcal{C}^\circ \to \text{Ass}^\circ$.

Then $F$ admits a left adjoint relative to $\mathcal{L}^\circ$ if and only if the underlying functor $\mathcal{M} \to \mathcal{N}$ admits a right adjoint $G : \mathcal{N} \to \mathcal{M}$ and for all objects $M \in \mathcal{M}, N \in \mathcal{N}$ the canonical morphism

$$[M, G(N)] \to [F(M), F(G(N))] \to [F(M), N]$$

is an equivalence.

Proof. Denote $\sigma \in \text{Mul}_{\mathcal{M}}(a; a; a)$ the operation and $a \in \text{Mul}_{\mathcal{M}}(a, ..., a)$.

As $\mathcal{M}$ respectively $\mathcal{N}$ are pseudo-enriched over a locally cocartesian fibration of operads $\mathcal{C}^\circ \to \text{Ass}^\circ$, for every $A_1, ..., A_n \in \mathcal{C}$ for some $n \in \mathcal{N}$ and $M \in \mathcal{M}, N \in \mathcal{N}$ the pullback of the canonical map

$$\text{Mul}_{\mathcal{N}}(A_1, ..., A_n, F(M); N) \to$$

$$\text{Mul}_{\mathcal{M}}(A_1, ..., A_n, G(F(M)); G(N)) \to \text{Mul}_{\mathcal{M}}(A_1, ..., A_n, M; G(N))$$

is an equivalence.
over $\text{Mul}_L(M; a, a, m; m)$ to $\{\sigma \circ (\alpha, m)\} \subset \text{Mul}_L(M; a, a, m; m)$ is equivalent to the map

$$\text{Mul}_N(\otimes_{\alpha}(A_1, ..., A_n), F(M); N) \to$$

$$\text{Mul}_M(\otimes_{\alpha}(A_1, ..., A_n), G(F(M)); G(N)) \to \text{Mul}_M(\otimes_{\alpha}(A_1, ..., A_n), M; G(N))$$

and the pullback of the canonical map

$$\text{Mul}_M(A_1, ..., A_n, M, G(N)) \to$$

$$\text{Mul}_N(A_1, ..., A_n, F(M), F(G(N))) \to \text{Mul}_N(A_1, ..., A_n, F(M), N)$$

over $\text{Mul}_L(M; a, a, m; m)$ to $\{\sigma \circ (\alpha, m)\} \subset \text{Mul}_L(M; a, a, m; m)$ is equivalent to the map

$$\text{Mul}_M(\otimes_{\alpha}(A_1, ..., A_n), M, G(N)) \to$$

$$\text{Mul}_N(\otimes_{\alpha}(A_1, ..., A_n), F(M), F(G(N))) \to \text{Mul}_N(\otimes_{\alpha}(A_1, ..., A_n), F(M), N).$$

As all operations of $\text{Mul}_L(M; a, a, m; m)$ are of the form $\sigma \circ (\alpha, m)$ for some $\alpha \in \text{Mul}_L(a, a; a)$, the statement follows from lemma ....

If $M^\circ \to L^\circ, N^\circ \to L^\circ$ exhibit $M$ respectively $N$ as enriched over the locally cocartesian fibration of operads $C^\circ \to \text{Ass}^\circ$, for all objects $A \in C, M \in M, N \in N$ the canonical map

$$\text{Mul}_N(A, F(M); N) \to \text{Mul}_M(A, G(F(M)); G(N)) \to \text{Mul}_M(A, M; G(N))$$

is equivalent to the canonical map

$$C(A, [F(M), N]) \to C(A, [G(F(M)), G(N)]) \to C(A, [M, G(N)])$$

and the canonical map

$$\text{Mul}_M(A, M; G(N)) \to \text{Mul}_N(A, F(M), F(G(N))) \to \text{Mul}_N(A, F(M), N)$$

is equivalent to the map

$$C(A, [M, G(N)]) \to C(A, [F(M), F(G(N))]) \to C(A, [F(M), N]).$$

\[\square\]

**Corollary 7.26.** Let $M^\circ \to L^\circ, N^\circ \to L^\circ$ be $\kappa$-small operads over $L^\circ$ that exhibit categories $M$ respectively $N$ as enriched over a locally cocartesian fibration of operads $C^\circ \to \text{Ass}^\circ$.

Let $F : M^\circ \to N^\circ$ be a lax $C^\circ$-linear functor.

Then $F$ is an equivalence of operads over $L^\circ$ if and only if the underlying functor $M \to N$ is essentially surjective and for all objects $M, M' \in M$ the canonical morphism

$$[M, M'] \to [F(M), F(M')]$$

is an equivalence.
Proof. Assume that the underlying functor \( M \to N \) of \( F \) is essentially surjective.

If for all objects \( M, M' \in M \) the canonical morphism
\[
\alpha : [M, M'] \to [F(M), F(M')]
\]
is an equivalence, then the underlying functor \( M \to N \) of \( F \) is fully faithful, using
the canonical equivalence \( C(1, [A, B]) \cong \mathcal{M}(A, B) \) for all \( A, B \in M \), and is thus
an equivalence.

Hence the underlying functor \( M \to N \) of \( F \) admits a right adjoint \( G : N \to M \)
such that unit and counit of the adjunction are equivalences.

So for all objects \( M \in M, N \in N \) the canonical morphism
\[
[M, G(N)] \to [F(M), F(G(N))] \to [F(M), N]
\]
is an equivalence.

Thus by corollary \( 7.25 \) \( F : M \to N \) admits a right adjoint relative to \( LM^{\otimes} \).

As unit and counit of the adjunction \( M \nRightarrow N : G \) are equivalences, unit and
counit of the adjunction \( F : M \nRightarrow N^{\otimes} \) relative to \( LM^{\otimes} \) are equivalences so that
\( F : M \to N^{\otimes} \) is an equivalence of operads over \( LM^{\otimes} \).

Remark 7.27. Let \( M^{\otimes} \to LM^{\otimes}, N^{\otimes} \to LM^{\otimes} \) be \( \kappa \)-small operads over \( LM^{\otimes} \)
that exhibit categories \( M \) respectively \( N \) as enriched over a locally cocartesian
fibration of operads \( C^{\otimes} \to Ass^{\otimes} \).

Let \( G : N^{\otimes} \to M^{\otimes} \) be a lax \( C^{\otimes} \)-linear functor that admits a left adjoint relative to \( LM^{\otimes} \).

The underlying functor \( N \to M \) of \( G \) is fully faithful if and only if for all
objects \( N, N' \in N \) the canonical morphism
\[
[N, N'] \to [G(N), G(N')]
\]
is an equivalence:

The if direction follows from the fact that the canonical map
\[
C(1, [N, N']) \to C(1, [G(N), G(N')])
\]
is equivalent to the canonical map \( N(N, N') \to M(G(N), G(N')) \).

The only if direction follows from the fact that for all objects \( A \) of \( C \) the map
\[
C(A, [N, N']) \to C(A, [G(N), G(N')])
\]
is equivalent to the canonical map
\[
\text{Mul}_N(A, N; N') \to \text{Mul}_M(A, G(N); G(N'))
\]
that factors as
\[
\text{Mul}_N(A, N; N') \to \text{Mul}_M(A, F(G(N)); N') \cong \text{Mul}_M(A, G(N); G(N'))
\]
and the fact that the counit \( F(G(N)) \to N \) is an equivalence if the underlying
functor \( N \to M \) of \( G \) is fully faithful.

Let \( L : M^{\otimes} \to N^{\otimes} : \iota \) be a \( C \)-enriched localization.
An object $M$ of $\mathcal{M}$ belongs to the essential image of $\iota$ if and only if for all local equivalences, i.e. for all morphisms $f : A \to B$ of $\mathcal{M}$ such that $L(f)$ is an equivalence, the induced morphism $[B, M] \to [A, M]$ is an equivalence:

If $M$ belongs to the essential image of $\iota$, i.e. $M \approx \iota(N)$ for some $N \in \mathcal{N}$, the induced morphism $[B, M] \to [A, M]$ is equivalent to the morphism $[L(B), N] \to [L(A), N]$ and is thus an equivalence.

On the other hand if for all local equivalences $f : A \to B$ the induced morphism $[B, M] \to [A, M]$ is an equivalence, for all local equivalences $f : A \to B$ the induced map $M(B, M) \to M(A, M)$ is an equivalence so that $M$ belongs to the essential image of $\iota$. 

7.3 Appendix C: Endomorphism objects

Proposition 7.28.  1. Let $\varphi : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a map of operads.

Denote $\text{Env}_{\mathcal{O}}(\mathcal{C})^\otimes \to \mathcal{O}^\otimes$ the enveloping $\mathcal{O}^\otimes$-monoidal category of $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$.

The following conditions are equivalent:

(a) $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a locally cocartesian fibration.
(b) For all objects $Y$ of $\mathcal{O}$ the full subcategory inclusion $\mathcal{C}_Y \subset \text{Env}_{\mathcal{O}}(\mathcal{C})_Y$ admits a left adjoint.

2. Let $\mathcal{M}^\otimes \to \mathcal{L}M^\otimes$ be a map of operads. Set $\mathcal{M} = \{m\} \times_{\mathcal{L}M^\otimes} \mathcal{M}^\otimes$ and $\mathcal{C}^\otimes = \text{Ass}^\otimes \times_{\mathcal{L}M^\otimes} \mathcal{M}^\otimes$. For every $n \in \mathbb{N}$ set $\text{Ass}^\otimes(n) := \text{Mul}_{\text{Ass}^\otimes}(m; \ldots, m)$. Denote $\sigma \in \text{Mul}_{\text{L}M^\otimes}(a; m; m)$ the unique object.

For every object $\alpha \in \text{Ass}^\otimes(n)$ for some $n \in \mathbb{N}$ denote $\alpha'$ the image of $\alpha$, the identity of $m$ and $\sigma$ under the composition

$\text{Mul}_{\text{L}M^\otimes}(a; m; m) \times (\text{Mul}_{\text{L}M^\otimes}(a; \ldots, a; m) \times \text{Mul}_{\text{L}M^\otimes}(m; m)) \to \text{Mul}_{\text{L}M^\otimes}(a; \ldots, m; m)$.

Let $X$ be an object of $\mathcal{M}$ and denote $\gamma : \text{Env}_{\text{Ass}^\otimes}(\mathcal{C})[X] \to \text{Env}_{\text{Ass}^\otimes}(\mathcal{C})$ the forgetful functor.

Assume that $\mathcal{C}^\otimes \to \text{Ass}^\otimes$ is a locally cocartesian fibration.

The following conditions are equivalent:

(a) For every objects $A_1, \ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and every $\alpha \in \text{Ass}_n$ the canonical map

$\text{Mul}_{\mathcal{M}^\otimes}(\otimes_\alpha(A_1, \ldots, A_n), X; X) \simeq \{\sigma\} \times_{\text{Mul}_{\mathcal{L}M^\otimes}(a; m, m)} \text{Mul}_{\mathcal{M}^\otimes}(\otimes_\alpha(A_1, \ldots, A_n), X; X) \to \{\alpha'\} \times_{\text{Mul}_{\mathcal{L}M^\otimes}(a; \ldots, a, m, m)} \text{Mul}_{\mathcal{M}^\otimes}(A_1, \ldots, A_n, X; X)$

is essentially surjective.
(b) For every object $Y$ of $\text{Env}_{\text{Ass}^\otimes}(\mathcal{C})[X]$ lying over $A$ of $\text{Env}_{\text{Ass}^\otimes}(\mathcal{C})$ and every local equivalence $\theta : A \to B$ in $\text{Env}_{\text{Ass}^\otimes}(\mathcal{C})$ with $B \in \mathcal{C}$ there is a lift $\phi : Y \to Z$ in $\text{Env}_{\text{Ass}^\otimes}(\mathcal{C})[X]$ of $\theta$.

3. The following conditions are equivalent:

(a) For every objects $A_1, \ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and every $\alpha \in \text{Ass}_n$ the canonical map

$\text{Mul}_{\mathcal{M}^\otimes}(\otimes_\alpha(A_1, \ldots, A_n), X; X) \simeq \{\sigma\} \times_{\text{Mul}_{\mathcal{L}M^\otimes}(a; m, m)} \text{Mul}_{\mathcal{M}^\otimes}(\otimes_\alpha(A_1, \ldots, A_n), X; X) \to \{\alpha'\} \times_{\text{Mul}_{\mathcal{L}M^\otimes}(a; \ldots, a, m, m)} \text{Mul}_{\mathcal{M}^\otimes}(A_1, \ldots, A_n, X; X)$

is fully faithful.
Proof. 1: We have a canonical equivalence $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C}) \simeq \text{Act}(\mathcal{O}^\otimes) \times_{\text{Fun}(\{1\}, \mathcal{O}^\otimes)} \mathcal{O}^\otimes$ over $\text{Fun}(\{1\}, \mathcal{O}^\otimes)$.

So for every object $Y$ of $\mathcal{O}$ we get a canonical equivalence

$$\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y \simeq ((\mathcal{O}^\otimes)_Y)_{\text{act}} \times_{\text{Fun}(\{1\}, \mathcal{O}^\otimes)} \mathcal{O}^\otimes$$

and given an object $B \in \mathcal{C}_Y$ and an object $A$ of $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y$ corresponding to objects $A_1, \ldots, A_n$ of $\mathcal{C}$ for some $n \in \mathbb{N}$ and an object $\alpha \in \text{Mul}_\mathcal{O}(\varphi(A_1), \ldots, \varphi(A_n), Y)$ we get a canonical equivalence

$$\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y(A, B) \simeq \{\alpha\} \times_{\text{Mul}_\mathcal{O}(\varphi(A_1), \ldots, \varphi(A_n), Y)} \text{Mul}_\mathcal{C}(A_1, \ldots, A_n, B).$$

To show that the full subcategory inclusion $\mathcal{C}_Y \subset \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y$ admits a left adjoint we have to find a morphism $\varphi : A \to B$ of $\mathcal{C}$ such that for every object $V$ of $\mathcal{C}_Y$ composition with $\varphi$ induces an equivalence

$$\mathcal{C}_Y(B, V) \simeq \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y(B, V) \to \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y(A, V) \simeq$$

$$\{\alpha\} \times_{\text{Mul}_\mathcal{O}(\varphi(A_1), \ldots, \varphi(A_n), Y)} \text{Mul}_\mathcal{C}(A_1, \ldots, A_n, V).$$

If $\varphi : \mathcal{O}^\otimes \to \mathcal{O}^\otimes$ is a locally cocartesian fibration, we have a locally $\varphi$-cocartesian lift $h : (A_1, \ldots, A_n) \to \otimes_\alpha (A_1, \ldots, A_n)$ in $\mathcal{C}^\otimes$ of the active morphism $\alpha$ of $\mathcal{O}^\otimes$.

Define $\theta : A \to \otimes_\alpha (A_1, \ldots, A_n)$ to correspond to the morphism $h$ under the equivalence $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y(A, \otimes_\alpha (A_1, \ldots, A_n)) \simeq \{\alpha\} \times_{\text{Mul}_\mathcal{O}(\varphi(A_1), \ldots, \varphi(A_n), Y)} \text{Mul}_\mathcal{C}(A_1, \ldots, A_n, \otimes_\alpha (A_1, \ldots, A_n)).$

For every object $V$ of $\mathcal{C}_Y$ composition with $\theta : A \to \otimes_\alpha (A_1, \ldots, A_n)$

$$\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y(\otimes_\alpha (A_1, \ldots, A_n), V) \to \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})_Y(A, V)$$

4. So the following conditions are equivalent:

(a) For every objects $A_1, \ldots, A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and every $\alpha \in \text{Ass}_{\mathcal{O}^\otimes}$ the canonical map

$$\text{Mul}_\mathcal{O}((\otimes_\alpha (A_1, \ldots, A_n), X; X) \simeq \{\sigma\} \times_{\text{Mul}_\mathcal{O}(a.m.m)} \text{Mul}_\mathcal{O}((\otimes_\alpha (A_1, \ldots, A_n), X; X) \to \{\alpha'\} \times_{\text{Mul}_\mathcal{O}(a.m.m)} \text{Mul}_\mathcal{O}(A_1, \ldots, A_n, X; X)$$

is an equivalence.

(b) For every object $Y$ of $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})[X]$ lying over $A$ of $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})$ and every local equivalence $\theta : A \to B$ in $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})$ with $B \in \mathcal{C}$ there is a $\gamma$-cocartesian lift $\varphi : Y \to Z$ in $\text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})[X]$ of $\theta$.

(c) The full subcategory inclusion $\mathcal{C}[X] \subset \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})[X]$ admits a left adjoint and $\gamma : \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})[X] \to \text{Env}_{\mathcal{O}^\otimes}(\mathcal{C})$ preserves local equivalences.
is equivalent to composition with \( h : (A_1, \ldots, A_n) \to \otimes_\alpha (A_1, \ldots, A_n) \)

\[
\zeta : \mathcal{C}_Y (\otimes_\alpha (A_1, \ldots, A_n), V) \to \{ \alpha \} \times_{\text{Mult}_\alpha (\varphi(A_1), \ldots, \varphi(A_n), Y)} \text{Mult}_\alpha (A_1, \ldots, A_n, V)
\]

as for \( V = \otimes_\alpha (A_1, \ldots, A_n) \) both maps send the identity to equivalent objects.

As \( h \) is locally \( \varphi \)-cartesian, \( \zeta \) is an equivalence.

So if \( \varphi : \mathcal{C}^\# \to \mathcal{O}^\# \) is a locally cocartesian fibration, for all objects \( Y \) of \( \mathcal{O} \) the full subcategory inclusion \( \mathcal{C}_Y \subset \text{Env}_{\mathcal{O}^\#} (\mathcal{C})_Y \) admits a left adjoint.

If for all objects \( Y \) of \( \mathcal{O} \) the full subcategory inclusion \( \mathcal{C}_Y \subset \text{Env}_{\mathcal{O}^\#} (\mathcal{C})_Y \) admits a left adjoint, then by lemma ... \( \varphi : \mathcal{C}^\# \to \mathcal{O}^\# \) is a locally cocartesian fibration.

2: Let \( A \) be an object of \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) \) corresponding to objects \( A_1, \ldots, A_n \) of \( \mathcal{C} \) for some \( n \in \mathbb{N} \) and an object \( \alpha \in \text{Ass}(n) \).

Denote \( h : (A_1, \ldots, A_n) \to \otimes_\alpha (A_1, \ldots, A_n) \) the unique cocartesian lift of \( \alpha \) and denote \( \theta : A \to \otimes_\alpha (A_1, \ldots, A_n) \) the morphism of \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) \) corresponding to \( h \) under the equivalence \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) (A, \otimes_\alpha (A_1, \ldots, A_n)) \cong \{ \alpha \} \times_{\text{Ass}(n)} \text{Mult}_\alpha (A_1, \ldots, A_n, \otimes_\alpha (A_1, \ldots, A_n)). \)

By 1. \( \theta \) is a local equivalence.

For every \( Y \in \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A \) and \( W \in \mathcal{C}[X]_{\otimes_\alpha (A_1, \ldots, A_n)} \) we have a canonical equivalence

\[
\{ \theta \} \times_{\text{Env}_{\text{Ass}^\#} (\mathcal{C}) (A, \otimes_\alpha (A_1, \ldots, A_n))} \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A (Y, W) \cong \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A (Y, \theta^* (W)) .
\]

Consequently there is a lift \( \phi : Y \to Z \) in \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A \) if and only if \( Y \) belongs to the essential image of the functor \( \theta^* : \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_{\otimes_\alpha (A_1, \ldots, A_n)} \to \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A \).

So the functor \( \theta^* : \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_{\otimes_\alpha (A_1, \ldots, A_n)} \to \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A \) is essentially surjective if and only if for every \( Y \in \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A \) there is a lift \( \phi : Y \to Z \) in \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X]_A \) of \( \theta \).

But the functor \( \theta^* \) is equivalent to the functor

\[
\text{Env}_{\text{LM}^\#} (M) (\otimes_\alpha (A_1, \ldots, A_n) \otimes X, X) \to \text{Env}_{\text{LM}^\#} (M) (A \otimes X, X)
\]

that is equivalent to the functor

\[
\text{Mult}_{\text{LM}^\#} (\otimes_\alpha (A_1, \ldots, A_n), X) \cong \{ \sigma \} \times_{\text{Mult}_{\text{LM}^\#} (a, m, m)}\text{Mult}_{\text{LM}^\#} (\otimes_\alpha (A_1, \ldots, A_n), X, X) \\
\to \{ \alpha' \} \times_{\text{Mult}_{\text{LM}^\#} (a, n, m, m)}\text{Mult}_{\text{LM}^\#} (A_1, \ldots, A_n, X, X).
\]

This shows 2.

3: Let \( \phi : Y \to Z \) be a morphism of \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X] \) lying over the morphism \( \theta : A \to \otimes_\alpha (A_1, \ldots, A_n) \) of \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) \).

For every object \( W \) of \( \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X] \) composition with \( \phi \) induces a commutative square

\[
\begin{array}{ccc}
\text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X] (Z, W) & \rightarrow & \text{Env}_{\text{Ass}^\#} (\mathcal{C}) [X] (Y, W) \\
\text{Env}_{\text{Ass}^\#} (\mathcal{C}) (\otimes_\alpha (A_1, \ldots, A_n), \gamma (W)) & \rightarrow & \text{Env}_{\text{Ass}^\#} (\mathcal{C}) (A, \gamma (W)) .
\end{array}
\]
As \( \gamma : \text{Env}_{\text{Ass}}(\mathcal{C})[X] \to \text{Env}_{\text{Ass}}(\mathcal{C}) \) is a right fibration, this square induces on the fiber over a morphism \( f : \otimes_{\alpha}(A_1, ..., A_n) \to \gamma(W) \) of \( \text{Env}_{\text{Ass}}(\mathcal{C}) \) the map

\[
\text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\otimes_{\alpha}(A_1, ..., A_n)}(Z,f^*(W)) \to \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\alpha}(\theta^*(Z), \theta^*(f^*(W)))
\]

\[
\to \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\alpha}(Y, \theta^*(f^*(W))),
\]

where the morphism \( Y \to \theta^*(Z) \) in \( \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\alpha} \) corresponding to \( \phi : Y \to Z \) is an equivalence as the fibers of the right fibration \( \gamma \) are spaces.

Hence \( \phi : Y \to Z \) is \( \gamma \)-cocartesian, i.e. square ... is a pullback square for all \( W \in \text{Env}_{\text{Ass}}(\mathcal{C})[X] \) if and only if for all \( W \in \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\otimes_{\alpha}(A_1, ..., A_n)} \) the map

\[
\text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\otimes_{\alpha}(A_1, ..., A_n)}(Z,W) \to \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\alpha}(\theta^*(Z), \theta^*(f^*(W)))
\]

is an equivalence, i.e \( \phi \) is locally \( \gamma \)-cocartesian.

So the map \( \theta^* : \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\otimes_{\alpha}(A_1, ..., A_n)} \to \text{Env}_{\text{Ass}}(\mathcal{C})[X]_{\alpha} \) is fully faithful if and only if every lift of the morphism \( \theta : A \to \otimes_{\alpha}(A_1, ..., A_n) \) is \( \gamma \)-cocartesian.

The functor \( \theta^* \) is equivalent to the functor

\[
\text{Env}_{\text{LM}^*}(\mathcal{C})(\otimes_{\alpha}(A_1, ..., A_n) \otimes X,X) \to \text{Env}_{\text{LM}^*}(\mathcal{C})(A \otimes X,X)
\]

that is equivalent to the functor

\[
\text{Mult}_{\text{LM}^*}(\otimes_{\alpha}(A_1, ..., A_n),X,X) \simeq \{\sigma\} \times_{\text{Mult}_{\text{LM}^*}(A,m,m)} \text{Mult}_{\text{LM}^*}(\otimes_{\alpha}(A_1, ..., A_n),X,X)
\]

\[
\to \{\sigma'\} \times_{\text{Mult}_{\text{LM}^*}(A,...,A,m,m)} \text{Mult}_{\text{LM}^*}(A_1, ..., A_n,X,X).
\]

4: In view of 2. and 3. condition (a) trivially implies (b) and condition (b) implies (a) as the fibers of the right fibration \( \gamma \) are spaces.

So (a) and (b) are equivalent.

c) follows immediately from b):

Let \( Y \) be an object of \( \text{Env}_{\text{Ass}}(\mathcal{C})[X] \). Then by 1. there is a local equivalence \( \gamma(Y) \to B \) of \( \text{Env}_{\text{Ass}}(\mathcal{C})[X] \) with \( B \in \mathcal{C} \) that admits a \( \gamma \)-cocartesian lift \( \phi : Y \to Z \) due to b).

Especially for every \( W \in \mathcal{C}[X] \) we have a pullback square

\[
\begin{array}{ccc}
\text{Env}_{\text{Ass}}(\mathcal{C})[X](Z,W) & \longrightarrow & \text{Env}_{\text{Ass}}(\mathcal{C})[X](Y,W) \\
\downarrow & & \downarrow \\
\text{Env}_{\text{Ass}}(\mathcal{C})(B, \gamma(W)) & \longrightarrow & \text{Env}_{\text{Ass}}(\mathcal{C})(\gamma(Y), \gamma(W))
\end{array}
\]

so that \( \phi : Y \to Z \) is a local equivalence.

Assume that c) holds.

Let \( Y \) be an object of \( \text{Env}_{\text{Ass}}(\mathcal{C})[X] \) and \( \theta : \gamma(Y) \to B \) a local equivalence in \( \text{Env}_{\text{Ass}}(\mathcal{C}) \) with \( B \in \mathcal{C} \).

There is a local equivalence \( Y \to V \) in \( \text{Env}_{\text{Ass}}(\mathcal{C})[X] \) with \( V \in \mathcal{C}[X] \) such that \( \gamma(\phi) \) is a local equivalence.

Consequently there is a local equivalence \( \phi : Y \to Z \) in \( \text{Env}_{\text{Ass}}(\mathcal{C})[X] \) lying over \( \theta : \gamma(Y) \to B \).
As $\gamma$ is a right fibration, every locally $\gamma$-cocartesian morphism is $\gamma$-cocartesian. Thus it is enough to see that $\phi : Y \to Z$ is locally $\gamma$-cocartesian.

But for every $W \in \text{Env}_{\text{Ass}}(\mathcal{E})[X]_B \simeq \mathcal{E}[X]_B$ we have a pullback square

$$
\begin{array}{ccc}
\text{Env}_{\text{Ass}}(\mathcal{E})[X](Z, W) & \xrightarrow{\gamma} & \text{Env}_{\text{Ass}}(\mathcal{E})[X](Y, W) \\
\downarrow & & \downarrow \\
\text{Env}_{\text{Ass}}(\mathcal{E})(B, B) & \xrightarrow{\gamma} & \text{Env}_{\text{Ass}}(\mathcal{E})(\gamma(Y), B)
\end{array}
$$

so that especially $\phi : Y \to Z$ is locally $\gamma$-cocartesian.

\[\square\]

**Lemma 7.29.** Let $M^\otimes$ be an operad over $LM^\otimes$. Set $M := \{m\} \times_{LM^\otimes} M^\otimes$ and $\mathcal{E}^\otimes := \text{Ass}^\otimes \times_{LM^\otimes} M^\otimes$.

Denote $\alpha : \Delta^1 \to LM^\otimes$ the morphism of $LM^\otimes$ corresponding to the unique object of $\text{Mul}_{LM^\otimes}(a, m, m)$.

$\alpha$ gives rise to a category $\text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes)$.

We have canonical functors $\text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes) \to M^\otimes_{(a, m)} \simeq \mathcal{E} \times M$ and $\text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes) \to M^\otimes_{\bullet} \simeq M$ evaluating at 0 respectively 1.

There is a canonical equivalence

$$
\text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes) \simeq (\mathcal{E} \times M \times M) \times_{(M^\otimes \times M^\otimes)} \text{Act}(M^\otimes)
$$

over $\mathcal{E} \times M \times M$.

In particular we have a canonical equivalence of right fibrations

$$
\{X\} \times_M \text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes) \simeq (\mathcal{E} \times M) \times_{M^\otimes} (M^\otimes)_{/X}^{\otimes}
$$

over $\mathcal{E} \times M$ and so a canonical equivalence of right fibrations

$$
\mathcal{E}[X] := \{(X, X)\} \times_M \text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes) \simeq (\mathcal{E} \times \{X\}) \times_{M^\otimes} (M^\otimes)_{/X}^{\otimes}
$$

over $\mathcal{E}$.

So the right fibration $\mathcal{E}[X] \to \mathcal{E}$ classifies the functor $\text{Mul}_{LM^\otimes}(-, X; X) : \mathcal{E}^{op} \to S(\kappa)$.

**Proof.** Set $X := \Delta^1 \times_{LM^\otimes} M^\otimes$.

By Lemma 7.30 we have a canonical equivalence

$$
\text{Fun}_{LM^\otimes}(\Delta^1, M^\otimes) \simeq \text{Fun}_{\Delta^1}(\Delta^1, X) \simeq (\mathcal{E} \times M \times M) \times_{\{(X, X)\}} \text{Fun}(\Delta^1, X)
$$

over $\mathcal{E} \times M \times M$.

Moreover we have a canonical equivalence

$$
((\mathcal{E} \times M \times M) \times_{\{(X, X)\}} \text{Fun}(\Delta^1, X)) \simeq (\mathcal{E} \times M \times M) \times_{(LM^\otimes \times LM^\otimes)} \text{Fun}(\Delta^1, LM^\otimes))
$$

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Lemma 7.31. Let $M$ be a $\kappa$-small category for a strongly inaccessible cardinal $\kappa$ and $\gamma : \Delta \to \Delta^1$ a functor with $M_0 = C$ and $M_1 = D$.

The commutative square

$$
\begin{array}{ccc}
\text{Fun}_{\Delta^1}(\Delta^1,M) & \longrightarrow & \text{Fun}(\Delta^1,M) \\
\text{Fun}_{\Delta^1}([0],M) \times \text{Fun}_{\Delta^1}([1],M) & \longrightarrow & \text{Fun}([0],M) \times \text{Fun}([1],M) \\
\end{array}
$$

is a pullback square.

Proof. We will show that the induced functor $\rho : \text{Fun}_{\Delta^1}(\Delta^1,M) \to (\mathcal{C} \times \Delta^1) \times_{(\mathcal{M} \times \mathcal{M})} \text{Fun}(\Delta^1,M)$ is an equivalence.

$\rho$ is essentially surjective because every morphism $X \to Y$ in $M$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ has to lie over the unique non-identity morphism of $\Delta^1$.

To see that $\rho$ is fully faithful, it is enough to see that $\beta : (\mathcal{C} \times \Delta^1) \times_{(\mathcal{M} \times \mathcal{M})} \text{Fun}(\Delta^1,M)$ and $\beta \circ \rho : \text{Fun}_{\Delta^1}(\Delta^1,M) \to \text{Fun}(\Delta^1,M)$ are fully faithful.

But we have pullback squares

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & M \\
\rho \downarrow & & \downarrow \beta \\
\Delta^1 & \longrightarrow & \text{Fun}_{\Delta^1}(\Delta^1,M) \\
\{0\} & \longrightarrow & \Delta^1 \\
\{1\} & \longrightarrow & \Delta^1 \\
\text{id}_{\Delta^1} & \longrightarrow & \text{Fun}(\Delta^1,\Delta^1),
\end{array}
$$

where the bottom and thus also the top functors are fully faithful.

Lemma 7.30. Let $\mathcal{M}$ be a $\kappa$-small monoidal category for a strongly inaccessible cardinal $\kappa$.

Let $A$ be an associative algebra of $\mathcal{C}$ and let $M$ be a left $A$-modal structure on $A$, i.e. $M \in \{A\} \times_{\mathcal{C}} \text{LMod}_A(\mathcal{C})$.

Denote $\Lambda' \in \{A\} \times_{\mathcal{C}} \text{LMod}_A(\mathcal{C})$ the left $A$-modal structure on $A$ that comes from $A$, i.e. $\Lambda'$ is the composition $\text{LMod}_A(\mathcal{C}) \to \text{Ass} \to \mathcal{C}$ of operads over $\text{Ass}$.

Denote $\mu_M : A \otimes A \to A$ the left action map provided by $M$ and similar for $\Lambda$. 

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Denote \( \eta : \mathbb{1}_C \to A \) the unit of \( A \) and \( \psi \) the composition \( \Lambda \simeq A \otimes \mathbb{1}_C \xrightarrow{\Lambda \otimes \eta} A \otimes A \xrightarrow{\mu_A} A \).

Then there is a canonical equivalence of spaces

\[
\{A\} \times_C \text{LMod}_A(C') \simeq \mathcal{C}(A, A)(id_A, \psi).
\]

In particular \( M \) is equivalent to \( A' \) in the category \( \{A\} \times_C \text{LMod}_A(C) \) if and only if the composition \( \psi : A \simeq A \otimes \mathbb{1}_C \xrightarrow{\Lambda \otimes \eta} A \otimes A \xrightarrow{\mu_M} A \) is the identity.

Especially \( M \) is equivalent to \( A' \) in the category \( \{A\} \times_C \text{LMod}_A(C) \) if and only if \( \mu_M \) is equivalent to \( \mu_{A'} \).

\[\textbf{Proof.}\] The morphism \( A \otimes \mathbb{1}_C \xrightarrow{\Lambda \otimes \eta} A \otimes A \xrightarrow{\mu_A} A \) is the canonical equivalence.

Thus \( \eta : \mathbb{1}_C \to A \) exhibits \( A \) as the free left \( A \)-module generated by \( \mathbb{1}_C \) so that the canonical map

\[
\gamma : \text{LMod}_A(C)(A', M) \to \mathcal{C}(A, A) \to \mathcal{C}(\mathbb{1}_C, A)
\]
is an equivalence.

Denote \( \beta \) the composition \( \mathcal{C}(\mathbb{1}_C, A) \to \mathcal{C}(A \otimes \mathbb{1}_C, A) \xrightarrow{\mathcal{C}(\Lambda \otimes \mathbb{1}_C, \mu_M)} \mathcal{C}(A, A) \).

Thus \( \gamma \) induces an equivalence \( \gamma' := \{id_A\} \times_{C}(A, A) \gamma : \{A\} \times_C \text{LMod}_A(C)(A', M) \xrightarrow{\{id_A\} \times_{C}(A, A)} \mathcal{C}(A, A) \).

The composition \( \{id_A\} \times_{C}(A, A) \text{LMod}_A(C)(A', M) \to \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \)
is equivalent to the map

\[
\{id_A\} \times_{C}(A, A) \text{LMod}_A(C)(A', M) \to \text{LMod}_A(C)(A', M) \to \mathcal{C}(\mathbb{1}_C, A) \to \mathcal{C}(\mathbb{1}_C, A)
\]

and is thus equivalent to the constant map with value \( \eta : \mathbb{1}_C \to A \).

Therefore \( \gamma' \) gives rise to a map

\[
\zeta : \{id_A\} \times_{C}(A, A) \text{LMod}_A(C)(A', M) \to \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \to \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A)
\]
such that the composition

\[
\{id_A\} \times_{C}(A, A) \text{LMod}_A(C)(A', M) \xrightarrow{\zeta} \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \to \mathcal{C}(1_{1_C}, 1_{1_C}) \to \mathcal{C}(\mathbb{1}_C, A)
\]
is equivalent to \( \gamma' \).

Thus \( \zeta \) admits a left inverse and it is enough to see that the composition

\[
\{id_A\} \times_{C}(A, A) \mathcal{C}(1_{1_C}, 1_{1_C}) \xrightarrow{\gamma'^{-1}} \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \to \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A)
\]
is equivalent to the identity.

This is equivalent to the condition that the composition

\[
\{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \xrightarrow{\gamma'^{-1}} \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A)
\]
is equivalent over \( \mathcal{C}(\mathbb{1}_C, A) \) to the canonical map

\[
\{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \to \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A)
\]

Choosing the inverse \( \gamma'^{-1} \) of \( \gamma' \) in \( \mathcal{S}(\kappa)_{\mathcal{C}(\mathbb{1}_C, A)} \) the composition \( \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \xrightarrow{\gamma'^{-1}} \{id_A\} \times_{C}(A, A) \mathcal{C}(\mathbb{1}_C, A) \)
is equivalent over \( \mathcal{C}(\mathbb{1}_C, A) \) to the identity.
7.4 Appendix D: General Appendix

Lemma 7.32. Let $\phi : C \to S$ be a functor between $\kappa$-small categories such that for all objects $s$ of $S$ the fiber $C_s$ admits a final object $X(s)$.

Assume that one of the following conditions is satisfied:

1. $\phi : C \to S$ is a locally cocartesian fibration.
2. $\phi : C \to S$ is a locally cartesian fibration such that the induced functors on the fibers preserve the final object.

The category $\text{Fun}_S(C, C)$ admits a final object $\alpha : S \to C$ such that for every $s \in S$ the image $\alpha(s)$ is the final object of $C_s$.

Especially a section $\alpha : S \to C$ of $\phi$ is a final object of $\text{Fun}_S(C, C)$ if and only if for every $s \in S$ the image $\alpha(s)$ is the final object of $C_s$.

Proof. Denote $W$ the full subcategory of $\text{Cat}_\kappa$ spanned by those categories $K$ with the property that for every functor $\psi : K \to C$ the following condition holds:

The category $\text{Fun}_S(K, C)$ admits a final object $\alpha : K \to C$ such that for every $k \in K$ the image $\alpha(k)$ is the final object of $C_{\psi(k)}$.

We will show that $W = \text{Cat}_\kappa(\kappa)$.

As $\text{Cat}_\kappa(\kappa)$ is the only full subcategory of $\text{Cat}_\kappa(\kappa)$ that contains the contractible category and $\Delta^1$ and is closed in $\text{Cat}_\kappa(\kappa)$ under $\kappa$-small colimits, it is enough to see that $W$ contains the contractible category and $\Delta^1$ and is closed in $\text{Cat}_\kappa(\kappa)$ under arbitrary coproducts and pushouts.

Tautologically the contractible category belongs to $W$.

Being right adjoint to the functor $C(-) : \text{Cat}_\kappa(\kappa) \to (\text{Cat}_\kappa(\kappa))^{\text{op}}$ the functor $\text{Fun}_S(-, C) : (\text{Cat}_\kappa(\kappa))^{\text{op}} \to \text{Cat}_\kappa(\kappa)$ sends $\kappa$-small colimits to limits.

So the case of coproducts follows from the fact that an object in an arbitrary product of categories is a final object if every component is final in each factor.

Let $X, Y, Z$ be objects of $W$ and $X \coprod_Y Z \to S$ a functor. Then $X \coprod_Y Z \to S$ is the pushout in $\text{Cat}_\kappa(\kappa)$ of the induced functors $\theta : Y \to X$ and $\varsigma : Y \to Z$ over $S$.

So the categories $\text{Fun}_S(X, C), \text{Fun}_S(Y, C), \text{Fun}_S(Z, C)$ admit final objects $\alpha, \beta$ respectively $\gamma$ that take values in final objects of each fiber.

Hence the unique morphisms $\alpha \circ \theta \to \beta$ and $\gamma \circ \varsigma \to \beta$ in $\text{Fun}_S(Y, C)$ are equivalences being levelwise equivalences.

Thus the category $\text{Fun}_S(X \coprod_Y Z, C) \simeq \text{Fun}_S(X, C) \times_{\text{Fun}_S(Y, C)} \text{Fun}_S(Z, C)$ admits a final object that takes values in final objects of each fiber using that every object of the pushout $X \coprod_Y Z \to S$ is the image of an object of $X$ or $Z$.

It remains to show that $\Delta^1$ belongs to $W$:

Let $f : s \to t$ be a morphism of $S$.

By assumption the fibers $C_s, C_t$ admit final objects $X(s)$ respectively $X(t)$.

If condition 1. holds, there is locally $\phi$-cocartesian lift $X(s) \to f_s(X(s))$ of $f$ in $C$, whose composition with the unique morphism $f_s(X(s)) \to X(t)$ in $C_t$ yields a morphism $\alpha : X(s) \to f_s(X(s)) \to X(t)$ in $C$ lying over $f$.

If condition 2. holds, there is locally $\phi$-cartesian lift $\beta : X(s) \to X(t)$ of $f$ in $C$. 

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Let $F : A \to B, G : X \to Y$ be morphisms of $\mathcal{C}$ lying over $f$.
We have a canonical equivalence

$$\text{Fun}_S(\Delta^1, \mathcal{C})(F, G) \simeq (\{f\} \times_{\text{Fun}(\Delta^1, S)} \text{Fun}(\Delta^1, \mathcal{C})) (F, G) \simeq$$

$$\{\text{id}_f\} \times_{\text{Fun}(\Delta^1, S)(t, f)} \text{Fun}(\Delta^1, \mathcal{C})(F, G) =$$

$$\{\text{id}_f\} \times_{\mathcal{C}(A, X) \times_{\mathcal{C}(A, Y)} \mathcal{C}(B, Y)} \mathcal{C}(A, X) \times_{\mathcal{C}(A, Y)} \mathcal{C}(B, Y) =$$

$$\begin{cases} \mathcal{C}(A, X) \times_{\mathcal{C}(f, A)} \mathcal{C}(B, Y) & \text{if } 1. \text{holds,} \\ \mathcal{C}(A, X) \times_{\mathcal{C}(A, \phi(Y))} \mathcal{C}(B, Y) & \text{if } 2. \text{holds.} \end{cases}$$

So for $G = \alpha$ or $G = \beta$ we see that $\alpha$ respectively $\beta$ is the final object of the category $\text{Fun}_S(\Delta^1, \mathcal{C})$.

Let $\phi : X \to S$ be a functor between $\kappa$-small categories and $\mathcal{E} \subset S$ a subcategory.
Denote $\tilde{X} \subset X$ the full subcategory spanned by the objects $A$ lying over some object $s$ of $S$ such that for every morphism $f : s \to t$ of $\mathcal{E}$ there exists a $\phi$-cocartesian lift $\lambda \to B$ of $f$.

Denote $\phi' : \tilde{X} \to S$ the restriction.
For every morphism of $\mathcal{E}$ the pullback $\Delta^1 \times_{S} \tilde{X} \to \Delta^1$ is a cocartesian fibration, whose cocartesian morphisms are $\phi'$-cocartesian and the full subcategory inclusion $\tilde{X} \subset X$ sends $\phi'$-cocartesian morphisms to $\phi$-cocartesian morphisms:

Let $f : s \to t$ be a morphism of $\mathcal{E}$ and $A \in \tilde{X}_s$. Then there is a $\phi$-cocartesian lift $\lambda \to \mathcal{E}_t(A)$ of $f$. We will show that $\mathcal{E}_t(A)$ belongs to $\tilde{X}$.

Let $g : t \to r$ be a morphism of $\mathcal{E}$. As $A$ belongs to $\tilde{X}_s$, there is a $\phi$-cocartesian lift $\lambda \to (g \circ f)\mathcal{E}_t(A)$ of $g \circ f : s \to r$.

Using that the morphism $A \to \mathcal{E}_t(A)$ is $\phi$-cocartesian, the morphism $A \to (g \circ f)\mathcal{E}_t(A)$ factors as the morphism $A \to \mathcal{E}_t(A)$ followed by a lift $\mathcal{E}_t(A) \to (g \circ f)\mathcal{E}_t(A)$ of $g : t \to r$.

As the morphisms $A \to \mathcal{E}_t(A)$ and $A \to (g \circ f)\mathcal{E}_t(A)$ are $\phi$-cocartesian, the morphism $\mathcal{E}_t(A) \to (g \circ f)\mathcal{E}_t(A)$ is $\phi$-cocartesian, too. Thus $\mathcal{E}_t(A)$ belongs to $\tilde{X}$.

**Lemma 7.33.** Let $\phi : X \to S$ be a functor between $\kappa$-small categories, $K$ a category and $\mathcal{E} \subset S$ a subcategory.

Denote $\tilde{X} \subset X$ the full subcategory spanned by the objects $A$ lying over some object $s$ of $S$ such that for every morphism of $f : s \to t$ of $\mathcal{E}$ there exists a $\phi$-cocartesian lift $A \to B$ of $f$.

Assume that the diagonal functor $X \to X^K$ over $S$ admits a left adjoint relative to $S$.

Then for every object $s$ of $S$ the fiber $\tilde{X}_s$ is closed in $X_s$ under colimits indexed by $K$. 

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Remark 7.34. We apply lemma 7.33 to the following situation:

Let \( \phi : X \to S \) and \( \varphi : Y \to S \) be functors between \( \kappa \)-small categories, \( \mathcal{E} \subset S \) a subcategory and \( \xi : Y \to X \) a functor over \( S \).

If for every morphism of \( \mathcal{E} \) the pullback \( \Delta^1 \times_S Y \to \Delta^1 \) is a cocartesian fibration, whose cocartesian morphisms are \( \varphi \)-cocartesian and \( \xi \) sends \( \varphi \)-cocartesian morphisms lying over morphisms of \( \mathcal{E} \) to \( \phi \)-cocartesian morphisms, then for every object \( s \) of \( S \) the fiber \( X_s \) contains the essential image of \( \xi_s : Y_s \to X_s \).

Assume that the diagonal functor \( X \to X^K \) over \( S \) admits a left adjoint relative to \( S \).

Then by lemma 7.33 \( X_s \) is closed in \( X_s \) under colimits indexed by \( K \).

Assume that for every object \( s \) of \( S \) the fiber \( X_s \) is the only full subcategory of \( X_s \) that contains the essential image of \( \xi_s : Y_s \to X_s \) and is closed in \( X_s \) under colimits indexed by \( K \).

Then we have \( X_s = X_s \) and so \( \tilde{X} = X \).

Thus for every morphism of \( \mathcal{E} \) the pullback \( \Delta^1 \times_S X \to \Delta^1 \) is a cocartesian fibration, whose cocartesian morphisms are \( \phi \)-cocartesian.

Proof. Let \( K^p \to X_s \) be a colimit diagram, whose restriction \( H : K \subset K^p \to X_s \) factors through \( \tilde{X}_s \). We want to see that \( \text{colim}(H) \) belongs to \( \tilde{X}_s \).

Let \( f : s \to t \) be a morphism of \( \mathcal{E} \). We have to find a \( \phi \)-cocartesian lift \( \text{colim}(H) \to Z \) of \( f \).

Denote \( \phi' : \tilde{X} \subset X \rightarrow S \) the restriction and \( \psi : \tilde{X}_K \simeq S \times_{\text{Fun}(K,S)} \text{Fun}(K,\tilde{X}) \to S \) the cotensor.

For every morphism \( \Delta^1 \to \mathcal{E} \) the pullback \( \Delta^1 \times_S \tilde{X} \) is a cocartesian fibration, whose cocartesian morphisms are \( \phi' \)-cocartesian.

Thus for every morphism \( \Delta^1 \to \mathcal{E} \) the pullback \( \Delta^1 \times_S \tilde{X}_K \) is a cocartesian fibration, whose cocartesian morphisms are \( \psi \)-cocartesian, i.e. are levelwise \( \phi' \)-cocartesian.

So we get a \( \psi \)-cocartesian morphism \( \alpha : H \to f_s(H) \) lying over \( f \).

By assumption the diagonal functor \( X \to X^K \) over \( S \) admits a left adjoint \( \chi : X^K \to X \) relative to \( S \).

\( \chi \) sends \( \alpha \) to a morphism \( \beta : \text{colim}(H) \to \text{colim}(f_s(H)) \) of \( X \) lying over \( f \).

The morphism \( \beta \) is \( \phi \)-cocartesian as the composition \( \tilde{X}_K \subset X^K \xrightarrow{\chi} X \) sends \( \psi \)-cocartesian morphisms to \( \phi \)-cocartesian morphisms:

Being a relative left adjoint the functor \( \chi : X^K \to X \) over \( S \) sends morphisms that are cocartesian with respect to the functor \( X^K \to S \) to \( \phi \)-cocartesian morphisms.

The full subcategory inclusion \( \tilde{X} \subset X \) sends \( \phi' \)-cocartesian morphisms to \( \phi \)-cocartesian morphisms so that the full subcategory inclusion \( \tilde{X}_K \subset X^K \) sends \( \psi \)-cocartesian morphisms to levelwise \( \phi \)-cocartesian morphisms, which are especially cocartesian with respect to the functor \( X^K \to S \) according to lemma 7.33.

Lemma 7.35. Let \( \phi : X \to S \) be a functor between \( \kappa \)-small categories and \( K \) a \( \kappa \)-small category. \( \phi \) induces a functor \( \text{Fun}(K,\phi) : \text{Fun}(K,X) \to \text{Fun}(K,S) \).

Let \( \tau \) be a morphism of \( \text{Fun}(K,X) \) that is levelwise \( \phi \)-cocartesian, i.e. such that for every \( k \in K \) the component \( \tau(k) \) is \( \phi \)-cocartesian.

Then \( \tau \) is \( \text{Fun}(K,\phi) \)-cocartesian.
Especially we have the following:

Denote $\psi : X^K \simeq S \times_{\text{Fun}(K,S)} \text{Fun}(K,X)$ the cotensor of the category $K$ with the category $X$ over $S$.

Every levelwise $\phi$-cocartesian morphism of $X^K$ is $\text{Fun}(K,\phi)$-cocartesian and thus especially $\psi$-cocartesian.

Proof. Denote $W$ the full subcategory of $\text{Cat}_\infty(\kappa)$ spanned by those categories $K$ with the property that every levelwise $\phi$-cocartesian morphisms of $\text{Fun}(K,X)$ is $\text{Fun}(K,\phi)$-cocartesian. We want to see that $W = \text{Cat}_\infty(\kappa)$.

As $\text{Cat}_\infty(\kappa)$ is the only full subcategory of $\text{Cat}_\infty(\kappa)$ that contains the contractible category and $\Delta^1$ and is closed in $\text{Cat}_\infty(\kappa)$ under $\kappa$-small colimits, it is enough to see that $W$ contains the contractible category and $\Delta^1$ and is closed in $\text{Cat}_\infty(\kappa)$ under $\kappa$-small colimits.

Tautologically the contractible category belongs to $W$.

To verify that $W$ is closed in $\text{Cat}_\infty(\kappa)$ under $\kappa$-small colimits, it is enough to check that $W$ is closed in $\text{Cat}_\infty(\kappa)$ under arbitrary coproducts and pushouts.

Using that the functor $\text{Fun}(\cdot,X) : \text{Cat}_\infty(\kappa)^{op} \to \text{Cat}_\infty(\kappa)$ sends $\kappa$-small colimits to limits the case of coproducts follows from the fact that given a family of functors $\theta_j : Y_j \to Z_j$, a morphism in the product $\prod_{j \in J} Y_j$ is $\prod_{j \in J} \theta_j$-cocartesian if for every $j \in J$ its image in $Y_j$ is $\theta_j$-cocartesian and the case of pushouts follows from the fact that given functors $\alpha : A \to X$, $\beta : B \to Y$, $\gamma : C \to Z$ and morphisms $\alpha \to \gamma$, $\beta \to \gamma$ in $\text{Fun}(\Delta^1, \text{Cat}_\infty(\kappa))$ a morphism in a pullback $\Lambda \times_C B$ is $\alpha \times \gamma$-cocartesian if its images in $A, B, C$ are $\alpha, \beta$ respectively $\gamma$-cocartesian.

So it remains to show that $\Delta^1$ belongs to $W$.

We want to see that every levelwise $\phi$-cocartesian morphism of $\text{Fun}(\Delta^1,X)$ corresponding to a commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow{g} & & \downarrow{h} \\
C & \longrightarrow & D
\end{array}
$$

in $X$, whose horizontal morphisms are $\phi$-cocartesian, is $\text{Fun}(\Delta^1,\phi)$-cocartesian.

Given a morphism $k : E \to F$ of $X$ the commutative square

$$
\begin{array}{ccc}
\text{Fun}(\Delta^1,X)(h,k) & \longrightarrow & \text{Fun}(\Delta^1,X)(g,k) \\
\downarrow & & \downarrow \\
\text{Fun}(\Delta^1,\text{S})(\phi(h),\phi(k)) & \longrightarrow & \text{Fun}(\Delta^1,\text{S})(\phi(g),\phi(k))
\end{array}
$$

is equivalent to the commutative square

$$
\begin{array}{ccc}
X(D,F) \times_{X(B,F)} X(B,E) & \longrightarrow & X(C,F) \times_{X(A,F)} X(A,E) \\
\downarrow & & \downarrow \\
\text{S}(\phi(D),\phi(F)) \times_{\text{S}(\phi(B),\phi(F))} \text{S}(\phi(B),\phi(E)) & \longrightarrow & \text{S}(\phi(C),\phi(F)) \times_{\text{S}(\phi(A),\phi(F))} \text{S}(\phi(A),\phi(E))
\end{array}
$$

and is thus a pullback square as the morphisms $A \to B$ and $C \to D$ of $X$ are $\phi$-cocartesian and taking pullback preserves pullbacks being a right adjoint. \qed
Corollary 7.36. Let $T \to S, \mathcal{C} \to T, \gamma : \mathcal{D} \to T$ be functors.

Denote $\psi : \text{Fun}_{T}^{S}(\mathcal{C}, \mathcal{D}) \to S$ the canonical functor and $\phi$ the composition $\mathcal{C} \to T \to S$.

Let $\mathcal{C}$ be a morphism of $\text{Fun}_{T}^{S}(\mathcal{C}, \mathcal{D})$ lying over a morphism $g$ of $S$ corresponding to a functor $\phi : \Delta^1 \times_{S} \mathcal{C} \to \Delta^1 \times_{S} \mathcal{D}$ over $\Delta^1 \times_{S} T$.

If $\phi : \mathcal{C} \to S$ is a cartesian fibration, then $f$ is $\psi$-cocartesian if $\phi : \Delta^1 \times_{S} \mathcal{C} \to \Delta^1 \times_{S} \mathcal{D}$ sends $\phi$-cartesian morphisms to $\gamma$-cocartesian morphisms.

Especially we have the following:

If the pullback $\Delta^1 \times_{S} \mathcal{C} \to \Delta^1$ is a cartesian fibration, then $f$ is locally $\psi$-cocartesian if $\phi : \Delta^1 \times_{S} \mathcal{C} \to \Delta^1 \times_{S} \mathcal{D}$ sends locally $\phi$-cartesian morphisms to locally $\gamma$-cocartesian morphisms.

Proof. Using Lemma 7.35 the statement follows as in the second part of the proof of Corollary 3.2.2.12. [2], where we need $\phi : \mathcal{C} \to S$ to be a cartesian fibration to represent $\phi$ by its mapping simplex.

The second part follows from the canonical equivalence $\Delta^1 \times_{S} \text{Fun}_{T}^{S}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}_{\Delta^1 \times_{S} T}(\Delta^1 \times_{S} \mathcal{C}, \Delta^1 \times_{S} \mathcal{D})$ over $\Delta^1$.

Lemma 7.37. Let $\kappa$ be a strongly inaccessible cardinal and $S, T$ be $\kappa$-small categories.

Let $\mathcal{C} \to S \times T$ be a map of locally cocartesian fibrations over $T$.

Then for all objects $t \in T$ the functor $\mathcal{C}_{t} \to \mathcal{C}$ over $S$ preserves cartesian morphisms.

Dually, let $\mathcal{C} \to S \times T$ be a map of locally cartesian fibrations over $T$.

Then for all objects $t \in T$ the functor $\mathcal{C}_{t} \to \mathcal{C}$ over $S$ preserves cocartesian morphisms.

Proof. Let $f : X \to Y$ be a cartesian morphism with respect to the functor $\mathcal{C}_{t} \to S$ lying over a morphism $s \to s'$ of $S$.

Let $Z$ be an object of $\mathcal{C}$ lying over an object $t'$ of $T$ and $s''$ of $S$.

For 1) we have to show that the commutative square

$$
\begin{array}{ccc}
\mathcal{C}(Z, X) & \longrightarrow & \mathcal{C}(Z, Y) \\
\downarrow & & \downarrow \\
S(s'', s) \times T(t', t) & \longrightarrow & S(s'', s') \times T(t', t)
\end{array}
$$

of spaces is a pullback square.

Considering square (17) as a square of spaces over $T(t', t)$ it is enough to see that square (17) induces on the fiber over every object $\varphi \in T(t', t)$ a pullback square.
Using that the functor $\mathcal{C} \to T$ is a locally cocartesian fibration, whose cocartesian morphisms get equivalences in $S$, the fiber of square $17$ over an object $\varphi \in T(t', t)$ is the following commutative square of spaces:

$$
\begin{array}{c}
\mathcal{C}(\varphi^*(Z), X) \\
\downarrow \\
S(s'', s) \\
\end{array} \quad \begin{array}{c}
\mathcal{C}(\varphi^*(Z), Y) \\
\downarrow \\
S(s'', s') \\
\end{array}
$$

But this square is a pullback square because $f : X \to Y$ is a cartesian morphism with respect to the functor $\mathcal{C}_t \to S$.

\[\square\]

**Corollary 7.38.**

1. Let $\mathcal{C} \to S \times T$ be a functor corresponding to a functor $\mathcal{C} \to S \times T$ over $T$ and a functor $\mathcal{C} \to S \times T$ over $S$ and $\mathcal{E} \in \text{Fun}(\Delta^1, S)$ a full subcategory.

If $\mathcal{C} \to S \times T$ is a map of (locally) cocartesian fibrations over $T$ which induces on the fiber over every $t \in T$ a cartesian fibration $\mathcal{C}_t \to S$ relative to $\mathcal{E}$, then $\mathcal{C} \to S \times T$ is a map of cartesian fibrations relative to $\mathcal{E}$ which induces on the fiber over every $s \in S$ a (locally) cocartesian fibration $\mathcal{C}_s \to T$.

Dually, if $\mathcal{C} \to S \times T$ is a map of (locally) cartesian fibrations over $T$ which induces on the fiber over every $t \in T$ a cocartesian fibration $\mathcal{C}_t \to S$ relative to $\mathcal{E}$, then $\mathcal{C} \to S \times T$ is a map of cocartesian fibrations relative to $\mathcal{E}$ which induces on the fiber over every $s \in S$ a (locally) cartesian fibration $\mathcal{C}_s \to T$.

2. Let $\mathcal{C} \to S \times T$ be a functor corresponding to a functor $\mathcal{C} \to S \times T$ over $T$ and a functor $\mathcal{C} \to S \times T$ over $S$.

Then the following two conditions are equivalent:

(a) $\mathcal{C} \to S \times T$ is a map of cocartesian fibrations over $T$ which induces on the fiber over every $t \in T$ a cartesian fibration $\mathcal{C}_t \to S$.

(b) $\mathcal{C} \to S \times T$ is a map of cartesian fibrations over $S$ which induces on the fiber over every $s \in S$ a cocartesian fibration $\mathcal{C}_s \to T$.

3. Let $\mathcal{C} \to S \times T, \mathcal{D} \to S \times T$ be functors satisfying the equivalent conditions of 4. and let $\mathcal{C} \to \mathcal{D}$ be a functor over $S \times T$ corresponding to a functor $\mathcal{C} \to \mathcal{D}$ over $T$ over the category $S \times T$ and a functor $\mathcal{C} \to \mathcal{D}$ over $S$ over the category $S \times T$.

Then the following two conditions are equivalent:

(a) $\mathcal{C} \to \mathcal{D}$ is a map of cocartesian fibrations over $T$ which induces on the fiber over every $t \in T$ a map of cartesian fibrations over $S$.

(b) $\mathcal{C} \to \mathcal{D}$ is a map of cartesian fibrations over $S$ which induces on the fiber over every $s \in S$ a map of cocartesian fibrations over $T$. 

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4. Consequently the following two subcategories of $\text{Cat}_\infty(\kappa)_{\mathcal{S} \times \mathcal{T}}$ coincide:

The subcategory with objects the functors $\mathcal{C} \rightarrow \mathcal{S} \times \mathcal{T}$ satisfying the condition of 4. (a) and with morphisms the functors $\mathcal{C} \rightarrow \mathcal{D}$ over $\mathcal{S} \times \mathcal{T}$ satisfying the condition of 5. (a).

The subcategory with objects the functors $\mathcal{C} \rightarrow \mathcal{S} \times \mathcal{T}$ satisfying the condition of 4. (b) and with morphisms the functors $\mathcal{C} \rightarrow \mathcal{D}$ over $\mathcal{S} \times \mathcal{T}$ satisfying the condition of 5. (b).

5. Given $\kappa$-small categories $\mathcal{C}, \mathcal{D}$ denote $\text{Fun}(\mathcal{C}, \text{Cat}_\infty(\kappa)_{\mathcal{D}})^{\text{cocom}}$ the subcategory of $\text{Cat}_\infty(\kappa)_{\mathcal{D}}$ with objects the functors $\mathcal{C} \rightarrow \text{Cat}_\infty(\kappa)_{\mathcal{D}}$ that send every object of $\mathcal{C}$ to a cocartesian fibration over $\mathcal{D}$ and with morphisms the natural transformations of functors $\mathcal{C} \rightarrow \text{Cat}_\infty(\kappa)_{\mathcal{D}}$ whose components are maps of cocartesian fibrations over $\mathcal{D}$.

Similarly we define $\text{Fun}(\mathcal{C}, \text{Cat}_\infty(\kappa)_{\mathcal{D}})^{\text{cocom}}$.

The category $\text{Fun}(\mathcal{T}, \text{Cat}_\infty(\kappa)_{\mathcal{S} \times \mathcal{T}})^{\text{cart}}$ is equivalent to the first subcategory of $\text{Cat}_\infty(\kappa)_{\mathcal{S} \times \mathcal{T}}$ of 6., the category $\text{Fun}(\mathcal{S}_\kappa, \text{Cat}_\infty(\kappa)_{\mathcal{T}})^{\text{cocom}}$ is equivalent to the second subcategory of $\text{Cat}_\infty(\kappa)_{\mathcal{S} \times \mathcal{T}}$ of 6.

Thus we obtain a canonical equivalence

$$\text{Fun}(\mathcal{T}, \text{Cat}_\infty(\kappa)_{\mathcal{S}})^{\text{cart}} \cong \text{Fun}(\mathcal{S}_\kappa, \text{Cat}_\infty(\kappa)_{\mathcal{T}})^{\text{cocom}}.$$

By composing the last equivalence with the equivalence $\text{Cat}_\infty(\kappa)_{\mathcal{S}} \cong \text{Cat}_\infty(\kappa)_{\mathcal{S}_\kappa}$ induced by the duality involution on $\text{Cat}_\infty(\kappa)$ (and replacing $\mathcal{S}$ by $\mathcal{S}_\kappa$) we get canonical equivalences

$$\text{Fun}(\mathcal{T}, \text{Cat}_\infty(\kappa)_{\mathcal{S}})^{\text{cart}} \cong \text{Fun}(\mathcal{S}_\kappa, \text{Cat}_\infty(\kappa)_{\mathcal{T}})^{\text{cocom}}$$

and

$$\text{Fun}(\mathcal{T}, \text{Cat}_\infty(\kappa)_{\mathcal{S}_\kappa})^{\text{cart}} \cong \text{Fun}(\mathcal{S}_\kappa, \text{Cat}_\infty(\kappa)_{\mathcal{T}_\kappa})^{\text{cocom}}.$$

Let $\kappa$ be a strongly inaccessible cardinal.

Let $\mathcal{S}$ be a $\kappa$-small category and $\mathcal{O}^\kappa$ be a $\kappa$-small operad.

Denote $\text{Mon}_{\mathcal{O}^\kappa}(\text{Cat}_\infty(\kappa)_{\mathcal{S}})^{\text{cocom}}$ the subcategory of $\text{Mon}_{\mathcal{O}^\kappa}(\text{Cat}_\infty(\kappa)_{\mathcal{S}})$ with objects the $\mathcal{O}^\kappa$-monoids of $\text{Cat}_\infty(\kappa)_{\mathcal{S}}$ that send every object $X$ of $\mathcal{O}$ to a cartesian fibration over $\mathcal{S}$ and with morphisms the natural transformations of functors $\mathcal{O}^\kappa \rightarrow \text{Cat}_\infty(\kappa)_{\mathcal{S}}$, whose components on objects of $\mathcal{O}$ are maps of cartesian fibrations over $\mathcal{S}$. Let $\text{Mon}_{\mathcal{O}^\kappa}(\text{Cat}_\infty(\kappa)_{\mathcal{S}})^{\text{cocom}}$ be defined similarly.

Denote $\text{Fun}(\mathcal{S}_\kappa, \text{Mon}_{\mathcal{O}^\kappa}(\text{Cat}_\infty(\kappa)))^{\text{cocom}}$ the subcategory of

$\text{Fun}(\mathcal{S}_\kappa, \text{Mon}_{\mathcal{O}^\kappa}(\text{Cat}_\infty(\kappa)))^{\text{cocom}}$ with the same objects and with morphisms the natural transformations of functors $\mathcal{S}_\kappa \rightarrow \text{Mon}_{\mathcal{O}^\kappa}(\text{Cat}_\infty(\kappa))$, whose components are $\mathcal{O}^\kappa$-monoidal functors.

**Corollary 7.39.** Let $\kappa$ be a strongly inaccessible cardinal.

Let $\mathcal{S}$ be a $\kappa$-small category and $\mathcal{O}^\kappa$ be a $\kappa$-small operad.
The canonical equivalence

\[ \text{Fun}(\mathcal{O}^\circ, \text{Cat}_\infty(\kappa)/S)^\text{cart} \simeq \text{Fun}(S^{\text{op}}, \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ})^{\text{cocart}} \]

of corollary 7.38 restricts to an equivalence

\[ \text{Mon}_{\mathcal{O}^\circ}(\text{Cat}_\infty(\kappa)/S)^{\text{cocart}} \simeq \text{Fun}(S^{\text{op}}, \text{Mon}_{\mathcal{O}^\circ, \text{lax}}(\text{Cat}_\infty(\kappa)))^{\text{cocart}}. \]

By composing this equivalence with the equivalence \( \text{Cat}_\infty(\kappa)/S \simeq \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ} \)

induced by the duality involution on \( \text{Cat}_\infty(\kappa) \) (and replacing \( S \) by \( S^{\text{op}} \)) we get an equivalence

\[ \text{Mon}_{\mathcal{O}^\circ}(\text{Cat}_\infty(\kappa)/S)^{\text{cocart}} \simeq \text{Fun}(S, \text{Mon}_{\mathcal{O}^\circ, \text{lax}}(\text{Cat}_\infty(\kappa)))^{\text{cocart}}. \]

Proof. Let \( \psi : \mathcal{O}^\circ \to \text{Cat}_\infty(\kappa)/S \) be an object of \( \text{Fun}(\mathcal{O}^\circ, \text{Cat}_\infty(\kappa)/S)^{\text{cart}} \) and \( H : S^{\text{op}} \to \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ} \) be an object of \( \text{Fun}(S^{\text{op}}, \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ})^{\text{cocart}} \) that correspond under the canonical equivalence

\[ \text{Fun}(\mathcal{O}^\circ, \text{Cat}_\infty(\kappa)/S)^{\text{cart}} \simeq \text{Fun}(S^{\text{op}}, \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ})^{\text{cocart}} \]

of corollary 7.38.

Then there is a functor \( \gamma : \mathcal{C} \to \mathcal{O}^\circ \times S \) that is a map of cocartesian fibrations over \( \mathcal{O}^\circ \) classifying \( \mathcal{O}^\circ \to \text{Cat}_\infty(\kappa)/S \) and is a map of cartesian fibrations over \( S \) classifying \( S^{\text{op}} \to \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ} \).

We have to see that \( \mathcal{O}^\circ \to \text{Cat}_\infty(\kappa)/S \) is an \( \mathcal{O}^\circ \)-monoid object of \( \text{Cat}_\infty(\kappa)/S \) if and only if \( S^{\text{op}} \to \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ} \) factors through the subcategory \( \text{Mon}_{\mathcal{O}^\circ, \text{lax}}(\text{Cat}_\infty(\kappa)) \) of \( \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ} \).

Let \( n \in \mathbb{N} \) and let for every \( i \in \{1, \ldots, n\} \) an inert morphism \( X \to X_i \) of \( \mathcal{O}^\circ \) be given lying over the unique inert morphism \( (n) \to (1) \) of \( \text{Fin}_* \) that sends \( i \) to \( 1 \).

Then the following two conditions are equivalent:

1. The induced functors \( \psi(X) \to \psi(X_i) \) over \( S \) for \( i \in \{1, \ldots, n\} \) form a product diagram in \( \text{Cat}_\infty(\kappa)/S \).

2. Each of the functors \( \psi(X) \to \psi(X_i) \) is a map of cartesian fibrations over \( S \) and for every \( s \in S \) the induced functors \( \psi(X)_s \to \psi(X_i)_s \) on the fiber over \( s \) form a product diagram.

By the naturality of the canonical equivalence \( \text{Fun}(\mathcal{O}^\circ, \text{Cat}_\infty(\kappa)/S)^{\text{cart}} \simeq \text{Fun}(S^{\text{op}}, \text{Cat}_\infty(\kappa)_{/\mathcal{O}^\circ})^{\text{cocart}} \) the induced functor \( \psi(X)_s \to \psi(X_i)_s \) on the fiber over \( s \) is classified by \( H(s)_{X_i} \to H(s)_{X} \).

Consequently it is enough to show that for every morphism \( h : t \to s \) of \( S \) and every inert morphism \( f : X \to Y \) of \( \mathcal{O}^\circ \) the following two conditions are equivalent:

1. The induced functor \( \psi(f) : \psi(X) \to \psi(Y) \) over \( S \) preserves cartesian morphisms lying over the morphism \( h : t \to s \).

2. The induced functor \( H(h) : H(s) \to H(t) \) over \( \mathcal{O}^\circ \) preserves cocartesian morphisms lying over the morphism \( f : X \to Y \).

This follows from lemma 7.40. \( \square \)
**Lemma 7.40.** Let $\kappa$ be a strongly inaccessible cardinal and $S, T, C$ be $\kappa$-small categories. Let $p : C \to T$ and $q : C \to S$ be functors.

Assume that the functor $\rho = (p, q) : C \to T \times S$ is a map of cocartesian fibrations over $T$, which is fiberwise a cartesian fibration, classifying a functor $\psi : T \to \text{Cat}_\infty(\kappa)^T$.

By corollary 7.39, $\rho = (q, p) : C \to S \times T$ is a map of cartesian fibrations over $S$, which is fiberwise a cocartesian fibration, classifying a functor $H : S^{op} \to \text{Cat}_\infty(\kappa)^T$.

Let $H(h)(B) \to B$ and $H(h)(\psi(f)(B)) \to \psi(f)(B)$ be $q$-cartesian lifts of the morphism $h : s \to s'$ of $S$ and let $H(h)(B) \to \psi(f)(H(h)(B))$ and $B \to \psi(f)(B)$ be $p$-cocartesian lifts of the morphism $f : t' \to t$ of $T$.

1. The morphisms $\psi(f)(H(h)(B)) \to \psi(f)(B)$ and $H(h)(B) \to H(h)(\psi(f)(B))$ induce the same morphism $\psi(f)(H(h)(B)) \to H(h)(\psi(f)(B))$ in the fiber $\psi(t)_s \simeq H(s)_t$.

Consequently the morphism $\psi(f)(H(h)(B)) \to \psi(f)(B)$ is $q$-cartesian if and only if $H(h)(B) \to H(h)(\psi(f)(B))$ is $p$-cocartesian.

2. This implies the following:

The functor $\psi(f) : \psi(t') \to \psi(t)$ sends $\rho_\kappa$-cartesian lifts of $h : s \to s'$ to $\rho_\kappa$-cartesian morphisms if and only if $H(h) : H(s) \to H(s')$ sends $\beta$-cocartesian lifts of $f : t' \to t$ to $\beta$-cocartesian morphisms.

**Proof.** Denote $\beta$ the composition $H(h)(B) \to B \to \psi(f)(B)$ of morphisms of $C$ so that $\beta$ lies over $f$ and $h$.

By definition the morphism $\psi(f)(H(h)(B)) \to H(h)(\psi(f)(B))$ in the fiber $\psi(t)_s \simeq H(s)_t \simeq C_{t, s}$ induced by $\psi(f)(H(h)(B)) \to \psi(f)(B)$ corresponds to $\beta$ under the top horizontal functor of the following diagram of pullback squares:

\[
\begin{array}{ccc}
\mathcal{C}(\psi(f)(H(h)(B)), H(h)(\psi(f)(B))) & \to & \mathcal{C}(\psi(f)(H(h)(B)), \psi(f)(B)) \\
\downarrow & & \downarrow \\
T(t, t) \times S(s, s) & \to & T(t, t) \times S(s, s')
\end{array}
\]

By definition the morphism $\psi(f)(H(h)(B)) \to H(h)(\psi(f)(B))$ in the fiber $\psi(t)_s \simeq H(s)_t \simeq C_{t, s}$ induced by $H(h)(B) \to H(h)(\psi(f)(B))$ corresponds to $\beta$ under the top horizontal functor of the following diagram of pullback squares:

\[
\begin{array}{ccc}
\mathcal{C}(\psi(f)(H(h)(B)), H(h)(\psi(f)(B))) & \to & \mathcal{C}(H(h)(B), H(h)(\psi(f)(B))) \\
\downarrow & & \downarrow \\
\mathcal{C}(H(h)(B), \psi(f)(B)) & \to & \mathcal{C}(H(h)(B), \psi(f)(B)) \\
\downarrow & & \downarrow \\
T(t, t) \times S(s, s) & \to & T(t', t) \times S(s, s')
\end{array}
\]

So both induced morphisms coincide as both outer squares of the two diagrams coincide. \[\square\]
Lemma 7.41. Let $S$ be a $\kappa$-small category, $p : X \to S$ and $q : Y \to S$ be $\kappa$-small cocartesian fibrations and $\phi : X \to Y$ a map of cocartesian fibrations over $S$ for a strongly inaccessible cardinal $\kappa$.

Let $f : s \to t$ be a morphism of $S$.

1. The following two conditions are equivalent to each other:

   (a) The induced functor $X_s \to Y_s \times_Y X_t$ is essentially surjective.

   (b) For every object $B$ of $X$ and every $q$-cocartesian lift $g : A \to \phi(B)$ of $f : s \to t$ there exists a $p$-cocartesian lift $D \to B$ of $g$.

2. The following two conditions are equivalent to each other:

   (a) For every morphism $f : s \to t$ of $S$ the induced functor $X_s \to Y_s \times_Y X_t$ is fully faithful.

   (b) Every $p$-cocartesian lift $h : D \to f_s(D)$ of $f$ is $\phi$-cartesian.

So if for every morphism $f : s \to t$ of $S$ the induced functor $X_s \to Y_s \times_Y X_t$ is an equivalence, for every object $B$ of $X$ and every $q$-cocartesian lift $g : A \to \phi(B)$ of $f : s \to t$ there exists a $\phi$-cartesian lift $D \to B$ of $g$.

Proof. We first show statement 1.

Assume (a) holds. Let $g : A \to \phi(B) \simeq f_s(A)$ be a $q$-cocartesian lift of $f : s \to t$.

As the induced functor $X_s \to Y_s \times_Y X_t$ is essentially surjective, there exists an object $D$ of $X_s$ an equivalence $\phi(D) \simeq A$ in $Y_s$ and a $p$-cocartesian morphism $h : D \to B$ lying over $f$.

As $\phi$ is a map of cocartesian fibrations over $S$, the morphism $\phi(h)$ is $q$-cocartesian and is thus equivalent to $g$.

Conversely assume that (b) holds and let $A$ be an object of $Y_s$, $B$ an object of $X_t$, and $g : A \to \phi(B) \simeq f_s(A)$ a $q$-cocartesian lift of $f$.

Then by (b) there is a $p$-cocartesian lift $D \to B \simeq f_s(D)$ of $g$.

So we get an equivalence $A \simeq \phi(D)$ such that the composition $\phi(B) \simeq \phi(f_s(D)) \simeq f_s(\phi(D)) \simeq f_s(A) \simeq \phi(B)$ is homotopic to the identity.

As next we verify 2.

Let $f : s \to t$ be a morphism of $S$ and $h : D \to f_s(D)$ a $p$-cocartesian lift of $f$.

Then for every object $Z$ of $X$ the fiber of the diagram

$$
\begin{array}{ccc}
X(Z,D) & \xrightarrow{X(f_s,\phi(D))} & X(Z,f_s(D)) \\
\downarrow & & \downarrow \\
Y(\phi(Z),\phi(D)) & \xrightarrow{Y(\phi(Z),f_s(\phi(D)))} & Y(\phi(Z),f_s(\phi(D)))
\end{array}
$$

over an object $\varphi$ of $S(p(Z),s)$ is equivalent to the diagram

$$
\begin{array}{ccc}
X_s(\varphi_*(Z),D) & \xrightarrow{X_s(f_*(\varphi_*(Z)),f_s(D))} & X_s(f_*(\varphi_*(Z)),f_s(D)) \\
\downarrow & & \downarrow \\
Y_s(\phi(\varphi_*(Z)),\phi(D)) & \xrightarrow{Y_t(f_*(\phi(\varphi_*(Z))),f_s(\phi(D)))} & Y_t(f_*(\phi(\varphi_*(Z))),f_s(\phi(D))).
\end{array}
$$
Remark 7.42. Let $\mathcal{M}^\otimes \to \mathcal{L}^\otimes$ be a $\mathcal{L}^\otimes$-monoidal category that exhibits a category $\mathcal{D}$ as left module over a monoidal category $\mathcal{C}$.

Denote $1_\mathcal{C}: \text{Ass}^\otimes \to \mathcal{C}^\otimes$ the initial object of $\text{Alg}(\mathcal{C})$ and $\mathcal{C}^\otimes_{/1_\mathcal{C}}$ the pullback of the cocartesian fibration $(\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{(1)}$ of operads along the map $1_\mathcal{C}: \text{Ass}^\otimes \to \mathcal{C}^\otimes$ of operads over $\text{Ass}^\otimes$.

Denote $\alpha : \mathcal{D} \times \mathcal{L}^\otimes \to \mathcal{M}^\otimes$ the functor over $\mathcal{L}^\otimes$ corresponding to the forgetful functor

$$\beta : \mathcal{D} = \text{LMod}_{/\mathcal{E}}(\mathcal{D}) \to \text{LMod}(\mathcal{D}) = \text{Alg}_{\mathcal{L}^\otimes/\mathcal{L}^\otimes}(\mathcal{M}^\otimes) \subset \text{Fun}_{\mathcal{L}^\otimes}(\mathcal{L}^\otimes, \mathcal{M}^\otimes).$$

Denote $\mathcal{X} \to \mathcal{D} \times \mathcal{L}^\otimes$ the pullback of the cocartesian fibration $(\mathcal{M}^\otimes)^{\Delta^1} \to (\mathcal{M}^\otimes)^{(1)}$ of operads over $\mathcal{L}^\otimes$ along $\alpha : \mathcal{D} \times \mathcal{L}^\otimes \to \mathcal{M}^\otimes$.

The functor $\mathcal{X} \to \mathcal{D} \times \mathcal{L}^\otimes$ is a cocartesian $\mathcal{D}$-family of $\mathcal{L}^\otimes$-monoidal categories and we have canonical equivalences $\text{Ass}^\otimes \times_{\mathcal{L}^\otimes} \mathcal{X} \simeq \mathcal{D} \times \mathcal{C}^\otimes_{/1_\mathcal{C}}$ over $\mathcal{D} \times \text{Ass}^\otimes$ and $\{m\} \times_{\mathcal{L}^\otimes} \mathcal{X} \simeq \text{Fun}(\Delta^1, \mathcal{D})$ of categories pseudoenriched over $\mathcal{C}^\otimes_{/1_\mathcal{C}}$.

In other words the functor $\text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}(\{1\}, \mathcal{D})$ is a left module over $\mathcal{D} \times \mathcal{C}^\otimes$ in $\text{Cat}^\text{cocart}_{\mathcal{D}/\mathcal{C}}(\kappa)$.

Especially the functor $\text{Fun}(\Delta^1, \mathcal{D}) \to \text{Fun}(\{1\}, \mathcal{D})$ can be promoted to a cocartesian $\mathcal{D}$-family of categories pseudo-enriched over $\mathcal{C}^\otimes_{/1_\mathcal{C}}$.

Given a functor $H: \mathcal{B} \to \mathcal{D}$ we have a canonical equivalence

$$\text{Fun}_\mathcal{D}(\mathcal{B}, \text{Fun}(\Delta^1, \mathcal{D}))^\otimes = \text{Fun}_{\mathcal{D} \times \mathcal{L}^\otimes}^{\mathcal{L}^\otimes}(\mathcal{B} \times \mathcal{L}^\otimes, \mathcal{X}) \simeq \text{Fun}_{(\mathcal{M}^\otimes)^{(1)}}^{\mathcal{L}^\otimes}(\mathcal{B} \times \mathcal{L}^\otimes, (\mathcal{M}^\otimes)^{\Delta^1})$$

$$\simeq \text{Map}_{\mathcal{L}^\otimes}(\mathcal{B}, \mathcal{M}^\otimes, (\mathcal{M}^\otimes)^{/(\alpha \otimes (H \times \mathcal{L}^\otimes)}) = \text{Fun}_\mathcal{D}(\mathcal{B}, \mathcal{D})^\otimes_{/(\alpha \otimes (H \times \mathcal{L}^\otimes))}$$

over $\mathcal{L}^\otimes$.

Proof. For every object $Z$ of $\mathcal{D}$ the fiber $\mathcal{X}_Z \to \mathcal{L}^\otimes$ is the pullback of the $\mathcal{L}^\otimes$-monoidal functor $(\mathcal{M}^\otimes)^{\Delta^1} \to (\mathcal{M}^\otimes)^{(1)}$ of operads over $\mathcal{L}^\otimes$ and is thus an operad over $\mathcal{L}^\otimes$.

The composition $\mathcal{D} \times \text{Ass}^\otimes \subset \mathcal{D} \times \mathcal{L}^\otimes \to \mathcal{M}^\otimes$ is equivalent to the composition $\mathcal{D} \times \text{Ass}^\otimes \to \text{Ass}^\otimes \xrightarrow{\alpha_\otimes} \mathcal{C}^\otimes \to \mathcal{M}^\otimes$.

Thus the pullback of $\mathcal{X} \to \mathcal{L}^\otimes$ to $\text{Ass}^\otimes$ is equivalent to the pullback of the $\mathcal{L}^\otimes$-monoidal functor $(\mathcal{M}^\otimes)^{\Delta^1} \to (\mathcal{M}^\otimes)^{(1)}$ along the functor $\mathcal{D} \times \text{Ass}^\otimes \to \text{Ass}^\otimes \xrightarrow{\alpha_\otimes} \mathcal{C}^\otimes \to \mathcal{M}^\otimes$ and is thus equivalent to the pullback of the monoidal functor $(\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{(1)}$ along the functor $\mathcal{D} \times \text{Ass}^\otimes \to \text{Ass}^\otimes \xrightarrow{\alpha_\otimes} \mathcal{C}^\otimes$ over $\text{Ass}^\otimes$, which is the functor $\mathcal{D} \times \mathcal{C}^\otimes_{/1_\mathcal{C}} \to \mathcal{D} \times \text{Ass}^\otimes$.

The composition $\mathcal{D} = \mathcal{D} \times \{m\} \subset \mathcal{D} \times \mathcal{L}^\otimes \to \mathcal{M}^\otimes$ is equivalent to the canonical functor $\mathcal{D} \to \mathcal{M}^\otimes$.  

\[\square\]
Proposition 7.43. Let $\mathcal{C}$ be a $\kappa$-small 2-category.

Let $X, Y$ be objects of $\mathcal{C}$ and $g : Y \to X$ a morphism of $\mathcal{C}$.

1. Let $f : X \to Y$ be a morphism of $\mathcal{C}$ and $\eta : \text{id}_X \to g \circ f$ a 2-morphism of $\mathcal{C}$.

Then there is a 2-morphism $\varepsilon : f \circ g \to \text{id}_Y$ of $\mathcal{C}$ satisfying the triangular identities $(\varepsilon \circ f) \circ (f \circ \eta) = \text{id}_f$ and $(g \circ \varepsilon) \circ (\eta \circ g) = \text{id}_g$ if and only if the following condition holds:

For every object $Z$ of $\mathcal{C}$ the induced natural transformation

$$\eta^Z := [Z, \eta] : \text{id}_{[Z, X]} \circ [Z, f] \circ [Z, g]$$

exhibits the functor $f^Z := [Z, f] : [Z, X] \to [Z, Y]$ as left adjoint to the functor $g^Z := [Z, g] : [Z, Y] \to [Z, X]$.

2. Let $\mathcal{C}$ be a $\kappa$-small closed and cotensored left module over $\mathcal{C}$.

Then the morphism $g : Y \to X$ of $\mathcal{C}$ admits a left adjoint $f : X \to Y$ if and only if the following two conditions hold:

(a) For every object $Z$ of $\mathcal{C}$ the induced functor $g^Z := [Z, g] : [Z, Y] \to [Z, X]$ admits a left adjoint $f^Z$.

(b) For every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced natural transformation $f^Z \circ [\varphi, X] \to f^Z \circ [\varphi, Y] \circ g^Z = f^Z \circ g^Z \circ [\varphi, Y] \circ f^Z$ is an equivalence.

Remark 7.44. The compatibility condition of (c) is equivalent to the condition that for every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced natural transformation

$$[\varphi, X] \circ g^Z \to g^Z \circ f^Z \circ [\varphi, X] \circ g^Z \circ f^Z \circ g^Z = \varphi \circ X \circ g^Z \circ f^Z$$

is an equivalence and is equivalent to the condition that for every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced functor $[\varphi, X] : [Z', X] \to [Z, X]$ preserves the unit of the adjunction in the following sense:

Let $\eta : \text{id} \to g^Z \circ f^Z$ be a unit of the adjunction $f^Z : [Z', X] \to [Z', Y] : g^Z$ and $H : Z' \to X$ be a morphism of $\mathcal{C}$.

Then the composition $H \circ \varphi \xrightarrow{\gamma(H) \circ \varphi} g^Z(f^Z(H)) \circ \varphi \simeq g^Z(f^Z(H) \circ \varphi)$ yields for every morphism $T : Z \to Y$ of $\mathcal{C}$ an equivalence

$$[Z, Y](f^Z(H) \circ \varphi, T) \to [Z, X](g^Z(f^Z(H) \circ \varphi), g^Z(T)) \to [Z, X](H \circ \varphi, g^Z(T)).$$

Proof. We show 1:

Denote $\varepsilon^Z : f^Z \circ g^Z \to \text{id}$ the counit of the adjunction $f^Z = [Z, f] : [Z, X] \to [Z, Y] : g^Z = [Z, g]$ and set $\varepsilon := \varepsilon^Y([\text{id}_Y] : f \circ g \to \text{id}_Y).$
In the following we will see that η and ε are related by the triangular identities.

The triangular identities of the adjunctions \( f^X = [X,f] : [X,X] \rightleftarrows [X,Y] : g^X = [X,g] \) and \( f^Y = [Y,f] : [X,Y] \rightleftarrows [Y,Y] : g^Y = [Y,g] \) imply that both compositions \((ε^X \circ f^X) \circ (f^X \circ η^X)\) and \((g^Y \circ ε^Y) \circ (η^Y \circ g^Y)\) of natural transformations of functors \([X,X] \rightleftarrows [X,Y]\) respectively \([Y,Y] \rightleftarrows [X,Y]\) are homotopic to the identity.

Evaluating at \( idX \) and \( idY \) we see that the compositions \( ε^X(f) \circ (f \circ η) \) and \((g \circ ε) \circ (η \circ g)\) are homotopic to the identity.

Therefore it is enough to show that \( ε^X(f) : f^X(g^X(f)) = f \circ g \circ f \rightarrow f\) is homotopic to \( ε \circ f : f \circ g \circ f \rightarrow f\).

This is equivalent to the condition that \( ε \circ f : f^X(g^X(f)) = f \circ g \circ f \rightarrow f\) is adjoint to the identity of \( g^X(f) = g \circ f \) with respect to the adjunction \( f^X = [X,f] : [X,X] \rightleftarrows [X,Y] : g^X = [X,g] \), in other words that \( g^X(ε \circ f) \circ η^X(g^X(f)) = (g \circ ε \circ f) \circ (η \circ g \circ f) = ((g \circ ε) \circ (η \circ g)) \circ f\) is homotopic to the identity of \( g \circ f \).

But we have already seen that \((g \circ ε) \circ (η \circ g)\) is homotopic to the identity of \( g \).

As we next prove 2.

Denote \( θ : C \rightarrow \text{Fun}(C^{op}, \text{Cat}_∞(κ)) \approx \text{Cat}^{cart}_∞(κ) \) the functorial adjoint to \([−,−] : C^{op} \times C \rightarrow \text{Cat}_∞(κ)\).

We have a canonical equivalence \( θ(Z^K) \approx θ(Z)^K \) natural in \( K \in \text{Cat}_∞(κ) \) and \( Z \in C \) classified by the canonical equivalence \([−,Z^K] \approx \text{Fun}(K,−) \circ [−,Z] \) of functors \( C^{op} \rightarrow \text{Cat}_∞(κ) \) that is represented by the natural equivalence \( \text{Cat}_∞(κ)(W,[T,Z^K]) \approx \text{Fun}(K,W) \circ (K \times T)/Z \approx \text{Cat}_∞(κ)(W,\text{Fun}(K,[T,Z]))\) for \( W \in \text{Cat}_∞(κ) \) and \( T \in C \).

Moreover we have a canonical equivalence \( ζ : ([\text{Cat}_∞(κ) \times C] × C^{−}) \circ θ \approx \text{Map}_{\text{Cat}_∞(κ)\times C}(U(κ) \times C,\text{Cat}_∞(κ) \times C^{−}) \circ θ \) of functors \( C^{−} \rightarrow \text{Cat}^{cart}_∞(κ) \) that classifies the canonical equivalence \([− \otimes C,−] \approx \text{Fun}(−,[−,C^{−}])\).

The functor \( (Δ^3)^{op} \approx Δ^1 \Rightarrow \text{Cat}^{cart}_∞(C^{−}) \) corresponding to \( θ(g) : θ(Y) \rightarrow θ(X) \) is classified by a map \( ρ : Z \rightarrow C \times Δ^1 \) of cartesian fibrations over \( Δ^1 \) such that \( ρ : Z \rightarrow C \times Δ^1 \) is itself a cartesian fibration.

By corollary \ref{cor:cart-fibrations} condition (a) and (b) of 2. together are equivalent to the condition that \( ρ : Z \rightarrow C \times Δ^1 \) is a map of bicartesian fibrations over \( Δ^1 \) encoding an adjunction relative to \( C \), where both the left adjoint \( F : θ(X) \rightarrow θ(Y) \) and right adjoint \( θ(g) : θ(Y) \rightarrow θ(X) \) are maps of cartesian fibrations over \( C \).

Let \( λ : \text{id}_{θ(X)} \rightarrow θ(g) \circ F \) be the unit of this adjunction relative to \( C \) and \( Φ : θ(X) \rightarrow θ(X)Δ^1 \approx θ(θ(X)Δ^1) \) the corresponding map of cartesian fibrations over \( C \) under the canonical equivalence \( \text{Cat}^{cart}_∞(C^{−})(θ(X),θ(θ(X)Δ^1)) \approx \text{Fun}(Δ^1,\text{Fun}^{cart}C(θ(X),θ(X)))\).

Set \( f := F_!(id_X) : X \rightarrow Y \) and \( φ := Φ_X(id_X) : X \rightarrow XΔ^1 \) and \( η := λ(id_X) : id_X \rightarrow g \circ f \) so that the morphism \( η \) of \([X,X] \) corresponds to \( φ : X \rightarrow XΔ^1 \).

Using the first part of the lemma it is enough to show that there is an equivalence \( α : F \rightarrow θ(f) \) of maps \( θ(X) \rightarrow θ(Y) \) of cartesian fibrations over \( C \) and
a commutative square

\[
\begin{array}{c}
\text{id}_{\theta(X)} \xrightarrow{\lambda} \theta(g) \circ F \\
\downarrow \cong \downarrow \theta(g) \circ \theta(f)
\end{array}
\]

in \(\text{Cat}_{\infty/\mathcal{C}}^\text{cart}(\kappa)\).

Such a commutative square considered as an equivalence in 
\(\text{Fun}(\Delta^1, \text{Fun}^\text{cart}_C(\theta(X), \theta(X))) \simeq \text{Cat}_{\infty/\mathcal{C}}^\text{cart}(\kappa)(\theta(X), \theta(X)^{\Delta^1})\)
between \(\lambda\) and \(\theta(\eta)\) corresponds to an equivalence \(\beta : \Phi \to \theta(\phi)\) of maps \(\theta(X) \to \theta(X)^{\Delta^1}\)
of cartesian fibrations over \(\mathcal{C}\) that is sent by the map \(\theta(X)^{\Delta^1} \to \theta(X)(1)\)
of cartesian fibrations over \(\mathcal{C}\) to the equivalence \(\theta(g) \circ \alpha : \theta(g) \circ F \to \theta(g) \circ \theta(f)\)
of maps \(\theta(X) \to \theta(X)\) of cartesian fibrations over \(\mathcal{C}\).

Denote \((-)^\circ : \text{Cat}_{\infty/\mathcal{C}}^\text{cart}(\kappa) \to \mathcal{R}(\kappa)\mathcal{C}\) the right adjoint of the full subcategory inclusion \(\mathcal{R}(\kappa)\mathcal{C} \subset \text{Cat}_{\infty/\mathcal{C}}^\text{cart}(\kappa)\) that takes fiberwise the core groupoid.

The Yoneda-embedding \(\text{Cat}_{\infty}(\kappa) \to \mathcal{P}(\text{Cat}_{\infty}(\kappa))\) of \(\text{Cat}_{\infty}(\kappa)\) induces a fully faithful functor \(\text{Cat}_{\infty/\mathcal{C}}(\kappa) = \text{Fun}(\mathcal{C}^\text{op}, \text{Cat}_{\infty}(\kappa)) \to \text{Fun}(\mathcal{C}^\text{op}, \mathcal{P}(\text{Cat}_{\infty}(\kappa))) \simeq \mathcal{P}(\mathcal{C} \times \text{Cat}_{\infty}(\kappa)) \simeq \mathcal{R}(\kappa)\mathcal{C} \times \text{Cat}_{\infty}(\kappa)\) that sends a cartesian fibration \(\mathcal{B} \to \mathcal{C}\) to the right fibration

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \mathcal{B}) \simeq \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}^\circ(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \mathcal{B})\]

of maps \(\mathcal{B} \to \mathcal{B}'\) over \(\mathcal{C}\) to the map \(\text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \mathcal{H})\) of right fibrations over \(\text{Cat}_{\infty}(\kappa) \times \mathcal{C}\).

Consequently it is enough to show that there is an equivalence

\[
\alpha' : \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \mathcal{F}^\circ) \simeq \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(f))^\circ
\]

of maps

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X))^\circ \simeq \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(Y))^\circ
\]

of right fibrations over \(\text{Cat}_{\infty}(\kappa) \times \mathcal{C}\) and an equivalence

\[
\beta' : \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \Phi)^\circ \simeq \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(\phi))^\circ
\]

of maps

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X)^{\Delta^1})^\circ \simeq \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X^{1}))^\circ
\]

of right fibrations over \(\text{Cat}_{\infty}(\kappa) \times \mathcal{C}\) that is sent by the map

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X^{\Delta^1}))^\circ \simeq \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X^{\Delta^1}))^\circ
\]

of right fibrations over \(\mathcal{C}\).

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(g)) \circ \alpha' : \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \mathcal{F})^\circ \circ \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \mathcal{F})^\circ
\]

\[
\circ \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(f))^\circ
\]

\[
\circ \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(f))^\circ
\]
of maps

\[ \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X))^\pi \to \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C}, \text{Cat}_{\infty}(\kappa) \times \theta(X))^\pi \]

of right fibrations over \( \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \).

The cartesian fibrations \( (\text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1)^{\mathcal{C} \times \Delta^1} \times \mathcal{C} \) are \( \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \) and \( \text{Map}_{\text{Cat}_{\infty}(\kappa) \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C} \times \Delta^1, \text{Cat}_{\infty}(\kappa) \times \mathcal{Z}) \to \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \) classify functors \( \text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C} \times \Delta^1 = \text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C} \times (\Delta^1)^{\text{op}} \to \text{Cat}_{\infty}(\kappa) \) that correspond to the natural transformations

\[
\text{Fun}(\mathcal{U} \times F, \mathcal{U} \times \mathcal{F}) : \text{Fun}(\mathcal{U} \times F, \mathcal{U} \times \mathcal{F}) \to \text{Fun}(\mathcal{U} \times \mathcal{F}, \mathcal{U} \times \mathcal{F})
\]

of functors \( \text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C} \times \Delta^1 \) relative to \( \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \).

As both natural transformations are canonically equivalent, we obtain an equivalence \( (\text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1)^{\mathcal{C} \times \Delta^1} \times \mathcal{C} \approx \text{Map}_{\text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C} \times \Delta^1, \text{Cat}_{\infty}(\kappa) \times \mathcal{Z}) \) of cartesian fibrations over \( \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \) that induces the equivalences \( \zeta_X : (\text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \times \mathcal{C} \times \Delta^1 \times \mathcal{C} \approx \text{Map}_{\text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C} \times \Delta^1, \text{Cat}_{\infty}(\kappa) \times \mathcal{Z}) \) and \( \zeta_Y : (\text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \times \mathcal{C} \times \Delta^1, \text{Cat}_{\infty}(\kappa) \times \mathcal{Z}) \) of cartesian fibrations over \( \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \) on the fibers over \( \{0\} \) and \( \{1\} \).

So the bicartesian fibrations \( (\text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1)^{\mathcal{C} \times \Delta^1} \times \mathcal{C} \) and \( \text{Map}_{\text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C} \times \Delta^1, \text{Cat}_{\infty}(\kappa) \times \mathcal{Z}) \) over \( \Delta^1 \) are equivalent over the bicartesian fibration \( \text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \to \Delta^1 \) and therefore encode equivalent adjunctions

\[
(\text{Cat}_{\infty}(\kappa) \times \mathcal{C} \times \Delta^1 \times \mathcal{C} \times \Delta^1 \times \mathcal{C} \approx \text{Map}_{\text{Cat}_{\infty}(\kappa)^{\text{op}} \times \mathcal{C}}(\mathcal{U}(\kappa) \times \mathcal{C} \times \Delta^1, \text{Cat}_{\infty}(\kappa) \times \mathcal{Z}) \)
\]
\[
\text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times F) : \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \theta(X)) \cong \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \theta(g)) \text{ relative to } \text{Cat}^\infty(\kappa) \times \mathcal{E}.
\]

Consequently there is a unique equivalence
\[
\sigma : (\zeta_Y)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times F) \circ \zeta_X \rightarrow (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \mathcal{E}
\]
of functors \((\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(X) \rightarrow (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(Y)\) such that
\[
(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \lambda : \text{id}_{(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(X)} \rightarrow ((\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(g)) \circ ((\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F F)
\]
is the composition
\[
\text{id}_{(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \lambda} : (\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \lambda) \circ \zeta_X
\]
\[
(\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times F) \circ \zeta_X
\]
\[
(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(X) \cong (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(Y) \cong (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(Y)^{[1]}
\]
of cartesian fibrations over \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\) that is sent by \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\) to the equivalence
\[
(\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \theta(g)) \circ \zeta_X
\]
\[
(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(g) \circ (\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times F) \circ \zeta_X
\]
of maps of cartesian fibrations over \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\).
\[
\gamma : (\zeta_X^{\Delta^1})^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \Phi) \circ \zeta_X \rightarrow (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \Phi
\]
between the corresponding maps \(\gamma : (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(X) \rightarrow (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(Y)^{[1]}\) of right fibrations over \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\) that is sent by \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\) to the equivalence
\[
(\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \theta(g)) \circ \zeta_X
\]
\[
(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(g) \circ (\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times F) \circ \zeta_X
\]
of maps of right fibrations over \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\).
\[
\gamma : (\zeta_X^{\Delta^1})^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \Phi) \circ \zeta_X
\]
between maps \(\gamma : (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(X)^{[1]} \rightarrow (\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F (\theta(X)^{\Delta^1})^{[1]}\) of right fibrations over \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\) that is sent by \(\text{Cat}^\infty(\kappa) \times \mathcal{E}\) to the equivalence
\[
(\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times \theta(g)) \circ \zeta_X
\]
\[
(\text{Cat}^\infty(\kappa) \times \mathcal{E}) \times_F \theta(g) \circ (\zeta_X)^{-1} \circ \text{Map}_{\text{Cat}^\infty(\kappa)}(\mathcal{U}(\kappa) \times \mathcal{E}, \text{Cat}^\infty(\kappa) \times F) \circ \zeta_X
\]
of maps of right fibrations over \( \text{Cat}_{\infty}(\kappa) \times C \).

The composition \( \mathcal{C} \overset{\theta}{\to} \text{Cat}_{\infty}(\kappa) \times C \overset{\Delta}{\to} \text{Cat}_{\infty}(\kappa) \times C \cong \mathcal{P}(\mathcal{C}) \) is equivalent to the Yoneda-embedding of \( \mathcal{C} \) and is thus fully faithful.

Therefore there is an equivalence \( \theta : F^{\gamma} \to \theta(f)^{\gamma} \) of maps \( \theta(X)^{\gamma} \to \theta(Y)^{\gamma} \) of right fibrations over \( \mathcal{C} \) and an equivalence \( \Phi^{\gamma} \to \theta(\phi)^{\gamma} \) of maps \( \theta(X)^{\gamma} \to \theta(X^{\Delta^{1}})^{\gamma} \) of right fibrations over \( \mathcal{C} \) that is sent by the map \( \theta(X^{\Delta^{1}})^{\gamma} \to \theta(X^{1})^{\gamma} \) of right fibrations over \( \mathcal{C} \) to the equivalence \( \theta(g)^{\gamma} \circ \theta : \theta(g)^{\gamma} \circ F^{\gamma} \to \theta(g)^{\gamma} \circ \theta(f)^{\gamma} \) of maps \( \theta(X)^{\gamma} \to \theta(X)^{\gamma} \) of right fibrations over \( \mathcal{C} \).

We define the equivalence \( \beta' \) as the composition

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \overset{\gamma \circ \phi^{-1}}{\to} \mathcal{U}(\kappa) \times C \cdot \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \cong \mathcal{C}_{\Delta^{1}} \cdot \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma},
\]

and the equivalence \( \alpha' \) as the composition

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \overset{\gamma \circ \phi^{-1}}{\to} \mathcal{U}(\kappa) \times C \cdot \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \cong \mathcal{C}_{\Delta^{1}} \cdot \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma}.
\]

Then \( \beta' \) is sent by the map

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \to \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma}
\]

of right fibrations over \( \text{Cat}_{\infty}(\kappa) \times C \) to the equivalence

\[
\text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \circ \alpha' : \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \circ \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \to \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma} \circ \text{Map}_{\text{Cat}_{\infty}(\kappa) \times C}(\mathcal{U}(\kappa) \times C, \text{Cat}_{\infty}(\kappa) \times C)^{\gamma}.
\]

\[\square\]

**Remark 7.45.** Let \( \mathcal{C} \) be a \( \kappa \)-small closed and cotensored left module over \( \text{Cat}_{\infty}(\kappa) \).

Let \( X, Y \) be objects of \( \mathcal{C} \) and \( f : X \to Y, \ g : Y \to X \) be morphisms of \( \mathcal{C} \).

Denote \( \mathcal{F} : \mathcal{Y} \to \mathcal{Y}, \mathcal{G} : \mathcal{Y} \to \mathcal{X} \) the maps of cartesian fibrations over \( \mathcal{C} \) classifying the natural transformations \([-, f] : [-, X] \to [-, Y] \) respectively \([-, g] : [-, Y] \to [-, X] \) of functors \( \mathcal{C}^{\text{op}} \to \text{Cat}_{\infty}(\kappa) \).

Then \( f \) is left adjoint to \( g \) if and only if \( \mathcal{F} \) is left adjoint to \( \mathcal{G} \).

**Corollary 7.46.** Let \( \mathcal{C} \) be a \( \kappa \)-small closed and cotensored left module over \( \text{Cat}_{\infty}(\kappa) \) for a strongly inaccessible cardinal \( \kappa \) and \( g : Y \to X \) a morphism of \( \mathcal{C} \).

Then \( g \) is a localisation if and only if the following two conditions are satisfied:
1. For every object \( Z \) of \( \mathcal{C} \) the induced functor \([Z, g] : [Z, Y] \to [Z, X]\) is a localisation.

2. For every morphism \( \varphi : Z \to Z' \) of \( \mathcal{C} \), the induced functor \([\varphi, X] : [Z', X] \to [Z, X]\) preserves local equivalences.

Proof. If \( g \) is a localisation or if condition 1. and 2. hold, \( g \) admits a left adjoint \( f : X \to Y \) in \( \mathcal{C} \) according to proposition 7.43.

Denote \( \varepsilon : f \circ g \to \text{id}_Y \) the counit of this adjunction.

As \([Z, -] : \mathcal{C} \to \text{Cat}_\infty(\kappa)\) is a \( \text{Cat}_\infty(\kappa) \)-enriched functor, the natural transformation \([Z, \varepsilon] : [Z, f] \circ [Z, g] \to \text{id}_{[Z, Y]}\) is the counit of the induced adjunction \([Z, f] : [Z, X] \rightleftarrows [Z, Y] : [Z, g]\).

Consequently \( \varepsilon : f \circ g \to \text{id}_Y \) is an equivalence if and only if for every object \( Z \) of \( \mathcal{C} \) the counit of the adjunction \([Z, f] : [Z, X] \rightleftarrows [Z, Y] : [Z, g]\) is an equivalence.

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