Junctions and thin shells in general relativity using computer algebra
I: The Darmois-Israel Formalism

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Abstract

We present the GRjunction package which allows boundary surfaces and thin-shells in general relativity to be studied with a computer algebra system. Implementing the Darmois-Israel thin shell formalism requires a careful selection of definitions and algorithms to ensure that results are generated in a straight-forward way. We have used the package to correctly reproduce a wide variety of examples from the literature. We present several of these verifications as a means of demonstrating the packages capabilities. We then use GRjunction to perform a new calculation - joining two Kerr solutions with differing masses and angular momenta along a thin shell in the slow rotation limit.

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1 Introduction

The Darmois-Israel junction/thin-shell formalism has found wide application in general relativity and cosmology \cite{1,2}. The junction of dust to Schwarzschild by Oppenheimer and Snyder allowed the first insights into the nature of gravitational collapse to a black hole \cite{3}. Since Israel’s landmark paper \cite{2} the formalism has been applied in a number of contexts ranging from further studies of gravitational collapse to the evolution of bubbles and domain walls in a cosmological setting.

In this paper we describe the GRjunction package we have developed to assist relativists in the evaluation of junction conditions and the parameters associated with thin-shells (the package is available free of charge \cite{4}). At the present time the package deals only with non-null surfaces - although efforts to extend this to null shells are underway. GRjunction runs under the computer algebra system Maple \cite{5} in conjunction with GRTensorII \cite{6}. Our goal in creating the package was to ensure that it could easily recover all the standard shell results in the literature (the bulk of which assume spherical symmetry) without biasing the package towards spherical symmetry in any way - allowing users to probe the relatively unstudied area of non-spherical shells and junctions. The package is necessarily interactive allowing users to manipulate results and determine the conditions for junctions.

We begin by outlining the shell formalism to establish notation and motivate choices of algorithms which we describe in the following section. We then demonstrate the package by repeating some standard junction and shell calculations. Next we present some new results relating to the study of shells around slowly spinning black holes. To validate the package we re-executed a number of the standard results in the literature. A summary of these tests appears in the final section.

The intention of this paper is to describe the junction package we have developed and not to review the vast literature on junctions and thin shells. The references we have chosen are not always the first or simplest treatment of a problem and in some cases we have deliberately selected examples which differ from the standard treatments to test the robustness of our package.

2 The Formalism

In this section we review the junction formalism to establish notation. For more detailed discussions see e.g.\cite{6-10}.

Consider two spacetimes (Lorentzian manifolds with signature (+ + + −)) \(M^+\) and \(M^-\) with metrics \(g^{\alpha \beta}_{\pm}(x^\pm_\gamma)\) and \(g^-_{\alpha \beta}(x^-_\gamma)\) in the coordinate systems \(x^+_\gamma\) and \(x^-_\gamma\). Within these spacetimes define two non-null 3-surfaces \(\Sigma^+\) and \(\Sigma^-\) (in
$M^+$ and $M^-$ respectively) with metrics $g^+_{ij}(\xi^c_\pm)$ and $g^-_{ij}(\xi^c_\mp)$ in the coordinates $\xi^c_+$ and $\xi^c_-$ which decompose each of the 4-manifolds into two distinct parts. (Greek indices range over the coordinates of the 4-manifold and Roman indices over the coordinates of the 3-surfaces). We label the distinct parts of $M^+$ created by $\Sigma^+$ as $M^+_1$ and $M^+_2$ and likewise for $M^-$. The junction/shell formalism constructs a new manifold $\mathcal{M}$ by joining one of the distinct parts of $M^+$ to one of the distinct parts of $M^-$ by the identification $\Sigma^+ = \Sigma^- \equiv \Sigma$. Clearly there are four possibilities, i.e. $M^+_1 \cup M^-_1$, $M^+_2 \cup M^-_1$, $M^+_1 \cup M^-_2$, $M^+_2 \cup M^-_2$. The assumed isometry between the points on the surface is often (but need not always be) the identification $\xi^c_+ = \xi^c_-$. What now follows holds simultaneously for $M^+$ and $M^-$ and so we drop the $\pm$ distinction in this paragraph. The parametric equation for $\Sigma$ is of the form

$$f(x^\alpha(\xi^a)) = 0.$$ \hfill (1)

We assume that $\Sigma$ is non-null. The unit 4-normals to $\Sigma$ in $M$ are given by

$$n_\alpha = \pm \frac{1}{(g^{\beta\gamma} \frac{\partial f}{\partial x^\beta} \frac{\partial f}{\partial x^\gamma})^{1/2}} \frac{\partial f}{\partial x^\alpha}.$$ \hfill (2)

We assume $n_\alpha \neq 0$ and label $\Sigma$ as timelike (spacelike) for $\Delta \equiv -n_\alpha n^\alpha = -1(1)$. The three basis vectors tangent to $\Sigma$ are

$$e^\alpha_{(a)} = \frac{\partial x^\alpha}{\partial \xi^a}$$ \hfill (3)

which give the induced metric on $\Sigma$ by

$$g_{ij} = \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} g_{\alpha\beta}.$$ \hfill (4)

The extrinsic curvature (second fundamental form) is given by

$$K_{ij} = \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \nabla_\alpha n_\beta$$ \hfill (5)

$$= -n_\gamma \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right).$$ \hfill (6)

We define $u^\alpha$ the four tangent to $\Sigma$ and $\dot{u}^\alpha(\equiv u^\beta \nabla_\beta u^\alpha)$ the four-acceleration. Combining these with the above relations gives

$$[n_\alpha \dot{u}^\alpha] = -u^i u^j [K_{ij}]$$ \hfill (7)

and

$$n_\alpha \ddot{u}^\alpha = -u^i u^j K_{ij}$$ \hfill (8)

where $[X] \equiv X^+ |_{\Sigma} - X^- |_{\Sigma}$ and $\overline{X} \equiv (X^+ |_{\Sigma} + X^- |_{\Sigma})/2$ with $X^\pm |_{\Sigma}$ denoting the limiting values of $X$ on $\Sigma$. 

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The Darmois conditions for the joining of a part of $M^+$ to a part of $M^-$ are

\[ [g_{ij}] = 0 \quad (9) \]

and

\[ [K_{ij}] = 0 \quad (10) \]

If both (9) and (10) are satisfied we refer to $\Sigma$ as a boundary surface. If only (9) is satisfied then we refer to $\Sigma$ as a thin-shell.

Conditions (9) and (10) require a common coordinate system on $\Sigma$ and this is easily done if one can set $\xi^a_+ = \xi^a_-$. Failing this, establishing (9) requires a solution to the three dimensional metric equivalence problem. Condition (10) as it stands is ambiguous since the orientation of the 4-vector field $n_\alpha \equiv n^\pm_\alpha |_\Sigma$ has not been specified. The Israel formalism requires the normals in $\mathcal{M}$ to point from $M^-_A$ to $M^+_B$ (where $A$ denotes the part of $M^-$ and $B$ denotes the part of $M^+$ we wish to use to form $\mathcal{M}$). Clearly the sign of the normal vectors are crucial since e.g. $n^-_\alpha$ points away from the portion of $M^-$ which will be used in forming $\mathcal{M}$. Hence an understanding of which side $n^-_\alpha$ points into is key. In general this can be done by considering a trajectory in $M^-$ through $\Sigma^-$ with tangent $n^-_\alpha$ (and likewise for $n^+_\alpha$). The majority of the existing literature deals with spherical symmetry where the direction of the normal is clear, but in more complicated examples (see below) great care must be taken. Note that while there are two normal vectors in each of $M^\pm$ once we have identified $\Sigma^-$ and $\Sigma^+$ there is a single unique normal field to $\Sigma$ in $\mathcal{M}$. There may be circumstances in which one can determine the differential relation between the coordinates of $M^-_A$ and $M^+_B$ via

\[ n^-_\alpha = \frac{\partial x^\alpha_+}{\partial x^-_\beta} n^+_\beta \quad (11) \]

in an open neighbourhood of $\Sigma$ in $\mathcal{M}$ and this will give the direction of one normal vector relative to the other. (In the case of a boundary surface often only the sign of $n_\alpha$ on one side of $\Sigma$ needs to be determined. Since $\nabla_\alpha n^\alpha = K^i_i$ in the case of a boundary surface (10) gives the useful relation $[\nabla_\alpha n^\alpha] = 0$.)

Some studies have left the sign of the normal vectors unspecified to exhaustively study the taxonomies of all possible combinations of $M^-$ and $M^+$ (usually excluding those which require shells which violate energy conditions e.g. [2, 3]). The junction package allows the user to leave the sign unspecified but we take the view that the “typical” starting point is to explicitly choose the signs of the normal vectors so that a particular combination of $M^-_A$ and $M^+_B$ can be studied.

Once we have selected signs in (2) there is no ambiguity in (10) and we use (9) and (10) in conjunction with the Einstein tensor $G_{\alpha\beta}$ to obtain the identities

\[ [G_{\alpha\beta} n^\alpha n^\beta] = 0 \quad (12) \]

\[ 4 \]
and

\[ G_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} n^\beta = 0. \]  
(13)

(Note that (12) and (13) do not guarantee (9) and (10).) This shows, for example, that for timelike \( \Sigma \) the flux through \( \Sigma \) (as measured comoving with \( \Sigma \)) is continuous.

The Israel formulation of thin shells follows from the Lanczos equation

\[ S_{ij} = \frac{\Delta}{8\pi} (\left[ K_{ij} - g_{ij} \left[ K^i \right] \right]) \]  
(14)

and we refer to \( S_{ij} \) as the surface stress-energy tensor of \( \Sigma \). The “ADM” constraint

\[ \nabla_j K^j - \nabla_i K = G_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} n^\beta \]  
(15)

along with Einstein’s equations then gives the conservation identity

\[ \Delta \nabla_i S^i_j = \left[ T_{\alpha\beta} n^\alpha \frac{\partial x^\beta}{\partial \xi^i} n^\beta \right]. \]  
(16)

The “Hamiltonian” constraint

\[ G_{\alpha\beta} n^\alpha n^\beta = \left( \Delta (\beta R) + K^2 - K_{ij} K^{ij} \right)/2 \]  
(17)

gives the evolution identity

\[ -S^{ij} K_{ij} = \left[ T_{\alpha\beta} n^\alpha n^\beta \right]. \]  
(18)

The identities (16) and (18) do not give information about the dynamics of the shell. The evolution of the shell stems from a phenomenological interpretation and the Lanczos equation (14).

### 2.1 Phenomenology

The standard phenomenology associated with \( \Sigma \) is introduced as follows: let \( \xi^a \) be the coordinates intrinsic to \( \Sigma \) and consider a trajectory \( \xi^a(\tau) \) with associated 3-tangent

\[ u^a = \frac{d\xi^a}{d\tau} \]  
(19)

where \( u^a u_a = \Delta \). For a timelike surface we view the curve as the worldline of a (possibly hypothetical) particle in the surface with \( \tau \) the proper time. (If the surface is spacelike the phenomenology is a formal analogy). The associated four tangent is

\[ u_+^a = \frac{\partial x^a}{\partial \xi^a} u^a = \frac{dx^a}{d\tau}. \]  
(20)

Equivalently

\[ u_a = u_+^a \frac{\partial x^a}{\partial \xi^a} \]  
(21)
We view $\Sigma$ as covered by a 3-vector field $u^a$. Note that $\tau$ is defined curve by curve on $\Sigma$ but not in general over the entire surface. If $\tau$ is defined over the entire surface then we label this as a "one-parameter surface". The surface energy density $\sigma$ associated with the trajectory $u^a$ on $\Sigma$ is defined by

$$S_{ab}u^b = -\sigma(\xi^c)u_a - q_a$$

(22)

where $u_a q^a = 0$. The inclusion of $q_a$ represents an intrinsic energy flux orthogonal to the prescribed intrinsic velocity field. The surface energy density $\sigma(\xi^c)$ is not in general an eigenvalue but is given by

$$\sigma(\xi^c) = -\Delta S_{ab}u^a u^b.$$  

(23)

In analogy to a perfect 4-fluid we suppose that $q^a = 0$ and that $S_{ab}$ takes the form

$$S_{ab} = -\Delta(\sigma(\xi^c) + p(\xi^c))u_au_b + p(\xi^c)g_{ab}.$$  

(24)

This defines the surface pressure (surface tension)

$$p(\xi^c) = (\sigma(\xi^c) + S_{a}^a)/2.$$  

(25)

It now follows from (16) that

$$\dot{\sigma} + (\sigma + p)\Phi = -\Delta \left[ T_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi_i} n^\beta \right]$$

(26)

where $\dot{\cdot}$ signifies the intrinsic 3-derivative along $u^a$ ( $\dot{\sigma} = u^a \nabla_a \sigma$) and $\Phi$ is the three-expansion of $u^a$ ($\Phi \equiv \nabla_a u^a$). From (14), the Lanczos equation (14) and the phenomenological equations (23) and (25) we have

$$[n_\alpha \dot{u}^\alpha] = 8\pi (p + \sigma/2).$$  

(27)

From (6), (14) and the identity (18) the phenomenology gives

$$\Delta(\sigma + p)n_\alpha \dot{u}^\alpha + pK^i_i = -\Delta \left[ T_{\alpha\beta} n^\alpha n^\beta \right].$$  

(28)

In spherical spacetimes the first integral of the evolution equation (28) is given by the identity (11)

$$\dot{R}^2 = \Delta + \left( \frac{[m]}{M} \right)^2 - \frac{2\Delta m}{R} + \left( \frac{M}{2R} \right)^2$$

(29)

where $R = R(\tau), \dot{\tau} \equiv d/d\tau$ and $M = 4\pi \sigma R^2$. The effective mass, $m_{\pm}$ is given by

$$m_{\pm} \equiv \frac{1}{2} \left( \frac{\theta^i}{\theta^i} \right)^{3/2} (4) R_{\theta\phi}^{\pm} \theta^b.$$  

(30)
3 Computer Implementation

Specifying the junction formalism for a computer algebra system requires a careful choice of specific object definitions and recognition of several standard calculus manipulations. Joining spacetimes requires the simultaneous consideration of four metrics, so the underlying general relativity software must permit this. In order to make the package reproduce the existing literature on spherical shells special consideration must be given to the choice of object definitions for one-parameter shells. We first present an overview of the package and then discuss the choices we have made in implementing the junction/thin shell package for the Maple version of GRTensorII.

3.1 Overview

The GRjunction package provides the user with a means to specify a surface, calculate intrinsic and extrinsic quantities on the surface, identify two such surfaces and evaluate whether a boundary surface or thin-shell results. The specification of a surface is done by invoking the command surf which then prompts the user for the necessary information. Once a surface has been specified objects defined on the surface (e.g. $K_{ij}$) can be calculated. The identification of two surfaces is performed via the command join which calculates $[g_{ij}]$ and displays the result. If $[g_{ij}] \neq 0$ the user can manipulate this expression in an attempt to determine restrictions on metric functions which will ensure $[g_{ij}] = 0$ (see the first example below). The jump or mean of any quantity in the joined manifold can be evaluated by means of the operators Jump and Mean. Hence to determine $[K_{ij}]$ the user would refer to the object Jump$[K(dn,dn)]$; to determine $\overline{K_{ij}}$, Mean$[K(up,up)]$. Standard quantities and equations for the joined spacetimes can be calculated in a straightforward manner. For example to determine $S^j_i$ the user refers to the object S3(dn,up). (By convention we list dn indices ahead of up indices for mixed two index objects).

While we have emphasised the Darmois-Israel formalism, GRjunction can trivially evaluate the Lichnerowicz junction conditions [14] which consist of

\[
\begin{align*}
[g_{\alpha\beta}] &= 0 \quad (31) \\
[\partial g_{\alpha\beta}/\partial x^\gamma] &= 0 \quad (32)
\end{align*}
\]

(Recall these conditions require admissible coordinates and consequently are not as general the Darmois-Israel conditions). The object $\partial g_{\alpha\beta}/\partial x^\gamma$ is referred to as $g(dn,dn,cdn)$ (cdn denoting the partial derivative index) and so (32) can be evaluated by referring to the object Jump$[g(dn,dn,cdn)]$. GRTensorII also allows users to work within the Newman-Penrose formalism and the GRjunction will allow users to evaluate jumps in the spin coefficients across a surface. (This technique is employed in e.g. [15]).
3.2 The surface and related quantities

In this section we describe how to specify a non-null surface to the junction package and the intrinsic (on Σ) and extrinsic quantities which can be calculated with the package. In this section we restrict attention to quantities which are independent of phenomenology.

All the expressions in this section should implicitly carry a ± designation (which we omit) with the exception of the $\xi^i$, the coordinates on Σ. For the package to compare first and second fundamental forms we must have $\xi^i_+ = \xi^i_-$. The package does not currently consider the three-metric equivalence problem.

In general to specify a surface Σ in a space $M$ with coordinates $x^\alpha$ in sufficient detail to allow calculation of the first and second fundamental forms we require

- the coordinates $\xi^i$ on Σ
- the coordinate definition of Σ: $x^\alpha = x^\alpha(\xi^i)$.
- the parametric definition of Σ: $f(x^\alpha) = 0$
- the choice of normal vector sign in (2)

The essential idea of the Darmois/Israel formalism is to use intrinsic quantities in the description of all objects of interest on Σ. However many of the definitions of objects on the surface are in terms of quantities in $M$. For example (3) uses the normal vector which involves partial derivatives with respect to the coordinates of $M$ and Christoffel symbols of $M$. This can frequently be resolved simply by substituting the coordinate definition of the surface (i.e. $x^\alpha = x^\alpha(\xi^i)$) but this is not always desirable. Consider a metric which has an arbitrary function $u(r, \theta)$ and a definition of Σ with coordinates $\xi^a = (\hat{\theta}, \hat{\phi}, \tau)$ and which has a coordinate definition which includes $r = f(\xi^i)$ and $\theta = \tilde{\theta}$. The Christoffel symbols used in defining $K_{ij}$ may contain partial derivatives of $u$ with respect to both $r$ and $\theta$. Substitution of $\theta = \hat{\theta}$ will merely change the variable, but the $r$ substitution will result in $\partial u(f(\xi^i), \hat{\theta})/\partial f(\xi^i)$. This is more than notationally ugly - it constitutes an error in Maple; you cannot take a partial derivative with respect to a function. In these cases we make use of the Maple operator $D$ (instead of the Maple procedure $\text{diff}$). For example the representation of the $r$ derivatives of $f(r)$ and $\nu(r, \theta)$ on the surface are represented in Maple as

\[
\frac{d}{dr} f(r) \bigg|_{r=R(t)} \to D(f)(R(t))
\]

\[
\frac{d}{dr} \nu(r, \theta) \bigg|_{r=R(t)} \to D[1](\nu)(R(t), \theta)
\]

(the $[1]$ indicates that the derivative acts on the first argument of the function $\nu$). It is essential that we make use of this form of representing derivatives to...
ensure that their time dependence is handled properly in subsequent calculations. While we could use the \(D\) derivative throughout we prefer to minimize it’s use since in the Maple output we find it preferable to have expressions with \(\partial f(r)/\partial r\) instead of \(D(f)(r)\). Consequently object definitions which take derivatives and then evaluate values on the surface make use of code which substitutes in the coordinate relations and ensures that the substitution does not cause errors in derivatives. This code then minimizes the number of \(D\) derivatives which appear by converting extraneous occurrences back into Maple’s usual \texttt{diff} derivative.

### 3.3 One-Parameter Surfaces

A large part of the existing shell literature has dealt with timelike spherical 3-surfaces within spherical 4-manifolds. The majority of these analyses define one of the \(\xi^a\) to be the proper time on the surface (what we referred to above as one-parameter shells). We must take some care in choosing algorithms for objects in these cases since some of the routine steps in a hand calculation are best avoided in a computer algebra approach. All of the algorithm issues arise in the specification of a surface in one of \(M^\pm\) so we limit our discussion to the specification of one surface.

First we describe two standard timelike surfaces we will make reference to in our discussion: a static spherically symmetric surface and the dynamic counterpart within a spherically symmetric 4-manifold. In the 4-manifold we take coordinates \((r, \theta, \phi, t)\) and on the 3-surface coordinates \((\tilde{\theta}, \tilde{\phi}, \tau)\). For the static shell we use the coordinate definitions

\[
r = R, \theta = \tilde{\theta}, \phi = \tilde{\phi}, t = T_s(\tau) \tag{36}
\]

and a surface equation \(r - R = 0\). For the dynamic shell we use

\[
r = R(\tau), \theta = \tilde{\theta}, \phi = \tilde{\phi}, t = T_d(\tau) \tag{37}
\]

and a surface equation \(r - R(\tau) = 0\).

In both cases it is conventional to ensure that the coefficient of \(d\tau^2\) on \(\Sigma\) is \(-1\) (hence \(\tau\) is the proper time on the shell). For this reason we express \(t\) as a function of \(\tau\) and then make use of the constraint \(u_\alpha u^\alpha = -1\) to eliminate e.g. \(\partial T_d(\tau)/\partial \tau\) from the quantities we calculate on \(\Sigma\). This facility is built into the \texttt{surf} command. The user is asked if the surface has a parameter which governs its evolution. If the user so indicates then the package evaluates \(u_\alpha u^\alpha\) and asks if there is a quantity to be eliminated. An attempt is then made to solve the tangent constraint so that this quantity can be eliminated. The tangent constraint is then automatically applied during the calculation of \(n_\alpha\) and \(K_{ij}\).

We must specify an algorithmic approach to calculating quantities such as \(K_{ij}\) and \(\dot{u}^\alpha\) which recover the results for one-parameter shells in a straightforward way. Not all object definitions are equivalent because some require
steps which are obvious in a hand calculation but are difficult for a computer algebra system. One example of this is the calculation of $\dot{u}^\alpha$. For a one parameter shell with $x^\alpha = x^\alpha(\tau)$ we would proceed with the definition:

$$u_\beta \nabla_\beta u^\alpha = u_\beta \left( \frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma \right)$$  \hfill (38)

$$= \frac{dx^\beta}{d\tau} \frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma u^\beta$$

$$= \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\gamma u^\beta$$  \hfill (39)

While in a hand calculation using the definition of a total derivative to get the last line is an obvious step this is not so in a computer algebra system. Once specific components are used in (38) and partial derivatives are taken, recognizing terms which can collected into total derivatives with respect to $\tau$ becomes a problem in pattern matching. To avoid such problems we require that the user indicate the variable to be used for total derivatives and we use the relation (39) instead of (38). The same issue arises in the calculation of $K_{ij}$ for the dynamic shell discussed above if we use (38) so we use the definition (39). This definition has the advantage that it is expressed in terms of derivatives with respect to intrinsic quantities (although we still need to evaluate the Christoffel symbols of the 4-manifold on $\Sigma$).

Another computer algebra issue arises in the calculation of the normal vector. For one-parameter surfaces it is customary to write the equation of the surface in terms of a function of a parameter (e.g. $r - R(\tau) = 0$). Yet the covariant derivative in $M$ will require partial derivatives with respect to the $x^\alpha$. The junction package requires that the surface equation be in terms of the $x^\alpha$ (so for the dynamic shell example above we specify $r - R(t)$ instead of $r - R(\tau)$). In these cases the package then applies the chain rule

$$\frac{dr}{dt} \rightarrow \frac{dR}{dT} \rightarrow \frac{dR(t)}{d\tau} \frac{d\tau}{dt} \rightarrow \frac{dR(\tau)}{dT(\tau)} \frac{dT(\tau)}{d\tau}.$$  \hfill (40)

where we have made use of the relations $r = R(\tau)$ and $t = T(\tau)$ and that $\dot{T} \neq 0$. If the surface equation is governed by a parameter other than $\tau$ then the normal vector and $K_{ij}$ can be expressed in terms of intrinsic quantities only if the function dependence on the other $\xi^i$ can be given explicitly in the surface equation (e.g. $r - r(t)\cos\theta = 0$). If all that is known is e.g. $r - R(t, \theta)$ then we cannot make use of (40) and $n_\alpha$ and $K_{ij}$ will be left in terms of the partial derivative with respect to the coordinates of $M$.

With these definitions and constraints we now have available the quantities: $n_\alpha, u_\alpha, \dot{u}^\alpha, g_{ij}$ and $K_{ij}$ in terms of intrinsic quantities. We require a few additional objects which are calculated in a straight-forward way. A summary of all the objects which can be calculated on a surface by the junction package is provided in Tables 1 and 2.
| Object | Definition                                                                 | Name | Comments                                                                 |
|--------|---------------------------------------------------------------------------|------|--------------------------------------------------------------------------|
| $x^\alpha(\xi^i)$ | $x^\alpha(\xi^i)$                                                       | xform(up) | coordinate definition of $\Sigma$                                      |
| $\Delta$ | $-n^\alpha n_\alpha$                                                    | utype | surface                                                                  |
| $f$    | $f(x^\alpha(\xi^i))$                                                    | surface | parametric surface equation                                               |
| $n_\alpha$ | $\partial f/\partial x^\alpha$ (normalized)                     | n(dn) | the normal to $\Sigma$                                                  |
| $u^\alpha$ | $dx^\alpha(\xi^i)/d\tau$                                           | u(up) | four tangent to $\Sigma$                                                |
| $\tau$  |                                                                               | totalVar | (one parameter shells only)                                               |
| $u^\alpha$ | $du^\alpha/d\tau + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$ | udot(up) | four acceleration of $\Sigma$                                          |
| $m$    | $\frac{1}{2}(4g_{\theta\theta})^{3/2}$ $(4)R_{\theta\phi} \theta\phi$ | mass  | (one parameter shells only)                                               |

Table 1: Objects in $M^\pm$

| Object | Definition                                                                 | Name | Comments                                                                 |
|--------|---------------------------------------------------------------------------|------|--------------------------------------------------------------------------|
| $\xi^i$ |                                                                               | x(up) | coordinates on $\Sigma$                                                |
| $g_{ij}$ | $\frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} + \frac{G_{\alpha\beta}}{2}$ | g(dn,dn) | first fundamental form                                                  |
| $K_{ij}$ | $-n_\gamma \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right)$ | K(dn,dn) | second fundamental form                                                 |
| $K_i^j$  |                                                                               | trK  | Hamiltonian constraint                                                  |
| $K_{ij}K^{ij}$ | $(3)R + K^2 + K_{ij}K^{ij})/2$ $= G_{\alpha\beta}n^\alpha n^\beta$ | Ksq  | Momentum constraint                                                     |
| $\nabla_i K - \nabla_j K_i^j = G_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} n^\beta$ | HCGeqn |                                             |
| $K_i^jK^{ij}$ | $(3)R + K^2 + K_{ij}K^{ij})/2 = 8\pi T_{\alpha\beta} n^\alpha n^\beta$ | HCTeqn |                                             |
| $\nabla_i K - \nabla_j K_i^j = 8\pi T_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} n^\beta$ | PCGeqn |                                             |
| $\nabla_i u^i$ |                                                                               | PCTeqn |                                             |
| $u^i$    | $\frac{\partial x^\alpha}{\partial \xi^i} u_\alpha$                   | u3(up) | three tangent                                                            |
| $\Phi$  | $\nabla_i u^i$                                                             | u3div | (one parameter surf. only)                                               |

Table 2: Objects on $\Sigma^\pm$
3.4 Objects in $\mathcal{M}$

Once the user has loaded two 4-manifolds and specified two 3-surfaces the command `join` is used to identify the two surfaces. A variety of objects relating to the boundary surface or shell can now be calculated (Tables 3 and 4).

The user now has four active metrics in the GRTensorII session ($\Sigma^\pm, M^\pm$). After using `join` the default metric is $\Sigma^+$. The operators `Jump` and `Mean` will by default make reference to objects in $\Sigma^\pm$. In general these operators can be used to take the jump or mean of objects from the default metric to any user specified metric. For example if the user had changed the default metric to $M^+$ (say `Schw`) and wished to evaluate (31) with $M^-$ as e.g. `SchwInterior` this would be done by using the metric name of $M^-$ as a second parameter to `Jump` i.e. `Jump[g(dn,dn),SchwInterior]`.

In practice users may wish to structure calculations and determine results in a variety of ways and this can require minor differences in the definitions of objects. Consider a user who wishes to study the dynamics of a Minkowski - shell - Schwarzschild scenario. After specifying the two surfaces and joining them they might elect to examine the density and pressures of the shell as given by (23) and (25) which is done by referring to the objects `sigma` and `P`. When considering the dynamics they may wish to have $\sigma$ and $P$ in explicit form or they might opt to have them appear simply as $\sigma(\xi^a)$ and $P(\xi^a)$ since this allows them to set $P(\xi^a) = 0$ to study the dust case. Consequently there are two versions of the history and conservation equations in Table 4 (e.g. `HGeqn` vs `H1Geqn`).

Further variations are provided to allow users to specify a stress-energy of the 4-manifold for use in the conservation and history equations. A user might opt to specify e.g. $T^\theta_\theta = p(r)$ instead of using $G^\theta_\theta/8\pi$ which might be a less transparent combination of functions of the 4-metric. This requires the definition of separate objects for each case (e.g. `HGeqn` vs `HTeqn`).

Table 3: Objects in $\mathcal{M}$ on $\Sigma$ (General Case)

| Object | Definition | Name | Comments |
|--------|------------|------|----------|
| $S_{ij}$ | $\frac{\Delta}{8\pi} ([K_{ij}] - g_{ij} [K^i_j])$ | S3(dn,dn) | intrinsic stress energy tensor |
| $[n_\alpha u^\alpha] = -u^i w^i [K_{ij}]$ | | nuJumpeqn | identity |
| $n_\alpha u^\alpha = -u^i w^i K_{ij}$ | | nuMeaneqn | identity |
| divergence (T) | $\Delta \nabla_j S^j_i = \left[ T_{\alpha\beta} \frac{\partial n^\alpha}{\partial \xi^i} n^\beta \right]$ | divSTeqn(dn) | identity |
| divergence (G) | $\Delta \nabla_j S^j_i = \frac{1}{8\pi} \left[ G_{\alpha\beta} \frac{\partial n^\alpha}{\partial \xi^i} n^\beta \right]$ | divSGeqn(dn) | identity |
| mean | $-S^{ij} K_{ij} = \Delta \left[ T_{\alpha\beta} n^\alpha n^\beta \right]$ | SKTeqn | identity |
| mean | $-S^{ij} K_{ij} = \frac{\Delta}{8\pi} \left[ G_{\alpha\beta} n^\alpha n^\beta \right]$ | SKGeqn | identity |
| Object | Definition | Name | Comments |
|--------|------------|------|----------|
| $\sigma$ | $-\Delta S_{ij}u^iu^j$ | sigma | surface energy density |
| $P$ | $(\sigma + S_i^i)/2$ | P | isotropic surface pressure |
| $\sigma_1$ | $\sigma(\xi^i)$ | sigmal | arbitrary energy density on $\Sigma$ |
| $P_1$ | $P(\xi^i)$ | P1 | arbitrary isotropic pressure on $\Sigma$ |
| conservation law (T) | $[n_\alpha u^\alpha] = 8\pi(P + \sigma/2)$ | nuPeqn | |
| conservation law (G) | $[n_\alpha u^\alpha] = 8\pi(P_1 + \sigma_1/2)$ | nuP1eqn | |
| $\dot{\sigma} + (\sigma + P)\Phi = -\Delta \left[T_{\alpha\beta}u^\alpha n^\beta\right]$ | CTeqn | |
| conservation law (G) | $\dot{\sigma} + (\sigma + P_1)\Phi = -\Delta \left[G_{\alpha\beta}u^\alpha n^\beta\right]/8\pi$ | CGeqn | |
| conservation law 1 (T) | $\dot{\sigma}_1 + (\sigma_1 + P_1)\Phi = -\Delta \left[T_{\alpha\beta}u^\alpha n^\beta\right]$ | C1Teqn | |
| conservation law 1 (G) | $\dot{\sigma}_1 + (\sigma_1 + P_1)\Phi = -\Delta \left[G_{\alpha\beta}u^\alpha n^\beta\right]/8\pi$ | C1Geqn | |
| history (T) | $\Delta(\sigma + P)n_\alpha u^\alpha + PK_i^i = -\Delta \left[T_{\alpha\beta}n^\alpha n^\beta\right]$ | HTeqn | |
| history (G) | $\Delta(\sigma + P_1)n_\alpha u^\alpha + P_1K_i^i = -\Delta \left[G_{\alpha\beta}n^\alpha n^\beta\right]/8\pi$ | HGeqn | |
| history (T) | $\Delta(\sigma_1 + P_1)n_\alpha u^\alpha + P_1K_i^i = -\Delta \left[T_{\alpha\beta}n^\alpha n^\beta\right]$ | H1Teqn | |
| history (G) | $\Delta(\sigma_1 + P_1)n_\alpha u^\alpha + P_1K_i^i = -\Delta \left[G_{\alpha\beta}n^\alpha n^\beta\right]/8\pi$ | H1Geqn | |
| evolution integral | $\dot{R}^2 = \Delta + \left(\frac{|m|}{4\pi R^2 \sigma_1}\right)^2 - \frac{2\Delta \pi}{R} + \left(\frac{4\pi R^2 \sigma_1}{2R}\right)^2$ | evInt1 | first integral of evolution equation (spherical one parameter shells only) |
| evolution integral | $\dot{R}^2 = \Delta + \left(\frac{|m|}{4\pi R^2 \sigma}\right)^2 - \frac{2\Delta \pi}{R} + \left(\frac{4\pi R^2 \sigma}{2R}\right)^2$ | evInt | first integral of evolution equation (spherical one parameter shells only) |

Table 4: Objects in $\mathcal{M}$ on $\Sigma$ (one parameter surfaces)
The implementation of (29) is complicated by the fact that the definition requires coordinates $\theta$ and $\phi$ in $M^{\pm}$ and makes reference to the shell function $R(\tau)$. Users must adhere to these conventions (by using coordinate names `theta`, `phi` and $r$ and specifying $r = R(\tau)$) if they wish to make use of this relation for the first integral of evolution equation. The junction package checks for compliance with these conventions before it will evaluate (29) to ensure that spurious results will not be generated.

4 Examples

4.1 Joining Schwarzschild to Uniform Dust

We begin the demonstration of the `GRjunction` package by deriving the junction of the Schwarzschild metric to an FRW spacetime [3]. (Many standard treatments merely verify that given certain values for $m$ and $R(\tau)$ in Schwarzschild a boundary surface exists, see e.g. [7]). The example also illustrates the interactive process which is required to determine junction conditions. The pedagogic comments make the example appear somewhat lengthy, however the example is computationally trivial (less than 3 CPU seconds on a SUN SPARC 5).

We seek to join the Schwarzschild metric in the form

$$ds^2 = \frac{dr^2}{1-2m/r} + r^2 d\theta_2^2 + r^2 \sin \theta_2^2 d\phi_2^2 - (1-2m/r) dt^2$$

(41)

to a portion of the closed FRW metric written as

$$ds^2 = a(t)^2(d\chi^2 + \sin \chi^2(d\theta_1^2 + \sin \theta_1^2 d\phi_1^2)) - dt^2$$

(42)

We take $(\theta, \phi, \tau)$ as coordinates on $\Sigma$. The definition of the surface in (41) is

$$r = R(\tau), \theta_2 = \theta, \phi_2 = \phi, t_2 = T(\tau)$$

(43)

with $r - R(\tau) = 0$ as the parametric form for the surface. For the FRW metric the surface is specified as

$$\chi = X, \theta_1 = \theta, \phi_1 = \phi, t = \tau$$

(44)

with parametric form $\chi - X = 0$.

The following session output (from this point to the end of this subsection) was taken directly from the Latex output provided by Maple.
We begin by loading the GRTensorII library into Maple.

```maple
> readlib(grii):
```

Next we load the junction package into GRTensorII.

```maple
> grlib(junction);
```

The GRJunction Package
Last modified September 28, 1995
Developed by Peter Musgrave and Kayll Lake, (c) 1995

The first step is to load the Schwarzschild metric via qload.

```maple
> qload(Schw);
```

*Default spacetime = Schw*

*For the Schw spacetime:*

**Coordinates**

\[ x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t \]

**Line element**

\[ ds^2 = \frac{d r^2}{1 - 2 \frac{m}{r}} + r^2 d \theta^2 + r^2 \sin(\theta)^2 d \phi^2 + \left( -1 + 2 \frac{m}{r} \right) d t^2 \]

Now we use the `surf` command to specify a surface. `surf` prompts the user for information which defines the surface and its normal vector.

```maple
> surf(Schw,ssurf);
```

First we are asked for the coordinates on the surface as a list.

Please enter the coordinates of the surface as a list e.g. `[theta, phi, tau];`

Enter a list >

```maple
> [theta, phi, tau];
```

Next we are asked to specify which of the \( \xi^i \) is the parameter for a one-parameter shell (if any).

For a one-parameter shell enter the parameter (0 for none) >

```maple
> tau;
```

Now we provide the coordinate definition of \( \Sigma \). (Note that in Maple [ ]’s do double duty; they denote lists and are used to indicate subscripts)

Please enter the coordinate definition of the surface (the \( x^\{\cdot a \} = x(\xi^\{\cdot b \}) \) ) as a LIST.

E.g. [ r=R(tau), theta=theta, phi=phi, t=T(tau) ]

> [r=R(tau), theta[2]=theta, phi[2]=phi, t[2]=T(tau)];

*CPU Time = .050*

The character of the normal vector is now entered.

Please indicate the nature of the surface normal vector (-1 = timelike, 1= spacelike) Enter +1,0 or -1 >
Since we have provided a shell parameter we are given the option of employing the relation $u_{\alpha} u^\alpha = -1$ to eliminate the derivative of one of the quantities used in the coordinate definition of the surface. We choose to eliminate $\partial T(\tau)/\partial \tau$ (in Maple parlance `diff(T(tau),tau)`). This choice will produce $d\tau^2 = -1$ on $\Sigma$.

Use $+/ -1 = u^{-a} u^a$ as a constraint? Enter 1 if yes

> 1;

Created definition for $u(dn)$

$$CPU Time = .100$$

The constraint equation is:

$$-1 = -r^2 \left( \frac{\partial}{\partial \tau} R(\tau) \right)^2 + \left( \frac{\partial}{\partial \tau} T(\tau) \right)^2 r^2 - 4 \left( \frac{\partial}{\partial \tau} T(\tau) \right)^2 r m + 4 \left( \frac{\partial}{\partial \tau} T(\tau) \right)^2 m^2$$

$$\frac{1}{(-r + 2 m) r}$$

Enter the term you wish to use the constraint to eliminate

Term

> `diff(T(tau),tau);`

In solving for $\partial T(\tau)/\partial \tau$ Maple determines that there are two choices (differing in sign). We are asked to choose one of them.

Solve returned multiple solutions. They are:

$$[1], - \sqrt{\frac{r^2 - 2 m + r \left( \frac{\partial}{\partial \tau} R(\tau) \right)^2}{r}}$$

$$\frac{r}{-r + 2 m} + 4 \frac{m}{-r + 2 m} - 4 \frac{m^2}{(-r + 2 m) r}$$

$$[2], - \sqrt{\frac{r^2 - 2 m + r \left( \frac{\partial}{\partial \tau} R(\tau) \right)^2}{r}}$$

$$\frac{r}{-r + 2 m} + 4 \frac{m}{-r + 2 m} - 4 \frac{m^2}{(-r + 2 m) r}$$

Please select a solution.

Enter choice

> 1;

Now we enter the parametric definition of the surface.

Default metric is now Schw

Please enter the equation for the surface.

The surface will be defined by setting the equation you enter to zero.
Finally, we are asked explicitly for the sign of the normal vector.
The definition of the normal vector is +/- grad(surface) please enter +1 or -1 to indicate the CHOICE of sign
Enter +1,-1 >

> 1;

Default metric is now ssurf

For the Schw spacetime:
The Equation of the surface
surface = r - R(t_2)

Coordinate transforms onto the surface
\[ x^r = R(\tau) \]
\[ x^\theta_2 = \theta \]
\[ x^\phi_2 = \phi \]
\[ x^t_2 = T(\tau) \]

For the ssurf spacetime:

Line element
\[ ds^2 = R(\tau)^2 d\theta^2 + R(\tau)^2 \sin(\theta)^2 d\phi^2 - d\tau^2 \]

The intrinsic metric and normal vector have been calculated.
You may wish to simplify them further before saving the surface or calculating K(dn,dn)

This completes the specification of the surface in the Schwarzschild exterior.
Next we load the FRW metric (which we will take as the interior).

> qload(frw);

Default spacetime = frw

For the frw spacetime:

Coordinates
\[ x^1 = \chi, x^2 = \theta_1, x^3 = \phi_1, x^4 = t \]
Line element

\[ ds^2 = a(t)^2 \, d\chi^2 + a(t)^2 \sin(\chi)^2 \, d\theta_1^2 + a(t)^2 \sin(\chi)^2 \sin(\theta_1)^2 \, d\phi_1^2 - dt^2 \]

Before defining the surface in the FRW spacetime we first calculate \( G^\alpha_\beta \) for later use.

> grcalc(G(dn,up));
Created definition for G(dn,up)

\[ CPU Time = .634 \]

We specify the surface in \( M^- \) in parametric form as \( \chi - X = 0 \). The process is identical to that followed for the Schwarzschild case above (except we do not use \( u^a u_a = -1 \)).

> surf(frw, fsurf);
Please enter the coordinates of the surface as a list
e.g. [theta, phi, tau];
Enter a list >

> [theta,phi,tau];
For a one-parameter shell enter the parameter (0 for none) >

> tau;

Please enter the coordinate definition of the surface
(the \( x^a = x^a(x^b) \)) as a LIST.
e.g. [ r=R(tau), theta=theta, phi=phi, t=T(tau)]

> [ chi = X, theta[1]=theta, phi[1]=phi, t=tau];
\[ CPU Time = .050 \]

Please indicate the nature of the surface normal vector
(-1 = timelike, 1= spacelike) Enter +1,0 or -1 >

> 1;
Use +/- 1 = u^a u_a as a constraint ? Enter 1 if yes >

> 0;
Default metric is now frw
Please enter the equation for the surface.
The surface will be defined by setting the equation you enter to zero.

surface

>

> chi-X;
\[ CPU Time = .016 \]
The definition of the normal vector is +/- grad(surface) please enter +1 or -1 to indicate the CHOICE of sign Enter +1,-1 >

> 1;

CPU Time = .050

Default metric is now fsurf

CPU Time = .066

For the frw spacetime:
The Equation of the surface
surface = \chi - X

For the frw spacetime:
Coordinate transforms onto the surface
\textit{xform} \chi = X
\textit{xform} \theta_1 = \theta
\textit{xform} \phi_1 = \phi
\textit{xform} t = \tau

For the fsurf spacetime:
Line element
\[ ds^2 = a(\tau)^2 \sin(X)^2 d\theta^2 + a(\tau)^2 \sin(X)^2 \sin(\theta)^2 d\phi^2 - d\tau^2 \]

The intrinsic metric and normal vector have been calculated. You may wish to simplify them further before saving the surface or calculating $K(\text{dn},\text{dn})$

Now we can identify the two surfaces we have specified by means of the command \texttt{join}. By convention the first surface name in the \texttt{join} command is taken as $\Sigma^+$ for the purposes of evaluating e.g. $[g_{ij}]$.

> join(ssurf,fsurf);

ssurf and fsurf are now joined.
The default metric name is ssurf.
The exterior metric is: ssurf
The interior metric is: fsurf

CPU Time = .034

For the ssurf spacetime:
Jump from defaultMetric − Mint
$Jump [g(dn,dn), fsurf]_{\theta \theta} = R(\tau)^2 - a(\tau)^2 \sin(X)^2$

$Jump [g(dn,dn), fsurf]_{\phi \phi} = R(\tau)^2 \sin(\theta)^2 - a(\tau)^2 \sin(X)^2 \sin(\theta)^2$

To obtain $[g_{ij}] = 0$ we require a particular value for $R(\tau)$. We use the GRTensor command `grcomponent` to extract $[g_{11}]$ (which we assign to `jump_g11`). We then set `jump_g11` equal to zero and solve for $R(\tau)$.

```maple
> jump_g11 := grcomponent(Jump[g(dn,dn)], [1,1]);

> sol := [solve(jump_g11 = 0, R(tau))];
```

We now substitute the positive solution back in to verify $[g_{ij}] = 0$. This is accomplished by using the routine `grmap` to map the Maple substitution command `subs` over the components of $Jump[g(dn,dn)]$. (The 'x' is a placeholder indicating which of the arguments to `subs` is to be filled in by the component value.)

```maple
> grmap(Jump[g(dn,dn)], subs, R(tau) = sol[1], 'x');
```

(Here we use a GRTensor short-cut. The _ refers to the last mentioned object. In this case `Jump[g(dn,dn)]` allowing us to save some typing.)

```maple
> grdisplay(_);
```

*For the ssurf spacetime:*

$Jump$ from defaultMetric $-$ Mint

$Jump[g(dn,dn), fsurf] = All$ components are zero

To establish a Schwarzschild-Dust boundary surface we next need to establish that $[K_{ij}] = 0$. We begin by calculating $Jump[K(dn,up)]$. We prefer the mixed form $K^j_i$ so that we can make later use of $G^j_a_i$ and the phenomenology of the FRW space.

```maple
> grcalc(Jump[K(dn,up)]);
```

*Created definition for K(dn,up)*

$CPU Time = 1.183$

```maple
> grdisplay(_);
```

*For the ssurf spacetime:*

$Jump$ from defaultMetric $-$ Mint

$Jump[K(dn,up), fsurf]_{\theta} =$

\[
- \frac{R(\tau) - 2m + R(\tau) \left( \frac{d}{d \tau} R(\tau) \right)^2}{R(\tau)^2} a(\tau) \sin(X) + \cos(X) R(\tau)
\]

\[- \frac{R(\tau) a(\tau) \sin(X)}{R(\tau)^2} a(\tau) \sin(X) + \cos(X) R(\tau)
\]
\[
\text{Jump } [K(dn, up), fsurf]_\phi \theta = \left\{-\sqrt{\frac{R(\tau) - 2 m + R(\tau) \left( \frac{\partial^2}{\partial \tau^2} R(\tau) \right)^2}{R(\tau)}} a(\tau) \sin(X) + \cos(X) R(\tau) \right\}\left\{-\sqrt{\frac{R(\tau) a(\tau) \sin(X)}{R(\tau) a(\tau) \sin(X)}} \right\}
\]

\[
\text{Jump } [K(dn, up), fsurf]_\tau \tau = \left\{\frac{R(\tau)^2 \left( \frac{\partial^2}{\partial \tau^2} R(\tau) \right) + m}{\sqrt{\frac{R(\tau) - 2 m + R(\tau) \left( \frac{\partial}{\partial \tau} R(\tau) \right)^2}{R(\tau)}} R(\tau) a(\tau) \sin(X) + \cos(X) R(\tau)} \right\}\left\{\frac{R(\tau) a(\tau) \sin(X)}{R(\tau) a(\tau) \sin(X)}} \right\}
\]

Setting \([K^\tau \tau] = 0\) gives a value for the Schwarzschild mass \(m\) in terms of \(R(\tau)\) and we substitute in this value. This leaves only the angular jump in \(K\) to contend with.

> grmap(_, subs, m = -R(tau)^2*diff(R(tau),tau,tau), 'x');

Applying routine subs to \(\text{Jump}[K(dn, up), fsurf]\)

Once again we make use of the value of \(R(\tau)\) which was required for \([g_{ij}] = 0\).

> grmap(_,subs, R(tau) = sol[1], 'x');

Applying routine subs to \(\text{Jump}[K(dn, up), fsurf]\)

> gralter(_,factor);

Component Alteration of a grtensor object:

Applying routine factor to object \(\text{Jump}[K(dn, up), fsurf]\)

\[
\text{CPU Time} = .150
\]

> grdisplay(_);

For the \text{ssurf} spacetime:

\(\text{Jump from defaultMetric} - \text{Mint}\)

\[
\text{Jump } [K(dn, up), fsurf]_\phi \theta = \left\{-\sqrt{1 + 2 a(\tau) \sin(X)^2 \left( \frac{\partial^2}{\partial \tau^2} a(\tau) \right) + \sin(X)^2 \left( \frac{\partial}{\partial \tau} a(\tau) \right)^2 + \cos(X)} \right\}\left\{\frac{a(\tau) \sin(X)}{a(\tau) \sin(X)}} \right\}
\]

\[
\text{Jump } [K(dn, up), fsurf]_\phi \phi = \left\{-\sqrt{1 + 2 a(\tau) \sin(X)^2 \left( \frac{\partial^2}{\partial \tau^2} a(\tau) \right) + \sin(X)^2 \left( \frac{\partial}{\partial \tau} a(\tau) \right)^2 + \cos(X)} \right\}\left\{\frac{a(\tau) \sin(X)}{a(\tau) \sin(X)}} \right\}
\]

At this point we need to make reference to the Einstein tensor for the FRW spacetime. To this point we have not imposed the restriction that the interior be dust. We do this by setting \(G^\theta_\theta = 0\) and solving for \(\partial^2 a(t)/\partial t^2\). This is then used in \(\text{Jump}[K(dn, up)]\) (with \(t = \tau\) on the surface as required). Recall that
by default object names refer to the ssurf spacetime (i.e $\Sigma^+$). The use of a metric name in square brackets after the tensor name below indicates which metric the object is to be taken from.

\[ grdisplay(G[frw](dn,up)); \]

For the frw spacetime:

\[
G(\frac{\partial}{\partial t}a(t))^2 + 2a(t)\sin(\chi)^2 \left( \frac{\partial^2}{\partial \tau^2}a(t) \right) + 1 - \cos(\chi)^2
\]

\[
G_{\theta_i \theta_1} = -\left( \frac{\partial}{\partial \tau}a(t) \right)^2 + 2a(t) \left( \frac{\partial^2}{\partial \tau^2}a(t) \right)
\]

\[
G_{\phi_i \phi_1} = -\left( \frac{\partial}{\partial \tau}a(t) \right)^2 + 2a(t) \left( \frac{\partial^2}{\partial \tau^2}a(t) \right)
\]

\[
G_{t \ t} = -\frac{2 \sin(\chi)^2 + 3 \sin(\chi)^2 \left( \frac{\partial}{\partial \tau}a(t) \right)^2 + 1 - \cos(\chi)^2}{a(t)^2 \sin(\chi)^2}
\]

To make use of the condition $p = 0$ we will extract the $G_{\theta_1}$ component (using grcomponent) and set it equal to zero and then isolate the $\frac{\partial^2 a(t)}{\partial \tau^2}$ term. We will then substitute this into $[K^j_i]$. (We need the Maple isolate library which we we now load)

\[ readlib(isolate); \]

The $\frac{\partial^2 a(t)}{\partial \tau^2}$ term is now isolated and we change $t$ to $\tau$ (via the subs command).

\[ da := \text{subs}( t=tau, \text{isolate( grcomponent(G[frw](dn,up),[2,2]) = 0,} \]
\[ \text{diff(a(t),t$2$)) }); \]

\[
da := \frac{\partial^2}{\partial \tau^2}a(\tau) = \frac{1}{2} \frac{-1 - \left( \frac{\partial}{\partial \tau}a(\tau) \right)^2}{a(\tau)}
\]

The resulting equation $da$ is substituted into $[K^j_i]$ and some simplification is then performed.

\[ grmap( Jump[K(dn,up)], \text{subs, da, 'x'}); \]

Applying routine subs to $Jump[K(dn,up),fsurf]$

\[ gralter(_,\text{expand, trig}); \]

Component Alteration of a grtensor object:

Applying routine expand to object $Jump[K(dn,up),fsurf]$ Applying routine simplify[trig] to object $Jump[K(dn,up),fsurf]$

\[ CPU Time = .267 \]

\[ grdisplay(_); \]

For the ssurf spacetime:
Jump from defaultMetric – Mint

\[
\text{Jump } [K(dn, up), fsurf]_\theta = \frac{-\sqrt{\cos(X)^2 + \cos(X) \cos(X)}}{a(\tau) \sin(X)}
\]

\[
\text{Jump } [K(dn, up), fsurf]_\phi = \frac{-\sqrt{\cos(X)^2 + \cos(X) \cos(X)}}{a(\tau) \sin(X)}
\]

We’re nearly there, but Maple will not collapse e.g. \(\sqrt{x^2}\) to \(x\) unless it is sure \(x\) is real, or explicitly told to do so. We tell it to go ahead by using the routine `simplify[sqrt,symbolic].`

> `gralter(_,sqrt);`

Component Alteration of a grtensor object:

Applying routine simplify[sqrt] to object Jump[K(dn,up),fsurf]

\[\text{CPU Time } = 0.066\]

So if we take the FRW solution to be dust then we can establish that \([g_{ij}] = 0\) and \(\left[K^j_i\right] = 0\) completing the junction of Schwarzschild to FRW.

> `grdisplay(_);`

For the ssurf spacetime:

\[\text{Jump from defaultMetric – Mint}\]

\[\text{Jump}[K(dn, up), fsurf] = \text{All components are zero}\]

With the FRW solution restricted to uniform dust then we can establish that \([g_{ij}] = 0\) and \(\left[K^j_i\right] = 0\) and hence a boundary surface exists provided we take \(R(\tau) = a(\tau) \sin(X)\) and \(m = -R(\tau)^2 \partial^2 R(\tau)/\partial r^2\) in Schwarzschild.

4.2 Shells in Spherically Symmetric Static Spacetimes

The evolution of thin shells in spherically symmetric spacetimes has been widely studied. Here we demonstrate how `GRJunction` is employed to determine the evolution of a thin-shell separating two spherically symmetric static spacetimes. This result contains as special case the classic analysis of Israel [3].

We take \(M^+\) and \(M^-\) as

\[
ds_{+}^2 = dr_{+}^2/\left(F(r_{+})\right) + r_{+}^2 d\Omega^2 - f(r_{-})dt_{+}^2,
\]

\[
ds_{-}^2 = dr_{-}^2/\left(f(r_{-})\right) + r_{-}^2 d\Omega^2 - f(r_{+})dt_{-}^2
\]

(45)

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\) (\(\theta\) and \(\phi\) are continuous through the surface). We define \(\Sigma\) in \(M^+\) by \(r_{+} = R(\tau)\) and in \(M^-\) as \(r_{-} = R(\tau)\). On \(\Sigma\) we choose coordinates \((\theta, \phi, \tau)\). We seek to determine the equation governing the evolution of the surface and the stress-energy of the surface.
We demonstrate how to achieve this in the session below. Once again every-thing from this point to the end of the subsection is Maple output. The specification of the surfaces follows exactly as in the Schwarzschild case in the previous example and so we omit the input portion of this process in the interests of brevity.

(Prior to this point we defined a surface $sout$ in the metric (46) which we labeled $\text{static}F$ and a surface $sint$ in (45), labeled $\text{static}f$.)

We now identify these two surfaces by using $\text{join}$. 

$$\text{join}(sout,sint);$$

$sout$ and $sint$ are now joined.
The default metric name is $sout$.
The exterior metric is: $sout$
The interior metric is: $sint$

\[CPU Time = 0.050\]

For the $sout$ spacetime:
Jump from defaultMetric – $Mint$

\[\text{Jump}[g(dn, dn), sint] = \text{All components are zero}\]
$\text{static}F$

We now calculate $S^i_j$ (the junction package object $S3(dn,up)$). Since this is non-zero it is clear that there is a thin shell seperating $\text{static}F$ and $\text{static}f$.

\[\text{grcalc}(S3(dn,up));\]
Created definition for $K(dn,up)$

\[CPU Time = 3.150\]

\[\text{gralter}(S3(dn,up), \text{factor});\]
Component Alteration of a grtensor object:

Applying routine factor to object $S3(dn,up)$

\[CPU Time = 0.167\]

To improve the appearance of the output we make use of GRTensor’s ability to represent derivatives as subscripts (so $dR(\tau)/d\tau \rightarrow R_\tau$) via the $\text{autoAlias}$ command. This command resides in the $\text{grtools}$ library, which we now load.

\[\text{readlib(grtools)};\]

Now we apply $\text{autoAlias}$ via $\text{grmap}$.

\[\text{grmap}(S3(dn,up), \text{autoAlias, 'x'});\]
Applying routine $\text{autoAlias}$ to $S3(dn,up)$

\[\text{grdisplay}(S3(dn,up));\]

For the $sout$ spacetime:

\[\text{Intrinsic stress – energy}\]
\begin{align*}
S_{\theta} \phi &= -\frac{1}{16} \left( -2 \sqrt{f(R(\tau))} + R_{\tau}^2 F(R(\tau)) - 2 \sqrt{f(R(\tau))} + R_{\tau}^2 R_{\tau}^2 \\
&\quad + 2 \sqrt{F(R(\tau))} + R_{\tau}^2 f(R(\tau)) + 2 \sqrt{F(R(\tau))} + R_{\tau}^2 R_{\tau}^2 \\
&\quad - \sqrt{f(R(\tau))} + R_{\tau}^2 R(\tau) D(F)(R(\tau)) - 2 \sqrt{f(R(\tau))} + R_{\tau}^2 R(\tau) R_{\tau,\tau} \\
&\quad + \sqrt{F(R(\tau))} + R_{\tau}^2 R(\tau) D(f)(R(\tau)) + 2 \sqrt{F(R(\tau))} + R_{\tau}^2 R(\tau) R_{\tau,\tau} \right) \left( R(\tau) \cdot \sqrt{F(R(\tau))} + R_{\tau}^2 \sqrt{f(R(\tau))} + R_{\tau}^2 \pi \right)
\end{align*}

\begin{align*}
S_{\phi} \phi &= -\frac{1}{16} \left( -2 \sqrt{f(R(\tau))} + R_{\tau}^2 F(R(\tau)) - 2 \sqrt{f(R(\tau))} + R_{\tau}^2 R_{\tau}^2 \\
&\quad + 2 \sqrt{F(R(\tau))} + R_{\tau}^2 f(R(\tau)) + 2 \sqrt{F(R(\tau))} + R_{\tau}^2 R_{\tau}^2 \\
&\quad - \sqrt{f(R(\tau))} + R_{\tau}^2 R(\tau) D(F)(R(\tau)) - 2 \sqrt{f(R(\tau))} + R_{\tau}^2 R(\tau) R_{\tau,\tau} \\
&\quad + \sqrt{F(R(\tau))} + R_{\tau}^2 R(\tau) D(f)(R(\tau)) + 2 \sqrt{F(R(\tau))} + R_{\tau}^2 R(\tau) R_{\tau,\tau} \right) \left( R(\tau) \cdot \sqrt{F(R(\tau))} + R_{\tau}^2 \sqrt{f(R(\tau))} + R_{\tau}^2 \pi \right)
\end{align*}

\begin{align*}
S_{\tau} \tau &= -\frac{1}{4} \left( -\sqrt{f(R(\tau))} + R_{\tau}^2 F(R(\tau)) - \sqrt{f(R(\tau))} + R_{\tau}^2 R_{\tau}^2 + \sqrt{F(R(\tau))} + R_{\tau}^2 f(R(\tau)) \\
&\quad + \sqrt{F(R(\tau))} + R_{\tau}^2 R_{\tau}^2 \right) \left( R(\tau) \cdot \sqrt{F(R(\tau))} + R_{\tau}^2 \sqrt{f(R(\tau))} + R_{\tau}^2 \pi \right)
\end{align*}

Note that since \( f(r) \) and \( F(r) \) are unspecified their derivatives cannot be evaluated. This results in the use of the Maple \( \text{D} \) derivative. We will demonstrate how to specify specific functions below.

Next we consider the equation for the history of the shell \( \text{HGeqn} \) and the first integral of this equation \( \text{evInt1} \) (which in the case of spherical symmetry stems from an identity).

\begin{verbatim}
> grcalc(HGeqn, evInt1);
Created definition for n(up)
Created definition for R(dn,dn,up,up)

CPU Time = 4.767
\end{verbatim}

Before displaying the results we factor the expressions.

\begin{verbatim}
> gralter(_,factor);
Component Alteration of a grtensor object:

Applying routine factor to object HGeqn
Applying routine factor to object evInt1

CPU Time = .317
\end{verbatim}

\begin{verbatim}
> grdisplay(_);
For the sout spacetime:
\end{verbatim}

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\[ HGeqn = \left( \frac{1}{4} \left( -R(\tau)\sigma(\theta,\phi,\tau)\sqrt{f(R(\tau))} + R_{\tau}^2 D(F)(R(\tau)) \right) \right. \]
\[ \left. - 2 R(\tau)\sigma(\theta,\phi,\tau)\sqrt{f(R(\tau))} + R_{\tau}^2 R_{\tau,\tau} - 2 R(\tau)\sigma(\theta,\phi,\tau)\sqrt{F(R(\tau))} + R_{\tau}^2 R_{\tau,\tau} \right) \]
\[ + 4 P(\theta,\phi,\tau)\sqrt{f(R(\tau))} + R_{\tau}^2 f(R(\tau)) + 4 P(\theta,\phi,\tau)\sqrt{F(R(\tau))} + R_{\tau}^2 R_{\tau,\tau} \]
\[ - \left. R(\tau)\sqrt{F(R(\tau))} + R_{\tau}^2 \sqrt{f(R(\tau))} + R_{\tau}^2 \right) \]
\[ \left/ \left( R(\tau)\sqrt{F(R(\tau))} + R_{\tau}^2 \sqrt{f(R(\tau))} + R_{\tau}^2 \right) \right) = \right. \]
\[ \left. - R(\tau) D(F)(R(\tau)) + F(R(\tau)) - R(\tau) D(f)(R(\tau)) - f(R(\tau)) \right) \]
\[ evInt1 = \left( R_{\tau}^2 = \frac{1}{64} \left( -64 \pi^2 \sigma(\theta,\phi,\tau)^2 R(\tau)^4 r^2 + r^4 F(r)^2 - 2 r^4 F(r)f(r) + r^4 f(r)^2 \right. \]
\[ + 64 \left( r^2 \right)^{3/2} \pi^2 \sigma(\theta,\phi,\tau)^2 R(\tau)^3 - 32 \left( r^2 \right)^{3/2} \pi^2 \sigma(\theta,\phi,\tau)^2 R(\tau)^3 F(r) + 32 \left( r^2 \right)^{3/2} \pi^2 \sigma(\theta,\phi,\tau)^2 R(\tau)^3 f(r) + 256 \pi^4 \sigma(\theta,\phi,\tau)^4 R(\tau)^6 r^2 \right) \]
\[ \left/ \left( \pi^2 \right) \sigma(\theta,\phi,\tau)^2 R(\tau)^4 r^2 \right) \right) \]

Now we define functions for \( f \) and \( F \). We consider the Israel thin-shell example and hence define \( f(r) = 1 \) and \( F(r) = 1 - 2m/r \) so we have a Minkowski interior and a Schwarzschild exterior. These functions are defined as Maple procedures. (This ensures that the deferred derivatives will evaluate. Note that a simple substitution of e.g. \( f(R(t)) = 1 - 2m/R(t) \) would not affect \( D(f)(R(t)) \) since \( f(R(t)) \) does not appear explicitly). A Maple procedure is declared in a statement of the form: procedureName := proc(arguments) procedure_body end:

\[
\text{> } f := \text{proc}(r) \text{ RETURN}(1); \text{ end;}
\]
\[
\text{> } F := \text{proc}(r) \text{ RETURN}(1-2*m/r); \text{ end;}
\]
To repeat Israel’s dust shell analysis we must set the pressure to zero, which we now do.

\[
\text{> } \text{grmap}(_, \text{subs, P(theta,phi,tau)=0,'x'});
\]
Applying routine subs to \( HGeqn \)
Applying routine subs to \( evInt1 \)

\[
\text{> } \text{gralter(_,power,expand);}
\]
Component Alteration of a grtensor object:

Applying routine simplify[power] to object \( HGeqn \)
Applying routine simplify[power] to object \( evInt1 \)
Applying routine expand to object \( HGeqn \)
Applying routine expand to object \( evInt1 \)
\[ CPU \text{ Time } = 1.116 \]

\[ > \text{grdisplay(_);} \]

For the sout spacetime:

\[ \text{History equation} \]

\[ HGeqn = \]

\[ \left( -\frac{1}{2} \frac{\sigma(\theta, \phi, \tau) m}{R(\tau)^2 \sqrt{1 - 2 \frac{m}{R(\tau)} + R^2}} - \frac{1}{2} \frac{\sigma(\theta, \phi, \tau) R_{\tau,\tau}}{\sqrt{1 - 2 \frac{m}{R(\tau)} + R^2}} - \frac{1}{2} \frac{\sigma(\theta, \phi, \tau) R_{\tau,\tau}}{\sqrt{1 + R^2}} = 0 \right) \]

\[ \text{evInt1} = \left( R_{\tau}^2 = -1 + \frac{1}{16} \frac{m^2}{\pi^2 \sigma(\theta, \phi, \tau) R(\tau)^4} + \frac{(r^2)^{3/2} m}{r^3 R(\tau)} + 4 \pi^2 \sigma(\theta, \phi, \tau)^2 R(\tau)^2 \right) \]

The above expressions correspond to the results given in [2]. Note that we did not have to integrate to obtain the first integral of the evolution equation.

### 4.3 A complicated Minkowski junction

In this section we describe the junction of the Minkowski metric in the form

\[ ds_{+}^2 = \frac{r^2 + u^2}{r^2 + a^2} dr^2 + \frac{r^2 + u^2}{a^2 - u^2} du^2 + \frac{(a^2 - u^2)(r^2 + a^2)}{a^2} d\phi^2 - dt^2 \]  

(47)

...to a metric \( ds_{+}^2 \) of the same form, but with coordinates \((R, U, \Phi, T)\) and parameter \( A \). These metrics arise from setting \( m = 0 \) in a Kerr metric and choosing \( u = a \cos \theta \). The coordinate names \( r \) and \( R \) while standard are quite misleading since surfaces of constant “radius” describe spheroids with oblateness governed by \( a \) or \( A \). We seek to join a surface of constant “radius” in \( M^- \) to \( M^+ \). The surface in \( M^+ \) will be some function of \( R \) and \( U \). This example demonstrates \textsc{GRJunction}'s ability to handle non-spherical surfaces in a case where we know a boundary surface must result.

The transformations from e.g. (17) to the Minkowski metric in spherical form (with coordinates \( \tilde{r} \) and \( \tilde{\theta} \)) are given by

\[ u = a \cos \theta \]

\[ r^2 = r^2 + a^2 \sin^2 \theta \]

\[ \tilde{r} \cos \tilde{\theta} = r \cos \theta. \]

(49)

If we take the definition of \( \Sigma^- \) as \( r = X \) (\( X \) a constant) then we can use (18) to determine that the the definition of the corresponding surface in \( M^+ \) has parametric equation:

\[ 0 = X - \left( R^2 + A^2 - U^2 - a^2 + \frac{1}{4}(-2R^2A^2 + 2a^2A^2 - 2A^4 + 2A^2U^2 + \right) \]

27
\[ 2(R^4 A^4 - 2R^2 A^4 a^2 + 2R^2 A^6 - 2R^2 A^4 U^2 + a^4 A^4 - 2a^2 A^6 + 2a^2 A^4 U^2 + A^8 - 2A^6 U^2 + A^4 U^4 + 4R^2 U^2 a^2 A^2)^{(1/2)} / A^2)^{(1/2)} \] (50)

and the relations \( x_+^\alpha = x_+^{\alpha}(\xi) \) are

\[
R = \left( -A^2 + \frac{1}{2a^2}(-X^2 a^2 + a^2 A^2 + a^2 u^2 - a^4 + (X^4 a^4 - 2X^2 a^4 A^2 - 2X^2 a^6 u^2 + 4X^2 u^2 A^2 a^2)^{(1/2)} + X^2 + a^2 - u^2)^{(1/2)} \right.
\]

\[
U = \frac{1}{a\sqrt{2}}(-X^2 a^2 + a^2 A^2 + a^2 u^2 - a^4 + (X^4 a^4 - 2X^2 a^4 A^2 - 2X^2 a^6 u^2 + 4X^2 u^2 A^2 a^2)^{(1/2)})^{1/2} + 2X^2 a^4 A^2 - 2X^2 a^6 u^2 + 4X^2 u^2 A^2 a^2)^{(1/2)} \right)
\]

\[
\Phi = \phi \quad (53)
\]

\[
t = T_+(\tau). \quad (54)
\]

In this case it not clear which sign we should choose for the normal vector in \( M^+ \). However since we know a priori that a boundary surface must result if the package does not reach this result then we can consider the other choice of normal sign for \( n^+_\alpha \). (In this case we choose the plus sign in (52)).

The package determines \([g_{ij}] = 0\) and that \( K^-_{ij} \) is:

\[
K^-_{\alpha\alpha} = \frac{X \sqrt{X^2 + a^2}}{(a^2 - u^2) \sqrt{X^2 + u^2}}
\]

\[
K^-_{\phi\phi} = \frac{X \sqrt{X^2 + a^2}(a^2 - u^2)}{a^2 \sqrt{X^2 + u^2}}. \quad (55)
\]

With careful direction of the computer simplification the junction package determines that \( K^+_{ij} \) is also given by (55). \[ \]

### 4.4 Joining Kerr to Kerr

In this section we consider joining the “interior” region of one Kerr spacetime \( \mathcal{M}^- \) in Boyer-Lindquist form

\[
\begin{align*}
ds^2 &= \rho \left( \frac{dr^2}{\delta} + d\theta^2 \right) + (r_-^2 + a^2) \sin \theta_-^2 d\phi_-^2 - dt_-^2 \\
&\quad + \frac{2mr_-}{\rho} (a \sin \theta_-^2 d\phi_- - dt_-)^2 \\
&\quad - \rho = r_-^2 + a^2 \cos \theta_-^2, \quad \delta = r_-^2 - 2mr_- + a^2 \quad (56)
\end{align*}
\]

\[ ^1 \text{In an earlier attempt to determine } K^+_{ij} \text{ (using Boyer-Lindquist coordinates) we encountered the Maple error “Object too Large” which occurs on 32-bit machines when an expression contains more than 64,000 terms. On present day workstations this limit can be reached in a matter of minutes and often little can be done to circumvent this Maple limitation.} \]
with mass \( m \) and angular momentum \( a \) to the “exterior” region of another Kerr spacetime \( M^+ \) with mass \( M \) and angular momentum \( A \) (we use coordinates \((r_\pm, \theta_\pm, \phi_\pm, t_\pm)\) for \( M^\pm \)). Such a problem would arise if a thin shell of matter was constructed around a Kerr black hole (i.e. a Dyson sphere). The general problem is an extremely difficult one as the “toy” problem \( m = M = 0 \) indicates (see section 4.3). In this section we limit the discussion to an expansion to first order in \( a \) and \( A \), since to this order the surface in \( M^\pm \) is spherical.

In the treatment of timelike spherical surfaces it is customary to use transformations such as \( r = R(\tau) \) and \( t = T(\tau) \) and then use \( u_\alpha u^\alpha = -1 \) to eliminate \( \partial T(\tau)/\partial \tau \). Note that in these spherical cases this produces \( g_{\tau\tau} = -1 \) on \( \Sigma \) as desired. In all spherical cases \( T \) will be strictly a function of \( \tau \) but this fails to be true in Kerr. For Kerr spacetimes to achieve \( g_{\tau\tau} = -1 \) we can use the same idea but if we blindly use \( t = T(\tau) \) we discover that in actuality \( t = T(\tau, \tilde{\theta}) \). We can try again - using \( \tilde{\theta} \) as an argument and we do get \( \tau \) as the proper time on the shell but now a \( \partial T(\tau, \tilde{\theta})/\partial \tilde{\theta} \) appears in \( g_{\tilde{\theta}\tilde{\theta}}, g_{\tilde{\theta}\tau} \), and \( g_{\tilde{\theta}\phi} \). Since we know \( \partial T/\partial \tau \) is merely a function of \( \tilde{\theta} \) we can integrate trivially but this makes those components with a \( \partial T/\partial \theta \) depend linearly on \( \tau \) and the metric components are explicitly dependent on proper time. Hence forcing \( \tau \) to be the proper time on the surface comes at considerable expense. Fortunately these problems do not arise in the order \((a, A)\) expansions.

To facilitate the matching of the \( g_{ij} \) on \( \Sigma \) we eliminate the \( g_{\tilde{\theta}\tau} \) terms on \( \Sigma \) by using a transformation to the zero angular momentum (ZAMO) frame in the definition of the surface. The transformations are

\[
r_\pm = R, \theta_\pm = \tilde{\theta}, \phi_\pm = \tilde{\phi} - \Omega_\pm T_\pm(\tau, \theta), t_\pm = T_\pm(\tau, \theta) \tag{57}
\]

where \( \Omega \equiv g_{\phi t}/g_{\phi\phi} \). Using \texttt{GRjunction} we calculate the metric and second fundamental form on the surface and only then expand to order \( a \), identify \( \Sigma^\pm \) and calculate \( S_{ij} \).

The package first determines

\[
\left. ds^2 \right|_{\Sigma^\pm} = R^2 d\tilde{\theta}^2 + R^2 \sin^2 \tilde{\theta} d\tilde{\phi}^2 - d\tau^2 \tag{58}
\]

and consequently \([g_{ij}] = 0\).

The resulting stress-energy of the shell is (from this point on we drop the tildes on \( \theta \) and \( \phi \)):

\[
\begin{align*}
S_\theta^\theta &= \frac{f(R - M) - F(R - m)}{8\pi R^2 f F} \\
S_\phi^\phi &= S_\theta^\theta \\
S_\tau^\phi &= R^2 \sin^2 \theta S_\tau^\phi \\
S_\phi^\tau &= \frac{3(am - AM)}{8\pi R^2} \\
\end{align*}
\]
\( S_{\tau}^\tau = \frac{f(R - 2M) - F(R - 2m)}{4\pi R^2 fF} \)

where \( f = \sqrt{1 - 2m/R}, \quad F = \sqrt{1 - 2M/R}. \)

Note the appearance of mixed term \( S_{\phi}^\phi \) which precludes a standard phenomenological interpretation (i.e. \( \sigma \) as given by (23) is not an eigenvector of \( S_{ij} \)). To first order we have

\[
\Omega_- = -\frac{am}{R^3}, \quad \Omega_+ = -\frac{AM}{R^3}
\]

and if we require \( \Omega_- = \Omega_+ \) then the mixed terms vanish. This now permits standard phenomenological interpretation of the shell and we can now interpret the density and pressures in the usual way. Note that in this case the pressure is isotropic and hoop stresses do not arise.

We can easily extend this calculation to the dynamic case with \( R \rightarrow R(\tau) \) in the above. The resulting stress energy tensor is:

\[
\begin{align*}
S_{\theta}^\theta &= \frac{R^2 \ddot{R}(f - F) + f(R - M) - F(R - m)}{8\pi R^2 fF} \\
S_{\phi}^\phi &= S_{\theta}^\theta \\
S_{\phi}^r &= R^2 \sin^2 \theta \ S_{\tau}^\phi \\
S_{\tau}^\phi &= \frac{3}{8\pi} \frac{AMf(R^5(\ddot{R}^2 R + R - 2M))^{1/2} - amF(R^5(\ddot{R}^2 R + R - 2m))^{1/2}}{R^7 fF} \\
S_{\tau}^r &= \frac{f(R - 2M) - F(R - 2m)}{4\pi R^2 fF}
\end{align*}
\]

where \( \dot{} = d/d\tau. \)

## 5 Verification

In developing the junction package we have re-executed a number of calculations performed in the literature to verify that the package can reproduce these results. We list some of the verifications we have performed in Table 5 (Maple worksheets for these verifications are available by ftp [4].)

In addition we have verified the identities (16) and (18) for a variety of spacetimes including a number of non-spherical wormhole spacetimes.
| Problem | Verified | Comments |
|---------|----------|----------|
| Dust shell | $K_{ij}$ | Minkowski-Shell-Schw. evolution equation |
| spherical symmetry | $K_{ij}$ | Checked form of $K_{ij}$ for general spherical symmetry |
| inhomogeneous slab cosmology | $[g_{ij}] = 0$ \ $[K_{ij}] = 0$ | planar symmetry |
| Schwarzschild wormhole | $S_{ij}$, $\sigma$, $p$ | “Thin-shell” wormhole formed by joining Schw. exterior to Schw. exterior |
| Schw- deSitter shell | $S_{ij}$ | Spherical symmetry but uses coordinates $(r, v, \theta, \phi)$ |
| Rotating dust shell | $[g_{ij}] = 0$, $K_{ij}^+ \text{ to order } a^3$ | match to Kerr exterior in small $a$ limit (non-spherical surface) |
| collapsing, rotating dust shell | $S_{ij}$ to order $a$ | Match to Kerr to order $a$ (dynamic shell) |

Table 5: Results verified with GRjunction
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