Abstract. We show that a subfactor planar algebra of finite depth $k$ is generated by a single $(k + 1)$-box.

The main result of [KdyTpr2010] shows that a subfactor planar algebra of finite depth is singly generated with a finite presentation. If $P$ is a subfactor planar algebra of depth $k$, it is shown there that a single $2k$-box generates $P$. It is natural to ask what the smallest $t$ is such that a single $t$-box generates $P$. While we do not resolve this question completely, we show in this note that $t \leq k + 1$ and that $k$ does not suffice in general. All terminology and unexplained notation will be as in [KdyTpr2010].

For the rest of the paper fix a subfactor planar algebra $P$ of finite depth $k$. Let $2t$ be such that it is the even number of $k$ and $k + 1$. We will show that some $2t$-box generates $P$ as a planar algebra. The main observation is the following proposition about complex semisimple algebras. We mention as a matter of terminology that we always deal with $\mathbb{C}$-algebra anti-automorphisms and automorphisms (as opposed to those that induce a non-identity involution on the base field $\mathbb{C}$).

**Proposition 1.** Let $A$ be a complex semisimple algebra and $S : A \to A$ be an involutive algebra anti-automorphism. There exists $a \in A$ such that $a$ and $Sa$ generate $A$ as an algebra.

We pave the way for a proof of this proposition by studying two special cases. In these, $n$ is a fixed positive integer.

**Lemma 2.** Let $S$ be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$. There is an algebra automorphism of $M_n(\mathbb{C})$ under which $S$ is identified with either (i) the transpose map or (ii) the conjugate of the transpose map by the matrix

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} (-J^T = -J^{-1}).$$

The second case may arise only when $n = 2k$ is even (and $I_k$ denotes, of course, the identity matrix of size $k$).

**Proof.** Let $T$ denote the transpose map on $M_n(\mathbb{C})$. The composite map $TS$ is then an algebra automorphism of $M_n(\mathbb{C})$ and is consequently given by conjugation with an invertible matrix, say $u$. Thus $Sx = (uxu^{-1})^T$. Involutivity of $S$ implies that $u$ is either symmetric or skew-symmetric. By Takagi’s factorization (see p204 and p217 of [HrnJhn1990]), $u$ is of the form $v^T v$ if it is symmetric and of the form $v^T J v$ if it is skew-symmetric for some invertible $v$. For the algebra automorphism

1991 Mathematics Subject Classification. Primary 46L37; Secondary 57M99.

Key words and phrases. Subfactor planar algebra, presentation.
of $M_n(\mathbb{C})$ given by conjugation with $v$, $S$ gets identified in the symmetric case with the transpose map and in the skew-symmetric case with the conjugate of the transpose map by $J$. 

**Corollary 3.** Let $S$ be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$. There is a non-empty Zariski open subset of $M_n(\mathbb{C})$ such that each $x$ in this subset is invertible and together with $Sx$ generates $M_n(\mathbb{C})$ as an algebra.

**Proof.** The elements $x$ and $Sx$ do not generate $M_n(\mathbb{C})$ as algebra if and only if the dimension of the span of all positive degree monomials in $x$ and $Sx$ is smaller than $n^2$. This is equivalent to saying that for any $n^2$ such monomials, the determinant of the $n^2 \times n^2$ matrix formed from the entries of these monomials vanishes. Since each of these entries is a polynomial in the entries of $x$, non-generation is a Zariski closed condition.

To show non-emptyness of the complement, it suffices, by Lemma 2, to check that when $S$ is the transpose map or the $J$-conjugate of the transpose map (when $n$ is even), some $x$ and $Sx$ generate $M_n(\mathbb{C})$. To see this note first that a direct computation shows that for non-zero complex numbers $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}$, the matrix whose only non-zero entries are $\alpha_1, \ldots, \alpha_{n-1}$ on the superdiagonal and the matrix whose only non-zero entries are $\beta_1, \ldots, \beta_{n-1}$ on the subdiagonal generate $M_n(\mathbb{C})$. Taking all $\alpha_i$ and $\beta_j$ to be 1 gives a pair of generators of $M_n(\mathbb{C})$ that are transposes of each other, while, if $n = 2k$ is even, taking all $\alpha_i$ to be 1 and all $\beta_j$ to be 1 except for $\beta_k = -1$ gives a pair of generators of $M_n(\mathbb{C})$ such that each is the $J$-conjugate of the transpose of the other.

Since invertibility is also a non-empty open condition, we are done. \hfill \Box

**Lemma 4.** Let $S$ be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections under which $S$ is identified with the map \( x \oplus y \mapsto y^T \oplus x^T \).

**Proof.** The map $x \oplus y \mapsto S(y^T \oplus x^T)$ is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections and is therefore given by $x \oplus y \mapsto xu^{-1} \oplus vyv^{-1}$ for invertible $u, v$. Hence $S(x \oplus y) = uy^T u^{-1} \oplus vx^T v^{-1}$. By involutivity of $S$, we may assume that $v = u^T$. It is now easy to check that under the algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ given by $x \oplus y \mapsto u^{-1} xu \oplus y$, $S$ is identified with $x \oplus y \mapsto y^T \oplus x^T$. \hfill \Box

In proving the analogue of Corollary 3 for $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, we will need the following lemma.

**Lemma 5.** Let $A$ and $B$ be finite dimensional complex unital algebras and let $a \in A$ and $b \in B$ be invertible. Then, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $a \oplus \lambda b \in A \oplus B$ contains both $a$ and $b$.

**Proof.** We may assume that $\lambda \neq 0$ and then it suffices to see that $a$ is expressible as a polynomial in $a \oplus \lambda b$. Note that since $a \oplus \lambda b$ is invertible and $A \oplus B$ is finite dimensional, the algebra generated by $a \oplus \lambda b$ is actually unital. In particular, it makes sense to evaluate any complex univariate polynomial on $a \oplus \lambda b$.

Let $p(X)$ and $q(X)$ be the minimal polynomials of $a$ and $b$ respectively. By invertibility of $a$ and $b$, neither $p$ nor $q$ has 0 as a root. The minimal polynomial of $\lambda b$ is $q(\lambda)$. Unless $\lambda$ is the quotient of a root of $p$ and a root of $q$, $p(X)$ and
Corollary 6. Let $S$ be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There is a non-empty Zariski open subset of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ such that each $x \oplus y$ in this subset is invertible and together with $S(x \oplus y)$ generates $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra.

Proof. As in the proof of Corollary 3, the set of $x \oplus y \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ such that $x \oplus y$ and $S(x \oplus y)$ do not generate $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra is Zariski closed. Thus the set of invertible elements in its complement is Zariski open.

To show non-emptiness, it suffices, by Lemma 4, to check that some invertible $x \oplus y$ and $y^T \oplus x^T$ generate $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra. Note that by Corollary 3, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $x \oplus \lambda x$ contains $x \oplus 0$ and $0 \oplus x$ and similarly the algebra generated by $\lambda x^T \oplus x^T$ contains $x^T \oplus 0$ and $0 \oplus x^T$. Thus the algebra generated by $x \oplus \lambda x$ and $\lambda x^T \oplus x^T$ is the whole of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$.

Proof of Proposition 7. Let $A$ denote the (finite) set of all inequivalent irreducible representations of $A$ and for $\pi \in A$, let $d_\pi$ denote its dimension. Since $S$ is an involutive anti-automorphism, it acts as an involution on the set of minimal central projections of $A$. It is then easy to see that there exist subsets $A_1$ and $A_2$ of $A$ and an identification

$$A \to \bigoplus_{\pi \in A_1} M_{d_\pi}(\mathbb{C}) \oplus \bigoplus_{\pi \in A_2} (M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C}))$$

such that each summand is $S$-stable.

Now, by Corollaries 3 and 4, in each summand of the above decomposition, either $M_{d_\pi}(\mathbb{C})$ or $M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C})$, there is an invertible element which together with its image under $S$ generates that summand.

Finally, an inductive application of Lemma 5 shows that if $a$ is a general linear combination of these generators, then $a$ and $Sa$ generate $A$ as an algebra.

Our main result now follows easily.

Proposition 7. Let $P$ be a subfactor planar algebra of finite depth $k$. Let $2t$ be the even number in $\{k, k + 1\}$. Then $P$ is generated by a single $2t$-box.

Proof. It clearly suffices to see that there is a $2t$-box such that the planar subalgebra generated by it contains $P_{2t}$, for then, this generated planar subalgebra contains $P_2$ as well (taking the right conditional expectation if $2t = k + 1$) and hence is the whole of $P$.

Let $S$ denote the map $Z(R_{2t})^{\otimes 2} : P_{2t} \to P_{2t}$ which is an involutive anti-automorphism of the semisimple algebra $P_{2t}$. By Proposition 4, there is an $a \in P_{2t}$ such that $a$ and $Sa$ generate $P_{2t}$ as an algebra. Since the planar subalgebra generated by $a$ certainly contains $Sa$, it follows that it contains all of $P_{2t}$.

We finish by showing that $k + 1$ might actually be needed.
Example 8. Let $P = P(V)$ be the tensor planar algebra (see [Jns1999] for details) of a vector space $V$ of dimension greater than 1. It is easy to see that $\text{depth}(P) = 1$. However, given any $a \in P_1 = \text{End}(V)$, if $Q$ is the planar subalgebra of $P$ generated by $a$, a little thought shows that $Q_1$ is just the algebra generated by $a$ and is hence abelian while $P_1$ is not.

Acknowledgments

We are grateful to Prof. T. Y. Lam for his remarks.

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