Strengthened Hardness for Approximating Minimum Unique Game and Small Set Expansion

Peng Cui

Key Laboratory of Data Engineering and Knowledge Engineering, MOE, School of Information Resource Management, Renmin University of China, Beijing 100872, P. R. China.

cuipeng@ruc.edu.cn

Abstract. In this paper, the author puts forward a variation of Feige’s Hypothesis, which claims that it is hard on average refuting Unbalanced Max 3-XOR against biased assignments on a natural distribution. Under this hypothesis, the author strengthens the previous known hardness for approximating Minimum Unique Game, $5/4 - \epsilon$, by proving that Min 2-Lin-2 is hard to within $3/2 - \epsilon$ and strengthens the previous known hardness for approximating Small Set Expansion, $4/3 - \epsilon$, by proving that Min Bisection is hard to approximate within $3 - \epsilon$. In addition, the author discusses the limitation of this method to show that it can strengthen the hardness for approximating Minimum Unique Game to $2 - \kappa$ where $\kappa$ is a small absolute positive, but is short of proving $\omega_k(1)$ hardness for Minimum Unique Game or Small Set Expansion, by assuming a generalization of this hypothesis on Unbalanced Max k-CSP with balanced pairwise independent predicate.

1 Preliminaries

In Unique Game (UG), we are given a graph $G = (V, E)$, and a set of labels, $[k]$. Each edge $e = (u, v)$ in the graph is equipped with a permutation $\pi_e : [k] \rightarrow [k]$. The solution of the problem is a labeling $f : V \rightarrow [k]$ that assigns a label to each vertex of $G$. An edge $e = (u, v)$ is said to be satisfied under $f$ if $\pi_e(f(u)) = f(v)$. The goal of the problem is to find a labeling such that the number of the satisfied edges under this labeling is maximized. The value of the instance $Val(I)$ is defined as the maximum fraction of the satisfied edges over all labeling. In the same situation of UG, the goal of Minimum Unique Game (Min UG) is to find a labeling such that the number of the unsatisfied edges under this labeling is minimized. The value of the instance $Val(I)$ is defined as the minimum fraction of the unsatisfied edges over all labeling.

In Max 2-Lin-2, we are given a set of linear equations over $GF_2$. Each equation contains exactly two variables. The goal of the problem is to seek an assignment of the variables such that the number of satisfied equations is maximized. In the same situation of Max 2-Lin-2, the goal of Min 2-Lin-2 is to seek an assignment
of the variables such that the number of unsatisfied equations is minimized. Note that Max 2-Lin-2 is a special case of UG, and Min 2-Lin-2 is special case of Min UG.

The Unique Game Conjecture (UGC)\(^8\) states: for every \(\zeta, \delta > 0\), there is a \(k = k(\zeta, \delta)\) such that given an instance \(I\) of UG with \(k\) labels it is NP-hard to distinguish whether \(Val(I) > 1 - \zeta\) or \(Val(I) < \delta\). The \((c, s)\)-approximation NP-hardness of UG is defined as: for some fixed \(0 < s < c < 1\), there is a \(k\) such that given an instance \(I\) of UG with \(k\) labels it is NP-hard to distinguish whether \(Val(I) > c\) or \(Val(I) < s + \epsilon\) for any \(\epsilon > 0\). For any fixed \(0 < c' < s' < 1\), the \((c', s')\)-approximation NP-hardness of Min UG is defined as: there is a \(k\) such that given an instance \(I\) of Min UG with \(k\) labels it is NP-hard to distinguish whether \(Val(I) > s' - \epsilon\) or \(Val(I) < c' + \epsilon\) for any \(\epsilon > 0\). Similarly, we can define the \((c', s')\)-approximation hardness of Min UG under Conjecture 2 or Conjecture 4.

In Small Set Expansion (SSE), we are given a graph \(G = (V, E)\) and a constant \(0 < \delta \leq 1/2\). The goal of the problem is to find a subset \(S \subseteq V\) satisfying \(|S|/|V| = \delta\) such that \(\Phi(S)\), the edge expansion of \(S\) is minimized. The edge expansion \(\Phi(S)\) of a subset \(S \subseteq V\) is defined as: \(\Phi(S) = \frac{|\{E \mid E \subseteq S\}|}{|E|}\). The expansion profile is defined as: \(\Phi_G(\delta) = \min_{|S|/|V| = \delta} \Phi(S)\), where \(0 < \delta \leq 1/2\). The Small Set Expansion Hypothesis (SSEH)\(^{11}\) states: for every \(\eta > 0\), there is a \(\delta\) such that it is NP-hard to distinguish whether \(\Phi_G(\delta) > 1 - \eta\) or \(\Phi_G(\delta) < \eta\).

As a special case of Small Set Expansion Problem, Min Bisection is defined as: given a graph \(G\) with \(n\) vertices, where \(n\) is even, find a set \(S\) of \(n/2\) vertices (a bisection) such that the number of edges connecting \(S\) and \(V \setminus S\) (the bisection width) is minimized.

The authors of \(^{10}\) show a new point of \((1/2, 3/8)\)-approximation NP-hardness of UG, which is an improvement of previously known \((3/4, 11/16)\)-approximation NP-hardness of UG\(^6\). Their result determines a two-dimensional region of for \((c, s)\)-approximation NP-hardness of UG, namely, the triangle with the three vertices \((0, 0)\), \((1/2, 3/8)\), and \((1, 1)\). All known points of \((c, s)\)-approximation NP-hardness of UG are in the triangle, plus an inferior bump area near the origin by \(^4\) (See Fig. 1). However, the best known hardness for approximating Min UG is still \(5/4 - \epsilon\) despite all the efforts. It is only known that we can rule out the possibility of PTAS for Min Bisection (and SSE) under a complexity assumption stronger than \(NP \neq P\)\(^9\), and that Min Bisection (and SSE) is hard to approximate within \(4/3 - \epsilon\) assuming Feige’s Hypothesis\(^3\). It would be interesting to answer the question whether we can further enlarge the hardness gap of Min UG and SSE.

The authors of \(^{31}\) has established connection between approximation complexity and average complexity. They use average complexity to prove inapproximability results for some famous problems, which has resisted discovery of meaningful inapproximability results under standard complexity assumptions. A recent example of such problems is Densest \(\kappa\)-Subgraph\(^2\).

Throughout this paper, let \(\alpha = \beta + \gamma - 2\beta\gamma\), and \(\epsilon\) generally denotes a negligible quantity.
2 Conjectures on Unbalanced 3-XOR and 3-AND

In this section, the author puts forward a variation of Feige’s Hypothesis, which claims it is hard on average refuting Unbalanced Max 3-XOR against biased assignments on a natural distribution. We can strengthen the previous known hardness for approximating Minimum Unique Game, $5/4 - \epsilon$, by proving that Min 2Lin-2 is hard to approximate within $3/2 - \epsilon$.

In Max 3-XOR, we are given a set of XOR clauses, each clause contains exactly three literals. The goal of the problem is to seek an assignment of the Boolean variables such that the number of satisfied clauses is maximized. In Max 3-AND, we are given a set of AND clauses, each clause contains exactly three literals. The goal of the problem is to seek an assignment of the Boolean variables such that the number of satisfied clauses is maximized.

In Random Unbalanced Max 3-XOR, we assume that formulas are generated by the following random process. Given parameters $n$ and $m$, each clause is generated independently at random by selecting the three variables in it independently at random and inserting the negative literal of the variable into the clause with probability $\beta < 1/2$ and inserting the positive literal of the variable into the clause with probability $1 - \beta$. $\beta$ is called imbalance of the instance. In addition, We are interested in the assignments such that the fraction of variables assigned to 0 is no more than $\gamma$, which is called bias of the assignments. In Random Unbalanced Max 3-AND, formulas are generated similarly, and we can define the notations, imbalance and bias, similarly.

In this paper, the author considers the average complexity of Random Unbalanced Max 3-XOR, and put forward a variation of Feige’s Hypothesis [31].

**Conjecture 1.** For every $0 < \gamma < \beta < 1/2$, for every fixed $\epsilon > 0$, for $\Delta$ a sufficiently large constant independent of $n$, there is no polynomial time algorithm that refutes most $\beta$-balanced Max 3-XOR formulas with $n$ variables and $m = \Delta n$ clauses, but never refutes a $1 - \epsilon$ satisfiable formula under $\gamma$-biased assignments.

The author also considers the average complexity of Random Unbalanced Max 3-AND, and put forward the following conjecture.

**Conjecture 2.** For every $0 < \gamma < \beta < 1/2$, for every fixed $\epsilon > 0$, for $\Delta$ a sufficiently large constant independent of $n$, there is no polynomial time algorithm that refutes most $\beta$-balanced Max 3-AND formulas with $n$ variables and $m = \Delta n$ clauses, but never refutes a $1 - \frac{3}{2} \alpha - \epsilon$ satisfiable formula under $\gamma$-biased assignments.

**Theorem 1.** Conjecture 1 implies Conjecture 2.

**Proof.** We rewrite a formula of $\beta$-balanced Max 3-XOR to a formula of $\beta$-balanced Max 3-AND. If the formula of Max 3-XOR is random, then the formula of Max 3-AND is also random. If the formula of Max 3-XOR $\phi$ is $1 - \epsilon$ satisfiable by $\gamma$-biased assignments, we show in the following that at least $1 - \frac{3}{2} \alpha - \epsilon$ fraction of clauses in $\phi$ have all the three literals satisfied.
On average, each positive literal has $3(1 - \beta)\Delta$ appearance in $\phi$, and each negative literal has $3\beta\Delta$ appearance in $\phi$. When $\Delta$ is large enough, standard bounds on large deviations show that with high probability, all but an $\epsilon$ fraction of the occurrences of positive literals correspond to positive literals that appear between $(3(1 - \beta) \pm \epsilon)\Delta$ times in $\phi$, and all but an $\epsilon$ fraction of the occurrences of negative literals correspond to negative literals that appear between $(3\beta \pm \epsilon)\Delta$ times in $\phi$.

If this does hold, observe that every $\gamma$-biased assignment $\psi$ does not satisfy on average at most $3(\beta(1 - \gamma) + \gamma(1 - \beta)) + \epsilon$ variables per clause in $\phi$. It then follows that at most $\frac{3}{2}(\beta(1 - \gamma) + \gamma(1 - \beta)) + \epsilon$ clauses have exactly one literal satisfied by $\psi$.

**Theorem 2.** Conjecture 2 holds for any $0 < \gamma < \beta < 1/2$ implies $(c', s')$-approximation hardness of Min UGP for $c' = \frac{1}{2}\alpha$ and $s' = \frac{1}{4}(1 - (1 - \beta)^3) - \epsilon$.

**Proof.** We use the three-dimensional cube gadget that is similar to the gadgets used by authors of [6,12].

Let $l_1 \land l_2 \land l_3$ be a clause in the formula of Max 3-AND, where $l_i$ is either a variable $x_i$ or its negation $\overline{x}_i$, for $i = 1, 2, 3$. The set of equations we construct have variables at the corners of a three-dimensional cube, which take value 1 or −1. For each $\mu \in \{0, 1\}^3$, we have a variable $v_{\mu}$. The variable $v_{000}$ is replaced by $w$ taking value −1. We let $u_1$ take the place of $v_{011}$, $u_2$ the place of $v_{101}$, and $u_3$ the place of $v_{110}$, where $u_i = -1$ if $x_i = 1$, and $u_i = 1$ if $x_i = 0$. For each edge $(u, v_{\mu})$ of the cube, we have the equation $wv_{\mu} = -1$. For each edge $(u, v_{\mu})$ of the cube, we have the equation $u_i v_{\mu} = 1$ if $l_i$ is positive, and the equation $u_i v_{\mu} = -1$ if $l_i$ is negative, for all $i = 1, 2, 3$.

If all $l_i$ are satisfied in the clause, we assign $v_{\mu}$ the value $(-1)^{\mu_1 + \mu_2 + \mu_3}$. All the twelve edge equation are satisfied and left no equation unsatisfied. Otherwise, an enumeration establishes that it is only possible to satisfy at most nine equations and left three equations unsatisfied, and that it is always possible to satisfy at least eight equations and left four equations unsatisfied.

Given a $\beta$-balanced Max 3-AND formulas $\phi$ that is at most $(1 - \beta)^3 + \epsilon$ satisfiable, at least $1 - (1 - \beta)^3 - \epsilon$ clauses in $\phi$ are unsatisfied.

Now we reduce a formula of $\beta$-balanced Max 3-AND to an instance of Min 2-Lin-2 using the gadget introduced above. If the formula is $1 - \frac{3}{2}\alpha - \epsilon$ satisfiable under $\gamma$-biased assignments, then the value of the instance of Min 2-Lin-2 is at most

$$\frac{3}{2}\alpha + \epsilon - \left(\frac{3}{2}\alpha + \epsilon\right)\frac{3}{4} = \frac{1}{2}\alpha + \epsilon.$$ 

If the formula is random, then it is at most $(1 - \beta)^3 + \epsilon$ satisfiable in high probability, which implies the value of the instance of Min 2-Lin-2 is at least

$$1 - (1 - \beta)^3 - \epsilon - (1 - (1 - \beta)^3 - \epsilon)\frac{3}{4} = \frac{1}{4}(1 - (1 - \beta)^3) - \epsilon.$$ 

**Corollary 1.** Conjecture 2 holds for arbitrarily small $\beta$ and $\gamma$ implies Min UGP is hard to approximate within $3/2 - \epsilon$. 

Lemma 1. For an integer $k \geq 3$ and every $\epsilon > 0$, there is some $\Delta_\epsilon > 0$ such that for every $\Delta > \Delta_\epsilon$, $n$ large enough, and $0 < \gamma < \beta < 1/2$, with high probability the following holds. Every set of $((1-\alpha)^k + \epsilon)m$ clauses in a random $\beta$-balanced Max 3-AND formula with $m = \Delta n$ clauses contains at least $\gamma n + 1$ different negative literals or $(1-\gamma)n + 1$ different positive literals.

Proof. Fix a set $S$ of $n$ literals with exactly $\gamma$ fraction of positive literals to be avoided. The probability that a random clause with three literals avoids these literals is $(1-\alpha)^3$. For large enough $\Delta$, standard bounds on large deviations implies that with probability greater than $1 - (1-\alpha)^3n$, less than $((1-\alpha)^k + \epsilon)m$ random clauses avoid the set $S$. As there are roughly $2^\gamma n$ ways of choosing the set $S$, the union bound implies that on one of them is avoided by a set of $((1-\alpha)^k + \epsilon)m$ clauses.

Theorem 3. Conjecture 2 holds for any $0 < \gamma < \beta < 1/2$ implies Small Set Expansion is hard to approximate within $\frac{2^{(1-(1-\beta)^3)}}{2^\alpha} - 1 - \epsilon$.

Proof. We reduce $\beta$-balanced Max 3-AND to Min Bisection. Given a Max 3-AND formula with $n'$ variables and $m' = \Delta n'$ clauses which we want to distinguish between the case at most $((1-\beta)^3 + \epsilon)m'$ clauses are satisfiable and the case that at least $(1 - \frac{3}{2} \alpha + \epsilon)m'$ clauses are satisfiable by $\gamma$-biased assignments, construct the following graph.

The left hand side (LHS) contains $2n'$ vertices, one for each literal. The right hand side (RHS) contains $m'$ clusters, one for each clause, where each cluster is a clique of size $m'$. In addition, the graph contains a clique of size $m'' = (1 - 3\alpha + \epsilon)m'^2$. In each cluster there is a unique vertex that is a "connecting vertex". Place an edge between a vertex that corresponds a literal and the connecting vertex of a cluster if the literal is in the clause that corresponds to the cluster. These are called the "bipartite" edges.

In this graph, find a minimum bisection, which contains exactly $n'$ LHS vertices, and $(1 - \frac{3}{2} \alpha - \epsilon)m'$ clusters. It suffices to consider only the connecting vertices from each of the $m'$ clusters, and we need to find a cut of minimum width that contains $n'$ vertices from the LHS, and $(1 - \frac{3}{2} \alpha - \epsilon)m'$ connecting vertices.

When the 3-AND formula has $(1 - \frac{3}{2} \alpha - \epsilon)m'$ satisfiable clauses by $\gamma$-biased assignments, we pick the set $S$ to contain the clauses corresponding to these clauses and the $n'$ literals corresponding to the assignments consistent with these clauses. The only edges cut by this bisection connect the satisfying literals to unsatisfied clauses. The number of bipartite edges within the set $S$ is $3(1 - \frac{3}{2} \alpha - \epsilon)m'$. The sum of degrees of the satisfied literals is $3(1 - \alpha)m'$. Hence the width of the bisection is $\frac{3}{2} \alpha + \epsilon$.

In a random 3-AND formula, we still need one side of the cut to contain $n'$ vertices and $(1 - \frac{3}{2} \alpha - \epsilon)m'$ clusters. This set of $n'$ literals has at most $((1 - \alpha)^3 + \epsilon)m'$ of these clauses 3-connected to it (by Lemma 1) and the other $(1 - \frac{3}{2} \alpha - (1-\alpha)^3 - 2\epsilon)m'$ clauses are 2-connected to it. Hence the width of the cut is at least

$$3(1 - \alpha)m' - 3((1-\alpha)^3 + \epsilon)m' - (1 - \frac{3}{2} \alpha - (1-\alpha)^3 - 2\epsilon)m' = (2(1-(1-\alpha)^3) - \frac{3}{2} \alpha - \epsilon)m'$$
which is lower bounded by \((2(1 - (1 - \beta)^3) - \frac{2}{3}\alpha - \epsilon)m'\).

**Corollary 2.** Conjecture 2 holds for and arbitrarily small \(\gamma\) and \(\beta\) implies SSE is hard to approximate within \(3 - \epsilon\).

### 3 Conjectures on Unbalanced k-CSP

In this section, the author discusses the limitation of our method. Conjecture 1 can be generalized to that it is hard on average refuting Unbalanced Max k-CSP with balanced pairwise independent predicate against biased assignments on a natural distribution. The largest strengthened hardness of Min 2-Lin-2 that Conjecture 3 can yield is \(2 - \kappa\) where \(\kappa\) is a small absolute positive. However, the author also shows that Conjecture 3 is not true for sufficiently large \(k\). Hence, we cannot further strengthen the hardness for approximating Minimum Unique Game to \(\omega_k(1)\), by proving that Min 2-Lin-2 is hard to approximate within any constant assuming Conjecture 3.

Let \(C\) be a \(k\)-ary predicate. In Max \(C\), we are given a set of clauses, each clause contains exactly \(k\) literals. A clause is satisfied if the values of literals satisfies \(C\). The goal of the problem is to seek an assignment of the Boolean variables such that the number of satisfied clauses is maximized. We consider the case Max \(C\) where \(C\) is a balanced pairwise independent predicate and the case Max k-AND where \(C\) is the predicate with one satisfying \(k\)-tuple \((1, \cdots, 1)\).

\(C\) is a balanced pairwise independent predicate, if for every two distinct coordinates \(i \neq j \in [k]\) and every two elements \(a_1, a_2 \in \{0, 1\}\),

\[\mathbb{P}[c_i = a_1, c_j = a_2] = 1/4,\]

where \(c = (c_1, ..., c_k)\) is a uniformly random element drawn from the support of \(C\).

In Random Unbalanced Max \(C\), we still assume that formulas are generated by the following random process. Given parameters \(n\) and \(m\), each clause is generated independently at random by selecting the \(k\) variables in it independently at random and inserting the negative literal of the variable into the clause with probability \(\beta < 1/2\) and inserting the positive literal of the variable into the clause with probability \(1 - \beta\). \(\beta\) is called imbalance of the instance. In addition, we are interested in the assignments such that the fraction of variables assigned to 0 is no more than \(\gamma\), which is called bias of the assignments. We also consider the case Max \(C\) where \(C\) is a balanced pairwise independent predicate and the case Max k-AND where \(C\) is the predicate with one satisfying \(k\)-tuple \((1, \cdots, 1)\).

In this section, the author considers the average complexity of Random Unbalanced Max \(C\) with balanced pairwise independent predicate, and puts forward a variation of Feige’s Hypothesis [31].

**Conjecture 3.** Let \(C\) be a predicate with balanced pairwise independent support. For every \(0 < \gamma < \beta < 1/2\), for every fixed \(\epsilon > 0\), for \(\Delta\) a sufficiently large
constant independent of $n$, there is no polynomial time algorithm that refutes most $\beta$-balanced Max $C$ formulas with $n$ variables and $m = \Delta n$ clauses, but never refutes a $1 - \epsilon$ satisfiable formula under $\gamma$-biased assignments.

The author also considers the average complexity of Random Unbalanced Max $k$-AND, and put forward the following conjecture.

**Conjecture 4.** For every $0 < \gamma < \beta < 1/2$, for every fixed $\epsilon > 0$, for $\Delta$ a sufficiently large constant independent of $n$, there is no polynomial time algorithm that refutes most $\beta$-balanced Max $k$-AND formulas with $n$ variables and $m = \Delta n$ clauses, but never refutes a $1 - 2\alpha - \epsilon$ satisfiable formula under $\gamma$-biased assignments.

We can prove that Conjecture 3 implies Conjecture 4 similarly as proof of Theorem 1.

**Theorem 4.** Conjecture 3 implies Conjecture 4.

**Proof.** We assume a clause in $\phi$ is satisfied, either it has all the $k$ literals satisfied, or it has at least $k/2$ literals unsatisfied.

We rewrite a formula of $\beta$-balanced Max $C$ to a formula of $\beta$-balanced Max $k$-AND. If the formula of Max $C$ is random, then the formula of Max $3$-AND is also random. If the formula of Max $C \phi$ is $1 - \epsilon$ satisfiable by $\gamma$-biased assignments, we show in the following that at least $1 - 2\alpha - \epsilon$ fraction of clauses in $\phi$ have all the $k$ literals satisfied.

On average, each positive literal has $k(1 - \beta)\Delta$ appearance in $\phi$, and each negative literal has $k\beta\Delta$ appearance in $\phi$. When $\Delta$ is large enough, standard bounds on large deviations show that with high probability, all but an $\epsilon$ fraction of the occurrences of positive literals correspond to positive literals that appear between $(k(1 - \beta) \pm \epsilon)\Delta$ times in $\phi$, and all but an $\epsilon$ fraction of the occurrences of negative literals correspond to negative literals that appear between $(k\beta \pm \epsilon)\Delta$ times in $\phi$.

Observe that every $\gamma$-biased assignment $\psi$ does not satisfy on average at most $k((\beta(1 - \gamma) + \gamma(1 - \beta)) + \epsilon)$ variables per clause in $\phi$. It then follows that at most $2(\beta(1 - \gamma) + \gamma(1 - \beta)) + \epsilon$ clauses have at least $k/2$ literals unsatisfied by $\psi$.

**Theorem 5.** Conjecture 4 holds for any $0 < \gamma < \beta < 1/2$ implies $(c', s')$-approximation hardness of Min $UG$ for $c' = O(1/\log k)\alpha$ and $s' = \Omega(1/k)(1 - (1 - \beta)^k) - o_k(\beta))$.

**Proof.** We use the $(\log k + 1)$-dimensional hypercube gadget that is similar to the gadgets used by authors of [6,12].

Let $l_1 \land \cdots \land l_k$ be a clause in the formula of Max $k$-AND, where $l_i$ is either a variable $x_i$ or its negation $\bar{x}_i$, for $i = 1, \cdots, k$. The set of equations we construct have variables at the corners of a $(\log k + 1)$-dimensional hypercube, which take value 1 or $-1$. For each $\mu \in \{0, 1\}^k$, we have a variable $v_\mu$. We let $u_1, \cdots, u_k$
take the place of $v_\mu$, for $\mu$'s that are length-(log $k + 1$) codes that have even number of 1. Let $u_i = -1$ if $x_i = 1$, and $u_i = 1$ if $x_i = 0$. For each edge $(u_\mu, v_\mu)$ of the cube, we have the equation $u_\mu v_\mu = 1$ if $l_\mu$ is positive, and the equation $u_\mu v_\mu = -1$ if $l_\mu$ is negative, for all $i = 1, \ldots, k$.

If all $l_\mu$ are satisfied in the clause, we assign $v_\mu$ the value $(-1)^{u_1 + \cdots + u_k}$. All the edge equations are satisfied and left no equation unsatisfied. Otherwise, it is only possible to satisfy at most $1 - \Omega(1/k)$ fraction of equations and left $\Omega(1/k)$ fraction of equations unsatisfied, and that it is always possible to satisfy at least $1 - O(1/\log k)$ equations and left $O(1/\log k)$ equations unsatisfied.

Given a $\beta$-balanced Max k-AND formulas $\phi$ that is at most $(1 - \beta)^k + o_k(\beta)$ satisfiable, at least $1 - (1 - \beta)^k - o_k(\beta)$ clauses in $\phi$ are unsatisfied.

Now we reduce a formula of $\beta$-balanced Max k-AND to an instance of Min 2-Lin-2 using the gadget introduced above. If the formula is $1 - 2\alpha - \epsilon$ satisfiable under $\gamma$-biased assignments, then the value of the instance of Min 2-Lin-2 is at most

$$2\alpha + \epsilon - (2\alpha + \epsilon)(1 - O(1/\log k)) = O(1/\log k)\alpha + \epsilon.$$  

If the formula is random, then it is at most $(1 - \beta)^k + o_k(\beta)$ satisfiable in high probability, which implies the value of the instance of Min 2-Lin-2 is at least

$$1 - (1 - \beta)^k - o_k(\beta) - (1 - (1 - \beta)^k - o_k(\beta))(1 - \Omega(1/k)) = \Omega(1/k)(1 - (1 - \beta)^k) - o_k(\beta).$$

**Theorem 6.** Conjecture 2 holds for any $0 < \gamma < \beta < 1/2$ implies Small Set Expansion is hard to approximate within

$$\frac{k - 1}{k} \frac{1 - (1 - \beta)^k}{\alpha} - \frac{k - 2}{k^2} - \epsilon.$$

**Proof.** We reduce $\beta$-balanced Max k-AND to Min Bisection. Given a Max k-AND formula with $n'$ variables and $m' = \Delta n'$ clauses in which we want to distinguish between the case at most $((1 - \beta)^k + \epsilon)m'$ clauses are satisfiable and the case that at least $(1 - 2\alpha + \epsilon)m'$ clauses are satisfiable by $\gamma$-biased assignments, construct the following graph.

The left hand side (LHS) contains $2n'$ vertices, one for each literal. The right hand side (RHS) contains $m'$ clusters, one for each clause, where each cluster is a clique of size $m'$. In addition, the graph contains a clique of size $m'' = (1 - 4\alpha + \epsilon)m'^2$. In each cluster there is a unique vertex that is a “connecting vertex”. Place an edge between a vertex that corresponds a literal and the connecting vertex of a cluster if the literal that corresponds the cluster. These are called the "bipartite" edges.

In this graph, find a minimum bisection, which contains exactly $n'$ LHS vertices, and $(1 - 2\alpha - \epsilon)m'$ clusters. It suffices to consider only the connecting vertices from each of the $m'$ clusters, and we need to find a cut of minimum width that contains $n'$ vertices from the LHS, and $(1 - 2\alpha - \epsilon)m'$ connecting vertices.

When the k-AND formula has $(1 - 2\alpha - \epsilon)m'$ satisfiable clauses by $\gamma$-biased assignments, we pick the set $S$ to contain the clauses corresponding to these clauses and the $n'$ literals corresponding to the assignments consistent with these clauses. The only edges cut by this bisection connect the satisfying literals to unsatisfied clauses. The number of bipartite edges within the set $S$ is $k(1 - 2\alpha - \epsilon)$.
The sum of degrees of the satisfied literals is \( k(1 - \alpha) m' \). Hence the width of the bisection is \( k\alpha + \epsilon \).

In a random k-AND formula, we still need one side of the cut to contain \( n' \) vertices and \( (1 - 2\alpha - \epsilon) m' \) clusters. This set of \( n' \) literals has at most \( ((1 - \alpha)^k + \epsilon)m' \) of these clauses \( k \)-connected to it (by Lemma 1) and the other \( (1 - 2\alpha - (1 - \alpha)^k - 2\epsilon)m' \) clauses are \( (k - 1) \)-connected to it. Hence the width of the cut is at least

\[
k(1 - \alpha)m' - k((1 - \alpha)^k + \epsilon)m' - (1 - 2\alpha - (1 - \alpha)^k - 2\epsilon)m' = ((k - 1)(1 - (1 - \alpha)^k) - (k - 2)\alpha - \epsilon)m',
\]

which is lower bounded by

\[
((k - 1)(1 - (1 - \beta)^k) - (k - 2)\alpha - \epsilon)m'.
\]

4 Discussion

Notice that for Theorem 5 to make sense, we have \( \beta = O(\log k/k) \). However, by the construction of gadgets in proof of Theorem 5, we can reduce the instance of Min 2-Lin-2 to an instance of Min 2-SAT, where at least \( 1 - O(\beta^2) \) fraction of the clauses is Horn. Min Horn-2-SAT can approximate with constant ratio by LP algorithms, see [5] (ratio 2) or [7] (ratio 3). Hence, the largest strengthened hardness of Min 2-Lin-2 that Conjecture 3 can yield is \( 2 - \kappa \) where \( \kappa \) is a small absolute positive, if we allow \( \log k \) to be a rational number and the gadgets in proof of Theorem 5 to be "fractional hypercubes". However, Conjecture 3 cannot be true when \( k \) is so large so that the hardness exceeds 2.

In Fig. 1, the dark gray area is the known region of \((c, s)\)-approximation NP-hardness of UGP, the light gray area at the top left corner is the region of \((c, s)\)-approximation hardness of UGP assuming Conjecture 3 for certain \( k \).

References

1. Alekhnovich, M. (2003, October). More on average case vs approximation complexity. In Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on (pp. 298-307). IEEE.
2. Alon, N., Arora, S., Manokaran, R., Moshkovitz, D., & Weinstei, O. (2011). Inapproximability of densest -subgraph from average case hardnes.
3. Feige, U. (2002, May). Relations between average case complexity and approximation complexity. In Proceedings of the thirty-fourth annual ACM symposium on Theory of computing (pp. 534-543). ACM.
4. Feige, U., & Reichman, D. (2004). On systems of linear equations with two variables per equation. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (pp. 117-127). Springer Berlin Heidelberg.
5. Guruswami, V., & Zhou, Y. (2011, January). Tight bounds on the approximability of almost-satisfiable Horn SAT and exact hitting set. In Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms (pp. 1574-1589). SIAM.
6. H?stad, J. (2001). Some optimal inapproximability results. Journal of the ACM (JACM), 48(4), 798-859.
Fig. 1. illustration of \((c, s)\)-approximation of UGP

7. Khanna, S., Sudan, M., Trevisan, L., & Williamson, D. P. (2001). The approx-
   imability of constraint satisfaction problems. SIAM Journal on Computing, 30(6),
   1863-1920.
8. Khot, S. (2002, May). On the power of unique 2-prover 1-round games. In Pro-
   ceedings of the thirty-fourth annual ACM symposium on Theory of computing (pp.
   767-775). ACM.
9. Khot, S. (2006). Ruling out PTAS for graph min-bisection, dense k-subgraph, and
   bipartite clique. SIAM Journal on Computing, 36(4), 1025-1071.
10. O’Donnell, R., & Wright, J. (2012, May). A new point of NP-hardness for Unique
     Games. In Proceedings of the 44th symposium on Theory of Computing (pp. 289-
     306). ACM.
11. Raghavendra, P., & Steurer, D. (2010, June). Graph expansion and the unique
     games conjecture. In Proceedings of the 42nd ACM symposium on Theory of com-
     puting (pp. 755-764). ACM.
12. Trevisan, L., Sorkin, G. B., Sudan, M., & Williamson, D. P. (2000). Gadgets, ap-
    proximation, and linear programming. SIAM Journal on Computing, 29(6), 2074-
    2097.