Quantum Error Correction with Uniformly Mixed State Ancillae

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It is often assumed that the ancilla qubits required for encoding a qubit in quantum error correction (QEC) have to be in pure states, |00...0⟩ for example. In this letter, we introduce an encoding scheme avoiding fully correlated errors, in which the ancillae may be in a uniformly mixed state. We demonstrate our scheme experimentally by making use of a three-qubit NMR quantum computer. Moreover, the encoded state has an interesting nature in terms of Quantum Discord, or purely quantum correlations between the data-qubit and the ancillae.

A quantum computer is vulnerable against environmental noise and it must be protected by one way or another. Quantum error correction (QEC) is one of the most successful approaches to this end [1]. Despite this great success, QEC requires expensive resources, or ancillae that are usually assumed to be in pure states [2,3]. However, it is not yet proved that ancillae in uniformly mixed states are useless. We extend previous works [4] and show an encoding scheme robust against fully correlated noise in which all the ancillae can be in uniformly mixed states. The encoded state has an interesting nature in terms of Quantum Discord, or purely quantum correlations between the data-qubit and the ancillae. Our QEC scheme also provides an example of Deterministic Quantum Computation with 1-Qubit (DQC-1) [6,7].

Suppose we have a single qubit in an arbitrary state ρ1, which we want to protect from noise. We introduce some additional qubits (ancillae) in order to protect the first qubit and suppose that all the qubits suffer from the same noise. Such a noise is called fully correlated and it may happen when the dimensions of the quantum computer are microscopic compared with the wavelength of external disturbances. Noiseless subsystem (NS) [8,9] and decoherence free subsystem (DFS) [10,11] are well known strategies to protect a system from such fully correlated noises [12,13]. These schemes, however, require ancillae in pure states and thus they are expensive.

In the following, we show that it is indeed possible to devise a cheaper QEC scheme employing ancillae in the uniformly mixed state. Let

\[ \rho_1 = \sigma_0/2 + (n_x,n_y,n_z) \cdot (\sigma_x,\sigma_y,\sigma_z)/2 \]

be the state of the qubit to be protected. Here σ0 is a unit matrix of dimension 2, \( \mathbf{n} = (n_x,n_y,n_z) \) is the Bloch vector, and σi is the ith component of the Pauli matrices. We introduce two ancillae in uniformly mixed states, whose Bloch vectors are \( \mathbf{0} \). The initial state of the three-qubit system is thus a tensor product state \( \rho_1 \otimes \sigma_0/2 \otimes \sigma_0/2 \). The unitary encoding operator \( U_E \) transforms the tensor product state to an entangled state \( \tilde{\rho}_3 \). If the state of the system is again \( \tilde{\rho}_3 \) even after the action of noises, a unitary recovery operator, \( U_R = U_E^{-1} \), transforms \( \tilde{\rho}_3 \) back to the initial tensor product state \( \rho_1 \otimes \sigma_0/2 \otimes \sigma_0/2 \) and \( \rho_1 \) can be recovered after tracing over the ancilla states.

It is highly counterintuitive that a QEC scheme works with ancillae in uniformly mixed states. The trick is that the uniformly mixed state \( (\sigma_0/2)^{\otimes 2} \) is rewritten as

\[
\frac{1}{4} | (n_2,n'_2) \rangle \langle n_2,n'_2 | + | -n_2,-n'_2 \rangle \langle -n_2,-n'_2 | \\
+ | -n_2,n'_2 \rangle \langle -n_2,n'_2 | + | n_2,-n'_2 \rangle \langle n_2,-n'_2 | \]  

where \( n_2 \) and \( n'_2 \) are arbitrary Bloch vectors \( |n_2| = |n'_2| = 1 \) and \( |n_2| \) and \( |n'_2| \) are pure states corresponding to \( n_2 \) and \( n'_2 \), respectively. If a QEC scheme works with arbitrary pure ancilla states, the superposition principle of quantum mechanics guarantees that ancillae in a uniformly mixed state do work as well.

A more formal description is given as follows. Suppose we have a single qubit in a state \( \rho_1 \), which we want to protect from noise operators \( \{\sigma_x,\sigma_y,\sigma_z\} \). To this end, we introduce two additional qubits, which may be in an arbitrary state \( \rho_2 \), and apply a suitable encoding operator \( U_E \) on \( \rho_1 \otimes \rho_2 \) to obtain a codeword \( \tilde{\rho}_3 = U_E (\rho_1 \otimes \rho_2) U_E^{\dagger} \). We introduce the fully correlated error channel \( \Phi \) represented by

\[ \Phi(\tilde{\rho}_3) = \sum_{i=0}^{3} p_i X_i \tilde{\rho}_3 X_i^\dagger, \]

where \( X_0 = \sigma_0^{\otimes 3}, X_1 = \sigma_x^{\otimes 3}, X_2 = \sigma_y^{\otimes 3}, X_3 = \sigma_z^{\otimes 3} \). Here \( p_i \geq 0 \) is the probability with which an error operator \( X_i \) acts on \( \tilde{\rho}_3 \) and we assume \( \sum_{i=0}^{3} p_i = 1 \). Suppose there is an encoding operator \( U_E \) satisfying

\[ U_E^{\dagger} X_i U_E = \sigma_0 \otimes M_i \]

for \( i = 1, 2, 3 \). Then, \( U_E \) defines the QEC scheme that we are seeking. We can show that

\[ U_R \Phi (U_E (\rho_1 \otimes \rho_2) U_E^{\dagger}) U_R^{\dagger} = \sum_{i=0}^{3} p_i (\rho_1 \otimes M_i \rho_2 M_i^\dagger) \]

\[ = \rho_1 \otimes \rho'_2, \]

where \( \rho'_2 = \sum_i p_i M_i \rho_2 M_i^\dagger \). This proves that, after decoding, the error channel \( \Phi \) affects only \( \rho_2 \) but not \( \rho_1 \).

There are infinitely many choices of \( U_E \) but careful inspection of the error operators reveals that \( U_E = \)
is summarized in Fig. 2.

Let us compare our QEC scheme with the NS encoding scheme discussed in [4]. This NS encoding scheme employs three qubits to encode a logical qubit robust against any noise of the form \( W^{\otimes 3} \), where \( W \) is an arbitrary element of the 2-dimensional representation of SU(2).

\[
U_{CNOT12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

The extreme case of the uniformly mixed state \( \rho_2 = \sigma_0^{\otimes 2}/4 \) is worth analyzing separately. When the data qubit is in a pure state, this provides an interesting example of DQC-1 [5]. Moreover, our QEC scheme equally works for a data qubit in a mixed initial state. It is readily found that

\[
U_{E} \Phi \left( U_{E} \left( \rho_1 \otimes \frac{\sigma_0^{\otimes 2}}{4} \right) U_{E}^\dagger \right) U_{E}^\dagger = \rho_1 \otimes \frac{\sigma_0^{\otimes 2}}{4}.
\]

This shows that \( \rho_1 \otimes \frac{\sigma_0^{\otimes 2}}{4} \) is a fixed point of this operation for any \( \rho_1 \in S_1 \).

Let us define the conditional entropy by

\[
S(\rho_{1|2}) = \sum_{m} p_+ S(\rho_{2|m}) + p_- S(\rho_{2|m^\perp})
\]

where \( p_\pm = \text{Tr}(\rho_{2|m}^\pm) \). Then, \( S(2|1) \) is defined similarly.

The explicit form of the ancilla state after the measurement of the data qubit with a basis \( \{| \pm m \rangle \} \) is

\[
\rho_{2|\pm} = \frac{1}{4} \left( \sigma_0 \otimes \sigma_0 \pm n_x m_x \sigma_x \otimes \sigma_0 \pm n_y m_y \sigma_y \otimes \sigma_z \pm n_z m_z \sigma_x \otimes \sigma_0 \right).
\]
The Bloch vector \( \mathbf{n} \) is defined as \((r, \theta_m, \phi_m) = (D(\pm \mathbf{m}))(2 : 1), \theta_m, \phi_m)\) is plotted, where \(\theta_m\) and \(\phi_m\) specify \(\mathbf{m} = (\sin \theta_m \cos \phi_m, \sin \theta_m \sin \phi_m, \cos \theta_m)\). \(D(\pm \mathbf{m}))(2 : 1) = 0\) for \(\mathbf{m} = \pm \hat{x}\). Therefore, \(D(2 : 1)\) vanishes for \(\mathbf{n} = \hat{x}\).

The corresponding conditional entropy is

\[
S_{(\pm \mathbf{m})}(2|1) = 2 - \frac{1}{8} \sum_{j=1}^{8} (1 + \mathbf{n} \cdot \mathbf{n}_j) \log_2 (1 + \mathbf{n} \cdot \mathbf{n}_j),
\]

where \(\mathbf{n}_j = (\pm m_x, \pm m_y, \pm m_z)\) are all eight combinations of three \(\pm\).

As an example, we show \(D_{(\pm \mathbf{m})}(2 : 1) = S(\hat{\rho}_1) - S(\hat{\rho}_3) + S_{(\pm \mathbf{m})}(2|1)\) for the initial state \(\rho_1\) with the Bloch vector \(\mathbf{n} = \hat{x}\) as a function of \(\mathbf{m} = (\sin \theta_m \cos \phi_m, \sin \theta_m \sin \phi_m, \cos \theta_m)\) in Fig. 3. Here, \(\hat{x}\), \(\hat{y}\), and \(\hat{z}\) are the unit vectors along the \(x\)-, \(y\)-, and \(z\)-axes, respectively. Note that when \(\mathbf{m} = \hat{x}\), \(D_{(\pm \mathbf{m})}(2 : 1) = 0\). Therefore, \(D(2 : 1) = 0\). The quantum discord \(D(2 : 1)\) as a function of \(\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) is shown in Fig. 4.

Although extensive optimization is necessary to evaluate QD in general, some initial states satisfying \(D(2 : 1) = 0\) are easily obtained by carefully inspecting the structure of \(\hat{\rho}_3\). Let us consider the case \(\mathbf{n} = \hat{x}\), for example. In this case, \(\hat{\rho}_3\) is reduced to \(\frac{1}{2} (\sigma_0 \otimes \sigma_0 \otimes \sigma_0 + \sigma_x \otimes \sigma_x \otimes \sigma_0)\), which contains only \(\sigma_0\) and \(\sigma_x\) for the data qubit. Therefore, it is reasonable to employ \(\hat{\rho}_3\) as a candidate for \(|\mu\rangle\) for obtaining \(S(2|1)\). With this choice, \(\hat{\rho}_3\) is reduced to a block-diagonal form

\[
\hat{\rho}_3 (\mathbf{n} = \hat{x}) = \sum_{\pm} |\pm \hat{x}\rangle \langle \pm \hat{x}| \otimes (p_\pm \rho_{2|\pm})
\]

and \(D(2 : 1) = 0\) is readily obtained \([5, 20]\).

\(D(1 : 2) = 0\) for an arbitrary initial state can be proved similarly. \(\hat{\rho}_3\) contains only \(\sigma_0 \otimes \sigma_0, \sigma_x \otimes \sigma_0, \sigma_0 \otimes \sigma_z\), and \(\sigma_x \otimes \sigma_z\) for the ancilla qubits. Therefore, we take

\[
|\Pi_{\pm\pm}\rangle / |\Pi_{\pm\pm}\rangle = |\pm \hat{x}, \pm \hat{z}\rangle \langle \pm \hat{x}, \pm \hat{z}| = \frac{\sigma_0 + \sigma_x}{2} \otimes \frac{\sigma_0 + \sigma_x}{2}
\]

as a complete set of four unit vectors that determine the projective measurement on the ancilla, although there are many other possibilities. \(\hat{\rho}_3\) is rewritten as

\[
\hat{\rho}_3 = \sum_{\pm \pm} (p_{\pm \pm} \rho_{1|\pm \pm}) \otimes |\Pi_{\pm\pm}\rangle / |\Pi_{\pm\pm}\rangle.
\]

When the data qubit and ancillae are rearranged, the density matrix is rewritten as a block-diagonal form and thus \(D(1 : 2) = 0\) is immediately obtained.

According to the classification introduced in \([21, 22]\), vanishing \(D(1 : 2)\) implies that our encoded state has a quantum-classical correlation. Furthermore, in case \(D(2 : 1)\) also vanishes, \(\hat{\rho}_3\) has a product eigenbasis as we have shown above, and the encoded state has a classical-classical correlation, or, in other words, is (properly) classically correlated.

When the ancillae are pure, we find \(D(2 : 1) = D(1 : 2)\), which is nothing but the entanglement entropy. For example,

\[
D(2 : 1) = D(1 : 2) = 2 - (1 - n_z) \log_2 (1 - n_z) - (1 + n_z) \log_2 (1 + n_z),
\]

when \(\rho_2 = |z\rangle \langle z| \otimes |z\rangle \langle z|\).

We demonstrate our QEC scheme with a NMR quantum computer. We employ a JEOL ECA-500 NMR spectrometer \([23]\), whose hydrogen Larmor frequency is approximately 500 MHz. We employ a linearly aligned three-spin molecule, \(^{13}\)C-labeled L-alanine (98% purity, Cambridge Isotope) solved in \(\text{D}_2\text{O}\).

We simplify the quantum circuit shown in Fig. 1 by taking into account the fact that the phases of states are not independently observed in a NMR quantum computer. Both the encoding and the decoding require only 5 pulses including refocusing pulses, taking approximately 25 ms.

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**FIG. 3:** Example of \(D_{(\pm \mathbf{m})}(2 : 1)\) when the initial state of the data qubit has the Bloch vector \(\mathbf{n} = \hat{x}\). \((r, \theta_m, \phi_m) = (D_{(\pm \mathbf{m})}(2 : 1), \theta_m, \phi_m)\) is plotted, where \(\theta_m\) and \(\phi_m\) specify \(\mathbf{m} = (\sin \theta_m \cos \phi_m, \sin \theta_m \sin \phi_m, \cos \theta_m)\). \(D_{(\pm \mathbf{m})}(2 : 1) = 0\) for \(\mathbf{m} = \pm \hat{x}\). Therefore, \(D(2 : 1)\) vanishes for \(\mathbf{n} = \hat{x}\).

**FIG. 4:** Quantum discord \(D(2 : 1)\) as a function of the initial state of the data qubit parameterized by the Bloch vector \(\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\). See Eq. 1. Coordinates \((r = D(2 : 1), \theta, \phi)\) depict QD as a function of \(\theta, \phi\). \(D(2 : 1)\) vanishes when \(\mathbf{n} = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\). The function \(D(2 : 1)\) takes the maximum value \((3/4) \log_2 3 - 1/2\) when \(\mathbf{n} = (\pm 1, \pm 1, 1)/\sqrt{3}\).
that our QEC scheme indeed eliminates the effects of the fully correlated noises. The table in Fig. 5 summarizes the entanglement fidelities.

It is noteworthy that one-qubit gate operations on the logical qubit take simple forms. Let $V$ be a one-qubit gate acting on the logical qubit. Then its action on the physical qubits is obtained by simplifying $U_E(V \otimes \sigma_0 \otimes \sigma_0) U_R$. For the simple gates $V = \sigma_x$, $\sigma_y$ and $\sigma_z$, the corresponding operations on the physical qubits are $\sigma_x \otimes \sigma_x \otimes \sigma_0$, $\sigma_y \otimes \sigma_0 \otimes \sigma_z$ and $\sigma_z \otimes \sigma_0 \otimes \sigma_0$, respectively. Note that these operators satisfy the ordinary $su(2)$ algebra. It is easy to obtain more general gate operations acting on the physical qubits by simply exponentiating these operators, e.g., $e^{-i\alpha \sigma_x} \otimes \sigma_0 \otimes \sigma_0$ implements $V = e^{-i\alpha \sigma_z}$. Note that $e^{-i\beta \sigma_y} \otimes \sigma_0 \otimes \sigma_z = e^{-i\pi(\sigma_z \otimes \sigma_0 \otimes \sigma_z)/4} e^{-i\beta(\sigma_y \otimes \sigma_0 \otimes \sigma_0 + i\pi(\sigma_z \otimes \sigma_0 \otimes \sigma_z))/4}$ in the case of $V = e^{-i\beta \sigma_y}$. From these operators, we can understand how the information of the data qubit is distributed in the encoded state. We note that direct operations on logical qubits in DFS/NS were discussed in [26].

In summary, we demonstrated a quantum error correction scheme avoiding fully correlated errors, in which the ancillae can be in a uniformly mixed state. Our results pave the way to new applications of DQC-1 to quantum computing. The analysis of quantum discord reveals that our encoding creates an interesting quantum correlation between the data qubit and the ancilla qubits; our encoded state has a quantum-classical correlation in general and has a classical-classical correlation when both left and right quantum discords vanish. We anticipate further progress both in the understanding of quantum correlations and the development of QEC schemes. Our QEC scheme admits simple one-qubit gate operations on the encoded qubits.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Error Operator & $X_0$ & $X_1$ & $X_2$ \\
\hline
$F_k(\sigma_0, \mathcal{M})$ & 0.68 & 0.73 & 0.75 \\
Tr($\mathcal{M}$) & 1.03 & 1.00 & 1.00 \\
\hline
\end{tabular}
\caption{Visualization of error correction performances. The surface of the Bloch sphere is mapped onto the surfaces in (a), (b) and (c) corresponding to three different error operators, $X_0, X_1, X_2$, respectively. We do not need to examine the density matrix of the thermal state is well approximated by
\[ \rho = (\sigma_0/2)^{\otimes 3} + \frac{\epsilon}{8} (\sigma_x \otimes \sigma_0 \otimes \sigma_0 + \sigma_0 \otimes \sigma_z \otimes \sigma_0 + \sigma_0 \otimes \sigma_0 \otimes \sigma_z) , \]
where $\epsilon \sim 10^{-6}$. Since $(\sigma_0/2)^{\otimes 3}$ is not visible in NMR, the density matrix of the thermal state is considered as a pseudo-pure state for DQC-1.

We perform three sets of experiments, in which we set
\begin{enumerate}
\item $\{p_i\} = (1, 0, 0, 0) : \text{no error}$
\item $\{p_i\} = (0, 1, 0, 0) : X_1 \text{ error}$
\item $\{p_i\} = (0, 0, 1, 0) : X_2 \text{ error}$
\end{enumerate}
respectively, in Eq. (3). We do not need to examine the $X_3$ error separately since $X_3 = iX_2X_1$. Each set starts with 4 different initial states in order to apply quantum process tomography [24]. The results are summarized in Fig. 5. Although the surfaces are distorted, it is clear
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