Compound Poisson Approximations in $\ell_p$-norm for Sums of Weakly Dependent Vectors

V. Čekanavičius · P. Vellaisamy

Received: 23 March 2020 / Revised: 10 August 2020 / Published online: 20 September 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
The distribution of the sum of 1-dependent lattice vectors with supports on coordinate axes is approximated by a multivariate compound Poisson distribution and by signed compound Poisson measure. The local and $\ell_\alpha$-norms are used to obtain the error bounds. The Heinrich method is used for the proofs.

Keywords Compound Poisson distribution · Expansion in the exponent · $\ell_p$ norm · Local norm · Multivariate distribution

Mathematics Subject Classification Primary 62E17 · Secondary 60F25

1 Introduction
Numerous papers are devoted to Poisson and compound Poisson approximations in one-dimensional case, see, for example, surveys [9,17]. Various metrics, such as local, Gini (Wasserstein) metric, Chi-square metric and analogues of $\ell_p$ norms, were used, see, for example, [4,5,11,18,27].

The multivariate case is less explored. For compound Poisson approximations in Lévy, Lévy–Prokhorov and Kolmogorov metrics, see [28–30]. Multivariate Poisson approximation in total variation for sums of independent lattice vectors concentrated on coordinate vectors is considered in [1,2,20], and compound Poisson and signed compound Poisson approximations are applied in [15,22–24]. The local point metric is also used in [15,20]. Necessary and sufficient conditions for the weak convergence

V. Čekanavičius
vydas.cekanavicius@mif.vu.lt

P. Vellaisamy
pv@math.iitb.ac.in

1 Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 03225 Vilnius, Lithuania
2 Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India
of the distribution of a sum of dependent random vectors to a compound Poisson vector are given in [16], Theorems 6.3 and 6.4. An estimate of Compound Poisson approximation to the sum of weakly dependent discrete random vectors is given in [16], Theorem 6.8.

In this paper, we investigate similarly the sum of random vectors (rvs) concentrated on $k$-dimensional unit vectors. We assume that rvs are 1-dependent and estimate the accuracy of approximations in local and $\ell_p$ norms. As usual, the 1-dependence means that sigma-algebras generated by $X_1, X_2, \ldots, X_k$ and $X_t, X_{t+1}, \ldots, X_s$ are independent for any $|k-t| > 1$. This implies, for example, that $X_1$ and $X_3, X_1$ and $X_4$, etc., are independent. Note that any sum of $m$-dependent dependent rvs can be reduced to the sum of 1-dependent rvs by grouping of consequent summands.

Let $\mathbf{0} = (0, \ldots, 0)$, and let $e_r$ denote the $r$’th coordinate vector in $\mathbb{R}^k$, $k$-dimensional Euclidean space, that is, $e_r = (0, \ldots, 1, \ldots, 0)$, $1 \leq r \leq k$. Further, let $X_1, X_2, \ldots, X_n$ be 1-dependent identically distributed $k$-dimensional rvs and $P(X_1 = e_r) = p_r, p_r \in (0, 1), r = 1, 2, \ldots, k$. $P(X_1 = \mathbf{0}) = 1 - (p_1 + p_2 + \cdots + p_k)$. The dependence of consequent summands is reflected in joint probabilities $p_{rj} = P(X_1 = e_r, X_2 = e_j)$, $p_{rjm} = P(X_1 = e_r, X_2 = e_j, X_3 = e_m)$ and so on. The distribution of $S_n = X_1 + X_2 + \ldots + X_n$ is denoted by $F_n$.

Our aim is to approximate the distribution of $S_n$. Let $\mathbb{Z}$ denote the set of integers. Approximations used in this paper are mostly defined by their Fourier transforms. If a measure $M$ is concentrated on $k$-dimensional set of integers $\mathbb{Z}^k = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, then its Fourier transform (characteristic function) is denoted by $\hat{M}(\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{Z}^k} M(\mathbf{m})e^{i\langle \mathbf{m}, \mathbf{t} \rangle}$. Here and henceforth, $(\mathbf{m}, \mathbf{t}) := m_1t_1 + m_2t_2 + \cdots + m_kt_k$ and $i$ is imaginary unit. Note that there is one-to-one correspondence between a distribution and its characteristic function. Observe also that $\hat{P}_n(\mathbf{t}) = E \exp\{i(S_n, \mathbf{t})\}$ and $Ee^{i\langle X_1, \mathbf{t} \rangle} = 1 + \sum_{r=1}^k p_r(e^{it_r} - 1)$.

We say rv $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_k)$ follows a $k$-dimensional Poisson if $\tilde{Y}_j$’s are independent and $\tilde{Y}_j$ follows the Poisson distribution with parameter $\mu_j$, $1 \leq j \leq k$. It is denoted by Pois$(\mu_j)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. In this paper, we consider the problem of approximating the distribution of $S_n$ to the $k$-dimensional Poisson distribution Pois$(\lambda)$, where $\lambda = (np_1, np_2, \ldots, np_k)$. Let $P(\lambda, s) = e^{-\lambda}s^s/s!$ denote probability mass function of the Poisson distribution. Then,

$$\text{Pois}(\lambda) = \prod_{j=1}^k P(np_j, m_j), \quad \hat{\text{Pois}}(\lambda)(\mathbf{t}) = \exp\left\{n \sum_{j=1}^k p_j(e^{it_j} - 1)\right\}.$$  

First-order asymptotic expansion is constructed as Pois$(\lambda) + A_1$, where measure $A_1$ has the following Fourier transform:

$$\hat{A}_1(\mathbf{t}) := \hat{\text{Pois}}(\lambda)(\mathbf{t})\left(-\frac{n}{2}\left(\sum_{j=1}^k p_j(e^{it_j} - 1)\right)^2 + (n - 1) \sum_{j,m=1}^k (p_{jm} - p_jp_m)(e^{it_j} - 1)(e^{it_m} - 1)\right). \quad (1)$$
Let $\Delta P(\lambda, s) := P(\lambda, s) - P(\lambda, s - 1)$, $\Delta^2 P(\lambda, s) = \Delta(\Delta P(\lambda, s))$, and let $d_{jr} = (n - 1)(p_{jr} - p_j p_r) - (n/2) p_j p_r$. From the formula of inversion, it follows that, for any $m \in \mathbb{Z}^k$,

$$A_1\{m\} = \sum_{j=1}^k d_{jj} \Delta^2 P(np_j, m_j) \prod_{l \neq j} P(np_l, m_l)$$

$$+ \sum_{j,r=1; j \neq r}^k d_{jr} \Delta P(np_j, m_j) \Delta P(np_r, m_r) \prod_{l \neq j,r} P(np_l, m_l).$$

If we construct similar asymptotic expansion in the exponent, then the result is signed compound Poisson measure $G$ with Fourier transform

$$\hat{G}(t) := \widehat{\text{Pois}}(\lambda)(t) \exp \left\{ - \frac{n}{2} \left( \sum_{j=1}^k p_j (e^{it_j} - 1) \right)^2 \right\}

+ (n - 1) \sum_{j,m=1}^k (p_{jm} - p_j p_m)(e^{it_j} - 1)(e^{it_m} - 1).$$

Formula of inversion also allows explicit expression for $G\{m\}$. However, the resulting formula is quite long and plays no role in further proofs and therefore is omitted. The idea to use compound Poisson-type signed measures by retaining a part of asymptotic expansion in the exponent goes back to [14]. In spite of obvious structural similarity, generally such measures ensure much better accuracy than Poisson or even the second-order Poisson approximations, see, for example, [8,17,22] and the references therein. Both the measures $A_1$ and $G$ can be written as convolutions of measures concentrated on various $e_r$'s.

In this paper, symbol $p$ is reserved for probabilities. Therefore, in the definition of norms, we instead use symbol $\alpha$, which is further on assumed to be fixed positive number. We define, respectively, local and $\ell_\alpha$-norms for finite measure $M$ concentrated on $\mathbb{Z}^k$ as

$$\|M\|_\infty = \sup_{m \in \mathbb{Z}^k} |M\{m\}|, \quad \|M\|_\alpha = \left( \sum_{m \in \mathbb{Z}^k} M^\alpha\{m\} \right)^{1/\alpha}, \quad \alpha \in [1, \infty).$$

The case $\alpha = 1$ corresponds to the total variation norm $\|M\| := \|M\|_1$. Local norm can be viewed as a limit case of $\|M\|_\alpha$ when $\alpha \to \infty$. Thus, total variation and local norms form natural boundaries for all $\ell_\alpha$ norms. In this paper, the emphasis is on local and $\ell_\alpha$, ($\alpha \geq 2$) norms. Note that total variation norm is equivalent to the total variation distance. More precisely, $d_{TV}(F, G) := \sup |F\{B\} - G\{B\}| = \frac{1}{2} \|F - G\|$, where supremum is taken over all $k$-dimensional Borel sets $B$.

We denote by $C$ positive absolute constants, the values of which may change from line to line, or even within the same line. Similarly, by $C(\cdot)$ we denote constants depending on the indicated argument only. Sometimes, to avoid possible ambiguity
we supply $C$ with index. Similarly, $\theta$ is used for a real or a complex number satisfying $|\theta| \leq 1$.

2 Some Known Results

The most part of multivariate results related to compound Poisson approximations are proved for independent rvs concentrated on $e_r$, $1 \leq r \leq k$. Set $p = p_1 + p_2 + \cdots + p_k$. The total variation metric is typically used to estimate the accuracy of approximations. If we assume that $\tilde{X}_j$ are independent copies of $X_j$, $j = 1, 2, \ldots, n$, and denote the distribution of $W_n = \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_n$ by $\mathcal{L}(W_n)$, then

$$\frac{1}{7} \max_{1 \leq r \leq k} \min(p_r, np_r^2) \leq \|\mathcal{L}(W_n) - \text{Pois}(\lambda)\| \leq 2 \min(p, np^2). \quad (3)$$

The upper bound follows directly from one-dimensional Poisson approximation to the binomial distribution; see [7], p. 29 and Equation (1.1) in [24]. The lower bound is a special case of Proposition 1.3 in [24]. Note that constant 2 on the right-hand side of (3) is not the optimal one and there exist other smaller constants, with long expressions, see, for example Eq. (50)–(51) in [17] or (1.3) in [24] or discussion in [18]. In [21], $\ell_\alpha$-norm was used for Krawtchouk expansions, though we are unaware about any similar multivariate Poisson approximation result. For the one-dimensional case, squared $\ell_2$-norm was used in the seminal paper of Franken [10] and the closeness of binomial and Poisson distributions was thoroughly investigated in [11] for an analogue of $\ell_\alpha$-norm for even more general case of $\alpha \in (0, \infty)$.

There are many local estimates for Poisson approximation to sums of random variables, see, for example, [3]. However, unlike in (3), they cannot be used for obtaining local estimate for $k$-dimensional vectors. The local bound for $\mathcal{L}(W_n)$ follows from Equation (26) in [20]. Let

$$v(r) = 2np_r^2 \min \left\{ \frac{1}{2np_r}, e \right\}.$$ 

If $\sum_{r=1}^k \sqrt{2v(r)} < 1$, then

$$\|\mathcal{L}(W_n) - \text{Pois}(\lambda)\|_\infty \leq 2 \prod_{j=1}^k \min \left\{ \frac{1}{2np_j}, e \right\} \left( \frac{\sum_{r=1}^k \sqrt{v(r)}}{1 - \sum_{r=1}^k \sqrt{2v(r)}} \right)^2. \quad (4)$$

If $1 - p > C > 0$ and all $np_r \geq 1$, then (4) implies

$$\|\mathcal{L}(W_n) - \text{Pois}(\lambda)\|_\infty \leq C(k) \prod_{j=1}^k \sqrt{np_j} \left( \sum_{r=1}^k \sqrt{p_r} \right)^2. \quad (5)$$
Observe that if \( p_1 = \cdots = p_k \), then the estimate (5) becomes \( C(k)p_1^2(np_1)^{-k/2} \), showing that the estimate depends on the dimension of vectors as well.

Approximation of sums of independent non-identically distributed vectors by analogues of \( G \) was thoroughly investigated by Roos [23]. For small probabilities, a different choice of parameters for \( G \) was proposed by Borovkov [6]. However, arguably the best estimate for identically distributed independent vectors follows from one-dimensional estimate from [27]: Let \( \omega := \sum_{r=1}^{k} p_r^2 / p < 1/2 \), then

\[
\| \mathcal{L}(W_n) - G \| \leq \frac{p^{3/2}}{\sqrt{n}} \left( \frac{\sqrt{6} \cdot 0.374}{(1 - \omega)^2} + \frac{\sqrt{3\omega}}{2\sqrt{2}(1 - \omega)^{5/2}} \right). \tag{6}
\]

Observe that the accuracy in (6) is at least of the order \( O(n^{-1/2}) \). Also, due to the independence, \( p_{rm} - p_rp_m = 0 \) and \( G \) has simpler structure than in (2).

From Corollary 6.9 in [16], it follows that for sums of 1-dependent rvs, Poisson approximation in total variation can be of order \( O(p \sqrt{n}) \), which is significantly weaker than (3) for moderate \( p \). To the best of our knowledge, there is no similar result for the 1-dependent vectors in \( \ell^\alpha \), \( \alpha > 1 \). Therefore, to get the general idea about what can be expected, we formulate two one-dimensional local estimates, which can be easily obtained from Lemmas 6.3, 6.4 and 6.7 in [8] and the inversion formula (Lemma 5.1) given in Sect. 4. Let \( k = 1 \) and \( p_1 < 0.01 \) and \( p_{12} < 0.05p_1 \), \( a := \max(np_1, 1) \). Then,

\[
\| F_n - \text{Pois}(np_1) \|_\infty \leq \frac{Cn(p_{12} + p_1^2)}{a \sqrt{a}} \tag{7}
\]

and

\[
\| F_n - G \|_\infty \leq \frac{Cn(p_{123} + p_{12}p + p_1^3)}{a^2}. \tag{8}
\]

Observe that summands \( p_{12} \) and \( p_{123} \) reflect the possible 1-dependence of random variables.

## 3 Results

We begin from assumptions on the smallness of probabilities and their interdependency. The magnitude of constants is determined by the method of proof. We assume that

\[
\max_{1 \leq j \leq k} p_j \leq \frac{1}{144k}, \quad \sum_{m=1}^{k} (p_{mj} + p_{jm}) \leq \frac{p_j}{5}, \quad j = 1, 2, \ldots, k. \tag{9}
\]

Note that, in general, it is allowed for all probabilities to depend on the number of summands \( n \), that is, \( X_1, X_2, \ldots \) can form triangular arrays (the scheme of series).
Poisson limit occurs when all $p_j = O(n^{-1})$. In our paper, all probabilities are small, though we nevertheless have included the case $p_j = O(1)$. The second assumption essentially reflects requirement for covariance between $X_1$ and $X_2$ to be small. In [8], similar assumptions are made for one-dimensional case.

Let $\gamma_j = \max(1, \sqrt{np_j}), j = 1, 2, \ldots, k,$

$$\begin{align*}
\varepsilon_1 &:= \sum_{r,j=1}^{k} \frac{pr_{rj} + pr_{r}p_{j}}{\gamma_r \gamma_j}, \\
\varepsilon_2 &:= \sum_{r,j,m=1}^{k} \frac{pr_{rjm} + pr_{rj}p_{m} + pr_{r}p_{j}p_{m}}{\gamma_r \gamma_j \gamma_m}, \\
\varepsilon_3 &:= \left| \sum_{r,m=1}^{k} \frac{2pr_{rm} - 3p_r p_m}{\sqrt{pr_{r}p_m}} \right|, \\
\varepsilon_4 &:= \sum_{r,m=1}^{k} \frac{pr_{rm} + pr_{r}p_{m}}{\sqrt{pr_{r}p_m}}, \\
\varepsilon_5 &:= \left| \sum_{r,j,m=1}^{k} \frac{3pr_{rjm} - 12pr_{rj}p_{m} + 10p_r p_{j}p_{m} + 2 \sum_{r=1}^{k} 3pr_{r}r - 12p_{rr}p_r + 10p_r^3}{\sqrt{pr_{r}p_{j}p_m}} \right|, \\
\varepsilon_6 &:= \sum_{r,j,m=1}^{k} \frac{pr_{rjm} + pr_{r}p_{jm} + pr_{r}p_{j}p_{m}}{\sqrt{pr_{r}p_{m}p_{j}}}. 
\end{align*}$$

We begin with a result for the Poisson approximation.

**Theorem 3.1** If the conditions in (9) are satisfied, then

$$\| F_n - \text{Pois}(\lambda) \|_\infty \leq C(k)n \varepsilon_1 \prod_{j=1}^{k} \gamma_j^{-1},$$

$$\| F_n - \text{Pois}(\lambda) \|_\alpha \leq C(k, \alpha)n \varepsilon_1 \prod_{j=1}^{k} \gamma_j^{-(\alpha-1)/\alpha}, \quad (\alpha \geq 2). \quad (10)$$

Observe that for $n\varepsilon_1 = o(1)$, it suffices that $p_r = o(1), pr_{rj} = O(pr_{r}p_{j}), (r, j = 1, \ldots, k)$). Unlike in [22,24], Kerstan’s method or other convolution technique cannot be applied for proofs, since we are dealing with dependent rvs. The Heinrich method, used in this paper, involves iterations of estimates and results in very large constants. Therefore, we concentrated our efforts on obtaining correct order of estimates, leaving the question about the magnitude of constants and their dependence on dimension $k$ for the future research. Note that asymptotic constants can be small, see Proposition 4.1.

As seen from the following corollary, the order of approximation in Theorem 3.1 is comparable to known results.

**Corollary 3.1** Let the conditions in (9) are satisfied and $np_j \geq 1, j = 1, 2, \ldots, k$. Then, from (10),

$$\| F_n - \text{Pois}(\lambda) \|_\infty \leq C(k) \prod_{j=1}^{k} \frac{1}{\sqrt{np_j}} \left( \sum_{r,m=1}^{k} \frac{pr_{rm}}{\sqrt{pr_{r}p_m}} + \left( \sum_{r=1}^{k} \sqrt{pr_r} \right)^2 \right). \quad (11)$$
Observe that for the case \( k = 1 \), the upper bound given in (11) coincides with (7), up to a constant. Moreover, if \( X_1, X_2, \ldots \) are independent rvs, then (11) is of the same order of accuracy as (5). Next, we consider probable improvements by short asymptotic expansion.

**Theorem 3.2** Under the assumptions in (9), we have

\[
\| F_n - \text{Pois}(\lambda) - A_1 \|_\infty \leq C(k)(n \varepsilon_2 + n^2 \varepsilon_1^2) \prod_{j=1}^{k} \gamma_j^{-1},
\]

\[
\| F_n - \text{Pois}(\lambda) - A_1 \|_\alpha \leq C(k, \alpha)(n \varepsilon_2 + n^2 \varepsilon_1^2) \prod_{j=1}^{k} \gamma_j^{-(\alpha-1)/\alpha}, \quad (\alpha \geq 2).
\]

It is easy to check that Theorem 3.2 is an improvement over Theorem 3.1, if \( pm = O(p_{r m}) \), \((r, m = 1, \ldots, k)\) and \( p = p_1 + \cdots + p_k = o(1)\), since then \( n \varepsilon_1 = O(n^{-k}) \) and \( n \varepsilon_2 = O(n^{-1/2} p^{3/2}) \). On the other hand, in the sense of order, there is no difference between corresponding estimates, if all \( p_j = O(1) \). Indeed, in the latter case \( n \varepsilon_1 = O(1) \) and the accuracy in both cases is of the order \( O(n^{-k(\alpha-1)/\alpha}) \).

Next, consider approximation with short asymptotic expansion in the exponent, that is signed compound Poisson measure \( G \) defined in Sect. 1.

**Theorem 3.3** Let the conditions in (9) hold. Then,

\[
\| F_n - G \|_\infty \leq C(k)n \varepsilon_2 \prod_{j=1}^{k} \gamma_j^{-1},
\]

\[
\| F_n - G \|_\alpha \leq C(k, \alpha)n \varepsilon_2 \prod_{j=1}^{k} \gamma_j^{-(\alpha-1)/\alpha}, \quad (\alpha \geq 2).
\]

In comparison with Poisson approximation, the signed compound Poisson approximation \( G \) is usually smaller by the factor \( n^{-1/2} \). Indeed, if all \( p_j = O(1) \), then \( n \varepsilon_2 = O(n^{-1/2}) \) and corresponding local estimate is of the order \( O(n^{-k(1+1)/2}) \) versus \( O(n^{-k/2}) \) for Poisson approximation. Observe also that when \( k = 1 \), the local estimate in Theorem 3.3 coincides with (8) up to constant.

**Corollary 3.2** If conditions (9) are satisfied and \( np_j \geq 1 \), \( i = 1, 2, \ldots, k \), then

\[
\| F_n - G \|_\infty \leq \frac{C(k)}{\sqrt{n}} \prod_{j=1}^{k} \frac{1}{\sqrt{np_j}} \sum_{r, j, m=1}^{k} \frac{p_{rjm} + p_{rj} p_m + p_r p_j p_m}{\sqrt{p_r p_j p_m}}.
\]

Are the estimates in Theorems 3.1 and 3.3 of the right order? To some extent, an affirmative answer is given by the lower bounds given below.

**Theorem 3.4** Let the conditions in (9) be satisfied and \( np_j \geq 1 \) for \( 1 \leq j \leq k \), \( \alpha \geq 1 \). There exist constants \( C_i(k) \), \( 1 \leq i \leq 6 \), such that, for any constant \( b \geq 1 \), the following lower bound estimates hold:
\[ \| F_n - \text{Pois}(\lambda) \|_\infty \geq \frac{C_1(k)}{b^2} \left( \varepsilon_3 - \frac{C_2(k)}{\min(b, n)} \varepsilon_4 \right) \prod_{j=1}^{k} (np_j)^{-1/2}, \]

\[ \| F_n - \text{Pois}(\lambda) \|_{\alpha} \geq \frac{C_3(k) 5^{-k(\alpha-1)/\alpha}}{b^2} \left( \varepsilon_3 - \frac{C_2(k)}{\min(b, n)} \varepsilon_4 \right) \prod_{j=1}^{k} (np_j)^{-(\alpha-1)/(2\alpha)}, \]

\[ \| F_n - G \|_{\infty} \geq \frac{C_4(k)}{b^3 \sqrt{n}} \left( \varepsilon_5 - \frac{C_5(k)}{\min(b, n)} \varepsilon_6 \right) \prod_{j=1}^{k} (np_j)^{-1/2}, \]

\[ \| F_n - G \|_{\alpha} \geq \frac{C_6(k) 5^{-k(\alpha-1)/\alpha}}{b^3 \sqrt{n}} \left( \varepsilon_6 - \frac{C_5(k)}{\min(b, n)} \varepsilon_6 \right) \prod_{j=1}^{k} (np_j)^{-(\alpha-1)/(2\alpha)}. \]

Note that, unlike in Theorems 3.1–3.3, we have \( \alpha \geq 1 \). Therefore, by taking \( \alpha = 1 \), we can establish the lower estimates for total variation norm. In some cases, the estimates in Theorem 3.4 can be negative. Therefore, Theorem 3.4 must be applied when \( \varepsilon_4 = O(\varepsilon_3) \). In this case, we can ensure non-triviality of estimates by choosing large enough constant \( b \), so that \( C_2(k) / \min(b, n) \) becomes small for all \( n \geq b \). We illustrate this approach by considering independent random vectors. Then, \( p_{rm} = prp_m \) and

\[ \varepsilon_3 = \left( \sum_{r=1}^{k} \sqrt{p_r} \right)^2, \quad \varepsilon_4 = 2 \left( \sum_{r=1}^{k} \sqrt{p_r} \right)^2. \]

Therefore, by choosing \( b = 4C_2(k) \), we get for all \( n \geq 4C_2(k) \),

\[ \varepsilon_3 - \frac{C_2(k)}{\min(b, n)} \varepsilon_4 \geq \frac{1}{2} \left( \sum_{r=1}^{k} \sqrt{p_r} \right)^2. \]

Similarly, we can estimate \( \varepsilon_5 \) and \( \varepsilon_6 \). We formulate lower estimates for independent vectors as a corollary. Let, as in previous section, \( W_n = \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_n \) be a sum of independent copies of \( X_1 \).

**Corollary 3.3** Let \( \max_{1 \leq j \leq k} p_j \leq 1/(144k) \) and \( np_j \geq 1, i = 1, 2, \ldots, k, \alpha \geq 1 \). Then, there exist absolute constants \( C_7(k), C_8(k), C_9(k) \), such that, for any \( n \geq C_7(k) \), the following estimates hold

\[ \| \mathcal{L}(W_n) - \text{Pois}(\lambda) \| \geq C_8(k) \left( \sum_{r=1}^{k} \sqrt{p_r} \right)^2 \prod_{j=1}^{k} (np_j)^{-1/2}, \]

\[ \| \mathcal{L}(W_n) - \text{Pois}(\lambda) \|_{\alpha} \geq C_8(k) 5^{-k(\alpha-1)/\alpha} \left( \sum_{r=1}^{k} \sqrt{p_r} \right)^2 \prod_{j=1}^{k} (np_j)^{-(\alpha-1)/(2\alpha)}, \]

\[ \| \mathcal{L}(W_n) - G \|_{\infty} \geq C_9(k) \frac{p^{3/2}}{\sqrt{n}} \prod_{j=1}^{k} (np_j)^{-1/2}. \]
\[ \| \mathcal{L}(W_n) - G \|_\alpha \geq \frac{C_9(k)}{\sqrt{n}} \prod_{j=1}^{k} (n p_j)^{-(\alpha - 1)/(2\alpha)}. \]

Comparing Corollary 3.3 with (5) and (6), we see that lower estimates have the same order as the upper estimates. The same reasoning applies for dependent vectors. For example, if \( p_r p_m = o(p_{rm}) \), then we always can choose \( b \) in such a way that, for sufficiently large \( n \),

\[ \varepsilon_3 - C_2(k) \varepsilon_4 \geq C(k) \sum_{r,m=1}^{k} \frac{p_{rm}}{\sqrt{p_r p_m}}. \]

Similarly, we can ensure the first right-hand side estimate in Theorem 3.4 to be positive if \( p_{mr} = o(p_m p_r) \). Application of Theorem 3.4 to a well-known statistic is given below.

## 4 An Application

Let \( \xi_1, \xi_2, \ldots \) be independent identically distributed Bernoulli variables with probability \( q \in (0, 1) \). One of the best known and thoroughly investigated examples of the sum of 1-dependent random variables is 2-runs statistic, that is, \( S = \xi_1 \xi_2 + \xi_2 \xi_3 + \cdots + \xi_n \xi_{n+1} \), see [8,19,26] and the references therein. We similarly construct two-dimensional parallel runs with random switching between them. More precisely, let \( \bar{\xi}_1, \bar{\xi}_2, \ldots \) be a sequence of independent identically distributed Bernoulli random variables with probability \( \bar{q} \in (0, 1) \) and let \( \eta_1, \eta_2, \ldots \) be another sequence of independent identically distributed Bernoulli variables with probability \( \delta \in (0, 1) \). Moreover, we assume the random variables in all three sequences to be mutually independent. Let us define a sequence of 1-dependent two-dimensional rvs in the following way:

\[
Y_j = (\eta_j \bar{\xi}_j, (1 - \eta_j) \bar{\xi}_j \bar{\xi}_{j+1}), \quad j = 1, 2, \ldots
\]

Let \( S = Y_1 + Y_2 + \cdots + Y_n \). It is easy to check that \( p_1 = \delta q^2, p_2 = (1 - \delta)q^2, \)
\( p_{11} = \delta^2 q^3, p_{12} = p_{21} = \delta(1 - \delta)q^2 \bar{q}, p_{22} = (1 - \delta)^2 \bar{q}^2, p_{111} = \delta^3 q^4, p_{112} = p_{211} = \delta^2 (1 - \delta) q^3 \bar{q}, p_{121} = \delta^2 (1 - \delta) q^2 \bar{q}^2, p_{122} = p_{221} = \delta (1 - \delta)^2 q^2 \bar{q}, p_{222} = (1 - \delta)^3 \bar{q}.
\]

Let us assume that \( q, \bar{q} \leq 1/17 \). Then, conditions (9) are satisfied. Observe also that

\[
\sum_{r,m=1}^{2} \frac{p_{rm}}{\sqrt{p_r p_m}} = \delta q + 2 \sqrt{\delta (1 - \delta) q \bar{q}} + (1 - \delta) \bar{q} \leq 2(q \delta + \bar{q}(1 - \delta)).
\]
Therefore, when \( np_1, np_2 \geq 1 \), it follows from (11) and Theorem 3.4 that there exists a constant \( C_{10} \) such that for \( n \geq C_{10} \),

\[
\frac{C_{11}(q\delta + \bar{q}(1-\delta))}{nq\bar{q}\sqrt{\delta(1-\delta)}} \leq \|L(S) - \text{Pois}(\lambda)\|_\infty \leq \frac{C_{12}(q\delta + \bar{q}(1-\delta))}{nq\bar{q}\sqrt{\delta(1-\delta)}},
\]

where \( \lambda = (n\delta q^2, n(1-\delta)\bar{q}^2) \). Similarly, for the case \( np_1, np_2 \geq 1 \) there exists \( C_{13} \) such that for all \( n \geq C_{13} \), we obtain

\[
\frac{C_{14}(q\delta\sqrt{\delta} + \bar{q}(1-\delta)\sqrt{1-\delta})}{n\sqrt{n} q\bar{q}\sqrt{\delta(1-\delta)}} \leq \|L(S) - G\|_\infty \leq \frac{C_{15}(q\delta\sqrt{\delta} + \bar{q}(1-\delta)\sqrt{1-\delta})}{n\sqrt{n} q\bar{q}\sqrt{\delta(1-\delta)}}.
\]

The condition for \( n \) to be larger than some absolute constant (which can be estimated with the help of Lemma 5.2) is needed for lower bounds only. The upper bounds in (12) and (13) hold for all \( n \geq 1 \). As expected, the benefits of expansion in the exponent are expressed through additional factor \( 1/\sqrt{n} \). In one-dimensional case, the local estimate for \( G \) is \( Cn^{-1} \), see [19], Theorem 2. The additional multiplier \( 1/\sqrt{nq\bar{q}} \) appears because we investigate two-dimensional vectors. When \( q \) and \( \bar{q} \) are slowly vanishing, the explicit form of the rvs allows to estimate asymptotic constant.

**Proposition 4.1** Let \( \delta \) be a constant, \( \max(q, \bar{q}) = o(1) \), and \( \min(nq, n\bar{q}) \to \infty \), as \( n \to \infty \). Then,

\[
\lim_{n \to \infty} \frac{nq\bar{q}\sqrt{\delta(1-\delta)}}{(q\delta + \bar{q}(1-\delta))} \|L(S) - \text{Pois}(\lambda)\|_\infty \\
\leq \frac{1}{e} \left(1 + \sqrt{\frac{\pi}{2}}\right) \left\{ \frac{1}{\sqrt{6}} \left(1 + \sqrt{\frac{\pi}{4}}\right) + \frac{1}{8} \left(1 + \sqrt{\frac{\pi}{2}}\right) \right\} \\
= 0.871 \ldots .
\]

Proposition 4.1 serves as an indicator that constants in the above theorems should not be very large.

## 5 Auxiliary Results

We begin from relating Fourier transforms to local and \( \ell_2 \) norms.

**Lemma 5.1** Let \( M \) be (signed) measure concentrated on \( \mathbb{Z}^k \). Then,

\[
\|M\|_\infty \leq \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |\hat{M}(t)| dt_1 \cdots dt_k,
\]

\[
\|M\|_2 = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |\hat{M}(t)|^2 dt_1 \cdots dt_k.
\]

\( \hat{\cdot} \) Springer
Proof. The first inequality follows directly from the inversion formula. The second is multidimensional Parseval’s identity. However, in order to keep the paper self-contained, we give an outline of the proof. First, we introduce measure \( M^-(m) = M(-m) \). One the one hand, convolution of both measures at point zero is equal to

\[
M * M^-(0) = \sum_{m \in \mathbb{Z}^k} M^2(m).
\]

On the other hand, by inversion formula

\[
M * M^-(0) = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \hat{M}(t) \hat{M}(-t) dt_1 \cdots dt_k.
\]

For the lower bounds, an appropriate inversion formula is needed. First, we introduce additional notation. Let

\[
a := (a_1, a_2, \ldots, a_k), \quad \beta := (\beta_1, \beta_2, \ldots, \beta_k), \quad t_\beta := \left( \frac{t_1}{\beta_1}, \frac{t_2}{\beta_2}, \ldots, \frac{t_k}{\beta_k} \right).
\]

Next, we define some weight functions. The lemma below holds true if

\[
\psi_j(t_j) = e^{-t_j^2/2} \quad \text{or} \quad \psi_j(t_j) = t_j e^{-t_j^2/2}.
\]

**Lemma 5.2** Let \( M \) be a finite measure concentrated on \( \mathbb{Z}^k \). Then, for any \( a \in \mathbb{R}^k \) and \( \beta_j \geq 1, (j = 1, 2, \ldots, k) \), the following inequalities hold:

\[
\|M\|_{\infty} \geq (4\sqrt{2\pi})^{-k} \prod_{j=1}^{k} \beta_j^{-1} |V(a, \beta)|, \quad \|M\| \geq (\sqrt{2\pi})^{-k} |V(a, \beta)|,
\]

\[
\|M\|_{\alpha} \geq (\sqrt{2\pi})^{-k} \beta^{-k(\alpha-1)/\alpha} \prod_{j=1}^{k} \beta_j^{-\alpha} |V(a, \beta)|, \quad (\alpha > 1).
\]

Here,

\[
V(a, \beta) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{m=1}^{k} \psi_m(t_m) e^{-i(t_\beta, a)} \hat{M}(t_\beta) dt_1 dt_2 \cdots dt_k.
\]

Proof. We adopt the proof of Lemma 10.1 from [7]. Observe that

\[
e^{-i(t_\beta, a)} \hat{M}(t_\beta) = \sum_{m \in \mathbb{Z}^k} e^{i(t_\beta, m-a)} M(m).
\]
By interchanging the order of integration and summation, we obtain

\[
V(\mathbf{a}, \beta) = \sum_{\mathbf{m} \in \mathbb{Z}^k} \prod_{j=1}^k \left( \int_{-\infty}^{\infty} \psi_j(t_j) e^{i t_j (m_j - a_j)/\beta_j} \, dt_j \right) M(\mathbf{m})
\]

\[
= (\sqrt{2\pi})^k \sum_{\mathbf{m} \in \mathbb{Z}^k} \prod_{j=1}^k \tilde{\psi}_j \left( \frac{m_j - a_j}{\beta_j} \right) M(\mathbf{m}).
\]

Here, depending on the choice of \( \psi_j(t_j) \),

\[
\tilde{\psi}_j(y) = e^{-y^2/2} \quad \text{or} \quad \tilde{\psi}_j(y) = iye^{-y^2/2}.
\]

For the norm \( \ell_\alpha, \alpha > 1 \), we apply Hölder’s inequality

\[
|V(\mathbf{a}, \beta)| \leq (\sqrt{2\pi})^k \|M\|_\alpha \left( \sum_{\mathbf{m} \in \mathbb{Z}^k} \prod_{j=1}^k \tilde{\psi}_j^{\alpha/(\alpha-1)} \left( \frac{m_j - a_j}{\beta_j} \right) \right)^{(\alpha-1)/\alpha}
\]

\[
= (\sqrt{2\pi})^k \|M\|_\alpha \left( \sum_{m_j \in \mathbb{Z}} \tilde{\psi}_j \left( \frac{m_j - a_j}{\beta_j} \right) \right)^{(\alpha-1)/\alpha}.
\]

Let \( q := \alpha/(\alpha-1) \), \( y_j := (m_j - a)/\beta_j \). Then, since \( q > 1 \),

\[
\sum_{j \in \mathbb{Z}} e^{-qy_j^2/2} \leq 1 + \beta_j \frac{\sqrt{2\pi}}{\sqrt{q}} \leq 1 + \beta_j \sqrt{2\pi}
\]

and

\[
\sum_{j \in \mathbb{Z}} |y_j|^q e^{-qy_j^2/2} \leq \frac{1}{2} \sum_{j \in \mathbb{Z}} \left( 1 + y_j^2 q \right) e^{-qy_j^2/2} \leq \frac{1}{2} \left( 1 + \beta_j \sqrt{2\pi} + \max_x x^q e^{-q x^2/2} \sum_{j \in \mathbb{Z}} e^{-q(2\pi x^2/2) y_j^2/2} \right)
\]

\[
\leq \frac{1}{2} \left( 1 + \beta_j \sqrt{2\pi} + 1 + \beta_j \sqrt{2\pi e/(e-2)} \right) < 5 \beta_j.
\]

Similarly, for the local norm,

\[
|V(\mathbf{a}, \beta)| \leq (\sqrt{2\pi})^k \|M\|_\infty \sum_{\mathbf{m} \in \mathbb{Z}^k} \prod_{j=1}^k \tilde{\psi}_j \left( \frac{m_j - a_j}{\beta_j} \right)
\]

\[
= (\sqrt{2\pi})^k \|M\|_\infty \prod_{j=1}^k \left( \sum_{m_j \in \mathbb{Z}} \tilde{\psi}_j \left( \frac{m_j - a_j}{\beta_j} \right) \right).
\[ \leq (\sqrt{2\pi})^k \| M \|_\infty \prod_{j=1}^k (4\beta_j). \]

For the total variation norm,
\[ |V(a, \beta)| \leq (\sqrt{2\pi})^k \sum_{m \in \mathbb{Z}^k} \prod_{j=1}^k \left| \tilde{\psi}_j \left( \frac{m_j - a_j}{\beta_j} \right) \right| |M(m)| \leq (\sqrt{2\pi})^k \| M \|. \]

Next, we formulate two technical results.

**Lemma 5.3** Let \( a > 0, m \geq 1 \). Then
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin(t/2)|^m \exp\{-a \sin^2(t/2)\} dt \leq \sqrt{\varepsilon} \left( 1 + \sqrt{\frac{\pi}{2}} \right) \left( \frac{m+1/2}{2\varepsilon a} \right). \]
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-a \sin^2(t/2)\} dt \leq \left( 1 + \sqrt{\frac{\pi}{2}} \right) \frac{1}{\sqrt{6a}}. \]

**Lemma 5.3** is essentially Lemma 6 in [21].

**Lemma 5.4** Let \( 0 \leq p \leq 1, a^2 + b^2 \leq 1 \). Then,
\[ |(1 - p) + p(a + ib)| \leq 1 + p(1 - p)(a - 1) \leq \exp\{p(1 - p)(a - 1)\}. \]

The proof is trivial and can be found, for example, in [15].

For \( X_j = (X_{j1}, X_{j2}, \ldots, X_{jk}), 1 \leq j \leq n \), we introduce accompanying complex-valued random variables
\[ Z_j := \exp\{i(t, X_j)\} - 1, \quad (t, X_j) = t_1X_{j1} + t_2X_{j2} + \cdots + t_kX_{jk}. \]

We assume that \( \hat{E}(Z_1) = EZ_1, \hat{E}(Z_1, Z_2) = EZ_1Z_2 - EZ_1EZ_2 \) and, for \( 3 \leq m \leq n \), define
\[ \hat{E}(Z_1, Z_2, \ldots, Z_m) = EZ_1Z_2 \cdots Z_m - \sum_{j=1}^{m-1} \hat{E}(Z_1, \ldots, Z_j)EZ_{j+1} \cdots Z_m. \]

The essence of Heinrich’s method is the following characterization lemma.

**Lemma 5.5** Let \( Z_1, \ldots, Z_n \) be defined as above, and let
\[ \sqrt{\varepsilon \langle Z_j \rangle^2} \leq \frac{1}{6}, \quad j = 1, 2, \ldots, n. \]

Then,
\[ \hat{F}_n(t) = \varphi_1(t)\varphi_2(t) \cdots \varphi_n(t), \]
where $\varphi_1(t) = EZ_1$ and, for $m = 2, \ldots, n$,

$$
\varphi_m(t) = 1 + EZ_m + \sum_{j=1}^{m-1} \frac{\hat{E}(Z_j, Z_{j+1}, \ldots, Z_m)}{\varphi_j(t)\varphi_{j+1}(t) \cdots \varphi_{m-1}(t)}.
$$

Lemma 5.5 follows from more general Lemma 3.1 in [12], see also Theorem 1 in [13]. Also, the next lemma also can be found in [12].

**Lemma 5.6** Let $Z_1, Z_2, \ldots, Z_n$ be 1-dependent complex-valued random variables with $E|Z_j|^2 < \infty$, $1 \leq j \leq n$. Then, for $1 \leq m \leq n$,

$$
|\hat{E}(Z_1, Z_2, \ldots, Z_m)| \leq 2^{m-1} \prod_{j=1}^{m} (E|Z_j|^2)^{1/2}.
$$

For the sake of convenience, we collect all the facts about $Z_j$ and present it in the following lemma. Let

$$
u(t) := \sum_{r=1}^{k} p_r \sin^2(t_r/2).
$$

**Lemma 5.7** Let the assumptions in (9) hold. Then,

$$
\begin{align*}
EZ_1 &= \sum_{r=1}^{k} p_r (e^{it_r} - 1), \quad EZ_1Z_2 = \sum_{r,m=1}^{k} p_{rm} (e^{it_r} - 1)(e^{it_m} - 1), \\
E|Z_1| &= 2 \sum_{r=1}^{k} p_r |\sin(t_r/2)|, \quad E|Z_1|^2 = 4u(t), \quad |Z_j| \leq 2, \\
E|Z_1Z_2| &\leq 0.4u(t), \quad E|Z_1|E|Z_2| \leq \frac{u(t)}{36}, \\
Re EZ_1 &= -2u(t), \quad |\hat{E}(Z_j, Z_{j+1}, \ldots, Z_m)| \leq 8u(t)(4u(t))^{m-j-1}.
\end{align*}
$$

Here, $Re EZ_1$ denotes the real part of complex number $EZ_1$.

**Proof** Let $a + ib$ be a complex number. Clearly, $|a + ib|^2 = a^2 + b^2$. Therefore, $|Z_j| \leq |\cos(t, X_j) + i\sin(t, X_j)| + 1 \leq 2$. Observe that due to the first assumption in (9)

$$
E|Z_1|E|Z_2| \leq 4 \sum_{r,m=1}^{k} p_r p_m |\sin(t_r/2)||\sin(t_m/2)|
$$

$$
\leq 2 \sum_{r,m=1}^{k} (p_r^2 \sin^2(t_r/2) + p_m^2 \sin^2(t_m/2))
$$

\(\square\) Springer
\[
= 4k \sum_{j=1}^{k} p_j^2 \sin^2(t_j/2) \leq \frac{u(t)}{36}.
\]

Similarly, due to the second assumption in (9),
\[
\mathbb{E}|Z_1 Z_2| = 4 \sum_{r,m=1}^{k} p_{rm} |\sin(t_r/2)||\sin(t_m/2)|
\]
\[
\leq 2 \sum_{r,m=1}^{k} p_{rm} (\sin^2(t_r/2) + \sin^2(t_m/2))
\]
\[
= 2 \sum_{r=1}^{k} \sin^2(t_r/2) \sum_{m=1}^{k} (p_{rm} + p_{mr}) \leq 0.4 \sum_{r=1}^{k} p_r \sin^2(t_r/2). \quad (22)
\]

Finally, by Lemma 5.6
\[
|\hat{\mathbb{E}}(Z_j, Z_{j+1}, \ldots, Z_m)| \leq 2^{m-j}(\sqrt{\mathbb{E}|Z_1|^2})^{m-j+1} = 2^{m-j}(2\sqrt{u(t)})^{m-j+1}.
\]

All other relations follow directly from the definition of \( Z_j \).

**Lemma 5.8** Assume the conditions in (9) hold and \(|t_j| \leq \pi\), for \(1 \leq j \leq k\). Then,
\[
|\varphi_m(t) - 1| \leq \frac{1}{10}, \quad \frac{1}{|\varphi_m(t)|} \leq \frac{10}{9}, \quad |\varphi_m(t) - 1 - \mathbb{E}Z_m| \leq 1.93 u(t), \quad (23)
\]
for \(1 \leq m \leq n\).

**Proof** Further in the proofs, for the sake of brevity, we write \( \varphi_m \) instead of \( \varphi_m(t) \) whenever no ambiguity can arise. The first two estimates follow from the third one. Indeed, due to (9), (19) and the definition of \( u(t) \),
\[
|\varphi_m - 1| \leq \mathbb{E}|Z_m| + |\varphi_m - 1 - \mathbb{E}Z_m| \leq 4 \sum_{r=1}^{k} p_r \leq \frac{4}{144} < \frac{1}{10}.
\]
Similarly,
\[
|\varphi_m| = |1 - \varphi_m - 1| \geq 1 - |\varphi_m - 1| \geq \frac{9}{10}.
\]

The proof of the third estimate in (23) is done by mathematical induction. Observe that due to (9) and (19), condition (17) is satisfied and we can apply Lemma 5.5. Let all the estimates hold for \( j = 1, 2, \ldots, m-1, m > 4 \). (For \( m = 2, 3, 4 \), the proof is similar and shorter.) Then,
\[
|\varphi_m - 1 - \mathbb{E}Z_m| \leq \frac{10}{9} |\hat{\mathbb{E}}(Z_1, Z_2)| + \left( \frac{10}{9} \right)^2 |\hat{\mathbb{E}}(Z_1, Z_2, Z_3)|
\]
Let the conditions stated in Lemma 5.9 be satisfied. Then,

\[ |\varphi_m(t)| \leq 1 - 0.05u(t) \leq \exp\{-0.05u(t)\}, \quad |\hat{F}_n(t)| \leq \exp\{-0.05nu(t)\}, \]

for \( 1 \leq m \leq n \).

\textbf{Proof} We have

\[ |\varphi_m(t)| \leq |1 + EZ_m + \varphi_m(t) - 1 - EZ_m| \leq |1 + EZ_m| + 1.93u(t). \]

It remains to apply (16) to the first summand. The second estimate in (5.9) follows from the Heinrich’s decomposition \( \hat{F}_n(t) = \varphi_1(t) \cdots \varphi_n(t) \).

Let us now denote the remainder terms by

\[ r_k(t) := \sum_{m,j=1}^k (pm_j + p_m p_j) |\sin(t_m/2)||\sin(t_j/2)|. \]
\[ r_2(t) := \sum_{i=1}^{k} |t_i| \sum_{m,j=1}^{k} (p_{mj} + p_m p_j) |\sin(t_m/2)||\sin(t_j)|, \]
\[ r_3(t) := \sum_{l,m,j=1}^{k} (p_{lmj} + p_{lm} p_j + p_l p_m p_j) |\sin(t_l/2)||\sin(t_m/2)||\sin(t_j)|, \]
\[ r_4(t) := \sum_{i=1}^{k} |t_i| \sum_{l,m,j=1}^{k} (p_{lmj} + p_{lm} p_j + p_l p_m p_j) |t_l||t_m||t_j|. \]

Lemma 5.10 Let conditions (9) be satisfied. Then, for all \( r = 2, \ldots, n \)
\[ \varphi_r(t) = 1 + E Z_r + \theta C(k) r_1(t), \]
\[ \varphi_r(t) = 1 + E Z_r - \sum_{j,m=1}^{k} (p_{jm} - p_j p_m) t_j t_m + \theta C(k) r_2(t), \]
\[ \varphi_r(t) = 1 + E Z_r + \hat{E}(Z_r, Z_{r-1}) + \theta C(k) r_3(t), \]
and for \( r = 3, 4, \ldots, n, \)
\[ \varphi_r(t) = 1 + E Z_r + \hat{E}(Z_r, Z_{r-1}) + \theta C(k) r_4(t). \]

Proof Let us assume that \( k \geq 5 \). Applying Lemma 5.5, we obtain
\[ \varphi_r = 1 + E Z_r + \frac{\hat{E}(Z_r, Z_{r-1})}{\varphi_j \cdots \varphi_{r-1}} + \sum_{j=1}^{r-5} \frac{\hat{E}(Z_j, \ldots, Z_r)}{\varphi_j \cdots \varphi_{r-1}}. \]

Estimating the absolute value of the second expression, as in the Proof of Lemma 5.8, we get
\[ \sum_{j=1}^{r-5} \frac{|\hat{E}(Z_j, \ldots, Z_r)|}{|\varphi_j \cdots \varphi_{r-1}|} \leq \sum_{j=1}^{r-5} \left( \frac{10}{9} \right)^{r-j} (4u(t))^3 \left( \frac{1}{6} \right)^{r-j-5} \leq Cu^3(t). \]

An application of Lemma 5.7 yields
\[ \frac{1}{\varphi_r} = \frac{1}{1 - (1 - \varphi_r)} = 1 + (1 - \varphi_r) + \theta C |1 - \varphi_r|^2 = -E Z_r + \theta C(k) u(t). \]

The first and the third part of the lemma follows by routinely applying Lemma 5.7, and therefore, we omit the detailed proof. For example,
\[ E|Z_r Z_{r-1} Z_{r-2} Z_{r-3}| \leq 2E|Z_1 Z_2 Z_3| \]

\( \square \) Springer
\[
= 16 \sum_{l,j,m=1}^{k} p_{ljm} |\sin(t_l/2)\sin(t_j/2)\sin(t_m/2)| \leq 16r_3(t),
\]
e etc. For the proof of second and fourth expansions, observe that since all coordinates of \(X_j\) are bounded by unity, we have

\[
|Z_j| \leq |(t, X_j)| \leq \sum_{j=1}^{k} |t_j|.
\]

Hence,

\[
E|Z_r Z_{r-1} Z_{r-2} Z_{r-3}| \leq \sum_{j=1}^{k} |t_j| E|Z_1 Z_2 Z_3| \leq \frac{1}{8} r_4(t).
\]

Here, we have used also inequality \(|\sin(t_j/2)| \leq |t_j|/2\). We also apply the trivial expansion

\[
(e^{it_j} - 1)(e^{it_m} - 1)(e^{it_l} - 1) = i^3 t_j t_m t_l + 2\theta(|t_l| + |t_j| + |t_m|)
\]

so that

\[
\hat{E}(Z_r, Z_{r-1}, Z_{r-2}) = i^3 \sum_{l,j,m=1}^{k} (p_{ljm} - 2 p_{l} p_{j} p_{m}) + \theta C(k)r_4(t).
\]

For \(k = 2, 3, 4\), the proof is similar, the only difference being finite number of estimated summands in Heinrich’s expansion.

Lemma 5.11 Assume the conditions in (9) hold. Then, for \(1 \leq r \leq n\),

\[
g_r(t) = 1 + EZ_r + \hat{E}(Z_r, Z_{r-1}) + \theta C(k) \sum_{l,j,m=1}^{k} (p_{l} p_{j} p_{m}) |\sin(t_l/2)\sin(t_j/2)\sin(t_m/2)|,
\]

\(\square\) Springer
\[ g_r(t) = 1 + EZ_r + \hat{E}(Z_r, Z_{r-1}) + \left( \frac{i^3}{3} \sum_{l,j,m=1}^{k} (3p_{lj}p_m - 4p_{lj}p_m)p_{ljm} \right) t_l t_j t_m + \theta C(k) \sum_{i=1}^{k} \sum_{l,j,m=1}^{k} (p_{lj}p_m + p_{lj}p_m)|t_l t_j t_m|, \]

\[ |g_r(t)| \leq \exp(-u(t)). \]

**Proof** Observe that by Lemma 5.7 (see also the Proof of Lemma 5.8):

\[ |g_r(t)| \leq \exp(-2u(t) + |\hat{E}(Z_r, Z_{r-1})| + 0.5|EZ_r|^2) \leq \exp(-2u(t) + 0.428u(t) + u(t)/72) \leq \exp(-u(t)). \]

Note also that the same estimate holds for \( r = 1 \). The Taylor series expansion gives us

\[ g_r(t) = 1 + EZ_r + \hat{E}(Z_r, Z_{r-1}) (1 + EZ) - \frac{1}{3} E^3 Z_r + \theta C(|\hat{E}(Z_r, Z_{r-1})||EZ_r| + |\hat{E}(Z_r, Z_{r-1})|^2 + |EZ_r|^4). \]

The rest of the arguments is similar to the proof of the previous lemma and, therefore, omitted. \( \square \)

6 Proofs

Proofs of Theorems 3.1 and 3.3 Let

\[ \pi_j(t) := \exp(EZ_j) = 1 + EZ_j + \theta C(k)|EZ_j|^2. \]

For simplicity, write \( \pi_j = \pi_j(t) \). Using Lemmas 5.7, 5.9 and 5.10, we have

\[ |\hat{F}_n(t) - \text{Pois}(\lambda)| = \left| \prod_{j=1}^{n} \varphi_j - \prod_{j=1}^{n} \pi_j \right| \leq \sum_{j=1}^{n} |\varphi_j - \pi_j| \sum_{m=1}^{j} |\varphi_j| \sum_{m=j+1}^{n} |\pi_j| \leq C(k)(r_1(t) + |EZ_j|^2)e^{-n0.05u(t)} \leq C(k)r_1(t)e^{-0.05nu(t)}. \]

Here, we have used also the trivial estimate \( \exp(0.04u(t)) \leq C(k) \). Consequently, applying Lemma 5.1 and a lemma of [25], we complete the proof for local and \( \ell_2 \) norms. Next, observe that for \( \alpha \geq 2 \), the following trivial estimate holds:

\[ \|M\|_\alpha \leq \left( \|M\|_\infty \right)^{(\alpha-2)/\alpha} \left( \|M\|_2 \right)^{2/\alpha}. \quad (25) \]

The proof of Theorem 3.3 is very similar. One needs to check that from Lemmas 5.10 and 5.11, it follows that \( |\varphi_m(t) - g_m(t)| \leq r_3(t) \), for all \( m = 2, \ldots, n \). It can
be directly verified that the same estimate holds also for $m = 1$. For the estimate of $|g_{j+1}(t) \cdots g_n(t)|$, one should apply the last estimate of Lemma 5.11.

**Proof of Theorem 3.2** It suffices to estimate the difference of corresponding Fourier transforms. We again apply Lemmas 5.7 and 5.10. By Bergström identity (see [7], p. 17),

$$\left| \prod_{m=1}^{n} \varphi_m - \prod_{m=1}^{n} \pi_m - \sum_{m=1}^{n} (\varphi_m - \pi_m) \prod_{j \neq m}^{n} \pi_j \right| \leq \sum_{r=1}^{n} |\varphi_r - \pi_r| \sum_{m=1}^{n} |\varphi_m - \pi_m| \prod_{j=r+1}^{n} |\varphi_j| \prod_{j \neq m}^{n} |\pi_j| \leq C(k) \exp\{-0.05nu(t)\}(nr(t))^2.$$

Next, observe that $|\pi_j - 1| \leq C(k)E|Z_m| \leq C(k) \sum_{r=1}^{n} p_r |\sin(t_r/2)|$. Therefore,

$$\left| \sum_{m=1}^{n} (\varphi_m - \pi_m) \prod_{j \neq m}^{n} \pi_j (1 - \pi_m) \right| \leq C(k)e^{-0.05nu(t)}r_1(t)n \sum_{r=1}^{n} p_r |\sin(t_r/2)|.$$

It is easy to check that $\pi_m = 1 + EZ_m + 0.5(EZ_k)^2 + \theta C(k)E|Z_k|^3$.

Applying Lemma 5.10, we get

$$\prod_{j=1}^{n} |\pi_j| \sum_{m=1}^{n} |\varphi_m - \pi_m + 0.5(EZ_m)^2 - \widehat{E}(Z_m, Z_{m-1})| \leq C(k)e^{-0.05nu(t)}r_2(t).$$

Collecting all the estimates given above and applying Lemmas 5.1, a lemma of [25] and (25), we complete the proof of theorem.

**Proof.** First, we deal with approximation by signed compound Poisson measure $G$. We apply Lemma 5.2 with $\mathbf{a} = (n - 1)\mathbf{t}, \mathbf{p} := (n - 1)(p_1t_1 + p_2t_2 + \cdots + p_kt_k)$ and $\beta_j = b\sqrt{np_j}$, $j = 1, 2, \ldots, n$. From Lemma 5.7, it follows that

$$|\varphi_m(t)e^{-i(t,p)} - 1| \leq |\varphi_m(t) - \pi_m(t)||e^{-i(t,p)}| + |\exp\left\{\sum_{r=1}^{k} p_r(e^{it_r} - 1 - it_r)\right\} - 1| \leq C(k)r_1(t) + C(k) \sum_{r=1}^{k} p_r t_r^2 \leq C(k) \sum_{r=1}^{k} p_r t_r^2.$$
Here, for the last step we argued as in (22). Similarly, we establish that the same estimate holds for $|g_m(t)e^{-i(t,p)} - 1|$. Therefore, for any $m = 1, 2, \ldots, n,$

$$\left| \prod_{j=1}^{m-1} \varphi_j(t_\beta) \prod_{j=m+1}^{n} g_j(t_\beta)e^{-i(a,t_\beta)} - 1 \right| \leq C(k)n \sum_{r=1}^{k} p_r \frac{t_r^2}{b^2 n p_r} = \frac{C(k)}{b^2} \sum_{r=1}^{k} t_r^2.$$  

Consequently, applying Lemmas 5.10 and 5.11 and using the above estimate, we get

$$\left| e^{-i(t_\beta,p)} \left( \prod_{j=1}^{n} \varphi_j(t_\beta) - \prod_{j=1}^{n} g_j(t_\beta) \right) - \sum_{j=1}^{n} (\varphi_j(t_\beta) - g_j(t_\beta)) \right| \leq \sum_{m=1}^{n} \sum_{r=1}^{k} |\varphi_j(t_\beta) - g_j(t_\beta)| C(k)b^2 \sum_{r=1}^{k} t_r^2 \leq \frac{C(k)}{b^2} \sum_{r=1}^{k} |t_r| r_4(t_\beta). \quad (26)$$

Expanding $g_1(t)$ in Taylor series and noting that the remainder term is smaller than $r_4(t)$, we obtain

$$\varphi_1(t) - g_1(t) = i^3 \sum_{r,j,m=1}^{k} p_r p_j p_m t_r t_j t_m + \theta C(k) r_4(t).$$

Similarly,

$$\varphi_2(t) - g_2(t) = i^3 \sum_{r,j,m=1}^{k} (6p_r p_j p_m - 5p_r p_j p_m) t_r t_j t_m + \theta C(k) r_4(t)$$

and from Lemmas 5.10 and 5.11, for $i = 3, 4, \ldots, n,$

$$\varphi_i(t) - g_i(t) = \frac{i^3}{3} \sum_{r,j,m=1}^{k} (3p_r p_m - 12p_r p_j p_m) t_r t_j t_m + \theta C(k) r_4(t).$$

Therefore,

$$\sum_{i=1}^{n} (\varphi_i(t_\beta) - g_i(t_\beta)) = \frac{i^3}{3 b^3 \sqrt{n}} \sum_{r,j,m=1}^{k} \frac{3p_r p_m - 12p_r p_j p_m}{\sqrt{p_r p_j p_m}} t_r t_j t_m$$

$$+ \theta \frac{1}{3 b^3 n \sqrt{n}} \sum_{r,j,m=1}^{k} \frac{-6p_r p_m + 30p_r p_j p_m - 22p_r p_j p_m}{\sqrt{p_r p_j p_m}} t_r t_j t_m \bigg| \bigg\rvert + \theta C(k)n r_4(t_\beta). \quad (27)$$

Combining the estimates in (26) and (27) and observing that integral of $|t_i|^3 \psi_1(t_i)$ is bounded by absolute constant, we can write
\[ |V(\mathbf{a}, \mathbf{b})| \geq \frac{1}{3b^3 \sqrt{n}} \left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{l=1}^{k} \psi_l(t_l) \sum_{r,j,m=1}^{k} v_{rjm} t_r t_j t_m \right| - \frac{C(k) \varepsilon_6}{b^3 \sqrt{n}}. \tag{28} \]

Here,
\[ v_{rjm} := \frac{3p_{rjm} - 12p_{rj} p_m + 10p_r p_j p_m}{\sqrt{p_r p_j p_m}}. \]

Now comes the tricky part, since all integrals with odd powers of \( t_l \) are equal zero. We can write
\[
\sum_{r,j,m=1}^{k} v_{rjm} = \sum_{r=1}^{k} t_r^3 v_{rrr} + \sum_{r \neq m}^{k} t_r^2 t_m (v_{rrm} + b_{rmr} + v_{mrr})
\]
\[ + \sum_{r \neq m \neq j}^{k} t_r t_j t_m (v_{rmj} + b_{rjm} + v_{mrj} + v_{mrl} + v_{jmr} + v_{jrm}). \]

Let us choose \( \psi_m(t_m) = t_m e^{-t_m^2/2} \) and \( \psi_r(t_r) = e^{-t_r^2/2}, \) for all \( r \neq m. \) Then, after integration, absolute value in (28) is equal to
\[
(\sqrt{2\pi})^k \left| 3v_{mmm} + \sum_{r=1, r \neq m}^{k} (v_{rrm} + v_{mrm} + v_{mmr}) \right|.
\]

By taking different \( m, \) we obtain \( k \) such integrals. Now, let us assume that \( \psi_m = t_m e^{-t_m^2/2}, \psi_r = t_r e^{-t_r^2/2}, \psi_j = t_j e^{-t_j^2/2} \) and all other \( \psi_l(t_l) = e^{-t_l^2/2}. \) Then, after integration, absolute value in (28) is equal to
\[
(\sqrt{2\pi})^k |v_{rmj} + v_{rjm} + v_{mrj} + v_{mrl} + v_{jmr} + v_{jrm}|.
\]

After taking all possible different combinations \( r, m, j, \) we arrive at the fact that absolute value in (28) can be taken equal to maximum of all these \( N = k + k(k - 1)(k - 2)/6 \) estimates. Next, observe that for any numbers \( B_1, \ldots, B_N, \) we have
\[
\max_{1 \leq j \leq N} |B_j| \geq \frac{1}{N} \sum_{j=1}^{N} |B_j| \geq \frac{1}{N} \left| \sum_{j=1}^{N} B_j \right|.
\]

Therefore,
\[
\left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{m=1}^{k} \psi_m(t_m) \sum_{r,j,m=1}^{k} v_{rjm} t_r t_j t_m \right|
\]
\[ \geq \frac{(\sqrt{2\pi})^k}{N} \left| \sum_{r,j,m=1}^{k} v_{rjm} + 2 \sum_{m=1}^{k} v_{mmm} \right|. \]
Collecting all the relevant estimates, we complete the proof for approximation \( G \).
The estimates for Poisson approximation are obtained by the similar arguments. Note
that as we need to integrate sums of the form
\[
\sum_{r,m=1}^k w_{rm} I_r I_m, \quad \text{the choice of } \psi_r(I_r), \psi_m(I_m)
\]
allows to estimate corresponding integral by
\[
(\sqrt{2\pi})^k \max \left\{ \left| \sum_{m=1}^k w_{mm} \right|, |w_{12}|, |w_{13}|, \ldots, |w_{k-1,k}| \right\}
\geq (\sqrt{2\pi})^k \frac{2}{k(k+1)} \left| \sum_{r,m=1}^k w_{rm} \right|. \quad \square
\]

**Proof.** By triangle inequality,
\[
\|F_n - \text{Pois}(\lambda)\|_\infty \leq \|F_n - \text{Pois}(\lambda) - A_1\|_\infty + \|A_1\|_\infty.
\]
Observe that
\[
-n^2 \left( \sum_{j=1}^2 p_j (e^{it_j} - 1) \right)^2 + (n-1) \sum_{j,m=1}^2 (p_{jm} - p_j p_m) (e^{it_j} - 1)(e^{it_m} - 1)
= n\delta^2 q^3 (e^{it_1} - 1)^2 (1 + o(1)) + n(1 - \delta)^2 \bar{q}^3 (e^{it_2} - 1)(1 + o(1))
- n\delta(1 - \delta)\bar{q}^2 q^2 (e^{it_1} - 1)(e^{it_2} - 1).}
\]

It remains to apply Lemmas 5.1, 5.3 and the obvious inequality
\[
\sqrt{\delta(1 - \delta)q\bar{q}} \leq \frac{\delta q^2 + (1 - \delta)\bar{q}^2}{2} \leq \frac{\delta q + (1 - \delta)\bar{q}}{2}. \quad \square
\]

**Acknowledgements** This paper was completed during the first author’s stay (January–February, 2020) at the Department of Mathematics, Indian Institute of Technology Bombay. The first author would like to thank the Department and the Institute for the invitation and the hospitality. The authors are also grateful to the referee for many useful remarks, which led to improvements of the paper.

**References**
1. Arenbaev, N.K.: Asymptotic behavior of the multinomial distribution. Theory Probab. Appl. 21(4), 805–810 (1977)
2. Barbour, A.D.: Multivariate Poisson-binomial approximation using Stein’s method. In: Barbour, A.D., Chen, L.H.Y. (eds.) Stein’s Method and Applications. IMS Lecture Note Series, vol. 5. pp. 131–142 (2005)
3. Barbour, A.D., Jensen, J.L.: Local and tail approximations near the Poisson limit. Scand J. Statist. 16, 75–87 (1989)
4. Barbour, A.D., Gan, H.L., Xia, A.: Stein factors for negative binomial approximation in Wasserstein distance. Bernoulli 21(2), 1002–1013 (2015)
5. Barbour, A.D., Röllin, A., Ross, N.: Error bounds in local limit theorems using Stein’s method. Bernoulli 25(2), 1076–1104 (2019)
6. Borovkov, K. A.: On the problem of improving Poisson approximation. Theory Probab. Appl. 33(2), 343–347 (1989). Translated from Teor. Veroyatnost. Primen. 33(2), 364–368 (1988)
7. Čekanavičius, V.: Approximation Methods in Probability Theory. Universitext, Springer Verlag, Berlin (2016)
8. Čekanavičius, V., Vellaisamy, P.: Discrete approximations for sums of m-dependent random variables. ALEA, Lat. Am. J. Probab. Math. Stat. 12, 765–792 (2015)
9. Chen, L.H.Y., Röllin, A.: Approximating dependent rare events. Bernoulli 19(4), 1243–1267 (2013)
10. Franken, P.: Approximation der Verteilungen von Summen unabhängiger nichtnegativer ganzzahliger Zufallsgrössen durch Poissonsche Verteilungen. Math. Nachr. 27, 303–340 (1964)
11. Hwang, H.-K., Zacharovas, V.: Uniform asymptotics of Poisson approximation to the Poisson-binomial distribution. Teor. Veroyatnost. Primenen. 55(2) 305–334, (2010). Reprinted in Theory Probab. Appl. 55(2), 198–224 (2011)
12. Heinrich, L.: A method for the derivation of limit theorems for sums of m-dependent random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete 60, 501–515 (1982)
13. Heinrich, L.: Some estimates of the cumulant-generating function of a sum of m-dependent random vectors and their application to large deviations. Math. Nachr. 120, 91–101 (1985)
14. Herrmann, H.: Variationsabstand zwischen der Verteilung einer Summe unabhängiger nichtnegativer ganzzahliger Zufallsgrössen und Poissonschen Verteilungen. Math. Nachr. 29(5), 265–289 (1965)
15. Kruopis, J., Čekanavičius, V.: Compound Poisson approximations for symmetric vectors. J. Multivariate Anal. 123, 30–42 (2014)
16. Novak, S.Y.: Extreme Value Methods with Applications to Finance. Chapman & Hall/CRC Press, London (2011)
17. Novak, S.Y.: Poisson approximation. Probab. Surveys 16, 228–276 (2019)
18. Novak, S.Y.: On the accuracy of poisson approximation. Extremes 22, 729–748 (2019)
19. Petrauskiené, J., Čekanavičius, V.: Compound poisson approximations for sums of one-dependent random variables I. Lith. Math. J. 50(3), 323–336 (2010)
20. Roos, B.: Metric multivariate Poisson approximation of the generalized multinomial distribution. Teor. Veroyatnost. Primen. 43(2), 404–413 (1998). Reprinted in Theory Probab. Appl. 43(2), 306–316 (1999)
21. Roos, B.: Multinomial and Krawtchouk approximations to the generalized multinomial distribution. Teor. Veroyatnost. Primen. 46(1), 117–133 (2001). Reprinted in Theory Probab. Appl. 46(1), 103–117 (2002)
22. Roos, B.: Kerstan’s method in the multivariate Poisson approximation: an expansion in the exponent. Teor. Veroyatnost. Primen. 47(2), 397–402 (2002). Reprinted in Theory Probab. Appl. 47(2), 358–363 (2003)
23. Roos, B.: Poisson approximation via the convolution with Kornya–Presman signed measures. Teor. Veroyatnost. Primen. 48(3), 628–632, (2003). Reprinted in Theor. Probab. Appl. 48(3), 555–560 (2004)
24. Roos, B.: Refined total variation bounds in the multivariate compound poisson approximation. ALEA, Lat. Am. J. Probab. Math. Stat. 14, 337–360 (2017)
25. Shorgin, S.Ya.: Approximation of a generalized binomial distribution. Theory Probab. Appl. 22(4), 846–850 (1978). Translated from Teor. Veroyatnost. Primen. 22(4), 867–871 (1977)
26. Wang, X., Xia, A.: On negative binomial approximation to k-runs. J. Appl. Probab. 45, 456–471 (2008)
27. Zacharovas, V., Hwang, H.-K.: A Charlier–Parseval approach to poisson approximations and its applications. Lith. Math. J. 50(1), 88–119 (2010)
28. Zaitsev, A. Yu.: Multidimensional version of the second uniform limit theorem of Kolmogorov. Theory Probab. Appl. 34(1), 108–128 (1989). Translated from Teor. Veroyatnost. Primen. 34(1), 128–151 (1989)
29. Zaitsev, A.Yu.: Approximation of convolutions of multi-dimensional symmetric distributions by accompanying laws. J. Soviet Mathematics 61(1), 1859–1872, (1992). Translated from Zapiski Nauchnykh Seminarov LOMI V. A. Steklova AN SSSR 177, 55–72 (1989)
30. Zaitsev, A. Yu.: On approximation of convolutions by accompanying laws in the scheme of series. J. Math. Sci. 199 162–167 (2014). Translated from Zapiski Nauchnykh Seminarov POMI 408, 175–186 (2012)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.