On the Black Holes in alternative theories of gravity: The case of non-linear massive gravity

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Abstract

I derive general conditions in order to explain the origin of the Vainshtein radius inside dRGT. The set of equations, which I have called "Vainshtein" conditions, are able to explain the coincidence between the Vainshtein radius in dRGT and the scale \( r_0 = \left( \frac{\mathcal{M}}{\mathcal{N}} \right)^{1/3} \), obtained naturally from the Schwarzschild de-Sitter (S-dS) space inside General Relativity (GR). In GR, this scale was interpreted as the maximum distance in order to get bound orbits. The same scale corresponds to the static observer position if we want to define the black hole temperature in an asymptotically de-Sitter space. In dRGT, the scale marks a limit after which the extra degrees of freedom of the theory become relevant. Finally, in order to finish the comparison, I derive the equations of motion for a massive test particle moving around the S-dS solution inside the dRGT theory. In unitary gauge, the resulting effective potential is velocity-dependent, suggesting then that the total energy is not conserved in the usual sense. This is the origin of the lost of predictability for the behavior of a test particle moving around the source unless some symmetry arguments are involved.

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I. INTRODUCTION

The Schwarzschild de-Sitter (S-dS) space in static coordinates has been widely studied in the past. Its analytic extension for S-dS space has been performed by Bażański and Ferrari [1]. They interpreted the scale \( r_0 = \left( \frac{3}{2} r_s r_\Lambda \right)^{1/3} \) as the distance where the 0-0 component of the S-dS metric takes a minimum value. As a consequence of this, it was found in [2] that \( r_0 \) represents a transition distance after which a photon suffers a gravitational blue shift when it moves away from a source. The same scale is used by Bousso and Hawking in order to find the appropriate expression for the temperature of a black hole immersed inside a de-Sitter space [3]. In such a case, the distance \( r_0 \) is interpreted as the position of the static observer in order to find the appropriate normalization for the time-like Killing vector. Then there exist a minimum temperature for the black hole given by \( T = \frac{1}{2\pi r_\Lambda} \) [3, 4]. This analysis differs in some details from the one done in [5] where the Black Hole thermodynamics inside the S-dS space was analyzed in detail, in that case however, \( r_0 \) does not play the central role for the definition of the Black Hole temperature. The role of \( r_0 \) as a static radius was also analyzed in [6] inside the Kerr-de Sitter space. In [7], Balaguera et al. found that \( r_0 \) represents the maximum distance within which we can find bound orbits solutions for a test particle moving around a source. In the same manuscript, the velocity bounds for a test particle inside the S-dS space were obtained, this work was then extended by Arraut et al. in [8] in order to incorporate other metric solutions. In [7], the authors also found that there exist a maximum angular momentum \( L_{\text{max}} \) for the test particle to be inside a bound orbit. If \( L = L_{\text{max}} \), then there exist a saddle point for the effective potential at the distance \( r_s \), this analysis was extended recently by the author [9]. In [10] and [11], the scale \( r_0 \) was derived by using a different method and some conditions for the circular orbits and its stabilities were obtained. However, in such a case, the conditions were not interpreted in terms of a maximum angular momentum \( L_{\text{max}} \). The scale \( r_0 \) plays a central role inside the \( \Lambda_3 \) version of the non-linear theory of massive gravity where it represents the distance below which non-linearities become important and General Relativity is restored [12]. In this paper I derive general conditions in order to explain the origin of the Vainshtein radius and its coincidence with the same scale obtained inside the General Relativity (GR) formulation. I have called them the "Vainshtein" conditions, which in general correspond to simple extremal conditions for the components of the dynamical metric in unitary gauge. I then analyze the equations of motion for a massive test particle under the influence of the S-dS metric in dRGT. I find that the equations of motion in unitary gauge contain a velocity-dependent effective potential, suggesting then that the total energy is not conserved in the usual sense. This explains the origin of the lost of predictability reported by Kodama and the author in a previous manuscript unless some symmetry protects the theory from this pathology. The paper is organized as follows: In Section (II), I introduce the basic aspects of the S-dS space in static coordinates and then derive the scale \( r_0 \) including its correction due to the angular momentum of a massive test particle moving around the source. In Section (III), I analyze the Black Hole temperature as defined by Bousso and Hawking. I explain the role of the scale \( r_0 \) in that situation. In Section (IV), I introduce the S-dS solution derived from the non-linear theory of massive gravity and then I explain the role of \( r_0 \) in this theory. In section (V), I write the S-dS solution inside dRGT gravity in unitary gauge as has been derived by Kodama and the author. In section (VI) I derive the Vainshtein conditions in order to explain why \( r_0 \) appears in both formulations, namely GR and dRGT. Although the results are obtained for the S-dS solution in dRGT, they can be applied to any solution in
unitary gauge. In section (VII), I use the Vainshtein conditions for deriving the Vainshtein scale inside the $\Lambda_5$ theory of massive gravity. In section (VIII), I derive the equations of motion of a massive test particle under the influence of the S-dS solution in dRGT. In section (IX), I write the equations of section (VIII) for the special case with two-free parameters but keeping the gauge transformation function constrained. Finally, in section (X), I conclude.

II. THE SCHWARZSCHILD DE-SITTER SPACE

The Schwarzschild-de Sitter metric in static coordinates, is given by:
\[ ds^2 = -e^{\nu(r)} dt^2 + e^{-\nu(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]  

where:
\[ e^{\nu(r)} = 1 - \frac{r_s}{r} - \frac{r^2}{3r_A^2}, \]

where $r_s = 2GM$ is the gravitational radius and $r_A = \frac{1}{\sqrt{\Lambda}}$ defines the cosmological constant scale. In this coordinate system, it has been demonstrated that the effective potential is given by:
\[ U_{\text{eff}}(r) = -\frac{r_s}{2r} - \frac{1}{6} \frac{r^2}{r_A^2} + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3}, \]

where $U_{\text{eff}}(r)$ is the effective potential which influences the motion of a massive test particle in S-dS space. The equation of motion of a massive test particle is given by:
\[ \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + U_{\text{eff}}(r) = \frac{1}{2} \left( E^2 + \frac{L^2}{3r_A^2} - 1 \right) = C, \]

where $C$ is a constant depending on the initial conditions of motion. This effective potential has three circular orbits. They correspond to the condition $\frac{dU_{\text{eff}}(r)}{dr} = 0$. In this manuscript, I focus on the scale $r_0$ which corresponds to one of the previously mentioned circular orbits. In [9], $r_0$ was derived and it is given by:
\[ r_0(\beta) = \left( \frac{3}{2} r_s r_A^2 \right)^{1/3} - \frac{1}{4\beta^2} (3r_s r_A^2)^{1/3}, \]

where we make explicit the angular momentum dependence of the massive test particle through the parameter $\beta = L/L_{\text{max}}$, with $L_{\text{max}} = \frac{3^{2/3}}{4} (r_s r_A)^{1/3}$ being the maximum angular momentum if we want to get bound orbits. This scale is the limit where the attractive effects due to gravity and the repulsive ones due to the cosmological constant ($\Lambda$) just cancel. This is the key point in the Bousso-Hawking definition of temperature as will be explained in the next section.

III. BLACK HOLE THERMODYNAMICS IN AN ASYMPTOTICALLY DE-SITTER SPACE

In agreement with Bousso and Hawking, the appropriate way to define the black Hole thermodynamics is by normalizing the time-like Killing vector such that the static observer
is located at the distance given by (5) with $\beta = 0$. If we assume that the observer does not have any angular momentum, then the surface gravity is defined as [3]:

$$\kappa_{BH,CH} = \left( \frac{(K^\mu \nabla_\mu K_\gamma)(K^\alpha \nabla_\alpha K_\gamma)}{-K^2} \right)^{1/2} \Bigg|_{r=r_{BH},r_{CH}}. \quad (6)$$

The subindices BH and CH, correspond to the Black Hole Horizon and the Cosmological one respectively. The event horizons are obtained from the condition:

$$g^{rr}(r_c) = 0, \quad (7)$$

and they are given explicitly by:

$$r_{CH} = -2r_A \cos \left( \frac{1}{3} \left( \cos^{-1} \left( \frac{3r_s}{2r_A} \right) + 2\pi \right) \right),$$

$$r_{BH} = -2r_A \cos \left( \frac{1}{3} \left( \cos^{-1} \left( \frac{3r_s}{2r_A} \right) + 4\pi \right) \right).$$

The two horizons become equal when the mass of the Black Hole reach its maximum value given by:

$$M_{max} = \frac{1}{3} \frac{m_{pl}^2}{m_A}, \quad (8)$$

where $m_{pl}$ corresponds to the Planck mass and $m_A = \sqrt{\Lambda}$. If the mass of a Black Hole is bigger than the value given by eq. (8), then there is no radiation at all and we have a naked singularity. As $M = M_{max}$, the two event horizons take the same value ($r_{BH} = r_{CH} = r_A = \frac{1}{\sqrt{\Lambda}}$), they are degenerate and a thermodynamic equilibrium is established. As has been explained by Bousso and Hawking [3], as $M \rightarrow M_{max}$, $V(r) \rightarrow 0$ between the two horizons (BH and Cosmological) and for that reason the Schwarzschild-like coordinates simply become inappropriate. In such a case we need a new coordinate system. In agreement with Ginsparg and Perry [13], we write:

$$9M^2 \Lambda = 1 - 3\epsilon^2, \quad 0 \leq \epsilon \ll 1, \quad (9)$$

where $\epsilon$ is a parameter related to the mass of the black-hole. In these coordinates, the degenerate case (when the two horizons become the same), corresponds to $\epsilon \rightarrow 0$. We must then define the new radial and the new time coordinates to be:

$$\tau = \frac{1}{\epsilon \sqrt{\Lambda}} \psi, \quad r = \frac{1}{\sqrt{\Lambda}} \left( 1 - \epsilon \cos \chi - \frac{1}{6} \epsilon^2 \right). \quad (10)$$

In these coordinates, the Black Hole horizon corresponds to $\chi = 0$ and the Cosmological horizon to $\chi = \pi$ [3]. The new metric obtained from the transformation is given by:

$$ds^2 = -r_A^2 \left( 1 + \frac{2}{3} \epsilon \cos \chi \right) \sin^2 \chi d\psi^2 + r_A^2 \left( 1 - \frac{2}{3} \epsilon \cos \chi \right) d\chi^2 + r_A^2 (1 - 2 \epsilon \cos \chi) d\Omega_2^2. \quad (11)$$

This metric has been expanded up to first order in $\epsilon$. Eq. (11) is of course the appropriate metric to be used as the mass of the Black Hole is near to its maximum value given by eq.
It has been found by Bousso and Hawking that the time-like Killing vector inside the
definition (6) has to be normalized in agreement with:

$$\gamma_t = \left(1 - \frac{3r_s}{2r_\Lambda} \right)^{2/3}$$

with $\gamma_t$ being the normalization factor for the time-like Killing vector defined as:

$$K = \gamma_t \frac{\partial}{\partial t}.$$  

In an asymptotically flat space, $\gamma_t \to 1$ when $r \to \infty$. But in the case of eq. (12), the
Killing vector is just normalized with respect to an observer at the position $r_0$ with $\beta = 0$
as has been defined previously. When the mass of the black hole reach its maximum value
defined as $\epsilon \to 0$ in eq. (9), the black hole temperature reach its minimum value given by:

$$2\pi T_{\text{min}} = \frac{1}{r_\Lambda},$$

where $\kappa$ is the surface gravity.

IV. BLACK HOLES IN DRGT NON-LINEAR THEORY OF MASSIVE GRAVITY

In agreement with Koyama and colleagues, it is possible to construct black hole solutions
inside the non-linear theory of massive gravity. It is natural to suspect that such solution
should be S-dS, although other solutions are possible in principle. In dRGT, we can get the
same solution given by eq. (1) but surrounded by a halo of helicity $0$ and $\pm 1$. The trick is
to use as a starting point a metric of the form [14]:

$$ds^2 = -dt^2 + (dr \pm \sqrt{f(r)}dt)^2 + r^2 d\Omega^2,$$

where $f(r)$ will be defined later. The previous metric is free of horizon singularities, such
that the invariant $g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab}$ (defined inside the dRGT theory) remains finite when all
the other standard relativistic invariants are also finite. The metric has to be a solution of
the Einstein’s equations, which in massive gravity are defined as:

$$G^{\mu\nu} = -m^2 X^{\mu\nu}.$$  

The solution (15), after the appropriate coordinate transformations, becomes the same
solution given by eq. (1) but surrounded by a St"uckelberg background defined by:

$$\Phi^0 = \frac{1}{\kappa}(t + f(r)),
\Phi^r = \left(1 + \frac{1}{\alpha} \right) r,
\Phi^\theta = \theta,
\Phi^\phi = \phi.$$  

The previous results correspond to a family of solutions satisfying a specific relation
between the two free parameters of the theory as has been explained in [14, 15]. The scale
$r_0$ defined before, inside the $\Lambda_3$ version of the theory, appears as the Vainshtein radius if we tune the mass of the graviton with the $\Lambda$ scale. For distances satisfying the condition $r \ll r_0$, non-linearities become relevant and General Relativity is recovered, avoiding in such a way the vDVZ discontinuity \cite{16}. The non-linear solution inside the dRGT theory, admits perturbative expansions in terms of the mass of the graviton for distances satisfying $r \ll r_0$. On the other hand, the same solutions admit perturbative expansions in terms of the Newtonian Constant for distances $r \gg r_0$. Then in some sense, $r_0$ is a scale which marks the transition between a solution dominated by the Newtonian constant and the one dominated by the graviton mass in direct analogy with what happens in General Relativity. The main difference is that $r_0$ in massive gravity is related to the existence of a strong coupling scale $\Lambda_3 = (M_{Pl}m^2)^{1/3}$ which appears in the Lagrangian when the theory is ghost-free \cite{15}. Later in this manuscript, I will explain the origin of such coincidence.

V. THE SCHWARZSCHILD DE-SITTER SOLUTION IN DRGT: UNITARY GAUGE

In \cite{17}, the S-dS solution in unitary gauge was derived for two different cases. The first one, corresponds to the family of solutions satisfying the condition $\beta = \alpha^2$, where $\beta$ and $\alpha$ correspond to the two free-parameters of the theory. In such a case, the gauge transformation function $T_0(r,t)$ becomes arbitrary. The second one, corresponds to the family of solutions with two-free parameters satisfying the condition $\beta \leq \alpha^2$ with the gauge transformation function $T_0(r,t)$ constrained. The generic solution is given explicitly as:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{rt}(drdt + dtdr) + r^2d\Omega_2^2,$$

(18)

where:

$$g_{tt} = -f(r)(\partial_t T_0(r,t))^2, \quad g_{rr} = -f(r)(\partial_r T_0(r,t))^2 + \frac{1}{f(r)}, \quad g_{tr} = -f(r)\partial_t T_0(r,t)\partial_r T_0(r,t),$$

(19)

where $f(r) = 1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2$. In this previous solution, all the degrees of freedom are inside the dynamical metric. The fiducial metric in this case is just the Minkowskian one given explicitly as:

$$f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{dr^2}{S_0^2} + \frac{r^2}{S_0^2}(d\theta^2 + r^2sin^2\theta),$$

(20)

where $S_0 = \frac{\alpha}{\alpha + 1}$. The St"uckelberg fields take the standard form defined in \cite{17}.

VI. THE VAINSHTEIN CONDITIONS

In S-dS space (in static coordinates), the scale $r_0$ with $\beta = 0$ can be obtained by solving the equation obtained from the condition $df(r)/dr = 0$. The solution shows the distance after which the slope of the function $f(r)$ changes its signature. This is not a coincidence since it is evident that the scale after which the cosmological constant becomes "dominant" has to be marked by an extremal condition. In the case where the metric components depend on both, position and time, the extremal conditions can be written in the following form:
\[ dg_{\mu\nu} = \left( \frac{\partial g_{\mu\nu}}{\partial r} \right)_t dr + \left( \frac{\partial g_{\mu\nu}}{\partial t} \right)_r dt = 0, \] (21)

where the notation \( \frac{\partial g_{\mu\nu}}{\partial w} \big|_x \) is just the partial derivative with respect to \( w \) but keeping the variable \( x \) constant. When applied to a static metric solution, eq. (21) is equivalent to \( \partial g_{\mu\nu}/\partial r = 0 \). In the case of S-dS space in GR, the condition (21) is satisfied as \( r = r_0 \) with \( \beta = 0 \). We can apply the same extremal conditions in dRGT theory in unitary gauge. In such a case, all the degrees of freedom (five in total) are inside the dynamical metric. In dRGT, the extremal condition (21) marks the scale after which the metric behavior begins to change. This scale is in fact the Vainshtein scale (Vainshtein radius for static metrics).

If we introduce the metric defined in (19) inside the condition (21), then we obtain the Vainshtein conditions for the metric under study. For the \( t - t \) component, we get:

\[
(f'(r)(\partial_T T_0(r,t))^2 + 2f(r)(\partial_T T_0(r,t))(\partial t, \partial_T T_0(r,t))) dr + 2f(r)(\partial_T T_0(r,t))\partial_t^2 T_0(r,t)dt = 0.
\] (22)

For the quasi-stationary case (i.e., the case where the metric can be translated to static coordinates), \( T_0(r,t) \sim t + A(r) \), where \( A(r) \) is an arbitrary function on space. Then the previous condition is reduced to \( f'(r) = 0 \). For the \( r - r \) component, we have to satisfy:

\[
\left( f'(r)(\partial_T T_0(r,t))^2 + 2f(r)(\partial_T T_0(r,t))(\partial^2_T T_0(r,t)) + \frac{f'(r)}{f(r)^2} \right) dr + 2f(r)(\partial_T T_0(r,t))(\partial_t, \partial_T T_0(r,t))dt = 0.
\] (23)

Again, if we assume quasi-stationary condition, then the previous result is reduced to:

\[
T_0'(r,t) (f'(r)T_0'(r,t) + 2f(r)T_0''(r,t)) + \frac{f'(r)}{f(r)^2} = 0.
\] (24)

Finally, for the \( t - r \) component, we get:

\[
(f'(r)(\partial_T T_0(r,t))(\partial_T T_0(r,t)) + f(r)(\partial_T T_0(r,t))(\partial_T T_0(r,t)) + f(r)(\partial_T T_0(r,t))(\partial^2_T T_0(r,t)) \times (\partial^2_T T_0(r,t))) dr + \left( f(r)(\partial^2_T T_0(r,t))(\partial_T T_0(r,t)) + f(r)(\partial_T T_0(r,t))(\partial_t, \partial_T T_0(r,t)) \right) dt = 0.
\] (25)

Once again, if the dynamical metric is quasi-stationary, then we get:

\[
\frac{T_0''(r,t)}{T_0'(r,t)} = -\frac{f'(r)}{f(r)} = C,
\] (26)

where \( C \) is an arbitrary constant and we assume \( \partial^2_T T_0(r,t) = \partial_t \partial_T T_0(r,t) = 0 \). Eq. (26) can be solved by separation of variables. By assuming a general exponential behavior, we would get:

\[
T_0(r,t) = Ae^{Cr}, \quad f(r) = Be^{-Cr}.
\] (27)

If we replace the condition (26) inside eq. (21), then we get:

\[
T_0'(r,t) = \pm f^{-1},
\] (28)
which is consistent with the result (27). This also implies that under the quasi-stationary condition, at the Vainshtein radius, \( T''_0(r,t) = 0 \) in agreement with the result obtained from (22). The set of conditions condensed in the single expression (21) is what I have called "Vainshtein conditions". From them we can find the Vainshtein scale which can be time-dependent for general backgrounds. The Vainshtein scale becomes equivalent to the already known Vainshtein radius when the metric is time-independent. In such a case, the result (21) is reduced to \( \partial g_{\mu \nu} / \partial r = 0 \), obtaining then as a Vainshtein radius the result \( r = r_0 \). This explains the origin of the coincidence with respect to the scale \( r_0 \) obtained from GR. Just for completing the previous arguments, we can specify that the Vainshtein scale is marked by three regimes:

\[
\begin{align*}
\partial_r T_0(r,t) = 0 & \rightarrow r << r_V, \\
\partial_r T_0(r,t) \neq 0 & \rightarrow r >> r_V,
\end{align*}
\]

\( T''_0(r,t) = 0 \rightarrow r = r_V, \tag{29} \)

where \( r_V \) is just the Vainshtein scale.

VII. THE VAINSHTEIN RADIUS IN \( \Lambda_5 \) THEORY

It is easy to extend the concepts of the previous section to the \( \Lambda_5 \) theory of non-linear massive gravity. In such a case, it is known that the Vainshtein scale is different with respect to the the case of \( \Lambda_3 \) and as a consequence, \( r_V \) will differ with respect to the case of GR with \( \Lambda_3 \). However, what is necessary to remark is that the origin of the Vainshtein scale again emerges as an extremal condition for the dynamical metric in unitary gauge. If we write the metric in a diagonal form as [16]:

\[
ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 C(r)d\Omega^2,
\]

(30)

where the fiducial metric still keeps the Minkowskian form. In this case, it is also easy to verify that the conditions:

\[
B'(r) = A'(r) = \frac{d(r^2 C(r))}{dr} = 0,
\]

(31)

reproduce the Vainshtein radius inside this theory. It is given by \( r_V = (GM/m^4)^{1/5} \). In fact, it is possible to verify that we can reproduce the Vainshtein scale for any massive theory of gravity by using the same principles. If gravity disappears, then the Vainshtein scale vanishes, namely, \( r_V = 0 \). This only means that the extra-degrees of freedom of the theory are relevant at any scale.

VIII. THE EFFECTIVE POTENTIAL IN DRGT MASSIVE GRAVITY

In order to compare massive gravity with General Relativity, it is important to derive the equations of motion for a massive test particle when it moves around a spherically symmetric source. The comparison can be better done if we work in unitary gauge such that all the degrees of freedom remain in the dynamical metric. The equations of motion can be written as:
\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 - E \left( \frac{g_{tt}}{g_{rr}g_{tt}} \right) \left( \frac{dr}{d\tau} \right) + \frac{L^2}{2r^2g_{rr}} = - \frac{1}{2g_{rr}} \left( \frac{E^2}{g_{tt}} + 1 \right), \quad (32)
\]

where \( g_{tt} \) and \( g_{rr} \) are defined in eq. (19). Note that as \( \partial_r T_0(r,t) = 0 \), the previous equation is reduced to the result (11) if we use the metric given by (19). Then the background degeneracy is generated by the fact that the function \( T_0(r,t) \) has a spatial dependence as was reported in [17] for the perturbation analysis of the S-dS solution inside dRGT. The degeneracy reproduces different equations of motion for each kind of gauge transformation function. If we replace the metric components (19) inside (32), then we get:

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + \frac{\partial_t T_0(r,t)f(r)E}{\partial_r T_0(r,t)(f(r)^2(\partial_r T_0(r,t))^2 - 1)} \left( \frac{dr}{d\tau} \right) - \frac{L^2}{2r^2} \left( \frac{f(r)}{f(r)^2(\partial_r T_0(r,t))^2 - 1} \right) \left( f(r)(\partial_r T_0(r,t)^2) - E^2 \right). \quad (33)
\]

In eq. (32), the energy and angular momentum have been introduced in the usual sense in agreement with the results of Sec. (11). The presence of a velocity dependent quantity in eq. (33) shows that the effective potential which influences the motion of a test particle, is velocity-dependent. This dependence cannot be gauged away as in GR. Any attempt of removing the velocity-dependence by coordinate transformation, just translates the degrees of freedom to the fiducial metric in dRGT. Then the origin of the velocity term inside the effective potential, comes from the St"uckelberg fields. The dependence of the effective potential with the velocity suggests that the total energy of a test particle is not conserved. This is true if we keep the standard definition of energy in agreement with the symmetry under time-translations. This result is true even if the dynamical metric in dRGT is static and as a consequence, the gauge-transformation function \( T_0(r,t) \) is linear in time. For the family of solutions with one-free parameter defined in [17], the gauge-transformation function \( T_0(r,t) \) is arbitrary. Another solution defined in [17] has two-free parameters, but the function \( T_0(r,t) \) is constrained in such a case. Here I will write explicitly the equations of motion (33) for this second case.

**IX. THE EFFECTIVE POTENTIAL IN DRGT: THE CASE OF TWO-FREE PARAMETERS WITH \( T_0(r,t) \) CONSTRAINED**

In [17], the S-dS solution was found for the family of solutions with one free parameter satisfying the condition \( \alpha = \beta^2 \). In such a case however, the gauge transformation function \( T_0(r,t) \) is not constrained and then the degeneracy of the background is expected. For that reason, the equations of motion found in (33) provide infinite solutions and the prediction of the future behavior of a test particle based in eq. (33) is impossible unless some symmetry protects the theory from being pathological at this level. On the other hand, for the family of solutions with two-free parameters found in [17], the gauge transformation function is constrained. Here I will focus on the gauge transformation function given by:

\[
T_0(r,t) = St \pm \int^{Sr} \left( \frac{1}{f(u)} - 1 \right) du. \quad (34)
\]

By defining the derivatives as:
\[ \partial_r T_0(r,t) = \pm S \left( \frac{1}{f(Sr)} - 1 \right), \quad \partial_t T_0(r,t) = S, \quad (35) \]

and taking into account the scale factor \( S \) inside the equation (32), the eq. (33) then becomes:

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 \pm \left( \frac{(1 - f(Sr))}{(1 - f(Sr))^2 - S^2} \right) \frac{Edr}{d\tau} - \frac{L^2}{2S^2r^2} \left( \frac{f(Sr)}{(1 - f(Sr))^2 - S^2} \right) = \frac{-E^2 + S^2 f(Sr)}{2S^2((1 - f(Sr))^2 - S^2)}. \quad (36)
\]

If we make the replacement \( f(Sr) = 1 - 2GM/(Sr) - \Lambda S^2r^2/3 \) inside the previous expression, then we get the following result:

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 \pm \left( \frac{(2GM/sr + \frac{1}{3}\Lambda S^2r^2)E}{(2GM/sr + \frac{1}{3}\Lambda S^2r^2)^2 - S^2} \right) \frac{dr}{d\tau} - \frac{L^2}{2S^2r^2} \left( \frac{1 - \frac{2GM}{Sr} - \frac{1}{3}\Lambda S^2r^2}{(2GM/sr + \frac{1}{3}\Lambda S^2r^2)^2 - S^2} \right) = \frac{-E^2 + S^2 \left( 1 - \frac{2GM}{Sr} - \frac{\Lambda S^2r^2}{3} \right)}{2S^2 \left( (\frac{2GM}{Sr} + \frac{\Lambda S^2r^2}{3})^2 - S^2 \right)}.
\quad (37)
\]

The previous equation is not trivially solved. If the test particle is far from the horizons of the S-dS space, then eq. (37) can be expanded as:

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 \mp \frac{xE}{S^2} \frac{dr}{d\tau} + \frac{L^2}{2r^2S^4}(1 - x) = \frac{1}{S^4}(E^2 - S^2(1 - x)) \quad (38)
\]

where \( x = 2GM/Sr + \Lambda S^2r^2/3 \). Far from the event horizons of S-dS space, the approximation \( x << 1 \) is valid. The previous equation clearly shows that the total energy \( E \) is not conserved in the usual sense and then the predictability of a test particle is lost. This apparent pathological behavior, first reported in [17] for the analysis of the S-dS black-hole stability in dRGT, might have deeper physical implications to be explained in future manuscripts.

X. CONCLUSIONS

I derived the general conditions in order to get the Vainshtein scale inside the dRGT formulation of massive gravity. The Vainshtein scale is just an extremal condition for the dynamical metric in unitary gauge when we analyze the dRGT formulation on the non-linear massive gravity theory. This explains the coincidence between the Vainshtein radius obtained from dRGT and the scale \( r_0 \) obtained in the standard GR theory for the S-dS solution. I have also derived the equations of motion for a massive test particle moving under the influence of the dynamical metric in dRGT. In particular, I have focused on the S-dS solution in unitary gauge such that all the degrees of freedom are inside the dynamical metric. The equations of motion revealed that the effective potential is velocity-dependent and as a consequence the total energy of the system is not conserved in the usual sense. This is the origin of the functional degeneracy already reported by Kodama and the author in a previous manuscript, unless the theory is protected by a symmetry. More physical
consequences of this apparent pathology are under study.

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