Second sound in Fermi gases at the BCS-BEC crossover

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The thermodynamic potential is calculated for a uniform superfluid gas of fermi atoms from the mean field BCS equations including corrections from induced interactions, Hartree-Fock energies and quasiparticle selfenergies. The entropy, specific heat and sound modes are calculated as function of temperature, density and interaction strength from the BCS to the unitarity limit and around the BCS-BEC crossover. The second sound speed is of particular interest as it is a clear signal of a superfluid component and it determines the critical temperature.

Recent experiments probe systems of fermions near Feshbach resonances by expansion [1–5], collective modes [6,7] and RF spectroscopy [8]. Interesting new strongly interacting or dense phases of fermions and bosons are created, e.g., that associated with the crossover from a superfluid or normal Fermi gas to a molecular BEC. The experiments provide strong evidence for a superfluid state at low temperatures. The next generation of experiments will measure sound velocities in these systems. Second sound is particularly interesting as it is a clear signal of a superfluid component and it also determines the critical temperature.

Ho [9] has studied the thermodynamics in the unitarity limit at the Feshbach resonance and expressed a number of thermodynamic quantities in terms of unknown universal parameters. The purpose of this work is to extend the analysis to all temperatures, densities and interaction strengths in the BCS limit and around the unitarity limit, and to calculate the thermodynamic quantities in terms the binding energy per particle $E/N$ and the pairing gap $\Delta$ at zero temperature. These quantities are known rather accurately as functions of density and interaction strength from Monte Carlo [10,11] and experimental results are compatible. These functions will be treated as input into a crossover model that is based on the mean field BCS equations of Refs. [12–15], however, including important corrections in the crossover model from induced interactions, self-energies and Hartree-Fock energies. We will then calculate the thermodynamic potential in the superfluid phase from which the entropy, specific heat, first and in particular second sound speeds can be calculated at low temperatures in the superfluid and normal phases from the dilute BCS limit up to and around the unitarity limit.

The mean field BCS equations of Refs. [12–15] have become a standard reference for at least a qualitative describing the BCS-BEC crossover as function of density, interaction strength and temperature. It describes a system of Fermi atoms of mass $m$ with two spin states in spin equilibrium $n_\uparrow = n_\downarrow = n/2$ interacting through a s-wave scattering length $a$. The two-body interaction range is assumed to be short as compared to $|a|$ leaving only two length scales $a$ and the interparticle spacing (or $k_F^{-1}$). At zero temperature all the physics depend only on one variable which is conveniently chosen as $x = 1/(ak_F)$. It varies from $-\infty$ in the dilute BCS limit through the unitarity limit at Feshbach resonance at $x = 0$ to $x \to +\infty$ in the molecular BEC limit.

The mean field BCS gap equation

$$\frac{1}{g} = \sum_k \left[ \frac{1}{2E_k} - \frac{m}{\hbar^2 k^2} - \frac{f_k}{E_k} \right],$$

(1)

is valid for any coupling strength $g = -4\pi \hbar^2 a/m$ and can thus at least qualitatively describe the smooth crossover from a BCS state to a molecular BEC. As usual $\epsilon_k = \hbar^2 k^2/2m - \mu_\Delta$ and $E_k = \sqrt{\epsilon_k^2 + \Delta^2}$. Note that the chemical potential $\mu_\Delta$ does not include Hartree-Fock energies and the quasiparticle energy does not contain any self-energy. The thermal distribution function is $f_k = (\exp(E_k/T) + 1)^{-1}$ in units where $k_B = 1$. With the equation for number density conservation

$$n = \sum_k \left[ 1 - \frac{\epsilon_k}{E_k} + \frac{\epsilon_k f_k}{E_k 2} \right],$$

(2)

the gap and chemical potential can be calculated as function of density, temperature and interaction strength. At zero temperature the last term in Eqs. (1) and (2) vanish and the gap and chemical potential are easily calculated as function of $x = 1/(ak_F)$ (see Refs. [13,15] and Fig. 1).

In the dilute BCS limit $\Delta << \mu_\Delta \simeq E_F$, where the Fermi energy is $E_F = \hbar^2 k_F^2/2m$, and the Fermi momentum $h k_F$ is given in terms of the density $n = k_F^3/3\pi^2$. The sum in the gap equation is at zero temperature simply $(mk_F^2/2\pi^2) \ln(\kappa E_F/\Delta_0)$, where $\kappa = 8/\epsilon^2$. The superfluid gap at zero temperature $\Delta_0 = \Delta(T = 0)$ becomes

$$\Delta_0 = \kappa E_F \exp \left( \frac{\pi}{2ak_F} \right),$$

(3)

Gorkov, however, found that induced interactions lead to a higher order correction such that: $a \rightarrow a^{-1} - 2k_F \ln(4e)/3\pi$ [16]. In the above gap equation, where the l.h.s. is proportional to $x = 1/(ak_F)$, this correction corresponds to adding or shifting $x$ by the amount $2 \ln(4e)/3\pi$. As result the Gorkov gap has $\kappa = (2/3)^{7/3}$ in Eq. (3). In the following we shall add this constant shift correction not only in the dilute BCS limit but generally at all $x$. Therefore the resulting gap and chemical potential as shown in Fig. 1 both deviate from standard
results by the above shift in $x$. The resulting gap is exact in the dilute BCS limit and generally in good agreement with Monte Carlo calculations [10] up to $x \lesssim 1$.

Neither self-energies nor Hartree-Fock energies are included in the mean field BCS equations. Monte Carlo calculations provide detailed insight in their contributions relative to the effect of pairing because pairing can be included or excluded in the trial wave functions. The energy per particle $E/N = (3/5)E_F(1 + \beta)$ is expressed in terms of the universal function $\beta(x) = E_{\text{int}}/E_{\text{kin}}$ [17]. In the unitarity limit $\beta(x = 0) = -0.46$ without pairing but $\beta(x = 0) = -0.56$ with pairing in the Monte Carlo calculations of Ref. [10] at zero temperature. Therefore, the dominant contribution to the binding energy and chemical potential does not come from pairing but from other correlations in the wave function that contribute to the Hartree-Fock energy. We shall therefore distinguish between $\mu_\Delta$, which includes only the pairing effects, and the correct chemical potential

$$\mu = \left(\frac{\partial E}{\partial N}\right)_{v,S} = E_F \left(1 + \beta - \frac{1}{5} x \frac{d\beta}{dx}\right),$$

which includes both pairing and Hartree-Fock energies. Both are shown in Fig. 1. On the BCS side and even in the unitarity limit $\mu_\Delta$ differs from $E_F$ by a small amount only. This is compatible with the small deviation between the chemical potentials calculated by Monte Carlo with and without pairing. On the BCS side $x = 1/(ak_F) > 0$ the pairing contribution to binding energies become increasingly important, and the chemical potential rapidly drops toward the molecular binding energy $\mu_\Delta \simeq \mu \simeq -h^2/2ma^2$ in the dilute BCS limit. Recent experiments on expansion energies [1-5] and collective modes in traps [6,7] are compatible with the Monte Carlo calculations.

In both the hydrodynamic limit and for a superfluid gas the first sound is given by the adiabatic sound speed

$$u_1^2 = \frac{n}{m} \left(\frac{\partial \mu}{\partial n}\right)_{v,S} = \frac{1}{3} v_F^2 \left[1 + \beta - \frac{3}{5} \frac{\partial \beta'}{\partial x} + \frac{1}{10} x^2 \frac{\partial \beta}{\partial x}\right],$$

where $v_F = \hbar k_F/m$, $\beta' = d\beta/dx$, etc. In the dilute BCS limit and at low temperature $u_1 = v_F \sqrt{(1 + (2/\pi)ak_F)/3}$. In the unitarity limit $u_1 = \sqrt{2\mu(x = 0)/3m} = v_F \sqrt{(1 + \beta(0))/3} \simeq 0.37v_F$. In the dilute BEC limit $u_1 = \sqrt{(\pi/2)\hbar^2 a_m/m^2}$, where $a_m \simeq 0.6a$ is the molecular scattering length [18,11]. The first sound speed from Monte Carlo calculations [11] is plotted in Fig. 1.

In a normal Fermi liquid the first sound speed is expressed in terms of the Landau parameters as: $u_1^2 = (v_F^2/3)(1 + F_0)/(1 + F_3)$, at low temperatures. The Fermi liquid theory can be generalized to finite temperatures $T \lesssim 0.5T_F$ and arbitrary relaxation times [19]. The effective mass $m^* = (1 + F_1/3)m$ may be determined from, e.g., the specific heat as shown below, and $F_0$ can then be determined from $u_1$ or equivalently $\beta(x)$. In the dilute BCS limit $m^*/m = 1 + (87\ln 2 - 1)/15\pi^2a^2k_F^2$. In the mean field BCS equation $m^* = m$ because selfenergies are not included. We shall in the following assume that selfenergies can be included in the quasiparticle energies such that they lead to an effective mass at the Fermi surface. Eventually $m^*$ will have to be measured and/or calculated by Monte Carlo as in the cases of $\Delta_0$ and $\mu$. The Landau parameter $F_1$ varies from zero in the BCS limit to $-1$ in the BEC limit where, however, the liquid will only be in the normal state for temperatures above the critical temperature. For a BEC this condition is $T > T_c = (n/\zeta(3/2))^{2/3}\pi/m \simeq 0.218E_F$. Fermi liquid theory also describes collisions and the transition between the hydrodynamic (first) sound and the collisionless (zero) sound as the collision rate decrease. The Landau damping can also be calculated and for $-1 \leq F_0 \leq 0$ the zero sound mode becomes purely imaginary in the collisionless limit.

In the following we shall refer to the mean field BCS equations with the above described corrections from induced interactions, Hartree-Fock energies and effective mass as simply the “crossover model”. It contains the correct pairing and mean field in the dilute BCS limit to leading orders, and is also correct in the unitarity limit to a good approximation. It does not solve the full many-body problem of strongly interacting Fermi gases but should regarded as an approximate model that includes some of the most important physics. As will shown in the following it has the virtue that it produces definite predictions for a number of observables which are about to be measured.

The gap at finite temperature is found by solving the gap equation. It is a major simplification that we for most purposes treat the pairing gap as being small as compared to the Fermi energy for $x \ll 0$. The next order correction to, e.g. the gap itself turns out to be
where the Hartree-Fock terms are of order \(s\) at zero temperature \(\Delta = \Delta_0\). It should be noted that the above thermodynamic potentials are the same. The difference is now included induced interactions and possible effective mass corrections such that the correct \(\Delta_0\) of Eq. (3) is reproduced. Inserting the coupling of Eq. (8) into the thermodynamic potential \(\Omega_s\) and again exploiting that \(\Delta \ll \mu_\Delta\), it reduces to

\[
\Omega_s = - N(0) \left[ \frac{\Delta^2}{2} + \ln \left( \frac{\Delta_0}{\Delta} \right) \right] - 4T \int_0^\infty d\varepsilon_k \ln (1 - f_k)
\]

The crossover model thus arrives at the standard expression for \(\Omega_s\) and therefore also the standard entropy \(S_s = - (\partial \Omega_s / \partial T)|_{\nu,\mu}\) and specific heat \(C_s = T (\partial S_s / \partial T)|_{\nu,\mu}\) in superfluid phase. In the normal phase \(S_n = C_n = N(0) 2\pi^2 T^3 / 3\). At low temperatures \(S_s = N(0) \sqrt{2\pi \Delta_0 / 3} \exp(-\Delta_0 / T)\) and \(C_s = N(0) \sqrt{2\pi \Delta_0 / 3} \exp(-\Delta_0 / T)\). Near \(T_c\): \(S_s / S_n(T) = 1 - (1 + \xi)(1 - T / T_c)\) and \(C_s / C_n(T_c) = \xi - 3.77(1 - T / T_c)\). Here \(\xi = 1 + 12/7(\Delta / T)\) is the superfluid specific heat relative to the normal one at \(T_c\).

Finally, we need the superfluid (London) density

\[
n_s = n \left( 1 + 2 \int_0^\infty d\varepsilon_k \frac{df_k}{dE_k} \right)
\]

where \(n = n_s + n_n\) is the total density. At low temperatures \(n_s / n = 1 - \sqrt{2\pi \Delta_0 / T} \exp(-\Delta_0 / T)\) whereas \(n_s / n = 2(1 - T / T_c)\) near \(T_c\).

\[
\text{FIG. 2. Thermodynamic quantities for a superfluid as calculated in the crossover model vs. } T / T_c. \text{ Shown are the gap } \Delta(T)/\Delta_0, \text{ entropy } S_s(T) / S_n(T), \text{ specific heat } C_s(T) / C_n(T), \text{ and superfluid density } n_s(T)/n. \text{ The second sound is plotted in units of } v_F \sqrt{m^* / m T_c / E_F}.\]

We now have all the thermodynamic quantities available for calculating the second sound speed

\[
u_2 = n_s \frac{S^2 T}{n_n m n C_s}.
\]

as shown in Fig. 2. At low temperatures the second sound is linear in temperature

\[
u_2 = \frac{3}{2} \sqrt{\frac{m^*}{m}} \frac{T}{E_F} v_F, \quad T \ll T_c.
\]

Around \(T \approx 0.7T_c\) the second sound speed has a broad maximum of

\[
u_2 \approx 0.53 v_F \sqrt{\frac{m^*}{m}} \frac{T}{E_F}, \quad T \sim 0.7T_c.
\]

Near the critical temperature \(T_c - T \ll T_c\) second sound
has the characteristic \( u_2 \propto \sqrt{1-T/T_c} \) behavior and the
vanishing point determines \( T_c \).

Ho [9] predicted the scaling behavior of the above thermodynamic quantities in the
unitarity limit \((x = 0)\) near \( T_c \) from universal scaling laws. In his analysis the
\( u_2 \propto \sqrt{1-T/T_c} \) dependence near \( T_c \) and \( x = 0 \) was
found but quantitatively it depended on unknown parameters. In comparison the
crossover model can be applied to all temperatures, densities and interaction strengths
as long as \( x \gtrsim 0.5 \). The results are given in terms of the functions \( \Delta_0(x) \), \( \mu(x) \) and \( m^*(x) \), of which the first two are known from
Monte Carlo calculations and experiments as discussed above.

The crossover model breaks down in the BEC limit, \( x \lesssim 0.5 \), because the gap
becomes large, the chemical potential becomes negative, gaussian fluctuations become
important at \( T_\mu \), and pseudogaps appear [20,21]. The second sound with
maximum \( u_2 \simeq 0.53u_F(T_c/E_F) \) can therefore not be trusted on the BEC
side for \( x \gtrsim 0.5 \) in the present model. If the first and second sound
continue to decrease and increase respectively as we go from the BCS to the BEC limit (see
Fig. 1), i.e. as \( x \lesssim 0 \), they would cross. However, the two sound modes are
coupled and undergo avoided crossing. This is known to occur in a BEC at the
temperature \( T^* = |g|/n \) [22,23]. Below this temperature \( (T \lesssim T^*) \) the
Bogoliubov condensate mode \( u^2 = |g|n_\nu(T)/m \) is first sound whereas above it is
second sound. Above the first sound speed is the classical result \( u_1 = 0.925 \sqrt{T/m} \). At
the crossings, where the normal and superfluid components couple strongly, we
may expect stronger damping of both sound modes.

In summary, the mean field BCS equations were corrected with induced interactions,
Hartree-Fock energies and self-energies. The resulting crossover model is
approximately correct at all temperatures, densities and interaction strengths up to
\( x \lesssim 0.5 \), and exact in the dilute BCS limit. A number of thermodynamic quantities
were calculated for gas of Fermi atoms with superfluid and normal components.
Second sound is particularly important because it only appears in a superfluid gas and
it reveals \( T_c \) where it vanishes. The predicted sound speeds in the superfluid phase can be
tested in upcoming experiments. The validity of crossover model will be put to
a test in the unitarity limit and measurements of the sound modes on the molecular
BEC side will provide important information on new phenomena as pseudogap,
quasiparticle energies, etc.

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