FRAME-TYPE FAMILIES OF TRANSLATES

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Abstract. We construct a uniformly discrete, and even sparse, sequence of real numbers \( \Lambda = \{ \lambda_n \} \) and a function \( g \in L^2(\mathbb{R}) \), such that for every \( q > 2 \), every function \( f \in L^2(\mathbb{R}) \) can be approximated with arbitrary small error by a linear combination \( \sum c_n g(t - \lambda_n) \) with an \( l_q \) estimate of the coefficients:

\[
\| \{ c_n \} \|_{l_q} \leq C(q) \|f\|.
\]

This can not be done for \( q = 2 \), according to [2].

Keywords: translates · frames · completeness with estimate of coefficients

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1. INTRODUCTION. RESULTS.

1.1. Let \( \Lambda \) be a uniformly discrete set of real numbers:

\[
\inf_{\lambda \neq \lambda'} |\lambda - \lambda'| = \delta > 0, \quad \lambda, \lambda' \in \Lambda.
\]

(1)

Given a function \( g \in L^2(\mathbb{R}) \), consider the family of translates

\[
\{ g(t - \lambda) \}_{\lambda \in \Lambda}.
\]

(2)

It is well known that for \( \Lambda = \mathbb{Z} \) this family cannot be complete in \( L^2(\mathbb{R}) \). It was conjectured that the same is true for every uniformly discrete set \( \Lambda \) (see, for example, [12], p.149, where even a stronger conjecture related to Gabor-type systems is discussed). However, this is not the case. The following theorem was proved in [9]:

Theorem A. Let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) be an "almost integer" spectrum:

\[
\lambda_n = n + \alpha_n : \quad 0 < |\alpha_n| \to 0 \quad (|n| \to \infty).
\]

(3)

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Then there exists a "generator" $g$ such that the family $(2)$ is complete in $L^2(\mathbb{R})$.

One may wish to construct a uniformly discrete set of translates $(2)$ with a stronger property then just completeness. However, one should keep in mind that no family $(2)$ can be a frame, see [2].

1.2. In [3] we introduced an intermediate property between completeness and frame, which is reproduced here in a slightly different form:

**Definition 1.** We say that a system of vectors $\{u_n\}$ in a Hilbert space $H$ is a $(QF)$-system if the following two conditions are fulfilled:

(i) for every $q > 2$ there is a constant $C(q)$, such that given $f \in H$ and $\varepsilon > 0$, one can find a linear combination

$$Q = \sum c_n u_n,$$

satisfying the conditions:

$$\| f - Q \| < \varepsilon,$$

and

$$\| \{c_n\} \|_{l_q} \leq C(q) \| f \|.$$  

(ii) (Bessel inequality):

$$\left( \sum |\langle f, u_n \rangle|^2 \right)^{\frac{1}{2}} \leq C' \| f \|, \quad \forall f \in H,$$

where the constant $C'$ does not depend on $f$.

Approximation property (i) above means "completeness with $l_q$ estimate of coefficients". Using the standard duality argument (see [3]), it can be reformulated as follows:

$$\| f \| \leq C(p) \left( \sum |\langle f, u_n \rangle|^p \right)^{\frac{1}{p}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$  

If the condition (i) in Definition 1 is required for $q=2$ then it is identical to usual definition of frames. So, one may regard $(QF)$-systems as a sort of "quasi-frames".

One may ask about the relation of $(QF)$-systems to $p$-frames defined in [1]. Notice that in the case of Hilbert space, $p$-frames may only exist for $p = 2$, when they are identical to usual frames.
1.3. In [3] we have constructed sparse exponential systems
\[ E(\Lambda) := \{e^{i\lambda x}\}_{\lambda \in \Lambda} \]
which are (QF)-systems in \( L^2(S) \) for "generic" sets \( S \) of large measure. Observe that these systems cannot be frames, due to celebrated Landau’s density theorem.

The goal of this work is to present a similar construction for the translates. Our main result is the following

**Theorem 1.** There exist a uniformly discrete sequence \( \Lambda = \{\lambda_n\} \) and a function \( g \in L^2(\mathbb{R}) \), such that the system (2) is a (QF)-system for \( L^2(\mathbb{R}) \).

We will prove this result in a stronger form, showing that \( \Lambda \) can be chosen sparse:

**Theorem 2.** Given a sequence of positive numbers \( \{\epsilon_n\} = o(1) \), one can choose \( \Lambda \) in Theorem 1 so that
\[ \frac{\lambda_{n+1}}{\lambda_n} > 1 + \epsilon_n. \]  
(8)

Clearly, if \( \epsilon_n \) decreases slowly enough then the gaps in the spectrum \( \Lambda \) grow "almost exponentially". This condition is sharp, see Remark 2.3 below. Observe that in the context of completeness, this lacunarity condition probably first appeared in [10], see also [6] and [3].

Some remarks on Theorems 1 and 2 are presented in section 2. In particular, we show (Propositions 1 and 2) that the generator \( g \) in these theorems can be chosen infinitely smooth but it cannot decrease fast at infinity.

### 2. PROOFS.

2.1. Following [9] we start with a reformulation of the result.

For \( f \in L^2(\mathbb{R}) \), we denote by \( \hat{f} \) its Fourier transform:
\[ \hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ixt} dx. \]

**Definition 2.** A function \( w(x) \in L^1(\mathbb{R}) \) is called a weight if it is strictly positive almost everywhere on \( \mathbb{R} \) (with respect to the Lebesgue measure).

Let us consider the weighted space
\[ L^2_w(\mathbb{R}) = \{ f : \int_{\mathbb{R}} |f(x)|^2 w(x) dx < \infty \}, \]
with the scalar product \( \langle f, g \rangle = \int f(x) \overline{g(x)} w(x) dx \).
Due to the Parseval equality for the Fourier transform, the transformation
\[ U_w : f \mapsto \left( \hat{f} \sqrt{w} \right) \]
is a unitary operator acting from \( L^2_w(\mathbb{R}) \) onto \( L^2(\mathbb{R}) \).
We set
\[ g(t) := \sqrt{w}. \]
Obviously, \( g \in L^2(\mathbb{R}) \), and we have:
\[ U_w(e^{i\lambda x}) = g(t - \lambda). \]
Since the \((QF)\)-property of a system of vectors is invariant with respect to unitary operators, we conclude that the system \( \{g(t - \lambda)\}_{\lambda \in \Lambda} \) satisfies this property in \( L^2(\mathbb{R}) \), if the system \( E(\Lambda) := \{e^{i\lambda x}\}_{\lambda \in \Lambda} \) does so in the space \( L^2_w(\mathbb{R}) \). Therefore, Theorem 2 is a consequence of the following

**Theorem 3.** There exist a weight \( w(x) \) and a uniformly discrete set \( \Lambda \) such that

(i) \( \Lambda \) satisfies (8) for a pre-given sequence \( \{\epsilon_n\} \);
(ii) The system \( E(\Lambda) \) is a \((QF)\)-system in \( L^2_w(\mathbb{R}) \).

**Remark 2.1.** Reversing the argument above, one can deduce Theorem 3 from Theorem 2, so that these results are equivalent.

Our goal now is to prove Theorem 3.

2.2. We need some lemmas.

**Lemma 1.** Let \( \Lambda \) be a uniformly discrete sequence and \( v \) be a weight such that \( h := \sqrt{v} \) is supported by \( [-\delta, \delta] \) where \( \delta \) is the separation constant defined in (1). Then \( E(\Lambda) \) is a Bessel system in \( L^2_v(\mathbb{R}) \).

**Proof.** The set of translates \( \{h(t-\lambda)\} \) is an orthogonal system of vectors in \( L^2(\mathbb{R}) \) with bounded norms, so it is a Bessel system in the space. The argument above shows that this system is obtained from \( E(\Lambda) \) by the action of the unitary operator
\[ U_v : L^2_v(\mathbb{R}) \to L^2(\mathbb{R}). \]
The lemma follows. \( \square \)

**Remark 2.2.** An equivalent definition of a Bessel system in \( H \):
\[ \| \sum a_n u_n \|_H \leq C' \|\{a_n\}\|_{l_2} \]
(see [13], p.155), clearly extends the result above to every weight \( w \), such that \( w(x) \leq v(x) \) almost everywhere.
Given a trigonometric polynomial $Q(x) = \sum_{\lambda \in \Lambda} c_\lambda e^{i\lambda x}$, we set
\[
\text{spec } Q = \{ \lambda \in \Lambda : c_\lambda \neq 0 \},
\]
and
\[
\|Q\|_q = \|\{c_\lambda\}\|_{l_q}.
\]

The following lemma is well known, it goes back to Menshov-type representation theorems.

**Lemma 2.** Given a segment $I \subset \mathbb{R}$ and a number $\mu > 0$, one can find a trigonometric polynomial $A(x) = \sum_{k=1}^{K} a_k e^{i k x}$ such that
\[
(i) \quad \|A\|_{(2+\mu)} < \mu; \\
(ii) \quad m\{x \in I : |A(x) - 1| > \mu\} < \mu.
\]

For proof see [5], Chapter 4 section 2.5, or [6], Lemma 4.1 and remark 2 on p.382.

**Lemma 3.** Given a segment $I \subset \mathbb{R}$, a number $\xi > 0$ and a function $f \in L^2(I)$, one can find a trigonometric polynomial $B(x) = \sum_{n=1}^{N} b_n e^{i\beta_n x}$ such that
\[
(i) \quad 0 \leq \beta_n - n < \xi \quad n = 1, 2, 3 \ldots N; \\
(ii) \quad m\{x \in I : |f(x) - B(x)| > \xi\} < \xi.
\]

This can be easily deduced from Landau’s theorem [8], or from [9].

**Lemma 4.** Let $I \subset \mathbb{R}$ be a segment. For every $\delta > 0$ and $f \in L^2(I)$, there exists a number $l > 0$ such that, given an integer $d > 0$, one can find a trigonometric polynomial $Q(x) = \sum_{m=1}^{M} c_{m} e^{i \lambda_{m} x}$ which satisfies
\[
(i) \quad \|Q\|_{(2+\delta)} < \delta; \\
(ii) \quad \lambda_1 \geq d; \\
(iii) \quad \frac{\lambda_{m+1}}{\lambda_{m}} > 1 + l, \quad m = 1, 2, 3 \ldots M; \\
(iv) \quad m\{x \in I : |f(x) - Q(x)| > \delta\} < \delta.
\]

**Proof.** We can assume that $I = [-\pi s, \pi s]$ for some integer $s > 0$.

Given $f \in L^2(I)$ and $0 < \delta < 1$, denote $\xi = \frac{\delta}{4}$ and use Lemma 3 to find a trigonometric polynomial,
\[
B(x) = \sum_{n=1}^{N} b_n e^{i\beta_n x},
\]
for which (11) and (12) hold.

Denote
\[
\mu = \frac{\delta}{2N \max\{1, \|B\|_{(2+\delta)}\}},
\]

(17)
and use Lemma 2 to find a trigonometric polynomial,

$$A(x) = \sum_{k=1}^{K} a_k e^{ikx},$$

for which (9) and (10) hold. Set

$$l = \frac{1}{2 + K}.$$  \hspace{1cm} (18)

Given a positive integer $d > 0$, denote

$$r_n = d(K + 1)^{n-1}$$  \hspace{1cm} (19)

and define

$$Q(x) = \sum_{m=1}^{M} c_m e^{i\lambda_m x} := \sum_{n=1}^{N} b_n e^{i\beta_n x} A(r_n x),$$

where $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$

Denote

$$J_n := \text{spec}(e^{i\beta_n x} A(r_n x))$$

and use (11) to check that $J_{n+1}$ follows $J_n$, for every $1 \leq n < N$. In particular, this fact combined with (9) and (17) means that

$$\|Q\|_{(2+\delta)} = \|A\|_{(2+\delta)} \|B\|_{(2+\delta)} \leq \|A\|_{(2+\mu)} \|B\|_{(2+\delta)} < \mu \|B\|_{(2+\delta)} < \delta.$$

So property (i) holds for $Q$.

From (19) we have $\lambda_1 = \beta_1 + r_1 = \beta_1 + d$, so property (ii) follows from (11).

To establish property (iii) note that there are two possible locations for $\lambda_m$ and $\lambda_{m+1}$ in the spectrum of $Q$. First, they can both belong to $J_n$, for some $1 \leq n < N$. In this case

$$\frac{\lambda_{m+1}}{\lambda_m} = \frac{\beta_n + (k + 1)r_n}{\beta_n + kr_n} \quad \text{for some } 1 \leq k < K.$$

On the other hand, $\lambda_m$ can be the last frequency in $J_n$, for some $1 \leq n \leq N$, while $\lambda_{m+1}$ is the first frequency in $J_{n+1}$. In this case

$$\frac{\lambda_{m+1}}{\lambda_m} = \frac{\beta_{n+1} + r_{n+1}}{\beta_n + Kr_n}.$$

In both cases it is easy to see that (11), (18) and (19) imply that (iii) holds for $Q$.

To finish the proof we need to show that property (iv) holds for $Q$. Note that $I = [-\pi s, \pi s]$, so from (10), (12) and (17), we have

$$m\{x \in I : |f(x) - Q(x)| > \delta\} \leq m\{x \in I : |f(x) - B(x)| > \frac{\delta}{2}\} + m\{x \in I : |B(x) - Q(x)| > \frac{\delta}{2}\} \leq \frac{\delta}{2} + m\{x \in I : |\sum_{n=1}^{N} b_n e^{i\beta_n x}(1 - A(r_n x))| > \frac{\delta}{2}\} \leq$$
\[ \frac{\delta}{2} + \sum_{n=1}^{N} m \{ x \in I : |b_{n}(1 - A(r_{n} x))| > \frac{\delta}{2N} \} \leq \]
\[ \frac{\delta}{2} + N m \{ x \in I : |1 - A(x)| > \frac{\delta}{2N\|b_{n}\|_{l^{2+\delta}}} \} < \delta \]

which completes the proof. \qed

We are now ready to prove Theorem 3.

Let \( 0 < \epsilon_{n} \to 0 \), we can assume that \( \epsilon_{n+1} < \epsilon_{n} \) for every \( n \). Fix an arbitrary weight \( v(x) \) with \( \left( \sqrt{v} \right) \) supported on \([-1, 1]\) and \( \int vdx = 1 \). Choose a sequence of functions \( f_{k} \in C(\mathbb{R}) \), \( f_{k}(x) = 0 \) for every \( |x| > k \), which is dense in \( L_{v}^{2}(\mathbb{R}) \) (as it will automatically be in every space \( L_{w}^{2}(\mathbb{R}) \), \( w \leq v \)).

We construct the sequence \( \Lambda \) by induction. At the \( k \)-th step assume that \( \{ \lambda_{n} \}_{1 \leq n < N} \) have already been defined (where \( N = N(k - 1) \)). For the segment \([-k, k]\), the function \( f_{k} \) and \( \delta = \frac{1}{2k} \), we use Lemma 4 to find a number \( l_{k} > 0 \). Let \( n_{k} \geq N \) be the first number for which

\[ \epsilon_{n_{k}} l_{k} \]

Choose arbitrary \( \lambda_{N}, \lambda_{N+1}, \lambda_{N+2}, \ldots, \lambda_{(n_{k} - 1)} \) so that (8) holds for every \( n < n_{k} \) and a number \( d_{k} \) for which

\[ \frac{d_{k}}{\lambda_{(n_{k} - 1)}} > 1 + \epsilon_{(n_{k} - 1)}. \]  \( \tag{21} \)

Use Lemma 4 to find a trigonometric polynomial \( Q_{k} \) so that properties (13) - (16) hold for \( f_{k}, \delta, l_{k}, d_{k} \) and \( Q_{k} \). Add all of spec \( Q_{k} \) as a block from the point \( n_{k} \) forward to form the sequence \( \{ \lambda_{n} \}_{1 \leq n < N(k)} \).

Set \( \Lambda = \{ \lambda_{n} \}_{n=1}^{\infty} \) and note that the combination of properties (14), (15), (20), and (21) ensures that (8) holds for \( \Lambda \).

To define the weight \( w \) denote

\[ E_{k} = \{ x \in [-k, k] : |f_{k}(x) - Q_{k}(x)| < \delta_{k} \}. \]  \( \tag{22} \)

From (16) we have

\[ m\{-k, k \} \setminus E_{k} < \delta_{k}. \]  \( \tag{23} \)

Define

\[ w(x) := v(x) \inf_{k} \{ \mathbb{I}_{E_{k}}(x) + \eta_{k} \mathbb{I}_{R/E_{k}}(x) \}, \]

where \( \mathbb{I}_{E} \) is the indicator function of \( E \) and \( \eta_{k} = (2k\|f_{k} - Q_{k}\|_{L_{v}^{2}(\mathbb{R})})^{-2} \).

Since \( \Sigma_{k} \delta_{k} < \infty \), (23) implies that \( 0 < w \leq v \) almost everywhere.

Moreover, from (22) we have

\[ \int |f_{k} - Q_{k}|^{2} wdx < (\delta_{k})^{2} \int_{E_{k}} vdx + \eta_{k} \int_{R/E_{k}} |f_{k} - Q_{k}|^{2} vdx, \]
so
\[ \| f_k - Q_k \|_{L^q_w(\mathbb{R})} < \frac{1}{k}. \] (24)

Clearly \( \Lambda \) is a uniformly discrete sequence. Moreover, we may suppose \( \{ \epsilon_n \} \) to decrease so slowly that the constant in (1) satisfies \( \delta > 1 \), so we can use Remark 2.2 to deduce that \( E(\Lambda) \) is a Bessel system in \( L^2_{w_w}(\mathbb{R}) \).

Fix \( q > 2 \). The proof will be complete if we show that, for any \( f \in L^2_{w_w}(\mathbb{R}) \) with \( \| f \| = 1 \) and \( \mu > 0 \), there exists a trigonometric polynomial \( Q \) such that \( \text{spec} Q \subset \Lambda \), \( \| f - Q \|_{L^q_w(\mathbb{R})} < \mu \) and \( \| Q \|_q \leq 1 \).

Given such \( f \) and \( \mu > 0 \), choose \( k \) large enough so that
\[ 2 + \delta_k < q \] (25)

\[ \| f - f_k \|_{L^q_w(\mathbb{R})} < \frac{\mu}{2} \] (26)

\[ k \mu > 2. \] (27)

We claim that for the polynomial \( Q = Q_k \) all of the properties described above hold. Indeed, from (13) and (25) we have
\[ \| Q \|_q \leq \| Q \|_{(2+\delta_k)} \leq \delta_k \leq 1, \]
while (24) (26) and (27) imply that \( \| f - Q_k \| < \mu \). This ends the proof. \( \square \)

2.3. Here we discuss the "time-frequency" localization of the generator \( g \).

**Proposition 1.** As in \([9]\), one can construct the function \( g \) in Theorems 1 and 2 to be infinitely smooth and even the restriction to \( \mathbb{R} \) of an entire function.

**Proof.** Indeed, in the proof of Theorem 3 it is enough to start with a weight \( v_0 \leq v \) with sufficiently fast decay, so the same will be true for \( w \). The relation
\[ g(t) := \sqrt[2]{(\sqrt{w})}. \]
implies the required property. \( \square \)

On the other hand, the weight \( w \), constructed in Theorem 3 is "irregular", which means that the generator \( g \) decreases slowly. This is inevitable, due to the following

**Proposition 2.** A generator \( g \) in Theorem 1 cannot belong to \( L^1(\mathbb{R}) \).
Proof. Indeed, suppose it does. Then the corresponding weight \( w(x) = |\hat{g}(x)|^2 \) is continues. The system \( E(\Lambda) \) is a \((QF)\)-system in \( L^2_w(\mathbb{R}) \). Clearly the same property holds in the space \( L^2_w(I) \) for an interval \( I \). The set

\[
\{ x \in I : w(x) > 0 \}
\]

is an open set of full measure. Take a finite union of intervals \( S \subseteq I \) such that:

\[
mS > \frac{mI}{2} ; \inf_{x \in S} w(x) > 0.
\]

Clearly \( E(\Lambda) \) is a \((QF)\)-system in \( L^2(S) \). Now we use Theorem 1 from [3], where it is proved that if \( S \) is a finite union of intervals and the system \( E(\Lambda) \) satisfies the condition \((i)\) in the Definition 1 (with some \( q > 2 \)), then the Beurling-Landau estimate:

\[
D^-(\Lambda) \geq \frac{mS}{2\pi}
\]

still holds. This contradicts (1), if \( mI \) is sufficiently large.

As a contrast, notice that in Theorem A, for an appropriate uniformly discrete \( \Lambda \), the generator \( g \) may belong to the Shwartz space \( S(R) \), see [11].

We conclude by a couple of other remarks.

**Remark 2.3.** The lacunarity condition in Theorems 2 and 3 is sharp. Indeed, it is well known (see [4]) that if \( \Lambda \) is lacunary in the Hadamard sense, that is \( \lambda_{n+1}/\lambda_n > c > 1 \), then the system \( \{ e^{i\lambda t} \}_{\lambda \in \Lambda} \) cannot be complete in \( L^2_w(\mathbb{R}) \).

**Remark 2.4.** By appropriate modification of the proof of Theorem 3, \( \Lambda \) in this result (as well as in Theorems 1 and 2) can be made a ”small perturbation” of integers, as in the equality (3). However, again, in contrast to Theorem A, the perturbations are not arbitrary. In particular \( \alpha_n \) cannot decrease as \( O(|n|^{-s}), s > 0 \). This can be proved similarly to the corresponding remark in [7].

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