UNIQUELY SEPARABLE EXTENSIONS

LARS KADISON

Abstract. The separability tensor element of a separable extension of noncommutative rings is an idempotent when viewed in the correct endomorphism ring; so one speaks of a separability idempotent, as one usually does for separable algebras. It is proven that this idempotent is full if and only the H-depth is one. Similarly, a split extension has a bimodule projection; this idempotent is full if and only if the ring extension has depth 1. The depth one Hopf algebroids are derived explicitly. If the separable idempotent is unique for some reason, then the separable extension is called uniquely separable. For example, a Frobenius extension with invertible E-index is uniquely separable if the centralizer equals the center of the over-ring. It is also shown that a uniquely separable extension of semisimple complex algebras with invertible E-index has depth 1. Earlier results for subalgebra pairs of group algebras are recovered, with corollaries on depth 1 over fields of characteristic zero and more general ground rings.

1. Introduction and Preliminaries

The classical notion of separable algebra is one of a semisimple algebra that remains semisimple under every base field extension. The approach of Hochschild to Wedderburn’s theory of associative algebras in the Annals was cohomological, and characterized a separable k-algebra A by having a separability idempotent in $A^e = A \otimes_k A^{\text{op}}$. A separable extension of noncommutative rings is characterized similarly by possessing separability elements in [14]: separable extensions are shown to be left and right semisimple extensions in terms of Hochschild’s relative homological algebra (1956). In related developments around 1960, also separable algebras over commutative rings, Galois and Brauer theory of commutative rings were first defined and studied by Auslander and Goldman. It is pointed out in Section 1 of this paper that the separability element of a (unital) ring extension of noncommutative rings $B \hookrightarrow A$ is an idempotent in $(A \otimes_B A)^B (\cong \text{End}_{A} A \otimes_B A_A)$. We prove

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that this idempotent is full if and only if the extension is H-separable, a strong condition generalizing the notion of Azumaya algebra.

Separability idempotents in $A \otimes_B A$ for a separable extension $A \supseteq B$ are generally not unique. For example, if $A = M_n(B)$ there are $n$ different separability idempotents defined as in Example 1.5 so that uniqueness can only happen for $n = 1$. For example, given a finite-dimensional group algebra $A = kG$ over an algebraically closed field $k$ with characteristic not dividing the order of $G$, one sees from the Wedderburn decomposition of semisimple algebras that $A$ has unique separability idempotent if and only if $G$ is abelian. The papers [35, 37] describe a condition on a subgroup $H$ of a finite group $G$ that is equivalent to $A = KG \supseteq B = KH$ having a unique separability element where $K$ is a commutative ring in which $|G : H|1$ is invertible: their condition is that each conjugacy class of $G$ be an $H$-orbit. For example, this happens if $G = HZ(G)$ where $Z(G)$ is the center of $G$. In Section 3 we generalize this to a separable Frobenius extension $A \supseteq B$ having invertible $E$-index in the center $Z(A)$. We show in Theorem 3.1 that $A$ has a unique separability element over $B$ if and only if the centralizer $A_B = Z(A)$. If $K$ is an algebraically closed field of characteristic zero, it follows from the trivial observation $Z(B) \subseteq A_B = Z(A)$ and Burciu’s characterization of depth one in [7] that $A$ is centrally projective over $B$: $B A_B \oplus \ast \cong n \cdot B B_B$, or $A$ as a natural $B$-bimodule is isomorphic to a direct summand of a finite direct sum of copies of $B$. It is shown in [25] that such extensions automatically satisfy $B B B_B \oplus \ast \cong B A_B$, so that a centrally projective extension $A \supseteq B$ is characterized by the bimodules $B A_B$ and $B B_B$ being similar [1].

Centrally projective extensions are the depth 1 case of odd minimal depth $d(B, A) = 2n + 1$ where a ring extension $A \supseteq B$ has similar (natural) $B$-$B$-bimodules $A \otimes B^n$ and $A \otimes B^{(n+1)} := A \otimes_B \cdots \otimes_B A$ ($n + 1$ times $A$). The H-separable extension $A \supseteq B$ defined in [13] is the H-depth 1 case of odd minimal H-depth $d_H(B, A) = 2n - 1$ where $A \otimes B^n \sim A \otimes B^{(n+1)}$ as $A$-bimodules. See below in this section for more details on depth and H-depth. We show in two propositions of Section 1 that H-depth one and depth one ring extensions are characterizable in terms of a full idempotent in an endomorphism ring above the ring extension (in fact, isomorphic to the centralizer rings of the Jones tower that receive Hopf structure in [21, 18, 19, 20]). In the last sections of this paper we show that unique separable subalgebra pairs of semisimple complex algebras with invertible $E$-index have depth 1. This is compared to results of [35, 37, Singh, Hanna] and [3, Boltje-Külshammer]
on uniquely separable group ring extensions and subgroups of finite groups having depth 1.

1.1. **Separable extensions and finite-dimensional algebra extensions.** A ring extension $R \supseteq S$ is a separable extension if $\mu : R \otimes_S R \to R$ splits as an $R$-$R$-bimodule epimorphism. This is clearly equivalent to there being an element $e \in R \otimes_S R$ such that $re = er$ for every $r \in R$ and $\mu(e) = 1$. (Briefly, one writes $e \in (R \otimes_S R)^R$ and $e^1 e^2 = 1$ and calls such an $e$ a separability element.) Separable extensions are characterized by having relative Hochschild cohomological dimension zero, and a separable extension $R \supseteq S$ satisfies the inequality in right global dimension of rings $D(S) \geq D(R)$ if $R_S$ is projective [14].

Note that the $S$-central elements in $R \otimes_S R$, denoted by $T : = (R \otimes_S R)^R$ are isomorphic to the endomorphism ring $\text{End}_R R \otimes_S R$ via $t \mapsto (r \otimes_S r') \mapsto rtr'$. The inverse mapping is of course $F \mapsto F(1 \otimes 1)$. This transfers a ring structure onto $T$ given in Sweedler-like notation by $t', t = t^1 \otimes t^2 \in (R \otimes_S R)^S$ and

$$tt' = t'^1 t^1 \otimes t^2 t'^2, \quad 1_T = 1_R \otimes_S 1_R.$$ 

A separability element $e = e^1 \otimes_S e^2 \in (R \otimes_S R)^R \subseteq T$ and satisfies $e^2 = e$ in this multiplication, since $e^1 e^2 = 1_R$. Thus it makes sense to continue calling it a *separability idempotent*, in continuation of the terminology in the special case of separable algebras (over a commutative ring) when $S \subseteq Z(R)$, the center of $R$.

The rest of the subsection studies theorems that are useful for identifying when subalgebras of a finite-dimensional algebra form a separable extension, or not. When the subalgebra is trivially one-dimensional, the important theorem for identifying a separable algebra states the following [33, 10.7]: "An algebra is separable if and only if it is semisimple with block matrix algebras over division algebras whose centers are separable field extensions of the ground field." The following lemma compiles the most useful results towards this end from [14] and [8], with easy proofs (for not necessarily finite-dimensional algebras).

**Lemma 1.1.** The following holds for a separable extension $A \supseteq B$.

- (A) If $\pi : A \to A'$ is an algebra epimorphism, with $\pi(B) = B'$, then $A'$ is a separable extension of $B'$.
- (B) If $T$ is an intermediate ring $A \supseteq T \supseteq B$, then $A$ is a separable extension of $T$.
- (C) If there is an algebra epimorphism $\pi : A \to B$, which splits $B \hookrightarrow A$, and $I$ denotes $\ker \pi$, then $I^2 = I$. 
Suppose Proposition 1.2. perfect field with subalgebra $B \supseteq A$. Then $A' \supseteq B'$ is a separable extension if and only if $A \supseteq B$ is a separable extension.

**Proposition 1.2.** Suppose $A$ is a finite-dimensional algebra over a perfect field with subalgebra $B$ with radical $J(B)$ an ideal of $A$. Then $A \supseteq B$ is a separable extension if and only if $J(A) = J(B)$.

**Proof.** Suppose $J(A) = J(B) = J$. Then $J^n = 0$ for some $n \in \mathbb{N}$. By the Wedderburn principal theorem [33, 11.6], there is a separable subalgebra $A'$ such that $A = A' \oplus J$. Let $B' = B \cap A'$, a subalgebra of $A'$ forming a separable extension by Lemma (B). Since $B = B' \oplus J$, it follows from Lemma (D) that $A$ is a separable extension of $B$.

Suppose $A$ is separable extension of $B$ where $J(B)$ is an ideal in $A$. Since $J(B)$ is nilpotent, one has $J(B) \subseteq J(A)$. Let $A' = A/J(B)$ and $B' = B/J(B)$. By Lemma (A), $A'$ is a separable extension of $B'$. The radical of $A'$ is $I := J(A)/J(B)$, and is the kernel of the canonical epi $A \rightarrow A'$. Since $B'$ is a separable algebra (over a perfect field), the canonical epi splits as an algebra mapping, so that Lemma (C) implies $I^2 = I$. Suppose $J(A)^n = 0$. Then $I = I^{2n} = 0$, whence $J(A) = J(B)$. □

The proposition also works without the hypothesis on the ground field if $A/J(A)$ and $B/J(B)$ are known to be separable algebras. This follows from Hochschild’s proof extending the Wedderburn principal theorem [33].

**Example 1.3.** Let $A = T_n(k)$ be the algebra upper-triangular $n$-by-$n$ matrices over any field $k$, $B = U(n)$ be the subalgebra strictly upper-triangular matrices with 1. It follows from $J(A) = J(B)$ and the proposition that $A$ is a separable extension of $B$.

Suppose $C = k1 + k(e_{12} + \cdots + e_{n-1,n}) + \cdots + ke_{1n}$ be the Jordan subalgebra $J_n(k)$ within $B$ and $A$. (Note that $D(A) = 1$ and $D(C) = \infty$, since $A$ is hereditary and $C \cong k[X]/(X^n)$ is Frobenius.) Except in the case $n = 1, 2$, it is a thornier problem to determine separability of the extension $A \supseteq C$, since $A \supseteq C$ may be either separable or inseparable in a general tower $A \supseteq B \supseteq C$, when $A \supseteq B$ is separable and $B \supseteq C$ is inseparable (i.e., not a separable extension).

Even when the radical of $B$ is not an $A$-ideal, it is often enough to maneuver as follows with the information in the Lemma above.

**Example 1.4.** Let $k$ be a field and $B$ be the Sweedler-Nakayama algebra $B = k(e_{11} + e_{44}) + k(e_{22} + e_{33}) + ke_{21} + ke_{43}$ in terms of matrix units in $M_4(k)$. Consider $B$ as a subalgebra of $A = ke_{11} + ke_{22} + \cdots + ke_{44}$.
$ke_{33} + ke_{44} + ke_{31} + ke_{41} + ke_{42} + ke_{43}$, a structural matrix subalgebra of $M_2(k)$. Note that $J(B)$ does not satisfy the hypothesis of the proposition, so we augment it to $J = ke_{21} + ke_{43} + ke_{41}$, an $A$-ideal such that $J \cap B = J(B)$. Then $A \to A/J$ maps $B$ onto $B' \cong k^2$, a separable $k$-algebra, but $A' := A/J$ has radical $I = ke_{31} + ke_{42}$. If $A'$ is separable over $B'$, then it is separable over its diagonal subalgebra $\cong k^2$; by Lemma (D) this implies $I = 0$ a contradiction. Then $A'$ is not separable over $B'$ and by Lemma (A), $A$ is not separable over $B$.

**Example 1.5.** Let $K$ be a commutative ring and $A = M_n(K)$ the full $K$-algebra of $n \times n$ matrices. Let $e_{ij}$ denote the matrix units ($i, j = 1, \ldots, n$). Any of the $n$ elements $e_j = \sum_{i=1}^{n} e_{ij} \otimes_K e_{ji}$ are separability idempotents for $A$.

1.2. Preliminaries on subalgebra depth. Let $A$ be a unital associative ring. The category of right modules over $A$ will be denoted by $\mathcal{M}_A$. Two modules $M_A$ and $N_A$ are $H$-equivalent (or similar) if $M \oplus \ast \cong N^q$ and $N \oplus \ast \cong M^r$ for some $r, q \in \mathbb{N}$ (sometimes briefly denoted by $M \sim N$). It is well-known that H-equivalent modules have Morita equivalent endomorphism rings.

Let $B$ be a subring of $A$ (always supposing $1_B = 1_A$). Consider the natural bimodules $A_A, B_A, A_B, B_A$ and $B_B$ where the last is a restriction of the preceding, and so forth. Denote the tensor powers of $B_A$ by $A_B^n = A \otimes_B \cdots \otimes_B A$ for $n = 1, 2, \ldots$, which is also a natural bimodule over $B$ and $A$ in any one of four ways; set $A_B^0 = B$ which is only a natural $B$-$B$-bimodule.

**Definition 1.6.** If $A_B^{(n+1)}$ is $H$-equivalent to $A_B^n$ as $X$-$Y$-bimodules, one says $B \subseteq A$ has

- **depth** $2n + 1$ if $X = B = Y$;
- **left depth** $2n$ if $X = B$ and $Y = A$;
- **right depth** $2n$ if $X = A$ and $Y = B$;
- **$H$-depth** $2n - 1$ if $X = A = Y$.

valid for even depth and $H$-depth if $n \geq 1$ and for odd depth if $n \geq 0$.

For example, $B \subseteq A$ has depth 1 iff $B_A$ and $B_B$ are $H$-equivalent. Equivalently,

$$B_A \oplus \ast \cong n \cdot B_B$$

for some $n \in \mathbb{N}$ [2]. This in turn is equivalent to there being $f_i \in \text{Hom}(B_A, B_B)$ and $r_i \in A^B$ such that $\text{id}_A = \sum_i f_i(-)r_i$, the classical central projectivity condition [3]. In this case, it is easy to show that $A$ is ring isomorphic to $B \otimes_{Z(B)} A^B$ where $Z(B), A^B$ denote the center of $B$ and centralizer of $B$ in $A$. From this one deduces that a centrally
projective ring extension $A \supseteq B$ (or depth 1 extension) has centers satisfying $Z(B) \subseteq Z(A)$.

Another example, $B \subset A$ has right depth 2 iff $A A_B$ and $A A \otimes_B A_B$ are similar. If $A = \mathbb{C} G$ is a group algebra of a finite group $G$ and $B = \mathbb{C} H$ is a group algebra of a subgroup $H$ of $G$, then $B \subseteq A$ has right depth 2 iff $H$ is a normal subgroup of $G$ [22]; a similar statement of normality is true for a Hopf subalgebra $R \subseteq H$ of finite index and over any field [3]. Depth two is the key condition that generates Hopf algebroid structures on certain endomorphism rings derived from the ring extension $A \supseteq B$ that occur in the next subsection.

Note that $A \otimes B \oplus \ast \cong A \otimes B(n+1)$ for all $n \geq 2$ and in any of the four natural bimodule structures: one applies $1_A$ and multiplication to obtain a split monic, or split epi oppositely. For three of the bimodule structures, it is true for $n = 1$; as $A$-$A$-bimodules, equivalently $A \oplus \ast \cong A \otimes_B A$ as $A^e$-modules, this is the separable extension condition on $B \subseteq A$. Experts will recognize

$$A A \otimes_B A \oplus \ast \cong q \cdot A A$$

for some $q \in \mathbb{N}$ as the H-separability condition, which implies $A$ is a separable extension of $B$ [17]. Somewhat similarly, $B A_B \mid q \cdot B B_B$ implies $B B_B \mid B A_B$ [25]. It follows that subalgebra depth and H-depth may be equivalently defined by replacing the similarity bimodule conditions for depth and H-depth in Definition [16] with the corresponding bimodules on

$$A \otimes_B (n+1) \oplus \ast \cong q \cdot A \otimes_B n$$

for some positive integer $q$ [4 24 25].

Note that if $B \subseteq A$ has H-depth $2n - 1$, the subalgebra has (left or right) depth $2n$ by restriction of modules. Similarly, if $B \subseteq A$ has depth $2n$, it has depth $2n + 1$. If $B \subseteq A$ has depth $2n + 1$, it has depth $2n + 2$ by tensoring either $- \otimes_B A$ or $A \otimes_B -$ to $A \otimes_B (n+1) \sim A \otimes_B n$. Similarly, if $B \subseteq A$ has left or right depth $2n$, it has H-depth $2n + 1$. Denote the minimum depth of $B \subseteq A$ (if it exists) by $d(B, A)$ [4]. Denote the minimum H-depth of $B \subseteq A$ by $d_h(B, A)$. Note that $d(B, A) < \infty$ if and only if $d_h(B, A) < \infty$; in fact, $|d(B, A) - d_h(B, A)| \leq 2$ if either is finite.

For example, for the permutation groups $\Sigma_n < \Sigma_{n+1}$ and their corresponding group algebras $B \subseteq A$ over any commutative ring $K$, one has depth $d(B, A) = 2n - 1$ [6 4]. Depths of subgroups in $PGL(2, q)$, twisted group algebras and Young subgroups of $\Sigma_n$ are computed in [10, 9, 11]. If $B$ and $A$ are semisimple complex algebras, the minimum odd depth is computed from powers of an order $r$ symmetric matrix
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with nonnegative entries \( S := MM^t \) where \( M \) is the inclusion matrix \( K_0(B) \to K_0(A) \) and \( r \) is the number of irreducible representations of \( B \) in a basic set of \( K_0(B) \); the depth is \( 2n + 1 \) if \( S^n \) and \( S^{n+1} \) have an equal number of zero entries \([6]\). Similarly, the minimum H-depth of \( B \subseteq A \) is computed from powers of an order \( s \) symmetric matrix \( T = M^tM \), where \( s \) is the rank of \( K_0(A) \), and the power \( n \) at which the number of zero entries of \( T^n \) stabilizes \([25]\). It follows that the subalgebra pair of semisimple complex algebras \( B \subseteq A \) always has finite depth.

2. FULL IDEMPOTENT CHARACTERIZATIONS OF H-SEPARABLE AND CENTRALLY PROJECTIVE RING EXTENSIONS

In this section, we give characterizations of H-depth 1 and depth 1 ring extension in terms of well-known idempotents being full idempotents. Recall the equivalent form of Morita theory, which states that for a ring \( A \) with idempotent \( e \in A \), \( eAe \) is Morita equivalent to \( A \) so long as \( e \) is a full idempotent, i.e. \( AeA = A \): the Morita context bimodules are \( eA \) and \( Ae \) \([1]\). Recall from the previous section the ring structure on \( T = (A \otimes_B A) \) of a ring extension \( A \supseteq B \).

**Proposition 2.1.** A separable extension \( R \supseteq S \) is H-separable if and only if a separability idempotent \( e \in T \) is full. In this case, the center \( Z(R) \) is Morita equivalent to \( \text{End}_S R \otimes_S R \) \([22, Corollary 4.3]\).

**Proof.** A ring extension \( R \supseteq S \) is H-separable if and only if there are \( q \) elements \( e_i \in T \) and \( q \) elements \( c_i \in R \) (the centralizer of \( S \) in \( R \)) such that \( 1_T = \sum_{i=1}^q c_i e_i \) \([7]\), which may be seen in an exercise using Eq. \( (2) \) and using \( q \) bimodule homomorphisms to and from \( R \otimes_S R \) composed and summing up to the identity mapping on \( R \otimes_S R \). It was pointed out that H-separable extensions are separable.

A separability idempotent \( e \) in \( T \) is full if \( TeT = T \) (equivalently, the left ideal \( (R \otimes_S R)^R = Te \) is a progenerator). But the two-sided ideal \( TeT = R^S(R \otimes_S R)^R \), since for any \( t \in T \), \( t^1t^2 \in R^S \) (where \( t = t^1 \otimes t^2 \) suppresses a possible summation). Thus, \( 1_T \in TeT \) if and only if \( R \) is an H-separable extension of \( S \).

Note that \( eTe \cong Z(R) \) via \( ef \mapsto f^1f^2 \) via the multiplication mapping \( \mu : R \otimes_S R \to R \), since \( Te = (R \otimes_S R)^R \) and for \( f \in (R \otimes_S R)^R \), \( ef = f^1f^2e = efe \). An inverse ring homomorphism is given by \( z \mapsto ze \). \( \square \)

Recall that a ring extension \( R \supseteq S \) is split if \( sS \oplus * \cong sRs \). Equivalently, there is a bimodule projection \( E \) from \( R \) onto \( S \) satisfying \( E^2 = E \in \text{End}_S R_S := U \).
Proposition 2.2. A split extension $R \supseteq S$ is centrally projective (or has depth one) if and only if there is a bimodule projection $E : R \to S$ that is a full idempotent in $U$. In this case, the center $Z(S)$ is Morita equivalent to $\text{End}_S(R_S)$.

Proof. Suppose $E$ is a full idempotent in $U$, i.e., $UEU = U$. Since $UEU = \{ \sum_i \alpha_i \circ E \circ \beta_i \mid \alpha_i, \beta_i \in U \} = \{ \sum_i \alpha_i(1_R)E(\beta_i(-)) \mid \alpha_i, \beta_i \in U \}$ and $\alpha_i(1_R) \in R^S$, then

$$\text{id}_R = \sum_{i=1}^q \alpha_i \circ E \circ \beta_i = \sum_i f_i \circ g_i$$

where $f_i := \alpha_i \circ E \in \text{Hom}(sS_S, sR_S)$ is in fact multiplication by $\alpha(1_R)$, and $g_i := E \circ \beta_i \in \text{Hom}(sR_S, sS_S)$. Since $\text{id}_R = \sum_i f_i \circ g_i$ is equivalent to Eq. (1), it follows that the ring extension has depth one.

Conversely, suppose $R \supseteq S$ has depth one. By [25, Lemma 1.9] there is a bimodule projection $E : R \to S$, which we view as an idempotent in $U$. Moreover, via Eq. (1) there are $f_i \in \text{Hom}(sS_S, sR_S), g_i \in \text{Hom}(sR_S, sS_S)$ such that $\text{id}_R = \sum_i f_i \circ g_i$. Then

$$\text{id}_R = \sum_i (f_i \circ E) \circ E \circ (E \circ g_i) \in U E U.$$

The final statement follows from $EUE = \{ E(\alpha(1_R))E \mid \alpha \in U \} = Z(S)E$, since $E(R^S) = Z(S)$, as well as $Z(S) \cong Z(S)E$. \qed

Recall that a ring $R$ with nontrivial idempotent $e$ has corner ring $eRe$, to which it passes on several properties studied in [28, Chapter 21] and [29, 16.25, 18.15]. This is captured in the following definition, which is related to the weaker notion of ”symmetric separably divides” in [27, Def. 2.1].

Definition 2.3. A ring $A$ Morita divides a ring $B$ if there is a bimodule homomorphism $\mu : BQ \otimes_A P_B \to _B B_B$ and a bimodule isomorphism (or epimorphism) $\nu : _A P \otimes_B Q_A \to _A A_A$, which are associative with respect to application to $Q \otimes P \otimes Q \to Q$ and $P \otimes Q \otimes P \to P$.

It follows that

(1) in case $\nu$ is an epi, then it is an isomorphism;
(2) $P_B$ and $BQ$ are finite projective modules;
(3) $_A P$ and $Q_A$ are generators;
(4) the (natural) $A$-$B$-bimodule homomorphism $P \to \text{Hom}(BQ, B_B)$, given by $p \mapsto \mu(- \otimes_A p)$, and the $B$-$A$-homomorphism $Q \to \text{Hom}(P_B, B_B)$ given by $q \mapsto \mu(q \otimes_A -)$, are both isomorphisms;
the natural ring homomorphisms $A \to \text{End} P_B$ and $A \to \text{End} _B Q$
are isomorphisms. Compare [1, Ex. 22.5]

Clearly, $A$ and $B$ are Morita equivalent if $A$ Morita divides $B$ and $B$ Morita divides $A$ with the same context bimodules. For example, a ring $R$ is Morita divided by a corner ring $eRe$ since the bimodule isomorphism $eR \otimes _R Re \cong eRe$, given by $er \otimes r'e \mapsto err'e$, is associative w.r.t. the bimodule homomorphism $\mu : eRe \otimes eRe \to R$ given by $re \otimes er' \mapsto rer'$. of course, $eR$ and $Re$ are projective ideals in $R$.

The corner ring $eRe$ Morita divides $R$ iff $e$ is a full idempotent, i.e., the mapping $\mu$ is epi. This example is well-representative of Morita divides, due to (5) above.

A special case of corner rings occurs for example in the finite-dimensional Hopf algebra $H$ action on algebra $A$, where there is an idempotent $e$ in the smash product $A \# H$ such that the subalgebra of invariants, $A^H \cong e(A \# H)e$, in case the trace function $A \to A^H$ is surjective [31, 4.3.4, 4.5].

A closer look at the last paragraphs of the proofs of Propositions 2.1 and 2.2 makes the following proposition obvious.

**Proposition 2.4.** Suppose $A$ is a ring extension of $B$. If $A$ is a separable extension of $B$, then $T$ Morita divides $Z(A)$. If $A$ is a split extension of $B$, then $U$ Morita divides $Z(B)$.

**Proof.** A second proof is obtained from noting $AA \otimes_B A \cong AA \oplus ^*$ for a separable extension, then apply $\text{End}$ and $\text{End} AA \cong Z(A)$, from which $Z(A)$ is a corner ring of $T$. Similarly, if $A$ is a split extension of $B$, we have $B \cong BB \oplus ^*$, then $\text{End} BA$ has corner ring $Z(B)$. □

**Example 2.5.** Consider the finite cyclic monoid
$$\{1, f, f^2, \ldots, f^n, \ldots, f^{2n}\}$$
subject to the relation $f^n = f^{2n}$. Choosing a field $k$, denote the commutative monoid algebra $M(2n, n)$ generated by $f$ of dim $2n$ with idempotent $e = f^n$. This is also a bialgebra [31]. Since $(ef)^n = ef^n = e$ and $[(1-e)f]^n = (1-e)e = 0$, the idempotent decomposition shows that

$$M(2n, n) = eM(2n, n) \oplus (1-e)M(2n, n) \cong k\mathbb{Z}_n \times k[X]/(X^n) \quad (4)$$

(an interesting contrast to the Taft Hopf algebra $H_n \cong k[X]/(X^n)\# k\mathbb{Z}_n$ of dimension $n^2$ if $k$ contains a primitive root-of-unity [31]). The subalgebra $B$ generated by $(1-e)f$ is isomorphic to $k[X]/(X^n)$, forms a separable extension with respect to $M(2n, n)$ by Proposition 1.2 if $k$ has characteristic zero. The algebra extension $M(2n, n) \supset B$ is
also a split extension (as $B$-modules) with a nonprojective direct summand $eM(2n,n)$. Both $M(2n,n)$ and $B$ are symmetric algebras, so this algebra extension is not Frobenius (an interesting supplement to the examples in [8]). In addition, the bialgebra $M(2n,n)$ has corner algebra the (Hopf) group algebra $k\mathbb{Z}_n$.

As a final remark, the endomorphism rings of the ring extension, in which $e$ or $E$ are full idempotents, are the centralizers $A$ and $B$, up to ring isomorphism and with suitable hypotheses on the ring extension, in the Hasse diagram of centralizers in [19 Figure 1]. The centralizer algebras $A$ and $B$ receive two nondegenerate pairings that provide dual Hopf algebra structures on these and show $R \supseteq S$ is a Hopf-Galois extension: the depth two condition required in all four publications [21, 18, 19, 20] is automatically satisfied by the depth one or H-depth one conditions above. The Hopf algebroid structures on $T$ and $U$ above are sketched in [21, 4.7, 4.8, 5.7, 5.8].

2.1. The Hopf Algebroid of a Centrally Projective Extension. Given rings $H$ and $R$ with commuting antimorphism and homomorphism of $R \to H$, a bialgebroid $(H, R, \Delta, \varepsilon)$ is an $R$-bimodule structure on $H$ derived from the commuting mappings, an $R$-coring $(H, \Delta, \varepsilon)$ that satisfy axioms of compatibility with the algebra structure of $H$ similar to a bialgebra (see for example [21]). An example of a bialgebroid, and another of a Hopf algebroid, is the following generalizations of the well-known Lu bialgebroids. Given an algebra $C$ over commutative ground ring $K$ such that $C$ is finitely generated projective as $K$-module, the following is technically a left bialgebroid over $C$ (with $\otimes = \otimes_K$).

Example 2.6. The endomorphism algebra $E := \text{End}_K C$ with $\bar{s}(c) = \lambda(c)$, $\bar{t}(c') = \rho(c')$, coproduct $\Delta(f)(c \otimes c') = f(cc')$ for $f \in \text{End}_K C$ after noting that $E \otimes_C E \cong \text{Hom}_K(C \otimes C, C)$ via $f \otimes g \mapsto (c \otimes c' \mapsto f(c)g(c'))$. The counit is given by $\varepsilon(f) = f(1)$. We see that this is the left bialgebroid $S$ above when $B = K$, a subring in the center of $A$.

Example 2.7. The ordinary tensor algebra $C \otimes C^\text{op}$ with $\bar{s}(c) = c \otimes 1$, $\bar{t}(c') = 1 \otimes c'$ with bimodule structure $c \cdot c' \otimes c'' \cdot c''' = cc' \otimes c''c'''$. Coproduct $\Delta(c \otimes c') = c \otimes 1 \otimes c'$ after a simple identification, with counit $\varepsilon(c \otimes c') = cc'$ for $c, c' \in C$. $C \otimes C^\text{op}$ is a left $C$-bialgebroid by arguing as in [30], or [21 $N = K$] since $C|K$ is D2. In addition, $\tau : C \otimes C^\text{op} \to C \otimes C^\text{op}$ defined as the twist $\tau(c \otimes c') = c' \otimes c$ is an antipode satisfying the axioms of a Hopf algebroid (in addition, $\tau^2 = \text{id}$, an involutive antipode).
Hopf algebroids are bialgebroids with antipode. The antipode may be transferred from one isomorphic bialgebroid to another by a tedious check of the axioms [21].

**Proposition 2.8.** If \( F : H_1 \to H_2 \) and \( f : R_1 \to R_2 \) are ring isomorphisms and \( \tau_1 \) is an antipode for \( H_1 \), then \( \tau_2 := F \tau_1 F^{-1} \) is an antipode for \( H_2 \).

As an example of bialgebroid homomorphism with fixed base ring, let \( C \) be the algebra introduced above and \( \tilde{F} : C \otimes C^{\text{op}} \to \text{End}_K C \) be defined by \( \tilde{F}(c \otimes c')(c'') = cc''c' \). The following is consequence of the well-known Azumaya theorem (cf. [17, 5.9]).

**Proposition 2.9.** \( F : C \otimes_K C^{\text{op}} \to \text{End}_K C \) is a bialgebroid isomorphism if \( C \) is an Azumaya \( K \)-algebra.

Let \( B \to A \) be a ring extension with centralizer subring \( R \), endomorphism ring \( S = \text{End}_{B|A}(A, B) \) and ring \( T = (A \otimes_B A)^B \). Suppose \( A|B \) is centrally projective, i.e.

\[
A \oplus * \cong n \cdot B
\]

as \( B\)-\( B \)-bimodules iff there are element \( f_i \in \text{Hom}_{B|B}(A, B) \) and \( r_i \in R \) (a so-called CP-dual basis) such that

\[
a = \sum_i r_i f_i(a),
\]

(5)

iff \( R \) is a f.g. projective algebra over \( Z := Z(B) \) and \( A \) is a tensor algebra, \( A \cong B \otimes_Z R \).

Let \( B \to A \) be a unital ring homomorphism (of associative rings). Equivalently, \( A|B \) is said to be a ring extension, with structure or unit mapping \( B \to A \). This mapping induces a natural bimodule \( B|A_B \) which is our most important means for studying the ring extension \( A|B \). In [21] a ring extension \( A|B \) is said to be of depth two if

\[
A \otimes_B A \oplus * \cong n \cdot A
\]

as natural \( B\)-\( A \) and \( A\)-\( B \)-bimodules. Equivalently, there are elements \( \beta_i \in S := \text{End}_{B|B}(A, B), t_i \in T := (A \otimes_B A)^B \) (called a left D2 quasibasis) such that

\[
a \otimes a' = \sum_i t_i \beta_i(a)a',
\]

(6)

and a right D2 quasibasis \( \gamma_j \in S, u_i \in T \) such that

\[
a \otimes a' = \sum_j a \gamma_i(u')u_i.
\]

We fix both D2 quasibases in our text below.
Now an $R$-coring structure $(U, \Delta, \varepsilon)$ is given by

$$\Delta(\alpha) := \sum_i \alpha(-t_1^i)t_2^i \otimes_R \beta_i$$

(7)

for every $\alpha \in U$, denoting $t_i = t_1^i \otimes t_2^i \in T$ by suppressing a possible summation, and

$$\varepsilon(\alpha) = \alpha(1)$$

(8)

satisfying the additional axioms of a bialgebroid [21, Section 4], such as multiplicativity of $\Delta$ and a condition that makes sense of this requirement.

For example, a centrally projective extension $A|B$ has depth two since the condition above on the tensor-square follows easily from its defining property

$$BA_B \oplus * \cong n \cdot B_B$$

(9)

by tensoring from the right by $- \otimes_B A_A$ and from the left by $A_B \otimes_B -$. This example includes a finitely generated (f.g.) projective algebra $A$ over commutative ring $B$. Alternatively, we can construct a left or right D2 quasibasis easily from a dual basis of central projectivity. For example, a f.g. projective $K$-algebra $A|K$ is centrally projective. If $\iota: B \to A$ denotes the structure mapping of the ring extension, we note that $f_i = \iota f_i \in T$, and for $a, a' \in A$

$$a \otimes a' = \sum_i r_i \otimes f_i(a)a' = \sum_i af_i(a') \otimes r_i,$$

whence $f_i, 1 \otimes r_i$ is a right D2 quasibasis and $f_i, r_i \otimes 1$ is a left D2 quasibasis for $A|B$.

**Proposition 2.10.** The two bijective maps $\Phi : S \xrightarrow{\cong} \text{End}_Z R$ via $\Psi(\alpha) = \alpha|_R$ and $R \otimes Z R^{\text{op}} \xrightarrow{\cong} T^{\text{op}}$ via $\phi(r \otimes r') = r \otimes_B r'$ are ring isomorphisms.

**Proof.** $\Psi$ and $\phi$ are clearly ring homomorphisms. The inverse of $\Psi$ is given by $\Psi^{-1}(\beta)(br) = b\beta(r)$ for each $b \in B, r \in R$ using the Lemma&Definition. The inverse of $\phi$ is given by

$$\phi^{-1}(t) = \sum_{i,j} f_i(t_1^i)f_j(t_2^j)r_i \otimes_Z r_j,$$

where $f_i, r_i$ is a CP-dual basis. Note that $f_i(t_1^i)f_j(t_2^j)$ is in $Z$ for each $i, j$. \qed

Centrally projective extensions coincide with depth one extensions [21, 7.1]. Since $S = \text{End}_{BA_B}$ and $T^{\text{op}}$ are left bialgebroids over $R$, we are led to the following in this special case of depth two.
Theorem 2.11. If $A|B$ is centrally projective, then $S \cong \text{End}_Z R$ and $T^{\text{op}} \cong R \otimes_Z R^{\text{op}}$ as $R$-bialgebroids via $\Psi$ and $\phi^{-1}$; whence $T^{\text{op}}$ is a Hopf algebroid and there is a natural bialgebroid homomorphism $T^{\text{op}} \to S$. Then $S$ is a Hopf algebroid if $R|Z$ is Azumaya.

Proof. We note that $(T^{\text{op}}, R, \bar{s}, \bar{t}, \Delta', \varepsilon')$ is a left bialgebroid where the product on $T^{\text{op}}$ is given by $tt' = t^1t'^1 \otimes t'^2t^2$, $\bar{s}(r) = r \otimes 1$, $\bar{t}(r) = 1 \otimes r$, which together induce from the left the ordinary $R$-bimodule structure on $(A \otimes_B A)^B$, $\Delta'$ derived from

$$\Delta(t) = \sum_i t_i \otimes_R (\beta_i(t^1) \otimes_B t^2) \quad (10)$$

and $\varepsilon'$ derived from

$$\varepsilon(t) = t^1t^2. \quad (11)$$

Denote $R \otimes_Z R^{\text{op}}$ by the Cartan-Eilenberg $R^e$ and $T^{\text{op}}$ by $T^o$. It suffices by Proposition 2.8 to check the commutativity of four diagrams for $\phi : (R^e, R, \bar{s}, \bar{t}, \Delta, \varepsilon) \to (T^o, R, \lambda, \rho, \Delta', \varepsilon')$ in the definition of bialgebroid homomorphism. The three diagrams are trivially commutative, while

$$\begin{array}{ccc}
R^e & \xrightarrow{\phi} & T^o \\
\Delta \downarrow & & \Delta' \downarrow \\
R^e \otimes_R R^e & \xrightarrow{\phi \otimes \phi} & T^o \otimes_R T^o
\end{array}$$

commutes since an application of Eq. (10) to the D2 left quasibase above (from CP-dual basis) gives

$$\Delta' \phi(r \otimes_Z r') = \sum_i (r_i \otimes_B 1) \otimes_R (f_i(r) \otimes_B r') = r \otimes 1 \otimes 1 \otimes r'$$

which is clearly equal to $(\phi \otimes \phi) \Delta(r \otimes r')$ by Example 2.7.

We next note that $E := \text{End}_Z R$ is a left $R$-bialgebroid by Example 2.6. Again three of the diagrams for establishing $\Psi : (S, R, \lambda, \rho, \Delta, \varepsilon) \to (E, R, \lambda', \rho', \Delta', \varepsilon')$ as a bialgebroid homomorphism are trivially commutative, the fourth commutative as above as a uniform choice of left D2 quasibases for $E|Z$ and $A|B$ and Eq. (7) yield

$$\Delta' \Psi(\alpha) = \sum_i \alpha|_R(-r_i) \otimes f_i|_R = (\Psi \otimes \Psi) \Delta(\alpha).$$

The bialgebroid homomorphism $\ell : T^{\text{op}} \to S$ is now the composition of these two bialgebroid isomorphisms with Lu’s homomorphism $R^e \to E$. 

for the f.g. projective $Z$-algebra $R$: this results in

$$\ell(t)(a) = \sum_i f_i(a)t^1r_it^2.$$ 

This is an isomorphism if $R|Z$ is Azumaya by Proposition 2.9.

If $Z \cdot 1_A$ is a field coinciding with the center of $A$, $T^{\text{op}}$ possesses a weak Hopf $Z$-algebra structure, since $R$ is separable $Z$-algebra [21 Prop. 9.4].

3. Uniquely separable Frobenius extensions

Recall that a Frobenius (ring) extension $A \supseteq B$ is characterized by having a (Frobenius) homomorphism $E : A \to B$ in $\text{Hom}(B\mathfrak{A}_B, B\mathfrak{B}_B)$ with elements (dual bases) $x_i, y_i \in A$ ($i = 1, \ldots, n$) such that id$_A = \sum^n_{i=1} E(-x_i)y_i = \sum^n_{i=1} x_iE(y_i-)$. Equivalently, $A_B$ is finite projective and $A \cong \text{Hom}(A_B, B_B)$ as natural $B$-$A$-bimodules: see [17] for more details. For example, given a group $G$ and subgroup $H$ of finite index $n$ with right coset representatives $g_1, \ldots, g_n$, $K$ an arbitrary commutative ring, the group algebra $A = KG$ is a Frobenius extension of the group subalgebra $B = KH$ with $E : A \to B$ the obvious projection defined by $E(\sum_{g \in G}agg) = \sum_{h \in H} ahh$ and dual bases $x_i = g_i^{−1}, y_i = g_i$.

The element $\sum_i x_iy_i := [A : B]_E$ is for all Frobenius extensions in the center $Z(A)$ by the short computation, $\sum_i x_iy_ia = \sum_{i,j} x_iE(y_iax_j)y_j = \sum_j ax_jy_j$. This element is sometimes called the $E$-index, independent of the choice of dual bases for the Frobenius homomorphism [16, 17]. In the group algebra extension example above note that $\sum^n_{i=1} x_iy_i = n1$, which is invertible if and only if $A \supseteq B$ is a separable extension [35]. In this case a separability idempotent is given by $\frac{1}{[A : B]} \sum^n_{i=1} g_i^{-1} \otimes KH g_i$.

We study in general terms when this separability idempotent is unique (e.g., in contrast to the $n$ different separability idempotents in Example [15]). Call a separable Frobenius extension satisfying the hypotheses in the theorem a uniquely separable Frobenius extension.

**Theorem 3.1.** Given Frobenius extension $A \supseteq B$ with Frobenius homomorphism $E : A \to B$ and dual bases $x_i, y_i \in A$ such that $\sum_i x_iy_i := [A : B]_E$ is an invertible element in $Z(A)$, then $Z(A) = A_B$ if and only if $A \supseteq B$ has the unique separability element $[A : B]E^{-1} \sum_i x_i \otimes_B y_i$.

**Proof.** Let $R$ denote the centralizer $A_B$. Recall that $A \otimes_B A \cong \text{End}_A A_B$ via $a \otimes_B a' \mapsto \lambda_a \circ E \circ \lambda_{a'}$ (with inverse $f \mapsto \sum_i f(x_i) \otimes_B y_i$). Consequently, $(A \otimes_B A)^A \cong \text{End}_AA_B \cong R$ via $e = e^1 \otimes_B e^2 \mapsto e^1 E(e^2-) \mapsto e^1 E(e^2)$. These have inverse mappings given by $r \mapsto \rho_r \mapsto \sum_i x_ir \otimes_B y_i$. 


Proof. The centralizer $R$ of an extension always contains the centers $Z(A)$ and $Z(B)$. From the hypothesis and the theorem, $Z(A) = R \supseteq Z(B)$, and apply Burciu’s result.

From [35, 37] and group theory, the following is a common setup.

**Corollary 3.2.** If $B \subseteq A$ is a uniquely separable Frobenius extension of semisimple complex algebras, then $d(B, A) = 1$.

**Proof.** The centralizer $R$ of an extension always contains the centers $Z(A)$ and $Z(B)$. From the hypothesis and the theorem, $Z(A) = R \supseteq Z(B)$, and apply Burciu’s result.

From [35, 37] and group theory, the following is a common setup.

**Proposition 3.3.** If the dual bases $x_i, y_i$ of a separable Frobenius extension $A \supseteq B$ (where $\sum i x_i y_i$ is invertible) may be chosen from $Z(A)$, then $A \supseteq B$ is uniquely separable Frobenius and has depth 1.

**Proof.** Since any (Casimir) element in $(A \otimes_B A)^A$ may be written as $\sum i x_i r \otimes_B y_i$ where $x_i, y_i \in Z(A)$, if we moreover assume that $\sum i x_i r y_i = 1$, then $r(\sum i x_i y_i) = 1$, whence $r$ is unique. This proves that $A \supseteq B$ is a uniquely separable Frobenius extension.

From $a = \sum E(ax_i) y_i$ for all $a \in A$, where $y_i \in R$ and $E(-x_i) \in \text{Hom}(B_A B, B_B)$ for each $i$. This characterizes central projectivity of $A$ over $B$, i.e., $d(B, A) = 1$. 

4. Group algebra extensions

Let $A = kG$ where $G$ is a finite group with subgroup $H < G$, and let $B = kH \subseteq A$ where $k$ is a field containing the inverse of $|G : H|$. Then $A \supseteq B$ is a split, separable Frobenius extension. If $g_1, \ldots, g_n \in$
$G$ is a right transversal of $H$ in $G$, then $e = \frac{1}{|G:H|} \sum_{i=1}^{n} g^{-1} \otimes_B g_i$ is a separability element. The next theorem is a consequence of a theorem by Singh-Hanna in [35], which we show is also a consequence of Theorem 3.1.

**Theorem 4.1** ([35]). A separable finite group algebra extension $A \supseteq B$ has unique separability element $e \in A \otimes_B A$ if and only if it satisfies the property:

**S** Any conjugacy class in $G$ is an $H$-orbit.

**Proof.** It is well-known and easy to check that the sum of elements in a conjugacy class is in the center of a group algebra $kG$, and that the dimension of the center is equal to the number of conjugacy classes of $G$. Similarly, the centralizer $R = A^B$ of $B$ in $A$ is the sum of elements in an $H$-orbit of $G$ and the dimension of $R$ is the number of distinct $H$-orbits in $G$. Thus the condition is equivalent to $R = Z(A)$, which in turn by Theorem 3.1 is equivalent to uniqueness of the separability element $e$. □

For example, $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \supseteq H = \{\pm 1, \pm i\}$ does not satisfy property (S). In [35] Theorem 3.8 it is shown that for groups $G$ of order less than 64, but different from 48, each subgroup $H$ satisfying the property (S) also satisfies $G = HZ(G)$; see also [36, Singh]. Notice that the following holds for centralizers of elements in a finite group $G$ with subgroup $H$.

**Lemma 4.2** ([35]). $H$ satisfies (S) in $G$ if and only if $G = HC_G(a)$ for every $a \in G$.

**Proof.** If for every $g, a \in G$, there is $h \in H$ such that $gag^{-1} = hah^{-1}$, then $h^{-1}ga = ah^{-1}g$, so that $h^{-1}g \in C_G(a)$. Then $g \in HC_G(a)$, whence $G = HC_G(a)$ for every $a \in G$.

Conversely, given arbitrary $g, a \in G = HC_G(a)$, there is $h \in H, x \in C_G(a)$ such that $g = hx$. Then $gag^{-1} = hxax^{-1}h^{-1} = hah^{-1}$. Thus $H < G$ satisfies (S). □

Compare this to the characterization of depth 1 in [3] of group algebra extensions over a field $k$ having characteristic zero: $d_k(H, G) = 1$ if and only if $G = HC_G(h)$ for every $h \in H$, a weaker condition than the one in the lemma. Thus for group algebras we may improve on Corollary 3.2 by noting the following.

**Proposition 4.3.** Suppose $k$ is a field of characteristic zero and $H$ a subgroup of a finite group $G$. If $kG \supseteq kH$ is uniquely separable, then it has depth 1.
Proposition 4.4. If a subgroup $H$ of a finite group $G$ satisfies $HZ(G) = G$, then $KG \supseteq KH$ is uniquely separable over any commutative ring $K$ where $|G : H|1$ is invertible, and $KG \supseteq KH$ has depth 1.

Proof. Given $g, x \in G$, there is $h \in H$ and $z \in Z(G)$, such that $g = hz$. Then $gxg^{-1} = hxh^{-1}$, so $H$ satisfies the property (S) in $G$. Also $Z(G) \subseteq C_G(H)$, so that $G = HZ(G) \subseteq HC_G(H)$. Then $G = HC_G(H)$, the sufficient condition in [3, 1.12] for $ZH \subseteq ZG$ to have depth 1, and therefore $d_K(H,G) = 1$ for any commutative ring $K$.  

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Department of Mathematics, 209 S. 33rd St., David Rittenhouse Lab, Philadelphia, PA 19104
E-mail address: lkadison@math.upenn.edu