RATIONALLY ELLIPTIC TORIC VARIETIES

INDRANIL BISWAS, VICENTE MUÑOZ, AND ANICETO MURILLO

ABSTRACT. We give a characterization of all complete smooth toric varieties whose rational homotopy is of elliptic type. All such toric varieties of complex dimension not greater than three are explicitly described.

1. Introduction

Toric varieties have been widely studied from diverse points of view. Since they have a combinatorial description in terms of polytopes and many of their topological properties (like the cohomology) are encoded combinatorially. Moreover, they furnish a large source of examples of algebraic varieties.

In this paper, we are interested in the behaviour of the rank of the homotopy groups of compact smooth toric varieties. Such a variety $X$ is always a formal algebraic manifold, which means that its rational homotopy type depends only on its rational cohomology $H^*(X, \mathbb{Q})$. In particular $\pi_\ast(X) \otimes \mathbb{Q}$, which is a rational vector space whose dimension is precisely the total rank of $\pi_\ast(X)$, is explicitly determined by $H^*(X, \mathbb{Q})$.

It is known that a simply connected finite CW-complex $X$ is either elliptic, that is, $\dim \pi_\ast(X) \otimes \mathbb{Q} < \infty$, or it is hyperbolic, in which case $\dim \pi_{\leq k}(X) \otimes \mathbb{Q}$ grows exponentially as $k$ increases.

Here we first notice that elliptic toric varieties are those whose rational cohomology algebra is a complete intersection, that is, a polynomial algebra truncated by an ideal generated by a regular sequence. Moreover, we prove that the Poincaré polynomial of these toric varieties coincides with that of a product of complex projective spaces (see Theorem 3.3). However, it may happen that the Poincaré polynomial of a smooth toric variety $X$ is that of a product of complex projective spaces, but $X$ is not elliptic (see Example 3.5(6)).

We illustrate the above result by describing all (algebraic isomorphism classes of) elliptic smooth toric varieties of dimension less than or equal to 3; see Theorems 4.1 and 4.2. In dimension 2, only $\mathbb{C}P^2$ and the Hirzebruch surfaces $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(b))$ are elliptic smooth toric varieties. Their homotopy types are $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

The 3-dimensional elliptic smooth toric varieties are, up to isomorphism, $\mathbb{C}P^3$, $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^2} \oplus \mathcal{O}_{\mathbb{C}P^2}(c))$, $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(a) \oplus \mathcal{O}_{\mathbb{C}P^1}(b))$, and $\mathbb{C}P^1$-bundles over Hirzebruch.

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surfaces. These varieties have the rational homotopy type of $\mathbb{C}P^3 \# \mathbb{C}P^3$, $\mathbb{C}P^1 \times \mathbb{C}P^2$ and a quotient $(S^3 \times S^3 \times S^3)/T^3$ respectively.

Elliptic complex manifolds of complex dimension less than or equal to 2 have been classified in [1]. Also, the rational homotopy type of moduli spaces of certain vector bundles over complex curves has been analysed in [3]. Our results here complement these references and show, in particular, the existence of a very large number of (non isomorphic) hyperbolic varieties.

We finally stress that our results and classification on toric varieties are always up to algebraic isomorphisms. If one loosens up this rigidity there are interesting results in the literature. A toric variety is in particular a torus manifold, that is, a smooth $2n$-dimensional manifold admitting an action of a real $n$-torus $\mathbb{T}^n = (S^1)^n$. In [23, Theorem 1.1], it is shown that an elliptic simply connected torus manifold whose integral cohomology is evenly graded is always homeomorphic to a quotient of a free linear torus action on a product of spheres. Moreover [23, Theorem 1.2], if the toric manifold is non-negatively curved then homeomorphism can be replaced by equivariant diffeomorphism. More generally, see [12, Theorem A], any elliptic simply connected torus orbifold has the rational homotopy type of a quotient of a product of spheres by a linear, almost free, torus action. Also, T. Bahri has drawn our attention to [13] for related results.

In low dimensions one can be more precise: any 4-dimensional torus manifold is equivariantly diffeomorphic to either the 4-sphere, an equivariant connected sum of copies of complex projective planes (possibly with reversed orientation) and Hirzebruch surfaces [20]. As mentioned above, the only ones that are rationally elliptic are the 4-sphere, the complex projective plane and the Hirzebruch surfaces. On the other hand, a simply connected 6-dimensional torus manifold whose cohomology is evenly graded is equivariantly diffeomorphic to either the 6-dimensional sphere, an equivariant connected sum of copies of 6-dimensional quasitoric manifolds, or $S^3$-bundles over $S^2$, see [15, Theorem 1.3]. Our results agree with this classification up to diffeomorphism.

After the completion of this work, Wiemeler has provided a classification of rationally elliptic toric orbifolds up to algebraic isomorphism in any dimension [24].

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2. TORIC VARIETIES AND RATIONAL HOMOTOPY

We recall some results from rational homotopy theory that will be used, and also summarize a brief introduction to toric varieties. More details can be found in [9] and [11] respectively.

A toric variety is a complex algebraic variety \( X \) of complex dimension \( N \) equipped with an algebraic action of the complex torus \( T = (\mathbb{C}^*)^N \) such that \( X \) contains a dense \( T \)-orbit on which the action of \( T \) is free.

Any toric variety can be described by a fan as follows. Let \( \Gamma \) be a lattice, meaning a group isomorphic to \( \mathbb{Z}^N \), and let \( \Delta \) be a fan of \( \Gamma \), that is, a collection of strongly convex rational polyhedral cones in the vector space \( V = \Gamma \otimes \mathbb{R} \), satisfying the conditions of a simplicial complex: every face of a cone in \( \Delta \) is a cone in \( \Delta \), and the intersection of two cones in \( \Delta \) is a face of each of the cones. Recall that a strongly convex rational polyhedral cone \( \sigma \) in \( F \) is a cone with apex at the origin, generated by a finite number of vectors in the lattice, and which intersects the opposite cone \( -\sigma \) only at the apex. A fan is complete if the union of its conforming cones is \( V \).

Let \( M = \text{Hom}(\Gamma, \mathbb{Z}) \) be the dual lattice. Every cone \( \sigma \) determines a finitely generated commutative semigroup \( S_\sigma = \{ u \in M \mid \langle u, v \rangle \geq 0, \text{ for all } v \in \sigma \} \), and an associated affine variety \( U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) \).

Any face \( \tau \) of a cone \( \sigma \) in a given fan \( \Delta \) induces an inclusion \( S_\sigma \subset S_\tau \) which, in turn, produces an open embedding \( U_\tau \to U_\sigma \). All these affine varieties fit together to form an algebraic variety which denoted by \( X(\Delta) \). As the apex \( (0) \) is a face of every cone and \( S_0 = M \), it follows that \( U_0 = \text{Spec}(\mathbb{C}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]) = (\mathbb{C}^*)^N = T \) is contained as an open subset of \( X(\Delta) \). Moreover, for every cone \( \sigma \) of \( \Delta \), the group \( T \) acts on \( U_\sigma \) via the map induced by the diagonal \( \mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[S_0] \). These also fit together to produce an action of \( T \) on \( X(\Delta) \).

A toric variety is compact if and only if its generating fan is complete [11 §3.5]. The criterion for smoothness is that for any collection of \( N \) spanning vectors \( v_1, \ldots, v_N \) of the fan, the following holds:

\[
|\det(v_1, \ldots, v_N)| = 1,
\]

that is, they span \( \mathbb{Z}^N \) (cf. [11 page 29]). Note that the sign of the determinant is positive if the basis \( \{v_1, \ldots, v_N\} \) is oriented. From now on we shall only consider compact and smooth toric varieties. Recall that these toric varieties are all simply connected [11 §3.2].

Concerning the rational homotopy theory, any topological space considered here shall be (of the homotopy type of) a simply connected CW-complex. Two such spaces \( X \) and \( Y \) have the same rational homotopy type if there is a continuous map
\( f : X \to Y \) such that \( \pi_\ast(f) \otimes \mathbb{Q} : \pi_\ast(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_\ast(Y) \otimes \mathbb{Q} \) is an isomorphism. Such a map \( f \) is called a rational homotopy equivalence. A space \( X \) is \emph{formal} if its rational homotopy type depends only on its rational singular cohomology algebra \( H^\ast(X, \mathbb{Q}) \). To give a more precise definition of this property the following notion is indispensable, see [9, §12]:

A \emph{Sullivan model} of a given a commutative differential graded algebra (cdga henceforth) \( A \) is a cdga quasi-isomorphism of the form

\[
(\Lambda V, d) \xrightarrow{\cong} A.
\]

where \( \Lambda V \) denotes the free commutative algebra generated by the graded vector space \( V \) and whose differential \( d \) satisfies a special recurrence property: there is a well ordered basis \( \{v_\alpha\} \) on \( V \) such that, for each \( \alpha \), \( dv_\alpha \) is a "polynomial" in \( \Lambda V \) which only involves the generators \( \{v_\beta\} \) \( \beta < \alpha \). If \( A \) is the cdga \( \Omega_{PL}(X) \) of "polynomial forms" on a given space \( X \), this is a \emph{Sullivan model} of \( X \). Whenever \( X \) is a manifold, \( \Omega_{PL}(X) \) can be replaced by the classical de Rham forms \( \Omega(X) \).

As a Sullivan model of a given cdga characterizes its homotopy type, a space \( X \) is \emph{formal} if a Sullivan model of \( H^\ast(X, \mathbb{Q}) \) (with trivial differential) is also a Sullivan model of \( X \). Classical examples of formal spaces are compact Kähler manifolds \([6]\). On the other hand, a formal space \( X \) is said to be \emph{intrinsically formal} if it is the only (up to rational homotopy equivalence) space with \( H^\ast(X, \mathbb{Q}) \) as rational cohomology algebra.

The \emph{elliptic-hyperbolic dichotomy} \([9, \S 33]\) asserts that given a simply finite CW-complex \( X \) then, either its homotopy groups have finite total rank,

\[
\dim \pi_\ast(X) \otimes \mathbb{Q} < \infty,
\]

or else, \( \dim \pi_i(X) \otimes \mathbb{Q} \) grows exponentially as \( i \) increases: there exists \( \lambda > 1 \) and an integer \( n \) such that

\[
\sum_{i \leq k} \pi_i(X) \otimes \mathbb{Q} \geq \lambda^k, \quad \text{for } k \geq n.
\]

In the first case the space \( X \) is said to be \emph{elliptic}; otherwise, it is \emph{hyperbolic}.

### 3. Ellipticity of toric varieties

We first check formality of compact toric varieties. This is not automatic as they are not Kähler in general: indeed, smooth compact toric varieties are not necessarily projective (see \([8]\) for examples). On the other hand, since \( H^2(X) \) is generated by divisors, it follows that \( H^2(X) = H^{1,1}(X) \). Thus, if \( X \) is Kähler then it is projective, and the assertion follows. For completeness, we remark that quasitoric manifolds, the topological analogue of projective toric varieties, are also known to be formal \([19, \text{Corollary 7.2}]\).

**Proposition 3.1.** A smooth compact toric variety \( X \) is formal.
Proof. We first recall the explicit description of the integer cohomology ring of a smooth toric variety $X$. Let $D_1, \ldots, D_d$ be the irreducible $T$-divisors of $X$, that is, the submanifolds of complex codimension $1$ which are invariant under the action of the complex torus $T$ \cite[s.3.3]{11}. Each $T$-divisor $D_i$ corresponds to an edge (1-cone) of the fan $\Delta$, and each of them is determined by the first point $v_i$ in the lattice which is touched by the edge. More concretely, $D_i = U_{v_i} - U_{(0)}$.

The cohomology of a compact smooth toric variety is

$$H^*(X) = \mathbb{Z}[D_1, \ldots, D_d]/I,$$

where each $D_i$ is of degree $2$ and $I$ is the ideal generated by the following:

(i) The products $D_{i_1} \cdots D_{i_k}$, for $v_{i_1}, \ldots, v_{i_k}$ distinct vertices which do not lie in a cone of $\Delta$, and

(ii) $\sum_{i=1}^d \langle u_j, v_i \rangle D_i$, with $\{u_j\}$ a basis of $M$.

This is proved in \cite{11} page 106] for projective toric varieties. The statement for any compact smooth toric variety follows from \cite{4}.

Now, we inductively build a Sullivan model of $X$ by means of this cohomology description. First, for each generator $D_i$ and each relation in (ii) fix a Thom form $\eta_i \in \Omega^2(X)$ representing $D_i$, and for each $u_j$ we fix a $1$-form $\xi_j \in \Omega^1(X)$ such that $d\xi_j = \sum_{i=1}^d \langle u_j, v_i \rangle \eta_i$. Consider the graded vector space $V_1 = \langle w_i, w_j \rangle$, where $|w_i| = 2$, $|w_j| = 1$, set $dw_i = 0$, $dw_j = \sum_{i=1}^d \langle u, v_i \rangle w_i$ and define

$$f: (\Lambda^1 V_1, d) \longrightarrow \Omega(X)$$

by $f(w_i) = \eta_i$, $f(w_j) = \xi_j$. Obviously $H(f)$ is surjective and $H^2(f)$ is an isomorphism.

Next, observe that if $v_{i_1}, \ldots, v_{i_k}$ are distinct vertices which do not lie in a cone of the fan, then the intersection of the divisors $D_{i_1}, \ldots, D_{i_k}$ is empty and thus, $\eta_{i_1} \wedge \ldots \wedge \eta_{i_k} = 0$. Hence, the kernel of $f$ is generated by the corresponding $w_{i_1} \cdots w_{i_k}$. Then, define the graded vector space $V_2 = \langle w_I \rangle$, where $I$ runs over the tuples $\{i_1, \ldots, i_k\}$ such that $v_{i_1}, \ldots, v_{i_k}$ do not lie in a cone, declare $dw_I = w_{i_1} \cdots w_{i_k}$ and extend $f$ to

$$f: (\Lambda(V_1 \oplus V_2), d) \longrightarrow \Omega(X),$$

by setting $f(V_2) = 0$. Again, $H(f)$ may fail to be injective as non-trivial kernel may appear with the new generators. Hence, we continue this process to find a quasi-isomorphism

$$f: (\Lambda V, d) \xrightarrow{\simeq} \Omega(X)$$

in which $V = \oplus_{n \geq 1} V_n$, $dV_n \subset \Lambda(V_1 \oplus \cdots \oplus V_{n-1})$ and $f(V_k) = 0$, for $k \geq 2$. By construction, this is a Sullivan model of $X$.

Finally, observe that $f$ readily produces also a quasi-isomorphism

$$\tilde{f}: (\Lambda V, d) \xrightarrow{\simeq} (H^*(X), 0)$$

which is trivial on any generator except $\tilde{f}(w_i) = D_i$. This proves the formality. $\square$
Remark 3.2. The proof of Proposition 3.1 generalizes to locally standard torus manifolds with integral cohomology generated in degree 2. This is because by [17] there is a similar presentation of the cohomology ring as in the case of toric manifolds.

In what follows, and as usual, the Betti numbers of a given space are \( b_k(X) = \dim H^k(X, \mathbb{Q}) \), the Euler characteristic of \( X \) is \( \chi(X) = \sum_k (-1)^k b_k \), and its Poincaré polynomial is given by
\[
P_X(t) = \sum_k b_k t^k.
\]

As for any local Noetherian ring, recall that a polynomial algebra \( K[x_1, \ldots, x_n]/I \) over a field \( K \) is a complete intersection if \( I \) is generated by a regular sequence \( p_1, \ldots, p_m \). That is, for each \( i = 2, \ldots, n \), the class of each \( p_i \) is not a zero divisor in \( K[x_1, \ldots, x_n]/(p_1, \ldots, p_{i-1}) \). If this polynomial algebra is finite dimensional, it is a complete intersection if and only if \( m = n \).

**Theorem 3.3.** A smooth compact toric variety \( X \) is elliptic if and only if its cohomology algebra \( H^*(X, \mathbb{Q}) \) is a complete intersection concentrated in even degrees.

When the above condition is satisfied, \( X \) is intrinsically formal, and its Poincaré polynomial coincides with that of a product of complex projective spaces.

**Proof.** By the general description in (3.1), \( H^*(X) \) is evenly graded. Assume that \( X \) is elliptic. By [9, Proposition 32.16], the cohomology \( H^*(X, \mathbb{Q}) \) is evenly graded if and only if it is of the form \( \mathbb{Q}[x_1, \ldots, x_n]/(p_1, \ldots, p_n) \) in which every \( x_i \) is of even degree while \( p_1, \ldots, p_n \) is a regular sequence.

Conversely, it is well-known (see for instance [10, §3]) that any space whose cohomology algebra is a complete intersection concentrated in even degrees is intrinsically formal and elliptic.

Let \( X \) be an elliptic toric variety whose rational cohomology is the complete intersection \( \mathbb{Q}[x_1, \ldots, x_n]/(p_1, \ldots, p_n) \) in which each \( x_i \) is of even degree. Again by [9, Proposition 32.16], this has two other equivalent re-formulations:

1. \( \chi(X) > 0 \), and
2. \( \dim \pi_{\text{even}} \otimes \mathbb{Q} = \dim \pi_{\text{odd}} \otimes \mathbb{Q} = n \).

Moreover, if \( 2\alpha_1, \ldots, 2\alpha_n \) and \( 2\beta_1 - 1, \ldots, 2\beta_n - 1 \) are the degrees of a basis of \( \pi_\ast(X) \otimes \mathbb{Q} \), then \( 2\alpha_i \) is precisely the degree of \( x_i \) for all \( i \), and the Poincaré polynomial of \( X \) is given by
\[
P_X(t) = \prod_{i=1}^{n} \frac{(1 - t^{2\beta_i})}{(1 - t^{2\alpha_i})}.
\]

Now, since \( H^*(X, \mathbb{Q}) \) is generated by elements of degree 2, every \( \alpha_i = 1 \) and
\[
P_X(t) = \frac{\prod_{i=1}^{n} (1 - t^{2\beta_i})}{(1 - t^2)^n}.
\]
But the Poincaré polynomial of the projective space $\mathbb{C}P^k$ is

$$P_{\mathbb{C}P^k}(t) = \frac{1 - t^{2k+2}}{1 - t^2} = 1 + t^2 + t^4 + \ldots + t^{2k},$$

and thus, the Poincaré polynomial of $X$ is the same as that of

$$\mathbb{C}P^{\beta_1-1} \times \ldots \times \mathbb{C}P^{\beta_n-1}.$$ 

In particular, we have $\dim_{\mathbb{C}} X = \beta_1 + \ldots + \beta_n - n$. \hfill $\square$

**Remark 3.4.** The proof of Theorem 3.3 generalizes to manifolds with cohomology generated in degree 2.

Note that Theorem 3.3 implies in particular that, for an elliptic toric variety $X$ of (complex) dimension $N$, we have

$$b_2 = n \leq \sum (\beta_i - 1) = N. \quad (3.3)$$

This is a classical fact, which is valid in general for elliptic 1-connected finite CW-complexes.

**Example 3.5.**

1. The projective space $\mathbb{C}P^N$ is a toric variety. The torus $T = (\mathbb{C}^*)^N$ acts by $(t_1, \ldots, t_N) \cdot [z_0, z_1, \ldots, z_N] = [z_0, t_1 z_1, \ldots, t_N z_N]$. Clearly $\mathbb{C}P^N$ is elliptic.

2. Since the product of toric varieties is a toric variety, the polynomial (3.2) is the Poincaré polynomial of (at least one) elliptic toric variety, namely $\mathbb{C}P^{\beta_1-1} \times \ldots \times \mathbb{C}P^{\beta_n-1}$.

3. If $X$ is a toric variety, and $Y \subset X$ is a $T$-invariant subvariety, then the blow-up $X' = \text{Bl}_Y X$ of $X$ along $Y$ is again a toric variety. One particular example of this is the blow-up of a $T$-fixed point.

For instance, take $X = \mathbb{C}P^2$ and blow-up at a $T$-fixed point $p$. Then $X' = \text{Bl}_p X$ is a toric variety, and as a $C^\infty$ manifold it is $X \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, where $\overline{\mathbb{C}P^2}$ means $\mathbb{C}P^2$ with the opposite orientation. The cohomology of $X'$ is

$$H^\ast(X', \mathbb{Q}) = \mathbb{Q}[x, y]/(xy, x^2 + y^2), \quad (3.4)$$

where $x$ is the cohomology class of a line in $\mathbb{C}P^2$, $y$ the cohomology class of the exceptional divisor, and $x^2$ is the volume form. The cohomology algebra (3.4) is a complete intersection, so by Theorem 3.3, the manifold $X'$ is elliptic.

This also follows from [1, Theorem 1.1] as $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is diffeomorphic to a Hirzebruch surface; it also follows from the more general result, [21, Lemma 3.2], that classifies the homeomorphic type of simply connected elliptic closed real manifolds of dimension 4.

4. Consider a toric variety $Y$ with equivariant line bundles $L_0, \ldots, L_k$. Then the projectivization of the total space of the bundle $L_0 \oplus \ldots \oplus L_k \longrightarrow Y$ is again a toric variety [11, page 42]; denote $X := \mathbb{P}(L_0 \oplus \ldots \oplus L_k)$. Moreover, if $Y$ is elliptic then $X$ is also elliptic. This follows from the fact that topologically $X$ is a fibration with fiber $\mathbb{C}P^k$ and base $Y$, and hence it
has finite-dimensional total rational homotopy. Furthermore, the Poincaré polynomial of $X$ is $P_X(t) = P_Y(t)P_{C^Pn}(t)$; this follows from [5].

In particular, taking $Y = C^P1$, we have the Hirzebruch surfaces $X_b = \mathbb{P}(O_{C^P1} \oplus O_{C^P1}(b))$, and these are (again) elliptic and toric. Topologically, for $b$ even we have a diffeomorphism $X_b \cong C^P1 \times C^P1$, and for $b$ odd we have $X_b \cong C^P2#CP^2$. These share the same Poincaré polynomial, but have different cohomology algebras. So they have different homotopy type although they have isomorphic homotopy groups.

(5) Let $X$ be the blow-up of $C^P^N$ at a point. As a smooth manifold $X \cong C^P^N#\overline{C^P^N}$. Recall also that $\overline{C^P^N} \cong C^P^N$ for $N$ odd, since such $C^P^N$ admits an orientation reversing diffeomorphism. The cohomology is

\[ H^*(X, \mathbb{Q}) = \mathbb{Q}[x, y]/(xy, x^N + y^N), \]

where $x$ is the class of the hyperplane of $C^P^N$ and $y$ is the class of the exceptional divisor. This algebra is a complete intersection, and thus $X$ is elliptic.

The variety $X$ can also be described as a $C^P1$-bundle over $C^P^N-1$. More concretely, $X = \mathbb{P}(O_{C^P^N-1} \oplus O_{C^P^N-1}(1))$. It follows again that it is toric and elliptic.

(6) Consider the toric variety $X$ given as the blow-up of $C^P3$ at two fixed points. As a smooth manifold $X \cong C^P3#\overline{C^P3}#\overline{C^P3} \cong C^P3#\overline{C^P3}#\overline{C^P3}$. Then,

\[ H^*(X, \mathbb{Q}) = \mathbb{Q}[x, y, z]/(xy, xz, yz, x^3 - y^3, x^3 - z^3). \]

Here, $x, y, z$ are the generators of $H^2(C^P3, \mathbb{Q})$ for each of the three connected summands. This algebra is not a complete intersection (by Poincaré duality it needs at least three relations in degree 4 and another two relations in degree 6). Thus, by Theorem 3.3, the manifold $X$ is hyperbolic. However, its Poincaré polynomial is $P_X(t) = (1+t^2)^3$ and it coincides with that of $C^P^1 \times C^P^1 \times C^P^1$. This example shows that the Poincaré polynomial does not characterize the rational homotopy type of a compact smooth toric variety. Also $b_2 = 3$, so it satisfies the inequality in (3.3) even though $X$ is not elliptic.

**Remark 3.6.** It is straightforward to check that any Poincaré duality algebra $H$ generated in degree 2 and concentrated in degrees $\leq 4$ is necessarily a complete intersection. In particular, as observed in the proof of Theorem 3.3, such an algebra is intrinsically formal and thus, it is the rational cohomology of a unique, up to rational homotopy, space $X$ whose Euler homotopy characteristic is zero. That is, $\dim \pi_{even} X \otimes \mathbb{Q} = \dim \pi_{odd} X \otimes \mathbb{Q}$. We then may apply [18, Theorem 1.2] to conclude that there are infinitely many different isomorphism classes of these algebras, namely:

\[ \mathbb{Q}[x]/(x^3), \quad \mathbb{Q}[x, y]/(x^2, y^2), \quad \mathbb{Q}[x, y]/(x^2 + \lambda y^2, xy), \quad \lambda \in \mathbb{Q}^*/(\mathbb{Q}^*)^2. \]
As stated, any of these algebras determines a unique and distinct rational homotopy type. However only the following four of them
\[ \mathbb{Q}[x]/(x^3), \quad \mathbb{Q}[x, y]/(x^2, y^2), \quad \mathbb{Q}[x, y]/(x^2 \pm y^2, xy), \]
are realized by manifolds (see [21, Lemma 3.2]), namely by
\[ \mathbb{C}P^2, \quad \mathbb{C}P^1 \times \mathbb{C}P^1, \quad \mathbb{C}P^2 \# \mathbb{C}P^2, \quad \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2. \]

The third one is not a complex manifold (from the Enriques-Kodaira classification [2]). All the other three are toric by Example 3.5.

On the other hand, Example 3.5(6) exhibits a hyperbolic toric variety \( X \) of complex dimension 3 with \( P_X(t) = (1 + t^2)^3 \). Such examples do not exist for the other possible choice of the Poincaré polynomial in the same dimension given by Theorem 3.3, namely the polynomial \((1 + t^2)(1 + t^2 + t^4)\).

**Proposition 3.7.** Let \( H \) be a Poincaré duality algebra generated in degree 2 with \( P_H(t) = (1 + t^2)(1 + t^2 + t^4) \). Then:

(i) \( H \) is a complete intersection.

(ii) There are countable different rational homotopy types of elliptic simply connected CW-complexes whose cohomology is of this kind.

(iii) The only manifolds with cohomology of this kind are \( \mathbb{C}P^1 \times \mathbb{C}P^2 \) and \( \mathbb{C}P^3 \# \mathbb{C}P^3 \).

**Proof.** (i) Choose a free presentation of \( H \) which is necessarily of the form \( H = \mathbb{Q}[x, y]/I \), for some ideal \( I \). In view of the given Poincaré polynomial, we conclude that \( I \) must contain only one quadratic polynomial \( p \) and only one cubic polynomial \( q \) not generated by \( p \). We will show that \( I = (p, q) \), from which it follows that \( H \) is a complete intersection.

We complexify \( H_\mathbb{C} = H \otimes \mathbb{Q} \mathbb{C} = \mathbb{C}[x, y]/I \). It is enough to show that \( I = (p, q) \) in \( \mathbb{C}[x, y] \). Factor \( p = p_1p_2, \quad q = q_1q_2q_3 \), and arrange variables (by a linear change of variables) so that \( p_1 = x \). If \( p_1 = p_2 \) then, unless some \( q_1 = x \), all the monomials \( x^4, x^3y, x^2y^2, xy^3, y^4 \) are contained in \( I \) and thus \( I = (p, q) \). If \( p_1 \neq p_2 \) then we may assume \( p_2 = y \) and again, unless one of the \( q_i \) equals either \( x \) or \( y \), all the monomials of degree 4 are in \( I \), which is necessarily generated by \( p, q \).

We finish the proof of (i) by showing that none of the \( q_i \) can match \( x \) or \( y \). By contradiction, we assume without losing generality that \( q_1 = p_1 = x \). As \( H_\mathbb{C} \) is Poincaré duality, let \( r \) be a quadratic polynomial so that \( q_1r \notin I \). If either \( q_2 \) or \( q_3 \) coincides with \( p_2 \), then \( q \) is a multiple of \( p \) which contradicts our hypothesis. If both \( q_2 \) and \( q_3 \) are different from \( p_2 \), then any quadratic polynomial, in particular \( r \), is generated by \( p_2, q_2q_3 \). Therefore, \( q_1r \) is in the ideal generated by \( p_1p_2, q_1q_2q_3 \) which is \( I \) and we again reach a contradiction.

(ii) Complete intersection algebras \( H \) with \( \dim H < 8 \) are classified in [18, Theorem 1.2]. An inspection shows that any such algebra as in the statement, i.e., generated in degree 2 and with the prescribed Poincaré polynomial, is necessarily of the form
\[ \mathbb{Q}[x, y]/(x^2 + \lambda y^2, \mu x^3 + \gamma x^2 y), \]
where the rational numbers $\lambda, \mu, \gamma$ run through a precise countable set. As any of this algebras is intrinsically formal, it is the cohomology algebra of exactly one elliptic space, up to rational homotopy.

(iii) By [14, Theorem 1.3], cf. [1, Theorem 1.3], the rational homotopy types in (ii) which can be realized by a manifold are $\mathbb{C}P^1 \times \mathbb{C}P^2$ and $\mathbb{C}P^3 \# \mathbb{C}P^3$. 

4. Elliptic toric varieties of dimension 2 and 3

There are various classifications of toric varieties in terms of their describing fans. In the general case, a $d$-dimensional toric variety with second Betti number $b_2$ is given by a polytope in $\mathbb{R}^d$ with $d + b_2$ spanning vertices. As $b_2$ can be arbitrarily large for toric varieties, there are infinitely many isomorphism classes of toric varieties of given dimension $d$. Among those, algebraic geometers have paid special attention on classifying Fano toric varieties, those whose anti-canonical line bundle is ample. Apart from other geometric considerations, the list of these varieties is finite as the Fano property produces a bound of $b_2$. Precisely, there are 5, 18, 124, 866 isomorphism classes of Fano $d$-dimensional toric varieties for $d = 2, 3, 4, 5$ (see [16]).

However, from the topological point of view, restricting to the Fano property is most unnatural. Nevertheless, the bound (3.3) on $b_2$ will let us, in particular, attack the classification of elliptic toric varieties in low dimensions.

Notice first that $\mathbb{C}P^1$ is the only toric variety of dimension 1. In dimensions 2 and 3 we have:

**Theorem 4.1.** Any elliptic smooth toric variety of dimension 2 is either $\mathbb{C}P^2$ or a Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(b))$ of invariant $b$.

**Proof.** Let $X$ be a 2-dimensional, elliptic toric variety and keep in mind that $b_2(X) = b_2 \leq 2$ and that the generating fan is necessary complete as $X$ is smooth. For $b_2 = 1$, let $v_1, v_2, v_3$ be vectors in $\mathbb{R}^2$ spanning the generating fan of $X$, which we label in the cyclic order. We can arrange coordinates so that $v_1 = (1, 0)$ and $v_2 = (0, 1)$, since any two vectors of these vectors constitute a basis of $\mathbb{Z}^2$ by smoothness of $X$. Again by smoothness, $\det(v_3, v_1) = \det(v_2, v_3) = 1$ and therefore $v_3 = (-1, -1)$. Hence $X = \mathbb{C}P^2$ which is elliptic.

For $b_2 = 2$, let the fan of $X$ be spanned by four vectors in $\mathbb{R}^2$, $v_1, v_2, v_3, v_4$ that we write in cyclic order. Therefore we may consider $v_1 = (1, 0), v_2 = (0, 1)$ and $\det(v_2, v_3) = \det(v_3, v_4) = \det(v_4, v_1) = 1$, again by the smoothness of $X$. The defining matrix is given by

$$
\begin{pmatrix}
1 & 0 & -1 & a \\
0 & 1 & b & -1
\end{pmatrix},
$$

with $ab = 0, a, b \in \mathbb{Z}$. For $a = 0$ the matrix is

$$
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & b
\end{pmatrix},
$$
which corresponds to the Hirzebruch surface $X_b = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(b))$ of invariant $b$. Topologically, it is the fiber bundle over $\mathbb{C}P^1$ whose fiber is $\mathbb{C}P^1$ and the Chern class is $c_1 = b$. This is always rationally elliptic as, in general, the total space of a fibration in which the base and the fiber are elliptic, is also elliptic. For $b = 0$, $a \neq 0$, we can swap the coordinates (and the order of $v_1, v_2$ and of $v_3, v_4$) to go back to the previous case.

Observe that, with the notation in (3.1),

$$H^*(X_b, \mathbb{Q}) = \mathbb{Q}[D_1, D_2, D_3, D_4]/I,$$

where

$$I = (D_1D_3, D_2D_4, D_1 - D_3, D_2 + bD_3 - D_4).$$

Setting $D_1 = x, D_2 = y, D_3 = x, D_4 = y + bx$ we obtain that the cohomology is:

$$H^*(X_b, \mathbb{Q}) = \mathbb{Q}[x, y]/(x^2, y(y + bx)).$$

Here $x$ is the class of the fiber, and $y$ is the class $\mathcal{O}_X(1)$. \hfill $\square$

Recall that there is a diffeomorphism $X_b \cong \mathbb{C}P^1 \times \mathbb{C}P^1$ for $b$ even, and $X_b \cong \mathbb{C}P^2 \# \mathbb{C}P^2$ for $b$ odd. Hence there are three smooth manifolds (and three rational homotopy types) corresponding to the varieties in Theorem 4.1 namely: $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$.

**Theorem 4.2.** Let $X$ be an elliptic smooth toric variety of dimension 3.

1. If $b_2 = 1$, then $X = \mathbb{C}P^3$.
2. If $b_2 = 2$, then $X$ is either $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^2} \oplus \mathcal{O}_{\mathbb{C}P^2}(c))$ or $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(a) \oplus \mathcal{O}_{\mathbb{C}P^1}(b))$ and thus, it has the rational homotopy type of $\mathbb{C}P^3 \# \mathbb{C}P^3$ or $\mathbb{C}P^1 \times \mathbb{C}P^2$ respectively.
3. If $b_2 = 3$, then $X$ is a $\mathbb{C}P^1$-bundle over a Hirzebruch surface and has the rational homotopy type of a quotient $(S^3 \times S^3 \times S^3)/T^3$.

**Proof.** We follow the notations and descriptions of [22]. For computing the cohomology of the resulting varieties we use repeatedly its presentation in (3.1). Again, in view of the inequality (3.3), $b_2 \leq 3$.

(1) For $b_2 = 1$, the generating fan is spanned by four vectors which can always be chosen as $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1), v_4 = (-1, -1, -1)$. This produces $X = \mathbb{C}P^3$.

(2) For $b_2 = 2$, choose the triangulation of the fan in [22 page 41] with (oriented) cones $(v_1, v_2, v_3), (v_1, v_3, v_4), (v_1, v_4, v_5), (v_1, v_5, v_2), (v_2, v_4, v_3), (v_2, v_5, v_4)$. Recall that $\det(v_i, v_j, v_k) = 1$ for each cone, by smoothness of $X$. We can arrange that $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)$. This produces the matrix

$$\begin{pmatrix}
1 & 0 & 0 & -1 & a \\
0 & 1 & 0 & -1 & b \\
0 & 0 & 1 & c & -1
\end{pmatrix}.$$
where \( ac = bc = 0, \ a, b, c \in \mathbb{Z} \). Therefore there are two possibilities:

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & c & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & -1 & a \\
0 & 1 & 0 & -1 & b \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}.
\] (4.1)

The first fan in (4.1) produces \( X = \mathbb{P}(O_{\mathbb{C}P^2} \oplus O_{\mathbb{C}P^2}(c)) \) which is a \( \mathbb{C}P^1 \)-bundle over \( \mathbb{C}P^2 \), always rationally elliptic with Poincaré polynomial \((1 + t^2)(1 + t^2 + t^4)\). A computation similar to the one in the proof of Theorem 1.1 shows that

\[
H^*(X, \mathbb{Q}) = \mathbb{Q}[x, y]/(x^3, y(y + cx)).
\]

Thus, as a smooth manifold, using Proposition 3.7, \( X \) has the rational homotopy type of (and hence it is diffeomorphic to) \( \mathbb{C}P^3 \# \mathbb{C}P^3 \) if \( c \neq 0 \), and that of \( \mathbb{C}P^1 \times \mathbb{C}P^2 \) if \( c = 0 \).

The second fan in (4.1) corresponds to \( X = \mathbb{P}(O_{\mathbb{C}P^1} \oplus O_{\mathbb{C}P^1}(a) \oplus O_{\mathbb{C}P^1}(b)) \), which is a \( \mathbb{C}P^2 \)-bundle over \( \mathbb{C}P^1 \) and again elliptic. We also obtain that

\[
H^*(X, \mathbb{Q}) = \mathbb{Q}[x, y]/(x^3, y(y + ax)(y + bx)) = \mathbb{Q}[x, y]/(x^3, y^3 - \lambda xy^2),
\]

for suitable \( \lambda \), and therefore \( X \) has always the rational homotopy type of (and hence it is diffeomorphic to) \( \mathbb{C}P^2 \times \mathbb{C}P^1 \).

(3) If \( b_2 = 3 \), by [22, page 42], there are two possible triangulations of the generating fan of \( X \):

Case (I). The (oriented) cones are \((v_1, v_2, v_3), (v_2, v_3, v_4), (v_1, v_5, v_2), (v_2, v_5, v_6), (v_1, v_4, v_5), (v_1, v_3, v_4), (v_3, v_6, v_4), (v_4, v_6, v_5)\). This produces the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & a & d & -1 \\
0 & 1 & 0 & -1 & c & f \\
0 & 0 & 1 & b & -1 & e
\end{pmatrix},
\]

where \( bc = de = af = 0 \) and \( ace = -dfb, a, b, c, d, e, f \in \mathbb{Z} \). There are six cases:

\[
a = c = d = 0, a = b = e = 0, c = e = f = 0, a = b = d = 0, c = d = f = 0,
\]

\[
b = e = f = 0.
\]

In all cases, there is at least one column with 0 in two of the entries. We reorder variables so that this happens in the last position, that is, \( v_6 = (-1, 0, 0) \). We also reorder the first and second coordinates so that \( c = 0 \). This gives the

\[
\begin{pmatrix}
1 & 0 & 0 & a & d & -1 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & b & -1 & 0
\end{pmatrix}.
\]

A computation shows that

\[
H^*(X, \mathbb{Q}) = \mathbb{Q}[D_1, D_2, D_3, D_4, D_5, D_6]/I.
\]

where

\[
I = (D_2D_4, D_3D_5, D_1D_6, D_1 - aD_4 + dD_5 - D_6, D_2 - D_4, D_3 + bD_4 - D_5).
\]
Choosing \( D_4 = D_2 = x, D_5 = y, D_3 = y - bx, D_6 = z \) and \( D_1 = z - ax - dy \), we obtain

\[
H^*(X, \mathbb{Q}) = \mathbb{Q}[x, y, z]/(x^2, y(y - bx), z(z - ax - dy)),
\]

which is the cohomology algebra of a \( \mathbb{C}P^1 \)-bundle over the Hirzebruch surface \( \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(b)) \), associated to the Chern class \( ax + dy \) (where \( x \) is the class of the fiber and \( y \) is the class of the section). In particular, \( X \) is rationally elliptic.

Moreover, being intrinsically formal, it has the rational homotopy type of the quotient \( (S^3 \times S^3 \times S^3)/T^3 \), via the action

\[
(u, v, w) \cdot ((p_1, p_2), (q_1, q_2), (r_1, r_2)) = ((up_1, up_2), (uq_1, u^b q_2), (ur_1, u^a v^d wr_2)),
\]

in which \( (u, v, w) \in T^3 \) and \( ((p_1, p_2), (q_1, q_2), (r_1, r_2)) \in S^3 \times S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \).

Indeed, see [7, Proposition 4.26], such manifold has precisely the above rational cohomology algebra.

Case (II). The (oriented) cones are \((v_1, v_2, v_3)\), \((v_1, v_3, v_4)\), \((v_3, v_5, v_4)\), \((v_1, v_4, v_6)\), \((v_1, v_6, v_2)\), \((v_4, v_5, v_6)\), \((v_2, v_6, v_3)\), \((v_3, v_6, v_5)\). The solutions are given by:

\[
\begin{pmatrix}
1 & 0 & 0 & a & 1 & a -1 & -1 \\
0 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & 1 & b & b & -1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 & a & -1 \\
0 & 1 & 0 & -1 & -a -1 & 1 \\
0 & 0 & 1 & 0 & b & -1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & a & 1 & -1 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & -1 & a & -1 \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & n & m & -1 \\
0 & 1 & 0 & -1 & a -1 & a \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & -1 & 1 -mp & p \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{pmatrix}
\]

Here, the first five families have parameters \( a, b \in \mathbb{Z} \), and the sixth matrix corresponds to 5 isolated cases, with parameters \( (n, m, p) = (-1, 1, -3), (-2, 1, -2), (-1, 2, -2), (-3, 2, -1), (-2, 3, -1) \), which are solutions to \( n(1 + mp) - m = 1 \) not appearing in the previous families.

The cohomology of is given this time by

\[
\mathbb{Q}[D_1, D_2, D_3, D_4, D_5, D_6]/I.
\]

where the common non linear generators of \( I \) are,

\[
D_1D_5, D_2D_4, D_2D_5, D_3D_4D_6, D_1D_3D_6
\]

and the linear ones are immediately obtained in view of the corresponding matrix.

Now we take \( x = D_4, y = D_5, z = D_6 \), to get that the cohomology rings in every one of the six cases is:

\[
\begin{align*}
\mathbb{Q}[x, y, z]/(x^2 + xy, y^2 + x^2 + yz, xz^2, (1 - a - b)y^3 + z^3), \\
\mathbb{Q}[x, y, z]/(x^2 + xy, y^2 + x^2 + yz, xz^2, (1 - a - b)y^3 + z^3), \\
\mathbb{Q}[x, y, z]/((1 - a)xy + y^2, x^2 - xz, xy - yz, x^3, yz^2 + z^3), \\
\mathbb{Q}[x, y, z]/((1 - a)xy + y^2, x^2 - xz, xy - yz, x^3, yz^2 + z^3), \\
\mathbb{Q}[x, y, z]/(y^2 + yz, x^2 - axz, (1 - a)xy + y^2, xz^2, z^3), \\
\mathbb{Q}[x, y, z]/(y^2 + yz, x^2 - axz, (1 - a)xy + y^2, xz^2, yz^2 + z^3), \\
\mathbb{Q}[x, y, z]/((n + mnp - m)xy - yz, \\
(x + (1 + mp)y - pz)x, (1 - np)xy + y^2, xz^2, z^3 - myz^2).
\end{align*}
\]
We now check that none of the quotient ideals in this list is generated by a regular sequence and thus, the corresponding toric variety is always hyperbolic. As the argument we follow is similar for all of them we only do the first one. As a vector space, the algebra is generated by

$$1, x, y, z, xy, xz, z^2, x^2 y, xyz, y^3, x^2 yz,$$

which gives the Poincaré polynomial $P_X(t) = 1 + 3t^2 + 3t^4 + t^6 = (1 + t^2)^3$. Hence, for the ideal of relations $I$, we need at least three relations of degree 4 (quadratic on $x, y, z$). But then $xz^2 \not\in (x^2 + xy, y^2 + xy, y^2 + yz)$, so we need at least another relation of degree 6 showing that the cohomology ring $\mathbb{Q}[x, y, z]/I$ is not a complete intersection, and hence the toric variety is hyperbolic. □

**Remark 4.3.**

(i) The above process can potentially be carried out for dimension $d = 4$, although the number of cases grows drastically. An alternative possibility is to restrict to finding which Fano toric varieties of dimensions $d = 4, 5$ are elliptic, using the classifications in [16]. Note that the Fano condition implies that there are finitely many and a bound of $b_2$.

(ii) The toric varieties described in Case (II) of the proof of Theorem 4.2 constitute a large class of hyperbolic manifolds of very special nature.

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**School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India**

*E-mail address: indranil@math.tifr.res.in*

**Departamento de Algebra, Geometría y Topología, Universidad de Málaga, Campus de Teatinos, s/n, 29071 Málaga, Spain**

*E-mail address: vicente.munoz@uma.es*

**Departamento de Algebra, Geometría y Topología, Universidad de Málaga, Campus de Teatinos, s/n, 29071 Málaga, Spain**

*E-mail address: aniceto@uma.es*