Research Article
Generalizations of the Lax-Milgram Theorem
Dimosthenis Drivaliaris and Nikos Yannakakis
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We prove a linear and a nonlinear generalization of the Lax-Milgram theorem. In particular, we give sufficient conditions for a real-valued function defined on the product of a reflexive Banach space and a normed space to represent all bounded linear functionals of the latter. We also give two applications to singular differential equations.

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1. Introduction

The following generalization of the Lax-Milgram theorem was proved recently by An et al. in [1].

**Theorem 1.1.** Let $X$ be a reflexive Banach space over $\mathbb{R}$, let $\{X_n\}_{n \in \mathbb{N}}$ be an increasing sequence of closed subspaces of $X$ and $V = \bigcup_{n \in \mathbb{N}} X_n$. Suppose that

$$A : X \times V \rightarrow \mathbb{R}$$

is a real-valued function on $X \times V$ for which the following hold:

(a) $A_n = A|_{X_n \times X_n}$ is a bounded bilinear form, for all $n \in \mathbb{N}$;
(b) $A(\cdot, v)$ is a bounded linear functional on $X$, for all $v \in V$;
(c) $A$ is coercive on $V$, that is, there exists $c > 0$ such that

$$A(v, v) \geq c\|v\|^2,$$

for all $v \in V$. 
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Then, for each bounded linear functional $v^*$ on $V$, there exists $x \in X$ such that

$$A(x, v) = \langle v^*, v \rangle,$$  \hspace{1cm} (1.3)

for all $v \in V$.

In this paper our aim is to prove a linear extension and a nonlinear extension of Theorem 1.1. In the linear case, we use a variant of a theorem due to Hayden [2, 3], and thus manage to substitute the coercivity condition in (c) of the previous theorem with a more general inf-sup condition. In the nonlinear case, we appropriately modify the notion of type $M$ operator and use a surjectivity result for monotone, hemicontinuous, coercive operators. We also present two examples to illustrate the applicability of our results.

All Banach spaces considered are over $\mathbb{R}$. Given a Banach space $X$, $X^*$ will denote its dual and $\langle \cdot, \cdot \rangle$ will denote their duality product. Moreover, if $M$ is a subset of $X$, then $M^\perp$ will denote its annihilator in $X^*$ and if $N$ is a subset of $X^*$, then $\perp N$ will denote its preannihilator in $X$.

2. The linear case

To prove our main result for the linear case, we need the following lemma which is a variant of [2, Theorem 12] and [3, Theorem 1].

**Lemma 2.1.** Let $X$ be a reflexive Banach space, let $Y$ be a Banach space and let

$$A : X \times Y \rightarrow \mathbb{R}$$  \hspace{1cm} (2.1)

be a bounded, bilinear form satisfying the following two conditions:

(a) $A$ is nondegenerate with respect to the second variable, that is, for each $y \in Y \setminus \{0\}$, there exists $x \in X$ with $A(x, y) \neq 0$;

(b) there exists $c > 0$ such that

$$\sup_{\|y\|=1} |A(x, y)| \geq c\|x\|, \hspace{1cm} (2.2)$$

for all $x \in X$.

Then, for every $y^* \in Y^*$, there exists a unique $x \in X$ with

$$A(x, y) = \langle y^*, y \rangle,$$  \hspace{1cm} (2.3)

for all $y \in Y$.

**Proof.** Let $T : X \rightarrow Y^*$ with $\langle Tx, y \rangle = A(x, y)$, for all $x \in X$ and all $y \in Y$. Obviously, $T$ is a bounded linear map. Since, by (b), $\|Tx\| \geq c\|x\|$, for all $x \in X$, $T$ is one to one. To complete the proof, we need to show that $T$ is onto.

Since $A$ is nondegenerate with respect to the second variable, we have that

$$\perp T(X) = \{ y \in Y \mid A(x, y) = 0, \forall x \in X \} = \{0\}. \hspace{1cm} (2.4)$$
Hence
\[(\perp T(X))^\perp = Y^*,\] (2.5)
and so by [4, Proposition 2.6.6],
\[\overline{T(X)}^{w^*} = Y^*.\] (2.6)

Thus to show that \(T\) maps \(X\) onto \(Y^*\), we need to prove that \(T(X)\) is \(w^*\)-closed in \(Y^*\). To see that, let \(\{ Tx_\lambda \}_{\lambda \in \Lambda}\) be a net in \(T(X)\) and let \(y^*\) be an element of \(Y^*\) such that
\[Tx_\lambda \rightharpoonup y^*.\] (2.7)

Without loss of generality, we may assume, using the special case of the Krein-Šmulian theorem on \(w^*\)-closed linear subspaces (see [4, Corollary 2.7.12]), the proof of which is originally due to Banach [5, Theorem 5, page 124] for the separable case and due to Dieudonné [6, Theorem 23] for the general case, that \(\{ Tx_\lambda \}_{\lambda \in \Lambda}\) is bounded. Thus, since \(\|Tx\| \geq c\|x\|\) for all \(x \in X\), the net \(\{ x_\lambda \}_{\lambda \in \Lambda}\) is also bounded. Hence, since \(X\) is reflexive, there exist a subnet \(\{ x_{\mu} \}_{\mu \in M}\) and an element \(x\) of \(X\) such that \(\{ x_{\mu} \}_{\mu \in M}\) converges weakly to \(x\). Since \(T\) is \(w - w^*\) continuous, \(Tx_{\mu} \rightharpoonup Tx\). Hence \(Tx = y^*\), and so \(T(X)\) is \(w^*\)-closed. \(\square\)

**Remark 2.2.** An alternative proof of the previous lemma can be obtained using the closed range theorem.

We are now in a position to prove our main result for the linear case.

**Theorem 2.3.** Let \(X\) be a reflexive Banach space, let \(Y\) be a Banach space, let \(\Lambda\) be a directed set, let \(\{ X_\lambda \}_{\lambda \in \Lambda}\) be a family of closed subspaces of \(X\), let \(\{ Y_\lambda \}_{\lambda \in \Lambda}\) be an upwards directed family of closed subspaces of \(Y\), and let \(V = \bigcup_{\lambda \in \Lambda} Y_\lambda\). Suppose that
\[A : X \times V \rightarrow \mathbb{R}\] (2.8)
is a function for which the following hold:
(a) \(A_\lambda = A|_{X_\lambda \times Y_\lambda}\) is a bounded bilinear form, for all \(\lambda \in \Lambda\);
(b) \(A(\cdot, v)\) is a bounded linear functional on \(X\), for all \(v \in V\);
(c) \(A_\lambda\) is nondegenerate with respect to the second variable, for all \(\lambda \in \Lambda\);
(d) there exists \(c > 0\) such that for all \(\lambda \in \Lambda\),
\[\sup_{\substack{y \in Y_\lambda, \|y\| = 1}} |A_\lambda(x, y)| \geq c\|x\|,\] (2.9)
for all \(x \in X_\lambda\).

Then, for each bounded linear functional \(v^*\) on \(V\), there exists \(x \in X\) such that
\[A(x, v) = \langle v^*, v \rangle,\] (2.10)
for all \(v \in V\).
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Proof. Let \( v^* \in V^* \), and for each \( \lambda \in \Lambda \), let \( v_\lambda^* = v^* |_{Y_\lambda} \). For all \( \lambda \in \Lambda \), \( v_\lambda^* \) is a bounded linear functional on \( Y_\lambda \). By hypothesis, for all \( \lambda \in \Lambda \), \( A_\lambda \) is a bounded bilinear form on \( X_\lambda \times Y_\lambda \) satisfying the two conditions of Lemma 2.1. Since for all \( \lambda \in \Lambda \), \( X_\lambda \) is a reflexive Banach space, we get that for each \( \lambda \in \Lambda \), there exists a unique \( x_\lambda \) such that \( A_\lambda(x_\lambda, y) = \langle v_\lambda^*, y \rangle \), for all \( y \in Y_\lambda \). Since \( A \) satisfies condition (d), we get that for all \( \lambda \in \Lambda \),

\[
c \|x_\lambda\| \leq \sup_{y \in Y_\lambda, \|y\| = 1} |A_\lambda(x_\lambda, y)| = \sup_{y \in Y_\lambda, \|y\| = 1} |\langle v_\lambda^*, y \rangle| \leq \|v^*\|. \tag{2.11}
\]

So \( \{x_\lambda\}_{\lambda \in \Lambda} \) is a bounded net in \( X \). Since \( X \) is reflexive, there exist a subnet \( \{x_{\lambda_\mu}\}_{\mu \in \mathcal{M}} \) of \( \{x_\lambda\}_{\lambda \in \Lambda} \) and \( x \) in \( X \) such that \( \{x_{\lambda_\mu}\}_{\mu \in \mathcal{M}} \) converges weakly to \( x \).

We are going to prove that \( A(x, v) = \langle v^*, v \rangle \), for all \( v \in V \). Take \( v \in V \). Then there exists some \( \lambda_0 \in \Lambda \) with \( v \in Y_{\lambda_0} \). Since \( \{x_{\lambda_\mu}\}_{\mu \in \mathcal{M}} \) is a subnet of \( \{x_\lambda\}_{\lambda \in \Lambda} \), there exists some \( \mu_0 \in \mathcal{M} \) with \( \lambda_{\mu_0} \geq \lambda_0 \). Hence, since the family \( \{Y_\lambda\}_{\lambda \in \Lambda} \) is upwards directed,

\[
v \in Y_{\lambda_\mu}, \tag{2.12}
\]

for all \( \mu \geq \mu_0 \). Thus, for all \( \mu \geq \mu_0 \),

\[
A_{\lambda_\mu}(x_{\lambda_\mu}, v) = \langle v_{\lambda_\mu}^*, v \rangle. \tag{2.13}
\]

Therefore

\[
\lim_{\mu \in \mathcal{M}} A(x_{\lambda_\mu}, v) = \langle v^*, v \rangle. \tag{2.14}
\]

Since \( A(\cdot, v) \) is a bounded linear functional on \( X \),

\[
\lim_{\mu \in \mathcal{M}} A(x_{\lambda_\mu}, v) = A(x, v). \tag{2.15}
\]

Hence \( A(x, v) = \langle v^*, v \rangle \). \hfill \Box

The following example illustrates the possible applicability of Theorem 2.3.

Example 2.4. Let \( a \in C^1(0, 1) \) be a decreasing function with \( \lim_{t \to 0} a(t) = \infty \) and \( a(t) \geq 0 \), for all \( t \in (0, 1) \). We will establish the existence of a solution for the following Cauchy problem:

\[
\begin{align*}
  u' + a(t)u &= f \quad \text{a.e. on } (0, 1), \\
  u(0) &= 0,
\end{align*}
\tag{2.16}
\]

where \( f \in L^2(0, 1) \).

Let \( X = \{u \in H^1(0, 1) \mid u(0) = 0\} \) be equipped with the norm \( \|u\| = \left( \int_0^1 |u'|^2 dt \right)^{1/2} \), which is equivalent to the original Sobolev norm, and \( Y = L^2(0, 1) \). Note that \( X \) is a reflexive Banach space, being a closed subspace of \( H^1(0, 1) \). Let \( \{\alpha_n\}_{n \in \mathbb{N}} \) be a decreasing sequence in \((0, 1)\) with \( \lim_{n \to \infty} \alpha_n = 0 \). Define

\[
X_n = \{u \in H^1(\alpha_n, 1) \mid u(\alpha_n) = 0\}, \quad Y_n = L^2(\alpha_n, 1) \tag{2.17}
\]
(we can consider $X_n$ and $Y_n$ as closed subspaces of $X$ and $Y$, resp., by extending their elements by zero outside $(\alpha_n, 1)$). Also let $V = \bigcup_{n=1}^\infty Y_n$.

Let $A : X \times V \to \mathbb{R}$ be the bilinear map defined by

$$A(u, v) = \int_0^1 u'v\,dt + \int_0^1 a(t)uv\,dt.$$  \hfill (2.18)

$A$ is well defined and $A(\cdot, v)$ is a bounded linear functional on $X$ for any $v \in V$.

Let $A_n = A|_{X_n \times Y_n}$. $A_n$ be a bounded bilinear form since

$$\left| A_n(u, v) \right| \leq (1 + M_n) \|u\|_{X_n} \|v\|_{Y_n},$$  \hfill (2.19)

where $M_n$ is the bound of $a$ on $[\alpha_n, 1]$. It should be noted that $A$ is not bounded on the whole of $X \times V$.

To show that $A_n$ is nondegenerate, let $v \in Y_n$ and assume that $A_n(u, v) = 0$ for all $u \in X_n$, that is,

$$\int_{\alpha_n}^1 (u' + a(t)u)v\,dt = 0, \quad \forall u \in X_n.$$  \hfill (2.20)

It is easy to see that the above implies that

$$\int_{\alpha_n}^1 wv\,dt = 0,$$  \hfill (2.21)

for any continuous function $w$, and therefore $v = 0$.

We next show that

$$\sup_{\|v\| = 1, v \in Y_n} \left| A_n(u, v) \right| \geq \|u\|_{X_n}.$$  \hfill (2.22)

Define $T_n : X_n \to Y_n^*$ by $\langle T_n u, v \rangle = A_n(u, v)$. $T_n$ is a well-defined bounded linear operator and $T_n u = u' + a(t)u$. Hence

$$\|T_n u\|^2 = \int_{\alpha_n}^1 \left| u' + a(t)u \right|^2\,dt$$

$$= \int_{\alpha_n}^1 |u'|^2\,dt + \int_{\alpha_n}^1 a^2(t)|u|^2\,dt + \int_{\alpha_n}^1 a(t)(u'^2)\,dt$$

$$= \int_{\alpha_n}^1 |u'|^2\,dt + \int_{\alpha_n}^1 (a^2(t) - a'(t))|u|^2\,dt + a(1)u'^2(1) \geq \|u\|_{X_n}^2,$$  \hfill (2.23)

since $u(\alpha_n) = 0$, $a$ is decreasing and $a(t) \geq 0$ for all $t \in (0, 1)$.

All the hypotheses of Theorem 2.3 are hence satisfied and so if $F \in V^*$ is defined by $F(v) = \int_0^1 f v\,dt$, then there exists $u \in X$ such that

$$A(u, v) = F(v), \quad \forall v \in V.$$  \hfill (2.24)

Thus $u$ satisfies (2.16).
3. The nonlinear case

We start by recalling some well-known definitions.

**Definition 3.1.** Let $T : X \to X^*$ be an operator. Then $T$ is said to be
(i) monotone if $\langle Tx - Ty, x - y \rangle \geq 0$, for all $x, y \in X$;
(ii) hemicontinuous if for all $x, y \in X$, $T(x + ty) \rightharpoonup Tx$ as $t \to 0^+$;
(iii) coercive if
\[
\lim_{\|x\| \to \infty} \frac{\langle Tx, x \rangle}{\|x\|} = \infty.
\] (3.1)

We also need the following generalization of the notion of type M operator (for the classical definition, see [7] or [8]).

**Definition 3.2.** Let $X$ be a Banach space, let $V$ be a linear subspace of $X$, and let $A : X \times V \to \mathbb{R}$ (3.2) be a function. Then $A$ is said to be of type M with respect to $V$ if for any net $\{v_\lambda\}_{\lambda \in \Lambda}$ in $V$, $x \in X$ and $v^* \in V^*$;
(a) $v_\lambda \rightharpoonup x$;
(b) $A(v_\lambda, v) \to \langle v^*, v \rangle$, for all $v \in V$;
(c) $A(v_\lambda, v_\lambda) \to \langle \hat{v}^*, x \rangle$, where $\hat{v}^*$ is the extension of $v^*$ on the closure of $V$,

imply that $A(x, v) = \langle v^*, v \rangle$, for all $v \in V$.

Our result is the following.

**Theorem 3.3.** Let $X$ be a reflexive Banach space, let $\Lambda$ be a directed set, let $\{X_\lambda\}_{\lambda \in \Lambda}$ be an upwards directed family of closed subspaces of $X$, and let $V = \bigcup_{\lambda \in \Lambda} X_\lambda$. Suppose that
\[
A : X \times V \to \mathbb{R}
\] (3.3)
is a function for which the following hold:
(a) $A$ is of type M with respect to $V$;
(b) $\lim_{\|x\| \to \infty} A(x, x)/\|x\| = \infty$;
(c) $A_\lambda(x, \cdot) \in X_\lambda^*$, for all $\lambda \in \Lambda$ and all $x \in X_\lambda$, where $A_\lambda$ is the restriction of $A$ on $X_\lambda \times X_\lambda$;
(d) the operator $T_\lambda : X_\lambda \to X_\lambda^*$, defined by $\langle T_\lambda x, y \rangle = A_\lambda(x, y)$ for all $x, y \in X_\lambda$, is monotone and hemicontinuous for all $\lambda \in \Lambda$.

Then for each $v^* \in V^*$, there exists $x \in X$ such that
\[
A(x, v) = \langle v^*, v \rangle,
\] (3.4)
for all $v \in V$.

**Proof.** As in the proof of Theorem 2.3, for each $\lambda \in \Lambda$, let $v^*_\lambda = v^* |_{X_\lambda}$. By the Browder-Minty theorem (see [8, Theorem 26.A]), a monotone, coercive, and hemicontinuous operator, from a real reflexive Banach space into its dual, is onto. Thus, by (b) and (d), for
each \( \lambda \in \Lambda \), the operator \( T_\lambda \) is onto and so there exists \( x_\lambda \in X_\lambda \) such that

\[
A_\lambda(x_\lambda, y) = \langle v_\lambda^*, y \rangle,
\]
for all \( y \in X_\lambda \). In particular \( A_\lambda(x_\lambda, x_\lambda) = \langle v_\lambda^*, x_\lambda \rangle \), and hence by (b), we get that the net \( \{x_\lambda\}_{\lambda \in \Lambda} \) is bounded. Continuing as in the proof of Theorem 2.3 and applying the fact that \( A \) is of type \( M \) with respect to \( V \), we get the required result. \( \square \)

Remark 3.4. It should be noted that since a crucial point in the above proof is the existence and boundedness of the net \( \{x_\lambda\}_{\lambda \in \Lambda} \), variants of the previous theorem could be obtained using in (b) and (d) alternative conditions corresponding to other surjectivity results.

We now apply Theorem 3.3 to a singular Dirichlet problem.

Example 3.5. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). We consider the Dirichlet problem

\[
- \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u}{\partial x_i} \right) + f(x, u) = 0 \quad \text{a.e. on } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

where \( a \in L^\infty_{\text{loc}}(\Omega) \) and there exists \( c_1 > 0 \) such that \( a(x) \geq c_1 \) a.e. on \( \Omega \), and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a monotone increasing (with respect to its second variable for each fixed \( x \in \Omega \)) Carathéodory function, for which there exist \( h \in L^2(\Omega) \) and \( c_2 > 0 \) such that

\[
|f(x, u)| \leq h(x) + c_2 |u|, \quad \forall x \in \Omega, u \in \mathbb{R}.
\]

We will show that if the above hypotheses on \( a \) and \( f \) hold, then problem (3.6) has a weak solution, that is, that there exists a function \( u \in H_0^1(\Omega) \) with

\[
\int_{\Omega} a(x) \nabla u \nabla v \, dx + \int_{\Omega} f(x, u) v \, dx = 0, \quad \forall v \in C_0^\infty(\Omega).
\]

To this end, let \( X = H_0^1(\Omega) \), let \( \{\Omega_n\}_{n \in \mathbb{N}} \) be an increasing sequence of open subsets of \( \Omega \) such that \( \overline{\Omega_n} \subseteq \Omega_{n+1} \) and

\[
\bigcup_{n=1}^\infty \Omega_n = \Omega
\]

and \( X_n = H_0^1(\Omega_n) \), for each \( n \in \mathbb{N} \). Observe that we can consider each \( X_n \) as a closed subspace of \( X \) by extending its elements by zero outside \( \Omega_n \) and let

\[
V = \bigcup_{n=1}^\infty X_n.
\]

Finally, let

\[
A : X \times V \to \mathbb{R}
\]
be the function defined by

\[ A(u, v) = \int_{\Omega} a(x) \nabla u \nabla v \, dx + \int_{\Omega} f(x, u) v \, dx. \quad (3.12) \]

By \( a(x) \geq c_1 \) a.e. on \( \Omega \), the monotonicity of \( f \), and the growth condition (3.7), we have

\[ A(u, u) = \int_{\Omega} a(x) |\nabla u|^2 \, dx + \int_{\Omega} f(x, u) u \, dx = \int_{\Omega} a(x) |\nabla u|^2 \, dx + \int_{\Omega} (f(x, u) - f(x, 0)) u \, dx + \int_{\Omega} f(x, 0) u \, dx \geq c_1 \|\nabla u\|_{L^2(\Omega)}^2 - \|h\|_{L^2(\Omega)} \|u\|_{H^1_0(\Omega)}. \quad (3.13) \]

Since by the Poincaré inequality \( \|\nabla u\|_{L^2(\Omega)} \) is equivalent to the norm of \( X \), it follows that \( A \) is coercive.

Let \( A_n = A|_{X_n \times X_n} \). Then, since \( a \in L^\infty_{\text{loc}}(\Omega) \), it follows that \( a \in L^\infty(\Omega_n) \), for all \( n \in \mathbb{N} \). Combining this with (3.7), we have that

\[ |A_n(u, v)| \leq c(u, n) \|v\|_{X_n}, \quad (3.14) \]

where \( c(u, n) \) is a positive constant depending on \( n \) and \( u \). So the operator

\[ T_n : X_n \rightarrow X_n^*, \quad (3.15) \]

with \( \langle T_n u, v \rangle_{X_n} = A_n(u, v) \), is well defined for all \( n \in \mathbb{N} \). Let

\[ T_{1,n}, T_{2,n} : X_n \rightarrow X_n^* \quad (3.16) \]

be the operators defined by

\[ \langle T_{1,n} u, v \rangle_{X_n} = \int_{\Omega_n} a(x) \nabla u \nabla v \, dx, \quad \langle T_{2,n} u, v \rangle_{X_n} = \int_{\Omega_n} f(x, u) v \, dx. \quad (3.17) \]

Then \( T_{1,n} \) is a monotone bounded linear operator. Using the monotonicity of \( f \), it is easy to see that \( T_{2,n} \) is monotone. Finally, recalling that the Nemytskii operator corresponding to \( f \) is continuous (see, e.g., [8, Proposition 26.7]) and that the embedding of \( X_n \) into \( L^2(\Omega_n) \) is compact, we have that \( T_{2,n} \) is hemicontinuous. Thus \( T_n = T_{1,n} + T_{2,n} \) is monotone and hemicontinuous for all \( n \in \mathbb{N} \).

To finish the proof, let \( u_n \rightharpoonup u \) in \( X \). Then since for all \( v \in V \),

\[ u \mapsto \int_{\Omega} a(x) \nabla u \nabla v \, dx \quad (3.18) \]

is a bounded linear functional and, by the continuity of the Nemytskii operator and the compactness of the embedding of \( X \) into \( L^2(\Omega) \),

\[ \int_{\Omega} f(x, u_n) v \, dx \rightarrow \int_{\Omega} f(x, u) v \, dx, \quad (3.19) \]
for all $v \in V$, we get that
\[
A(u, v) \longrightarrow A(u, v), \quad \forall v \in V.
\] (3.20)
Thus $A$ is of type $M$ with respect to $V$. Applying now Theorem 3.3 we get that there exists $u \in X$ such that $A(u, v) = 0$ for all $v \in V$. Observing that $C_0^\infty(\Omega)$ is contained in $V$, we get that $u$ is the required weak solution of (3.6).

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Dimosthenis Drivaliaris: Department of Financial and Management Engineering, University of the Aegean, 31 Fostini Street, 82100 Chios, Greece
Email address: d.drivaliaris@fme.aegean.gr

Nikos Yannakakis: Department of Mathematics, School of Applied Mathematics and Natural Sciences, National Technical University of Athens, Iroon Polytechneiou 9, 15780 Zografou, Greece
Email address: nyian@math.ntua.gr