Equations of Geodesic Deviation and the Inverse Scattering Transform

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Abstract
Solutions of equations of geodesic deviation in three- and four-dimensional spaces are obtained via the inverse scattering transform. It is shown that in the case of three-dimensional space solutions of geodesic deviation equations are reduced to solutions of the well-known Zakharov-Shabat problem. In four-dimensional space system of geodesic deviation equations is associated with a $3 \times 3$ matrix Schrödinger equation, and dependence on parameters is defined by the nonlinear equations of three-wave interaction.

Keywords: geodesic deviation, mKdV equation, Chandrasekhar metrics, matrix Schrödinger equation.

1 Introduction

It is well-known that a $m \times m$ matrix Schrödinger equation on $-\infty < x < \infty$ is defined by the following expression

$$L \psi(x, k) = \lambda \psi(x, k), \quad \lambda = k^2,$$

where

$$L = -(\partial^2/\partial x^2)I + U(x),$$

$$I = (\delta_{ij}), \quad U(x) = (u_{ij}(x)); \quad i, j = 1, \ldots, m,$$

$$\psi(x, k) = [\psi_1(x, k), \psi_2(x, k), \ldots, \psi_m(x, k)].$$

Further, let $\eta^i$ be the components of deviation vector between two infinitesimally nearby geodesic lines. Then the components $\eta^i$ satisfy to the Jacby equation

$$v^i \nabla_i (v^j \nabla_j \eta^l) = -v^j R^l_{ikm} v^m \eta^k,$$  \hspace{1cm} (1.1)

where $v^i$ are the components of the tangent vector along a geodesic line $\gamma$, $R^l_{ikm}$ is the curvature tensor of the metrics

$$ds^2 = g_{ij} dx^i dx^j.$$

In a special system of coordinates, where axis $x^j$ is a geodesic line, equation (1.1) has the following form

$$d^2 \eta^l/dx^2 + R^l_{ikm} \eta^m = 0.$$  \hspace{1cm} (1.2)

In the paper it has been shown that in the case of three-dimensional space with the metrics

$$ds^2 = dx^2 + A(x, y, z) dy^2 + 2B(x, y, z) dy dz + C(x, y, z) dz^2$$  \hspace{1cm} (1.3)
the equations of geodesic deviations
\[
\frac{d^2 \eta^2}{dx^2} + R^2_{i21} \eta^2 + R^2_{i31} \eta^3 = 0,
\]
\[
\frac{d^2 \eta^3}{dx^2} + R^3_{i21} \eta^2 + R^3_{i31} \eta^3 = 0
\]
(1.4)
may be represented in the form of the 2 × 2 matrix Schrödinger equation
\[
-\frac{d^2 \eta^2}{dx^2} + (-R^2_{i21} + \lambda^2) \eta^2 + (-R^2_{i31}) \eta^3 = \lambda^2 \eta^2,
\]
\[
-\frac{d^2 \eta^3}{dx^2} + (-R^3_{i21}) \eta^2 + (-R^3_{i31} + \lambda^2) \eta^3 = \lambda^2 \eta^3.
\]
(1.5)
On the other hand, it is known that AKNS-system [6]
\[
\frac{\partial \psi_1}{\partial x} + i\lambda \psi_1 = q(x, y, z) \psi_2,
\]
\[
\frac{\partial \psi_2}{\partial x} - i\lambda \psi_2 = r(x, y, z) \psi_1
\]
(1.6)
can be rewritten in the form of a Schrödinger-like equation [1]
\[
\left[ -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \frac{\partial^2}{\partial x^2} + \begin{pmatrix} rq & qx \\ rx & rq \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]
(1.7)
The comparison of the systems (1.7) and (1.5) gives the following conditions on the curvature tensor
\[
\lambda^2 - R^2_{i21} = rq, \quad \lambda^2 - R^3_{i31} = rq,
\]
\[
R^2_{i31} = -qx, \quad R^3_{i21} = -rx.
\]
(1.8)
Analogously, in the case of 4-dimensional space with a geodesic coordinate system
\[
ds^2 = dt^2 + g_{ab} dx^a dx^b
\]
(1.9)
the geodesic deviations equation has the form [5]
\[
\frac{d^2 \eta^1}{dt^2} + R^1_{010} \eta^1 + R^1_{020} \eta^2 + R^1_{030} \eta^3 = 0,
\]
\[
\frac{d^2 \eta^2}{dt^2} + R^2_{010} \eta^1 + R^2_{020} \eta^2 + R^2_{030} \eta^3 = 0,
\]
\[
\frac{d^2 \eta^3}{dt^2} + R^3_{010} \eta^1 + R^3_{020} \eta^2 + R^3_{030} \eta^3 = 0.
\]
(1.10)
In the present paper we consider solutions of the equations (1.4) and (1.10) obtained by the inverse scattering transform. Our consideration is realized on the basis of a Chandrasekhar metrics [7, 8] (the so-called space-time of a sufficiently general structure), which includes as particular cases the static and spherically symmetric solutions (Schwarzschild and Reissner-Nordström metrics), and also stationary and axially symmetric solutions (Kerr and Kerr-Newman metrics) and so on. In section 2 we introduce a three-dimensional analog of the Chandrasekhar metrics, the particular case of which is coincide with the metrics (1.3). It is shown that in the orthonormal basis, related with this metrics, solutions of the system (1.4) are reduced to the solutions of the Zakharov-Shabat problem [9]. Thus, a dependence of the potential \(u\) on parameters \(y\) and \(z\) is described by the modified Korteweg-de Vries (mKdV) equations. Different particular cases, in which the vector of geodesic deviation \(\eta\) is explicitly expressed via the fundamental solutions (Jost functions) of
the Zakharov-Shabat problem, are considered at the end of section 2. In section 3
we introduce a $3 \times 3$ matrix Schrödinger equation which then is associated with the
system of type (1.10). Further, a dependence on parameters is reduced to evolution
equations of the well-known problem of three-wave interaction, the explicit solutions
of which was obtained by Zakharov and Manakov in 1973 [10, 11, 12]. It is shown
that in the case of decay instability and reality of potential matrix, the system of
equations of geodesic deviation (1.10) has a wide class of particular solutions.

2 Three-dimensional space

2.1 The three-dimensional Chandrasekhar metrics

Let us consider in the three-dimensional space with a signature ($-,-,-$) a metrics
of the following form

$$ds^2 = -\sum_A e^{2\mu_A} (dx^A)^2 - e^{2\psi}(dx^3 - \sum_A q_A dx^A)^2, \quad (2.1)$$

where $A = 1, 2$. $\psi, \mu_A$ and $q_A$ are the functions on variables $x^1, x^2, x^3$.

The orthonormal basis, related with this metrics, is defined by the following
covariant and contravariant vectors

$$e_{(1)i} = (0, 0, -e^{\mu_1}), \quad e_{(2)i} = (0, -e^{\mu_2}, 0),$$

$$e_{(3)i} = (-e^\psi, q_1 e^\psi, q_2 e^\psi). \quad (2.2)$$

$$e^{i(1)} = (q_2 e^{-\mu_1}, 0, e^{-\mu_1}), \quad e^{i(2)} = (q_1 e^{-\mu_2}, e^{-\mu_2}, 0),$$

$$e^{i(3)} = (e^{-\psi}, 0, 0). \quad (2.3)$$

From (2.2) and (2.3) it follows that

$$e^{(a)} e^{(b)i} = \eta_{(a)(b)} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}. \quad (2.4)$$

Let

$$\omega^A = e^{\mu_A} dx^A, \quad \omega^3 = e^\psi(dx^3 - \sum_A q_A dx^A) \quad (2.4)$$

be the basis 1-forms. It is easy to see that inverse relations for (2.4) have the form

$$dx^A = e^{-\mu_A} \omega^A, \quad dx^3 = e^{-\psi} \omega^3 + \sum_A e^{-\mu_A} q_A \omega^A. \quad (2.5)$$

Expressing the exterior derivatives of the forms $\omega^i$ via the basis 2-forms
$\omega^j \wedge \omega^j$ ($i \neq j, i, j = 1, 2, 3$), we have

$$d\omega^A = \sum_B e^{\mu_A} \mu_{A,B} dx^B \wedge dx^A + e^{\mu_A} \mu_{A,3} dx^3 \wedge dx^A =$$

$$= \sum_B e^{-\mu_B} \mu_{A,B} \omega^B \wedge \omega^A + \mu_{A,3} \left[ e^{-\psi} \omega^3 + \sum_B e^{-\mu_B} q_B \omega^B \right] \wedge \omega^A =$$

$$= \sum_B e^{-\mu_B} (\mu_{A,B} + q_B \mu_{A,3}) \omega^B \wedge \omega^A + e^{-\psi} \mu_{A,3} \omega^3 \wedge \omega^A. \quad (2.6)$$
Comparing the equations (2.8) and (2.9) with the equations (2.12) and (2.13), we have

\[
\frac{1}{2}T^j = d\omega^j + \omega_j^i \wedge \omega^i = \Omega^j_i, \\
\frac{1}{2}R^j_{ikm} \omega^k \wedge \omega^m = \Omega^j_i
\]

are called respectively the first and second Cartan structure equations, where the Cartan 2-form \( \Omega^j_i \) is

\[
\Omega^j_i = d\omega^j + \omega_j^i \wedge \omega^i.
\]

Owing to absence of torsion \( (T^j = 0) \) the first Cartan structure equation gives

\[
d\omega^3 = - \sum_A \omega_A^3 \wedge \omega^A, \\
d\omega^A = - \sum_B \omega_B^A \wedge \omega^B - \omega_3^A \wedge \omega^3.
\]

These equations allow us to define the connection 1-forms \( \omega_A^3 \) and \( \omega^A_3 \) if the forms \( d\omega^3 \) and \( d\omega^A \) are known. Since the 1-forms \( \omega^3 \) and \( \omega^A \) are the basis forms, then

\[
\omega^j_i = -\omega_i^j \quad (i, j = 1, 2, 3).
\]

Comparing the equations (2.8) and (2.9) with the equations (2.12) and (2.13), we obtain

\[
\omega_A^3 = - \omega_3^A = e^{-\mu_A} \Psi_A \omega^3 + e^{-\mu_A} \mu_A \omega^A + \frac{1}{2} \sum_B e^{-\mu_A - \mu_B} Q_{AB} \omega^B, \\
\omega_B^A = - \omega_A^B = - \frac{1}{2} e^{-\mu_A - \mu_B} Q_{AB} \omega^3 + e^{-\mu_B} \mu_A \omega^A - e^{-\mu_A} \mu_B \omega^B
\]

where

\[
Q_{AB} = q_{A:B} - q_{B:A}, \\
\Psi_A = \psi_A + q_{A,3}
\]

From (2.15) and (2.16) for the different connection forms we have

\[
\omega_1^3 = e^{-\mu_1} \omega_1^3 - \frac{1}{2} e^{-\mu_1 - \mu_2} Q_{12} \omega^2 - e^{-\mu_1} \Psi_1 \omega^3, \\
\omega_2^3 = - \frac{1}{2} e^{-\mu_1 - \mu_2} Q_{21} \omega^1 + e^{-\mu_2} \mu_2 \omega^2 - e^{-\mu_2} \Psi_2 \omega^3, \\
\omega_1^1 = e^{-\mu_2} \mu_2 \omega^1 - e^{-\mu_1} \mu_2 \omega^2 - \frac{1}{2} e^{-\mu_1 - \mu_2} Q_{12} \omega^3.
\]
Further, in order to calculate the components of the Riemann tensor from the second Cartan structure equation

\[ \frac{1}{2} R^j_{ikl} \omega^k \wedge \omega^l = \Omega^j_i = d \omega^j_i + \omega^j_k \wedge \omega^k_j, \]  

\[(2.20)\]

it is necessary at first to calculate the exterior derivatives of the connection forms \[(2.19).\]

**Lemma (Chandrasekhar [8]).** If \( F \) is an arbitrary functions of the arguments \( x^1, x^2 \) and \( x^3 \), then

\[ d(F \omega^3) = \sum_A e^{-\psi - \mu A} D_A (F e\psi) \omega^A \wedge \omega^3 + \frac{1}{2} \sum_{A,B} F e^{-\mu A - \mu B} Q_{AB} \omega^A \wedge \omega^B, \]  

\[(2.21)\]

\[ d(F \omega^A) = \sum_B e^{-\mu A - \mu B} (e^{\mu A} F)_{;B} \omega^B \wedge \omega^A + e^{-\psi - \mu A} (e^{\mu A} F)_{,3} \omega^3 \wedge \omega^A, \]  

\[(2.22)\]

where \( D_A \) is an operator, the action of which on an arbitrary function \( f(x^1, x^2, x^3) \) is defined by the following expression

\[ D_A f = f_{,A} + q_{A,3} f = f_{,A} + (q_A f)_{,3}. \]  

\[(2.23)\]

Using this lemma, we obtain

\[ d \omega^1_2 = - \sum_A e^{-\psi - \mu A} D_A \left( \frac{1}{2} e^{2\psi - \mu_1 - \mu_2 Q_{12}} \omega^A \wedge \omega^3 \right. \]

\[ - e^{-\psi - \mu_1} (e^{\mu_1 - \mu_2} \omega_{12})_{,3} \omega^1 \wedge \omega^3 + e^{-\psi - \mu_2} (e^{\mu_2 - \mu_1} \omega_{21})_{,3} \omega^2 \wedge \omega^3 + \]

\[ + \omega^1 \wedge \omega^2 \left\{ - \frac{1}{4} e^{2\psi - 2\mu_1 - 2\mu_2 Q_{12}} - \right. \]

\[ - e^{-\mu_1 - \mu_2} \left[ (e^{\mu_1 - \mu_2} \omega_{12})_{,2} + (e^{\mu_2 - \mu_1} \omega_{21})_{,1} \right] \}. \]  

\[(2.24)\]

\[ \omega^3_1 \wedge \omega^3_2 = \left[ e^{-2\psi} \omega_{1,3} \mu_{2,3} + \frac{1}{4} e^{2\psi - 2\mu_1 - 2\mu_2 Q_{12} Q_{21}} \omega^1 \wedge \omega^2 - \right. \]

\[ - e^{-\psi - \mu_2} \omega_{1,3} \mu_{2,3} + \frac{1}{4} e^{2\psi - 2\mu_1 - 2\mu_2} \omega_{1,3} \mu_{2,3} \]

\[ + \left. + \frac{1}{2} e^{2\psi - 2\mu_1 - 2\mu_2} Q_{12} \omega_{2} + e^{-\psi - \mu_1} \omega_{1,3} \mu_{2,3} \right] \omega^2 \wedge \omega^3, \]  

\[(2.25)\]

\[ d \omega^3_3 = - \sum_A e^{-\psi - \mu A} D_A (e^{\psi - \mu_2} \omega_2) \omega^A \wedge \omega^3 + \]

\[ + e^{-\psi} (e^{\mu_2} Q_{21})_{,3} \omega^1 \wedge \omega^3 - e^{-\psi - \mu_2} (e^{\mu_2 - \mu_3} \omega_{2,3})_{,3} \omega^2 \wedge \omega^3 + \]

\[ + \omega^1 \wedge \omega^2 \left\{ - \frac{1}{2} e^{-\psi - \mu_1 - 2\mu_2} Q_{12} - \right. \]

\[ + e^{-\mu_1 - \mu_2} \left[ (e^{\mu_2 - \mu_3} \omega_{2,3})_{,1} + \frac{1}{2} e^{2\psi - 2\mu_2 Q_{21}} \right] \}. \]  

\[(2.26)\]
\[ \omega^1_2 \wedge \omega^3_3 = \left[ e^{-\psi - \mu_1} \mu_{2:1} \mu_{1:3} - \frac{1}{2} e^{\psi - \mu_1 - 2 \mu_2} Q_{12 \mu_{1:2}} \right] \omega^1 \wedge \omega^2 + \]
\[ + \left[ \frac{1}{2} e^{-\mu_1 - \mu_2} Q_{12 \mu_{1:3}} - e^{-\mu_1 - \mu_2} \Psi_{1 \mu_{1:2}} \right] \omega^1 \wedge \omega^3 + \]
\[ + \left[ e^{-2 \mu_1} \Psi_{1 \mu_{2:1}} - \frac{1}{4} e^{-2 \psi - 2 \mu_1 - 2 \mu_2} Q_{12}^2 \right] \omega^2 \wedge \omega^3, \quad (2.27) \]

\[ d \omega^1_3 = - \sum_A e^{-\psi - \mu_A} D_A (e^{\psi - \mu_1} \Psi_1) \omega^A \wedge \omega^3 - \]
\[ - e^{-\psi - \mu_3} (e^{\mu_1 - \psi} \mu_{1:3}, \omega^1 \wedge \omega^3 + e^{-\psi - \mu_2} \left( \frac{1}{2} e^{\psi - \mu_1} Q_{12} \right) \omega^2 \wedge \omega^3 + \]
\[ + \omega^1 \wedge \omega^2 \left\{ - \frac{1}{2} e^{2 \psi - 2 \mu_1 - 2 \mu_2} \Psi_{1 Q_{12} \mu_{1:2}} - \right. \]
\[ - e^{-\mu_1 - \mu_2} \left[ (e^{\mu_1 - \psi} \mu_{1:3}, \omega^1 \wedge \omega^2 + \frac{1}{2} e^{\psi - \mu_1} Q_{12} \right] \right\}, \quad (2.28) \]

\[ \omega^1 \wedge \omega^3 = \left[ e^{-\psi - \mu_3} e^{\mu_1 - \psi} \mu_{1:3} \mu_{2:3} - \frac{1}{2} e^{\psi - 2 \mu_1 - 2 \mu_2} \mu_{1:2} Q_{21} \right] \omega^2 \wedge \omega^2 - \]
\[ - \left[ e^{-2 \mu_1 - \mu_2} \mu_{1:2} \mu_{2:3} - \frac{1}{4} e^{2 \psi - 2 \mu_1 - 2 \mu_2} Q_{12} \right] \omega^1 \wedge \omega^3 + \]
\[ + \left[ e^{-\mu_1 - \mu_2} \mu_{2:1} \mu_{2:3} + \frac{1}{2} e^{-\mu_1 - \mu_2} Q_{12} \right] \omega^2 \wedge \omega^3. \quad (2.29) \]

Further, from the equations (2.24) and (2.26) we obtain

\[ \frac{1}{2} R^1_{2:kl} \omega^k \wedge \omega^l = \Omega^1_2 = d \omega^1_2 - \omega^3_3 \wedge \omega^2_3. \quad (2.30) \]

Substituting (2.24) - (2.26) into this equation and collecting the coefficients at \( \omega^k \wedge \omega^l \), we obtain the components \( R^1_{2:kl} \) of the curvature tensor. For example, with the object to calculate the component \( R^1_{212} \), we must collect the coefficients at \( \omega^1 \wedge \omega^1 \) in the expression for \( \Omega^1_2 \). Analogously, the components \( R^1_{2:13} \) and \( R^1_{2:23} \) are obtained from \( \Omega^1_3 \) via the comparison of the coefficients at \( \omega^1 \wedge \omega^2 \) and \( \omega^2 \wedge \omega^3 \). In like manner from the equation

\[ \frac{1}{2} R^2_{3:kl} \omega^k \wedge \omega^l = \Omega^2_3 = d \omega^2_3 - \omega^1_3 \wedge \omega^3 \quad (2.31) \]

and equations (2.26) - (2.27) we obtain the components \( R^2_{2:23} \) and \( R^2_{2:13} \). Analogously, from equation

\[ \frac{1}{2} R^2_{3:kl} \omega^k \wedge \omega^l = \Omega^3_2 = d \omega^3_2 + \omega^1_2 \wedge \omega^3 \quad (2.32) \]

and equations (2.27) - (2.28) we have the component \( R^3_{3:13} \). Finally, we have the following six essential components of the curvature tensor:

\[ R^1_{212} = - \frac{1}{4} e^{2 \psi - 1 \mu_1 - 2 \mu_2} Q_{12}^2 - \frac{1}{4} e^{2 \psi - 1 \mu_1 - 2 \mu_2} \left[ (e^{\mu_1 - \mu_2} \mu_{1:2})_2 + (e^{\mu_2 - \mu_1} \mu_{2:1})_1 \right] - \frac{1}{4} e^{2 \psi - 1 \mu_1 - 2 \mu_2} Q_{12} \omega_{12} Q_{21}, \quad (2.33) \]

\[ R^1_{213} = e^{-\psi - \mu_1} D_1 (1/2 e^{2 \psi - 1 \mu_1 - 2 \mu_2} Q_{12}^2) - e^{-\psi - \mu_1} (e^{\mu_1 - \mu_2} \mu_{1:3})_3 + \]
\[ + e^{-\psi - \mu_2} \Psi_{2 \mu_{1:3}} + \frac{1}{2} e^{-2 \psi - 2 \mu_1 - 2 \mu_2} \Psi_{1 Q_{12} \mu_{1:3}}, \quad (2.34) \]
the metrics (2.1) is coincide with the three-dimensional metrics considered in [5]. Let us consider a particular case ($\mu_2 \neq 0$).

2.2 Solutions of equations of geodesic deviation in the three-dimensional space

Let us consider a particular case ($\mu_1 = q_1 = 0$) of the metrics (2.1). In this case the metrics (2.1) is coincide with the three-dimensional metrics considered in [5] if we suppose

$$A(x, y, z) = - \left( e^{2\mu_2} + q_2^2 e^{2\psi} \right), \quad B(x, y, z) = q_2 e^{2\psi},$$

$$C(x, y, z) = - e^{2\psi}. \quad (2.39)$$

At the condition $\mu_1 = q_1 = 0$ the covariant and contravariant vectors take the form

$$e_{(1)i} = (0, 0, -1), \quad e_{(2)i} = (0, -e^{\mu_2}, 0),$$

$$e_{(3)i} = (-e^{\psi}, 0, q_2 e^{\psi}); \quad e_{(j)} = (q_2, 0, 1), \quad e_{(j)} = (0, e^{-\mu_2}, 0),$$

$$e_{(3)i} = (e^{-\psi}, 0, 0). \quad (2.41)$$

It is easy to see that in this orthonormal basis for the components of the curvature tensor we have

$$R^a_{ijkl} = -R_{ijkl}. \quad (2.42)$$

It is well-known that the Riemann tensor $R_{ijkl}$ has the following symmetry properties:

$$R_{ijkl} = R_{klij},$$

$$R_{ijkl} = -R_{ijlk},$$

$$R_{ijkl} = -R_{ijlk}. \quad (2.43)$$

It is easy to show that the symmetry properties (2.43) decrease the number of independent (essential) components of the Riemann tensor from $n^4$ to $n^2(n^2 - 1)/12$, where $n$ is a dimensionality of the space. In the case of three-dimensional space we have six independent components of the curvature tensor: $R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{2313}, R_{2323}$. Further, using (2.35)-(2.38), we see that
in the system among the four components of the curvature tensor only three are independent, namely, \( R_{121}^2, R_{131}^3 \) and \( R_{131}^2 \) (or \( R_{121}^3 \)). The latter two components are coincide with each other in virtue of \( 2.42 - 2.43 \). Therefore,

\[
R_{121}^3 = -R_{113}^2.
\]

Hence it immediately follows that the conditions \( 1.5 \) and the system \( 1.7 \) are reduced to the form

\[
\lambda^2 - R_{121}^2 = -u^2, \quad \lambda^2 - R_{131}^3 = -u^2,
\]

\[
R_{121}^3 = -R_{113}^2 = u_x;
\]

\[
\left[ - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} -u^2 & u_x \\ -u_x & -u^2 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]

(2.45)

It is easy to see that the matrix equation (2.45) corresponds to the Zakharov-Shabat system \([9]\)

\[
\frac{\partial \psi_1}{\partial x} + i\lambda \psi_1 = u \psi_2,
\]

\[
\frac{\partial \psi_2}{\partial x} - i\lambda \psi_2 = -u \psi_1.
\]

(2.46)

Thus, in the orthonormal basis \( 2.40 - 2.41 \), related with the metrics \( 2.1 \), at the condition \( \mu_1 = q_1 = 0 \) the AKNS-system for the equations of geodesic deviation is reduced to the Zakharov-Shabat system. Moreover, instead the two potentials in AKNS-system we have now only one potential in ZS-system.

Let us calculate the independent components of the curvature tensor in the system \( 1.4 \) for the metrics \( 2.1 \) at the condition \( \mu_1 = q_1 = 0 \). From \( 2.38 \), \( 2.39 \) and \( 2.38 \) we have

\[
R_{212}^1 = -\mu_{2,1} - \mu_{2,1}^2 - \frac{1}{4} e^{-2\mu_2 q_{2,1}} (e^{2\psi} + 1),
\]

(2.47)

\[
R_{213}^1 = \frac{1}{2} e^{\psi - 2\mu_2 q_{2,1}} + \frac{3}{2} e^{\psi - 2\mu_2 \psi_{1q_{2,1}}} - \frac{1}{2} e^{\psi - 2\mu_2} \mu_{2,1} q_{2,1},
\]

(2.48)

\[
R_{313}^1 = -\psi_{,1} - \psi_{,1}^2 + \frac{1}{4} e^{2\psi - 2\mu_2} q_{2,1}^2.
\]

(2.49)

So, in the case of the metrics \( 2.4 \) our problem of solving of the equations of geodesic deviation is reduced to the Zakharov-Shabat problem \( 2.46 \). It is known that fundamental solutions (Jost functions) of ZS-problem are defined by the following expressions \([6, 13, 14]\)

\[
\varphi_1^-(x, \lambda) = e^{-i\lambda x} + \int_0^x dx' A_1(x, x') e^{-i\lambda x'},
\]

(2.50)

\[
\varphi_2^-(x, \lambda) = \int_0^x dx' A_2(x, x') e^{-i\lambda x'};
\]

\[
\varphi_1^+(x, \lambda) = \int_x^\infty dx' B_1(x, x') e^{i\lambda x'},
\]

(2.51)

\[
\varphi_2^+(x, \lambda) = e^{i\lambda x} + \int_x^\infty dx' B_2(x, x') e^{i\lambda x'}.
\]
These solutions are linearly dependent:

\[
\varphi^-(x, \lambda) = c_{11}(\lambda) \varphi^+(x, \lambda) + c_{12}(\lambda) \tilde{\varphi}^+(x, \lambda),
\]

\[
\varphi^+(x, \lambda) = c_{21}(\lambda) \tilde{\varphi}^-(x, \lambda) + c_{22}(\lambda) \varphi^-(x, \lambda),
\]

(2.52)

where

\[
\varphi^\pm(x, \lambda) = \frac{\varphi^\mp(x, \lambda)}{c_2^{\pm}(x, \lambda)} , \quad \tilde{\varphi}^\mp(x, \lambda) = \begin{pmatrix} \varphi_1^\mp(x, -\lambda) \\ -\varphi_1^\mp(x, -\lambda) \end{pmatrix}.
\]

(2.53)

Further, the pair of Gel’fand-Levitan-Marchenko integral equations can be derived from (2.52) by means of the Fourier transform:

\[
-A_2(x,y) + \Omega_L(x+y) + \int_{-\infty}^{x} dx' A_1(x, x') \Omega_L(x' + y) = 0,
\]

\[
A_1(x,y) + \int_{-\infty}^{x} dx' A_2(x, x') \Omega_L(x' + y) = 0,
\]

(2.55)

where

\[
\Omega_L = r_L(z) - i \sum_{l=1}^{N} \frac{c_{22}(\lambda_l)}{\bar{c}_{12}(\lambda_l)} e^{-i\lambda_l z},
\]

\[
\Omega_R = r_R(z) + i \sum_{l=1}^{N} \frac{c_{11}(\lambda_l)}{\bar{c}_{21}(\lambda_l)} e^{i\lambda_l z}.
\]

(2.56)

Thus, the potential \( u(x) \) is expressed via the kernels \( A_1, A_2 \) and \( B_1, B_2 \) as follows

\[
u = -2A_2(x,x), \quad u = -2B_1(x,x),
\]

\[
u^2 = 2 \frac{dA_1(x,x)}{dx}, \quad u^2 = -2 \frac{dB_2(x,x)}{dx}.
\]

(2.57)

In the case of a reflectionless potential \( (r_L(z) = 0) \) the system of Gel’fand-Levitan-Marchenko integral equations may be solved explicitly. In this case we obtain

\[
\Omega_L = -i \sum_{l=1}^{N} \frac{c_{22}(\lambda_l)}{\bar{c}_{12}(\lambda_l)} e^{-i\lambda_l z}.
\]

(2.58)

Thus, the potential \( u(x) \) is defined by a following expression

\[
u = 2 \frac{d}{dx} \arctan \left[ \frac{\text{Im} \det(I - iM)}{\text{Re} \det(I - iM)} \right],
\]

(2.59)
Here $\kappa_i$ are the poles of the transmission coefficient $T_L(\lambda) = \frac{1}{c_{12}(\lambda)}$. Analogous relations take place in the case of system (2.66).

Let us return to the equations of geodesic deviation. From the conditions on the curvature tensor (2.44) it follows that

$$R_{121}^2 = R_{131}^3,$$

$$R_{121}^3 = u_x.$$

Substituting the expressions (2.47) - (2.49) into the latter equations, we obtain

$$\psi_{11} + \psi_{1}^2 - \mu_{2,11} - \mu_{1}^2 - \frac{1}{2} e^{-2\mu_{2,11}} (e^{2\psi} + \frac{1}{2}) = 0,$$

$$\frac{1}{2} e^{-\mu_{2,11}} \psi_{1,2,1} + 3 e^{-\mu_{2,1}} \psi_{1,1,1} - \frac{1}{2} e^{-\mu_{2,11}} \mu_{2,1,1} = u_{,1}.$$

Thus, we have the system of differential equations (2.67) - (2.68) as the conditions on the potential $u(x)$. The explicit form of $u(x)$ we will find by means of the inverse scattering problem. Moreover, the potential $u(x)$ depends parametrically on variables $y$ and $z$. According to widely accepted methods [6, 13, 14], the dependence on variables $y$ and $z$ may be represented by a nonlinear integrable equation. Indeed, the dependence on $y$ for $\psi_1$ and $\psi_2$ from (2.66) may be expressed in general form

$$\psi_{1y} = A \psi_1 + B \psi_2,$$

$$\psi_{2y} = C \psi_1 + D \psi_2.$$

The compatibility conditions of (2.67) with (2.66) give (at this point, $\lambda_y = 0$):

$$A_x = u(B + C),$$

$$B_x + 2iA = -2u A + u_y,$$

$$C_x - 2i\lambda C = -2u A - u_y,$$

where $D_x = -A_x$. Further, let us suppose that $A = \sum_0^3 a_n \lambda^n$, $B = \sum_0^3 b_n \lambda^n$ and $C = \sum_0^3 c_n \lambda^n$. Substituting these series into (2.67), (2.68) we obtain for the coefficients $a_n$, $b_n$ and $c_n$ the following expressions

$$a_3 = a_3(y), \quad b_3 = c_3 = 0,$$

$$a_2 = a_2(y), \quad b_2 = -c_2 = ia_3 u,$$

$$a_1 = -\frac{1}{2} a_3 u^2, \quad b_1 = -\frac{1}{2} a_3 u_x + ia_2 u, \quad c = -\frac{1}{2} a_3 u_x - ia_2 u,$$

$$a_0 = -\frac{1}{2} a_2 u^2, \quad b_0 = c_0 = \frac{i}{4} a_3 (u_{xx} + 2u^3) - \frac{1}{2} a_2 u_x.$$

In the equations (2.67) - (2.68) the components independent on $\lambda$ give the evolution equation $u_y = b_{0x} + 2a_0 u$. Using the obtained above expressions (2.69), we obtain for the coefficients $a_0$ and $b_0$:

$$u_y = \frac{1}{4} i a_3 (6u^2 u_x + u_{xxx}) - a_2 (u^3 + \frac{1}{2} u_{xx}).$$
Suppose $a_2 = 0$ and $a_3 = 4i$ we have the modified Korteweg-de Vries equation

$$u_y + 6u^2u_x + u_{xxx} = 0. \quad (2.70)$$

Thus, the dependence on parameter $y$ for the potential $u$ is defined by the mKdV equation. Thus, the system \textbf{[2.66]} has a form

$$\psi_{1y} = 2i\lambda(u^2 - 2\lambda^2)\psi_1 + (4\lambda^2u + 2i\lambda u_x - 2u^3 - u_{xx})\psi_2,$$

$$\psi_{2y} = (-4\lambda^2u + 2i\lambda u_x + 2u^3 + u_{xx})\psi_1 - 2i\lambda(u^2 - 2\lambda^2)\psi_2. \quad (2.71)$$

Solutions of the modified Korteweg-de Vries equation can be found by the standard procedure \textbf{[6, 13, 14]}. When $u \to 0$ we see that the dependence on $y$ is described by a limiting form of the equations \textbf{[2.71]}

$$\psi_{1y} = -4i\lambda^3\psi_1,$$

$$\psi_{2y} = 4i\lambda^3\psi_2. \quad (2.72)$$

Let us assume that $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is proportional to the fundamental solution $\varphi^-$ at $x \to -\infty$. Then, at $x \to -\infty$ we have $\psi(x, y) = f(y)\varphi^- \to f(y)e^{-i\lambda x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Substituting $\psi = f(y)e^{-i\lambda x}$ into the first equation from \textbf{[2.72]}, we obtain $f(y) = f(0)\exp(-4i\lambda^3 y)$. From \textbf{[2.62]} at $x \to +\infty$ it follows that

$$\psi = f(y)\varphi^- \to f(0)e^{-4i\lambda^3 y} \left[ c_{11}(\lambda, y)e^{i\lambda x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_{12}(\lambda, y)e^{-i\lambda x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \quad (2.73)$$

Substituting it again into \textbf{[2.72]}, we obtain that $\dot{c}_{12} = 0, \dot{c}_{11} = 8i\lambda^3$, whence

$$c_{12}(\lambda, y) = c_{12}(\lambda, 0),$$

$$c_{11}(\lambda, y) = c_{11}(\lambda, 0)e^{8i\lambda^3 y}. \quad (2.74)$$

The analogous calculations for $\psi \sim \varphi^+$ give

$$c_{22}(\lambda, y) = c_{22}(\lambda, 0)e^{-8i\lambda^3 y}. \quad (2.75)$$

Using the dependence on parameter $y$ given by the relations \textbf{[2.11]- [2.14]}, we have for \textbf{[2.02]} and \textbf{[2.57]} the following expressions

$$m_L(\kappa_i, y) = -i\frac{c_{22}(\kappa_i, y)}{c_{12}(\kappa_i, 0)} = m_L(\kappa_i, 0)e^{-8i\lambda^3 y}, \quad (2.76)$$

$$\Omega_L(z, y) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} R_L(\lambda, 0)e^{-8i\lambda^3 y - i\lambda z} + \sum_{l=1}^{N} m_L(\kappa_l, 0)e^{-8i\lambda_l y - i\kappa_l z}, \quad (2.77)$$

where $R_L(\lambda, 0) = -c_{22}(\lambda)/c_{21}(\lambda)$. Further, from \textbf{[2.59]} it follows that the potential $u(x, y)$ is expressed by the kernel of Gel’fand-Levitan-Marchenko equations \textbf{[2.40]} as

$$u(x, y) = -2A_2(x, x, y).$$

In case of the reflectionless potential ($R_L(\lambda, 0) = 0$) the integral equations \textbf{[2.55]} are solved explicitly. In this case, the potential $u(x, y)$ is defined by the formula \textbf{[2.01]} with the matrix $M$ of the following form

$$M_{ij} = \frac{i}{\kappa_i + \kappa_j} \frac{m_L(\kappa_j, y)}{e^{-i(\kappa_i + \kappa_j)x}}. \quad (2.78)$$
In the simplest case of the one-soliton solution ($N = 1$), the matrix $M$ is reduced to a scalar $M = i(m_1/2\kappa_1)\exp(-2i\kappa_1x)$. Taking into account the relation (2.76), we see that the matrix $M$ at $\kappa = i\lambda$ can be written as

$$M = \frac{m_1(0)}{2\lambda}e^{2\lambda x - 8\lambda^3 y}.$$  

Thus, in case of the one-soliton solution, the potential, defined by the expression (2.61) with the matrix (2.78), is reduced to the form

$$u(x, y) = -2\frac{\partial}{\partial x} \arctan \left[ \frac{m_1(0)}{2\lambda}e^{2\lambda x - 8\lambda^3 y} \right]$$  

(2.79)

or

$$u(x, y) = \pm 2\lambda \mathrm{sech}(2\lambda x - 8\lambda^3 y + \delta),$$  

(2.80)

where $\delta = \ln [m_1(0)/2\lambda]$. For $m_1(0) < 0$ we take the upper sign, for $m_1(0) > 0$ the lower sign.

Supposing the analogous dependence on the parameter $z$, that is, defining it by the modified Korteweg-de Vries equation of the form

$$u_z + 6u^2ux + uxxx = 0,$$  

(2.81)

we came in the case of the one-soliton solution to the following dependence

$$u(x, y, z) = \pm 2\lambda \mathrm{sech}(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta).$$  

(2.82)

Thus, we see that dependence of the potential $u$ on the parameters $y$ and $z$ is given by the mKdV equations (2.70) and (2.81). Let us consider now how the vector of geodesic deviation $\eta$ may be expressed via the fundamental solutions $\varphi^{\pm}$ of the Zakharov-Shabat problem (2.46). We will consider here two particular cases of the system (2.63)-(2.64).

2.2.1 $\psi = 0$

In this case, the coefficients (2.39) of the metrics (1.3) are

$$A = -\left(e^{2\mu^2} + q_2^2\right), \quad B = q_2, \quad C = -1.$$  

(2.83)

And the system (2.63)-(2.64) is reduced to the form

$$\mu_{2,11} + \mu_{2,1}^2 + \frac{3}{4}e^{-2\mu_2}q_{2,1}^2 = 0,$$  

(2.84)

$$\frac{1}{2}e^{-\mu_2}q_{2,11} - \frac{1}{2}e^{-\mu_2}\mu_{2,1}q_{2,1} = u_{,1}.$$  

(2.85)

The latter equation, obviously, may be written as

$$\frac{1}{2}\left(e^{-\mu_2}q_{2,1}\right)_{,1} = u_{,1},$$

whence

$$u = \frac{1}{2}e^{-\mu_2}q_{2,1},$$  

(2.86)

On the other hand, in virtue of (2.82), we have

$$\frac{1}{2}e^{-\mu_2}q_{2,1} = \pm 2\lambda \mathrm{sech}(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta).$$  

(2.87)
Using the well-known relation $\sinh 2A = 2 \cosh A \sinh A$, we can write (choosing the upper sign) (2.87) in the form

$$e^{-\mu^2} q_{2,1} = \frac{8\lambda \sinh(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta)}{\sinh(4\lambda x - 16\lambda^3 y - 16\lambda^3 z + 2\delta)}. \quad (2.88)$$

Whence, supposing $e^{-\mu^2} = 1$, we obtain

$$q_{2,1} = 8\lambda \sinh(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta), \quad (2.89)$$

or

$$q_{2,1} = -4\lambda \cosh(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta). \quad (2.90)$$

in the case $m_1(0) > 0$.

Thus, using (2.50)-(2.51) and (2.59), we obtain that solutions of the equations of geodesic deviation (1.4) in the case of the metrics (2.83) are expressed via the fundamental solutions of the Zakharov-Shabat problem as follows

$$\eta^2 \sim \varphi^-_1(x, \lambda) = e^{-i\lambda x} + \frac{1}{2} \int_{-\infty}^{x} dx' \int u^2 dx e^{-i\lambda x'}, \quad (2.93)$$

$$\eta^3 \sim \varphi^-_2(x, \lambda) = -\frac{1}{2} \int_{-\infty}^{x} dx' ue^{-i\lambda x'}, \quad (2.94)$$

or

$$\eta^2 \sim \varphi^+_1(x, \lambda) = -\frac{1}{2} \int_{-\infty}^{x} dx' ue^{i\lambda x'}, \quad (2.95)$$

$$\eta^3 \sim \varphi^+_2(x, \lambda) = e^{i\lambda x} - \frac{1}{2} \int_{-\infty}^{x} dx' \int u^2 dx e^{i\lambda x}, \quad (2.96)$$

where

$$u = \frac{1}{2} e^{-\mu^2} q_{2,1}, \quad (2.97)$$

and the functions $\mu^2$ and $q_2$ are related by the equation

$$\mu^2_{2,11} + \mu^2_{2,1} + \frac{1}{4} e^{-2\mu^2} q_{2,1}^2 = 0. \quad (2.98)$$

For example, in the case of the parameter dependence on $y$ and $z$ described by the mKdV equations (2.70) and (2.81) at $N = 1$ (one-soliton solution) we have the following integral representations

$$\eta^2 \sim e^{-i\lambda x} + \lambda \int_{-\infty}^{x} dx \tanh(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta)e^{-i\lambda x}, \quad (2.99)$$

$$\eta^3 \sim +\lambda \int_{-\infty}^{x} dx \sech(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta)e^{-i\lambda x}. \quad (2.99)$$
\( \eta^2 \sim \mp \lambda \int \frac{dx}{x} \text{sech}(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta) e^{i\lambda x}, \)  
\( \eta^3 \sim e^{i\lambda x} - \lambda \int \frac{dx}{x} \text{tanh}(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta) e^{i\lambda x}, \)

where for \( m_1(0) < 0 \) we take the upper sign and the lower sign for \( m_1(0) > 0. \) The constraint (2.98) gives (here the functions \( \mu^2 \) and \( q^2 \) are defined by (2.89)-(2.90) or (2.91)-(2.92)):

\[
\cosh^2(4\lambda x - 16\lambda^3 y - 16\lambda^3 z + 2\delta) + 3 \sinh^2(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta) = 1.
\]

### 2.2.2 \( \mu_2 = 0 \)

In this case, the coefficients (2.39) of the metrics (1.3) are

\[
A(x, y, z) = -(1 + q_2^2 e^{2\psi}), \quad B(x, y, z) = q_2 e^{2\psi}, \quad C(x, y, z) = -e^{2\psi}.
\]

The system (2.63)-(2.64) is reduced to the form

\[
\psi_{,11} + \psi_{,1} - \frac{1}{2} q_{2,1}(e^{2\psi} + \frac{1}{2}) = 0,
\]

\[
\frac{1}{2} e^{\psi} q_{2,1} + \frac{3}{2} e^{\psi} \psi_{,1} q_{2,1} = u_{,1}.
\]

Using the substitution \( \theta = \frac{\psi}{3} \), we obtain from the latter equation

\[
\frac{1}{2} \left(e^{\theta/3} q_{2,1}\right)_{,1} = u_{,1}.
\]

Therefore, in this case the vector of geodesic deviation \( \eta \) is also expressed via the fundamental solutions \( \varphi^\pm(x, \lambda) \) of the form (2.93)-(2.94) or (2.95)-(2.96). At this point,

\[
u = \frac{1}{2} e^{\theta/3} q_{2,1}
\]

and the functions \( \theta \) and \( q_2 \) are related by the equation

\[
\theta_{,11} + \frac{1}{3} \theta_{,1}^2 - \frac{3}{2} q_{2,1}(e^{2/3} + \frac{1}{2}) = 0.
\]

In the case of the one-soliton solution, a potential \( u \) is defined by the expression (2.82), and the functions \( \theta \) and \( q_2 \) are respectively equal to

\[
\theta = 3 \ln \cosh(4\lambda x - 16\lambda^3 y - 16\lambda^3 z + 2\delta),
\]

\[
q_2 = \pm \cosh(2\lambda x - 8\lambda^3 y - 8\lambda^3 z + \delta).
\]

More complicated case \( \mu_2 \neq 0, \psi \neq 0 \) and also multi-soliton solutions will be considered in a separate paper.

## 3 Four-dimensional space

### 3.1 3 × 3 matrix Schrödinger equation

Let us consider the following linear problem

\[
\psi_{,4} = i\zeta D \psi + N \psi,
\]
where $\psi_4 = \frac{\partial}{\partial x^4} \psi$, $x^4 = it$; $\zeta$ is a spectral parameter and $\psi$ is a $3 \times 1$ matrix (vector) of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$  

The $3 \times 3$ matrices $D$ and $N$ (a potential matrix) are

$$D = \begin{pmatrix} \pm d_1 & 0 & 0 \\ 0 & \pm d_2 & 0 \\ 0 & 0 & \pm d_3 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & N_{12} & N_{13} \\ N_{21} & 0 & N_{23} \\ N_{31} & N_{32} & 0 \end{pmatrix}.$$  

The system (3.1) may be rewritten (see Appendix) in the following form (3 $\times$ 3 matrix Schrödinger equation)

$$-i \psi_{44} + \Re \psi = d^2 \zeta^2 \psi, \quad (3.2)$$

where $I$ is a $3 \times 3$ unit matrix, $d = (d_1, d_2, d_3)$ and

$$\Re = \begin{pmatrix} N_{12}N_{21} + N_{13}N_{31} & N_{12,4} + N_{13}N_{32} & N_{13,4} + N_{12}N_{23} \\ N_{21,4} + N_{23}N_{31} & N_{21,N_{21}} + N_{23}N_{32} & N_{23,4} + N_{21}N_{13} \\ N_{31,4} + N_{32}N_{21} & N_{32,4} + N_{31}N_{12} & N_{31}N_{13} + N_{32}N_{23} \end{pmatrix}.$$  

Further, it is easy to see that for the metrics (1.12) the geodesic deviation equation (1.10),

$$\eta_{44}^1 + R_{4144}^1 \eta^1 + R_{4244}^1 \eta^2 + R_{4344}^1 \eta^3 = 0,$$

$$\eta_{44}^2 + R_{4144}^2 \eta^1 + R_{4244}^2 \eta^2 + R_{4344}^2 \eta^3 = 0,$$

$$\eta_{44}^3 + R_{4144}^3 \eta^1 + R_{4244}^3 \eta^2 + R_{4344}^3 \eta^3 = 0,$$  

can be rewritten in the form of the $3 \times 3$ matrix Schrödinger operator

$$-\eta_{44}^1 + (-R_{4144}^1 + d_1^2 \zeta^2) \eta^1 + (-R_{4244}^1) \eta^2 + (-R_{4344}^1) \eta^3 = d_1^2 \zeta^2 \eta^1,$$

$$-\eta_{44}^2 + (-R_{4144}^2) \eta^1 + (-R_{4244}^2 + d_2^2 \zeta^2) \eta^2 + (-R_{4344}^2) \eta^3 = d_2^2 \zeta^2 \eta^2,$$

$$-\eta_{44}^3 + (-R_{4144}^3) \eta^1 + (-R_{4244}^3) \eta^2 + (-R_{4344}^3 + d_3^2 \zeta^2) \eta^3 = d_3^2 \zeta^2 \eta^3.$$  

Comparing these equations with equations (3.2), we obtain the following conditions on the curvature tensor:

$$d_1^2 \zeta^2 - R_{4144}^1 = N_{12}N_{21} + N_{13}N_{31},$$

$$d_2^2 \zeta^2 - R_{4244}^2 = N_{21}N_{12} + N_{23}N_{32},$$

$$d_3^2 \zeta^2 - R_{4344}^3 = N_{31}N_{13} + N_{32}N_{23},$$

$$-R_{4424}^1 = N_{12,4} + N_{13}N_{32},$$

$$-R_{4434}^1 = N_{13,4} + N_{12}N_{23},$$

$$-R_{4414}^2 = N_{21,4} + N_{23}N_{31},$$

$$-R_{4423}^2 = N_{23,4} + N_{21}N_{13},$$

$$-R_{4414}^3 = N_{31,4} + N_{32}N_{21},$$

$$-R_{4414}^3 = N_{31,4} + N_{32}N_{21}. \quad (3.5)$$  

### 3.2 Chandrasekhar metrics

In the 4-dimensional space with a signature (−, −, −, −) the Chandrasekhar metrics is defined by the following expression [8]

$$ds^2 = - \sum_A e^{2\mu_A} (dx^A)^2 - e^{2\psi} (dx^1 - \sum_A q_A dx^A)^2, \quad (3.6)$$

15
where $A = 2, 3, 4$, $\psi$, $\mu_A$, and $q_A$ are the functions on variables $x^1, x^2, x^3, x^4$.

The orthonormal tetrad, related with the metrics (3.6), is defined by the following covariant basis vectors:

\[
\begin{align*}
    e_{(4)i} &= (-e^{\nu_4}, 0, 0, 0), \\
    e_{(1)i} &= (q_4 e^{\nu_4}, -e^{\nu_4}, q_2 e^{\nu_4}, q_3 e^{\nu_4}), \\
    e_{(2)i} &= (0, 0, -e^{\mu_2}, 0), \\
    e_{(3)i} &= (0, 0, 0, -e^{\mu_3}).
\end{align*}
\]  

(3.7)

And also the contravariant basis vectors are

\[
\begin{align*}
    e^i_{(4)} &= (e^{\mu_4}, q_4 e^{-\mu_4}, 0, 0), \\
    e^i_{(1)} &= (0, e^{-\psi}, 0, 0), \\
    e^i_{(2)} &= (0, q_2 e^{-\mu_2}, e^{-\mu_2}, 0), \\
    e^i_{(3)} &= (0, q_3 e^{-\mu_3}, 0, e^{-\mu_3}).
\end{align*}
\]  

(3.8)

From (3.7) and (3.8) it is easy to see that

\[
e^{i(\alpha)}_{}e^{(b)i} = \eta_{(\alpha)(b)} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}.
\]

Therefore, in this orthonormal basis for the components of the curvature tensor we have always

\[
R^m_{klm} = -R^m_{klm}.
\]  

(3.9)

Moreover, at $d\xi = -i\xi^2 dx^4$, $\nu = \mu_4$ and $\omega = \varphi_4$ there exists an analytic continuation of the basis (3.7) - (3.8) with the signature $(-, -, -)$ onto a basis with a signature $(-, -, -, +)$, the covariant and contravariant vectors of which have the form

\[
\begin{align*}
    e_{(1)i} &= (\varphi_4 e^{\psi}, -e^{\nu_4}, q_2 e^{\psi}, q_3 e^{\psi}), \\
    e_{(2)i} &= (0, 0, 0, -e^{\mu_2}), \\
    e_{(3)i} &= (0, 0, 0, -e^{\mu_3}), \\
    e_{(4)i} &= (e^{\nu_4}, 0, 0, 0),
\end{align*}
\]  

(3.10)

It is obvious that in the orthonormal basis (3.7) - (3.8) among the nine components of the curvature tensor of the system (3.6) only six are independent, namely,

\[
R^1_{414}, R^2_{424}, R^3_{434}, R^1_{424}, R^1_{434}, R^2_{434}.
\]

In the orthonormal basis (3.7) - (3.8) for the metrics (3.6) these components have the form

\[
\begin{align*}
    - R_{1414} &= -e^{-\psi_4-\mu_4}D_4\big(e^{\psi_4-\mu_4}\Psi_4\big) - e^{-2\mu_2}\Psi_2\mu_4;2 - e^{-2\mu_3}\Psi_3\mu_4;3 - \\
    &\quad - e^{-\psi_4-\mu_4}\big(e^{\mu_4-\psi_4}\mu_4;1\big)_1 + \frac{1}{4} e^{2\psi_4-2\mu_4}\left[e^{-2\mu_2}Q^2_{24} + e^{-2\mu_3}Q^2_{34}\right],
\end{align*}
\]  

(3.11)

\[
\begin{align*}
    - R_{2424} &= -e^{-\mu_2-\mu_4}\left[(e^{\mu_2-\mu_4}\mu_4;2)_2 + (e^{\mu_2-\mu_4}\mu_4;2)_2\right] - \\
    &\quad - e^{-2\mu_3}\mu_4\mu_4;2_3 - \frac{3}{4} e^{2\psi_2-2\mu_2-2\mu_4}Q^2_{24} - e^{-2\psi_2}\mu_2\mu_4;1,
\end{align*}
\]  

(3.12)

\[
\begin{align*}
    - R_{3434} &= -e^{-\mu_3-\mu_4}\left[(e^{\mu_3-\mu_4}\mu_4;3)_3 + (e^{\mu_3-\mu_4}\mu_4;3)_3\right] - \\
    &\quad - e^{-2\mu_2}\mu_4\mu_4;2_3 - \frac{3}{4} e^{2\psi_2-2\mu_3-2\mu_4}Q^2_{34} - e^{-2\psi_2}\mu_3\mu_4;1,
\end{align*}
\]  

(3.13)
\[ R_{1424} = e^{\psi - 2\mu_4 - \mu_2}Q_{24} \left( \Psi_4 - \frac{1}{2}e^{\mu_2} \right) + \frac{1}{2}e^{\mu_4 - \mu_2} \left( e^{\psi - \mu_4}Q_{24} \right) + \]
\[ + \frac{1}{2}e^{\psi - \mu_3}Q_{32} \mu_4 + e^{-\mu_4 - \mu_2} \left( e^{-\psi + \mu_4} \mu_4 \right) + e^{-\psi - \mu_2} \mu_2, \quad (3.14) \]

\[ R_{1434} = e^{\psi - 2\mu_4 - \mu_3}Q_{34} \left( \Psi_4 - \frac{1}{2}e^{\mu_3} \right) + \frac{1}{2}e^{\mu_4 - \mu_3} \left( e^{\psi - \mu_4}Q_{34} \right) + \]
\[ + \frac{1}{2}e^{\psi - \mu_2 - \mu_3}Q_{23} \mu_4 + e^{-\mu_4 - \mu_3} \left( e^{-\psi + \mu_4} \mu_4 \right) + e^{-\psi - \mu_3} \mu_3, \quad (3.15) \]

\[ R_{2434} = e^{-\mu_2 - \mu_3} \left[ \mu_4 q_4 - \mu_4 q_4 - \mu_4 q_4 \right] - \]
\[ - \frac{3}{4}e^{2\psi - \mu_3 - 2\mu_4 - \mu_2}Q_{34}Q_{42} - \frac{1}{2}e^{-\mu_3 - \mu_2}Q_{32} \mu_4, \quad (3.16) \]

where

\[ D_A f = f, A + (q_A f),_1, \]
\[ \Psi_A = \psi, A + q_A, \]
\[ Q_{AB} = q_{A:B} - q_{B:A}, \]
\[ f_A = f, A + q_A f, _1. \]

### 3.3 Solutions of equations of geodesic deviation in the four-dimensional space

It is easy to see that the Chandrasekhar metrics coincide with the metrics at \( \mu_4 = q_4 = 0 \). In this case, the orthonormal basis is reduced to the form

\[
\begin{align*}
  c_{(1)i} & = (0, -e^\psi, q_2 e^\psi, q_3 e^\psi), \\
  c_{(2)i} & = (0, 0, -e^{\mu_2}, 0), \\
  c_{(3)i} & = (0, 0, 0, -e^{\mu_3}), \\
  c_{(4)i} & = (-1, 0, 0, 0); \\
  e^{(i)} & = (0, e^\psi, 0, 0), \\
  e^{(2)} & = (0, q_2 e^{-\mu_2}, e^{-\mu_2}, 0), \\
  e^{(3)} & = (0, q_3 e^{-\mu_3}, 0, 0), \\
  e^{(4)} & = (1, 0, 0, 0).
\end{align*}
\]

It is obvious that in this basis we have \( R_{jk} = -R_{ijk l} \), and the components of the curvature tensor are

\[ -R_{1414} = -\psi,_{44} - \psi^2 + \frac{1}{4}e^{2\psi} \left[ e^{-2\mu_2} q_{2,4}^2 + e^{-2\mu_3} q_{3,4}^2 \right], \quad (3.17) \]
\[ -R_{2424} = -\mu_2,_{24} - \mu_2^2 - 3 e^{2\psi - 2\mu_2} q_{2,4}^2, \quad (3.18) \]
\[ -R_{3434} = -\mu_3,_{34} - \mu_3^2 - 3 e^{2\psi - 2\mu_3} q_{3,4}^2, \quad (3.19) \]
\[ R_{1424} = -\frac{1}{2}e^{\psi - \mu_2} \left[ q_{2,44} - \mu_2 q_{2,4} + 3 \psi,_{4} q_{2,4} \right], \quad (3.20) \]
\[ R_{1434} = -\frac{1}{2}e^{\psi - \mu_3} \left[ q_{3,44} - \mu_3 q_{3,4} + 3 \psi,_{4} q_{3,4} \right], \quad (3.21) \]
\[ R_{2434} = \frac{3}{4}e^{2\psi - \mu_2 - \mu_3} q_{2,4} q_{3,4}. \quad (3.22) \]

Further, let us define now the evolution equations related with the problem. We consider the following system

\[ \psi,_{4} = i\zeta D\psi + N\psi, \]
\[ \psi,_{1} = Q\psi, \]
where $\psi_1 = \frac{D}{\partial x^1}$, $x^1$ is a parameter of the considered problem. $D$, $N$, $Q$ be the $3 \times 3$ matrices. At this point, $D$ is diagonal: $D = d_i \delta_{ij}$, $d_i = \text{const}$; $N$ is such a matrix that $N_{ii} = 0$. From the compatibility condition $\psi_{14} = \psi_{41}$ and the requirement $\zeta_1 = 0$ we obtain

$$Q_4 = N_{11} + i\zeta [D, Q] + [N, Q].$$

Decomposing $Q$ in the form

$$Q = Q^{(1)} + Q^{(0)},$$

we have $Q_4^{(0)} = N_{11} + [N, Q^{(0)}]$, whence we obtain the system of $n(n - 1)$ equations (see [6]):

$$N_{ij,1} - a_{ij} N_{ij,4} = \sum_k (a_{ik} - a_{kj}) N_{ik} N_{kj},$$

where

$$a_{ij} = \frac{1}{d_i - d_j} q_j = a_{ji}.$$

Equations (3.23) may be reduced to the standard system of nonlinear equations of three-wave interaction. Namely, we obtain

$$Q_{1,1} + C_1 Q_{1,4} = i\gamma_1 Q_2^* Q_3^*,$$
$$Q_{2,1} + C_2 Q_{2,4} = i\gamma_2 Q_1^* Q_3^*,$$
$$Q_{3,1} + C_3 Q_{3,4} = i\gamma_3 Q_1^* Q_2^*,$$

where $\gamma_1 \gamma_2 \gamma_3 = -1$, $\gamma_i = \pm 1$ and

$$N_{12} = -i Q_3 / \sqrt{\beta_1 \beta_3}, \quad N_{21} = -i Q_2 / \sqrt{\beta_1 \beta_3},$$
$$N_{23} = +i Q_1 / \sqrt{\beta_1 \beta_3}, \quad N_{13} = -\gamma_1 \gamma_3 N_{31}^*,$$
$$N_{32} = \gamma_2 \gamma_3 N_{23}^*, \quad N_{21} = \gamma_1 \gamma_2 N_{12}^*,$$

here

$$q_j = -i \frac{C_1 C_2 C_3}{C_j}, \quad \beta_{ij} = d_i - d_j = C_j - C_i,$$
$$C_3 > C_2 > C_1.$$

In the system (3.24) there is a decay instability (for the waves with positively defined energy) if the sign of one $\gamma_n$ is different from the other, and also there is an explosive instability when $\gamma_1 = \gamma_2 = \gamma_3 = -1$. Solutions of the system (3.24) was obtained by Zakharov and Manakov in 1973 [10] [11] [12]. They have the form

$$Q_1 = \sqrt{\beta_1 \beta_{13}} \frac{2 \chi_3}{\mathcal{D}} e^{i \xi_3 (x^4 - C_1 x^1 - \xi_3)} e^{\chi_1 (x^4 - C_3 x^1 - \xi_3)},$$

$$Q_2 = -\frac{4 \chi_1 \chi_3 \beta_{13}^2 \gamma_3}{\sqrt{\beta_1 \beta_{13} (\xi_1 - \xi_3)}} e^{-i \xi_3 (x^4 - C_3 x^1 - \xi_3)} e^{-i \xi_3 (x^4 - C_1 x^1 - \xi_3)},$$

$$Q_3 = \sqrt{\beta_1 \beta_{13} \gamma_2 \gamma_3} \frac{2 \chi_1}{\mathcal{D}} e^{i \xi_3 (x^4 - C_3 x^1 - \xi_3)} e^{\chi_3 (x^4 - C_1 x^1 - \xi_3)},$$

$$Q_4 = -\frac{4 \chi_1 \chi_3 \beta_{13}^2 \gamma_3}{\sqrt{\beta_1 \beta_{13} (\xi_1 - \xi_3)}} e^{-i \xi_3 (x^4 - C_3 x^1 - \xi_3)} e^{-i \xi_3 (x^4 - C_1 x^1 - \xi_3)},$$

$$Q_5 = \sqrt{\beta_1 \beta_{13} \gamma_2 \gamma_3} \frac{2 \chi_1}{\mathcal{D}} e^{i \xi_3 (x^4 - C_3 x^1 - \xi_3)} e^{\chi_3 (x^4 - C_1 x^1 - \xi_3)},$$

$$Q_6 = -\frac{4 \chi_1 \chi_3 \beta_{13}^2 \gamma_3}{\sqrt{\beta_1 \beta_{13} (\xi_1 - \xi_3)}} e^{-i \xi_3 (x^4 - C_3 x^1 - \xi_3)} e^{-i \xi_3 (x^4 - C_1 x^1 - \xi_3)}.$$
where
\[
\mathcal{D} = \left[ e^{\chi_1(x^4 - C_3 x^1 - \varphi_3)} - \gamma_1 \gamma_2 e^{\chi_1(x^4 - C_3 x^1 - \varphi_3)} \right] \times \\
\times \left[ e^{\chi_3(x^4 - C_1 x^1 - \varphi_1)} - \gamma_2 \gamma_3 e^{\chi_3(x^4 - C_1 x^1 - \varphi_1)} \right] + \\
+ \gamma_1 \gamma_3 \frac{\tilde{\zeta}_1 - \tilde{\zeta}_1}{(\zeta_1 - \zeta_3)(\zeta_1 - \zeta_3)} e^{\chi_1(x^4 - C_3 x^1 - \varphi_3)} e^{\chi_3(x^4 - C_1 x^1 - \varphi_1)}, \quad (3.29)
\]

Supposing now that the matrix \( N \) is real (\( N^* = N \)) and choosing \( \gamma_1 = \gamma_3 = 1, \gamma_2 = -1 \), we obtain from (3.29)
\[
N_{12} = -\text{Re}(iQ_3/\sqrt{\beta_{12}/\beta_{23}}), \quad N_{31} = -\text{Re}(iQ_2/\sqrt{\beta_{12}/\beta_{23}}), \\
N_{23} = +\text{Re}(iQ_1/\sqrt{\beta_{12}/\beta_{23}}), \quad N_{13} = -N_{31}, \\
N_{32} = -N_{23}, \quad N_{21} = -N_{12}. \quad (3.30)
\]

It is obvious that the latter three conditions in (3.30) are equivalent to antisymmetry of the matrix \( N \).

Thus, we assume that the potential matrix \( N \) is real and antisymmetric. Taking it into account and also the expressions (3.31)-(3.32), we obtain from the conditions on the curvature tensor (3.3) the following system of differential equations:
\[
\begin{align*}
-\psi_{,4} - \psi_{,4}^2 + \frac{1}{4} e^{2\psi} \left[ e^{-2\mu_2} q_{2,4}^2 + e^{-2\mu_3} q_{3,4}^2 \right] &= N_{12}^2 + N_{13}^2 + d_1^2 \zeta^2, \quad (3.31) \\
-\mu_{2,44} - \mu_{2,4}^2 - \frac{3}{4} e^{2\psi - 2\mu_2} q_{2,4}^2 &= N_{12}^2 + N_{23}^2 + d_2^2 \zeta^2, \quad (3.32) \\
-\mu_{3,44} - \mu_{3,4}^2 - \frac{3}{4} e^{2\psi - 2\mu_3} q_{3,4}^2 &= N_{13}^2 + N_{23}^2 + d_3^2 \zeta^2, \quad (3.33) \\
\frac{1}{2} e^{\psi - \mu_2} [q_{2,4} - \mu_{2,4} q_{2,4} + 3\psi_{,4} q_{2,4}] &= N_{23} N_{31}, \quad (3.34) \\
\frac{1}{2} e^{\psi - \mu_3} [q_{3,4} - \mu_{3,4} q_{3,4} + 3\psi_{,4} q_{3,4}] &= N_{12} N_{23}, \quad (3.35) \\
\frac{3}{4} e^{2\psi - \mu_2 - \mu_3} q_{2,4} q_{3,4} &= N_{31} N_{12}. \quad (3.36)
\end{align*}
\]

Obviously, this system has a great number of particular cases. For example, let us consider one simplest case.

### 3.3.1 \( q_3 = \text{const}, \, \psi = \mu_2 = 0 \)

In this case for the metrics (1.4) we have
\[
\begin{align*}
g_{11} &= -1, \quad g_{22} = -(1 + g_2^2), \quad g_{33} = -(e^{2\mu_3} + \text{const}^2), \\
g_{12} &= 2q_2, \quad g_{13} = 2 \text{const}, \quad g_{23} = -2 \text{const} q_2.
\end{align*}
\]
The system (3.31)-(3.36) is reduced to the form
\[
q_{2,4}^2 = N_{12}^2 + N_{13}^2 + d_2^4 \zeta^2, \quad (3.37)
\]
\[
-\frac{3}{4} q_{2,4}^2 = N_{12}^2 + N_{23}^2 + d_2^4 \zeta^2, \quad (3.38)
\]
\[-\mu_{3,44} - \mu_{3,4}^2 = N_{13}^2 + N_{23}^2 + d_3^4 \zeta^2, \quad (3.39)\]
\[-\frac{1}{2} q_{2,44} = N_{23}N_{31}, \quad (3.40)\]
\[N_{12}N_{23} = 0, \quad (3.41)\]
\[N_{31}N_{12} = 0. \quad (3.42)\]

From the latter two equations it follows that \(N_{12} = 0\). Therefore, in this case, the potential matrix \(N\) has a form
\[
\begin{pmatrix}
0 & 0 & -N_{31} \\
0 & 0 & N_{23} \\
N_{31} & -N_{23} & 0
\end{pmatrix}.
\]

Further, at \(d_1^2 = -4/3d_2^2\) from (3.37)-(3.42) it follows that
\[
-\frac{1}{2} q_{2,44} = N_{23}N_{31}, \quad (3.43)
\]
\[-\mu_{3,44} - \mu_{3,4}^2 = -\frac{1}{3} N_{23}^2 + d_3^2 \zeta^2, \quad (3.44)\]
\[N_{13}^2 + \frac{4}{3} N_{23}^2 = 0. \quad (3.45)\]

In accordance with (3.30), the components \(N_{23}\) and \(N_{31}\) are defined as
\[
N_{23} = \frac{2\chi_3}{\mathcal{D}} \left[ \frac{2(\beta_{23}\chi_1 - \beta_{12}\chi_3)}{\beta_{23}\xi_1 - \beta_{12}\xi_3)^2} \right. \\
\times \cos \xi_3 \left( x^4 - C_1x^1 - \bar{\varphi}_1 \right) e^{-\chi_1(x^4 - C_3x^1 - \varphi_3)} - \\
- 2 \sin \xi_3 \left( x^4 - C_1x^1 - \bar{\varphi}_1 \right) \cosh \chi_1 \left( x^4 - C_3x^1 - \varphi_3 \right) \right]. \quad (3.46)
\]

\[
N_{31} = -\frac{4\chi_1\chi_3\beta_{13}(\beta_{23}\xi_1 - \beta_{12}\xi_3)}{\beta_{23}\xi_1 - \beta_{12}\xi_3)^2} \times \\
\left[ \sin \xi_1 \left( x^4 - C_3x^1 - \bar{\varphi}_3 \right) \cos \xi_3 \left( x^4 - C_1x^1 - \bar{\varphi}_1 \right) + \\
+ \cos \xi_1 \left( x^4 - C_3x^1 - \bar{\varphi}_3 \right) \sin \xi_3 \left( x^4 - C_1x^1 - \bar{\varphi}_1 \right) \right], \quad (3.47)
\]

where
\[
\mathcal{D} = 4 \cosh \chi_1 \left( x^4 - C_3x^1 - \varphi_3 \right) \cosh \chi_3 \left( x^4 - C_1x^1 - \varphi_1 \right) + \\
+ \frac{4\chi_1\chi_3}{\beta_{12}^2(\xi_1^2 + \chi_3^2) + 2\beta_{12}\beta_{23}(\xi_1\chi_3 + \chi_1\chi_3) + \beta_{23}^2(\xi_1^2 + \chi_1^2)} \times \\
\times e^{-\chi_1(x^4 - C_3x^1 - \varphi_3)} e^{-\chi_3(x^4 - C_1x^1 - \varphi_1)}. \quad (3.48)
\]

After very cumbersome but elementary calculations it is easy to verify that solutions of the system (3.31)-(3.36) exist.
Appendix

Let us consider the system \((3.1)\) with the matrix

\[
D = \begin{pmatrix}
-d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & -d_3
\end{pmatrix}.
\]

Differentiating \((3.1)\) and excluding the first derivatives \(\psi_4\), we obtain the following system

\[
- \psi_{1,44} + (N_{12}N_{21} + N_{13}N_{31})\psi_1 + (N_{12,4} + N_{13}N_{32} - i\zeta d_1 N_{12} + i\zeta d_2 N_{12})\psi_2 +
\]

\[
+ (N_{13,4} + N_{12}N_{23} - i\zeta d_1 N_{13} - i\zeta d_3 N_{13})\psi_3 = \zeta^2 d_1^2 \psi_1, \quad (A.1)
\]

\[
- \psi_{2,44} + (N_{21,4} + N_{23}N_{31} + i\zeta d_2 N_{21} - i\zeta d_1 N_{21})\psi_1 + (N_{21}N_{12} + N_{23}N_{32})\psi_2 +
\]

\[
+ (N_{23,4} + N_{21}N_{13} + i\zeta d_2 N_{23} - i\zeta d_3 N_{23})\psi_3 = \zeta^2 d_2^2 \psi_2, \quad (A.2)
\]

\[
- \psi_{3,44} + (N_{31,4} + N_{32}N_{21} - i\zeta d_3 N_{31} - i\zeta d_1 N_{31})\psi_1 + (N_{32,4} + N_{31}N_{12} - i\zeta d_3 N_{32} +
\]

\[
+ i\zeta d_2 N_{32})\psi_2 + (N_{31}N_{13} + N_{32}N_{23})\psi_3 = \zeta^2 d_3^2 \psi_3. \quad (A.3)
\]

Analogously, for the matrix

\[
D = \begin{pmatrix}
d_1 & 0 & 0 \\
0 & -d_2 & 0 \\
0 & 0 & d_3
\end{pmatrix}
\]

we have the system

\[
- \psi_{1,44} + (N_{12}N_{21} + N_{13}N_{31})\psi_1 + (N_{12,4} + N_{13}N_{32} + i\zeta d_1 N_{12} - i\zeta d_2 N_{12})\psi_2 +
\]

\[
+ (N_{13,4} + N_{12}N_{23} + i\zeta d_1 N_{13} + i\zeta d_3 N_{13})\psi_3 = \zeta^2 d_1^2 \psi_1, \quad (A.4)
\]

\[
- \psi_{2,44} + (N_{21,4} + N_{23}N_{31} - i\zeta d_2 N_{21} + i\zeta d_1 N_{21})\psi_1 + (N_{21}N_{12} + N_{23}N_{32})\psi_2 +
\]

\[
+ (N_{23,4} + N_{21}N_{13} - i\zeta d_2 N_{23} + i\zeta d_3 N_{23})\psi_3 = \zeta^2 d_2^2 \psi_2, \quad (A.5)
\]

\[
- \psi_{3,44} + (N_{31,4} + N_{32}N_{21} + i\zeta d_3 N_{31} + i\zeta d_1 N_{31})\psi_1 + (N_{32,4} + N_{31}N_{12} + i\zeta d_3 N_{32} -
\]

\[
- i\zeta d_2 N_{32})\psi_2 + + (N_{31}N_{13} + N_{32}N_{23})\psi_3 = \zeta^2 d_3^2 \psi_3. \quad (A.6)
\]

Adding the systems \((A.1)-(A.3)\) and \((A.4)-(A.6)\), we obtain in the result

\[
- \psi_{1,44} + (N_{12}N_{21} + N_{13}N_{31})\psi_1 + (N_{12,4} + N_{13}N_{32})\psi_2 +
\]

\[
+ (N_{13,4} + N_{12}N_{23})\psi_3 = \zeta^2 d_1^2 \psi_1, \quad (A.7)
\]

\[
- \psi_{2,44} + (N_{21,4} + N_{23}N_{31})\psi_1 + (N_{21}N_{12} + N_{23}N_{32})\psi_2 +
\]

\[
+ (N_{23,4} + N_{21}N_{13})\psi_3 = \zeta^2 d_2^2 \psi_2, \quad (A.8)
\]

\[
- \psi_{3,44} + (N_{31,4} + N_{32}N_{21})\psi_1 + (N_{32,4} + N_{31}N_{12})\psi_2 +
\]

\[
+ (N_{31}N_{13} + N_{32}N_{23})\psi_3 = \zeta^2 d_3^2 \psi_3. \quad (A.9)
\]

It is easy to see that the latter system can be rewritten in the form of \(3 \times 3\) matrix Schrödinger equation \((3.2)\).
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