ON THE INITIAL BOUNDARY VALUE PROBLEM OF A NAVIER-STOKES/\(Q\)-TENSOR MODEL FOR LIQUID CRYSTALS

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ABSTRACT. This work is concerned with the solvability of a Navier-Stokes/\(Q\)-tensor coupled system modeling the nematic liquid crystal flow on a bounded domain in three dimensional Euclidian space with strong anchoring boundary condition for the order parameter. We prove the existence and uniqueness of local in time strong solutions to the system with an anisotropic elastic energy. The proof is based on mainly two ingredients: first, we show that the Euler-Lagrange operator corresponding to the Landau-de Gennes free energy with general elastic coefficients fulfills the strong Legendre condition. This result together with a higher order energy estimate leads to the well-posedness of the linearized system, and then a local in time solution of the original system which is regular in temporal variable follows via a fixed point argument. Secondly, the hydrodynamic part of the coupled system can be reformulated into a quasi-stationary Stokes type equation to which the regularity theory of the generalized Stokes system, and then a bootstrap argument can be applied to enhance the spatial regularity of the local in time solution.

1. Introduction. Nematic liquid crystal is a sort of material which may flow as a conventional liquid while the molecules are oriented in a crystal-like way. One of the successful continuum theories modeling nematic liquid crystals is the \(Q\)-tensor theory, also referred to as Landau-de Gennes theory, which uses a \(3 \times 3\) traceless and symmetric matrix-valued function \(Q(x)\) as order parameters to characterize the orientation of molecules near material point \(x\) (cf. [14]). The matrix \(Q\), also called \(Q\)-tensor, can be interpreted as the second momentum of a number density function

\[
Q(x) = \int_{S^2} (m \otimes m - \frac{1}{3} I_3) f(x,m) dm,
\]

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where \( f(x, m) \) corresponds to the number density of liquid crystal molecules which orient along the direction \( m \) near material point \( x \). The configuration space for \( Q \)-tensor will be denoted by

\[
Q = \{ Q \in \mathbb{R}^{3 \times 3} \mid Q_{ij} = Q_{ji}, \sum_{i=1}^{3} Q_{ii} = 0 \}. \tag{1}
\]

If \( Q(x) \) has three equal eigenvalues, it must be zero and this corresponds to the isotropic phase. When \( Q(x) \) has two equal eigenvalues, it can be written as

\[
Q(x) = s(x) \left( n(x) \otimes n(x) - \frac{1}{3} I_3 \right),
\]

for some \( n(x) \in S^2 \) and \( s(x) \in \mathbb{R} \) and it is said to be uniaxial. If all three eigenvalues of \( Q(x) \) are distinct, it is called biaxial and can be written as

\[
Q(x) = s(x) \left( n(x) \otimes n(x) - \frac{1}{3} I_3 \right) + r(x) \left( m(x) \otimes m(x) - \frac{1}{3} I_3 \right),
\]

where \( n(x), m(x) \in S^2 \) and \( s(x), r(x) \in \mathbb{R} \).

The classic Landau-de Gennes theory associates to each \( Q(x) \) a free energy of the following form:

\[
F(Q, \nabla Q) = \int_{\Omega} \left( \frac{a}{2} \text{tr} Q^2 - \frac{b}{3} \text{tr} Q^3 + \frac{c}{4} (\text{tr} Q^2)^2 \right) \, dx
\]

\[
+ \frac{1}{2} \int_{\Omega} \left( L_1 |\nabla Q|^2 + L_2 Q_{ij,j} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} + L_4 Q_{ik}Q_{ij,k}Q_{ij,\ell} \right) \, dx
\]

\[= F_b(Q) + F_c(\nabla Q, Q). \tag{2}\]

Here and in the sequel, we shall adopt the Einstein’s summation convention by summing over repeated greek letters. In (2), \( F_b(Q) \) is the bulk energy, describing the isotropic-nematic phase transition while \( F_c(\nabla Q, Q) \) is the elastic energy which characterizes the distortion effect. The parameters \( a, b, c \) are temperature dependent constants with \( b, c > 0 \), and \( L_1, L_2, L_3, L_4 \) are elastic coefficients. In the sequel, we shall call \( F(Q, \nabla Q) \) isotropic if \( L_2 = L_3 = L_4 = 0 \) and anisotropic if at least one of \( L_2, L_3, L_4 \) does not vanish. This work is devoted to the later case. Note that the term in (2) corresponding to \( L_4 \) is cubic and will lead to severe analytic difficulties: it is shown in [4] that \( F(Q, \nabla Q) \) with \( L_4 \neq 0 \) is not bounded from below. In this work, we follow [5, 25] and assume:

\[
L_1 > 0, \quad L_1 + L_2 + L_3 =: L_0 > 0, \quad L_4 = 0. \tag{3}\]

In order to introduce the system under consideration, we need some notation. For any \( Q \in Q \) defined by (1), \( S_Q(M) \) will be a linear operator acting on any \( 3 \times 3 \) matrix \( M \)

\[
S_Q(M) = \xi \left( \frac{1}{2} (M + M^T) \cdot (Q + \frac{1}{3} I_3) + \frac{1}{2} (Q + \frac{1}{3} I_3) \cdot (M + M^T) - 2(Q + \frac{1}{3} I_3) (Q + \frac{1}{3} I_3) : M \right). \tag{4}\]

Note that \( S_Q(M) \) is a traceless symmetric matrix and if \( M \) is symmetric and traceless additionally, it reduces to

\[
S_Q(M) = \xi \left( M \cdot (Q + \frac{1}{3} I_3) + (Q + \frac{1}{3} I_3) \cdot M - 2(Q + \frac{1}{3} I_3) (Q : M) \right).
\]
Here $A : B = \text{tr}(AB^T)$ and $A \cdot B$ denotes the usual matrix product of $A, B$ and the ‘dot’ will be sometimes omitted if it is clear from the context. The parameter $\xi$ is a constant depending on the molecular details of a given liquid crystal and measures the ratio between the tumbling and the aligning effect that a shear flow would exert over the liquid crystal directors. Concerning the hydrodynamic part, for any vector field $u$, its gradient $\nabla u$ can be written as the sum of the symmetric and anti-symmetric parts:

$$\nabla u = D(u) + W(u),$$
where

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

With these notation, the Navier-Stokes/Q-tensor system, proposed in Beris-Edwards [6], can be written as:

$$u_t + u \cdot \nabla u = \nabla P + \nabla \cdot (\sigma^a + \sigma^d),$$

$$\nabla \cdot u = 0,$$

$$Q_t + u \cdot \nabla Q + Q \cdot W(u) - W(u) \cdot Q = \Gamma H(Q) + S_Q(D(u)).$$

In (5c), $\Gamma$ is the rotational diffusion constant and without loss of generality, we shall assume $\Gamma = 1$ in the sequel. The unknowns $(u, P)$ correspond to the velocity/pressure of the hydrodynamics respectively. The stress terms $\sigma^a$, $\sigma^d$ and $\sigma^d$ on the right hand side of (5a) are symmetric viscous stress, anti-symmetric viscous stress and distortion stress respectively:

$$\sigma^a(u, Q) := \nu D(u) - S_Q(H(Q)),$$

$$\sigma^d(Q) := Q \cdot H(Q) - H(Q) \cdot Q,$$

$$\sigma^d(Q) := (\sigma^d_{ij}(Q))_{1 \leq i, j \leq 3} = -\left(\frac{\partial F(Q, \nabla Q)}{\partial Q_{kt,j}} Q_{kt,i}\right)_{1 \leq i, j \leq 3}. \quad (6c)$$

$H(Q)$ is the molecular field, defined as the variational derivative of (2) and is written as the sum of the bulk part and the elastic part:

$$H(Q) := -\frac{\delta F(Q, \nabla Q)}{\delta Q} := -\mathcal{L}(Q) - \mathcal{J}(Q). \quad (7)$$

The operator $\mathcal{L}$ and $\mathcal{J}$ can be written explicitly by

$$-\mathcal{L}_{ij}(Q) = L_1 \Delta Q_{ij} + \frac{L_2 + L_3}{2} (Q_{ik,kj} + Q_{jk,ki} - 2Q_{kt,k\ell} \delta_{ij}),$$

$$-\mathcal{J}_{ij}(Q) = -a Q_{ij} + b(Q_{jk}Q_{ki} - \frac{1}{3} \text{tr}(Q^2) \delta_{ij}) - c \text{tr}(Q^2) Q_{ij}. \quad (8)$$

We note that the operator defined via (8) can not be considered as a perturbation of $L_1 \Delta$ as we only assumes (3). Actually, one of the key results in this work is Lemma 2.2 below, showing that (8) fulfills the strong Legendre condition.

The coupled system (5) has been recently studied by several authors. For the case $\xi = 0$, which corresponds to the situation when the molecules only tumble in a shear flow but are not aligned by the flow, the existence of global weak solutions to the Cauchy problem in $\mathbb{R}^d$ with $d = 2, 3$ is proved in [21]. Moreover, solutions with higher order regularity and the weak-strong uniqueness for $d = 2$ is discussed. Later, these results are generalized in [22] to the case when $|\xi|$ is sufficiently small. Large time behavior of the solution to the Cauchy problem in $\mathbb{R}^3$ with $\xi = 0$ is recently discussed in [11]. The global well-posedness and long-time behavior of system with
nonzero $\xi$ in the two-dimensional periodic setting are studied in [8]. The uniqueness of weak solutions to the Cauchy problem in $\mathbb{R}^2$ was proved in [12, 13].

In [19], the authors considered Beris-Edwards system with anisotropic elastic energy (2) (with $L_2 + L_3 > 0$ and $L_4 = 0$). They proved the existence of global weak solutions as well as the existence of a unique global strong solution for the Cauchy problem in $\mathbb{R}^3$ provided that the fluid viscosity is sufficiently large. In [7, 9, 20], the weak solution of the gradient flow generated by the general Landau-de Gennes energy (2) with $L_4 \neq 0$ is established for small initial data.

Some recent progresses have also been made on the analysis of certain modified versions of Beris-Edwards system. In [26], when $\xi = 0$ and the polynomial bulk energy is replaced by a singular potential derived from molecular Maier-Saupe theory, the author proved, under periodic boundary conditions, the existence of global weak solutions in space dimension two and three. Moreover, the existence and uniqueness of global regular solutions for dimension two is obtained. In [15, 16], the authors derived a nonisothermal variants of (5) and proved the existence of global weak solutions in the case of a singular potential under periodic boundary conditions for general $\xi$. We also mention that a rigorous derivation of the general Ericksen-Leslie system from the small elastic limit of Beris-Edwards system (with arbitrary $\xi$) is recently given in [25] using the Hilbert expansion.

In the aforementioned works, the domain under consideration is either the whole space or the tori. The initial-boundary value problems of (5) have been also investigated by several authors, see for instance [1, 2, 17, 18], in which the existences of weak solutions has been studied. In addition, in [1, 17], the authors proved the existence of local in time solution with higher order time regularity for (5) through different approaches. However, the higher order spacial regularity is not obtained due to the lack of effective energy estimate in the presence of inhomogenous boundary condition for $Q$.

The main goal of the presented work is to improve the results in [1, 17] to nature regularity in space variable. This gives a full answer to the construction of local in time strong solution of (5) in the presence of inhomogenous boundary condition for $Q$.

We shall consider the initial-boundary conditions
\begin{align}
(u|_{t=0} = (u_0, Q_0), \quad (9a) \\
(u, Q)|_{\partial \Omega} = (0, Q_0|_{\partial \Omega}), \quad (9b)
\end{align}
where $Q_0 = Q_0(x)$ is time-independent. Note that such a result requires a compatibility condition on the initial data $Q_0$. To see that, we write (5) in an abstract form
\begin{equation}
\frac{d}{dt}(u, Q) = \mathcal{E}(u, Q), \quad (10)
\end{equation}
where $\mathcal{E} : H_{0,\sigma}^1(\Omega) \times H^2(\Omega) \rightarrow H_{\sigma}^{-1}(\Omega) \times L^2(\Omega)$ is defined by
\begin{align*}
\langle \mathcal{E}(u, Q), (\varphi, \Psi) \rangle &= -\int_{\Omega} (-u \otimes u + \sigma^s + \sigma^a + \sigma^d) : \nabla \varphi \, dx \\
&+ \int_{\Omega} (-u \cdot \nabla Q - Q \cdot W(u) + W(u) \cdot Q + \mathcal{H}(Q) + \mathcal{S}_Q(D(u))) : \Psi \, dx,
\end{align*}
for all $(\varphi, \Psi) \in H_{0,\sigma}^1(\Omega) \times L^2(\Omega; Q)$. Note that the functional spaces used here are defined in Section 2. Since (9b) specifies a time-independent boundary condition, it follows that $\partial_t Q|_{\partial \Omega} = 0$ which leads to the compatibility condition that the trace of
the second component on the right-hand side of (10) vanishes on ∂Ω. This motives
to define the admissible class for the initial data
\[ \mathcal{I} = \{(u_0, Q_0) \in H^1_{0,\sigma}(\Omega) \times H^2(\Omega; Q) \mid \mathcal{E}(u_0, Q_0) \in L^2_\sigma(\Omega) \times H^1(\Omega)\}. \] (11)
It is not hard to see \( \mathcal{I} \) is not empty. For example, for any \( Q \) solving \( H(Q) = 0 \) and any \( u \in H^3_0(\Omega) \), we have \( (u, Q) \in \mathcal{I} \). We note that such a compatibility condition is
natural in the sense that it can not be disregarded by changing the function spaces
unless one considers very weak solution.

The main result of this paper can be stated as follows.

**Theorem 1.1.** Assume the coefficients of elastic energy satisfy (3). Then for any
\( (u_0, Q_0) \in \mathcal{I} \) with \( Q_0|_{\partial \Omega} \in H^{5/2}(\partial \Omega) \), there exists some \( T > 0 \) such that the system
(5) and (9) has a unique solution
\[ u \in H^2(0, T; H^{-1}(\Omega)) \cap H^1(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \]
\[ Q \in H^2(0, T; L^2(\Omega; Q)) \cap H^1(0, T; H^2(\Omega; Q)) \cap L^\infty(0, T; H^3(\Omega; Q)). \] (12)

Theorem 1.1 essentially improves the spatial regularity of solution obtained in [1]
and generalizes their result to the case of anisotropic elastic energy. This is accom-
plished by the crucial observation that the terms containing third order derivatives
on \( Q \) in (5a) can be eliminated and the system can be reduced into a Stokes-type
system with positive definite viscosity coefficient. Moreover, under the general ass-
sumption (3), the operator \( L \) defined by (8) is strongly elliptic. This fact leads
to \( W^{2,p} \)-estimates for the solution so that we can work with the general case of
anisotropic energy rather than the isotropic energy case \( (L_2 = L_3 = 0) \). The strong
ellipticity of \( L \) is proved in Lemma 2.2 by an explicit construction of the coefficient
matrix, which involves a fairly sophisticated anisotropic tensor of order six. We
also mention that the local in time strong solution constructed here is valid for any
\( \xi \in \mathbb{R} \).

The rest parts of the work is organized as follows. In Section 2, we introduce
notation and analytic tools that will be used throughout the paper. The most im-
portant results involve the solvability theorem on the generalized Stokes system,
due to Solomnikov [24], as well as Lemma 2.2 on the analysis of the operator \( L \). In
Section 3 an abstract evolution equation that incorporates (5), (9) and a compati-
bility condition is introduced and the functional analytic framework is established.
The core part, Section 4, is devoted to the proof of Theorem 1.1 by showing that the
abstract evolution equation has a local in time solution. This is accomplished by
proving the existence of a local in time solution that is regular in temporal variable
in the first stage, following the method in [1], and then using the structure of (5) to
eliminate the higher order terms in the additional stress tensors of (5a) and recast
it into a generalized Stokes system. Afterwards, the spatial regularity of (5) with
initial-boundary condition (9) is improved using the \( L^p \)-estimate of the generalized
Stokes system together with bootstrap arguments.

2. Preliminaries.

2.1. **Notations.** Throughout this paper, the Einstein’s summation convention
will be adopted. That is, we shall sum over repeated greek letters. For any \( 3 \times 3 \)
matrix \( A, B \in \mathbb{R}^{3 \times 3} \), their usual matrix product will be denoted by \( A \cdot B \) or even
shortly by \( AB \) if it is clear from the context. The Frobenius product of two matrices
corresponds to \( A : B = \text{tr}(AB^T) = A_{ij}B_{ij} \) and this induces a norm \( |A| = \sqrt{A_{ij}A_{ij}} \).
For any matrix-valued function $F = (F_{ij})_{1 \leq i,j \leq 3}$, we denote $F_{ij,k} = \partial_k F_{ij}$ and $\text{div} F = \nabla \cdot F = (\partial_i F_{ij})_{1 \leq i \leq 3}$.

In tensor analysis, the Levi-Civita symbol $\{\varepsilon^{ijk}\}_{1 \leq i,j,k \leq 3}$ and Kronecker symbol $\{\delta^i_j\}_{1 \leq i,j \leq 3}$ are very useful to deal with operations involving inner and wedge product: for any $a, b \in \mathbb{R}^3$, their inner and wedge products are given by

$$a \cdot b = a_i b^i \delta^i_j, \quad a \wedge b = (a_j b_k \varepsilon^{ijk})_{1 \leq i \leq 3}$$

respectively. The following identity is well known:

$$\varepsilon^{ijk} \varepsilon^{imn} = (\delta^j_m \delta^k_n - \delta^j_n \delta^k_m).$$

(13)

For any vector field $u$, its divergence and curl can be calculated by

$$\nabla \cdot u = \delta^i_j u_{i,j}, \quad \nabla \wedge u = (\varepsilon^{ijk}u_{k,j})_{1 \leq i \leq 3}.$$

(14)

2.2. Function spaces and the generalized Stokes system. Throughout this work, $\Omega \subset \mathbb{R}^3$ will be a bounded domain with smooth boundary and $\Omega_F = \Omega \times (0, T)$ will denote the parabolic cylinder. Standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{s,p}(\Omega)$ as well as $L^p(\Omega; M)$ and $W^{s,p}(\Omega; M)$ for the corresponding spaces for $M$-valued functions will be employed. Sometimes the domain and the range are omitted for simplicity if it is clear from the context. The $L^2$-based Sobolev spaces are denoted by $H^s(\Omega; M)$ or simply by $H^s(\Omega)$. For any Banach space $\mathcal{X}$, $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ denotes the dual product between $\mathcal{X}$ and its dual space $\mathcal{X}^*$ and we shall simply write $\langle \cdot, \cdot \rangle$ if the function spaces under consideration are clear from the context.

An important result related to the incompressible Navier-Stokes equation is the Helmholtz decomposition

$$L^2(\Omega; \mathbb{R}^3) = L^2_\sigma(\Omega) \oplus (L^2_\sigma(\Omega))^\perp,$$

where $L^2_\sigma(\Omega)$ denotes the space of solenoidal vector field and its orthogonal space is given by

$$(L^2_\sigma(\Omega))^\perp = \{ u \in L^2(\Omega; \mathbb{R}^3), u = \nabla q \text{ for some } q \in H^1(\Omega) \}.$$

The Helmholtz projection (also referred to as Leray projection), i.e., the orthogonal projection $L^2(\Omega; \mathbb{R}^3) \mapsto L^2_\sigma(\Omega)$, is denoted by $P_\sigma$. The readers can refer to [23] for its basic properties. For any $f \in H^{-1}(\Omega; \mathbb{R}^3)$, $P_\sigma f \in H^{-1}_\sigma(\Omega)$ is interpreted by

$$P_\sigma f = f|_{H^1_\sigma(\Omega)}.$$ Moreover, for $F \in L^2(\Omega; \mathbb{R}^{3 \times 3})$, $\text{div} F \in H^{-1}(\mathbb{R}^3)$ is defined by

$$\langle \text{div} F, \Phi \rangle_{H^{-1}, H^1_\sigma} = -\int_\Omega F : \nabla \Phi \, dx, \quad \text{for all } \Phi \in H^1_\sigma(\Omega; \mathbb{R}^3).$$

We end this subsection by the following result due to Solonnikov [24], which is crucial in the discussion of the spatial regularity during the proof of Theorem 1.1.

**Proposition 1.** Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with smooth boundary and $p > 3/2$. Assume that the tensor-valued function $A^{kl}(x, t) \in C(\overline{\Omega_T})$ satisfies $A^{kl}(\cdot, t) \in W^{1,q}(\Omega)$ for almost every $t \in [0, T]$, with $\frac{1}{q} < \frac{1}{d} + \min\{\frac{p-1}{p}, \frac{1}{d}\}$, and the strong Legendre condition, i.e. there exists two positive constants $\Lambda > \lambda > 0$ such that

$$\lambda |\xi|^2 \leq \xi_{kl} A^{kl}_{ij}(x, t) \xi_{j\ell} \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{3 \times 3}, \quad \forall (x, t) \in \overline{\Omega} \times (0, T),$$

(15)
then for any \( v^0 \in W^{2-2/p,p}(\Omega; \mathbb{R}^3) \) with \( \text{div} \ v^0 = 0 \) and \( f \in L^p(\Omega_T; \mathbb{R}^3) \), the system

\[
\begin{align*}
\partial_t v_k &= A^{kt}_{ij}(x,t) \partial_i \partial_j v_k + \partial_k P + f_k, \quad 1 \leq k \leq d, \\
\text{div} v &= 0, \\
v|_{t=0} &= v^0, \\
v|_{\partial \Omega} &= 0,
\end{align*}
\]

has a unique solution \((v, P)\) such that
\[
v \in L^p(0,T; W^{2,p}(\Omega)), \quad v_t, \nabla P \in L^p(\Omega_T).
\]

Moreover, the following estimate holds for some constant \( C \) that is independent of \( v^0 \) and \( f \):
\[
\|v_t\|_{L^p(\Omega_T)} + \|v\|_{L^p(0,T; W^{2,p}(\Omega))} + \|\nabla P\|_{L^p(\Omega_T)} \leq C(\|v^0\|_{W^{2-2/p,p}(\Omega)} + \|f\|_{L^p(\Omega_T)}).
\]

The result is still valid when \( p = 3/2 \) and \( v^0 \equiv 0 \).

In the sequel, we shall also need the stationary version of the above result when \( p = \frac{3}{2} \):

**Corollary 1.** Assume \( A^{kt}_{ij}(x) \in C(\overline{\Omega}) \cap W^{1,6}(\Omega) \) and there exists two positive constants \( \Lambda > \lambda > 0 \) such that
\[
\lambda |\xi|^2 \leq \xi_k A^{kt}_{ij}(x) \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{3 \times 3}, \quad \forall x \in \Omega, \tag{16}
\]

then for any \( f \in L^{3/2}(\Omega; \mathbb{R}^3) \), the system

\[
\begin{align*}
\partial_i (A^{kt}_{ij}(x) \partial_j v_k) + \partial_k P &= f_k, \quad (1 \leq k \leq d) \\
\text{div} v &= 0, \\
v|_{\partial \Omega} &= 0 \tag{17}
\end{align*}
\]

has a unique solution \((v, P)\) with \( v \in W^{2,3/2}(\Omega), \nabla P \in L^{3/2}(\Omega) \) and the following estimate holds
\[
\|v\|_{W^{2,3/2}(\Omega)} + \|\nabla P\|_{L^{3/2}(\Omega)} \leq C(1 + \|A\|_{W^{1,6}(\Omega)})\|f\|_{L^{3/2}(\Omega)}.
\]

Moreover, if \( f \in L^2(\Omega) \), we have the improved estimate
\[
\|v\|_{W^{2,3/2}(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \leq C(1 + \|A\|_{W^{1,6}(\Omega)})\|f\|_{L^2(\Omega)}.
\]

In the above two inequalities, \( C \) only depends on the continuous modulus of \( A^{kt}_{ij}(x) \) and geometric information of \( \Omega \).

**Proof.** Since \( H^1(\Omega) \hookrightarrow L^6(\Omega) \), it follows from duality that \( L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega) \) and this implies that \( f \in L^{3/2}(\Omega) \hookrightarrow H^{-1}(\Omega) \). The assumption on \( A^{kt}_{ij}(x) \), especially \((16)\), implies that the bilinear form
\[
a(u, v) := \int_{\Omega} \partial_i u_k A^{kt}_{ij}(x) \partial_j v_k dx
\]

is coercive on \( H^1_{0,\sigma}(\Omega) \) and the existence of solution \( u \in H^1_{0,\sigma}(\Omega) \) to \((17)\) follows from Lax-Milgram theorem. In order to obtain the \( L^p \)-estimate of \((17)\), we set \( v(x, t) = \zeta(t)u(x) \) where \( \zeta \) is a non-negative smooth function such that \( \zeta(t) = 0 \) for \( t \leq 0 \) and \( \zeta(t) = 1 \) for \( t \geq 1 \). It can be verified that \( v(x, t) \) satisfies the following equation on \( \Omega \times [0,2] \) in the sense of distribution:

\[
\begin{align*}
\partial_t v_k - A^{kt}_{ij}(x) \partial_i \partial_j v_k - \partial_k (\zeta(t)P) &= \zeta(t) \partial_i A^{kt}_{ij}(x) \partial_j u_k + \zeta'(t)u - \zeta(t)f(x), \\
\nabla \cdot v &= 0, \\
v|_{(\partial \Omega \times [0,2]) \cup \{t=0\} \times (\Omega)} &= 0.
\end{align*} \tag{18}
\]
It follows from $A_{ij}^{k\ell} \in W^{1,0}(\Omega)$ and $u \in H_0^1(\Omega)$ that
\[
\tilde{f} := \zeta(t)\partial_t A_{ij}^{k\ell}(x)\partial_j u_{\ell} + \zeta'(t)u - \zeta(t)f(x) \in L^{3/2}(\Omega_T).
\]
Using Proposition 1
\[
\|v\|_{L^{3/2}(0,1;W^{2,3/2}((\Omega)))} + \|\nabla(\zeta(t)P)\|_{L^{3/2}(\Omega \times (0,2))} \leq C\|\tilde{f}\|_{L^{3/2}(\Omega \times (0,2))},
\]
and this leads to
\[
\|u\|_{W^{2,3/2}(\Omega)} + \|\nabla P\|_{L^{3/2}(\Omega)} \leq C(1 + \|\nabla A\|_{L^6(\Omega)}) \|f\|_{L^{3/2}(\Omega)}.
\]
To prove the second equality in the statement, note that $u \in W^{2,3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega)$ and together with $f \in L^2(\Omega)$ improves the estimate for $\tilde{f}$
\[
\|\tilde{f}\|_{L^2(\Omega \times (0,2))} \leq C(1 + \|\nabla A\|_{L^6(\Omega)}) \|f\|_{L^2(\Omega)}.
\]
So applying Proposition 1 to solve (18) again leads to the second inequality. \qed

2.3. Abstract parabolic equation. Following the method in [1], we shall prove the regular in time solution with the aid of the following result:

**Proposition 2.** Suppose that $\mathbb{V}$ and $\mathbb{H}$ are two separable Hilbert spaces such that the embedding $\mathbb{V} \hookrightarrow \mathbb{H}$ is injective, continuous, and dense. Fix $T \in (0,\infty)$. Suppose that a bilinear form $a(\cdot, \cdot): \mathbb{V} \times \mathbb{V} \mapsto \mathbb{R}$ is given which satisfies for all $\phi, \psi \in \mathbb{V}$ the following assumptions:

(a) there exists a constant $c > 0$, independent of $\phi$ and $\psi$, with
\[
|a(\phi, \psi)| \leq c\|\phi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}};
\]

(b) there exist $k_0, \alpha > 0$ independent of $\phi$, with
\[
a(\phi, \phi) + k_0\|\phi\|^2_{\mathbb{H}} \geq \alpha\|\phi\|^2_{\mathbb{V}};
\]

Then there exists a representation operator $L: \mathbb{V} \mapsto \mathbb{V}'$ with $a(\phi, \psi) = \langle L\phi, \psi \rangle_{\mathbb{V}', \mathbb{V}}$, which is continuous and linear. Moreover, for all $f \in L^2((0,T);\mathbb{V})$ and $y_0 \in \mathbb{H}$, there exists a unique solution
\[
y \in \{v: [0,T] \mapsto \mathbb{V} \text{ with } v \in L^2(0,T;\mathbb{V}), \partial_tv \in L^2(0,T;\mathbb{V}')\}
\]
solving the equation
\[
\partial_t y + Ly = f \quad \text{ in } \mathbb{V}' \text{ for a.e. } t \in (0,T),
\]
subject to the initial condition $y(0) = y_0$. Finally, assume additionally that $y_0 \in \mathbb{V}$. Then $L: H^1((0,T);\mathbb{V}) \mapsto H^1((0,T);\mathbb{V}')$ is continuous and for all $f \in H^1((0,T);\mathbb{V}')$ which satisfy the compatibility condition $f(0) \in \mathbb{H}$, the solution $y$ satisfies
\[
y \in H^1((0,T);\mathbb{V}) \quad \text{ and } \quad \partial_t^2 y \in L^2((0,T);\mathbb{V}').
\]

The proof of Proposition 2 can be found in [27, Lemma 26.1 and Theorem 27.2].

2.4. Anisotropic Laplacian. We consider the following bilinear form
\[
a(\Psi, \Phi) = \int_{\Omega} \left( L_1\Psi_{ij,k}\Phi_{ij,k} + L_2\Psi_{ij,j}\Phi_{ik,k} + L_3\Psi_{ij,k}\Phi_{ik,j} \right)dx. \tag{19}
\]
for $\Psi, \Phi \in H^1(\Omega; \mathbb{Q})$. 
**Lemma 2.1.** For any \( f \in H^{-1}(\Omega; Q) \), there exists a unique \( Q \in H^1_0(\Omega; Q) \) such that

\[
a(Q, \Phi) = \langle f, \Phi \rangle \text{ for any } \Phi \in H^1_0(\Omega; Q),
\]
and there exists a constant \( C \) depending on the geometry of \( \Omega \) such that

\[
\|Q\|_{H^1_0(\Omega)} \leq C \|f\|_{H^{-1}(\Omega)}.
\]

This lemma can be proved using the construction in the proof of Lemma 2.2 below. However, we present a simpler proof here:

**Proof of Lemma 2.1.** One can verify that

\[
a(Q, \Phi) = \langle L(Q), \Phi \rangle_{H^{-1,1}_0(\Omega)}
\]

where \( L \) is the operator defined by (8). In order to apply Lax-Milgram theorem to deduce the existence of solution to (19), we need to show that \( a(\cdot, \cdot) \) is coercive in \( H^1_0(\Omega; Q) \):

\[
a(Q, Q) \geq \lambda \int_{\Omega} |\nabla Q|^2 \, dx,
\]

for some \( \lambda > 0 \). Note that, it suffices to prove the above inequality for smooth functions that vanishes on \( \partial \Omega \). Actually, for any \( Q \in H^1_0(\Omega; Q) \), we choose \( Q_n \in C^\infty_0(\Omega) \) such that \( Q_n \rightarrow Q \) strongly in \( H^1_0(\Omega; Q) \). Then the conclusion follows from the continuity of \( a(\cdot, \cdot) \).

Now we focus on the proof of (20) for \( Q \in C^\infty_0(\Omega) \). It follows from (13) and (14) that

\[
|\nabla \wedge Q|^2 = |\varepsilon^{ijk} Q_{ij,k}|^2 = \varepsilon^{ijk} Q_{ij,k} \varepsilon^{jmn} Q_{m,n} = (\delta^{ij}_m \delta^{jk}_n - \delta^{ij}_n \delta^{jk}_m) Q_{ij,k} Q_{m,n}
\]

\[
= Q_{ij,k} Q_{ij,k} - Q_{ij,k} Q_{ik,j} = |\nabla Q|^2 - Q_{ij,k} Q_{ik,j}.
\]

Therefore by \( Q|_{\partial \Omega} = 0 \),

\[
\int_{\Omega} Q_{ij,k} Q_{ik,k} \, dx = \int_{\Omega} (Q_{ij,k} Q_{ik,k} + \partial_j (Q_{ij} Q_{ik,k}) - \partial_k (Q_{ij} Q_{ik,j})) \, dx
\]

\[
= \int_{\Omega} (|\nabla Q|^2 - |\nabla \wedge Q|^2) \, dx + \int_{\partial \Omega} (Q_{ij} Q_{ik,j} \nu_k - Q_{ij} Q_{ik,k} \nu_j) \, dS
\]

\[
= \int_{\Omega} (|\nabla Q|^2 - |\nabla \wedge Q|^2) \, dx.
\]

The above two formula together implies

\[
a(Q, Q) = \int_{\Omega} \left( L_1 |\nabla Q|^2 + L_2 Q_{ij,k} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} \right) \, dx
\]

\[
= \int_{\Omega} L_1 |\nabla Q|^2 \, dx + \int_{\Omega} (L_2 + L_3) (|\nabla Q|^2 - |\nabla \wedge Q|^2) \, dx.
\]

Therefore, it is easy to see that

\[
a(Q, Q) \geq \int_{\Omega} L_1 |\nabla Q|^2 \, dx
\]

when \( L_2 + L_3 \geq 0 \), and

\[
a(Q, Q) \geq \int_{\Omega} (L_1 + L_2 + L_3) |\nabla Q|^2 \, dx = L_0 \int_{\Omega} |\nabla Q|^2 \, dx
\]

when \( L_2 + L_3 \leq 0 \). \( \square \)
The validity of the $L^p$-estimate requires the verification of the strong Legendre condition for $-\mathcal{L}$ in (7). To this end, we consider the following second order operator defined for $Q \in H^2(\Omega; \mathbb{R}^{3 \times 3})$:

$$(\tilde{\mathcal{L}}(Q))_{ij} = -L_i \Delta Q_{ij} - \frac{1}{4}(L_2 + L_3)\left(\partial_i \partial_k Q_{jk} + \partial_j \partial_k Q_{ik} + \partial_i \partial_k Q_{kj} + \partial_j \partial_k Q_{ki}\right) - \frac{4}{3} \partial_i \partial_j Q_{kk} - \frac{4}{3} \delta^i_j \partial_k \partial_l Q_{kl} + \frac{4}{9} \delta^i_j \Delta Q_{kk}.$$ 

**Lemma 2.2.** Let $p > 1$ be fixed. For any $F \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ and $g \in W^{2-1/p,p}(\partial \Omega; \mathbb{R}^{3 \times 3})$, there exists a unique $Q \in W^{2,p}(\Omega; \mathbb{R}^{3 \times 3})$ that solves $\tilde{\mathcal{L}}Q = F$ with boundary condition $Q|_{\partial \Omega} = g$. Moreover, there exists $C > 0$ depending only on $\Omega$ such that

$$||Q||_{W^{2,p}(\Omega)} \leq C \left(||g||_{W^{2-1/p,p}(\partial \Omega)} + ||F||_{L^p(\Omega)}\right).$$

Especially, when $F, g \in Q$, we have $Q \in W^{2,p}(\Omega; Q)$ satisfying $\tilde{\mathcal{L}}Q = F$.

**Proof.** It suffices to verify the strong Legendre condition (see (15) for instance) for $\tilde{\mathcal{L}}$ and then the conclusion follows from standard theory of elliptic system (cf. [3, Chapter IV] or [10, Chapter 10]). To this end, we first note that $\tilde{\mathcal{L}}$ can be written as

$$(\tilde{\mathcal{L}}(Q))_{ij} = -L_i \partial_k \left(\delta^k_i \delta^j_l \delta^j_l \partial_k Q_{ij}ight) - \frac{1}{4}(L_2 + L_3)\partial_k \left(\left(\delta^k_i \delta^j_l \delta^j_l + \delta^j_i \delta^k_l \delta^i_l + \delta^j_i \delta^k_l \delta^j_l + \delta^j_i \delta^k_l \delta^j_l\right)\partial_k Q_{ij}ight) - \frac{4}{3} \delta^j_i \delta^k_l \partial_k Q_{ij} - \frac{4}{3} \delta^j_i \delta^j_l \partial_k Q_{ij} + \frac{4}{9} \delta^j_i \delta^j_l \partial_k Q_{ij} = -\partial_k \left(A_{(ij)(i'j')ij} \partial_k Q_{ij'}\right),$$

where

$$A_{(ij)(i'j')} = L_i \delta^j_l \delta^k_l + \frac{1}{4}(L_2 + L_3)\left(\delta^j_l \delta^j_l + \delta^j_l \delta^j_l + \delta^j_l \delta^j_l + \delta^j_l \delta^j_l\right)\partial_k Q_{ij} - \frac{4}{3} \delta^j_i \delta^k_l \partial_k Q_{ij} - \frac{4}{3} \delta^j_i \delta^j_l \partial_k Q_{ij} + \frac{4}{9} \delta^j_i \delta^j_l \partial_k Q_{ij}.$$  

To verify the strong Legendre condition for $A_{(ij)(i'j')}$, we need to compute

$$A_{(ij)(i'j')} \xi_{ij} = L_1 \sum_{i,j} \xi_{ij}^2 + \frac{1}{4}(L_2 + L_3)\left(\xi_{ij}^2 + \xi_{ij}^2 + \xi_{ij}^2 + \xi_{ij}^2\right)\partial_k Q_{ij} + \frac{4}{3} \delta^j_i \partial_k Q_{ij} - \frac{4}{3} \delta^j_i \partial_k Q_{ij} + \frac{4}{9} \delta^j_i \partial_k Q_{ij} = -\sum_{i,j} |\xi_{ij}|^2.$$ 

To this end, we define a new tensor by $\zeta_{ij} = \xi_{ij} - \frac{1}{2} \delta^j_i \delta^j_k$. Then it is easy to verify that

$$\sum_{i,j} |\zeta_{ij}|^2 \leq \sum_{i,j} |\xi_{ij}|^2.$$ 

Thus we have, for the case $L_2 + L_3 \leq 0$

$$A_{(ij)(i'j')} \xi_{ij} = L_1 \sum_{i,j} \left|\xi_{ij}\right|^2 + \frac{1}{4}(L_2 + L_3)\left(\zeta_{ij}^2 + \zeta_{ij}^2 + \zeta_{ij}^2 + \zeta_{ij}^2\right)\partial_k Q_{ij} + \frac{4}{3} \delta^j_i \partial_k Q_{ij} - \frac{4}{3} \delta^j_i \partial_k Q_{ij} + \frac{4}{9} \delta^j_i \partial_k Q_{ij}$$

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Corollary 2. The operator $\mathcal{L} : H^2(\Omega; \mathbb{Q}) \cap H^1_0(\Omega; \mathbb{Q}) \mapsto L^2(\Omega; \mathbb{Q})$ defined by (8) is an isomorphism.

3. Abstract form of the system. The task of this section is to setup the functional analytic framework for (5) and (9). We first remark that the Beris-Edward system (5) obeys the basic energy dissipation law

$$\frac{d}{dt} \left( F(Q, \nabla Q) + \int_{\Omega} \frac{1}{2} |u|^2 \, dx \right) + \int_{\Omega} \left( |\nabla u|^2 + |\mathcal{H}(Q)|^2 \right) \, dx = 0. \tag{22}$$
This can be formally done by first testing equation (5a) by the velocity field \( u \) and testing (5c) by \( \mathcal{H}(Q) \) in (7), then simple integration by parts lead to:

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = - \int_{\Omega} (-S_Q(\mathcal{H}) + Q \cdot \mathcal{H} - \mathcal{H} \cdot Q : \nabla u) \, dx - \int_{\Omega} \partial_j \left( \frac{\partial F}{\partial Q_{kl,j}} Q_{kl,i} \right) u_i \, dx,
\]

(23)

and

\[
\begin{align*}
- \frac{d}{dt} F(Q, \nabla Q) &+ \int_{\Omega} u \cdot \nabla Q : \mathcal{H}(Q) \, dx \\
&= \int_{\Omega} \left( |\mathcal{H}(Q)|^2 + (S_Q(D(u)) + W(u) \cdot Q - Q \cdot W(u)) : \mathcal{H} \right) \, dx.
\end{align*}
\]

(24)

Since we have

\[
\int_{\Omega} \partial_j \left( \frac{\partial F}{\partial Q_{kl,j}} Q_{kl,i} \right) u_i \, dx = \int_{\Omega} \left( \partial_j \left( \frac{\partial F}{\partial Q_{kl,j}} \right) Q_{kl,i} + \frac{\partial F}{\partial Q_{kl,j}} Q_{kl,ij} \right) u_i \, dx
\]

\[
= \int_{\Omega} \left( \mathcal{H}_{kl}(Q) Q_{kl,i} + \frac{\partial F}{\partial Q_{kl}} Q_{kl,i} + \frac{\partial F}{\partial Q_{kl,j}} Q_{kl,ij} \right) u_i \, dx
\]

\[
= \int_{\Omega} \mathcal{H}_{kl}(Q) Q_{kl,i} u_i \, dx
\]

thus subtracting (23) from (24) and using the following cancellation yields (22).

The following lemma indicates the important cancellation law between nonlinear terms:

**Lemma 3.1.** For any \( Q \in \mathcal{Q} \), the linear operator \( S_Q(M) \) defined by (4) is symmetric and traceless. Moreover, for any \( 3 \times 3 \) matrices \( P \) and \( M \), it holds

\[
S_Q(M) : P = S_Q(P) : M.
\]

(25)

In addition, if \( M \) is symmetric, then

\[
(-S_Q(M) + Q \cdot M - M \cdot Q) : P = (-S_Q(S) + Q \cdot A - A \cdot Q) : M,
\]

(26)

where \( S \) and \( A \) are symmetric and anti-symmetric parts of \( P \) respectively.

**Proof.** Note that the space consisting of all \( k \times k \) matrices under the Frobenius product \( A : B = \text{tr}(AB^T) = A_{ij}B_{ij} \) is a Hilbert space and it allows the following direct product decomposition:

\[
\mathbb{R}^{n \times n} = \{ M \in \mathbb{R}^{n \times n}, M_{ij} = M_{ji} \} \oplus \{ M \in \mathbb{R}^{n \times n}, M_{ij} = -M_{ji} \}.
\]

Actually, any \( M \) can be uniquely written as the sum of two orthogonal parts

\[
M = \frac{M + M^T}{2} + \frac{M - M^T}{2}.
\]
Then direct calculations implies the identity (25). Formula (26) is trickier and can be proved using (25):

\[
\begin{align*}
(-S_Q(M) + Q \cdot M - MT \cdot Q) : P &= (-S_Q(M) + Q \cdot M - MT \cdot Q) : (S + A) \\
&= -S_Q(M) : S + (Q \cdot M - MT \cdot Q) : A \\
&= -S_Q(S) : M + (Q \cdot A - A \cdot Q) : M.
\end{align*}
\]

\[\Box\]

The typical situation for the application of (26) is when \( P = \nabla u = D(u) + W(u) \) for some vector field \( u \) and \( Q, M \in \mathbb{Q} \):

\[
(-S_Q(M) + Q \cdot M - MT \cdot Q) : \nabla u = (-S_Q(D(u)) + Q \cdot W(u) - W(u) \cdot Q) : M.
\]

Let \((u_0, Q_0) \in \mathcal{J}\), defined by (11). As usual, the first equation in (5) will be formulated by testing with divergence-free vector fields, or equivalently, by applying the Leray’s projector

\[
\begin{align*}
&\left\{ \begin{array}{l}
u_t = P_\sigma \text{ div } (-u \otimes u + \sigma^s(u, Q) + \sigma^a(Q) + \sigma^d(Q)), \\
Q_t + u \cdot \nabla Q + Q \cdot W(u) - W(u) \cdot Q = H(Q) + S_Q(D(u)),
\end{array} \right.
\end{align*}
\]

where \( P_\sigma : H^{-1}(\Omega; \mathbb{R}^3) \mapsto H^{-1}(\Omega) \) and \( \text{div}: L^2(\Omega; \mathbb{R}^{3 \times 3}) \mapsto H^{-1}(\Omega; \mathbb{R}^3) \) are defined in Section 2. The idea is to rewrite the nonlinear system (27) as an abstract evolution equation in a suitable Banach space. With the notation introduced in Section 2, we define the linearized operator at the initial director field \( Q_0 \) by

\[
\mathcal{L}_{Q_0} \left( \begin{array}{c}
u \\
Q \end{array} \right) := \frac{d}{dt} \left( \begin{array}{c}
u \\
Q \end{array} \right) - \left( P_\sigma \text{ div } [D(u) + S_{Q_0}(\mathcal{L}(Q)) - Q_0 \cdot \mathcal{L}(Q) + \mathcal{L}(Q) \cdot Q_0] - \mathcal{L}(Q) + S_{Q_0}(D(u)) - Q_0 \cdot W(u) + W(u) \cdot Q_0 \right)
\]

and nonlinear part is given by

\[
\tilde{\mathcal{N}}_{Q_0} \left( \begin{array}{c}
u \\
Q \end{array} \right) := \left( P_\sigma \text{ div } [S_Q(\mathcal{J}(Q)) - Q \cdot \mathcal{J}(Q) + \mathcal{J}(Q) \cdot Q - u \otimes u - \frac{\partial F}{\partial \mathcal{Q} Q} : \nabla Q] - \mathcal{J}(Q) - u \cdot \nabla Q \\
+ \left( P_\sigma \text{ div } [S_Q(\mathcal{L}(Q)) - S_{Q_0}(\mathcal{L}(Q)) - (Q - Q_0) \cdot \mathcal{L}(Q) + \mathcal{L}(Q) \cdot (Q - Q_0)] - \mathcal{L}(Q) + S_{Q_0}(D(u)) + Q_0 - Q \cdot W(u) - W(u) \cdot (Q_0 - Q) \right) \right).
\]

So if \((u, Q)\) is a solution to (5) satisfying initial-boundary conditions (9), then \((u_h, Q_h) = (u - u_0, Q - Q_0)\) satisfies the operator equation

\[
\mathcal{L}_{Q_0} \left( \begin{array}{c}
u_h + u_0 \\
Q_h + Q_0 \end{array} \right) = \tilde{\mathcal{N}}_{Q_0} \left( \begin{array}{c}
u_h + u_0 \\
Q_h + Q_0 \end{array} \right),
\]

as well as the homogeneous initial-boundary conditions.

Due to the inhomogenous boundary conditions, the operator \( \tilde{\mathcal{N}}_{Q_0} \) is defined on an affine space. For the purpose of applying classical result in functional analysis and operator theory, we shall rewrite it as an nonlinear operator between two Banach spaces. To this end, we denote the stationary version of (28) by

\[
\mathcal{J}_{Q_0} \left( \begin{array}{c}
u \\
Q \end{array} \right) := \left( P_\sigma \text{ div } [D(u) + S_{Q_0}(\mathcal{L}(Q)) - Q_0 \cdot \mathcal{L}(Q) + \mathcal{L}(Q) \cdot Q_0] - \mathcal{L}(Q) + S_{Q_0}(D(u)) - Q_0 \cdot W(u) + W(u) \cdot Q_0 \right).
\]
Then it follows from the linearity of (28) and the assumption that \((u_0, Q_0)\) is time-
dependent that, equation (30) is equivalent to
\[
\mathcal{L}_Q u = p \frac{\partial u}{\partial t} + \mathcal{Q}_u u + \mathcal{R}_Q u = \mathcal{Q}_0 u + \mathcal{P}_Q (u_0).
\]

The right-hand side of (31) is a translated version of (29) and is a mapping between
linear spaces rather than affine spaces:
\[
\mathcal{N}_{(u_0, Q_0)} u = \mathcal{Q}_0 u + \mathcal{P}_Q (u_0).
\]

So we end up with the following abstract parabolic system that is equivalent to
(30):
\[
\mathcal{L}_Q u = \mathcal{N}_{(u_0, Q_0)} u.
\]

To incorporate the initial-boundary condition (9), we turn to the definition of
functional spaces \(X_0\) and \(Y_0\) such that \(\mathcal{L}_Q, \mathcal{N}_{(u_0, Q_0)} : X_0 \to Y_0\) with \(\mathcal{L}_Q\) being
an isomorphism. Motivated by the idea to construct solutions which are twice
differentiable in time and the precise assertions in Theorem 1.1, we need to prove
the existence of regular solutions of the linear equation
\[
\mathcal{L}_Q u = (f, g)
\]
subject to homogeneous initial data with right-hand side \((f, g) \in Y_0\). The general
linear theory requires a compatibility condition which is taken care of by the
definition of \(Y_0\) as
\[
Y_0 = \{ (f, g) \in H^1(0, T; H^{-1}(\Omega)) \times H^1(0, T; L^2(\Omega; \mathbb{Q})) : \]
\[
(f, g)|_{t=0} \in L^2(\Omega) \times H^1(\Omega) \}. \tag{34}
\]

These spaces are equipped with the usual norms in product spaces and for spaces
of functions of one variable with values in a Banach space together with the correct
norm of the initial data. More precisely, the norm of \(Y_0\) is given by
\[
\|(f, g)\|_{Y_0}^2 = \|f\|^2_{H^{-1}(0, T; H^{-1}(\Omega))} + \|g\|^2_{H^{-1}(0, T; L^2(\Omega; \mathbb{Q}))}.
\]

Note that the last part of the norm is not controlled by applying trace theorems to
the first two parts. Now we turn to the domain of \(\mathcal{L}_Q\):
\[
X_u = H^2(0, T; H^{-1}(\Omega)) \cap H^2(0, T; H^1(\Omega)),
\]
\[
X_Q = H^2(0, T; L^2(\Omega; \mathbb{Q})) \cap H^2(0, T; H^2(\Omega; \mathbb{Q})),
\]

together with the norms
\[
\|u\|_{X_u} = \|u\|_{H^2(0, T; H^{-1}(\Omega))} + \|u\|_{H^{-1}(0, T; H^1(\Omega))} + \|\partial_t u\|_{H^1(\Omega)} + \|\partial_{tt} u\|_{L^2(\Omega)},
\]
\[
\|Q\|_{X_Q} = \|Q\|_{H^2(0, T; L^2(\Omega; \mathbb{Q}))} + \|Q\|_{H^{-1}(0, T; H^2(\Omega; \mathbb{Q}))} + \|\partial_t Q\|_{H^1(\Omega)} + \|\partial_{tt} Q\|_{H^1(\Omega)}.
\]

Note that the last two terms in the norms are important to obtain in the sequel
constants that are uniformly bounded as \(T \to 0\). The corresponding subspaces
related to the homogeneous initial and boundary conditions in the formulation of
the problem are defined by
\[
X_0 = \{ (u, Q) \in X_u \times X_Q \mid Q|_{\partial \Omega} = 0, (u, Q)|_{t=0} = (0, 0) \},
\]
which is equipped with the product norm
\[
\|(u, Q)\|_{X_0} = \|(u, Q)\|_{X_u \times X_Q}.
\]
Proposition 3. If \((u_h, Q_h)\) is a strong solution to (33), then \((u, Q) = (u_h, Q_h) + (u_0, Q_0)\) is a solution to (5) and (9a).

4. Proof of Theorem 1.1. The first step towards the proof of the local in time existence of strong solutions is to construct a regular in time solution, following the method in [1]. The following result establishes the invertibility of the linear operator equation. Note that we are seeking a solution of the linear equation in \(X_0\), i.e., a solution with homogeneous initial and boundary conditions.

Proposition 4. For any fixed \(T \in (0, 1]\), \(\mathcal{L}_{Q_0}\) defined by (28) is a bounded linear operator \(X_0 \mapsto Y_0\) and for every \((f, g) \in Y_0\), the operator equation

\[
\mathcal{L}_{Q_0}(u, Q) = (f, g)
\]

has a unique solution \((u, Q) \in X_0\) satisfying

\[
\|\mathcal{L}_{Q_0}^{-1}(f, g)\|_{X_0} = \|(u, Q)\|_{X_0} \leq C_{\mathcal{L}}\|(f, g)\|_{Y_0},
\]

where \(C_{\mathcal{L}}\) is independent of \(T \in (0, 1]\). In particular \(\mathcal{L}_{Q_0} : X_0 \mapsto Y_0\) is invertible and \(\mathcal{L}_{Q_0}^{-1}\) is a bounded linear operator with norm independent of \(T \in (0, 1]\).

Proof. In order to apply Proposition 2, we define the Hilbert spaces

\[
\mathbb{H} = \mathbb{H}_1 \times \mathbb{H}_2 = L_2^2(\Omega) \times H_0^1(\Omega; Q),
\]

\[
\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 = H_0^1(\Omega; \sigma) \times (H^2(\Omega; Q) \cap H_0^1(\Omega; Q)),
\]

and equip them with standard product Sobolev norm. The dual spaces of \(\mathbb{V}\) with respect to pivot space \(\mathbb{H}\) is

\[
\mathbb{V}' = \mathbb{V}_1' \times \mathbb{V}_2' = H_{\sigma}^{-1}(\Omega) \times L^2(\Omega; Q),
\]

and the dual product is given by

\[
\langle (u, Q), (\varphi, \Phi) \rangle_{\mathbb{V}', \mathbb{V}} := \langle u, \varphi \rangle_{\mathbb{V}_1', \mathbb{V}_1} + \langle Q, \Phi \rangle_{\mathbb{V}_2', \mathbb{V}_2} = \langle u, \varphi \rangle_{H_{\sigma}^{-1}, H_\sigma^1} + \int_{\Omega} Q : \mathcal{L}(\Phi) \, dx,
\]

according to Lemma 2. As a result, the space \(Y_0\) defined by (34) can be written by

\[
Y_0 = \{(f, g) \in H^1(0, T; \mathbb{V}') \mid (f, g)|_{t=0} \in \mathbb{H}\}.
\]

We shall define the bilinear form \(a(\cdot, \cdot)\) on \(\mathbb{V}\) by

\[
a((u, Q), (v, P)) = \int_{\Omega} \left(D(u) + S_{Q_0}(\mathcal{L}(Q))\right) - Q_0 \cdot \mathcal{L}(Q) + \mathcal{L}(Q) \cdot Q_0 \right) : Dv \, dx
\]

\[
+ \int_{\Omega} \left(\mathcal{L}(P) - S_{Q_0}(D(u)) + Q_0 \cdot W(u) - W(u) \cdot Q_0 \right) : \mathcal{L}(P) \, dx.
\]

One can verify that this bilinear form satisfies the hypothesis for applying Proposition 2. Especially, coerciveness follows from cancellation law (26),

\[
a((u, Q), (u, Q)) = \int_{\Omega} \left(D(u) + S_{Q_0}(\mathcal{L}(Q))\right) - Q_0 \cdot \mathcal{L}(Q) + \mathcal{L}(Q) \cdot Q_0 \right) : Du \, dx
\]

\[
+ \int_{\Omega} \left(\mathcal{L}(Q) - S_{Q_0}(D(u)) + Q_0 \cdot W(u) - W(u) \cdot Q_0 \right) : \mathcal{L}(Q) \, dx
\]

\[
= \int_{\Omega} (|D(u)|^2 + |\mathcal{L}(Q)|^2) \, dx \geq C\|(u, Q)\|_{Y_0}^2.
\]
In the last step, we employed Corollary 2. So there exists a bounded linear operator $L: \mathcal{V} \mapsto \mathcal{V}'$ such that
\[ \langle L(u, Q), (\varphi, \Phi) \rangle_{\mathcal{V}', \mathcal{V}} = a((u, Q), (\varphi, \Phi)) . \]
Moreover, for any $(f, g) \in Y_0$, the abstract evolution equation
\[ \langle (\partial_t u, \partial_t Q), (\varphi, \Phi) \rangle_{\mathcal{V}', \mathcal{V}} + \langle L(u, Q), (\varphi, \Phi) \rangle_{\mathcal{V}', \mathcal{V}} = \langle (f, g), (\varphi, \Phi) \rangle_{\mathcal{V}', \mathcal{V}} \]  
has a unique solution $(u, Q)$ satisfying
\[ (u, Q) \in H^1((0, T); \mathcal{V}) \quad \text{and} \quad (\partial_t^2 u, \partial_t^2 Q) \in L^2((0, T); \mathcal{V}') . \]
or shortly $(u, Q) \in X_0$. Now we need to show that (38) is equivalent to (35): it is evident that, choosing $(0, \Phi) \in \mathcal{V}$ in (38) implies the first equation in (35). To identify the equation for $Q$, we choose $(0, \Phi) \in \mathcal{V}$ as test function and deduce from (37) that
\[
\int_{\Omega} g(t, x) : \mathcal{L}(\Phi(x)) \, dx \\
= \langle \partial_t Q, \Phi \rangle_{\mathcal{V}', \mathcal{V}_2} + \int_{\Omega} \left( \mathcal{L}(Q) - \mathcal{S}_Q_0(D(u)) + Q_0 \cdot W(u) - W(u) \cdot Q_0 \right) : \mathcal{L}(\Phi) \, dx \\
= \int_{\Omega} \left( \partial_t Q + \mathcal{L}(Q) - \mathcal{S}_Q_0(D(u)) + Q_0 \cdot W(u) - W(u) \cdot Q_0 \right) : \mathcal{L}(\Phi) \, dx .
\]

In view of Lemma 2, $\mathcal{L}: \mathcal{V}_2 \mapsto L^2(\Omega; \mathcal{Q})$ is bijective and thus
\[ \partial_t Q + \mathcal{L}(Q) - \mathcal{S}_Q_0(D(u)) + Q_0 \cdot W(u) - W(u) \cdot Q_0 = g, \quad \text{a.e. in } \Omega \times (0, T) . \]

Altogether, we have proven that $\mathcal{L}_Q_0: X_0 \mapsto Y_0$ is an isomorphism. Since $\mathcal{L}_Q_0$ is also a bounded linear operator and the operator norm only depends on $Q_0$ and geometry of $\Omega$, the boundedness of its inverse operator $\mathcal{L}_Q_0^{-1}: Y_0 \mapsto X_0$ follows from inverse mapping theorem. The assertion that $C_{\mathcal{F}}$ is independent of $T$ follows from standard energy estimate and the cancellation law (26). Here we omit the details. \hfill \Box

**Proposition 5.** Fix $0 < T \leq 1$, $R > 0$, $(u_0, Q_0) \in \mathcal{F}$. Let $\mathcal{N}(u_0, Q_0)$ be the nonlinear operator defined in (32) and $B_{X_0}(0, R) = \{(v, P) \in X_0, \| (v, P) \|_{X_0} \leq R \}$. Then the following assertions hold for all $(u_i, Q_i) \in B_{X_0}(0, R)$, $i = 1, 2$:

(i) $\mathcal{N}(u_0, Q_0)$ maps $X_0$ to $Y_0$.

(ii) Local Lipschitz continuity: there exists a constant $C_{\mathcal{N}}(T, R, Q_0, u_0) > 0$ such that
\[ \| \mathcal{N}(u_0, Q_0)(u_1, Q_1) - \mathcal{N}(u_0, Q_0)(u_2, Q_2) \|_{Y_0} \leq C_{\mathcal{N}}(T, R, Q_0, u_0) \| (u_1 - u_2, Q_1 - Q_2) \|_{X_0} \| . \] (39)

(iii) Local boundedness:
\[ \| \mathcal{N}(u_0, Q_0)(u_1, Q_1) \|_{Y_0} \leq C_{\mathcal{N}}(T, R, Q_0, u_0) \| (u_1, Q_1) \|_{X_0} + \| \mathcal{E}(u_0, Q_0) \|_{Y_0} . \] (40)

where $\mathcal{E}$ is given by (10).

(iv) For any fixed $R > 0$, $\lim_{T \to 0} C_{\mathcal{N}}(T, R, Q_0, u_0) = 0$.

The proof of this result can be adapted line by line from [1, Proposition 4.3]. Actually, the proof in [1] is slightly more general since they work with variable viscosity in the fluid equation and mixed boundary condition for the $Q$-tensor field.
Proof of Theorem 1.1. The proof will be divided into two steps. First, we shall use Proposition 4 and 5 to prove the existence and uniqueness of a regular in time solution. Based on this, in the second step, owning to a special structure of the system, we improve the spatial regularity of $u$ and also $Q$ and this leads to the strong solution of (5).

**Step 1.** Regularity in time.

We first show that

$$\mathcal{A} := \mathcal{L}_{Q_0}^{-1} \mathcal{N}(u_0, Q_0) : X_0 \mapsto X_0$$

has a unique fixed-point. By (36) and (39) we find for all $(u_{h1}, Q_{h1}) \in B_{X_0}(0, R)$ that

$$\|\mathcal{L}_{Q_0}^{-1} \mathcal{N}(u_0, Q_0)(u_{h1}, Q_{h1}) - \mathcal{L}_{Q_0}^{-1} \mathcal{N}(u_0, Q_0)(u_{h2}, Q_{h2})\|_{X_0}$$

$$\leq C \|\mathcal{N}(u_0, Q_0)(u_{h1}, Q_{h1}) - \mathcal{N}(u_0, Q_0)(u_{h2}, Q_{h2})\|_{Y_0}$$

$$\leq C \mathcal{C}_\mathcal{N}(T, R, u_0, Q_0)\|(u_{h1} - u_{h2}, Q_{h1} - Q_{h2})\|_{X_0}.$$ 

Therefore $\mathcal{A}$ is a contraction mapping for $T \ll 1$. A similar argument shows that $\mathcal{A}$ maps $B_{X_0}(0, R)$ into itself. In fact, by (40), we deduce that

$$\|\mathcal{A}(u_{h1}, Q_{h1})\|_{X_0} \leq C \mathcal{C}_\mathcal{N}(u_0, Q_0)(u_{h1}, Q_{h1})\|_{Y_0}$$

$$\leq C \mathcal{C}_\mathcal{N}(C, T, R, u_0, Q_0)\|(u_{h1}, Q_{h1})\|_{X_0} + \|\mathcal{E}(u_0, Q_0)\|_{Y_0}.$$ 

So we can fix $R \gg 1$ large enough and then choose $T \ll 1$ small enough in such a way that

$$\|\mathcal{A}(u_{h1}, Q_{h1})\|_{X_0} \leq C \mathcal{C}_\mathcal{N}(C, T, R, u_0, Q_0)\|(u_{h1}, Q_{h1})\|_{X_0} + \frac{R}{2} \leq R.$$ 

We conclude from Banach’s fixed-point theorem that $\mathcal{A}$ possess a unique fixed-point $(u_h, Q_h) \in X_0$ and it is a solution of the system (5), according to (33) and Proposition 3.

The argument implies the uniqueness as well. Suppose that there was another solution $(\hat{u}_h, \hat{Q}_h)$ in $B_{X_0}(0, R)$ with $R_1 > R$. Choose $\hat{T} \leq T$ and repeat the above argument to show the uniqueness of fixed-points of $\mathcal{A}$, which implies $(u_h, Q_h) = (\hat{u}_h, \hat{Q}_h)$ on $(0, \hat{T}) \times \Omega$. Then the uniqueness follows by the continuity argument.

So $(u, Q) = (u_h, Q_h) + (u_0, Q_0)$ is a solution of (5) with

$$u \in H^2(0, T; H^2(\Omega)) \cap H^1(0, T; H^1(\Omega)),$$

$$Q \in H^2(0, T; L^2(\Omega; Q)) \cap H^1(0, T; H^2(\Omega; Q)).$$

and it follows from standard interpolation result that

$$u \in C([0, T]; H^1(\Omega)), \quad Q \in C([0, T]; H^2(\Omega))$$

(42)

and

$$u_t \in C([0, T]; L^2(\Omega)), \quad Q_t \in C([0, T]; H^1(\Omega)).$$

(43)

These also imply, together with Sobolev embedding that

$$Q_t + u \cdot \nabla Q \in C([0, T]; W^{1,3/2}(\Omega)).$$

(44)

**Step 2.** Spatial regularity.

For any vector field $v \in H^1(\Omega; \mathbb{R}^3)$, we denote the symmetric matrix

$$T(Q, \nabla v) := S(Q(D(v)) - Q \cdot W(v) + W(v) \cdot Q,$$
where the operator $S_Q$ is defined by (4). We also denote
\[ \sigma(Q, \nabla v) := S_Q(T(Q, \nabla v)) - Q \cdot T(Q, \nabla v) + T(Q, \nabla v) \cdot Q. \]
Note that, for any vector field $v \in H^1(\Omega; \mathbb{R}^3)$, not necessarily divergence-free, $T(Q, \nabla v)$ is traceless and symmetric according to (4). Then we have from $Q$-tensor equation (5c) that
\[ \mathcal{H}(Q) = -T(Q, \nabla u) + (\partial_t Q + u \cdot \nabla Q). \]
In addition, if we define
\[ f := -S_Q(Q_t + u \cdot \nabla Q) + Q \cdot (Q_t + u \cdot \nabla Q) - (Q_t + u \cdot \nabla Q) \cdot Q, \]
then the following identity holds:
\[ D(u) + \sigma(Q, \nabla u) + f = \sigma^s + \sigma^a. \] (46)
Substituting (46) into (5a) leads to
\[ \nabla \cdot (D(u) + \sigma(Q, \nabla u)) + \nabla P = -\nabla \cdot (f + \sigma^d) + u \cdot \nabla u + u_t. \] (47)
If we denote
\[ \tilde{f} := -\nabla \cdot (f + \sigma^d) + u \cdot \nabla u, \]
where $f$ is defined by (45), then due to the regularity result (41), we can show that
\[ \tilde{f} \in C([0, T]; L^{3/2}(\Omega)). \] (48)
Actually, it follows from (44), (4) and Sobolev embedding that
\[ S_Q(Q_t + u \cdot \nabla Q) \in C([0, T]; W^{1,3/2}(\Omega) \cap L^3(\Omega)) \]
and also $f \in C([0, T]; W^{1,3/2}(\Omega))$. These together with (42) imply (48).

The crucial observation is that, (47) is a Stokes system with variable coefficient. To show this, we claim that, the bilinear form
\[ a(u, v) := (D(u) + \sigma(Q, \nabla u), \nabla v) \]
defines a symmetric positive definite bilinear form on $H^1(\Omega; \mathbb{R}^3)$. Actually, note that $S_Q(T)$ is symmetric while $Q \cdot T - T \cdot Q$ is antisymmetric, we infer from (26) that
\[ \sigma(Q, \nabla v) : \nabla u = \left( S_Q(T(Q, \nabla v)) - Q \cdot T(Q, \nabla v) + T(Q, \nabla v) \cdot Q \right) : \nabla u \]
\[ = \left( S_Q(D(u)) - Q \cdot W(u) + W(u) \cdot Q \right) : T(Q, \nabla v) \]
\[ = T(Q, \nabla u) : T(Q, \nabla v). \]
This formula together with the definition of $T(Q, \nabla v)$ implies that, there exists a smooth tensor-valued function
\[ \left\{ \mathbb{A}^{k\ell}_{ij}(z) : Q \rightarrow \mathbb{R} \right\}_{1 \leq i,j,k,\ell \leq 3} \]
with
\[ \xi^k_j \mathbb{A}^{k\ell}_{ij}(z) \xi^\ell_j \geq 0, \forall \xi \in \mathbb{R}^3, \forall z \in \mathcal{Q} \]
such that the following identity holds almost everywhere for \((x, t) \in \Omega_T:\)
\[
\sigma(Q(x, t), \nabla u) : \nabla v = \partial_k u_i \hat{A}^{ij}_{kl}(Q(x, t)) \partial_l v_j, \quad \forall u, v \in H^1(\Omega; \mathbb{R}^3).
\]
Consequently, the system (47) can be reduced to
\[
\begin{aligned}
\partial_t \left( (\delta^i_j \delta^j_k + \hat{A}^{ij}_{kl}(Q(x, t))) \partial_j u_i \right) + \partial_k P &= \tilde{f}_k + \partial_t u_k \\
\nabla \cdot u &= 0, \\
u|_{\partial \Omega} &= 0.
\end{aligned}
\]
Then it follows from Corollary 1 as well as (42) that for a.e. \(t \in [0, T],\)
\[
\|u\|_{W^{2,3/2}(\Omega)} + \|\nabla P\|_{L^{3/2}(\Omega)} \leq C \left( 1 + \|Q\|_{C([0,T];H^2(\Omega))} \right) \|\tilde{f} + \partial_t u\|_{L^{3/2}(\Omega)},
\]
and this yields the second order derivative estimate for the velocity field:
\[
\|u\|_{L^{\infty}(0,T;W^{2,3/2}(\Omega))} + \|\nabla P\|_{L^{\infty}(0,T;L^{3/2}(\Omega))} \leq C \left( 1 + \|Q\|_{C([0,T];H^2(\Omega))} \right) \|\tilde{f} + \partial_t u\|_{C([0,T];L^{3/2}(\Omega))}.
\]
On the other hand, we can write (5c) as
\[
Q_t - \mathcal{L}(Q) = -\mathcal{J}(Q) - u \cdot \nabla Q - Q \cdot W(u) + W(u) \cdot Q + \mathcal{S}_Q(D(u)) = : \mathcal{N}(u, \nabla u, Q, \nabla Q).
\]
We claim that
\[
\mathcal{N}(u, \nabla u, Q, \nabla Q) \in L^\infty(0, T; W^{1,3/2}(\Omega)).
\]
Indeed, it follows from (50) that \(\nabla u \in L^\infty(0, T; W^{1,3/2} \cap L^{3}(\Omega))\) and this together with (42), Sobolev embedding and Hölder’s inequality implies (52). Consequently, we can apply Lemma 2.2 to deduce the higher order regularity of \(Q:\)
\[
\|Q\|_{L^{\infty}(0,T;W^{3,3/2}(\Omega))} \leq C \left( \|\mathcal{N}(u, \nabla u, Q, \nabla Q)\|_{L^\infty(0,T;W^{1,3/2}(\Omega))} + \|Q_t\|_{L^\infty(0,T;W^{1,3/2}(\Omega))} + \|Q\|_{W^{3-2,3/2}(\partial \Omega)} \right).
\]
Combining (50), (53) and (41), one can verify as previously that (44) can be improved to be
\[
Q_t + u \cdot \nabla Q \in L^\infty(0, T; H^1(\Omega)),
\]
and this will in turn improve (49) to be
\[
\mathcal{S}_Q(Q_t + u \cdot \nabla Q) \in L^\infty(0, T; H^1(\Omega)).
\]
So we end up with an improved estimate \(\tilde{f} \in L^\infty(0, T; L^2(\Omega)),\) in contrast to (48). So one can argue in the same manner as in the previous step by employing the second part of Corollary 1:
\[
\|u\|_{L^\infty(0,T;H^2(\Omega))} \leq C \|\tilde{f}, u_t\|_{L^\infty(0,T;L^2(\Omega))}.
\]
This implies \(\mathcal{S}_Q(D(u)) \in L^\infty(0, T; H^1(\Omega))\) and thus \(\mathcal{N}(u, \nabla u, Q, \nabla Q) \in L^\infty(0, T; H^1(\Omega))\). Together with (43) and the boundary condition \(Q|_{\partial \Omega} = Q_0|_{\partial \Omega} \in H^{5/2}(\partial \Omega),\) we can employ Lemma 2.2 to enhance the regularity of (51) by \(Q \in L^\infty(0, T; H^3(\Omega))\). This completes the proof of the main theorem. \(\square\)

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