A SYMMETRY THEOREM ON A MODIFIED JEU DE TAQUIN

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Abstract. For their bijective proof of the hook-length formula for the number of
standard tableaux of a fixed shape Novelli, Pak and Stoyanovskii [2] define a modi-
fied jeu de taquin which transforms an arbitrary filling of the Ferrers diagram with
1, 2, . . . , n (tabloid) into a standard tableau. Their definition relies on a total order
of the cells in the Ferrers diagram induced by a special standard tableau, however, this
definition also makes sense for the total order induced by any other standard tableau.
Given two standard tableaux P, Q of the same shape we show that the number of
tabloids which result in P if we perform modified jeu de taquin with respect to the
total order induced by Q is equal to the number of tabloids which result in Q if we
perform modified jeu de taquin with respect to P. This symmetry theorem extends
to skew shapes and shifted skew shapes.

1. Introduction

A partition of a positive integer n is a sequence of integers λ = (λ₁, λ₂, . . . , λᵣ)
with λ₁ + λ₂ + · · · + λᵣ = n and λ₁ ≥ λ₂ ≥ · · · ≥ λᵣ ≥ 0. The (unshifted) Ferrers
diagram of shape λ is an array of cells with r left-justified rows and λᵢ cells in row i.
(See Figure 1.a.) If λ is a partition with distinct components then the shifted Ferrers
diagram of shape λ is an array of cells with r rows, each row indented by one cell to
the right with respect to the previous row and λᵢ cells in row i. (See Figure 1.b.) If
µ = (µ₁, µ₂, . . . , µᵦ), λ = (λ₁, λ₂, . . . , λᵣ) are partitions (resp. partitions with distinct
components) such that λᵢ ≤ µᵢ for 1 ≤ i ≤ r then the unshifted (resp. shifted) skew
Ferrers diagram of shape µ/λ is the diagram we obtain if we remove the cells of the
unshifted (resp. shifted) Ferrers diagram of shape λ from the unshifted (resp. shifted)
Ferrers diagram of shape µ. (See Figure 2)

Definition 1. A tabloid of (shifted) skew shape µ/λ is an arbitrary filling of the
(shifted) skew Ferrers diagram of shape µ/λ with the integers 1, 2, . . . , |µ/λ|.

As usual a (shifted) standard skew tableau is a tabloid with increasing rows and
columns. For the rest of the article we fix an unshifted or shifted skew shape µ/λ and
set n = |µ/λ|. 

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Example 1. Observe that

\[
R = \begin{array}{cccc}
8 & 3 & 6 \\
9 & 5 & 1 & 4 \\
2 & 7 \\
\end{array}
\]

is a tabloid of shifted skew shape \((6, 5, 4, 2)/(5, 3)\) and that

\[
P = \begin{array}{cccc}
2 & 1 & 5 \\
3 & 4 & 6 & 8 \\
7 & 9 \\
\end{array}
\]

is a shifted standard skew tableau of shifted skew shape \((6, 5, 4, 2)/(5, 3)\).

Next we define forward jeu de taquin in a tabloid \(T\) of (shifted) skew shape \(\mu/\lambda\). If \((i, j)\) is a cell in the (shifted) skew Ferrers diagram of shape \(\mu/\lambda\) let \(T_{i,j}\) denote its entry in \(T\). In order to simplify the description of forward jeu de taquin we define \(T_{i,j} = \infty\) if \((i, j)\) is a cell outside of \(T\).

**Definition 2** (Forward jeu de taquin). Let \(T\) be a tabloid and \(e\) an entry in \(T\). Forward jeu de taquin in \(T\) with \(e\) is defined as follows: Consider the neighbour of \(e\) to the right and the neighbour of \(e\) below and if \(e\) is greater than the minimum of these neighbours we exchange \(e\) with the minimum. Next consider the new neighbours of \(e\) to the right and below and exchange \(e\) with the minimum of these two if \(e\) is greater than...
this minimum. We repeat this procedure with $e$ until $e$ is stable, i.e. smaller than its neighbour to the right and its neighbour below.

**Example 2. Performing forward jeu de taquin with 8 in the tabloid**

\[
\begin{array}{ccc}
8 & 1 & 4 \\
2 & 3 & 5 \\
6 & 7 \\
\end{array}
\]

results in the sequence

\[
\begin{array}{ccc}
1 & 8 & 4 \\
2 & 3 & 5 , \\
6 & 7 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 8 & 5 , \\
6 & 7 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & 8 . \\
6 & 7 \\
\end{array}
\]

Next we define an ‘ordering procedure’ which assigns a (shifted) standard skew tableau to every tabloid $T$. The procedure depends on another (shifted) standard skew tableau $S$ of the same shape.

**Definition 3 (Modified jeu de taquin).** Let $T$ be a tabloid and $S$ be a (shifted) standard skew tableau of the same shape. Modified jeu de taquin in $T$ with respect to $S$ is defined as the step by step performance of forward jeu de taquin with the entries in $T$ in the order the (shifted) standard skew tableau $S$ predicts, starting with the entry in $T$, whose cell has the greatest label in $S$. We denote the resulting (shifted) standard skew tableau by $\text{MJ}_S(T)$.

Observe that for tabloids of normal shape $(3,3,2)$ the ‘ordering procedure’ from \[2\] is the modified jeu de taquin with respect to the standard tableau $\begin{array}{ccc}1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 \end{array}$ from Definition 3. In their paper Novelli, Pak and Stoyanovskii show that their ‘ordering procedure’ has the nice property that the number of tabloids that are mapped to a fixed standard tableau is independent of this output standard tableau and that this number is equal to the product over all hook-lengths.

In our running example (Example 1) modified jeu de taquin with respect to $P$ applied to the tabloid $R$ gives the intermediate tabloids

\[
\begin{array}{ccc}
8 & 8 & 8 \\
3 & 4 & 3 & 4 \quad 3 & 4 \\
9 & 5 & 1 & 6 ' & 9 & 1 & 5 & 6 ' \quad 1 & 2 & 5 & 6 ' \\
2 & 7 & 2 & 7 \quad 7 & 9 \\
\end{array}
\]

before we finally obtain

\[
\text{MJ}_P(R) = \begin{array}{ccc}4 \\
3 & 6 \\
1 & 2 & 5 & 8 =: Q. \\
7 & 9 \end{array}
\]
2. The symmerty theorem

If $P, Q$ are two (shifted) standard skew tableaux of the same shape $\mu/\lambda$ let $A_{P,Q}$ denote the number of tabloids $T$ of shape $\mu/\lambda$ with the property that the application of modified jeu de taquin to $T$ with respect to $P$ results in $Q$. For the normal shape $(3,3,2)$ the matrix $(A_{P,Q})_{(P,Q)}$ is presented in Figure 3, where '1' stands for 936, '2' stands for 944, '3' stands for 960, '4' stands for 976, '5' stands for 984 and '6' stands for 996. Moreover every row and column corresponds to one of the 42 standard tableaux of shape $(3,3,2)$, which are ordered lexicographically if we identify a standard tableau with the permutation we obtain by reading the standard tableau row wise from top to bottom and within a row from left to right.

By this and other computer experiments\footnote{Those computer experiments were originally intended to find a total order of the cells in the shifted Ferrers diagram such that the number of tabloids that are mapped to a fixed shifted standard tableau by modified jeu de taquin with respect to the order is independent of the output shifted standard tableau and with this a proof of the shifted hook-length formula similar to \cite{2}. In \cite{1} we show that the rowwise performance of modified jeu de taquin, from bottom to top and within a row from right to left, has this nice property in the shifted case.} we were led to the conjecture that $A_{P,Q} = A_{Q,P}$. However, we discovered that a more general theorem is the key to this observation. In order to state it, we need another definition.

**Definition 4.** Let $S$ be a (shifted) standard skew tableau and $\pi$ a permutation of \{1, 2, $\ldots$, $n$\}. Then $S_\pi$ denotes the tabloid we obtain from $S$ by replacing every entry $i$ in $S$ by $\pi(i)$.

Let $\pi = 389561247$. Observe that $R = P_\pi$ and

$$Q_{\pi^{-1}} = \begin{pmatrix} 8 \\ 1 & 5 \\ 6 & 7 & 4 & 2 \\ 9 & 3 \end{pmatrix}.$$  

The fact that $MJ_Q(Q_{\pi^{-1}}) = P$ does not come by chance.

**Theorem 1.** Let $P, Q$ be two (shifted) standard skew tableaux of the same shape and let $\pi$ be a permutation of \{1, 2, $\ldots$, $n$\}. Then

$$MJ_P(P_\pi) = Q \iff MJ_Q(Q_{\pi^{-1}}) = P.$$ 

Before we are able to prove the theorem we need the definition of backward jeu de taquin which is in some sense the inverse operation of forward jeu de taquin. In order to simplify the description we set $T_{i,j} = 0$ if $(i,j)$ is a cell outside of $T$, where $T$ is a tabloid.

**Definition 5** (Backward jeu de taquin). Let $T$ be a tabloid and $e$ an entry in $T$. Backward jeu de taquin in $T$ with $e$ is defined as follows: Exchange $e$ with the maximum of its neighbour to the left and its neighbour above if $e$ has either a neighbour to the left or above in the fixed shape. We repeat this procedure with $e$ until $e$ has no neighbour to the left and no neighbour above in the fixed (shifted) skew shape.
Figure 3. The Matrix $(A_{P,Q})_{(P,Q)}$ for the normal shape $(3, 3, 2)$. 
Observe that in Example 2 the input tabloid can be obtained from the output tabloid by performing backward jeu de taquin with 8.

Proof of Theorem 7. We only have to show one direction of the assertion, for the other follows by symmetry.

Let $S$ be a (shifted) standard skew tableau and $T$ be a tabloid. We define $\text{FJ}(T, S) = (S', T')$, where $T' = \text{MJ}_S(T)$ and $S'$ is the tabloid we obtain from $S$ after the performance of modified jeu de taquin in $T$ with respect to $S$, if we simultaneously apply the transpositions we apply during modified jeu de taquin to $T$ also to $S$. (If we exchange $T_{i,j}$ and $T_{i,j+1}$ in $T$ we exchange $S_{i,j}$ and $S_{i,j+1}$ in $S$ and if we exchange $T_{i,j}$ and $T_{i+1,j}$ in $T$ we exchange $S_{i,j}$ and $S_{i+1,j}$ in $S$.)

In our running example: If we perform modified jeu de taquin in $R$ with respect to $P$ and perform the transpositions simultaneously in $P$ we obtain the intermediate tabloids

\[
\begin{bmatrix}
2 & 2 & 2 \\
1 & 8 & 1 & 8 \\
3 & 4 & 6 & 5 & 1 & 2 & 5 & 6 \\
7 & 9 & 7 & 9 & 1 & 5 & 6 & 7 & 9 & 4 & 5
\end{bmatrix},
\begin{bmatrix}
8 & 3 & 4 & 1 & 2 & 5 & 6 \\
6 & 7 & 4 & 2 & 7 & 9 \\
2 & 1 & 5 & 6 & 7 & 9 & 4 & 5
\end{bmatrix}
\]

before we finally obtain

\[
\begin{bmatrix}
8 \\
1 & 5 \\
6 & 7 & 4 & 2 \\
9 & 3
\end{bmatrix}
\]

Note that the output tabloid is equal to $Q_{\pi^{-1}}$. This is because in the course of applying FJ to a pair $(P_{\pi}, P)$ the first tabloid of the current pair can always be obtained from the second tabloid in the pair by applying $\pi$.

Observe that the following operation BJ is the inverse of FJ. Let $T'$ be a (shifted) standard skew tableau and $S'$ be a tabloid. For $i = 1$ to $i = n$ perform backward jeu de taquin in $T'$ in the subshape of $\mu/\lambda$ consisting of the cells of $S'$ whose entries are greater or equal than $i$ and with the entry of $T'$ that is in the cell of the entry $i$ in $S'$. Again perform the transpositions simultaneously in $S'$. If $T'$ results in $T$ and $S'$ results in $S$ then we define $\text{BJ}(S', T') = (T, S)$. Observe that $S$ is a (shifted) standard skew tableau by construction. Furthermore $\text{BJ} \cdot \text{FJ} = \text{id}$ and $\text{FJ} \cdot \text{BJ} = \text{id}$.

If we apply BJ to $S' = Q_{\pi^{-1}}$ and $T' = Q$ we obtain the following pairs of intermediate tabloids:

\[
\begin{bmatrix}
2 & 8 \\
1 & 8 & 3 & 4 \\
6 & 7 & 4 & 5 & 1 & 2 & 5 & 6 \\
9 & 3 & 7 & 9
\end{bmatrix},
\begin{bmatrix}
2 & 8 \\
1 & 8 & 3 & 4 \\
3 & 6 & 4 & 5 & 1 & 2 & 5 & 6 \\
9 & 3 & 7 & 9 & 1 & 5 & 6 & 2 & 7
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 8 \\
1 & 8 & 3 & 4 \\
3 & 4 & 6 & 5 & 1 & 2 & 5 & 6 \\
7 & 9 & 2 & 7
\end{bmatrix},
\begin{bmatrix}
2 & 8 \\
1 & 5 & 3 & 6 \\
3 & 4 & 6 & 8 & 1 & 2 & 5 & 6 \\
7 & 9 & 2 & 7
\end{bmatrix}
\]

\[
= (P, R)
\]
Clearly \( MJ_P(P_\pi) = Q \) is equivalent with \( FJ(P_\pi, P) = (Q_{\pi^{-1}}, Q) \). Thus the assertion of the theorem is that \( FJ \) is an involution. This is equivalent to \( FJ = BJ \).

In order to show that we decompose \( FJ \) and \( BJ \) into its elementary steps. Let \( S, T \) be tabloids of the same shape and \( 1 \leq i \leq n \). Then the pair \((T', S') = J^i(T, S)\) is defined as follows: Let \( T' \) be the tabloid we obtain by performing forward jeu de taquin with the entry in the cell of \( T \) which is labelled with \( i \) in \( S \) and perform the corresponding transpositions in \( S \) also in order to obtain \( S' \). Observe that

\[
FJ(T, S) = \text{SWITCH } J^1 J^2 \ldots J^n (T, S)
\]

with \( \text{SWITCH}(X, Y) = (Y, X) \) if \( T \) is a tabloid and \( S \) is a (shifted) standard skew tableau. The pair \((S, T) = J_i(S', T')\) is defined as follows, where \( S', T' \) are tabloids of the same shape and the cells of the entries of \( S' \) greater or equal to \( i \) form a subshape: Let \( T \) be the tabloid we obtain by performing backward jeu de taquin in \( T' \) in the subshape of \( \mu/\lambda \) consisting of the cells of \( S' \) whose entries are greater or equal than \( i \) and with the entry in the cell of \( T' \) which is labelled with \( i \) in \( S' \). Again perform the corresponding transpositions in \( S' \) also in order to obtain \( S \). Observe that

\[
BJ(S', T') = \text{SWITCH } J_n J_{n-1} \ldots J_1 (S', T')
\]

if \( S' \) is a tabloid and \( T' \) is a (shifted) standard skew tableau.

Next we show the following identity

\[
FJ(T, S) = \text{SWITCH } J^1 \ldots J^{k-1} J^{k+1} \ldots J^n J_1 (T, S),
\]

for a tabloid \( T \) and a (shifted) standard skew tableau \( S \), where \( k \) is the entry in the cell of \( S \) which is labelled with \( 1 \) in \( T \). Let \((i, j)\) be the cell of \( 1 \) in \( T \). If the cell \((i, j)\) has no neighbour to the left and no neighbour above in \( T \) then \( J_1(T, S) = (T, S) \) and since

\[
J^k J^{k+1} \ldots J^n (T, S) = J^{k+1} \ldots J^n (T, S)
\]

(1 is always stable), this proves the assertion in this case. Now suppose that \((i, j-1), (i-1, j)\) are both cells in the fixed shape and \( S_{i,j-1} > S_{i-1,j} \) for the other cases are similar. Then the entry 1 in \( T \) is first involved in a transposition in the application of \( FJ \) to \((T, S)\) when performing the first step of \( J^{S_{i,j-1}} \) to the current pair. In this case the entries in cells \((i, j-1)\) and \((i, j)\) are exchanged in both tabloids. Note that this is also the first time that an entry in cell \((i, j-1)\) is involved in a transposition in the application of \( FJ \). But this transposition is also the first step in the application of \( J_1 \) to \((T, S)\). If \((i, j-1)\) has no neighbour to the left and no neighbour above in the fixed shape the assertion is proved, for 1 in \( T \) is neither involved in another transposition of \( FJ \) nor of \( J_1 \) and the application of \( J_1 \) terminates. Otherwise suppose \((i-1, j-1), (i, j-2)\) are both cells in the fixed shape and \( S_{i-1,j-1} > S_{i,j-2} \) for the other cases are similar. Then the entry 1 in \( T \) is involved in a transposition in the application of \( FJ \) to \((T, S)\) for the second time when performing the first transposition of \( J^{S_{i-1,j-1}} \) to the current pair. In this step the entries in cells \((i-1, j-1)\) and \((i, j-1)\) are exchanged in both tabloids. Again this is the first time an entry in \((i-1, j-1)\) is involved in a transposition of \( FJ \). But this transposition is also the second step in the application of \( J_1 \) to \((T, S)\) etc. Roughly speaking the backward path in the application of \( J_1 \) to \((T, S)\), which we obtain by performing backward jeu de taquin with the entry
in \( S \) in the cell labelled with 1 in \( T \), is equal to the 'backward path' of 1 in \( T \), which we obtain indirectly in the application of FJ to \((T, S)\) by performing forward jeu de taquin to all entries in \( T \).

Now we show \( \text{FJ} = \text{BJ} \) by induction with respect to \( n \). For \( n = 1 \) there is nothing to prove. Suppose that \((T', S')\) is a pair of tabloids, where entry 1 in \( T' \) has no neighbour to the left and no neighbour above in \( T' \) and where \( S' \) without the entry in the cell of 1 in \( T' \) is standard. Let \( \text{FJ}'(T', S') \) and \( \text{BJ}'(T', S') \) denote the output pairs after the application of FJ and BJ to the pair \((T', S')\) with the cell of 1 in \( T' \) omitted in both tabloids. By induction \( \text{FJ}'(T', S') = \text{BJ}'(T', S') \). Let \((T, S)\) be a pair of a tabloid \( T \) and a (shifted) standard skew tableau \( S \) and observe that the pair \((T', S') = J_1(T, S)\) has the property of \((T', S')\) above. Thus

\[
\text{FJ}(T, S) = \text{FJ}'(J_1(T, S)) = \text{BJ}'(J_1(T, S)) = \text{BJ}(T, S),
\]

where the first equality follows from (1).

**Corollary 1.** Let \( P, Q \) be two (shifted) standard skew tableaux of the same shape. Then \( A_{P,Q} = A_{Q,P} \).

**Proof.** By the theorem

\[
\{ \pi \in S_n \mid \text{MJ}_P(P_\pi) = Q \} = \{ \pi \in S_n \mid \text{MJ}_Q(Q_{\pi^{-1}}) = P \},
\]

where \( S_n \) denotes the symmetric group of order \( n \).

Thus

\[
A_{P,Q} = |\{ T \text{ tabloid} \mid \text{MJ}_P(T) = Q \}| = |\{ \pi \in S_n \mid \text{MJ}_P(P_\pi) = Q \}| = |\{ \pi \in S_n \mid \text{MJ}_Q(Q_{\pi^{-1}}) = P \}| = |\{ T \text{ tabloid} \mid \text{MJ}_Q(T) = P \}| = A_{Q,P}.
\]

**References**

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[2] J.-C. Novelli, I. Pak and A.V. Stoyanovskii, A direct bijective proof of the hook-length formula, *Discrete Math. Theoret. Computer Science* 1 (1997), 53–67.