On the unique solution of the generalized absolute value equation

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Abstract

In this paper, some useful necessary and sufficient conditions for the unique solution of the generalized absolute value equation (GAVE) $Ax - B|x| = b$ with $A, B \in \mathbb{R}^{n \times n}$ from the optimization field are first presented, which cover the fundamental theorem for the unique solution of the linear system $Ax = b$ with $A \in \mathbb{R}^{n \times n}$. Not only that, some new sufficient conditions for the unique solution of the GAVE are obtained, which are weaker than the previous published works.

Keywords Generalized absolute value equation · Unique solution · Necessary and Sufficient condition

Mathematics Subject Classification 90C05 · 90C30 · 65F10

1 Introduction

In this paper, we concentrate on the generalized absolute value equation (GAVE), whose form is below

$$Ax + B|x| = b,$$  \hspace{1cm} (1.1)
with \( A, B \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \). When \( B = I \), where \( I \) stands for the identity matrix, the GAVE (1.1) reduces to the absolute value equation (AVE)

\[
Ax + |x| = b. \tag{1.2}
\]

The GAVE (AVE) have received considerable attention because they are used as a useful tool in the optimization field, such as the complementarity problem, linear programming and convex quadratic programming, see [1–5]. Especially, for solving the well-known linear complementarity problem (LCP), the LCP is to find \( z \) such that

\[
w = Mz + q \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T w = 0, \quad \text{with} \quad M \in \mathbb{R}^{n \times n}, \quad q \in \mathbb{R}^n, \tag{1.3}
\]

many efficient numerical methods can be established on the base of the form of the GAVE (1.1), such that the modulus-type iteration method [6,7] and its various versions [8,9], the generalized Newton method [10], and so on.

The research of the unique solution is a very important branch of theoretical analysis of the GAVE (AVE). By observing the structure of the GAVE (1.1), it is not difficult to see that the nonlinear and nondifferentiable term \( B|x| \) often leads to the nondeterminacy of the solution of the GAVE (1.1). In this case, we have to give some constraints to guarantee that the GAVE (1.1) has a unique solution. Recently, some sufficient conditions for the unique solution of the GAVE (1.1) have been obtained in the literatures. For example, in [13], Rohn et al. found that the GAVE (1.1) for any \( b \in \mathbb{R}^n \) has a unique solution if \( \rho(|A^{-1}B|) < 1 \), where \( \rho(\cdot) \) denotes the spectral radius of the matrix. From the singular value of the matrix, Rohn in [14] showed that the GAVE (1.1) for any \( b \in \mathbb{R}^n \) has a unique solution when \( \sigma_1(|B|) < \sigma_n(A) \), where \( \sigma_1 \) and \( \sigma_n \), respectively, denote the maximal and minimal singular value of the matrix. Based on the work in [14], Wu and Li in [15] obtained an improved result, i.e., if \( \sigma_1(B) < \sigma_n(A) \), then the GAVE (1.1) for any \( b \in \mathbb{R}^n \) has a unique solution. Other sufficient conditions for the unique solution of the GAVE (1.1), one can see [1,11,12] for more details.

By investigating the previous published works in [1,11,13–15], the presented conditions for the unique solution of the GAVE (1.1) just are sufficient conditions. Although Wu and Li in [16] presented some necessary and sufficient conditions for the unique solution of the AVE (1.2), it is regretful that these results in [16] are not suitable for the GAVE (1.1) because matrix \( B \in \mathbb{R}^{n \times n} \) in (1.1) is free. So far, to our knowledge, the necessary and sufficient condition for the unique solution of the GAVE (1.1) is void, which is our motivation for this paper. That is to say, the intent of the present paper is to address this question, and present some necessary and sufficient conditions for the unique solution of the the GAVE (1.1). Although these results in [16] for the GAVE (1.1) is invalid, the idea employed in [16] is used to establish the necessary and sufficient condition for the unique solution of the GAVE (1.1). Not only that, these necessary and sufficient conditions for the unique solution of the GAVE (1.1) can contain the fundamental theorem for the unique solution of the linear system \( Ax = b \) with \( A \in \mathbb{R}^{n \times n} \), and also yield some new sufficient conditions for the unique solution of the GAVE (1.1). These new sufficient conditions are weaker than the previous published works.
The rest of the paper unfolds below. Section 2 consists of some useful lemmas and the definition of $P$-matrix. Section 3 contains some necessary and sufficient conditions for the unique solution of the GAVE (1.1). It also contains some new sufficient conditions for the unique solution of the GAVE (1.1). In Sect. 4, some conclusions are given to end the paper.

2 Preliminaries

In this section, we remind the definition of $P$-matrix and some useful lemmas for the later discussion.

Definition 2.1 [5] Matrix $A \in \mathbb{R}^{n \times n}$ is called a $P$-matrix if all its principal minors are positive.

Lemma 2.1 [5] The linear complementarity problem, which finds $z \in \mathbb{R}^n$ such that

$$w = Mz + q \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T w = 0 \quad \text{with} \quad M \in \mathbb{R}^{n \times n},$$

has a unique solution for any $q \in \mathbb{R}^n$ if and only if the matrix $M$ is a $P$-matrix.

Lemma 2.2 [17] Matrix $M$ is a $P$-matrix if and only if matrix $MD + I - D$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$.

By the way, Lemma 2.2 implies that matrix $M$ is a $P$-matrix, which is equivalent:

1. matrix $M(I - D) + D$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$.
2. matrix $MF_0 + F_1$ is nonsingular, where $F_0, F_1 \in \mathbb{R}^{n \times n}$ are two arbitrary nonnegative diagonal matrices with $F = F_0 + F_1 > 0$.

Lemma 2.3 [18] Let $A, B \in \mathbb{R}^{n \times n}$. Then

$$\sigma_i(A + B) \geq \sigma_i(A) - \sigma_1(B), \quad i = 1, 2, \ldots, n,$$

where $\sigma_1 \geq \ldots \geq \sigma_n(\geq 0)$ are the singular values of matrix.

3 Main results

In this section, based on the above results in Sect. 2, we shall address the problem of the necessary and sufficient condition for the unique solution of the GAVE (1.1).

First, based on Lemma 2.1, we can obtain Theorem 3.1.

Theorem 3.1 Let $A + B$ be nonsingular in (1.1). Then the GAVE (1.1) has a unique solution for any $b \in \mathbb{R}^n$ if and only if matrix $(A + B)^{-1}(A - B)$ is a $P$-matrix.
Theorem 3.2 The GAVE (1.1) has a unique solution for any \( b \in \mathbb{R}^n \) if and only if matrix \( A + B \bar{D} \) is nonsingular for any diagonal matrix \( \bar{D} = \text{diag}(\bar{d}_i) \) with \( \bar{d}_i \in [-1, 1] \).

Proof Since we can express matrix \( \bar{D} \) as

\[
\bar{D} = I - 2D,
\]

where \( D = \text{diag}(d_i) \) with \( 0 \leq d_i \leq 1 \), then matrix \( A + B \bar{D} \) is nonsingular for any diagonal matrix \( \bar{D} = \text{diag}(\bar{d}_i) \) with \( \bar{d}_i \in [-1, 1] \) (it implies that \( A + B \) is nonsingular), if and only if the matrix \( (A + B)^{-1}(A + B - 2BD) \) with \( A, B \in \mathbb{R}^{n \times n} \) is nonsingular for any diagonal matrix \( D = \text{diag}(d_i) \) with \( 0 \leq d_i \leq 1 \).

By the simple computation, we obtain

\[
(A + B)^{-1}(A + B - 2BD) = (A + B)^{-1}(AD - BD + A + B - AD - BD) = (A + B)^{-1}[(A - B)D + A + B - AD - BD] = (A + B)^{-1}[(A - B)D + (A + B)(I - D)] = (A + B)^{-1}(A - B)D + I - D.
\]

Based on Lemma 2.2, it is easy to know that \( (A + B)^{-1}(A - B) \) is a \( P \)-matrix. Further, based on Theorem 3.1, the GAVE (1.1) has a unique solution for any \( b \in \mathbb{R}^n \). \( \square \)

As it is known that all the matrices \( A \) and \( B \) of the GAVE (1.1) in general are two arbitrary \( n \times n \) real matrices. In this case, if we take \( B = 0 \) in Theorem 3.2, Theorem 3.2 reduces to the fundamental theorem of the linear system \( Ax = b \) for \( A \in \mathbb{R}^{n \times n} \): the linear system \( Ax = b \) has a unique solution for any \( b \in \mathbb{R}^n \) if and only if matrix \( A \in \mathbb{R}^{n \times n} \) is nonsingular. In a way, Theorem 3.2 generalizes the necessary and sufficient condition for the unique solution of the linear system \( Ax = b \) with \( A \in \mathbb{R}^{n \times n} \). Of course, we also know that the linear system \( Ax = b \) has a unique solution for any \( b \in \mathbb{C}^n \) if and only if matrix \( A \in \mathbb{C}^{n \times n} \) is nonsingular. In addition, it is noted that Theorem 3.2 implies that matrix \( A \in \mathbb{R}^{n \times n} \) in (1.1) should be nonsingular, whereas, matrix \( B \in \mathbb{R}^{n \times n} \) in (1.1) is free. That is to say, matrix \( B \in \mathbb{R}^{n \times n} \) in (1.1) may be nonsingular or singular.

When \( B = I \) in Theorem 3.2, we immediately obtain the necessary and sufficient condition for the unique solution of the AVE (1.2), see Corollary 3.1.
Corollary 3.1  The AVE (1.2) has a unique solution for any \( b \in \mathbb{R}^n \) if and only if \( A + \tilde{D} \) is nonsingular for any diagonal matrix \( \tilde{D} = \text{diag}(\tilde{d}_i) \) with \( \tilde{d}_i \in [-1, 1] \).

In Corollary 3.1, if we express \( \tilde{D} \) as
\[
\tilde{D} = -I + 2D \text{ or } \tilde{D} = I - 2D
\]
where any diagonal matrix \( D = \text{diag}(d_i) \) with \( 0 \leq d_i \leq 1 \), then Corollary 3.1, respectively, reduces to Theorem 3.2 and Theorem 3.3 in [16], which are main results in [16].

Based on Theorem 3.2, a series of new sufficient conditions for the GAVE (1.1) and the AVE (1.2) can be obtained. Some of new sufficient conditions for the unique solution of the GAVE (1.1) or the AVE (1.2) for any \( b \in \mathbb{R}^n \) are weaker than the previously published works. For example, if we express matrix \( A + B \tilde{D} \) in Theorem 3.2 as
\[
A + B \tilde{D} = A(I + A^{-1} B \tilde{D}), \quad (3.2)
\]
then Theorem 3.3 can be obtained, which is stated below and its proof is omitted.

**Theorem 3.3** If matrix \( A \) in (1.1) is nonsingular and satisfies
\[
\rho(A^{-1} B \tilde{D}) < 1 \quad (3.3)
\]
for any diagonal matrix \( \tilde{D} = \text{diag}(\tilde{d}_i) \) with \( \tilde{d}_i \in [-1, 1] \), then the GAVE (1.1) for any \( b \in \mathbb{R}^n \) has a unique solution. In addition, if matrix \( A \) in (1.2) is nonsingular and satisfies
\[
\rho(A^{-1} \tilde{D}) < 1 \quad (3.4)
\]
for any diagonal matrix \( \tilde{D} = \text{diag}(\tilde{d}_i) \) with \( \tilde{d}_i \in [-1, 1] \), then the AVE (1.2) for any \( b \in \mathbb{R}^n \) has a unique solution.

In addition, if we use \( \rho(\tilde{D} A^{-1}) < 1 \) and \( \rho(\tilde{D} A^{-1}) < 1 \) instead of \( \rho(A^{-1} B \tilde{D}) < 1 \) and \( \rho(A^{-1} \tilde{D}) < 1 \) in Theorem 3.3, respectively, the corresponding results still hold.

The following example shows that sometimes the GAVE does not satisfy the condition (3.3) in Theorem 3.3, it still has a unique solution for some \( b \).

**Example 1** The GAVE
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
|x_1| \\
|x_2|
\end{bmatrix}
= \begin{bmatrix}
0 \\
2
\end{bmatrix}
\]
has a unique solution \( x_1 = 0 \) and \( x_2 = 1 \).

In [13], Rohn et al. gave the following sufficient condition for the unique solution of the GAVE (1.1), see Theorem 3.4.
Theorem 3.4 [13] If matrix $A$ in (1.1) is nonsingular and satisfies
\[ \rho(|A^{-1}B|) < 1, \] (3.5)
then the GAVE (1.1) for any $b \in \mathbb{R}^n$ has a unique solution.

It is easy to know that the condition (3.3) in Theorem 3.3 is slightly weaker than the condition (3.5) in Theorem 3.4 ([13]). That is to say,
\[ \rho(A^{-1}B\bar{D}) \leq \rho(|A^{-1}B|). \]

By the simple calculation, we have
\[ A^{-1}B\bar{D} \leq |A^{-1}B\bar{D}| \leq |A^{-1}B||\bar{D}| \leq |A^{-1}B|. \]

Further, the condition (3.4) of Theorem 3.3 is slightly weaker than $\rho(|A^{-1}|) < 1$ in [13] and $\|A^{-1}\|_2 < 1$ in [3]. Since
\[ \rho(A^{-1}B\bar{D}) \leq \sigma_1(A^{-1}B\bar{D}) \leq \sigma_1(A^{-1}B)\sigma_1(\bar{D}) \leq \sigma_1(A^{-1}B) \]
for any diagonal matrix $\bar{D} = \text{diag}(\bar{d}_i)$ with $\bar{d}_i \in [-1, 1]$, the somewhat stronger sufficient condition is obtained, see Corollary 3.2.

Corollary 3.2 If matrix $A$ in (1.1) is nonsingular and satisfies
\[ \sigma_1(A^{-1}B) < 1, \] (3.6)
then the GAVE (1.1) for any $b \in \mathbb{R}^n$ has a unique solution.

Example 2 The GAVE
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 
\end{bmatrix}
+ 
\begin{bmatrix}
-0.9 & 0.4 \\
-0.4 & -0.9 
\end{bmatrix}
\begin{bmatrix}
|x_1| \\
|x_2| 
\end{bmatrix}
= 
\begin{bmatrix}
0.5 \\
-0.3 
\end{bmatrix}
\]
has a unique solution $x_1 = x_2 = 1$. Theorem 3.4 ([13]) is invalid to judge the unique solution of Example 2 because of $\rho(|A^{-1}B|) = 1.3 > 1$. Whereas, when using Corollary 3.2, we can check that $\sigma_1(A^{-1}B) < 1$. In fact, by the simple computations, we obtain that $\sigma_1(A^{-1}B) = 0.9849 < 1$.

Remark 3.1 When $B = I$, Corollary 3.2 is the Proposition 4 in [3] and Corollary 3.1 in [16], but our proof is different from the proof in [3] and [16].

Remark 3.2 Noting that $\sigma_1(X)\sigma_n(X^{-1}) = 1$ for non-singular matrix $X$. When $B$ in (1.1) is nonsingular, we can use
\[ \sigma_n(B^{-1}A) > 1, \] (3.7)
instead of the condition (3.6). In this case, the GAVE (1.1) for any $b \in \mathbb{R}^n$ has a unique solution as well. It is not difficult to find that the condition (3.7) is slighter weaker than the condition

$$\sigma_1(B) < \sigma_n(A),$$

which was provided in [15]. By the simple computations, we have

$$\sigma_n(B^{-1}A) \geq \sigma_n(B^{-1})\sigma_n(A) = \sigma_n(A) / \sigma_1(B).$$

Under the condition (3.8), we get the condition (3.7). Since the condition (3.8) is slighter weaker than the following condition

$$\sigma_1(|B|) < \sigma_n(A),$$

which was provided in [14], it follows that the condition (3.7) is the weakest condition, compared with the conditions (3.8) and (3.9). In addition, if $B = I$ in (3.7), then the condition (3.7) reduces to the Proposition 3 (i) in [3] and Theorem 3.6 in [16].

It is known that when the smallest singular value $\sigma_n$ of matrix $A$ is greater than 0, matrix $A$ is nonsingular. Based on this fact, we have Corollary 3.3.

**Corollary 3.3** If matrix $A$ in (1.2) satisfies $\sigma_n(A + I) > 2$, then the AVE (1.2) has a unique solution for any $b \in \mathbb{R}^n$.

**Proof** Based on Corollary 3.1, when the matrix $A + I - 2D$ is nonsingular for any diagonal matrix $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$, the AVE (1.2) has a unique solution for any $b \in \mathbb{R}^n$. Let $\sigma_n(A + I - 2D)$ indicate the minimal singular value of the matrix $A + I - 2D$. Based on Lemma 2.3, we have

$$\sigma_n(A + I - 2D) \geq \sigma_n(A + I) - 2\sigma_1(D).$$

Since $\sigma_1(D) \leq 1$, clearly, when $\sigma_n(A + I) > 2$, we have $\sigma_n(A + I - 2D) > 0$. This implies that matrix $A + I - 2D$ is nonsingular. \qed

Of course, combining matrix $A + \tilde{D}$ of Corollary 3.1 with the approach of the proof of Corollary 3.3, the sufficient condition $\sigma_n(A) > 1$ can be obtained as well. It is noted that the condition $\sigma_n(A + I) > 2$ is slighter stronger than the condition $\sigma_n(A) > 1$. By the simple calculation, we have

$$\sigma_n(A) = \sigma_n(A + I - I) \geq \sigma_n(A + I) - \sigma_1(I) > 1.$$
4 Conclusions

In this paper, we have presented some necessary and sufficient conditions for the unique solution of the generalized absolute value equation (GAVE) $Ax - B|x| = b$ with $A, B \in \mathbb{R}^{n \times n}$. These results not only address the question of the necessary and sufficient condition for the unique solution of the GAVE, not only contain the fundamental theorem for the unique solution of the linear system $Ax = b$ with $A \in \mathbb{R}^{n \times n}$. Moreover, some presented new sufficient conditions for the unique solution of the GAVE are weaker than the previous published works.

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