Schrödinger equation of general potential

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Abstract

It is well known that the Schrödinger equation is only suitable for the particle in common potential $V(\vec{r}, t)$. In this paper, a general Quantum Mechanics is proposed, where the Lagrangian is the general form. The new quantum wave equation can describe the particle which is in general potential $V(\vec{r}, \vec{\dot{r}}, t)$. We think these new quantum wave equations can be applied in many fields.

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1. Introduction

It is well known that quantum mechanics (QM) acquired its final formulation in 1925-1926 through fundamental papers of Schrödinger and Heisenberg. Originally these papers appeared as two independent views of the structure of quantum mechanics, but in 1927 Schrödinger established their equivalence, and since then one or the other of the papers mentioned have been used to analyze quantum mechanical systems, depending on which method gave the most convenient way of solving the problem. Thus the existence of alternative procedures to solve a given problem can be quite fruitful in deriving solutions of it. Quantum Mechanics has become one of the most important foundations of physics, and achieved great success, physicists had begun to consider the possibility to generalize the traditional framework of it[1]. Up to now, although various generalizations of QM are all not very useful or successful, this kind of attempts have never stopped[2-6].

In the 1940’s Richard Feynman, and later many others, derived a propagator for quantum mechanical problems through a path integration procedure[7-9]. In contrast with the Hamiltonian emphasis in the original formulation of quantum mechanics, Feynman’s approach could be referred to as Lagrangian and it emphasized the propagator $K(x, t; x', t')$ which takes the wave function $\psi(x', t')$ at the point $x'$ and time $t'$ to the point $x$ at time $t$. While this propagator could be derived by the standard methods of quantum mechanics, Feynman invented a procedure by summing all time dependent paths connecting points $x, x'$ and this became an alternative formulation of quantum mechanics whose results coincided with the older version when all of them where applicable, but also became relevant for problems that the original methods could not solve.

At present, quantum mechanics is only suitable for common potential $V(\vec{r}, t)$, but not suitable for general potential $V(\vec{r}, \vec{r}, t)$. In this paper, we extend the Schrödinger equation to suit general potential.
2. The Schrödinger wave equation for general Lagrangian

It’s well known that the Schrödinger wave equation can be obtained by Feynman path integral. For the Lagrangian function

\[ L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m \dot{\vec{r}}^2 - V(r, t). \]  (1)

For each path \( x(t) \) connecting \((x_i, t_i)\) and \((x_f, t_f)\), calculate the action \( S[\vec{r}(t)] \) defined by

\[ S[\vec{r}(t)] = \int_{t_i}^{t_f} L(\vec{r}(t), \dot{\vec{r}}(t), t) dt, \]  (2)

with the help of the formula of Feynman path integral

\[ \psi(\vec{r}_2, t_2) = \int d\vec{r}_1 K(\vec{r}_2 t_2; \vec{r}_1 t_1) \psi(\vec{r}_1, t_1), \]  (3)

where

\[ K(\vec{r}_2 t_2; \vec{r}_1 t_1) = \int \exp[i S[\vec{r}(t)]/\hbar] D[\vec{r}(t)], \]  (4)

we can obtain the Schrödinger wave equation

\[ i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = [-\frac{\hbar}{2m} \nabla^2 + V(r, t)] \psi(\vec{r}, t). \]  (5)

In the following, we will give the general Schrödinger wave equation for the general Lagrangian function

\[ L(x, \dot{x}, t) = a(t) \dot{x}^2 + b(t) x \ddot{x} + c(t) x^2 + d(t) \dot{x} + f(t) x + g(t), \]  (6)

we have already found that as a consequence of Eq. (3), we have the equation

\[ \psi(x_2, t_2) = \int_{-\infty}^{\infty} K(x_2, t_2; x_1, t_1) \psi(x_1, t_1) dx_1, \]  (7)

Eq. (7) gives the wave function at a time \( t_2 \) in terms of the wave function at a time \( t_1 \). In order to obtain the differential equation, we apply this relationship in the special case that the time \( t_2 \) differs only by an infinitesimal interval \( \varepsilon \) from \( t_1 \). The propagator \( K(x_2, t_2; x_1, t_1) \) is proportional to the exponential of \( i/\hbar \) times the action for the interval \( t_1 \) to \( t_2 \). For a short interval \( \varepsilon \) the action is approximately \( \varepsilon \) times the Lagrangian for this interval, we have

\[ \psi(x, t + \varepsilon) = C \int_{-\infty}^{\infty} \exp[i \frac{\varepsilon}{\hbar} L(\frac{x - x'}{\varepsilon}, \frac{x + x'}{2})] \psi(x', t) dx', \]  (8)
substituting Eq.(6) into (8), one can obtain

$$\psi(x, t + \varepsilon) = C \int_{-\infty}^{\infty} \exp \left[ \frac{i\varepsilon}{\hbar} (a(t) \left( \frac{x-x'}{\varepsilon} \right)^2 + b(t) \frac{x+x'}{2} + c(t) \left( \frac{x+x'}{2} \right)^2 \right]$$

$$+ d(t) \frac{x-x'}{\varepsilon} + f(t) \frac{x+x'}{2} + g(t)] \psi(x', t) dx'$$

$$= C \int_{-\infty}^{\infty} \exp \left[ \frac{i}{\hbar} (a(t) \left( \frac{x-x'}{\varepsilon} \right)^2 + b(t) \frac{x+x'}{2} (x-x') + c(t) \varepsilon \left( \frac{x+x'}{2} \right)^2 \right]$$

$$+ d(t)(x-x') + f(t) \varepsilon \frac{x+x'}{2} + g(t)] \psi(x', t) dx'. \quad (9)$$

The quantity $\frac{(x-x')^2}{\varepsilon}$ appear in the exponent of the first factor. It is clear that if $x'$ is appreciably different from $x$, this quantity is very large and the exponential consequently oscillates very rapidly as $x'$ varies, when this factor oscillates rapidly, the integral over $x'$ gives a very small value. Only if $x'$ is near $x$ do we get important contributions. For this reason we make the substitution $x' = x + \eta$ with the expectation that appreciable contribution to the integral will occur only for small $\eta$, we obtain

$$\psi(x, t + \varepsilon) = C \int_{-\infty}^{\infty} \exp \left[ \frac{i\eta^2}{\hbar} a(t) \right] \cdot \exp \left[ \frac{i}{\hbar} (x + \frac{\eta}{2}) (-\eta) b(t) \right] \cdot \exp \left[ \frac{i\varepsilon}{\hbar} (x + \frac{\eta}{2}) c(t) \right]$$

$$\cdot \exp \left[ \frac{i}{\hbar} (-\eta) d(t) \right] \cdot \exp \left[ \frac{i\varepsilon}{\hbar} (x + \frac{\eta}{2}) f(t) \right] \cdot \exp \left[ \frac{i\varepsilon}{\hbar} g(t) \right] \psi(x + \eta, t) d\eta$$

$$= C \int_{-\infty}^{\infty} \exp \left[ \frac{i\eta^2}{\hbar} a(t) \right] \cdot \exp \left[ -\frac{i\eta}{\hbar} b(t) x \right] \cdot \exp \left[ -\frac{i\eta^2}{2\hbar} b(t) \right] \cdot \exp \left[ \frac{i\varepsilon}{\hbar} c(t) x^2 \right]$$

$$\cdot \exp \left[ \frac{i\varepsilon}{\hbar} x \eta c(t) \right] \cdot \exp \left[ \frac{i\varepsilon^2}{4\hbar} c(t) \right] \cdot \exp \left[ -\frac{i\eta}{\hbar} d(t) \right] \cdot \exp \left[ \frac{i\varepsilon}{\hbar} f(t) x \right]$$

$$\cdot \exp \left[ \frac{i\varepsilon}{2\hbar} g(t) \right] \psi(x + \eta, t) d\eta. \quad (10)$$

The phase of the first exponential charges by the order of 1 radian when $\eta$ is of the order $\sqrt{\frac{n\varepsilon}{a(t)}}$, so that most of the integral is contributed by values of $\eta$ in this order. We may expand $\psi$ in a power series, we need only keep terms of order $\varepsilon$. This implies keeping second-order terms in $\eta$. Expanding the left-hand side to first order in $\varepsilon$ and the right-hand side to first order in $\varepsilon$ and second order in $\eta$, we have

$$e^{-\frac{i\varepsilon}{\hbar} b(t) x} = 1 - \frac{i\eta}{\hbar} b(t) x + \frac{1}{2} \frac{\eta^2}{\hbar^2} b^2(t) x^2, \quad (11)$$

$$e^{-\frac{i\eta^2}{2\hbar} b(t)} = 1 - \frac{i\eta^2}{2\hbar} b(t), \quad (12)$$

$$e^{\frac{i\varepsilon}{\hbar} c(t) x^2} = 1 + \frac{i\varepsilon}{\hbar} c(t) x^2, \quad (13)$$
\[ e^{i\frac{\eta^2}{2\hbar} c(t)} = 1 + \frac{i\varepsilon \eta^2}{4\hbar} c(t), \quad (14) \]
\[ e^{i\frac{\eta}{\hbar} c(t)x} = 1 + \frac{i\varepsilon \eta}{\hbar} c(t)x, \quad (15) \]
\[ e^{-\frac{i\eta}{\hbar} d(t)} = 1 - \frac{i\eta}{\hbar} d(t) + \frac{1}{2} \frac{\eta^2}{\hbar^2} d^2(t), \quad (16) \]
\[ e^{i\frac{\eta}{\hbar} f(t)x} = 1 + \frac{i\varepsilon \eta}{\hbar} f(t)x, \quad (17) \]
\[ e^{i\frac{\eta}{\hbar} f(t)} = 1 + \frac{i\varepsilon \eta}{2\hbar} f(t), \quad (18) \]
\[ e^{i\frac{\eta}{\hbar} g(t)} = 1 + \frac{i\varepsilon \eta}{\hbar} g(t), \quad (19) \]
\[ \psi(x + \eta, t) = \psi(x, t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2}, \quad (20) \]

and

\[ \psi(x, t) + \varepsilon \frac{\partial \psi}{\partial t} = C \int_{-\infty}^{\infty} \exp\left[ i\eta^2 \frac{a(t)}{\hbar \varepsilon} \right] (1 - i\eta \frac{a(t)}{\hbar} + \frac{1}{2} \frac{\eta^2}{\hbar^2} a^2(t) x^2) (1 - i\eta \frac{b(t)}{\hbar}) (1 - i\eta \frac{c(t)}{\hbar} + \frac{1}{2} \frac{\eta^2}{\hbar^2} c^2(t)) \right] \]
\[ (1 + i\frac{\varepsilon}{\hbar} f(t) x) (1 + i\frac{\varepsilon}{\hbar} g(t)) \left( \psi(x, t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2} \right) \eta d\eta, \quad (21) \]

when \( \varepsilon \to 0 \) and \( \eta \to 0 \), the Eq. (21) becomes

\[ \psi(x, t) = C \int_{-\infty}^{\infty} \exp\left[ i\frac{\eta^2}{\hbar \varepsilon} a(t) \right] \psi(x, t) d\eta, \quad (22) \]

and the constant \( C \) is

\[ C = \frac{1}{\int_{-\infty}^{\infty} \exp\left[ i\frac{\eta^2}{\hbar \varepsilon} a(t) \right] d\eta} = \sqrt{\frac{a(t)}{i\pi \hbar \varepsilon}}. \quad (23) \]

In order to evaluate the right-hand side of Eq. (21), we shall have to use three integrals

\[ \int_{-\infty}^{\infty} e^{i\frac{\eta^2}{\hbar \varepsilon} a(t)} \eta d\eta = 0, \quad (24) \]
\[ \int_{-\infty}^{\infty} e^{i\frac{\eta^2}{\hbar \varepsilon} a(t)} \eta^2 d\eta = \frac{i\hbar \varepsilon \sqrt{i\pi \hbar \varepsilon}}{2a(t)}, \quad (25) \]
\[ \int_{-\infty}^{\infty} e^{i\frac{\eta^2}{\hbar \varepsilon} a(t)} \eta^4 d\eta = \frac{3}{4} (i\hbar \varepsilon)^2 \sqrt{i\pi \hbar \varepsilon} \frac{a(t)}{a(t)^2}. \quad (26) \]
In Eq. (21), we can easily find the terms $\frac{i\eta^2}{\hbar}c(t)$, $\frac{i\eta x}{\hbar}c(t)$ and $\frac{i\eta}{\hbar}f(t)$ integral are either zero or $O(\varepsilon^2)$ from Eqs. (24)-(26), and they can be neglected in Eq. (21). The Eq. (21) becomes

$$
\psi(x, t) + \varepsilon \frac{\partial \psi}{\partial t} = C \int_{-\infty}^{\infty} \exp\left[\frac{i\eta^2}{\hbar}a(t)\right] (1 - \frac{i\eta}{\hbar}b(t)x + \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x^2)(1 - \frac{i\eta}{2\hbar}b(t)) \\
+ (1 + \frac{i\varepsilon}{\hbar}c(t)x^2)(1 - \frac{i\eta}{\hbar}d(t) + \frac{1}{2} \frac{\eta^2}{\hbar^2}d^2(t))(1 + \frac{i\varepsilon}{\hbar}f(t)x) \\
+ (1 + \frac{i\varepsilon}{\hbar}g(t))(\psi(x, t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2})d\eta,
$$

(27)

we defined

$$
I = (1 - \frac{i\eta}{\hbar}b(t)x + \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x^2)(1 - \frac{i\eta}{2\hbar}b(t))(1 + \frac{i\varepsilon}{\hbar}c(t)x^2) \\
(1 - \frac{i\eta}{\hbar}d(t) + \frac{1}{2} \frac{\eta^2}{\hbar^2}d^2(t))(1 + \frac{i\varepsilon}{\hbar}f(t)x)(1 + \frac{i\varepsilon}{\hbar}g(t)) \\
= (1 - \frac{i\eta}{\hbar}b(t)x + \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x^2 - \frac{i\eta^2}{2\hbar}b(t)x - \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x - \frac{i\eta^4}{4\hbar^3}b^3(t)x^2)(1 + \frac{i\varepsilon}{\hbar}c(t)x^2) \\
(1 - \frac{i\eta}{\hbar}d(t) + \frac{1}{2} \frac{\eta^2}{\hbar^2}d^2(t))(1 + \frac{i\varepsilon}{\hbar}g(t) + \frac{i\varepsilon}{\hbar}f(t)x - \frac{\varepsilon^2}{\hbar^2}f(t)g(t)x).
$$

(28)

In Eq. (28), the terms $\frac{i\eta}{\hbar}b^2(t)x$, $\frac{i\eta}{\hbar}b^2(t)x^2$ and $\frac{i\eta}{\hbar}f(t)g(t)$ integral are either zero or $O(\varepsilon^2)$ from Eqs. (24)-(26), and they can be neglected in Eq. (27). The function $I$ becomes

$$
I = (1 - \frac{i\eta}{\hbar}b(t)x + \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x^2 - \frac{i\eta^2}{2\hbar}b(t))(1 + \frac{i\varepsilon}{\hbar}c(t)x^2) \\
(1 - \frac{i\eta}{\hbar}d(t) + \frac{1}{2} \frac{\eta^2}{\hbar^2}d^2(t))(1 + \frac{i\varepsilon}{\hbar}g(t) + \frac{i\varepsilon}{\hbar}f(t)x) \\
= (1 - \frac{i\eta}{\hbar}b(t)x + \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x^2 - \frac{i\eta^2}{2\hbar}b(t))(1 - \frac{i\eta}{\hbar}d(t) + \frac{1}{2} \frac{\eta^2}{\hbar^2}d^2(t)) \\
(1 + \frac{i\varepsilon}{\hbar}g(t) + \frac{i\varepsilon}{\hbar}f(t)x + \frac{i\varepsilon}{\hbar}c(t)x^2 - \frac{\varepsilon^2}{\hbar^2}c(t)g(t)x^2 - \frac{\varepsilon^2}{\hbar^2}c(t)f(t)x^3).
$$

(29)

In Eq. (29), we neglect terms including $\varepsilon^2, \eta^3$ and $\eta^4$, and $I$ can be written as

$$
I = (1 - \frac{i\eta}{\hbar}b(t)x + \frac{1}{2} \frac{\eta^2}{\hbar^2}b^2(t)x^2 - \frac{i\eta^2}{2\hbar}b(t) - \frac{i\eta^2}{\hbar^2}b^2(t)x^2) \\
(1 + \frac{i\varepsilon}{\hbar}g(t) + \frac{i\varepsilon}{\hbar}f(t)x + \frac{i\varepsilon}{\hbar}c(t)x^2) \\
= (1 - \frac{i\eta}{\hbar}(b(t)x + d(t)) + \frac{1}{2} \frac{\eta^2}{\hbar^2}(b^2(t)x^2 - i\hbar b(t) + d^2(t) - 2b(t)d(t)v)) \\
(1 + \frac{i\varepsilon}{\hbar}(g(t) + f(t)x + c(t)x^2)) \\
= (1 - \frac{i\eta}{\hbar}A(x, t) + \frac{1}{2} \frac{\eta^2}{\hbar^2}B(x, t))(1 + \frac{i\varepsilon}{\hbar}C(x, t)),
$$

(30)
where the functions $A(x,t)$, $B(x,t)$ and $C(x,t)$ are
\begin{align*}
A(x,t) &= b(t)x + d(t), \\
B(x,t) &= b^2(t)x^2 - ihb(t) + d^2(t) - 2b(t)d(t)x, \\
C(x,t) &= g(t) + f(t)x + c(t)x^2,
\end{align*}
(31)

substituting Eq. (30) into (27), we have
\begin{align*}
\psi(x,t) + \varepsilon \frac{\partial \psi}{\partial t} &= C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}}(1 - \frac{i\eta}{h} A(x,t) + \frac{\eta^2}{2h^2} B(x,t)) (1 + \frac{i\varepsilon}{h} C(x,t)) \\
&\quad \cdot (\psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2}) d\eta \\
&= C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}}(1 - \frac{i\eta}{h} A(x,t) + \frac{\eta^2}{2h^2} B(x,t) + \frac{i\varepsilon}{h} C(x,t) + \frac{\eta^2}{h^2} A(x,t) \\
&\quad \cdot C(x,t) + \frac{i\eta^2 \varepsilon}{2h^2} B(x,t) C(x,t)) \cdot (\psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2}) d\eta. \quad (32)
\end{align*}

In Eq. (32), the terms including $\eta \varepsilon$ and $\eta^2 \varepsilon$ integral are either zero or $O(\varepsilon^2)$, and they can be neglected in Eq. (32). The Eq. (32) becomes
\begin{align*}
\psi(x,t) + \varepsilon \frac{\partial \psi}{\partial t} &= C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}}(1 - \frac{i\eta}{h} A(x,t) + \frac{\eta^2}{2h^2} B(x,t) + \frac{i\varepsilon}{h} C(x,t)) \\
&\quad \cdot (\psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2 \psi}{\partial x^2}) d\eta. \quad (33)
\end{align*}

In Eq. (33), the terms including $\eta$, $\eta^3$ and $\eta^4$ integral are either zero or $O(\varepsilon^2)$, and they can be neglected in Eq. (33). The Eq. (33) becomes
\begin{align*}
\psi(x,t) + \varepsilon \frac{\partial \psi}{\partial t} &= C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}} \psi(x,t) + \frac{\eta^2}{2h^2} B(x,t) \psi(x,t) + \frac{i\varepsilon}{h} C(x,t) \psi(x,t) \\
&\quad - \frac{i\eta^2}{h} A(x,t) \frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} d\eta \\
&= \psi(x,t) + \frac{1}{2h^2} B(x,t) \psi(x,t) \cdot C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}} \eta^2 d\eta + \frac{i\varepsilon}{h} C(x,t) \psi(x,t) \\
&\quad \cdot C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}} \eta^2 d\eta - \frac{i\eta^2}{h} A(x,t) \frac{\partial \psi}{\partial x} \cdot C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}} \eta^2 d\eta + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \\
&\quad \cdot C \int_{-\infty}^{\infty} e^{\frac{\eta^2}{a(t)}} \eta^2 d\eta \\
&= \psi(x,t) + \frac{1}{2h^2} B(x,t) \psi(x,t) \frac{i\hbar \varepsilon}{2a(t)} + \frac{i\varepsilon}{h} C(x,t) \psi(x,t) \\
&\quad - \frac{i\hbar \varepsilon}{2a(t)} \frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \frac{i\hbar \varepsilon}{2a(t)}. \quad (34)
\end{align*}
Equating the coefficient of powers of $\varepsilon$, we have
\[
\frac{\partial \psi(x,t)}{\partial t} = \frac{i}{4a(t)} B(x,t)\psi(x,t) + \frac{i}{\hbar} C(x,t)\psi(x,t) + \frac{1}{2a(t)} A(x,t) \frac{\partial \psi}{\partial x} + \frac{i\hbar}{4a(t)} \frac{\partial^2 \psi}{\partial x^2}, \tag{35}
\]
multipled the coefficient of $i\hbar$, we have
\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{4a(t)} B(x,t)\psi(x,t) - C(x,t)\psi(x,t) + \frac{\hbar}{2a(t)} A(x,t) \frac{\partial \psi}{\partial x} - \frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi}{\partial x^2}, \tag{36}
\]
substituting the function $A(x,t), B(x,t)$ and $C(x,t)$ into Eq. (36), we obtain the general Schrödinger equation
\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi(x,t)}{\partial x^2} + i\frac{\hbar}{2a(t)} (b(t)x + d(t)) \frac{\partial \psi(x,t)}{\partial x} - \frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi(x,t)}{\partial x^2}.
\] 
\[
-(\frac{b^2(t)}{4a(t)})^2x^2 - \frac{i\hbar b(t)}{4a(t)} + \frac{d^2(t)}{4a(t)} - \frac{b(t)d(t)}{2a(t)} x + g(t) + f(t)x + c(t)x^2 \psi(x,t). \tag{37}
\]
The Eq. (37) is a general Schrödinger equation for one-dimensional. We can discuss Eq. (37) as follows:
(a) When $b(t) = c(t) = d(t) = g(t) = 0$, the Lagrangian function (6) is
\[
L(x,\dot{x},t) = a(t)\dot{x}^2 + f(t)x,
\]
Eq. (37) becomes
\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi(x,t)}{\partial x^2} - f(t)x\psi(x,t). \tag{39}
\]
(b) When $b(t) = d(t) = f(t) = g(t) = 0$, the Lagrangian function (6) is
\[
L(x,\dot{x},t) = a(t)\dot{x}^2 + c(t)x^2,
\]
Eq. (37) becomes
\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi(x,t)}{\partial x^2} - c(t)x^2\psi(x,t). \tag{41}
\]
(c) When $c(t) = d(t) = f(t) = g(t) = 0$, the Lagrangian function (6) is
\[
L(x,\dot{x},t) = a(t)\dot{x}^2 + b(t)x \cdot \dot{x},
\]
Eq. (37) becomes
\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi(x,t)}{\partial x^2} + i\hbar \frac{b(t)x}{2a(t)} \frac{\partial \psi(x,t)}{\partial x} - \frac{b^2(t)}{4a(t)}x^2 - \frac{i\hbar b(t)}{4a(t)} \psi(x,t). \tag{43}
\]
(d) When \( b(t) = c(t) = f(t) = g(t) = 0 \), the Lagrangian function (6) is

\[
L(x, \dot{x}, t) = a(t)x^2 + d(t)\dot{x},
\]

Eq. (37) becomes

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{4a(t)} \frac{\partial^2 \psi(x, t)}{\partial x^2} + i\frac{\hbar}{2a(t)} \frac{\partial \psi(x, t)}{\partial x} - \frac{d^2(t)}{4a(t)} \psi(x, t).
\]

For the general Lagrangian function in three-dimensional, it is

\[
L(\vec{r}, \dot{\vec{r}}, t) = a(t)\vec{r}^2 + b(t)\vec{r} \cdot \dot{\vec{r}} + c(t)\vec{r}^2 + g(t),
\]

We can easily obtain the general Schrödinger equation for three-dimensional, it is

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{4a(t)} \nabla^2 \psi(\vec{r}, t) + i\frac{\hbar}{2a(t)} b(t)\vec{r} \cdot \nabla \psi(\vec{r}, t)
\]

\[
- \left( \frac{b^2(t)}{4a(t)} - \frac{i\hbar b(t)}{4a(t)} + c(t)\vec{r}^2 + g(t) \right) \psi(\vec{r}, t).
\]

Eq. (47) is a general Schrödinger equation, which is suitable for a general potential. From the equation, we can study the general system.

3. Conclusion

We know Schrödinger equation is quantum wave equation, which is only suitable for a particle in common potential \( V(\vec{r}, t) \). When a particle is in general potential \( V(\vec{r}, \vec{r}, t) \), we need new quantum wave equation. In this paper, we apply the approach of path integral to obtain the general Schrödinger equation, which is suitable for the general potential. We think the general Schrödinger equation will be used widely in many fields.
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