Renormalization Group Flow in Algebraic Holography

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An approach to the Holographic Renormalization Group in the context of Rehren duality – a structural form of the AdS-CFT correspondence, in the context of Local Quantum Physics (Algebraic QFT) – is proposed. Special attention to the issue of UV/IR connection is paid.

1. Introduction

The remarkable scaling properties of the AdS-CFT correspondence have been proven extremely useful as a calculational device for the scaling behaviour of holographic pairs, known as the Holographic Renormalization Group (RG). In this communication, the aforementioned scaling properties will be studied in a somewhat different context. We'll adopt the formalism of Local Quantum Physics, in terms of C*-algebras of local observables, for which a structural (i.e., model-independent) form of the AdS-CFT correspondence has been proven by Rehren. This form, called Algebraic Holography or Rehren duality, is, however, a statement for local observables in a fixed spacetime background: some aspects that are linked to unrestricted general covariance, such as conformal anomalies, are quite obscure in our setting. Nevertheless, it will be shown that other phenomena, such as the UV/IR connection and the very duality between the dynamics between different leaves of the Poincaré foliation of anti-de Sitter (AdS) spacetime and the RG flow of the boundary (Minkowski) theory, here appear naturally.

2. Algebraic Holography

Here we shall review the main ideas underlying Rehren duality.

2.1. Geometry

Let the d-dimensional Minkowski space be denoted by $\mathcal{M}_d$, $d \geq 2$, and its conformal compactification by $c\mathcal{M}_d$. Anti-de Sitter space-time with $d+1$ dimensions (denoted $AdS_{d+1}$, or simply $AdS$) can be described by its embedding in $\mathbb{R}^{d+2}$, given by the quadric (let $R = 1$)

$$X^0X^0 - \mathbf{X} \cdot \mathbf{X} - X^d X^d + X^{d+1} X^{d+1} = R^2$$

($\mathbf{X} = (X^1, X^2, \ldots, X^{d-1})$). (1)

Its boundary at spatial infinity is timelike, and conformal to $c\mathcal{M}_d$. The (identity component of the) isometry group of $AdS_{d+1}$ is $SO_e(d, 2)$ (AdS group). A crucial feature of it is that it coincides with the conformal group of $\mathcal{M}_d$. In $AdS_{d+1}$, we can define a causally complete region

$$\mathcal{W}_0 = \{ X \in AdS_{d+1} : X^d + X^{d+1} - \sqrt{1 + \mathbf{X} \cdot \mathbf{X}} > |X^0| \},$$

(2)

called a (standard) wedge. The class of all wedges is obtained by the action of the AdS group on $\mathcal{W}_0$. Using a Poincaré coordinate patch $(z, x^\mu) : z > 0, x^\mu \in \mathcal{M}_d, \mu = 0, \ldots, d-1$

$$\begin{align*}
X^\mu &= \frac{1}{2z} x^\mu \\
X^d &= \frac{1}{2z} x^2 + \frac{1}{2z} x_\mu x^\mu \\
X^{d+1} &= \frac{1}{2z} x^2 - \frac{1}{2z} x_\mu x^\mu
\end{align*}$$

(3)

in the region \{ $X \in AdS_{d+1} : X^d + X^{d+1} > 0$\},
we can write the AdS metric as

$$ds^2 = \frac{1}{z^2} (dx_\mu dx^\mu - dz^2)$$

(4)
and the standard wedge as 
\[ \mathcal{W}_0 := \{ (x^\mu, z) : \sqrt{z^2 + x \cdot x} < 1 - |x^0| \}. \] 

By taking the limit \( z \to 0 \), we reach the boundary \( \mathcal{M}_d \), yielding the intersection of \( \mathcal{W}_0 \) with it: 
\[ \mathcal{K}_0 = \alpha(\mathcal{W}_0) = \{ x^\mu \in \mathcal{M}_d : |x| < 1 - |x^0| \}. \]

This is the (standard) diamond in Minkowski space. Let \( \alpha \) be a bijection from the set of all wedges in \( AdS_{d+1} \) to their intersections with \( AdS_{d+1} \) boundary (namely, the conformal class of diamonds), such that:

- \( \alpha(\mathcal{W}_0) = \mathcal{K}_0 \), and
- It intertwines both actions of \( SO_c(d, 2) \) on \( AdS_{d+1} \) and \( c.\mathcal{M}_d \).

Such an \( \alpha \) preserves inclusions and causal complements. Moreover, wedges and the conformal class of diamonds generate all open sets in their respective spaces.

### 2.2. Observables

We work with theories of local observables, defined as a (isotonous, anti-de Sitter covariant and causal) net of unital C*-algebras \( \mathfrak{A}(\mathcal{O}) \) indexed by all regions \( \mathcal{O} \) in \( AdS \). Notice that we mean causal with respect to the covering of \( AdS \), since pure \( AdS \) possesses closed timelike curves. The correspondence \( \alpha \) built above between the set of wedges \( \mathcal{W} = \{ \Lambda \mathcal{W}_0 \subset AdS_{d+1} : \Lambda \in SO(d, 2) \} \) and the conformal class of diamonds \( \mathbf{K} = \{ \alpha(\Lambda \mathcal{W}_0) \subset \mathcal{M}_d : \Lambda \in SO(d, 2) \} \) allows us to build a (isotonous, conformally covariant and causal) net of unital C*-algebras \( \mathfrak{B}(\mathcal{O}) \) over the regions \( \mathcal{O} \subset \mathcal{M}_d \) by defining
\[ \mathfrak{B}(\alpha(\mathcal{W})) := \mathfrak{A}(\mathcal{W}), \mathcal{W} \in \mathcal{W}. \] 

Therefore, we can state the

**Theorem (Rehren Duality)** [2]. From an anti-de Sitter covariant and local net of observables in \( AdS_{d+1} \), one can build an one-to-one correspondence to a conformally covariant and local net of observables in \( c.\mathcal{M}_d \), intertwining the action of \( SO_c(d, 2) \) and preserving inclusions and causal complements (net isomorphism).

### 3. Leaf nets and holographic RG flow

Consider again the Poincaré coordinate patch [3], which can be seen as to define a foliation (warped product) of the region \( AdS_{d+1}^+ = \{ X \in AdS_{d+1} : X^d + X^{d+1} > 0 \} \). For each fixed \( z \), the (timelike) leaf \( (z, x^\mu) \) is conformal to Minkowski spacetime by a conformal factor of \( 1/z^2 = (X^d + X^{d+1})^2 \). Two other crucial properties of these leaves\(^2\) are:

1. The subgroup of \( SO_c(d, 2) \) defined by
   \[ (z, x^\mu) \mapsto (z, \Lambda^\mu_\nu x^\nu + a^\mu), \quad \Lambda \in SO(d, 1), a^\mu \in \mathcal{M}_d \]
   (Poincaré subgroup) leaves every leaf invariant and acts as the \( (d\text{-dimensional}) \) Poincaré group on each leaf, including the boundary \( (z = 0) \);

2. The subgroup of \( SO_c(d, 2) \) defined by
   \[ (z, x^\mu) \mapsto (\lambda z, \lambda x^\mu), \quad \lambda > 0 \]
   (dilation subgroup) leaves the region \( AdS_{d+1}^+ \) invariant, and acts as the dilation subgroup of \( \mathcal{M}_d \) only at the boundary.

Using Rehren’s theorem as a guide, one can think of the regions
\[ \alpha_z(\mathcal{W}_0) = \{ x^\mu : (z, x^\mu) \in \mathcal{W}_0 \}, 0 < z < 1 \]

as images of \( \mathcal{W}_0 \) under a family of maps \( \alpha_z \) that intertwine the actions of the Poincaré (sub)group with respect to each leaf, just as \( \alpha \) intertwines the actions of the full \( AdS \) group. Furthermore, by assigning to each region in \( \mathcal{M}_d \) the C*-algebra \( \mathfrak{C}_z(\alpha_z(\mathcal{W}_0)) := \mathfrak{A}(\mathcal{W}_0) \) and extending the nets to all \( \mathcal{M}_d \) by Poincaré covariance, we get a family of (isotonous, Poincaré covariant) leaf nets of observables, indexed by \( z \). The leaf nets possess, relatively to the boundary net, the following properties:

\(^2\)Also called branes in [2], after the work of Randall and Sundrum [4]. Here, however, we find such terminology inadequate, since, in our case, such “branes” are not considered as being dynamical objects; thus, we’ll stick to the name “leaf” throughout the entire text.
Scaling dynamics: \( \forall 0 < \lambda \leq 1, \, 0 < z < 1, \)
\[
\mathfrak{B}(\lambda \alpha(\mathcal{W}_0)) = \mathfrak{C}_z(\lambda \alpha_z(\mathcal{W}_0)) \tag{11}
\]
(remember that the dilations act covariantly on the boundary net); and, as a natural consequence of this,

Algebraic UV/IR connection: \( \forall 0 < z < \lambda \leq 1, \)
\[
\mathfrak{B}(\lambda \alpha(\mathcal{W}_0)) = \mathfrak{C}_z(\lambda \alpha_z(\mathcal{W}_0)). \tag{12}
\]

The last property requires some explanation. The physical content of the leaf nets is merely a description of the relation between the energy-momentum behaviour of the bulk and the boundary theories. To show this, we’ll perform the construction of the respective scaling algebras over Minkowski spacetime, following the work of Buchholz and Verch[3].

**Definition**[3]. Let \( \mathfrak{A}(\mathcal{O}) \) be a net of local observables in \( \mathcal{M}_d \). The corresponding scaling net is given by assigning to each region \( \mathcal{O} \subset \mathcal{M}_d \) the scaling algebra \( \mathfrak{A}(\mathcal{O}) \) of (uniformly continuous and bounded) functions \( A : \lambda \in (0, a) \mapsto \Lambda_\lambda \in \mathfrak{A}(\mathcal{O}) \) (a can be infinite), such that:

1. \( \mathfrak{A}(\mathcal{O}) \) is a C*-algebra under pointwise C*-algebraic operations, with the C*-norm \( \|A\| = \sup_{\lambda \in (0,a)} \|\Lambda_\lambda\| \);
2. The action \( \delta_{\Lambda,x} \) of the Poincaré group on the underlying net can be lifted to the scaling net by taking \( \hat{\delta}_{\Lambda,x}(A)_\lambda = \delta_{\Lambda,\Lambda x}(\Lambda_\lambda) \); we assume that it’s norm continuous: \( \|\hat{\delta}_{\Lambda,x}(A) - A\| \rightarrow 0, \forall A. \)
3. Scaling transformations act in a covariant and norm continuous manner, through \( \hat{\delta}_{\Lambda}(A)_\lambda := \Lambda_{\lambda 1}, \forall \mu : \mu \lambda < a. \)

Physically, the scaling net corresponds to all admissible (multiplicative) renormalization prescriptions in the underlying theory, which, of course, must be all physically equivalent if the theory has sufficiently good UV behaviour. An important question to be asked is: how do the scaling algebras \( \mathfrak{B}(\alpha(\mathcal{W}_0)) \) and \( \mathfrak{C}_z(\alpha_z(\mathcal{W}_0)) \) compare? The answer is simple and is given by [12]: the scaling properties coincide up to the scale \( \lambda = z \), for each corresponding leaf net. Should we ignore the contribution of the scaled observables below \( \lambda = z \) (UV cutoff), the corresponding leaf net will behave just like the boundary theory dual to a “IR-cutoff” AdS net. This somewhat loose explanation suggests the name “algebraic UV/IR connection”, since a similar phenomenon occurs in the (stringy) AdS/CFT correspondence[8].

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