CANONICAL 3+1 DESCRIPTION
OF RELATIVISTIC MEMBRANES

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ABSTRACT

$M$-dimensional extended objects $\Sigma$ can be described by projecting a Diff $\Sigma$ invariant
Hamiltonian of time-independent Hamiltonian density $\mathcal{H}$ onto the Diff $\Sigma$- singlet sector,
which after Hamiltonian reduction, using $\mathcal{H}$ itself for one of the gauge-fixing conditions,
results in a non-local description that may enable one to extend the non-local symmetries
for strings to higher dimensions and make contact with gravity at an early stage.

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Whereas the Hamiltonian light-cone description of relativistic membranes (cf.[1]) has widely been used [2], a corresponding $3 + 1$ formulation, that would e.g. provide a Hamiltonian structure for the 'generalized $su(\infty)$– Nahm-equations' (derived in [3]) as well as the steady state irrotational isentropic inviscid Kármán-Tsien gas (see [4]), appears to be missing. Filling this gap turns out to reveal a number of rather interesting features of membrane theory (more generally, the theory of massless extended objects of arbitrary dimension). The disappearance of one of the light-cone coordinates from the light-cone Hamiltonian, e.g., finds its correspondence in the time-independence of the Hamiltonian density(!) in the ‘$n + 1$’-formulation (which can then be used to partly fix the invariance under time-independent reparametrisations of the extended object). The Hamiltonian equations of motion can be shown to be implied by $n$ infinite sets of conservation laws. Though the complete Hamiltonian reduction is difficult to perform explicitly, it is likely that a non-local description similar to the loop-representation in general relativity (see e.g. [5]) will result. As most of the considerations apply to general $M$-dimensional extended objects moving in $M + 1$ dimensional euclidean space ($M = 2$ for membranes), I will start by considering the relativistic minimal hypersurface- problem in arbitrary space-time dimensions $D$, i.e. embeddings of $n$-dimensional manifolds $\mathcal{M}$ (of signature $(+, -, \cdots, -)$) into $n + 1$-dimensional Minkowski-space, for which the first variation of their volume $S$,

\[
S = \int_{\mathcal{M}} d^n \varphi \sqrt{G} \\
G = (-)^M \det \left( \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\mu\nu} \right)_{\alpha,\beta=0, \cdots, M=n-1} \\
\eta_{\mu\nu} = \text{diag}(1, -1, \cdots, -1), 
\]

vanishes. In order to simplify the equations of motion,

\[
\frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta x^\mu = 0 \quad \mu = 0 \cdots n, 
\]

one may choose $\varphi^0$ to be the time $t = x^0$ (leaving the time-dependent shape of $\Sigma$, $\tilde{x} = (x^1, \cdots, x^n) = \tilde{x}(t, \varphi^1, \cdots, \varphi^M)$ to be determined), and use the remaining invariance under time-dependent spatial reparametrisations, $\varphi^r \to \varphi^r(t, \varphi^1 \cdots \varphi^M)$ to
The demand

$$\dot{x} \partial_r x = 0 \ , \ r = 1, \ldots, M \ .$$

(3)

The special feature of a hypersurface is that the $\mu = i = 1, \ldots, n = M + 1$ part of (2) is automatically satisfied, provided (3), and the $\mu = 0$ part of (2),

$$\partial_t \left( \frac{g}{1 - x} \right)^{1/2} = 0$$

(4)

holds; $g$ denotes the determinant of the $M \times M$ matrix formed by $g_{rs} := \partial_r x \partial_s x$. Minimal $n$ dimensional hypersurfaces can therefore be described by the $n$ first-order equations(4), once integrated, and (3). These may be put into Hamiltonian form by restricting the Diff $\Sigma$ invariant Hamiltonian

$$H = \int_{\Sigma} d^M \varphi \sqrt{\dot{p}^2 + g}$$

(5)

to the 'singlet-sector',

$$C_r := \dot{p} \frac{\partial_r x}{\sqrt{\dot{p}^2 + g}} = 0$$

(6)

To check this, one first makes sure that, due to the canonical equations of motion,

$$\dot{x} = \frac{\dot{p}}{\sqrt{\dot{p}^2 + g}}, \quad \dot{p} = \partial_r \left( \frac{g g^{rs} \partial_s x}{\sqrt{\dot{p}^2 + g}} \right),$$

(7)

$C_r$ is time-independent. Using (6), one then finds that the Hamiltonian density $\mathcal{H}$ is also(!) conserved:

$$\partial_t \left( \sqrt{\dot{p}^2 + g} \right) = 0$$

(8)

As before, the second order equations for $\dot{x}$ are then automatically satisfied:

$$\ddot{x} = \frac{1}{\mathcal{H}} \partial_r \left( \frac{g g^{rs} \partial_s x}{\mathcal{H}} \right) .$$

(9)

Note that (6) and (8) ( hence (9) ) is also consistent with the equations of motion derived by choosing as Hamiltonian density any non-linear function of $\mathcal{H}$ ( in these
cases, however, the equations of motion alone will not be sufficient to make \( p \cdot \partial_r \dot{x} \) proportional to some conserved quantity.

The Hamiltonian form (5)/(6) may be derived in a less ad hoc way, by the standard canonical procedure, just choosing \( \varphi^0 = x^0 \) (and \(-S\) instead of \(S\)), leaving

\[
G_{\alpha\beta} = \begin{pmatrix}
1 - \dot{x}^2 & -\dot{x} \partial_r \dot{x} \\
-\dot{x} \partial_r \dot{x} & -g_{rs}
\end{pmatrix},
\]

(10)

\[
\mathcal{L} = -\sqrt{G} = -\sqrt{g} \sqrt{1 - \dot{x}^2} + (\dot{x} \partial_r \dot{x}) g^{rs} (\dot{x} \partial_s \dot{x}) .
\]

(11)

Defining canonical momenta,

\[
p_i = \frac{\delta \mathcal{L}}{\delta \dot{x}_i} = \sqrt{\frac{g}{1 - \dot{x}^2} + (\dot{x} \partial_r \dot{x}) g^{rs} (\dot{x} \partial_s \dot{x})} \left( \dot{x}_i - \partial_r x_i g^{rs} (\dot{x} \partial_s \dot{x}) \right),
\]

(12)

it is easy to see that

\[
\mathcal{H} := \dot{x} \cdot \dot{p} - \mathcal{L} = \sqrt{p^2 + g}
\]

(13)

and

\[
\phi_r := \dot{p} \partial_r \dot{x} \equiv 0 \quad \text{for} \quad r = 1, \ldots, M
\]

(14)

(as a consequence of (12), i.e. without assuming (3)). The \( \phi_r \) are primary first class constraints (their Poissonbrackets among themselves, and with \( \mathcal{H} \) vanish on the constraint surface). According to Dirac [6], one should use

\[
H_T := \int_\Sigma \sqrt{p^2 + g} + \int_\Sigma u^r \phi_r ,
\]

(15)

leading to the equations of motion

\[
\begin{align*}
\dot{x} &= \frac{\dot{p}}{\sqrt{p^2 + g}} + u^r \partial_r \dot{x} \\
\dot{p} &= \partial_r \left( \frac{g g^{rs} \partial_s \dot{x}}{\sqrt{p^2 + g}} + u^r \dot{p} \right),
\end{align*}
\]

(16)
from which \( u^r \) can be determined as

\[
  u^r = g^{rs} \hat{x} \partial_s \hat{x} .
\]  

(17)

(16)/(17) are equivalent to the Lagrangian equations of motion,

\[
  \partial_t \left( \frac{\delta L}{\delta \dot{x}^i} \right) + \partial_r \left( \frac{\delta L}{\delta (\partial_r x^i)} \right) = 0 .
\]  

(18)

The effect of choosing the time-dependence of \( \varphi = (\varphi_1, \ldots, \varphi^M) \) such that \( \hat{x} \partial_r \hat{x} \equiv 0 \) is therefore not (6), but putting \( u^r = 0 \), in (15).

Note that (5) and (6) are therefore valid for arbitrary codimension (i.e. \( M \)-dimensional extended objects in \( D \)-dimensional Minkowski-space).

In any case, the question is how to proceed from (5)/(6). At first, one may hope that the existence of \( n \) time-independent, \( \varphi \) dependent functionals (of the \( n \) fields \( \hat{x} \) and their conjugate momenta, \( \hat{p} \)) will be sufficient to have some kind of 'infinite-dimensional Liouville integrability'. However, whereas \( H(\varphi) \) commutes (weakly) with itself,

\[
  \int \Sigma f(\varphi) H(\varphi), \quad \int \Sigma h(\tilde{\varphi}) H(\tilde{\varphi}) = \int \Sigma (f \partial_r h - h \partial_r f) g g^{rs} \frac{C_s}{H} ,
\]  

(19)

it does not commute with \( C_r \):

\[
  \{ H(\varphi), \ C_r(\tilde{\varphi}) \} \approx \partial_r \delta^{(M)}(\varphi, \tilde{\varphi}) .
\]  

(20)

One may try to subtract from \( C_r \) a term \( \partial_r Y \), \( Y \) conjugate to \( H \), or enlarge the phase-space by a pair of conjugate fields, or argue, that the (weak) commutativity of \( H \) with itself already provides a separation of variables for the extended object ('up to projecting onto \( C_r = 0' \)). However, if one performs the Hamiltonian reduction by choosing, e.g.,

\[
  \Pi^r := \varphi^r - x^r \equiv 0 , \quad r = 1, \ldots, M
\]  

(21)

(strictly speaking, this way of gauge-fixing is globally possible only for certain infinitely
extended surfaces) one would have \( \{ C_r, \Pi^s \} = \partial_r x^s \approx \delta^s_r \), hence

\[
\{ F, G \}^* = \{ F, G \} - \int_\Sigma \{ F, C_r(\varphi) \} \{ \Pi^r(\varphi), G \} d^M \varphi
+ \int_\Sigma \{ G, C_r(\varphi) \} \{ \Pi^r(\varphi), F \} d^M \varphi
\]  

(22)

for the Dirac-bracket on the reduced phase-space, and due to (20), \( H \) will no longer be conserved, as

\[
\mathcal{H} := \{ \mathcal{H}, H \}^* \approx \partial_r(p_a \partial_r x_a) \neq 0 \quad a = M + 1, \ldots, n .
\]  

(23)

The left-over fields, however, remain conjugate in the reduced Hamiltonian,

\[
H = \int \sqrt{p_a p_a + p_a p_b \partial_r x_a \partial_r x_b + \det(\delta_{rs} + \partial_r x_a \partial_s x_a)}
\]  

(24)

which for the hypersurface case simplifies \((z = x^n)\) to

\[
H = \int \sqrt{1 + p^2} \sqrt{1 + \partial_r z \partial_r z} ,
\]  

(25)

and agrees, in the case of membranes, with the one derived, (in a rather different way) in [3]. A more interesting way to fix the gauge is to use \( H \) itself, by demanding e.g.

\[
\Pi := \mathcal{H} - \rho(\varphi) H \equiv 0 \quad \int_\Sigma \rho(\varphi) = 1 .
\]  

(26)

When following the Dirac-procedure [6], it is probably best to split (6) into \( C = \partial_r C_r = 0 \), and the complement (the elimination of \( C \) and \( \Pi \) will then involve the Green’s function for the Laplacian on \( \Sigma \) ) However, one may also work with the symplectic form \( \int_\Sigma d \vec{x}(\varphi) \wedge d \vec{p}(\varphi) \) in the following way: Restricting to the hyper-surface case, one first solves (6) by

\[
\vec{p} = p \cdot \hat{m}
\]

\[
M \cdot (\hat{m})_i = \epsilon_{i_1 \ldots i_M} \epsilon^{r_1 \ldots r_M} \partial_{r_1} x^{i_1} \cdots \partial_{r_M} x^{i_M}
\]  

(27)
parameter class of canonical transformations,

\[
\left( \vec{x}, \vec{p} \right) \rightarrow \left( \vec{x}^\lambda := \vec{x}, \ vec{p}^\lambda := \vec{p} + \lambda \vec{m} \right),
\]

(28)
generated by

\[
Q_\lambda = \frac{1}{(M + 1)} \int_\Sigma x \cdot \vec{m}
\]

(29)
- which allows e.g. to express \( H \) as \( \frac{1}{\sqrt{2}} \int_\Sigma \sqrt{p^2 + p_-^2} \). In any case, upon (27) the symplectic form becomes

\[
- \int_\Sigma \epsilon^{ri_1 \ldots ri_M} (\partial_r x^{i_1} \ldots \partial_{r_M} x^{i_M} \frac{dp \wedge dx^i}{M} + \partial_{r_1} p \partial_{r_2} x^{i_2} \ldots \partial_{r_M} x^{i_M} dx^i \wedge dx^{i_1}).
\]

(30)

Changing variables from \( (\vec{p}, \vec{x}) \) to \( (\mathcal{H} = \sqrt{p^2 + 1} \sqrt{g}, \vec{x}) \), and then eliminating part of the degeneracy of (30) by using (26), (30) will be of the form

\[
\int_\Sigma w_i(\varphi) dE \wedge dx^i(\varphi) + \int_{\Sigma \times \Sigma} w_{ij}(\varphi, \tilde{\varphi}) dx^i(\varphi) \wedge dx^j(\tilde{\varphi})).
\]

(31)

After eliminating the remaining \( 1 + (M - 1) \cdot \infty \) degenerate directions in (31), one must find \( Q \) and \( P \) ( in terms of \( E \) and \( \vec{x} \) ) such that, at least locally,

\[
w = dQ \wedge dP.
\]

(32)
The dynamics will then be given by

\[
E = E(Q, P).
\]

(33)
Even for the string case ( closed strings in 3-dimensional flat Minkowski space ) this point of view should be quite interesting ( while it seems still difficult to find \( Q \) and \( P \), one knows that there must exist infinitely many conserved quantities for (33) ). In this
case, \( \{ \Pi(\varphi), C(\tilde{\varphi}) \} \approx \delta'(\varphi - \tilde{\varphi}) =: \chi(\varphi, \tilde{\varphi}) \). Apart from having to take proper care of the zero eigenvalue of \( \chi \) the difference of the Dirac- and the original bracket will be

\[
\int_{\Sigma \times \Sigma} \{ \cdot, \mathcal{H}(\varphi) - H\rho(\varphi) \} \theta(\varphi, \tilde{\varphi}) \{ C(\tilde{\varphi}), \cdot \} =: \chi(\varphi, \tilde{\varphi}) \quad \text{(antisymmetrized)}.
\]

However, as both \( C \) and \( \mathcal{H} \) commute with \( H \), the time-evolution of \( x_1 \) and \( x_2 \) will be unaltered:

\[
\dot{x}_r = p_r = p_{rs}x'_s = \sqrt{\frac{1}{x} - \frac{1}{E^2\rho^2}} \epsilon_{rs}x'_s \quad \text{and} \quad \dot{t} = \frac{\partial}{\partial \varphi}.
\]

(35)

In the membrane-case one will get the generalized \( su(\infty) \) Nahm equations [3] this way. Somewhere in between (in complexity) are axially symmetric membranes (for which a zero-curvature-condition was given in [4], and derived to be equivalent to strings in a curved 3-dimensional space, in [7]), with equations of motion

\[
\dot{r} = \sqrt{\frac{1}{g} - \frac{1}{E^2\rho^2}} z' \cdot r
\]

\[
\dot{z} = -\sqrt{\frac{1}{g} - \frac{1}{E^2\rho^2}} r' \cdot r \quad (g = r^2 (r'^2 + z'^2)).
\]

(36)

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