FUNDAMENTAL REPRESENTATIONS OF QUANTUM AFFINE
SUPERALGEBRAS AND $R$–MATRICES

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Abstract. We study a certain family of finite-dimensional simple representations over
quantum affine superalgebras associated to general linear Lie superalgebras, the so-called
fundamental representations: the denominators of rational $R$–matrices between two fun-
damental representations are computed; a cyclicity (and so simplicity) condition on tensor
products of fundamental representations is proved.

1. Introduction

Fix $M, N$ two natural numbers and $q$ a non-zero complex number which is not a root of
unity. Let $\mathfrak{g} := \mathfrak{gl}(M,N)$ be the general linear Lie superalgebra. Let $U_q(\mathfrak{g})$ be the associated
quantum affine superalgebra. This is a Hopf superalgebra neither commutative nor co-
commutative, and it can be seen as a deformation of the universal enveloping algebra of the
affine Lie superalgebra $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. In this paper we are mainly concerned with the
structure of tensor products of finite-dimensional simple $U_q(\mathfrak{g})$-modules.

Quantum affine superalgebras, as supersymmetric generalization of quantum affine al-
gebras, were defined previously by Yamane [Ya] with Drinfeld–Jimbo generators (and with
Drinfeld loop generators for $U_q(\mathfrak{g})$). They appeared as the algebraic supersymmetries of
solvable models such as the $q$–state vertex model [PS] and the $t$–$J$ models [Ko]; their
highest weight representations were identified in these models with the spaces of states to
calculate correlation functions. Recently, various quantum superalgebras (finite type, affine
type, Yangian) together with their finite-dimensional representations associated to the sim-
ple Lie superalgebra $psl(2,2)$ are linked to the integrability structures in the context of the
AdS/CFT correspondence and in Hubbard model (see [BGM] and its references). Quantum
affine superalgebra associated to the exceptional Lie superalgebra $D(2,1;x)$ is related to
generalized hypergeometric equations [BL].

Compared to the rich literature on quantum affine algebras (see the review papers [CH, Le]), quantum affine superalgebras have been less studied. Technical difficulties already arise
in the situation of finite-dimensional simple Lie superalgebras: all the Borel subalgebras are
not conjugated, Weyl groups are not enough to characterize linkage, etc.

The series of papers [Zh1, Zh2, Zh3] studied systematically finite-dimensional repre-
sentations of $U_q(\mathfrak{g})$. In [Zh1], there is a similar highest $\ell$-weight classification [CP2] of
finite-dimensional simple modules adapted to the Drinfeld new realization of $U_q(\mathfrak{g})$. Our
motivating questions are as follows. Let $S_1, S_2, \cdots, S_n$ be such $U_q(\mathfrak{g})$-modules.

(I) Construct $U_q(\mathfrak{g})$-module morphisms from $S_1 \otimes S_2$ to $S_2 \otimes S_1$.

(II) Determine when $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ is a highest $\ell$-weight module.
In the non-graded case, (I) and (II) are related to each other by the notion of normalized $R$-matrix $R_{S_1,S_2}$ proposed in [AK]. This is a matrix-valued rational function depending on the ratio $\frac{a}{b}$ of spectral parameters $a, b \in \mathbb{C}^\times$ of $S_1$ and $S_2$ respectively. Whenever it is well-defined (in other words the denominator of $R_{S_1,S_2}$ is non-zero when specialized to $S_1,S_2$), $R_{S_1,S_2}$ composed with the flip map is a module morphism from $S_1 \otimes S_2$ to $S_2 \otimes S_1$. It was proved in [Ka] (first conjectured in [AK]) that the tensor product in (II) is of highest $\ell$-weight if $R_{S_i,S_j}$ is well-defined for all $i < j$, under the assumption that the $S_i$ are good modules within the framework of crystal base theory. Similar results were obtained by Varagnolo–Vasserot [VV] for fundamental modules over simply-laced quantum affine algebras via Nakajima quiver varieties, and by Chari [Ch] in general situations via the braid group action on affine Cartan subalgebras. Here fundamental modules are certain simple modules whose highest $\ell$-weights are of particular forms [CI] Definition 3.4).

Quite recently, normalized $R$-matrices were used to establish generalized Schur–Weyl duality between representations of quantum affine algebras and those of quiver Hecke algebras and monoidal categorifications of (quantum) cluster algebras [HL, KKK, KKKO]. We refer to the table in [Oh, Appendix A] for a summary of poles with multiplicity of normalized $R$-matrices between two fundamental modules over quantum affine algebras. We mention earlier works of Chari–Pressley [CP1] on zeros and poles of $R$-matrices for Yangians.

In this paper, we study (I) and (II) for fundamental modules over $U_q(\hat{\mathfrak{g}})$. The fundamental modules $V_{r,a}^\Sigma$ over $U_q(\hat{\mathfrak{g}})$ are defined by a fusion procedure (Definition 2.4). They depend on a spectral parameter $a \in \mathbb{C}^\times$ and a Dynkin node together with signature $(r, \varepsilon)$: positive if $(\varepsilon = +, 1 \leq r \leq M)$ and negative if $(\varepsilon = -, 1 \leq r \leq N)$. When $N = 0$ they are the fundamental modules in the non-graded case [DO]. The main results of this paper are:

(A) denominators of $R$-matrices between two fundamental modules (Theorems 4.1, 4.2);
(B) a sufficient condition for a tensor product of fundamental modules $\otimes_{i=1}^n V_{r_i,a_i}^\varepsilon_i$ to be of highest $\ell$-weight when $(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n) = (+ + \cdots + - \cdots -)$ (Theorem 6.1).

In §8 we indicate a general idea to study tensor products of arbitrary signatures. Eventually it is enough to solve a problem of linear algebra (Question 11). (B) has the following two consequences. Let $S_1, S_2, \ldots, S_n$ be fundamental modules.

(C) $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ is simple if and only if so is $S_i \otimes S_j$ for all $i < j$ (Theorem 7.2).
(D) if the parities of the $S_i$ are the same, then $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ is simple if and only if it is of highest $\ell$-weight and of lowest $\ell$-weight (Corollary 8.4).

Let us make comparisons of (A)–(D) with related results in literature.

1. When restricted to the finite type quantum superalgebra $U_q(\mathfrak{g})$, a positive (resp. negative) fundamental module is in the category $\mathcal{O}$ (resp. its dual category $\mathcal{O}^*$) of [BKK], and their tensor products may not be semi-simple. In deducing (A) we make tricky use of a fact (Lemma 2.2) on the tensor product of a highest $\ell$-weight module and a lowest $\ell$-weight module, as opposed to the non-graded case [CP1, DO, KOS, Oh] where tensor product decompositions and spectral decompositions were usually needed.

Our arguments can be applied to the non-graded case. However it seems that even in the situation [DO] of quantum affine algebras of type A the calculations would become more involved than those in Theorems 4.1, 4.2. By the fusion procedure lowest $\ell$-weight vectors of fundamental modules are pure tensors in our situation, while they are alternating sums.
over symmetric groups in [DO]. The denominators in Theorems 4.1–4.2 are simpler than those in [DO, Equation (2.8)]. Notably, if $S_1$ and $S_2$ are fundamental modules of different signatures, then the denominator of $R_{S_1, S_2}$ is a polynomial of degree 1.

We expect similar simplification of denominators of $R_{S_1, S_2}$ for more general simple $U_q(\hat{\mathfrak{g}})$-modules $S_1$ in the category $\mathcal{O}$ and $S_2$ in the category $\mathcal{O}^*$. This might be related to the crystal base theory developed in [BKK] for $U_q(\hat{\mathfrak{g}})$-modules.

2. (B) can be viewed as a super version of cyclicity results in [Ch, Ka, VV]. As explained in the introduction of [Zh2], the methods in the non-graded case do not admit straightforward generalizations. Nevertheless a weaker result has been proved in [Zh2] under the assumption that in the tensor product of (B) the $(r_i, \varepsilon_i)$ must be the same. This weaker result has been used in [Zh3] to construct asymptotic modules in the sense of Hernandez–Jimbo [HJ], and it will again be needed in the present paper to validate the fusion procedure (in the proof of Proposition 2.5).

The idea of proof in [Zh2] is a modification of Chari’s reduction arguments in [Ch]: to restrict $U_q(\mathfrak{g})$-modules to $U_q(\mathfrak{gl}(1,1))$-modules. Every step of reduction therein resulted in tensor products of two-dimensional simple modules. An essential improvement in this paper is to view these tensor products as Weyl modules (Lemma 6.6). From this viewpoint the reduction arguments in [Zh2] work equally well even if the $(r_i, \varepsilon_i)$ change.

The Weyl modules over $U_q(\mathfrak{g})$ were defined in [Zh1]; they are super analog of Weyl modules over quantum affine algebras [CP3]. The case of $\mathfrak{gl}(1,1)$ is already useful enough to prove (B). It would be interesting to look at general case of $\mathfrak{gl}(M, N)$.

We mention the recent works of Guay–Tan [GT] on a similar cyclicity result where the braid group action in [Ch] was defined for Yangians. It is question to construct similar braid group (or groupoid) action in the super case in order to study more general simple modules. For this, it might be useful to look at different RTT realizations of $U_q(\mathfrak{g})$ (by permuting the parity of the base vectors in $V$ [Zh2, Definition 3.5]).

3. (C) is true for all finite-dimensional simple modules over quantum affine algebras. Its proof in [He] utilized deep theory of $q$-characters of Frenkel–Reshetikhin. In our situation, since we are restricted to fundamental modules, up to some duality arguments, (C) is a direct consequence of (B).

(D) is special in the super case, and has been proved in [Zh2] for all finite-dimensional simple modules over $U_q(\mathfrak{gl}(1,1))$. In the non-graded case, due to the action of Weyl groups, such a tensor product is of highest $\ell$-weight if and only if it is of lowest $\ell$-weight.

The paper is organized as follows. §2 prepares the necessary background on highest $\ell$-weight modules and on fundamental modules. §3 constructs the normalized $R$-matrices between two fundamental modules from elementary ones via a fusion procedure. §4 computes the denominators of the normalized $R$-matrices. §5 proves some easy but important properties of Weyl modules. §6 proves (B) by a series of reductions. §7 then discusses the consequences of (B). §8 reduces the general case of (B) without assumption on signature to a question of linear algebra (Question [1]).

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Fix $M, N \in \mathbb{Z}_{>0}$. This section collects basic facts on quantum superalgebras associated to the general linear Lie superalgebra $\mathfrak{gl}(M, N)$ and their representations.

2.1. Quantum superalgebras. Set $\kappa := M + N$, $I := \{1, 2, \cdots, \kappa\}$ and

\[
| \cdot : I \to \mathbb{Z}_2, i \mapsto |i| = \begin{cases} 0 & (i \leq M), \\ 1 & (i > M), \\ -1 & (i > M), \end{cases}
\]

defined by \(d_{i} : I \to \mathbb{Z}, i \mapsto d_{i} = \begin{cases} 1 & (i \leq M), \\ -1 & (i > M). \end{cases}\)

Set $q_{i} := q^{d_{i}}$. Set \(\mathcal{P} := \oplus_{i \in I} \mathbb{Z} \varepsilon_{i}\). Let \((, ) : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}\) be the bilinear form defined by \((\varepsilon_{i}, \varepsilon_{j}) = \delta_{ij} d_{i}\). Let \(| \cdot | : \mathcal{P} \to \mathbb{Z}_2\) be the morphism of abelian groups such that \(|\varepsilon_{i}| = |i|\).

In the following, we only consider the parity \(|x| \in \mathbb{Z}_2\) of \(x\) when either \(x \in I\), \(x \in \mathcal{P}\) or \(x\) is a \(\mathbb{Z}_2\)-homogeneous vector of a vector superspace. Associated to two vector superspaces \(V, W\) and \(W\) is the graded permutation \(c_{V, W} : \mathcal{V} \otimes \mathcal{W} \to \mathcal{W} \otimes \mathcal{V}\) defined by \(v \otimes w \mapsto (-1)^{|v||w|} w \otimes v\). Except in \(q\) always denotes \(\mathfrak{gl}(M, N)\), while \(q' = \mathfrak{gl}(N, M)\).

Let \(V = \oplus_{i \in I} \mathcal{C}v_i\) be the vector superspace with \(\mathbb{Z}_2\)-grading \(|v_i| = |i|\). For \(i, j \in I\), let \(E_{ij} \in \text{End}(V)\) be the endomorphism \(v_k \mapsto \delta_{jk} v_i\). Introduce the Perk–Schultz matrix \(R(z, w) \in \text{End}(V \otimes V)[[z, w]]:\)

\[
R(z, w) = \sum_{i \in I} (z q_i - w q_i^{-1}) E_{ii} \otimes E_{ii} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{jj} + z \sum_{i < j} (q_i - q_j^{-1}) E_{ij} \otimes E_{ij} + w \sum_{i < j} (q_j - q_i^{-1}) E_{ji} \otimes E_{ji}.
\]

It is well-known that \(R(z, w)\) satisfies the quantum Yang–Baxter equation:

\[
R_{12}(z_1, z_2) R_{13}(z_1, z_3) R_{23}(z_2, z_3) = R_{23}(z_2, z_3) R_{13}(z_1, z_3) R_{12}(z_1, z_2).
\]

Here we use the following convention for the tensor subscripts. Let \(n \geq 2\) and \(A_1, A_2, \cdots, A_n\) be unital superalgebras. Let \(1 \leq i < j \leq n\). If \(x \in A_i\) and \(y \in A_j\), then

\[
(x \otimes y)_{ij} := (\otimes_{k=i+1}^{j-1} 1_{A_k}) \otimes x \otimes (\otimes_{k=i+1}^{j-1} 1_{A_k}) \otimes y \otimes (\otimes_{k=j+1}^{n} 1_{A_k}) \in \otimes_{k=1}^{n} 1_{A_k}.
\]

Now we can define the quantum affine superalgebra associated to \(\mathfrak{g}\).

**Definition 2.1.** [Zhu] The quantum affine superalgebra \(U_q(\hat{\mathfrak{g}})\) is the superalgebra defined by

1. RTT-generators \(s_{ij}^{(n)}, t_{ij}^{(n)}\) for \(i, j \in I\) and \(n \in \mathbb{Z}_{>0}\);
2. \(\mathbb{Z}_2\)-grading \(|s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i| + |j|\);
3. RTT-relations in \(U_q(\hat{\mathfrak{g}}) \otimes (\text{End} V^{\otimes 2})[[z, z^{-1}, w, w^{-1}]\]:

\[
R_{23}(z, w) T_{12}(z) T_{13}(w) = T_{13}(w) T_{12}(z) R_{23}(z, w),
\]

\[
R_{23}(z, w) S_{12}(z) S_{13}(w) = S_{13}(w) S_{12}(z) R_{23}(z, w),
\]

\[
R_{23}(z, w) T_{12}(z) T_{13}(w) = S_{13}(w) T_{12}(z) R_{23}(z, w),
\]

\[
t_{ij}^{(0)} = s_{ji}^{(0)} = 0 \quad \text{for} \quad 1 \leq i < j \leq \kappa,
\]

\[
t_{ii}^{(0)} s_{ii}^{(0)} = 1 = s_{ii}^{(0)} t_{ii}^{(0)} \quad \text{for} \quad i \in I.
\]

Here \(T(z) = \sum_{i, j \in I} t_{ij}(z) \otimes E_{ij} \in (U_q(\hat{\mathfrak{g}}) \otimes \text{End} V)[[z^{-1}]]\) and \(t_{ij}(z) = \sum_{n \in \mathbb{Z}_{>0}} t_{ij}^{(n)} z^{-n} \in U_q(\hat{\mathfrak{g}})[[z^{-1}]],\) and similar definition of \(S(z)\) except that the \(z^{-n}\) is replaced by the \(z^n\).
$U_q(\hat{g})$ has a Hopf superalgebra structure with counit $\varepsilon : U_q(\hat{g}) \rightarrow \mathbb{C}$ defined by $\varepsilon(s_{ij}^{(n)}) = \varepsilon(t_{ij}^{(n)}) = \delta_{ij} \delta_{n0}$, and coproduct $\Delta : U_q(\hat{g}) \rightarrow U_q(\hat{g}) \otimes U_q(\hat{g})$:

$$\Delta(s_{ij}^{(n)}) = \sum_{m=0}^{n} \sum_{k \in I} \varepsilon_{ijk} s_{ik}^{(m)} \otimes s_{kj}^{(n-m)}, \quad \Delta(t_{ij}^{(n)}) = \sum_{m=0}^{n} \sum_{k \in I} \varepsilon_{ijk} t_{ik}^{(m)} \otimes t_{kj}^{(n-m)}.$$  

(2.2)

Here $\varepsilon_{ijk} := (-1)^{|i|+|k|+|k||j|}$. The antipode $S : U_q(\hat{g}) \rightarrow U_q(\hat{g})$ is determined by

$$\Delta(S(z)) = S(z)^{-1}.$$  

(2.3)

Here the RHS of the above formulas are well-defined owing to the last two relations in Definition 2.1. The subalgebra of $U_q(\hat{g})$ generated by the $s_{ij}^{(0)}, t_{ij}^{(0)}$ is a sub-Hopf-superalgebra denoted by $U_q(g)$. To simplify notations, write $s_{ij} := s_{ij}^{(0)}, t_{ij} := t_{ij}^{(0)}$.

We recall symmetry properties of $U_q(\hat{g})$, following mainly [Zh2] [Zh3].

For $\mathfrak{gl}(N,M) =: g'$, let us define the quantum superalgebras $U_q(\hat{g}')$, $U_q(g')$ in exactly the same way as $U_q(\hat{g}), U_q(g)$, except that we interchange $M,N$ everywhere. Let $s_{ij}^{(n)}, t_{ij}^{(n)}$ for $i,j \in I$ and $n \in \mathbb{Z}_{\geq 0}$ be the corresponding RTT generators of $U_q(\hat{g}')$, so that their $\mathbb{Z}_2$-degrees are $|s_{ij}^{(n)}| = |t_{ij}^{(n)}| = |i|' + |j|'$ where $|i|^' = \overline{\mathfrak{t}}$ for $1 \leq i \leq N$ and $\mathfrak{t}$ otherwise. For $i \in I$, set $\hat{i} := \kappa + 1 - i$. Let $\alpha \in \mathbb{C}^\times$. The following are isomorphisms of Hopf superalgebras $(\varepsilon_{ij} := (-1)^{|i|'+|i||j|'}$ and $\varepsilon_{ij}' := (-1)^{|i|'+|i||j|'})$

$$\Phi : U_q(\hat{g}) \rightarrow U_q(\hat{g}), \quad s_{ij}^{(n)} \mapsto a^n s_{ij}^{(n)}, \quad t_{ij}^{(n)} \mapsto a^{-n} t_{ij}^{(n)},$$  

(2.4)

$$\Psi : U_q(\hat{g}) \rightarrow U_q(\hat{g})^{\text{cop}}, \quad s_{ij}^{(n)} \mapsto \varepsilon_{ji} t_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_{ji} s_{ji}^{(n)},$$  

(2.5)

$$f : U_q(\hat{g}) \rightarrow U_q(\hat{g})^{\text{cop}}, \quad s_{ij}^{(n)} \mapsto i_{ji} s_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto i_{ji} t_{ji}^{(n)}.$$  

(2.6)

Here $A^{\text{cop}}$ of a Hopf superalgebra $A$ takes the same underlying superalgebra but the twisted coproduct $\Delta^{\text{cop}} := c_{A,A} \Delta$ and antipode $S^{-1}$. The $\Psi, f$ restrict naturally to isomorphisms of $U_q(g)$ and $U_q(g')$, still denoted by $\Psi, f$. Let $f(z) \in 1 + z\mathbb{C}[[z]]$ and $g(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$. The following are morphisms of superalgebras:

$$\text{ev}_a : U_q(\hat{g}) \rightarrow U_q(g), \quad s_{ij}(z) \mapsto s_{ij} - z a t_{ij}, \quad t_{ij}(z) \mapsto t_{ij} - z^{-1} a^{-1} s_{ij},$$  

(2.7)

$$\phi_{[f(z),g(z)]} : U_q(\hat{g}) \rightarrow U_q(g), \quad s_{ij}(z) \mapsto f(z) s_{ij}(z), \quad t_{ij}(z) \mapsto g(z) t_{ij}(z).$$  

(2.8)

These morphisms satisfy natural compatibility relations. For example,

$$\Psi \circ \text{ev}_a = \text{ev}_{a^{-1}} \circ \Psi : U_q(\hat{g}) \rightarrow U_q(g), \quad f \circ \text{ev}_a = \text{ev}_a \circ f : U_q(\hat{g}) \rightarrow U_q(g).$$

2.2. Highest $\ell$-weight modules. The Hopf superalgebra $U_q(\hat{g})$ is $P$-graded: an element $x \in U_q(\hat{g})$ is of weight $\lambda \in P$ if $s_{ij}^{(0)} x t_{ij}^{(0)} = q^{(\lambda, \epsilon_i)} x$ for all $i \in I$. Indeed, $s_{ij}^{(n)}$ and $t_{ij}^{(n)}$ are of weight $\epsilon_i - \epsilon_j$. Such a $P$-grading descends to $U_q(g)$. For a $U_q(g)$-module $V$ and $\lambda \in P$, we set $V_\lambda$ to be the subspace of $V$ formed of vectors $v$ such that $s_{ii}^{(0)} v = q^{(\lambda, \epsilon_i)} v$ for all $i \in I$, and call it the weight space of weight $\lambda$.

Let $V$ be a $U_q(\hat{g})$-module. A non-zero vector $v \in V$ is called a highest $\ell$-weight vector if it is a common eigenvector for the $s_{ii}^{(n)}, t_{ii}^{(n)}$ and it is annihilated by the $s_{ij}^{(n)}, t_{ij}^{(n)}$ with $i < j$. 

V is called a highest ℓ-weight module if it is generated as a $U_q(\mathfrak{g})$-module by a highest ℓ-weight vector. Similarly, there are the notions of lowest ℓ-weight vector/module by replacing $(i < j)$ with $(i > j)$. By dropping the $(n)$, we obtain the notions of highest/lowest weight vector/module related to $U_q(\mathfrak{g})$-modules. According to Equation (2.2), a tensor product of highest/lowest (ℓ)-weight vectors is again a highest/lowest (ℓ)-weight vector. This is not necessarily true when replacing “vector” with “module”, yet we have the following:

Lemma 2.2. [Zh2 Lemma 4.5] Let $V_+$ (resp. $V_-$) be a $U_q(\mathfrak{g})$-module of highest (resp. lowest) ℓ-weight. Let $v_+, v_+ \in V_+$ (resp. $v_-, v_- \in V_-$) be a highest (resp. lowest) ℓ-weight vector. Then the $U_q(\mathfrak{g})$-module $V_+ \otimes V_-$ (resp. $V_- \otimes V_+$) is generated by $v_+ \otimes v_-$ (resp. $v_- \otimes v_+$).

The proof of this lemma in [Zh2] utilized the Drinfeld new realization of $U_q(\mathfrak{g})$.

For $\lambda = \sum_1^\kappa \lambda_i \epsilon_i \in \mathbb{P}$, let $L(\lambda)$ be the simple $U_q(\mathfrak{g})$-module of highest weight $\lambda$; it is finite-dimensional if and only if $\lambda_i \geq \lambda_{i+1}$ for $i \neq M$; see [Za].

Example 1. The vector representation $\rho_{(1)}$ of $U_q(\mathfrak{g})$ on $V$ is defined by

$$
\rho_{(1)}(s_{ii}) = q_i E_{ii} + \sum_{j \neq i} E_{jj}, \quad \rho_{(1)}(t_{ii}^{-1}) \quad \text{for } i \in I,
$$

$$
\rho_{(1)}(s_{ij}) = (q_i - q_i^{-1}) E_{ij}, \quad \rho_{(1)}(t_{ji}) = -(q_i - q_i^{-1}) E_{ji} \quad \text{for } i < j.
$$

$\nu_i$ (resp. $\nu_\kappa$) is a highest (resp. lowest) weight vector and $v_i$ is of weight $\epsilon_i$ for $i \in I$. The resulting $U_q(\mathfrak{g})$-module $V$ is $L(\epsilon_1)$. For $a \in \mathbb{C}^{\times}$, let $W(a)$ denote the $U_q(\mathfrak{g})$-module $v_a^* V$.

Example 2. The pull back $\Psi^* V$ of the $U_q(\mathfrak{g})$-module $V$ by the isomorphism $\Psi$ in Equation (2.5) defines another representation $\rho_{(1)} \Psi$ of $U_q(\mathfrak{g})$ on $V$. Let $W$ be the corresponding $U_q(\mathfrak{g})$-module. For distinction, let us write $w_i := \Psi^* v_i$ for $i \in I$. Now $w_\kappa$ (resp. $w_1$) is a highest (resp. lowest) weight vector and $w_i$ is of weight $- \epsilon_i$. So $W = L(- \epsilon_\kappa)$ as $U_q(\mathfrak{g})$-modules. For $a \in \mathbb{C}^{\times}$, let $W(a)$ denote the $U_q(\mathfrak{g})$-module $v_a^* W$.

To motivate the definition fundamental modules, let us recall the highest ℓ-weight classification of finite-dimensional simple $U_q(\mathfrak{g})$-modules from [Za1, Za3]. Let $S$ be such a module. Firstly $S$ contains a unique (up to scalar multiple) highest ℓ-weight vector $v$. Secondly, for $1 \leq i < \kappa$, the eigenvalues of $s_{ii}(z)s_{i+1,i+1}(z)^{-1}, t_{ii}(z)t_{i+1,i+1}(z)^{-1}$ associated to $v$ turn out to be the $z = 0, z = \infty$ Taylor expansions of a rational function $f_i(z) \in \mathbb{C}(z)$ satisfying:

(1) if $i \neq M$, then $f_i(z)$ is a product of the $q_i^{1-zaq_i^2}$ with $a \in \mathbb{C}^{\times}$;

(2) $f_M(z)$ is a product of the $c \frac{1-za}{1-zaq_i^2}$ with $c, a \in \mathbb{C}^{\times}$.

Thirdly $S \mapsto \Pi(S) := (f_i(z))_{1 \leq i < \kappa} \in \mathbb{C}(z)^{\kappa-1}$ establishes a bijection between the isomorphism classes of finite-dimensional simple $U_q(\mathfrak{g})$-modules up to tensor products with one-dimensional modules and elements in $\mathbb{C}(z)^{\kappa-1}$ with conditions (1)–(2). Lastly, if $S, S'$ are two finite-dimensional simple $U_q(\mathfrak{g})$-modules, then by Equation (2.2), $S \otimes S'$ contains a highest ℓ-weight vector which gives rise to another simple module $S''$ with

$$
\Pi(S'') = \Pi(S)\Pi(S') \in \mathbb{C}(z)^{\kappa-1}.
$$

Notes:

1. We use rational functions instead of Drinfeld polynomials in [Zh1 Prop. 4.12] and [Zh3 Prop. 6.7]; they are indeed equivalent under the Ding–Frenkel homomorphism reviewed in [Zh2 Theorem 3.12].
Remark 2.3. Let $\lambda = \sum_i \lambda_i \epsilon_i \in P$ and $a \in \mathbb{C}^\times$. When $L(\lambda)$ is finite-dimensional,

$$\Pi(\text{ev}_a^* L(\lambda)) = (q^{\lambda_1} - zaq^{-\lambda_1}, q^{\lambda_2} - zaq^{-\lambda_2}, \ldots, q^{\lambda_{M-1}} - zaq^{-\lambda_{M-1}}, q^{\lambda_M} - zaq^{-\lambda_M}, q^{-\lambda_{M+1}} - zaq^{\lambda_{M+1}}, \ldots, q^{-\lambda_{N+1}} - zaq^{\lambda_{N+1}}).$$

2.3. Fundamental representations. We are interested in such simple $U_q(\mathfrak{g})$-modules $S$ that all but one components of $\Pi(S)$ are 1. They can be constructed by fusion procedures.

Definition 2.4. Let $a \in \mathbb{C}^\times$ and $s,t \in \mathbb{Z}_{>0}$ be such that $s \leq M$ and $t \leq N$. The sub-$U_q(\mathfrak{g})$-module of $\otimes_{j=1}^t V(aq^{2j})$ generated by $v_{\kappa}^{\otimes s}$ is called a positive fundamental module and denoted by $V_{s,a}^+$. The sub-$U_q(\mathfrak{g})$-module of $\otimes_{j=1}^t W(aq^{2j})$ generated by $w_1^{\otimes t}$ is called a negative fundamental module and denoted by $V_{t,a}^-$. The terminology “positive/negative” will be justified at the end of this section. The following proposition will be proved in [Zh3, Prop.4.6] when twisted duals are introduced.

Proposition 2.5. $V_{s,a}^+$ and $V_{t,a}^-$ are simple $U_q(\mathfrak{g})$-modules for $1 \leq s \leq M, 1 \leq t \leq N$.

The following theorem is a special case of more general results in [BKK].

Theorem 2.6. Let $1 \leq s \leq M$. The $U_q(\mathfrak{g})$-module $L(\epsilon_1)^{\otimes s} = V^{\otimes s}$ is completely reducible. Its submodule generated by $v_{\kappa}^{\otimes s}$ is isomorphic to $L(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_s)$ whose weight spaces are one-dimensional and whose weights are the $\epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_s}$ where: $i_1 \leq i_2 \leq \cdots \leq i_s$; if $i_k = i_{k+1}$ then $i_k > M$. Denote this sub-$U_q(\mathfrak{g})$-module by $V_{s,a}^+$.

In [BKK], to certain $\lambda \in P$ is associated an $(M, N)$-hook Young diagram $Y^\lambda$ (an ordinary Young diagram without box at the $(M+1, N+1)$-position). Such $L(\lambda)$ has a crystal basis in the sense of Kashiwara labeled by semi-standard tableaux (assignment of numbers between 1 and $\kappa$ to the boxes according to certain rules) in $Y^\lambda$. In the above theorem, $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_s$ corresponds to the Young diagram with $s$ boxes in one column. The conditions of the $i_k$ are exactly those of being a semi-standard tableau. For example, when $M = N = 2$, the weights (and crystal basis vectors) are indexed by the following tableaux:

$$M = N = 2, V_2^+ = L(\epsilon_1 + \epsilon_2) :$$

1 2 1 3 1 4 2 3 2 4 3 4 3 4 4

Lemma 2.7. For $1 \leq s \leq M$, the sub-$U_q(\mathfrak{g})$-module $V_s^+$ has a highest weight vector

$$v^{(s)} := \sum_{\sigma \in \mathfrak{S}_s} (-q)^{l(\sigma)} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)} \in V^{\otimes s}.$$
using the formulas in Example 11 we are reduced to the case \( s = 2 = k \) and \( j = 1 \). Now \( \mathbf{v}_1 \otimes \mathbf{v}_2 - q \mathbf{v}_2 \otimes \mathbf{v}_1 \) is easily shown to be of highest weight.

\( v^{(s)} \) generates a sub-\( U_q(\mathfrak{g}) \)-module \( S \). It is of highest weight \( \epsilon_1 + \epsilon_2 + \cdots + \epsilon_8 \) and completely reducible by Theorem 2.6. \( S \) must be simple, and \( (S)_{s \kappa} = (V_{s \kappa})_{s \kappa} = \mathbb{C} \mathbf{v}_{s \kappa}^{(s)} \). This implies that \( V_{s \kappa}^{+} \subseteq S \). Since \( S \) is a simple \( U_q(\mathfrak{g}) \)-module, \( V_{s \kappa}^{+} = S \) and \( v^{(s)} \in V_{s \kappa}^{+} \).

**Lemma 2.8.** Let \( a \in \mathbb{C}^{\times} \) and \( 1 \leq s \leq M \). as \( U_q(\mathfrak{g}) \)-modules, \( V_{s,a}^{+} \cong ev_{aq^{-2s}}(V_{s}^{+}) \otimes D \) for some one-dimensional module \( D \). As sub-\( U_q(\mathfrak{g}) \)-modules of \( V_{s \kappa}^{(s)} \), we have \( V_{s,a}^{+} = V_{s \kappa}^{+} \).

**Proof.** It is enough to prove the first part, as \( V_{s \kappa}^{+} \subseteq V_{s,a}^{+} \). Let \( f_{i}(z) \) be the eigenvalue of \( s_{ii}(z) \) associated to the lowest \( \ell \)-weight vector \( v_{s \kappa}^{(s)} \) in \( \otimes_{j=1}^{s} V(aq^{-2j}) \). Then \( f_{i}(z) = \prod_{j=1}^{s}(1 - zq^{-2j}) \) for \( i < \kappa \) and \( f_{s}(z) = \prod_{j=1}^{s}(q^{-1} - zq^{-2j} + 1) \). Similar statements hold for \( t_{ii}(z) \). Now set \( D = \phi_{g(z)}^{*} \mathbb{C} \) where \( \mathbb{C} \) is the one-dimensional trivial \( U_q(\mathfrak{g}) \)-module and \( f(z) = \prod_{j=1}^{s}(1 - zq^{-2j}) \). Then the lowest \( \ell \)-weight vector of \( ev_{aq^{-2s}}(V_{s}^{+}) \otimes D \) and \( v_{s \kappa}^{(s)} \) have the same eigenvalues of the \( s_{ii}(z) \), \( t_{ii}(z) \). Since \( V_{s,a}^{+} \) is a simple \( U_q(\mathfrak{g}) \)-module, it must be isomorphic to \( ev_{aq^{-2s}}(V_{s}^{+}) \otimes D \).

Similar results as in the above two lemmas hold true for negative fundamental modules.

**Lemma 2.9.** Let \( a \in \mathbb{C}^{\times} \) and \( 1 \leq t \leq N \). Let \( V_{t}^{-} \) be the sub-\( U_q(\mathfrak{g}) \)-module of \( W_{t} \) generated by \( w_{t}^{(t)} \). Then as sub-\( U_q(\mathfrak{g}) \)-modules \( V_{t,a}^{-} = V_{t}^{-} \), as \( U_q(\mathfrak{g}) \)-modules \( V_{t,a}^{-} \cong ev_{aq^{-2t}}(V_{t}^{-}) \otimes D \) for some one-dimensional module \( D \), and \( V_{t}^{-} \) has a highest weight vector

\[
\sum_{\sigma \in \Sigma} (-q)^{\ell(\sigma)} w_{\kappa - t + \sigma(1)} \otimes w_{\kappa - t + \sigma(3)} \otimes \cdots \otimes w_{\kappa - t + \sigma(t)} \in W_{t}^{(t)}.
\]

**Proof.** Let \( v'_{i}, V', c', P', V'_{t,a}^{+} \) be the corresponding objects for \( \mathfrak{g}' \). By comparing the highest \( \ell \)-weight vectors in \( \otimes_{j=1}^{s} V(aq^{-2j}) \), we get a \( U_q(\mathfrak{g}') \)-linear isomorphism \( \theta : f^{*}W(a) \cong V'(a) \otimes \mathbb{C}_{\pi} \), \( u_{i} \mapsto x_{i}v' \otimes [\pi] \) where \( x_{i} \in \mathbb{C}^{\times} \) for \( i \in I \) and \( \mathbb{C}_{\pi} = \mathbb{C}[[\pi]] \) is the one-dimensional odd module over \( U_q(\mathfrak{g}') \). The graded permutation \( V'(a) \otimes \mathbb{C}_{\pi} \rightarrow \mathbb{C}_{\pi} \otimes V'(a) \), being \( U_q(\mathfrak{g}') \)-linear, induces an isomorphism of \( U_q(\mathfrak{g}') \)-modules

\[
\Sigma_{t} : \otimes_{j=1}^{1} (V'(aq)^{2j}) \otimes \mathbb{C}_{\pi} \rightarrow \otimes_{j=1}^{1} (V'(aq)^{2j}) \otimes \mathbb{C}_{\pi}.
\]

The assignment \( \otimes_{j=1}^{1} u_{j} \mapsto (-1)^{1} \sum_{k} u_{k} \otimes u_{j} \otimes_{j=1}^{1} u_{j} \) extends to a \( U_q(\mathfrak{g}') \)-linear isomorphism \( \sigma_{t} : f^{*}(\otimes_{j=1}^{1} W(aq^{2j})) \rightarrow \otimes_{j=1}^{1} f^{*}W(aq^{2j}) \) by Equation (2.6). The composition

\[
f^{*}(\otimes_{j=1}^{1} W(aq^{2j})) \leftrightarrow \otimes_{j=1}^{1} f^{*}W(aq^{2j}) \rightarrow \otimes_{j=1}^{1} V'(aq^{2j}) \otimes \mathbb{C}_{\pi} \rightarrow \otimes_{j=1}^{1} T T V'(aq^{2j}) \otimes \mathbb{C}_{\pi} \rightarrow \mathbb{C}_{\pi}\]

restricts to a \( U_q(\mathfrak{g}') \)-module isomorphism \( \vartheta_{t} : f^{*}V_{t,a}^{-} \rightarrow V_{t,a}^{-} \otimes \mathbb{C}_{\pi}^{\otimes 2} \otimes \mathbb{C}_{\pi} \rightarrow \mathbb{C}_{\pi} \). Lemmas 2.7, 2.8 for the \( U_q(\mathfrak{g}') \)-module \( V_{t,a}^{+} \) can be translated into those for the \( U_q(\mathfrak{g}) \)-module \( V_{t,a}^{-} \) via \( \vartheta_{t} \).

By comparing the weights, we see that as \( U_q(\mathfrak{g}) \)-modules, \( V_{t}^{-} \cong L(\pi_{t}) \). Here the dual space \( V^{*} = \text{hom}(V, \mathbb{C}) \) of a \( U_q(\mathfrak{g}) \)-module \( V \) is endowed with the \( U_q(\mathfrak{g}) \)-module structure

\[
\langle xl, v \rangle := (-1)^{|x||l|} \langle l, S(x)v \rangle \quad \text{for } x \in U_q(\mathfrak{g}), l \in V^{*}, v \in V.
\]
thought of as (up to a scalar factor which is a meromorphic function in $\tau$ from the weight grading. The $U$ by the $(s_{a,b})_{\tau}$.

Example 3. $V_2^+$ is spanned by the $v_i^{(2)}$ and $v_i \otimes v_j - (-1)^{|i||j|} q v_j \otimes v_i$ with $M < l \leq \kappa$ and $1 \leq i < j \leq \kappa$; see [Zh2] §2. By Lemma 2.9, $V_2^-$ is spanned by the $w_i^{(2)}$ and $w_i \otimes w_j + (-1)^{|i||j|} q w_j \otimes w_i$ with $1 \leq l \leq M$ and $1 \leq i < j \leq \kappa$. In general, $V_s^+$ (resp. $V_s^-$) is seen as quantum exterior power $\Lambda_s V$ (resp. symmetric power $S_s^* W$).

From Lemmas 2.8–2.9 and Remark 2.3 we see that for $1 \leq s \leq M$ and $1 \leq t \leq N$,

$$\Pi(V_{s,a}^+((\sigma))) = (1^{s-1}, q^{1} - zaq^{-2s}, 1^{1-s})$$

$$\Pi(V_{a,b}^-((\sigma))) = (1^{s-1}, q^{1} - zaq^{-2s}, 1^{1-s}).$$

$\pm$ indicate the positive/negative powers of $q$ in $\Pi(V_{s,a}^+)_{z=0}$. Our definition of fundamental modules, viewed in terms of the highest $\ell$-weight classification, is then in accordance with that in the non-graded case [CTII Definition 3.4]; see also Footnote 3.

3. $R$–MATRICES OF FUNDAMENTAL REPRESENTATIONS

The aim of this section is to construct $U_q(\mathfrak{g})$-linear maps between fundamental modules. The following lemma is our starting point. Its proof, postponed to [4] (page 24), is independent of denominators of normalized $R$-matrices; see the remark before Theorem 7.2.

Lemma 3.1. Let $V, W$ be two fundamental modules with highest $\ell$-weight vectors $v, w$ respectively. For $a, b \in \mathbb{C}^\times$, denote $V_a := \Phi_a^* V$ and $W_b := \Phi_b^* W$. There exists a finite set $\Sigma \subset \mathbb{C}^\times$ such that: if $\frac{a}{b} \notin \Sigma$, then $V_a \otimes W_b$ is a simple $U_q(\mathfrak{g})$-module and there exists a unique $U_q(\mathfrak{g})$-module isomorphism $V_a \otimes W_b \rightarrow W_b \otimes V_a$ sending $v \otimes w$ to $w \otimes v$.

Remark 3.2. Presumably the quantum affine superalgebra $U_q(\mathfrak{g})$ admits a universal $R$-matrix $\mathcal{R}(z) \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})[[z]]$. Here $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) is the subalgebra generated by the $(s_{i,j}^{(n)})_{i,j}$ (resp. the $(t_{i,j}^{(n)})_{i,j}$), and $\otimes$ is a completed tensor product arising from the weight grading. The $U_q(\mathfrak{g})$-module isomorphism in the above lemma can then be thought of as (up to a scalar factor which is a meromorphic function in $\frac{a}{b}$) the specialization $c_{V,W} \mathcal{R}(\frac{a}{b})_{\mathcal{V},W}$. In [Zh2] §3.3.6 a Hopf pairing $\mathcal{V}$ between these two subalgebras was constructed. The author believes that $\mathcal{V}$ is non-degenerate and its Casimir element gives $\mathcal{R}(z)$. See [Zh2] for a proof in the case $\mathfrak{gl}(1,1)$.

The following result is taken from [Zh2] Lemma 4.6.

Lemma 3.3. For $a, b \in \mathbb{C}^\times$, $c_{\mathcal{V},W} R(a,b) : V(a) \otimes W(b) \rightarrow V(b) \otimes V(a)$ is $U_q(\mathfrak{g})$-linear.

Let $F : V(a) \otimes W(b) \rightarrow W(b) \otimes V(a)$ be a $U_q(\mathfrak{g})$-linear map sending $v_i \otimes w_i$ to $w_i \otimes v_i$; by Lemma 3.3 $F$ exists when $\frac{a}{b}$ is generic. We shall compute the $F(v_i \otimes w_j)$.

Step I. For $k \neq l$, $C v_k \otimes w_l$ is the weight space of $V(a) \otimes W(b)$ of weight $\epsilon_k - \epsilon_l$. The zero weight space is spanned by the $v_i \otimes w_i$. Similar statements hold for $W(b) \otimes V(a)$. Since $F$
respects the weight spaces, there exist $\lambda_{ij}, \theta_{kl}$ for $i, j, k, l \in I$ and $k \neq l$ such that $\theta_{1k} = 1$,

$$F(v_j \otimes w_j) = \sum_{i \in I} \lambda_{ij}w_i \otimes v_i, \quad F(v_k \otimes w_l) = \theta_{kl}w_l \otimes v_k \quad \text{for} \quad k \neq l.$$  

**Step II.** Let $i, j, k \in I$ be such that $i < j < k$. Compare $\theta_{ij}$ with $\theta_{ik}$. We have

(2.a) $F(s_{jk}(v_i \otimes w_j)) = s_{jk}F(v_i \otimes w_j) = \theta_{ij}s_{jk}(w_j \otimes v_i)$.

By Equation (2.2) and Examples 1–2,

$$s_{jk}(v_i \otimes w_j) = \sum_{l=j}^{k} (s_{jl} \otimes s_{lk})(v_i \otimes w_j) = (s_{jj} \otimes s_{jk})(v_i \otimes w_j) = (-1)^{|i|(|j|+|k|)}s_{jj}v_i \otimes s_{jk}w_j$$

$$= v_i \otimes s_{jk}w_j = (q_k^{-1} - q_i)v_i \otimes w_k,$$

$$s_{jk}(w_j \otimes v_i) = \sum_{l=j}^{k} (s_{jl} \otimes s_{lk})(w_j \otimes v_i) = (s_{jj} \otimes s_{kk})(w_j \otimes v_i) = (q_k^{-1} - q_i)w_k \otimes v_i.$$

It follows from (2.a) that $\theta_{ij} = \theta_{ik}$. Next compare $\theta_{ik}$ and $\theta_{jk}$ by using

$$t_{ji}(v_i \otimes w_k) = (t_{ji} \otimes t_{ii})(v_i \otimes w_k) = (q_i^{-1} - q_i)v_i \otimes w_k,$$

$$t_{ji}(w_k \otimes v_i) = (t_{jj} \otimes t_{ji})(w_k \otimes v_i) = (-1)^{|i|+|j|}(q_i^{-1} - q_i)w_k \otimes v_j.$$  

Applying $Ft_{ji}$ and $t_{ji}F$ to $v_i \otimes w_k$ we get $\theta_{ik} = (-1)^{|i|+|j|}\theta_{jk}$. Now compute

$$t_{kj}(v_j \otimes w_i) = (t_{kj} \otimes t_{jj})(v_j \otimes w_i) = (q_j^{-1} - q_j)v_k \otimes w_i,$$

$$t_{kj}(w_i \otimes v_j) = (t_{kk} \otimes t_{kj})(w_i \otimes v_j) = (q_j^{-1} - q_j)w_i \otimes v_k.$$  

Applying $Ft_{kj}$ and $t_{kj}F$ to $v_i \otimes w_k$ we get $\theta_{ji} = \theta_{ki}$. At last consider

$$s_{ij}(v_k \otimes w_i) = (s_{ii} \otimes s_{ij})(v_k \otimes w_i) = (-1)^{|i|+|j|}(q_j^{-1} - q_j)v_k \otimes w_j,$$

$$s_{ij}(w_i \otimes v_k) = (s_{ij} \otimes s_{jj})(w_i \otimes v_k) = (q_j^{-1} - q_j)w_j \otimes v_k.$$  

Applying $Fs_{ij}$ and $s_{ij}F$ to $v_k \otimes w_i$ we get $\theta_{ki} = (-1)^{|i|+|j|}\theta_{kj}$.  

**Step III.** We assume that $j < k$. Let us compare $\theta_{jk}$ and $\theta_{kj}$. Compute

$$s_{jk}(v_k \otimes w_j) = (s_{jj} \otimes s_{jk} + s_{jk} \otimes s_{kk})(v_k \otimes w_j) = (q_j - q_j^{-1})(v_j \otimes w_j - v_k \otimes w_k),$$

$$t_{kj}(v_j \otimes w_k) = (t_{kk} \otimes t_{kj} + t_{kj} \otimes t_{jj})(v_j \otimes w_k) = (q_j - q_j^{-1})(v_j \otimes w_j - v_k \otimes w_k).$$  

By applying $F$ to the above identities, we get $\theta_{kj}s_{jk}(w_j \otimes v_k) = \theta_{jk}t_{kj}(w_k \otimes v_j)$. On the other hand, a straightforward calculation indicates that

$$s_{jk}(w_j \otimes v_k) = t_{kj}(w_k \otimes v_j)$$

$$= (q_j - q_j^{-1})(q_j^{-1}w_j \otimes v_j + \sum_{j < l < k} (q_j^{-1} - q_j)w_l \otimes v_l - (-1)^{|j|+|k|}q_kw_k \otimes v_k),$$

It follows that $\theta_{jk} = \theta_{kj}$ and $F(v_j \otimes w_j - v_k \otimes w_k) = \theta_{jk}(q_j - q_j^{-1})^{-1}s_{jk}(w_j \otimes v_k)$. In conclusion, for all $i, j \in I$, we have $\theta_{ij} = \theta_{1k}(-1)^{|i|j|} = (-1)^{|i|j|}$.  


Step IV. Assume that \( j < k \). Let us apply \( Fs_{jk} \) and \( s_{jk}F \) to \( v_k \otimes w_k \). We have
\[
s_{jk}(v_k \otimes w_k) = (s_{jk} \otimes s_{kk})(v_k \otimes w_k) = q_k^{-1}(q_j^{-1} - q_k^{-1})v_j \otimes w_k.
\]
So \( Fs_{jk}(v_k \otimes w_k) = \theta_{jk}q_k^{-1}(q_j^{-1} - q_k^{-1})w_k \otimes v_j \). If \( i \neq j, k \), then \( s_{jk}(w_i \otimes v_i) = 0 \). Otherwise,
\[
s_{jk}(v_j \otimes v_j) = (s_{jk} \otimes s_{kk})(v_j \otimes v_j) = (q_k^{-1} - q_k)v_k \otimes v_j,
\]
\[
s_{jk}(w_k \otimes v_k) = (s_{jj} \otimes s_{jk})(w_k \otimes v_k) = (-1)^{|j| + |k|}(q_j^{-1} - q_k^{-1})w_k \otimes v_j.
\]
It follows that \( \lambda_{kk} - \lambda_{jk} = \theta_{jk}(-1)^{|j| + |k|}q_k^{-1} \).

Step V. Let us apply \( Fs_{\kappa_1}(z) \) and \( s_{\kappa_1}(z)F \) to \( v_1 \otimes w_\kappa \) by developing
\[
s_{\kappa_1}(z)(v_1 \otimes w_\kappa) = (s_{\kappa_1}(z) \otimes s_{11}(z) + s_{\kappa\kappa}(z) \otimes s_{\kappa_1}(z))(v_1 \otimes w_\kappa)
\]
\[
= (q - q^{-1})za(1 - zb)v_\kappa \otimes w_\kappa + (1 - za)(q^{-1} - q)zbv_1 \otimes w_1,
\]
\[
s_{\kappa_1}(z)(w_\kappa \otimes v_1) = \sum_l (s_{\kappa_1}(z) \otimes s_{l1}(z))(w_\kappa \otimes v_1)
\]
\[
= (q^{-1} - q)zb(q - zaq^{-1})w_1 \otimes v_1 - (q - zbq^{-1})(q - q^{-1})zw_1 \otimes v_\kappa
\]
\[+ \sum_{1 < \kappa < l} (-1)^{|l|}(q_l^{-1} - q_l)zb(q - q^{-1})zw_l \otimes v_l.
\]
From the identity \( Fs_{\kappa_1}(z)(v_1 \otimes w_\kappa) = s_{\kappa_1}(z)(w_\kappa \otimes v_1) \) we deduce that:
\[
F(v_1 \otimes w_1) = \frac{bq - aq^{-1}}{b - a}w_1 \otimes v_1 + \frac{a(q - q^{-1})}{b - a} \sum_{l > 1} w_l \otimes v_l,
\]
\[
F(w_\kappa \otimes v_1) = \frac{b(q - q^{-1})}{b - a} \sum_{l < \kappa} w_l \otimes v_1 - \frac{bq^{-1} - aq}{b - a} w_\kappa \otimes v_1.
\]
By using the identity for \( F(v_i \otimes w_i - v_\kappa \otimes w_\kappa) \) in Step III, we obtain that for \( i \in I \),
\[
F(v_i \otimes w_i) = \sum_{l < i} \frac{b(q - q^{-1})}{b - a} w_l \otimes v_i + (-1)^{|i|} \frac{bq_i - aq_i^{-1}}{b - a} w_i \otimes v_i + \sum_{l > i} \frac{a(q - q^{-1})}{b - a} w_l \otimes v_i.
\]

Now let us introduce the matrix \( R_{a,b}^+: c_{\mathbf{V}, \mathbf{W}}F \in \text{End}(\mathbf{V} \otimes \mathbf{W}) \):
\[
R_{a,b}^+ = \sum_{i \neq j} E_{ii} \otimes E_{jj} + \sum_{i \in I} \frac{bq_i - aq_i^{-1}}{b - a} E_{ii} \otimes E_{ii}
\]
\[+ \sum_{i < j} \frac{a(q_j^{-1} - q_j)}{b - a} E_{ji} \otimes E_{ji} + \sum_{i > j} \frac{b(q_i - q_i^{-1})}{b - a} E_{ji} \otimes E_{ji}.
\]
(3.9)

Here by abuse of language \( E_{ij} \) is also in \( \text{End}(\mathbf{W}) \) sending \( w_k \) to \( \delta_{jk}w_i \).

Lemma 3.4. \( c_{\mathbf{V}, \mathbf{W}}R_{a,b}^+ : \mathbf{V}(a) \otimes \mathbf{W}(b) \rightarrow \mathbf{W}(b) \otimes \mathbf{V}(a) \) is a morphism of \( U_q(\mathfrak{g}) \)-modules provided that \( a, b \in \mathbb{C}^\times \) and \( a \neq b \).

Proof. Let \( \pi_1, \pi_2 \) denote the representations of \( U_q(\mathfrak{g}) \) on \( \mathbf{V}(a) \otimes \mathbf{W}(b) \) and \( \mathbf{W}(b) \otimes \mathbf{V}(a) \) respectively. We need to show that for \( x \) an arbitrary RTT generator of \( U_q(\mathfrak{g}) \),
\[
(*)_x : \quad (a - b)c_{\mathbf{V}, \mathbf{W}}R_{a,b}^+ \pi_1(x) = (a - b)\pi_2(x)c_{\mathbf{V}, \mathbf{W}}R_{a,b}^+.
\]
By Examples 12 and Equation (2.2), $\pi_1(x), \pi_2(x)$ are polynomials in $a, b$. Combining Equation (3.9), we see that $(\ast)_x$ is a polynomial equation in $a, b$. Lemma 3.3 and the above explicit computation of $F$ prove $(\ast)_x$ when $\frac{a}{b}$ is in the complementary of a finite subset of $\mathbb{C}^\times$. By polynomiality $(\ast)_x$ is true for all $a, b \in \mathbb{C}^\times$.

Let us define two classes of fusion $R$-matrices: $s, t \in \mathbb{Z}_{>0}$.

\begin{equation}
R_{a,b}^{t,s} := \prod_{j=1}^{s} \prod_{i=1}^{t} (R_{aq^{-2i}, bq^{-2j}})_{i,s+j} \in \operatorname{End}(V^{\otimes s} \otimes W^{\otimes t}),
\end{equation}

(3.10)\begin{equation}
R_{a,b}^{t,s} := \prod_{j=1}^{s} \prod_{i=1}^{t} R(aq^{-2i}, bq^{-2j})_{i,s+j} = \prod_{i=1}^{s} \prod_{j=1}^{t} R(aq^{-2i}, bq^{-2j})_{i,s+j} \in \operatorname{End}(V^{\otimes s} \otimes W).\end{equation}

(3.11)\begin{equation}
\text{The last equation holds by definition of the tensor subscripts in (2.1).}
\end{equation}

Lemma 3.5. Let $1 \leq s \leq M$ and $t \in \mathbb{Z}_{>0}$.

(A) Suppose $t \leq M$. Then for all $a, b \in \mathbb{C}^\times$, the linear map 

\begin{equation}
\chi_{V, W} : R_{a,b}^{t,s} \rightarrow \mathbb{C} [a,b]^t \otimes \mathbb{C} [a,b]^s, \end{equation}

restricts to a $U_q(\mathfrak{g})$-module map $V^{\otimes s} \otimes V^{\otimes t}_a \rightarrow V^{\otimes s}_t \otimes V^{\otimes t}_a$. In particular, there exist $X, Y \in \mathbb{C}[a,b]$ such that $X_{v(s) \otimes v(t)} = \chi_{V, W}(v(s) \otimes v(t)), R_{a,b}^{t,s}(v(s) \otimes v(t)) = Y_{v(s) \otimes v(t)}$.

(B) Suppose $t \leq N$. Let $a, b \in \mathbb{C}^\times$ be such that $aq^{-2i} \neq bq^{-2j}$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$. Then the linear map 

\begin{equation}
\chi_{V, W} : R_{a,b}^{t,s} \rightarrow \mathbb{C} [a,b]^t \otimes \mathbb{C} [a,b]^s, \end{equation}

restricts to a $U_q(\mathfrak{g})$-module map $V^{\otimes s} \otimes V^{\otimes t}_a \rightarrow V^{\otimes s}_t \otimes V^{\otimes t}_a$. Furthermore we have 

\begin{equation}
R_{a,b}^{t,s}(v(s) \otimes w(t)) = v(s) \otimes w(t), R_{a,b}^{t,s}(v(s) \otimes w(t)) = v(s) \otimes w(t). \end{equation}

Proof. We shall prove (A); the same idea goes for (B). By Lemma 3.3 $F_{a,b} := \chi_{V, W} : R_{a,b}^{t,s}$ is indeed $U_q(\mathfrak{g})$-linear. By Equations (2.1) and (3.11), 

\begin{equation}
\begin{aligned}
(1) & : \quad F_{a,b}(v_{\kappa}^{\otimes s+t}) = Y v_{\kappa}^{\otimes s+t}, \quad Y := \prod_{i=1}^{s} \prod_{j=1}^{t} (aq^{-2i-1} - bq^{-2j+1}).
\end{aligned}
\end{equation}

It is therefore enough to show the following (by Lemmas 2.8, 2.9 $V^{\otimes s}_a = V^{\otimes s}_a$) 

\begin{equation}
(2) : \quad F_{a,b}(V^{\otimes s}_s \otimes V^{\otimes t}_a) \subseteq V^{\otimes s}_t \otimes V^{\otimes t}_a, \quad F_{a,b}(v(s) \otimes v(t)) \in \mathbb{C} [v(t) \otimes v(s)].
\end{equation}

By Equation (2.1) the matrix coefficients of $F_{a,b} \in \operatorname{End}(V^{\otimes s+t})$ are polynomial in $a, b$. Let $\Sigma \subset \mathbb{C}^\times$ be as in Lemma 3.1 so that $S_1 := V^{\otimes s}_s \otimes V^{\otimes t}_a \cong V^{\otimes t}_a \otimes V^{\otimes s}_s =: S_2$ are simple $U_q(\mathfrak{g})$-modules whenever $\frac{a}{b} \notin \Sigma$. We show that (2) holds for $\frac{a}{b} \in \mathbb{C}^\times \setminus \Sigma$. This will imply (2) for all $a, b \in \mathbb{C}^\times$ by polynomiality, as in the proof of Lemma 3.4.

Let $\frac{a}{b} \notin \Sigma$. Then the simple $U_q(\mathfrak{g})$-modules $S_1$ and $S_2$ are both generated by $v_{\kappa}^{\otimes s+t}$. By (1), $F_{a,b}$ restricts to $U_q(\mathfrak{g})$-linear map $F_{a,b} : S_1 \rightarrow S_2$, and the first relation in (2) is proved. Since $v(s) \otimes v(t)$ and $v(t) \otimes v(s)$ are highest $\ell$-weight vectors of $S_1, S_2$ respectively, they must be stable by $F_{a,b}$. This proves the second relation in (2). \( \square \)
4. Denominators of $R$–matrices

Lemma 3.5 together with its proof gives us three types of rational functions of $\frac{a}{q}$: the End$(V^+_s \otimes V^+_t)$-valued $X^{-1}R_{a,b}^{s,t}|_{V^+_s \otimes V^+_t}$ and $Y^{-1}R_{a,b}^{s,t}|_{V^+_s \otimes V^+_t}$ for $1 \leq s, t \leq M$; the End$(V^+_s \otimes V^-_t)$-valued $R_{a,b}^{s,t}|_{V^+_s \otimes V^-_t}$ for $1 \leq s \leq M, 1 \leq t \leq N$. The denominator of such a rational function $R(a, b)$ is defined as a homogeneous polynomial $D(a, b)$ in $a, b$ of minimal degree such that $D(a, b)R(a, b)$ is polynomial; it is well-defined up to scalar product by a non-zero complex number. In this section, we shall compute these denominators.

In the following, if $v, w$ belong to the same vector space and $v \in C^t w$, then we write $v \doteq w$. The denominator of the third rational function is fairly easy.

Theorem 4.1. Let $1 \leq s \leq M$ and $1 \leq t \leq N$. The denominator of the End$(V^+_s \otimes V^-_t)$-valued rational function $R_{a,b}^{s,t}|_{V^+_s \otimes V^-_t}$ is $bq^2 - aq^{-2s}$.

Proof. By definition $v^s \otimes x$ is a lowest $\ell$-weight vector generating the simple $U_q(\tilde{g})$-module $V^+_s$. Owing to Lemmas 2.2 and 2.8 $v^s \otimes w(t)$ generates the $U_q(\tilde{g})$-module $V^+_s \otimes V^-_t$. By Lemma 3.5 $R_{a,b}^{s,t}$ respects the $U_q(\tilde{g})$-module structures. We are reduced to consider the rational function $x_{a,b} := R_{a,b}^{s,t}(v^s \otimes w(t)) \in V^s \otimes V^-_t(a, b)$.

Claim. A vector in the subspace $V^-_t$ of $W^s \otimes t^{-1}$ is uniquely determined by its components $w_i \otimes W^{s-1}_i$ with $i = \kappa$ or $i \leq \kappa - t$.

Proof. We prove the equivalent following statement $P(t)$ by induction on $1 \leq t \leq N$:

$$P(t) : V^-_t \cap (\sum_{\kappa - t < j < \kappa} w_j \otimes W^{s-1}_j) = 0.$$ 

For $t = 1$, this is obvious. $P(2)$ comes from Example 3. Assume that $t > 2$. Suppose that LHS of $P(t)$ contains a non-zero vector $y = \sum_{j=\kappa - t+1}^{\kappa} w_j \otimes x_j$. By Lemma 2.9

$$V^-_t = U_q(\tilde{g})(w_1^{s-1} \otimes w_1) \subseteq U_q(\tilde{g})w_1^{s-1} \otimes W = V^-_t \otimes W.$$ 

The induction hypothesis $P(t - 1)$ implies that $x_{\kappa-t+1} = 0$. A careful analysis of the first two tensor factors of $y$ in view of $V^-_t \subseteq V^-_2 \otimes V^-_{t-2}$ leads to

$$y = w_{\kappa-t+1} \otimes \sum_{j=\kappa-t+2}^{\kappa-1} w_j \otimes y_j + \sum_{j=\kappa-t+2}^{\kappa-1} w_j \otimes (-q)w_{\kappa-t+1} \otimes y_j.$$ 

Since $V^-_1 \subseteq W \otimes V^-_{t-1}$, we must have $0 \neq x_{\kappa-t+1} = \sum_{j=\kappa-t+2}^{\kappa-1} w_j \otimes y_j \in V^-_{t-1}$, in contradiction with $P(t - 1)$. This proves $P(t)$. 

Let us determine the components $V^s \otimes w_i \otimes W^{s-1}$ in $x_{a,b}$. By Equation (3,10),

$$R_{a,b}^{s,t} = F_t F_{t-1} \cdots F_2 F_1, \quad F_j = \prod_{l=1}^{s} (R_{aq^{-2l}, bq^{2j}}^{+,-})_{l, s+j}.$$ 

2To illustrate this claim, let $A$ be the algebra generated by the $w_i$ for $i \in I$ and subject to relations $w_i w_j - (-1)^{|i||j|} w_j w_i = w_i^2 = 0$ for $1 \leq i < j \leq \kappa$ and $M < l \leq \kappa$; see Example 3. Let $1 \leq t \leq N$ and $m$ be a non-zero product of $t$ $w_i$’s. Then up to scalar multiple $m = w_i m'$ with $i = \kappa$ or $1 \leq i \leq \kappa - t$. 


Let $\tau \in \mathcal{S}_t$. If $\tau(1) \neq t$, then $F_t$ fixes the term $v_{\kappa}^{\otimes k} \otimes w_{\kappa-t+r(1)} \otimes \cdots \otimes w_{\kappa-t+r(t)}$ in $v_{\kappa}^{\otimes k} \otimes w_t$. Applying $F_t F_{t-1} \cdots F_2$ to this term results in irrelevant components $V_{\kappa}^{\otimes k} \otimes w_t \otimes W^{\otimes t-1}$ with $\kappa-t < j < \kappa$. We are reduced to consider the case $\tau(1) = t$ and to evaluate $R_{a,b}^t(v_{\kappa}^{\otimes k} \otimes w_a \otimes x)$, where $x$ is a sum of $(t-1)$-fold tensor products of the $w_j$ with $\kappa-t < j < \kappa$. By Equation (3.9), the term $V_{\kappa}^{\otimes k} \otimes w_{\kappa-t} \otimes W^{\otimes t-1}$ in $F_t(v_{\kappa}^{\otimes k} \otimes w_{\kappa} \otimes x)$ is

$$
\prod_{i=1}^{s} \frac{bq^{2}q^{-1} - aq^{-2i}x_{k}^{(i)} \otimes w_{\kappa} \otimes x}{bq^{2} - aq^{-2i}x_{k}^{(i)} \otimes w_{\kappa} \otimes x} = q^{-s} \frac{bq^{2} - a}{bq^{2} - aq^{-2}x_{k}^{(i)} \otimes w_{\kappa} \otimes x}.
$$

Notice that the $F_j$ with $2 \leq j \leq t$ fix $v_{\kappa}^{\otimes k} \otimes w_{\kappa} \otimes x$. So the above term is exactly the component of $V_{\kappa}^{\otimes k} \otimes w_{\kappa} \otimes W^{\otimes t-1}$ in $x_{a,b}$. For $1 \leq i \leq \kappa - t$, again by Equation (3.9), the terms $V_{\kappa}^{\otimes k} \otimes w_{i} \otimes W^{\otimes t-1}$ in $F_t(v_{\kappa}^{\otimes k} \otimes w_{\kappa} \otimes x)$ and in $x_{a,b}$ are the same:

$$
\sum_{k=1}^{s} (-1)^{(s-k)(|v|+1)} \frac{bq^{2}(q_{i} - q_{i-1})}{bq^{2} - aq^{-2k}} \prod_{l=k+1}^{s} \frac{bq^{2}q^{-1} - aq^{-2l}q_{i}^{(l)} \otimes v_{\kappa}^{(k)} \otimes w_{i} \otimes v_{\kappa}^{(s-k)} \otimes w_{i} \otimes x}{bq^{2} - aq^{-2l}q_{i}^{(l)} \otimes v_{\kappa}^{(k)} \otimes w_{i} \otimes v_{\kappa}^{(s-k)} \otimes w_{i} \otimes x}. \tag{4.2}
$$

The coefficients are $(-1)^{(s-k)(|v|+1)} \frac{bq^{2}(q_{i} - q_{i-1})}{bq^{2} - aq^{-2k}} \prod_{l=k+1}^{s} \frac{bq^{2}q^{-1} - aq^{-2l}q_{i}^{(l)} \otimes v_{\kappa}^{(k)} \otimes w_{i} \otimes v_{\kappa}^{(s-k)} \otimes w_{i} \otimes x}{bq^{2} - aq^{-2l}q_{i}^{(l)} \otimes v_{\kappa}^{(k)} \otimes w_{i} \otimes v_{\kappa}^{(s-k)} \otimes w_{i} \otimes x}$. Together with the claim, we conclude that the denominator of $x_{a,b}$ is $(bq^{2} - aq^{-2s})$. \hfill \Box

The denominators of the first two rational functions are given as follows.

**Theorem 4.2.** Let $1 \leq s, t \leq M$. Let $u = \min(s, t)$. In the situation of Lemma 3.3 (A), we have $X = \frac{N}{\prod_{j=1}^{u}(a - bq^{-2j(t-u+j)})}$, where $N = \prod_{j=1}^{u}(a - bq^{-2j(s-u+j)})$. Moreover, $N$ (resp. $D$) is the denominator of the $\mathrm{End}(V_{s}^{+} \otimes V_{t}^{+})$-valued rational function $\frac{X}{R_{a,b}^{s,t}|V_{s}^{+} \otimes V_{t}^{+}}$. \hfill \[\frac{1}{(s-t)}\]

**Proof.** The idea is similar to that of Theorem 4.1. We shall compute $\frac{X}{R_{a,b}^{s,t}|V_{s}^{+} \otimes V_{t}^{+}}$ and prove the statement for $\frac{1}{R_{a,b}^{s,t}|V_{s}^{+} \otimes V_{t}^{+}}$. Notice that $Y$ is computed in the proof of Lemma 3.3.

Step I. By definition $X$ is the coefficient of $(-q)^{(t_u)} v_{\kappa}^{(s)} \otimes v_{\kappa}^{(t_u)} \otimes v_{\kappa}^{(t_u)} \otimes v_{\kappa}^{(t_u)}$ in $R_{a,b}^{s,t}(v_{\kappa}^{(s)} \otimes v_{\kappa}^{(t_u)})$; here $v_{\kappa}^{(s)}$ is the permutation $j \rightarrow t + 1 - j$.

Claim 1. For $1 \leq i, j \leq \kappa$, the term $V_{\kappa}^{\otimes k} \otimes v_{i}$ appears in $R_{a,b}^{s,t}(v_{\kappa}^{(s)} \otimes v_{j})$ only if $i \leq j$.

This comes from the fact that $R_{a,b}^{s,t}(V_{n,a} \otimes V_{m,c})$ is a $\mathbb{C}$-valued polynomial of degree $\min(s, t)$. For $1 \leq j \leq t$, let $X_j$ denote the coefficient of $v_{\kappa}^{(s)} \otimes v_{j}$ in $R_{a,b}^{s,t}(v_{\kappa}^{(s)} \otimes v_{j})$.

Then $X = X_{t}X_{t-1} \cdots X_{1}$ by Equation (3.3). \hfill \[\frac{1}{R_{a,b}^{s,t}|V_{s}^{+} \otimes V_{t}^{+}}\]

If $j > s$, then the coefficient of $v_{s} \otimes v_{s+1} \otimes \cdots \otimes v_{s} \otimes v_{s+1} \otimes \cdots \otimes v_{1} \otimes v_{j}$ in $R_{a,b}^{s,t}(v_{s} \otimes v_{s+1} \otimes \cdots \otimes v_{s} \otimes v_{s+1} \otimes \cdots \otimes v_{1} \otimes v_{j})$

$X_{j} = \prod_{i=1}^{s} (aq^{-2i} - bq^{-2(t+1-i)})$.

If $j \leq s$, then the value of $u_{1} := v_{j} \otimes \cdots \otimes v_{j+1} \otimes \cdots \otimes v_{j-1} \otimes v_{j-2} \otimes \cdots \otimes v_{j}$ in $R_{a,b}^{s,t}(v_{s} \otimes v_{s+1} \otimes \cdots \otimes v_{s} \otimes v_{s+1} \otimes \cdots \otimes v_{1} \otimes v_{j})$

$X_{j} = (aq^{-1} - bq^{-2(t+1-j)}) \prod_{i=2}^{s} (aq^{-2i} - bq^{-2(t+1-j)})$.

In the following, we mainly treat the case $s \leq t$ so that $u = s$. The case $s > t$ will be sketched at Step V. Consider the $V_{s}^{\otimes k} \otimes v_{\kappa}^{(s)} \otimes v_{\kappa}^{(s)}$-valued polynomial $u_{2} := R_{a,b}^{s,t}(v_{s} \otimes v_{\kappa}^{(s)} \otimes v_{\kappa}^{(s)})$. As

3 In the non-graded case, $U_{n}(g_{M})$ has fundamental modules $V_{s,a}$ where $1 \leq s < M$ and $a \in \mathbb{C}$ such that $\Pi(V_{s,a})$ is of the form $(s-1, \frac{1}{1-s}, 1^{M-1-s})$; see [22.2]. Up to shifts of spectral parameters, $V_{s,a}$ is $V_{a}^{(s)}$ in [DOI §2.2], and the denominator for $V_{a}^{(s)}$ and $V_{a}^{(s)}$ therein is a polynomial of degree $\min(s, t, M-s, M-t)$. \hfill \[\frac{1}{R_{a,b}^{s,t}|V_{s}^{+} \otimes V_{t}^{+}}\]
By Equations (2.1) and (3.11), the denominator of $\frac{1}{X} u_2$ is that of $\frac{1}{X} R^s_{a,b} V^+_s \otimes V^+_i$. Already $X = \frac{a-bq^{2i}}{a-bq^{2i-r}} \prod_{l=1}^{i} \prod_{i=1}^{t} (aq^{-2i} - bq^{-2j})$ and $\frac{1}{X} = \frac{1}{N}$.

Step II. By Equations (2.1) and (3.11), $u_2$ is a linear combination of the $v_{j_1} \otimes \cdots \otimes v_{j_{s+t}}$, where $v_i$ appears once if $i \leq s$ and $t$ times if $i = \kappa$. Similar to the claim in the proof of Theorem [1.1] it is enough to determine for a given pair $(0 \leq r \leq s, \sigma \in \mathfrak{S}_s)$ the coefficient \( k_{r,\sigma} \) in $u_2$ of the vector $v^\otimes_r \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)} \otimes v^\otimes_{s-r}$.

Step III. Let $1 \leq i \leq s$. Define $W_i$ to be the set of $\sigma \in \mathfrak{S}_s$ such that $\sigma(s) = i$. Set

$$v_i^{(s)} := \sum_{\sigma \in W_i} (-q)^{l(\sigma)} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s-1)} \in V^+_s \otimes V^+_i.$$ 

V making $\mathfrak{S}_{s-1}$ as the subgroup of $\mathfrak{S}_s$ formed of permutations fixing $s$. The multiplication $\mathfrak{S}_{s-1} \to \mathfrak{S}_s, \sigma \mapsto \tau_1 \tau_1 + \cdots + \tau_{s-1} \sigma$ induces a bijective map $\mathfrak{S}_{s-1} \to W_i$ which increases the length of permutations by $s - i$; here the $\tau_j := (j, j+1)$ denote simple transpositions. Now set the next two claims comes from Equations (2.11) and (3.11).

Claim 2. $R^s_{a,b} v^\otimes_1 v^\otimes_2 \otimes \cdots \otimes v^\otimes_t$ is obtained from $(-q)^{s-i} R^s_{a,b} v^{(i)} \otimes v^\otimes_i$ by replacing the $v_j$ in the tensor factors with $v_{j+1}$ whenever $i < j < i + 1$.

Claim 3. The term $V^\otimes s-1 \otimes v_i \otimes V^\otimes t$ in $u_2$ is obtained by inserting $f_s v_i$ at the $s$-th position of the tensor factors of $R^{s-1}(s) v^\otimes_1 v^\otimes_t$. Here

$$(a). \quad f_s = \prod_{j=1}^{t} (aq^{-2s} - bq^{-2j}).$$

The next claim reduces the problem of $V^+_s \otimes V^+_t$ to the case $V^+_s \otimes V^+_s$.

Claim 4. Let $g_r$ be the coefficient of $v^\otimes_r \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)}$ in $R^r_{a,b} (v^\otimes_r \otimes v^\otimes s)$. Then for $\sigma \in \mathfrak{S}_r$, the coefficient of $v^\otimes_r \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)} \otimes v^\otimes_{s-r}$ in $R^r_{a,b} (v^\otimes_r \otimes v^\otimes s)$ is

$$(b). \quad (-q)^{l(\sigma)} g_r \times \prod_{i=1}^{r} \prod_{j=r+1}^{s} (aq^{-2i} - bq^{-2j}) =: (-q)^{l(\sigma)} h_r.$$ 

Indeed, bases on the explicit formula of $v^\otimes_r \in V^+_r$, the coefficient of $v^\otimes_r \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)}$ in $R^r_{a,b} (v^\otimes_r \otimes v^\otimes s) \in V^+_s \otimes V^+_s$ should be $(-q)^{l(\sigma)} g_r$. Combining Claims 2–4, we obtain the following: for $0 \leq r \leq s$ and $\sigma \in \mathfrak{S}_s$, the coefficient in $v_2$ of the vector $v^\otimes_r \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)} \otimes v^\otimes_{s-r}$ is

$$(c). \quad (-q)^{x_\sigma} f_{s-1} \cdots f_{r+1} h_r = k_{r,\sigma}$$ 

for certain $x_{\sigma, l} \in \mathbb{Z}_{\geq 0}$ defined inductively by Claims 2 and 4.

Step IV. Compute $g_r$. Let $\sigma \in \mathfrak{S}_s$, with $i_l := \sigma^{-1}(l)$ for $1 \leq l \leq s$. By Equations (2.1) and (3.11), the coefficient of $v^\otimes_s \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_s$ in $R^{s,s}_{a,b} (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(s)} \otimes v^\otimes s)$ is

$$(d). \quad (-1)^{l(\sigma)} \prod_{l \neq i_1} (aq^{-2l} - bq^{-2j}) \times aq^{-2i_1} (q - q^{-1})$$ 

$$(e). \quad \times \prod_{l \neq i_1, i_2} (aq^{-2l} - bq^{-2j}) \times aq^{-2i_2} (q - q^{-1}) \times (aq^{-2i_1-1} - bq^{-3})$$
\[
\times \prod_{l \neq i_1, i_2, i_3} (aq^{-2l} - bq^{-6}) \times aq^{-2i_1}(q - q^{-1}) \times (aq^{-2i_1-1} - bq^{-5})(aq^{-2i_2-1} - bq^{-5}) \\
\times \cdots \times aq^{-2i_s}(q - q^{-1}) \times \prod_{l \neq i_s} (aq^{-2l-1} - bq^{-2s+1}) \\
= (-1)^l(\sigma) a^s(q - q^{-1})^s q^{-s(s+1)} - \frac{s(s-1)}{2} \prod_{i=1}^s \prod_{j=1}^{s-1} (aq^{-2i} - bq^{-2j}).
\]

From the explicit formula of \(v^{(s)}\) it follows that \(g_s = a^s C_s \prod_{i=1}^s \prod_{j=1}^{s-1} (aq^{-2i} - bq^{-2j})\) where
\[
C_s := (q - q^{-1})^s q^{-s(s+1)} \frac{s(s-1)}{2} \prod_{i=1}^s q^{\frac{i(i-1)}{2}} q^{s-s+1} - \frac{s(s-1)}{2} \prod_{i=1}^s q^{i} - 1 = 0.
\]

Combining with the formulas (a)–(c) above, we have:
\[
k_{r,\sigma} = f_s f_{s-1} \cdots f_{t+1} h_r \Rightarrow \prod_{i=1}^t (aq^{-2i} - bq^{-2t}) \prod_{i=1}^t (aq^{-2i} - bq^{-2j}).
\]

It follows that \(k_{r,\sigma} = a^r(a - bq^{-2(t-s)}) \prod_{j=t-r}^s (a - bq^{-2j})^{-1}\).

Step V. Finally, let us consider the case \(s > t\). We determine the \(V^{\otimes s+t}\)-valued function \(R_{a,b}^{s,t}(v^{(s,t)}(v^{(t)}) = u_3\). We have \(X = \frac{a-bq^{-2t}}{a-bq^{-2s}} \prod_{i=1}^s \prod_{j=1}^t (aq^{-2i} - bq^{-2j})\) and again \(\frac{v_r}{X} = \frac{Q}{N}\). As in Step II, we are reduced to determine the coefficients \(k_{r,\sigma}\) in \(u_3\) of the vectors \(v_{k}^{\otimes s-r} \otimes v_{\sigma(1)} \otimes \cdots v_{\sigma(t)} \otimes v_{k}^{\otimes r}\) where \(0 \leq r \leq t, \sigma \in S_t\). Similar arguments as Claims 2–4 and Step IV indicate that
\[
k_{r,\sigma} = b^r(a - b) \prod_{j=t-r+1}^t (a - bq^{-2j})^{-1}.
\]

Hence \(k_{r,\sigma} = b^r(a - b) \prod_{j=t-r+1}^t (a - bq^{-2j})^{-1}\). \(\square\)

5. Weyl modules over quantum affine \(\mathfrak{gl}(1,1)\)

In this section \(M = N = 1\) and \(g = \mathfrak{gl}(1,1)\). We discuss Weyl modules over \(U_q(\widehat{\mathfrak{g}})\), which were previously defined in [Zh1] 4.

Let \(R_0\) be the set of rational functions \(f(z) \in \mathbb{C}(z)\) which are products of the \(c_{\frac{1-za}{1-za}}\) with \(a, c \in \mathbb{C}^\times\). Let \(R_1\) be the set of pairs \((f(z), P(z)) \in R_0 \times \mathbb{C}[z]\) such that \(P(z) \in 1 + z\mathbb{C}[z]\) and \(\frac{P(z)}{f(z)} \in \mathbb{C}[z]\). (So \(\deg P = \deg P\).) For \((f, P) \in R_1\), the Weyl module \(\mathcal{W}(f; P)\) is the \(U_q(\widehat{\mathfrak{g}})\)-module generated by an even vector \(w\) and subject to relations:

\(\text{(W1)}\) \(s_{12}(z)w = t_{12}(z)w = 0, s_{22}(z)w = t_{22}(z)w = w, s_{11}(z)w = f(z)w = t_{11}(z)w;\)

\(\text{(W2)}\) \(\frac{P(z)}{f(z)} s_{21}(z)w \in \mathcal{W}(f; P)[[z]]\) is a polynomial of degree \(\leq \deg P\).

4In [Zh1] §4.1 Weyl modules were defined in terms of Drinfeld loop generators. It is not difficult to translate it by RTT generators, using the Ding-Frenkel homomorphism reviewed in [Zh2] Theorem 3.12.
In the last two equations of (W1), \( f(z) \in \mathbb{C}(z) \) is to be developed at \( z = 0, \infty \) accordingly. Let \( V(f) \) be the simplest \( \ell \)-weight \( U_q(\mathfrak{g}) \)-module whose highest \( \ell \)-weight vector is even and verifies (W1). Let \( f = \prod_i c_i \frac{1-z a_i}{1-z a_i c_i^2} \) be such that \( a_i, c_i \in \mathbb{C}^\times \) and \( a_i \neq a_j c_j^2 \) whenever \( i \neq j \). Then \( V(f) \cong \otimes_i V(c_i \frac{1-z a_i}{1-z a_i c_i^2}) \) as \( U_q(\mathfrak{g}) \)-modules [Zh2 Theorem 5.2].

Recall from [Zh4 §2] the Drinfeld generators of \( U_q(\mathfrak{g}) \): \( E_n, F_n, h_s, \phi_+^{(0)}, (s_{11}^{(0)})^{\pm 1} \) with \( n \in \mathbb{Z} \) and \( s \in \mathbb{Z}_{\neq 0} \). Set \( F^+(z) := -\sum_{n>0} F_n z^n \) and \( \phi^\pm(z) = \sum_{s \geq 0} \phi_\pm^s z^s \). We have:

\[
\Delta F^+(z) = \phi^+(z) \otimes F^+(z) + F^+(z) \otimes 1, \quad \Delta \phi^\pm(z) = \phi^\pm(z) \otimes \phi^\pm(z),
\]

\[
[h_s, F_n] = \frac{1 - q^{2s}}{s(q - q^{-1})} F_{n+s}, \quad F_n F_m + F_m F_n = h_s h_t - h_t h_s, \quad F^+(z) = s_{21}(z)s_{11}(z)^{-1}.
\]

The \( \phi_n^\pm \) are central. Let \( U^- \) (resp. \( U^\geq \)) be the subalgebra generated by the \( F_n \) (resp. the other Drinfeld generators). Then \( U_q(\mathfrak{g}) = U^- U^\geq \).

**Lemma 5.1.** For \((f, P) \in R_1 \), \( W(f; P) \) has a simple quotient \( V(f) \) and is spanned by the \( F_{n_1} F_{n_2} \cdots F_{n_s} w \) with \( s \geq 0 \) and \( n_j > 0 \) for \( 1 \leq j \leq s \). Furthermore, for all \( w' \in W \), (W2) holds and \( \phi^+(z)x = f(z)^{-1}w' \).

**Proof.** Let \( V(f) \cong \otimes_i (f_i) \) be a decomposition with \( f_i = c_i \frac{1-z a_i}{1-z a_i c_i^2} \) for \( 1 \leq i \leq l \). Then \( P(z) \) is divisible by \( \prod_{i=1}^l (1 - za_i) \). Let \( x_i \in V(f_i) \) be a highest \( \ell \)-weight vector. Set \( x := \otimes_{i=1}^l x_i \). From the coproduct of \( F^+(z) \) and \( \phi^+(z) \) we get

\[
F^+(z)x = \sum_{j=1}^l (\otimes_{i<j} \phi^+(z)x_i) \otimes F^+(z)x_j \otimes (\otimes_{i>j} x_i).
\]

In view of the explicit construction of \( V(f_i) \) in [Zh4 §5], both \((1 - za_i)\phi^+(z)x_i = c_i^{-1}(1 - za_i c_i^2)x_i \) and \((1 - za_i)F^+(z)x_i \) are polynomials of degree 1. It follows that \( \prod_{i=1}^l (1 - za_i)F^+(z)x = \) a polynomial of degree \( \leq l \), implying (W2) for \( x \in V(f) \). So \( V(f) \) is a quotient of \( W(f; P) \). Observe from [Zh4 §2] that for the highest \( \ell \)-weight vector \( x \) we have \( \phi^+(z)x = s_{21}(z)s_{11}(z)^{-1}x \in \mathbb{C}[z][x] \). The remaining statements then come from \( U_q(\mathfrak{g}) = U^- U^\geq \) and from the commuting relations of Drinfeld generators listed above. \( \square \)

**Proposition 5.2.** Let \((f, P), (g, Q) \in R_1 \). If the polynomials \( P \) and \( Q \) are co-prime, then \( W(f; P) \otimes W(g; Q) \) is of highest \( \ell \)-weight and is a quotient of \( W(fg; PQ) \).

**Proof.** Let \( \deg P = l \) and \( \deg Q = u \). Let \( x' \in W(f; P) \) and \( y' \in W(g; Q) \) be homogeneous vectors. From the above lemma, \( P(z)F^+(z)x' = \sum_{i=1}^l z^i x_i \) and \( Q(z)F^+(z)y' = \sum_{j=1}^u z^j y_j \) for certain \( x_i \in W(f; P) \) and \( y_j \in W(g; Q) \). The polynomial\(^4\)

\[
F^+(z)(x' \otimes y') = F^+(z)x' \otimes y' + (-1)^{|x'|} \phi^+(z)x' \otimes F^+(z)y'
\]

\[
= \frac{1}{P(z)Q(z)} (Q(z) \sum_{i=1}^l z^i x_i \otimes y' + (-1)^{|x'|} P(z) \sum_{j=1}^u z^j x' \otimes y_j).
\]



\(^4\) The above formula is obtained by carefully expanding the coproduct of \( F^+(z) \) and \( \phi^+(z) \) with \( x' \otimes y' \).
Theorem 6.1. \( P(z)Q(z)F^+(z)(x' \otimes y') \) is a polynomial of degree \( l + u \). Introduce \( T(z) := (-1)^{|x'| P(z)} f(z) \); it is a polynomial of degree \( l \). Since \( Q(z) \) and \( T(z) \) are co-prime, the polynomials \( z^i Q(z), z^j T(z) \) with \( 1 \leq i \leq \ell \) and \( 1 \leq j \leq u \) are linearly independent, and the \( x_i \otimes y_j, x' \otimes y_j \) are in the subspace spanned by the coefficients of \( P(z)Q(z)F^+(z)(x' \otimes y') \).

We have proved that \( F_s x' \otimes y', x' \otimes F_y y' \in U^-(x' \otimes y') \) for all \( s \in \mathbb{Z}_{>0} \) and \( x' \otimes y' \in W(f; P) \otimes W(g; Q) \). This implies by induction on \( s, t \in \mathbb{Z}_{>0} \) that

\[
F_{n_1} F_{n_2} \cdots F_{n_s} x' \otimes F_{m_1} F_{m_2} \cdots F_{m_t} y' \in U^-(x' \otimes y') \quad \text{for } n_1, \ldots, n_s, m_1, \ldots, m_t \in \mathbb{Z}_{>0}.
\]

Take \( x', y' \) to be the highest \( \ell \)-weight generators of \( W(f; P) \) and \( W(g; Q) \). From the above lemma we see that \( W(f; P) \otimes W(g; Q) = U^-(x' \otimes y') \). Moreover, \( x' \otimes y' \) satisfies the conditions (W1)–(W2) in the definition of \( W(fg; PQ) \). \( \square \)

6. Cyclicity of tensor products

In this section we provide sufficient conditions for a tensor product of fundamental modules to be of highest \( \ell \)-weight, improving previously established ones in \( [Zh2] \).

The main result of this section is as follows. For \( r_1, r_2 \in \mathbb{Z}_{>0} \), set

\[
\Sigma(r_1, r_2) := \{ q^{2l} \in \mathbb{C} \mid r_2 - \min(r_1, r_2) < l \leq r_2 \}.
\]

**Theorem 6.1.** Let \( k, l \in \mathbb{Z}_{>0} \). Let \( 1 \leq r_1, r_2, \ldots, r_k \leq M \) and \( 1 \leq s_1, s_2, \ldots, s_l \leq N \). Let \( a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in \mathbb{C}^* \). The \( U_q(\hat{g}) \)-module \( (\otimes_{i=1}^k V_{r_i}^+) \otimes (\otimes_{j=1}^l V_{s_j}^-) \) =: \( S \) is of highest \( \ell \)-weight if the following three conditions are satisfied:

1. \( C1 \) \( \frac{a_i}{b_j} \notin \Sigma(r_i, r_j) \) for \( 1 \leq i < j \leq k \);
2. \( C2 \) \( \frac{b_j}{a_j} \notin \Sigma(s_i, s_j) \) for \( 1 \leq i < j \leq l \);
3. \( C3 \) \( a_i q^{-2r_i} \notin b_j q^2 \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq l \).

The proof of the theorem needs a series of reduction lemmas. In the following, for \( V_1, V_2 \) two \( U_q(\hat{g}) \)-modules, we write \( V_1 \simeq V_2 \) if as \( U_q(\hat{g}) \)-modules \( V_1 \cong V_2 \otimes D \) for a one-dimensional \( U_q(\hat{g}) \)-module \( D \). Let \( A, B \) be two Hopf superalgebras. Let \( g : A \to B \) be a morphism of superalgebras. (In general \( g \) does not respect coproduct structures.) Let \( V \) be a \( B \)-module and \( W \) a sub-vector-superspace of \( V \). Suppose that \( W \) is stable by \( g(A) \). The action of \( g(A) \) endows \( W \) with an \( A \)-module structure, denoted by \( g^W \).

From now on, set \( U := U_q(\hat{g}(1, 1)) \). Let \( g_1 : U \to U_q(\hat{g}) \) be the superalgebra morphism defined by \( \left( \begin{array}{cc} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{array} \right) \mapsto \left( \begin{array}{cc} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{array} \right) \) and similar formulas for the \( t_{ij}(z) \). The following special property of fundamental modules is crucial in our reduction arguments. It was used implicitly in the proof of \( [Zh2] \) Theorem 4.2. We think of the trivial module also as fundamental modules: \( \mathbb{C} = V_{0,0}^+ \).

**Lemma 6.2.** Let \( X_1 \) (resp. \( X_3 \)) be a positive (resp. negative) fundamental module with \( x_1 \) (resp. \( x_3 \)) a lowest \( \ell \)-weight vector and let \( Y_j := g_1(U) x_j \subseteq X_j \) for \( j = 1, 3 \). Let \( X_2 \) be another \( U_q(\hat{g}) \)-module and \( Y_2 \) a sub-vector-superspace of \( X_2 \) stable by \( g_1(U) \). Then:

1. \( Y_1 \otimes Y_2 \otimes Y_3 \), as a subspace of the \( U_q(\hat{g}) \)-module \( X_1 \otimes X_2 \otimes X_3 \), is stable by \( g_1(U) \);
2. The identity map \( \text{Id} : g_1^*(Y_1 \otimes Y_2 \otimes Y_3) \cong g_1^* Y_1 \otimes g_1^* Y_2 \otimes g_1^* Y_3 \) is \( U \)-linear.
Lemma 6.3. Set modules; for example be in the situation of Theorem 6.1, and add sub-indexes to emphasize the fundamental module \( W \). The general case is just a combination of the above two cases. Let us fix three distinguished vectors of a positive fundamental module \( V \) as follows: \( v^1 \) is a highest \( \ell \)-weight vector; \( v^3 \) is a lowest \( \ell \)-weight vector; \( v^2 = s(0)_1 v^3 \). For a negative fundamental module \( W \), the three vectors \( w^1, w^2, w^3 \) are defined in the same way. We shall be in the situation of Theorem 6.3 and add sub-indexes to emphasize the fundamental modules; for example \( v^1_{ri,aj} \in V_{ri,ai} \) and \( w^3_{rj,bj} \in V_{sj,bj} \). The following lemma comes from the proof of Lemma 6.2. It is the reduction from \( \mathfrak{gl}(M, N) \) to \( \mathfrak{gl}(1, 1) \).

**Lemma 6.3.** Set \( W_i := g_1(U)v^1_i \subseteq V^+_r \) and \( W_j := g_1(U)w^3_j \subseteq V^-_s \). Then \( W_i = CW^2_i + CW^3_i \) and \( W_j = CW^2_j + CW^3_j \). As \( U \)-modules

\[
g_1^* W_i \simeq V(q^r \frac{1 - z a_i q^{-2r - 1} - z a_i q^{-2}}{1 - z a_i q^{-2}}), \quad g_1^* W_j' \simeq V(q^{-s_j} \frac{1 - z b q^{2s_j}}{1 - z b_j})
\]

and \( v^2_i, w^2_j \) (resp. \( v^3_i, w^3_j \)) are highest (resp. lowest) \( \ell \)-weight vectors.

**Proof.** We prove the negative case. Replace \( V^- \) by \( ev_{s, q} V^- \) according to Lemma 2.9. From the second part of the proof of Lemma 6.2, we see that: \( w^2, w^3 \) are of weights \( (1 - s) \epsilon_1 - \epsilon_\kappa, -s \epsilon_1 \) respectively; \( W' = CW^2 + CW^3 \) and \( w^2 \) is a highest \( \ell \)-weight vector of \( g_1^* W' \);

\[
s_{11}(z)s_{1\kappa}(z)w^2 = \frac{q^r - z b q^{2s + 1}}{q - z b q^{2s + 1}}w^2 = q^{-s}s \frac{1 - z b q^{2s}}{1 - z b}w^2 = t_{11}(z)t_{1\kappa}(z)w^2.
\]

This proves the second isomorphism in the lemma. \( \square \)

Let \( U := U_q(\mathfrak{gl}(M - 1, N)) \) and \( g_2 : U_2 \rightarrow U_q(\mathfrak{g}) \) be the superalgebra morphism defined by \( s_{ij}(z) \mapsto s_{i+1,j+1}(z) \) and similar formula for the \( t_{ij}(z) \). Let \( V^\pm_{r,a} \) denote the positive/negative fundamental modules over \( U_2 \), with \((+, 1 \leq r < M)\) or \((-1 \leq r \leq N)\).

**Lemma 6.4.** Set \( K_i := g_2(U_2)v^1_i \subseteq V^+_r \) and \( K_j' := g_2(U_2)w^1_j \subseteq V^-_s \). Then \( v^2_i, w^2_j \) are highest \( \ell \)-weight vectors of \( g_2^* K_i \), respectively. Furthermore, \( K := (\bigotimes_{i=1}^k K_i) \otimes (\bigotimes_{j=1}^l K_j') \) is stable by \( g_2(U_2) \), and identity map \( l : g_2^* K \cong (\bigotimes_{i=1}^k g_2^* K_i) \otimes (\bigotimes_{j=1}^l g_2^* K_j') \) is \( U_2 \)-linear.

**Proof.** Let us assume first that \( X_3 \) is the trivial module. We shall prove that \( s_{ij}(z)Y_1 = 0 = t_{ij}(z)Y_1 \) whenever \( i \in \{1, \kappa\} \) and \( j \notin \{1, \kappa\} \); this will imply (1) and that the operators \( g_1^\otimes 2(\Delta_U(y)) \) and \( \Delta_{U_q}(g_1(y)) \) on \( Y_1 \otimes Y_2 \) are identical for \( y \) an arbitrary RTT generator of \( U \), which proves (2). Let \( X_1 = V_{s,a}^+ \) with \( 1 \leq s \leq M \). By Lemma 2.8 and Theorem 2.6, \( Y_1 \) is two-dimensional, and its weights are \( \lambda_1 := s \epsilon_\kappa, \lambda_2 := \epsilon_1 + (s - 1) \epsilon_\kappa \). Let \( u \in Y_1 \) be of weight \( \lambda_k \) with \( k = 1, 2 \). Then \( s_{ij}(z)u \) and \( t_{ij}(z)u \) are of weight \( \lambda_k + \epsilon_i - \epsilon_j \), which is not a weight of \( V_{s,a}^+ \) by Theorem 2.6. So \( s_{ij}(z)u \neq 0 = t_{ij}(z)u \), as desired.

Secondly let us assume that \( X_1 \) is the trivial module. We prove that \( s_{ij}(z)Y_3 = 0 = t_{ij}(z)Y_3 \) whenever \( j \in \{1, \kappa\} \) and \( i \notin \{1, \kappa\} \); this will also imply (2). Let \( X_3 = V_{r,a}^- \) with \( 1 \leq r \leq N \). The proof of Lemma 2.9 and Theorem 2.6 show that: \( Y_3 \) is two-dimensional with weights \( \mu_1 := -r \epsilon_1, \mu_2 := (r - 1) \epsilon_1 - \epsilon_\kappa \). Now \( \mu_k + \epsilon_i - \epsilon_j \) is not a weight of \( V_{r,a}^- \) for \( k = 1, 2 \), leading to the desired result.

The general case is just a combination of the above two cases. \( \square \)
Lemma 6.5. Set \( L_i := g_3(U_3)v_i^1 \subseteq V_{r_i,a_i}^+ \) and \( L'_j := g_3(U_3)w_j^1 \subseteq V_{s_j,b_j}^- \). Then \( w_j^2 \in L'_j \) and \( g_3^* L_i \simeq V_{r_i,a_i}^+ \), \( g_3^* L'_j \simeq V_{s_j-1,b_j}^- \) as \( U_3 \)-modules, with \( w_j^1 \) and \( w_j^2 \) being highest and lowest \( \ell \)-weight vectors of \( g_3^* L'_j \) respectively. Furthermore, \( L := (\otimes_{i=1}^k L_i) \otimes (\otimes_{j=1}^l L'_j) \) is stable by \( g_3(U_3) \), and the identity map \( \text{Id} : g_3^*(L) \cong (\otimes_{i=1}^k g_3^* L_i) \otimes (\otimes_{j=1}^l g_3^* L'_j) \) is \( U_3 \)-linear.

Lemmas 6.4 and 6.5 can be deduced from Theorem 2.6 and Lemmas 2.8, 2.9. We have used the above three reductions in \([Zh2]\) to prove a weaker version of Theorem 6.1. The following lemma is new and is a crucial step in the proof of Theorem 6.1.

Lemma 6.6. Set \( W := g_1(U)((\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1)) \subseteq (\otimes_{i=1}^k V_{r_i,a_i}^+) \otimes (\otimes_{j=1}^l V_{s_j,b_j}^-) \). There exists a one-dimensional \( U \)-module \( D \) making \( W \) a quotient of \( D \otimes \text{W}(\prod_{i=1}^k \frac{q - za_i q^{-2r_i - 1}}{1 - za_i q^{-2r_i}} \times \prod_{j=1}^l \frac{1 - zb_j q^2}{q - zb_j q} \prod_{i=1}^k (1 - za_i q^{-2r_i - 2}) \times \prod_{j=1}^l (1 - zb_j q^2)) \).

Proof. We can replace \( V_{r_i,a_i}^+ \) and \( V_{s_j,b_j}^- \) by \( \text{ev}_{a_q^{-2r_i} V_{r_i,a_i}^+}^* \) and \( \text{ev}_{b_q^{-2} V_{s_j,b_j}^-}^* \) respectively. By Lemmas 2.8, 2.9 \( v_i^1, w_j^1 \) are of weights \( \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{r_i}, -\epsilon_{s_j} - 1 - \cdots - \epsilon_{s_j}, 1 + 1 \). Let \( v \) be ordered tensor product of the \( v_i^1, w_j^1 \) and \( W = g_1(U)v \). By Equation (2.7) : \( s_{i \kappa}(z)v = 0 = t_{i \kappa}(z)v; s_{i \kappa}(z)v \) is a polynomial of degree \( \leq k + l \); \( s_{ii}(z)v = f_i(z)v \) and \( t_{ii}(z)v = g_i(z)v \) where

\[
\begin{align*}
  f_1 &= \prod_{i=1}^k (q - za_i q^{-2r_i - 1}) \times \prod_{j=1}^l (1 - zb_j q^2), \\
  g_1 &= \prod_{i=1}^k (a_i^{-1} q^{2r_i}) \times \prod_{j=1}^l (b_j^{-1} q^2), \\
  f_\kappa &= \prod_{i=1}^k (1 - za_i q^{-2r_i}) \times \prod_{j=1}^l (q - zb_j q), \\
  g_\kappa &= \prod_{i=1}^k (a_i^{-1} q^{2r_i}) \times \prod_{j=1}^l (b_j^{-1} q^2).
\end{align*}
\]

Let \( D_1 = \phi_{[f_{1 \kappa}^{-1} g_{1 \kappa}^{-1}]}^*(C) = \text{Cd} \) be the one-dimensional \( U \)-module. Then the tensor product of \( U \)-modules \( D_1 \otimes W \) is generated by \( d \otimes v \). Moreover, \( d \otimes v \) satisfies all the relations in the definition of \( W((\frac{q}{f_2} : k q^{-f_1}) \); the last one therefore has \( D_1 \otimes W \) as a quotient.

Corollary 6.7. We have \( v_i^3 \otimes (\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1) \in g_1(U)(v_i^3 \otimes (\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1)) \) if \( a_i \neq a_q^{-2r_i}, b_q^4 \) for all \( i, j \). Similarly, if \( b_i \neq a_q^{-2r_i - 2s_i}, b_j q^{-2} \) for all \( i, j \), then \( (\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1) \otimes w_j^2 \in g_1(U)((\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1) \otimes w_j^2) \).

Proof. Let us prove the second part, the first part being similar. We are in the situation of Lemma 6.2 where \( X_1 \) is trivial, \( X_3 := V_{s_j,b_j}^- \), \( Y_3 := W'_j \subseteq X_3 \) (see Lemma 6.3) and

\[
X_2 := (\otimes_{i=1}^k V_{r_i,a_i}^+) \otimes (\otimes_{i=1}^l V_{s_j,b_j}^-), \quad Y_2 := g_1(U)((\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1)) \subseteq X_2.
\]

It follows that \( g_1^*(Y_2 \otimes Y_3) \subseteq g_1^* Y_2 \otimes g_1^* Y_3 \). The \( U \)-modules \( g_1^* Y_2 \) and \( g_1^* Y_3 \) (see Lemma 6.3) are generated by highest \( \ell \)-weight vectors \( (\otimes_{i=1}^k v_i^1) \otimes (\otimes_{j=1}^l w_j^1) \) and \( w_2^2 \) respectively. It
is therefore enough to prove that \( g_1^r Y_2 \otimes g_1^s Y_3 \) is of highest \( \ell \)-weight. Up to tensor products by one-dimensional modules, by Lemma 6.6 \( g_1^r Y_2 \) is a quotient of the Weyl module

\[
W_2 := \mathcal{W} \left( \prod_{i=1}^k \frac{q - za_i q^{-2r_i - 1}}{1 - za_i q^{-2r_i}} \right) \times \prod_{j=1}^l \frac{1 - zb_j q^2}{q - zb_j q} \cdot \prod_{i=1}^k (1 - za_i q^{-2r_i - 2}) \times \prod_{j=1}^l (1 - zb_j q^2);
\]

by Lemmas 6.3 and 5.1 \( g_1^r Y_3 \) is a quotient of the Weyl module

\[
W_3 := \mathcal{W} \left( q^{-s_l} \frac{1 - zbq^{2s_l}}{1 - zb}; 1 - zbq^{2s_l} \right).
\]

By assumption \( b q^{2s_l} \notin \{ b_j, a_i q^{-2r_i} : 1 \leq i \leq k, 1 \leq j \leq l \} \). We deduce from Proposition 5.2 that \( W_2 \otimes W_3 \) is of highest \( \ell \)-weight. Its quotient \( g_1^r Y_2 \otimes g_1^s Y_3 \) is also of highest \( \ell \)-weight. \( \square \)

Now we can prove three special cases of Theorem 6.1.

**Corollary 6.8.** Under conditions (1)–(2) in Theorem 6.1, the tensor products \( \otimes_{i=1}^l V_{r_i, a_i}^+ \) and \( \otimes_{j=1}^l V_{s_j, b_j}^- \) are both of highest \( \ell \)-weight.

**Proof.** We shall prove the positive case by induction on \( M \) and \( k \); the negative case uses essentially the same arguments. It is useful to include the case \( M = 0 \) where the \( r_i = 0 \) and \( S \) is trivial. Let \( M > 0 \). (C1) implies the conditions of the \( a_i \) in the above corollary. So

\[(1) : \; v_1^2 \otimes (\otimes_{i=2}^k v_1^1) \in g_1(U)(v_1^2 \otimes (\otimes_{i=2}^k v_1^1)).\]

Now consider the \( U_2 \)-module \( g_2^r(K) \) in Lemma 6.4 with \( K = g_2(U_2)(\otimes_{i=1}^k v_1^1) \) (so \( l = 0 \)). Since (C1) stays the same when replacing \( M \) by \( M - 1 \), the induction hypothesis applied to \( M - 1 \) indicates that \( g_2^r(K) \) is of highest \( \ell \)-weight and

\[(2) : \; v_1^2 \otimes (\otimes_{i=2}^k v_1^1) \in g_2(U_2)(\otimes_{i=1}^k v_1^1).\]

Next the induction hypothesis applied to \( k - 1 \) together with Lemma 2.2 indicates that

\[(3) : \; U_q(\widehat{g})(v_1^3 \otimes (\otimes_{i=2}^k v_1^1)) = \otimes_{i=1}^k V_{r_i, a_i}^+.\]

From (1)–(3) it follows that \( \otimes_{i=1}^k V_{r_i, a_i}^+ \) is of highest \( \ell \)-weight. \( \square \)

**Corollary 6.9.** Let \( 1 \leq r \leq M, 1 \leq s \leq N \) and \( a, b \in \mathbb{C}^\times \). If \( a q^{-2r} \neq b q^2 \) then \( V_{r, a}^+ \otimes V_{s, b}^- \) is of highest \( \ell \)-weight.

**Proof.** Firstly use induction on \( M \). By Corollary 6.7 \( v_1^3 \otimes w^1 \in g_1(U)(v_1^2 \otimes w^1) \) if \( a \neq b q^4 \). Consider \( K = g_2(U_2)v_1^1 \) and \( K' = g_2(U_2)w^1 \) in Lemma 6.3; the induction hypothesis applied to \( M - 1 \) shows that if \( a q^{-2r} \neq b q^2 \) then \( v_1^2 \otimes w^1 \in g_2(U_2)(v_1^1 \otimes w^1) \). Combining with Lemma 2.2, we see that \( V_{r, a}^+ \otimes V_{s, b}^- \) is of highest \( \ell \)-weight if \( a \neq b q^4 \) and \( a q^{-2r} \neq b q^2 \).

Secondly use induction on \( N \). By Corollary 6.7 \( v_1^1 \otimes w^3 \in g_1(U)(v_1^1 \otimes w^2) \) if \( b \neq a q^{-2r-2s} \). Consider \( L = g_3(U_3)v_1^1 \) and \( L' = g_3(U_3)w^1 \) in Lemma 6.3; the induction hypothesis applied to \( N - 1 \) shows that if \( a q^{-2r} \neq b q^2 \) then \( v_1^1 \otimes w^3 \in g_3(U_3)(v_1^1 \otimes w^1) \). Thus \( V_{r, a}^+ \otimes V_{s, b}^- \) is of highest \( \ell \)-weight if \( b \neq a q^{-2r-2s} \) and \( a q^{-2r} \neq b q^2 \).

Conclude as \( \{ a \neq b q^4, a q^{-2r} \neq b q^2 \} \cup \{ b \neq a q^{-2r-2s}, a q^{-2r} \neq b q^2 \} = \{ a q^{-2r} \neq b q^2 \} \). \( \square \)
Corollary 6.10. If \( \frac{a_i}{a_j} \notin \Sigma(r_1, r_2) \) and \( \frac{a_i}{a_j} \notin \Sigma(r_2, r_1) \), then \( V_{r_1, a_1}^+ \otimes V_{r_2, a_2}^+ \cong V_{r_2, a_2}^+ \otimes V_{r_1, a_1}^+ \) as \( U_q(\mathfrak{g}) \)-modules. Similarly, the \( U_q(\mathfrak{g}) \)-modules \( V_{s_1, b_1}^- \otimes V_{s_2, b_2}^- \) and \( V_{s_2, b_2}^- \otimes V_{s_1, b_1}^- \) are isomorphic if \( \frac{b_i}{b_j} \notin \Sigma(s_1, s_2) \) and \( \frac{b_i}{b_j} \notin \Sigma(s_2, s_1) \).

Proof. It is enough to consider positive fundamental modules as \( f^*(V_{r,a}) \cong V_{r,a}^{2r+2} \) in view of the proof of Lemma 2.9. By Lemma 3.5 (A) and Theorem 4.2, there exists a \( U_q(\mathfrak{g}) \)-linear map \( R_{r_1,r_2} : V_{r_1,a_1}^+ \otimes V_{r_2,a_2}^+ \to V_{r_2,a_2}^+ \otimes V_{r_1,a_1}^+ \) sending \( v_1^1 \otimes v_2^1 \) to \( v_1^2 \otimes v_1^1 \). By Corollary 6.8 \( v_1^1 \otimes v_1^1 \) generates \( V_{r_2,a_2}^+ \otimes V_{r_1,a_1}^+ \). So \( R_{r_1,r_2} \) is an isomorphism of \( U_q(\mathfrak{g}) \)-modules. \( \square \)

Corollary 6.11. Under conditions (C1)–(C2) in Theorem 6.1 we have \( \sigma \in \mathfrak{S}_k, \tau \in \mathfrak{S}_l \):

1. as \( U_q(\mathfrak{g}) \)-modules \( \otimes_{i=1}^k V_{r_i,a_i}^+ \cong \otimes_{i=1}^k V_{r_i,a_i}^+ \) and \( \otimes_{j=1}^l V_{s_j,b_j}^- \cong \otimes_{j=1}^l V_{s_j,b_j}^- \);

2. \( \frac{a_{r(i)}}{a_{r(j)}}, \frac{b_{r(i)}}{b_{r(j)}} \notin q^{2Z_{<0}} \) whenever \( i < j \).

One can copy the proof of [AK, Corollary 2.2] using Corollaries 6.8 and 6.10.

Proof of Theorem 6.1. We use induction on \( M + N \) and \( k + l \). Owing to Corollary 6.11 we can assume that for \( 1 \leq i < j \leq k \), either \( \frac{a_i}{a_j} \in q^{2Z_{\geq 0}} \) or \( \frac{a_i}{a_j} \notin q^{2Z} \). If one of the two assumptions in Corollary 6.7 is satisfied, then using reduction to either \( U_2 \) in Lemma 6.4 or to \( U_3 \) in Lemma 6.5 together with Lemma 2.2, we can conclude as in the proof of Corollary 6.8 that \( S \) is of highest \( \ell \)-weight.

Suppose that first assumption in Corollary 6.7 fails, so that \( a_1 = b_0q^4 \) for some \( 1 \leq t \leq l \). Now let us rearrange the tensor product \( \otimes_{j=1}^l V_{s_j,b_j}^- \) as in Corollary 6.11 in such a way that \( \frac{b_{r(l)}}{b_{r(t)}} \in q^{2Z_{\geq 0}} \). (This is possible by Corollary 6.10 and possibly \( \tau(l) = t \).) Assume next that the second assumption in Corollary 6.7 fails, so that \( b_2(l) = a_0q^{-2r_1-2s_\tau(l)} \) for some \( 1 \leq i \leq k \). It follows that \( \frac{b_{r(i)}}{b_{r(t)}} = \frac{a_{r(i)}}{a_{r(t)}}q^{4-2r_1-2s_\tau(l)} \). From \( \frac{b_{r(l)}}{b_{r(t)}} \in q^{2Z_{\geq 0}} \) we obtain \( \frac{a_{r(i)}}{a_{r(t)}} \in q^{2Z} \) and so \( a_i \in q^{2Z_{<0}} \). This forces \( r_i = s_\tau(l) = 1, a_i = a_1 \) and \( b_0q^4 = a_0q^{-2} = a_0q^{-2r_i} \), in contradiction with (C3). This completes the proof of Theorem 6.1. \( \square \)

The following lemma (actually only the case \( k + l = 2 \)) will be used in the next section.

Lemma 6.12. In Theorem 6.1 if \( k = l = 1 \), or \( l = 0 \) or \( k = 0 \), then (C3), or (C1) or (C2) is necessary for \( S \) to be of highest \( \ell \)-weight.

Proof. The case \( k + l = 2 \) comes from the proof of Lemma 2.9, Lemma 3.5 and Theorems 4.1 4.2 and the case \( kl = 0 \) from Corollary 6.10 by induction on \( k + l \) as in the proof of Theorem 6.1 on page 22. \( \square \)

7. Simplicity of Tensor Products

We give equivalent conditions for a tensor product of fundamental modules to be simple. Let us recall the notion of twisted dual to pass from “highest/lowest \( \ell \)-weight” to “simple”. Let \( V \) be a finite-dimensional \( U_q(\mathfrak{g}) \)-module. Its twisted dual, is the dual space \( \text{hom}(V, \mathbb{C}) \) endowed with a \( U_q(\mathfrak{g}) \)-module structure, denoted by \( V^\vee \), as follows:

\[ \langle xl, v \rangle := (-1)^{|x||l|} \langle l, x\Psi(x)\rangle \] for \( x \in U_q(\mathfrak{g}), l \in \text{hom}(V, \mathbb{C}), v \in V \).

For \( V, W \) finite-dimensional \( U_q(\mathfrak{g}) \)-modules, by Equation 2.5, we have a natural isomorphism of \( U_q(\mathfrak{g}) \)-modules \( (V \otimes W)^\vee \cong V^\vee \otimes W^\vee \). For \( 1 \leq i \leq n \), let \( V_i \) be a finite-dimensional
simple $U_q(\mathfrak{g})$-module generated by a highest $\ell$-weight vector $v_i$. Then $V_i^{\vee}$ is again simple and contains a highest $\ell$-weight vector $v_i^*$ such that $v_i^*(v_i) = 1$. By duality argument, the tensor product $\otimes_{i=1}^n V_i$ is of highest $\ell$-weight if and only if the submodule $S$ of $\otimes_{i=1}^n V_i^{\vee}$ generated by $\otimes_{i=1}^n v_i^{\vee}$ is contained in all the other non-zero submodules. ($S$ must then be simple and is the socle of $\otimes_{i=1}^n V_i^{\vee}$.) The tensor product $\otimes_{i=1}^n V_i$ is simple if and only if $\otimes_{i=1}^n V_i$ and $\otimes_{i=1}^n V_i^{\vee}$ are both of highest $\ell$-weight. Similar statements hold for lowest $\ell$-weight modules.

The twisted dual of $V(a)$ has been computed in [Zh3, Eq.(3.26)]:

$$V(a)^\vee \simeq V(a^{-1}q^{2M-2N}).$$

Next let us compute the twisted dual of $W(a)$ in Example 2. Denote by $\rho_a$ the representation of $U_q(\mathfrak{g})$ on $W(a)$. As in [Zh3, §3.2], introduce

$$X(z) = (\rho_a \otimes \text{Id}_{\text{End} W})(\sum_{i,j} s_{ij}(z) \otimes E_{ij})$$

$$= \sum_i (q_i^{-1} - zaq_i) E_{ii} \otimes E_{ii} + (1 - za) \sum_{i \neq j} E_{ii} \otimes E_{jj}$$

$$+ \sum_{i < j} (q_j^{-1} - q_j) E_{ji} \otimes E_{ij} + za \sum_{i > j} (q_j^{-1} - q_j) E_{ji} \otimes E_{ij} \in \text{End}(W^{\otimes 2})[[z]].$$

Set $A := (1 - zaq^2)(1 - zaq^{-2})$. By Equation (2.23), we have

$$X(z)^{-1} = (\rho_a \otimes \text{Id}_{\text{End} W})(\sum_{i,j} S(t_{ij}(z)) \otimes E_{ij})$$

$$= \frac{1}{A} \bigl( (q_i - zaq_i^{-1}) E_{ii} \otimes E_{ii} + (1 - za) \sum_{i \neq j} E_{ii} \otimes E_{jj}$$

$$+ \sum_{i < j} (q_j - q_j^{-1}) E_{ji} \otimes E_{ij} + za \sum_{i > j} (q_j - q_j^{-1}) E_{ji} \otimes E_{ij} \bigr).$$

Similarly we can find the $\rho_a(S(t_{ij}(z)))$. By comparing highest $\ell$-weights, we obtain

$$W(a)^\vee \simeq W(a^{-1}).$$

Proof of Proposition 2.5 Let us consider the positive case; the negative case can then be implied as in the proof of Lemma 2.9. Recall the following fact in [Zh2, Prop.4.7]: $\otimes_{i=1}^s V(aq^{2i})$ is of lowest $\ell$-weight for all $a \in \mathbb{C}^\times$ and $s \in \mathbb{Z}_{>0}$. By taking twisted dual and using the formula of $V(a)^\vee$, we see that the lowest $\ell$-weight vector $v_0^{\otimes s}$ generates the simple socle of $\otimes_{i=1}^s V(aq^{-2i})$ for all $a \in \mathbb{C}^\times$. In particular, $V_{s,a}^+$ in Definition 2.4 is simple.

Lemma 7.1. $(V_{r,a}^+)^\vee \simeq V_{r,a^{-1}q^{2(M-\kappa+1+r)}}^+$ and $(V_{s,a}^-)^\vee \simeq V_{s,a^{-1}q^{-2s-2}}^-$ for $a \in \mathbb{C}^\times$.

Proof. Let us prove the positive case. Recall from the proof of Proposition 2.5 that $\otimes_{j=\tau}^1 V(aq^{-2j})$ is of lowest $\ell$-weight. This gives rise to a diagram of $U_q(\mathfrak{g})$-modules

$$\otimes_{j=\tau}^1 V(aq^{-2j}) \xrightarrow{\theta} V_{r,a}^+ \xrightarrow{\tau} \otimes_{j=1}^r V(aq^{-2j})$$

where $\theta, \tau$ are $U_q(\mathfrak{g})$-linear and they both fix $v_0^{\otimes s}$. Taking the twisted dual and using the formula of $V(a)^\vee$, one obtains a similar diagram where the $U_q(\mathfrak{g})$-linear maps fix lowest $\ell$-weight vectors. One can use Definition 2.4 to conclude $(V_{r,a}^+)^\vee \simeq V_{r,a^{-1}q^{2(M-\kappa+1+r)}}^+$. □
Corollary 7.4. A tensor product of positive fundamental modules is simple if and only if $q$.

The "only if" part is trivial as in the non-graded case in [He, §3–4]. They use essentially Weyl modules in [Zh2 Prop. 4.7] on lowest $\ell$-weight classification in [2.2] we must have a unique $U_q(\mathfrak{g})$-linear isomorphism $V_a \otimes W_b \cong W_b \otimes V_a$ fixing highest $\ell$-weight vectors $v \otimes w \mapsto w \otimes v$. Such an isomorphism also resolves the case where the signature of $(V, W)$ is $(-+)$. □

We would like to emphasize that the above proofs of Proposition 2.5, Lemmas 7.1 and 7.12 are independent of the results in §4 [Zh2]. They use essentially Weyl modules in [Zh2 Prop. 4.7] on lowest $\ell$-weight modules, and the twisted dual formula in [Zh3 Eq. (3.26)].

Theorem 7.2. A tensor product of fundamental modules $V_1 \otimes V_2 \otimes \cdots \otimes V_s$ is simple if and only if so is $V_i \otimes V_j$ for all $1 \leq i < j \leq s$.

Proof. The "only if" part is trivial as in the non-graded case in [He, §6]: if $\otimes_{i=1}^s V_i$ is simple, then so are $S_i \otimes S_i$ and $\otimes_{i=1}^s S_{\sigma(j)}$ for $1 \leq i < s$ and $\sigma \in S_s$ by comparing highest $\ell$-weights. For the "if" part, since $V_i \otimes V_j$ is simple, it is isomorphic to $V_j \otimes V_i$. Without loss of generality we can assume that $\otimes_{i=1}^s V_i =: S$ is of the form in Theorem 6.11; a tensor product of positive fundamental modules followed by that of negative fundamental modules. By Lemma 6.12 such a tensor product verifies the conditions (C1)–(C3) in Theorem 6.11 and is therefore of highest $\ell$-weight. Similar arguments adapted to $(\otimes_{i=1}^s V_i)^\vee \cong \otimes_{i=1}^s V_i^\vee$ by Lemma 7.1 lead to $S^\vee$ is of highest $\ell$-weight. So $S$ is simple. □

Remark 7.3. Let us make explicit the simplicity condition. Index $i = (r_i, \varepsilon_i)$ where $1 \leq r_i \leq M$ if $\varepsilon_i = +$ and $1 \leq r_i \leq N$ if $\varepsilon_i = -$. Define

$$\Delta_{ij} := \left\{ \begin{array}{ll}
(\prod_{i=1}^{\min(r_i, r_j)} (a_i - a_j q^{\pm 2(r_j - \min(r_i, r_j))}) & \text{if } \varepsilon_i = \varepsilon_j = \pm,
(a_i - a_j q^{2r_i + 2}) & \text{if } (\varepsilon_i, \varepsilon_j) = (+, -),
(a_i - a_j q^{-2M + 2N - 2r_i - 2}) & \text{if } (\varepsilon_i, \varepsilon_j) = (-, +). \end{array} \right. $$

Then $\otimes_{i=1}^s V_{r_i, \varepsilon_i}$ is simple if and only if $\Delta_{ij} \neq 0$ for all $i \neq j$.

Corollary 7.4. A tensor product of positive fundamental modules is simple if and only if it is both of highest $\ell$-weight and of lowest $\ell$-weight.

Proof. The "only if" part is trivial by definition. The "if" part is a direct consequence of Theorem 7.2 Lemma 6.12 and the above theorem. □

The above corollary remains true for tensor products of negative fundamental modules, by using the pull back $f^*$ in the proof of Lemma 2.9. In [Zh2 §5], the above corollary was proved for all finite-dimensional simple modules over a Borel subalgebra of $U_q(\mathfrak{gl}(1, 1))$ (and so over the full quantum affine superalgebra), the so-called $q$-Yangian.

Example 4. Corollary 7.4 fails if "positive" is removed. Consider $S := V(a) \otimes W(b) = V_1^{+a} \otimes V_{1, b}^{-2}$. We have $v_1^1 = v_1^2 = v_1$ and $w_1^1 = w_2^2 = w_\kappa$. Set $W = g_1(U)v_1^1$ and $W' = g_2(U)w_2^1$. By Lemmas 6.2, 6.3 and 6.6 as $U$-modules

$$g_1^* (W \otimes W') \cong g_1^* W \otimes g_1^* W' \cong V(\frac{q - zaq^{-1}}{1 - za}) \otimes V(\frac{1 - zb}{q - zbq^{-1}}).$$
It follows that $S$ is of highest $\ell$-weight and of lowest $\ell$-weight if $a \neq b$. On the other hand, $S$ is simple if and only if $a \neq b$ and $b \neq aq^{-2M+2N}$.

**Example 5.** Let $1 \leq s \leq M$ and $1 \leq t \leq N$ be such that $s - t = M - N$. The tensor product $V_{s,a}^+ \otimes V_{t,b}^-$ is simple if and only if it is of highest $\ell$-weight.

8. Final remarks

In this final section, we make remarks which are not used in the proof of main results.

We use the convention in Remark 7.3 associated to an index $1 \leq i \leq s$ is a couple $(r_i, \varepsilon_i)$ where either $(\varepsilon_i = +, 1 \leq r_i \leq M)$ or $(\varepsilon_i = -, 1 \leq r_i \leq N)$. Consider the tensor product $S:= \otimes_{i=1}^s V_{r_i,a_i}^\varepsilon$. We want to know when $S$ is of highest $\ell$-weight.

Let us define $S$. Theorem 6.1 gives a criteria for $S$ to be of highest $\ell$-weight in signature $(+ + \cdots - - \cdots)$. In the proof of Theorem 6.1 Corollary 6.7 is the crucial step to go from $s$:

$$g \boxtimes \sum_{i,j} s_{ij}(z) \otimes E_{ij} \mapsto \sum_{i,j} t_{ij}(z) \otimes E_{ij} \mapsto \sum_{i,j} t_{ij}(z) \otimes E_{ij}^{-1}.$$

**Lemma 8.1.** There is an isomorphism of Hopf superalgebras $h : U_{q^{-1}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})\text{cop}$

$$\sum_{i,j} \tilde{s}_{ij}(z) \otimes E_{ij} \mapsto \sum_{i,j} s_{ij}(z) \otimes E_{ij}^{-1}, \quad \sum_{i,j} \tilde{t}_{ij}(z) \otimes E_{ij} \mapsto \sum_{i,j} t_{ij}(z) \otimes E_{ij}^{-1}.$$

**Proof.** The idea is the same as that of [Zh2] Prop.3.4, based on the identity

$$R_q(z,w)^{-1} = \frac{1}{(zq-w)^{-1}(wq^{-1}-wq)}R_{-1}(z,w) \in \text{End}(V^2)(z,w).$$

and on the definition of $U_q(\mathfrak{g})$ in [Zh2] Definition 3.5].

$h$ is inspired by the involution of quantum affine algebras in [AK] Appendix A.

Let $\tilde{V}_{r,a}^\varepsilon$ be the corresponding fundamental modules over $U_{q^{-1}}(\mathfrak{g})$. From the definition of twisted dual and from Lemma 6.1 we obtain

\begin{equation}
(8.13) \quad h^*V_{r,a}^+ \simeq \tilde{V}_{r,aq^{-2M+2N-2r+2}}^+, \quad h^*V_{r,a}^- \simeq \tilde{V}_{r,aq^{2r+2}}^-.
\end{equation}

Now Lemma 6.2 and Corollary 6.7 can be generalized accordingly. For this purpose, let us define the $K_{ij}^l$ and $K_{ij}^r$ associated to the tensor product $S := \otimes_{i=1}^s V_{r_i,a_i}^{\varepsilon_i}$ as follows:

\begin{equation}
(8.14) \quad (K_{ij}^l, K_{ij}^r) := \begin{cases} (1, 1), & \text{if } \varepsilon_i = \varepsilon_j, \\ (a_i - a_jq^4, a_i - a_jq^{2r_i+2r_j}), & \text{if } (\varepsilon_i, \varepsilon_j) = (+, -), \\ (a_i - a_jq^{-2M+2N-4}, a_i - a_jq^{-2M+2N-2r_i-2r_j}), & \text{if } (\varepsilon_i, \varepsilon_j) = (-, +), \\ (a_i - a_jq^{2M+2N-4}, a_i - a_jq^{-2M+2N-2r_i-2r_j}), & \text{if } (\varepsilon_i, \varepsilon_j) = (-, +), \end{cases}
\end{equation}

\begin{equation}
(8.15) \quad f_i^l := \prod_{j<i} \Delta_{ij} \times \prod_{j \neq i} K_{ij}^l, \quad f_i^r := \prod_{j \neq i} K_{ij}^r \times \prod_{j > i} \Delta_{ji}.
\end{equation}
To unify notations in [6] let $u_1^i$ and $u_3^i$ be highest and lowest $\ell$-weight vectors of $V_{r_i,a_i}^{\varepsilon_i}$ and $u_2^i = s_{1i}(0)u_3^i$. From Theorem 2.6 and Lemmas 2.8, 2.9 we see that $u_1^i = h(s_{1i})u_3^i$. Corollary 6.7 together with its proof is now generalized as follows.

**Corollary 8.2.** Assume that $\Delta_{ij} \neq 0$ for all $1 \leq i < j \leq s$. If $f_l^i \neq 0$, then $S = \bigotimes_{j \neq i} V_{r_j,a_j}^{\varepsilon_j} =: S^i_l$ as $U_q(\widehat{g})$-modules and $u_1^i \otimes (\bigotimes_{j \neq i} u_1^j) \in S^i_l$. Similarly, if $f_r^i \neq 0$, then $S = \bigotimes_{j \neq i} V_{r_j,a_j}^{\varepsilon_j} =: S^i_r$ as $U_q(\widehat{g})$-modules and $(\bigotimes_{j \neq i} u_1^j) \otimes u_3^i \in U_q(\widehat{g})((\bigotimes_{j \neq i} u_1^j) \otimes u_3^i) \subseteq S^i_r$.

In the corollary, $\bigotimes_{j \neq i}$ means the ordered tensor product $(\otimes_{j=1}^{i-1})(\otimes_{j=i+1}^{s})$. We arrive at the following problem of linear algebra.

**Question 1.** For $1 \leq i \leq s$, let $(r_i, \varepsilon_i)$ be as above and let $a_i \in \mathbb{C}^x$. Define $\Delta_{ij}, f_l^i, f_r^i$ by Equations (7.12), (8.14) and (8.15). Suppose that $f_l^i = f_r^i = 0$ for all $1 \leq i \leq s$. Then is it necessarily true that $\prod_{i<j} \Delta_{ij} = 0$?

**Remark 8.3.** Suppose that the answer to the above question is positive. We can argue as in the proof of Corollary 6.8 to conclude that $S = \bigotimes_{i=1}^{s} V^{\varepsilon_i}_{r_i,a_i}$ is of highest $\ell$-weight if $\prod_{i<j} \Delta_{ij} \neq 0$.

The proof of Theorem 6.1 together with Lemma 8.1 and Equation (8.13) actually affirms the cases $(\varepsilon_1 \cdots \varepsilon_s) = (++ \cdots + - - - -) \text{ and } (- - \cdots - + + + +)$. So Theorem 6.1 remains true when the tensor product is of signature $(- - \cdots + + + +)$.

**Example 6.** Let $s = 3$. The answer to Question 1 is affirmative. Indeed the only essential difficulty appears when $(\varepsilon_1 \varepsilon_2 \varepsilon_3) = (++-)$ and $\prod_{i<j} \Delta_{ij} \neq 0 = \Delta_{21} = \Delta_{32}$. (If $\Delta_{21} \neq 0$ then one can exchange $\varepsilon_1$ and $\varepsilon_2$ to arrive at the known signature $(-++)$; similar arguments for $\Delta_{32}$.) Suppose $f_l^i = f_r^i = 0$. By definition, $f_l^i = K_{12} = 0$ and $f_r^3 = K_{33} = 0$. From $K_{12} = 0 = \Delta_{21} = a_1 - a_2 q^4 = a_2 - a_1 q^{-2M+2N-2r_2-2}$

we get $2 = 2M - 2N + 2r_2$. Next from $K_{23}^r = a_2 - a_3 q^{-2M+2N-2r_2-2r_3} = 0$ we get $a_1 = a_3 q^{2-2r_3}$. But $\Delta_{13} \neq 0$, we have $\min(r_1, r_3) = 1$. Since $\Delta_{12} = a_1 - a_2 q^{2r_1+2} \neq 0$ and $a_1 = a_2 q^{r_1}$, $r_1 > 1$. Since $\Delta_{23} = a_2 - a_3 q^{-2M+2N-2r_2-2} \neq 0$ and $a_2 = a_3 q^{-2M+2N-2r_2-2r_3}$, $r_3 > 1$. It follows that $\min(r_1, r_3) > 1$, a contradiction. As a consequence, $V_{r_1,a_1}^{\varepsilon_1} \otimes V_{r_2,a_2}^{\varepsilon_2} \otimes V_{r_3,a_3}^{\varepsilon_3}$ is of highest $\ell$-weight if $\prod_{i<j} \Delta_{ij} \neq 0$.

**Example 7.** Assume that $r_i = 1$ for all $1 \leq i \leq s$. Since $f_l^i = 0 = \prod_{i=2}^{s} K_{1i}$, there exists $1 < i \leq s$ such that $K_{1i} = 0$. It follows that $\varepsilon_1 \neq \varepsilon_i$ and $\Delta_{1i} = K_{1i} = 0$. As a consequence, the tensor product $\bigotimes_{i=1}^{s} V^{\varepsilon_i}_{r_i,a_i}$ is of highest $\ell$-weight if $\prod_{i<j} \Delta_{ij} \neq 0$.

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