On the Existence of Pareto Efficient and Envy Free Allocations

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Abstract

Envy-freeness and Pareto Efficiency are two major goals in welfare economics. The existence of an allocation that satisfies both conditions has been studied for a long time. Whether items are indivisible or divisible, it is impossible to achieve envy-freeness and Pareto Efficiency ex post even in the case of two people and two items. In contrast, in this work, we prove that, for any cardinal utility functions (including complementary utilities for example) and for any number of items and players, there always exists an ex ante mixed allocation which is envy-free and Pareto Efficient, assuming the allowable assignments are closed under swaps, i.e. if given a legal assignment, swapping any two players allocations produces another legal assignment. The problem remains open in the divisible case.

1 Introduction

Efficiency and fairness are two important goals in welfare economics. Pareto Efficiency and envy-freeness are the foremost notions of, respectively, efficiency and fairness for the allocation problem. A given allocation is Pareto Efficient if there is no other allocation in which no one loses and at least one person gains, and it is envy-free if no person can gain by exchanging her bundle with someone else's.

The question of whether there exists an allocation that is both Pareto Efficient and envy-free has been studied for a long time. Unfortunately, for general utility functions, in both the divisible and indivisible cases, solutions that are simultaneously Pareto Efficient and envy-free cannot be ensured. In the indivisible case, in which items cannot be split, allocating one item among two people who both value the item will never be envy-free and, in the divisible case, there is a well-known example which comprises two items
and two players such that there is no simultaneously Pareto Efficient and envy-free allocation.

However, in these counter-examples, the allocations are deterministic. In other words, there is no randomness. Randomness is often the enabler for existence; for instance, the existence of a Nash equilibrium. So, for the allocation problem, what happens if we consider a mixed allocation instead of a pure allocation? Does there exist a mixed Pareto Efficient and envy-free allocation?

In this paper, we focus on the indivisible case, and our answer is YES. We prove that, for any cardinal utility functions and for any number of items and players, there always exists an ex ante mixed allocation which is envy-free and Pareto Efficient, assuming the allowable assignments are swappable. An allocation set is swappable if the allocation that results from any single pair of players exchanging their allocated bundles is also allowable. Clearly, the allocation set that can allocate any subset of items to any player is swappable.

Our approach is to construct a mapping from the space of mixed allocations and weight vectors to itself. We then apply the Kakutani fixed-point theorem [19] to obtain a fixed point. Finally, we prove that the fixed point corresponds to a mixed Pareto Efficient and envy-free allocation. The proof is inspired by [27, 2, 31].

2 Related Work

A detailed survey on fairness and further background can be found in [6, 7, 24].

Research on fair allocation research dates back to at least [26]. A fair allocation is defined as a Pareto Efficient allocation in which everyone prefers their own bundle to other players’ bundles, which is exactly the notion of envy-freeness proposed in [17].

The existence of Pareto Efficient and envy-free allocations has been studied in both the divisible and indivisible cases.

When items are divisible, previous work [29, 27, 13, 28, 30] showed that Pareto Efficient and envy-free allocations exist under a variety of assumptions, including that utility functions are strictly monotone, continuous, or convex. In contrast, Vohra [30] showed that when the economy has increasing-marginal-returns, there exist cases such that no Pareto Efficient and envy-free allocation exists. Also, Maniquet [23] gave an example with two items and three players for which there is no Pareto Efficient and envy-
In the indivisible setting, for the case of mixed allocations, Bogomolnaia and Moulin [5] introduced the Probabilistic Serial mechanism and showed this new mechanism results in an *ordinally efficient* expected matching which is envy-free in their setting. Ordinal efficiency is a notion which is slightly weaker than Pareto Efficiency. Budish et al. [9] gave a Pareto Efficient and envy-free allocation when the allocation constraints satisfy a bihierarchy assumption which applies to multi-item allocation problems with possibly non-linear utility functions.

For the deterministic case, because of the simple counter-example mentioned above, researchers have proposed many other notions of fairness. The two most closely related notions are *EF1* (*envy free up to one good*) [8] and *EFX* (*envy free up to any good*) [10]. Recall that the idea in the definition of envy-freeness is that each player will compare their bundle to those of the other players. These alternate notions also have players compare their bundle to the other players’ bundles, but in *EF1*, players delete their favorite item from the other bundle before doing the comparison, and in *EFX*, players will not envy another bundle after deleting their least favorite item. Lipton et al. [22] showed that an *EF1* allocation always exists. For the *EFX* allocation, Plaut and Roughgarden [25] showed that in some situations (utility functions are identical or additive) existence is guaranteed, while for general utility functions, there exist examples such that no *EFX* allocation is Pareto Efficient.

In addition, Dickerson et al. [16] showed that if the number of items is at least a logarithmic factor larger than the number of players, then with high probability, an envy free allocation exists.

Other fairness notions include Nash Social Welfare [10, 13, 12], max-min fairness [20], and CEEI [29].

There has been considerable recent work [13, 1, 3, 4, 11, 14, 18, 21] on the computational complexity of computing the Nash Social Welfare, both exactly and approximately, for divisible and indivisible items.

### 3 Notations and Result

There are $m$ items and $n$ players. Each player have a positive utility function $u_i(x_i)$ on each subset $x_i \subseteq \{1, 2, \ldots, m\}$. And there are $k$ allocations, $A^{(1)}, A^{(2)}, \ldots, A^{(k)}$. Allocation $A^{(j)}$ allocates $A^{(j)}_i$ to player $i$, where $A^{(j)}_i \subseteq \{1, 2, \ldots, m\}$ and $A^{(j)}_i \cap A^{(j)}_{i'} = \emptyset$ for any $i$ and $i'$. We say the allocation set is *swappable* if and only if for any allocation $A^{(l)}$...
and any pair \( \tilde{i} \) and \( \tilde{i} \), there exists one allocation \( A^{(l)} \) such that \( A^{(l)}_i = A^{(r)}_i \) for any \( i \neq \{ \tilde{i}, \tilde{i} \} \).

We define a mixed allocation to be a probability distribution on allocation: \( p = (p_1, p_2, \ldots, p_k) \in P \) such that \( \sum_j p_j = 1 \) and \( p_j \geq 0 \) for any \( j \). Given \( p \), the expected utility for player \( i \) is \( \sum_j p_j u_i(A^{(l)}_i) \).

A mixed allocation \( p \) is Pareto Efficient (PE) if there doesn’t exist another mixed allocation \( p' \) such that for all \( i \), \( \sum_j p'_j u_i(A^{(l)}_i) \geq \sum_j p_j u_i(A^{(l)}_i) \) and there exist one \( i \) such that this inequality is strict.

A mixed allocation \( p \) is Envy Free (EF) if for every pair \( i \) and \( i' \) of players, \( \sum_j p_j u_i(A^{(l)}_i) \geq \sum_j p_j u_i(A^{(l)}_{i'}) \).

**Theorem 3.1.** If the allocation set is swappable, then there exists a Pareto Efficient and Envy Free mixed allocation.

## 4 Proof

WLOG, we assume that \( 1 \leq u_i(x_i) \leq 2 \) for all \( i \) and \( x_i \). We will use a fixed point argument. To this end, we construct a mapping from \( P \times W \) to itself.

Here, \( P \) is the set of mixed allocations and \( W \) is the set of weighted vector \( \{ w = (w_1, w_2, \ldots, w_n) \mid \sum w_i = 1 \) and \( w_i \geq \epsilon \} \). We will specify \( \epsilon \) later. Now we construct a mapping from \((p, w)\) to \((P(w), \nu(p, w))\), where \( P(w) \) is a subset of \( P \) and \( \nu(p, w) \in W \).

\[
P(w) = \{ p' \mid p' \in P \text{ and } p' \in \arg \max \sum_i w_i \sum_j p'_j u_i(A^{(l)}_i) \};
\]

\[
\nu(p, w) = \text{proj}_W (\nu(p, w));
\]

\[
\nu_i(p, w) = w_i + \frac{\max_j \sum_j p_j u_i(A^{(l)}_i)}{\sum_{i'} \max_j \sum_j p_j u_{i'}(A^{(l)}_i)} - \frac{\sum_j p_j u_i(A^{(l)}_i)}{\sum_j \sum p_j u_{i'}(A^{(l)}_i)}.
\]

**Lemma 4.1.** There exists a fixed point, \((p^*, w^*)\), such that \( p^* \in P(w^*) \) and \( \nu(p^*, w^*) = w^* \).

We prove this via the following three claims.

**Claim 4.2.** Let \( A(w) = \{ j \mid A^{(l)}_j \text{ maximizes } \sum_i w_i u_i(A^{(l)}_i) \text{ over all allocations} \} \). Then, \( P(w) \) is a simplex on \( A(w) \), that for any \( p'' \in P(w) \), \( p''_j > 0 \) only if \( j \in A(w) \) (and, of course \( \sum_j p''_j = 1 \) and \( p''_j \geq 0 \)).

**Proof of Claim 4.2.** It’s not hard to see that in the definition of \( P(w) \), we can rewrite \( \sum_i w_i \sum_j p'_j u_i(A^{(l)}_i) \) as \( \sum_j p'_j \sum_i w_i u_i(A^{(l)}_i) \). So any probability
\[ p_j' > 0 \text{ on an allocation } A^{(j)} \text{ that does not maximize } \sum_i w_i u_i(A_i^{(j)}) \text{ will contradict the definition of } \mathcal{P}(w). \]

Claim 4.3. Give a mixed allocation \( p' \in P \), the set of \( w \) such that \( p' \in \mathcal{P}(w) \) is a convex closed set.

Proof of Claim 4.3. The set of \( w \) such that \( p' \in \mathcal{P}(w) \) can be written as a linear program: for any \( j \) and \( j' \) such that \( p_j > 0 \), \( \sum_i w_i u_i(A_i^{(j)}) \geq \sum_i w_i u_i(A_i^{(j')}) \).

Claim 4.4. For any series \((w^{(t)}, p^{(t)})\) with \( \lim_{t \to \infty} w^{(t)} = \bar{w} \) and \( \lim_{t \to \infty} p^{(t)} = p \), if for every \( t \), \( p^{(t)} \in \mathcal{P}(w^{(t)}) \), then \( p \in \mathcal{P}(w) \).

Proof of Claim 4.4. Consider the set \( S = \{ j | p_j > 0 \} \). Since the dimension of \( p \), which is the number of allocations, is finite, there must exist a \( t' \) such that \( p_j^{(t)} > 0 \) for all \( j \in S \) and all \( t \geq t' \). Consider the set \( W \) of \( w \) such that \( A^{(j)} \) maximizes \( \sum_i w_i u_i(A_i^{(j)}) \) for all \( j \in S \). Since \( p^{(t)} \in \mathcal{P}(w^{(t)}) \), \( p^{(t)} \in \arg\max \sum_j p_j^{(t)} \sum_i w_i^{(t)} u_i(A_i^{(j)}) \). This implies \( w^{(t)} \in W \) for any \( t \geq t' \). Furthermore, \( W \) is convex and closed by Claim 4.3 which implies \( w \in W \). By Claim 4.2, \( p \in \mathcal{P}(w) \).

Proof of Lemma 4.1. The result follows by using the Kakutani fixed-point theorem.

\[ p^* \] is a Pareto Efficient allocation.

Proof. Note that the fact that \( p^* \in \mathcal{P}(w^*) \) means \( p^* \) maximizes \( \sum_i w^*_i \sum_j p_j^* u_i(A_i^{(j)}) \).

So, there cannot be another \( p \) such that for every \( i \), \( \sum_j p_j u_i(A_i^{(j)}) \geq \sum_j p_j^* u_i(A_i^{(j)}) \), with the inequality being strict for some \( i \).

The following two lemmas prove the theorem.

Lemma 4.6. If \( \nu(p^*, w^*) \in W \), then \( p^* \) is a Pareto Efficient and envy free allocation.

Lemma 4.7. if \( \nu(p^*, w^*) \) is not in \( W \), then \( w^* \neq \varnothing(p^*, w^*) \).

Proof of Theorem 3.1. The theorem follows from Lemmas 4.6, 4.7 and 4.4.
Next, we start to prove Lemma 4.6. We first construct an envy graph \((V, E)\) based on \(p^*\). \(V\) is the set of players and \((i, i') \in E\) if and only if \(i\) envies \(i'\), which means \(\sum_j p_{j}^* u_i(A_i^{(j)}) < \sum_j p_{j}^* u_i(A_i^{(j')})\).

**Claim 4.8.** Let \(p^*\) be a Pareto Efficient mixed allocation. The corresponding envy graph is acyclic.

**Proof of Claim 4.8.** If the graph has a cycle, then we can improve everyone’s utility functions in this cycle by exchanging the allocations along the cycle, contradicting Pareto Efficiency.

Given the Pareto Efficient mixed allocation \(p^*\), we define the set of envy free players to be \(I(p^*) = \{i|\text{player } i \text{ does not envy } j \text{ for all } j\}\).

**Claim 4.9.** Suppose \(p^*\) be a Pareto Efficient mixed allocation. Then \(I(p^*)\) is not empty.

**Proof of Claim 4.9.** This follows from the fact that the graph is acyclic.

**Claim 4.10.** Let \(p^*\) be a Pareto Efficient mixed allocation. Then, for any \(i \in I(p^*)\),

\[
\max_j \frac{\sum_j p_j u_i(A_i^{(j)})}{\sum_{i'} \max_i \sum_j p_j u_i'(A_i^{(j')})} \leq \frac{\sum_j p_j u_i(A_i^{(j)})}{\sum_{i'} \sum_j p_j u_i'(A_i^{(j')})}.
\]

and equality holds if and only if \(p^*\) is envy free.

**Proof of Claim 4.10.** The inequality follows from the following two facts:

- \(\max_i \sum_j p_j u_i(A_i^{(j)}) \geq \sum_j p_j u_i(A_i^{(j)})\);
- for any \(i \in I(p^*)\), \(\max_i \sum_j p_j u_i(A_i^{(j)}) = \sum_j p_j u_i(A_i^{(j)})\).

Equality holds if and only if for all players, \(\max_i \sum_j p_j u_i(A_i^{(j)}) = \sum_j p_j u_i(A_i^{(j)})\) and thus no one envies anyone else.

**Proof of Lemma 4.6.** It is not hard to see that if \(\nu(p^*, w^*) \in W\) and \(w^*\) is a fixed point, then \(w^* = \omega(p^*, w^*) = \nu(p^*, w^*)\), and

\[
\frac{\sum_{i'} \sum_j p_{j} u_{i'}(A_{i'}^{(j)})}{\sum_{i'} \sum_j p_{j} u_{i'}(A_{i'}^{(j)})} = \sum_j p_j u_i(A_i^{(j)}) \leq \sum_j p_j u_i(A_i^{(j)}) = \sum_{i'} \sum_j p_{j} u_{i'}(A_{i'}^{(j)})
\]

In additional, by Claim 4.5 \(p^*\) is a Pareto Efficient allocation and so from Claim 4.10 \(p^*\) is envy free.
Next, we will show Lemma 4.11 namely that if \( \nu(p^*, w^*) \) is not in \( W \), then \( w^* \neq \varpi(p^*, w^*) \). We will use the following claims.

**Claim 4.11.** \( \varpi(p^*, w^*) \leq \max\{\nu(p^*, w^*), \epsilon\} \).

**Proof of Claim 4.11.** For simplicity, let \( x^* = \varpi(p^*, w^*) \) and \( y^* = \nu(p^*, w^*) \). Then \( x^* = \text{proj}_W y^* \) means \( x^* \) is the result of the following optimization program: \( \min_x \frac{1}{2}\|x-y^*\|^2 \) such that \( x \in W \). Note that \( W = \{(w_1, \ldots, w_n) \mid \sum_i w_i = 1 \text{ and } w_i \geq \epsilon\} \). The Lagrange form is \( \frac{1}{2}\|x-y^*\|^2 - \lambda(\sum_i x_i - 1) - \sum_i \beta_i(x_i - \epsilon) \). From the KKT condition, we know that \( x_i^* - y_i^* - \lambda - \beta_i = 0, \beta_i \geq 0, x_i^* \geq \epsilon, \beta_i(x_i^* - \epsilon) = 0 \) and \( \sum_i x_i^* = 1 \). From the construction of \( \nu(p^*, w^*) \), we know that \( \sum_i y_i^* = 1 = \sum_i x_i^* \). Since for the KKT condition \( x_i^* - y_i^* - \lambda - \beta_i = 0 \), by summing over all \( i \), we have \( \lambda = -\frac{1}{n} \sum_i \beta_i \leq 0 \). If \( x_i^* = \epsilon \) then the result follows. Otherwise, \( \beta_i = 0 \), which implies \( x_i^* = y_i^* + \lambda \leq y_i^* \).

**Claim 4.12.** Let \( p \in \mathcal{P}(w) \) be a Pareto Efficient allocation. Suppose that \( w_j \leq \rho w_i \), where

\[
\rho = \frac{1}{2} \min_{i,j,s} \frac{u_i(A_j^{(s)}) - u_i(A_i^{(s)})}{u_j(A_i^{(s)}) - u_j(A_j^{(s)})},
\]

where \( u_i(A_i^{(s)}) < u_i(A_j^{(s)}) \) and \( u_j(A_i^{(s)}) < u_j(A_j^{(s)}) \).

Then player \( i \) will not envy player \( j \).

**Proof.** Consider \( A(w) \) defined in Claim 4.2. Suppose that for any allocation in this set, player \( i \) does not envy player \( j \). Then player \( i \) will not envy player \( j \) in the mixed allocation \( p \in \mathcal{P}(w) \). We show by contradiction that for any allocation in this set, player \( i \) will not envy player \( j \). Suppose player \( i \) envies player \( j \) in an allocation \( A^{(s)} \). Then, since \( s \in A(w) \) and the allocation set is swappable,

\[
w_i u_i(A_i^{(s)}) + w_j u_j(A_j^{(s)}) \geq w_i u_i(A_j^{(s)}) + w_j u_j(A_i^{(s)}).
\]

(1)

Since player \( i \) envies player \( j \),

\[
u_i(A_i^{(s)}) < u_i(A_j^{(s)}).
\]

(2)

From (1) and (2),

\[
u_j(A_i^{(s)}) < u_j(A_j^{(s)}).
\]

Since \( 0 < w_j \leq \rho w_i \) and from (1),

\[
\rho(u_j(A_i^{(s)}) - u_j(A_j^{(s)})) \geq u_i(A_j^{(s)}) - u_i(A_i^{(s)}),
\]

which contradicts the definition of \( \rho \). \( \square \)
Claim 4.13. Given $p^* \in \mathcal{P}(w)$, if $\epsilon < \frac{\rho}{n}$, then there exists a player $i$ such that $w_i > \epsilon$ and $i \in I(p^*)$.

Proof. Since $\epsilon < \frac{\rho}{n}$, there exists an $\xi$ such that $\frac{1}{n} > \xi > \epsilon$ and, for each $i$, $w_i$ is not in the interval between $\xi$ and $\frac{\xi}{p}$. Therefore, we can divide all the players into two sets $D = \{i|w_i < \xi\}$ and $U = \{i|w_i > \frac{\xi}{p}\}$. Note that $D \cap U = \emptyset$, $D \cup U = \{1, 2, \ldots, n\}$, and $U$ is not empty since there exists one player $i$ with $w_i \geq \frac{1}{n}$. By Claim 4.12, we know that players in $U$ will not envy players in $D$, and we know that from Claim 4.8, the envy graph is acyclic, so the result follows. \hfill \Box

Proof of Lemma 4.4. Since $\nu(p^*, w^*)$ is not in $W$, then $\max_i \sum_j p_j u_i(A_i^{(j)}) \neq \sum_j p_j u_i(A_i^{(j)})$ for some $i$. So $p^*$ is not an envy-free allocation, which implies that for $i \in I(p^*)$,

\[
\frac{\max_i \sum_j p_j u_i(A_i^{(j)})}{\sum_{i'} \max_{i'} \sum_j p_j u_{i'}(A_i^{(j)})} < \frac{\sum_j p_j u_i(A_i^{(j)})}{\sum_{i'} \sum_j p_j u_{i'}(A_i^{(j)})}.
\]

From Claim 4.13, we know that there exists one player $i^* \in I(p^*)$ with $w_i^* > \epsilon$. Therefore, by (3), $\nu_i^*(p^*, w^*) < w_i^*$. By Claim 4.11, $\omega_i^*(p^*, w^*) \leq \max\{\nu_i^*(p^*, w^*), \epsilon\} < w_i^*$, the result follows. \hfill \Box

5 Discussion

In this paper, we showed that for any utility functions, a Pareto Efficient and envy-free allocation always exists if the allocation set is swappable. It needs to be noted that for the divisible case, the problem still remains open. Our proof cannot be simply generalized to the divisible case. This is because our $\epsilon$ will tend to 0 as $\rho$ tends to 0, in which case $\mathcal{P}(w)$ will no longer ensure Pareto Efficiency as for some $i$, $w_i = 0$.

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