Magic Neutrino Mass Matrix and the Bjorken-Harrison-Scott Parameterization

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Abstract

Observed neutrino mixing can be described by a tribimaximal MNS matrix. The resulting neutrino mass matrix in the basis of a diagonal charged lepton mass matrix is both 2-3 symmetric and magic. By a magic matrix, I mean one whose row sums and column sums are all identical. I study what happens if 2-3 symmetry is broken but the magic symmetry is kept intact. In that case, the mixing matrix is parameterized by a single complex parameter $U_{e3}$, in a form discussed recently by Bjorken, Harrison, and Scott.
I. INTRODUCTION

A global fit on neutrino oscillation experiments has established the mixing parameters to be \( \sin^2 \theta_{13} = 0.9^{+2.3}_{-0.9} \times 10^{-2} \), \( \sin^2 \theta_{12} = 0.314(1^{+0.18}_{-0.15}) \), and \( \sin^2 \theta_{23} = 0.44(1^{+0.41}_{-0.22}) \), all at 95\% C.L. Squared mass differences are also known but neither the Dirac phase nor the two Majorana phases has been measured. The mixing angles are consistent with a tribimaximal structure first proposed by Harrison, Perkins, and Scott (HPS) \(^2\), whose mixing matrix is shown in eq. (1). The corresponding mixing parameters are \( \sin^2 \theta_{13} = 0 \), \( \sin^2 \theta_{12} = 0.333 \), and \( \sin^2 \theta_{23} = 0.50 \). Many dynamical models have been proposed to explain the mixing \(^3\), usually through a horizontal group spontaneously broken by appropriately chosen scalar fields and parameters. Often these models also predict some deviations from the HPS mixing. We shall not discuss such models, but shall pursue possible deviations through a symmetry structure of the neutrino mass matrix.

The HPS mixing possesses two remarkable properties. In the basis where the charged lepton mass matrix is diagonal, it gives rise to a neutrino mass matrix that is 2-3 symmetric, so that when we interchange the second and the third rows, and simultaneously the second and the third columns, the mass matrix remains the same. See eq. (2) below. Such a 2-3 symmetry has been extensively studied \(^4\). It leads to maximal atmospheric mixing with \( \sin^2 \theta_{23} = 1 \), and a vanishing reactor angle with \( \sin^2 \theta_{13} = 0 \). The mixing matrix is real, so no Dirac phase is present, but the three eigen-masses are unknown and could be complex, thus admitting arbitrary Majorana phases and neutrino masses.

As we shall show, the HPS neutrino mass matrix is also magic, in the sense that the sum of each column and the sum of each row are all identical. I call such a matrix *magic* because it reminds me of magic squares, though the latter also have identical diagonal sums.

It is this magic property of the mass matrix that we want to investigate, with or without 2-3 symmetry. For specific examples, see, for example, Ref. \(^5\). It turns out that the mixing matrix for magic mass matrices can be parameterized in the way proposed recently by Bjorken, Harrison, and Scott (BHS) \(^6\), with arbitrary Majorana phases and neutrino masses.
II. HPS MIXING MATRIX

The HPS form of the MNS mixing matrix is

\[ U_{HPS} = \begin{pmatrix} 
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} 
\end{pmatrix}. \] (1)

If \( m_i \) are the neutrino masses, possibility complex to absorb the two Majorana phases, and if \( m = \text{diag}(m_1, m_2, m_3) \), then the neutrino mass matrix may be written

\[ M_{HPS} = U_{HPS}^T m U_{HPS} = \frac{1}{6} \begin{pmatrix} 
4m_1 + 2m_2 & -2m_1 + 2m_2 & -2m_1 + 2m_2 \\
-2m_1 + 2m_2 & m_1 + 2m_2 + 3m_3 & m_1 + 2m_2 - 3m_3 \\
-2m_1 + 2m_2 & m_1 + 2m_2 - 3m_3 & m_1 + 2m_2 + 3m_3 
\end{pmatrix}. \] (2)

It is 2-3 symmetric. It is also magic, because the row sums and column sums are each equal to \( m_2 \). Conversely, we shall show in the next section that a magic and 2-3 symmetric matrix gives rise to the mixing structure (1), so that a magic 2-3 symmetry for the mass matrix is synonymous to a HPS mixing structure.

III. MAGIC MATRIX

An \( n \times n \) matrix \( A \) will be called magic if the row sums and the column sums are all equal to a common number \( \alpha \):

\[ \sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = \alpha. \] (3)

For example, every permutation matrix of \( n \) objects has a single 1 in every row and every column, and 0 elsewhere, so it is a magic matrix with \( \alpha = 1 \).

It is easy to see that magic matrices are closed under addition, multiplication, inverse, and scalar multiplication. In other words, if \( c \) is a number, and \( A, B \) are magic matrices with common sums \( \alpha, \beta \), then \( A + B, AB, A^{-1}, \) and \( cA \) are all magic, with common sums \( \alpha + \beta, \alpha \beta, \alpha^{-1}, \) and \( \alpha c \). Such closure properties are also true for 2-3 symmetric matrices.

Eq. (3) tells us that \( A \) has an eigenvector \( v \), with eigenvalue \( \alpha \). Namely, if all the \( n \) components \( v_i \) of the vector \( v \) are equal, then

\[ \sum_j A_{ij}v_j = \alpha v_i, \quad \sum_i v_iA_{ij} = \alpha v_j. \] (4)
This eigenvector is normalized if we choose $v_i = 1/\sqrt{n}$.

We specialize now to $3 \times 3$ matrix $M$ that is magic. To satisfy (3), it must have the general form

$$M = \begin{pmatrix} a & b & c \\ e & d & a+b+c-d-e \\ b+c-e & a+c-d & d+e-c \end{pmatrix},$$

so it is described by 5 complex parameters. If $M$ is symmetric, as Majorana neutrino mass matrices are, then $e = b$, and we are left with 4 complex parameters. As we shall see, they are the three complex masses $m_i$ and $U_{e3}$. If $M$ is 2-3 symmetric, then $b = c = e$, making $M$ automatically symmetric, and we are left with only 3 parameters, the three complex masses. This is the case with the HPS mass matrix in (2).

Our task is to find the mixing matrix $U$ for a symmetric magic mass matrix $M$, i.e., a unitary matrix $U$ so that $U^T M U$ becomes a diagonal matrix $m = \text{diag}(m_1, m_2, m_3)$. Since $U$ is unitary, we can write this relation as $MU = U^m$, so that

$$Mu_i = m_i u_i^*$$

if $u_1, u_2, u_3$ are the three column vectors of $U$. Since the normalized eigenvector $v$ in (4) is real, it satisfies (6) and is one of the three column vectors $u_i$. By comparing with the HPS mixing matrix (1), we see that $v = u_2$. In other words, the second neutrino mass eigenstate $\nu_2$ always mixes democratically with all three neutrino flavor states, thus a magic mass matrix leads to a trimaximal mixing for $\nu_2$. This is the basic assumption involved in the Bjorken-Harrison-Scott (BHS) parameterization of the mixing matrix. The rest of the matrix elements in $U$ is determined by unitarity and allowed phase choices. The result, as given by BHS, is

$$U = \begin{pmatrix} 2C/\sqrt{6} & 1/\sqrt{3} & U_{e3} \\ -C/\sqrt{6} - \sqrt{3}U_{e3}^*/2 & 1/\sqrt{3} & C/\sqrt{2} - U_{e3}/2 \\ -C/\sqrt{6} + \sqrt{3}U_{e3}^*/2 & 1/\sqrt{3} & -C/\sqrt{2} - U_{e3}/2 \end{pmatrix},$$

where $U_{e3}$ is complex so it contains the Dirac phase, allowing CP violation, and $C = \sqrt{1 - 3|U_{e3}|^2/2}$.

Note that

$$\sum_i U_{ij} = 0$$
for \( j = 1 \) and \( 3 \), because these two columns must be orthogonal to the second column.

If \( M \) is 2-3 symmetric as well, then \( U_{e3} = 0 \), so \( U = U_{HPS} \) as claimed.

Conversely, in terms of the neutrino masses \( m_i \) (possibly complex after absorbing the Majorana phases), the mass matrix \( M = U m U^T \) can be calculated from (7). Such a matrix is necessarily magic on account of (8):

\[
\sum_i M_{ij} = \sum_i U_{ik} m_k U_{jk} = \sum_i U_{i2} m_2 U_{j2} = m_2 = \sum_j M_{ij}.
\]  

(9)

IV. CONCLUSION

In conclusion, present experimental data are consistent with the HPS mixing of eq. (1), which exhibits a trimaximal mixing for \( \nu_2 \) and a bimaximal mixing for \( \nu_3 \). In the basis where the charged-lepton mass matrix is diagonal, the neutrino mass matrix \( M \) possesses both magic symmetry and 2-3 symmetry. If magic symmetry is broken but 2-3 symmetry is kept, then the bimaximal structure of \( \nu_3 \) is retained, but the trimaximal nature of \( \nu_2 \) is broken, thus keeping atmospheric mixing maximal and the reactor angle zero, but the solar mixing is left as a free parameter. If 2-3 symmetry is broken but magic symmetry is kept, then it retains the trimaximal structure of \( \nu_2 \) while breaking the bimaximal nature of \( \nu_3 \). The mixing matrix in this case is parameterized by Bjorken, Harrison, and Scott [6], with a single complex parameter \( U_{e3} \), thus allowing CP violation. There are also two relations between the mixing parameters, arising from the trimaximal structure \( U_{e2} = U_{\mu 2} = U_{\tau 2} = 1/\sqrt{3} \). In terms of the usual Chau-Keung parameterization of the mixing angles [8], one relation is \( \sin \theta_{12} \cos \theta_{13} = 1/\sqrt{3} \), and the other involves the Dirac phase angle.

If both the 2-3 and the magic symmetries are violated, then we are back to the general case with a full blown Chau-Yeung parameterization.

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[1] G.L. Fogli, E. Lisi, A. Marrone, and A. Palazzo, hep-ph/0506083
[2] P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. B458, (1999) 79, hep-ph/9904297
 Phys. Lett. B530, (2002) 167, hep-ph/0202074
[3] Some recent papers include, E. Ma, [hep-ph/0606039] Y. Koide, [hep-ph/0605074] R.N. Mohapatra, S. Nasri, and H.-B. Yu, [hep-ph/0605020] W. Grimus and L. Lavoura, JHEP 0508 (2005) 013, [hep-ph/0509239] P.F. Harrison and W.G. Scott, Phys. Lett. B557 (2003) 76, [hep-ph/0302025] Please consult these paper for earlier references.

[4] T. Fukuyama and H. Nishiura, in Proceeding of 1997 Shizuoka Workshop on Masses and Mixings of Quarks and Leptons, [hep-ph/9702253] C.S. Lam, Phys. Lett. B507 (2001) 214, [hep-ph/0104116] W. Grimus and L. Lavoura, Phys. Lett. JHEP 0107 (2001) 045, [hep-ph/0105212] E. Ma, Phys. Rev. D 66 (2002) 117301, [hep-ph/0207352] P.F. Harrison, and W.G. Scott, Phys. Lett. B547 (2002) 219, [hep-ph/0210197] R. N. Mohapatra, SLAC Summer Inst. lecture; [http://www-conf.slac.stanford.edu/ssi/2004] JHEP 0410 (2004) 027, [hep-ph/0408187]

[5] P.F. Harrison and W.G. Scott, Phys. Lett. B594 (2004) 324, [hep-ph/0403278]

[6] J.D. Bjorken, P.F. Harrison, and W.G. Scott, [hep-ph/0511201]

[7] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theo. Phys. 28 (1962) 870.

[8] L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett. 53 (1984) 1802.