AN ORIENTED HYPERGRAPHIC APPROACH TO ALGEBRAIC
GRAPH THEORY

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Abstract. An oriented hypergraph is a hypergraph where each vertex-edge incidence is
given a label of +1 or −1. We define the adjacency, incidence and Laplacian matrices of an
oriented hypergraph and study each of them. We extend several matrix results known for
graphs and signed graphs to oriented hypergraphs. New matrix results that are not direct
generalizations are also presented. Finally, we study a new family of matrices that contains
walk information.

1. Introduction

Researchers have studied the adjacency, Laplacian, normalized Laplacian and signless
Laplacian matrices of a graph [1, 3]. Recently there has been a growing study of matrices
associated to a signed graph [7]. In this paper we hope to set the foundation for a study of
matrices associated with an oriented hypergraph.

An oriented hypergraph is a hypergraph with the additional structure that each vertex-edge
incidence is given a label of +1 or −1 [4, 5]. A consequence of studying oriented hypergraphs
is that graphs and signed graphs can be viewed as specializations. A graph can be thought
of as an oriented hypergraph where each edge is contained in two incidences, and exactly
one incidence of each edge is signed +1. A signed graph can be thought of as an oriented
hypergraph where each edge is contained in two incidences.

We define the adjacency, incidence, and Laplacian matrices of an oriented hypergraph and
examine walk counting to extend classical matrix relationships of graphs and signed graphs
to the setting of oriented hypergraphs. New relationships involving the dual hypergraphic
structure are also presented, including a result on line graphs of signed graphs. Finally,
we study a new matrix which encapsulates walk information on an oriented hypergraph to
provide a unified combinatorial interpretation of the Laplacian matrix entries.

2. Background

2.1. Oriented Hypergraphs. Throughout, V and E will denote disjoint finite sets whose
respective elements are called vertices and edges. An incidence function is a function
\( \iota : V \times E \to \mathbb{Z}_{\geq 0} \). A vertex \( v \) and an edge \( e \) are said to be incident (with respect to \( \iota \)) if
\( \iota(v, e) \neq 0 \). An incidence is a triple \( (v, e, k) \), where \( v \) and \( e \) are incident and \( k \in \{1, 2, 3, \ldots, \iota(v, e)\} \); the value \( \iota(v, e) \) is called the multiplicity of the incidence.

Let \( \mathcal{I} \) be the set of incidences determined by \( \iota \). An incidence orientation is a function
\( \sigma : \mathcal{I} \to \{+1, -1\} \). An oriented incidence is a quadruple \( (v, e, k, \sigma(v, e, k)) \). An oriented hypergraph is a quadruple \( (V, E, \mathcal{I}, \sigma) \), and its underlying hypergraph is the triple \( (V, E, \mathcal{I}) \).
A hypergraph is simple if $\iota(v, e) \leq 1$ for all $v$ and $e$, and for convenience we will write $(v, e)$ instead of $(v, e, 1)$ if $G$ is a simple hypergraph. Two, not necessarily distinct, vertices $v$ and $w$ are said to be adjacent with respect to edge $e$ if there exist incidences $(v, e, k_1)$ and $(w, e, k_2)$ such that $(v, e, k_1) \neq (w, e, k_2)$. An adjacency is a quintuple $(v, k_1; w, k_2; e)$, where $v$ and $w$ are adjacent with respect to $e$ using incidences $(v, e, k_1)$ and $(w, e, k_2)$. The degree of a vertex $v$, denoted by $\deg(v)$, is equal to the number of incidences containing $v$. A $k$-regular hypergraph is a hypergraph where every vertex has degree $k$. The size of an edge is the number of incidences containing that edge. A $k$-edge is an edge of size $k$. A $k$-uniform hypergraph is a hypergraph where all of its edges have size $k$.

A walk is a sequence $W = a_0, i_1, a_1, i_2, a_2, i_3, a_3, ..., a_{n-1}, i_n, a_n$ of vertices, edges and incidences, where $\{a_k\}$ is an alternating sequence of vertices and edges, $i_h$ is an incidence containing $a_{h-1}$ and $a_h$, and $i_{2h-1} \neq i_{2h}$. The first and last elements of this sequence are called anchors. A walk where both anchors are vertices is called a vertex-walk. A walk where both anchors are edges is called an edge-walk. A walk where one anchor is a vertex and the other is an edge is called a cross-walk. The length of a walk is half the number of incidences that appear in that walk. Let $a_i, a_j \in V \cup E$ and $k \in \frac{1}{2} \mathbb{Z}$, define $w(a_i, a_j; k)$ to be the number of walks of length $k$ from $a_i$ to $a_j$. Observe that vertex-walks and edge-walks have integral length, while cross-walks have half-integral length.

Given a hypergraph $G$ the incidence dual $G^*$ is the hypergraph obtained by reversing the roles of the vertices and edges. Similarly, for an oriented hypergraph $G = (V, E, I, \sigma)$, the incidence dual $G^*$ is the oriented hypergraph $(E, V, I^*, \sigma^*)$, where $I^* = \{(e, v, k) : (v, e, k) \in I\}$, and $\sigma^* : I^* \to \{+1, -1\}$ such that $\sigma^*(e, v, k) = \sigma(v, e, k)$. The set $I^*$ determines an incidence function $\iota^*$ where $\iota^*(e, v) = \iota(v, e)$.

**Proposition 2.1.** If $G$ is an oriented hypergraph, then $G^{**} = G$.

The concept of signed adjacency was introduced in [1] as a hypergraphic alternative to signed edges. The sign of the adjacency $(v, k_1; w, k_2; e)$ is defined as

$$sgn_e(v, k_1; w, k_2) = -\sigma(v, e, k_1)\sigma(w, e, k_2).$$

This will oftentimes be shortened to $sgn_e(v, w) = -\sigma(v, e)\sigma(w, e)$ when the adjacency is understood, as in the case of simple oriented hypergraphs. We will assume $sgn_e(v, k_1; w, k_2) = 0$ if $v$ and $w$ are not adjacent.

The dual to signed adjacency is the sign of the co-adjacency

$$sgn_v(e, k_1; f, k_2) = -\sigma(v, e, k_1)\sigma(f, k_2).$$

The sign of a walk is the product of the signs of all adjacencies in the walk if it is a vertex-walk, the product of the signs of all co-adjacencies if it is an edge-walk, and is the product of the signs of all adjacencies in the walk along with the extra incidence sign if it is a cross-walk. That is, the sign of a walk $W = a_0, i_1, a_1, i_2, a_2, i_3, a_3, ..., a_{n-1}, i_n, a_n$ is

$$\text{sgn}(W) = (-1)^p \prod_{h=1}^{n} \sigma(i_h),$$

where $p = \lfloor n/2 \rfloor$.

There are a number of additional signed walk counts we need besides $w(a_i, a_j; k)$. Let $w^+(a_i, a_j; k)$ be the number of positive walks of length $k$ between anchors $a_i$ and $a_j$, and $w^-(a_i, a_j; k)$ be the number of negative walks of length $k$ between anchors $a_i$ and $a_j$. Finally, let $w^\pm(a_i, a_j; k) = w^+(a_i, a_j; k) - w^-(a_i, a_j; k)$. 
A vertex-switching function is any function $\theta : V \rightarrow \{-1, +1\}$. Vertex-switching the oriented hypergraph $G$ means replacing $\sigma$ by $\sigma^\theta$, defined by: $\sigma^\theta(v, e, k_1) = \theta(v)\sigma(v, e, k_1)$; producing the oriented hypergraph $G^\theta = (V, E, I, \sigma^\theta)$. The vertex-switching produces an adjacency signature $sgn^\theta$, defined by: $sgn^\theta_e(v, k_1; w, k_2) = \theta(v)sgn_e(v, k_1; w, k_2)\theta(w)$.

2.2. Signed Graphs. A 2-uniform simple oriented hypergraph has been called a bidirected graph, and was first studied by Edmonds and Johnson [2]. Since each 2-edge forms a unique adjacency we can regard the adjacency sign as the sign of the 2-edge, and a bidirection as an orientation of the signed edges. Zaslavsky introduced this concept to signed graphs in [6].

A signed graph is a graph with the additional structure that each edge is given a sign of either $+1$ or $−1$. Formally, a signed graph is a pair $\Sigma = (\Gamma, sgn)$ consisting of an underlying graph $\Gamma = (V, E)$ and a signature $sgn : E \rightarrow \{+1, −1\}$. An unsigned graph can be thought of as a signed graph with all edges signed $+1$. The sign of an edge $e$ is denoted $sgn(e)$.

An oriented signed graph is a pair $(\Sigma, \tau)$, consisting of a signed graph $\Sigma$ and an orientation $\tau : V \times E \rightarrow \{-1, 0, +1\}$, where

$$\tau(v, e)\tau(w, e) = -sgn(e) \text{ if } v \text{ and } w \text{ are adjacent via } e, \quad \tau(v, e) = 0 \text{ if } v \text{ and } e \text{ are not incident;}$$

furthermore $\tau(v, e) \neq 0$ if $v$ is incident to $e$. By convention, $\tau(v, e) = +1$ is thought as an arrow pointing toward the vertex $v$, and $\tau(v, e) = -1$ is thought as an arrow pointing away from the vertex $v$. Thus, an orientation can be viewed as a bidirection on a signed graph.

**Proposition 2.2.** $G$ is a 2-uniform oriented hypergraph if, and only if, $G$ is an oriented signed graph.

The line graph of a signed graph can be defined via oriented signed graphs. The following definition was introduced by Zaslavsky [7]. Let $\Lambda(\Gamma)$ denote the line graph of the unsigned graph $\Gamma$. A line graph of an oriented signed graph $(\Sigma, \tau)$ is the oriented signed graph $(\Lambda(\Gamma), \tau_\Lambda)$, where $\tau_\Lambda$ is defined by

$$\tau_\Lambda(e_{ij}, e_{ij}e_{jk}) = \tau(v_j, e_{ij}).$$

The signature of this line graph is denoted by $sgn_{\Lambda(\tau)}$.

3. **Adjacency matrix**

The adjacency matrix $A_G = [a_{ij}]$ of a simple oriented hypergraph $G$ is defined by

$$a_{ij} = \sum_{e \in E} sgn_e(v_i, v_j).$$

Vertex switching an oriented hypergraph $G$ can be described as matrix conjugation of the adjacency matrix. For a vertex switching function $\theta$, we define a diagonal matrix $D(\theta) := \text{diag}(\theta(v_i) : v_i \in V)$.

**Proposition 3.1.** Let $G$ be an oriented hypergraph. If $\theta$ is a vertex switching function on $G$, then $A_{G^\theta} = D(\theta)^T A_G D(\theta)$.

If $A_\Gamma$ is the adjacency matrix of an unsigned graph $\Gamma$, it is well known that the $(i, j)$-entry of $A_\Gamma^k$ counts the number of walks of length $k$ from $v_i$ to $v_j$. Zaslavsky generalized this result to signed graphs [7]. Here we present a generalization to oriented hypergraphs.
Theorem 3.2. If \( G \) is a simple oriented hypergraph and \( k \) a non-negative integer, then the \((i, j)\)-entry of \( A_G^k \) is \( w^+(v_i, v_j; k) \).

**Proof.** We prove this using mathematical induction.

**Base case:** If \( k = 0 \), then \( A_G^0 = I \), which is consistent with a 0-walk travelling nowhere.

If \( k = 1 \), then \((A_G^1)_{ij} = a_{ij}\), where

\[
a_{ij} = \sum_{e \in E} \text{sgn}_e(v_i, v_j) = w^+(v_i, v_j; 1) - w^-(v_i, v_j; 1) = w^±(v_i, v_j; 1).
\]

**Induction hypothesis:** Suppose that \((A_G^k)_{ij} = w^±(v_i, v_j; k)\).

We calculate the \((i, j)\)-entry of \( A_G^{k+1} \) as follows:

\[
(A_G^{k+1})_{ij} = (A_G A_G^k)_{ij} = \sum_{l=1}^{n} a_{il} \cdot w^±(v_i, v_j; k) \quad \text{(by the Induction Hypothesis)}
\]

\[
= \sum_{l=1}^{n} \left( \sum_{e \in E} \text{sgn}_e(v_i, v_l) \right) [w^+(v_l, v_j; k) - w^-(v_l, v_j; k)]
\]

\[
= \sum_{l=1}^{n} (w^+(v_l, v_i; 1) - w^-(v_l, v_i; 1)) [w^+(v_l, v_j; k) - w^-(v_l, v_j; k)]
\]

\[
= \sum_{l=1}^{n} [w^+(v_l, v_i; 1)w^+(v_l, v_j; k) + w^-(v_l, v_i; 1)w^-(v_l, v_j; k)]
\]

\[
- w^-(v_l, v_i; 1)w^+(v_l, v_j; k) - w^+(v_l, v_i; 1)w^-(v_l, v_j; k)].
\]

Notice that the number of positive walks of length \( k + 1 \) from \( v_i \) to \( v_j \) with \( v_l \) as the second vertex is

\[
w^+(v_i, v_l; 1)w^+(v_l, v_j; k) + w^-(v_i, v_l; 1)w^-(v_l, v_j; k).
\]

Similarly, the number of negative walks of length \( k + 1 \) from \( v_i \) to \( v_j \) with \( v_l \) as the second vertex is

\[
w^-(v_i, v_l; 1)w^+(v_l, v_j; k) + w^+(v_i, v_l; 1)w^-(v_l, v_j; k).
\]

Since any walk of length \( k + 1 \) from \( v_i \) to \( v_j \) must have one of the vertices \( v_1, v_2, \ldots, v_n \) as its second vertex, it must be that

\[
w^+(v_i, v_j; k + 1) = \sum_{l=1}^{n} [w^+(v_l, v_i; 1)w^+(v_l, v_j; k) + w^-(v_l, v_i; 1)w^-(v_l, v_j; k)]
\]

and

\[
w^-(v_i, v_j; k + 1) = \sum_{l=1}^{n} [w^-(v_l, v_i; 1)w^+(v_l, v_j; k) + w^+(v_l, v_i; 1)w^-(v_l, v_j; k)].
\]

Thus, the \((i, j)\)-entry of \( A_G^{k+1} \) simplifies to

\[
w^+(v_i, v_j; k + 1) - w^-(v_i, v_j; k + 1) = w^±(v_i, v_j; k + 1).
\]

Similarly, using the incidence dual \( G^\ast \), we can obtain information about walks between edges of \( G \).

Theorem 3.3. If \( G \) is a simple oriented hypergraph and \( k \) a non-negative integer, then

\[(i, j)\)-entry of \( A_G^k = w^±(v_i^\ast, v_j^\ast; k) = w^±(e_i, e_j; k).\]
Theorem 3.4. Let $\Sigma = (\Gamma, sgn)$ be a simple signed graph. Let $G = (\Sigma, \tau)$ be an oriented signed graph. Then

$$A_{(\Lambda(\Gamma), \tau)} = A_{G^*}.$$  

Proof. By Proposition 2.2, $G$ may be treated as both a signed graph and a 2-uniform simple oriented hypergraph. Therefore, $G$ has an incidence orientation $\sigma$ defined by $\sigma(v, e) = \tau(v, w)$, for all $(v, e) \in I$.

The $(i, j)$-entry of $A_{G^*}$ is

$$sgn_{e^*}(v^*_i, v^*_j) = sgn_v(e_i, e_j) = -\sigma(v, e_i)\sigma(v, e_j) = -\tau(v, e_i)\tau(v, e_j) = -\tau(e_i, e_j)\tau(e_i, e_j) = sgn_{\Lambda(e_i, e_j)}(e_i, e_j),$$

which is the $(i, j)$-entry of $A_{(\Lambda(\Gamma), \tau)}$. Since both $A_{(\Lambda(\Gamma), \tau)}$ and $A_{G^*}$ are $m \times m$ matrices, the proof is complete. \qed

4. Incidence, degree, and Laplacian matrices

Given a labeling $v_1, v_2, \ldots, v_n$ of the elements of $V$, and $e_1, e_2, \ldots, e_m$ of the elements of $E$ of an oriented hypergraph $G$, the incidence matrix $H_G = [\eta_{ij}]$ is the $n \times m$ matrix defined by

$$(4.1) \quad \eta_{ij} = \sum_{k=1}^{\iota(v_i, e_j)} \sigma(v_i, e_j, k).$$

If $G$ is simple, then Equation (4.1) is equivalent to

$$\eta_{ij} = \begin{cases} \sigma(v_i, e_j) & \text{if } (v_i, e_j) \in I, \\ 0 & \text{otherwise}. \end{cases}$$

For signed and unsigned graphs, the line graph is the graphical approximation of incidence duality. One advantage of incidence duality is that it is an involution and provides a hypergraphic analog to transposition of matrices.

Theorem 4.1. If $G$ is an oriented hypergraph, then $H_G^T = H_{G^*}$. 

Another familiar graphic matrix that has a natural extension to oriented hypergraphs is the degree matrix. The degree matrix of an oriented hypergraph $G$ is $D_G = [d_{ij}] := \text{diag}(\deg(v_1), \ldots, \deg(v_n))$.

The Laplacian matrix is defined as $L_G := D_G - A_G$.

Theorem 4.2. If $G$ is a simple oriented hypergraph, then $L_G = H_GH_G^T$. 

Proof. The $(i, j)$-entry of $H_GH_G^T$ corresponds to the $i$th row of $H_G$, indexed by $v_i \in V$, multiplied by the $j$th column of $H_G^T$, indexed by $v_j \in V$. Therefore, this entry is precisely

$$\sum_{e \in E} \eta_{i,e} \eta_{e,j} = \sum_{e \in E} \sigma(v_i, e)\sigma(v_j, e).$$

If $i = j$, then the sum simplifies to

$$\sum_{e \in E} |\sigma(v_i, e)|^2 = \deg(v_i),$$
since $|\sigma(v_i, e)| = 1$ if $v_i$ is incident to $e$. If $i \neq j$, then the sum simplifies to
\[ \sum_{e \in E} \sigma(v_i, e)\sigma(v_j, e) = \sum_{e \in E} -\text{sgn}_e(v_i, v_j) = -a_{ij}. \]

Hence, $H_G H_G^T = D_G - A_G = L_G$. \hfill \Box

Vertex switching an oriented hypergraph $G$ can be described as matrix multiplication of the incidence matrix, and matrix conjugation of the Laplacian matrix.

**Proposition 4.3.** Let $G$ be an oriented hypergraph. If $\theta$ is a vertex switching function on $G$, then
\begin{enumerate}
  \item $H_G^{\theta} = D(\theta)H_G$, and
  \item $L_G^{\theta} = D(\theta)^T L_G D(\theta)$.
\end{enumerate}

The following corollary is immediate from Theorems 4.1 and 4.2. This generalizes the classical relationship between the incidence, degree, adjacency and Laplacian matrices known for graphs and signed graphs.

**Corollary 4.4.** The following relations hold for a simple oriented hypergraph $G$.
\begin{enumerate}
  \item $L_G = D_G - A_G = H_G H_G^T$,
  \item $L_G^* = D_G^* - A_G^* = H_G^T H_G$.
\end{enumerate}

If $G$ is a simple $k$-uniform oriented hypergraph, we can specialize the above as follows.

**Corollary 4.5.** If $G$ is a simple $k$-uniform oriented hypergraph, then
\[ H_G^T H_G = kI - A_G^*. \]

*Proof.* Since every edge has size $k$, the incidence dual is $k$-regular. Hence, $D_G^* = kI$. \hfill \Box

Now we have enough machinery to show that Corollaries 4.4 and 4.5 are generalizations of signed graphic relations, and can also be viewed as a generalization of the classical relationship known for graphs (see [7] for details).

**Corollary 4.6.** Let $\Sigma = (\Gamma, \text{sgn})$ be a simple signed graph. Let $G = (\Sigma, \tau)$ be an oriented signed graph. Then
\begin{enumerate}
  \item $H_G H_G^T = D_G - A_G = L_G$,
  \item $H_G^T H_G = 2I - A_{(\Lambda(\Gamma), \tau_{\Lambda})}$.
\end{enumerate}

*Proof.* By Corollary 4.4 (1) is immediate. Since $G$ is 2-uniform, by Corollaries 4.4 and 4.5 $H_G^T H_G = 2I - A_G^*$. Also, by Theorem 4.5 $A_G^* = A_{(\Lambda(\Gamma), \tau_{\Lambda})}$. The result follows. \hfill \Box

5. **Walk and weak walk matrices**

Here we study the underlying structure of the adjacency, incidence, and degree matrices to provide a unified combinatorial interpretation of the entries of $L$.

Let $A_1, A_2 \in \{V, E\}$ and $X_{(G, A_1, A_2, k)} = [x_{ij}]$ be the $A_1 \times A_2$ matrix where $x_{ij} = w^\pm(a_i, a_j; k)$. The matrix $X_{(G, A_1, A_2, k)}$ is called a $k$-walk matrix of an oriented hypergraph $G$. If $A_1 = A_2$, then we will assume that $k$ is a nonnegative integer. If $A_1 \neq A_2$, then we will assume that $k$ is a nonnegative half-integer.

**Lemma 5.1.** If $G$ is a simple oriented hypergraph, then $X_{(G, V, V, k)} = A^k$.

*Proof.* The result is immediate by Theorem 3.2. \hfill \Box
Walk matrices allow us to present the incidence matrix as a measure of 1/2-walks.

**Lemma 5.2.** If $G$ is an oriented hypergraph, then $X_{(G,V,E,1/2)} = H_G$.

*Proof.* The $(v, e)$-entry of $X_{(G,V,E,1/2)}$ is $x_{ve} = w^\pm(v, e; 1/2) = \sigma(v, e) = \eta_{ve}$. □

The incidence dual again provides new walk information on $G$.

**Lemma 5.3.** The following duality relationships hold for $k$-walk matrices.

1. $X_{(G,V,V,k)} = X_{(G^*,E^*,E^*,k)}$.
2. $X_{(G,E,E,k)} = X_{(G^*,V^*,V^*,k)}$.
3. $X^T_{(G,V,E,k)} = X_{(G,E,V,k)} = X_{(G^*,V^*,E^*,k)}$.

The following corollary translates the Laplacian matrix in terms of 1/2-walks. It says that if $G$ is an oriented hypergraph, then the $(i, j)$-entry of $L_G$ represents the number of half walks from $v_i$ to some edge $e$ multiplied by the number of half walks from edge $e$ to $v_j$.

**Corollary 5.4.** If $G$ is an oriented hypergraph, then $L_G = X_{(G,V,E,1/2)}X_{(G,E,V,1/2)}$.

*Proof.* By Theorem 4.2 and Lemma 5.2,

$$L_G = H_G(H_G)^T = X_{(G,V,E,1/2)}(X_{(G,V,E,1/2)})^T = X_{(G,V,E,1/2)}X_{(G,E,V,1/2)}.$$ □

A *weak walk* is a walk in which the condition $i_{2h-1} \neq i_{2h}$ is removed and you are allowed to immediately return along the same incidence at any point. The sign of a weak walk is given by Equation (2.1).

Paralleling the notation $w(a_i, a_j; k)$ for the number of walks of length $k$ from $a_i$ to $a_j$, we let $\tilde{w}(a_i, a_j; k)$ denote the number of weak walks of length $k$ between $a_i$ and $a_j$, and define $\tilde{w}^+(a_i, a_j; k)$, $\tilde{w}^-(a_i, a_j; k)$, $\tilde{w}^\pm(a_i, a_j; k)$ for weak walks analogously. A weak walk of length 1 of the form $v, i, e, i, v$ is called a *backstep*.

**Theorem 5.5.** If $G$ is an oriented hypergraph, then the $(i, j)$-entry of $D_G$ is $\tilde{w}(v_i, v_j; 1) - w(v_i, v_j; 1)$, the number of backsteps at $v_i$.

*Proof.* The weak walks of length 1 from a fixed vertex $v_i$ must contain two incidences. These naturally partition into those walks that return along the same incidence, or backsteps, and those that form adjacencies. The value $w(v_i, v_j; 1)$ is precisely those that form adjacencies, so $\tilde{w}(v_i, v_j; 1) - w(v_i, v_j; 1)$ counts the incidences at vertex $v_i$. □

The next theorem provides an alternative way to calculate the entries of the Laplacian matrix of an oriented hypergraph.

**Theorem 5.6.** If $G$ is an oriented hypergraph, then the $(i, j)$-entry of $L_G$ is

$$\ell_{ij} = \tilde{w}(v_i, v_j; 1) - 2w^+(v_i, v_j; 1).$$

*Proof.* The $(i, j)$-entry of $L(G)$ is

$$\ell_{ij} = d_{ij} - a_{ij} = [\tilde{w}(v_i, v_j; 1) - w(v_i, v_j; 1)] - w^\pm(v_i, v_j; 1) \quad \text{(by Lemma 5.1 and Theorem 5.5)}$$

$$= \tilde{w}(v_i, v_j; 1) - 2w^+(v_i, v_j; 1).$$ □

**Lemma 5.7.** The sign of every backstep is negative.
The weak \(k\)-walk matrix \(W_{(G,A_1,A_2,k)} = [w_{ij}]\) is defined similar to the \(k\)-walk matrix except \(w_{ij} = \tilde{w}^\pm(a_i,a_j;k)\).

Theorem 5.6 allows us to calculate the entries of the Laplacian matrix of an oriented hypergraph via walk information. Here we present a unified combinatorial interpretation of the Laplacian matrix entries as weak walks of length 1.

**Theorem 5.8.** If \(G\) is a simple oriented hypergraph, then \(L_G = -W_{(G,V,V,1)}\).

**Proof.** If \(i = j\), then every weak 1-walk is a backstep since \(G\) is simple. Thus,

\[
\begin{align*}
\tilde{w}_{ii} &= \tilde{w}^+(v_i,v_i;1) - \tilde{w}^-(v_i,v_i;1) = 0 = \deg(v_i),
\end{align*}
\]

If \(i \neq j\), then there are no backsteps and each weak walk is a walk. Thus,

\[
\begin{align*}
w_{ij} &= \tilde{w}^+(v_i,v_j;1) = a_{ij}.
\end{align*}
\]

Hence, \(W_{(G,V,V,1)} = -D_G + A_G = -L_G\). \(\Box\)

Theorems 5.6 and 5.8 are summarized here.

**Corollary 5.9.** If \(G\) is a simple oriented hypergraph, then the \((i,j)\)-entry of \(L_G\) is

\[
\ell_{ij} = \tilde{w}(v_i,v_j;1) - 2w^+(v_i,v_j;1) = -\tilde{w}^+(v_i,v_j;1).
\]

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