Ricci-flat Kähler Manifolds
from Supersymmetric Gauge Theories

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Abstract

Using techniques of supersymmetric gauge theories, we present the Ricci-flat metrics on non-compact Kähler manifolds whose conical singularity is repaired by the Hermitian symmetric space. These manifolds can be identified as the complex line bundles over the Hermitian symmetric spaces. Each of the metrics contains a resolution parameter which controls the size of these base manifolds, and the conical singularity appears when the parameter vanishes.

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1 Introduction

$\mathcal{N} = 2$ supersymmetric nonlinear sigma models in two dimensions on Ricci-flat Kähler manifolds can be considered as the model of the superstring theory on curved space. Ricci-flat Kähler manifolds are also important ingredient for D-branes in curved space. In the previous letter, we presented the simple derivation of the $O(N)$ symmetric Ricci-flat metric, which actually coincides with the Stenzel metric on the cotangent bundle over $S^{N-1}$. The conical singularity is resolved by $S^{N-1}$ with a radius being the deformation parameter. It reduces to the Eguchi-Hanson gravitational instanton and the six-dimensional deformed conifold in the cases of $N = 3$ and $N = 4$, respectively. In a new metric for the six-dimensional conifold, in which the conical singularity is repaired by $S^2 \times S^2$, was found. It was generalized in our previous letter to the higher dimensional conifold, in which the singularity is resolved by the complex quadric surface $Q^{N-2} = SO(N)/[SO(N - 2) \times U(1)]$. The new manifold can be regarded as the complex line bundle over $Q^{N-2}$, which is a Hermitian symmetric space.

In this paper, we present the new Ricci-flat metrics replacing the base manifold by other Hermitian symmetric spaces, the Grassmann manifold $G_{N,M} = SU(N)/[SU(N - M) \times U(M)]$, $SO(2N)/U(N)$ and $Sp(N)/U(N)$. To do this, we apply the technique of the gauge theory formulation of supersymmetric nonlinear sigma models on the Hermitian symmetric spaces, which was used for the study of non-perturbative effects. We note that our manifolds are natural generalizations of the Calabi metric on the complex line bundle over $CP^{N-1}$.

This paper is organized as follows. In Sec. 2, we recapitulate the construction of compact Kähler manifolds $G_{N,M}$, $SO(2N)/U(N)$ and $Sp(N)/U(N)$ by supersymmetric gauge theories, and extend this to non-compact Kähler manifolds. In Sec. 3, we impose the Ricci-flat condition on these non-compact manifolds. Symmetry plays a crucial role to reduce partial differential equations to ordinary differential equations of one variable. In Sec. 4, we present explicit expressions of Kähler metrics and their Kähler potentials. It is found that these manifolds contain resolution parameter $b$ as an integration constant, and the conical singularity is resolved by $G_{N,M}$, $SO(2N)/U(N)$ or $Sp(N)/U(N)$ of a radius expressed in terms of $b$. These manifolds are complex line bundles over $G_{N,M}$, $SO(2N)/U(N)$ and $Sp(N)/U(N)$. Sec. 5 is devoted to Conclusion and Discussions. In Appendix, we summarize the isomorphisms between the lower dimensional base manifolds and the duality between the Grassmann manifolds, and show that they hold for total spaces.
2 Construction by Supersymmetric Gauge Theories

2.1 Compact Kähler Manifolds from Gauge Theories

In this section we recapitulate the construction of \( G_{N,M} \), \( SO(2N)/U(N) \) and \( Sp(N)/U(N) \), using supersymmetric gauge theories \([11]\). Such a method was first found for the projective space \( \mathbb{C}P^{N-1} \) \([14]\) and the Grassmann manifold \( G_{N,M} \) \([17]\), and then recognized as the symplectic or the Kähler quotient \([18]\).

Construction of \( G_{N,M} \) \([17,18]\). Let \( \Phi(x, \theta, \theta) \) be an \( N \times M \) matrix-valued chiral superfield. The group \( SU(N) \times U(M) \) can act on it as

\[
\Phi \rightarrow \Phi' = g_L \Phi g_R^{-1}, \quad (g_L, g_R) \in (SU(N), U(M)).
\]  

(2.1)

We promote the right action of \( U(M) \) to a gauge symmetry by introducing a vector superfield \( V(x, \theta, \theta) \), taking a value in the Lie algebra of \( U(M) \). The gauge transformation is given by

\[
\Phi \rightarrow \Phi' = \Phi e^{-i\Lambda}, \quad e^{V} \rightarrow e^{V'} = e^{i\Lambda} e^{V} e^{-i\Lambda},
\]  

(2.2)

where \( \Lambda(x, \theta, \theta) \) is a parameter chiral superfield, taking a value in the Lie algebra of \( U(M) \). Note that the local invariance group is enlarged to the complexification of the gauge group, \( U(M)^C = GL(N, \mathbb{C}) \), since the scalar component of \( \Lambda(x, \theta, \theta) \) is complex. The Lagrangian invariant under the global \( SU(N) \) and the local \( U(M) \) symmetries is given by

\[
\mathcal{L} = \int d^4 \theta \, \mathcal{K}(\Phi, \Phi^\dagger, V) = \int d^4 \theta \left[ \text{tr} (\Phi^\dagger \Phi e^{V}) - \text{c tr} \, V \right],
\]

(2.3)

where \( \mathcal{K} \) is the Kähler potential. Here \( \text{c} \) is a real positive constant, called the Fayet-Iliopoulos (FI) parameter, and \( \text{c tr} \, V \) is called the FI D-term.

Since \( V \) is an auxiliary field, it can be eliminated by its equation of motion \([1]\)

\[
\delta \mathcal{L}/\delta V = \Phi^\dagger \Phi e^V - \text{c} \, 1_M = 0,
\]

(2.4)

where \( 1_M \) is an \( M \times M \) unit matrix. Substituting the solution, \( V(\Phi, \Phi^\dagger) = -\log (\Phi^\dagger \Phi/\text{c}) \), back into the Lagrangian \( (2.3) \), we obtain

\[
\mathcal{K}(\Phi, \Phi^\dagger, V(\Phi, \Phi^\dagger)) = \text{c tr} \, \log(\Phi^\dagger \Phi) = \text{c} \, \log \det(\Phi^\dagger \Phi)
\]

\[= \text{c} \, \log \det(1_M + \varphi^\dagger \varphi). \]

(2.5)

Since the gauge group is complexified, we have chosen the gauge fixing as

\[
\Phi = \begin{pmatrix} 1_M \\ \varphi \end{pmatrix},
\]

(2.6)

\[\text{1 We regard } e^{-V} \delta e^V \text{ as an infinitesimal parameter: } \delta \mathcal{L} = \text{tr} [\Phi^\dagger \Phi e^V (e^{-V} \delta e^V)] - \text{c tr} (\delta \log e^V) = \text{tr} [\Phi^\dagger \Phi e^V - \text{c} 1_M] X^{-1} \delta X], \text{ where } X = e^V. \]
where $\varphi(x, \theta, \bar{\theta})$ is an $(N - M) \times M$ matrix-valued chiral superfield. The constant terms in (2.5) have been omitted, since they disappear under the superspace integral $\int d^4\theta$. (2.5) is the Kähler potential of $G_{N,M} = SU(N)/[SU(N - M) \times U(M)]$, whose complex dimension is $M(N - M)$. It becomes one of $CP^{N-1} = SU(N)/[SU(N - 1) \times U(1)]$ if we set $M = 1$, in which case the gauge group is $U(1)$.

*Construction of $SO(2N)/U(N)$ and $Sp(N)/U(N)$*. Let us replace the size of the matrix $\Phi$, $(N, M)$, by $(2N, N)$, corresponding to $G_{2N,N}$, and introduce the invariant tensor of $SO(2N)$ or $Sp(N)$:

$$J = \begin{pmatrix} 0 & 1_N \\ \epsilon 1_N & 0 \end{pmatrix},$$

(2.7)

where $\epsilon = 1$ for $SO(2N)$, and $\epsilon = -1$ for $Sp(N)$. The invariant Lagrangian is given by

$$\mathcal{L} = \int d^4\theta \left[ \text{tr} (\Phi^\dagger \Phi e^V) - c \text{tr} V \right] + \left[ \int d^2\theta \text{tr} (\Phi_0 \Phi^T J \Phi) + c.c. \right],$$

(2.8)

where $\Phi_0(x, \theta, \bar{\theta})$ is an auxiliary chiral superfield of an $N \times N$ matrix, belonging to (anti-)symmetric tensor representation of the gauge group $U(N)$ for $SO(2N)/U(N)$ [$Sp(N)/U(N)$] with the suitable $U(1)$ charge.

By the integration over $V$, we obtain (2.5) with the same gauge fixing as (2.6). The integration over $\Phi_0$ gives the constraint

$$\Phi^T J \Phi = \varphi + \epsilon \varphi^T = 0,$$

(2.9)

which implies that the $N \times N$ matrix-valued chiral superfield $\varphi$ is anti-symmetric or symmetric for $SO(2N)/U(N)$ or $Sp(N)/U(N)$, respectively. The Kähler potential (2.5) with the constraints (2.5) is one of $SO(2N)/U(N)$ or $Sp(N)/U(N)$, whose complex dimension is $\frac{1}{2}N(N - 1)$ or $\frac{1}{2}N(N + 1)$, respectively.

Instead of the Kähler potential of the Lagrangian (2.3) and (2.8), we can start from

$$K(\Phi, \Phi^\dagger, V) = f(\text{tr} (\Phi^\dagger \Phi e^V)) - c \text{tr} V,$$

(2.10)

where $f$ is an arbitrary function. We can show that we obtain the same results even if we start from (2.10) [12]. Let us make some comments. We have used the classical equation of motion of $V$ to eliminate it. We can promote this to the quantum level in the path integral formalism [12]. If we add the kinetic term for $V$ rather than regarding $V$ as auxiliary, our manifolds are obtained as the classical moduli space of the gauge theories [19].
2.2 Non-compact Kähler Manifolds from Gauge Theories

Let us construct the non-compact Kähler manifolds, by restricting the gauge degrees of freedom from $U(M)$ to $SU(M)$. To do this, we promote the FI-parameter $c$ in (2.10) to an auxiliary vector superfield $C(x, \theta, \overline{\theta})$:

$$K_0(\Phi, \Phi^\dagger, V, C) = f(\text{tr}(\Phi^\dagger \Phi^e V)) - C \text{tr} V ,$$  

(2.11)

where $f$ is an arbitrary function.\(^2\) Note that $V(x, \theta, \overline{\theta})$ in this Lagrangian is still taking a value in the Lie algebra of $U(M)$. The equations of motion of $V$ and $C$ read

$$\frac{\delta L}{\delta V} = f'(\text{tr}(\Phi^\dagger \Phi^e V)) \Phi^\dagger \Phi^e V - C M = 0 ,$$  

(2.12a)

$$\frac{\delta L}{\delta C} = \text{tr} V = 0 ,$$  

(2.12b)

respectively, where the prime denotes the differentiation with respect to the argument of $f$. The gauge group is restricted to $SU(M)$ by (2.12b). The trace and the determinant of (2.12a) are

$$f'(\text{tr}(\Phi^\dagger \Phi^e V)) \text{tr} (\Phi^\dagger \Phi^e V) = MC ,$$  

(2.13a)

$$\left[ f'(\text{tr}(\Phi^\dagger \Phi^e V)) \right]^M \det(\Phi^\dagger \Phi) = C^M ,$$  

(2.13b)

since $\det e^V = 1$ for the $SU(M)$ gauge field $V$. Eliminating $C$ from these equations, the solution of $V$ reads

$$\text{tr} (\Phi^\dagger \Phi^e V) = M \left[ \det(\Phi^\dagger \Phi) \right]^{\frac{1}{M} .}$$  

(2.14)

Substituting this back into (2.11) and taking account of (2.12b), we obtain the nonlinear Kähler potential

$$K_0(\Phi, \Phi^\dagger, V(\Phi, \Phi^\dagger)) = f \left( M \left[ \det(\Phi^\dagger \Phi) \right]^{\frac{1}{M} \right) \equiv K(X(\Phi, \Phi^\dagger)) ,$$  

(2.15)

where $X(\Phi, \Phi^\dagger)$ is a vector superfield, invariant under the global $U(N)$ [$SO(2N)$ or $Sp(N)$] and the local $SU(M)$ [$SU(N)$] symmetries, defined by

$$X(\Phi, \Phi^\dagger) = \log \det \Phi^\dagger \Phi ,$$  

(2.16)

and $K(X)$ is a real function of $X$ related to $f$. Here the logarithm in the definition of $X$ is just a convention. (Note that this definition of the invariant is different from the one in [5, 10].) From the viewpoint of the algebraic variety, $X$ is the gauge invariant parameterizing the moduli space of supersymmetric gauge theories [13].

\(^2\) There exist independent invariants $\text{tr}[(\Phi^\dagger \Phi^e V)^2], \cdots, \text{tr}[(\Phi^\dagger \Phi^e V)^M]$, besides $\text{tr}(\Phi^\dagger \Phi^e V)$. We can show that, even if these are included as the arguments of the arbitrary function of (2.11), we obtain the same result (2.13). The situation is the same for the cases of the $U(M)$ gauge field, (2.16), for compact manifolds.
Since the gauge group is complexified to $SU(M)^C = SL(M, \mathbb{C})$, we can choose a gauge fixing as
\[ \Phi = \sigma \left( \begin{array}{c} 1_M \\ \varphi \end{array} \right), \]  
(2.17)
where $\varphi$ is an $(N - M) \times M$ matrix-valued chiral superfield, and $\sigma(x, \theta, \bar{\theta})$ is a chiral superfield. Comparing (2.17) with (2.6), we find that the superfield $\sigma$ is parameterizing a fiber, while $\varphi$ is parameterizing a base manifold, with the total space being a complex line bundle. Under this gauge fixing, the invariant superfield $X$ is decomposed as
\[ X = M \log |\sigma|^2 + \log \det(1_M + \varphi^\dagger \varphi) = M \log |\sigma|^2 + \Psi, \]  
(2.18)
where we have defined
\[ \Psi \equiv \log \det(1_M + \varphi^\dagger \varphi). \]  
(2.19)
Note that $\Psi$ is a Kähler potential of $G_{N,M} [SO(2N)/U(N) \text{ or } Sp(N)/U(N)]$ obtained in (2.5) [with the constraint (2.9)].

Let us introduce some notations. We denote the elements of the matrix-valued chiral superfield $\varphi$ by $\varphi_{Aa}$, where the upper case and the lower case indices, $A$ and $a$, run from 1 to $N - M$ and from 1 to $M$, respectively. Since the size of the matrix $\varphi$ is $N \times N$ in the cases of $Sp(N)/U(N)$ and $SO(2N)/U(N)$, we denote its elements by $\varphi_{ab}$. In this case, only the components $\varphi_{ab}$ with $b \geq a$ ($b > a$) are considered as independent. When we discuss the total space, we use the coordinates $z^\mu \equiv (\sigma, \varphi_{Aa})$. It should be noted that from now on we use the same letters for chiral superfields and their complex scalar components.

We make a comment on the symmetry breaking. These non-compact manifolds can be regarded as
\[ \mathbb{R} \times \frac{G}{H} = \mathbb{R} \times \frac{SU(N)}{SU(N - M) \times SU(M)}, \quad \mathbb{R} \times \frac{SO(2N)}{SU(N)}, \quad \mathbb{R} \times \frac{Sp(N)}{SU(N)}, \]  
(2.20)
at least locally. The part of $G/H$ is parametrized by the Nambu-Goldstone bosons arising from the spontaneous breaking of the global symmetry $G$ down to $H$, whereas the factor of $\mathbb{R}$ is parametrized by the so-called quasi-Nambu-Goldstone boson (see e.g. [20, 21, 11]).

3 Ricci-flat Conditions

We would like to determine the function $K(X)$ in (2.15), by imposing the Ricci-flat condition on the manifold. The metric of the Kähler manifold is given by $g_{\mu\nu} = \partial_\mu \partial_\nu K$, where $\partial_\mu = \partial / \partial z^\mu$.
\[ g_{\mu^* \nu^*} = \begin{pmatrix} g_{\sigma \sigma^*} & g_{\sigma (Bb)^*} \\ g_{(Aa) \sigma^*} & g_{(Aa)(Bb)^*} \end{pmatrix}, \] (3.1a)

with each block being

\[ g_{\sigma \sigma^*} = \mathcal{K}'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*}, \quad g_{(Aa) \sigma^*} = \mathcal{K}'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*}, \] (3.1b)

\[ g_{\sigma (Bb)^*} = \mathcal{K}'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*}, \quad g_{(Aa)(Bb)^*} = \mathcal{K}'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*}, \] (3.1c)

where the prime denotes the differentiation with respect to the argument \( X \) of the function \( \mathcal{K}(X) \).

Here, we have used equations, \( \frac{\partial^2 X}{\partial \sigma \partial \sigma^*} = \frac{\partial^2 X}{\partial \sigma \partial \phi^*} = \frac{\partial^2 X}{\partial X \partial \phi^*} = 0 \) (\( \sigma \neq 0 \)), which follow from (2.18).

The determinant of the metric is calculated, to yield

\[ \det g_{\mu^* \nu^*} = M^2 |\sigma|^2 \mathcal{K}'' \cdot \det_{(Aa)(Bb)^*} \left( \mathcal{K}' \frac{\partial^2 X}{\partial \phi_{Aa} \partial \phi_{Bb}^*} \right), \] (3.2)

where \( \det_{(Aa)(Bb)^*} \) denotes the determinant of the matrix of the tensor product, spanned by \( (Aa) \) and \( (Bb)^* \). Since the Ricci-form is given by \( (Ric)_{\mu^* \nu^*} = -\partial_{\mu} \partial_{\nu^*} \log \det g_{\kappa \lambda^*} \), the Ricci-flat condition \( (Ric)_{\mu^* \nu^*} = 0 \) implies

\[ \det g_{\mu^* \nu^*} = (\text{constant}) \times |F|^2, \] (3.3)

with \( F \) being a holomorphic function.

### 3.1 Line Bundle over \( G_{N,M} \)

In this section let us obtain the explicit solution of the Ricci-flat metric on the line bundle over the Grassmann manifold. Let us calculate the \( X \) differentiated by matrix fields \( \phi_{Aa} \) once and twice, needed for the calculation of the Kähler metric (3.1). By noting

\[ \frac{\partial \phi_{Bb}}{\partial \phi_{Aa}} = \delta_{AB} \delta_{ab}, \] (3.4)

we obtain

\[ \frac{\partial X}{\partial \phi_{Aa}} = \partial_{(Aa)} \Psi = \left[ (1_M + \phi^\dagger \phi)^{-1} \phi^\dagger \right]_{aA}, \] (3.5a)

\[ \frac{\partial X}{\partial \phi_{Aa}^*} = \partial_{(Aa)^*} \Psi = \left[ \phi (1_M + \phi^\dagger \phi)^{-1} \right]_{Aa}, \] (3.5b)

\[ \frac{\partial^2 X}{\partial \phi_{Aa} \partial \phi_{Bb}^*} = \partial_{(Aa)} \partial_{(Bb)^*} \Psi = (1_M + \phi^\dagger \phi)^{-1} \left[ 1_{(N-M)} - \phi (1_M + \phi^\dagger \phi)^{-1} \phi^\dagger \right]_{BA}. \] (3.5c)
Here we have used the definition of $\Psi$ in (2.19), and $\partial_{(Aa)}$ and $\partial_{(Aa)^*}$ represent differentiations with respect to $\varphi_{Aa}$ and $\varphi_{Aa}^*$, respectively. Note that (3.5c) is just the Kähler metric of the Grassmann manifold $G_{N,M}$. The determinant of $g_{\mu\nu^*}$ can be calculated as

$$\det g_{\mu\nu^*} = \frac{M^2}{|\sigma|^2} \mathcal{K}''(\mathcal{K}')^{M(N-M)} \cdot \det_{(Aa)(Bb)^*} \left[ \partial_{(Aa)}\partial_{(Bb)^*} \Psi \right].$$

To obtain the concrete expression of this determinant, we use a symmetry transformation preserving the value of the determinant. Under the transformation of the complex isotropy $[SU(N-M) \times SU(M)]^C = SL(N-M,C) \times SL(M,C)$, the coordinates transform linearly as $z^\mu \rightarrow z'^\mu = V_{\mu\nu^*} z^\nu$. Since the transformation matrix $V$ belongs to a subgroup of $SL(N-M,C)$, the equation $\det V = 1$ holds and the $\det g_{\mu\nu^*}$ is invariant: $\det g_{\mu\nu^*} \rightarrow \det g'_{\mu\nu^*} = \det g_{\mu\nu^*}|\det V|^2 = \det g_{\mu\nu^*}$. For an arbitrary matrix-valued chiral superfield $\varphi$, there exists a complex isotropy which permits the transformation of $\varphi$ to the form of

$$\varphi = \left( \begin{array}{c} \varphi_0 \\ \vdots \\ 0 \end{array} \right),$$

where the dots denote zero elements, and the only non-zero element is $\varphi_{11} \equiv \varphi_0$. The matrix of the tensor product (3.5c) is diagonalized as

$$\frac{\partial^2 X}{\partial \varphi_{Aa} \partial \varphi_{Bb}^*} = \text{diag. } \left( \xi^2, \xi, \ldots, \xi; \xi, 1, \ldots, 1; \ldots; \xi, 1, \ldots, 1 \right),$$

where $\xi \equiv (1 + |\varphi_0|^2)^{-1}$. Each block separated by the semicolons is labeled by the indices $A = B$, which run from 1 to $N - M$, and in each block the indices $a = b$ run from 1 to $M$. With noting $\xi = (1 + |\varphi_0|^2)^{-1} = [\det(1_N + \varphi^\dagger \varphi)]^{-1} = |\sigma|^2 M e^{-X}$, the determinant (3.6) can be calculated as

$$\det g_{\mu\nu^*} = M^2 |\sigma|^{2(MN-1)} e^{-NX} \mathcal{K}''(\mathcal{K}')^{M(N-M)}.$$ 

The Ricci-flat condition (3.3) becomes

$$e^{-NX} \frac{d}{dX} (\mathcal{K}')^{M(N-M)+1} = a,$$

where $a$ is a real constant.

### 3.2 Line Bundles over $SO(2N)/U(N)$ and $Sp(N)/U(N)$

In this section, we construct the Ricci-flat metrics on the line bundles over $SO(2N)/U(N)$ and $Sp(N)/U(N)$. These cases are obtained by imposing the constraint (2.9) on the Grassmann manifold $G_{2N,N}$. Under the condition (2.9), the differentiations with respect to the matrix elements $\varphi_{ab}$, corresponding to (3.4) for the Grassmann case, become

$$\frac{\partial \varphi_{cd}}{\partial \varphi_{ab}} = (\delta_{ca} \delta_{db} - \epsilon \delta_{cb} \delta_{da}) \left( 1 - \frac{1}{2} \delta_{ab} \right),$$

where $\epsilon = 1$ for $Sp(N)/U(N)$. The determinant (3.6) of $g_{\mu\nu^*}$ becomes

$$\det g_{\mu\nu^*} = M^2 |\sigma|^{2(MN-1)} e^{-NX} \mathcal{K}''(\mathcal{K}')^{M(N-M)}.$$ 

The Ricci-flat condition (3.3) becomes

$$e^{-NX} \frac{d}{dX} (\mathcal{K}')^{M(N-M)+1} = a,$$

where $a$ is a real constant.
where we do not take a sum over the index $a$ or $b$. Using this, the $X$ differentiated by one or two $\varphi$’s can be calculated, to yield

\[
\frac{\partial X}{\partial \varphi_{ab}} = \partial_{(ab)}\Psi = \sum_{c,d=1}^{N} \left[ (1_N + \varphi^\dagger \varphi)^{-1} \varphi_{cd} \right] \left( \delta_{ca} \delta_{db} - \epsilon \delta_{cb} \delta_{da} \right) \left( 1 - \frac{1}{2} \delta_{ab} \right),
\]

(3.12a)

\[
\frac{\partial X}{\partial \varphi_{ab}^\ast} = \partial_{(ab)^\ast}\Psi = \sum_{c,d=1}^{N} \left[ \varphi(1_N + \varphi^\dagger \varphi)^{-1} \right] \left( \delta_{ca} \delta_{db} - \epsilon \delta_{cb} \delta_{da} \right) \left( 1 - \frac{1}{2} \delta_{ab} \right),
\]

(3.12b)

\[
\frac{\partial^2 X}{\partial \varphi_{ab} \partial \varphi_{cd}^\ast} = \partial_{(ab)} \partial_{(cd)^\ast}\Psi = \left( 1 - \frac{1}{2} \delta_{ab} \right) \left( 1 - \frac{1}{2} \delta_{cd} \right) \left\{ \varphi(1_N + \varphi^\dagger \varphi)^{-1} \left[ (1_N + \varphi^\dagger \varphi)^{-1} \varphi_{cd} \right] \right\} \left( \delta_{ca} \delta_{db} - \epsilon \delta_{cb} \delta_{da} \right) \left( 1 - \frac{1}{2} \delta_{ab} \right),
\]

(3.12c)

Here the last term in the last line implies adding the preceding two terms with the exchange of the indices. Note again that (3.12c) is just the Kähler metric of $SO(2N)/U(N)$ or $Sp(N)/U(N)$. The determinant (3.2) can be calculated as

\[
\det g_{\mu\nu} = \frac{N}{\sigma^2} \kappa''(\kappa')^{1/2}N(N-1) \cdot \det_{(ab)(cd)^\ast} \partial_{(ab)} \partial_{(cd)^\ast}\Psi.
\]

(3.13)

We again use the complex isotropy transformation of $SU(N)^C = SL(N, \mathbb{C})$, preserving the determinant. We first discuss $SO(2N)/U(N)$ followed by $Sp(N)/U(N)$.

The line bundle over $SO(2N)/U(N)$. Using the complex isotropy transformation of $SL(N, \mathbb{C})$, the arbitrary $\varphi$ can be put

\[
\varphi = \begin{pmatrix}
0 & \varphi_0 & 0 \\
-\varphi_0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

(3.14)

where non-zero elements are $\varphi_{12} = -\varphi_{21} \equiv \varphi_0$. The matrix of the tensor product (3.12d) is diagonalized as

\[
\frac{\partial^2 X}{\partial \varphi_{ab} \partial \varphi_{cd}^\ast} \bigg|_{b>a, d>c} = \text{diag} \left( 2\xi^2; 2\xi; \cdots; 2\xi; 2\xi; \cdots; 2\xi; 2\xi; \cdots; 2\xi; \cdots; 2\xi; \cdots; 2\xi; \cdots; 2\xi; \cdots; 2\xi; \cdots; 2; \cdots; 2 \right),
\]

(3.15)

where $\xi \equiv (1 + |\varphi_0|^2)^{-1}$. Each block separated by the semicolons is labeled by the indices $a = c$, which run from 1 to $N$, and the indices $b = d$ run from $a + 1 = c + 1$ to $N$ in the $a$-th block, by the conditions $b > a$ and $d > c$. Noting $\xi = (1 + |\varphi_0|^2)^{-1} = [\det(1_N + \varphi^\dagger \varphi)]^{-1/2} = |\sigma|^N e^{-X/2}$, we can calculate the determinant (3.13), given by

\[
\det g_{\mu\nu} = N^2 2^{1/2}N(N-1)|\sigma|^2N(N-1) e^{-(N-1)^2} \kappa''(\kappa')^{1/2}N(N-1).
\]

(3.16)
The Ricci-flat condition \( (3.3) \) becomes
\[
e^{-\frac{1}{2}(N-1)N} \frac{d}{dX} (\mathcal{K}')^{\frac{1}{2}(N-1)+1} = a .
\] (3.17)

The line bundle over \( \text{Sp}(N)/U(N) \). There exists an isotropy transformation which transforms an arbitrary matrix \( \varphi \) to the form of \( (3.7) \). The matrix of the tensor product \( (3.12c) \) is diagonalized as
\[
\frac{\partial^2 X}{\partial \varphi_{ab} \partial \varphi_{cd}} \begin{array}{ccc}
\xi^2 & \cdots & \xi^2 \\
N-1 & \cdots & N-1 \\
N-2 & \cdots & N-2 \\
N-3 & \cdots & N-3
\end{array} = \text{diag} \begin{array}{ccc}
(1 + \frac{2}{2}; 1, 2, \cdots, 2; 1, 2, \cdots, 2; \cdots; 1, 2; 1)
\end{array},
\] (3.18)

where \( \xi \equiv (1 + |\varphi_0|^2)^{-1} \). Each block separated by the semicolons is labeled by the indices \( a = c \), which run from 1 to \( N \), and the indices \( b = d \) run from \( a = c \) to \( N \) in the \( a \)-th block by the conditions \( b \geq a \) and \( d \geq c \). Noting \( \xi = (1 + |\varphi_0|^2)^{-1} = [\det(1_N + \varphi^\dagger \varphi)]^{-1} = |\sigma|^{2N} e^{-X} \), we can calculate the determinant \( (3.13) \), given by
\[
\det g_{\mu \nu} = N^2 2^{\frac{1}{2}}(N-1)|\sigma|^{2N(N+1)-2} e^{-(N+1)X} \mathcal{K}''(\mathcal{K}')^{-\frac{1}{2}(N+1)} .
\] (3.19)

The Ricci-flat condition \( (3.3) \) becomes
\[
e^{-\frac{1}{2}(N+1)N} \frac{d}{dX} (\mathcal{K}')^{\frac{1}{2}(N+1)+1} = a .
\] (3.20)

4 Ricci-flat Metrics and Kähler Potentials

4.1 Kähler Potentials

We can immediately solve \( (3.10) \), \( (3.17) \) and \( (3.20) \):
\[
d\mathcal{K} = \begin{cases}
(\lambda e^{NX} + b)^{\frac{1}{2}}, & g = M(N - M) + 1 \quad \text{for } G_{N,M} , \\
(\lambda e^{(N-1)X} + b)^{\frac{1}{2}}, & f = \frac{1}{2}N(N-1) + 1 \quad \text{for } \text{SO}(2N)/U(N) , \\
(\lambda e^{(N+1)X} + b)^{\frac{1}{2}}, & h = \frac{1}{2}N(N + 1) + 1 \quad \text{for } \text{Sp}(N)/U(N) ,
\end{cases}
\]

(4.1)

where \( \lambda \) is a constant related to \( a, N \) and \( M \), and \( b \) is an integration constant interpreted as a resolution parameter of the conical singularity. Although these are sufficient to obtain the Kähler metrics using \( (3.1) \), we can calculate Kähler potentials themselves:
\[
\mathcal{K}(X) = \begin{cases}
\frac{g}{N} \left[ (\lambda e^{NX} + b)^{\frac{1}{2}} + b^{\frac{1}{2}} \cdot I \left( b^{-\frac{1}{2}} (\lambda e^{NX} + b)^{\frac{1}{2}} ; g \right) \right] & \text{for } G_{N,M} , \\
\frac{f}{N-1} \left[ (\lambda e^{(N-1)X} + b)^{\frac{1}{2}} + b^{\frac{1}{2}} \cdot I \left( b^{-\frac{1}{2}} (\lambda e^{(N-1)X} + b)^{\frac{1}{2}} ; f \right) \right] & \text{for } \text{SO}(2N)/U(N) , \\
\frac{h}{N+1} \left[ (\lambda e^{(N+1)X} + b)^{\frac{1}{2}} + b^{\frac{1}{2}} \cdot I \left( b^{-\frac{1}{2}} (\lambda e^{(N+1)X} + b)^{\frac{1}{2}} ; h \right) \right] & \text{for } \text{Sp}(N)/U(N) .
\end{cases}
\]

(4.2)
Here the function $I(y; n)$ is defined by

$$I(y; n) \equiv \int_{t_0}^y \frac{dt}{t^n - 1} = \frac{1}{n} \left[ \log (y - 1) - \frac{1 + (-1)^n}{2} \log (y + 1) \right]$$

$$+ \frac{1}{n} \sum_{r=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \cos \frac{2\pi r}{n} \log \left( y^2 - 2y \cos \frac{2\pi r}{n} + 1 \right)$$

$$+ \frac{2}{n} \sum_{r=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \sin \frac{2\pi r}{n} \cdot \arctan \left[ \frac{\cos (2\pi r/n) - y}{\sin (2\pi r/n)} \right].$$

In the limit of $b \to 0$, these manifolds become (generalized) conifolds with their Kähler potentials,

$$K = \begin{cases} \frac{\alpha^+}{N^2} \left[ |\sigma|^{2M} \text{det}(1_M + \varphi^\dagger \varphi) \right]^{N/2} & \text{for } G_{N,M}, \\ \frac{\alpha^+}{(N-1)^2} \left[ |\sigma|^{2N} \text{det}(1_N + \varphi^\dagger \varphi) \right]^{-N-1} & \text{for } SO(2N)/U(N), \\ \frac{\alpha^+}{N(N+1)} \left[ |\sigma|^{2N} \text{det}(1_N + \varphi^\dagger \varphi) \right]^{-N+1} & \text{for } Sp(N)/U(N). \end{cases}$$

4.2 Ricci-flat Kähler Metrics

We can calculate the Ricci-flat Kähler metrics substituting the solutions (4.1) into (3.1). The component $g_{a\sigma*}$ is

$$g_{a\sigma*} = \begin{cases} \lambda_{M+2N}^{\frac{N-1}{g}} (\lambda e^{NX} + b)^{\frac{1}{2} - 1} e^{N\Psi} |\sigma|^{2MN-2} & \text{for } G_{N,M}, \\ \lambda_{N^2(N-1)}^{\frac{N-1}{g}} (\lambda e^{(N-1)X} + b)^{\frac{1}{2} - 1} e^{(N-1)\Psi} |\sigma|^{2(N-1)-2} & \text{for } SO(2N)/U(N), \\ \lambda_{N(N+1)}^{\frac{N}{g}} (\lambda e^{(N+1)X} + b)^{\frac{1}{2} - 1} e^{(N+1)\Psi} |\sigma|^{2(N+1)-2} & \text{for } Sp(N)/U(N), \end{cases}$$

where $\Psi$ was defined in (2.19). These are singular at the surface $\sigma = 0$: $g_{a\sigma*}|_{\sigma=0} = 0$. This singularity is just a coordinate singularity of $z^\mu = (\sigma, \varphi_{Aa})$. To find a regular coordinate, we perform coordinate transformations,

$$\rho \equiv \begin{cases} \sigma^{MN/MN} & \text{for } G_{N,M}, \\ \sigma^{N(N-1)/N(N-1)} & \text{for } SO(2N)/U(N), \\ \sigma^{N(N+1)/N(N+1)} & \text{for } Sp(N)/U(N), \end{cases}$$

with $\varphi_{Aa}$ (or $\varphi_{ab}$) being unchanged. The metrics in the regular coordinates $z^{\mu'} = (\rho, \varphi_{Aa})$ can be calculated, to give

$$g_{\rho\rho*} = \lambda_{M+2N}^{2N} \left( \lambda e^{NX} + b \right)^{\frac{1}{2} - 1} e^{N\Psi},$$

$$g_{\rho(Bb)*} = \lambda_{M+2N}^{2N^2} \left( \lambda e^{NX} + b \right)^{\frac{1}{2} - 1} e^{N\Psi} \rho^* \cdot \partial_{(Bb)*} \Psi,$$

$$g_{(Aa)(Bb)*} = \lambda_{M+2N}^{2N^3} \left( \lambda e^{NX} + b \right)^{\frac{1}{2} - 1} e^{N\Psi} (\rho)^2 \cdot \partial_{(Aa)} \Psi \partial_{(Bb)*} \Psi.$$
for $G_{N,M}$,
\[
g_{pp} = \frac{N^2(N-1)}{f} (\lambda e^{(N-1)X} + b)^{\frac{1}{N-1}} e^{(N-1)\Psi},
\]
\[
g_{cd} = \frac{N^2(N-1)^2}{f} (\lambda e^{(N-1)X} + b)^{\frac{1}{N-1}} e^{(N-1)\Psi} \rho^c \cdot \partial_{cd} \Psi,
\]
\[
g_{ab} = \frac{N^2(N-1)^3}{f} (\lambda e^{(N-1)X} + b)^{\frac{1}{N-1}} e^{(N-1)\Psi} |\rho|^2 \cdot \partial_{ab} \Psi \partial_{cd} \Psi,
\]
\[
+ (\lambda e^{(N-1)X} + b)^{\frac{1}{N-1}} \partial_{ab} \partial_{cd} \Psi,
\]
for $SO(2N)/U(N)$, where $\Psi$ differentiated by the base coordinates $\varphi_{Aa}$ or $\varphi_{ab}$ are given in (3.5) and (3.12).

The metrics of the submanifold defined by $\rho = 0$ ($d\rho = 0$) are
\[
g_{(Aa)(Bb)} |_{\rho=0} (\varphi, \varphi^*) = b^{\frac{1}{N}} \partial_{(Aa)} \partial_{(Bb)} \Psi \quad \text{for } G_{N,M},
\]
\[
g_{(ab)(cd)} |_{\rho=0} (\varphi, \varphi^*) = b^{\frac{1}{N}} \partial_{(ab)} \partial_{(cd)} \Psi \quad \text{for } SO(2N)/U(N),
\]
\[
g_{(ab)(cd)} |_{\rho=0} (\varphi, \varphi^*) = b^{\frac{1}{N}} \partial_{(ab)} \partial_{(cd)} \Psi \quad \text{for } Sp(N)/U(N),
\]
where $\Psi$ is the Kähler potential of these manifolds found in (2.5), and $\Psi$ differentiated by two fields is given in (3.5d) or (3.12c). Therefore we find that the total spaces are the complex line bundles over these Hermitian symmetric spaces as base manifolds, with $\rho$ (or $\sigma$) being a fiber. Actually, it was shown in [22] that there exists a Ricci-flat metric on the complex line bundle over any Kähler-Einstein manifolds. In the limit of $b \to 0$, these base manifolds shrink and the manifolds become conifolds. The conical singularities are resolved by $G_{N,M}$, $SO(2N)/U(N)$ and $Sp(N)/U(N)$ of the radii $b^{1/2g}$, $b^{1/2f}$ and $b^{1/2h}$, respectively.

From the relation of $G_{N,1} = \mathbb{C}P^{N-1}$, we also have the complex line bundle over $\mathbb{C}P^{N-1}$. The Kähler potential (4.2) in the case of $M = 1$ ($g = N$) coincides with the flat one in the limit of $b \to 0$, but with a coordinate identification $\rho = \sigma^N/N$. The singular limit is $\mathbb{C}^N/\mathbb{Z}_N$, and this orbifold singularity is resolved by $\mathbb{C}P^{N-1}$. This coincides with the Calabi metric [15], so our manifolds can be considered as natural generalizations of the Calabi metric.
5 Conclusion and Discussions

We have constructed non-compact Kähler manifolds, modifying the Kähler quotient construction of the Hermitian symmetric spaces of the classical groups by restricting the gauge group $U(M)$ to $SU(M)$. We have presented the Ricci-flat metrics and their Kähler potentials on these manifolds. The essential point was that the partial differential equation (3.3) was reduced to the ordinary differential equations (3.10), (3.17) and (3.20), using the isotropy transformation. These metrics contain the resolution parameter as an integration constant, and the conical singularities are resolved by the Hermitian symmetric spaces with the radii of the resolution parameter. They have been recognized as the line bundle over the Hermitian symmetric spaces, and contain the Calabi metrics on the line bundle over $\mathbb{C}P^{N-1}$. Our manifolds in lower dimensions are discussed in Appendix, which are not included in the list of [24].

Our method can be applied to the cases in which all non-compact directions can be transformed to each other by the isotropy. Such a viewpoint was discussed in [21] in terms of the supersymmetric nonlinear realization. We can also construct other conifolds with the isometry of the exceptional groups from the Hermitian symmetric spaces of the exceptional groups [23]. We would like to discuss whether the deformation parameter exists, as in the case of the conifold [5]. We also would like to clarify the relation between our manifolds of the line bundle over $Sp(2)/U(2) \simeq Q^3$ and the Spin(7) manifold in [25], since both manifolds can be written in the form of $\mathbb{R} \times Sp(2)/SU(2)$. The investigation of superconformal field theories corresponding to our manifolds is also an interesting task.

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A Isomorphism

We have sets of the isomorphism between the lower dimensional base manifolds,

\begin{align}
\text{i)} & \quad \mathbb{C}P^1 \simeq SO(4)/U(2) \simeq Sp(1)/U(1) \simeq Q^1, \\
\text{ii)} & \quad \mathbb{C}P^3 \simeq SO(6)/U(3), \\
\text{iii)} & \quad Sp(2)/U(2) \simeq Q^3, \\
\text{iv)} & \quad G_{4,2} \simeq Q^4,
\end{align}

(A.1)
in addition to the novel duality relation

\[ G_{N,M} \simeq G_{N,N-M}. \quad (A.1e) \]

In this appendix we show that total spaces on these base manifolds coincide. It gives a nontrivial check for our results.

Before doing that we quote the results of the line bundle over \( Q^{N-2} = SO(N)/[SO(N-2) \times U(1)] \), as a conifold \( [10] \). The superfields \( \sigma \) and \( w_i (i = 1, \cdots N-2) \) constitute an \( N \)-vector as \( \Phi^T = (\sigma, w_i, -\frac{1}{2} \sum_{i=1}^{N-2} (w_i)^2) \). The Kähler potential differentiated by the invariant \( X = \log \Phi^\dagger \Phi \) and a coordinate transformation are

\[
\frac{dK}{dX} = (\lambda e^{(N-2)X} + b)^{\frac{N-1}{N-2}}, \quad \sigma = \rho^{N-2} / N-2. \quad (A.2)
\]

Note that the notation of \( X \) is different from that of \([10]\).

i) \textit{Eguchi-Hanson space}. All of the lowest dimensional manifolds of \((A.1d)\) coincide with the Eguchi-Hanson gravitational instanton \([7] \). The matrix field \( \Phi_I (I = 1, 2, 3, 4) \) and invariants \( X_I \equiv \log \det \Phi_I \Phi_I \) are

\[
\Phi_1 = \sigma_1 \begin{pmatrix} 1 \\ \varphi_1 \end{pmatrix}, \quad X_1 = \log |\sigma_1|^2 + \log(1 + |\varphi_1|^2), \quad (A.3a)
\]

\[
\Phi_2 = \sigma_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \varphi_2 \\ -\varphi_2 & 0 \end{pmatrix}, \quad X_2 = 2 \log |\sigma_2|^2 + 2 \log(1 + |\varphi_2|^2), \quad (A.3b)
\]

\[
\Phi_3 = \sigma_3 \begin{pmatrix} 1 \\ \varphi_3 \end{pmatrix}, \quad X_3 = \log |\sigma_3|^2 + \log(1 + |\varphi_3|^2), \quad (A.3c)
\]

\[
\Phi_4 = \sigma_4 \begin{pmatrix} 1 \\ \varphi_4 \\ -\frac{1}{2} (\varphi_4)^2 \end{pmatrix}, \quad X_4 = \log |\sigma_4|^2 + 2 \log \left(1 + \frac{1}{2} |\varphi_4|^2\right), \quad (A.3d)
\]

for the cases of the base manifolds \( CP^1, SO(4)/U(2), Sp(1)/U(1) \) and \( Q^1 \), respectively. Relations of the base and the fiber coordinates of these four manifolds are \( \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 / \sqrt{2} \) and \( (\sigma_1)^2 \sim (\sigma_2)^2 \sim (\sigma_3)^2 \sim (\sigma_4)^2 \sim \rho \), respectively. Note that each fiber \( \sigma_I (I = 1, 2, 3, 4) \) consistently defines the same \( \rho \) as the regular coordinate from \((4.6)\) and \((A.2)\). The Kähler potentials coincide up to a overall constant:

\[
\frac{dK}{dX_I} \sim \sqrt{\lambda e^{2X_1} + b} \sim \sqrt{\lambda e^{X_2} + b} \sim \sqrt{\lambda e^{2X_3} + b} \sim \sqrt{\lambda e^{X_4} + b}. \quad (A.4)
\]

The orbifold singularity in \( C^2/Z_2 \) is resolved by \( S^2 \).
ii) Complex four-dimensional Calabi metric. The matrix field $\Phi_I (I = 1, 2)$ and invariants $X_I \equiv \log \det \Phi^*_I \Phi_I$ are $(i = 1, 2, 3)$

$$
\Phi_1 = \sigma_1 \begin{pmatrix} 1 \\ w_i \end{pmatrix}, \quad X_1 = \log |\sigma_1|^2 + \log \left( 1 + \sum_{i=1}^{3} |w_i|^2 \right), 
\Phi_2 = \sigma_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varphi_1 & \varphi_2 \\ -\varphi_1 & 0 & \varphi_3 \\ -\varphi_2 & -\varphi_3 & 0 \end{pmatrix}, \quad X_2 = 3 \log |\sigma_2|^2 + 2 \log \left( 1 + \sum_{i=1}^{3} |\varphi_i|^2 \right),
$$

(A.5a) (A.5b)

for the cases of the base manifolds $\mathbb{C}P^3$ and $SO(6)/U(3)$, respectively. Identifications of the base and the fiber coordinates are $w_i = \varphi_i$ and $(\sigma_1)^4 \sim (\sigma_2)^6 \sim \rho$, respectively, where each fiber $\sigma_I (I = 1, 2)$ consistently defines the same $\rho$ as the regular coordinate from (4.4). The Kähler potentials coincide up to an overall constant:

$$
\frac{dK}{dX_I} \sim (\lambda e^{4X_1} + b)^{\frac{1}{4}} \sim (\lambda e^{2X_2} + b)^{\frac{1}{4}}.
$$

(A.6)

The orbifold singularity in $\mathbb{C}^4/\mathbb{Z}_4$ is resolved by $\mathbb{C}P^3 \simeq SO(6)/U(3)$.

iii) Another metric with complex four dimensions. The matrix field $\Phi_I (I = 1, 2)$ and invariants $X_I \equiv \log \det \Phi^*_I \Phi_I$ are $(i = 1, 2, 3)$

$$
\Phi_1 = \sigma_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_1 = 2 \log |\sigma_1|^2 + \log \left( 1 + \sum_{i=1}^{3} |\varphi_i|^2 + \frac{1}{2} |\varphi_3 - \varphi_1 \varphi_2|^2 \right),
\Phi_2 = \sigma_2 \begin{pmatrix} 1 \\ w_i \end{pmatrix}, \quad X_2 = \log |\sigma_2|^2 + \log \left( 1 + \sum_{i=1}^{3} |w_i|^2 + \frac{1}{4} \sum_{i=1}^{3} \sum_{i=1}^{3} |w_i|^2 \right),
$$

(A.7a) (A.7b)

for the cases of the $Sp(2)/U(2)$ and $Q^3$ base manifolds. Identifications of the base and the fiber coordinates are

$$
\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},
$$

(A.8)

and $(\sigma_1)^6 \sim (\sigma_2)^3 \sim \rho$, respectively. Again, each fiber $\sigma_I (I = 1, 2)$ consistently defines the same $\rho$ as the regular coordinate from (4.6) and (A.2). The Kähler potentials coincide up to an overall
constant:

\[ \frac{dK}{dX_I} \sim (\lambda e^{3X_1} + b)^\frac{1}{4} \sim (\lambda e^{3X_2} + b)^\frac{1}{4}. \]  

(A.9)

iv) **The line bundle over the Klein quadric (with complex five dimensions).** The embedding of \( G_{4,2} \) into \( CP^5 \) is known as the Plücker embedding. The matrix field \( \Phi_I \) \((I = 1, 2) \) and invariants \( X_I \equiv \log \det \Phi_I^\dagger \Phi_I \) are \((i = 1, 2, 3, 4) \):

\[
\Phi_1 = \sigma_1 \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\varphi_1 & \varphi_3 \\
\varphi_4 & \varphi_2
\end{pmatrix}, \quad X_1 = 2 \log |\sigma_1|^2 + \log \left(1 + \sum_{i=1}^{4} |\varphi_i|^2 + |\varphi_1\varphi_2 - \varphi_3\varphi_4|^2\right),  
\]

(A.10a)

\[
\Phi_2 = \sigma_2 \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\frac{1}{2} \sum_{i=1}^{4} w_i^2 & w_i
\end{pmatrix}, \quad X_2 = \log |\sigma_2|^2 + \log \left(1 + \sum_{i=1}^{4} |w_i|^2 + \frac{1}{4} \sum_{i=1}^{4} w_i^2 \right)^2, \quad (A.10b)
\]

for the cases of the base manifolds \( G_{4,2} \) and \( Q^4 \). Identifications of the base and the fiber coordinates are

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & i/\sqrt{2} & 0 & 0 \\
\frac{1}{\sqrt{2}} & -i/\sqrt{2} & 0 & 0 \\
0 & 0 & i/\sqrt{2} & -1/\sqrt{2} \\
0 & 0 & i/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix} \begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{pmatrix}, \quad (A.11)
\]

and \((\sigma_1)^8 \sim (\sigma_2)^4 \sim \rho\), respectively, from (4.6) and (A.2). Each fiber defines the same the regular fiber coordinate \( \rho \). The Kähler potentials coincide up to an overall constant:

\[ \frac{dK}{dX_I} \sim (\lambda e^{4X_1} + b)^\frac{1}{8} \sim (\lambda e^{4X_2} + b)^\frac{1}{8}. \]  

(A.12)

v) **Duality between the Grassmann manifolds.** The matrix field \( \Phi_I \) \((I = 1, 2) \) and invariants \( X_I \equiv \log \det \Phi_I^\dagger \Phi_I \) are

\[
\Phi_1 = \sigma_1 \begin{pmatrix}
1_M \\
\varphi_1
\end{pmatrix}, \quad X_1 = M \log |\sigma_1|^2 + \log \det(1_M + \varphi_1^\dagger \varphi_1), \quad (A.13a)
\]

\[
\Phi_2 = \sigma_2 \begin{pmatrix}
1_{N-M} \\
\varphi_2
\end{pmatrix}, \quad X_2 = (N - M) \log |\sigma_2|^2 + \log \det(1_{N-M} + \varphi_2^\dagger \varphi_2), \quad (A.13b)
\]

for the cases of the base manifolds \( G_{N,M} \) and \( G_{N,N-M} \), respectively. Here \( \varphi_1 \) and \( \varphi_2 \) are \([N - M] \times M \) and \([M \times (N - M)] \)-matrices, respectively. Identifications of the base and the fiber coordinates are \( \varphi_1 = \varphi_2^T \), and \((\sigma_1)^{N,M} \sim (\sigma_2)^{N(N-M)} \sim \rho\), respectively, due to (4.6), in which
each fiber defines the same regular coordinate $\rho$. The Kähler potentials coincide up to an overall constant:

$$ \frac{dK}{dX_I} \sim (\lambda e^{NX_1} + b)^{\frac{1}{\theta}} \sim (\lambda e^{NX_2} + b)^{\frac{1}{\theta}}. \quad (A.14) $$

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