Separability for lattice systems at high temperature

Heide Narnhofer *
Institute for Theoretical Physics
University of Vienna, Austria

Abstract

Equilibrium states of infinite extended lattice systems at high temperature are studied with respect to their entanglement. Two notions of separability are offered. They coincide for finite systems but differ for infinitely extended ones. It is shown that for lattice systems with localized interaction for high enough temperature there exists no local entanglement. Even more quasifree states at high temperature are also not distillably entangled for all local regions of arbitrary size. For continuous systems entanglement survives for all temperatures. In mean field theories it is possible, that local regions are not entangled but the entanglement is hidden in the fluctuation algebra.

PACS numbers: 03.65Ud, 03.67Hk, 05.30Fk
Keywords: localized entanglement, lattice systems

* E–mail address: narnh@ap.univie.ac.at
1 Introduction

Entanglement of quantum systems means that it is impossible to describe the state over the combined system only by its local constituents. This fact has been started to be investigated in systems built by many constituents, especially for systems over a lattice $\mathbb{Z}^\nu$ with $\nu$ the finite dimension of the lattice in a state invariant under translation [1], [2]. As it is discussed in [3] translational invariance restricts the possibility of entanglement between two lattice points. On the other hand the entanglement of subsystems can increase with the size of the subsystems essentially to the maximal possible value [4].

In this note we will consider two definitions for separability: on one hand we can demand that the state is a linear superposition of product states. On the other hand we can demand that for any two localized subsystems the entanglement between them vanishes. It will be shown that for finite systems, i.e. for systems consisting of a finite number of tensor products, the two definitions coincide. In the thermodynamic limit however the equilibrium states of the Ising model that is essentially a classical model can not be written as linear superposition of tensor product states if we do not accept measures over the state space that ask for some care (compare [6]). However the state restricted to all local subsystems is separable and we will take this fact as characteristic for the absence of entanglement [12].

We will study whether interacting systems at high temperature become separable. Based on a proof of the uniqueness of equilibrium states at high temperature by [7], [8] the control, how equilibrium states can be constructed as perturbation of the tracial state also suffices to prove that the local entanglement vanishes. However the estimate to find a bound on the critical temperature above which no entanglement can show up depends on the size of the subsystems. Therefore non localized entanglement cannot be excluded.

The above argument does not work for continuous systems. In the frame work of relativistic quantum field theory the analogue of the Reeh- Schlieder theorem in temperature states [9] together with the entanglement witness found in [5] guarantees also for equilibrium states that every two local regions are entangled. For the nonrelativistic free time evolution a scaling argument shows that again entanglement survives, but it becomes more and more localized in space and therefore more and more energy is needed for its detection. For quasifree lattice systems however, where the state is determined by the two point function we have enough control to show, that at least the entanglement cannot be distillable. It is also hard to imagine how with exponentially decreasing spacial correlation functions a delocalized entanglement witness can be constructed.

New features appear in mean field theories as the BCS- model. The state factorizes over local regions and therefore considered as state over the quasilocal algebra the state is not entangled in our sense. If we move however to the algebra of fluctuations [16] correlations between nearest neighbors disappear in such a weak way that correlations between macroscopic bulks remain and entanglement can occur for the fluctuation algebra.
2 Separable states and localized entanglement

For a finite multipartite system separable states are defined as a linear combination of tensor product states

$$ω = \sum_i \lambda_i \omega_i^1 \otimes \ldots \otimes \omega_i^k \quad \lambda_i \geq 0 \quad \sum_i \lambda_i = 1 \quad (1)$$

where $k$ is the number of constituents. If in the one dimensional situation $\nu = 1$ we order the points $i = 1, \ldots, k$ on a circle and assume periodicity, i.e. $A_{k+1} = A_k$ and take the shift $\tau A = A_{i+1}$ then we can construct translationally invariant separable states

$$\bar{ω} = \frac{1}{k} \sum_{l=1}^{k} ω \circ \tau^l \quad (2)$$

Generalization to higher dimensions is obviously possible. However this procedure does not work when $k \to \infty$. Separable states, i.e. linear combinations of product states are only translationally invariant if they are finite combinations of periodic states so that it is sufficient to reduce the sum in (2) over the period.

If an equilibrium state is a product state then necessarily $H = \sum_{i \in \mathbb{Z}^\nu} \ldots 1 \otimes h_i \otimes 1 \otimes \ldots$. Therefore there is no entanglement for all temperatures. Assume we can divide the state into product states over $A_\Lambda \otimes \tau^{|\Lambda|} A_\Lambda$ where $\Lambda = \cup(1, \ldots, k), A_\Lambda = A_1 \otimes A_2 \otimes \ldots A_k, |\Lambda| = k < \infty$. With the same argument $H = \sum_I \tau^{\delta_I} h$ for some $h \in A_\Lambda$. This forbids interaction between neighbors. Together with the assumption that the Hamiltonian is invariant under space translation the class of permitted Hamiltonians cannot be generalized. Therefore separable equilibrium states for finite temperature can only exist if there is no interaction between lattice points. An exception is the BCS-model, but the effective Hamiltonian in the thermodynamic limit again reduces to a Hamiltonian of the above type [10].

The Ising model $H = -\sum_{j,k} J_{j,k} \sigma_j^x \sigma_k^x, |j - k| = 1$ can be considered both as classical and also as quantum mechanical model, depending on the choice of the quasilocal algebra, on which $H$ acts. The classical equilibrium state can be extended to the quantum model. It contains classical spacial correlations. Therefore for every finite subsystem it is the sum over product states, but it is not separable in the above sense. Nevertheless restricted to any local subsystems of finite size the systems will not be quantum mechanically entangled. This suggests the definition:

**Definition 1:** Let $\Lambda_1 = (i_1, \ldots, i_n), |\Lambda_1| = n, \Lambda_2 = (k_1, \ldots, k_r), |\Lambda_2| = r, i_s \neq k_t$. A state is not entangled if restricted to any two local subsystems $A_{\Lambda_1} \otimes A_{\Lambda_2}$ it is not entangled.

**Definition 2:** A state is not entangled to order $N$ if for any two local subsystems $A_{\Lambda_1} \otimes A_{\Lambda_2}$ with $|\Lambda_1| < N, |\Lambda_2| < N$ the state is not entangled.

**Definition 3:** An invariant state is delocalized entangled of order $N$ if it is entangled in the sense of Definition 1 but not in the sense of Definition 2, i.e. if all algebras $A_{\Lambda_1}, A_{\Lambda_2}$
with $|\Delta 1| < N + 1$ or $|\Delta 2| < N + 1$ are not entangled, but there exist algebras $A_{\Delta 1}, A_{\Delta 2}$ with $|\Delta 1| = N + 1, |\Delta 2| = N + 1$, such that $A_{\Delta 1}$ and $A_{\Delta 2}$ are entangled.

In this way we classify how entanglement witnesses have to be localized. Of course it is always possible to take a linear superposition of entanglement witnesses and in this way to obtain a new delocalized one. But only if we search for the one that is localized as much as possible we get information about local correlations. In fact examples of translationally invariant states that are delocalized entangled can be constructed, and these examples are pure states [11] with weak decay of the spacial correlations.

In [12] it was argued that according to the fact, that the set of entangled states is open, a state is either not entangled or delocalized entangled of some order $N \geq 1$. This corresponds to the fact, that we consider only entanglement witnesses that can be approximated by local operators. In chapter 5 we will extend the definition of entanglement to mesoscopic observables and will give an example, in which entanglement appears, that is not observable on the local level.

It remains to argue that for finite system we have not changed the definition of separability, that is that for finite systems every state that is not entangled to every order is also separable. This follows from the following observations: the states that restricted to two subsystems are not entangled and therefore separable states form a convex set in state space. The intersection of convex sets is again convex. A convex set is characterized by its boundary: in every neighborhood of a point of the boundary there lies a point that does not belong to the set. Therefore it is not separable with respect to some subalgebras $A_\Lambda \otimes A_\Lambda^c$ where we can choose the second algebra to be the complement, because entanglement is monotonically increasing with the algebras [12]. Therefore an extremal not entangled state is a product state for two subalgebras. We continue by induction. This state over $A_\Lambda$ is again not entangled for the subalgebras of $A_\Lambda$. The boundary of all these states consists of states that are products of states over some subalgebras of $A_\Lambda$. We can continue by induction reducing in every step the size of the algebra. Therefore in a finite number of steps we reach the algebra over a lattice point. It follows that in fact the state can be written as linear combination of factorizing states.

### 3 Entanglement at high temperature

We consider equilibrium states for hamiltonians of the form $H = \sum_{k \in \mathbb{Z}} \tau^k H_\Lambda$ where $\Lambda$ is some finite region. Since $\Lambda$ is not just a point we permit interaction between neighboring regions. For simplicity we assume finite range interaction but the results can be extended for interactions that decrease sufficiently with the distance [8]. Therefore our considerations are applicable to all popular examples as the ferromagnetic and antiferromagnetic Heisenberg model, Hubbard model etc. For all these models the time evolution $\alpha_t$ is well defined as automorphism on the quasilocal algebra, i.e. the norm closure of the strictly local operators. Their equilibrium states satisfy the KMS condition

$$\omega(AB) = \omega(B\alpha_{it}A)$$
with $\beta$ the inverse temperature. For $\beta \to 0$ we obtain the tracial state, which is separable. That this is not only formally so but that the equilibrium state can be obtained by perturbation of the tracial state was shown in [3]. Of course the perturbation series in $\beta$ has in general finite convergence range, otherwise phase transitions would be forbidden. The possibility of perturbation theory guarantees that for high enough temperature the equilibrium state is close to the tracial state. We have to specify the way of convergence. We need norm convergence in an appropriate norm, because the limit must exist for all operators. But this norm can not be the operator norm because representations corresponding to states at different temperature are not equivalent. For our entanglement definition however we need uniformity for operators that are localized to the same amount. All these requirements are met by the calculations in [8].

To give a taste of the calculation we repeat the definitions and estimates following [8]: With the matrix units $e(I_X, J_X) = \prod_{l=1}^{n} e(i_{x_l}, j_{x_l})$, $X = (x_1, \ldots, x_n)$, $I_X = (i_{x_1}, \ldots, i_{x_n})$, $J_X = (j_{x_1}, \ldots, j_{x_n})$ the state is determined by $\bar{\omega}(I_X, J_X) = \omega(e(I_X, J_X))$. Let $|\omega| = \sup |\omega(e(I_X, J_X))|$. Let

$$\bar{\delta}(I_X, J_X) = 1 \text{ for } X = \Phi, = \frac{1}{d+1} \delta_{i_x,j_x} \text{ for } X = (x), = 0 \text{ otherwise}$$

Here $d$ is the dimension of the algebra at a lattice point. Then the KMS condition implies

$$\bar{\omega} = \bar{\delta} + K\bar{\omega} + L_{\beta,H_\Lambda}\bar{\omega}$$

with

$$(Kf)(I_X, J_X) = \frac{1}{d+1} \delta_{i_{x_1},j_{x_1}} f(I_{X'}, J_{X'}), \quad X = (x_1, \ldots, x_n), X' = (x_2, \ldots, x_n)$$

and $L_{\beta,H_\Lambda}$ is a lengthy expression (see [8]) reflecting the complex time evolution of the matrix units. We only need the result that $|K| = \frac{1}{d+1}$ and $|L_{\beta,H_\Lambda}|$ can be bounded by $(d+1)^2|H_\Lambda| ||(1 - 2\beta||H_\Lambda||)^{-1}$ where $||H_\Lambda||$ is some appropriate norm on the interaction [8]. With $\bar{\omega} = \sum_n (K + L_{\beta,H_\Lambda})^n \bar{\delta}$ we notice that for sufficiently small $\beta$ the series converges and the resulting state will be close to the tracial state for matrix units, if only $\beta$ is sufficiently small.

Further we know that in a neighborhood of the tracial state (whose size $\epsilon_N$ again depends on the dimension of the algebra) every state is separable. According to the estimates in [8] for every $N$ there exists a critical temperature $\beta(N, \epsilon)$, such that $|\omega_{\beta}(A) - tr(A)| < \epsilon_N ||A|| \quad \forall \beta < \beta(N, \epsilon_N), A \in \mathcal{A}_\Lambda, |\Lambda| < N$. This guarantees the absence of entanglement for sufficiently localized regions and sufficiently high temperature, where the critical value for the temperature depends on the size of the localization and of course on the details of the interaction.

4 Entanglement in quasifree states

The estimates in [7], [8] are very general and take into account that the non commutativity effects increase with increasing size of the algebra. However we know that in
every extremal equilibrium state the correlations tend to infinity with increasing distance. Therefore it is plausible to expect that in general entanglement occurs already for reasonably small subalgebras or not at all. We will examine this behavior in the example of Fermi systems on a lattice with a quasifree time evolution. Though here only the even part of the localized algebras commute and the odd part has to be handled with care [13] entanglement of even states is not effected. Equilibrium states are even, therefore we can rely on the characterization of entanglement offered in [14]. Since the time evolution by assumption is quasifree also the equilibrium states are quasifree. A generale state can be calculated if we know all expectation values of $e^{iA}$, where $A$ runs over all selfadjoint operators quadratic in creation and annihilation operators. Applying the transposed these operators remain quadratic. For a quasifree state the expectation value of $e^{iA}$ is in one to one correspondence to the two point function. Therefore it is sufficient to examine whether under the application of a partial transposed the two point function determines a state.

More precisely we consider a hamiltonian $H = \sum_{x,y} a^+_x V(x - y) a_y$. We can regard $V$ to be an operator in $l^2$. Then the equilibrium state is given by

$$\omega(a^\dagger(f)a(g)) = \langle g| \frac{1}{1+e^{\beta V}} |f \rangle$$

where $f, g \in l^2$. All other expectation values can be written as polynomial over these two point functions. Especially (we assume $\langle f|g \rangle = 0$)

$$\omega(a^\dagger(f)a(g) + a(g)a^\dagger(\bar{f}) + a^\dagger(g)a(f) + a^\dagger(f)a(f) + a(\bar{f})a^\dagger(\bar{f}) + a^\dagger(g)a(g) + a(\bar{g})a^\dagger(\bar{g}) =$$

$$\begin{bmatrix}
  f & a_{11} & a_{12} & 0 & 0 & f  \\
  g & a^\dagger_{12} & a_{22} & 0 & 0 & \bar{g}  \\
  \bar{f} & 0 & 0 & 1 - a_{11} & a^\dagger_{12} & \bar{f}  \\
  \bar{g} & 0 & 0 & a_{12} & 1 - a_{22} & \bar{g}
\end{bmatrix}$$

where $a_{11}, a_{12}, a_{22}$ are determined by (3). More precisely the matrix in (4) consists of the parts

$$\begin{pmatrix}
  A & B  \\
  B^* & 1 - A
\end{pmatrix}$$

where $\omega(a^+(f)a(g)) = \langle g| A|f \rangle$, $\omega(a(g)a^+(f)) = \langle g| 1 - A|f \rangle$, $\omega(a(f)a(g)) = \langle \bar{g}| B|f \rangle$.

In a gauge invariant equilibrium state, as it corresponds to our hamiltonian $H$, $B = 0$. In general $B \neq 0$, but in order that (4) defines a state the above expression has to be positive definite which corresponds to $A(1 - A) \geq B^*B$. If we now apply the transposition only on one part, i.e. $a(f) \rightarrow a(f), a(g) \rightarrow a^\dagger(\bar{g}), a^\dagger(g)a(g) \rightarrow a^\dagger(g)a(g)$ then (4) becomes

$$\begin{bmatrix}
  f & a_{11} & 0 & 0 & 0 & 0 & a_{12} & f  \\
  g & 0 & 0 & 0 & a_{22} & a_{21} & 0 & \bar{g}  \\
  \bar{f} & 0 & 0 & a^\dagger_{21} & 1 - a_{11} & 0 & 0 & \bar{f}  \\
  \bar{g} & a^\dagger & 0 & a_{12} & 0 & 1 - a_{22} & \bar{g}
\end{bmatrix}$$
The positivity requirement becomes

\[ a_{11}(1 - a_{22}) \geq a_{12}a_{12}^*, \tag{7} \]

and the inequality has to hold with respect to all pairs \( f, g \). When we vary \( f \in H_{A_1} \subset l^2 \) and \( g \in H_{A_2} \subset l^2 \) where \( H_{A_1} \) and \( H_{A_2} \) are orthogonal subspaces corresponding to two localized regions then we have to read the equation as equation between operators. If the positivity condition is violated, then according to the results of [14] the state between the algebras \( A_{A_1} \) and \( A_{A_2} \) is entangled, if not then it is at least not distillably entangled. However \( A = \frac{1}{1 + e^\mu} \) converges in norm to \( \frac{1}{2} \) for \( \beta \to 0 \). Therefore (7) is satisfied for sufficiently high temperatures. Especially also the [CHSH] inequality for any choice of operators cannot serve as entanglement witness.

A similar analysis holds, if we consider bosons in a quasi-free state where

\[ \omega(a^*(f)a(g)) = \langle g | \frac{1}{e^{\beta V + \mu} - 1} | f \rangle \tag{8} \]

Then

\[ \omega(a^*(f)a(g) + a(\bar{g})a^*(\bar{f}) + a^*(f)a(f) + a^*(f)a(f) + a(\bar{f})a^*(\bar{f}) + a^*(g)a(g) + a(\bar{g})a^*(\bar{g}) = \]

\[ \begin{pmatrix} f & a_{11} & a_{12} & 0 & 0 & 0 \\ g & a_{12}^* & a_{22} & 0 & 0 & 0 \\ \bar{f} & 0 & 0 & 1 + a_{11} & a_{12}^* & \bar{f} \\ \bar{g} & 0 & 0 & a_{12}^* & 1 + a_{22} & \bar{g} \end{pmatrix} \]

(9)

If we take partitions of the algebra into account that correspond to (6) then the condition on positivity is always satisfied. Entanglement between subalgebras only occurs if the partition into the subalgebras includes a non trivial Bogoliubov transformation, including a mixture of creation and annihilation operators. But even then in the limit \( \beta \to 0 \) \( a_{11}, a_{22} \to \frac{1}{1 + e^\mu}, a_{12} \to 0 \) which again guarantees that for high enough temperature right and left region are not entangled.

The situation is different for continuous systems and vanishing chemical potential. Consider first free fermions. Then

\[ \omega(a^*(f)a(\bar{g})) = \int dp \tilde{f}(p) \tilde{g}(p) \frac{1}{1 + e^{\beta p^2}} \tag{10} \]

Since we can observe entanglement between two appropriate modes \( f, g \) for \( \beta = 1 \) we can observe entanglement for \( \tilde{f}(p) = \sqrt{\beta} \tilde{f}(\beta p), \tilde{g}(p) = \sqrt{\beta} \tilde{g}(\beta p) \). If \( f \) was localized in the right \([0, \infty)\) and \( g \) in the left \((-\infty, 0]\) also the scaled functions are localized and entanglement cannot disappear in the high temperature limit. The same effect occurs in temperature states in relativistic quantum field theories. Here [9] has shown that the Reeh- Schlieder theorem is still valid, the GNS vector is cyclic and separating for every local subalgebra. But this implies entanglement [3]. It means, that the effect of any local manipulation of the state can also be achieved by manipulations done in another local region, which is exactly what entanglement enables.
5 Delocalized entanglement

As stated in [12] on the basis of the quasilocal algebra entanglement can either be observed already on the local level, if we only choose the local region sufficiently large, or it cannot be observed at all. This is in contradiction to the result in [15], where mesoscopic entanglement was measured. However in this context the entanglement witnesses have been no quasilocal operators and also not the mean of quasilocal operators as in [1]. They belong to the algebra of fluctuations. The precise definition of this algebra is given in [16].

If one defines an operator $\vec{S}$ by

$$\hat{\omega}(e^{i\vec{a}\vec{S}}) = \lim_{N \to \infty} \omega_\beta(e^{i\vec{a}\sum \frac{\sigma_k \omega(\sigma_k)}{\sqrt{N}}})$$

then the operators satisfy commutation relations similar to the Weyl algebra

$$\hat{\omega}(e^{iaS_x}e^{ibS_y}) = e^{-iab_{xy}}\hat{\omega}(e^{ibS_y}e^{iaS_x}).$$

where $s_z = \omega(\sigma_z^k)$ in a state invariant under translations. Therefore we can interpret $e^{iaS_x}, e^{iaS_y}$ as operators in a Weyl algebra, that we call fluctuation algebra, and $\hat{\omega}$ as state over it. We can divide the fluctuation algebra into a right and a left part

$$(e^{i\vec{a}\vec{S}_l}) = \lim_{N \to \infty} (e^{i\vec{a}\sum_{k=-N_{\omega}}^{0} \frac{\sigma_k \omega(\sigma_k)}{\sqrt{N}}})$$

$$(e^{i\vec{a}\vec{S}_r}) = \lim_{N \to \infty} (e^{i\vec{a}\sum_{k=1}^{(1-\omega)N} \frac{\sigma_k \omega(\sigma_k)}{\sqrt{N}}})$$

and get in this way two commuting Weyl algebras. As in [17] we consider the equilibrium states corresponding to the two hamiltonians

$$H_N^{(1)} = \sum_k a\sigma_z^k$$

$$H_N^{(2)} = \sum_k c\sigma_z^k + \frac{1}{N} \sum_{k,l} (\sigma_x^k \sigma_x^l + \sigma_y^k \sigma_y^l + \sigma_z^k \sigma_z^l).$$

Both hamiltonians lead to the same type of equilibrium state on the quasilocal algebra, namely

$$\omega_\beta(\sigma^h \sigma^l) = \omega_\beta(\bar{\sigma}^h \bar{\sigma}^l)$$

with $\omega_\beta(\sigma_z^k) = s_z(\beta), \omega_\beta(\sigma_x^k) = \omega_\beta(\sigma_y^k) = 0$, where $a = a(c, \beta) = c + \omega_\beta(\sigma_z^k)$ in (15) is an effective field strength. If we evaluate the right and the left fluctuation algebra in this limit state, then the state factorizes and we have no entanglement. If however we couple the limit in the state with the limit in the operator

$$\hat{\omega}(e^{i\vec{a}\vec{S}}) = \lim_{N \to \infty} \frac{1}{\text{Tr}e^{-\frac{H_N^{(2)}}{N}}} \text{Tr}e^{-\frac{H_N^{(2)}}{N}}(e^{i\vec{a}\sum \frac{\sigma_k \omega(\sigma_k)}{\sqrt{N}}})$$

7
then we are considering the limit of an equilibrium state with respect to a time evolution that converges in the limit to the one determined by $H^{(2)}_N$ and differs from the one given by $H^{(1)}_N$:

$$\frac{d}{dt}S_{rx} = -(c + \frac{2}{N}S_{lz})S_{ry} - \frac{2}{N}S_{rz}S_{ly}, \quad \frac{d}{dt}S_{ry} = (c + \frac{2}{N}S_{lz})S_{rx} - \frac{2}{N}S_{rz}S_{ly} \quad (19)$$

$$\frac{d}{dt}S_{lx} = -(c + \frac{2}{N}S_{rz})S_{ly} - \frac{2}{N}S_{lz}S_{ry}, \quad \frac{d}{dt}S_{ly} = (c + \frac{2}{N}S_{rz})S_{lx} - \frac{2}{N}S_{lz}S_{ry}. \quad (20)$$

$\frac{2}{N}S_{rz}, \frac{2}{N}S_{lz}$ can be replaced by their expectation value. Then the evolution becomes linear and is given by a Hamiltonian of the form

$$H = \nu_1(x^2 + p^2) + \nu_2(y^2 + q^2), \quad [x, p] = i, \quad [y, q] = i \quad (21)$$

with

$$x = a_1S_{rx} + b_1S_{lx}, \quad y = a_2S_{rx} + b_2S_{lx} \quad (22)$$

where the constants can be calculated from (19),(20) and are determined by the effective interaction between right and left. Therefore they depend on $\omega_\beta(\sigma_k z)$ and thus on the temperature. Analyticity properties guarantee that also the limit state over the fluctuation algebra is an equilibrium state with respect to the limit-time evolution of the fluctuation algebra. More precisely we can apply Theorem 5.3.12, 6.3.27 and 6.3.28 of [8]. Since the time derivative is uniformly bounded we have convergence on the boundary of the analyticity strip. On this boundary the Hamiltonian corresponds to the quadratic Hamiltonian (21) with expectation value taken over all coherent states. This domain is sufficient to define the Hamiltonian uniquely. Therefore all conditions of [8] are met. On the fluctuation algebra the equilibrium states respectively the groundstate are those of two harmonic oscillators.

Since we have two different rotation velocities the state, which is a Gaussian state over the two Weyl algebras, does in general not factorize. Depending on the difference between the two velocities, i.e. on the temperature, we can examine entanglement of the right and left fluctuation algebra on the basis of the results in [12], [18]. More explicitly we know that as a consequence of the Weyl relations [19] and if $a_1b_1 + a_2b_2 = 0$ (which happens if $\alpha = \frac{1}{2}$, i.e. if the subalgebras have the same size) that

$$< S_{rx}^2 > + < S_{ry}^2 > \geq |\omega(\sigma^k_z)|, \quad < S_{lx}^2 > + < S_{ly}^2 > \geq |\omega(\sigma^k_z)|$$

For a separable state this implies

$$<(S_{lx} + S_{rx})^2> + <(S_{ly} - S_{lx})^2> \geq |\omega(\sigma^k_z)| \quad (23)$$

Inserting the expectation value corresponding to the temperature this inequality will always hold in the example with $\alpha = \frac{1}{2}$. Nevertheless also in this situation we have non trivial classical correlations. This is also in agreement with the calculation (9). If however we vary the size of the left and right fluctuation algebra we can violate (23) for small temperatures. Especially in the groundstate we have entanglement on the basis of the fluctuation algebra.
References

[1] V. Vedral: High temperature macroscopic entanglement, arXiv quant-ph/0405102
    C. Brukner, v. Vedral: Macroscopic witnesses of quantum entanglement, arXiv quant-ph/0406040

[2] F. Verstraete, M. Popp, J.I. Cirac: Entanglement versus correlations in spin systems, Phys. Rev. Lett 92,027901, 82004)

[3] W.K. Wootters: Entangled chains, Contemporary math. 305, 299 (2002)

[4] M. Fannes, B. Haegeman, M. Mosony: Entropy growth of shift invariant states on a quantum spin chain , J. Math. Phys. 44 6005-6019 (2003), B. Haegeman: Local aspects of quantum entropy, thesis, Leuven (2004)

[5] H.Narnhofer: The role of transposition and CPT operation for entanglement, Phys. Lett.a 310 423-433 (2003)

[6] A.W. Majewski: On the measure of entanglement, J. Phys. A Math. Gen. 35, 123 (2002)

[7] G. Gallavotti, S. Miracle-Sole, D.W. Robinson: Analyticity properties of a lattice gas, Phys. lett.A 25, 493-494 (1967), W. Greenberg: Critical temperature bounds for quantum lattice gases, Commun. Math. Phys. 13, 335-344 (1969)

[8] O. Bratteli, D.W. Robinson: Operator Algebras and Quantum Statistical Mechanics 2, Springer Berlin Heidelberg (1996)

[9] C. Jaekel: The Reeh- Schlieder property for thermal field theories, J. Math. Phys. 41, 1745-1754 (2000)

[10] W. Thirring: On the mathematical structure of the BCS-model II, Commun. mat. Phys. 7, 181-189 (1968)

[11] B Hiesmayr, H. Narnhofer: in preparation

[12] H. Narnhofer: Entanglement, split and nuclearity in quantum field theory, Rep. Math. P hys. 50/1,111-123 (2002)

[13] H. Moriya: Separability condition for the states of fermion lattice systems and its characterization, arXiv:quant-ph/0405166

[14] M. Horodecki, P. Horodecki, R. Horodecki: Separability of mixed states. necessary and sufficient conditions, Phys. Lett. A 223,1-8, (1996)

[15] B Julsgard, A. Koshekin, E.S. Polzik: Lett. to Nature 413, 400 (2001)

[16] D. Goderis, A. Verbeure, P. Vets, Commun. Math. Phys. 128, 533 (1990)
[17] H. Narnhofer: The Time Evolution of the Fluctuation Algebra in Mean Field Theories, Found. of Phys. Lett., 17, 3, 235-253 (2004)

[18] R. Simon: Phys. Rev. Lett. 84, 2726 (2000)

[19] H. Narnhofer, W. Thirring: Phys. Rev. A 66, 031211 (2002)