Delay-dependent Asymptotic Stability of Highly Nonlinear Stochastic Differential Delay Equations Driven by $G$-Brownian Motion

Chen Fei$^2$, Weiyin Fei$^1$, Xuerong Mao$^3$, Litan Yan$^2$

1 School of Mathematics and Physics, Anhui Polytechnic University, Wuhu, Anhui 241000, China.
2 Department of Statistics, College of Science, Donghua University, Shanghai, 200051, China.
3 Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK.

Abstract

Based on the classical probability, the stability of stochastic differential delay equations (SDDEs) whose coefficients are either linear or nonlinear but bounded by linear functions have been investigated intensively. Moreover, the delay-dependent stability of highly nonlinear hybrid stochastic differential equations has also been studied recently. In this paper, by using the nonlinear expectation theory, we explore the delay-dependent stability of a class of highly nonlinear hybrid stochastic differential delay equations driven by $G$-Brownian motion ($G$-SDDEs). Firstly, we give preliminaries on sublinear expectation. Then, the delay-dependent criteria of the stability and boundedness of solutions to $G$-SDDEs is developed. Finally, an illustrative example is analyzed by the $\varphi$-max-mean algorithm.

Key words: stochastic differential delay equation (SDDE); sublinear expectation; $G$-Brownian motion; asymptotic stability; Lyapunov functional.

MR Subject Classification: 60H10.

1 Introduction

A sublinear expectation (c.f., [38, 39, 41]) can be represented as the upper expectation of a subset of linear expectations. In most cases, this subset is often treated as an uncertain model of probabilities. Peng introduced the $G$-expectation theory ($G$-framework) (see [38] and the references therein) in 2006, where the notion of $G$-Brownian motion and the corresponding stochastic calculus of Itô’s type were established. Moreover, the stochastic calculus on $G$-Brownian motion is explored by Denis et al. [4], Fei and Fei [13], Hu and Peng [22], Li and Peng [26], Peng and Zhang [43], Soner et al. [47] and Zhang [52], etc.

So far, there is large amount of literature on the problem of asset pricing and financial decisions under model uncertainty. Chen and Epstein [1] put forward to the model of an intertemporal recursive utility, where risk and ambiguity are differentiated, but uncertainty is only a mean uncertainty without a volatility uncertainty. The model of the optimal consumption and portfolio with ambiguity are also investigated in Fei [11, 12]. Epstein and Ji [5, 6] generalized the Chen-Epstein model and maintained a separation between risk aversion and intertemporal substitution. We know that equivalence of priors is an optional assumption in

*Corresponding author. E-mail: wyfe@ahpu.edu.cn
Gilboa and Schmeidler [18]. Apart from very recent developments, the stochastic calculus presumes a probability space framework. However, from an economics perspective, the assumption of equivalence seems far from innocuous. Informally, if her environment is complex, how could the decision-maker come to be certain of which scenarios regarding future asset prices and rates of return, for example, are possible? In particular, ambiguity about volatility implies ambiguity about which scenarios are possible, at least in a continuous time setting. A large amount of literature has argued that the stochastic time varying volatility is important for understanding features of asset returns, and particularly empirical regularities in the derivative markets.

On the other hand, it is well to be known that the study of differential equations from both the theory and its applications is important. The classical stochastic differential equation with Brownian motion doesn’t take an ambiguous factor into consideration. Thus, in some complex environments, these equations are too restrictive for describing some phenomena. Recently, under uncertainty, a kind of stochastic differential equations driven by $G$-Brownian motion has been investigated by Deng et al. [3], Fei and Fei [7], Fei et al. [9], Gao [17], Hu et al. [23], Li and Yang [27], Lin [28], Lin [29], Luo and Wang [32], Peng [41], Ren et al. [44], Xu et al. [49], Yao and Zong [50], and Yin et al. [51].

We know that the stability of the classical stochastic differential equations is an important topic in the study of stochastic systems. Since Lyapunov [33] and LaSalle [24] obtained the stability for nonlinear systems, with the help of Itô’s breakthrough work about Itô calculus, Hasminskii [19] first studied the stability of the linear Itô equation perturbed a noise. Since then, the stability analysis for stochastic differential systems has been done by many researchers. The stabilization and destabilization of hybrid systems of stochastic differential equations is explored by many researchers, such as Mao et al. [35]. In Luo and Liu [31], the stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps is investigated.

Since a system described by the stochastic differential equation driven by $G$-Brownian motion provides a characterization of the real world with both randomness and ambiguity, it is necessary to investigate its stability like a classical stochastic differential equation. One kind of exponential stability for stochastic differential equations driven by $G$-Brownian motion is discussed by Zhang and Chen [53] where the quasi-sure analysis is used. Fei and Fei [14] investigated the quasi sure exponential stability by $G$-Lyapunov functional method for obtaining different results.

However, in many real systems, such as science, industry, economics and finance etc., we will run into time lag. Differential delay equations (DDEs) have been used to set up such time-delay systems. However, since the time-delay often causes the instability of systems, the stability of DDEs has been investigated intensively for more than 50 years. In 1980’s, the stochastic delay differential equations were developed in order to model real systems which are subject to external noises. Since then, in the study of SDDEs including hybrid SDDEs the stability analysis has been one of the most important topics (see, e.g., [25, 30, 34]).

The existing delay-dependent stability criteria are mainly provided for the hybrid SDDEs whose coefficients are either linear or nonlinear but bounded by linear functions (or, satisfy the linear growth condition). Recently, [20, 21] initiate the investigation on the stability of the hybrid highly nonlinear stochastic delay differential equations driven by Brownian noise. Based on highly nonlinear hybrid SDDEs (see, e.g., [20, 21]), [15] has further established the delay-dependent stability criterion. Recently, the delay-dependent stability of the highly nonlinear hybrid SDDEs is investigated in [8, 10, 16, 37, 45, 46].

Moreover, to our best knowledge, the current states of the considered real systems mainly follow an SDDE disturbed by Brownian motion. In fact, due to uncertainty including probabilistic and Knightian ones [11], many real systems may be disturbed by $G$-Brownian motion. Thus we can characterize our some real systems by Peng’s sublinear expectation framework. Moreover, in this paper, we will explore the dependent stability criteria of SDDEs driven by
G-Brownian motion (G-SDDEs), where the coefficients of G-SDDEs are highly nonlinear without linear growth conditions. Our results extend the ones under linear expectation framework discussed in Fei et al.\textsuperscript{[15]}.

We now give an example on the stability of the following highly nonlinear G-SDDE

\[ dX(t) = [-X^3(t) - X(t - \delta(t))]dt + \frac{1}{2} \epsilon^{-t}dB(t), \] (1.1)

where \( \delta(t) \geq 0 \) is a time lag, \( X(t) \in \mathbb{R} \) is the state of the highly non-linear hybrid system, \( B(t) \) is a scalar G-Brownian motion (related notions see \[41\]).

For the above system (1.1), if the time delay \( \delta(t) = 0.01 \), the computer simulation shows it is asymptotically stable (see Figure 4.1). If the time-delay is large, say \( \delta(t) = 2 \), the computer simulation shows that G-SDDE (1.1) is unstable (see Figure 4.2). In other words, whether the delay G-SDDE is stable or not depends on how small or large the time-delay is. On the other hand, both drift and diffusion coefficients of G-SDDE affect the stability of the system due to highly nonlinear. However, there is no delay dependent criterion which can be applied to G-SDDE to derive a sufficient bound on the time-delay \( \delta(t) \) such that G-SDDE is stable. The aim of this paper is to establish the delay dependent criteria for highly nonlinear G-SDDEs.

The main contributions of this paper are presented as follows:

(i). We first try to give the criteria on delay-dependent stability of G-SDDEs with highly nonlinear coefficients.

(ii). The delay-dependent criteria are established by new mathematical techniques, such as constructing Lyapunov functional.

(iii). The stochastic calculus on G-Brownian motion is applied to solve the stability of the systems with ambiguity.

The arrangement of the paper is as follows. In Section 2, we give preliminaries on sublinear expectations and G-Brownian motions. Furthermore, we characterize the properties of G-Brownian motions and G-martingales. In Section 3, the quasi sure exponential stability of the solutions to G-SDDE is studied. Next, we give an illustrative example in Section 4, where we use the \( \varphi \)-max-mean algorithm suggested by Peng \[40\]. Finally, Section 5 concludes this paper.

2 Notations and Preliminaries

In this section, we first give the notion of sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\), where \( \Omega \) is a given state set and \( \mathcal{H} \) a linear space of real valued functions defined on \( \Omega \). The space \( \mathcal{H} \) can be considered as the space of random variables. The following concepts come from Peng \[41\].

**Definition 2.1** A sublinear expectation \( \hat{E} \) is a functional \( \hat{E} : \mathcal{H} \rightarrow \mathbb{R} \) satisfying

(i) Monotonicity: \( \hat{E}[X] \geq \hat{E}[Y] \) if \( X \geq Y \);

(ii) Constant preserving: \( \hat{E}[c] = c \);

(iii) Sub-additivity: For each \( X,Y \in \mathcal{H}, \hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y] \);

(iv) Positivity homogeneity: \( \hat{E}[\lambda X] = \lambda \hat{E}[X] \) for \( \lambda \geq 0 \).

**Definition 2.2** Let \((\Omega, \mathcal{H}, \hat{E})\) be a sublinear expectation space. \((X(t))_{t \geq 0}\) is called a d-dimensional stochastic process if for each \( t \geq 0 \), \( X(t) \) is a d-dimensional random vector in \( \mathcal{H} \).

A d-dimensional process \((B(t))_{t \geq 0}\) on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called a G-Brownian motion if the following properties are satisfies:

(i) \( B_0(\omega) = 0 \);

(ii) for each \( t,s \geq 0 \), the increment \( B(t+s) - B(t) \) is \( N(\{0\} \times s\Sigma) \)-distributed and is independent from \( (B(t_1), B(t_2), \cdots, B(t_n)) \), for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \).
More details on the notions of $G$-expectation $\hat{E}$ and $G$-Brownian motion on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ may be found in Peng [11].

We now give the definition of Itô integral. For simplicity, in the rest of the paper, we introduce Itô integral with respect to 1-dimensional $G$-Brownian motion with $G(\alpha) := \frac{1}{2}\hat{E}[\alpha B(1)^2] = \frac{1}{2}(\sigma^2\alpha^2 - \alpha^2\alpha^2)$, where $\hat{E}[B(1)^2] = \sigma^2, \hat{E}[B(1)^2] := -\hat{E}[-B(1)^2] = \sigma^2$, $0 < \sigma \leq \sigma < \infty$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = (t_0, \cdots, t_N)$ of $[0, T]$, where $T$ can take infinite, we get

$$\eta(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L^p_G(\Omega)$, $k = 0, 1, \cdots, N - 1$ are given. The collection of these processes is denoted by $M^p_G(0, T)$. We denote by $M^p_G(0, T)$ the completion of $M^p_G(0, T)$ with the norm

$$\|\eta\|_{M^p_G(0, T)} := \left\{ \int_0^T \hat{E}\big|\eta(t)\big|^p dt \right\}^{1/p} < \infty.$$ We provide the following property which comes from Denis et al. [4] or Zhang and Chen [53].

**Proposition 2.3** Let $\hat{E}$ be $G$-expectation. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that for all $X \in \mathcal{H}, \hat{E}[X] = \max_{P \in \mathcal{P}} E^P[X]$, where $E^P[\cdot]$ is the linear expectation with respect to $P$.

From the above Proposition 2.3, we know that the weakly compact family of probability measures $\mathcal{P}$ characterizes the degree of Knightian uncertainty. Especially, if $\mathcal{P}$ is singleton, i.e. $\{P\}$, then the model has no ambiguity. The related calculus reduces to a classical one.

The definition of stochastic integral $\int_0^T \eta(t)dB(t)$ can see Peng [11]. By [13, Proposition 2.5] and Peng [11, Lemma 3.5 on page 43], we easily obtain the following lemma.

**Lemma 2.4** For all $s < t \in [0, T]$, we have

$$\hat{E}\left| \int_s^t \zeta(v)dv < B(v) \right|^2 \leq (t-s)\hat{E}\left( \int_s^t |\zeta(v)|^2dv \right),$$

$$\hat{E}\left| \int_s^t \zeta(v)dB(v) \right|^2 \leq \hat{E}\left( \int_s^t |\zeta(v)|^2dv \right).$$

The following Burkholder-Davis-Gundy inequality provides the explicit bounds based on those in Gao [17, Theorems 2.1-2.2].

**Lemma 2.5** (Burkholder-Davis-Gundy inequality) Let $p > 0$ and $\zeta = \{\zeta(s), s \in [0, T]\} \in M^p_G(0, T)$. Then, for all $t \in [0, T]$,

$$\hat{E}\sup_{s \leq u \leq t} \left| \int_s^u \zeta(v)dv < B(v) \right|^p \leq (t-s)^{p-1}C_1(p, \tilde{\sigma})\hat{E}\left( \int_s^t |\zeta(v)|^2dv \right)^{p/2},$$

$$\hat{E}\sup_{s \leq u \leq t} \left| \int_s^u \zeta(v)dB(v) \right|^p \leq C_2(p, \tilde{\sigma})\hat{E}\left( \int_s^t |\zeta(v)|^2dv \right)^{p/2},$$
where \( C_1(p, \sigma) = \bar{\sigma}^p, \) \( C_2(p, \sigma) = C_p \bar{\sigma}^p, \) and the constant \( C_p \) is defined as follows

\[
C_p = \left( \frac{32}{p} \right)^{p/2}, \quad \text{if } 0 < p < 2;
\]

\[
C_p = 4, \quad \text{if } p = 2;
\]

\[
C_p = \left( \frac{p^p + 1}{2(p - 1)^{p-1}} \right)^{p/2}, \quad \text{if } p > 2.
\]

**Proof:** Based on the idea from Peng et al. [42], we can characterize the probability measure family \( \mathcal{P} \) as follows:

\[
\mathcal{P} = \{ P(\sigma) ; (\sigma_t)_{t \in [0, \infty]} \text{ is a continuous process}, \sigma_t \in [\bar{\sigma}, \bar{\sigma}] \}.
\]

For each probability measure \( P(\sigma) \in \mathcal{P}, \) we have \( dB(t) = \sigma_t dw(t), \) where \( (w(t))_{t \geq 0} \) is a standard Brownian motion under \( P(\sigma). \) The corresponding linear expectation is denoted by \( \hat{E}^{P(\sigma)}. \) Thus, by the Hölder inequality, we get

\[
\hat{E} \sup_{s \leq u \leq t} \left| \int_s^u \zeta(v) d B(v) \right|^p
\]

\[
= \sup_{P(\sigma) \in \mathcal{P}} \hat{E}^{P(\sigma)} \sup_{s \leq u \leq t} \left| \int_s^u \zeta(v) d w(v) \right|^p
\]

\[
\leq (t - s)^{p - 1} \bar{\sigma}^{p} \hat{E}^{P(\sigma)} \left( \int_s^t |\zeta(v)|^2 d v \right)^{p/2}
\]

\[
= (t - s)^{p - 1} \bar{\sigma}^{p} \hat{E} \left( \int_s^t |\zeta(v)|^2 d v \right)^{p/2}.
\]

From the classical Burkholder-Davis-Gundy inequality (see, e.g. Mao and Yuan [36, Theorem 2.13 on page 70]), we get

\[
\hat{E} \sup_{s \leq u \leq t} \left| \int_s^u \zeta(v) d B(v) \right|^p
\]

\[
= \sup_{P(\sigma) \in \mathcal{P}} \hat{E}^{P(\sigma)} \sup_{s \leq u \leq t} \left| \int_s^u \zeta(v) d w(v) \right|^p
\]

\[
\leq C_p \sup_{P(\sigma) \in \mathcal{P}} \hat{E}^{P(\sigma)} \sup_{s \leq u \leq t} \left| \int_s^u |\zeta(v)|^2 d v \right|^{p/2}
\]

\[
\leq C_p \bar{\sigma}^{p} \hat{E} \left( \int_s^t |\zeta(v)|^2 d v \right)^{p/2}.
\]

Thus, the proof is complete. \( \square \)

We now define \( G \)-upper capacity \( \mathcal{V}(\cdot) \) and \( G \)-lower capacity \( \mathcal{V}(\cdot) \) by

\[
\mathcal{V}(A) = \sup_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{B}(\Omega),
\]

\[
\mathcal{V}(A) = \inf_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{B}(\Omega).
\]

Thus a property is called to hold quasi surely (q.s.) if there exists a polar set \( D \) with \( \mathcal{V}(D) = 0 \) such that it holds for each \( \omega \in D^c \). We say that a property holds \( P \)-q.s. means that it holds \( P \)-a.s. for each \( P \in \mathcal{P} \). If an event \( A \) fulfills \( \mathcal{V}(A) = 1 \), then we call the event \( A \) occurs \( \mathcal{V} \)-a.s.

## 3 Delay-Dependent Asymptotic Stability of G-SDDEs

For the convenience of presentation, all processes take values in \( \mathbb{R} \). Let \( \mathbb{R}_+ = [0, \infty) \). If \( A \) is a subset of \( \Omega \), denote by \( I_A \) its indicator function. Let \( (\Omega, \mathcal{H}, \{\Omega_t\}_{t \geq 0}, \mathbb{E}, \mathcal{V}) \) be a generalized
sublinear expectation space. Let \((B(t))_{t \geq 0}\) be one dimensional \(G\)-Brownian motion defined on the sublinear expectation space.

Let \(f, g : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}\) be Borel measurable functions and \(h : \mathbb{R}_+ \to \mathbb{R}\) be deterministic continuous function. For \(\tau > 0\), let \(\delta(t) \in [0, \tau], t \geq 0\) satisfy \(d\delta(t)/dt = \dot{\delta}(t) \leq \delta\). Consider one dimensional highly nonlinear variable delay SDE driven by \(G\)-Brownian motion (\(G\)-SDDE)

\[
dx(t) = f(X(t), X(t - \delta(t)), t)dt + g(X(t), X(t - \delta(t)), t)d < B(t) > + h(t)dB(t) \tag{3.1}
\]
on \(t \geq 0\) with nonrandom initial data

\[
\{X(t) : -\tau \leq t \leq 0\} = \eta \in C([-\tau, 0]; \mathbb{R}). \tag{3.2}
\]

The uniqueness of solutions to stochastic differential equations driven by \(G\)-Brownian motion has been proved under the coefficients satisfying non-Lipschitzian conditions, where the coefficients is often bounded by a linear function (see, e.g., Lin [28]). In this paper, however, the coefficients of \(G\)-SDDE (3.1) cannot be bounded by a linear function. The coefficients in (3.1) are called highly nonlinear in terms of Fei et al. [15] and Hu et al. [20], the corresponding equations (3.1) are called highly nonlinear \(G\)-SDDEs. In Fei et al. [9], the existence and uniqueness of solutions to \(G\)-SDDEs (3.1) are proved, while the stability and boundedness of solutions to \(G\)-SDDE (3.1) is investigated as well. We will consider highly nonlinear \(G\)-SDDEs which, in general, do not satisfy the linear growth condition in this paper. Therefore, we impose the polynomial growth condition, instead of the linear growth condition. Let us provide these conditions as an assumption for our aim.

**Assumption 3.1** Assume that for any \(b > 0\), there exists a positive constant \(K_b\) such that

\[
|f(x, y, t) - f(\bar{x}, y, t)| \vee |g(x, y, t) - g(\bar{x}, y, t)| \\
\leq K_b(|x - \bar{x}| + |y - \bar{y}|)
\]
for all \(x, \bar{x}, y, \bar{y} \in \mathbb{R}^d\) with \(|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq b\) and all \(t \in \mathbb{R}_+.\) Assume moreover that there exist three constants \(K > 0, q_1, q_2\) such that

\[
|f(x, y, t)| \leq K(1 + |x|^{q_1} + |y|^{q_1}) \\
|g(x, y, t)| \leq K(1 + |x|^{q_2} + |y|^{q_2}) \\
h(t) \leq K
\]
for all \(x, y \in \mathbb{R}, t \in \mathbb{R}_+.\)

The condition (3.3) with \(q_1 = q_2 = 1\) is the familiar linear growth condition. But, we emphasise that we are here interested in highly nonlinear \(G\)-SDDEs which mean \(q_1 > 1\) or \(q_2 > 1\). The condition (3.3) is referred as the polynomial growth condition. It is known that Assumption 3.1 only guarantees that the \(G\)-SDDE (3.1) with the initial data (3.2) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition by Lyapunov functions. To this end, we need more notation.

We denote \(C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+)\) as the family of non-negative functions \(U(x, t)\) defined on \((x, t) \in \mathbb{R} \times \mathbb{R}_+\) which are continuously twice differentiable in \(x\) and once in \(t\). Now we can state another assumption.
**Assumption 3.2** Let $H(\cdot) \in C(\mathbb{R} \times [-\tau, \infty); \mathbb{R}_+).$ Assume that there exists a function $\tilde{U} \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+),$ nonnegative constants $c_1, c_2, c_3$ and $q = 2(q_1 \lor q_2)$ such that

$$c_3 < c_2(1 - \delta), |x|^q \leq \tilde{U}(x, t) \leq H(x, t)$$

for $\forall (x, t) \in \mathbb{R} \times \mathbb{R}_+,$ and

$$\mathbb{L}\tilde{U}(x, y, t) := \tilde{U}_t(x, t) + \tilde{U}_x(x, t)f(x, y, t) + G(2\tilde{U}_x(x, t)g(x, y, t) + \tilde{U}_{xx}(x, t)h^2(t))$$

$$\leq c_1 - c_2 H(x, t) + c_3 H(y, t - \delta(t))$$

for all $x, y \in \mathbb{R}, t \in \mathbb{R}_+.$

The following result gives the boundedness of the solution to $G$-SDDE (3.1) (see, e.g., Fei et al. [9] Theorem 5.2).

**Proposition 3.3** Under Assumptions 3.1 and 3.2, the highly nonlinear $G$-SDDE (3.1) with the initial data (3.2) has unique global solution satisfying

$$\sup_{-\tau \leq t < \infty} \mathbb{E}|X(t)|^q < \infty.$$

Next, we will use the method of Lyapunov functionals to investigate the delay-dependent asymptotic stability. We define two segments $\bar{X}(t) := \{X(t + s) : -2\tau \leq s \leq 0\}$ for $t \geq 0.$ For $\bar{X}(t)$ to be well defined for $0 \leq t < 2\tau,$ we set $X(s) = \eta(-\tau)$ for $s \in [-2\tau, -\tau).$ We construct the Lyapunov functional as follows

$$V(\bar{X}(t), t) = U(X(t), t)$$

$$\theta \int_{t-\tau}^{t} \int_{t+s}^{t} \left[ \tau |f(X(u), X(u - \delta(u)), u)|^2 + \tau \delta^2 |g(X(u), X(u - \delta(u), u))|^2 + \delta^2 |h(u)|^2 \right] duds$$

for $t \geq 0,$ where $U \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+)$ such that

$$\lim_{|x| \to \infty} \left[ \inf_{t \in \mathbb{R}_+} U(x, t) \right] = \infty,$$

and $\theta$ is a positive number to be determined later while we set

$$f(x, s) = f(x, 0), \ g(x, s) = g(x, 0)$$

for all $x \in \mathbb{R}, s \in [-2\tau, \infty).$ Applying the Itô formula for $G$-Brownian motion (see [10]) to $U(X(t), t),$ we get, for $t \geq 0,$

$$dU(X(t), t)$$

$$= \left( U_t(X(t), t) + U_x(X(t), t)f(X(t), X(t - \delta(t)), t) \right) dt$$

$$+ \left( U_x(X(t), t)g(X(t), X(t - \delta(t)), t) + \frac{1}{2} h^2(t)U_{xx}(X(t), t) \right) dB(t)$$

$$+ U_x(X(t), t)h(t)dB(t)$$

$$\leq LU(X(t), t, X(t - \delta(t), t)dt + U_x(X(t), t)h(t)dB(t)$$
by [13 Proposition 2.5]. Rearranging terms gives

\[
dU(X(t), t) \\
\leq \left( U_x(X(t), t)[f(X(t), X(t - \delta(t)), t) - f(X(t), X(t), t)] \\
+ LU(X(t), X(t - \delta(t)), t) \right) dt + U_x(X(t), t)h(t)dB(t),
\]

where the function \( LU : \mathbb{R}^2 \times C([-\delta(t), 0]; \mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R} \) is defined by

\[
LU(x, y, t) = U_t(x, t) + U_x(x, t)f(x, x, t) + G(2g(x, y, t)U_x(x, x) + h(t)^2U_{xx}(x, x)).
\]

Moreover, the fundamental theory of calculus shows

\[
d\left( \int_{-\tau}^{0} \int_{t+s}^{t} \left[ \tau|f(X(u), X(u - \delta(u)), u)|^2 \\
+ \tau\bar{\sigma}^2|g(X(u), X(u - \delta(u)), u)|^2 + \sigma^2|h(u)|^2 \right] duds \right) \\
= \left( \int_{-\tau}^{t} \left[ \tau|f(X(u), X(u - \tau), u)|^2 \\
+ \tau\bar{\sigma}^2|g(X(u), X(u - \delta(u)), u)|^2 + \sigma^2|h(u)|^2 \right] duds \right) dt.
\]

**Lemma 3.4** With the notation above, \( V(\bar{X}(t), t) \) is G-Itô process on \( t \geq 0 \) with its Itô differential

\[
dV(\bar{X}(t), t) \leq LV(\bar{X}(t), t)dt + dM(t) \quad q.s.,
\]

where \( M(t) \) is a G-continuous martingale with \( M(0) = 0 \) and

\[
LV(\bar{X}(t), t) = \\
U_x(X(t), t)[f((X(t), X(t - \delta(t)), t) - f(X(t), X(t), t)] \\
+ LU(X(t), X(t - \delta(t)), t) \\
+ \theta\tau\left[ \tau|\bar{f}(X(t), X(t - \delta(t)), t)|^2 \\
+ \tau\bar{\sigma}^2|\bar{g}(X(t), X(t - \delta(t)), t)|^2 + \sigma^2|h(t)|^2 \right] \\
- \theta\int_{t-\tau}^{t} \left[ \tau|\bar{f}(X(u), X(u - \tau), u)|^2 \\
+ \tau\bar{\sigma}^2|\bar{g}(X(u), X(u - \delta(u)), u)|^2 + \sigma^2|h(u)|^2 \right] du.
\]

To study the delay-dependent asymptotic stability of the G-SDDE (3.1), we need to impose two new assumptions.

**Assumption 3.5** Assume that there are functions \( U \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+), \) \( U_1 \in C(\mathbb{R} \times [-\tau, \infty); \mathbb{R}_+), \) and positive numbers \( \alpha_1, \alpha_2 \) and \( \beta_k \) \((k = 1, 2, 3, 4)\) such that

\[
\alpha_2 < \alpha_1(1 - \delta)
\]

(3.7)
and
\[ \mathcal{L}U(x, y, t) + \beta_1 |U_x(x, t)|^2 + \beta_2 |f(x, y, t)|^2 + \beta_3 |g(x, y, t)|^2 + \beta_4 |h(t)|^2 \leq -\alpha_1 U_1(x, t) + \alpha_2 U_1(y, t - \delta(t)), \]

for all \( x \in \mathbb{R}, t \in \mathbb{R}_+ \).

**Assumption 3.6** Assume that there exists a positive number \( \varpi \) such that
\[ |f(x, x, t) - f(x, y, t)| \leq \varpi |x - y| \]
for all \( x \in \mathbb{R}, t \in [-2\tau, \infty) \).

**Theorem 3.7** Let Assumptions 3.1, 3.2, 3.5 and 3.6 hold, where \( h(\cdot) \) is a deterministic function. Assume also that
\[ \tau \leq \sqrt{\frac{4\beta_1 \beta_2}{3\varpi^2}} \wedge \sqrt{\frac{4\beta_1 \beta_3}{3\varpi^2 \sigma^2}} \wedge \sqrt{\frac{4\beta_1 \beta_4}{3\varpi^2 \sigma^2}}. \] (3.10)
Then for any given initial data (3.2), the solution of the G-SDDE (3.1) has the properties that
\[ \mathbb{E} \int_0^\infty U_1(X(t), t) dt < \infty \] (3.11)
and
\[ \sup_{0 \leq t < \infty} \hat{\mathbb{E}} U(X(t), t) < \infty. \]

**Proof:** Fix the initial data \( \eta \in C([\tau, 0]; \mathbb{R}) \) arbitrarily. Let \( k_0 > 0 \) be a sufficiently large integer such that \( \|\eta\| := \sup_{-\tau \leq s \leq 0} |\eta(s)| < k_0 \). For each integer \( k > k_0 \), define the stopping time
\[ \sigma_k = \inf\{t \geq 0 : |x(t)| \geq k\}. \]
It is easy to see that \( \sigma_k \) is increasing as \( k \to \infty \) and \( \lim_{k \to \infty} \sigma_k = \infty \) a.s. By the Itô formula for G-Brownian motion we obtain from Lemma 3.4 that
\[ \hat{\mathbb{E}} V(\bar{X}(t \wedge \sigma_k), t \wedge \sigma_k) \leq V(\bar{X}(0), 0) + \hat{\mathbb{E}} \int_0^{t \wedge \sigma_k} L V(\bar{X}(s), s) ds \] (3.12)
for any \( t \geq 0 \) and \( k \geq k_0 \). Let \( \theta = \frac{3\varpi^2}{(4\beta_1)} \). By Assumption 3.6 and Cauchy-Schwartz inequality, it is easy to see that
\[ U_x(X(t), t)[f(X(t), X(t), t) - f(X(t), X(t - \delta(t)), t)] \leq \beta_1 |U_x(X(t), t)|^2 + \frac{\varpi^2}{4\beta_1} |X(t) - X(t - \delta(t))|^2. \] (3.13)
By condition (3.10), we also have
\[ \theta \tau^2 \leq \beta_2, \ \theta \sigma^2 \tau^2 \leq \beta_3, \ \theta \sigma^2 \tau \leq \beta_4. \]
By Assumption 3.8, we then have

\[ LV(\bar{X}(s), s) \leq LU(X(s), X(s - \delta(t)), s) + \beta_1|U_x(X(s), s)|^2 + \beta_2|f(X(s), X(s - \delta(s)), s)|^2 + \beta_3|g(X(s), X(s - \delta(s)), s)|^2 + \beta_4|h(s)|^2 + \frac{\sigma^2}{4\beta_1}|X(s) - X(s - \delta(s))|^2 \]

\[ - 3\frac{\sigma^2}{4\beta_1} \int_{s-\delta(s)}^{s} \left[ \tau|f(X(u), X(u - \delta(u)), u)|^2 + \tau\sigma^2|g(X(u), X(u - \delta(u)), u)|^2 + \bar{\sigma}^2|h(u)|^2 \right] du. \]

By Assumption 3.8, we then have

\[ LV(\bar{X}(s), s) \leq -\alpha_1U_1(X(s), s) + \alpha_2U_1(X(s - \delta(s)), s - \delta(s)) + \frac{\sigma^2}{4\beta_1} \int_{s-\delta(s)}^{s} |X(s) - X(s + u)|^2 du \]

\[ - 3\frac{\sigma^2}{4\beta_1} \int_{s-\delta(s)}^{s} \left[ \tau|f(X(u), X(u - \delta(u)), u)|^2 + \tau\sigma^2|g(X(u), X(u - \delta(u)), u)|^2 + \bar{\sigma}^2|h(u)|^2 \right] du. \]

Substituting this into (3.12) implies

\[ EV(\bar{X}(t \wedge \sigma_k), t \wedge \sigma_k) \leq V(\bar{X}(0), 0) + I_1 + I_2, \quad (3.14) \]

where

\[ I_1 = \mathbb{E} \int_{0}^{t \wedge \sigma_k} \left[ -\alpha_1U_1(X(s), s) + \alpha_2U_1(X(s - \delta(s)), s - \delta(s)) \right] ds, \]

\[ I_2 = \frac{\sigma^2}{4\beta_1} \mathbb{E} \int_{0}^{t \wedge \sigma_k} \left[ |X(s) - X(s - \delta(s))|^2 + 3 \int_{s-\delta(s)}^{s} \left( \tau|f(X(u), X(u - \delta(u)), u)|^2 + \tau\sigma^2|g(X(u), X(u - \delta(u)), u)|^2 + \bar{\sigma}^2|h(u)|^2 \right) du \right] ds. \]

We notice the following fact

\[ \int_{0}^{\sigma_k \wedge t} U_1(X(s - \delta(s)), s - \delta(s)) ds \leq \frac{1}{1 - \delta} \int_{-\tau}^{0} U_1(\eta(s), s) ds + \frac{1}{1 - \delta} \int_{0}^{\sigma_k \wedge t} U_1(x(s), s) ds. \]

Thus we get

\[ I_1 \leq \frac{\alpha_2}{1 - \delta} \int_{-\tau}^{0} U_1(\eta(u), u) du + \frac{\bar{\alpha}}{1 - \delta} \mathbb{E} \left[ - \int_{0}^{t \wedge \sigma_k} U_1(X(s), s) ds \right], \]

10
where \( \bar{\alpha} = (1 - \delta)\alpha_1 - \alpha_2 > 0 \) by Assumption 3.5. Substituting this into (3.14) yields
\[
\frac{\bar{\alpha}}{1 - \delta} \mathbb{E} \int_{0}^{t \wedge \sigma_k} U_1(X(s), s) ds \leq C_1 + I_2,
\]
(3.15)
where \( C_1 \) is a constant defined by
\[
C_1 = V(\bar{X}(0), 0) + \frac{\alpha_2}{1 - \delta} \int_{0}^{0} U_1(\eta(s), s) ds.
\]
Let \( k \to \infty \) in (3.15) to obtain
\[
\frac{\bar{\alpha}}{1 - \delta} \mathbb{E} \int_{0}^{t} U_1(X(s), s) ds \leq C_1 + \bar{I}_2,
\]
(3.16)
where
\[
\bar{I}_2 = \frac{\omega^2}{4 \beta_1} \mathbb{E} \left[ \int_{0}^{t} \left[ |X(s) - X(s - \delta(s))|^2 - 3 \int_{s - \delta(s)}^{s} \left( \tau |f(X(u), X(u - \delta(u)), u)|^2 + \tau \sigma^2 |g(X(u), X(u - \delta(u)), u)|^2 + \bar{\sigma}^2 |h(u)|^2 \right) du \right] ds \right].
\]
For \( t \in [0, \tau] \), we have
\[
\bar{I}_2 \leq \frac{\omega^2}{2 \beta_1} \int_{0}^{\tau} \left( \mathbb{E} |X(s)|^2 + \mathbb{E} |X(s - \delta(s))|^2 \right) ds \leq \frac{\tau \omega^2}{\beta_1} \left( \sup_{-\tau \leq u \leq \tau} \mathbb{E} |X(u)|^2 \right).
\]
(3.17)
For \( t > \tau \), we have
\[
\bar{I}_2 \leq \frac{\tau \omega^2}{\beta_1} \left( \sup_{-\tau \leq u \leq \tau} \mathbb{E} |X(u)|^2 \right) + \frac{\omega^2}{4 \beta_1} \mathbb{E} \left[ \int_{\tau}^{t} \left[ |X(s) - X(s - \delta(s))|^2 - 3 \int_{s - \delta(s)}^{s} \left( \tau |f(X(u), X(u - \delta(u)), u)|^2 + \tau \sigma^2 |g(X(u), X(u - \delta(u)), u)|^2 + \bar{\sigma}^2 |h(u)|^2 \right) du \right] ds \right].
\]
(3.18)
Noting
\[
|X(s) - X(s - \delta(s))| = \left| \int_{s - \delta(s)}^{s} f(X(u), X(u - \delta(u)), u) du + \int_{s - \delta(s)}^{s} g(X(u), X(u - \delta(u)), u) d < B(u) > + \int_{s - \delta(s)}^{s} h(u) dB(u) \right|.
\]
11
by [13 Proposition 2.5] and Cauchy-Schwartz inequality, we have

\[
|X(s) - X(s - \delta(s))|^2 \\
\leq 3 \int_{s-\delta(s)}^{s} \tau |f(X(u), X(u - \delta(t)), u)|^2 du \\
+ 3\bar{\sigma}^2 \int_{s-\delta(s)}^{s} \tau |g(X(u), X(u - \delta(s)), u)|^2 du \\
+ 3 \left( \int_{s-\delta(s)}^{s} h(u) dB(u) \right)^2 \text{ q.s.}
\]

Thus we get

\[
\hat{\mathbb{E}} \int_{t}^{t+\tau} \left[ |X(s) - X(s - \delta(s))|^2 - 3 \int_{s-\delta(s)}^{s} \tau |f(X(u), X(u - \delta(u)), u)|^2 du \\
+ \bar{\sigma}^2 \tau |g(X(u), X(u - \delta(u)), u)|^2 + \bar{\sigma}^2 |h(u)|^2 \right] ds \\
\leq 3 \hat{\mathbb{E}} \int_{t}^{t+\tau} \left( \int_{s-\delta(s)}^{s} h(u) dB(u) \right)^2 ds \\
- 3\bar{\sigma}^2 \int_{t}^{t+\tau} \int_{s-\delta(s)}^{s} |h(u)|^2 duds = 0
\]

by Lemma 2.4 and the properties of sublinear expectation and $G$-Brownian motion.

Thus from (3.17) and (3.18) we get

\[
\tilde{I}_2 \leq \frac{\tau \bar{\omega}^2}{\beta_1} \left( \sup_{-\tau \leq u \leq \tau} \hat{\mathbb{E}}|X(u)|^2 \right). \tag{3.19}
\]

Substituting (3.19) into (3.16), together with (3.10), yields

\[
\frac{\bar{\alpha}}{1-\delta} \mathcal{E} \int_{0}^{t} U_1(X(s), s) ds \leq C_1 + \frac{4 \beta_3}{3\bar{\sigma}^2} \sup_{-\tau \leq u \leq \tau} \hat{\mathbb{E}}|X(u)|^2 := C_2.
\]

Letting $t \to \infty$ gives

\[
\mathcal{E} \int_{0}^{\infty} U_1(X(s), s) ds \leq \frac{(1-\delta)C_2}{\bar{\alpha}}.
\]

We deduce easily from (3.14) that

\[
\hat{\mathbb{E}}U \left( X(t \wedge \sigma_k), t \wedge \sigma_k \right) \leq C_1 + I_2.
\]

Letting $k \to \infty$ we get

\[
\hat{\mathbb{E}}U(X(t), t) \leq C_2 < \infty,
\]

which shows

\[
\sup_{0 \leq t < \infty} \hat{\mathbb{E}}U(X(t), t) < \infty.
\]

Thus the proof is complete.\[\square\]
Corollary 3.8 Let the conditions of Theorem 3.7 hold. If there moreover exists a pair of positive constants \( c \) and \( p \) such that
\[
c|x|^p \leq U_1(x), \quad \forall x \in \mathbb{R} \times \mathbb{R}_+, \]
then for any given initial data (3.2), the solution of G-SDDE (3.1) satisfies
\[
\mathcal{E} \left[ \int_0^\infty |X(t)|^p dt \right] < \infty,
\]
which means
\[
\int_0^\infty \mathcal{E}|X(t)|^p dt < \infty. \quad (3.20)
\]
Noting the property of sublinear \( \mathcal{E} \), this corollary follows from Theorem 3.7 obviously. However, it does not follow from (3.20) that \( \lim_{t \to \infty} \hat{\mathcal{E}}|X(t)|^p = 0 \). The following proposition provides a stronger claim.

Proposition 3.9 Let the conditions of Corollary 3.8 hold. If, moreover,
\[
p \geq 2 \quad \text{and} \quad (p + q_1 - 1) \vee (p + q_2 - 1) \leq q,
\]
then the solution of G-SDDE (3.1) satisfies
\[
\lim_{t \to \infty} \hat{\mathcal{E}}|X(t)|^p = 0
\]
for any initial data (3.2). That is, G-SDDE (3.1) is \( p \)-moment asymptotically stable under the sublinear expectation \( \hat{\mathcal{E}} \).

Proof: Fix the initial data (3.2) arbitrarily. For any \( 0 \leq t_1 < t_2 < \infty \), by the Itô formula for G-Brownian motion, we get
\[
d|X(t)|^p = p|X(t)|^{p-1} f(X(t), X(t-\delta(t)), t) dt
+ p|X(t)|^{p-2} [X(t)g(X(t), X(t-\delta(t)))
+ \frac{1}{2} (p-1)h^2(t)] dt < B(t) > + p|X(t)|^{p-1} h(t) dB(t),
\]
Noting \( d < B(t) > \leq \tilde{\sigma}^2 dt \) q.s. (see, e.g., Fei and Fei [14]), along with the polynomial growth condition (3.3), we get
\[
|\hat{\mathcal{E}}|X(t_2)|^p - \hat{\mathcal{E}}|X(t_1)|^p |
\leq \hat{\mathcal{E}} \int_{t_1}^{t_2} \left( p|X(t)|^{p-1} |f(X(t), X(t-\delta(t)), t)| 
+ p|X(t)|^{p-2} |X(t) g(X(t), X(t-\delta(t)))| 
+ \frac{1}{2} p(p-1)\tilde{\sigma}^2 |X(t)|^{p-2} h^2(t) 
\right) dt
\leq \hat{\mathcal{E}} \int_{t_1}^{t_2} \left( pK|X(t)|^{p-1} [1 + |X(t)|^{q_1} + |X(t-\delta(t))|^{q_1}] 
+ pK \tilde{\sigma}^2 |X(t)|^{p-1} [1 + |X(t)|^{q_2} + |X(t-\delta(t))|^{q_2}] 
+ \frac{1}{2} p(p-1)K^2 \tilde{\sigma}^2 |X(t)|^{p-2} \right) dt.
\]
By inequalities,
\[ |X(t)|^{p-1}|X(t - \delta(t))|^{q_1} \leq |X(t)|^{p+q_1-1} + |X(t - \delta(t))|^{p+q_1-1}, \]
\[ |X(t)|^{p-1} \leq 1 + |X(t)|^q, \]
etc., and noting that for any \(1 \leq \bar{p} \leq q\), by Proposition 3.3 we have
\[ \hat{E}|X(t - \delta(t))|^{\bar{p}} \leq 1 + \sup_{-\tau \leq s < \infty} \hat{E}|X(s)|^q < \infty, \]
we can obtain
\[ \left| \hat{E}|X(t_2)|^{\bar{p}} - \hat{E}|X(t_1)|^{\bar{p}} \right| \leq C_3(t_2 - t_1), \]
where
\[ C_3 = \left( 4pK(1 + \bar{\sigma}^2) \right) + \frac{1}{2}K^2p(p - 1)\bar{\sigma}^2(1 + \sup_{-\tau \leq s < \infty} \hat{E}|X(s)|^q) < \infty. \]
Thus we have \( \hat{E}|X(t)|^{\bar{p}} \) is uniformly continuous in \( t \) on \( \mathbb{R}_+ \). By (3.20), there is a sequence \( \{t_l\}_{l=1}^\infty \) in \( \mathbb{R} \) such that \( \hat{E}|X(t_l)|^{\bar{p}} \to 0 \), which easily show the claim. Thus, the proof is complete. □

**Proposition 3.10** Let the conditions of Theorem 3.7 hold. Assume also that there are positive constants \( p \) and \( c \) such that
\[ c|x|^p \leq U(x, t), \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+. \]
Moreover assume there exists a function \( W : \mathbb{R} \to \mathbb{R}_+ \) such that
\[ W(x) = 0 \text{ if and only if } x = 0 \]
and
\[ W(x) \leq U_1(x), \forall x \in \mathbb{R}. \]
Then for any given initial data (3.2), the solution \( X(\cdot) \) to Eq. (3.1) obeys that
\[ \lim_{t \to \infty} X(t) = 0 \quad \mathbb{V}\text{-a.s.} \]

**Proof:** Let \( X(\cdot) \) be the solution to Eq. (3.1) with initial data \( \eta \) defined in (3.2). Since the conditions in Theorem 3.7 hold, we can show that
\[ C_4 := \mathcal{E} \left[ \int_0^\infty W(X(t))dt \right] < \infty, \]
which implies
\[ \int_0^\infty W(X(t))dt < \infty \quad \mathbb{V}\text{-a.s.} \] (3.22)
Set \( \sigma_k := \inf \{t \geq 0 : |X(t)| \geq k\} \). We observe from (3.22) that
\[ \lim_{t \to \infty} \inf W(X(t)) = 0 \quad \mathbb{V}\text{-a.s.} \] (3.23)
Moreover, in the same way as Theorem 3.7 was proved, we can show that
\[ \hat{E}|X(T \wedge \sigma_k)|^{\bar{p}} \leq C, \quad \forall T > 0, \]
which implies, by Chebyshev inequality (see, e.g., Chen et al. [2], Proposition 2.1 (2)),
\[ k_p \mathbb{P}(\sigma_k \leq T) \leq C. \]

Letting \( T \to \infty \) yields
\[ k_p \mathbb{P}(\sigma_k < \infty) \leq C. \] (3.24)

We now claim that
\[ \lim_{t \to \infty} W(X(t)) = 0 \quad \mathbb{P}\text{-a.s.} \] (3.25)

In fact, if this is false, then we can find a number \( \varepsilon \in (0, 1/4) \) such that
\[ \mathbb{P}(\Omega_1) \geq 4\varepsilon, \] (3.26)
where \( \Omega_1 = \{ \limsup_{t \to \infty} W(X(t)) > 2\varepsilon \} \). Recalling (3.24), we can find an integer \( m \) sufficiently large for \( \mathbb{P}(\sigma_m < \infty) \leq \mathbb{P}(\sigma_m < \infty) \leq \varepsilon \). This means that
\[ \mathbb{P}(\Omega_2) = 1 - \mathbb{P}(\Omega_2^c) \geq 1 - \varepsilon. \] (3.27)

where \( \Omega_2 := \{|X(t)| < m \text{ for } \forall t \geq -\tau\} \), and \( \Omega_2^c \) is the complement of \( \Omega_2 \). By (3.26) and (3.27) we get
\[ \mathbb{P}(\Omega_1 \cap \Omega_2) \geq \mathbb{P}(\Omega_1) - \mathbb{P}(\Omega_2^c) \geq 3\varepsilon. \] (3.28)

Let us now define the stopped process \( \zeta(t) = X(t \wedge \sigma_m) \) for \( t \geq -\tau \). Clearly, \( \zeta(t) \) is a bounded Itô process with its differential
\[ d\zeta(t) = \phi(t)dt + \psi(t)d < B(t) > + h(t)dB(t), \]
where
\[ \phi(t) = f(X(t), X(t - \delta(t)), t)I_{[0, \sigma_m]}(t), \]
\[ \psi(t) = g(X(t), X(t - \delta(t)), t)I_{[0, \sigma_m]}(t), \]
where \( f, g, h \) are defined by (3.1). Recalling the polynomial growth condition (3.3) and the boundedness of \( h(\cdot) \), we know that \( \phi(\cdot), \psi(\cdot) \) and \( h(\cdot) \) are bounded processes, say
\[ |\phi(t)| \vee |\psi(t)| \vee h(t) \leq C_5 \quad \mathbb{P}\text{-a.s.} \] (3.29)
for all \( t \geq 0 \) and some \( C_5 > 0 \). Moreover, we also observe that \( |\zeta(t)| \leq m \) for all \( t \geq -\tau \). Define a sequence of stopping times
\[ \rho_1 = \inf\{t \geq 0 : W(\zeta(t)) \geq 2\varepsilon\}, \]
\[ \rho_{2j} = \inf\{t \geq \rho_{2j-1} : W(\zeta(t)) \leq \varepsilon\}, \quad j = 1, 2, \cdots, \]
\[ \rho_{2j+1} = \inf\{t \geq \rho_{2j} : W(\zeta(t)) \geq 2\varepsilon\}, \quad j = 1, 2, \cdots. \]

From (3.23) and the definition of \( \Omega_1 \) and \( \Omega_2 \), we have
\[ \Omega_1 \cap \Omega_2 \subset \{ \sigma_m = \infty \} \bigcap \left( \bigcap_{j=1}^{\infty} \{ \rho_j < \infty \} \right). \] (3.30)

We also note that for all \( \omega \in \Omega_1 \cap \Omega_2 \), and \( j \geq 1 \),
\[ W(\zeta(\rho_{2j-1})) - W(\zeta(\rho_{2j})) = \varepsilon \quad \text{and} \]
\[ W(\zeta(t)) \geq \varepsilon \quad \text{when } t \in [\rho_{2j-1}, \rho_{2j}]. \] (3.31)
Since $W(\cdot)$ is uniformly continuous in the close ball $S_m = \{x \in \mathbb{R}^d : |x| \leq m\}$. We can choose $\delta = \delta(\epsilon) > 0$ small sufficiently for which

$$|W(\zeta) - W(\tilde{\zeta})| < \epsilon, \zeta, \tilde{\zeta} \in \bar{S}_m, \text{ with } |\zeta - \tilde{\zeta}| < \delta. \quad (3.32)$$

We emphasize that for $\omega \in \Omega_1 \cap \Omega_2$, if $|\zeta(\rho_{2j-1} + u) - \zeta(\rho_{2j-1})| < \delta$ for all $u \in [0, \lambda]$ and some $\lambda > 0$, then $\rho_{2j} - \rho_{2j-1} \geq \lambda$. Choose a sufficiently small positive number $\lambda$ and then a sufficiently large positive integer $j_0$ such that

$$3C_5^2 \lambda (\lambda + \lambda C_1(2, \bar{\sigma}) + C_2(2, \bar{\sigma})) \leq \epsilon \delta^2 \quad \text{and} \quad C_4 < \epsilon^2 \lambda j_0. \quad (3.33)$$

By (3.28) and (3.30) we can further choose a sufficiently large number $T$ for

$$\forall (\rho_{2j_0} \leq T) \geq 2\epsilon. \quad (3.34)$$

In particular, if $\rho_{2j_0} \leq T$, then $|\zeta(\rho_{2j_0})| < m$, and hence $\rho_{2j_0} < \sigma_m$ by the definition of $\zeta(t)$. We hence have

$$\zeta(t, \omega) = X(t, \omega) \quad \text{for all } 0 \leq t \leq \rho_{2j_0} \text{ and } \omega \in \{\rho_{2j_0} \leq T\}.$$ 

By the Burkholder-Davis-Gundy inequality under sublinear expectation (see, e.g., Lemma [2.5]), we can have that, for $1 \leq j \leq j_0$,

$$\mathbb{E} \left( \sup_{0 \leq t \leq \lambda} |\zeta(\rho_{2j-1} \wedge T + t) - \zeta(\rho_{2j-1} \wedge T)|^2 \right) \leq 3\lambda \mathbb{E} \int_{\rho_{2j-1} \wedge T}^{\rho_{2j-1} \wedge T + \lambda} |\phi(s)|^2 ds$$

$$+ 3C_1(2, \bar{\sigma}) \lambda \mathbb{E} \int_{\rho_{2j-1} \wedge T}^{\rho_{2j-1} \wedge T + \lambda} |\psi(s)|^2 ds$$

$$+ 3C_2(2, \bar{\sigma}) \int_{\rho_{2j-1} \wedge T}^{\rho_{2j-1} \wedge T + \lambda} |h(s)|^2 ds$$

$$\leq 3C_5^2 \lambda (\lambda + \lambda C_1(2, \bar{\sigma}) + C_2(2, \bar{\sigma})),$$

which, together with (3.33) and Chebyshev inequality for sublinear expectation $\mathbb{E}$, we can obtain that

$$\forall \left( \sup_{0 \leq t \leq \lambda} |\zeta(\rho_{2j-1} \wedge T + t) - \zeta(\rho_{2j-1} \wedge T)| \geq \delta \right) \leq \epsilon.$$

Noting that $\rho_{2j-1} \leq T$ if $\rho_{2j_0} \leq T$, we can derive from (3.34) and the above inequality that

$$\forall \left( \rho_{2j_0} \leq T \right) \cap \left\{ \sup_{0 \leq t \leq \lambda} |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| < \delta \right\}$$

$$= \forall (\rho_{2j_0} \leq T)$$

$$- \forall \left( \rho_{2j_0} \leq T \right) \cap \left\{ \sup_{0 \leq t \leq \lambda} |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| \geq \delta \right\}$$

$$\geq \forall (\rho_{2j_0} \leq T) - \forall \left( \rho_{2j_0} \leq T \right) \cap \left\{ \sup_{0 \leq t \leq \lambda} |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| \geq \delta \right\}$$

$$\geq \epsilon.$$

This, together with (3.32), implies easily that

$$\forall \left( \rho_{2j_0} \leq T \right) \cap \{ \rho_{2j} - \rho_{2j-1} \geq \lambda \} \geq \epsilon. \quad (3.35)$$
By (3.31) and (3.35), and the sub-additivity and Chebyshev inequality for sublinear expectation $\mathcal{E}$, we derive

$$C_4 \geq \sum_{j=1}^{j_0} \mathcal{E} \left( I_{(\rho_{2j_0} \leq t)} \int_{\rho_{2j-1}}^{\rho_{2j}} W(X(t))dt \right)$$

$$\geq \varepsilon \sum_{j=1}^{j_0} \mathcal{E} \left( I_{(\rho_{2j_0} \leq t)} (\rho_{2j} - \rho_{2j-1}) \right)$$

$$\geq \varepsilon \lambda \sum_{j=1}^{j_0} \mathcal{V} \left( \{\rho_{2j_0} \leq t\} \cap \{\rho_{2j} - \rho_{2j-1} \geq \lambda\} \right)$$

$$\geq \varepsilon^2 \lambda j_0.$$  

This contradicts the second inequality in (3.33). Thus (3.25) must hold.

We now claim $\lim_{t \to \infty} X(t) = 0 \ \text{V-a.s.}$ If this were not true, then

$$\varepsilon_1 := \mathcal{V}(\Omega_3) > 0,$$

where $\Omega_3 = \{\limsup_{t \to \infty} |X(t)| > 0\}$. On the other hand, by (3.24), we can find a positive integer $m_0$ large enough for $\mathcal{V}(\sigma_{m_0} < \infty) \leq 0.5\varepsilon_1$. Let $\Omega_4 = \{\sigma_{m_0} = \infty\}$. Then

$$\mathcal{V}(\Omega_3 \cap \Omega_4) = \mathcal{V}(\Omega_3) - \mathcal{V}(\Omega_3 \cap \Omega_4^c) \geq \mathcal{V}(\Omega_3) - \mathcal{V}(\Omega_4) \geq 0.5\varepsilon_1.$$

For any $\omega \in \Omega_3 \cap \Omega_4$, $X(t, \omega)$ is bounded on $t \in \mathbb{R}_+$. We can then find a sequence $\{t_j\}_{j \geq 1}$ such that $t_j \to \infty$ and $X(t_j, \omega) \to \tilde{X}(\omega) \neq 0$ as $j \to \infty$. This, together with the continuity of $W$, implies

$$\lim_{j \to \infty} W(X(t_j, \omega)) = W(\tilde{X}(\omega)) > 0,$$

which show

$$\limsup_{t \to \infty} W(X(t, \omega)) > 0 \ \text{for all} \ \omega \in \Omega_3 \cap \Omega_4.$$  

But this contradicts (3.25). We therefore must have the assertion $\lim_{t \to \infty} X(t) = 0 \ \text{V-a.s.}$ Thus, the proof is complete. □

4 Stability Example for G-SDDEs

For a highly nonlinear G-SDDE (3.1) with $g(t) \equiv 0$, by the equal-spaced partition of time, it follows that

$$X(t_i) = X(t_{i-1}) + f(X(t_{i-1}), X(t_{i-1} - \delta(t_{i-1})), t_{i-1}) \Delta t_i$$

$$+ h(t_{i-1})(B(t_i) - B(t_{i-1}))$$  

(4.1)

with initial data $\eta = \{X(t) = \eta(t), t \in [-\tau, 0]\}, \Delta = \Delta t_i = t_i - t_{i-1} = 1/N, i = 1, \cdots, N$, where $N$ is a positive integer.

Let us first introduce the simulation algorithm for G-Brownian motion $(B(t))_{t \geq 0}$. Now consider a random variable $\zeta = B(t_i) - B(t_{i-1}) \sim \mathcal{N}(0, [\sigma^2, \sigma^2 \Delta], i = 1, \cdots, N)$, we construct an experiment as follows. We take equal-step points $\sigma^k, k = 1, \cdots, m$, such that $\bar{\sigma} = \sigma^1 < \sigma^2 < \cdots < \sigma^k < \cdots < \sigma^m = \bar{\sigma}$. For the $k$th-random sampling $(k = 1, \cdots, m)$, $\zeta_{k}^{ij}(i = 1, \cdots, N; j = 1, \cdots, n)$ are from the classical normal distribution $\mathcal{N}(0, (\sigma^k)^2 \Delta)$. Define $X^{kj}(t_i)$ by

$$X^{kj}(t_i) = X^{kj}(t_{i-1})$$

$$+ f(X^{kj}(t_{i-1}), X^{kj}(t_{i-1} - \delta(t_{i-1})), t_{i-1}) \Delta + h(t_{i-1})\zeta_i^{kj}$$  

(4.2)
for \( i = 1, \ldots, N; j = 1, \ldots, n; k = 1, \ldots, m \). By the \( \varphi \)-max-mean algorithm from Peng [40, Section 3.3], for \( p \geq 1 \) we can obtain the estimator \( \hat{\mu}_{m,n}(p) \) of \( \mathbb{E}|X(t_i)|^p \) equals to \( \max \left\{ \frac{1}{m} \sum_{k=1}^{m} |X^{kj}(t_i)|^p \right\} \), and the estimator \( \hat{\mu}_{m,n}(p) \) of \( \mathbb{E}|X(t_i)|^p \) to \( \min \left\{ \frac{1}{m} \sum_{k=1}^{m} |X^{kj}(t_i)|^p \right\} \), \( i = 1, \ldots, N \), respectively.

Also, in terms of Theorem 24 in Peng [40], we know \( \hat{\mu}_{m,n}(p) = \max \left\{ \frac{1}{m} \sum_{k=1}^{m} |X^{kj}(t_i)|^p \right\} \) is the unbiased estimator of \( \mathbb{E}|X(t_i)|^p \), and \( \hat{\mu}_{m,n}(p) = \min \left\{ \frac{1}{m} \sum_{k=1}^{m} |X^{kj}(t_i)|^p \right\} \) is the unbiased estimator of \( \mathbb{E}|X(t_i)|^p \). Moreover, we easily know that if \( \mathbb{E}|X(t)| \) with the solution \( X(t) \) to highly nonlinear G-SDDE (3.1) is stable, then the solution \( X(t) \) to G-SDDE (3.1) is quasi surely stable, i.e., \( \mathbb{V}(\lim_{t \to \infty} |X(t)| = 0) = 1 \). Meanwhile, if \( \mathbb{E}|X(t)| \) is unstable under sublinear expectation, then the solution \( X(t) \) to G-SDDE (3.1) is unstable, i.e., \( \mathbb{V}(\lim_{t \to \infty} |X(t)| \neq 0) = 1 \).

Through utilizing the above discussion on the simulation algorithm, we now investigate an example to illustrate our theory.

**Example 4.1** Let us consider the G-SDDE (1.1), where \( f(x, y, t) = -x^3 - y, h(t) = \frac{1}{2}e^{-t}, \sigma^2 = 0.5, \bar{\delta} = 0.1 \). We consider two case: \( \delta(t) = 0.01 \) and \( \delta(t) = 2 \) for all \( t \geq 0 \). In case \( \delta(t) = 0.01 \), let the initial data \( \eta(u) = 2 + \sin(u) \) for \( u \in [-0.01, 0] \), the sample paths of the solution of the G-SDDE are shown in Figure 4.1, which indicates that the G-SDDE is asymptotically stable. In the case \( \delta(t) = 2 \), let the initial data \( \eta(u) = 2 + \sin(u) \) for \( u \in [-2, 0] \), the solution of the G-SDDE (1.1) are plotted in Figure 4.2, which indicates that G-SDDE is asymptotically unstable. From the example we can see G-SDDE (1.1) is stable or not depends on how long or short the time-delay is.

![Figure 4.1: The computer simulation of the upper expectation a(t) = \( \mathbb{E}|X(t)| \) and the lower expectation b(t) = \( \mathbb{E}|X(t)| \) of the solution to the G-SDDE (1.1) with \( \delta(t) = 0.01 \) using the Euler–Maruyama method with step size 10^{-3}.](image)

We can see coefficients defined by (1.1) satisfy Assumption 3.1 with \( q_1 = 3 \) and \( q_2 = 0 \). Define \( \bar{U}(x, t) = |x|^6 \) for \( (x, t) \in \mathbb{R} \times \mathbb{R}_+ \). It is easy to show that

\[
\mathbb{L}\bar{U}(x, y, t) = 6x^5 f(x, y, t) + 15x^4 |h(t)|^2
\]
Figure 4.2: The computer simulation of the lower expectation $\mathcal{E}|X(t)|$ of the solution to the $G$-SDDE \((1.1)\) with $\delta(t) = 2$ using the Euler–Maruyama method with step size $10^{-3}$.

for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. Thus we get

\[
\mathcal{L} \tilde{U}(x, y, t) = 6x^5(-x^3 - y) + \frac{15}{4}x^4e^{-t}
\leq 4x^4 + 5x^6 - 6x^8 + y^6
\leq c_1 - 2(1 + 1.5x^6 + 2.5x^8) + (1 + 1.5y^6 + 2.5y^8),
\]

where

\[
c_1 = \sup_{x \in \mathbb{R}} (1 + 4x^4 + 8x^6 - x^8) > 0.
\]

Thus, we can set $H(x, t) = 1 + 1.5x^6 + x^8$. Due to $1 = c_3 < (1 - \bar{\delta})c_2 = 1.8$, we know Assumption \(3.2\) holds. From Proposition \(3.3\) the solution of the $G$-SDDE \((1.1)\) satisfies

\[
\sup_{-\tau \leq t < \infty} \mathcal{E}|X(t)|^6 < \infty.
\]

To verify Assumption \(3.5\) we define

\[
U(x, t) = e^{-t} + x^2 + x^4,
\] (4.3)

which shows

\[
U_t = -e^{-t}, U_x(x, t) = 2x + 4x^3, U_{xx} = 2 + 12x^2,
\]

for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. By equation \(3.6\), we have

\[
\mathcal{L}U(x, y, t) =
- e^{-t} + (2x + 4x^3)(-x^3 - x) + \frac{1}{8}e^{-2t}(2 + 12x^2)
\leq \frac{3}{4}e^{-t} - 0.5x^2 - 6x^4 - 4x^6.
\]
Moreover
\[ |U_x(x, t)|^2 = 4x^2 + 16x^4 + 16x^6, \quad (4.4) \]
\[ |f(x, y, t)|^2 = | - x^3 - y |^2 \leq 1.5x^4 + x^6 + y^2 + 0.5y^4. \quad (4.5) \]

Since \( g(\cdot) \equiv 0, \beta_3 \) can take an arbitrary large number. Next, setting \( \beta_1 = 0.1, \beta_2 = 0.05, \beta_4 = 1, \)
we get that
\[
\mathcal{L}U(x, y, t) + \beta_1 |U_x(x, t)|^2 + \beta_2 |f(x, y, t)|^2 + \beta_4 |h(t)|^2 \\
\leq -0.5e^{-t} - 0.1x^2 - 4x^4 - 2x^6 + 0.05y^2 + 0.0025y^4 \\
\leq -((0.5e^{-t} + 0.1x^2 + 4x^4 + 2x^6)) \\
+ 0.5(0.5e^{-t-\delta(t)}) + 0.1y^2 + 4y^4 + 2y^6) \\
=: -U_1(x, t) + 0.05U_1(y, t - \delta(t)).
\]

Letting \( U_1(x, t) = 0.5e^{-t} + 0.1x^2 + 4x^4 + 2x^6. \) Due to \( 0.9 = \alpha_1(1 - \tilde{\delta}) > \alpha_2 = 0.05, \)
we get condition (3.7). Noting that \( \varpi = 1, \)
then condition (3.10) becomes
\[ \delta(t) \leq 0.08. \]

![Figure 4.3](image-url)

Figure 4.3: The computer simulation of the upper expectation \( a(t) = \hat{\mathbb{E}}|X(t)| \) and the lower \[ b(t) = \mathbb{E}|X(t)| \] of the solution to the G-SDDE (1.1) with \( \delta(t) = 0.08 \) using the \[ \text{Euler–Maruyama method with step size } 10^{-3}. \]

By Theorem 3.7, we can therefore conclude that the solution of the G-SDDE (1.1) has the properties that
\[
\int_0^\infty (X^2(t) + X^4(t) + X^6(t))dt < \infty \text{ q.s.,} \\
\mathbb{E} \int_0^\infty (X^2(t) + X^4(t) + X^6(t))dt < \infty.
\]

Moreover, as \( |X(t)|^p \leq X^2(t) + X^4(t) + X^6(t) \) for any \( p \in [2, 6], \)
we have
\[
\mathbb{E} \int_0^\infty |X(t)|^p dt < \infty.
\]
Recalling \( q_1 = 3 \), \( q_2 = 0 \) and \( q = 6 \), we see that for \( p = 4 \), all conditions of Theorem 3.9 are satisfied and hence we have

\[
\lim_{t \to \infty} \mathbb{E}|X(t)|^4 = 0.
\]

We perform a computer simulation with the time-delay \( \delta(t) = 0.08 \) for all \( t \geq 0 \) and the initial data \( X(u) = 2 + \sin(u) \) for \( u \in [-0.08, 0] \). The sample paths of the solution to the G-SDDE (1.1) are plotted in Figure 4.3. The simulation supports our theoretical results.

5 Conclusion

In real systems, we are often faced with two kinds of uncertainties: probability and Knightian ones, respectively. By using Peng’s sublinear expectation framework, we can characterize the systems with ambiguity. This paper gives a description of this kind of system with delay through G-Brownian motion. Moreover, the boundedness and stability of solutions to G-SDDEs are discussed. We first give the delay-dependent criteria of stability and boundedness of the solutions to highly nonlinear G-SDDEs by the method of Lyapunov functional. Then, in Section 4 we give a simulation algorithm related G-Brownian motion, moreover an illustrative example is explored. In fact, under classical probability, the stability of highly nonlinear hybrid SDDEs has been studied by some researchers. However, our idea and framework with sublinear expectation provide a new perspective for further investigating the stability and boundedness of highly nonlinear stochastic systems driven by G-Brownian motion.

Acknowledgements

The authors would also like to thank the Natural Science Foundation of China (71571001) for their financial support.

References

[1] Chen, Z. and Epstein, L. (2002). Ambiguity, risk and asset returns in continuous time, *Econometrica* 70(4), 1403-1443.

[2] Chen, Z., Wu, P. and Li, B. (2013). A strong law of large numbers for non-additive probabilities, *International Journal of Approximate Reasoning* 54, 365-377.

[3] Deng, S., Fei, C., Fei, W, and Mao, X. (2019). Stability equivalence between the stochastic differential delay equations driven by G-Brownian motion and the Euler-Maruyama method, *Applied Mathematics Letters* 96, 138-146.

[4] Denis, L., Hu, M., and Peng, S. (2008). Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, arXiv:0802.1240v1.

[5] Epstein, L. and Ji, S. (2013). Ambiguity volatility, possibility and utility in continuous time, *Review of Financial Studies* 26(7), 1740-1786.

[6] Epstein, L. and Ji, S. (2014). Ambiguity volatility and asset pricing in continuous time, *Journal of Mathematical Economics* 50, 269-282.

[7] Fei, C. and Fei, W. (2019). Consistency of least squares estimation to the parameter for stochastic differential equations under distribution uncertainty, *Acta Mathematica Scientia* 39A(6), 1499-1513 (see arXiv:1904.12701v1).
[8] Fei, C., Fei, W., Mao, X., Xia, D., and Yang, L. (2019). Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, *IEEE Trans. Automat. Control*, DOI:10.1109/TAC.2019.2933604.

[9] Fei, C., Fei, W., and Yan, L. (2019). Existence-uniqueness and stability of solutions to highly nonlinear stochastic differential delay equations driven by $G$-Brownian motions, *Appl. Math. J. Chinese Univ.* 34(2), 184-204.

[10] Fei, C., Shen, M., Fei, W., Mao, X., and Yan L. (2019). Stability of highly nonlinear hybrid stochastic integro-differential delay equations, *Nonlinear Analysis: Hybrid Systems* 31, 180-199.

[11] Fei, W. (2007). Optimal consumption and portfolio choice with ambiguity and anticipation, *Information Sciences* 177, 5178-5190.

[12] Fei, W. (2009). Optimal portfolio choice based on $\alpha$-MEU under ambiguity, *Stochastic Models* 25, 455-482.

[13] Fei, W. and Fei, C. (2013). Optimal stochastic control and optimal consumption and portfolio with $G$-Brownian motion, [arXiv:1309.0209v1](https://arxiv.org/abs/1309.0209).

[14] Fei, W. and Fei, C. (2013). On exponential stability for stochastic differential equations disturbed by $G$-Brownian motion, see [arXiv:1311.7311](https://arxiv.org/abs/1311.7311).

[15] Fei, W., Hu, L., Mao, X., and Shen, M. (2017). Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica* 28, 165-170.

[16] Fei, W., Hu, L., Mao, X. and Shen, M. (2018). Structured robust stability and boundedness of nonlinear hybrid delay systems, *SIAM Contr. Opt.* 56(4), 2662-2689.

[17] Gao, F.Q. (2009). Pathwise properties and homeomorphic flows for stochastic differential equations driven by $G$-Brownian motion, *Stoch. Proc. Appl.* 119 (10), 3356-3382.

[18] Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique priors, *J. Mathematical Economics* 18, 141-153.

[19] Hasminskii, R.Z. (2012). *Stochastic Stability of Differential Equations, Second edition*, Springer-Verlag, Berlin, Heidelberg.

[20] Hu, L., X. Mao, and Shen, Y. (2013). Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Systems & Control Letters* 62, 178-187.

[21] Hu, L., X. Mao, and Zhang, L. (2013). Robust stability and boundedness of nonlinear hybrid stochastic delay equations, *IEEE Trans. Automat Control* 58(9), 2319-2332.

[22] Hu, M. and Peng, S. (2009). On representation theorem of $G$-expectations and paths of $G$-Brownian motion, *Acta Mathematicae Applicatae Sinica, English Series* 25(3), 539-546.

[23] Hu, M., Ji, S., Peng, S. and Song, Y. (2014). Backward stochastic differential equations driven by $G$-Brownian motion, *Stochastic Processes and their Applications* 124, 759-784.

[24] LaSalle, J.P. (1968). Stability theory of ordinary differential equations, *J. Differential Equations* 4, 57-65.

[25] Lei, J. and Mackey, M. (2007). Stochastic differential delay equation, moment stability, and application to hematopoietic stem cell regulation systems, *SIAM J. Appl. Math.* 67(2), 387-407.
[26] Li, X and Peng, S. (2009). Stopping times and related Itô's calculus with G-Brownian motion, *Stoch. Proc. Appl.* 121, 1492-1508.

[27] Li, Y. and Yan, L. (2018). Stability of delayed Hopfield neural networks under a sublinear expectation framework, *Journal of the Franklin Institute* 355(10), 4268-4281.

[28] Lin, Q. (2013). Some properties of stochastic differential equations driven by G-Brownian motion, *Acta Math. Sin. (Engl. Ser.)* 29, 923-942.

[29] Lin, Y. (2018). Stability of delayed Hopfield neural networks under a sublinear expectation framework, *Journal of the Franklin Institute* 355(10), 4268-4281.

[30] Liu, J. (2012). On asymptotic convergence and boundedness of stochastic systems with time-delay, *Automatica* 48, 3166-3172.

[31] Luo, J. and Liu, K. (2008). Stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps, *Stoch. Proc. Appl.* 118, 864-895.

[32] Luo, P. and Wang, F. (2014). Stochastic differential equations driven by G-Brownian motion and ordinary differential equations. *Stoch. Proc. Appl.* 124, 3869-3885.

[33] Lyapunov, A.M. (1907). Problème général de la stabilité du movement, Comm. Soc. Math. Kharkov 2 (1892) 265-272; Translated in: Annales de la faculté des sciences de Toulouse 9, 203-474.

[34] Mao, X. (2007). *Stochastic Differential Equations and Their Applications, 2nd Edition*, Horwood Pub. Chichester.

[35] Mao, X., Yin G., and Yuan C. (2007). Stabilization and destabilization of hybrid systems of stochastic differential equations, *Automatica* 43, 264-273.

[36] Mao, X. and Yuan C. (2006). *Stochastic Differential Equations with Markovian Switching*, Imperical College Press, London.

[37] Mei, C., Fei, C., Fei, W., Mao, X. (2020). Stabilisation of highly non-linear continuous-time hybrid stochastic differential delay equations by discrete-time feedback control, *IET Control Theor. Appl.* 14 (2), 313-323.

[38] Peng, S. (2006). G-expectation, G-Brownian motion and related stochastic calculus of Itô's type, The Abel Symposium 2005, Abel Symposia 2, Edit. Benth et. al., pp. 541-567, Springer-Verlag.

[39] Peng, S. (2008). Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, *Stoch. Proc. Appl.* 118(12), 2223-2253.

[40] Peng, S. (2016). Theory, methods and meaning of nonlinear expectation theory, *Sci. China Math.* 47(10), 1223-1254. (in Chinese)

[41] Peng, S. (2019). *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, Springer, Berlin.

[42] Peng, S., Song, Y., and Zhang, J. (2014), A complete representation theorem for G-martingales, *Stochastics* 86(4), 609-631.

[43] Peng, S. and Zhang, H. (2017). Stochastic calculus with respect to G-Brownian motion viewed through rough paths. *Sci. China Math.* 60(1), 1-20.
[44] Ren, Y., Yin, W., Sakthivel, R. (2018). Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation, *Automatica* 95, 146-151.

[45] Shen, M., Fei, C., Fei, W., and Mao, X. (2019). Boundedness and stability of highly nonlinear hybrid neutral stochastic systems with multiple delays, *Sci. China Inf. Sci.* 62, 202205.

[46] Shen, M., Fei, C., Fei, W., and Mao, X. (2020). Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations, *Systems & Control Letters* 137, 104645.

[47] Soner, M., Touzi, N., and J. Zhang (2011). Quasi-sure stochastic analysis through aggregation, *Electronic Journal of Probability* 16, 1844-1879.

[48] Song, Y. (2011), Some properties on G-evaluation and its applications to G-martingale decomposition, *Sci. China Math.* 54 (2), 287-300.

[49] Xu, L., Ge Sam S., and Hu, H. (2019), Boundedness and stability analysis for impulsive stochastic differential equations driven by G-Brownian motion, *International Journal of Control* 92(3), 642-652.

[50] Yao, Z.H. and Zong, X.F. (2020). Delay-dependent stability of a class of stochastic delay systems driven by G-Brownian motion, *IET Control Theory & Applications* 14(6), 834-842.

[51] Yin, W, Cao, J, and Ren, Y. (2019). Quasi-sure exponential stabilization of stochastic systems induced by G-Brownian motion with discrete time feedback control, *Journal of Mathematical Analysis and Applications* 474(1), 276-289.

[52] Zhang, L. (2016). Rosenthal’s inequalities for independent and negatively dependent random variables under sub-linear expectations with applications, *Sci. China Math.* 59(4),751-768.

[53] Zhang, D. F. and Chen, Z. (2012). Exponential stability for stochastic differential equation driven by G-Brownian motion, *Applied Mathematics Letters* 25, 1906-1910.