The parameterfree Comprehension does not imply the full Comprehension in the 2nd order Peano arithmetic

Vladimir Kanovei† Vassily Lyubetsky‡

September 19, 2022

Abstract

The parameter-free part \( \text{PA}_2^* \) of \( \text{PA}_2 \), the 2nd order Peano arithmetic, is considered. We make use of a product/iterated Sacks forcing to define an \( \omega \)-model of \( \text{PA}_2^* + \text{CA}(\Sigma^1_2) \), in which an example of the full Comprehension schema \( \text{CA} \) fails. Using Cohen’s forcing, we also define an \( \omega \)-model of \( \text{PA}_2^* \), in which not every set has its complement, and hence the full \( \text{CA} \) fails in a rather elementary way.

Contents

1 Introduction 2
2 Preliminaries 3
3 Extension by Cohen reals 4
4 Generalized Sacks iterations 5
5 Iterated perfect sets 6
6 The forcing and the basic extension 8
7 The subextension 10
8 Remarks and questions 12
References 12

---

*This paper was written under the support of RFBR (Grant no 20-01-00670).
†Institute for Information Transmission Problems (Kharkevich Institute) of Russian Academy of Sciences (IITP), Moscow, Russia, kanovei@iitp.ru
‡Institute for Information Transmission Problems (Kharkevich Institute) of Russian Academy of Sciences (IITP), Moscow, Russia, lyubetsk@iitp.ru
1 Introduction

Discussing the structure and deductive properties of the second order Peano arithmetic $\text{PA}_2$, Kreisel [9, §III, page 366] wrote that the selection of subsystems “is a central problem”. In particular, Kreisel notes, that

[...] if one is convinced of the significance of something like a given axiom schema, it is natural to study details, such as the effect of parameters.

Recall that parameters in this context are free variables in various axiom schemata in $\text{PA}$, $\text{ZFC}$, and other similar theories. Thus the most obvious way to study “the effect of parameters” is to compare the strength of a given axiom schema $S$ with its parameter-free subschema $S^\ast$. (The asterisk will mean the parameter-free subschema in this paper.)

Some work in this direction was done in the early years of modern set theory. In particular Guzicki [6] proved that the Levy-style generic collapse (see, e.g., Levy [11] and Solovay [18]) of all cardinals $\omega^L_{\alpha}$, $\alpha < \omega_1^L$, results in a generic extension of $L$ in which the (countable) choice schema $\text{AC}$, in the language of $\text{PA}_2$, fails but its parameter-free subschema $\text{AC}^\ast$ holds, so that $\text{AC}^\ast$ is strictly weaker than $\text{AC}$. This can be compared with an opposite result for the dependent choice schema $\text{DC}$, in the language of $\text{PA}_2$, which is equivalent to its parameter-free subschema $\text{DC}^\ast$ by a simple argument given in [6].

Some results related to parameter-free versions of the Separation and Replacement axiom schemata in $\text{ZFC}$ also are known from [3, 12, 14].

This paper is devoted to the role of parameters in the comprehension schema $\text{CA}$ of $\text{PA}_2$. Let $\text{PA}_2^\ast$ be the subtheory of $\text{PA}_2$ in which the full schema $\text{CA}$ is replaced by its parameter-free version $\text{CA}^\ast$, and the Induction principle is formulated as a schema rather than one sentence. The following Theorems 1.1 and 1.2 are our main results.

**Theorem 1.1.** Let Cohen be the Cohen forcing for adding a generic subset of $\omega$. Let $\text{Cohen}^\omega$ be the finite-support product. Suppose that $\langle x_i \rangle_{i<\omega_1}$ is a sequence Cohen$^\omega$-generic over $L$, the constructible universe.

Let $X = (\mathcal{P}(\omega) \cap L) \cup \{ x_i : i < \omega \}$. Then $\langle \omega ; X \rangle$ is a model of $\text{PA}_2^\ast$, but not a model of $\text{CA}$ as $X$ does not contain the complements $\omega \setminus x_i$.

Thus $\text{CA}$, even in the particular form claiming that every set has its complement, is not provable in $\text{PA}_2^\ast$.

It is quite obvious that a subtheory like $\text{PA}_2^\ast$, that does not allow such a fundamental thing as the complement formation, is unacceptable. This is why we adjoin $\text{CA}(\Sigma^1_2)$, i.e., the full $\text{CA}$ (with parameters) restricted to $\Sigma^1_2$ formulas, in the next theorem, to obtain a more plausible subsystem.
**Theorem 1.2.** There is a generic extension $L[G]$ of $L$ and a set $M \in L[G]$, such that $\mathcal{P}(\omega) \cap L \subseteq M \subseteq \mathcal{P}(\omega)$ and $(\omega; M)$ is a model of $\text{PA}_2^* + \text{CA}(\Sigma_2^1)$ but not a model of $\text{PA}_2$.

Therefore $\text{CA}$ is not provable even in $\text{PA}_2^* + \text{CA}(\Sigma_1^2)$.

Theorem 1.2 will be established by means of a complex product/iteration of the Sacks forcing and the associated coding by degrees of constructibility, approximately as discussed in [13, page 143], around Theorem T3106.

Identifying the theories with their deductive closures, we may present the concluding statements of Theorems 1.1 and 1.2 as resp.

$$\text{PA}_2^* \nsubseteq \text{PA}_2 \text{ and } (\text{PA}_2^* + \text{CA}(\Sigma_2^1)) \nsubseteq \text{PA}_2.$$ 

Studies on subsystems of $\text{PA}_2$ have discovered many cases in which $S \nsubseteq S'$ holds for a given pair of subsystems $S, S'$, see e.g. [17]. And it is a rather typical case that such a strict extension is established by demonstrating that $S'$ proves the consistency of $S$. One may ask whether this is the case for the results in the displayed line above. The answer is in the negative: namely the theories $\text{PA}_2^*$, $\text{PA}_2^* + \text{CA}(\Sigma_2^1)$, and the full $\text{PA}_2$ happen to be equiconsistent by a result in [4], also mentioned in [15]. This equiconsistency result also follows from a somewhat sharper theorem in [16, 1.5].

2 Preliminaries

Following [1, 9, 17] we define the second order Peano arithmetic $\text{PA}_2$ as a theory in the language $\mathcal{L}(\text{PA}_2)$ with two sorts of variables – for natural numbers and for sets of them. We use $j, k, m, n$ for variables over $\omega$ and $x, y, z$ for variables over $\mathcal{P}(\omega)$, reserving capital letters for subsets of $\mathcal{P}(\omega)$ and other sets. The axioms are as follows:

1. Peano’s axioms for numbers.

2. The Induction schema $\Phi(0) \land \forall k (\Phi(k) \implies \Phi(k + 1)) \implies \forall k \Phi(k)$, for every formula $\Phi(k)$ in $\mathcal{L}(\text{PA}_2)$, and in $\Phi(k)$ we allow parameters, i.e., free variables other than $k$.

3. Extensionality for sets.

4. The Comprehension schema $\text{CA}$: $\exists x \forall k (k \in x \iff \Phi(k))$, for every formula $\Phi$ in which the variable $x$ does not occur, and in $\Phi$ we allow parameters.

---

1 The authors are thankful to Ali Enayat for the references to [4, 15, 16] in matters of this equiconsistency result.

2 We cannot use Induction as one sentence because the Comprehension schema $\text{CA}$ is not assumed in full generality in the context of Theorem 1.1.
We let $\text{CA}(\Sigma^1_3)$ be the full $\text{CA}$ restricted to $\Sigma^1_2$ formulas $\Phi$.\(^3\)

We let $\text{CA}^*$ be the parameter-free sub-schema of $\text{PA}$ (that is, $\Phi(k)$ contains no free variables other than $k$).

We let $\text{PA}_2^*$ be the subsistem of $\text{PA}_2$ with $\text{CA}$ replaced by $\text{CA}^*$.

Remark 2.1. In spite of Theorem 1.1, $\text{PA}_2^*$ proves $\text{CA}$ with parameters over $\omega$ (but not over $\mathcal{P}(\omega)$) allowed. Indeed suppose that $\Phi$ is $\Phi(k,m)$ in (4) and $\Phi$ has no other free variables. Arguing in $\text{PA}_2$, assume towards the contrary that the formula $\psi(m) := \exists x \forall k (k \in x \iff \Phi(k,m))$ holds not for all $m$. By Induction, take the least $m$ for which $\psi(m)$ fails. This $m$ is definable, and therefore it can be eliminated, and hence we have $\psi(m)$ for this $m$ by $\text{CA}^*$. This is a contradiction. \hfill $\square$

3 Extension by Cohen reals

Here we prove Theorem 1.1. We assume some knowledge of forcing and generic models, as e.g. in Kunen [10], especially Section IV.6 there on the “forcing over the universe” approach.

Recall that the Cohen forcing notion $\text{Cohen} = 2^{<\omega}$ consists of all finite dyadic tuples including the empty tuple $\Lambda$. If $u, v \in 2^{<\omega}$ then $u \subseteq v$ means that $v$ is a proper extension of $u$, whereas $u \subseteq v$ means $u \subseteq v \lor u = v$. The finite-support product $\mathcal{P} = (2^{<\omega})^\omega$ consists of all maps $p : \omega \to 2^{<\omega}$ such that $p(i) = \Lambda$ (the empty tuple) for all but finite $i < \omega$. The set $\mathcal{P}$ is ordered opposite to the componentwise extension, so that $p \leq q$ ($p$ is stronger as a forcing condition) iff $q(i) \subseteq p(i)$ for all $i < \omega$. The condition $\Lambda^\omega$ defined by $\Lambda^\omega(i) = \Lambda, \forall i$, is the $\leq$-largest (the weakest) element of $\mathcal{P}$.

We consider the set $\text{Perm}$ of all idempotent permutations of $\omega$, that is, all bijections $\pi : \omega \xrightarrow{\text{onto}} \omega$ such that $\pi = \pi^{-1}$ and the domain of nontriviality $|\pi| = \{i : \pi(i) \neq i\}$ is finite. If $\pi \in \text{Perm}$ and $p$ is a function with $\text{dom} \pi = \omega$, then $\pi p$ is defined by $\text{dom}(\pi p) = \omega$ and $(\pi p)(\pi(i)) = p(i)$ for all $i < \omega$, so formally $\pi p = p \circ \pi^{-1} = p \circ \pi$ (the superposition). In particular if $p \in \mathcal{P}$ then $\pi p \in \mathcal{P}$ and $|\pi p| = \pi^{-1}|p| = \{\pi(i) : i \in |p|\}$.

Proof (Theorem 1.1). We make use of Gödel’s constructible universe $L$ as the ground model for our forcing constructions. Suppose that $G \subseteq \mathcal{P}$ is a set $\mathcal{P}$-generic over $L$. If $i < \omega$ then

- $G_i = \{p(i) : p \in G\} \subseteq 2^{<\omega}$ is a set $2^{<\omega}$-generic (Cohen generic) over $L$,

- $a_i[G] = \bigcup G_i \subseteq 2^\omega$ is a real Cohen generic over $L$, and

- $x_i[G] = \{n : a_i(n) = 1\} \subseteq \omega$ is a subset of $\omega$ Cohen generic over $L$.

\(^3\) A $\Sigma^1_2$ formula is any $L(\text{PA}_2)$ formula of the form $\forall x \exists y \Psi$, where $\Psi$ does not contain quantified variables over $\mathcal{P}(\omega)$. 

4
Thus $X[G] \in L[G]$ and $X[G]$ consists of all subsets of $\omega$ already in $L$ and all Cohen-generic sets $x_i[G], i < \omega$.

We assert that the model $\langle \omega; X[G] \rangle$ proves Theorem 1.1.

The only thing to check is that $\langle \omega; X[G] \rangle$ satisfies $\text{CA}'$. For that purpose, assume that $\Phi(k)$ is a parameter-free $L(\text{PA}_2)$ formula with $k$ the only free variable. Consider the set $y = \{ k < \omega : \langle \omega; X[G] \rangle \models \Phi(k) \}$; then $y \subseteq L[G], y \subseteq \omega$. We claim that in fact $y$ belongs to $L$, and hence to $X[G]$.

Let $\forces$ be the forcing relation associated with $P$. In particular, if $p \in P$ and $\psi$ is a parameter-free formula then $p \forces \psi$ iff $\psi$ holds in any $P$-generic extension $L[H]$ of $L$ such that $p \in H$.

Let $G$ be a canonical $P$-name for $G$. We assert that

$$y = \{ k < \omega : \Lambda^\omega \forces \langle \omega; X[G] \rangle \models \Phi(k) \}.$$  

Indeed assume that the condition $\Lambda^\omega$ $P$-forces $\langle \omega; X[G] \rangle \models \Phi(k)$. But $\Lambda^\omega \in G$ since $\Lambda^\omega$ is the weakest condition in $P$. Therefore $\langle \omega; X[G] \rangle \models \Phi(k)$ by the forcing theorem, thus $k \in y$, as required.

To prove the converse, assume that $k \in y$. Then by the forcing theorem there is a condition $p \in G$ forcing $\langle \omega; X[G] \rangle \models \Phi(k)$. We claim that then $\Lambda^\omega$ forces the same as well.

Indeed otherwise there is a condition $q \in P$ which forces $\langle \omega; X[G] \rangle \models \neg \Phi(k)$. There is a permutation $\pi \in \text{Perm}$ satisfying $|r| \cap |p| = \emptyset$, where $r = \pi q \in P$. We claim that $r$ forces $\langle \omega; X[G] \rangle \models \neg \Phi(k)$. Indeed assume that $H \subseteq P$ is a set $P$-generic over $L$, and $r \in H$. We have to prove that $\langle \omega; X[H] \rangle \models \neg \Phi(k)$. The set $K = \{ \pi r' : r' \in H \}$ is $P$-generic over $L$ along with $H$ since $\pi \in L$. Moreover $K$ contains $q$. It follows that $\langle \omega; X[K] \rangle \models \neg \Phi(k)$ by the forcing theorem and the choice of $q$. However the sequence $\langle x_i[K] \rangle_{i < \omega}$ is equal to the permutation of the sequence $\langle x_i[H] \rangle_{i < \omega}$ by $\pi$. It follows that $X[H] = X[K]$, and hence $\langle \omega; X[H] \rangle \models \neg \Phi(k)$, as required. Thus indeed $r$ forces $\langle \omega; X[G] \rangle \models \neg \Phi(k)$.

However $p$ forces $\langle \omega; X[G] \rangle \models \Phi(k)$, and $p, r$ are compatible in $P$ because $|r| \cap |p| = \emptyset$. This is a contradiction.

We conclude that $\Lambda^\omega$ forces $\langle \omega; X[G] \rangle \models \Phi(k)$, and this completes the proof of (1).

But it is known that the forcing relation $\forces$ is expressible in $L$, the ground model. Therefore it follows from (1) that $y \in L$, hence $y \in X[G], as required.

4 Generalized Sacks iterations

Here we begin the proof of Theorem 1.2. The proof involves the engine of generalized product/iterated Sacks forcing developed in [7, 8] on the base of earlier
papers [2, 5] and others. We still consider the constructible universe $\mathbf{L}$ as the ground model for the extension, and define, in $\mathbf{L}$, the set

$$I = (\omega_1 \times 2^{<\omega}) \cup \omega_1; \quad I \in \mathbf{L}, \quad (2)$$

partially ordered so that $\langle \gamma, s \rangle \preceq \langle \beta, t \rangle$ iff $\gamma = \beta$ and $s \subseteq t$ in $2^{<\omega}$, while the ordinals in $\omega_1$ (the second part of $I$) remain $\preceq$-incomparable.

Our plan is to define a product/iterated generic Sacks extension $\mathbf{L}[\vec{a}]$ of $\mathbf{L}$ by an array $\vec{a} = \langle a_i \rangle_{i \in I}$ of reals $a_i \in 2^\omega$, in which the structure of “sacksness” is determined by this set $I$, so that in particular each $a_i$ is Sacks-generic over the submodel $\mathbf{L}[\langle a_j \rangle_{j < i}]$.

Then we define the set $J \subseteq \mathbf{L}[\vec{a}]$ of all elements $i \in I$ such that:

— either $i = \langle \gamma, 0^m \rangle$, where $\gamma < \omega_1$ and $m < \omega$,
— or $i = \langle \gamma, 0^m \setminus 1 \rangle$, where $\gamma < \omega_1$ and $m < \omega$, $a_\gamma(m) = 1$.

This any $i = \langle \gamma, 0^m \rangle \in J$ is a splitting node in $J$ if $a_\gamma(m) = 1$, or in other words

$$a_\gamma(m) = 1 \iff \langle \gamma, 0^m \rangle \text{ is a splitting node in } J, \quad (3)$$

We’ll finally prove that the according set

$$M = \mathcal{P}(\omega) \cap \bigcup_{i_1, \ldots, i_n \in J} \mathbf{L}[a_{i_1}, \ldots, a_{i_n}] \quad (4)$$

leads to the model $\langle \omega; M \rangle$ for Theorem 1.2. The reals $a_\gamma$ will not belong to $M$ by the choice of $J$, but will be definable in $\langle \omega; M \rangle$ (with $a_{\langle \gamma, \Lambda \rangle} \subseteq \omega$ as a parameter) via the characterization of the splitting nodes in $J$ by (3).

## 5 Iterated perfect sets

**Arguing in $\mathbf{L}$ in this section.** we define $I = \langle I; \preceq \rangle$ as above.

Let $\Xi$ be the set of all countable (including finite) sets $\zeta \subseteq I$.

If $\zeta \subseteq \Xi$ then IS$_\zeta$ is the set of all initial segments of $\zeta$.

Greek letters $i, j, \xi, \eta, \zeta, \psi$ will denote sets in $\Xi$.

Characters $i, j$ are used to denote elements of $I$.

For any $i \in \zeta \subseteq \Xi$, we consider initial segments $\zeta[<i] = \{ j \in \zeta : j < i \}$ and $\zeta[\neq i] = \{ j \in \zeta : j \neq i \}$, and $\zeta[= i], \zeta[\neq i]$ defined analogously.

Further, $\omega^\omega$ is the Baire space. Points of $\omega^\omega$ will be called reals.

Let $\mathcal{D} = 2^\omega \subseteq \omega^\omega$ be the Cantor space. For any countable set $\xi$, $\mathcal{D}^\xi$ is the product of $\xi$-many copies of $\mathcal{D}$ with the product topology. Then every $\mathcal{D}^\xi$ is a compact space, homeomorphic to $\mathcal{D}$ itself unless $\xi = 0$.

Assume that $\eta \subseteq \xi \subseteq \Xi$. If $x \in \mathcal{D}^\xi$ then let $x|\eta \in \mathcal{D}^\eta$ denote the usual restriction. If $X \subseteq \mathcal{D}^\xi$ then let $X|\eta = \{ x|\eta : x \in X \}$. To save space, let $X|\neq i$ mean $X|\xi[<i]$, $X|\neq i$ mean $X|\xi[\neq i]$, etc.
But if \(Y \subseteq \mathcal{D}^n\) then we put \(Y^{-1} = \{x \in \mathcal{D}^\xi : x|\eta \in Y\}\).

To describe the idea behind the definition of iterated perfect sets, recall that the Sacks forcing consists of perfect subsets of \(\mathcal{D}\), that is, sets of the form \(H^n\mathcal{D} = \{H(a) : a \in \mathcal{D}\}\), where \(H : \mathcal{D}^\text{onto} \to X\) is a homeomorphism.

To get a product Sacks model, with two factors (the case of a two-element unordered set as the length of iteration), we have to consider sets \(X \subseteq \mathcal{D}^2\) of the form \(X = H^n\mathcal{D}^2\), where \(H\), a homeomorphism defined on \(\mathcal{D}^2\), splits in obvious way into a pair of one-dimensional homeomorphisms.

To get an iterated Sacks model, with two stages of iteration (the case of a two-element ordered set as the length of iteration), we have to consider sets \(X \subseteq \mathcal{D}^2\) of the form \(X = H^n\mathcal{D}^2\), where \(H\), a homeomorphism defined on \(\mathcal{D}^2\), satisfies the following: if \(H(a_1, a_2) = \langle x_1, x_2 \rangle\) and \(H(a'_1, a'_2) = \langle x'_1, x'_2 \rangle\) then \(a_1 = a'_1 \iff x_1 = x'_1\).

The combined product/iteration case results in the following definition.

**Definition 5.1** (iterated perfect sets, [7, 8]). For any \(\zeta \in \Xi, \text{Perf}_\zeta\) is the collection of all sets \(X \subseteq \mathcal{D}^\zeta\) such that there is a homeomorphism \(H : \mathcal{D}^\zeta \overset{\text{onto}}{\longrightarrow} X\) satisfying

\[
x_0|\xi = x_1|\xi \iff H(x_0)|\xi = H(x_1)|\xi
\]

for all \(x_0, x_1 \in \text{dom} H\) and \(\xi \in \Xi, \xi \subseteq \zeta\). Homeomorphisms \(H\) satisfying this requirement will be called projection–keeping. In other words, sets in \(\text{Perf}_\zeta\) are images of \(\mathcal{D}^\zeta\) via projection–keeping homeomorphisms. \(\Box\)

**Remark 5.2.** Note that \(\emptyset\), the empty set, formally belongs to \(\Xi\), and then \(\mathcal{D}^\emptyset = \{\emptyset\}\), and we easily see that \(\emptyset = \{\emptyset\}\) is the only set in \(\text{Perf}_\emptyset\). \(\Box\)

For the convenience of the reader, we now present five lemmas on sets in \(\text{Perf}_\zeta\) established in [7, 8].

**Lemma 5.3** (Proposition 4 in [7]). Let \(\zeta \in \Xi\). Every set \(X \in \text{Perf}_\zeta\) is closed and satisfies the following properties:

**P-1.** If \(i, j \in \zeta\) and \(z \in X|_{<i}\), then \(D_{X|_{<i}}(j) = \{x(i) : x \in X \land x|_{<i} = z\}\) is a perfect set in \(\mathcal{D}\).

**P-2.** If \(\xi \in \text{IS}_\zeta\), and a set \(X' \subseteq X\) is open in \(X\) (in the relative topology) then the projection \(X'|\xi\) is open in \(X|\xi\). In other words, the projection from \(X\) to \(X|\xi\) is an open map.

**P-3.** If \(\xi, \eta \in \text{IS}_\zeta\), \(x, y \in X|\xi, y \in X|\eta\), and \(x|\xi \cap \eta = y|\xi \cap \eta\), then \(x \cup y \in X|\xi \cup \eta\).

**Proof** (sketch). Clearly \(\mathcal{D}^\zeta\) satisfies P-1, P-2, P-3, and one easily shows that projection–keeping homeomorphisms preserve the requirements. \(\Box\)

**Lemma 5.4** (Lemma 6 in [7]). If \(\zeta \in \Xi, X \in \text{Perf}_\zeta, \xi \in \text{IS}_\zeta\), then \(X|\xi \in \text{Perf}_\xi\).
**Lemma 5.5** (Lemma 8 in [7]). If $\zeta \in \mathfrak{E}$, $X \in \text{Perf}_{\zeta}$, a set $X' \subseteq X$ is open in $X$, and $x_0 \in X'$, then there is a set $X'' \in \text{Perf}_{\zeta}$, $X'' \subseteq X'$, clopen in $X$ and containing $x_0$.

**Lemma 5.6** (Lemma 10 in [7]). Suppose that $\zeta \in \mathfrak{E}$, $\eta \in IS_{\zeta}$, $X \in \text{Perf}_{\zeta}$, $Y \in \text{Perf}_{\eta}$, and $Y \subseteq X \upharpoonright \eta$. Then $Z = X \cap (Y \upharpoonright -1) \in \text{Perf}_{\zeta}$.

**Lemma 5.7** (Lemma 10 in [8]). Suppose that $\zeta \in \mathfrak{E}$, $\xi \subseteq \zeta$, $X \in \text{Perf}_{\zeta}$. Then $X \upharpoonright -1 \zeta$ belongs to $\text{Perf}_{\zeta}$.

### 6 The forcing and the basic extension

This section introduces the forcing notion we consider and the according generic extension called the basic extension.

We continue to argue in $L$. Recall that a partially ordered set $I \in L$ is defined by (2) in Section 4, and $\mathfrak{E}$ is the set of all at most countable initial segments $\xi \subseteq I$ in $L$. For any $\zeta \in \mathfrak{E}$, let $P_{\zeta} = (\text{Perf}_{\zeta})^L$.

The set $P = P_I = \bigcup_{\zeta \in \mathfrak{E}} P_{\zeta} \in L$ will be the forcing notion.

To define the order, we put $\|X\| = \zeta$ whenever $X \in P_{\zeta}$. Now we set $X \leq Y$ (i.e. $X$ is stronger than $Y$) iff $\zeta = \|Y\| \subseteq \|X\|$ and $X \upharpoonright \zeta \subseteq Y$.

**Remark 6.1.** We may note that the set $\mathbb{1} = \{\varnothing\}$ as in Remark 5.2 belongs to $P$ and is the $\leq$-largest (i.e., the weakest) element of $P$. 

Now let $G \subseteq P$ be a $P$-generic set (filter) over $L$.

**Remark 6.2.** If $X \in P_{\zeta}$ in $L$ then $X$ is not even a closed set in $\mathcal{D}_{\zeta}$ in $L[G]$. However we can transform it to a perfect set in $L[G]$ by the closure operation. Indeed the topological closure $X^\#$ of such a set $X$ in $\mathcal{D}_{\zeta}$ taken in $L[G]$ belongs to $\text{Perf}_{\zeta}$ from the point of view of $L[G]$. 

It easily follows from Lemma 5.5 that there exists a unique array $a_i[G] = \langle a_i[G] \rangle_{i \in I}$, all $a_i[G]$ being elements of $2^\omega$, such that $a_i[G] \upharpoonright \xi \in X^\#$ whenever $X \in G$ and $\|X\| = \zeta \in \mathfrak{E}$. Then $L[G] = L[\langle a_i[G] \rangle_{i \in I}] = L[a[G]]$ is a $P$-generic extension of $L$.

**Theorem 6.3** (Theorems 24, 31 in [7]). Every cardinal in $L$ remains a cardinal in $L[G]$. Every $a_i[G]$ is Sacks generic over the model $L[a[G] \upharpoonright \xi]$.

We now present several lemmas on reals in $P$-generic models $L[G]$, established in [7]. In the lemmas, we let $G \subseteq P$ be a set $P$-generic over $L$.

**Lemma 6.4** (Lemma 22 in [7]). Suppose that sets $\eta$, $\xi \in \mathfrak{E}$ satisfy $\forall j \in \eta \exists i \in \xi (j \prec i)$. Then $a[G] \upharpoonright \eta \in L[a[G] \upharpoonright \xi]$.

**Lemma 6.5** (Lemma 26 in [7]). Suppose that $K \in L$ is an initial segment in $I$, and $i \in I \setminus K$. Then $a_i[G] \notin L[a[G] \upharpoonright K]$.
Lemma 6.6 (Corollary 27 in [7]). If \( i \neq j \) then \( a_i[G] \neq a_j[G] \) and even \( L[a_i[G]] \neq L[a_j[G]] \). \qed

Lemma 6.7 (Lemma 29 in [7]). If \( K \in L \) is an initial segment of \( I, \) and \( r \) is a real in \( L[G] \), then either \( r \in L[x \upharpoonright K] \) or there is \( i \notin K \) such that \( a_i[G] \in L[r] \).

We apply the lemmas in the proof of the next theorem. Let \( \leq_L \) denote the Gödel wellordering on \( 2^\omega \), so that \( x \leq_L y \) iff \( x \in L[y] \). Let \( x <_L y \) mean that \( x \leq_L y \) but \( y \notin L[x] \), and \( x \equiv_L y \) mean that \( x \leq_L y \) and \( y \leq_L x \).

**Theorem 6.8.** Assume that \( i \in I \) and \( r \in L[G] \cap 2^\omega \). Then

(i) if \( j \in I \) and \( j \preceq i \) then \( a_j[G] \leq_L a_i[G] \);  

(ii) if \( j \in I \) and \( j \not\preceq i \) then \( a_j[G] \nleq_L a_i[G] \);  

(iii) if \( r \leq_L a_i[G] \) then \( r \in L \) or \( r \equiv_L a_j[G] \) for some \( j \in I, j \preceq i \);  

(iv) if \( i = \langle \gamma, s \rangle \in I, e = 0,1 \), and \( i \upharpoonright e = \langle \gamma, s \upharpoonright e \rangle \) then \( a_{i \upharpoonright e}[G] \) is a true successor of \( a_i[G] \) in the sense that \( a_i[G] <_L a_{i \upharpoonright e}[G] \) and any real \( y \in 2^\omega \) satisfies \( y <_L a_{i \upharpoonright e}[G] \iff y \leq_L a_i[G] \);  

(v) if \( i = \langle \gamma, s \rangle \in I, x \in 2^\omega \cap L[G] \) is a true successor of \( a_i[G] \) in the sense of (iv), then there is \( e = 0 \) or \( 1 \) such that \( x \equiv_L a_{i \upharpoonright e}[G] \).

**Proof.** (i) Apply Lemma 6.4 with \( \eta = \{ j \} \) and \( \xi = \{ i \} \).  

(ii) Apply Lemma 6.5 with \( K = [i] \).  

(iii) If there are elements \( j \in I, j \preceq i, \) such that \( a_j[G] \in L[r] \), then let \( j \) be the largest such one, and let \( \xi = [j] \) (a finite initial segment of \( I \)). Then, by Lemma 6.7, either \( r \in L[a_i[G]] \upharpoonright [\xi] \), or there is \( i' \notin \xi \) such that \( a_{i'}[G] \in L[r] \).

In the “either” case, we have \( r \in L[a_j[G]] \) by (i), so that \( L[r] = L[a_j[G]] \) by the choice of \( j \). In the “or” case we have \( a_{i'}[G] \in L[a_i[G]] \), hence \( i' \preceq i \) by (ii). But this contradicts the choice of \( j \) and \( i' \).

Finally if there is no \( j \in I, j \preceq i, \) such that \( a_j[G] \in L[r] \), then the same argument with \( \xi = \emptyset \) gives \( r \in L \).

(iv) The relation \( a_j[G] <_L a_{i \upharpoonright e}[G] \) is implied by Lemmas 6.4 and 6.5. If now \( y <_L a_{i \upharpoonright e}[G] \) then \( y \in L \) or \( y \equiv_L a_j[G] \) for some \( j \preceq i \) by (iii), and in the latter case in fact \( j < i \upharpoonright e \), hence \( j \preceq i \), and then \( y \equiv_L a_i[G] \).  

(v) By (iv), it suffices to prove that \( x \leq_L a_{i \upharpoonright 0}[G] \) or \( x \leq_L a_{i \upharpoonright 1}[G] \). Assume that \( x \not\leq_L a_{i \upharpoonright 0}[G] \). Then by Lemma 6.7 there is an element \( j \in I \) such that \( j \neq i \upharpoonright 0 \) and \( a_{i \upharpoonright 0}[G] \leq_L x \). If \( a_j[G] <_L x \) strictly then \( a_j[G] \leq_L a_i[G] \) by the true successor property, hence \( i_0 \preceq i \), contrary to \( i_0 \neq i \upharpoonright 0 \), see above. Therefore in fact \( a_{i_0}[G] \equiv_L x \). Then we must have \( i_0 = i \upharpoonright 0 \) or \( i_0 = i \upharpoonright 1 \) as \( x \) is a true successor, but then \( i_0 = i \upharpoonright 1 \), as \( x \not\leq_L a_{i \upharpoonright 0}[G] \) was assumed, and we are done. \qed

9
7 The subextension

Following the arguments above, assume that $G \subseteq \mathbb{P}$ is a set $\mathbb{P}$-generic over $L$, and consider the set $J[G] \subseteq L[G]$ of all elements $i \in I$ such that either $i = \langle \gamma, m \rangle$, where $\gamma < \omega_1$ and $m < \omega$, or $i = \langle \gamma, 0^m \rangle$, where $\gamma < \omega_1$ and $m < \omega$, $a_\gamma[G](m) = 1$. Following (4), we define

$$M[G] = \mathcal{P}(\omega) \cap \bigcup_{i_1, \ldots, i_n \in J[G]} L[a_{i_1}[G], \ldots, a_{i_n}[G]],$$

Lemma 7.1. If $i \notin J[G]$ then $a_i[G] \notin M[G]$.

Proof. This is not immediately a case of Lemma 6.5 because $J[G] \notin L$. However the set $K = \{j \in I : i \neq j\}$ belongs to $L$ and satisfies $J[G] \subseteq K \subseteq I$. We have $i \notin K$, and hence $a_i[G] \notin L[a[G] | K]$ by Lemma 6.5. On the other hand, we easily check $X \subseteq L[a[G] | K]$, and we are done.

We are going to prove that $\langle \omega; M[G] \rangle$ is a model of $\text{PA}_2 + \text{CA}(\Sigma^1_2)$, but the full $\text{CA}$ fails in $\langle \omega; M[G] \rangle$.

Part 1: $\langle \omega; M[G] \rangle$ is a model of all axioms of $\text{PA}_2$ except for $\text{CA}$, trivial.

Part 2: $\langle \omega; M[G] \rangle$ is a model of $\text{CA}(\Sigma^1_2)$ (with parameters). This is also easy by the Shoenfield absoluteness theorem.

Part 3: $\langle \omega; M[G] \rangle$ fails to satisfy the full $\text{CA}$. Here we need some work. Let $\gamma < \omega_1^L$, so that both $\gamma$ and each pair $\langle \gamma, s \rangle$, $s \in 2^{<\omega}$, belong to $I$ by (2) in Section 4, in particular $i_0 = \langle \gamma, \Lambda \rangle \in I$, where $\Lambda$ is the empty tuple. In addition $\gamma$ (as an element of $I$) does not belong to $J[G]$. Our plan is to prove that $a_\gamma[G] \notin M[G]$ but $a_\gamma[G]$ is definable in $\langle \omega; M[G] \rangle$.

Subpart 3.1: $a_\gamma[G] \notin M[G]$ by Lemma 7.1 just because $\gamma \notin J[G]$.

Subpart 3.2: $a_\gamma[G]$ is definable in $\langle \omega; M[G] \rangle$ with $a_{i_0}[G]$ as a parameter, where $i_0 = (\gamma, \Lambda) \in J[G]$. Namely we claim that for any $m < \omega$:

$$a_\gamma[G](m) = 1 \text{ iff there is an array of reals } b_0, b_1, \ldots, b_m, b_{m+1} \text{ and }$$

$$b'_m + 1 \text{ in } 2^\omega \text{ such that } b_0 = a_{i_0}, \text{ each } b_{k+1} \text{ is a true successor of } b_k (k \leq m), \text{ } b'_{m+1} \text{ is a true successor of } b_m \text{ as well, and } b'_{m+1} \not\equiv_L b_{m+1}. \tag{6}$$

The formula in the right-hand side of (6) is based on the Gödel canonical $\Sigma^1_2$ formula for $\leq_L$, which is absolute for $M[G]$ by the definition of $M[G]$. Therefore (6) implies that $a_\gamma[G]$ is definable in $\langle \omega; M[G] \rangle$ with $a_{i_0}[G]$ as a parameter. Thus it remains to establish (6).

Direction $\implies$. Assume that $a_\gamma[G](m) = 1$. Then $J[G]$ contains the elements $i_k = \langle \gamma, 0^k \rangle$, $k \leq m + 1$, along with an element $i'_{m+1} = \langle \gamma, 0^m \rangle$. Therefore the reals $b_k = a_{i_k}[G], k \leq m + 1$, and $b'_{m+1} = a_{i'_{m+1}}[G]$ belong to $M[G]$. Now
Theorem 6.8(iv),(ii) implies that the reals $b_k$ and $b'_{m+1}$ satisfy the right-hand side of (6), as required.

Direction $\iff$. Assume that the reals $b_k$, $k \leq m + 1$, and $b'_{m+1}$ satisfy the right-hand side of (6). By Theorem 6.8(v), there is an array of bits $e_1, \ldots, e_m, e_{m+1}$ and $e'_{m+1}$ such that $b_k = a_i[k]$ for all $k \leq m + 1$ and $b'_m = a_{i'[m+1]}[G]$, where $i_k = \langle \gamma, \langle e_1, \ldots, e_k \rangle \rangle$ and $i'[m+1] = \langle \gamma, \langle e_1, \ldots, e_{m+1} \rangle \rangle$.

However we must have $i_k \in J[G]$ for all $k \leq m + 1$, and $i'[m+1] \in J[G]$, by Lemma 7.1, since the reals $b_k$ and $b'_{m+1}$ belong to $M[G]$.

Part 4: $\langle \omega; M[G] \rangle$ satisfies the parameter-free schema $\text{CA}^*$. This is rather similar to the verification of $\text{CA}^*$ in $\langle \omega; X[G] \rangle$ in Section 3.

Assume that $\Phi(k)$ is a parameter-free $L(PA_2)$ formula with $k$ the only free variable. Consider the set $y = \{ k < \omega : \langle \omega; M[G] \rangle \models \Phi(k) \}$; then $y \in L[G]$, $y \subseteq \omega$. We claim that $y$ even belongs to $L$, and hence to $M[G]$.

By Theorem 6.8(v), there is an array of bits $e_1, \ldots, e_m, e_{m+1}$ such that $\mathbb{P}$ forces the same as well.

In the nontrivial direction, assume that $k \in y$. Then by the forcing theorem there is a condition $X \in G$ forcing $\langle \omega; M[G] \rangle \models \Phi(k)$. We claim that then $\mathbb{1}$ forces the same as well.

To prove this reduction, we define, still in $L$, the set $\text{Perm} \subseteq L$ that consists of all bijections $\pi : \omega_1 \rightarrow \omega_1$ such that $\pi = \pi^{-1}$ and the domain of nontriviality $|\pi| = \{ \alpha : \pi(\alpha) \neq \alpha \}$ is at most countable, i.e., bounded in $\omega_1$. Any $\pi \in \text{Perm}$ acts on:

- elements $i = \gamma$ or $i = \langle \gamma, s \rangle$ of $I$, by $\pi i = \pi(\gamma)$, resp. $i = \langle \pi(\gamma), s \rangle$;
- maps $g$ with $\text{dom} g \subseteq I$, by $\text{dom}(\pi g) = \pi^{-1}\text{dom} g$ and $(\pi g)(\pi(\alpha)) = g(\alpha)$ for all $\alpha \in \text{dom} g$;
- thus if $\xi \subseteq I$ and $x \in \mathcal{D}^\xi$ then $\pi x \in \mathcal{D}^{\pi^{-1}\xi}$ and $(\pi x)(\pi(\alpha)) = x(\alpha)$;
- sets $X \in \text{Perf}_\xi$, $\xi \in \Xi$, by $\pi X = \{ \pi x : x \in X \} \in \text{Perf}_{\pi^{-1}\xi}$.

We return to the nontrivial direction $\implies$ of (7), where we have to prove that the condition $\mathbb{1}$ forces $\langle \omega; M[G] \rangle \models \Phi(k)$”. Let this be not the case.

\footnote{See Kunen [10] on forcing, especially Section IV.6 there on the “forcing over the universe” approach.}
Then there is a condition \( Y \in \mathcal{P} \) which forces \( \langle \omega; M[G] \rangle \models \neg \Phi(\kappa) \). There is a permutation \( \pi \in \text{Perm} \) satisfying \( \|Z\| \cap \|X\| = \emptyset \), where \( Z = \pi Y \in \mathcal{P} \). We claim that \( Z \) forces \( \langle \omega; M[G] \rangle \models \neg \Phi(\kappa) \). Indeed assume that \( H \subseteq \mathcal{P} \) is a set \( \mathcal{P} \)-generic over \( L \), and \( Z \in H \). We have to prove that \( \langle \omega; M[H] \rangle \models \neg \Phi(\kappa) \). The set \( K = \{ \pi Z' : Z' \in H \} \) is \( \mathcal{P} \)-generic over \( L \) along with \( H \) since \( \pi \in L \). Moreover \( K \) contains \( Y \). It follows that \( \langle \omega; M[K] \rangle \models \neg \Phi(\kappa) \) by the forcing theorem and the choice of \( Y \).

However the array \( a[K] \) is equal to the permutation of the array \( a[H] \) by \( \pi \). It follows that \( M[H] = M[K] \), and hence \( \langle \omega; M[H] \rangle \models \neg \Phi(\kappa) \), as required. Thus indeed \( Z \) forces \( \langle \omega; M[G] \rangle \models \neg \Phi(\kappa) \).

Recall that \( X \) forces \( \langle \omega; M[G] \rangle \models \Phi(\kappa) \). On the other hand, \( X, Z \) are compatible in \( \mathcal{P} \) because \( \|Z\| \cap \|X\| = \emptyset \). This is a contradiction.

We conclude that \( 1 \) forces \( \langle \omega; M[G] \rangle \models \Phi(\kappa) \), and this completes the proof of (7). But it is known that the forcing relation \( \models \) is expressible in \( L \), the ground model. Therefore it follows from (7) that \( y \in L \), hence \( y \in M[G] \), as required.

8 Remarks and questions

Here we present three questions related to possible extensions of Theorem 1.2.

**Problem 8.1.** Is the parameter-free countable choice schema \( \text{AC}^* \) true in the language \( \mathcal{L}(\text{PA}_2) \) true in the models \( \langle \omega; M[G] \rangle \) defined in Section 7?

**Problem 8.2.** Can we sharpen the result of Theorem 1.2 by specifying that \( \text{CA}(\Sigma^1_3) \) is violated? The combination \( \text{CA}(\Sigma^1_3) \) plus \( \neg \text{CA}(\Sigma^1_{n+1}) \) would be optimal. The counterexample to \( \text{CA} \) defined in Section 7 (Part 3) definitely is more complex than \( \Sigma^1_3 \).

**Problem 8.3.** As a generalization of the above, prove that, for any \( n \geq 2 \), \( \text{PA}_n^* + \text{CA}(\Sigma^1_n) \) does not imply \( \text{CA}(\Sigma^1_{n+1}) \). In this case, we’ll be able to conclude that the full schema \( \text{CA} \) is not finitely axiomatizable over \( \text{PA}_n^* \). Compare to Problem 9 in [1, §11].

**Acknowledgement.** The authors are thankful to Ali Enayat for his enlightening comments that made it possible to accomplish this research.

**References**

[1] Krzysztof R. Apt and W. Marek. Second order arithmetic and related topics. *Ann. Math. Logic*, 6:177–229, 1974.

[2] James E. Baumgartner and Richard Laver. Iterated perfect-set forcing. *Ann. Math. Logic*, 17:271–288, 1979.

[3] Manuel Corrada. Parameters in theories of classes. Mathematical logic in Latin America, Proc. Symp., Santiago 1978, 121-132 (1980), 1980.
[4] Harvey Friedman. On the necessary use of abstract set theory. *Advances in Mathematics*, 41(3):209–280, 1981.

[5] Marcia J. Groszek. Applications of iterated perfect set forcing. *Ann. Pure Appl. Logic*, 39(1):19–53, 1988.

[6] Wojciech Guzicki. On weaker forms of choice in second order arithmetic. *Fundam. Math.*, 93:131–144, 1976.

[7] Vladimir Kanovei. Non-Glimm-Effros equivalence relations at second projective level. *Fund. Math.*, 154(1):1–35, 1997.

[8] Vladimir Kanovei. On non-wellfounded iterations of the perfect set forcing. *J. Symb. Log.*, 64(2):551–574, 1999.

[9] Georg Kreisel. A survey of proof theory. *J. Symb. Log.*, 33:321–388, 1968.

[10] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic*. College Publications, London, 2011.

[11] Azriel Levy. Definability in axiomatic set theory II. In Yehoshua Bar-Hillel, editor, *Math. Logic Found. Set Theory, Proc. Int. Colloq., Jerusalem 1968*, pages 129–145, Amsterdam-London, 1970. North-Holland.

[12] Azriel Levy. Parameters in comprehension axiom schemes of set theory. Proc. Tarski Symp., internat. Symp. Honor Alfred Tarski, Berkeley 1971, Proc. Symp. Pure Math. 25, 309-324 (1974), 1974.

[13] A. R. D. Mathias. Surrealist landscape with figures (a survey of recent results in set theory). *Period. Math. Hung.*, 10:109–175, 1979.

[14] Ralf Schindler and Philipp Schlicht. ZFC without parameters (a note on a question of Kai Wehmeier). https://ivv5hpp.uni-muenster.de/u/rds/ZFC_without_parameters.pdf. Accessed: 2022-09-06.

[15] Thomas Schindler. A disquotational theory of truth as strong as \( Z_2^- \). *J. Philos. Log.*, 44(4):395–410, 2015.

[16] James H. Schmerl. Peano arithmetic and hyper-Ramsey logic. *Trans. Am. Math. Soc.*, 296:481–505, 1986.

[17] Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Logic. Cambridge: Cambridge University Press; Urbana, IL: ASL, 2nd edition, 2009. Pages xvi + 444.

[18] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math. (2)*, 92:1–56, 1970.