Weighted Sobolev $L^p$ Estimates for Homotopy Operators on Strictly Pseudoconvex Domains with $C^2$ boundary

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Abstract
We derive estimates in a weighted Sobolev space $W^{k,p}_{\beta}(D)$ for a homotopy operator on a bounded strictly pseudoconvex domain $D$ of $C^2$ boundary in $\mathbb{C}^n$. As a result, we show that given any $2n < p < \infty$, $k > 1$, $q \geq 1$, and a $\bar{\partial}$-closed $(0,q)$ form $\varphi$ of class $W^{k,p}(D)$, there exists a solution $u$ to $\bar{\partial} u = \varphi$, such that $u \in W^{k,1}_{\frac{p}{2}}(D)$ for any $\epsilon > 0$. If $k = 1$, then we can take $p$ to be any value between 1 and $\infty$. In other words, the solution gains almost $\frac{1}{2}$-derivative in a suitable sense.

Keywords
Strictly pseudoconvex domains · Homotopy formula · Sobolev estimates

Mathematics Subject Classification 32A26 · 32T15 · 32W05

1 Introduction

In this paper we prove a regularity result concerning the solution of $\bar{\partial}$-equation on a strictly pseudoconvex domain $D$ with respect to a weighted Sobolev norm, assuming the boundary $bD$ is $C^2$. We define the weighted Sobolev space $W^{k,p}_{\beta}(D)$ for a bounded domain $D \subset \mathbb{R}^N$ to be the subspace of $W^{k,p}(D)$ with norm

$$
\|u\|_{W^{k,p}_{\beta}(D)} = \sum_{|\alpha| \leq k+1} \left( \int_D |\bar{\partial}^\alpha u(x)|^p d(x)^{(1-\beta)p} \, dx \right)^{\frac{1}{p}}.
$$

(1.1)

Here $k$ is a non-negative integer, $1 \leq p < \infty$, $0 < \mu < 1$, and $d(x) = \text{dist}(x, bD)$. These are Banach spaces with the norm defined as above. The reader can refer to [8]

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for some properties of $W^{k,p}_\beta(D)$. For $p = \infty$, the appropriate space to consider is the Hölder–Zygmund space.

We mention some brief history regarding the “$1/2$-estimate” for $\bar{\partial}$ solution $u$ of $\bar{\partial}u = \varphi$, for a $\bar{\partial}$-closed $(0,q)$ form $\varphi$ on bounded strictly pseudoconvex domains. Regarding Sobolev space estimates, Greiner and Stein [7] showed that for $q = 1$, Kohn’s canonical solution $\bar{\partial}^* N \varphi$ is in $L^p_{k+1/2}(D)$, if $\varphi$ in $L^p_k(D)$, for $1 < p < \infty$, and $k$ is any non-negative integer. Here $L^p_k(D)$ is the Bessel potential space, as defined in [18, p. 135]. Chang [3] extended this result for all $q \geq 1$. Greiner–Stein and Chang assume that $bD$ is smooth.

On the Hölder estimate side of $\bar{\partial}$ solutions, Henkin and Romanov [16] first achieved the $C^{1/2}$ estimate of $\bar{\partial}$ solutions for continuous $(0,1)$ form $\varphi$. Siu [17] proved the $C^{k+1/2}$ estimate for $q = 1$ and $k \geq 1$. Lieb–Range [10] constructed a $\bar{\partial}$ solution operator $H_q$, $q \geq 1$ and proved the $C^{k+1/2}$ estimate when the boundary is $C^{k+2}$. In both results of Siu and Lieb–Range, $\varphi$ is assumed to be $\bar{\partial}$ closed. When $bD$ is smooth, Greiner and Stein (for $q = 1$) [7] showed that Kohn’s canonical solution is in $\Lambda^{r+1/2}_r$ if $\varphi \in \Lambda_r$, for all $r > 0$. Here $\Lambda_r$ stands for the Zygmund space, as defined in [7, p. 141]. Chang [3] extended this result for any $q \geq 1$ on the Siegel upper-half space.

Recently Gong [6] derived a new homotopy formula [see (2.14) and (2.15) below],

$$
\varphi = \bar{\partial}H_q \varphi + H_{q+1} \bar{\partial} \varphi, \quad q \geq 1, \quad (1.2)
$$

$$
\varphi = H_0 \varphi + H_1 \bar{\partial} \varphi, \quad q = 0, \quad (1.3)
$$

for a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with the minimal smoothness condition of $C^2$ boundary. He proved that $1/2$ estimate holds for the homotopy operator $H_q$ in the Zygmund space. More specifically, he showed that for $q \geq 1$, $H_q \varphi$ is in $\Lambda^{r+1/2}_r$ if $\varphi \in \Lambda_r$, $r > 1$, and $H_q \varphi$ is in $C^{3/2}(\bar{D})$ if $\varphi \in C^1(\bar{D})$, and the estimates do not require $\varphi$ to be $\bar{\partial}$-closed.

There are two main features in the above homotopy formula in [6]. The first is the regularized Leray map, introduced in [6]. The second feature is the commutator $[\bar{\partial}, E]$, where $E$ is an extension operator bounded in $\Lambda_r$-norm. This commutator was introduced by Peters [14] and it has been used by Michel [11], Range [15], Michel-Shaw [13], Alexandre [2], and others.

We shall derive an analog of the $1/2$ estimate with respect to the weighted Sobolev space (1.1) for the homotopy operators $H_q$ and $H_0$. In fact, one can view Gong’s result as the $p = \infty$ analog of our estimates. In Sect. 2 we prove that homotopy formulas (1.2) and (1.3) hold in the distribution sense if $\varphi, \bar{\partial} \varphi \in W^{1,1}(D)$; see Proposition 2.8. Our goal is to prove the following:

**Theorem 1.1** Let $D \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with $C^2$ boundary. Let $k$ be a positive integer, and $q$ be a non-negative integer. 
(i) Let $1 < p < \infty$, and $q > 0$. Then for any $\beta, 0 < \beta < 1/2$,

$$
\|H_q \varphi\|_{W^{1,p}_\beta(D)} \leq C \|\varphi\|_{W^{1,p}(D)}.
$$
(ii) Let \(2n < p < \infty, k \geq 2\), and \(q > 0\). Then for any \(\beta, 0 < \beta < \frac{1}{2}\),

\[
\|H_q \varphi\|_{W^{k,p}_\beta(D)} \leq C \|\varphi\|_{W^{k,p}(D)}.
\]

(iii) Let \(1 < p < \infty\). Then for any \(\beta, 0 < \beta < 1\),

\[
\|H_0 \varphi\|_{W^{0,p}_\beta(D)} \leq C \|\varphi\|_{W^{1,p}(D)}.
\]

(iv) Let \(2n < p < \infty, k \geq 2\). Then for any \(\beta, 0 < \beta < 1\),

\[
\|H_0 \varphi\|_{W^{k-1,p}_\beta(D)} \leq C \|\varphi\|_{W^{k,p}(D)}.
\]

Here we denote by \(C\) some positive constants which depend on \(D, n, p,\) and \(\beta\).

We emphasize that \(\varphi\) in the above estimates are not necessarily \(\overline{\partial}\)-closed. As a consequence, we have the following corollary:

**Corollary 1.1.1** Let \(D \subset \mathbb{C}^n\) be a bounded strictly pseudoconvex domain with \(C^2\) boundary. Let \(q\) be a positive integer. There exist a solution operator \(H_q\) to the \(\overline{\partial}\)-equation \(\overline{\partial} u = \varphi\) in \(D\), for a given \(\overline{\partial}\)-closed \((0, q)\) form \(\varphi\), such that the estimates in (i) and (ii) of Theorem 1.1 hold. In other words the solution \(u\) gains \(\frac{1}{2} - \varepsilon\) derivative.

The paper is organized as follows. In Sect. 2 we collect a few facts about the Stein extension operator, Sobolev space, and the trace operator. We then derive the homotopy formula for Sobolev classes. We also recall from [6] the regularized Leray map and its properties. In Sect. 3 we prove the estimates for \(H_q, q \geq 1\) (part (i) and (ii) of Theorem 1.1). The main technical part involves a subtle use of integration by parts to move derivatives from the kernel to \(\varphi\). In Sect. 4 we prove the estimates for \(H_0\) (part (iii) and (iv) of Theorem 1.1).

## 2 Homotopy Formula for Sobolev Space

In this section we derive the homotopy formula introduced in [6] for the Sobolev classes. We shall need some standard facts about Sobolev spaces. For reader’s convenience we state them here. We use \(W^{k,p}(D)\) to denote the usual Sobolev space with norm

\[
\|u\|_{W^{k,p}(D)} = \left( \sum_{|\alpha| \leq k} \left( \int_D |\partial^\alpha u(x)|^p \, dx \right)^\frac{1}{p} \right)^{\frac{1}{p}}.
\]

We remind the reader that the \(\overline{\partial}\) solution space \(W^{k,p}_\beta(D)\) defined in Sect. 1 has actually \(k + 1\) interior derivatives. Thus \(W^{k,p}_\beta(D) \subset W^{k+1,p}(D')\), for any relatively compact subdomain \(D'\) of \(D\).
Proposition 2.1 Let $D \subset \mathbb{R}^N$ be a bounded domain with $C^1$ boundary. Assume $N < p < \infty$ and $u \in W^{k,p}(D)$. Then up to a set of measure 0, $u \in C^{k-1,\alpha}(\bar{D})$, for $\alpha = 1 - \frac{N}{p} > 0$, and $u$ satisfies the estimate

$$\|u\|_{C^{k-1,\alpha}(\bar{D})} \leq C \|u\|_{W^{k,p}(D)},$$

where $C$ depends on $k$, $p$, $N$ and $D$.

The proof can be found in [9, p. 335].

We need an extension operator due to E. Stein.

Proposition 2.2 Let $D$ be a bounded domain whose boundary satisfies the minimal smoothness condition as defined in [18, p. 189], (in particular, a bounded domain is minimally smooth if its boundary is locally given by graphs of Lipschitz functions.)

Then

(i) There is a continuous linear operator $E : W^{k,p}(D) \rightarrow W^{k,p}(\mathbb{R}^N)$ so that $Ef = f$ on $D$, for all $p$, $1 \leq p \leq \infty$, and all non-negative integer $k$.

(ii) There is a continuous linear operator $E : C^0(\bar{D}) \rightarrow C^0(\mathbb{R}^N)$ so that $Ef = f$ on $D$ and

$$|Ef|_{C^r(\mathbb{R}^N)} \leq C(r,D)|f|_{C^r(\bar{D})}, \quad \forall r \in [0, \infty).$$

The proof of (i) can be found in [18, p. 181], and the proof of (ii) can be found in [6].

In what follows we denote $\mathbb{R}_+^N = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}, x_N > 0\}$.

Definition 2.3 The boundary $\partial \omega$ of an open set $\omega \subset \mathbb{R}^N$ is uniformly Lipschitz if there exist $\epsilon, L > 0, M \in \mathbb{N}$, and a locally finite countable open cover $U_l$ of $\partial \omega$ satisfying

(i) If $x \in \partial \omega$, then $B(x, \epsilon) \subset U_l$ for some $l \in \mathbb{N}$.

(ii) No point of $\mathbb{R}^N$ is contained in more than $M$ of the $U_l$’s.

(iii) For each $k$ there exist local coordinates $y = (y_1, \ldots, y_{\alpha_l}, \ldots, y_N)$ and a Lipschitz function $f_l : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ with $\text{Lip } f_l \leq L$, such that

$$U_l \cap \omega = U_l \cap A_l, \quad A_l = \{y \in \mathbb{R}^N : y_{\alpha_l} > f_l(y'_{\alpha_l})\},$$

where $y'_{\alpha_l} = (y_1, \ldots, \hat{y}_{\alpha_l}, \ldots, y_N)$, and $\hat{\cdot}$ means $\cdot$ is omitted.

We now define the trace operator for $W^{1,1}(\omega)$. First we define it on $W^{1,1}(\mathbb{R}_+^N)$.

Proposition 2.4 Let $N \geq 2$ and let $W^{1,1}_0(\mathbb{R}_+^N)$ be the family of all functions $u \in W^{1,1}(\mathbb{R}_+^N)$ with bounded support. Then there exist a linear operator

$$Tr : W^{1,1}_0(\mathbb{R}_+^N) \longrightarrow L^1(\mathbb{R}^{N-1})$$

such that

(i) $Tr(u)(x') = u(x', 0)$ for all $x' \in \mathbb{R}^{N-1}$, and for all $u \in W^{1,1}_0(\mathbb{R}_+^N) \cap C(\mathbb{R}_+^N \cup \{x_N = 0\})$. 

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For all \( u \in W^{1,1}_0(\mathbb{R}^N) \),
\[
\int_{\mathbb{R}^{N-1}} |Tr(u)(x')| \, dx' \leq \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_N} (x) \right| \, dx. \tag{2.2}
\]

For all \( v \in C^1_c(\mathbb{R}^N) \), \( u \in W^{1,1}_0(\mathbb{R}^N) \), and \( i = 1, \ldots, N \),
\[
\int_{\mathbb{R}^N} \frac{\partial(uv)}{\partial x_i} \, dx = \int_{\mathbb{R}^{N-1}} uTr(v)v_i \, dx' \tag{2.3}
\]
where \( v = -e_N = (0, \ldots, 0, -1) \) is the outer unit normal on \( \mathbb{R}^{N-1} = \{x_N = 0\} \), \( dx = dx_1 \ldots dx_{N-1} \) and \( dx' = dx_1 \ldots dx_{N-1} \).

For proof see [9, p. 452].

**Proposition 2.5** Let \( \omega \subset \mathbb{R}^N \), \( N \geq 2 \), be an open set whose boundary \( b\omega \) is uniformly Lipschitz, with the corresponding \( \varepsilon, L, M \) given as in Definition 2.3. There exist a continuous linear operator
\[
Tr : W^{1,1}(\omega) \longrightarrow L^1(b\omega, ds_{b\omega})
\]
satisfying the following

- (i) \( Tr(u) = u \) on \( b\omega \) for all \( u \in W^{1,1}(\omega) \cap C(\overline{\omega}) \).
- (ii) Denote by \( ds_{b\omega} \) the surface element of \( b\omega \). We have
\[
\int_{b\omega} |Tr(u)| \, ds_{b\omega} \leq \frac{CM}{\varepsilon} \sqrt{1 + L^2} \int_\omega |u| \, dx + \sqrt{1 + L^2} \int_\omega |\nabla u| \, dx. \tag{2.4}
\]

The reader can refer to [9, pp. 460–462] for the proof of Proposition 2.5. For later use we recall the construction of the above trace operator. Let \( \{U_l\} \) be an open cover of \( b\omega \) as given in Definition 2.3, and let \( \chi_l \) be smooth partition of unity such that \( \text{supp} \chi_l \subset \subset U_l \). Then \( u = \sum_l \chi_l u : = \sum_l u_l \) in a neighborhood of \( b\omega \), and \( u_l \) has compact support in \( U_l \). Since \( \omega \cap U_l = A_l \cap U_l \) by (2.1), we can extend \( u_l \) to be 0 in \( A_l \setminus U_l \) to obtain \( u_l \in W^{1,1}(A_l) \). Define
\[
Tr(u) = \sum_l Tr(u_l), \quad Tr(u_l) = Tr(u_l \circ \psi_l) \circ \psi_l^{-1}, \quad u_l := \chi_l u. \tag{2.5}
\]

where \( \psi_l : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is given by \( \psi_l(y) = (y_1, y_{a_l-1}, y_{a_l} + f_l(y_{a_l}', y_{a_l+1}, \ldots, y_{N})) \). Furthermore we can choose the partition of unity \( \chi_l \) so that \( Tr(u_l) \) is compactly supported in \( b\omega \cap U_l \).

Let \( \phi = \sum_l \phi^l dx_l \) be a differential form of degree \( q \), for \( q \geq 1 \). We say that \( \phi \in W^{1,1}(\omega) \) if each component function \( \phi^l \) belongs to the class \( W^{1,1}(\omega) \). We define the trace of \( \phi \) on \( b\omega \) to be
\[
Tr(\phi) = \sum_{|I|=q} Tr(\phi^I) dx_I. \tag{2.6}
\]
Proposition 2.6 Let \( \omega \subset \mathbb{R}^N \) be a bounded domain with uniformly Lipschitz boundary. Suppose that \( \phi \) is a differential form and \( \phi \in W^{1,1}(\omega) \). We have

\[
\int_{\partial \omega} \text{Tr}(\phi) \wedge \alpha = \int_{\omega} d(\phi \wedge \alpha) \tag{2.7}
\]

for any \( \alpha \) which is a \( C^1(\mathbb{R}^N) \) form.

Formula (2.7) can be proved by pulling back the forms to the upper half plane \( \mathbb{R}^N_+ \) by Lipschitz maps, smoothing out the Lipschitz maps and using (2.3). We leave the details to the reader.

Lemma 2.7 Let \( \omega \subset \mathbb{R}^N \) be a bounded domain with \( C^1 \) boundary, and \( \omega' \subset \subset \omega \). Suppose \( k(z, \xi) \) is uniformly bounded for \( z \in \omega' \) and \( \xi \) in some neighborhood of \( \partial \omega \), and is uniformly continuous in \( \xi \). Suppose \( u \in W^{1,1}(\omega) \). Let \( \omega_j \) be a sequence of smooth domains approximating \( \omega \) from inside, i.e. \( \omega_j \subset \subset \omega \), \( \omega_j + 1 \subset \subset \omega \), and such that locally the defining functions of \( \partial \omega_j \) converge uniformly to that of \( \partial \omega \) in \( C^1 \)-norm. Then

\[
\int_{\partial \omega_j \cap U_l} k(z, \xi) \text{Tr}(u)(\xi) \, d\sigma_{\omega_j}(\xi) \xrightarrow{j \to \infty} \int_{\partial \omega \cap U_l} k(z, \xi) \text{Tr}(u)(\xi) \, d\sigma_{\omega}(\xi)
\]

uniformly on \( z \in \omega' \). Here \( d\sigma_{\omega}(\xi) \) and \( d\sigma_{\omega_j}(\xi) \) denote the surface elements of \( \partial \omega \) and \( \partial \omega_j \), respectively.

Proof Let \( \{U_l\} \) be a (finite) open cover of \( \partial \omega \) and \( \partial \omega_j \) as given in Definition 2.3, for \( j \) sufficiently large. By the way we define trace (2.5), it suffices to prove that for each \( l \),

\[
\int_{\partial \omega_j \cap U_l} k(z, \xi) \text{Tr}(u)(\xi) \, d\sigma_{\omega_j}(\xi) \xrightarrow{j \to \infty} \int_{\partial \omega \cap U_l} k(z, \xi) \text{Tr}(u)(\xi) \, d\sigma_{\omega}(\xi)
\]

where \( u \) has compact support in \( U_l \). There exist local coordinates \( x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \), and \( C^1 \) functions \( f, f_j, \mathbb{R}^{N-1} \to \mathbb{R} \), such that \( \omega \cap U_l = A \cap U_l \), and \( \omega_j \cap U_l = A_j \cap U_l \), where

\[
A = \{(x', x_N) \in \mathbb{R}^N : x_N > f(x')\}, \quad A_j = \{(x', x_N) \in \mathbb{R}^N : x_N > f_j(x')\}.
\]

Since \( \omega_j \subset \subset \omega \), we can assume \( A_j \subset \subset A \). Since \( u \) has compact support in \( U_l \), we can extend \( u \) to be 0 in \( A \setminus U_l \) (Thus \( u \) is also 0 in \( A_j \setminus U_l \)) to obtain \( u \in W^{1,1}(A) \) and \( u \in W^{1,1}(A_j) \). By assumption, \( f_j \) converges uniformly to \( f \) in \( C^1(\mathbb{R}^{N-1}) \). The surface area element on \( \partial \omega \cap U_l \) is given by

\[
ds(\partial \omega) = \sqrt{1 + \left| \nabla f(x') \right|^2} \, dx', \quad dx' = dx_1 \cdots dx_{N-1},
\]

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and similarly $ds(b\omega_j) = \sqrt{1 + |\nabla f_j(x')|^2} \, dx'$. Define $C^1$ diffeomorphisms $\psi_j : B_0 \to U_0$ by

$$
\psi_j(x) = (x', x_N + f(x')), \\
\psi_j(x) = (x', x_N + f_j(x')).
$$

Let $\tilde{u} = u \circ \psi$, $\tilde{u}_j = u \circ \psi_j$. Note that $\psi : \mathbb{R}^N_+ \to A$, and $\psi_j : \mathbb{R}^N_+ \to A_j$, and $\tilde{u}$ and $\tilde{u}_j$ are functions in $W^{1,1}_0(\mathbb{R}^N_+)$. By (2.5), $\text{Tr}(u)|_{bA}(x', f(x')) = \text{Tr}(\tilde{u})(x')$ and $\text{Tr}(u)|_{bA_j}(x', f_j(x')) = \text{Tr}(\tilde{u}_j)(x')$, for $x' \in \mathbb{R}^{N-1}$. As remarked before, $\text{Tr}(u)|_{bA}$ (resp. $\text{Tr}(u)|_{bA_j}$) is compactly supported in $bA \cap U_l$ (resp. $bA_j \cap U_l$). Since $\text{Tr}(\tilde{u}) = \text{Tr}(u \circ \psi) = \text{Tr}|_{bA}(u \circ \psi)$, $\text{Tr}(\tilde{u}_j) = \text{Tr}(u \circ \psi_j) = \text{Tr}(u)|_{bA_j} \circ \psi_j$, we have $\text{Tr}(\tilde{u})$ and $\text{Tr}(\tilde{u}_j)$ are compactly supported in $\mathbb{R}^{N-1}$. Thus

$$
\left| \int_{b\omega_j \cap U_l} k(z, \xi) \text{Tr}(u)(\xi) ds(b\omega_j) - \int_{b\omega \cap U_l} k(z, \xi) \text{Tr}(u)(\xi) ds(b\omega) \right|
$$

$$
= \left| \int_{\mathbb{R}^{N-1}} k(z, (x', f_j(x')) | Tr(\tilde{u}_j)(x') g_j(x') - k(z, (x', f(x')) | Tr(\tilde{u})(x') g(x') dx' \right|
$$

$$
\leq F_j(z) + G_j(z) + H_j(z), \quad (2.8)
$$

where

$$
F_j(z) = \int_{\mathbb{R}^{N-1}} |k(z, (x', f_j(x'))| - k(z, (x', f(x')))| | Tr(\tilde{u}_j)(x') g_j(x')| \, dx',
$$

$$
G_j(z) = \int_{\mathbb{R}^{N-1}} |k(z, (x', f(x'))| | Tr(\tilde{u}_j - \tilde{u})(x')| | g_j(x')| \, dx',
$$

$$
H_j(z) = \int_{\mathbb{R}^{N-1}} |k(z, (x', f(x'))| | Tr(\tilde{u})(x')| |(g_j - g)(x')| \, dx',
$$

and

$$
g(x') = \sqrt{1 + |\nabla f(x')|^2}, \quad g_j(x') = \sqrt{1 + |\nabla f_j(x')|^2}.
$$

By assumption, $|f_j - f|$ converges to 0 uniformly on $\mathbb{R}^{N-1}$ and $k(z, \xi)$ is uniformly continuous in $\xi$ in a neighborhood of $b\omega$. So we have

$$
|k(z, (x', f_j(x'))| - k(z, (x', f(x')))| \xrightarrow{j \to \infty} 0, \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}.
$$

Hence to show $F_j$ converges to 0 uniformly in $z \in \omega'$, it suffices to show

$$
\int_{B_{1}^{N-1}} |Tr(\tilde{u}_j)(x')| \, |g_j(x')| \, dx' \leq C \quad (2.9)
$$
for some $C$ independent of $j$. By (2.2), we have

$$
\int_{\mathbb{R}^{N-1}} |\text{Tr}(\tilde{u}_j)(x')g_j(x')| \, dx' \leq C \int_{\mathbb{R}^N} \left| \frac{\partial \tilde{u}_j}{\partial x_N}(x) - \frac{\partial \tilde{u}_j}{\partial x_N}(x) \right| \, dx
$$

$$
\leq \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_N}(x', x_N + f_j(x')) - \frac{\partial u}{\partial x_N}(x', x_N + f(x')) \right| \, dx
$$

$$
+ \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_N}(x', x_N + f(x')) \right| \, dx.
$$

Below we show the first integral in the last inequality converges to 0 as $j \to \infty$. This proves (2.9) and thus $F_j$ converges to 0 uniformly in $z \in \omega'$. By assumption, $|g_j - g|$ converges to 0 uniformly on $\mathbb{R}^{N-1}$, $|k(z, (x', f(x'))| \leq C$ for $x' \in \mathbb{R}^{N-1}$, and $\text{Tr}(\tilde{u}) \in L^1(\mathbb{R}^{N-1})$, it follows that $H_j$ converges to 0 uniformly on $z \in \omega'$. For $G_j$, by (2.2) we have

$$
\int_{\mathbb{R}^{N-1}} |\text{Tr}(\tilde{u} - \tilde{u}_j)(x')| \, dx' \leq \int_{\mathbb{R}^N} \left| \frac{\partial \tilde{u}}{\partial x_N}(x) - \frac{\partial \tilde{u}_j}{\partial x_N}(x) \right| \, dx
$$

$$
= \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial y_N}(\psi(x)) - \frac{\partial u}{\partial y_N}(\psi_j(x)) \right| \, dx
$$

$$
= \int_{A} \left| \frac{\partial u}{\partial y_N}(y) - \frac{\partial u}{\partial y_N}(y, y_N + (f_j - f)(y')) \right| \, dy.
$$

Since $f_j$ converges to $f$ uniformly on $\mathbb{R}^{N-1}$, we can show the last integral converges to 0 by a standard smoothing argument. Since

$$
|k(z, (x', f(x'))| \leq C, \quad |g_j(x')| \leq C,
$$

we have proved that $G_j$ converges to 0 uniformly on $z \in \omega'$. The conclusion of the lemma then follows from estimate (2.8). \hfill \Box

We now extend the homotopy formula in [6] to $\varphi$ satisfying $\varphi, \bar{\partial} \varphi \in W^{1,1}(D)$.

**Proposition 2.8** Let $D \subset \mathbb{C}^n$ be a bounded domain with $C^1$ boundary and let $U$ be a bounded neighborhood of $\overline{D}$. Let $g^0 = \zeta - z$. Let $g^1 = W(z, \zeta)$, where $W \in C^1(D \times (U \setminus D))$ is a Leray mapping, that is, $W$ is holomorphic in $z \in D$ and satisfies

$$
\Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z) \neq 0, \quad z \in D, \quad \zeta \in U \setminus D.
$$

Let $\varphi$ be a $(0, q)$-form. Suppose that $\varphi$ and $\bar{\partial} \varphi$ are in $W^{1,1}(D)$. (That is, all the coefficient functions of $\varphi$ and $\bar{\partial} \varphi$ are in $W^{1,1}(D)$). Then we have the following:

(i) The Bochner–Martinelli formula

$$
\varphi = \bar{\partial} z \int_D \Omega_{0,q-1}^0(z, \zeta) \wedge \varphi + \int_D \Omega_{0,q}^0(z, \zeta) \wedge \bar{\partial} \varphi + \int_{bD} \Omega_{0,q}^0(z, \zeta) \wedge Tr(\varphi)
$$

(2.11)
holds in the distribution sense in D.

(ii) The following homotopy formula holds in D in the distribution sense:

\[ \varphi = \overline{\partial} H_q \varphi + H_{q+1} \overline{\partial} \varphi, \quad 1 \leq q \leq n \]  
\[ \varphi = H_0 \varphi + H_1 \overline{\partial} \varphi, \quad q = 0 \]

where

\[ H_q \varphi := \int_U \Omega_{0,q-1}^0 \wedge E \varphi + \int_{U \setminus D} \Omega_{0,q-1}^W \wedge [\overline{\partial}, E] \varphi, \quad 1 \leq q \leq n \]
\[ H_0 \varphi := \int_{U \setminus D} \Omega_{0,0}^1 \wedge [\overline{\partial}, E] \varphi, \quad [\overline{\partial}, E] \varphi = \overline{\partial} E \varphi - E \overline{\partial} \varphi. \]

Here \( \Omega_{0,q} \) stands for the \((0, q)\) component of \( \Omega \) of type \((0, q)\) in \( z \), and

\[ \Omega^0(z, \zeta) = \frac{1}{(2\pi i)^n} \left( \frac{\zeta - z}{|\zeta - z|^2} \right)^n, \quad \overline{\partial}_{\xi, \zeta} = \overline{\partial}_\xi + \overline{\partial}_\zeta; \]
\[ \Omega^W(z, \zeta) = \frac{1}{(2\pi i)^n} \left( \frac{W, d\zeta}{\Phi(z, \zeta)} \right)^n, \quad \Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z); \]
\[ \Omega^{0,W}(z, \zeta) = \frac{1}{(2\pi i)^n} \left( \frac{d\zeta - d\zeta}{|\zeta - z|^2} \right)^n \wedge \left( \frac{W, d\zeta}{(W, \zeta - z)} \right)^n \]
\[ \wedge \sum_{i+j=n-2} \left[ \frac{(d\zeta - d\zeta)}{|\zeta - z|^2} \right]^i \wedge \left[ \frac{(W, d\zeta)}{(W, \zeta - z)} \right]^j. \]

We set \( \Omega^{0,W}_{0,-1} = 0 \) and \( \Omega^{0,W}_{0,-1} = 0 \).

**Proof** (i) By some abuse of notation, we shall denote the coefficient functions of \( \varphi \) by \( \varphi \), and our smoothing is done componentwise. Let \( \{ \psi_\varepsilon \}_{\varepsilon > 0} \) be the standard mollifier which satisfies \( \psi_\varepsilon \in C^\infty_0(B_\varepsilon(0)) \), \( \psi_\varepsilon \geq 0 \), and \( \int_{\mathbb{C}^n} \psi_\varepsilon = 1 \). Let \( \varphi_\varepsilon = \varphi \ast \psi_\varepsilon \) be defined by

\[ \varphi_\varepsilon(z) = (z) = \int_{\mathbb{C}^n} \varphi(z - \zeta) \psi_\varepsilon(\zeta) \, dV(\zeta) = \int_{B_\varepsilon(0)} \varphi(z - \zeta) \psi_\varepsilon(\zeta) \, dV(\zeta). \]

Then we can show that for any \( D' \subset \subset D \), and \( \varepsilon < \varepsilon_0 \) sufficiently small,

\[ \varphi_\varepsilon \xrightarrow{\varepsilon \to 0} \varphi \text{ in } W^{1,1}(D'), \quad \overline{\partial} \varphi_\varepsilon \xrightarrow{\varepsilon \to 0} \overline{\partial} \varphi \text{ in } W^{1,1}(D'). \]  

When \( bD \subset C^1 \) and \( \varphi \in C^1(\overline{D}) \), the proof of formula (2.11) can be found in [4, p. 265]. Let \( D_j \) be a sequence of domains with \( C^\infty \) boundary approximating \( D \) from
inside, \(D_j \subset D_{j+1} \subset \cdots \subset D\), so that locally the defining functions of \(D_j\) converge uniformly in \(C^1\)-norm. Fix \(j\) and \(\varepsilon_0 > 0\) such that \(dist(D_j, D) > \varepsilon_0\). The formula (2.11) then holds for \(\varphi_\varepsilon\) on \(D_j\) for any \(\varepsilon < \varepsilon_0\)

\[
\varphi_\varepsilon(z) = \overline{\partial} \int_{D_j} \Omega^0_{0,q-1}(z, \xi) \wedge \varphi_\varepsilon + \int_{D_j} \Omega^0_{0,q}(z, \xi) \wedge \overline{\partial} \varphi_\varepsilon + \int_{\partial D_j} \Omega^0_{0,q}(z, \xi) \wedge \varphi.
\]

By Sobolev embedding \([9, p. 312]\), \(W^{1,1}(D) \subset L^{2n/(2n-1)}(D)\). Applying this and the Calderón–Zygmund estimate for the Newtonian potential \([5, p. 230]\), we have for any \(D' \subset D_j\),

\[
\left| \int_{D_j} \Omega^0_{0,q-1}(z, \xi) \wedge (\varphi_\varepsilon - \varphi) \right| \leq C(n) \| \varphi_\varepsilon - \varphi \|_{L^{2n/(2n-1)}(D')} \leq C(n) \| \varphi_\varepsilon - \varphi \|_{W^{1,1}(D_j)},
\]

and similarly,

\[
\left| \int_{D_j} \Omega^0_{0,q}(z, \xi) \wedge (\overline{\partial} \varphi_\varepsilon - \overline{\partial} \varphi) \right| \leq C(n) \| \overline{\partial} \varphi_\varepsilon - \overline{\partial} \varphi \|_{W^{1,1}(D_j)}.
\]

Note that the above constants \(C\) depend only on the dimension \(n\) and is independent of \(j\). Now, \(\| \Omega^0_{0,q}(z, \xi) \| \leq C\) for \(z \in D'\) and \(\xi \in \partial D_j, D' \subset D_j\). By estimate (2.4), there exists a constant \(C\) independent of \(j\) such that

\[
\left| \int_{bD_j} \Omega^0_{0,q}(z, \xi) \wedge (Tr(\varphi_\varepsilon) - Tr(\varphi)) \right|_{C^0(D')} \leq C \| Tr(\varphi_\varepsilon) - Tr(\varphi) \|_{L^1(bD_j)} \leq C \| \varphi_\varepsilon - \varphi \|_{W^{1,1}(D_j)}.
\]

As \(\varepsilon \to 0\), all these expressions in (2.20), (2.21) and (2.22) converge to 0. Thus

\[
\varphi(z) = \overline{\partial} \int_{D_j} \Omega^0_{0,q-1}(z, \xi) \wedge \varphi + \int_{D_j} \Omega^0_{0,q}(z, \xi) \wedge \varphi + \int_{bD_j} \Omega^0_{0,q}(z, \xi) \wedge \varphi
\]

holds in the distribution sense. (In fact, we only need to show convergence in \(L^1(D')\).)

Finally we let \(j \to \infty\). For some constant \(C\) independent of \(j\), we have

\[
\left| \int_{D \setminus D_j} \Omega^0_{0,q-1}(z, \xi) \wedge \varphi \right| \leq C \| \varphi \|_{L^1(D \setminus D_j)} \to 0, \quad j \to \infty,
\]

\[
\left| \int_{D_j \setminus D_j} \Omega^0_{0,q}(z, \xi) \wedge \overline{\partial} \varphi \right| \leq C \| \overline{\partial} \varphi \|_{L^1(D_j \setminus D_j)} \to 0, \quad j \to \infty.
\]
where the convergence is uniform on \( D' \). For \( z \in D' \), and \( \zeta \) in a small neighborhood of \( bD, \Omega_{0,q}^0(z, \zeta) \) is smooth in both variable. Hence we can apply Lemma 2.7 to get

\[
\int_{bD} \Omega_{0,q}^0(z, \zeta) \wedge Tr(\varphi) \xrightarrow{j \rightarrow \infty} \int_{bD} \Omega_{0,q}^0(z, \zeta) \wedge Tr(\varphi). \tag{2.23}
\]

Consequently the Bochner–Martinelli formula (2.11) holds in the distribution sense for any \( \varphi \) satisfying \( \varphi, \partial \varphi \in W^{1,1}(D) \).

(ii) We prove formula (2.12). The proof for (2.13) is similar and we shall omit the proof. First let us derive (2.12) under the assumptions \( bD \in C^2, W \in C^2(D \times (U \setminus D)) \) and \( \varphi, \bar{\partial} \varphi \in C^1(B) \). This part of the proof is the same as presented in [6], and we put it here since later on we shall prove the same formula under weaker assumptions. In this case, the Bochner–Martinelli formula holds:

\[
\varphi = \bar{\partial}_z \int_D \Omega_{0,q-1}^0(z, \zeta) \wedge \varphi + \int_D \Omega_{0,q}^0(z, \zeta) \wedge \bar{\partial} \varphi + \int_{bD} \Omega_{0,q}^0(z, \zeta) \wedge \varphi. \tag{2.24}
\]

By the Koppelman lemma [4, p. 264], one has, for \( q \geq 1 \),

\[
\Omega_{0,q}^0 - \Omega_{0,q}^W = \bar{\partial}_z \Omega_{0,q}^0 - \bar{\partial}_z \Omega_{0,q-1}^0, \quad (z, \zeta) \in D \times (U \setminus D). \tag{2.25}
\]

Applying this to the boundary integral in (2.24) we get

\[
\varphi(z) = \bar{\partial}_z B_q \varphi(z) + B_{q+1} \varphi(z) + \int_{bD} \Omega_{0,q}^W(z, \zeta) \wedge \varphi
- \bar{\partial}_z \int_{bD} \Omega_{0,q-1}^W(\zeta, z) \wedge \varphi(\zeta) - \int_{bD} \Omega_{0,q}^0(\zeta, z) \wedge \bar{\partial} \varphi(\zeta), \quad z \in D. \tag{2.26}
\]

where we denote

\[
B_q \varphi = \int_D \Omega_{0,q-1}^0(z, \zeta) \wedge \varphi.
\]

We denote by \( B_q; \Omega(\varphi) \) for the above integral when the domain of integration is \( \Omega \), and \( B_q \varphi = B_q; D(\varphi) \). Since \( W \) is a Leray map and it is holomorphic in \( z \), in view of expression (2.17), we have

\[
\Omega_{0,q}^W(z, \zeta) = 0, \quad \text{for } q \geq 1. \tag{2.27}
\]

For the last two integrals in (2.26) we first extend \( \varphi, \bar{\partial} \varphi \) to \( E \varphi, E \bar{\partial} \varphi \in W^{1,1}(U) \) by means of Proposition 2.2. Applying Stokes theorem to the domain \( U \setminus D \) we get

\[
\int_{bD} \Omega_{0,q-1}^0 \wedge \varphi = - \int_{U \setminus D} \bar{\partial}_z \Omega_{0,q-1}^0 \wedge E \varphi - \int_{U \setminus D} \Omega_{0,q-1}^0 \wedge \bar{\partial}_z E \varphi
= - \int_{U \setminus D} \Omega_{0,q-1}^0 \wedge E \varphi + \int_{U \setminus D} \Omega_{0,q-1}^0 \wedge E \varphi
\]

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and
\[
\int_{bD} \Omega_{0,q}^W \wedge E \overline{\partial}_\zeta \varphi = - \int_{U \backslash D} \overline{\partial}_\zeta \Omega_{0,q}^W \wedge E \overline{\partial}_\zeta \varphi - \int_{U \backslash D} \Omega_{0,q}^W \wedge \overline{\partial}_\zeta E \overline{\partial}_\zeta \varphi
\]
\[
= - \int_{U \backslash D} \Omega_{0,q}^0 \wedge E \overline{\partial}_\zeta \varphi + \overline{\partial}_\zeta \int_{U \backslash D} \Omega_{0,q-1}^W \wedge E \overline{\partial}_\zeta \varphi - \int_{U \backslash D} \Omega_{0,q}^W \wedge \overline{\partial}_\zeta E \overline{\partial}_\zeta \varphi.
\]

By (2.27), \(\Omega_{0,q-1}^W = 0\) if \(q \geq 2\). If \(q = 1\), we have \(\Omega_{0,q-1}^W = \Omega_{0,0}^W\) is holomorphic in \(z\), and \(\Omega_{0,0}^W = \Omega_{0,0}^W = 0\). Using these facts and substituting (2.28) and (2.29) into (2.26), we obtain (2.12).

Suppose now that \(bD \in C^1, W \in C^1(D \times (U \backslash D))\) and \(\varphi, \overline{\partial}_\zeta \varphi \in W^{1,1}(D)\). We shall derive the homotopy formula (2.12) in the distribution sense. We need to justify (2.26), (2.28) and (2.29).

As before, we take a sequence of domains \(D_j\) with smooth boundary approximating \(D\) from inside, such that locally the defining functions of \(D_j\) converge in the \(C^1\) norm. Consider sufficiently large \(j\) such that \(D_j \subset \subset D_j \subset \subset D\). Let \(\varphi \varepsilon\) be a sequence of smooth forms so that \(\varphi \varepsilon \to \varphi\), and \(\overline{\partial}_\zeta \varphi \varepsilon \to \overline{\partial}_\zeta \varphi\) in \(W^{1,1}(D_j)\). Since \(D \times (U \backslash D)\) has \(C^1\) boundary, by Proposition 2.2 (ii) we can extend \(W\) to get \(EW \in C^1(\mathbb{C}^n \times \mathbb{C}^n)\), such that \(EW(z, \zeta) = W(z, \zeta)\) for \(z \in D\) and \(\zeta \in \overline{U \backslash D}\). Note that \(EW(\cdot, \zeta)\) may not be holomorphic for \(\zeta \in D\).

For \(z \in D\) and \(\zeta \in U\), define
\[
(EW)_{\varepsilon'}(z, \zeta) = \int_{\mathbb{C}^n \times \mathbb{C}^n} \psi_{\varepsilon'}(z' - z, \zeta' - \zeta) EW(z', \zeta') dV(z')dV(\zeta')
\]
where \(\psi_{\varepsilon'}\) is the standard mollifier. Then \((EW)_{\varepsilon'}\) is \(C^\infty\) in \(\mathbb{C}^n \times \mathbb{C}^n\). Also \((EW)_{\varepsilon', \zeta - z} \neq 0\) for \(z \in D'\) and \(\zeta \in bD_j\), if \(\varepsilon'\) is sufficiently small and \(j\) is sufficiently large. Indeed, by assumption \((W, \zeta - z) \neq 0\) on \(D \times (U \backslash D)\). Since \(D' \times bD\) is a compact subset of \(D \times (U \backslash D)\), we have \(|(EW, \zeta - z)| = |(W, \zeta - z)| \geq \delta\) on \(D' \times bD\), and \((EW)_{\varepsilon', \zeta - z} \geq \delta'\) if \(\varepsilon'\) is small and \(j\) is large. Let
\[
\Omega_{0, (EW)_{\varepsilon'}}^0(\zeta, \zeta) = \frac{1}{(2\pi i)^n} \frac{\langle \zeta - \overline{z}, d\zeta \rangle}{|\zeta - z|^2} \wedge \frac{\langle (EW)_{\varepsilon'}, d\zeta \rangle}{(EW)_{\varepsilon'}, \zeta - z}
\]
\[
\wedge \sum_{i+j=n-2} \left[ \frac{\langle d\overline{\zeta} - \overline{d\zeta}, d\zeta \rangle}{|\zeta - z|^2} \right]^i \wedge \left[ \overline{\partial}_{\zeta, \overline{\zeta}} \frac{\langle (EW)_{\varepsilon'}, d\zeta \rangle}{(EW)_{\varepsilon'}, \zeta - z} \right]^j.
\]

Then the Koppleman lemma says
\[
\Omega_{0,q}^0 - \Omega_{0,q}^{(EW)_{\varepsilon'}} = \overline{\partial}_\zeta \Omega_{0,q}^{0,(EW)_{\varepsilon'}} + \overline{\partial}_z \Omega_{0,q-1}^{0,(EW)_{\varepsilon'}}\text{, for } (z, \zeta) \in D' \times bD_j.
\]
We have for \( z \in D' \),
\[
\varphi(\epsilon)(z) = \overline{\partial}B_{q; D_j}(\varphi(\epsilon))(z) + B_{q+1; D_j}(\overline{\partial}\varphi(\epsilon))(z) + \int_{bD_j} \Omega_{0,q}^{(EW)}(z, \xi) \wedge \varphi(\epsilon) - \overline{\partial}_z \int_{bD_j} \Omega_{0,q-1}^{(EW)}(z, \xi) \wedge \varphi(\xi) - \int_{bD_j} \Omega_{0,q}^{(EW)}(z, \xi) \wedge \overline{\partial}_\xi \varphi(\xi).
\]

As shown in (i), \( B_{q; D_j}(\varphi(\epsilon)) \) and \( B_{q+1; D_j}(\overline{\partial}\varphi(\epsilon)) \) converge to \( B_q \varphi \) and \( B_{q+1}\overline{\partial} \varphi \), respectively, in \( W^{1, \frac{2n}{n-1}}(D') \)-norm as \( \epsilon \to 0 \) and \( j \to \infty \). By estimate (2.4) and (2.19),
\[
\| Tr(\varphi(\epsilon)) - Tr(\varphi) \|_{L^1(bD_j)} \leq C \| \varphi(\epsilon) - \varphi \|_{W^{1,1}(D_j)} \xrightarrow{\epsilon \to 0} 0
\]
where \( C \) can be chosen independent of \( j \). Also \( \Omega_{0,q}^{(EW)}(z, \xi) \) converges uniformly to \( \Omega_{0,q}^{EW}(z, \xi) \) on \( D' \times bD_j \) as \( \epsilon' \to 0 \). Hence we have
\[
\int_{bD_j} \Omega_{0,q}^{(EW)}(z, \xi) \wedge \varphi(\epsilon) \xrightarrow{\epsilon, \epsilon' \to 0} \int_{bD_j} \Omega_{0,q}^{EW}(z, \xi) \wedge Tr(\varphi)
\]
uniformly on \( z \in D' \). Since \( \Omega_{0,q}^{EW}(z, \xi) \) is uniformly bounded in the first variable and uniformly continuous in the second variable for \( z \in D' \) and \( \xi \) in a small neighborhood of \( bD \), applying Lemma 2.7 we get
\[
\int_{bD_j} \Omega_{0,q}^{EW}(z, \xi) \wedge Tr(\varphi) \xrightarrow{j \to \infty} \int_{bD} \Omega_{0,q}^{EW}(z, \xi) \wedge Tr(\varphi) = 0, \quad (q \geq 1)
\]
where the convergence is uniform for \( z \in D' \). This shows that the third term in (2.32) converges to 0 as \( \epsilon, \epsilon' \to 0 \) and \( j \to \infty \). Similarly, by taking the limit as \( \epsilon, \epsilon' \to 0 \) and then \( j \to \infty \), we can show
\[
\int_{bD_j} \Omega_{0,q}^{0,(EW)}(z, \xi) \wedge \varphi(\epsilon)(\xi) \xrightarrow{\epsilon \to 0} \int_{bD} \Omega_{0,q-1}^{0,(EW)}(z, \xi) \wedge Tr(\varphi)(\xi),
\]
\[
\int_{bD_j} \Omega_{0,q}^{0,(EW)}(z, \xi) \wedge \overline{\partial}_\xi \varphi(\xi) \xrightarrow{\epsilon \to 0} \int_{bD} \Omega_{0,q}^{0,(EW)}(z, \xi) \wedge Tr(\overline{\partial}_\xi \varphi)(\xi),
\]
where the convergence is uniform on \( z \in D' \). Putting together above results we obtain
\[
\varphi(z) = \overline{\partial}B_q(\varphi)(z) + B_{q+1}(\overline{\partial}\varphi)(z)
\]
\[
- \overline{\partial}_z \int_{bD} \Omega_{0,q-1}^{W}(z, \xi) \wedge Tr(\varphi) - \int_{bD} \Omega_{0,q}^{W}(z, \xi) \wedge Tr(\overline{\partial}_\xi \varphi), \quad z \in D'
\]
in the distribution sense.
Finally we check (2.28) and (2.29). Let \((EW)_{\epsilon'}\) and \(\Omega^{0,(EW)}_{\epsilon'}\) be defined as in (2.30) and (2.31). Set \(\phi = E\varphi\) or \(E\overline{\varphi}\), so \(\phi \in W^{1,1}(U)\). By Proposition 2.6, we have for \(z \in D'\),

\[
\int_{bD} \Omega^{0,(EW)_{\epsilon'}}(z, \zeta) \wedge \text{Tr}(\phi) = - \int_{U \setminus D} d\left(\Omega^{0,(EW)_{\epsilon'}}(z, \zeta) \wedge \phi\right)
\]

\[
= - \int_{U \setminus D} \overline{\partial}_\zeta \Omega^{0,(EW)_{\epsilon'}} \wedge \phi - \int_{U \setminus D} \Omega^{0,(EW)_{\epsilon'}} \wedge \overline{\partial}_\zeta \phi
\]

\[
= - \int_{U \setminus D} \Omega^{0}(z, \zeta) \wedge \phi + \int_{U \setminus D} \Omega^{(EW)_{\epsilon'}}(z, \zeta) \wedge \phi
\]

\[
+ \overline{\partial}_z \int_{U \setminus D} \Omega^{0,(EW)_{\epsilon'}}(z, \zeta) \wedge \phi - \int_{U \setminus D} \Omega^{0,(EW)_{\epsilon'}}(z, \zeta) \wedge \overline{\partial} \phi. \tag{2.34}
\]

As \(\epsilon' \to 0\), the \(\Omega^{(EW)_{\epsilon'}}\), \(\Omega^{0,(EW)_{\epsilon'}}\) converge uniformly to \(\Omega^W = \Omega^W\) and \(\Omega^{0,EW} = \Omega^{0,W}\) for \((z, \zeta) \in D' \times U \setminus D\), respectively. Thus

\[
\int_{U \setminus D} \Omega^{(EW)_{\epsilon'}}(z, \zeta) \wedge \phi \xrightarrow{\epsilon' \to 0} \int_{U \setminus D} \Omega^{W}(z, \zeta) \wedge \phi = 0.
\]

Letting \(\epsilon' \to 0\) in (2.34) we get

\[
\int_{bD} \Omega^{0,W}(z, \zeta) \wedge \phi = - \int_{U \setminus D} \Omega^D(z, \zeta) \wedge \phi + \overline{\partial}_z \int_{U \setminus D} \Omega^{0,W}(z, \zeta) \wedge \phi
\]

\[
- \int_{U \setminus D} \Omega^{0,W}(z, \zeta) \wedge \overline{\partial} \phi \tag{2.35}
\]

in the distribution sense. This completes the proof of formula (2.12) for \(bD \in C^1\), \(W \in C^1(D \times (U \setminus D))\) and \(\varphi, \overline{\partial} \varphi \in W^{1,1}(D)\). \(\square\)

The key to our estimate is the control of the blow-up order of derivatives of the Leray map \(W(z, \zeta)\) as \(\zeta\) approaches the boundary from outside the domain. Let \(D\) be a bounded domain in \(\mathbb{C}^n\). Define for \(\delta > 0\),

\[
D_\delta = \{z \in \mathbb{C}^n : \text{dist}(z, \overline{D}) < \delta\}, \quad D_{-\delta} = \{z \in D : \text{dist}(z, bD) > \delta\}.
\]

Gong [6] proved the following result:

**Proposition 2.9** Let \(D\) be a bounded domain in \(\mathbb{C}^n\) with \(C^2\) boundary. Let \(\rho_0\) be a \(C^2\) defining function of \(D\). That is, there exists a neighborhood \(U\) of \(\overline{D}\) such that \(D = \{z \in U : \rho_0 < 0\}\) and \(\nabla \rho_0 \neq 0\) on \(bD\). Then there exists a real function \(\tilde{\rho}_0 \in C^2(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n \setminus \overline{D})\) such that \(\tilde{\rho}_0 = \rho_0\) in \(\overline{D}\), and for \(0 < d(x) := \text{dist}(x, D) < 1\), we have

\[
|\partial_x^i \tilde{\rho}_0(x)| \leq C_i |\rho_0|_{C^2(\overline{D})}(1 + d(x)^{2-i}) \tag{2.36}
\]
for $i = 0, 1, 2, \ldots$. We call $\tilde{\rho}_0$ the regularized defining function with respect to $\rho_0$.

Assume further that $D$ is strictly pseudoconvex. Let $\rho_1 = e^{L_0 \rho_0 - 1}$, where $L_0$ is sufficiently large so that $\rho_1$ is strictly plurisubharmonic in a neighborhood $\omega$ of $bD$. Let $\rho$ be the regularized defining function with respect to $\rho_1$. Then there exist $\delta > 0$ and a function $W$ (called regularized Leray map) in $D_\delta \times (D_\delta \setminus D_{-\delta})$ satisfying the following.

(i) $W : D_\delta \times (D_\delta \setminus D_{-\delta}) \to \mathbb{C}^n$ is a $C^1$ mapping, $W(z, \zeta)$ is holomorphic in $z \in D_\delta$, and $\Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z) \neq 0$ for $\rho(z) < \rho(\zeta)$.

(ii) If $|\zeta - z| < \varepsilon$, and $z \in D_\delta \setminus D_{-\delta}$, then $\Phi(z, \zeta) = F(z, \zeta)M(z, \zeta)$, $M(z, \zeta) \neq 0$ and

$$F(z, \zeta) = -\sum \frac{\partial \rho}{\partial \zeta_j}(z_j - \zeta_j) + \sum a_{jk}(\zeta)(z_j - \zeta_j)(z_k - \zeta_k),$$

$$\text{Re} F(z, \zeta) \geq \rho(\zeta) - \rho(z) + |\zeta - z|^2/C,$$

with $M, F \in C^1(D_\delta \times (D_\delta \setminus D_{-\delta}))$ and $a_{jk} \in C^\infty(\mathbb{C}^n)$.

(iii) For each $z \in D_\delta$, $\zeta \in D_\delta \setminus \bar{D}$, $0 \leq i, j \leq \infty$, the following holds:

$$|\partial^i_z \partial^j_\zeta W(z, \zeta)| \leq C_{i,j}(D, |\rho_0|_{\bar{D}, 2}, \delta)(1 + \text{dist}^{-1-j}(\zeta, D)). \quad (2.37)$$

The corresponding holomorphic support function $\Phi(z, \zeta) = W(z, \zeta) \cdot (z - \zeta)$ satisfies the following estimate: near every $\zeta^* \in bD$, there exists a neighborhood $V$ of $\zeta^*$ such that for all $z \in V$, there exists a coordinate map $\phi_z : V \to \mathbb{R}^{2n}$ given by $\phi_z : \zeta \in V \to (s, t) = (s_1, s_2, t_3, \ldots, t_{2n})$. Furthermore, for $z \in V \cap D$, $\zeta \in V \setminus D$:

$$|\Phi(z, \zeta)| \geq c \left( d(z) + s_1 + |s_2| + |t|^2 \right), \quad (2.38)$$

$$|\Phi(z, \zeta)| \geq c|z - \zeta|^2, \quad |\zeta - z| \geq c|s_2, t|, \quad (2.39)$$

where $c > 0$ is a constant. In particular,

$$\Phi(z, \zeta) \neq 0, \quad \text{for } z \in D \text{ and } \zeta \in D_\delta \setminus D.$$

**Lemma 2.10** Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^2$ boundary. Let $\rho$ be the regularized defining function as in Proposition 2.9. Assume $\partial_z \rho(\zeta^*) \neq 0$, for some $\zeta^* \in bD$. Then $\phi_{\zeta^*} = (\phi^1, \phi^2, \ldots, \phi^{2n})$ given by

$$s_1 = \phi^1(\zeta) = \rho(\zeta), \quad s_2 = \phi^2(\zeta) = 1m(\rho_\zeta \cdot (\zeta - \zeta^*),$$

$$(t_{2k-1}, t_{2k}) = (\phi^{2k-1}(\zeta, \zeta), \phi^{2k}(\zeta, \zeta)) = (\text{Re}(\zeta_k - \zeta_k^*), 1m(\zeta_k - \zeta_k^*)) \quad (k = 2, \ldots, n) \quad (2.40)$$
defines a $C^1$ coordinate transformation in some neighborhood $V_0$ of $\zeta^*$. Furthermore, $\phi^{-1}$ satisfies for $m = 0, 1, 2, \ldots$,

$$\left| \partial_s^m \phi^{-1}(s) \right| \leq C(1 + d(\phi^{-1}(s)))^{1-m}, \quad s \in \phi(V_0 \setminus D).$$  \quad (2.41)

**Proof** From Proposition 2.9, we have $\rho \in C^2(\mathbb{C}^n), \phi \in C^1(\mathbb{C}^n)$. Up to a nonzero scalar multiple, the Jacobian matrix at $\zeta = \zeta^*$ is:

$$D\phi \mid_{\zeta = \zeta^*} = \left( \begin{array}{cccccc}
\frac{\partial \rho}{\partial \zeta_1} & \frac{\partial \rho}{\partial \zeta_2} & \cdots & \frac{\partial \rho}{\partial \zeta_n} \\
\frac{\partial \rho}{\partial \zeta_1} & \frac{\partial \rho}{\partial \zeta_2} & \cdots & \frac{\partial \rho}{\partial \zeta_n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array} \right)_{2n \times 2n}.$$

If $n = 1$, it holds that $\det(D\phi) \mid_{\zeta = \zeta^*} = -2 \frac{\partial \rho}{\partial \zeta_1} \frac{\partial \rho}{\partial \zeta_1} \neq 0$. Suppose we have proved for $k \geq 1$. Denote by $D_k$ and $D_{k+1}$ the determinants of $D\phi \mid_{\zeta = \zeta^*}$ when $n = k$ and $n = k + 1$. Computing the determinant using row expansion of second to the last row (0, 0, $\cdots$, 1, 1) in the above matrix, we get

$$D_{k+1} = -D_k - D_k = -2D_k \neq 0.$$  

Thus $\det(D\phi) \mid_{\zeta = \zeta^*} \neq 0$ for all $n \geq 1$. By the inverse function theorem, there exists a neighborhood $V_0$ of $\zeta^*$ such that $\phi : V_0 \to \phi(V_0)$ is a $C^1$ diffeomorphism and $\phi^{-1} \in C^1(\phi(V_0))$.

Next, we analyze the inverse of $D\phi$. Replacing the second row in the above matrix by

$$\left( \begin{array}{cccc}
\frac{\partial \phi^2}{\partial \zeta_1} & \frac{\partial \phi^2}{\partial \zeta_2} & \cdots & \frac{\partial \phi^2}{\partial \zeta_n} \\
\frac{\partial \phi^2}{\partial \zeta_1} & \frac{\partial \phi^2}{\partial \zeta_2} & \cdots & \frac{\partial \phi^2}{\partial \zeta_n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array} \right),$$

we obtain the Jacobian matrix $D\phi$. Leaving out the constant $\frac{1}{2^l}$, we compute for $i = 1, \ldots, n$,

$$\begin{align*}
\frac{\partial \phi^2}{\partial \zeta_i} &= \frac{\partial}{\partial \zeta_i} \left( \rho \zeta \cdot (\zeta - \zeta^*) - \rho \zeta^* \cdot (\zeta - \zeta^*) \right) \\
&= \sum_j \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j} (\zeta_j - \zeta_j^*) + \frac{\partial \rho}{\partial \zeta_i} (\zeta) - \sum_j \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j} (\zeta_j - \zeta_j^*), \quad (2.42)
\end{align*}$$

and

$$\frac{\partial \phi^2}{\partial \zeta_i} = \frac{\partial}{\partial \zeta_i} \left( \rho \zeta \cdot (\zeta - \zeta^*) - \rho \zeta^* \cdot (\zeta - \zeta^*) \right).$$
\[
= \sum_j \frac{\partial^2 \rho}{\partial \xi_i \partial \xi_j} (\xi_j - \xi_j^*) - \sum_j \frac{\partial^2 \rho}{\partial \xi_i \partial \xi_j} (\xi_j - \xi_j^*) - \frac{\partial \rho}{\partial \xi_i}(\xi) . \tag{2.43}
\]

By the inverse function theorem \[ \frac{D\phi^{-1}}{D\phi^{-1}} - \frac{1}{D\phi^{-1}} \circ \phi^{-1} \] in \( \phi(V_0) \). Recall the formula

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A), \tag{2.44}
\]

where \( \text{adj}(A) \) is the adjugate of \( A \). Set \( A = D\phi \). Then the entries of \( \text{adj}(A) \) and \( \det(A) \) are linear combinations with constant coefficients of

\[
\frac{\partial \rho}{\partial \xi_i} \frac{\partial \phi^2}{\partial \xi_j}, \quad \frac{\partial \rho}{\partial \xi_i} \frac{\partial \phi^2}{\partial \xi_j}, \quad \frac{\partial \rho}{\partial \xi_i} \frac{\partial \phi^2}{\partial \xi_j}, \tag{2.45}
\]

where \( \frac{\partial \phi^2}{\partial \xi_j} \) and \( \frac{\partial \phi^2}{\partial \xi_j} \) are given by (2.42) and (2.43). In view of (2.42) and (2.43), these expressions are products of the form

\[
(D\rho)(\xi)(D^2\rho(\xi))N(\xi - \xi^*), \quad (D\rho)(\xi)(D\rho(\xi))N(\xi - \xi^*), \tag{2.46}
\]

where \( D\rho \) and \( D^2\rho \) denote the first and second derivatives of \( \rho \) and \( N(\xi - \xi^*) \) takes the form \( \xi_j - \xi_j^* \) or \( \overline{\xi}_j - \xi_j^* \). By (2.44) the entries of \( [D\phi^{-1}(s)] = [D\phi]^{-1} \circ \phi^{-1}(s) \) take the form \( \frac{P(\xi)}{Q(\xi)} \circ \phi^{-1} \), where \( Q(\xi) \circ \phi^{-1} \neq 0 \) in \( \phi(V_0) \), and \( P(\xi) \) and \( Q(\xi) \) are some linear combination of expressions in (2.46).

By (2.36) the following estimates hold for \( \zeta \in V_0 \cap (\mathbb{C}^n \setminus D) \):

\[
\left| \frac{\partial^i \rho}{\partial \zeta^i}(\xi) \right| \leq C \left( 1 + d(\xi)^{2-i} \right), \quad \left| \frac{\partial^i \phi}{\partial \zeta^i}(\xi) \right| \leq C \left( 1 + d(\xi)^{1-i} \right), \tag{2.47}
\]

where \( d(\xi) = dist(\zeta, D) \). We show that \( \phi^{-1} \) satisfies the estimate:

\[
\left| \frac{\partial^m \phi^{-1}}{\partial \zeta^m}(s) \right| \leq C (1 + d(\phi^{-1}(s))^{1-m}) \tag{2.48}
\]

for \( s \in \phi(V_0 \cap (\mathbb{C}^n \setminus D)) \) and \( m = 0, 1, 2, \ldots \). Since \( \phi^{-1} \in C^1(\phi(V_0)) \), it follows that (2.48) holds for \( m = 1 \). We have

\[
\frac{\partial \phi^{-1}}{\partial \zeta}(s) = \frac{P(\xi)}{Q(\xi)} \circ \phi^{-1}(s) .
\]

Applying chain rule, we get

\[
\frac{\partial^2 \phi^{-1}}{\partial \zeta^2}(s) = \left( \frac{\partial \zeta}{P} - \frac{P(\zeta)}{Q^2}(\phi^{-1}(s)) \right) \cdot \frac{\partial \phi^{-1}}{\partial \zeta}(s) .
\]

\[
= \frac{(\partial \zeta P)Q - P(\partial \zeta Q)(\phi^{-1}(s))}{Q^3} \cdot \frac{\partial \phi^{-1}}{\partial \zeta}(s) .
\]
In general, we can write $\partial_\xi^m \phi^{-1}(s)$ as a finite linear combination of

$$ \left[ \partial_\xi^{j_1} P \cdots \partial_\xi^{j_l} P \right] \left[ \partial_\xi^{k_1} Q \cdots \partial_\xi^{k_{l'}} Q \right] P^{m_1} Q^{m_2} \circ \phi^{-1}(s) $$

$$ \sum_l j_l + \sum_{l'} k_{l'} = m - 1, \quad m_1, m_2 \geq 1. $$

(2.49)

Since $P$ and $Q$ are linear combinations of expressions in (2.46), by the first inequality in (2.47) we obtain that the expression in (2.49) is bounded by

$$ C (1 + d(\phi^{-1}(s))^{2-2-2-(m-1)}) = C (1 + d(\phi^{-1}(s))^{1-m}), $$

for $s \in \phi(V_0 \cap (\mathbb{C}^n \setminus D))$ and $m = 1, 2, \ldots$. This proves (2.41).

We now construct the coordinate system $(V, \phi)$ mentioned in the remark after Proposition 2.9. Since $d\rho \neq 0$ on $bD$, by a linear change of coordinates we can assume that $\partial_{\xi^*} \rho \neq 0$ at $\xi^* \in bD$. By Lemma 2.40 we can define a $C^1$ coordinate transformation $\phi_{\xi^*}$ [given by (2.40)] in some ball $B_{\varepsilon}(\xi^*)$ of small radius $\varepsilon > 0$. We can find $\varepsilon_0 > 0$ sufficiently small, such that $\phi_{\xi}$ defined by replacing $\xi^*$ by $z$ in (2.40) is a $C^1$ coordinate transformation in $B_{\varepsilon_0}(z)$, for all $z$ in some neighborhood $\omega_{\xi^*}$ of $\xi^*$. Define $V = \omega_{\xi^*} \cap B_{\varepsilon_0/2}(\xi^*)$. Then $|\xi - z| < \varepsilon_0$ for $z, \xi \in V$, and thus $\phi_{\xi}$ defines a coordinate transformation on $V \subset B_{\varepsilon_0}(z)$.

We end the section with a trivial estimate for the top form $q = n$:

**Proposition 2.11** Let $1 \leq p < \infty$. Let $D$ be a bounded domain in $\mathbb{C}^n$ whose boundary is locally given by graphs of Lipschitz functions. Let $k \geq 0$ be an integer. Suppose that $\varphi$ is a $\overline{\partial}$-closed $(0, n)$-form. Then there exists a linear operator $S$ so that $\overline{\partial}S\varphi = \varphi$ and

$$ \|S\varphi\|_{W^{k+1, p}(D)} \leq C(n, p)\|\varphi\|_{W^{k, p}(D)}. $$

(2.50)

**Proof** Let $B_R(0)$ be some ball centered at $0$ of radius $R$ such that $D \subset B_R(0)$. Extend each component of $\varphi$ to a $W^{k, p}(B_R(0))$ function with compact support in $B_R(0)$. Denote the resulting extended form by $\widetilde{\varphi}$. Since $\widetilde{\varphi}$ is a $(0, n)$-form, it follows $\overline{\partial}\widetilde{\varphi}$ is $\overline{\partial}$-closed. Applying the homotopy formula for $B_R(0)$ (see [19, p. 314]) and Proposition 3.2, we obtain the desired estimate.

**3 Estimates for $H_q$**

We first prove a lemma which will be used in our main estimate.

**Lemma 3.1** Let $0 < \delta < \frac{1}{2}$. 

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(i) We have
\[
\int_0^1 \int_0^1 \frac{s^{1+\alpha} \, dt \, ds}{(\delta + s + t^2)^3} \leq \begin{cases} 
C(\alpha)\delta^{\alpha - \frac{1}{2}} & \text{if } 0 \leq \alpha < \frac{1}{2}, \\
C(\alpha)(1 + |\log \delta|) & \text{if } \alpha = \frac{1}{2}, \\
C(\alpha) & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

(ii) If \(0 < \alpha < 1\), we have
\[
\int_0^1 \int_0^1 \int_0^1 \frac{s_1^{\alpha-1} t^{2n-3} \, ds_1 \, ds_2 \, dt}{(\delta + s_1 + s_2 + t^2)^3(\delta + s_1 + s_2 + t)^{2n-3}} \leq C(n, \alpha)\delta^{-\frac{3}{2} + \alpha}.
\]

Proof (i) Denote the integral by \(I\) and split the domain of integration into three regions.

- **\(R_1\):** \(\delta + s > t > t^2\). We have
  \[
  I \leq \int_{s=0}^1 \int_{t=0}^{\sqrt{\delta + s}} \frac{s^{1+\alpha} \, dt \, ds}{(\delta + s)^3} \leq \int_{s=0}^1 (\delta + s)^{-1+\alpha} \, ds \leq \frac{C}{\alpha}.
  \]

- **\(R_2\):** \(t^2 < \delta + s < t\). We have
  \[
  I \leq \int_{s=0}^1 \int_{t=\sqrt{\delta + s}}^1 \frac{s^{1+\alpha} \, dt \, ds}{(\delta + s)^3} \leq \int_{s=0}^1 (\delta + s)^{-\frac{3}{2} + \alpha} \, ds
  \leq \begin{cases} 
C(\alpha)\delta^{\alpha - \frac{1}{2}} & \text{if } 0 \leq \alpha < \frac{1}{2}, \\
C(\alpha)(1 + |\log \delta|) & \text{if } \alpha = \frac{1}{2}, \\
C(\alpha) & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

- **\(R_3\):** \(\delta + s < t^2 < t\). We have
  \[
  I \leq \int_{s=0}^1 \int_{t=\sqrt{\delta + s}}^1 \frac{s^{1+\alpha} \, dt \, ds}{t^6} \leq \int_{s=0}^1 (\delta + s)^{-\frac{3}{2} + \alpha} \, ds
  \leq \begin{cases} 
C(\alpha)\delta^{\alpha - \frac{1}{2}} & \text{if } 0 \leq \alpha < \frac{1}{2}, \\
C(\alpha)(1 + |\log \delta|) & \text{if } \alpha = \frac{1}{2}, \\
C(\alpha) & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

Put together the estimates we obtain (3.1).

(ii) Denote the integral by \(I\), and split the domain of integration into seven regions.

- **\(R_1\):** \(t > t^2 > \delta, s_1, s_2\). We have
  \[
  I \leq \int_{\sqrt{\delta}}^1 \int_{t=\sqrt{\delta+\delta}}^1 \left( \int_0^{t^2} s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^{t^2} ds_2 \right) \, dt \leq C(\alpha) \int_{\sqrt{\delta}}^1 t^{-4+2\alpha} \, dt \leq C(\alpha)\delta^{-\frac{3}{2} + \alpha}.
  \]

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\( R_2 : t > \delta > t^2, s_1, s_2 \). We have
\[
I \leq \delta^{-\frac{3}{2}} \left( \int_0^\delta \frac{t^{2n-3}}{t^{2n-3}} \, dt \right) \left( \int_0^\delta s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^\delta ds_2 \right) \leq C(\alpha) \delta^{-\frac{3}{2} + \alpha}.
\]

\( R_3 : t > s_1 > \delta, t^2, s_2 \). We have
\[
I \leq \int_0^1 \frac{1}{s_1^{s_1}} \left( \int_0^\delta \frac{t^{2n-3}}{t^{2n-3}} \, dt \right) \left( \int_0^\delta s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^\delta ds_2 \right) \leq C(\alpha) \delta^{-\frac{3}{2} + \alpha}.
\]

\( R_4 : t > s_2 > \delta, t^2, s_1 \). We have
\[
I \leq \delta^{-\frac{3}{2}} \left( \int_0^\delta \frac{t^{2n-3}}{t^{2n-3}} \, dt \right) \left( \int_0^\delta s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^\delta ds_2 \right) \leq C(n, \alpha) \delta^{-1 + \alpha}.
\]

\( R_5 : \delta > t, t^2, s_1, s_2 \). We have
\[
I \leq \delta^{-\frac{3}{2}} \left( \int_0^\delta \frac{t^{2n-3}}{s_2^{2n-3}} \, dt \right) \left( \int_0^\delta s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^\delta ds_2 \right) \leq C(n, \alpha) \delta^{-1 + \alpha}.
\]

\( R_6 : s_1 > \delta, t, t^2, s_2 \). We have
\[
I \leq \delta^{-\frac{3}{2}} \left( \int_0^\delta \frac{t^{2n-3}}{s_1^{2n-3}} \, dt \right) \left( \int_0^\delta s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^\delta ds_2 \right) \leq C(n, \alpha) \delta^{-1 + \alpha}.
\]

\( R_7 : s_2 > \delta, t, t^2, s_1 \). We have
\[
I \leq \delta^{-\frac{3}{2}} \left( \int_0^\delta \frac{t^{2n-3}}{s_2^{2n-3}} \, dt \right) \left( \int_0^\delta s_1^{\alpha-1} \, ds_1 \right) \left( \int_0^\delta ds_2 \right) \leq C(n, \alpha) \delta^{-1 + \alpha}.
\]

Put together the estimates we obtain (3.2).

For \( q \geq 1 \), we can write the solution operator as
\[
H_q \varphi = u_0 + u_1,
\]
where
\[
u_0(z) = \int_U \Omega_{0, \alpha-1}^0(z, \zeta) \wedge E \varphi, \quad u_1(z) = \int_{U \setminus D} \Omega_{0, \alpha-1}^{01}(z, \zeta) \wedge [\overline{\partial}, E] \varphi.
\]

**Proposition 3.2** Let \( 1 < p < \infty \), and let \( U \) be a domain in \( \mathbb{C}^n \), with \( n > 1 \). Let \( u_0 \) be defined as in (3.4). Suppose \( \varphi \in W^{k, p}(U) \), for some nonnegative integer \( k \). Then \( u_0 \in W^{k+1, p}(U) \), and
\[
\|u_0\|_{W^{k+1, p}(U)} \leq C(n, p) \|\varphi\|_{W^{k, p}(U)}.
\]
**Proof** Let $f$ be a coefficient function of $\phi$. Up to a constant, $u_0$ can be written as a finite linear combination of

$$
\int_U \frac{\xi^i - z^i}{|\xi - z|^{2n}} f(\xi) \, dV(\xi) = \frac{1}{n-1} \int_U \partial_{\xi^i} \left( |\xi - z|^{2-2n} \right) f(\xi) \, dV(\xi) = c_0 \partial_{\xi^i} N f(z),
$$

where $N$ denotes the Newtonian potential. Thus we just have to show that

$$
\| Nf \|_{W^{k+2,p}(U)} \leq C(n, p) \| f \|_{W^{k,p}(U)}.
$$

The proof is by Calderón–Zygmund theory. The $k = 0$ case is proved in Theorem 9.9 in [5, p. 230]. Assume $k \geq 1$, we would like to move the derivatives onto $f$. Since $f$ is compactly supported in $U$, we can trivially extend $f$ to a function $\tilde{f}$ in $W^{k+2,p}(\mathbb{C}^n)$. Denoting by $\Gamma_1$ the kernel of the Newtonian potential, we have

$$
D^k_x N f(x) = D^k_x \int_{\mathbb{C}^n} \Gamma(x - y) \tilde{f}(y) \, dy
$$

$$
= D^k_x \int_{\mathbb{C}^n} \Gamma(y) \tilde{f}(x - y) \, dy
$$

$$
= (-1)^k \int_{\mathbb{C}^n} \Gamma(y) D^k_x \tilde{f}(x - y) \, dy
$$

$$
= (-1)^k \int_U \Gamma(x - y) D^k_y f(y) \, dy = (-1)^k N(D^k f).
$$

Thus

$$
\| D^{k+2} N f \|_{L^p(\Omega)} = \| D^2 N(D^k f) \|_{L^p(\Omega)} \leq C(n, p) \| D^k f \|_{L^p(\Omega)}. \quad \Box
$$

For our estimate of $u_1$ given by (3.4), we need a lemma on integration by parts.

**Lemma 3.3** Let $D$ be a bounded domain in $\mathbb{R}^N$ with $C^1$ boundary. Let $\alpha > 0$, and let $j, i_k$ be nonnegative integers. Suppose $f \equiv 0$ on $bD$.

(i) Suppose $f \in C^{j+\alpha}(\overline{D})$. Let $g_1, g_2$ be functions in $C^\infty(D)$ satisfying

$$
|g_k(\xi)| \leq C|d(\xi)|^{-i_k}, \quad |\partial_{\xi^j} g_k(\xi)| \leq C|d(\xi)|^{-i_k - 1}, \quad k = 1, 2 \quad (3.6)
$$

for $\xi \in D$, $d(\xi) = \text{dist}(\xi, bD)$, $1 \leq l \leq n$, and some $i_k \geq 0$. Suppose $i_k, j$ satisfy $j \geq i_1 + i_2$. Then we have

$$
\int_D f \partial_{\xi^j} (g_1 g_2) \, dV(\xi) = \int_D f (\partial_{\xi^j} g_1) g_2 \, dV(\xi) + \int_D g_1 (\partial_{\xi^j} g_2) \, dV(\xi).
$$

(ii) Suppose $f \in W^{1,1}(D) \cap C^{j+\alpha}(\overline{D})$, and let $g_1$ be as in (i) satisfying the estimate (3.6), such that $j \geq i_1$. We have

$$
\int_D f (\partial_{\xi^j} g_1) \, dV(\xi) = - \int_D (\partial_{\xi^j} f) g_1 \, dV(\xi).
$$
(iii) Let \( \rho \) be a \( C^1 \) defining function of \( D \). Suppose \( f \in W^{1,p}(D) \cap C^\alpha(\overline{D}) \), for \( p > 1 \). Let \( \phi(\zeta) = (s_1, \hat{s}) \), \( \hat{s} = (s_2, \ldots, s_{n-2}) \) be a coordinate system in a neighborhood \( V \) of some \( p \in bD \), i.e. \( \phi: V \to \phi(V) \) is a \( C^1 \) diffeomorphism. Define \( \tilde{f}(s) = f(\phi^{-1}(s)) \) for \( s \in \phi(V) \). Suppose \( g \) is a function in \( C^\infty(\phi(D \cap V)) \) satisfying

\[
|g(s)| \leq C(1 + |\log s_1|), \quad |\partial_{s_1} g(s)| \leq C s_1^{-1}
\]

for all \( s_1 < 1 \). Then

\[
\int_{\phi(D \cap V)} \tilde{f}(s) \partial_{s_1} g(s) \, ds = -\int_{\phi(D \cap V)} (\partial_{s_1} \tilde{f}(s)) g(s) \, ds.
\]

**Proof** (i) Let \( D_{-\delta} = \{ z \in D : d(z) > \delta \} \), with \( d(z) = \text{dist}(z, bD) \). Take a sequence of cut-off functions \( \chi_n \in C_0^\infty(D_{-\frac{\delta}{n}}) \) such that \( 0 \leq \chi_n \leq 1 \), \( \chi_n \equiv 1 \) on \( D_{-\frac{\delta}{n}} \) and \( |\nabla \chi_n(\zeta)| \leq C|d(\zeta)|^{-1} \) for \( \zeta \in D \).

It suffices to show that

\[
\int_D f \partial_{\tilde{\zeta}_i}(g_1 g_2)(1 - \chi_n) \, dV(\zeta) \xrightarrow{n \to 0} 0;
\]

\[
\int_D f (\partial_{\tilde{\zeta}_i} g_1)(1 - \chi_n) \, dV(\zeta) \xrightarrow{n \to 0} 0;
\]

\[
\int_D f g_1(\partial_{\tilde{\zeta}_i} g_2)(1 - \chi_n) \, dV(\zeta) \xrightarrow{n \to 0} 0.
\]

Since \( f \) vanishes on \( bD \) and \( f \in C^{j+\alpha}(\overline{D}) \), then \( |f(\zeta)| \leq C d(\zeta)^{j+\alpha} \), for \( \zeta \in D \).

In view of (3.6) and that \( j > i_1 + i_2 \), the integrands in the above expression are bounded above in absolute value by a positive constant times \( |d(\zeta)|^{-1+\alpha} \in L^1(D) \).

Since \( 1 - \chi_n \) converges to 0 pointwise on \( D \), the result follows from the dominated convergence theorem.

(ii) Let \( \chi_n \) be defined as above. It suffices to show that

\[
\int_D f \partial_{\tilde{\zeta}_i}((1 - \chi_n) g_1) \, dV(\zeta) \xrightarrow{n \to 0} 0;
\]

\[
\int_D (\partial_{\tilde{\zeta}_i} f)(1 - \chi_n) g_1 \, dV(\zeta) \xrightarrow{n \to 0} 0.
\]

The first statement follows from the dominated convergence theorem applied to the estimate \( |f \partial_{\tilde{\zeta}_i}((1 - \chi_n) g_1)| \leq C d(\zeta)^{j+\alpha-(i_1+1)} \leq C d(\zeta)^{-1+\alpha} \in L^1(D) \). For the second statement, there are two cases. If \( j \geq 1 \), we have \( |(\partial_{\tilde{\zeta}_i} f)(1 - \chi_n) g_1| \leq C d(\zeta)^{j-1+\alpha-i_1} \leq C d(\zeta)^{-1+\alpha} \in L^1(D) \). If \( j = i_1 = 0 \), then \( |(\partial_{\tilde{\zeta}_i} f)(1 - \chi_n) g_1| \leq C |\partial_{\tilde{\zeta}_i} f| \in L^1(D) \), by the assumption that \( f \in W^{1,1}(D) \).

(iii) We can assume that \( f \) is compactly supported in \( V \), and thus \( \tilde{f} = f(\phi^{-1}(s)) \) is compactly supported in \( \phi(V) \). Let \( \{\chi_n\} \) be defined as above. Define \( \tilde{\chi}_n(s) = \chi_n(\phi^{-1}(s)) \) for \( s \in \phi(V) \). Then \( 1 - \tilde{\chi}_n \equiv 0 \) on \( \phi(D_{-\frac{\delta}{n}}) \cap \phi(V) \). Since \( f \equiv 0 \)
on \( bD \) and \( f \in C^\alpha(D) \), we have \( |\tilde{f}(s)| \leq C s_1^\alpha \) and

\[
\int_{\phi(D \cap V)} |\tilde{f}(s)| \left| \partial_{s_1} [(1 - \tilde{x}_n)g(s)] \right| \, ds \leq C \int_{\phi(D \setminus D_{\frac{1}{2}}) \cap \phi(V)} s_1^{1+\alpha - \varepsilon} \, ds \xrightarrow{n \to 0} 0,
\]

where \( \varepsilon > 0 \) is some arbitrary small number. By Hölder’s inequality, we have

\[
\int_{\phi(D \cap V)} \left| \partial_{s_1} \tilde{f} \right| \left| (1 - \tilde{x}_n) \right| \left| g(s_1) \right| \, ds \leq C \left[ \int_{\phi(D \setminus D_{\frac{1}{2}}) \cap \phi(V)} \left| \partial_{s_1} \tilde{f} \right|^p \, ds \right]^{\frac{1}{p}} \left[ \int_{\phi(D \setminus D_{\frac{1}{2}}) \cap \phi(V)} \left( (1 - \tilde{x}_n) (1 + \left| s_1 \right|) \right)^{p'} \, ds \right]^{\frac{1}{p'}},
\]

which converges to 0 since \( \tilde{f} \in W^{1,p}(\phi(D) \cap \phi(V)) \), for \( p > 1 \).

We are now ready for the proof of our main theorem.

**Theorem 3.4** Let \( D \subset \subset \mathbb{C}^n \) be a bounded strictly pseudoconvex domain with \( C^2 \) boundary. For \( q \geq 1 \), let \( H_q \varphi \) be given by (3.3)–(3.4).

(i) Let \( 1 < p < \infty \). Suppose \( \varphi \in W^{1,p}(D) \). Then \( H_q \varphi \in W^{1,p}_\beta(D) \), for any \( 0 < \beta < \frac{1}{2} \), and

\[
\|H_q \varphi\|_{W^{1,p}_\beta(D)} \leq C(D, p, \beta) \|\varphi\|_{W^{1,p}(D)}.
\]

(ii) Let \( k \geq 2 \), and \( 2n < p < \infty \). Suppose \( \varphi \in W^{k,p}(D) \). Then \( H_q \varphi \in W^{k,p}_\beta(D) \), for any \( 0 < \beta < \frac{1}{2} \), and

\[
\|H_q \varphi\|_{W^{k,p}_\beta(D)} \leq C(D, p, \beta) \|\varphi\|_{W^{k,p}(D)}.
\]

**Proof** (i) We have \( H_q \varphi = u_0 + u_1 \), where \( u_0 \) and \( u_1 \) are given by formula (3.4). By Proposition 3.2, \( u_0 \in W^{k+1,p}(D) \), and the following estimate holds:

\[
\|u_0\|_{W^{k+1,p}(D)} \leq C(n, p) \|E \varphi\|_{W^{k,p}(U)} \leq C(n, p, D) \|\varphi\|_{W^{k,p}(D)},
\]

for any non-negative integer \( k \). So we only need to estimate \( u_1 \). Choose \( U = D_{\delta} \) as in Proposition 2.9. We will estimate

\[
F(z) = \int_D d(z)^\gamma \left| D_z^2 \int_{\Omega^1_{0,q}(z, \xi) \setminus [\partial, E] \varphi(\xi) \, dV(\xi) \right|^p \, dV(z),
\]

where we set \( \gamma = 1 - \beta \). For \( z \in D \), we estimate

\[
\left| D_z^2 \int_{\Omega^1_{0,q}(z, \xi) \setminus [\partial, E] \varphi \, dV(\xi) \right|,
\]
where in the definition of $\Omega^{0,w}$ given by (2.18) we set $W$ to be the regularized Leray map in Proposition 2.9. We can write the above integral as a linear combination of

$$Kf(z) := \int_{U\setminus D} f_1(z, \xi) \frac{\tilde{N}_{1-2l}(\xi - z)}{\Phi^{n-l}(z, \xi)} \, dV(\xi), \quad 1 \leq l \leq n - 1,$$

where

$$f_1(z, \xi) = f(\xi) P_1(W_1(z, \xi), z, \xi), \quad W_1 = (\hat{D}_\xi W, \partial_\xi^{p_0} \hat{D}_\xi W(z, \xi)), \quad p_0 \leq 2,$$

(3.10)

$$\Phi(z, \xi) = W(z, \xi) \cdot (\xi - z), \quad \tilde{N}_{1-2l}(\xi - z) = \frac{N_1(\xi - z)}{|\xi - z|^{2l}}.$$

(3.11)

Here $f$ is a coefficient function of $[\partial, E] \varphi$, and $f \equiv 0$ on $\overline{D}$. $P_1(w)$ denotes a polynomial in $w$ and $\overline{w}$, $\hat{D}_\xi W$ denotes $W$ and its first-order $\xi$ derivatives, and $N_i$ denotes a monomial of degree $i$ in $\xi - z$ and $\overline{\xi} - \overline{z}$. $N_i$ and $P_1$ may differ when they recur.

Let $V$ be a small neighborhood of a fixed boundary point $\xi^* \in bD$, as given in the remarks after Proposition 2.9. By a linear change of coordinates we can assume that $\partial_{\xi^1} \rho(\xi^*) \neq 0$. For $z \in V$, let $\phi_z : V \rightarrow \phi(V)$ be the coordinate transformation given by (2.40). Using a partition of unity in $\xi$ space and replacing $f$ by $\chi f$ for a $C^\infty$ cut-off function $\chi$, we may assume

$$\text{supp}_\xi f \subset V \setminus D.$$ 

Similarly by a partition of unity in $z$ space and replacing $\Omega^{0,1}_{0,q} \Omega^{0,1}_{0,q}$ by $\chi \Omega^{0,1}_{0,q}$ we may assume

$$\text{supp}_z \Omega^{0,1}_{0,q}(z, \xi) \subset V \cap D.$$ 

Since $\varphi \in W^{1,p}(D)$, we have $f \in L^p(U)$. By (2.37), we have

$$|\partial^l_\xi \tilde{N}_{1-2l}(\xi - z)| \leq |\xi - z|^{1-2l-j},$$

(3.12)

$$\partial^l_\xi \Phi^{-(n-l)}(z, \xi) \leq C_j(D, |\rho_0|_{\mathcal{D}_2})|\Phi^{-(n-l)-j}(z, \xi)|.$$  

(3.13)

Write

$$\partial^2_\xi Kf(z) = \int_{U\setminus D} A(z, \xi) f(\xi) \, dV(\xi),$$

where $A(z, \xi)$ is a sum of three kinds of terms:

$$A_1(z, \xi) = \frac{P_1(z, \xi)}{\Phi^{n-l}(z, \xi)} \partial^2_\xi \tilde{N}_{1-2l}(\xi - z),$$

$$A_2(z, \xi) = \frac{P_1(z, \xi)}{\Phi^{n-l+1}(z, \xi)} \partial_\xi \tilde{N}_{1-2l}(\xi - z).$$
\[ A_3(z, \zeta) = \frac{P_1(z, \zeta)}{\Phi_{n-l+2}(z, \zeta)} \left\{ \tilde{N}_{1-2l}(\zeta - z) \right\}, \]

and \( P_1 \) has the same form as (3.10). We have

\[ |\partial_z^2 Kf(z)| \leq \int_{U \setminus D} |A(z, \zeta)|^{\frac{1}{p}} |A(z, \zeta)|^{\frac{1}{p'}} |f(\zeta)| \, dV(\zeta), \]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Applying Hölder’s inequality, we get

\[ |\partial_z^2 Kf(z)|^p \leq \left[ \int_{U \setminus D} |A(z, \zeta)| |f(\zeta)|^p \, dV(\zeta) \right] \left[ \int_{U \setminus D} |A(z, \zeta)| \, dV(\zeta) \right]^{\frac{p}{p'}}. \quad (3.14) \]

By (2.39) and (3.11), \( C'|\zeta - z| \geq |\Phi(z, \zeta)| \geq C|\zeta - z|^2 \). In view of (3.12) and (3.13), we have \( |A_1| \leq C|A_3|, |A_2| \leq C|A_3| \), and it suffices to estimate \( A_3(z, \zeta) \) for \( l = n - 1 \). From now on we just take \( A \) to be

\[ A(z, \zeta) = \frac{P_1(z, \zeta)}{\Phi(z, \zeta)^3} \left\{ \tilde{N}_{-(2n-3)}(\zeta - z) \right\}. \]

By estimate (2.38), for \( z \in V \cap D \) and \( \zeta \in V \setminus D \):

\[ |\Phi(z, \zeta)| \geq c(d(z) + s_1 + |s_2| + |t|^2), \quad |\zeta - z| \geq c|(s_2, t)|, \quad (3.15) \]

where \( (s_1, s_2, t) = (\phi_{1}(\zeta), \phi_{2}^{2}(\zeta), \phi_{3}^{t}(\zeta) = \phi_{3}(\zeta), \ldots, \phi_{2n}^{2n}(\zeta)) \). By (3.15) and integrating by polar coordinates for \( s = (s_1 = \rho, s_2) \in \mathbb{R}^2 \) and \( t = (t_1, \ldots, t_{2n-2}) \in \mathbb{R}^{2n-2} \), we have

\[
\int_{U \setminus D} |A(z, \zeta)| \, dV(\zeta) \leq C_0 \int_{s=0}^{1} \int_{t=0}^{1} \frac{st^{2n-3} \, ds \, dt}{(d(z) + s + t^2)^3 t^{2n-3}} \\
\leq C_0 \int_{s=0}^{1} \int_{t=0}^{1} s \, ds \, dt (d(z) + s + t^2)^3 \\
\leq C_0' d(z)^{-\frac{1}{2}},
\]

where we used Lemma 3.1(i) for the last inequality. The constant \( C_0 \) depends only on \( D \), the defining function \( \rho_0 \) and \( C_0 \) is independent of \( z \in D \). Using this estimate in (3.14) we get

\[
\int_{D} d(z)^{\gamma p} |\partial_z^2 Kf(z)|^p \, dV(z) \\
\leq (C_0)^{\frac{p}{p'}} \int_{D} \int_{U \setminus D} d(z)^{\gamma'} |A(z, \zeta)||f(\zeta)|^p \, dV(\zeta) \, dV(z)
\]
\[ \leq (C_0)^{\gamma_1} \int_{U \setminus D} \left[ \int_D d(z)^\gamma |A(z, \xi)| dV(z) \right] |f(\xi)|^p dV(\xi), \quad (3.16) \]

where

\[ \gamma' = \gamma p - \left( \frac{1}{2} \right) \left( \frac{p}{p'} \right) = \left( \gamma - \frac{1}{2} \right) p + \frac{1}{2}. \quad (3.17) \]

For each \( z \in V \), the \( C^1 \) coordinate transformation \( \phi_z \) is given by (2.40):

\[ \phi_z^1(\xi) = \rho(\xi), \quad \phi_z^2(\xi) = \Im(\rho \cdot (\xi - z)), \quad \phi_z^\gamma(\xi) = (\gamma' - \gamma', \gamma' - z'). \]

where we set \( \gamma' \equiv (\Re(\xi), \Im(\xi), \ldots, \Re(\xi_n), \Im(\xi_n)) \) and similarly for \( z' \). For \( \xi \in V \), we define \( \tilde{\phi}_\xi : V \rightarrow \phi(V) \) to be

\[ \tilde{\phi}^1_\xi(z) = \rho(z), \quad \tilde{\phi}^2_\xi(z) = \Im(\rho \cdot (\xi - z)), \]

\[ \tilde{\phi}^\gamma_\xi(z) = (\gamma' - z'), \quad \gamma' = (\Re(\xi), \Im(\xi), \ldots, \Re(\xi_n), \Im(\xi_n)) \quad (3.18) \]

which is a coordinate system for \( z \in V \). Write \( (\tilde{s}_1, \tilde{s}_2, \tilde{t}) = (\tilde{\phi}^1_\xi(z), \tilde{\phi}^2_\xi(z), \tilde{\phi}^\gamma_\xi(z)) \). By (3.15) we have for \( z \in V \cap D \) and \( \xi \in V \setminus D \),

\[ |\Phi(z, \xi)| \geq c(d(z) + \phi_z^1(\xi) + |\phi_z^2(\xi)| + |\phi_z^\gamma(\xi)|^2) \]

\[ \geq c(d(\xi) + |\tilde{\phi}^1_\xi(z)| + |\tilde{\phi}^2_\xi(z)| + |\tilde{\phi}^\gamma_\xi(z)|^2) \]

\[ = c(d(\xi) + |\tilde{s}_1| + |\tilde{s}_2| + |\tilde{t}|^2), \quad (3.19) \]

and

\[ |\xi - z| \geq c|s_2, t| = c|s_2, \tilde{t}|. \quad (3.20) \]

Writing in polar coordinates and using that \( d(z) \leq C \rho(z) = C|s_\xi| \leq C|\tilde{s}| \), we have by Lemma 3.1(i) again

\[ \int_D d(z)^\gamma |A(z, \xi)| dV(z) \leq C \int_{\tilde{s}}^{1} \int_{\tilde{t}}^{1} \frac{\tilde{s}^{\gamma' - \frac{1}{2}} d\tilde{s} d\tilde{t}}{(d(\xi) + \tilde{s} + \tilde{t})^{\gamma n - \frac{3}{2}}} \]

\[ \leq \begin{cases} C d(\xi)^{\gamma' - \frac{1}{2}} & \text{if } 0 \leq \gamma' < \frac{1}{2}, \\ C(1 + |\log d(\xi)|) & \text{if } \gamma' = \frac{1}{2}, \\ C(\gamma') & \text{if } \gamma' > \frac{1}{2}. \end{cases} \quad (3.21) \]

If \( \gamma > \frac{1}{2} \), then by (3.17), \( \gamma' > \frac{1}{2} \). Using (3.21) in (3.16), we get

\[ \left[ \int_D d(z)^\gamma |A(z, \xi)| dV(z) \right]^{\frac{1}{p}} \leq C'(D, \gamma') \left[ \int_{U \setminus D} |f(\xi)|^p dV(\xi) \right]^{\frac{1}{p}} \]

\[ \leq C'(D, \gamma') \|\varphi\|_{W^{1,p}(D)}. \]
Thus we have shown that
\[ \|u_1\|_{W^{1,p}_\beta(D)} \leq C(D, \beta)\|\varphi\|_{W^{1,p}(D)}. \]

for any \(0 < \beta < \frac{1}{2}\).

(ii) As in (i) we only need to prove the estimate (3.8) for \(u_1\) which is given by
\[ u_1(z) = \int_{U \setminus D} \Omega_{0,q-1}^0(z, \zeta) \wedge [\partial, E]\varphi. \]

Suppose \(\varphi \in W^{k,p}(D)\), for \(k \geq 2\) and \(2n < p < \infty\). We show that \(u_1 \in W^{k,p}_\beta(D)\),
for any \(0 < \beta < \frac{1}{2}\). Let \(f\) be a coefficient function of \([\partial, E]\varphi\). As before take \(U = D_b\)
as in Proposition 2.9. Then \(f \in W^{k-1,p}(U)\). By Proposition 2.1, \(f \in C^{k-2+\alpha}(U)\) for \(n = 1 - \frac{2n}{p}\). Since \(f \equiv 0\) on \(D\), for \(\zeta \in U\) the following holds:
\[ |\partial_{\zeta}^q f(\zeta)| \leq C_q |f|_{U;k-2+\alpha} d(\zeta)^{k-2+\alpha-q}, \quad 0 \leq q \leq k-2, \quad (3.22) \]
where \(d(\zeta) = \text{dist}(\zeta, bD)\). We have
\[ \int_D |\partial_{\zeta}^{k+1} u_1(z)|^p d(z)^{\gamma p} dV(z) \]
\[ = \int_D |\partial_{\zeta}^2 \int_{U \setminus D} \partial_{\zeta}^{k-1} \Omega_{0,q}^0(z, \zeta) \wedge [\partial, E]\varphi(\zeta) dV(\zeta)|^p d(z)^{\gamma p} dV(z). \]

We can write the inner integral above as a linear combination of
\[ K_i f(z) = \int_{U \setminus D} f_1(z, \zeta) \frac{N_{1-\mu_0+\mu_2}(\zeta - z)}{\Phi_{n-\mu_0+\mu_2}(z, \zeta)|\zeta - z|^{2l+2\mu_2}} dV(\zeta), \quad 1 \leq l \leq n-1, \]
\[ f_1(z, \zeta) = f(\zeta) P_1(W(z, \zeta), \zeta, \zeta), \quad W = (\partial_{\zeta} W, \partial_{\zeta}^k \partial_{\zeta} W(z, \zeta)), \]
\[ \mu_0 + \mu_1 + \mu_2 \leq k-1, \quad 1 - \mu_0 + \mu_2 \geq 0, \quad k_0 \leq k-1. \quad (3.23) \]

We apply integration by parts in two stages. In the first stage, we integrate by parts to reduce the exponent of \(\Phi\) in the denominator to \(n - l\), as in Ahern–Schneider [1], Lieb–Range [10], and Gong [6]. See also Michel–Perotti [12] for estimates without using integration by parts for piecewise smooth strictly pseudoconvex domains via Seeley extension.

Let \(V\) be a small neighborhood of a fixed boundary point \(\zeta^* \in bD\) as in (i). Suppose that for \(z \in V \cap D\) and \(\zeta \in V \setminus D\),
\[ u(z, \zeta) := \partial_{\zeta_i} \Phi(z, \zeta) \neq 0, \quad \text{for some } i. \]

By (2.37), for fixed \(z \in D\) the following estimates hold for \(\zeta \in V \setminus D\) if \(bD\) is \(C^2\):
\[ |\partial_{\zeta_i}^q \Phi^{-k}(z, \zeta)| \leq C(D, z) \left(1 + d(\zeta)^{1-q}\right), \quad (3.24) \]
for \( q = 0, 1, 2, \ldots \) and \( k = 1, 2, \ldots \). Up to a constant multiple, we rewrite \( K_1 f \) (3.23) as

\[
K_1 f (z) = \int_{U \setminus D} f(\zeta) h(z, \zeta) \Phi^{-(n-l+\mu_1)}(z, \zeta) \, dV(\zeta)
\]

\[
= \int_{U \setminus D} f(\zeta) u(z, \zeta)^{-1} h(z, \zeta) \partial_{\xi_a} \Phi^{-(n-l+\mu_1-1)}(z, \zeta) \, dV(\zeta) \quad (3.26)
\]

where we set

\[
h(z, \zeta) = P_1(W_1(z, \zeta), z, \zeta) \frac{N_1 - \mu_0 + \mu_2 (\zeta - z)}{|\zeta - z|^{2l+2\mu_2}}. \quad (3.27)
\]

For fixed \( z \in D \) the following holds for \( \zeta \in U \setminus D \),

\[
|\partial_{\xi_a}^q h(z, \zeta)| \leq C(D, z) d(\zeta)^{-q}, \quad q = 0, 1, 2, \ldots 
\]

again by (2.37). Then we get

\[
K_1 f (z) = \int_{U \setminus D} f(\zeta) \partial_{\xi_a} \left[ u^{-1}(z, \zeta) h(z, \zeta) \Phi^{-(n-l+\mu_1-1)}(z, \zeta) \right] \, dV(\zeta)
\]

\[
- \int_{U \setminus D} f(\zeta) \partial_{\xi_a} \left[ u^{-1}(z, \zeta) h(z, \zeta) \Phi^{-(n-l+\mu_1-1)}(z, \zeta) \right] \, dV(\zeta) \quad (3.29)
\]

\[
= - \int_{U \setminus D} \left[ \partial_{\xi_a} f(\zeta) \right] \left[ u^{-1}(z, \zeta) h(z, \zeta) \Phi^{-(n-l+\mu_1-1)}(z, \zeta) \right] \, dV(\zeta)
\]

\[
- \int_{U \setminus D} f(\zeta) \partial_{\xi_a} \left[ u^{-1}(z, \zeta) h(z, \zeta) \right] \Phi^{-(n-l+\mu_1-1)}(z, \zeta) \, dV(\zeta). \quad (3.30)
\]

We now justify the above steps. We have \( f \in W^{1,p}(U) \cap C^{j+q}(U) \), for \( j = k - 2 \). Apply Lemma 3.3 to the domain \( U \setminus D \) with \( f \equiv 0 \) on \( b(U \setminus D) \). By (3.25), (3.28), we know that \( u^{-1}, h \) satisfy the estimates (3.6) with \( i_k = 0 \). In view of (2.38), for fixed \( z \in D \) we have \( |\Phi^{-(n-l+\mu_1)}(z, \zeta)| \leq C(z) \) and

\[
|\partial_{\xi_a} \Phi^{-(n-l+\mu_1-1)}(z, \zeta)| = |Cu(z, \zeta) \Phi^{-(n-l+\mu_1)}(z, \zeta)| \leq C(D, z).
\]

Thus \( \Phi^{-(n-l+\mu_1-1)} \) also satisfies the estimate (3.6) with \( i_k = 0 \). Then the first equality (3.29) follows from Lemma 3.3(i) and the second equality (3.30) follows from Lemma 3.3(ii).

We can repeat this procedure \( \mu_1 (\leq k - 1) \) times. Indeed, suppose we have done \( m \) times, \( 1 \leq m \leq k - 2 \). Then the integral is a linear combination of terms of the form

\[
\int_{U \setminus D} \left( \partial_{\xi_a}^{m_1} f \right) \partial_{\xi_a}^{m_2} \left[ u^{-1}, h \right] \Phi^{-(n-l+\mu_1-m)} \, dV(\zeta) \quad (m_1 + m_2 = m)
\]
\[
\int_{U \setminus D} \left( \partial_{\xi_0}^{m_1} f \right) \partial_{\xi_1}^{m_2} \left\{ u^{-1}, h \right\} u^{-1} \partial_{\xi_0} \Phi^{-(n-l+\mu_1-m-1)} \, dV(\zeta),
\]

where \( \partial_{\xi_0}^{m_1} \{ u^{-1}, h \} \) denotes a linear combination with constant coefficients of the terms

\[
\partial_{\xi_0}^{\lambda_1} (u^{-1}) \partial_{\xi_1}^{\lambda_2} (u^{-1}) \cdots \partial_{\xi_1}^{\lambda_p} (u^{-1}) \partial_{\xi_0}^{\lambda_0} (h),
\]

\( \lambda_i \geq 0, \sum_{i=0}^p \lambda_i = m_2. \) (3.31)

Then \( \partial_{\xi_0}^{m_1} f \in W^{1,p}(U \setminus D) \cap C^{k-2-m_1+\alpha}(U \setminus D). \) Also \( \partial_{\xi_0}^{m_2} \{ u^{-1}, h \} \), \( u^{-1} \) satisfy estimates (3.6) for \( k = m_2 \), and

\[
\left| \Phi_1 - (n-l) \right| \leq C(z). \] Since \( k-2-m_1-m_2 = k-2-m \geq 0 \), the hypothesis of Lemma 3.3(i) and (ii) holds, and we can do the procedure one more time.

From the above argument, we can now write \( K_1 f \) given by (3.26) as a linear combination of

\[
K_2 f(z) = \int_{U \setminus D} \partial_{\xi_0}^{\tau_0} f(\zeta) \partial_{\xi_1}^{\tau_1} \left\{ u^{-1}, h \right\} (\zeta, \zeta) \Phi^{-(n-l)}(z, \zeta) \, dV(\zeta)
\]

\( \tau_0 + \tau_1 = \mu_1, \tau_0 < k-1, \) (3.32)

and

\[
K_2' f(z) = \int_{U \setminus D} \partial_{\xi_0}^{\tau_0} f(\zeta) h'(z, \zeta) u^{-(k-1)} \Phi^{-(n-l)}(z, \zeta) \, dV(\zeta),
\]

(3.33)

where

\[
h'(z, \zeta) = P_1(W_1(z, \zeta), z, \zeta) \frac{N_1(z - \zeta)}{|z - \zeta|^{2l}}, \quad (\mu_0 = \mu_2 = 0).
\]

In the case, all \( k-1 \) derivatives fall onto \( f \), we have the integral \( K_2' f \). Since \( \partial_{\xi_0}^{\tau_0} f \in L^p(U \setminus D) \), this reduces to the earlier \( k = 1 \) case, and we obtain

\[
\int_{U \setminus D} \left| \partial_{\zeta}^2 K_2' f(z, \zeta) \right|^p \, d(z)^{\gamma} \, dV(z) \leq C \| \varphi \|^p_{W^{k,p}(D)}, \quad (3.34)
\]

for any \( \gamma > \frac{1}{2} \).

The above integration by parts suffices to derive the estimates in [10] and [6]. For our estimates, we must go through a second stage of integration by parts for \( K_2 f \) to avoid unnecessary loss in regularity. We integrate by parts with respect to the normal direction, and again we rely on the regularized Leray map.

\( \subseteq \) Springer
In view of (3.31) and (3.27), we can write \( \partial_{z_{0}}^{q} u^{-1}(z, \xi) \) as a linear combination of

\[
\begin{align*}
\hat{\partial}_{z_{0}}^{q} u^{-1}(z, \xi) &= \partial_{z_{0}}^{1} u^{-1} \cdots \partial_{z_{0}}^{p}(u^{-1}) \\
\hat{\partial}_{z_{0}}^{q}(P_{1}(W_{1}, z, \xi)) \hat{\partial}_{z_{0}}^{v_{1}} \left( \frac{N_{1}^{-\mu_{0}+\mu_{2}}(\xi - z)}{|\xi - z|^{2j+2\mu_{2}}} \right) \\
\sum_{j=0}^{p} i_{j} = v_{0}, \quad v_{0} + v_{1} = \tau_{1}.
\end{align*}
\] (3.35)

For \( z \in V \cap D \), let \( \phi_{z} : U_{0} \to H^{+} \) be given by (2.40), where we denote

\[ H^{+} = [0, 1] \times [-1, 1]^{2n-1}. \]

For simplicity we write \( \phi \) and \( \phi^{-1} \) in place of \( \phi_{z} \) and \( \phi_{z}^{-1} \). Define

\[
\begin{align*}
\hat{\partial}_{z_{0}}^{q} f(s) &= \hat{\partial}_{z_{0}}^{q} f(\phi^{-1}(s)), \quad \hat{\Phi}(z, s) = \Phi(z, \phi^{-1}(s)), \\
g^{(z, s)} &= \left( \hat{\partial}_{z_{0}}^{q} u^{-1}, h \right)(z, \phi^{-1}(s)) \left| \det(D\phi^{-1})(s) \right|,
\end{align*}
\] (3.37)

where \( D\phi^{-1} \) denotes the Jacobian of \( \phi^{-1} \). Then \( K_{2} f \) given by (3.32) can be written as a linear combination of

\[
K_{3} f(z) = \int_{H^{+}} \hat{\partial}_{z_{0}}^{q} f(s) \frac{g^{(z, s)}}{\Phi^{n-1}(z, s)} \, dV(s)
\] (3.38)

\[
= \int_{H^{+}} \hat{\partial}_{z_{0}}^{q} f(s) \partial_{s_{1}} I_{1}(z, s) \, dV(s),
\] (3.39)

where

\[
I_{1}(z, s) = \int_{1}^{s_{1}} \frac{g^{(z, (\eta_{1}, \acute{s}))}}{\Phi^{n-1}(z, (\eta_{1}, \acute{s}))} \, d\eta_{1},
\] (3.40)

and \( s = (s_{1}, \acute{s}) \), \( \acute{s} = (s_{2}, t_{3} \ldots, t_{2n}) \). Observe that for a fixed \( z \in D \), by (2.38) the \( \hat{\Phi}(z, s) \) and \( |\phi^{-1}(s) - z| \) are bounded below by a constant depending on \( d(z) \). By definition of \( \phi \) and (2.36), the following holds for \( \zeta \in V \setminus D \):

\[
| \partial_{\zeta}^{q} \phi(\zeta) | \leq C_{q}(1 + d(\zeta)^{1-q}), \quad q = 0, 1, 2, \ldots
\] (3.41)

By estimate (2.41), the following holds for \( s \in H^{+} \):

\[
| \partial_{s}^{q} \phi^{-1}(s) | \leq C_{q}(1 + d(\phi^{-1}(s))^{1-q}) \leq C_{q}'(1 + s_{1}^{1-q}), \quad q = 0, 1, 2, \ldots
\] (3.42)
In particular,

\[ |\partial_\nu^q \det(D\phi^{-1})(s)| \leq C_q s_1^{-q}, \quad q = 0, 1, 2, \ldots, \]  

(3.43)

and thus

\[ |\partial_\nu^q \det(D\phi^{-1})(\eta_1, \hat{s})| \leq C_q \eta_1^{-q}, \quad q = 0, 1, 2, \ldots \]  

(3.44)

Writing \( Q_1(z, \xi) = P_1(W_1(z, \xi), z, \xi) \), we have by estimate \((2.37)\),

\[ |\partial_\nu^q Q_1(z, \xi)| \leq C(D, z)d(\xi)^{-q}, \quad q = 0, 1, 2, \ldots. \]  

(3.45)

By estimates \((3.25)\) and \((3.45)\), we have

\[
\begin{align*}
|\partial_\xi^q Q_1(z, \phi^{-1}(\eta_1, \hat{s}))| + |\partial_\xi^q (u^{-1})(z, \phi^{-1}(\eta_1, \hat{s}))| & \leq C(D) \left[d(\phi^{-1}(\eta_1, \hat{s}))\right]^{-q} \\
& \leq C(D) \left[\phi_1(\phi^{-1}(\eta_1, \hat{s}))\right]^{-q} \\
& = C(D)\eta_1^{-q} 
\end{align*}
\]

(3.46)

for \( q = 0, 1, 2 \ldots \). Applying \((3.44), (3.46)\) to \((3.35), (3.37)\) we get

\[
\begin{align*}
|g^i(z, (\eta_1, \hat{s}))| & \leq C \left[\partial_{\xi h}^i \left\{u^{-1}, h\right\}(z, \phi^{-1}((\eta_1, \hat{s})))\right]\left|\det(D\phi^{-1})((\eta_1, \hat{s}))\right| \\
& \leq C(D, z)\eta_1^{-v_0} |\phi^{-1}((\eta_1, \hat{s}))|^{-1} - 2l - \mu - \mu_2 - v_1 \\
& \leq C(D, z)\eta_1^{-v_0}, 
\end{align*}
\]

(3.47)

where \( v_0 = \sum_{j=0}^p i_j \). In view of \((3.40)\) and \((3.47)\), we have

\[
|I_1(z, s)| \leq C(D, z) \int_{s_1}^{1} \eta_1^{-v_0} d\eta_1 \leq \begin{cases} 
C(D, z)s_1^{-(v_0-1)} & \text{if } v_0 > 1, \\
C(D, z) \log s_1 & \text{if } v_0 = 1, \\
C(D, z) & \text{if } v_0 = 0,
\end{cases}
\]

and

\[
|\partial_{s_1} I_1(z, s)| = \left|g^i(z, (s_1, \hat{s}))\right| / \Phi^{n-l}(z, (s_1, \hat{s})) \leq \begin{cases} 
C(D, z)s_1^{v_0} & \text{if } v_0 > 1, \\
C(D, z)s_1^{-1} & \text{if } v_0 = 1, \\
C(D, z) & \text{if } v_0 = 0.
\end{cases}
\]

If \( v_0 > 1 \), we can apply Lemma 3.3 (ii) to \( K_3 f \) given by \((3.39)\), where \( \partial_{\xi h}^{(v_0)} f(s) \in W^{1, p}(H^+) \cap C^{j+\alpha}(H^+) \) for \( j = k - 2 - \tau_0 \geq 0 \) \((\tau_0 \leq k - 2)\) and \( i_1 = v_0 - 1 \). We
have

\[ j - i = (k - 1) - \tau_0 - \nu_0 \geq (k - 1) - \tau_0 - \tau_1 \geq (k - 1) - \mu_1 \geq 0. \]

Thus we can integrate by parts in (3.39) to get

\[ K_3 f(z) = - \int_{H^+} \partial_{s_1} \rho_{\xi_0} f(s) I_1(z, s) dV(s). \]  \hspace{0.5cm} (3.48)

If \( \nu_0 = 1 \), we can apply Lemma 3.3(iii) for \( \partial_{\xi_0} f(s) \in W^{1, p}(H^+) \cap C^\alpha(H^+) \), and integrate by parts in (3.39) to get (3.48). Finally if \( \nu_0 = 0 \), we can again apply Lemma 3.3(ii) with \( \partial_{\xi_0} f \in C^{k-2-\tau_0+\alpha}(H^+) \) and \( i = 0 \).

We claim that we can integrate by parts in this fashion \( k - 1 - \tau_0 \) times. Suppose we did it for \( m \) times, for \( 1 \leq m \leq (k - 2) - \tau_0 \), and we have

\[ K_3 f(z) = \pm \int_{H^+} \partial_{s_1}^m \rho_{\xi_0} f(s) I_m(z, s) dV(s), \]

where

\[ I_m(z, s) = \int_{1}^{s_1} \int_{1}^{\eta_1} \cdots \int_{1}^{\eta_{m-1}} \frac{g^l(z, (\eta_m), (\hat{s})) [d\eta]^m}{\Phi^{n-l}(z, (\eta_m), (\hat{s}))}, \]

and we denote \([d\eta]^m := d\eta_m \cdots d\eta_1\). Recall (3.42),

\[ |\partial_s^q \phi^{-1}(s)| \leq C_q (1 + s_1^{-q}), \quad s \in \phi(V \setminus D), \quad q = 0, 1, 2, \ldots \]  \hspace{0.5cm} (3.49)

In particular \( |\partial_s^1 \phi^{-1}(s)| \leq C \). By the chain rule, we observe that \( \partial_{s_1}^m \rho_{\xi_0} f(s) = \partial_{s_1}^m \rho_{\xi_0} f(\phi^{-1}(s)) \) is a sum of terms of the form

\[ \left[ \partial_{s_1}^{m_0 + \tau_0} f(\phi^{-1}(s)) \right] \left[ \partial_s^{m_1} \phi^{-1}(s) \right] \cdots \left[ \partial_s^{m_\ell} \phi^{-1}(s) \right] \left[ \partial_s \phi^{-1} \right]^{m_{\ell+1}}, \]

\[ m_0 + m_1 + \cdots + m_\ell \leq m + 1, \quad m_0 \leq m, \quad m_{\ell+1} \leq m. \]

In view of this and (3.49), we have the estimate

\[ \left| \partial_{s_1}^m \rho_{\xi_0} f(s) \right| \leq C \left[ d(\phi^{-1}(s)) \right]^{k-2+\alpha-m-\tau_0} \leq C \phi^1(\phi^{-1}(s))^{k-2+\alpha-m-\tau_0} \leq C s_1^{k-2+\alpha-m-\tau_0}. \]  \hspace{0.5cm} (3.50)
Write $K_3 f$ as

$$K_3 f (z) = \pm \int_{H^+} \left( \frac{\partial_{s_1}^m \partial_{\mathbf{i}_0} \bar{f} (s)}{\partial_{s_1}^m I_{m+1} (z, s)} \right) dV(s) \quad (3.51)$$

where

$$I_{m+1} (z, s) = \int_{\eta_1}^{s_1} \int_{\eta_1}^{\eta_2} \cdots \int_{\eta_m}^{\eta_m} \frac{g^i (\eta, (\eta_{m+1}, \hat{s}))}{\Phi_{n-l} (\eta, (\eta_{m+1}, \hat{s}))} [d\eta]^{m+1}.$$  

We have by (3.47),

$$|I_{m+1} (z, s)| \leq C (D, z) \int_{s_1}^{1} \int_{\eta_1}^{1} \cdots \int_{\eta_m}^{1} (\eta_{m+1})^{-\nu_0} [d\eta]^{m+1}$$

$$\leq \begin{cases} C (D, z) s_1^{-\nu_0 + m}, & \text{if } m + 1 < \nu_0 = \sum \iota_j, \\ C (D, z) (1 + | \log s_1 |), & \text{if } m + 1 = \nu_0, \\ C (D, z), & \text{if } m + 1 > \nu_0, \end{cases}$$

where $\iota_j$ are defined in (3.36). The derivative satisfies:

$$|\partial_{s_1} I_{m+1} (z, s)| = |I_m (z, s)|$$

$$\leq \begin{cases} C (D, z) s_1^{-\nu_0 + m}, & \text{if } m + 1 < \nu_0, \\ C (D, z) s_1^{-1}, & \text{if } m + 1 = \nu_0, \\ C (D, z) (1 + | \log s_1 |), & \text{if } m + 1 > \nu_0. \end{cases}$$

If $m + 1 < \nu_0 = \sum \iota_j$, then $I_{m+1}$ satisfies estimate (3.6) with $i_k$ replaced by $i = \nu_0 - (m + 1)$. We can apply Lemma 3.3(ii) to (3.51) for $\partial_{s_1}^m \partial_{\mathbf{i}_0} \bar{f} (s) \in W^{1,p} (H^+) \cap C^{,j+\alpha} (H^+)$, with $j = k - 2 - m - \tau_0 \geq 0$. We have

$$j - i = (k - 1) - \tau_0 - \nu_0 \geq (k - 1) - \tau_0 - \tau_1 \geq (k - 1) - \mu_1 \geq 0.$$  

Thus we can integrate by parts in (3.51) to get

$$K_3 f (z) = \pm \int_{H^+} \left( \frac{\partial_{s_1}^{m+1} \partial_{\mathbf{i}_0} \bar{f} (s)}{\partial_{s_1}^{m+1}} I_{m+1} (z, s) dV(s). \right. \quad (3.52)$$

If $m + 1 = \nu_0 = \sum \iota_j$, then $I_{m+1}$ satisfies estimate (3.7), and we can apply Lemma 3.3 (iii) to obtain (3.52). If $m + 1 > \nu_0 = \sum \iota_j$, then $I_{m+1}$ satisfies estimate (3.6) with $i_k$ replaced by $i = 0$, and we again apply Lemma 3.3 (ii) to obtain (3.52). In conclusion,
we can transform $K_3f$ given by (3.38) via integration by parts to the form

$$\int_{H^+} F(s) I_{k-1-\tau_0}(z, s) \, dV(s),$$  \hspace{1cm} (3.53)

where

$$F(s) = \partial_s^{k-1-\tau_0} \tilde{\Phi}^{-\tau_0}_0 f(s) \in L^p(H^+),$$

and

$$I_{k-1-\tau_0}(z, s) = \int_1^{s_1} \int_1^{s_1} \cdots \int_1^{s_1} \frac{g'(z, (\eta_{k-1-\tau_0}, \hat{s}))}{\Phi^{n-1}(z, (\eta_{k-1-\tau_0}, \hat{s}))} \, d\eta, \quad \text{with } d\eta = d\eta_{k-1-\tau_0} \cdots d\eta_1.$$  

At this point, we can no longer apply integration by parts. Taking two more $z$ derivatives directly on the integral (3.53), we see that $\partial_z^2 F(s)$ is a sum of three terms:

$$\int_{H^+} F(s) \left[ \int_1^{s_1} \int_1^{s_1} \cdots \int_1^{s_1} \frac{\partial_z^2 g'(z, (\eta_{k-1-\tau_0}, \hat{s}))}{\Phi^{n-1}(z, (\eta_{k-1-\tau_0}, \hat{s}))} \, d\eta \right] \, dV(s),$$

$$\int_{H^+} F(s) \left[ \int_1^{s_1} \int_1^{s_1} \cdots \int_1^{s_1} \frac{\psi_1 \partial_z g'(z, (\eta_{k-1-\tau_0}, \hat{s}))}{\Phi^{n-1+1}(z, (\eta_{k-1-\tau_0}, \hat{s}))} \, d\eta \right] \, dV(s),$$

$$\int_{H^+} F(s) \left[ \int_1^{s_1} \int_1^{s_1} \cdots \int_1^{s_1} \frac{\psi_2 g'(z, (\eta_{k-1-\tau_0}, \hat{s}))}{\Phi^{n-1+2}(z, (\eta_{k-1-\tau_0}, \hat{s}))} \, d\eta \right] \, dV(s),$$

where $\psi_1$ is a multiple of $\partial_z \tilde{\Phi}(z, s)$ and $\psi_2(z, s)$ is a linear combination of $(\partial^2_z \tilde{\Phi}(z, s))^2$ and $\partial_z^2 \tilde{\Phi}(z, s)\tilde{\Phi}(z, s)$. Since $\tilde{\Phi}(z, s) = \Phi(z, \phi^{-1}_z(s))$, and $\Phi$, $\phi$ [given by (2.40)] are holomorphic in $z \in V$, we see that $\psi_1$ and $\psi_2$ are smooth functions in $z \in V$. In view of expressions (3.35), (3.37), and estimates (2.37), (3.44), we obtain

$$|\partial_z^q g'(z, s)| \leq C s_1^{-v_0} |\phi^{-1}(s) - z|^{1-2l-q-\mu_0-\mu_2-v_1}, \quad q = 0, 1, 2.$$  \hspace{1cm} (3.55)
Since \( \tilde{\Phi}(z, s) = \Phi(z, \phi^{-1}(s)) \), by (2.39) and (3.11), we have
\[
C|\phi^{-1}(s) - z| \geq |\tilde{\Phi}(z, s)| \geq c \left| \phi^{-1}(s) - z \right|^2. \tag{3.56}
\]
Replacing \( s \) in (3.55) and (3.56) by \((\eta_{k-1-\tau_0}, \hat{s})\), we see that in order to estimate \( \partial_z^2 K_3 f \) it suffices to estimate the integral in (3.54) for \( l = n - 1 \), i.e.,
\[
I(z) = \int_{H^+} F(s) J_{k-1-\tau_0}(z, s) \, dV(s) \tag{3.57}
\]
where \( J_{k-1-\tau_0}(z, s) \) is
\[
\int_{k-1-\tau_0 \text{ integrals}} \int_{1}^{\eta_{k-1-\tau_0}} \int_{1}^{\eta_{k-2-\tau_0}} \cdots \int_{1}^{\eta_{k-2-\tau_0}} \left( \frac{\Phi^{-1}(\eta_{k-1-\tau_0}, \hat{s}) - z}{|\phi^{-1}(\eta_{k-1-\tau_0}, \hat{s}) - z|^{(2n-3)+\mu_0+\mu_2+\nu_1}} \right) \, d\eta.
\tag{3.58}
\]
We now estimate each part of the integrand. Since
\[
\eta_{k-1-\tau_0} \geq \eta_{k-2-\tau_0} \geq \cdots \geq \eta_1, \tag{3.59}
\]
we have, by writing \( \phi = (\phi_1, \phi = (\phi_2, \ldots, \phi_{2n})) \),
\[
\left| \phi^{-1}(\eta_{k-1-\tau_0}, \hat{s}) - z \right|
\geq c \left\{ |\eta_{k-1-\tau_0} - \phi_1(z)| + |\hat{s} - \hat{\phi}(z)| \right\}
\geq c \left\{ |\eta_{k-1-\tau_0} + |\phi_1(z)| + |\hat{s} - \phi(z)| \right\}, \quad \eta_{k-1-\tau_0} > 0, \quad \phi_1(z) = \rho(z) < 0
\geq c \left\{ s_1 + |\phi_1(z)| + |\hat{s} - \phi(z)| \right\}, \quad \eta_{k-1-\tau_0} > 0, \quad \phi_1(z) = \rho(z) < 0
\geq c \left\{ s_1 + |\phi_1(z)| + |\hat{s} - \phi(z)| \right\}, \quad \eta_{k-1-\tau_0} > 0, \quad \phi_1(z) = \rho(z) < 0
\geq c|s - \phi(z)|, \quad s = (s_1, \hat{s}) = (s_1, s_2, t_3, \ldots, t_{2n}). \tag{3.60}
\]
By (2.38) and writing \( \phi = (\phi_1, \phi_2, \phi' = (\phi_3, \ldots, \phi_{2n})) \), we have,
\[
\left| \Phi(z, \eta_{k-1-\tau_0}, \hat{s}) \right|
= \left| \Phi \left( z, \phi^{-1}(\eta_{k-1-\tau_0}, \hat{s}) \right) \right|
\geq c \left( d(z) + |\phi_1(\phi^{-1}(\eta_{k-1-\tau_0}, \hat{s}))| + |\phi_2(\phi^{-1}(\eta_{k-1-\tau_0}, \hat{s}))| \right)
\geq c \left( d(z) + \eta_{k-1-\tau_0} + |s_2| + |t|^2 \right)
\geq c \left( d(z) + s_1 + |s_2| + |t|^2 \right), \tag{3.61}
\]
where \( c \) is independent of \( z \in V \cap D \). Since

\[
|\phi^{-1}((\eta_{k-1-\tau_0}, \tilde{s})) - z| \geq c \eta_{k-1-\tau_0},
\]

we have

\[
|\phi^{-1}(s) - z|^{-\mu_0 - \mu_2 - \nu_1} \leq C (\eta_{k-1-\tau_0})^{-\mu_0 - \mu_2 - \nu_1}.
\]

Using (3.60) and (3.61) we can estimate the integral \( J_{k-1-\tau_0}(z, s) \) given by (3.58) by pulling out \( |\phi^{-1}((\eta_{k-1-\tau_0}, \tilde{s})) - z|^{-(2n-3)} \) and \( \tilde{\Phi}^{-3}(z, (\eta_{k-1-\tau_0}, \tilde{s})) \) from the integral sign. In view of (3.62) and \( \nu_0 + \nu_1 + \mu_0 + \mu_2 = k - 1 - \tau_0 \), we obtain for \( F(s) \in L^p(H^+), \)

\[
|I(z)| \leq C(D) \int_{H^+} \frac{|F(s)||s - \phi(z)|^{-(2n-3)}}{(d(z) + s_1 + |s_2| + |t|^2)^3} \times \left[ \int_{s_1}^1 \int_{\eta_1}^1 \cdots \int_{\eta_{k-2-\tau_0}}^{1} \frac{d\eta}{(\eta_{k-1-\tau_0})^{k-1-\tau_0}} \right] dV(s),
\]

\[
\leq C(D) \int_{H^+} |F(s)|A(z, s)(1 + |\log s_1|) dV(s),
\]

where we denote

\[
A(z, s) = \frac{|s - \phi(z)|^{-(2n-3)}}{(d(z) + s_1 + |s_2| + |t|^2)^3}, \quad s = (s_1, s_2, t).
\]

By the second inequality in (2.39), we have

\[
|s - \phi(z)| \geq c|\phi^{-1}(s) - z|
\]

\[
\geq c[d(z) + s_1 + |\phi_2(\phi^{-1}(s))| + |\phi'(\phi^{-1}(s))|]
\]

\[
\geq c[d(z) + s_1 + |s_2| + |t|].
\]

(3.63)

We estimate

\[
|I(z)| \leq C(D) \int_{H^+} |F(s)| \left[ A(z, s) \right]^{\frac{1}{p}} \left[ A(z, s) \right]^{\frac{1}{p'}} (1 + |\log s_1|) dV(s).
\]

By Hölder’s inequality,

\[
|I(z)|^p \leq \left[ \int_{H^+} A(z, s)|F(s)|^p ds \right] \left[ \int_{H^+} A(z, s)(1 + |\log s_1|)^{p'} ds \right]^{\frac{p}{p'}}.
\]
Using polar coordinates for \( t = (t_1, \ldots, t_{2n-2}) \), and (3.63), we have

\[
\int_{H^+} A(z, s)(1 + |\log s_1|)^{p'} \, ds \\
\leq \int_0^1 \int_0^1 \int_0^1 \frac{(1 + |\log s_1|)^{p'} t^{2n-3}}{(d(z) + s_1 + s_2 + t^2)^3(d(z) + s_1 + s_2 + t)^{2n-3}} \, ds_1 \, ds_2 \, dt \\
\leq C(\varepsilon) \int_0^1 \int_0^1 \int_0^1 \frac{s_1^{-\varepsilon p'} t^{2n-3}}{(d(z) + s_1 + s_2 + t^2)^3(d(z) + s_1 + s_2 + t)^{2n-3}} \, ds_1 \, ds_2 \, dt \\
\leq C'(\varepsilon) d(z)^{-\frac{1}{2} - \varepsilon p'},
\]

where for the last inequality we used Lemma 3.1(ii) with \( \alpha = 1 - \varepsilon p' \). Summarizing the results we have

\[
\int_D \left| \frac{d^2}{dz^2} K_{3f} \right|^p \, dV(z) \leq C(D) \int_D |I(z)|^p \, dV(z) \\
\leq C(D) [C(\varepsilon)]^{\frac{p'}{p}} \int_D \int_{H^+} d(z)^{\gamma'} A(z, s) |F(s)|^p \, dV(s) \, dV(z) \\
= C(D) [C(\varepsilon)]^{\frac{p'}{p}} \int_{H^+} \left[ \int_D d(z)^{\gamma'} A(z, s) \, dV(z) \right] |F(s)|^p \, dV(s), \quad (3.64)
\]

where

\[
\gamma' = \gamma p - \left( \frac{1}{2} \right) \left( \frac{p}{p'} \right) - \varepsilon p = \left( \gamma - \frac{1}{2} - \varepsilon \right) p + \frac{1}{2}.
\]

Pick \( \gamma \) and \( \varepsilon \) with \( \gamma > \frac{1}{2} + \varepsilon \), then \( \gamma' > \frac{1}{2} \). For each \( s \in H^+ \), let \( \tilde{\phi}_{\phi^{-1}(s)} : V \to \mathbb{R}^{2n} \) be the coordinate map given by (3.18):

\[
\tilde{\phi}_{\phi^{-1}(s)}^1(z) = \rho(z), \quad \tilde{\phi}_{\phi^{-1}(s)}^2(z) = Im(\rho_{\phi^{-1}(s)} \cdot (\phi^{-1}(s) - z)), \\
\tilde{\phi}_{\phi^{-1}(s)}^3(z) = (|\phi^{-1}(s)|' - z')
\]

where \([\phi^{-1}(s)]' = ([\phi^{-1}(s)]_3, \ldots, [\phi^{-1}(s)]_{2n})\) the vector of the last \( 2n - 2 \) components, and similarly \( z' = (Re(z_2), Im(z_2), \ldots, Re(z_n), Im(z_n)) \). Write \( \tilde{\phi}_{\phi^{-1}(s)}(z) = (\tilde{s}_1, \tilde{s}_2, \tilde{t}) \). Using polar coordinates for \((\tilde{s}_1, \tilde{s}_2) \in \mathbb{R}^2, \tilde{t} \in \mathbb{R}^{2n-2}, \) and \( cd(z) \leq |\tilde{s}_1| \leq C d(z) \), we get for \( \gamma' > \frac{1}{2} \),

\[
\int_D d(z)^{\gamma'} A(z, s) \, dV(z) \\
\leq C(D) \int_{\tilde{t} \in [-1, 1]^{2n-2}} \int_{\tilde{s}_2 = -1}^{0} \int_{\tilde{s}_1 = -1}^{1} \frac{|\tilde{s}_1|^{\gamma'} d\tilde{s}_1 d\tilde{s}_2 d\tilde{t}}{(d(\xi) + |\tilde{s}_1| + |\tilde{s}_2| + \tilde{t}^2)^{n+1}} \\
\leq C(D) \int_0^1 \int_0^1 \frac{1}{(d(\xi) + \tilde{s} + \tilde{t}^2)^{n+1}} d\tilde{s} d\tilde{t} \leq C(D, \gamma'),
\]
where for the last inequality we used Lemma 3.1(i). It follows from (3.64) that for any $\gamma > \frac{1}{2}$,

$$
\int_{D} \left| \frac{\partial^{2}}{\partial z} K z f \right|^{p} d(z)^{\gamma} \ dV(z) \leq C(D, \gamma) \left[ \int_{U\backslash D} |F(s)|^{p} \ dV(s) \right]^{{\frac{1}{p}}} \\
\leq C(D, \gamma) \| \varphi \|_{W^{k, p}(D)},
$$

Thus we have shown that $u_1$ given in (3.4) satisfies the estimate:

$$
\| u_1 \|_{W^{k, p}(D)} \leq C(D, \beta) \| \varphi \|_{W^{k, p}(D)}, \quad \text{for any } 0 < \beta < \frac{1}{2}.
$$

\[ \Box \]

### 4 Estimates for $H_0$

**Lemma 4.1** (i) Let $0 < \delta < 1$, and $n \geq 2$. Then

$$
\int_{0}^{1} \int_{0}^{1} \frac{st^{2n-3}}{(\delta + s + t^{2})^{n+1}} \ ds \ dt \leq C(n)(1 + | \log \delta |).
$$

(ii) Let $\alpha > 0$, $0 < \delta < 1$, and $n \geq 2$. Then

$$
\int_{0}^{1} \int_{0}^{1} \frac{s^{1+\alpha}t^{2n-3}}{(\delta + s + t^{2})^{n+1}} \ ds \ dt \leq C(n, \alpha).
$$

(iii) Let $0 < \alpha < 1$. Then

$$
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{s^{1-\alpha}t^{2n-3}}{(\delta + s_1 + s_2 + t^{2})^{n+1}} \ ds_1 ds_2 dt \leq C(n, \alpha)\delta^{-1+\alpha}.
$$

**Proof** (i) Denote the integral by $I$ and split the domain of integration $[0, 1] \times [0, 1]$ into six regions: $R_1 : \delta \leq t^{2} \leq s$. We have

$$
I \leq \int_{\delta}^{1} \int_{t=0}^{\sqrt{s}} \frac{st^{2n-3}}{s^{n+1}} \ dt \ ds \leq C(n) \int_{\delta}^{1} s^{-1} \ ds \leq C(n)(1 + | \log \delta |).
$$

$R_2 : t^{2} \leq \delta \leq s$. We have

$$
I \leq \int_{0}^{\sqrt{\delta}} \int_{s=\delta}^{1} \frac{st^{2n-3}}{s^{n+1}} \ ds \ dt \leq \int_{0}^{\sqrt{\delta}} \delta^{-n+1}t^{2n-3} \ dt \leq C.
$$
$R_3 : \delta \leq s \leq t^2$. We have

$$I \leq \int_{t=\sqrt{\delta}}^{\sqrt{\delta}} \int_{s=0}^{t^2} \frac{s t^{2n-3}}{t^{2n+2}} \, ds \, dt \leq C \int_{t=\sqrt{\delta}}^{\sqrt{\delta}} t^{-1} \, dt \leq C (1 + |\log \delta|).$$

$R_4 : s \leq \delta \leq t^2$. We have

$$I \leq \int_{s=0}^{\delta} \int_{t=0}^{1} \frac{s t^{2n-3}}{t^{2n+2}} \, dt \, ds \leq C \int_{s=0}^{\delta} \delta^{-2} s \, ds \leq C.$$

$R_5 : t^2 \leq s \leq \delta$. We have

$$I \leq \int_{s=0}^{\delta} \int_{t=\sqrt{\delta}}^{1} \frac{s t^{2n-3}}{\delta^{n+1}} \, dt \, ds \leq C (n) \int_{s=0}^{\delta} \delta^{-(n+1)} s^n \, ds \leq C (n).$$

$R_6 : s \leq t^2 \leq \delta$. We have

$$I \leq \int_{s=0}^{\sqrt{\delta}} \int_{t=0}^{\sqrt{\delta}} \frac{s t^{2n-3}}{\delta^{n+1}} \, ds \, dt \leq C \int_{s=0}^{\sqrt{\delta}} \delta^{-(n+1)} t^{2n+1} \, dt \leq C (n).$$

(ii) Split the domain of integration $[0, 1] \times [0, 1]$ into six regions.

$R_1 : \delta \leq t^2 \leq s$. We have

$$I \leq \int_{t=0}^{1} \int_{s=\delta}^{\sqrt{\delta}} \frac{s^{1+\alpha} t^{2n-3}}{s^{n+1}} \, ds \, dt \leq C (n) \int_{s=\delta}^{\sqrt{\delta}} s^{\alpha-1} \, ds \leq C (n, \alpha).$$

$R_2 : \delta \leq s \leq t^2$. We have

$$I \leq \int_{t=0}^{1} \int_{s=\delta}^{\sqrt{\delta}} \frac{s^{1+\alpha} t^{2n-3}}{s^{n+1}} \, ds \, dt \leq C \int_{t=0}^{1} t^{2\alpha-1} \, dt \leq C (\alpha).$$

$R_3 : t^2 \leq \delta \leq s$. We have

$$I \leq \int_{s=\delta}^{\sqrt{\delta}} \int_{t=0}^{1} \frac{s^{1+\alpha} t^{2n-3}}{s^{n+1}} \, ds \, dt \leq \int_{s=\delta}^{\sqrt{\delta}} \delta^{-n+1+\alpha} t^{2n-3} \, dt \leq C (n).$$

$R_4 : s \leq \delta \leq t^2$. We have

$$I \leq \int_{t=0}^{1} \int_{s=\delta}^{\sqrt{\delta}} \frac{s^{1+\alpha} t^{2n-3}}{s^{n+1}} \, ds \, dt \leq C \int_{t=0}^{\delta} \delta^{-2} s^{1+\alpha} \, ds \leq C (n) \delta^{\alpha}.$$

$R_5 : t^2 \leq s \leq \delta$. We have

$$I \leq \int_{s=0}^{\delta} \int_{t=0}^{\sqrt{\delta}} \frac{s^{1+\alpha} t^{2n-3}}{s^{n+1}} \, ds \, dt \leq C (n) \int_{s=0}^{\delta} \delta^{-(n+1)} s^{\alpha} \, ds \leq C (n) \delta^{\alpha}.$$
\( R_6 : s \leq t^2 \leq \delta \). We have
\[
I \leq \int_0^{\sqrt{\delta}} \int_{s=0}^{t^2} \frac{s^{1+\alpha} t^{2n-3}}{\delta^{n+1}} ds \, dt \leq C \int_0^{\sqrt{\delta}} \delta^{-(n+1)} t^{2n+2\alpha+1} dt \leq C(n)\delta^{\alpha}.
\]

(iii) Divide the domain of integration \([0, 1] \times [0, 1]\) into four regions:

\( R_1 : t^2 > \delta, s_1, s_2 \). We have
\[
I \leq \int_0^{1} \left( \int_0^{\delta} s_1^{n-1+\alpha} ds_1 \right) \left( \int_0^{\delta} ds_2 \right) dt \\
\leq \int_0^{1} t^{2n+2\alpha} dt \leq C(\alpha)\delta^{1+\alpha}.
\]

\( R_2 : \delta > t^2, s_1, s_2 \). We have
\[
I \leq \delta^{-(n+1)} \int_0^{\delta} t^{2n-3} \left( \int_0^{\delta} s_1^{-1+\alpha} ds_1 \right) \left( \int_0^{\delta} ds_2 \right) dt \\
\leq C(n, \alpha)\delta^{-(n+1)\alpha} \leq C(n, \alpha)\delta^{1+\alpha}.
\]

\( R_3 : s_1 > \delta, t^2, s_2 \). We have
\[
I \leq \int_0^{\delta} \frac{s_1^{-1+\alpha}}{s_1^{n+1}} \left( \int_0^{\delta} t^{2n-3} dt \right) \left( \int_0^{s_1} ds_2 \right) ds_1 \\
\leq C \int_0^{\delta} s_1^{\alpha-2} \leq C\delta^{1+\alpha}.
\]

\( R_4 : s_2 > \delta, t^2, s_1 \). We have
\[
I \leq \int_0^{\delta} s_2^{-(n+1)} \left( \int_0^{\delta} t^{2n-3} dt \right) \left( \int_0^{s_2} s_1^{1+\alpha} ds_1 \right) ds_2 \\
\leq C(\alpha) \int_0^{\delta} s_2^{\alpha-2} ds_2 \leq C(\alpha)\delta^{1+\alpha}.
\]

We now prove the estimate for the holomorphic projection operator \( H_0 \). In this case, we have a loss which is arbitrarily small in the exponent of the weight.

**Theorem 4.2** Let \( D \subset C^n \) be a bounded strictly pseudoconvex domain with \( C^2 \) boundary. Let \( H_0 \varphi \) be defined by formula (2.15).

(i) For any \( 1 < p < \infty \), we have
\[
\| H_0 \varphi \|_{W^{0,p}(D)} \leq C(D, \beta) \| \varphi \|_{W^{1,p}(D)}, \quad \text{for any } \beta, \ 0 < \beta < 1.
\]
(ii) Suppose \(2n < p < \infty\), and \(k \geq 2\) is an integer. We have

\[
\|H_0 \varphi\|_{W_{\beta}^{k-1,p}(D)} \leq C(D, \beta)\|\varphi\|_{W_{\beta}^{k,p}(D)}, \quad \text{for any } \beta, \ 0 < \beta < 1.
\]

**Proof** (i) In view of (2.17), \(H_0 \varphi\) can be written as a linear combination of

\[
K f(z) = \int_{U \setminus D} f(\xi) \frac{\hat{\partial}_z W(z, \xi)}{\Phi^n(z, \xi)} \, dV(\xi), \quad \Phi(z, \xi) = W \cdot (\xi - z),
\]

where \(f\) denotes a coefficient function of \([\partial, E]\) \(\varphi\). Thus \(f \in L^p(U \setminus D)\), and \(f \equiv 0\) in \(D\). Let \(\hat{\partial}_z W\) denote the products of \(W\) and its first derivatives in \(\xi\). Let \(W_1 = (\hat{\partial}_z W, \hat{\partial}_z^{\nu_0} \hat{\partial}_z W)\).

Let \(V\) be a neighborhood of \(\xi_0 \in bD\), as given by the remark after Proposition 2.9. Using a partition of unity in \(\xi\) and \(z\) space, we can assume

\[
supp_\xi f \subset V \setminus D, \quad supp_\xi \Omega_{0,q}^1(z, \xi) \subset V \cap D.
\]

We have

\[
\int_D d(z)^n |\partial_z K f(z) \varphi(z)|^p \, dV(z)
\]

\[
= \int_D d(z)^n \left| \int_{U \setminus D} f(\xi) k(z, \xi) dV(\xi) \right|^p \, dV(z)
\]

\[
\leq \int_D d(z)^n \left| \int_{U \setminus D} |f(\xi)||k(z, \xi)| \frac{1}{p} |k(z, \xi)| \frac{1}{p} dV(\xi) \right|^p \, dV(z)
\]

\[
\leq \int_D d(z)^n \left[ \int_{U \setminus D} |f(\xi)||k(z, \xi)| dV(\xi) \right] \left[ \int_{U \setminus D} |k(z, \xi)| dV(\xi) \right] \frac{1}{p} \, dV(z),
\]

(4.1)

where we set

\[
k(z, \xi) = \partial_z \left( \frac{\hat{\partial}_z W(z, \xi)}{\Phi^n(z, \xi)} \right) = \frac{\partial_z \hat{\partial}_z W(z, \xi)}{\Phi^n(z, \xi)} - n \frac{\hat{\partial}_z W(z, \xi) \partial_z \Phi(z, \xi)}{\Phi^{n+1}(z, \xi)}.
\]

For fixed \(z \in V \cap D\), define the coordinate map \(\phi_z : V \rightarrow \phi(V)\) as in (2.40). Write \(\phi_z(\xi) = (s_1, s_2, t)\). Then from (2.37) and (3.15), we have

\[
|k(z, \xi)| \leq \frac{C(D)}{(d(z) + s_1 + |s_2| + |t|^2)^{n+1}}.
\]

(4.2)
Integrating using polar coordinates for $s = (s_1, s_2) \in \mathbb{R}^2$ and $t = (t_1, \ldots, t_{2n-2}) \in \mathbb{R}^{2n-2}$, we have by Lemma 4.1(i),

$$
\int_{U \setminus D} |k(z, \zeta)| \, dV(\zeta) \leq C(D) \int_{t \in [-1,1]^{2n-2}} \int_0^1 \frac{ds_1 ds_2 dt}{(d(z) + s_1 + |s_2| + t^2)^{n+1}} 
$$

$$
\leq C(D) \int_0^1 \frac{\int_{t \in [-1,1]^{2n-2}} s t^{2n-3} \, dt}{(d(z) + s + t^2)^{n+1}} \, ds dt 
\leq C(D)(1 + |\log d(z)|) 
\leq C(D, \varepsilon) d(z)^{-\varepsilon},
$$

for any $\varepsilon > 0$. Substituting the above estimate into the last line of (4.1) we get

$$
\int_D d(z)^{\gamma'} |\partial_2 H_0 \varphi(z)|^p \, dV(z) 
\leq [C(D, \varepsilon)]^{\frac{p}{p'}} \int_D d(z)^{\gamma'} \left[ \int_{U \setminus D} |f(\zeta)|^p |k(z, \zeta)| \, dV(\zeta) \right] \, dV(z) 
= [C(D, \varepsilon)]^{\frac{p}{p'}} \int_{U \setminus D} |f(\zeta)|^p \left[ \int_D (z)^{\gamma'} |k(z, \zeta)| \, dV(z) \right] \, dV(\zeta),
$$

where we set

$$
\gamma' = \gamma p - \varepsilon \frac{p'}{p'} = p(\gamma - \varepsilon) + \varepsilon. \quad (4.3)
$$

Choose $\gamma$ and $\varepsilon$ with $\gamma > \varepsilon > 0$ so that $\gamma' > 0$. For $\zeta \in V \setminus D$, let $\tilde{\zeta}$ be the coordinate map given by (3.18). Writing $\tilde{\varphi}_\zeta(z) = (\tilde{s}_1, \tilde{s}_2, \tilde{t})$, and using polar coordinates for $\tilde{s} = (\tilde{s}_1, \tilde{s}_2) \in \mathbb{R}^2$, $\tilde{t} \in \mathbb{R}^{2n-2}$, and $cd(z) \leq \tilde{s}_1 \leq Cd(z)$, we get for $\gamma' > 0$,

$$
\int_D d(z)^{\gamma'} |k(z, \zeta)| \, dV(z) 
\leq C(D) \int_{\tilde{t} \in [-1,1]^{2n-2}} \int_{\tilde{s}_2 = -1}^{\tilde{s}_2 = 1} \int_{\tilde{s}_1 = 0}^{\tilde{s}_1 = 1} \frac{\tilde{s}_1^{\gamma'} d\tilde{s}_1 d\tilde{s}_2 d\tilde{t}}{(d(\zeta) + \tilde{s}_1 + |\tilde{s}_2| + \tilde{t})^{n+1}} 
\leq C(D) \int_0^1 \frac{\int_{\tilde{t} \in [-1,1]^{2n-2}} \tilde{s}_1^{1+\gamma'/2n-3} d\tilde{s} d\tilde{t}}{(d(\zeta) + \tilde{s} + |\tilde{t}|^2)^{n+1}} \leq C(D, \gamma') \quad (4.4)
$$

where in the last inequality we used Lemma 4.1(ii). Consequently

$$
\left[ \int_D d(z)^{\gamma'} |\partial_2 H_0 \varphi(z)|^p \, dV(z) \right]^{\frac{1}{p'}} \leq C(D, \gamma) \left[ \int_{U \setminus D} |f(\zeta)|^p \, dV(\zeta) \right]^{\frac{1}{p}},
$$

i.e.

$$
\|H_0 \varphi\|_{W_0^{0,p}(D)} \leq C(D, \beta) \|\varphi\|_{W^{1,p}(D)},
$$

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for any $\beta$, $0 < \beta < 1$.

(ii) Assume

$$\text{supp}_\xi f \subset V \setminus D, \quad \text{supp}_\xi \Omega_{0, q}^\alpha(z, \xi) \subset V \cap D,$$

where $V$ is the same as in (i). We can write $\partial_z^k H_{0\xi}$ as a linear combination of

$$K_1 f(z) = \int_{U \setminus D} f(\xi) \frac{W_1(z, \xi)}{\Phi_{n+1}(z, \xi)} dV(\xi), \quad 0 \leq l \leq k,$$

where $W_1(z, \xi)$ denotes some polynomial in $\partial_z^{k_0} W(z, \xi)$ and $\partial_z^{k_1} \Phi(z, \xi)$, for $k_0, k_1 \geq 0$.

We now integrate by parts to reduce the exponent of $\Phi$ in the denominator to $n + 1$. Let $\xi_{i_n}$ be such that $u(z, \xi) := \partial_{\xi_{i_n}} \Phi(z, \xi) \neq 0$ for $z \in V \cap D$ and $\xi 
\in V \setminus D$. Write

$$K_1 f(z) = \int_{U \setminus D} f(\xi) W_1(z, \xi) u^{-1}(z, \xi) \partial_{\xi_{i_n}} \Phi^{-(n+1)}(z, \xi) dV(\xi).$$

By Proposition 2.1, we have $f \in W^{k-1, p}(U) \subset C^{k-2+\alpha}(U)$, for $\alpha = 1 - \frac{2n}{p} \in (0, 1)$. Since $f \equiv 0$ in $\overline{D}$, we have $|f(\xi)| \leq |f|_{U; k-2+\alpha} d(\xi)^{k-2+\alpha}$, for $\xi \in U \setminus \overline{D}$. Here $d(\xi) = \text{dist}(\xi, D)$. By (2.37), $|\partial_\xi^1 W_1(z, \xi)| \leq C(D) d(\xi)^{-i}$, and $|\partial_\xi^1 u^{-1}(z, \xi)| \leq C(D) d(\xi)^{-i}$, for $i = 0, 1, 2, \ldots$. In particular,

$$\left| W_1 u^{-1} \right| \leq C(D), \quad \left| \partial_{\xi_{i_n}} \left( W_1 u^{-1} \right) \right| \leq C(D) d(\xi)^{-1}.$$

In view of (2.38) for fixed $z \in D$, we have $|\Phi^{-(n+k-1)}(z, \xi)| \leq C(z)$ and

$$\left| \partial_{\xi_{i_n}} \Phi^{-(n+k-1)} \right| = \left| \Phi^{-(n+k-1)} u \right| \leq C(z, D).$$

Thus $W_1 u^{-1}$ and $\Phi^{-(n+k-1)}$ satisfy the estimate (3.6) for $i_k = 0$. Applying Lemma 3.3 (i) and (ii) we obtain

$$K_1 f(z) = \int_{U \setminus D} f(\xi) \partial_{\xi_{i_n}} \left( W_1(z, \xi) u^{-1}(z, \xi) \Phi^{-(n+1)}(z, \xi) \right) dV(\xi)$$

$$- \int_{U \setminus D} f(\xi) \partial_{\xi_{i_n}} \left( W_1(z, \xi) u^{-1}(z, \xi) \right) \Phi^{-(n+1)}(z, \xi) dV(\xi)$$

$$= - \int_{U \setminus D} \left( \partial_{\xi_{i_n}} f(\xi) \right) W_1(z, \xi) u^{-1}(z, \xi) \Phi^{-(n+1)}(z, \xi) dV(\xi)$$

$$- \int_{U \setminus D} f(\xi) \partial_{\xi_{i_n}} \left( W_1(z, \xi) u^{-1}(z, \xi) \right) \Phi^{-(n+1)}(z, \xi) dV(\xi).$$
We can repeat this procedure \( l - 1 (\leq k - 1) \) times. Indeed, suppose we have done \( m \) times, \( 1 \leq m \leq k - 2 \). Then the integral is a linear combination of

\[
\int_{U \setminus D} \left( \partial_{\xi_{i_{a}}}^{m_{1}} f \right) \partial_{\xi_{i_{a}}}^{m_{2}} \left\{ u^{-1}, W_{1} \right\} \Phi^{- (n + l - m)} dV(\zeta), \quad m_{1} + m_{2} = m,
\]

\[
= \int_{U \setminus D} \left( \partial_{\xi_{i_{a}}}^{m_{1}} f \right) \partial_{\xi_{i_{a}}}^{m_{2}} \left\{ u^{-1}, W_{1} \right\} u^{-1} \partial_{\xi_{i_{a}}} \Phi^{- (n + l - m - 1)} dV(\zeta),
\]

where \( \partial_{\xi_{i_{a}}}^{m_{2}} \left\{ u^{-1}, W_{1} \right\} \) is a linear combination of

\[
\partial_{\xi_{i_{a}}}^{\lambda_{1}} (u^{-1}) \partial_{\xi_{i_{a}}}^{\lambda_{1}} (u^{-1}) \cdots \partial_{\xi_{i_{a}}}^{\lambda_{p}} (u^{-1}) \partial_{\xi_{i_{a}}}^{\lambda_{0}} (W_{1}(z, \zeta)),
\]

\[
\lambda_{i} \geq 0, \quad \sum_{i=0}^{p} \lambda_{i} = m_{2}.
\]

We have \( \partial_{\xi_{i_{a}}}^{m_{1}} f \in W^{1, p}(U \setminus D) \cap C^{k - 2 - m_{1} + \alpha}(U \setminus D) \), and \( \partial_{\xi_{i_{a}}}^{m_{2}} \left\{ u^{-1}, W_{1} \right\} \) satisfies the estimate (3.6) for \( i_{k} = m_{2} \), and \( u^{-1}, \Phi^{- (n + l - m - 1)} \) satisfy estimates (3.6) for \( i_{k} = 0 \). Since \( k - 2 - m_{1} - m_{2} = k - 2 - m \geq 0 \), the hypothesis of Lemma 3.3(i) and (ii) hold, and we can do the procedure one more time. In the end, we can write \( K_{1} f \) as a linear combination of

\[
\widetilde{K}_{1} f(z) = \int_{U \setminus D} [\partial_{\xi_{i_{a}}}^{k-1} f(\zeta)] W_{1}(u^{-1})^{k-1} \Phi^{- (n+1)}(z, \zeta) dV(\zeta),
\]

and

\[
K_{2} f(z) = \int_{U \setminus D} \partial_{\xi_{i_{a}}}^{\tau_{0}} f(\zeta) \partial_{\xi_{i_{a}}}^{\tau_{1}} \left\{ u^{-1}, W_{1} \right\} \Phi^{- (n+1)}(z, \zeta) dV(\zeta),
\]

\[
\tau_{0} + \tau_{1} = l - 1 \leq k - 1.
\]

As \( \partial_{\xi_{i_{a}}}^{k-1} f \in L^{p}(U) \), \( \widetilde{K}_{2} f \) can be estimated in the same way as part (i):

\[
\left[ \int_{D} d(z)^{\gamma p} |\widetilde{K}_{2} f(z)|^{p} dV(z) \right]^{1/p} \leq C(\gamma, D) \| f \|_{W^{k-1,p}(U \setminus D)} \leq C(\gamma, D) \| \phi \|_{W^{k,p}(D)}
\]

for any \( \gamma > 0 \). For \( K_{2} f \), we integrate by parts in the direction \( \xi_{1} \). Take \( V \) and \( \phi \) as in (2.40), and set \( \widetilde{\phi}(\phi^{2}, \ldots, \phi^{2n}) \). Let \( U_{0} = V \cap (U \setminus D) \). Define

\[
\widetilde{\partial}_{\xi_{i_{a}}}^{\tau_{0}} f(s) = \partial_{\xi_{i_{a}}}^{\tau_{0}} f(\phi^{-1}(s)), \quad \widetilde{\Phi}(z, s) = \Phi(z, \phi^{-1}(s)),
\]

\[
g(z, s) = \partial_{\xi_{i_{a}}}^{\tau_{1}} \left\{ u^{-1}, W_{1} \right\} (z, \phi^{-1}(s)) \left| \det(D\phi^{-1}(s)) \right|.
\]

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Then we have, for $s = (s_1, \hat{s})$, $\hat{s} = (s_2, t_3, \ldots, t_{2n})$,

$$K_2 f(z) = \int_{H^+} \hat{\partial}_{\zeta}^{\infty} f(s) g(z, s) \Phi^{-(n+1)}(z, s) \, dV(s)$$

$$= \int_{H^+} \hat{\partial}_{\zeta}^{\infty} f(s) \partial_{s_1} I_1(z, s) \, dV(s), \quad (4.7)$$

where

$$I_1(z, s) = \int_1^{s_1} g(z, (\eta_1, \hat{s})) \Phi^{-(n+1)}(z, (\eta_1, \hat{s})) \, d\eta_1.$$ 

By (2.37) and (3.44), we have

$$|g(z, (\eta_1, \hat{s}))| \leq C(D)\eta_1^{-\tau_1}.$$ 

(4.8)

Thus

$$|I_1(z, s)| \leq C(D, z) \int_{s_1}^1 \eta_1^{-\tau_1} \, d\eta_1 \leq \begin{cases} C(D, z) s_1^{-(\tau_1-1)} & \text{if } \tau_1 > 1, \\ C(D, z) \log s_1 & \text{if } \tau_1 = 1, \\ C(D, z) & \text{if } \tau_1 = 0, \end{cases}$$

and

$$|\partial_{s_1} I_1(z, s)| \leq \frac{|g(z, (\eta_1, \hat{s}))|}{\Phi^{n+1}(z, (\eta_1, \hat{s}))} \leq \begin{cases} C(D, z) s_1^{-\tau_1} & \text{if } \tau_1 > 1, \\ C(D, z) s_1^{-1} & \text{if } \tau_1 = 1, \\ C(D, z) & \text{if } \tau_1 = 0. \end{cases}$$

If $\tau_1 > 1$, we have $\hat{\partial}_{\zeta}^{\infty} f \in W^{1,p}(H^+) \cap C^{j+\alpha}(H^+)$ for $j = k - 2 - \tau_0 \geq 0$ $(\tau_0 \leq k - 2)$ and $I_1$ satisfies estimate (3.6) with $i_k$ replaced by $\tau_1 - 1$. Furthermore, we have

$$j - (\tau_1 - 1) = k - 1 - \tau_0 - \tau_1 \geq k - 1 - l \geq 0.$$ 

Thus we can apply Lemma 3.3(ii) and integrate by parts in (4.7) to get

$$K_2 f(z) = -\int_{H^+} \partial_{s_1} \hat{\partial}_{\zeta}^{\infty} f(s) I_1(z, s) \, dV(s). \quad (4.9)$$

If $\tau_1 = 1$, then $I_1$ satisfies estimate (3.7). Thus we can apply Lemma 3.3(iii) and integrate by parts to get (4.9). If $\tau_0 = 0$, then $I_1$ satisfies estimate (3.6) with $i_k$ being replaced by 0 and again we can integrate by parts by Lemma 3.3(ii) to obtain (4.9).

We can integrate by parts $k - 1 - \tau_0 = \tau_1$ times. Indeed, suppose we have done it $m$ times, $1 \leq m \leq k - 2 - \tau_0$. We have

$$K_2 f = \pm \int_{H^+} \partial_{s_1}^m \hat{\partial}_{\zeta}^{\infty} f(s) I_m(z, s) \, dV(s)$$

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\[= \pm \int_{H^+} \partial_{s_1}^{m} \partial_{\xi_0}^{\tau_0} f(s) \partial_{x_1} I_{m+1}(z, s) \, dV(s),\]

where
\[I_m(z, s) = \int_1^{s_1} \int_1^{\eta_1} \cdots \int_1^{\eta_{m-1}} g(z, (\eta_m, \hat{s})) \frac{[d\eta]^m}{\Phi^m+1(z, (\eta_m, \hat{s}))},\]

and we denote \([d\eta]^m := d\eta_m \cdots d\eta_1\). By (4.8) and (2.38),
\[|I_{m+1}(z, s)| \leq C(D, z) \int_1^{s_1} \int_1^{\eta_1} \cdots \int_1^{\eta_{m+1}} [d\eta]^{m+1}\]
\[\begin{cases} C(D, z)s_1^{-\tau_1+(m+1)}, & \text{if } m+1 < \tau_1, \\ C(D, z)(1 + |\log s_1|), & \text{if } m+1 = \tau_1, \\ C(D, z), & \text{if } m+1 > \tau_1. \end{cases}\]

and
\[|\partial_{s_1} I_{m+1}(z, s)| = |I_m(z, s)| \leq \begin{cases} C(D, z)s_1^{-\tau_1+m}, & \text{if } m+1 < \tau_1, \\ C(D, z)s_1^{-1}, & \text{if } m+1 = \tau_1, \\ C(D, z)(1 + |\log s_1|), & \text{if } m+1 > \tau_1. \end{cases}\]

Applying Lemma 3.3(ii) and (iii) to these cases we obtain
\[K_2 f(z) = \pm \int_{H^+} \left( \partial_{s_1}^{m+1} \partial_{\xi_0}^{\tau_0} f \right) (s) I_{m+1}(z, s) \, dV(s).\]

In conclusion, we can integrate by part \(k - 1 - \tau_0\) times to transform \(K_2 f\) given by (4.5) to the form
\[K_2 f(z) = -\int_{H^+} F(s) I_{k-1-\tau_0}(z, s) \, dV(s),\]

where
\[F(s) = \partial_{s_1}^{k-1-\tau_0} \partial_{\xi_0}^{\tau_0} f(z, s) \in L^p(U \setminus D),\]

and
\[I_{k-1-\tau_0} = \int_1^{s_1} \int_1^{\eta_1} \cdots \int_1^{\eta_{k-2-\tau_0}} g(z, (\eta_{k-1-\tau_0}, \hat{s})) \frac{[d\eta]}{\Phi^{m+1}(z, (\eta_{k-1-\tau_0}, \hat{s}))} \]

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with \([d\eta] = d\eta_{k-1-\tau_0} \ldots d\eta_1\). By (3.61), we have
\[
|\tilde{\Phi}(z, (\eta_{k-1-\tau_0}, \delta))| \geq c \left(d(z) + s_1 + |s_2| + |t|^2\right),
\] (4.10)
where \(c\) is independent of \(z \in V\). From (4.8) and (4.10) we see that \(|K_2 f(z)|\) is bounded by

\[
C(D) \int_{H^+} \frac{|F(s)| \, dV(s)}{(d(z) + s_1 + |s_2| + |t|^2)^{n+1}} \left[ \int_{s_1}^1 \int_{\eta_1}^1 \int_{\eta_{k-1-\tau_0}}^{(\eta_{k-1-\tau_0})^{-\tau_1}} \, d\eta \right]^{\frac{1}{k-1-\tau_0} \text{ integrals}} \leq C(D) \int_{H^+} |F(s)| \, [k(z, s)] \left(1 + |\log s_1|\right) \, dV(s), \quad \tau_1 \leq k - 1 - \tau_0,
\]
where we denote
\[
k(z, s) = \frac{1}{(d(z) + s_1 + |s_2| + |t|^2)^{n+1}}.
\]
Using polar coordinates for \(t = (t_1, \ldots, t_{2n-2})\), and applying Lemma 4.1(iii) for \(x = 1 - \varepsilon p'\), we have for any \(\varepsilon > 0\),
\[
\int_{H^+} [k(z, s)](1 + |\log s_1|)^{p'} \, dV(s) \leq C(\varepsilon) \int_0^1 \int_0^1 \int_0^1 \frac{s_1^{-\varepsilon p'} t^{2n-3} \, ds_1 \, ds_2 \, dt}{(d(z) + s_1 + s_2 + t^2)^{n+1}} \leq C(\varepsilon) d(z)^{-\varepsilon p'}.
\]
Thus for any \(\gamma > 0\), we have
\[
\int_D d(z)^{\gamma p} |K_2 f(z)|^p \, dV(z)
\]
\[
\leq C(D) \int_D d(z)^{\gamma p} \left[ \int_{H^+} |F(s)| \, [k(z, s)](1 + |\log s_1|) \, dV(s) \right]^p \, dV(z)
\]
\[
= C(D) \int_D d(z)^{\gamma p} \left[ \int_{H^+} |F(s)| \, [k(z, s)]^{\frac{1}{p}} [k(z, s)]^{\frac{1}{p'}} (1 + |\log s_1|) \, dV(s) \right]^p \, dV(z)
\]
\[
\leq C(D) \int_D d(z)^{\gamma p} \left[ \int_{H^+} |F(s)|^{\frac{1}{p}} [k(z, s)] \, dV(s) \right]
\]
\[
\times \left[ \int_{H^+} k(z, s)(1 + |\log s_1|)^{\frac{1}{p'}} \, dV(s) \right]^{\frac{p}{p'}} \, dV(z)
\]
\[
\leq C(D)[C(\varepsilon)]^{\frac{p}{p'}} \int_D d(z)^{\gamma p'} \left[ \int_{H^+} |F(s)|^p [k(z, s)] \, dV(s) \right] \, dV(z)
\]
\[
\leq C(D)[C(\varepsilon)]^{\frac{p}{p'}} \int_{H^+} |F(s)|^p \left[ \int_D d(z)^{\gamma p'} k(z, s) \, dV(z) \right] \, dV(s),
\]
where we denote
\[ \gamma' = \gamma_p - \epsilon_p = p(\gamma - \epsilon). \] (4.11)

Choose \( \gamma \) and \( \epsilon \) such that \( \gamma > \epsilon \). Then \( \gamma' > 0 \). For each \( s \in H^+ \), let \( \tilde{\phi}_{(\gamma)} : V \rightarrow \mathbb{R}^{2n} \) be the coordinate map given by (3.18) and write \( \tilde{\phi}_{(\gamma)} = (\tilde{s}_1, \tilde{s}_2, \tilde{t}) \). Using polar coordinates for \((\tilde{s}_1, \tilde{s}_2) \in \mathbb{R}^2, \tilde{t} \in \mathbb{R}^{2n-2}\), and \( |d\tilde{t}| \leq s_1(z) \leq C d(z) \), we get
\[
\int_D d(z)^{\gamma'} k(z, s) \, dV(z) \leq C \int_{\tilde{t} \in [-1,1]^{2n-2}} \int_{\tilde{s}_2 = -1}^{1} \int_{\tilde{s}_1 = 0}^{1} \frac{\tilde{s}_1^\gamma \tilde{d}\tilde{s}_{1} \tilde{d}\tilde{s}_{2} \tilde{d}\tilde{t}}{(d(\xi) + |\tilde{s}_1| + |\tilde{s}_2| + |\tilde{t}|)^{n+1}} \\
\leq C \int_{0}^{1} \int_{0}^{1} \frac{\tilde{s}_1^{\gamma'} \tilde{d}\tilde{s}_{1} \tilde{d}\tilde{s}_{2} \tilde{d}\tilde{t}}{(d(\xi) + \tilde{s}_1 + \tilde{s}_2 + |\tilde{t}|)^{n+1}} \\
\leq C(n, \gamma'),
\]
where in the last inequality we used Lemma 4.1(ii). Hence
\[
\left[ \int_D d(z)^{\gamma'} |K_2 f(z)|^p \, dV(z) \right]^\frac{1}{p} \leq C(D, \gamma) \left[ \int_{U \setminus D} |F(\xi)|^p \, dV(\xi) \right]^\frac{1}{p}.
\]
Combining this and (4.6) we obtain for any \( \gamma > 0 \),
\[
\left[ \int_D d(z)^{\gamma'} |\partial^k_{\xi} H_0 \varphi(z)|^p \, dV(z) \right]^\frac{1}{p} \leq C(D, \gamma) \left[ \int_{U \setminus D} |F(\xi)|^p \, dV(\xi) \right]^\frac{1}{p},
\]
or
\[
\| H_0 \varphi \|_{W^{-1,p}(D)} \leq C(D, \beta) \| \varphi \|_{W^{k,p}(D)}, \quad 0 < \beta < 1.
\]

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