Analysis of finite temperature phase transition using level spacing

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Let $B$ be the largest spacing between adjacent eigenvalues of the Polyakov loop. We propose to employ the distribution of $B$ as an order parameter for the finite temperature phase transition in $SU(N)$ lattice gauge theories.

Using smeared links to reduce ultraviolet fluctuations, we carry out a test for the gauge group $SU(3)$.

The lattice regulator is a powerful non-perturbative framework to study the thermal properties of gauge theories. Polyakov loops provide a useful order parameter for the study of the finite temperature confinement-deconfinement phase transition of non-Abelian $SU(N)$ gauge theories. Polyakov loops are not properly renormalized quantities and in four dimensions are strongly affected by ultraviolet fluctuations. It is well known that in the continuum limit at fixed temperature the expectation value of an unrenormalized Polyakov loop vanishes even in the deconfined phase, due to divergent self-energy contributions to the free energy.

For $N \leq 3$, a lattice theory with Wilson action undergoes a bulk cross-over, separating the lattice strong- and weak-coupling regimes [3]. The bulk cross-over is characterized by a steep increase in the average plaquette value and occurs at a fixed lattice coupling $\beta_B$. This steep increase occurs when the probability of $1 \times 1$ Wilson loops to have eigenvalues near $-1$ decreases dramatically (for $N = \infty$ a true gap opens up). For $N \geq 5$ the cross-over becomes a discontinuity and there is a phase transition, but no symmetry breaks and there is no continuum counterpart to this transition. However, the bulk cross-over can affect the finite temperature transition [4] on coarse lattices.

The Polyakov loop $L$ is the parallel transporter around a closed loop in the “temperature” direction and gets multiplied by a $Z_N$ phase under certain action preserving changes of link variables in the path integral. In the confined regime (the disordered phase of a gauge theory) the eigenvalues of the Polyakov loops are randomly distributed over the unit circle. At some critical $\beta_c$ the $Z_N$ symmetry gets broken and the eigenvalues of the Polyakov loop favor one of the $N$ roots of unity on the unit circle. $\beta_c$ scales with $N_t$, the length of the lattice in the temperature direction. In order to describe the physical finite temperature transition it is necessary to have $\beta_c > \beta_B$. In our

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{plaq_dist.png}
\caption{Eigenvalue distribution of the plaquette in the confined and deconfined phase.}
\end{figure}
SU(3) simulations the plaquette distribution already had an apparent gap for \( N_t = 4 \) over the whole range of \( \beta \) that we studied as seen in Fig. 1.

Let the gauge invariant eigenvalues of a given Polyakov loop \( L \) be \( e^{i\theta_j} (j = 1, 2, \ldots, N) \), \(-\pi \leq \theta_1 < \theta_2 < \ldots < \theta_N < \pi\). The maximal level spacing \( B \) is defined as
\[
\max \{2\pi - \theta_N + \theta_1, \theta_N - \theta_{N-1}, \ldots, \theta_2 - \theta_1\}.
\]

The distribution of \( B \) changes through the finite temperature transition. It is neither affected by the particular location on the unit circle where the spectrum condenses, nor by tunneling between the different \( Z_N \) vacua. If the eigenvalues \( \theta_j \) are randomly distributed on the unit circle \( B \) will be of order \( \sim 2\pi/N \); when the Polyakov loops get ordered, the concentration of angles at one location creates an effective gap, increasing \( B \). Random matrix theory can be used to describe the distribution of \( B \) in the confined phase where it has one peak value. This is no longer true in the deconfined phase where the distribution tends to have two distinct peaks. The development of two peaks in the deconfined phase is signaled by the single peak distribution developing a more populated tail as one goes through the phase transition. Deep in the confined phase one can compute the average value of \( B \) by averaging over a very large number of random matrices. In the case of \( SU(3) \) we get \( \langle B_{\text{random}} \rangle = 0.9564(1) \) and \( \langle B_{\text{random}} \rangle \to 0 \) for \( N \to \infty \) in the case of \( SU(N) \).

Following [1], we denote by Re\((L)\) the projection of the trace of \( L \) onto the nearest \( Z_3 \) axis. The critical \( \beta_c \) is generally determined by locating the peak in \( \chi_L = \langle \text{Re}(L)\text{Re}(L) \rangle - \langle \text{Re}(L) \rangle^2 \). Projection onto the nearest \( Z_N \) works reasonably well for \( N = 3 \) but can become a problem close to the transition for larger \( N \). The observable \( B \) does not suffer from such a problem since it is defined as the biggest level spacing without any reference to its location on the unit circle.

Figure 2. Effect of APE smearing the links.

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We performed various Monte Carlo simulations to study the finite temperature transition using the standard Wilson gauge action. A generic $SU(N)$ gauge fields update was implemented, even though the present results are restricted to $SU(3)$. We used the heat-bath algorithm following Creutz-Kennedy-Pendleton [5], where the update of one link corresponds to updates by all the $N(N - 1)/2$ $SU(2)$ subgroups. In our Markov chain a step between two measurements consisted of one heat-bath sweep plus one over-relaxation sweep to avoid being trapped in a locally metastable state. Each over-relaxation update was done on the full $SU(N)$ group [6]. We considered lattices with $N_t = 4$ and $L = 12, 16, 24$ and typically we collected at least 15,000 measurements. The errors were estimated with the jack-knife method.

In order to suppress ultra-violet fluctuations we used APE smeared links following [7]. We smeared both the time-like and the space-like links on the lattice in parallel and repeated the process $N_t$ times. Subsequently, we used the resulting links to compute the Polyakov loops. In Fig. (2) one can see that smeared links provide a largely enhanced signal in the Polyakov loop susceptibility.

The biggest level spacing turned out to be a good order parameter showing transitions consistent with traces of Polyakov loops as one can see in Fig. (3) (the horizontal line represents $\langle B_{\text{random}} \rangle$, the limiting value as $\beta \to 0$). By studying $\chi_B$ in Fig. (4) we infer $\beta_c = 5.6925(25)$ and this is consistent with [1]. Both $L = 16$ and $L = 24$ show well defined peaks in $\chi_B$ and our estimate of $\beta_c$ was done without resorting to re-weighting and by working at relatively small volumes. The lack of monotonicity seen in Fig. (4) in the deconfined phase is due to the system being trapped into one of the $Z_3$ vacua for a long time during the measurement history.

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