The McMahon pseudo-metrics of minimal semiflows with invariant measures

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Abstract
Let \((T, X)\) with phase mapping \((t, x) \mapsto tx\) be any minimal semiflow with phase semigroup \(T\) and with compact \(T_2\) phase space \(X\). When \((T, X)\) admits an invariant Borel probability measure, using McMahon pseudo-metrics we mainly show in this paper the following results:

(1) \(S_{eq} = Q\), where \(S_{eq}\) is the equicontinuous structure relation and \(Q\) the regionally proximal relation of \((T, X)\).

(2) \((T, X)\) is weak-mixing iff \((T, X \times Y)\) is T.T. for all \((T, Y)\) which is T.T. and admits of an invariant measure of full support, iff \((T, X')\) is weak-mixing for all cardinality \(c \geq 1\), and iff \(Q = X \times X\).

(3) If \((T, X)\) is point-distal with \(X\) non-trivial, it has a non-trivial equicontinuous factor.

(4) If \((T, X)\) is in addition such that each \(t\) of \(T\), \(x \mapsto tx\) is an invertible self-map of \(X\),
   - \(Q\) is equal to Veech’s relation \(D(X)\).
   - \((T, X)\) is almost automorphic iff \(\pi: (T, X) \to (T, X/S_{eq})\) is of almost 1-1 type.

(5) \((T, X)\) has no non-trivial distal factor if and only if it is weakly disjoint from every \(E\)-semiflow with phase semigroup \(T\).

Here (3) partially answers an open question of Veech 1977 [30, p. 802] that is for any point-distal flow on non-metrizable compact \(T_2\)-space \(X\).

Keywords: Semiflow · equicontinuous structure · regional proximity · minimality · weak-mixing · \(E\)-semiflow

2010 MSC: 37A25 · 37B05 · 54H15

1. Introduction

As usual, a semiflow with phase space \(X\) and with phase semigroup \(T\) is understood as a pair \((T, X)\) where, unless otherwise specified, we assume that
• $X$ is a compact $T_2$-space, **not necessarily metrizable**, and $T$ a topological semigroup with the identity $e$; moreover, $T$ acts jointly continuously on $X$ by phase mapping $(t, x) \mapsto tx$ such that $e x = x$ and $(s t) x = s (t x)$ for all $x \in X$ and $s, t \in T$.

When $T$ is a topological group here, $(T, X)$ will be called a flow. Given $x \in X$, $U \subseteq X$, and $V \subseteq X$, for $(T, X)$, if no confusion, we write for convenience

1. $N_T(x, U) = \{ t | tx \in U \}$ and $N_T(U, V) = \{ t | U \cap t^{-1} V \neq \emptyset \}$.

We shall say that:

2. $(T, X)$ is minimal if and only if $T x$ is dense in $X$: $\text{cls}_XT x = X$, for all $x \in X$; An $x \in X$ is called a minimal point or an a.p. point of $(T, X)$ if $\text{cls}_XT x$ is a minimal subset of $X$.

3. $(T, X)$ admits of an invariant measure $\mu$ if $\mu$ is a Borel probability measure on $X$ such that $\mu(B) = \mu(t^{-1}B)$ for all Borel set $B \subseteq X$ and each $t \in T$. If every non-empty open subset of $X$ is of positive $\mu$-measure, then we say $\mu$ is of full support.

4. $(T, X)$ is surjective if for each $t$ of $T$, $x \mapsto tx$ is an onto self-map of $X$.

5. $(T, X)$ is invertible if for each $t \in T$, $x \mapsto tx$ is a 1-1 onto self-map of $X$. In this case, by $\langle T \rangle$ we mean the smallest group of self-homeomorphisms of $X$ with $T \subseteq \langle T \rangle$.

6. $T$ is called amenable if any semiflow on a compact $T_2$-space with the phase semigroup $T$ admits an invariant measure. Particularly, each abelian semigroup is amenable.

Let $A_X = \{(x, x) | x \in X\}$ and $\mathcal{B}_X$ the compatible symmetric uniform structure of $X$. For $\varepsilon \in \mathcal{B}_X$ and $x \in X$, set

7. $\varepsilon[x] = \{ y \in X | (x, y) \in \varepsilon \}$, which is a neighborhood of $x$ in $X$.

A set-valued map $f : X \rightharpoonup X$ is said to be continuous at $x_0 \in X$ if and only if given $\varepsilon \in \mathcal{B}_X$ there is a $\delta \in \mathcal{B}_X$ such that $x \in \delta[x_0]$ implies $f(x) \subseteq \varepsilon[f(x_0)]$ and $f(x_0) \subseteq \varepsilon[f(x)]$.

8. A surjective semiflow $(T, X)$ is called bi-continuous if for each $t \in T$, $x \mapsto t^{-1} x$ is continuous at every point of $X$.

Clearly, if $(T, X)$ is invertible, it is bi-continuous. In addition, if let $X = [0, 1]^\mathbb{Z}$ with the usual topology and $\sigma : X \to X$ defined by $(x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}$; then the cascade $(\sigma, X)$ is bi-continuous.

In this paper, we shall study the equicontinuous structure relation, weak-mixing and weak disjointness, and Veech’s relations by using the McMahon pseudo-metric of minimal semiflows with invariant measures.

**1.1. Equicontinuous structure relation**

First of all we recall a basic notion — equicontinuity, which is valid for any semiflow $(T, X)$ and for which there are some equivalent conditions in [5].

**Definition 1.1.** $(T, X)$ is equicontinuous if and only if given $\varepsilon \in \mathcal{B}_X$, there exists some $\delta \in \mathcal{B}_X$ such that $T \delta \subseteq \varepsilon$, i.e., if $(x, x') \in \delta$ then $(tx, tx') \in \varepsilon$ for all $t \in T$.

As is proved in [14, 30, 3], for $(T, X)$ there exists on $X$ a least closed $T$-invariant equivalence relation, denoted $S_{eq}(X)$, such that $(T, X/S_{eq}(X))$, defined by

$$(t, S_{eq}[x]) \mapsto S_{eq}[tx] \quad \forall (t, x) \in T \times X,$$

is an equicontinuous semiflow.
Definition 1.2. $S_{eq}(X)$ is called the equicontinuous structure relation of $(T, X)$. Simply write $X_{eq} = X/S_{eq}(X)$. $(T, X_{eq})$ is called the maximal equicontinuous factor of $(T, X)$. Let $\pi : X \rightarrow X_{eq}$, $x \mapsto S_{eq}[x]$, be the canonical projection.

In a number of situations the equicontinuous structure relation of minimal flows is known explicitly (cf., e.g., [28, 3, 6, 4, 7]). Particularly, if $Q$ is the “regionally proximal relation” (cf. Definition 1.3 below), then the following two facts are well known:

Theorem A (cf. [30, Theorem 2.6.2]). Let $(T, X)$ be a minimal flow such that $(T, X \times X)$ has a dense set of minimal points; then $S_{eq}(X) = Q(X)$. (This is due to R. Ellis and W.A. Veech independently.)

Theorem B (cf. McMahon [24] and also see [3, Theorem 9.8]). If $(T, X)$ is a minimal flow admitting an invariant measure, then $S_{eq}(X) = Q(X)$.

Based on the recent work [12, 5, 9], we will generalize the above classical Theorem B to semiflows on compact $T_2$-spaces.

Definition 1.3. Let $(T, X)$ be a semiflow with phase semigroup $T$.

1. $(T, X)$ is called distal if given $x, x' \in X$ with $x \neq x'$, there is an $\varepsilon \in \mathcal{B}_X$ such that $t(x, x') \notin \varepsilon$ for all $t \in T$. Then by [5, Theorem 1.15], whenever $(T, X)$ is distal, then it is invertible.

Theorem C (cf. [5]). If $(T, X)$ is a (minimal) distal semiflow, then $(T, X)$ is a (minimal) distal flow. Hence by Furstenberg’s theorem [15], $(T, X)$ admits invariant measures.

2. An $x \in X$ is called a distal point of $(T, X)$ if no point of $\text{cls}_X T x$ other than $x$ is proximal to $x$; i.e., if $y \in \text{cls}_X T x$ is such that $t_n(x, y) \rightarrow (z, z)$ for some $z \in X$ and some net $\{t_n\}$ in $T$ then $y = x$. It is easy to verify that:

Lemma D. $(T, X)$ is distal iff every point of $X$ is distal for $(T, X)$.

$(T, X)$ is called point-distal if there is a distal point whose orbit is dense in $X$.

3. We say $x \in X$ is regionally proximal to $x' \in X$ for $(T, X)$, denoted by $(x, x') \in Q(X)$ or $x' \in Q[x]$, if there are nets $\{x_n\}, \{x'_n\}$ in $X$ and $\{t_n\}$ in $T$ with $x_n \rightarrow x, x'_n \rightarrow x'$ and $\lim t_n t_n^{-1} = \lim t_n t_n^{-1}$.

Then by the equality:

$$Q(X) = \bigcap_{\varepsilon \in \mathcal{B}_X} \text{cls}_X \times \bigcup_{t \in T} t^{-1} \varepsilon$$

$Q(X)$ is a closed reflexive symmetric relation. An example due to McMahon [23] shows it is not always an equivalence relation even in minimal flows.

If $(T, X)$ is a flow and $(x, x') \in Q(X)$, then for all $t \in T$, by $\lim t_n(t_n^{-1})(t x_n) = \lim t_n(t_n^{-1})(t x'_n)$ and $t_n t_n^{-1} \in T$, it easily follows that $t(x, x') \in Q(X)$ and so $Q(X)$ is invariant.

In general, $Q(X)$ is neither invariant nor transitive in semiflows; yet if $(T, X)$ is minimal admitting an invariant measure, as we will see, $Q(X)$ is an invariant closed equivalence relation on $X$, and in fact

Theorem 1.4. Let $(T, X)$ be a minimal semiflow, which admits an invariant measure, then it holds that $S_{eq}(X) = Q(X)$. 

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It should be noted that the condition “admitting an invariant measure” is important for our statement above. For example, let $X = [0, 1]$ the unit interval with the usual topology and for every $\alpha$ with $0 < \alpha < 1$, define two injective mappings of $X$ into itself:

$$f_\alpha : X \to X, \ x \mapsto \alpha x \quad \text{and} \quad g_\alpha : X \to X, \ x \mapsto 1 - \alpha(1 - x).$$

Now let

$$T = \{ f_{\alpha}^{n_{1}} \circ f_{\alpha}^{n_{2}} \circ \cdots \circ f_{\alpha}^{n_{k}} \mid 0 < \alpha_i < 1, 0 < \gamma_i < 1, \epsilon_i = 0 \text{ or } 1, \delta_i = 0 \text{ or } 1, n = 1, 2, \ldots \}$$

be the discrete semigroup generated by $f_\alpha, g_\alpha, 0 < \alpha < 1$. It is easy to see that each $t \in T$ is injective and that $(T, X)$ is equicontinuous minimal so $S_{eq}(X) = A_X$, but $Q(X) = X \times X$. See [5, Example 1.2].

Thus, as consequences of Theorem 1.4, we can easily obtain the following two corollaries.

**Corollary 1.5.** Let $(T, X)$ be a minimal semiflow with $T$ amenable; then $S_{eq}(X) = Q(X)$.

Thus if $f : X \to X$ is a minimal continuous transformation of $X$, then $S_{eq}(X) = Q(X)$ associated to the natural $\mathbb{Z}_+\cdot$-action.

Next, replacing amenability of $T$ in Corollary 1.5 by distality of $(T, X)$, we can obtain the following result:

**Corollary 1.6.** Let $(T, X)$ be a minimal distal semiflow; then $S_{eq}(X) = Q(X)$.

**Proof.** Since $(T, X)$ is distal, each $t \in T$ is a self-homeomorphism of $X$. Then by Theorem C, $(T, X)$ admits an invariant measure. Then Corollary 1.6 follows from Theorem 1.4. □

§§2 and 3 of this paper will be devoted to proving Theorem 1.4. Notice that although we will see that $(T, X_{eq})$ is invertible in the situation of Theorem 1.4, yet each $t \in T$ itself need not be invertible with respect to $(T, X)$. This will cause the difficulty.

### 1.2. Weak-mixing semiflows

Let $(T, X)$ be a semiflow. By $l(X)$ we denote the collection of non-empty open subsets of $X$. Then, as usual, we introduce four basic notions.

1) $(T, X)$ is called **topologically transitive** (T.T.) iff every $T$-invariant set with non-empty interior is dense in $X$ iff $N_T(U, V) \neq \emptyset$ for $U, V \in l(X)$.

2) $(T, X)$ is called **point-transitive** (P.T.) if there is some point $x \in X$ with $Tx$ dense in $X$. In this case, write $x \in \text{Tran}(T, X)$.

3) $(T, X)$ is called **weak-mixing** if $(T, X \times X)$ is a T.T. semiflow.

4) $(T, X)$ is called **discretely thick** T.T. if given $U, V \in l(X), N_T(U, V)$ is discretely thick (i.e. for any finite subset $K$ of $T$ there is $t \in T$ such that $Kt \subseteq N_T(U, V)$).

5) $A \subseteq T$ is called an **IP-set** of $T$ if there exists a sequence $\{ t_n \}$ in $T$ such that $t_n, t_n, \ldots, t_n \in A$ for all $1 \leq n_1 < n_2 < \cdots < n_k < \infty$ and $k \geq 1$. $(T, X)$ is said to be **IP-T.T.** if $N_T(U, V)$ is an IP-set of $T$ for $U, V \in l(X)$.

In [16, p. 26], T.T. is called “ergodic”. It should be noted that in our situation, T.T. $\Rightarrow$ P.T., and weak-mixing $\Rightarrow$ P.T. as well.

In §4, we will characterize “weak-mixing” of minimal semiflow with invariant measures by using the McMahon pseudo-metric; see Theorem 4.12.
Let \( X^n = X \times \cdots \times X \) (\( n \)-times), for any integer \( n \geq 2 \). Based on \((T, X)\), \((T, X^n)\) is also a semiflow, which is defined by \( t : (x_i)_{i=1}^n \mapsto (tx_i)_{i=1}^n \). Given any cardinality \( \varepsilon \geq 1 \), we can similarly define \((T, X^n)\).

It is a well-known important fact that (cf. [16, Proposition II.3] for cascades, [18, Theorem 1.11] for \( T \) abelian groups, and [11, Lemma 3.2] for abelian semigroups):

**Theorem E.** If \((T, X)\) is weak-mixing with \( T \) abelian, then \((T, X^n)\) is \( T \cdot T \) for all integer \( n \geq 2 \).

In fact, this very important theorem can be extended to amenable semigroups as follows, which is new even for flows.

**Theorem 1.7.** Let \((T, X)\) be a minimal semiflow, which admits of an invariant measure (e.g. \( T \) is an amenable semigroup). Then \((T, X)\) is weak-mixing if and only if \((T, X \times Y)\) is \( T \cdot T \) for all \((T, Y)\) which is \( T \cdot T \) and admits of a full ergodic center if and only if \( Q(X) = X \times X \).

The “ergodic center” will be defined in Definition 4.3. This theorem will be proved in §4.1 under the guises of Corollary 4.8 for the first “if and only if” and Theorem 4.12 for the second “if and only if”.

**Theorem 1.8.** Let \((T, X)\) be a minimal semiflow, which admits an invariant measure (e.g. \( T \) an amenable semigroup). Then \((T, X)\) is weak-mixing if and only if \((T, X^\ast)\) is \( T \cdot T \) for all cardinality \( \varepsilon \geq 2 \) if and only if \((T, X)\) is a discretely thickly \( T \cdot T \) semiflow.

Theorem 1.8 will be proved in §4.1 under the guise of Theorem 4.11. It is not known whether these assertions still hold for a non-minimal semiflow with amenable phase semigroup.

**Definition 1.9** ([9]). An \( x \in X \) is called locally almost periodic (l.a.p.) for \((T, X)\) if for each neighborhood \( U \) of \( x \) there are a neighborhood \( V \) of \( x \) and a discretely syndetic subset \( A \) of \( T \) such that \( AV \subseteq U \).

Here by a “discretely syndetic” set \( A \) we mean that there is a finite subset \( K \) of \( T \) such that \( K_t \cap A \neq \emptyset \) \( \forall t \in T \). A subset of \( T \) is discretely syndetic if and only if it intersects non-voidly each discretely thick subset of \( T \) (cf., e.g., [5]).

**Corollary 1.10.** Let \((T, X)\) be a minimal semiflow with \( T \) amenable. If \((T, X)\) is weak-mixing, then there is no l.a.p. point of \((T, X)\).

**Proof.** Suppose the contrary that there is an l.a.p. point \( x_0 \). Since \( X \) is a non-singleton \( T_2 \)-space, we can choose disjoint \( U, U' \in \text{Int}(X) \) with \( x_0 \in U \). Then there are a discretely syndetic set \( A \subseteq T \) and an open neighborhood \( V \) of \( x \) such that \( AV \subseteq U \). But this contradicts that \( N_T(V, U') \) is discretely thick in \( T \) by Theorem 1.8. This proves Corollary 1.10. \( \square \)

Recall that an \( x \in X \) is referred to as an \( IP^\ast \)-recurrent point of \((T, X)\) if and only if given a neighborhood \( U \) of \( x \), \( N_T(x, U) \) is an \( IP^\ast \)-set in the sense that it intersects non-voidly every \( IP \)-set of \( T \) (cf. [17, 10]).

**Theorem F** (cf. [17, Theorem 9.11] and [10, Theorem 4]). Let \((T, X)\) be any semiflow and \( x \in X \); then \( x \) is a distal point of \((T, X)\) if and only if \( x \) is an \( IP^\ast \)-recurrent point of \((T, X)\).

**Corollary 1.11.** If \((T, X)\) is a minimal weak-mixing semiflow with \( T \) amenable, then there exists no distal point of \((T, X)\).
Notes.

1. See Veech [29] and Furstenberg [17, Theorem 9.12] for cascade \((f, X)\) with \(X\) a compact metric space and Dai-Tang [11, Proposition 3.5] for IP-T.T. semiflow \((T, X)\) with first countable phase space \(X\). Here without any countability our proof is completely new.

2. If \(T\) is a group in place of “\(T\) amenable”, the consequence of Corollary 1.11 still holds; see Theorem 2.10.

3. By Corollary 1.11, a point-distal semiflow is T.T. but it is never an IP-T.T. semiflow.

Proof. Assume for a contradiction that \((T, X)\) has a distal point, say \(x_0 \in X\). Let \(W\) be a closed subset of \(X\) with \(x_0 \notin W\) and \(\text{Int} W \neq \emptyset\). Let \(\beta = [0, 1) \in \beta X\) be such that \(\beta[x_0] \cap W = \emptyset\). For all \(\varepsilon \in \beta X\) with \(\varepsilon \leq \beta\), define a closed subset of \(X\) as follows: \(W_\varepsilon = \{x \in W \mid \exists t \in T \land tx_0 \in \text{cl}_{\beta X}[x_0]\}\). Since \(x_0\) is IP-recurrent by Theorem 2, \(N_T(x_0, \varepsilon[x_0])\) is an IP-set. On the other hand, by Theorem 1.8, \(N_T(W, \varepsilon[x_0])\) is discretely thick and so \(N_T(W, \varepsilon[x_0])\) is an IP-set (cf., e.g., [17, Lemma 9.1] and [9, (1.2a)]). Therefore, \(W_\varepsilon \neq \emptyset\) for all \(\varepsilon \in \beta X\). Given any \(\varepsilon_1, \ldots, \varepsilon_n \leq \beta\), let \(\varepsilon = \varepsilon_1 \cap \cdots \cap \varepsilon_n\); then \(W_{\varepsilon_1} \cap \cdots \cap W_{\varepsilon_n} \supseteq W_\varepsilon \neq \emptyset\). Thus \(\{W_\varepsilon \mid \varepsilon \leq \beta\}\) has the finite intersection property. This shows that \(\bigcap_{\varepsilon \leq \beta} W_\varepsilon \neq \emptyset\). Now let \(y \in \bigcap_{\varepsilon \leq \beta} W_\varepsilon\) be any given. Clearly, \(x_0 \neq y\). Since for each \(\varepsilon \leq \beta\) there is some \(t \in T\) such that \(tx_0, ty \in \text{cl}_{\beta X}[x_0]\), hence \(y\) is proximal to \(x_0\). This is a contradiction to the assumption that \(x_0\) is a distal point of \((T, X)\). This proves Corollary 1.11.

Theorem 1.12 (Veech 1977 [30, Question, p. 802] for any point-distal flows). Let \((T, X)\) be a point-distal semiflow with \(T\) amenable and \(X\) non-trivial. Then \((T, X)\) has a non-trivial equicontinuous factor.

Proof. By Theorem 1.4, \(S_{eq}(X) = Q(X)\). Suppose the contrary that \((T, X)\) has no non-trivial equicontinuous factor, then \(Q(X) = X \times X\) and so \((T, X)\) is weakly mixing by Theorem 1.7. But this contradicts Corollary 1.11. The proof is complete.

It should be noticed that “\(T\) amenable” may be relaxed by “\((T, X)\) admitting of an invariant measure” in Corollaries 1.10 and 1.11 and Theorem 1.12.

As other application of Theorem 1.8 we will consider the chaotic dynamics of weak-mixing amenable semiflows in §4.2; see Theorems 4.17 and 4.19 and Corollary 4.21.

1.3. Veech’s relations of surjective dynamics

Definition 1.13 (cf. [27, 28, 8, 7, p. 741] for \(T\) in groups and [9] for any semiflows). Let \((T, X)\) be a surjective semiflow.

1. An \(x \in X\) is called almost automorphic (a.a.) for \((T, X)\), denoted \(x \in P_{aa}(T, X)\), if and only if \(t_nx \to y, x_n^\prime \to x^\prime, t_nx_n^\prime = y\) implies \(x = x^\prime\) for every net \(\{t_n\}\) in \(T\).

2. If \(x \in P_{aa}(T, X)\) and \(\text{cl}_{\beta X} T x = X\), then \((T, X)\) is called an a.a. semiflow.

Since an a.a. point is a distal point of any surjective \((T, X)\) (cf. [9]), hence an a.a. semiflow is point-distant and so minimal.

3. We say that \((x, x^\prime)\) is in Veech’s relation \(V\) of \((T, X)\), denoted \((x, x^\prime) \in V(X)\) or \(x^\prime \in V[x]\), if there exist a net \(x_n^\prime \to x^\prime\), a point \(y \in X\), and a net \(\{t_i\}\) in \(T\) such that \(t_i x \to y\) and \(t_i x_n^\prime = y\).

Then:

- \(V[x] = \{x\}\) if and only if \(x \in P_{aa}(T, X)\).
(4) Given \( x \in X \), define \( D[x] \) for \((T, X)\) by

\[
D[x] = \bigcap_{e \in \mathbb{X}} D(x, e), \quad \text{where } D(x, e) = \text{cls}_X A^{-1} Ax \text{ and } A = N_T(x, e[x]).
\]

\( D[x] \) is closed, and, of course, \( x \in D[x] \). Then we will say \((x, y) \in D(X)\) if and only if \( y \in D[x] \).

We notice here that \( D(x, e) \) is originally defined by \( D(x, e) = \text{cls}_X AA^{-1} x \) in [28]. However, since \( T \) is an abelian group in [28], so our definition agrees with Veech’s in flows of abelian groups.

**Definition 1.14.** Let \((T, X)\) and \((T, Y)\) be two any semiflows.

1. \( \pi : X \to Y \) is called an epimorphism between \((T, X)\) and \((T, Y)\), denoted \((T, X) \xrightarrow{\pi} (T, Y) \), if \( \pi \) is continuous surjective with \( \pi(tx) = \pi(t)x \) for all \( t \in T \) and \( x \in X \). In this case, \((T, Y)\) is called a factor of \((T, X)\), and \((T, X)\) an extension of \((T, Y)\).

2. \( (T, X) \xrightarrow{\pi} (T, Y) \) is said to be of almost 1-1 type if there exists a point \( y \in Y \) such that \( \pi^{-1}(y) \) is a singleton set.

It is clear that if \((T, X)\) is surjective (resp. minimal), then its factors are also surjective (resp. minimal). Of course, its factor need not be invertible if \((T, X)\) is invertible. Moreover, even if \((T, X)\) has an invertible non-trivial factor, \((T, X)\) itself need not be invertible. As mentioned before, we will be mainly concerned with the maximal equicontinuous factor \((T, X_{eq})\) of a minimal semiflow \((T, X)\).

Using a variation of a theorem of Bogoliouboff and Fölner [28, Theorem 4.1] that is valid only for discrete abelian groups, Veech proved the following theorem:

**Theorem G** (cf. [28, Theorem 1.1]). *If \((T, X)\) is a minimal flow with \( T \) an abelian group, then \( D(X) = S_{eq}(X) \).*

Although Veech’s proof of Theorem G does not work for non-abelian flows, yet using different approaches we can obtain the following generalization in §5.

**Theorem 1.15** (Veech’s Structure Theorem for a.a. semiflows; cf. [9, Theorem 5.9] for \( T \) abelian). Let \((T, X)\) be a minimal bi-continuous semiflow, which admits an invariant measure. Then:

1. When \((T, X)\) is invertible, then \( Q[x] = D[x] = \overline{V[x]} \) for all \( x \in X \).
2. \((T, X)\) is a.a. if and only if \( \pi : (T, X) \to (T, X_{eq}) \) is of almost 1-1 type.

Hence if \( X \) is compact metric and \((T, X)\) is a.a., then \( P_{eq}(T, X) \) is a residual subset of \( X \).

**Corollary 1.16** (Veech for \( T \) in amenable groups; unpublished notes). *Let \((T, X)\) be an invertible minimal semiflow with \( T \) an amenable semigroup. Then:*

1. \( Q[x] = D[x] = \overline{V[x]} \) for all \( x \in X \).
2. \((T, X)\) is a.a. if and only if \( \pi : (T, X) \to (T, X_{eq}) \) is of almost 1-1 type.

**Proof.** Since \( T \) is amenable, \((T, X)\) admits an invariant measure. Then Corollary 1.16 follows from Theorem 1.15.

**Corollary 1.17.** *Let \((T, X)\) be a minimal bi-continuous semiflow, which admits an invariant measure. Then \((T, X)\) is equicontinuous iff all points are a.a. points.*
This will be proved in §5. We note that Corollary 1.17 in the case that $T$ is an abelian group and $X$ a compact metric space is [8, Corollary 21]. Moreover, the statement of Corollary 1.17 still holds without the condition “which admits an invariant measure” by using different approaches (cf. [9, Theorem 4.5]).

**Corollary 1.18.** Let $(T, X)$ be a minimal bi-continuous semiflow, which admits an invariant measure. If $(T, X)$ is a.a., then it has l.a.p. points.

*Proof.* By Theorem 1.15, $\pi : X \twoheadrightarrow X_{eq}$ is of almost 1-1 type. Since $(T, X_{eq})$ is equicontinuous invertible, it is l.a.p. and so $(T, X)$ has l.a.p. points. The proof is complete.

Theorem 1.15 will be proved in §5. In fact, (1) of Theorem 1.15 is Theorem 5.3 and (2) of Theorem 1.15 is Theorem 5.5. In fact, since $(T, X)$ is bi-continuous, $(T, X)$ must be invertible when it is a.a by [9, Lemma 1.1].

### 2. Preliminary lemmas

To prove our main results Theorem 1.4, Theorem 1.7 and Theorem 1.15, we will need some preliminary lemmas and theorems. Among them, Theorem 2.7 asserts that an epimorphism can transfer $Q(X)$ of a minimal semiflow $(T, X)$ onto the regionally proximal relation of its invertible factor.

#### 2.1. Preliminary notions

Let $(T, X)$ be a semiflow with phase semigroup $T$ and with compact $T_2$ phase space $X$. We have introduced some necessary notions in §1. Here we need to introduce another one.

**Definition 2.1.** We shall say that $(T, X)$ admits an invariant quasi-regular Borel probability measure $\mu$, provided that $\mu$ is an invariant Borel probability measure on $X$ such that

- $\mu$ is quasi-regular in the sense that for any Borel set $B$ and $\varepsilon > 0$ one can find an open set $U$ with $B \subseteq U$ and $\mu(U - B) < \varepsilon$, and for any open set $U$ and $\varepsilon > 0$ one can find a compact set $K$ with $K \subset U$ and $\mu(U - K) < \varepsilon$.

Since $X$ is compact $T_2$ here, a quasi-regular Borel probability measure must be regular. Moreover, it is well known that if $T$ is an amenable semigroup or $(T, X)$ is distal, then $(T, X)$ always admits an invariant Borel probability measure.

Recall that a point is called minimal if its orbit closure is a minimal subset. Since $X$ is compact, then by Zorn’s lemma, minimal points always exist. A minimal point is also called an “almost periodic point” or a “uniformly recurrent point” in some works like [13, 17, 3]. If $(T, X)$ is a minimal semiflow, then $\Delta X$ is a minimal set of $(T, X \times X)$.

If the set of minimal points is dense in $X$, then we will say $(T, X)$ has a dense set of minimal points or with dense almost periodic points.

#### 2.2. Preliminary lemmas

Now we will introduce and establish in this subsection some preliminary lemmas needed in our later discussion.

**Lemma 2.2** ([5]). Let $(T, X)$ be a semiflow. Then:

1. If $(T, X)$ is distal, then it is invertible.
2. If $(T, X)$ is surjective equicontinuous, then it is distal.
3. If $(T, X)$ is invertible equicontinuous, then $(T, X)$ is an equicontinuous flow.
Lemma 2.3 ([12, 5]). \( (T, X) \) is an equicontinuous surjective semiflow if and only if \( Q(X) = A_X \) if and only if \( (T, X) \) is an equicontinuous flow.

Lemma 2.4 ([5]). Let \( (T, X) \) be a minimal semiflow. If \( (T, X) \) admits an invariant measure, then \( (T, X) \) is surjective.

Lemma 2.5 (cf. [3, Lemma 7.6] for \( T \) a group). Let \( \pi: (T, X) \to (T, Y) \) be an epimorphism where \( (T, X) \) contains a dense set of minimal points and \( (T, Y) \) is minimal invertible, and let \( \varepsilon \in \mathcal{U}_Y \). Then \( (\pi \times \pi)(T^{-1}\varepsilon) \) belongs to \( \mathcal{U}_Y \).

Proof. Let \( V \in \mathfrak{l}(X) \) such that \( V \times V \subseteq \varepsilon \) and take a minimal point \( x \) of \( X \) with \( x \in V \). Since \( (T, Y) \) is minimal, hence \( \pi(\text{cls}(TX)) = Y \). Thus there is some \( \tau \in T \) such that \( \text{Int}\{\pi(\tau^{-1}V)\} \neq \emptyset \). Let \( \eta = T^{-1}(\text{Int}\{\pi(\tau^{-1}V)\} \times \text{Int}\{\pi(\tau^{-1}V)\}) \). Then \( \eta \) is a non-empty \( T^{-1} \)-invariant open set which meets \( A_Y \). Since \( A_Y \) is minimal for \( (T, Y) \), and \( (T, Y) \setminus \eta \) is \( T \)-invariant, it follows that \( A_Y \subseteq \eta \) and so \( \eta \in \mathcal{U}_Y \). Then by \( (\pi \times \pi)(T^{-1}\varepsilon) \supseteq T^{-1}(\pi(\tau^{-1}V) \times \pi(\tau^{-1}V)) \supseteq \eta \), we can conclude that \( (\pi \times \pi)(T^{-1}\varepsilon) \in \mathcal{U}_Y \). \( \square \)

The following result is useful, which generalizes [15, Corollary to Theorem 8.1] and [29, Proposition 2.3] that are in minimal distal flows with compact metric phase spaces by different approaches.

Corollary 2.6. Let \( \pi: (T, X) \to (T, Y) \) be an epimorphism of semiflows where \( (T, X) \) has a dense set of minimal points and \( (T, Y) \) is minimal invertible. Assume \( \{(y_n, y'_n)\} \) is a net in \( Y \times Y \) such that \( \lim_n (y_n, y'_n) = \Delta_Y \). Then there are nets \( \{(x_i, x'_i)\} \) and \( \{(t_i)\} \) in \( T \) such that \( x_i \to x \) and \( (x_i, x'_i) \in \Delta_X \) and that \( \{(\pi(x_i), \pi(x'_i))\} \) is a subnet of \( \{(y_n, y'_n)\} \).

Proof. Given any \( \varepsilon \in \mathcal{U}_X \), since \( (\pi \times \pi)(T^{-1}\varepsilon) \in \mathcal{U}_Y \) by Lemma 2.5 and \( \lim_n (y_n, y'_n) = \Delta_Y \), then we can take some \( n \in \mathbb{N} \) with \( (y_n, y'_n) \) such that \( (y_n, y'_n) \in (\pi \times \pi)(T^{-1}\varepsilon) \). Let \( t \in T \) and \( x_n, x'_n \in X \) such that \( t(x_n, x'_n) \in \varepsilon \) and \( \pi(x_n) = y_n \) and \( \pi(x'_n) = y'_n \). This completes the proof of Corollary 2.6. \( \square \)

Corollary 2.6 is very useful for our later Theorem 2.7. Moreover, it should be noted that we are not permitted to use \( (\pi \times \pi)(T\varepsilon) \in \mathcal{U}_Y \) instead of \( (\pi \times \pi)(T^{-1}\varepsilon) \in \mathcal{U}_Y \) for \( T \varepsilon \) is not necessarily open in our semiflow context.

2.3. Regional proximity relations of extensions and factors

Now we will be concerned with the relationship of \( Q(X) \) with the same relation of its factors. The point of Theorem 2.7 below is that \( (T, X) \) need not be invertible.

Theorem 2.7. Let \( \pi: (T, X) \to (T, Y) \) be an epimorphism, where \( (T, X) \) is surjective with a dense set of minimal points and \( (T, Y) \) is invertible minimal. Then \( (\pi \times \pi)Q(X) = Q(Y) \).

Proof. Clearly \( (\pi \times \pi)Q(X) \subseteq Q(Y) \). For the other direction inclusion, let \( (y, y') \in Q(Y) \). Then there are nets \( \{(y_n, y'_n)\} \subseteq Y \times Y \) and \( \{(t_n)\} \subseteq T \) such that \( (y_n, y'_n) \to (y, y') \) and \( \lim_n \tau_n(y_n, y'_n) = \Delta_Y \). By Corollary 2.6, choose \( x_n, x'_n \in X \) and \( t_n \in T \) (passing to a subnet if necessary) with \( \lim_n t_n(x_n, x'_n) = \Delta_X \) and \( \pi(x_n) = \tau_n y_n \) and \( \pi(x'_n) = \tau_n y'_n \). Then \( \lim_n \tau_n(x_n, x'_n) = \Delta_X \) and \( \pi(x_n) = y_n \) and \( \pi(x'_n) = y'_n \). By taking a subnet of \( \{(x_n, x'_n)\} \) if necessary, let \( x_n \to x \) and \( x'_n \to x' \). Thus \( (x, x') \in Q(X) \) with \( \pi(x) = y \) and \( \pi(x') = y' \). Thus \( Q(Y) \subseteq (\pi \times \pi)Q(X) \). This proves Theorem 2.7. \( \square \)
Corollary 2.8 (cf. [14, Lemma 12] for \( T \) a group). Let \( \pi: (T, X) \to (T, Y) \) be an epimorphism such that \( (T, X) \) is surjective with dense minimal points and \( (T, Y) \) is minimal invertible. Then \( (T, Y) \) is equicontinuous if and only if \( (\pi \times \pi)Q(X) = \Delta_Y \).

**Proof.** The “only if” follows from Lemma 2.3 and Theorem 2.7. Conversely, by Theorem 2.7, \( Q(Y) = \Delta_Y \) and so the “if” follows from Lemma 2.3.

Now applying Theorem 2.7 with \( \pi: (T, X) \to (T, X_{eq}) \), we can easily obtain the following result, which is very useful in surjective semiflows and which partially generalizes [13, Proposition 4.20].

**Corollary 2.9.** Let \( (T, X) \) be a surjective minimal semiflow. Then \( S_{eq}(X) \) is the smallest closed invariant equivalence relation containing \( Q(X) \).

**Proof.** By Lemma 2.2, \( (T, X_{eq}) \) is minimal invertible, where \( X_{eq} = X/S_{eq} \). Then by Lemma 2.2, \( \Delta_{X_{eq}} \subseteq (\pi \times \pi)Q(X) \subseteq Q(X_{eq}) = P(X_{eq}) = \Delta_{X_{eq}} \). Thus \( Q(X) \subseteq S_{eq} \).

Let \( R \) be a closed invariant equivalence relation on \( X \) with \( Q(X) \subseteq R \). \( \pi: (T, X) \to (T, X/R) \) the canonical epimorphism and \( \pi_*: E(X) \to E(X/R) \) the standard semigroup homomorphism between the Ellis enveloping semigroups induced by \( \pi \). If \( u^2 = u \in E(X/R) \) then \( \pi^{-1}(u) \) is a closed subsemigroup of \( E(X) \). Hence there exists \( v^2 = v \in E(X) \) with \( \pi_*(v) = u \). Since \( (x, v(x)) \in P(X) \subseteq Q(X) \subseteq R \), then \( \pi(x) = \pi(v(x)) = u\pi(x) \) for all \( x \in X \). Thus \( u = \text{id}_{X/R} \) is an identity of \( X/R \).

This implies that \( (T, X/R) \) is distal and so invertible (cf. Lemma 2.2). Then by Theorem 2.7 and \( Q(X) \subseteq R \), \( \Delta_{X/R} = (\pi \times \pi)Q(X) = Q(X/R) \). Thus \( (T, X/R) \) is equicontinuous by Lemma 2.3 or 2.2. Whence \( S_{eq} \subseteq R \). This proves Corollary 2.9.

We note here that when \( T \) is a group, the proof of Corollary 2.9 may be simplified much as follows:

**Another proof of Corollary 2.9 for \( T \) a group.** Since \( \pi: X \to X_{eq} \) is an epimorphism of minimal flows, by \( (\pi \times \pi)Q(X) \subseteq Q(X_{eq}) = \Delta \) it follows that \( Q(X) \subseteq S_{eq} \). On the other hand, if \( R \) is an invariant closed equivalence relation on \( X \) with \( Q(X) \subseteq R \), then by Theorem 2.7 we can conclude that \( Q(X/R) = \Delta \) so \( (T, X/R) \) is equicontinuous. Thus \( S_{eq} \subseteq R \). The proof is complete.

Recall that Lemma 2.3 is proved in [12, pp. 46-47] by using Lemma 2.2, a.a. points and Veech’s relation \( V(X) \) in (1) of Definition 1.13. In fact, we can simply reprove it by only using Lemma 2.2 and Corollary 2.9.

**Another proof of Lemma 2.3.** Let \( (T, X) \) be equicontinuous surjective; then \( \Delta_X = P(X) = Q(X) \) by Lemma 2.2. Conversely, assume \( Q(X) = \Delta_X \); then \( (T, X) \) is invertible and pointwise minimal.

By Corollary 2.9, it follows that \( S_{eq} = \Delta_X \) so \( (T, X) \) is equicontinuous surjective.

**Theorem 2.10** (cf. [17, Theorem 9.12] for \( T = \mathbb{Z} \)). Let \( (T, X) \) be a minimal weak-mixing flow with \( X \) a non-trivial compact metric space. Then no point of \( X \) is distal.

**Proof.** Since \( (T, X) \) is weak-mixing, \( Q(X) = X \times X \) and so \( S_{eq} = X \times X \) by Corollary 2.9.

This shows that \( (T, X) \) has no non-trivial equicontinuous factor. However, if \( (T, X) \) had a distal point it were point-distal. So by Veech’s [29, Theorem 6.1], \( (T, X) \) would have a non-trivial equicontinuous factor. Thus \( (T, X) \) has no distal point. The proof is complete.

In addition, if \( X \) is metrizable and \( (T, X) \) is invertible or \( T \) is a group, Corollary 1.11 can be proved by using Corollary 2.9 as follows:
Another proof of Corollary 1.11 when \((T, X)\) invertible and \(X\) metrizable. First \(S_{eq} = X \times X\) by Corollary 2.9 and so \((T, X)\) has no non-trivial equicontinuous factor. However, if \((T, X)\) has a distal point with \(T\) amenable, then by \([5, \text{Corollary 4.6}]\) it follows that \((T, X)\) has a non-trivial equicontinuous factor. Thus \((T, X)\) has no distal point and the proof is complete. □

2.4. Quasi-regular invariant measure

Since \(X\) is not necessarily metrizable, hence a Borel probability measure does not need be regular. However, next we will show that \((T, X)\) admits an invariant Borel probability measure if and only if it admits an invariant quasi-regular (regular) Borel probability measure. Let \(C_c(X)\) be the space of continuous real-valued functions with compact support.

Riesz-Markov theorem. Let \(X\) be a locally compact \(T_2\) space and \(I\) a positive linear functional on \(C_c(X)\). Then there is a Borel measure \(\mu\) on \(X\) such that

\[
I(f) = \int_X f \, d\mu \quad \forall f \in C_c(X).
\]

The measure \(\mu\) may be taken to be quasi-regular. In this case, it is then unique.

Lemma 2.11. \((T, X)\) admits an invariant measure if and only if it admits an invariant quasi-regular Borel probability measure.

Proof. The “if” part is trivial. So we now assume \((T, X)\) admits an invariant Borel probability measure \(v\). Then by the Riesz-Markov theorem, we can obtain a positive linear functional

\[
I(f) = \int_X f \, dv \quad \forall f \in C(X).
\]

Since \(v\) is \(T\)-invariant, \(I\) is also \(T\)-invariant in the sense that \(I(f) = I(ft)\) for all \(f \in C(X)\) and \(t \in T\). Further by the Riesz-Markov theorem again, we can find a unique quasi-regular Borel probability measure \(\mu\) such that

\[
I(f) = \int_X f \, d\mu \quad \forall f \in C(X).
\]

Since \(I\) is \(T\)-invariant, so \(\mu\) is also invariant. Therefore the “only if” part holds. □

In view of Lemma 2.11, we will identify an invariant Borel probability measure with an invariant quasi-regular Borel probability measure in our later arguments if no confusion arises.

2.5. Furstenberg’s structure theorem

Let \(T\) be any discrete semigroup with identity \(e\) and let \(\theta \geq 1\) be some ordinal. Following \([15]\), a projective system of minimal semiflows with phase semigroup \(T\) is a collection of minimal semiflows \((T, X_\lambda)\) on compact \(T_2\) spaces \(X_\lambda\) indexed by ordinal numbers \(\lambda \leq \theta\), and a family of epimorphisms, \(\pi^0_\lambda: (T, X_\lambda) \to (T, X_\theta)\), for \(0 \leq \nu < \lambda \leq \theta\), satisfying:

(1) If \(0 \leq \nu < \lambda < \eta \leq \theta\), then \(\pi^\eta_\lambda = \pi^\nu_\lambda \circ \pi^\eta_\nu\).

(2) If \(\mu \leq \theta\) is a limit ordinal, then \(X_\mu\) is the minimal subset of the Cartesian product semiflows \((T, X_{k<\mu})\) consisting of all \(x = (x_\lambda)_{k<\mu} \in \times_{k<\mu} X_\lambda\) with \(x_\nu = \pi^\nu_\lambda(x_\lambda)\) for all \(\nu < \lambda < \mu\) and then for \(\lambda < \mu\), \(\pi^\nu_\lambda: X_\mu \to X_\lambda\) is just the projection map. In this case, we say that \((T, X_\mu)\) is the projective limit of the family of minimal semiflows \(\{(T, X_\lambda)\}_{\lambda < \mu}\).
Let \( \pi: (T, X) \rightarrow (T, Y) \) be an epimorphism of two semiflows. Then \( \pi \) is called relatively equicontinuous if given \( \varepsilon \in \mathcal{V} \) there is a \( \delta \in \mathcal{V} \) such that whenever \((x, x') \in \delta \) with \( \pi(x) = \pi(x') \), then \((ix, ix') \in \varepsilon \) for all \( t \in T \) (cf. [15] and [3, p. 95]). In this case, \( (T, X) \) is also called a relatively equicontinuous extension of \( (T, Y) \).

Now based on these definitions, we are ready to state the Furstenberg structure theorem for minimal distal semiflows as follows:

**Furstenberg’s Structure Theorem** (cf. [5, Theorem 5.14]). Let \( \pi: (T, X) \rightarrow (T, Y) \) be an epimorphism between distal minimal semiflows. Then there is a projective system of minimal semiflows \(|(T, X_\lambda)\}_{\lambda \leq \theta}|, for some ordinal \( \theta \geq 1 \), such that \( X_\theta = X \), \( X_0 = Y \) such that if \( 0 \leq \lambda < \theta \), then \( (T, X_\lambda) \xrightarrow{\pi_*} (T, X_\theta) \) is a relatively equicontinuous extension.

In fact, we will only need the special case that \( (T, Y) \) is the semiflow with \( Y \) a singleton space. See Lemma 4.1 below.

3. **McMahon pseudo-metric and \( S_{eq} = Q \)**

This section will be mainly devoted to proving Theorem 1.4 and considering another proximity relation of Veech. The McMahon pseudo-metric \( D_t \) and the induced equivalence relation \( K_t \) based on an invariant closed subset \( J \) of \( X \times X \) are useful techniques for our aim here.

3.1. **McMahon pseudo-metric**

In this subsection, let \( (T, X) \) be a minimal semiflow; and suppose \( (T, Y) \) is any semiflow, which admits an invariant (quasi-regular Borel probability) measure \( \mu \). If \( J \) is a closed invariant subset of \( X \times Y \) and \( x \in X \), then the section \( J_x \) is defined by

\[
J_x = \{ y \in Y \mid (x, y) \in J \}.
\]

Such \( J \) is called a *joining* of \( (T, X) \) and \( (T, Y) \) in some works if we additionally require that \( \pi_Y(J) = Y \) where \( \pi_Y : (x, y) \mapsto y \).

**Lemma 3.1** (cf. [3, Lemma 9.4] for \( T \) a group). Let \( J \) be a closed invariant subset of \( X \times Y \). If \( x, x' \in X \), then \( \mu(J_x) = \mu(J_{x'}) \).

**Proof.** Let \( \varepsilon > 0 \) and let \( V \) be an open set in \( Y \) with \( J_x \subset V \) and \( \mu(V) < \mu(J_x) + \varepsilon \). Let \( (t_n) \) be a net in \( T \) for which \( t_n x \to x \). It is easy to see that \( J_{t_n x} \subset V \) for \( n \geq n_0 \), for some \( n_0 \). Hence \( \mu(J_{x'}) \leq \mu(J_{t_n x'}) \leq \mu(V) \leq \mu(J_x) + \varepsilon \). Letting \( \varepsilon \to 0 \), we have \( \mu(J_{x'}) \leq \mu(J_x) \). By symmetry, \( \mu(J_x) \leq \mu(J_{x'}) \) and the lemma is thus proved. \( \Box \)

**Definition 3.2.** If \( J \) is a closed invariant subset of \( X \times Y \), we define the *McMahon pseudo-metric* \( D_J \) on \( X \) by

\[
D_J(x, x') = \mu(J_x \triangle J_{x'}) \quad \forall x, x' \in X.
\]

It is easy to check that \( D_J \) is a pseudo-metric on \( X \). Moreover, \( D_J(x, x') = 0 \) if \( J_x \subset J_{x'} \) or if \( J_x \supseteq J_{x'} \) by Lemma 3.1.

**Lemma 3.3** (cf. [3, Lemma 9.5] for \( T \) a group). The pseudo-metric \( D_J \) on \( X \) is continuous and satisfies \( D_J(x, x') = D_J(tx, tx') \) for all \( t \in T \) and any \( x, x' \in X \).
Proof. Let $x, x' \in X$ and $t \in T$. First note that $tJ_x \subseteq J_{tx}$ and $\mu(tJ_x) = \mu(J_{tx})$ by Lemma 3.1. So $tJ_x = J_{tx} \pmod{0}$. Similarly, $tJ_{x'} \subseteq J_{tx'}$ and $tJ_{x'} = J_{tx'} \pmod{0}$. Thus

$$\gamma^{-1}(tJ_x) = J_x \pmod{0} \quad \text{and} \quad \gamma^{-1}(tJ_{x'}) = J_{x'} \pmod{0}.$$  

This implies that $J_x \cap J_{x'} \supseteq \gamma^{-1}(tJ_x \cap tJ_{x'}) = \gamma^{-1}(J_{tx} \cap J_{tx'}) \supseteq J_x \cap J_{x'} \pmod{0}$ and thus it holds that $D_t(x, x') = D_t(tx, tx')$.

To show that $D_J$ is continuous, we first note that if $U$ is an open set in $Y$, with $J_x \subseteq U$, and $\mu(U - J_x) < \epsilon$, then $\mu(U - J_{x'}) < \epsilon$, for $x'$ sufficiently close to $x$. Now, let $\{x_n\}$ be a net in $X$ with $x_n \to x$. Let $\epsilon > 0$, $U$ open in $Y$ with $J_x \subseteq U$ and $\mu(U - J_x) < \epsilon$. Then if $n \geq n_0$, $J_{x_n} \subseteq U$ and then $\mu(J_x - J_{x_n}) < \epsilon$, so $D_J(x, x_n) < 2\epsilon$. Thus, if $x_n \to x$, then $D_J(x, x_n) \to 0$. It follows immediately that $D_J$ is continuous. In fact, let $(x_n, x'_n) \to (x, x')$, then

$$\lim_n |D_J(x, x') - D_J(x_n, x'_n)| \leq \lim_n |D_J(x, x_n) + D_J(x_n, x'_n) + D_J(x'_n, x') - D_J(x_n, x'_n)| = 0.$$  

The proof of Lemma 3.3 is therefore completed. \qed

Standing notation 3.4. Let $K_J$ be the equivalence relation on $X$ defined by the pseudo-metric $D_J: (x, x') \in K_J$ if and only if $D_J(x, x') = 0$.

Lemma 3.5 (cf. [3, Lemma 9.6] for $T$ a group). $K_J$ is a closed $T$-invariant equivalence relation which contains $Q(X)$.

Proof. Lemma 3.3 implies that $K_J$ is a closed invariant equivalence relation. Now if $(x, x')$ belongs to $Q(X)$, let $x_n \to x, x'_n \to x'$ and $(x_n, x'_n) \to (z, z)$ for some $z \in X$, as in Definition 1.3. Then $D_J(x_n, x'_n) \to D_J(x, x')$ and $D_J(x, x') = D_J(t_n x_n, t_n x'_n) \to D_J(z, z) = 0$. So $D_J(x, x') = 0$ and $(x, x') \in K_J$. \qed

Theorem 3.6. If $J$ is a closed invariant subset of $(T, X \times Y)$ and if $(T, X)$ is surjective. Then $S_{eq}(X) \subseteq K_J$.

Proof. First, we claim that $(T, X/K_J)$ is invertible. Indeed, let $x, x' \in X$ with $(x, x') \notin K_J$. If $(tx, tx') \in K_J$ for some $t \in T$, then $J_{tx} = J_{tx'} \pmod{0}$. Thus $tJ_x = J_{tx} = J_{tx'} = tJ_{x'} \pmod{0}$ and so

$$J_x = \gamma^{-1}[J_x] = \gamma^{-1}[tJ_{x'}] = J_{x'} \pmod{0}$$

and then $(x, x') \in K_J$. This contradiction shows that $(tx, tx') \notin K_J$. Thus each $t \in T$ is invertible for $(T, X/K_J)$.

Let $\pi: X \to X/K_J$ be the canonical projection. Since $Q(X) \subseteq K_J$ by Lemma 3.5, then from Theorem 2.7 and Lemma 2.3 we can see $(T, X/K_J)$ is equicontinuous and so $S_{eq} \subseteq K_J$. \qed

It should be noticed that in Theorem 3.6, we do not require that $\mu$ is such that $\mu(V) > 0$ for all non-empty open subset $V$ of $Y$. For example, if $\mu$ is exactly concentrated on a fixed point, then $K_J = X \times X$ and so $(T, X/K_J)$ is trivial.

3.2. $S_{eq}(X) = Q(X)$

Now we will specialize to the case $(T, Y) = (T, X)$, which admits an invariant measure $\mu$. There is no loss of generality in assuming $\mu$ is quasi-regular by Lemma 2.11.

Theorem 1.4. Let $(T, X)$ be a minimal semiflow, which admits an invariant quasi-regular measure $\mu$. Then $S_{eq}(X) = Q(X)$.
Proof. First by Lemma 2.4, it follows that \((T, X)\) and \((T, X_{eq})\) both are surjective. Since \((T, X_{eq})\) is equicontinuous surjective, then \((T, X_{eq})\) is invertible by Lemma 2.2. Thus \(Q(X) \subseteq S_{eq}\) by Theorem 2.7. To prove Theorem 1.4, it is sufficient to show that \(S_{eq} \subseteq Q(X)\).

Let \((x, y) \in S_{eq}\), and let \(V\) be a neighborhood of \(x\). Consider the closed invariant subset \(J\) of \(X \times X\) defined by

\[
J = \bigcup_{x' \in V} T(y, x') = \bigcup_{x' \in V} (T_y(x')).
\]

Hence

\[
J = \text{cls}_{X \times X} \{ (z, w) \mid \exists x' \in V \text{ and net } \{t_n\} \text{ in } T \text{ s.t. } t_n(y, x') \to (z, w) \}.
\]

Then \(V \subseteq J\), (by taking \(\{t_n\} = \{e\}\)). Now \((x, y) \in S_{eq}\), so \((x, y) \in K\) by Theorem 3.6, and it follows that \(V \subseteq J\) \((\text{mod } 0)\) and so \(V \subseteq J\) for \(\text{supp } (\mu) = X\).

Summarizing, if \((x, y) \in S_{eq}\), and \(V\) is a neighborhood of \(x\), then there is an \(x' \in V\) and a \(\tau \in T\) such that \(\tau x'\) and \(ry\) are in \(V\). Since \(V\) is arbitrary, it follows that \((x, y) \in Q(X)\). Moreover, \((x, x) \in J\) by \(V \subseteq J\).

The proof of Theorem 1.4 is thus completed. \(\square\)

Note that the proof just given shows that in the definition of \(Q(X)\) one can take one of the nets in \(X\) to be constant. Precisely, we have:

**Lemma 3.7.** Under the hypotheses of Theorem 1.4, the following conditions are pairwise equivalent:

1. \((x, y) \in Q(X)\).
2. There are nets \([x_n] \in X\) and \([t_n] \in T\) with \(x_n \to x\), \(t_n x_n \to x\) and \(t_n y \to x\).
3. There are nets \([x_n] \in X\) and \([s_n]\) in \(T\) such that \(x_n \to x\), \(s_n x_n \to y\) and \(s_n y \to y\).

**Note.** \((1) \Leftrightarrow (2)\) in flows is due to McMahon [24].

**Proof.** \((1) \Leftrightarrow (2)\) and \((2) \Rightarrow (3)\) are obvious. For \((3) \Rightarrow (1)\), as in the proof of Theorem 1.4, since \((x, x) \in J\) and \(J\) is closed invariant, so by minimality of \((T, X)\), it follows that \((y, y) \in J\). Thus, we can find nets \([x_n]\) and \([s_n]\) such that \(x_n \to x\), \(s_n x_n \to y\) and \(s_n y \to y\). This proves Lemma 3.7. \(\square\)

The \((1) \Leftrightarrow (2)\) of Lemma 3.7 implies the following (cf. [3, Corollary 9.10] for \(T\) a group).

**Corollary 3.8.** Let \((T, X)\) be a minimal semiflow admitting an invariant measure. Then,

\[
Q[y] = \bigcap_{\alpha \in \mathbb{B}_X} \bigcup_{t \in T} (r^{-1}\alpha)[y],
\]

for all \(y \in X\).

**Proof.** First, if \(x \in \bigcap_{\alpha \in \mathbb{B}_X} \bigcup_{t \in T} (r^{-1}\alpha)[y]\), then for every \(\alpha \in \mathbb{B}_X\), there are \(x_n \in \alpha[x]\) and \(t_n \in T\) with \((t_n y, t_n x_n) \in \alpha\). This shows that \(x \in Q[y]\). Conversely, if \(x \in Q[y]\), then by Lemma 3.7, \(x \in \bigcup_{\alpha \in R} (r^{-1}\alpha)[y]\) for all \(\alpha \in \mathbb{B}_X\). Thus \(x \in \bigcap_{\alpha \in \mathbb{B}_X} \bigcup_{t \in T} (r^{-1}\alpha)[y]\). \(\square\)

The \((1) \Leftrightarrow (3)\) of Lemma 3.7 is very useful for proving Theorem 1.15 in §5. The relation in (3) of Lemma 3.7 was first introduced and studied by Veech in [30]. See §3.3 for the details.
3.3. Another proximity relation

Following Veech [30, p. 806 and p. 819], we introduce the following notation.

**Definition 3.9.** Let \((T, X)\) be a minimal semiflow.

1. We say \((T, X)\) satisfies the Bronstein condition if \((T, X \times X)\) contains a dense set of minimal points.
2. Given \(x \in X\), define \(U[x]\) to be the set of \(z \in X\) for which there exist nets \(t_n \in T\) and \(z_n \in X\) such that \(z_n \to z\), \(t_n z_n \to x\) and \(t_n x \to x\).

It is clear that \(P[x] \subseteq U[x] \subseteq Q[x]\) for all \(x \in X\). In [30] Veech proved the following theorem:

**Theorem** (cf. [30, Theorem 2.7.6]). Let \((T, X)\) be a minimal flow satisfying the Bronstein condition. Then \(S_{eq}[x] = U[x] = Q[x]\) for all \(x \in X\).

Therefore, it holds that

**Corollary** (cf. [30, Theorem 2.7.5]). If \((T, X)\) is a minimal distal flow, then \(S_{eq}[x] = U[x] = Q[x]\) for all \(x \in X\).

With an invariant measure instead of the Bronstein condition, by using Lemma 3.7 and Theorem 1.4 we can easily obtain the following.

**Theorem 3.10.** Let \((T, X)\) be a minimal semiflow, which admits an invariant measure. Then \(U[x] = Q[x] = S_{eq}[x]\) for all \(x \in X\).

Since every distal semiflow always has an invariant measure by Furstenberg’s theorem, we can easily obtain the following corollary.

**Corollary 3.11.** If \((T, X)\) be a minimal distal semiflow, then \(S_{eq}[x] = U[x] = Q[x]\) for all \(x \in X\).

In fact, we have the following

**Lemma 3.12** (cf. [9, Theorem 5.6]). If \((T, X)\) is a minimal bi-continuous semiflow, then we have \(D[x] = U[x]\) for all \(x \in X\).

4. Weak-mixing minimal semiflows

In this section, we will characterize a minimal weak-mixing semiflow by using the McMahon pseudo-metric \(D_J\) introduced in §3.1. Moreover, we will consider the chaotic dynamics of minimal weak-mixing semiflow with amenable phase semigroup.

4.1. Characterizations of minimal weak-mixing semiflows

Let \((T, X)\) be a surjective semiflow. Recall that \(U(X)\) is the collection of non-empty open subsets of \(X\). Then it is clear that the weak-mixing is “highly non-equicontinuous”. In fact, if \((T, X)\) is weak-mixing, then \(Q(X) = X \times X\); and further its factor \((T, X_{eq})\) is trivial by Corollary 2.9.

**Lemma 4.1.** Let \((T, X)\) be a minimal surjective semiflow. Then the following two conditions are equivalent:
(1) \((T, X)\) has no non-trivial distal factor.
(2) \((T, X)\) has no non-trivial equicontinuous factor.

Proof. (1) \(\Rightarrow\) (2) by Lemma 2.2. And (2) \(\Rightarrow\) (1) follows easily from Furstenberg’s structure theorem stated in §2.5.

Therefore, any minimal surjective semiflow is “highly non-equicontinuous” if and only if it is “highly non-distal.”

**Definition 4.2** (cf. [25]). Let \((T, X)\) and \((T, Y)\) be two semiflows with compact \(T_2\) phase spaces and with the same phase semigroup \(T\). We will say \((T, X)\) is weakly disjoint from \((T, Y)\), denoted \(X \perp^w Y\), if \((T, X \times Y)\) is a T.T. semiflow.

It should be noted here that since our phase spaces need not be metrizable, Definition 4.2 is not equivalent to requiring that \((T, X \times Y)\) is point-transitive even for \(T\) in groups as in [3, p. 150].

Let \((f, X)\) be a cascade where \(f\) is a homeomorphism on a compact metric space, and assume \((f, X)\) has no non-trivial equicontinuous factor. Then \((f, X \times Y)\) is T.T. ([22]). Next we will consider an open question of Furstenberg.

Let \(T\) be a semigroup; then the class of minimal semiflow with phase semigroup \(T\) will be denoted by \(SF_{\text{min}}\), and we write

\[
SF_{\text{f}} = \{(T, X) \mid (T, X)\text{ is T.T.}\} \quad \text{and} \quad SF_{\text{ann}} = \{(T, X) \mid (T, X \times X)\text{ is T.T.}\}.
\]

Note that all the phase spaces are compact \(T_2\) for our semiflows. Moreover, \(SF_{\text{min}} \subset SF_{\text{f}}\) for all semigroup \(T\).

Furstenberg’s [16, Proposition II.11] asserts that \(SF_{\text{ann}} \times SF_{\text{min}} \subset SF_{\text{f}}\) if \(T = \mathbb{Z}_+\). In view of this, he further asked the following problem:

*Is it true for \(T = \mathbb{Z}_+\) that \(SF_{\text{ann}} \times SF_{\text{min}} \subset SF_{\text{f}}\)?* (See [16, Problem F].)

Next we will introduce a class of semiflows which are weaker than minimal semiflows with amenable phase semigroups.

**Definition 4.3.** Let \(T\) be a discrete semigroup and let \(X\) vary in the set of compact \(T_2\) spaces.

1. By the **ergodic center** of \((T, X)\), denoted by \(\mathcal{C}_{\text{erg}}(T, X)\), we mean the smallest closed invariant subset of \(X\) of \(\mu\)-measure 1, for all invariant measure \(\mu\) of \((T, X)\). If \((T, X)\) has no invariant measure, then we shall say \(\mathcal{C}_{\text{erg}}(T, X) = \emptyset\).
2. \((T, X)\) is called an **E-semiflow**, denoted \((T, X) \in SF_e\), if \((T, X) \in SF_{\text{f}}\) and \(\mathcal{C}_{\text{erg}}(T, X) = X\).

Equivalently, \((T, X) \in SF_e\) if and only if it is T.T. with full ergodic center.

The following lemma is a simple observation and so we will omit its proof details.

**Lemma 4.4.** Let \(T\) be an amenable semigroup. If \((T, X)\) is T.T. such that the set of minimal points is dense in \(X\), then \((T, X) \in SF_e\).

\(SF_e\) is an extension of the class of \(E\)-systems of Glasner and Weiss [19, 20]. If \((T, X) \in SF_e\) with \(T\) a countable discrete semigroup, then \(N_f(U, V)\) is syndetic in \(T\) for all \(U, V \in \mathcal{L}(X)\).

**Lemma 4.5.** If \((T, X) \in SF_e\) and \(U \in \mathcal{L}(X)\), then there exists an invariant measure \(\mu\) of \((T, X)\) such that \(\mu(U) > 0\). Hence every \((T, X)\) in \(SF_e\) is surjective.
Proof. First we note that \( \mathcal{C}_{erg}(T, X) \) is just equal to the closure of the union of the supports of all invariant measures of \((T, X)\). Then there is some invariant measure \( \mu \) such that \( \text{supp}(\mu) \cap U \neq \emptyset \) so that \( \mu(U) > 0 \). Finally, let \( t \in T \). Since \( X \setminus tX \) is open and \( X \) is Borel, so if \( tX \neq X \) we have \( 1 > \mu(tX) = \mu(t^{-1}tX) = \mu(X) = 1 \) a contradiction. Thus \((T, X)\) is surjective. The proof is complete. \( \square \)

**Theorem 4.6.** Let \((T, X) \in \mathcal{SF}_{\text{min}}\) be surjective. Then \((T, X)\) has no non-trivial distal factor if and only if \( X \perp^w Y \) for all \((T, Y) \in \mathcal{SF}_r\).

**Proof.** The “only if” part: By Lemma 4.1 we may assume \((T, X)\) has no non-trivial equicontinuous factor \( X_{eq} \). Let \((T, Y)\) be TT with \( \mathcal{C}_{erg}(T, Y) = Y \). If \((T, X \times Y) \notin \mathcal{SF}_r\), then there would be an invariant closed subset \( J \) of \( X \times Y \) such that \( \text{Int}_{X \times Y} J \neq \emptyset \) and that \( J \neq X \times Y \). So there could be a point \( x \in X \) and an \( U \in \mathbb{U}(Y) \) such that \( U \subset J \). Moreover, \( J \neq Y \). Let \( V = Y - J \). Then \( V \neq \emptyset \). Since \((T, Y)\) is TT, so there is \( \tau \in T \) such that \( \tau U \cap V \neq \emptyset \).

Since \( U \cap \tau^{-1}V \in \mathbb{U}(Y) \), we can find an invariant quasi-regular Borel probability \( \mu \) on \( Y \) such that \( \mu(U \cap \tau^{-1}V) > 0 \). Since \( \tau U \cap V = (U \cap \tau^{-1}V) \), hence \( \mu(\tau J \cap V) > 0 \) and then \( \mu(J \cap V) > 0 \). Having let \( D_J \) be the McMahon pseudo-metric as in Definition 3.2 associated to \( J \) and \( \mu \), let \( D_J(x, \tau x) > 0 \). Let \( K_J \) be the closed invariant equivalence relation defined by \( D_J \) on \( X \). Then \( K_J \neq X \times X \), and by Theorem 3.6, \((T, X/K_J)\) is a non-trivial equicontinuous factor of \((T, X)\). This contradiction shows that \( J = X \times Y \) and therefore \((T, X \times Y)\) must be TT and so \( X \perp^w Y \).

The “if” part: Let \( X \perp^w Y \) for all \((T, Y) \in \mathcal{SF}_r\). To be contrary, assume \( X_{eq} \) is not a singleton set. Then \((T, X_{eq})\) is minimal distal by Lemma 2.2. So \((T, X_{eq}) \in \mathcal{SF}_r\) and thus \( X \perp^w X_{eq} \). Therefore \( X_{eq} \perp^w X_{eq} \) and \((T, X_{eq})\) is weak-mixing. This is a contradiction to Lemma 2.3. Thus \( X_{eq} \) must be a singleton set.

Therefore the proof of Theorem 4.6 is completed. \( \square \)

It should be noted that \((T, X)\) need not admit an invariant measure in the above Theorem 4.6.

**Lemma 4.7.** Let \((T, X) \in \mathcal{SF}_{\text{min}}\) admits an invariant measure. Then it has no non-trivial equicontinuous factor if and only if it is weak-mixing.

**Proof.** First \((T, X)\) is surjective by Lemma 2.4, and moreover, \((T, X) \in \mathcal{SF}_r\). If \((T, X)\) has no non-trivial equicontinuous factor, then it is weak-mixing from Theorem 4.6. Conversely, if \((T, X)\) is weak-mixing, then \( Q(X) = X \times X \) so that \((T, X)\) has no non-trivial equicontinuous factor by Corollary 2.9. The proof is complete. \( \square \)

The following Corollary 4.8 implies the first “if and only if” of Theorem 1.7.

**Corollary 4.8.** Let \((T, X) \in \mathcal{SF}_{\text{min}}\) admits an invariant measure. Then \((T, X)\) is weak-mixing iff \( X \perp^w Y \) for all \((T, Y) \in \mathcal{SF}_r\).

**Corollary 4.9** (cf. [3, Corollary 11.8] for \( T \) in abelian groups). Let \( T \) be an amenable semigroup and \((T, X) \in \mathcal{SF}_{\text{min}}\). Then \((T, X)\) is weak-mixing iff \( X \perp^w Y \) for all \((T, Y) \in \mathcal{SF}_{\text{min}}\) iff \( X \perp^w Y \) for all \((T, Y) \in \mathcal{SF}_r\).

Comparing with [3, Corollary 11.8] here our phase spaces are not required to be a metric space and further ‘T.T.’ does not imply ‘point-transitive’ in our setting. Moreover, the minimality of \((T, X)\) is important for Corollary 4.9. In fact, following [1, 21] there are non-minimal non-weak-mixing cascade \((T, X)\), which are weakly disjoint every \( E \)-system.
Corollary 4.10. Let \((T, X) \in SF_{\min}\) be surjective. If \((T, X)\) is weak-mixing, then \(X \perp^n Y\) for all \((T, Y) \in SF_v\).

Proof. By Corollary 2.9, \(S_{\nu}(X) \supseteq Q(X)\). Then the statement follows from Theorem 4.6 and the fact that \(Q(X) = X \times X\). □

Now Theorem 1.8 stated in §1.2 easily follows from the following more general theorem. Our proof below is of interest because there exists no Furstenberg’s intersection lemma ([16, Proposition II.3] and [11, Lemma 3.2]) in the literature for non-abelian phase semigroup \(T\).

Theorem 4.11. Let \((T, X)\) be a minimal semiflow, which admits an invariant measure. Then the following conditions are pairwise equivalent:

1. \((T, X)\) is weak-mixing.
2. \((T, X^n)\) is T.T. for all \(n \geq 2\).
3. \(N_T(U, V)\) is discretely thick in \(T\) for all \(U, V \in \mathfrak{U}(X)\).
4. Let \(I\) be any set with \(\text{card } I \geq 2\) and \((T, X_i) = (T, X)\) for all \(i \in I\); then \((T, \times_{i \in I} X_i)\) is weak-mixing.

Proof. (1) ⇒ (2). We will proceed to show that \((T, X^n)\) is T.T. for all \(n \geq 2\) by induction on \(n\). First, by definition of weak-mixing, \((T, X \times X)\) is T.T. so the case of \(n = 2\) holds. Now, assume the statement holds for all integer \(n = k \geq 2\). Let \(Y = X^k\) and then \((T, Y)\) is a T.T. semiflow. Since \((T, X)\) is minimal with an invariant measure, then \((T, X)\) is surjective by Lemma 2.4 and there is an invariant measure \(\nu\) on \(X\) with \(\text{supp}(\nu) = X\). Define a Borel probability measure \(\mu = \nu^k\) on \(Y\) by \(\mu(V_1 \times \cdots \times V_k) = \nu(V_1) \cdots \nu(V_k)\) for all open subsets \(V_1, \ldots, V_k\) of \(X\). Clearly, \(\mu\) is \(T\)-invariant on \(Y\) and moreover every open subset of \(Y\) has positive \(\mu\)-measure. Then by Theorem 4.6, \((T, X \times Y)\) must be T.T. and thus \((T, X^{k+1})\) is a T.T. semiflow.

(2) ⇒ (3). Let \(U, V \in \mathfrak{U}(X)\) and \(K = \{k_1, \ldots, k_n\}\) a finite subset of \(T\). Since by Lemma 2.4 each \(k_i^{-1}V \neq \emptyset\), \(U \times \cdots \times U\) and \(k_1^{-1}V \times \cdots \times k_n^{-1}V\) both are non-empty open subsets of \(X^k\), then there is some \(t \in T\) such that \(U \times \cdots \times U \cap t^{-1}(k_1^{-1}V \times \cdots \times k_n^{-1}V) \neq \emptyset\). Thus \(Kt \subseteq N_T(U, V)\) and so \(N_T(U, V)\) is discretely thick in \(T\).

(3) ⇒ (1). Let \(U, V, U', V' \in \mathfrak{U}(X)\). Then \(N_T(U, V)\) is discretely thick in \(T\); moreover, \(N_T(U', V')\) is discretely syndetic in \(T\) by [11, Lemma 2.3]. So \(N_T(U, V) \cap N_T(U', V') \neq \emptyset\). This shows that \(N_T(U \times U', V \times V') \neq \emptyset\) and thus \((T, X \times X)\) is a T.T. semiflow.

(2) ⇔ (4). (4) ⇒ (2) is obvious; and (2) ⇒ (4) follows from the definition of the product topology of \(\times_{i \in I} X_i\).

The proof of Theorem 4.11 is thus completed. □

The following Theorem 4.12 implies that:

If \((T, X)\) is minimal with \(T\) an amenable semigroup, then \((T, X)\) is weak-mixing if and only if it has no non-trivial distal factor.

This would just answer a question of Petersen in [26, p. 280]. Moreover Theorem 4.12 is a semigroup version of Theorem 2.10 by different approaches.

Theorem 4.12. Let \((T, X)\) be a minimal semiflow, which admits an invariant measure. Then, the following five statements are pairwise equivalent:

1. \((T, X)\) is weak-mixing.

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(2) $Q(X) = X \times X$.
(3) $S_{c_0}(T, X) = X \times X$.
(4) $(T, X)$ has no non-trivial distal factor.
(5) $(T, X)$ has no non-trivial equicontinuous factor.

In particular, each of the above five conditions implies that $(T, X)$ has no distal point.

**Proof.** First of all, $(T, X)$ is surjective by Lemma 2.4. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (5) both are trivial; (2) $\Rightarrow$ (3) is by Corollary 2.9 and (4) $\Leftrightarrow$ (5) follows easily from Lemma 4.1. Finally (5) $\Rightarrow$ (1) follows from Lemma 4.7 at once.

Finally let any of (1) through (5) hold. By Theorem 4.11, $(T, X)$ is thickly transitive, i.e., for all $U, V \in \mathcal{U}(X)$, $N_T(U, V)$ is a thick subset of $T$. Then no point of $X$ is distal for $(T, X)$ by Corollary 1.11. Precisely speaking, this follows from a proof similar to that of Corollary 1.11 with Theorem 4.11 in place of Theorem 1.8.

The proof of Theorem 4.12 is thus completed. $\blacksquare$

Note here that comparing with [3, Theorem 9.13] the only new ingredients of Theorem 4.12 are that our phase space $X$ is not required to be metrizable and that $T$ is not necessarily a group. The non-metrizable condition of $X$ is just the point for Theorem 1.12.

4.2. Chaos of minimal weak-mixing semiflow

Let $(T, X)$ be a semiflow on a uniform non-singleton space $(X, \mathcal{U}_X)$, where $\mathcal{U}_X$ is a compatible symmetric uniform structure on $X$. We first introduce the notion of sensitivity.

**Definition 4.13** (cf. [11]). $(T, X)$ is said to be sensitive to initial conditions if there exists an $\varepsilon \in \mathcal{U}_X$ such that for all $x \in X$ and $\delta \in \mathcal{U}_X$ there are $y \in \delta[x]$ and $t \in T$ with $(tx, ty) \notin \varepsilon$.

Given any $\varepsilon \in \mathcal{U}_X$, let

$$\text{Equiv}_\varepsilon(T, X) = \{x \in X | \exists \delta \in \mathcal{U}_X \text{ s.t. } t(\delta[x]) \subseteq \varepsilon[tx] \forall t \in T\}.$$ 

Then,

$(T, X)$ is not sensitive to initial conditions iff $\text{Equiv}_\varepsilon(T, X) = \emptyset$ for all $\varepsilon \in \mathcal{U}_X$.

It is already known that “weak-mixing is highly non-equicontinuous” (cf. Theorem 4.12). In fact, weak-mixing is even incompatible with “$\varepsilon$-equicontinuity” as follows:

**Lemma 4.14.** If $(T, X)$ is a weak-mixing semiflow on a uniform $T_2$-space $(X, \mathcal{U}_X)$, then $(T, X)$ is sensitive to initial conditions.

**Proof.** By contradiction, suppose $(T, X)$ were not sensitive to initial conditions; then it holds that $\text{Equiv}_\varepsilon(T, X) \neq \emptyset$ for all $\varepsilon \in \mathcal{U}_X$. By the weak-mixing property, $(T, X \times X)$ is a T.T. semiflow. Since $X \times X \neq \Delta_X$, there is an $\varepsilon \in \mathcal{U}_X$ such that $U := X \times X \setminus \text{cls}_{X \times X} \varepsilon \neq \emptyset$. Take an $x_0 \in \text{Equiv}_\varepsilon(T, X)$ where $\varepsilon/3$ is an entourage in $\mathcal{U}_X$ such that $\varepsilon/3 \circ \varepsilon/3 \subseteq \varepsilon$. Then there is a $\delta \in \mathcal{U}_X$ such that $t(\delta[x_0] \times \delta[x_0]) \subseteq \varepsilon$ for all $t \in T$. Thus $N_T(\delta[x_0] \times \delta[x_0], U) = \emptyset$, which is a contradiction to that $(T, X \times X)$ is a T.T. semiflow. The proof of Lemma 4.14 is therefore completed. $\blacksquare$

Let $\mathcal{H}(T)$ be the collection of non-empty compact subsets of $T$. Given $\varepsilon \in \mathcal{U}_X$ and $x \in X$, we define the $\varepsilon$-stable set of $(T, X)$ at $x$ as follows:

$$W_\varepsilon^c(T, X; x) = \{y \in X | \exists K \in \mathcal{H}(T) \text{ s.t. } (tx, ty) \in \varepsilon \forall t \in T \setminus K\}.$$ 

Then there holds the following lemma.
**Lemma 4.15** (cf. [11, Lemma 3.15]). Let \((T, X)\) be a semiflow such that \(T\) is \(\sigma\)-compact. If \((T, X)\) is sensitive to initial conditions, then there is an \(\varepsilon \in \mathcal{U}_X\) such that \(W^0_\varepsilon(T, X; x)\) is of the first category in \(X\) for all \(x \in X\).

The following notion is stronger than the above sensitivity to initial conditions.

**Definition 4.16.** Let \((T, X)\) be a semiflow and \(\varepsilon, \delta \in \mathcal{U}_X\) with \(\varepsilon > \delta\). For \(x \in X\) define a set

\[
LY_{\varepsilon, \delta}[x] = \left\{ y \in X \mid y \notin W^0_\varepsilon(T, X; x) \text{ and } \exists t \in T \text{ s.t. } (tx, ty) \in \delta \right\}.
\]

We say that \((T, X)\) is \(\varepsilon-\delta\) Li-Yorke sensitive if \(LY_{\varepsilon, \delta}[x]\) is of the second category for all \(x \in X\).

**Theorem 4.17.** Let \((T, X)\) be a minimal semiflow with \(T\) \(\sigma\)-compact, which admits of an invariant measure (e.g. \(T\) is amenable). If \((T, X)\) is weak-mixing, then there exists an \(\varepsilon \in \mathcal{U}_X\) such that for all \(\delta \in \mathcal{U}_X\) with \(\delta < \varepsilon\), \((T, X)\) is \(\varepsilon-\delta\) Li-Yorke sensitive.

**Proof.** By Lemma 4.14, \((T, X)\) is sensitive to initial conditions. So by Lemma 4.15, it follows that there exists an \(\varepsilon \in \mathcal{U}_X\) such that \(W^0_\varepsilon(T, X; x)\) for all \(x \in X\) is of the first category.

Let \(\delta \in \mathcal{U}_X\) with \(\delta < \varepsilon\) be any given. Given any \(x \in X\), let \(P[x]\) be the proximal cell at \(x\); i.e., \(y \in P[x]\) if \(\exists z \in X\) and \(\{t_n\} \subseteq T \text{ s.t. } (t_nx, t_ny) \to (z, z)\). Then by a slight modification of the proof of Corollary 1.11 in §1.2, we can conclude that \(P[x]\) is dense in \(X\). In addition,

\[
P[x] = \bigcap_{\alpha \in \mathcal{U}_T} \bigcup_{t \in \mathbb{R}} r^{-1}(\alpha[tx]).
\]

Then \(\bigcup_{t \in \mathbb{R}} r^{-1}(\alpha[tx])\) is an open dense subset of \(X\) and \(\bigcup_{t \in T} r^{-1}(\delta[tx]) \subseteq LY_{\varepsilon, \delta}[x]\).

Thus, \(LY_{\varepsilon, \delta}[x]\) is of the second category. This thus completes the proof of Theorem 4.17.

**Definition 4.18.** Let \((T, X)\) be a semiflow and \(\varepsilon \in \mathcal{U}_X\). For \(x \in X\) define a set

\[
LY_{\varepsilon, 0}[x] = \left\{ y \in X \mid y \notin W^0_\varepsilon(T, X; x) \text{ and } \exists \{t_n\} \in T \text{ s.t. } \lim_{n \to \infty} d(t_nx, t_ny) \in A_X \right\}.
\]

We say that \((T, X)\) is \(\varepsilon-0\) Li-Yorke sensitive if \(LY_{\varepsilon, 0}[x]\) is of the second category for all \(x \in X\).

**Theorem 4.19** (cf. [2] for \(T = \mathbb{Z}_+\) and [11] for \(T\) in abelian semigroups). Let \((T, X)\) be a minimal semiflow with \(T\) \(\sigma\)-compact and \(X\) a compact metric space, which admits of an invariant measure (e.g. \(T\) is amenable). If \((T, X)\) is weak-mixing, then there exists an \(\varepsilon \in \mathcal{U}_X\) such that \((T, X)\) is \(\varepsilon-0\) Li-Yorke sensitive.

**Proof.** This follows from the proof of Theorem 4.17 by noting that \(P[x]\) is a dense \(G^\circ\)-set in \(X\) when \(X\) is a compact metric space.

**Definition 4.20.** Let \(T\) be \(\sigma\)-compact with a sequence of compact subsets \(F_1 \subset F_2 \subset F_3 \subset \cdots\) such that \(T = \bigcup F_n\) and \(\varepsilon > 0\). \((T, X)\) with \(X\) a metric space is called densely Li-Yorke \(\varepsilon\)-chaotic if there is a dense Cantor set \(\Theta \subseteq X\) such that for all \(x, y \in \Theta\) with \(x \neq y\), there are \(\{t_n\}\) and \(\{s_n\}\) in \(T\) with the properties:

1. \(\lim_{n \to \infty} d(t_nx, t_ny) = 0;\)
2. \(s_n \notin F_n\) and \(\lim_{n \to \infty} d(s_nx, s_ny) \geq \varepsilon.\)
Now we can easily obtain a Li-Yorke chaos result from Theorem 4.19.

**Corollary 4.21** (cf. [11, Proposition 3.18] for $T$ in abelian semigroups). Let $(T, X)$ be a minimal semiflow with $T$ $\sigma$-compact and $X$ a compact metric space, which admits of an invariant measure (e.g., $T$ is amenable). If $(T, X)$ is weak-mixing, then $(T, X)$ is densely Li-Yorke $\epsilon$-chaotic for some $\epsilon > 0$.

5. **Veech’s structure theorem of a.a. semiflows**

Let $V$ and $D$ be Veech’s relations defined as in Definition 1.13 in §1.3. This section will be devoted to proving Theorem 1.15.

For that, we need to introduce two important lemmas. The first one is borrowed from [9].

**Lemma 5.1** (cf. [9, Theorem 5.11]; also [28, Theorem 1.2] for $T$ in abelian groups). Let $(T, X)$ be an invertible minimal semiflow. Then $V[x]$ is a dense subset of $D[x]$ for each $x \in X$. If $X$ is metrizable, then $V[x] = D[x]$ for all $x \in X$.

**Lemma 5.2.** Let $(T, X)$ be a bi-continuous semiflow and let $x_0 \in P_{ad}(T, X)$. Then $V[x_0]$ is a dense subset of $D[x_0]$. If $X$ is metrizable, then $V[x_0] = D[x_0]$.

**Proof.** Let $\Sigma$ be the set of all continuous pseudo-metrics on the compact $T_2$ space $X$. Let $\rho \in \Sigma$ be arbitrarily fixed. First, for every $\delta > 0$ and all $x, y \in X$, we shall say $(x, y) \in \delta$ if $\rho(x, y) < \delta$.

For closed subsets $A, B$ of $X$, we say $\rho_H(A, B) < \delta$ if and only if $\rho(a, b) < \delta$ for all $a \in A$ and $b \in B$. We will say $\rho(a, x) < \delta$ if and only if $\rho(a, x) < \delta$ for some $a \in A$.

For all $\rho \in \Sigma$ we can select a sequence $(\rho_n)$ such that $\rho_n \rightarrow \rho$. We will say $(\rho_n)$ is syndetic in $\rho$ if $\rho_n \rightarrow \rho$.

Since $x_0$ is a distal point, we can choose $\delta > 0$, $x_0 \in D[x_0]$ and $\rho_H(\delta) < \delta$. Moreover, if $F$ is any finite subset of $(T^{-1} \circ T)$ it also can be arranged that $\rho_H(\delta, x_0, x_0) < \delta$.

By the choice of $\delta$, we can select a sequence $(\sigma_1, \tau_1), (\sigma_2, \tau_2), \ldots$ inductively as follows. First we can choose $\sigma_1, \tau_1 \in N_T(x_0, \delta)$. Let $F_0 = \{e\}$ with $\rho(\delta, x_0, x_0) = \delta$. Having chosen $(\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)$ let $F_n$ be the finite set of elements of $(T^{-1} \circ T)$ which are representable as

$$t = \tau_n^{-1} \sigma_n^{-1} \cdots \sigma_1^{-1} x_0$$

where $e_1 = 0$ or $-1$ and $e_i = 0$ or $1$.

Then choose $\sigma_{n+1}, \tau_{n+1} \in N_T(x_0, \delta_{n+1})$ such that

(a) $\rho_H(\gamma \sigma_1 x_0, x_0) < \delta_{n+1}$ for all $\gamma \in \sigma_{n+1}, \tau_{n+1}$; and

(b) $\rho(\delta_{n+1}, \sigma_{n+1} x_0, x_0) < \delta_{n+1}$.

Based on the sequence $((\sigma_n, \tau_n))$, we define $\alpha_1 = \tau_1, \alpha_2 = \sigma_1 \tau_2$ in $T$, and in general,

$$\alpha_n = \sigma_1 \cdots \sigma_{n-1} \tau_n \in T \quad (n = 2, 3, \ldots).$$

If $m < n$ we have from (a) that

$$\rho(\alpha_m x_0, \alpha_n x_0) \leq \sum_{j=0}^{n-m-1} \rho(\alpha_{m+j} x_0, \alpha_{m+j+1} x_0) \leq \sum_{j=0}^{n-m-1} (2 \delta_{m+j} + \delta_{m+j+1})$$

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tends to 0 as \( m \to \infty \). Thus \( \rho\lim_{n} \alpha_{n}x_{0} \) exists because \( \sum_{m=1}^{\infty} \delta_{m} < \infty \). Letting \( y \) be the \( \rho \)-limit, we now claim \( x' \in \rho\lim_{m\to\infty} \alpha_{m}^{-1}y \) (that is, \( \exists x'_{m} \in \alpha_{m}^{-1}y \) s.t. \( \rho\lim x'_{m} = x' \)). To see this we note that if \( n > m \), then
\[
\alpha_{m}^{-1}\alpha_{n}x_{0} = \tau_{m}^{-1}\alpha_{m}^{-1}\cdots\alpha_{n-1}^{-1}\tau_{n}x_{0} = \tau_{m}^{-1}\alpha_{m}^{-1}\cdots\alpha_{n-1}^{-1}\tau_{n}x_{0}
\]
because \( \tau_{n}x = x \forall t \in T \) for all \( x \in P_{\text{inv}}(T, X) \) and \( \rho\lim_{m} 1.13 \) is invariant; and therefore by argument as above, for \( n > m \),
\[
\rho_{H} \left( \alpha_{m}^{-1}\alpha_{n}x_{0}, \tau_{m}^{-1}\alpha_{m}x_{0} \right) \leq \sum_{k=m+1}^{n} \delta_{k} \to 0 \quad \text{as } m \to \infty.
\]
Thus by (b),
\[
\lim_{m \to \infty} \rho \left( \alpha_{m}^{-1}y, x' \right) = \lim_{m \to \infty} \rho \left( \alpha_{m}^{-1}x, x' \right) = 0
\]
and then we can choose \( x'_{m} \in \alpha_{m}^{-1}y \) such that \( x'_{m} \to x' \). By choosing a subnet \( \beta_{i} \) from the sequence \( \{x_{i}\} \) in \( T \), there are points \( z, x'' \in X \) such that \( \beta_{i}x_{0} \to z \) and \( x'' \in V[x_{0}] \) s.t., \( \lim \rho_{H} \beta_{i}x_{0} \) in \( X \); and moreover, \( \rho(x'', x') = 0 \). Of course, if \( \rho \) is just a metric on \( X \), then \( x'' = x' \).

This shows that \( V[x_{0}] \) is dense in \( D[x_{0}] \) and \( V[x_{0}] = D[x_{0}] \) when \( X \) is metrizable. The proof of Lemma 5.3 is thus complete.

The following result is exactly (1) of Theorem 1.15, which generalizes Veech’s [28, Theorem 1.1] from abelian groups to invertible semiflows admitting invariant measures by using completely different approaches. In fact, our proof is much more simpler than Veech’s one presented in [28] that depends on harmonic analysis on locally compact abelian groups.

**Theorem 5.3** (cf. [7, (iii) of Theorem 16] for \( T \) in groups). *Let \( (T, X) \) be an invertible minimal semiflow admitting an invariant measure. Then \( Q[x] = D[x] = V[x] \) for all \( x \in X \).*

**Proof.** Given \( x \in X \), \( D[x] \subseteq Q[x] \) is obvious. In fact, \( V[x] \subseteq Q[x] \) and \( Q[x] \) is closed by Definitions 1.3 and 1.13. So \( D[x] = V[x] \subseteq Q[x] \) by Lemma 5.1.

To show \( Q[x] \subseteq D[x] \) for all \( x \in X \), we fix an \( x \in X \) and let \( y \in Q[x] \). Then by (3) of Lemma 3.7, there are nets \( \{y_{n}\}, \{z_{n}\} \) in \( X \) and \( \{t_{n}\} \) in \( T \) such that
\[
y_{n} \to y, \quad z_{n} \to x, \quad z_{0} = t_{0}y_{n}, \quad t_{n}x \to x.
\]
Then \( Q[x] \subseteq D[x] \) by Lemma 3.12. Indeed, let \( \varepsilon \in \mathbb{B}_{T} \) be arbitrary. It is sufficient to show that \( y \in D(x, \varepsilon) = N_{T}(x, \varepsilon[x])^{-1}N_{T}(x, \varepsilon[x]) \). There is no loss of generality in assuming \( z_{0} \in \varepsilon[x] \) and \( t_{n} \in N_{T}(z_{0}, \varepsilon[x]) \) for all \( n \). Since \( (T, X) \) is minimal, we can take a \( t \in N_{T}(x, \varepsilon[x]) \) such that \( tx \) is close sufficiently to \( z_{0} \) so that \( t^{-1}tx \) is close sufficiently to \( y_{n} \). This means that \( y \in D(x, \varepsilon) \).

Therefore, we have concluded that \( D[x] = Q[x] \) for all \( x \in X \). This proves Theorem 5.3.

Using Lemma 5.2 in place of Lemma 5.1, we can obtain the following, which is important for (2) of Theorem 1.15 stated in §1.3.

**Lemma 5.4.** *Let \( (T, X) \) be a minimal semiflow admitting an invariant measure such that \( x \mapsto \tau_{x}x \) is continuous for \( t \in T \). If \( x_{0} \in P_{\text{inv}}(T, X) \), then \( \{x_{0}\} \subseteq Q[x_{0}] \subseteq D[x_{0}] = Q[x_{0}] \).*

**Proof.** At first, by [5, Lemma 3.5] we see \( (T, X) \) is surjective. By Lemma 5.2, it follows that \( \{x_{0}\} = V[x_{0}] = D[x_{0}] \subseteq Q[x_{0}] \). Moreover, \( Q[x_{0}] \subseteq D[x_{0}] \) by Lemma 3.12. Thus \( \{x_{0}\} = V[x_{0}] = D[x_{0}] = Q[x_{0}] \) and the proof is complete.
Based on Theorem 1.4 and Lemma 5.4 we can obtain the next result which is just the second part of Theorem 1.15 stated in §1.3.

**Theorem 5.5.** Let \((T, X)\) be a minimal bi-continuous semiflow admitting an invariant measure. Then \((T, X)\) is a.a. if and only if \(\pi: (T, X) \rightarrow (T, X_{eq})\) is of almost 1-1 type.

**Proof.** Since \((T, X)\) is a minimal semiflow admitting an invariant measure, by Theorem 1.4 we have \(S_{eq}(X) = Q(X)\).

1. Let \((T, X)\) be a.a.; then there exists an \(x_0 \in X\) such that \(V[x_0] = \{x_0\}\). Then \(Q[x] = \{x_0\}\) by Lemma 5.4. Thus, \(\pi: (T, X) \rightarrow (T, X_{eq})\) is of almost 1-1 type at \(x_0\). This has concluded the “only if” part.

2. Assume \(\pi: (T, X) \rightarrow (T, X_{eq})\) is of almost 1-1 type. Then we can take an \(x \in X\) such that \(\pi^{-1}(x) = \{x\}\). Set \(y = \pi(x)\). Since \((T, X_{eq})\) is minimal equicontinuous invertible by Lemmas 2.4 and 2.2, hence \(V[y] = \{y\}\). This implies that \(V[x] = \{x\}\) and so \((T, X)\) is an a.a. semiflow. This proves the “if” part. The proof is complete.

**Corollary 1.17.** Let \((T, X)\) be a minimal bi-continuous semiflow, which admits an invariant measure. Then \((T, X)\) is equicontinuous iff all points are a.a. points.

**Proof.** (1) The “only if” part. By Lemma 2.3, \(P(X) = Q(X) = \Delta_X\). Thus \((T, X)\) is pointwise a.a. by Theorem 1.15.

(2) The “if” part. By Theorem 1.15, \(Q(X) = \Delta_X\). Then \((T, X)\) is equicontinuous by Lemma 2.3. This proves Corollary 1.17.

**Acknowledgments**

The author would like to thank Professor Joe Auslander and Professor Eli Glasner for their many helpful comments on the first version of this paper.

This work was partly supported by National Natural Science Foundation of China (Grant Nos. 11431012, 11271183) and PAPD of Jiangsu Higher Education Institutions.

**References**

[1] E. Akin and E. Glasner, *Residual properties and almost equicontinuity*. J. Anal. Math. 84 (2001), 243–286.

[2] E. Akin and S. Kolyada, *Li-Yorke sensitivity*. Nonlinearity 16 (2003), 1421–1433.

[3] J. Auslander, *Minimal Flows and their Extensions*. North-Holland Math. Studies, vol. 153. North-Holland, Amsterdam, 1988.

[4] J. Auslander, *Minimal flows with a closed proximal cell*. Ergod. Th. & Dynam. Sys. 21 (2001), 641–645.

[5] J. Auslander and X. Dai, *Minimality, distality and equicontinuity for semigroup actions on compact Hausdorff spaces*. Discret. Contin. Dyn. Syst. (in press).

[6] J. Auslander, D. Ellis and R. Ellis, *The regionally proximal relation*. Trans. Amer. Math. Soc. 347 (1995), 2139–2146.

[7] J. Auslander, G. Greschonig and A. Nagar, *Reflections on equicontinuity*. Proc. Amer. Math. Soc. 142 (2014), 3129–3137.

[8] J. Auslander and N. Markley, *Locally almost periodic minimal flows*. J. Difference Equ. Appl. 15 (2009), 97–109.

[9] X. Dai, *Almost automorphism of surjective semiflows with compact Hausdorff phase spaces*. J. Math. Anal. Appl. (in press).

[10] X. Dai and H. Liang, *On Galvin’s theorem for compact Hausdorff right-topological semigroups with dense topological centers*. Sci. China Math. 60 (2017), 2421–2428.

[11] X. Dai and X. Tang, *Devaney chaos, Li-Yorke chaos, and multi-dimensional Li-Yorke chaos for topological dynamics*. J. Differential Equations 263 (2017), 5521–5553.
[12] X. Dai and Z. Xiao, Equicontinuity, uniform almost periodicity, and regionally proximal relation for topological semiflows. Topology Appl. 231 (2017), 35–49.

[13] R. Ellis, Lectures on topological dynamics. Benjamin, New York, 1969.

[14] R. Ellis and W. H. Gottschalk, Homomorphisms of transformation groups. Trans. Amer. Math. Soc. 94 (1960), 258–271.

[15] H. Furstenberg, The structure of distal flows. Amer. J. Math. 85 (1963), 477–515.

[16] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. Math. Systems Theory 1 (1967), 1–49.

[17] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory. Princeton University Press, Princeton, New Jersey, 1981.

[18] E. Glasner, Proximal Flows. Lecture Notes in Math. 517, Springer-Verlag, 1976.

[19] E. Glasner and B. Weiss, Sensitive dependence on initial conditions. Nonlinearity 6 (1993), 1067–1075.

[20] E. Glasner and B. Weiss, Locally equicontinuous dynamical systems. Colloq. Math. 84–85 (2000), 345–361.

[21] W. Huang and X. Ye, An explicit scattering, non-weakly mixing example and weak disjointness. Nonlinearity 15 (2002), 849–862.

[22] H. Keynes and J. B. Robertson, Eigenvalue theorems in topological transformation groups. Trans. Amer. Math. Soc. 139 (1969), 359–369.

[23] D. C. McMahon, Weak mixing and a note on the structure theorem for minimal transformation groups. Illinois J. Math. 20 (1976), 186–197.

[24] D. C. McMahon, Relativized weak disjointness and relatively invariant measures. Trans. Amer. Math. Soc. 236 (1978), 225–237.

[25] R. Peleg, Weak disjointness of transformation groups. Proc. Amer. Math. Soc. 33 (1972), 165–170.

[26] K. E. Petersen, Disjointness and weak mixing of minimal sets. Proc. Amer. Math. Soc. 24 (1970), 278–280.

[27] W. A. Veech, Almost automorphic functions on groups. Amer. J. Math. 87 (1965), 719–751.

[28] W. A. Veech, The equicontinuous structure relation for minimal abelian transformation groups. Amer. J. Math. 90 (1968), 723–732.

[29] W. A. Veech, Point-distal flows. Amer. J. Math. 92 (1970), 205–242.

[30] W. A. Veech, Topological dynamics. Bull. Amer. Math. Soc. 83 (1977), 775–830.