Self-phase modulation of spherical gravitational waves

J.T. Mendonça and V. Cardoso

GoLP and CENTRA, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

M. Marklund

Department of Electromagnetics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden

M. Servin and G. Brodin

Department of Plasma Physics, SE-901 87 Umeå, Sweden

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Abstract

Self-phase modulation of spherical gravitational wavepackets propagating in a flat space-time in the presence of a tenuous distribution of matter is considered. Analogies with respect to similar effects in nonlinear optics are explored. Self phase modulation of waves emitted from a single source can eventually lead to an efficient energy dilution of the gravitational wave energy over an increasingly large spectral range. An explicit criterium for the occurrence of a significant spectral energy dilution is established.

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I. INTRODUCTION

It is well known that the Einstein’s equation describing gravitational waves is strongly nonlinear\textsuperscript{1,2}. Nonlinear wave processes similar to those observed in Optics can then eventually occur. Recently, the possible occurrence of self-phase modulation, harmonic generation and nonlinear wave mixing was considered\textsuperscript{3,4}.

This is an important issue in two different aspects. First, from a theoretical point of view it is important to explore and to understand the similarities and differences between gravitational wave phenomena and effects in Nonlinear Optics that have been tested in laboratory. Second, and in more practical terms, this is important in what concerns the possible detection of gravitational waves. Detectors have been designed and built under the assumption that the frequency spectrum of gravitational waves emitted by astronomical objects is conserved, and that the wave intensities decrease as the inverse of the square of the distance. However, the nonlinear processes can eventually lead to spectral energy dilution such that the energy density received within the detectable frequency bandwidth is significantly decreased. Other processes eventually contributing to energy dilution could be due to the coupling with plasma waves\textsuperscript{5} and with photons\textsuperscript{6,7}.

For waves emitted from a single astronomical source, the main nonlinear effects that can occur are self-phase modulation (for short pulses, of the order of a few cycles) and harmonic cascades (for longer pulses). Attention was however called to the fact that, for parallel propagation, the strong nonlinearities associated with empty and flat space-time exactly cancel each other\textsuperscript{4}. The existence of such a negative result is apparently due to the absence of gravitational wave dispersion. Note, however, that anti-parallel wave configurations lead to a nonlinear coupling\textsuperscript{8}, but clearly these are not relevant to waves emitted by single sources.

In this work, we return to the problem of parallel wave interactions. We will focus on self-phase modulation of spherical waves emitted by isolated sources. For simplicity, we will consider a flat space-time, filled with a tenuous distribution of matter\textsuperscript{9}. The matter distribution will guarantee the existence of wave dispersion, which is an important ingredient of self-phase modulation in Nonlinear Optics\textsuperscript{10,11}. The nonlinear wave equation is established in Section II, and the dispersion properties of linear waves is discussed in Section III. Nonlinear evolution of a spherical gravitational wavepacket is studied in Section IV, where
a necessary criterium for the occurrence of a significant amount of self-phase modulation is established. Finally, in Section V we state the conclusions.

II. NONLINEAR WAVE EQUATION

We consider propagation of small amplitude gravitational waves, in a region of space-time where we have a tenuous distribution of matter \[9\]. We can then, in a first approximation, neglect the background field curvature.

We are considering a flat space-time, perturbed by a small amplitude gravitational wave. This can be described by the metric tensor elements

\[ g_{ij} = \eta_{ij} + h_{ij}, \]  

where \(|h_{ij}| \ll 1\) represent the gravitational wave, and \(\eta_{ij}\) are the metric tensor elements of flat space-time

\[ \eta_{00} = 1 , \quad \eta_{ii} = -1 \quad (i = 1, 2, 3) , \quad \eta_{ij} = 0 \quad (i \neq j). \]

In this case, we can derive from Einstein’s equation the following nonlinear wave equation

\[ \Box^2 h_{ik} = -2\kappa S_{ij} + 2R_{ik}^{(3)}. \]  

where \(\kappa = (8\pi/c^2)G\), and \(G\) is the gravitational constant, and \(\Box^2\) is the d’Alembert operator in the usual form

\[ \Box^2 = \partial^i \partial_j = \eta^{ij} \partial_i \partial_j, \]  

where we use \(\partial_i \equiv \partial/\partial x^i\). The nonlinear term \(R_{ik}^{(3)}\) contains third order nonlinearities of the Ricci tensor. It can be explicitly written as

\[ R_{ik}^{(3)} = -\frac{1}{4} \eta^{nm} h^{lp} \left[ (\partial_j h_{mi} + \partial_i h_{mj} - \partial_m h_{ij}) (\partial_n h_{pk} + \partial_k h_{pn} - \partial_p h_{kn}) \right. \]

\[ \left. - (\partial_k h_{pi} + \partial_i h_{pk} - \partial_p h_{ik}) \partial_l h_{np} \right] . \]

Here we have neglected the second-order nonlinear term, \(R_{ik}^{(2)} = 0\). Note that the second order part of the Ricci tensor does not vanish identically unless the response due to the
pseudo energy-momentum tensor is taken into account \cite{4}. Combined with the original perturbation, these second order terms contribute to the self phase modulation of the wave. The equation studied here may therefore be regarded as a model equation for the nonlinear dynamics of gravitational waves. In equation (3) we have also included the linear dispersion term associated with the matter distribution $\kappa S_{ij}$, where $S_{ij}$ is related with the energy-momentum tensor $T_{ij}$ by

$$S_{ij} = T_{ij} - \frac{1}{2} \eta_{ij} T$$

where $T = T^i_i$ is the trace.

\section*{III. LINEAR DISPERSION RELATION}

Let us first consider the properties of a linear wave, propagating radially from a given point source. In order to discuss the dispersion properties of this wave we first consider the linearized wave equation

$$\Box^2 h_{ik} = -\kappa S_{ij},$$

and assume a plane wave solution of the form

$$h_{ij} = \epsilon_{ij} A \exp[i q_n x^n],$$

where $i = \sqrt{-1}$, $A$ is the amplitude, $\epsilon_{ij}$ is unit polarization tensor such that $\epsilon^i_j \epsilon_{ij} = 1$, and $q_n$ are the components of the four-wavevector. If the scale of variation of the amplitude $A$ is much larger than the typical wavelength, we can use $\partial_j h_{ik} = i q_j h_{ik}$, and write the linear wave equation as

$$\eta^{jn} q_j q_n h_{ik} = 2 \kappa S_{ik}.$$  

The perturbed energy-momentum tensor can be considered proportional to the local amplitude of the gravitational wave: $S_{ik} = w_{ik} A$, where the tensor $w_{ik}$ depends on the properties of the medium. We are then lead to the following linear dispersion relation

$$\eta^{jn} q_j q_n = w,$$  

(10)
where we have used \( w = 2\kappa e^{ikx}w_{ik} \). Particular examples of \( w \) can be found in the literature. For instance, the cases of a cold dust cloud \([12]\) and of a magnetized plasma \([13]\) are well-established and don’t need to be explicitly given here. The contribution to dispersion from the background curvature has also been calculated for various space-times, see e.g. \([12]\).

In order to deal with spherical waves from a localized source it is appropriate to use a spherical coordinate system \((r, \theta, \phi)\), such that:

\[
x^0 = ct, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi.
\]

For waves propagating in the radial direction we have to replace solution \((8)\) by the following wave solution

\[
h_{ik} = \epsilon_{ik} \frac{a}{r} \exp(iq_0 x^0 + iq_1 x^1) = \epsilon_{ik} \frac{a}{r} \exp(iqr - i\Omega t),
\]

where \( a \) is the new amplitude, \( q = q_1 \) and \( \Omega = -q_0c \). Because the nondiagonal components \( \eta^{ik} \), with \( i \neq k \), are equal to zero, we can easily transform the dispersion relation \((10)\) into

\[
\left( \frac{\Omega^2}{c^2} - q^2 \right) = w(r, \theta, \phi),
\]

where we have retained the possibility of a non-uniform distribution of matter. We can see that the matter distribution can change the phase velocity of the gravitational wave, according to

\[
v_f = \frac{\Omega}{q} = \sqrt{c^2 + \frac{w}{k^2}} \simeq c + c \frac{w}{2q^2}.
\]

For the group velocity, we have

\[
v = \frac{\partial \Omega}{\partial q} = \frac{qc^2}{\Omega} \left( 1 + \frac{1}{2q} \frac{\partial w}{\partial q} \right) \simeq c \left( 1 + \frac{1}{2q} \frac{\partial w}{\partial q} \right) \sqrt{1 - wc^2/\Omega^2}.
\]

It is clear that, even if \( w \) is independent of \( q \), the presence of a small amount of matter leads to a group velocity wave dispersion. It is known from Nonlinear Optics that wave dispersion is an essential ingredient of self-phase modulation \([10]\). It is the absence of dispersion that eventually explains the otherwise counter-intuitive result that self-phase modulation is absent for plane gravitational waves propagating in empty flat space-time \([4]\). The inclusion of matter is thus an essential ingredient of the present study.
IV. NONLINEAR WAVE PROPAGATION

We can now examine the possibility of a given radial wave, satisfying the above linear dispersion relation, to interact with itself, due to the nonlinear contributions contained in the term \( R_{ik}^{(3)} \). The nonlinear contributions of \( S_{ik} \) could equally be included, but for simplicity they are neglected here. The existence of nonlinear wave coupling implies that the wave amplitude \( a \) in equation (12) can no longer be a constant, and is replaced by a slowly varying function of \( r \) and \( t \). This means that we now have

\[
\partial_j h_{ik} = (iq_j + \frac{1}{a} \partial_j a)h_{ik}.
\]

So, we can write

\[
\Box^2 h_{ik} \simeq [-\eta^{jn} q_j q_n + i\eta^{jn} q_j \ln a]h_{ik}.
\]

Assuming that the above linear dispersion relation still holds, we can cancel the first of these terms with the linear contribution from \( S_{ik} \). For the second term, we can write it as

\[
i\eta^{jn} q_j \partial_n = -iq\partial_r - i\frac{\Omega}{c^2} \partial_t = -iq(\partial_r + \frac{1}{v} \partial_t)
\]

where \( v = c^2/v_f \) is the group velocity.

In order to establish the nonlinear equation for the slowly varying amplitude \( a \) we now use an approximate expression for \( R_{ik}^{(3)} \), where only the terms oscillating at the frequency of the wave \( \Omega \) are retained

\[
R_{ik}^{(3)} \simeq \frac{1}{2} \left( \frac{\Omega^2}{c^2} - q^2 \right) \frac{|a|^2}{r^2} a \exp(iqr - i\Omega t).
\]

In deriving this expression we have added to the solution (12) its complex conjugate, in order to adequately describe a real wavepacket. For a given line of sight between the source at \( r \simeq 0 \) and the eventual observer at a finite distance \( r \), we can use equation (14) with fixed values of \( \theta \) and \( \phi \). Replacing in the nonlinear wave equation (3) we obtain

\[
iq(\partial_r + \frac{1}{v} \partial_t)a = w(r) \frac{|a|^2}{r^3} a
\]

This nonlinear equation for the slow wave amplitude clearly shows that the nonlinear effects disappear in the absence of matter, \( w(r) = 0 \), as noticed previously [4]. Let us make
a variable transformation from the pair \((r, t)\) to \((z, \tau)\), where we define \(z = r - vt\) and \(\tau = t\). We have then \(\partial_r = \partial_z\) and \(\partial_t = \partial_\tau - v \partial_z\). Replacing this in the above equation, we get

\[
\partial_\tau a = -i \frac{w(z, \tau)}{q} v \frac{|a|^2}{r^2(z, \tau)} a
\]

This equation is satisfied by a solution of the form

\[
a(z, \tau) = a(z) \exp[i\phi(z, \tau)]
\]

with the phase function determined by

\[
\phi(z, \tau) = \phi_0 - \int_{\tau'}^{\tau} \frac{w(z, \tau')}{q} v \frac{|a|^2}{r^2(z, \tau')} d\tau'
\]

This solution represents a gravitational wavepacket propagating spherically with a nearly constant envelope \(a(z)\) and a variable nonlinear phase. The wave frequency shift \(\Delta \Omega\) will be given by the derivative of this phase with respect to the time variable \(t\). Neglecting the small variation of the distance \(r(z, \tau)\) and the matter distribution \(w(z, \tau)\) inside the wavepacket envelope, this means that \(\Delta \Omega\) will be essentially due to the variation of the energy distribution \(|a(z)|^2\) with respect to time. But this envelope is only a function of \(z = r - vt\) and we can use

\[
\partial_t |a(z)|^2 = -v \partial_z |a(z)|^2
\]

We can then state that

\[
\Delta \Omega(\tau) = -v \partial_z \phi(z, \tau).
\]

Noting that, for short wavepackets, the variation of the matter dispersion term \(w(z, \tau)\) and distance with respect to the source \(r(z, \tau)\) can be negligible, we replace them in the expression of the phase by their central values \(w(z) = w(z = 0, \tau)\) and \(r(\tau) = r(z = 0, \tau)\). This leads to the following expression of the frequency shift occurring inside the wavepacket envelope

\[
\Delta \Omega(\tau) = \int_{\tau'}^{\tau} \frac{w(\tau')}{q} \frac{v^2}{r^2(\tau')} \partial_z |a(z)|^2 d\tau'
\]
Here we notice that the distance travelled by the wavepacket can be written as $r(\tau) = \int_0^\tau v(\tau')d\tau' \simeq c\tau$. Neglecting the possible slow change on the shape of the envelope over distance, we can finally write the above expression as

$$\Delta \Omega(\tau) \simeq \frac{1}{q} \partial_z |a(z)|^2 \int_0^\tau \frac{w(\tau')}{\tau'^2} d\tau'$$

(27)

In order to understand the physical meaning of this result, let us consider the simple case of a uniform distribution of matter along the entire line of sight. We can use $w(\tau) \simeq w_0 = \text{const.}$, and get for the frequency shift, after a distance $r \simeq c\tau$ travelled by the wakepacket

$$\Delta \Omega(r) \simeq -\frac{c}{q} w_0 \partial_z |A(z)|^2 r,$$

(28)

where we have used the local spherical wave amplitude $A(z) = a(z)/r$, observed at a distance $r$ from the source. This linear dependence of the frequency shift with time, or with the travelled distance, was found previously for linear propagation [3] and is well known from Nonlinear Optics [10]. Here, however, the frequency shift is proportional to the square of the local amplitude, which means that this effect can only the significant if it occurs over short distances, not far away from the emitter. This feature is specific of spherical waves propagating in uniform media.

Another interesting case is that of a non-uniform matter distribution where the wavepacket propagates across a succession of $N$ localized clouds, at distances $r_i$ (with $i = 1...N$) from the source, and widths $\Delta r_i \ll r_i$. We can then transform equation (27) in

$$\Delta \Omega(r) \simeq -\frac{c}{q} \sum_i w_i \partial_z |A_i(z)|^2 \Delta r_i$$

(29)

where the local envelope amplitudes are determined by $A_i(z) = a(z)/r_i$. Again, the strongest contribution to the total frequency shift will result from the clouds located nearest to the source, supposing that they all have similar matter densities.

Let us assume that the source emits a gravitational wavepacket with $n$ cycles. Its width with be $\delta z \simeq 2\pi n/q$. And the maximum frequency shift associated with the closest cloud ($i = 1$) will be of the order of

$$\Delta \Omega_{\text{max}} \simeq \frac{c}{q} w_1 |A_1(z = 0)|^2 \frac{\Delta r_1}{\delta z}$$

(30)
We can also write \( w_1 = (\Omega/c)^2 \alpha \), where \( \alpha \ll 1 \) is a small dimensionless factor. Noting that \( \Omega \simeq qc \), this allows us to establish the necessary condition for a large frequency shift, leading to a significant spectral energy dilution, as \( \Delta \Omega \geq \Omega \), or equivalently

\[
\alpha |A_1|^2 \frac{\Delta r_1}{\delta z} = \frac{\alpha}{2\pi n} |A_1|^2 (q\Delta r_1) \geq 1 \tag{31}
\]

Notice here that \((q\Delta r_1/2\pi)\) is the number of wavelengths over the cloud width. This can be a very large number. For instance, for \( \Omega = 10^4 \) (wave in the KHz range) and a width of \( \Delta r_1 = 10^{-4} \) parsec, we get \((q\Delta r_1) = 10^{13}\). This means that, for a short pulse \((n < 10)\), dense enough cloud \((\alpha > 10^{-4})\) and close enough source \((|A_1| > 10^{-4})\), the above criterium could be satisfied. This confirms, in more solid grounds, the suggestion previously made \[3\] that self-phase modulation could eventually take place.

Some further discussion on the size of the effect of self-phase modulation may be in place. In principle, close to the source the self-phase modulation could be stronger than the value presented above, not only because of larger gravitational wave amplitude, but mainly because of the significant angular dependence of the metric. This can be seen as follows. Observe that terms proportional to \( \Omega^2 - c^2 q^2 \) in Eq. \[19\] implicitly contains an angular dependence, due to the fact that for spherical waves \( q = q(r) \) is found from the spherical Bessel functions. Separating variables in spherical coordinates, it can be inferred that \( \Omega^2 - c^2 q^2 \sim \ell(\ell-1)c^2/r^2 \), \( \ell \) being the mode number of the associated Legendre polynomial. Thus, for high mode numbers, the nonlinear modification may increase by a substantial amount.

V. CONCLUSIONS

Nonlinear wave propagation of spherical gravitational waves was considered in this work. The possible occurrence of self-phase modulation was discussed. The case of short wavepackets emitted from a point source in flat space-time was examined, where a tenuous distribution of matter was retained in order to guarantee linear wave dispersion, which is a necessary condition for self-phase modulation to occur. An explicit criterium for a significant spectral energy dilution due to self-phase modulation was established. It leads to the conclusion that the occurrence of self-phase modulation due to matter distribution very close to the gravitational wave source is plausible.

In contrast with what could occur with plane wave propagation, for spherical waves the
contributions of phase modulation over distance decay very rapidly with distance from the
source, due to wave amplitude decrease. For this reason, self-phase modulation is dominantly
occurring very close to the emitter. Notice however that the region where this effect takes
place is not necessarily the region where it can be observed, because the expanded wave
spectrum will then propagate far away without further changes.

The efficiency of the self-phase modulation process is directly dependent on wave dis-

currence, which is a consequence of matter distribution. Curvature of space-time would also
contribute to wave dispersion and would enhance the process. If, instead of spherical wave
emission we have some kind of directionality, the wave amplitude decay will be smaller and
phase modulation will also increase. Another source of nonlinearity is the energy-momentum
tensor, or the matter distribution itself, which was not retained here. Space-time curvature,
directionality effects and energy-momentum nonlinearities will eventually lead to more fa-
vorable criteria for the occurrence of self-phase modulation of gravitational waves, and will
be considered in a future work.

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