Two-loop QCD vertices, WST identities and RG quantities

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Abstract

Recent results from the study of the two-loop three-gluon and ghost-gluon vertices are reviewed. The relevant Ward–Slavnov–Taylor (WST) identities and Renormalization Group (RG) quantities are discussed.
1 Introduction

The one-loop QCD vertices have been known for quite some time. In the symmetric case, \( p_1^2 = p_2^2 = p_3^2 \), the one-loop result for the three-gluon vertex in an arbitrary covariant gauge was calculated by Celmaster and Gonsalves \[1\] and confirmed by Pascual and Tarrach \[2\]. The general off-shell case was considered in the Feynman gauge by Ball and Chiu \[3\]. Later, various on-shell results have also been given, by Brandt and Frenkel \[4\], and by Nowak, Praszalowicz and Slomiński \[5\]. The most general results, valid for arbitrary values of the space-time dimension and the covariant-gauge parameter, have been presented in our paper \[6\]. The one-loop quark-gluon vertex (or its Abelian part which is related to the QED vertex) was studied by several authors \[7\].

Two-loop corrections to the three-gluon and ghost-gluon vertices have been studied in the zero-momentum limit, which refers to the case when one gluon has vanishing momentum. In this limit, the renormalized expressions for QCD vertices in the Feynman gauge have been presented by Braaten and Leveille \[8\]. In an arbitrary covariant gauge, the relevant results have been presented in our previous paper \[9\]; they will be reviewed here. It should be noted that the zero-momentum limit of the three-gluon vertex, as well as the relevant limits of the ghost-gluon vertex, are infrared finite, i.e. we do not get any singularities of infrared (on-shell) origin. The main argument \[9\] is just power counting.

Another limit of interest corresponds to an on-shell configuration, when two external gluons are on the mass shell, \( p_1^2 = p_2^2 = 0 \). This limit possesses essential infrared (on-shell) singularities \[10\]. The relevant on-shell results for the three-gluon and ghost-gluon vertices have been recently presented by two of us \[11\].

Information about Green functions is also required for the calculation of certain quantities related to the renormalization group equations, such as the \( \beta \) function and anomalous dimensions. The two-loop-order contributions to these quantities were calculated by Caswell \[12\], Jones \[13\], Vladimirov and Tarasov \[14\], and Egoryan and Tarasov \[15\], whereas the three-loop-order results were obtained by Tarasov, Vladimirov and Zharkov \[16\], and by Larin and Vermaseren \[17\]. Moreover, recently the four-loop-order expressions became available, due to Larin, van RItbergen and Vermaseren \[18\].

2 Preliminaries

The lowest-order gluon propagator in a general covariant gauge is

\[
\left( \delta^{a_1 a_2} / p^2 \right) \left( g_{\mu_1 \mu_2} - \xi_B p_{\mu_1} p_{\mu_2} / p^2 \right),
\]

where \( \xi_B \equiv 1 - \alpha_B \) is the bare gauge parameter.

The three-gluon vertex is defined as

\[
\Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2, p_3) \equiv -i g_B f^{a_1 a_2 a_3} \Gamma_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2, p_3),
\]

where \( f^{a_1 a_2 a_3} \) are the totally antisymmetric colour structures corresponding to the adjoint representation of the gauge group (for example, \( SU(N) \)); \( g_B \) is the bare coupling constant. When one of the momenta is zero, the three-gluon vertex contains only two tensor
The ghost-gluon vertex can be represented as
\[ \Gamma_{\mu_1\mu_2\mu_3}(p, -p, 0) = (2g_{\mu_1\mu_2}p_{\mu_3} - g_{\mu_1\mu_3}p_{\mu_2} - g_{\mu_2\mu_3}p_{\mu_1}) T_1(p^2) \]
\[ - p_{\mu_3} (g_{\mu_1\mu_2} - p_{\mu_1}p_{\mu_2}/p^2) T_2(p^2). \]  

At the lowest, “zero-loop” order, we have \( T_1^{(0)} = 1, \ T_2^{(0)} = 0. \)

The ghost-gluon vertex can be represented as
\[ \tilde{\Gamma}_{\mu_1}^{a_1a_2a_3}(p_1, p_2; p_3) \equiv -ig_B f^{a_1a_2a_3} p_1^\mu \tilde{\Gamma}_{\mu\mu\mu}(p_1, p_2; p_3), \]  

where \( p_1 \) is the out-ghost momentum, \( p_2 \) is the in-ghost momentum, \( p_3 \) and \( \mu_3 \) are the momentum and the Lorentz index of the gluon (all momenta are ingoing). For \( \tilde{\Gamma}_{\mu\mu\mu} \), the following decomposition \([3]\) is useful:
\[ \tilde{\Gamma}_{\mu\mu\mu}(p_1, p_2; p_3) = g_{\mu\mu_3} a(p_3, p_2, p_1) + \{ \text{terms with } p_{i\mu}p_j p\}_{\mu_3}. \]  

At the “zero-loop” level, \( \tilde{\Gamma}_{\mu\mu\mu}^{(0)} = g_{\mu\mu_3}, a^{(0)} = 1. \)

In particular, when \( p_3 = 0 \) or \( p_2 = 0 \) we get
\[ \tilde{\Gamma}_{\mu\mu_3}(-p, p; 0) = g_{\mu\mu_3} a_3(p^2) + \{ \text{terms with } p_{\mu}p_{\mu_3} \text{ term} \}, \]
\[ \tilde{\Gamma}_{\mu\mu_3}(p, 0; -p) = g_{\mu\mu_3} a_2(p^2) + \{ \text{terms with } p_{\mu}p_{\mu_3} \text{ term} \}, \]
\[ a_3(p^2) \equiv a(0, p, -p), \quad a_2(p^2) \equiv a(-p, 0, p). \]  

The gluon polarization operator and the ghost self energy are defined as
\[ \Pi_{\mu_1\mu_2}^{a_1a_2}(p) \equiv -\delta^{a_1a_2} \left( p^2 g_{\mu_1\mu_2} - p_{\mu_1}p_{\mu_2} \right) J(p^2), \]
\[ \tilde{\Pi}_{a_1a_2}(p^2) = \delta^{a_1a_2} p^2 \left[ G(p^2) \right]^{-1}. \]  

In the lowest-order approximation \( J^{(0)} = G^{(0)} = 1. \)

We shall use dimensional regularization \([19]\), with the space-time dimension \( n = 4-2\varepsilon, \ \varepsilon \to 0. \) In this paper we adopt the modification of the renormalization prescription by \'t Hooft \([20]\), corresponding to the so-called \( \overline{\text{MS}} \) scheme \([21]\). In what follows, the notations \( \xi, \alpha, g \) (without subscript) correspond to the renormalized (in the \( \overline{\text{MS}} \) scheme) quantities. In eqs. \([1]\), \([2]\), \([3]\) they are understood as the bare quantities \( \xi_B, \alpha_B, g_B. \)

The renormalization constants \( Z_T \) relating the dimensionally-regularized one-particle-irreducible Green functions to the renormalized ones,
\[ \Gamma^{(\text{ren})} \left( \left\{ \frac{p_j^2}{\mu^2} \right\}, \alpha, g^2 \right) \overset{\varepsilon \to 0}{\longrightarrow} \left\{ Z_T \left( \frac{1}{\varepsilon}, \alpha, g^2 \right) \right\} \Gamma \left( \left\{ p_j^2 \right\}, \alpha_B, g_B^2, \varepsilon \right) \]

look in this scheme like
\[ Z_T \left( \frac{1}{\varepsilon}, \alpha, g^2 \right) = 1 + \sum_{j=1}^{\infty} C_{\alpha}^{[j]}(\alpha, g^2) \frac{1}{\varepsilon j}, \]

where \( \alpha = 1 - \xi. \) In eq. \([3]\) \( \mu \) is the renormalization parameter with the dimension of mass. It is assumed that on the r.h.s. of eq. \([4]\) the squared bare charge \( g_B^2 \) and the bare
gauge parameter $\alpha_B$ must be substituted in terms of renormalized ones, multiplied by appropriate $Z$ factors (cf. eqs. ([10]) and ([17])).

We use the following definitions for renormalization factors:

$$
\Gamma^{(\text{ren})}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = Z_1 \Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3),
$$

$$
\Pi^{(\text{ren})}_{\mu_1\mu_2}(a_1 a_2)(p) = Z_2 \Pi_{\mu_1\mu_2}(a_1 a_2)(p),
$$

$$
\tilde{\Gamma}^{(\text{ren})}_{\mu}(a_1 a_2 a_3)(p_1, p_2, p_3) = \tilde{Z}_1 \tilde{\Gamma}_{\mu}(a_1 a_2 a_3)(p_1, p_2, p_3),
$$

$$
\tilde{\Pi}^{(\text{ren})}_{a_1 a_2}(p^2) = \tilde{Z}_3 \tilde{\Pi}_{a_1 a_2}(p^2).
$$

The WST identity requires that

$$
Z_3 / Z_1 = \tilde{Z}_3 / \tilde{Z}_1.
$$

The results for these renormalization factors in the pure Yang–Mills theory were first presented by Jones ([13]) (Feynman gauge) and by Vladimirov and Tarasov ([14]) (an arbitrary covariant gauge). The complete results in an arbitrary covariant gauge, including the fermionic contributions, were presented by Egorian and Tarasov ([13]) (cf. also in refs. [10, 22]).

Using ([13]), the bare coupling constant $g_B^2$ can be chosen (in the $\overline{\text{MS}}$ scheme) as

$$
g_B^2 = \left[ \mu^2 e^\gamma/(4\pi) \right]^\epsilon g^2 Z_1^2 Z_3^{-1} \tilde{Z}_3^{-2} = \left[ \mu^2 e^\gamma/(4\pi) \right]^\epsilon g^2 Z_1^2 Z_3^{-3},
$$

where $\gamma$ is the Euler constant. The gauge parameter $\alpha = 1 - \xi$ is renormalized as

$$
\alpha_B = Z_3 \alpha, \quad \text{so that} \quad \xi_B = 1 - Z_3(1 - \xi).
$$

Below we shall also use the notation

$$
h \equiv g^2/(4\pi)^2 = \alpha_s/(4\pi), \quad \text{where} \quad \alpha_s \equiv g^2/(4\pi).
$$

### 3 WST identities

In a covariant gauge, the Ward–Slavnov–Taylor (WST) identity ([13]) for the three-gluon vertex is of the following form ([24]):

$$\begin{align*}
p_3^{\mu_3} \Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) &= -J(p_3^2)G(p_3^2) \left( g_{\mu_1}^{\mu_3} p_1^2 - p_{1\mu_1} p_1^{\mu_3} \right) \tilde{\Gamma}_{\mu_3\mu_2}(p_1, p_3; p_2) \\
&\quad + J(p_2^2)G(p_3^2) \left( g_{\mu_2}^{\mu_3} p_2^2 - p_{2\mu_2} p_2^{\mu_3} \right) \tilde{\Gamma}_{\mu_3\mu_1}(p_2, p_3; p_1).
\end{align*}
$$

Consider what follows from ([13]) in the limit when one of the momenta vanishes. Contracting with a non-zero momentum, we get

$$
p^{\mu_3} \Gamma_{\mu_1\mu_2\mu_3}(p, -p, 0) = -J(p^2)G(p^2)a_3(p^2) \left( g_{\mu_2\mu_3} p^2 - p_{\mu_2} p_{\mu_3} \right),
$$

where $a_3(p^2)$ is defined in eq. ([3]). Considering contraction with the vanishing momentum, we get a differential WST identity. It can be written in a way which involves just the $a$...
functions from the ghost-gluon vertex \[9\]. For the scalar functions \( T_1(p^2) \) and \( T_2(p^2) \), the differential WST identity reads

\[
T_1(p^2) = a_3(p^2) G(p^2) J(p^2),
\]

\[
T_2(p^2) = 2T_1(p^2) - 2G(0) \left[ a_2(p^2) \frac{d}{dp^2} \left( p^2 J(p^2) \right) - p^2 J(p^2) \frac{da_2(p^2)}{dp^2} + \tilde{a}_2(p^2) J(p^2) \right],
\]

where the function \( \tilde{a}_2(p^2) \) is defined as

\[
\tilde{a}_2(p^2) \equiv p_1 \sigma \frac{\partial}{\partial p_1 \sigma} a(p_3, -p_1 - p_3, p_1) \bigg|_{p_1 = -p_3 = p}.
\]

It can be calculated directly at the diagrammatic level (see below).

\[\text{(21)}\]

\[\text{(22)}\]

\[\text{(23)}\]

Therefore, the differential WST identity makes it possible to define the whole three-gluon vertex (not only its longitudinal part) in terms of two-point functions and the ghost-gluon vertex. Moreover, it can be used as another independent way, in addition to the direct calculation, to obtain results for the three-gluon vertex.

4 Results for the three-gluon vertex

The results for unrenormalized one-loop contributions to the scalar functions \( T_1(p^2) \) and \( T_2(p^2) \) (in arbitrary space-time dimension) can be found in ref. \[6\], eqs. (4.30), (4.31), (4.33) and (4.34).

The diagrams contributing to the three-gluon vertex at the two-loop level are shown in Fig. 1 of ref. \[9\]. Note that the non-planar diagrams do not contribute, since their colour structures vanish \[25\]. When one external momentum vanishes, technically the problem reduces to the calculation of two-point two-loop Feynman integrals. To calculate the occurring integrals with higher powers of the propagators, the integration-by-parts procedure \[26\] has been used. For the integrals with numerators, some other known algorithms \[24, 27\] were employed. Straightforward calculation of the sum of all these contributions yields the results for the unrenormalized scalar functions which are presented in eqs. (4.8)–(4.11) of ref. \[9\].

Using eq. (11) with

\[
Z_1 = 1 + \frac{h}{\xi} \left[ C_A \left( \frac{\xi}{2} + \frac{3}{4} \xi \right) - \frac{4}{3} T \right] + h^2 \left\{ C_A T \left[ \frac{1}{\xi^2} \left( \frac{5}{2} - \xi \right) - \frac{25}{12 \xi} \right] - \frac{2}{\xi} C_F T \right\} + C_A^2 \left[ \frac{1}{\xi^2} \left( -\frac{13}{8} \xi + \frac{15}{32} \xi^2 \right) + \frac{1}{\xi} \left( \frac{21}{16} + \frac{45}{32} \xi - \frac{3}{16} \xi^2 \right) \right] + \mathcal{O}(h^3),
\]

we obtain the renormalized scalar amplitudes appearing in the three-gluon vertex (cf. eq. \[8\]),

\[
T_1^{\text{(ren)}} = 1 + h \left[ C_A \left( -\frac{25}{18} + \frac{1}{2} \xi - \frac{1}{4} \xi^2 \right) + \frac{20}{9} T \right] + h^2 \left[ C_A^2 \left( - \frac{401}{288} - \frac{1}{4} \xi + \frac{2317}{576} \xi + \frac{13}{8} \xi^2 \xi_3 + \frac{113}{144} \xi^2 \xi_3 + \frac{1}{16} \xi^3 + \frac{1}{16} \xi^4 \right) \right] + C_A T \left( \frac{777}{72} + 8 \xi_3 + \frac{20}{9} \xi + \frac{10}{9} \xi^2 \right) + C_F T \left( \frac{55}{3} - 16 \xi_3 \right) + \mathcal{O}(h^3),
\]

\[\text{(24)}\]

\[\text{(25)}\]
\[ T_2^{\text{(ren)}} = h \left[ C_A \left( -\frac{4}{3} - 2\xi + \frac{1}{4}\xi^2 + \frac{8}{3}T \right) + h^2 \left[ C_A T \left( \frac{157}{9} - \frac{37}{3}\xi - \frac{2}{3}\xi^2 \right) + 8C_FT \right. \right. \
\left. \left. + C_A^2 \left( -\frac{641}{36} - \zeta_3 + \frac{5}{36}\xi - \frac{1}{3}\xi\zeta_3 - \frac{297}{144}\xi^2 + \frac{19}{18}\xi^3 - \frac{1}{4}\xi^4 \right) \right] + O(h^3). \]  

Here and henceforth, we put \( p^2 = -\mu^2 \) in the renormalized expressions. In Feynman gauge our expressions agree with those by Braaten and Leveille [8]. 

In eqs. (25) and (26) we use the standard notation \( C_A \) for the eigenvalue of the quadratic Casimir operator in the adjoint representation, \( f^{a\cd}f^{bcd} = C_A\delta_{ab} \) (\( C_A = N \) for the SU\((N)\) group), whereas \( C_F \) is the eigenvalue of this operator in the fundamental representation \( \left( C_F = (N^2 - 1)/(2N) \right) \) for the SU\((N)\) group. Furthermore, \( T \equiv N_f T_R, \) \( T_R = \frac{1}{8} \text{Tr}(I) = \frac{1}{2} \) where \( I \) is the “unity” in the space of Dirac matrices, \( N_f \) is the number of quarks, \( \zeta_3 \equiv \zeta(3) \) is the value of Riemann’s zeta function.

5 Results for the ghost-gluon vertex

In order to check the WST identity, we need results for the ghost-gluon vertex in two limits corresponding to eq. (3). We shall also need the derivative \( \tilde{a}_2(p^3) \), eq. (23). The relevant one-loop results (for an arbitrary \( n \)) are listed in Appendix A of ref. [8]. Two-loop contributions to the ghost-gluon vertex are shown in Fig. 2 of ref. [9]. Straightforward calculation gives the unrenormalized results presented in eqs. (5.6)–(5.13) of ref. [9]. 

The derivative (23) has been calculated in the following way [9]. Let us consider the momenta \( p_1 \) and \( p_3 \) as independent variables, whereas \( p_2 = -p_1 - p_3 \). Therefore, the momentum \( p_1 \) flows from the in-ghost leg to the out-ghost leg. An unambiguous \( p_1 \) path inside the diagram can be chosen as the one coinciding with the ghost line. 

This is convenient, since all we need to differentiate are just two types of objects: ghost propagators and ghost-gluon vertices occurring along this path. In this way, we avoid differentiating gluon propagators and three-gluon vertices. We also avoid getting third powers of propagators. Technically, the propagators and vertices along the ghost path were “marked” by introducing an extra argument (say, \( z \)). Then, the derivative with respect to \( z \) was considered, and the rules for differentiating the ghost-gluon vertex and the ghost propagator (with subsequent contraction with \( p_{1\mu_1} \)) were supplied. In this way, we just formally differentiate along the ghost line, and then perform all calculations for \( p_1 = -p_3 = p, \) \( p_2 = 0 \). Finally, extracting the coefficient of \( g_{\mu\nu3} \) we arrived at the results for the function (23) presented in eqs. (5.14), (5.15) of ref. [9]. 

Using eq. (12) with
\[ \tilde{Z}_1 = 1 - \frac{h}{2\varepsilon} C_A (1 - \xi) + h^2 C_A^2 (1 - \xi) \left[ \frac{1}{\varepsilon^2} \left( \frac{5}{8} - \frac{1}{4}\xi \right) + \frac{1}{\varepsilon} \left( \frac{1}{8} + \frac{3}{4}\xi \right) \right] + O(h^3), \]  

we obtain the renormalized expressions for the scalar functions occurring in the ghost-gluon vertex:

\[ a_3^{\text{(ren)}} = 1 + \frac{1}{4} h C_A \left( 1 - \xi \right) + h^2 \left[ C_A^2 \left( \frac{137}{18} - \frac{1}{2}\zeta_3 - \frac{291}{36}\xi - \frac{1}{6}\xi\zeta_3 + \frac{7}{16}\xi^2 + \frac{3}{16}\xi^2\zeta_3 \right) + \frac{1}{4} C_AT \right] + O(h^3). \]  

\[ a_2^{\text{(ren)}} = 1 + \frac{1}{4} h C_A \xi \left( 1 - \xi \right) + h^2(1 - \xi) \left[ C_A^2 \left( \frac{167}{72} - \zeta_3 - \frac{43}{144}\xi - \frac{1}{16}\xi^2 - \frac{3}{16}\xi^3 \right) - \frac{5}{8} C_AT(1 - \xi) \right] + O(h^3). \]
6 Results for the two-point functions

One-loop results in arbitrary space-time dimension are available e.g. in refs. [6, 28] (see also in Appendix A of ref. [1]). Calculating the sum of one-particle irreducible two-loop diagrams contributing to the gluon polarization operator and the ghost self energy (shown in Fig. 3 and Fig. 4 of ref. [9], respectively), we arrive at the results which are presented in eqs. (6.10)–(6.15) of ref. [9].

Using eqs. (13) and (14) with

\[
\dot{Z}_3 = 1 + \frac{h}{\xi} \left[ C_A \left( \frac{2}{3} + \frac{1}{2} \xi \right) - \frac{5}{2^3 \xi} \right] + h^2 \left[ C_A T \left( \frac{1}{\xi^2} \left( \frac{5}{3} - \frac{2}{3} \xi \right) - \frac{5}{2^3 \xi} \right) - \frac{2}{\xi} C_F T \right.
\]
\[
+ C_A^2 \left( \frac{1}{\xi^2} \left( -\frac{25}{12^2} + \frac{5}{2^3 \xi} + \frac{1}{2^2 \xi^2} \right) + \frac{1}{\xi} \left( \frac{23}{8^2} + \frac{15}{2^3 \xi} \right) \right) \bigg\} + O(h^3), \tag{30}
\]

we obtain the renormalized expressions for two-point functions

\[
J^{(\text{ren})} = 1 + h \left[ C_A \left( \frac{2}{3} + \frac{1}{2} \xi \right) \right] + h^2 \left[ C_A T \left( \frac{1}{\xi^2} \left( \frac{5}{3} - \frac{2}{3} \xi \right) \right) \right. \bigg\} + O(h^3), \tag{32}
\]

\[
G^{(\text{ren})} = 1 + h C_A + h^2 \left[ C_A^2 \left( \frac{997}{26^2} + \frac{10}{3^2 \xi} \right) \right. \left. \right\} + O(h^3). \tag{33}
\]

In Feynman gauge, these results agree with those presented in ref. [9].

7 Renormalization group quantities

Using the $1/\xi$ term of the renormalization factor $Z_\Gamma$ (cf. eq. (10)), one can obtain the corresponding anomalous dimension $\gamma_\Gamma$ via

\[
\gamma_\Gamma \left( \alpha, g^2 \right) = g^2 \frac{\partial}{\partial g^2} C_{\Gamma,1} \left( \alpha, g^2 \right). \tag{34}
\]

We have checked that in the Feynman gauge $\xi = 0 \ (\alpha = 1)$ the results for the anomalous dimensions $\tilde{\gamma}_1$, $\gamma_3$ and $\tilde{\gamma}_3$ coincide (in the two-loop approximation) with those from ref. [16]:

\[
\gamma_1 = h \left[ C_A \left( \frac{2}{3} + \frac{1}{2} \xi \right) - \frac{5}{2^3 \xi} \right] + h^2 \left[ C_A^2 \left( \frac{21}{2^3} + \frac{45}{2^2 \xi} \right) - \frac{25}{6} C_A T + 4 C_F T \right] + ..., \\
\tilde{\gamma}_1 = -\frac{3}{2} h C_A (1 - \xi) + h^2 C_A^2 (1 - \xi) \left( \frac{5}{3} + \frac{1}{3} \xi \right) + ..., \\
\gamma_3 = h \left[ C_A \left( \frac{2}{3} + \frac{1}{2} \xi \right) - \frac{5}{2^3 \xi} \right] + h^2 \left[ C_A^2 \left( \frac{23}{2^3} \right) - \frac{15}{2^3 \xi} \right] - 5 C_A T + 4 C_F T \right] + ..., \\
\tilde{\gamma}_3 = h C_A \left( \frac{1}{2} + \frac{1}{2} \xi \right) + h^2 \left[ C_A^2 \left( \frac{49}{2^3} \right) - \frac{15}{2^3 \xi} \right] + ... .
\]
One can easily check that $\gamma_1 - \gamma_3 = \tilde{\gamma}_1 - \tilde{\gamma}_3$ (this follows from the WST identity (15) and the definition (34)). Moreover, since $\beta(g^2) = g^2 [2\tilde{\gamma}_1 - \gamma_3 - 2\tilde{\gamma}_3]$ (cf. in ref. [16]) we obtain the same result for the two-loop $\beta$ function as those given in refs. [12–15], namely:

$$\frac{1}{g^2} \beta(g^2) = h \left[ -\frac{11}{3} C_A + \frac{4}{3} T \right] + h^2 \left[ -\frac{34}{3} C_A^2 + \frac{20}{3} C_A T + 4 C_F T \right] + O(h^3).$$

Higher terms of the $\beta$ function are available in refs. [16–18].

### 8 Conclusion

We have discussed the calculation of two-loop QCD vertices in an arbitrary covariant gauge, mainly in the zero-momentum limit [9]. For the three-gluon vertex, we needed to calculate two scalar functions, $T_1(p^2)$ and $T_2(p^2)$, associated with different tensor structures, cf. eq. (3). Two independent ways of calculating these scalar functions have been realized. One of them is just the straightforward calculation of all diagrams (cf. Fig. 1 of ref. [9]). Another way is based on exploiting the differential WST identity. In this way, we obtain representations of the scalar functions $T_1(p^2)$ and $T_2(p^2)$, eqs. (21) and (22), in terms of the functions occurring in the ghost-gluon vertex, its derivative (23), the gluon polarization operator and the ghost propagator. We have calculated all these functions and confirmed the result of the straightforward calculation.

We have constructed renormalized expressions for all Green functions involved. Note that in the zero-momentum limit the three-gluon vertex has no infrared (on-shell) singularities, this is a “pure” case for performing the ultraviolet renormalization. In particular, the $Z$ factors (e.g., in the $\overline{\text{MS}}$ scheme) can be constructed just by using the condition of finiteness of the renormalized Green functions.

In principle, the techniques for calculating scalar integrals corresponding to the two-loop QCD vertices in the general off-shell case are already available [20]. However, some work is still needed to construct the complete algorithm.

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