Output Feedback Control for Active Suspension Electro-Hydraulic Actuator Systems With a Novel Sampled-Data Nonlinear Extended State Observer

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ABSTRACT In this paper, an output feedback controller based on a novel sampled-data nonlinear extended state observer (SDNLESO) is proposed for an active suspension electro-hydraulic actuator (ASEHA) system to address the heavy nonlinearities, model uncertainties and sampled-data behavior. The designed controller only requires little prior knowledge about the controlled system for the purpose of position tracking. The SDNLESO, integrating a novel nonlinear extended state observer and an output predictor, is developed to simultaneously and continuously estimate the unmeasurable states and total disturbance for an error dynamic system rather than the original ASEHA system, where the actual discrete displacement tracking error between measurement output and the reference signal serves as the observer input. Then, a compensated controller is synthesized based on the obtained estimates, which is robust against the matched and mismatched disturbances of the system while being able to ensure an expected transient tracking performance and final tracking accuracy. By constructing weighted error system and using the geometric homogeneity theory, the SDNLESO convergence is proven, while the maximum allowable sampling period is derived theoretically for guiding the selection of the actual sampling interval. Moreover, the closed-loop system stability and the tracking error exponential convergence are guaranteed within the Lyapunov framework. Finally, a number of practical experiments are carried out on the ASEHA system test platform. The results show that the proposed control method is applicable and valid for nonlinear and uncertain ASEHA system despite the existence of a large actuator area ratio.

INDEX TERMS Active suspension electro-hydraulic actuator (EHA), nonlinearity, output feedback, sampled-data observer, uncertainty.

I. INTRODUCTION Suspension systems play crucial roles in vehicle chassis, which are responsible for the ride comfort, maneuverability, and road holding [1]–[3]. In terms of the three types of suspension systems, active suspensions possess greater potential in improving the performance objectives than passive suspensions and semi-active suspensions; because additional actuators, which can add and dissipate energy from the system, are placed in parallel with the passive components (i.e., springs and dampers) [4]. Electro-hydraulic actuators (EHA) are preferentially embedded in active suspension systems due to their advantages of a high power-to-weight ratio, low maintenance cost and fast response characteristics [5], [6]. Among most reported studies concerning active suspension control, active suspension electro-hydraulic actuators (ASEHAs) are regarded as force-output elements, no matter their nonlinear dynamics are considered [6]–[8] or not [2], [3], [9]. In practical scenarios, complicated road conditions will lead to inevitable and drastic work pressure fluctuations in an ASEHA system, which make it difficult to guarantee the force control precision.
and the expected suspension control requirements. Given this considerable obstacle in the implementation of active suspensions with force-output control, the current study considers the ASEHA as a displacement-output actuator. That is, controlling each ASEHA to achieve a high-quality track towards the desired displacement signal, which is calculated based on the full vehicle dynamics and is able to ensure a stable vehicle body attitude. Thus, it is necessary to develop an effective displacement tracking control approach for the ASEHA system.

It is worth mentioning that EHA systems are subjected to heavy nonlinearities [10] and numerous uncertainties [11]–[13] (e.g., parametric variations, matched and mismatched modeling errors and uncertain external disturbances), which pose considerable challenges to the high-performance control of ASEHA systems. Despite its dominance, the traditional PID (proportional integral derivative) is restricted to cope with the sophisticated nonlinear and uncertain behaviors in EHA systems. Thus, many advanced control technologies have been developed, such as adaptive backstepping control [14]–[16], adaptive robust control [5], [11], adaptive integral robust control [12], [13] and sliding mode control [17], [18]. These control methods have realized enhanced tracking performance for EHA systems in the presence of parametric uncertainties and external disturbances. However, all the aforementioned approaches are full state feedback approaches, where every state signal is supposed to be measurable. Obviously, this assumption is not the case for real physical systems. In many practical EHA systems, especially in ASEHA systems, only the position information can be easily measured, and the other states, such as velocity and hydraulic pressure, are unavailable due to cost or space restrictions. Therefore, much effort has been made to develop effective output feedback control methods for EHA systems from both theoretical and practical aspects.

The concept behind most output feedback control technologies is to design an observer that can (partially or completely) estimate the unmeasurable states and disturbances in a controlled system. Then, a compensated controller is synthesized to realize the desired control target based on the obtained estimates. Recently, a number of observer-based output feedback controllers have been developed for active suspension systems [4], [9], [19], [20] and EHA systems [21]–[24], which are able to deal with the nonlinearities and uncertainties effectively. Unfortunately, many methods designed for EHA systems suffer from limitations. For instance, in [21], [22], the designed observers estimated only the unmeasurable states of EHA systems, which were constructed on a deterministic system model without consideration of the matched and mismatched disturbances. In [23], [24], observers were developed that assume the required states are measurable and estimate only the mechanical or hydraulic disturbances affecting the EHA systems. In this sense, the active disturbance rejection control (ADRC) proposed by Han [25], has received tremendous attention since it can simultaneously estimate the unmeasurable states and unmodeled disturbances while requiring little prior knowledge about the controlled systems. Recently, according to ADRC theory, [16] introduced a linear extended state observer (LESO)-based output feedback control methodology for nonlinear EHA systems, which is able to estimate the state information and time-varying matched disturbances arising from parameter variation and interior leakage while showing high tracking accuracy towards the desired signal, but it can not addressed the mismatched disturbances such as unmodeled friction forces and load disturbances effectively. Furthermore, an output feedback controller with the integration of a LESO and a disturbance observer was proposed for an EHA system [26], [27], successfully improving the displacement tracking performance while being robust against strong matched and mismatched modeling uncertainties.

As asserted by Han [28], in an ADRC scheme, a properly designed nonlinear extended state observer (NLESO) would yield better performance than a LESO. Guo and Zhao [29], [30] contributed much effort to the theoretical research of this issue, and their simulation results have proven that the proposed NLESO has the advantages of a smaller estimation error peak value and higher observation accuracy over the LESO. However, in these works, NLESO convergence was proven on the basis of strong prerequisite assumptions concerning system states and observer estimation error states, which are difficult to verify in use. Zhao et al. [31], Zhao and Guo [32], and Zhao and Jiang [33] performed further exploration in this field. However, a careful review of the existing literature shows that few studies have conducted systematical NLESO-based ADRC research on EHA systems, let alone experimental verification. Therefore, developing an effective NLESO-based controller for EHA systems along with comprehensive stabilization proof and experimental validation constitutes one of the motivations of this study.

Note that among most of the reported observer-based control strategies, the observers are designed in the continuous time domain. In fact, when it comes to the matter of implementation in digital signal processors (DSPs), the measurable outputs of the controlled systems are sampled; that is, only the discrete sampling instants can be fed into the observer. To maintain the observation performance and convergence of the continuous-time observer, the sampling period must be very small, which poses great challenges to the hardware capability. A general technique is to perform a discretization on the system model and then design a discrete-time observer for the discretized model [34], [35]. However, the exact discretization for a nonlinear system model is an overcomplicated problem. Meanwhile, if the continuous-time model is substituted by a discrete-time description, the system inter-sample dynamic behavior will be lost. To overcome these problems, authors in [36] designed a sampled-data observer for a class of nonlinear systems, in which a LESO is coupled with an output predictor to obtain the system dynamics between two consecutive sampling instants. The sampled-data observer
subjected to discrete measurements preserved the benefits of continuous-time LESO, which was successfully applied to the EHA systems [37], [38]. Recently, authors in [39] constructed a nonlinear sampled-data ESO (NSESO)-based ADRC scheme for a pneumatic muscle actuator (PMA) by considering the discrete sampling behavior of system output. However, the convergence of the NSESO was proven under a series of prior assumptions concerning the Lyapunov function, which is difficult to verify in practice. Moreover, the dynamic model of a PMA is significantly different from the EHA systems. To the best of our knowledge, the employment of a NLESO with discrete sampled data onto EHA systems has yet to be attempted, which formulates another motivation for the current investigation.

In this paper, by referring the NLESO in [32] and the sampled-data observer in [40], a novel sampled-data NLESO (SDNLESO) is proposed for a nonlinear and uncertain ASEHA system where the discreteness of the measurement output is considered. In order to attenuate the impacts of matched and mismatched disturbances, the SDNLESO is designed for an error dynamic system rather than the original ASEHA system, using only the discrete displacement tracking error between sampling output and the given reference signal. Instead of employing the zero-order holder (which holds the most recent measurement), an inter-sampled output predicator is plugged into the SDNLESO to acquire a copy of the system output between two consecutive sampling instants so that continuous states and disturbance estimation can be ensured. Then, a compensated control law is synthesized based on the estimates produced by SDNLESO for the purpose of position tracking. The main contributions of this paper are listed as follows:

- The proposed output feedback controller is able to deal with the complex nonlinearities, model uncertainties and sampled-data behavior of the output signal in the ASEHA system, requiring only the displacement tracking error between discrete measurement output and the reference signal.
- A SDNELSO coupled with a novel NLESO (not a LESO) and an output predictor is developed. Unlike previous methods observing the matched and mismatched disturbances respectively to attenuate their impacts, the SDNLESO is devoted to continuously estimating the unmeasurable states and total disturbance of the new constructed error dynamic system based on the discrete sampled data.
- The SDNLESO convergence is proved by using the geometric homogeneity theory without the need for strong prerequisite state assumptions, and the maximum allowable sampling period is derived theoretically to guide the sampling period selection in real applications. Moreover, the whole closed-loop system stability is systematically established within the Lyapunov framework.
- Extensive practical experiments have verified the feasibility and validation of the proposed control method although the actuator area ratio is rather large. This is the first time that such an SDNLESO-based controller is designed and tested experimentally for an ASEHA system.

The rest of this paper is organized as follows. In Sect. II, the nonlinear and uncertain dynamic model of the ASEHA system is derived. The SDNLESO-based output feedback controller along with the observer convergence and closed-loop system stability analysis are presented in Sect. III. A number of practical experiments showing the effectiveness of the proposed control scheme are conducted in Sect. IV. Sect. V offers some conclusions and suggestions for future work.

II. ASEHA SYSTEM MODEL

It is well recognized that vehicle body attitude constantly changes along with road surface fluctuations. For a vehicle equipped with a active suspension system, if the attitude deviation is counteracted by the movement of ASEHAs in real time, then the vehicle body will remain stable. Therefore, the vehicle body attitude control includes two aspects, namely, inversely calculating the ideal expansion and contraction value of each ASEHA based on the full vehicle active suspension system model and controlling the ASEHA to track its desired value. The entire control system is considerably complicated, so this article focuses on the second aspect.

The schematic of the ASEHA system under study is depicted in Fig. 1, whose dynamics are slightly different from the common model of the EHA system presented in [21], [22], [38]. In Fig. 1, the ASEHA is placed vertically to support the vehicle body weight, so it is necessary to consider the gravity effect of the mass. By driving the servo-valve with proper input current, the hydraulic oil flowing into the two chambers can be controlled. In this manner, the mass will move along with the desired trajectory and the expected vehicle body attitude can be achieved.

![FIGURE 1. The schematic of the ASEHA system.](image-url)
The motion dynamics of the mass obeys Newton’s law, which can be described by

\[ m\ddot{x}_v = P_1A_1 - P_2A_2 - mg - b_2\dot{x}_v + f_1(t, x_v, \dot{x}_v) \]  

(1)

where \( m \) denotes the mass that simulate the vehicle body weight, whose displacement is the only measurable signal of the ASEHA system and be denoted as \( x_v \); \( P_1 \) and \( P_2 \) represent the pressures in non-rod chamber and rod chamber, respectively; \( A_1 \) and \( A_2 \) are the equivalent areas of non-rod piston and rod piston, respectively, which can be calculated according to the piston diameter \( D \) and the rod diameter \( d \). Besides, \( b_2 \) denotes the viscous damping coefficient of hydraulic fluid and \( g \) denotes the gravitational acceleration; \( f_1 \) stands for the mechanical disturbances caused by unmodeled friction forces and other external disturbances.

Neglecting the external leakage, the pressure dynamics inside the two chambers are expressed as \([10],[13]\)

\[
\dot{P}_1 = \frac{\beta_e}{V_1}[-A_1\dot{x}_v - C_i(P_1 - P_2) + Q_1 + q_1(t)]
\]

\[
\dot{P}_2 = \frac{\beta_e}{V_2}[A_2\dot{x}_v + C_i(P_1 - P_2) - Q_2 + q_2(t)]
\]

(2)

where \( V_1 = V_{01} + A_1 x_v \) and \( V_2 = V_{02} - A_2 x_v \) are the control volumes of the two chambers, which are varying along with the mass displacement \( x_v \); \( V_{01} \) and \( V_{02} \) are the initial control volumes of the actuator chambers; \( Q_1 \) is the supplied flow rate to the non-rod chamber and \( Q_2 \) is the return flow rate of the rod chamber; \( q_1(t) \) and \( q_2(t) \) denote the modeling errors of pressure dynamics \( P_1 \) and \( P_2 \), respectively; \( C_i \) is the internal leakage coefficient of the actuator due to oil pressure; \( \beta_e \) is the effective oil bulk modulus. The flow rates \( Q_1 \) and \( Q_2 \) are proportional to the spool valve position and nonlinear with the pressure difference, which can be modeled by \([10],[14]\)

\[
Q_1 = k_q s(x_v)\sqrt{P_s - P_1} + s(-x_v)\sqrt{P_1 - P_r}
\]

\[
Q_2 = k_q s(x_v)\sqrt{P_2 - P_r} + s(-x_v)\sqrt{P_s - P_2}
\]

(3)

where \( k_q \) is the flow gain with respect to the spool valve displacement; \( P_s \) and \( P_r \) are the supply pressure and return pressure of the hydraulic system, respectively; and the function \( s(x_v) \) is defined as

\[
s(x_v) = \begin{cases} 
1, & x_v \geq 0 \\
0, & x_v < 0
\end{cases}
\]

(4)

Note that the dynamic response of the servo-valve is much faster than that of ASEHA system, so the valve model can be approximated as a proportional element with regard to the control current \( u \), which is given as

\[
x_v = k_i u
\]

(5)

By Defining the state vector as \( x = [x_1, x_2, x_3] = [x_v, \dot{x}_v, \frac{P_1A_1 - P_2A_2}{m}] \), the ASEHA system dynamics (1)-(5) can be converted to a state-space form

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = x_3 - \gamma x_2 - g + f_1(t, x_1, x_2)/m
\]

\[
\dot{x}_3 = -\alpha x_2 - \beta x_3 + bu + q(t)
\]

(6)

where

\[
y = \frac{b_2}{m} \dot{x}_1 \quad \alpha = \frac{\beta_e}{m} \left( \frac{A_1^2}{V_1} + \frac{A_2^2}{V_2} \right),
\]

\[
\beta = \beta_e C_i \left( \frac{A_1}{V_1} + \frac{A_2}{V_2} \right), 
\]

\[
b = \frac{\beta_e k_i}{m} \left( \frac{A_1}{V_1} R_1 + \frac{A_2}{V_2} R_2 \right), 
\]

\[
R_1 = s(x_v)\sqrt{P_s - P_1} + s(-x_v)\sqrt{P_1 - P_r},
\]

\[
R_2 = s(x_v)\sqrt{P_2 - P_r} + s(-x_v)\sqrt{P_s - P_2},
\]

\[
q(t) = \frac{\beta_e k_i}{m} \left[ \frac{A_1}{V_1} q_1(t) - \frac{A_2}{V_2} q_2(t) \right].
\]

And \( \frac{P_1 - P_2}{m} \approx x_3 \), the causal error is attached to \( q(t) \).

Remark 1: The ASEHA system possesses severe nonlinear characteristics in pressure and flow dynamics due to the compressibility of hydraulic fluid and the complicated flow features of servo-valve.

Remark 2: The ASEHA system suffers from a variety of model uncertainties. For example, in (6), \( f_1 \) arising from the unmodeled Coulomb friction, Strubeck effect as well as the external disturbances is always existing. The complicated internal leakage behavior makes it almost impossible to capture the exact model characteristics of pressure dynamics. Meanwhile, the ASEHA model is subjected to heavy parametric uncertainties. Specifically, the vehicle body mass \( m \) may vary with the passengers number and the payload; the hydraulic parameters \( b_2 \), \( \beta_e \) and \( C_i \) are rather sensitive to the temperature inside the actuator; \( V_1 \), \( V_2 \), \( R_1 \) and \( R_2 \) are also constantly changing as the system operates.

Accordingly, we set \( m_n, b_n, \beta_{en} \) and \( C_{in} \) as the nominal values of system parameters \( m, b_2, \beta_e \) and \( C_i \). And the nominal values of time-varying parameters \( \gamma, \alpha, \beta \) and \( b \) shown in (6) are denoted as \( \gamma_n, \alpha_n, \beta_n \) and \( b_n \). Additionally, the values of \( \alpha \), \( \beta \) and \( b \) when the piston is in the intermediate position are regarded as their nominal values. Then the lumped disturbances caused by \( \gamma \) deviation and the modeling error \( f_1(t, x_1, x_2)/m \) are denoted as \( d_1(t) \). The uncertainties arising from the changes of \( \alpha \), \( \beta \) and \( b \) and the modeling error \( q(t) \) are lumped together, which are denoted as \( d_2(t) \).

In addition, the ASEHA system output \( y = x_1 \) is only available at each discrete sampling instant \( t_k \). Thus, the ASEHA system dynamics are expressed as

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = x_3 - \gamma_n x_2 - g + d_1(t)
\]

\[
\dot{x}_3 = -\alpha_n x_2 - \beta_n x_3 + b_n u + d_2(t)
\]

\[
y(t_k) = x_1(t_k), \quad k \in N
\]

(7)

where \( \gamma_n = \frac{b_n}{m_n}, \alpha_n = \frac{2b_n(A_1+A_2)}{m_nL}, \beta_n = \frac{4b_nC_{in}}{L} \) and \( b_n = \frac{2b_2k_i}{m_nD} \sqrt{P_s}, \) in which \( L \) is the stroke of the hydraulic actuator. In (7), \( d_1(t) \) is regarded as mismatched disturbances, and \( d_2(t) \) is considered as matched disturbances. As seen, all modeling uncertainties are lumped to matched and mismatched disturbances.

Note that the system (7) is a hybrid uncertain system, which combines the continuous dynamic behaviors of states \( x_1, x_2, \)
$x_3$ between two consecutive sampling instants $[t_k, t_{k+1}]$ and a discrete measurable output $y$ which is updated at sampling time $t = t_k$.

### III. CONTROLLER DESIGN AND STABILITY ANALYSIS

In this section, an output feedback controller is designed for the purpose of position tracking towards the desired signal of the ASEHA system, which is tolerant against the mismatched disturbances $d_1(t)$ and matched disturbances $d_2(t)$. Inspired by [32], [40], we propose a novel SDNLESO to simultaneously estimate the unmeasurable states and total disturbance of the error dynamics (rather than the original ASEHA system) using only the displacement tracking error between output signal and the reference; at the same time, the sampled-data effect is taken into account. The compensated control law is then synthesized on the basis of the SDNLESO, where the total disturbance is canceled by its estimated value. Moreover, the stability analyses are presented systematically.

### A. SDNLESO AND CONTROL LAW DESIGN

For the sake of SDNLESO design, we define the following error vector, i.e., $e(t) = [e_1(t), e_2(t), e_3(t)]^T$, where $e_1(t) = y(t) - v(t)$ is the displacement tracking error between system output $y(t)$ and the desired reference signal $v(t)$, which is available at discrete sampling instant. Besides, $e_i(t) = ˙e_{i-1}(t), (i = 2, 3), ²d(t) = [d_1(t), ²d_1(t), ²d_2(t)]^T$. Then the following error dynamics can be derived according to the ASEHA system (7).

\[
\begin{align*}
\dot{e}(t) &= A_r e(t) + B_r \phi(e_2(t), e_3(t)) - B_r r(\dot{v}(t), \dot{v}(t), \ddot{v}(t)) \\
&\quad + B_r ²b_n u(t) + B_r ²d_1(t), ²d_2(t)²d(t) \\
y_e(t_k) &= e_1(t_k), \quad k \in N \\
\end{align*}
\]

where

\[
A_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \\
\phi(e_2(t), e_3(t)) = -(a_n - \gamma_n e_2(t) - (\beta_n + \gamma_n) e_3(t) - \beta_n g_r, \\
r(\dot{v}(t), \dot{v}(t), \ddot{v}(t)) = (a_n - \gamma_n \dot{v}(t) + (\beta_n + \gamma_n) \ddot{v}(t) + \dddot{v}(t), \\
²d(t) = -(\beta_n + \gamma_n) ²\alpha_2(t) + \beta_n ²d_1(t) + ²d_1(t) + ²d_2(t).
\]

Note that the desired trajectory-based term $r(\dot{v}(t), \dot{v}(t), \ddot{v}(t))$ is known, which can be computed off-line. The unmatched disturbances and the matched disturbances are transferred to the total disturbance $²d(t)$, so if the controller can overcome the impact of the total disturbance, it will be robust against the matched and unmatched disturbances. Meanwhile, the displacement tracking performance can be guaranteed by the convergence of tracking error $e_1(t)$.

In what follows, our task is to design a SDNLESO which is able to simultaneously estimate the unmeasurable states $e_2$, $e_3$ and system total disturbance $²d(t)$ under the measurable sampled-data $e_1(t_k)$.

In this paper, the observer and controller are designed according to the following assumptions [16], [37].

**Assumption 1:** The desired position trajectory $v(t) \in C^3$ and is bounded; $P_1$ and $P_2$ are both bounded by $P_3$ and $P_4$, i.e., $0 < P_r < P_1, P_2 < P_r$. Besides, the disturbances $d_1(t)$ and $d_2(t)$ are smooth enough and both bounded such that the new defined total disturbance $²d(t)$ and its first-order derivative $h(t) = ²d(t)$ are bounded by positive constants, i.e., $|²d(t)| \leq M_1, |h(t)| \leq M_2$.

The ASEHA system operates under practical physical limitation, which is BIBS (bounded input bounded state), so the above assumption is considerably reasonable.

**Assumption 2:** The function $\phi(e_2(t), e_3(t))$ is global Lipschitz with respect to $e_2$ and $e_3$, while there exist constants $l_1, l_2 > 0$ such that

\[
\begin{align*}
|\phi(e_2(t), e_3(t)) - \phi(\dot{e}_2(t), \dot{e}_3(t))| &\leq l_1 |e_2(t) - \dot{e}_2(t)| + l_2 |e_3(t) - \dot{e}_3(t)| \\
\end{align*}
\]

Note that the partial derivatives of $\phi(e_2(t), e_3(t))$ and $\frac{\partial \phi(e_2(t), e_3(t))}{\partial e_2}$ are both uniformly bounded. Hence, it follows from the global Lipschitz Lemma in [41] that the function $\phi(e_2(t), e_3(t))$ is global Lipschitz with respect to $e_2$ and $e_3$. See Appendix A for more details.

In error dynamic system (8), the total disturbance $²d(t)$ is regarded as an extended state $e_4(t)$, then the SDNLESO is constructed as

\[
\begin{align*}
\dot{e}(t) &= \tilde{A}_e \dot{e}(t) + \tilde{B}_e [\phi(\dot{e}_2(t), \dot{e}_3(t)) + r(\dot{v}(t), \dot{v}(t), \ddot{v}(t) + \dot{v}(t))] \\
&\quad + \tilde{B}_e ²b_n u(t) + \tilde{B}_e ²d_1(t), ²d_2(t)²d(t) \\
ξ(t) &= \dot{e}_2(t) + \dot{e}_3(t) + \dot{e}_4(t), \quad t \in [t_k, t_{k+1}] \\
ξ(t_{k+1}) &= e_1(t_{k+1})
\end{align*}
\]

As presented in (10), the SDNLESO is composed of a NLESO and an inter-sampled output predictor. $\dot{e}(t) = [\dot{e}_1(t), \dot{e}_2(t), \dot{e}_3(t), \dot{e}_4(t)]^T$ is the continuous-time estimate for the error vector $e(t)$. $ξ(t)$ denotes the continuous-time prediction for the system output, which serves as the input of the NLESO. Besides, the matrices $\tilde{A}_e$ and $\tilde{B}_e$ are given as

\[
\tilde{A}_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B}_e = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T.
\]

In (10), $ρ$ is a constant tuning parameter, and $k_i (i = 1, 2, 3, 4)$ are selected such that the following matrix is Hurwitz

\[
\Xi = \begin{bmatrix} -k_1 & 1 & 0 & 0 \\ -k_2 & 0 & 1 & 0 \\ -k_3 & 0 & 0 & 1 \\ -k_4 & 0 & 0 & 0 \end{bmatrix}.
\]
Additionally, the function $g_i(q) = |q|^\theta \text{sign}(q)$, $\theta_i = i\theta - (i-1), \theta \in (0, 1)$.

**Assumption 3:** The function $g_i(q)$ is assumed to be Lipschitz with Lipschitz constant $c_i$, then there is

$$|g_i(q_1(t)) - g_i(q_2(t))| \leq c_i |q_1(t) - q_2(t)|, \quad i = 1, 2, 3, 4$$

**Remark 3:** Unlike the traditional zero-order-hold (ZOH) method that discretizes the system dynamics and then make directly use of the sampled-data, the inter-sampled output predictor in (10) provides a continuous time output estimation by predicting the system output between two consecutive sampling instants and re-initializing it at each sampling instant $t = t_k$. In this sense, the inter-sample dynamic behavior of the controlled system is retained while the properties of continuous-time observer are able to be well maintained.

Based on the estimates acquired by SDNLESO, a compensated controller is designed such that an excellent tracking effect toward the reference signal $\nu(t)$ can be realized. The output feedback control law is synthesized as

$$u = \frac{1}{b_0} \left[ \sum_{j=1}^{3} \alpha_j \hat{\nu}(t) - \hat{\nu}_4(t) - \varphi(\hat{\nu}_2(t), \hat{\nu}_3(t)) \right] + r(\hat{\nu}(t), \hat{\nu}(t), \hat{\nu}(t))$$

(12)

where the parameters $\alpha_j (j = 1, 2, 3)$ are chosen such that the following matrix $A$ is Hurwitz,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}.$$

Now, we will state the main results of this paper.

**Theorem 1:** Suppose that the matrix $\Xi_e$ is Hurwitz, and Assumptions 1-3 are satisfied. If the sampling period $\tau \Delta \sup_k (t_{k+1} - t_k)$ meets $\tau < \tau_{\text{max}}$ with

$$\tau_{\text{max}} = \frac{\sqrt{2} \pi}{\rho^4 \sum c_i}$$

then there exist constants $\theta^* \in \left(\frac{3}{4}, 1\right)$ and $\rho^* > 1$ ($\rho^*$ will be discussed later) such that for any $\theta \in (\theta^*, 1)$ and $\rho > \rho^*$, (10) is a sampled-data observer for hybrid system (8), and the observer estimate errors satisfy

$$|\xi(t) - \tilde{\xi}(t)| \leq \Pi_1 \left(\frac{1}{\rho}\right)^{9-2i}, \quad i = 1, 2, 3, 4$$

(14)

for any $t > t_r$, where $t_r > 0$ is a $\rho$-dependent constant, and $\Pi_1$ is a $\rho$-independent positive constant. Additionally, there is $\lim_{\rho \to \infty i \to \infty} |\xi(t) - \tilde{\xi}(t)| = 0$.

**Theorem 2:** Considering the error dynamic system (8) with the sampled-data observer (10) and the output feedback control law (12), if matrix $A$ is Hurwitz, then the output tracking error of system (7) is exponentially convergent while satisfying

$$|e_1(t)| \leq \Pi_3 \exp^{-\epsilon_1(t-t_0)} + \frac{1}{\rho^2} \Pi_2$$

(15)

Namely, $\lim_{t \to \infty} |\eta(t)| = 0$, where $\epsilon_1$ is defined in (52), $\Pi_2$ and $\Pi_3$ are $\rho$-independent constants. This implies that the displacement tracking performance can be guaranteed.

**B. CONVERGENCE PROOF OF SDNLESO**

In what follows, we will establish the convergence of the proposed SDNLESO in the presence of discrete-time measurement signal. Meanwhile, the maximum allowable sampling period is derived theoretically, which serves as the basis of sensor sampling frequency selection in practical applications.

In view of the error dynamic system (8) and its SDNLESO (10), let

$$\dot{\hat{\xi}}(t) = e_i(t) - \hat{\xi}(t), \quad i = 1, 2, 3, 4$$

$$\eta(t) = \rho^{9-2i} \hat{\eta}(t)$$

$$\eta(t) = \rho^2 \hat{\eta}(t) - \epsilon_i(t)$$

$$\eta(t) = [\eta_1(t) \eta_2(t) \eta_3(t) \eta_4(t)]^T$$

(16)

Then the weighted error system can be written as

$$\dot{\eta}(t) = \rho^2 G(\eta(t), \eta(t)) + \rho^3 \Psi(\hat{\nu}_2(t), \hat{\nu}_3(t)) + \rho \Delta(t)$$

$$\dot{\eta}(t) = -\rho^2 \eta_2(t), \quad t \in [t_k, t_{k+1})$$

$$\eta(t) = 0$$

(17)

where

$$G(\eta(t), \eta(t)) = \begin{bmatrix} \eta_2(t) - k_{g1} \eta_1(t) + 1(t) \\ \eta_2(t) - k_{g2} \eta_1(t) + 1(t) \\ \eta_2(t) - k_{g3} \eta_1(t) + 1(t) \\ -k_{g4} \eta_1(t) + 1(t) \end{bmatrix}$$

$$\Delta(t) = \begin{bmatrix} \eta_2(t) - \hat{\nu}_2(t) & \hat{\nu}_3(t) \end{bmatrix}$$

$$\tilde{\eta}(t) = 0$$

Before proceeding, we need some preliminary lemmas. First, define the following vector field

$$F_0(\tilde{z}) = \begin{bmatrix} F_{01}(\tilde{z}) & F_{02}(\tilde{z}) & F_{03}(\tilde{z}) & F_{04}(\tilde{z}) \end{bmatrix}^T$$

$$F_{0j}(\tilde{z}) = \tilde{z}_{j+1} - k_{gj}(\tilde{z}_1), \quad j = 1, 2, 3$$

$$F_{04}(\tilde{z}) = -k_{g4}(\tilde{z}_1), \quad \tilde{z} = [\tilde{z}_1 \tilde{z}_2 \tilde{z}_3 \tilde{z}_4]^T$$

(18)

It can be verified that $F_0(\tilde{z})$ is homogeneous of degree $\chi = 0 \rightarrow 1$ with respect to the weights $\{\nu(t) = (i-1)\theta - (i-2)t\}_{i=1}^{[42]}$. Then there split into several lemmas for vector field $F_0(\tilde{z})$. 
Lemma 1: If $\mathbb{E}^\theta$ is Hurwitz, for any $\theta \in (\theta^+, 1)$, $\theta^+ \in (\frac{3}{2}, 1)$, the system $\hat{\mathbb{F}} = F_\theta(\hat{z})$ is finite-time stable [43]. Meanwhile, it follows from [32], [44] that there exist a positive definite, and radially unbounded Lyapunov function $V_\theta(\hat{z})$, which is homogeneous of degree $\lambda > 1$ with respect to the weights $\{v_i\}_{i=1}^4$. Besides, the Lie derivative of $V_\theta(\hat{z})$ along with the vector $F_\theta(\hat{z})$ is negative definite.

Lemma 2: If Lemma 1 is satisfied, it can conclude from [31], [33], [42] that $\frac{\partial V_\theta(\hat{z})}{\partial \hat{z}}$ and $L_{F_\theta} V_\theta(\hat{z})$ are homogeneous of degrees $\lambda - v_i$ and $\lambda + \kappa$ with respect to the weights $\{v_i\}_{i=1}^4$. Furthermore, we can get that

$$\frac{\partial V_\theta(\hat{z})}{\partial \hat{z}} \leq \tilde{B}_1(V_\theta(\hat{z}))^{\frac{\lambda-v_i}{\lambda}} + \tilde{B}_2(V_\theta(\hat{z}))^{\frac{\lambda}{\lambda}}$$

$$L_{F_\theta} V_\theta(\hat{z}) \leq -\tilde{B}_3(V_\theta(\hat{z}))^{\frac{\lambda+\kappa}{\lambda}}$$

(19)

where $\tilde{B}_j > 0$ ($j = 1, 2, 3$) are positive constants.

Note that the weighted error system is associated with the state estimation errors and the output predicted error, so the Lyapunov function $V_1(t)$ is chosen as

$$V_1(\eta(t), \eta_\xi(t)) = V_\theta(\eta(t)) + V_L(\eta_\xi(t))$$

(20)

in which $V_\theta(\eta(t))$ is chosen satisfying Lemma 1, whose form can refer to the literature [33]. $V_L(\eta_\xi(t)) = \kappa \phi(t)\eta_\xi^2(t)$, herein, $\kappa$ is a positive constant, $\phi(t)$ is a bounded positive function for correcting the output predicted error, which is given by

$$\begin{cases} 
\phi(t) < 0, & t \in [t_k, t_k+1) \\
\phi(t_k) = \varepsilon, & \forall k \in N \\
\phi(t_k + \tau_{max}) = e^{-1}, & \varepsilon > 1
\end{cases}$$

(21)

where $\tau_{max}$ is the maximum sampling interval.

For Lyapunov function $V_1(\eta(t), \eta_\xi(t))$, the initial value

$$V_1(\eta(0), \eta_\xi(0)) = V_\theta(\eta(0)) + V_L(\eta_\xi(0)) \Delta \tilde{M}_\eta$$

(22)

Define the following compact sets

$$\mathcal{H}_1 = \left\{ (\eta(t), \eta_\xi(t)) \in \mathbb{R}^2 \mid V_1(\eta(t), \eta_\xi(t)) \leq \tilde{M}_\eta \right\}$$

(23)

$$\mathcal{H}_2 = \left\{ \eta(t) \in \mathbb{R}^2 \mid V_\theta(\eta(t)) \leq \tilde{M}_\eta \right\}$$

(24)

It is easy to see that $(\eta(0), \eta_\xi(0)) \in \mathcal{H}_1$. Without loss of generality, we assume $\mathcal{H}_2 \subset \mathcal{H}_1$, $\mathcal{H}_1 - \mathcal{H}_2 \neq \emptyset$. Then we will prove that if $(\eta(t), \eta_\xi(t))$ leaves from $\mathcal{H}_1 - \mathcal{H}_2$, it will enter into compact set $\mathcal{H}_2$ in finite time.

Taking the derivative of $V_\theta(\eta(t))$ between $[t_k, t_k+1)$, we can obtain

$$\frac{dV_\theta(\eta(t))}{dt} = \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \dot{\eta}_i(t)$$

$$= \rho^2 \left( \sum_{i=1}^3 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \left[ \eta_{i+1}(t) - k_i \left( \eta_1(t) + \eta_\xi(t) \right) \right] \right)$$

$$+ \rho \frac{\partial V_\theta(\eta(t))}{\partial \eta_4} \left( \eta_4(t) - \eta_\xi(t) \right)$$

$$+ \rho^2 \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \left[ \eta_i(t) - g_i \left( \eta_1(t) + \eta_\xi(t) \right) \right]$$

$$+ \rho^2 \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \left( \eta_i(t) - \eta_\xi(t) \right)$$

$$+ \rho^2 \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \left( \eta_i(t) - \eta_\xi(t) \right)$$

(25)

According to Assumptions 1-3 and Lemma 2, (25) can be rewritten as

$$\frac{dV_\theta(\eta(t))}{dt} \leq -\rho^2 \tilde{B}_3(\eta(t))^{\frac{\lambda+\kappa}{\lambda}} + \rho \tilde{B}_1 M_2(\eta(t))^{\frac{\lambda-v_i}{\lambda}}$$

$$+ \rho^2 \left( \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} k_i \left[ g_i \left( \eta_1(t) - \eta_\xi(t) \right) \right] \right)$$

$$+ \rho^2 \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \left( \eta_i(t) - \eta_\xi(t) \right)$$

(26)

If $(\eta(t), \eta_\xi(t)) \in \mathcal{H}_1 - \mathcal{H}_2$, according to $v_1 > v_2 > v_3 > v_4$, (26) is derived as

$$\frac{dV_\theta(\eta(t))}{dt} \leq -\rho^2 \tilde{B}_3(\eta(t))^{\frac{\lambda+\kappa}{\lambda}} + \rho \tilde{B}_1 M_2(\eta(t))^{\frac{\lambda-v_i}{\lambda}}$$

$$+ 2\tilde{B}_1^2 \left( \sum_{i=1}^4 \eta_i(t) \right) \frac{v_4}{v_3} + \rho^2 \left( \sum_{i=1}^4 \frac{\partial V_\theta(\eta(t))}{\partial \eta_i} \eta_i(t) \right)$$

$$+ \tilde{B}_1 \tilde{B}_2 \left( l_1 + l_2 \right) \left( \eta(t) \right)$$

(27)

On the other hand, the derivative of $V_L(\eta_\xi(t))$ between $[t_k, t_k+1)$ is calculated and gives

$$\dot{V}_L(\eta_\xi(t)) = \kappa \phi(t) \eta_\xi^2(t) + 2\kappa \phi(t) \eta_\xi(t) \eta_\xi(t)$$

$$= \kappa \phi(t) \eta_\xi^2(t) - 2\kappa \rho^2 \phi(t) \eta_\xi(t) \eta_\xi(t)$$

(28)

Choosing

$$\dot{\phi}(t) = -\kappa \left( \rho^4 \phi^2(t) + 1 \right), \quad t \in [t_k, t_k+1)$$

(29)

leads to

$$\dot{V}_L(\eta_\xi(t)) = -\left( \kappa \rho^2 \phi(t) \eta_\xi(t) + \eta_\xi(t) \right)^2$$

$$+ \eta_\xi^2(t) - \kappa^2 \eta_\xi^2(t)$$

(30)

In terms of (27) and (30), we have

$$\dot{V}_1(\eta(t), \eta_\xi(t)) \leq -\rho^2 \tilde{B}_3(\eta(t))^{\frac{\lambda+\kappa}{\lambda}} + 2\tilde{B}_1^2 \left( \sum_{i=1}^4 \eta_i(t) \right) \frac{v_4}{v_3} + \rho \tilde{B}_1 M_2(\eta(t))^{\frac{\lambda-v_i}{\lambda}}$$

$$+ \tilde{B}_1 \tilde{B}_2 \left( l_1 + l_2 \right) \left( \eta(t) \right)$$

$$- \kappa^2 \frac{\rho^4}{8} \left( \sum_{i=1}^4 \eta_i(t) \right)^2 \left| \eta(t) \right|^2$$

(31)
From Lemma 2, it is easy to obtain that for any \( \rho \geq \left[ \frac{16\bar{B}_1^2}{B_3} (V_0 (\eta(t))) \right]^{\frac{1}{2}} \), then there is
\[
2\bar{B}_1^2 (V_0 (\eta(t)))^{\frac{2\lambda - \nu_4}{\lambda}} \leq \frac{1}{8} \rho^2 B_3 (V_0 (\eta(t)))^{\frac{1}{2}} + \frac{\rho^4}{8} \left( \sum_{i=1}^{4} k_i c_i \right)^2 |\eta_\xi(t)|^2
\]
(32)
Similarly, if
\[
\rho \geq \max \left\{ 1, \left[ \frac{16\bar{B}_1^2}{B_3} (V_0 (\eta(t))) \right]^{\frac{1}{2}} \right\},
\]
we derive
\[
\dot{V}_1 (\eta(t), \eta_\xi(t)) \\
\leq -\frac{1}{2} \rho^2 \bar{B}_3 (V_0 (\eta(t)))^{\frac{1}{2}} + \left( \kappa^2 - \frac{\rho^4}{8} \left( \sum_{i=1}^{4} k_i c_i \right)^2 \right) |\eta_\xi(t)|^2
\]
(34)
Since \((\eta(t), \eta_\xi(t))\) leaves from \(\mathcal{H}_2 - \mathcal{H}_1\), we can obtain that for any \( \rho > \rho_1^* \)
\[
\rho_1^* = \left\{ 1, \frac{4\bar{B}_1}{\sqrt{B_3}} \bar{M}_\eta^{\frac{1}{2} - \frac{2}{\nu_4} - \lambda}, \sqrt{\frac{8}{B_3}} \bar{B}_1 (l_1 + l_3) \bar{M}_\eta^{\frac{1}{2} - \frac{2}{\nu_4} - \lambda}, \frac{8\bar{B}_1^2}{B_3} (V_0 (\eta(t)))^{\frac{1}{2}} \right\}
\]
(35)
the inequality (34) apparently holds.
Choosing
\[
\kappa = \sqrt{\frac{\rho^4}{8} \left( \sum_{i=1}^{4} k_i c_i \right)^2 + \epsilon}
\]
(36)
where \( \epsilon \) is a sufficiently small positive constant, then we can obtain
\[
\dot{V}_1 (\eta(t), \eta_\xi(t)) \leq -\frac{1}{2} \rho^2 \bar{B}_3 (V_0 (\eta(t)))^{\frac{1}{2}} - \epsilon |\eta_\xi(t)|^2 < 0
\]
(37)
Inequality (37) means that \( V_1 (\eta(t), \eta_\xi(t)) \) strictly decreases as \( t \) increases when \((\eta(t), \eta_\xi(t)) \in \mathcal{H}_1 - \mathcal{H}_2\). Thus we can get that there exists a constant \( t_p > 0 \) such that
\[
\{(\eta(t), \eta_\xi(t)) \mid t \in [t_p, \infty) \} \subset \mathcal{H}_2, \forall \rho > \rho_1^*
\]
(38)
Then (26) can be deduced into
\[
\frac{dV_0 (\eta(t))}{dt} \leq -\rho^2 \bar{B}_3 (V_0 (\eta(t)))^{\frac{1}{2}} + 2\bar{B}_1^2 + (l_1 + l_2) V_0 (\eta(t)) (\eta(t)))^{\frac{1}{2}} - \epsilon |\eta_\xi(t)|^2
\]
(39)
Combining with (30), we obtain
\[
\dot{V}_2 (\eta(t), \eta_\xi(t)) \\
\leq -\rho^2 \bar{B}_3 (V_0 (\eta(t)))^{\frac{1}{2}} + 2\bar{B}_1^2 (V_0 (\eta(t)))^{\frac{2\lambda - \nu_4}{\lambda}} + \rho^4 \sum_{i=1}^{4} k_i c_i |\eta_\xi(t)|^2
\]
(40)
Due \((\eta(t), \eta_\xi(t)) \in \mathcal{H}_2\), it follows from (40) that for any \( \rho > \rho_2^* \)
\[
\rho_2^* = \left\{ 1, \frac{4\bar{B}_1}{\sqrt{B_3}}, \frac{8\bar{B}_1^2}{B_3}, \sqrt{\frac{8\bar{B}_1^2 (l_1 + l_2)}{B_3}}, \sqrt{\frac{8}{B_3}} \right\}
\]
(41)
there is
\[
\dot{V}_1 (\eta(t), \eta_\xi(t)) \leq -\frac{1}{2} \rho^2 \bar{B}_3 (V_0 (\eta(t)))^{\frac{1}{2}} - \epsilon |\eta_\xi(t)|^2 - \left( \kappa^2 - \frac{\rho^4}{8} \left( \sum_{i=1}^{4} k_i c_i \right)^2 \right) |\eta_\xi(t)|^2
\]
(42)
By (36), the inequality (37) still holds. As seen, \( \dot{V}_1 (\eta(t), \eta_\xi(t)) \) is negative definite for any \( \rho > \rho^* \)
\[
\rho^* = \max \{ \rho_1^*, \rho_2^* \}
\]
(43)
while \( V_0 (\eta(t)) \) and \( V_1 (\eta_\xi(t)) \) are both positive. Then it can be concluded that \( V_0 (\eta(t)) \to 0, V_1 (\eta_\xi(t)) \to 0 \) as \( t \to \infty \) for any initial value \((\eta(0), \eta_\xi(0))\). Obviously, there exist positive constants \( \rho^*, \Pi_1 \) and \( t_r > t_p \) satisfy
\[
V_0 (\eta(t)) \leq \Pi_1, \forall \rho > \rho^*, \forall t > t_r
\]
(44)
It together with (16) and (19) yields
\[
|e_i (t) - \hat{e}_i (t)| \leq \Pi_1 \left( \rho \right)^{9 - 2i}, \quad i = 1, 2, 3, 4
\]
(45)
which implies that the observer estimated errors can fast converge to zero for any \( t > t_r \) as long as \( \rho \) is large.
For computing the maximum allowable sampling time \( \tau_{\max} \), we take the integral of (29) between \( t_k \) and \( t_k + \tau_{\max} \) with \( t \to 0 \).
\[
\tau_{\max} = \lim_{t \to 0} \frac{1}{\rho} \left[ \frac{\rho}{\kappa \rho^2} \left( \frac{\pi}{\rho^2} \right) \right]
\]
\[ \dot{e}(t) = \mathcal{A}e(t) + B\tilde{\varphi}(\tilde{e}_2(t), \tilde{e}_3(t)) \]
\[ + B\left(-\sum_{i=1}^{3} \alpha_i \tilde{e}_i(t) + \tilde{e}_4(t)\right) \]  
(47)

where \( \mathcal{A} \) have been presented in previous section, and \( B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \).

Consider the Lyapunov candidate \( V_2(t) = e^T(t)P e(t) \), in which \( P \) is the positive definite solution to Lyapunov equation

\[ \mathcal{A}^TP + P\mathcal{A} = -Q \]
(48)

and \( Q \) is a positive definite matrix.

The time derivative of \( V_2(t) \) gives

\[ \dot{V}_2(t) = -e^T(t)Q e(t) + 2e^T(t)PB\tilde{\varphi}(\tilde{e}_2(t), \tilde{e}_3(t)) \]
\[ + 2e^T(t)PB\left(-\sum_{i=1}^{3} \alpha_i \tilde{e}_i(t) + \tilde{e}_4(t)\right) \]
\[ \leq -\lambda_{\min}(Q)\|e(t)\|^2 + 2\lambda_{\max}(P)\|e(t)\| \]
\[ \times (l_1|\tilde{e}_2(t)| + l_2|\tilde{e}_3(t)|) \]
\[ + 2\lambda_{\max}(P)\|e(t)\|\left(\sum_{i=1}^{3} \alpha_i |\tilde{e}_i(t)| + |\tilde{e}_4(t)|\right) \]  
(49)

where \( \lambda_{\max}(\cdot) \) and \( \lambda_{\min}(\cdot) \) denote the maximum and minimum eigenvalue of matrix \( \cdot \).

By (45), there exist positive constants \( \rho^*, \Gamma_i \) and \( T \) such that for any \( \rho > \rho^* \)

\[ |\tilde{e}_i(t)| \leq \frac{\Gamma_i}{\rho}, \quad \forall t > T, \quad i = 1, 2, 3, 4 \]  
(50)

Then (49) can be involved into

\[ \dot{V}_2(t) \leq -\lambda_{\min}(Q)\|e(t)\|^2 + 2\lambda_{\max}(P)\frac{\Gamma}{\rho}\|e(t)\| \]
\[ \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V_2(t) + 2\lambda_{\max}(P)\frac{\Gamma}{\rho}\sqrt{\frac{V_2(t)}{\lambda_{\min}(P)}} \]  
(51)

in which \( \Gamma = \frac{2}{i=1} l_i \Gamma_{i+1} + \frac{3}{i=1} \alpha_i \Gamma_i + \Gamma_4 \).

It follows that

\[ \frac{d}{dt}\sqrt{V_2(t)} \leq -\epsilon_1\sqrt{V_2(t)} + \frac{\Gamma}{\rho}\sqrt{\frac{V_2(t)}{\lambda_{\min}(P)}} \]  
(52)

with \( \epsilon_1 = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \epsilon_2 = \frac{\lambda_{\max}(P)}{\rho\epsilon_1\sqrt{\lambda_{\min}(P)}} \).

Solving the differential equation (52), then there is

\[ \|e(t)\| \leq \left(\frac{V_2(t_0)}{\lambda_{\min}(P)} - \frac{\epsilon_2\Gamma}{\rho\epsilon_1\sqrt{\lambda_{\min}(P)}}\right) \exp^{-\epsilon_1(t-t_0)} \]
\[ + 1 - \frac{\epsilon_2\Gamma}{\rho\epsilon_1\sqrt{\lambda_{\min}(P)}} \]  
(53)

Note that if \( t \to \infty \), the first term will diminish. The ultimate tracking errors \( e_1(t) \) will converge toward a ball whose radius depends on the SDNLESO estimate errors and the control parameter \( \rho \). By taking \( \Pi_2 = \frac{\epsilon_1\Gamma}{\epsilon_1\sqrt{\lambda_{\min}(P)}}, \Pi_3 = \sqrt{\frac{V_2(t_0)}{\lambda_{\min}(P)}} - \frac{\epsilon_2\Gamma}{\rho\epsilon_1\sqrt{\lambda_{\min}(P)}} \), the assertion (15) of Theorem 2 is proven.

Remark 4: In view of (45), (46) and (53), we can easily see that increasing the control parameter \( \rho \) will lead to a reduced maximum sampling period \( \tau_{\max} \). Meanwhile, a large \( \rho \) is conductive to decreasing the observer estimate errors \( \tilde{e}_i(t) \) and the final tracking error \( e_1(t) \). However, excessive \( \rho \) may result in undesirable chattering phenomenon thence to deteriorate the control effect. Therefore, an appropriate control parameter \( \rho \) should be selected for trading off the tracking performance and the maximum sampling interval.

Remark 5: We should mention that, if the practical sampling period is sufficiently small, it is not necessary to consider the sampled-data influence. In this way, the observer and controller design procedures in continuous time domain are still applicable.

**IV. EXPERIMENTS AND RESULTS**

**A. EXPERIMENT SETUP**

To demonstrate the performance of the proposed control method, an experimental test platform is set up and photographed in Fig.2. The test platform is comprised of three interconnected subsystems; that is, hydraulic, mechanical, and electric control parts. The hydraulic part mainly includes a set of high-pressure pump, a relief valve, a servo-valve, an asymmetrical hydraulic cylinder, and other auxiliary components. An actuator (i.e., the asymmetric hydraulic cylinder) and its supporting mass, which simulates the vehicle body weight, make up the mechanical subsystem. Additionally, the electronic control part consists of an industrial computer, in which an A/D card and a D/A card are plugged, a signal converter, and a displacement sensor that is installed inside the hydraulic cylinder. The specifications of the above hardware components are provided in Table 1. In the constructed ASEHA system, the actuator displacement can be controlled by offering proper hydraulic oil flow to the cylinder chambers. For this intention, the computer needs to calculate specific command signal using only the displacement measurement signal received from the A/D card. Then, the control signal is sent from the D/A card and transformed by the
signal converter so as to drive the servo-valve. Based on the above test platform, the nominal values of the main system parameters are listed in Table 2.

From the perspective of engineering experience, it is difficult to guarantee the displacement tracking quality of EHA systems if the actuator area ratio is more than 3.0. However, it is worth noting that the experimental actuator area ratio is $A_1/A_2 \approx 4.5$. In what follows, we will conduct experiments on this test platform to verify the properties of the proposed control scheme.

### B. PERFORMANCE INDEXES

The following three performance indices, i.e., the maximum, average, as well as standard deviation of the tracking errors are employed to evaluate the tracking ability of the proposed control approach. Their specific definitions are presented as follows [13], [16].

1) Maximum absolute value of the tracking errors is defined as
   \[
   M_e = \max_{i=1,\ldots,N} |e_1(i)| \tag{54}
   \]
   where $N$ is the number of the recorded digital signals. It is used to assess the transient tracking accuracy.

2) Average tracking error is defined as
   \[
   \mu = \frac{1}{N} \sum_{i=1}^{N} |e_1(i)| \tag{55}
   \]
   which is used as an objective numerical measure of average tracking performance.

3) Standard deviation of the tracking error is defined as
   \[
   \sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (|e_1(i)| - \mu)^2} \tag{56}
   \]
   which is utilized to evaluate the deviation level of the tracking errors.

### C. EFFECTS OF CONTROL PARAMETER

In this section, we investigate the influence of control parameter on the displacement tracking performance. The sinusoidal signal is considered as the reference trajectory and the actuator initial displacement is set as $0.015$ m. Through repeated experiment debugging, the related parameters of SDNLESO-based controller are selected as $[k_1, k_2, k_3, k_4] = [4, 6, 4, 1]$, $\theta = 0.9$, $[\alpha_1, \alpha_2, \alpha_3] = [-19800, -1600, -3]$. In accordance with the theorems presented above, we find that the control parameter $\rho$ yields an important impact on the tracking effect. For comparison, the tracking error curves of $\rho = 2, 3.5, 5, 6$ under the same sampling period $\tau = 7$ ms are depicted.
in Fig.3(a)-(d), at the same time, their transient responses are amplified. In order to display the final tracking accuracy more clearly, three cycles’ steady-state tracking errors are compared in Fig.3(e). In addition, the performance indexes during the three cycles are collected in Table 3.

As shown in the enlarged views of Fig.3(a)-(c), the initial error peak values and convergence rates for the three different control parameters are 0.0414 m, 0.0260 m, 0.0189 m and 0.85 s, 0.6 s, 0.5 s. It is clear that the transient tracking performance has improved significantly as $\rho$ increases from 2 to 5. After carefully observing the final tracking errors from Fig.3(a)-(c) and Fig. 3(e), we see that the steady-state performance of $\rho = 3.5$ is better than $\rho = 2$, but not as good as $\rho = 5$, and the performance indexes in Table 3 show the same results. Thus, it can be concluded that a reasonably

![Figure 3](image-url)

**FIGURE 3.** Tracking errors for different control parameters $\rho$. (a) Tracking errors for $\rho = 2$. (b) Tracking errors for $\rho = 3.5$. (c) Tracking errors for $\rho = 5$. (d) Tracking errors for $\rho = 6$. (e) Detailed tracking errors comparisons for different parameters.

**TABLE 3.** Performance indexes of different control parameters.

| Indexes | $M_e$ (mm) | $\mu$ (mm) | $\sigma$ (mm) |
|---------|------------|------------|---------------|
| $\rho = 2$ | 0.0230     | 0.0079     | 0.0054        |
| $\rho = 3.5$ | 0.0182     | 0.0065     | 0.0042        |
| $\rho = 5$ | 0.0131     | 0.0052     | 0.0030        |
| $\rho = 6$ | 0.0154     | 0.0060     | 0.0039        |
larger control parameter will result in faster convergence speed and smaller steady-state error, which further validates the statements in Theorem 2. However, if the control parameter continuously increases to $ρ = 6$, as shown in Fig.3(d) and Table 3, an obvious deterioration occurs in the transient process when compared with the results of $ρ = 5$; meanwhile, the maximum absolute value $M_e$, average tracking error $μ$, and standard deviation $σ$ of the steady-state stage also exhibit a slight increase. This is just because the oversized $ρ$ will bring about a chattering phenomenon. Therefore, the value of $ρ$ that is either too small or too large will degrade the control effect.

D. EFFECTS OF SAMPLING PERIOD

To demonstrate the effect of sampling interval on the control performance, experiments are conducted using the proposed method under $τ = 7$ ms, 8 ms, 9 ms, respectively. The control parameter is fixed as $ρ = 5$, and the other design parameters are chosen the same as Sect. IV-C. The corresponding tracking errors are shown in Fig.4(a)-(c), and three cycles’ steady-state tracking errors curves are depicted in Fig.4(d), whose performance indexes are provided in Table4. In addition, Fig.5 presents the control inputs of different sampling intervals.

| Indexes | $M_e$ (mm) | $μ$ (mm) | $σ$ (mm) |
|---------|------------|----------|----------|
| $τ = 7$ ms | 0.0131 | 0.0052 | 0.0030 |
| $τ = 8$ ms | 0.0142 | 0.0063 | 0.0038 |
| $τ = 9$ ms | 0.0176 | 0.0078 | 0.0047 |

It can be seen from Fig.4(a)-(c) that the initial error peak values for $τ = 7$ ms, 8 ms, and 9 ms are 0.0189 m, 0.0235 m, and 0.0305 m, respectively; thus, the transient tracking performance of $τ = 7$ ms is better than that of the other two sampling periods. Fig.4(d) indicates that the final tracking performance degrades obviously as $τ$ increases, and the quantized values in Table 4 further confirm this result. What’s more, Fig.5 shows that the oscillations of the control input have become increasingly intense with the augmentation of the sampling interval. As drawn in Fig.5(c), when the sampling period $τ$ increases to 9 ms, severe control chattering occurs; at the same time, the experimental process is accompanied by considerable vibration noise. Therefore, we can deduce that the practical allowable maximum sampling period is approximately 9 ms. There is a slight deviation between the actual test result and the theoretical calculation ($τ_{max} = 7.48$ ms) acquired by formula (13). Nevertheless, the result still provides important guiding significance for the sampling period selection in real applications.
Figure 5. Control inputs for different sampling periods. (a) Control inputs for $\tau = 7$ ms. (b) Control inputs for $\tau = 8$ ms. (c) Control inputs for $\tau = 9$ ms.

E. COMPARATIVE RESULTS OF DIFFERENT CONTROL METHODS

To verify the superiority of the proposed control method, the SDNLESO-based controller (SDNLESO) is compared with traditional PI controller and SDLESO-based controller (SDLESO) introduced in [8], [15]. The sampling interval is selected as $\tau = 7$ ms. The control parameters of SDNLESO are set as $\rho = 5$, and the other design parameters are chosen the same as Sect. IV-C. The tuning parameters of SDLESO are given as $[k_1, k_2, k_3, k_4] = [4, 6, 4, 1]$, $[\alpha_1, \alpha_2, \alpha_3] = [-14050, -1100, -10]$ and the bandwidth gain is taken as $\omega_0 = 18$. The PI controller gains are $k_p = 180$, $k_i = 0.075$.

Figure 6. Tracking errors for different control methods. (a) Tracking errors for SDNLESO. (b) Tracking errors for SDLESO. (c) Tracking errors for PI. (d) Detailed tracking errors comparisons for different control method.
Fig.6(a)-(c) show the tracking errors of the three controllers. In addition, more intuitive steady-state error contrast curves are shown in Fig.6(d), and the corresponding performance indexes are collected in Table 5. As shown in Fig.6(a)-(d), the final steady-state tracking errors of the SDNLESO and SDLESO are less than those of the PI controller. Meanwhile, it can be seen from Table 5 that the steady-state maximum absolute value $M_e$, average tracking error $\mu$, and standard deviation $\sigma$ of the SDLESO are reduced by 39.57%, 50.76%, and 38.24% compared to the PI controller, whereas the reductions of the SDNLESO are 52.89%, 60.60%, and 55.88%, respectively. Thus, the proposed controller can obtain better steady-state performance than the SDLESO. In addition, the enlarged views in Fig.6(a)-(b) indicate that the transient performance of the SDNLESO is superior to that of the the SDLESO since it produces a smaller initial peak value and faster convergence speed. These findings suggest that the SDNLESO can achieve a more favorable control effect than the SDLESO and PI controllers.

Moreover, Fig.7 presents the control inputs for the three controllers. It is clear that the input currents of the PI controller possess smaller magnitudes while with smaller oscillations. To a certain extent, these factors explain the dominance of the PID in control field. Nevertheless, this is not sufficient to remedy the defect of low control accuracy. As shown in Fig.7(a)-(b), although the input amplitudes of the SDLESO are similar to those of the SDNLESO, its curve oscillations are considerably more severe. Meanwhile, from the perspective of the experimental phenomenon, the SDLESO generates more serious vibration sounds. These results further confirm the merits of the proposed control strategy even in the presence of such a large actuator area ratio.

V. CONCLUSION

In this study, a novel SDNLESO-based output feedback control scheme was proposed for ASEHA systems subjected to complex nonlinearities, matched and mismatched modeling uncertainties. Considering the discrete nature of the system output signal, a SDNLESO was constructed to continuously estimate the unmeasurable states and total disturbances of the new constructed error dynamic system. A compensated control law was further synthesized to guarantee the strong robustness and the displacement tracking performance based on the obtained estimates. By using the geometric homogeneity theory and the Lyapunov theory, the SDNLESO convergence and the whole closed-loop system stability were systematically proven. Finally, extensive practical experiments were conducted on the ASEHA system test platform, which verified the applicability and effectiveness of the proposed method despite the rather large actuator area ratio.

Our future works will consider 1) applying the proposed control scheme to actual vehicle attitude adjustment; 2) combining the novel SDNLESO with the other advanced control technologies such as sliding mode, or neural network in order to further improve the position tracking quality.

APPENDIX A

Proof of Assumption 2: Taking the partial derivatives of $\varphi(e_2(t), e_3(t))$ with respect to $e_2$ and $e_3$, respectively,
we obtain
\[
\frac{\partial \varphi (e_2(t), e_3(t))}{\partial e_2} = -\alpha_n + \gamma_n^2
\]
(57)
and
\[
\frac{\partial \varphi (e_2(t), e_3(t))}{\partial e_3} = -(\beta_n + \gamma_n)
\]
(58)

Since \( \gamma_n, \alpha_n, \) and \( \beta_n \) are all nominal constants, \( \frac{\partial \varphi (e_2(t), e_3(t))}{\partial e_2} \) and \( \frac{\partial \varphi (e_2(t), e_3(t))}{\partial e_3} \) are both uniformly bounded. According to the global Lipschitz Lemma in [41], we can conclude that \( \varphi (e_2(t), e_3(t)) \) is global Lipschitz with respect to \( e_2 \), and \( \varphi (e_2(t), e_3(t)) \) is also global Lipschitz with respect to \( e_3 \), while satisfying
\[
\begin{align*}
|\varphi (e_2(t), e_3(t)) - \varphi (\hat{e}_2(t), e_3(t))| & \leq l_1 |e_2(t) - \hat{e}_2(t)| \\
|\varphi (\hat{e}_2(t), e_3(t)) - \varphi (\hat{e}_2(t), \hat{e}_3(t))| & \leq l_2 |e_3(t) - \hat{e}_3(t)|
\end{align*}
\]
(59)
(60)
in which \( l_1 \) and \( l_2 \) are the Lipschitz constants.

From (59) and (60), we have
\[
|\varphi (e_2(t), e_3(t)) - \varphi (\hat{e}_2(t), e_3(t))| \\
= |\varphi (e_2(t), e_3(t)) - \varphi (\hat{e}_2(t), e_3(t)) + \varphi (\hat{e}_2(t), e_3(t)) - \varphi (\hat{e}_2(t), e_3(t))| \\
\leq |\varphi (e_2(t), e_3(t)) - \varphi (\hat{e}_2(t), e_3(t))| + |\varphi (\hat{e}_2(t), e_3(t)) - \varphi (\hat{e}_2(t), \hat{e}_3(t))| \\
\leq l_1 |e_2(t) - \hat{e}_2(t)| + l_2 |e_3(t) - \hat{e}_3(t)|
\]
(61)

Thus, the function \( \varphi (e_2(t), e_3(t)) \) is global Lipschitz with respect to \( e_2 \) and \( e_3 \).

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