Homogenousness and Specificity *

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Abstract

We interpret homogenousness as a second order property and base it on the same principle as nonmonotonic logic: there might be a small set of exceptions. We use this idea to analyse fundamental questions about defeasible inheritance systems.

In an appendix, we discuss the concept of the core of a (model) set.

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1 Introduction

1.1 Homogeneousness as a default meta-rule

Homogeneousness was discussed as an important - though rarely explicitly addressed - concept by the author in [Sch97-2], section 1.3.11, page 32, and treated in more detail in [GS16], chapter 11, see also [Sch18b], section 5.7.

It is a second order hypothesis about the world, more precisely about the adequacy of our concepts analysing the world, and discussed in an informal way in [GS16] and [Sch18b].

The aim of these notes is to make the discussion more formal, treating it as a second order application of the fundamental concept of nonmonotonicity - that the set of exceptions is small - and, in particular, to base the intuitively very appealing idea of specificity - a way of solving conflicts between contradictory homogeneousness requirements - on that same fundamental concept.

The author recently discovered (reading [SEP13], section 4.3) that J. M. Keynes’s Principle of the Limitation of Independent Variety, see [Key21] expresses essentially the same idea as homogeneousness. (It seems, however, that the epistemological aspect, the naturalness of our concepts, is missing in his work.) By the way, [SEP13] also mentions “inference pressure” (in section 3.5.1) discussed in [Sch97-2], section 1.3.4, page 10. Thus, the ideas are quite interwoven.

Our main formal contribution here is to analyse a size relation \(<\) (or \(<'\)) between sets, generated by a relation \(\prec\) between elements - similarly to Definition 2.6 and Fact 2.7 in [Sch97-2].

We use these ideas to take a new look at defeasible inheritance systems in Section 7 (page 11), and analyse two fundamental decisions

(1) Upward vs. downward chaining
(2) Extensions vs. direct scepticism

Moreover we outline principles for a formal semantics based on our ideas.

1.2 A general comment

The reader will see that we treat here again semantics based on the notions of distance and size. These notions seem very natural, perhaps also because they have a neurological correspondence: semantically close neurons or groups of neurons tend to fire together, and a large number of neurons has a potentially bigger effect than a small number, as their effect on other neurons might add up.

In an appendix, we discuss a different, we think important, concept, the core of a set, base it on distance, and find it by repeated application of standard theory revision.

2 Filters and Ideals

Definition 2.1

Let \(X \neq \emptyset\).

(1) \(\mathcal{F}(X) \subseteq \mathcal{P}(X)\) is called a filter on \(X\) iff

1. \(X \in \mathcal{F}(X), \emptyset \not\in \mathcal{F}(X)\)

2. \(A \subseteq B \subseteq X, A \in \mathcal{F}(X) \implies B \in \mathcal{F}(X)\)

3. \(A,B \in \mathcal{F}(X) \implies A \cap B \in \mathcal{F}(X)\) (finite intersection suffices here)

(2) If there is \(A \subseteq X\) such that \(\mathcal{F}(X) = \{A' \subseteq X : A \subseteq A'\}\), we say that \(\mathcal{F}(X)\) is the (principal) filter generated by \(A\).
(3) $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ is called an ideal on $X$ iff

1. $X \notin \mathcal{I}(X)$, $\emptyset \in \mathcal{I}(X)$
2. $A \subseteq B \subseteq X$, $B \in \mathcal{I}(X) \Rightarrow A \in \mathcal{I}(X)$
3. $A, B \in \mathcal{I}(X) \Rightarrow A \cup B \in \mathcal{I}(X)$ (finite union suffices here)

**Definition 2.2**

Let $X \neq \emptyset$, $\mathcal{F}(X)$ a filter over $X$, then

$\{ A \subseteq X : X - A \in \mathcal{F}(X) \}$ is the corresponding ideal $\mathcal{I}(X)$ (and $\mathcal{F}(X) \cap \mathcal{I}(X) = \emptyset$).

Given $\mathcal{F}(X)$ and the corresponding $\mathcal{I}(X)$, we set

$\mathcal{M}(X) := \{ A \subseteq X : A \notin \mathcal{F}(X) \cup \mathcal{I}(X) \}$.

The intuition is that elements of the filter are big subsets, of the idea small subsets, and subsets in $\mathcal{M}$ have medium size.

$\square$

When speaking about $\mathcal{F}, \mathcal{I}, \mathcal{M}$ over the same set $X$, we will always assume that they correspond to each other as just defined.

**Remark 2.1**

$X \in \mathcal{I}(X \cup Y) \Rightarrow Y \in \mathcal{F}(X \cup Y)$, but not necessarily the converse.

**Proof**

$X \in \mathcal{I}(X \cup Y) \Rightarrow (X \cup Y) - X \in \mathcal{F}(X \cup Y)$, and $(X \cup Y) - X \subseteq Y$, so $Y \in \mathcal{F}(X \cup Y)$.

For the converse: Consider $X = Y$, then $Y \in \mathcal{F}(X \cup Y)$, but $X \notin \mathcal{I}(X \cup Y)$.

$\square$

**Definition 2.3**

Given $X$, $\mathcal{F}(X)$ (and corresponding $\mathcal{I}(X)$, $\mathcal{M}(X)$), and $A, B \subseteq X$, we define:

1. $A <_X B \iff A \in \mathcal{I}(X), B \in \mathcal{F}(X)$
2. $A <'_X B \iff$
   - (a) $B \in \mathcal{F}(X)$ and $A \in \mathcal{I}(X) \cup \mathcal{M}(X)$
   - or
   - (b) $B \in \mathcal{M}(X)$ and $A \in \mathcal{I}(X)$
3. If $X = A \cup B$, we write $A < B$ and $A <' B$, instead of $A <_X B$ and $A <'_X B$.

Note that case (2)(b) of the definition is impossible if $X = A \cup B$: By Remark 2.1 (page 4), if $A \in \mathcal{I}(A \cup B)$, then $B \in \mathcal{F}(A \cup B)$.

Obviously, $<_X$ and $<_' X$ are irreflexive.

We define two coherence properties:
Definition 2.4
(Coh1) \( X \subseteq Y \Rightarrow \mathcal{I}(X) \subseteq \mathcal{I}(Y) \).
(Coh2) \( A, B \in \mathcal{I}(X), A \cap B = \emptyset \Rightarrow A \in \mathcal{I}(X - B) \)

These properties will be discussed in more detail below in Section 3 (page 6), as they are closely related to properties of a preferential relation \( \prec \) between elements of \( X \), see [Sch18a].

First, an initial remark: if (Coh1) and (Coh2) hold, \( \mathcal{F} \) and \( \mathcal{I} \) behave well:

Fact 2.2
(Coh1) + (Coh2) imply:
(1) Let \( X \in \mathcal{F}(X') \), then \( (X \cap A) \in \mathcal{F}(X) \Leftrightarrow X' \cap A \in \mathcal{F}(X') \)
(2) Let \( X' \in \mathcal{F}(X), Y' \in \mathcal{F}(Y) \), then the following four conditions are equivalent:
\( X < Y, X' < Y, X < Y', X' < Y' \)

Proof
(1)
\( \Rightarrow \): \( X \in \mathcal{F}(X') \), so \( X' - X \in \mathcal{I}(X') \), \( X - A \in \mathcal{I}(X) \subseteq \mathcal{I}(X') \) by (Coh1), so \( (X' - X) \cup (X - A) \in \mathcal{I}(X') \Rightarrow X' - ((X' - X) \cup (X - A)) = X' \cap X \cap A \subseteq X' \cap A \in \mathcal{F}(X') \).
\( \Leftarrow \): \( X' \cap A \in \mathcal{F}(X') \), so \( X' - A \in \mathcal{I}(X') \), and \( X - A \subseteq X' - A \), so \( X - A \in \mathcal{I}(X') \). Moreover \( X' - X \in \mathcal{I}(X') \), and \( (X' - X) \cap (X - A) = \emptyset \), so \( X - A \in \mathcal{I}(X) \) by (Coh2).

(2)
By Remark 2.4, it suffices to show \( X' \in \mathcal{I}(X' \cup Y) \) etc.
We use the finite union and downward closure properties of \( \mathcal{I} \) without mentioning.

We also use the following without further mentioning:
(a) \( (X \cup Y) - (X - X') \subseteq X' \cup Y \)
(b) \( (X \cup Y) - (Y - Y') \subseteq X \cup Y' \)
(c) \( (X \cup Y) - ((X' - X') \cup (Y - Y')) \subseteq X' \cup Y' \)
(d) \( X - X', Y - Y', (X - X') \cup (Y - Y') \in \mathcal{I}(X \cup Y) \) by (Coh1)

We now show the equivalences.
(2.1) \( X < Y \Rightarrow X' < Y' \):
\( X' \in \mathcal{I}(X \cup Y), X - X' \in \mathcal{I}(X \cup Y) \), so by (Coh2) \( X' \in \mathcal{I}((X \cup Y) - (X - X')) \subseteq \mathcal{I}(X \cup Y) \) by (Coh1).

(2.2) \( X < Y \Rightarrow X < Y' \):
\( X \in \mathcal{I}(X \cup Y), ((Y - Y') - X) \in \mathcal{I}(Y) \subseteq \mathcal{I}(X \cup Y) \) by (Coh1). \( X \cap ((Y - Y') - X) = \emptyset \), so \( X \in \mathcal{I}((X \cup Y) - ((Y - Y') - X)) \) by (Coh2), but \( (X \cup Y) - ((Y - Y') - X) = X \cup Y' \).

Note that we did not use \( X', X \) is just an arbitrary set.

(2.3) \( X < Y \Rightarrow X' < Y' \):
Let \( X < Y \); by (2.1) \( X' < Y' \), so by (2.2) \( X' < Y' \).

(2.4) \( X < Y' \Rightarrow X < Y \):
Trivial by (Coh1).

(2.5) \( X' < Y \Rightarrow X < Y' \):
\( X' \in \mathcal{I}(X' \cup Y) \subseteq \mathcal{I}(X \cup Y), X - X' \in \mathcal{I}(X) \subseteq \mathcal{I}(X \cup Y) \), so \( X \in \mathcal{I}(X \cup Y) \).

Note that we did not use \( Y', Y \) is just an arbitrary set.

(2.6) \( X' < Y' \Rightarrow X < Y \):
Let $X' < Y'$, so $X' < Y$ by (2.4), so $X < Y$ by (2.5).

\[ \square \]

**Definition 2.5**
Let $X \neq \emptyset$, $\prec$ a binary relation on $X$, we define for $\emptyset \neq A \subseteq X$
\[ \mu(A) := \{ x \in A : \exists x' \in A. x' \prec x \} \]
We assume in the sequel that for any such $X$ and $A$, $\mu(A) \neq \emptyset$.

**Fact 2.3**
Let $\mathcal{F}(A) := \{ A' \subseteq A : \mu(A) \subseteq A' \}$ the filter over $A$ generated by $\mu(A)$, then the corresponding $\mathcal{I}(A) = \{ A' \subseteq A : A' \cap \mu(A) = \emptyset \}$, and $\mathcal{M}(A) = \{ A' \subseteq A : A' \cap \mu(A) \neq \emptyset \}$.

When we discuss $\prec$ on $U$, and $<_X$, $<_X', <'$ for subsets of $U$, we implicitly mean the filters, ideals, etc. generated by $\mu$ on subsets of $U$, as discussed in Fact 2.3 (page 6).

It is now easy to give examples:

**Example 2.1**
$<$ is neither upward nor downward absolute. Intuitively, in a bigger set, formerly big sets might become small, conversely, in a smaller set, formerly small sets might become big.

Let $A, B \subseteq X \subseteq Y$. Then
(1) $A <_X B$ does not imply $A <_Y B$
(2) $A <_Y B$ does not imply $A <_X B$

(1): Let $Y := \{ a, b, c \}$, $X := \{ a, b \}$, $c \prec b \prec a$. Then $\{ a \} <_X \{ b \}$, but both $\{ a \}, \{ b \} \in \mathcal{I}(Y)$.
(2): Let $Y := \{ a, b, c \}$, $X := \{ a, c \}$, $c \prec b \prec a$, but NOT $c \prec a$. Then $\{ a \} <_Y \{ c \}$, but both $\{ a \}, \{ c \} \in \mathcal{M}(X)$.

We discuss now properties of $<$ and $<'$, and their relation to properties of $\prec$, when $<$ ($<'$) are generated by $\prec$ as in Fact 2.3 (page 6).

## 3 $<$ on $U$ and $< (<')$ on $\mathcal{P}(U)$

**Definition 3.1**
We define the following standard properties for $<$:
(1) Transitivity (trivial)
(2) Smoothness
If $x \in X - \mu(X)$, there there is $x' \in \mu(X). x' \prec x$
(3) Rankedness
If neither $x \prec x'$ nor $x' \prec x$, and $x \prec y$ ($y < x$), then also $x' \prec y$ ($y \prec x'$).
(Rankedness implies transitivity.)
See, e.g. Chapter 1 in [Sch18a].
3.1 Simple and smooth \( \prec \)

Recall:

**Definition 3.2**

(\(\mu PR\)) \(X \subseteq Y \Rightarrow \mu(Y) \cap X \subseteq \mu(X)\)

(\(\mu CUM\)) \(\mu(X) \subseteq Y \subseteq X \Rightarrow \mu(X) = \mu(Y)\)

Again, see, e.g. Chapter 1 in [Sch18a](#).

**Fact 3.1**

(1) (Coh1) is equivalent to the basic property of preferential structures, (\(\mu PR\)).

(2) The basic property of smooth preferential structures, (\(\mu Cum\)), implies (Coh2), and (Coh1) + (Coh2) imply (\(\mu Cum\)).

**Proof**

As there is a biggest \(A \in \mathcal{I}(X), A = X - \mu(X)\), we can argue with elements.

(1) (\(\mu PR\)) \(\Rightarrow\) (Coh1): \(x \in \mathcal{I}(X) \Rightarrow x \in X, x \notin \mu(X) \Rightarrow x \in Y, x \notin \mu(Y)\).

(Coh1) \(\Rightarrow\) (\(\mu PR\)): \(x \in \mu(Y) \cap X\), suppose \(x \notin \mu(X) \Rightarrow x \in \mathcal{I}(X) \subseteq \mathcal{I}(Y) \Rightarrow x \in Y, x \notin \mu(Y)\), contradiction.

(2) (\(\mu CUM\)) \(\Rightarrow\) (Coh2): Let \(A, B \in \mathcal{I}(X), A \cap B = \emptyset\), so \(\mu(X - B) = \mu(X) \Rightarrow A \in \mathcal{I}(X - B)\).

(Coh1) + (Coh2) \(\Rightarrow\) (\(\mu CUM\)): Let \(\mu(X) \subseteq Y \subseteq X, X - Y, Y - \mu(X) \in \mathcal{I}(X)\), and \((X - Y) \cap (Y - \mu(X)) = \emptyset\), so \(Y - \mu(X) \in \mathcal{I}(X - (X - Y)) = \mathcal{I}(Y)\), so \(\mu(Y) \subseteq \mu(X)\). \(\mu(X) \subseteq \mu(Y)\) follows from (Coh1)

\(\square\)

**Example 3.1**

(1) Consider \(a \prec b \prec c\), but not \(a \prec c\), with \(Y := \{b, c\}, X := \{a, b\}, Z := \{a, c\}\). Then \(\{b\} <_{X} \{a\}, \{c\} <_{Y} \{b\}\), but \(\{c\} \not<_{Z} \{a\}\). Non-transitivity of \(\prec\) is crucial here.

(2) Consider \(a, a_{i} : i \in \omega, b, c\) with \(c \prec b, a \succ a_{0} \succ a_{1} \succ \ldots\), and close under transitivity. Then \(\{a\} <_{\{a, b, a_{i} : i \in \omega\}} \{b\}, \{b\} <_{\{b, c\}} \{c\}, \) but \(\{a\} \not<_{\{a, c\}} \{c\}\). Note that this structure is not smooth, but transitive.

\(\square\)

**Fact 3.2**

\(\prec\) is transitive, if \(\prec\) is smooth.

**Proof**

By Fact [3.1](#) (page 7), we may use (Coh1) and (Coh2).

Let \(X < Y < Z\), so \(X \in \mathcal{I}(X \cup Y)\) and \(Y \in \mathcal{I}(Y \cup Z)\). We have to show \(X <_{X \cup Z} Z\), i.e. \(X \in \mathcal{I}(X \cup Z)\), \(Z \in \mathcal{F}(X \cup Z)\).

Consider \(X \cup Y \cup Z\), then by \(X \in \mathcal{I}(X \cup Y)\), \(X \in \mathcal{I}(X \cup Y \cup Z)\). By the same argument, \(Y \in \mathcal{I}(X \cup Y \cup Z)\), thus \(Y - (X \cup Z) \in \mathcal{I}(X \cup Y \cup Z)\). As \((X \cup Y \cup Z) - (Y - (X \cup Z)) = X \cup Z\), and \(X \cap (Y - (X \cup Z)) = \emptyset\), \(X \in \mathcal{I}(X \cup Z)\) by (Coh2), and \(Z \in \mathcal{F}(X \cup Z)\) by Remark [2.1](#) (page 4)

\(\square\)
3.2 Ranked≺

Rankedness speaks about \( \mathcal{M} \), so it is not surprising that \(<'\) behaves well for ranked≺.

**Definition 3.3**

We define \( rk(X) := rk(\mu(X)) \).

This is well-defined.

**Fact 3.3**

(1) Let \( A, B \subseteq X \).

Then \( A <' X B \) iff

(a) \( rk(B) < rk(A) \) and \( rk(B) = rk(X) \) or
(b) \( rk(B) = rk(A) = rk(X) \) and \( \mu(A) \subseteq \mu(B) = \mu(X) \).

(2) \( A <' B \) iff

(a) \( rk(B) < rk(A) \) or
(b) \( rk(B) = rk(A) \) and \( \mu(A) \subseteq \mu(B) \).

(Recall that case (2) (b) in Definition 2.3 (page 4) is impossible, if \( X = A \cup B \)).

**Proof**

(1)

\( A <' X B \) iff

\( B \in \mathcal{F}(X) \) and \( A \in \mathcal{I}(X) \) or
\( B \in \mathcal{F}(X) \) and \( A \in \mathcal{M}(X) \) or
\( B \in \mathcal{M}(X) \) and \( A \in \mathcal{I}(X) \).

(2)

The case \( A <' B \) is immediate.

\( \square \)

**Example 3.2**

Here, \( \prec \) is transitive and smooth, but not ranked, and \(<'\) is not transitive.

Consider \( X := \{x_2, x_3, x_4\}, Y := \{x_1, x_2, y\}, \)
\( x_4 \prec x_2, y \prec x_3, y \prec x_1. \)
\( \mu(X) = \{x_3, x_4\}, \{x_3\} \in \mathcal{M}(X), \{x_2\} \in \mathcal{I}(X), \{x_2\} <' X \{x_3\}. \)
\( \mu(Y) = \{x_2, y\}, \{x_2\} \in \mathcal{M}(Y), \{x_1\} \in \mathcal{I}(Y), \{x_1\} <' Y \{x_2\}. \)

Let \( x_1, x_3 \in Z, \{x_1\} <' Z \{x_3\}? \)
If \( y \in Z, \{x_1\}, \{x_3\} \in \mathcal{I}(Z). \)
If \( y \notin Z, \{x_1\}, \{x_3\} \in \mathcal{M}(Z). \)

So \( \{x_1\}, \{x_3\} \) have same size in \( Z. \) \( \square \)
Fact 3.4
Let the relation $\prec$ be ranked. Then $\prec'$ is transitive.

Proof
Let $A \prec' B \prec' C$. If both $A \prec' B$ and $B \prec' C$ hold by case (2) (b) in Fact 3.3 (page 8), then $A \prec' C$ again by case (2) (b), otherwise $A \prec' C$ by case (2) (a).

\[\blacksquare\]

Fact 3.5
Let $X < Y' \in M(Y)$, then $X < Y'$ (and $X' < Y'$ for $X' \in M(X) \cup I(X)$).
(This does not hold for $X <' Y$, of course.)

Proof
By $X < Y$, $rk(Y) \prec rk(X)$, but $rk(Y) = rk(Y')$, so $X < Y'$. \[\blacksquare\]

4 Specificity and Differentiation of Size

We now base the specificity criterion on the same notion of size as nonmonotonicity.
In this section, $\rightarrow$ and $\not\rightarrow$ are the positive or negative arrows of defeasible inheritance diagrams.

Fact 4.1
Suppose (Coh1) holds. If $C \subseteq B$ or $C \rightarrow B$, and $B \rightarrow A$, $C \not\rightarrow A$ for some $A$, then $C < B$.
(Likewise, if $B \not\rightarrow A$, $C \rightarrow A$, only the contradiction matters.)

Proof
It suffices to show $C \in I(B \cup C)$, as then $B \in F(B \cup C)$ by Remark 2.1 (page 4).
If $C \subseteq B$ or $C \rightarrow B$, then $C \cap B \in F(C)$, moreover $C \cap \neg A \in F(C)$, so $C \cap B \cap \neg A \in F(C)$ by the finite intersection property. Thus $C - (B \cap \neg A) \in I(C) \subseteq I(B \cup C)$ by (Coh1). $C \cap (B \cap \neg A) \subseteq B \cap \neg A \in I(B)$ by $B \rightarrow A$, so $C \cap (B \cap \neg A) \in I(B) \subseteq I(B \cup C)$ by (Coh1), thus $C \in I(B \cup C)$ by the finite union property. \[\blacksquare\]

Corollary 4.2
If $B \rightarrow A$, $C \not\rightarrow A$, then $C \rightarrow B$ and $B \rightarrow C$ together are impossible by irreflexivity. \[\blacksquare\]

We may now base the specificity principle, like nonmonotonicity itself, on small exception sets, but this time, the exceptions are second order. We thus have a uniform background principle for reasoning.
4.1 Summary

It seems best to illustrate the situation with an example.

Consider the Tweety Diagram: \( D \rightarrow B \rightarrow A, D \rightarrow C \not\rightarrow A, C \rightarrow B \). If we treat \( D \) like \( C \), we violate a comparatively smaller subset: As \( C < B \), and \( D \rightarrow B \), \( D \rightarrow C \), \( D \) is a comparatively smaller subset of \( B \) than of \( C \), and smaller exception sets are more tolerable than bigger exception sets.

Thus, size is a very strong concept for the foundation of nonmonotonic reasoning.

(In addition, otherwise, the chain \( D \rightarrow C \rightarrow B \) has two changes: \( B \rightarrow A \), \( C \not\rightarrow A \), \( D \rightarrow A \), but this way, we have only one change: \( B \rightarrow A \), \( C \not\rightarrow A \), \( D \not\rightarrow A \).)

4.2 Differentiation of Size

Our basic approach has only three sizes: small, medium, big.

Using above ideas, we may further differentiate: if \( A \in \mathcal{I}(B) \), and \( A' \in \mathcal{I}(A) \), then \( A' \) is doubly small in \( B \), etc.

5 Homogenousness

We have now the following levels of reasoning:

1. Classical logic:
   - monotony, no exceptions, clear semantics
2. Preferential logic:
   - small sets of exceptions possible, clear semantics, strict rules about exceptions, like \((\mu CUM)\), no other restrictions
3. Meta-Default rules (Homogenousness):
   - They have the form: \( \alpha \sim \beta \), and even if \( \alpha \land \alpha' \not\sim \beta \) in the nonmonotonic sense of (2), we prefer those models where \( \alpha \land \alpha' \sim \beta \), but exceptions are possible by nonmonotonicity itself, as, e.g., \( \alpha \land \alpha' \sim \neg \beta \) in (2).

   We minimize those exceptions, and resolve conflicts whenever possible, as in Fact 4.1 (page 9), by using the same principle as in level (2): we keep exception sets small. This is summarized in the specificity criterion.

   (We might add a modified length of path criterion as follows: Let \( x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n \), \( x_i \rightarrow y_i \), \( x_{i+1} \not\rightarrow y_i \). We know by Fact 4.1 (page 9) that \( x_0 < \ldots < x_n \), then any shorter chain \( a \rightarrow \ldots \rightarrow b \) has a shorter possible size reduction (if there are no other chains, of course!), and we can work with this. This is the same concept as in \([\text{Sch18e}], \text{section 4}.\)

   This has again a clear (preferential) semantics, as our characterisations are abstract, see e.g. \([\text{Sch18a}]\).

Remark: Inheritance diagrams and Reiter defaults are based on homogenousness, e.g. in the concatenation by default.

6 Extensions

All distance based semantics, like theory revision, counterfactuals, have a natural notion of size: the set of closest elements is the smallest set in the filter. Thus, we can apply our techniques to them, too.

Analogical reasoning and induction also operate with distance (analogical reasoning) and size (comparison of the sample size with the target size), so we may apply our principles here, too.
7 Defeasible Inheritance Diagrams

We discuss two of the main questions about defeasible inheritance diagrams in the light of our above analysis.

(1) Upward versus downward chaining
(2) Extension based versus directly sceptical approaches

7.1 Upward versus downward chaining

The problem of downward chaining

![Diagram 7.1](image)

**Diagram 7.1**

Discussion of Diagram 7.1 (page 11):

We assume (Coh1) and (Coh2).

By Fact 1.1 (page 9), we know that $X < V$, and by Fact 2.2 (page 5) (2), we know that $X' < V'$ for any $X' \in \mathcal{F}(X)$, $V' \in \mathcal{F}(V)$, etc.

By specificity, there is $U' \in \mathcal{F}(U)$ with $U' \subseteq X$, $U' \subseteq V$, $U' \subseteq \neg Y$.

Of course, only (a big subset of) $U \cap X$ is affected by $X \not\rightarrow Y$, as we do only downward reasoning, no analogical (sideways) reasoning, or so.

There is $Z' \in \mathcal{F}(Z)$ with $Z' \subseteq U$, $Z' \subseteq \neg X$, thus, $Z'$ is not affected by $X \not\rightarrow Y$.

But there is no information that $Z' \not\subseteq V$.

So there is $Z'' \subseteq Z'$, $Z'' \in \mathcal{F}(Z)$, $Z'' \subseteq U \cap V \cap \neg X$, and $Z''$ inherits from $V$ that $Z'' \subseteq Y$ (or, better: there is $Z''' \subseteq Z''$, $Z''' \in \mathcal{F}(Z)$, $Z''' \subseteq Y$).

So, in this example, our (downward) approach coincides with upward chaining, see [Sch97-2], section 6.1.3. Basically, the reason is that we look inside $U$, at $U \cap X$, $U - X$, and not only globally at $U$, which would involve (hidden) analogous reasoning.
7.2 Extension based versus directly sceptical approaches

Consider Diagram 7.2 (page 12).

Extension based approaches branch into different extensions when a conflict cannot be solved by specificity. Each extension violates the homogenousness assumption rather drastically. Assuming that half of $U$ (a medium size subset) is in $Y$, the other half in $\neg Y$, is a less drastic violation, thus it corresponds to the overall strategy to minimize violation of homogenousness.

But, there is no principal difference between two medium size subsets and one big and one small subset which are in conflict. So they should be treated the same way. Thus, in one single picture we have not only conflicting big and small subsets, but also conflicting medium size subsets, so this is more in the directly sceptical colour - without saying it is strictly the same as the traditional directly sceptical approach.

Thus, we have $U' \in \mathcal{F}(U)$ with $U' \subseteq V \cap X$, and $U'_1, U'_2 \in \mathcal{M}(U')$, $U'_1 = U' - U'_2$, $U'_1 \subseteq Y$, $U'_2 \subseteq \neg Y$.

Of course, any $U'' \subseteq U - X$ is NOT affected by $X \not\rightarrow Y$.

Thus, considering Diagram 7.3 (page 12), there is $Z' \in \mathcal{F}(Z)$, $Z' \subseteq U - X$, and this inherits only from $V$, i.e. that it is mostly in $Y$.

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**The Nixon Diamond**

![Diagram 7.2](image-url)

Diagram 7.2
Example 7.1
Let \( U \) be the bottom node of two Nixon diamonds, e.g. add to Diagram 7.2 (page 12) three nodes \( V', X', Y' \) with \( U \to V' \to Y', U \to X' \nleftrightarrow Y' \).

Then we have to split \( U \) into two sets \( U_0, U_1 \in \mathcal{M}(U) \), with \( U_0 \subseteq Y, U_1 \cap Y = \emptyset \) (basically) and two sets \( U_2, U_3 \in \mathcal{M}(U) \), with \( U_2 \subseteq Y', U_3 \cap Y' = \emptyset \), and, as they are independent, we split \( U \) into four sets, all in \( \mathcal{M}(U) \), \( U_{00}, U_{01}, U_{10}, U_{11} \), with e.g. \( U_{00} \subseteq Y, Y' \), \( U_{01} \subseteq Y \), but \( U_{01} \cap Y' = \emptyset \), \( U_{10} \cap Y = \emptyset \), but \( U_{10} \subseteq Y' \), and \( U_{11} \cap Y = \emptyset \), \( U_{11} \cap Y' = \emptyset \).

Note that it unimportant in which order we treat the conflicts or if we treat them simultaneously - as should be the case.

7.3 Ideas for a Semantics

If we want to treat the Nixon Diamond, we have to consider \( \mathcal{M}(X) \). So far, we have not considered abstract coherence properties for \( \mathcal{M} \). We do this now.

Definition 7.1

(\( \mu = \)) This is a condition for ranked preferential structures.

\( X \subseteq Y, X \cap \mu(Y) \neq \emptyset \Rightarrow \mu(X) = \mu(Y) \cap X. \)

We re-write this:

\( X \in \mathcal{M}(Y) \cup \mathcal{F}(Y) \Rightarrow \mathcal{F}(X) = \{ Y' \cap X : Y' \in \mathcal{F}(Y) \} \)

(The case \( X \in \mathcal{F}(Y) \) does not interest here very much.)

Illustration of (the main part of) (\( \mu = \)):

Suppose we add to Diagram 7.2 (page 12) an arrow \( U \to W \), then we know that \( U \cap W \in \mathcal{F}(U) \), and \( U \cap Y \in \mathcal{M}(U) \), \( U - Y \in \mathcal{M}(U) \), so \( U \cap W \cap Y \in \mathcal{F}(U \cap Y) \), \( U \cap W - Y \in \mathcal{F}(U - Y) \) by (\( \mu = \)).
If, in addition, we add \( Z \), and \( Z \rightarrow U \), and a negative arrow \( Z \notightarrow Y \) (not \( Z \notightarrow X \), as in Diagram 7.3 (page 12)), \( Z \) is mostly in \( U' \in M(U) \) with \( U' \cap Y = \emptyset \), and we may still conclude by the above that \( Z \) inherits to be (mostly) in \( W \) from \( U \cap W - Y \in \mathcal{F}(U - Y) \).

### 7.3.1 Principles

We work with very few background principles:

1. For many purposes, reasoning with abstract size seems the adequate approach.

2. As always in nonmonotonic reasoning, small sets of (first-level) exceptions are possible, so we work with \( \mathcal{F} \) and \( \mathcal{I} \), instead of the full or empty set (we used \( \emptyset \) above as an abbreviation).

3. The hard rules of the background logic and of the filter/ideal properties tell us how to treat big/small/medium subsets on the first level.

4. This is complemented by the homogenousness principle and conflict resolution by specificity on the second level.

5. We treat all subsets the same way, not medium size sets differently by branching into different possibilities.

6. Specificity is based on the same idea as nonmonotonicity itself: we tolerate (better) small exception sets (than bigger ones).

Based on these principles, we proceed as follows:

1. We decide for a background logic, i.e. for coherence conditions. (Coh1), (Coh2), \( \mu = \) seems a good choice.

2. We respect the coherence conditions, and inherit properties strictly downward, not by analogy. Contradictions are either solved by specificity, or we chose one half for property \( \phi \), the other for \( \neg \phi \).

   Independence is respected as in Example 7.1 (page 13).

   It is important to chose the (sub)sets from which we inherit carefully, there is no analogical reasoning here. This was illustrated in above examples.
8 Appendix - the Core of a Set

The following remarks are only abstractly related to the main part of these notes. The concept of a core is a derivative concept to the notion of a distance, and the formal approach is based on theory revision, see e.g. [LMS01], or [Sch18b], section 4.3.

We define the core of a set as the subset of those elements which are “sufficiently” far away from elements which are NOT in the set. Thus, even if we move a bit, we still remain in the set.

This has interesting applications. E.g., in legal reasoning, a witness may not be very sure about colour and make of a car, but if he errs in one aspect, this may not be so important, as long as the other aspect is correct. We may also use the idea for a differentiation of truth values, where a theory may be “more true” in the core of its models than in the periphery, etc.

In the following, we have a set $U$, and a distance $d$ between elements of $U$. All sets $X, Y, etc.$ will be subsets of $U$. $U$ will be finite, the intuition is that $U$ is the set of models of a propositional language.

**Definition 8.1**

Let $x \in X \subseteq U$.

1. $\text{depth}(x) := \min \{d(x, y) : y \in U - X\}$
2. $\text{depth}(X) := \max \{\text{depth}(x) : x \in X\}$

**Definition 8.2**

Fix some $m \in \mathbb{N}$, the core will be relative to $m$. One might write $\text{Core}_m$, but this is not important here, where the discussion is conceptual.

Define $\text{core}(X) := \{x \in X : \text{depth}(x) \geq \text{depth}(X)/m\}$

(We might add some constant like $1/2$ for $m = 2$, so singletons have a non-empty core - but this is not important for the conceptual discussion.)

It does not seem to be easy to describe the core operator with rules e.g. about set union, intersection, etc. It might be easier to work with pairs $(X, \text{depth}(X))$, but we did not pursue this.

We may, however, base the notion of core on repeated application of the theory revision operator $*$ (for formulas) or $|$ (for sets) as follows:

Given $X \subseteq U$ (defined by some formula $\phi$), and $Y := U - X$ (defined by $\neg \phi$), the outer elements of $X$ (those of depth 1) are $Y \mid X (M(\neg \phi \ast \phi))$. The elements of depth 2 are $(Y \cup (Y \mid X)) \mid (X - (Y \mid X)), M((\neg \phi) \lor (\neg \phi \ast \phi)) * (\phi \land \neg (\neg \phi \ast \phi)))$ respectively, etc.

We make this formal.

**Fact 8.1**

1. The set version
   Consider $X_0$, we want to find its core.
   Let $Y_0 := U - X_0$
   Let $Z_0 := Y_0 \mid X_0$
   Let $X_1 := X_0 - Z_0$
   Let $Y_1 := Y_0 \cup Z_0$
   Continue $Z_1 := Y_1 \mid X_1$ etc. until it becomes constant, say $Z_n = X_n$
   Now we go back: $\text{Core}(X_0) := X_n \cup \ldots \cup X_{n/2}$
(2) The formula version
Consider \( \phi_0 \), we want to find its core.
Let \( \psi_0 := \neg \phi_0 \)
Let \( \tau_0 := \psi_0 * \phi_0 \)
Let \( \phi_1 := \phi_0 \land \neg \tau_0 \)
Let \( \psi_1 := \psi_0 \lor \tau_0 \)
Continue \( \tau_1 := \psi_1 * \phi_1 \) etc. until it becomes constant, say \( \tau_n = \phi_n \)
Now we go back: \( Core(\phi_0) := \phi_n \lor \ldots \lor \phi_{n/2} \)

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