Symmetry breaking and Goldstone theorem in de Sitter space

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We consider an \( O(N) \) symmetric scalar field model in the mean field (Hartree) approximation and show that the symmetry can be broken in de Sitter space. We find that the phase transition can be of first order, and that its strength depends non-analytically on the parameters of the model. We also show that the would-be Goldstone bosons acquire a mass, effectively becoming pseudo-Goldstone bosons, thus breaking the \( O(N) \) symmetry. Our results imply that topological defects can form during inflation.

I. INTRODUCTION

Ever since Kirzhnits and Linde [1] pointed out that thermal radiative effects can induce phase transitions in the early Universe, phase transitions have played a central role in the early Universe cosmology. In particular, they have been used to drive out-of-equilibrium phenomena which can lead to creation of the matter-antimatter asymmetry, preheating, formation of topological defects, etc. The effects induced by particle creation in an expanding Universe setting are quite delicate and have not yet been fully understood, albeit there is a large literature on the subject, a non-representative sample includes Refs. [2–17]. Based upon a mean field (Hartree) analysis of a scalar self-interacting theory, Ford and Vilenkin [18, 19] pointed out a long time ago that the infrared effects in de Sitter space may restore symmetries spontaneously broken by the vacuum. A similar conclusion was reached by Ratra in Ref. [20]. However, in these works it was not realised that mass generation can regulate the infrared divergences of de Sitter space. The infrared effects in (quasi-)de Sitter spaces have received a considerable attention in recent literature, and several papers have been published [21–25] which have – just as Ford and Vilenkin – treated the problem in the mean...
field – or Hartree – approximation (see also Refs. [26, 27] for some recent mean field results on flat space). The current consensus is that the infrared effects in de Sitter space are strong enough to restore the broken $O(N)$ symmetry. While this is correct if the corresponding effective action is averaged over infinite distances, from the observational point of view the more relevant question is whether the symmetry gets broken or restored when averaged over some fixed physical scale [39]. In this work we take the point of view that the effective action should be averaged over some fixed physical scale and we show that symmetries are then generally not restored in de Sitter space. We also give a simple criterion for symmetry restoration. For simplicity, we consider here only the global $O(N)$ symmetric scalar field model.

For pedagogical reasons we begin in section II by analysing a real scalar field ($O(1)$ model). The central part of the paper is section III where we analyse an $O(N)$ symmetric model on de Sitter space. Section IV is reserved for a discussion, and the Appendix for technical details on the de Sitter space scalar field propagator.

II. A REAL SCALAR FIELD

The free action of a real scalar field $\phi(x)$ in $D$ space-time dimensions reads,

$$S[\phi] = \int d^Dx \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right],$$

(1)

where $m_0$ and $\lambda$ denote the field mass and quartic coupling, respectively, $g_{\mu\nu}$ is the metric tensor, $g^{\mu\nu}$ its inverse, and $g = \text{det}[g_{\mu\nu}]$. The metric signature we use is $(-,+,+,...)$. The vacuum is given by $\phi^2 = 0$ for $m_0^2 > 0$ and $\phi^2 = -6m_0^2/\lambda$ when $m_0^2 < 0$. In the case when $\phi^2 > 0$ the $Z_2$ symmetry ($\phi \rightarrow -\phi$) of the action (1) is (completely) broken by the vacuum in which $\phi^2 = -6m_0^2/\lambda$. If one goes through a quench from high temperatures (e.g. in an early Universe setting), when $m_0^2 < 0$ domain walls form by the Kibble mechanism [33], such that the state spontaneously breaks translation invariance. By causality, at least of the order of one domain wall for $a$ Hubble volume. Once formed, their energy density scales as $\propto 1/a^2$ ($a$ denotes the scale factor), such that in decelerating spacetimes domain walls will dominate the energy density at late times (a menace to get rid of), while in accelerating spacetimes (such as inflation) they get diluted.

Here we shall perform a mean field (Hartree) analysis, for which the effective potential (up to two loop order) is of the form,

$$V_{\text{MF}} = V_0 + \frac{1}{2} m_0^2 (\phi^2 + i \Delta(x;x)) + \frac{\lambda}{4!} \left( \phi^4 + 6\phi^2 i \Delta(x;x) + 3[i \Delta(x;x)]^2 \right) + \frac{\lambda}{2} \text{Tr} \ln[i \Delta(x;x)],$$

(2)
where \( m_0^2 \) denotes a bare mass term, \( \lambda > 0 \) a quartic coupling, \( \phi(x) = \langle \Omega | \hat{\phi}(x) | \Omega \rangle \) is a mean field, \( | \Omega \rangle \) is a state, and \( i\Delta(x;x') = \langle \Omega | T[\delta \hat{\phi}(x')\delta \hat{\phi}(x)] | \Omega \rangle \) is the Feynman propagator for the field fluctuations \( \delta \hat{\phi}(x) = \hat{\phi}(x) - \phi(x) \), where \( T \) stands for time ordering. For simplicity, we have assumed that gravity is nondynamical and that all quantities \( \phi(x) \) and \( i\Delta(x;x) \) are either constant or adiabatically varying in time (on a de Sitter background). Varying the mean field action

\[
S_{MF}[\phi, \Delta] = \int d^Dx \sqrt{-g(x)} \left[ -\frac{1}{2} (\partial_\mu \phi)(\partial_\nu \phi)g^{\mu\nu} + \frac{1}{2}[\Box x i\Delta(x;x)]_{x'\to x} - V_{MF}(\phi, \Delta) \right]
\]  

(3)

with respect to \( \phi(x) \) and \( i\Delta(y;x) \) results in

\[
\left[ \Box_x - m_0^2 - \lambda \frac{1}{2} i\Delta(x;x) \right] \phi - \frac{\lambda}{6} \phi^3 = 0
\]

(4)

\[
\sqrt{-g} \left[ \Box_x - m_0^2 - \lambda \left( \phi^2 + i\Delta(x;x) \right) \right] \delta^D(x-y) = i[i\Delta(x;y)]^{-1},
\]

(5)

where \( \Box_x = (-g)^{-1/2} \partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu \) denotes the d’Alembertian \( (\Box_x = g^{\mu\nu} \nabla_\mu \nabla_\nu) \) as it acts on a scalar quantity. Since the coincident propagator \( i\Delta(x;x) \) is in general divergent (see the Appendix), these equations need to be renormalised. From the structure of equations \( (4, 5) \), a mass renormalisation suffices \( \text{cf.} \ [40] \). When viewed as a function of the ultraviolet cutoff \( \Lambda \), the coincident propagator exhibits power law divergences \( \propto \Lambda^{D-2}, \Lambda^{D-4} \), etc. \( \text{cf.} \ [26, 27] \), which are automatically subtracted in dimensional regularisation. In the special cases when the spacetime dimension is even \( (D = 2n, \ n = 1, 2, \ldots) \), there is a logarithmic divergence in \( \Lambda \), that manifests itself as a simple pole in the coincident propagator, \( i\Delta(x;x)_{\text{div}} \propto 1/(D-2n) \) \( (n = 1, 2, \ldots) \), see Eqs. \( (58-62) \) in the Appendix. When this divergence is absorbed in the bare mass \( m_0^2 \), one gets a finite, renormalised mass term,

\[
m^2 = m_0^2 + \lambda \frac{1}{2} i\Delta(x;x)_{\text{div}}
\]

\[
i\Delta(x;x)_{\text{div}} = \frac{H^{D-2} \Gamma \left( \frac{D-1}{2} \right)}{4\pi^{(D-1)/2}} \left[ \psi \left( \frac{D}{2} \right) - \psi(D-1) - \psi \left( 1 - \frac{D}{2} \right) - \gamma_E + \frac{1}{D-1} \right],
\]

(6)

where \( m^2 \) can be either positive or negative. Hence, the renormalised, manifestly finite, form of Eqs. \( (4, 5) \) is,

\[
\left[ \Box_x - m^2 - \lambda \frac{1}{2} i\Delta(x;x)_{\text{fin}} \right] \phi - \frac{\lambda}{6} \phi^3 = 0
\]

(7)

\[
\sqrt{-g} \left[ \Box_x - m^2 - \lambda \left( \phi^2 + i\Delta(x;x)_{\text{fin}} \right) \right] i\Delta(x;x') = i\delta^D(x-x'),
\]

(8)

where \( i\Delta(x;x)_{\text{fin}} = i\Delta(x;x) - i\Delta(x;x)_{\text{div}} \) is the finite part of the coincident correlator, \textit{cf.} Eqs. \( (59-62) \).
Since $\phi$ is (by assumption) slowly varying, we have $\square \phi \approx 0$, and Eq. (8) yields

$$\left[ m^2 + \frac{\lambda}{2} \Delta(x; x)_{\text{fin}} + \frac{\lambda}{6} \phi^2 \right] \phi = 0 .$$

This is solved by $\phi = 0$ or, when $\phi^2 > 0$, by

$$\phi^2 = -\frac{6m^2}{\lambda} - 3\iota \Delta(x; x)_{\text{fin}} > 0 ,$$

(10)

At early times $\iota \Delta(x; x)_{\text{fin}}$ grows as given in (60–62), reaching at late times the de Sitter invariant limit (59), and the condition (9) becomes,

$$\phi^2 = -\frac{6m^2}{\lambda} - \frac{3\Gamma (D+1)}{2\pi^{(D+1)/2} m_{\text{MF}}^2} > 0 ,$$

(11)

where $m_{\text{MF}}^2 = \frac{\partial^2 V_{\text{MF}}}{\partial \phi^2}$ is the mean field mass satisfying the mass gap equation

$$m_{\text{MF}}^2 = m^2 + \frac{\lambda}{2} \left( \phi^2 + \iota \Delta(x; x)_{\text{fin}} \right) = \begin{cases} -2m^2 - \lambda \iota \Delta(x; x)_{\text{fin}}, & \text{if } \phi^2 > 0 \\ m^2 + \frac{\lambda}{2} \iota \Delta(x; x)_{\text{fin}}, & \text{if } \phi^2 = 0 . \end{cases}$$

(12)

Note that the mean field mass $m_{\text{MF}}$ is also the mass of field fluctuations in the correlator equation (8).

When Eq. (59) is inserted into (12) one gets,

$$m_{\text{MF}}^2 + 2m^2 + \frac{\lambda \Gamma (D+1)}{2\pi^{(D+1)/2} m_{\text{MF}}^2} H^D = 0 \quad (\phi^2 > 0) .$$

(13)

This is solved by,

$$m_{\text{MF}}^2 = -m^2 \pm \sqrt{m^4 - m_{\text{cr}}^4} , \quad m_{\text{cr}}^4 = (\lambda m_{\text{MF}}^2) \iota \Delta(x; x)_{\text{fin}} = \frac{\lambda H^D \Gamma (D+1)}{2\pi^{(D+1)/2}} \quad (\phi^2 > 0) .$$

(14)

We see that – when the symmetry is broken, $\phi^2 > 0$ – there is a minimum $|m^2|$ for which the gap equation (14) permits a meaningful (real) solution [41]:

$$|m^2| > m_{\text{cr}}^2 = \sqrt{\frac{\lambda H^D \Gamma (D+1)}{2\pi^{(D+1)/2}}} .$$

(15)

In $D = 2, 3$ and 4, $m_{\text{cr}}^2 = \sqrt{\lambda/(4\pi)H}, \sqrt{\lambda/2H^{3/2}/\pi},$ and $\sqrt{3\lambda/2H^2/(2\pi)}$, respectively. There is a simple way to determine the physically correct sign in Eq. (14). In the limit when $H \to 0$ the infrared enhanced fluctuations are absent, such that one should recover the tree level mass, $m_{\text{MF}}^2 \to -2m^2$. This then implies that the physical mean field mass corresponds to the positive branch in (14),

$$m_{\text{MF}}^2 = -m^2 + \sqrt{m^4 - m_{\text{cr}}^4} \quad (\phi^2 > 0) .$$

(16)
On the other hand, when the symmetry is unbroken, \( \phi^2 = 0 \), Eq. (12) implies,

\[
m^4_{\text{MF}} - m^2 m^2_{\text{MF}} - \frac{1}{2} m^4_{\text{cr}} = 0
\]  

which is solved by,

\[
m^2_{\text{MF}} = \frac{m^2}{2} + \sqrt{\frac{m^4}{4} + \frac{m^4_{\text{cr}}}{2}} \quad (\phi^2 = 0).
\]  

This formula agrees with Eq. (5.9) of Ref. [18] (provided one makes the replacement \(-2\lambda \rightarrow \lambda\) in [18]), and also with the results of Refs. [24, 25]. In Eq. (18) we have dropped the solution with a negative sign in front of the square root, because for that solution \( m^2_{\text{MF}} < 0 \), which is unacceptable on physical grounds (in this case the de Sitter invariant state would be unstable under small perturbations). When the solution (18) is inserted into Eq. (13), one can show that the only real solution for \( \phi \) is \( \phi = 0 \), consistent with the assumption of unbroken symmetry made in deriving Eq. (18).

To summarise, we have found that, when

\[
m^2 < -m^2_{\text{cr}} = -\sqrt{\frac{\lambda H^D \Gamma (\frac{D+1}{2})}{2\pi^{(D+1)/2}}} \quad \text{(vacuum breaks the } Z_2 \text{ symmetry)},
\]

the infrared fluctuations on de Sitter space may not be able to restore the broken \( Z_2 \) symmetry (\( \phi \rightarrow -\phi \)) of the vacuum of a real scalar field (1). In this case \( \phi^2 = 3 m^2_{\text{MF}}/\lambda > 0 \) and Eq. (16) applies. Otherwise, when \( m^2 \geq -m^2_{\text{cr}} \) the \( Z_2 \) symmetry is restored and Eq. (18) applies. From Eq. (16) we see that, in the broken symmetry case, there is a minimum mean field mass, given by \( (m^2_{\text{MF}})_{\text{cr}} = -m^2 = m^2_{\text{cr}} \), implying that for any finite coupling \( \lambda \), as \( |m^2| \) increases from \( |m^2| < m^2_{\text{cr}} \) to \( |m^2| > m^2_{\text{cr}} \), the order parameter \( \phi^2 = 3 m^2_{\text{MF}}/\lambda \) will experience a jump,

\[
\Delta \phi^2 = \frac{3 m^2_{\text{cr}}}{\lambda} = \sqrt{\frac{9 H^D \Gamma (\frac{D+1}{2})}{2\pi^{(D+1)/2}}}.
\]

Notice that increasing \( H \) and decreasing \( \lambda \) strengthens the transition.

An interesting question is what the above analysis implies for the history of the Universe, and in particular for inflationary cosmology. In order to address that question, we shall consider two scenarios. In Scenario A inflation starts from a vacuum state with a non-zero vacuum energy, and \( H(t) \) adiabatically decreases in time. In Scenario B inflation starts after an early radiation era, with a temperature \( T \gg H \) \((T \propto 1/a)\), and \( H \approx \text{const.} \) during inflation.

In Scenario A, early in inflation the expansion rate \( H \) is large and the criterion (19) is not met, and hence the symmetry is unbroken. As \( H(t) \) decreases, at some moment \( t_T \) during inflation
|m^2| = m_{cF}^2(t_T), for $t > t_T$ the criterion (19) is met, and the symmetry gets broken. We shall now argue that in this case the transition is not first order. Based on our de Sitter invariant analysis, one would expect that at the transition the field acquires the expectation value given by (20), but the details of the dynamics of the phase transition are unclear, especially since this has to evoke a de Sitter breaking physics, which we have so far not discussed. Nevertheless, based on the existing literature [28–31] one can make a crude quantitative analysis. Close to the transition, the condition for stable defect formation $m_{MF}^2 > 8H^2$ [30, 31] is not met. Moreover, since the semiclassical action is not much greater than one, semiclassical methods do not apply, and one should make use of Starobinsky’s stochastic inflation. According to Vilenkin [28] and Linde [29], the probability for nucleation of nearly homogeneous regions (over the horizon size) of a horizon volume is of the order of one per horizon time and volume [42]. This then means that the transition will not be first order but some higher order or crossover [43]. Once they nucleate, these regions will grow in size superluminally, and hence cannot decay; very quickly the whole Universe will be filled by nearly homogeneous super-Hubble size regions of broken phase. The field $\phi$ in the Starobinsky’s stochastic picture exhibits a Brownian motion. The field exhibits random jumps $\delta \phi \sim H/(2\pi)$ over time scales $\sim 1/H$, such that it will achieve the broken phase value $\phi \sim H/\lambda^{1/4}$ after $O(1/\sqrt{\lambda})$ jumps. This conclusion is supported by a more quantitative analysis which follows from Eq. (61), according to which (in $D = 4$) $m_{MF}^2 \simeq (\lambda/2)[H^2/(4\pi^2)] \ln(a)$ and will reach $m_{ct}^2 = \sqrt{(3\lambda/2)H^2/(2\pi)}$ after $\ln[a(t)/a(t_T)] \simeq 4\pi \sqrt{3/(2\lambda)}$ e-foldings (assuming it started from zero), at which point the field will settle to its critical value (20). Of course, as $H$ adiabatically evolves, $\phi$ will adjust to $\phi^2 = 3m_{MF}^2/\lambda$, with $m_{MF}^2$ given in (16).

The above analysis shows that there is effectively no barrier to growth of $\phi$, and it will grow as required by the local dynamics. But, there is still the concern that causality will prevent growth of a condensate of constant value across causally disconnected regions of de Sitter space. Indeed, the size of domains of constant $\phi$ will at any given time in inflation be finite, and averaging over the whole Universe will necessarily give $\phi = 0$. The physical size of domains at time $t > t_T$ will be $\sim e^{H(t-t_T)/H}$, and as long as this size is larger than any physical size of relevance, one can take $\phi$ to be constant (spatially independent) across the whole Universe. But, what about the correlator, which is formally obtained by averaging field fluctuations over the whole Universe? For a massive scalar, a typical physical scale over which field fluctuations significantly contribute is given by $m_{MF}^{-1}$ or smaller, and contribution of larger scales is suppressed. Hence, as long as the scale $(1/H) \exp[H(t - t_T)]$ is much larger than $1/m_{MF}$, there is a well defined separation of scales.
in between the condensate $\phi$ and the fluctuations $i\Delta(x; x) = \langle \delta\phi^2 \rangle$, and the averaging procedure advocated in this paper is justified. Moreover, when the field is massive, one can extend the infrared cutoff $\sim m_{\text{MF}}^{-1}$ (which was there due to the finite size of averaging domain) to zero without significantly changing the result for $\langle \delta\phi^2 \rangle$, and with the bonus of restoring de Sitter invariance.

In Scenario $B$ inflation is preceded by a radiation era characterised by a temperature $T \gg H$, such that, at early stages of inflation, $m_{\text{MF}}^2 \approx m^2 + \lambda T^2/24 > 0$ and the symmetry is unbroken. During inflation the temperature drops rapidly as $T \propto 1/a \propto e^{-Ht}$, and after some time $m^2 + \lambda T^2/24 < 0$. This scenario is a realisation of quenched transition mentioned at the beginning of this section. If at that moment the criterion for domain wall formation $|m^2| > 4H^2$ is met, the semiclassical analysis of phase transition applies \cite{44}, and the transition will be of first order: bubbles of broken phase will nucleate and moreover topological defects will form. A detailed description of production and evolution of spherical domain walls, loops of (global) cosmic strings and global monopole-antimonopole pairs during inflation is given in Refs. \cite{30, 32}.

In summary, we have considered two inflationary scenarios. In Scenario $A$ the transition is a crossover, a temporary breakdown of de Sitter symmetry occurs, and a (approximate) de Sitter invariant state is reached after some time after the transition. In Scenario $B$ inflation is preceded by a radiation era, a first order transition can occur and topological defects such as domain walls may form.

### III. THE $O(N)$ MODEL

We shall now consider the symmetry breaking in an $O(N)$ symmetric scalar field theory in the early Universe setting. Recall that this model allows for formation of (global) cosmic strings (when $N = 2$), global monopoles (when $N = 3$), global cosmic textures (when $N = 4$), etc. The free action of an $O(N)$ symmetric scalar field reads,

$$S = \int d^Dx \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \sum_{a=1}^{N} (\partial_\mu \phi_a)(\partial_\nu \phi_a) - \frac{1}{2} m_0^2 \sum_{a=1}^{N} \phi_a^2 - \frac{\lambda}{4N} \left[ \sum_{a=1}^{N} \phi_a^2 \right]^2 \right]. \tag{21}$$

Similarly as in the Brout-Englert-Higgs (BEH) mechanism, when $m_0^2 < 0$ and $\lambda > 0$ the vacuum breaks the $O(N)$ symmetry to the $O(N-1)$ symmetry, such that the resulting vacuum manifold is the $N$ dimensional sphere, $S^N \sim O(N)/O(N-1)$ \cite{33}. As a result, one of the scalars acquires a mass, while the other $N-1$ scalars remain massless. These massless excitations are known as Goldstone bosons. Unlike in the simple $O(N)$ model \cite{21}, at low energies the Goldstone bosons
in the BEH mechanism acquire a mass and become the longitudinal excitations of the $W^\pm$ and $Z$ bosons. For that reason they are known as pseudo-Goldstone bosons. We shall now see that the Goldstone bosons of the $O(N)$ model in de Sitter space (more generally in inflationary spacetimes) become massive due to the infrared (super-Hubble) enhancement of scalar correlations, and in that respect they can be considered as pseudo-Goldstone bosons.

The mean field (two loop) effective potential of an $O(N)$ symmetric field $\phi_a$ corresponding to the tree level action [21] reads,

$$V_{\text{MF}} = \frac{1}{2} m_0^2 \left[ \sum_{a=1}^{N} \left( \phi_a^2 + i \Delta_{aa}(x; x) \right) \right] + \frac{\lambda}{4N} \left[ \left( \sum_{a=1}^{N} \phi_a^2 \right)^2 + 2 \left( \sum_{a=1}^{N} \phi_a^2 \right) \sum_{b=1}^{N} i \Delta_{bb}(x; x) \right] + 4 \sum_{a,b=1}^{N} \phi_a \phi_b i \Delta_{ab}(x; x) + \left( \sum_{a=1}^{N} i \Delta_{aa}(x; x) \right)^2 + \frac{1}{2} \sum_{a,b=1}^{N} \left( i \Delta_{ab}(x; x) \right)^2 + \frac{1}{2} \text{Tr} \ln (i \Delta_{aa}(x; x)), \tag{22}$$

resulting in the following two loop effective action,

$$S_{\text{MF}}[\phi_a, \Delta_{bc}] = \int d^D x \sqrt{-g} \left[ -\frac{1}{2} \sum_{a=1}^{N} g^{\mu\nu}(\partial_\mu \phi_a)(\partial_\nu \phi_a) + \frac{1}{2} \sum_{a=1}^{N} \left[ \Box_x i \Delta_{aa}(x; x') \right]_{x' \rightarrow x} - V_{\text{MF}} \right]. \tag{23}$$

Varying the action with respect to $\phi_a(x)$ and $\Delta_{aa}(x'; x)$ gives the following (mean field) equations of motion (cf. Eqs. [7-8]),

$$\left[ \Box_x - m_0^2 - \frac{\lambda}{N} \sum_{b=1}^{N} \left( \phi_b^2 + i \Delta_{bb}(x; x) \right) \right] \phi_a(x) - \frac{2\lambda}{N} \sum_{b=1}^{N} i \Delta_{ab}(x; x) \phi_b(x) = 0 \tag{24}$$

$$\left[ \Box_x - m_0^2 - \frac{\lambda}{N} \sum_{c=1}^{N} \left( \phi_c^2 + i \Delta_{cc}(x; x) \right) \right] i \Delta_{ab}(x; x')
- \frac{2\lambda}{N} \sum_{c=1}^{N} \left[ \phi_a(x) \phi_c(x) + i \Delta_{ac}(x; x) \right] i \Delta_{cb}(x; x') = \delta_{ab} \frac{i \delta^D(x - x')}{\sqrt{-g}}, \tag{25}$$

where

$$i \Delta_{cb}(x; x') = \left\langle \Omega | T\left[ \delta \hat{\phi}_b(x') \delta \hat{\phi}_a(x) \right] \right| \Omega \right\rangle \tag{26}$$

denotes the time-ordered scalar field propagator for scalar field fluctuations, $\delta \hat{\phi}_a(x) = \hat{\phi}_a(x) - \langle \Omega | \hat{\phi}_a(x) | \Omega \rangle$. Just as in the one scalar case, Eqs. (25) can be renormalised by absorbing the infinite part of the coincident scalar propagator [59] into the bare mass $m_0^2$,

$$m^2 = m_0^2 + \frac{(N + 2) \lambda H^{D-2} \Gamma \left( \frac{D+1}{2} \right)}{4\pi^{(D-1)/2}} \left[ \psi \left( \frac{D}{2} \right) - \psi(D-1) - \psi \left( 1 - \frac{D}{2} \right) - \gamma_E \right], \tag{27}$$

resulting in manifestly finite renormalised equations analogous to Eqs. [7-8].
In the limit when the fields are slowly varying ($\Box \phi_a \simeq 0$), the renormalised form of Eq. (24) yields the following criterion for symmetry breaking,

$$
[m^2 + \frac{\lambda}{N} \sum_{b=1}^{N} (\phi_b^2 + t \Delta_{bb}(x; x)_{\text{fin}})] \phi_a(x) + \frac{2\lambda}{N} \sum_{b=1}^{N} t \Delta_{ab}(x; x)_{\text{fin}} \phi_b(x) = 0. 
$$

(28)

The field mass matrix $M^2$ is obtained by taking a second field derivative of $V_\text{MF}$ (or equivalently by taking a single derivative with respect to $t \Delta_{ba}$),

$$
M^2_{ab} = \left[ m^2 + \frac{\lambda}{N} \sum_{c=1}^{N} (\phi_c^2 + t \Delta_{cc}(x; x)_{\text{fin}}) \right] \delta_{ab} + \frac{2\lambda}{N} [\phi_a \phi_b + t \Delta_{ab}(x; x)_{\text{fin}}].
$$

(29)

Notice that this result can be read off also from Eq. (25), as $M^2$ is the mass term of the propagator $t \Delta_{ab}$. Both Eq. (28) and (29) contain in general off-diagonal terms. One can diagonalize them by an $N \times N$ dimensional orthonormal matrix $R = (R_{ab})$, $R \cdot R^T = I$, for which

$$
\phi^d_a = \sum_b R_{ab} \phi_b = \phi \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad t \Delta^d_{ab} = \sum_{ce} R_{ac} t \Delta_{ce} R_{be} = \begin{pmatrix} t \Delta^d_{11} & 0 & 0 & \cdots & 0 \\ 0 & t \Delta^d_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & t \Delta^d_{NN} \end{pmatrix}.
$$

(30)

Because of the unbroken $O(N - 1)$ symmetry, $t \Delta^d_{ii}$ are all equal for $2 \leq i \leq N$. The diagonal form of Eqs. (28, 29) is:

$$
\left[ m^2 + \frac{\lambda}{N} (\phi^2 + 3t \Delta^d_{11}(x; x)_{\text{fin}} + (N-1)t \Delta^d_{22}(x; x)_{\text{fin}}) \right] \phi^d_i(x) = 0.
$$

(31)

($\phi^d_i = 0$ for $2 \leq i \leq N$) and

$$
M^2_1 \equiv (M^d_{11})^2 = m^2 + \frac{\lambda}{N} [3\phi^2 + 3t \Delta^d_{11}(x; x)_{\text{fin}} + (N-1)t \Delta^d_{22}(x; x)_{\text{fin}}],
$$

$$
M^2_g \equiv (M^d_{ii})^2 = m^2 + \frac{\lambda}{N} [\phi^2 + t \Delta^d_{11}(x; x)_{\text{fin}} + (N+1)t \Delta^d_{22}(x; x)_{\text{fin}}], \quad (2 \leq i \leq N).
$$

(32)

Eq. (31) implies that the $O(N)$ symmetry is broken when

$$
\phi^2 = (\phi^d)^2 = -\frac{N m^2}{\lambda} - 3t \Delta^d_{11}(x; x)_{\text{fin}} - (N - 1)t \Delta^d_{22}(x; x)_{\text{fin}} > 0.
$$

(33)

Otherwise, $\phi^d_i = 0$ and the symmetry is unbroken. When this is inserted into (32), we get that the mass terms (in the broken phase) become,

$$
M^2_1 \equiv (M^d_{11})^2 = \frac{2\lambda}{N} \phi^2 = -2m^2 - \frac{2\lambda}{N} [3t \Delta^d_{11}(x; x)_{\text{fin}} + (N-1)t \Delta^d_{22}(x; x)_{\text{fin}}],
$$

$$
M^2_g \equiv (M^d_{ii})^2 = \frac{2\lambda}{N} [t \Delta^d_{ii}(x; x)_{\text{fin}} - t \Delta^d_{11}(x; x)_{\text{fin}}], \quad (2 \leq i \leq N).
$$

(34)
In the special case when $N = 1$ the first equation agrees with Eq. (12) (provided, of course, one rescales the $\lambda$ in (31) as $\lambda \rightarrow \lambda/6$).

With this, the renormalised and diagonalised form of Eq. (25) becomes,
\[ \Box x - M_{1i}^2 \, i\Delta_{ii}(x; x') = \frac{i\delta^D(x-x')}{\sqrt{-g}}, \quad \Box x - M_{gi}^2 \, i\Delta_{ii}(x; x') = \frac{i\delta^D(x-x')}{\sqrt{-g}} \quad (2 \leq i \leq N). \] (35)

The implied stability of de Sitter space then demands that both $(M_{1i}^2)^2 = M_{1i}^2 > 0$ ($N \geq i \geq 2$) and $M_{gi}^2 > 0$. Next we insert the coincident propagator (59) into (34) to obtain:
\[ M_{1i}^2 = -2m^2 - \frac{\lambda H^D \Gamma\left(\frac{D+1}{2}\right)}{N\pi^{(D+1)/2}} \left[ N - 1 \frac{M_{2g}^2 - 1}{M_{2g}^2} + 3 \right] \]
\[ M_{gi}^2 = \frac{\lambda H^D \Gamma\left(\frac{D+1}{2}\right)}{N\pi^{(D+1)/2}} \left[ 1 \frac{M_{2g}^2 - 1}{M_{2g}^2} \right]. \] (36)

Notice that positivity of $M_{gi}^2$ implies that $M_{1i}^2 > M_{gi}^2 > 0$. The fact that the Goldstone bosons become massive on de Sitter background is reminiscent of the BEH mechanism, in which the Goldstones are ‘eaten up’ by the longitudinal degrees of freedom of the $W^\pm$ and $Z$ bosons, thus becoming massive. Moreover, massive pseudo-Goldstones imply a complete breaking of the original $O(N)$ symmetry of the model.

Equations (36) are the main result of this work. In order to analyse them, it is convenient to work with the following dimensionless quantities,
\[ \mu_1^2 = \frac{M_{1i}^2}{(-2m^2)}, \quad \mu_2^2 = \frac{M_{gi}^2}{(-2m^2)}, \quad \lambda_D = \frac{\lambda H^D \Gamma\left(\frac{D+1}{2}\right)}{N\pi^{(D+1)/2}(-2m^2)^2}, \] (37)

after which Eqs. (36) become
\[ \mu_1^2 = 1 - \lambda_D \left( \frac{N-1}{\mu_2^2} + 3 \right), \quad \mu_2^2 = \lambda_D \left( \frac{1}{\mu_2^2} - 1 \right). \] (38)

Before we perform a general analysis of these equations, notice that in the case when $N = 1$, the second equation decouples, and one gets
\[ \mu_1^4 - \mu_1^2 + 3\lambda_D = 0, \] (39)
whose (physical) root is
\[ \mu_1^2 = \frac{1}{2} + \sqrt{\frac{1}{4} - 3\lambda_D}. \] (40)

The minimum critical mass is then determined by $\lambda_D < (\lambda_D)_{cr} = 1/12$ ($1 \geq \mu_1^2 \geq (\mu_1^2)_{cr} = 1/2$), which accords with Eq. (15) (when one takes account of the different definition of $\lambda$ in the Lagrangians (1) and (21)).
In the general case the gap equations (38) admit a small coupling expansion. Similarly as in the thermal case, the expansion parameter is $\sqrt{\lambda_D}$, and hence non-perturbative,

\[
\begin{align*}
\mu_1^2 &= 1 - (N-1)\sqrt{\lambda_D} - \frac{N+5}{2} \lambda_D - \frac{N-1)(4N-21)}{8} \lambda_D^{3/2} + O(\lambda_D^2) \\
\mu_g^2 &= \sqrt{\lambda_D} - \frac{1}{2} \lambda_D - \frac{4N-5}{8} \lambda_D^{3/2} + O(\lambda_D^2).
\end{align*}
\]

(41)

In figure 1 we show the rescaled masses $\mu_1^2$ and $\mu_g^2$ as a function of $\lambda_D$. Both, the small coupling series solutions (41) (dashed) as well as the full solutions of (38) (solid) are shown. We have plotted the heavy (Higgs) scalar mass $\mu_1^2 = M_1^2/(-2m^2)$ (the upper curves starting at one when $\lambda_D = 0$) and the pseudo-Goldstone mass $\mu_g^2 = M_g^2/(-2m^2)$ (the lower curves starting at zero when $\lambda_D = 0$) for $N = 1$ (the most extended green curves), $N = 2$ (the intermediate red curves), $N = 4$ (the intermediate blue curves) and $N = 10$ (the most squeezed gray curves). While the approximate solutions (41) continue for a while longer, the exact solutions end suddenly at a critical point, at which a minimum

![FIG. 1: The rescaled masses $\mu_1^2 = M_1^2/(-2m^2)$ (upper curves) and $\mu_g^2 = M_g^2/(-2m^2)$ (lower curves) as a function of the rescaled (dimensionless) coupling $\lambda_D$ defined in (37). Both the exact solutions (solid) of Eqs. (38) and the approximate solutions (41) (dashed) are shown. When viewed from the right to left, we have plotted the cases: $N = 1$ (green), $N = 2$ (red), $N = 4$ (blue) and $N = 10$ (gray).](image-url)
(critical) mass is reached. Just like for the case when \( N = 1 \), where the mass parameter decreases monotonically from \( \mu_1^2 = 1 \) (at \( \lambda_D = 0 \)) to \((\mu_1^2)_{cr} = 1/2 \) (at \( \lambda_D = (\lambda_D)_{cr} = 1/12 \)), see Eq. (40), for \( N > 1 \), \( \mu_1^2 \) evolves monotonically from \( \mu_1^2 = 1 \) (at \( \lambda_D = 0 \)) to some \((\mu_1^2)_{cr} > 0 \), the end point being a function of \( N \).

\[
\frac{(M_{1,s}^2)_{cr}}{-2m^2}
\]

\[
\frac{(M_{1,g}^2)_{cr}}{-2m^2}
\]

\[
N \quad 0.00 \quad 0.02 \quad 0.04 \quad 0.06 \quad 0.08 \quad (\lambda_D)_{cr}
\]

\[
0.0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1.0
\]

FIG. 2: The rescaled critical (minimum) masses for the Higgs-like excitation \((\mu_1^2)_{cr} = (M_{1,s}^2)_{cr} / (-2m^2) \) (the upper solid blue curve) and for the pseudo-Goldstones \((\mu_g^2)_{cr} = (M_{1,g}^2)_{cr} / (-2m^2) \) (the lower dashed red curve) as a function of \( N \) (left panel) and as a function of the rescaled critical coupling \((\lambda_D)_{cr} \) (defined as the maximum allowed \( \lambda_D \) for a given \( N \)) (right panel). Individual points where \( N \) is an integer are shown as gray dots.

As we have seen in the analysis of the one scalar field case, this end point plays an important role as it tells us how the system behaves at the critical point (where the transition takes place). In order to study the critical behaviour in some detail, in figure 2 we plot the critical mass parameters \((\mu_1^2)_{cr} \) (upper solid blue curve) and \((\mu_g^2)_{cr} \) (lower dashed red curve) as a function of \( N \) (left panel) and as a function of \((\lambda_D)_{cr} = \max[\lambda_D] \) for a fixed \( N \) (right panel) (corresponding to the end points on figure 1). One can show that, to a good approximation, \((\lambda_D)_{cr} \simeq 3/[2(N + 2)^2] \) (more precisely \((\lambda_D)_{cr} \simeq 1.62/[N+2]^{1.92} \), such that \((\lambda_D)_{cr} \) approaches approximately quadratically zero as \( N \to \infty \). A manifestation of this is the fractional power behaviour of \( \mu_{1,g}^2 \) close to the origin for small \((\lambda_D)_{cr} \) (large \( N \)) seen on the right panel in figure 2.

Just as in the one field case, the transition in the \( O(N) \) model is such that, imposing de Sitter symmetry, implies a jump in the order parameter at the transition. Indeed, from Eq. (31) we see
that
\[ \Delta \phi^2 = \frac{N(M_1^2)}{2\lambda} = \frac{N(-m^2)(\mu_1^2)}{\lambda} > 0, \tag{42} \]
with \((\mu_1^2)_{cr}\) plotted in figure\[2\]. In fact, from the left and right panels on figure\[2\] one can read off that \((\mu_1^2)_{cr} \approx 2/[3\sqrt{N}]\) and \((\mu_1^2)_{cr} \propto (\lambda_D)^{1/4} \propto (\lambda/N^2)^{1/4}|m^2|^{-1/2}\), implying that \(\Delta \phi^2 \propto |m^2|N^{1/2}/\lambda^{3/4}\), which is to be compared with the single field result \((20)\), where we found that \((\Delta \phi^2)_{N=1} \propto \lambda^{-1/2}\).

Hence, in the large \(N\) limit the strength of the transition \((42)\) exhibits a qualitatively different dependence on \(\lambda\) and \(m^2\) than in the single field case \((20)\). Ref. \[24\] considered the analogous problem in the large \(N\) limit and found that no jump in the order parameter is possible. The results can be related to ours by noticing that the masses we found scale as \(M_{1,g}^2 \propto 1/\sqrt{N}\) (cf. figure\[2\]) and thus vanish as \(N \to \infty\), and hence the (rescaled) order parameter \((42)\), \(\Delta \phi^2/N \propto 1/\sqrt{N}\) also vanishes as \(N \to \infty\).

For completeness, we shall now briefly analyse the unbroken symmetry case. In this case \(\phi_i^d = 0\) \((i = 1, 2, ..., N)\) and Eq. \((31)\) implies,
\[ m^2 + \frac{\lambda}{N} \left(3i\Delta_{11}^d(x;x)_{fin} + (N-1)i\Delta_{22}^d(x;x)_{fin}\right) > 0 \tag{43} \]
and analogous steps as above yield,
\[ \mu_1^2 = -\frac{1}{2} + \frac{\lambda_D}{2} \left(\frac{N-1}{\mu_g^2} + \frac{3}{\mu_1^2}\right), \quad \mu_g^2 = \mu_1^2 + \lambda_D \left(\frac{1}{\mu_g^2} - \frac{1}{\mu_1^2}\right). \tag{44} \]

Just as in the broken case, when \(N = 1\) we have,
\[ \mu_1^2 = -\frac{1}{4} - \sqrt{\frac{1}{16} + \frac{3\lambda_D}{2}}, \tag{45} \]
which agrees with Eq. \((18)\). In fact, it is quite easy to obtain the general solution of equations \((44)\). Indeed, observe that the second equation can be written as,
\[ (\mu_1^2 - \mu_g^2)(\lambda_D + \mu_1^2\mu_g^2) = 0. \tag{46} \]
The positivity of \(\lambda_D\), \(\mu_1^2\) and \(\mu_g^2\) immediately implies that the only consistent solution is
\[ \mu_g^2 = \mu_1^2. \tag{47} \]
It is not surprising that in this case all particles must have the same mass since the symmetry is unbroken. With this, the first equation in \((44)\) is easily solved,
\[ \mu_1^2 = -\frac{1}{4} - \sqrt{\frac{1}{16} + \frac{(N+2)\lambda_D}{2}}, \tag{48} \]
which can be also written as,

\[ M_1^2 = M_g^2 = \frac{m^2}{2} + \sqrt{\frac{m^4}{4} + \frac{(N+2)m^4_{cr}}{6}}, \]  

(49)

where \( m_{cr} \) is given in (15). This generalizes the real field result (18) to the \( O(N) \) symmetric case. From Eq. (49) one can easily see that \( M_1^2 = M_g^2 > 0 \), as it should be. The solution (49) can be expanded in powers of \( \lambda \). When \( m^2 \gg \sqrt{Nm^2_{cr}} \) the expansion is analytic in \( \lambda \), \( M_1^2 = M_g^2 \simeq m^2 + (N+2)m^4_{cr}/(3m^2) = m^2 + (N+2)\lambda H^4/(8\pi^2 m^2) \), while in the limit when \( m \to 0 \), the expansion is non-analytic in \( \lambda \), \( M_1^2 = M_g^2 \simeq \sqrt{[(N+2)/6]m^2_{cr}} = \sqrt{[(N+2)\lambda] H^2/(4\pi)} \), where in the latter equalities for simplicity we took \( D = 4 \).

An important question is what does the jump in the order parameter (42) imply for the nature of the transition. We can answer this question by considering the analogous two inflationary scenarios from the end of section II. Just like in the real scalar field case, in Scenario A, in which inflation begins from a false vacuum state, the transition is a crossover and proceeds via temporary breaking of de Sitter symmetry, until a de Sitter invariant state (over finite, but very large domains) gets established. In Scenario B, in which inflation is preceded by a radiation era, the transition can be of first order (if \( |m^2| \gg H^2 \)), bubbles of the broken phase nucleate and the field tunnels to the broken phase minimum. During the transition topological defects in general form (global cosmic strings when \( N = 2 \), global monopoles when \( N = 3 \) and higher order defects such as cosmic texture when \( N \geq 3 \)).

IV. DISCUSSION

We have analysed the \( O(N) \) symmetric scalar field model (21) in the mean field (Hartree) approximation (22) on de Sitter space. We have shown that symmetry breaking can occur, and that the would-be Goldstone bosons acquire a mass (see figure I) due to the enhanced infrared correlations in de Sitter space, and that the \( O(N) \) symmetry gets completely broken by the ground state of the theory. Next we have studied the strength of the transition and shown that, depending on the inflationary scenario assumed, the transition can proceed either as a crossover or as a first order phase transition. Curiously, the jump in the order parameter (12) exhibits a non-analytic dependence on the parameters of the model, \( \Delta \phi^2 \propto |m|N^{1/4} \lambda^{-3/4} \), where \( |m| \) and \( \lambda \) denote the mass parameter and the quartic self-coupling of the model.

While the mean field results are of their own interest, it would be desirable to investigate whether (and how) the mean field results presented here change when one includes higher loop corrections.
A first step in this direction is taken in Ref. [25] where the local contribution to the self-mass from the two loop (sun-set) diagram was estimated, and where it was found that, in the massless limit, the mean field mass-squared gets reduced by a factor $1/\sqrt{2}$.

Second, it is instructive to compare our results with the (old) stochastic theory results of Starobinsky and Yokoyama [34], which is known to resum the leading $\log(a)$ corrections to infrared correlators on de Sitter space, see e.g. [9]. From Eq. (23) of Ref. [34] we read (upon a rescaling, $\lambda \to \lambda/6$),

$$m_{\text{stoch}}^2 = \frac{\lambda}{2} \langle \phi^2 \rangle = \frac{3 \Gamma(3/4)}{2\pi \Gamma(1/4)} \sqrt{\lambda H^2} \approx 0.1614 \sqrt{\lambda H^2},$$

which is to be compared with Eqs. (18) and (15), which in the limit when $m^2 \to 0$ and in $D = 4$ yield $m_{\text{MF}}^2 \to \sqrt{3\lambda H^2}/(4\pi)$. This then implies,

$$\frac{m_{\text{stoch}}^2}{m_{\text{MF}}^2} = 2\sqrt{3} \frac{\Gamma(3/4)}{\Gamma(1/4)} \approx 1.17.$$ (51)

Even though the difference in the results is modest, the question – which result is correct? – is, nevertheless, important. In the derivation of the stochastic result (50), one assumes that the tree level potential remains unchanged, i.e. that for the late time behaviour the tree level potential should be used when stochastic theory is applied to inflation. At the moment there is no fundamental understanding concerning whether the tree level potential or some effective potential should be used in stochastic formalism. We close this discussion by noting that one can recover exactly the mean field result $(m_{\text{MF}}^2)_{m\to 0}$ from stochastic formalism, provided one replaces the tree level potential $V = (\lambda/4!)\phi^4$ by its Gaussian counterpart, $V \to V_{\text{Gauss}} = (\lambda/4)\langle (\phi)^2 \rangle \phi^2$. While this is suggestive, it does not ultimately tell us what is the correct answer.

Furthermore, it is useful to mention the well understood thermal case, where also non-analytic behaviour in the coupling constant occurs when a self-consistent Hartree approximation (daisy resummation) is employed in the model considered in this paper. Up to a logarithmic correction, in the symmetric case ($m^2 > 0$) the resummed mass of a real scalar field of section II is of the form,

$$m_{\text{MF}}(T) = \sqrt{m^2 + \frac{\lambda T^2}{24} + \frac{\lambda^2 T^2}{(16\pi)^2}} - \frac{\lambda T}{16\pi},$$ (52)

which, when $m^2 \to 0$ yields, $m_{\text{MF}}^2(T) \simeq \frac{\lambda T^2}{24} \left(1 - \frac{\sqrt{3\lambda}}{2\sqrt{2\pi}}\right)$. The crucial difference with the de Sitter result (19) is that the thermal series for $m_{\text{MF}}^2$ begins at $\sim \lambda T^2$, and not at $\sim \sqrt{\lambda H^2}$ as it is the case in the de Sitter case. This is because the infrared sector of de Sitter space is more infrared divergent than the thermal infrared sector of bosonic field theory.
Finally, after the original version of this work was completed, an interesting paper by Boyanovsky appeared \[37\], which used somewhat different techniques and confirmed the main results of this work, namely the symmetry breaking in an $O(N)$ scalar model and the mass generation mechanism of (would-be) Goldstone bosons.

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Appendix: The de Sitter space propagator for massive and massless scalar fields

Here we review some of the basic properties of the scalar propagator on de Sitter background in $D$ space time dimensions. The time ordered (Feynman) propagator $i\Delta \equiv i\Delta^{++}$ obeys the equation,

$$\sqrt{-g} \left[ \Box_D - m^2 \right] i\Delta^{++}(x; x') = i\delta^D(x - x'),$$  \(53\)

where $m$ is a mass and $\Box_x = (-g)^{-1/2} \partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu$ is the scalar d’Alembertian in D spacetime dimensions. The de Sitter invariance allows one to write the d’Alembertian in a de Sitter invariant form,

$$\left[ (\bar{\eta} - \bar{\eta}')(\eta - \eta')^2 + \| \vec{x} - \vec{x}' \|^2 \right] i\Delta(x; x') = \frac{i}{\sqrt{-g} H^2} \delta^D(x - x'),$$  \(54\)

where $\bar{\eta} = a(\eta')a(\eta) H^2[ - (\eta - \eta')^2 + \| \vec{x} - \vec{x}' \|^2 ]$ is related to the geodesic distance on de Sitter space $\ell(x; x')$ as, $\bar{\eta} = 4 \sin^2(H \ell/2)$. Here $a$ denotes the scale factor, $\eta$ is conformal time and $\vec{x}$ comoving coordinate. The unique solutions for the relevant propagators of the Schwinger-Keldysh (or in-in) formalism can be written in terms of the Gauss’ hypergeometric function $\, _2F_1$ as follows,

$$i\Delta^{\alpha\beta}(x; x') = \frac{H^{D-2} \Gamma(D-1) \Gamma(D-1 - \nu_D) \Gamma(D-1 + \nu_D)}{(4\pi)^{D/2} \Gamma(D/2)} \times _2F_1 \left( \frac{D-1}{2} + \nu_D, \frac{D-1}{2} - \nu_D; \frac{D}{2}; 1 - \frac{\eta^{\alpha\beta}}{4} \right),$$  \(55\)

where

$$\nu_D = \left( \frac{D-1}{2} \right)^2 - \frac{m^2}{H^2}. $$  \(56\)

Here $m^2 > 0$ represents the (renormalised) field mass parameter, which includes the renormalised mass, the mean field correction (the finite part of $(\lambda/2)i\Delta(x; x)$) and possibly also the term that
originates from a nonminimal coupling, \( \Delta m^2 = \xi D (D - 1) H^2 \) in the lagrangian, \( \Delta \mathcal{L} = -\xi R \phi^2 \), where \( R = D (D - 1) H^2 \) is the Ricci scalar in de Sitter space. The functions \( y^{\alpha \beta} (\alpha, \beta = \pm) \) in Eq. (55) denote,

\[
\begin{align*}
    y^{++} &= a(\eta)a(\eta') H^2 [-(|\eta - \eta'| + i \epsilon)^2 + \| \vec{x} - \vec{x}' \|^2] \\
    y^{+-} &= a(\eta)a(\eta') H^2 [-(|\eta - \eta'| - i \epsilon)^2 + \| \vec{x} - \vec{x}' \|^2] \\
    y^{-+} &= a(\eta)a(\eta') H^2 [-(|\eta - \eta'| - i \epsilon)^2 + \| \vec{x} - \vec{x}' \|^2] \\
    y^{--} &= a(\eta)a(\eta') H^2 [-(|\eta - \eta'| + i \epsilon)^2 + \| \vec{x} - \vec{x}' \|^2],
\end{align*}
\]

with \( \epsilon > 0 \) infinitesimal. All propagators in (55) have the same coincident limit,

\[
\iota \Delta(x; x) = \iota \Delta^{\alpha \beta}(x; x) = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D-1}{2} + \nu_d \right) \Gamma \left( \frac{D-1}{2} - \nu_d \right)}{\Gamma \left( \frac{1}{2} + \nu_d \right) \Gamma \left( \frac{1}{2} - \nu_d \right)} \Gamma \left( 1 - \frac{D}{2} \right).
\]

Due to the last \( \Gamma \) function, this propagator exhibits a simple pole in even dimensions, \( D = 2, 4, 6, \ldots \), which reflects an ultraviolet (UV) logarithmic divergence. Of course, the leading UV divergence of the coincident propagator in de Sitter space is the same as that in Minkowski space, and it is of a degree \( D - 2 \), the subleading is of a degree \( D - 4 \), etc., the degree zero representing a logarithmic divergence. As it is well known, dimensional regularisation is blind to power law divergences (they are automatically subtracted by analytic extension), and exhibits only logarithmic divergences. The effect of the propagator (58) can be considered in the weak curvature (Minkowski) limit, when \( m^2 \gg H^2 \) (in which case one recovers the Minkowski space result plus small corrections) and in a strong curvature regime, in which \( m^2 \ll H^2 \). Ignoring the UV divergence in (58) one can naively expand it in powers of \( m^2/H^2 \), and one obtains [47],

\[
\begin{align*}
\iota \Delta(x; x) &= \frac{H^{D-2}}{4\pi^{D-1/2}} \left[ \psi \left( \frac{D}{2} \right) - \psi(D-1) - \psi \left( 1 - \frac{D}{2} \right) - \gamma_E + \frac{1}{D-1} \right] \\
&\quad + \frac{\Gamma \left( \frac{D+1}{2} \right) H^D}{2\pi^{D+1/2} m^2} + \mathcal{O} \left( \frac{m^2}{H^2} \right),
\end{align*}
\]

such that in \( D = 2, 3, 4 \) the \( \mathcal{O}(m^{-2}) \) terms are \( H^2/(4\pi^2 m^2) \), \( H^3/(2\pi^2 m^2) \), and \( 3H^4/(8\pi^2 m^2) \), respectively. In our analysis in the main text we assume that both finite and infinite \( m \)-independent terms in (59) are absorbed in the physical definition of the mass term.

The de Sitter invariant limit will be attained after some time during inflation. If the mass is very small \( m^2 \ll H^2 \), the propagator will at early times grow logarithmically with the scale factor (linearly with cosmological time). This can be seen by recalling that in the infrared [11] the
coincident propagator satisfies,

\[
\imath \Delta(x; x) = \frac{H^{D-2}}{2D \pi \Gamma(D-3/2) \Gamma(D-1)} \int_{k_0/(Ha)}^{\infty} dz z^{D-2} |H^{(1)}_{\nu D}(z)|^2 \\
= \frac{H^{D-2}}{(4\pi)^D/2} \frac{\Gamma\left(\frac{D-1}{2} + \nu_D\right) \Gamma\left(\frac{D-1}{2} - \nu_D\right)}{\Gamma\left(\frac{1}{2} + \nu_D\right) \Gamma\left(\frac{1}{2} - \nu_D\right)} \Gamma\left(1 - \frac{D}{2}\right) - \frac{H^{D-2} \Gamma(\nu)^2}{8\pi (D-3/2) \Gamma\left(\frac{D-1}{2}\right)} \times \frac{\left(k_0 \right)^{D-1-2\nu_D}}{D - 1 - 2\nu_D} \\
+ \mathcal{O}\left(k_0^{D+1-2\nu_D}, k_0^{D-1}, k_0^{D-1+2\nu_D}\right),
\]

(60)

where we took \(a(t_0) = a_0 = 1\) and \(k_0\) is an infrared (comoving) momentum cut-off. Notice that at early times (and in the limit when \(m \rightarrow 0\) and \(\nu_D \rightarrow (D - 1)/2\)) the coincident propagator grows logarithmically with time as (see also Refs. [35, 36])

\[
\imath \Delta(x; x) = \frac{H^{D-2} \Gamma\left(\frac{D-1}{2}\right)}{4\pi (D-1)^{1/2}} \left[\psi\left(\frac{D}{2}\right) - \psi(D-1) - \psi\left(1 - \frac{D}{2}\right) - \gamma_E + \frac{1}{D-1}\right] \\
+ \frac{H^{D-2} \Gamma\left(\frac{D-1}{2}\right)}{2\pi (D+1)^{1/2}} \left[\ln(a) - \ln\left(\frac{k_0}{2H}\right)\right] + \mathcal{O}(m^2/H^2).
\]

(61)

This is to be compared with the Onemli-Woodard coincident propagator for a massless scalar field [6, 48]:

\[
[\imath \Delta(x; x)]_{OW} = \frac{H^{D-2}}{4\pi (D-1)^{1/2}} \Gamma\left(\frac{D-1}{2}\right) \Gamma\left(1 - \frac{D}{2}\right) \frac{H^{D-2} \Gamma\left(\frac{D-1}{2}\right)}{2\pi (D+1)^{1/2}} \ln(a).
\]

(62)

The logarithmic growth saturates when the propagator reaches the de Sitter invariant value \([59]\), which characterizes the time scale at which the propagator (and thereby the state) becomes de Sitter invariant.

[1] D. A. Kirzhnits, A. D. Linde, “Macroscopic Consequences of the Weinberg Model,” Phys. Lett. B42 (1972) 471-474.
[2] N. C. Tsamis and R. P. Woodard, “The quantum gravitational back-reaction on inflation,” Annals Phys. 253 (1997) 1 [arXiv:hep-ph/9602316].
[3] N. C. Tsamis and R. P. Woodard, “Quantum Gravity Slows Inflation,” Nucl. Phys. B 474 (1996) 235 [arXiv:hep-ph/9602315].
[4] V. F. Mukhanov, L. R. W. Abrao and R. H. Brandenberger, “On the back reaction problem for gravitational perturbations,” Phys. Rev. Lett. 78 (1997) 1624 [arXiv:gr-qc/9609026].
[5] V. K. Onemli, R. P. Woodard, “Superacceleration from massless, minimally coupled phi**4,” Class. Quant. Grav. 19 (2002) 4607. [gr-qc/0204065].
[6] V. K. Onemli, R. P. Woodard, “Quantum effects can render \( w < -1 \) on cosmological scales,” Phys. Rev. D70 (2004) 107301. [gr-qc/0406098].

[7] T. Brunier, V. K. Onemli and R. P. Woodard, “Two loop scalar self-mass during inflation,” Class. Quant. Grav. 22, 59 (2005) [arXiv:gr-qc/0408080].

[8] T. Prokopec, N. C. Tsamis, R. P. Woodard, “Two Loop Scalar Bilinears for Inflationary SQED,” Class. Quant. Grav. 24 (2007) 201-230. [gr-qc/0607094].

[9] T. Prokopec, N. C. Tsamis, R. P. Woodard, “Stochastic Inflationary Scalar Electrodynamics,” Annals Phys. 323 (2008) 1324-1360. [arXiv:0707.0847 [gr-qc]].

[10] T. Prokopec, N. C. Tsamis, R. P. Woodard, “Two loop stress-energy tensor for inflationary scalar electrodynamics,” Phys. Rev. D78 (2008) 043523. [arXiv:0802.3673 [gr-qc]].

[11] T. M. Janssen, S. P. Miao, T. Prokopec and R. P. Woodard, “Infrared Propagator Corrections for Constant Deceleration,” Class. Quant. Grav. 25 (2008) 245013 [arXiv:0808.2449 [gr-qc]].

[12] T. Janssen and T. Prokopec, “The Graviton one-loop effective action in cosmological space-times with constant deceleration,” Annals Phys. 325 (2010) 948 [arXiv:0807.0447 [gr-qc]].

[13] T. Prokopec, O. Tornkvist, R. P. Woodard, “Photon mass from inflation,” Phys. Rev. Lett. 89 (2002) 101301. [astro-ph/0205331].

[14] T. Prokopec, O. Tornkvist, R. P. Woodard, “One loop vacuum polarization in a locally de Sitter background,” Annals Phys. 303 (2003) 251-274. [gr-qc/0205130].

[15] T. Prokopec, R. P. Woodard, “Dynamics of superhorizon photons during inflation with vacuum polarization,” Annals Phys. 312 (2004) 1-16. [gr-qc/0310056].

[16] E. O. Kahya, V. K. Onemli, R. P. Woodard, “A Completely Regular Quantum Stress Tensor with \( w < -1 \),” Phys. Rev. D81 (2010) 023508. [arXiv:0904.4811 [gr-qc]].

[17] T. M. Janssen, S. P. Miao, T. Prokopec and R. P. Woodard, “The Hubble Effective Potential,” JCAP 0905 (2009) 003 [arXiv:0904.1151 [gr-qc]].

[18] A. Vilenkin, L. H. Ford, “Gravitational Effects upon Cosmological Phase Transitions,” Phys. Rev. D26 (1982) 1231.

[19] L. H. Ford, A. Vilenkin, “Global Symmetry Breaking In Two-dimensional Flat Space-time And In De Sitter Space-time,” Phys. Rev. D33 (1986) 2833.

[20] B. Ratra, “Restoration of Spontaneously Broken Continuous Symmetries in de Sitter Space-Time,” Phys. Rev. D31 (1985) 1931-1955.

[21] A. Riotto, M. S. Sloth, “On Resumming Inflationary Perturbations beyond One-loop,” JCAP 0804.
[22] C. P. Burgess, L. Leblond, R. Holman, S. Shandera, “Super-Hubble de Sitter Fluctuations and the Dynamical RG,” JCAP 1003 (2010) 033. [arXiv:0912.1608 [hep-th]].

[23] C. P. Burgess, R. Holman, L. Leblond, S. Shandera, “Breakdown of Semiclassical Methods in de Sitter Space,” JCAP 1010 (2010) 017. [arXiv:1005.3551 [hep-th]].

[24] J. Serreau, “Effective potential for quantum scalar fields on de Sitter,” [arXiv:1105.4539 [hep-th]].

[25] B. Garbrecht, G. Rigopoulos, “Self Regulation of Infrared Correlations for Massless Scalar Fields during Inflation,” [arXiv:1105.0418 [hep-th]].

[26] U. Reinosa and Z. Szep, “Broken phase scalar effective potential and Phi-derivable approximations,” Phys. Rev. D 83 (2011) 125026 [arXiv:1103.2689 [hep-ph]].

[27] U. Reinosa and Z. Szep, “A critical look at the role of the bare parameters in the renormalization of Phi-derivable approximations,” [arXiv:1109.1232 [hep-th]].

[28] A. Vilenkin, “The Birth of Inflationary Universes,” Phys. Rev. D 27 (1983) 2848.

[29] A. D. Linde, “Hard art of the universe creation (stochastic approach to tunneling and baby universe formation),” Nucl. Phys. B 372 (1992) 421 [hep-th/9110037].

[30] R. Basu, A. H. Guth and A. Vilenkin, “Quantum creation of topological defects during inflation,” Phys. Rev. D 44 (1991) 340.

[31] R. Basu and A. Vilenkin, “Nucleation of thick topological defects during inflation,” Phys. Rev. D 46 (1992) 2345.

[32] R. Basu and A. Vilenkin, “Evolution of topological defects during inflation,” Phys. Rev. D 50 (1994) 7150 [gr-qc/9402040].

[33] T. W. B. Kibble, “Some Implications of a Cosmological Phase Transition,” Phys. Rept. 67 (1980) 183.

[34] A. A. Starobinsky, J. Yokoyama, “Equilibrium state of a self-interacting scalar field in the De Sitter background,” Phys. Rev. D50 (1994) 6357-6368. [astro-ph/9407016].

[35] T. Prokopec, E. Puchwein, “Photon mass generation during inflation: de Sitter invariant case,” JCAP 0404 (2004) 007. [astro-ph/0312274].

[36] T. Prokopec, E. Puchwein, “Nearly minimal magnetogenesis,” Phys. Rev. D70 (2004) 043004. [astro-ph/0403335].

[37] D. Boyanovsky, “Spontaneous symmetry breaking in inflationary cosmology: on the fate of Goldstone Bosons,” Phys. Rev. D 86 (2012) 023509 [arXiv:1205.3761 [astro-ph.CO]].
[38] C. Pathinayake and L. H. Ford, “Growth Of Scalar Field Quantum Fluctuations In Robertson - Walker Universes,” Phys. Rev. D 37 (1988) 2099.

[39] As early as 1986 Ford and Vilenkin [19] correctly reasoned: ”[.] it would be completely consistent with all observations for our present Universe to be a de Sitter space with $H^{-1} \sim 10^{10}$ yr. It would be very surprising if this were to have any effect upon symmetry breaking on terrestrial or subatomic scales.” While this observation is correct, it does not follow from their analysis.

[40] To renormalise the mean field effective potential (2), one also needs to renormalise $V_0$. Since $V_0$ does not affect our analysis, we shall not renormalise $V_0$ here.

[41] A simple algebra shows that both mean field masses-squared $m_{\text{MF}}^2$ in (14) are positive, and moreover both are consistent with broken symmetry $\phi^2 > 0$, see (11), which is to be contrasted to the conclusion in [24].

[42] The argument goes as follows. In stochastic inflation, the probability that the field has a value that is homogeneous over a Hubble volume is, $P \sim \exp\left(-\phi^2/(2\langle \delta \phi^2 \rangle)\right)$. During the transition the value of $\langle \delta \phi^2 \rangle$ will grow according to (61) until it reaches its de Sitter invariant value, $\langle \delta \phi^2 \rangle = 3H^4/(8\pi^2 m_{\text{MF}}^2)$. A typical value of the field condensate that is unsuppressed will then be $\phi^2 \sim 3H^4/(4\pi^2 m_{\text{MF}}^2)$, which is $2/3$-rds of $\Delta \phi^2$ in (20) at the transition.

[43] If initial conditions are $O(N)$ symmetric, the phase transition will be of some higher (than first) order. If initial conditions are such to respect $O(N - 1)$ (but not $O(N)$ symmetry), then the transition will be a crossover.

[44] The semiclassical treatment predics the following probability for nucleation of Hubble-size domains, $P \sim \exp\left(-\frac{2\pi^2 m_{\text{MF}}^2}{3\lambda H^4}\right) \approx \exp\left(-\frac{8\pi^2 |m|^2}{3\lambda H^4}\right) \ll 1$ [29], where to get the last equality we used $m_{\text{MF}}^2 \simeq 2|m|^2$. Since $|m|^2 > 4H^2$, the expression in the exponent is much greater than one, signaling applicability of the semiclassical approximation.

[45] Recall that, in the limit when $\lambda_D \rightarrow 0$, the physical branch yields $\mu_1^2 = M_1^2/(-2m^2) = 1$.

[46] Formally, one can also solve Eq. (38) for $\mu_2^2$ in the $N = 1$ case, and one finds for the critical value, $\langle \mu_2^2 \rangle_{\text{cr}} = (\sqrt{13} - 1)/12 \simeq 0.217$, which agrees with the results plotted in figures 1 and 2.

[47] The $O(m^2)$ term in Eq. (59) has a divergent coefficient. To make the renormalised gap equation consistent, the divergent part of the $O(m^2)$ term would have to be absorbed in the renormalised mass term $m^2$.

[48] Not surprisingly, the coincident propagator of a light scalar field (60) and the Onemli-Woodard coincident massless scalar propagator (61) possess identical late time logarithmically growing terms $\propto \ln(a)$.
The time-independent parts do not agree, however. But this was to be expected, since these constant pieces do not have an independent physical meaning, as they can be absorbed in the mass counterterm of the self-interacting scalar theory.