ON FINITE-DIMENSIONAL SEMISIMPLE AND COSEMISIMPLE
HOPF ALGEBRAS IN POSITIVE CHARACTERISTIC

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July 8, 1998

Introduction

Recently, important progress has been made in the study of finite-dimensional
semisimple Hopf algebras over a field of characteristic zero (see [Mo] and references
therein). Yet, very little is known over a field $k$ of positive characteristic. In this
paper we first prove in Theorem 2.1 that any finite-dimensional semisimple and
cosemisimple Hopf algebra over $k$ can be lifted to a semisimple Hopf algebra of the
same dimension over a field of characteristic zero. Moreover, we prove in Theorems
2.2 and 2.3 that this lifting is functorial, and that it carries a (quasi)triangular
object to a (quasi)triangular object. We then use these lifting theorems to prove
some results on finite-dimensional semisimple and cosemisimple Hopf algebras $A$
over $k$, notably Kaplansky’s 5th conjecture from 1975 on the order of the antipode
of $A$ [K]. These results have already been proved over a field of characteristic zero, so
in a sense we demonstrate that it is sufficient to consider semisimple Hopf algebras
over such a field (they are also cosemisimple [LR2]), and then to use our Lifting
Theorems 2.1, 2.2 and 2.3 to prove them for semisimple and cosemisimple Hopf
algebras over a field of positive characteristic. In our proof of Lifting Theorems
2.1 and 2.2 we use standard arguments of deformation theory from positive to zero
characteristic. The key ingredient of the proof is the theorem that the bialgebra
cohomology groups vanish. We conclude the paper by proving in Theorem 4.2 that
any semisimple Hopf algebra $A$ of dimension $d > 2$ over a field $k$ of characteristic
$p > d^{\varphi(d)/2}$ (here $\varphi$ is the Euler function), is also cosemisimple. This result was
known in characteristic 0 [LR2], and in characteristic $p > (2d^2)^{2d^2-4}$ [So].

1. The bialgebra cohomology

In the proof of our lifting theorems we will use the bialgebra cohomology [GS].
Let $A, B$ be bialgebras over any field $F$, and let $\phi : A \rightarrow B$ be a homomorphism of
bialgebras. We define $C^{p,q} = C^{p,q}(A,B,\phi) = \text{Hom}_F(A^{\otimes(p+1)}, B^{\otimes(q+1)})$, $p, q \geq 0$,
and two differentials, the algebra differential \( d^p_a : C^{p,q} \to C^{p+1,q} \) and the coalgebra differential \( d^p_c : C^{p,q} \to C^{p+1,q+1} \), by the following formulas:

\[
(d^p_a f)(a_1 \otimes \cdots \otimes a_{p+2}) =
\]
\[
(-1)^{p+1} \phi^{(q+1)}(\Delta_{q+1}(a_1)) f(a_2 \otimes \cdots \otimes a_{p+2}) + (-1)^{p+2} f(a_1 a_2 \otimes \cdots \otimes a_{p+2}) + \cdots
\]  
\[
(1.1) \quad + f(a_1 \otimes \cdots \otimes a_{p+1} a_{p+2}) - f(a_1 \otimes \cdots \otimes a_{p+1}) \phi^{(q+1)}(\Delta_{q+1}(a_{p+2})),
\]

\[
(d^p_c f)(a_1 \otimes \cdots \otimes a_{p+1}) =
\]
\[
\sum \phi(a_1^{(1)} \cdots a_{p+1}^{(1)} \otimes f(a_1^{(2)} \otimes \cdots \otimes a_{p+1}^{(2)})) -
\]
\[
-(\Delta \otimes I^{\otimes q})(f(a_1 \otimes \cdots \otimes a_{p+1})) + \cdots + (-1)^{q+1}(I^{\otimes q} \otimes \Delta)(f(a_1 \otimes \cdots \otimes a_{p+1}))+
\]
\[
(1.2) \quad + (-1)^{q+2} \sum f(a_1^{(1)} \otimes \cdots \otimes a_{p+1}^{(1)} \otimes \phi(a_1^{(2)} \cdots a_{p+1}^{(2)}),
\]

where \( \Delta(a) = \sum d^{(1)} \otimes d^{(2)}, \Delta_q : A \to A^{\otimes q}, q \geq 2, \) is the iterated coproduct \((\Delta_1 = I \text{ is the identity map})\). It is straightforward to check that \( d^0_a = 0, d^0_c = 0 \) and \( d_a d_c = d_c d_a \). Thus, \((C^\bullet, d_{a,c})\) forms a bicomplex. Consider the corresponding total complex with \( C^n(A,B,\phi) = \oplus_{p+q=n} C^{p,q} \), and the differential \( d \) determined by \( d_{C^{p,q}} = d^{p,q}_a + (-1)^p d^{p,q}_c \). We call the total complex \((C^\bullet(A,B,\phi), d)\) the bialgebra cochain complex of \((A,B,\phi)\). The cohomology of \( C^\bullet(A,B,\phi)\) is called the bialgebra cohomology of \((A,B,\phi)\) and denoted by \( \hat{H}^\bullet(A,B,\phi) \). In the special case in which \( A = B \) and \( \phi = I \), we write \( C^\bullet(A), \hat{H}^\bullet(A) \) etc.

**Theorem 1.1.** [St] If \( A \) is a finite-dimensional semisimple and cosemisimple Hopf algebra over any field, then \( \hat{H}^\bullet(A) = 0 \).

This theorem has the following generalization.

**Theorem 1.2.** Let \( A \) be a finite-dimensional semisimple Hopf algebra and \( B \) a finite-dimensional cosemisimple Hopf algebra over any field. If \( \phi : A \to B \) is a homomorphism of Hopf algebras, then \( \hat{H}^\bullet(A,B,\phi) = 0 \).

**Proof.** Enlarge the bicomplex \( C^{p,q} \) to \( \hat{C}^{p,q} \) by setting \( \hat{C}^{p,q} = C^{p,q} \) for \( p,q \geq 0 \), \( \hat{C}^{-1,q} = B^{\otimes(q+1)} \) and \( \hat{C}^{p,q} = 0 \) otherwise, and taking the differentials \( d_a, d_c \) determined in (1.2). Let \( \hat{C}^\bullet = \hat{C}^\bullet(A,B,\phi) \) denote the corresponding total complex. Also set \( \hat{D}^q(B) = C^{-1,q}(B) \) and let \((D^\bullet(B), d_c)\) be the corresponding coalgebra complex. One has the following exact sequence of complexes:

\[
0 \to C^n(A,B,\phi) \to \hat{C}^n(A,B,\phi) \to D^{n+1}(B) \to 0,
\]

and thus the following long exact sequence of cohomology:

\[
\cdots \to \hat{H}^{i+1}(D^\bullet(B)) \to \hat{H}^i(C^\bullet(A,B,\phi)) \to \hat{H}^i(\hat{C}^\bullet(A,B,\phi)) \to \hat{H}^{i+2}(D^\bullet(B)) \to \cdots.
\]

A theorem of Hochschild [Ho, Theorems 3.1 and 4.1] states that the cohomology of a finite-dimensional semisimple algebra over an algebraically closed field with coefficients in any bimodule vanishes in positive dimensions. In particular, this implies that \( \hat{H}^i(D^\bullet(B)) = 0 \) for all \( i \), and hence that \( \hat{H}^\bullet(C^\bullet(A,B,\phi)) = \hat{H}^\bullet(\hat{C}^\bullet(A,B,\phi)) \). Thus it suffices to show that \( \hat{H}^\bullet(\hat{C}^\bullet(A,B,\phi)) = 0 \).
We shall apply the following standard lemma in homological algebra (a dual version of [Lo, Lemma 1.0.12]): Let $E^{*,*}$ be a cochain double complex with $E^{p,q}=0$ unless $q \geq 0$, $p \geq -1$, and let $E^*$ be the corresponding total complex. Suppose that the columns of $E^{*,*}$ are acyclic except in degree $p = -1$. Then $H^i(E^*) = H^i(K^*)$, where $K^q = \ker(d : E^{-1,q} \to E^{0,q})$.

Apply the above to $E^{p,q} = \hat{C}^{p,q}$. In this case $K^q = (B^{\otimes (q+1)})^A$ (the space of elements of $B^{\otimes (q+1)}$ which commute with $\Delta_{q+1}(\phi(a))$ for all $a \in A$). Indeed, the operator $d : E^{-1,q} = B^{\otimes (q+1)} \to E^{0,q} = A^* \otimes B^{\otimes (q+1)}$ is given by $d(b_1 \otimes \cdots \otimes b_{q+1})(a) = [\phi^{(q+1)}(\Delta_{q+1}(a)), b_1 \otimes \cdots \otimes b_{q+1}]$. Since the $q$-th column of $E^{p,q}$ is the Hochschild complex of $A$ with coefficients in $B^{\otimes (q+1)}$ (with degree shifted down by 1), by Hochschild’s theorem it is acyclic in $p \geq 0$. Therefore, $H^i(C^*(A,B,\phi)) = H^i((B^{*+1})^A)$ which implies that

(1.3) $H^i(C^*(A,B,\phi)) = H^i(D^*(B)^A)$.

Let $H = \text{Im}(\phi)$, and $\phi_1 : H \to B$ be the corresponding injective Hopf algebra map. Notice that $H$ is both semisimple and cosemisimple. Now, since $H = \text{Im}(\phi)$ it follows by (1.3) that $H^i(C^*(A,B,\phi)) = H^i(D^*(B)^H)$. By (1.3), this equals $H^i(C^*(H,B,\phi_1))$, which equals by duality to $H^i(C^*(B^*,H^*,\phi_1^*)$. Finally, by (1.3), this equals $H^i(D^*(H^*)^B^*)$, and since $H^* = \text{Im}(B^*)$, this equals $H^i(D^*(H^*)^B^*)$ which equals 0 by Theorem 1.1. □

As an example, let us prove a simple corollary of this theorem.

**Corollary 1.3.** Let $A$ be a finite-dimensional semisimple Hopf algebra and $B$ a finite-dimensional cosemisimple Hopf algebra over any field. Then $\text{Hom}_{Hopf}(A,B)$ is finite. In particular the group of Hopf automorphisms of a finite-dimensional semisimple and cosemisimple Hopf algebra $A$ is finite.

**Proof.** Consider the variety $\text{Hom}_{Hopf}(A,B)$. The Zariski tangent space to a point $\phi \in \text{Hom}_{Hopf}(A,B)$ equals $H^0(A,B,\phi)$ which equals 0 by Theorem 1.2. This implies that $\text{Hom}_{Hopf}(A,B)$ is finite (see [Sh, Ex. 3, p.54 and Th. 3, p.78]), and the corollary is proved. (Notice that to prove the finiteness of the automorphism group it is enough to use Theorem 1.1, which states that the Lie algebra of this group is zero; so in this case Theorem 1.2 is not needed). □

**Remark 1.4.** Let us give an alternative proof to the fact that $\text{Aut}_{Hopf}(A)$ is finite, which does not use cohomology. Let $f$ be a bialgebra derivation of the Hopf algebra $A$. Since $A$ is semisimple $f$ is inner. Let $x \in A$ be such that $f(a) = xa - ax$ for all $a \in A$. Note that $\Delta(f(a)) = (I \otimes f + f \otimes I)\Delta(a)$ for all $a \in A$, is equivalent to $[\Delta(x) - 1 \otimes x - x \otimes 1, \Delta(a)] = 0$ for all $a \in A$. By Drinfeld’s theorem [D], if $\lambda$ is an integral of $A^*$ and $b \in A \otimes A$ is such that $[b, \Delta(a)] = 0$ for all $a \in A$, then $(\lambda \otimes I)(b)$ is central in $A$. Applying this to $b = \Delta(x) - x \otimes 1 - 1 \otimes x$ we get that $(\lambda \otimes I)(\Delta(x) - 1 \otimes x - x \otimes 1) = -\lambda(1)x$ is central in $A$. Since $A$ is cosemisimple we can choose $\lambda$ such that $\lambda(1) \neq 0$. Hence $x$ is central and $f = 0$. This implies that the Lie algebra of the group of Hopf automorphisms is zero, and hence that it is finite. □

**Corollary 1.5.** Let $A$ be a finite-dimensional cosemisimple Hopf algebra over any field. Then $A$ admits finitely many quasitriangular structures.
Proof. A quasitriangular structure on $A$ determines a homomorphism of Hopf algebras $A^{*\cop} \to A$, where $A^{*\cop}$ is the Hopf algebra obtained from $A^*$ by taking the opposite comultiplication. Therefore the result follows from Corollary 1.3. □

Corollaries 1.3, 1.5 were proved by Radford [R1, R2] in characteristic 0. He also proved them in characteristic $p$ bigger than the dimension of the Hopf algebra, under some additional assumptions. Schneider gave a proof of the finiteness of $\text{Aut}_{\text{Hopf}}(A)$ which is the same as our proof [Sc], and Waterhouse proved the same result without using cohomology [W].

2. The Lifting Theorems

Let $p$ be a prime number and $k$ be a perfect field of characteristic $p$. Let $O = W(k)$ be the ring of Witt vectors of $k$ [Se, Sections 2.5, 2.6], and $K$ the field of fractions of $O$. Recall that $O$ is a local complete discrete valuation ring, and that the characteristic of $K$ is zero. Let $m$ be the maximal ideal in $O$, which is generated by $p$. One has $m^n/m^{n+1} = k$ for any $n \geq 0$ (here $m^0 = O$). In the case $k = F_p$ one has that $O = \mathbb{Z}_p$ is the ring of $p$–adic integers, and $K = \mathbb{Q}_p$.

Theorem 2.1. Let $A$ be a semisimple and cosemisimple $N$-dimensional Hopf algebra over an algebraically closed field $k$ of characteristic $p$. Then:

(i) There exists a unique (up to isomorphism) Hopf algebra $\overline{A}$ over $O$ which is free of rank $N$ as an $O$-module, and such that $\overline{A}/p\overline{A}$ is isomorphic to $A$ as a Hopf algebra.

(ii) The $K$-Hopf algebra $A_0 = \overline{A} \otimes_O K$ is semisimple and cosemisimple. All irreducible $A_0$-modules and comodules over $K$ are defined already over $K$. The dimensions of irreducible modules and comodules and the Grothendieck rings of the categories of modules and comodules for $A$ are the same as for $A_0$.

Proof. (i) To show the existence of $\overline{A}$ we will prove that there exists a sequence of Hopf algebras $A_n$ over $O_n := O/p^n$ which are free modules of rank $N$ over $O_n$ such that $A_1 = A$ and $A_{n+1}/p^n$ is isomorphic to $A_n^*$ as a Hopf algebra. Then we can fix isomorphisms $f_n : A_{n+1}/p^n \to A_n$ and define $\overline{A}$ as $\lim A_n$.

Our proof is by induction on $n$. The case $n = 1$ is clear. Suppose we have constructed $A_i$ for $i \leq n$, and let us construct $A_{n+1}$. Take $A_{n+1}$ to be a free rank $N$ module over $O_{n+1}$ and fix a module isomorphism $f_n : A_{n+1}/p^n \to A_n$. Let $E$ be the set of extensions of the product and coproduct of $A_n$ to $A_{n+1}$ (just as module maps). That is, $E$ is the set of pairs $(m', \Delta') \in \text{Hom}_{O_{n+1}}(A_{n+1}^{\otimes 2}, A_{n+1}) \oplus \text{Hom}_{O_{n+1}}(A_{n+1}, A_{n+1}^{\otimes 2})$ such that $m'$, $\Delta'$ are mapped to the product and coproduct of $A_n$ under $f_n$.

Let $A_{ij} = \text{Hom}_k(A^{\otimes i}, A^{\otimes j})$. We have an action of the additive group $A_{21} \oplus A_{12}$ on $E$ which is defined as follows. Let $(\mu, \delta) \in A_{21} \oplus A_{12}$. Lift them in any way to $(\mu', \delta') \in \text{Hom}_{O_{n+1}}(A_{n+1}^{\otimes 2}, A_{n+1}) \oplus \text{Hom}_{O_{n+1}}(A_{n+1}, A_{n+1}^{\otimes 2})$. Now define $(\mu, \delta) \circ (m', \Delta') = (m' + p^n\mu', \Delta' + p^n\delta')$. Clearly, this does not depend on the lifting, so it defines a desired group action. It is clear that the constructed group action is free. It is also easy to see that it is transitive. Indeed, if $(m', \Delta'), (m'', \Delta'')$ are two elements of $E$ then $(m' - m'', \Delta' - \Delta'')$ is zero modulo $p^n$, so it has the form $p^n(\mu', \delta')$. Let $\mu \in A_{21}$, $\delta \in A_{12}$ be the reductions of $\mu', \delta'$ mod $p$. Then $(m'', \Delta'') = (\mu, \delta) \circ (m', \Delta')$. Thus, $E$ is a principal homogeneous space of $A_{21} \oplus A_{12}$.

Now define a map $c : E \to A_{31} \oplus A_{32} \oplus A_{13}$, which measures the failure of $(m', \Delta') \in E$ to satisfy Hopf algebra axioms. This map is defined by the following...
rule: Consider the element
\[ a(m', \Delta') = (m'(I \otimes m') - m'(m' \otimes I), \Delta'm' - m'_{13} m'_{24} (\Delta' \otimes \Delta'), (I \otimes \Delta')(\Delta' - (\Delta' \otimes I) \Delta'). \]

It is clear that \( a \) is zero modulo \( p^n \). So there exists \( b \) such that \( a = p^n b \). The element \( b \) is not unique but unique modulo \( p \), so we set \( c(m', \Delta') = b \mod p \).

Observe that \( c \) takes values in \( C^2(A) \) of the bialgebra cochain complex. Moreover, it is straightforward to check that \( d \circ c = 0 \).

By Theorem 1.1, \( H^2(A) = 0 \). This implies that for any \( (m', \Delta') \in E \) there exists \( (\mu, \delta) \in C^1(A) = A_{21} \oplus A_{12} \) such that \( c(m', \Delta') = d(\mu, \delta) \). It is easy to check that \( c(x + p^n y) = c(x) + d y \) for \( x \in E \) and \( y \in A_{21} \oplus A_{12} \), so if we set \( m'' = m' - p^n \mu, \Delta'' = \Delta' - p^n \delta \) then we get \( c(m'', \Delta'') = 0 \).

It remains to show that \( (A_{n+1}, m'', \Delta'') \) has the unit, counit, and antipode which satisfy the axioms of a Hopf algebra, and equal the unit, counit, and antipode of \( A_{n} \) modulo \( p^n \). For unit and counit it is trivial, since it is well known that existence of a unit is preserved under algebra deformations. To show the existence of the antipode, we have to show that there exists \( S'' : A_{n+1} \to A_{n+1} \) such that the antipode equation \( m''(S'' \otimes id) \Delta'' = i \varepsilon \) holds, where \( i \) is the unit and \( \varepsilon \) the counit. Consider the map \( T : \text{End}_{O_{n+1}}(A_{n+1}) \to \text{End}_{O_{n+1}}(A_{n+1}) \) given by \( T(S'') = m''(S'' \otimes id) \Delta'' \).

This map is a linear isomorphism modulo \( p \) (since the antipode in a Hopf algebra is unique), so it is a module isomorphism. Thus, the antipode equation has a unique solution. The uniqueness of this solution implies that it gives the antipode of \( A_{n+1} \) when reduced modulo \( p^n \). Thus, \( A_{n+1} \) is a Hopf algebra which is isomorphic to \( A_{n} \) modulo \( p^n \). The existence part of the theorem is proved.

We now prove uniqueness. Let \( \mathcal{A}, \mathcal{A}' \) be two algebras satisfying the conditions of the theorem. Let \( A'_{n+1}, A''_{n+1} \) be their reductions modulo \( p^n \). We will show that for any isomorphism of Hopf algebras \( f_n : A'_{n} \to A''_{n} \) there exists an isomorphism of Hopf algebras \( f_{n+1} : A'_{n+1} \to A''_{n+1} \) such that \( f_{n+1} = f_n \mod p^n \). Since \( A'_1 = A''_1 = A \), this implies the uniqueness part of the theorem.

We identify \( A'_n \) and \( A''_n \) by \( f_n \) and assume \( A'_n = A''_n, f_n = id \). We also assume that \( A'_{n+1}, A''_{n+1} \) are the same as modules. As we saw, the set of extensions of the product and coproduct on \( A_n \) one step higher (with axioms satisfied) is the set of points \( (m', \Delta') \in E \) such that \( c(m', \Delta') = 0 \). Thus, if we have two such extensions \( (m', \Delta') \) and \( (m'', \Delta'') \) corresponding to \( A'_{n+1}, A''_{n+1} \) then the element \( (\mu, \delta) = (m'' - m', \Delta' - \Delta'') \in C^1(A) \) (here we use the transitivity of the action of \( A_{12} \oplus A_{21} \) on \( E \)) satisfies \( d(\mu, \delta) = 0 \). But by Theorem 1.1, \( H^1(A) = 0 \). Thus, \( (\mu, \delta) = d \gamma \), where \( \gamma \in \text{End}_k(A) \). Let \( \hat{\gamma} \) be an extension of \( \gamma \) to a module map \( A'_{n+1} \to A''_{n+1} \) (i.e. \( \hat{\gamma} \) equals \( \gamma \mod p \)), and let \( \eta = id - p^n \hat{\gamma} : A'_{n+1} \to A''_{n+1} \) (it is clear that this depends only on \( \gamma \) and not on \( \hat{\gamma} \)). This is the required isomorphism.

(ii) It remains to show that \( A_0 \) is semisimple and cosemisimple, and that dimensions of irreducible modules and comodules and Grothendieck rings are the same. To do this, we let \( n_i \) be the dimensions of irreducible representations of \( A \). Then \( A = \oplus M_{n_i}(k) \), where \( M_n \) is the matrix algebra of size \( n \). By Hochschild’s theorem, the algebra \( A \) has zero Hochschild cohomology, so it has a unique lifting to \( O \), namely \( \oplus M_{n_i}(O) \). Thus, \( \mathcal{A} = \oplus M_{n_i}(O) \) as an \( O \)-algebra. This implies that \( A_0 = \oplus M_{n_i}(K) \) and hence it is semisimple. The fact that \( A_0 \) is cosemisimple is shown in the same way by taking the duals (it also follows from [LR2]). The fact that all \( A_0 \) modules and comodules over \( \mathcal{A} \) are defined over \( K \) and the identity of dimensions of the irreducible modules and comodules and of Grothendieck rings is now clear. \( \square \)
In the following we consider lifting of homomorphisms.

**Theorem 2.2.** Let \( A, B \) be finite-dimensional semisimple and cosemisimple Hopf algebras over an algebraically closed field \( k \) of characteristic \( p \), \( \phi : A \to B \) be a homomorphism of Hopf algebras, and \( \bar{A}, \bar{B} \) the corresponding liftings to \( \mathcal{O} \). Then there exists a unique homomorphism of Hopf algebras \( \bar{\phi} : \bar{A} \to \bar{B} \) such that \( \bar{\phi} = \phi \) modulo \( p \).

**Proof.** We first prove existence. It is enough to prove that there exists a sequence of Hopf algebra maps \( \phi_n : \bar{A}/p^n \to \bar{B}/p^n, n \geq 1 \), such that \( \phi_1 = \phi \) and \( \phi_n = \phi_{n-1} \) modulo \( p^{n-1} \). We construct it by induction. For \( n = 1 \) there is nothing to construct as \( \phi_1 = \phi \). Suppose \( \phi_n \) has been constructed, and choose an \( \mathcal{O}_{n+1} \)-map \( \tilde{\phi}_{n+1} : \bar{A}/p^{n+1} \to \bar{B}/p^{n+1} \) such that \( \tilde{\phi}_{n+1} = \phi_n \) modulo \( p^n \). Write \( \tilde{\phi}_{n+1}(xy) - \tilde{\phi}_{n+1}(x)\tilde{\phi}_{n+1}(y) = p^n \tilde{\psi}(x,y) \) with \( \tilde{\psi} = \psi \) modulo \( p \), for some \( \psi : A \otimes A \to B \).

Also write \( (\phi_{n+1} \otimes \phi_{n+1})(\Delta(x)) - \Delta(\tilde{\phi}_{n+1}(x)) = p^n \tilde{\eta}(x) \) with \( \tilde{\eta} = \eta \) modulo \( p \), for some \( \eta : A \to B \otimes B \). The pair \( (\psi, \eta) \) is uniquely determined by \( \phi \), and it is straightforward to check that it is a 1-cocycle in the total complex \( C^*(A, B, \phi) \).

Hence by Theorem 1.2, there exists a linear map \( \chi : A \to B \) such that \( (\psi, \eta) = d\chi \). Choose a lifting \( \tilde{\chi} : \bar{A}/p^{n+1} \to \bar{B}/p^{n+1} + \) of \( \chi \), and set \( \phi_{n+1} = \tilde{\phi}_{n+1} - p^n\tilde{\chi} \). Then \( \phi_{n+1} \) is a Hopf algebra map.

We now prove uniqueness. Let \( \bar{\phi}, \bar{\phi}' : \bar{A} \to \bar{B} \) be two liftings of \( \phi : A \to B \). We prove by induction on \( n \) that \( \phi_n, \phi'_n : \bar{A}/p^n \to \bar{B}/p^n \) (the reductions of \( \bar{\phi}, \bar{\phi}' \) modulo \( p^n \) ) are equal. Indeed, this is clear for \( n = 1 \); suppose it is known for \( n = m \). Let \( \phi_{m+1} - \phi'_{m+1} = p^m\tilde{\chi} \), where \( \tilde{\chi} = \chi \) modulo \( p \), where \( \chi : A \to B \) is a linear map. It is straightforward to check that \( \chi \) is a 0-cocycle in \( C^*(A, B, \phi) \), which implies by Theorem 1.2 that \( \chi = 0 \) and thus \( \phi_{m+1} = \phi'_{m+1} \), as desired. This concludes the proof of the theorem. \( \Box \)

**Theorem 2.3.** Let \( (A, R) \) be a finite-dimensional quasitriangular semisimple and cosemisimple Hopf algebra over an algebraically closed field \( k \) of characteristic \( p \). Then there exists a unique \( \bar{R} \in A \otimes A \) such that \( (\bar{A}, \bar{R}) \) is quasitriangular, and \( \bar{R} = R \) modulo \( p \). Furthermore, if \( (A, R) \) is triangular then so is \( (\bar{A}, \bar{R}) \).

**Proof.** Let \( R = \sum x_i \otimes y_i \), and let \( D(A) \) denote the quantum double of \( A \). Since \( (A, R) \) is quasitriangular, the map \( \phi : (D(A), R) \to (A, R) \) determined by the formula \( \phi(q \otimes a) = \sum q(x_i) y_i a \) is a surjective homomorphism of Hopf algebras such that \( (\phi \otimes \phi)(R) = R \) (here \( R \) is the universal R-matrix of \( D(A) \)). Now consider the Hopf algebra \( D(\bar{A}) \). This Hopf algebra is a lifting of \( D(A) \), so by Theorems 2.1 and 2.2, there is a unique isomorphism \( D(\bar{A}) \to D(A) \) which is the identity modulo \( p \). We identify \( D(\bar{A}) \) with \( D(A) \) using this isomorphism. Now, by Theorem 2.2, \( \phi \) can be lifted to a surjective homomorphism of Hopf algebras \( \bar{\phi} : D(\bar{A}) \to A \). Let \( \bar{R} \) be the quasitriangular structure of \( D(\bar{A}) \), and define \( \bar{R} = (\bar{\phi} \otimes \bar{\phi})(\bar{R}) \). It is straightforward to check that \( \bar{R} \) is quasitriangular structure of \( \bar{A} \) which equals \( R \) modulo \( p \) (i.e. is a lifting of \( R \)). This lifting is unique because any lifting \( \bar{R} \) of \( R \) defines a homomorphism of Hopf algebras \( \theta_R : A^{*\text{cop}} \to \bar{A} \), which is a lifting of \( \theta_R : A^{*\text{cop}} \to A \) corresponding to \( R \), and by Theorem 2.2 the lifting of \( \theta_R \) is unique.

Now, suppose \( (A, R) \) is triangular; that is, \( (R^{21})^{-1} = R \). Then \( (\bar{R}^{21})^{-1} \) and \( \bar{R} \) are both quasitriangular structures on \( \bar{A} \) which are liftings of \( R \). By uniqueness of lifting they are equal, and \( \bar{R} \) is triangular as well. \( \Box \)
Corollary 2.4. The assignment $A \to A_0$ determines a functor between the categories of finite-dimensional semisimple and cosemisimple Hopf algebras over $k$ and finite-dimensional semisimple Hopf algebras over $K$. It also determines a functor between the categories of quasitriangular Hopf algebras, and those of triangular ones.

Proof. The first statement follows from Theorems 2.1 and 2.2. As for the second statement, let $A, B$ be finite-dimensional semisimple and cosemisimple Hopf algebras over $k$. Suppose $f : (A,R_A) \to (B,R_B)$ is a map of quasitriangular Hopf algebras. Then we have to show that $(\bar{f} \otimes \bar{f})(\bar{R}_A) = \bar{R}_B$. Indeed, set $R_B' = (\bar{f} \otimes \bar{f})(\bar{R}_A)$. Clearly, the associated map $\theta_{R_B'} : \bar{B}^* \to \bar{B}$ is a Hopf algebra map which equals $\theta_{\bar{R}_B}$ modulo $p$. But, $\theta_{\bar{R}_B} = \theta_{\bar{R}_B}$ modulo $p$ too, hence by Theorem 2.2 $\bar{R}_B = R_B'$. □

3. Applications of the lifting theorems

The lifting theorems provide a simple way to prove results about semisimple and cosemisimple Hopf algebras in characteristic $p$ which are known in characteristic 0. In this section we give a few applications of this sort. We start with Kaplansky’s 5th conjecture from 1975.

Theorem 3.1. Let $A$ be a finite-dimensional semisimple and cosemisimple Hopf algebra over any field $k$. Then the square of the antipode of $A$ is the identity.

Proof. In characteristic 0, the result is known and due to Larson and Radford [LR1]. Suppose $k$ is of characteristic $p$. We can assume that $k$ is algebraically closed. Using Theorem 2.1(i), we can construct the Hopf algebras $\overline{A}$ and $A_0$. By the characteristic 0 result, the square of the antipode is the identity in $A_0$. Thus it is so in $\overline{A} \subset A_0$ and hence in $A = \overline{A}/p\overline{A}$. □

This result was known in characteristic $p > (\dim A)^2$ [LR1].

Corollary 3.2. Let $A$ be a finite-dimensional Hopf algebra with antipode $S$ over any field $k$. Then:

(i) $A$ is semisimple and cosemisimple if and only if $S^2 = I$ and $\dim A \neq 0$ in $k$.
(ii) If $A$ is semisimple and cosemisimple and $k$ is algebraically closed, then for any irreducible representation $V$ of $A$, $\dim V \neq 0$ in $k$.

Proof. (i) Suppose $A$ is semisimple and cosemisimple. Then by Theorem 3.1, $S^2 = I$, and by [LR1, Theorem 2], $\dim A \neq 0$ in $k$. The ”only if” part follows from [R4], since $tr(S^2) = tr(I) = \dim A \neq 0$.

(ii) Follows from [La, Theorem 4.3], which state the same as Corollary 3.2 (ii) under the assumption that $A$ is finite-dimensional semisimple and $S^2 = I$. □

Corollary 3.3. (i) Let $A$ be a finite-dimensional semisimple or cosemisimple minimal quasitriangular Hopf algebra over any field $k$. Then the square of the antipode of $A$ is the identity. If $\text{char}(k) = p > 0$, then $p$ does not divide $\dim A$. (ii) Let $A$ be a finite-dimensional semisimple triangular Hopf algebra over any field $k$. Then the Drinfeld element $u$ satisfies $u = S(u)$ and $u^2 = 1$.

Proof. (i) By [R3, Proposition 14], $A$ is also cosemisimple, and hence by Theorem 3.1, $S^2 = I$. The last statement follows now from Corollary 3.2.

(ii) Consider the corresponding minimal Hopf subalgebra $A_R$ of $A$. Clearly, $u \in A_R$. By (i), $(S|_{A_R})^2 = I$ and hence $u$ is central in $A_R$. It is straightforward to check that
tr(u) = tr(S(u)) in any irreducible representation of $A_R$ (see e.g. [EG1]), hence by Corollary 3.2 (or Theorem 4.3 of [L]), $u = S(u)$. But $S(u) = u^{-1}$ since $u$ is a grouplike element. Thus $u = u^{-1}$ and $u^2 = 1$. □

**Theorem 3.4.** Let $A$ be a semisimple and cosemisimple Hopf algebra of prime dimension $p$ over a field $k$. Then $A$ is commutative and cocommutative.

**Proof.** In characteristic zero, the result is known [Z]. For positive characteristic (which has to be different from $p$ by Corollary 3.2), it suffices to assume that $k$ is algebraically closed. In this case $A_0$ is commutative and cocommutative by the characteristic zero result, thus so are $\overline{A}$ and $A$. □

**Corollary 3.5.** Let $A$ be a Hopf algebra of prime dimension $p$ over a field $k$ with characteristic $q$ such that $q > p$. Then $A$ is commutative and cocommutative.

**Proof.** We follow the ideas of [Z]. Let $G(A), G(A^*)$ denote the groups of grouplike elements of $A, A^*$ respectively. By [NZ], $|G(A)|, |G(A^*)| \in \{1, p\}$. If either $|G(A)| = p$ or $|G(A^*)| = p$ then $A = k\mathbb{Z}_p$ and we are done. Suppose $|G(A)| = |G(A^*)| = 1$. Then $A$ and $A^*$ are unimodular and hence by Radford’s formula [R5], $S^4 = I$. Suppose that $A$ is not semisimple. Then by [R4], $tr(S^2) = 0$ in $k$. But since $S^4 = I$, this implies that there exist integers $0 \leq a, b \leq p$ such that $a - b = 0$ in $k$, and $a + b = p$. Since $q > p$ this is impossible, and hence $A$ is semisimple. Similarly, $A^*$ is semisimple and the result follows from Theorem 3.4. □

**Theorem 3.6.** Let $A$ be a semisimple and cosemisimple Hopf algebra of dimension $pq$ over a field $k$, where $p, q$ are distinct primes. Then $A$ is commutative or cocommutative.

**Proof.** In characteristic zero, the result is known [EG2]. For positive characteristic (which has to be different from $p$ and $q$ by Corollary 3.2), it suffices to assume that $k$ is algebraically closed. In this case $A_0$ is commutative or cocommutative by the characteristic zero result, thus so are $\overline{A}$ and $A$. □

**Theorem 3.7.** Let $A$ be a finite-dimensional semisimple and cosemisimple Hopf algebra over an algebraically closed field, and let $D(A)$ be the quantum double of $A$. Then the dimension of any irreducible representation of $D(A)$ divides the dimension of $A$.

**Proof.** In characteristic zero, the result is known [EG1]. For positive characteristic, first note that the double of $A$ is semisimple and cosemisimple, and that taking the double commutes with lifting (as we mentioned in the proof of Theorem 2.3). By the characteristic zero result, the statement is true for $D(A_0) \otimes_K \overline{K}$. By Theorem 2.1(ii), all representations of $D(A_0) \otimes_K \overline{K}$ split already over $K$. Therefore, applying Theorem 2.1(ii) again we get that the result also holds for $D(A)$. □

**Corollary 3.8.** Let $A$ be a finite-dimensional semisimple and cosemisimple quasitriangular Hopf algebra over an algebraically closed field. Then the dimension of any irreducible representation of $A$ divides the dimension of $A$.

**Proof.** We have a surjective homomorphism of Hopf algebras $D(A) \rightarrow A$ (as we mentioned in the proof of Theorem 2.3), so any irreducible $A$-module is also an irreducible $D(A)$-module. Now apply Theorem 3.7. □
Theorem 3.9. Let $A$ be a finite-dimensional semisimple and cosemisimple triangular Hopf algebra over any field $k$. Then the dimensions of irreducible modules and the Grothendieck ring of the category $\text{Rep}(A)$ coincide with those of some finite group $G$.

Proof. Consider the Hopf algebra $A_0$. Its category of representations is equivalent to the one of some finite group $G$ by [EG1, Theorem 2.1]. But the dimensions of irreducible representations and the Grothendieck ring are the same for $A$ and $A_0$ by Theorem 2.1. □

Theorem 3.10. Let $A$ be a semisimple and cosemisimple Hopf algebra of dimension $p^n$, where $p$ is a prime, over an algebraically closed field $k$. Then $A$ has a non-trivial central grouplike element.

Proof. In characteristic zero, the result is known [Ma]. For positive characteristic (which has to be different from $p$ by Corollary 3.2), the statement is true for $A_0 \otimes_K \overline{K}$ by the characteristic zero result. Let $g$ be a non-trivial central grouplike element of $A_0 \otimes_K \overline{K}$. Then $g : A_0^* \to \overline{K}$ is a 1-dimensional representation such that $(g \otimes 1) \circ \Delta = (1 \otimes g) \circ \Delta$. But by Theorem 2.1(ii), this representation is obtained by lifting of a unique representation $g' : A^* \to k$, which therefore must satisfy the same equation. Then $g'$ is a nontrivial central grouplike element in $A$. □

4. When Finite-Dimensional Semisimple Hopf Algebras are Cosemisimple

Let $A$ be any semisimple Hopf algebra of dimension $d > 2$ over a field $k$ of characteristic $p > d^{\varphi(d)/2}$ (here $\varphi$ is the Euler function i.e. $\varphi(d) = \#\{1 \leq m \leq d | (d, m) = 1\}$). In this section we prove that $A$ is also cosemisimple. This result was known in characteristic 0 [LR2], and in characteristic $p > (2d^2)^{2d^2-4}$ [So]. So our result slightly improves Sommerhauser’s result. Our proof is based on combining the ideas of Larson and Radford [LR2] with a trivial number theoretic lemma.

Lemma 4.1. Let $P = \sum_{m=0}^{r-1} a_m x^m \in \mathbb{Z}[x]$ be a polynomial of degree $r - 1 > 1$. Let $\sum_{m=0}^{r-1} |a_m| = D$, and suppose that $P(e^{2\pi i/r})$ is a non-zero real number. Then for all $p > D^{\varphi(r)/2}$ such that $p$ does not divide $r$, and any primitive $r$th root of unity $\zeta \in \mathbb{F}_p$ (the algebraic closure of the field of $p$ elements), one has $P(\zeta) \neq 0$.

Proof. Let $N = \prod_{l=1, l<r/2} P(e^{2\pi i/l})$ be the product of all the conjugates of $P(e^{2\pi i/r})$. It is easy to see that $N \in \mathbb{Q}^*$ and is an algebraic integer, so we have that $N$ is a non-zero integer. Clearly, $|N| \leq D^{\varphi(r)/2}$, and hence $p$ does not divide $N$. This implies that $N$ is a unit in the ring $\mathcal{O} = W(\mathbb{F}_p)$ of Witt vectors of $\mathbb{F}_p$. Let $\hat{\zeta}$ be the lifting of $\zeta$ to $\mathcal{O}$; that is, $\hat{\zeta}^r = 1$ and $\hat{\zeta} = \zeta$ modulo $p$. Then $P(\hat{\zeta})$ is a unit in $\mathcal{O}$ since $\prod_{l=1, l<r/2} P(\hat{\zeta}^l) = N$ and $P(\hat{\zeta}^l)$ is an algebraic integer (hence an integer in $\mathcal{O}$) for all $l$. Therefore, $P(\zeta) \neq 0$ in $\mathbb{F}_p$. □

Theorem 4.2. Let $A$ be a $d$-dimensional Hopf algebra, $d > 2$, with antipode $S$ over a field $k$ of characteristic $p > d^{\varphi(d)/2}$. Then the following are equivalent:

(i) $A$ is semisimple.

(ii) $A$ is cosemisimple.
(iii) \( S^2 = I \).

**Proof.** (i) implies (ii): We may assume that \( k \) is algebraically closed, and start as in [LR2]. Since \( A \) is semisimple we have \( A = \bigoplus \text{End}_k(V_i) \) where \( V_i \) are the irreducible representations of \( A \). Let \( I \) be the set of all \( i \)'s for which \( S^2(\text{End}_k(V_i)) = \text{End}_k(V_i) \) (in fact, it is easy to show that all \( i \) are such, but we do not need it). Since \( A \) is unimodular it follows by [R5] that for all \( a \in A \), \( S^4(a) = gag^{-1} \) for some grouplike element \( g \in A \). Thus \( S^4d = I \), so if \( s \) is the order of \( S^4 \) then \( s \) divides \( d \). Let \( r = 2s \) be the order of \( S^2 \). If \( r = 2 \), then \( S^4 = I \) and the result follows from [LR1] (for \( p > d \)). If \( r = 2d \) then \( A \) is a group algebra, so there is nothing to prove. Therefore we may assume \( 2 < r \leq d \), thus \( p > r \). Since for all \( i \in I \), \( S^2(\text{End}_k(V_i)) \) is an algebra automorphism, it is inner; that is \( S^2(\text{End}_k(V_i))(B) = Bg_i^{-1} \) for some \( g_i \in \text{Aut}_k(V_i) \). Since \( g_i^r \) is central in \( \text{End}_k(V_i) \) we may assume that \( g_i^r = I \). Let \( \zeta \) be a primitive \( r \)th root of unity in \( k \), and write \( \text{End}_k(V_i) \) as \( \bigoplus_{j=0}^{r-1} V_{ij} \) where \( V_{ij} \) is the eigenspace corresponding to \( \zeta^j \) (possibly \( V_{ij} = 0 \)). Set \( d_{ij} = \dim V_{ij} \). Then,

\[
T = \text{tr}(S^2) = \sum_{i \in I} \text{tr}(S^2_{\text{End}_k(V_i)}) = \sum_{i \in I} \text{tr}(g_i)\text{tr}(g_i^{-1}) = \sum_{i \in I} (\sum_{j=0}^{r-1} d_{ij}\zeta^j)(\sum_{j=0}^{r-1} d_{ij}\zeta^{-j}).
\]

By [R4], it is enough to show that \( T \neq 0 \) in \( k \).

Now, since \( \zeta^r = 1 \), we can write

\[
T = \sum_{m=0}^{r-1} a_m\zeta^m, \quad a_m = \sum_{i \in I} (\sum_{j-l=m \text{ (mod } r)} d_{ij}d_{il}) \in \mathbb{Z}_+.
\]

Consider the polynomial \( P(x) = \sum_{m=0}^{r-1} a_m x^m \in \mathbb{Z}[x] \). Since in \( \mathbb{Q}(e^{2\pi i/r}) \),

\[
(\sum_{j=0}^{r-1} d_{mj}e^{2\pi ij/r})(\sum_{j=0}^{r-1} d_{mj}e^{-2\pi ij/r}) = |\sum_{j=0}^{r-1} d_{mj}e^{2\pi ij/r}|^2 \geq 0,
\]

and \( \text{tr}(S^2_{\text{End}_k(V_k)}) = 1 \), it follows that \( P(e^{2\pi i/r}) \in \mathbb{R}^+ \). Finally, \( D = \sum_{m} a_m \leq d \) and \( \varphi(r) \leq \varphi(d) \) (as \( r = 2d/n \), where \( n \geq 2 \) and divides \( d \)). Therefore the result follows from Lemma 4.1.

(ii) implies (i): Follows from (i) implies (ii) for \( A^* \).

(i) and (ii) imply (iii): Follows from [LR1] where the result is proved for \( p > d^2 \).

(iii) implies (i) and (ii): Follows from [R1] since \( d \neq 0 \) in \( k \).

**Acknowledgments.** The authors are grateful to David Kazhdan and Murray Gerstenhaber for useful discussions and explanations, and to Dima Arinkin for reading the manuscript.

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