The Schrödinger functional coupling in quenched QCD at low energies

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Existing non-perturbative computations of the running coupling of quenched QCD in the Schrödinger functional scheme are extended to scales \( \mu \) lying much deeper in the low-energy regime. We are able to reach \( \mu^{-1} \approx 0.9 \) fm, where a significant deviation from its perturbative evolution is observed.

1. MOTIVATION

The Schrödinger functional (SF) of QCD provides a suitable tool to compute the running of the strong coupling \( \alpha_s \) \cite{1,2} and to solve other renormalization problems \cite{3,4} non-perturbatively by means of numerical simulations. The basic idea to cover the scales involved – often differing by many orders of magnitude – is to use a recursive finite-size scaling technique \cite{5} to relate the low-energy sector with the scaling regime at high energies, where the evolution of \( \alpha_s \) follows the perturbative renormalization group.

A particular property of the coupling in the SF scheme, \( \alpha_{SF} \), which however should not be regarded as universal for QCD couplings, is that its non-perturbative running is quite accurately described by perturbation theory (PT) down to surprisingly low energies. Therefore, it appears interesting to investigate \( \alpha_{SF}(\mu) \) in quenched QCD at much lower scales \( \mu = 1/L \) \((L \) the linear size of the finite system) than considered so far \cite{1,3}, in order to see a deviation from PT.

As an aside we would like to remark that during the year 2000 the present study at the same time was designed to test the APEmille computers installed at DESY Zeuthen \cite{6} and to gain experiences with code adaptation and optimization for this architecture in the context of a first, clear-cut physics project.

2. SF SETUP

The (lattice regularized) SF is given in terms of the effective action (free energy) \( \Gamma \) of QCD satisfying Dirichlet boundary conditions in Euclidean time \cite{7}. A renormalized coupling, \( g_{SF}^2 \equiv \bar{g}^2 \), is introduced as the response to an infinitesimal variation of a specific 1–parameter (\( \eta \)) family of prescribed constant abelian boundary fields. Taking into account the perturbative series \( \Gamma = \Gamma_0 g_0^{-2} + \Gamma_1 + \Gamma_2 g_0^2 + \cdots \), we define \( \bar{g}^2 \) by

\[
\frac{\partial \Gamma}{\partial \eta} \bigg|_{\eta=0} = \frac{\partial \Gamma_0}{\partial \eta} \bigg|_{\eta=0} \frac{1}{\bar{g}^2(L)}.
\]

(1)

It genuinely depends only on one renormalization scale, \( L = 1/\mu \), and the quantity that has been devised to map out the scale evolution of \( \bar{g} \) is the step scaling function (SSF) \( \sigma(s, u) \):

\[
\sigma(s, u) = \bar{g}^2(sL) \bigg|_{\bar{g}^2(L)=u}.
\]

(2)

It represents an integrated beta function for finite scale transformations with rescaling factor \( s \) \cite{5}. In principle, if one has control over the exact SF, one can trace the non-perturbative evolution of the coupling in discrete steps \( \bar{g}^2(L) \to \bar{g}^2(sL) \to \bar{g}^2(s^2L) \to \cdots \). For one rescaling step, \( \sigma \) is computed as the continuum limit \( a \to 0 \) of the SSF at finite resolution, \( \Sigma(s, u, a/L) \). So far the method of construction for each \( u \)–value has been to choose \( L/a \), determine \( \beta \) to match \( \bar{g}^2 = u \) on the corresponding lattice and to then simulate on an \( sL/a \)–lattice with the same \( \beta \) and read off \( \Sigma = \bar{g}^2(sL) \). For further details we refer to \cite{1,3}.
3. SIMULATIONS AND RESULTS

3.1. New simulation strategy

In our previous computations \( L_{\text{max}} \) defined by \( g^2(L_{\text{max}}) = 3.48 \) has been both an intermediate reference scale and the low energy bound of our scaling investigations. In ref. [8] this scale was connected with the scale \( r_0 \approx 0.5 \text{ fm} \) from the static potential yielding \( L_{\text{max}}/r_0 = 0.718(16) \) in the continuum limit. In the same reference a relation connecting \( \beta \) with \( r_0/a \) was given for the range \( 5.7 \leq \beta \leq 6.57 \). These combined informations enable us to directly determine \( \beta \)-values corresponding to \( L = sL_{\text{max}} \) for a series of chosen resolutions \( L/a \) for continuum extrapolation. We just use the cited relation for \( r_0/a = (r_0/L_{\text{max}})(L/a)(1/s) \) in the allowed range. In this way we construct an alternative lattice estimator \( \tilde{\Sigma}(s, 3.48, a/L) \) with the same continuum limit \( \sigma(s, 3.48) \) but cutoff effects different from \( \Sigma \).

In fig. 1 we demonstrate consistency between the two approaches for \( s = 1.5 \). In fig. 2 continuum extrapolations of \( \tilde{\Sigma} \) are shown for other \( s \)-values, which now take us down to \( 2.5L_{\text{max}} \) in energy.

For the \( O(a) \) boundary improvement term of the SF we use the 2–loop approximation \( \tilde{\Sigma}(s, 3.48, a/L) \). For our Monte Carlo simulations we employed a similar ‘hybrid-overrelaxation’ algorithm as in \( \tilde{\Sigma}(s, 3.48, a/L) \). Each iteration consists of 1 heatbath update and \( N_{\text{OR}} \) subsequent overrelaxation sweeps (typically \( N_{\text{OR}} = 5, 7 \)). As detailed there, non-gaussian tails and long autocorrelations in the coupling at low energy are overcome by a modified sampling procedure that enhances the tail contributions and is compensated by reweighting.

3.2. Results

The SSF \( \tilde{\Sigma} \)-values in fig. 2 approach the continuum at rate compatible with \( a^2 \). This shows that lattice artifacts linear in \( a \), which in the SF a priori exist, are invisible to our accuracy with 2–loop boundary improvement and allowed us to fit by the form \( \tilde{\Sigma} = \sigma + \rho_2 (a^2/r_0^2) \). As a safeguard against underestimating the uncertainty in \( \sigma \) we omitted the two coarsest lattices. This is in contrast with the extrapolation of \( L_{\text{max}}/r_0 \), where a small linear component was found [10].

\[
\sigma(\mu) = \frac{g^2}{4\pi}
\]

Fig. 3. Running of \( \alpha \equiv \alpha_{\text{SF}} = g^2/4\pi \).
Fig. 4. $L$-dependence of $\overline{g}^2$ versus PT. The artificial "4–loop" $\beta$–function uses $b_3 = 1/(4\pi)^4$.

Fig. 5. Estimates of $m$ via leading-order strong coupling (full line) and our data (dotted line).

Figs. 3 and 4 display the scale evolution of the SF coupling over the whole energy range that is available from refs. [1,3] together with this work and confront it with results from integrations with the perturbative $\beta$–function. With decreasing energy a substantial deviation from PT now becomes clearly visible. The steep growth of $\ln \{\overline{g}^2(L)\}$ with $L$ reveals the non-perturbative behaviour to set in for $L > 0.7 r_0$.

3.3. Discussion of the large–$L$ asymptotics

In the low-energy domain $\overline{g}^2$ is dominated by non-perturbative contributions. Since the boundary fields are locally pure gauge configurations, any dependence of the effective action $\Gamma$ on the background field is caused by correlations around the spatially periodic lattice. At large volume they are exponentially suppressed in a theory with a mass gap, and one expects

$$\overline{g}^2(L) \propto \exp\{mL\} \quad \text{at large } L.$$  \hspace{1cm} (3)

The mass $m$ is characteristic for the gauge field dynamics close to the boundaries (but not obviously related to the bulk correlation length). Based on this argument we attempt to compare the behaviour of $\overline{g}^2$ as a function of $L$ with the leading-order prediction from the strong-coupling expansion [11]: $\overline{g}^{-2}(L) \sim A \times \exp\{-m(\beta)L\}$, with $m = (3/4)m_G = -3\ln \beta$ and $m_G$ the $0^{++}$ glueball mass. Thus, assuming this ansatz, we fit:

$$\ln \{\overline{g}^2(L)\} = A' + (mL_{\text{max}}) \times (L/L_{\text{max}}).$$ \hspace{1cm} (4)

Fig. 5 confronts the outcome of this analysis applied to the two highest points with the slope obtained in leading-order strong coupling,

$$mr_0 = (3/4) \times (m_{G} r_0) \Leftrightarrow mL_{\text{max}} \approx 2.3,$$ \hspace{1cm} (5)

with $m_{G} r_0 \approx 4.3$ from [12]. Exploring the possibility, whether in eq. (3) a prefactor $L^c$, $c \neq 0$, can build up when higher orders in the strong-coupling expansion are summed, would demand a calculation of at least a few further orders.

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