On the recovering of acoustic attenuation in 2D acoustic tomography

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Abstract. The inverse problem of recovering the acoustic attenuation in the inclusions inside the human tissue is considered. The coefficient inverse problem is formulated for the first-order system of PDE. We reduce the inverse problem to the optimization of the cost functional by gradient method. The gradient of the functional is determined by solving a direct and conjugate problem. Numerical results are presented.

Introduction

In this paper the problem of detecting inclusions in human soft tissues using ultrasound tomography is considered [19, 20, 25, 31, 33, 34, 36].

The system of hyperbolic equations of the first order is considered as the mathematical model of the acoustic tomography, because these equations are obtained directly from the conservation laws of continuum mechanics. This allows us to control the basic invariants when solving direct [1] and inverse [26] problems. This is important for solving unstable problems, since the conservation laws of basic invariants are the only criterion for the well-posedness of the solution. Some considerations on the choice of such a mathematical model based on a first-order system and a method for solving such a system can be found in [32, 35, 37, 38].

Numerical methods based on the Godunov scheme are widely used, and there are a huge number of their variants and implementations[18, 29].

For the solution of the coefficient inverse problems gradient methods [9, 12, 17, 23, 27] and global-convergence [14, 15, 16, 21, 24] are applied. A family of Newton-type methods should also be mentioned. However, their disadvantage is the solution of an additional linear inverse problem, which must be solved at each iteration. When considering multidimensional problems, this need to deal with this additional linear inverse problem usually becomes too complicated.

Inverse problems for hyperbolic systems were investigated theoretically in [6].

The modelling of radiation pattern of acoustic sources was developed in [39]. The problem is formulated as a control problem.
1. Inverse Problem

Let us consider the direct problem of acoustic wave propagation through the 2D medium in the domain \( \Omega = (x, y) \in [0, L] \times [0, L] \):

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \tag{1}
\]

\[
\frac{\partial p}{\partial t} + \sigma p + \rho c^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \theta_\Omega(x, y) I(t), \quad (x, y) \in \Omega, \tag{2}
\]

\[
u, v \big|_{(x,y) \in \partial \Omega} = 0, \tag{3}
\]

\[
u, v \big|_{t=0} = 0. \tag{4}
\]

Here \( u = u(x,y,t) \) is the velocity vector with respect to \( x \), \( v = v(x,y,t) \) is the velocity vector with respect to \( y \), \( p = p(x,y,t) \) is the exceeded pressure, \( c = c(x,y) \) is the wave speed, \( \rho = \rho(x,y) \) is the density of the medium, \( \sigma = \sigma(x,y) \) is the acoustic attenuation. \( \theta_\Omega(x,y) \) is the characteristic function of the source location, \( I(t) \) is the time pulse.

In the inverse problem it is required to find \( \sigma(x,y) \) by known the following additional information:

\[
p(x,y,t) = f_k(x,y,t), \quad (x, y) \in \Omega_k, \quad k = 1, \ldots, N. \tag{5}
\]

This means that the data of the inverse problem is the pressure that is measured in the receivers located in \( \Omega_k, k = 1, \ldots, N \).

Let us reformulate inverse problem (1)—(4), (5) in operator form

\[
A(\sigma) = f, \quad \sigma(x,y) \to f_k(x,y,t), \quad k = 1, \ldots, N. \tag{6}
\]

We reduce the inverse problem (1)—(4), (5) to minimization of the following cost functional:

\[
J(\sigma) = \sum_{k=1}^{N} \int_{0}^{T} \int_{\Omega_k} \left[ p(x,y,t;\sigma) - f_k(x,y,t) \right]^2 dx dy dt \to \min_{\sigma} \tag{7}
\]

by gradient method

\[
\sigma^{(n+1)} = \sigma^{(n)} - \alpha_n J'(\sigma^{(n)}). \tag{8}
\]

Here \( \sigma^{(0)} \) is initial guess, \( \alpha_n > 0 \) is descent parameter, \( J'(\sigma) \) is the gradient of the functional, which can be calculated by the formula [36]

\[
J'(\sigma)(x,y) = \int_{0}^{T} \frac{p(x,y,t) \Psi_3(x,y,t)}{\rho(x,y)c^2(x,y)} dt. \tag{9}
\]

Here function \( \Psi_3(x,y,t) \) is the solution of the conjugate problem [13, 37]:

\[
\frac{\partial \Psi_1}{\partial t} + \frac{1}{\rho} \frac{\partial \Psi_3}{\partial x} = 0; \tag{10}
\]

\[
\frac{\partial \Psi_2}{\partial t} + \frac{1}{\rho} \frac{\partial \Psi_3}{\partial y} = 0; \tag{11}
\]

\[
\frac{\partial \Psi_3}{\partial t} - \sigma \Psi_3 + \rho c^2 \left( \frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial y} \right) = 2\rho c^2 \sum_{k=1}^{N} \theta_{\Omega_k}(x,y) \left[ p(x,y,t) - f_k(x,y,t) \right]; \tag{12}
\]

\[
\Psi_i(x,y,T) = 0, \quad i = 1, 2, 3; \tag{13}
\]

\[
\Psi_i|_{(x,y) \in \partial \Omega} = 0, \quad i = 1, 2, 3. \tag{14}
\]
The inverse problem of recovering conductivity using the gradient method was considered in [23] and recovering attenuation in acoustic [30].

At each iteration of the gradient method we solve the direct and conjugate problems. In [40] it was presented an approach to save twice memory on the stage of adjoint problem and gradient calculation and compare it with usual approach in memory and CPU time cost. Convergence of gradient methods for hyperbolic equations was investigated in [9, 12]. It was shown [13] that if we add a priori information about the solution of the inverse problem to the gradient method, the number of iterations will decrease significantly. Work [28] presented a stopping criterion the the gradient method, consistent with the accumulation of machine round-off errors.

2. The numerical results

The acoustic parameters of human body are taken from [2, 11, 22].

Let us consider how acoustic attenuation influences the inverse problem data. On the 1 we solve the direct problem using different $\sigma$ and find the pressure that we measure in the receiver. Numerical calculations have shown that an increase $\sigma$ in the leads to a decrease in the amplitude of the pressure wave (see figure 1).

We apply the MUSCL scheme [4, 5, 7, 8, 10] for solving direct and conjugate problems.

![Figure 1](image1.png)

**Figure 1.** Test — influence of acoustic attenuation on the inverse problem data.

We solve the inverse problems of recovering acoustic attenuation with known the wave speed $c(x, y)$ and the density $\rho(x, y)$ of the medium. We consider the uniform mesh $100 \times 100$ for each space variable and two systems of 8 and 16 transducers, and use Pusyrev wavelet with frequency $\nu_0 = 100$ KHz. True model of acoustic attenuation is presented on figure 2.

On figure 3 the numerical solution of inverse problem of recovering $\sigma(x, y)$ is presented. Left it is shown the inverse problem solution for 8 and 1000 iteration, Right — for 16 transducers and 1000 iteration. Recovering the acoustic attenuation for 16 transducers is much better, the discrepancy is less by an order of magnitude then for 8 transducers.
Conclusion
In this paper, we considered the inverse problem of recovering acoustic attenuation for 2D hyperbolic system of equations of the first order. An algorithm for solving a direct problem is implemented using MUSCL-Hancock scheme [3] taking into account acoustic attenuation. A gradient method for solving a two-dimensional coefficient inverse problem of determining the acoustic attenuation in the medium is developed and implemented. The results of numerical calculations are presented.

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