RELATIVE NONHOMOGENEOUS QUADRATIC DUALITY

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Abstract. This paper contains a detailed exposition of the nonhomogeneous quadratic duality theory in the relative situation over a noncentral, noncommutative, nonsemisimple base ring, as announced in [14, Section 0.4]. We prove the Poincaré–Birkhoff–Witt theorem in this context. The duality between the ring of differential operators and the de Rham DG-algebra, with the ring of functions as the base ring, is the thematic example. The moderate generality level makes the exposition in this paper more accessible than the very heavily technical [14, Section 11].

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Introduction

0.0. Let $A$ be an associative ring and $R \subset A$ be a subring. Derived Koszul duality is the functor $\text{Ext}^*_A(-, R)$, or $\text{Tor}^*_A(R, -)$, or $\text{Ext}^*_A(R, -)$, enhanced to an equivalence of derived categories of modules.

The above definition raises many questions. To begin with, $R$ is not an $A$-module. So what does this Ext and Tor notation even mean?

Secondly, let us consider the simplest example where $R = k$ is a field and $A = k[x]$ is the algebra of polynomials in one variable. Then $k$ indeed can be viewed as an $A$-module. There are many such module structures, indexed by elements $a$ of the field $k$: given $a \in k$, one can let the generator $x \in A$ act in $k$ by the multiplication with $a$. Denote the resulting $A$-module by $k_a$.

To be specific, let us choose $k = k_0$ as our preferred $A$-module structure on $k$. Then the functors $\text{Ext}^*_A(-, k_0)$, $\text{Tor}^*_A(k_0, -)$, and $\text{Ext}^*_A(-, k_0)$ are indeed well-defined on the category of $A$-modules. But these functors are far from being faithful or conservative: all of them annihilate the $A$-modules $k_a$ with $a \neq 0$. How, then, can one possibly hope to enhance such cohomological functors to derived equivalences?
0.1. Koszul duality has to be distinguished from the comodule-contramodule correspondence, which is a different, though related, phenomenon.

In the simplest possible form, the comodule-contramodule correspondence is the functor \( \text{Ext}_{A}^{\ast}(-, A) \) enhanced to a derived equivalence (while Koszul duality is \( \text{Ext}_{A}^{\ast}(-, k) \), where \( k \) is the ground field). In a more realistic covariant and relative situation, comparable to the discussion of Koszul duality in Section 0.0, the comodule-contramodule correspondence would be a derived equivalence enhancement of a functor like \( \text{Ext}_{A}^{\ast}(\text{Hom}_{R}(A, R), -) \) or \( \text{Tor}_{A}^{\ast}(-, \text{Hom}_{R}(A, R)) \).

0.2. In the present author’s research, the desire to understand Koszulity and Koszul duality was the starting point. Then the separate existence and importance of comodule-contramodule correspondence was realized, particularly in the context of semi-infinite homological algebra [14]. The derived nonhomogeneous Koszul duality over a field was formulated as a “Koszul triality” picture, which is a triangle diagram of derived equivalences with the comodule-contramodule correspondence present as one side of the triangle and two versions of Koszul duality as two other sides [15].

The comodule-contramodule correspondence, its various versions, generalizations, and philosophy, are now discussed in several books and papers of the present author, including [14, 15, 17, 19, 20] and others. On the other hand, the derived nonhomogeneous Koszul duality over a field attracted interest of a number of authors, starting from early works [3, 8, 7] and to very recent, such as [2, 9]; there is even an operadic version of it in [4].

Still, there is a void in the literature concerning relative nonhomogeneous Koszul duality. Presently, the only source of information on this topic known to this author is his own book [14], which contains an introductory discussion without proofs or details in [14, Section 0.4] and a heavily technical treatment in a very general and complicated setting in [14, Section 11]. (The memoir [16] represents a very different point of view.) The present paper is intended to fill the void by providing a reasonably accessible, detailed exposition on a moderate generality level.

0.3. Let us start to explain the meaning of the terms involved. In the notation of Section 0.0, relative means that \( R \) is an arbitrary ring rather than simply the ground field. Homogeneous Koszul duality means that \( A = \bigoplus_{n=0}^{\infty} A_{n} \) is a nonnegatively graded ring and \( R = A_{0} \) is the degree-zero grading component. In this case, \( R \) is indeed naturally both a left and a right \( R \)-module, so the meaning of the Ext and Tor notation in Section 0.0 is clear. Nonhomogeneous Koszul duality is the situation when there is no such grading on the ring \( A \).

The main specific aspect of the homogeneous case is that one can consider graded \( A \)-modules with a bounding condition on the grading, that is, only positively graded or only negatively graded modules. If \( M \) is a positively graded left \( A \)-module, then \( R \otimes_{A} M = 0 \) implies \( M = 0 \), while if \( P \) is a negatively graded left \( A \)-module, then \( \text{Hom}_{A}(R, P) = 0 \) implies \( P = 0 \). Hence the second problem described in Section 0.0 does not occur, either.
In the nonhomogeneous situation, the solution to the second problem from Section 0.0 is to consider derived categories of the second kind. This means that certain complexes or DG-modules are declared to be nonzero objects in the derived category even though their cohomology modules vanish.

As to the first problem, it may well happen that $R$ has a (left or right) $A$-module structure even though $A$ is not graded. When such an module structure (extending the natural $R$-module structure on $R$) has been chosen, one says that the ring $A$ is augmented. In this case, the related Ext or Tor functor is well-defined. One wants to enhance it to a functor with values in DG-modules over a suitable DG-ring in such a way that it would induce a triangulated equivalence.

Generally speaking, the solution to the first problem is to consider curved DG-modules (CDG-modules), whose cohomology modules are undefined. So the Ext or Tor itself has no meaning, but the related curved DG-module has. In the augmented case, this DG-module becomes uncurved, and indeed computes the related Ext or Tor.

0.4. Let us now begin to state what our assumptions and results are. We assume that a ring $\tilde{A}$ is endowed with an increasing filtration $R = F_0\tilde{A} \subset F_1\tilde{A} \subset F_2\tilde{A} \subset \cdots$ which is exhaustive ($\tilde{A} = \bigcup_n F_n\tilde{A}$) and compatible with the multiplication in $\tilde{A}$. Furthermore, the successive quotients $\text{gr}_n\tilde{A} = F_n\tilde{A}/F_{n-1}\tilde{A}$ are assumed to be finitely generated projective left $R$-modules. Finally, the associated graded ring $A = \text{gr}^F\tilde{A} = \bigoplus_n \text{gr}_n^F\tilde{A}$ has to be Koszul over its degree-zero component $A_0 = R$; this means, in particular, that the ring $A$ is generated by its degree-one component $A_1$ over $A_0$ and defined by relations of degree 2.

In these assumptions, we assign to $(\tilde{A}, F)$ a curved DG-ring (CDG-ring) $(B, d, h)$, which is graded by nonnegative integers, $B = \bigoplus_{n=0}^\infty B^n$, $B^0 = R$, has a differential (odd derivation) $d: B^n \to B^{n+1}$ of degree 1, and a curvature element $h \in B^2$. The CDG-ring $(B, d, h)$ is defined uniquely up to a unique isomorphism of CDG-rings, which includes the possibility of change-of-connection transformations. The grading components $B^n$ are finitely generated projective right $R$-modules. In particular, one has $B^1 = \text{Hom}_R(A_1, R)$ and $A_1 = \text{Hom}_{R^{op}}(B^1, R)$.

Furthermore, to any left $\tilde{A}$-module $P$ we assign a CDG-module structure on the graded left $B$-module $B \otimes_R P$, and to any right $\tilde{A}$-module $M$ we assign a CDG-module structure on the graded right $B$-module $\text{Hom}_{R^{op}}(B, M)$. These constructions are then extended to complexes of left and right $\tilde{A}$-modules $P^\bullet$ and $M^\bullet$, assigning to them left and right CDG-modules $B \otimes_R P^\bullet$ and $\text{Hom}_{R^{op}}(B, M^\bullet)$ over $(B, d, h)$. A certain (somewhat counterintuitive) way to totalize bigraded modules is presumed here. The resulting functors induce the derived equivalences promised in Section 0.0. The functor $P^\bullet \mapsto B \otimes_R P^\bullet$ is a CDG-enhancement of the (possibly nonexistent) $\text{Ext}_A^\bullet(R, P)$, and the functor $M^\bullet \mapsto \text{Hom}_{R^{op}}(B, M^\bullet)$ is a CDG-enhancement of the (possibly nonexistent) $\text{Tor}_A^\bullet(M, R)$. However, there are further caveats.
One important feature of the nonhomogeneous Koszul duality over a field, as developed in the memoir [15], is that it connects modules with comodules or contramodules. In fact, the “Koszul triality” of [15] connects modules with comodules and contramodules. In the context of relative nonhomogeneous Koszul in the generality of the present paper, the Koszul triality picture splits into two dualities. A certain exotic derived category of right $\tilde{A}$-modules is equivalent to an exotic derived category of right $B$-comodules, while another exotic derived category of left $\tilde{A}$-modules is equivalent to an exotic derived category of left $B$-contramodules.

What are the “comodules” and “contramodules” in our context? First of all, we have complexes of $\tilde{A}$-modules on the one side and CDG-modules over $B$ on the other side; so both the comodules and the contramodules are graded $B$-modules. In fact, the (graded) right $B$-comodules are a certain full subcategory in the graded right $B$-modules, and similarly the (graded) left $B$-contramodules are a certain full subcategory in the graded left $B$-modules.

Which full subcategory? A graded right $B$-module $N$ is called a graded right $B$-comodule if for every element $x \in N$ there exists an integer $n \geq 1$ such that $xB_m = 0$ for all $m \geq n$. The definition of $B$-contramodules is more complicated and, as usually, involves certain infinite summation operations. A graded left $B$-module $Q$ is said to be a graded left $B$-contramodule if, for every integer $m$, every sequence of elements $q_i \in Q^{m-i}$, $i \geq 0$, and every sequence of elements $b_i \in B^i$, an element denoted formally by $\sum_{i=0}^{\infty} b_i q_i \in Q^m$ is defined. One imposes natural algebraic axioms on such infinite summation operations, and then proves that an infinite summation structure on a given graded left $B$-module $Q$ is unique if it exists.

In particular, this discussion implies that (somewhat counterintuitively), in the notation of Section 0.4, the bigraded module $\text{Hom}_{\text{R}^{op}}(B, M^\bullet)$ has to be totalized by taking infinite direct sums along the diagonals (to obtain a graded right $B$-comodule), while the bigraded module $B \otimes_{R} P^\bullet$ needs to be totalized by taking infinite products along the diagonals (to obtain a graded left $B$-contramodule).

The explanation for the counterintuitive totalization procedures mentioned in Section 0.5, from our perspective, is that $B$ is a “fake” graded ring. It really “wants” to be a coring, but this point of view is hard to fully develop. It plays a key role, however, in (at least) one of our two proofs of the Poincaré–Birkhoff–Witt theorem for nonhomogeneous Koszul rings.

The graded coring in question is $C = \text{Hom}_{\text{R}^{op}}(B, R)$, that is, the result of applying the dualization functor $\text{Hom}_{\text{R}^{op}}(B, R)$ to the graded ring $B$. The point is that we have already done one such dualization when we passed from the $R$-$R$-bimodule $A_1$ to the $R$-$R$-bimodule $B^1 = \text{Hom}_R(A_1, R)$, as mentioned in Section 0.4. The two dualization procedures are essentially inverse to each other, so the passage to the coring $C$ over $R$ returns us to the undualized world, depending covariantly-functorially on the ring $A$.

Experience teaches that the passage to the dual vector space is better avoided in derived Koszul duality. This is the philosophy utilized in the memoir [15] and the
book [14]. This philosophy strongly suggests that the graded coring $C$ is preferable to the graded ring $B$ as a Koszul dual object to a Koszul graded ring $A$.

The problem arises when we pass to the nonhomogeneous setting. In the context of the discussion in Section 0.4, the odd derivation $d$, which is a part of the structure of a CDG-ring $(B, d, h)$, is not $R$-linear. In fact, the restriction of $d$ to the subring $R = B^0 \subset B$ may well be nonzero, and in the most interesting cases it is. This is a distinctive feature of the relative nonhomogeneous Koszul duality. So how does one apply the functor $\text{Hom}_R(\cdot, R)$ to a non-$R$-linear map?

0.7. The duality between the ring of differential operators and the de Rham DG-algebra of differential forms is the thematic example of relative nonhomogeneous Koszul duality. Let $X$ be a smooth affine algebraic variety over a field (or a smooth real manifold). Let $O(X)$ denote the ring of functions and $\text{Diff}(X)$ denote the ring of differential operators on $X$. Endow the ring $\text{Diff}(X)$ with an increasing filtration $F$ by the order of the differential operators. So the associated graded ring $\text{Sym}_{O(X)}(T(X)) = \text{gr}^F \text{Diff}(X)$ is the symmetric algebra of the $O(X)$-module $T(X)$ of vector fields on $X$.

In this example, $R = O(X)$ is our base ring, $\tilde{A} = \text{Diff}(X)$ is our nonhomogeneous Koszul ring over $R$, and $A = \text{Sym}_{O(X)}(T(X))$ is the related homogeneous Koszul ring. The graded ring Koszul to $A$ over $R$ is the graded ring of differential forms $B = \Omega(X)$. There is no curvature in the CDG-ring $(B, d, h)$ (one has $h = 0$; a nonzero curvature appears when one passes to the context of twisted differential operators, e.g., differential operators acting in the sections of a vector bundle $E$ over $X$; see [14, Section 0.4.7] or [15, Section B]). The differential $d: B \to B$ is the de Rham differential, $d = d_{dR}$; so $(B, d)$ is a DG-algebra over $k$.

But the de Rham DG-algebra is not a DG-algebra over $O(X)$ (and neither the ring $\text{Diff}(X)$ is an algebra over $O(X)$). In fact, the restriction of the de Rham differential to the subring $O(X) \subset \Omega(X)$ is quite nontrivial.

0.8. So the example of differential operators and differential forms is a case in point for the discussion in Section 0.6. In this example, $C = \text{Hom}_{O(X)}(\Omega(X), O(X))$ is the graded coring of polyvector fields over the ring of functions on $X$. Certainly there is no de Rham differential on polyvector fields. What structure on polyvector fields corresponds to the de Rham differential on the forms?

Here is what we do. We adjoin an additional generator $\delta$ to the de Rham DG-ring $(\Omega(X), d_{dR})$, or more generally to the underlying graded ring $B$ of a CDG-ring $(B, d, h)$. The new generator $\delta$ is subject to the relations $[\delta, b] = d(b)$ for all $b \in B$ (where the bracket denotes the graded commutator) and $\delta^2 = h$. Then there is a new differential on the graded ring $\hat{B} = B[\delta]$, which we denote by $\partial = \partial/\partial\delta$.

The differential $\partial$ is $R$-linear (and more generally, $B$-linear with signs), so we can dualize it, obtaining a coring $\hat{C} = \text{Hom}_{R^\text{op}}(\hat{B}, R)$ with the dual differential $\text{Hom}_{R^\text{op}}(\partial, R)$. This is the structure that was called a quasi-differential coring in [14]. It plays a key role in the exposition in [14, Section 11].
Of course, the odd derivation $\partial = \partial/\partial \delta$ is acyclic, and so is the dual odd coderivation on the coring $\hat{C}$. This may look strange; but in fact, this is how it should be. Recall that we started with a curved DG-ring $(B, d, h)$. Its differential $d$ does not square to zero, and its cohomology is undefined. So there is no cohomology ring in the game, and it is not supposed to suddenly appear from the construction.

0.9. In the present version of this paper, we discuss the homogeneous quadratic duality over a base ring in Section 1, flat and finitely projective Koszul graded rings over a base ring in Section 2, relative nonhomogeneous quadratic duality in Section 3, and the Poincaré–Birkhoff–Witt theorem for nonhomogeneous Koszul rings over a base ring in Section 4. We do not yet cover the derived Koszul duality for modules.

Acknowledgment. Parts of the material presented in this paper go back more than a quarter century. This applies to the content of Sections 1–2 and the computations in Section 3 (with the notable exception of the 2-category story), which I worked out sometime around 1992. The particular case of duality over a field, which is much less complicated, was presented in the paper [12], and the possibility of extension to the context of a base ring was mentioned in [12, beginning of Section 4]. Subsequently, I planned and promised several times over the years to write up a detailed exposition. This paper partially fulfills that promise. I would like to thank all the people, too numerous to be mentioned here by name, whose help and encouragement contributed to my survival over the decades. Speaking of more recent events, I am grateful to Andrey Lazarev, Julian Holstein, and Bernhard Keller for stimulating discussions and interest to this work. The author was supported by research plan RVO: 67985840 when writing the paper up.

1. Homogeneous Quadratic Duality over a Base Ring

All the associative rings in this paper are unital. We will always presume unitality without mentioning it; so all the left and ring modules over associative rings are unital, all the ring homomorphisms take the unit to the unit, all the subrings contain the unit, and all the gradings and filtrations are such that the unit element belongs to the degree-zero grading/filtration component.

Given an associative ring $R$, we denote by $R\text{-mod}$ the abelian category of left $R$-modules and by $\text{mod-}R$ the abelian category of right $R$-modules.

Let $R$, $S$, and $T$ be three associative rings. For any left $R$-modules $L$ and $M$, we denote by $\text{Hom}_R(L, M)$ the abelian group of all left $R$-module morphisms $L \to M$. If $L$ is an $R$-$S$-bimodule and $M$ is an $R$-$T$-bimodule, then the group $\text{Hom}_R(L, M)$ acquires a natural structure of $S$-$T$-bimodule. Similarly, for any right $R$-modules $Q$ and $N$, the abelian group of all right $R$-module morphisms $Q \to N$ is denoted by $\text{Hom}_{R^{op}}(Q, N)$ (where $R^{op}$ stands for the ring opposite to $R$). If $Q$ is an $S$-$R$-bimodule and $N$ is a $T$-$R$-bimodule, then $\text{Hom}_{R^{op}}(Q, N)$ is a $T$-$S$-bimodule.

In particular, for any $R$-$S$-bimodule $U$, the abelian group $\text{Hom}_R(U, R)$ is naturally an $S$-$R$-bimodule. If $U$ is a finitely generated projective left $R$-module, then
Lemma 1.1. Let $U$ be an $R$-$S$-bimodule and $V$ be an $S$-$T$-bimodule. Then there is a natural morphism of $T$-$R$-bimodules

$$\Hom_S(V, S) \otimes_S \Hom_U(U, R) \longrightarrow \Hom_R(U \otimes_S V, R),$$

which is an isomorphism whenever the left $S$-module $V$ is finitely generated and projective.

(b) Let $M$ be an $S$-$R$-bimodule and $N$ be a $T$-$S$-bimodule. Then there is a natural morphism of $R$-$T$-bimodules

$$\Hom_{R \otimes S}(M, R) \otimes_S \Hom_{S \otimes R}(N, S) \longrightarrow \Hom_{R \otimes S}(N \otimes_S M, R),$$

which is an isomorphism whenever the right $S$-module $N$ is finitely generated and projective.

Proof. Part (a): the desired map takes an element $g \otimes f \in \Hom_S(V, S) \otimes_S \Hom_U(U, R)$ to the map $U \otimes_S V \longrightarrow R$ taking an element $u \otimes v$ to the element $f(u)g(v) \in R$, for any $g \in \Hom_S(V, S)$, $f \in \Hom_U(U, R)$, $u \in U$, and $v \in V$. The second assertion does not depend on the $T$-module structure on $V$, so one can assume $T = \mathbb{Z}$ and, passing to the finite direct sums and direct summands in the argument $V \in S$-$\text{mod}$, reduce to the obvious case $V = S$. Part (b): the desired map takes an element $h \otimes k \in \Hom_{R \otimes S}(M, R) \otimes_S \Hom_{S \otimes R}(N, S)$ to the map $N \otimes_S M \longrightarrow R$ taking an element $n \otimes m$ to the element $h(k(n)m) \in R$, for any $h \in \Hom_{R \otimes S}(M, R)$, $k \in \Hom_{S \otimes R}(N, S)$, $n \in N$, and $m \in M$. The second assertion does not depend on the $T$-module structure on $N$, so it reduces to the obvious case $N = S$. 

Let $R$ be an associative ring and $V$ be an $R$-$R$-bimodule. The tensor ring of $V$ over $R$ (otherwise called the ring freely generated by an $R$-$R$-bimodule $V$) is the graded ring $T_R(V) = \bigoplus_{n=0}^{\infty} T_{R,n}(V)$ with the components $T_{R,0}(V) = R$, $T_{R,1}(V) = V$, $T_{R,2}(V) = V \otimes_R V$, and $T_{R,n}(V) = V \otimes_R V \otimes_R \cdots \otimes_R V$ (n factors) for $n \geq 2$. The multiplication in $T_R(V)$ is defined by the obvious rules $r(v_1 \otimes \cdots \otimes v_n) = (rv_1) \otimes v_2 \otimes \cdots \otimes v_n$, $(v_1 \otimes \cdots \otimes v_s)s = v_1 \otimes \cdots \otimes v_{s-1} \otimes (v_s s)$, and $(v_1 \otimes \cdots \otimes v_n)(v_{n+1} \otimes \cdots \otimes v_{n+m}) = v_1 \otimes \cdots \otimes v_{n+m}$ for all $r, s \in R$ and $v_i \in V$. 

[7]
Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree-zero component $A_0 = R$. Denote by $V$ the $R$-$R$-bimodule $V = A_1$. Then there exists a unique homomorphism of graded rings $\pi_A: T_R(V) \to A$ acting by the identity maps on the components of degrees 0 and 1. The ring $A$ is said to be generated by $A_1$ (over $A_0$) if the map $\pi_A$ is surjective. Furthermore, denote by $J_A = \ker(\pi_A)$ the kernel ideal of the ring homomorphism $\pi_A$. Then $J_A$ is a graded ideal in $T_R(V)$, so we have $J_A = \bigoplus_{n=2}^{\infty} J_{A,n}$, where $J_{A,n} \subset T_{R^n}(V)$. Set $I_A = J_{A,2} \subset V \otimes_R V$. A graded ring $A$ generated by $A_1$ over $A_0$ is said to be quadratic (over $R = A_0$) if the two-sided ideal $J_A \subset T_R(V)$ is generated by $I_A$, that is $J_A = (I_A)$, or explicitly

$$J_{A,n} = \sum_{i=1}^{n-1} T_{R,i-1}(V) \cdot I_A \cdot T_{R,n-i-1}(V) \quad \text{for all } n \geq 3.$$  

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a quadratic graded ring with the degree-zero component $R = A_0$. We will say that $A$ is $2$-left finitely projective if the left $R$-modules $A_1$ and $A_2$ are projective and finitely generated. Furthermore, $A$ is $3$-left finitely projective if the same applies to the left $R$-modules $A_1$, $A_2$, and $A_3$. Similarly, a quadratic graded ring $B = \bigoplus_{n=0}^{\infty} B_n$ with the degree-zero component $R = B_0$ is $2$-right finitely projective if the right $R$-modules $B_1$ and $B_2$ are finitely generated projective, and $B$ is $3$-right finitely projective if the same applies to the right $R$-modules $B_1$, $B_2$, and $B_3$.

Now we fix an associative ring $R$ and consider the category of graded rings over $R$, denoted $R$-\textit{rings}_{gr}, defined as follows. The objects of $R$-\textit{rings}_{gr} are nonnegatively graded associative rings $A = \bigoplus_{n=0}^{\infty} A_n$ endowed with a fixed ring isomorphism $R \cong A_0$. Morphisms $A \to \tilde{A}$ in $R$-\textit{rings}_{gr} are graded ring homomorphisms forming a commutative triangle diagram with the isomorphisms $R \cong A_0$ and $\tilde{R} \cong \tilde{A}_0$. Various specific classes of graded rings defined above in this section (and below in the next one) are viewed as full subcategories in $R$-\textit{rings}_{gr}.

**Proposition 1.2.** There is an anti-equivalence between the categories of 2-left finitely projective quadratic graded rings $A$ over $R$ and 2-right finitely projective quadratic graded rings $B$ over $R$, called the quadratic duality and defined by the following rules. Given a ring $A$, the ring $B$ is constructed as $B = T_R(B_1)/(I_B)$, where $B_1 = \text{Hom}_R(A_1, R)$ and $I_B = \text{Hom}_R(A_2, R) \subset B_1 \otimes_R B_1$. Then the $R$-$R$-bimodule $B_2$ is naturally isomorphic to $\text{Hom}_R(I_A, R)$. Conversely, given a ring $B$, the ring $A$ is constructed as $A = T_R(A_1)/(I_A)$, where $A_1 = \text{Hom}_R(B_1, R)$ and $I_A = \text{Hom}_R(B_2, R) \subset A_1 \otimes_R A_1$. Then the $R$-$R$-bimodule $A_2$ is naturally isomorphic to $\text{Hom}_R(I_B, R)$.

**Proof.** The category of quadratic graded rings $A$ over $R$ is equivalent to the category of $R$-$R$-bimodules $V = A_1$ endowed with a subbimodule $I = I_A \subset V \otimes_R V$. Here morphisms in the category of pairs $(V, I)$ are defined as $R$-$R$-bimodule morphisms $f: (V, I) \to (\tilde{V}, \tilde{I})$ are defined as $R$-$R$-bimodule morphisms $f: V \to \tilde{V}$ such that $(f \otimes f)(I) \subset \tilde{I}$.

A quadratic graded ring $A$ is 2-left finitely projective if and only if in the related pair $(V, I)$ the left $R$-modules $V$ and $A_2 = (V \otimes_R V)/I$ are finitely generated and projective. Assuming the former condition, the left $R$-module $V \otimes_R V$ is then finitely generated and projective, too, so the latter condition is equivalent to the $R$-$R$-subbimodule $I \subset V \otimes_R V$ being split as a left $R$-submodule.
Now we have a short exact sequence of $R$-$R$-bimodules
\[
0 \longrightarrow I_A \longrightarrow A_1 \otimes_R A_1 \longrightarrow A_2 \longrightarrow 0,
\]
which is split as a short exact sequence of left $R$-modules. Applying the functor $\text{Hom}_R(-, R)$ and taking into account Lemma 1.1(a), we obtain a short exact sequence of $R$-$R$-bimodules
\[
0 \longrightarrow \text{Hom}_R(A_2, R) \longrightarrow \text{Hom}_R(A_1, R) \otimes_R \text{Hom}_R(A_1, R) \longrightarrow \text{Hom}_R(I_A, R) \longrightarrow 0,
\]
which is split as a short exact sequence of right $R$-modules. It remains to set $B_1 = \text{Hom}_R(A_1, R)$ and $I_B = \text{Hom}_R(A_2, R)$, so that $B_2 = \text{Hom}_R(I_A, R)$. According to the discussion in the beginning of this section, $B_1$ and $B_2$ are finitely generated projective right $R$-modules. It is clear that this construction is a contravariant functor between the categories of 2-left finitely projective and 2-right finitely projective quadratic graded rings over $R$, and that the similar construction with the left and right sides switched provides the inverse functor in the opposite direction. \hfill \Box

The 2-left finitely projective quadratic ring $A$ and the 2-right finitely projective quadratic ring $B$ as in Proposition 1.2 are said to be quadratic dual to each other.

**Proposition 1.3.** The anti-equivalence of categories from Proposition 1.2 takes 3-left finitely projective quadratic graded rings to 3-right finitely projective quadratic graded rings and vice versa.

**Proof.** For any quadratic graded ring $A$, the grading component $A_3$ is the cokernel of the map $(A_1 \otimes_R I_A) \oplus (I_A \otimes_R A_1) \longrightarrow A_1 \otimes_R A_1 \otimes_R A_1$ induced by the inclusion map $I_A \longrightarrow A_1 \otimes_R A_1$. When the components $A_1$ and $A_2$ are projective as (say, left) $R$-modules, the maps $A_1 \otimes_R I_A \longrightarrow A_1 \otimes_R A_1 \otimes_R A_1$ and $I_A \otimes_R A_1 \longrightarrow A_1 \otimes_R A_1 \otimes_R A_1$ are injective, so we have a four-term exact sequence of $R$-$R$-bimodules
\[
(2) \quad 0 \longrightarrow I_A^{(3)} \longrightarrow (A_1 \otimes_R I_A) \oplus (I_A \otimes_R A_1) \longrightarrow A_1 \otimes_R A_1 \otimes_R A_1 \longrightarrow A_3 \longrightarrow 0,
\]
where $I_A^{(3)} = (A_1 \otimes_R I_A) \cap (I_A \otimes_R A_1) \subset A_1 \otimes_R A_1 \otimes_R A_1$. When the component $A_3$ is a projective left $R$-module, too, we observe that all the terms of this exact sequence, except perhaps the leftmost one, are projective left $R$-modules. It follows that the sequence (2) splits as an exact sequence of left $R$-modules, and the leftmost term $I_A^{(3)}$ is a projective left $R$-module, too.

Furthermore, when $A_1$, $A_2$, and $A_3$ are finitely generated projective left $R$-modules, all the terms of the sequence (2) are also finitely generated projective left $R$-modules. Applying the functor $\text{Hom}_R(-, R)$ to (2), we obtain a four-term exact sequence of $R$-$R$-bimodules
\[
(3) \quad 0 \longrightarrow \text{Hom}_R(A_3, R) \longrightarrow B_1 \otimes_R B_1 \otimes_R B_1 \longrightarrow (B_1 \otimes_R B_2) \oplus (B_2 \otimes_R B_1) \longrightarrow \text{Hom}_R(I_A^{(3)}, R) \longrightarrow 0.
\]
Now for any quadratic graded ring $B$, the cokernel of the map $B_1 \otimes_R B_1 \otimes_R B_1 \longrightarrow (B_1 \otimes_R B_2) \oplus (B_2 \otimes_R B_1)$ induced by the (surjective) multiplication map $B_1 \otimes_R
Proposition 1.3 does not hold in degrees higher than 3 for quadratic graded rings in general. It holds under the Koszulity assumption, though, as we will see in the next section.

Remark 1.4. The above discussion of the categories of nonnegatively graded and quadratic rings can be modified or expanded by including non-identity isomorphisms (in particular, automorphisms) in the degree-zero component. Denote by $\text{Rings}_{gr}$ the category whose objects are nonnegatively graded associative rings $A = \bigoplus_{n=0}^{\infty} A_n$, and morphisms are defined as follows. A morphism $A \rightarrow A'$ in $\text{Rings}_{gr}$ is a morphism of graded rings $f: A \rightarrow A'$ whose degree-zero component $f_0: A_0 \rightarrow A'_0$ is an isomorphism. Then the same classes of 2- and 3-left/right finitely projective quadratic rings as in Propositions 1.2 and 1.3 can be viewed as full subcategories in $\text{Rings}_{gr}$. The assertions of the two propositions remain valid with this modification.

The inclusion of the full subcategory of quadratic graded rings over $R$ into the category of (nonnegatively) graded rings over $R$ has a right adjoint functor, which we denote by $A \mapsto qA$. For any nonnegatively graded ring $A$, the quadratic graded ring $A' = qA$ together with the graded ring homomorphism $A' \rightarrow A$ is characterized by the properties that the maps $A'_0 \rightarrow A_0$ and $A'_1 \rightarrow A_1$ are isomorphisms and the map $A'_2 \rightarrow A_2$ is injective. Explicitly, the ring $qA$ is constructed as the ring with degree-one generators and quadratic relations $qA = T_{A_0}(A_1)/(I_A)$, where $I_A \subset A_1 \otimes_R A_1$ is the kernel of the multiplication map $A_1 \otimes_R A_1 \rightarrow A_2$.

### 2. Flat and Finitely Projective Koszulity

#### 2.1. Graded and ungraded $\text{Ext}$ and $\text{Tor}$

So far in this paper we denoted a graded ring by $A = \bigoplus_{n=0}^{\infty} A_n$ (and for the most part we will continue to do so in the sequel), but this is a colloquial abuse of notation. A graded abelian group $U$ is properly thought of as a collection of abelian groups $U = (U_n)_{n \in \mathbb{Z}}$. Then there are several ways to produce an ungraded group from a graded one.

Two of them are important for us in this section. One can take the direct sum of the grading components, which we denote by $\Sigma U = \bigoplus_{n \in \mathbb{Z}} U_n$; or one can take the product of the grading components, which we denote by $\Pi U = \prod_{n \in \mathbb{Z}} U_n$.

In particular, let $A = (A_n)_{n \in \mathbb{Z}}$ be a graded ring and $M = (M_n)_{n \in \mathbb{Z}}$ be a graded left $A$-module. Then $\Sigma A = \bigoplus_{n \in \mathbb{Z}} A_n$ is the underlying ungraded ring of $A$; and there are two underlying ungraded $\Sigma A$-modules associated with $M$. Namely, both the abelian groups $\Sigma M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $\Pi M = \prod_{n \in \mathbb{Z}} M_n$ have natural structures of left $\Sigma A$-modules. Denoting the category of graded left $A$-modules by $A \text{-mod}_{gr}$, we have two forgetful functors $\Sigma$ and $\Pi: A \text{-mod}_{gr} \rightarrow \Sigma A \text{-mod}$.
The tensor product of a graded right \( A \)-module \( N \) and a graded left \( A \)-module \( M \) is naturally a graded abelian group \( N \otimes_A M \), and applying the functor \( \Sigma \) to \( N \otimes_A M \) produces the tensor product of the ungraded \( \Sigma A \)-modules \( \Sigma N \) and \( \Sigma M \),
\[
\Sigma(N \otimes_A M) \simeq \Sigma N \otimes_{\Sigma A} \Sigma M.
\] (4)

Similarly, for any graded left \( A \)-modules \( L \) and \( M \) one can consider the graded abelian group \( \text{Hom}_A(L, M) \) with the components \( \text{Hom}_{A,n}(L, M) \) consisting of all the homogeneous left \( A \)-module maps \( L \to M \) of degree \( n \). The purpose of introducing the functor \( \Pi \) above was to formulate the comparison between the graded and ungraded \( \text{Hom} \), which has the form
\[
\Pi \text{Hom}_A(L, M) \simeq \text{Hom}_{\Sigma A}(\Sigma L, \Pi M).
\] (5)

Furthermore, the functor \( \Sigma \) takes projective graded \( A \)-modules to projective \( \Sigma A \)-modules, while the functor \( \Pi \) takes injective graded \( A \)-modules to injective \( \Sigma A \)-modules (as one can see from the description of projective and injective modules as the direct summands of the free and cofree modules, respectively). In addition, the functor \( \Sigma \) takes flat graded \( A \)-modules to flat \( \Sigma A \)-modules (as one can see from the Govorov–Lazard description of flat modules as the filtered direct limits of finitely generated free modules). We define the graded versions of \( \text{Tor} \) and \( \text{Ext} \) as the derived functors of the graded tensor product and \( \text{Hom} \), computed in the abelian categories of graded (right and left) modules.

So, for any graded right \( A \)-module \( N \) and any graded left \( A \)-module \( M \) there is a bigraded abelian group
\[
\text{Tor}_i^A(N, M) = (\text{Tor}_{i,j}^A(N, M))_{i,j}, \quad i \geq 0, \ j \in \mathbb{Z},
\]
where \( i \) is the usual homological grading and \( j \) is the internal grading (induced by the grading of \( A, N \), and \( M \)). In order to compute the bigraded group \( \text{Tor}_i^A(M, N) \), one chooses a graded projective (or flat) resolution of one of the \( A \)-modules \( M \) and \( N \) and takes its tensor product over \( A \) with the other module; then the grading \( i \) is induced by the homological grading of the resolution and the grading \( j \) comes from the grading of the tensor product of any two graded modules. In view of the above considerations concerning projective/flat graded modules, the formula (4) implies a similar formula for the \( \text{Tor} \) groups,
\[
\Sigma \text{Tor}_i^A(N, M) \simeq \text{Tor}_{i,j}^{\Sigma A}(\Sigma N, \Sigma M) \quad \text{for every } i \geq 0,
\]
or more explicitly,
\[
\text{Tor}_{i,j}^{\Sigma A}(\Sigma M, \Sigma N) \simeq \bigoplus_{j \in \mathbb{Z}} \text{Tor}_{i,j}^A(M, N).
\] (6)

Similarly, for any graded left \( A \)-modules \( L \) and \( M \) there is a bigraded abelian group
\[
\text{Ext}_i^A(L, M) = (\text{Ext}_{i,n}^A(L, M))_{i,n}, \quad i \geq 0, \ n \in \mathbb{Z},
\]
where \( i \) is the usual cohomological grading and \( n \) is the internal grading. In order to compute the bigraded group \( \text{Ext}_i^A(L, M) \), one chooses either a graded projective resolution of the \( A \)-module \( L \), or a graded injective resolution of the \( A \)-module \( M \), and
takes the graded Hom; then the grading $i$ is induced by the (co)homological grading of the resolution and the grading $n$ comes from the grading of the Hom groups.

In the context of the internal grading of the Ext, we will put $n = -j$ and use the notation $\text{Ext}^{i,j}_{A}(L, M) = \text{Ext}^{i}_{A,L}(L, M)$. By abuse of terminology, the grading $j$ will be also called the internal grading of the Ext.

In view of the above considerations concerning projective and injective graded modules, the formula (5) implies a similar formula for the Ext groups,

$$\Pi \text{Ext}^{i}_{A}(L, M) \simeq \text{Ext}^{i}_{\Sigma A}(\Sigma L, \Pi M) \quad \text{for every } i \geq 0,$$

or more explicitly,

$$\text{Ext}^{i}_{\Sigma A}(\Sigma L, \Pi M) \simeq \prod_{j \in \mathbb{Z}} \text{Ext}^{i,j}_{A}(L, M).$$

For any three graded left $A$-modules $K$, $L$, and $M$, there are natural associative, unital composition/multiplication maps

$$\text{Ext}^{i,j}_{A}(L, M) \times \text{Ext}^{i',j'}_{A}(K, L) \longrightarrow \text{Ext}^{i+i',j+j'}_{A}(K, M), \quad i, i' \geq 0, \ j, j' \in \mathbb{Z}$$

on the bigraded Ext groups. Whenever the graded $A$-module $L$ only has a finite number of nonzero grading components (so $\Sigma L = \Pi L$), the passage to the infinite products with respect to the internal gradings $j'$ and $j''$ makes the multiplications (8) on the graded Ext groups agree with the similar multiplications on the ungraded Ext (between the $\Sigma A$-modules $\Sigma K$, $\Sigma L = \Pi L$, and $\Pi M$).

2.2. Relative bar resolution. Given an $R-R$-bimodule $V$, we will use the notation $V^\otimes_R n = \tau_{R,n}(V)$ for the tensor product $V \otimes_R \cdots \otimes_R V$ ($n$ factors).

Let $R \longrightarrow A$ be an injective homomorphism of associative rings. Denote by $A_{+}$ the $R-R$-bimodule $A/R$. Let $L$ be a left $A$-module. The reduced relative bar resolution of $L$ is the complex of left $A$-modules

$$\cdots \longrightarrow A \otimes_R A_+ \otimes_R A_+ \otimes_R L \longrightarrow A \otimes_R A_+ \otimes_R L \longrightarrow A \otimes_R L \longrightarrow L \longrightarrow 0$$

with the differential given by the standard formula $\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes l) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes l - a_0 \otimes a_1 a_2 \otimes a_3 \otimes \cdots \otimes l + \cdots + (-1)^n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n l$. One can easily check that the image of the right-hand side in $A \otimes_R A_{+}^{\otimes_R n-1} \otimes_R L$ does not depend on the arbitrary choice of liftings $a_i \in A$ of the given elements $\bar{a}_i \in A_{+}$, $1 \leq i \leq n$; so the differential is well-defined.

The complex (9) is contractible as a complex of left $R$-modules; the contracting homotopy is given by the formulas $t(l) = 1 \otimes l$, $l \in L$, and $t(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes l) = 1 \otimes a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes l$, where $\bar{a}_i \in A_{+}$, $1 \leq i \leq n$, and $\bar{a}_0 \in A_{+}$ is the image of the element $a_0 \in A$ under the natural surjection $A \longrightarrow A_{+}$. Hence it follows that the complex (9) is acyclic.

If both $L$ and $A_{+}$ are flat left $R$-modules, then all the left $A$-modules $A \otimes_R A_{+}^{\otimes_R n} \otimes_R L$ are flat, so (9) is a flat resolution of the left $A$-module $L$. Similarly, (9) is a projective resolution of the left $A$-module $L$ whenever both the left $R$-modules $L$ and $A_{+}$ are projective.
Assume that $L$ and $A_+$ are flat left $R$-modules, and let $N$ be an arbitrary right $A$-module. Then one can use the flat resolution (9) of the left $A$-module $L$ in order to compute the groups $\text{Tor}_i^A(N, L)$. Thus the groups $\text{Tor}_i^A(N, L)$ are naturally isomorphic to the homology groups of the bar complex

\[
\cdots \to N \otimes_R A_+ \otimes_R A_+ \otimes_R L \to N \otimes_R A_+ \otimes_R L \to N \otimes_R L \to 0.
\]

Switching the roles of the left and right modules and using the reduced relative bar resolution of $N$, we conclude that the same bar complex (10) computes the groups $\text{Tor}_i^A(N, L)$ whenever a right $A$-module $N$ is a flat right $R$-module, $A_+$ is a flat right $R$-module, and $L$ is an arbitrary left $A$-module.

Let $M$ be a left $A$-module. The reduced relative cobar resolution of $M$ is the complex of left $A$-modules

\[
0 \to M \to \text{Hom}_R(A, M) \to \text{Hom}_R(A_+ \otimes_R A, M) \to \cdots
\]

with the differential given by the formula $(\partial f)(\bar{a}_n \otimes \cdots \otimes \bar{a}_1 \otimes a_0) = a_n f(a_{n-1} \otimes \cdots \otimes a_0) - f(a_n a_{n-1} \otimes a_{n-2} \otimes \cdots \otimes a_0) + \cdots + (-1)^n f(a_n \otimes \cdots \otimes a_2 \otimes a_1 a_0)$, where $f \in \text{Hom}_R(A^{\otimes n+1}_+ \otimes_R A, M)$ and $a_i \in A$ are arbitrary liftings of elements $\bar{a}_i \in A_+$, $1 \leq i \leq n$. One easily checks that the expression in the right-hand side vanishes on the kernel of the natural surjection $A^{\otimes n+1}_+ \to A^{\otimes n}_+ \otimes_R A$; so the differential is well-defined. The left $A$-module structure on $\text{Hom}_R(A^{\otimes n}_+ \otimes_R A, M)$ is induced by the right $A$-module structure on $A$.

The complex (11) is contractible as a complex of left $R$-modules; the contracting homotopy is given by the formula $t(f)(\bar{a}_{n-1} \otimes \cdots \otimes \bar{a}_1 \otimes a_0) = (-1)^n f(\bar{a}_{n-1} \otimes \cdots \otimes \bar{a}_1 \otimes \bar{a}_0 \otimes 1)$. In particular, it follows that the complex (11) is acyclic. If $M$ is an injective left $R$-module and $A_+$ is a flat right $R$-module, then all the left $A$-modules $\text{Hom}_R(A^{\otimes n}_+ \otimes_R A, M), \ n \geq 0$, are injective; so (11) is an injective resolution of the left $A$-module $M$.

Let $L$ and $M$ be left $A$-modules. Assuming that $L$ is a projective left $R$-module and $A_+$ is a projective left $R$-module, one can use the projective resolution (9) of the left $A$-module $L$ in order to compute the groups $\text{Ext}^i_A(L, M)$. Thus the groups $\text{Ext}^i_A(L, M)$ are naturally isomorphic to the cohomology groups of the cobar complex

\[
0 \to \text{Hom}_R(L, M) \to \text{Hom}_R(A_+ \otimes_R L, M) \to \cdots
\]

Alternatively, assuming that $M$ is an injective left $R$-module and $A_+$ is a flat right $R$-module, one can use the injective resolution (11) of the left $A$-module $M$ in order to compute the groups $\text{Ext}^i_A(L, M)$. Under these assumptions, one comes to the same conclusion that the groups $\text{Ext}^i_A(L, M)$ are naturally isomorphic to the cohomology groups of the cobar complex (12).

When $R$ and $A$ are graded rings, $R \to A$ is a graded ring homomorphism, and $L$ is a graded left $A$-module, one can interpret (9) as a graded resolution of the graded $A$-module $L$. When $L$ and $A_+$ are flat graded left $R$-modules, (9) is a graded flat
resolution of the graded $A$-module $L$; and when $L$ and $A_+$ are projective graded left $R$-modules, (9) is a graded projective resolution.

For a graded left $A$-module $M$, one can also interpret the Hom notation in (11) as the graded Hom, obtaining a graded resolution of the graded $A$-module $M$. When $M$ is an injective graded left $R$-module and $A_+$ is a flat graded right $R$-module, (11) is a graded injective resolution.

It follows that, under the graded versions of the above flatness, projectivity, and/or injectivity assumptions, the graded bar complex (10) computes the bigraded $\text{Tor}^A_i(N, L)$, and the graded version of the cobar complex (12) computes the bigraded $\text{Ext}^A_i(L, M)$.

The functor $\Sigma$ transforms the graded versions of (9) and (10) (for graded rings $R$ and $A$ and graded modules $L$ and $N$) into the ungraded ones (for the ungraded rings $\Sigma R$ and $\Sigma A$ and the ungraded modules $\Sigma L$ and $\Sigma N$). The functor $\Pi$ transforms the graded versions of (11) and (12) (for graded rings $R$ and $A$ and graded modules $L$ and $M$) into the ungraded ones (for the ungraded rings $\Sigma R$ and $\Sigma A$ and the ungraded modules $\Sigma L$ and $\Pi M$).

The cobar complexes also compute the composition/multiplication on the Ext groups. Let $K$, $L$, and $M$ be left $A$-modules; assume that the left $R$-modules $K$, $L$, and $A_+$ are projective. Then the natural composition/multiplication maps on the cobar complexes

$$\text{Hom}_R(A^+_i \otimes_R L, M) \times \text{Hom}_R(A^+_i \otimes_R K, L) \longrightarrow \text{Hom}_R(A^+_i \otimes_R K, M)$$

agree with the cobar differentials and induce the Yoneda multiplication maps

$$\text{Ext}^i_A(L, M) \times \text{Ext}^j_A(K, L) \longrightarrow \text{Ext}^{i+j}_{A}(K, M)$$
on the Ext groups. In the case of graded rings $R$ and $A$ and graded $A$-modules $K$, $L$, and $M$, the same assertion applies to the graded Ext.

2.3. Diagonal Tor and Ext. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree-zero component $R = A_0$. Then the projection onto the degree-zero component is a graded ring homomorphism $A \longrightarrow R$. Using this homomorphism, one can consider $R$ as a left and right graded module over $A$.

So we can consider the bigraded Tor groups $\text{Tor}^A_{i,j}(R, R)$ and the bigraded Ext ring $\text{Ext}^i_{\Sigma A}(R, R)$, with the (co)homological grading $i$ and the internal grading $j$. In fact, as $R$ is an $R$-$A$-bimodule and an $A$-$R$-bimodule, the groups $\text{Tor}^A_{i,j}(R, R)$ have natural structures of $R$-$R$-bimodules.

First of all, we notice the connection between the graded and ungraded Tor and Ext. As the graded (left or right) $A$-module $R$ is concentrated in the internal grading 0, we have $\Sigma R = R = \Pi R$. Hence the formulas (6) and (7) reduce to

$$\text{Tor}^\Sigma_A(R, R) \simeq \bigoplus_{j \in \mathbb{Z}} \text{Tor}^A_{i,j}(R, R)$$

$$\text{Ext}^i_{\Sigma A}(R, R) \simeq \prod_{j \in \mathbb{Z}} \text{Ext}^i_{A,j}(R, R).$$
Proposition 2.1. Assume that the grading components $A_n$, $n \geq 1$, are flat left $R$-modules, and the multiplication map $A_1 \otimes_R A_1 \to A_2$ is surjective with the kernel $I_A$. Then there are natural isomorphisms of $R$-$R$-bimodules

(a) $\operatorname{Tor}^A_{i,j}(R, R) = 0$ whenever $i < 0$, or $i = 0$ and $j > 0$, or $i > j$;
(b) $\operatorname{Tor}^A_{0,0}(R, R) = R$, $\operatorname{Tor}^A_{1,1}(R, R) \simeq A_1$, $\operatorname{Tor}^A_{2,2}(R, R) \simeq I_A$, and

$$
\operatorname{Tor}^A_{n,n}(R, R) \simeq \bigcap_{k=1}^{n-1} A_1^{\otimes n-k-1} \otimes_R I_A \otimes_R A_1^{\otimes k-n-1} \subset A_1^{\otimes n}, \quad n \geq 2.
$$

Proof. The $R$-$R$-bimodules $\operatorname{Tor}^A_{i,j}(R, R)$ can be computed as the homology bimodules of the bar complex (10) for $N = R = L$,

$$
\cdots \to A_+ \otimes_R A_+ \otimes_R A_+ \to A_+ \otimes_R A_+ \to A_+ \to 0 \to R \to 0.
$$

More explicitly, this means that, for every fixed $n \geq 1$, the bimodules $\operatorname{Tor}^A_{i,n}(R, R)$ are the homology bimodules of the complex of $R$-$R$-bimodules

$$
0 \to A_1^{\otimes_R n} \to \bigoplus_{k=1}^{n-1} A_1^{\otimes n-k-1} \otimes_R I_A \otimes_R A_1^{\otimes k-n-1} \to \cdots \\
\cdots \to (A_+^{\otimes_R 1})_n \to \cdots \\
\cdots \to \bigoplus_{k=1}^{n-1} A_k \otimes_R A_{n-k} \to A_n \to 0.
$$

The assertion (a) follows immediately; and to prove (b), it remains to compute, for every $1 \leq k \leq n - 1$, the kernel of the map $A_1^{\otimes_R n} \to A_1^{\otimes n-k-1} \otimes_R A_2 \otimes_R A_1^{\otimes n-k-1}$ induced by the multiplication map $A_1 \otimes_R A_1 \to A_2$.

Specifically, we have to check that the natural short sequence of $R$-$R$-bimodules

$$
0 \to A_1^{\otimes_R n-k-1} \otimes_R I_A \otimes_R A_1^{\otimes n-k-1} \to A_1^{\otimes_R n} \to A_1^{\otimes n-k-1} \otimes_R A_2 \otimes_R A_1^{\otimes n-k-1} \to 0
$$

is exact. Indeed, by assumption we have a short exact sequence of $R$-$R$-bimodules

$$
0 \to I_A \to A_1 \otimes_R A_1 \to A_2 \to 0,
$$

whose terms are flat left $R$-modules. Furthermore, $A_1^{\otimes_R n-k-1}$ is a flat left $R$-module, too. It follows that tensoring (15) (with any right $R$-module, and in particular) with $A_1^{\otimes_R k-1}$ on the left does not affect exactness; and neither does tensoring (any exact sequence of right $R$-modules, and in particular) the resulting short sequence with $A_1^{\otimes_R n-k-1}$ on the right. \qed

Concerning the Ext, our notation $\operatorname{Ext}^A_R(R, R)$ presumes, as above, that the Ext is taken in the category of left $R$-modules. So, in particular, the ring $\operatorname{Ext}^A_R(R, R) = \operatorname{Hom}_R(R, R) = R^{op}$ is the opposite ring to $R$.

Proposition 2.2. Assume that the grading components $A_n$, $n \geq 1$, are finitely generated projective left $R$-modules, and the multiplication map $A_1 \otimes_R A_1 \to A_2$ is surjective with the kernel $I_A$. Then

(a) $\operatorname{Ext}^A_{i,j}(R, R) = 0$ whenever $i < 0$, or $i = 0$ and $j > 0$, or $i > j$;
(b) the diagonal Ext ring $\bigoplus_{n=0}^{\infty} \operatorname{Ext}^A_{n,n}(R, R)$ is naturally isomorphic, as a graded ring, to the opposite ring $B^{op}$ to the 2-right finitely projective quadratic dual ring.
\(B = T_R(B_1)/(I_B), \ B_1 = \text{Hom}_R(A_1, R), \ I_B = \text{Hom}_R(A_2, R)\) to the 2-left finitely projective quadratic ring \(qA = T_R(A_1)/(I_A)\).

**Proof.** Part (a) does not depend on the finite generatedness assumptions on the \(R\)-modules \(A_n\), and requires only the projectivity assumptions. The bigraded ring \(\text{Ext}_A(R, R)\) can be computed as the cohomology ring of the DG-ring (12) for \(L = R = M\). The latter can be obtained by applying the functor \(\text{Hom}_R(\cdot, R)\) to the bar complex (13),

\[
0 \longrightarrow R \longrightarrow \text{Hom}_R(A_+, R) \longrightarrow \text{Hom}_R(A_+ \otimes_R A_+, R) \longrightarrow \cdots
\]

More specifically, for every fixed \(n \geq 1\), the groups \(\text{Ext}_A^n(R, R)\) are the cohomology groups of the complex obtained by applying \(\text{Hom}_R(\cdot, R)\) to the complex (14). This proves part (a).

When all the grading components \(A_j\) are finitely generated and projective left \(R\)-modules, the complex obtained by applying the functor \(\text{Hom}_R(\cdot, R)\) to the complex (14) can be computed using Lemma 1.1(a) as

\[
0 \longrightarrow A_\vee^n \longrightarrow \bigoplus_{k=1}^{n-1} A_\vee^{n-k} \otimes_R A_k \longrightarrow \cdots \longrightarrow (A_\vee^1 \otimes R) \longrightarrow \cdots
\]

\[
\longrightarrow \bigoplus_{k=1}^{n-1} A_\vee^{n-k-1} \otimes_R A_2 \otimes_R A_\vee^{k-1} \longrightarrow A_\vee \otimes_R 1 \longrightarrow 0,
\]

where the notation is \(U^\vee = \text{Hom}_R(U, R)\). Setting \(B_1 = A_\vee^1, \ I_B = A_\vee^2, \) and \(B = T_R(B_1)/(I_B)\), we obtain the desired isomorphism \(\text{Ext}_A^{n,n}(R, R) \simeq B_n\). It follows immediately from the construction of the multiplication on the cobar complex that this is a graded ring isomorphism between \(\bigoplus_n \text{Ext}_A^{n,n}(R, R)\) and \(B^{\text{op}}\). \(\square\)

**Remark 2.3.** The bar complex \(\text{Bar}_R(A)\) computing Tor\(^A(R, R)\) (13) has a natural structure of a (coassociative, counital) DG-coring over the ring \(R\), with the obvious maps of counit \(\text{Bar}_R(A) \longrightarrow R\) and a comultiplication \(\text{Bar}_R(A) \longrightarrow \text{Bar}_R(A) \otimes_R \text{Bar}_R(A)\) compatible with the bar differential. However, as the functor of tensor product over \(R\) is not left exact, this DG-coring structure, generally speaking, does not descend to a coring structure on the homology modules. When \(A_+\) is a flat left \(R\)-module and Tor\(^A(R, R)\) is a flat left \(R\)-module, the nonexactness problem does not interfere, and the (bi)graded \(R\)-bimodule Tor\(^A(R, R)\) is a (coassociative, counital) coring over \(R\). In particular, the diagonal Tor bimodule \(C = \bigoplus_{n=0}^\infty \text{Tor}_{n,n}^A(R)\) becomes a graded coring over \(R\) in these assumptions, with the obvious counit \(C \longrightarrow R\) and the induced comultiplication map \(C \longrightarrow C \otimes_R C\).

### 2.4. Low-dimensional Tor, degree-one generators and quadratic relations.

As in the previous section, we consider a nonnegatively graded ring \(A = \bigoplus_{n=0}^\infty A_n\) with the degree-zero component \(R = A_0\).

**Proposition 2.4.** Assume that the grading components \(A_n, \ n \geq 1, \) are flat left \(R\)-modules. Then
(a) the graded ring $A$ is generated by $A_1$ over $R$ if and only if $\text{Tor}_{1,j}^A(R, R) = 0$ for all $j > 1$;

(b) the graded ring $A$ is quadratic if and only if $\text{Tor}_{1,j}^A(R, R) = 0$ for all $j > 1$ and $\text{Tor}_{2,j}^A(R, R) = 0$ for all $j > 2$.

**Proof.** Part (a): following the proof of Proposition 2.1, the group $\text{Tor}_{1,n}^A(R, R)$ can be computed as the rightmost homology group of the complex (14), that is, the cokernel of the multiplication map $\bigoplus_{k=1}^n A_k \otimes_R A_{n-k} \longrightarrow A_n$. Now it is clear that a nonnegatively graded ring $A$ is generated by $A_1$ over $R = A_0$ if and only if the latter map is surjective for all $n \geq 2$.

Part (b): by part (a), we can assume that $A$ is generated by $A_1$; then we have to show that $A$ is quadratic if and only if $\text{Tor}_{2,j}^A(R, R) = 0$ for all $j > 2$. Once again, we compute the group $\text{Tor}_{2,n}^A(R, R)$ using the complex (14). So we have to show that $A$ is quadratic if and only if the short sequence

$$
\bigoplus_{k+l+m=n}^{k,l,m\geq 1} A_k \otimes_R A_l \otimes_R A_m \longrightarrow \bigoplus_{k+l=n}^{k,l\geq 1} A_k \otimes_R A_l \longrightarrow A_n \longrightarrow 0
$$

is right exact for all $n \geq 3$.

Given an $R$-$R$-bimodule $V$ and an $R$-$R$-subbimodule $I \subset V \otimes_R V$, one can construct the quadratic ring $A' = T_R(V)/(I)$ by the following inductive procedure. Set $A_0' = R$, $A_1' = V$, and $A_2' = (V \otimes_R V)/I$; then there are the obvious multiplication maps $A_k' \times A_l' \longrightarrow A_{k+l}'$ for $k, l \geq 0$, $k + l \leq 2$. For every $n \geq 3$, set $A_n'$ to be the cokernel of the $R$-$R$-bimodule morphism

$$
\bigoplus_{k+l+m=n}^{k,l,m\geq 1} A_k' \otimes_R A_l' \otimes_R A_m' \longrightarrow \bigoplus_{k+l=n}^{k,l\geq 1} A_k' \otimes_R A_l'.
$$

Then we have biadditive multiplication maps $A_k' \times A_l' \longrightarrow A_{k+l}'$ defined for all $k + l = n$, $k, l \geq 0$ and satisfying the associativity equations $(ab)c = a(bc)$ for all $a \in A_k'$, $b = A_l'$, $c \in A_m'$, $k + l + m = n$, $k, l, m \geq 0$. So we obtain a graded ring $A'$. Obviously, $A'$ is the graded associative ring freely generated by $A_1' = V$ over $A_0' = R$ with the relations $I \subset V \otimes_R V$; so in other words, $A' \cong T_R(V)/(I)$.

Returning to the original graded ring $A$, put $V = A_1$ and let $I$ be the kernel of the multiplication map $A_1 \otimes_R A_1 \longrightarrow A_2$. Then there is a unique homomorphism of graded rings $A' \longrightarrow A$ acting by the identity map in degrees 0 and 1. The ring homomorphism $A' \longrightarrow A$ is surjective by assumption. Arguing by induction in $n$, one can easily see that this ring homomorphism is an isomorphism if and only if the short sequence (18) is right exact for every $n \geq 3$.

2.5. **Koszul complexes.** Let $A = \bigoplus_{n=0}^{\infty} A_n$ and $B = \bigoplus_{n=0}^{\infty} B_n$ be two nonnegatively graded rings with the same degree-zero component $A_0 = R = B_0$. Suppose that $B_1$ is a finitely generated projective right $R$-module, and that we have an $R$-$R$-bimodule morphism $\tau : \text{Hom}_{R^\text{op}}(B_1, R) \longrightarrow A_1$.

Denote by $K^\tau(B, A)$ the following bigraded $A$-$B$-bimodule endowed with an $A$-$B$-bimodule endomorphism $\partial^\tau$. The bigrading components of $K^\tau(B, A)$ are $K^\tau_{i,n}(B, A) = \text{Hom}_{R^\text{op}}(B_i, A_{n-i})$, $i \geq 0$, $n \geq i$. Here $i$ is interpreted as the
homological grading and $n$ is the internal grading. The homogeneous $A$-$B$-bimodule endomorphism $\partial^r : K^r(B, A) \to K^r(B, A)$ of bidegree $(i, n) = (-1, 0)$ is constructed as the composition

$$
\text{Hom}_{R^\text{op}}(B_i, A_j) \longrightarrow \text{Hom}_{R^\text{op}}(B_{i-1} \otimes_R B_1, A_j)
$$

$$
\simeq \text{Hom}_{R^\text{op}}(B_{i-1}, \text{Hom}_{R^\text{op}}(B_1, A_j)) \simeq \text{Hom}_{R^\text{op}}(B_{i-1}, A_j \otimes_R \text{Hom}_{R^\text{op}}(B_1, R))
$$

$$
\longrightarrow \text{Hom}_{R^\text{op}}(B_{i-1}, A_j \otimes_R A_1) \longrightarrow \text{Hom}_{R^\text{op}}(B_{i-1}, A_{j+1}).
$$

Here the map $\text{Hom}_{R^\text{op}}(B_i, A_j) \to \text{Hom}_{R^\text{op}}(B_{i-1} \otimes_R B_1, A_j)$ is induced by the multiplication map $B_{i-1} \otimes_R B_1 \to B_i$, the map $\text{Hom}_{R^\text{op}}(B_{i-1}, A_j \otimes_R \text{Hom}_{R^\text{op}}(B_1, R)) \to \text{Hom}_{R^\text{op}}(B_{i-1}, A_j \otimes_R A_1)$ is induced by the given map $\tau : \text{Hom}_{R^\text{op}}(B_1, R) \to A_1$, and the map $\text{Hom}_{R^\text{op}}(B_{i-1}, A_j \otimes_R A_1) \to \text{Hom}_{R^\text{op}}(B_{i-1}, A_{j+1})$ is induced by the multiplication map $A_j \otimes_R A_1 \to A_{j+1}$.

**Lemma 2.5.** Assume that both $B_1$ and $B_2$ are finitely generated projective right $R$-modules. Then the following conditions are equivalent:

(a) $(\partial^r)^2 = 0$ on the whole bigraded $A$-$B$-bimodule $K^r(B, A)$;

(b) the composition $\text{Hom}_{R^\text{op}}(B_2, R) \xrightarrow{\partial^r} \text{Hom}_{R^\text{op}}(B_1, A_1) \xrightarrow{\partial^r} \text{Hom}_{R^\text{op}}(R, A_2) = A_2$

vanishes;

(c) the composition of maps $\text{Hom}_{R^\text{op}}(B_2, R) \longrightarrow \text{Hom}_{R^\text{op}}(B_1 \otimes_R B_1, R) \simeq \text{Hom}_{R^\text{op}}(B_1, R) \otimes_R \text{Hom}_{R^\text{op}}(B_1, R)$ $\longrightarrow$ $\tau \otimes_R A_1 \otimes_R A_1 \longrightarrow A_2$

vanishes.

**Proof.** (a) $\implies$ (b) is obvious.

(b) $\iff$ (c) holds because the two maps in question are the same.

(c) $\implies$ (a) is straightforward, using the isomorphism $\text{Hom}_{R^\text{op}}(B_2, A_j) \simeq A_j \otimes_R \text{Hom}_{R^\text{op}}(B_2, R)$. $\square$

When the equivalent conditions of the lemma hold, the bigraded $A$-$B$-bimodule $K^r(B, A)$ with the differential $\partial^r$ can be viewed as a complex of graded left $A$-modules. We call it the **Koszul complex** and denote by $K^r(B, A)$.

Specifically, the following particular case is important. Let $A$ be a nonnegatively graded ring with the degree-zero component $R = A_0$. Assume that $A_1$ and $A_2$ are finitely generated projective left $R$-modules and the multiplication map $A_1 \otimes_R A_1 \to A_2$ is surjective. Denote the kernel of the latter map by $I_A$, and consider the 2-left finitely projective quadratic graded ring $qA = T_R(A_1)/(I_A)$. Let $B$ be the 2-right projectively quadratic graded ring quadratic dual to $qA$. Then we have an isomorphism of $R$-$R$-bimodules $\text{Hom}_{R^\text{op}}(B_1, R) \simeq A_1$, which we use as our choice of the map $\tau$. The assumptions of Lemma 2.5 are satisfied, and condition (c) holds by the construction of quadratic duality.

Therefore, the Koszul complex $K^r(B, A)$ is well-defined. In this particular case, we omit $\tau$ from the notation for the Koszul complex, as the structure isomorphism of quadratic duality is presumed.

### 2.6. Distributive collections of subobjects.

Let $C$ be an abelian category in which all the subobjects of any given object form a set, and let $W \in C$ be an object. A set $\Omega$ of subobjects of $W$ is said to be a **lattice of subobjects** if $0 \in \Omega$, $W \in \Omega$, and...
for any \( X, Y \in \Omega \) one has \( X \cap Y \in \Omega \) and \( X + Y \in \Omega \). Any lattice of subobjects is modular, i.e., one has \((X + Y) \cap Z = X + (Y \cap Z)\) whenever \( X, Y, Z \in \Omega \) and \( X \subseteq Z \). A lattice of subobjects \( \Omega \) is said to be distributive if the identity \((X + Y) \cap Z = X \cap (Y \cap Z)\) holds for all \( X, Y, Z \in \Omega \), or equivalently, the identity \((X + Y) \cap (X + Z) = X + (Y \cap Z)\) holds for all \( X, Y, Z \in \Omega \).

Let \( n \geq 2 \) be an integer and \( X_1, \ldots, X_{n-1} \subseteq W \) be a collection of \( n - 1 \) subobjects in \( W \). The lattice \( \Omega \) of subobjects of \( W \) generated by \( X_1, \ldots, X_{n-1} \) consists of all the subobjects of \( W \) that can be obtained from \( X_1, \ldots, X_{n-1} \) by applying iteratively the operations of finite sum and finite intersection.

The collection of subobjects \((X_1, \ldots, X_{n-1})\) in \( W \) is said to be distributive if the lattice \( \Omega \) of subobjects of \( W \) generated by \( X_1, \ldots, X_{n-1} \) is distributive. Any pair of subobjects \( X_1, X_2 (n = 3) \) forms a distributive collection, but a triple of subobjects \( X_1, X_2, X_3 (n = 4) \) does not need to be distributive. A collection of subobjects \((X_1, \ldots, X_{n-1})\) in \( W \) is said to be almost distributive if all its proper subcollections \((X_1, \ldots, \hat{X}_k, \ldots, X_{n-1})\), \( 1 \leq k \leq n - 1 \), are distributive.

The following two lemmas hold in any modular lattice \( \Omega \), but we state them for lattices of subobjects only.

**Lemma 2.6.** A triple of subobjects \( X, Y, Z \subseteq W \) is distributive if and only if \((X + Y) \cap Z = X \cap (Y \cap Z)\), and if and only if \((X + Y) \cap (X + Z) = X + (Y \cap Z)\). Any permutation of \( X, Y, Z \) replaces these equations by equivalent ones.

**Proof.** This is an easy exercise; see [5, Lemma 1] or [11, Lemma 6.1 in Chapter 1]. \( \square \)

**Lemma 2.7.** An almost distributive collection of subobjects \((X_1, \ldots, X_{n-1})\) in \( W \) is distributive if and only if, for every \( 2 \leq k \leq n - 2 \), the triple of subobjects \( X_1 + \cdots + X_{k-1}, X_k, X_{k+1} \cap \cdots \cap X_{n-1} \subseteq W \) is distributive.

**Proof.** This is the result of the paper [10], improving upon the previous work in [5]. See also [11, Theorem 6.3 in Chapter 1]. \( \square \)

For any collection of subobjects \( X_1, \ldots, X_{n-1} \subseteq W \), we consider the following three complexes in \( C \). The bar complex \( B_\ast = B_\ast(W; X_1, \ldots, X_{n-1}) \) has the form

\[
W \longrightarrow \bigoplus_t W/X_t \longrightarrow \bigoplus_{t_1 < t_2} W/(X_{t_1} + X_{t_2}) \longrightarrow \cdots \longrightarrow \bigoplus_{t_1 < \cdots < t_s} W/(X_{t_1} + \cdots + X_{t_s}) \longrightarrow \cdots \longrightarrow W/(X_1 + \cdots + X_{n-1}).
\]

Here the indices \( t_s \) range from 1 to \( n - 1 \). The leftmost term \( W \) is placed in the homological degree \( n \), the term with the summation over \( 1 \leq t_1 < \cdots < t_s \leq n - 1 \) is placed in the homological degree \( n - s \), and the rightmost term \( W/(X_1 + \cdots + X_{n-1}) \) is placed in the homological degree 1. The component of the differential acting from the direct summand \( W/(X_{t_1} + \cdots + X_{t_s}) \) in \( B_{n-s+1} \) to the direct summand \( W/(X_1 + \cdots + X_{n-1}) \) in \( B_{n-1} \) is the natural epimorphism taken with the sign \((-1)^{s-1}\).
The cobar complex $B^\bullet = B^\bullet(W; X_1, \ldots, X_{n-1})$ has the form
\begin{align*}
(20) \quad X_1 \cap \cdots \cap X_{n-1} & \longrightarrow \bigoplus_{1 \leq t_1 < \cdots < t_i \leq n-1} X_{t_1} \cap \cdots \cap X_{t_i} \longrightarrow \cdots \\
& \longrightarrow \bigoplus_{1 \leq t_1 < t_2 \leq n-1} X_{t_1} \cap X_{t_2} \longrightarrow \bigoplus_{1 \leq t \leq n} X_t \longrightarrow W.
\end{align*}
Here the leftmost term $X_1 \cap \cdots \cap X_{n-1}$ is placed in the cohomological degree 1, the term with the summation over $1 \leq t_1 < \cdots < t_i \leq n-1$ is placed in the cohomological degree $n-i$, and the rightmost term $W$ is placed in the cohomological degree $n$. The component of the differential acting from the direct summand $X_{t_1} \cap \cdots \cap X_{t_i}$ in $B^{n-i}$ to the direct summand $X_{t_1} \cap \cdots \cap X_{t_i}$ in $B^{n-i+1}$ is the natural monomorphism taken with the sign $(-1)^{i-1}$.

The Koszul complex $K_\bullet(W; X_1, \ldots, X_{n-1})$ is
\begin{align*}
(21) \quad X_1 \cap \cdots \cap X_{n-1} & \longrightarrow X_2 \cap \cdots \cap X_{n-1} \longrightarrow X_3 \cap \cdots \cap X_{n-1}/X_1 \\
& \longrightarrow \cdots \longrightarrow (X_{i+1} \cap \cdots \cap X_{n-1})/(X_1 + \cdots + X_{i-1}) \longrightarrow \cdots \\
& \longrightarrow X_{n-1}/(X_1 + \cdots + X_{n-3}) \longrightarrow W/(X_1 + \cdots + X_{n-2}) \longrightarrow W/(X_1 + \cdots + X_{n-1}).
\end{align*}
Here the notation is $Y/Z = Y/(Y \cap Z) = (Y + Z)/Z$. The term $X_1 \cap \cdots \cap X_{n-1}$ is placed in the homological degree $n$, and the term $W/(X_1 + \cdots + X_{n-1})$ is placed in the homological degree 0.

**Lemma 2.8.** Let $X_1, \ldots, X_{n-1}$ be an almost distributive collection of subobjects of an object $W \in \mathcal{C}$. Then the following conditions are equivalent:

(a) the collection $(X_1, \ldots, X_{n-1})$ is distributive;

(b) the Koszul complex $K_\bullet(W; X_1, \ldots, X_{n-1})$ is exact;

(c) the bar complex $B_\bullet(W; X_1, \ldots, X_{n-1})$ is exact everywhere except for its leftmost term $W$;

(c*) the cobar complex $B^\bullet(W; X_1, \ldots, X_{n-1})$ is exact everywhere except for its rightmost term $W$.

**Proof.** This is [11, Proposition 7.2 in Chapter 1]. The assertion in [11] is stated for collections of subspaces in vector spaces, but the same argument applies to subobjects of any object in an abelian category. \qed

Let $\mathcal{F} \subseteq \mathcal{C}$ be a class of objects closed under extensions and the passages to the kernels of epimorphisms. We will say that a lattice $\Omega$ of subobjects in an object $W$ is an $\mathcal{F}$-lattice if one has $Y/Z \in \mathcal{F}$ for any pair of subobjects $Y, Z \in \Omega$ such that $Z \subseteq Y$. In particular, existence of an $\mathcal{F}$-lattice of subobjects in $W$ implies that $W = W/0 \in \mathcal{F}$.

A collection of subobjects $X_1, \ldots, X_{n-1} \subseteq W$ is said to be $\mathcal{F}$-distributive if the lattice $\Omega$ of subobjects in $W$ generated by $X_1, \ldots, X_{n-1}$ is a distributive $\mathcal{F}$-lattice.

**Lemma 2.9.** A distributive collection of subobjects $X_1, \ldots, X_{n-1} \subseteq W$ is $\mathcal{F}$-distributive if and only if one has $W/(X_{t_1} + \cdots + X_{t_i}) \in \mathcal{F}$ for all $1 \leq t_1 < \cdots < t_i \leq n-1$, $0 \leq i \leq n-1$. 20
Proof. This is a generalization of [14, Lemma 11.4.3.2], and the same proof applies. Alternatively, one can argue by induction in \( n \) in the following way. For any three subobjects \( Z \subset Y \) and \( X \) in \( W \), there is a short exact sequence \( 0 \rightarrow (X \cap Y)/(X \cap Z) \rightarrow Y/Z \rightarrow (X + Y)/(X + Z) \rightarrow 0 \). Taking \( Y, Z \in \Omega \) and \( X = X_{n-1} \), we observe that the object \((X_{n-1} + Y)/(X_{n-1} + Z)\) belongs to \( F \) by the induction assumption applied to the collection of subobjects \((X_{t} + X_{n-1})/X_{n-1} \subset W/X_{n-1}, 1 \leq t \leq n - 2\). Since \( F \) is closed under extensions, it suffices to show that \((X_{n-1} \cap Y)/(X_{n-1} \cap Z) \in F \). Applying the same argument to \( X = X_{n-2}, \ldots \), we reduce the question to showing that the object \( X_{1} \cap X_{2} \cap \cdots \cap X_{n-1} \) belongs to \( F \). The argument finishes similarly to the proof in [14], by invoking Lemma 2.8(a) \( \Rightarrow \) (c) and the assumption that the class \( F \) is closed under the kernels of epimorphisms. \( \square \)

We will say that a lattice \( \Omega \) of subobjects in an object \( W \) is split if for any pair of subobjects \( Y, Z \in \Omega \) such that \( Z \subset Y \) we have that \( Z \) is a split subobject of \( Y \). Equivalently, \( \Omega \) is split if every \( Z \in \Omega \) is a split subobject of \( W \). A collection of subobjects \( X_1, \ldots, X_{n-1} \subset W \) is said to be split distributive if the lattice \( \Omega \) of subobjects in \( W \) generated by \( X_1, \ldots, X_{n-1} \) is split and distributive.

Lemma 2.10. A collection of subobjects \( X_1, \ldots, X_{n-1} \subset W \) is split distributive if and only if there exists a finite direct sum decomposition \( W = \bigoplus \eta W_\eta \) of the object \( W \) such that each of the subobjects \( X_1 \) is the sum of a set of subobjects \( W_\eta \).

Proof. The proof of [11, Proposition 7.1 (a) \( \Leftrightarrow \) (b) in Chapter 1] is applicable. \( \square \)

2.7. Collections of subbimodules. Let \( R \) and \( S \) be associative rings and \( W \) be an \( R\ltimes S \)-bimodule. Following the terminology of Section 2.6, we say that a collection of \( R\ltimes S \)-subbimodules \( X_1, \ldots, X_{n-1} \subset W \) is distributive if the lattice of \( R\ltimes S \)-subbimodules \( \Omega \) generated by \( X_1, \ldots, X_{n-1} \) in \( W \) is distributive. A collection of subbimodules \( X_1, \ldots, X_{n-1} \subset W \) is said to be left flat distributive if it is distributive and for every pair of subbimodules \( Y, Z \in \Omega \) such that \( Z \subset Y \) the quotient bimodule \( Y/Z \) is a flat left \( R \)-module.

Lemma 2.11. Let \( R, S, \) and \( T \) be associative rings, \( W \) be an \( R\ltimes S \)-bimodule, and \( U \) be an \( S\ltimes T \)-bimodule. Let \( X_1, \ldots, X_{n-1} \subset W \) be a left flat distributive collection of \( R\ltimes S \)-subbimodules, and let \( Y_1, \ldots, Y_{m-1} \subset U \) be a left flat distributive collection of \( S\ltimes T \)-subbimodules. Then

\[
X_1 \otimes_S U, \ldots, X_{n-1} \otimes_S U, W \otimes_S Y_1, \ldots, W \otimes_S Y_{m-1} \subset W \otimes_S U
\]

is a left flat distributive collection of \( R\ltimes T \)-subbimodules in the \( R\ltimes T \)-bimodule \( W \otimes_S U \).

Proof. First of all, the map \( X_i \otimes_S U \rightarrow W \otimes_S U \) induced by the inclusion \( X_i \rightarrow W \) is injective for every \( 1 \leq i \leq n-1 \), since \( U \) is a flat left \( S \)-module. The map \( W \otimes_S Y_j \rightarrow W \otimes_S U \) induced by the inclusion \( Y_j \rightarrow U \) is also injective for every \( 1 \leq j \leq m-1 \), since \( U/Y_j \) is a flat left \( S \)-module. So \( X_i \otimes_S U \) and \( W \otimes_S Y_j \) are indeed subbimodules in \( W \otimes_S U \).
Furthermore, arguing by induction in $n + m$, we can assume that our collection of $n + m - 2$ subbimodules in $W \otimes_S U$ is almost distributive. Up to a homological shift by $[-1]$, the bar complex $B_* = B_*(W \otimes_S U; X_1 \otimes_S U, \ldots, X_{m-1} \otimes_S U, W \otimes_S Y_1, \ldots, W \otimes_S Y_{m-1})$ is isomorphic to the tensor product of two bar complexes $B_* (W; X_1, \ldots, X_{m-1}) \otimes_S B_* (U; Y_1, \ldots, Y_{m-1})$. The only nonzero homology bimodule of the bar complex $B_* (U; Y_1, \ldots, Y_{m-1})$ is $Y_1 \cap \ldots \cap Y_{m-1} \subset U$, and it is a flat left $S$-module by assumption. So are all the terms of the bar complex $B_* (U; Y_1, \ldots, Y_{m-1})$. Thus the bar complex $B_*$ is exact everywhere except for its leftmost term $W \otimes_S U$. Applying Lemma 2.8 (c) $\Rightarrow$ (a), we can conclude that our collection of $n + m - 2$ subbimodules in $W \otimes_S U$ is distributive.

Finally, the quotient bimodule of $W \otimes_S U$ by the sum of any subset of $X_1 \otimes_S U, \ldots, X_{m-1} \otimes_S U$, $W \otimes_S Y_1, \ldots, W \otimes_S Y_{m-1}$ is isomorphic to the tensor product of the quotient bimodules of $W$ and $U$ by the sums of the respective subsets of $X_1, \ldots, X_{m-1}$ and $Y_1, \ldots, Y_{m-1}$. As the tensor product of an $R$-flat $R$-$S$-bimodule and an $S$-flat $S$-$T$-bimodule is an $R$-flat $R$-$T$-bimodule, any such quotient bimodule of $W \otimes_S U$ is a flat left $R$-module. It remains to apply Lemma 2.9 (for the class $F$ of $R$-flat $R$-$S$-bimodules in the abelian category $C$ of $R$-$S$-bimodules) in order to finish the proof of the lemma. □

**Lemma 2.12.** In the context of Lemma 2.11, for any two subbimodules $X', X'' \subset W$ belonging to the lattice of subbimodules generated by $X_1, \ldots, X_{m-1}$ in $W$ and any two subbimodules $Y', Y'' \subset U$ belonging to the lattice of subbimodules generated by $Y_1, \ldots, Y_{m-1}$ in $U$, the following equations for subbimodules in $W \otimes_S U$ hold:

(a) $(X' + X'') \otimes_S U = (X' \otimes_S U) + (X'' \otimes_S U)$;
(b) $(X' \cap X'') \otimes_S U = (X' \otimes_S U) \cap (X'' \otimes_S U)$;
(c) $W \otimes_S (Y' + Y'') = (W \otimes_S Y') + (W \otimes_S Y'')$;
(d) $W \otimes_S (Y' \cap Y'') = (W \otimes_S Y') \cap (W \otimes_S Y'')$;
(e) $X' \otimes_S Y' = (W \otimes_S Y') \cap (X' \otimes_S U)$;
(f) $(X' \cap X'') \otimes_S (Y' \cap Y'') = (X' \otimes_S Y') \cap (X'' \otimes_S Y'')$.

**Proof.** The maps $X' \otimes_S U \to W \otimes_S U$ and $W \otimes_S Y' \to W \otimes_S U$ induced by the inclusions $X' \to W$ and $Y' \to U$ are injective, as explained in the proof of Lemma 2.11. Furthermore, the map $X' \otimes_S Y' \to W \otimes_S U$ is injective as the composition of injective maps $X' \otimes_S Y' \to W \otimes_S Y' \to W \otimes_S U$ or $X' \otimes_S Y' \to X' \otimes_S U \to W \otimes_S U$. So these are indeed subbimodules in $W \otimes_S U$.

Now the equations (a) and (c) are obvious. The equation (b) holds since $U$ is flat left $S$-module. To prove (d), it suffices to consider the four-term exact sequence of flat left $S$-modules

$$0 \to Y' \cap Y'' \to Y' \oplus Y'' \to U \to U/(Y' + Y'') \to 0$$

and tensor it with $W$ over $S$ on the left.

To prove (e), one has to check that that the short sequence $0 \to X' \otimes_S Y' \to W \otimes_S U \to (W/X' \otimes_S U) \oplus (W \otimes_S U/Y')$ is left exact. This follows from exactness of the sequences $0 \to X' \otimes_S Y' \to W \otimes_S Y' \to (W/X') \otimes_S Y'$ and $0 \to W \otimes_S Y' \to$
Lemma 2.13. Let \( R \) and \( S \) be associative rings, \( W \) be an \( R-S \)-bimodule, and \( U \) be an \( S-T \)-bimodule. Let \( X_1, \ldots, X_{n-1} \subset W \) be a left projective distributive collection of \( S \)-subbimodules, and let \( Y_1, \ldots, Y_{m-1} \subset U \) be a left projective distributive collection of \( S \)-subbimodules. Then
\[
X_1 \otimes_S U, \ldots, X_{n-1} \otimes_S U, W \otimes_S Y_1, \ldots, W \otimes_S Y_{m-1} \subset W \otimes_S U
\]
is a left projective distributive collection of subbimodules in the \( R-T \)-bimodule \( W \otimes_S U \).

Proof. Similar to the proof of Lemma 2.11.

Lemma 2.14. Let \( X_1, \ldots, X_{n-1} \subset W \) be a left split distributive collection of subbimodules in an \( R-S \)-bimodule \( W \). Then \( \text{Hom}_R(W/X_1, R), \ldots, \text{Hom}_R(W/X_{n-1}, R) \subset \text{Hom}_R(W, R) \) is a right split distributive collection of subbimodules in the \( S-R \)-bimodule \( \text{Hom}_R(W, R) \). If the left \( R \)-module \( W \) is projective and finitely generated, then the above collection of subbimodules in \( \text{Hom}_R(W, R) \) is right projective distributive. In this case, the map \( W \ni Y \mapsto \text{Hom}_R(W/Y, R) \subset \text{Hom}_R(W, R) \) is an anti-isomorphism between the lattice of subbimodules in \( W \) generated by \( X_1, \ldots, X_{n-1} \) and the lattice of subbimodules in \( \text{Hom}_R(W, R) \) generated by \( \text{Hom}_R(W/X_1, R), \ldots, \text{Hom}_R(W/X_{n-1}, R) \) (i.e., a bijection of sets transforming the sums into the intersections and vice versa).

Proof. Follows easily from Lemma 2.10 (applied in the category of left \( R \)-modules \( C = R\text{-mod} \)).

2.8. Left flat Koszul rings. Let \( A = \bigoplus_{n=0}^\infty A_n \) be a nonnegatively graded ring with the degree-zero component \( R = A_0 \). The graded ring \( A \) is said to be left flat Koszul if one of the equivalent conditions of the next theorem is satisfied. (Right flat Koszul graded rings are defined similarly.)

Theorem 2.15. The following three conditions are equivalent:

(a) \( A_n \) is a flat left \( R \)-module for every \( n \geq 1 \) and \( \text{Tor}_{ij}^A(R, R) = 0 \) for all \( i \neq j \);
(b) \( A_n \) is a flat left \( R \)-module for every \( n \geq 1 \), the graded ring \( A \) is quadratic, and, setting \( V = A_1 \) and denoting by \( I \subset V \otimes_R V \) the kernel of the multiplication map \( A_1 \otimes_R A_1 \rightarrow A_2 \), for every \( n \geq 4 \) the collection of \( n - 1 \) submodules
\[
I \otimes_R V \otimes_R \cdots \otimes_R V, V \otimes_R I \otimes_R V \otimes_R \cdots \otimes_R V, \ldots, V \otimes_R \cdots \otimes_R V \otimes_R I
\]
in the $R$-$R$-bimodule $W = V^\otimes n$ is distributive;

c $A_1$ and $A_2$ are flat left $R$-modules, the graded ring $A$ is quadratic, and for every $n \geq 1$ the collection of $R$-$R$-submodules $V^\otimes R^{k-1} \otimes_R I \otimes_R V^\otimes R^{n-k-1}$, $1 \leq k \leq n-1$ (22) in the $R$-$R$-bimodule $W = V^\otimes n$ is left flat distributive.

Proof. First of all, by Proposition 2.4(b), condition (a) implies that $A$ is quadratic. So we can assume that $A = T_R(V)/(I)$. Furthermore, any collection of less than three subobjects is distributive; so the distributivity condition in (b) is trivial for $n \leq 3$. It is clear from the formula (1) from Section 1 that the left flat distributivity condition in (c) implies flatness of the left $R$-modules $A_n$ for all $n \geq 1$. This suffices to prove the implication (c) $\Rightarrow$ (b).

To deduce (b) $\Rightarrow$ (c), we observe that the quotient bimodule of $V^\otimes n$ by any subset of the submodules (22) is isomorphic to the tensor product $A_{j_1} \otimes_R \cdots \otimes_R A_{j_s}$ for some $j_1, \ldots, j_s \geq 1$, $j_1 + \cdots + j_s = n$. Flatness of the left $R$-modules $A_j$ implies flatness of such tensor products (as left $R$-modules), and it remains to invoke Lemma 2.9 for the abelian category $C$ of $R$-$R$-bimodules and the class $F \subset C$ of all $R$-$R$-bimodules that are flat as left $R$-modules.

It remains to prove the equivalence (a) $\iff$ (b). Arguing by induction in $n \geq 1$, we can assume that the collection of $j - 1$ submodules (22) in the $R$-$R$-bimodule $V^\otimes R$ is left flat distributive for all $1 \leq j \leq n - 1$ and $\text{Tor}^A_{i,j}(R, R) = 0$ for all $i \neq j \leq n - 1$. Under this induction assumption, we will prove the equivalence of conditions (a) and (b) for the fixed value of $j = n$.

By Lemma 2.11 (applied to the lattice of $k-1$ submodules in the $R$-$R$-bimodule $W = V^\otimes R^k$ and the lattice of $n-k-1$ submodules in the $R$-$R$-bimodule $U = V^\otimes R^{n-k}$, $1 \leq k \leq n - 1$), the induction assumption implies that the collection of $n - 1$ submodules (22) in the $R$-$R$-bimodule $V^\otimes R$ is almost distributive.

Finally, the bar complex (14) computing the $R$-$R$-bimodules $\text{Tor}^A_{i,n}(R, R)$ is isomorphic to the lattice bar-complex $B_n(W; X_1, \ldots, X_{n-1})$ (19) for the collection of $n - 1$ submodules $X_k = V^\otimes R^{k-1} \otimes_R I \otimes_R V^\otimes R^{n-k-1}$ in the $R$-$R$-bimodule $W = V^\otimes R$. Hence Lemma 2.8(a) $\iff$ (c) implies the desired equivalence (a) $\iff$ (b).

The construction of the Koszul complex in Section 2.5 was using double dualization: first we passed from $A_1$ to $B_1 = \text{Hom}_R(A_1, R)$, and then set $K_1(B, A) = \text{Hom}_{R^e}(B_1, A)$. Therefore, the assumption that $A_1$ is a finitely generated projective $R$-module was needed. The following alternative approach allows to produce the Koszul complex for any left flat Koszul ring.

Let $A$ be a left flat Koszul graded ring. Denote the kernel of the (surjective) multiplication map $A_1 \otimes_R A_1 \rightarrow A_2$ by $I_A \subset A_1 \otimes_R A_1$. Set $I_A^{(0)} = R$, $I_A^{(1)} = A_1$, $I_A^{(2)} = I_A$, and

$$I_A^{(n)} = \bigcap_{k=1}^{n-1} A_1 \otimes R^{k-1} \otimes_R I_A \otimes_R A_1 \otimes R^{n-k-1} \subset A_1 \otimes R.$$  

(23)

So $I_A^{(n)}$ is an $R$-$R$-submodule in $A_1 \otimes R$. 
The intersection of all the subbimodules indexed by \( k = 2, \ldots, n-1 \) (i.e., of all but the first one) in (23) is the subbimodule

\[
\bigcap_{k=2}^{n-1} A_1^{\otimes R k-1} \otimes_R I_A \otimes_R A_1^{\otimes R n-k-1} = A_1 \otimes_R I_A^{(n-1)} \subset A_1^{\otimes R n}
\]

by Lemma 2.12(d). Hence we obtain an injective \( R-R \)-bimodule morphism \( I_A^{(n)} \rightarrow A_1 \otimes_R I_A^{(n-1)} \), which is defined for all \( n \geq 1 \). For every \( n \geq 2 \), the image of the composition \( I_A^{(n)} \rightarrow A_1 \otimes_R I_A^{(n-1)} \rightarrow A_1 \otimes_R A_1 \otimes_R I_A^{(n-2)} \) is contained in the subbimodule \( I_A \otimes_R I_A^{(n-2)} \subset A_1 \otimes_R A_1 \otimes_R I_A^{(n-2)} \).

Put \( K_{i,n}(A) = A_{n-i} \otimes_R I_A^{(i)} \) for every \( i \geq 0 \), \( n \geq i \), and define the differential \( \partial: K_i(A) \rightarrow K_{i-1}(A) \) as the composition

\[
A_j \otimes_R I_A^{(i)} \rightarrow A_j \otimes_R A_1 \otimes_R I_A^{(i-1)} \rightarrow A_{j+1} \otimes_R I_A^{(i-1)}
\]

of the map \( A_j \otimes_R I_A^{(i)} \rightarrow A_j \otimes_R A_1 \otimes_R I_A^{(i-1)} \) induced by the map \( I_A^{(i)} \rightarrow A_1 \otimes_R I_A^{(i-1)} \) and the map \( A_j \otimes_R A_1 \otimes_R I_A^{(i-1)} \rightarrow A_{j+1} \otimes_R I_A^{(i-1)} \) induced by the multiplication map \( A_j \otimes_R A_1 \rightarrow A_{j+1} \). Since the composition \( I_A \rightarrow A_1 \otimes_R A_1 \rightarrow A_2 \) vanishes, we have \( \partial^2 = 0 \). So we obtain a complex of graded left \( A \)-modules \( K_\bullet(A) \) with the homological grading \( i \) and the internal grading \( n = i + j \).

**Proposition 2.16.** For any left flat Koszul graded ring \( A \), the Koszul complex \( K_\bullet(A) \) is a graded flat resolution of the left \( A \)-module \( R \).

**Proof.** The graded left \( A \)-modules \( A \otimes_R I_A^{(i)} \) are flat, because the left \( R \)-modules \( I_A^{(i)} \) are. Furthermore, for every \( n \geq 1 \) the internal degree \( n \) component of the Koszul complex \( K_\bullet(A) \) is isomorphic to the lattice Koszul complex \( K_\bullet(W; X_1, \ldots, X_{n-1}) \) (21) for the collection of \( n-1 \) subbimodules \( X_k = A_1^{\otimes R k-1} \otimes_R I_A \otimes_R A_1^{\otimes R n-k-1} \) in the \( R-R \)-bimodule \( W = A_1^{\otimes R n} \). Thus it remains to apply Lemma 2.8(a) \( \Rightarrow \) (b). \( \square \)

2.9. Finitely projective Koszul rings. Let \( A = \bigoplus_{n=0}^\infty A_n \) be a nonnegatively graded ring with the degree-zero component \( R = A_0 \). Assume that \( A_1 \) and \( A_2 \) are finitely generated projective left \( R \)-modules. When the multiplication map \( A_1 \otimes_R A_1 \rightarrow A_2 \) is surjective, we denote its kernel by \( I \subset V \otimes_R V \), where \( V = A_1 \), and consider the 2-left projective quadratic graded ring \( qA = T_R(V)/(I) \) together with its quadratic dual 2-right finitely projective quadratic graded ring \( B \).

As above in this paper, we denote by \( \text{Ext}_A(R,R) \) the bigraded \( \text{Ext} \) ring of the graded left \( A \)-module \( R \), and by \( \text{Ext}_{B^e}(R,R) \) the bigraded \( \text{Ext} \) ring of the graded right \( B \)-module \( R \). The graded ring \( A \) is said to be left finitely projective Koszul if one of the equivalent conditions of the next theorem is satisfied.

**Theorem 2.17.** For any nonnegatively graded ring \( A \), the following four conditions are equivalent:

(a) \( A_n \) is a finitely generated projective left \( R \)-module for every \( n \geq 1 \) and the graded ring \( A \) is left flat Koszul;
(b) $A_n$ is a finitely generated projective left $R$-module for every $n \geq 1$, the multiplication map $A_1 \otimes_R A_1 \to A_2$ is surjective, $B_n$ is a finitely generated projective right $R$-module for every $n \geq 1$, and $\text{Ext}^{i,j}_A(R, R) = 0$ for all $i \neq j$;

(c) the graded ring $A$ is quadratic, $A_1$ and $A_2$ are finitely generated projective left $R$-modules, and for every $n \geq 1$ the collection of $R$-$R$-subbimodules $V \otimes_R^{a,k-1} \otimes_R I \otimes_R V \otimes_R^{n-k-1}$, $1 \leq k \leq n-1$ in the $R$-$R$-bimodule $W = V \otimes_R^{n}$ is left projective distributive;

(d) $A_1$ and $A_2$ are finitely generated projective left $R$-modules, the multiplication map $A_1 \otimes_R A_1 \to A_2$ is surjective, $B_i$ is a finitely generated projective right $R$-module for every $i \geq 1$, and the Koszul complex $K_*(B, A)$ from Section 2.5 is exact in all the internal degrees $n \geq 1$.

Proof. (a) $\iff$ (c) Take the condition of Theorem 2.15(b) as the definition of left flat Koszulity and argue similarly to the proof of Theorem 2.15 (b) $\iff$ (c), using Lemma 2.9 for the abelian category $C$ of $R$-$R$-bimodules and the class $F \subset C$ of all $R$-$R$-bimodules that are projective as left $R$-modules.

(a) $\iff$ (b) Take the condition of Theorem 2.15(a) as the definition of left flat Koszulity. The $R$-$R$-bimodules $\text{Tor}^A_{i+n}(R, R)$ can be computed as the homology of the complex (14), while the $R$-$R$-bimodules $\text{Ext}^i_A(R, R)$ can be computed as the cohomology of the complex (17). Under (a), the graded ring $A$ is quadratic; so any one of the conditions (a) or (b) implies surjectivity of the multiplication map $A_1 \otimes_R A_1 \to A_2$. By Proposition 2.2, we have $\text{Ext}^n_{A,n}(R, R) \cong B_n$.

Now the following lemma shows that, given a nonnegatively graded algebra $A$ with the degree-zero component $R = A_0$ such that $A_j$ is a finitely generated projective left $R$-module for every $j \geq 1$, and given a fixed $n \geq 1$, one has $\text{Tor}^A_{i+n}(R, R) = 0$ for all $1 \leq i \leq n-1$ if and only if $\text{Ext}^i_A(R, R) = 0$ for all $1 \leq i \leq n-1$ and the right $R$-module $\text{Ext}^n_{A,n}(R, R)$ is projective.

**Lemma 2.18.** Let $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to 0$ be a complex of finitely generated projective left $R$-modules, and let $0 \to \text{Hom}_R(C_1, R) \to \cdots \to \text{Hom}_R(C_n, R) \to 0$ be the dual complex of finitely generated right $R$-modules. Then the following four conditions are equivalent:

(a) $H_i(C_\bullet) = 0$ for all $1 \leq i \leq n-1$;

(a') the complex of left $R$-modules $C_\bullet$ is homotopy equivalent to the one-term complex $H_n(C_\bullet)[n]$;

(b) $H^i(\text{Hom}_R(C_\bullet, R)) = 0$ for all $1 \leq i \leq n-1$ and $H^n(\text{Hom}_R(C_\bullet, R))$ is a (finitely generated) projective right $R$-module;

(b') the complex of right $R$-modules $\text{Hom}_R(C_\bullet, R)$ is homotopy equivalent to the one-term complex $H^n(\text{Hom}_R(C_\bullet, R))[-n]$.

If any one of these equivalent conditions holds, then $H_n(C_\bullet)$ is a finitely generated projective left $R$-module and $H^n(\text{Hom}_R(C_\bullet, R)) \cong \text{Hom}_R(H_n(C_\bullet), R)$. □

(c) $\implies$ (d) Under (c), the results of Lemmas 2.12 and 2.14 are available for the collections of subbimodules $V \otimes_R^{a,k-1} \otimes_R I \otimes_R V \otimes_R^{j-k-1}$ in the $R$-$R$-bimodules $V \otimes_R^{j}$, $1 \leq k \leq j-1$, allowing to compute the internal degree $n$ component of the
Koszul complex $K_*(B, A)$ as the lattice Koszul complex $K_*(W; X_1, \ldots, X_{n-1})$ for the collection of subbimodules $X_k = V \otimes R^{k-1} \otimes_R I \otimes_R V \otimes R^{n-k-1} \subset V \otimes_R W$. Then it remains to apply Lemma 2.8 (a) $\Rightarrow$ (b).

(d) $\implies$ (a), (b) Under (d), the Koszul complex $K_*(B, A)$ is a graded projective resolution of the graded left $A$-module $R$. Computing $\text{Tor}_A(R, R)$ and $\text{Ext}_A(R, R)$ in terms of this resolution immediately yields (a) and (b) (where (a) is interpreted as the condition of Theorem 2.15(a)).

The definition of right finitely projective Koszul graded ring is obtained from the above definition of a left finitely projective Koszul ring by switching the roles of the left and right sides.

**Proposition 2.19.** Let $A$ be a 2-left finitely projective quadratic graded ring with the degree-zero component $R = A_0$, and let $B$ be the 2-right finitely projective quadratic graded ring quadratic dual to $A$. Then the ring $A$ is left finitely projective Koszul if and only if the ring $B$ is right finitely projective Koszul. If this is the case, one has $B \simeq \text{Ext}_A(R, R)^{\text{op}}$ and $A \simeq \text{Ext}_B(R, R)^{\text{op}}$.

**Proof.** The first assertion is provable using the condition of Theorem 2.17(c) as the definition of projective Koszulity and the result of Lemma 2.14. Then, by Theorem 2.17(b), we have $\text{Ext}_A^{i,j}(R, R) = 0$ for $i \neq j$, and similarly, $\text{Ext}_B^{i,j}(R, R) = 0$ for $i \neq j$. Finally, the graded ring isomorphism $B^{\text{op}} \simeq \bigoplus_{n \geq 0} \text{Ext}_A^{n,n}(R, R)$ is provided by Proposition 2.2. The graded ring isomorphism $A \simeq \bigoplus_{n \geq 0} \text{Ext}_B^{n,n}(R, R)$ can be obtained by applying the same proposition to the ring $B^{\text{op}}$ and observing that the 2-left finitely projective quadratic graded ring $B^{\text{op}}$ is quadratic dual to the 2-right finitely projective quadratic graded ring $A^{\text{op}}$.

**Corollary 2.20.** The anti-equivalences of categories from Propositions 1.2–1.3 restrict to an anti-equivalence between the category of left finitely projective Koszul graded rings $A$ and the category of right finitely projective Koszul graded rings $B$ over any fixed base ring $R$.

### 3. Relative Nonhomogeneous Quadratic Duality

Nonhomogeneous quadratic rings $\tilde{A}$ can be informally described as rings defined by nonhomogeneous quadratic relations over a fixed base ring $R$. Not every system of nonhomogeneous quadratic relations is good enough to define a nonhomogeneous quadratic ring (see the general discussion in the introductions to [12, 11], [11, Section 5], and the counterexamples in [18, Section 5] and [12, Section 3.4]). For a system of nonhomogeneous quadratic relations to “make sense”, its coefficients must, in turn, satisfy a certain system of equations, called the self-consistency equations.

Dualizing the degree-one and degree-zero parts of the nonhomogeneous quadratic relations and imposing the equations dual to the self-consistency equations on the
relations’ coefficients produces a curved DG-ring structure \((B, d, h)\) on the dual quadratic graded ring \(B\) to the quadratic graded ring \(A\) defined by the homogeneous quadratic parts of the nonhomogeneous quadratic relations.

3.1. Nonhomogeneous quadratic rings. Let \(\tilde{A}\) be an associative ring with a subring \(R \subset \tilde{A}\). Consider the \(R\)-\(R\)-bimodule \(\tilde{A}/R\) and suppose that we have chosen a subbimodule \(V \subset \tilde{A}/R\). Denote by \(\tilde{V} \subset \tilde{A}\) the full preimage of \(V\) under the surjective \(R\)-\(R\)-bimodule morphism \(\tilde{A} \to \tilde{A}/R\).

Consider the tensor ring \(T_R(\tilde{V}) = \bigoplus_{n=0}^{\infty} T_{R,n}(\tilde{V})\), \(T_{R,n}(\tilde{V}) = \tilde{V} \otimes_n \tilde{V}\), as in Section 1. Then the \(R\)-\(R\)-bimodule morphism of identity inclusion \(\tilde{V} \to \tilde{A}\) extends uniquely to a ring homomorphism \(\pi_{\tilde{A}} : T_R(\tilde{V}) \to \tilde{A}\) forming a commutative triangle diagram with the subring inclusions \(R \to T_R(\tilde{V})\) and \(R \to \tilde{A}\).

Assume that the ring \(\tilde{A}\) is generated by its subgroup \(\tilde{V}\). In other words, the ring homomorphism \(\pi_{\tilde{A}} : T_R(\tilde{V}) \to \tilde{A}\) is surjective. Let \(\tilde{J}_\tilde{A} \subset T_R(\tilde{V})\) be the kernel of \(\pi_{\tilde{A}}\).

Define an increasing filtration \(F\) on the ring \(T_R(\tilde{V})\) by the rule \(F_n T_R(\tilde{V}) = \bigoplus_{i=0}^{n} T_{R,i}(\tilde{V}) \subset T_R(\tilde{V})\). Furthermore, put \(\tilde{F}_n \tilde{A} = \pi_{\tilde{A}}(F_n T_R(\tilde{V})) \subset \tilde{A}\). So we have \(F_0 T_R(\tilde{V}) = R\) and \(F_1 T_R(\tilde{V}) = R \oplus \tilde{V}\), hence \(F_0 \tilde{A} = R\) and \(F_1 \tilde{A} = \tilde{V} \subset \tilde{A}\).

Clearly, \(F\) is an exhaustive multiplicative filtration on the ring \(\tilde{A}\), that is \(\tilde{A} = \bigcup_{n \geq 0} F_n \tilde{A}\) and \(F_n \tilde{A} \cdot F_m \tilde{A} = F_{n+m} \tilde{A}\) holds for all \(n, m \geq 0\). (Sometimes we will say that the filtration \(F\) on \(\tilde{A}\) is generated by \(F_1 \tilde{A}\) over \(R\), which means that \(F\) is generated by \(F_1\) and \(F_0 \tilde{A} = R\).) The filtration \(F\) on the graded ring \(T_R(\tilde{V})\) has similar properties.

Put \(\tilde{I}_\tilde{A} = F_2 T_R(\tilde{V}) \cap \tilde{J}_{\tilde{A}}\); so \(\tilde{I}_\tilde{A}\) is an \(R\)-\(R\)-subbimodule in \(F_2 T_R(\tilde{V}) = R \oplus \tilde{V} \oplus \tilde{V} \otimes R^2\).

We will say that the ring \(\tilde{A}\) with a fixed subring \(R \subset \tilde{A}\) and a fixed \(R\)-\(R\)-subbimodule of generators \(R \subset \tilde{V} \subset \tilde{A}\) is weak nonhomogeneous quadratic over \(R\) if the ideal \(\tilde{J}_\tilde{A} \subset T_R(\tilde{V})\) is generated by its subgroup \(\tilde{I}_\tilde{A}\).

Let \(\prime \tilde{A}\) and \(\prime' \tilde{A}\) be two weak nonhomogeneous quadratic rings over the same base ring \(R\), and let \(R \subset \prime \tilde{V} \subset \prime \tilde{A}\) and \(R \subset \prime' \tilde{V} \subset \prime' \tilde{A}\) be their fixed subbimodules of generators. A morphism of weak nonhomogeneous quadratic rings \(f : \prime \tilde{A} \to \prime' \tilde{A}\) is a ring homomorphism forming a commutative triangle diagram with the subring inclusions \(R \to \prime \tilde{A}\) and \(R \to \prime' \tilde{A}\) and satisfying the condition that \(f(\prime \tilde{V}) \subset \prime' \tilde{V}\).

Conversely, suppose that we are given an associative ring \(\tilde{A}\) with an exhaustive multiplicative increasing filtration \(0 = F_{-1} \tilde{A} \subset F_0 \tilde{A} \subset F_1 \tilde{A} \subset F_2 \tilde{A} \subset \cdots\). Then \(R = F_0 \tilde{A}\) is a subring in \(\tilde{A}\). Suppose that the filtration \(F\) on \(\tilde{A}\) is generated by \(F_1\), and put \(\tilde{V} = F_1 \tilde{A}\). Then we have \(R \subset \tilde{V} \subset \tilde{A}\) and the ring \(\tilde{A}\) is generated by its subgroup \(\tilde{V}\).

Consider the associated graded ring \(gr^F \tilde{A} = \bigoplus_{n=0}^{\infty} gr^F_n \tilde{A}\), where \(gr^F_n \tilde{A} = F_n \tilde{A} / F_{n-1} \tilde{A}\). We will say that the filtered ring \(\tilde{A}\) is nonhomogeneous quadratic if the ring \(A = gr^F \tilde{A}\) is quadratic (in the sense of Section 1).
Let \( (\tilde{A}, F) \) and \( (\tilde{A}', F') \) be two nonhomogeneous quadratic rings with the same degree-zero filtration component \( F_0' \tilde{A} = R = F_0' \tilde{A} \). A morphism of nonhomogeneous quadratic rings \( f: \tilde{A} \to \tilde{A}' \) is a morphism of filtered rings (that is, a ring homomorphism such that \( f(F_n' \tilde{A}) \subset F_n' \tilde{A} \) for all \( n \geq 0 \)) forming a commutative triangle diagram with the subring inclusions \( R \to \tilde{A} \) and \( R \to \tilde{A}' \).

**Lemma 3.1.** The above construction assigning the submodule of generators \( \tilde{V} = F_1 \tilde{A} \) to a filtration \( F \) on a ring \( \tilde{A} \) defines a fully faithful functor from the category of nonhomogeneous quadratic rings to the category of weak nonhomogeneous quadratic rings (over any fixed base ring \( R \)). In particular, any nonhomogeneous quadratic ring is weak nonhomogeneous quadratic (so our terminology is consistent).

**Proof.** We will only prove the second assertion. Let \( (A, F) \) be a nonhomogeneous quadratic ring. We have to show that the ideal \( \tilde{I}_A \subset T_R(\tilde{V}) \) is generated by \( \tilde{I}_A = F_2 T_R(\tilde{V}) \cap \tilde{J}_A \). Indeed, denote the ideal generated by \( I_A \) by \( J' \subset T_R(\tilde{V}) \), and consider the ring \( \tilde{A}' = T_R(\tilde{V}) / J' \). Then \( J' \subset \tilde{J}_A \), so there is a unique surjective ring homomorphism \( \tilde{A}' \to \tilde{A} \) forming a commutative triangle diagram with the surjective ring homomorphisms \( T_R(\tilde{V}) \to A' \) and \( T_R(\tilde{V}) \to \tilde{A} \).

For every \( n \geq 0 \), denote by \( F_n A' \subset \tilde{A}' \) the image of the subgroup \( F_n T_R(\tilde{V}) \subset T_R(\tilde{V}) \) under the ring homomorphism \( T_R(\tilde{V}) \to \tilde{A}' \). Then \( F \) is an exhaustive multiplicative filtration on the ring \( A' \). Furthermore, the image of \( F_n A' \) under the ring homomorphism \( \tilde{A}' \to \tilde{A} \) coincides with \( F_n \tilde{A} \). As the maps \( F_n T_R(\tilde{V}) \to F_n \tilde{A} \) are isomorphisms for \( n = 0 \) and 1, it follows that so are the maps \( F_n A' \to F_n \tilde{A} \). Moreover, we have \( \tilde{I}_A \subset F_2 T_R(\tilde{V}) \cap J' \) by construction, hence \( F_2 T_R(\tilde{V}) \cap J' = F_2 T_R(\tilde{V}) \cap \tilde{J}_A \). Therefore, the map \( F_2 A' \to F_2 \tilde{A} \) is an isomorphism, too.

We need to show that \( J' = J_{\tilde{A}} \); equivalently, this means that the ring homomorphism \( A' \to \tilde{A} \) is an isomorphism. It suffices to check that the induced homomorphism of graded rings \( \text{gr}^F A' \to \text{gr}^F \tilde{A} \) is an isomorphism. Set \( A' = \text{gr}^F \tilde{A} \) and \( \tilde{A} = \text{gr}^F A \). The graded ring \( A' \) is generated by \( A_1' \) over \( A_0' = R \), since the filtration \( F \) on the ring \( A' \) is generated by \( F_1 \) by construction. The graded ring \( \tilde{A} \) is quadratic by assumption. The graded rings homomorphism \( A' \to \tilde{A} \) is an isomorphism in the degrees 0, 1, and 2, as we have shown. It remains to apply the next lemma. \( \square \)

**Lemma 3.2.** Let \( f: A' \to A \) be a homomorphism of nonnegatively graded rings such that the maps \( f_n: A'_n \to A_n \) are isomorphisms for \( n = 0, 1, \) and 2. Assume that the graded ring \( A' \) is generated by \( A'_1 \) over \( A'_0 \), while the graded ring \( A \) is quadratic. Then the map \( f \) is an isomorphism of graded rings.

**Proof.** Set \( A'_0 = R = A_0 \) and \( A'_1 = V = A_1 \). Then there is a unique graded ring homomorphism \( \pi_{A'}: T_R(V) \to A' \) acting by the chosen isomorphisms \( R \simeq A_0 \) and \( V \simeq A'_1 \) on the components of degrees 0 and 1, and a similar unique graded ring homomorphism \( \pi_A: T_R(V) \to A \). The triangle diagram of ring homomorphisms \( T_R(V) \to A' \to A \) is commutative, i.e., \( \pi_A = f \pi_{A'} \). Furthermore, both the graded
rings $A$ and $A'$ are generated by their degree-one components (over their degree-zero components) by assumption, hence both the maps $\pi_{A'}$ and $\pi_A$ are surjective.

Denote by $J_{A'}$ and $J_A \subset T_R(V)$ the kernels of the graded ring homomorphisms $\pi_{A'}$ and $\pi_A$. Then we have $J_{A'} \subset J_A$ and $J_{A',2} = J_{A,2}$, since the map $f_2: A'_2 \to A_2$ is an isomorphism by assumption. The algebra $A$ is quadratic, so the ideal $J_A$ is generated by $I_A = J_{A,2}$. It follows that $J_{A'} = J_A$, hence $f_n$ is an isomorphism for all $n$. □

**Remark 3.3.** We will see below in Section 4.6 that under the left finitely projective Koszulity assumption the classes of weak nonhomogeneous quadratic rings and non-homogeneous quadratic rings coincide. Specifically, if $\tilde{A}$ is a weak nonhomogeneous quadratic ring such that the quadratic graded ring $q\text{gr}^F\tilde{A}$ is left finitely projective Koszul (where the filtration $F$ on $\tilde{A}$ is generated by $F_1\tilde{A} = \tilde{V}$ over $F_0\tilde{A} = R$, as above), then the graded ring $\text{gr}^F\tilde{A}$ is quadratic (so the filtered ring $\tilde{A}$ is nonhomogeneous quadratic).

### 3.2. Curved DG-rings

A CDG-ring (curved differential graded ring) $B = (B, d, h)$ is a graded associative ring $B = \bigoplus_{n \in \mathbb{Z}} B^n$ endowed with a sequence of maps $d_n: B^n \to B^{n+1}$, $n \in \mathbb{Z}$, and an element $h \in B^2$ satisfying the following conditions:

(i) $d$ is an odd derivation of $B$, that is $d(bc) = d(b)c + (-1)^{|b|}bd(c)$ for all $b \in B^{[b]}$ and $c \in B^{[c]}$, $|b|, |c| \in \mathbb{Z}$;

(ii) $d^2(b) = [h, b]$ for all $b \in B$ (where $[h, b] = hb - bh$ is the commutator);

(iii) $d(h) = 0$.

In the context of the present paper, all CDG-rings will be nonnegatively graded, that is $B = \bigoplus_{n=0}^{\infty} B^n$. We denote the grading of $B$ by upper indices, because the differential $d$ has degree 1.

Let $'B = ('B, d', h')$ and $''B = (''B, d'', h'')$ be two CDG-rings. A morphism of CDG-rings $''B \to 'B$ is a pair $(f, a)$ consisting of a morphism of graded rings $f: ''B \to 'B$ and an element $a \in 'B^1$ such that

(iv) $f(d''(b)) = d'(f(b)) + [a, f(b)]$ for all $b \in ''B^{[b]}$ (where $[x, y] = xy - (-1)^{|x||y|}yx$, $x \in 'B^{[x]}$, $y \in ''B^{[y]}$, is the supercommutator); and

(v) $f(h'') = h' + d'(a) + a^2$.

The composition of morphisms is defined by the formula $(f, a)(g, b) = (fg, a + f(b))$. The identity morphism is the morphism $(id, 0)$. These rules define the category of CDG-rings. We will denote the category of nonnegatively graded CDG-rings $(B, d, h)$ with the fixed degree-zero component $B^0 = R$ by $R\text{-rings}_{cdg}$. Morphisms $''B \to 'B$ in $R\text{-rings}_{cdg}$ are CDG-ring morphisms $(f, a): ''B \to 'B$ such that the graded ring homomorphism $f: ''B \to 'B$ forms a commutative triangle diagram with the fixed isomorphisms $R \cong ''B^0$ and $R \cong 'B^0$.

The element $h \in B^2$ is called the curvature element. The element $a \in 'B^1$ is called the change-of-connection element.

For any CDG-ring $B = (B, d, h)$ and any element $a \in 'B^1$, the triple $'B = (B, d', h')$ with $d' = d + [a, -]$ and $h' = h + d(a) + a^2$ is also a CDG-ring. The CDG-rings $B$ and $'B$ are connected by the isomorphism $(id, a): 'B \to B$. Such isomorphisms
will be called *change-of-connection isomorphisms*, while CDG-ring morphisms of the form \((f, 0)\) will be called *strict morphisms*. Any morphism of CDG-rings \((f, a)\): "\(B = (\'B, d', h') \longrightarrow \'B = (B, d', h')\)" decomposes uniquely into a strict morphism followed by a change-of-connection isomorphism, \((f, a) = (\text{id}, a)(f, 0)\).

Furthermore, one can define the 2-category of CDG-rings as follows. Let \((f, a)\) and \((g, b)\): "\(B = (\'B, d'', h'') \longrightarrow \'B = (B, d', h')\)" be two CDG-ring morphisms with the same domain and codomain. A 2-morphism \((f, a) \xrightarrow{\sim} (g, b)\) is an invertible element \(z \in \'B^0\) satisfying the equations

\[
\begin{align*}
\text{(vi)} & \quad g(c) = zf(c)z^{-1} \text{ for all } c \in \'B; \text{ and} \\
\text{(vii)} & \quad b = za z^{-1} - d'(z)z^{-1}.
\end{align*}
\]

The element \(z \in \'B^0\) is called the *gauge transformation element*.

The vertical composition of two 2-morphisms \((f', a') \xrightarrow{w} (f'', a'') \xrightarrow{w} (f''' , a''' )\) is the 2-morphism \((f', a') \xrightarrow{z} (f'', a'') \xrightarrow{z} (f''' , a''' )\). The identity 2-morphism is the 2-morphism \((f, a) \xrightarrow{1} (f, a)\). The horizontal composition of two 2-morphisms \((g', b') \xrightarrow{w} (g'', b'')\): \((C, d_C, h_C) \longrightarrow (B, d_B, h_B)\) and \((f', a') \xrightarrow{z} (f'', a'')\): \((B, d_B, h_B) \longrightarrow (A, d_A, h_A)\) with the element \(w = z f'(w) = f''(w)z \in \'A^0\).

All the 2-morphisms of CDG-rings are invertible. If \((f, a)\): "\(B \longrightarrow \'B\)" is a morphism of CDG-rings and \(z \in \'B^0\) is an invertible element, then the pair \((g, b)\) defined by the formulas (vi–vii) is also a morphism of CDG-rings \((g, b)\): "\(B \longrightarrow \'B\)". The morphisms \((f, a)\) and \((g, b)\) are connected by the 2-isomorphism \((f, a) \xrightarrow{\sim} (g, b)\).

The 2-category \(\text{Rings}_{\text{cdg}2}\) of nonnegatively graded CDG-rings is defined as the following subcategory of the 2-category of CDG-rings (cf. Remark 3.1). The objects of \(\text{Rings}_{\text{cdg}2}\) are nonnegatively graded CDG-rings \((B, d, h)\), \(B = \bigoplus_{n=0}^{\infty} B^n\). Morphisms \((\"B, d', h'\) \longrightarrow (\'B, d'', h'')\) in \(\text{Rings}_{\text{cdg}2}\) are morphisms of CDG-rings \((f, a)\): \((\"B, d', h'\) \longrightarrow (\'B, d'', h'')\) such that the degree-zero component \(f_0\): \((\"B^0 \longrightarrow (\'B^0 \approx \'B'\). 2-morphisms \((f, a) \xrightarrow{z} (g, b)\) in \(\text{Rings}_{\text{cdg}2}\), between morphisms \((f, a)\) and \((g, b)\) belonging to \(\text{Rings}_{\text{cdg}2}\), are arbitrary 2-morphisms from \((f, a)\) to \((g, b)\) in the 2-category of CDG-rings.

3.3. **Self-consistency equations.** Let \(\tilde{A}\) be a weak nonhomogeneous quadratic ring over its subring \(R \subset \tilde{A}\) with the \(R\)-\(R\)-subbimodule of generators \(\tilde{V} \subset \tilde{A}\). Consider the related increasing filtration \(F\) on the ring \(\tilde{A}\), as constructed in Section 3.1, and let \(A = \text{gr}^F \tilde{A}\) be the associated graded ring. Then \(A\) is a nonnegatively graded ring generated by its degree-one component \(A_1 = \tilde{V} / R\) over the degree-zero component \(A_0 = R\).

Denote by \(I \subset V \otimes_R V\) the kernel of the multiplication map \(A_1 \otimes_R A_1 \longrightarrow A_2\), and consider the quadratic graded ring \(qA = T_R(V)/I\). Then we have a natural (adjunction) homomorphism of graded rings \(qA \longrightarrow A\), which is an isomorphism in the degrees \(n = 0, 1, 2\), and a surjective map in the degrees \(n \geq 3\). We will say that a weak nonhomogeneous quadratic ring \(A\) is *3-left finitely projective* if such is the quadratic graded ring \(qA = q \text{gr}^F \tilde{A}\).
Let \( \tilde{A} \) be a weak nonhomogeneous quadratic ring. In the rest of this section, we will assume that the \( R-R \)-bimodule \( V = \tilde{V}/R = A_1 \) is projective as a left \( R \)-module and the \( R-R \)-bimodule \( F_2A/F_1\tilde{A} = A_2 \) is flat as a left \( R \)-module.

Then, in particular, in the short exact sequence of \( R-R \)-bimodules \( 0 \rightarrow R \rightarrow \tilde{V} \rightarrow V \rightarrow 0 \) all the bimodules are projective as left \( R \)-modules. Therefore, this sequence splits as a short exact sequence of left \( R \)-modules, and we can choose a splitting \( \tilde{V} \rightarrow V \). Let \( V' \subset \tilde{V} \) be the image of \( V \) under such a splitting map; so \( V' \) is a left (but not a right) \( R \)-submodule in the \( R-R \)-bimodule \( \tilde{V} \subset \tilde{A} \) such that \( \tilde{V} = R \oplus V' \) as a left \( R \)-module. We will call \( V' \) a submodule of strict generators of \( \tilde{A} \).

In the rest of this section, we will identify \( V \) with \( V' \). Let \( v \in V \) and \( r \in R \) be two elements. Let \( rv \) and \( vr \in V \) denote the elements obtained by applying to \( v \in V \) the left and right action of the element \( r \in R \) in the \( R-R \)-bimodule \( \tilde{V} \), and let \( r \ast v \) and \( v \ast r \in \tilde{V} \) denote the products of the elements \( v \in V \simeq V' \subset \tilde{V} \subset \tilde{A} \) and \( r \in R \subset \tilde{A} \) in the ring \( \tilde{A} \). Then we have

\[
(24) \quad r \ast v = rv \quad \text{and} \quad v \ast r = vr + q(v, r) \in \tilde{V},
\]

where \( q(v, r) \in R \) is a certain uniquely defined element. The map \( q: V \times R \rightarrow R \) is obviously biadditive, so it can be (uniquely) extended to an abelian group homomorphism \( q: V \otimes R \rightarrow R \). Since \( R \) is a subring in \( \tilde{A} \), we also have \( r \ast s = rs \) (where the left-hand side denotes the product in \( \tilde{A} \) and the right-hand side is the product in \( R \)) for any pair of elements \( r, s \in R \).

Furthermore, let \( i \in I \) be an element. Consider the natural surjective map \( V \otimes R \rightarrow V \otimes_R V \) from the tensor product of two copies of \( V \) over the ring of integers \( \mathbb{Z} \) to their tensor product over \( R \), and denote by \( \hat{I} \subset V \otimes R V \) the full preimage of the subbimodule \( I \subset V \otimes_R V \) under the map \( V \otimes R V \rightarrow V \otimes_R V \). Let \( i \in \hat{I} \) denote some preimage of the element \( i \in I \) under the natural surjective map \( \tilde{I} \rightarrow I \).

The element \( i \in I \subset V \otimes_R V \) can be presented as a finite sum of decomposable tensors, \( i = \sum_{\alpha} i_{1,\alpha} \otimes_R i_{2,\alpha}, \ i_{1,\alpha}, \ i_{2,\alpha} \in V \). We will suppress the notation for the sum over \( \alpha \), and write simply \( i = i_1 \otimes i_2 \). Similarly, the element \( i \in \hat{I} \subset V \otimes R V \) can be presented as a finite sum \( i = \sum_{\alpha} \hat{i}_{1,\alpha} \otimes \hat{i}_{2,\alpha}, \ hat{i}_{1,\alpha}, \ hat{i}_{2,\alpha} \in V \). Once again, we will omit the notation for the sum over \( \alpha \), and write \( i = \hat{i}_1 \otimes \hat{i}_2 \). All our formulas will be biadditive in \( \hat{i}_1 \) and \( \hat{i}_2 \) (or, as may be the case, appropriately \( R \)-bilinear in \( \hat{i}_1 \) and \( \hat{i}_2 \)), so such simplification of the notation will be harmless.

For any two elements \( u \) and \( v \in V \), we identify \( u \) and \( v \) with their images under the embedding \( V \simeq V' \hookrightarrow \tilde{V} \subset \tilde{A} \) and consider their product \( u \ast v \in \tilde{A} \) in the ring \( \tilde{A} \). The assignment of the element \( u \ast v \) to a pair of elements \( u \) and \( v \) can be uniquely extended to well-defined homomorphism of abelian groups (or left \( R \)-modules) \( V \otimes R V \rightarrow \tilde{A} \), which does not, generally speaking, factorize through \( V \otimes_R V \).

In particular, for any element \( i = i_1 \otimes i_2 \in I \) and any its preimage \( \hat{i} = \hat{i}_1 \otimes \hat{i}_2 \in \hat{I} \), we consider the element \( \hat{i}_1 \ast \hat{i}_2 \in \tilde{A} \). We have \( \hat{i}_1, \hat{i}_2 \in V' \subset V = F_1\tilde{A} \), so \( \hat{i}_1 \ast \hat{i}_2 \in F_2\tilde{A} \). Moreover, the image of the element \( \hat{i}_1 \ast \hat{i}_2 \) under the natural surjection \( F_2\tilde{A} \twoheadrightarrow F_2\tilde{A}/F_1\tilde{A} \) is equal to the product \( i_1i_2 \) computed in the associated graded
ring $A = \text{gr}^F \hat{A}$. Now we have $i_1 i_2 = 0$ in $F_2 \hat{A}/F_1 \hat{A} = A_2$, since the composition $I \rightarrow V \otimes_R V \rightarrow A_2$ vanishes by construction. Thus $i_1 i_2 \in F_1 \hat{A} = \hat{V} \subset \hat{A}$. Therefore, there exist uniquely defined elements $p(i) \in V$ and $h(i) \in R$ such that
\begin{equation}
  i_1 i_2 = p(i) - h(i) \in \hat{V} \subset \hat{A},
\end{equation}
where the element $p(i) \in V$ is identified with its image under the splitting $V \simeq V' \hookrightarrow \hat{V}$ and the element $h(i) \in R$ is identified with its image under the subring inclusion $R \hookrightarrow \hat{V} \subset \hat{A}$.

Furthermore, denote by $I^{(3)}$ the intersection $I \otimes_R V \cap V \otimes_R I \subset V \otimes_R V \otimes_R V$. Notice that, since the left $R$-modules $V$ and $A_2 = (V \otimes_R V)/I$ are flat by assumption, the tensor products $I \otimes_R V$ and $V \otimes_R I$ are subbimodules in the triple tensor product $V \otimes_R V \otimes_R V$ (see the proofs of Propositions 1.3 and 2.1).

We are also interested in “the intersection $\hat{I} \otimes V \cap V \otimes \hat{I} \subset V \otimes V \otimes V$”, but here we cannot claim that $\hat{I} \otimes V$ and $V \otimes \hat{I}$ are subgroups in $V \otimes V \otimes V$. So the intersection as such is not well-defined. Instead, we have two abelian group homomorphisms $\hat{I} \otimes V \rightarrow V \otimes V \otimes V$ and $V \otimes \hat{I} \rightarrow V \otimes V \otimes V$ induced by the inclusion $\hat{I} \rightarrow V \otimes V$. We denote by $\hat{I}^{(3)}$ the fibered product of the abelian groups $\hat{I} \otimes V$ and $V \otimes \hat{I}$ over the group $V \otimes V \otimes V$.

**Lemma 3.4.** The natural surjections $\hat{I} \otimes V \rightarrow I \otimes V$, $V \otimes \hat{I} \rightarrow V \otimes I$, and $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ induce a surjective map $\hat{I}^{(3)} \rightarrow I^{(3)}$.

**Proof.** Denote by $\hat{I} \otimes V$ and $V \otimes \hat{I}$ the images of the maps $\hat{I} \otimes V \rightarrow V \otimes V \otimes V$ and $V \otimes \hat{I} \rightarrow V \otimes V \otimes V$, and let $\hat{I}^{(3)} \subset V \otimes V \otimes V$ stand for the intersection $\hat{I} \otimes V$ and $V \otimes \hat{I}$. Then there is a natural surjective map $\hat{I}^{(3)} \rightarrow \hat{I}^{(3)}$. The surjection $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ restricts to a map $\hat{I}^{(3)} \rightarrow I^{(3)}$, and the map $\hat{I}^{(3)} \rightarrow I^{(3)}$ is equal to the composition $\hat{I}^{(3)} \rightarrow \hat{I}^{(3)} \rightarrow I^{(3)}$. It remains to prove that the map $\hat{I}^{(3)} \rightarrow I^{(3)}$ is surjective.

We have a short exact sequence of abelian groups $0 \rightarrow \hat{I} \rightarrow V \otimes V \rightarrow A_2 \rightarrow 0$. Hence the subgroups $X_1 = \hat{I} \otimes V$ and $X_2 = V \otimes \hat{I} \subset V \otimes V \otimes V$ are the kernels of the surjective maps $V \otimes V \otimes V \rightarrow A_2 \otimes V$ and $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$, respectively. Denote by $Y_1$ and $Y_2 \subset V \otimes V \otimes V$ the kernels of the natural surjective maps $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ and $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$. Then we have $X_1 \subset Y_1$ and $X_2 \subset Y_2 \subset V \otimes V \otimes V$.

Furthermore, $Y_1 + Y_2$ is the kernel of the natural surjective $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$. Similarly, $X_1 + Y_2$ is the kernel of the surjective map $V \otimes V \otimes V \rightarrow A_2 \otimes V$, and $Y_1 + X_2$ is the kernel of the surjective map $V \otimes V \otimes V \rightarrow V \otimes A_2$. It follows that the subbimodule $I \otimes V \subset V \otimes V \otimes V$ is the image of $X_1 \subset V \otimes V \otimes V$ under the map $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$, and the subbimodule $V \otimes I \subset V \otimes V \otimes V$ is the image of $X_2 \subset V \otimes V \otimes V$ under the same map.
The assertion that the map $\tilde{T}^{(3)} \to I^{(3)}$ is surjective is now expressed by the distributivity equation

$$X_1 \cap X_2 + (Y_1 + Y_2) = (X_1 + Y_2) \cap (Y_1 + X_2)$$
on subgroups in the abelian group $W = V \otimes_\mathbb{Z} V \otimes_\mathbb{Z} V$. What we have here is two filtrations $0 \subset Y_1 \subset X_1 \subset W$ and $0 \subset Y_2 \subset Y_2 \subset W$ of an abelian group $W$. It remains to observe that any two filtrations of an abelian category object generate a distributive lattice of its subobjects, which is a particular case of [5, Theorem 5] or [11, Corollary 6.4 in Chapter 1].

Let $j \in I^{(3)}$ be an element and $\hat{j} \in \hat{T}^{(3)}$ be one of its preimages. The element $j \in I^{(3)} \subset V^{\otimes 3}$ can be presented as a finite sum of decomposable tensors, $j = \sum_\alpha j_{1,\alpha} \otimes R j_{2,\alpha} \otimes R j_{3,\alpha}$. As above, we will suppress the notation for the sum over $\alpha$, and write simply $j = j_1 \otimes j_2 \otimes j_3$. Moreover, the image of the element $j \in \hat{T}^{(3)}$ under the natural map $\hat{T}^{(3)} \to V^{\otimes 3}$ can be presented as a finite sum $\sum_\alpha \hat{j}_{1,\alpha} \otimes \hat{j}_{2,\alpha} \otimes \hat{j}_{3,\alpha}$, where $\hat{j}_{1,\alpha}, \hat{j}_{2,\alpha}, \hat{j}_{3,\alpha} \in V$. We will write simply $\hat{j} = \hat{j}_1 \otimes \hat{j}_2 \otimes \hat{j}_3$, omitting the notation for the sum over $\alpha$ and ignoring the distinction between an element of $\hat{T}^{(3)}$ and its image in $V^{\otimes 3}$ in this notation.

For any three elements $u$, $v$, and $w \in V$, we will identify $u$, $v$, and $w$ with their images under the embedding $V \simeq V' \hookrightarrow \hat{V} \subset \hat{A}$ and consider the triple product $(u * v) * w = u * v * w = u * (v * w) \in \hat{A}$ in the ring $\hat{A}$. In particular, for any element $j \in \hat{T}^{(3)}$, the element $j_1 \otimes j_2 \otimes j_3 \in \hat{A}$ is well-defined.

**Proposition 3.5.** Let $\hat{A}$ be a weak nonhomogeneous quadratic ring such that the left $R$-module $V = A_1$ is projective and the left $R$-module $A_2$ is flat. Suppose that a left $R$-linear splitting $V \simeq V' \hookrightarrow \hat{V}$ of the surjective $R$-$R$-bimodule map $V \to \hat{V} / R = V$ has been chosen. Then the maps $q: V \times R \to R$, $p: \hat{T} \to V$, and $h: \hat{T} \to R$ defined above in (24–25) satisfy the following self-consistency equations:

\begin{itemize}
  \item[(a)] $q(rv, s) = rq(v, s)$ for all $r, s \in R$ and $v \in V$;
  \item[(b)] $q(v, rs) = q(rv, s) + q(v, r)s$ for all $r, s \in R$ and $v \in V$;
  \item[(c)] $p(r\hat{i}_1 \otimes \hat{i}_2) = rp(\hat{i}_1 \otimes \hat{i}_2)$ for all $r \in R$ and $\hat{i} \in \hat{T}$;
  \item[(d)] $h(r\hat{i}_1 \otimes \hat{i}_2) = rh(\hat{i}_1 \otimes \hat{i}_2)$ for all $r \in R$ and $\hat{i} \in \hat{T}$;
  \item[(e)] $p(u \otimes rv - ur \otimes v) = q(u, r)v$ for all $r \in R$ and $u, v \in V$;
  \item[(f)] $h(u \otimes rv - ur \otimes v) = 0$ for all $r \in R$ and $u, v \in V$;
  \item[(g)] $p(\hat{i}_1 \otimes \hat{i}_2 r) = p(\hat{i}_1 \otimes \hat{i}_2) - \hat{i}_1 q(\hat{i}_2, r)$ for all $r \in R$ and $\hat{i} = \hat{i}_1 \otimes \hat{i}_2 \in \hat{T}$;
  \item[(h)] $h(\hat{i}_1 \otimes \hat{i}_2 r) = h(\hat{i}_1 \otimes \hat{i}_2) - q(p(\hat{i}_1 \otimes \hat{i}_2), r) + q(\hat{i}_1, q(\hat{i}_2, r))$ for all $r \in R$ and $\hat{i} \in \hat{T}$;
  \item[(i)] $p(j_1 \otimes j_2 \otimes j_3) \in \hat{T} \subset V \otimes_\mathbb{Z} V$ for all $j = j_1 \otimes j_2 \otimes j_3 \in \hat{T}^{(3)}$;
  \item[(j)] $p(p(j_1 \otimes j_2 \otimes j_3) - j_1 \otimes p(j_2 \otimes j_3)) = (j_1 \otimes j_2)j_3 - j_1h(j_2 \otimes j_3)$ for all $j \in \hat{T}^{(3)}$;
  \item[(k)] $h(p(j_1 \otimes j_2 \otimes j_3) - j_1 \otimes p(j_2 \otimes j_3)) = q(j_1, h(j_2 \otimes j_3))$ for all $j \in \hat{T}^{(3)}$.
\end{itemize}

**Proof.** All these equations follow, in one way or another, from the associativity of multiplication in the ring $\hat{A}$. The specific computations proving each of the formulas are presented below one by one.
Part (a): compare \( r * v * s = (r * v) * s = (rv) * s = rvs + q(rv, s) \) with \( r * v * s = r * (vs + q(v, s)) = rvs + rq(v, s) \).

Part (b): compare \( v * r * s = v * (r * s) = v * (rs) = vrs + q(v, rs) \) with \( v * r * s = (v * r) * s = (vr + q(v, r)) * s = vrs + q(vr, s) + q(v, r)s \).

Parts (c) and (d): notice first of all that \( \sum r_i \hat{a}_1 \otimes \hat{a}_2, \alpha \in \widetilde{I} \) whenever \( r \in R \) and \( \sum \hat{a}_1 \otimes \hat{a}_2, \alpha \in \widetilde{I} \). So the left-hand sides of both the equations are well-defined. To deduce the equations, compare

\[
\begin{align*}
  r * \hat{i}_1 * \hat{i}_2 &= (r * \hat{i}_1) * \hat{i}_2 = p(r \hat{i}_1 \otimes \hat{i}_2) - h(r \hat{i}_1 \otimes \hat{i}_2)
\end{align*}
\]

with

\[
\begin{align*}
  r * \hat{i}_1 * \hat{i}_2 &= r * (\hat{i}_1 * \hat{i}_2) = r * (p(\hat{i}_1 \otimes \hat{i}_2) - h(\hat{i}_1 \otimes \hat{i}_2)) = rp(\hat{i}_1 \otimes \hat{i}_2) - rh(\hat{i}_1 \otimes \hat{i}_2)
\end{align*}
\]

and equate the terms belonging to \( V \simeq V' \subset \widetilde{V} \) separately and the terms belonging to \( R \subset \widetilde{V} \) separately.

Parts (e) and (f): first of all, one has \( u \otimes rv - ur \otimes v = \hat{I} \) for all \( r \in R \) and \( u, v \in V \). So the left-hand sides of the equations are well-defined. Furthermore,

\[
\begin{align*}
  0 &= u * (r * v) - (u * r) * v = u * (rv) - ur * v - q(u, r) * v \\
  &= p(u \otimes rv - ur \otimes v) - h(u \otimes rv - ur \otimes v) - q(u, r)v
\end{align*}
\]

and it remains to equate separately the terms belonging to \( V' \) and to \( R \).

Parts (g) and (h): we have \( \sum r_i \hat{a}_1 \otimes \hat{a}_2, \alpha \in \widetilde{I} \) whenever \( r \in R \) and \( \sum \hat{a}_1 \otimes \hat{a}_2, \alpha \in \widetilde{I} \); so the left-hand sides of both the equations are well-defined. Now compare

\[
\begin{align*}
  \hat{i}_1 * \hat{i}_2 * r &= (\hat{i}_1 * \hat{i}_2) * r = p(\hat{i}_1 \otimes \hat{i}_2) - h(\hat{i}_1 \otimes \hat{i}_2)) * r \\
  &= p(\hat{i}_1 \otimes \hat{i}_2)r + q(p(\hat{i}_1 \otimes \hat{i}_2), r) - h(\hat{i}_1 \otimes \hat{i}_2)r
\end{align*}
\]

with

\[
\begin{align*}
  \hat{i}_1 * \hat{i}_2 * r &= \hat{i}_1 * (\hat{i}_2 * r) = \hat{i}_1 * (\hat{i}_2r + q(\hat{i}_2, r)) \\
  &= p(\hat{i}_1 \otimes \hat{i}_2r) - h(\hat{i}_2 \otimes \hat{i}_2r) + \hat{i}_1q(\hat{i}_2, r) + q(\hat{i}_1, q(\hat{i}_2, r))
\end{align*}
\]

and equate separately the terms belonging to \( V' \) and to \( R \).

Parts (i–k): given an element \( j \in \hat{I}^{(3)} \), we have \( (j_1 * j_2) * j_3 = j_1 * j_2 * j_3 = \hat{j}_1 * (j_2 \hat{j}_3) \) in \( F_3 \hat{A} \subset \hat{A} \). By construction, the value of this triple product in \( A \) only depends on the image of the element \( j \) in the group \( V \otimes_2 V \otimes_2 V \). We will compute the value of \( (j_1 * j_2) \hat{j}_3 \) in terms of the image of \( j \) in \( \hat{I} \otimes_2 V \) and the value of \( \hat{j}_1 * (j_2 * \hat{j}_3) \) in terms of the image of \( j \) in \( V \otimes_2 \hat{I} \), and then equate the two expressions.

Specifically, we have

\[
(j_1 * j_2) \hat{j}_3 = p(j_1 \otimes j_2) * \hat{j}_3 - h(j_1 \otimes j_2) * j_3
\]

and

\[
\hat{j}_1 * (j_2 * \hat{j}_3) = \hat{j}_1 * p(j_2 \otimes \hat{j}_3) - \hat{j}_1 * h(j_2 \otimes \hat{j}_3).
\]

hence

\[
p(j_1 \otimes j_2) * \hat{j}_3 - \hat{j}_1 * p(j_2 \otimes \hat{j}_3) = h(j_1 \otimes j_2) * \hat{j}_3 - \hat{j}_1 * h(j_2 \otimes \hat{j}_3).
\]
Now, first of all, the right-hand side of (26) belongs to $\tilde{V} \subset \tilde{A}$, hence so does the left-hand side. Both summands in the left-hand side belong to $F_2\tilde{A}$. So the image of the left-hand side in $A_2 = F_2A/F_1A$ has to vanish, which means that the expression $p(\hat{j}_1 \otimes \hat{j}_2) \otimes_R \hat{j}_3 - \hat{j}_1 \otimes_R p(\hat{j}_2 \otimes \hat{j}_3)$ belongs to $I \subset V \otimes_R V$. This proves part (i).

It remains to compute both sides of (26) as

$$p(\hat{j}_1 \otimes \hat{j}_2) \ast \hat{j}_3 - \hat{j}_1 \ast p(\hat{j}_2 \otimes \hat{j}_3) = p(p(\hat{j}_1 \otimes \hat{j}_2) \otimes \hat{j}_3 - \hat{j}_1 \otimes p(\hat{j}_2 \otimes \hat{j}_3)) - h(p(\hat{j}_1 \otimes \hat{j}_2) \otimes \hat{j}_3 - \hat{j}_1 \otimes p(\hat{j}_2 \otimes \hat{j}_3))$$

and

$$h(\hat{j}_1 \otimes \hat{j}_2) \ast \hat{j}_3 - \hat{j}_1 \ast h(\hat{j}_2 \otimes \hat{j}_3) = h(\hat{j}_1 \otimes \hat{j}_2) \hat{j}_3 - \hat{j}_1 h(\hat{j}_2 \otimes \hat{j}_3) - q(\hat{j}_1, h(\hat{j}_2 \otimes \hat{j}_3)).$$

Comparing and equating separately the terms belonging to $V'$ and to $R$ produces the desired formulas (j–k).

3.4. The CDG-ring corresponding to a nonhomogeneous quadratic ring.
Let $R$ and $S$ be associative rings, $U$ be an $R$-$S$-bimodule, and $U^\vee = \text{Hom}_R(U, R)$ be the dual $S$-$R$-bimodule. We will use the notation

$$\langle u, f \rangle = f(u) \in R \quad \text{for any } u \in U \text{ and } f \in U^\vee.$$

Then the condition that $f: U \to R$ is a left $R$-module homomorphism is expressed by the identity

$$\langle ru, f \rangle = r\langle u, f \rangle \quad \text{for all } r \in R, \ u \in U, \text{ and } f \in U^\vee,$$

while the construction of the left $S$-module structure on $U^\vee$ is expressed by the identity

$$\langle u, sf \rangle = \langle us, f \rangle \quad \text{for all } u \in U, \ s \in S, \text{ and } f \in U^\vee,$$

and the construction of the right $R$-module structure on $U^\vee$ is expressed by

$$\langle u, fr \rangle = \langle u, f \rangle r \quad \text{for all } u \in U, \ f \in U^\vee, \text{ and } r \in R.$$

Furthermore, given three rings $R$, $S$, and $T$, an $R$-$S$-bimodule $U$, and an $S$-$T$-bimodule $V$, the construction of the natural homomorphism of $T$-$R$-bimodules

$$\text{Hom}_S(V, S) \otimes_S \text{Hom}_R(U, R) \longrightarrow \text{Hom}_R(U \otimes_S V, R),$$

from Lemma 1.1(a) can be expressed by the formula

$$\langle u \otimes v, g \otimes f \rangle = \langle u(v, g), f \rangle = \langle u, \langle v, g \rangle f \rangle$$

for all $u \in U, \ v \in V, \ g \in \text{Hom}_S(V, S)$, and $f \in \text{Hom}_R(U, R)$.

**Proposition 3.6.** Let $\tilde{A}$ be a 3-left finitely projective weak nonhomogeneous quadratic ring over its subring $R \subset \tilde{A}$ with the $R$-$R$-bimodule of generators $R \subset \tilde{V} \subset \tilde{A}$. Denote by $B$ the 3-right finitely projective quadratic graded ring quadratic dual to the 3-left finitely projective quadratic graded ring $qA = q\text{gr}^F \tilde{A}$. Suppose that a left $R$-linear splitting $V \simeq V' \hookrightarrow \tilde{V}$ of the surjective $R$-$R$-bimodule map $\tilde{V} \longrightarrow \tilde{V}/R = V$ has been chosen. Then the formulas

$$\langle v, d_0(r) \rangle = q(v, r)$$

(27)
(28) \[ \langle i, d_1(b) \rangle = \langle p(\hat{i}_1 \otimes \hat{i}_2), b \rangle - q(\hat{i}_1, \langle \hat{i}_2, b \rangle) \]

for all \( r \in R, \ v \in V, \ i \in I, \) and \( b \in B^1, \) where the maps \( q \) and \( p \) are given by \((24-25),\) specify well-defined abelian group homomorphisms \( d_0: B^0 \rightarrow B^1 \) and \( d_1: B^1 \rightarrow B^2. \) Furthermore, the map \( h: \hat{I} \rightarrow R \) descends uniquely to a well-defined left \( R \)-linear map \( I \rightarrow R, \) providing an element \( h \in \text{Hom}_R(\hat{I}, R) = B^2. \) The maps \( d_0 \) and \( d_1 \) satisfy the equations

(a) \( d_0(rs) = d_0(r)s + rd_0(s) \) for all \( r, s \in R; \)
(b) \( d_1(rb) = d_0(r)b + rd_1(b) \) for all \( r \in R, \ b \in B^1; \)
(c) \( d_1(br) = d_1(b)r - bd_0(r) \) for all \( r \in R, \ b \in B^1; \)
(d) \( d_1(d_0(r)) = hr - rh \) for all \( r \in R; \)
(e) \( \sum_{\alpha} d_1(e_{1,\alpha})e_{2,\alpha} - \sum_{\alpha} e_{1,\alpha}d_1(e_{2,\alpha}) = 0 \) in \( B^3 \) for all tensors \( e = \sum_{\alpha} e_{1,\alpha} \otimes_R e_{2,\alpha} \in B^1 \otimes_R B^1 \) such that the image of \( e \) vanishes in \( B^2. \)

The formula

\[ d_2(e_1e_2) = d_1(e_1)e_2 - e_1d_1(e_2) \]

for all \( e_1, e_2 \in B^1 \)

specifies a well-defined abelian group homomorphism \( d_2: B^2 \rightarrow B^3, \) which satisfies the equations

(f) \( d_2(d_1(b)) = hb - bh \) for all \( b \in B^1; \) and

(g) \( d_2(h) = 0. \)

In other words, the maps \( d_0 \) and \( d_1 \) admit a unique extension to an odd derivation \( d: B \rightarrow B \) of degree 1, and the triple \((B, d, h)\) is a CDG-ring.

Proof. Recall that, by the definition of quadratic duality, we have \( B^0 = R, \ B^1 = \text{Hom}_R(V, R), \) and \( B^2 = \text{Hom}_R(\hat{I}, R) \) (where \( \hat{I} \) is the kernel of the surjective multiplication map \( A_1 \otimes_R A_1 \rightarrow A_2 \simeq (qA)_2; \) the latter isomorphism holds since the graded ring \( A \) is generated by \( A_1 \) over \( R = A_0. \) The grading on the ring \( B \) was denoted by lower indices in Sections 1–2, but we denote it by upper indices here; the convention is \( B^n = B_n \) for all \( n \geq 0 \) (and \( B^n = 0 = B_n \) for \( n < 0). \)

Firstly we have to check that the maps \( d_0 \) and \( d_1 \) are well-defined by the formulas \((27-28).\) Concerning \( d_0, \) it needs to be checked that \( v \mapsto \langle v, d_0(r) \rangle \) is a left \( R \)-linear map \( V \rightarrow R \) for every \( r \in R. \) Indeed, we have

\[ \langle sv, d_0(r) \rangle = q(sv, r) = sq(v, r) = s\langle v, d_0(r) \rangle \]

for all \( r, \ s \in R \) and \( v \in V \) by Proposition 3.5(a).

Concerning \( d_1, \) it needs to be checked that, for every element \( b \in B^1, \) the map \( \hat{I} \rightarrow R \) defined by the formula \( i \mapsto \langle p(\hat{i}_1 \otimes \hat{i}_2), b \rangle - q(\hat{i}_1, \langle \hat{i}_2, b \rangle) \) descends to a left \( R \)-linear map \( I \rightarrow R. \) Indeed, for all \( u, v \in V \) and \( r \in R \) we have

\[ \langle p(u \otimes rv - ur \otimes v), b \rangle - q(u, \langle rv, b \rangle) + q(ur, \langle v, b \rangle) = \langle q(u, r)v, b \rangle - q(u, \langle rv, b \rangle) + q(ur, \langle v, b \rangle) = q(u, r)v, b - q(u, r\langle v, b \rangle) + q(ur, \langle v, b \rangle) = 0 \]
by Proposition 3.5(e) and (b), the latter of which is being applied to the elements $u \in V$ and $r$, $\langle v, b \rangle \in R$. Since $\hat{I} \rightarrow I$ is a surjective map with the kernel spanned as an abelian group, by the elements $u \otimes rv - ur \otimes v$, it follows that our map $\hat{I} \rightarrow R$ descends uniquely to a map $d_{1}(b) : I \rightarrow R$. To prove that the latter map is left $R$-linear, we compute

$$\langle ri, d_{1}(b) \rangle = \langle p(r\hat{1} \otimes \hat{2}), b \rangle - q(r\hat{1}, \langle \hat{2}, b \rangle)$$

$$\quad = \langle rp(\hat{1} \otimes \hat{2}), b \rangle - rq(\hat{1}, \langle \hat{2}, b \rangle)$$

$$\quad = r\langle p(\hat{1} \otimes \hat{2}), b \rangle - rq(\hat{1}, \langle \hat{2}, b \rangle) = r\langle i, d_{1}(b) \rangle$$

using Proposition 3.5(c) and (a).

Similarly, the map $h : \hat{I} \rightarrow R$ descends to a well-defined map $I \rightarrow R$ by Proposition 3.5(f), and the latter map is left $R$-linear by Proposition 3.5(d).

Now we have to prove the equations (a–g). Part (a): for every element $v \in V$, one has

$$\langle v, d_{0}(rs) \rangle = q(v, rs) = q(vr, s) + q(v, r)s = \langle vr, d_{0}(s) \rangle + \langle v, d_{0}(r) \rangle s$$

$$\quad = \langle v, rd_{0}(s) \rangle + \langle v, d_{0}(r) \rangle s = \langle v, rd_{0}(s) + d_{0}(r)s \rangle$$

by Proposition 3.5(b).

Part (b): for every element $i \in I$ and its preimage $\hat{i} \in \hat{I}$, one has

$$\langle i, d_{1}(rb) \rangle = \langle p(\hat{1} \otimes \hat{2}), rb \rangle - q(\hat{1}, \langle \hat{2}, rb \rangle) = \langle p(\hat{1} \otimes \hat{2}), rb \rangle - \langle \hat{1}, \langle \hat{2}, rb \rangle \rangle$$

$$\quad = \langle i, q(\hat{2}, r), b \rangle + \langle p(\hat{1} \otimes \hat{2}r), b \rangle - q(\hat{1}, \langle \hat{2}r, b \rangle \rangle$$

$$\quad = \langle i, q(\hat{2}, r), b \rangle + \langle i, d_{1}(b) \rangle = \langle i, d_{0}(r) \rangle b + \langle i, rd_{1}(b) \rangle$$

by Proposition 3.5(g).

Part (c): for every element $i \in I$ and its preimage $\hat{i} \in \hat{I}$, one has

$$\langle i, d_{1}(br) \rangle = \langle p(\hat{1} \otimes \hat{2}), br \rangle - q(\hat{1}, \langle \hat{2}, br \rangle) = \langle p(\hat{1} \otimes \hat{2}), br \rangle - \langle \hat{1}, \langle \hat{2}, br \rangle \rangle$$

$$\quad = \langle p(\hat{1} \otimes \hat{2}), br \rangle - \langle \hat{1}, \langle \hat{2}, b \rangle r - q(\hat{1}, \langle \hat{2}, b \rangle \rangle r - q(\hat{1}, \langle \hat{2}, b \rangle \rangle r$$

$$\quad = \langle i, d_{1}(b) \rangle r - \langle i, d_{1}(b) \rangle d_{0}(r) = \langle i, d_{1}(b) \rangle r - \langle i, bd_{0}(r) \rangle$$

by Proposition 3.5(b) applied to the elements $\hat{i} \in V$ and $\langle \hat{2}, b \rangle$, $r \in R$.

Part (d): for every element $i \in I$ and its preimage $\hat{i} \in \hat{I}$, one has

$$\langle i, d_{1}(d_{0}(r)) \rangle = \langle p(\hat{1} \otimes \hat{2}), d_{0}(r) \rangle - q(\hat{1}, \langle \hat{2}, d_{0}(r) \rangle)$$

$$\quad = q(p(\hat{1} \otimes \hat{2}), r) - q(\hat{1}, q(\hat{2}, r)) = h(\hat{1} \otimes \hat{2})r - h(\hat{1} \otimes \hat{2})r$$

$$\quad = \langle i, h \rangle r - \langle ir, h \rangle = \langle i, hr \rangle - \langle i, rh \rangle$$

by Proposition 3.5(h).

In order to prove parts (e–g), we recall the natural isomorphism $B^{3} \simeq \text{Hom}_{R}(I^{(3)}, R)$, which holds for a 3-left finitely projective quadratic algebra $qA$ and its quadratic dual 3-right finitely projective quadratic algebra $B$ according to the proof of Proposition 1.3. In view of this isomorphism, in order to verify an
equation in the group $B^3$, it suffices to evaluate it on every element $j \in I^{(3)}$ and check that the resulting equation in $R$ holds.

Furthermore, for any tensor $e = \sum_{\alpha} e_{1,\alpha} \otimes e_{2,\alpha} = e_1 \otimes e_2 \in B^1 \otimes_R B^1$, and for every element $j \in I^{(3)}$, its preimage $\tilde{j} \in \tilde{I}^{(3)}$, we compute

$$
\langle j, d_1(e_1)e_2 - e_1d_1(e_2) \rangle = \langle j_1(j_2 \otimes j_3), d_1(e_1) \rangle - \langle j_1 \otimes j_2, j_3, e_1 \rangle, d_1(e_2) \rangle
$$

by Proposition 3.5(g) applied to the tensor $\hat{j}$, the multiplication map $\otimes_{B^1 \otimes R B^1}$, the tensor $\hat{B}$ under the multiplication map $\otimes_{B^1 \otimes R B^1}$, $I \langle \hat{2} \rangle$, $\langle \hat{3} \rangle$, $\hat{1}$, and the element $r = \langle \hat{j}, e_1 \rangle$.

Now, tensors $e \in B^1 \otimes_R B^1$ whose image vanishes in $B^2$ form the $R$-$R$-bimodule $I_B \subset B^1 \otimes_R B^1$ of quadratic relations in the quadratic graded ring $B$. By Proposition 1.2, we have $I_B = \text{Hom}_R(A_2, R) = \text{Hom}_R((V \otimes_R V)/I, R)$. To prove part (e), it remains to observe that, in view of Proposition 3.5(i), both the summands in the final expression in (29) involve the pairing of an element of $I \subset V \otimes_R V$ with the element $e$. Therefore, both the summands vanish for $e \in I_B \subset B^1 \otimes_R B^1$.

Part (I): choose an element $e_1 \otimes e_2 = \sum_{\alpha} e_{1,\alpha} \otimes e_{2,\alpha} \in B^1 \otimes_R B^1$ whose image under the multiplication map $B^1 \otimes_R B^1 \rightarrow B^2$ is equal to $d_1(b)$. Then for every element $j \in I^{(3)}$ and its preimage $\tilde{j} \in \tilde{I}^{(3)}$ we have

$$
\langle j, d_2(d_1(b)) \rangle = \langle j, d_1(e_1)e_2 - e_1d_1(e_2) \rangle
$$

by (29), (28), and Proposition 3.5(j) and (h), the latter of which is being applied to the tensor $\hat{i}_1 \otimes \hat{i}_2 = \hat{j}_1 \otimes \hat{j}_2$ and the element $r = \langle \hat{j}_3, b \rangle$.

Part (g): choose an element $h_1 \otimes h_2 = \sum_{\alpha} h_{1,\alpha} \otimes h_{2,\alpha} \in B^1 \otimes_R B^1$ whose image under the multiplication map $B^1 \otimes_R B^1 \rightarrow B^2$ is equal to $h$. Then for every element $j \in I^{(3)}$ and its preimage $\tilde{j} \in \tilde{I}^{(3)}$ we have

$$
\langle j, d_2(h) \rangle = \langle j, d_1(h_1)h_2 - h_1d_1(h_2) \rangle
$$

by (29), (28), and Proposition 3.5(j) and (h), the latter of which is being applied to the tensor $\hat{i}_1 \otimes \hat{i}_2 = \hat{j}_1 \otimes \hat{j}_2$ and the element $r = \langle \hat{j}_3, b \rangle$.
by (29) and Proposition 3.5(k).

Finally, for any quadratic graded ring \( B \), any pair of maps \( d_0: B^0 \to B^1 \) and \( d_1: B^1 \to B^2 \) satisfying (a–c) and (e) can be extended to an odd derivation \( d: B \to B \) of degree 1 in a unique way. The equations (d) and (f) imply that \( d(d(x)) = [h, x] \) for all \( x \in B \), since \( B \) is generated by \( B^1 \) over \( B^0 \).

3.5. Change of strict generators. Let \( \tilde{A} \) be a weak nonhomogeneous quadratic ring over a subring \( R \subset A \) with the \( R\)-\( R \)-subbimodule of generators \( \tilde{V} \subset \tilde{A} \). Assume that the \( R\)-\( R \)-bimodule \( V = \tilde{V}/R \) is projective as a left \( R \)-module and the \( R\)-\( R \)-bimodule \( F_2\tilde{A}/F_1\tilde{A} \) is flat as a left \( R \)-module.

Furthermore, assume that we are given two left \( R \)-linear splittings \( V \cong V'' \subset \tilde{V} \) and \( V \cong V' \subset \tilde{V} \) of the surjective \( R\)-\( R \)-bimodule morphism \( \tilde{V} \to V \). Given an element \( v \in V \), we will denote by \( v'' \in V'' \) and \( v' \in V' \) its images under the two splittings. Then \( v \mapsto v'' - v' \) is a left \( R \)-linear map \( V \to R \), which we will denote by \( a \). Conversely, given a left \( R \)-linear splitting \( V \cong V' \subset \tilde{V} \) and a left \( R \)-linear map \( a: V \to R \), one can construct a second splitting \( V \cong V'' \subset \tilde{V} \) by the rule

\[
V'' = \{ v' + a(v) \mid v \in V \} \subset \tilde{V}.
\]

Denote the maps \( q, p, h \) by the formulas (24–25) using the splitting \( V' \subset \tilde{V} \) by

\[
q: V \times R \to R, \quad p': \hat{I} \to V, \quad h': \hat{I} \to R
\]
and the similar maps constructed using the splitting \( V'' \subset \tilde{V} \) by

\[
q'': V \times R \to R, \quad p'': \hat{I} \to V, \quad h'': \hat{I} \to R.
\]

**Proposition 3.7.** The maps \( q'', p'' \), and \( h'' \) can be obtained from the maps \( q', p', \) and \( h' \) and the map \( a: V \to R \) by the formulas

(a) \( q''(v, r) = q'(v, r) + a(v)r - a(v)r' \) for all \( r \in R \) and \( v \in V \);

(b) \( p''(i_1 \otimes i_2) = p'(i_1 \otimes i_2) + a(i_1)i_2 + i_1a(i_2) \) for all tensors \( i = i_1 \otimes i_2 \in \hat{I} \);

(c) \( h''(i_1 \otimes i_2) = h'(i_1 \otimes i_2) + a(p'(i_1 \otimes i_2)) - q'(i_1, a(i_2)) + a(i_1a(i_2)) \) for all \( i \in \hat{I} \).

**Proof.** In our new notation, the formulas (24–25) take the form

\[
r' * v' = (rv)', \quad r * v'' = (rv)'',
\]
\[
v' * r = (vr)' + q'(v, r) \quad \text{and} \quad v'' * r = (vr)'' + q''(v, r),
\]
and

\[
i_1' * i_2' = p'(i_1 \otimes i_2)' - h'(i_1 \otimes i_2) \quad \text{and} \quad i_1'' * i_2'' = p''(i_1 \otimes i_2)' - h''(i_1 \otimes i_2)
\]
for all \( r \in R, v \in V, \) and \( i \in \hat{I} \). Here the elements \( rv, vr \in V \) correspond to the elements \( (rv)', (vr)' \in V' \) and \( (rv)'', (vr)'' \in V'' \). Similarly, \( p'(i_1 \otimes i_2), p'(i_1 \otimes i_2) \in V' \) while \( p'(i_1 \otimes i_2)' \in V' \) and \( p''(i_1 \otimes i_2)' \in V'' \).

Part (a): one has, on the one hand,

\[
v' * r = (vr)' + q'(v, r) = (vr)'' - a(vr) + q'(v, r),
\]
and on the other hand,  
\[ v' \ast r = (v'' - a(v)) \ast r = (vr)'' + q''(v, r) - a(v)r, \]
since \( v' = v'' - a(v) \) for all \( v \in V \).

Parts (b–c): we have, on the one hand

(31) \[ \hat{1}_1 \ast \hat{1}_2 = p'(\hat{1}_1 \otimes \hat{1}_2) - h'(\hat{1}_1 \otimes \hat{1}_2) = p'(\hat{1}_1 \otimes \hat{1}_2)'' - a(p'(\hat{1}_1 \otimes \hat{1}_2)) - h'(\hat{1}_1 \otimes \hat{1}_2), \]
and on the other hand,

(32) \[ \hat{1}_1 \ast \hat{1}_2 = (\hat{1}_1'' - a(\hat{1}_1)) \ast (\hat{1}_2'' - a(\hat{1}_2)) \]
\[ = \hat{1}_1'' \ast \hat{1}_2'' - a(\hat{1}_1) \ast \hat{1}_2'' - \hat{1}_1'' \ast a(\hat{1}_2) + a(\hat{1}_1) \ast a(\hat{1}_2) \]
\[ = p''(\hat{1}_1 \ast \hat{1}_2)'' - h''(\hat{1}_1 \ast \hat{1}_2) - (a(\hat{1}_1 \hat{1}_2)')'' - q''(\hat{1}_1, \hat{1}_2) + a(\hat{1}_1)a(\hat{1}_2). \]

Comparing (31) with (32) and equating separately the terms belonging to \( V'' \subset \hat{V} \) and to \( R \subset \hat{V} \), we obtain part (b) as well as the equation

\[ h''(\hat{1}_1 \otimes \hat{1}_2) = h'(\hat{1}_1 \otimes \hat{1}_2) + a(p'(\hat{1}_1 \otimes \hat{1}_2)) - q''(\hat{1}_1, \hat{1}_2) + a(\hat{1}_1)a(\hat{1}_2). \]

In order to deduce part (c), it remains to take into account the equation

\[ -q''(\hat{1}_1, \hat{1}_2) + a(\hat{1}_1)a(\hat{1}_2) = -q'(\hat{1}_1, \hat{1}_2) + a(\hat{1}_1)a(\hat{1}_2) \]

obtained by substituting \( v = \hat{1}_1 \) and \( r = a(\hat{1}_2) \) into part (a). \(\square\)

**Proposition 3.8.** Let \( \hat{A} \) be a 3-left finitely projective weak nonhomogeneous quadratic ring over its subring \( R \subset \hat{A} \) with the \( R-R \)-bimodule of generators \( R \subset \hat{V} \subset \hat{A} \). Let \( V \simeq V' \hookrightarrow \hat{V} \) and \( V \simeq V'' \rightarrow \hat{V} \) be two left \( R \)-linear splittings of the surjective \( R-R \)-bimodule map \( \hat{V} \twoheadrightarrow \hat{V}/R = V \). Denote by \( 'B = (B, d', h') \) and \( "B = (B, d'', h'') \) the two related CDG-ring structures on the 3-right finitely projective graded ring \( B \) quadratic dual to \( qA \), as constructed in Proposition 3.6. Let \( a \in \hom_{R}(V, R) = B^{1} \) be the element for which the two splittings \( V' \subset V \) and \( V'' \subset V \) are related by the rule \( V' = \{ v' \mid v \in V \} \) and \( V'' = \{ v'' \mid v \in V \} \) with \( v', v'' \mapsto v \) under the map \( \hat{V} \twoheadrightarrow V \) and \( v'' = v' + a(v) \) (30). Then the equations

(a) \( d''(a) = d''(a) + ara \) for all \( r \in R \);
(b) \( d''(a) = d''(a) + ab + ba \) for all \( b \in B^{1} \); and
(c) \( h'' = h' + d''(a) + a^{2} \)

hold in \( B \), showing that \( (\id, a) : "B \twoheadrightarrow 'B \) is a CDG-ring isomorphism.

**Proof.** In our new notation, the formulas (27–28) take the form

\[ \langle v, d''(a) \rangle = q''(v, r) \quad \text{and} \quad \langle v, d''(a) \rangle = q''(v, r), \]
\[ \langle i, d_{1}''(b) \rangle = \langle p''(\hat{1}_1 \otimes \hat{1}_2), b \rangle - q''(\hat{1}_1, \hat{1}_2, b), \]
\[ \langle i, d_{1}''(b) \rangle = \langle p''(\hat{1}_1 \otimes \hat{1}_2), b \rangle - q''(\hat{1}_1, \hat{1}_2, b). \]

Part (a): for every element \( v \in V \), one has

\[ \langle v, d''(a) \rangle = q''(v, r) = q'(v, r) + a(v)r - a(v)r \]
by Proposition 3.7(a).

Part (b): for every element \( i \in I \) and its preimage \( \hat{i} \in \hat{I} \), one has
\[
\langle i, d''_1(b) \rangle = \langle p''(i \otimes \hat{i}_2), b \rangle - q''(\hat{i}_1, \langle i, b \rangle) = \langle \langle i_1, b \rangle \rangle + \langle \hat{i}_1 \langle \hat{i}_2, b \rangle \rangle - a(i_1) \langle \hat{i}_2, b \rangle + a(i_1 \langle i_2, b \rangle)
\]
by Proposition 3.7(b) and (a), the latter of which is being applied to the elements \( v = i_1 \) and \( r = \langle i_2, b \rangle \).

Part (c): for every element \( i \in I \) and its preimage \( i \in \hat{I} \), one has
\[
\langle i, h'' \rangle = \langle i, h' \rangle + \langle p'(i \otimes \hat{i}_2), a \rangle - q'(\hat{i}_1, \langle i_2, a \rangle) + \langle \hat{i}_1 \langle i_2, a \rangle, a \rangle = \langle i, d'_1(b) \rangle + \langle i, a \rangle + \langle i, ba \rangle
\]
by Proposition 3.7(c).

Finally, the equations (a) and (b) imply that \( d''(x) = d'(x) + [a, x] \) for all \( x \in B \), since \( B \) is generated by \( B^1 \) over \( B^0 = R \).

3.6. The nonhomogeneous quadratic duality functor. Let \( R \) be an associative ring. We denote by \( R\text{-rings}_{\text{fil}} \) the category of filtered rings \((\hat{A}, F)\) with increasing filtrations \( F \) and the filtration component \( F_0 \hat{A} \) identified with \( R \).

So the objects of \( R\text{-rings}_{\text{fil}} \) are associative rings \( \hat{A} \) endowed with a filtration \( 0 = F_{-1} \hat{A} \subset F_0 \hat{A} \subset F_1 \hat{A} \subset F_2 \hat{A} \subset \cdots \) such that \( \hat{A} = \bigcup_{n=0}^{\infty} F_n \hat{A} \), the filtration \( F \) is compatible with the multiplication in \( \hat{A} \), and an associative ring isomorphism \( R \simeq F_0 \hat{A} \) has been chosen. Morphisms \( (\hat{A}, F) \to (\hat{A}', F) \) in \( R\text{-rings}_{\text{fil}} \) are ring homomorphisms \( f: \hat{A} \to \hat{A}' \) such that \( f(F_n \hat{A}) \subset F_n \hat{A}' \) for all \( n \geq 0 \) and the ring homomorphism \( F_0 f: F_0 \hat{A} \to F_0 \hat{A}' \) forms a commutative triangle diagram with the fixed isomorphisms \( R \simeq F_0 \hat{A} \) and \( R \simeq F_0 \hat{A}' \).

The category of 3-left finitely projective weak nonhomogeneous quadratic rings over \( R \), denoted by \( R\text{-rings}_{\text{wmlq}} \), is defined as the full subcategory in \( R\text{-rings}_{\text{fil}} \) whose objects are the 3-left finitely projective weak nonhomogeneous quadratic rings \( R \subset \hat{V} \subset \hat{A} \) endowed with the filtration \( F \) generated by \( F_1 \hat{A} = \hat{V} \) over \( F_0 \hat{A} = R \). In other words, this means that a morphism \( (\hat{A}, \hat{V}) \to (\hat{A}', \hat{V}') \) in \( R\text{-rings}_{\text{wmlq}} \) is a ring homomorphism \( f: \hat{A} \to \hat{A}' \) forming a commutative triangle diagram with the inclusions \( R \simeq F_0 \hat{A} \to \hat{A} \) and \( R \simeq F_0 \hat{V} \to \hat{V} \) and satisfying the condition that \( f(\hat{V}) \subset \hat{V}' \) (cf. the discussion in Section 3.1).

Furthermore, the category of 3-right finitely projective quadratic CDG-rings over \( R \), denoted by \( R\text{-rings}_{\text{cdg, rq}} \), is the full subcategory in the category \( R\text{-rings}_{\text{cdg}} \) (as defined in Section 3.2) consisting of all the CDG-rings \((B, d, h)\) whose underlying nonnegatively graded ring \( B \) is 3-right finitely projective quadratic over \( R \).

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Theorem 3.9. The constructions of Propositions 3.6 and 3.8 define a fully faithful contravariant functor

\[(R\text{-rings}_{\text{wnlq}})^{\text{op}} \longrightarrow R\text{-rings}_{\text{cdg, rq}}\]

from the category of 3-left finitely projective weak nonhomogeneous quadratic rings to the category of 3-right finitely projective quadratic CDG-rings over \(R\).

Proof. Let \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) be the category whose objects are 3-left finitely projective weak nonhomogeneous quadratic rings \(R \subset \tilde{V} \subset \tilde{A}\) with a chosen submodule of strict generators \(V' \subset \tilde{V}\). So \(V'\) is a left \(R\)-submodule in \(\tilde{V}\) such that \(\tilde{V} = R \oplus V'\) as a left \(R\)-module. Morphisms \((\tilde{A}, \tilde{V}, V) \rightarrow (\tilde{A}', \tilde{V}', V')\) in \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) are the same as morphisms \((\tilde{A}, \tilde{V}) \rightarrow (\tilde{A}', \tilde{V}')\) in \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\); so a morphism in \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) has to take \(\tilde{V}\) into \(\tilde{V}'\), but it does not need to respect the chosen submodules of strict generators \(V' \subset \tilde{V}\) and \(V' \subset \tilde{V}'\) in any way. Then the functor \(R\text{-rings}^{\text{sg}}_{\text{wnlq}} \rightarrow R\text{-rings}^{\text{sm}}_{\text{wnlq}}\) forgetting the choice of the submodule of strict generators \(V' \subset \tilde{V}\) is fully faithful and surjective on objects; so it is an equivalence of categories.

Furthermore, let \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) be the subcategory in \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) whose objects are all the objects of \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) and whose morphisms are the morphisms \(f: (\tilde{A}, \tilde{V}, V) \rightarrow (\tilde{A}', \tilde{V}', V')\) in \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) such that \(f(V) \subset V\). The category \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) can be called the category of 3-left finitely projective weak nonhomogeneous quadratic rings with strict generators chosen, while the category \(R\text{-rings}^{\text{sm}}_{\text{wnlq}}\) is the category of 3-left finitely projective weak nonhomogeneous quadratic rings with strict generators and strict morphisms.

Finally, let \(R\text{-rings}_{\text{cdg}}\) denote the subcategory in \(R\text{-rings}_{\text{cdg, rq}}\) whose objects are all the objects of \(R\text{-rings}_{\text{cdg}}\) and whose morphisms are the strict morphisms only, i.e., all morphisms of the form \((g, 0): (B, d', h') \rightarrow (B, d', h')\) in \(R\text{-rings}_{\text{cdg}}\). We denote by \(R\text{-rings}_{\text{cdg, rq}}\) the intersection \(R\text{-rings}_{\text{cdg}} \cap R\text{-rings}_{\text{cdg, rq}} \subset R\text{-rings}_{\text{cdg}}\); that is, the category of 3-right finitely projective quadratic CDG-rings over \(R\) and strict morphisms between them.

Then the construction of Proposition 3.6 assigns an object \((B, d, h) \in R\text{-rings}_{\text{cdg, rq}}\) to every object \((\tilde{A}, \tilde{V}, V') \in R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) in a natural way. Given a morphism \(f: (\tilde{A}, \tilde{V}, V) \rightarrow (\tilde{A}', \tilde{V}', V')\) in \(R\text{-rings}^{\text{sg}}_{\text{wnlq}}\) such that \(f(V) \subset V\), we consider the induced morphism of 3-left finitely projective quadratic graded rings \(q\text{gr}^{F}f: q\text{gr}^{F} \tilde{A} \rightarrow q\text{gr}^{F} \tilde{A}'\). According to Propositions 1.2 and 1.3, the morphism \(q\text{gr}^{F}f\) induces a morphism in the opposite direction between the quadratic dual 3-right finitely projective quadratic graded rings, \(g: B \rightarrow B\).

Let \((B, d', h')\) and \((B, d'', h'')\) denote the 3-right finitely projective quadratic CDG-rings assigned to the 3-left finitely projective weak nonhomogeneous quadratic rings \((\tilde{A}, \tilde{V})\) and \((\tilde{A}', \tilde{V}')\) with the submodules of strict generators \(V' \subset \tilde{V}\) and \(V' \subset \tilde{V}'\) by the construction of Proposition 3.6. Assigning the morphism \((g, 0): (B, d'', h'') \rightarrow (B, d', h')\) to the morphism \(f: (\tilde{A}, \tilde{V}, V) \rightarrow (\tilde{A}', \tilde{V}', V')\),
we obtain a contravariant functor

\[(\text{R-rings}_{\text{sm}}^{\text{sgsm}})_{\text{op}} \longrightarrow \text{R-rings}_{\text{cdg}, \text{rq}}^{\text{sm}}.\]

We still have to check that the functor \((34)\) is well-defined, i.e., that \((g, 0): ("B, d", h") \longrightarrow ("B, d", h')\) is indeed a morphism of CDG-rings. Simultaneously we will see that the functor \((34)\) is fully faithful.

Indeed, specifying a morphism \(f: (\tilde{A}, \tilde{V}, \tilde{V}) \longrightarrow (\tilde{A}, \tilde{V}, \tilde{V})\) in \((\text{R-rings}_{\text{sm}}^{\text{sgsm}})_{\text{op}}\) means specifying an \(R-R\)-bimodule map \(f_1: \tilde{V}/R \longrightarrow \tilde{V}/R\) which, interpreted as a map \(\tilde{V} \longrightarrow \tilde{V}\) and taken together with the identity map \(R \longrightarrow R\), extends (necessarily uniquely) to a ring homomorphism \(\tilde{A} \longrightarrow \tilde{A}\). The latter condition is equivalent to the following two:

(i) the map \(\tilde{f}_1 = \text{id}_R \oplus f_1: \tilde{V} = R \oplus \tilde{V} \longrightarrow R \oplus \tilde{V} = \tilde{V}\) agrees with the right \(R\)-module structures on \(\tilde{V}\) and \(\tilde{V}\);

(ii) the tensor ring homomorphism \(T_R(\tilde{f}_1): T_R(\tilde{V}) \longrightarrow T_R(\tilde{V})\) induced by the map \(\tilde{f}_1: \tilde{V} \longrightarrow \tilde{V}\) takes the \(R-R\)-submodule \(I_{\tilde{A}} \subset R \oplus \tilde{V} \oplus \tilde{V} \otimes \tilde{V}\) of nonhomogeneous quadratic relations in the ring \(\tilde{A}\) into the \(R-R\)-submodule \(\tilde{I}_{\tilde{A}} \subset R \oplus \tilde{V} \oplus \tilde{V} \otimes \tilde{V}\) of nonhomogeneous quadratic relations in the ring \(\tilde{A}\) (see Section 3.1 for the notation).

Denote the maps defined by the formulas \((24–25)\) for \((\tilde{A}, \tilde{V}, \tilde{V})\) and \((\tilde{A}, \tilde{V}, \tilde{V})\) by

\[
q': \tilde{V} \times R \longrightarrow R, \quad p': \tilde{I} \longrightarrow \tilde{V}, \quad h': \tilde{I} \longrightarrow R,
\]

\[
q": \tilde{V} \times R \longrightarrow R, \quad p": \tilde{I} \longrightarrow \tilde{V}, \quad h": \tilde{I} \longrightarrow R,
\]

where the notation \(\tilde{V} = \tilde{V}/R\) and \(\tilde{V} = \tilde{V}/R\) is presumed, while \(\tilde{I} \subset \tilde{V} \otimes \tilde{V}\) and \(\tilde{I} \subset \tilde{V} \otimes \tilde{V}\) are the full preimages of the \(R-R\)-submodules \(I \subset V \otimes V\) and \(I \subset V \otimes V\) of quadratic relations in the graded rings \(A = \text{gr}^F \tilde{A}\) and \(A = \text{gr}^F \tilde{A}\), respectively. Then condition (i) is equivalent to the equation

\[
(35) \quad q'(v, r) = q"(f_1(v), r) \quad \text{for all } v \in \tilde{V}, \ r \in R.
\]

Assuming (i) or \((35)\), condition (ii) is equivalent to the combination of the inclusion

\[
(36) \quad (f_1 \otimes f_1)(I) \subset I
\]

with the equations

\[
(37) \quad f_1(p'(i_1 \otimes i_2)) = p"(f_1(i_1) \otimes f_1(i_2)) \quad \text{for all } i_1 \otimes i_2 \in \tilde{I},
\]

\[
\quad h'(i_1 \otimes i_2) = h"(f_1(i_1) \otimes f_1(i_2)) \quad \text{for all } i_1 \otimes i_2 \in I.
\]

Finally, the inclusion \((36)\) holds if and only if the \(R-R\)-bimodule morphism \(g_1 = \text{Hom}_R(f_1, R): \tilde{B}^1 = \text{Hom}_R(\tilde{V}, R) \longrightarrow \text{Hom}_R(\tilde{V}, R) = \tilde{B}^1\) together with the identity map \(\tilde{B}^0 = R \longrightarrow R = \tilde{B}^0\) can be extended to a graded ring homomorphism
\(g: \"B \rightarrow \ 'B\). Assuming (36) or (equivalently) the existence of \(g = (g_n)_{n=0}^{\infty}\): the equations (35) and (37) are equivalent to the equations
\[d'_0(r) = g_1(d'_0(r)) \quad \text{for all } r \in R,\]
(38)
\[d'_1(g_1(b)) = g_2(d'_1(b)) \quad \text{for all } b \in \"B^1,}\]
\[h' = g_2(h''),\]
which mean that \((\"B, d'', h'') \rightarrow (\'B, d', h')\) is a CDG-ring morphism.

Conversely, given a strict morphism \((g, 0): (\"B, d'', h'') \rightarrow (\'B, d', h')\) between 3-right finitely projective quadratic CDG-rings coming from 3-left finitely projective nonhomogeneous quadratic rings \((\tilde{\Lambda}, \tilde{V}, \tilde{\phi})\) and \((\tilde{A}, \tilde{V}, \tilde{\sigma})\), we put \(f_1 = \text{Hom}_{\text{R}^{op}}(g_1, R): \tilde{V} = \text{Hom}_{\text{R}^{op}}(\tilde{B}^1, R) \rightarrow \text{Hom}_{\text{R}^{op}}(\tilde{B}^1, R) = \"V\) and extend the map \(f_1\) together with the identity map \(\tilde{A} \rightarrow \tilde{A}\) to a ring homomorphism \(g: \"B \rightarrow \ 'B\) extending the given map \(g_1: \"B^1 \rightarrow \ 'B^1\) together with the identity map \(\"B^0 = R \rightarrow R = \ 'B^0\) and satisfying the equations (38).

Now we will construct a fully faithful functor
\[(R-\text{rings}_{\text{sm}}^{\text{cdg}})_{\text{op}} \rightarrow \text{R-\text{rings}_{\text{cdg}, \text{rq}}^{\text{sm}}}}\]
(39)
extending the functor (34) to nonstrict morphisms. For 3-left finitely projective weak nonhomogeneous quadratic ring \((\tilde{\Lambda}, \tilde{V}) \in R-\text{rings}_{\text{sm}}^{\text{cdg}}\) and any two choices of a submodule of strict generators \(V', V'' \subset \tilde{V}\), we have two objects \((\tilde{\Lambda}, \tilde{V}, V')\) and \((\tilde{\Lambda}, \tilde{V}, V'')\) \(\in R-\text{rings}_{\text{sm}}^{\text{cdg}}\) connected by an isomorphism \((\tilde{\Lambda}, \tilde{V}, V') \rightarrow (\tilde{\Lambda}, \tilde{V}, V'')\) corresponding to the identity map id: \(\tilde{\Lambda} \rightarrow \tilde{\Lambda}\). Let us call such isomorphisms in \(R-\text{rings}_{\text{sm}}^{\text{cdg}}\) the change-of-strict-generators isomorphisms.

Let \(f: (\tilde{\Lambda}, \tilde{V}, V) \rightarrow (\tilde{\Lambda}, \tilde{V}, V)\) be an arbitrary morphism in \(R-\text{rings}_{\text{sm}}^{\text{cdg}}\). Denote by \(V'\) a submodule of \(\tilde{V}\) the full preimage of the left \(R\)-submodule \(V \subset \tilde{V}\) under the \(R-R\)-bimodule morphism \(F_1f: \tilde{V} \rightarrow \tilde{V}\). Then one has \(V' = R \oplus V'\); so \(V' \subset \tilde{V}\) is another choice of a submodule of strict generators in \(\tilde{\Lambda}\), alternative to \(V \subset \tilde{V}\). Any morphism \(f: (\tilde{\Lambda}, \tilde{V}, V) \rightarrow (\tilde{\Lambda}, \tilde{V}, V)\) in \(R-\text{rings}_{\text{sm}}^{\text{cdg}}\) decomposes uniquely into a change-of-strict-generators isomorphism \((\tilde{\Lambda}, \tilde{V}, V) \rightarrow (\tilde{\Lambda}, \tilde{V}, V)\) followed by a strict morphism \((\tilde{\Lambda}, \tilde{V}, V) \rightarrow (\tilde{\Lambda}, \tilde{V}, V)\).

The construction of Proposition 3.8 assigns a change-of-connection isomorphism in \(R-\text{rings}_{\text{cdg}, \text{rq}}^{\text{sm}}\) to every change-of-strict-generators isomorphism in \(R-\text{rings}_{\text{sm}}^{\text{cdg}}\). Decomposing any morphism in \(R-\text{rings}_{\text{cdg}, \text{rq}}^{\text{sm}}\) into a change-of-strict-generators-isomorphism followed by a strict morphism, one extends the functor (34) to a contravariant functor (39). We omit further details, which are straightforward.

We still have to show that the functor (39) is fully faithful. It is clear from the construction of Proposition 3.8 that this functor functor restricts to a fully faithful functor from the subcategory of change-of-strict-generators isomorphisms in \(R-\text{rings}_{\text{sm}}^{\text{cdg}}\).
to the subcategory of change-of-connection isomorphisms in $R \text{-} \text{rings}^{\text{cdg}, \text{rq}}$. Since the functor (34) is fully faithful as well, it follows that so is the functor (39).

In order to construct the desired fully faithful functor (33), it remains to choose any quasi-inverse functor to the category equivalence $R \text{-} \text{rings}^{\text{sg}}_{\text{wnlq}} \longrightarrow R \text{-} \text{rings}^{\text{wnlq}}$ and compose it with the fully faithful functor (39).

Remark 3.10. The counterexample in [12, Section 3.4] shows that the fully faithful contravariant functor in Theorem 3.9 is not an anti-equivalence of categories (even when $R = k$ is the ground field). We will see below in Section 4.6 that this functor becomes an anti-equivalence when restricted to the full subcategories of, respectively, left and right finitely projective Koszul rings in $R \text{-} \text{rings}^{\text{wnlq}}$ and $R \text{-} \text{rings}^{\text{cdg}, \text{rq}}$.

3.7. Nonhomogeneous duality 2-functor. We define the 2-category of filtered rings $\text{Rings}_{\text{fil}}^{\text{cdg}}$ as follows. The objects of $\text{Rings}_{\text{fil}}^{\text{cdg}}$ are associative rings $\hat{A}$ endowed with an exhaustive increasing filtration $0 = F_{-1} \hat{A} \subset F_0 \hat{A} \subset F_1 \hat{A} \subset F_2 \hat{A} \subset \cdots$ compatible with the multiplication on $A$. Morphisms $(\hat{A}, F) \longrightarrow (\hat{\hat{A}}, F)$ in $\text{Rings}_{\text{fil}}^{\text{cdg}}$ are ring homomorphisms $f: \hat{A} \longrightarrow \hat{\hat{A}}$ such that $f(F_n \hat{A}) \subset F_n \hat{\hat{A}}$ for all $n \geq 0$ and the map $F_0 f: F_0 \hat{A} \longrightarrow F_0 \hat{\hat{A}}$ is an isomorphism. 2-morphisms $f \xrightarrow{\sim} g$ between a pair of parallel morphisms $f, g: (\hat{A}, F) \longrightarrow (\hat{\hat{A}}, F)$ are invertible elements $z \in F_0 \hat{\hat{A}}$ such that $g(c) = zf(c)z^{-1}$ for all $c \in \hat{A}$.

The vertical composition of two 2-morphisms $f' \xrightarrow{w} f''$ is the 2-morphism $f' \xrightarrow{z} f''$. The identity 2-morphism is the 2-morphism $f \xrightarrow{1} f$. The horizontal composition of two 2-morphisms $g' \xrightarrow{w} g'': (\hat{A}, F) \longrightarrow (\hat{B}, F)$ and $f' \xrightarrow{z} f'': (\hat{B}, F) \longrightarrow (\hat{C}, F)$ is the 2-morphism $f'g' \xrightarrow{z_0 w} f''g'': (\hat{A}, F) \longrightarrow (\hat{C}, F)$ with the element $z \circ w = zf'(w) = f''(w)z$.

All the 2-morphisms of filtered rings are invertible. If $f: (\hat{\hat{A}}, F) \longrightarrow (\hat{\hat{\hat{A}}}, F)$ is a morphism of filtered rings and $z \in F_0 \hat{\hat{A}}$ is an invertible element, then $g: c \longrightarrow zf(c)z^{-1}$ is also a morphism of filtered rings $g: (\hat{\hat{A}}, F) \longrightarrow (\hat{\hat{\hat{A}}}, F)$. The morphisms $f$ and $g$ are connected by the 2-isomorphism $f \xrightarrow{\sim} g$.

The condition about the map $F_0 f: F_0 \hat{A} \longrightarrow F_0 \hat{\hat{A}}$ being an isomorphism could be harmlessly dropped from the above definition (which makes sense without this condition just as well), but we need it for the purposes of the next definition. The 2-category of 3-left finitely projective weak nonhomogeneous quadratic rings, denoted by $\text{Rings}_{\text{wnlq}}^{\text{sg}}$, is defined as the following 2-subcategory in $\text{Rings}_{\text{fil}}^{\text{cdg}}$. The objects of $\text{Rings}_{\text{wnlq}}^{\text{sg}}$ are the 3-left finitely projective weak nonhomogeneous quadratic rings $R \subset \hat{V} \subset A$ with the filtration $F$ generated by $F_1 \hat{A} = \hat{V}$ over $F_0 \hat{A} = R$. All morphisms in $\text{Rings}_{\text{fil}}^{\text{cdg}}$ between objects of $\text{Rings}_{\text{wnlq}}^{\text{sg}}$ are morphisms in $\text{Rings}_{\text{wnlq}}^{\text{sg}}$, and all 2-morphisms in $\text{Rings}_{\text{fil}}^{\text{cdg}}$ between morphisms of $\text{Rings}_{\text{wnlq}}^{\text{sg}}$ are 2-morphisms in $\text{Rings}_{\text{wnlq}}^{\text{sg}}$.

Furthermore, the 2-category of 3-right finitely projective quadratic CDG-rings, denoted by $\text{Rings}_{\text{cdg}, \text{rq}}^{\text{rg}}$, is the following 2-subcategory in the 2-category $\text{Rings}_{\text{cdg}, \text{rq}}^{\text{cdg}}$ (as defined in Section 3.2). The objects of $\text{Rings}_{\text{cdg}, \text{rq}}^{\text{cdg}}$ are all the CDG-rings $(B, d, h)$ whose underlying nonnegatively graded ring $B$ is 3-right finitely projective quadratic (over its degree-zero component $B^0$). All morphisms in $\text{Rings}_{\text{cdg}, \text{rq}}^{\text{cdg}}$ between objects of
Lemma 3.11. Let $R \subset \tilde{V} \subset \tilde{A}$ be a 3-left finitely projective weak nonhomogeneous quadratic ring, and let $V \subset \tilde{V}$ be a submodule of strict generators of $\tilde{A}$. Let $(B, d, h)$ be the 3-right finitely projective quadratic CDG-ring corresponding to $(\tilde{A}, \tilde{V}, V)$ under the construction of Proposition 3.6. Let $z \in R$ be an invertible element. Consider the conjugation morphism $f_{z^{-1}}: \tilde{A} \to \tilde{A}$ taking any element $c \in \tilde{A}$ to the element $f_{z^{-1}}(c) = z^{-1}cz$. Then the CDG-ring morphism $(B, d, h) \to (B, d, h)$ corresponding to the morphism $f_{z^{-1}}$ under the duality functor of Theorem 3.9 is equal to

$$(g_z, a_z): (B, d, h) \to (B, d, h),$$

where $g_z: B \to B$ is the conjugation map taking any element $b \in B$ to the element $g_z(b) = zbz^{-1}$, and the element $a_z \in B^1$ is given by the formula $a_z = -d(z)z^{-1}$.

Proof. Strictly speaking, the assertion of the lemma does not literally make sense as stated, and we need to make it more precise before proving it. The problem is that $f_{z^{-1}}: \tilde{A} \to \tilde{A}$ is not a morphism in the category $\text{Rings}_{\text{cdg}, \text{rq}}$ as defined in Section 3.6, because it does not restrict to the identity map $\tilde{A} \subset R \to R \subset \tilde{A}$, but rather to the map $R \to R$ of conjugation with $z^{-1}$. For the same reason, $g_z: B \to B$ is not a morphism in $\text{Rings}_{\text{gr}}$, and consequently $(g_z, a_z): (B, d, h) \to (B, d, h)$ is not a morphism in $\text{Rings}_{\text{cdg}}$ or $\text{Rings}_{\text{cdg}, \text{rq}}$. So some preparatory work is needed before the functor (33) could be applied to the map $f_{z^{-1}}$.

Denote by $t_z: R \to R$ the conjugation map $r \mapsto zrz^{-1}$, $r \in R$. Let $(\tilde{A}(z), \tilde{V}(z))$ denote the following 3-left finitely projective weak nonhomogeneous quadratic ring over $R$. As an associative ring, $\tilde{A}(z)$ coincides with $\tilde{A}$. Denoting by $\iota: R \to A$ the embedding of the ring $R$ as a subring of the ring $\tilde{A}$, the map $\iota^z = \iota t_z: R \to \tilde{A} = \tilde{A}(z)$ makes $R$ a subring of $\tilde{A}(z)$. For any element $c \in \tilde{A}$, we will denote by $c^z \in \tilde{A}(z)$ the corresponding element of the ring $\tilde{A}(z)$. So in particular, we have $\iota^z(r) = (\iota r)(z^{-1})^z$ for all $r \in R$. Furthermore, as a subgroup of $A(z) = A$, the group $V^z$ coincides with $V$. Then the map $f_{z^{-1}}: \tilde{A}(z) \to \tilde{A}$ taking an element $c^z \in \tilde{A}(z)$ to the element $z^{-1}cz \in \tilde{A}$ is a morphism in the category $\text{Rings}_{\text{wmlq}}$.

Let $(B(z), d(z), h(z))$ denote the following 3-right finitely projective quadratic CDG-ring over $R$. As a graded associative ring, $B(z)$ coincides with $B$; and both the differential $d(z): B(z) \to B(z)$ and the curvature element $h \in B(z)$ coincide with the differential $d$ and the curvature element $h$ in $B$. However, denoting by $\iota: R \to B^0$ the identification of the ring $R$ with the degree-zero component of the graded ring $B$, the identification of the ring $R$ with the degree-zero component of the graded ring $B(z)$ is provided by the map $\iota^z = \iota t_z: R \to B(z), 0 = B^0$.

For any element $b \in B$, we denote by $b^z \in B(z)$ the corresponding element of the ring $B(z)$. So, in particular, we have $d(z)(b^z) = (d(b))^z$ for all $b \in B$, and our notation for the element $h(z)$ is consistent: $h(z) \in B(z)$ is the element corresponding to $h \in B^2$. Then the map $g_z: B \to B(z)$ taking an element $b \in B$ to
the element \((z^{-1}b)z \in B^1\) is a morphism in the category \(R\)-rings\(_{gr}\). Furthermore, let \(a_z^0\) be the element corresponding to the element \(a_z = -d(z)z^{-1} \in B^1\) under the identity isomorphism \(B = B^0\). Then the pair \((g_z, a_z^0)\) is a morphism of CDG-rings \((B, d, h) \to (B^0, d^0, h^0)\), as one can readily check. Therefore, it is also a morphism in the category \(R\)-rings\(_{cdg}\) and in the category \(R\)-rings\(_{cdg, rq}\).

Now the promised precise formulation of the lemma claims that the duality functor (33) takes the morphism \(f_{z^{-1}}: \widehat{A}(z) \to \tilde{A}\) in the category \(R\)-rings\(_{wmlq}\) to the morphism \((g_z, a_z^0): (B, d, h) \to (B^0, d^0, h^0)\) in the category \(R\)-rings\(_{cdg, rq}\).

To be even more precise, we need to establish first that the functor (33) takes the object \(\widehat{A}(z) \in R\)-rings\(_{wmlq}\) to the object \((B^0, d^0, h^0) \in R\)-rings\(_{cdg, rq}\). Put \(A(z) = gr F(A(z))\); then the identity isomorphism of rings \(A_0^0 = B_0^0\), commuting with the identifications \(A_0^0 \cong R \cong B_0^0\) together with the \(A_0^0 - A_0^0\)-bimodule isomorphism \(\text{Hom}_{A_0^0}(A_1^0, A_0^0) = B_0^1\) allow to consider \(B^0\) as the quadratic dual ring to \(A(z)\). Let the subgroup \(\{V(z) \subset \tilde{V}(z)\) coincide with the subgroup \(\{V \subset \tilde{V}\) then the construction of Proposition 3.6 takes the 3-left finitely projective weak nonhomogeneous quadratic ring \((\tilde{A}(z), \tilde{V}(z))\) with the submodule of strict generators \((V(z) \subset \tilde{V}(z))\) to the 3-right finitely projective quadratic CDG-ring \((B^0, d^0, h^0)\).

At last, we can perform the computation proving the lemma. We have to check that the functor (39) takes the morphism \(f_{z^{-1}}: (\widehat{A}(z), V^0(z), V(z)) \to (\tilde{A}, \tilde{V}, V)\) to the morphism \((g_z, a_z^0): (B, d, h) \to (B^0, d^0, h^0)\).

Set \(\{V(z) = z^{-1}V(z)z \in \tilde{V}(z)\). Then our morphism \(f_{z^{-1}}: (\widehat{A}(z), \tilde{V}(z), V(z)) \to (\tilde{A}, \tilde{V}, V)\) decomposes into a change-of-strict-generators morphism \((\tilde{A}(z), \tilde{V}(z), V(z)) \to (\tilde{A}, \tilde{V}, V)\), which acts by the identity map on the underlying associative ring \(\tilde{A}(z)\), followed by the strict morphism \(f_{z^{-1}}: (\tilde{A}(z), \tilde{V}(z), V(z)) \to (\tilde{A}, \tilde{V}, V)\). Denote by \((B^0, d^0, h^0)\) the CDG-ring associated to the 3-left finitely projective weak nonhomogeneous quadratic ring \((\tilde{A}(z), \tilde{V}(z))\) with the submodule of strict generators \((V(z) \subset \tilde{V}(z))\) by the construction of Proposition 3.6.

Let \(q': V(z) \times R \to R\) be the map defined by the formula (24) using the splitting \(\{V(z) \subset \tilde{V}(z)\) of the bimodule of generators of the 3-left finitely projective weak nonhomogeneous quadratic ring \(\tilde{A}(z)\). Denoting by \(u' \in V(z)\) and \(u'' \in V(z)\) the elements corresponding to an element \(u = v(z) \in V(z) = \tilde{V}(z)/R\), we have

\[ u'' = z * (z^{-1}u) = (uz, z^{-1}) = u' + q'(uz, z^{-1}) \]

Furthermore, by Proposition 3.5(b),

\[ 0 = q'(u, 1) = q'(u, zz^{-1}) = q'(uz, z^{-1}) + q'(u, z)z^{-1} \]

hence

\[ u'' = u' - q'(u, z)z^{-1} = u' - \langle u, d_0(z)z^{-1} = u' - \langle u, d_0(z)z^{-1} = u' + a_z^0(u) \]

By Proposition 3.8, it follows that \((id, a_z^0): (B^0, d^0, h^0) \to (B^0, d^0, h^0)\) is a change-of-connection morphism of CDG-rings over \(R\). Hence one can compute
that \( d(z)(zb(z)^{-1}) = zd(z)(b(z)^{-1}) = z(b(z)^{-1}) \) for all \( b \in B \), and \( h(z) = zh(z)^{-1} \).

By construction, \( (id, a_z^{2}): (B(z), d(z), h(z)) \rightarrow (B(z), d(z), h(z)) \) is the change-of-connection morphism of CDG-rings over \( R \) assigned to the change-of-generator morphism \( \tilde{A}(z), \tilde{V}(z), 'V'(z) \rightarrow \tilde{A}(z), \tilde{V}(z), ''V'(z) \) by the functor (39).

Finally, we need to show that the functor (34) takes the strict morphism \( f_{z^{-1}}: (\tilde{A}(z), \tilde{V}(z), ''V'(z)) \rightarrow (\tilde{A}, \tilde{V}, 'V) \) to the strict morphism \( (g_z, 0): (B, d, h) \rightarrow (B(z), d(z), h(z)) \) for this purpose, it suffices to check that the morphism \( \tilde{f}_{z^{-1}} = grF f_{z^{-1}}: A(z) = grF \tilde{A}(z) \rightarrow grF \tilde{A} = A \) corresponds to the morphism \( g_z: B \rightarrow B(z) \) under the homogeneous quadratic duality of Propositions 1.2 and 1.3. All we need to do is to observe that

\[
\langle v(z), g_z(b) \rangle = \langle v(z), zb(z)^{-1} \rangle = z\langle z^{-1}v(z)z, b(z)^{-1} \rangle = t_z(\langle z^{-1}v(z)z, b(z)^{-1} \rangle) = \langle z^{-1}vz, b \rangle = \langle \tilde{f}_{z^{-1}}(v(z)), b \rangle
\]

for all \( v(z) \in V(z) \) and \( b \in B^1 \).

It remains to compute the image of the composition of our morphisms in the category \( R\text{-rings}_{\text{wlnq}} \)

\[
(\tilde{A}(z), \tilde{V}(z), 'V'(z)) \rightarrow (\tilde{A}(z), \tilde{V}(z), ''V'(z)) \xrightarrow{f_{z^{-1}}} (\tilde{A}, \tilde{V}, 'V)
\]

under the functor (39). This is equal, by construction, to the composition of morphisms of CDG-rings

\[
(id, a_z^{2}) \circ (g_z, 0) = (g_z, a_z^{2})
\]

as desired.

In the context of the functor (41) below instead of the functor (33), the assertion of the lemma becomes literally true as stated, without the additional discussion in the first half of the above proof.

**Theorem 3.12.** The constructions of Theorem 3.9 and Lemma 3.11 define a fully faithful strict contravariant 2-functor

\[
(\text{Rings}_{\text{wlnq}))^{op} \longrightarrow \text{Rings}_{\text{cdg2, rq}}
\]

from the 2-category of 3-left finitely projective weak nonhomogeneous quadratic rings to the 2-category of 3-right finitely projective quadratic CDG-rings.

**Proof.** The 2-functor (40) is fully faithful in the strict sense: for any two objects \((\tilde{A}, \tilde{V})\) and \((''A, ''V)\) in \(\text{Rings}_{\text{wlnq}}\) and the corresponding CDG-rings \((B, d', h')\) and \((''B, d'', h'')\) in \(\text{Rings}_{\text{cdg2, rq}}\), the 2-functor (40) induces a bijection between morphisms \((\tilde{A}, \tilde{V}) \rightarrow (''A, ''V)\) in \(\text{Rings}_{\text{wlnq}}\) and morphisms \((B, d'', h'') \rightarrow (''B, d', h')\) in \(\text{Rings}_{\text{cdg2, rq}}\). Furthermore, for any pair of parallel morphisms \(f', f'': (\tilde{A}, \tilde{V}) \rightarrow (''A, ''V)\) in \(\text{Rings}_{\text{wlnq}}\) and the corresponding pair of parallel morphisms \(g', g'': (B, d', h') \rightarrow (''B, d'', h'')\) in \(\text{Rings}_{\text{cdg2, rq}}\), the 2-functor (40) induces a bijection between 2-morphisms \(f' \rightarrow f''\) in \(\text{Rings}_{\text{wlnq}}\) and 2-morphisms \(g'' \rightarrow g'\) in \(\text{Rings}_{\text{cdg2, rq}}\).
To construct the desired 2-functor, denote by $\text{Rings}_{\text{wnlq}} \subset \text{Rings}_{\text{wnlq}2}$ the category whose objects are the objects of the 2-category $\text{Rings}_{\text{wnlq}2}$ and whose morphisms are the morphisms of the 2-category $\text{Rings}_{\text{wnlq}2}$ (but there are no 2-morphisms in $\text{Rings}_{\text{wnlq}}$). Similarly, denote by $\text{Rings}_{\text{cdg}, \text{rq}}$ the category whose objects are the objects of the 2-category $\text{Rings}_{\text{cdg}2, \text{rq}}$ and whose morphisms are the morphisms of the 2-category $\text{Rings}_{\text{cdg}2, \text{rq}}$ (but there are no 2-morphisms in $\text{Rings}_{\text{cdg}, \text{rq}}$). Then essentially the same construction that was used to define the functor (33) in Theorem 3.9 provides a fully faithful contravariant functor

$$\text{(Rings}_{\text{wnlq}})^{\text{op}} \longrightarrow \text{Rings}_{\text{cdg}, \text{rq}}.$$  

In order to extend the functor (41) to a 2-functor (40), we notice that, in the notation of the first paragraph of this proof, for every 2-morphism $f' \overset{\sim}{\longrightarrow} f''$ in $\text{Rings}_{\text{wnlq}2}$, both the morphisms $F_0f': F_0'\mathbb{A} \longrightarrow F_0''\mathbb{A}$ and $F_0f'': F_0''\mathbb{A} \longrightarrow F_0'\mathbb{A}$ are ring isomorphisms and the preimages of the element $z \in F_0''\mathbb{A}$ under these two isomorphisms coincide, $(F_0f')^{-1}(z) = (F_0f'')^{-1}(z)$, because the element $z$ is preserved by the conjugation with $\mathbb{A}$. Similarly, for any 2-morphism $g' \overset{\sim}{\longrightarrow} g''$ in $\text{Rings}_{\text{cdg}2, \text{rq}}$, both the morphisms $g_0': B^0 \longrightarrow B^0$ and $g_0'': B^0 \longrightarrow B^0$ are ring isomorphisms and the preimages of the element $w \in B^0$ under these two isomorphisms coincide, $g_0'(w) = g_0''(w)$. By construction, we have $B^0 = F_0'\mathbb{A}$ and $B^0 = F_0''\mathbb{A}$. The maps $F_0f': F_0'\mathbb{A} \longrightarrow F_0''\mathbb{A}$ and $g_0': B^0 \longrightarrow B^0$ are mutually inverse under this identification, and so are the two maps $F_0f'': F_0''\mathbb{A} \longrightarrow F_0'\mathbb{A}$ and $g_0'': B^0 \longrightarrow B^0$, that is $g_0' = (F_0f')^{-1}$ and $g_0'' = (F_0f'')^{-1}$.

We assign a 2-morphism $g'' \overset{w}{\longrightarrow} g'$ in $\text{Rings}_{\text{cdg}2, \text{rq}}$ to a 2-morphism $f' \overset{\sim}{\longrightarrow} f''$ in $\text{Rings}_{\text{wnlq}2}$ if $g_0'(z) = w = g_0''(z)$, or equivalently, if $(F_0f')(w) = z = (F_0f'')(w)$. It only needs to be checked that $g'' \overset{w}{\longrightarrow} g'$ is a 2-morphism in $\text{Rings}_{\text{cdg}2, \text{rq}}$ if and only if $f' \overset{\sim}{\longrightarrow} f''$ is a 2-morphism in $\text{Rings}_{\text{wnlq}2}$. Then the compatibility with the vertical and horizontal compositions of 2-morphisms will be clear from the construction of such compositions in the beginning of this section and in Section 3.2.

Let us define basic 2-morphisms in the 2-category $\text{Rings}_{\text{wnlq}2}$ as 2-morphisms of the form $f_x^{-1} \overset{\sim}{\longrightarrow} \text{id}_{(\mathbb{A}, \mathbb{V})}$, where $(\mathbb{A}, \mathbb{V})$ is an object of $\text{Rings}_{\text{wnlq}2}$. $z \in F_0\mathbb{A}$ is an invertible element, $f_x^{-1}: (\mathbb{A}, \mathbb{V}) \longrightarrow (\mathbb{A}, \mathbb{V})$ is the morphism taking an element $c \in \mathbb{A}$ to the element $f_x(c) = z^{-1}cz \in \mathbb{A}$, and $\text{id}_{(\mathbb{A}, \mathbb{V})}: (\mathbb{A}, \mathbb{V}) \longrightarrow (\mathbb{A}, \mathbb{V})$ is the identity morphism. Then any 2-morphism in $\text{Rings}_{\text{wnlq}2}$ decomposes uniquely as a morphism followed by a basic 2-morphism, and also as a basic 2-morphism followed by a morphism. Specifically, a 2-morphism $f' \overset{\sim}{\longrightarrow} f''$ as above is the composition of the morphism $f''$ followed by the basic 2-morphism $f_x^{-1} \overset{\sim}{\longrightarrow} \text{id}_{(\mathbb{A}, \mathbb{V})}$; and it is also the composition of the basic 2-morphism $f_w^{-1} \overset{w}{\longrightarrow} \text{id}_{(\mathbb{A}, \mathbb{V})}$ followed by the 2-morphism $f''$.

Similarly, we define basic 2-morphisms in the 2-category $\text{Rings}_{\text{cdg}2, \text{rq}}$ as 2-morphisms of the form $(\text{id}_B, 0) \overset{\sim}{\longrightarrow} (g_z, a_z): (B, d, h) \longrightarrow (B, d, h)$, where $(B, d, h)$ is an object of $\text{Rings}_{\text{cdg}, \text{rq}}$. $z \in B^0$ is an invertible element, $g_z: B \longrightarrow B$ is the graded ring homomorphism taking an element $b \in B$ to the element $g_z(b) = zbz^{-1} \in B$, and
\[ a_z = -d(z)z^{-1} \in B^1. \] Then any 2-morphism in \( \text{Rings}_{\text{wlnq2}} \) decomposes uniquely as a morphism followed by a basic 2-morphism, and also as a basic 2-morphism followed by a morphism. Specifically, a 2-morphism \( g'' \xrightarrow{w} g' \) as above is the composition of the morphism \( g'' \) followed by the basic 2-morphism \( (\text{id}_B, 0) \xrightarrow{w} (g_w, a_w): (B, d', h') \rightarrow (B, d', h') \), and it is also the composition of the basic 2-morphism \( (\text{id}_B, 0) \xrightarrow{z} (g_z, a_z): (B, d', h') \rightarrow (B, d', h') \) followed by the morphism \( g'' \).

In other words, \( f' \xrightarrow{z} f'' \) is a 2-morphism in \( \text{Rings}_{\text{wlnq2}} \) if and only if \( f' = f_{z-1}f'' \), or equivalently, \( f' = f''f_{w-1} \). Similarly, \( g'' \xrightarrow{w} g'' \) is a 2-morphism in \( \text{Rings}_{\text{cdg2,rq}} \) if and only if \( g' = (g_w, a_w) \circ g'' \), or equivalently, \( g' = g'' \circ (g_z, a_z) \).

It remains to refer to Lemma 3.11 for the assertion that, for any object \((\tilde{A}, \tilde{V}) \in \text{Rings}_{\text{wlnq2}} \) and the corresponding object \((B, d, h) \in \text{Rings}_{\text{cdg2,rq}} \), basic 2-morphisms \( f_{z-1} \xrightarrow{z} \text{id}_{(\tilde{A},\tilde{V})} \) in \( \text{Rings}_{\text{wlnq2}} \) correspond to basic 2-morphisms \( (\text{id}_B, 0) \xrightarrow{z} (g_z, a_z): (B, d, h) \rightarrow (B, d, h) \) in \( \text{Rings}_{\text{cdg2,rq}} \).

3.8. **Augmented nonhomogeneous quadratic rings.** Let \( \tilde{A} \) be an associative ring and \( R \subset \tilde{A} \) be a subring. A left augmentation of \( \tilde{A} \) over \( R \) is a left ideal \( \tilde{A}^+ \subset \tilde{A} \) such that \( \tilde{A} = R \oplus \tilde{A}^+ \). Equivalently, a left augmentation is a left action of \( \tilde{A} \) in \( R \) extending the regular left action of \( R \) on itself.

Given a left augmentation ideal \( \tilde{A}^+ \subset \tilde{A} \), such a left action of \( \tilde{A} \) in \( R \) is obtained by identifying \( R \) with the quotient left \( \tilde{A} \)-module \( \tilde{A}/\tilde{A}^+ = R \). Conversely, given a left augmentation action of \( \tilde{A} \) in \( R \), the left augmentation ideal \( \tilde{A}^+ \subset \tilde{A} \) is recovered as the annihilator of the element \( 1 \in R \).

The category of left augmented rings over \( R \), denoted by \( \text{Rings}^\text{aug}_R \), is defined as follows. The objects of \( \text{Rings}^\text{aug}_R \) are associative rings \( \tilde{A} \) endowed with a subring identified with \( R \) and a left augmentation ideal \( \tilde{A}^+ \subset \tilde{A} \). Morphisms \((\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{A}', \tilde{A}^+) \) in \( \text{Rings}^\text{aug}_R \) are ring homomorphisms \( f: \tilde{A} \rightarrow \tilde{A}' \) forming a commutative triangle diagram with the embeddings \( R \rightarrow \tilde{A} \) and \( R \rightarrow \tilde{A}' \) and satisfying the condition of compatibility with the augmentations, namely, that \( f(\tilde{A}^+) \subset \tilde{A}'^+ \). Equivalently, both the conditions on \( f \) can be expressed by saying that the left action of \( \tilde{A}' \) in \( R \) coincides with the action obtained from the left action of \( \tilde{A} \) in \( R \) by the restriction of scalars via \( f \).

Moreover, one can define the 2-category of left augmented rings, denoted by \( \text{Rings}^\text{aug}_R \), in the following way. The objects of \( \text{Rings}^\text{aug}_R \) are associative rings \( \tilde{A} \) endowed with a subring \( F_0\tilde{A} \subset \tilde{A} \) and a left ideal \( \tilde{A}^+ \subset \tilde{A} \) such that \( \tilde{A} = F_0\tilde{A} \oplus \tilde{A}^+ \). Morphisms \((\tilde{A}, F_0\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{A}', F_0\tilde{A}', \tilde{A}'^+) \) in \( \text{Rings}^\text{aug}_R \) are ring homomorphisms \( f: \tilde{A} \rightarrow \tilde{A}' \) such that \( f \) restricts to an isomorphism \( F_0f: F_0\tilde{A} \rightarrow F_0\tilde{A}' \) and \( f(\tilde{A}^+) \subset \tilde{A}'^+ \). 2-morphisms \( f \xrightarrow{z} g \) between a pair of parallel morphisms \( f, g: (\tilde{A}, F_0\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{A}', F_0\tilde{A}', \tilde{A}'^+) \) are invertible elements \( z \in F_0\tilde{A} \) such that \( g(c) = f(ycz^{-1}) \) for all \( c \in \tilde{A} \).
Notice that it follows from the latter condition that $z'\overline{A}^+z^{-1} = \overline{A}^+$, or equivalently, $\overline{A}^+z = \overline{A}^+$. Indeed, $zcz^{-1} = r + a$ for $c, a \in \overline{A}^+$ and $r \in F_0\overline{A}$ implies $g(c) = f(zcz^{-1}) = f(r) + f(a) \in \overline{A}$ with $f(r) \in F_0\overline{A}$ and $f(a) \in \overline{A}^+$, hence $f(r) = 0$ and $r = 0$. Similarly, $z^{-1}cz = r + a$ for $c, a \in \overline{A}^+$ and $r \in F_0\overline{A}$ implies $f(c) = g(z^{-1}cz) = g(r) + g(a) \in \overline{A}$, hence $r = 0$.

The vertical composition of two 2-morphisms $f \xrightarrow{w} f'' \xrightarrow{z} f'''$ is the 2-morphism $f \xrightarrow{wz} f'''$. The identity 2-morphism is the 2-morphism $f \xrightarrow{1} f$. The horizontal composition of two 2-morphisms $g^\prime \xrightarrow{w} g''$: $(\tilde{A}, f_0\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{B}, f_0\tilde{B}, \tilde{B}^+)$ and $f^\prime \xrightarrow{z} f''$: $(\tilde{B}, f_0\tilde{B}, \tilde{B}^+) \rightarrow (\tilde{C}, f_0\tilde{C}, \tilde{C}^+)$ is the 2-morphism $f^\prime g^\prime \xrightarrow{wz} f''g''$: $(\tilde{A}, f_0\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{C}, f_0\tilde{C}, \tilde{C}^+)$ with the element $z \circ w = (f_0g''^{-1}(z)w = w(f_0g'')^{-1}(z))$, where $(f_0g'')^{-1}, (f_0g'')^{-1}: f_0\tilde{B} \rightarrow f_0\tilde{A}$ are the inverse maps to the ring isomorphisms $f_0g', f_0g'': f_0\tilde{A} \rightarrow f_0\tilde{B}$.

All the 2-morphisms of left augmented rings are invertible. If $f: (\tilde{A}, f_0\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{A}', f_0\tilde{A}', \tilde{A}'^+)$ is a morphism of left augmented rings and $z \in f_0\tilde{A}$ is an invertible element such that $\tilde{A}^+z = \overline{A}^+$, then $g: c \mapsto f(zcz^{-1})$ is also a morphism of left augmented rings $g: (\tilde{A}, f_0\tilde{A}, \tilde{A}^+) \rightarrow (\tilde{A}', f_0\tilde{A}', \tilde{A}'^+)$, and the morphisms $f$ and $g$ are connected by the 2-isomorphism $f \xrightarrow{z} g$.

Let $(\tilde{A}, F)$ be a filtered ring with an increasing filtration $0 = F_{-1}\tilde{A} \subset F_0\tilde{A} \subset F_1\tilde{A} \subset F_2\tilde{A} \subset \cdots$ (which, as above, is presumed to be exhaustive and compatible with the multiplication in $\tilde{A}$). The filtered ring $(\tilde{A}, F)$ is said to be left augmented if the ring $\tilde{A}$ is left augmented over its subring $F_0\tilde{A}$. In other words, this means that a left ideal $\tilde{A}^+ \subset \tilde{A}$ is chosen such that $\tilde{A} = F_0\tilde{A} \oplus \tilde{A}^+$.

We denote by $R\text{-}\text{rings}_{\text{aug}}$ the category of left augmented filtered rings with the filtration component $F_0\tilde{A}$ identified with $R$. So the objects of $R\text{-}\text{rings}_{\text{aug}}$ are left augmented filtered rings $(\tilde{A}, F, \tilde{A}^+)$ for which a ring isomorphism $R \simeq F_0\tilde{A}$ has been chosen. Morphisms $(\tilde{A}, F, \tilde{A}^+) \rightarrow (\tilde{A}', F', \tilde{A}'^+)$ in $R\text{-}\text{rings}_{\text{aug}}$ are ring homomorphisms $f: \tilde{A} \rightarrow \tilde{A}'$ such that $f(F_n\tilde{A}) \subset F_n\tilde{A}'$ for all $n \geq 0$, $f(\tilde{A}^+) \subset \tilde{A}'^+$, and the ring homomorphism $F_0f: F_0\tilde{A} \rightarrow F_0\tilde{A}'$ forms a commutative triangle diagram with the fixed isomorphisms $R \simeq F_0\tilde{A}$ and $R \simeq F_0\tilde{A}'$.

The definition of the 2-category of left augmented filtered rings, denoted by $\text{Rings}_{\text{aug}}$, is similar to the above. The objects of $\text{Rings}_{\text{aug}}$ are left augmented filtered rings $(\tilde{A}, F, \tilde{A}^+)$. Morphisms $(\tilde{A}, F, \tilde{A}^+) \rightarrow (\tilde{A}', F', \tilde{A}'^+)$ are ring homomorphisms $f: \tilde{A} \rightarrow \tilde{A}'$ such that $f(F_n\tilde{A}) \subset F_n\tilde{A}'$ for all $n \geq 0$, the map $F_0f: F_0\tilde{A} \rightarrow F_0\tilde{A}'$ is an isomorphism, and $f(\tilde{A}^+) \subset \tilde{A}'^+$. 2-morphisms $f \xrightarrow{z} g$ between a pair of parallel morphisms $f, g: (\tilde{A}, F, \tilde{A}^+) \rightarrow (\tilde{A}, F, \tilde{A}^+)$ are invertible elements $z \in F_0\tilde{A}$ such that $g(c) = f(zcz^{-1})$ for all $c \in \tilde{A}$. The composition of 2-morphisms in $\text{Rings}_{\text{aug}}$ is defined in the same way as in the category $\text{Rings}_{\text{aug}}$, so there is an obvious forgetful strict 2-functor $\text{Rings}_{\text{aug}} \rightarrow \text{Rings}_{\text{aug}}$. 
A weak nonhomogeneous quadratic ring \( R \subset \tilde{V} \subset \tilde{A} \) is said to be left augmented if the ring \( \tilde{A} \) is endowed with a left augmentation over its subring \( R \). The category of 3-left finitely projective left augmented weak nonhomogeneous quadratic rings over \( R \), denoted by \( R^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \), is defined as the full subcategory in \( R^{\text{aug}_{\text{fil}}}_{\text{fil}} \) whose objects are the left augmented weak nonhomogeneous quadratic rings over \( R \) that are 3-left finitely projective as weak nonhomogeneous quadratic rings.

The 2-category of 3-left finitely projective left augmented weak nonhomogeneous quadratic rings, denoted by \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \), is defined as the following 2-subcategory in \( \text{Rings}^{\text{aug}_{\text{fil}}} \). The objects of \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \) are the 3-left finitely projective left augmented weak nonhomogeneous quadratic rings \( R \subset \tilde{V} \subset \tilde{A} \subset \tilde{A}^+ \) with the filtration \( F \) generated by \( F_i \tilde{A} \) over \( F_0 \tilde{A} \). All morphisms in \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{fil}} \) between objects of \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \) are morphisms in \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \), and all 2-morphisms in \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{fil}} \) between morphisms of \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \) are 2-morphisms in \( \text{Rings}^{\text{aug}_{\text{fil}}}_{\text{wnlq}} \).

A DG-ring \((B, d)\) is a graded associative ring \( B = \bigoplus_{n \in \mathbb{Z}} B^n \) endowed with an odd derivation \( d: B \to B \) of degree 1 such that \( d^2 = 0 \). In other words, one can say a CDG-ring is a CDG-ring \((B, d, h)\) with \( h = 0 \). In this section, we consider nonnegatively graded DG-rings, that is \( B = \bigoplus_{n=0}^{\infty} B^n \).

A morphism of DG-rings \( f: \((B, d')\) \to \((B, d')\) \) is a morphism of graded rings \( f: \ B \to B' \) such that \( fd' = d'f \). In other words, one can say that a morphism of DG-rings is a morphism of CDG-rings \((f, a): \((B, d', 0)\) \to \((B, d', 0)\) \) with \( a = 0 \). Notice that there exist CDG-ring morphisms \((f, a)\) with \( a \neq 0 \) both the domain and codomain of which are DG-rings. In other words, DG-rings form a subcategory in CDG-rings, but it is not a full subcategory.

We will denote the category of nonnegatively graded DG-rings \((B, d)\) with the fixed degree-zero component \( B^0 = R \) by \( \text{Rings}_{\text{dg}}^{\text{aug}_{\text{fil}}} \). Morphisms \( f: \((B, d')\) \to \((B, d')\) \) in \( \text{Rings}_{\text{dg}}^{\text{fil}} \) are DG-ring morphisms such that the graded ring homomorphism \( f: \ B \to B' \) forms a commutative triangle diagram with the fixed isomorphisms \( R \cong B^0 \) and \( R \cong B'^0 \).

One can define the 2-category of DG-rings as follows. Let \( f, g: \((B, d')\) \to \((B, d')\) \) be a pair of parallel morphisms of DG-rings. A 2-morphism \( f \xrightarrow{\tilde{z}} g \) is an invertible element \( z \in B^0 \) such that \( d'(z) = 0 \) and \( g(c) = zf(c)z^{-1} \) for all \( c \in B \).

The vertical composition of two 2-morphisms \( f' \xrightarrow{w} f'' \xrightarrow{z} f''' \) is the 2-morphism \( f' \xrightarrow{wz} f''' \). The identity 2-morphism is the 2-morphism \( f \xrightarrow{1} f \). The horizontal composition of two 2-morphisms \( g' \xrightarrow{w} g'' : (C, dC) \to (B, dB) \) and \( f' \xrightarrow{z} f'' : (B, dB) \to (A, dA) \) is the 2-morphism \( f'g' \xrightarrow{zo1} f''g'' : (C, dC) \to (A, dA) \) with the element \( z \circ w = zf'(w) = f''(w)z \in A^0 \).

All the 2-morphisms of DG-rings are invertible. If \( f: \((B, d')\) \to \((B, d')\) \) is a morphism of DG-rings and \( z \in B^0 \) is an invertible element such that \( d'(z) = 0 \), then \( g: c \mapsto zf(c)z^{-1} \) is also a morphism of DG-rings \( g: \((B, d')\) \to \((B, d')\) \). The morphisms \( f \) and \( g \) are connected by the 2-isomorphism \( f \xrightarrow{\tilde{z}} g \).
It is clear from these definitions that the 2-category of DG-rings is a 2-subcategory of the 2-category of CDG-rings. Notice the difference, however: the 2-morphisms of CDG-rings correspond to arbitrary invertible elements $z \in B^0$. The 2-morphisms of DG-rings correspond to invertible cocycles $z \in B^0$, $d'(z) = 0$.

The 2-category $\mathbf{Rings}$ of nonnegatively graded DG-rings is defined as the following subcategory of the 2-category of DG-rings. The objects of $\mathbf{Rings}$ are nonnegatively graded DG-rings $(B, d)$, $B = \bigoplus_{n=0}^\infty B^n$. Morphisms $f : (\langle B, d' \rangle) \rightarrow (\langle B, d \rangle)$ in $\mathbf{Rings}$ are morphisms of DG-rings such that the map $f_0 : \langle B^0 \rangle \rightarrow \langle B^0 \rangle$ is an isomorphism. 2-morphisms $f \xrightarrow{z} g$ in $\mathbf{Rings}$ between morphisms $f$ and $g$ belonging to $\mathbf{Rings}$ are arbitrary 2-morphisms from $f$ to $g$ in the 2-category of DG-rings.

The category of 3-right finitely projective quadratic DG-rings over $R$, denoted by $\mathbf{Rings}^{\text{dg}}$, is the full subcategory in the category $\mathbf{Rings}^{\text{dg}}$ consisting of all the DG-rings $(B, d)$ whose underlying nonnegatively graded ring $B$ is 3-right finitely projective quadratic over $R$.

The 2-category of 3-right finitely projective quadratic DG-rings, denoted by $\mathbf{Rings}_{\text{dg},\text{rq}}$, is the similar 2-subcategory in the 2-category $\mathbf{Rings}^{\text{dg}}$. The objects of $\mathbf{Rings}_{\text{dg},\text{rq}}$ are all the DG-rings $(B, d)$ whose underlying nonnegatively graded ring $B$ is 3-right finitely projective quadratic over $B^0$. All morphisms in $\mathbf{Rings}_{\text{dg}}$ between objects of $\mathbf{Rings}_{\text{dg},\text{rq}}$ are morphisms in $\mathbf{Rings}_{\text{dg},\text{rq}}$, and all 2-morphisms in $\mathbf{Rings}_{\text{dg}}$ between morphisms of $\mathbf{Rings}_{\text{dg},\text{rq}}$ are 2-morphisms in $\mathbf{Rings}_{\text{dg},\text{rq}}$.

**Theorem 3.13.** The nonhomogeneous quadratic duality functor of Theorem 3.9 restricts to a fully faithful contravariant functor

\[(\mathbf{Rings}^{\text{laug}}_{\text{wnlq}})^{\text{op}} \longrightarrow \mathbf{Rings}^{\text{dg},\text{rq}}\]

from the category of 3-left finitely projective left augmented weak nonhomogeneous quadratic rings to the category of 3-right finitely projective quadratic DG-rings over $R$.

**Proof.** It is clear from the above discussion that $\mathbf{Rings}^{\text{dg},\text{rq}}$ is a subcategory in $\mathbf{Rings}_{\text{dg},\text{rq}}$. Moreover, the category of 3-right finitely projective quadratic DG-rings $\mathbf{Rings}^{\text{cdg},\text{rq}}$ is a full subcategory in the category $\mathbf{Rings}^{\text{sm},\text{cdg},\text{rq}}$ of 3-right finitely projective quadratic CDG-rings over $R$ and strict morphisms between them (which was introduced in the proof of Theorem 3.9).

Similarly, we observe that the category of 3-left finitely projective left augmented weak nonhomogeneous quadratic rings $\mathbf{Rings}^{\text{laug}}_{\text{wnlq}}$ is a full subcategory in the category $\mathbf{Rings}^{\text{sasm}}_{\text{wnlq}}$ of 3-left finitely projective weak nonhomogeneous quadratic rings $(\tilde{A}, \tilde{V})$ with a fixed submodule of strict generators $V' \subset \tilde{V}$ and morphisms $f : (\langle \tilde{A}, \tilde{V}, 'V \rangle) \rightarrow (\langle 'A, 'V, 'V \rangle)$ preserving the submodule of strict generators. Indeed, given a 3-left finitely projective left augmented weak nonhomogeneous quadratic ring $(\tilde{A}, \tilde{V}, \tilde{A}^+)$ over $R$, we choose the left $R$-submodule $V' = \tilde{A}^+ \cap \tilde{V} \subset \tilde{V}$ as the submodule of strict generators of $\tilde{A}$. The left augmentation ideal $\tilde{A}^+ \subset \tilde{A}$ can be then recovered as the left ideal (equivalently, the subring without unit) generated by $V'$ in $\tilde{A}$.
Moreover, the essential image of the fully faithful functor $R\text{-rings}^\text{aug}_{\text{wnlq}} \rightarrow R\text{-rings}^\text{sgsm}_{\text{wnlq}}$ can be explicitly described as follows. Given an object $(\widetilde{A}, \widetilde{V}, V') \in R\text{-rings}^\text{sgsm}_{\text{wnlq}}$, consider the related maps $q: V \times R \rightarrow R$, $p: \widetilde{I} \rightarrow R$, and $h: \widetilde{I} \rightarrow R$ defined by the formulas (24–25). Then the object $(\widetilde{A}, \widetilde{V}, V')$ corresponds to a (3-left finitely projective) left augmented weak nonhomogeneous quadratic ring if and only if one has $i_1 * i_2 \in V' \subset \widetilde{V}$ for all $i \in \widetilde{I}$, that is, $h = 0$.

Indeed, the “only if” assertion is obvious. To prove the “if”, one observes that, for any weak nonhomogeneous quadratic ring $(\widetilde{A}, \widetilde{V})$ satisfying the assumptions of Section 3.3 and any chosen submodule of strict generators $V' \subset \widetilde{V}$, the ring $\widetilde{A}$ is generated by the ring $R$ and the abelian group $V'$ with the defining relations (24–25). It is clear from the form of these relations that the left ideal generated by $V'$ in $\widetilde{A}$ does not intersect $R$ whenever $h = 0$.

The latter condition means exactly that the quadratic CDG-ring $(B, d, h)$ assigned to $(\widetilde{A}, \widetilde{V}, V')$ by the functor (34) is a DG-ring. The desired fully faithful contravariant functor (42) can be now obtained as a restriction of the fully faithful contravariant functor (34) to the full subcategory $R\text{-rings}^\text{aug}_{\text{wnlq}} \subset R\text{-rings}^\text{sgsm}_{\text{wnlq}}$. □

**Theorem 3.14.** The nonhomogeneous quadratic duality 2-functor of Theorem 3.12 restricts to a fully faithful strict contravariant 2-functor

$$\text{(Rings}^\text{aug}_2)\text{op} \rightarrow \text{Rings}_{\text{dg}2,\text{rq}}$$

from the 2-category of 3-left finitely projective left augmented weak nonhomogeneous quadratic rings to the 2-category of 3-right finitely projective quadratic DG-rings.

**Proof.** Similarly to the functor (40), the functor (43) is fully faithful in the strict sense. For any two objects $(A', V', A^+)$ and $(A'', V'', A''^+) \in \text{Rings}^\text{aug}_2$ and the corresponding DG-rings $(B', d')$ and $(B'', d'') \in \text{Rings}_{\text{dg}2,\text{rq}}$, the 2-functor (43) induces a bijection between morphisms $(A', V', A^+) \rightarrow (A'', V'', A''^+)$ in $\text{Rings}^\text{aug}_2$ and morphisms $(B', d') \rightarrow (B'', d'')$ in $\text{Rings}_{\text{dg}2,\text{rq}}$. Furthermore, for any pair of parallel morphisms $f', f'': (A', V', A^+) \rightarrow (A'', V'', A''^+)$ in $\text{Rings}^\text{aug}_2$ and the corresponding pair of parallel morphisms $g', g'': (B', d') \rightarrow (B'', d'')$ in $\text{Rings}_{\text{dg}2,\text{rq}}$, the 2-functor (43) induces a bijection between 2-morphisms $f' \rightarrow f''$ in $\text{Rings}^\text{aug}_2$ and 2-morphisms $g' \rightarrow g''$ in $\text{Rings}_{\text{dg}2,\text{rq}}$.

It is clear from the discussion above in this section that $\text{Rings}_{\text{dg}2,\text{rq}}$ is a 2-subcategory in $\text{Rings}_{\text{cdg}2,\text{rq}}$. Similarly, the 2-category $\text{Rings}^\text{aug}_2$ can be viewed as a 2-subcategory of the 2-category $\text{Rings}_{\text{wnlq}2}$ in the following way. To any object $(\widetilde{A}, \widetilde{V}, A^+) \in \text{Rings}^\text{aug}_2$ one assigns the object $(\widetilde{A}, \widetilde{V}) \in \text{Rings}_{\text{wnlq}2}$; and to any morphism $f: (A', V', A^+) \rightarrow (A'', V'', A''^+)$ in $\text{Rings}^\text{aug}_2$ one assigns the morphism $f: (\widetilde{A}', \widetilde{V}') \rightarrow (\widetilde{A}'', \widetilde{V}'')$ in $\text{Rings}_{\text{wnlq}2}$. Finally, to any 2-morphism $f \rightarrow g: (A', V', A^+) \rightarrow (A'', V'', A''^+)$ in $\text{Rings}^\text{aug}_2$ one assigns the 2-morphism $f \rightarrow g: (\widetilde{A}', \widetilde{V}') \rightarrow (\widetilde{A}'', \widetilde{V}'')$ in $\text{Rings}_{\text{wnlq}2}$.
with the element \( w = (F_0 f)(z) = (F_0 g)(z) \). Here \( z \in F_0 'A \) is an invertible element such that \( 'A^+ z = 'A^+ \) and \( w \in F_0 ''A \) is an invertible element, while \( F_0 f \) and \( F_0 g : F_0 'A \rightarrow F_0 ''A \) are two ring isomorphisms whose values coincide on the element \( z \).

Denote by \( \text{Rings}_{\text{wlnq}}^{\text{aug}} \subset \text{Rings}_{\text{wlnq}}^{\text{aug2}} \) the category whose objects are the objects of the 2-category \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \) and whose morphisms are the morphisms of the 2-category \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \) (but there are no 2-morphisms in \( \text{Rings}_{\text{wlnq}}^{\text{aug}} \)). Similarly, denote by \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug}} \) the category whose objects are the objects of the 2-category \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \) and whose morphisms are the morphisms of the 2-category \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \) (but there are no 2-morphisms in \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug}} \)). Then essentially the same argument that was used to restrict the functor (33) to the functor (42) in Theorem 3.13 shows that the fully faithful functor (41) restricts to a fully faithful functor

\[
(\text{Rings}_{\text{wlnq}}^{\text{aug}})^{\text{op}} \longrightarrow \text{Rings}_{\text{dg}}^{\text{aug}}.
\]

To deduce the existence of a fully faithful functor (43) from the existence of the fully faithful functors (40) and (44), one can observe that both the embeddings of 2-categories \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \longrightarrow \text{Rings}_{\text{wlnq}}^{\text{aug}} \) and \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \longrightarrow \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug}} \) are fully faithful on the level of 2-morphisms. In other words, this means that any 2-morphism in \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \) between a pair of parallel morphisms in \( \text{Rings}_{\text{wlnq}}^{\text{aug}} \) belongs to \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \); and similarly, any 2-morphism in \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \) between a pair of parallel morphisms in \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug}} \) belongs to \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \).

More generally, \( \text{Rings}_{\text{dfi}}^{\text{aug2}} \) is a 2-subcategory in \( \text{Rings}_{\text{dfi}}^{\text{aug}} \) such that any 2-morphism in \( \text{Rings}_{\text{dfi}}^{\text{aug2}} \) belongs to \( \text{Rings}_{\text{dfi}}^{\text{aug}} \); and \( \text{Rings}_{\text{dg}}^{\text{aug2}} \) is a 2-subcategory in \( \text{Rings}_{\text{dg}}^{\text{aug}} \) such that any 2-morphism in \( \text{Rings}_{\text{dg}}^{\text{aug2}} \) belongs to \( \text{Rings}_{\text{dg}}^{\text{aug}} \). This suffices to prove the theorem.

Alternatively, one can construct the fully faithful strict 2-functor (43) in the way similar to the construction of the fully faithful strict 2-functor (40) in the proof of Theorem 3.12. For this purpose, one defines the “basic 2-morphisms” in the 2-categories \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \) and \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \), and observes that any 2-morphism in \( \text{Rings}_{\text{wlnq}}^{\text{aug2}} \) decomposes uniquely as a basic 2-morphism followed by a morphism (but not in the other order), while any 2-morphism in \( \text{Rings}_{\text{dg}, \text{rq}}^{\text{aug2}} \) decomposes uniquely as a morphism followed by a basic 2-morphism (but not in the other order).

The reason is, essentially, that the cocycle equation \( d'(z) = 0 \) in the definition of a 2-morphism of DG-rings \( f \overset{z}{\rightarrow} g : ("B, d") \rightarrow ("B', d') \) does not imply the equation \( d''(w) = 0 \) for the element \( w = f_0^{-1}(z) = g_0^{-1}(z) \). The latter equation is stronger than the former one, and does not need to hold. Similarly, the condition \( 'A^+ z = 'A^+ \) related to the definition a 2-morphism of left augmented rings \( f \overset{z}{\rightarrow} g : (\tilde{A}, F_0 \tilde{A}, 'A^+) \rightarrow (\tilde{A}', F_0 \tilde{A}', 'A^+) \) does not imply the condition \( "A^+ w = "A^+ \) for the element \( w = (F_0 f)(z) = (F_0 g)(z) \). The latter condition is stronger than the former one, and does not need to hold. \( \square \)
4.1. **Central element theorem.** Let $\hat{A} = \bigoplus_{n=0}^{\infty} \hat{A}_n$ be a nonnegatively graded ring with the degree-zero component $R = \hat{A}_0$, and let $t \in \hat{A}_1$ be a central element. Let $A = \hat{A}/\hat{A}t$ denote the quotient ring of $\hat{A}$ by the homogeneous ideal generated by $t$. So $A = \bigoplus_{n=0}^{\infty} A_n$ is also a nonnegatively graded ring with the degree-zero component $A_0 = R$, the degree-one component $A_1 = \hat{A}_1/\hat{A}t$, and the degree $n$ component $A_n = \hat{A}_n/\hat{A}_{n-1}t$ for all $n \geq 1$. We will say that $t$ is a **nonzero-divisor** in $\hat{A}$ if $at = 0$ implies $a = 0$ for any $a \in \hat{A}_n$, $n \geq 0$.

**Proposition 4.1.** Let $\hat{A}$ be a nonnegatively graded ring and $t \in \hat{A}_1$ be a central element. Then

(a) the graded ring $\hat{A}$ is generated by $\hat{A}_1$ over $\hat{A}_0$ if and only if the graded ring $A = \hat{A}/\hat{A}t$ is generated by $A_1$ over $A_0$; (b) assuming that $t$ is a nonzero-divisor in $\hat{A}$, the graded ring $\hat{A}$ is quadratic if and only if the graded ring $A = \hat{A}/\hat{A}t$ is quadratic.

**Proof.** Part (a): for any nonnegatively graded ring $C = \bigoplus_{n=0}^{\infty} C_n$ generated by $C_1$ over $C_0$, and for any homogeneous ideal $H \subset C$, the quotient ring $A = C/H$ is generated by $A_1$ over $A_0$. This proves the implication “only if”.

To prove the “if”, suppose that we are given a nonnegatively graded ring $C$ and a homogeneous ideal $H \subset C$ which is generated, as a two-sided ideal, by its degree-one component $H_1$. Suppose further that the quotient ring $A = C/H$ is generated by $A_1$ over $A_0$. Let $C' \subset C$ denote the subring in $C$ generated by $C'_1 = C_1$ over $C'_0 = C_0$. Then the composition $C' \rightarrow C \rightarrow A$ is surjective, so we have $C = C' + H$. Arguing by induction, we will prove that $C'_n = C_n$ for every $n \geq 2$. Indeed, assume that $C'_k = C_k$ for all $k \leq n-1$. Then $H_n = \sum_{k=1}^{n-1} C_{k-1} H_1 C_{n-k} = \sum_{k=1}^{n-1} C_{k-1} C_{n-k} H_1 C_k \subset C_n$, hence $C_n = C'_n + H_n = C'_n$. Thus $C' = C$, so $C$ is generated by $C_1$ over $C_0$.

Part (b): for any quadratic graded ring $C = \bigoplus_{n=0}^{\infty} C_n$ and any homogeneous ideal $H \subset C$ that is generated, as a two-sided ideal, by its components $H_1$ and $H_2$, the quotient ring $A = C/H$ is quadratic. This proves the implication “only if” (which does not depend on the assumption that $t$ is a nonzero-divisor).

To prove the “if”, suppose that $t \in \hat{A}_1$ is a central nonzero-divisor and the graded ring $A = \hat{A}/\hat{A}t$ is quadratic. Then, by part (a), the graded ring $\hat{A}$ is generated by $\hat{A}_1$ over $R = \hat{A}_0$.

Let $\tilde{I} \subset \hat{A}_1 \otimes_R \hat{A}_1$ be the kernel of the multiplication map $\hat{A}_1 \otimes_R \hat{A}_1 \rightarrow \hat{A}_2$ and $q\hat{A} = T_R(\hat{A}_1)/\tilde{I}$ be the quadratic graded ring generated by $\hat{A}_1$ with the relations $I$ over $R$. Then we have a unique surjective homomorphism of graded rings $q\hat{A} \rightarrow \hat{A}$ acting by the identity maps on the components of degree 0 and 1. By construction, the graded ring map $q\hat{A} \rightarrow \hat{A}$ is also an isomorphism in degree 2.

The isomorphism $q\hat{A}_1 \simeq \hat{A}_1$ allows to consider $t$ as an element of the ring $q\hat{A}$. Moreover, $t \in q\hat{A}_1$ is a central element, since $q\hat{A}$ is generated by $q\hat{A}_1$ over $R$ and the
relations of commutativity of \( t \) with the elements of \( R \) and \( q \hat{A}_1 \) have degree \( \leq 2 \), so they hold in \( \hat{A} \) whenever they hold in \( \hat{A} \).

Furthermore, by the “only if” assertion (which we have already explained) the quotient ring \( A' = q\hat{A}/(q\hat{A})t \) is quadratic. We have the induced homomorphism of graded rings \( A' = q\hat{A}/(q\hat{A})t \to \hat{A}/At = A \). Since the map \( q\hat{A} \to \hat{A} \) is an isomorphism in degree \( \leq 2 \), so is the map \( A' \to A \). Since the graded ring \( A \) is quadratic by assumption, it follows by virtue of Lemma 3.2 that the map \( A' \to A \) is an isomorphism of graded rings.

It follows that the kernel \( H \subset q\hat{A} \) of the graded ring homomorphism \( q\hat{A} \to \hat{A} \) is contained in \((q\hat{A})t \subset q\hat{A} \). Now we will prove by induction in \( n \geq 3 \) that \( H_n = 0 \). Indeed, let \( h \in H_n \) be an element. Then \( h = h't \) for some \( h' \in q\hat{A}_{n-1} \). The image of \( h \) under the ring homomorphism \( q\hat{A} \to \hat{A} \) vanishes, and since \( t \) is a nonzero-divisor in \( \hat{A} \), it follows that the image of \( h' \) under the same homomorphism vanishes as well. Hence \( h' \in H_{n-1} = 0 \) by the induction assumption and \( h = h't = 0 \).

We have shown that \( q\hat{A} \to \hat{A} \) is an isomorphism of graded rings, and it follows that the graded ring \( \hat{A} \) is quadratic. \( \square \)

**Lemma 4.2.** Let \( \hat{A} \) be a nonnegatively graded ring, \( t \in \hat{A}_1 \) be a central nonzero-divisor, and \( A = \hat{A}/\hat{A}t \) be the quotient ring. Let \( n \geq 0 \) be an integer. Assume that \( A_n \) is a finitely generated projective (projective, or flat) left module over the ring \( R = \hat{A}_0 = A_0 \) for all \( 0 \leq j \leq n \). Then \( \hat{A}_j \) is a finitely generated projective (resp., projective or flat) left \( R \)-module for all \( 0 \leq j \leq n \).

**Proof.** Provable by induction in \( n \) using the short exact sequences of \( R \)-\( R \)-bimodules \( 0 \to \hat{A}_{n-1} \xrightarrow{t} \hat{A}_n \to A_n \to 0 \). \( \square \)

The following theorem extends to the relative context a very specific particular case of the result of [13, second assertion of Theorem 6.1].

**Theorem 4.3.** Let \( \hat{A} \) be a nonnegatively graded ring and \( t \in \hat{A}_1 \) be a central nonzero-divisor. Assume that \( A_n = \hat{A}_n/\hat{A}_{n-1}t \) is a flat left \( R \)-module for every \( n \geq 1 \). Then the graded ring \( \hat{A} \) is left flat Koszul if and only if the graded ring \( A \) is left flat Koszul.

**Proof.** For any homomorphism of (graded) rings \( C \to A \), any (graded) right \( C \)-module \( N \), and any (graded) left \( A \)-module \( M \), the isomorphism of left derived functors of tensor product

\[
(N \otimes_C^L A) \otimes_A^L M \simeq N \otimes_C^R M
\]

on the derived categories of modules leads to a spectral sequence of (internally graded) abelian groups

\[
E^2_{p,q} = \text{Tor}^A_p(\text{Tor}^C_q(N, A), M) \implies E^\infty_{p,q} = \text{gr}^F_p \text{Tor}^C_{p+q}(N, M)
\]

with the differentials \( d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1} \).

In particular, for any homomorphism of nonnegatively graded rings \( C \to A \) acting by the identity map on their degree-zero components \( C_0 = R = A_0 \), we have a spectral
sequence of internally graded $R$-$R$-bimodules

\[(45) \quad E^2_{p,q} = \text{Tor}^A_p(\text{Tor}^C_q(R, A), R) \implies E^\infty_{p,q} = \text{gr}_p \text{Tor}^C_{p+q}(R, R).\]

In the situation at hand with $C = \hat{A}$ and $A = \hat{A}/t$, where $t \in \hat{A}_1$ is a central nonzero-divisor, we have

$$\text{Tor}^\hat{A}_q(R, A) = \begin{cases} R & \text{for } q = 0, \\ Rt & \text{for } q = 1, \\ 0 & \text{for } q \geq 2, \end{cases}$$

where the $R$-$R$-bimodule $\text{Tor}^\hat{A}_0(R, A) = R$ is situated in the internal degree 0 and the $R$-$R$-bimodule $\text{Tor}^\hat{A}_1(R, A) = Rt$ is situated in the internal degree 1.

Furthermore, the assumption that $A_n$ is a flat left $R$-module for every $n \geq 1$ implies that $\hat{A}_n$ is a flat left $R$-module as well, as one can show arguing by induction in $n$ and using the short exact sequences of $R$-$R$-bimodules $0 \rightarrow \hat{A}_{n-1} \overset{t}{\rightarrow} \hat{A}_n \rightarrow A_n \rightarrow 0$.

Now if $\text{Tor}^A_{i,j}(R, R) = 0$ for all $i \neq j$, then every term $E^2_{p,q}$ of the spectral sequence (45) is concentrated in the internal degree $p + q$; hence so is the term $E^\infty_{p,q}$. It follows immediately that $\text{Tor}^\hat{A}_{i,j}(R, R) = 0$ for all $i \neq j$. So the conditions of Theorem 2.15(a) hold for $\hat{A}$ whenever they hold for $A$. This proves the implication “if”.

To prove the “only if”, one can proceed by induction in $i$. Assume that the graded $R$-$R$-bimodule $\text{Tor}^A_{i,j}(R, R)$ is concentrated in the internal degree $j = p$ for all $p \leq i - 1$. Then the terms $E^2_{p,q}$ are concentrated in the internal degree $p + q$ for all $p \leq i - 1$.

Furthermore, if the graded $R$-$R$-bimodule $\text{Tor}^\hat{A}_{i,i}(R, R)$ is concentrated in the internal degree $i$, then so are the $R$-$R$-bimodules $E^\infty_{p,q}$ for all $p + q = i$. In particular, the term $E^\infty_{i,0}$ is concentrated in the internal degree $i$.

The only possibly nontrivial differentials passing through $E^r_{i,0}$ with $r \geq 2$ are $d^2_{i,0} : E^2_{i,0} \rightarrow E^2_{i-2,1}$. As the term $E^2_{i-2,1}$ is concentrated in the internal degree $i - 1$ by the induction assumption and the above discussion, and the term $E^\infty_{i,0}$ is concentrated in the internal degree $i$, it follows that the term $E^2_{i,0} = \text{Tor}^\hat{A}_{i,i}(R, R)$ can only have nonzero components in the internal degrees $j = i - 1$ and $i$. It remains to recall that $\text{Tor}^A_{i,j}(R, R) = 0$ for $j < i$ by Proposition 2.1(a). Thus the graded $R$-$R$-bimodule $\text{Tor}^\hat{A}_{i,i}(R, R)$ is concentrated in the internal degree $j = i$.

The next result is a kind of Poincaré–Birkhoff–Witt theorem (see first proof of Theorem 4.19 in Section 4.6).

**Theorem 4.4.** Let $\hat{A}$ be a quadratic graded ring and $t \in \hat{A}_1$ be a central element. Assume that the left $R$-modules $\hat{A}_n$ are flat for all $n \geq 1$ and the graded ring $A = \hat{A}/\hat{A}t$ is left flat Koszul. Assume further that the three maps $R \overset{t}{\rightarrow} \hat{A}_1 \overset{t}{\rightarrow} \hat{A}_2 \overset{t}{\rightarrow} \hat{A}_3$ are injective. Then the central element $t$ is a nonzero-divisor in $\hat{A}$.

**Proof.** For any graded module $M$ over a graded ring $C$, let us denote by $M(1)$ the same module with the shifted grading, $M(1)_n = M_{n-1}$. Denote by $H \subset \hat{A}$ the kernel
of the multiplication map \( \hat{A} \rightarrow \hat{A} \). Then we have a four-term exact sequence of graded \( \hat{A} \)-\( \hat{A} \)-bimodules

\[
0 \rightarrow H(1) \rightarrow \hat{A}(1) \xrightarrow{t} \hat{A} \rightarrow A \rightarrow 0.
\]

It follows that \( \text{Tor}^\hat{A}_n(R, A) \simeq R \otimes \hat{A} H(1) \) and \( \text{Tor}^\hat{A}_n(\text{Tor}^\hat{A}_n(R, A), R) \simeq R \otimes \hat{A} H(1) \otimes \hat{A} R \)

as an internally graded \( R \)-\( R \)-bimodule. We also have \( \text{Tor}^\hat{A}_1(R, A) = Rt \simeq R(1) \) an \( \text{Tor}^\hat{A}_0(R, A) = R \).

By assumption, we have \( H_n = 0 \) for \( n \leq 2 \). Arguing by induction in \( n \geq 3 \), we will prove that \( H_n = 0 \) for all \( n \). Assuming that \( H_{n-1} = 0 \), the \( R \)-\( R \)-bimodule \( H_n \) is isomorphic to the degree \( n \) component of the \( R \)-\( R \)-bimodule \( R \otimes \hat{A} H \otimes \hat{A} R \), or which is the same, the degree \( n+1 \) component of the \( R \)-\( R \)-bimodule \( R \otimes \hat{A} H(1) \otimes \hat{A} R \).

We will make use of the spectral sequence (45) for the graded ring homomorphism \( C = \hat{A} \rightarrow A \). In the induction assumption as above, we need to check that the degree \( n+1 \) component of the term \( E^2_{0,2} \) vanishes. Indeed, we have \( \text{Tor}^\hat{A}_{2j}(R, R) = 0 \) for \( j \geq 3 \) by Proposition 2.4(b) (since the graded ring \( \hat{A} \) is quadratic and its grading components are flat left \( R \)-modules by assumption), hence the term \( E^\infty_{p,q} \) has no grading components in the internal degrees \( j \geq 3 \) when \( p+q = 2 \). In particular, this applies to the term \( E^2_{0,2} \).

The only possibly nontrivial differentials passing through \( E^2_{0,2} \) with \( r \geq 2 \) are \( d^2_{2,1} : E^2_{2,1} \rightarrow E^2_{0,2} \) and \( d^3_{3,0} : E^3_{3,0} \rightarrow E^3_{0,2} \). Since the graded ring \( A \) is left flat Koszul by assumption, we have \( \text{Tor}^\hat{A}_{2j}(R, R) = 0 \) for \( j \geq 3 \) and \( \text{Tor}^\hat{A}_{3j}(R, R) = 0 \) for \( j \geq 4 \) by Theorem 2.15(a). Therefore, both the terms \( E^2_{2,1} \) and \( E^2_{3,0} \) have no grading components in the internal degrees \( j \geq 4 \). It follows that the term \( E^2_{0,2} \) cannot have a nonzero grading component in the degree \( n+1 \geq 4 \), and we are done. \( \square \)

4.2. Quasi-differential graded rings and CDG-rings. A quasi-differential graded ring \( (\hat{B}, \partial) \) is a graded associative ring \( \hat{B} = \bigoplus_{n \in \mathbb{Z}} \hat{B}^n \) endowed with an odd derivation \( \partial : \hat{B} \rightarrow \hat{B} \) of degree \(-1\) such that \( \partial^2 = 0 \) and the homology ring \( H_\partial(\hat{B}) = \ker \partial / \im \partial \) vanishes. The latter condition holds (for an odd derivation \( \partial \) of degree \(-1\) satisfying \( \partial^2 = 0 \)) if and only if the homology class of the unit element \( 1 \in \hat{B}^0 \) vanishes, that is, \( 1 \in \partial(\hat{B}^1) \).

The underlying graded ring \( B \) of a quasi-differential graded ring \( (\hat{B}, \partial) \) is defined as the kernel of the differential: \( \hat{B} \rightarrow \hat{B} \), that is \( B = \ker \partial \subset \hat{B} \). A quasi-differential structure on a graded ring \( B \) is the datum of a quasi-differential graded ring \( (\hat{B}, \partial) \) together with a graded ring isomorphism \( B \simeq \ker \partial \subset \hat{B} \).

A morphism of quasi-differential graded rings \( \hat{f} : (\"\hat{B}, \partial\") \rightarrow (\"\hat{B}, \partial\") \) is a morphism of graded rings \( f : \"\hat{B} \rightarrow \"\hat{B} \) such that \( \partial f = f \partial \). The composition of morphisms of quasi-differential graded rings is defined in the obvious way. These rules define the category of quasi-differential graded rings.

A quasi-differential graded ring \( (\hat{B}, \partial) \) is said to be nonnegatively graded if its underlying graded ring \( B \) is nonnegatively graded, \( B = \bigoplus_{n=0}^\infty B^n \), or equivalently,
the graded ring $\hat{B}$ is nonnegatively graded, $\hat{B} = \bigoplus_{n=0}^{\infty} \hat{B}^n$. For a nonnegatively graded quasi-differential ring $(\hat{B}, \partial)$, one has $B^0 = \hat{B}^0$.

We will denote the category of nonnegatively graded quasi-differential rings $(\hat{B}, \partial)$ with the fixed degree-zero component $\hat{B}^0 = B^0 = R$ by $\text{Rings}_{\text{qdg}}$. Morphisms $\hat{f} : (\hat{B}, \partial') \to (\hat{C}, \partial)$ in $\text{Rings}_{\text{qdg}}$ are morphisms of quasi-differential rings such that the graded ring homomorphism $\hat{f} : \hat{B} \to \hat{C}$ forms a commutative triangle diagram with the fixed isomorphisms $R \simeq \hat{B}^0$ and $R \simeq \hat{C}^0$, or equivalently, the induced homomorphism $\hat{f} : \hat{B} \to \hat{C}$ between the graded rings $\hat{B} = \text{ker} \partial'' \subset \hat{B}$ and $\hat{C} = \text{ker} \partial' \subset \hat{C}$ forms a commutative triangle diagram with the fixed isomorphisms $R \simeq \hat{B}^0$ and $R \simeq \hat{C}^0$.

**Theorem 4.5.** The category of quasi-differential graded rings is equivalent to the category of CDG-rings. In particular, for any fixed ring $R$, the category $R \text{– Rings}_{\text{qdg}}$ is equivalent to the category $R \text{– Rings}_{\text{cdg}}$.

**Proof.** To a CDG-ring $(B, d, h)$, one assigns the following quasi-differential graded ring $(\hat{B}, \partial)$. The graded ring $\hat{B} = B[\delta]$ is obtained by adjoining to the graded ring $B$ a new element $\delta \in \hat{B}$ satisfying the relations

$$[\delta, b] = \delta b - (-1)^{|b|}b\delta = d(b) \quad \text{for all } b \in B$$

and

$$\delta^2 = h.$$  

In other words, the elements of the grading component $\hat{B}^n$ are all the formal expressions $b + c\delta$ with $b \in B^n$ and $c \in B^{n-1}$. The unit element in $\hat{B}$ is $1_{\hat{B}} = 1_B + 0\delta$. The multiplication in $\hat{B}$ is given by the formula

$$(b' + c'\delta)(b'' + c''\delta) = (b'b'' + c'd(b'') + (-1)^{|c'|}c'db'' + (b'c'' + (-1)^{|b'|}c'b'' + c'd(b''))\delta.$$  

The odd derivation $\partial$ on $\hat{B}$ can be described informally as the partial derivative $\partial = \partial / \partial \delta$. Explicitly, we put $\partial (b + c\delta) = (-1)^{|c|}c + 0\delta$. So the kernel $\ker \partial \subset \hat{B}$ clearly coincides with the subring $B \subset \hat{B}$ embedded by the obvious rule $b \mapsto b + 0\delta$.

Let $(id, a) : (B, d', h') \to (B, d, h)$ be a change-of-connection isomorphism of CDG-rings; so $d'(b) = d(b) + [a, b]$ for all $b \in B$ and $h' = h + d'(a) + a^2$. Let $\hat{B} = B[\delta']$ and $\hat{C} = B[\delta]$ denote the quasi-differential graded rings corresponding to $(B, d', h')$ and $(B, d, h)$, respectively. Then the rules $b \mapsto b$ for all $b \in B$ and $\delta' \mapsto \delta' + a$ define an isomorphism of graded rings $\hat{B} \simeq \hat{C}$ forming a commutative diagram with the odd derivations $\partial' : \hat{B} \to \hat{C}$ and $\partial : \hat{C} \to \hat{B}$.

Generally, to a morphism of CDG-rings $(f, a) : (B, d', h') \to (B, d, h')$ one assigns the morphism of quasi-differential graded rings $\hat{f} : (\hat{B}, \partial') \to (\hat{C}, \partial)$, where the graded ring homomorphism $\hat{f} : \hat{B} = B[\delta'] \to B[\delta]$ is $\hat{f}$ takes any element $b \in \hat{B} \subset B$ to the element $\hat{f}(b) = f(b) \in B \subset \hat{B}$ and the element $\delta' \in B$ to the element $\hat{f}(\delta') = \delta' + a \in \hat{B}$. This construction defines a functor from the category of
CDG-rings to the category of quasi-differential graded rings, and one can easily see that this functor is fully faithful.

To construct the inverse functor, one needs to choose, for each quasi-differential graded ring \((\hat{B}, \partial)\), an element \(\delta \in \hat{B}^1\) such that \(\partial(\delta) = 1\). Then the CDG-ring \((B, d, h)\) is recovered by the rules \(B = \ker \partial \subset \hat{B}\), \(d(b) = [\delta, b]\) for all \(b \in B \subset \hat{B}\), and \(h = \delta^2\). One has \(\partial([\delta, b]) = [\partial(\delta), b] - [\delta, \partial(b)] = [1, b] - [\delta, 0] = 0\) for all \(b \in B\) and \(\partial(\delta^2) = [\partial(\delta), \delta] = [1, \delta] = 0\), hence \([\delta, b] \in B\) and \(\delta^2 \in \hat{B}\), as desired. The construction of the morphism of CDG-rings assigned to a morphism of quasi-differential graded rings is obvious from the above. \(\square\)

One can define the 2-category of quasi-differential graded rings in the way similar to the definition of the 2-category of DG-rings in Section 3.8. Let \(\hat{f}: (\hat{B}, \partial) \longrightarrow (\hat{B}', \partial')\) be a pair of parallel morphisms of quasi-differential graded rings. A 2-morphism \(\hat{f} \rightarrow \hat{g}\) is an invertible element \(z \in \hat{B'}^0 = \ker(\partial_0': \hat{B}' \rightarrow \hat{B}'^{-1})\) such that \(\hat{g}(c) = z \hat{f}(c)z^{-1}\) for all \(c \in \hat{B}\). The composition of 2-morphisms is defined in the same way as in Section 3.8.

The 2-category \(\text{Rings}_{qdg}^{+}\) of nonnegatively graded quasi-differential rings is defined as the following subcategory of the 2-category of quasi-differential graded rings. The objects of \(\text{Rings}_{qdg}^{+}\) are nonnegatively graded quasi-differential rings \((\hat{B}, \partial)\). Morphisms \(\hat{f}: (\hat{B}, \partial) \longrightarrow (\hat{B}', \partial')\) in \(\text{Rings}_{qdg}^{+}\) are morphisms of quasi-differential graded rings such that the map \(\hat{f}_0: \hat{B}^0 \longrightarrow \hat{B}'^0\) is an isomorphism. 2-morphisms \(\hat{f} \rightarrow \hat{g}\) in \(\text{Rings}_{qdg}^{+}\) between morphisms \(\hat{f}\) and \(\hat{g}\) belonging to \(\text{Rings}_{qdg}^{+}\) are arbitrary 2-morphisms from \(\hat{f}\) to \(\hat{g}\) in the 2-category of quasi-differential graded rings.

**Theorem 4.6.** The equivalence of categories from Theorem 4.5 can be extended to a strict equivalence between the 2-category of CDG-rings and the 2-category of quasi-differential graded rings. In particular, the 2-category of nonnegatively graded CDG-rings \(\text{Rings}_{qdg}^{+}\) is strictly equivalent to the 2-category of quasi-differential graded rings \(\text{Rings}_{qdg}^{+}\).

**Proof.** Let \((f, a)\) and \((g, b): (\hat{B}, \partial', h') \longrightarrow (\hat{B}', \partial', h')\) be a pair of parallel morphisms of CDG-rings and \(\hat{f}, \hat{g}: \hat{B} = \hat{B}'[\delta'] \longrightarrow \hat{B}'[\delta'] = \hat{B}'\) be the corresponding pair of parallel morphisms of quasi-differential graded rings. Let \(z \in \hat{B'}^0\) be an invertible element. We leave it to the reader to check that \((f, a) \rightarrow (g, b)\) is a 2-morphism in the 2-category of CDG-rings if and only if \(\hat{f} \rightarrow \hat{g}\) is a 2-morphism in the 2-category of quasi-differential graded rings. \(\square\)

**4.3. Quadratic quasi-differential graded rings.** We will say that a nonnegatively graded quasi-differential ring \((\hat{B}, \partial)\) is quadratic if its underlying graded ring \(B = \ker \partial \subset \hat{B}\) is quadratic.

**Lemma 4.7.** Let \((\hat{B}, \partial)\) be a nonnegatively graded quasi-differential ring and \(B = \ker \partial \subset \hat{B}\) be its underlying graded ring. Then
(a) the graded ring \( \hat{B} \) is generated by \( \hat{B}^1 \) over \( \hat{B}^0 \) whenever the graded ring \( B \) is generated by \( B^1 \) over \( B^0 \);

(b) the graded ring \( \hat{B} \) is quadratic whenever the graded ring \( B \) is quadratic.

Proof. Part (a): following the proof of Theorem 4.5, we have \( \hat{B} = B[\delta] \), where \( \delta \in \hat{B}^1 \).
So the ring \( \hat{B} \) is generated by \( B \) and \( \delta \); and if \( B \) is generated by \( B^1 \) over \( B^0 \), then it follows that \( \hat{B} \) is generated by \( \hat{B}^1 \) over \( \hat{B}^0 = B^0 \).

Part (b): the ring \( \hat{B} = B[\delta] \) is generated by the ring \( B \) and the adjoined element \( \delta \) with the defining relations (46–47). The relation (47) has degree 2; and it remains to observe that, whenever the ring \( B \) is generated by \( B^1 \) over \( B^0 \), it suffices to impose the relation (46) for elements \( b \in B^0 \) and \( B^1 \) only. This makes the ring \( \hat{B} \) defined by relations of degree \( \leq 2 \) between generators of degree \( \leq 1 \) provided that the ring \( B \) can be so defined. \( \square \)

Lemma 4.8. Let \( (\hat{B}, \partial) \) be a nonnegatively graded quasi-differential ring and \( B = \ker \partial \subset \hat{B} \) be its underlying graded ring.

(a) Assume that the graded ring \( \hat{B} \) is generated by \( \hat{B}^1 \) over \( \hat{B}^0 \). Then the graded ring \( \hat{B} \) is generated by \( B^1 \) over \( B^0 \) if and only if the component \( B^2 \) is generated by \( B^1 \), that is, the multiplication map \( B^1 \otimes_{B^0} B^1 \rightarrow B^2 \) is surjective.

(b) Assume that the graded ring \( \hat{B} \) is generated by \( B^1 \) over \( B^0 \) and the graded ring \( \hat{B} \) is quadratic. Then the graded ring \( B \) is quadratic if and only if it has no relations of degree 3, that is, in other words, the natural homomorphism of graded rings \( qB \rightarrow \hat{B} \) is injective in degree 3.

Proof. The “only if” assertion in both (a) and (b) is obvious. To the “if”, choose an element \( \delta \in \hat{B}^1 \) such that \( \partial(\delta) = 1 \) and consider the CDG-ring \( (B, d, h) \) corresponding to \( (\hat{B}, \delta) \) under the equivalence of categories from Theorem 4.5.

Part (a): let \( 'B \subset B \) denote the graded subring in \( B \) generated by \( 'B^1 = B^1 \) over \( 'B^0 = B^0 \). By assumption, we have \( 'B^2 = B^2 \). Hence the curvature element \( h \in B^2 \) belongs to \( 'B \). Moreover, we have \( d(\partial(B^1)) \subset 'B^2 \), and it follows that \( d(\partial(B)) \subset 'B \subset B \). Thus \( ('B, d|_{'B}, h) \) is a CDG-ring. Let \( \hat{\hat{B}} = B[\delta] \) be the quasi-differential graded ring corresponding to the CDG-ring \( ('B, d|_{'B}, h) \) under the construction of Theorem 4.5.

Then \( \hat{\hat{B}} \) is naturally a graded subring in \( \hat{B} \). Furthermore, we have \( \hat{\hat{B}}^0 = \hat{B}^0 \) and \( \hat{\hat{B}}^1 = \hat{B}^1 \) since \( B^0 = B^0 \) and \( 'B^1 = B^1 \). Since the graded ring \( \hat{B} \) is generated by \( \hat{B}^1 \) over \( \hat{B}^0 \) by assumption, it follows that \( \hat{\hat{B}} = \hat{B} \). In view of the equivalence of categories from Theorem 4.5, we can conclude that \( 'B = B \).

Part (b): the natural homomorphism of graded rings \( qB \rightarrow B \), which we will denote by \( f \), is surjective in our assumptions. If it is injective in degree 3, this means that it is an isomorphism in degrees 2 and 3. So we can consider the curvature element \( h \in B^2 \) as an element of \( qB^2 \).

Furthermore, the components \( d_0 : B^0 \rightarrow B^1 \) and \( d_1 : B^1 \rightarrow B^2 \) of the odd derivation \( d : B \rightarrow B \) can be considered as maps \( d_0 : qB^0 \rightarrow qB^1 \) and \( d_1 : qB^1 \rightarrow qB^2 \). The possibility of extending these maps to an odd derivation \( d' : qB \rightarrow qB \) presents itself if and only if they respect the defining relations of the graded ring \( qB \), which all

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Lemma 4.9. Let $B$ be a nonnegatively graded quasi-differential ring and $B = \ker \partial \subset \hat{B}$ be its underlying graded ring. Let $n \geq 0$ be an integer. Then $\hat{B}^j$ is a finitely generated projective (projective, or flat) right module over the ring $R = \hat{B}^0 = B^n$ for all $0 \leq j \leq n$ if and only if $B^j$ is a finitely generated projective (resp., projective or flat) right $R$-module for all $0 \leq j \leq n$.

Proof. First we observe that, for any the odd derivation $\partial$ of degree $-1$ on a nonnegatively graded ring $\hat{B}$, the component $\partial_0; \hat{B}^0 \longrightarrow \hat{B}^{-1}$ vanishes, since $\hat{B}^{-1} = 0$. In other words, we have $\partial(R) = 0$; hence $\partial$ is an $R$-$R$-bimodule map.

Returning to the situation at hand, the case $n = 0$ is obvious. Arguing by induction in $n$, it suffices to assume that $B^{n-1}$ is a finitely generated projective (resp., projective or flat) right $R$-module, and prove that $\hat{B}^n$ is a finitely generated projective (resp., projective or flat) right $R$-module if and only if $B^n$ is. This is clear from the short exact sequence of $R$-$R$-bimodules $0 \longrightarrow B^n \longrightarrow \hat{B}^n \overset{\partial}{\longrightarrow} B^{n-1} \longrightarrow 0$. □

The following theorem is the version of a very specific particular case of [13, Corollary 6.2(b)].

Theorem 4.10. Let $(\hat{B}, \partial)$ be a nonnegatively graded quasi-differential ring and $B = \ker \partial \subset \hat{B}$ be its underlying graded ring. Then
(a) the graded ring $\hat{B}$ is right flat Koszul whenever the graded ring $B$ is;
(b) the graded ring $\hat{B}$ is right finitely projective Koszul whenever the graded ring $B$ is.

Proof. In view of Lemma 4.9, it suffices to prove part (a). Then, by the same lemma, we know that $\hat{B}^n$ is a flat right $R$-module for every $n \geq 0$.

In the notation of the proof of Theorem 4.4, we have a short exact sequence of $R$-$B$-bimodules $0 \rightarrow B \rightarrow \hat{B} \xrightarrow{\partial} B(1) \rightarrow 0$. So the graded right $B$-module $\hat{B}$ is projective (in fact, free). Hence the spectral sequence (45) for the morphism of graded rings $B \rightarrow \hat{B}$ (with the roles of the left and right sides switched) degenerates to an isomorphism

$$\text{Tor}^B_i(R, R) \simeq \text{Tor}^\hat{B}_i(R, \text{Tor}^B(\hat{B}, R)) \quad \text{for all } i \geq 0.$$  

Furthermore, the internally graded $R$-$R$-bimodule $\text{Tor}^B_1(\hat{B}, R)$ can be computed as

$$\hat{B}/\hat{BB} = \text{Tor}^B_0(\hat{B}, R) = R \oplus R(1),$$

where $B^+ = \bigoplus_{n=1}^\infty B^n$. The $R$-$R$-bimodule direct summand $R(1) \subset \text{Tor}^B_0(\hat{B}, R)$ is a left $\hat{B}$-subbimodule, since the ring $\hat{B}$ is nonnegatively graded. So we have a short exact sequence of graded $\hat{B}$-$R$-bimodules

$$0 \rightarrow R(1) \rightarrow \text{Tor}^0(\hat{B}, R) \rightarrow R \rightarrow 0,$$

where $\hat{B}$ acts in $R(1)$ and in $R$ via the augmentation map $B \rightarrow B/B^+ = R$. We are arriving to a long exact sequence of $R$-$R$-bimodules

$$\cdots \rightarrow \text{Tor}^\hat{B}_{i,j-1}(R, R) \rightarrow \text{Tor}^\hat{B}_{i,j}(R, R) \rightarrow \text{Tor}^\hat{B}_{i,j-1}(R, R) \rightarrow \cdots,$$

where the index $i \geq 0$ denotes the homological grading, while the index $j \geq 0$ denotes the internal grading of the Tor induced by the grading of $B$ and $\hat{B}$.

Now if $\text{Tor}^\hat{B}_{i,j}(R, R) = 0$ for $i \neq j$ and $\text{Tor}^\hat{B}_{i,j-1}(R, R) = 0$, then it follows that $\text{Tor}^\hat{B}_{i,j}(R, R) = 0$. Arguing by induction in $i \geq 0$ (or in $j \geq 0$), one proves that $\text{Tor}^\hat{B}_{i,j}(R, R) = 0$ for $i \neq j$. \qed

4.4. Central elements and acyclic derivations. Let $\hat{A} = \bigoplus_{n=0}^\infty \hat{A}_n$ be a 2-left finitely projective quadratic graded ring with the degree-zero component $R = \hat{A}_0$, and let $\hat{B} = \bigoplus_{n=0}^\infty \hat{B}_n$, $\hat{B}^0 = R$, be the 2-right finitely projective quadratic graded ring quadratic dual to $\hat{A}$. The aim of this section is to establish and study a correspondence between central elements $t \in \hat{A}_1$ and odd derivations $\partial: \hat{B} \rightarrow \hat{B}$ of degree $-1$.

Lemma 4.11. For any element $t \in \hat{A}_1 \simeq \text{Hom}_{R^e}(\hat{B}^1, R)$, consider the related right $R$-module morphism $\partial_1: \hat{B}^1 \rightarrow R$. Then

(a) the element $t$ commutes with all the elements of $\hat{A}_0 = R$ in $\hat{A}$ if and only if the map $\partial_1$ is an $R$-$R$-bimodule morphism;
(b) the element \(t\) is central in \(\hat{A}\) if and only if the map \(\partial_1\) can be (uniquely) extended to an odd derivation \(\partial: \hat{B} \to \hat{B}\) of degree \(-1\).

**Proof.** Part (a): in the notation of Section 3.4, we have \(\partial_1(b) = \langle t, b \rangle\) for all \(b \in \hat{B}^1\). Furthermore,

\[
\langle rt, b \rangle = r\langle t, b \rangle \quad \text{and} \quad \langle tr, b \rangle = \langle t, rb \rangle \quad \text{for all} \quad r \in R \text{ and } b \in \hat{B}^1.
\]

Hence one has \(rt = tr\) for all \(r \in R\) if and only if \(r\partial_1(b) = \partial_1(rb)\) for all \(r \in R\) and \(b \in \hat{B}^1\), i.e., if and only if \(\partial_1\) is a left \(R\)-module map.

Part (b): any odd derivation \(\partial: \hat{B} \to \hat{B}\) of degree \(-1\) is an \(R\)-\(R\)-bimodule map, as it was explained in the proof of Lemma 4.9. Moreover, the odd derivation \(\partial\) is uniquely determined by its component \(\partial_1\), since the graded ring \(\hat{B}\) is generated by \(\hat{B}^1\) over \(\hat{B}^0\). Similarly, the element \(t\) is central in \(\hat{A}\) if and only if it commutes with all the elements of \(\hat{A}_0\) and \(\hat{A}_1\), since the graded ring \(\hat{A}\) is generated by \(\hat{A}_1\) over \(\hat{A}_0\).

So we can assume that both the equivalent conditions of part (a) hold, and it remains to prove that the element \(t\) commutes with all the elements of \(\hat{A}_1\) if and only if the \(R\)-\(R\)-bimodule map \(\partial_1: \hat{B}^1 \to \hat{B}^0\) can be extended to an odd derivation of degree \(-1\). The latter condition can be equivalently restated as follows. Consider the \(R\)-\(R\)-bimodule morphism \(\partial_2: \hat{B}^1 \otimes_R \hat{B}^1 \to \hat{B}^2\) defined by the formula

\[
\partial_2(b_1 \otimes b_2) = \partial_1(b_1)b_2 - b_1\partial_1(b_2) \quad \text{for all} \quad b_1, b_2 \in \hat{B}^1.
\]

For any \(R\)-\(R\)-bimodule map \(\partial_1: \hat{B}^1 \to R\), the corresponding map \(\partial_2\) is well-defined. Since the graded ring \(\hat{B}\) is quadratic, the map \(\partial_1\) extends to an odd derivation of \(\hat{B}\) if and only if the map \(\partial_2\) annihilates the kernel \(\hat{J} \subset \hat{B}^1 \otimes_R \hat{B}^1\) of the surjective multiplication map \(\hat{B}^1 \otimes_R \hat{B}^1 \to \hat{B}^2\).

On the other hand, the condition that the element \(t\) commutes with all the elements of \(\hat{A}_1\) can be restated as follows. Consider the \(R\)-\(R\)-bimodule morphism

\[
\tilde{t}: \hat{A}_1 \longrightarrow \hat{A}_1 \otimes_R \hat{A}_1
\]

defined by the formula

\[
\tilde{t}(a) = a \otimes t - t \otimes a.
\]

Then the element \(t \in \hat{A}_1\) commutes with all the elements of \(\hat{A}_1\) if and only if the composition of the map \(\tilde{t}\) with the surjective multiplication map \(\hat{A}_1 \otimes_R \hat{A}_1 \to \hat{A}_2\) vanishes. In order to prove the desired equivalence, it remains to recall and observe that the contravariant functor \(\text{Hom}_R(-, R)\) transforms the \(R\)-\(R\)-bimodule \(\hat{A}_1\) into the \(R\)-\(R\)-bimodule \(\hat{B}^1\), the \(R\)-\(R\)-bimodule \(\hat{A}_1 \otimes_R \hat{A}_1\) into the \(R\)-\(R\)-bimodule \(\hat{B}^1 \otimes_R \hat{B}^1\), the map \(\tilde{t}\) into the map \(\hat{R}_2\), the \(R\)-\(R\)-bimodule \(\hat{A}_2\) into the \(R\)-\(R\)-bimodule \(\hat{J}\), and the surjection \(\hat{A}_1 \otimes_R \hat{A}_1 \to \hat{A}_2\) into the inclusion \(\hat{J} \to \hat{B}^1 \otimes_R \hat{B}^1\). Besides, the \(R\)-\(R\)-bimodules \(\hat{A}_1, \hat{A}_1 \otimes_R \hat{A}_1, \) and \(\hat{A}_2\) are finitely generated and projective as left \(R\)-modules, while the \(R\)-\(R\)-bimodules \(\hat{B}^1, \hat{B}^1 \otimes_R \hat{B}^1\), and \(\hat{J}\) are finitely generated and
Lemma 4.12. Let \( \partial : \hat{B} \to \hat{B} \) be an odd derivation of degree \(-1\). Then

(a) \( \partial^2 = 0; \)

(b) the homology ring \( H_0(\hat{B}) \) vanishes if and only if, for the central element \( t \in \hat{A}_1 \) corresponding to \( \partial \), the multiplication map \( \hat{A}_0 \to \hat{A}_1 \) is injective and its cokernel \( A_1 = \hat{A}_1/\hat{A}_0 t \) is a projective left \( R \)-module.

Proof. Part (a): the compositions \( \partial_1 \circ \partial_0 : \hat{B}^0 \to \hat{B}^{-1} \to \hat{B}^{-2} \) and \( \partial_0 \partial_1 : \hat{B}^1 \to \hat{B}^0 \to \hat{B}^{-1} \) vanish, since \( \hat{B}^{-1} = 0 \). Since the square of an odd derivation is a derivation and the graded ring \( \hat{B} \) is generated by \( \hat{B}^1 \) over \( \hat{B}^0 \), it follows that \( \partial^2 = 0 \) on the whole graded ring \( \hat{B} \). Part (b) is a particular case of the assertion that a morphism of finitely generated projective left \( R \)-modules \( t : \hat{A}_0 \to \hat{A}_1 \) is injective with a projective cokernel if and only if the dual morphism of finitely generated projective right \( R \)-modules \( \partial_1 : \hat{B}^1 \to \hat{B}^0 \) is surjective.

Proposition 4.13. Let \( \hat{A} \) be a 2-left finitely projective quadratic graded ring and \( \hat{B} \) be the quadratic dual 2-right finitely projective quadratic graded ring. Let \( t \in \hat{A}_1 \) be a central element and \( \partial : \hat{B} \to \hat{B} \) be the corresponding odd derivation of degree \(-1\) (see Lemma 4.11(b)). Assume that the equivalent conditions of Lemma 4.12(b) are satisfied. Let \( B = \text{ker} \partial \subset \hat{B} \) be the underlying graded ring of the quasi-differential graded ring \( (\hat{B}, \partial) \), and let \( A = \hat{A}/At \) be the quotient ring. Then

(a) the graded ring \( B \) is generated by \( B^1 \) over \( B^0 \) if and only if the multiplication map \( \hat{A}_1 \to \hat{A}_2 \) is injective and its cokernel \( A_2 = \hat{A}_2/\hat{A}_1 t \) is a projective left \( R \)-module;

(b) if the equivalent conditions of part (a) hold, then the 2-right finitely projective quadratic graded ring \( \partial B \) is quadratic dual to the 2-left finitely projective quadratic graded ring \( A \). The composition of graded ring homomorphisms \( \text{q}B \to B \to \hat{B} \) is the morphism of 2-right finitely projective quadratic graded rings corresponding to the surjective morphism of 2-left finitely projective quadratic graded rings \( \hat{A} \to A \) under the equivalence of categories from Proposition 1.2.

Proof. Part (a): according to Lemma 4.8(a), the graded ring \( B \) is generated by \( B^1 \) over \( B^0 \) if and only if the multiplication map \( B^1 \otimes_R B^1 \to B^2 \) is quadratic. In the notation of the proof of Lemma 4.11(b), we have

\[
B^1 \otimes_R B^1 \subset \ker \hat{\partial}_2 \subset \hat{B}^1 \otimes_R \hat{B}^1.
\]

Here the map \( B^1 \otimes_R B^1 \to \hat{B}^1 \otimes_R \hat{B}^1 \) induced by the inclusion of the \( R-R \)-subbimodule \( B^1 \) into the \( R-R \)-bimodule \( \hat{B}^1 \) is injective, because the right \( R \)-module \( \hat{B}^1 \) is projective and the quotient bimodule \( \hat{B}^1/B^1 \), being naturally isomorphic to \( R \) via the surjective \( R-R \)-bimodule map \( \partial_1 : \hat{B}^1 \to \hat{B}^0 \), is also projective as a right \( R \)-module. The inclusion \( B^1 \otimes_R B^1 \subset \ker \hat{\partial}_2 \) is clear from the construction of the map \( \hat{\partial}_2 \).
Furthermore, following the proof of Lemma 4.11(b), we have $\tilde{J} \subset \ker \tilde{\partial}_2$, since $\partial$ is an odd derivation of $\hat{B}$ by assumption. The two $R$-$R$-bimodule maps

$$\hat{B}^1 \otimes_R \hat{B}^1 \rightarrow \hat{B}^1, \quad b_1 \otimes_R b_2 \mapsto d(b_1)b_2 \text{ and } b_1d(b_2)$$

restrict to one and the same map $u$: $\ker \tilde{\partial}_2 \rightarrow \hat{B}_1$. The subbimodule $B^1 \otimes_R B^1 \subset \ker \tilde{\partial}_2$ is the kernel of the map $u$.

The map $\tilde{\partial}_2$ is equal to the composition of the surjective multiplication map $\hat{B}^1 \otimes_R \hat{B}^1 \rightarrow \hat{B}^2$ with the map $\partial_2: \hat{B}^2 \rightarrow \hat{B}^1$. By the definition, $B^2 \subset \hat{B}^2$ is the kernel of the latter map. It follows that the multiplication map $B^1 \otimes_R B^1 \rightarrow B^2$ is surjective if and only if one has $\ker \tilde{\partial}_2 = \tilde{J} + B^1 \otimes_R B^1$. The latter condition holds if and only if the map

$$\bar{u} = u|_{\tilde{J}}: \tilde{J} \rightarrow \hat{B}^1$$

is surjective.

It remains to observe that the functor $\text{Hom}_R(-, R)$ transforms the map $\hat{A}_1 \xrightarrow{t} \hat{A}_2$ into the map $\bar{u}$. Thus the map $\bar{u}$ is surjective if and only if the map $t$ is injective and its cokernel $A_2$ is a projective left $R$-module.

In part (b), the graded ring $A$ is quadratic by (the proof of) Proposition 4.1(b) and 2-left finitely projective by assumptions; while the quadratic graded ring $qB$ is 2-right finitely projective by Lemma 4.9. Furthermore, whenever the equivalent conditions of Lemma 4.12(b) hold, the functor $\text{Hom}_R(-, R)$ transforms the $R$-$R$-bimodule $A_1$ into the $R$-$R$-bimodule $B^1$. This is clear from the proof of Lemma 4.12(b). Finally, whenever the equivalent conditions of part (a) of the present lemma hold, the functor $\text{Hom}_R(-, R)$ transforms the $R$-$R$-bimodule $A_2$ into the $R$-$R$-bimodule

$$\ker u = \tilde{J} \cap (B^1 \otimes_R B^1) \subset \hat{B}^1 \otimes_R \hat{B}^1.$$

The latter $R$-$R$-bimodule is the kernel of the surjective multiplication map $B^1 \otimes_R B^1 \rightarrow B^2$. Then one has to check that the same functor transforms the surjective multiplication map $A_1 \otimes_R A_1 \rightarrow A_2$ into the inclusion map $\ker u \rightarrow B^1 \otimes_R B^1$. $\square$

**Proposition 4.14.** Let $\hat{A}$ be a 3-left finitely projective quadratic graded ring and $\hat{B}$ be the quadratic dual 3-right finitely projective quadratic graded ring. Let $t \in \hat{A}_1$ be a central element and $\partial: \hat{B} \rightarrow \hat{B}$ be the corresponding odd derivation of degree $-1$. Assume that the equivalent conditions of Lemma 4.12(b) are satisfied and the equivalent conditions of Proposition 4.13(a) are satisfied as well. Then the graded ring $B$ is quadratic if and only if the multiplication map $\hat{A}_2 \xrightarrow{t} \hat{A}_3$ is injective and its cokernel $A_3 = \hat{A}_3/\hat{A}_2t$ is a projective left $R$-module.

**Proof.** By Lemma 4.9, the right $R$-module $B^3$ is projective in our assumptions. So, if the graded ring $B$ is quadratic, then it is 3-right finitely projective quadratic. Since $qB$ is quadratic dual to $A$ by Proposition 4.13(b), it then follows by virtue of Proposition 1.3 that the quadratic graded ring $A$ is 3-left finitely projective. Thus we can assume that $A_3$ is a projective left $R$-module in all cases.

Then, again by Propositions 4.13(b) and 1.3, it follows that the quadratic graded ring $qB$ is 3-right finitely projective. According to Lemma 4.8(b), the graded ring $B$
is quadratic if and only if the surjective homomorphism of graded rings $qB \to B$ is an isomorphism in degree 3. Equivalently, this means that the composition $qB^3 \to B^3 \to \hat{B}^3$ is an injective map. Furthermore, since both $qB^3$ and $B^3$ are projective right $R$-modules, and since the right $R$-module $\hat{B}^3/B^3 \simeq B^2$ is projective as well, we can conclude that the map $qB^3 \to B^3$ is an isomorphism if and only if applying the functor $\text{Hom}_R(-, R)$ to the composition $qB^3 \to B^3 \to \hat{B}^3$ produces a surjective map $f: \text{Hom}_R(\hat{B}^3, R) \to \text{Hom}_R(qB^3, R)$.

Denote by $I \subset A_1 \otimes_R A_1$ and $\hat{I} \subset \hat{A}_1 \otimes_R \hat{A}_1$ the kernels of the multiplication maps $A_1 \otimes_R A_1 \to A_1$ and $\hat{A}_1 \otimes_R \hat{A}_1 \to \hat{A}_1$. Both the graded rings $A$ and $\hat{A}$ are 3-left finitely projective quadratic in our assumptions; $qB$ and $\hat{B}$, respectively, are their quadratic dual 3-right finitely projective quadratic rings. Put

$$I^{(3)} = (I \otimes_R A_1) \cap (A_1 \otimes_R I) \subset A_1 \otimes_R A_1 \otimes_R A_1$$

and

$$\hat{I}^{(3)} = (\hat{I} \otimes_R \hat{A}_1) \cap (\hat{A}_1 \otimes_R \hat{I}) \subset \hat{A}_1 \otimes_R \hat{A}_1 \otimes_R \hat{A}_1.$$

Following the proof of Proposition 1.3, we have natural isomorphisms of $R$-$R$-bimodules $\text{Hom}_R(qB^3, R) \simeq I^{(3)}$ and $\text{Hom}_R(\hat{B}^3, R) \simeq \hat{I}^{(3)}$. The map

$$f: \hat{I}^{(3)} = \text{Hom}_R(\hat{B}^3, R) \longrightarrow \text{Hom}_R(qB^3, R) = I^{(3)}$$

that we are interested in is induced by the surjective morphism of quadratic graded rings $\hat{A} \to A$ (which is quadratic dual to the morphism of quadratic graded rings $qB \to \hat{B}$ according to Proposition 4.13(b)).

We have shown that the graded ring $B$ is quadratic if and only if the natural map $f: \hat{I}^{(3)} \to I^{(3)}$ is surjective. Let us show that the latter condition holds if and only if the multiplication map $\hat{A}_2 \to \hat{A}_3$ is injective.

Consider the three-term complex

$$\hat{A}_1 \otimes_R \hat{A}_1 \otimes_R \hat{A}_1 \longrightarrow (\hat{A}_1 \otimes_R \hat{A}_2) \oplus (\hat{A}_2 \otimes_R \hat{A}_1) \longrightarrow \hat{A}_3 \longrightarrow 0$$

and denote it by $\hat{C}_\bullet$. For the sake of certainty of notation, let us place the complex $\hat{C}_\bullet$ in the homological degrees 1, 2, and 3, so that $\hat{C}_3 = \hat{A}_1^{(3)}$ and $\hat{C}_1 = \hat{A}_3$. Endow the graded ring $\hat{A}$ with a multiplicative decreasing filtration by homogeneous ideals $\hat{A} = G^0\hat{A} \supset G^1\hat{A} \supset G^2\hat{A} \supset \cdots$ defined by the rule $G^p\hat{A} = \mathfrak{p}^p\hat{A}$, and endow the complex $\hat{C}_\bullet$ with the induced decreasing filtration. So, in particular, one has $\hat{A}/G^1\hat{A} = A$, and the complex $\hat{C}_\bullet/G^1\hat{C}_\bullet$ is isomorphic to the complex

$$A_1 \otimes_R A_1 \otimes_R A_1 \longrightarrow (A_1 \otimes_R A_2) \oplus (A_2 \otimes_R A_1) \longrightarrow A_3 \longrightarrow 0,$$

which we denote by $C_\bullet = C_\bullet(A)$.

Since the graded rings $\hat{A}$ and $A$ are quadratic, we have $H_i(\hat{C}_\bullet) = 0 = H_i(C_\bullet)$ for all $i \neq 3$, while $H_3(\hat{C}_\bullet) = \hat{I}^{(3)}$ and $H_3(C_\bullet) = I^{(3)}$. It is clear from the homological long exact sequence related to the short exact sequence of complexes

$$0 \longrightarrow G^1\hat{C}_\bullet \longrightarrow \hat{C}_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

(48)
that the natural map \( f : \hat{I}^{(3)} \to I^{(3)} \) is surjective if and only if \( H_2(G^1\hat{C}_*) = 0 \).

Consider the nonnegatively graded ring \( \mathbb{Z}[\hat{t}] \) of polynomials in one variable \( \hat{t} \) of degree 1 with integer coefficients. Let \( A[\hat{t}] = A \otimes_\mathbb{Z} \mathbb{Z}[\hat{t}] \) be the tensor product of the graded rings \( A \) and \( \mathbb{Z}[\hat{t}] \), taken over the ring \( \mathbb{Z} \) and endowed with the induced grading. Obviously, the graded ring \( A[\hat{t}] \) is quadratic. Furthermore, there is a natural morphism of graded rings

\[
A[\hat{t}] \longrightarrow gr_G \hat{A} = \bigoplus_{p=0}^{\infty} G^p \hat{A}/G^{p+1} \hat{A}
\]

whose restriction to \( A \subset A[\hat{t}] \) is equal to the inclusion \( A \simeq \hat{A}/G^1 \hat{A} \hookrightarrow gr_G \hat{A} \) and which takes the element \( \hat{t} \in A[\hat{t}] \) to the coset \( t + G^2 \hat{A} \in G^1 \hat{A}/G^2 \hat{A} \).

One easily observes that the map (49) is surjective. Furthermore, the map \( A[\hat{t}]_n \longrightarrow (gr_G \hat{A})_n \) is injective (equivalently, an isomorphism) if and only if the map \( \hat{A}_{j-1} \longrightarrow \hat{A}_j \) is injective for all \( j \leq n \). In our assumptions, we know that this injectivity holds in the internal degrees \( j = 1 \) and \( 2 \), and we are interested in knowing whether it holds for \( j = 3 \). So the map \( \hat{A}_2 \longrightarrow \hat{A}_3 \) is injective if and only if the map \( A[\hat{t}]_3 \longrightarrow gr_G \hat{A}_3 \) is.

Put \( \hat{A} = A[\hat{t}] \) and consider the three-term complex

\[
\hat{A}_1 \otimes_R \hat{A}_1 \longrightarrow (\hat{A}_1 \otimes_R \hat{A}_2) \oplus (\hat{A}_2 \otimes_R \hat{A}_1) \longrightarrow \hat{A}_3 \longrightarrow 0,
\]

denoted by \( \tilde{C}_* = C_* (\hat{A}) \). Since the graded ring \( \hat{A} \) is 2-left finitely projective, the complex \( gr_G \tilde{C}_* \) is isomorphic to the complex \( C_* (gr_G \hat{A}) \). The surjective morphism of graded rings \( \hat{A} \longrightarrow gr_G \hat{A} \) induces a surjective morphism of complexes \( \tilde{C}_* \longrightarrow gr_G \tilde{C}_* \).

Moreover, the maps \( \tilde{C}_i \longrightarrow gr_G \tilde{C}_i \) are isomorphisms for \( i \neq 1 \), since the map \( \hat{A}_j \longrightarrow gr_G \hat{A}_j \) is an isomorphism for \( j < 3 \). Furthermore, the complex \( \tilde{C}_i \) is acyclic in the homological degrees \( i \neq 3 \), since the graded ring \( \hat{A} = A[\hat{t}] \) is quadratic. Since the map \( \tilde{C}_3 \longrightarrow gr_G \tilde{C}_3 \) is surjective, it follows that the map \( A[\hat{t}]_3 \longrightarrow gr_G \hat{A}_3 \) is an isomorphism if and only if \( H_2(gr_G \tilde{C}) = 0 \).

It remains to show that \( H_2(G^1\tilde{C}_*) = 0 \) if and only if \( H_2(gr_G \tilde{C}) = 0 \). The implication “if” is obvious. To prove the “only if”, we assume that \( H_2(G^1\tilde{C}_*) = 0 \), the map \( f : \hat{I}^{(3)} \to I^{(3)} \) is surjective, and the graded ring \( B \) is quadratic.

Consider the additional grading \( p \) on the ring \( gr_G \hat{A} \) and the complex \( gr_G \tilde{C}_* \) induced by the indexing of the filtration \( G \). The related grading \( p \) on the ring \( \hat{A} = A[\hat{t}] \) and the complex \( \tilde{C}_* = C_* (\hat{A}) \) is induced by the grading of the ring \( \mathbb{Z}[\hat{t}] \). The additional grading \( p \) on the rings \( gr_G \hat{A} \) and \( \hat{A} \) takes values in the monoid of nonnegative integers \( p \geq 0 \). On the complexes \( gr_G \tilde{C}_* \) and \( \tilde{C}_* \), the additional grading takes values \( 0 \leq p \leq 3 \).

In the additional grading \( p = 0 \), the morphism of graded rings \( A[\hat{t}] \longrightarrow gr_G \hat{A} \) is an isomorphism, since \( A = \hat{A}/\hat{A}t \). Therefore, the morphism of complexes \( \tilde{C} \longrightarrow gr_G \tilde{C} \) is an isomorphism in the additional grading \( p = 0 \). Any possible homology classes in \( H_2(gr_G \tilde{C}) \) would occur in the additional grading \( p = 1, 2, \) or 3.
We have $H_1(\text{gr}_G\hat{A}) = 0$, as the graded ring $\text{gr}_G\hat{A}$ is generated by its component $A \oplus R[\bar{t}]$ of degree $n = 1$ over its component $R$ of degree $n = 0$. Let us compute the $p$-graded abelian group $(R-R\text{-bimodule}) \tilde{I}^{(3)} = H_3(\hat{C}) = H_3(\text{gr}_G\hat{C})$.

The quadratic graded ring $\hat{A} = A[\bar{t}]$ (in the grading $n$) is 3-left finitely projective, and its quadratic dual 3-right finitely projective graded ring $\hat{B}$ can be computed as the graded ring $\hat{B} = B[\delta] = B \otimes_{\mathbb{Z}} \mathbb{Z}[\delta]$ obtained by adjoining to $B$ a generator $\delta$ of degree $n = 1$ (and $p = 1$) with the relations $\delta b + b \delta = 0$ for all $b \in B^1 \delta$. Here the notation $B \otimes_{\mathbb{Z}} \mathbb{Z}[\delta]$ stands for the relations $\delta b = q b \delta, \ b \in B^1$, with $q = -1$ [11, Section 1 of Chapter 3]. Consequently, $\hat{B}^3 = B^3 \oplus B^2 \delta$ and $\tilde{I}^{(3)} \cong \text{Hom}_{\text{qr}}(\hat{B}^3, R) = I^{(3)} \oplus \bar{\epsilon} I$, where $\bar{\epsilon}$ is a variable dual to $\delta$. The direct summand $I^{(3)}$ of the $p$-graded group $\tilde{I}^{(3)}$ sits in the additional grading $p = 0$ and the direct summand $\bar{\epsilon} I \simeq I$ sits in the additional grading $p = 1$.

Thus we have

$$H_3(G^p\hat{C}_*/G^{p+1}\hat{C}_*) = \begin{cases} I^{(3)} & \text{for } p = 0 \\ \bar{\epsilon} I & \text{for } p = 1 \\ 0 & \text{for } p \geq 2. \end{cases}$$

In view of the spectral sequence connecting the homology groups of the complexes $G^1\hat{C}_*$ and $\text{gr}_G\hat{C}_*$, it remains to check that the map

$$(50) \quad H_3(G^1\hat{C}_*) \longrightarrow H_3(G^1\hat{C}_*/G^2\hat{C}_*) = \bar{\epsilon} I$$

is surjective (or equivalently, an isomorphism). Here the group $H_3(G^1\hat{C}_*)$ is the kernel of the natural map $f: \tilde{I}^{(3)} \longrightarrow I^{(3)}$ (see short exact sequence (48)).

Following the discussion in the beginning of this proof, the map $f: \tilde{I}^{(3)} \longrightarrow I^{(3)}$ can be obtained by applying the functor $\text{Hom}_{\text{qr}}(-, R)$ to the composition of maps $qB^3 \longrightarrow B^3 \longrightarrow \hat{B}^3$. In our present assumptions, the map $f$ is surjective and the graded ring $\hat{B}$ is quadratic. The kernel of the map $f$ is $\text{ker}(f) \cong \text{Hom}_{\text{qr}}(\hat{B}^3/B^3, R) = \text{Hom}_{\text{qr}}(B^2 \delta, R) \simeq \bar{\epsilon} I$, as desired. Notice that existence of a well-defined odd derivation $\bar{\partial} = \partial/\partial \delta$ on $\hat{B}$ ensures injectivity (equivalently, bijectivity) of the multiplication map $B^2 \delta \longrightarrow \hat{B}^3/B^3$. □

4.5. \textbf{Nonhomogeneous quadratic duality via quasi-differential graded rings.}

The construction of the nonhomogeneous quadratic duality functors

$$(\text{Rings}_{\text{wnlg}})^{\text{op}} \longrightarrow \text{Rings}_{\text{cdg}, \text{qr}} \quad \text{and} \quad (\text{Rings}_{\text{wnlg}}^{\text{op}}) \longrightarrow \text{Rings}_{\text{cdg}, 2, \text{qr}}$$

in Theorems 3.9 and 3.12 are based on the computations in Sections 3.3–3.5, which are beautiful, but quite involved. The definitions and results above in Section 4 allow to produce the nonhomogeneous duality functors in a more conceptual fashion.

A nonnegatively graded quasi-differential ring $(\hat{B}, \bar{\partial})$ is said to be $3$-right finitely projective quadratic if its underlying graded ring $B = \text{ker} \bar{\partial} \subset \hat{B}$ is $3$-right finitely projective quadratic. The results of Section 4.3 explain how such properties of the graded ring $B$ are related to the similar properties of the graded ring $\hat{B}$. 

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Let $R$ be an associative ring. The category of 3-right finitely projective quadratic quasi-differential graded rings over $R$, denoted by $R$–$\text{rings}_{qdg,rq}$, is the full subcategory in the category $R$–$\text{rings}_{qdg}$ (as defined in Section 4.2) consisting of all the 3-right finitely projective quadratic quasi-differential graded rings.

Furthermore, the 2-category of 3-right finitely projective quadratic quasi-differential graded rings, denoted by $\text{Rings}_{qdg2,rq}$, is the following 2-subcategory in the 2-category $\text{Rings}_{qdg2}$ (which was also defined in Section 4.2). The objects of $\text{Rings}_{qdg2,rq}$ are all the 3-right finitely projective quadratic quasi-differential graded rings. All morphisms in $\text{Rings}_{qdg2}$ between objects of $\text{Rings}_{qdg2,rq}$ are morphisms in $\text{Rings}_{qdg2,rq}$, and all 2-morphisms in $\text{Rings}_{qdg2}$ between morphisms of $\text{Rings}_{qdg2,rq}$ are 2-morphisms in $\text{Rings}_{qdg2,rq}$.

Restricting the equivalence of categories $R$–$\text{rings}_{qdg} \simeq R$–$\text{rings}_{cdg}$ provided by Theorems 4.5 to the full subcategories of 3-right finitely projective quadratic CDG-rings and quasi-differential graded rings over $R$, we obtain an equivalence of categories

\begin{equation}
R$–$\text{rings}_{qdg,rq} \simeq R$–$\text{rings}_{cdg,rq}.
\end{equation}

Similarly, a restriction of the strict equivalence of 2-categories $\text{Rings}_{qdg2} \simeq \text{Rings}_{cdg2}$ provided by Theorem 4.6 produces a strict equivalence between the 2-categories of 3-right finitely projective quadratic CDG-rings and quasi-differential graded rings,

\begin{equation}
\text{Rings}_{qdg2,rq} \simeq \text{Rings}_{cdg2,rq}.
\end{equation}

Let $(\hat{\mathcal{A}}, F)$ be a filtered ring with an increasing filtration $0 = F_{-1} \mathcal{A} \subset F_0 \mathcal{A} \subset F_1 \mathcal{A} \subset F_2 \mathcal{A} \subset \cdots$. Consider two graded rings related to such a filtration: the associated graded ring $A = \text{gr}^F \mathcal{A}$ and the Rees ring $\hat{\mathcal{A}} = \bigoplus_{n=0}^{\infty} F_n \mathcal{A}$.

The unit element $1 \in F_0 \mathcal{A}$, viewed as an element of $F_1 \mathcal{A}$, represents a central nonzero-divisor $t \in \hat{\mathcal{A}}_1$. The quotient ring $\hat{\mathcal{A}}/\hat{\mathcal{A}}t$ is naturally isomorphic to the associated graded ring $A$ of the filtration $F$ on the ring $\mathcal{A}$. By Proposition 4.1, the graded ring $\hat{\mathcal{A}}$ is generated by $\hat{\mathcal{A}}_1$ over $\hat{\mathcal{A}}_0$ if and only if the graded ring $A$ is generated by $A_1$ over $A_0$; and moreover, the graded ring $\hat{\mathcal{A}}$ is quadratic if and only if the graded ring $A$ is.

Let $\hat{\mathcal{A}}$ be a weak nonhomogeneous quadratic ring over a subring $R \subset \mathcal{A}$ with the $R$-$R$-bimodule of generators $\mathcal{V} \subset \hat{\mathcal{A}}$. Let $F$ be the related increasing filtration on the ring $\hat{\mathcal{A}}$, as defined in Section 3.1. Then the graded ring $\hat{\mathcal{A}}$ is generated by $\hat{\mathcal{A}}_1$ over $\hat{\mathcal{A}}_0$, and the graded ring $A$ is generated by $A_1$ over $A_0$. The graded rings $\hat{\mathcal{A}}$ and $A$ do not need to be quadratic (as we only assume $\hat{\mathcal{A}}$ be a weak nonhomogeneous quadratic ring); so let us consider the quadratic graded rings $q\hat{\mathcal{A}}$ and $qA$.

**Lemma 4.15.** Let $\hat{\mathcal{A}}$ be a nonnegatively graded ring, and let $t \in \hat{\mathcal{A}}$ be a central element. Let $A = \hat{\mathcal{A}}/\hat{\mathcal{A}}t$ be the quotient ring. Then

(a) $t \in q\hat{\mathcal{A}}_1 = \hat{\mathcal{A}}_1$ is a central element in the quadratic graded ring $q\hat{\mathcal{A}}$;

(b) for each $n = 1, 2, \text{ or } 3$, the multiplication map $q\hat{\mathcal{A}}_{n-1} \rightarrow q\hat{\mathcal{A}}_n$ is injective whenever the multiplication map $\hat{\mathcal{A}}_{n-1} \rightarrow \hat{\mathcal{A}}_n$ is;

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(c) there is a natural isomorphism of quadratic graded rings \( q\hat{A}/(q\hat{A})t \simeq qA. \)

**Proof.** In part (a), the element \( t \in q\hat{A}_1 \) commutes with all the elements of \( q\hat{A}_0 \) and \( q\hat{A}_1 \), since the graded ring homomorphism \( q\hat{A} \rightarrow \hat{A} \) is an isomorphism in degree \( n \leq 1 \) and a monomorphism in degree \( n = 2 \), and the element \( t \in \hat{A}_1 \) is central in \( \hat{A} \). Since the graded ring \( q\hat{A} \) is generated by \( q\hat{A}_1 \) over \( q\hat{A}_0 \), it follows that the element \( t \in q\hat{A}_1 \) is central in \( qA \).

Part (b) follows from injectivity of the map \( q\hat{A}_{n-1} \rightarrow \hat{A}_{n-1} \) for \( n \leq 3 \) and commutativity of the square diagram \( q\hat{A}_{n-1} \rightarrow q\hat{A}_n \rightarrow \hat{A}_n \rightarrow q\hat{A}_{n-1} \).

In part (c), the graded ring homomorphism \( q\hat{A} \rightarrow \hat{A} \) induces a graded ring homomorphism between the quotient rings \( q\hat{A}/(q\hat{A})t \rightarrow \hat{A}/\hat{A}t = A \). Since the graded ring \( q\hat{A}/(q\hat{A})t \) is quadratic by Proposition 4.1, the latter morphism, in turn, induces the desired graded ring homomorphism \( f: q\hat{A}/(q\hat{A})t \rightarrow qA \). The map \( f \) is an isomorphism in the degrees \( n \leq 2 \) by construction, hence it is an isomorphism of quadratic graded rings by Lemma 3.2.

**Proposition 4.16.** Let \( R \subset \tilde{V} \subset \hat{A} \) be a 3-left finitely projective weak nonhomogeneous quadratic ring. Consider the corresponding graded rings \( \hat{A} = \bigoplus_{n=0}^{\infty} F_n \hat{A} \) and \( A = \text{gr}^F \hat{A} \). Then the quadratic graded rings \( q\hat{A} \) and \( qA \) are 3-left finitely projective.

Let \( \tilde{B} \) be the 3-right finitely projective quadratic graded ring quadratic dual to \( q\hat{A} \), and let \( \partial: \tilde{B} \rightarrow \tilde{B} \) be the odd derivation of degree \(-1\) corresponding to the central element \( t \in \hat{A}_1 \), as in Lemma 4.11(b). Then \((\tilde{B}, \partial)\) is a quasi-differential graded ring. The underlying graded ring \( B = \ker \partial \subset \tilde{B} \) is 3-right finitely projective quadratic and quadratic dual to \( qA \).

**Proof.** The quadratic graded ring \( qA \) is 3-left finitely projective by the definition of what it means for a weak nonhomogeneous quadratic ring \( \hat{A} \) to be 3-left finitely projective (see Section 3.3). To prove that the left \( R \)-module \( q\hat{A}_n \) is finitely generated projective for \( n \leq 3 \), one argues by induction in \( 0 \leq n \leq 3 \), using the short exact sequences of \( R\)-\( R \)-bimodules \( 0 \rightarrow q\hat{A}_{n-1} \rightarrow q\hat{A}_n \rightarrow q\hat{A}_n \rightarrow 0 \) provided by Lemma 4.15. Here the multiplication maps \( q\hat{A}_{n-1} \rightarrow q\hat{A}_n \) are injective for \( n \leq 3 \) by Lemma 4.15(b), since \( t \) is a nonzero-divisor in \( \hat{A} \).

The pair \((\tilde{B}, \partial)\) is a quasi-differential graded ring by Lemma 4.12 (applied to the quadratic graded ring \( q\hat{A} \) with its central element \( t \)). The graded ring \( B = \ker \partial \subset \tilde{B} \) is generated by \( B^1 \) over \( B^0 \) by Proposition 4.13(a) and quadratic by Proposition 4.14. The quadratic graded ring \( B \) is 3-right finitely projective by Lemma 4.9. It is quadratic dual to \( qA \) by Proposition 4.13(b).

**Theorem 4.17.** The construction of Proposition 4.16 defines a fully faithful contravariant functor

\[
(R\text{-rings}_{wnq})^{\text{op}} \longrightarrow R\text{-rings}_{qd, rq}
\]

from the category of 3-left finitely projective weak nonhomogeneous quadratic rings to the category of 3-right finitely projective quasi-differential quadratic graded rings.
over $R$. The functor (53) forms a commutative triangle diagram with the fully faithful contravariant functor (33) and the equivalence of categories (51).

The same construction also defines a fully faithful strict contravariant 2-functor (54)

$$\text{(Rings}_{\text{wnlq}2})^{\text{op}} \rightarrow \text{Rings}_{\text{qdgr}2_{\text{rq}}}$$

from the 2-category of 3-left finitely projective weak nonhomogeneous quadratic rings to the 2-category of 3-right finitely projective quasi-differential quadratic graded rings. The strict 2-functor (54) forms a commutative triangle diagram with the fully faithful strict contravariant 2-functor (40) and the strict equivalence of 2-categories (52).

Proof. The proof is straightforward.

4.6. PBW theorem. The Poincaré–Birkhoff–Witt theorem in nonhomogeneous quadratic duality tells that, when restricted to the left/right finitely projective Koszul rings on both sides, the fully faithful contravariant nonhomogeneous quadratic duality functors from Theorems 3.9, 3.12, 3.13, 3.14, and 4.17 become anti-equivalences of categories (or strict anti-equivalences of 2-categories). In other words, every right finitely projective Koszul CDG-ring (or quasi-differential graded ring) arises from a left finitely projective nonhomogeneous Koszul ring.

A (weak) nonhomogeneous quadratic ring $R \subset \tilde{V} \subset \tilde{A}$ is said to be left finitely projective Koszul if the quadratic graded ring $q\tilde{A} = qgr^F\tilde{A}$ is left finitely projective Koszul (in the sense of the definition in Section 2.9). A nonnegatively graded CDG-ring $(B, d, h)$ is said to be right finitely projective Koszul if the nonnegatively graded ring $B$ is right finitely projective Koszul. A nonnegatively graded quasi-differential ring $(\hat{B}, \partial)$ is said to be right finitely projective Koszul if its underlying graded ring $B = \ker \partial \subset \hat{B}$ is right finitely projective Koszul.

The following result was mentioned in Remark 3.3 in Section 3.1.

**Theorem 4.18.** If a weak nonhomogeneous quadratic ring is left finitely projective Koszul, then it is nonhomogeneous quadratic.

So we will call such filtered rings as in Theorem 4.18 left finitely projective nonhomogeneous Koszul rings. Notice that a filtered ring $(\tilde{A}, F)$ with an increasing filtration $F$ such that $A_n = \text{gr}_n^F \tilde{A}$ is a projective left $A_0$-module for all $n \geq 0$ is left finitely projective Koszul (i.e., the graded ring $A = \text{gr}^F \tilde{A}$ is left finitely projective Koszul) if and only if the Rees ring $\hat{A} = \bigoplus_{n=0}^{\infty} F_n \tilde{A}$ is left finitely projective Koszul. This is clear from Lemma 4.2 and Theorem 4.3.

The proof of Theorem 4.18 will be given at the end of Section 4.6.

**Theorem 4.19.** Every right finitely projective Koszul CDG-ring arises from a left finitely projective nonhomogeneous Koszul ring via the construction of Proposition 3.6. Equivalently, every right finitely projective Koszul quasi-differential graded ring arises from a left finitely projective nonhomogeneous Koszul ring via the construction of Proposition 4.16.

The two assertions in Theorem 4.19 are equivalent in view of Theorems 4.5 and 4.17. We will prove the second assertion.
First proof of Theorem 4.19. Let \((\hat{B}, \partial)\) be a right finitely projective Koszul quasi-differential graded ring. Then the graded ring \(B\) is right finitely projective Koszul by definition and the graded ring \(\hat{B}\) is right finitely projective Koszul by Theorem 4.10(b). Let \(\hat{A}\) be the left finitely projective Koszul ring quadratic dual to \(\hat{B}\) (see Proposition 2.19), and let \(t \in \hat{A}_1\) be the central element corresponding to the odd derivation \(\partial: \hat{B} \to \hat{B}\) (see Lemma 4.11). Let \(A = \hat{A}/\hat{A}t\) be the quotient ring.

By Lemma 4.12, the multiplication map \(\hat{A}_0 \overset{t}{\to} \hat{A}_1\) is injective and \(A_1\) is a projective left \(R\)-module. By Proposition 4.13(a), the multiplication map \(\hat{A}_1 \rightarrowtail \hat{A}_2\) is injective and \(A_2\) is a projective left \(R\)-module. By Proposition 4.13(b), the right finitely projective Koszul ring \(B\) is quadratic dual to the \(2\)-left finitely projective quadratic ring \(A\). Therefore, the quadratic graded ring \(A\) is left finitely projective Koszul. By Proposition 4.14, the multiplication map \(\hat{A}_2 \rightarrowtail \hat{A}_3\) is injective.

Applying Theorem 4.4, we conclude that \(t\) is a nonzero-divisor in \(\hat{A}\). It follows that \(\hat{A}\) is the Rees ring of the filtered ring \(A = \hat{A}/\hat{A}(t - 1)\) with the filtration \(F_nA = \hat{A}_n + \hat{A}(t - 1)\), and \(A\) is the associated graded ring, \(A \simeq gr^{\hat{F}}\hat{A}\). We have obtained the desired left finitely projective nonhomogeneous Koszul ring \(\hat{A}\). □

Second proof of Theorem 4.19. This is a particular case of the argument in [14, proof of Theorem 11.6]. Let \((\hat{B}, \partial)\) be a right finitely projective Koszul quasi-differential graded ring and \(B = \ker \partial \subset \hat{B}\) be its underlying right finitely projective Koszul graded ring. Then the right \(R\)-modules \(B^n\) are finitely generated projective by definition and the right \(R\)-modules \(\hat{B}^n\) are finitely generated projective by Lemma 4.9.

The (essentially) spectral sequence argument below is to be compared with, and distinguished from, a quite different (and simpler) spectral sequence argument proving the Poincaré–Birkhoff–Witt theorem for nonhomogeneous quadratic algebras over the ground field in [12, Section 3.3] and [11, Proposition 7.2 in Chapter 5].

Put \(D_n = \text{Hom}_{R^{op}}(B^n, R)\). Then \(D = \bigoplus_{n=0}^{\infty} D_n\) is a graded coring over the ring \(R\) with the counit map \(\varepsilon: D \to D_0 \simeq R\) dual to the inclusion map \(R \simeq B^0 \to B\) and the comultiplication maps \(\mu_{ij}: D_{i+j} \to D_i \otimes_R D_j\) obtained by dualizing the multiplication maps \(B^i \otimes_R B^j \to B^{i+j}\) in the graded ring \(B\),

\[
D_{i+j} = \text{Hom}_{R^{op}}(B^{i+j}, R) \longrightarrow \text{Hom}_{R^{op}}(B^i \otimes_R B^j, R) \simeq \text{Hom}_{R^{op}}(B_i, R) \otimes_R \text{Hom}_{R^{op}}(B_j, R) = D_i \otimes_R D_j.
\]

In the pairing notation of Section 3.4, we have \(\langle \mu(f), b_1 \otimes b_2 \rangle = \langle f, b_1 b_2 \rangle\) for all \(f \in D\) and \(b_1, b_2 \in B\).

Similarly, we set \(\hat{D}_n = \text{Hom}_{R^{op}}(B^n, R)\); so \(\hat{D} = \bigoplus_{n=0}^{\infty} \hat{D}_n\) is also a graded coring over \(R\). The odd derivation \(\partial: \hat{B} \to \hat{B}\) is an \(R-R\)-bilinear map, so it dualizes to an \(R-R\)-bilinear map \(\text{Hom}_{R^{op}}(\partial, R): \hat{D} \to \hat{D}\), which we denote for brevity also by \(\partial\). There is sign rule involved: in the pairing notation, we put \(\langle \partial(f), b \rangle = (-1)^{|f|} \langle f, \partial(b) \rangle\) for \(f \in \hat{D}\) and \(b \in \hat{B}\). The map \(\partial: \hat{D} \to \hat{D}\) is an odd coderivation of degree 1 on the coring \(\hat{D}\), in the sense that its components act as \(\partial_n: \hat{D}_n \to \hat{D}_{n+1}\) and, in the
symbolic notation $\mu(f) = \mu_1(f) \otimes \mu_2(f)$ for the comultiplication, one has

$$\partial(\mu(f)) = \partial \mu_1(f) \otimes \mu_2(f) + (-1)^{[\mu_1(f)]} \mu_1(f) \otimes \partial \mu_2(f)$$

for all $f \in \hat{D}$.

The odd coderivation $\partial$ on the graded coring $\hat{D}$ has zero square, $\partial^2 = 0$ (so one can say that $\hat{D}$ is a DG-coring over $R$). Furthermore, the odd coderivation $\partial$ on $\hat{D}$ has vanishing cohomology, $H_\partial(\hat{D}) = 0$. The cokernel of $\partial$ is the quotient coring $D$ of the coring $\hat{D}$, that is $\hat{D} \twoheadrightarrow \text{coker}(\partial) \simeq D$.

Consider the bigraded $R$-$R$-bimodule $K$ with the components $K^{p,q} = \hat{D}_{q-p}$ for $p \leq 0$, $q \leq 0$, and $K^{p,q} = 0$ otherwise. The $R$-$R$-bimodule $K$ can be viewed as a bigraded coring over $R$ with the comultiplication inherited from the comultiplication of $\hat{D}$. Considered as a graded coring in the total grading $p + q$, the coring $K$ has an odd coderivation $\partial_K$ of degree 1 with the components $\partial_K^{p,q} : K^{p,q} \rightarrow K^{p,q+1}$ given by the rule $\partial_K^{p,q} = \partial_{q-p} : \hat{D}_{q-p} \rightarrow \hat{D}_{q-p+1}$. So for every fixed $p = -n$, $n \geq 0$, the components $K^{p,q}$ with varying $q$ form a complex of $R$-$R$-bimodules

$$0 \longrightarrow K^{p,p} \overset{\partial_K^{p,p}}{\longrightarrow} K^{p,p+1} \overset{\partial_K^{p,p+1}}{\longrightarrow} \cdots \overset{\partial_K^{p,-2}}{\longrightarrow} K^{p,-1} \overset{\partial_K^{p,-1}}{\longrightarrow} K^{p,0} \longrightarrow 0$$

isomorphic to

$$0 \longrightarrow \hat{D}_0 \overset{\partial_0}{\longrightarrow} \hat{D}_1 \overset{\partial_1}{\longrightarrow} \cdots \overset{\partial_{n-2}}{\longrightarrow} \hat{D}_{n-1} \overset{\partial_{n-1}}{\longrightarrow} \hat{D}_n \longrightarrow 0.$$
All the three derivations are constructed to be odd derivations in the parity $p+q+r$. The derivations $\partial_T$, $d_T$, and $\delta_T$ have tridegrees $(0,1,0)$, $(0,0,1)$, and $(1,1,-1)$, respectively, in the trigrading $(p,q,r)$ of the tensor ring $T_R(K_+)$. All the three derivations have zero squares, and they pairwise anti-commute.

There is an increasing filtration $F$ on the graded ring $T_R(K_+)$ whose component $F_nT_R(K_+)$ is the direct sum of all the trigrading components $T_R(K_+)^{p+q+r}$ with $-p \leq n$. This filtration is compatible with the differentials $\partial_T$, $d_T$, and $\delta_T$; the graded ring $\text{gr}^F T_R(K_+) = \bigoplus_{n=0}^{\infty} F_nT_R(K_+)/F_{n-1}T_R(K_+)$ with the differential induced by $\partial_T + d_T + \delta_T$ is naturally isomorphic to $T_R(K_+)$ with the differential $\partial_T + d_T$.

Let us also consider the tensor ring $T_R(D_+)$ of the $R$-$R$-bimodule $D_+ = D/R$. Similarly to the above, we endow $T_R(D_+)$ with the grading $p$ coming from the grading $D^p = D_n = D_{-p}$ of $D_+$ and the grading $r$ by the number of tensor factors. As to the grading $q$, we set it to be identically zero on $T_R(D_+)$. We will consider $T_R(D_+)$ as a graded ring in the total grading $p + r$. Let $d'_T$ be the only odd derivation of $T_R(D_+)$ which maps the subbimodule $D_+ \subset T_R(D_+)$ to $D_+ \otimes_R D_+$ by the comultiplication map $\mu$ with the sign rule similar to the above.

The DG-ring $T_R(D_+)$ is naturally isomorphic to the cobar complex (16) (with the roles of the left and right sides switched) computing the bigraded $\text{Ext}_{L_R^p}(R,R)$. There is some sign rule involved in this isomorphism, which is discussed in [14, proof of Theorem 11.6]. By Proposition 2.19, it follows that the bigraded ring of cohomology of the DG-ring $T_R(D_+)$ is isomorphic to the left finitely projective Koszul graded ring $A = \bigoplus_{n=0}^{\infty} A_n$ quadratic dual to $B$, placed in the diagonal bigrading $-p = r = n$. In the total grading $p + r$, the cohomology ring of the DG-ring $T_R(D_+)$ is the whole ring $A$ placed in the degree $p + r = 0$.

Consider the morphism of DG-rings $(T_R(K_+), \partial_T + d_T) \to (T_R(D_+), d'_T)$ induced by the surjective morphism of graded corings $K \to D$. The morphism of DG-corings $K \to D$ is a quasi-isomorphism, so the induced morphism of DG-rings is a quasi-isomorphism, too, due to the right flatness/projectivity conditions imposed on the $R$-$R$-bimodule $D$ and the presence of a nonpositive (essentially, negative) internal grading $p$. The fact that the components of fixed grading $p$ in the DG-co-ring $K$ are finite complexes (of projective right $R$-modules, with projective right $R$-modules of cohomology) is relevant here. Thus we have $H^i_{\partial_T + d_T}(T_R(K_+)) \simeq A$ and $H^i_{\partial_T + d_T}(T_R(K_+)) = 0$ for $i \neq 0$.

Finally, we put $\tilde{A} = H^0_{\partial_T + d_T}(T_R(K_+))$. Then the ring $\tilde{A}$ is endowed with an increasing filtration $F$ induced by the filtration $F$ of the DG-ring $(T_R(K_+), \partial_T + d_T + \delta_T)$. Since $H^i_{\partial_T + d_T}(T_R(K_+)) = 0$ for $i \neq 0$, we can conclude that the associated graded ring of the ring $A$ is naturally isomorphic to the graded ring $A$, that is $\text{gr}^F \tilde{A} \simeq A$, while $H^i_{\partial_T + d_T}(T_R(K_+)) = 0$ for $i \neq 0$.

Since the graded ring $A = \text{gr}^F \tilde{A}$ is left finitely projective Koszul, the graded ring $\tilde{A} = \bigoplus_{n=0}^{\infty} F_n \tilde{A}$ is left finitely projective Koszul as well (by Lemma 4.2 and Theorem 4.3). Let $\tilde{B}'$ be the right finitely projective Koszul graded ring quadratic dual.
to $\hat{\tilde{A}}$. By Proposition 4.16, the graded ring $\hat{\tilde{B}}'$ is endowed with an odd derivation $\partial': \hat{\tilde{B}}' \to \hat{\tilde{B}}'$ of degree $-1$, making it a right finitely projective Koszul quadratic graded ring. The underlying graded ring $\hat{\tilde{B}}' = \ker \partial' \subset \hat{\tilde{B}}'$ of the quadratic differential graded ring $(\hat{\tilde{B}}', \partial')$ is quadratic dual to $A$, so we have $\hat{\tilde{B}}' \simeq B$. It remains to construct a natural isomorphism of quasi-differential graded rings $(\hat{\tilde{B}}', \partial') \simeq (\tilde{B}, \partial)$.

For this purpose, let us consider the dual graded corings $\hat{\tilde{D}}' = \text{Hom}_{\text{op}}(\hat{\tilde{B}}', R)$ and $\hat{\tilde{D}} = \text{Hom}_{\text{op}}(\hat{\tilde{B}}, R)$. As in the beginning of this proof, we have an odd coderivation $\partial: \hat{\tilde{D}}' \to \hat{\tilde{D}}'$ of degree $1$ dual to the odd derivation $\partial': \hat{\tilde{B}}' \to \hat{\tilde{B}}'$. It suffices to construct a natural isomorphism $(\hat{\tilde{D}}, \partial) \to (\hat{\tilde{D}}', \partial')$ of DG-corings over $R$.

The embedding of the component $\hat{\tilde{D}}_1 = T_R(K_+)^{-1,0,1} \to T_R(K_+)$ induces an isomorphism of $R$-$R$-bimodules $\hat{\tilde{D}}_1 \simeq F_1\hat{\tilde{A}}$. The composition $\hat{\tilde{D}}_2 \to \hat{\tilde{D}}_1 \otimes_R \hat{\tilde{D}}_1 \simeq F_1\hat{\tilde{A}} \otimes_R F_1\hat{\tilde{A}} \to F_2\hat{\tilde{A}}$ of the comultiplication and multiplication maps vanishes, being killed by the differential $(\partial_T + d_T + \delta_T)^{-2,0,1} = d_T^{-2,0,1}: T_R(K_+)^{-2,0,1} \to T_R(K_+)^{-2,0,2}$. So there is a natural morphism of graded corings $\hat{\tilde{D}} \to \hat{\tilde{D}}'$. Since the embedding $R = F_0\hat{\tilde{A}} \to F_1\hat{\tilde{A}}$ corresponds to the map $\tilde{D}_0: R = \tilde{D}_0 \to \tilde{D}_1$ under the isomorphisms $F_0\hat{\tilde{A}} = R = \tilde{D}_0$ and $F_1\hat{\tilde{A}} \simeq \tilde{D}_1$, the graded coring morphism $\hat{\tilde{D}} \to \hat{\tilde{D}}'$ forms a commutative square diagram with the differentials $\partial$ and $\partial'$ on $\hat{\tilde{D}}$.

The induced morphism $\text{coker}(\partial) \to \text{coker}(\partial')$ coincides with the natural isomorphism $D \to D'$ on the components of degree $1$, and hence on the other components as well. Therefore, the morphism of corings $\hat{\tilde{D}} \to \hat{\tilde{D}}'$ is also an isomorphism.

**Proof of Theorem 4.18.** Let $R \subset \tilde{V} \subset \hat{\tilde{A}}$ be a weak nonhomogeneous quadratic ring and $A = \text{gr}^F\hat{\tilde{A}}$ be its associated graded ring with respect to the filtration $F$ generated by $F_1\hat{\tilde{A}} = \tilde{V}$ over $F_0\hat{\tilde{A}} = R$. Assume that the quadratic graded ring $qA = q\text{gr}^F\hat{\tilde{A}}$ is left finitely projective Koszul. Let $(B, d, h)$ be the CDG-ring corresponding to $\hat{\tilde{A}}$ under the construction of Proposition 3.6. Equivalently, one can consider the quasi-differential graded ring $(\hat{\tilde{B}}, \partial)$ with the underlying graded ring $B = \ker \partial \subset \hat{\tilde{B}}$ corresponding to $\hat{\tilde{A}}$ under the construction of Proposition 4.16.

Whichever one of these two points of view one takes, the graded ring $B$ is quadratic and quadratic dual to the quadratic graded ring $qA$ by construction. Hence the graded ring $B$ is right finitely projective Koszul by Proposition 2.19. Applying Theorem 4.19, we see that the CDG-ring $(B, d, h)$ or the quasi-differential ring $(\hat{\tilde{B}}, \partial)$ comes from a left finitely projective nonhomogeneous Koszul ring $(\tilde{A}', F)$. Following either one of the two proofs of Theorem 4.19, the graded ring $\text{gr}^F\tilde{A}'$ is quadratic and left finitely projective Koszul.

It remains to observe that the nonhomogeneous quadratic duality functor assigns the same CDG-ring $(B, d, h)$ (or the same quasi-differential graded ring $(\hat{\tilde{B}}, \partial)$) to the two $3$-left finitely projective (weak) nonhomogeneous quadratic rings $\tilde{A}$ and $\tilde{A}'$. Since the nonhomogeneous quadratic duality functor is fully faithful by Theorem 3.9 or 4.17, it follows that the two (weak) nonhomogeneous quadratic rings $(\tilde{A}, F)$ and $(\tilde{A}', F)$ are isomorphic. Hence the graded ring $A = \text{gr}^F\tilde{A} \simeq \text{gr}^F\tilde{A}'$ is quadratic (and
left finitely projective Koszul). In other words, the weak nonhomogeneous quadratic ring $R \subset \tilde{V} \subset \tilde{A}$ is actually nonhomogeneous quadratic. □

4.7. Anti-equivalences of Koszul ring categories. Let $R$ be an associative ring. The category of left finitely projective nonhomogeneous Koszul rings over $R$, denoted by $R$–rings$_{nlk}$, is defined as the full subcategory of the category of filtered rings $R$–rings$_{fil}$ (see Section 3.6) whose objects are the left finitely projective nonhomogeneous Koszul rings $(\tilde{A}, F)$. The category of right finitely projective Koszul CDG-rings over $R$, denoted by $R$–rings$_{cdg, rk}$, is the full subcategory in the category of nonnegatively graded CDG-rings $R$–rings$_{cdg}$ whose objects are the right finitely projective Koszul CDG-rings. The category of right finitely projective Koszul quasi-differential rings over $R$, denoted by $R$–rings$_{qdg, rk}$, is the full subcategory in the category of nonnegatively graded quasi-differential rings $R$–rings$_{qdg}$ whose objects are the right finitely projective Koszul quasi-differential graded rings.

Corollary 4.20. The constructions of Theorems 3.9, 4.5, and 4.17 define natural (anti)-equivalences

$$(R$–rings$_{nlk})^{\text{op}} \simeq R$–rings$_{qdg, rk} \simeq R$–rings$_{cdg, rk}$$

between the categories of left finitely projective nonhomogeneous Koszul rings, right finitely projective Koszul quasi-differential rings, and right finitely projective CDG-rings over $R$.

Proof. Follows from the mentioned theorems and Theorem 4.19. □

The 2-category of left finitely projective nonhomogeneous Koszul rings, denoted by $\text{Rings}_{nk2}$, is defined as the following 2-subcategory in the 2-category of filtered rings $\text{Rings}_{fil2}$ (see Section 3.7). The objects of $\text{Rings}_{fil2}$ are the left finitely projective nonhomogeneous Koszul rings $(\tilde{A}, F)$. All morphisms in $\text{Rings}_{fil2}$ between objects of $\text{Rings}_{nk2}$ are morphisms in $\text{Rings}_{nk2}$, and all 2-morphisms in $\text{Rings}_{fil2}$ between morphisms of $\text{Rings}_{nk2}$ are 2-morphisms in $\text{Rings}_{nk2}$.

The 2-category of right finitely projective Koszul CDG-rings, denoted by $\text{Rings}_{cdg, rk}$, is the following 2-subcategory in the 2-category of nonnegatively graded CDG-rings $\text{Rings}_{cdg2}$. The objects of $\text{Rings}_{cdg2}$ are all the right finitely projective Koszul CDG-rings $(B, d, h)$. All morphisms in $\text{Rings}_{cdg2}$ between objects of $\text{Rings}_{cdg, rk}$ are morphisms in $\text{Rings}_{cdg, rk}$, and all 2-morphisms in $\text{Rings}_{cdg2}$ between morphisms of $\text{Rings}_{cdg, rk}$ are 2-morphisms in $\text{Rings}_{cdg, rk}$.

The 2-category of right finitely projective Koszul quasi-differential graded rings, denoted by $\text{Rings}_{qdg, rk}$, is the following 2-subcategory in the 2-category of nonnegatively graded quasi-differential rings $\text{Rings}_{qdg2}$ (see Section 4.2). The objects of $\text{Rings}_{qdg2}$ are all the right finitely projective Koszul quasi-differential graded rings $(\hat{B}, \partial)$. All morphisms in $\text{Rings}_{qdg2}$ between objects of $\text{Rings}_{qdg, rk}$ are morphisms in $\text{Rings}_{qdg, rk}$, and all 2-morphisms in $\text{Rings}_{qdg2}$ between morphisms of $\text{Rings}_{qdg, rk}$ are 2-morphisms in $\text{Rings}_{qdg, rk}$. 

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Corollary 4.21. The constructions of Theorems 3.12, 4.6, and 4.17 define natural strict (anti)-equivalences

\[ (\text{Rings}_{\text{nlk}})^{\text{op}} \simeq \text{Rings}_{\text{qdg}}^{\text{rk}} \simeq \text{Rings}_{\text{cdg}}^{\text{rk}} \]

between the 2-categories of left finitely projective nonhomogeneous Koszul rings, right finitely projective Koszul quasi-differential rings, and right finitely projective CDG-rings.

Proof. Follows from the mentioned theorems and Theorem 4.19. \(\square\)

A left finitely projective nonhomogeneous Koszul ring \((\widetilde{A}, F)\) is said to be left augmented if the ring \(\widetilde{A}\) is left augmented over its subring \(F_{0}\). In other words, this means that a left ideal \(\widetilde{A}^{+} \subset \widetilde{A}\) is chosen such that \(\widetilde{A} = F_{0}\widetilde{A} \oplus \widetilde{A}^{+}\) (see Section 3.8). The category of left augmented left finitely projective nonhomogeneous Koszul rings over \(R\), denoted by \(\text{R}^{\text{laug}}_{\text{nlk}}\), is defined as the full subcategory in the category of left augmented filtered rings \(\text{R}^{\text{fil}}\) whose objects are the left augmented left finitely projective nonhomogeneous Koszul rings.

A nonnegatively graded DG-ring \((B, d)\) (in the sense of Section 3.8) is said to be right finitely projective Koszul if the graded ring \(B\) is right finitely projective Koszul. The category of right finitely projective Koszul DG-rings over \(R\), denoted by \(\text{R}^{\text{rk}}_{\text{dg}}\), is defined as the full subcategory in the category of nonnegatively graded DG-rings \(\text{R}^{\text{dg}}\) whose objects are the right finitely projective Koszul DG-rings.

Corollary 4.22. The construction of Theorem 3.13 defines a natural anti-equivalence

\[ (\text{R}^{\text{laug}}_{\text{nlk}})^{\text{op}} \simeq \text{R}^{\text{rk}}_{\text{dg}} \]

between the category of left augmented left finitely projective nonhomogeneous Koszul rings and the category of right finitely projective Koszul DG-rings.

Proof. Follows from Theorem 4.19 and the discussion in the proof of Theorem 3.13. \(\square\)

The 2-category of left augmented left finitely projective nonhomogeneous Koszul rings, denoted by \(\text{R}^{\text{laug}}_{\text{nlk}}\), is defined as the following 2-subcategory in the 2-category of left augmented filtered rings \(\text{R}^{\text{fil}}\) (see Section 3.8). The objects of \(\text{R}^{\text{laug}}_{\text{nlk}}\) are the left augmented left finitely projective nonhomogeneous Koszul rings \((\widetilde{A}, F, \widetilde{A}^{+})\). All morphisms in \(\text{R}^{\text{laug}}_{\text{fil}}\) between objects of \(\text{R}^{\text{laug}}_{\text{nlk}}\) are morphisms in \(\text{R}^{\text{laug}}_{\text{fil}}\), and all 2-morphisms in \(\text{R}^{\text{laug}}_{\text{fil}}\) between morphisms of \(\text{R}^{\text{laug}}_{\text{nlk}}\) are 2-morphisms in \(\text{R}^{\text{laug}}_{\text{fil}}\).

The 2-category of right finitely projective Koszul DG-rings, denoted by \(\text{R}^{\text{rk}}_{\text{dg}}\), is defined as the following 2-subcategory in the 2-category of nonnegatively graded DG-rings \(\text{R}^{\text{dg}}\). The objects of \(\text{R}^{\text{rk}}_{\text{dg}}\) are the right finitely projective Koszul DG-rings \((B, d)\). All morphisms in \(\text{R}^{\text{dg}}\) between objects of \(\text{R}^{\text{dg}}\) are morphisms in \(\text{R}^{\text{dg}}\), and all 2-morphisms in \(\text{R}^{\text{dg}}\) between morphisms of \(\text{R}^{\text{dg}}\) are 2-morphisms in \(\text{R}^{\text{dg}}\).
Corollary 4.23. The construction of Theorem 3.14 defines a natural strict anti-equivalence
\[(\text{Rings}^{\text{aug}}_{\text{nlk}})^{\text{op}} \simeq \text{Rings}_{\text{dg}^2,\text{rk}}\]
between the 2-category of left augmented left finitely projective nonhomogeneous Koszul rings and the 2-category of right finitely projective Koszul DG-rings. □

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