MODEL CATEGORY EXTENSIONS OF THE PIRASHVILI-SŁOMIŃSKA THEOREMS

RANDALL D. HELMSTUTLER

ABSTRACT. We describe the class of semi-stable model categories, which generalize the equivalence of finite products and coproducts in abelian and stable model categories, and use this to establish Morita equivalences among categories of functors. We provide a construction of pairs of small categories—known as conjugate pairs—whose associated categories of diagrams are Quillen equivalent in the semi-stable setting. We frame our development in the context of Morita theory, following Słomińska’s work on similar questions for categories of functors enriched over and taking values in $R$-modules.

1. INTRODUCTION

1.1. Background. Given a model category $C$ and any small category $D$, one can form the category $[D, C]$ of functors from $D$ to $C$. When $C$ is cofibrantly generated, this category of diagrams inherits a model structure. A basic question is then to determine when two such model categories of diagrams in $C$ are Quillen equivalent. In this article we develop an approach to this problem for the case when $C$ shares characteristics common to abelian and stable model categories. Precisely, in both types of categories the natural map $X \vee Y \to X \times Y$ is an equivalence of the right kind: an isomorphism in the abelian case and a weak equivalence in the stable case. Suitably generalized, this leads to the notion of a semi-stable model category.

The ideas of classical Morita theory have been used for some time now in describing equivalences of categories of functors; an early example may be found in [10]. More recently, Słomińska has developed a robust set of sufficient conditions that yield Morita equivalences for categories of functors taking values in $R$-modules [12]. Up to this point, such work has been primarily an algebraist’s affair. Our approach is to deduce a Morita theorem for diagrams in a semi-stable model category by suitably adapting a portion of Słomińska’s framework.

The genesis of both Słomińska’s original work and our current homotopy-theoretic version may be found in the results of Pirashvili in [11]. Let $E$ denote the category with objects the sets $n = \{1, 2, \ldots, n\}$ and with the
surjective maps as the morphisms (where $\mathbf{0}$ is the empty set). Let $\Gamma$ denote the category with objects the finite based sets $\mathbf{n}_+ = \{0, 1, 2, \ldots, n\}$ and morphisms the based maps, where $0$ acts as the basepoint. Paraphrased, Pirashvili proves in [11] that the functor categories $[\Gamma^{\text{op}}, \mathcal{C}]$ and $[\mathcal{E}^{\text{op}}, \mathcal{C}]$ are equivalent when $\mathcal{C}$ is an abelian category. The equivalence is given by the cross effect construction of Eilenberg and MacLane in [5].

At about the same time, similar equivalences were being noticed in stable homotopy theory, again arising from a (homotopical) cross effect. The categories $\mathcal{E}$ and $\Gamma$ appear together in several instances, such as in [1] and [2]. In his work on generalizations of the infinite symmetric product to categories of $S$-algebras, Kuhn [8] explicitly mentions an apparent connection to Pirashvili’s algebraic equivalence of categories. All of this suggests that the categories $[\Gamma^{\text{op}}, \mathcal{C}]$ and $[\mathcal{E}^{\text{op}}, \mathcal{C}]$ are Quillen equivalent when $\mathcal{C}$ is a stable model category. This is indeed the case, as proven and generalized in the author’s doctoral dissertation (independent of the work of Slomińska).

At this point it is an attractive idea to attempt to unify the algebraic theorem with the stable homotopy analogue. This is the motivation for considering semi-stable model categories: our main result proves both theorems at once. Hence en route to the proof of the stable homotopy result we recover the abelian case as well, but we do so via model-categorical methods. While we would never maintain that the techniques of model categories are a more sensible approach to abelian category questions, the point is that we can prove the stable homotopy version of the theorem while capturing the similarities between these disparate classes of categories.

Slomińska’s Morita theory [12] is a generalization of the behavior exhibited by the categories $\mathcal{E}$ and $\Gamma$. She gives conditions on pairs $(\mathcal{B}, \mathcal{A})$ of small categories enriched over $R$-modules so that there is an adjoint equivalence between $[\mathcal{B}^{\text{op}}, R^{\text{mod}}]$ and $[\mathcal{A}^{\text{op}}, R^{\text{mod}}]$. Suitably enriched, the pair $(\Gamma, \mathcal{E})$ is an example of such a Morita equivalent pair of small categories. Not only does this succeed in recovering Pirashvili’s theorem (at least for the category of $R$-modules), it also lends itself to numerous additional examples.

In this article we give similar conditions on pairs $(\mathcal{B}, \mathcal{A})$ of small categories so that the associated functor categories $[\mathcal{B}^{\text{op}}, \mathcal{C}]$ and $[\mathcal{A}^{\text{op}}, \mathcal{C}]$ are Quillen equivalent when $\mathcal{C}$ is semi-stable. Like Slomińska’s work, this is achieved by abstracting certain relations between the categories $\mathcal{E}$ and $\Gamma$, although we take a different approach necessitated by the non-additive nature of model categories. In Slomińska’s framework all categories and functors were additive, and because of this we will necessarily lose a couple of her examples.

Fortunately, the main feature of the pair $(\Gamma, \mathcal{E})$ to be generalized is not hard to describe. In fact, it closely parallels the First Isomorphism Theorem in group theory. Every $\Gamma$-map $\gamma : \mathbf{m}_+ \to \mathbf{n}_+$ admits a uniquely determined
three-fold factorization

\[
\begin{array}{ccc}
m_+ & \overset{\gamma}{\longrightarrow} & n_+ \\
\downarrow q & & \downarrow i \\
r_+ & \overset{\alpha}{\longrightarrow} & s_+
\end{array}
\]

where

- \( q \) is the quotient map that collapses the “kernel” \( \gamma^{-1}(0) \) of \( \gamma \),
- \( i \) represents the inclusion of the image of \( \gamma \) into its codomain, and
- \( \alpha \) is the unique epimorphism making the diagram commute.

Note that \( \alpha \) has the additional property that only the basepoint is mapped to the basepoint. Thus we may forget the basepoints and view \( \alpha \) as a morphism \( \alpha : r \to s \) in \( \mathcal{E} \).

Thus we see that every \( \Gamma \)-map gives rise to a uniquely determined map in \( \mathcal{E} \), and we can keep track of this assignment by recording the “cokernel” \( q \) and the inclusion map \( i \). Furthermore, note that the category of inclusion maps is in some sense dual to the category of quotients. Abstracting all of this structure leads to our notion of a conjugate pair of small categories, of which \( (\Gamma, \mathcal{E}) \) is the prime example. This is made precise in Section 4.

The primary goal of this paper is to show that every conjugate pair is in fact a Morita equivalent pair of small categories in the semi-stable setting. Paraphrasing to avoid some currently undefined notation, our main result asserts the following:

**Theorem 7.2.** Suppose that \( \mathcal{C} \) is a model category. Every conjugate pair \((\mathcal{B}, \mathcal{A})\) of small categories gives rise to an adjoint pair

\[
[B^{\text{op}}, \mathcal{C}] \xrightarrow{\mathcal{L}} [A^{\text{op}}, \mathcal{C}] \xleftarrow{\mathcal{R}}
\]

and this is a Quillen equivalence when \( \mathcal{C} \) is semi-stable.

This article is a generalization of several results found in the author’s doctoral dissertation, written under the direction of Prof. Nicholas Kuhn at the University of Virginia. We are grateful for all the helpful suggestions he has offered.

1.2. **Organization of the paper.** In Section 2.1 we lay our foundation for Morita equivalence for categories of functors. We will take classical Morita theory as inspiration for both the terminology and the notation. Section 2.2 sets in stone some blanket technical assumptions we must impose on model categories for the purposes of this paper, and cofibrant generation is quickly reviewed. In Section 2.3 we give the definition of a semi-stable model category.
In Section 3 we begin the development of the small category side of the story, where we carefully define the notion of an *indexing category*. Such categories will play the role of the category of inclusions in the three-fold factorizations. This type of category will also index some fundamental (co)product decompositions, hence the name.

Section 4 formalizes three-fold factorization to the notion of a *conjugate pair* of small categories; plenty of examples follow in Section 4.3. The bulk of the hard work lies in Section 5 where we show that a conjugate pair allows for the creation of a special functor, called the *regular bimodule*. Here, the term *bimodule* means a functor of the form $P : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Sets}_*$ (we are using $\text{Sets}_*$ to denote the category of based sets). The regular bimodule conveniently encodes many naturality properties which are necessary for the proof of our theorem, and these are detailed in Section 5.

Every bimodule creates an adjoint pair between functor categories. For a regular bimodule, the resulting right adjoint admits a nice product decomposition; this is the content of Section 6. In Section 6.2 we analyze the behavior of free functors and their pushouts under this adjunction. The remainder of the paper is devoted to proving Theorem 7.2, our main result.

## 2. Categorical Preliminaries

### 2.1. The Morita theory context

For this discussion, let us fix a pointed category $\mathcal{C}$ having all limits and colimits. In the applications in this article, $\mathcal{C}$ will always be a pointed model category.

The constructions involved in our Morita theory naturally arise from products and coproducts indexed by based sets. As an initial technicality we must be clear about how we will handle the basepoint. Given a based set $S$ and an object $C$ of $\mathcal{C}$, the product $\prod_S C$ will always stand for the ordinary product of copies of $C$, one copy for each non-basepoint element of $S$. That is, the factor corresponding to the basepoint of $S$ will always be taken to be the zero object of $\mathcal{C}$. The same convention applies to coproducts $\bigsqcup_S C$ indexed by based sets (as all of our categories will be pointed, we will be using the wedge symbol $\vee$ for the coproduct). It will be convenient to denote $\bigsqcup_S C$ by $C \otimes S$ on occasion.

Let us now fix a small category $\mathcal{A}$. Given functors $G : \mathcal{A}^{\text{op}} \to \mathcal{C}$ and $P : \mathcal{A}^{\text{op}} \to \text{Sets}_*$, we can form a new functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{C}$ by the assignment

$$(x, y) \mapsto \prod_{P(y)} G(x).$$

We define $\text{Hom}^\mathcal{A}(P, G)$ to be the end of this functor (see [9] for a refresher on ends, if necessary). We are writing $\mathcal{A}$ as a superscript to distinguish this from a set of morphisms in a category. It is entirely formal that this construction has all of the expected properties of a Hom-like object: functoriality of the expected variances, correct behavior with (co)products, and
a Yoneda lemma. For example, if \( P = A(−, a)_+ \) is a representable (think: free) functor, we have a natural isomorphism \( \text{Hom}^A(P, G) = G(a) \).

Likewise, we can use coends of coproducts to define a tensor product of functors. Fix a small category \( B \) and functors \( F : \mathcal{B}^{\text{op}} \to \mathcal{C} \) and \( Q : \mathcal{B} \to \text{Sets}_∗ \). From this we can form a functor \( \mathcal{B}^{\text{op}} \times \mathcal{B} \to \mathcal{C} \) via

\[
(x, y) \mapsto F(x) \otimes Q(y),
\]

the coend of which we denote \( F \otimes B Q \). Again, this has all of the expected properties of a tensor product.

The starting point in classical Morita theory is the simple fact that every bimodule gives an adjoint pair between categories of modules. In our situation, every functor of the form \( P : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Sets}_∗ \) will give rise to an adjoint pair between categories of contravariant functors; we will therefore call such functors bimodules.

To that end, fix a bimodule \( P : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Sets}_∗ \). Our \( \text{Hom}^A(−, −) \) and \( − \otimes B − \) constructions yield an adjoint pair

\[
[A^{\text{op}}, \mathcal{C}] \xrightarrow{\mathcal{L}} [B^{\text{op}}, \mathcal{C}] \xleftarrow{\mathcal{R}} [A^{\text{op}}, \mathcal{C}]
\]

described as follows. Given \( F : \mathcal{B}^{\text{op}} \to \mathcal{C} \) we define the functor \( \mathcal{L}F : A^{\text{op}} \to \mathcal{C} \) by

\[
\mathcal{L}F(a) = F \otimes B P(a, −).
\]

Dually, given \( G : \mathcal{A}^{\text{op}} \to \mathcal{C} \) we define \( \mathcal{R}G : \mathcal{B}^{\text{op}} \to \mathcal{C} \) by

\[
\mathcal{R}G(b) = \text{Hom}^A(P(−, b), G).
\]

It is entirely formal that \((\mathcal{L}, \mathcal{R})\) is an adjoint pair. All of the Quillen equivalences in our work will arise from this type of adjunction.

2.2. Model categories. We will take as the root definition of model category that which is set forth in Chapter 1 of Hovey’s Model Categories [7]. In particular, we will assume that the functorial factorizations are fixed as part of the underlying model structure (others assume that such factorizations merely exist). We will also be required to make some additional technical assumptions. For brevity’s sake, we will bundle these assumptions into the term model category.

Convention 2.1. In this paper, the term model category shall always mean a pointed, cofibrantly generated model category \( \mathcal{C} \) as in [7], possessing the following additional properties:

- \( \mathcal{C} \) is proper.
- The projections \( X \xrightarrow{p_X} X \times Y \xleftarrow{p_Y} Y \) are always fibrations.
- If \( f : A → B \) and \( g : C → D \) are acyclic fibrations then the map \( \xrightarrow{p_A \times qpc} A × C → B × D \) is a weak equivalence.
- If \( f : A → B \) and \( g : C → D \) are acyclic cofibrations then the map \( \xrightarrow{i_B f ∨ i_D g} A ∨ C → B ∨ D \) is a weak equivalence.
For the most part these assumptions are fairly mild. The properness assumption will guarantee that we have well-behaved homotopy pullbacks and pushouts, as in [6, Chapter 13]. Note that all projections will be fibrations whenever all objects in \( \mathcal{C} \) are fibrant. We also gain the following two results which will be needed in Section 6.

**Proposition 2.2.** Whenever \( f : A \to B \) and \( g : C \to D \) are fibrations, so is the map \( f p_A \times g p_C : A \times C \to B \times D \).

*Proof.* Since \( p_A \) and \( p_C \) are fibrations, so are \( f p_A \) and \( g p_C \). The result follows since products of fibrations are fibrations. \( \square \)

**Proposition 2.3.** Let \( \mathcal{C} \) be a model category in which the natural map \( X \vee Y \to X \times Y \) is a weak equivalence for all cofibrant objects \( X \) and \( Y \). Suppose that \( A \) and \( C \) are cofibrant objects. Then maps \( f : A \to B \) and \( g : C \to D \) are weak equivalences if and only if \( f p_A \times g p_C : A \times C \to B \times D \) is a weak equivalence.

*Proof.* The “if” direction \( (\Leftarrow) \) follows directly from the retract axiom in a general model category. For the other implication, suppose that \( f \) and \( g \) are weak equivalences. We may factor \( f \) and \( g \) as cofibrations followed by acyclic fibrations, say

\[
A \xrightarrow{f'} B' \sim B
\]

and

\[
C \xrightarrow{g'} D' \sim D
\]

respectively. Note that \( f' \) and \( g' \) are necessarily weak equivalences, and that \( B' \) and \( D' \) are cofibrant.

We now have a commutative diagram

\[
\begin{array}{c}
A \vee C \xrightarrow{(1)} B' \vee D' \\
\downarrow \quad \downarrow \\
A \times C \quad B' \times D' \\
\downarrow \quad \downarrow \\
B \times D
\end{array}
\]

in which map (4) is the map in question. Maps (1) and (5) are weak equivalences by our conventions, while maps (2) and (3) are weak equivalences by hypothesis. Hence map (4) is a weak equivalence, as desired. \( \square \)

We close this section with a few remarks about cofibrant generation and categories of functors; see [6, Chapter 11] for complete details. When \( \mathcal{C} \) is a cofibrantly generated model category, the category \( [\mathcal{D}^{\text{op}}, \mathcal{C}] \) of functors \( \mathcal{D}^{\text{op}} \to \mathcal{C} \) inherits a model structure (here \( \mathcal{D} \) can be any small category). This model structure is also cofibrantly generated, and weak equivalences...
and fibrations of diagrams are defined objectwise. We will need a complete
description of the cofibrations.

**Definition 2.4.** Fix an object \(d\) of \(D\) and an object \(C\) of \(C\). The *free functor* generated by \(d\) and \(C\) is the functor \(F_d^C : D^{op} \to C\) given by

\[
F_d^C(x) = C \otimes D(x, d)_+.
\]

Clearly every map \(i : B \to C\) in \(C\) induces a natural transformation \(i_* : F_d^B \to F_d^C\) of free functors. When \(C\) is cofibrantly generated, the cofibrations in \([D^{op}, C]\) are generated by the maps of the form \(i_* : F_d^B \to F_d^C\), where \(i\) is a generating cofibration of \(C\). Thus all cofibrations in the category of
diagrams are obtained through transfinite compositions of pushouts of such
maps (and retracts thereof).

In order to handle these transfinite compositions, we require one more
technical tool before we proceed, namely a reasonable sequential homotopy
colimit functor. Let \(C\) be a model category and fix an ordinal \(\lambda\). By a \(\lambda\)-
sequence in \(C\) we mean a functor \(X : \lambda \to C\), where \(\lambda\) is made into a category
in the usual way. Since \(C\) is cofibrantly generated, the category \([\lambda, C]\) of all
such sequences inherits a model structure with objectwise weak equivalences
and fibrations. This is even true if \(C\) is not cofibrantly generated, though we
will not need this fact. In this model structure, a \(\lambda\)-sequence is cofibrant if
it is objectwise cofibrant and each map in the diagram is a cofibration; see
\cite[Example 4.3]{Hirschhorn}.

We will use the letter \(Q\) to denote cofibrant replacement. According
to Hirschhorn in \cite[Proposition 17.9.1]{Hirschhorn}, the map \(\text{colim}(g) : \text{colim}(X) \to \text{colim}(Y)\) is a weak equivalence whenever \(g : X \to Y\) is a weak equivalence of cofibrant \(\lambda\)-sequences. This of course implies that the natural map
\(\text{colim}(QX) \to \text{colim}(X)\) is a weak equivalence whenever \(X\) is a cofibrant
\(\lambda\)-sequence.

One would like to define the homotopy colimit of a \(\lambda\)-sequence \(X\) as the
colimit of its cofibrant replacement. This turns out to be fine for objects,
but some care must be taken with the morphisms. We remark that cofibrant
replacements of objects are well-defined since the functorial factorizations
are fixed as part of the model structure. However, cofibrant replacements of
maps are unique only up to (left) homotopy. By passing to the homotopy
category \(\text{ho}(C)\), this problem is remedied.

**Proposition 2.5.** Forming colimits of cofibrant replacements yields a well-
deﬁned functor \(\text{hocolim} : [\lambda, C] \to \text{ho}(C)\). If \(X\) is a cofibrant \(\lambda\)-sequence, the
natural map \(\text{hocolim}(X) \to \text{colim}(X)\) is an isomorphism in the homotopy
category.

### 2.3. Semi-stable model categories

We can now describe the properties
model categories must have in order to develop our Morita equivalences. It
is helpful to keep in mind that each axiom below is a statement about the
 equivalence of finite products and coproducts.
Definition 2.6. We say that a model category $\mathcal{C}$ is *semi-stable* if the following product-coproduct coherence axioms are satisfied:

- **The Lower Triangular Axiom:** Suppose that $A_1$ and $A_2$ are cofibrant objects and that the map $f : A_1 \vee A_2 \to A_1 \times A_2$ has components $f_{ij} : A_i \to A_j$. If the diagonal components $f_{ii}$ are weak equivalences and $f_{12} = 0$, then $f$ is a weak equivalence.

- **Pushout-Product Coherence:** Given two homotopy pushout diagrams $(i = 1, 2)$

  \[
  \begin{array}{ccc}
  X_i & \longrightarrow & Y_i \\
  \downarrow & & \downarrow \\
  Z_i & \longrightarrow & P_i
  \end{array}
  \]

  the product square

  \[
  \begin{array}{ccc}
  X_1 \times X_2 & \longrightarrow & Y_1 \times Y_2 \\
  \downarrow & & \downarrow \\
  Z_1 \times Z_2 & \longrightarrow & P_1 \times P_2
  \end{array}
  \]

  is also a homotopy pushout.

- **Colimit-Product Coherence:** Whenever $X$ and $Y$ are cofibrant $\lambda$-sequences in $\mathcal{C}$, the natural map

  \[
  \text{hocolim}(X \times Y) \longrightarrow \text{hocolim}(X) \times \text{hocolim}(Y)
  \]

  is an isomorphism in the homotopy category.

Remark. By applying the Lower Triangular Axiom to the “identity matrix,” it is immediate that Proposition 2.3 holds in any semi-stable model category.

Proposition 2.7. Every complete and cocomplete abelian category admits the structure of a semi-stable model category.

Proof. Suppose that $\mathcal{C}$ is abelian with all limits and colimits. It is well-known that declaring the weak equivalences to be the isomorphisms gives a model structure on $\mathcal{C}$ (and here, every map is both a fibration and a cofibration). In this case, all “homotopy adjectives” become vacuous. It is immediate that all parts of Convention 2.1 are satisfied. Moreover, Pushout-Product and Colimit-Product Coherence follow from the fact that the natural map $X \oplus Y \to X \times Y$ is always an isomorphism.

For the Lower Triangular Axiom, suppose that $f = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is such a map with $a$ and $d$ isomorphisms. We can write down the inverse of $f$ explicitly...
as
\[ f^{-1} = \begin{pmatrix} a^{-1} & 0 \\ -d^{-1}ca^{-1} & d^{-1} \end{pmatrix} \]
as one can check. Hence the Lower Triangular Axiom holds and \( \mathcal{C} \) is semi-stable.

Many stable model categories also satisfy the axioms for a semi-stability. Recall that a pointed model category is *stable* if its homotopy category is triangulated under its natural loop and suspension. For a definition that avoids reference to the homotopy category, Mark Hovey points out in [7, Chapter 7] that this is equivalent to the coincidence of homotopy pullback and pushout squares in the underlying model category. Hence Pushout-Product Coherence is to be expected as products and pullbacks interact nicely. The third axiom will follow whenever products of weak equivalences are weak equivalences and so are the maps \( X \vee Y \to X \times Y \) for \( X \) and \( Y \) are cofibrant.

**Proposition 2.8.** The Lower Triangular Axiom holds in any stable model category in which all objects are fibrant.

*Proof.* Take a map \( f = \begin{pmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{pmatrix} \) as in the statement of the axiom. First one notes that the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{i_1} & A_1 \vee A_2 \\
\downarrow{f_{11}} & & \downarrow{f} \\
A_1 \times \{0\} & \xrightarrow{1 \times 0} & A_1 \times A_2 \xrightarrow{p_2} A_2
\end{array}
\]

commutes precisely because \( f_{12} = 0 \). As \( A_1 \) and \( A_2 \) are cofibrant, the top row is a homotopy cofiber sequence in the homotopy category. Likewise, since all objects are fibrant, the bottom row is a homotopy fiber sequence. Of course in the homotopy category, fiber and cofiber sequences coincide. Since \( f_{11} \) and \( f_{22} \) are weak equivalences, we now have a map of fiber sequences that is an isomorphism on the fiber and the base. Thus \( f \) represents an isomorphism in the homotopy category, and is therefore a weak equivalence. \( \square \)

**Example 2.9.** The simplest example of such a stable model category is the category of chain complexes of \( R \)-modules. Here the weak equivalences are the homology isomorphisms and the fibrations are the surjections. Of course this category is also abelian and thus may be regarded as semi-stable with the isomorphisms as the weak equivalences. Our main theorem says something very different in each of these cases.
3. Indexing Categories

Categories with a certain kind of rigid structure will be instrumental throughout the development of our main result. The tight structural properties of these categories will give us control over the indexing of crucial (co)product decompositions. In essence, these indexing categories serve to parameterize images and cokernels.

For this, we will need to recall some basic facts about EI-categories. First, in an EI-category all retracts are isomorphisms. In any category with pullbacks, the condition that all retracts are isomorphisms is equivalent to every map being monic. Hence all maps are monic in an EI-category with pullbacks. An additional finiteness assumption is all that we require in the following.

Definition 3.1. A small EI-category $I$ with pullbacks is an indexing category if each comma category $I \downarrow a$ has a skeleton with a finite number of objects.

The last condition simply asserts that for each object $a$ there are only finitely many maps to $a$ up to “covering” equivalence. Moreover, such covering equivalences are unique since all maps in an indexing category are monic. We will denote a skeleton of $I \downarrow a$ by $\text{sk}(I \downarrow a)$. One important structural aspect of indexing categories is given in the next proposition.

Proposition 3.2. Let $I$ be an indexing category. For each object $a$, the category $\text{sk}(I \downarrow a)$ has the structure of a finite partially ordered set.

Proof. This is even true before taking skeleta (except for the finiteness). The ordering $\leq$ is defined as follows: for maps $i : x \to a$ and $j : y \to a$, we declare $i \leq j$ if there is a commutative triangle

$$
\begin{array}{ccc}
  x & \xrightarrow{k} & y \\
  \downarrow{i} & & \downarrow{j} \\
  & a & \\
\end{array}
$$

Checking that this gives a partial ordering is routine. □

Example 3.3. Let $P$ be a partially ordered set with unique greatest lower bounds. Suppose that the segment $\{ x \in P \mid x \leq a \}$ is finite for each $a \in P$. Then the category formed from the poset $P$ (in the usual way) is an indexing category. Hence any finite tree forms an indexing category, as does the subgroup lattice of a finite group.

Example 3.4. Given a subset $A$ of the natural numbers (without 0), let $A_+$ denote $A \cup \{0\}$, where 0 will always play the role of the basepoint. Let $\mathcal{I}$ denote the category with objects the finite sets $A_+$ and morphisms the based injective functions. As pullbacks here are given by intersections, it is easy to see that $\mathcal{I}$ is an indexing category.
Example 3.5. (After example 11.2 of [14].) Fix a finite group $G$ and a homogeneous $G$-set $G/H$. Denote by $\Gamma(G/H)$ the category with $G/H$ as its only object and equivariant $G$-maps as the morphisms. It is well known that all equivariant maps $G/H \to G/H$ are automorphisms, so $\Gamma(G/H)$ is an EI-category with pullbacks. The finiteness condition is obviously met, so $\Gamma(G/H)$ is an indexing category. Functors from this category into the category of $R$-modules give left $R\text{Aut}(G/H)$-modules.

Example 3.6. Let $G$ be a group and let $\mathcal{M}_G$ be the category of finite $G$-sets and equivariant monomorphisms. Pullbacks correspond to intersections and the finiteness condition on the skeleta is clearly satisfied. Thus $\mathcal{M}_G$ is an indexing category.

Example 3.7. Call a $\Gamma$-map $i : m_+ \to n_+$ ordered if $i(x) < i(y)$ whenever $x < y$. Letting $\mathcal{O}$ denote the subcategory of $\Gamma$ consisting of the ordered maps, we see that $\mathcal{O}$ is an indexing category. It is clear that $\mathcal{O}(m_+, n_+)$ is in one-to-one correspondence with the subsets of $n = \{1, 2, \ldots, n\}$ of order $m$. Under this correspondence, the pullback of two maps in $\mathcal{O}$ corresponds to the intersection of the subsets they represent. Moreover, we have $\text{sk}(\mathcal{O} \downarrow n_+) = \mathcal{O} \downarrow n_+$ since $\mathcal{O}$ has no non-identity isomorphisms.

This last example has some additional structure that should be emphasized. Every ordered map $i : m_+ \to n_+$ has a natural dual $i^* : n_+ \to m_+$ that collapses the complement of the image of $i$ to the basepoint and satisfies $i^* \circ i = 1$. These two properties in fact characterize the “collapse” map $i^*$. All such collapse maps give a subcategory of $\Gamma$.

It is obvious that $(k \circ i)^* = i^* \circ k^*$, so that the category $\mathcal{O}^*$ of collapse maps is isomorphic to $\mathcal{O}^{\text{op}}$. Moreover, a commutative square

\[ \begin{array}{ccc}
    m_+ & \xrightarrow{i} & n_+ \\
    j \downarrow & & \downarrow k \\
    r_+ & \xrightarrow{l} & s_+
\end{array} \]

in $\mathcal{O}$ is a pullback if and only if $j \circ i^* = l^* \circ k$ (recall that pullbacks are intersections here). This “interchange law” implies that all composites of the form $j \circ i^*$ ($i, j \in \mathcal{O}$) yield a subcategory of $\Gamma$: in order to compose two such maps, one must use the appropriate pullback to swap the two middle terms. Note that this new parent category contains both $\mathcal{O}$ and $\mathcal{O}^*$ as subcategories. In the next section we will see that every indexing category allows for a construction of this sort; see Example 4.7 below.

4. Conjugate Pairs of Small Categories

4.1. Categories admitting conjugation. Let $\mathcal{U}$ be a category with subcategories $\mathcal{P}$ and $\mathcal{Q}$, all three having the same objects. We say that $\mathcal{U}$ factors as $\mathcal{Q} \circ \mathcal{P}$ if every morphism of $\mathcal{U}$ is expressible as a composition $q \circ p$ for some maps $q \in \mathcal{Q}$ and $p \in \mathcal{P}$. In this case we shall write $\mathcal{U} = \mathcal{Q} \circ \mathcal{P}$. For
us, such factorizations will not necessarily be unique on the nose, but only up to the correct notion of equivalence of maps in an indexing category (see the first axiom below). We will write $\text{Iso}(C)$ for the class of isomorphisms in a category $C$.

**Definition 4.1.** Suppose that $U$ is a small category which factors as $U = I \circ A$, where $I$ is an indexing category with $\text{Iso}(I) \subseteq \text{Iso}(A)$. We say that this factorization *admits conjugation* if the following two axioms hold:

- The factorization $U = I \circ A$ is unique up to lifting isomorphisms in $I$. Precisely, for each commutative square

$$
\begin{array}{ccc}
  a & \xrightarrow{\alpha} & b \\
  \beta \downarrow & & \downarrow i \\
  c & \xrightarrow{j} & d
\end{array}
$$

with $\alpha, \beta \in A$ and $i, j \in I$, the indicated lift exists and is an isomorphism in $I$.

- Given $\alpha : a \to b$ in $A$ and $i : c \to b$ in $I$, there is a pullback in $U$ of the form

$$
\begin{array}{ccc}
  p & \xrightarrow{i'} & a \\
  \alpha' \downarrow & & \downarrow \alpha \\
  c & \xrightarrow{i} & b
\end{array}
$$

with $\alpha' \in A$ and $i' \in I$. Moreover, the natural isomorphism relating two such pullback squares is given by an isomorphism in $I$.

In essence, these axioms serve to generalize the notion of image and inverse image, ensuring each is unique up to the correct kind of isomorphism. The motivation for this definition comes from the following example.

**Example 4.2.** Let us say that a map $\gamma$ in $\Gamma$ is *regular* if $\gamma^{-1}(0) = \{0\}$. That is, $\gamma$ is regular if it sends only the basepoint to the basepoint. Let $U$ denote the subcategory of regular maps. By forgetting the basepoints, we see that $U$ is equivalent to the category of finite (unbased) sets.

Take as our indexing category the category $\mathcal{O}$ of ordered maps in $\Gamma$ (see Example 3.7). By adding disjoint basepoints, we will consider the category $\mathcal{E}$ of unbased epimorphisms as a subcategory of $\Gamma$. Every map in $U$ clearly factors as an epimorphism followed by the inclusion of the image into the original codomain. Once this image subset is uniquely represented by an $\mathcal{O}$-map, we see that $U$ factors as $\mathcal{O} \circ \mathcal{E}$. It is immediate that the first axiom holds since such factorizations are unique on the nose (note that $\mathcal{O}$ has no non-identity isomorphisms).

For the second axiom, it is instructive to check that the pullback of an $\mathcal{E}$-map $\alpha : n_+ \to n_+$ along an $\mathcal{O}$-map $i : k_+ \to n_+$ is the inverse image under $\alpha$ of the subset of $n_+$ represented by $i$. It is then clear that parallel
partners in the pullback square belong to the same subcategory. Again, $O$ is sparse enough that the uniqueness requirement is trivially met, so the second axiom holds. Hence the factorization $U = O \circ E$ admits conjugation.

We will see that whenever a factorization $U = I \circ A$ admits conjugation, one can construct a new category $B$, obtained from $U$ by attaching formal pre-compositions by maps in $T^{op}$. In this way $B$ itself factors as $B = I \circ A \circ T^{op}$, giving the three-fold factorizations alluded to in the introduction.

4.2. Conjugate pairs. Throughout, suppose that $U = I \circ A$ is a factorization admitting conjugation. We will describe how to construct a new category $B$ containing all of the original data as subcategories, with $B$ factoring as $B = I \circ A \circ T^{op}$. We will refer to the pair of categories $(B,A)$ as a conjugate pair. This construction is a generalization of the induction categories of [14], or the category $\omega(G)$ of [13], both well-known to representation theorists. In fact, if one takes $A = \text{Iso}(I)$ then our two axioms for conjugation are trivially satisfied, and the construction we give reduces to the others.

The category $B$ will have the same objects as $U$. A morphism $\beta : a \to b$ in $B$ will be represented by a diagram

$$
a \leftarrow a_1 \xrightarrow{\alpha} b_1 \xrightarrow{i} b
$$

where $i,j \in I$ and $\alpha \in A$. Writing $i^* : a \to a_1$ for the formal opposite of $i$, we shall write $\beta = j \circ \alpha \circ i^*$. In order to get an honest category, we will have to identify some of these morphisms. We will declare the morphism above to be equivalent to

$$
a \leftarrow a_2 \xrightarrow{k} b_2 \xrightarrow{l} b
$$

if there exist isomorphisms $\varphi, \psi \in I$ making the entire diagram commute.

It is easy to check that this gives an equivalence relation, and one can take the morphisms in $B$ to be equivalence classes of such diagrams. Alternatively, we may take the morphisms of $B$ to be all formal composites of the form $\beta = j \circ \alpha \circ i^*$ as above, with the understanding that such representations are not unique. It is convenient to think of $\beta$ as admitting a three-fold factorization

$$
a \xrightarrow{\beta} b
$$

$$
i^* \downarrow \quad \downarrow j
$$

$$
a_1 \xrightarrow{\alpha} b_1
$$
with such factorizations unique only up to adjustments by isomorphisms in \( \mathcal{I} \). The latter point of view leads to simpler notation, so this is the approach we will take. We will refer to \( i^* \) as the cokernel of \( \beta \), and likewise we will call \( j \) its image.

**Remark.** In any two three-fold factorizations of the given map \( \beta \), the cokernels are equivalent morphisms in the comma category \( \mathcal{I} \downarrow a \). Likewise, the images are equivalent in \( \mathcal{I} \downarrow b \). Hence three-fold factorizations are unique if we require the \( \mathcal{I} \)-components to lie in a fixed skeleton of the relevant comma category.

Composition of such morphisms is defined in terms of pullbacks in the indexing category \( \mathcal{I} \) and the two axioms for conjugation. The composition of \( a \leftarrow i \ w \rightarrow^\alpha x \rightarrow^j b \) with \( b \leftarrow k \ y \rightarrow^\gamma z \rightarrow c \) is displayed in the following diagram:

\[
\begin{array}{ccccccc}
p & \rightarrow^\alpha' & q & \rightarrow^\gamma' & r \\
\downarrow^{k''} & & \downarrow^{k'} & & \downarrow^{j''} \\
w & \rightarrow^\alpha & x & \rightarrow^y & z \\
\downarrow^i & & \downarrow^j & & \downarrow^l \\
a & \rightarrow^b & c.
\end{array}
\]

The middle diamond is the pullback of \( j \) along \( k \), formed in \( \mathcal{I} \). The upper-left square is the pullback of \( \alpha \) along \( k' \) per the second axiom for conjugation; hence \( \alpha' \in \mathcal{A} \) and \( k'' \in \mathcal{I} \). According to the first axiom, the map \( \gamma \circ j' \) admits a factorization of the form \( j'' \circ \gamma' \) where \( j'' \in \mathcal{I} \) and \( \gamma' \in \mathcal{A} \); this is the upper-right square. Hence the composition is given by \( (lj'') \circ (\gamma' \alpha') \circ (ik'')^* \).

Of course, none of the steps in the composition are necessarily uniquely determined. However, it is easy to check that different choices would lead to equivalent morphisms (thanks to the axioms for conjugation), so the composition is in fact well-defined. This completes the description of the category \( \mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{\text{op}} \).

**Definition 4.3.** A pair \((\mathcal{B}, \mathcal{A})\) of small categories is a conjugate pair if there exists a factorization \( \mathcal{U} = \mathcal{I} \circ \mathcal{A} \) admitting conjugation with \( \mathcal{B} \) equivalent to the category \( \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^{\text{op}} \).

**Proposition 4.4.** Suppose that \((\mathcal{B}, \mathcal{A})\) is a conjugate pair arising from the factorization \( \mathcal{U} = \mathcal{I} \circ \mathcal{A} \).

(a) For maps \( i, j \in \mathcal{I} \), we have \( (j \circ i)^* = i^* \circ j^* \) whenever the composition is defined.

(b) For any map \( i \in \mathcal{I} \), we have \( i^* \circ i = 1 \).

(c) If \( i \in \mathcal{I} \) and both \( \alpha, i^* \circ \alpha \in \mathcal{A} \), then \( i \) must be an isomorphism.
Proof. The first two assertions follow immediately from the law of composition in $\mathcal{B}$. For the second, one needs only to recall that all maps in $\mathcal{I}$ are monic, hence the pullback (formed in $\mathcal{I}$) of $i$ along itself may be given by completing the square with identity maps.

For the third claim, suppose that

\[
\begin{array}{ccc}
p & \xrightarrow{j} & a \\
\downarrow & & \downarrow \\
c & \xrightarrow{i} & b
\end{array}
\]

is a pullback as in Definition 4.1, so that $i^* \circ \alpha = \gamma \circ j^*$. Since $\gamma \circ j^* \in A$, the diagram

\[
\begin{array}{ccc}
p & \xrightarrow{j} & c \\
\downarrow & & \downarrow \\
a & \xrightarrow{1} & a \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\gamma j^*} & c \\
\end{array}
\]

shows that $j$ must be an isomorphism in $\mathcal{I}$, hence $\alpha = i \circ \gamma \circ j^{-1}$. Since $\text{Iso}(\mathcal{I}) \subseteq \text{Iso}(A)$, we see that $\gamma \circ j^{-1} \in A$, and the same sort of equivalence diagram shows that $i$ is an isomorphism. $\square$

4.3. Examples. We start with the smallest and perhaps most instructive example.

Example 4.5. Let $\mathcal{I}$ be the category consisting of two objects and only one non-identity map, say $0 \xrightarrow{i} 1$.

Taking $A$ to be discrete, the resulting category $\mathcal{B} = \mathcal{I} \circ \mathcal{I}^{\text{op}}$ has diagrammatic representation

\[
\begin{array}{ccc}
0 & \xrightarrow{i} & 1 \\
\end{array}
\]

where $i^* \circ i = 1$. Note then that $i \circ i^*$ is an idempotent.

At this point it is instructive to recall what our main theorem would say here. It would assert that, for semi-stable model categories $\mathcal{C}$, the functor category $[\mathcal{B}^{\text{op}}, \mathcal{C}]$ is equivalent to $[\mathcal{A}^{\text{op}}, \mathcal{C}]$. As $A$ is discrete, the latter is simply $\mathcal{C} \times \mathcal{C}$. Hence the theorem says that to give a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i} & N \\
\end{array}
\]

in $\mathcal{C}$ with $i \circ i^*$ an idempotent is equivalent to giving two objects in $\mathcal{C}$ (namely, $M$ and the “kernel” of $i^*$). Hence our main theorem is essentially a statement about the ability to split idempotents in $\mathcal{C}$; this is the elegant explanation of our work. (Of course, we are concerned with obtaining a Quillen equivalence and so we are splitting idempotents in the homotopy category, not $\mathcal{C}$ itself.)
Example 4.6. Let $\mathcal{I}$ be the category of a partially ordered set $P$ as in Example 3.3. With $\mathcal{A}$ discrete, the category $\mathcal{B} = \mathcal{I} \circ \mathcal{I}^{\text{op}}$ has the following description. A morphism $a : x \to y$ in $\mathcal{B}$ is just the statement that $a \leq x$ and $a \leq y$. The composition of $a : x \to y$ and $b : y \to z$ is the greatest lower bound of $a$ and $b$. The category $\mathcal{I}$ appears in $\mathcal{B}$ as the maps $x : x \to y$ and similarly the maps $x : y \to x$ represent $\mathcal{I}^{\text{op}}$ as a subcategory. We now see that each map $a : x \to y$ in $\mathcal{B}$ factors uniquely as

$$
\begin{array}{ccc}
x & \xrightarrow{a} & y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
 & a & \\
\end{array}
$$

Hence $(\mathcal{B}, \mathcal{I})$ is a conjugate pair.

Example 4.7. The two previous examples generalize: every indexing category gives rise to a conjugate pair. Given an indexing category $\mathcal{I}$, let $\mathcal{A}$ be the category of isomorphisms in $\mathcal{I}$. It is immediate that the axioms for conjugation are satisfied, and so we obtain a category $\mathcal{B}$ with $(\mathcal{B}, \text{Iso}(\mathcal{I}))$ a conjugate pair. This is the non-additive version of the induction categories of [14].

Example 4.8. Here we obtain Pirashvili’s example [11], namely that $(\Gamma, \mathcal{E})$ is a conjugate pair. Recall the terminology and notation of Example 4.2, where we saw that $\mathcal{U} = \mathcal{O} \circ \mathcal{E}$ admits conjugation. We claim that the category $\mathcal{B}$ is equivalent to $\Gamma$. Recalling that $\mathcal{O}^{\text{op}}$ is equivalent to the category $\mathcal{O}^{*}$ of collapse maps, we show that every map in $\Gamma$ admits an internal three-fold factorization of the type $\mathcal{O} \circ \mathcal{E} \circ \mathcal{O}^{*}$.

Fix a $\Gamma$-map $\gamma : m_+ \to n_+$ and let $i : r_+ \to m_+$ be the $\mathcal{O}$-map representing the complement of the “kernel” $\gamma^{-1}(0)$ of $\gamma$. Upon using $i^*$ to collapse the kernel to a point, $\gamma$ induces a regular map $\overline{\gamma} : r_+ \to n_+$. This regular map $\overline{\gamma}$ then admits a factorization by an $\mathcal{E}$-map $\gamma' : r_+ \to s_+$ followed by the map $j : s_+ \to n_+$ representing the image of $\gamma$ (which is also the image of $\overline{\gamma}$). Thus $\gamma$ admits a three-fold factorization as $\gamma = j \circ \gamma' \circ i^*$:

$$
\begin{array}{ccc}
m_+ & \xrightarrow{\gamma} & n_+ \\
i^* & & j \\
r_+ & \xrightarrow{\gamma'} & s_+.
\end{array}
$$

Hence $(\Gamma, \mathcal{E})$ is a conjugate pair. By assuming all maps are weakly monotone one can also obtain a simplicial version of this example.

Example 4.9. The previous example may be fattened up a bit. Let $\mathcal{B}$ denote the category with objects the finite based subsets $A_+$ of the natural numbers (with 0 acting as the basepoint) and morphisms the based maps. With $\mathcal{A}$ the subcategory of regular based surjections and $\mathcal{I}$ as in Example 3.4, we have that $(\mathcal{B}, \mathcal{A})$ forms a conjugate pair.
Example 4.10. Let $\mathcal{B}$ denote the category of $\Gamma$-maps $\beta$ such that the inverse image $\beta^{-1}(x)$ of each nonzero point $x$ is either empty or a singleton. That is, $\beta \in \mathcal{B}$ may send lots of elements to the basepoint, but modulo this, it is injective. With $\Sigma$ denoting the category of regular permutations, $\mathcal{B}$ factors as $\mathcal{O} \circ \Sigma \circ \mathcal{O}^*$ and $(\mathcal{B}, \Sigma)$ is a conjugate pair.

5. Two Natural Decompositions

For the remainder of this paper, $(\mathcal{B}, \mathcal{A})$ will denote a fixed conjugate pair arising from a factorization $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$. In this section we construct the bimodule $\mathcal{U}_\alpha : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Sets}_*$ which will induce our Quillen equivalence. In Propositions 5.6 and 5.8 we show that this bimodule is “free” as a right $\mathcal{A}$-module and a “generator” in its left $\mathcal{B}$-module structure.

Definition 5.1. The maps in $\mathcal{B}$ lying in the subcategory $\mathcal{U} = \mathcal{I} \circ \mathcal{A}$ will be called the regular maps. If a map is not regular, we will say it is singular.

We remark that a map $\beta = j \circ \alpha \circ i^*$ is regular if and only if $i$ is an isomorphism in $\mathcal{I}$. Since $\text{Iso}(\mathcal{I}) \subseteq \text{Iso}(\mathcal{A})$, such a map is always equal to one with the identity map as its cokernel, as shown by the diagram

\[
\begin{array}{ccc}
  a & \xrightarrow{i} & a_1 \\
  \downarrow i & & \downarrow i \\
  1 & \xrightarrow{\alpha^{-1}} & b_2 \\
  \downarrow 1 & & \downarrow j \\
  a & \xrightarrow{\alpha i} & b_1 \\
  \downarrow j & & \downarrow j \\
  b & & b
\end{array}
\]

Notation 5.2. Let $\mathcal{S}(a,b)$ denote the set of singular maps in $\mathcal{B}(a,b)$. We let $\mathcal{U}(a,b)_+$ denote the quotient set

\[
\mathcal{U}(a,b)_+ = \mathcal{B}(a,b)/\mathcal{S}(a,b)
\]

where we take the singular maps as a basepoint. As $\mathcal{B}(a,b)$ is the disjoint union of the regular and singular maps, this may also be regarded as the set of regular maps together with a disjoint basepoint (thus this notation is sensible).

Proposition 5.3. In any conjugate pair $(\mathcal{B}, \mathcal{A})$, the singular maps have the following “ideal-like” properties:

(a) If $\alpha \in \mathcal{A}$ and $\gamma$ is singular, then $\gamma \circ \alpha$ is singular when defined.
(b) If $\beta \in \mathcal{B}$ and $\sigma$ is singular, then $\beta \circ \sigma$ is singular when defined.

Proof. We prove only the first statement; the proof of the second is similar (and simpler). Factor $\gamma$ as $\gamma = j \circ \gamma' \circ i^*$ and suppose that $\gamma \circ \alpha$ is regular. If $i^* \circ \alpha$ factors as $i^* \circ \alpha = l \circ \delta \circ k^*$ then the cokernel of $\gamma \circ \alpha$ is still $k^*$. Since $\gamma \circ \alpha$ is regular, $k$ must be an isomorphism in $\mathcal{I}$, and we can now arrange to take $k$ to be the identity by our previous remarks. From $i^* \circ \alpha = l \circ \delta$ we compose on the left with $l^*$, and Proposition 4.3 gives that $(i \circ l)^* \circ \alpha = \delta \in \mathcal{A}$, and hence $i \circ l$ must be an isomorphism in $\mathcal{I}$. We now see that $i$ is a retract in the EI-category $\mathcal{I}$, and hence $i$ is an isomorphism. Thus $\gamma$ is regular. \qed
Proposition 5.4. Given a conjugate pair \((B, A)\), the construction \(U(\cdot, \cdot)_+\) defines a functor \(A^{\text{op}} \times B \to \text{Sets}_*\).

Proof. Suppose we are given morphisms \(\alpha : a \to b\) in \(A\) and \(\beta : c \to d\) in \(B\). We then get an associated map \(B(b, c) \to B(a, d)\) which sends a map \(\gamma : b \to c\) to the composite \(\beta \circ \gamma \circ \alpha\). By Proposition 5.3 this sends singular maps to singular maps. Hence this passes down to quotients, as desired. \(\square\)

The functor \(U_+ : A^{\text{op}} \times B \to \text{Sets}_*\) is the bimodule desired for our Morita equivalence. We will refer to this functor as the regular bimodule.

Lemma 5.5. Suppose that \((B, A)\) is a conjugate pair and fix an object \(a\) of \(A\). Every map \(\beta : b \to c\) in \(B\) induces a map \(\beta_* : \bigvee_{i \in \text{sk}(I \downarrow b)} A(a, \text{dom}(i))_+ \to \bigvee_{j \in \text{sk}(I \downarrow c)} A(a, \text{dom}(j))_+\). Furthermore, this assignment is functorial, so that wedge sums of the form

\[\bigvee_{i \in \text{sk}(I \downarrow b)} A(a, \text{dom}(i))_+\]

give a functor \(B \to \text{Sets}_*\) in the \(b\)-variable.

Verifying that this defines a functor is fairly straightforward, and at worst consists of checking a few special cases. The proof uses only three-fold factorizations and Proposition 5.3, therefore we will only describe the induced map \(\beta_*\).

Suppose we are in the summand corresponding to \(i : b' \to b\) in \(I\), and let \(\alpha : a \to b'\) be a map in \(A\). We consider the composite

\[a \xrightarrow{\alpha} b' \xrightarrow{i} b \xrightarrow{\beta} c.\]

If this composite is singular, we define \(\beta_*(\alpha)\) to be the basepoint. Otherwise it is regular, and thus admits a factorization

\[
\begin{array}{ccc}
a & \xrightarrow{\beta i \alpha} & c \\
\downarrow{} & & \downarrow{}
\end{array}
\]

\[
\begin{array}{ccc}
1_* & & j \\
a & \xrightarrow{\alpha'} & c'
\end{array}
\]

where \(\alpha' \in A\) and \(j \in \text{sk}(I \downarrow c)\). Note that when factored in this form, \(\alpha'\) is uniquely determined. We then let \(\beta_*(\alpha) = \alpha'\), corresponding to the summand indexed by \(j\).

The next result states that the bimodule \(U_+ : A^{\text{op}} \times B \to \text{Sets}_*\) is free as a right \(A\)-module. Note that the naturality claim in the following is well-posed by the lemma.
Proposition 5.6. Suppose that \((\mathcal{B}, \mathcal{A})\) is a conjugate pair. For each pair of objects \(a\) and \(b\) of \(\mathcal{B}\), there is an isomorphism of based sets

\[
\mathcal{U}(a, b)_+ = \bigvee_{i \in \text{sk}(\mathcal{I}|b)} \mathcal{A}(a, \text{dom}(i))_+
\]

which is natural in both variables. In other words, for each object \(b\) of \(\mathcal{B}\) there is a natural equivalence

\[
\mathcal{U}(-, b)_+ = \bigvee_{i \in \text{sk}(\mathcal{I}|b)} \mathcal{A}(-, \text{dom}(i))_+
\]

of functors \(\mathcal{A}^{\text{op}} \to \text{Sets}_*\) and these equivalences vary naturally with \(b\).

The idea of the proof is simple: every regular map \(\gamma : a \to b\) admits a factorization by a map \(\alpha : a \to b'\) in \(\mathcal{A}\) followed by the inclusion \(i : b' \to b\) of the image of \(\gamma\) back into the codomain. Once a skeleton has been fixed, such a factorization is unique. Hence under our isomorphism, \(\gamma\) corresponds to \(\alpha \in \mathcal{A}(a, b')_+\), landing in the summand indexed by \(i\).

Next we carry out a similar analysis in the other variable.

Lemma 5.7. Suppose that \((\mathcal{B}, \mathcal{A})\) is a conjugate pair and fix an object \(c\) of \(\mathcal{B}\). Every map \(\beta : a \to b\) in \(\mathcal{B}\) induces a map

\[
\beta^* : \bigvee_{i \in \text{sk}(\mathcal{I}|b)} \mathcal{U}(\text{dom}(i), c)_+ \to \bigvee_{j \in \text{sk}(\mathcal{I}|a)} \mathcal{U}(\text{dom}(j), c)_+.
\]

Furthermore, this assignment is functorial, so that wedge sums of the form

\[
\bigvee_{i \in \text{sk}(\mathcal{I}|b)} \mathcal{U}(\text{dom}(i), c)_+
\]

give a functor \(\mathcal{B}^{\text{op}} \to \text{Sets}_*\) in the \(b\)-variable.

Proposition 5.8. Suppose that \((\mathcal{B}, \mathcal{A})\) is a conjugate pair. For each pair of objects \(b\) and \(c\) of \(\mathcal{B}\), there is an isomorphism of based sets

\[
\mathcal{B}(b, c)_+ = \bigvee_{i \in \text{sk}(\mathcal{I}|b)} \mathcal{U}(\text{dom}(i), c)_+
\]

which is natural in the first variable.

We remark that in general the isomorphisms of Proposition 5.8 are not natural in the second variable. Furthermore, by the finiteness condition on the indexing category \(\mathcal{I}\), the wedge sums of Propositions 5.6 and 5.8 consist of only a finite number of summands.

As before, verifying the assorted claims is rather formal (yet very tedious) once the definition of the induced map \(\beta^*\) is made clear. To that end, suppose
that \( \beta : a \to b \) has three-fold factorization

\[
\begin{array}{c}
a \\ \downarrow ^{i_1} \\ a_1
\end{array}
\xleftarrow{\beta_1}
\begin{array}{c}
\uparrow ^{j_1} \\ b_1
\end{array}
\xrightarrow{\beta}
\begin{array}{c}
b \\ \downarrow \\ b' \end{array}
\]

chosen with respect to the skeleta. We shall describe the map

\[
\beta^* : \bigvee_{i \in \text{sk}(I \uparrow b)} \mathcal{U}(\text{dom}(i), c)_+ \longrightarrow \bigvee_{j \in \text{sk}(I \downarrow a)} \mathcal{U}(\text{dom}(j), c)_+
\]

on the summand corresponding to a fixed map \( i : b' \to b \) in \( I \). (This map will always send regular maps to regular maps, so the basepoint is of no concern here.) The composite

\[
a_1 \xrightarrow{\beta_1} b_1 \xrightarrow{j_1} b \xrightarrow{i^*} b'
\]

admits a factorization as a map \( i^*_2 : a_1 \to a_2 \) followed by a regular map \( \beta_2 : a_2 \to b' \) (so \( \beta_2 \) is simply the last two legs of the three-fold factorization). Given a regular map \( \gamma : b' \to c \), we define

\[
\beta^*(\gamma) = \gamma \circ \beta_2
\]

landing in the summand corresponding to \( j = i_1 \circ i_2 : a_2 \to a \).

6. The Induced Adjoint Pair; Free Functors

6.1. First properties. Thus far, from a conjugate pair \((\mathcal{B}, \mathcal{A})\) we obtain an associated bimodule \( \mathcal{U}_+ : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Sets}_+ \). For a fixed model category \( \mathcal{C} \), this in turn gives rise to an adjoint pair

\[
\begin{array}{c}
\mathcal{B}^{\text{op}}, \mathcal{C} \\ \xleftarrow{\mathcal{L}} \\ \xrightarrow{\mathcal{R}} \\ \mathcal{A}^{\text{op}}, \mathcal{C}
\end{array}
\]

between model categories of functors, as described in Section 2.1. Before we can show that this is a Quillen equivalence when \( \mathcal{C} \) is semi-stable, we need some basic properties of this adjoint pair.

**Proposition 6.1.** Suppose that \( \mathcal{C} \) is a model category and that \((\mathcal{B}, \mathcal{A})\) is a conjugate pair of small categories. For each \( G : \mathcal{A}^{\text{op}} \to \mathcal{C} \) and \( b \) in \( \mathcal{B} \), there is an isomorphism

\[
\mathcal{R}G(b) = \prod_{i \in \text{sk}(I \uparrow b)} G(\text{dom}(i))
\]

and this is natural in both \( G \) and \( b \).

**Proof.** Recall that \( \mathcal{R}G(b) = \text{Hom}^A(\mathcal{U}(-, b)_+, G) \). Now use the natural isomorphism

\[
\mathcal{U}(-, b)_+ = \bigvee_{i \in \text{sk}(I \uparrow b)} \mathcal{A}(-, \text{dom}(i))_+
\]
supplied by Proposition 5.6 and the formal properties of $\text{Hom}^A(\cdot, G)$ to obtain the desired isomorphism.

Corollary 6.2. Suppose that $\mathcal{C}$ is a model category and that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. Then the adjoint pair $(\mathcal{L}, \mathcal{R})$ associated to the regular bimodule is a Quillen pair.

Proof. It is enough to check that $\mathcal{R}$ preserves fibrations and acyclic fibrations. This follows immediately from Proposition 2.2 and our conventions regarding model categories.

Corollary 6.3. Suppose that $\mathcal{C}$ is a semi-stable model category and that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. Let $\tau: F \to G$ be a natural transformation of functors $\mathcal{A}^{\text{op}} \to \mathcal{C}$ with $F$ cofibrant. Then $\tau$ is a weak equivalence if and only if $\mathcal{R}\tau$ is a weak equivalence.

Proof. This follows directly from Proposition 2.3.

Corollary 6.3 is one of two major ingredients in the proof of our main theorem. The missing ingredient is Proposition 7.1, which states that the unit map $\eta_F: F \to \mathcal{R}(\mathcal{L}F)$ is a weak equivalence whenever $F$ is cofibrant. In the next section we build the necessary machinery to complete this final step.

6.2. Free functors and pushouts. Our ultimate goal is to establish that the unit map $\eta_F: F \to \mathcal{R}(\mathcal{L}F)$ is a weak equivalence whenever $F$ is cofibrant. We prove this by an induction argument using the cofibrant generation hypothesis. As a first step we examine the free functors. Throughout this section, $\mathcal{C}$ denotes a semi-stable model category.

Lemma 6.4. Suppose that $F: \mathcal{B}^{\text{op}} \to \mathcal{C}$ is the free functor $F^C_\mathcal{b}$ with $C$ a cofibrant object of $\mathcal{C}$. Then $\mathcal{L}F: \mathcal{A}^{\text{op}} \to \mathcal{C}$ is given by $\mathcal{L}F(-) = C \otimes \mathcal{U}(-, b)_+.$

Proof. This is just the associativity of our various tensor products, together with the Yoneda lemma. By hypothesis, $F(x) = C \otimes \mathcal{B}(x, b)_+$ so that

$$
\mathcal{L}F(a) = F \otimes_\mathcal{B} \mathcal{U}(a, -)_+ \\
= (C \otimes \mathcal{B}(-, b)_+) \otimes_\mathcal{B} \mathcal{U}(a, -)_+ \\
= C \otimes (\mathcal{B}(-, b)_+ \otimes_\mathcal{B} \mathcal{U}(a, -)_+) \\
= C \otimes \mathcal{U}(a, b)_+.
$$

Proposition 6.5. Suppose that $F: \mathcal{B}^{\text{op}} \to \mathcal{C}$ is a free functor of the form $F^C_\mathcal{b}$ with $C$ a cofibrant object of $\mathcal{C}$. Then the unit map $\eta_F: F \to \mathcal{R}(\mathcal{L}F)$ is a weak equivalence.

Proof. We must show that for each object $a$, the natural map $\eta_F: F(a) \to \mathcal{R}(\mathcal{L}F)(a)$ is a weak equivalence. Recall that Proposition 5.8 supplies the isomorphism

$$
\mathcal{B}(a, b)_+ = \bigvee_{i \in \text{sk}(Ia)} \mathcal{U}(\text{dom}(i), b)_+.
$$
which is natural in the first variable. Using this in conjunction with Lemma 6.4 we obtain natural isomorphisms

\[ F(a) = C \otimes B(a, b)_+ \]
\[ = C \otimes \bigoplus_{i \in \operatorname{sk}(\mathcal{I} \downarrow a)} \mathcal{U}(\operatorname{dom}(i), b)_+ \]
\[ = \bigoplus_{i \in \operatorname{sk}(\mathcal{I} \downarrow a)} C \otimes \mathcal{U}(\operatorname{dom}(i), b)_+ \]
\[ = \bigoplus_{i \in \operatorname{sk}(\mathcal{I} \downarrow a)} \mathcal{L}F(\operatorname{dom}(i)). \]

Now consider the composite

\[ \bigoplus_{i \in \operatorname{sk}(\mathcal{I} \downarrow a)} \mathcal{L}F(\operatorname{dom}(i)) \simeq F(a) \xrightarrow{\eta^F} \mathcal{R}(\mathcal{L}F)(a) \simeq \prod_{j \in \operatorname{sk}(\mathcal{I} \downarrow a)} \mathcal{L}F(\operatorname{dom}(j)). \]

Fix maps \( i : x \to a \) and \( j : y \to a \) in the indicated skeleton. Then \( i \) determines the inclusion of the summand \( \mathcal{L}F(x) = C \otimes \mathcal{U}(x, b)_+ \) into the above coproduct, while \( j \) determines the projection onto the factor \( \mathcal{L}F(y) = C \otimes \mathcal{U}(y, b)_+ \) out of the product. Let \( f_{ij} \) denote the corresponding composite through the map (*) above. Recall that the index set \( \operatorname{sk}(\mathcal{I} \downarrow a) \) is a finite poset.

We claim that \( f_{ii} \) is the identity and that \( f_{ij} = 0 \) when either \( i < j \) or when \( i \) and \( j \) are incomparable. Writing \( F(a) = C \otimes B(a, b)_+ \), one checks that \( f_{ij} \) is induced by the composite

\[ \mathcal{U}(x, b)_+ \to B(a, b)_+ \to \mathcal{U}(y, b)_+ \]

which sends a regular map \( \gamma : x \to b \) to the composite \( \gamma \circ i^* \circ j \). Note that when \( i = j \), we have \( i^* \circ j = i^* \circ i = 1 \), hence \( f_{ii} \) is the identity. To prove that \( f_{ij} \) is the zero map in the other cases it suffices by Proposition 5.3 to show that \( i^* \circ j \) is singular.

In the case that \( i < j \) there is a diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{k} & y \\
  & \downarrow{i} & \downarrow{j} \\
  & a
\end{array}
\]

in \( \mathcal{I} \) where \( k \) is not an isomorphism, and from \( i = j \circ k \) we obtain \( i^* \circ j = k^* \). As \( k \) is not an isomorphism, \( i^* \circ j \) is singular, hence \( f_{ij} \) is the zero map.

The last case we must consider is when \( i \) and \( j \) are not comparable. If the pullback of \( i \) and \( j \) is

\[
\begin{array}{ccc}
  z & \xrightarrow{k} & y \\
  l & \downarrow{i} & \downarrow{j} \\
  x & \xrightarrow{i} & a
\end{array}
\]

as a diagram.
then we have \( l \circ k^* = i^* \circ j \). If \( k \) is an isomorphism we obtain \( j \leq i \), a contradiction. Hence \( i^* \circ j \) is singular, and \( f_{ij} \) is again the zero map.

We are almost in a position to apply the Lower Triangular Axiom. We need only turn the finite partially ordered set \( sk(I \downarrow a) \) into a linearly ordered set, consistent with the original partial ordering (finiteness is crucial here). This is achieved by inserting relations \(<\) between the incomparable maps; see, for instance, Theorem 4.5.2 of [3] for details. There is some choice here of course, but this does not matter.

In such a linear ordering, \( i < j \) means one of two things: either \( i < j \) in the original poset, or \( i \) and \( j \) were not comparable in the original partial ordering. In either event, \( f_{ij} \) is the zero map by our previous remarks. Hence under this linear ordering our map \((\ast)\) is lower triangular. Since \( C \) is cofibrant, the Lower Triangular Axiom shows that \((\ast)\) is a weak equivalence. Therefore \( \eta_F \) is a weak equivalence, as desired. \( \square \)

**Proposition 6.6.** Suppose that \( F_1, F_2 : \mathcal{B}^{\text{op}} \to \mathcal{C} \) are free functors and we have a pushout square

\[
\begin{array}{ccc}
F_1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]

in which

(a) the map \( F_1 \to F_2 \) is a generating cofibration in the category of functors, and

(b) the natural map \( \eta_X : X \to \mathcal{R}(LX) \) is a weak equivalence.

Then the map \( \eta_Y : Y \to \mathcal{R}(LY) \) is a weak equivalence.

**Proof.** As \( F_1 \to F_2 \) is a cofibration between cofibrant objects, the given pushout square is also a homotopy pushout. Since \( \mathcal{L} \) is left Quillen, the same is true of

\[
\begin{array}{ccc}
\mathcal{L}F_1 & \longrightarrow & \mathcal{L}F_2 \\
\downarrow & & \downarrow \\
\mathcal{L}X & \longrightarrow & \mathcal{L}Y.
\end{array}
\]

Applying \( \mathcal{R} \) to this yields an (objectwise) finite product of homotopy pushouts, so by Pushout-Product Coherence the square

\[
\begin{array}{ccc}
\mathcal{R}(\mathcal{L}F_1) & \longrightarrow & \mathcal{R}(\mathcal{L}F_2) \\
\downarrow & & \downarrow \\
\mathcal{R}(\mathcal{L}X) & \longrightarrow & \mathcal{R}(\mathcal{L}Y)
\end{array}
\]
is also a homotopy pushout. We now have a map
\[
\begin{array}{c}
\mathcal{R}(\mathcal{L}F_1) \\ \\
\downarrow \\ \\
F_1 \\ \\
\downarrow \\ \\
\mathcal{R}(\mathcal{L}X) \\ \\
\downarrow \\ \\
X \\
\end{array}
\rightarrow
\begin{array}{c}
\mathcal{R}(\mathcal{L}F_2) \\ \\
\downarrow \\ \\
F_2 \\ \\
\downarrow \\ \\
\mathcal{R}(\mathcal{L}Y) \\ \\
\downarrow \\ \\
Y \\
\end{array}
\]

of homotopy pushout squares in which the maps \( F_i \to \mathcal{R}(\mathcal{L}F_i) \) and \( X \to \mathcal{R}(\mathcal{L}X) \) are weak equivalences. It now follows (say, from Proposition 13.5.10 of \([6]\)) that the map \( Y \to \mathcal{R}(\mathcal{L}Y) \) is also a weak equivalence. \(\square\)

7. The Main Theorem

We are now in a position to prove the final missing ingredient. Again, \( C \) denotes a semi-stable model category throughout.

**Proposition 7.1.** For an arbitrary cofibrant functor \( F : \mathcal{B}^{op} \to C \), the unit map \( \eta_F : F \to \mathcal{R}(\mathcal{L}F) \) is a weak equivalence.

**Proof.** If \( F \) is cofibrant, it is either a cell complex or a retract thereof. The retract case follows easily from the cell complex case, so we assume that \( F \) is a cell complex. This means that \( 0 \to F \) is a relative cell complex, so that there is an ordinal \( \lambda \) and a transfinite diagram \( X : \lambda \to [\mathcal{B}^{op}, C] \) such that

1. \( X_0 = 0 \),
2. \( \text{colim}_\alpha X_\alpha = F \),
3. \( 0 \to F \) is the composition of \( X \), and
4. for each ordinal \( \alpha < \lambda \), the map \( X_\alpha \to X_{\alpha+1} \) fits into a pushout square

\[
\begin{array}{c}
F_1 \\ \\
\downarrow \\ \\
X_\alpha \\
\end{array}
\rightarrow
\begin{array}{c}
\mathcal{R}(\mathcal{L}X_\alpha) \\
\downarrow \\
\mathcal{R}(\mathcal{L}F) \\
\end{array}
\]

where \( F_1 \) and \( F_2 \) are free functors and \( i : F_1 \to F_2 \) is a generating cofibration in the category of diagrams.

For each ordinal \( \alpha < \lambda \) we have a commutative square

\[
\begin{array}{c}
X_\alpha \\
\downarrow \\
F \end{array}
\rightarrow
\begin{array}{c}
\mathcal{R}(\mathcal{L}X_\alpha) \\
\downarrow \\
\mathcal{R}(\mathcal{L}F) \\
\end{array}
\]
Taking homotopy colimits gives a commutative square

\[
\begin{array}{ccc}
\text{hocolim}_{\alpha} X_{\alpha} & \xrightarrow{(C)} & \text{hocolim}_{\alpha} \mathcal{R}(\mathcal{L}X_{\alpha}) \\
\downarrow & & \downarrow \\
F & \xrightarrow{\gamma(\eta_F)} & \mathcal{R}(\mathcal{L}F)
\end{array}
\]

in the homotopy category, where \(\gamma : \mathcal{C} \rightarrow \text{ho}(\mathcal{C})\) is the canonical localization. We claim that the maps labelled (A), (B) and (C) are isomorphisms. This would make \(\gamma(\eta_F)\) an isomorphism, so that \(\eta_F\) is then a weak equivalence. In fact, (A) is readily seen to be an isomorphism: since \(X\) is necessarily a cofibrant diagram, the map \(\text{hocolim}_{\alpha} X_{\alpha} \rightarrow \text{colim}_{\alpha} X_{\alpha} = F\) is an isomorphism.

Next, we show that map (B) is an isomorphism in the homotopy category. The key fact is that \(\mathcal{L}X\) is again a cofibrant diagram, so that \(\text{hocolim}_{\alpha} \mathcal{L}X_{\alpha} \cong \text{colim}_{\alpha} \mathcal{L}X_{\alpha} \cong \mathcal{L}F\). Invoking Colimit-Product Coherence, upon evaluation at an object \(b\) of \(\mathcal{B}\) we have a sequence of natural isomorphisms

\[
\text{hocolim}_{\alpha} \mathcal{R}(\mathcal{L}X_{\alpha})(b) = \text{hocolim}_{\alpha} \prod_{i \in \text{sk}(\mathcal{I})(b)} \mathcal{L}X_{\alpha}(\text{dom}(i)) = \prod_{i \in \text{sk}(\mathcal{I})(b)} \text{hocolim}_{\alpha} \mathcal{L}X_{\alpha}(\text{dom}(i)) = \prod_{i \in \text{sk}(\mathcal{I})(b)} \text{colim}_{\alpha} \mathcal{L}X_{\alpha}(\text{dom}(i)) = \mathcal{R}(\mathcal{L}F)(b).
\]

Hence map (B) is an isomorphism.

All that remains to be shown is that map (C) is an isomorphism. It suffices to prove that \(X_{\beta} \rightarrow \mathcal{R}(\mathcal{L}X_{\beta})\) is a weak equivalence for each ordinal \(\beta < \lambda\), and for this we argue by transfinite induction. Fix an ordinal \(\beta\) and assume that \(X_{\alpha} \rightarrow \mathcal{R}(\mathcal{L}X_{\alpha})\) is a weak equivalence for each \(\alpha < \beta\). There are two cases: \(\beta\) is a limit ordinal, or it is not.

In one case, \(\beta\) is not a limit ordinal, so that \(\beta = \alpha + 1\) for some \(\alpha\). By hypothesis, \(X_{\alpha} \rightarrow \mathcal{R}(\mathcal{L}X_{\alpha})\) is a weak equivalence. By examining statement (iv) above, we see that Proposition 6.6 implies that \(X_{\beta} \rightarrow \mathcal{R}(\mathcal{L}X_{\beta})\) is indeed a weak equivalence.

In the last case, \(\beta\) is a limit ordinal, so that \(\text{colim}_{\alpha<\beta} X_{\alpha} \rightarrow X_{\beta}\) is an isomorphism in \(\mathcal{C}\). An argument exactly like that above for map (B) shows that \(\text{hocolim}_{\alpha<\beta} \mathcal{R}(\mathcal{L}X_{\alpha}) \rightarrow \mathcal{R}(\mathcal{L}X_{\beta})\) is an isomorphism in \(\text{ho}(\mathcal{C})\). Moreover, the inductive hypothesis implies that the map \(\text{hocolim}_{\alpha<\beta} X_{\alpha} \rightarrow \text{hocolim}_{\alpha<\beta} \mathcal{R}(\mathcal{L}X_{\alpha})\) is an isomorphism as well. We then have a commutative...
square
\[
\begin{array}{ccc}
\text{hocolim } X_\alpha & \overset{\sim}{\longrightarrow} & \text{hocolim } \mathcal{R}(\mathcal{L}X_\alpha) \\
\downarrow & & \downarrow \\
X_\beta = \text{colim } X_\alpha & \longrightarrow & \mathcal{R}(\mathcal{L}X_\beta)
\end{array}
\]

in \text{ho}(\mathcal{C}) with isomorphisms as indicated. Thus, \(X_\beta \to \mathcal{R}(\mathcal{L}X_\beta)\) is a weak equivalence. Transfinite induction now shows that map (C) is an isomorphism, as desired. \(\square\)

**Theorem 7.2.** Suppose that \((\mathcal{B}, \mathcal{A})\) is a conjugate pair of small categories and that \(\mathcal{C}\) is a semi-stable model category. Then the adjoint pair
\[
\begin{array}{ccc}
[\mathcal{B}^{\text{op}}, \mathcal{C}] & \overset{\mathcal{L}}{\longrightarrow} & [\mathcal{A}^{\text{op}}, \mathcal{C}] \\
\downarrow & & \downarrow \\
\mathcal{R} & \longrightarrow & \mathcal{R} \mathcal{L}
\end{array}
\]

associated to the regular bimodule \(\mathcal{U}_+: \mathcal{A}^{\text{op}} \times \mathcal{B} \to \text{Sets}_*\) is a Quillen equivalence.

**Proof.** By the well-known criteria, it suffices to show that for each cofibrant functor \(F: \mathcal{B}^{\text{op}} \to \mathcal{C}\) and each fibrant functor \(G: \mathcal{A}^{\text{op}} \to \mathcal{C}\), a map \(\tau: \mathcal{L}F \to G\) is a weak equivalence if and only if its adjoint \(\tau^\#: F \to \mathcal{R}G\) is as well. The crucial observation is that the adjoint \(\tau^\#\) is simply the composite
\[
\tau^\#: F \overset{\eta_F}{\longrightarrow} \mathcal{R}(\mathcal{L}F) \overset{\mathcal{R}\tau}{\longrightarrow} \mathcal{R}G.
\]
As \(F\) is cofibrant, \(\eta_F\) is then a weak equivalence. Hence \(\tau^\#\) is a weak equivalence if and only if \(\mathcal{R}\tau\) is, and the result follows immediately from Corollary 6.3. \(\square\)

In the case that \(\mathcal{C}\) is abelian and given the trivial model structure, the theorem reduces to the following.

**Corollary 7.3.** Suppose that \(\mathcal{C}\) is a complete and cocomplete abelian category. Given any conjugate pair \((\mathcal{B}, \mathcal{A})\), the adjoint pair
\[
\begin{array}{ccc}
[\mathcal{B}^{\text{op}}, \mathcal{C}] & \overset{\mathcal{L}}{\longrightarrow} & [\mathcal{A}^{\text{op}}, \mathcal{C}] \\
\downarrow & & \downarrow \\
\mathcal{R} & \longrightarrow & \mathcal{R}
\end{array}
\]

associated to the regular bimodule is an equivalence of categories.

**Example 7.4.** Applying the theorem to Example 4.5 gives the well-known fact that idempotents split in stable homotopy and abelian categories. This is not new of course, but it’s satisfying to see this manifest itself here.

**Example 7.5.** Let \(\mathcal{C}\) be any complete and cocomplete abelian category. Applying the theorem to the conjugate pair \((\Gamma, \mathcal{E})\), we recover Pirashvili’s first main result in [11]. Taking \(\mathcal{C}\) to be a stable model category satisfying our technical assumptions, we confirm the conjecture that started this project: the categories of functors indexed by \(\Gamma^{\text{op}}\) and \(\mathcal{E}^{\text{op}}\) are in fact Quillen equivalent when taking values in a stable model category.
In conclusion, we remark that this is not a complete Morita theory for semi-stable model categories of diagrams. The best possible result would be a complete characterization of the pairs \((B, A)\) yielding a Quillen equivalence. We have given sufficient—but not necessary—conditions for the existence of such pairs of categories. Moreover, classical Morita equivalence is characterized by finitely generated projective generators. In our case, the regular bimodule is more than projective: it is free. If the analogy is to be believed, one would think there must be a suitable projective version of this development.

References

[1] G. Arone, A generalization of Snaith-type filtration, Trans. Amer. Math. Soc. 351 (1999), 1123–1150.
[2] M. Basterra and R. McCarthy, \(\Gamma\)-homology, topological André-Quillen homology and stabilization, Topology and its Applications 121 (2002), 551–566.
[3] R. A. Brualdi, Introductory Combinatorics, Prentice Hall, 1999.
[4] W. Chachólski and J. Scherer, Homotopy Theory of Diagrams, Memoirs of the American Mathematical Society, no. 736, American Mathematical Society, 2002.
[5] S. Eilenberg and S. MacLane, On the groups \(H(\Pi, n)\). II. Methods of computation, Ann. of Math. (2) 60 (1954), 49–139.
[6] P. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs, no. 99, American Mathematical Society, 2003.
[7] M. Hovey, Model Categories, Mathematical Surveys and Monographs, no. 63, American Mathematical Society, 1999.
[8] N. J. Kuhn, The McCord model for the tensor product of a space and a commutative ring spectrum, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhauser, 2004, pp. 213–236.
[9] S. MacLane, Categories for the Working Mathematician, 2 ed., Graduate Texts in Mathematics, no. 5, Springer-Verlag, 1998.
[10] D. C. Newell, Morita theorems for functor categories, Trans. Amer. Math. Soc. 168 (1972), 423–433.
[11] T. Pirashvili, Dold-Kan type theorem for \(\Gamma\)-groups, Math. Ann. 318 (2000), 277–298.
[12] J. Słomińska, Dold-Kan type theorems and Morita equivalences of functor categories, J. Algebra 274 (2004), 118–137.
[13] J. Thévenaz and P. Webb, The structure of Mackey functors, Trans. Amer. Math. Soc. 347 (1995), no. 6, 1865–1961.
[14] T. tom Dieck, Transformation groups, de Gruyter Studies in Mathematics, vol. 8, Walter de Gruyter & Co., Berlin, 1987.

Department of Mathematics, University of Mary Washington, 1301 College Avenue, Fredericksburg, VA 22401

E-mail address: rheinstu@umw.edu