Anomalous transport and quantum-classical correspondence

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We present evidence that anomalous transport in the classical standard map results in strong enhancement of fluctuations in the localization length of quasienergy states in the corresponding quantum dynamics. This generic effect occurs even far from the semiclassical limit and reflects the interplay of local and global quantum suppression mechanisms of classically chaotic dynamics. Possible experimental scenarios are also discussed.

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It is generally accepted that quantum mechanics suppresses chaotic classical motion. Numerous studies have identified mechanisms for suppression which, for our purposes, fall into two broad classes. One class is exemplified by ‘dynamical localization’ where quantum eigenstates are localized in an action variable like momentum despite the deterministic diffusion exhibited in the limiting classical dynamics\(^1\). This effect is analogous to the insulator phase or Anderson localization in tight-binding models\(^2\) and, in one-dimension, the localized wavefunction has a characteristic exponential form. An associated scale is the localization length \(\xi\) which is related to the classical diffusion constant, an important fact for the work reported here. There also exists a demarcating timescale \(t^*\) beyond which quantum and classical dynamics deviate. Despite a few counterexamples, dynamical localization provides a ‘global’ mechanism for quantum suppression.

The second class can be motivated by the fast deviation of wavepacket dynamics from the classical motion even in the semiclassical limit. This effect is contained in the logarithmic break time \(\tau_h = \ln(I/\hbar)/\Gamma\) for the breakdown of quantum-classical correspondence\(^3\), relative to a characteristic Lyapunov exponent \(\Gamma\) and action \(I\). However, the quantum dynamics retains features of the classical dynamics for times beyond this estimate. The phenomenon of ‘scarring’\(^4\) refers to quantum coherences associated with local, unstable and marginally stable, classical invariant structures. Though there are still many open questions, this effect has been numerically and experimentally observed in a wide variety of strongly coupled quantum systems.

In this letter, we present evidence for another example of ‘classical persistence’ related to anomalous diffusion and accelerator modes in chaotic dynamics. Recently, there has been considerable activity in the area of anomalous classical transport\(^5\)\(^6\). What is of particular relevance to the present work is that even in an established paradigm like the standard map, described by \(q_{n+1} = q_n + p_{n+1}\) and \(p_{n+1} = p_n + K \sin q_n\), some surprising new results were reported.

In the limit of large \(K\), it is well-known that the classical standard map dynamics is diffusive in the action variable \(p\) with diffusion constant \(<p^2>/t = K^2/2 \equiv D_{ql}\). What is also established is that away from this limit, the diffusion constant exhibits peaks with changing \(K\) and varies in time\(^6\). Recent results\(^6\) with improved precision and detail show that there are values of \(K\) where the effective diffusion constant (over some time) can be different from \(D_{ql}\) by many orders of magnitude. Figure 1 shows these windows of anomalous diffusion which are not very narrow and exhibit sub-structure in each of the peaks.

It was shown that there is no diffusion at all and that the real random walk process corresponds to a Lévy-like wandering characterized by a transport exponent \(\mu\) given by \(<p^2> \approx t^\mu\), where \(\mu > 1\). \(\mu\) varies with \(K\) and has the ‘normal’ value \(\mu = 1\) only at special values of \(K\). Superdiffusion occurs for \(\mu > 1\) leading to anomalous growth in momentum. It was explicitly shown\(^5\) that this could result from an island hierarchy with a peculiar topological structure near the ‘accelerator’ modes that appear at special values of \(K\). Near these values, classical

![Figure 1](image-url)
trajectories stick to the boundaries of these islands which, in turn, leads to flights of arbitrary length. The net result is strong intermittency and superdiffusion.

Another signature of anomalous transport results from the fact that a classical trajectory originating in a region of phase space recurs infinitely often. From the set of recurrence times \( \{\tau_j\} \), \( j = 0, 1, \cdots \), Poincaré cycles \( \{\tau_j\} = \{\tau_{j+1} - \tau_j\} \) and an associated probability distribution function \( R(t) \) can be constructed. In the case of ‘perfect’ mixing and normal diffusion, \( R(t) \) is strictly Poissonian whereas it exhibits power-like asymptotics for anomalous kinetics. This classical characteristic of anomalous transport is illustrated in Fig. \( \text{FIG. } 2 \) and is important in the corresponding quantum dynamics as well.

\[
\log_{10}[R(t)] = \log_{10}[R(t)] \\
10^{-5} \leq t \leq 40
\]

**FIG. 2.** Distribution of Poincaré cycles. The inset clearly displays the power-law tail of the distribution.

We raise several questions, arising from the classical anomalous diffusion, in the corresponding quantum dynamics of the standard map or, equivalently, the ‘delta-kicked rotor’, \( H = p^2/2 + K \cos q \sum_n \delta(t - n) \). The scale length \( \tilde{\xi} \) for dynamical localization is related to the classical diffusion constant and even follows the variations in \( D \) with \( K \) \( \text{[11]} \). Thus, we anticipate that the strong enhancement in \( D \) resulting from anomalous diffusion should also be reflected in \( \tilde{\xi} \). The nature of dynamical localization in these special windows is a related question. Further, as small local structures in the classical phase space lead to the superdiffusion, these issues directly pertain to the interplay of local and global aspects of quantum suppression. This idea is reinforced by the enhanced diffusion that ought to be visible in the quantum dynamics even far from the semiclassical limit, where scattering is significant. Other authors have considered quantum dynamics in the presence of anomalous transport \( \text{[11]} \) though in a different regime. We concentrate exclusively on the situation where the quantization scale \( 2\pi \hbar \) is much larger than the size of the islands. Thus, any coherences associated with the islands would be due to ‘scarred’ quantum states.

Consider the evolution of \( \langle p^2 \rangle \) with time \( t \) (measured as the number of kicks) in the quantum dynamics for two values of the classical stochasticity parameter \( K \).

The computation uses fast-Fourier transforms to evolve a plane wave initial state under repeated application of the single-kick evolution operator

\[
U = \exp (-ip^2/2\hbar) \exp (-ik\cos(q)/\hbar).
\]

The expectation values are then computed from the time-evolved wavefunction. The value \( K_c = 6.908745 \cdots \) was found \( \text{[1]} \) to be a critical value in a wide parametric window dominated by anomalous transport. We illustrate this regime by considering \( K^* = 6.905 \) and contrast it with \( K = 11 \), where the effective diffusion coefficient is actually below the quasilinear value \( D_{ql} \) (seen in Fig. \( \text{FIG. } 3 \)).

\[
\log_{10}[P^2(t)] =\log_{10}[P^2(t)] \\
0 \leq t \leq 5000
\]

**FIG. 3.** Variation of \( \langle p^2 \rangle \) (normalized to the saturation value \( D_{ql}t^\ast \)) with the number of kicks \( t \). Two representative values of \( a \), one large (\( Q, a \approx 61803399 \)) and the other small (\( SC, a \approx 0.042340526 \)), are shown for \( K = 11 \) (dashed lines) and \( K^* = 6.905 \) (solid lines). Note that the larger a value was time-averaged to remove fluctuations that mask the trends. The inset shows a log-log plot (at longer times, \( \approx 5000 \) kicks) for \( a = 0.042340526, K^* = 6.905 \) (upper curve) and \( K = 11.0 \) (lower curve). The corresponding slopes are 0.21 (solid line) and 0.06 (dashed line) respectively.

Figure, \( \text{FIG. } 3 \) shows \( \langle p^2 \rangle \) for \( (D_{ql}t^\ast) \) as a function of time for two representative values of \( a \). For large \( a \), both values of \( K \) result in strong saturation with \( K = 11 \) having a higher saturation value than \( K^* = 6.905 \). However, with decreasing \( a \) the enhanced diffusion for \( K^* = 6.905 \) becomes evident and there is a slowdown in growth in lieu of saturation (over the time considered). By contrast, \( K = 11 \) appears to saturate at a value which is about an order of magnitude lower. This difference is considerably larger than earlier expectations based on quasilinear analysis. The inset in Fig. \( \text{FIG. } 3 \) displays the existence of power-law behavior in \( \langle p^2 \rangle \) for longer times at \( K^* \) as compared with saturation at \( K = 11 \). Note that the slope points to sub-diffusive rather than to super-diffusive growth for which we have no clear explanation at present.
A stronger signature appears on considering the variation of localization length $\xi$ with $K$. It was shown that $\xi$ tracks the variations in the classical diffusion coefficient $D$ when plotted with respect to $K_q = K \sin (h/2)/(h/2)$. It is clear that this can be an important correction for larger values of $h$. Figure 4 shows the variation in $\xi$, normalized to the quasilinear estimate, with respect to $K_q$. $\xi$ is obtained by fitting an exponential to the time-dependent wavefunction for long times. The peaks in the inset coincide with the classical acceleration modes at $K = 2\pi$ and $4\pi$ and $\xi$ computed at different times reflect the same features. The more detailed scan displays a reasonable correspondence with the results shown in Fig. 1, despite being far from the semiclassical limit. It should be noted that the highest value occurs at $K_q \approx K_c$ defined earlier.

![Figure 4](image_url)

**FIG. 4.** Oscillations in the localization length $\xi$ normalized to the quasilinear estimate as a function of $K_q$. $a \approx 0.131267463$ which makes the corrections to the classical stochasticity parameter $K$ significant. The inset shows peaks associated with the accelerator modes while the main figure zooms in on the region considered in Fig. 1. The inset also shows $\xi$ computed after 3000 kicks (points) as well as 9000 (line) kicks. Note that the wavefunction averaged over 600 kicks is used to determine $\xi$.

We now consider the nature of the localized wavefunction when the classical transport is predominantly anomalous. Figure 5 shows the evolved momentum distribution averaged over 600 kicks. Panel (a) corresponds to $K_q$ at a peak in Fig. 1 while (b) sits in the valley. Localization does occur though the characteristic lineshape is strongly non-exponential for $K_q$ in the neighborhood of the peaks. The contrast with the clearly exponential lineshape shown in (b) is striking. It should be noted that the central region in (a) also displays curvature and an exponential fit results in a $\xi$ which is considerably larger than the quasilinear prediction. However, in case (b), the computed $\xi$ is well-approximated by the quasilinear estimate. Moving away from the peak in $K_q$ decreases the prominence of the shoulders in the lineshape though these are still clearly visible. We also note that the shoulders develop over a time (which depends on $\hbar$) which is in the neighborhood of the cross-over time from exponential to power-law behavior shown in Fig. 1.

![Figure 5](image_url)

**FIG. 5.** Lineshapes after 6000 kicks starting from a plane wave initial condition. (a) corresponds to $K_q \approx K_c = 6.90845 \cdots$ while (b) corresponds to $K_q = 10.69$. $a \approx 0.131267463$ and the noise in the lineshapes was reduced by averaging over 600 kicks.

Arguments for deviations of the localization length from the simple estimates proceed from the long-time averaged momentum distribution $\langle f_n \rangle = \sum_m |\psi_m(n_0)|^2 |\psi_m(n)|^2$, starting from a plane wave $n = n_0$ initial condition. $\psi_m(n)$ is the plane wave representation of the eigenfunction with quasienergy $\omega_m$. The analysis of localization assumes that all quasienergy states are exponentially localized and that the fluctuations in $\xi$ are small. If this were true in the case of anomalous transport, the oscillations in Fig. 4 would be directly correlated with the number of quasienergy states participating in the dynamics. To address this view, we computed that participation ratio $PR$ from the time averaged autocorrelation function

$$PR = \lim_{T \to \infty} \frac{1}{T} |\langle \psi(T)|\psi(0) \rangle|^2.$$  

(2)

As seen from Fig. 5, the only correlation with Fig. 4 occurs in the valley where the lineshape is exponentially localized. At the peaks, these results strongly support the idea that anomalous transport results in strong fluctuations in $\xi$ among the individual quasienergy states. We have verified this by constructing the ‘entire’ spectrum of the quantum map $U$, in the momentum representation, under conditions necessary for dynamical localization. The average value of $\xi$ does not reflect the large changes in $D$ though dispersion $\Delta \xi$ exhibits large fluctuations. This view is being explored further at present. As we argued earlier, a similar scenario may apply to the interplay of scarring with dynamical localization.
The classical flights of arbitrary length result (for the standard map) from a hierarchy of boundary-layer islands [1]. The island area $S_n$, associated period $T_n$, and Lyapunov exponent $\Gamma_n$ scale as: $S_n = \lambda_n^\alpha S_0$, $T_n = \lambda_n^\beta T_0$ and $\Gamma_n = \lambda_n^\gamma \Gamma_0$ with the generation $n$ of the island chain. $\lambda_S < 1$, $\lambda_T > 1$ and $\lambda_\Gamma \approx 1/\lambda_T$. It is straightforward to construct a semiclassical break-time appropriate to anomalous transport using these scaling rules.

Quantum flights are possible as long as $S_n \geq \hbar$ leading to a critical value $\tau_n = |\ln \hbar/S_0|/|\ln \lambda_S|$. The corresponding time is $T_n = T_0 (S_0/\hbar)^{1/\mu}$ where $\mu = |\ln \lambda_S|/|\ln \lambda_T| > 1$ is a classical transport exponent [3]. We generalize the break time $\tau_n$ defined earlier to $\ln (I/\hbar)/\Gamma_n$ where $I$ was some characteristic action. The longest time corresponds to the smallest $\Gamma_n = \Gamma_h = 1/T_h$ from which we have

$$\tau_n = T_n \ln (I/\hbar) = T_0 (S_0/\hbar)^{1/\mu} \ln (I/\hbar). \quad (3)$$

The result points to a power-law dependence on $\hbar$ rather than the logarithmic one as well as a possible crossover in scaling behavior. Note that the saturation time $T_h$ for flights is a pure quantum effect and one that is necessary for localization, which cannot occur if arbitrarily long flights exist with high probability (not exponentially small). Theoretical estimates are considerably harder when $\hbar > S_0$.

The issue of fluctuations in $\xi$ is relevant to an atom optics realization of the kicked pendulum [4]. Here, the localization properties are similar to those in the quantum standard map, with the complication that small islands persist in the classical dynamics. As we have illustrated, this could lead to strong fluctuations in the localized wavefunction. More recent experiments have realized the delta-kicked rotor system [13] where the measurement of diffusion, over timescales considerably longer than those predicted earlier, can provide direct evidence of anomalous transport in select windows of $K$. We note that the signatures of anomalous transport are similar to those resulting from external noise, in both the line-shape and the time evolution of the energy. This may be especially relevant in the weak noise limit and will be considered in detail elsewhere.

In conclusion, we have presented evidence of the role of anomalous diffusion in enhancing quantum fluctuations. Note that the standard map is not ideal for isolating the effects of anomalous transport. The kicked harmonic oscillator, which results in the classical web map, is a more interesting candidate where super diffusion dominates the classical dynamics [7]. The absence of localization in that system [14] and a recent suggestion for an experimental realization in the context of ion traps [13] makes this an attractive direction to pursue.

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