SINGULARITIES OF VARIATIONS OF
MIXED HODGE STRUCTURE

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Abstract. We give a condition for a variation of mixed Hodge structure on a curve
to be admissible. It involves the asymptotic behavior of a grading of the weight
filtration, supplementing exactly the description of the graded variation and its mon-
odromy given by Schmid’s Orbit Theorems. In many salient cases the condition is
equivalent to admissibility.

1. Introduction

By the work of Griffiths and Schmid, the Gauss–Manin connection of a variation
of pure, polarized Hodge structure \( V \to \Delta \) with unipotent monodromy has a
regular singular point at the origin. Accordingly [5], the Hodge filtration \( \mathcal{F} \) of such a
variation extends to a system of holomorphic subbundles of the canonical extension
\( \tilde{V} \to \Delta \) of \( V \) [8]. Moreover, as a consequence of the \( SL_2 \) Orbit Theorem [15], the
limiting Hodge filtration of such a variation pairs with the (shifted) monodromy
weight filtration of \( V \) to define a limiting mixed Hodge structure on the central fiber
of \( \tilde{V} \).

In contrast, the situation for variations of graded-polarized mixed Hodge struc-
ture is markedly different [16]: irregular singularities and monodromies occur in
the simplest of non-geometric examples. For variations with unipotent monodromy,
such irregularities can be tamed by imposing the following admissibility conditions:

1. the limiting Hodge filtration \( F_\infty \) of \( V \) exists;
2. the relative weight filtration \( rW = rW(N,W) \) exists;

which in turn imply:

3. the pair \( (F_\infty, rW) \) is a mixed Hodge structure, relative to which the mon-
odromy logarithm \( N \) is morphism of type \((-1, -1)\).

Admissible variations are closed under degenerations. Most notably, variations
of pure structure are automatically admissible: this is a major consequence of
Schmid’s Orbit Theorems. Yet, this very fact limits their use to situations where
the Orbit Theorems can be ignored \textit{ab initio}, that is, when the limiting structure
can be constructed from other data, as in [16], where it comes from geometry.
It is clear that a mixed version of Griffith’s program and applications like those

We wish to thank P.Deligne, whose ideas on limits of mixed Hodge structures were generously
shared and play an important role here. The significance of the grading \( Y \) was discussed in
correspondence of his with E. Cattani and one of us (Kaplan) dating back to 1992 [6]. Related
results were presented in [16], [12] and [14].
of [2], [4] and [11], require a condition supplementary to Schmid’s Theorems to determine admissibility. For elementary reasons, these theorems do hold for the graded variation, hence the condition must be on the degeneration of the extension data.

In this article we propose a condition—that certain smooth grading $\mathbb{C}$-grading of the weight filtration extends continuously over the puncture- and prove its equivalence with admissibility in a number of cases. This equivalence is a mixed analog of the SL(2)-Orbit Theorem, since the way the logarithmic monodromy breaks up under the limiting grading characterizes “the monodromies that can occur”.

To state the results in detail, recall that a mixed Hodge structure $(F, W)$ on a complex vector space $V = V_R \otimes \mathbb{C}$ defines a unique bigrading $V = \bigoplus_{p,q} I^{p,q}$ of $V$ with the following properties:

$$F^p = \oplus_{a \geq p, b} I^{a,b}, \quad W_k = \oplus_{a+b \leq k} I^{a,b}$$

$$\tilde{F}^{p,q} \equiv \bar{I}^{n-p} \mod \bigoplus_{r<q, s<p} I^{r,s} \quad (1.1)$$

In particular, a mixed Hodge structure determines a grading $Y_{(F,W)}$ of the underlying weight filtration $W$ by the rule

$$Y_{(F,W)}(v) = kv \iff v \in \bigoplus_{p+q=k} I^{p,q} \quad (1.2)$$

These identification of a $\mathbb{Z}$-grading of $V$ with the semisimple endomorphism with degrees as eigenvalues, will be used throughout. Under it, the action of $GL(V)$ on gradings becomes the adjoint action on $\text{End}(V)$.

Our first result specifies the condition and asserts the equivalence in the case of variations that are unipotent in the sense of [9].

**Theorem I.** Let $\mathcal{V} \to \Delta^*$ be a variation of graded-polarized mixed Hodge structure such that the graded variation is constant. Then, $\mathcal{V}$ is admissible if and only if the $C^\infty$-grading $\tilde{Y}$ of $W$ defined pointwise by (1.2) extends continuously to $\tilde{\mathcal{V}} \to \Delta$.

The implications in

$\mathcal{V}$ admissible $\iff \tilde{Y}$ extends

can be established in many other cases. Here, we also establish the full equivalence when the limiting objects are real in the sense specified below.

In spite of the existing evidence, it is not obvious that this reality assumption can be dropped altogether. One cannot just replace a variation of mixed structure by a split one, as one does in the pure case: in the terms of [12], the monopole $g(z)$ become singular at infinity. In any case, the real situation arises often enough in practice (e.g., mirror symmetry) and is a necessary step towards the general case—because it is so for pure structures.
Recall that a mixed Hodge structure \((F, ^rW)\) is split over \(\mathbb{R}\) if and only if
\[
I_{p,q}^{\alpha}(F, ^rW) = I_{q,p}^{\alpha}(F, ^rW)
\]
Accordingly, we shall say that an admissible nilpotent orbit
\[(e^{zN}.F, W)\]
of mixed Hodge structure is split if the mixed Hodge structure
\[(F, ^rW(N, W))\]
splits over \(\mathbb{R}\). Note that if a nilpotent orbit \((e^{zN}.F, W)\) is admissible, that is precisely its limiting mixed Hodge structure. Our second result is:

**Theorem II.** Let
\[z \to e^{zN}.F\]
be a nilpotent orbit of graded-polarized mixed Hodge structure such that the graded orbits are split. Then \(e^{zN}.F\) is admissible and split if and only if the limit
\[
Y_\infty = \lim_{t \to \infty} Ad(e^{-zN})Y(e^{zN}.F, W)
\]
exists and is defined over \(\mathbb{R}\).

For a general variation, the limiting mixed Hodge structure is only defined once a coordinate on the base -more specifically, an element of \(T_0(\Delta)^*\), has been chosen. We can then refer to split admissible variations, as those whose limiting mixed Hodge structure is split. With this understood, one has

**Theorem III.** Let \(V \to \Delta^*\) be a variation of graded-polarized mixed Hodge structure with unipotent monodromy, and \(\mathcal{Y}\) denote the \(C^\infty\) grading the weight filtration \(W\) obtained by applying (1.2) to each fiber of \(V\). Suppose that the graded variations are split. Then, the variation \(V\) is admissible and split if and only if \(\mathcal{Y}\) extends continuously to a grading of \(W\) in \(\tilde{V}\), with limiting value \(\mathcal{Y}(0) = \lim_{s \to 0} \mathcal{Y}(s)\) defined over \(\mathbb{R}\).

In any of the above situations, let
\[N = N_0 + N_{-2} + \cdots\]
be the decomposition of the monodromy logarithm relative to the grading \(Y_\infty\) (i.e., \(\mathcal{Y}(0))\). Let
\[\rho : sl_2(\mathbb{C}) \to End(V)\]
be the representation determined by the Orbit Theorems for pure structures applied to \(Gr^W\), together with the identification \(Gr^W \approx V\) provided by \(Y_\infty\). Let
\[n_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad n_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\]
In the situation of Theorems II and III where \(Y_\infty\) is real, \(\rho\) is defined over \(\mathbb{R}\):
\[\rho : sl_2(\mathbb{R}) \to gl(V_\mathbb{R}).\]
In Theorem I instead, \(\rho\) is trivial in \(Gr^W\) and \(N = N_{-2}\), but \(Y_\infty\) need not be real. In any case, the following holds
Theorem IV.

- $N_0 = \rho(n_-)$,
- $N_{-1} = 0$

and for all $k > 1$, $N_{-k}$ is a highest weight vector for $\rho$ of weight $k - 2$, i.e.,

- $\rho(h)N_{-k} = (k - 2)N_{-k}$
- $\rho(n_+)N_{-k} = 0$.

Next, we sketch the proofs of these theorems. Theorem I is proved in §3. One first writes the period map $F: U \to \mathcal{M}$ as

$$F(z) = e^{zN} e^{\Gamma(s)} F_\infty$$

where $F_\infty$ is the limiting Hodge filtration of $\mathcal{V}$, and $\Gamma(s)$ is a holomorphic function of $s = e^{2\pi iz}$ which takes values in the nilpotent Lie algebra

$$\text{Lie}_{-1} = \{ \alpha \in \text{End}(V) \mid \alpha : W_k \to W_{k-1} \forall k \}$$

and vanishes at $s = 0$. To finish the proof we apply Theorem (2.9), which provides a description of how the decomposition $V = \bigoplus_{p,q} I_{p,q}$ varies under the action of $\exp(\text{Lie}_{-1})$ on $\mathcal{M}$.

Theorems II and III are proved in §4 and §5, respectively. One of the implications in the first depends heavily upon the results of Deligne [6] discussed in the appendix at the end of our paper. They imply that if $W$ and $N$ arise from a nilpotent orbit of graded-polarized mixed Hodge structure $e^{zN} F$ which is admissible and split, then

$$Y(e^{zN}.F.W) = \text{Ad}(e^{zN}) \hat{Y}$$

where $\hat{Y} = Y(N, Y)$, with $Y = Y(F, W)$, is a grading of $W$ defined over $\mathbb{R}$. In particular, whenever $e^{zN}.F$ is a split admissible nilpotent orbit, the associated grading

$$e^{-zN}.Y(e^{zN}.F.W)$$

in Theorem II has constant value $Y_\infty = \hat{Y}$. Likewise, if $(F, W)$ is split over $\mathbb{R}$ then so are each of the induced mixed Hodge structures in the graded, and the corresponding nilpotent orbits are split.

To prove the converse, we recall that split nilpotent orbits of mixed Hodge structure are just $\text{SL}(2, \mathbb{R})$-orbits, assemble the corresponding representations in the graded into one on $V$ as already described, define

$$(r)Y := Y_\infty + \rho(h), \quad (r)W_k = \bigoplus_{j \leq k} E_j((r)Y)$$

and prove that $(r)W_k$ is actually the relative weight filtration of $N$ and $W$. This is the most technically complicated part of the proof, involving the use of the $\mathfrak{sl}_2$ symmetry to deduce properties of certain Laurent series. Theorem IV is a corollary of these proofs.

Some of the material from [6] and [10] is being published here for the first time.
2. Preliminaries

In this section we reformulate our theorems in terms of period maps, review
the definition of admissibility, discuss the geometry of the associated classifying
spaces and describe an analog of the Nilpotent Orbit Theorem for variations of
graded-polarized mixed Hodge structure.

Definition. A variation of graded-polarized mixed Hodge structure consists of a
$\mathbb{Q}$-local system $\mathcal{V}_\mathbb{Q} \to S$ endowed with the following additional data:

1. A rational, increasing weight filtration

$$0 \subseteq \cdots \subseteq W_{k-1} \subseteq W_k \subseteq \cdots \subseteq \mathcal{V}_\mathbb{C}$$

of $\mathcal{V}_\mathbb{C} = \mathcal{V}_\mathbb{Q} \otimes \mathbb{C}$.

2. A decreasing Hodge filtration

$$0 \subseteq \cdots \subseteq \mathcal{F}_p \subseteq \mathcal{F}^{p-1} \subseteq \cdots \subseteq \mathcal{V}_\mathbb{C} \otimes \mathcal{O}_S$$

which is holomorphic and horizontal with respect to the Gauss-Manin con-
nnection $\nabla$ of $\mathcal{V}_\mathbb{C} \otimes \mathcal{O}_S$ (i.e. $\nabla \mathcal{F}_p \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}$) and pairs with the weight
filtration $\mathcal{W}$ to define a mixed Hodge structure on each fiber of $\mathcal{V}$.

3. A collection of rational, non-degenerate bilinear forms

$$S_k : Gr^W_k(\mathcal{V}_\mathbb{Q}) \otimes Gr^W_k(\mathcal{V}_\mathbb{Q}) \to \mathbb{Q}, \quad Gr^W_k := W_k/W_{k-1}$$

of alternating parity $(-1)^k$ which polarize the corresponding variations of
pure Hodge structure

$$FGr^W_k := \frac{\mathcal{F}_p \cap W_k + W_{k-1}}{W_{k-1}}$$

Actually, we shall make no reference to either the graded-polarizations or the
rational structure itself, so the results hold for arbitrary real variations of graded-
polarizable mixed Hodge structure.

Now, as discussed in [10], [14] and elsewhere, the data of such a variation $\mathcal{V} \to S$
may be reformulated in terms of the monodromy representation

$$\rho : \pi_1(S, s_0) \to Aut(\mathcal{V}_{s_0}), \quad \text{Image}(\rho) = \Gamma$$

of $\mathcal{V}$ on a given fiber $\mathcal{V}_{s_0}$ and the corresponding period map

$$\varphi : S \to \mathcal{M}/\Gamma$$

obtained by parallel translating the data of $\mathcal{V}$ to $\mathcal{V}_{s_0}$. Alternatively, upon passage
to the universal cover $\tilde{S}$ of $S$, such a variation is equivalent to the monodromy
representation $\rho$ defined above together with a $\pi_1$-equivariant map

$$F : \tilde{S} \to \mathcal{M}$$
from $\tilde{S}$ into a suitable classifying space $\mathcal{M}$ of graded-polarized mixed Hodge structure. $F$ is both holomorphic and horizontal: i.e.

$$\frac{\partial}{\partial z_j} F^p(z) \subseteq F^p(z), \quad \frac{\partial}{\partial \bar{z}_j} F^p(z) \subseteq F^{p-1}(z).$$

In particular, if $V \to \Delta^*$ is a variation of graded-polarized mixed Hodge structure with unipotent monodromy action $T = e^{-N}$, its period map $\varphi$ may be viewed as a holomorphic, horizontal map

$$F : U \to \mathcal{M}$$

from the upper half-plane $U$ into $\mathcal{M}$ which satisfies the quasi-periodicity condition

$$F(z + 1) = e^N.F(z).$$

Our choice of sign for the monodromy logarithm reflects the fact that we are pulling back the data of $V$ to a fixed reference fiber.

More specifically, parallel translation under $\nabla$ endows our fixed reference fiber $V = V_{s_0}$ with a choice of rational structure $V_\mathbb{Q}$ as well as a constant, rational weight filtration $W$ and a collection of constant, rational, non-degenerate bilinear forms $S_k : Gr^W_k (V_{\mathbb{Q}}) \otimes Gr^W_k (V_{\mathbb{Q}}) \to \mathbb{Q}$ of alternating parity $(-1)^k$. Modulo the action of the monodromy group $\Gamma$, we also obtain a Hodge filtration $F$, which pairs with $W$ to define a graded-polarized mixed Hodge structure with constant graded Hodge numbers

$$h^{p,q} = \dim H^{p,q}$$

Consider the corresponding classifying space

$$\mathcal{M} = \mathcal{M}(W, S, h^{p,q}),$$

consisting of all filtrations $F$ of $V$ such that $(F, W)$ is a mixed Hodge structure, which is graded-polarized by $S$ and such that

$$\dim_C F^p Gr^W_k = \sum_{r \geq p} h^{r,k-r}.$$ 

As shown in [10] and [14], $\mathcal{M}$ is a homogeneous complex manifold which fits into an ascending sequence of homogeneous spaces

$$M_{\mathbb{R}} \subseteq \mathcal{M} \subseteq \tilde{M} \subseteq \tilde{\mathcal{F}}(W) \subseteq \mathcal{F},$$

defined as follows:

- $\mathcal{F}$ is the flag variety consisting of all decreasing filtrations $F$ such that $\dim_C F^p = \sum_{r \geq p} h^{r,s}$.
- $\tilde{\mathcal{F}}(W)$ is the submanifold of $\tilde{\mathcal{F}}$ consisting of those filtrations $F \in \tilde{\mathcal{F}}$ such that $\dim_C F^p Gr^W_k = \sum_{r \geq p} h^{r,k-r}$.
- $\tilde{\mathcal{M}}$ is the submanifold of $\tilde{\mathcal{F}}(W)$ consisting of all filtrations $F \in \tilde{\mathcal{F}}(W)$ which satisfy Riemann’s first bilinear relation with respect to the graded-polarizations $S$.
- $\mathcal{M}_{\mathbb{R}}$ is the $C^\infty$-submanifold of $\mathcal{M}$ consisting of the filtration $F \in \mathcal{M}$ for which the associated mixed Hodge structure $(F, W)$ is split over $\mathbb{R}$.
The corresponding sequence of Lie groups is

\[ G_R \subseteq G \subseteq G_C \subseteq GL(V)^W \subseteq GL(V) \]

where

- \( GL(V)^W = \{ g \in GL(V) \mid g : W_k \to W_k \ \forall k \} \).
- \( G_C = \{ g \in GL(V)^W \mid Gr(g) \in Aut_C(S) \} \).
- \( G = \{ g \in G_C \mid Gr(g) \in Aut_R(S) \} \).
- \( G_R = \{ g \in G \mid g \in GL(V_R) \} \).

i.e. \( GL(V) \) acts transitively on \( \tilde{\mathcal{F}} \), \( GL(V)^W \) acts transitively on \( \tilde{\mathcal{F}}(W) \), \( G_C \) acts transitively on \( \tilde{\mathcal{M}} \), \( G \) acts transitively on \( \mathcal{M} \) and \( G_R \) acts transitively on \( \mathcal{M}_R \).

Next we describe the canonical extension of \( V \to \Delta^* \) in more detail [8]. Given a flat vector bundle \( E \to \Delta^* \) with unipotent monodromy, there exists a unique extension \( E^c \to \Delta^* \) relative to which the flat connection of \( E \) has at worst simple poles with nilpotent residues along the divisor \( D = \Delta^n / \Delta^* \). Alternatively, given a choice of local coordinates \((s_1, \ldots, s_n)\) on \( \Delta^* \) relative to which the divisor \( D \) assumes the form \( s_1 \cdot \ldots \cdot s_n = 0 \), the canonical extension \( E^c \) described above may be identified with the local free sheaf generated by the sections

\[ \tilde{\sigma} = \exp(\frac{1}{2\pi i} \sum_{j=1}^n \log s_j N_j) \sigma \]

where \( \sigma \) is a flat, multivalued section of \( E \) and

\[ N_j = -\log(T_j) \]

is the monodromy logarithm associated to pulling back along the \( j \)'th loop

\[ \gamma_j(t) = (s_1, \ldots, e^{2\pi i t} s_j, \ldots, s_n). \]

Remark. We shall denote the adjoint action of \( GL(V) \) upon \( End(V) \) by \( g.\alpha \) and the action of \( G_C \) [resp. \( G \)] upon \( \tilde{\mathcal{M}} \) [resp. \( \mathcal{M} \)] by \( g.F \).

**Lemma 2.1.** Let \( V \to \Delta^* \) be a variation of graded-polarized mixed Hodge structure with unipotent monodromy action \( T = e^{-N} \), and \( F : U \to \mathcal{M} \) be a lift of the associated period map \( \phi : \Delta^* \to \mathcal{M}/\Gamma \) to the upper half-plane. Then, the functions

\[ \psi(z) = e^{-zN} F(z), \quad Y(z) = e^{-zN} Y_{(F(z), W)} \]

satisfy the periodicity conditions

\[ \psi(z + 1) = \psi(z), \quad Y(z + 1) = Y(z) \]

and hence descend to well defined functions \( \psi(s) \) and \( Y(s) \) on the punctured disk \( \Delta^* \) via the covering map \( z \to s = e^{2\pi i z} \). Moreover,

- The Hodge filtration \( F \) of \( V \) extends to the canonical extension \( \tilde{V} \to \Delta \) if and only if the limiting Hodge filtration

\[ F_\infty = \lim_{s \to 0} \psi(s) \]
exists as an element of $\mathcal{M}$.

- The grading $Y$ of the underlying weight filtration $W$ defined by the $I^{p,q}$'s of $V$ extends continuously to the canonical extension of $V$ as a grading of $W$ if and only if

$$Y_\infty = \lim_{s \to 0} Y(s)$$

exists, and grades $W$.

**Proof.** One simply identifies $\tilde{V}$ with the locally free sheaf generated by the sections $\tilde{\sigma}$ described above.

For a coordinate-free description, both the limiting Hodge filtration $F_\infty$ and the limiting grading $Y_\infty$ defined above should actually be viewed as objects defined on the cotangent space of $\Delta$ at zero. More precisely, if $\tilde{s}$ and $s$ are holomorphic coordinates on $\Delta$ which vanish at 0, then corresponding limiting objects $F_\infty$ and $Y_\infty$ defined above will in general agree if and only if $(d\tilde{s})_0 = (ds)_0$.

Recall that a variation of graded-polarized mixed Hodge structure with unipotent monodromy action $T = e^{-N}$ is admissible [16] if and only if the following hold:

1. The Hodge filtration $F$ of $V \to \Delta^*$ extends holomorphically to the canonical extension $\tilde{V}$.
2. The relative weight filtration $r^W = r^W(N,W)$ exists.

The relative weight filtration is defined by the Lemma below. In the appendix to [16], Deligne proved that whenever a variation of graded-polarized mixed Hodge structure is admissible, then the limiting Hodge filtration $F_\infty$ of $V$ pairs with the corresponding relative weight filtration $r^W$ to define a mixed Hodge structure for which $N$ is a $(-1,-1)$-morphism.

To define the filtration $r^W = r^W(N,W)$, recall first that given a nilpotent endomorphism $N$ of a finite dimensional complex vector space $V$, the corresponding (monodromy) weight filtration $W(N)$ is the unique increasing filtration of $V$ such that

- $N : W_\ell(N) \to W_{\ell-2}(N)$.
- The induced map $N^W : \text{Gr}^W_\ell(N) \to \text{Gr}^W_{\ell-2}(N)$ is an isomorphism.

Given an increasing filtration $W$ of a finite dimensional vector space $V$ and an index $\ell \in \mathbb{Z}$, the corresponding shifted object is the filtration

$$W[\ell]_j = W_{j+\ell}$$

**Lemma 2.2.** (cf. [16]) Let $W$ be an increasing filtration of a finite dimensional vector space $V$ and $N$ be a nilpotent endomorphism of $V$ which preserves $W$. Then, there exists at most one increasing filtration $r^W = r^W(N,W)$ of $V$ such that for each index $k$:

1. $N : r^W_k \to r^W_{k-2}$.
2. $r^W$ induces on $\text{Gr}^W_k$ the shifted monodromy weight filtration

$$W(N : \text{Gr}^W_k \to \text{Gr}^W_k)[-k]$$

of $N$ on $\text{Gr}^W$.

To close this section we discuss some of the geometry of the classifying space $\mathcal{M}$. 

Lemma 2.3. Let $F(z)$ be the period map of a variation of graded-polarized mixed Hodge structure with unipotent monodromy for which the limiting Hodge filtration $F_\infty = \lim_{\Im(z)\to\infty} e^{-zN}.F(z)$ exists. Then, given a choice of a vector space decomposition

$$\text{Lie}(G_C) = \text{Lie}(G_{C\infty}^F) \oplus q$$

there exists a unique holomorphic function $\Gamma : \Delta \to q$, $\Gamma(0) = 0$, $s = e^{2\pi iz}$, such that

$$F(z) = e^{zN}e^{\Gamma(s)}.F_\infty$$

for $\Im(z) >> 0$.

Proof. One simply notes that the map $e^{-zN}.F(z)$ takes values in the homogeneous space $\mathcal{M}$, upon which $G_C$ acts transitively. To obtain $\Gamma(0) = 0$, note the definition of $F_\infty$.

Remark. If $F(z)$ is unipotent in the sense of [9], i.e. the induced variations on Gr$W$ are constant, the function $\Gamma(s)$ of (2.3) assumes values in the subalgebra

$$\eta_+ = \bigoplus_{r\geq 0, s<0} gl(V)^{r,s}, \quad \eta_- = \bigoplus_{s\geq 0, r<0} gl(V)^{r,s}, \quad \eta_0 = gl(V)^{0,0}, \quad \Lambda^{-1,-1} = \bigoplus_{r,s<0} gl(V)^{r,s}. $$

Accordingly, $\text{Lie}(G_C) = \text{Lie}(G_{C\infty}^F) \oplus q$, where

$$\text{Lie}(G_C) = \eta_+ \oplus \eta_0, \quad q = \eta_- \oplus \Lambda^{-1,-1}$$

(2.4)

and

$$\text{Lie}(G_C) = \eta_+ \oplus \eta_0 \oplus \eta_- \oplus \Lambda^{-1,-1}$$

determines projection operators

$$\pi_+ : \text{Lie}(G_C) \to \eta_+, \quad \pi_- : \text{Lie}(G_C) \to \eta_-,$$
$$\pi_0 : \text{Lie}(G_C) \to \eta_0, \quad \pi_\Lambda : \text{Lie}(G_C) \to \Lambda^{-1,-1}.$$ (2.5)

The congruence (1.1) is reflected in the following relations

$$\eta_+ \subseteq \eta_- \oplus \Lambda^{-1,-1}, \quad \eta_0 \subseteq \eta_0 \oplus \Lambda^{-1,-1},$$
$$\eta_- \subseteq \eta_+ \oplus \Lambda^{-1,-1}, \quad \Lambda^{-1,-1} = \Lambda^{-1,-1}.$$ (2.6)

Note that the subalgebra $\Lambda^{-1,-1}$ depends only upon the mixed Hodge structure $(F,W)$ and not the choice of graded-polarization.
Lemma 2.7. ([10], [14]) Let \((F, W)\) be a mixed Hodge structure. Then,
\[
g \in \exp(\Lambda_{(F, W)}^{-1, -1}) \implies I_{(g, F, W)}^{p, q} = gI_{(F, W)}^{p, q}
\]
In particular, \(F \in \mathcal{M}\) and \(g \in G_R \cup \exp(\Lambda_{(F, W)}^{-1, -1})\) implies \(I_{(g, F, W)}^{p, q} = gI_{(F, W)}^{p, q}\).

Given \(F \in \mathcal{M}\), we can smoothly decompose
\[
g_C = g_Re^\lambda f \tag{2.8}
\]
for \(g_C \in G_C, g \approx 1, \) with
\[
g_R \in G_R, \quad e^\lambda \in \exp(\Lambda^{-1, -1}), \quad f \in G_C^F.
\]
Similarly,
\[
g = g_Re^\lambda f
\]
for \(g \in G, \) with
\[
g_R \in G_R, \quad e^\lambda \in \exp(\Lambda^{-1, -1}), \quad f \in G_{-1}^F = G_C^F \cap \exp(\text{Lie}_{-1})
\]
Therefore (2.8) holds whenever \(g_C.F \in \mathcal{M}\).

For \(g \in \exp(\text{Lie}_{-1})\) we can be more precise. Define an extended Lie monomial to be a non-zero function \(\mu : \text{Lie}_{-1} \to \text{Lie}_{-1}\) which can be obtained from the primitive monomial \(m(\alpha) = \alpha\) in a finite number of steps via the operations of complex conjugation, taking Hodge components with respect to \((F, W)\) and forming Lie brackets (e.g. \(\mu(\alpha) = [\bar{\alpha}, \alpha^{-1.0}]^{-1, -1}\)). In particular, if \(\mu\) is an extended Lie monomial then so is any non-zero function of the form \(\pi_+ (\mu), \pi_\Lambda (\mu)\) or \(\pi_- (\mu)\). Define
\[
\text{Lie}_{-r} = \{ \zeta \in gl(V) \mid \zeta : W_k \to W_{k-r} \}
\]
which are ideals of \(\text{Lie}_{-1}\).

Theorem (2.9). Let \(F \in \mathcal{M}\) and \(\alpha \in \text{Lie}_{-1}\). Then,
\[
e^\alpha = e^\gamma e^\lambda e^\phi
\]
where \(\gamma, \lambda, \) and \(\phi\) are extended Lie polynomials in \(\alpha\) which take values in the respective subalgebras \(\text{Lie}(G_R)_{-1} = \text{Lie}(G_R) \cap \text{Lie}_{-1}, \Lambda_{-1, -1}^{1, -1}\) and \(\text{Lie}(G_C^F)_{-1} = \text{Lie}(G_C^F) \cap \text{Lie}_{-1}\). Moreover,
\[
\begin{align*}
\gamma &= \pi_-(\alpha) + \overline{\pi_-(\alpha)} \mod \text{Lie}_{-2}, \\
\lambda &= \pi_\Lambda (\alpha) - \overline{\pi_\Lambda (\alpha)} \mod \text{Lie}_{-2}, \\
\phi &= \pi_+ (\alpha) - \overline{\pi_+ (\alpha)} \mod \text{Lie}_{-2}. \tag{2.10}
\end{align*}
\]

Proof. Since \(\text{Lie}_{-1}\) is nilpotent, the Campbell–Baker–Hausdorff formula for it terminates after finitely many terms. Set
\[
\gamma_1 = \pi_- (\alpha) + \overline{\pi_- (\alpha)}, \quad \lambda_1 = \pi_\Lambda (\alpha) - \overline{\pi_\Lambda (\alpha)}, \quad \phi_1 = \pi_+ (\alpha) - \overline{\pi_+ (\alpha)}.
\]
Because of (2.6) and \([\text{Lie}_{-p}, \text{Lie}_{-q}] \subseteq \text{Lie}_{-p-q}\), one has \(\gamma_1 + \lambda_1 + \phi_1 = \alpha\). Therefore
\[
e^{-\gamma_1}e^{\lambda_1}e^{\phi_1} = e^{\alpha+\beta_1}
\]
where \(\beta_1\) is an extended Lie polynomial with values in \(\text{Lie}_{-2}\). Inductively, suppose that there exist extended Lie polynomials \(\gamma_k\) with values in \(\text{Lie}(G_{\mathbb{R}})_{-1}\), \(\lambda_k\) with values in \(\Lambda^{-1,1}\) and \(\phi_k\) with values in \(\text{Lie}(G_{\mathbb{C}}^F)_{-1}\) such that
\[
e^{-\gamma_k}e^{\lambda_k}e^{\phi_k} = e^{\alpha+\beta_k}
\]
for some extended Lie polynomial \(\beta_k\) with values in \(\text{Lie}_{-1}\). Let
\[
\begin{align*}
\gamma_{k+1} &= \gamma_k + \gamma', & \gamma' &= -\pi_-(\beta_k) - \overline{\pi_-(\beta_k)} \\
\lambda_{k+1} &= \lambda_k + \lambda', & \lambda' &= -\pi_{\Lambda}(\beta_k) + \pi_{\Lambda}(\overline{\pi_-(\beta_k)}) \\
\phi_{k+1} &= \phi_k + \phi', & \phi' &= -\pi_+(\beta_k) + \pi_+(\overline{\pi_-(\beta_k)})
\end{align*}
\]
Then \(\gamma' + \lambda' + \phi' = -\beta_k\), and hence
\[
e^{-\gamma_{k+1}}e^{\lambda_{k+1}}e^{\phi_{k+1}} = e^{\alpha+\gamma'+\lambda'+\phi'+\beta_{k+1}} = e^{\alpha+\beta_{k+1}}
\]
for some extended Lie polynomial \(\beta_{k+1} \in \text{Lie}_{-2}\). As \(\text{Lie}_{-r} = 0\) for some index \(r = r_0\), this completes the proof.

3. Unipotent Variations

Let \(X\) be a smooth, complex algebraic variety. Then, by the work of Hain, Morgan et. al. (cf. [1] for an overview) for each positive integer \(k\), there exists a canonical admissible, variation of graded-polarized mixed Hodge structure \(\mathcal{V} \to X\) with fiber
\[
\mathcal{V}_x = \mathbb{C}\pi_1(X,x)/(J_x)^{k+1}
\]
where \(J_x\) is the augmentation ideal of \(\mathbb{C}\pi_1(X,x)\). The monodromy representation
\[
\rho : \pi_1(X,x) \to \text{Aut}(\mathcal{V}_x)
\]
of such a variation is unipotent and the variations of pure, polarized Hodge structure induced by \(\mathcal{V}\) on \(\text{Gr}^W\) are constant in this situation. Such variations are called unipotent. With this motivation in mind, we consider here problem of determining the admissibility for such variations in terms of the grading \(Y\) as discussed in [§1], beginning with the following observations:

(a) By virtue of Schmid’s Nilpotent Orbit Theorem, the monodromy logarithm \(\mathcal{N}\) of a unipotent variation \(\mathcal{V} \to \Delta^*\) must act trivially on \(\text{Gr}^W\).
(b) On account of (a), the relative weight filtration \(\mathcal{W}\) of a unipotent variation exists if and only if \(\mathcal{N} : W_k \to W_{k-2}\) for each index \(k\), i.e. \(\mathcal{W} = W\).
Theorem 3.1. Let $\mathcal{V} \to \Delta^*$ be a unipotent variation of graded-polarized mixed Hodge structure which is admissible. Then, $\mathcal{V}$ extends continuously, as a grading of $\mathcal{W}$, to the canonical extension $\tilde{\mathcal{V}} \to \Delta$.

Proof. Let $F: U \to \mathcal{M}$ be a lift the period map of $\mathcal{V}$ to the upper half-plane $U$ and let $N$ be its monodromy logarithm. Then, as discussed in §2, the claim is equivalent to the assertion that

$$Y_\infty = \lim_{\text{Im}(z) \to \infty} e^{-zN}Y_{(F(z),W)}$$

exists and grades $W$.

To verify the existence of the grading $Y_\infty$, observe that on account of the unipotency of $\mathcal{V}$, the function

$$\psi(s) = e^{-zN}F(z), \quad s = e^{2\pi i z}$$

considered in §2 takes values in $\mathcal{M}$ (and not $\tilde{\mathcal{M}}$ as is a priori the case). Indeed, in our case $\psi(s)$ induces the same filtration as $F(z)$ on $\text{Gr}^W$, and hence is an element of $\mathcal{M}(\text{Gr}^W)$. Moreover, since the induced filtration $F(z)\text{Gr}^W$ of a unipotent variation is by definition constant, the limiting Hodge filtration $F_\infty = \lim_{s \to 0} \psi(s)$ of a $\mathcal{V}$ is likewise an element of $\mathcal{M}$. By (b) above, the relative weight filtration of our admissible variation $\mathcal{V}$ must coincide with the weight filtration $W$ of $\mathcal{V}$. By Deligne’s theorem, $N$ is $(-1,-1)$–morphism of $(F_\infty,W)$. We now apply Lemma (2.3), with $q = \eta_- \oplus \Lambda^{-1,-1}$ as defined by (2.4) and $F = F_\infty$. This gives

$$F(z) = e^{zN}e^{\Gamma(s)}F_\infty, \quad \Gamma(0) = 0$$

relative to a holomorphic function $\Gamma(s)$ taking values in the subalgebra

$$q \cap \text{Lie}_{-1} = \bigoplus_{r < 0, r + s \leq -1} \text{gl}(V)^{r,s}$$

(3.3)

Consequently,

$$e^{-zN}Y_{(F(z),W)} = e^{-zN}Y_{(e^{zN}e^{\Gamma(s)},F_\infty,W)}$$

(3.2)

To finish the proof, let $e^X(z) = e^{zN}e^{\Gamma(s)}$. Then

$$e^X(z) = e^{zN+\Gamma(s)+(\text{brackets of } zN \text{ and } \Gamma(s))}$$

(3.4)

while, by Theorem (2.9),

$$e^X(z) = e^{\gamma(z)}e^{\lambda(z)}e^{\phi(z)}$$

where $\gamma(z)$, $\lambda(z)$ and $\phi(z)$ extended Lie polynomials in $X(z)$ with respect to $(F_\infty,W)$. Accordingly,

$$e^{-zN}Y_{(e^{zN}e^{\Gamma(s)},F_\infty,W)} = e^{-zN}e^{\gamma(z)}e^{\lambda(z)}Y_{(F_\infty,W)}$$

(3.5)

$$= e^{-zN}e^{zN}e^{\Gamma(s)}e^{-\phi(z)}Y_{(F_\infty,W)}$$

$$= e^{\Gamma(s)}e^{-\phi(z)}Y_{(F_\infty,W)}$$
Since $\phi(z)$ is an extended Lie polynomial in $X(z)$ which takes values in $\text{Lie}(C^\infty_{\mathbb{C}})_{-1}$, the Hodge components $X(z)^{r,s}$ of $X(z)$ with both $r, s < 0$ can only appear in $\phi(z)$ inside of Lie brackets which contain some Hodge component $X(z)^{p,q}$ with either $p$ or $q$ greater than zero. Indeed, since

$\Lambda^{-1,-1} = \bigoplus_{r,s<0} \text{gl}(V)^{r,s}$

is a subalgebra of $\text{Lie}_{-1}$ which is closed under complex conjugation, and the extended Lie monomials are exactly the non-zero functions $\mu: \text{Lie}_{-1} \rightarrow \text{Lie}_{-1}$ which can be constructed in a finite number of steps from the primitive monomial $m(\alpha) = \alpha$ via the operations of complex conjugation, taking Hodge components and forming Lie brackets, any Lie monomial $\mu(\alpha)$ which depends only on the Hodge components $\alpha^{r,s}$, with both $r, s < 0$ must preserve $\Lambda^{-1,-1}$.

Returning to (3.4) and noting that $N \in \text{gl}(V)^{-1,-1}$, it follows from the above considerations that

$$||\phi(z)|| \leq K|z|^b e^{-2\pi i z}$$

for some constants $K$ and $b$ (and $|| \ast ||$ a norm on $\text{Lie}_{-1}$). Consequently,

$$\lim_{\text{Im}(z) \rightarrow \infty} \phi(z) = 0.$$  (3.6)

On account of (3.5), (3.6) and the fact that $\Gamma(0) = 0$, the limit (3.2) exists and equals $Y(F_{\infty}, W)$.

This finishes the proof of (3.1).

**Corollary 3.7.** Let $\mathcal{V} \rightarrow \Delta^*$ be a unipotent variation of graded-polarized mixed Hodge structure which is admissible. Then, $\mathcal{Y}$ extends continuously, as a grading of $\mathcal{W}$, to the canonical prolongation of $\mathcal{V}$, with limiting value $Y_{\infty} = Y(F_{\infty}, W)$.

We now establish the converse of Theorem (3.1):

**Theorem (3.8).** Let $\mathcal{V} \rightarrow \Delta^*$ be a unipotent variation of graded-polarized mixed Hodge structure, and suppose that the grading $\mathcal{Y}$ of the underlying weight filtration $\mathcal{W}$ defined by the $P^{r,s}$'s of $\mathcal{V}$ extends continuously, as a grading of $\mathcal{W}$, to the canonical extension $\mathcal{V}$. Then, $\mathcal{V}$ is admissible.

**Proof.** As in the proof of Theorem (3.1), we select a lifting $F : U \rightarrow \mathcal{M}$ of the period map of $\mathcal{V}$. Lemma (2.1), implies that $\mathcal{Y}$ extends continuously to a grading of $\mathcal{W}$ in the canonical extension of $\mathcal{V}$, that

$$Y_{\infty} = \lim_{s \rightarrow 0} Y(s), \quad Y(s) = e^{-zN}Y(F(z), W)$$  (3.9)

with $s = e^{2\pi i z}$, exists and grades $W$.

To see that (3.9) implies the existence of the limiting Hodge filtration

$$F_{\infty} = \lim_{s \rightarrow 0} \psi(s), \quad \psi(s) = e^{-zN}F(z)$$

observe that $Y(F(z), W)$ preserves the filtration $F(z)$, so $Y(s)$ preserves $\psi(s)$. Accordingly, $\psi(s)$ can be obtained by pulling back the induced filtration $\psi(s)Gr^W$ via the isomorphism $V \cong Gr^W$ determined by $Y(s)$, namely

$$Gr^W_k \cong E_k(Y(s))$$
As the induced filtrations $\psi(s)G^W$ are constant, the existence of the limiting grading (3.9) imply the existence of the limiting Hodge filtration $F_\infty$.

By (b), the existence of the relative weight filtration is equivalent (in the present case) to the statement

$$N : W_k \to W_{k-2}.$$ 

As in the proof of Theorem (3.1), the unipotency of $\mathcal{V}$ implies that $F_\infty \in \mathcal{M}$. We can then apply Theorem (2.3) with $F = F_\infty$ and $q = \eta_- \oplus \Lambda^{-1,-1}$. This gives

$$F(z) = e^{zN}e^{\Gamma(s)}F_\infty, \quad \Gamma(s) = 0$$

relative to a holomorphic function $\Gamma(s)$ taking values in the subalgebra (3.3). Letting

$$e^{X(z)} = e^{zN}e^{\Gamma(s)},$$

we may write (cf. (3.5))

$$Y(s) = e^{\Gamma(s)}e^{-\phi(z)}.Y(F_\infty,W)$$

where $\phi(z)$ is the extended Lie polynomial in $X(z)$ obtained by decomposing $e^{X(z)}$ in accord with (2.9). Moreover, by (2.10),

$$\phi(z) = \pi_+(X(z)) - \pi_+(\overline{\pi_-(X(z))}) \mod Lie_{-2} \quad (3.10)$$

Claim:

$$N : F^p_\infty \to F^{p-1}_\infty \quad (3.11)$$

Indeed, by the horizontality of $F(z)$,

$$\frac{d}{dz}F^p(z) \subseteq F^{p-1}(z)$$

Inserting the formula $F(z) = e^{zN}e^{\Gamma(s)}$ in the last equality and simplifying, one then obtains

$$e^{-\Gamma(s)}N + 2\pi is e^{-\Gamma(s)} \frac{d}{ds}e^{-\Gamma(s)} : F^p_\infty \to F^{p-1}_\infty$$

Setting $s = 0$, (3.11) follows.

Because of (3.11), the mixed-Hodge decomposition of $N$ relative to $(F_\infty, W)$ satisfies

$$N = N^{0,-1} + N^{-1,0} \mod Lie_{-2} \quad (3.12)$$

with

$$N^{-1,0} = N^{0,-1} \mod Lie_{-2} \quad (3.13)$$

In addition, by (3.4),

$$X(z) = zN + \Gamma(s) \mod Lie_{-2} \quad (3.14)$$

Taking note of (3.10) and (3.12)–(3.14), it then follows that

$$\phi(z) = 2\pi isN^{0,-1} + \pi_+(\Gamma(s)) - \pi_+(\overline{\pi_-(\Gamma(s))}) \mod Lie_{-2} \quad (3.15)$$
Accordingly, the limiting grading

\[ Y_\infty = \lim_{s \to 0} Y(s) = \lim_{\text{Im}(z) \to \infty} e^{-zN}Y_{(F(z), W)} \]

\[ = \lim_{\text{Im}(z) \to \infty} e^{\Gamma(s)}e^{-\phi(z)}Y_\infty \]  

exists only if \( N^{0,-1} = 0 \). Indeed, by [3], the group \( \exp(\text{Lie}_{-1}) \) acts simply transitively upon the set of all gradings \( Y \) of \( W \). Therefore, in order for the limit (3.16) to exist, \( ||\phi(z)|| \) must remain bounded as \( \text{Im}(z) \to \infty \), and hence \( N^{0,-1} \) must be equal to zero by (3.16) [recall \( \Gamma(0) = 0 \)]. Since \( N^{-1,0} = N^{0,-1} \) by (3.13), it then follows that \( N = 0 \mod \text{Lie}_{-2} \), i.e. \( N : W_k \to W_{k-2} \).

Combining Theorems (3.1) and (3.8), we then obtain:

**Theorem (I).** Let \( \mathcal{V} \to \Delta^* \) be a variation of graded-polarized mixed Hodge structure which is unipotent. Then, \( \mathcal{V} \) is admissible if and only if the \( C^\infty \)-grading \( Y \) of \( W \) defined by the \( I_p,q \)'s of \( \mathcal{V} \) extends continuously, as a grading of \( W \), to the canonical extension \( \tilde{\mathcal{V}} \).

### 4. Split Orbits

Let \( \mathcal{M} = \mathcal{M}(W, S, h^{p,q}) \) be a classifying space of graded-polarized mixed Hodge structure, with “compact dual” \( \hat{\mathcal{M}} \) and associated Lie groups \( G_{\mathbb{R}}, G \) and \( G_{\mathbb{C}} \), as described in section 2.

**Definition.** A nilpotent orbit of graded-polarized mixed Hodge structure (modeled on \( \mathcal{M} \)) consists of a filtration \( F \in \hat{\mathcal{M}} \) and an element \( N \in \text{Lie}(G_{\mathbb{R}}) \) such that

- \( N : F^p \to F^{p-1} \) for each index \( p \).
- There exists a constant \( \alpha \) such that \( \text{Im}(z) > \alpha \implies e^{zN}F \in \mathcal{M} \).

In this section we prove the following version of Theorem II:

**Theorem 4.1.** A nilpotent orbit of graded-polarized mixed Hodge structure \( e^{zN}.F \) is admissible and split if and only if

(a) The limit

\[ Y_\infty = \lim_{\text{Im}(z) \to \infty} \text{Ad}(e^{-zN})Y_{(e^{zN}.F, W)} \]

exists, grades \( W \) and is defined over \( \mathbb{R} \).

(b) Each of the induced orbits \( e^{zN}.F\text{Gr}_k^W \) is split

As noted in the introduction, the fact that an admissible split nilpotent orbit satisfies the conditions of the Theorem is an immediate consequence of results of [6] discussed in the attached appendix.

To prove the converse, we let

\[ \rho : \text{sl}_2(\mathbb{R}) \to \text{End}_{\mathbb{R}}(V) \]

denote the representation of \( \text{sl}_2(\mathbb{R}) \) constructed in §1 by simply pulling back the representations \( \rho_k : \text{sl}_2(\mathbb{R}) \to \text{End}_{\mathbb{R}}(\text{Gr}_k^W) \) to \( V \) via the isomorphism \( \text{Gr}_k^W \cong V \) induced by \( Y_\infty \), and define

\[ ^{(r)}Y = Y_\infty + Y_o, \quad Y_o = \rho(h) \]
Then, as discussed in §1, in order for Theorem (4.1) to hold, \((r)Y\) must grade the relative weight filtration of \(N\) and \(W\), i.e. the associated filtration
\[
(r)W_k = \bigoplus_{\ell \leq k} E_{\ell}(r)Y
\]
must satisfy the following two conditions (cf. (2.2)):

(i) \(N: (r)W_k \to (r)W_{k-2}\).
(ii) \((r)W\) induces on \(Gr^W_k\) the shifted monodromy weight filtration
\[
W^\#(k) = W(N: Gr^W_k \to Gr^W_k)[-k]
\]
of \(N\) on \(Gr^W\).

To facilitate the proof we record the following observations:

(1) The limiting grading \(Y_\infty\) from (a) in Theorem (4.1), preserves \(F\).
(2) \(Y_o = \rho(h)\) preserves \(F\).
(3) Suppose that \((r)Y\) does indeed grade the relative weight filtration of \(N\) and \(W\). Then \((F, rW)\) is split over \(\mathbb{R}\). In particular, the proof of Theorem (4.1) will be complete once we verify (i) and (ii) above.
(4) The representation \(\rho\) defined above acts on \(Gr^W\) by infinitesimal isometries (i.e. \(\rho\) takes values in \(\text{Lie}(G_R)\)).

To prove (1), one simply notes that \(e^{-zN}.Y_{(e^zN,F,W)}\) preserves \(F\) whenever \((e^zN,F,W)\) is a mixed Hodge structure. To prove (2), we simply recall from [15] that:

\[
\rho_k(h) = Y(Y_{(FG^W_k,F,W^\#(k))}) - k
\]

To prove (3), observe that items (1) and (2) together with the hypothesis of Theorem, imply that \((r)Y = Y_\infty + Y_o\) is a semisimple endomorphism of \(V\) which preserves \(F\). Accordingly, if \((r)Y\) also grades \(W\) then \((F, rW)\) must be a split mixed Hodge structure. The proof of (4) can be found in [12] and boils down to the assertion that the monodromy weight filtration \(W(N: Gr^W_k \to Gr^W_k)\) is self-dual with respect to the polarization \(\gamma_k\).

To verify (ii), one simply observes that on account of the formula for \(\rho_k\) given above, \(\rho(k)Y = Y_\infty + \rho(h)\) acts on \(Gr^W_k\) as \(Y_{(F,W^\#(k))}\).

To establish (i), we shall actually verify a (seemingly) stronger condition, namely
\[
[(r)Y, N] = -2N
\]

To this end, we let
\[
N = N_0 + N_{-1} + \cdots
\]
denote the decomposition of \(N\) with respect to the eigenvalues of \(ad Y_\infty\) and recall that in the present context:
\[
Y_o = \rho(h), \quad N_0 = \rho(n_+)
\]
Accordingly, (4.2) holds if and only if
\[
[Y_o, N_{-k}] = (k - 2)N_{-k}
\]
for each index \(k > 0\). Consequently, it will suffice to prove the following assertion:
Lemma 4.2. Relative to the representation \( \rho \) of \( sl_2(\mathbb{R}) \) defined above, any non-zero component \( N_{-k} \) with \( k > 0 \) is a highest weight vector of weight exactly \((k - 2)\).

For the remainder of this section, we shall write \( Y_F \) in place of \( Y_{(F,W)} \) whenever \( F \in \mathcal{M} \).

By virtue of (a), we know that:

\[
Y_{\infty} = \lim_{y \to \infty} e^{-iyN} Y_{e^{iyN}F} \tag{4.3}
\]

Furthermore, we also know that \( N \) is horizontal with respect to \( F \), and hence

\[
N_{-k} : F^p \to F^{p-1} \tag{4.4}
\]

since \( Y_{\infty} \) preserves \( F \). Accordingly, we can prove Lemma (4.2) inductively by explicitly computing the right hand side of (4.3) and then imposing the horizontality condition (4.4).

To fill in the details, observe that Schmid’s \( SL_2 \) Orbit Theorem applied to \( Gr^W \) (a sum of variations of pure structures) together with (b), implies that the pair

\[
(e^{iyN_0}F, W)
\]

is a mixed Hodge structure for all \( y > 0 \). In particular, the base point

\[
F_o = e^{iN_0}F
\]

is an element of \( \mathcal{M} \). Moreover, since \( Y_{\infty} \) is both real and preserves the filtration \( e^{iyN_0}F \),

\[
Y_{\infty} = Y_{(e^{iyN_0}F, W)}
\]

for all \( y > 0 \).

Next, we introduce the \( \exp(Lie_{-1}) \)-valued function

\[
e^{Q(y)} = e^{iyN}e^{-iyN_0} = e^{iN_0+iyN_0}e^{-iyN_0}, \quad N' = N - N_0 \tag{4.5}
\]

and note that

\[
e^{iyN} = e^{Q(y)}e^{iyN_0} = y^{-\frac{1}{2}Y_o}e^{P(y)}e^{iN_0}y^{\frac{1}{2}Y_o} \tag{4.7}
\]

upon setting \( P(y) = Ad(y^{\frac{1}{2}Y_o})Q(y) \). In particular, since \( y^{\frac{1}{2}Y_o} \) preserves \( F \):

\[
e^{iyN_o}F = y^{-\frac{1}{2}Y_o}e^{P(y)}e^{iN_0}y^{\frac{1}{2}Y_o}F = y^{-\frac{1}{2}Y_o}e^{P(y)}F_o
\]

Thus, upon applying Theorem (2.9) to \( e^{P(y)} \) with \( F = F_o \), we obtain a distinguished decomposition

\[
e^{P(y)} = e^{\gamma(y)}e^{\lambda(y)}f(y), \quad f(y) = e^{\phi(y)}
\]

such that

\[
I_{e^{P(y)}}^{p,q} f^{-1}(y) I_{e^{P(y)}}^{p,q} = e^{P(y)} f^{-1}(y) I_{e^{P(y)}}^{p,q}
\]
Consequently,
\[
Y_{e^{iyN}, F} = y^{-\frac{1}{2}Y_o}Y_{e^{P(y)}, F_o} = y^{-\frac{1}{2}Y_o}e^{P(y)}f^{-1}(y).Y_{(F_o, W)}
\]

\[
= y^{-\frac{1}{2}Y_o}e^{P(y)}f^{-1}(y).Y_{\infty}
\]

wherefrom [cf. (4.7)]
\[
e^{-iyN}Y_{e^{iyN}, F} = e^{-iyN}y^{-\frac{1}{2}Y_o}e^{P(y)}f^{-1}(y).Y_{\infty}
\]

\[
= (y^{-\frac{1}{2}Y_o}e^{P(y)}e^{iN_0y\frac{1}{2}Y_o})^{-1}y^{-\frac{1}{2}Y_o}e^{P(y)}f^{-1}(y).Y_{\infty}
\]

\[
= y^{-\frac{1}{2}Y_o}e^{-iN_0f^{-1}(y)}Y_{\infty}
\]

\[
= y^{-\frac{1}{2}Y_o}e^{-iN_0f^{-1}(y)}e^{iN_0}.Y_{\infty}
\]

with the very last step being justified by the fact that \(e^{iN_0}.Y_{\infty} = Y_{\infty}\).

Since \([Y_o, Y_{\infty}] = 0\),
\[
e^{-iyN}Y_{e^{iyN}, F} = [Ad(y^{-\frac{1}{2}Y_o})Ad(e^{-iN_0})f^{-1}(y)].Y_{\infty}
\]

In particular, since \(f^{-1}(y)\) takes values in \(\exp(\text{Lie}_{-1})\), (4.3) holds if and only if
\[
\lim_{y \to \infty} Ad(y^{-\frac{1}{2}Y_o})Ad(e^{-iN_0})f^{-1}(y) = 1
\]

Equivalently, if \(f(y) = e^{B(y)}\), then
\[
\lim_{y \to \infty} Ad(y^{-\frac{1}{2}Y_o})Ad(e^{-iN_0})B(y) = 0
\]

Write
\[
B(y) = \sum B_my^m
\]

Then,
\[
Ad(y^{-\frac{1}{2}Y_o})Ad(e^{-iN_0})B(y) = \sum_{m} \sum_{j} [Ad(e^{-iN_0})B_m]jY_o y^{m-j}
\]

and hence (4.7) holds if and only if
\[
[e^{-i\text{ad}N_0}B_m]jY_o = 0, \quad \forall \ j \leq m
\]

Let \([A]^S_{\lambda}\) denote the component of \(A\) in the \(\lambda\)-eigenspace of a semisimple element \(S \in \text{End}(V)\).

To compute \(B(y)\), note that [6] (cf. Appendix):
\[
e^{Q(y)} = e^{(iyN_0+iyN')}e^{-iyN_0}
\]

\[
= \exp(\Psi(iyN', \ldots, \frac{1}{m!}\text{ad}(iyN_0)^m(iyN'), \ldots))
\]

for some Lie polynomial \(\Psi\) such that
\[
Q(y) = \frac{e^{\text{ad}(iyN_0)}}{\text{ad}(iyN_0)}(iyN') + O((iyN')^2)
\]
where $O((iyN')^2)$ is of “lower weight”, i.e. if $N' = N_{-k}$, this is a decomposition of
$Q(y)$ according to $E_{-k}(Y_{\infty}) + \text{Lie}_{-2k}$. Consequently,

$$P(y) = \frac{e^{Ad(iN_{0})}}{ad(iN_{0})} - (iN'(y)) + U[-2](y)$$

where

$$N'(y) = yAd(y \frac{Y}{y})N' = \sum_{k \geq 1} \sum_{j} [N_{-k}](y)^{1+\frac{j}{2}}$$

and $U[-2] = y^2Ad(y \frac{Y}{y})O((N')^2)$ is of lower weight.

Now, $\text{Lie}(G_C)$ decomposes as a direct sum of irreducible $sl_2$-modules $U$ under
the adjoint action of $\rho$. Moreover, since $Y_{\infty}$ is both defined over $\mathbb{R}$ and commutes
with $\rho$, $U$ is likewise defined over $\mathbb{R}$ and contained in some eigenspace of $ad Y_{\infty}$. In
order to better understand that decomposition, let us assume $U \subseteq \text{Lie}(G_C)$ to be
an irreducible submodule contained in the $-k$ eigenspace of $ad Y_{\infty}$ for some positive
value of $k$. To see that $U$ inherits a pure Hodge structure of weight $-k$ from

$$\text{Lie}(G_C) = \bigoplus_{r,s} \text{gl}(V)_{(r,s)^{\mathbb{Z}}_W}$$

we observe that in the present context, the adjoint representation $\rho : sl_2(\mathbb{C}) \to
\text{Lie}(G_C)$ becomes a morphism of mixed Hodge structures, upon endowing $sl_2(\mathbb{C})$
with the following pure Hodge structure of weight zero:

$$x^+ = \frac{1}{2}(h - in_+ - in_+) \in H^{1,-1}, \quad x^+ = \frac{1}{2}(h + in_+ + in_+) \in H^{1,-1},$$

$$z = i(n_+ - n_+) \in H^{0,0},$$

with $N_o = \rho(n_-)$ and $Y_o = \rho(h)$. Since the action of $Z = \rho(z)$ stabilizes each
summand of (4.17), the Hodge decomposition

$$\cdots + \text{gl}(V)^{0,-k} + \text{gl}(V)^{-1,-k+1} + \cdots + \text{gl}(V)^{-k,1} + \cdots$$

of $U$ must be exactly parallel to the corresponding decomposition

$$\cdots + E^Z_{-k} + (E^Z_{-2} + \cdots + E^Z_{-k+2}) + E_{-k} + \cdots$$

of $U$ with respect to the eigenvalues of $\rho(Z)$.

Remark. To avoid confusion regarding the Hodge types of the summands appearing
in the preceding equation, we note that (4.10) is equivalent to the assertion that
$E^Z_{k+2p}$ has Hodge type $(p, -k - p)$. We also note that for $k > 1$, the parenthesized
terms appearing in (4.10) correspond to $U \cap \Lambda_{(p,0)^{\mathbb{Z}}}$.

We may now proceed with the proof of Lemma (4.2). Inductively, we may
assume that it holds for eigenvalues of $ad Y_{\infty}$ bigger than $-k$. In particular, since
the $su_2$-basis $(X^-, Z, X^+)$ given by

$$X^- = \rho(x^-), \quad Z = \rho(z), \quad X^+ = \rho(x^+)$$

defines the same representation of $\mathfrak{sl}_2(\mathbb{C})$ as $(N_0, Y_0, N_0^+)$, the monomials

$$(ad N_0)^j N_{-\ell}, \quad -\ell > -k, \quad 0 \leq j \leq \ell - 2 \tag{4.14}$$

belong to $\Lambda_{(F_0, W)}^{-1,-1}$ on account of (4.10). Consequently, if we decompose

$$P(y) = P_{-1}(y) + P_{-2}(y) + \cdots$$

according to the eigenvalues of $ad Y_\infty$, (4.14) and (4.11) imply that

$$P(y) = P_\mathbb{R}(y) + P_\Lambda(y) + B(y) \mod Lie_{-k-1} \tag{4.15}$$

with $P_\mathbb{R}(y) \in Lie(G_\mathbb{R})_{-1}$, $P_\Lambda(y) \in \Lambda_{(F_0, W)}^{-1,-1}$ and

$$P_{-k}(y) = \frac{e^{ad(iN_0)} - 1}{ad(iN_0)} \sum_m y^{1+\frac{1}{2}m}[N_{-k}]_{m}Y_\infty \mod \Lambda_{(F_0, W)}^{-1,-1} \tag{4.16}$$

Indeed: (4.11) gives a formula for $P(y)$ in terms of the monomials $(ad N_0)^r N_{-s}$ and their projections. By inductive hypothesis and the fact that $\Lambda_{(F_0, W)}^{-1,-1}$ is an associative subalgebra of $gl(V)$, the monomials listed in (4.14) can only contribute factors which belong to $\Lambda_{(F_0, W)}^{-1,-1}$.

Together with (4.12), this shows

$$B(y) = \text{proj}_{F_0} \left( \frac{e^{ad(iN_0)} - 1}{ad(iN_0)} \sum_m iy^{1+\frac{1}{2}m}[N_{-k}]_{m}Y_\infty \right) \mod Lie_{-k-1} \tag{4.17}$$

where, in general, we shall let

$$\text{proj}_{F_0} : Lie(G_\mathbb{C}) \to Lie(G_{\mathbb{C}}^{F_0})$$

denote the projection operator defined by (4.11) via the rule:

$$\text{proj}_{F_0}(\sum_{r,s} a^{r,s}) = \sum_{r \geq 0, s} a^{r,s}$$

Accordingly, by (4.17), the coefficients $B_m$ of the series expansion $B(y) = \sum_m B_m y^{\frac{1}{2}m}$ are given by

$$B_m = \text{proj}_{F_0} \left( \frac{e^{ad(iN_0)} - 1}{ad(N_0)} [N_{-k}]_{m-2}Y_\infty \right) \mod Lie_{-1-k}$$

Returning now to the setting of (4.10), let us restrict our attention to an irreducible $\mathfrak{sl}_2$-module

$$U \subseteq E_{-k}(ad Y_\infty)$$

of highest weight $d$. Then, by virtue of our preceding remarks, we obtain the identity

$$B_m = \text{proj}_{F_0} \left( \frac{e^{ad(iN_0)} - 1}{ad(N_0)} [N_{-k}]_{m-2}Y_\infty \right)$$
To continue, we note that for positive values of $k$, we may compute $\text{proj}_{F_o}(T)$ for any element $T \in E_{-k}(Y_\infty)$ via the formula

$$\text{Lie}(G_C) = \text{Lie}(G_R) + \Lambda^{-1,-1} + \text{Lie}(G^F_o)$$

by determining its Hodge components: replacing $T$ by $\frac{1}{2}(T - \overline{T})$ gets rid of the real part, and taking just the components of type ..., $(1, -k - 1), (0, -k)$ removes the contribution from $\Lambda^{-1,-1}$. Thus,

$$\text{proj}_{F_o}(T) = \frac{1}{2} \sum_{a \geq 0} (T - \overline{T})^{a,-k-a}$$

In particular,

$$B_m = \text{proj}_{F_o} \left( \frac{e^{ad(iN_0)} - 1}{ad(N_0)} [N_{-k}]^{Y_o}_{m-2} \right)$$

$$= \frac{1}{2} \sum_{a \geq 0} \left( \frac{e^{ad(iN_0)} - 1}{ad(N_0)} - \frac{e^{-ad(iN_0)} - 1}{ad(N_0)} \right) [N_{-k}]^{Y_o}_{m-2} a,-k-a$$

$$= i \sum_{a \geq 0} \left( \frac{\sin ad(N_0)}{ad(N_0)} [N_{-k}]^{Y_o}_{m-2} \right) a,-k-a$$

Therefore, by (4.10):

$$B_m = i \sum_{\ell \geq k} \left( \frac{\sin ad(N_0)}{ad(N_0)} [N_{-k}]^{Y_o}_{m-2} \right)^Z_{\ell}$$  (4.18)

Since $F_o = e^{iN_0}F$, the horizontality of $N_{-k}$ at $F$ together with (4.10) imply that

$$e^{iad N_0} N_{-k} \in \bigoplus_{a \geq -1} gl(V)^a,-k-a \bigoplus_{\ell \geq k-2} E^Z_{\ell}$$  (4.19)

and hence

$$d \geq k - 2.$$  (4.20)

(4.19) is equivalent to the assertion that

$$N_{-k} \in \bigoplus_{\ell \geq k-2} E^Y_{\ell},$$  (4.20)

since $e^{iad N_0}$ is actually an isomorphism from $E^Y_{\ell}$ to $E^Z_{\ell}$ (see proof of Lemma (4.23) below).

**Corollary.** By virtue of horizontality, $B_m = 0$ for $m < k$.

**Proof.** Indeed, by (4.20), $[N_{-k}]^{Y_o}_{j} = 0$ unless $j \geq k - 2$. Therefore, by (4.18), $B_m = 0$ unless $m \geq k$.

To prove that the remaining coefficients $B_m = 0$, we note that by (4.8), in order for the limiting grading $Y_\infty$ to exist, we must have

$$e^{-iad N_0} B_j \in \bigoplus_{\ell \geq j+2} E^Y_{\ell}$$

for any non-zero coefficient $B_j$, and hence

$$B_j \in \bigoplus_{\ell \geq j+2} E^Z_{\ell}$$  (4.21)

since $e^{iad N_0}$ is an isomorphism from $E^Y_{\ell}$ to $E^Z_{\ell}$.
Corollary. \( B_j = 0 \) if \( j \geq d \).

To dispense with the remaining coefficients

\[ B_k, \ldots, B_{d-2} \quad (4.22) \]

we will use that, in the present context,

\[ \left[ \sin N \frac{E_a}{N} \right]_b ^Z \neq 0 \]

iff both \( a \) and \( b \) are weights and either \( a \) is a highest weight, or \( a < |b| \). This will be proved in (4.23) below.

To see that all of the coefficients in (4.22) must then vanish, note that by (4.18), such a coefficient \( B_j \) could be non-zero if and only if:

\[ [N-k]_{j-2} ^Y \neq 0 \]

If \( d \) was bigger than \( d-2 \), (4.23) would imply

\[ [B_j]_Z ^Y \neq 0 \]

on account of (4.18). As this contradicts (4.21), all the coefficients listed in (4.22) must vanish. Accordingly, \( N-k \) must be of exactly highest weight \( k-2 \).

It only remains to prove

**Lemma 4.23.** Let \( (\rho, V) \) be a finite-dimensional representation of \( sl_2(C) \),

\[
Y = \rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \rho \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad N = \rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\( y_a, z_a \), the natural projections to the \( a \)-weight spaces of \( Y \) and \( Z \), respectively, and

\[ T = \frac{\sin N}{N} = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} N^{2n} \in \text{End}(V) \]

Then

\[ z_b T y_a \neq 0 \]

iff both \( a \) and \( b \) are weights, and either \( a \) is a highest weight or \( a < |b| \).

**Proof.** Can take \( \rho \) irreducible of highest weight \( d \) and realized on the space of homogeneous polynomials of degree \( d \) in two real variables \( u, v \). The elements \( u^p v^{d-p} \) form a basis adapted to \( sl_2 \), the elements \( w^p \bar{w}^{d-p} \) (\( w = u + iv \)) form a basis adapted to \( su_2 \) and \( N \) acts by \( v \frac{\partial}{\partial w} \). Since

\[
eq \frac{1}{m!} \left( \frac{\partial}{\partial u} \right)^m u^p (iv)^m = \sum_{m=0}^{p} \frac{p(p-1) \cdots (p-m+1)}{m!} u^{p-m} (iv)^m \]

one has

\[ e^{iu} \frac{\partial}{\partial u} u^p v^{d-p} = u^p v^{d-p} = \frac{1}{(2k)^{d-p}} u^p (w - \bar{w})^{d-p}, \]

\[ e^{-iu} \frac{\partial}{\partial u} u^p v^{d-p} = \frac{1}{(2i)^{d-p}} u^p (w - \bar{w})^{d-p} \]

Therefore
\[ T u^p v^{d-p} = \frac{\sin N}{N} u^p v^{d-p} \]
\[ = (2i)^{-1} \left( e^{iv\frac{\partial}{\partial u}} - e^{-iv\frac{\partial}{\partial u}} \right) (e^{u\frac{\partial}{\partial v}})^{-1} u^p v^{d-p} \]
\[ = (2i)^{-1} \left( e^{iv\frac{\partial}{\partial u}} - e^{-iv\frac{\partial}{\partial u}} \right) \frac{1}{p+1} v^{p+1} u^{d-p-1} \]
\[ = \frac{1}{(2i)^{d-p+1}(p+1)} (w^{p+1} - \bar{w}^{p+1})(w - \bar{w})^{d-p-1} \]

For \( p = d \) one gets

\[ T u^d = \frac{1}{2i(d+1)} (w^{d+1} - \bar{w}^{d+1})(w - \bar{w})^{-1} = \frac{1}{2i(d+1)} \sum_{p=0}^{d} w^p \bar{w}^{d-p} \]

which has non-zero projections on all \( E^Z_j \)'s. On the other hand, if \( p < d \), then \( d-p-1 \geq 0 \) and

\[ (2i)^{d-p+1}(p+1) Tu^p v^{d-p} = w^{p+1} (w - \bar{w})^{d-p-1} - \bar{w}^{p+1} (w - \bar{w})^{d-p-1} = \sum_{j=0}^{d-p-1} \binom{d-p-1}{j} (w^{d-j} \bar{w}^j - w^j \bar{w}^{d-j}) \]

which has non-zero components exactly in \((E^Z_0 \oplus \cdots \oplus E^Z_{2p-d+2}) \oplus (E^Z_{2p-d+2} \oplus \cdots \oplus E^Z_{d})\). This finishes the proof of the Lemma.

5. Split Variations

Here we will prove the following version of Theorem III:

**Theorem 5.1.** Let \( V \rightarrow \Delta^* \) be a variation of graded-polarized mixed Hodge structure with unipotent monodromy, and \( \mathcal{Y} \) denote the \( C^\infty \) grading the weight filtration \( \mathcal{W} \) obtained by applying (1.2) to each fiber of \( V \). Then, the variation \( V \) is admissible and split if and only if

(a) \( \mathcal{Y} \) extends continuously to a grading of \( \mathcal{W} \) in the canonical prolongation of \( \mathcal{V} \), with limiting value \( \lim_{s \rightarrow 0} \mathcal{Y}(s) \) defined over \( \mathbb{R} \).

(b) Each of the induced variations \( FGr_k^\mathcal{Y} \) is split.

As noted in the introduction, in order to make the various reality conditions well defined, one must select a non-zero reference element \( \theta \in T_0(\Delta) \).

First, the limiting Hodge filtration \( F_\infty \) of such a variation always exists. If \( V \) is admissible, this is true by definition. On the other hand, if (a) and (b) hold, then \( F_\infty \) is obtained by extending the graded limiting Hodge filtrations to \( V \) via \( Y_\infty \), since \( \mathcal{Y} \) preserves \( F \). More explicitly, if \( F : U \rightarrow M \) represents the period map of \( V \) then

\[ F_\infty = \lim_{s \rightarrow 0} \psi(s) \]

where \( \psi(s) : \Delta^* \rightarrow \hat{M} \) is the map associated to the periodic function

\[ \psi(z) = e^{-zN} F(z). \]
Likewise, the hypothesis assert the existence of the grading 

\[ Y_{\infty} = \lim_{\text{Im}(z) \to \infty} Y(z) = \lim_{s \to 0} Y(s) \]  

(5.3)

where \( Y(s) \) is the grading of \( W \) defined by the periodic function

\[ Y(z) = e^{-zN}Y(F(z),W). \]

Since the graded variation is a sum of variations of pure Hodge structures, \( \lim_{s \to 0} \psi(s)Gr^W \) exists. Since \( Y(s) \) preserves the filtration \( \psi(s) \), it follows that whenever \( Y_{\infty} \) exists, so does \( F_{\infty} \). We may therefore write

\[ F(z) = e^{zN}e^{\Gamma(s)}F_{\infty}, \quad \Gamma(0) = 0 \]  

(5.4)

relative to a suitable holomorphic function \( \Gamma(s) \) (cf. (2.3)).

To complete the proof, we will show that under the hypothesis of the Theorem,

\[ \lim_{\text{Im}(z) \to \infty} e^{-zN}Y(F(z),W) = \lim_{\text{Im}(z) \to \infty} e^{-zN}Y(e^{zN}F_{\infty},W) \]  

(5.5)

Explanation: if (5.5) holds and \( V \) is admissible and split, \( Y_s \) extends as described in the introduction. Conversely, if both (5.5) and the hypothesis of the Theorem hold, we can invoke (4.1) to prove that \( V \) is admissible, with limiting mixed Hodge structure \((F_{\infty}, rW)\) which splits over \( \mathbb{R} \).

Let \( C \) to be the class of real-analytic functions \( q(z) \) which take values in \( GL(V)^W \) and satisfy an estimate of the form

\[ q(z) = 1 + o(e^{-ky}), \quad k > 0 \]

for \( \text{Im}(z) >> 0 \), and denote membership in \( C \) by the shorthand \( q \approx 1 \). The correspondence

\[ (F,W) \leftrightarrow (Y(F,W), FGr^W) \]

is (real) birational and smooth along \( \mathcal{M} \), since the \( I^{p,q} \) are obtained from \( F \) and \( W \) by taking finite intersections and sums. Consequently, an \( \mathcal{M} \)-valued function \( \phi(z) \) is of the form

\[ \phi(z) = q(z).F_o \]

relative to a fixed base point \( F_o \in \mathcal{M} \) and some function \( q(z) \) of class \( C \) iff

\[ (Y(\phi(z),W), \phi(z)Gr^W) \to (Y(F_o,W), F_oGr^W) \]

exponentially fast. By smoothness, this conclusion remains valid for variable \( F_o \), provided the limiting value of \( F_o \) belongs to \( \mathcal{M} \). Recall the notation \( Y_F = Y(F,W) \) for \( F \in \mathcal{M} \).
Theorem 5.6. Let \( F(z) \) be the period map of a split admissible variation of graded-polarized mixed Hodge structure with unipotent monodromy. Then, the limiting grading (5.2) may be computed in terms of the corresponding nilpotent orbit, i.e.

\[
\lim_{\text{Im}(z) \to \infty} e^{-zN}Y_F(z) = \lim_{\text{Im}(z) \to \infty} e^{-zN}Y_{e^{zN}F_\infty}
\]

Proof. By (5.3), we may write our period map \( F(z) \) in the form

\[
F(z) = e^{-yN}q(z).F_\infty
\]

relative to a function \( q(z) \approx 1 \).

Next, we note that splitting \('Y\) of \((F_\infty, 'W)\) preserves \(F_\infty\) and satisfies \([Y, N] = -2N\). Consequently,

\[
e^{iyN}.F_\infty = y^{-\frac{1}{2}}Y.e^{iN}F_\infty
\]

Therefore, upon writing \( q_1(z) = \text{Ad}(e^{iyN})q(z) \) and \( q_2(z) = y^{\frac{1}{2}}Y.q_1(z) \), we obtain

\[
F(z) = e^{-yN}e^{iyN}q(z).F_\infty = e^{-yN}q_1(z)e^{iyN}.F_\infty
\]

\[
= e^{-yN}e^{-\frac{1}{2}Y}e^{iN}F_\infty = e^{-yN}e^{-\frac{1}{2}Y}q_2(z)e^{iN}.F_\infty
\]

Since \( e^{iyN} \) and \( y^{-\frac{1}{2}}Y \) preserve \( W \) and are polynomials in positive and negative powers of \( y^{\frac{1}{2}} \), \( q_1 \) and \( q_2 \approx 1 \). Since \( e^{iN}.F_\infty \in \mathcal{M} \) by virtue of Schmid’s \( SL_2 \) Orbit Theorem:

\[
Y_F(z) = Y_{e^{-yN}e^{-\frac{1}{2}Y}q_2(z)e^{iN}.F_\infty} = e^{-yN}e^{-\frac{1}{2}Y}Y(q_2(z))e^{iN}.F_\infty
\]

(5.7)

The map

\[
\mathcal{M} \to Y(W), \quad F \in \mathcal{M} \to Y_{(F, W)}
\]

into the space \( Y(W) \) consisting of all gradings \( Y \) of \( W \) is real-algebraic. Since \( q_2 \approx 1, e^{iN}.F_\infty \in \mathcal{M} \) and \( G_C \) acts transitively on \( Y(W) \), we conclude:

\[
Y_{q_2(z)e^{iN}.F_\infty} = q_3(z).Y_{e^{iN}.F_\infty}
\]

for some \( q_3 \approx 1 \). Consequently, (5.7) implies that

\[
e^{-yN}.Y_F(z) = e^{-iyN}e^{-\frac{1}{2}Y}q_3(z).Y_{e^{iN}.F_\infty}
\]

\[
= q_4(z)e^{-iyN}e^{-\frac{1}{2}Y}Y_{e^{iN}.F_\infty} = q_4(z)e^{-iyN}.Y_{e^{iN}.F_\infty}
\]

for some \( q_4 \approx 1 \), and hence:

\[
\lim_{\text{Im}(z) \to \infty} e^{-yN}.Y_F(z) = \lim_{\text{Im}(z) \to \infty} e^{-yN}Y_{e^{yN}F_\infty}
\]
Corollary. Under the hypothesis of Theorem 5.6, the period map $F(z)$ satisfies (a) and (b). Moreover, in this setting the limiting grading $Y_\infty$ coincides with the grading of $W$ obtained by applying the first theorem of [6] (cf. Appendix) to the pair

$$(N, Y), \quad Y = Y(e_{F_\infty}, W)$$

To prove the converse, let $\rho$ be the representation of $\mathfrak{sl}_2(\mathbb{R})$ attached to a variation $V$ which satisfies (a) and (b) of Theorem 5.1, i.e. pull back the representations $\rho_k$ which define the associated nilpotent orbits $e^{zN}_* FGr_k W$ which are $SL_2$ Orbits, on account of (b), via the grading $Y_\infty$, which is defined over $\mathbb{R}$. Then, as in §4:

1. $N_0 = \rho(h)$, where $N = N_0 + N_{-1} + \ldots$

   denotes the decomposition of $N$ with respect to the eigenvalues of $ad Y_\infty$.

2. $Y_o = \rho(h)$ preserves $F_\infty$.

Likewise, conditions (a) and (b) imply that $Y_\infty$ preserves $F_\infty$, since the grading $Y(s)$ from (5.2) preserves the filtration $\psi(s)$.

Theorem 5.8. Let $F(z)$ be the period map of a variation of graded-polarized mixed Hodge structure with unipotent monodromy for which conditions (a) and (b) of Theorem 5.1 hold. Then,

$$Y_\infty = \lim_{Im(z) \to \infty} e^{-zN} Y e^{zN} F_\infty$$

Proof. By (5.3),

$$F(z) = e^{zN} g(z) F_\infty$$

for some function $g(z) \approx 1$. Since $C$ is closed under conjugation by $e^{zN}$,

$$F(z) = q(z)e^{zN} F_\infty$$

for some function $q \approx 1$. Next, we prove that for a suitable integer $m \in \mathbb{Z}$, the endomorphism

$$T = Y_o + mY_\infty, \quad Y_o = \rho(H)$$

satisfies the estimate

$$yAd(y^T)N = N_0 + o(y^{-1})$$

Indeed, decompose $N = N_0 + N_{-1} + \ldots$ according to the eigenvalues of $Y_\infty$. Then,

$$yAd(y^T)N = Ad(y^{2Y_o} yAd(y^mY_\infty) N = N_0 + \sum_{j \geq 1} y^{1-mj} Ad(y^{2Y_o} N_{-j})$$

Since $j \geq 1$ and only finitely many eigenvalues of $Y_o$ appear non-trivially in each $N_j$, we may force the leftmost sum to involve only negative powers of $y$, by making $m$ sufficiently large.

By hypothesis, $T$ is defined over $\mathbb{R}$. The remarks immediately preceding (5.8), $T$ also preserves $F_\infty$. Hence

$$e^{iyN} F_\infty = e^{iyN} y^{-T} F_\infty = y^{-T} e^{iyAd(y^T)} N = y^{-T} P(y) e^{iN_0} F_\infty$$
Theorem 1. Let \( N \) be a nilpotent endomorphism of \( V \) for which the corresponding relative weight filtration \( ^rW = ^rW(N, W) \) exists, and \( ^rY \) be a grading of \( ^rW \) which preserves \( W \) and satisfies the additional condition \( [^rY, N] = -2N. \) Then, there exists a unique grading \( Y \) of \( W \) such that

1. \( [^rY, Y] = 0. \)
2. \( N = N_0 + N_{-2} + \cdots \) when decomposed relative to the eigenvalues of \( \text{ad}Y. \)
3. \( (\text{ad}N_0)^{k-1} N_{-k} = 0 \) for all \( k > 0. \)

Sketch of Proof. The desired grading \( Y \) may be constructed as follows: Let \( Y_0 \) be a grading of \( W \) for which \( [^rY, Y] = 0, \) and note that the group

\[
G_\alpha = \{ g \in GL(V) \mid [g, ^rY] = 0, \ (g-1) : W_k \to W_{k-1} \ \forall k \}
\]

acts transitively on the set of all such gradings. Assume by induction that the initial grading \( Y_0 \) satisfies (1)–(3) modulo the ideal

\[
\text{Lie}_{-r} = \{ \alpha \in \text{End}(V) \mid \alpha : W_k \to W_{k-r} \ \forall k \}
\]

Then one may construct an element \( g \in G_\alpha \) such that, relative to the grading \( Y' = \text{Ad}(g)Y_0, \) (1)–(3) hold modulo the ideal \( \text{Lie}_{-(r+1)} \subseteq \text{Lie}_{-r}. \) Since \( \text{Lie}_{-r} = 0 \) for some finite index \( r, \) the desired grading \( Y \) will be obtained after finitely many steps. The details are discussed in [12].

Remark. The preceding result may be reformulated as the statement that given the existence of \( ^rW = ^rW(N, W) \) and a grading \( ^rY \) of the type described above, there exists a unique grading \( Y \) of \( W \) which commutes with \( ^rY \) and has the property that the associated \( sl_2 \)-triple \( (N_0, ^rY - Y, N_0^+) \) satisfies the commutativity condition \( [N - N_0, N_0^+] = 0. \)
Lemma. The construction of Theorem 1 is both functorial and compatible with the operations of direct sum, tensor product and dual.

To state Deligne’s second theorem, we begin with a triple \((F, W, N)\) of the type arising from the degeneration of an admissible variation of graded-polarizable mixed Hodge structure, i.e.

- \(V = V_\mathbb{R} \otimes \mathbb{C}\) relative to some underlying real form \(V_\mathbb{R}\) to which both \(W\) and \(N\) descend.
- The relative weight filtration \(^rW = ^rW(N, W)\) exists.
- The pair \((F, ^rW)\) is a mixed Hodge structure relative to which \(N\) is a \((-1, -1)\)-morphism and \(W\) is a filtration by sub-mixed Hodge structures.

In particular, given any such triple \((F, W, N)\), we may construct an associated grading \(Y = Y(F, W, N)\) of \(W\) via application of Theorem 1 to the grading \(^rY\) of \(^rW\) determined by the \(I_{p,q}\)’s of the mixed Hodge structure \((F, ^rW)\).

Remark. Inspection of the proof of Theorem 1 shows that whenever the mixed Hodge structure \((F, ^rW)\) associated to the triple \((F, W, N)\) is split, the corresponding grading \(Y = Y(F, W, N)\) lies in \(\text{End}(V_\mathbb{R})\).

Lemma. The grading \(Y = Y(F, W, N)\) constructed above preserves \(F\).

Sketch of Proof. (cf. [12] for details). One uses the preceding lemma to reduce the problem to the case where \((F, ^rW)\) is a split mixed Hodge structure which is either of the form

- \(V = \bigoplus_p I^{p,p}\) with \(N(I^{p,p}) \subseteq I^{p-1,p-1}\) or,
- \(V = I^{p,0} \oplus I^{0,p}\) and \(N = 0\).

Theorem 2. Let \((F, W, N)\) be a triple of the type described above for which the mixed Hodge structure \((F, ^rW)\) is split. Then, the pair \((e^{iyN}F, ^rW)\) defines a mixed Hodge structure for all \(y > 0\). Moreover, the grading \(Y_{(e^{iyN}F, W)}\) of \(W\) determined by the \(I_{p,q}\)’s of \((e^{iyN}F, W)\) is given by the formula

\[
Y_{(e^{iyN}F, W)} = e^{iyN}Y(F, W, N)
\]

Proof. As a consequence of [3], the pair \((e^{iyN}F, W)\) is a mixed Hodge structure for all \(y > 0\). To establish the stated formula, we define \(Y\) to be \(Y(F, W, N)\), and note that

\[
Y = Y_{(e^{iyN}F, W)}
\]

for all \(y > 0\) since, in the present context, \(Y\) is a grading of \(W\) which is both defined over \(\mathbb{R}\) and preserves \(e^{iyN}F\). Accordingly, in order to establish Theorem 2, it will suffice to show that

\[
e^{\phi(y)} := e^{iyN}e^{-iyN_0} \in \exp(\Lambda^{(1)}_{(e^{iyN_0}F, W)})
\]

Indeed, if the preceding equation is true, then

\[
Y_{(e^{iyN}F, W)} = Y_{(e^{iyN}e^{-iyN_0}e^{iyN_0}F, W)} = e^{iyN}e^{-iyN_0}Y_{(e^{iyN_0}F, W)} = e^{iyN}e^{-iyN_0}Y = e^{iyN}Y
\]
To verify (•), note that φ(y) is a Lie polynomial in the monomials \((ad N_0)^a N_{-k}\). In particular φ(y) contains no \(N_0\) term. Moreover, by conditions (2) and (3) of Theorem 1, \((ad N_0)^a N_{-k}\) is zero unless \(k > 1\) and \(0 \leq a \leq k - 2\). It then follows from (4.10) that

\[
(ad N_0)^a N_{-k} \in \bigoplus_{0 < r < k} gl(V)^{-r,r-k}_{(e^{iyN_0},W)} \subseteq \Lambda^{-1,-1}_{(e^{iyN_0},F,W)}
\]

Consequently, φ(y) takes values in \(\Lambda^{-1,-1}_{(e^{iyN_0},F,W)}\) since the latter is a Lie subalgebra of \(gl(V)\).

It is clear that Theorem (IV) in the introduction follows as a corollary of these proofs.

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