A Study of Axiom of Choice for Fs-Sets

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Abstract
In this paper, based upon Fs-set theory [Yogesara V, Srinivas G, Rath B. A theory of Fs-sets, Fs-complements and Fs-de Morgan laws. IJARCS. 2013;4(10)], we define Fs-Cartesian product of given family Fs-subsets of given Fs-set and we prove Axiom of choice for Fs-sets and we study the validity of converse of the Axiom of choice for Fs-sets.

1. Introduction

Ever since Zadeh [1] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Murthy [2] introduced f-set in order to prove Axiom of choice for fuzzy sets which is not true for L-fuzzy sets introduced by Goguen [3]. The collection of all f-subsets of given f-set with his definition f-complement [4] could not form a compete Boolean algebra also for any f-subset \( B = (B, \overline{B}, L_B) \) and for any \( b \in B \) the complement of \( B_b \) denoted by \( \overline{(B_b)} \) is not discussed in the f-set theory interdicted by Neog and Sut [5, 6]. Recently many researchers put their efforts in order to prove collection of all fuzzy subsets of given fuzzy set is Boolean algebra under suitable operations and it seems among them the efforts of Neog and Sut [5, 6] and Mamoni [7] are most successful. The definition of fuzzy set given by Neog is based on the definition of fuzzy set given by Baruah [8]. Particularly in the definition of membership function of Neog and Sut [5] namely, \( \mu_1(x) - \mu_2(x) \) and \( -\mu_2(x) \) will not be in the real interval \([0, 1]\). To eliminate those lacunae Vaddiparthi Yogeswara, G. Srinivas and Biswajit Rath introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions. They are successful in their efforts in proving that result with some conditions. In papers [9] and [10] Vaddiparthi Yogeswara, Biswajit Rath and S. V. G. Reddy introduced the concept of Fs-Function between two Fs-subsets of given Fs-set and defined an image of an Fs-subset under a given Fs-function. Also they studied the properties of images under various kinds of Fs-functions.

In this paper, we introduced the concept of Fs-Cartesian product of given family of Fs-subsets of give Fs-set and prove Axiom of choice for Fs-sets also we study the validity of...
converse of the Axiom of choice for Fs-sets For smooth reading of paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra \( L_A \) [1.1] by \( M_A \) or 1. We denote Fs-union and crisp set union by same symbol \( \cup \) and similarly Fs-intersection and crisp set intersection by the same symbol \( \cap \) [11]. For all lattice theoretic properties and Boolean algebraic properties one can refer Szşz [12], Garret [13], Steven and Paul [14], James [15] and Thomas [16].

2. Preliminaries

Fs-Set

**Definition 2.1:** Let \( U \) be a universal set, \( A_1 \subseteq U \) and let \( A \subseteq U \) be non-empty. A four tuple \( \mathcal{A} = (A_1, A, \mu_{1A_1}, \mu_{2A}) \), \( L_A \) is said to be an Fs-set if, and only if

1. \( A \subseteq A_1 \)
2. \( L_A \) is a complete Boolean Algebra
3. \( \mu_{1A_1}: A_1 \to L_A \), \( \mu_{2A}: A \to L_A \), are functions such that \( \mu_{1A_1}|A = \mu_{2A} \)
4. \( A: A \to L_A \) is defined by

\[ \overline{A} = \mu_{1A_1} \setminus (\mu_{2A} \cap \cdot \cdot \cdot) , \text{for each } x \in A \]

**Definition 2.2:** Let \( \mathcal{A} = (A_1, A, \mu_{1A_1}, \mu_{2A}) \), \( L_A \) and \( \mathcal{B} = (B_1, B, \mu_{1B_1}, \mu_{2B}) \), \( L_B \) be a pair of Fs-sets. \( \mathcal{B} \) is said to be an Fs-subset of \( \mathcal{A} \), denoted by \( \mathcal{B} \subseteq \mathcal{A} \), if, and only if

1. \( B_1 \subseteq A_1 \), \( A \subseteq B \)
2. \( L_B \) is a complete subalgebra of \( L_A \) or \( L_B \leq L_A \)
3. \( \mu_{1B_1} = \mu_{1A_1}|B_1 \), and \( \mu_{2B}|A = \mu_{2A} \)

**Proposition 2.3:** Let \( \mathcal{B} \) and \( \mathcal{A} \) be a pair of Fs-sets such that \( \mathcal{B} \subseteq \mathcal{A} \). Then \( \overline{\mathcal{B}} = \overline{\mathcal{A}} \) is true for each \( x \in A \)

**Remark 2.1:** For some \( L_X \), such that \( L_X = L_A \) a four tuple \( \mathcal{X} = (X_1, X, \overline{X}(\mu_{1X_1}, \mu_{2X}) \), \( L_X \) is not an Fs-set if, and only if

(a) \( X \notin X_1 \)
(b) \( \mu_{1X_1}, X \notin \mu_{2X} \), for some \( x \in X_1 \)

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of \( \mathcal{B} \) for any \( \mathcal{B} \subseteq \mathcal{A} \).

**Definition 2.4:** An Fs-subset \( \mathcal{Y} = (Y_1, Y, \overline{Y}(\mu_{1Y_1}, \mu_{2Y}) \), \( L_Y \) of \( \mathcal{A} \), is said to be an Fs-empty set of second kind if, and only if

(a) \( Y_1 = Y \)
(b) \( L_Y = L_A \)
(c) \( \overline{Y} = 0 \)
Remark 2.2: We denote Fs-empty set of first kind or Fs-empty set of second kind by $\Phi_\mathcal{A}$.

Definition 2.5: Let $\mathcal{B}_1 = (B_{11}, B_1, B_1, B_1, \mu_{B_1}, L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, B_2, \mu_{B_2}, L_{B_2})$ be a pair of Fs-sets. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal, denoted by $\mathcal{B}_1 = \mathcal{B}_2$ if, and only if

1. $B_{11} = B_{12}, B_1 = B_2$
2. $L_{B_1} = L_{B_2}$
3. (a) $(\mu_{B_{11}} = \mu_{B_{12}}, \mu_{B_1} = \mu_{B_2})$, or (b) $B_1 = B_2$

Remark 2.3: We can easily observe that 3(a) and 3(b) are not equivalent statements.

Proposition 2.6: $\mathcal{B}_1 = (B_{11}, B_1, B_1, \mu_{B_1}, L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, B_2, \mu_{B_2}, L_{B_2})$ are equal if, and only if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$

Proposition 2.7: Fs-union for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B} = (B_1, B_1, B_1, \mu_{B_1}, L_B)$ and $\mathcal{C} = (C_1, C_1, \mu_{C_1}, L_C)$ be a pair of Fs-subsets of $\mathcal{A}$. Then, the Fs-union of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B} \cup \mathcal{C}$ is defined as $\mathcal{B} \cup \mathcal{C} = (D_1, D_1, D_1, \mu_{D_1}, L_D)$, where

(a) $D_1 = B_1 \cup C_1, D = B \cap C$
(b) $L_D = L_B \lor L_C = \text{complete subalgebra generated by } L_B \cup L_C$
(c) $\mu_{1D_1}: D_1 \rightarrow L_D$ is defined by

$$\mu_{1D_1}x = (\mu_{1B_1} \lor \mu_{1C_1})x.$$

Proposition 2.8: $\mathcal{B} \cup \mathcal{C}$ is an Fs-subset of $\mathcal{A}$.

Definition 2.9: Fs-intersection for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B} = (B_1, B_1, B_1, \mu_{B_1}, L_B)$ and $\mathcal{C} = (C_1, C_1, \mu_{C_1}, L_C)$ be a pair of Fs-subsets of $\mathcal{A}$ satisfying the following conditions:

(a) $B_1 \cap C_1 \supseteq B \cup C$
(b) $\mu_{1B_1}x \land \mu_{1C_1}x = (\mu_{2B} \lor \mu_{2C})x$, for each $x \in A$

Then, the Fs-intersection of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B} \cap \mathcal{C}$ is defined as $\mathcal{B} \cap \mathcal{C} = (E_1, E_1, E_1, \mu_{1E_1}, L_E)$, where

1. $E_1 = B_1 \cap C_1, E = B \cup C$
2. $L_E = L_B \land L_C = L_B \cap L_C$
3. $\mu_{1E_1}: E \rightarrow L_E$ is defined by $\mu_{1E_1}x = \mu_{1B_1}x \land \mu_{1C_1}x$

$\mu_{2E}: E \rightarrow L_E$
is defined by

$$\mu_{2E} = (\mu_{2B} \lor \mu_{2C})x.$$ 

$\bar{E} : E \rightarrow L_E$ is defined by

$$\bar{E}x = \mu_{1E}x \land (\mu_{2E}x)' .$$

**Remark 2.4:** If (a) or (b) fails we define $B \cap C$ as $B \cap C = \Phi$, which is the Fs-empty set of first kind.

**Proposition 2.10:** For any Fs-subsets $B, C$ and $D$ of $\mathcal{A} = (A, A, \bar{A}, (\mu_{1A}, \mu_{2A}), L_A)$, the following associative laws are true:

(i) $B \cup (C \cup D) = (B \cup C) \cup D$

(ii) $B \cap (C \cap D) = (B \cap C) \cap D$, whenever Fs-intersections exist.

**Definition 2.11 (Arbitrary Fs-unions and arbitrary Fs-intersections):** Given a family $(B_i)_{i \in I}$ of Fs-subsets of $\mathcal{A} = (A, A, \bar{A}, (\mu_{1A}, \mu_{2A}), L_A)$, where $B_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$, for any $i \in I$

Definition of Fs-union is as follows 2.12:

**Case (1):** For $I = \Phi$, define Fs-union of $(B_i)_{i \in I}$, denoted by $\bigcup_{i \in I} B_i$ as $\bigcup_{i \in I} B_i = \Phi$, which is the Fs-empty set

**Case (2):** Define for $I \neq \Phi$, Fs-union of $(B_i)_{i \in I}$ denoted by $\bigcup_{i \in I} B_i$ as follow

$$\bigcup_{i \in I} B_i = B = (B_1, B_\bar{B}, (\mu_{1B_1}, \mu_{2B}), L_B),$$

where,

1. $B_1 = \bigcup_{i \in I} B_{1i}, B = \bigcap_{i \in I} B_i$
2. $L_B = \bigvee_{i \in I} L_{B_i} = \text{complete subalgebra generated by } \bigcup L_i(L_i = L_{B_i})$
3. $\mu_{1B_1} : B_1 \rightarrow L_B$ is defined by

$$\mu_{1B_1}x = (\bigvee_{i \in I} \mu_{1B_{1i}})x = \bigvee_{i \in I_x} \mu_{1B_{1i}}x, \text{ where }$$

$$I_x = \{i \in I \mid x \in B_i\}$$

$\mu_{2B} : B \rightarrow L_B$ is defined by

$$\mu_{2B}x = \left(\bigwedge_{i \in I} \mu_{2B_i}\right)x = \bigwedge_{i \in I} \mu_{2B_i}x$$

$\bar{B} : B \rightarrow L_B$ is defined by $\bar{B}x = \mu_{1B_1}x \land (\mu_{2B}x)'$

**Remark 2.5:** We can easily show that (d) $B_1 \supset B$ and $\mu_{1B_1}B \geq \mu_{2B}$.
Definition 2.13 (Fs-intersection):

Case (1): For \( i = \Phi \), we define Fs-intersection of \((\mathcal{B}_i)_{i \in I}\), denoted by \( \bigcap_{i \in I} \mathcal{B}_i \), as

\[
\bigcap_{i \in I} \mathcal{B}_i = \mathcal{C} = (C_1, C, \overline{C} (\mu_{1C}, \mu_{2C}), \mathcal{L}_C)
\]

Case (2): Suppose \( \bigcap_{i \in I} B_1 \supseteq \bigcup_{i \in I} B_i \) and \( \bigwedge_{i \in I} \mu_{1B_i} (\bigcup_{i \in I} B_i) \supseteq \bigvee_{i \in I} \mu_{2B_i} \),

Then, we define Fs-intersection of \((\mathcal{B}_i)_{i \in I}\), denoted by \( \bigcap_{i \in I} \mathcal{B}_i \), as follows

\[
\bigcap_{i \in I} \mathcal{B}_i = \mathcal{C} = (C_1, C, \overline{C} (\mu_{1C1}, \mu_{2C}), \mathcal{L}_C)
\]

(1) \( C_1 = \bigcap_{i \in I} B_{1i} \), \( C = \bigcup_{i \in I} B_i \)

(2) \( L_C = \bigwedge_{i \in I} L_{B_i} \)

(3) \( \mu_{1C1} : C_1 \rightarrow L_C \) is defined by \( \mu_{1C1} x = (\bigwedge_{i \in I} \mu_{1B_i}) x = \bigwedge_{i \in I} \mu_{1B_i} x \)

\( \mu_{2C} : C \rightarrow L_C \)

is defined by \( \mu_{2C} x = (\bigvee_{i \in I} \mu_{2B_i}) x = \bigvee_{i \in I} \mu_{2B_i} x \),

where \( I_x = \{ i \mid x \in B_i \} \)

\( \mu_{2C} x \) is defined by \( \mu_{2C} x = \mu_{2C} x \cap (\mu_{2C} x) \)

Case (3): \( \bigcap_{i \in I} B_{1i} \not\subseteq \bigcup_{i \in I} B_i \) or \( \bigwedge_{i \in I} \mu_{1B_i} (\bigcup_{i \in I} B_i) \not\supseteq \bigvee_{i \in I} \mu_{2B_i} \)

We define \( \bigcap_{i \in I} \mathcal{B}_i = \Phi \).

Lemma 2.1: For any Fs-subset \( \mathcal{B} = (B_1, B, \overline{B} (\mu_{1B}, \mu_{2B}), L_B) \) and \( \mathcal{B} \subseteq \mathcal{B} = (B_{1i}, B_i, \overline{B}_i (\mu_{1B_{1i}}, \mu_{2B_{1i}}), L_{B_i}) \) for each \( i \in I \), \( \bigcap_{i \in I} \mathcal{B}_i \) exists and \( \mathcal{B} \subseteq \bigcap_{i \in I} \mathcal{B}_i \)

Proposition 2.14: \((\mathcal{L}(\mathcal{A}), \cap)\) is \( \wedge \)-complete lattices.

Corollary 2.1: For any Fs-subset \( \mathcal{B} \) of \( \mathcal{A} \), the following results are true

(1) \( \Phi_{\mathcal{A}} \cup \mathcal{B} = \mathcal{B} \)

(2) \( \Phi_{\mathcal{A}} \cap \mathcal{B} = \Phi_{\mathcal{A}} \).

Proposition 2.15: \((\mathcal{L}(\mathcal{A}), \cup)\) is \( \vee \)-complete lattices.

Corollary 2.2: \((\mathcal{L}(\mathcal{A}), \cup, \cap)\) is a complete lattice with \( \vee \) and \( \wedge \).

Proposition 2.16: Let \( \mathcal{B} = (B_1, B, \overline{B} (\mu_{1B}, \mu_{2B}), L_B) \), \( \mathcal{C} = (C_1, C, \overline{C} (\mu_{1C}, \mu_{2C}), L_C) \) and \( \mathcal{D} = (D_1, D, \overline{D} (\mu_{1D}, \mu_{2D}), L_D) \). Then \( \mathcal{B} \cup (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D}) \) provided \( \mathcal{C} \cap \mathcal{D} \) exists.

Proposition 2.17: Let \( \mathcal{B} = (B_1, B, \overline{B} (\mu_{1B}, \mu_{2B}), L_B) \), \( \mathcal{C} = (C_1, C, \overline{C} (\mu_{1C}, \mu_{2C}), L_C) \) and \( \mathcal{D} = (D_1, D, \overline{D} (\mu_{1D}, \mu_{2D}), L_D) \). Then \( \mathcal{B} \cap (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{D}) \) provided in R.H.S \( (\mathcal{B} \cap \mathcal{C}) \) and \( (\mathcal{B} \cap \mathcal{C}) \) exist.

3. Fs-functions

Definition 3.1: A Triplet \((f_1, f, \Phi)\) is said to be is an Fs-Function between two given Fs-subsets \( \mathcal{B} = (B_1, B, \overline{B} (\mu_{1B}, \mu_{2B}), L_B) \) and \( \mathcal{C} = (C_1, C, \overline{C} (\mu_{1C}, \mu_{2C}), L_C) \) of \( \mathcal{A} \), denoted by
Figure 1. Fs-function $\tilde{f} : \mathcal{B} \to \mathcal{C}$.

$$(f_1, f, \Phi) : \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \to \mathcal{C} = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$$ if, and only if (using the diagrams shown in Figure 1).

1. $f_1|_B = f$ is onto
2. $\Phi : L_B \to L_C$ is complete homomorphism

(f$_1$, f, $\Phi$) is denoted by $\tilde{f}$

**Proposition 3.2:**

(i) $\mu_{1C_1}|_C \circ f_1|_B \geq \mu_{2C} \circ f$

(ii) $\Phi \circ \mu_{1B_1} \geq \Phi \circ \mu_{2B}$

**Definition 3.3:** Increasing Fs-function $\tilde{f}$ is said to be an increasing Fs-function, and denoted by $\tilde{f}_i$ if, and only if (using Figure 1)

1. $\mu_{1C_1}|_C \circ f_1|_B \geq \Phi \circ \mu_{1B_1}$

2. $\mu_{2C} \circ f \leq \Phi \circ \mu_{2B}$

**Proposition 3.4:** $\Phi \circ (\mu_{2B} x)^c = [(\Phi \circ \mu_{2B} x)^c]^c$

**Proposition 3.5:** $\Phi \circ \overline{B} \leq \overline{C} \circ f$, provided $\tilde{f}$ is an increasing Fs-function

**Definition 3.6:** Decreasing Fs-function $\tilde{f}$ is said to be decreasing Fs-function denoted as $\tilde{f}_d$ if and only if

1. $\mu_{1C_1}|_C \circ f_1|_B \leq \Phi \circ \mu_{1B_1}$

2. $\mu_{2C} \circ f \geq \Phi \circ \mu_{2B}$

**Proposition 3.7:** $\Phi \circ \overline{B} \geq \overline{C} \circ f$, provided $\tilde{f}$ is a decreasing Fs-function

**Definition 3.8:** Preserving Fs-function $\tilde{f}$ is said to be preserving Fs-function and denoted as $\tilde{f}_p$ if, and only if

1. $\mu_{1C_1}|_C \circ f_1|_B = \Phi \circ \mu_{1B_1}$

2. $\mu_{2C} \circ f = \Phi \circ \mu_{2B}$
**Proposition 3.9:** \( \Phi \circ B = C \circ f \), provided \( f \) is Fs-preserving function

**Definition 3.10 (Composition of two Fs-function):**

Given two Fs-functions \( \bar{f} : \mathcal{B} \to \mathcal{C} \) and \( \bar{g} : \mathcal{C} \to \mathcal{D} \). We denote composition of \( \bar{g} \) and \( \bar{f} \) as \( \bar{g} \circ \bar{f} \) and define as \( (\bar{g} \circ \bar{f}) = (g_1, g, \Psi) \circ (f_1, f, \Phi) = [g_1 \circ f_1, g \circ f, \Psi \circ \Phi] \)

**Proposition 3.11:**

(a) Composition of two increasing Fs-function are increasing.

(b) Composition of two decreasing Fs-function are decreasing.

(c) Composition of two preserving Fs-function are preserving.

**Remark 3.1:** \((f_1, f, \Phi)\) is preserving if, and only if \((f_1, f, \Phi)\) simultaneously both increasing and decreasing.

**Proposition 3.12:** The class of all Fs-sets as objects together with morphism sets Fs-functions under the partial operation denoted by \( \circ \) is called composition between Fs-functions whenever it exists is a category denoted by \( \mathbb{F}_s\text{-SET} \).

Here \((g_1, g, \Psi) \circ (f_1, f, \Phi) = (g_1 \circ f_1, g \circ f, \Psi \circ \Phi)\)

**Proposition 3.13:** The class of all Fs-sets as objects together with morphism sets increasing Fs-functions under the partial operation denoted by \( \circ \) is called composition between increasing Fs-functions whenever it exists is a category denoted by \( \mathbb{F}_s\text{-SET}_i \).

**Proposition 3.14:** The class of all Fs-sets as objects together with morphism sets decreasing Fs-functions under the partial operation denoted by \( \circ \) is called composition between decreasing Fs-functions whenever it exists is a category denoted by \( \mathbb{F}_s\text{-SET}_d \).

**Proposition 3.15:** The class of all Fs-sets as objects together with morphism sets preserving Fs-functions under the partial operation denoted by \( \circ \) is called composition between preserving Fs-functions whenever it exists is a category denoted by \( \mathbb{F}_s\text{-SET}_p \).

### 4. Fs-cartesian product

**Definition 4.1:** Let \((\mathcal{A}_i)_{i \in I}\) be a non-empty family of non-empty Fs-sets.

Define Fs-Cartesian Product of \((\mathcal{A}_i)_{i \in I}\) denoted by \(\prod_{i \in I} A_i\) as follows.

Here \(\mathcal{A}_i = (A_{1i}, A_i, \overline{A}_i, (\mu_1 A_{1i}, \mu_2 A_i), L_A)\), \(L_A\) is a non-degenerating complete Boolean algebra and \(\overline{A}_a \neq 0\) for at least one \(a_i \in A_i\).

Here \(\prod_{i \in I} \mathcal{A}_i = \mathcal{X}' = (X_1, X_\overline{X}, (\mu_1 X_1, \mu_2 X_\overline{X}), L_X)\), where

\(X_1 = \prod_{i \in I} A_{1i}\) such that \((\prod_{i \in I} A_{1i}, (P_{1i})_{i \in I})\) is the product of \((A_{1i})_{i \in I}\) in \(\mathbb{SET}\), the category of sets with usual maps between crisp sets.

\(X = \prod_{i \in I} A_i\) such that \((\prod_{i \in I} A_i, (P_{i})_{i \in I})\) is the product of \((A_i)_{i \in I}\) in \(\mathbb{SET}\), the category of sets with usual maps between crisp sets.

\(L_X = \prod_{i \in I} L_A\) such that \((\prod_{i \in I} L_A, (\pi_i)_{i \in I})\) is the product of \((L_A)_{i \in I}\) in \(\mathbb{CBOO}\), the category of complete Boolean algebras with complete homomorphism between complete Boolean
algebra.

\[
\mu_{1X_1} = \prod_{i \in I} \mu_{1A_i} : \prod_{i \in I} A_{i} \rightarrow \prod_{i \in I} L_{A_i}
\]

\[
(a_i)_{i \in I} \mapsto (\mu_{1A_i}P_{1i}(a_i))_{i \in I} = (\mu_{1A_i}a_i)_{i \in I}
\]

\[
\mu_{2X_1} = \prod_{i \in I} \mu_{2A_i} : \prod_{i \in I} A_{i} \rightarrow \prod_{i \in I} L_{A_i}
\]

\[
(a_i)_{i \in I} \mapsto (\mu_{2A_i}P_{2i}(a_i))_{i \in I} = (\mu_{2A_i}a_i)_{i \in I}
\]

\[
X = \prod_{i \in I} A_i : \prod_{i \in I} A_{i} \rightarrow \prod_{i \in I} L_{A_i}
\]

\[
(a_i)_{i \in I} \mapsto (\bar{A}_iP_{1i}(a_i))_{i \in I} = (\bar{A}_ia_i)_{i \in I} = [\mu_{1A_i}a_i \land (\mu_{2A_i}a_i)^c]_{i \in I}
\]

is an Fs-set

The Fs-function \((P_{1i}, P_{2i}, \pi_i) : X \rightarrow A_i\) are Fs-projections

In particular \(\prod_{i \in I} A_i = X = A_I\) where \(A_i = A, \forall i \in I\).

**Definition 4.2 (Fs-Cartesian Product of non-empty family of non-empty Fs-subsets of \(\mathcal{A}\))**:

Let \((\mathcal{B}_i)_{i \in I}\) be a non-empty family of non-empty Fs-subset of \(\mathcal{A}\).

Define Fs-Cartesian Product of \((\mathcal{B}_i)_{i \in I}\), denoted by \(\prod_{i \in I} \mathcal{B}_i\) as follows.

Here \(\mathcal{B}_i = (B_{1i}, B_{2i}, \bar{B}_i (\mu_{1B_{1i}}, \mu_{2B_{2i}}, L_{B_i}), L_{B_i})\) in a non-degenerating complete Boolean algebra and \(\bar{B}_ib_i \neq 0\) for at least one \(b_i \in B_i\).

Here \(\prod_{i \in I} \mathcal{B}_i = C = (C_1, C\bar{C}, (\mu_{1C_1}, \mu_{2C}), L_C)\), where \(C_1 = \prod_{i \in I} B_{1i}\) such that \((\prod_{i \in I} B_{1i}, (P_{1i})_{i \in I})\) is the product of \((B_{1i})_{i \in I}\) in \(\text{SET}\), the category of sets with usual maps.

\(C = \prod_{i \in I} B_i\) such that \((\prod_{i \in I} B_{i}, (\pi_i)_{i \in I})\) is the product of \((B_i)_{i \in I}\) in \(\text{SET}\), the category of sets with usual maps.

\(L_C = \prod_{i \in I} L_{B_i}\) such that \((\prod_{i \in I} B_{i}, (\pi_i)_{i \in I})\) is the product of \((L_{B_i})_{i \in I}\) in \(\text{CBOO}\), the category of complete Boolean algebras.

\[
\mu_{1C_1} = \prod_{i \in I} \mu_{1B_{1i}} : \prod_{i \in I} B_{1i} \rightarrow \prod_{i \in I} L_{B_i}
\]

\[
(b_i)_{i \in I} \mapsto (\mu_{1B_{1i}}P_{1i}(b_i))_{i \in I} = (\mu_{1B_{1i}}b_i)_{i \in I}
\]

\[
\mu_{2C} = \prod_{i \in I} \mu_{2B_{2i}} : \prod_{i \in I} B_{2i} \rightarrow \prod_{i \in I} L_{B_i}
\]

\[
(b_i)_{i \in I} \mapsto (\mu_{2B_{2i}}P_{2i}(b_i))_{i \in I} = (\mu_{2B_{2i}}b_i)_{i \in I}
\]

\[
C = \prod_{i \in I} \bar{B}_i : \prod_{i \in I} B_i \rightarrow \prod_{i \in I} L_{B_i}
\]

\[
(b_i)_{i \in I} \mapsto (\bar{B}_iP_{1i}(b_i))_{i \in I} = (\bar{B}_ib_i)_{i \in I} = [\mu_{1B_{1i}}b_i \lor (\mu_{2B_{2i}}b_i)^c]_{i \in I}
\]

is an Fs-subset of \(\mathcal{A}'\)
(P_{1i}, P_i, \pi_i) : C \rightarrow B_i.

(P_{1i}, P_i, \pi_i) : C \rightarrow B_i is an Fs-function is a preserving function, where P_{1i}\mid_C = P_i. I.e following diagrams shown in Figure 2 are commutative.

\[ (\mu_{1B_{1i}} \circ P_{1i}) (b_i)_{i\in I} = \mu_{1B_{1i}} b_i \]
\[ \pi_j \circ (\mu_{1B_{1i}}) (b_i)_{i\in I} = \mu_{1B_{1i}} b_i \]
\[ (\mu_{2B_i} \circ P_i) (b_i)_{i\in I} = \mu_{2B_i} b_i \]
\[ \pi_j \circ (\mu_{2B_i}) (b_i)_{i\in I} = \mu_{2B_i} b_i \]

Remark 4.1: Observe that \( \pi_i : \prod_{i\in I} L_{B_i} \rightarrow L_{B_i} \) is a complete homomorphism.

Proof: Let \((b_i)_{i\in I}\) and \((d_i)_{i\in I}\) \(\in\) \(\prod_{i\in I} L_{B_i}\)

LHS: \(\prod_{i\in I} ((b_i)_{i\in I} \land (d_i)_{i\in I}) = \prod_{i\in I} ((b_i \land d_i)_{i\in I}) = b_i \land d_i \)

RHS: \(\prod_{i\in I} (b_i)_{i\in I} \land \prod_{i\in I} (d_i)_{i\in I} = b_i \land d_i \)

LHS = RHS

Similarly LHS: \(\prod_{i\in I} ((b_i)_{i\in I} \lor (d_i)_{i\in I}) = \prod_{i\in I} ((b_i \lor d_i)_{i\in I}) = b_i \lor d_i \)

RHS: \(\prod_{i\in I} (b_i)_{i\in I} \lor \prod_{i\in I} (d_i)_{i\in I} = b_i \lor d_i \)

LHS = RHS

LHS: \(\prod_{i\in I} ((b_i)_{i\in I})^c = \prod_{i\in I} (b_i^c)_{i\in I} = b_i^c \)

RHS: \(\prod_{i\in I} (b_i)_{i\in I}^c = (b_i^c)_{i\in I} = b_i^c \)

Hence \(\pi_i : \prod_{i\in I} L_{B_i} \rightarrow L_{B_i}\) is a complete homomorphism.

Axiom of choice for Fs-Set

Theorem 4.3: Let \((B_i)_{i\in I}\) be a non-empty family of non-empty Fs-subsets of \(A\), then Fs-Cartesian Product of \((B_i)_{i\in I}\), namely \(\prod_{i\in I} B_i\) is a non-empty Fs-subset, that is, \(\prod_{i\in I} B_i \neq \Phi_A\)

Proof: Given \(B_i \neq \Phi_A\) for each \(i \in I\)

\[ \Rightarrow B_{1i} \supseteq B_i, \mu_{1B_{1i}} |_{B_i} = \mu_{2B_i}, 0, 1 \in L_{B_i} \text{ such that } 0 \neq 1, \text{ for each } i \in I \]

\[ \Rightarrow \prod_{i\in I} B_{1i} = C_1 \supseteq \prod_{i\in I} B_i = C \quad \cdots (1) \]

And \(\mu_{1B_{1i}} |_{B_i} = (\mu_{2B_i})_{i\in I}\) \(\mu_{1B_{1i}} > \mu_{2B_i} b_i\) for \(b_i \in B_i\) \(\therefore B_i \neq \Phi_A\) for each \(i \in I\) imply \(\mu_{1B_{1i}} x > \mu_{2B_i} x\) for at least one \(b_i \in B_i\)

Hence \(\mu_{1B_{1i}} (b_i)_{i\in I} > (\mu_{2B_i} b_i)_{i\in I} (b_i)_{i\in I} \cdots (2)\)

Also observed that \(\prod_{i\in I} L_{B_i}\) is non-degenerate because \((a_i)_{i\in I} < (b_i)_{i\in I}\) where \(a_i = 0\) and \(b_i = 1\) for each \(i \in I\) \(\cdots (3)\)
Hence from (1), (2) and (3) we get

$$\prod_{i \in I} B_i \neq \Phi_{\mathcal{A}}$$

**Converse of the above theorem is true conditionally**

**Theorem 4.4:** Let \((B_i)_{i \in I}\) be a family of Fs-subsets such that \(\prod_{i \in I} B_i \neq \Phi_{\mathcal{A}}\). Then, for each \(i \in I\), \(B_i\) are NON-Fs-empty set of first kind.

**Proof:** Suppose if possible \(B_{i_0} = \Phi_{\mathcal{A}}\) = Fs-empty set of first kind, for some \(i_0 \in I\).

**Case (I):** \(B_{i_0} \supseteq B_i\)

\[\prod_{i \in I} B_{i_0} \supseteq \prod_{i \in I} B_i\] (\because \(\prod_{i \in I} B_i \supseteq \prod_{i \in I} B_i\) if, and only if \(B_i \supseteq B_i\) for each \(i \in I\))

Hence \(\prod_{i \in I} B_{i_0} = \Phi_{\mathcal{A}}\) Fs-empty set of first is contradiction.

**Case (II):** \(\mu_{B_{i_0}} B_i \neq \mu_{2B_0} B_i\) for some \(B_i \in \prod_{i \in I} B_i \neq \prod_{i \in I} B_i\) for any \(i_0 \in I\)

\[\Rightarrow (\mu_{B_{i_0}} B_i)_{i \in I} \neq (\mu_{2B_0} B_i)_{i \in I}\]

Hence \(\prod_{i \in I} B_i = \Phi_{\mathcal{A}}\) is contradiction.

From case (I) and (II) we get

\(B_i\) are NON-Fs-empty set of first kind for each \(i \in I\)  

**Example 4.1:** A non-empty Fs-Cartesian product of \(X\) of Fs-subset such that one of the member in \(X\) is Fs-empty set of second kind

Let \(\mathcal{B} = (B_1, B_2, (\mu_{B_1}, \mu_{B_2}), L_B)\) and \(\mathcal{C} = (C_1, C_2, (\mu_{C_1}, \mu_{C_2}), L_C)\) are Fs-subsets of an Fs-set \(\mathcal{A} = (A_1, A_2, (\mu_{A_1}, \mu_{A_2}), L_A)\) where \(A_1 = B_1 = C_1 = B = C = A\), \(\mu_{B_1} a = 1, \mu_{2B_1} a = 0\) and \(\mu_{C_1} a = \mu_{2C_1} a = 1\), \(\forall a \in A\) and \(L_B = L_C = \{0, 1\}\). Observe that \(\mathcal{C}\) is an Fs-empty set of second kind.

Suppose Fs-Cartesian product of \(\mathcal{B}\) and \(\mathcal{C}\), denoted by \(\mathcal{B} \times \mathcal{C} = X\) is given by \(\mathcal{B} \times \mathcal{C} = X = (X_1, X_2, (\mu_{X_1}, \mu_{X_2}), L_X)\), where \(X_1 = X = A \times A = A^2\), \(\mu_{X_1} = \mu_{B_1} \times \mu_{C_1}\), \(\mu_{X_2} = \mu_{B_2} \times \mu_{C_2}\) and \(L_X = L_B \times L_C\).

\(\mu_{X_1} = \mu_{B_1} \times \mu_{B_2}: X \longrightarrow L_B\) is defined as

\[\mu_{X_1}(a_1, a_2) = (\mu_{B_1} \times \mu_{C_1})(a_1, a_2) = (\mu_{B_1} a_1, \mu_{C_1} a_2) = (1, 1)\]

\(\mu_{X_2} = \mu_{B_2} \times \mu_{C_2}: X \longrightarrow L_C\) is defined as

\[\mu_{X_2}(a_1, a_2) = (\mu_{B_2} \times \mu_{C_2})(a_1, a_2) = (\mu_{B_2} a_1, \mu_{C_2} a_2) = (0, 1)\]

\[\overline{X}(a_1, a_2) = \mu_{X_1} (a_1, a_2) \land (\mu_{X_2}(a_1, a_2))^c\]

\[= (1, 1) \land (0, 1)^c\]

\[= (1, 1) \land (0^c, 1^c)\]

\[= (1, 1) \land (1, 0)\]

\[= (1 \land 1 \land 0) = (1, 0)\]

Hence \(\overline{X} \neq 0\) also \(L_X = L_B \times L_C = \{(0, 0), (0, 1), (1, 0), (1, 1)\}\)

Here \(X \neq \Phi_{\mathcal{A}}\), but \(\mathcal{C}\) is an Fs-empty set of second kind.
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