RATIONAL POINTS ON CURVES OVER FUNCTION FIELDS

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Abstract. We provide in this paper an upper bound for the number of rational points on a curve defined over a one variable function field over a finite field. The bound only depends on the curve and the field, but not on the Jacobian variety of the curve.

Let \( k \) be a finite field of positive characteristic \( p \) and \( C \) a smooth, projective, geometrically connected curve defined over \( k \) of genus \( g \). Denote by \( K = k(C) \) its function field. Let \( K_s \) be a separable closure of \( K \). Given a smooth, projective, geometrically connected curve \( X \) defined over \( K \) of genus \( d \geq 2 \), the analogue of the Mordell's conjecture asks whether the set \( X(K) \) is finite.

This does not come without a constraint, otherwise this question would have a trivial negative answer. One has to assume that \( X \) is non-isotrivial. This means that there does not exist a smooth projective geometrically connected curve \( X_0 \) defined over a finite extension \( l \) of \( k \) and a common extension \( L \) of both \( K \) and \( l \) such that \( X \times_K L \cong X_0 \times_l L \) (cf. [Sa66]). Under the aforementioned condition the finiteness of \( X(K) \) is a theorem due to Samuel [Sa66].

Our purpose in this note is to give an effective bound for the cardinality of the set \( X(K) \) in terms of the minimal number of invariants associated with our given geometric situation. Namely, our upper bound will depend just on \( d \), \( g \) and the conductor of \( X \) (this will be defined later in the text). Let us insist on the fact that this bound does not depend on the Jacobian of the curve \( X \), in particular there is no need of calculating the \( K \)-rank of the Jacobian to effectively use the bound.

The history of explicit upper bounds for \( \#X(K) \) starts with the work of Szpiro [Sz81] which in fact gives an explicit upper bound for the height of points in \( X(K) \). This however depends on the geometry of a semi-stable fibration on curves \( \phi : X \to C \) which gives a minimal model of \( X/K \) over \( C \).

We will follow another approach. This is based on a work of Buium and Voloch [BuVo96] that gives an explicit bound for a conjecture of Lang. This conjecture has Mordell’s conjecture as a special case. Another very important ingredient is an inequality relating the conductor of the curve and the conductor of its Jacobian \( J \) which originates in [Bl87] (see Proposition 7). Denote by \( j : X \hookrightarrow J \) an embedding of \( X \) into \( J \).

In the next theorem the authors need the hypothesis: \( X \) is defined over \( K \) but not over \( K^p \). It turns out that this is equivalent to the Kodaira-Spencer map of the
Jacobian variety $J$ of $X$ being non-zero (for more details see [Vo91] and [Sz81]). This hypothesis also implies that $X/K$ is non-isotrivial.

**Theorem 1** (Buium-Voloch). [BuVo96] Theorem Let $k$ be a finite field of characteristic $p$, $K$ a one variable function field over $k$, $X/K$ a smooth, projective, geometrically curve defined over $K$ of genus $d \geq 2$. We suppose that $X$ is not defined over $K^p$. Let $\Gamma$ a subgroup of $\text{J}(K_s)$ such that $\Gamma/p\Gamma$ is finite. The following inequality holds:

$$\#(X \cap \Gamma) \leq \#(\Gamma/p\Gamma).p^d.3^d.(8d - 2).d!$$

**Remark 2.** Take $\Gamma = J(K)$. Then by the Mordell-Weil theorem $J(K)/pJ(K)$ is a finite group. Writing $J(K) = \mathbb{Z}^r \times J(K)_{\text{tor}}$ where $r = \text{rk}(J(K))$, one has $J(K)/pJ(K) = (\mathbb{Z}/p\mathbb{Z})^r \times J(K)_{\text{tor}}/pJ(K)_{\text{tor}}$. Its order is bounded from above by $p^{d+r}$. Next we discuss an upper bound for the rank.

**Remark 3.** Let $k$ be any field and $\mathcal{C}$ smooth projective geometrically connected curve over $k$. Denote by $K = k(\mathcal{C})$ its function field. Let $A/K$ be a non-constant abelian variety over $K$ and denote by $(\tau, B)$ its $K/k$-trace (cf. [La83]). Let $k$ be an algebraic closure of $k$. A theorem due to Lang and Néron ([La83], [LaNe59]) states that the quotient group $A(k(\mathcal{C}))/\tau B(\bar{k})$ is a finitely generated abelian group. A fortiori, the quotient group $A(K)/\tau B(k)$ is also finitely generated. Ogg in the 60’s (cf. [Ogg62], [Ogg67]) produced the following upper bound for the rank of the geometric quotient $A(k(\mathcal{C}))/\tau B(\bar{k})$ (hence of $A(K)/\tau B(k)$). Below we define the conductor $f_{A/K}$ of $A/K$. Let $d_0 = \dim(B)$. Then the upper bound is $2d(2g - 2) + f_{A/K} + 4d_0 \leq 4dg + f_{A/K}$. In particular, if $K$ is a one variable function field over a finite field, then $\text{rk}(A(K)) \leq 4dg + f_{A/K}$.

**Definition 4.** Let $\ell$ be a prime number different from the characteristic of $k$. Denote by $T_{\ell}(A)$ the $\ell$-adic Tate module of $A$ and define $V_\ell(A) = T_{\ell}(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. For each place $v$ of $K$ denote by $I_v$ an inertia group at $v$ (well-defined up to conjugation). Let $e_v$ be the codimension of the subgroup of $I_v$-invariants $V_\ell(A)^I_v$ in $V_\ell(A)$. Let $\delta_v$ be the Swan conductor of $H^1_{et}(A_{K_v}, \mathbb{Q}_\ell)$ (cf. [Sc69]). Define the conductor divisor $\hat{S}_{A/K} = \sum_v (e_v + \delta_v) \cdot v$. Its degree is denoted by $f_{A/K}$.

**Definition 5.** A model of $X/K$ over $\mathcal{C}$ is a smooth projectively connected surface $\mathcal{X}$ defined over $k$ and a proper flat morphism $\phi : \mathcal{X} \rightarrow \mathcal{C}$. Each place $v$ of $K$ is identified with a point of $\mathcal{C}$. Denote by $\kappa_v$ the residue field at $v$ (which is a finite field) and let $\kappa_v$ be an algebraic closure of $\kappa_v$. Denote by $\mathcal{X}_v$ the fiber of $\phi$ at $v$. For an algebraic variety $Z$ defined over a field $l$ and for an extension $L$ of $l$ denote $Z_L = Z \times_{l} L$.

**Definition 6.** Fix a place $v$ of $K$. The Artin conductor of the curve $X$ over $K$ at $v$ is defined as $f_{X/K,v} = -\chi(X_{K_v}) + \chi(\mathcal{X}_{v,\kappa_v}) + \delta_v$, where $\chi(X_{K_v})$, respectively $\chi(\mathcal{X}_{v,\kappa_v})$ denotes the Euler-Poincaré characteristic of $X_{K_v}$, respectively $\mathcal{X}_{v,\kappa_v}$. The term $\delta_v$ denotes the Swan conductor of $H^1(X_{\bar{K}_v}, \mathbb{Q}_\ell)$ at $v$ (cf. [LiSa00] end of p. 414) for the definition, [Sc69] for the definition of the Swan conductor, as well as [Bl87, 1]). Define $f_{X/K} = \sum_v f_{X/K,v} \cdot \deg v$ to be the global conductor of the curve $X/K$.

The following result is a consequence of the subsequent lemma in [Bl87].

**Proposition 7.** We have the inequality $f_{J/K} \leq f_{X/K}$. 
Lemma 8. [Bl87, Lemma 1.2] Fix a place \( v \) of \( K \) and let \( I_v \) be an inertia subgroup of \( \text{Gal}(K_s/K) \) at \( v \). Then:

1. \( H^1_{\text{ét}}(X_{K_s}, \mathbb{Q}_\ell)^I_v \cong H^1_{\text{ét}}(X_{v, \bar{\kappa}_v}, \mathbb{Q}_\ell) \) for \( i = 0, 1 \).
2. Let \( M_v \) be the free abelian group generated by the irreducible components of \( X_{v, \bar{\kappa}_v} \). Since the individual components are not necessarily defined over \( \kappa_v \), there is an action of \( \hat{\mathbb{Z}} \cong \text{Gal}(\bar{\kappa}_v/\kappa_v) \) on \( M_v \). Moreover, there is an exact sequence of \( \hat{\mathbb{Z}} \)-modules:

\[
0 \to \mathbb{Q}_\ell(-1) \to M_v \otimes \mathbb{Q}_\ell(-1) \to H^2_{\text{ét}}(X_{v, \bar{\kappa}_v}, \mathbb{Q}_\ell) \to H^2_{\text{ét}}(X_{K_s}, \mathbb{Q}_\ell)^I_v \to 0.
\]

Remark 9. The definition of the conductor given in [LiSa00] agrees with that given in [Bl87] (up to sign).

Proof of Proposition 9. It follows from the definition of \( f_{X/K,v} \), Lemma 8 and the fact that the action of the Galois group \( \text{Gal}(K_s/K) \) on the étale cohomology groups \( H^i_{\text{ét}}(X_{K_s}, \mathbb{Q}_\ell) \) (for \( i = 0, 2 \)) is trivial that we have an equality:

\[
f_{X/K,v} = \dim(H^1_{\text{ét}}(X_{K_s}, \mathbb{Q}_\ell)) - \dim(H^1_{\text{ét}}(X_{K_s}, \mathbb{Q}_\ell)^I_v) + m_v - 1 + \delta_v,
\]

where \( m_v \) denotes the number of the irreducible components of \( X_{v, \bar{\kappa}_v} \). The proposition now follows from observing that \( H^2_{\text{ét}}(X_{K_s}, \mathbb{Q}_\ell) \cong H^2_{\text{ét}}(J_{K_s}, \mathbb{Q}_\ell) \) (cf. [Mi85, Corollary 9.6]).

Theorem 10. Let \( k \) be a finite field of characteristic \( p \), \( C \) a smooth, projective, geometrically connected curve defined over \( k \) of genus \( g \) and denote by \( K = k(C) \) its function field. Let \( X/K \) be a smooth, projective, geometrically connected curve defined over \( K \) of genus \( d \geq 2 \). We assume that \( X \) is not defined over \( K^p \). Then the following inequality holds:

\[
\#X(K) \leq p^{2d(2g+1)+f_{X/K}}.3^d.(8d-2).d!.
\]

Proof. Denote by \( X(K) = \{ x_1, \ldots, x_m \} \) the finite set of \( K \)-rational points of \( X \). Let \( \Gamma \) be the subgroup of \( J(K) \) generated by the images \( \{ j(x_1), \ldots, j(x_m) \} \) of these points under the embedding \( j : X \hookrightarrow J \) of \( X \) into its Jacobian variety \( J \). Observe that \( \#(\Gamma/p\Gamma) \leq \#(J(K)/pJ(K)) \leq p^{r+d} \leq p^{(d(2g+1)+f_{X/K})/d} \) by Remark 3 and Proposition 7. The result is now a consequence of Theorem 1.

Remark 11. We would now like to compare our result with a result similar in nature when we replace the one variable function field \( K \) defined over a finite field \( k \) by a number field \( K \). In order to do this we refer to the work of Rémond (cf. [Re10]).

Theorem 12 (Rémond). Let \( X \) be a smooth, projective, geometrically connected curve of genus \( d \geq 2 \) defined over a number field \( K \), then one has

\[
\#X(K) \leq (2^{3s+2d} \cdot [K : \mathbb{Q}] \cdot d \cdot \max(1, h_\theta))^{(r+1)}d^{2^9},
\]

where \( h_\theta \) is the theta height of \( J \) and \( r = \text{rk}(J(K)) \).

Remark 13. Using Proposition 5.1 page 775 of [Re10], one has \( r \ll \log(f_{J/K}) \), as in the function field case, but the bound on the number of points is still depending on the height of the Jacobian variety. To be more precise, Rémond shows in loc. cit. how to produce a bound depending on the height of a model of the curve (and not of its Jacobian variety), but it seems difficult to get rid of this height. It would be a consequence of a conjecture of Lang and Silverman, as explained in
the introduction of [Pa12]. Note that in the function field case, the height of the variety \( J \) is comparable to the degree of the conductor \( f_{J/K} \), as shown in [HiPa11 Corollary 6.12].

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