Q-operators are 't Hooft lines

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ABSTRACT: We study 't Hooft lines in four-dimensional holomorphic-topological Chern-Simons theory. We relate them to Q-operators in the theory of integrable systems. We give a physical interpretation of the fundamental TQ and QQ relations satisfied by Q-operators and conventional transfer matrices.
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1 Introduction

Four-dimensional Chern-Simons theory [1–3] is a unified approach to studying integrability which has been able to explain many examples of integrable systems: spin chains [1, 2], integrable field theories [4–8], spin chains with boundary [9, 10], and integrable string theories [11].

In this paper we generalize the analysis of [2, 3] relating integrable spin chains and four-dimensional Chern-Simons theory to include an ingredient previously missing: the Q-operator. Before we state the problem we solve, let us review how four-dimensional Chern-Simons theory is related to integrable spin chains.

Consider the four-manifold $\mathbb{R}^2 \times \mathbb{C}$ with coordinates $x, y$ on $\mathbb{R}^2$ and holomorphic coordinate $z$ on $\mathbb{C}$. The fundamental field of four-dimensional Chern-Simons theory is a gauge-field $A$ with three components: $A_z, A_x, A_y$. The Lagrangian is

$$\int \text{CS}(A) dz,$$

where $\text{CS}(A)$ is the Chern-Simons three-form. The gauge field $A$ is required to tend to 0 as $z \to \infty$.

The XXX spin chain systems are generated in this setup as follows. Take the gauge group to be $SL_2(\mathbb{C})$, and make one of the two topological directions, say $y$, periodic. Insert $N$ fundamental Wilson lines wrapping the $x$-direction. We can take all the fundamental Wilson lines to live at $z = 0$. The Hilbert space of the spin system is $(\mathbb{C}^2)^{\otimes N}$, which is the tensor product of the spaces of states living at the end of each Wilson line.\footnote{For this analysis, it is important to use the boundary condition that $A \to 0$ as $z \to \infty$. This breaks the gauge symmetry and makes the effective two-dimensional theory on the $x$-$y$ plane obtained by integrating out the gauge field massive.}

The transfer matrix $T(z)$ of the spin chain is realized by also placing a fundamental Wilson line wrapping the (periodic) $y$-direction, at some point $z$.

An important ingredient in the theory of integrable models was missing from the analysis of [2, 3]: Baxter’s Q-operator [12], first introduced in his seminal analysis of the 8-vertex model. The Q-operators $Q_{\pm}(z)$ are the two linearly independent solutions to Baxter’s TQ relation

$$T(z, \phi)Q_{\pm}(z, \phi) = (z - \frac{i}{2} \hbar)^N Q_{\mp}(z + \hbar, \phi) + (z + \frac{i}{2} \hbar)^N Q_{\pm}(z - \hbar, \phi).$$

The Q-operators also commute with the transfer matrix:

$$[Q_{\pm}(z), T(z')] = 0.$$
The Q-operator, and the relations between the T- and Q-operators, are essential for analyzing the spectrum of an integrable model, and indeed one can derive the Bethe ansatz from the relations between the T- and Q-operators (see [13] for a discussion).

In this paper we will analyze another class of topological line defects in four-dimensional Chern-Simons theory: ‘t Hooft lines. We will find that ‘t Hooft lines produce naturally Q-operators.

Our main results are the following.

1. We show that the Q-operator for $SL_2$, up to an important normalizing factor, is the fractional ‘t Hooft line of charge $1/2$ for $SL_2$ (or of charge 1 for $PSL_2$). More generally, we show that Q-operators for $SL_n$ [14] and $SO(n)$ [15] arise from ‘t Hooft lines whose charge is a minuscule coweight of the adjoint form of the group. We do this by reproducing, from an analysis of ‘t Hooft lines, the “oscillator” construction of Q-operators presented in these works. We also generalize these constructions to include the minuscule coweights of $E_6$, $E_7$ (which also give rise to oscillator representations).

2. We derive the TQ and QQ relations from a first-principles analysis of ‘t Hooft lines. The TQ relation follows from the Witten effect [16] in four-dimensional gauge theory.

3. We analyze ‘t Hooft-Wilson lines whose charge is an arbitrary coweight of an arbitrary simple group. We argue that these ‘t Hooft-Wilson lines are classified by representations of a certain shifted Yangian [17–19], just as in [2, 3] we found that Wilson lines are classified by representations of the Yangian. We show how to construct these line defects by using the quantum Coulomb branches of three-dimensional $N = 4$ quiver gauge theories [20], which are linked to four-dimensional Chern-Simons theory by string dualities [21]. In a sense, this gives a string theory motivation for the known relation between Coulomb branches and shifted Yangians [20].

4. We derive new QQ relations valid for the classical groups and the exceptional groups $E_6$, $E_7$. In each case, the Q-operator is the ‘t Hooft line labelled by a minuscule coweight, and the QQ relation expresses the product of two Q-operators of opposite weight as the Wilson lines associated to a parabolic Verma module.

5. We build Q-operators for some of the class of integrable field theories constructed in [4]. We prove that, for $g = sl_2$, these Q-operators satisfy Baxter’s TQ relation.

2 ‘t Hooft lines in four-dimensional Chern-Simons theory

Before we turn to linking the Q-operator and ‘t Hooft lines, we will introduce ‘t Hooft lines in four-dimensional Chern-Simons theory. The analysis in this section is mostly at the classical level. An analysis of ‘t Hooft lines at the quantum level is quite subtle, and will be discussed from two different perspectives in sections 9 and 10.
Let us start by recalling the definition of ’t Hooft line in ordinary four-dimensional \( U(1) \) Yang-Mills theory. In this case, a line operator is said to have magnetic charge \( k \) if, in the presence of the line operator, the gauge field on the complement of the line defines a topologically non-trivial \( U(1) \) bundle on the two-sphere surrounding the line. This bundle should have first Chern class \( k \).

An ’t Hooft line of charge \( k \) is by definition a line operator of magnetic charge \( k \) and electric charge 0.

In order to satisfy the Yang-Mills equations, the field strength in the presence of the ’t Hooft line should look like a Dirac monopole:

\[
F \sim \|x\|^{-3} \epsilon_{ijk} x_i dx_j dx_k + \text{regular terms},
\]

(2.1)

where \( x_i \) are the coordinates normal to the line.

For a non-Abelian Yang-Mills theory with gauge group \( G \), we can define a Dirac monopole by choosing a cocharacter \( \rho: U(1) \to G \) to embed the \( U(1) \) Dirac monopole into \( G \). A Dirac monopole of this coweight has the feature that on the \( S^2 \) surrounding the line, the gauge field defines a \( G \)-bundle which is the bundle associated to the \( U(1) \) bundle on \( S^2 \) of first Chern class \( k \), by the map \( \rho \). This means that the field strength takes the form

\[
F \sim \|x\|^{-3} \epsilon_{ijk} x_i dx_j dx_k \rho + \text{regular terms},
\]

(2.2)

where we view \( \rho \) as an element of \( \mathfrak{g} \).

In order to define an ’t Hooft line one should require the field strength to approach the Dirac monopole, up to some appropriate class of gauge transformations. This is a subtle problem; one reason is that for a non-Abelian gauge group the field strength is not gauge invariant. Further, “bubbling” gauge field configurations can potentially collapse on the ’t Hooft line and change its charge. Even semiclassically, a proper UV definition of the ’t Hooft line requires one to understand and quantize the phase space of such field configurations. This is a rather theory-dependent procedure.

Let us now try to understand what gauge fields in four-dimensional Chern-Simons theory look like when we ask that they have monopole charge on the 2-sphere surrounding a line in the topological direction. Let us first recall that four-dimensional Chern-Simons theory is an analytically-continued theory as in [22], where the space of fields is a complex manifold and the action functional is a holomorphic function. The functional integral should be performed over a contour, and in perturbation theory the contour is irrelevant. Instead of writing the gauge group as a compact group, we will write it as a complex group, as is more natural in our setting. The choice of contour might involve a choice of real form of the complex group.

Because of the mixed holomorphic/topological nature of the theory, studying the behaviour of the gauge field on the unit two-sphere will not be convenient. Instead we will consider a topologically equivalent region, which is the boundary of the solid cylinder where \( |z| \leq \epsilon \) and \( |y| \leq \epsilon \).
This boundary is divided into the three regions:

\[ y = -\epsilon, \quad |z| \leq \epsilon, \quad (2.3) \]
\[ -\epsilon \leq y \leq \epsilon, \quad |z| = \epsilon, \quad (2.4) \]
\[ y = \epsilon, \quad |z| \leq \epsilon. \quad (2.5) \]

A solution to the equations of motion (modulo gauge transformation) for four-dimensional Chern-Simons theory on this region is described by:

1. A holomorphic bundle on the disc \( |z| \leq \epsilon \) and \( y = -\epsilon \) and at \( y = \epsilon \).

2. An isomorphism between these two bundles when restricted to \( |z| = \epsilon \). This isomorphism is provided by parallel transport in the \( y \)-direction on the cylinder \( |z| = \epsilon, -\epsilon \leq y \leq \epsilon \).

Trivializing the holomorphic bundles at \( y = \pm \epsilon \), the parallel transport from \( y = -\epsilon \) to \( y = \epsilon \) is an element of the loop group \( LG \) of the complex gauge group \( G \). The change of trivialization at \( y = -\epsilon \) acts by left multiplication with an element of \( L_+ G \subset LG \), where \( L_+ G \) is the group of loops which extend to a holomorphic map from the disc to \( G \). Change of trivialization at \( y = \epsilon \) acts by right multiplication by \( L_+ G \). Therefore the moduli space of solutions to the equations of motion on the boundary of the solid cylinder is

\[ L_+ G \backslash LG / L_+ G. \quad (2.6) \]

We will define an ’t Hooft line (at least on-shell) by declaring that our solutions to the equations of motion is locally gauge equivalent to the Dirac singularity given by a coweight \( \rho \) of \( G \), which is the point

\[ z^\rho \in LG. \quad (2.7) \]

Let us try to connect this prescription with the more familiar ’t Hooft lines of Yang-Mills theory. In four-dimensional \( N = 2 \) gauge theories, BPS ’t Hooft operators impose an additional singularity in an extra scalar field \( \Phi \), so that Bogomolny equations \( F = *d\Phi \) are satisfied \cite{23, 24}. In particular,

\[ [D_{x_3} + \Phi, D_{x_1} + iD_{x_2}] = 0. \quad (2.8) \]

In four-dimensional Chern-Simons theory, one has only three of the four components of the gauge field: \( A_x, A_y, A_z \). Here we work on \( \mathbb{R}^2 \times \mathbb{C} \), with coordinates \( x, y, z \) and align the ’t Hooft line along the \( x \)-direction.

If we identify \( A_y = A_{x_3} + i\Phi \) and \( A_z = A_{x_1} + iA_{x_2} \) from a BPS ’t Hooft operator we obtain an complexified, \( x \)-independent solution of the equations of motion which is a good candidate Dirac singularity for four-dimensional Chern-Simons theory. One can then check that the BPS ’t Hooft operator does indeed give us a solution of the four-dimensional Chern-Simons equations of motion with the behaviour described above. See e.g. \cite{25}.
2.1 ‘t Hooft lines and the affine Grassmannian

In the mathematics literature, the space $LG/L_+G$ is known as the *affine Grassmannian*. Strictly speaking, the affine Grassmannian is a slight variant of this, where we replace the group $LG$ of loops into $G$ by the group $G((z))$, whose elements should be viewed as Laurent series valued in $G$.\(^2\)

If we want to consider a very small cylinder surrounding the ‘t Hooft line, the affine Grassmannian version of this space is in fact the correct thing to use. We should use Laurent series so that we have only finite-order poles at $z = 0$ and we can expand in series in $z$ because we are working in a region where $z$ is very small.

What this analysis tells us is that to define an ‘t Hooft line, we should ask that our fields extend across some submanifold of the affine Grassmannian $Gr_G = G((z))/G[[z]]$ which is stable under the action of left multiplication by $G[[z]]$ and contains $z^\rho$. Satisfyingly, it is well-known \([26]\) that the set of $G[[z]]$ orbits in the affine Grassmannian is precisely parametrized in this manner in terms of the set of dominant coweights $G$. This is our semiclassical definition of an ‘t Hooft line.

Most $G[[z]]$-orbits in the affine Grassmannian are not closed, and their closures are singular algebraic varieties. It is technically difficult to work with these varieties, but the difficulties can be overcome with some input from the theory of Coulomb branches of three-dimensional $N = 4$ gauge theories, as we will see in section 9. Furthermore, the orbits corresponding to minuscule coweights are always smooth and closed. For the groups $PSL_n$, the singularities of more general ‘t Hooft operators can be resolved or deformed by splitting them into a collection of minuscule ones \([27]\). (For other groups there are not enough minuscule coweights to do this.)

2.2 ‘t Hooft lines as generated by singular gauge transformations

A convenient ansatz for us is that an ‘t Hooft line of charge $\rho$ at $y = 0$, $z = 0$ in four-dimensional Chern-Simons theory is generated by a gauge transformation with a singularity at $z = 0$, $y = 0$. A gauge field on a topologically non-trivial bundle, like that sourced by an ‘t Hooft line, is always obtained from gluing trivial bundles on patches by using gauge transformations on the overlap. The unusual feature about four-dimensional Chern-Simons theory, as opposed to ordinary Yang-Mills theory, is that locally all solutions to the equations of motion are trivial.

Therefore, we can engineer the field sourced by an ‘t Hooft line by taking a trivial gauge field on the region $y \leq 0$, $y \geq 0$, and gluing these two trivial gauge fields by the gauge transformation $z^\mu$.

\(^2\)For example $GL_n((z))$ is the group of invertible $n \times n$ matrices whose entries are Laurent series. For a general group $G$, we can define $G((z))$ by viewing $G$ as the subgroup of $GL_n$ cut out by some polynomial equations, like $AA^T = 1$ in the case of $O(n)$; and then defining $G((z))$ to be the subgroup of $GL_n((z))$ cut out by the same polynomial equations, e.g. $A(z)A^T(z) = 1$ for $O(n)((z))$. 
If we do this, then the parallel transport of the gauge field from the region $y < 0$ to the region $y > 0$ is of the form

$$g_1(z)z^\mu g_2(z),$$

(2.9)

where $g_1(z)$, $g_2(z)$ are arbitrary $G$-valued holomorphic functions of $z$, regular at 0. Note that the possible values for the parallel transport only depend on the Weyl orbit of $\mu$, and that we can derive the same expression by considering the $G[[z]]$ orbit in the affine Grassmannian containing $z^\mu$.

If we allow monopole bubbling (which does not happen for minuscule coweights), then, if $\mu$ is a dominant coweight, we also allow field configurations such that the parallel transport takes the form

$$g_1(z)z^{\mu'} g_2(z),$$

(2.10)

where $\mu' \leq \mu$.

### 2.3 't Hooft lines at infinity

Let us define an 't Hooft line at 0 as above, by a singular gauge transformation or equivalently by specifying the singularity in the parallel transport of the gauge field past the 't Hooft line. We can define an 't Hooft line at $\infty$ in a similar way.

In [1, 2], the boundary conditions at $z = \infty$ is that the gauge fields and gauge transformations go as $1/z$ near infinity. We extend the plane $\mathbb{C}$ into $\mathbb{CP}^1$, with the point at $\infty$ being special because the one-form $dz$ has a second-order pole there.

We define the 't Hooft line at $\infty$ to be a modification of the boundary conditions, as follows. We ask that for $y \leq 0$ and for $y \geq 0$, the gauge field and all gauge transformations go as $1/z$. The gauge fields on the regions $y \leq 0, y \geq 0$, are related by the gauge transformation $z^\mu$. Because $z^\mu$ has a singularity at $z = \infty$, this gauge transformation is not one of the permitted ones at $\infty$, so that we have modified the boundary conditions.

With an 't Hooft line at $\infty$, the parallel transport of the gauge field on a path from $y < 0$ to $y > 0$ takes the form $g_1(z)z^\mu g_2(z)$, where $g_1(z), g_2(z)$ are gauge transformations of the kind allowed at $\infty$. This means that $g_1(z)$ is regular in a neighbourhood of $z = \infty$ and $g_1(\infty) = 1$.

The Dirichlet boundary conditions we use for four-dimensional Chern-Simons theory completely break the gauge symmetry at $z = \infty$. In the bulk, two 't Hooft lines associated to coweights in the same Weyl orbit are equivalent, as the corresponding singularities are related by a gauge transformation. At the boundary, this is no longer true, so that 't Hooft lines at $z = \infty$ are labelled by coweights, and not by Weyl orbits of coweights.

Instead, at $z = \infty$ we have a $G$-global symmetry. In particular an element $w$ in the Weyl group gives a global symmetry of the theory which sends the 't Hooft line at $\infty$ of charge $\mu$ to that of charge $w(\mu)$.

This will turn out to be essential because we will identify a Q-operator with an 't Hooft line of charge $\mu$ in the bulk together with the corresponding 't Hooft line at $\infty$ of charge $-\mu$. Such configurations are labelled by coweights and not by Weyl orbits of coweights. Q-operators are also labelled by (minuscule) coweights, and coweights related by a Weyl
transformation give different Q-operators. For example, for $G = PSL_2$, there are two Q-operators $Q_+, \ Q_-$ which correspond to the two minuscule coweights of $PSL_2$ (or better, to the two fractional minuscule coweights of $SL_2$).

3 Background on Q-operators

We will follow [13] for definitions and conventions on Q-operators. In this section we will mostly discuss gauge groups $SL_2$ and $PSL_2$.

Consider an integrable spin chain with $N$ sites and with periodic boundary conditions. The boundary condition can be modified by a twist, where we apply an element of the Cartan $H$ of the group $G$ when we identify the two sides of the open spin chain to make it periodic. The twist parameter will be modelled by $e^{i\phi}$, for $\phi \in \mathfrak{h}$, the Lie algebra of $H$. The Q-operators are only well-defined with a non-zero twist parameter $\phi$.

From the point of view of four-dimensional Chern-Simons theory, the twist parameter is given asking that the gauge field $A$ does not tend to zero at $z = \infty$, but is the element $i\phi$ in the Cartan. For $SL_2(\mathbb{C})$, the Cartan is of rank 1, so $\phi$ is a scalar, and

$$A = \text{Diag}(i\phi, -i\phi)dx,$$

(3.1)

where $x$ is the periodic direction.

Let us write down the TQ and QQ relations for $SL_2$. The TQ relation is

$$T(z, \phi)Q_\pm(z, \phi) = (z - \frac{1}{2}h)^N Q_\pm(z + h, \phi) + (z + \frac{1}{2}h)^N Q_\pm(z - h, \phi).$$

(3.2)

In addition, if $T_j^+(z, \phi)$ is the transfer matrix for the Verma module of highest weight $j$, we have the QQ relation

$$2i \sin(\phi/2)T_{j - \frac{1}{2}}^+(z) = Q_+(z + hj)Q_-(z - hj).$$

(3.3)

Here $\phi$ is the twist parameter.

A final identity we will need is that $Q_+$ and $Q_-$ are related by the Weyl group of $SL_2(\mathbb{C})$:

$$wQ_+(z, \phi)w = Q_-(z, -\phi).$$

(3.4)

This is equation (3.55) of [13].

4 Q-operators as ’t Hooft lines

Let us introduce the particular ’t Hooft lines that we will match with the Q-operators for the group $SL_2$. We let $H_{\pm \frac{1}{2}}(z_0)$ be the ’t Hooft lines of fractional charge $\pm \frac{1}{2}$ at $z_0$ and of charge $\mp \frac{1}{2}$ at $\infty$. By definition, the field sourced by the ’t Hooft line has a singularity which near
both \( z_0 \) and \( \infty \) is given by gluing the trivial field configurations along \( y \leq 0 \) and \( y \geq 0 \) using the singular gauge transformation

\[
\begin{pmatrix}
(z - z_0)^{\pm \frac{1}{2}} & 0 \\
0 & (z - z_0)^{\mp \frac{1}{2}}
\end{pmatrix}.
\]

There are two important points to note about this. First, this gauge transformation has a branch cut. This reflects the fact that we are dealing with fractional ’t Hooft lines, which live at the end of a Dirac string.

Since gauge transformations are broken at \( z = \infty \), the operators \( H_{\pm} \) are not equivalent.

Our proposal, for \( G = SL_2 \), is that the two operators \( Q_+(z), Q_-(z) \) are related to the operators \( H_{\pm} \) by an equation of the form

\[
Q_{\pm}(z) = G(z)H_{\pm}^{\frac{1}{2}}(z)
\]

for a normalizing function \( G(z) \).

If ’t Hooft lines are to be related to Q-operators in this way, then the TQ and QQ relations should be uplifted to relations involving line defects.

The TQ relation says that the Wilson line, which becomes \( T \), when fused with an ’t Hooft line \( Q \), must become a sum of two ’t Hooft lines, with certain prefactors.

The prefactors, however, are problematic. For the TQ equation (3.2) to hold in the category of line defects, the prefactors \( (z \pm \frac{1}{2} \hbar)^N \) must be represented by a line defect, say \( L \), which, when we cross it with the \( N \) fundamental Wilson lines at \( z = 0 \), yields the factor \( (z \pm \frac{1}{2} \hbar)^N \). Such a line defect does not exist with gauge group \( SL_2 \). Indeed, matching magnetic charges on both sides of the TQ relation, this putative line defect must have no magnetic charge, and so be a Wilson line. The exchange of gluons between two Wilson lines goes like \( 1 + O(\hbar/z) \), so can not give the factor of \( z \pm \frac{1}{2} \hbar \).

The solution to this problem lies in the fact that the T-operator that arises from four-dimensional Chern-Simons theory has a very particular normalization, arising from the quantum determinant. Let us explain how this arises, and how it solves the problem just discussed.

### 4.1 Normalizing the T-operator

In the standard theory of integrable spin chains, the L-operator is a local quantity from which one can build the transfer matrix. In the language of [2] the L-operator arises from a fundamental Wilson line of \( SL_2(\mathbb{C}) \) at \( z = 0 \) crossing some other Wilson line at \( z \). In [2], the gauge theory setup forces one to have an L-operator normalized so that the quantum determinant is 1. In standard references, the L-operator does not have this normalization. We will refer to the unnormalized L-operator as \( \mathring{L} \), and the corresponding transfer matrix as \( \mathring{T} \), whereas the normalized L-operators and transfer matrices will be \( L \) and \( T \).

\footnote{In fact there is a short exact sequence of line defects, rather than an equality: at the level of the trace this distinction is irrelevant.}
There is a Wilson line associated to any representation of $\mathfrak{sl}_2$. Let $\rho(e)$, $\rho(f)$, $\rho(h)$ be the matrices defining this representation, where as usual $[e, f] = h$, $[h, e] = 2e$, $[h, f] = 2f$. Then the unnormalized $L$-operator corresponding to this Wilson line is

$$\tilde{L}(z) = \begin{pmatrix} z + \frac{1}{2}h\rho(h) & h\rho(f) \\ h\rho(e) & z - \frac{1}{2}h\rho(h) \end{pmatrix}. \quad (4.3)$$

The normalized $L$-operator is

$$L(z) = F(z, j) \begin{pmatrix} z + \frac{1}{2}h\rho(h) & h\rho(f) \\ h\rho(e) & z - \frac{1}{2}h\rho(h) \end{pmatrix}. \quad (4.4)$$

If we have a highest-weight representation of $\mathfrak{sl}_2(\mathbb{C})$ of weight $j$, then, as we show in appendix A, the prefactor $F(z, j)$ must satisfy the difference equation

$$F(z, j)F(z + h, j) = 1 \quad z^2 + hz - h^2j(j + 1). \quad (4.5)$$

This equation states that the quantum determinant is one.

For integral $j$, this equation has a unique rational solution, which we can write as a ratio of $\Gamma$-functions

$$F(z, j) = \frac{1}{2h} \frac{\Gamma \left( \frac{1}{2}(z + h(j + 1)) \right) \Gamma \left( \frac{1}{2}(z - hj) \right) \Gamma \left( \frac{1}{2}(z + h(j + 2)) \right) \Gamma \left( \frac{1}{2}(z - hj + h) \right)}{\Gamma \left( \frac{1}{2}(z + h(j + 1)) \right) \Gamma \left( \frac{1}{2}(z - hj) \right) \Gamma \left( \frac{1}{2}(z + h(j + 2)) \right) \Gamma \left( \frac{1}{2}(z - hj + h) \right)} \quad (4.6)$$

with a nice perturbative expansion in odd powers of $h/z$.

For all $j$, there is a unique solution to (4.5) which is $z^{-1}$ times a series in $h/z$, and this is the the prefactor determined by gauge theory considerations. This perturbative solution is the asymptotic expansion of (4.6) in a Stokes sector of width $2\pi$ in the $h/z$ plane centered on the positive real axis, but it is not odd under $z \rightarrow -z$. The alternative solution $-F(-z, j)$ agrees with the perturbative solution in a Stokes sector of width $2\pi$ in the $h/z$ plane centered on the negative real axis.

This shows that, for non-integral $j$, there are two natural non-perturbative completions of the perturbative Wilson line, with the normalizing factors $F(z, j)$ and $-F(-z, j)$. (More generally, we could multiply $F(z, j)$ by an appropriate trigonometric function of $z/h$ which does not affect the asymptotic expansion of $F(z, j)$ in the Stokes sector where it is defined)). The physical meaning of this ambiguity is obscure, but perhaps it could be explained in the string-theoretic embedding of the system [21, 28, 29]. In any case, it is only relevant in a non-perturbative completion of four-dimensional Chern-Simons theory, which is outside of the scope of this paper.

The normalized and unnormalized T-operators in a spin chain system with $L$ sites are related by

$$T(z, \phi) = F(z)^N \tilde{T}(z, \phi) \quad (4.7)$$

This T-operator is the one obtained by the perturbative analysis of four-dimensional Chern-Simons theory, because the quantum determinant relation can be derived from QFT considerations [3].

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*Our conventions are such that $h$ acts by $2j$ on the highest weight vector.*
4.2 Normalizing the ’t Hooft line

Let
\[ G(z) = \frac{1}{(2\hbar)^{1/2}} \frac{\Gamma\left(\frac{1}{2h}(z + \frac{1}{2}j)\right)}{\Gamma\left(\frac{1}{2h}(z + \frac{3j}{2})\right)}. \]  

Our proposed normalization for the relation between the Q-operators and the ’t Hooft lines is that
\[ H_{\pm \frac{1}{2}}(z) = G(z)^N Q_{\pm}(z), \]
where the ’t Hooft line crosses \( N \) Wilson lines, with periodic boundary conditions. Let us check that this normalization renders the statement that the TQ and QQ relations are relations among line defects plausible.

Let us first consider what form the QQ relation might take for ’t Hooft lines. The QQ relation for the normalized T-operator \( T^+_j \) for a Verma module of highest weight \( j \) takes the form
\[ 2i\sin(\phi/2)F(z, j - \frac{1}{2})^{-N} T^+_j(z) = Q_+(z + hj)Q_-(z - hj). \]

Writing this in terms of the ’t Hooft lines, the equation is
\[ T^+_j(z) = \frac{F(z, j - \frac{1}{2})^N}{G(z + hj)G(z - hj)} \frac{1}{2i\sin(\phi/2)} H_{\frac{1}{2}}(z + hj)H_{-\frac{1}{2}}(z - hj). \]

This is supposed to come from an identity between line operators in four-dimensional Chern-Simons theory. For this to hold, the prefactor \( F(z, j - \frac{1}{2})^N/G(z + hj)G(z - hj) \) must be independent of \( z \), as in a theory with a simple gauge group, we are not free to multiply the transfer matrix associated to a line defect by a function of \( z \). Fortunately, we have
\[ G(z + hj)G(z - hj) = F(z, j - \frac{1}{2}). \]

Further, the factor \((2i\sin(\phi/2))^{-1}\) in (4.11) arises from the collision of the two ’t Hooft lines at \( z = \infty \). We will derive the QQ relation in this form from first principles in section 16.

Note that Stirling’s formula implies that \( G(z) \) has an asymptotic expansion\(^5\) as \( h \to 0 \) which is
\[ G(z) = z^{-1/2}(1 + C_2h^2z^{-2} + C_3h^3z^{-3} + \cdots) \]
for certain constants \( C_i \). That is, \( G(z) \) is a series in \( h/z \) times \( z^{-1/2} \). The factor of \( z^{-1/2} \) introduces a branch cut, consistent with the fact that we are working with a fractional ’t Hooft line, which lives at the end of a Dirac string.

What about the TQ relation? In the usual normalization, this takes the form
\[ \tilde{T}(z, \phi) Q_\pm(z, \phi) = (z - \frac{1}{2}h)^N Q_{\pm}(z + h, \phi) + (z + \frac{1}{2}h)^N Q_{\pm}(z - h, \phi). \]

As we have discussed, the factors of \((z \pm \frac{1}{2}h)^N\) can not be present if we would like a relation that holds at the level of line defects. For TQ relation to make sense in our context, we need

\(^5\)This expansion is valid if \( h \) is a small positive real number and \(|\text{Arg}(z)| < \pi\).
the normalization factor relating the ’t Hooft lines to the Q-operators and that relating $T$ to $\tilde{T}$ to cancel the factors of $(z \pm \frac{1}{2} \hbar)^N$. The normalization we have derived does exactly this.

Inserting the prefactors relating the ’t Hooft lines and the Q-operators, the normalized version of the TQ relation that we need to prove is

$$F(z, \frac{1}{2} \hbar) - N G(z) T(z, \phi) H_{\frac{1}{2}}(z, \phi) = G(z + h)^{-N} (z - \frac{1}{2} \hbar)^N H_{\frac{1}{2}}(z + h, \phi) + G(z - h)^{-N} (z + \frac{1}{2} \hbar)^N H_{\frac{1}{2}}(z - h, \phi). \quad (4.15)$$

Using the fact that $F(z, \frac{1}{2} \hbar) = G(z + h) G(z - h)$, and multiplying both sides by $G(z + h)^N G(z - h)^N G(z)^N$ we find that the TQ relation takes the form

$$T(z, \phi) H_{\frac{1}{2}}(z, \phi) = G(z - h)^N G(z)^N (z - \frac{1}{2} \hbar)^N H_{\frac{1}{2}}(z + h, \phi) + G(z)^N G(z + h)^N (z + \frac{1}{2} \hbar)^N H_{\frac{1}{2}}(z - h, \phi). \quad (4.16)$$

Using the difference equation

$$G(z)^{-1} G(z + h)^{-1} = z + \frac{1}{2} \hbar \quad (4.17)$$

we find that, expressed in terms of the operators $H_{\pm \frac{1}{2}}$, the normalized TQ relation takes the form

$$T(z, \phi) H_{\frac{1}{2}}(z, \phi) = H_{\frac{1}{2}}(z + h, \phi) + H_{\frac{1}{2}}(z - h, \phi). \quad (4.18)$$

As desired, the factors of $(z \pm \frac{1}{2} \hbar)^N$ have been cancelled.

What we have shown is that, when we introduce the normalizing factor in the T-operator required by the quantum determinant relation, the TQ relation takes the simpler form (4.18). This is essential for our story to work, because only an equation with integral coefficients can come from a relation between line defects.

Note that, as in the case of Wilson lines for $SL_2$, there are several possible non-perturbative completions of the normalization factor $G(z)$. One trivial variation is to simply use $-G(z)$, which will of course still satisfy the TQ and QQ relations. This is simply the parity-reversal of the ’t Hooft line. More interestingly, we can replace $G(z)$ by $iG(-z)$. This still satisfies (4.17) and so will satisfy the TQ relation. If we use $iG(-z)$ in place of $G(z)$, then the QQ relation will still hold as long as we replace $F(z)$ by $-F(-z)$ in the normalization of the Wilson line.

As in the analysis of Wilson lines, $G(z)$ and $iG(-z)$ have the same asymptotic expansion, as one can see from the fact that

$$iG(-z) = G(z) \frac{1 - i e^{i z/\hbar}}{1 + i e^{i z/\hbar}}. \quad (4.19)$$

In the region where $\hbar \to 0$ and $z/\hbar$ is close to the positive imaginary axis, the two expressions coincide up to exponentially suppressed terms.
This suggests there are two (or more) natural non-perturbative completions of the ’t Hooft line, just as there are two non-perturbative completions of the Wilson line. If we take the pair of Q-operators built from $G(z)$, their product yields the Wilson line normalized by $F(z,j)$. Similarly, the Q-operators built from $iG(-z)$ yields the Wilson line normalized by $-F(-z,j)$. However, this is not all we can do: if we multiply the Q-operator normalized by $G(z)$ with that normalized by $iG(-z)$, we get the T-operator normalized by $F$ times a trigonometric function of $z/\hbar$. Again, we don’t have a good physical understanding of the various non-perturbative completions of ’t Hooft and Wilson lines, but it would be fascinating to understand it from a string-theoretic UV completion of the theory.

5 The TQ relation from the Witten effect

We will now show that our formulation of the TQ equation in terms of ’t Hooft lines (4.18) follows from the Witten effect [16]. Let us first recall some background on the Witten effect.

Let $G$ be the gauge group, and $T$ the maximal torus. Let $\Gamma = \text{Hom}(U(1),T)$ be the coweight lattice, and $\Gamma^\vee = \text{Hom}(T,U(1))$ be the weight lattice. The magnetic charge of a line operator lives in $\Gamma$, and the electric charge lives in $\Gamma^\vee$. The Killing form gives a map $\rho: \Gamma \rightarrow \Gamma^\vee$.

In the case of $G = SU(2)$, the electric and magnetic charge lattices are both $\mathbb{Z}$, but the map from the magnetic to the electric charge lattices is given by multiplication by 2. We can take a basis for the magnetic charge lattice to be associated to the element $h \in \mathfrak{sl}_2(\mathbb{C})$.

The Witten effect states the following. Suppose we have a four-dimensional gauge theory with gauge group $G$, and a line defect with magnetic charge $m \in \Gamma$. Then, the electric charge is not in $\Gamma^\vee$; rather, it is in $\Gamma^\vee + \rho(m)\theta/2\pi$, (5.1)

where $\theta$ is the $\theta$-angle. In particular, if we continuously vary $\theta$ to $\theta'$, then the electric charge of a line defect of magnetic charge $m$ shifts by $\rho(m)(\theta' - \theta)/2\pi$.

The Lagrangian of four-dimensional Chern-Simons theory, in the normalization we are using, is

$$\frac{1}{2\hbar\pi} \int dz \text{CS}(A) = \frac{1}{4\hbar\pi} \int dz \left( \langle A, dA \rangle + \frac{1}{3} \langle A, [A, A] \rangle \right). \quad (5.2)$$

By integration by parts, we can rewrite this as

$$-\frac{1}{4\hbar\pi} \int z \text{Tr} F(A)^2. \quad (5.3)$$

The usual $\theta$-angle in a four-dimensional gauge theory is defined so that we add on a term

$$-\frac{\theta}{8\pi^2} \int \text{Tr} F(A)^2. \quad (5.4)$$

Thus, the Lagrangian describes varying $\theta$-angle. If we move an ’t Hooft line from $z$ to $z_0 + \hbar$, then the $\theta$-angle seen by the ’t Hooft line shifts by $2\pi$. This, combined with the Witten
effect, tells us that a shift $z \rightarrow z + \hbar$ in the position of a line defect of magnetic charge $m$ shifts the electric charge $e$ to $e + \rho(m)$.

We are interested in this in the case when the gauge group is $SU(2)$, and we have a fractional 't Hooft line at $z$ of magnetic charge $\frac{1}{2}$. We call this line $H_{\frac{1}{2}}(z)$. We see that the 't Hooft line $H_{\frac{1}{2}}(z + \hbar)$ is the same as the dyonic line at $z$ of magnetic charge $\frac{1}{2}$ and electric charge $\rho(\frac{1}{2}) = 1$.

Let us now derive the TQ relations. We propose that the operator $Q_+(z)$ is given by the fractional 't Hooft line $H_{\frac{1}{2}}(z)$ of charge $1/2$, up to the factor $G(z)$. The T-operator is, of course, associated to a Wilson line. On the 't Hooft line, gauge symmetry is broken to the Borel subgroup of $SL_2$.

This means that bringing a fundamental Wilson line to the 't Hooft line gives a dyonic line obtained by coupling the fundamental representation of $SL_2$ to the Borel gauge symmetry. As a representation of the Borel, the fundamental representation of $SL_2$ is an extension of the charge 1 and $-1$ representations of the Cartan. At the level of traces, this representation is equivalent to the sum of the charge $\pm 1$ representations of the Cartan.

The Witten effect tells us that the 't Hooft line $H_{\frac{1}{2}}(z)$ of magnetic charge $\frac{1}{2}$ coupled to a representation of charge $\pm 1$ is the 't Hooft line $H_{\frac{1}{2}}(z \pm \hbar)$ with shifted value of the spectral parameter.

We conclude that

$$T(z)H_{\frac{1}{2}}(z) = H_{\frac{1}{2}}(z + \hbar) + H_{\frac{1}{2}}(z - \hbar).$$

This is precisely the TQ relation (4.18), when we identify as before $H_{\frac{1}{2}}(z) = G(z)^N Q_+(z)$.

This concludes our derivation of the TQ relation from the Witten effect.

5.1 The Witten effect in four-dimensional Chern-Simons theory

We have derived the TQ relation from the Witten effect. Although the Witten effect is very robust, a cautious reader might be concerned about the application of the Witten effect in the non-standard gauge theory we are using. In this section we will verify the Witten effect directly to leading order, by a Feynman diagram computation.

For simplicity, we will present the analysis for the group $G = SL_2$ and a fractional 't Hooft line of charge $\frac{1}{2}$, although it is not difficult to generalize to an arbitrary group.

On the 't Hooft line, the gauge group is broken to the Borel $B \subset SL_2$. Since there is a homomorphism from the Borel $B$ to the Cartan $H$, we can couple a Wilson line in a rank one representation of $H$ to give a (classically) gauge-invariant Wilson-'t Hooft line. A rank one representation of $H$ is specified by the weight, which is the electric charge of the Wilson-'t Hooft line and which we denote by $e$.

We let $H_{(e, \frac{1}{2})}(z)$ be the Wilson-'t Hooft line of electric charge $e$ and magnetic charge $\frac{1}{2}$.

\footnote{The easiest way to see this is to note that, viewing the 't Hooft line as a line defect of charge 1 for $PSL_2$, the corresponding orbit in the affine Grassmannian is $\mathbb{C}P^1$. The stabilizer of a point in $\mathbb{C}P^1$ is the Borel subgroup.}
The Witten effect in four-dimensional Chern-Simons theory states that

\[ H_{(e, \frac{1}{2})}(z) = H_{\frac{1}{2}}(z + eh). \] (5.6)

Let us derive this, to leading order in \( h \).

Consider an \('t\) Hooft line wrapping \( y = 0, z = 0 \). The field sourced by an \('t\) Hooft line is, essentially by definition, the field given by gluing the trivial field configuration at \( y \leq 0 \) and \( y \geq 0 \) by the singular gauge transformation \( \text{Diag}(z^{1/2}, z^{-1/2}) \) at \( y = 0 \). This means that the effect of an \('t\) Hooft line wrapping \( y = 0, z = 0 \) on a Wilson line wrapping \( x = 0 \), at \( z \) is simply the matrix

\[ M(z) = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} (1 + O(h/z)). \] (5.7)

Adding electric charge to the \('t\) Hooft line means that we add the term

\[ e \int_{y=0,z=0} A_h \] (5.8)

to the Lagrangian of the theory. To leading order in \( h \), the effect of this modification on a Wilson line passing the \('t\) Hooft line will be given by the exchange of a single gluon between the Wilson line and the dyonic line. We can compute this by using the same Feynman diagrammatics as in [2].

Only the \( h\)-component of the gauge field is coupled when we deform the \('t\) Hooft line to a Wilson-\('t\) Hooft line. Therefore the group theory factor in the propagator connecting the Wilson and dyonic lines will have \( h \) on each end.

The analytic factor is identical to that considered in the calculations in [2] of the single gluon exchange between two Wilson lines. In [2], we found that the analytic factor is \( h/z \). The component of the quadratic Casimir involving \( h \in \mathfrak{sl}_2(\mathbb{C}) \) is \( \frac{1}{2} h \otimes h \) (as usual \( h = \text{Diag}(1, -1) \)). We find that introducing electric charge \( e \) to the \('t\) Hooft line changes the matrix \( M \) by

\[ M(z) \mapsto \left( 1 + \frac{eh}{2z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) M(z) + O(h^2). \] (5.9)

Now, it is obvious that

\[ eh\partial_z M(z) = \frac{eh}{2z} h M(z) + O(h^2). \] (5.10)

This verifies the Witten effect, to leading order in \( h \):

\[ H_{(1, \frac{1}{2})}(z) = H_{\frac{1}{2}}(z + h) + O(h^2). \] (5.11)

It is more challenging to show that this result holds to all orders in \( h \). To see this, we will first use the fact, proved in section 13, that the \('t\) Hooft line has only one parameter, which we can take to be the position in the \( z\)-plane.
This implies that
\[ H_{(e, \frac{1}{2})}(z) = H_{\frac{1}{2}}(z + f(h, e)) \] (5.12)
for some unknown function \( f \). Both the 't Hooft line and the dyonic line placed at \( z = 0 \) are preserved by the symmetry of the system which scales both \( z \) and \( h \). This implies that the shift in \( z \) on the right hand side of the equation must be linear in \( h \), so that, by the argument we have already provided,
\[ f(h, e) = eh. \] (5.13)

6 Phase spaces of 't Hooft operators

In this section we will carefully analyze the phase space of Chern-Simons theory in the presence of 't Hooft lines at 0 and at \( \infty \). There are a number of subtle issues that arise when we discuss 't Hooft lines that do not appear for Wilson lines, so we will first discuss the phase space in the absence of 't Hooft lines.

6.1 Moduli of bundles on \( \mathbb{CP}^1 \)

Recall that we treat four-dimensional Chern-Simons theory as a theory on \( \mathbb{R}^2 \times \mathbb{CP}^1 \) with the one-form \( dz \) on \( \mathbb{CP}^1 \), where all components of the gauge field and all gauge transformations go like \( 1/z \) near \( z = \infty \). In [2, 3] we analyzed the gauge theory perturbatively, working near the trivial field configuration \( A = 0 \). In that case, the moduli space of solutions to the equations of motion is a point. The key point here is that the trivial holomorphic bundle on \( \mathbb{CP}^1 \) admits no infinitesimal deformations or automorphisms which are trivial at \( z = \infty \).

A perturbative analysis is not sufficient for the treatment of 't Hooft lines, so here we must be more careful. If the group \( G \) is simply connected, then the moduli space of \( G \)-bundles on \( \mathbb{CP}^1 \) is connected, and the only stable bundle is the trivial bundle. In this case, the previous analysis suffices.

If \( G \) is not simply connected, then the moduli space of \( G \)-bundles on \( \mathbb{CP}^1 \) has components labelled by \( \pi_1(G) \), and a non-perturbative treatment of four-dimensional Chern-Simons theory must take account of these components. For example, for the group \( PSL_2 \), the moduli of bundles has two components, one containing the trivial bundle and one containing \( \mathcal{O} \oplus \mathcal{O}(1) \). In the non-trivial component, the moduli\(^7 \) of bundles trivialized at \( \infty \) is \( \mathbb{CP}^1 \) is \( \mathbb{CP}^1 \).

To see this, note that the group \( PSL_2 \) acts on the moduli space of bundles trivialized at \( \infty \). The stabilizer of \( \mathcal{O} \oplus \mathcal{O}(1) \) consists of those projective automorphisms of the fibre at \( z = \infty \) which extend to bundle automorphisms on all \( \mathbb{CP}^1 \). This is the Borel subgroup \( B \subset PSL_2 \). Thus, the moduli of bundles is \( PSL_2/B = \mathbb{CP}^1 \). More generally, take any simple Lie group \( G \) of adjoint type. The components of the moduli space of \( G \)-bundles on \( \mathbb{CP}^1 \), trivialized at \( \infty \), consist of the trivial bundle, plus a component for each Weyl orbit of minuscule coweights of \( G \). Given any minuscule coweight \( \mu \), we form a bundle on \( \mathbb{CP}^1 \) whose transition function

\(^7\) As a stack, the moduli space is bigger and contains bundles such as \( \mathcal{O}(1) \oplus \mathcal{O}(-2) \). However, these do not contribute to the path integral, as the bundles isomorphic to \( \mathcal{O} \oplus \mathcal{O}(1) \) form an open substack.
around $\infty$ is $z^\mu$. This bundle, together with its trivialization at $\infty$, lives in a moduli space which is $G/P_\mu$, where $P_\mu \subset G$ is a parabolic subgroup associated to $\mu$. (The Lie algebra $p_\mu$ is defined to be the subalgebra of $\mathfrak{g}$ of elements of charge $\geq 0$ under $\mu$, and $P_\mu$ is the exponential of $p_\mu$.)

To see that the moduli space is $G/P_\mu$, we note that $G$ acts on this moduli space by changing the framing at infinity, and at the level of the Lie algebra, a symmetry at $\infty$ extends to a bundle automorphism on all of $\mathbb{C}P^1$ if and only if it is of non-negative charge under $\mu$.

In [1, 2] it was shown that spin chain systems arise by placing Wilson lines in four-dimensional Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{C}P^1$. The analysis here only used the trivial bundle on $\mathbb{C}P^1$, and did not take into account the other components of the moduli space. Because of this, we conclude that this analysis is complete only in the case that $G$ is simply connected. When $G$ is not simply connected, the other components of the moduli space will give rise to other sectors of the Hilbert space of the spin chain system, which have not been analyzed.

Since our goal in this paper is to make contact with the $Q$-operators in ordinary spin chain systems, we see that we are forced to always work with the simply-connected form of the group.

### 6.2 Solutions to the equations of motion in the presence of an ’t Hooft line

Consider the four-dimensional Chern-Simons setup on $\mathbb{R}^2 \times \mathbb{C}P^1$ for a simply connected group. Place an ’t Hooft line wrapping $y = 0, z = 0$ of charge $\mu$. This may be a fractionally charged ’t Hooft line, in which case we view it as living at the end of a Dirac string at $y = 0$ stretched between $z = 0$ and $z = \infty$.

In the region $y < 0$, since $G$ is a simply-connected group, the $G$-bundle on $\mathbb{C}P^1$ must be trivial, and in fact trivialized because the trivialization at $z = \infty$ extends across uniquely all $\mathbb{C}P^1$. The same holds for $y > 0$. This means that we can form a gauge-invariant classical observable by measuring the parallel transport of the gauge field from $y \ll 0$ to $y \gg 0$:

$$L(z) = \text{PExp} \int_y A_y(z) \in G.$$  \hfill (6.1)

This is a holomorphic $G$-valued function of $z$, with possible poles and zeroes at $z = \infty, z = 0$ arising from the ’t Hooft lines at 0 and $\infty$.

The space of solutions to the equations of motion will be a sub-space of the space of holomorphic maps from $\mathbb{C}^\times$ to $G$, specified by the allowed poles at 0 and $\infty$. By expanding in Laurent series near $z = 0$, we will view it as a subgroup of the loop group $G((z))$.

As we have seen, the singularity in the ’t Hooft line of charge $\mu$ at $z = 0$ means that the parallel transport takes the form

$$G[[z]]z^{\mu'}G[[z]] \subset G((z)),$$  \hfill (6.2)

where $\mu$ is dominant and $\mu' \leq \mu$. 

At \( z = \infty \), the singularity from an 't Hooft line of charge \( \eta \) means the parallel transport takes the form
\[
G_0[z^{-1}]z^{-\eta}G_0[z^{-1}],
\]
(6.3) where \( G_0[z^{-1}] \) is the group of polynomial maps from a neighbourhood of \( \infty \) to \( G \) which takes the value 1 at \( \infty \).

We thus conclude that the phase space of the 't Hooft line is the set of operators \( L(z) \in G((z)) \) which satisfy both of these constraints. It is known [20] that if \( G \) is an ADE group, this space is the Coulomb branch of an ADE quiver gauge theory. This gives us a perspective on quantizing these phase spaces that we will explore in section 9.

In the case of fractional 't Hooft lines, we will also allow L-operators in \( G(z^{1/n}) \) where \( n \) is the rank of the center of \( G \), and the branch cuts live in the center. (We could also of course pass to the adjoint form of the group, in which case the 't Hooft line is no longer fractional, but we prefer not to do that for the reasons mentioned above).

### 6.3 Poisson bracket on the phase space

The Poisson bracket on the phase space of the theory in the presence of an 't Hooft line can be computed using the semiclassical version of the RTT relation. The idea is the following. Fix an 't Hooft line at \( z = 0 \) crossed by a Wilson line in some representation \( R \) at \( z \). Let \( A \) be the algebra of functions on the phase space of the 't Hooft line. The L-operator is an element
\[
L(z) \in A \otimes \text{End}(R).
\]
(6.4)
The matrix entries of \( L(z) \) are thus entries in the algebra \( A \). These matrix entries \( L^i_j(z) \) are obtained by asking that the Wilson line has initial state \( i \) and final state \( j \).

\[
\langle i | \quad z \quad | j \rangle
\]

**Figure 1.** A vertical Wilson line, equipped with an incoming state \( \langle i \rangle \) and an outgoing state \( | j \rangle \), gives rise to a function \( L_j^i(z) \) on the phase space of the horizontal 't Hooft line.

These functions satisfy some commutation relations (or, semiclassically, Poisson brackets) which arise from the R-matrix appearing when Wilson lines cross. These commutation relations are called the RTT or RLL relations, and were derived from a field theory analysis.
Figure 2. Two vertical Wilson lines are “bent” to cross each other, above or below a given horizontal 't Hooft line line.

in \[3\]. We will review the analysis and apply it to understand the Poisson bracket on the phase space of 't Hooft lines.

The commutation relation given is described diagrammatically from figure 2. The diagram holds simply because the Wilson lines do not touch the 't Hooft line or each other in the four-dimensional space-time, and so can be freely moved. Translated into symbols, the relation becomes

$$\sum_{r,s} R_{rs}^i(z-z')L^i_r(z)L^k_s(z') = \sum_{r,s} L^i_r(z)L^k_s(z')R_{rs}^{ik}(z-z'). \quad (6.5)$$

In interpreting this equation, note that the entries $R_{rs}^{ij}(z-z')$ of the R-matrix are scalar functions, whereas the entries $L^i_j$ of the T-operator are functions on the phase space in the presence of the 't Hooft line.

To understand the Poisson bracket between 't Hooft lines, we should take the semiclassical $r$-matrix, under the expansion

$$R(z) = 1 + \hbar \frac{1}{z}c + O(\hbar^2), \quad (6.6)$$

where $c \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir. In the above formula, we should use $c$ acting on the representation $R \otimes R$, which we write $c_{rs}^{ik}$. By examining the order $\hbar$ term in the RLL relation, we find the Poisson bracket between the operators $L^i_j(z)$ is

$$\{L^i_j(z), L^k_l(z')\} = \sum_{r,s} \frac{1}{z-z'}c_{rs}^{ik}L^i_r(z)L^k_s(z') - \frac{1}{z-z'}L^i_r(z)L^k_s(z')c_{jl}^{rs}. \quad (6.7)$$

This expression, in general, describes the Poisson bracket on the phase space of the 't Hooft line, where we put arbitrary charge $\mu$ at 0 and $\eta$ at $\infty$. It is shown in \[30\] that in the case
of an ADE group, this Poisson bracket matches the natural Poisson bracket on the Coulomb branch of a three-dimensional \( N = 4 \) quiver gauge theory. We will discuss the quiver gauge theory picture in more detail later.

### 7 Oscillator realizations of Q-operators

In the sequence of papers \cite{13,14} it was shown that the Q-operators for the group \( SL_n \) can be realized by a more fundamental object: an L-operator valued in an oscillator representation. For the group \( SL_2 \), the L-operator takes the form

\[
L(z) = \frac{1}{\sqrt{|z|}} \left( \begin{array}{cc} z + bc & b \\ c & 1 \end{array} \right),
\]

where \( b, c \) are the generators of a Weyl algebra, satisfying the commutation relations \([b, c] = 1\).

In this section we will show how this L-operator arises from the analysis of four-dimensional Chern-Simons theory. We will show that the phase space for the group \( SL_2 \), in the presence of an 't Hooft line of charge \((\frac{1}{2}, -\frac{1}{2})\) at 0 and \((-\frac{1}{2}, \frac{1}{2})\) at \( \infty \), is \( \mathbb{C}^2 \) with Darboux coordinates \( b, c \). Then we will calculate the L-operator that arises when we cross a Wilson line at \( z \) with these 't Hooft operators, and we will find the result written above.

Our analysis also yields the Q-operators for \( SL_n \) as constructed in \cite{14}, and for \( SO_n \) as analyzed in \cite{15}. Our approach gives a uniform construction of oscillator-valued L-operators associated to 't Hooft lines of minuscule charge for any simple group with a minuscule coweight, namely the classical groups and \( E_6, E_7 \).

#### 7.1 Phase space for the minuscule coweight of \( sl_2 \)

Let us start by analyzing the phase space for a pair of 't Hooft operators for \( SL_2 \) of charge \( \mu = (\frac{1}{2}, -\frac{1}{2}) \) at 0 and \(-\mu\) at \( \infty \). As we discussed earlier, the boundary conditions at \( z = \infty \) on the region \( y < 0 \) and \( y > 0 \) force the bundle to be trivial, so that parallel transport from \( y < 0 \) to \( y > 0 \) is a well-defined matrix valued function of \( z \).

We claim that the permitted singularities at 0 and \( \infty \) show that the possible matrices are of the form

\[
L(z) = \frac{1}{\sqrt{|z|}} \left( \begin{array}{cc} z + bc & b \\ c & 1 \end{array} \right)
\]

for arbitrary \( b, c \). Thus, the phase space is a copy of \( \mathbb{C}^2 \).

The proof of this is an explicit computation. To perform it, we work in \( PSL_2 \) instead of \( SL_2 \) and drop the normalizing prefactor, so that \( z^\mu = \text{Diag}(z, 1) \). Suppose that there are elements \( A(z), B(z) \) in \( PSL_2[[z]] \) so that

\[
A(z)z^\mu B(z) = L(z).
\]

Then, clearly, \( L(z) \) has no poles at \( z = 0 \), and \( M(0) \) is a matrix of rank 1.
Similarly, at $z = \infty$, we have
\[
\tilde{A}(z)z^\mu \tilde{B}(z) = L(z),
\] (7.4)
where $\tilde{A}(z), \tilde{B}(z)$ go to the identity at $z = \infty$. From this we see that $L(z)$ has at most a first order pole at infinity, and the polar part is $\text{Diag}(z, 0)$. Further, the term regular at infinity is of the form
\[
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \tilde{A}_{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{B}_{-1},
\] (7.5)
where $\tilde{A}_{-1}, \tilde{B}_{-1}$ are the coefficients of $z^{-1}$. Therefore the coefficient of $z^0$ in $L(z)$ is of the form
\[
L_0 = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}.
\] (7.6)
The constraint that $L_0$ has determinant 0 forces $a = bc$, so that
\[
L(z) = \begin{pmatrix} z + bc & b \\ c & 1 \end{pmatrix}
\] (7.7)
as desired.

We have shown that the ‘t Hooft line, with charge $\left(\frac{1}{2}, -\frac{1}{2}\right)$ at 0 and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ at 0, has phase space $\mathbb{C}^2$ and classical L-operator as above.

This is a classical limit of the L-operator which gives rise to $Q$-functions [13] in $\mathfrak{sl}_2$ spin chains. For this L-operator, $b$ and $c$ live in an oscillator algebra with commutation relations $[b, c] = \hbar$. We will show shortly that for the ‘t Hooft line, the Poisson bracket is $\{b, c\} = 1$.

This computation is a proof, to leading order, that the Q-operator is given by the ‘t Hooft line.

### 7.2 A general minuscule coweight

Now let us consider an ‘t Hooft line of charge $\mu$ where $\mu$ is a minuscule coweight in the adjoint form (necessarily in the simply connected form $\mu$ is fractional). We place charge $-\mu$ at $\infty$. We will find that the phase space is, as in the case of $SL_2$ discussed above, a symplectic vector space, which quantizes into an oscillator algebra. The L-operator obtained from a Wilson line crossing the ‘t Hooft line has a very simple general form.

Let us first recall the definition of minuscule coweight [31]. A minuscule coweight $\mu$ in the Cartan of a simple Lie algebra $\mathfrak{g}$ is characterized by the fact that the only eigenvalues of $\mu$ on $\mathfrak{g}$ are $-1, 0, 1$. We denote the $\pm 1$ eigenspaces by $\mathfrak{n}^\pm$, and the 0 by $\mathfrak{t}_\mu$. Note that $[\mathfrak{n}^\pm, \mathfrak{n}^\pm] = 0$. We can form a parabolic subalgebra $\mathfrak{p} = \mathfrak{t} \oplus \mathfrak{n}^+$ consisting of the charge $\geq 0$ elements of $\mathfrak{g}$. The exponential $P$ of $\mathfrak{p}$ is a parabolic subgroup of $G$, and the quotient space $G/P$ plays an important role in the theory.

Minuscule coweights are closely related to symmetric spaces, since we get a $\mathbb{Z}/2$ grading on $\mathfrak{g}$ where given by reducing the eigenvalues of $\mu$ modulo 2. The symmetric spaces associated to minuscule coweights are Hermitian.
For example, the minuscule coweights of $PSL_n$ are of the form
\[
\mu_k = \text{Diag}(z, \ldots , z, 1, \ldots , 1)
\] (7.8)
for $k = 1, \ldots , n - 1$. When viewed as fractional cocharacters of $SL_n$, they are
\[
\mu_k = \text{Diag}(z^{1-k/n}, \ldots , z^{1-k/n}, z^{-k/n}, \ldots , z^{-k/n}).
\] (7.9)

The 0 eigenspace of the minuscule coweight we call $l_\mu$; it is a Levi factor of the parabolic $p_\mu$ consisting of the 0 and 1 eigenspaces. The Levi factor corresponding to $\mu_k$ is
\[
l_k = \mathfrak{sl}_k \oplus \mathfrak{sl}_{n-k} \oplus \mathbb{C} \cdot \mu_k,
\] (7.10)
where here we view $\mu_k$ as an element of the Cartan Lie algebra $\mathfrak{h}$. The parabolic subalgebra $p$ is the subalgebra of block-upper triangular matrices
\[
p = \mathfrak{sl}_k \oplus \mathfrak{sl}_{n-k} \oplus \mathbb{C} \cdot \mu_k \oplus \text{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k}).
\] (7.11)
This has the feature that, when we exponentiate to a subgroup $P \subset SL_n$, we have
\[
SL_n/P = \text{Gr}(k, n).
\] (7.12)

Now let us analyze the phase space of an 't Hooft line with charge $\mu$ at 0 and $-\mu$ at $\infty$, where $\mu$ is minuscule. The singularity at 0 means that we can write
\[
L(z) = A(z)z^\mu B(z),
\] (7.13)
where $A(z), B(z)$ are regular at 0. We can use the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus l \oplus \mathfrak{n}^+$ to decompose the elements $A(z), B(z)$:
\[
A(z) = e^{a^+(z)}A_0(z)e^{a^-(z)},
\] (7.14)
\[
B(z) = e^{b^+(z)}B_0(z)e^{b^-(z)},
\] (7.15)
where $a^\pm(z), b^\pm(z)$ are valued in $\mathfrak{n}^\pm$, and $A_0, B_0$ are valued in $L$, the exponentiation of $l$. Now,
\[
z^\mu e^{a^\pm(z)} = e^{z^\pm 1 a^\pm(z)}z^\mu.
\] (7.16)
Also, $z^\mu$ commutes with $A_0(z), B_0(z)$. Then we can write
\[
L(z) = e^{a^+(z)}A_0(z)e^{a^-(z)}z^\mu e^{b^+(z)}B_0(z)e^{b^-(z)}
\]
\[
= e^{a^+(z)}A_0(z)e^{a^-(z)}e^{b^+(z)}z^\mu B_0(z)e^{b^-(z)}.
\] (7.17)
Since $e^{a(z)}e^{zb(z)}$ is an element of $G[[z]]$ which is in $\text{Exp}_-$ at $z = 0$, we can write it in the form

$$e^{a(z)}e^{zb(z)} = e^{zb(z)}M_0(z)e^{a(z)},$$

(7.18)

where $M_0(z)$ is valued in $L$ and is the identity at $z = 0$. Further, we can move elements $M_0(z)$, $B_0(z)$, $A_0(z)$ past the elements $e^{a(z)}$, $e^{b(z)}$ at the price of conjugating them by $B_0(z)$ or $A_0(z)$. Thus, by re-defining $a^\pm(z)$, $b^\pm(z)$ we find that $L(z)$ can be written in the form

$$L(z) = e^{a^+(z)+zb^+(z)}z^\mu C_0(z)e^{a^-(z)+b^-(z)}$$

(7.19)

Here, all expressions are regular at 0, so that we can absorb $zb^+(z)$ into $a^+(z)$ and $za^-(z)$ into $b^-(z)$ to give an expression of the form

$$L(z) = e^{a^+(z)}z^\mu C_0(z)e^{b^-(z)}.$$  

(7.20)

Using the same analysis with the decomposition

$$L(z) = \tilde{A}(z)z^\mu \tilde{B}(z),$$

(7.21)

where $\tilde{A}(z)$, $\tilde{B}(z)$ take value 1 at $\infty$ we get

$$L(z) = e^{zb^+(z)}z^\mu \tilde{C}_0(z)e^{z\tilde{a}^-(z)},$$

(7.22)

where $\tilde{a}^-(z)$, $\tilde{b}^+(z)$ vanish at $z = \infty$.

In the analysis at 0 and $\infty$, we have decomposed $L(z)$ as a product of an element of $N^+((z))$, $L((z))$, and $N^-((z))$. This decomposition is unique. Since

$$e^{zb^+(z)}z^\mu \tilde{C}_0(z)e^{z\tilde{a}^-(z)} = e^{a^+(z)}z^\mu C_0(z)e^{b^-(z)},$$

(7.23)

we have

$$\tilde{z}b^+(z) = a^+(z),$$

(7.24)

$$\tilde{z}a^- = b^-(z),$$

(7.25)

$$\tilde{C}_0(z) = C_0(z).$$

(7.26)

Imposing the constraints that $\tilde{C}_0(z)$ is the identity at $\infty$ and $C_0(z)$ is regular at 0, we find that $C_0(z) = 1$.

Next, $\tilde{a}^\pm(z)$, $\tilde{b}^\pm(z)$ have series expansions in $n^\pm$ as series in $z^{-1}$, with no constant term. Similarly, $a^\pm(z)$, $b^\pm(z)$ have series expansions in $z$, where the constant term is allowed. Identifying both sides, we find that $a^+(z)$ and $b^-(z)$ are constant. We let $X^+$, $X^-$ be the values of $a^+(z)$, $b^-(z)$ at 0. We find that we can write $L(z)$ uniquely as

$$L(z) = e^{X^+}z^\mu e^{-X^-}$$

(7.27)

$$= e^{X^-}z^\mu e^{X^+}/z.$$  

(7.28)
Let us now compute this for minuscule coweights of $\mathfrak{s}l_n$. We will focus on those which are in the Weyl orbits of $\mu = (1, 0, \ldots, 0)$, and, because the L-operator becomes conjugate by a Weyl transformation when we choose a different element in the Weyl orbit of $\mu$, we only need to compute the L-operator for this element. In this case, $\mathfrak{n}^+$ consists of those matrices $M^+_j$ whose only non-zero entries are $M^+_j, j > 1$; and $\mathfrak{n}^-$ consists of the matrices with non-zero entries $M^+_1, j > 1$.

The decomposition of $L(z)$ as $e^{X^+} z^{\mu} e^{X^-}$ is

$$L(z) = \begin{pmatrix}
1 & b_1 & \cdots & b_{n-1} \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
z & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
c_1 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
c_{n-1} & 0 & \cdots & 1
\end{pmatrix}
= \left( \begin{array}{cccc}
z + \sum b_i c_i & b_1 & \cdots & b_{n-1} \\
c_1 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
c_{n-1} & 0 & \cdots & 1
\end{array} \right),$$

(7.29)

This is the same L-operator (in the semiclassical limit) as derived in [14]. This paper also included more general L-operators, but these are simply obtained from the one we have derived by a Weyl group transformation which reorders the rows and columns. From our perspective, they are obtained by applying a Weyl group element to the charge at $\infty$, and so replacing it by a different coweight in the same Weyl orbit.

### 7.3 The Poisson bracket on the phase space for $\mathfrak{gl}_n$

Let us consider the Poisson bracket on the phase space in the case of the coweight of $\mathfrak{gl}_n$ of charge $(1, 0, \ldots, 0)$. We have seen that the L-operator (7.29) is the same as that studied by Bazhanov et al [14], with $L^1_i(z) = b_i$, $L^1_j(z) = c_j$. The quadratic Casimir for $\mathfrak{gl}_n$ is

$$c^{ij}_{kl} = \delta^i_k \delta^j_l.$$  

(7.30)

Therefore the Poisson bracket takes the form

$$(z - z') \{L^1_i(z), L^1_j(z')\} = L^1_i(z)L^1_j(z') - L^1_j(z)L^1_i(z').$$  

(7.31)

Since $L^1_i(z) = z + b_i c^i$, and $L^1_j(z) = \delta^i_j$ if $i, j > 1$, we find

$$\{b_i, c_j\} = \delta^i_j$$  

(7.32)

as in [14].
7.4 The Poisson bracket in general

Now let us return to the case of a general minuscule coweight $\mu$ in a simple group $G$ in which we have a minuscule coweight $\mu$ at 0 and we have $-\mu$ at $\infty$. In this case, it is convenient to take the Wilson line to live in the adjoint representation.\(^8\) We decompose $\mathfrak{g}$ as above into $\mathfrak{n}^+ \oplus \mathfrak{l} \oplus \mathfrak{n}^-$ and take a basis $X^m$ for $\mathfrak{n}^+$, and $Y_n$ for $\mathfrak{n}^-$. We then write, as before

$$L(z) = e^{b_m X^m} z^\mu e^{c_n Y_n}.$$  

(7.33)

The $b_m$, $c_n$ are coordinates on the phase space. We choose a basis so that the non-degenerate pairing between $\mathfrak{n}^+$ and $\mathfrak{n}^-$ coming from the Killing form is $\delta_{mn}^n$. The coordinates $b_m$, $c_n$ can be recovered by studying how $L(z)$ acts on the elements $X^m$, $Y_n$ and $\mu$ in the adjoint representation:

$$\langle \mu, L(z) Y_n \rangle = z^{-1} \langle \mu, b_m [X^m, Y_n] \rangle = -z^{-1} b_n,$$  

(7.34)

$$\langle X^m, L(z) \mu \rangle = z^{-1} c^m,$$  

(7.35)

where $\langle -, - \rangle$ is the Killing form.

We will write the matrix elements of the components of $L(z)$ which act on the space $\mathfrak{n}_- \oplus \mathbb{C} \cdot \mu$, using the index 0 to indicate $\mu$.

$$\langle \mu, \mu \rangle L^0_n(z) = -z^{-1} b_n,$$  

(7.36)

$$L_0^n(z) = z^{-1} c^n,$$  

(7.37)

$$L^0_0(z) = 1 - c^m b_m z^{-1} \langle \mu, \mu \rangle^{-1},$$  

(7.38)

$$L^m_n = z^{-1} \delta^m_n.$$  

(7.39)

The quadratic Casimir has three terms, living in $\mathfrak{l} \otimes \mathfrak{l}$ and $\mathfrak{n}^\pm \otimes \mathfrak{n}^\mp$. Only the last two terms will enter the Poisson bracket relation. These terms are

$$X^m \otimes Y_m + Y_m \otimes X^m.$$  

(7.40)

Acting on the elements $Y_m$, $\mu$, we have

$$c^{m0}_{00} = \frac{1}{\langle \mu, \mu \rangle} \delta^m_n, \quad c^{0m}_{00} = \frac{1}{\langle \mu, \mu \rangle} \delta^m_n.$$  

(7.41)

The Poisson bracket relation in this case becomes

$$\langle \mu, \mu \rangle (z - z') \{ L^0_n(z), L^m_0(z') \} = L^m_n(z) L^0_0(z') - L^0_0(z) L^m_n(z')$$  

(7.42)

$$\delta^m_n \frac{1}{zz'} (z' - c^k b_k) - \delta^m_n \frac{1}{zz'} (z - c^k b_k) = \delta^m_n \frac{z' - z}{zz'}. $$  

(7.43)

From which we conclude,

$$\{ b_n, c^m \} = \delta^m_n.$$  

(7.44)

\(^8\)Note that outside of type $A$, the adjoint representation does not lift to a quantum Wilson line. The anomaly, as analyzed in [2], occurs at two loops and does not affect the Poisson bracket computation.
7.5 Fundamental coweight of $SO(2n)$

There are three Weyl orbits of minuscule coweights for $SO(2n)$. The first is associated to an embedding

$$\mu: SO(2) \to SO(2n)$$

(7.45)

corresponding to a choice of an orthogonal decomposition $\mathbb{C}^{2n} = \mathbb{C}^2 \oplus \mathbb{C}^{2n-2}$ of the vector representation. There are $2n$ elements in the Weyl orbit of this minuscule coweight. Indeed, if we identify the maximal torus of $SO(2n)$ with $SO(2)^n$ under the obvious embedding, the minuscule coweights in this Weyl orbit are of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$.

The Levi subgroup in this case is

$$I = \mathfrak{so}(2) \oplus \mathfrak{so}(2n-2) \subset \mathfrak{so}(2n).$$

(7.46)

The subalgebras $\mathfrak{n}_+, \mathfrak{n}_-$ are the elements of $\mathfrak{so}(2n)$ of charge $\pm 1$ under the action of $SO(2)$. Take an orthonormal basis $x_1, x_2, y_1, \ldots, y_{2n-2}$ of the vector representation $\mathbb{C}^{2n}$. The adjoint representation is $\wedge^2 \mathbb{C}^{2n}$. The subspaces $\mathfrak{n}_\pm$ are each of dimension $2n-2$ and are spanned by $x_\pm \wedge y_j$, $j = 1, \ldots, 2n-2$, where $x_\pm = \frac{1}{\sqrt{2}}(x_1 \pm iz_2)$.

The L-operator for a Wilson line crossing a minuscule 't Hooft line is, according to our calculations earlier,

$$L(z) = e^{(x_+ \wedge y_j)b_j} z^\mu e^{(x_- \wedge y_k)c_k},$$

(7.47)

where $\{b_j, c_k\} = \delta_{jk}$, $j = 1, \ldots, 2n-2$.

We will calculate this in the vector representation. First note that

$$e^{(x_+ \wedge y_j)b_j}x_- = x_- - y_j b_j - \frac{1}{2}x_+ b_j b_j,$$

(7.48)

and similarly with $x_+$ and $x_-$ switched. From this we have

$$L(z)x_+ = zx_+ - y_j c_j - x_+ b_j c_j - z^{-1} \frac{1}{2} x_- c_j c_j + z^{-1} \frac{1}{2} y_k b_k c_j c_j + z^{-1} x_+ b_j c_k c_k,$$

(7.50)

$$L(z)x_- = z^{-1} (x_+ - b_j y_j - \frac{1}{2} x_+ b_j b_j),$$

(7.51)

$$L(z)y_j = y_j + x_+ b_j + z^{-1} c_j (x_- + y_k b_k + \frac{1}{2} x_+ b_j b_j).$$

(7.52)

This matrix is the same as the L-operator studied in [32], equation (4.1), up to a change of basis and multiplying by an overall factor of $z$. They use a basis of the form $x^1_\pm, \ldots, x^r_\pm$ with $x^1_\pm = x_\pm$, $x^{2k}_\pm = y_{2k-1} \pm iy_{2k}$.

7.6 Spinor coweights of $SO(2n)$

The other two Weyl orbits of minuscule coweights for $SO(2n)$ are associated to the two spinor nodes of the Dynkin diagram. Choose a basis $x_1, \ldots, x_n, y^1, \ldots, y^n$ of the vector representation $\mathbb{C}^{2n}$ under which $\langle x_i, y_j \rangle = \delta_{ij}$. As usual we identify $\mathfrak{so}(2n)$ with $\wedge^2 \mathbb{C}^{2n}$. A basis for the coweight lattice is given by the expressions $x_1 \wedge y^1, \ldots, x_n \wedge y^n$. This is the
standard basis of the maximal torus associated to the natural embedding $SO(2)^n \subset SO(2n)$, where each copy of $SO(2)$ rotates one of the planes spanned by $x_i, y_i$.

In this basis, the spinorial minuscule coweights are given by

$$\sum(-1)^{k_i} \frac{1}{2}(x_i \wedge y_i), \quad (7.53)$$

where $k_i = 0, 1$. These are minuscule, because the basis vectors $x_i, y_j$ have charge $\pm \frac{1}{2}$ so that every element in the adjoint representation $\wedge^2 \mathbb{C}^{2n}$ has charge $0$ or $\pm 1$.

Thus, there are $2^n$ minuscule coweights of this form. An element of the Weyl group can switch an even number of the signs, so that there are two Weyl orbits of minuscule coweights, each with $2^{n-1}$ elements. The two Weyl orbits are related by the outer automorphism of $SO(2n)$.

We will calculate the L-operator associated to the coweight $\sum \frac{1}{2} x_i \wedge y_i$, with all signs $+1$. The L-operator for other coweights in the same Weyl orbit can be determined by conjugating by an element of the Weyl group, and the L-operator for elements in the other Weyl orbit can be obtained by conjugating with the matrix in $O(2n)$ which switches $x_1$ and $y_1$.

For the coweight $\sum \frac{1}{2} x_i \wedge y_i$, the elements $x_i \wedge x_j$ are of charge $+1$, $x_i \wedge y^1$ are of charge $0$, and $y^i \wedge y^j$ are of charge $-1$. Thus, the Levi $l$ is $\mathfrak{gl}(n)$, and $\mathfrak{n}_\pm$ are the exterior squares of the fundamental and antifundamental representations of $\mathfrak{gl}(n)$, and are of dimension $\binom{n}{2}$. As usual, we introduce oscillators $b^{rs} \in \mathfrak{n}^\vee_+$, $c_{rs} \in \mathfrak{n}^\vee_-$, which are antisymmetric in their indices and satisfy \{ $b^{rs}, c_{mn}$ \} = $\delta^r_m \delta^s_n - \delta^m_r \delta^s_n$.

The L-operator is

$$L(z) = e^{x_r \wedge x_s b^{rs}} z^\mu e^{y^m \wedge y^n c_{mn}}, \quad (7.54)$$

where $z^\mu x_r = z^{1/2} x_r$, $z^\mu y_r = z^{-1/2} y_r$.

Thus,

$$L(z) y^r = z^{-1/2} (y^r + 2b^{sr} x_s), \quad (7.55)$$

$$L(z) x_r = z^{1/2} x_r + z^{-1/2} c_{sr} y^s + 4z^{-1/2} x_s b^{sm} c_{mr}. \quad (7.56)$$

Multiplying by $z^{1/2}$ and re-ordering the basis elements we find the expression computed in [32], equation (4.11).

### 7.7 A sketch of the L-operator for the minuscule coweight of $E_6$

So far, we given a presentation of the L-operators associated to minuscule ‘t Hooft lines of $SL_n$ and $SO(2n)$, recovering known expressions for Q-operators. As a final example in this section, we will present something a little new, which is the L-operator for the minuscule coweight of $E_6$. There is only one Weyl orbit of minuscule coweight in this case. The Levi factor is $l = \mathfrak{so}(10) \oplus \mathfrak{so}(2)$, and the subspaces $\mathfrak{n}_\pm$ are the two spin representations of $\mathfrak{so}(10)$. We choose our coweight so that $\mathfrak{n}_\pm = S_\pm$. The 27 dimensional representation of $E_6$ decomposes, under $\mathfrak{so}(10) \oplus \mathfrak{so}(2)$, as the vector representation of $\mathfrak{so}(10)$ of charge $2/3$ under $\mathfrak{so}(2)$; the spin representation $S_+$ of charge $-1/3$; and the trivial representation of charge $-4/3$. 

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Let $x_i$ be a basis for the vector representation of $\mathfrak{so}(10)$, and $\psi_\alpha$, $\psi^\alpha$ bases for the spin representations $S_\pm$. A basis for the 27 dimensional representation of $E_6$ is given by $v_i$ in the trivial representation, $x_i$, and $\psi_\alpha$. A basis for the adjoint representation of $E_6$ is $x_i \wedge x_j$, $\Psi_\alpha$, $\bar{\Psi}_\alpha$. The action on the subspace of $E_6$ spanned by $\Psi_\alpha$, $\bar{\Psi}_\beta$ on the 27 dimensional representation is given by

\begin{align}
\Psi_\alpha \cdot v &= \psi_\alpha, \\
\Psi_\alpha \cdot \psi_\beta &= \Gamma^i_{\alpha\beta} x_i, \\
\Psi_\alpha \cdot x_i &= \Gamma^i_{\alpha\beta} \psi_\beta, \\
\Psi^\alpha \cdot \psi_\beta &= \delta_\alpha^\beta v.
\end{align}

Here $\Gamma$ is the ten dimensional $\Gamma$ matrix intertwining the vector representation with $S_+ \otimes S_-$ (appropriately normalized).

For the L-operator, there are oscillators $b^\alpha$, $c_\beta$ satisfying $\{b^\alpha, c_\beta\} = \delta_\alpha^\beta$. The L-operator takes the form

\begin{equation}
L(z) = e^{\Psi_\alpha b^\alpha} z^\mu e^{\Psi^\beta c_\beta},
\end{equation}

where $z^\mu v = z^{-4/3} v$, $z^\mu x_i = z^{2/3} x_i$, $z^\mu \psi_\alpha = z^{-4/3} \psi_\alpha$. We will not attempt to give a more explicit form for $L$, leaving it to the interested reader to compute further.

We have given a uniform presentation of the semiclassical L-operator associated to a minuscule 't Hooft line (with opposite charge at $\infty$). In each case, we find that the L-operator lives in the semiclassical limit of an oscillator algebra. These oscillator representations generalize those studied in type A by [14] and for type D in [32].

## 8 \ 't Hooft lines and the shifted Yangian

It has recently become clear [19] that Q-operators arise from representations of a certain shifted Yangian. One can view the starting point for this development as the work of Bazhanov et al [13], reviewed above, where it was shown that the Q-operator arises from an L-operator valued in an oscillator algebra. These authors viewed this L-operator as providing an unusual representation of the Yangian. This, however, is an error in terminology: the Yangian algebra has an RLL presentation where the L-operator is required to have leading term, as a series in $1/z$, the identity. As we have seen, the L-operators giving rise to Q-operators have leading term $z^\mu$, where $-\mu$ is the charge of the 't Hooft line at $\infty$.

In [19] (see also [33]) it was shown that the correct interpretation of the L-operators of [13] is that they provide a representation of the antidominant shifted Yangian.

Here, we will review their work and explain how the antidominant shifted Yangian appears naturally in four-dimensional Chern-Simons theory with an 't Hooft line at $\infty$. We will also propose a conjectural RLL description of the antidominant shifted Yangian for all groups except $E_8$, generalizing the description of the shifted Yangian given in type A in [19] and of the ordinary Yangian in [3].
First, let us make a comment on the terminology. In the literature [17, 18], there is a shifted Yangian associated to any coweight $\mu$ of the Lie algebra. Somewhat confusingly, this construction is not covariant under the action of the Weyl group. This means that the shifted Yangian associated to two coweights in the same Weyl orbit are not isomorphic. This is in contrast to the gauge-theory constructions of this paper, where everything is always covariant under the $G$ global symmetry.

Our construction will give an algebra associated to any coweight $\eta$, which is isomorphic to the antidominant shifted Yangian associated to the antidominant coweight in the Weyl orbit of $\eta$.

Let us now turn to our description of the shifted Yangian. Let $G$ be a simple and simply-connected group, and let $\eta$ be a (possibly fractional) coweight. Consider four-dimensional Chern-Simons theory with an 't Hooft line of charge $\eta$ at $z = \infty, y = 0$. Consider an arbitrary collection of line defects at generic values of $z$, again at $y = 0$. To absorb the monopole charge at $\infty$, these can not be pure Wilson lines, but must be Wilson-'t Hooft lines.

Let us compactify this four-dimensional system to two dimensions along the $\mathbb{CP}^1$ with coordinate $z$. In the region $y < 0$ and $y > 0$, the result is the trivial theory. This is because we are working with a simple group, so that the only solution to the equations of motion is the trivial one, and all fields are infinitely massive. Along the line $y = 0$ we find an effective quantum mechanical system with some algebra of operators $A$.

We will show that $A$ acquires a homomorphism from an infinite-dimensional algebra called the antidominant shifted Yangian.

To do this, we will follow the analysis of [3], and consider a Wilson line along $x = 0$ at some value of $z$ near $\infty$. We must specify the representation this Wilson line lives in. For the classical groups $SL_n, SO_n, Sp_n$ we take the vector representation. For $G_2$ we take the 7-dimensional representation, $F_4$ the 26, $E_6$ the 27, and $E_7$ the 56. The Wilson line corresponding to each of these representations exists at the quantum level. Further, in each case the group can be realized as the group of invertible matrices acting on the chosen representation which preserves certain tensors. As explained in [3], in each case the tensors lift to junctions of Wilson lines.

Now consider placing an 't Hooft line at $y = 0$ at $z = \infty$, a Wilson line at $x = 0$ at $z$ near $\infty$, and some arbitrary line defects at other points in the $z$-plane, along $y = 0$. As above, algebra of operators on the horizontal line defects is $A$. A vertical line defect, with incoming state $\langle i |$ and outgoing state $| j \rangle$, gives an operator $L^i_j(z) \in A$ as in Figure 3.

The presence of the 't Hooft line constrains the behaviour of $L^i_j(z)$ near $z = \infty$. In the absence of the 't Hooft line, we have $L^i_j(z) = \delta^i_j 1_A + O(z^{-1})$. In the presence of the 't Hooft line at $\infty$, this is changed as follows. Assume that we have a basis of the representation $R$ where the Wilson line lives in which the coweight $\eta$ of the 't Hooft line acts on the $i$th basis element by $\eta_i$. We ask that, near $z = \infty$, there is an expansion of the form

$$L^i_j(z) = \alpha^i_k(z) \delta^k_i z^{-\eta_k} \beta^j_l(z), \quad (8.1)$$
Figure 3. A vertical Wilson line, equipped with an incoming state $\langle i \vert$ and an outgoing state $\vert j \rangle$, gives rise to an operator $L^i_j(z) \in A$ in the quantum algebra of the horizontal ’t Hooft line.

where $\alpha^i_k(z)$, $\beta^j_l(z)$ are series in $1/z$ with entries in $A$, whose leading term is $\delta^i_j 1_A$.

We can think of this expression as saying that $L^i_j(z)$ looks, near $\infty$, like the monopole singularity $z^\mu$, multiplied on either side by “perturbative” contribution $s$ arising from gluon exchange between the vertical Wilson line and whatever horizontal line defects we have. The gauge theory construction implies that, in perturbation theory, this is the most general possible form of $L(z)$. Indeed, we are clearly free to multiply on the left and the right by the monopole singularity $z^\mu$, and further any small variation of the monopole singularity $z^\mu$ coming from quantum effects can be absorbed into a left or right multiplication by $\alpha(z)$, $\beta(z)$ as above.

For example, with a coweight of charge $(\frac{1}{2}, -\frac{1}{2})$ for $SL_2$, the operator $L^i_j(z)$ will have expansion at $\infty$,

$$L(z) = G(z) \left( z + l^1_1[-1] + l^1_1[0]z^{-1} + \ldots \quad l^1_1[0]z^{-1} + \ldots \right) ,$$

where the overall normalizing factor $G(z)$ will be determined by the requirement that the quantum determinant is one, as we will see later. Thus, the generators of the shifted Yangian algebra we find are $l^1_1[k]$, $k \geq -1$; $l^2_2[k]$, $k \geq 1$; and $l^1_2[k]$, $l^2_1[k]$ for $k \geq 0$.

The gauge-theory construction implies that the generating function $L(z)$ satisfies the RLL relation, as we have already discussed semiclassically (see Figure 2). As explained in this diagram, this is a consequence of the fact that Wilson lines crossing above or below the ’t Hooft line give the same elements of the algebra $L$. The algebraic form of the relation is

$$\sum_{r,s} R^{ik}_{rs}(z - z') L^i_j(z') L^i_k(z) = \sum_{r,s} L^i_r(z) L^k_s(z') R^{rs}_{ij}(z - z').$$

It was shown in [19] that, with gauge group $GL_n$, this RLL relation, together with the boundary condition on the behaviour of $L(z)$ at $\infty$, gives rise to the shifted Yangian for $\mathfrak{gl}_n$. 

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8.1 Extra relations from other groups

To understand the algebra we get for other groups, we need to introduce extra relations that come from junctions of Wilson lines [3]. Suppose we have $k$ Wilson lines in the gauge theory with gauge Lie algebra $\mathfrak{g}$, in representations $V_1, \ldots, V_k$. Suppose that there is a $\mathfrak{g}$-invariant element of the tensor product $V_1 \otimes \cdots \otimes V_k$. Classically, this provides a junction between the Wilson lines, when they are all placed at the same value of the spectral parameter. For example, if we have $n$ copies of the fundamental representation of $\mathfrak{sl}_n$, then the determinant defines an invariant vector in $V \otimes^n$ and so a (classical) junction between Wilson lines.

In [3], we analyzed when we can lift these junctions to the quantum level. The general story is involved: sometimes you can, and sometimes there is an anomaly. When the junctions can be lifted to the quantum level, the spectral parameters of the various lines have to have certain very special shifts. For example, for the determinant junction between $n$ fundamental Wilson lines of $\mathfrak{sl}_n$, there is a lift to a junction at the quantum level provided that the spectral parameters of the $n$ lines are $z, z + 2h, \ldots, z + 2(n - 1)h$.

Consider a simple group presented as a matrix group preserving some additional invariant tensors. For example, we can present $SO(n)$ as the group of automorphisms of $\mathbb{C}^n$ preserving a symmetric pairing, or $G_2$ as the group of automorphisms of $\mathbb{C}^7$ preserving an antisymmetric cubic tensor and a non-degenerate symmetric pairing. We can attempt to lift this presentation of the group to the gauge theory setting, by lifting the invariant tensors defining the group to junctions between the defining Wilson line.

In [3], we succeeded in doing this for all groups except $E_8$, with particular shifts. The results are as follows (here $h^\vee$ is the dual Coxeter number).

1. For type $A$, the determinant junction lifts with the spectral parameters $z, z + 2h, \ldots, z + 2(n - 1)h$.

2. For $\mathfrak{so}_n, \mathfrak{sp}_n$ the junction defined by the symmetric or antisymmetric pairing lifts to the quantum level with spectral parameters $z, z + hh^\vee$.

3. For $G_2$, the group is defined by preserving the quadratic and cubic invariant tensors on $\mathbb{C}^7$. The corresponding junctions exist at the quantum level with shifts $z, z + hh^\vee$ for the quadratic vertex and $z, z + \frac{2}{3}hh^\vee, z + \frac{4}{3}hh^\vee$ for the cubic vertex.

4. The group $F_4$ is the automorphisms of $\mathbb{C}^{26}$ preserving a symmetric pairing and a cubic tensor. The vertices lift to the quantum level with spectral parameters $z, z + hh^\vee$ for the quadratic vertex and $z, z + \frac{2}{3}hh^\vee, z + \frac{4}{3}hh^\vee$ for the cubic vertex.

5. For $E_6$, the group is the automorphisms of $\mathbb{C}^{27}$ which preserve a cubic tensor. This vertex lifts to the quantum level with spectral parameters $z, z + \frac{2}{3}hh^\vee, z + \frac{4}{3}hh^\vee$ for the cubic vertex.

6. For $E_7$, the group is the automorphisms of $\mathbb{C}^{56}$ preserving a symplectic pairing and a quartic tensor. These vertices lift to the quantum level with the shifts $z, z + hh^\vee$ for the quadratic vertex and $z, z + \frac{1}{2}hh^\vee, z + hh^\vee, z + hh^\vee, z + hh^\vee$ for the quartic vertex.
Figure 4. Topological invariance allows us to move the position of the horizontal ’t Hooft line defects past the collection of Wilson lines attached to the vertex without affecting the result (We do not have a similar construction for $E_8$, because the smallest representation of $E_8$, which is the adjoint, does not lift to a quantum Wilson line).

The existence of the vertex between Wilson lines implies extra relations, beyond the RLL relation, on the universal algebra that acts on a line defect when we have an ’t Hooft line at $\infty$. The extra relation is sketched in Figure 4: we can contemplate moving an ’t Hooft line past a network of Wilson lines. This will have no effect, by topological invariance of the theory in the $x$-$y$ plane, leading to a relation among the algebra elements in the series $L^i_j(z)$.

For example, for $\mathfrak{sl}_n$, the relation is the quantum determinant relation

$$\sum_{k_r} \text{Alt}(k_0, \ldots, k_{N-1}) L^0_0(z) L^1_1(z + 2\hbar) \cdots L^{k_{N-1}}_{N-1}(z + 2(N-1)\hbar) = 1.$$  \hfill (8.4)

For $\mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, the extra relation we get is

$$\omega_{ij} L^k_i(z) L^j_j(z + \hbar v_e) = \omega_{ij},$$  \hfill (8.5)

where $\omega_{ij}$ is the invariant pairing on the vector representation (symmetric or antisymmetric as appropriate).

For the other groups, we have a similar presentation. For instance, for $E_7$, we have one extra relation which is

$$\Omega_{ijkl} L^i_m(z) L^j_n(z + \frac{1}{2} \hbar v \epsilon e) L^k_o(z + \hbar v) L^l_p(z + \frac{3}{2} \hbar v) = \Omega_{mnop}.$$  \hfill (8.6)

In each case, the RLL relation, the extra relations coming from vertices between Wilson lines, and the boundary condition at $z = \infty$, define an associative algebra $Y^\mu(\mathfrak{g})$.

**Conjecture 1.** The algebra $Y^\mu(\mathfrak{g})$ is isomorphic to the antidominant shifted Yangian.

At the classical level, this holds because of the results of [30]. In general, a sufficiently strong uniqueness theorem for the shifted Yangian as a quantization of its classical limit (compatible with additional structures such as coproducts) will prove that our algebra $Y^\mu(\mathfrak{g})$ is isomorphic to the shifted Yangian, but such a result seems not to be currently available.
8.2 Coproducts

It is known [34] that the shifted Yangian admits left and right coproducts relating it to the ordinary Yangian:

\[ \Delta_L : Y^\mu(g) \to Y(g) \otimes Y^\mu(g), \]
\[ \Delta_R : Y^\mu(g) \to Y^\mu(g) \otimes Y(g). \]  

(8.7)

The left and right coproducts commute with each other and are co-associative, making the category of \( Y^\mu(g) \)-modules into a bimodule category over the category of \( Y(g) \)-modules.

From the field theory perspective, the category of \( Y^\mu(g) \) modules is the category of line defects\(^9\) in the bulk with a parallel 't Hooft line of charge \( \mu \) at \( \infty \).

The coproduct tells us that we can fuse an ordinary line defect coming from the left or the right with a line defect parallel to the 't Hooft line at \( \infty \).

From our definition of \( Y^\mu(g) \) and the presentation of \( Y(g) \) presented in [3], the left and right coproducts are easy to define. We present \( Y(g) \) as being generated by the coefficients of an L-operator \( L^0(z) \) which near \( \infty \) goes like \( 1 + O(1/z) \). These are subject to the RLL relation, as well as the extra relations coming from junctions between line defects.

Similarly, \( Y^\mu(g) \) is presented as above as generated by the coefficients of \( L^\mu(z) \) subject to the RLL relation as well as the relations arising from junctions between Wilson lines.

To give a coproduct

\[ \Delta_R : Y^\mu(g) \to Y^\mu(g) \otimes Y(g) \]  

we need to give an L-operator \( \Delta^\mu_R(z) \in Y^\mu(g) \otimes Y(g) \) satisfying the relations defining \( Y^\mu(g) \). Since \( Y(g) \) and \( Y^\mu(g) \) are defined in terms of the coefficients of the L-operators \( L^0(z), L^\mu(z) \) we must write \( \Delta_R L^\mu(z) \) in terms of \( L^0(z) \) and \( L^\mu(z) \).

We calculate \( \Delta_R L^\mu(z) \) by considering a vertical Wilson line passing two horizontal line defects representing representations of \( Y(g) \) and \( Y^\mu(g) \), as in Figure 5.

From this diagram, it is clear that the coproduct must be given by

\[ \Delta_R L^\mu_j(z) = L^\mu_j(z) L^0_{k,j}(z), \]  

(8.9)

where we treat both sides as series in \( 1/z \) and identify coefficients.

In order to show that this defines a homomorphism from \( Y^\mu(g) \) to \( Y^\mu(g) \otimes Y(g) \), we need to show that \( \Delta_R L^\mu(z) \) as defined above satisfies the relations defining \( Y^\mu(g) \). That is, \( \Delta_R L^\mu(z) \) must satisfy the RLL relation, the boundary behaviour at \( z = \infty \), and the extra relations coming from vertices between Wilson lines, assuming that \( L^0(z) \) and \( L^\mu(z) \) do.

This is not hard to check from the field theory picture. For example, in Figure 6 we present the diagram indicating why the coproduct \( \Delta_R L^\mu(z) \) respects the relations coming from vertices between Wilson lines, assuming that \( L^0(z) \) and \( L^\mu(z) \) do.

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\(^9\)At various points we need to distinguish between analytically-continued line defects, where the one-dimensional system has an algebra of operators but not a Hilbert space; and full line defects, where there is a Hilbert space too. Here for simplicity we are discussing full line defects.
8.3 More general coproducts

We can, of course, study more general coproducts, by trying a formula

$$\triangle L^{\mu + \mu'}(z) \overset{?}{=} L^{\mu}(z) L^{\mu'}(z). \quad (8.10)$$

It is not at all obvious that something like this will work, because the expression on the right hand side may not have the correct behaviour at $z = \infty$. However, we can check that if $\mu, \mu'$ both have the property that $\langle \mu, \alpha \rangle \geq 0, \langle \mu', \alpha \rangle \geq 0$ for all positive roots $\alpha$, then $\triangle L^{\mu + \mu'}(z)$ as defined by this equation has the behaviour at $z = \infty$ required of an $L$-operator of charge $\mu + \mu'$.

To see this, we note that we can write

$$L^{\mu}(z)L^{\mu'}(z) = A(z) z^{\mu} B(z) z^{\mu'} C(z), \quad (8.11)$$

where $A, B, C$ are matrices acting on the representation where our Wilson line lives which look like $1 + O(1/z)$. We can write

$$B(z) = e^{\sum_{\alpha < 0} b_{\alpha}(z)} e^{b_0(z)} e^{\sum_{\alpha > 0} b_{\alpha}(z)}, \quad (8.12)$$
where \( b_\alpha(z) \) is in the \( \alpha \) root space, and \( b_0(z) \) is in the Cartan.

Thus,

\[
L^\mu(z)L^{\mu'}(z) = A(z)e^{\sum_{\alpha<0} z^{(\mu,\alpha)}b_\alpha(z)}z^{\mu}z^{\mu'}e^{b_0(z)}e^{-\sum_{\alpha>0} z^{(\mu',\alpha)}b_\alpha(z)}C(z).
\]  

(8.13)

We have \( \langle \mu, \alpha \rangle \leq 0 \) if \( \alpha < 0 \) and \( \langle \mu', \alpha \rangle \geq 0 \) if \( \alpha > 0 \). Therefore \( A(z)e^{\sum_{\alpha<0} z^{(\mu,\alpha)}b_\alpha(z)} \) and \( e^{b_0(z)}e^{-\sum_{\alpha>0} z^{(\mu',\alpha)}b_\alpha(z)}C(z) \) are both of the form \( 1 + O(1/z) \), so that \( L^\mu(z)L^{\mu'}(z) \) is of the form required to define an \( L \)-operator of charge \( \mu + \mu' \).

As in the case when \( \mu' = 0 \), it is easy to see that \( \Delta L^{\mu+\mu'}(z) \) satisfies the RLL relation and the relations coming from vertices between Wilson lines.

One can ask if this coproduct matches the “standard” one on antidominant shifted Yangians \([17, 18, 34]\). For type \( A \), this was shown in [19], but it appears to be unknown in general (although surely true).

9 ‘t Hooft operators and Coulomb branches of three-dimensional \( \mathbb{N} = 4 \) gauge theories

In section 6.2, we have discussed the classical phase space for four-dimensional Chern-Simons theory in the presence of an ‘t Hooft operator of charge \( \mu \) at 0 and \( \eta \) at infinity. We described the phase space as the space of element in \( G((z)) \) which satisfy certain properties at 0 and \( \infty \).

The quantum version of such a description would be to describe the operator algebra \( A^\eta_\mu \) for the quantum mechanical system arising from the quantization of the phase space, as well as an algebra morphism \( Y^\eta \rightarrow A^\eta_\mu \) describing the effective coupling between the system and the internal degrees of freedom of a generic transverse Wilson line.

In this section, we will propose an answer to this question by a somewhat circuitous route. It remains to show that this answer actually coincides with the result of a perturbative calculation in the presence of the ‘t Hooft line, but we show that the answer passes important consistency checks.

The proposal is simple: the phase space coincides with the Coulomb branch of vacua of certain three-dimensional \( \mathbb{N} = 4 \) ADE quiver gauge theories, which admit a natural quantization via \( \Omega \)-deformation. This deformation gives a “quantum Coulomb branch algebra” which is known to admit an algebra morphism from \( Y^\eta \), which is nicely compatible with coproducts in a way we will review \([20]\). It is our candidate for \( A^\eta_\mu \).

It is worth pointing out that the existence of the algebra morphism from \( Y^\eta \) and the compatibility with the coproducts are not obvious from the Coulomb branch description of the algebra and appear rather miraculous. A proper justification of our proposal would thus explain the appearance of these surprising extra structures.

Recall that four-dimensional Chern-Simons theory can be embedded into a physical gauge theory in several different ways, related by string dualities \([21, 28, 29]\). For our purposes, it is most useful to focus on the treatment of \([21]\), where the theory is identified with an \( \Omega \)-deformation of six-dimensional \( \mathbb{N} = (1,1) \) super Yang-Mills theory with gauge algebra \( \mathfrak{g} \). In
that context, ’t Hooft lines in the four-dimensional Chern-Simons theory are naturally the image of three-dimensional BPS ’t Hooft defects in the physical theory. Such defects preserve a three-dimensional $N = 4$ supersymmetry subalgebra, and the $\Omega$-deformation of the ambient theory is accompanied by a $\Omega$-deformation of the defect theory.

The six-dimensional $N = (1, 1)$ super Yang-Mills theory is IR free and requires a UV completion. An ’t Hooft defect has a somewhat hybrid status: although it is defined using the IR free degrees of freedom, the singularities associated to monopole bubbling may signal the need for an extra UV input. The situation is somewhat analogous to the treatment of instanton particles in five-dimensional maximally supersymmetric super Yang-Mills theory: although much of the phase space consists of semiclassical solitons described by instanton solutions of the IR free gauge theory, a proper treatment of the singularities associated to zero-size instantons require some UV input.

The analogy becomes even sharper if we add a couple of directions to space-time, and discuss instanton membranes in seven-dimensional ADE super Yang-Mills theory. If the gauge theory is realized at an ADE singularity in M-theory, then small instantons are M2-branes which can leave the singularity along the ADE direction. This UV process is not visible in the naive IR gauge theory description, but it is fully captured by adding the information of the low-energy supersymmetric quantum field theory which lives on zero-size instantons: a three-dimensional $N = 4$ affine ADE quiver gauge theory with no flavours. This theory has a Coulomb branch which reproduces the smooth instanton moduli space, as well as a Higgs branch which describes the motion of M2-branes away from the singularity.

In the case of ’t Hooft defects in six-dimensional $N = (1, 1)$ super Yang-Mills theory, we can ask for a low energy description of an ’t Hooft defect of charge $\mu$ “covered” by bubbling gauge configurations leading to a charge $\eta$ at infinity. We want to claim that this will be a three-dimensional $N = 4$ ADE quiver gauge theory whose Coulomb branch reproduces the space of supersymmetric gauge configurations with these charges, i.e. the Bogomolny moduli space which coincides with the four-dimensional Chern-Simons phase space discussed before. The Higgs branch of the theory, which is only present when the Coulomb branch has singularities, will describe dynamical processes which are invisible in the six-dimensional super Yang-Mills description.

The ADE quiver gauge theory is characterized by two collections of non-negative integers: the ranks $N_i$ of the $U(N_i)$ gauge groups at each node and the numbers $M_i$ of fundamental flavours attached to each node. The $M_i$ encode the charge $\mu = \sum_i M_i w_i$ of the ’t Hooft operator. The Weyl orbit of the charge at infinity $\eta$ is encoded by the imbalance at each node

$$\delta_i = \sum_j C_{ij} N_j - M_i,$$

where $C_{ij}$ is the Cartan matrix of the ADE diagram. The correct phase space is reproduced only if the imbalance at each node is non-negative.\(^\text{10}\)

\(^{10}\)In order for the quantized Coulomb branch to admit a trace and/or interesting weight modules, the imbalance should not be too positive. We will come back to this later.
We should also mention that the theory has a collection of mass deformation parameters, one for each individual flavour. Turning on the mass parameters corresponds to fragmenting the ‘t Hooft defect into a collection of defects of smaller magnetic charge. By sending a mass parameter at infinity one can send an ‘t Hooft line to infinity as well.

When we turn on the Ω-deformation, the masses are redefined by amounts proportional to ℏ and the notion of “mass equal to 0” is ambiguous. In a sense, once we quantize the system the ‘t Hooft operators are always fragmented into pieces of minimal charge.

We will not attempt to give a full derivation for this effective worldvolume theory. There are several equivalent UV completions of six-dimensional super Yang-Mills theory in string theory. ‘t Hooft operators for some basic charges can be engineered with the help of extra branes, but the details are cumbersome. Obtaining more general charges may require further manipulations, possibly done after restricting the system to an intermediate UV completion such as $N = (1,1)$ little string theory [29].

Notice that this type of analysis is subtle even for Wilson lines. Naively, supersymmetric Wilson lines exist in six-dimensional super Yang-Mills theory for any representation, but we know that only those that can be lifted to Yangian representations are available in four-dimensional Chern-Simons theory. This must mean that the “bad” lines, when placed in the Ω-deformation, do not admit counterterms which are required to preserve supersymmetry quantum mechanically. In any case, they must be absent in the UV completion. This is challenging to verify directly.

From this point of view, the fact that the Ω-deformation of “antidominant” three-dimensional $N = 4$ ADE quiver gauge theories always yields a gauge-invariant line defect in four-dimensional Chern-Simons theory is rather striking.

### 9.1 ADE Quantum Coulomb branches

An Ω-deformation reduces a three-dimensional $N = 4$ gauge theory to a one-dimensional system governed by an operator algebra denoted as the quantum Coulomb branch algebra. Mathematically, the quantum Coulomb branch algebra is given by the BFN construction [35]. The algebra is a quantization of the algebra of holomorphic functions on the Coulomb branch of the three-dimensional $N = 4$ gauge theory. Besides the quantization/Ω-deformation parameter ℏ, the algebra depends on certain equivariant parameters (aka masses), which become the position of ‘t Hooft lines in the $z$-plane in the four-dimensional Chern-Simons picture.

The quantum Coulomb branch algebra has a complicated mathematical definition, which encodes the properties of generic BPS monopole operators in the three-dimensional gauge theory. Some elements in the algebra have a particularly simple definition:

- Some infinite collection of $H_i^{(n)}$ generators built from the vectormultiplet scalars, with no monopole charge.

- Some infinite collection of $E_i^{(n)}$ generators of minimal positive monopole charge at the $i$-th node.
• Some infinite collection of $F_i^{(n)}$ generators of minimal negative monopole charge at the $i$-th node.

For the ADE quivers relevant to the description of the 't Hooft defects, these generators are known to satisfy the commutation relations for the antidominant shifted Yangian $Y(g)^\eta$ [20] and thus give an algebra morphism from $Y(g)^\eta$ to the quantum Coulomb branch algebra.

This morphism is actually surjective, so that the quantum Coulomb branch algebra is a truncation of the Yangian. The specific form of the truncation depends on the mass parameters $m_i$. Alternatively, one can present the Yangian as a quantization of the Beilinson-Drinfeld affine Grassmannian associated to points $m_i$ in $\mathbb{C}$, and truncate that in a more canonical way.

The corresponding classical statement is that there are holomorphic functions $H_i^{(n)}$, $E_i^{(n)}$, $F_i^{(n)}$, together with other functions defined by their Poisson brackets, which can be assembled together into a formal generating series $g(z)$ which satisfies the classical limit of the shifted Yangian commutation relations and lies in a certain submanifold determined by the $N_i$ and $m_i$ parameters.

This is precisely the phase space discussed in section 6. In the language we used in that section, $g(z)$ represents the monodromy of a Wilson line passing elementary 't Hooft operators at positions $z_i = m_i$ and charges $\rho_i$ determined by the node of the quiver the flavour is attached at.

Quantum mechanically, the analogue of the classical holonomy $g(z)$ evaluated in some representation $r$ is the collection of R-matrices $g_R(z)$ between a generic representation of the Yangian and a finite-dimensional representation $R$ with spectral parameter $z$.

9.2 Coproduct

The coproduct for the Yangian is a quantization of the simple classical relation

$$\Delta g(z) = g_1(z) \otimes g_2(z),$$

where we are using the group composition law. As we have seen in section 8, this expression defines the coproduct for the (shifted or unshifted) Yangian at the quantum level as well, as long as we treat $g(z)$ as a matrix acting in an appropriate representation.

In the context of Bogomolny equations, we can consider a situation where the collection of Dirac singularities is split into two sub-collections whose charge still lies in the root lattice. If we separate the two collections by a large amount in the $x$-direction, we can produce a BPS solution for the composite system by combining BPS solutions for the subcollections. This leads precisely to the above coproduct.

This assertion is known to hold quantum mechanically for the quantized Coulomb branches: there is a map from the quantized Coulomb branch for ranks $N_i + N_i'$ and flavours $M_i + M_i'$ to the tensor product of the quantized Coulomb branches for ranks $N_i$ and $N_i'$ and flavours $M_i$ and $M_i'$. This map is compatible with the maps from the Yangian and the Yangian coproduct [34]. With some abuse of notation, we will denote this map between quantized Coulomb branches as a coproduct as well.
Because the Yangian coproduct arises in four-dimensional Chern-Simons perturbation theory precisely from the topological fusion of line defects, this assertion is an implicit test of the duality between the order and disorder definitions of the defects: it is compatible with the topological fusion of line defects.

9.3 $A_1$ examples

In the case of $\mathfrak{sl}_2$, the quiver has one node and we have three-dimensional SQCD, with $U(N)$ gauge group and $M$ flavours. This corresponds to $\mu = M\sigma_3$ and $\eta = (2N - M)\sigma_3$.

9.3.1 A single ’t Hooft operator

Placing a minimal ’t Hooft charge at $\infty$, we have $N = M = 1$: SQED with one flavour. This theory has no Higgs branch, and a Coulomb branch isomorphic to $\mathbb{C}^2$. The quantum Coulomb branch algebra is a Weyl algebra. The map from the shifted Yangian is given by the familiar oscillator representation

$$L(z) = \begin{pmatrix} z + bc & b \\ c & 1 \end{pmatrix} \quad (9.3)$$

as expected.

9.3.2 Two ’t Hooft operators, zero charge at infinity

Next, we can look at $N = 1, M = 2$: SQED with two flavours. Both Higgs and Coulomb branches of this theory take the form of $A_1$ singularities. The quantum Coulomb branch algebra is a central quotient of $U(\mathfrak{sl}_2)$. The map from the Yangian is given by the familiar representation

$$\tilde{L}(z) = \begin{pmatrix} z + h^{1/2}p(h) & h\rho(f) \\ h\rho(e) & z - h^{1/2}p(h) \end{pmatrix} \quad (9.4)$$

appropriately normalized.

We thus recognize that a “Verma module” Wilson line is the same as two ’t Hooft lines, separated by a distance $h(2j + 1)$. This statement is clearly close to the QQ relation, but it is subtly different: the charges at infinity have already been cancelled against each other.

9.3.3 Two ’t Hooft operators, non-zero charge at infinity

Finally, we can look at the case $N = 2, M = 2$. The map from the shifted Yangian is given by the representation

$$L(z) = \begin{pmatrix} z^2 + a_1z + a_0 & b_1z + b_0 \\ c_1z + c_0 & d_0 \end{pmatrix} \quad (9.5)$$

Classically, we want to impose $\det L(z) = z^2$, or $\det L(z) = z(z - m)$ if we turn on a mass deformation. Quantum-mechanically, we impose quantum determinant relations.

This is the first case where we can write a reasonable coproduct:

$$L(z) = \begin{pmatrix} z + bc & b \\ c & 1 \end{pmatrix} \begin{pmatrix} z' + b'c' & b' \\ c' & 1 \end{pmatrix} \quad (9.6)$$
9.4 A choice of traces

In order to define a closed line defect, we need to equip the auxiliary one-dimensional system with a trace. This can be done in multiple ways, as the quantum Coulomb branch algebra has a non-trivial linear space of traces. More precisely it has a non-trivial space of traces twisted by elements of the Cartan of the global $g$ symmetry, which can be defined simply as (analytic continuation of) traces on highest weight modules for the algebra. Only certain linear combinations of the twisted traces remain finite as the twisting is turned off.

An important observation is that the Coulomb branch may not admit any trace, twisted or not. By the quantum Hikita conjecture \cite{36}, the space of twisted traces is isomorphic to the quantum cohomology ring of the Higgs branch. A balanced ADE quiver has a rich Higgs branch and a correspondingly rich space of traces. As we make the nodes unbalanced, the expected dimension of the Higgs branch $\sum_i N_i (M_i - \delta_i)$ diminishes. In order to build transfer matrices, we thus want our quivers not to be too unbalanced.

As traces are multiplicative under the coproduct, we can produce many traces starting from simple examples, simply by grouping the elementary 't Hooft operators in different sub-collections and splitting the charge at infinity accordingly, as long as each individual collection admits a trace.

Given a trace $\text{Tr}_a$ on the quantized Coulomb branch algebra, we can define generalized transfer matrices

$$L_{a}^{\mu,\eta} = \text{Tr}_a g_{R_1}[z_1] \cdots g_{R_L}[z_L]$$

acting on a spin chain with sites in representations $R_i$ and impurity parameters $z_i$. These transfer matrices will commute with the transfer matrices built from finite-dimensional representations of the Yangian, essentially by construction. They depend on the charge $\mu$ at 0, the charge $\eta$ at infinity and the choice of trace $a$.

When $\mu$ is minuscule and $\eta$ is in the Weyl orbit of $\mu$, one can verify that the quantized Coulomb branch algebras are Weyl algebras and $g_R$ are the oscillator L-operators. The unique twisted trace on the Weyl algebra gives generalized transfer matrices which coincide with the Q-operators.

9.5 't Hooft-Wilson lines and Coulomb branches

An 't Hooft-Wilson line lifts in six dimensions to a Wilson line sitting on an 't Hooft defect. In the enhanced low energy description, that would be an half-BPS line defect in the three-dimensional $N = 4$ theory, in the sense studied by \cite{37, 38}. These line defects give rise to tri-holomorphic sheaves on the Coulomb branch. In the presence of $\Omega$-deformation, they support algebras of local operators which are a sort of matrix generalization of the quantized Coulomb branch algebra.

Line defects which correspond to well-defined 't Hooft-Wilson lines must have the property that the corresponding algebras still admit algebra morphisms from the shifted Yangians, so that they can represent line defects in four-dimensional Chern-Simons theory. We thus predict the existence of a sub-category of line defects with this property.
Physically, there is a general strategy to produce interesting line defects: the “vortex construction”. See [39] for a four-dimensional analogue. The basic idea is to consider a three-dimensional quiver theory with the same shape and \( \eta \), but with some extra flavours, whose masses are tuned in such a way to allow for a Higgs branch to open up. If the setting is appropriate, the Higgs branch vev will trigger a flow to the original three-dimensional theory. A position-dependent Higgs branch vev will trigger a “vortex” flow which ends producing an extra line defect in the desired three-dimensional theory.

These operations survive the \( \Omega \)-deformation. The main difference is that the tuning of the masses is modified by a multiple of \( \hbar \), which governs which vortex flow we follow. At the level of quantum Coulomb branches, the adjustment of the mass parameters allows for a further truncation of the algebra.

There is a simple interpretation of these manipulations: the extra flavours represent a collection of extra ’t Hooft lines and the adjusted masses tell us how to tune the spectral parameter to allow the ’t Hooft lines to fuse into the original line and be truncated to a more general ’t Hooft-Wilson line. We leave a careful treatment of this construction to future work.

10 Building ’t Hooft lines using Gukov-Witten type surface operators

Above we described the ’t Hooft line in four-dimensional Chern-Simons theory in the standard way, by specifying a singularity in the gauge field. From this, we derived the phase space of the theory in the presence of the ’t Hooft operator. While convenient for many purposes, this description has some disadvantages.

For one thing, it is rather difficult to perform Feynman diagram computations in the presence of an ’t Hooft line, described as a singularity in the gauge field. In addition, it is rather difficult to use this description to understand the effect of an ’t Hooft line on other defects in the theory, such as the surface defects that lead to integrable field theories [4].

A full microscopic understanding of ’t Hooft lines will allow us to build a Q-operator in all of the integrable field theories constructed using the method of [4]. This would be a line defect in each such theory which commutes with the T-operator and satisfies the TQ and QQ relations, in the normalized form we have been using.

In this section we will present a more complete description, where the ’t Hooft line lives on the boundary of an explicit Lagrangian surface defect. (In contrast to the surface defects considered in [4], these surface defects are topological.) This description has many advantages, and is quite easy to compute with. We did not introduce it earlier as it is a description that is rather unfamiliar to most physicists, and is quite special to the situation we are considering. This description also allows us to prove the QQ relation, expressing the fusion of certain ’t Hooft lines in terms of Wilson lines.

The surface defect we introduce is reminiscent of the Gukov-Witten defect [40] in \( N = 2 \) supersymmetric gauge theories. This defect has the feature that it can be removed by a singular gauge transformation. It thus can be thought of as a Dirac string. By making this surface defect end, we will find the ’t Hooft line.
All our computations will be local, and apply equally well to the rational, trigonometric, and elliptic setups, and indeed to the higher-genus integrable field theories considered in [4]. Because of this, we will work on $\mathbb{R}^2 \times \mathbb{C}$, and consider a surface operator at $0 \in \mathbb{C}$.

10.1 Defining the surface defect

Let $X$ be a complex manifold with a $G$-action. We consider the following $\sigma$-model with target $X$. The fields of the theory are a map $\sigma : \mathbb{R}^2 \to X$, and a one-form

$$\eta \in \Omega^1(\mathbb{R}^2, \sigma^* T^* X).$$

(10.1)

Here and below, all tensors on $X$ are defining in the holomorphic sense, only using $(1, 0)$ vectors and covectors.

The Lagrangian is

$$\int_{\mathbb{R}^2} \eta d\sigma.$$  

(10.2)

The field $\eta$ has a gauge transformation generated by

$$\chi \in \Omega^0(\mathbb{R}^2, \sigma^* T^* X).$$

(10.3)

The gauge transformation sends $\eta \mapsto \eta + d\chi$.

This theory arises in two ways:

1. It is the $B$-twist of the $\sigma$-model with $(2, 2)$ supersymmetry.

2. It is the analytically-continued version of the Poisson $\sigma$-model with target $X$ [41], with zero Poisson tensor. That is, if we wrote the same formulae for the Lagrangian but where $X$ was a real manifold, we would find the Poisson $\sigma$-model. Here, we are simply complexifying the fields.

Since this is an analytically-continued theory, we need to use an appropriate contour. To describe the algebra of operators of the theory, which is our main concern, the contour doesn’t matter, so we will not spend much time on this point.

The natural contour from the point of view of the Poisson $\sigma$-model is to choose a real slice of $X$. This is also the natural thing to do in our context: we choose a contour for four-dimensional Chern-Simons theory associated to a real form of $G$, and we choose a real slice of $X$ which is acted on by the real form of $G$.

We will couple this surface defect to four-dimensional Chern-Simons theory. The surface defect is placed at the plane $z = 0$. The Lagrangian for the coupled theory is simply

$$\int_{\mathbb{R}^2 \times 0} \eta d\Delta \sigma.$$  

(10.4)

We can write this more explicitly. Let $a$ be a Lie algebra index and $i$ an index for local coordinates on $X$. Let $V_i^a$ be the holomorphic vector fields on $X$ giving the action of $g$. Then the Lagrangian is

$$\int_{\mathbb{R}^2 \times 0} \left( \eta d\sigma + \eta^i (\sigma^* V_i^a) A_a \right).$$

(10.5)
The gauge transformation for \( \eta \) is now given by the covariant derivative:
\[
\delta \eta = \eta + d_A \chi.
\] (10.6)
The bulk gauge transformations act on the fields \( \eta, \sigma \) in the obvious way. The Lagrangian is invariant under bulk gauge transformations, but not under the gauge transformation generated by \( \chi \). By integration by parts, the variation under this gauge transformation is given by
\[
- \int_{\mathbb{R}^2 \times 0} \chi^i F(A)_a \sigma^* (V^a_i).
\] (10.7)
To correct for this, we will need to vary the four-dimensional gauge-field by the gauge transformation \( \chi \). The required variation is
\[
\delta A_a = \delta_{z=0} \chi^i \sigma^* (V^b_i) g_{ba},
\] (10.8)
where \( g_{ab} \) is the inverse of the Killing form on \( g \).

The variation of the Chern-Simons action is
\[
\delta \text{CS}(A) = \delta_{z=0} \chi^i \sigma^* (V^a_i) F(A)_a,
\] (10.9)
which cancels the variation (10.7). Note that only the \( \mathfrak{z} \) component of the four-dimensional gauge field acquires this extra variation.

10.2 The equations of motion in the presence of the surface defect

Let us assume that \( G \) acts transitively on \( X \), and that further, the stabilizer of a point in \( X \) is a parabolic subgroup \( P \subset G \). We will describe the solutions to the equations of motion in this context.

By a gauge symmetry, we can assume that \( \sigma \) is constant with value \( x \). The tangent space to \( X \) at \( x \) is the quotient \( g/p \), where \( p \) is the Lie algebra of \( P \). We let \( n^- \subset p \) be the nilpotent subalgebra. Choose a nilpotent subalgebra \( n^+ \) complementary to \( p \), and a Levi factor \( l \subset p \). Thus, we have a triangular decomposition
\[
g = n^- \oplus l \oplus n^+.
\] (10.10)
We will choose a basis of \( g \) with indices \( t^-_i \) for elements of \( n^- \), \( t^+_i \) for elements of \( n^+ \), and \( t^0_i \) for elements of the Levi factor \( l \). The Killing form is \( \langle t^-_i, t^+_j \rangle = \delta_{ij}, \langle t^0_i, t^0_m \rangle = \delta_{lm} \). We will denote the corresponding components of the gauge field by \( A^-, A^+, A^0 \).

There is a canonical isomorphism
\[
T_x X = n^+.
\] (10.11)
Hence, we can view the field \( \eta \) as a one-form valued in \( n^- \).

In this gauge, the Lagrangian takes the form
\[
\int_{\mathbb{R}^2 \times 0} \eta_i(x) A^+_i + \int_{\mathbb{R}^2 \times \mathbb{C}} d z \text{CS}(A).
\] (10.12)
Varying $\eta$ tells us that, at $z = 0$, $A^+$ vanishes so that $A$ is the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$.

Next, let us vary $A$. We find

$$\eta_i \delta_{z=0} + dz F(A)^-_i = 0.$$  (10.13)

This tells us that $A^-$ can have poles at $z = 0$.

To understand this better, we work in an axial gauge in which $A^z = 0$. In the absence of the surface defect, the equations of motion in this gauge tell us that $\bar{\partial}_z A_x = 0$ and $\bar{\partial}_z A_y = 0$.

In the presence of the surface defect, these equations are modified to saying that

$$\bar{\partial}_z A^-_x + \eta_x = 0,$$  (10.14)

$$\bar{\partial}_z A^-_y + \eta_y = 0.$$  (10.15)

This tells us that $A^-$ can have a first-order pole at $z = 0$, whose residue is given by $\eta$.

The result of this analysis is that *all* of the fields on the surface defect have been absorbed into a modification of the gauge field: $A^+$ has a zero at $z = 0$, and $A^-$ has a pole.

In a similar way, the bulk gauge transformation $c^+$ must have a zero at $z = 0$. The gauge transformation $c^-$ can have a pole, whose residue is given by the defect gauge transformation $\chi$.

For a concrete example, consider the case when $G = PSL_2(\mathbb{C})$ and $X = \mathbb{CP}^1$. In this case, the surface defect is equivalent to allowing the component $A^1_2$ of the gauge field has a pole at $z = 0$, and requiring that $A^2_1$ has a corresponding zero.

10.3 Removing the surface defects corresponding to minuscule coweights

Now let us analyze when we can remove the surface defect. As above, our surface defect is built using a parabolic subgroup $P \subset G$, and we continue to use the notation $n^\pm$, $l$, where $\mathfrak{p} = n^- \oplus l$.

The condition we need in order to be able to remove the surface defect is that there is a cocharacter $\rho: \mathbb{C}^\times \to G$ such that the $l$ is the zero eigenspace of $\rho$ in the adjoint representation, and $n^\pm$ are the $\pm 1$ eigenspaces. This condition means that $\rho$ is a minuscule coweight.

In this situation, we can perform a gauge transformation by the singular gauge field $z^\rho$ (where as before $z$ is the coordinate in the holomorphic direction of the four-dimensional space-time). Conjugating the gauge field by $\rho(z)$ will give the field $A^+$ a first-order zero at $z = 0$, and $A^-$ a first-order pole. This is because $A^\pm$ are the $\pm 1$ eigenspaces of the cocharacter $\rho$.

In the example where $G = PSL_2(\mathbb{C})$ and $X = \mathbb{CP}^1$, we have $z^\rho = \text{Diag}(z, 1)$. Conjugating by $\rho(z)$ gives $A^1_2$ a pole at $z = 0$ and $A^2_1$ a zero, which is equivalent to introducing the surface defect.

10.4 The surface defect as a monodromy defect

Suppose we have a simply-connected gauge group $G$, and we have a surface defect as above associated to a minuscule coweight $\rho$ of the adjoint form $G_{\text{ad}}$ of $G$. The Weyl group orbits
of minuscule coweights are in bijection with the non-identity elements of the center of $G$. We will show that in this situation, the surface defect associated to a minuscule coweight is equivalent to a “monodromy defect”, where the gauge field has monodromy around the location of the surface given the corresponding element of the center of $G$.

Strictly speaking, the phrase monodromy defect is not quite correct when the surface wraps the topological plane, because we only have a partial gauge field in the holomorphic plane. In this case, the “monodromy” means that the bundle is a $G_{ad}$-bundle with lifts to a $G$-bundle away from the location of the surface defect but there is an obstruction to such a lift near the defect.

Let us explain these obstructions. Let $\mathcal{G}$ be the sheaf on $\mathbb{C}$ of smooth maps to $G$. For any smooth manifold, $H^1(M, \mathcal{G})$ is the set of isomorphism classes of smooth $G$-bundles on $M$. There is a short exact sequence

$$1 \rightarrow Z(G) \rightarrow \mathcal{G} \rightarrow \mathcal{G}_{ad} \rightarrow 1,$$

where $Z(G)$ is the constant sheaf with value the center of $G$. This leads to an exact sequence of cohomology groups

$$H^1(U, \mathcal{G}) \rightarrow H^1(U, \mathcal{G}_{ad}) \rightarrow H^2(U, Z(G))$$

for any open $U$ in $\mathbb{C}$. Thus, any smooth $G_{ad}$ bundle has a characteristic class living in $H^2$ with coefficients in $Z(G)$, and this characteristic class is the obstruction to lifting to a $G$-bundle.

In the case $G = SL_n$, this characteristic class is the first Chern class of a $PSL_n$ bundle, taken modulo $n$.

The surface defect we introduce, when we take the gauge group to be the simply connected form $G$, is equivalent to working with the adjoint form $G_{ad}$ but where we specify the value of the characteristic class in $H^2_c(U, Z(G))$ (where $U$ is a neighbourhood of the location of the surface defect and the subscript $c$ indicates compact support). Suppose the surface defect is at $z = 0$, and is associated to a Weyl orbit of minuscule coweights corresponding to an element $\rho \in Z(G)$. Then we will show that the characteristic class is

$$\delta_{z=0} \rho \in H^2_c(U, Z(G)).$$

To see this, we recall that we can remove the surface defect using the gauge transformation $\rho(z) \in G_{ad}$. This gives us a Čech description of the principle $G_{ad}$ bundle in the presence of the surface defect, as follows. We let $U$ be a neighbourhood of $z = 0$. The transition function gluing a bundle near $z = 0$ to a bundle away from $z = 0$ is given by a smooth map

$$U \setminus \{z = 0\} \rightarrow G_{ad}.$$

The $G_{ad}$ bundle sourced by the surface defect has transition function given by $\rho(z)$. This will define a $G$-bundle if and only $\rho(z)$ lifts to a continuous map to $G$. The obstruction to doing so is the homotopy class of the map $\rho(z) : S^1 \rightarrow G_{ad}$ in $\pi_1(G_{ad}) = Z(G)$.

This tells us that the obstruction to lifting the bundle sourced by the surface defect is the class $\delta_{z=0} \rho \in H^2(U, Z(G))$. 

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This description of the surface defect makes it clear that it is entirely topological, and can be placed on any two-cycle $\Sigma$ on the four-dimensional spacetime of four-dimensional Chern-Simons theory. In the presence of the surface defect, we take our gauge group to be the adjoint form $G_{ad}$ but only include topological types of bundles with characteristic class given by $\delta_{\Sigma} \rho \in H^2(X, Z(G))$.

10.5 Monodromy defects for $SL_n(\mathbb{C})$

In the case $G = SL_n(\mathbb{C})$ and we use the surface defect associated to $\mathbb{CP}^{n-1}$, there is a convenient way to rephrase this. In perturbation theory, our surface defect has the effect of moving us from the trivial bundle $\mathcal{O}^n$ on the holomorphic plane to the bundle $\mathcal{O}(1) \oplus \mathcal{O}^{n-1}$. This bundle has determinant $\mathcal{O}(1)$, and so is not an $SL_n(\mathbb{C})$ bundle. More generally, if we start by perturbing around some bundle $E$ with trivial determinant, and introduce the surface defect, we will find the configuration is equivalent to working with a bundle $E'$ with determinant $\det E' = \mathcal{O}(1)$.

The introduction of the surface defect means that instead of considering bundles with trivial determinant, we are working with bundles with determinant $\mathcal{O}(1)$. Such bundles are principle bundles for a twisted form of $SL_n(\mathbb{C})$, which forms a non-trivial bundle of groups on the holomorphic plane $\mathbb{C}$. This is the group of automorphisms of the bundle $\mathcal{O}(1) \oplus \mathcal{O}^{n-1}$ which act as the identity on the determinant bundle $\mathcal{O}(1)$. Introducing the surface defect means we are working with this twisted form of $SL_n(\mathbb{C})$.

Asking that a bundle has determinant $\mathcal{O}(1)$ is the same as asking that its first Chern class is $\delta = 0$. The obstruction to a $PSL_n(\mathbb{C})$ bundle lifting to an $SL_n(\mathbb{C})$ bundle is the first Chern class, modulo $n$. Working with bundles with fixed determinant $\mathcal{O}(1)$ is then equivalent to working $PSL_n(\mathbb{C})$ bundles whose modulo $n$ Chern class is $\delta = 0$.

10.6 Surface defects and the dynamical Yang-Baxter equation

In [2], it was shown that if we consider four-dimensional Chern-Simons theory with holomorphic curve $\Sigma$, and we choose boundary conditions so that the holomorphic bundles we consider on $\Sigma$ have no moduli, then the theory leads to solutions of the Yang-Baxter equation with no dynamical parameter. In [3], we discussed the case when the holomorphic bundles on $\Sigma$ have moduli, in which case the solution to the Yang-Baxter equation has a dynamical parameters given by the moduli.

Let us consider the case that $\Sigma$ is an elliptic curve, with gauge group $SL_n(\mathbb{C})$. The moduli of semistable holomorphic $SL_n(\mathbb{C})$ bundles on the elliptic curve is isomorphic to $\mathbb{CP}^{n-1}$. Thus, we find solutions to the Yang-Baxter equation with dynamical parameter.

Introducing surface defects changes the moduli. In some cases, a surface defect adds moduli; in other cases, moduli are removed.

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\[ ^{11} A \text{PSL}_n(\mathbb{C}) \text{ bundle is a complex vector bundle taken up to tensoring with a line bundle. Tensoring with a line bundle adds a multiple of } n \text{ to the first Chern class, so the first Chern class of a PSL}_n(\mathbb{C}) \text{ bundle is well-defined modulo } n. \]
For example, let us introduce a single $\mathbb{CP}^{n-1}$ surface defect in the case when we work on an elliptic curve with gauge group $E$. In this case, we are no longer considering the moduli of $SL_n(\mathbb{C})$ bundles on $E$, but the moduli of $PSL_n(\mathbb{C})$ bundles whose first Chern class modulo $n$ is 1. This moduli space is a point [42] (at least when we focus on stable bundles, as is physically reasonable). The unique stable bundle is the rigid $PSL_n(\mathbb{C})$ bundle studied in [3]. In this example, we find that introducing the surface defect has moved us from a situation with zero modes to one without zero modes.

In [3], it was shown that the path integral defined in perturbation theory around this rigid $PSL_n(\mathbb{C})$ bundle yields Belavin’s elliptic R-matrix, which does not have a dynamical parameter. This analysis was only perturbative, and neglected other possible $PSL_n(\mathbb{C})$ bundles. Non-perturbatively, it would seem that four-dimensional Chern-Simons theory for the group $PSL_n(\mathbb{C})$ does not lead to Belavin’s elliptic R-matrix, because of the presence of bundles of other degree.

The correct, non-perturbative statement is that the elliptic R-matrix without dynamical parameter arises from four-dimensional Chern-Simons theory for the group $SL(n, \mathbb{C})$ with a single surface defect.

10.7 The dynamical Yang-Baxter equation in the rational case

Now let us consider four-dimensional Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{C}$, with simply-connected gauge group $G$. The boundary condition is that the gauge field tends to zero at $z = \infty$, so that the bundle extends to a bundle on $\mathbb{R}^2 \times \mathbb{CP}^1$. This bundle is trivialized at $\mathbb{R}^2 \times \infty$.

Introducing a surface defect corresponding to an element of the center of $G$ means that we use $G_{ad}$ bundles on $\mathbb{R}^2 \times \mathbb{CP}^1$, which have non-trivial topology on $\mathbb{CP}^1$. We expect that having a surface defect built from the $\sigma$-model on $G/P$, for a parabolic $P \in G$ corresponding to a minuscule coweight, we should find dynamical parameters living in $G/P$. After all, the only extra degrees of freedom we have introduced live in $G/P$.

We can also see this from the geometry of the moduli space of $G$-bundles on $\mathbb{CP}^1$. Connected components of the moduli stack of holomorphic $G_{ad}$ bundles on $\mathbb{CP}^1$ are labelled by the center of $G$, or equivalently, by Weyl orbits of minuscule coweights. If we ask that such bundles are trivialized at $\infty$, then each connected component has a $G$-action. In each connected component, there is an open $G$-orbit which is of the form $G/P$, with parabolic $P$ corresponding to the minuscule coweight labelling the component.

We expect this setup to lead to rational R-matrices with a dynamical parameter living in $G/P$.

11 Making the surface defect end, and ’t Hooft lines

We would like to have an interface between gauge-theory configurations with this surface defect at $z = 0$, and configurations without the surface defect. This turns out to be very simple: all we have to do is impose a boundary condition for our surface defect.
We coordinatize the topological plane by \(x, y\) and consider the surface defect on the region where \(y \geq 0\). The most natural boundary condition for the Poisson \(\sigma\)-model with target \(X\) is to set \(\eta = 0\) on the line \(y = 0\).

More generally, we can introduce a boundary condition where we set \(\eta = 0\) and at the boundary place a holomorphic vector bundle on \(X\). In order to be able to couple to the gauge field, we will need this holomorphic vector bundle to be \(G\)-equivariant.

In the case that the surface defect is of the form \(G/P\) for a parabolic associated to a minuscule coweight, we have seen that we can remove the surface defect by a gauge transformation \(\rho(z)\). When the surface defect has boundary at \(y = 0\) we should use a gauge transformation which is \(\rho(z)\) for \(y < -\epsilon\) and the identity for \(y > 0\). When we do this, we are left with a line defect.

This line defect is an 't Hooft line (or an 't Hooft-Wilson line if we include a non-trivial vector bundle on \(X\) into the boundary conditions). To see this, we note that classically, the field sourced by this line defect is obtained from the trivial field configuration by applying a gauge transformation which is \(\rho(z)\) for \(y \ll 0\) and the identity for \(y \gg 0\). The effect of such a gauge field on a Wilson line in the \(y\)-direction is \(\rho(z)\), which is precisely what we expect from an 't Hooft line.

There is a subtlety here, in that the precise matrix \(\rho(z)\) that arises depends on which point on the manifold \(X = G/P\) we use. (The conjugacy class of the matrix does not). To specify the field sourced by the 't Hooft line defect, we need to say what the line defect does at \(x = \pm \infty\). In this analysis we are giving our surface defect Neumann boundary conditions \(\sigma = \sigma_0\) for a point \(\sigma_0 \in G/P\) at \(x = \pm \infty\). When we use a gauge transformation to write the surface defect as a line defect, the choice of boundary condition for the surface defect specifies states at the end of the 't Hooft line.

11.1 Equivalence between the surface defect and affine Grassmannian definitions of the 't Hooft line

The definition of the 't Hooft line using surface defects is equivalent to that using the affine Grassmannian. To see this, we need to describe the moduli space of solutions to the equations of motion of the theory when we have a surface defect that ends.

We will take \(G = G_{\text{ad}}\) to be of the adjoint form, so that there are no Dirac strings. We will place the surface defect given by a \(G/P\ \sigma\)-model at \(z = 0, y \leq 0\). The boundary condition at \(y = 0\) is \(\eta = 0\).

We will consider the solutions to the equations of motion in a neighbourhood of \(z = 0\). When \(y > 0\), we can trivialize the bundle. The group \(G[[z]]\) acts on the space of solutions by change of the trivialization on the region where \(y > 0\).

Away from \(y = 0, z = 0\), the moduli space of solutions is the same as in the absence of the surface defect, because the surface defect can be removed by a gauge transformation. Thus, away from \(y = 0, z = 0\), the moduli space of solutions is described by the affine Grassmannian \(G((z))/G[[z]]\).
Including the locus when \( y = 0, z = 0 \), we find that the moduli space of solutions to the equations of motion is given by some \( G[[z]] \)-orbit in the affine Grassmannian. This orbit contains the point in affine Grassmannian given by the minuscule coweight defining the surface defect. This proves the equivalence between the surface-defect picture and the affine Grassmannian picture.

11.2 't Hooft lines as interfaces

We have defined a fractional 't Hooft line for the adjoint form of a group \( G \) as the line operator living at the end of the surface defect. Tautologically, the 't Hooft line is an interface between the theory in the presence of the surface defect and the theory without a surface defect.

One can ask why an 't Hooft line would be expected to behave in this way. Let us consider the rational case, so the holomorphic plane is \( \mathbb{C} \), and take the gauge group to be \( SL_2(\mathbb{C}) \). A fractional 't Hooft line at \( z = 0, y = 0 \) gives rise to a Hecke transformation which will turn the trivial bundle at \( y < 0 \) into a bundle isomorphic to \( \mathcal{O}(1) \oplus \mathcal{O} \) on \( y > 0 \). A gauge field for the bundle \( \mathcal{O}(1) \oplus \mathcal{O} \) has non-trivial zero modes, leading to a dynamical parameter.

From this we see that an 't Hooft operator provides an interface for four-dimensional Chern-Simons theory on \( \mathbb{R}^2 \times \mathbb{C} \) where there is no dynamical parameter, to a setting where there is a dynamical parameter.

There is an important difference between \( PSL_2(\mathbb{C}) \) and \( SL_2(\mathbb{C}) \), or more generally between the adjoint and simply connected form of a group. For \( PSL_2(\mathbb{C}) \), the non-perturbative path integral for four-dimensional Chern-Simons theory should sum over the different topological types of gauge field on \( \mathbb{C} \) (trivialized at \( \infty \)). For \( PSL_2(\mathbb{C}) \), there are two components, and the 't Hooft line associated to the minuscule coweight moves us from one component to the other.

For \( SL_2(\mathbb{C}) \), the two connected components of the moduli of \( PSL_2(\mathbb{C}) \) bundles have a different interpretation. The component with trivial \( w_2 \) gives us \( SL_2(\mathbb{C}) \) bundles, and this appears in the theory without a surface defect. The component with non-trivial \( w_2 \) occurs when we have a surface defect.

The fact that minuscule 't Hooft lines always move us from a setting with no dynamical parameter to one with a dynamical parameter will cause us difficulties when we define Baxter's Q-operator in terms of 't Hooft lines. We will find that to define Baxter's Q-operator we will need to have a line defect at \( z = \infty \) as well as an 't Hooft line at \( z = 0 \).

12 't Hooft lines at \( \infty \) and Q-operators

In this section we will re-derive the phase space of minuscule 't Hooft lines from the surface defect picture. As we have seen, the Q-operator arises when we have an 't Hooft line in the bulk but also modify the boundary condition at \( z = \infty \) along a line parallel to the 't Hooft line. Here, we will describe this modification of the boundary condition at \( z = \infty \) as introducing a surface defect at \( \infty \).
Thus, fix a minuscule coweight $\rho$. Let us decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ according to the eigenvalues of $\rho$. We define a parabolic $P$ by saying that its Lie algebra is $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. We define unipotent subgroups of $G$ by $G^\pm = \exp(\mathfrak{g}_\pm)$. We let the corresponding components of $A$ be $A^{-1}$, $A^0$, $A^1$.

The defect at $z = \infty$ is described by coupling the gauge theory to the analytically-continued Poisson $\sigma$-model on the vector space $\mathfrak{g}^1$. The fields of this theory are

\(\tilde{\sigma}: \mathbb{R}^2 \to \mathfrak{g}^1,\)
\(\tilde{\eta} \in \Omega^1(\mathbb{R}^2, \mathfrak{g}^{-1}).\) (12.1)

The Lagrangian is \(\int \tilde{\eta} d\tilde{\sigma}.\) As before, \(\tilde{\eta}\) has a gauge symmetry, \(\tilde{\eta} \mapsto \tilde{\eta} + d\tilde{\chi} + \partial_u A^1\).

We would like to couple this to the gauge theory. The boundary conditions for the gauge theory require our gauge field $A$ is divisible by $1/z$ near $\infty$. To couple in a non-trivial way to the $\sigma$-model on $\mathfrak{g}^1$, we need to involve the derivative of $A$ in $u = 1/z$ at $z = \infty$.

We couple the topological $\sigma$-model with target $\mathfrak{g}^1$ to the gauge field $\partial_u A^1$ acting by translation, so that the Lagrangian is

\(\int (\tilde{\eta} d\tilde{\sigma} + \tilde{\eta} \partial_u A^1).\) (12.3)

(Recall $u = 1/z$.) The other components of the gauge field $A$ do not couple to the surface defect.

Only the transformations corresponding to the $\partial_u c^1$ ghosts will act in a non-trivial way. These act by translation.

As before, the gauge transformation of $\tilde{\eta}$ is now

\(\tilde{\eta} \mapsto \tilde{\eta} + d\tilde{\chi} + \partial_u A^1.\) (12.4)

For the coupled system to be gauge invariant, we need to make $A^{-1}$ vary under the $\chi$ gauge transformation:

\(A^{-1} \mapsto A^{-1} + \delta_{u=\epsilon} \chi,\) (12.5)

where $\epsilon$ is a small parameter. With this term, the variation of the term $\int A^{-1} dA^1 u^{-2} du$ under $\chi$ gives us the term $\int_{u=0} \partial_u dA^1 \chi$, which cancels the variation of $\int_{u=0} \tilde{\eta} \partial_u A^1$ under $\chi$.

A repeat of the argument we gave before shows that, if we solve the equations of motion in the presence of the defect at $z = \infty$, then $A^1$ is forced to have a second-order zero, whereas $A^{-1}$ is allowed to be regular at $u = 0$. The value of $A^1$ at $u = 0$ is given by $\tilde{\eta}$, whereas $\tilde{\sigma}$ can be set to zero by a gauge transformation.

Before we introduced the defect, the boundary condition stated that $A$ vanishes at $u = 0$. Conjugating by $z^{-\rho} = u^\rho$ introduces an extra factor of $u$ to $A^1$, so that it has a second order zero at $u = 0$; and introduces an extra factor of $u^{-1}$ to $A^{-1}$, so that it can be regular. Thus, applying the gauge transformation $z^{-\rho}$ moves us from the setting without the defect at $z = \infty$ to one with the defect.
Next, suppose we introduced a defect both at 0 and at \( \infty \). At \( z = 0 \), we will introduce the topological \( \sigma \)-model on the partial flag manifold \( G/P \), and work in perturbation theory around the field configuration given by the image of the identity in \( G \). The stabilizer of this point is \( P \), so that, following our earlier analysis, the defect is equivalent to asking that \( A^{-1} \) has a zero at \( z = 0 \) and \( A^1 \) has a first-order pole. Applying \( z^{-\rho} \) moves us from the trivial defect to this defect.

We conclude that the gauge transformation \( z^{-\rho} \) removes the defects at 0, \( \infty \) simultaneously.

We can make the defect at \( z = \infty \) end at \( y = 0 \), by using the boundary condition where \( \eta = 0 \). If we remove the defect at \( y < 0 \) by a gauge transformation, we are left with a line defect, as we considered in section 4.

If we make both the bulk and boundary surface defects end at \( y = 0 \), we are left with the configuration that we found earlier gives the Q-operator. We will re-derive the phase space of the Q-operator from this perspective shortly.

At a first pass, however, consider a Wilson line at a point \( z \) along \( x = 0 \). Let us consider the effect of the two surface defects, ending at \( y = 0 \), on this Wilson line. To remove the surface defects on \( y < 0 \), we apply the gauge transformation \( z^{-\rho} \). As we have seen, this removes both surface defects at 0 and \( \infty \), leaving us with line defects at \( y = 0 \), \( z = \infty \).

To leading order in \( \hbar \), the state in the Wilson line at \( x = 0 \) transforms by \( z^{-\rho} \) when it passes the line defects at \( y = 0 \). This is because we have applied the gauge transformation \( z^{-\rho} \) at one side and not the other. This is precisely the behaviour that characterizes an ’t Hooft line. The quantum corrections to this expression can in principle be calculated by the exchange of gluons between the surface defects and the Wilson line, giving an operator which is \( z^{-\rho} \) times a series in \( \hbar z^{-1} \). In practice, calculating these quantum corrections is quite non-trivial.

12.1 The phase space in the presence of surface defects

Let us consider the situation above, with surface defects at 0 and \( \infty \) both ending at \( y = 0 \). In this section we will show that we can recover the phase space of minuscule ’t Hooft lines that we discussed in section 6.

Let us consider the effective two-dimensional theory obtained by compactifying on the \( z \)-plane \( \mathbb{CP}^1 \), with defects at 0 and \( \infty \). Before the introduction of the defects, this is the trivial theory. The defects make the effective theory the topological \( \sigma \)-model with target \( G/P \times g_1 \). This is simply the product of the topological \( \sigma \)-models we have inserted at \( z = 0 \), \( z = \infty \).

The exchange of a single gluon between the defects at \( z = 0 \) and \( z = \infty \) couples the topological \( \sigma \)-model on \( G/P \) to that on \( g_1 \). As before, we use the notation \( \eta, \sigma \) for the fields of the topological \( \sigma \)-model on \( G/P \), and \( \tilde{\eta}, \tilde{\sigma} \) for the \( \sigma \)-model on \( g_1 \). The defect at \( z = \infty \) is only coupled to the components \( A^1 \) of the gauge field in \( g_1 \).

We will compute the coupling between the two defects using the technique of [4]. Let us explain the general method. Choose a basis \( t_a \) of \( g \), and let \( J_a, \tilde{J}_a \) be the currents which couple the defects at 0 and \( \infty \) to the gauge field. If \( c^{ab} \) denotes the quadratic Casimir, then
the coupling between defects at $z, z'$ is
\[ c^{ab} J_a \tilde{J}_b \frac{1}{z - z'} . \] (12.6)

In our case, the defect at $\infty$ is only coupled to elements of $g_1$. Because of the form of the quadratic Casimir, the coupling can only involve the components of the current in $g_{-1}$ at $z = 0$. We can take a basis $X_i$ of $g_1$ and the dual basis $Y^i$ of $g_{-1}$. Let $V^i$ denote the vector fields on $G/P$ giving the action of $g_{-1}$. Then,
\[ J^i = \langle \eta, V^i \rangle . \] (12.7)

The derivative $-z^2 \partial_z$ of $A_1$ couples to $\tilde{\eta}_i$ at $z = \infty$.

From this, we see that the exchange of a gluon couples the two surface defects by
\[ \lim_{z \to \infty} \int \langle \eta, V^i \rangle \tilde{\eta}_i z^2 \partial_z \frac{1}{z} = \int \tilde{\eta}_i \langle \eta, V^i \rangle . \] (12.8)

This coupling also modifies the gauge transformations, so that $\sigma, \tilde{\sigma}$ also transform by the gauge transformations $\chi, \tilde{\chi}$:
\[ \tilde{\sigma}_i \mapsto \tilde{\sigma}_i - \langle \chi, V_i \rangle , \]
\[ \sigma \mapsto \sigma + \tilde{\chi}^i V_i . \] (12.9)

This describes the effective two-dimensional model obtained by integrating out the four-dimensional gauge fields, at the classical level. This model is the analytically-continued Poisson $\sigma$-model [41] on $G/P \times g_1$. This manifold has a holomorphic Poisson tensor, coming from the map $g_{-1} \to \text{Vect}(G/P)$ and the component of the quadratic Casimir which lies in $g_1 \otimes g_{-1}$.

For example, if $G = SL_2(\mathbb{C})$, then the holomorphic Poisson manifold is $\mathbb{C}P^1 \times \mathbb{C}$, with coordinates $u, v$. The Poisson tensor is $u^2 \partial_u \partial_v$.

The boundary condition at $y = 0$ that gives rise to the ’t Hooft defects is that where $\eta = 0, \tilde{\eta} = 0$.

We can choose a reality condition for the analytically-continued Poisson $\sigma$-model for which the fields only see the open subset where the Poisson tensor is non-degenerate.

To describe this, note that there is an open orbit of $\exp(g_{-1})$ in $G/P$ containing the image of the identity in $G/P$. This open orbit has no stabilizer, and is isomorphic to $g_{-1}$. If we assume that $\sigma$ is in this orbit, then topological $\sigma$-model on $G/P$ is replaced by that on $g_{-1}$. In this orbit, the action is simply
\[ \int (\tilde{\eta}_i \eta_i + \eta_i d\sigma^i + \tilde{\eta}_i d\tilde{\sigma}_i) \] (12.10)
and the gauge transformations are $\delta \sigma^i = \tilde{\chi}^i, \delta \tilde{\sigma}_i = \chi_i, \delta \eta_i = d\chi_i, \delta \tilde{\eta}_i = d\tilde{\chi}_i$.

This is the Poisson $\sigma$-model with target the symplectic manifold $g_{-1} \oplus g_1$. As is well-known, the bulk degrees of freedom of the Poisson $\sigma$-model with target a symplectic manifold are entirely massive, and the model localizes onto the boundary.
Concretely, we can see this as follows. Under the field redefinition \( \gamma_i = \eta_i + d\tilde{\sigma}_i \), \( \tilde{\gamma}_i = \tilde{\eta}_i - d\sigma_i \), the action becomes simply

\[
\int_{\mathbb{R} \times \mathbb{R}_{\geq 0}} \tilde{\gamma}_i \gamma_i - \int_{\mathbb{R}} \tilde{\sigma}_i d\sigma_i.
\]

The second term is the Lagrangian for topological quantum mechanics with values in the symplectic manifold \( g_1 \oplus g_{-1} \), and so describes the phase space of the 't Hooft line.

The bulk system is entirely massive. Indeed, the fields \( \gamma_i, \tilde{\gamma}_i \) are invariant under the \( \chi, \tilde{\chi} \) gauge transformation, and \( \sigma_i, \tilde{\sigma}_i \) transform by a shift. We can choose a gauge where \( \sigma_i(x, y) \) is independent of \( y \) (where the boundary is at \( y = 0 \)). In this gauge, the fields \( \sigma_i(x, y) \) are only boundary fields, and the fields \( \gamma_i, \tilde{\gamma}_i \) are massive with no kinetic term.

We have shown that the effective theory obtained by compactifying four-dimensional Chern-Simons theory to \( \mathbb{R}^2 \), in the presence of two surface defects which end, gives the trivial theory on \( \mathbb{R}^2 \) together with a line defect describing topological quantum mechanics with target \( g_1 \oplus g_{-1} \). This manifold is therefore the phase space. Thus, we have re-derived from this perspective the phase space of the theory in the presence of an 't Hooft line at 0 and at \( \infty \).

13 Quantization of the 't Hooft line

Classical Wilson lines in four-dimensional Chern-Simons theory sometimes have an anomaly to quantization. This was studied by an explicit Feynman diagram computation in \([3]\). One can ask if the same holds for 't Hooft lines.

If we describe the 't Hooft line as a singularity in the gauge field, this question is very difficult to answer. An advantage of the surface-defect description is that we can answer this question directly by using standard cohomological techniques.

For any topological line defect in a partially topological theory, there is a differential-graded associative algebra \( A \) of local operators on the defect. (At the classical level, \( A \) will be commutative.) By descent, possible deformations of the defect are given by elements of \( A \) of ghost number 1. Thus, to compute the possible first-order deformations of the defect, we need to compute \( H^1(A) \). Similarly, anomalies to quantizing the system will be described by \( H^2(A) \). This perspective was taken in \([3]\) in the analysis of Wilson lines.

Here we will prove the following result.

**Theorem 1.** Let \( A \) be the dg algebra of local operators on an 't Hooft line associated to a minuscule coweight of any simple group. then \( H^2(A) = 0 \) and \( H^1(A) \) is one dimensional. This implies that there are no anomalies to constructing the 't Hooft line at the quantum level, and that there is only one possible counterterm, which corresponds to moving the spectral parameter.

**Proof.** Let us describe the algebra \( A \) that appears in the analysis of a minuscule 't Hooft line. (In the language of \([43]\), this will be the factorization algebra of operators in the coupled system.) The surface defect is the topological \( \sigma \)-model with target \( G/P \). On the boundary,
we ask that $\eta = 0$, and the gauge symmetries of the system also vanish. Thus, the boundary operators are just built from the field $\sigma$. Further, the equations of motion show that $d\sigma = 0$, so derivatives of $\sigma$ do not appear. Therefore the algebra of operators on the boundary of the defect is simply $O(G/P)$, the algebra of polynomial functions on $G/P$. Since we are doing a perturbative analysis, we can work in a neighbourhood of a point in $G/P$, which is $g_1$. The algebra of operators is then the algebra of functions on $g_1$, which is the symmetric algebra on $g_1$. This is before coupling to the bulk gauge theory. The algebra of operators of the bulk system is generated by the ghost $c$ and its $z$-derivatives. The generators $\partial_z^k c$ are in ghost number 1, anticommute with each other, and have usual BRST operator. In mathematical terms, this algebra is the Lie algebra cochain complex of $g[[z]]$.

When we couple the two systems, we have the operators $\sigma$ in $g_1^\vee$ in ghost number 0, and $\partial_z^k c \in g_1^\vee$ in ghost number 1. Because the field in $g_1^\vee$ transforms by a shift under the action of gauge symmetry, we find a term in the BRST operator whereby

$$Q\sigma^i = c^i.$$  \hfill (13.1)

(Here $i$ runs over a basis of $g_1^\vee$.)

This makes it clear that the cohomology of the coupled system is generated in ghost number 1 by $\partial_z^k c^a$, for $k > 0$ and all values of $a$, and by $c^a$ for $a$ corresponding to elements of $g_0 \oplus g_{-1}$ In mathematical terms, the algebra of operators is the Lie algebra cochains of the algebra

$$z g[[z]] \oplus g_0 \oplus g_{-1}.$$  \hfill (13.2)

Algebras of this type are called (parabolic) Iwahori algebras in the mathematics literature. It is the Lie algebra of the group of maps from the formal disc to $G$, which at the origin land in the parabolic subalgebra $P$ which exponentiates $g_0 \oplus g_{-1}$.

Our task is to compute the cohomology of this Lie algebra in degrees 1 and 2. The task is simplified by noting that all cohomology classes must live in the trivial representation of the subalgebra $g_0$, which acts semi-simply on the Lie algebra cochain complex. Let us use $c^i$ for the ghosts in $g_{-1}$ and $c_i$ in $g_1$. We can decompose $g_0$ into an Abelian algebra, spanned by the minuscule coweight $\mu$, and a semi-simple algebra $l$. We let $c^0$ indicate the component of the ghost corresponding to the Abelian algebra, and $c^a$ the components corresponding to $l$. The spaces $g_{\pm 1}$ are both irreducible representations of $g_0$.

This discussion shows that $g_0$ invariant cochains of ghost number $\leq 2$ are of the form:

1. $\partial_z^k c^i \partial_z^l c_i$ for $l > 0$. Cochains of this nature are in ghost number 2.

2. Cochains which only involve $g_0[[z]] = l[[z]] + C\mu[[z]]$.

Let us first consider the cochains in $l[[z]]$. These must be $l$ invariant. Since $l$ is a sum of simple Lie algebras, there are no $l$-invariant cochains in degree 1, as there are no $l$-invariant elements in $l$. There are, however, $l$-invariant cochains in degree 2, and we need to show that these are never closed.
The results of [44] compute the cohomology of $l[[z]]$ and show that it is the same as the cohomology of $l$. This vanishes in degrees 1 and 2. Therefore, no $l$-invariant cochain in degree 2 can be closed (because there is no cohomology and also no exact cochains).

Next, we need to consider whether linear combinations of the expressions $\partial_z^k c^i \partial_z^l c^i$ are BRST closed. The charge under rotating $z$ is a symmetry of the problem, so we can fix $k + l$.

One of the terms in the BRST variation of $c^i$ is $c^0 c^i$, and one of the terms in the BRST variation of $c_i$ is $-c^0 c_i$. Let us compute the coefficient of $(\partial_z^k c^0) c^i \partial_z^l c_i$ in the BRST variation of $\partial_z^k c^i \partial_z^l c_i$, assuming $l > 0$. We find

$$Q \partial_z^k c^i \partial_z^l c_i = \partial_z^k (c^0 c^i) \partial_z^l c_i - \partial_z^k c^i \partial_z^l (c^0 c_i) + \cdots$$

$$= (\partial_z^k c^0) c^i \partial_z^l c_i + \cdots$$

assuming that $l > 0$. The only other expression whose BRST variation can produce a term like $(\partial_z^k c^0) c^i \partial_z^l c_i$ is $c^i \partial_z^{k+l} c_i$.

We conclude that in any linear combination

$$\sum A_{k,n} \partial_z^k c^i \partial_z^n c_i$$

for scalars $A_{k,n}$ which is BRST invariant, the coefficient $A_{k,n}$ is determined by $A_{0,n}$, so that there is at most one BRST invariant term for each value of $n$.

However, there is also one BRST exact term for each value of $n > 0$, namely

$$Q \partial_z^n c^0 = \partial_z^n (\sum c^i c_i).$$

This is not zero if $n > 1$. This BRST exact term must therefore be the same as the unique BRST closed term.

This completes the proof that there is no cohomology in degree 2. Therefore, the 't Hooft line exists at the quantum level.

Let us now finish by computing the degree 1 cohomology. The only $g_0$-invariant cochains in degree 1 are $\partial_z^k c^0$ for $k \geq 0$. We have seen that $\partial_z^k c^0$ is not closed if $k > 0$. However, $c^0$ is BRST closed but not BRST exact. This implies that the defect has at most one deformation, given by the descendent of $c^0$. This, in turn, is given by the internal $\int A^0$ of the component of the gauge field proportional to $\mu$.

As we have already seen, the Witten effect (section 5) tells us that adding this expression to the boundary has the same effect as shifting the spectral parameter. This tells us that, as desired, the only modification the minuscule 't Hooft line has is that we can shift its spectral parameter.

14 The Koszul dual of the algebra of operators on the 't Hooft line is the dominant shifted Yangian

Our analysis above showed that, classically, the algebra of operators on the 't Hooft line is the Lie algebra cochains of $p \oplus z g[[z]]$. This is in contrast to the algebra of operators of the bulk system on its own, which is the Lie algebra cochains of $g[[z]]$. 

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We would like to ask what the Koszul dual of the algebra of operators on the ’t Hooft line is.

We do not have space here to go into detail on the role of Koszul duality in quantum field theory: see for instance [45]. Denote by \( A_H \) the dg algebra of operators on the ’t Hooft line, and the Koszul dual algebra by \( A_H^! \). Then, for any analytically-continued quantum mechanical system with algebra of operators \( B \), coupling \( B \) to the ’t Hooft line is the same as giving an algebra homomorphism \( A_H^! \to B \).

In [1] it was proven by an abstract argument that the Koszul dual of the algebra of operators of the bulk system is the Yangian, \( Y(\mathfrak{g}) \). Classically, this is easy to see, as the Koszul dual of the Lie algebra cochains of \( \mathfrak{g}[[z]] \) is the universal enveloping algebra of \( \mathfrak{g}[[z]] \), which is the classical limit of the Yangian. The argument at the quantum level presented in [1] relied on a uniqueness theorem of Drinfeld.

This was made more explicit in [3], where we studied explicitly the Feynman diagrams which contribute to anomalies to coupling a line defect to four-dimensional Chern-Simons theory. We found that these anomalies cancel exactly when the algebra \( B \) of operators on the line defect has a homomorphism from the Yangian algebra \( Y(\mathfrak{g}) \).

Here we will prove the following result.

**Proposition 1.** The Koszul dual of the algebra \( A_H \) of local operators on a minuscule ’t Hooft line is the dominant shifted Yangian \( Y^\mu(\mathfrak{g}) \), associated to the dominant coweight in the same Weyl orbit as the coweight defining the ’t Hooft line.

**Proof.** To see this, let us first look at the classical level. The classical Koszul dual is the universal enveloping algebra of \( \oplus \mathfrak{g}[[z]] \). Physically, this is reasonable: it says that to couple a line defect to the ’t Hooft line, we need to be able to couple the components of the gauge field \( A \) which are in , and all components of the derivatives \( \partial^k A \) for \( k > 0 \).

What happens at the quantum level? In principle, one could try to reproduce the Feynman diagram analysis of [3] in the presence of the ’t Hooft line. This, however, could be quite challenging. Instead, we can use the relationship between the bulk and boundary algebras.

Let us denote the algebra of bulk operators by \( A_{\text{Bulk}} \). We give this an \( (A_\infty) \) structure using the operator product in the direction wrapped by the ’t Hooft line.

By taking an operator in the bulk to one on the defect, we get a homomorphism

\[
A_{\text{Bulk}} \to A_H.
\] (14.1)

This leads to a homomorphism of Koszul dual algebras in the other direction:

\[
A_H^! \to A_{\text{Bulk}}^! = Y(\mathfrak{g}).
\] (14.2)

This homomorphism has a simple explanation in gauge theory terms. If we have an analytically-continued quantum mechanical system with algebra of operators \( B \), then coupling this system to the bulk gauge theory is the same as giving a homomorphism \( Y(\mathfrak{g}) \to B \). If we do this, then we can bring the line defect to the ’t Hooft line, giving us a way of coupling the quantum
mechanical system to the ’t Hooft line. This gives us a homomorphism $A^1_H \to B$. Since this holds for any $B$, we can take $B = Y(g)$ and so get a natural map $A^1_H \to Y(g)$.

This map is injective. To see this, it is enough to check it classically. At the classical level, $A^1_H$ is the universal enveloping algebra of $\mathfrak{p} \oplus z\mathfrak{g}[[z]]$, and $Y(g)$ becomes the universal enveloping algebra of $\mathfrak{g}[[z]]$. The homomorphism just comes from the embedding $\mathfrak{p} \oplus z\mathfrak{g}[[z]] \hookrightarrow \mathfrak{g}[[z]]$, so it is injective.

It is clear that the fact that $A^1_H$ is a subalgebra of $Y(g)$ constrains it greatly. As we will see, this suffices to show that the Koszul dual algebra $A^1_H$ is the dominant shifted Yangian:

$$A^1_H = Y_\mu(g). \quad (14.3)$$

Indeed, in the work of [18], the dominant shifted Yangian $Y_\mu(g)$ is shown to be subalgebra of $Y(g)$ with the same classical limit as that of $A^1_H$.

We need to show that this is enough to constrain the algebra uniquely. This seems very plausible: after all, it is hard to lift a subalgebra of $\mathcal{U}(\mathfrak{g}[[z]])$ to one of the Yangian, and it seems unlikely that there is more than one way to do so.

Let us explain the simple algebraic argument for this uniqueness. At the classical level, the subalgebra $A^1_H$ contains all the level one generators $t^i[1]$ of the Yangian, and those level zero generators $t^i[0]$, $t_\alpha[0]$ corresponding to $\mathfrak{g}_1 \oplus \mathfrak{g}_0$. The algebra is generated by $t^i[0]$, $t_\alpha[0]$ and $t_i[1]$, because commutators with $t^i[0]$ applied to $t_j[1]$ give the rest of the level 1 generators. Commutators of level 1 generators give level 2 and higher generators, so this description completely characterizes $A^1_H$ as a subalgebra of $\mathcal{U}(\mathfrak{g}[[z]])$, at the classical level. This description also holds also for the dominant shifted Yangian.

We need to show that there is (up to a shift in the spectral parameter) only one way to lift this to a subalgebra of the quantum Yangian. At the quantum level, these generators can be modified. There is a symmetry which rotates $z$ and $\hbar$ simultaneously. The level 0 generators are uncharged under this symmetry, and the level 1 generators have the same charge as $\hbar$. This means that only level 1 generators can be modified, and only by adding on $\hbar$ times a polynomial in the level 0 generators. If we perform such a modification, then $A^1_H$ will be generated by the level 0 generators in $\mathfrak{g}_1 \oplus \mathfrak{g}_0$, and some modification

$$t_i[1] + \hbar f_i \quad (14.4)$$

of the level 1 generators in $\mathfrak{g}_{-1}$. Here $f_i$ is some word in the level zero generators. Since the algebra has $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ symmetry, $f_i$ must transform in $\mathfrak{g}_{-1}$ viewed as a $\mathfrak{g}_0$ representation.

If $f_i$ is simply $t_i$, we can absorb it into a shift in the spectral parameter, which shifts the level 1 generators by level 0 generators. Therefore $f_i$ is an expression which is at least quadratic in the level 0 generators.

Because the $f_i$ must generate a copy of $\mathfrak{g}$ as a $\mathfrak{p}$-representation, we must have

$$[t^j, [t^k, [t^l, f_i]]] = 0. \quad (14.5)$$

One of the terms in the operation of bracketing with $[t^j, -]$ is to remove $t_l$ and replace it with $\delta_l^i t_0$, where $t_0$ represents the generator corresponding to the coweight $\mu$. This implies that
$f_i$ can contain at most two generators from $g_{-1}$, because if it contained more than two, then (14.5) can not hold.

In the case that $f_i$ contains two generators from $g_{-1}$, then it can be written as

$$f_i = t^i C_{ij}^{kl} t_k t_l,$$

where $C_{ij}^{kl}$ is in the universal enveloping algebra of $g_0$. Let us use equation (14.5) to show that this can not arise. First, we note that the only element in $U(g_0) \subset U(g)$ which commutes with all elements of $g_1$ is the identity. From this, it is easy to check that (14.5) can only hold if the tensor $C_{ij}^{kl}$ is the identity, in which case we see, by asking that $f_i$ transforms in the representation $g_{-1}$ of $g_0$, that

$$f_i = t^i t_j t_i.$$

An explicit calculation shows that this expression can not satisfy (14.5).

Similarly, if $f_i$ has one generator in $g_{-1}$, then it is of the form

$$f_i = C_i^{j} t_j,$$

where again $C_i^{j} \in U(g_0)$. If $C_i^{j}$ is not the identity, then, since it does not commute with all elements in $g_1$, we can exclude this possibility using (14.5). If $C_i^{j}$ is the identity, then $f_i$ is a linear expression and so can be absorbed by a shift in the spectral parameter.

This completes the proof that the Koszul dual of the algebra of operators on the 't Hooft line is the dominant shifted Yangian.

15 Q-operators in integrable field theories

In this section we will analyze what happens if we study the surface defects of the type considered in [4] together with an 't Hooft line.

The main result is a construction of a Q-operator in a class of integrable field theories. This Q-operator is an interface instead of a line defect, and it satisfies the TQ relation, to leading order in $\hbar$. It would be very interesting to pursue this analysis to all orders in $\hbar$, including quantum effects in the integrable field theory, but that is beyond the scope of this work.

Holomorphic and antiholomorphic surface defects as in [4] give rise, when we compactify to on the $z$-plane, to a two-dimensional integrable field theory. The expectation value of the $x$ and $y$ components at $z \in \mathbb{C}$ is the Lax matrix:

$$\mathcal{L}(z) = \langle A(z) \rangle.$$

We will focus on the class of defects called order defects in [4], where we couple chiral or antichiral degrees of freedom at various values of $z$. Typically, these auxiliary systems are given by some system of fermions, or some $\beta$-$\gamma$ systems. The details of the systems will not matter for our analysis, however.
We will let $J_{(i)}$ denote the currents in the $i$’th chiral or antichiral system, placed at $z_i$. The Lax matrix is then

$$\mathcal{L}^a(z) = \sum c^{ab} \frac{1}{z - z_i} J_{(i)},$$  \hspace{1cm} (15.2)

where $c^{ab}$ is the inverse of the quadratic Casimir.

Each order defect has a Lagrangian $S_{(i)}$, typically that of a free chiral or antichiral theory. Integrating out the gauge fields of four-dimensional Chern-Simons theory couples these order defects giving a Lagrangian of the form

$$\frac{1}{\hbar} \sum_i S_{(i)} + \frac{1}{\hbar} \sum_{i,j} \frac{1}{z_i - z_j} J_{(i)} \wedge J_{(j)},$$  \hspace{1cm} (15.3)

where we treat the current in a chiral theory as a $(1,0)$ form and an antichiral theory as a $(0,1)$-form, so that the $J_{(i)} \wedge J_{(j)}$ term vanishes unless one defect is chiral and the other antichiral.

The simplest models of this form have only two defects, one a system of chiral fermions and one a system of antichiral fermions. In this case, the model is the Gross-Neveu or Thirring model, or some similar model with a $\psi \overline{\psi} \psi \overline{\psi}$ interaction.

### 15.1 Coupling an integrable field theory to the topological $\sigma$-model

View an ’t Hooft line at $z$ as living at the boundary of the topological surface defect wrapping the region $y \leq 0$. Then, the analysis of [4] immediately implies that, in the effective two-dimensional theory, the topological surface defect is coupled to the integrable field theory using the Lax operator $\mathcal{L}(z)$. Concretely, if $V_a$ are the vector fields on $G/P$ generating the $g$ action, then the coupling is by

$$\frac{1}{\hbar} \int_{x,y \geq 0} \langle \eta, V_a \rangle \wedge \mathcal{L}^a(z).$$  \hspace{1cm} (15.4)

(The gauge transformations of the topological $\sigma$-model are also modified, as we will see explicitly below in the case $G = SL_2$.)

This modification of the integrable field theory plays the role of the Q-operator. It is an interface between the original integrable field theory and the theory coupled to the topological $\sigma$-model on $G/P$. The main goal of this section is to justify this, by deriving the TQ relation to leading order in $\hbar$.

Take $G = SL_2$, and take the basis $e, f, h$ of $SL_2$ given $[e,f] = h, [h,e] = 2e, [h,f] = -2f$. Explicitly,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (15.5)

The flag variety $G/P$ in this case is $\mathbb{C}P^1$ and we will work in a neighbourhood of the origin. The topological $\sigma$-model has two fields $\eta, \sigma$ and the Lagrangian is

$$\frac{1}{\hbar} \int_{x,y \geq 0} \left( \eta \wedge d\sigma + \mathcal{L}^e(z) \wedge \eta - 2\mathcal{L}^h(z) \wedge \eta \sigma - \mathcal{L}^f(z) \wedge \eta \sigma^2 \right).$$  \hspace{1cm} (15.6)
In our basis, we have
\[ L_e(z) = \sum_i \frac{1}{z - z_i} J_{f, (i)} , \quad L^h(z) = \frac{1}{2} \sum_i \frac{1}{z - z_i} J_{h, (i)} , \quad L^f(z) = \frac{1}{2} \sum_i \frac{1}{z - z_i} J_{e, (i)}. \] (15.7)

If we denote the parameter for gauge transformations of \( \eta \) as \( \chi \), then the gauge transformations of \( \eta \) are modified by
\[ \delta \eta = d\chi - 2\chi L^h(z) - 2\chi L^f(z)\sigma. \] (15.8)

Further, the fields of the integrable field theory also transform under the gauge transformation. Consider the system at \( z_i \). The fields of this system couple to the 1-form \( \eta \) by the current
\[ \frac{1}{z - z_i} (J_{f, (i)} - J_{h, (i)}\sigma - J_{e, (i)}\sigma^2)^{\sigma}. \] (15.9)

Since \( \eta \) is the gauge field associated to the gauge transformation \( \chi \), then the fields in the system at \( z_i \) transform under \( \chi \) according to this current.

We note that the Lagrangian is gauge invariant. To see this, we note our conventions are such that
\[ dL^e = -2L^h \wedge L^e , \quad dL^h = -L^e \wedge L^f , \quad dL^f = 2L^h \wedge L^f . \] (15.10)

Then gauge variation has two terms: variation of the field \( \eta \), and variation of the fields in the chiral or antichiral theories we are coupling to. Each term vanishes independently. The variation of \( \eta \) gives (after integrating by parts and using the fact that the gauge parameter vanishes on the boundary)
\[ (-2\chi L^h - 2\chi \sigma L^f) d\sigma + \chi dL^e - 2\chi L^e (L^h + \sigma L^f) - 2\chi d(\sigma L^h) + 4\chi \sigma^2 L^h L^f - \chi d(L^f \sigma^2) + 2\chi \sigma^2 L^f L^h. \] (15.11)

This expression simplifies to
\[ \chi dL^e - 2\chi L^e \wedge L^h - 2\chi \sigma dL^h - 2\chi \sigma L^e \wedge L^f - \chi \sigma^2 dL^f + 2\chi \sigma^2 L^h \wedge L^f , \] (15.12)

which vanishes by the Lax equation.

The variation of the field in the defect at \( z_i \) gives
\[ \frac{1}{(z - z_i)^2} \eta \chi [J_{f, (i)} - \sigma J_{h, (i)} - \sigma^2 J_{f, (i)}, J_{f, (i)} - \sigma J_{h, (i)} - \sigma^2 J_{f, (i)}], \] (15.13)

which vanishes.

15.2 The interface is conserved

So far, we have explained how to build a gauge-invariant coupling between the integrable field theory and the new topological degrees of freedom. We will make the topological system end at \( y = 0 \), with boundary condition \( \eta = 0 \) (we also require that the gauge parameter \( \chi \) vanishes
on the boundary). This gives an interface between the original integrable field theory, and that with the topological fields added.

We would like this interface to play the role of the Q-operator. For this to happen, we need to first verify that this interface is conserved. This amounts to checking that we get the same system if, instead of placing the topological degrees of freedom on the region $x, y$ where $y \geq 0$, we place them on the region $x, y$ where $y \geq \epsilon$.

We can work in an axial gauge for the gauge field of the Poisson $\sigma$-model. This sets the $y$-component $\eta_y$ to zero. The boundary condition is that $\eta_x = 0$. If we modify the location of the boundary, then to first order in $\epsilon$ the boundary condition becomes $\eta_x + \epsilon \partial_y \eta_x = 0$.

If we modify the location of the boundary condition, we add on the term

$$
\int_x \int_{y=0}^\epsilon \left( \eta \wedge d\sigma + \mathcal{L}^e(z) \wedge \eta - 2\mathcal{L}^h(y) \wedge \eta \sigma - \mathcal{L}^f(z) \wedge \eta \sigma^2 \right).
$$

To leading order in $\epsilon$, this is

$$
\epsilon \int_{y=0}^\epsilon \iota_{\partial_y} \left( \eta \wedge d\sigma + \mathcal{L}^e(z) \wedge \eta - 2\mathcal{L}^h(y) \wedge \eta \sigma - \mathcal{L}^f(z) \wedge \eta \sigma^2 \right),
$$

where $\iota_{\partial_y}$ indicates contraction with this vector field. This vanishes to leading order in $\epsilon$, because our gauge choice is $\eta_y = 0$ so that we need only consider terms that only depend on $\eta_x$; by $\eta_x$ is proportional to $\epsilon$ by the boundary condition $\eta_x + \epsilon \partial_y \eta_x = 0$.

To finish the analysis, we need to check that the modification of the boundary condition to $\eta_x + \epsilon \partial_y \eta_x = 0$ has no effect. The equations of motion for $\eta_x$ are (using the gauge condition that $\eta_y = 0$)

$$
\partial_y \eta_x = \eta_x (\mathcal{L}^e_y - 2\sigma \mathcal{L}^h_y - \sigma^2 \mathcal{L}^f_y).
$$

Since this is divisible by $\eta_x$, we see that the condition $\eta_x + \epsilon \partial_y \eta_x = 0$ is equivalent to the condition that $\eta_x = 0$.

We have shown that the interface is conserved, meaning that we can move its location freely. In that sense, it behaves like the T-operator which we will now discuss.

### 15.3 The TQ relation

The T-operator is the path ordered exponential of the Lax matrix:

$$
T(z) = \text{PExp} \int_{y=0}^z \left( \begin{array}{cc} \mathcal{L}^h(z) & \mathcal{L}^e(z) \\ \mathcal{L}^f(z) & -\mathcal{L}^h(z) \end{array} \right).
$$

The T-operator is also conserved, because of the flatness of the Lax connection. (Note that there is no factor of $\hbar$ here, unlike in the Lagrangian of the topological surface defect or the integrable field theory.)

Let us consider placing the T-operator at the same location as the Q-operator, that is, at the boundary of the topological surface defect.
Varying the field $\eta$ in (15.6) tells us that
\[ \mathcal{L}^\epsilon(z) = d\sigma + 2\mathcal{L}^h(z)\sigma + \mathcal{L}^f(z)\sigma^2. \] (15.18)

When the transfer matrix is placed in the same location as the Q-operator, we can $\mathcal{L}^\epsilon(z)$ in the transfer matrix by this expression. This gives us
\[ T(z)H(z) = P\exp \int_{y=0} \begin{pmatrix} \mathcal{L}^h(z) & \mathcal{L}^h(z) \\ \mathcal{L}^f(z) & -\mathcal{L}^h(z) \end{pmatrix}. \] (15.19)

Here, $H(z)$ is the interface given by the boundary of the surface defect with fields $\sigma, \eta$.

The T-operator is the parallel transport of the connection given by the matrix in the equation, and as such it is invariant under gauge transformations. There is a gauge transformation
\[ \begin{pmatrix} \mathcal{L}^h(z) & \mathcal{L}^h(z) + \sigma \mathcal{L}^f(z) \\ \mathcal{L}^f(z) & -\mathcal{L}^h(z) - \sigma \mathcal{L}^f(z) \end{pmatrix} = d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (15.20)

Using this gauge transformation, the T-operator (in the presence of the 't Hooft line) becomes
\[ T(z)H(z) = H(z) \exp \left( \int \mathcal{L}^h(z) + \sigma \mathcal{L}^f(z) \right) \exp \left( \int -\mathcal{L}^h(z) - \sigma \mathcal{L}^f(z) \right). \] (15.21)

(On the right hand side we include the factor $H(z)$ to emphasize that the path-ordered exponential is taking place at the boundary of the topological surface defect.)

If we make the $x$-direction periodic, and take the trace of the transfer matrix, we can rewrite this as
\[ T(z)H(z) = H(z) \exp \left( \int \mathcal{L}^h(z) + \sigma \mathcal{L}^f(z) \right) + H(z) \exp \left( \int -\mathcal{L}^h(z) - \sigma \mathcal{L}^f(z) \right). \] (15.22)

On the right hand side, we have two copies of the 't Hooft line, dressed by the operators $\pm(\mathcal{L}^h(z) + \sigma \mathcal{L}^f(z))$.

This is very close to the TQ relation, with the normalization we are using. The operators $\pm(\mathcal{L}^h(z) + \sigma \mathcal{L}^f(z))$ are problematic, however. The final step is to show that we can trade these operators at the boundary for a variation in the parameter $z$.

15.4 Varying $z$

Let us consider the effect of varying the position $z$ of the topological surface defect. We will see that this can be compensated for by a field redefinition.

Varying $z$ modifies the coupling between the topological theory and the integrable field theory by adding the term
\[ \int \eta \left( \partial_z \mathcal{L}^\epsilon(z) - 2\sigma \partial_z \mathcal{L}^h(z) - \sigma^2 \partial_z \mathcal{L}^f(z) \right). \] (15.23)
(The gauge variation of the chiral and antichiral fields is modified in a similar way.) We will show that this can be compensated for by a field redefinition.

On the chiral or antichiral defect at $z_i$, we can perform a field redefinition given by the global $\mathfrak{sl}_2$ symmetry. We will label the field redefinition by the associated current $J_{e,(i)}$, $J_{f,(i)}$, $J_{h,(i)}$. We are interested in these field redefinitions as first-order perturbations of the identity.

For instance, if we have a set of complex chiral fermions at $z_i$ in the fundamental representation of $\mathfrak{sl}_2$, then $1 + \epsilon J_{e,(i)}^c$ sends $\psi_1 \to \psi_1 + \epsilon \psi_2$, $\psi_2 \to \psi^2 - \epsilon \psi^1$.

We can also make these field redefinitions depend on the field $\sigma$, of the topological surface defect. For example, the field redefinition associated to $1 + \epsilon \sigma J_{(i)}^c$ sends $\psi_1 \to \psi_1 + \epsilon \sigma \psi_2$, $\psi^2 \to \psi^2 - \epsilon \sigma \psi^1$.

Varying $z$ to $z + \epsilon$ can be compensated for by the field redefinition

$$1 + \epsilon \left( \sum \frac{1}{z - z_i} (\frac{1}{2} J_{h,(i)} + \sigma J_{e,(i)}) \right).$$

Under this field redefinition, the components of the Lax matrix vary as:

$$\delta \mathcal{L}^e(z) = \delta \sum \frac{1}{z - z_i} \delta J_{f,(i)} = \sum \frac{1}{(z - z_i)^2} (-J_{f,(i)} + \sigma J_{h,(i)}) = \epsilon \partial_z \mathcal{L}^e(z) - 2 \epsilon \sigma \partial_z \mathcal{L}^h(z),$$

$$\delta \mathcal{L}^h(z) = \frac{1}{2} \delta \sum \frac{1}{z - z_i} \delta J_{h,(i)} = - \sum \frac{1}{(z - z_i)^2} \sigma J_{e,(i)} = \epsilon \partial_z \mathcal{L}^f(z),$$

$$\delta \mathcal{L}^f(z) = \delta \sum \frac{1}{z - z_i} \delta J_{e,(i)} = \sum \frac{1}{(z - z_i)^2} J_{e,(i)} = - \epsilon \partial_z \mathcal{L}^f(z).$$

Thus, this field redefinition sends

$$\mathcal{L}^e(z) - 2 \sigma \mathcal{L}^h(z) - \sigma^2 \mathcal{L}^f(z) \to \mathcal{L}^e(z + \epsilon) - 2 \sigma \mathcal{L}^h(z + \epsilon) - \sigma^2 \mathcal{L}^f(z + \epsilon).$$

That is, it has the same effect as varying $z$.

This shows us that the topological surface defect, when coupled to the integrable field theory, does not change when we vary $z$.

What happens when the surface defect has a boundary? In that case, in the bulk of the surface defect, we can compensate for the variation of $z$ by the field redefinition described above. However, this introduces an extra term in on the boundary.

Let us see this concretely in the case when we perform the required field redefinition to a system of chiral fermions $\psi_1$, $\psi_2$, $\psi^1$, $\psi^2$. The transformation corresponding to the element $e \in \mathfrak{sl}_2$ sends $\psi_1 \to \psi_2$, $\psi^2 \to -\psi^1$. This, of course, preserves the Lagrangian $\frac{1}{2} \int \psi_i \overline{\psi}^i$. When the surface defect has a boundary, we only need to perform the field redefinition on the region $y \geq 0$. This sends $\psi_1 \to \delta_{y\geq0} \psi_2$, $\psi^2 \to -\delta_{y\geq0} \psi^1$. This does not preserve the Lagrangian: we pick up a term

$$- \frac{1}{\hbar} \int \psi_2(\overline{\partial} \delta_{y\geq0}) \psi^1 = \frac{1}{\hbar} \int_{y=0} \psi_2 \psi^1 = \frac{1}{\hbar} \int_{y=0} J_e.$$

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In the same way, if we perform the field redefinition associated to any current \( J \) in the region \( y \geq 0 \) in the chiral or antichiral theories that we place at \( z_i \), we pick up a boundary term given by \( h^{-1} \int_{y=0} J \).

This tells us that if, in the bulk of the surface defect, we vary \( z \) to \( z + \hbar \), we can compensate for this variation by the field redefinition \( \hbar \left( \sum \frac{1}{z - z_i} \left( \frac{1}{2} J_{h,(i)} + \sigma J_{e,(i)} \right) \right) ; \) but that this also picks up a boundary term of the form

\[
\int_{y=0} \left( \sum \frac{1}{z - z_i} \left( \frac{1}{2} J_{h,(i)} + \sigma J_{e,(i)} \right) \right) = \int_{y=0} \left( \mathcal{L}^h(z) + \sigma \mathcal{L}^f(z) \right).
\]

Note that the factors of \( \hbar \) have cancelled out, and we have shown that

\[
H(z + \hbar) = H(z) \exp \left( \int_{y=0} \mathcal{L}^h(z) + \sigma \mathcal{L}^f(z) \right).
\]

(Our analysis here is valid to leading order in \( \hbar \).)

Finally, this allows us to prove the TQ relation:

\[
T(z)H(z) = H(z) \exp \left( \int \mathcal{L}^h(z) + \sigma \mathcal{L}^f(z) \right) + H(z) \exp \left( \int \mathcal{L}^h(z) - \sigma \mathcal{L}^f(z) \right)
\]

\[
= H(z + \hbar) + H(z - \hbar)
\]

(15.30)

to leading order in \( \hbar \).

We have derived this for integrable field theories obtained as order defects in four-dimensional Chern-Simons theory. The analysis is valid, with some small variations, in the trigonometric and elliptic cases also. However, a more involved analysis will be necessary to study the Q-operators when we have disorder defects.

16 Composing ’t Hooft lines, QQ relations and and parabolic Verma modules

The most fundamental relation among the T and Q operators is the QQ relation, which expresses the product of two Q operators in terms of the T-operator associated to a Verma module. Here we will proof the ’t Hooft line version of this relation. If \( \mu \) is a minuscule coweight, we let \( H_\mu(z) \) be the corresponding ’t Hooft line, defined as before in terms of a surface defect for \( G/P \) with a boundary condition.

In this section we will relate the product of two ’t Hooft lines with a certain analytically-continued quantum mechanics. The quantum mechanics lives on \( T^* G/P \) viewed as a holomorphic symplectic manifold. In analytically-continued quantum mechanics, we have access to the algebra of operators of the quantum mechanical system, but we have not specified which representation of this algebra will be the Hilbert space. The algebra of operators is what mathematicians would call \( \text{Diff}(G/P, K^{1/2}) \), the algebra of holomorphic differential operators on \( G/P \) twisted by \( K^{1/2} \). The factor of \( K^{1/2} \) is included to ensure that the system is symmetric under time-reversal. We include a factor of \( \hbar \) in our definition of \( \text{Diff}(G/P, K^{1/2}) \),
viewing it as a deformation quantization of $T^*G/P$. The factor of $\hbar$ can be removed by redefinition of the generators.

The system admits a deformation, where the algebra of operators is $\text{Diff}(G/P, K^c)$. This deformation comes from a holomorphic symplectic deformation of the target manifold $T^*G/P$. We will describe this algebra explicitly momentarily.

We will let $T^\mu_j(z)$ be the analytically-continued line defect in four-dimensional Chern-Simons theory obtained by coupling analytically-continued quantum mechanics, with algebra of operators $\text{Diff}(G/P, K^c)$, at the point $z$ in the spectral parameter plane. This algebra acts on a parabolic Verma module for $\mathfrak{g}$ with highest weight $j = c \dim \mathfrak{g}_1$; it is more convenient to parametrize in terms of $j$ rather than $c$.

Here we will show the following proposition.

**Proposition 2.** For any minuscule coweight $\mu$ of any simple group, there is an isomorphism of analytically-continued line defects

$$H_{-\mu}(a\hbar)H_{\mu}(a\hbar) \cong T^\mu_{\dim \mathfrak{g}_1 \left(\frac{1}{2} + \frac{2a}{h^\vee}\right)}(0),$$

where $h^\vee$ is the dual Coxeter number.

### 16.1 Partial flag varieties and oscillator algebras

To make the proposition more concrete, let us describe in more detail the algebra of twisted differential operators on $G/P$, and how it relates to oscillator algebras. As usual, write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $U \subset G/P$ be the open subset which is the orbit of the base point under $\exp(\mathfrak{g}_{-1})$. If $q_i, p^j$ are bases of $\mathfrak{g}_1, \mathfrak{g}_{-1}$, the algebra $\text{Diff}(U)$ is generated by variables $q_i, p^j$ with canonical commutation relations

$$[p^i, q_j] = \hbar \delta^i_j.$$  

(16.2)

For every complex number $c$, there is an injective homomorphism

$$\text{Diff}(G/P, K^c) \to \text{Diff}(U)$$

(16.3)

defined, geometrically, by restricting a global differential operator to $U$ and using a trivialization of the bundle $K$ on $U$.

A natural Fock representation of $\text{Diff}(U)$ on $\mathbb{C}[p^i]$, where $q_i$ acts as $-\hbar \partial_{p^i}$, gives rise to a representation of $\text{Diff}(G/P, K^c)$. This representation is a parabolic Verma module. It has a highest weight vector annihilated by $\mathfrak{g}_1$ in a rank one representation of $\mathfrak{g}_0$. It will be convenient to use the highest weight of the parabolic Verma module, rather than the twist parameter $c$, as our parameter.

To describe $\text{Diff}(G/P, K^c)$ as a subalgebra of the oscillator algebra, we will choose a basis of $\mathfrak{g}$ compatible with the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. Our basis elements will be $t_i$, 

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for $g_{-1, g_1}$, and $t_\alpha$ for the orthogonal complement of $\mu$ in $g_0$, and $t_0$ corresponding to the coweight $\mu$. The commutation relations are

\begin{align}
[t^i, t_j] &= -\frac{1}{\langle \mu, \mu \rangle} \delta^i_j t_0 + f^{i\alpha}_j t_\alpha, \\
[t^0, t^i] &= -t^i, \\
[t^0, t_i] &= t_i, \\
[t_\alpha, t^i] &= c_{\alpha\beta} f^{i\beta}_j t^j, \\
[t_\alpha, t_i] &= -c_{\alpha\beta} f^{j\beta}_i t^j.
\end{align}

(16.4)

Here $c_{\alpha\beta}$ is invariant pairing on $g$.

These act on symplectic manifold with coordinates $q_i, p^j$ by the Hamiltonians

\begin{align}
t^i &\mapsto p^i, \\
t_0 &\mapsto -q_i p^i + h a, \\
t_\alpha &\mapsto -c_{\alpha\beta} f^{i\beta}_j p^j q_j, \\
t_i &\mapsto -\frac{1}{\langle \mu, \mu \rangle} h a q_i - C_{il}^{jk} q_j q_k p^l
\end{align}

for some tensor $C$ whose precise form doesn’t matter. Here $a$ is the highest weight of the parabolic Verma module.

We can write the Lagrangian for the analytically-continued Wilson line as

\begin{align}
\frac{1}{h} \int_x p^i dq_i + \frac{1}{h} \int_x A^a_\mu H_a(p, q),
\end{align}

(16.6)

where $H_a$ are the Hamiltonians written above.

If we multiply all the Lie algebra generators by $1/h$, this gives a homomorphism from $g$ to the oscillator algebra. The algebra of twisted differential operators on $G/P$ is the subalgebra of the oscillator algebra generated by $g$, where the parameter $a$ encodes the twist.

The only parameter in this expression is the constant $a$. We can transform the system by sending $p \mapsto -p$ and reversing $x \mapsto -x$, which reverses the order of the line defect. This sends $a \mapsto -a - \dim g_1$. To see this, we note that sending $x \mapsto -x$ reverses the ordering of the operators, and so sends

\begin{align}
- q_i p^i &\mapsto p^i q_i = q_i p^i + h (\dim g_1).
\end{align}

(16.7)

(Similarly for the Hamiltonian associated to $t_i$ which also depends on $a$.) We also pick up a sign from reversing the order of the integral along $x$, leading to the transformation

\begin{align}
- q_i p^i + h a &\mapsto -q_i p^i - h (\dim g_1 + a).
\end{align}

(16.8)

From this, we see that Wilson line is invariant under sending $x \mapsto -x$ has $a = -\frac{1}{2} \dim g_1$. When describing the algebra as twisted differential operators on $G/P$, this corresponds to the twist by $K^{1/2}$, which is self-dual.
16.2 Colliding ’t Hooft lines

Now let us turn to the proof of the proposition. As before, at \( z = z_0 \), we couple to the \( \sigma \)-model at \( G/P \) wrapping \( y \leq 0 \), giving the ’t Hooft line \( H_\mu(z_0) \).

We would like to compare this to the same surface wrapping \( y \geq 0 \). This gives the ’t Hooft line of charge \(-\mu\) at \( z_0 \). To see this, we recall that the presence of the surface defect is equivalent to allowing certain singularities in the gauge field. Having the surface defect at \( y \leq 0 \) gives us a Hecke transformation removing the singularities, and having it at \( y \geq 0 \) creates them.

Consider the configuration where we have the \( G/P \) surface defect at \( z_0 \), in the range \(-\epsilon \leq y \leq \epsilon\), with the same boundary conditions as before at \( y = \pm \epsilon\). This configuration is given by the composition of line defects

\[
H_{-\mu}(z_0)H_\mu(z_0). 
\]  
(16.9)

By applying a gauge transformation, we can map this to a configuration where the surface defects are at \( y \leq -\epsilon, y \geq \epsilon \). Therefore, to understand the composition of ’t Hooft lines, we have to understand the simpler question of the effective quantum mechanical system obtained from our topological \( \sigma \)-model on an interval of small width.

In general, consider the Poisson \( \sigma \)-model with target \( X \) on the strip \([-\epsilon, \epsilon] \times \mathbb{R}\), with Neumann boundary conditions at both ends, and with zero Poisson tensor. As above, coordinates are \( y, x \) with \(-\epsilon \leq y \leq \epsilon\). Neumann boundary condition means that the field \( \eta \) and its gauge transformation \( \chi \), both vanish at the boundary.

It is not difficult to check that the resulting model is equivalent to topological quantum mechanics on \( T^*X \). Recall that the fields are \( \sigma: [-\epsilon, \epsilon] \times \mathbb{R} \to X \) and \( \eta \in \Omega^1([-\epsilon, \epsilon] \times \mathbb{R}, \sigma^*T^*X) \). We note that varying \( \eta_x \) tells us that \( \partial_y \sigma = 0 \), so that \( \sigma \) can be viewed as a map \( \sigma: \mathbb{R} \to X \). If we vary \( \sigma \) by adding on a term that only depends on \( y \), we find that \( \eta_x = 0 \). By a gauge transformation, we can set \( \eta_y \) to be \( \eta^0 \delta_{y=0} \), for some \( \eta^0 \in \Omega^0(\mathbb{R}, \sigma^*T^*X) \). The resulting action is \( \int_\mathbb{R} \eta^0 \delta_{y=0} \), as desired.

Applying this to our situation, we find that \( H_{-\mu}(z_0)H_\mu(z_0) \) is equivalent to a Wilson line where we couple topological quantum mechanics on \( T^*G/P \).

This analysis is valid at the classical level. To lift it to the quantum level, we need to know what possible deformations this Wilson line has. In appendix B we prove the following.

**Theorem 2.** The analytically-continued Wilson line given by topological quantum mechanics on \( T^*G/P \) exists at the quantum level. Furthermore, this line defect has only two deformations, one given by shifting the position in the \( z \)-plane, and one by replace differential operators by differential operators twisted by \( K^c \).

This is proved by simply computing the cohomology groups which describe obstructions to quantizing the Wilson line and deformations of the Wilson line.

From this, we conclude that

\[
H_{-\mu}(0)H_\mu(0) = T^\mu_j(\lambda) 
\]  
(16.10)
for some values of $\lambda, j$. More generally, we have
\[ H_{-\mu}(-ha)H_{\mu}(ha) = T^\mu_j(\lambda), \] (16.11)
where $j, \lambda$ depend on $a$.

We can fix the parameter $\lambda$ by symmetries. Recall that all the line defects are at $y = 0$. Consider the symmetry of the system which sends $y \rightarrow -y$ and $z \rightarrow -z$. This sends
\[ H_{-\mu}(-ha)H_{\mu}(ha) \rightarrow H_{\mu}(-ha)H_{-\mu}(ha) \] (16.12)
and
\[ T^\mu_j(\lambda) \rightarrow T^\mu_j(-\lambda). \] (16.13)
We conclude that
\[ H_{\mu}(-ha)H_{-\mu}(ha) = T^\mu_j(-\lambda). \] (16.14)
In the case when $\mu$ and $-\mu$ are in the same Weyl orbit, $H_{\mu} = H_{-\mu}$ by a gauge transformation. Therefore,
\[ H_{-\mu}(-ha)H_{\mu}(ha) = T^\mu_j(-\lambda) = T^\mu_j(\lambda) \] (16.15)
by (16.11). This tells us that $\lambda = 0$.

In certain cases, $\mu$ and $-\mu$ are not in the same Weyl orbit. This happens for the spinor coweights of $\mathfrak{so}(4n)$ and the minuscule coweights of $\mathfrak{sl}(n)$ with $n > 2$. In this case, there is an outer automorphism which switches $\mu$ and $-\mu$. Further, the analytically-continued line defects $T^\mu$ and $T^{-\mu}$ are actually isomorphic in these cases. In these cases, we can also conclude that $\lambda = 0$.

Next, we need to fix the value of the constant $j$ in (16.11). Let us first take $a = 0$. The surface defect wrapping $-\epsilon \leq y \leq \epsilon$ at $z = 0$ is invariant under the symmetry which sends $x \rightarrow -x$ and which sends $\eta \mapsto -\eta$ where, as above, $\eta$ is the one-form field on the surface defect. This symmetry preserves the boundary conditions at $y = \pm \epsilon$, and also extends to a symmetry of the coupled 2d-4d system, where we also send $z \mapsto -z$.

When we send $\epsilon \rightarrow 0$ to turn the surface defect into a line defect, the line defect is invariant under the symmetry sending $x \rightarrow -x$ and $z \mapsto -z$. This switches the orientation of the line, and so sends the algebra of operators to its opposite. This fixes $j = -1/2 \dim \mathfrak{g}_1$, as this is the only value of $j$ for which the line defect is isomorphic to its opposite.

We have proved, as desired, that
\[ H_{-\mu}(0)H_{\mu}(0) = T_{-1/2 \dim \mathfrak{g}_1}(0). \] (16.16)

If we include the shifts in the 't Hooft line, we must have
\[ H_{-\mu}(-ha)H_{\mu}(ha) = T^\mu_{-1/2 \dim \mathfrak{g}_1 + F(a)}(0) \] (16.17)
for some function of $a$. The transformation $z \mapsto -z, x \mapsto -x$ leads to the equality
\[ H_{-\mu}(ha)H_{\mu}(-ha) = T^\mu_{-\dim \mathfrak{g}_1/2 - F(a)}(0) \] (16.18)
so that $F$ is an odd function of $a$.

Further, the system with 't Hooft lines has a symmetry which scales $z$ and $\hbar$ simultaneously. For this symmetry to extend to the line defect $T^{\mu} - \frac{1}{2} \dim g_1 + F(a)(0)$, the function $F(a)$ must be linear in $a$, leading to the conclusion.

To fix the coefficient, we use the framing anomaly. The 't Hooft line is invariant under the reflection $x \to -x$ which reverses the ordering along the line. As shown in [2], for any line defect, to construct the dual line, we first reflect and then shift in the $z$ plane by $\hbar^\vee/2$. There is an interface linking the trivial line with the fusion of the original line and its dual.

From this we see that $H_{-\mu}(-\hbar h^\vee/4)H_{\mu}(\hbar h^\vee/4)$ has an interface with the trivial line. Since

$$H(-\hbar h^\vee/4)H(\hbar h^\vee/4) = T^{\mu} - \frac{\dim g_1}{\hbar^\vee} - F(h^\vee/4),$$

the right hand side must have an interface with the trivial line. This only happens when the highest weight is zero, so $F(h^\vee/4) = -\frac{\dim g_1}{\hbar^\vee}$, so that in general

$$H_{-\mu}(-ha)H_{\mu}(ha) = T^{\mu} - \frac{\dim g_1}{\hbar^\vee} - 2a\frac{\dim g_1}{\hbar^\vee}(0).$$

In particular, for $g = sl_2$, we have $\hbar^\vee = 2$ and $\dim g_1 = 1$, so that

$$H_{-\mu}(-ha)H_{\mu}(ha) = T^{\mu} - \frac{\dim g_1}{\hbar^\vee} - 2a\frac{\dim g_1}{\hbar^\vee}(0),$$

where on the right hand side we have the analytically-continued Verma module of spin $j = a - \frac{1}{2}$.

Evidently, this is reminiscent of the standard QQ relation [13]

$$2i \sin(h\phi/2)T^+_{j - \frac{1}{2}}(z, \phi)Q_+(z + hj, \phi)Q_-(z - hj, \phi)$$

except for the dependence on the twist parameter $\phi$. We will explain shortly how the dependence on the parameter arises from placing 't Hooft lines at $\infty$, and also explain how the normalizations work out.

### 16.3 The QQ relation for a minuscule coweight

In this section we will derive the QQ relation for the 't Hooft line of an arbitrary minuscule coweight of an arbitrary group. Recall that the Q-operator, appropriately normalized, comes from 't Hooft operator placed at 0 and at $\infty$. The 't Hooft operator at $\infty$ is given by a surface operator which is the $\sigma$-model whose target $g_1$ and which couples only to the components of the gauge field in $g_1$.

By the analysis above, colliding the two 't Hooft lines at $\infty$ gives us an analytically-continued line defect whose algebra of operators is the oscillator algebra with generators $\tilde{p}_i$, $\tilde{q}^j$ with commutation relations

$$[\tilde{p}_i, \tilde{q}^j] = \hbar.$$
This line defect is only coupled to the components $A^i$ of the gauge field corresponding to $g_1$, via
\[ \frac{1}{\hbar} \int_{z=\infty} z^2 \partial_z A^i \bar{p}_i. \] (16.24)
We can compute the L-operator from a probe Wilson line at $z$ crossing this line defect at $\infty$, by computing the exchange of a single gluon. The exchange of a single gluon is
\[ t^i \bar{p}_i \lim_{z \to \infty} z^2 \partial_z \frac{1}{z} = -t^i \bar{p}_i \] (16.25)
(where $t^i$ acts in the representation associated to the probe Wilson line). Note that this expression is independent of $\hbar$, because the coupling with the line defect at $\infty$ has a factor of $1/\hbar$, and the propagator has a factor of $\hbar$.

This tells us that, to leading order in $\hbar$, we have
\[ L(z) = \exp \left( -t^i \bar{p}_i \right). \] (16.26)
Note that the L-operator is independent of $z$.

For example, for $g = sl_2$ with a Wilson line in the fundamental representation, we find
\[ L(z) = \begin{pmatrix} 1 & \bar{p} \\ 0 & 1 \end{pmatrix} \] (16.27)
to leading order in $\hbar$.

We will denote this line defect at $\infty$ by $T^\mu_\infty$, where $\mu$ is the minuscule coweight. In the case of $sl_2$, we will write it as $T^+_\infty$, $T^-_\infty$ where $+$, $-$ are the minuscule coweights $\pm \frac{1}{2}$.

In [13], they show that this L-operator appears in the QQ relation. They find the composition of two Q-operators gives rise to two oscillator line defects. One is the Verma module for $sl_2$, and one the oscillator representation (16.27). In symbols, before taking the trace, their relation is
\[ T^-_\infty T^+_{j-\frac{1}{2}}(z) = Q^+(z + h j, \phi) Q^-(z - h j, \phi). \] (16.28)
Here, $\bar{T}$ is the unnormalized T-operator.

This is exactly what we find in our analysis. The Q-operator, up to normalization, is given by an ’t Hooft line at $z$ and an ’t Hooft line at $\infty$. As before, we will abuse notation and let $H_{\pm}(z)$ be the ’t Hooft line of charge $\pm \frac{1}{2}$ at $z$ and $\mp \frac{1}{2}$ at $\infty$. Our relation from colliding ’t Hooft lines is
\[ T^-_\infty T^+_{j-\frac{1}{2}}(z) = H^+(z + h j) H^-(z - h j), \] (16.29)
which is identical to (16.28). In our prescription, the extra oscillator representation comes from the collision of ’t Hooft lines at $\infty$.

The only difference between what we found and what is found in [13] is in the normalization. Recall that the Q-operators are normalized with respect to the ’t Hooft line by
\[ Q_{\pm}(z) = G(z)^L H_{\pm}(z), \] (16.30)
where \( L \) is the number of sites on the spin chain and

\[
G(z) = \frac{1}{(2\hbar)^{1/2}} \frac{\Gamma \left( \frac{1}{2\hbar} (z + \frac{h}{2}) \right)}{\Gamma \left( \frac{1}{2\hbar} (z + 3\hbar) \right)}, \tag{16.31}
\]

and the T-operators are normalized so that

\[
\tilde{T}_j(z) = F(j, j)^L T_j(z), \tag{16.32}
\]

where

\[
F(j, j) = \frac{1}{2\hbar} \frac{\Gamma \left( \frac{1}{2\hbar} (z + h(j + 1)) \right)}{\Gamma \left( \frac{1}{2\hbar} (z + h(j + 2)) \right)} \frac{\Gamma \left( \frac{1}{2\hbar} (z - hj) \right)}{\Gamma \left( \frac{1}{2\hbar} (z - hj + h) \right)} \tag{16.33}
\]

is the normalizing function we used earlier.

We have

\[
G(z + hj)G(z - hj) = F(z, j - \frac{1}{2}) \tag{16.34}
\]

so that all the normalizing factors work out to match our relation between 't Hooft lines with the QQ relation (16.28) derived in [13].

16.4 The QQ relation after taking the trace

The line defect \( T^- \) at \( \infty \) couples to a Wilson line only by an upper-triangular matrix. Therefore, at first sight, it seems that it disappears after taking the trace.

This is not quite correct, however, because the line defect has algebra of operators the oscillator algebra. A trace for the oscillator algebra only exists once we introduce a twist parameter. The twist parameter is obtained by modifying the boundary condition so that our gauge field does not vanish at \( z = \infty \), but instead takes constant value \( \phi \) (which we can take to be in the Cartan). The background field \( \phi \) couples non-trivially to the quantum mechanical system at \( z = \infty \). The coupling is quadratic. If we take a basis \( \phi_r \) of the Cartan, and a basis \( \tilde{p}_i \) of \( g_1 \) with weights \( w^r_i \), then the coupling is

\[
- \frac{1}{\hbar} \sum \phi_r w^r_i \frac{1}{2} (\tilde{p}_i \tilde{q}^i + \tilde{q}^i \tilde{p}_i). \tag{16.35}
\]

This expression is time-reversal invariant.

This coupling gives a non-trivial Hamiltonian to the topological quantum mechanics. We can absorb the factor of \( \hbar \) into the normalization of \( \tilde{p}_i \). If we take the \( x \)-direction, which the defect wraps, to be of period 1, then we can compute the partition function of the system by taking the trace on the Fock module \( \mathbb{C}[\tilde{p}_1, \ldots, \tilde{p}_k] \).

Since the Hilbert space is a tensor product of the spaces \( \mathbb{C}[\tilde{p}_i] \), we see the partition function is a product. Each factor in the tensor product gives us

\[
e^{\frac{1}{2} \phi_r w^r_i} + e^{\frac{3}{2} \phi_r w^r_i} + \cdots = \frac{1}{e^{-\frac{1}{2} \phi_r w^r_i} - e^{\frac{1}{2} \phi_r w^r_i}} = - \frac{1}{2 \sinh(\frac{1}{2} \phi_r w^r_i)}. \tag{16.36}
\]
Therefore the trace of the line defect at infinity is

\[ \prod_i \frac{-1}{2 \sinh(\frac{1}{2} \phi_i w_i^r)} = \prod_{\alpha, \langle \mu, \alpha \rangle < 0} \frac{-1}{2 \sinh(\frac{1}{2} \phi_\alpha)}. \tag{16.37} \]

On the right hand side, we have written the expression as a product over all roots \( \alpha \) with \( \langle \mu, \alpha \rangle < 0 \).

Including this factor from the collision of 't Hooft lines at \( \infty \), we find the general form of the QQ relation, valid for any minuscule coweight, is

\[ \prod_{\alpha, \langle \mu, \alpha \rangle < 0} (-2) \sinh(\frac{1}{2} \phi_\alpha) T^\mu_{- \frac{\dim g_1}{2} + \alpha} (z) = H_{-\mu} \left( z - h \frac{ah^\vee}{\dim g_1} \right) H_\mu \left( z + h \frac{ah^\vee}{\dim g_1} \right). \tag{16.38} \]

We can compare this with the relation derived in [13] in the case of \( \mathfrak{sl}_2 \), which is

\[ 2i \sin(\phi/2) \tilde{T}_{j - \frac{1}{2}}^+ (z) = Q_+ (z + hj) Q_- (z - hj). \tag{16.39} \]

The relations are the same after rotating the basis of the Cartan by \(-i\) and including the normalizing factors relating the Q-operators with the 't Hooft lines.

This completes our derivation of the QQ relation from the collision of 't Hooft lines. It is worth pointing out that the last step, of taking the trace, requires choosing a contour for the path integral of the 't Hooft line. This corresponds to replacing the analytically-continued line defects by an actual line defect, by choosing a module for the algebra of operators.

Although we do not give the detailed analysis here, we note that the method applies equally well in the trigonometric case. In the trigonometric case, the 't Hooft line at \( \infty \) can be placed at either \( z = 0 \) or \( z = \infty \), leading to extra possibilities.

In the elliptic case, things are a little more complicated as there is no way to place an 't Hooft line at \( \infty \). A single 't Hooft line must be viewed as an interface between integrable models with dynamical parameter corresponding to different components of the moduli of \( G \)-bundles on an elliptic curve.

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A The prefactor in the L-operator

Consider an arbitrary Wilson line associated to a representation of $\mathfrak{sl}_2$ in which the quadratic Casimir $\frac{1}{4} h^2 + \frac{1}{2}(e f + f e)$ acts by a constant $C$, which in a highest weight representation with highest weight vector of spin $j$, is $j(j + 1)$. (Our conventions are such that $h$ acts by $2j$.) Let $\rho(e), \rho(f), \rho(h)$ be the matrices defining this representation, where as usual $[e, f] = h$, $[h, e] = 2e$, $[h, f] = 2f$. Then the L-operator is

$$L(z) = F(z, j) \begin{pmatrix} z + \frac{1}{4} \rho(h) & \rho(f) \\ \rho(e) & z - \frac{1}{2} \rho(h) \end{pmatrix}. \quad (A.1)$$

The quantum determinant relation states that

$$L_{11}(z)L_{22}(z + h) - L_{12}(z)L_{21}(z + h) = 1. \quad (A.2)$$

(It can be tricky to track down the conventions which determine the precise shift in $z$. However, there is only one possible shift for which a term in the quantum determinant proportional to $h$ drops out, and it is proportional to the identity operator.) This becomes

$$(z + \frac{1}{4} \rho(h))(z + h - \frac{1}{4} \rho(h)) - h^2 \rho(e)\rho(f) = F(z, j)^{-1}F(z + h, j)^{-1}. \quad (A.3)$$

Expanding the left hand side we find

$$z^2 + h z - h^2 \frac{1}{4} \rho(h)^2 + h^2 \frac{1}{8} \rho(h) - h^2 \rho(e)\rho(f)$$

$$= z^2 + h z - h^2 \frac{1}{4} \rho(h)^2 + h^2 \frac{1}{8} \rho(h) - h^2 \frac{1}{2} (\rho(e)\rho(f) + \rho(f)\rho(e) - [\rho(f), \rho(e)]). \quad (A.4)$$

The terms $\rho(h)$ and $[\rho(f), \rho(e)]$ cancel, and $\frac{1}{4}\rho(h)^2 + \frac{1}{2}(\rho(e)\rho(f) + \rho(f)\rho(e))$ is $C = j(j + 1)$. Therefore the equation becomes

$$z^2 + h z - h^2 j(j + 1) = F(z, j)^{-1}F(z + h, j)^{-1}. \quad (A.5)$$

Note

$$z^2 + h z - h^2 j(j + 1) = (z + h(j + 1))(z - hj). \quad (A.6)$$

A solution to this equation is given by

$$F(z, j) = \frac{1}{2h} \frac{\Gamma \left( \frac{1}{2h}(z + h(j + 1)) \right) \Gamma \left( \frac{1}{2h}(z - hj) \right)}{\Gamma \left( \frac{1}{2h}(z + h(j + 2)) \right) \Gamma \left( \frac{1}{2h}(z - hj + h) \right)}. \quad (A.7)$$

Note that factorize $F(z, j)$ is a product of two factors, one of which only depends on $z_+ = z + h(j + 1/2)$ the other on $z_- = z - h(j + 1/2)$:

$$F(z, j) = G(z_+)G(z_-), \quad (A.8)$$

where

$$G(z) = \frac{1}{(2h)^{1/2}} \frac{\Gamma \left( \frac{1}{2h}(z + \frac{h}{2}) \right)}{\Gamma \left( \frac{1}{2h}(z + 2\frac{h}{2}) \right)}. \quad (A.9)$$

This factorization corresponds to the factorization of the T-operator into a product of two ’t Hooft lines.
B Quantization of analytically-continued Wilson lines

Here we will prove the following.

Theorem 3. The analytically-continued Wilson line given by topological quantum mechanics on $T^*G/P$ exists at the quantum level. Furthermore, this line defect has only two deformations, one given by deforming the topological quantum mechanics by a Wess-Zumino term associated to the class in $H^2(G/P) = \mathbb{C}$, and one given by the position in the $z$-plane.

Proof. Let us be a little more precise about the statement. The Lagrangian for the quantum-mechanical system is

$$\frac{1}{\hbar} \int p \, dq.$$  \hspace{1cm} (B.1)

We will first fix some quantization of the quantum mechanical system, compatible with the $\mathbb{C}^\times$ action which scales $p$ and $\hbar$. Algebraically, this amounts to deforming the Poisson algebra of functions $\mathcal{O}(T^*G/P)$ into a non-commutative algebra equipped with a filtration whose associated graded is the original Poisson algebra. It is known [46] that there is a bijection between such quantizations and classes in $H^2(G/P)$. In terms of the Lagrangian, this term is the Wess-Zumino term associated to a class in $H^2(G/P)$.

Let us fix one such quantization, with quantum algebra $B$. This is the algebra of local operators on the line defect. We now analyze whether we can couple this to four-dimensional Chern-Simons theory. Algebraically, this amounts to finding a homomorphism from the Yangian $Y(g)$ to $B$, extending the known homomorphism $Ug \to B$.

When we treat the gauge field classically, we can couple the line defect with algebra of operators $B$. The algebra of operators of the coupled system is $C^*(g[[z]],B)$, the Lie algebra cochain complex of $g[[z]]$ with coefficients in $B$.

Anomalies to the line defect existing at the quantum level are given by $H^2$ with coefficients in the algebra of local operators of the couple along the line. Deformations of the line defect are given by $H^1$.

Consider any line defect with algebra of operators $B$. In [2], appendix C, it was proved that the cohomology group $H^2(g[[z]],B)$ vanishes unless there is a $G$-invariant map $\wedge^2 g \to B$ which does not factor through the adjoint representation. Further, the group $H^1(g[[z]],B)$ has dimension the number of copies of the adjoint representation in $B$. Each copy of the adjoint representation gives a deformation of the line defect where we couple $\partial_z A$ to that copy of the adjoint.

If $B$ is any of the quantum algebras deforming functions on $T^*G/P$, then, as a representation of $G$, it is isomorphic to functions on $T^*G/P$. We need to check that:

1. There is only one copy of the adjoint representation in $\mathcal{O}(T^*G/P)$.

2. Every map of $G$-representations $\wedge^2 g \to \mathcal{O}(T^*G/P)$ factors through the adjoint representation.
The case $G = SL_2(\mathbb{C})$, $G/P = \mathbb{CP}^1$ is rather special. Here, since $\bigwedge^2 \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$, the second condition is vacuous, and it is rather easy to see that there is only one copy of the adjoint representation in $\text{Diff}(\mathbb{CP}^1, K^{1/2})$. We will assume that $G \neq SL_2(\mathbb{C})$ in what follows.

As a $G$-representation, the space $\mathfrak{g}(T^*G/P)$ is isomorphic to the $P$-invariants in $\mathfrak{g}(G) \times \text{Sym}^* \mathfrak{g}_1$, where as above we decompose $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the Lie algebra of $P$ is $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. Also $P$ acts on the right on $\mathfrak{g}(G)$, and $G$ acts on the left on $\mathfrak{g}(G)$ and $\text{Sym}^* \mathfrak{g}_1$ has only a $P$-action, not a $G$-action.

By the Peter-Weyl theorem, $\mathfrak{g}(G) = \oplus V_L \otimes V_R^*$ where the direct sum is over all irreducible representations $V$ of $G$, and $V_L$ has the left action of $G$, $V_R^*$ is the dual representation with the right action of $G$. Therefore, functions on $T^*G/P$ can be written as

$$\oplus_V V \otimes (V^* \otimes \text{Sym}^* \mathfrak{g}_1)^P, \quad (B.2)$$

where we are summing over all irreducible representations of $G$. Each irreducible representation $V^*$ of $G$ is restricted to a $P$-representation, and each $\text{Sym}^k \mathfrak{g}_1$ is viewed as a $P$-representation because $\mathfrak{g}_1 \subset \mathfrak{g}$ is a sub-$P$ representation of $\mathfrak{g}_1$. (Recall that $P = \exp(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$).

The number of copies of the adjoint representation appearing in $\mathfrak{g}(T^*G/P)$ is

$$\dim(\mathfrak{g} \otimes \text{Sym}^* \mathfrak{g}_1)^P. \quad (B.3)$$

Note that

$$\mathfrak{g}_0 = \mathfrak{l} \oplus \mathbb{C} \cdot \rho, \quad (B.4)$$

where $\mathfrak{l}$ is a semi-simple Lie algebra with no Abelian factors, $\rho$ is the coweight defining $P$. The eigenvalues of $\rho$ on $\mathfrak{g}_1, \mathfrak{g}_0, \mathfrak{g}_{-1}$ are $1, 0, -1$. We can first pass to $\rho$-invariants of $\mathfrak{g} \otimes \text{Sym}^* \mathfrak{g}_1$, which are

$$\mathfrak{g}_{-1} \otimes \text{Sym}^1 \mathfrak{g}_1 \oplus \mathfrak{l} \otimes \text{Sym}^0 \mathfrak{g}_1 \oplus \mathbb{C} \cdot \rho \otimes \text{Sym}^0 \mathfrak{g}_1. \quad (B.5)$$

Next, let us pass to $\mathfrak{l}$-invariants. In every case, $\mathfrak{g}_{-1}$ is an irreducible representation of $\mathfrak{l}$. Then the space of $\mathfrak{l}$-invariants is two dimensional, given by the invariants in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1$ and by the third factor $\mathbb{C} \cdot \rho$. But, $\rho$ is not invariant under $\mathfrak{g}_1 \subset \mathfrak{p}$, so that there is only at most a one-dimensional subspace of $P$-invariants, and hence at most one copy of the adjoint representation. Since there is evidently at least one copy of the adjoint representation appearing in functions on $T^*G/P$, this proves there is exactly one copy, as desired.

Next, we need to check that the space of $P$-invariants in $\bigwedge^2 \mathfrak{g} \otimes \text{Sym}^* \mathfrak{g}_1$ is one-dimensional (spanned by the $P$-invariants in the adjoint representation). This is required to show that there are no obstructions to quantizing the line defect, i.e. to show that the algebra of twisted differential operators on $G/P$ has a homomorphism from the Yangian.

The $\rho$-invariants in $\bigwedge^2 \mathfrak{g} \otimes \text{Sym}^* \mathfrak{g}_1$ are

$$\bigwedge^2 \mathfrak{g}_0 \otimes \text{Sym}^0 \mathfrak{g}_1 \oplus (\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}) \otimes \text{Sym}^0 \mathfrak{g}_1 \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0) \otimes \text{Sym}^1 \mathfrak{g}_1 \oplus (\bigwedge^2 \mathfrak{g}_{-1}) \otimes \text{Sym}^2 \mathfrak{g}_1. \quad (B.6)$$
Let us now pass to $\mathfrak{l}$-invariants. There are no $\mathfrak{l}$-invariants in

$$\wedge^2 g_0 = \wedge^2 (I \oplus \mathbb{C} \cdot \rho) = I \oplus \wedge^2 I \tag{B.7}$$

because $I$ is semi-simple with no Abelian factors. The possible $\mathfrak{l}$ invariants live in

$$\begin{align*}
\mathfrak{g}_{-1} \otimes I \otimes \text{Sym}^1 \mathfrak{g}_1, & \quad \text{(B.8)} \\
\mathfrak{g}_{-1} \otimes \mathbb{C} \cdot \rho \otimes \text{Sym}^1 \mathfrak{g}_1, & \quad \text{(B.9)} \\
\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \otimes \text{Sym}^0 \mathfrak{g}_1, & \quad \text{(B.10)} \\
\wedge^2 \mathfrak{g}_{-1} \otimes \text{Sym}^2 \mathfrak{g}_1. & \quad \text{(B.11)}
\end{align*}$$

The first three lines manifestly contain at least one $\mathfrak{l}$-invariant element. The last line may or may not.

Next, let us impose the constraint of invariance under $\mathfrak{g}_1$, recalling that we are interested in invariants for $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Since $\mathfrak{g}_1$ acts trivially on each symmetric power $\text{Sym}^k \mathfrak{g}_1$, we find that $\mathfrak{g}_1$ invariance imposes independent constraints for each power $\text{Sym}^k \mathfrak{g}_1$ that appears.

We will treat the $\text{Sym}^0 \mathfrak{g}_1$ factor (B.10) first. Using the fact that the component of the Lie bracket $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_0$ landing in $\mathbb{C} \cdot \rho$ provides a non-degenerate pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, we see that there are no $\mathfrak{g}_1$ invariants in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1$.

For the $\text{Sym}^2 \mathfrak{g}_1$ factor (B.11), the same argument implies that there are no $\mathfrak{g}_1$ invariants in $\wedge^2 \mathfrak{g}_{-1}$.

Only the $\text{Sym}^1 \mathfrak{g}_1$ factors (B.8), (B.9) remain. Let us make the assumption that the adjoint representation of $I$ appears only once in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1$. We will check this on a case by case basis shortly.

Under this assumption, there are two $\mathfrak{l}$ invariant tensors in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \otimes \text{Sym}^1 \mathfrak{g}_1$. One is in $\mathfrak{g}_{-1} \otimes \mathbb{C} \cdot \rho \otimes \text{Sym}^1 \mathfrak{g}_1$ and one is in $\mathfrak{g}_{-1} \otimes I \otimes \text{Sym}^1 \mathfrak{g}_1$. This first of these two tensor is not $\mathfrak{g}_1$ invariant, so that we are left with at most one $\mathfrak{p}$ invariant element in $\wedge^2 \mathfrak{g} \otimes \text{Sym}^* \mathfrak{g}_1$, as desired.

It remains to check that there is only one copy of the adjoint representation of $I$ in $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$.

There are minuscule coweights for the Lie algebras of all classical types and $E_6, E_7$. In type $A$, the minuscule coweights of $\mathfrak{sl}_n(\mathbb{C})$ are given by Lie algebra elements $\rho = \text{Diag}(1^k, 0^{n-k})$ for $1 \leq k \leq (n-1)/2$. The simple Lie algebra is $I = \mathfrak{sl}_k \oplus \mathfrak{sl}_{n-k}$. If we decompose $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$, then the representation $\mathfrak{g}_1, \mathfrak{g}_{-1}$ of $I$ are $\mathfrak{g}_1 = \text{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})$ and $\mathfrak{g}_{-1} = \text{Hom}(\mathbb{C}^{n-k}, \mathbb{C}^k)$. Thus, $\mathfrak{g}_{\pm 1}$ are the bifundamental representations of $\mathfrak{sl}_k \oplus \mathfrak{sl}_{n-k}$.

Manifestly, there is only one copy of $I$ in the tensor product $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$ in this case.

For $SO(2n, \mathbb{C})$, the spinorial minuscule coweight is defined as follows. (There are two Weyl orbits of coweights like this related by an outer automorphism of $SO(2n, \mathbb{C})$; the argument is the same in each case.) We decompose $\mathbb{C}^{2n} = \mathbb{C}^n_+ \oplus \mathbb{C}^n_-$ where $\mathbb{C}^n_\pm$ are null. There is an element $\rho \in \mathfrak{so}(2n)$ where $\rho$ acts on $\mathbb{C}^n_+$ by $1/2$ and on $\mathbb{C}^n_-$ by $-1/2$. The subgroup $I$ is $\mathfrak{sl}_n$, where $\mathbb{C}^n_\pm$ are the fundamental and antifundamental representations of $I$. The subspaces $\mathfrak{g}_{\pm 1}$ of $\mathfrak{so}(2n, \mathbb{C})$ are $\wedge^2 \mathbb{C}^n_{\pm}$, the exterior square of the fundamental and antifundamental representations of $\mathfrak{sl}_n(\mathbb{C})$. 

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Again, there is only one copy of the adjoint of $\mathfrak{sl}_n(\mathbb{C})$ in the tensor product $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$.

For $SO(n+2, \mathbb{C})$ there is another minuscule coweight obtained from the orthogonal decomposition $\mathbb{C}^{n+2} = \mathbb{C}^2 \oplus \mathbb{C}^n$. The coweight $\rho$ is the generator of the $SO(2, \mathbb{C})$ rotating $\mathbb{C}^2$. The simple algebra $\mathfrak{l}$ is $\mathfrak{so}(n, \mathbb{C})$. The spaces $\mathfrak{g}_{\pm 1}$ are both the vector representation $\mathbb{C}^n$ of $\mathfrak{so}(n, \mathbb{C})$. Again, there is only one copy of $\mathfrak{so}(n, \mathbb{C})$ in $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$.

For $Sp(2n, \mathbb{C})$ there is a minuscule coweight coming from the decomposition $\mathbb{C}^{2n} = \mathbb{C}^n_+ \oplus \mathbb{C}^n_-$ where $\mathbb{C}^n_\pm$ are Lagrangians, and $\rho$ acts on $\mathbb{C}^n_\pm$ with weights $\pm 1/2$. The Lie algebra $\mathfrak{l}$ is $\mathfrak{sl}_n$, and $\mathfrak{g}_{\pm}$ are $\text{Sym}^2 \mathbb{C}^n_\pm$. These are the symmetric squares of the fundamental and antifundamental representations. Again, it is easy to see that there is only one $\mathfrak{sl}_n$ invariant element in $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$.

The remaining simple algebras with minuscule coweights are the exceptional algebras $\mathfrak{e}_6$ and $\mathfrak{e}_7$. The algebra $\mathfrak{e}_6$ has a minuscule coweight where $\mathfrak{l} = \mathfrak{so}(10)$, and $\mathfrak{g}_{\pm 1}$ are the spin representations $S_{\pm}$. If $V$ denotes the vector representation of $\mathfrak{so}(10)$, then

$$S_+ \otimes S_- = \wedge^0 V \oplus \wedge^2 V \oplus \wedge^4 V$$

so that the adjoint representation of $\mathfrak{so}(10)$ appears exactly once.

For $\mathfrak{e}_7$, there is a minuscule coweight where $\mathfrak{l} = \mathfrak{e}_6$ and where $\mathfrak{g}_{\pm}$ are the two irreducible 27 dimensional representations of $\mathfrak{e}_6$, which we denote $\mathbf{27}$ and $\overline{\mathbf{27}}$. One can check that there is only one copy of the adjoint in $\mathbf{27} \otimes \overline{\mathbf{27}}$. □

References

[1] K. Costello, *Supersymmetric gauge theory and the Yangian*, 1303.2632.

[2] K. Costello, E. Witten and M. Yamazaki, *Gauge theory and integrability, I*, *ICCM Not.* 6 (2018) 46 [1709.09993].

[3] K. Costello, E. Witten and M. Yamazaki, *Gauge theory and integrability, II*, *ICCM Not.* 6 (2018) 120 [1802.01579].

[4] K. Costello and M. Yamazaki, *Gauge Theory And Integrability, III*, 1908.02289.

[5] B. Vicedo, *Holomorphic Chern-Simons theory and affine Gaudin models*, 1908.07511.

[6] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *A unifying 2D action for integrable $\sigma$-models from 4D Chern-Simons theory*, *Lett. Math. Phys.* 110 (2020) 1645 [1909.13824].

[7] O. Fukushima, J.-i. Sakamoto and K. Yoshida, *Yang-Baxter deformations of the AdS$_5 \times$S$^5$ supercoset sigma model from 4D Chern-Simons theory*, *JHEP* 09 (2020) Art. 100 [2005.04950].

[8] D. Bykov, *Quantum flag manifold $\sigma$-models and Hermitian Ricci flow*, 2006.14124.

[9] R. Bittleston and D. Skinner, *Gauge theory and boundary integrability*, *JHEP* 05 (2019) 195, 52 [1903.03601].

[10] R. Bittleston and D. Skinner, *Gauge theory and boundary integrability. Part II. Elliptic and trigonometric cases*, *JHEP* 06 (2020) 080, 34 [1912.13441].

[11] K. Costello and B. Stefañski, Jr., *Chern-Simons origin of superstring integrability*, *Phys. Rev. Lett.* 125 (2020) 121602, 6 [2005.03064].
12. R.J. Baxter, *Partition function of the eight-vertex lattice model*, *Ann. Phys.* **70** (1972) 193.

13. V.V. Bazhanov, T. Lukowski, C. Meneghelli and M. Staudacher, *A shortcut to the Q-operator*, *J. Stat. Mech.* **1011** (2010) P11002 [1005.3261].

14. V.V. Bazhanov, R. Frassek, T. Lukowski, C. Meneghelli and M. Staudacher, *Baxter Q-operators and representations of Yangians*, *Nuclear Phys. B* **850** (2011) 148 [1010.3699].

15. R. Frassek, *Oscillator realisations associated to the D-type Yangian: towards the operatorial Q-system of orthogonal spin chains*, *Nuclear Phys. B* **956** (2020) 115063, 22 [2001.06825].

16. E. Witten, *Dyons of charge eθ/2π*, *Phys. Lett. B* **86** (1979) 283.

17. J. Brundan and A. Kleshchev, *Shifted Yangians and finite W-algebras*, *Adv. Math.* **200** (2006) 136 [math/0407012].

18. J. Kamnitzer, B. Webster, A. Weekes and O. Yacobi, *Yangians and quantizations of slices in the affine Grassmannian*, *Algebra Number Theory* **8** (2014) 857.

19. R. Frassek, V. Pestun and A. Tsymbaliuk, *Lax matrices from antidominantly shifted Yangians and quantum affine algebras*, 2001.04929.

20. A. Braverman, M. Finkelberg and H. Nakajima, *Coulomb branches of 3d N = 4 quiver gauge theories and slices in the affine Grassmannian*, *Adv. Theor. Math. Phys.* **23** (2019) 75 [1604.03625].

21. K. Costello and J. Yagi, *Unification of integrability in supersymmetric gauge theories*, 1810.01970.

22. E. Witten, *A new look at the path integral of quantum mechanics*, in *Surveys in differential geometry. Volume XV. Perspectives in mathematics and physics*, vol. 15 of *Surv. Differ. Geom.*, p. 345, Int. Press, Somerville, MA (2011), DOI [1009.6032].

23. A. Kapustin, *Wilson–’t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev. D (3)* **74** (2006) 025005.

24. A. Kapustin, *Holomorphic reduction of N = 2 gauge theories, Wilson–’t Hooft operators, and S-duality*, hep-th/0612119.

25. A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, *Commun. Number Theory Phys.* **1** (2007) 1 [hep-th/0604151].

26. X. Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, in *Geometry of moduli spaces and representation theory*, vol. 24 of *IAS/Park City Math. Ser.*, pp. 59–154, Amer. Math. Soc., Providence, RI (2017).

27. J. Gomis, T. Okuda and V. Pestun, *Exact results for ’t Hooft loops in gauge theories on S4*, *JHEP* **05** (2012) 141 [1105.2568].

28. M. Ashwinkumar, M.-C. Tan and Q. Zhao, *Branes and categorifying integrable lattice models*, *Adv. Theor. Math. Phys.* **24** (2020) 1 [1806.02821].

29. N. Nekrasov, “Open-closed (little) string duality and Chern-Simons-Bethe/gauge correspondence.” Talk at String Math 2017, July 24–28, 2017.

30. J. Kamnitzer, K. Pham and A. Weekes, *Hamiltonian reduction for affine grassmannian slices and truncated shifted yangians*, 2009.11791.
[31] B.H. Gross, *On minuscule representations and the principal SL₂*, Represent. Theory 4 (2000) 225.

[32] G. Ferrando, R. Frassek and V. Kazakov, *QQ-system and Weyl-type transfer matrices in integrable SO(2r) spin chains*, 2008.04336.

[33] H. Zhang, *Yangians and Baxter’s relations*, Lett. Math. Phys. 110 (2020) 2113 [1808.02294].

[34] M. Finkelberg, J. Kamnitzer, K. Pham, L. Rybnikov and A. Weekes, *Comultiplication for shifted Yangians and quantum open Toda lattice*, Adv. Math. 327 (2018) 349 [1608.03331].

[35] A. Braverman, M. Finkelberg and H. Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional N = 4 gauge theories, II*, Adv. Theor. Math. Phys. 22 (2018) 1071 [1601.03586].

[36] J. Kamnitzer, M. McBreen and N. Proudfoot, *The quantum hikita conjecture*, 1807.09858.

[37] B. Assel and J. Gomis, *Mirror symmetry and loop operators*, JHEP 11 (2015) 055 [1506.01718].

[38] T. Dimofte, N. Garner, M. Geracie and J. Hilburn, *Mirror symmetry and line operators*, JHEP 02 (2020) 075, 147 [1908.00013].

[39] D. Gaiotto, L. Rastelli and S.S. Razamat, *Bootstrapping the superconformal index with surface defects*, JHEP 01 (2013) 022 [1207.3577].

[40] S. Gukov and E. Witten, *Gauge theory, ramification, and the geometric Langlands program*, in *Current developments in mathematics, 2006*, p. 35, Int. Press, Somerville, MA (2008) [hep-th/0612073].

[41] A.S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. 212 (2000) 591 [math/9902090].

[42] L.W. Tu, *Semistable bundles over an elliptic curve*, Adv. Math. 98 (1993) 1.

[43] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, vol. 31 of *New Mathematical Monographs*, Cambridge University Press, Cambridge (2017), 10.1017/9781316678626.

[44] S. Fishel, I. Grojnowski and C. Teleman, *The strong Macdonald conjecture and Hodge theory on the loop Grassmannian*, Ann. of Math. (2) 168 (2008) 175 [math/0411355].

[45] K. Costello and N.M. Paquette, *Twisted supergravity and Koszul duality: a case study in AdS₃*, 2001.02177.

[46] R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in algebraic context*, Mosc. Math. J. 4 (2004) 559 [math/0309290].