HEAT CONTENT ASYMPTOTICS FOR OPERATORS OF
LAPLACE TYPE WITH SPECTRAL BOUNDARY CONDITIONS

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Abstract. Let $P$ be an operator of Dirac type and let $D = P^2$ be the asso-
ciated operator of Laplace type. We impose spectral boundary conditions and
study the leading heat content coefficients for $D$.

1. introduction

Let $P$ be an operator of Dirac type on a vector bundle $V$ over a compact Rie-
mannian manifold $M$ of dimension $m$ with smooth boundary $\partial M$. Let $D := P^2$
be the associated operator of Laplace type. The leading symbol $\gamma$ of $P$ defines a
Clifford module structure on $V$. Choose an auxiliary connection $\nabla$ on $V$ so that
$\nabla \gamma = 0$. Adopt the Einstein convention and sum over repeated indices; indices $i,j$
will range from 1 to $m$ and index a local orthonormal frame $\{e_i\}$ for $TM$. Expand

$P = \gamma_i \nabla e_i + \psi_P$.

We must impose suitable boundary conditions. As $P$ need not admit local bound-
ary conditions, we shall consider spectral boundary conditions; these were first in-
troduced by Atiyah et. al. [1] to study the index theorem for manifolds with
boundary. Near the boundary, normalize the local frame so that $e_m$ is the inward
unit geodesic vector field. Let indices $a,b$ range from 1 to $m-1$ and index the
induced orthonormal frame for $T\partial M$. Let

$A = -\gamma_m \gamma_a \nabla e_a + \psi_A$

for some endomorphism $\psi_A$ of $V|_{\partial M}$; $A$ is of Dirac type on $V|_{\partial M}$ with respect to
the induced tangential Clifford module structure $\gamma^T_a := -\gamma_m \gamma_a$.

For the sake of simplicity, we shall assume $A$ has no purely imaginary eigenval-
ues. Let $\Pi^+_A$ be spectral projection on the span of the generalized eigenspaces of
$A$ corresponding to eigenvalues with positive real part. This spectral projection
defines a boundary condition for $P$; the associated boundary operator for $D$ is

$\mathcal{B} := \Pi^+_A \oplus \Pi^+_A P$.

Let $D_{\mathcal{B}}$ be the associated realization. The fundamental solution $e^{-tD_{\mathcal{B}}}$ of the heat
equation is well defined; $u = e^{-tD_{\mathcal{B}}} \phi$ is characterized by the relations

$(\partial_t + D)u = 0, \quad Bu = 0, \quad \text{and} \quad u|_{t=0} = \phi$.

We refer to Grubb [10, 11] and Grubb and Seeley [12, 13, 14] for further details.

Let $\langle \cdot, \cdot \rangle$ be the natural pairing between $V$ and the dual bundle $V^*$, let $dx$ be the
Riemannian measure on $M$, and let $\rho \in C^\infty(V^*)$ be the specific heat. The total
heat energy content of the manifold with initial temperature $\phi$ is given by

$\beta(\phi,\rho,D,\mathcal{B})(t) := \int_M \langle u(x; t), \rho(x) \rangle dx$.

If we had imposed local boundary conditions such as Dirichlet or Robin, then it is
well known that there is a complete asymptotic series for $\beta$ with locally computable

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coefficients $\beta_n$. We refer to \cite{2, 3, 4, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18} for a discussion of this case. Thus we assume there exists a complete asymptotic series as $t \downarrow 0$ of the form

$$
\beta(\phi, \rho, D, B)(t) \sim \sum_{n=0}^{\infty} \beta_n(\phi, \rho, D, B)t^{n/2}.
$$

If $\Psi$ is an operator on $C^\infty(V)$ or on $C^\infty(V^*)$, let $\tilde{\Psi}$ be the formal adjoint on $C^\infty(V^*)$ or on $C^\infty(V)$, respectively. Similarly, if $\Psi$ is an operator on $C^\infty(V|_{BM})$ or on $C^\infty(V^*|_{BM})$, let $\tilde{\Psi}$ be the adjoint on $C^\infty(V^*|_{BM})$ or on $C^\infty(V|_{BM})$, respectively. Let $\bar{\nabla}$ be the dual connection on $V^*$, let $L_{ab}$ be the second fundamental form of the boundary, and let $dy$ be the Riemannian measure on the boundary. We omit the proof of the following Lemma in the interests of brevity as it is straightforward.

**Lemma 1.1.**

1. $\bar{P} = -\tilde{\gamma}_1^i\nabla e_a + \tilde{\psi}_p$ and $\bar{A} = -\tilde{\gamma}_1^i\nabla e_a + \tilde{\psi}_A$.
2. Let $A^# := \tilde{\gamma}_m^i\tilde{\nabla}_{e_m}$ on $C^\infty(V^*|_{BM})$. Then:
   a. $\tilde{\beta}^+$ defines the adjoint boundary condition for $\bar{P}$.
   b. $\tilde{B} := \tilde{\beta}^+\otimes \tilde{\beta}^+$ defines the adjoint boundary condition for $\bar{B} := \tilde{P}^2$.
   c. Let $\psi_{a#} := \tilde{\gamma}_m\tilde{\gamma}_a^i + L_{aa}\text{Id}$. Then $A^# = -\tilde{\gamma}_1^i\tilde{\nabla}_{e_m} + \psi_{a#}$.
3. $\int_M \{\langle D\phi, \rho \rangle - \langle \phi, D\rho \rangle \} dx = -\int_{BM} \{\langle \tilde{\gamma}_m^i\Pi^\perp_A P\phi, \rho \rangle + \langle P\phi, \tilde{\gamma}_m^i\Pi^\perp_A P \rangle + \langle \phi, \tilde{\gamma}_m^i\Pi^\perp_A \tilde{P} \rangle + \langle \tilde{\gamma}_m^i\Pi^\perp_A \tilde{P} \rangle \} dy$.
4. We have $(\text{Id} - \tilde{\beta}^+\otimes \tilde{\beta}^+)\tilde{\gamma}_m = \tilde{\gamma}_m\Pi^\perp_A$.
5. If $B\phi = 0$ and if $\tilde{B}\rho = 0$, then $\int_M \{\langle D\phi, \rho \rangle - \langle \phi, D\rho \rangle \} dx = 0$.

We now state the main result of this paper:

**Theorem 1.2.** Adopt the notation established above.

1. $\beta_0(\phi, \rho, D, B) = \int_M \langle \phi, \rho \rangle dx$.
2. $\beta_1(\phi, \rho, D, B) = -\frac{1}{\sqrt{2}} \int_{BM} \langle \Pi^\perp_A \phi, \Pi^\perp_A \rho \rangle dy$.
3. $\beta_2(\phi, \rho, D, B) = -\int_M \langle D\phi, \rho \rangle dx + \int_{BM} \{ -\langle \tilde{\gamma}_m^i\Pi^\perp_A P\phi, \rho \rangle - \langle \tilde{\gamma}_m^i\Pi^\perp_A P \rangle \} dy$.
   \[ + \frac{1}{2} \{ \langle L_{aa} + A + A^# - \tilde{\gamma}_m\tilde{\gamma}_a^i + \psi_p + \psi_p\tilde{\gamma}_m - \psi_A - \psi_{a#} \Pi^\perp_A \phi, \Pi^\perp_A \rho \rangle \} dy. \]

A variant of this result was established in \cite{9} using a special case calculation. In this paper, we extend this result by using functorial methods. In Section 2, we derive various naturality properties of these invariants. These properties are then used in Section 2 to complete the proof.

2. Properties of the heat content invariants

We begin with some general observations:

**Lemma 2.1.**

1. $\beta_0(\phi, \rho, D, B) = \int_M \langle \phi, \rho \rangle dx$.
2. $\beta_n(\phi, \rho, D, B) = \beta_n(\rho, \phi, \bar{D}, \bar{B})$.
3. If $B\phi = 0$, then $\tilde{\beta}_n(\phi, \rho, D, B) = -\beta_{n-2}(D\phi, \rho, D, B)$.
4. If $M$ is closed, then $\beta_n(\phi, \rho, D, B) = 0$ if $n$ is odd while if $n = 2k$ is even, then $\tilde{\beta}_{2k}(\phi, \rho, D, B) = (-1)^k \frac{1}{k!} \int_M \langle D^k \phi, \rho \rangle dx$.
5. $\beta_n(\phi, \rho, (-P)^2, B) = \beta_n(\phi, \rho, P^2, B)$.

**Proof.** The first assertion is immediate since $u|_{t=0} = \phi$. To prove the second assertion, we set $u(t) := e^{-tD\phi}$ and $\tilde{u}(t) := e^{-t\tilde{D}\rho}$. We then have

$$
\partial_t u = -Du \quad \text{and} \quad \partial_t \tilde{u} = -\tilde{D}\tilde{u}.
$$

Fix $t > 0$. For $0 \leq s \leq t$, set $f(s) := \int_M \langle u(x; s), \tilde{u}(x; t-s) \rangle dx$. By Lemma \ref{lem:1}(5),

$$
\partial_s f(s) = \int_M \{ \langle \partial_s u(x; s), \tilde{u}(x; t-s) \rangle - \langle u(x; s), \partial_s \tilde{u}(x; t-s) \rangle \} dx
$$

$$
= \int_M \{ -\langle Du(x; s), \tilde{u}(x; t-s) \rangle + \langle u(x; s), \tilde{D}\tilde{u}(x; t-s) \rangle \} dx
$$

$$
= 0.
$$
since $Bu = 0$ and $\tilde{B}u = 0$. Because $f(s)$ is constant, Assertion (2) follows as
\[
0 = f(t) - f(0) = \int_M \{u(x;t), \tilde{u}(x;0)\} - \{u(x;0), \tilde{u}(x;t)\} \, dx
= \beta(\phi, \rho, D, B)(t) - \beta(\rho, \phi, \tilde{D}, \tilde{B})(t).
\]

To establish the third assertion, we suppose that $B\phi = 0$. Since $\tilde{B}u = 0$, we can use Lemma 1.1 (5) and Assertion (2) to see
\[
\partial_t \beta(\phi, \rho, D, B)(t) = \partial_t \beta(\rho, \phi, \tilde{D}, \tilde{B})(t)
= \int_M \{\partial_t \tilde{u}(x; t), \phi(x)\} \, dx - \int_M \{\tilde{D}u(x; t), \phi(x)\} \, dx
= -\int_M \{\tilde{u}(x; t), D\phi(x)\} \, dx - \beta(\rho, D\phi, \tilde{D}, \tilde{B})(t)
= -\beta(\rho, D, D, B)(t).
\]

We equate terms in the asymptotic expansions to derive Assertion (3). Assertion (4) follows by induction from Assertions (1) and (3); Assertion (5) is immediate. □

Mixed boundary conditions will play an important role in our discussion. Assume given an endomorphism $\gamma_0$ of $V|_{\partial M}$ so that $\gamma_0^2 = \text{Id}$. Let
\[
\Xi_\pm := \frac{1}{2}(\text{Id} \pm \gamma_0) \quad \text{and} \quad V_\pm := \Xi_\pm \{V|_{\partial M}\}
\]
be the associated spectral projections and eigenspaces. For $S \in \text{End}(V|_{\partial M})$, set
\[
B_{\gamma_0, S} \phi := \Xi_\pm (\nabla_{\epsilon_m} + S)\phi|_{\partial M} \oplus \Xi_\mp \phi|_{\partial M}.
\]

The operator $D$ determines a natural connection $\nabla^D$. The following Theorem, after taking into account our sign conventions, follows from results of [1].

**Theorem 2.2.** Adopt the notation established above. Then:

1. $\beta_0(\phi, \rho, D, B_{\gamma_0, S}) = \int_M \{\phi, \rho\} \, dx$.
2. $\beta_1(\phi, \rho, D, B_{\gamma_0, S}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M} \{\Xi_+ \phi, \Xi_+ \Sigma\rho\} \, dy$.
3. $\beta_2(\phi, \rho, D, B_{\gamma_0, S}) = -\int_M \{D\phi, \rho\} \, dx + \int_{\partial M} \{\Xi_\pm (\nabla_{\epsilon_m} \phi + S\phi), \rho\} + (\frac{1}{2}I_{\gamma_0} \Xi_+ \phi, \rho) - (\Xi_+ \phi, \nabla^D_{\epsilon_m} \rho\}$.

**Example 2.3.** We can relate spectral and mixed boundary conditions in the following special setting. Let $(\theta_1, ..., \theta_{m-1})$ be the usual periodic parameters on the torus $\mathbb{T}^{m-1}$ and let $r$ be the radial parameter on the interval $[0, 1]$. Let $f \in C^\infty([0, 1]$ with $f(0) = f(1) = 0$. Take a warped product on $M := \mathbb{T}^{m-1} \times [0, 1]$ of the form:
\[
ds_M^2 := e^{2f(r)} d\theta^a \wedge d\theta^a + dr^2.
\]
The volume element is then given by $dx = gdr d\theta_1 ... d\theta_{m-1}$ where $g := e^{(m-1)f}$. Let $\epsilon(0) := +1$ and $\epsilon(1) := -1$ so that the inward unit normal is $\epsilon \partial_r$. Let $\Theta_i \in M_{\ell}(\mathbb{C})$ satisfy the Clifford commutation relations $\Theta_i \Theta_j + \Theta_j \Theta_i = -2\delta_{ij}$. Set
\[
\gamma_m := \Theta_m, \quad \gamma_a := e^{f}\Theta_a, \quad \text{and} \quad \gamma^a := e^{-f}\Theta_a.
\]
Let $\gamma_0 \in M_{\ell}(\mathbb{C})$ satisfy $\gamma_0^2 = \text{Id}$ and $\gamma_0 \gamma_m + \gamma_m \gamma_0 = 0$. Let $\delta_1 > 0$ and $\delta_2 > 0$ be real parameters. Let $V = M \times \mathbb{C}^\ell$, let $\phi = \phi(r)$ and let $\rho = \rho(r)$. Set
\[
P := \gamma_m \partial_r + \gamma_a \partial_a + \delta_1 \gamma_m \gamma_0 \quad \text{on} \quad C^\infty(V),
\]
\[
A := -\epsilon \gamma_m \gamma_a \partial_a + \delta_2 \gamma_0 \quad \text{on} \quad C^\infty(V|_{\partial M}).
\]
For generic values of $\delta_2$, $\text{ker}(A) = \{0\}$.

Let $V_0 := [0, 1] \times \mathbb{C}^\ell$. Let $\Xi_\pm$ be projection on the $\pm 1$ eigenspaces of $\gamma_0$; we have that $\gamma_m \Xi_\pm = \Xi_\pm \gamma_m$. Let $P_0 := \gamma_m \partial_r + \delta_1 \gamma_m \gamma_0$ on $C^\infty(V_0)$, and let $D_0 := P_0^*$. Set
\[
B_0 \phi := \{\Xi_+ \phi \oplus \Xi_+ P_0 \phi\}|_{[0, 1]} \quad \text{and} \quad S := \delta_1 \gamma_0.$
We show that \( B_0 \) and \( B_{m,s} \) define the same boundary conditions for \( D_0 \) by checking:

\[
\Xi_+(\partial|_{[0,1]}) = 0 \quad \text{and} \quad \Xi_+ P_0 \partial|_{[0,1]} = 0, \\
\Xi_+(\partial|_{[0,1]}) = 0 \quad \text{and} \quad \Xi+ \gamma_m (\partial_0 \phi + \delta_1 \gamma_0 \phi)|_{[0,1]} = 0, \\
\Xi_+(\phi|_{[0,1]}) = 0 \quad \text{and} \quad \Xi_+ \gamma_m (\partial_0 \phi + \delta_1 \gamma_0 \phi)|_{[0,1]} = 0, \\
\Xi_+(\phi|_{[0,1]}) = 0 \quad \text{and} \quad \Xi+ (\partial_0 + S)\partial|_{[0,1]} = 0.
\]

**Lemma 2.4.** \( \beta_n(\phi,g^{-1} \rho,D,B) = (2\pi)^{m-1} \beta_n(\phi,\rho,D_0,B_{m,s}) \) in Example 2.3.

**Proof.** Let \( u_0 := e^{-tD_0} \partial_0 \phi \). Set \( u(r,\theta;t) := u_0(r,t) \). If \( \Phi = \Phi(\rho) \) and \( \Psi = \Psi(\rho) \), then \( \Pi^+_A \Phi = \Xi_+ \Phi \) and \( \Pi^+_A \Psi = \Xi_+ \Psi \). Thus we may show \( u = e^{-tD_\rho} \phi \) by checking

\[
(\partial_t + D) u = (\partial_t + D_0) u = 0, \\
B u = \Pi^+_A u \oplus \Pi^+_A P u = \Xi_+ u \oplus \Xi_+ P_0 u = B_0 u_0 = 0, \\
u|_{t=0} = u_0|_{t=0} = \phi.
\]

After taking into account the change in volume elements and equating terms in the asymptotic expansions, the result follows.

**Remark 2.5.** The coefficients appearing in Theorem 1.2 are independent of the dimension \( m \). This observation is quite general. Let \( P_0 \) be an operator of Dirac type on a bundle \( V_0 \) over an \( m \) dimensional manifold \( M_0 \). By doubling the rank of \( V_0 \) and by replacing \( P_0 \) by \( P_0 \oplus -P_0 \) if necessary, we may suppose there exists \( \gamma_0 \) so

\[
\gamma_0 P_0 + P_0 \gamma_0 = 0 \quad \text{and} \quad \gamma_0^2 = -\text{Id}.
\]

Define analogous structures on \( M := M_0 \times S^1 \) by setting

\[
P := P_0 + \gamma_0 \partial_\theta \quad \text{and} \quad A := A_0 - \gamma_m \gamma_0 \partial_\theta.
\]

Let \( \phi(x,\theta) = \phi_0(x) \) and let \( u_0 := e^{-tD_0} \partial_0 \phi_0 \). Set \( u(x,\theta;t) := u_0(x;t) \). Then one has \( \phi = e^{-tD_\rho} \phi \); thus the formulæ in dimension \( m + 1 \) restrict to the corresponding formulæ in dimension \( m \). By contrast, the corresponding formulæ for the heat trace asymptotics \([5,7]\) exhibit a very complicated dependence on \( m \).

### 3. Proof of Theorem 1.2

Since \( u|_{t=0} = \phi \), Assertion (1) of Theorem 1.2 is immediate. The interior integrands in \( \beta_1 \) and \( \beta_2 \) are determined by Lemma 2.1 (4). Thus we need only determine the boundary integrands. We apply Lemma 2.1 throughout. Dimensional analysis shows that the boundary integrands are homogeneous of total weight \( n - 1 \) in the jets of \( \phi \), of \( \rho \), and of the derivatives of the symbols of \( A \) and of \( P \). The spectral projections \( \Pi^+_A \) and \( \Pi^+_{A_\#} \) and the endomorphisms \( \gamma_1 \) have weight 0; the second fundamental form \( L \), the operators \( A \) and \( P \), and the endomorphisms \( \psi_A \) and \( \psi_P \) have weight 1.

We begin by studying \( \beta_1 \); there is no interior contribution. If \( B \phi = 0 \), then by Lemma 2.1 (3), \( \beta_1(\phi,\rho,D,B) = 0 \). Dually by Lemma 2.1 (2),

\[
\beta_1(\phi,\rho,D,B) = \beta_1(\rho,\phi,D,B) = 0 \quad \text{if} \quad B \rho = 0.
\]

Since the boundary integrand for \( \beta_1 \) must be homogeneous of weight 0, there exist universal constants so

\[
|\beta_1(\phi,\rho,D,B) = \int_{\partial M} \{ c_0(m) \langle \Pi^+_A \phi, \Pi^+_{A_\#} \rho \rangle + c_1(m) \langle \gamma_m \Pi^+_A \phi, \Pi^+_{A_\#} \rho \rangle \} \, dy.
\]

By Lemma 2.1 (5), \( \beta_1(\phi,\rho,P^2,B) = \beta_1(\phi,\rho,(-P)^2,B) \). Replacing \( P \) by \( -P \) replaces \( \gamma_0 \) by \( -\gamma_0 \). Thus \( \langle \gamma_m \Pi^+_A \phi, \Pi^+_{A_\#} \rho \rangle \) plays no role so we may take \( c_1(m) = 0 \).

We apply Lemma 2.1 with \( f = 0 \). By Theorem 1.2 and Equation (3.a),

\[
\begin{align*}
\beta_1(\phi,\rho,D,B) &= (2\pi)^{m-1} \beta_1(\phi,\rho,D_0,B_0) = -\frac{\pi}{\sqrt{2}} \int_{\partial M} \langle \Xi_+ \phi, \Xi_+ \rho \rangle \, dy \\
&= c_0(m) \int_{\partial M} \langle \Xi_+ \phi, \Xi_+ \rho \rangle \, dy.
\end{align*}
\]
This shows that $c_0(m) = -\frac{2}{\sqrt{m}}$ which completes the proof of Theorem 1.2 (2).

**Remark 3.1.** The constant $c_0(m)$ was determined in [9] using a special case computation and the present calculation should be regarded as providing a useful cross check on that calculation.

To prove the final assertion of Theorem 1.2 we express $\beta_2$ in terms of invariants with undetermined universal coefficients.

**Lemma 3.2.** There exist universal constants $c_i$ so that

$$\beta_2(\phi, \rho, D, B) = \int_M (D\phi, \rho) dx + \int_{\partial M} \{-\langle \gamma_m \Pi_A^\perp \rho, \phi \rangle + \langle c_2(A + A^\#) + c_3 L_{aa} + c_4 (\gamma_m \psi_p - \psi_p \gamma_m) + c_5 (\psi_A + \psi_A^\#) \Pi_A^\perp \phi, \Pi_A^\perp \rho \} \} dy.$$  

**Proof.** We argue heuristically. By Lemma 2.1, the interior integral for $\beta_2$ is given by $-\langle D\phi, \rho \rangle$. We define the normalized invariant $C$, which is given by a suitable boundary integral, by the identity

$$\beta_2(\phi, \rho, D, B) = C(\phi, \rho, D, B) - \int_M (D\phi, \rho) dx + \int_{\partial M} \{-\langle \gamma_m \Pi_A^\perp \rho, \phi \rangle - \langle \gamma_m \Pi_A^\perp \phi, \Pi_A^\perp \rho \} \} dy.$$  

As we must replace $\gamma_m$ by $-\tilde{\gamma}_m$ in passing to the dual structures, we have

$$0 = \beta_2(\phi, \rho, D, B) - \beta_2(\rho, \phi, \tilde{D}, \tilde{B}) = C(\phi, \rho, D, B) - C(\rho, \phi, \tilde{D}, \tilde{B}) - \int_M \langle D\phi, \rho \rangle - \langle \phi, \tilde{D}\rho \} \} dy + \int_{\partial M} \{ -\langle \gamma_m \Pi_A^\perp \rho, \phi \rangle - \langle \gamma_m \Pi_A^\perp \phi, \Pi_A^\perp \rho \} \} - \langle P\phi, \tilde{\gamma}_m \Pi_A^\perp \rho \} \} dy.$$  

We now use the Greens formula given in Lemma 1.1 (3) to see that

$$C(\phi, \rho, D, B) = C(\rho, \phi, \tilde{D}, \tilde{B}).$$  

Thus we can assume that the integral expressions for $C$ are symmetric in $\phi$ and $\rho$.

If $B\phi = 0$, then $C(\phi, \rho, D, B) = 0$ by Lemma 2.1 (3). Similarly $C(\phi, \rho, D, B) = 0$ if $B\rho = 0$. Thus after eliminating divergence terms, the integral formula for $C$ is bilinearly expressible in terms of tangential operators applied to

$$\{ \Pi_A^\perp \phi, \Pi_A^\perp \phi \} \} and \{ \Pi_A^\perp \rho, \Pi_A^\perp \rho \} \}.$$  

Since the boundary integrals defining $C$ have total weight 1, terms which are bilinear in $\Pi_A^\perp \phi$ and $\Pi_A^\perp \rho$ do not appear. By Lemma 1.1 (4),

$$\gamma_m \Pi_A^\perp = (Id - \Pi_A^\perp) \gamma_m \} so \} \int_{\partial M} \langle \gamma_m \Pi_A^\perp \phi, \Pi_A^\perp \phi \} \} = 0$$  

for any $\Phi$, $\tilde{\Phi}$. This shows that terms which are bilinear in $\Pi_A^\perp \phi$ and $\Pi_A^\perp \rho$ or in $\Pi_A^\perp \phi$ and $\Pi_A^\perp \rho$ do not involve $\gamma_m$. Taking into account the symmetry of $C$, we see that these terms would have the form

$$\int_{\partial M} b_0(\langle \Pi_A^\perp \phi, \Pi_A^\perp \phi \} \} + \langle \Pi_A^\perp \phi, \Pi_A^\perp \rho \} \} dy.$$  

Lemma 2.1 (5) now shows $b_0 = 0$. Consequently

$$C(\phi, \rho, D, B) = \int_{\partial M} \langle \Pi_A^\perp \phi, \Pi_A^\perp \phi \} \} dy$$  

where

$$T = b_1 A + b_2 \gamma_m A + b_3 A \gamma_m + b_4 \gamma_m A \gamma_m + c_5 L_{aa} \} Id + b_5 \psi_p + b_6 \gamma_m \psi_p + b_7 \psi_p \gamma_m + b_8 \gamma_m \psi_p \gamma_m + b_9 \psi_A + b_{10} \gamma m \psi_A + b_{11} \psi_A \gamma m + b_{12} \psi_A \gamma m.$$  

It is worth while making a few remarks about what invariants do not appear. Modulo terms in $L_{aa}$, we can replace $\gamma_m \psi_A \gamma m$ by $\psi_A \gamma m$ and $\psi_A$ by $-\psi_A \gamma m$. By Lemma
By Lemma 2.4, \( \gamma_m L_{aa} \) cannot appear. Furthermore, the invariants \( \gamma_a \psi_p \gamma_a \) and \( \gamma_a \gamma_m \psi_p \gamma_a \) would violate Remark 2.5.

Replacing \( P \) by \( \tilde{P} \) replaces \( \gamma_m \) by \( -\tilde{\gamma}_m \), \( A \) by \( A^\#: \psi_p \) by \( \tilde{\psi}_p \), and \( \psi_A \) by \( \psi_{A^\#} \). Thus the symmetry of Lemma 2.1 (2) yields \( b_1 = b_4 \), \( b_6 = -b_7 \), and \( b_9 = b_{12} \).

Lemma 2.1 (5) implies \( b_2 = b_3 = b_5 = b_6 = b_{10} = b_{11} = 0 \). Setting \( b_1 = b_4 = c_2 \), \( b_6 = -b_7 = c_4 \), and \( b_9 = b_{12} = c_5 \) then yields the formula of the Lemma; we use Remark 2.5 to see the coefficients \( c_i \) are universal.

We complete the proof of Theorem 2.2 (3) by showing:

**Lemma 3.3.**

(1) \( c_2 = \frac{1}{12}, \) \( c_4 = -\frac{1}{12}, \) \( c_5 = -\frac{1}{27} \).

(2) \( c_3 = \frac{1}{4} \).

**Proof.** Again, we take the flat metric in Example 2.6. The flat connection is compatible with the Clifford module structure. It is not, however, the only possible compatible connection. Let \( g_a \) be auxiliary real constants. We define a compatible connection by setting \( \omega_a := g_a \Id \). As \( \gamma_a \) and \( \gamma_0 \) anti-commute with \( \gamma_m \),

\[
\psi_p = \delta_1 \gamma_m \gamma_0 - g_a \gamma_a \quad \text{so} \quad \gamma_m \psi_p - \psi_p \gamma_m = -2 \delta_1 \gamma_0 - 2 \gamma_m \gamma_0 \gamma_a,
\]

\[
\psi_A = \delta_2 \gamma_0 + \varepsilon \gamma_m \gamma_0 \gamma_a \quad \text{so} \quad \psi_A + \gamma_m \psi_A \gamma_m = 2 \delta_2 \gamma_0 + 2 \gamma_m \gamma_0 \gamma_a.
\]

We take \( \rho = 0 \) near \( r = 1 \) so only the component where \( r = 0 \) is relevant in integrating over \( \partial M \). On this component, \( \partial_r \) is the inward geodesic normal and we set \( \varepsilon = 1 \). By Lemma 2.2

\[
\beta_2(\phi, \rho, D, B) = -\int_M \langle D_\phi \rho \rangle + \int_{\partial M} \{ -\langle \gamma_m \Xi + \gamma_m (\partial_r + \delta_1 \gamma_0) \phi, \rho \rangle + \langle \gamma_m \Xi + \gamma_m (\partial_r + \delta_1 \gamma_0) \phi, \rho \rangle + (2c_2 + 2c_5) \delta_2 \gamma_0
\]

+ \( (2c_5 - 2c_4) \gamma_m \gamma_0 \gamma_a - 2c_5 \delta_1 \gamma_0 \Xi + \phi, \Xi + \rho \} \}
\]

\[
= -\int_M \langle D_\phi \rho \rangle + \int_{\partial M} \{ -\langle \Xi - (\partial_r + \delta_1 \gamma_0) \phi, \rho \rangle - \langle \Xi + \phi, \partial_r \rho \rangle + \langle -\langle \Xi - (\partial_r + \delta_1 \gamma_0) \phi, \rho \rangle - \langle \Xi + \phi, \partial_r \rho \rangle + \langle (2c_2 + 2c_5) \delta_2 \gamma_0 + (2c_5 - 2c_4) \gamma_m \gamma_0 \gamma_a \Xi + \phi, \rho \rangle \}
\]

On the other hand, since \( P_0^2 = -(\delta_r^2 - \delta_\theta^2) \Id \), the connection defined by \( D_0 \) is the trivial connection. Thus by Theorem 2.2

\[
(2\pi)^{m-1} \beta_2(\phi, \rho, D_0, B_0) = -\int_M \langle D_\phi \rho \rangle \} dx + \int_{\partial M} \{ -\langle \Xi - (\partial_r + \delta_1 \gamma_0) \phi, \rho \rangle - \langle \Xi + \phi, \partial_r \rho \rangle \} \}
\]

By Lemma 2.4 \( \beta_2(\phi, \rho, D, B) = (2\pi)^{m-1} \beta_2(\phi, \rho, D_0, B_0) \). We may now complete the proof of Assertion (1) by deriving the relations

\[
2c_2 + 2c_5 = 0, \quad 2c_5 - 2c_4 = 0, \quad \text{and} \quad 2c_4 = -1.
\]

To study the coefficient of \( L_{aa} \), we let \( f \) be arbitrary in Example 2.6 but set \( \delta_1 = 0 \). We have \( A^\# = A \); thus \( \Pi^+_A = \Pi^+_A \). Let \( \phi = \phi(r) \) and \( \rho = \rho(r) \). We suppose \( \rho \) vanishes identically near \( r = 1 \) and suppress \( \varepsilon \). We may then apply Lemma 2.4 and Theorem 2.2 with \( S = 0 \) to compute:

\[
\beta_2(\phi, g^{-1} \rho, D, B) = (2\pi)^{m-1} \beta_2(\phi, \rho, D_0, B_0)
\]

\[
= \int_M \{ -\langle D_0 \phi, \rho \rangle dr d\theta + \int_{\partial M} \{ -\langle \Xi - (\partial_r + \delta_1 \gamma_0) \phi, \rho \rangle - \langle \Xi + \phi, \partial_r \rho \rangle \} \}
\]

We have \( \Pi^+_A = \Xi + \) and \( \Pi^+_A = \Xi + \). Since \( \tilde{P} g^{-1} \rho = -g^{-1} \gamma_m \partial_r \rho \) and \( dy = gd\theta \),

\[
\int_{\partial M} \{ -\langle \gamma_m \Pi^+_A P \phi, g^{-1} \rho \rangle - \langle \gamma_m \Pi^+_A \phi, \tilde{P} g^{-1} \rho \rangle \} \}
\]

Consequently we have

\[
(3.b) \quad 0 = \{ c_2 (A + A^\#) + c_3 L_{aa} + c_4 (\gamma_m \psi_p - \psi_p \gamma_m) + c_5 (\psi_A + \psi_{A^\#}) \} \Pi^+_A \phi.
\]
We must define a compatible connection. Let $\omega_m = 0$ and $\omega_a = \frac{i}{2} \partial_r f \cdot \gamma_m \gamma_a$ define a connection $\nabla$ on $V$. We have

$$\Gamma_{mab} = \Gamma_{amb} = -\Gamma_{a bm} = \delta_{ab} e^{2f} \partial_r f,$$

$$\Gamma_{ma}^b = \Gamma_{am}^b = \delta_{ab} \partial_r f, \quad \text{and} \quad \Gamma_{ab}^m = -\delta_{ab} e^{2f} \partial_r f.$$  

We have $\gamma_{ji} = \partial_r^i \gamma_j - \Gamma_{ij} \partial_r^k \gamma_k + [\omega_i, \gamma_j]$. We show $\nabla \gamma = 0$ by checking:

$$\gamma_{m;m} = 0,$$

$$\gamma_{m;ma} = \partial_r f \cdot \gamma_m - \Gamma_{m;ma} \gamma_b = 0,$$

$$\gamma_{m;ab} = -\Gamma_{ab}^m \gamma_m + [\omega_b, \gamma_a] = -\partial_r f \cdot \gamma_a + \frac{1}{2} \partial_r f [\gamma_m \gamma_a, \gamma_m] = 0,$$

$$\gamma_{m;ab} = -\Gamma_{ba}^m \gamma_m + [\omega_b, \gamma_a] = \partial_r f \cdot e^{2f} \gamma_m \delta_{ab} + \frac{1}{2} \partial_r f [\gamma_m \gamma_b, \gamma_a] = 0.$$  

Consequently

$$\psi_P = -\gamma^a \omega_a = -\frac{1}{2} \partial_r f \gamma^a \gamma_m \gamma_a = -\frac{1}{2} (m-1) \partial_r f \gamma_m,$$

$$\psi_A = \gamma_m \gamma^a \omega_a + \partial_r f \gamma_m = -\frac{1}{2} (m-1) \partial_r f + \partial_r f \gamma_0.$$  

Thus $\gamma_m \psi_P - \psi_P \gamma_m = 0$ and $\psi_A + \psi_A^\# = L_{aa} \text{Id} + 2\partial_r \gamma_0$. Furthermore as $\phi$ and $\rho$ are independent of $\theta$, $A + A^\# = 2 \partial_r \gamma_0$. Thus Equation (3.13) yields $c_3 + c_5 = 0$ so, by Assertion (1), $c_3 = -c_5 = \frac{1}{2}$. This completes the proof of Lemma 3.3 and thereby the proof of Theorem 1.2.  

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