A SIMPLIFICATION OF APÉRY’S PROOF OF THE IRRATIONALITY OF $\zeta(3)$

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Abstract. A simplification of Apéry’s proof of the irrationality of $\zeta(3)$ is presented. The construction of approximations is motivated from the viewpoint of 2-dimensional recurrence relations which simplifies many of the details of the proof. Conclusive evidence is also presented that these constructions arise from a continued fraction due to Ramanujan.

1. Introduction

In 1978, R. Apéry [1] presented his famous proof of the irrationality of $\zeta(3)$. His method involved the explicit construction of two solutions $a_n$ and $b_n$ of the recurrence

\[(n + 1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1},\]

for $n \geq 1$ such that $a_n/b_n \to \zeta(3)$ as $n \to \infty$. These solutions also satisfied the arithmetic properties $b_n, [1, 2, \ldots, n]^3 a_n \in \mathbb{Z}$. Put together, these facts turned out to be sufficient to complete the proof of irrationality of $\zeta(3)$. For an account of the history and the “miraculous” nature of this construction, see van der Poorten [7].

Many proofs of the irrationality of $\zeta(3)$ have followed since then, all of which construct the same sequences, $a_n$ and $b_n$, by vastly different methods (see Fischler [6] for a survey). One of these proofs is by Apéry himself [2] (arguably his only complete proof of this result). This paper deals with a method of interpolation for continued fractions and constructs a series of continued fractions for $\zeta(3)$ from which the sequences $a_n$ and $b_n$ are obtained. We also remark that none of these proofs have generalisations to higher zeta values. For example, it is still unknown whether $\zeta(5)$ is irrational.

In Vol. 2 of Ramanujan’s notebooks, Berndt [5] suggests that a certain continued fraction of Ramanujan is related to the proof in [2]. Recall that theHurwitz zeta function, $\zeta(s, x)$ is defined for $\text{Re} \ s > 1, \text{Re} \ x > 0$ by

\[\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.\]

The continued fraction of Ramanujan in consideration ([5], Entry 32(iii), p. 153) is

\[\zeta(3, x + 1) = \frac{1}{P(0, x) + \frac{-1^6}{P(1, x) + \frac{-2^6}{P(2, x) + \frac{-3^6}{P(3, x) + \cdots}}}}\]

for $\text{Re} \ x > -\frac{1}{2}$ where $P(n, x) = n^3 + (n + 1)^3 + (4n + 2)x(x + 1)$. In the discussion following this entry in [5], it is stated that the specialisation $x = 1$ yields a continued fraction for $\zeta(3)$ which is “of crucial importance” in the work of Apéry [2].

Around the same time, F. Apéry [3] in a biographical note on his father R. Apéry, states that the construction in [1] is motivated from a “number table due to Ramanujan”.

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In this note, we take the view that a more detailed analysis of the method in [2] leads one to the conclusion that the constructions are indeed based on Ramanujan’s continued fraction [2]. We present here such an analysis, which also allows us to give a simplified proof of Apéry’s result, which we state as

**Theorem.** \(\zeta(3)\) is irrational.

This note is organised as follows. We first present the constructions of [2] from the viewpoint of 2-dimensional recurrence relations in Sec. 2. Section 3 is devoted to the proof of the theorem. In Sec. 4, we end with some concluding remarks on the comparison with Apéry’s approach, the relation to the Ramanujan’s continued fraction and generalisations to other constants.

## 2. Construction of the tables

We start by defining the homogenous polynomials

\[
\begin{align*}
  f(i, j) &= i^3 + 2i^2j + 2ij^2 + j^3, \\
  g(i, j) &= i^3 - 2i^2j + 2ij^2 - j^3.
\end{align*}
\]

We identify here the key properties of these polynomials that will be used in the construction. They are

\[
\begin{align*}
  (4) \quad f(i, j)g(i, j) &= f(i, 0)g(i, 0) + f(0, j)g(0, j), \\
  (5) \quad f(i + 1, j) - f(i, j + 1) &= g(i + 1, j + 1) - g(i, j).
\end{align*}
\]

Now we present the 2-dimensional recurrence which will play a central role in our construction.

**Proposition 1.** *The recurrence*

\[
\begin{align*}
  \left( \begin{array}{c}
  f(i, j) \\
  f(0, j)
  \end{array} \right) \left( \begin{array}{c}
  u_{i-1,j} \\
  u_{i-1,j-1}
  \end{array} \right) &= f(i, 0) \left( \begin{array}{c}
  u_{i,j} \\
  u_{i,j-1}
  \end{array} \right),
\end{align*}
\]

*for integers* \(i, j \geq 1\) *has a rational valued solution* \(u_{i,j}\) *for each of the following boundary conditions*

\(\begin{array}{l}
(a) \; \forall i, j \geq 0, \; u_{0,j} = u_{0,i} = 1, \\
(b) \; u_{0,0} = 0 \; \text{and} \; \forall i, j \geq 1,
\end{array}\)

\[
\begin{align*}
  u_{0,j} &= \sum_{n \leq j} \frac{1}{f(0,n)} \prod_{k<n} \frac{-g(0,k)}{f(0,k)} = \sum_{n \leq j} n^{-3}, \\
  u_{i,0} &= \sum_{n \leq i} \frac{1}{f(n,0)} \prod_{k<n} \frac{g(k,0)}{f(k,0)} = \sum_{n \leq i} n^{-3}.
\end{align*}
\]

*Proof.** We first derive additional conditions that any solution of (6) has to satisfy. The top entry on the right in (6) for \(i, j\) is the same as the bottom entry for \(i, j + 1\). Hence the solution has to satisfy the recurrence

\[
\begin{align*}
  (7) \quad f(0, j + 1)u_{i-1,j+1} &= (f(i, j) - g(i, j + 1))u_{i-1,j} + g(0, j)u_{i-1,j-1}.
\end{align*}
\]

This is a condition on the solution for row \(i - 1\) with \(i \geq 1\). Next by inverting (6) and using property (4), we get

\[
\begin{align*}
  (8) \quad \left( \begin{array}{cc}
  f(i, j) & -g(0, j) \\
  -f(0, j) & f(i, j)
  \end{array} \right) \left( \begin{array}{c}
  u_{i,j} \\
  u_{i,j-1}
  \end{array} \right) &= g(i, 0) \left( \begin{array}{c}
  u_{i-1,j} \\
  u_{i-1,j-1}
  \end{array} \right).
\end{align*}
\]

This equation likewise leads to the condition that \(u_{i,j}\) satisfies

\[
\begin{align*}
  (9) \quad f(0, j + 1)u_{i,j+1} &= (f(i, j + 1) - g(i, j))u_{i,j} + g(0, j)u_{i,j-1}.
\end{align*}
\]
This is a condition on the solution for row $i$, which, by property (5), is the same as (7) with $i - 1$ replaced by $i$.

Conversely, the recurrence (6) can be used to construct row $i$ from row $i - 1$ in a well-defined manner, if (7) is satisfied for row $i - 1$. We will use this observation, inductively along the rows, to construct the required solutions.

First it can be easily verified that both of the given boundary conditions (a) and (b) satisfy (7) for the row $i = 0$. This will be the base case.

We now assume that we have constructed the solution up to row $i - 1$ and that the row $i - 1$ satisfies (7). This implies that the recurrence (6) can be used to construct row $i$ from row $i - 1$ in a well-defined manner. Hence (8) holds and by the discussion above, we can then conclude that row $i$ also satisfies (7) and the induction step is complete.

Hence there exist solutions to (6) satisfying the given boundary conditions (a) and (b) along the row $i = 0$. The only step remaining is to verify that these solutions satisfy the respective boundary conditions along the column $j = 0$. This can be verified by using (6) and (8) to get

$$
\begin{pmatrix}
    f(i, j) & -g(i, 0) \\
    f(i, 0) & -g(i, j)
\end{pmatrix}
\begin{pmatrix}
    u_{i,j-1} \\
    u_{i-1,j-1}
\end{pmatrix}
= f(0, j)
\begin{pmatrix}
    u_{i,j} \\
    u_{i-1,j}
\end{pmatrix}.
$$

This gives us the condition

$$f(i + 1, 0)u_{i+1,j-1} = (f(i, j) + g(i + 1, j))u_{i,j-1} - g(i, 0)u_{i-1,j-1}.$$  

Now it can be easily verified that the given boundary conditions are solutions of this for $j = 1$ and the initial values $u_{1,0}$ and $u_{0,0}$ agree with our previous construction.  

Call the solutions to (6) corresponding to the initial conditions (a) and (b) of Prop. 1 as $q_{i,j}$ and $p_{i,j}$ respectively. Now we explore the arithmetic properties of these tables of rational numbers. Here we use an additional property of $f(i, j)$, namely

$$f(0, x), f(x, 0) \in \{x^3, -x^3\}.$$  

We shall use the notation $d_n = [1, 2, \ldots n]$ for the rest of this note.

**Proposition 2.** Any solution $u$ of (6) has the property that $u_{i,j}$ is a $\mathbb{Z}$-linear combination of $u_{i-1,j}$, $u_{i,j-1}$ and $u_{i-1,j-1}$ for $i, j \geq 1$. Hence $q_{i,j}$ and $d_{\max(i,j)} p_{i,j}$ are integers.

**Proof.** For the proof of the first statement of the proposition, we start with the assumption that the gcd $(i, j) = 1$. By (10), this means that the gcd $(f(0, j), f(i, 0)) = 1$. Hence, there exists an integer $x$ such that

$$f(i, j) \equiv xf(0, j) \mod f(i, 0).$$

Multiplying this by $g(i, j)$, we get

$$f(i, j)g(i, j) \equiv xf(0, j)g(i, j) \mod f(i, 0).$$

Using (11) makes the left side $\equiv f(0, j)g(0, j)$ and cancelling $f(0, j)$ from both sides gives

$$g(0, j) \equiv xg(i, j) \mod f(i, 0).$$

Hence, by subtracting $x$ times the second row from the first row in (6), we get coefficients which are divisible by $f(i, 0)$. Thus we conclude that $u_{i,j} - xu_{i,j-1}$ is a $\mathbb{Z}$-linear combination of $u_{i-1,j}$ and $u_{i-1,j-1}$. Since $x$ is an integer, we get the first part of the proposition, for the special case $(i, j) = 1$.  

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The general case \((i, j) = d\) is handled by calling \(i' = i/d, j' = j/d\), dividing \((6)\) by \(d^3\) and using the homogenity of \(f(i, j)\) and \(g(i, j)\) to get
\[
\begin{pmatrix}
  f(i', j') \\
  f(0, j')
\end{pmatrix}
\begin{pmatrix}
  g(0, j') \\
  g(i', j')
\end{pmatrix}
= u_{i-1,j}
\begin{pmatrix}
  u_{i-1,j-1} \\
  u_{i,j-1}
\end{pmatrix}
\]
This reduces to the previous case as \((i', j') = 1\) and we proceed as before and complete the proof of the first statement of the proposition.

For the second statement of the proposition, we use the first statement recursively to obtain that \(u_{i,j}\) is a \(\mathbb{Z}\)-linear combination of \(u_{0,0}, u_{0,1}, \ldots u_{0,j}, u_{1,0}, u_{2,0}, \ldots u_{i,0}\). Thus the second statement on \(q_{i,j}\) and \(p_{i,j}\) follows from the arithmetic properties of the boundary values (a) and (b) in Prop. 1.

Now, we will prove that \(p_{i,j}/q_{i,j}\) converge to \(\zeta(3)\) uniformly in \(i\) and \(j\). For simplicity of presentation we shall use one more property of \(f, g\) namely
\[
f(i, j) - f(0, j) > g(i, j) - g(0, j), \quad i, j \geq 1.\tag{11}
\]
We define the table \(\epsilon_{i,j}\) as
\[
\epsilon_{i,j} = q_{i,j}\zeta(3) - p_{i,j}.\tag{12}
\]
and prove

**Proposition 3.** The table \(\epsilon_{i,j} \to 0\) uniformly as \(i, j \to \infty\).

**Proof.** First we note that condition \((11)\) implies that for \(x \geq 1\),
\[
f(i, j)x + g(0, j) > f(0, j)x + g(i, j).
\]
Using this in \((3)\) with \(x = q_{i-1,j}/q_{i-1,j-1}\), gives the implication \(q_{i-1,j} \geq q_{i-1,j-1} \Rightarrow q_{i,j} > q_{i,j-1}\). Hence, we conclude that \(q_{i,j}\) is monotonically increasing along rows \(i \geq 1\).

Define the following quantities:
\[
\delta_{\text{row}}^{i,j} = p_{i-1,j}q_{i-1,j-1} - p_{i-1,j-1}q_{i-1,j}, \quad \delta_{\text{col}}^{i,j} = p_{i,j-1}q_{i-1,j-1} - p_{i-1,j-1}q_{i,j-1}.\tag{13}
\]
Now, we take the second row of \((3)\) for both \(p_{i,j}\) and \(q_{i,j}\), multiply by \(q_{i-1,j-1}\) and \(p_{i-1,j-1}\) resp. and subtract to get
\[
-j^3\delta_{\text{row}}^{i,j} = i^3\delta_{\text{col}}^{i,j},\tag{14}
\]
Similarly using the second rows of \((3)\) gives
\[
-j^3\delta_{\text{row}}^{i+1,j} = -i^3\delta_{\text{col}}^{i,j}.
\]
Using \((13)\) and \((14)\) recursively with the initial values \(\delta_{\text{row}}^{i,j} = j^{-3}\) (verified directly) gives us \(\delta_{\text{row}}^{i,j} = j^{-3}\) and \(\delta_{\text{col}}^{i,j} = i^{-3}\) for \(i, j \geq 1\).

Now, we deduce the difference in the ratios \(r_{i,j} = p_{i,j}/q_{i,j}\) along the columns,
\[
r_{i,j} - r_{i-1,j} = \frac{\delta_{\text{col}}^{i,j+1}}{q_{i,j}q_{i-1,j}} = \frac{1}{j^3q_{i,j}q_{i-1,j}},
\]
tends to 0 for \(i\) fixed and \(j \to \infty\). This, coupled with the fact that \(r_{0,j} \to \zeta(3)\) (from the boundary conditions in Prop. 1), implies that each row in \(r_{i,j}\) has \(\zeta(3)\) as limit.

As for the difference in \(r_{i,j}\) along the rows,
\[
r_{i,j} - r_{i,j-1} = \frac{\delta_{\text{row}}^{i,j+1}}{q_{i,j}q_{i,j-1}} = \frac{1}{j^3q_{i,j}q_{i,j-1}}.
\]
Hence for any \(i, j\) we have
\[
|\zeta(3) - r_{i,j}| \leq \frac{1}{q_{i,j}} \sum_{k \geq j} \frac{1}{k^2}.
\]

Multiplying by \(q_{i,j}\) on both sides, we get that \(|\epsilon_{i,j}| \leq \zeta(3)/q_{i,j}\) which tends to zero uniformly in \(i, j\) by monotonically increasing integers \(q_{i,j}\) (see Prop. 2). \(\Box\)

3. PROOF OF THE THEOREM

For the proof of the theorem we shall use the criterion that a number \(\alpha\) is irrational if there exists a sequence of integers \(a_n\) and \(b_n\) such that
\[
0 \neq |a_n - b_n\alpha| \to 0 \text{ as } n \to \infty \tag{15}
\]
For, if not, let \(\alpha = r/s\) with coprime integers \(r, s\). Then the modulus of the \(\mathbb{Z}\)-linear form in 1 and \(\alpha\) in (15) is either 0 or \(\geq 1/s\) for any choice of integers \(a_n\) and \(b_n\). This contradicts (15) and hence \(\alpha\) is irrational.

The linear forms in 1 and \(\zeta(3)\) needed to use the above criterion will come from the diagonal \(\epsilon_{n,n}\). For estimating the decay of these forms, we shall need Poincaré’s theorem in vector form. For a discussion of the history of this theorem and references, see Aptekarev et. al. [4], Ch.3, Sec.1. Poincaré’s theorem is usually used in the simpler setting of 1-dimensional recurrence relations and the following ([4], pp.1104) is a generalisation.

**Proposition 4.** (Poincaré-Perron, in vector form) Let \(x^n = (x^n_1, x^n_2, \ldots x^n_k)\) be a sequence of \(k\)-dimensional vectors which is a solution of
\[
x^n = A_n x^{n-1}, \tag{16}
\]
where the vectors are taken to be column vectors and \(A_n\) is a \(k \times k\) matrix. Let \(A_n \to A\) as \(n \to \infty\), where \(A\) is a diagonalizable matrix with eigenvalues of distinct magnitude.

Then, either \(x^n = 0\) eventually or there exists a component \(j\) of \(x^n\) such that
\[
\lim_{n \to \infty} x^n_j x^{n-1}_j = \lambda, \quad \text{and} \quad \lim_{n \to \infty} x^n_j x^{n-1}_j = e, \tag{17}
\]
where \((\lambda, e)\) is an eigenpair of \(A\).

If the system (16) is nondegenerate \((A_n\) is nonsingular for all \(n))\), then for any eigenpair \((\lambda, e)\) of \(A\), there exists a solution \(x^n\) of (16) and component \(j\) such that (17) holds.

We use Prop. 4 by defining \(\mathbf{x}^n = (\epsilon_{n,n+1}, \epsilon_{n,n})\) and
\[
A_n = \begin{pmatrix} \frac{6n^2+9n^2+5n+1}{(n+1)^2} & -\frac{n^2}{(n+1)^2} \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \tag{18}
\]

We note that the difference equation (16) is satisfied. This is because, by the definition (12) of \(\epsilon_{i,j}\) it is also a solution of the recurrence (6) of Prop. 1. The second matrix on the right in (18) is from (9) with \(i = n, j = n\) which transforms \((\epsilon_{n-1,n}, \epsilon_{n-1,n-1})\) to \((\epsilon_{n,n}, \epsilon_{n,n-1})\). The first matrix is from (7) with \(i = n+1, j = n\) which transforms \((\epsilon_{n,n}, \epsilon_{n,n-1})\) to \((\epsilon_{n,n+1}, \epsilon_{n,n})\) as required.

We have \(A_n \to A\) where
\[
A = \begin{pmatrix} 35 & -6 \\ 6 & -1 \end{pmatrix},
\]
with the eigenvalues of \(A\) being \(17 \pm 12\sqrt{2}\) and the corresponding eigenvectors being \((18 \pm 12\sqrt{2}, 6)\).
Now, we apply Prop. 4 and note that the system is nondegenerate for \( n \geq 1 \). Hence, the eventually zero case cannot occur because the vector \((\epsilon_1, \epsilon_1, 1)\) is not the zero vector. Thus, there exists \( j \) and an eigenpair \((\lambda, e)\) such that (17) holds.

Note that in (17), we can always choose \( j \) to be any of the choices where \( e_j \neq 0 \). Since neither of the eigenvectors of \( A \) has any zero entries, we can conclude that (17) holds for both values \( j = 1, 2 \). We choose \( j = 2 \) and get

\[
|\epsilon_{n,n}| = e^{n(\log \lambda + o(1))},
\]

where \( \lambda = 17 \pm 12 \sqrt{2} \). However, \( \lambda = 17 + 12 \sqrt{2} \) is not possible because that will contradict Prop. 3. Hence, \( \lambda = 17 - 12 \sqrt{2} \) in the decay estimate (19).

Recall that the prime number theorem implies that \( d_n = e^{n(1+o(1))} \) ([7], p.198). From Prop. 2, we obtain that \( d_n^3 p_{n,n} \) and \( q_{n,n} \) are integers. Hence, the decay estimate for \( d_n^3 \epsilon_{n,n} \) is

\[
|d_n^3 q_{n,n} \zeta(3) - d_n^3 p_{n,n}| = e^{n(3+\log \lambda + o(1))}.
\]

Since \( \log(17 - 12 \sqrt{2}) < -3 \), we conclude that the \( \mathbb{Z} \)-linear form in 1 and \( \zeta(3) \) in (20) is nonzero and tends to 0 as \( n \to \infty \). By the criterion (15), we conclude that \( \zeta(3) \) is irrational.

4. Concluding remarks

Comparison with Apéry’s approach:

At this point, we would like to stress that the construction of the tables \( p_{i,j} \) and \( q_{i,j} \) is entirely from [2] and it is Apéry’s method that we have adapted for the proof of Prop. 1. The only novelty in our approach is in viewing these tables as the solutions of a 2-dimensional recurrence, whereas in [2] they appear as a sequence of continued fractions (indexed by row).

The advantage for us is that the proof of the arithmetic properties in Prop. 2 is natural and elementary, whereas in [2], a proof for \( q_{i,j} \in \mathbb{Z} \) is given using differential equations and there is no indication of the proof for the properties of \( p_{i,j} \). This is a reasonable achievement for our approach, given that these properties are considered deep (“a fundamental miracle” [7], p. 202). Proposition 3 is also an easy consequence of the recurrence (6), while there is not even a mention of this fact in [2].

The last step in [2] is the claim that the “diagonal” sequences \( p_{i,i} \) and \( q_{i,i} \) satisfy the recurrence (1) and hence the (usual) Poincaré’s theorem gives the irrationality of \( \zeta(3) \). We remark that his claim can easily be verified from our recurrence relations. We have preferred to follow the vector based approach because it is more natural and the vector form of the Poincaré’s theorem is not more difficult than the 1-dimensional version (see [4]).

Relation to Ramanujan’s continued fraction:

Now we sketch a proof that Apéry’s approach of viewing each row as a continued fraction recovers the intimate connection with Ramanujan’s continued fraction [2]. First we note that along each row where \( i \) is fixed, \( p_{i,j} \) and \( q_{i,j} \) satisfy (7) (with \( i - 1 \) replaced by \( i \)). Hence the recurrence satisfied by \( j!^3 p_{i,j} \) and \( j!^3 q_{i,j} \) is

\[
u_{i,j+1} = P(j, i) u_{i,j} - j^6 u_{i,j-1},
\]

for \( j \geq 1 \). Here we used the fact that \( f(i+1, j) - g(i+1, j+1) = P(j, i) \) where \( P(j, i) = j^3 + (j+1)^3 + (4j+2)i(i+1) \) is the same polynomial appearing in [2].
Hence if we define the continued fraction $\omega(i)$ by
\[
\omega(i) = \frac{-1^6}{P(1, i)} + \frac{-2^6}{P(2, i)} + \frac{-3^6}{P(3, i)} + \ldots
\]
then from the theory of continued fractions, we can obtain
\[
\zeta(3) = \frac{\omega(i)p_{i,0} + p_{i,1}}{\omega(i)q_{i,0} + q_{i,1}}
\]
We recall that $q_{i,0} = 1$ and $p_{i,0} = \sum_{n \leq i} n^{-3}$. It is also easy to obtain, using the methods of Sec. 2, that $q_{i,1} = P(0, i)$ and $p_{i,1} = P(0, i)p_{i,0} + 1$. Putting all these values in (22), we get exactly the identity of Ramanujan (2) specialized at $x = i$.

Now, in principle, Carlson’s theorem can be used to derive (2) for the entire halfplane $\text{Re } x > -1/2$ from the equality at all positive integers, if we show that the growth of the continued fraction is sufficiently slow (the Hurwitz zeta function is bounded in this range).

Thus, we see that the point of view in [2] gives a clear link with (2) and bolsters the claim that Apéry’s constructions were indeed motivated from (2). We must mention at this point that the idea of using (2) to produce such a startling diophantine application is a testimony of the genius of Apéry.

**Generalisations:**

As a final remark, these methods can be used as in [2] to show that $\log 2$ and $\zeta(2)$ are irrational by suitable choices for $f(i, j)$ and $g(i, j)$. For $\log 2$, the choice $f(i, j) = i + j$, $g(i, j) = i - j$ satisfies all the conditions specified in Sec. 2 and yields a proof of irrationality as in Sec. 3. This is indeed related to continued fraction Entry 29 (Cor.) of [5]. Similarly for $\zeta(2)$, the choice $f(i, j) = i^2 + ij + \frac{1}{2}j^2$, $g(i, j) = -i^2 + ij - \frac{1}{2}j^2$ yields a proof of irrationality and is related to Entry 30 (Cor.) of [5]. Note that (10) is not satisfied in this case, and more work is needed to establish the analogue of Prop. 2.

However, there exist no nontrivial homogeneous polynomials $f(i, j)$, $g(i, j)$ of degree $> 3$ which satisfy the conditions (11) и (15). Since these conditions are crucial in the construction, we conclude that our approach, in its current form, fails for higher zeta values. One may need to look beyond $2 \times 2$ matrix recurrence relations to attack these constants!

**Acknowledgements:** I wish to thank Professors R. Balsubramanian, M. R. Murty, Y. Nesterenko, T. Rivoal, K. Srinivas and M. Waldschmidt for their suggestions and encouragement during various stages of this project.

**References**

[1] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Astérisque* 61 1979, 11–13.

[2] R. Apéry, Interpolation de fractions continues et irrationalité de certaines constantes. *C.T.H.S., Bull. de la Sect. des Sc. (Math.)* III Bibl. Nationale, Paris, 1981, 37–53.

[3] F. Apéry, Roger Apéry, 1916–1994: a radical mathematician. *Math. Intelligencer* 18 (2) 1996, 54–61.

[4] A. I. Aptekarev, V. I. Buslaev, A. Martínez-Finkelshtein and S. P. Suetin, Padé approximants, continued fractions, and orthogonal polynomials *Russ. Math. Surv.* 66 (6) 2011, 1049–1131.

[5] B. C. Berndt, Ramanujan’s Notebooks, Part II. *Springer-Verlag* New York, 1989.

[6] S. Fischler, Irrationalité de valeurs de zêta [d’après Apéry, Rivoal, ...]. *Séminaire Bourbaki* exposé numéro 910, 2002–2003.

[7] A. van der Poorten, A proof that Euler missed... Apéry’s proof of the irrationality of $\zeta(3)$ (An informal report). *Math. Intelligencer* 1 (4) 1978/79, 195–203.