Branching laws on the metaplectic cover of $\text{GL}_2$

A Thesis

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Professor Dipendra Prasad, at the Tata Institute of Fundamental Research, Mumbai.

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In my capacity as supervisor of the candidate’s thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Dipendra Prasad

Date:
Dedicated to

My Parents
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Chapter 1

Introduction

1.1 Some background and history

Let $G$ be a group and $H$ a subgroup of $G$. Let $\pi_1$ and $\pi_2$ be irreducible representations of $G$ and $H$ respectively. By ‘branching laws’ one refers to rules describing the space $\text{Hom}_H(\pi_1, \pi_2)$ of $H$-equivariant linear maps from $\pi_1$ to $\pi_2$. The sixties saw the birth of the celebrated Langlands conjectures that predicted deep connections between the representation theory of reductive groups (over local fields and over the adeles of a global field), which may be called the ‘automorphic or harmonic analytic side’, and the study of Galois representations, which may be referred to as the ‘arithmetic side’. In the nineties, B. Gross and D. Prasad systematically investigated branching laws for certain classical groups and came up with a conjectural answer to many branching problems in terms of objects on the arithmetic side, known as the Langlands parameters and ‘$\epsilon$-factors’. This has been extended to cover many cases in \cite{2}.

In another direction, Shimura’s work on modular forms of half-integral weight \cite{29}, and various follow-ups had suggested that an analogue of Langlands’ program should be there not only for reductive groups over local and global fields, but also for certain covering groups of these. Since then many people have studied representation theory for these covering groups, and recently there have been many attempts (e.g., \cite{33}, \cite{34}) to adapt
Chapter 1, Section 1.1

Langlands’ conjectures to the setting of covering groups. This task appears formidable but also seems to be a potential source for a rich theory. Thus, it seems natural to investigate branching laws in the context of covering groups, and this is what this thesis endeavors to do.

One of the first observations of Gross and Prasad was that the restriction problem for a \((p\text{-adic})\) pair \((G, H)\) should not be studied in isolation but together with that for various pairs \((G', H')\), as \((G', H')\) runs over groups closely related to \((G, H)\), known as the (pure) inner forms of \(G, H\). In an early work [23], Prasad studied the restriction problem for the pairs \((\text{GL}_2(E), \text{GL}_2(F))\) and \((\text{GL}_2(E), D_F^\times)\), where \(F\) is a non-Archimedean local field of characteristic zero, \(D_F\) is the unique quaternion division algebra over \(F\) and \(E\) is a quadratic extension of \(F\). In hindsight, this case already captures many of the subtleties that show up in the general case.

For \(p\text{-adic}\) groups, where most representations are infinite dimensional, precise questions about branching have been formulated only in contexts where we have a theorem of ‘multiplicity one’, or at least ‘finite multiplicity’. More precisely, one restricts to pairs \((G, H)\) of \(p\text{-adic}\) groups and a class of pairs \((\pi_1, \pi_2)\) of representations that ensure that the dimension of \(\text{Hom}_H(\pi_1, \pi_2)\) is 0 or 1, or at least finite.

Prasad proved a multiplicity one theorem for the pair \((\text{GL}_2(E), \text{GL}_2(F))\) and gave a classification of pairs \((\pi_1, \pi_2)\) of irreducible ‘admissible’ representations \(\pi_1\) of \(\text{GL}_2(E)\) and \(\pi_2\) of \(\text{GL}_2(F)\) such that

\[ \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) \neq 0. \]

Further, he showed that there is a certain dichotomy relating the restriction problem for the pairs \((\text{GL}_2(E), \text{GL}_2(F))\) and \((\text{GL}_2(E), D_F^\times)\). More precisely, the following theorem was proved in [23].

**Theorem 1.1.1 (Prasad).** Let \(\pi_1\) and \(\pi_2\) be irreducible infinite dimensional representations of \(\text{GL}_2(E)\) and \(\text{GL}_2(F)\) respectively. Assume that the central character of \(\pi_1\) restricted to the center of \(\text{GL}_2(F)\) is the same as the central character of \(\pi_2\). Then
1.2 Description of the problem

1. for a principal series representation \( \pi_2 \) of \( \text{GL}_2(F) \), we have

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) = 1,
\]

2. for a discrete series representation \( \pi_2 \) of \( \text{GL}_2(F) \), letting \( \pi'_2 \) be the finite dimensional irreducible representation of \( D_F^\times \) associated to \( \pi_2 \) by the Jacquet-Langlands correspondence, we have

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) + \dim \text{Hom}_{D_F^\times}(\pi_1, \pi'_2) = 1.
\]

3. There is a criterion in terms of a certain epsilon factor attached to \( \pi_1, \pi_2 \) that determines when \( \dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) = 1 \).

1.2 Description of the problem

The first covering group to be considered is \( \widetilde{\text{GL}}_2(E) \), a two fold cover of \( \text{GL}_2(E) \) known as the metaplectic cover of \( \text{GL}_2(E) \), which will be defined in Section 2.4 by an explicit (Kubota) cocycle with values in \( \mu_2 = \{ \pm 1 \} \), giving rise to an exact sequence of groups

\[
1 \rightarrow \mu_2 \rightarrow \widetilde{\text{GL}}_2(E) \rightarrow \text{GL}_2(E) \rightarrow 1.
\]

To consider the restriction problem for \( (\widetilde{\text{GL}}_2(E), \text{GL}_2(F)) \) in analogy with that for the pair \( (\text{GL}_2(E), \text{GL}_2(F)) \), we need \( \text{GL}_2(F) \) and \( D_F^\times \) to be realized in a suitable manner as subgroups of \( \widetilde{\text{GL}}_2(E) \). The first question to be analysed is whether \( \text{GL}_2(F) \) and \( D_F^\times \) are subgroups of \( \widetilde{\text{GL}}_2(E) \). More specifically, the question is whether the covering \( \widetilde{\text{GL}}_2(E) \) splits when restricted to \( \text{GL}_2(F) \) and \( D_F^\times \). We shall discuss this question in Chapter 3. It turns out to be rather easy to prove this for \( \text{GL}_2(F) \) but not so for \( D_F^\times \). Actually, we consider \( \mathbb{C}^\times \)-covering of \( \widetilde{\text{GL}}_2(E) \) obtained from \( \widetilde{\text{GL}}_2(E) \), namely \( \widetilde{\text{GL}}_2(E) \times_{\mu_2} \mathbb{C}^\times \), and it is this covering that splits when restricted to \( D_F^\times \), see Theorem 3.1.1. An admissible representation of \( \widetilde{\text{GL}}_2(E) \) (respectively, \( \widetilde{\text{GL}}_2(E) \times_{\mu_2} \mathbb{C}^\times \)) is called genuine if the action of \( \mu_2 \) (respectively, \( \mathbb{C}^\times \)) is non-trivial (respectively, \( \mathbb{C}^\times \) acts by identity) on the representation space. It is
clear that the category of genuine representations of $GL_2(E)$ and that of $GL_2(E) \times_{\mu_2} \mathbb{C}^\times$ are equivalent. Hence there is no harm in replacing the group $GL_2(E)$ by $GL_2(E) \times_{\mu_2} \mathbb{C}^\times$. We shall abuse the notation and often write $GL_2(E)$ for $GL_2(E) \times_{\mu_2} \mathbb{C}^\times$. After Theorem 5.1.1, we will know that $GL_2(F)$ and $D_F^\times$ are subgroups of $GL_2(E)$ but not canonically, as there are many ‘inequivalent’ embeddings of these two subgroups inside $GL_2(E)$. In fact, the set of splittings of the map $p : GL_2(E) \to GL_2(E)$ restricted to either of $GL_2(F)$ or $D_F^\times$ is a principal homogeneous space over the character group of $F^\times$. Before we begin the study of restriction of representations from $GL_2(E)$ to $GL_2(F)$ and to $D_F^\times$, we need to fix a splitting of these two subgroups inside $GL_2(E)$. We also require that these fixed embeddings of $GL_2(F)$ and $D_F^\times$ inside $GL_2(E)$ are compatible in the sense that they satisfy a ‘technical’ condition, see “Working Hypothesis 5.1.2” in Section 5.1 formulated by D. Prasad. We are not able to prove this hypothesis at the moment and hence we assume it, and put it to use in Chapter 5.

For $X \subset GL_2(E)$, let $\tilde{X}$ denote the inverse image of $X$ in $GL_2(E)$. Let $Z$ be the center of $GL_2(E)$. It is important to note that the subgroup $\tilde{Z}$ is an abelian group containing the center of $GL_2(E)$ but it is not the center of $GL_2(E)$. The center of $GL_2(E)$ is $\tilde{Z}^2$. Let $\pi$ be an irreducible admissible genuine representation of $GL_2(E)$. In the study of the restriction of a representation of $GL_2(E)$ to the subgroups $GL_2(F)$ and $D_F^\times$, the space of Whittaker functionals of representations of $GL_2(E)$ plays an important role. Let $\omega_\pi$ be the central character of $\pi$. Define $\Omega(\omega_\pi) = \{\mu : \tilde{Z} \to \mathbb{C}^\times | \mu|_{\tilde{Z}^2} = \omega_\pi\}$. Sometimes we regard $\Omega(\omega_\pi)$ as a $\tilde{Z}$-module $\oplus_{\mu \in \Omega(\omega_\pi)} \mu$. Let $\psi$ be a non-trivial additive character of $E$. The $\psi$-twisted Jacquet functor $\pi_{N,\psi}$ is a finite dimensional completely reducible $\tilde{Z}$-module, each character of $\tilde{Z}$ appearing with multiplicity $\leq 1$ (by [5, Theorem 4.1]). With these notations, we wish to prove the following theorem which is analogous to Theorem 1.1.1.

**Theorem 1.2.1.** Let $\pi_1$ be an irreducible admissible genuine representation of $GL_2(E)$ and let $\pi_2$ be an infinite dimensional irreducible admissible representation of $GL_2(F)$. Assume that the central characters $\omega_{\pi_1}$ of $\pi_1$ and $\omega_{\pi_2}$ of $\pi_2$ agree on $E^\times 2 \cap F^\times$. Fix a non-trivial
1.3. The strategy of proofs

additive character \( \psi \) of \( E \) such that \( \psi|_F = 1 \). Then:

1. For a principal series representation \( \pi_2 \) of \( \text{GL}_2(F) \), (except for a few pairs \( (\pi_1, \pi_2) \) for a given \( \pi_1 \) to be described explicitly in Chapter [3]) we have

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) = \dim \text{Hom}_{\text{Z}(F)}((\pi_1)_{N,\psi}, \omega_{\pi_2}).
\]

2. For a principal series representation \( \pi_1 \) of \( \tilde{\text{GL}}_2(E) \) and a discrete series representation \( \pi_2 \) of \( \text{GL}_2(F) \), let \( \pi'_2 \) be the finite dimensional representation of \( D_F^\chi \) associated to \( \pi_2 \) by the Jacquet-Langlands correspondence. Then (except for a few pairs \( (\pi_1, \pi_2) \) for a given \( \pi_1 \) to be described explicitly in Chapter [3]) we have

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) + \dim \text{Hom}_{D_F^\chi}(\pi_1, \pi'_2) = [E^\times : F^\times E^{\times 2}].
\]

3. For an irreducible admissible genuine representation \( \pi_1 \) of \( \tilde{\text{GL}}_2(E) \) and an irreducible supercuspidal representation \( \pi_2 \) of \( \text{GL}_2(F) \), let \( \pi'_1 \) be an admissible genuine representation of \( \tilde{\text{GL}}_2(E) \) which has the same central character as \( \pi_1 \) and \( (\pi_1)_{N,\psi} \oplus (\pi'_1)|_{N,\psi} = \Omega(\omega_{\pi_1}) \) as \( \tilde{Z} \)-modules. Let \( \pi'_2 \) be the finite dimensional representation of \( D_F^\chi \) associated to \( \pi_2 \) by the Jacquet-Langlands correspondence. Then

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1 \oplus \pi'_1, \pi_2) + \dim \text{Hom}_{D_F^\chi}(\pi_1 \oplus \pi'_1, \pi'_2) = [E^\times : F^\times E^{\times 2}].
\]

\[ \text{1.3 The strategy of proofs} \]

The strategy to prove this theorem is similar to that in [23], which we briefly recall here. Part 1 of Theorem [1.2.1] is proved by looking at the Kirillov model of an irreducible admissible genuine representation of \( \tilde{\text{GL}}_2(E) \) and its Jacquet module restricted to \( \text{GL}_2(F) \). Part 2 of Theorem [1.2.1] is proved using Mackey theory. Part 3 of Theorem [1.2.1] is proved using a trick of Prasad in [23], where we ‘transfer’ results of principal series representations (as in Part 2) to the representations which do not belong to principal series (Prasad ‘transfers’ the results from a principal series representation to a discrete series representation). This
is done by using character theory and an analogue of a result of Casselman and Prasad [23, Theorem 2.7] for $\widetilde{\text{GL}_2}(E)$. The theorem of Casselman-Prasad is as follows.

**Theorem 1.3.1.** Let $\pi_1$ and $\pi_2$ be two irreducible admissible infinite dimensional representations of $\text{GL}_2(E)$ which have the same central character. Then the virtual representation $\pi_1 - \pi_2$ of $\text{GL}_2(E)$ restricted to any compact modulo central subgroup of $\text{GL}_2(E)$ is finite dimensional.

Let $\Theta_\pi$ denote the character of an admissible representation $\pi$ (see Section 2.3). Prasad gives a proof of Theorem 1.3.1 by observing that $\Theta_\pi_1 - \Theta_\pi_2$ is an everywhere smooth function on $\text{GL}_2(E)$. We need an analogue of this result for $\widetilde{\text{GL}_2}(E)$, which is as follows:

**Theorem 1.3.2.** Let $\Pi_1$ and $\Pi_2$ be two irreducible admissible genuine representations of $\widetilde{\text{GL}_2}(E)$ with the same central character and such that $(\Pi_1)_N,\psi \cong (\Pi_2)_N,\psi$ as $\tilde{\mathbb{Z}}$-modules, where $\psi$ is a non-trivial additive character of $E$. Then $\Theta_{\Pi_1} - \Theta_{\Pi_2}$ is a smooth function on $\widetilde{\text{GL}_2}(E)$.

This theorem is an application of a theorem of F. Rodier [25], generalized by C. Mœglin and J.-L. Waldspurger [20] and is extended by this author to the setting of covering groups (see Theorem 4.1.2). We prove the theorem of Mœglin-Waldspurger for any covering group $\tilde{G}$ of a connected reductive group $G$ in Chapter 4. This is an important part of the thesis and we give an overview of this result below.

### 1.4 A theorem of Mœglin-Waldspurger for covering groups

Let $G$ be a connected reductive group defined over $E$ and $\tilde{G} = G(E)$. Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of $G$ and $\mathfrak{g} = \mathfrak{g}(E)$. A theorem of F. Rodier [25] for connected reductive split groups relates the dimension of a certain space of non-degenerate Whittaker forms of
1.4. A theorem of Mœglin-Waldspurger for covering groups

an irreducible admissible representation \((\pi, W)\) to a certain coefficient in the character expansion \(\Theta_\pi\) of \(\pi\) around the identity. Rodier, for his proof, had to assume that the residual characteristic is large enough. The theorem of Rodier was generalised by C. Mœglin and J.-L. Waldspurger [20] in several directions, yielding in particular a statement for arbitrary connected reductive group over \(p\)-adic field of odd residual characteristic. The theorem of Mœglin-Waldspurger is a more precise statement about certain coefficients in the character expansion around identity and certain spaces of ‘degenerate’ Whittaker forms (see Section 4.4 for the definition of degenerate Whittaker forms). In the case of even residual characteristic, the theorem of Mœglin-Waldspurger has been recently proved by S. Varma [27]. We generalize this theorem of Mœglin-Waldspurger to the setting of a locally compact topological central extension of an arbitrary connected reductive group defined over a \(p\)-adic field of arbitrary residual characteristic in Chapter 4.

Let \(\mu_r := \{z \in \mathbb{C}^\times : |z|^r = 1\}\). Let \(\tilde{G}\) be a locally compact topological central extension of \(G\) by \(\mu_r\). Let \(Y \in \mathfrak{g}\) be a nilpotent element and \(\varphi : \mathbb{G}_m \to G\) be a one parameter subgroup of \(G\) satisfying

\[
\text{Ad}(\varphi(s))Y = s^{-2}Y. \tag{1.1}
\]

Let \((\pi, W)\) be an irreducible admissible genuine representation of \(\tilde{G}\). We fix a non-trivial additive character \(\psi\) of \(E\) with conductor \(\mathcal{O}_E\), where \(\mathcal{O}_E\) is ring of integers of \(E\). Associated to a pair \((Y, \varphi)\) as in \([\text{1.1}]\) one can define a certain space \(\mathcal{W}_{(Y, \varphi)}\), called the space of degenerate Whittaker forms of \((\pi, W)\) relative to \((Y, \varphi)\) (see Section \([4.4]\)). Let \(\mathcal{N}_{\text{Wh}}(\pi)\) denote the set of nilpotent orbits \(\mathcal{O}\) of \(\mathfrak{g}\) for which there exists an element \(Y \in \mathcal{O}\) and a one parameter subgroup \(\varphi\) satisfying Equation \([\text{1.1}]\) such that \(\mathcal{W}_{(Y, \varphi)} \neq 0\). Recall that the Harish-Chandra-Howe character expansion (as extended by Wen-Wei Li in \([\text{15}]\) in the setting of a covering group) of \((\pi, W)\) around the identity is a sum \(\sum_{\mathcal{O}} c_{\mathcal{O}} \widetilde{\mu}_\mathcal{O}\), where \(\mathcal{O}\) varies over the set of nilpotent orbits of \(\mathfrak{g}\), each \(c_{\mathcal{O}}\) is a complex number and \(\widetilde{\mu}_\mathcal{O}\) denotes the Fourier transform of a suitably normalized invariant measure \(\mu_\mathcal{O}\) on \(\mathcal{O}\). Let \(\mathcal{N}_{\text{tr}}(\pi)\) denote the set of nilpotent orbits \(\mathcal{O}\) of \(\mathfrak{g}\) such that the corresponding coefficient \(c_{\mathcal{O}}\) is non-zero in
the character expansion of $\pi$ around the identity. There is a natural partial order on the set of nilpotent orbits in $\mathfrak{g}$: $\mathcal{O}_1 \leq \mathcal{O}_2$ if $\bar{\mathcal{O}}_1 \subset \bar{\mathcal{O}}_2$. Let $\text{Max}(\mathcal{N}_\text{Wh}(\pi))$ and $\text{Max}(\mathcal{N}_{\text{tr}}(\pi))$ denote the sets of maximal elements in $\mathcal{N}_\text{Wh}(\pi)$ and $\mathcal{N}_{\text{tr}}(\pi)$ respectively, with respect to the above partial order. Then we prove the following theorem which is a generalization of the main theorem of Mœglin-Waldspurger in Chapter I of [20].

**Theorem 1.4.1.** Let $\pi$ be an irreducible admissible genuine representation of $\tilde{G}$. Then

$$\text{Max}(\mathcal{N}_\text{Wh}(\pi)) = \text{Max}(\mathcal{N}_{\text{tr}}(\pi)).$$

Moreover, if $\mathcal{O}$ is an element in either of these sets, then for any $(Y, \varphi)$ as above with $Y \in \mathcal{O}$ we have

$$c_{\mathcal{O}} = \dim \mathcal{W}(Y, \varphi).$$

### 1.5 A question

Let us come back to Theorem 1.2.1. In the proof of part 2 of the theorem, the number $[E^\times : F^\times E^\times 2]$ is related to the fact that $(\pi_1)_{N, \psi} = \Omega(\omega_{\pi_1})$ for a principal series representation $\pi_1$ (see Proposition 2.6.4). But for a representation $\pi_1$, which is not a principal series, $(\pi_1)_{N, \psi}$ is a proper $\tilde{Z}$-submodule of $\Omega(\omega_{\pi_1})$. To ‘compensate’, we add another genuine representation $\pi_1'$ which has the same central character and satisfies $(\pi_1)_{N, \psi} \oplus (\pi_1')_{N, \psi} = \Omega(\omega_{\pi_1})$. Then we can utilize Theorem 1.3.2 for $\pi_1 \oplus \pi_1'$ and a suitable principal series representation $P$s. It is not clear in general how to describe a ‘natural’ $\pi_1'$ for a given $\pi_1$ with the above properties. The question of describing a ‘natural’ $\pi_1'$ for a given $\pi_1$ reduces to the following question about the representations of $\tilde{\text{SL}_2}(E)$.

**Question 1.5.1.** Let $\tau$ be an irreducible admissible genuine representation of $\tilde{\text{SL}_2}(E)$. Is there a ‘natural’ choice of an genuine admissible representation of finite length $\tau'$ with a central character (not necessarily irreducible) such that $\omega_\tau = \omega_{\tau'}$, and $\tau$ admits a non-zero $\psi$-Whittaker functional if and only if $\tau'$ does not admit a non-zero $\psi$-Whittaker functional for any non-trivial additive character $\psi$ of $E$?
1.5. A question

We remark that the Waldspurger involution defined on the set of isomorphism classes of irreducible admissible genuine representations of $\widetilde{\text{SL}_2(E)}$, written as $\tau \mapsto \tau_W$, has the property that $\tau$ admits a non-zero $\psi$-Whittaker functional if and only if $\tau_W$ does not admit a non-zero $\psi$-Whittaker functional for any non-trivial character $\psi$ of $E$, but the central characters of $\tau$ and $\tau_W$ are different. However, the question above requires the central characters of $\tau$ and $\tau'$ to be the same. The fact that $\tau$ and $\tau_W$ have different central characters also makes it difficult to extend the Waldspurger involution from $\widetilde{\text{SL}_2(E)}$ to $\widetilde{\text{GL}_2(E)}$.

This question will be discussed in Chapter 6 where we provide $\pi'_1$ for certain representations $\pi_1$. We are not able to construct a ‘natural’ $\pi'_1$ for all irreducible admissible genuine supercuspidal representations $\pi_1$ of $\widetilde{\text{GL}_2(E)}$ (equivalently, $\tau'$ for all irreducible admissible genuine supercuspidal representation $\tau$ of $\widetilde{\text{SL}_2(E)}$). It is not clear if the inability to do so is a reflection on us, or if there is a more fundamental reason for this.

In Chapter 6 we also consider the question of restriction of an irreducible admissible genuine representation of $\widetilde{\text{GL}_2(E)}$ to $\widetilde{\text{SL}_2(E)}$ and show that this restriction may not satisfy ‘multiplicity one’. In fact, we prove that the multiplicity can be either 0, 1, 2 or 4. The results in Chapter 6 are consequences of a theorem of Waldspurger in [31] which involve the so called Waldspurger involution and $\theta$-correspondence.
Chapter 2

Preliminaries

2.1 Linear algebraic groups

A linear algebraic group $G$ over a field $k$ is a closed subgroup of $GL_N$ for some non-negative integer $N$. A linear algebraic group is called a torus if it is isomorphic to $(\mathbb{G}_m)^n$ over $\bar{k}$ for some non-negative integer $n$, where $\mathbb{G}_m = GL_1$. The radical $Rad(G)$ of $G$ is defined to be the identity component of the maximal normal solvable subgroup of $G$. A maximal connected solvable closed subgroup $B$ of $G$ is called a Borel subgroup. A closed subgroup $P$ of $G$ is called a parabolic subgroup if $G/P$ is a projective algebraic variety. Any Borel subgroup $B$ is a parabolic subgroup of $G$. An element $x \in G$ is called unipotent if for any algebraic injective morphism $i : G \hookrightarrow GL_N$, $(i(x) - Id)^N = 0$. A linear algebraic group is called unipotent if every element in it is a unipotent element. The unipotent radical of a linear algebraic group $G$ is the subvariety of unipotent elements in $Rad(G)$ which can be shown to be a subgroup. The group $G$ is called reductive (resp. semi-simple ) if $Rad(G)$ is a torus (resp. trivial). Equivalently $G$ can be defined to be reductive if its unipotent radical is trivial. A reductive linear algebraic group $G$ defined over $k$ is called quasi-split if there exists a Borel subgroup $B$ of $G$ which is defined over $k$. Moreover $G$ is called split if it is quasi-split and $B/U$ is a split torus (i.e. isomorphic to $(\mathbb{G}_m)^n$ over $k$), where $U$ is the unipotent radical of $B$ and $n$ is a non-negative integer.
2.2 Covering groups and genuine representations

Let $A$ be a finite abelian group. Let $E$ be a non-Archimedean local field of characteristic zero and $G$ a connected reductive group defined over $E$. A locally compact topological central extension $\tilde{G}$ of $G = G(E)$ by $A$ gives rise to the following exact sequence of groups

$$1 \to A \to \tilde{G} \xrightarrow{p} G \to 1,$$

where image of $A$ lies in the center of $\tilde{G}$. Such an extension can be described by a suitable element in $H^2(G, A)$, where $A$ is considered to be a $G$-module with trivial action. If $\beta : G \times G \to A$ is a 2-cocycle corresponding to the central extension $\tilde{G}$, then the group $\tilde{G}$ can be described more explicitly; namely $\tilde{G}$ can be identified with $G \times A$ as a set such that modulo this identification the multiplication in $\tilde{G}$ is described by $(g_1, a_1) \cdot (g_2, a_2) = (g_1 g_2, a_1 a_2 \beta(g_1, g_2))$. We refer to these groups as covering groups. A covering group $\tilde{G}$ is locally compact and totally disconnected like $G$. We remark that the topology on $\tilde{G}$ is not obtained by transferring the product topology on $G \times A$. It is well known that the covering $\tilde{G} \to G$ splits when restricted to a small enough open subgroup of $G$ and let $U$ be such a subgroup. If we fix a splitting $s : U \to \tilde{G}$ then we define $s(U)$ to be an open subgroup of $\tilde{G}$ and $s$ being homeomorphic onto its image, which in turn defines a topology on $\tilde{G}$ making it a topological group and it is this topology on $\tilde{G}$ with which we work.

Let $\tilde{x} \in \tilde{G}$ and $y \in G$. For $\tilde{y} \in p^{-1}\{y\}$, the element $\tilde{y} \tilde{x} \tilde{y}^{-1}$ is independent of the choice of $\tilde{y}$ in $p^{-1}\{y\}$. We abuse the notation and write $y \tilde{x} y^{-1}$ for $\tilde{y} \tilde{x} \tilde{y}^{-1}$.

Let $(\pi, W)$ be a representation of $\tilde{G}$, where $W$ is a complex vector space. The representation $(\pi, W)$ is called smooth if the stabilizer of every element $w \in W$ is an open subgroup of $\tilde{G}$. The representation $(\pi, W)$ is called admissible if $(\pi, W)$ is smooth and $\pi^K := \{w \in W \mid \pi(k)w = w, \forall k \in K\}$ is finite dimensional for all open compact subgroup $K$ of $\tilde{G}$. If $(\pi, W)$ is an irreducible admissible representation of $\tilde{G}$ with central character $\omega_\pi$, then
2.3. Character expansion and Whittaker functionals

\(\omega_\pi(A)\) is a finite cyclic (sub)group \(\mu_r\) of \(\mathbb{C}^\times\). Such a representation will factor through a representation of \(\tilde{G}/\ker(\omega_\pi|_A)\), which can be identified with a central extension of \(G\) by \(\mu_r\). Thus it suffices to consider only those central extensions of \(G\) for which \(A = \mu_r\) for \(r \geq 1\). From now onward, we will consider only such extensions. A representation \((\pi, W)\) of \(\tilde{G}\) is called genuine if the action of \(\mu_r\) is given by scalar multiplication, which makes sense as \(\mu_r \subset \mathbb{C}^\times\).

2.3 Character expansion and Whittaker functionals

In this section, we recall some facts about the character distribution of an admissible genuine representation of locally compact topological central extension \(\tilde{G}\) of \(G = G(E)\) by \(\mu_r\) with \(r \geq 1\), where \(G\) is a connected reductive group defined over \(E\), see [16, Chapter 2]. Let \(C^\infty_c(\tilde{G})\) be the space of smooth (locally constant) functions with compact support and let \(f \in C^\infty_c(\tilde{G})\). Let \(\mu_r := \text{Hom}(\mu_r, \mathbb{C}^\times)\) and for \(\xi \in \hat{\mu}_r\) let

\[C^\infty_{c,\xi}(\tilde{G}) := \{ f \in C^\infty_c(\tilde{G}) | f(\epsilon \tilde{x}) = \xi(\epsilon)f(\tilde{x}), \forall \epsilon \in \mu_r \text{ and } \forall \tilde{x} \in \tilde{G}\}.

We have the following canonical decomposition

\[C^\infty_c(\tilde{G}) = \bigoplus_{\xi \in \hat{\mu}_r} C^\infty_{c,\xi}(\tilde{G}).\]

Let \(\text{Rep}(\tilde{G})\) be the set of isomorphism classes of irreducible admissible representations of \(\tilde{G}\). For \(\xi \in \hat{\mu}_r\), let \(\text{Rep}_\xi(\tilde{G}) = \{ \pi \in \text{Rep}(\tilde{G}) | \pi(\epsilon) = \xi(\epsilon)\text{id}\}\). Let \(\xi, \chi \in \hat{\mu}_r\). Let \(\pi \in \text{Rep}_\chi(\tilde{G})\) and \(f \in C^\infty_{c,\xi}(\tilde{G})\). Then the operator \(\pi(f) : \pi \longrightarrow \pi\) given by \(v \mapsto \int_{\tilde{G}} f(\tilde{x})\pi(\tilde{x})v d\tilde{x}\) is zero except for \(\xi = \chi\). Moreover, \(\pi(f)\) has finite rank as \(\pi\) is admissible and hence its trace is well defined. Then

\[f \mapsto \text{trace}(\pi(f))\]

defines a distribution on \(\tilde{G}\), called the character distribution of \(\pi\). Let \(\text{gen}\) be the genuine character of \(\mu_r\), i.e. \(\text{gen}(\epsilon) = \epsilon\). If \(\pi \in \text{Rep}_{\text{gen}}(\tilde{G})\) then the character distribution is determined by its restriction to the subspace \(C^\infty_{c,\text{gen}}(\tilde{G})\) of \(C^\infty_c(\tilde{G})\), the so called space of
anti-genuine functions on $\tilde{G}$. From [15, Theorem 4.3.2], the character distribution of an irreducible admissible genuine representation of $\tilde{G}$ is represented by a locally integrable function $\Theta_\pi$ on $\tilde{G}$, i.e.,

$$\text{trace}(\pi(f)) = \int_{\tilde{G}} \Theta_\pi(\tilde{x}) f(\tilde{x}) d\tilde{x}.$$  

This function $\Theta_\pi$ is conjugation invariant in the sense that it satisfies $\Theta_\pi(x^y) = \Theta_\pi(x)$. The function $\Theta_\pi$ is called the character of the representation $\pi$. The function $\Theta_\pi$ is known to be smooth at each regular semi-simple element in $\tilde{G}$. We have an asymptotic description (or Harish-Chandra-Howe character expansion) of $\Theta_\pi$ in a neighbourhood of any singular semisimple element of $\tilde{G}$. In a neighbourhood of the identity, the Harish-Chandra-Howe character expansion is of the form

$$\sum_{O} c_O \hat{\mu}_O$$

where $O$ runs over the set of $\text{Ad}(G)$-orbits of nilpotents elements in the Lie algebra $\mathfrak{g}$, the $c_O \in \mathbb{C}$ are constants and $\hat{\mu}_O$ is the Fourier transform of a suitably chosen $\text{Ad}(G)$-invariant measure on the orbit $O$. More precisely, there exists a neighbourhood $U$ of the identity such that if $f \in C^\infty_{c,\text{gen}}(\tilde{G})$ be such that support of $f$ lies in $U$, then

$$\Theta_\pi(f) = \sum_{O} c_O \int_{O} f \circ \exp(X) d\mu_O.$$  

Note that we have implicitly used the fact that the covering $\tilde{G}$ splits when restricted to small enough open subgroup. In fact, there exists an exponential map $\exp : L \to \tilde{G}$, where $L$ is a sufficiently small open set containing 0 in the Lie algebra $\mathfrak{g}$.

Let $N$ be a maximal unipotent subgroup of $G$. The covering restriction of the covering $\tilde{G} \to G$ to $N$ splits in a unique way. Let $\chi$ be a non-degenerate character of $N = N(E)$. Then the pair $(N, \chi)$ is called a non-degenerate Whittaker datum.

**Definition 2.3.1.** Let $(N, \chi)$ be a non-degenerate Whittaker datum. A non-zero linear functional $\ell : W \to \mathbb{C}$ is called a Whittaker functional with respect to $(N, \chi)$ if it satisfies the following condition:

$$\ell(\pi(n)v) = \chi(n)\ell(v), \text{ for all } n \in N \text{ and } v \in W.$$
2.4 Two fold covers of $\text{SL}_2(E)$ and $\text{GL}_2(E)$

We give an explicit description of two fold covers of $\text{SL}_2(E)$ and $\text{GL}_2(E)$ by describing an explicit 2-cocycle defining each of these covers \[11]. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(E)$, set

$$x(g) = \begin{cases} 
  c & \text{if } c \neq 0 \\
  d & \text{if } c = 0.
\end{cases}$$

Define

$$\beta(g_1, g_2) = (x(g_1), x(g_2))(-x(g_1)^{-1}x(g_2), x(g_1g_2))$$

(2.1)

for $g_1, g_2 \in \text{SL}_2(E)$, where $(\ast, \ast)$ denotes quadratic Hilbert symbol of the field $E$. For $g \in \text{GL}_2(E)$, denote by $p(g)$ the element of $\text{SL}_2(E)$ which satisfies $g = \begin{pmatrix} 1 & 0 \\ 0 & \det(g) \end{pmatrix} p(g)$.

For $y \in E^\times$ write $g^y = \begin{pmatrix} 1 & 0 \\ 0 & y^{-1} \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$. Define

$$v(y, g) = \begin{cases} 
  1 & \text{if } c \neq 0 \\
  (y, d) & \text{if } c = 0.
\end{cases}$$

The formula of (2.1) can be extended to $\text{GL}_2(E) \times \text{GL}_2(E)$ by

$$\beta(g_1, g_2) = \beta(p(g_1)^{\det(g_2)}, p(g_2))v(\det(g_2), p(g_1)).$$

(2.2)

It can be verified that $\beta : \text{GL}_2(E) \times \text{GL}_2(E) \to \{\pm 1\}$ defined by Equation (2.2) is a 2-cocycle. We define $\widetilde{\text{GL}_2(E)}$ to be $\text{GL}_2(E) \times \{\pm 1\}$ as a set, but with the group law given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\beta(g_1, g_2))$$

for $g_1, g_2 \in \text{GL}_2(E)$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. This gives the following short exact sequence of groups

$$1 \to \{\pm 1\} \to \widetilde{\text{GL}_2(E)} \xrightarrow{p} \text{GL}_2(E) \to 1.$$
One can verify that the restriction of the cocycle to the subgroup of upper triangular matrices is given as follows:

$$\beta \left( \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix} \right) = (a_1, b_2). \quad (2.3)$$

For any subset $X$ of $\text{GL}_2(E)$, let $\tilde{X}$ be the inverse image of $X$ in $\tilde{\text{GL}}_2(E)$. Let $A = A(E)$ be the group of diagonal matrices in $\text{GL}_2(E)$ and $N(E)$ the group of upper triangular unipotent matrices in $\text{GL}_2(E)$. Write $B(E) = A(E) \cdot N(E)$ for the group of upper triangular matrices in $\text{GL}_2(E)$. Then from Equation (2.3) it is clear that the covering of $\text{GL}_2(E)$ splits when restricted to $N(E)$, since the cocycle is identically 1. Moreover, $\tilde{N}(E) = N(E) \times \{\pm 1\}$ as groups and hence we will regard $\tilde{N}(E)$ as a subgroup of $\tilde{\text{GL}}_2(E)$ with the obvious splitting. The group $\tilde{A} \cong \tilde{B}(E)/\tilde{N}(E)$ is not abelian. By the non-degeneracy of the quadratic Hilbert symbol, it follows that the subgroup $\tilde{A}^2 = \left\{ \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \epsilon \right\} : a, b \in E^\times, \epsilon \in \{\pm 1\}$ of $\tilde{A}$ is the center of $\tilde{A}$. Further, $\tilde{A}^2 \cong A^2 \times \{\pm 1\}$. We have the following short exact sequence of groups

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{A} \rightarrow A \rightarrow 1.$$

The following proposition will make it easier to describe the genuine representations of the group $\tilde{A}$.

**Proposition 2.4.1.** Let $G$ be a locally compact topological group with center $Z(G)$ of finite index. Let $Z_1(G)$ be a normal abelian subgroup of $G$ containing $Z(G)$ such that $[G : Z(G)] = [Z_1(G) : Z(G)]^2$. Note that the inner conjugation action of $G$ on $Z_1(G)$ induces an action of $G/Z_1(G)$ on $\tilde{Z_1(G)}$ the group of character of $Z_1(G)$. Assume that this action of $G/Z_1(G)$ on $\tilde{Z_1(G)}$ is transitive on the set of characters of $Z_1(G)$ with a given non-trivial restriction on $Z(G)$. Let $\chi$ be a non-trivial character of $Z(G)$, and $\chi_1$ a character of $Z_1(G)$ with $\chi_1|_{Z_1(G)} = \chi$. Then $\text{ind}^G_{Z_1(G)}(\chi_1)$ is an irreducible representation of $G$. Moreover, an irreducible representation $\pi$ of $G$ with a non-trivial central character $\chi$ is $\text{ind}^G_{Z_1(G)}(\chi_1)$ where $\chi_1$ is a character of $Z_1(G)$ such that $\chi_1|_{Z_1(G)} = \chi$. 

Proof. The proof of the first assertion follows from noting that ind\textsubscript{G(Z)}(\chi_1) must be irreducible since any G-module containing \chi_1 must contain \chi_1^g for all \( g \in G \). The second assertion follows from taking any character of \( Z_1(G) \) appearing in the irreducible representation \( \pi \).

Let \( Z = Z(E) \) be the center of \( GL_2(E) \). Recall that center of \( \tilde{A} \) is \( \tilde{A}^2 \).

**Lemma 2.4.2.** The group \( G = \tilde{A} \) with \( Z_1(G) = \tilde{Z}\tilde{A}^2 \) and \( \chi \) any genuine character of \( \tilde{A}^2 \) satisfies the hypothesis in Proposition 2.4.1, i.e. the induced action of \( \tilde{A}/\tilde{Z}\tilde{A}^2 \) on \( \tilde{Z}\tilde{A}^2 \) is transitive on the set of characters of \( \tilde{Z}\tilde{A}^2 \) whose restriction to \( \tilde{A}^2 \) is a given genuine character.

**Proof.** As quadratic Hilbert symbol satisfies \( (a,b) = (b,a) \), it can be easily verified that \( \tilde{Z}\tilde{A}^2 \) is a maximal abelian subgroup of \( \tilde{A} \). Clearly \( [\tilde{A} : \tilde{A}^2] = [\tilde{Z}\tilde{A}^2 : \tilde{A}^2]^2 = [E^\times : E^\times 2]^2 \). Let \( \chi \) be a genuine character of \( \tilde{A}^2 \). Let \( \mu_1 \) and \( \mu_2 \) be two extensions of \( \chi \) to \( \tilde{Z}\tilde{A}^2 \). Then \( \mu_1\mu_2^{-1} \) is trivial on \( \tilde{A}^2 \) and hence descends to a quadratic character of \( Z \). There are \([E^\times : E^\times 2]\) quadratic characters of \( E^\times \) given by \( x \mapsto (x, a) \) where \( a \in E^\times \) is determined modulo \( E^\times 2 \).

So there exists \( a \in E^\times /E^\times 2 \) such that \( \mu_2(\tilde{z}) = (a, z)\mu_1(\tilde{z}) \) for all \( \tilde{z} \in \tilde{Z} \) with \( p(\tilde{z}) = z \). As the character \( \mu_1 \) is a genuine character of \( \tilde{Z}\tilde{A}^2 \), it can be easily verified that \( \mu_2 = \mu_1^{g(a)} \) where \( g(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \) is a representative of \( \tilde{A}/\tilde{Z}\tilde{A}^2 \cong A/ZA^2 \). Thus the induced action of \( \tilde{A}/\tilde{Z}\tilde{A}^2 \) on the set of characters of \( \tilde{Z}\tilde{A}^2 \) which extend the character \( \chi \) of \( \tilde{A}^2 \) is transitive.

As \( \tilde{A}^2 \cong A^2 \times \{\pm 1\} \) as groups, genuine characters of \( \tilde{A}^2 \) are in obvious bijection with characters of \( A^2 \). Then the following corollary is immediate.

**Corollary 2.4.3.** The set of genuine irreducible representations of \( \tilde{A} \) is parametrized by the set of characters of \( A^2 \). The dimension of an irreducible genuine representation of \( \tilde{A} \) is \([E^\times : E^\times 2]\). ☐

Hence, any irreducible genuine representation of \( \tilde{A} \) can be constructed as follows. Let \( \chi_1, \chi_2 \)
be a pair of characters of $E^\times$. Define a character $\chi$ of $\tilde{A}^2$ given by

$$
\chi \left( \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \epsilon \right) = \epsilon \chi_1(a^2)\chi_2(b^2).
$$

(2.4)

Choose any extension of this character to $\tilde{Z}\tilde{A}^2 = \tilde{Z}\tilde{A}$ and denote this extended character by the same letter $\chi$. Let $\tilde{\tau} = \operatorname{ind}_{\tilde{A}} \tilde{Z} (\chi)$. By Proposition 2.4.1 we know that $\tilde{\tau}$ is irreducible. By the same proposition any irreducible genuine representation of $\tilde{A}$ is of this type. We note that $\tilde{\tau}$ does not depend on the choice of the character of $\tilde{Z}\tilde{A}^2$ which extends the character $\chi$. The following lemma is immediate.

**Lemma 2.4.4.** Let $\tilde{\tau} = \operatorname{ind}_{\tilde{Z}\tilde{A}} (\chi)$. Then $\tilde{\tau}|_{\tilde{Z}}$ contains all the possible characters $\mu$ of $\tilde{Z}$ such that $\mu|_{\tilde{Z}^2} = \chi|_{\tilde{Z}^2}$. Moreover, $\tilde{\tau}|_{\tilde{Z}}$ is an $[E^\times : E^\times^2]$ dimensional representation which is a direct sum of distinct characters of $\tilde{Z}$.

### 2.5 Representations of $\tilde{\operatorname{GL}}_2(E)$

The first observation about admissible genuine representation of $\tilde{\operatorname{GL}}_2(E)$ is that they are all infinite dimensional. Indeed suppose $(\pi, W)$ is a finite dimensional admissible representation of $\tilde{\operatorname{GL}}_2(E)$. Since $\pi$ is admissible, the kernel of $\pi : \tilde{\operatorname{GL}}_2(E) \to \operatorname{GL}(W)$ is an open normal subgroup of $\tilde{\operatorname{GL}}_2(E)$. In particular, $\ker(\pi)$ contains $N(E)$, $wN(E)w^{-1}$ and $\tilde{\operatorname{SL}}_2(E)$, where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus $\ker(\pi)$ contains $\mu_2$ and hence $\pi$ cannot be genuine.

We first describe the principal series representations, which are analogous to principal series representations of $\tilde{\operatorname{GL}}_2(E)$ [3]. Recall that although $\tilde{Z}$ is abelian, it does not lie in the center of $\tilde{\operatorname{GL}}_2(E)$. The center of $\tilde{\operatorname{GL}}_2(E)$ is $\tilde{Z}^2$. Let $(\tilde{\tau}, V)$ be an irreducible genuine representation of $\tilde{A}$. Extend this representation to a representation of $\tilde{B}(E)$ by defining the action of $N(E)$ on $V$ to be trivial. Then the normalised induction $\operatorname{Ind}_{\tilde{A}}^{\tilde{\operatorname{GL}}_2(E)} (\tilde{\tau})$ is called a principal series. As in the case of $\tilde{\operatorname{GL}}_2(E)$, there is an analogous criterion for the irreducibility of a principal series representation. If a principal series is reducible, it is of length two, and both the Jordan-Hölder factors are infinite dimensional (as these
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are genuine representations) unlike for $\text{GL}_2(E)$. We recall the criterion of irreducibility of a principal series now. Let $\tilde{\tau} = \text{Ind}_{\tilde{Z}A}^{\tilde{\text{GL}_2}(E)}(\chi)$ be an irreducible representation of $\tilde{A}$, where $\chi$ is as given in Equation 2.4. From [4], the principal series representation $\text{Ind}_{B(E)}^{\tilde{\text{GL}_2}(E)}(\tilde{\tau})$ is irreducible if and only if $\chi_1^2/\chi_2^2 \neq |\cdot|^1$, where $\chi_1, \chi_2$ are characters of $E^\times$ satisfying

$$\chi \left( \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \epsilon \right) = \epsilon \chi_1(a^2)\chi_2(b^2)$$

and $|\cdot|$ is the normalised absolute value on $E$.

An irreducible admissible genuine representation of $\tilde{\text{GL}_2}(E)$ which is not a Jordan-Hölder factor of a principal series is called a supercuspidal representation. Thus there are two types of irreducible admissible genuine representations of $\tilde{\text{GL}_2}(E)$, those which arise as Jordan-Hölder factors of principal series representations on the one hand supercuspidal representations on the other.

Another way to look at an irreducible representation of $\tilde{\text{GL}_2}(E)$ is via a representation of $\tilde{\text{SL}_2}(E)$, which will be useful to us later, e.g. in Section 4.6 and Chapter 6. Let

$$\text{GL}_2(E)_+ := \{ g \in \text{GL}_2(E) : \det(g) \in E^\times \} = Z \cdot \text{SL}_2(E).$$

**Lemma 2.5.1.** The centralizer of $\tilde{Z}$ in $\tilde{\text{GL}_2}(E)$ is $\tilde{\text{GL}_2}(E)_+$. 

**Proof.** Consider the following map

$$\phi : \tilde{Z} \times \tilde{\text{GL}_2}(E) \longrightarrow \{ \pm 1 \}$$

given by

$$\phi(\tilde{z}, \tilde{g}) := \tilde{z}\tilde{g}\tilde{z}^{-1}\tilde{g}^{-1}.$$ 

Note that this map is a ‘bi-character’ and $\phi(\tilde{z}, \tilde{g})$ depends only on $p(\tilde{z})$ and $p(\tilde{g})$. To prove the proposition we shall prove that the right kernel of the map $\phi$ is $\text{GL}_2(E)_+$. 

**Observation 1:** As $\tilde{Z}^2$ is the center of $\tilde{\text{GL}_2}(E)$ so it is the left kernel $\phi$.

**Observation 2:** As $\tilde{Z}$ is abelian it lies in the right kernel of the map $\phi$. Moreover, the commutator of $\text{GL}_2(E)$ is $\text{SL}_2(E)$ so $\tilde{\text{SL}_2}(E)$ also lies in the right kernel of $\phi$. And the group generated by $\tilde{Z}$ and $\tilde{\text{SL}_2}(E)$ is $\tilde{\text{GL}_2}(E)_+$. So centralizer of $\tilde{Z}$ in $\tilde{\text{GL}_2}(E)$ contains
From these observations we conclude that the map \( \phi \) factors through
\[
\bar{\phi} : \hat{Z}/\hat{Z}^2 \times \hat{\GL}_2(E)/\hat{\GL}_2(E)_+ \to \{ \pm 1 \}.
\]

We now write this map \( \bar{\phi} \) more explicitly using self-explanatory notation.
\[
\hat{Z}/\hat{Z}^2 \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in E^\times/E^{\times 2} \right\}, \quad \hat{\GL}_2(E)/\hat{\GL}_2(E)_+ \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} : a \in E^\times/E^{\times 2} \right\}
\]

Both the sets involved in \( \bar{\phi} \) are isomorphic to \( E^\times/E^{\times 2} \) and using the description of the Kubota cocycle \( \beta \) on diagonal elements we get
\[
\bar{\phi} \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right) = (a, b).
\]

From the non-degeneracy of the quadratic Hilbert symbol, it follows that the right kernel of the map \( \phi \) is \( \hat{\GL}_2(E)_+ \).

We have that \( \hat{\GL}_2(E)_+ = \hat{Z} \cdot \hat{\SL}_2(E) \) and that the center of \( \hat{\GL}_2(E)_+ \) is \( \hat{Z} \). Note that \( \hat{Z} \cap \hat{\SL}_2(E) = \{ \pm 1 \} \), which is the center of \( \hat{\SL}_2(E) \) and the index \( [\hat{\GL}_2(E) : \hat{\GL}_2(E)_+] = [E^\times : E^{\times 2}] < \infty \).

**Definition 2.5.2.** Let \( \tau \) be an irreducible admissible genuine representation of \( \hat{\SL}_2(E) \) and \( \mu \) a genuine character of \( \hat{Z} \). We say that \( \mu \) and \( \tau \) are compatible if the central character of \( \tau \) (i.e. \( \tau \) restricted to \( \{ \pm 1 \} \)) is the same as \( \mu|_{\{ \pm 1 \}} \).

If \( \mu \) and \( \tau \) are compatible, we can define an irreducible representation of \( \hat{\GL}_2(E)_+ \) on the space of \( \tau \) with central character \( \mu \) and on which \( \hat{\SL}_2(E) \) acts by \( \tau \). Denote this representation by \( \mu \tau \) and consider
\[
\pi := \text{ind}_{\hat{\GL}_2(E)_+}^{\hat{\GL}_2(E)} (\mu \tau), \tag{2.5}
\]

For \( a \in E^\times \), let \( \mu^a \) denote the genuine character of \( \hat{Z} \) defined by
\[
\mu^a(x, \epsilon) := (x, a) \mu(x, \epsilon) \quad \forall x \in E^\times, \epsilon \in \{ \pm 1 \}. \tag{2.6}
\]
2.5. Representations of $\widetilde{GL}_2(E)$

By the commutation relation in $\widetilde{A}$, it follows that conjugation by $\text{diag}(a, 1) \in \text{GL}_2(E)$ on a genuine character $\mu$ of $\widetilde{Z}$ takes $\mu$ to $\mu^a$. By non-degeneracy of the quadratic Hilbert symbol, if $a$ represents a non-trivial coset of $E^\times/E^{\times 2}$, then $x \mapsto (x, a)$ is a non-trivial character of $E^\times$. It follows that $\mu = \mu^a$ if and only if $a \in E^{\times 2}$. One may choose the representatives of the quotient $\widetilde{GL}_2(E)/\widetilde{GL}_2(E)_+$ to be $g(a) := \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$ for $a \in E^\times$ representing cosets of $E^\times/E^{\times 2}$. If we write $(\mu\tau)^a$ for the conjugate representation of $\mu\tau$ by the element $g(a)$, then it follows that

$$\mu\tau \not\cong (\mu\tau)^{g(a)} \cong \mu^a\tau^{g(a)} \quad (2.7)$$

as representations of $\widetilde{GL}_2(E)_+$ if $a \notin E^\times$, since the central characters of $\mu\tau$ and $(\mu\tau)^{g(a)}$, namely $\mu$ and $\mu^a$, are different. By Clifford theory, the representation $\pi$ of $\widetilde{GL}_2(E)$ defined by equation 2.5 is irreducible. Moreover, for all $a \in E^\times$ we have

$$\pi := \text{ind}_{\widetilde{GL}_2(E)_+}^{\widetilde{GL}_2(E)} (\mu\tau) \cong \text{ind}_{\widetilde{GL}_2(E)_+}^{\widetilde{GL}_2(E)} (\mu\tau)^a \quad (2.8)$$

and

$$\pi|_{\widetilde{GL}_2(E)_+} \cong \bigoplus_{a \in E^\times/E^{\times 2}} (\mu\tau)^a \quad (2.9)$$

and

$$\pi|_{\widetilde{SL}_2(E)} \cong \bigoplus_{a \in E^\times/E^{\times 2}} \tau^a. \quad (2.10)$$

Conversely, using Frobenius reciprocity and the fact that there exists an irreducible $\widetilde{SL}_2(E)$-subrepresentation of an irreducible admissible genuine representation of $\widetilde{GL}_2(E)$, it is easy to prove that any irreducible admissible genuine representation of $\widetilde{GL}_2(E)$ arises as in Equation 2.5 for some choice of $\mu$ and $\tau$.

**Remark 2.5.3.** From the analysis above, it follows that $\pi$ restricted to $\widetilde{GL}_2(E)_+$ is multiplicity free. Later we will see in Section 6.3 that the restriction of $\pi$ to $\widetilde{SL}_2(E)$ may not be multiplicity free.
2.6 Whittaker functionals for $\widetilde{GL}_2(E)$

Let $\psi$ be a non-trivial character of $E$. Following [5], we recall the definition of a $\psi$-Whittaker functional of a representation $(\pi, W)$ of any of $\widetilde{GL}_2(E), GL_2(E), \widetilde{SL}_2(E)$ or $SL_2(E)$. We identify $N(E)$ with $E$ as a topological group in obvious way.

**Definition 2.6.1.** A linear functional $\Lambda : W \rightarrow \mathbb{C}$ is called a $\psi$-Whittaker functional if it satisfies the following:

$$\Lambda \left( \pi \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} v \right) = \psi(n)\Lambda(v), \forall n \in E \text{ and } v \in V. \quad (2.11)$$

The representation $(\pi, W)$ is called $\psi$-generic if admits a non-zero $\psi$-Whittaker functional.

It is known that an irreducible admissible infinite dimensional representation of any of $\widetilde{GL}_2(E), GL_2(E), \widetilde{SL}_2(E)$ or $SL_2(E)$ is $\psi$-generic for some non-trivial character $\psi$ of $E$. Moreover, all infinite dimensional irreducible admissible representations of $\widetilde{GL}_2(E)$ or $GL_2(E)$ are $\psi$-generic for any non-trivial character $\psi$, see [5]. In particular, genuine representations of $\widetilde{GL}_2(E)$ are $\psi$-generic for any non-trivial character $\psi$. Recall that the space of Whittaker functionals for an irreducible admissible infinite dimensional representation of $GL_2(E)$ is one dimensional, an assertion which is known as the uniqueness of Whittaker model. But this need not be true for an irreducible genuine representation of $\widetilde{GL}_2(E)$.

Let $\omega_\pi$ denote the central character of an irreducible genuine representation $(\pi, W)$ of $\widetilde{GL}_2(E)$. For a genuine character $\chi$ of $\tilde{Z}^2$, we define a $\tilde{Z}$-module $\Omega(\chi)$ on which $\tilde{Z}^2$ acts by $\chi$ and any genuine character $\mu$ of $\tilde{Z}$ with $\mu|_{\tilde{Z}^2} = \chi$ appears in $\Omega(\chi)$ with multiplicity one.

We abuse the notation and write $\mu \in \Omega(\chi)$ if $\mu$ appears in $\Omega(\chi)$, i.e. $\text{Hom}_{\tilde{Z}}(\Omega(\chi), \mu) = 1$. Let $\mathcal{L}$ be the space of all $\psi$-Whittaker functionals for $(\pi, W)$. Then $\tilde{Z}$ has a natural action on $\mathcal{L}$ given by $(\tilde{z}, \Lambda)(v) := \Lambda(\pi(\tilde{z})v)$. As the action of $\tilde{Z}^2$ on $\mathcal{L}$ is by $\omega_\pi$, a character of $\tilde{Z}$ appearing in $\mathcal{L}$ belongs to $\Omega(\omega_\pi)$. For $\mu \in \Omega(\omega_\pi)$, let $\mathcal{L}_\mu := \{ \Lambda \in \mathcal{L} \mid \tilde{z} \cdot \Lambda = \mu(\tilde{z})\Lambda, \forall \tilde{z} \in \tilde{Z} \}$. Call $\mathcal{L}_\mu$ the space of $(\psi, \mu)$-Whittaker functionals.
2.6. Whittaker functionals for $\widetilde{\text{GL}}_2(E)$

**Theorem 2.6.2.** [5, Theorem 4.1] For an irreducible admissible genuine representation $\pi$ of $\widetilde{\text{GL}}_2(E)$, we have $\dim L_\mu \leq 1$ for all $\mu \in \Omega(\omega_\pi)$.

**Definition 2.6.3.** Let $N$ be a group, $\pi$ a representation of $N$ and $\psi$ a character of $N$. Let $\pi(N, \psi)$ be the vector space spanned by $\{\pi(n)v - \psi(n)v \mid n \in N \text{ and } v \in \pi\}$. Then $\pi_{N,\psi} := \pi/\pi(N, \psi)$ is called $\psi$-twisted Jacquet module of $\pi$. If $\psi = 1$ then we write $\pi_N$ for $\pi_{N,\psi}$ and call it the Jacquet module of $\pi$.

If $B$ is a group, $N \subset B$ a normal subgroup and $\pi$ a representation of $B$ then $\pi_N$ has an induced action of $B/N$ hence is a $B/N$-module. Thus $\pi \mapsto \pi_N$ defines a functor from the category of $B$-modules to the category of $B/N$-modules. For a non-trivial character $\psi$ of $N$, $\pi_{N,\psi}$ has an induced action of $\text{Norm}(N, \psi)/N$, where $\text{Norm}(N, \psi) = \{b \in B \mid \psi(bnb^{-1}) = \psi(n)\}$ and hence $\pi \mapsto \pi_{N,\psi}$ defines a functor from the category of $B$-modules to the category of $\text{Norm}(N, \psi)/N$-modules.

Note that, $L$, as a vector space, is dual of $\pi_{N(E),\psi}$. From Theorem 2.6.2 it follows that the multiplicity of a character $\mu \in \Omega(\omega_\pi)$ in $\pi_{N(E),\psi}$ is at most one, i.e. $\dim \text{Hom}_{\tilde{Z}}(\pi_{N(E),\psi}, \mu) \leq 1$. As a $\tilde{Z}$-module we have

$$\pi_{N(E),\psi} \subset \Omega(\omega_\pi).$$

**Proposition 2.6.4.** Let $\pi$ be a principal series representation of $\widetilde{\text{GL}}_2(E)$ with central character $\omega_\pi : \tilde{Z}^2 \to \mathbb{C}^\times$. Let $\psi$ be a non-trivial additive character of $E$. Then all the characters of $\tilde{Z}$ which extend $\omega_\pi$ appear in $\pi_{N(E),\psi}$, i.e. as a $\tilde{Z}$-module

$$\pi_{N(E),\psi} \cong \Omega(\omega_\pi).$$

We prove this proposition in the next few lemmas.

**Lemma 2.6.5.** Let $N(E)^-$ be the group of lower triangular unipotent matrices. Let $V_0$ be the subspace of functions in the space of the principal series representation $V(\tilde{\tau}) = \text{Ind}_{B(E)}^{\widetilde{\text{GL}}_2(E)}(\tilde{\tau})$ which have compact support when restricted to $N(E)^-$, where $(\tilde{\tau}, V)$ is an irreducible genuine representation of $\tilde{A}$. Then $V_0$ is of finite codimension in $V(\tilde{\tau})$. Moreover,
we have $V(\tilde{\tau})/V_0 \cong V$. On this quotient space $V$ the induced action of $N(E)^-$ is trivial and the induced action of $\tilde{A}$ is $\tilde{\tau}^w$.

Proof. Recall that $V(\tilde{\tau})$ is space of $V$ valued functions $f$ on $\widetilde{GL}_2(E)$ which are locally constant and satisfy $f(\tilde{b}g) = \tilde{\tau}(\tilde{b})f(g)$ for all $\tilde{b} \in \widetilde{B}(E)$ and $g \in \text{GL}_2(E)$. Because of the Bruhat decomposition $\widetilde{GL}_2(E) = \widetilde{wB(E)} \cup N^-(E)\widetilde{B}(E)$, a function $f \in V(\tilde{\tau})$ is determined by its values on $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and on the set $N(E)^-$. Define the evaluation map at $w$, $e : V(\tilde{\tau}) \longrightarrow V$, by $f \mapsto f(w)$. It is easy to verify that $V_0 = \ker(e)$. Note that $V_0$ is stable under the action of $B(E)^- := \tilde{A}N(E)^-$, so we have the following short exact sequence on $\tilde{A}N(E)^-$-modules

$$0 \longrightarrow V_0 \longrightarrow V(\tilde{\tau}) \longrightarrow V(\tilde{\tau})/V_0 \cong V \longrightarrow 0.$$  

The induced action of $N(E)^-$ is trivial on the quotient $V$. For $\tilde{a} \in \tilde{A}$ we have

$$(\pi(\tilde{a})f)(w) = f(\tilde{w}\tilde{a}) = f(\tilde{w}\tilde{a}^{-1}w) = \tilde{\tau}(\tilde{w}\tilde{a}^{-1})f(w) = \tilde{\tau}^w(\tilde{a})f(w)$$

proving that the action of $\tilde{A}$ on the quotient $V$ is same as $\tilde{\tau}^w$. \qed

The next two lemmas are immediate and these will complete the proof of Proposition 2.6.4.

**Lemma 2.6.6.** Following the notation of the above lemma, $V_0$ and $V(\tilde{\tau})$ are genuine $\widetilde{B(E)}^-$-modules. Let $\psi$ be a non-trivial character of $E$. Then

$$V(\tilde{\tau})_{N(E)^-,\psi} \cong (V_0)_{N(E)^-,\psi} \cong \Omega(\omega_\pi).$$

**Lemma 2.6.7.** If $\psi^{-1}$ is given by $x \mapsto \psi(-x)$ then as $\tilde{Z}$-modules we have

$$(V_0)_{N(E),\psi^{-1}} \cong (V_0)_{N^-(E),\psi}.$$
2.7 The Jacquet module and the Kirillov model for \( \widetilde{\text{GL}}_2(E) \)

2.7.1 The Jacquet module with respect to \( N(E) \)

Let \( \pi \) be an irreducible admissible genuine representation of \( \widetilde{\text{GL}}_2(E) \). We will describe \( \pi_{N(E)} \) in this section. It is well known that \( \pi_{N(E)} = 0 \) if and only if \( \pi \) is a supercuspidal representation, i.e. it does not appear as a subquotient of a principal series representation. So we need to consider only those representations which arise as Jordan-Hölder factors of principal series representations. Let \( \pi = \text{ind}^{\widetilde{\text{GL}}_2(E)}_{\widetilde{B}(E)}(\tilde{\tau}) \) be a principal series representation. As \( \widetilde{\text{GL}}_2(E) = \widetilde{B}(E) \sqcup \widetilde{B}(E)w\widetilde{B}(E) \) and \( \widetilde{B}(E)w\widetilde{B}(E) \) is open in \( \widetilde{\text{GL}}_2(E) \), we have the following filtration of \( \widetilde{B}(E) \)-modules \( 0 \subseteq \pi_w \subseteq \pi \), where \( \pi_w \) is space of functions supported on \( \widetilde{B}(E)w\widetilde{B}(E) \). This gives us the following filtration of the Jacquet modules \( \pi_{N(E)} \) (as \( \tilde{A} \)-modules):

\[
0 \to (\tilde{\tau})^w \cdot \delta^{1/2} \to \pi_{N(E)} \to \tilde{\tau} \cdot \delta^{1/2} \to 0. \tag{2.12}
\]

Both the Jordan-Hölder factors are genuine representations of \( \tilde{A} \) of dimension \( \dim(\tilde{\tau}) \). By Lemma 2.4.3 both are irreducible \( \tilde{A} \)-modules. Its semi-simplification \( \pi_{N(E)}^{ss} \) equals \( \tilde{\tau} \cdot \delta^{1/2} \oplus \tilde{\tau}^w \cdot \delta^{1/2} \). Note that \( \tilde{\tau} \) is determined by its restriction to \( \tilde{A}^2 \), i.e. a pair \( (\chi_1^2, \chi_2^2) \), where \( \chi_1, \chi_2 \) are characters of \( E^\times \). The restriction of \( \tilde{\tau}^w \) to \( \tilde{A}^2 \) is \( (\chi_2^2, \chi_1^2) \). The two Jordan-Hölder factors are isomorphic to each other if and only if \( \chi_1^2 = \chi_2^2 \). So the short exact sequence of \( \tilde{A} \)-modules in Equation 2.12 splits whenever \( \chi_1^2 \neq \chi_2^2 \). In particular, when \( \pi \) is a reducible principal series representation, the short exact sequence in Equation 2.12 splits as \( \chi_1^2/\chi_2^2 = | \cdot |^{1+} \).

If \( \pi \) is an irreducible principal series, then we know its Jacquet module \( \pi_{N(E)} \) in the sense that we know its Jordan-Hölder factors. Moreover \( \pi_{N(E)} \) is of length two as \( \tilde{A} \)-module. Let us assume that the principal series representation \( \pi \) is reducible and its Jordan-Hölder factors are \( \pi_1 \) and \( \pi_2 \) giving rise to the following exact sequence of \( \widetilde{\text{GL}}_2(E) \)-modules

\[
0 \to \pi_1 \to \pi \to \pi_2 \to 0.
\]
As the Jacquet functor is exact [1, Proposition 2.35], we get the following short exact sequence of $\tilde{A}$-modules

$$0 \rightarrow (\pi_1)_{N(E)} \rightarrow \pi_{N(E)} \rightarrow (\pi_2)_{N(E)} \rightarrow 0.$$ 

As we know that $(\pi_1)_{N(E)}$ and $(\pi_2)_{N(E)}$ are non-zero, one of these is $\tilde{\tau} \cdot \delta^{1/2}$ and the other is $\tilde{\tau}^w \cdot \delta^{1/2}$. As $\pi_1$ is a subrepresentation of $\pi$, by Frobenius reciprocity we have

$$\text{Hom}_{\tilde{GL}_2(E)}(\pi_1, \pi) = \text{Hom}_A((\pi_1)_{N(E)}, \tilde{\tau} \cdot \delta^{1/2}),$$

therefore $(\pi_1)_{N(E)} \cong \tilde{\tau} \cdot \delta^{1/2}$ and hence $(\pi_2)_{N(E)} = \tilde{\tau}^w \cdot \delta^{1/2}$.

### 2.7.2 The Kirillov model

Now we describe the Kirillov model of an irreducible admissible genuine representation $\pi$ of $\tilde{\text{GL}}_2(E)$ [4]. Recall $\pi_{N(E),\psi} = \pi/\pi(N(E),\psi)$. Let $l : \pi \rightarrow \pi_{N(E),\psi}$ be the canonical map. Let $C^\infty(E^\times, \pi_{N(E),\psi})$ denote the space of smooth functions on $E^\times$ with values in $\pi_{N(E),\psi}$. Define the Kirillov mapping

$$K : \pi \rightarrow C^\infty(E^\times, \pi_{N(E),\psi})$$

given by $v \mapsto \xi_v$ where $\xi_v(x) = l \left( \pi \left( \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) v \right)$. We summarize some of the properties of the Kirillov mapping in the following proposition.

**Proposition 2.7.1.**

1. If $v' = \pi \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, 1 \right) v$, then

   $$\xi_{v'}(x) = (x, d) \psi(b d^{-1} x) \pi \left( \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, 1 \right) \xi_v(a d^{-1} x).$$

2. For $v \in W$ the function $\xi_v$ is a locally constant function on $E^\times$ which vanishes outside a compact subset of $E$.

3. The map $K$ is an injective linear map.
2.7. The Jacquet module and the Kirillov model for $\widetilde{GL}_2(E)$

4. The image $K(\pi)$ of the map $K$ contains the space $S(E^\times, \pi_{N(E),\psi})$ of smooth functions with compact support in $E^\times$.

5. The Jacquet module $\pi_{N(E)}$ of $\pi$ is isomorphic to $K(\pi)/S(E^\times, \pi_{N(E),\psi})$.

6. The representation $\pi$ is supercuspidal if and only if $K(\pi) = S(E^\times, \pi_{N(E),\psi})$.

Proof. Part 1 follows from the definition. The proofs of part 2 and 3 are verbatim those of Lemma 2 and Lemma 3 in [7]. The proofs of part 4, 5 and 6 follow from the proofs of the corresponding statements of [24, Theorem 3.1].

Since the map $K$ is injective, we can transfer the action of $\widetilde{GL}_2(E)$ on $W$ (via $\pi$) to $K(\pi)$ using the map $K$. The realization of $(\pi, W)$ on $K(\pi)$ is called the Kirillov model, on which the action of $\widetilde{B}(E)$ is explicitly given by part 1 in Proposition 2.7.1. It is clear that $S(E^\times, \pi_{N(E),\psi})$ is $\widetilde{B}(E)$ stable, which gives rise to the following short exact sequence of $\widetilde{B}(E)$-modules

$$0 \to S(E^\times, \pi_{N(E),\psi}) \to K(\pi) \to \pi_{N(E)} \to 0. \quad (2.13)$$

2.7.3 The Jacquet module with respect to $N(F)$

Now restrict an irreducible admissible genuine representation $\pi$ of $\widetilde{GL}_2(E)$ to $B(F)$. $N(F) \subset B(F)$ is a normal subgroup. To simplify notation we write $N$ for $N(F)$ in the rest of this section. We describe the Jacquet module $\pi_N$ of $\pi$, which we will need in Chapter 5. We utilize the short exact sequence in Equation 2.13 of $\widetilde{B}(E)$-modules arising from the Kirillov model of $\pi$, which is also a short exact sequence of $B(F)$-modules. By the exactness of the Jacquet functor with respect to $N$, we get the following short exact sequence from Equation 2.13

$$0 \to S(E^\times, \pi_{N(E),\psi})_N \to K(\pi)_N \to (\pi_{N(E),\psi})_N \cong \pi_{N(E)} \to 0. \quad (2.13)$$

Let us first describe $S(E^\times, \pi_{N(E),\psi})_N$, the Jacquet module of $S(E^\times, \pi_{N(E),\psi})$ with respect to $N = N(F)$. Let $S(F^\times, \pi_{N(E),\psi})$ be the space of locally constant functions with compact support from $F^\times$ with values in $\pi_{N(E),\psi}$ be trivial $N(F)$-module.
Proposition 2.7.2. $S(E^\times, \pi_{N,\psi})_N \cong S(F^\times, \pi_{N(E),\psi})$.

The Proposition 2.7.2 follows from the proposition below. The author thanks Professor D. Prasad for suggesting the proof below.

Proposition 2.7.3. Let $S(E^\times)$ be a representation space for $N \cong E$ with the action of $N$ given by $(n \cdot f)(x) = \psi(nx)f(x)$ for all $x \in E^\times$ where $\psi$ is a non-trivial additive character of $E$ such that $\psi|_F = 1$. Then the restriction map

$$S(E^\times) \longrightarrow S(F^\times)$$

(2.14)

gives the Jacquet module, i.e. the above map realizes $S(E^\times)_N$ as $S(F^\times)$.

Proof. Note that $S(E^\times) \hookrightarrow S(E)$. For a fixed Haar measure $dw$ on $E$, we define the Fourier transform $\mathcal{F}_\psi : S(E) \to S(E)$ with respect to the character $\psi$ by

$$\mathcal{F}_\psi(f)(z) := \int_E f(w)\psi(zw) \, dw.$$ 

$\mathcal{F}_\psi$ is an isomorphism of vector spaces and image of $S(E^\times)$ can be identified with those functions whose integral on $E$ is zero. The Fourier transform takes the action of $N(E)$ on $S(E^\times)$ to the restriction of the action of $N(E)$ on $S(E)$ given by $(n \cdot f)(x) = f(x+n)$. Here we have identified $N(E)$ with $E$. Thus the maximal quotient of $S(E)$ on which $N(F)$ acts trivially can be identified with $S(F)$ by integrating along the fibres (defined below) of the mapping $\phi : E \to F$ given by $\phi(e) = \frac{e + \sqrt{d}}{2\sqrt{d}}$ if $E = F(\sqrt{d})$. Note that $\phi(z + x) = \phi(z)$ for all $z \in E$ and $x \in F$. We define the integration along the fibres of the map $\phi$, $I : S(E) \to S(F)$ as follows:

$$I(f)(y) := \int_F f(x + \sqrt{d}y) \, dx \text{ for all } y \in F.$$ 

It can be checked that $I(f)$ belongs to $S(F)$. Note that $\psi_{\sqrt{d}} = \psi_{\sqrt{d}}|_F : x \mapsto \psi(\sqrt{d}x)$ is a non-trivial character of $F$. The proposition will follow if we prove the commutativity of the following diagram:

$$
\begin{array}{ccc}
S(E) & \xrightarrow{\mathcal{F}_\psi} & S(E) \\
\text{Res} & & \text{Res} \\
\downarrow & & \downarrow \\
S(F) & \xrightarrow{\mathcal{F}_\psi, \pi} & S(F)
\end{array}
$$
where $\mathcal{F}_\psi$ (respectively, $\mathcal{F}_{\psi,\sqrt{d}}$) is the Fourier transform on $\mathcal{S}(E)$ (respectively, $\mathcal{S}(F)$) with respect to the character $\psi$ (respectively, $\psi_{\sqrt{d}} = (\psi\sqrt{d})|_F$), $\text{Res}$ denotes the restriction mapping and $I$ denote the integration along the fibres mentioned above. $\mathcal{F}_{\psi,\sqrt{d}} : \mathcal{S}(F) \to \mathcal{S}(F)$ is defined by $\mathcal{F}_{\psi,\sqrt{d}}(\phi)(x) := \int_F \phi(y) \psi_{\sqrt{d}}(xy) dy$ for all $x \in F$. We claim that the above diagram is commutative. Let $f \in \mathcal{S}(E)$. We want to show that $I \circ \mathcal{F}_\psi(f)(y) = \mathcal{F}_{\psi,\sqrt{d}} \circ \text{Res}(f)(y)$ for all $y \in F$. We write an element of $E$ as $x + \sqrt{d}y$ with $x, y \in F$. We choose a measure $dx$ on $F$ which is self dual with respect to $\psi_{\sqrt{d}}$ in the sense that $\mathcal{F}_{\psi,\sqrt{d}}(\mathcal{F}_{\psi,\sqrt{d}}(\phi))(x) = \phi(-x)$ for all $\phi \in \mathcal{S}(F)$ and $x \in F$. We identify $E$ with $F \times F$ as vector space. Consider the product measure $dx dy$ on $E = F \times F$. Then using Fubini’s theorem and $\int_F \int_F \phi(z_2) \psi_{\sqrt{d}}(xz_2) dz_2 dx = \mathcal{F}_{\psi,\sqrt{d}}(\mathcal{F}_{\psi,\sqrt{d}}(\phi))(0) = \phi(0)$ for $\phi \in \mathcal{S}(F)$, we get the following:

\[
I \circ \mathcal{F}_\psi(f)(y) = \int_F \mathcal{F}_\psi(f)(x + \sqrt{d}y) dx \\
= \int_F \int_{E \times F} f(z_1 + \sqrt{d}z_2) \psi((x + \sqrt{d}y)(z_1 + \sqrt{d}z_2)) dz_1 dz_2 dx \\
= \int_F \int_F f(z_1 + \sqrt{d}z_2) \psi_{\sqrt{d}}(y z_1 + x z_2) dz_1 dz_2 dx \\
= \int_F \left( \int_F f(z_1 + \sqrt{d}z_2) \psi_{\sqrt{d}}(x z_2) dz_2 dx \right) \psi_{\sqrt{d}}(y z_1) dz_1 \\
= \int_F f(z_1) \psi_{\sqrt{d}}(y z_1) dz_1 \\
= \mathcal{F}_{\psi,\sqrt{d}} \circ \text{Res}(f)(y).
\]

This proves the commutativity of the above diagram. \qed
Chapter 3

Splitting questions

3.1 Introduction

Let $E$ be a non-Archimedean local field. This chapter will be concerned with a specific 2-fold covers of $\text{GL}_2(E)$, to be called the metaplectic covering of $\text{GL}_2(E)$, which was defined in Section 2.4. We recall that there is a unique (up to isomorphism) non-trivial 2-fold cover of $\text{SL}_2(E)$ called the metaplectic cover and denoted by $\widetilde{\text{SL}}_2(E)$, but there are many inequivalent 2-fold coverings of $\text{GL}_2(E)$ which extend this 2-fold covering of $\text{SL}_2(E)$. The covering $\widetilde{\text{GL}}_2(E)$ of $\text{GL}_2(E)$ can be described as follows. Observe that $\text{GL}_2(E)$ is the semi-direct product of $\text{SL}_2(E)$ and $E \times$, where $E \times$ sits inside $\text{GL}_2(E)$ as $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$. This action of $E \times$ on $\text{SL}_2(E)$ lifts uniquely to an action of $E \times$ on $\widetilde{\text{SL}}_2(E)$. The group $\widetilde{\text{SL}}_2(E) \rtimes E \times$ is the metaplectic cover $\widetilde{\text{GL}}_2(E)$ of $\text{GL}_2(E)$. Thus the metaplectic cover of $\text{GL}_2(E)$ that we consider in this chapter is that cover of $\text{GL}_2(E)$ which extends the metaplectic cover of $\text{SL}_2(E)$ and is further split on the subgroup $\left\{ \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} : e \in E \times \right\}$.

Given a central extension of a group $G$ by $\mathbb{Z}/2\mathbb{Z}$, say

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G' \longrightarrow G \longrightarrow 1$$
there is a natural central extension, say $G''$, of $G$ by $\mathbb{C}^\times$, given by
\[ G'' := G' \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}^\times := \frac{G' \times \mathbb{C}^\times}{\langle -1, -1 \rangle}, \]
which sits in the following exact sequence
\[
\begin{array}{ccccccccc}
1 & \to & \mathbb{Z}/2\mathbb{Z} & \to & G'' & \to & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mathbb{C}^\times & \to & G' & \to & G & \to & 1
\end{array}
\]
This $\mathbb{C}^\times$-central extension of $G$ is said to be obtained from the 2-fold cover $G' \to G$ of $G$. It is well known that $\mathbb{C}^\times$-covers tend to be easier to analyse and this is what we shall do in this chapter.

Let $F$ be a non-Archimedean local field of characteristic zero. Let $D_F$ denote the unique quaternion division algebra with center $F$. Note that $D_F \otimes F \cong M_2(E)$. By Skolem-Noether theorem, such an embedding is uniquely determined up to conjugation by elements of $GL_2(E)$. The main theorem of this chapter is the following:

**Theorem 3.1.1.** Let $E$ be a quadratic extension of a non-Archimedean local field $F$ and $\widetilde{GL_2(E)}$ the two-fold metaplectic covering of $GL_2(E)$. Then:

1. The two-fold metaplectic covering splits over the subgroup $GL_2(F)$.

2. The $\mathbb{C}^\times$-covering obtained from $\widetilde{GL_2(E)}$ splits over the subgroup $D_F^\times$.

From now onward we abuse the notation and write $\widetilde{GL_2(E)}$ for the $\mathbb{C}^\times$-covering obtained from the metaplectic cover $\widetilde{GL_2(E)}$. Note that a quadratic extension $L$ of $F$ gives rise to two embeddings of $L$ in $M(2, E)$ as in the diagram below:

\[
\begin{array}{ccc}
D_F & \to & M(2, E) \\
\downarrow & & \downarrow \\
L & \to & M(2, F)
\end{array}
\]
3.1. Introduction

By Skolem-Noether theorem, any two embeddings of $L \otimes E$ in $M(2, E)$ and hence of $L$ are conjugate in $M(2, E)$ by $GL_2(E)$.

Let $GL_2(E)_{\mathbb{C}^\times}$ denote the $\mathbb{C}^\times$-covering of $GL_2(E)$ obtained from 2-fold cover $GL_2(E)$.

**Refined Question 3.1.2.** Does there exist a natural identification of the set of splittings of the $\mathbb{C}^\times$-cover $\widetilde{GL_2(E)}_{\mathbb{C}^\times}$ of $GL_2(E)$ restricted to $GL_2(F)$ and set of splittings restricted to $D_F^\times$ (in either of the two cases the set of splittings is a principal homogeneous space over the character group of $F^\times$) such that for any quadratic extension $L$ of $F$, the two embeddings of $L^\times$ in $\widetilde{GL_2(E)}_{\mathbb{C}^\times}$

are conjugate in $\widetilde{GL_2(E)}_{\mathbb{C}^\times}$?

We are not able to handle the refined question, and will only content with the proof of the existence of a splitting of the metaplectic cover of $GL_2(E)$ restricted to $D_F^\times$. However the above refined question plays an important role in harmonic analysis relating the pair $(\widetilde{GL_2(E)}, GL_2(F))$ with the pair $(\widetilde{GL_2(E)}, D_F^\times)$.

We briefly say a few words about the proofs. The proof for $GL_2(F)$ is straightforward from the explicit knowledge of the cocycle defining the metaplectic cover. For any quadratic extension $L$ of $F$, we know that the embedding $L^\times \hookrightarrow D_F^\times$ is conjugate inside $GL_2(E)$ to the embedding of $L^\times$ inside $GL_2(E)$ realized as $L^\times \hookrightarrow GL_2(F) \hookrightarrow GL_2(E)$ (Skolem-Noether theorem). Since the metaplectic cover of $GL_2(E)$ splits when restricted to $GL_2(F)$, it is split in particular over $L^\times$ for any quadratic extension $L$ of $F$. Thus we know that the
restriction of the metaplectic cover of $GL_2(E)$ to $D_F^\times$ has the property that it splits over $L^\times$ for any quadratic extension $L$ of $F$. This is the key property to be used in the proofs below.

### 3.2 Splitting over $GL_2(F)$

We prove the following proposition:

**Proposition 3.2.1.** Let $E$ be a quadratic extension of a non-Archimedean local field $F$. Then the metaplectic 2-fold cover $\tilde{GL}_2(E)$ of $GL_2(E)$, as described in the introduction, splits over the subgroup $GL_2(F)$.

*Proof.* To prove that the covering $\tilde{GL}_2(E)$ of $GL_2(E)$ splits over $GL_2(F)$, it suffices to show that the 2-cocycle $\beta$ which defines the 2-fold metaplectic cover satisfies $\beta(\sigma, \tau) = 1$ for all $\sigma, \tau \in GL_2(F)$, i.e., the cocycle is identically 1 when restricted to $GL_2(F)$. One knows that the defining expression of the cocycle $\beta$ involves only quadratic Hilbert symbols of the field $E$. The proposition will follow once we prove that the restriction of the quadratic Hilbert symbol of $E$ to $F$ is identically 1, which is the content of the next lemma. \hfill \Box

**Lemma 3.2.2.** If we denote the quadratic Hilbert symbol of the field $E$ by $(\cdot, \cdot)_E$, then

$$(a, b)_E = 1 \text{ for all } a, b \in F^\times.$$

*Proof.* Let $(\cdot, \cdot)_F$ denotes the quadratic Hilbert symbol of the field $F$. Then it is well known that for $a \in F^\times$ and $b \in E^\times$, we have

$$(a, b)_E = (a, Nb)_F.$$

Hence for $a, b \in F^\times$ we have

$$(a, b)_E = (a, Nb)_F = (a, b^2)_F = 1.$$

\hfill \Box
3.3 Splitting over $\text{SL}_1(D_F)$

Recall that $D_F$ denotes the unique quaternion division algebra over the field $F$ and that $\text{SL}_1(D_F)$ is the subgroup of norm 1 elements in $D_F^\times$. Fix an embedding $E \hookrightarrow D_F$ through which $D_F$ can be realized as a two dimensional vector space over $E$ with $E$ acting on $D_F$ on the left and $D_F$ acting on itself on the right. This gives rise to an embedding $D_F^\times \hookrightarrow \text{GL}_2(E)$. Since $\text{SL}_1(D_F)$ is compact, we can assume that $\text{SL}_1(D_F) \subset \text{GL}_2(O_E)$. It is well known that if the residue characteristic of $F$ is odd, then the two-fold metaplectic cover $\tilde{\text{GL}_2(E)}$ of $\text{GL}_2(E)$ splits over $\text{GL}_2(O_E)$ and hence over $\text{SL}_1(D_F)$. Such a simple minded proof does not work for $p = 2$. However, we prove in this section that the $\mathbb{C}^\times$-metaplectic cover of $\text{SL}_2(E)$ does split when restricted to $\text{SL}_1(D_F)$.

**Proposition 3.3.1.** The restriction of the non-trivial 2-fold cover of $\text{SL}_4(F)$ to $\text{SL}_2(E)$ remains non-trivial, hence gives the unique non-trivial 2-fold cover of $\text{SL}_2(E)$.

**Proof.** The proposition amounts to the assertion that there is a commutative diagram involving the unique 2-fold covers of $\text{SL}_2(E)$ and $\text{SL}_4(F)$ as follows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \tilde{\text{SL}_2(E)} & \longrightarrow & \text{SL}_2(E) & \longrightarrow & 1 \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \tilde{\text{SL}_4(F)} & \longrightarrow & \text{SL}_4(F) & \longrightarrow & 1
\end{array}
$$

This follows from the generality that the transfer map

$$
tr : K_2(E)/2K_2(E) \longrightarrow K_2(F)/2K_2(F)
$$

is an isomorphism \cite{18}.

**Corollary 3.3.2.** The $\mathbb{C}^\times$-cover of $\text{SL}_2(E)$ obtained from $\tilde{\text{SL}_2(E)}$ splits over $\text{SL}_1(D_F)$.

**Proof.** From Proposition 3.3.1 the restriction of the 2-fold cover from $\text{SL}_4(F)$ to $\text{SL}_2(E)$ remains non-trivial. Since we have an inclusion of groups

$$
\begin{array}{ccc}
\text{SL}_2(E) & \hookrightarrow & \text{Sp}_4(F) & \hookrightarrow & \text{SL}_4(F),
\end{array}
$$


and all these groups have a unique non-trivial 2-fold cover, we deduce that the unique non-trivial 2-fold cover of $\text{SL}_4(F)$ restricts to give the unique non-trivial 2-fold cover of $\text{Sp}_4(F)$ which in turn restricts to the unique non-trivial 2-fold cover of $\text{SL}_2(E)$. Now we use the inclusion of the groups

$$\text{SL}_1(D_F) \hookrightarrow \text{SL}_2(E) \hookrightarrow \text{Sp}_4(F)$$

and use a result of Kudla [12, Theorem 3.1] according to which the restriction of the $\mathbb{C}^\times$-covering of $\text{Sp}_4(F)$ to $U(2)$ splits. (The result of Kudla is valid for any unitary group $U(n)$ defined by a skew hermitian form in $n$ variables over $E$ and hence comes with a natural embedding in $\text{Sp}_{2n}(F)$). If we take a non-degenerate hermitian form in 2 variables which is anisotropic, then the corresponding unitary group is $U(2)$ and, $SU(2) \cong \text{SL}_1(D_F)$. As a result, the restriction of the $\mathbb{C}^\times$-covering from $\text{SL}_2(E)$ to $\text{SL}_1(D) = SU(2) \subset U(2)$ splits.

$$\square$$

### 3.4 Splitting over $D_F^\times$

In this section we prove the splitting of the $\mathbb{C}^\times$-cover of $\text{GL}_2(E)$ obtained from $\widetilde{\text{GL}}_2(E)$ over $D_F^\times$.

#### 3.4.1 The case of even residue characteristic

Note the following short exact sequence

$$1 \longrightarrow \text{SL}_1(D_F) \longrightarrow D_F^\times \longrightarrow F^\times \longrightarrow 1.$$  \hspace{1cm} (A)

Let $\mathbb{C}^\times$ be the trivial $D_F^\times$-module. Then $H^2(D_F^\times, \mathbb{C}^\times)$ classifies central extensions of $D_F^\times$ by the group $\mathbb{C}^\times$. The Hochschild-Serre spectral sequence arising from (A) gives a filtration on $H^2(D_F^\times, \mathbb{C}^\times)$:

$$H^2(D_F^\times, \mathbb{C}^\times) = F^0 \supseteq F^1 \supseteq F^2 \supseteq 0$$

with $F^0/F^1 = E_{\infty}^{0,2}$, $F^1/F^2 = E_{\infty}^{1,1}$ and $F^2 = E_{\infty}^{2,0}$, where
Consider the embedding $D_F^\times \hookrightarrow \text{GL}_2(E)$ and denote the restriction of the central extension of $\text{GL}_2(E)$ to $D_F^\times$ as well as the corresponding element of $H^2(D_F^\times, \mathbb{C})$ by $\beta$. In Section 3.3 we proved that $\beta$ restricted to $\text{SL}_1(D_F)$ is trivial, therefore $\beta \in F^1$. In even residue characteristic, since we are dealing with a cohomology class of order 2 (or 1), the following result of C. Riehm [26] implies that $\beta$ must be trivial in $F^1/F^2$.

**Proposition 3.4.1.** Let $G_0 = \text{SL}_1(D_F)$ and for $i \geq 1$, let $G_i$ denote the $i$-th standard congruence subgroup of $G_0$. Then

$$[G_0, G_0] = G_1.$$  

In particular, the character group of $\text{SL}_1(D_F)$ is a finite cyclic group of order prime to $p$.

Thus, in the case of even residue characteristic, an element of $H^2(D_F^\times, \mathbb{C})$ of order 2 which is trivial when restricted to $\text{SL}_1(D_F)$ arises by inflation from an element of $H^2(F^\times, \mathbb{C})$. An element of $H^2(F^\times, \mathbb{C})$ is represented by a central extension

$$1 \rightarrow \mathbb{C} \rightarrow \tilde{F}^\times \rightarrow F^\times \rightarrow 1.$$  

The proof of the splitting of the $\mathbb{C}^\times$-metaplectic cover of $\text{GL}_2(E)$ restricted to $D_F^\times$ will be completed in the case of even residue characteristic once we prove the following lemma.

**Lemma 3.4.2.** A $\mathbb{C}^\times$-covering of $D_F^\times$ coming from a $\mathbb{C}^\times$-covering of $F^\times$ via the norm map, which is trivial on $L^\times$ for all quadratic extensions $L$ of $F$, is trivial.

**Proof.** Suppose there exists a non-trivial $\mathbb{C}^\times$-covering of $D_F^\times$ coming from a $\mathbb{C}^\times$-central extension $\tilde{F}^\times$ of $F^\times$ via the norm map, which is trivial on $L^\times$ for all quadratic extension $L$ of $F$. If the cover $\tilde{F}^\times$ is non-trivial, then it is non-abelian. Thus there are two elements $e_1, e_2 \in \tilde{F}^\times$ which do not commute. Look at the images, say, $f_1, f_2$ of $e_1, e_2$ in $F^\times$. Let $\bar{f}_1,$
\(\bar{f}_2\) be images of \(f_1, f_2\) in \(F^\times/F^{\times 2}\). Since the residue characteristic of \(F\) is even, \(F^\times/F^{\times 2}\) is a vector space over \(\mathbb{Z}/2\mathbb{Z}\) of dimension \(\geq 3\). Therefore given any two elements \(\bar{f}_1, \bar{f}_2 \in F^\times/F^{\times 2}\), there exist a subgroup \(F_1 \hookrightarrow F^\times\) of index 2 containing \(f_1, f_2\). By local class field theory, there exists a unique quadratic extension \(M\) of \(F\) with \(\text{Norm}_{M/F}(M^\times) = F_1\). Now we use the fact given to us that the central extension of \(D_F^\times\) that we are considering is trivial on \(L^\times\) for any quadratic extension \(L\) of \(F\), in particular on \(M^\times\). Hence the inverse image of \(M^\times\) in the central extension must be abelian, a contradiction to the construction of \(M\).

\(\square\)

### 3.4.2 The case of odd residue characteristic

In this subsection we assume that the residue characteristic \(p\) of \(F\) is odd. We first introduce more notation. Let \(\mathcal{O}_{D_F}\) be the maximal compact subring of \(D_F\) and \(\mathcal{P}_{D_F}\) the maximal ideal of \(\mathcal{O}_{D_F}\). Let \(D_F^\times(1) := 1 + \mathcal{P}_{D_F}\). Note that \(D_F^\times(1)\) is a normal pro-\(p\) subgroup in \(D_F^\times\).

Since \(p\) is odd and \(D_F^\times(1)\) is a normal pro-\(p\) subgroup

\[H^2(D_F^\times, \mathbb{Z}/2\mathbb{Z}) \cong H^2(D_F^\times/D_F^\times(1), \mathbb{Z}/2\mathbb{Z}).\]

In other words, every 2-fold central extension of \(D_F^\times\) arises as a pull back of a 2-fold central extension \(D_F^\times/D_F^\times(1)\). The group \(D_F^\times/D_F^\times(1)\) is (non-canonical) isomorphic to \(\mathbb{F}_{q^2}^\times \rtimes \mathbb{Z}\), where \(\mathbb{F}_{q^2}\) is the finite field with \(q^2\) elements and \(\mathbb{Z}\) operates on \(\mathbb{F}_{q^2}^\times\) by powers of the Frobenius map \(x \mapsto x^q\). This group sits in the following short exact sequence

\[0 \rightarrow \mathbb{F}_{q^2}^\times \rightarrow G' := D_F^\times/D_F^\times(1) \rightarrow \mathbb{Z} \rightarrow 0.\]

Using this description of the group we prove the following proposition.

**Proposition 3.4.3.** (A) We have

\[H^2(D_F^\times, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.\]

(B) If we denote the subgroup of 2-torsion elements of \(H^2(D_F^\times, \mathbb{C}^\times)\) by \(H^2(D_F^\times, \mathbb{C}^\times)[2]\) then

\[H^2(D_F^\times, \mathbb{C}^\times)[2] = \mathbb{Z}/2\mathbb{Z}.\]
Proof. (A) Since $G' = \mathbb{F}_q^\times \rtimes \mathbb{Z}$ and $\mathbb{Z}$ has cohomological dimension 1, the Hochschild-Serre spectral sequence $E_2^{i,j} = H^i(\mathbb{Z}, H^j(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z}))$ calculating the cohomology of $G'$ satisfies $E_1^{1,1} = E_\infty^{1,1}$, $E_2^{0,2} = E_\infty^{0,2}$ and $E_2^{2,0} = E_\infty^{2,0} = 0$. Therefore

$$0 \longrightarrow H^1(\mathbb{Z}, H^1(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z})) \longrightarrow H^2(G', \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0.$$  

Since $H^1(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^2(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and since $\mathbb{Z}$ must act trivially on $\mathbb{Z}/2\mathbb{Z}$, we get

$$0 \rightarrow H^1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(G', \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$  

which proves part (A) of the proposition.

(B) This is evident from the next lemma, namely Lemma 3.4.4. This proves the proposition.

By Proposition 3.4.3 there are four non-isomorphic two-fold coverings of the group $D_F^\times$. The lemma below proves that one of the three non-trivial 2-fold covers becomes trivial as a $\mathbb{C}^\times$-cover.

**Lemma 3.4.4.** We have a short exact sequence

$$0 \longrightarrow \frac{H^1(D_F^\times, \mathbb{C}^\times)}{2H^1(D_F^\times, \mathbb{C}^\times)} \longrightarrow H^2(D_F^\times, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(D_F^\times, \mathbb{C}^\times)[2] \longrightarrow 0$$

with

$$\frac{H^1(D_F^\times, \mathbb{C}^\times)}{2H^1(D_F^\times, \mathbb{C}^\times)} \cong \mathbb{Z}/2\mathbb{Z}$$

where for any abelian group $A$, $A[2] = \{a \in A : 2a = 0\}$.

**Proof.** The short exact sequence can be deduced from the long exact sequence of cohomology groups of $D_F^\times$ arising from the following short exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \longrightarrow 0.$$
Since \([D_F^\times, D_F^\times] = \text{SL}_1(D_F)\) and \(D_F^\times / \text{SL}_1(D_F) \cong F^\times\), the second statement follows from the fact that the character group of \(D_F^\times\), i.e. \(H^1(D_F^\times, \mathbb{C}^\times)\), is the same as the character group of \(F^\times\), and using that \(F\) has odd residue characteristic, it is easy to see that

\[
\frac{H^1(F^\times, \mathbb{C}^\times)}{2H^1(F^\times, \mathbb{C}^\times)} \cong \mathbb{Z}/2\mathbb{Z}.
\]

\[\Box\]

**Proposition 3.4.5.** Let \(M\) be the quadratic unramified extension of \(F\) with \(M \hookrightarrow D_F\). Then a two-fold cover of \(D_F^\times\) which remains non-trivial with \(\mathbb{C}^\times\) coefficients does not split over the subgroup \(M^\times \hookrightarrow D_F^\times\).

**Proof.** Let \(M^\times(1) = 1 + \mathcal{P}_M\). As \(M\) is a quadratic unramified extension of \(F\), we have

\[
M^\times / M^\times(1) \cong \mathbb{F}_q^\times \times \mathbb{Z}.
\]

Since \(\mathbb{Z}\) has cohomological dimension 1, by the Kunneth theorem

\[
H^2(M^\times / M^\times(1), \mathbb{Z}/2\mathbb{Z}) \cong H^2(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z}) \oplus \left( H^1(\mathbb{F}_q^\times, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \right)
\]

\[
\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.
\]

Since \(M^\times / M^\times(1) \cong \mathbb{F}_q^\times \times \mathbb{Z}\), its character group is isomorphic to \(\mathbb{F}_q^\times \times \mathbb{C}^\times\). So once again as in lemma 3.4.4, we get the following short exact sequence:

\[
0 \to \mathbb{Z}/2\mathbb{Z} = \frac{H^1(M^\times / M^\times(1), \mathbb{C}^\times)}{2H^1(M^\times / M^\times(1), \mathbb{C}^\times)} \to H^2(\mathbb{M}^\times, \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{M}^\times, \mathbb{C}^\times) \to 0
\]

By considering the embedding \(M^\times / M^\times(1) \hookrightarrow D_F^\times / D_F^\times(1) = G'\), we get the following exact sequences with connecting homomorphisms

\[
0 \to \frac{H^1(G', \mathbb{C}^\times)}{2H^1(G', \mathbb{C}^\times)} \to H^2(G', \mathbb{Z}/2\mathbb{Z}) \to H^2(G', \mathbb{C}^\times)[2] \to 0 \quad (**)
\]

In the next lemma we prove that \(h\) is injective. This proves the proposition. \[\Box\]
Lemma 3.4.6. The right most vertical map $h : H^2(G', \mathbb{C}^\times)[2] \to H^2(M^\times, \mathbb{C}^\times)[2]$ in the above diagram is an isomorphism.

Proof. Consider the short exact sequence which appeared in the proof of proposition 3.4.3 with $\mathbb{Z}/2\mathbb{Z}$ replaced by $\mathbb{C}^\times$,

$$0 \longrightarrow H^1(\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)) \longrightarrow H^2(G', \mathbb{C}^\times) \longrightarrow H^2(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)^\mathbb{Z} \longrightarrow 0.$$ 

This combined with the fact that the second cohomology of a cyclic group with coefficients in $\mathbb{C}^\times$ is zero, implies that

$$H^2(G', \mathbb{C}^\times) = H^1(\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)) = H^1(\mathbb{Z}, \mathbb{F}_{q^2}^\times).$$

Similarly

$$H^2(M^\times, \mathbb{C}^\times) = H^1(2\mathbb{Z}, H^1(\mathbb{F}_{q^2}^\times, \mathbb{C}^\times)) = H^1(2\mathbb{Z}, \mathbb{F}_{q^2}^\times).$$

We need to prove that the restriction map

$$H^1(\mathbb{Z}, \mathbb{F}_{q^2}^\times)[2] \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow H^1(2\mathbb{Z}, \mathbb{F}_{q^2}^\times)[2] \cong \mathbb{Z}/2\mathbb{Z}$$

is injective. For this, consider the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{2}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

The above exact sequence gives rise to the following inflation-restriction exact sequence

$$0 \longrightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_{q^2}^\times) \longrightarrow H^1(\mathbb{Z}, \mathbb{F}_{q^2}^\times) \longrightarrow H^1(2\mathbb{Z}, \mathbb{F}_{q^2}^\times).$$

By the next lemma there is an isomorphism of $\mathbb{F}_{q^2}^\times$ with $\mathbb{F}_{q^2}^\times$ preserving the natural $Gal(\mathbb{F}_{q^2}/\mathbb{F}_q)$ action on these groups. Hence by Hilbert’s Theorem 90 we get that $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_{q^2}^\times) = 0$. So the map

$$H^1(\mathbb{Z}, \mathbb{F}_{q^2}^\times) \to H^1(2\mathbb{Z}, \mathbb{F}_{q^2}^\times)$$

is injective and hence in particular on 2-torsions

$$H^1(\mathbb{Z}, \mathbb{F}_{q^2}^\times)[2] \to H^1(2\mathbb{Z}, \mathbb{F}_{q^2}^\times)[2].$$

This proves that the map $h$ is non-zero and an isomorphism.  \qed
Lemma 3.4.7. There is an isomorphism of $\widehat{\mathbb{F}^\times_{q^d}}$ with $\mathbb{F}^\times_{q^d}$ such that the natural Galois action of $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ on $\widehat{\mathbb{F}^\times_{q^d}}$ becomes the inverse of the natural action of $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ on $\mathbb{F}^\times_{q^d}$ (where by “inverse” of an action of an abelian group $G$ on a module $M$, we mean $g \ast m = (g^{-1})m$).

Proof. Since the $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ operates by $x \mapsto x^q$ on $\mathbb{F}^\times_{q^d}$, the proof of the lemma is clear. □
Chapter 4

A theorem of Mœglin-Waldspurger for covering groups

4.1 Introduction

Let $E$ be a non-Archimedean local field of characteristic zero, $G$ a connected split reductive group defined over $E$ and $G = G(E)$. Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of $G$ and $\mathfrak{g} = \mathfrak{g}(E)$. Let $(\pi, W)$ be an irreducible admissible representation of $G$. A theorem of F. Rodier, in [25], relates the dimension of the space of non-degenerate Whittaker functionals of $\pi$ with respect to a non-degenerate Whittaker datum and coefficients in the character expansion of $\pi$ around the identity. More precisely, Rodier proves that if the residue characteristic of $E$ is large enough and the group $G$ is split then the dimension of the space of non-degenerate Whittaker functionals for $(\pi, W)$ with respect to any Whittaker datum equals the coefficient in the character expansion of $\pi$ at the identity corresponding to an appropriate maximal nilpotent orbit in the Lie algebra $\mathfrak{g}$. Rodier proved his theorem assuming that the residue characteristic of $E$ is large enough, in fact, greater than a constant which depends only on the root datum of $G$. A theorem of C. Mœglin and J.-L. Waldspurger [20] generalizes this theorem of Rodier, in particular proving the theorem of Rodier for the fields $E$ whose residue characteristic is odd. Their version of the theorem does not require $G$ to be split.
The theorem of Mœglin-Waldspurger is a more precise statement about the coefficients appearing in the character expansion around the identity and certain spaces of ‘degenerate’ Whittaker forms. In a recent work of S. Varma [27] this theorem has been proved for fields with even residue characteristic. So the theorem of Mœglin-Waldspurger is true for all connected reductive groups without any restriction on the residue characteristic of the field $E$. We now recall the theorem of Mœglin-Waldspurger. To state the theorem we need to introduce some notation. Let $Y$ be a nilpotent element in $\mathfrak{g}$ and suppose $\varphi : \mathbb{G}_m \rightarrow G$ is a one parameter subgroup satisfying
\begin{equation}
Ad(\varphi(t))Y = t^{-2}Y.
\end{equation}

Associated to such a pair $(Y, \varphi)$ one can define a certain space $W_{(Y, \varphi)}$, called the space of degenerate Whittaker forms of $(\pi, W)$ relative to $(Y, \varphi)$ (see Section 4.4 for the definition). Define $N_{Wh}(\pi)$ to be the set of nilpotent orbits $\mathcal{O}$ of $\mathfrak{g}$ for which there exists an element $Y \in \mathcal{O}$ and a $\varphi$ satisfying (4.1) such that the space $W_{(Y, \varphi)}$ of degenerate Whittaker forms relative to the pair $(Y, \varphi)$ is non-zero.

Recall that the character expansion of $(\pi, W)$ around the identity is a sum $\sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$, where $\mathcal{O}$ varies over the set of nilpotent orbits of $\mathfrak{g}$, $c_{\mathcal{O}} \in \mathbb{C}$ and $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of a suitably chosen measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$. One defines $N_{tr}(\pi)$ to be the set of nilpotent orbits $\mathcal{O}$ of $\mathfrak{g}$ such that the corresponding coefficient $c_{\mathcal{O}}$ in the character expansion of $\pi$ around the identity is non-zero.

We have the standard partial order on the set of nilpotent orbits in $\mathfrak{g}$: $\mathcal{O}_1 \leq \mathcal{O}_2$ if $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$. Let $\text{Max}(N_{Wh}(\pi))$ and $\text{Max}(N_{tr}(\pi))$ denote the set of maximal element in $N_{Wh}(\pi)$ and $N_{tr}(\pi)$ respectively with respect to this partial order. Then the main theorem of Chapter I of [20] is as follows:

**Theorem 4.1.1 (Mœglin-Waldspurger).** Let $G$ be a connected reductive group defined over
4.1. Introduction

E. Let \( \pi \) be an irreducible admissible representation of \( G = G(E) \). Then

\[
\text{Max}(\mathcal{N}_\text{Wh}(\pi)) = \text{Max}(\mathcal{N}_\text{tr}(\pi)).
\]

Moreover, if \( \mathcal{O} \) is an element in either of these sets, then for any \((Y, \varphi)\) as above with \( Y \in \mathcal{O} \) we have

\[
c_\mathcal{O} = \dim \mathcal{W}_{(Y, \varphi)}.
\]

If one considers the case of the pair \((Y, \varphi)\) with \( Y \) a ‘regular’ nilpotent element then the above theorem of Mœglin-Waldspurger specializes to Rodier’s theorem.

In this Chapter, we generalize the theorem of Mœglin-Waldspurger to the setting of a covering group \( \tilde{G} \) of \( G \). Let \( \mu_r \) be the group of \( r \)-th roots of unity in \( \mathbb{C}^\times \). An \( r \)-fold covering group \( \tilde{G} \) of \( G \) is a locally compact topological central extension of \( G \) by \( \mu_r \) giving rise to the following short exact sequence

\[
1 \rightarrow \mu_r \rightarrow \tilde{G} \rightarrow G \rightarrow 1.
\]

The representations of \( \tilde{G} \) on which \( \mu_r \) acts via the natural embedding \( \mu_r \hookrightarrow \mathbb{C}^\times \) are called genuine representations. The definition of the space of degenerate Whittaker forms of a representation of \( G \) involves only unipotent groups. Since the covering \( \tilde{G} \rightarrow G \) splits over any unipotent subgroup of \( G \) in a unique way, see \cite{21}, this makes it possible to define the space of degenerate Whittaker forms for any genuine smooth representation \((\pi, W)\) of \( \tilde{G} \). In particular, it makes sense to talk of the set \( \mathcal{N}_\text{Wh}(\pi) \).

The existence of a character expansion of an admissible genuine representation \((\pi, W)\) of \( \tilde{G} \) has been proved by Wen-Wei Li in \cite[Theorem 4.1.10]{15}. At the identity, the Harish-Chandra-Howe character expansion of an irreducible genuine representation has the same form as the character expansion of an irreducible admissible representation of a linear group and therefore it makes sense to talk of \( \mathcal{N}_\text{tr}(\pi) \). This makes it possible to have an analogue of Theorem \[4.1.1\] in the setting of covering groups. The main aim of this paper is to prove the following.
Theorem 4.1.2. Let $\pi$ be an irreducible admissible genuine representation of $\tilde{G}$. Then

$$\text{Max}(\mathcal{N}_{Wh}(\pi)) = \text{Max}(\mathcal{N}_{tr}(\pi)).$$

Moreover, if $O$ is an element in either of these sets, then for any $(Y, \varphi)$ as above with $Y \in O$ we have

$$c_O = \dim \mathcal{W}_{(Y, \varphi)}.$$

We will use the work of Mœglin-Waldspurger [20], and to accommodate the case of even residue characteristic, we follow Varma [27]. Let us describe some of the ideas involved in the proof. Let $Y$ be a nilpotent element in $g$ and $\varphi$ a one parameter subgroup as above. Let $g_i$ be the eigenspace of weight $i$ under the action of $G_m$ on $g$ via $\text{Ad} \circ \varphi$. One can attach a parabolic subgroup $P$ with unipotent radical $N$ whose Lie algebras are $p := \oplus_{i \geq 0} g_i$ and $n := \oplus_{i > 0} g_i$. The one parameter subgroup $\varphi$ also determines a parabolic subgroup $P^-$ opposite to $P$ with Lie algebra $p^- := \oplus_{i \leq 0} g_i$. For simplicity, assume $g_1 = 0$ for the purpose of the introduction. Then $n = \oplus_{i \geq 2} g_i$, and $\gamma \mapsto \psi(B(Y, \log \gamma))$ defines a character of $N = N(E)$, where $B$ is an $\text{Ad}(G)$-invariant non-degenerate symmetric bilinear form on $g$ and $\psi$ is an additive character of $E$. In this case (i.e., $g_1 = 0$), the space of degenerate Whittaker forms $\mathcal{W}_{(Y, \varphi)}$ is defined to be the twisted Jacquet module of $\pi$ with respect to $(N, \chi)$. In the case where $g_1 \neq 0$, the definition of $\mathcal{W}_{(Y, \varphi)}$ needs to be appropriately modified (see Section 4.4).

On the other hand, to the pair $(Y, \varphi)$ one attaches certain open compact subgroups $G_n$ of $G$ for large $n$ and certain characters $\chi_n$ of $G_n$. One then proves that the covering $\tilde{G} \rightarrow G$ splits over $G_n$ for large $n$, so that $G_n$ can be seen as subgroups of $\tilde{G}$ as well. Let $t := \varphi(\varpi)$ and $\tilde{t}$ be any lift of $t$ in $\tilde{G}$. It turns out that $\tilde{t}^{-n}G_n\tilde{t}^n \cap N$ becomes an “arbitrarily large” subgroup of $N$ and $\tilde{t}^{-n}G_n\tilde{t}^n \cap P^-$ an “arbitrarily small” subgroup of $P^-$, as $n$ becomes large. For large $n$, the characters $\chi_n$ have been so defined that the character $\chi'_n := \chi_n \circ \text{Int}(\tilde{t}^n)$ restricted to $\tilde{t}^{-n}G_n\tilde{t}^n \cap N$ agrees with $\chi$. Using the Harish-Chandra-Howe character expansion one proves that the dimension of $(G_n, \chi_n)$-isotypic component of $W$ is
equal to $c_{\mathcal{O}}$ for large enough $n$, where $\mathcal{O}$ is the nilpotent orbit of $Y$ in $\mathfrak{g}$. Note that the $(G_n, \chi_n)$-isotypic component of $W$ and the $(\tilde{t}^{-n}G_n\tilde{t}^n, \chi_n \circ \text{Int}(\tilde{t}^n))$-isotypic component of $W$ are isomorphic as vector spaces. Finally one proves that there is a natural isomorphism between $(\tilde{t}^{-n}G_n\tilde{t}^n, \chi_n \circ \text{Int}(\tilde{t}^n))$-isotypic component of $W$ and $\mathcal{W}_{(Y, \phi)}$.

Remark 4.1.3. The definition of $\mathcal{W}_{(Y, \phi)}$ (hence that of $\mathcal{N}_{\text{Wh}}(\pi)$) depends on a choice of an additive character $\psi$ of $E$ and a choice of an $\text{Ad}(G)$-invariant non-degenerate symmetric bilinear form $B$ on $\mathfrak{g}$. On the other hand, in the character expansion, the $c_{\mathcal{O}}$’s (hence $\mathcal{N}_{\text{tr}}(\pi)$) depend on $\psi$, $B$, a measure on $\tilde{G}$ and a measure on $\mathfrak{g}$. However by requiring the measures on $\tilde{G}$ and $\mathfrak{g}$ to be compatible via the exponential map $\exp$ one gets rid of the dependency of the $c_{\mathcal{O}}$ on the measures on $\tilde{G}$ and $\mathfrak{g}$. Therefore the $c_{\mathcal{O}}$’s depend only on $\psi$ and $B$. For a more detailed discussion about the dependency on $B$ and $\psi$ on the results here, see Remark 4 in [27].

Remark 4.1.4. One aspect in Varma’s proof for $p = 2$, which does not obviously generalise from the case when $p \neq 2$ is the prescription of the character $\chi_n$ of $G_n$ given in [20], which is due to somewhat bad behaviour of the Campbell-Hausdorff formula in the $p = 2$ case. Using Kirillov theory for compact $p$-adic groups Varma prescribes a $\chi_n$ (although not unique) which will serve our purpose.

Although the methods of the proof of Theorem 4.1.2 are not new and heavily depend on the proofs in the case of linear groups [20, 27], the result is useful in the study of the representation theory of covering groups. We will make use of an application (Theorem 4.6.3) to this result in the next chapter, when we generalize a result of D. Prasad [23] in the setting of covering groups, namely, in the harmonic analysis relating the pairs $(\widetilde{\text{GL}_2}(E), \text{GL}_2(F))$ and $(\widetilde{\text{GL}_2}(E), D_F^\times)$, where $E/F$ is a quadratic extension of non-Archimedian local fields, $\widetilde{\text{GL}_2}(E)$ is a certain two fold cover of $\text{GL}_2(E)$ defined in Section 2.4 and $D_F$ is the quaternion division algebra with center $F$ for suitable embeddings $\text{GL}_2(F) \hookrightarrow \widetilde{\text{GL}_2}(E)$ and $D_F^\times \hookrightarrow \widetilde{\text{GL}_2}(E)$. 
4.2 Subgroups $G_n$ and characters $\chi_n$

In this section, we recall a certain sequence of subgroups $G_n$ of $G$, which form a basis of neighbourhoods at identity and certain characters $\chi_n : G_n \to \mathbb{C}^\times$. Although the objects involved in this section were defined for linear groups in \cite{20, 27}, we will lift them to our covering groups in a suitable way in Section 4.3 and work with these lifts.

Let $\mathfrak{O}_E$ denote the ring of integers in $E$. We fix an additive character $\psi$ of $E$ with conductor $\mathfrak{O}_E$. Fix an $\text{Ad}(G)$-invariant non-degenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to E$. Let $Y$ be a nilpotent element in $\mathfrak{g}$. Choose a one parameter subgroup $\varphi : \mathbb{G}_m \to G$ satisfying

$$\text{Ad}(\varphi(s))Y = s^{-2}Y, \forall s \in \mathbb{G}_m. \quad (4.3)$$

We note that for a given nilpotent element $Y \in \mathfrak{g}$ the existence of $\varphi$ is guaranteed by the theory of $\mathfrak{sl}_2$-triplets but there are examples of $\varphi$ which do not come from $\mathfrak{sl}_2$-triplets.

For $i \in \mathbb{Z}$, define

$$\mathfrak{g}_i = \{ X \in \mathfrak{g} : \text{Ad}(\varphi(s))X = s^iX, \forall s \in \mathbb{G}_m \}.$$ 

Set

$$\mathfrak{n} := \mathfrak{n}^+ := \bigoplus_{i > 0} \mathfrak{g}_i, \mathfrak{n}^- := \bigoplus_{i < 0} \mathfrak{g}_i, \mathfrak{p}^- := \bigoplus_{i \leq 0} \mathfrak{g}_i.$$ 

The parabolic subgroup $\mathfrak{P}^-$ of $G$ normalizing $\mathfrak{n}^-$ has $\mathfrak{p}^-$ as its Lie algebra. Let $N = N^+$ be the unipotent subgroup of $G$ having $\mathfrak{n}$ as the Lie algebra.

Let $G(Y)$ be the centralizer of $Y$ in $G$ and $Y^\#$ the centralizer of $Y$ in $\mathfrak{g}$. The $G$-orbit $\mathcal{O}_Y$ of $Y$ can be identified with $G/G(Y)$ and therefore its tangent space at $Y$ can be identified with $\mathfrak{g}/Y^\#$. Note that

$$Y^\# = \{ X \in \mathfrak{g} : [X, Y] = 0 \}$$

$$= \{ X \in \mathfrak{g} : B([X, Y], Z) = 0, \forall Z \in \mathfrak{g} \}$$

$$= \{ X \in \mathfrak{g} : B(Y, [X, Z]) = 0, \forall Z \in \mathfrak{g} \}.$$
4.2. Subgroups $G_n$ and characters $\chi_n$

The bilinear form $B$ gives rise to a non-degenerate alternating form $B_Y : \mathfrak{g}/Y^\# \times \mathfrak{g}/Y^\# \rightarrow E$ defined by $B_Y(X_1, X_2) = B(Y, [X_1, X_2])$.

Let $L \subset \mathfrak{g}$ be a lattice satisfying the following conditions:

1. $[L, L] \subset L$,
2. $L = \oplus_{i \in \mathbb{Z}} L_i$, where $L_i = L \cap \mathfrak{g}_i$,
3. The lattice $L/L_Y$, where $L_Y = L \cap Y^\#$, is self dual (i.e. $(L/L_Y)^\perp = L/L_Y$) with respect to $B_Y$. (For any vector space $V$ with a non-degenerate bilinear form $B'$ and a lattice $M$ in $V$, $M^\perp := \{X \in V : B'(X, Y) \in \Sigma_E, \forall Y \in V\}$.)

A lattice $L$ satisfying the above properties can be chosen by taking a suitable basis of all $\mathfrak{g}_i$'s, see [20]. Now we summarize a few well known properties of the exponential map, and use them to define subgroups $G_n$ and their Iwahori decompositions.

Lemma 4.2.1. 1. There exists a positive integer $A$ such that the exponential map $\exp$ is defined and injective on $\varpi^A L$, with inverse log.

2. $\exp |_{\varpi^n L}$ is a homeomorphism of $\varpi^n L$ onto its image $G_n := \exp(\varpi^n L)$, which is an open subgroup of $G$ for all $n \geq A$.

3. Set $P_n^- = \exp(\varpi^n L \cap p^-)$ and $N_n = \exp(\varpi^n L \cap n)$. Then we have an Iwahori factorization

$$G_n = P_n^- N_n.$$  

We will be working with a certain character $\chi_n$ of $G_n$, which we recall in the next lemma.

Lemma 4.2.2. For large $n$ there exists a character $\chi_n$ of $G_n$, whose restriction to $\exp((Y^\# \cap \varpi^n L) + \varpi^{n+\text{val}_2} L)$ coincides with $\gamma \mapsto \psi(B(\varpi^{-2n} Y, \log \gamma))$. If $P_n^-$ is as in Lemma 4.2.1, the character $\chi_n$ can be chosen so that

$$\chi_n(p) = 1, \forall p \in P_n^-.$$
For a proof of this lemma and other properties of this character $\chi_n$, see [27, Lemma 5].

**Remark 4.2.3.** If $p \neq 2$, then the map $\gamma \mapsto \psi(B(\varpi^{-2n}Y, \log \gamma))$ itself defines a character of $G_n$ for large $n$ and satisfies the properties stated in Lemma 4.2.2. But when $p = 2$, the number of such characters is greater than one, for more details see [27].

### 4.3 Covering groups

Let $\mu_r := \{z \in \mathbb{C}^\times \mid z^r = 1\}$. Consider an $r$-fold covering $\tilde{G}$ of $G$ which is a locally compact topological central extension by $\mu_r$, giving rise to the following short exact sequence

$$1 \rightarrow \mu_r \rightarrow \tilde{G} \overset{p}{\rightarrow} G \rightarrow 1.$$ 

As $\mu_r$ is central in $\tilde{G}$ for any $x \in G$ and $y \in \tilde{G}$ the element $\tilde{x}yx^{-1}$ does not depend on the choice of $\tilde{x}$ for $\tilde{x} \in p^{-1}(\{x\})$. We may write this element as $xyx^{-1}$.

**Lemma 4.3.1.** 1. The covering $\tilde{G} \overset{p}{\rightarrow} G$ splits over any unipotent subgroup of $G$ in a unique way.

2. For large enough $n$ the covering $\tilde{G} \overset{p}{\rightarrow} G$ splits over $G_n$. In fact, for large $n$, there is a splitting $s$ of $\tilde{G} \overset{p}{\rightarrow} G$ restricted to $\cup_{g \in G} gG_n g^{-1}$ such that $s(\text{hth}^{-1}) = hs(t)h^{-1}$ for all $h \in G$.

**Proof.** 1. This is well known, see [21]. For a simpler proof, in the case when $E$ has characteristic zero, see [16, Section 2.2].

2. Recall that the subgroups $G_n$ form a basis of neighbourhoods of the identity. It is well known that the covering $\tilde{G} \overset{p}{\rightarrow} G$ splits over a neighbourhood of the identity. Therefore for large enough $n$, the covering splits over $G_n$. There is more than one possible splitting for the cover $\tilde{G} \overset{p}{\rightarrow} G$ over $G_n$. If a splitting is fixed, then any other splitting over $G_n$ will differ from the above splitting by a character $G_n \rightarrow \mu_r$.

Fix some $m$ such that the covering $\tilde{G} \overset{p}{\rightarrow} G$ splits over $G_{m} = \exp(\varpi^m L)$. As mentioned
above, any two splittings over the subgroup $G_m$ will differ by a character $G_m \to \mu_r$ and any such character is trivial over

$$G^r_m := \{g^r : g \in G_m\}.$$ 

Hence all the possible splittings over $G_m$ agree on $G^r_m$. The subset $G^r_m$ is a subgroup of $G_m$ as it equals $\exp(r \cdot \pi^m L)$. Let $g, h \in G$. Then

$$(gG_mg^{-1} \cap hG_mh^{-1}) \supset (gG^r_mg^{-1} \cap hG^r_mh^{-1}).$$

This implies that any two splittings of $\tilde{G} \rightarrow G$ restricted to $gG^r_mg^{-1} \cap hG^r_mh^{-1}$, one of which comes from the restriction of a splitting of $\tilde{G} \rightarrow G$ over $gG_mg^{-1}$ and the other of which comes from the restriction of a splitting over $hG мh^{-1}$, are the same.

Now choose $A'$ so large such that $G_n \subset G^r_m$ for $n \geq A'$. We fix the splitting of $G_n$ which comes from that of the restriction to $G^r_m$. This gives us a splitting over $\cup_{g \in G} gG_n g^{-1}$.

Using this splitting we get that an exponential map is defined from a small enough neighbourhood of $g$ to $\tilde{G}$, namely the usual exponential map composed with this splitting, which one can use to define the character expansion of an irreducible admissible genuine representation $(\pi, W)$ of $\tilde{G}$, which has been done by Wen-Wei Li in [15].

**Remark 4.3.2.** If $r$ is co-prime to $p$, then as $G_n$ is a pro-$p$ group and $(r, p) = 1$, there is no non-trivial character from $G_n$ to $\mu_r$. In that situation, there is a unique splitting in the above lemma.

From now onwards, for large enough $n$, we treat $G_n$ not only as a subgroup of $G$ but also as one of $\tilde{G}$, with the above specified splitting. In other words, for the covering group $\tilde{G}$ (as in the linear case) we have a sequence of pairs $(G_n, \chi_n)$ using the splitting specified above which satisfies the properties described in Section 2.
Definition 4.3.3. Let $H \subset G$ be an open subgroup and $s : H \hookrightarrow \tilde{G}$ be a splitting. Then for any $\phi \in C_c^\infty(G)$ with $\text{supp}(\phi) \subset H$, define $\tilde{\phi}_s \in C_c^\infty(\tilde{G})$ as follows:

$$
\tilde{\phi}_s(g) := \begin{cases} 
\phi(g'), & \text{if } g = s(g') \in s(H) \\
0, & \text{if } g \in \tilde{G}\setminus s(H)
\end{cases}
$$

Note that this definition depends upon the choice of splitting. Whenever the splitting is clear in the context or it has been fixed and there is no confusion we write just $\tilde{\phi}$ instead of $\tilde{\phi}_s$ and $H$ for $s(H)$. Recall that the convolution $\phi \ast \phi'$ for $\phi, \phi' \in C_c^\infty(G)$ is defined by

$$
\phi \ast \phi'(x) = \int_G \phi(xy^{-1})\phi'(y) \, dy.
$$

Observe that

$$
\text{supp}(\phi \ast \phi') \subset \text{supp}(\phi) \cdot \text{supp}(\phi'),
$$

which implies the lemma below.

Lemma 4.3.4. Let $H$ be an open subgroup of $G$ such that the covering $\tilde{G} \to G$ has a splitting over $H$, say, $s : H \hookrightarrow \tilde{G}$, satisfying $s(xy) = s(x)s(y)$ whenever $x, y$ are in $H$. If $\phi, \phi' \in C_c^\infty(G)$ are such that $\text{supp}(\phi)$ and $\text{supp}(\phi')$ are contained in $H$, then we have

$$
\tilde{\phi} \ast \tilde{\phi}' = \tilde{\phi} \ast \tilde{\phi}'.
$$

4.4 Degenerate Whittaker forms

In this section we give the definition of degenerate Whittaker forms for a smooth genuine representation $\pi$ of $\tilde{G}$. This is an adaptation of Section I.7 of [20] and Section 5 of [27].

Define $N := \exp(n) = \exp(\oplus_{i \geq 1} g_i), N^2 := \exp(\oplus_{i \geq 2} g_i)$ and $N' = \exp(g_1 \cap Y\#)N^2$. It is easy to see that $N^2, N'$ are normal subgroups of $N$. Let $H$ be the Heisenberg group defined with $g_1/(g_1 \cap Y\#) \times E$ as underlying set using the symplectic form induced by $B_Y$, i.e. for $X, Z \in g_1/(g_1 \cap Y\#)$ and $a, b \in E$,

$$
(X, a)(Y, b) = \left( X + Y, a + b + \frac{1}{2}B_Y(X, Z) \right).
$$

(4.4)
Consider the map $N \to H$ given by

$$\exp(X) \mapsto (\bar{X}, B(Y, X)),$$

where $\bar{X}$ is the image of the $\mathfrak{g}_1$ component of $X$ in $\mathfrak{g}_1/(\mathfrak{g}_1 \cap Y^\#)$. The Campbell-Hausdorff formula implies that the above map is a homomorphism with the following kernel

$$N'' = \{ n \in N' : B(Y, \log n) = 0 \}.$$

Let $\chi : N' \to \mathbb{C}^\times$ be defined by

$$\chi(\gamma) = \psi \circ B(Y, \log \gamma).$$

(4.5)

Note that $\gamma \mapsto B(Y, \log \gamma) \in E \cong \{0\} \times E \subset H$ induces an isomorphism $N'/N'' \cong E$.

We note that the cover $\tilde{G} \twoheadrightarrow G$ splits uniquely over the subgroups $N, N'$ and $N''$. We denote the images of these splittings inside $\tilde{G}$ by the same letters. For a smooth genuine representation $(\pi, W)$ of $\tilde{G}$ we define

$$N^2\chi W = \{ \pi(n)w - \chi(n)w : w \in W, n \in N^2 \}$$

and

$$N'\chi W = \{ \pi(n)w - \chi(n)w : w \in W, n \in N' \}.$$ 

Note that $N$ normalizes $\chi$. Therefore $H = N/N''$ acts on $W/N'\chi W$ in a natural way. This action restricts to $N'/N''$ (the center of $N/N''$) as multiplication by the character $\chi$. Let $S$ be the unique irreducible representation of the Heisenberg group $H$ with central character $\chi$.

**Definition 4.4.1.** Define the space of degenerate Whittaker forms for $(\pi, W)$ associated to $(Y, \varphi)$ to be

$$\mathcal{W} = \mathcal{W}_{(Y, \varphi)} := \text{Hom}_H(S, W/N'\chi W).$$

**Remark 4.4.2.** If $\mathfrak{g}_1 = 0$, then $N = N' = N^2$. In this case, $\mathcal{W} \cong W/N\chi W$ is the $(N, \chi)$-twisted Jacquet functor.
**Definition 4.4.3.** For a smooth representation \((\pi, W)\) of \(\tilde{G}\) define \(N_{\text{Wh}}(\pi)\) to be the set of nilpotent orbits \(O\) of \(\mathfrak{g}\) such that there exists \(Y \in O\) and \(\varphi\) as in Equation 4.3, such that the space of degenerate Whittaker forms for \(\pi\) associated to \((Y, \varphi)\) is non-zero.

As \(\mathfrak{g}_1/(\mathfrak{g}_1 \cap Y^\#)\) is a symplectic vector space and \(L/L_Y\) is self dual, it follows that \(L_H := (L \cap \mathfrak{g}_1)/(L \cap \mathfrak{g}_1 \cap Y^\#)\) is a self dual lattice in the symplectic vector space \(H/Z(H) \cong \mathfrak{g}_1/(\mathfrak{g}_1 \cap Y^\#)\).

Recall the definition of the Heisenberg group \(H\) (see Equation 4.4) and as \(\psi\) is trivial on \(\mathcal{O}_E\), it follows that one can extend the character \(\psi\) of \(E \cong Z(H)\) to a character of the inverse image of \(2L_H\) under \(H \to \mathfrak{g}_1/(\mathfrak{g}_1 \cap Y^\#)\) by defining it to be trivial on \(2L_H \times \{0\} \subset H\). From Lemma 4 in [27], this character can be extended to a character \(\tilde{\chi}\) on the inverse image \(H_0\) of \(L_H\) under the natural map \(H \to \mathfrak{g}_1/(\mathfrak{g}_1 \cap Y^\#)\).

**Remark 4.4.4.** There are one parameter subgroups \(\varphi\) which do not arise from \(\mathfrak{sl}_2\)-triplets. If \(\varphi\) arises from \(\mathfrak{sl}_2\)-triplets, then it is easy to see that \(Y^\# \subset \oplus_{i \leq 0} \mathfrak{g}_i\). In particular, we have \(\mathfrak{g}_1 \cap Y^\# = \{0\}\). As \(\mathfrak{g}_1\) is a symplectic vector space, the Heisenberg group \(H \cong \mathfrak{g}_1 \times E\).

Then, by Chapter 2, Section I.3 of [19], one knows that \(\mathcal{S} = \text{ind}^H_{H_0} \tilde{\chi}\), \(\text{ind}\) denoting the induction with compact support. Since \(H_0\) is an open subgroup of the locally profinite group \(H\), we have the following form of the Frobenius reciprocity law:

\[
\text{Hom}_H(\mathcal{S}, \tau) = \text{Hom}_H(\text{ind}^H_{H_0} \tilde{\chi}, \tau) = \text{Hom}_{H_0}(\tilde{\chi}, \tau |_{H_0})
\]

for any smooth representation \(\tau\) of \(H\). Thus, in the category of representations of \(N\) on which \(N'\) acts via the character \(\chi\), the functor \(\text{Hom}_H(\mathcal{S}, -)\) amounts to taking the \(\tilde{\chi} |_{H_0}\)-isotypic component. Since \(H_0\) is compact modulo the center, this functor is exact. Thus we have

\[
\mathcal{W} = \text{Hom}_H(\mathcal{S}, W/N^\prime_{\chi} W) \cong (W/N_{\chi}^\prime W)^{(H_0, \tilde{\chi})},
\]

where \((W/N_{\chi}^\prime W)^{(H_0, \tilde{\chi})}\) denotes the \((H_0, \tilde{\chi})\)-isotypic component of \(W/N_{\chi}^\prime W\).
Recall that we have defined certain characters $\chi_n$ in Section 4.2, and now we have a character $\tilde{\chi}$. We need to choose them in a compatible way. First we fix a character $\tilde{\chi}$ as above and consider it as a character of $\exp(g_1 \cap L)N'$ in the obvious way (as $\exp(g_1 \cap L)N'$ is the inverse image of $H_0$ under $N \to H$). Let $t := \varphi(\varpi) \in G$. Let $\tilde{t} \in \tilde{G}$ be any lift of $t$ in $\tilde{G}$. Let

$$G'_n = \text{Int}(\tilde{t}^{-n})(G_n), \quad P'_n = \text{Int}(\tilde{t}^{-n})(P'_n) \text{ and } V'_n = \text{Int}(\tilde{t}^{-n})(N_n).$$

It can be easily verified that $V'_n$ contains $\exp(g_1 \cap L)$. We also have $V'_n \subset V'_m$ for large $m, n$ with $n \leq m$. Moreover

$$\exp(g_1 \cap L)N^2 = \bigcup_{n \geq 0} V'_n.$$  

It can also be verified easily that $\tilde{\chi} \circ \text{Int}(\tilde{t}^{-n})$ restricts to a character of $N_n$ that extends the character on $N_n + \text{val}_2 N'_n$ given by $\gamma \mapsto \psi(B(\varpi^{-2n}Y, \log \gamma))$. Now define

$$\chi_n(p \bar{v}) = \tilde{\chi}(\tilde{t}^{-n} p \bar{v}^{-n}), \forall p \in P'_n \text{ and } \forall v \in V'_n.$$  

(4.7) \hfill \text{Lemma 4.4.5 (Lemma 6 in [27]). Let $\chi_n$ be as defined in Equation 4.7. Then $\chi_n$ is a character of $G_n$ and satisfies the properties stated in Lemma 4.2.2.} 

Define a character $\chi'_n$ on $G'_n$ as follows:

$$\chi'_n := \chi_n \circ \text{Int}(\tilde{t}^n).$$

**Remark 4.4.6.** The characters $\chi_n$ have been so defined that for large $n$, $\chi'_n$ agrees with $\chi$ on the intersection of their domains, namely, for large $n$ we have,

$$\chi'_n |_{V'_n} = \tilde{\chi} |_{V'_n}.$$  

In particular, $\chi'_n |_{\exp(L \cap g_1)} = \tilde{\chi} |_{\exp(L \cap g_1)}$. One can also see that $\chi'_n$ and $\chi'_m$ (for large $n, m$) agree on $G'_n \cap G'_m$, because they agree on $V'_n \cap V'_m$ and also on $P'_n \cap P'_m$ (being trivial on it).

Set

$$W_n := \{w \in W \mid \pi(\gamma)w = \chi_n(\gamma)w, \forall \gamma \in G_n\}$$  

(4.8)
and

$$W'_n := \{ w \in W \mid \pi(\gamma)w = \chi'_n(\gamma)w, \forall \gamma \in G'_n \} = \pi(\tilde{t}^{-n})W_n \quad (4.9)$$

For large $m, n$ define the map $I'_{n,m} : W'_n \to W'_m$ by

$$I'_{n,m}(w) = \int_{G'_m} \chi'_m(\gamma^{-1})\pi(\gamma)w \, d\gamma. \quad (4.10)$$

Let $m, n$ be large with $m > n$. Since $\chi'_n$ is trivial on $P'_n \supset P'_m$ and since $G'_m = P'_mV'_m$, for a convenient choice of measures we have

$$I'_{n,m}(w) = \int_{V'_m} \chi'_m(x^{-1})\pi(x)w \, dx$$

$$= \int_{\exp(g_1 \cap L)} \tilde{\chi}^{-1}(\exp X)\pi(\exp X) \int_{N^2 \cap G'_m} \chi(x^{-1})\pi(x)w \, dx \, dX.$$ 

Now using the fact that $\exp(g_1 \cap L)$ lies in $G'_n$ for large $n$ and that it normalizes the character $\chi|_{N^2}$, we get

$$I'_{n,m}(w) = \int_{N^2 \cap G'_m} \chi(x^{-1})\pi(x)w \, dx$$

$$= \int_{N \cap G'_m} \chi(x^{-1})\pi(x)w \, dx.$$ 

From this the following is clear. For large $n, m$ with $m > n$ and suitable choice of measures we have

$$I'_{n,m} = I'_{n+1,m} \circ I'_{n,n+1}. \quad (4.11)$$

For large $n$, the above equation gives that $\ker I'_{n,m} \subset \ker I'_{n,l}$ for $n < m \leq l$. Set $W'_{n,\chi} := \bigcup_{m>n} \ker I'_{n,m}$. Recall that for any unipotent subgroup $U$, a character $\chi : U \to \mathbb{C}^\times$ and $w \in W$, $\int_K \chi(x)^{-1}\pi(x)w \, dx = 0$ for some open compact subgroup $K$ of $U$ if and only if $w \in U_\chi W$, where $U_\chi W$ is the span of $\{ \pi(u)w - \chi(u)w \mid w \in W, u \in U \}$. Thus we have $W'_{n,\chi} \subset N^2\chi W \subset N'_\chi W$, which gives rise to the following natural maps

$$j_n : W'_n/W'_{n,\chi} \to W/N^2\chi W \quad \text{and} \quad j'_n : W'_n/W'_{n,\chi} \to W/N'_\chi W$$

and these maps give the following diagram:

$$\begin{array}{ccc}
W'_n/W'_{n,\chi} & \xrightarrow{j'_n} & W/N'_\chi W \\
\downarrow j_n & \scriptstyle{\exists \text{natural}} & \\
W/N^2\chi W & \end{array} \quad (4.12)$$
By the compatibility between \( \chi'_n \) and \( \tilde{\chi} \), it is easy to see that the image of \( j'_n \) is contained in \( (W/N'_\chi W)^{(H_0,\tilde{\chi})} \). Let \( w \in W \) be such that the image \( \bar{w} \) of \( w \) in \( W/N'_\chi W \) belongs to \( (W/N'_\chi W)^{(H_0,\tilde{\chi})} \). For large \( n \), \( P'_n \) acts trivially on \( w \), as \( (\pi,W) \) is smooth. Since \( G'_n = P'_n V'_n = V'_n P'_n \), the element
\[
\int_{V'_n} \chi'_n(x^{-1})\pi(x)w \, dx
\]
belongs to \( W'_n \). As \( \chi'_n \) and \( \tilde{\chi} \) are compatible, it can be seen that its image in \( W/N'_\chi W \) is \( \bar{w} \). This gives us the following lemma.

**Lemma 4.4.7.** Let \( (Y,\varphi) \) be arbitrary. Then any element of \( (W/N'_\chi W)^{(H_0,\chi)} \) belongs to \( j'_n(W'_n) \) for all sufficiently large \( n \). In particular, if \( W \neq 0 \) then, for large \( n \), \( W_n \) and \( W'_n \) are non-zero.

### 4.5 The main theorem

Now recall that, by the work of Wen-Wei Li [15, Theorem 4.1.10], the Harish-Chandra-Howe character expansion of an irreducible admissible genuine representation of \( \tilde{G} \) at the identity element has an expression of the same form as that of an irreducible admissible representation of a linear group. The proof of the following lemma for a covering group is verbatim that of [20, Proposition I.11], or equivalently, of [27, Proposition 1].

**Proposition 4.5.1.** Let \( \mathcal{W} \) be the space of degenerate Whittaker forms for \( \pi \) with respect to a given \( (Y,\varphi) \). If \( \mathcal{W} \neq 0 \) then there exists a nilpotent orbit \( \mathcal{O} \) in \( \mathcal{N}_u(\pi) \) such that \( \mathcal{O}_Y \leq \mathcal{O} \) (i.e., \( Y \in \bar{\mathcal{O}} \)).

Let the function \( \phi_n : G \to \mathbb{C} \) be defined by
\[
\phi_n(\gamma) = \begin{cases} 
\chi_n(\gamma^{-1}), & \text{if } \gamma \in G_n \\
0, & \text{otherwise.}
\end{cases}
\]

Consider the corresponding function \( \tilde{\phi}_n : \tilde{G} \to \mathbb{C} \) (see Definition 4.3.3). Write the character expansion of \( \pi \) at the identity element as follows:
\[
\Theta_\pi \circ \exp = \sum_{\mathcal{O}} c_\mathcal{O} \hat{\mu}_\mathcal{O}.
\]
Choose \( n \) large enough so that the above expansion is valid over \( G_n \) and then evaluate \( \Theta_\pi \) at the function \( \tilde{\phi}_n \). As \( \pi(\tilde{\phi}_n) \) is a projection from \( W \) to \( W_n \), by definition we get \( \Theta_\pi(\tilde{\phi}_n) = \text{trace} \pi(\tilde{\phi}_n) = \dim W_n \). Now assume that \((Y, \varphi)\) is such that \( O_Y \) is a maximal element in \( \mathcal{N}_{tr}(\pi) \). On the other hand, if we evaluate \( \sum_{\mathcal{O}} c_{\mathcal{O}} \mu_{\mathcal{O}}(\tilde{\phi}_n) \), it turns out that \( \mu_{\mathcal{O}}(\tilde{\phi}_n) \) is zero unless \( \mathcal{O} = O_Y \). In addition, if we fix a \( G \)-invariant measure on \( O_Y \) as in I.8 of \[20\] (for more details about this invariant measure see Section 3 of \[27\]), we get the following lemma.

**Lemma 4.5.2.** ([20, Lemma I.12] and [27, Lemma 7])
Suppose \((Y, \varphi)\) is such that \( O_Y \) is a maximal element of \( \mathcal{N}_{tr}(\pi) \). Then for large \( n \),

\[
\dim W_n = c_{O_Y}.
\]

In particular, the dimension of \( W_n \) is finite and independent of \( n \), for large \( n \).

From Lemma 4.4.7 we know that every vector in \( \mathcal{W} \) is in the image of \( j_n' \) for large \( n \). In particular, if \( W_n \) is finite dimensional, we get that the map \( j_n' \) is surjective. Moreover, we have the following lemma whose proof is verbatim that of Corollary I.14 in \[20\] and Lemma 8 in \[27\] in the case of a linear group.

**Lemma 4.5.3.** Let \((Y, \varphi)\) be such that \( O_Y \) is a maximal element of \( \mathcal{N}_{tr}(\pi) \). Then for large \( n \), the maps \( j_n \) and \( j_n' \) are injections and the image of \( j_n' \) is \( (W/N'\chi W)^{(H_0, \tilde{\chi})} \).

Let \( \phi'_n : G \rightarrow \mathbb{C} \) be defined by

\[
\phi'_n(\gamma) = \begin{cases} 
\chi'_n(\gamma^{-1}), & \text{if } \gamma \in G'_n \\
0, & \text{otherwise.}
\end{cases}
\]

Consider the corresponding function \( \tilde{\phi}'_n : \tilde{G} \rightarrow \mathbb{C} \). Thus, \( \tilde{\phi}'_n = \tilde{\phi}_n \circ \text{Int}(\tilde{t}^n) \).

**Lemma 4.5.4.** Consider a pair \((Y, \varphi)\) such that \( \mathcal{O} = O_Y \) is a maximal element of \( \mathcal{N}_{tr}(\pi) \). Then for large enough \( n \):
4.5. The main theorem

1. Let \( \mathcal{Y}_n \subset G'_{n+1} \cap G(Y) \) be a set of representatives for the \( G'_{n+1} \) double cosets in \( G'_n (G_{n+1} \cap G(Y)) G'_n \). Then for large enough \( n \),

\[
\tilde{\phi}'_n \ast \tilde{\phi}'_{n+1} \ast \tilde{\phi}'_n (g) = \begin{cases} 
\lambda \cdot (\chi'_n)^{-1} (h_1 h_2), & \text{if } g = h_1 y h_2 \text{ with } y \in \mathcal{Y}_n, h_1, h_2 \in G'_n \\
0, & \text{if } g \notin G'_n \mathcal{Y}_n G'_n,
\end{cases}
\]

where \( \lambda = \text{meas}(G'_n \cap G'_{n+1}) \text{meas}(G'_n) \).

2. For large \( n \), \( I'_{n,n+1} \) is injective.

Proof. From part (a) of Lemma 9 in [27], we have

\[
\phi'_n \ast \phi'_{n+1} \ast \phi'_n (g) = \begin{cases} 
\lambda \cdot (\chi'_n)^{-1} (h_1 h_2), & \text{if } g = h_1 y h_2 \text{ with } y \in \mathcal{Y}_n, h_1, h_2 \in G'_n \\
0, & \text{if } g \notin G'_n \mathcal{Y}_n G'_n
\end{cases}
\]

where \( \lambda = \text{meas}(G'_n \cap G'_{n+1}) \text{meas}(G'_n) \). Now part 1 follows from Lemma 4.3.4 which gives that for large \( n \),

\[
\tilde{\phi}'_n \ast \tilde{\phi}'_{n+1} \ast \tilde{\phi}'_n = (\tilde{\phi}'_n \ast \tilde{\phi}'_{n+1} \ast \tilde{\phi}'_n).
\]  \( (4.13) \)

Now we prove part 2. It is enough to show that \( \pi(\tilde{\phi}'_n \ast \tilde{\phi}'_{n+1} \ast \tilde{\phi}'_n) \) acts by a non-zero multiple of identity on \( W'_n \), since that would imply that \( I'_{n+1,n} \circ I'_{n,n+1} \) is a non-zero multiple of the identity on \( \text{W}_n' \). From part 1 we get that \( \tilde{\phi}'_n \ast \tilde{\phi}'_{n+1} \ast \tilde{\phi}'_n \) is a positive linear combination of functions \( \tilde{\phi}'_{n,y} : \gamma \mapsto \tilde{\phi}'_n (\gamma y^{-1}) \), where \( y \in G'_{n+1} \cap G(Y) \) is fixed and \( G(Y) \) is centralizer of \( Y \in G \). Then the lemma follows from the fact that \( \pi(y) \) acts trivially on \( W'_n \) for large \( n \), so that

\[
\pi(\tilde{\phi}'_{n,y})|_{W'_n} = \pi(\tilde{\phi}'_n)\pi(y)|_{W'_n} = \pi(\tilde{\phi}'_n)|_{W'_n}. \quad \square
\]

Theorem 4.5.5. Let \( (\pi, W) \) be an irreducible admissible genuine representation of \( \hat{G} \).

1. The set of maximal elements in \( \mathcal{N}_{\text{tr}}(\pi) \) coincides with the set of maximal elements in \( \mathcal{N}_{\text{Wh}}(\pi) \).

2. Let \( \mathcal{O} \) be a maximal element in \( \mathcal{N}_{\text{tr}}(\pi) \). Then the coefficient \( c_{\mathcal{O}} \) equals the dimension of the space of degenerate Whittaker forms with respect to any pair \( (Y, \varphi) \) such that \( Y \in \mathcal{O} \) and \( \varphi : \mathbb{G}_m \rightarrow G \) satisfies \( \text{Ad}(\varphi(s))Y = \gamma^{-2}Y \) for all \( s \in E^\times \).
Proof. Let $\mathcal{O}$ be a maximal element in $\mathcal{N}_{\text{tr}}(\pi)$. Choose $(Y, \varphi)$ such that $Y \in \mathcal{O}$ and such that $\varphi : \mathbb{G}_m \to \mathbf{G}$ satisfies $\text{Ad}(\varphi(s))Y = s^{-2}Y$. Then, from Lemma 4.5.2, for large $n$ we have

$$\dim W_n = c_\mathcal{O}.$$ 

Therefore $W_n \neq 0$ (resp $W'_n \neq 0$) for large $n$. By Lemma 4.5.3, the map $j'_n$ is injective and maps surjectively onto $(W/N'_X)^{(H_0, \tilde{\chi})}$. But by the second part of Lemma 4.5.4 and Equation 4.11, $I'_{n,m}$ is injective for large $n$ and $m > n$ which implies that $W'_{n,\chi} = \bigcup_{m>n} \ker(I'_{n,m}) = 0$. From Equation 4.3, we have $\mathcal{W} \cong (W/N'_X)^{(H_0, \tilde{\chi})}$. Hence $\dim \mathcal{W} = \dim W'_n = \dim W_n = c_\mathcal{O}$, which proves part 2 of the theorem. In particular, $\mathcal{W} \neq 0$ and hence $\mathcal{O} \in \mathcal{N}_{\text{Wh}}(\pi)$. Now we claim that $\mathcal{O}$ is maximal in $\mathcal{N}_{\text{Wh}}(\pi)$. If not, there is a maximal orbit $\mathcal{O}' \in \mathcal{N}_{\text{Wh}}(\pi)$ such that $\mathcal{O} \subsetneq \mathcal{O}'$. From Proposition 4.5.1, there exists $\mathcal{O}'' \in \mathcal{N}_{\text{tr}}(\pi)$ such that $\mathcal{O}' \leq \mathcal{O}''$. Therefore $\mathcal{O} \subsetneq \mathcal{O}''$ and $\mathcal{O}, \mathcal{O}'' \in \mathcal{N}_{\text{tr}}(\pi)$, a contradiction to the maximality of $\mathcal{O}$ in $\mathcal{N}_{\text{tr}}(\pi)$.

Let $\mathcal{O}$ be a maximal element in $\mathcal{N}_{\text{Wh}}(\pi)$. By Proposition 4.5.1 there exists an element in $\mathcal{O}' \in \mathcal{N}_{\text{tr}}(\pi)$ such that $\mathcal{O} \leq \mathcal{O}'$. We may assume $\mathcal{O}'$ to be maximal in $\mathcal{N}_{\text{tr}}(\pi)$. Then by the result in the above paragraph, $\mathcal{O}'$ is a maximal element in $\mathcal{N}_{\text{Wh}}(\pi)$. But $\mathcal{O}$ is also maximal in $\mathcal{N}_{\text{Wh}}(\pi)$. Hence $\mathcal{O} = \mathcal{O}'$. This proves that $\mathcal{O}$ is a maximal element in $\mathcal{N}_{\text{tr}}(\pi)$ too. 

**4.6 An application: a theorem of Casselman-Prasad**

Let $\mathbf{G}$ be a connected reductive quasi-split group and $\mathbf{N}$ a maximal unipotent subgroup of $\mathbf{G}$. Let $\chi$ be a non-degenerate character of $N = \mathbf{N}(E)$. Then the pair $(N, \chi)$ is called a non-degenerate Whittaker datum. It is well known that there is a bijection between the set of regular nilpotent $\text{Ad}(\mathbf{G})$-orbits in $\mathfrak{g}$ and the set of conjugacy classes of non-degenerate Whittaker data. We state this bijection explicitly in the case where $\mathbf{G} = \text{SL}_2$ and $\mathfrak{g} = \mathfrak{sl}_2$. For any non-zero nilpotent orbit there is a lower triangular nilpotent matrix
4.6. An application: a theorem of Casselman-Prasad

Let $\tau$ be an irreducible admissible genuine representation of $\widetilde{\text{SL}_2}(E)$. Recall that for an irreducible admissible genuine representation $\tau$ of $\widetilde{\text{SL}_2}(E)$ the character distribution $\Theta_\tau$ is a smooth function on the set of regular semisimple elements. The Harish-Chandra-Howe
character expansion of $\Theta_\tau$ in a neighbourhood of identity is given as follows:

$$\Theta_\tau \circ \exp = c_0(\tau) + \sum_{a \in \mathbb{E} / \mathbb{E} \times 2} c_a(\tau) \cdot \hat{\mu}_{N_a}$$

where $c_0(\tau), c_a(\tau)$ are constants and $\hat{\mu}_{N_a}$ is the Fourier transform of a suitably chosen $\text{Ad}(\text{SL}_2(E))$-invariant measure on $N_a$. It follows from Theorem 2.6.2 and (2.10) that for any non-trivial additive character $\psi'$ of $N$, the dimension of the space of $(N, \psi')$-Whittaker functionals for $\tau$ is at most one. Therefore, from the theorem of Rodier, as extended in Theorem 4.1.2 each $c_a(\tau)$ is either 1 or 0 depending on whether $\tau$ admits a non-zero Whittaker functional corresponding to the non-degenerate Whittaker datum $(N, \psi_a)$ or not.

**Remark 4.6.1.** For $g \in \tilde{G}$, there exists a semisimple element $g_s \in \tilde{G}$ such that $g$ belongs to any conjugation invariant neighbourhood of $g_s$ in $\tilde{G}$.

Let $\tau_1$ and $\tau_2$ be two irreducible admissible genuine representations of $\text{SL}_2(E)$. As $\{\pm 1\}$ is the center of $\text{SL}_2(E)$ and $\Theta_{\tau_1}, \Theta_{\tau_2}$ are given by smooth functions at regular semisimple points, by Remark 4.6.1 it follows that if $\Theta_{\tau_1} - \Theta_{\tau_2}$ is a smooth function in a neighbourhood of the identity then it is smooth function on the whole of $\text{SL}_2(E)$ provided both $\tau_1, \tau_2$ have the same central characters.

For any non-trivial additive character $\psi'$ of $E$, let us assume that $\tau_1$ admits a non-zero Whittaker functional for $(N, \psi')$ if and only if $\tau_2$ does so too. Under this assumption $c_a(\tau_1) = c_a(\tau_2)$ for all $a \in E^\times / E^\times 2$. Then we have the following result.

**Theorem 4.6.2.** Let $\tau_1, \tau_2$ be two irreducible admissible genuine representations of $\text{SL}_2(E)$ with the same central characters. For a non-trivial additive character $\psi'$ of $E$ assume that $\tau_1$ admits a non-zero Whittaker functional with respect to $(N, \psi')$ if and only if $\tau_2$ admits a non-zero Whittaker functional with respect to $(N, \psi')$. Then $\Theta_{\tau_1} - \Theta_{\tau_2}$ is constant in a neighbourhood of identity and hence smooth on $\text{SL}_2(E)$.

Using Theorem 4.6.2, we prove an extension of a theorem of Casselman-Prasad [23, Theorem 5.2].
**Theorem 4.6.3.** Let \( \psi \) be a non-trivial character of \( E \). Let \( \pi_1 \) and \( \pi_2 \) be two irreducible admissible genuine representations of \( \widetilde{GL_2(E)} \) with the same central characters such that \((\pi_1)_{N,\psi} \cong (\pi_2)_{N,\psi}\) as \( \hat{Z} \)-modules. Then \( \Theta_{\pi_1} - \Theta_{\pi_2} \) is a smooth function on \( GL_2(E) \).

**Proof.** We already know that \( \Theta_{\pi_1} \) and \( \Theta_{\pi_2} \) are smooth on the set of regular semisimple elements, so is \( \Theta_{\pi_1} - \Theta_{\pi_2} \). To prove the smoothness on whole of \( \widetilde{GL_2(E)} \), we need to prove the smoothness at every point in \( \tilde{Z} \). As \( \tilde{Z} \) is not the center, the smoothness at the identity is not enough to imply the smoothness at every point in \( \tilde{Z} \). Note that \( \tilde{Z} \) is the center of \( \widetilde{GL_2(E)}_+ \) and \( \widetilde{GL_2(E)}_+ \) is an open and normal subgroup of \( \widetilde{GL_2(E)} \) of index \([E^\times : E^{\times 2}]\), cf. Section 2.5. Let \( \tilde{z} \in \tilde{Z} \). The character expansion of \( \pi_1 \) (respectively, of \( \pi_2 \)) at \( \tilde{z} \) is the same as that of \( \pi_1 \big|_{\widetilde{GL_2(E)}_+} \) (respectively, of \( \pi_2 \big|_{\widetilde{GL_2(E)}_+} \)) at \( \tilde{z} \). Let \( \mu \) be a genuine character in \( \Omega(\omega_{\pi_1}) = \Omega(\omega_{\pi_2}) \). Choose irreducible admissible genuine representations \( \tau_1 \) and \( \tau_2 \) of \( \widetilde{SL_2(E)} \) which are compatible with \( \mu \) (see Section 2.5), and such that

\[
\pi_1 = \text{ind}_{\widetilde{GL_2(E)}_+}^{\widetilde{GL_2(E)}} (\mu \tau_1) \quad \text{and} \quad \pi_2 = \text{ind}_{\widetilde{GL_2(E)}_+}^{\widetilde{GL_2(E)}} (\mu \tau_2). \tag{4.14}
\]

By Equation 2.9

\[
\pi_1 \big|_{\widetilde{GL_2(E)}_+} = \bigoplus_{a \in E^\times/E^{\times 2}} (\mu \tau_1)_a \quad \text{and} \quad \pi_2 \big|_{\widetilde{GL_2(E)}_+} = \bigoplus_{a \in E^\times/E^{\times 2}} (\mu \tau_2)_a, \tag{4.15}
\]

where we abuse notation to let \( a \) denote the matrix \( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \). Let \( \Theta_{\rho,g} \) denote the character expansion of an irreducible admissible representation \( \rho \) in a neighbourhood of the point \( g \), then

\[
\Theta_{\pi_1, \tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \Theta_{(\mu \tau_1)_a, \tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \mu^a(\tilde{z}) \Theta_{\tau_1^a,1}
\]

and

\[
\Theta_{\pi_2, \tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \Theta_{(\mu \tau_2)_a, \tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \mu^a(\tilde{z}) \Theta_{\tau_2^a,1},
\]

where \( \mu^a \) is the character of \( \tilde{Z} \) given by \( \mu^a(\tilde{z}) = (a, z) \mu(\tilde{z}) \) where \( z = p(\tilde{z}) \). The smoothness of \( \Theta_{\pi_1} - \Theta_{\pi_2} \) will follow if we prove the smoothness of \( \Theta_{\tau_1^a} - \Theta_{\tau_2^a} \) for all \( a \in E^\times/E^{\times 2} \). For any non-trivial character \( \psi' \) of \( E \), we note that \( \tau_1 \) (resp. \( \tau_2 \)) admits a non-zero \( \psi' \)-Whittaker
function if and only if $\tau_2^a$ (resp. $\tau_2^a$) admits a non-zero $\psi'_a$-Whittaker functional. By Theorem 4.6.2, the smoothness of $\Theta_{\tau_1} - \Theta_{\tau_2}$ is equivalent to the smoothness of $\Theta_{\tau_1} - \Theta_{\tau_2}$. From the expression 4.15 $\mu^a \in (\pi_1)_{N,\psi}$ (respectively, $\mu^a \in (\pi_2)_{N,\psi}$) if and only if $\tau_1^a$ (respectively, $\tau_2^a$) admits a $\psi$-Whittaker functional. It can also be seen easily that $\tau_1$ (respectively, $\tau_2$) admits a $\psi$-Whittaker functional if and only if $\tau_1^a$ (respectively $\tau_2^a$) admits $\psi_a$-Whittaker functional.

It follows that $(\pi_1)_{N,\psi} = (\pi_2)_{N,\psi}$ is equivalent to the following: for all non-trivial characters $\psi'$ of $E$, $\tau_1$ has a $\psi'$-Whittaker functional if and only if $\tau_2$ have a $\psi'$-Whittaker functional.

Now from Theorem 4.6.2 $\Theta_{\tau_1} - \Theta_{\tau_2}$ is smooth on $\widetilde{\text{SL}_2(E)}$. □

**Corollary 4.6.4.** Let $\pi_1, \pi_2$ be two irreducible admissible genuine representations of $\widetilde{\text{GL}_2(E)}$ with the same central character such that $(\pi_1)_{N,\psi} \cong (\pi_2)_{N,\psi}$ as $\tilde{\mathbb{Z}}$-modules. Let $H$ be a subgroup of $\widetilde{\text{GL}_2(E)}$ that is compact modulo center. Then there exist finite dimensional representations $\sigma_1, \sigma_2$ of $H$ such that

$$\pi_1|_H \oplus \sigma_1 \cong \pi_2|_H \oplus \sigma_2.$$ 

In other words, this corollary says that the virtual representation $(\pi_1 - \pi_2)|_H$ is finite dimensional and hence the multiplicity of an irreducible representation of $H$ in $(\pi_1 - \pi_2)|_H$ will be finite.
Chapter 5

Restriction from $\widetilde{GL_2(E)}$ to $GL_2(F)$
and $D_F^\times$

5.1 Introduction

Let $F$ be a non-Archimedian local field of characteristic zero and let $E$ be a quadratic extension of $F$. The problem of decomposing a representation of $GL_2(E)$ restricted to $GL_2(F)$ was considered and solved by D. Prasad in [23], proving a multiplicity one theorem, and giving an explicit classification of representations $\pi_1$ of $GL_2(E)$ and $\pi_2$ of $GL_2(F)$ such that there exists a non-zero $GL_2(F)$ invariant linear form:

$$l : \pi_1 \otimes \pi_2 \to \mathbb{C}.$$ 

This problem is closely related to a similar branching law from $GL_2(E)$ to $D_F^\times$, where $D_F$ is the unique quaternion division algebra which is central over $F$, and $D_F^\times \hookrightarrow GL_2(E)$. We recall that the embedding $D_F^\times \hookrightarrow GL_2(E)$ is given by fixing an isomorphism $D_F \otimes E \cong M_2(E)$, and by the Skolem-Noether theorem, such an embedding of $D_F^\times$ inside $GL_2(E)$ is unique upto conjugation by elements of $GL_2(E)$. Henceforth, we fix one such embedding of $D_F^\times$ inside $GL_2(E)$. The restriction problems for the pair $(GL_2(E), GL_2(F))$ and $(GL_2(E), D_F^\times)$ are related by a certain dichotomy. More precisely, the following result was proved in [23]:
Theorem 5.1.1 (D. Prasad). Let \( \pi_1 \) and \( \pi_2 \) be irreducible admissible infinite dimensional representations of \( \text{GL}_2(E) \) and \( \text{GL}(2,F) \) respectively such that the central character of \( \pi_1 \) restricted to center of \( \text{GL}_2(F) \) is the same as the central character of \( \pi_2 \). Then

1. For a principal series representation \( \pi_2 \) of \( \text{GL}_2(F) \), we have
   \[
   \dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) = 1.
   \]

2. For a discrete series representation \( \pi_2 \) of \( \text{GL}_2(F) \), let \( \pi'_2 \) be the finite dimensional representation of \( D_F^\times \) associated to \( \pi_2 \) by the Jacquet-Langlands correspondence, then
   \[
   \dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) + \dim \text{Hom}_{D_F^\times}(\pi_1, \pi'_2) = 1.
   \]

In this chapter, we will study the analogous problem in the metaplectic setting. More precisely, instead of considering \( \text{GL}_2(E) \) we will consider the group \( \widetilde{\text{GL}}_2(E)_{C^\times} \) which is a topological central extension of \( \text{GL}_2(E) \) by \( C^\times \), which is obtained from the two fold topological central extension \( \widetilde{\text{GL}}_2(E) \) that has been defined in Section 2.4 explicitly. In Chapter 3, we have proved that the covering \( \widetilde{\text{GL}}_2(E)_{C^\times} \) of \( \text{GL}_2(E) \) splits when restricted to \( \text{GL}_2(F) \) or \( D_F^\times \). Recall that the splittings over \( \text{GL}_2(F) \) and \( D_F^\times \) are not unique. As there is more than one splitting in each case, to study the problem of decomposing a representation of \( \widetilde{\text{GL}}_2(E)_{C^\times} \) restricted to \( \text{GL}_2(F) \) and \( D_F^\times \), we will fix one splitting of each of the subgroups \( \text{GL}_2(F) \) and \( D_F^\times \), related to each other as in the following working hypothesis formulated by D. Prasad.

Working Hypothesis 5.1.2. Let \( L \) be a quadratic extension of \( F \). The sets of splittings
are principal homogeneous spaces over the Pontrjagin dual of $F^\times$. We work under the hypothesis that there is a natural identification between these two sets of splittings in such a way that for any quadratic extension $L$ of $F$, any two embeddings of $L^\times$ in $\widetilde{GL_2(E)}_{\mathbb{C}^\times}$ as in the following diagrams are conjugate in $\widetilde{GL_2(E)}_{\mathbb{C}^\times}$.

Here $L^\times \hookrightarrow \text{GL}_2(F)$ (respectively, $L^\times \hookrightarrow D_F^\times$) are obtained by identifying a suitable maximal torus of $\text{GL}_2(F)$ (respectively, $D_F^\times$ viewed as an algebraic group) with $\text{Res}_{L/F}\mathbb{G}_m$.

Let $B(E), A(E)$ and $N(E)$ be the Borel subgroup, maximal torus and maximal unipotent subgroup of $\text{GL}_2(E)$ consisting of all upper triangular matrices, diagonal matrices and upper triangular unipotent matrices respectively. Let $B(F), A(F)$ and $N(F)$ denote the corresponding subgroups of $\text{GL}_2(F)$. Let $Z$ be the center of $\text{GL}_2(E)$ and $\tilde{Z}$ the inverse image of $Z$ in $\widetilde{GL_2(E)}$. Note that $\tilde{Z}$ is an abelian subgroup of $\widetilde{GL_2(E)}$ but is not the center.
of $\widetilde{\text{GL}}_2(E)$; the center of $\widetilde{\text{GL}}_2(E)$ is $\tilde{Z}^2$, the inverse image of $Z^2 := \{ z^2 \mid z \in \mathbb{Z} \}$.

Let $\psi$ be a non-trivial additive character of $E$. Let $\pi$ be an irreducible admissible genuine representation of $\widetilde{\text{GL}}_2(E)$ and recall that the $\psi$-twisted Jacquet module $\pi_{N,\psi}$ is a $\tilde{Z}$-module. Moreover $\pi_{N,\psi}$ is a $\tilde{Z}$-submodule of $\Omega(\omega_\pi)$.

The main theorem proved in this chapter is the following:

**Theorem 5.1.3.** Let $\pi_1$ be an irreducible admissible genuine representation of $\widetilde{\text{GL}_2}(E)$ and let $\pi_2$ be an infinite dimensional irreducible admissible representation of $\text{GL}_2(F)$. Assume that the central characters $\omega_{\pi_1}$ of $\pi_1$ and $\omega_{\pi_2}$ of $\pi_2$ agree on $E^\times \cap F^\times$. Fix a non-trivial additive character $\psi$ of $E$ such that $\psi|_F = 1$. Let $Q = (\pi_1)_{N(E)}$ be the Jacquet module of $\pi_1$. Then

1. Let $\pi_2 = \text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi)$ be a principal series representation of $\text{GL}_2(F)$. Assume that $\text{Hom}_{A(F)}(Q, \chi \cdot \delta^{1/2}) = 0$. Then

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) = \dim \text{Hom}_{Z(F)}((\pi_1)_{N,\psi}, \omega_{\pi_2}).
\]

2. Let $\pi_1 = \text{Ind}_{B(E)}^{\text{GL}_2(E)}(\tilde{\tau})$ be a principal series representation of $\widetilde{\text{GL}}_2(E)$ and $\pi_2$ a discrete series representation of $\text{GL}_2(F)$. Let $\pi'_2$ be the finite dimensional representation of $D_F^\times$ associated to $\pi_2$ by the Jacquet-Langlands correspondence. Assume that $\text{Hom}_{\text{GL}_2(F)}\left(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\tilde{\tau}), \pi_2\right) = 0$. Then

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) + \dim \text{Hom}_{D_F^\times}(\pi_1, \pi'_2) = [E^\times : F^\times E^\times].
\]

3. Let $\pi_1$ an irreducible admissible genuine representation of $\widetilde{\text{GL}}_2(E)$ and $\pi_2$ a supercuspidal representation of $\text{GL}_2(F)$. Let $\pi'_1$ be a genuine representation of $\widetilde{\text{GL}}_2(E)$ which has the same central character as that of $\pi_1$ and as a $\tilde{Z}$-module $\pi_1 \otimes (\pi'_1)|_{N,\psi} = \Omega(\omega_{\pi_1})$. Let $\pi'_2$ be the finite dimensional representation of $D_F^\times$ associated to $\pi_2$ by the Jacquet-Langlands correspondence. Then

\[
\dim \text{Hom}_{\text{GL}_2(F)}(\pi_1 \oplus \pi'_1, \pi_2) + \dim \text{Hom}_{D_F^\times}(\pi_1 \oplus \pi'_1, \pi'_2) = [E^\times : F^\times E^\times].
\]
5.2 Part 1 of Theorem 5.1.3

Let \( \pi_2 = \text{Ind}_{B(F)}^{GL_2(F)}(\chi) \) be a principal series representation of \( GL_2(F) \) where \( \chi \) is a character of \( A(F) \). By Frobenius reciprocity [11, Theorem 2.28], we get

\[
\text{Hom}_{GL_2(F)}(\pi_1, \pi_2) = \text{Hom}_{GL_2(F)}(\pi_1, \text{Ind}_{B(F)}^{GL_2(F)}(\chi)) = \text{Hom}_{A(F)}((\pi_1)_{N(F)}, \chi \cdot \delta^{1/2})
\]

where \((\pi_1)_{N(F)}\) is the Jacquet module of \( \pi_1 \) with respect to \( N(F) \). Now depending on whether \( \pi_1 \) is a supercuspidal representation or not, we consider them separately.

First we consider the case when \( \pi_1 \) is a supercuspidal representation of \( \widetilde{GL_2(E)} \). Then one knows that the functions in the Kirillov model have compact support in \( E^\times \) and one has \( \pi_1 \cong S(F^\times, (\pi_1)_{N, \psi}) \), see Theorem 2.7.1. Now using Proposition 2.7.2 we get the following:

\[
\text{Hom}_{GL_2(F)}(\pi_1, \pi_2) = \text{Hom}_{A(F)}((\pi_1)_{N(F)}, \chi \cdot \delta^{1/2}) = \text{Hom}_{A(F)}(S(E^\times, (\pi_1)_{N, \psi})_{N(F)}, \chi \cdot \delta^{1/2}) = \text{Hom}_{A(F)}(S(F^\times, (\pi_1)_{N, \psi}), \chi \cdot \delta^{1/2})
\]

From the Kirillov model of \( \pi_1 \), it follows that the action of \( A(F) \) on \( S(F^\times, (\pi_1)_{N, \psi}) \) is given by

\[
\left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \cdot \xi = \left( \begin{array}{cc} d & 0 \\ 0 & d \end{array} \right) \cdot (\xi(ad^{-1}x)),
\]

for all \( a, d, x \in F^\times \) and \( \xi \in S(F^\times, (\pi_1)_{N, \psi}) \). From this explicit action of \( A(F) \) on \( S(F^\times, (\pi_1)_{N, \psi}) \) it can be checked that as an \( A(F) \)-module \( S(F^\times, (\pi_1)_{N, \psi}) \cong \text{ind}_{Z(F)}^{A(F)}(\pi_1)_{N, \psi} \). Using Frobenius reciprocity [11, Proposition 2.29], we get the following:

\[
\text{Hom}_{GL_2(F)}(\pi_1, \pi_2) = \text{Hom}_{A(F)}(\text{ind}_{Z(F)}^{A(F)}(\pi_1)_{N, \psi}, \chi \cdot \delta^{1/2}) = \text{Hom}_{Z(F)}((\pi_1)_{N, \psi}, (\chi \cdot \delta^{1/2})|_{Z(F)}) = \text{Hom}_{Z(F)}((\pi_1)_{N, \psi}, \omega_{\pi_2})
\]

Now we consider the case when \( \pi_1 \) is not a supercuspidal representation of \( \widetilde{GL_2(E)} \). Then from Equation 2.13 we get the following short exact sequence of \( A(F) \)-modules

\[
0 \rightarrow S(F^\times, (\pi_1)_{N, \psi}) \rightarrow (\pi_1)_{N(F)} \rightarrow Q \rightarrow 0.
\]
Now applying the functor $\text{Hom}_{A(F)}(-, \chi \delta^{1/2})$, we get the following long exact sequence

$$
0 \rightarrow \text{Hom}_{A(F)}(Q, \chi \delta^{1/2}) \rightarrow \text{Hom}_{A(F)}((\pi_1)_{N(F)}, \chi \delta^{1/2}) \rightarrow \text{Hom}_{A(F)}(S(F^\times, (\pi_1)_{N,\psi}), \chi \delta^{1/2}) \rightarrow \text{Ext}^1_{A(F)}(Q, \chi \delta^{1/2}) \rightarrow \cdots
$$

**Lemma 5.2.1.** $\text{Hom}_{A(F)}(Q, \chi \delta^{1/2}) = 0$ if and only if $\text{Ext}^1_{A(F)}(Q, \chi \delta^{1/2}) = 0$.

*Proof.* The space $Q$ is finite dimensional and completely reducible. So it is enough to prove the lemma for one dimensional representation, i.e., for characters of $A(F)$. Moreover, one can regard these representations as representation of $F^\times$ (after tensoring by a suitable character of $A(F)$ so that it descends to a representation of $A(F)/Z(F) \cong F^\times$). Then our lemma follows from the following lemma due to D. Prasad: $\square$

**Lemma 5.2.2.** If $\chi_1$ and $\chi_2$ are two characters of $F^\times$, then

$$
\dim \text{Hom}_{F^\times}(\chi_1, \chi_2) = \dim \text{Ext}^1_{F^\times}(\chi_1, \chi_2).
$$

*Proof.* Since $F^\times \cong O^\times \times \mathbb{Z}$ where $O$ is the ring of integers in $F$ and $O^\times$ is compact, $\text{Ext}^i_{F^\times}(\chi_1, \chi_2) = H^i(Z, \text{Hom}_{O^\times}(\chi_1, \chi_2))$. If $\text{Hom}_{O^\times}(\chi_1, \chi_2) = 0$, then the lemma is obvious. Hence suppose that $\text{Hom}_{O^\times}(\chi_1, \chi_2) \neq 0$. Then $\text{Hom}_{O^\times}(\chi_1, \chi_2)$ is certain one dimensional vector space with an action of $\mathbb{Z}$. If the action of $\mathbb{Z}$ on $\text{Hom}_{O^\times}(\chi_1, \chi_2)$ is non-trivial then $H^i(Z, \text{Hom}_{O^\times}(\chi_1, \chi_2)) = 0$ for all $i \geq 0$. Whereas if the action of $\mathbb{Z}$ on $\text{Hom}_{O^\times}(\chi_1, \chi_2)$ is trivial, then $H^0(Z, Z) \cong H^1(Z, Z) \cong \mathbb{Z}$. $\square$

We have made an assumption that $\text{Hom}_{A(F)}(Q, \chi \delta^{1/2}) = 0$ and hence by the lemma above $\text{Ext}^1_{A(F)}(Q, \chi \delta^{1/2}) = 0$. So in this case

$$
\text{Hom}_{A(F)}((\pi_1)_{N,\psi}, \chi \delta^{1/2}) \cong \text{Hom}_{A(F)}(S(F^\times, (\pi_1)_{N,\psi}), \chi \delta^{1/2}) = \text{Hom}_{Z(F)}((\pi_1)_{N,\psi}, \omega_{\pi_2}).
$$

Hence

$$
\dim \text{Hom}_{GL_2(F)}(\pi_1, \pi_2) = \dim \text{Hom}_{Z(F)}((\pi_1)_{N,\psi}, \omega_{\pi_2}).
$$
Remark 5.2.3. As $Q$ is a finite dimensional representation of $\widetilde{A}(E)$, only finitely many characters of $A(F)$ appear in $Q$. For a given $\pi_1$ there are only finitely many characters $\chi$ such that $\text{Hom}_{A(F)}(Q, \chi \cdot \delta^{1/2}) \neq 0$. We are leaving out at most $2[E^\times : E^\times_2]$ many principal series representations $\pi_2$ for a given $\pi_1$. Note that $2[E^\times : E^\times_2]$ is the maximum possible dimension of $Q$, i.e. the case of a principal series representation $\pi_1$.

5.3 Part 2 of Theorem 5.1.3

In this section, we consider the case when $\pi_1$ is a principal series representation of $\widetilde{\text{GL}}_2(E)$ and $\pi_2$ a discrete series representation of $\text{GL}_2(F)$.

Let $\pi_1 = \text{Ind}_{\overline{\text{B}(E)}}^{\widetilde{\text{GL}}_2(E)}(\tilde{\tau})$, where $(\tilde{\tau}, V)$ is a genuine irreducible representation of $\tilde{A} = A(E)$. Now as in [23], we use Mackey theory to understand its restriction to $\text{GL}_2(F)$. We have $\overline{\text{GL}}_2(E)/\overline{\text{B}(E)} \cong \mathbb{P}^1_F$ and this has two orbits under the left action of $\text{GL}_2(F)$. One of the orbits is closed, and naturally identified with $\mathbb{P}^1_F \cong \text{GL}_2(F)/B(F)$. The other orbit is open, and can be identified with $\mathbb{P}^1_F - \mathbb{P}^1_F \cong \text{GL}_2(F)/E^\times$. By Mackey theory, we get the following exact sequence of $\text{GL}_2(F)$-modules:

$$0 \to \text{ind}_{E^\times}^{\text{GL}_2(F)}(\tilde{\tau}|_{E^\times}) \to \pi_1 \to \text{Ind}_{B(F)}^{\text{GL}_2(F)}(\tilde{\tau}|_{B(F)} \delta^{1/2}) \to 0, \quad (5.1)$$

where $\tilde{\tau}|_{E^\times}$ is the representation of $E^\times$ obtained from the embedding $E^\times \hookrightarrow \tilde{A}$ which comes from conjugating the embedding $E^\times \hookrightarrow \text{GL}_2(F) \hookrightarrow \overline{\text{GL}}_2(E)$ by an element in $\overline{\text{SL}}_2(E)$. We now identify $E^\times$ with its image inside $\tilde{A}$ which is given by $x \mapsto \left( \begin{array}{cc} x & 0 \\ 0 & \bar{x} \end{array} \right)$, $\epsilon(x)$ where $\bar{x}$ is the non-trivial $\text{Gal}(E/F)$-conjugate of $x$ and $\epsilon(x) \in \{\pm 1\}$. Conjugation by an element in $\overline{\text{SL}}_2(E)$ ensures that this embedding of $E^\times$ in $\tilde{A}$ restricted to $F^\times$ is the one given by $f \mapsto (f, 1)$ for all $f \in F^\times$. Now let $\pi_2$ be any irreducible admissible representation of $\text{GL}_2(F)$. By applying the functor $\text{Hom}_{\text{GL}_2(F)}(-, \pi_2)$ to the short exact sequence (5.1), we
get the following long exact sequence:

\[
\begin{array}{cccccccc}
0 & \to & \text{Hom}_{\text{GL}_2(F)}(\tau|_{B(F)}\delta^{1/2}), \pi_2) & \to & \text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) \\
& & \to & \text{Hom}_{\text{GL}_2(F)}(\tau|_{E^\times}), \pi_2) & \to & \text{Ext}_{\text{GL}_2(F)}^1(\tau|_{B(F)}\delta^{1/2}), \pi_2) \\
& & \to & \cdots & \end{array}
\]

(5.2)

From [22, Corollary 5.9] we know that

\[
\text{Hom}_{\text{GL}_2(F)}(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi, \delta^{1/2}), \pi_2) = 0
\]

\[
\Downarrow
\]

\[
\text{Ext}_{\text{GL}_2(F)}^1(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi, \delta^{1/2}), \pi_2) = 0.
\]

Then from the exactness of (5.2), it follows that

\[
\text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) = 0
\]

\[
\Downarrow
\]

\[
\text{Hom}_{\text{GL}_2(F)}(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\tau|_{B(F)}\delta^{1/2}), \pi_2) = 0 \text{ and } \text{Hom}_{\text{GL}_2(F)}(\text{ind}_{E^\times}^{\text{GL}_2(F)}(\tau'|_{E^\times}), \pi_2) = 0.
\]

Note that the representation \(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\tau|_{B(F)})\) consists of finitely many principal series of \(\text{GL}_2(F)\). We have made the assumption that \(\text{Hom}_{\text{GL}_2(F)}(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\tau|_{B(F)}), \pi_2) = 0\), it follows that

\[
\text{Ext}_{\text{GL}_2(F)}^1(\text{Ind}_{B(F)}^{\text{GL}_2(F)}(\tau, \delta^{1/2}), \pi_2) = 0.
\]

This gives

\[
\text{Hom}_{\text{GL}_2(F)}(\pi_1, \pi_2) \cong \text{Hom}_{\text{GL}_2(F)}(\text{ind}_{E^\times}^{\text{GL}_2(F)}(\tau'|_{E^\times}), \pi_2)
\]

\[
\cong \text{Hom}_{E^\times}(\tau'|_{E^\times}, \pi_2|_{E^\times})
\]

The following lemma describes \(\tau'|_{E^\times}\).

**Lemma 5.3.1.** If we identify \(E^\times\) with its image \(\left\{ \left( \begin{array}{cc} x & 0 \\ 0 & \bar{x} \end{array} \right), \epsilon(x) \mid x \in E^\times \right\} \) inside \(\tilde{A}\) as above then the subgroup \(E^\times \cdot \tilde{A}^2\) inside \(\tilde{A}\) is a maximal abelian subgroup. Moreover, \(\tau'|_{E^\times}\) contains all the characters of \(E^\times\) which are same as \(\omega_\tilde{\tau}|_{E^\times^2}\) when restricted to \(E^\times^2\), where \(\omega_\tilde{\tau}\) is the central character of \(\tilde{\tau}\).
5.3. Part 2 of Theorem 5.1.3

Proof. From the explicit cocycle description and the non-degeneracy of quadratic Hilbert symbol, it is easy to verify that $E^\times \cdot \tilde{A}^2$ is a maximal abelian subgroup of $\tilde{A}$. The rest of the proof is similar to that of Lemma 2.4.2. 

As $\pi_2$ is a discrete series representation, it is not always true (unlike what happens in case of a principal series representation) that any character of $E^\times$, whose restriction to $F^\times$ is the same as the central character of $\pi_2$, appears in $\pi_2$. Let $\pi'_2$ be the finite dimensional representation of $D_F^\times$ associated to $\pi_2$ by the Jacquet-Langlands correspondence. Considering the left action of $D_F^\times$ on $\mathbb{P}_1^E \cong \widetilde{\text{GL}_2(E)/B(E)}$ induced by $D_F^\times \hookrightarrow \widetilde{\text{GL}_2(E)}$ it is easy to verify that $\mathbb{P}_1^E \cong D_F^\times/E^\times$. Then by Mackey theory, the principal series representation $\pi_1$ when restricted to $D_F^\times$, becomes isomorphic to $\text{ind}_{E^\times}^{D_F^\times} (\tilde{\tau}'|E^\times)$.

$$\text{Hom}_{D_F^\times} [\pi_1, \pi_2'] \cong \text{Hom}_{D_F^\times} [\text{ind}_{E^\times}^{D_F^\times} (\tilde{\tau}'|E^\times), \pi_2'] \cong \text{Hom}_{E^\times} (\tilde{\tau}'|E^\times, \pi_2'|E^\times)$$

In order to prove

$$\dim \text{Hom}_{\text{GL}_2(F)} [\pi_1, \pi_2] + \dim \text{Hom}_{D_F^\times} [\pi_1, \pi_2'] = [E^\times : F^\times E^{\times 2}]$$  \hspace{1cm} (5.3)$$

we shall prove

$$\dim \text{Hom}_{E^\times} (\tilde{\tau}'|E^\times, \pi_2|E^\times) + \dim \text{Hom}_{E^\times} (\tilde{\tau}'|E^\times, \pi_2'|E^\times) = [E^\times : F^\times E^{\times 2}].$$  \hspace{1cm} (5.4)$$

By Remark 2.9 in [23], a character of $E^\times$ whose restriction to $F^\times$ is the same as the central character of $\pi_2$ appears either in $\pi_2$ with multiplicity one or in $\pi'_2$ with multiplicity one, and exactly one of the two possibilities hold. Note that we are assuming that the two embeddings of $E^\times$, one via $\text{GL}_2(F)$ and other via $D_F^\times$ are conjugate in $\widetilde{\text{GL}_2(E)}$. Then the left hand side of Equation (5.4) is the same as the number of characters of $E^\times$ appearing in $(\tilde{\tau}, V)$ which upon restriction to $F^\times$ coincide with the central character of $\pi_2$, which equals $\dim \text{Hom}_{F^\times} (\tilde{\tau}|F^\times, \omega|F^\times)$. We are reduced to the following lemma.

Lemma 5.3.2. Let $(\tilde{\tau}, V)$ be an irreducible genuine representation of $\tilde{A}$ and, let $\chi$ be a character of $Z(F) = F^\times$ such that $\chi|_{E^\times \cap F^\times} = \tilde{\tau}|_{E^\times \cap F^\times}$. Then

$$\dim \text{Hom}_{F^\times} (\tilde{\tau}, \chi) = [E^\times : F^\times E^{\times 2}].$$
Proof. Note that $E^2 \cap F^x = Z^2 \cap F^x$. From Lemma 2.4.4, $\tilde{\tau}|Z \cong \Omega(\omega_{\pi_1})$. If a character $\mu \in \Omega(\omega_{\pi_1})$ is specified on $F^x$ then it is specified on $F^x E^2$. Therefore the number of characters in $\Omega(\omega_{\pi_1})$ which agree with $\chi$ when restricted to $F^x$ is equal to $[E^x : F^x E^2]$. □

5.4 Part 3 of Theorem 5.1.3

Let $\pi_1$ be an irreducible admissible genuine representation of $\widetilde{GL}_2(E)$. We take another admissible genuine representation $\pi_1'$ having the same central character as that of $\pi_1$ and satisfying $(\pi_1)_{N,\psi} \oplus (\pi_1')_{N,\psi} \cong \Omega(\omega_{\pi_1})$ as $\tilde{Z}$-modules. From Proposition 2.6.4 if $\pi_1$ is a principal series representation then we can take $\pi_1' = 0$. We will see in Theorem 6.5.1 that if $\pi_1$ is not a principal series representation then $(\pi_1)_{N,\psi}$ is a proper $\tilde{Z}$-submodule of $\Omega(\omega_{\pi_1})$ forcing $\pi_1' \neq 0$. In particular, if $\pi_1$ is one of the Jordan-Hölder factors of a reducible principal series representation then one can take $\pi_1'$ to be the other Jordan-Hölder factor of the principal series representation. For a supercuspidal representation $\pi_1$ we do not have any obvious choice for $\pi_1'$, and this issue will be taken up in the next chapter.

Let $\pi_2$ be a supercuspidal representation of $GL_2(F)$. To prove Theorem 5.1.3 in this case, we use character theory and deduce the result by using the result of restriction of a principal series representation of $\widetilde{GL}_2(E)$ which has already been proved in Section 5.3. We can assume, if necessary after twisting by a character of $F^x$, that $\pi_2$ is minimal representation. The representation $\pi_2$ is called minimal if the conductor of $\pi_2$ is less than or equal to the conductor of $\pi_2 \otimes \chi$ for any character $\chi$ of $F^x$. Then by a theorem of Kutzko [13], we can take $\pi_2$ to be $\text{ind}_K^{GL_2(F)}(W_2)$, where $W_2$ is a representation of a maximal compact modulo center subgroup $K$ of $GL_2(F)$. By Frobenius reciprocity,

$$\text{Hom}_{GL_2(F)}(\pi_1 \oplus \pi_1', \pi_2) = \text{Hom}_{GL_2(F)}\left(\pi_1 \oplus \pi_1', \text{ind}_K^{GL_2(F)}(W_2)\right)$$
$$= \text{Hom}_K((\pi_1 \oplus \pi_1')|_K, W_2).$$

To prove Theorem 5.1.3 it suffices to prove that:

$$\dim \text{Hom}_K((\pi_1 \oplus \pi_1')|_K, W_2) + \dim \text{Hom}_{D_K}[\pi_1 \oplus \pi_1', \pi_2'] = [E^x : F^x E^2].$$
5.4. Part 3 of Theorem 5.1.3

For any (virtual) representation \( \pi \) of \( \widetilde{\text{GL}}_2(E) \), let \( m(\pi, W_2) = \dim \text{Hom}_K[\pi|_K, W_2] \) and \( m(\pi, \pi'_2) = \dim \text{Hom}_{D^\times_F}[\pi, \pi'_2] \). With these notations we will prove:

\[
m(\pi_1 \oplus \pi'_1, W_2) + m(\pi_1 \oplus \pi'_1, \pi'_2) = [E^\times : F^\times E^\times 2].
\]

(5.5)

Let \( Ps \) be an irreducible principal series representation of \( \widetilde{\text{GL}}_2(E) \) whose central character \( \omega_{Ps} \) is same as the central character \( \omega_{\pi_1} \) of \( \pi_1 \) (it is clear that there exists one such). By Proposition 2.6.4, we know that \((Ps)_{N,\psi} \cong \Omega(\omega_{Ps})\) as \( \tilde{Z} \)-module. On the other hand, the representation \( \pi'_1 \) has been chosen in such a way that \((\pi_1)_{N,\psi} \oplus (\pi'_1)_{N,\psi} = \Omega(\omega_{\pi_1})\) as \( \tilde{Z} \)-module. Then, as a \( \tilde{Z} \)-module we have

\[
(\pi_1 \oplus \pi'_1)_{N,\psi} = (\pi_1)_{N,\psi} \oplus (\pi'_1)_{N,\psi} = \Omega(\omega_{\pi_1}) = \Omega(\omega_{Ps}) = (Ps)_{N,\psi}.
\]

We have already proved in Section 5.3 that

\[
m(Ps, W_2) + m(Ps, \pi'_2) = [E^\times : F^\times E^\times 2].
\]

In order to prove Equation 5.5, we prove

\[
m(\pi_1 \oplus \pi'_1 - Ps, W_2) + m(\pi_1 \oplus \pi'_1 - Ps, \pi'_2) = 0.
\]

(5.6)

The relation in Equation 5.6 follows form the following theorem:

**Theorem 5.4.1.** Let \( \Pi_1, \Pi_2 \) be two genuine representations of \( \widetilde{\text{GL}}_2(E) \) of finite length with a central character such that \((\Pi_1)_{N,\psi} \cong (\Pi_2)_{N,\psi}\) as \( \tilde{Z} \)-modules for a non-trivial additive character \( \psi \) of \( E \). Let \( \pi_2 \) be an irreducible supercuspidal representation of \( \text{GL}_2(F) \) such that the central characters \( \omega_{\Pi_1} \) of \( \Pi_1 \) and \( \omega_{\pi_2} \) of \( \pi_2 \) agree on \( F^\times \cap E^\times 2 \). Let \( \pi'_2 \) be the finite dimensional representation of \( D^\times_F \) associated to \( \pi_2 \) by the Jacquet-Langlands correspondence. Then

\[
m(\Pi_1 - \Pi_2, \pi_2) + m(\Pi_1 - \Pi_2, \pi'_2) = 0.
\]

We will use character theory to prove this relation following very closely. First of all, by Theorem 4.6.3, \( \Theta_{\Pi_1-\Pi_2} \) is given by smooth function on \( \text{GL}_2(E) \). Now we recall the Weyl integration formula for \( \text{GL}_2(F) \).
5.4.1 Weyl integration formula

**Lemma 5.4.2.** [9, Formula 7.2.2] For a smooth and compactly supported function \( f \) on \( \text{GL}_2(F) \) we have

\[
\int_{\text{GL}_2(F)} f(y)\,dy = \sum_{E_i} \int_{E_i} \Delta(x) \left( \frac{1}{2} \int_{E_i\setminus\text{GL}_2(F)} f(\bar{g}^{-1}x \bar{g})\,d\bar{g} \right) \,dx
\]  

(5.7)

where the \( E_i \)'s are representatives for the distinct conjugacy classes of maximal tori in \( \text{GL}_2(F) \) and

\[
\Delta(x) = \left\| \frac{(x_1 - x_2)^2}{x_1x_2} \right\|_F
\]

where \( x_1 \) and \( x_2 \) are the eigenvalues of \( x \).

We will use this formula to integrate the function \( f(x) = \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{W_2}(x) \) on \( \mathcal{K} \) which is extended to \( \text{GL}_2(F) \) by setting it to be zero outside \( \mathcal{K} \). In addition, we also need the following result of Harish-Chandra, cf. [23, Proposition 4.3.2].

**Lemma 5.4.3** (Harish-Chandra). Let \( F(g) = (gv, v) \) be a matrix coefficient of a supercuspidal representation \( \pi \) of a reductive \( p \)-adic group \( G \) with center \( Z \). Then the orbital integrals of \( F \) at regular non-elliptic elements vanish. Moreover, the orbital integral of \( F \) at a regular elliptic element \( x \) contained in a torus \( T \) is given by the formula

\[
\int_{T \setminus G} F(gx\bar{g})d\bar{g} = \frac{(v, v) \cdot \Theta_\pi(x)}{d(\pi) \cdot \text{vol}(T/Z)},
\]

(5.8)

where \( d(\pi) \) denotes the formal degree of the representation \( \pi \).

Since \( \pi_2 \) is obtained by induction from \( W_2 \), a matrix coefficient of \( W_2 \) (extended to \( \text{GL}_2(F) \) by setting it to be zero outside \( \mathcal{K} \)) is also a matrix coefficient of \( \pi_2 \). It follows that

1. for the choice of Haar measure on \( \text{GL}_2(F)/F^\times \) giving \( \mathcal{K}/F^\times \) measure 1, we have

\[
\dim W_2 = d(\pi_2),
\]

2. for a separable quadratic field extension \( E_i \) of \( F \) and a regular elliptic element \( x \) of \( \text{GL}_2(E) \) which generates \( E_i \), and for the above Haar measure \( d\bar{g} \),

\[
\int_{E_i^\times \setminus \text{GL}_2(F)} \Theta_{W_2}(\bar{g}^{-1}x \bar{g})d\bar{g} = \frac{\Theta_{\pi_2}(x)}{\text{vol}(E_i^\times /F^\times)}.
\]

(5.9)
5.4.2 Completion of the proof of Theorem \(5.1.3\)

We recall the following important observation from Section \(5.4.1\) and Theorem \(4.6.3\):

1. the virtual representation \((\Pi_1 - \Pi_2)|_K\) is finite dimensional,

2. \(\Theta_{\mathcal{W}_2}\) is also a matrix coefficient of \(\pi_2\) (extended to \(\text{GL}_2(F)\) by zero outside \(\mathcal{K}\)),

3. there is Haar measure on \(\text{GL}_2(F)/F^\times\) giving \(\text{vol}(\mathcal{K}/F^\times) = 1\) such that the Equation \(5.9\) is satisfied.

4. the orbital integral in Equation \(5.8\) vanishes if \(T\) is maximal split torus.

Let \(E_i\)'s be the quadratic extensions of \(F\). Then these observations together with Lemma \(5.4.3\) imply the following

\[
m(\Pi_1 - \Pi_2, W_2) = \frac{1}{\text{vol}(\mathcal{K}/F^\times)} \int_{\mathcal{K}/F^\times} \Theta_{\Pi_1 - \Pi_2} \cdot \Theta_{W_2}(x) \, dx
\]

\[
= \frac{1}{\text{vol}(\mathcal{K}/F^\times)} \int_{\text{GL}_2(F)/F^\times} \Theta_{\Pi_1 - \Pi_2} \cdot \Theta_{W_2}(x) \, dx
\]

\[
= \frac{1}{\text{vol}(\mathcal{K}/F^\times)} \sum_{E_i} \int_{E_i^\times/F^\times} \Delta(x) \left[ \frac{1}{2} \int_{E_i^\times \setminus \text{GL}_2(F)} \Theta_{\Pi_1 - \Pi_2} \cdot \Theta_{W_2}(\bar{g}^{-1}x\bar{g}) \, dg \right] \, dx
\]

\[
= \sum_{E_i} \frac{1}{2\text{vol}(E_i^\times/F^\times)} \int_{E_i^\times/F^\times} (\Delta \cdot \Theta_{\Pi_1 - \Pi_2} \cdot \Theta_{\pi_2})(x) \, dx.
\]

Similarly, we have the equality

\[
m(\Pi_1 - \Pi_2, \pi_2') = \sum_{E_i} \frac{1}{2\text{vol}(E_i^\times/F^\times)} \int_{E_i^\times/F^\times} (\Delta \cdot \Theta_{\Pi_1 - \Pi_2} \cdot \Theta_{\pi_2'})(x) \, dx.
\]

Note that \(E_i\)'s correspond to quadratic extensions of \(F\) and the embeddings of \(\text{GL}_2(F)\) and \(D_F^\times\) have been fixed so that the working hypothesis (as stated in the introduction of this chapter) is satisfied, i.e. the embeddings of the \(E_i\)'s in \(\text{GL}_2(F)\) and in \(D_F^\times\) are conjugate in \(\text{GL}_2(E)\). Then the value of \(\Theta_{\Pi_1 - \Pi_2}(x)\) for \(x \in E_i\), does not depend on the inclusion of \(E_i\)
inside $\widetilde{\text{GL}_2(E)}$, i.e. on whether inclusion is via $\text{GL}_2(F)$ or via $D_F^\times$. Now using the relation

$$\Theta_{\pi_2}(x) = -\Theta_{\pi'_2}(x)$$

on regular elliptic elements $x$ [9, Proposition 15.5], we conclude the following, which proves the Equation 5.6

$$m(\Pi_1 - \Pi_2, W_2) + m(\Pi_1 - \Pi_2, \pi'_2) = 0.$$
Chapter 6

Some consequences of Waldspurger’s theorem

6.1 Introduction

Let $E$ be a non-Archimedean local field of characteristic zero and $\psi$ a non-trivial character of $E$. Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be two admissible genuine representations of $\widehat{\text{GL}_2(E)}$ with the same central character. By abuse of notation, we say that $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have “complementary Whittaker model” if $(\tilde{\pi}_1)_{N,\psi} \oplus (\tilde{\pi}_2)_{N,\psi} = \Omega(\omega_{\tilde{\pi}_1})$ as $\tilde{Z}$-module for a non-trivial character $\psi$ of $E$. It can be easily seen that this notion does not depend on the choice of the character $\psi$ of $E$. By another abuse of notation, we say that a representation (of either of $\widehat{\text{GL}_2(E)}$ or of $\widehat{\text{SL}_2(E)}$) admits a $\psi$-Whittaker model if it admits a non-zero $\psi$-Whittaker functional.

Let $\tilde{\pi}$ be an irreducible admissible genuine supercuspidal representation of $\widehat{\text{GL}_2(E)}$. In this chapter, we wish to construct another representation $\tilde{\pi}'$ of $\widehat{\text{GL}_2(E)}$ with a central character such that

1. the central characters of $\tilde{\pi}$ and $\tilde{\pi}'$ are same, and

2. $\tilde{\pi}$ and $\tilde{\pi}'$ have “complementary Whittaker models”.

By Proposition 2.6.4 if $\tilde{\pi}$ is an irreducible principal series representation of $\tilde{GL}_2(E)$ then $\tilde{\pi}' = 0$. We will prove in Proposition 6.5.1 that if $\tilde{\pi}$ is an irreducible discrete series representation then $\tilde{\pi}_{N,\psi}$ is a proper $\tilde{Z}$-submodule of $\Omega(\omega_{\tilde{\pi}})$ and hence $\tilde{\pi}' \neq 0$. If $\tilde{\pi}$ is a Jordan-Hölder factor of a reducible principal series then $\tilde{\pi}'$ can be taken to be the other Jordan-Hölder factor. If $\tilde{\pi}$ is an irreducible supercuspidal representation then there is no obvious choice for $\tilde{\pi}'$. The main question which we take up in this chapter is the following.

**Question 6.1.1.** Let $\tilde{\pi}$ is an irreducible admissible genuine supercuspidal representation of $\tilde{GL}_2(E)$ and $\psi$ a non-trivial additive character of $E$. Is there a ‘natural’ choice of a genuine admissible representation of finite length $\tilde{\pi}'$ with a central character as that of $\tilde{\pi}$ (not necessarily irreducible) such that $\tilde{\pi}_{N,\psi} \oplus \tilde{\pi}'_{N,\psi} \cong \Omega(\omega_{\tilde{\pi}})$ as $\tilde{Z}$-modules?

We are able to answer this question only for a certain class of representations $\tilde{\pi}$ which we will describe in Section 6.5. We are not able to describe a ‘natural’ choice of $\tilde{\pi}'$ for all supercuspidal representations; it is not clear if the inability to do so is a reflection on us, or if there is a more fundamental reason for this inability. Recall from Section 2.5 that $\tilde{\pi} = \text{ind}_{\tilde{GL}_2(E)}^{\tilde{GL}_2(E)}(\mu\tau)$, where $\tau$ is an irreducible admissible genuine representation of $\tilde{SL}_2(E)$ and $\mu$ is a genuine character of $\tilde{Z}$ which is compatible with $\tau$. Further, from Equation 2.9 it is clear that Question 6.1.1 is equivalent to the following question.

**Question 6.1.2.** Let $\tau$ be an irreducible admissible genuine representation of $\tilde{SL}_2(E)$ and $\psi$ a non-trivial additive character of $E$. Is there a ‘natural’ choice of an genuine admissible representation of finite length $\tau'$ with a central character same as that of $\tau$ (non necessarily irreducible) such that $\tau$ admits a non-zero $\psi$-Whittaker functional if and only if $\tau'$ does not admit a non-zero $\psi$-Whittaker functional?

**Remark 6.1.3.** Write the Waldspurger involution on $\tilde{SL}_2(E)$ as $\tau \leftrightarrow \tau_W$. It does not fix the isomorphism classes of any discrete series representation of $\tilde{SL}_2(E)$ (see next Section). Then $\tau$ admits a non-zero $\psi$-Whittaker model if and only if $\tau_W$ does not admit a non-zero $\psi$-Whittaker model, and the central character of $\tau_W$ is opposite to that of $\tau$. On the other hand, in Question 6.1.1 we require the same central character for $\tau$ and $\tau'$. 
Remark 6.1.4. We note the following facts about the space of $\psi$-Whittaker functionals for $\tilde{\pi}$ and $\tau$, which follow easily from Theorem 2.6.2, Equation 2.9 and 2.10.

1. The dimension of the space of $\psi$-Whittaker functionals for an irreducible admissible genuine representation $\tau$ of $\widetilde{SL}_2(E)$ is at most one dimensional.

2. There exists $a \in E^\times$ such that $\tau^a$ has a non-zero $\psi$-Whittaker functional.

3. $\tilde{\pi}$ has a non-trivial $(\mu^a, \psi)$-Whittaker functional if and only if $\tau^a$ has a non-trivial $\psi$-Whittaker functional.

Another question which we take up in this chapter is the question of restriction of an irreducible admissible genuine representation $\tilde{\pi}$ of $\widetilde{GL}_2(E)$ to the subgroup $\widetilde{SL}_2(E)$. We use the identification in Equation (2.10) to calculate the multiplicity of a representation of $\widetilde{SL}_2(E)$ in an irreducible admissible genuine representation of $\widetilde{GL}_2(E)$. We prove that the multiplicity may be greater than one, in fact one of 1, 2 or 4. We make use of theta correspondence and the Waldspurger involution [31] to prove the results in this chapter.

6.2 $\theta$-correspondence and the Waldspurger involution

In this section, we recall some results of Waldspurger from [31], related to the $\theta$-correspondence between $\widetilde{SL}_2(E)$ and $\text{PGL}_2(E)$ and that between $\widetilde{SL}_2(E)$ and $PD^\times$, where $D$ is the unique quaternion division algebra over $E$. We will use these results repeatedly.

For any non-trivial additive character $\psi$ of $E$ one has the Weil index $\gamma(\psi)$ which is an eighth root of unity. For $a \in E^\times$, let $\psi_a$ be the additive character of $E$ defined by $\psi_a(x) = \psi(ax)$. For $a, b \in E^\times$, the Weil index satisfies the following property:

$$\gamma(\psi_a)\gamma(\psi_b) = (a, b)\gamma(\psi_{ab})\gamma(\psi).$$

(6.1)

Set $\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$ and define $\chi_\psi : \mathbb{Z} \rightarrow \mathbb{C}^\times$ by

$$\chi_\psi(a, \epsilon) = \epsilon \cdot \gamma(a, \psi).$$

(6.2)
By Equation 6.1, $\chi_\psi$ is a genuine character of $\tilde{Z}$.

**Definition 6.2.1.** Let $\tau$ be an irreducible admissible genuine representation of $\tilde{\text{SL}_2}(E)$. We define the central sign $z_\psi(\tau)$ of $\tau$ by

$$z_\psi(\tau) = \omega_\tau(-1)/\chi_\psi(-1) \in \{\pm 1\},$$

where $\omega_\tau$ denotes the central character of $\tau$ and $-1$ denotes any element of $\tilde{Z}$ lying over $-1 \in Z$. Note that the quotient above does not depend on the choice of $\tilde{Z}$.

It follows from the definition of the central sign that

$$z_\psi(\tau^x) = z_\psi(\tau)\chi_x(-1) = z_\psi(\tau)(x, -1)$$

for any $x \in E^\times$, where $\chi_x(z) := (x, z)$ for $z \in E^\times$.

**Definition 6.2.2.** Let $\tau_1$ and $\tau_2$ be two irreducible admissible genuine representations of $\tilde{\text{SL}_2}(E)$. We say that $\tau_1$ and $\tau_2$ have opposite central characters if $z_\psi(\tau_1) = -z_\psi(\tau_2)$.

Now fix a non-trivial additive character $\psi$ of $E$. With respect to this choice of $\psi$, one has a $\theta$-correspondence between the isomorphism classes of irreducible admissible genuine representations of $\tilde{\text{SL}_2}(E)$ and the isomorphism classes of irreducible admissible representations of $\text{PGL}_2(E)$

$$\text{Irrep}(\tilde{\text{SL}_2}(E)) \xrightarrow{\theta(-, \psi)} \text{Irrep}(\text{PGL}_2(E))$$

as well as between irreducible admissible genuine representations of $\tilde{\text{SL}_2}(E)$ and irreducible admissible representations of $\text{PD}^\times$

$$\text{Irrep}(\tilde{\text{SL}_2}(E)) \xrightarrow{\theta(-, \psi)} \text{Irrep}(\text{PD}^\times).$$

Though this correspondence $\tau \mapsto \theta(\tau, \psi)$ depends on the choice of $\psi$, it will be abbreviated to $\tau \mapsto \theta(\tau)$ as $\psi$ has been fixed. The $\theta$-correspondence between $\tilde{\text{SL}_2}(E)$ and $\text{PGL}_2(E)$ gives a one to one mapping from the set of isomorphism classes of irreducible admissible
genuine representations of $\widetilde{\text{SL}_2(E)}$ which have a $\psi$-Whittaker model onto the set of isomorphism classes of all irreducible admissible representations of $\text{PGL}_2(E)$. Similarly, the $\theta$-correspondence between $\widetilde{\text{SL}_2(E)}$ and $PD^\times$ gives a one to one mapping from the set of isomorphism classes of irreducible admissible genuine representations of $\widetilde{\text{SL}_2(E)}$ which do not have a $\psi$-Whittaker model onto the set of isomorphism classes of all irreducible representations of $PD^\times$. Thus the $\theta$-correspondence defines a bijection (which depends on the choice of $\psi$):

$$\text{Irrep}(\widetilde{\text{SL}_2(E)}) \longleftrightarrow \text{Irrep}(\text{PGL}_2(E)) \sqcup \text{Irrep}(PD^\times).$$

(6.5)

Now we can describe the Waldspurger involution \[31\] $W : \text{Irrep}(\widetilde{\text{SL}_2(E)}) \to \text{Irrep}(\widetilde{\text{SL}_2(E)})$ which is defined using

1. the $\theta$-correspondence from $\widetilde{\text{SL}_2(E)}$ to $\text{PGL}_2(E)$,
2. the $\theta$-correspondence from $\widetilde{\text{SL}_2(E)}$ to $PD^\times$ and
3. the Jacquet-Langlands correspondence viewed as a map from $\text{Irr}(\text{PGL}_2(E)) \sqcup \text{Irr}(PD^\times)$ to itself.

The Waldspurger involution is the unique map $W : \text{Irrep}(\widetilde{\text{SL}_2(E)}) \to \text{Irrep}(\widetilde{\text{SL}_2(E)})$ that makes the following diagram commutative:

This involution is defined on the set of all representations of $\widetilde{\text{SL}_2(E)}$, and its fixed points are precisely the irreducible admissible genuine representations which are not discrete series representations. Denote this involution by $\tau \mapsto \tau_W$. This involution is independent of the character $\psi$ chosen to define it \[31\]. For $\tau \in \text{Irrep}(\widetilde{\text{SL}_2(E)})$, we say that $\tau$ has a non-zero $\theta$ lift to $\text{PGL}_2(E)$ (respectively, $PD^\times$) if $\theta(\tau) \neq 0$ in $\text{PGL}_2(E)$ (respectively, $PD^\times$). For $\pi \in \text{Irrep}(\text{PGL}_2(E))$, let $\epsilon(\pi)$ denote the value at $\frac{1}{2}$ of standard $\epsilon$-factor, i.e. $\epsilon(\pi, \frac{1}{2}, \psi)$. 
Theorem 6.2.3 (Waldspurger [31]). Let $\tau$ be an irreducible admissible genuine representation of $\widetilde{SL}_2(E)$. Let $\psi$ be a non-trivial additive character of $E$. Then

1. $\tau$ has a $\psi$-Whittaker model if and only if $\tau_W$ does not have a $\psi$-Whittaker model. Moreover, $\tau$ and $\tau_W$ have opposite central characters.

2. $\tau$ has a non-zero $\theta$ lift to $PGL_2(E)$ with respect to $\psi$ if and only if one of the following equivalent conditions is satisfied:

   (a) $z_\psi(\tau) = \epsilon(\theta(\tau, \psi))$.

   (b) $\tau$ has a $\psi$-Whittaker model.

   (c) $\tau_W$ does not have a $\psi$-Whittaker model.

3. $\tau$ has a non-zero $\theta$ lift to $PD^\times$ with respect to $\psi$ if and only if one of the following equivalent conditions is satisfied:

   (a) $z_\psi(\tau) = -\epsilon(\theta(\tau, \psi))$.

   (b) $\tau$ does not have $\psi$-Whittaker model.

   (c) $\tau_W$ has $\psi$-Whittaker model.

Theorem 6.2.4 (Waldspurger [31]). Let $\tau$ be an irreducible admissible genuine representation of $\widetilde{SL}_2(E)$ and $\psi$ a non-trivial additive character of $E$. Then

1. For $a \in E^\times$, let $\chi_a$ be the quadratic character of $E^\times$ defined by $\chi_a(x) = (a, x)$. Both the representations $\tau$ and $\tau^a$ of $\widetilde{SL}_2(E)$ have a non-zero $\theta$ lift (with respect to the character $\psi$) either to $PGL_2(E)$ or to $PD^\times$ if and only if

   $$\epsilon(\theta(\tau) \otimes \chi_a) = \chi_a(-1)\epsilon(\theta(\tau)),$$

   and if this condition is satisfied,

   $$\theta(\tau^a) \cong \theta(\tau) \otimes \chi_a.$$
6.3 Higher multiplicity in restriction from $\widetilde{\text{GL}}_2(E)$ to $\widetilde{\text{SL}}_2(E)$

If $\epsilon(\theta(\tau) \otimes \chi_a) = -\chi_a(-1)\epsilon(\theta(\tau))$, then $\theta(\tau)$ is a representation of $\text{PGL}_2(E)$ if and only if $\theta(\tau^a)$ is a representation of $\text{PD}^\times$; and

$$\theta(\tau^a) = \theta(\tau)^J \otimes \chi_a.$$ 

2. For $a \in E^\times$, let $\psi_a$ be the additive character of $E$ given by $\psi_a(x) = \psi(ax)$. Assume that $\tau$ admits a $\psi$-Whittaker model. Then the following conditions are equivalent

(a) $\tau$ admits a $\psi_a$-Whittaker model.

(b) $\epsilon(\pi \otimes \chi_a) = \chi_a(-1)\epsilon(\pi)$.

(c) $\theta(\tau^a, \psi_a) = \theta(\tau, \psi)$.

6.3 Higher multiplicity in restriction from $\widetilde{\text{GL}}_2(E)$ to $\widetilde{\text{SL}}_2(E)$

Let $\tilde{\pi}$ be an irreducible admissible genuine representation of $\widetilde{\text{GL}}_2(E)$. Let $\mu$ be a character of $\tilde{Z}$ and $\tau$ an irreducible representation of $\widetilde{\text{SL}}_2(E)$, which are compatible, such that $\mu\tau$ appears in $\tilde{\pi}$ restricted to $\widetilde{\text{GL}}_2(E)_+$. We have

$$\tilde{\pi}|_{\widetilde{\text{GL}}_2(E)_+} = \bigoplus_{a \in E^\times/E^\times 2} (\mu^a \tau^a)$$

where $a \in E^\times/E^\times 2$ is regarded as an elements in the split torus ($\cong E^\times \times E^\times$) of the form $\text{diag}(a, 1)$. Since the restriction of $\mu\tau$ from $\widetilde{\text{GL}}_2(E)_+$ to $\widetilde{\text{SL}}_2(E)$ is $\tau$, the multiplicity with which the representation $\tau$ appears in $\tilde{\pi}$, to be denoted by $m(\tilde{\pi}, \tau)$, is given by

$$m(\tilde{\pi}, \tau) = \#\{a \in E^\times/E^\times 2 : \tau^a \cong \tau\}.$$ 

We have the following immediate corollaries to part 1 of Theorem 6.2.4.

Lemma 6.3.1. For an irreducible admissible genuine representation $\tau$ of $\widetilde{\text{SL}}_2(E)$, and $a \in E^\times$, we have

$$\tau \cong \tau^a \iff \begin{cases} (1) \quad \theta(\tau) \otimes \chi_a \cong \theta(\tau) \\ (2) \quad \chi_a(-1) = 1. \end{cases}$$
Proof. If $\tau \cong \tau^a$, then considering the central characters on both sides, we find that $\chi_a(-1) = 1$. Further, if $\tau \cong \tau^a$, then in particular, either they both have $\theta$ lifts to $\text{PGL}_2(E)$ or they both have $\theta$ lift to $PD^\times$, and $\theta(\tau) \cong \theta(\tau^a)$. Thus from part 1 of Theorem 6.2.4, we get $\theta(\tau) \otimes \chi_a \cong \theta(\tau)$. To prove the converse, note that $\theta(\tau) \otimes \chi_a \cong \theta(\tau) \Rightarrow \epsilon(\theta(\tau) \otimes \chi_a) = \epsilon(\theta(\tau))$. As $\chi_a(-1) = 1$, we get $\epsilon(\theta(\tau) \otimes \chi_a) = \chi_a(-1)\epsilon(\theta(\tau))$. From loc. cit., $\theta(\tau) = \theta(\tau^a)$ and hence $\tau \cong \tau^a$. \hfill $\Box$

Corollary 6.3.2. The multiplicity of $\tau$ in $\tilde{\pi}$ is given by

$$m(\tilde{\pi}, \tau) = \# \{ a \in E^\times/E^\times_2 : \theta(\tau) \otimes \chi_a \cong \theta(\tau) \text{ and } \chi_a(-1) = +1 \}.$$ 

It is well-known that for a representation $\pi$ of $\text{GL}_2(E)$, cf. [14, Section 2]

$$m(\pi) = \# \{ a \in E^\times/E^\times_2 : \pi \cong \pi \otimes \chi_a \} \in \{1, 2, 4\}.$$ 

The condition $\chi_a(-1) = 1$ is automatic in some situations, for example if $-1 \in E^\times_2$. Thus we get

$$m(\tilde{\pi}, \tau) \in \{1, 2, 4\}$$

even when the residue characteristic of $E$ is 2.

6.4 A lemma on Waldspurger involution

We recall that for an irreducible admissible genuine representation $\tau$ of $\text{SL}_2(E)$, the central characters of $\tau$ and $\tau_W$ are different. The group $\text{GL}_2(E)$ acts on the set of isomorphism classes of irreducible admissible genuine representations of $\text{SL}_2(E)$ by conjugation. This action reduces to an action of $E^\times$, by identifying $E^\times$ into $\text{GL}_2(E)$ as $\left\{ \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} : e \in E^\times \right\}$.

We denote this action by $\tau \mapsto \tau^a$ for $a \in E^\times$. Since a similar action produces an $L$-packet for $\text{SL}_2(E)$, whereas for $\text{SL}_2(E)$, one defines an $L$-packet by taking $\tau$ and $\tau_W$, we investigate in this section if it can happen that $\tau_W \cong \tau^a$ for some $a \in E^\times$ and $\tau$ a discrete series representation of $\text{SL}_2(E)$. 

Lemma 6.4.1. Let \( \tau \) be a discrete series representation of \( \widetilde{\text{SL}}_2(E) \). Let \( \psi \) be a non-trivial additive character of \( E \) such that \( \tau \) has a \( \psi \)-Whittaker model. Then there exists \( a \in E^\times \) with \( \tau^a \cong \tau_W \) if and only if for \( \pi = \theta(\tau, \psi) \), we have

(i) \( \pi \cong \pi \otimes \chi_a \)

(ii) \( \chi_a(-1) = -1 \).

Proof. Let \( \pi = \theta(\tau, \psi) \) and \( \theta(\tau_W, \psi) = \pi^{JL} \), where \( \pi^{JL} \) denotes the representation of \( PD^\times \) which is associated to \( \pi \) via the Jacquet-Langlands correspondence. From part 2 of Theorem 6.2.4 it follows that if \( \epsilon(\pi \otimes \chi_a) = \chi_a(-1)\epsilon(\pi) \), then \( \tau^a \) lifts to \( \text{PGL}_2(E) \) and not to \( PD^\times \) and hence \( \tau^a \) cannot be isomorphic to \( \tau_W \). Thus if \( \tau^a \) were isomorphic to \( \tau_W \), then we must have \( \epsilon(\pi \otimes \chi_a) = -\chi_a(-1)\epsilon(\pi) \). In this case, by Theorem 6.2.4, \( \tau^a \) lifts to \( PD^\times \), and in fact to the representation \( \pi^{JL} \otimes \chi_a \) of \( PD^\times \). Therefore

\[
\tau^a \cong \tau_W \iff \begin{cases} 
(i) & \epsilon(\pi \otimes \chi_a) = -\chi_a(-1)\epsilon(\pi) \\
(ii) & \pi^{JL} \cong \pi^{JL} \otimes \chi_a.
\end{cases}
\]  

(6.6)

The conditions (i) and (ii) in (6.6) can be combined to say that

\[
\tau^a \cong \tau_W \iff \begin{cases} 
(i) & \pi \cong \pi \otimes \chi_a \\
(ii) & \chi_a(-1) = -1.
\end{cases}
\]

This completes the proof of the lemma.

As a consequence of Lemma 6.3.1 and Lemma 6.4.1, we obtain:

Corollary 6.4.2. Let \( \tau \) be an irreducible genuine discrete series representation of \( \widetilde{\text{SL}}_2(E) \). Let \( m_1 = \#\{\tau^a, (\tau_W)^a \mid a \in E^\times\} \) and let \( m_2 \) be the cardinality of the \( L \)-packet of \( \text{SL}_2(E) \) determined by \( \theta(\tau, \psi) \). Then

\[
m_1 \cdot m_2 = 2[E^\times : E^{\times 2}].
\]

If \( \pi \) is a principal series representation of \( \text{PGL}_2(E) \) with \( \pi \otimes \chi_a \cong \pi \), then \( \pi \) must be the principal series representation \( Ps(\mu, \mu \chi_a) \) with \( \mu^2 = \chi_a \), and as a result

\[
\chi_a(-1) = \mu^2(-1) = 1.
\]
Corollary 6.4.3. Let $\tau$ be an irreducible admissible genuine representation of $\widetilde{SL}_2(E)$ such that $\theta(\tau)$ an irreducible principal series representation of $\text{PGL}_2(E)$. Let $m_1 = \#\{\tau^a \mid a \in E^\times\}$, and $m_2$ the cardinality of the $L$-packet of $\text{SL}_2(E)$ determined by $\theta(\tau)$,

$$m_1 \cdot m_2 = [E^\times : E^{\times 2}].$$

6.5 Complementary Whittaker models

Let $\tilde{\pi}$ be an irreducible admissible genuine representation of $\widetilde{GL}_2(E)$. By Theorem 2.6.2, we know that the space of all $\psi$-Whittaker functionals is finite dimensional and that the characters of $\tilde{Z}$ appear with multiplicity at most one in this space of $\psi$-Whittaker functionals. The characters of $\tilde{Z}$ which appear in the space of $\psi$-Whittaker functionals are extensions of the central characters of the representation $\tilde{\pi}$. In other words, $\tilde{\pi}_{N,\psi} \subset \Omega(\omega_{\tilde{\pi}})$. Thus there are at most $\#(E^\times / E^{\times 2})$ characters appearing in the space of $\psi$-Whittaker functionals of $\tilde{\pi}$. We know that if $\tilde{\pi}$ is a principal series representation then $\tilde{\pi}_{N,\psi} \cong \Omega(\omega_{\tilde{\pi}})$, by Theorem 2.6.4.

Proposition 6.5.1. If $\tilde{\pi}$ is a discrete series representation of $\widetilde{GL}_2(E)$ then as a $\tilde{Z}$-module

$$\tilde{\pi}_{N,\psi} \subset \Omega(\omega_{\tilde{\pi}}).$$

Proof. Write $\tilde{\pi} = \text{ind}_{\text{GL}_2(E)_+}^{\tilde{\text{GL}}_2(E)} (\mu \tau)$. Observe that $\tilde{\pi}$ is a discrete series representation if and only if $\tau$ is a discrete series representation. Consider the set $\{\tau, \tau_W\}$. Since the Waldspurger involution does not fix any discrete series representation, $\tau \not\equiv \tau_W$. Let $\psi$ be a non-trivial additive character of $E$ such that $\tau_W$ admits a $\psi$-Whittaker model. Then $\tau$ does not have a $\psi$-Whittaker model. Since $\tilde{\pi} = \text{ind}_{\text{GL}_2(E)_+}^{\tilde{\text{GL}}_2(E)} (\mu \tau)$, we have $\tilde{\pi}|_{\text{GL}_2(E)_+} = \bigoplus_{a \in E^\times / E^{\times 2}} \mu^a \tau^a$ with $\mu^a = \mu \cdot \chi_a$ where the $\chi_a$, defined by $\chi_a(x) = (x, a)$, are distinct characters of $E^\times$. Therefore $\mu$ does not appear $\tilde{\pi}_{N,\psi}$. \hfill \Box
6.5. Complementary Whittaker models

6.5.1 Case 1: $-1 \in E^{\times 2}$

In this subsection, suppose $-1 \in E^{\times}$. Suppose that a genuine character $\mu$ of $\tilde{Z}$ and an irreducible admissible genuine representation $\tau$ of $\text{SL}_2(E)$ are compatible. As $-1 \in E^{\times 2}$, $\mu^a$ is also compatible with $\tau$ for all $a \in E^{\times}$.

Lemma 6.5.2. Let $-1 \in E^{\times 2}$ and let $\psi$ be a non-trivial additive character of $E$. Let $\tilde{\pi} = \text{ind}_{\text{GL}_2(E)_+}^{\text{GL}_2(E)} (\mu \tau)$. For $a \in E^{\times}/E^{\times 2}$, if we write $\tilde{\pi}_a := \text{ind}_{\text{GL}_2(E)_+}^{\text{GL}_2(E)} (\mu^a \tau)$, then for all $\mu \in \Omega(\omega_{\tilde{\pi}})$, the multiplicity of $\mu$ in $\bigoplus_{a \in E^{\times}/E^{\times 2}} \tilde{\pi}_a$ is $\dim \tilde{\pi}_{N,\psi}$. In particular, if $\tilde{\pi}_{N,\psi}$ is one dimensional then $\bigoplus_{a \in E^{\times}/E^{\times 2}} \tilde{\pi}_a \cong \Omega(\omega_{\tilde{\pi}})$.

Proof. For $\mu \in \Omega(\omega_{\tilde{\pi}})$, it is clear that $\mu$ appears in $\tilde{\pi}_{N,\psi}$ if and only if $\mu^a \in \tilde{\pi}_a$. The lemma follows easily by Remark 6.1.4. \qed

Now we assume that residue characteristic of $E$ is odd, so we have $\#(E^{\times}/E^{\times 2}) = 4$.

Proposition 6.5.3. Let $-1 \in E^{\times 2}$ and suppose that the residual characteristic of $E$ is odd. Let $\tilde{\pi}$ be an irreducible admissible genuine representation of $\text{GL}_2(E)$ such that $\dim \tilde{\pi}_{N,\psi} = 2$. Assume that $\tilde{\pi} := \text{ind}_{\text{GL}_2(E)_+}^{\text{GL}_2(E)} (\mu \tau)$ for some compatible $\mu$ and $\tau$ such that $\tau$ admits a non-zero $\psi$-Whittaker functional. Then there exists $b \in E^{\times} - E^{\times 2}$ such that for $\tilde{\pi}' := \text{ind}_{\text{GL}_2(E)_+}^{\text{GL}_2(E)} (\mu^b \tau)$ we have

$$(\tilde{\pi})_{N,\psi} \oplus (\tilde{\pi}_b)_{N,\psi} \cong \Omega(\omega_{\tilde{\pi}}).$$

Proof. Write $E^{\times}/E^{\times 2} = \{1, a, b, ab\}$. Assume $(\tilde{\pi})_{N,\psi} = \mu \oplus \mu^a$. Then each of $\tau^a$ and $\tau$ admits a non-zero $\psi$-Whittaker functional. Equivalently, $\tau$ admits a non-zero $\psi$-Whittaker functional as well as a non-zero $\psi^a$-Whittaker functional. Therefore $\tau^b$ and $\tau^{ab}$ have $\psi_b$ and $\psi_{ab}$-Whittaker models. Therefore for

$$\tilde{\pi}' := \text{ind}_{\text{GL}_2(E)_+}^{\text{GL}_2(E)} (\mu^b \tau),$$

Then, by Remark 6.1.4, we have $(\tilde{\pi}')_{N,\psi} = \mu^b \oplus \mu^{ab}$. Therefore we have

$$(\tilde{\pi})_{N,\psi} \oplus (\tilde{\pi}_b)_{N,\psi} \cong (\mu \oplus \mu^a) \oplus (\mu^b \oplus \mu^{ab}) = \Omega(\omega_{\tilde{\pi}}).$$
Thus $\tilde{\pi}'$ is a representation of $\tilde{\text{GL}}_2(E)$ which has the same central character as that of $\tilde{\pi}$ and complementary Whittaker model to that of $\tilde{\pi}$.

\[ \square \]

### 6.5.2 Case 2: $-1 \notin E^{\times 2}$

In this subsection we assume that $-1 \notin E^{\times 2}$.

**Proposition 6.5.4.** Let $\tau$ be an irreducible admissible supercuspidal genuine representation of $\tilde{\text{SL}}_2(E)$. Assume that $p$ is odd and that $-1$ is not a square in $E$. Let $\psi$ be a non-trivial character of $E$ such that $\tau$ admits $\psi$-Whittaker model. Assume that for $\pi = \theta(\tau; \psi)$, $\pi \cong \pi \otimes \chi_b$ where $\chi_b$ corresponds to a quadratic ramified extension of $E$. Then for $a = -b$, the representations $\tau$ and $\tau^a$ have opposite central characters. Moreover, for any non-trivial character $\psi'$ of $E$, $\tau$ admits a non-zero $\psi'$-Whittaker functional if and only if $\tau^a$ admits a non-zero $\psi'$-Whittaker functional.

**Proof.** As $\chi_b$ corresponds to a quadratic ramified extension of $E$, for $a = -b$ we have $\chi_a(-1) = -1$. Hence, $\tau$ and $\tau^a$ have opposite central character. Therefore, it is enough to show that the following holds for all $x \in E^{\times}$:

$$\tau \text{ has } \psi^x\text{-Whittaker model } \iff \tau^a \text{ has } \psi^x\text{-Whittaker model.}$$ (6.7)

The condition in (6.7) translates into

$$\theta(\tau, \psi^x) \neq 0 \iff \theta(\tau^a, \psi^x) \neq 0, \quad \forall x \in E^{\times}$$

i.e.,

$$\theta(\tau^x, \psi) \neq 0 \iff \theta(\tau^{ax}, \psi) \neq 0, \quad \forall x \in E^{\times}.$$ (6.8)

Let $V^+$ and $V^-$ be 3-dimensional quadratic spaces such that $O(V^+) = \text{PGL}_2(E) \times \{\pm 1\}$ and $O(V^-) = \text{PD}^\times \times \{\pm 1\}$. Set $\epsilon(V^+) = 1$ and $\epsilon(V^-) = -1$. Let $\epsilon \in \{\pm\}$ be such that the theta lift $\theta(\tau^x, \psi)$ is non-zero on $O(V^\epsilon)$. By parts 2, 3 of Theorem 6.2.3 and Theorem 6.2.4

$$\frac{z_\psi(\tau^x)}{z_\psi(\tau^{ax})} = \frac{\epsilon(\pi \otimes \chi_x)\epsilon(V^\epsilon)}{\epsilon(\pi \otimes \chi_{ax})\epsilon(V^\epsilon)} = \frac{\epsilon(\pi \otimes \chi_x)}{\epsilon(\pi \otimes \chi_{ax})}. \quad (6.8)$$

Recall that in odd residue characteristic

$$\chi_x(-1) = (x, -1) = (-1)^{\text{val}(x)}, \quad (6.9)$$
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therefore in our case, \( \chi_a(-1) = -1 \). By Equation 6.4, \( z_\psi(\tau^{ax}) = z_\psi(\tau^x)\chi_a(-1) \), and hence we have \( z_\psi(\tau^{ax}) = -z_\psi(\tau^x) \). Therefore the Equation 6.8 simplifies to

\begin{equation}
\varepsilon(\pi \otimes \chi_a) = -\varepsilon(\pi \otimes \chi_{ax}) \quad \forall x \in E^\times.
\end{equation}

(6.10)

Let \( \chi_u = \chi - 1 \) be the unramified quadratic character of \( E^\times \). Let \( \text{cond}(\pi) \) denote the conductor of \( \pi \). By [30, Equation 3.2.1], we have

\begin{equation}
\varepsilon(\pi \otimes \chi_u) = (-1)^{\text{cond}(\pi)}\varepsilon(\pi).
\end{equation}

(6.11)

Thus if the conductor of \( \pi \), is odd

\begin{equation}
\varepsilon(\pi \otimes \chi_u) = -\varepsilon(\pi).
\end{equation}

(6.11)

By [30, proposition 3.5], it follows that if \( \pi = \pi \otimes \chi_b \) then the conductor of \( \pi \) is odd and hence Equation (6.11) is satisfied. The assumption \( \pi \cong \pi \otimes \chi_b \) is equivalent to

\begin{equation}
\pi \otimes \chi_{-1} \cong \pi \otimes \chi_a.
\end{equation}

(6.12)

It follows from (6.11) that

\begin{equation}
\varepsilon(\pi \otimes \chi_a) = \varepsilon(\pi \otimes \chi_{-1}) = -\varepsilon(\pi).
\end{equation}

(6.13)

Now (6.10) follows from (6.13) by direct verification for each element

\[ x \in E^\times/E^\times_2 = \{1, -1, a, b = -a\}. \]

\[ \square \]

**Corollary 6.5.5.** Assume that the residue characteristic of \( E \) is odd and that \(-1 \notin E^\times_2\).

Let \( \tilde{\pi} = \text{ind}_{\text{GL}_2(E)}^{\text{GL}_2(E)}(\mu \tau) \) where \( \mu \) and \( \tau \) are as before. Assume that for \( \pi = \theta(\tau, \psi) \), \( \pi \cong \pi \otimes \chi \) for some quadratic character \( \chi \) of \( E^\times \) corresponding to a quadratic ramified extension of \( E \). Then there exists \( a \in E^\times \) such that \( \tau \) and \( \tau^a_W \) have same central character, and for any non-trivial character \( \psi^f \) of \( E \), \( \tau \) admits a non-zero \( \psi^f \)-Whittaker functional if and only if \( \tau^a_W \) does not admit a non-zero \( \psi^f \)-Whittaker functional. Thus,

\[ \tilde{\pi}' := \text{ind}_{\text{GL}_2(E)}^{\text{GL}_2(E)}(\mu \tau^a_W) \]

has a complementary set of Whittaker models to that of \( \tilde{\pi} \).
Proof. By Proposition 6.5.4, there exists an \( a \in E^\times/E^\times_2 \) such that \( \tau \) and \( \tau^a \) have opposite central characters, and for any non-trivial character \( \psi' \) of \( E \), \( \tau \) admits a non-zero \( \psi' \)-Whittaker functional if and only if \( \tau^a \) admits a non-zero \( \psi' \)-Whittaker functional. By part 1 of Theorem 6.2.3, \( \tau \) and \( \tau^a_W \) have the same central character, and for any non-trivial character \( \psi' \) of \( E \), \( \tau \) admits a non-zero \( \psi' \)-Whittaker functional if and only if \( \tau^a_W \) does not admit a non-zero \( \psi' \)-Whittaker functional. Now the corollary follows immediately. \( \square \)
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