ON THE UNIQUENESS OF ADS
SPACE-TIME IN HIGHER DIMENSIONS

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ABSTRACT. In this paper, based on an intrinsic definition of asymptotically AdS
space-times, we show that the standard anti-de Sitter space-time is the unique strictly
stationary asymptotically AdS solution to the vacuum Einstein equations with neg-
ative cosmological constant in dimension less than 7. Instead of using the positive
energy theorem for asymptotically hyperbolic spaces our approach appeals to the
classic positive mass theorem for asymptotically flat spaces.

1. INTRODUCTION

Recently, there has been some interest in the study of space-times that satisfy
Einstein equations with negative cosmological constant in association with the so-
called AdS/CFT correspondence. With the presence of a negative cosmological
constant, the anti-de Sitter space-time replaces the Minkowski space-time as the
ground state of the theory. Bocher, Gibbons and Horowitz showed that in 3 + 1 di-
dimensions, the only strictly stationary asymptotically AdS space-time that satisfies
the vacuum Einstein equations with negative cosmological constant is the anti-de
Sitter space-time in [BGH] (see also [CS]). Another class of globally static asymp-
totically locally AdS space-times, the AdS solitons, are also important in the theory.
In [GSW], Galloway, Surya and Woolgar proved a uniqueness theorem of the AdS
solitons. Later, in [ACD], Anderson, Chrusciel and Delay improved the uniqueness
theorem of AdS solitons.

In [Wa1], the uniqueness result of [BGH] was generalized to higher dimensions
when the space-time is static and the static slice is of spin structure. Proofs in
[BGH] and [Wa1] all appeal to the positive energy theorem for asymptotically hy-
perbolic spaces (see [CH], [Wa3] and some early references therein). Our proof of
the uniqueness of the AdS space-time instead appeals to the classic positive mass
Theorem for asymptotically flat spaces. By this approach we may use the classic positive mass theorem of Schoen and Yau [SY] to drop the spin structure assumption in dimension less than 7.

The anti-de Sitter space-time in \( n+1 \) dimensions is given by \((R^{n+1}, g_{AdS})\) where

\[
g_{AdS} = -(1 + r^2)dt^2 + \frac{1}{1 + r^2} dr^2 + r^2 d\sigma_0
\]

in coordinates \((t, r, \theta) \in R \times [0, \infty) \times S^{n-1}\) and \(d\sigma_0\) is the standard round metric on a unit \((n-1)\)-sphere. It is a static solution to the vacuum Einstein equation

\[
\text{Ric} - \frac{1}{2} R g + \Lambda g = 0
\]

with negative cosmological constant \(\Lambda = -\frac{1}{2} n(n-1)\). The staticity means that \((R^{n+1}, g_{AdS})\) can be constructed by a triple \((R^n, g_H, \sqrt{1 + r^2})\) where \(g_H\) is the hyperbolic metric on \(R^n\) and

\[
\nabla^2 \sqrt{1 + r^2} = \sqrt{1 + r^2} g_H
\]
on the hyperbolic space \((R^n, g_H)\). The simplest examples of space-times that are asymptotically the same as the anti-de Sitter space-time at the infinity are the so-called Schwarzschild-AdS space-times whose metrics are given by

\[
g^+_M = -(1 + r^2 - \frac{M}{r^{n-2}})dt^2 + \frac{1}{1 + r^2 - \frac{M}{r^{n-2}}} dr^2 + r^2 d\sigma_0.
\]

They also satisfy the vacuum Einstein equation (1.2), but the difference is that on the AdS space-time there is an everywhere time-like Killing field \(\frac{\partial}{\partial t}\) while this is not so on the Schwarzschild-AdS space-times. In other words, the AdS space-time is strictly stationary, but the Schwarzschild-AdS space-times are not.

We will follow the idea in [AM] to give a definition of asymptotically AdS space-times (see Definition 2.1). One can find a good discussion of the comparisons of different definitions of asymptotically AdS space-times in [CS]. Then we show

**Theorem 1.1.** Suppose that \((Y^{n+1}, g)\) is a strictly stationary asymptotically AdS space-time. And suppose that \(g\) satisfies the vacuum Einstein equation with negative cosmological constant. Then \((Y^{n+1}, g)\) is static, i.e.

\[
\begin{align*}
Y^{n+1} &= R \times \Sigma \\
g &= -V dt^2 + h
\end{align*}
\]
where $V > 0$ on $\Sigma$ and

$$\begin{cases}
\Delta \sqrt{V} = n \sqrt{V} \\
Ric[h] + nh = (\sqrt{V})^{-1} \nabla^2 \sqrt{V}
\end{cases}$$

(1.6)
on the Riemannian manifold $(\Sigma, h)$.

By the Frobenius Theorem, staticity is locally equivalent to $\theta = \omega \wedge d\omega = 0$ where $\omega$ is the dual of the given Killing vector field $X$. Instead of using topological assumptions to write $*\theta = d\psi$ to prove the vanishing of $\theta$ in classic Lichnerowicz argument (see [BGH], [Ca]) we observe that

$$d(\frac{1}{V} \omega) = - \frac{1}{V^2} i_X \theta,$$

(1.7)

which allows us to calculate the boundary integral to show the vanishing of $\theta$ from the behavior of $X$ near the infinity. We adopt the method of Fefferman and Graham [FG], [G] to construct a preferable coordinate system near the infinity which allows us to know the asymptotic behavior of both the metric $g$ and the Killing field $X$ near the infinity rather precisely.

Our next goal is to prove the static solution $(\Sigma, h, \sqrt{V})$ must be the same as $(\mathbb{R}^n, g_H, \sqrt{1 + r^2})$ for some choice of coordinates. Namely,

**Theorem 1.2.** Suppose that an asymptotically AdS space-time is a static solution satisfying (1.5) and (1.6). Then $(\Sigma, h, \sqrt{V}) = (\mathbb{R}^n, g_H, \sqrt{1 + r^2})$ for some choice of coordinates in dimension between 3 and 7.

Our approach is similar to the one used to prove the uniqueness of conformally compact Einstein manifolds in [Q]. We use the global defining function $(\sqrt{V}+1)^{-1}$ to turn $(\Sigma, h)$ into a compact manifold $(\Sigma, \bar{h})$ which has the round sphere as its totally umbilical boundary and whose scalar curvature is nonnegative. The nonnegativity of the scalar curvature follows from the application of the strong maximum principle and the following Bochner formula:

**Lemma 1.3.** Suppose that an asymptotically AdS space-time is a static solution satisfying (1.5) and (1.6). Then

$$-\Delta (V - |\nabla \sqrt{V}|^2 - 1) = 2|\nabla^2 \sqrt{V} - \sqrt{V} h|^2 - \frac{\nabla \sqrt{V}}{\sqrt{V}} \cdot \nabla (V - |\nabla \sqrt{V}|^2 - 1).$$

Then we appeal to the recent work in [Mi](see also [ST]) to conclude that it has to be scalar flat, which implies

$$\nabla^2 \sqrt{V} = \sqrt{V} h.$$

(1.9)

Note that [Mi] relies on the classic positive mass theorem of Scoen and Yau [SY]. Theorem 1.2 then follows from the following lemma similar to a theorem of Obata in [Ob].
Lemma 1.4. Suppose that \((M, g)\) is a complete Riemannian manifold. And suppose that there is a positive function \(\phi\) such that \(\nabla^2 \phi = \phi g\). Then \((M, g)\) is isometric to \((\mathbb{R}^n, g_H)\).

2. Asymptotically AdS space-times

In this section we will start with an intrinsic definition of asymptotically AdS space-times and derive some properties of a strictly stationary asymptotically AdS space-time. We will then prove a lemma of Lichnerowicz type similar to the one in [BGH]. We note that, in fact, it was asked whether the uniqueness theorem in their paper [BGH] still holds if one uses the definition of asymptotically AdS space-times proposed by Ashtekar and Magnon in [AM] (see also [Ha]).

Let us first introduce the AdS space-time in general dimensions. The anti-de Sitter space-time in \((n+1)\) dimensions can be given by \((\mathbb{R}^{n+1}, g_{\text{AdS}})\) where

\[
 g_{\text{AdS}} = -(1 + r^2) dt^2 + \frac{1}{1 + r^2} dr^2 + r^2 d\sigma_0^n
\]

\(d\sigma_0^n\) is the unit round metric on \(S^{n-1}\), \(t \in (-\infty, +\infty)\), and \(r \in [0, +\infty)\). In the following we will adopt the definition of an asymptotically AdS space-time given by Ashtekar and Magnon in [AM]. To do so, let us first discuss what a conformal completion for a space-time is following the idea of Penrose in [Pe]. Suppose that \(Y^{n+1}\) is a manifold with boundary \(\partial Y^{n+1} = X^n\). Then \(\Omega\) is said to be a defining function of \(X^n\) in \(Y^{n+1}\) if

a) \(\Omega > 0\), in \(Y^{n+1}\);

b) \(\Omega = 0\), on \(X^n\); and

c) \(d\Omega \neq 0\) on \(X^n\).

A space-time \((Y^{n+1}, g)\) has a \(C^k\) conformal completion if \(Y^{n+1}\) is a manifold with boundary \(X^n\) and the metric \(\Omega^2 g\) for a defining function \(\Omega\) of \(X^n\) in \(Y^{n+1}\) extends in \(C^k\) to the closure of \(Y^{n+1}\).

Definition 2.1. A space-time \((Y^{n+1}, g)\) of dimension \((n+1)\) is said to be asymptotically AdS if

1) \((Y^{n+1}, g)\) has a \(C^k\) conformal completion and its boundary \(\partial Y^{n+1} = X^n\) is topologically \(R \times S^{n-1}\);

2) the space-time \((Y^{n+1}, g)\) satisfies the Einstein equation with a negative cosmological constant \(\Lambda\)

\[
 R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}
\]

where \(\Omega^{-n} T_{ab}\) admits a \(C^k\) extension to the closure of \(Y^{n+1}\);
3) $(X^n, \Omega^2 g|_{TX^n})$ is conformal to $(R \times S^{n-1}, g_0)$ where $g_0 = -dt^2 + d\sigma_0$.

For the convenience, from now on, we will always assume

\begin{equation}
\Lambda = -\frac{n(n-1)}{2}.
\end{equation}

First, by definition, defining functions for $X^n$ in $Y^{n+1}$ are not unique and two different defining functions differ by a positive function on the closure of $Y^{n+1}$. Therefore only the class of quadratic forms $\Omega^2 g|_{TX^n}$ up to a conformal factor is determined by $g$. Second, by requiring the fall-off of the energy-stress tensor, one can compute that the sectional curvature of $g$ would asymptotically go to $-|d\Omega|^2_{\Omega^2 g}$, and conclude

\[ |d\Omega|^2_{\Omega^2 g}|_{X^n} = 1 > 0. \]

Therefore $X^n$ is a time-like hypersurface in $(Y^{n+1}, \Omega^2 g)$. Finally the conformal flatness of the Lorentz metric $\Omega^2 g|_{TX^n}$ depends only on the Lorentz metric $g$. In fact, Hawking in [Ha] had already observed that locally conformal flatness is an appropriate boundary condition.

We next want to choose a coordinate system near the boundary for an asymptotically AdS space-time. What we will do is mostly an analogue to the Euclidean cases which have been established in [FG], [G]. First, we construct a special defining function, at least in a tubular neighborhood of the boundary for each given metric in the class $[-dt^2 + d\sigma_0]$ on the boundary by solving a first order PDE. Namely,

**Lemma 2.1.** Suppose $(Y^{n+1}, g)$ is an asymptotically AdS space-time, and $\Omega$ is a defining function. Then, for each metric $\hat{g} = e^{2\phi} g_0$ where $g_0 = -dt^2 + d\sigma_0$, there is a unique defining function $s$ in a tubular neighborhood of the boundary $X^n$ in $Y^{n+1}$ such that

a) $s^2 g|_{TX^n} = \hat{g}$;

b) $|ds|_{s^2 g} = 1$ in the tubular neighborhood.

**Proof.** Set $s = e^w \Omega$. Then

\[ ds = e^w (d\Omega + \Omega dw) \]

and

\[ |ds|_{s^2 g}^2 = |ds|_{e^{2w} \Omega^2 g}^2 = e^{-2w} |ds|_{\Omega^2 g}^2 = |d\Omega + \Omega dw|_{\Omega^2 g}^2 = |d\Omega|_{\Omega^2 g}^2 + 2\Omega(d\Omega, dw)_{\Omega^2 g} + \Omega^2 |dw|_{\Omega^2 g}^2. \]
Thus, the requirement $|ds|_{s^2g} = 1$ is equivalent to solving

$$2(d\Omega, dw)_{\Omega^2g} + \Omega|dw|^2_{\Omega^2g} = \frac{1 - |d\Omega|^2_{\Omega^2g}}{\Omega}.$$  

The boundary condition is determined as follows: if we denote $\Omega^2g|_{TX^n} = e^{2\psi}g_0$, then

$$w|_{X^n} = \phi - \psi.$$  

It is easily seen that (2.4) and (2.5) is non-characteristic. Notice that both $d\Omega$ and $dw$ are space-like.

**Lemma 2.2.** Suppose $(Y^{n+1}, g)$ is an asymptotically AdS space-time. Suppose that $s$ is the special defining function obtained in Lemma 2.1 for which $s^2g|_{TX^n} = g_0$. Then

$$g = s^{-2}(ds^2 + g_s)$$  

where

$$g_s = -(1 + \frac{s^2}{4})^2dt^2 + (1 - \frac{s^2}{4})^2d\sigma_0 + O(s^n).$$

**Proof.** The proof again is adopted from the argument given in [FG], [G]. By the fall-off condition of the energy-momentum tensor $T_{ab}$ one can rewrite the equation (2.2) in coordinates $R \times S^{n-1} \times [0, \epsilon)$ near the boundary as

$$h''_{ab} + (1 - n)h'_{ab} - h^{cd}h'_{cd}h_{cd} - sh^{cd}h'_{ac}h'_{bd} + \frac{1}{2}sh^{cd}h'_{cd}h'_{ab} - 2sR_{ab}[h] = O(s^n).$$

where $h$ stands for $g_s$ for convenience. The signature of $g_s$ here does not make any difference in terms of solving the expansion of $g_s$. Therefore, similar to what is known for Euclidean case, all odd order terms of order $\leq n-1$ vanish and all even order terms of order $\leq n-1$ is determined by the metric $g_0$ on $X^n$. Moreover, when $n$ is odd, the $n$th order term is traceless; when $n$ is even, in general one would need to add one more term in the order of $s^n \log s$ which is traceless and determined by $g_0$ while the trace part of the $n$th order is also determined by $g_0$. By comparing to the AdS space

$$g_{AdS} = s^{-2}(ds^2 - \frac{s^2}{4})^2dt^2 + (1 - \frac{s^2}{4})^2d\sigma_0)$$

which is of the same boundary metric $-dt^2 + d\sigma_0$, we may complete the proof.
Remark 2.3. In the above argument, it is clear that a weaker fall-off condition of the energy-momentum would imply a weaker control of the asymptotic of the metric $g$.

Next we will follow [BGH] to restrict ourselves to the so-called strictly stationary space-time. That is to assume, for an asymptotically AdS space-time, there is a global everywhere time-like Killing field which approaches $\frac{\partial}{\partial t}$ asymptotically towards the boundary. In [BGH] it was shown that a strictly stationary asymptotically AdS space-time (by their definition in dimension 3) which solves the vacuum Einstein equations with negative cosmological constant must be a static one. Before proceeding to prove the staticity we want to study the asymptotic behavior of a Killing field that approaches $\frac{\partial}{\partial t}$ at the infinity. We will use the favorable coordinates constructed in Lemma 2.2. Denote the Killing field by

$$(2.9) \quad X = a(s, t, \sigma) \frac{\partial}{\partial s} + b(s, t, \sigma) \frac{\partial}{\partial t} + c^i(s, t, \sigma) \frac{\partial}{\partial \theta^i},$$

where $(s, t, \theta_1, \cdots, \theta_{n-1}) \in [0, \epsilon) \times \mathbb{R} \times S^{n-1}$. For similar computations, please see [Wa2]. First of all, by the boundary condition, we know that

$$b(0, t, \theta) = 1, \quad c^i(0, t, \theta) = 0, \forall i = 1, \cdots, n - 1. \quad (2.10)$$

Computing $X g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ one gets

$$\frac{\partial a}{\partial s} = \frac{a}{s}. \quad (2.11)$$

Therefore $a(s, t, \theta) = sa(t, \theta)$. For convenience we denote by $t = \theta_0$ and $b = c^0$, and use Greek letters to include zero. Computing $X g(\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\beta})$ one gets

$$g_s(\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\beta}) \frac{\partial c^\beta}{\partial s} + s \frac{\partial a}{\partial \theta^\alpha} = 0. \quad (2.12)$$

Therefore

$$\begin{cases} b(s, t, \theta) = 1 - \int_0^s u g_0^{0\beta} du \frac{\partial a}{\partial \theta^\beta} \\ c^\alpha(s, t, \theta) = -\int_0^s u g_{u\alpha}^{\beta\gamma} du \frac{\partial a}{\partial \theta^\beta} \end{cases}. \quad (2.13)$$

In the other directions one gets

$$g_s(\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\gamma}) c^\beta_{\gamma, \beta} + g_s(\frac{\partial}{\partial \theta^\beta}, \frac{\partial}{\partial \theta^\gamma}) c^\gamma_{\alpha, \gamma} = 0.$$
Notice that, if we denote the Christoffel symbols of metric $g_s$ on the slices by $\Gamma^\alpha_{\beta\gamma}$ and $\bar{\Gamma}^a_{bc}$ for the ones of metric $g$, then

\begin{equation}
\Gamma^\alpha_{\beta\gamma} = \bar{\Gamma}^\alpha_{\beta\gamma}, \quad \bar{\Gamma}^a_{\beta\gamma} = \frac{1}{2} h^\alpha_{\gamma\delta} \frac{\partial}{\partial s} h_{\gamma\beta} - \frac{1}{s} \delta_{\alpha\beta},
\end{equation}

where again, for convenience, we use $h = g_s$. Thus

\begin{equation}
 h_{\alpha\gamma}(\frac{\partial c^\gamma}{\partial \theta^\beta} - c^\delta \Gamma^\gamma_{\delta\beta}) + h_{\beta\gamma}(\frac{\partial c^\gamma}{\partial \theta^\alpha} - c^\delta \Gamma^\gamma_{\delta\alpha}) = 2ah_{\alpha\beta} - sa \frac{\partial}{\partial s} h_{\alpha\beta}.
\end{equation}

Taking $s = 0$ in (2.15) we immediately see that $a(t, \theta) = 0$ and surprisingly get $X = \frac{\partial}{\partial t}$ in this neighborhood. Moreover (2.15) implies $g_s$ is independent of $t$ in this neighborhood. Let us summarize what we obtained in a lemma.

**Lemma 2.4.** Suppose that $(Y^{n+1}, g)$ is an asymptotically AdS space-time and that $s$ is the special defining function such that $s^2 g|_{TX} = - dt^2 + d\sigma_0$. If $X$ is a Killing field that approaches to $\frac{\partial}{\partial t}$ at the infinity, then $X = \frac{\partial}{\partial t}$ and $g_s$ is independent on $t$ in a tubular neighborhood of the boundary.

A strictly stationary asymptotically AdS space-time $(Y^{n+1}, g)$ comes with a proper and free $R^1$ action. Here properness of the action comes from the causality axiom (cf. [Ca]). Therefore, by Theorem 1.11.4 in [DK], we know that $Y$ is a principle bundle over the smooth orbit space $Y/R$. Thus topologically, $Y = R \times \Sigma$ for some smooth $n$-manifold $M$ with boundary $S^{n-1}$ in the light of the discussion above on the asymptotic behavior of the Killing field $X$.

Now let us discuss the staticity of a space-time. A good reference for this discussion is [Ca]. A space-time is said to be static if there is an everywhere time-like Killing field whose trajectories are everywhere orthogonal to a family of space-like hypersurfaces. Let us introduce some notations. Let $\{e_a\}$ be an orthonormal frame and $\{w^a\}$ be its co-frame. Suppose $X = k^a e_a$ is an everywhere time-like Killing field and let $\omega = k_a w^a$. In this notation $X$ is a Killing field if

\begin{equation}
k_{a,b} + k_{b,a} = 0.
\end{equation}

By Frobenius Theorem, staticity is equivalent to asking that the differential ideal generated by the differential $\omega$ be closed under exterior differentiation, i.e.

\begin{equation}
\theta = \omega \wedge d\omega = 0.
\end{equation}

A connected static space-time becomes $R \times \Sigma$ where $\Sigma$ is a static slice (topologically the same as $M$), and the metric $g = -V dt^2 + g_\Sigma$ where $V = -k^a k_a$ and $g_\Sigma$ is metric on the slice $\Sigma$ of Euclidean signature. The following lemma of Lichnerowicz type is a straightforward generalization of a lemma in [BGH] (see also [Ca] for more details), but the proof is adapted for general dimension and uses no additional topological assumptions. Notice that the definition of an asymptotically AdS space-time in [BGH] is different from ours in this note.
Lemma 2.5. Any strictly stationary asymptotically AdS space-time \((Y^{n+1}, g)\) which satisfies the vacuum Einstein equations with negative cosmological constant \(\Lambda\) is static.

To prove this lemma we observe

Lemma 2.6. \((2.18)\)

\[
\frac{i_X \theta}{V^2} = -d\left(\frac{\omega}{V}\right)
\]

where

\[V = -k^a k_a.\]

Proof. We simply compute

\[
d\left(\frac{\omega}{V}\right) = \frac{d\omega}{V} - \frac{1}{V^2} dV \wedge \omega = -\frac{1}{V^2} (d\omega(-V) + dV \wedge \omega) = -\frac{i_X \theta}{V^2}.
\]

Because

\[i_X \omega = \omega(X) = k^a k_a = -V\]

and

\[i_X dw = i_X (k_{a,b} w^b \wedge w^a) = k_{a,b} i_X w^b w^a - k_{a,b} w^b i_X w^a = k_{a,b} k^b w^a - k_{a,b} k^a w^b = -2k^b k_{b,a} w^a = dV.\]

Proof of Lemma 2.5. Let us consider the Hodge dual \(*\theta\) of \(\theta\). Since

\((2.19)\)

\[d^* \theta = (k_{[a,b} k_c)]^c w^a \wedge w^b = \frac{2}{3} k^c R_{c[a} k_{b]} w^a \wedge w^b = 0\]

due to the fact that \(R_{ab} = n \eta_{ab}\) where \(\eta_{ab}\) is the standard Minkowski metric (please see Chapter 6 in Part II of [Ca]) and

\((2.20)\)

\[i_X (\ast \theta) = \ast (\theta \wedge \omega) = 0,\]

it follows

\[(2.21)\]

\[d\left(\frac{\omega}{V} \wedge \ast \theta\right) = -\frac{i_X \theta \wedge \ast \theta}{V^2} = -\frac{i_X (\theta \wedge \ast \theta)}{V^2}.
\]

The next step is to integrate over a space-like hypersurface \(\Sigma_\epsilon\) whose boundary is a large \((n-1)\)-sphere \(S_{\epsilon}^{n-1} = \{s = \epsilon, t = c\}\) in the preferable coordinates. We therefore have

\[(2.22)\]

\[\int_{\Sigma_\epsilon} \frac{\theta}{V^2} \wedge \omega(N) \, d\sigma = -\int_{S_{\epsilon}^{n-1}} \frac{\omega}{V} \wedge \ast \theta\]
where $N$ is the unit normal of $\Sigma_\varepsilon$, and $d\sigma$ is the volume element of $\Sigma_\varepsilon$ in the space-time. Notice that $\omega(N) > 0$ and $\theta$ is space-like since $\theta \wedge \omega = 0$. Now let us recall that the Killing field $X$ is just $\frac{\partial}{\partial t}$ in the preferable coordinates. Thus

$$\omega|_{S^{n-1}} = -V dt + g_{0k} d\theta^k = -V dt + s^{n-2}\tau_{0k} d\theta^k = e^{n-2}\tau_{0k} d\theta^k$$

where

$$g = s^{-2}(ds^2 + (1 + \frac{s^2}{4})^2 dt^2 + (1 - \frac{s^2}{4})^2 d\sigma_0 + s^n \tau)$$

and

$$V = s^{-2}(1 + \frac{s^2}{4})^2 - s^{n-2}\tau_{00}.$$ 

Then

$$d\omega = -dV \wedge dt + dg_{0k} \wedge d\theta^k,$$

$$\theta = -V dt \wedge dg_{0k} \wedge d\theta^k - s^{n-2}\tau_{0k}d\theta^k \wedge dV \wedge dt + s^{n-2}\tau_{0k}d\theta^k \wedge dg_{0l} \wedge \theta^l,$$

and

$$*\theta|_{S^{n-1}} = (-V \frac{\partial g_{0k}}{\partial s} + g_{0k} \frac{\partial V}{\partial s}) * (dt \wedge ds \wedge d\theta^k)$$

$$= C s^{n-5} * (dt \wedge ds \wedge d\theta^k)$$

$$= C d\theta^1 \wedge \ldots \wedge d\theta^{k} \wedge \ldots \wedge d\theta^{n-1}$$

where $C$ stands for some function on $S^{n-1}$. Therefore

$$\frac{\omega}{V} \wedge *\theta|_{S^{n-1}} = O(e^n) d\theta^1 \wedge d\theta^2 \wedge \ldots \wedge d\theta^{n-1}.$$ 

This implies that $\theta = 0$ on the hypersurface $\Sigma$. But $\Sigma$ is arbitrary, so $\theta = 0$ on $Y^{n+1}$, which finishes the proof.

Let us conclude this section by making it clear what a static asymptotically AdS space-time which satisfies the vacuum Einstein equations with negative cosmological constant $\Lambda$ is. We state our observation in the following lemma.
Lemma 2.7. Under the assumption of Lemma 2.5, in the preferable coordinate system at the infinity, indeed, a slice of constant $t$ is a static slice, i.e. $\frac{\partial}{\partial t}$ is orthogonal to the slice of constant $t$.

Proof. Consider the conformal completion $(Y^{n+1}, \bar{g})$ where $\bar{g} = ds^2 + g_s$. By the construction of the preferable coordinate system, each curve $\gamma(s) = (s, t, \theta_1, \cdots, \theta_{n-1})$ is a geodesic from the point $(0, t, \theta_1, \cdots, \theta_{n-1})$ in the space-time $(Y^{n+1}, \bar{g})$. On the other hand, a static slice $\Sigma$ of $(Y^{n+1}, g)$ is still a maximum integral hypersurface which is orthogonal to $\frac{\partial}{\partial t}$ everywhere with respect to $\bar{g}$. Because $\bar{g} = ds^2 + g_s$ is independent of $t$, such $\Sigma$ is totally geodesic in $(Y^{n+1}, \bar{g})$. Therefore a geodesic emanating from a boundary point $(0, t_0, \theta_1, \cdots, \theta_{n-1})$ with respect to the metric $\bar{g}$ stays in a static slice. Thus a slice of constant $t$ coincides with a static slice. So the proof is complete.

We summarize our result in the following theorem:

Theorem 2.8. Suppose $(Y^{n+1}, g)$ is a strictly stationary asymptotically AdS space-time that satisfies the vacuum Einstein equations with negative cosmological constant $\Lambda = -\frac{n(n-1)}{2}$. Then $Y^{n+1} = R \times \Sigma^n$, 

\begin{equation}
(2.29) \quad g = -V dt^2 + h
\end{equation}

and, on $\Sigma$,

\begin{equation}
(2.30) \quad \left\{ \begin{array}{l}
\Delta \sqrt{V} = n \sqrt{V} \\
\text{Ric}[h] + nh = (\sqrt{V})^{-1} \nabla^2 \sqrt{V},
\end{array} \right.
\end{equation}

where $h$ is the metric induced from $g$ on a static slice $\Sigma$ of Euclidean signature. Moreover $(\Sigma, h)$ is conformally compact of the same regularity as of the conformal completion of $(Y^{n+1}, g)$ with the conformal infinity $(S^n, [d\sigma_0])$ where

$$V^{-1} h|_{TS^{n-1}} = d\sigma_0.$$

3. Static asymptotically AdS space-times

In this section we study static asymptotically AdS space-times. We will prove the uniqueness of static asymptotically AdS space-times. In dimension $3+1$, with a bit restrictive definition of asymptotically AdS space-times, the uniqueness was first proved in [BGH] (see also [CS]). Then, assuming spin structure for $n > 3$, the uniqueness of static solutions $(M^n, g, V)$ to the vacuum Einstein equations with negative cosmological constant was established in [Wa1](see the definition of a static
solution \((M, g, V)\) in [CS], [Wa1]). Our proof will not use the spin structure in dimensions higher than three, but instead will rely on a recent work of Miao [Mi] (see also [ST]) which in turn depends on the classic positive theorem of Schoen and Yau [SY] for asymptotically flat manifolds.

By Theorem 2.8 in the previous section, a static asymptotically AdS space-time which satisfies the vacuum Einstein equations with negative cosmological constant is given by a static solution \((\Sigma, h, \sqrt{V})\) in our notation. Therefore, by Lemma 2.2 in the previous section, we know that

\[
(3.1) \quad h = s^{-2}(ds^2 + (1 - \frac{s^2}{4})^2d\sigma_0 + \tau s^n + o(s^n)),
\]

\[
(3.2) \quad V = s^{-2}((1 + \frac{s^2}{4})^2 - \alpha s^n + o(s^n)),
\]

and

\[
(3.3) \quad V - |\nabla \sqrt{V}|^2 - 1 = n\alpha s^{-2} + o(s^n-2)
\]

where \(\alpha = -\text{Tr}_d\sigma_0 \tau\) (these were known in [Wa1]).

To motivate our argument in this section we recall the following fact about the static solution \((B^n, \frac{2}{1-|x|^2}|dx|^2, \frac{1+|x|^2}{1-|x|^2})\) associated with the AdS space-time \((R^{n+1}, g_{AdS})\). Namely, if one uses the global defining function

\[
u = \frac{1}{\sqrt{V} + 1} = \frac{1 - |x|^2}{2},
\]

then

\[
u^2 h = |dx|^2.
\]

Therefore, for a static solution \((\Sigma, h, \sqrt{V})\), if we denote \(\nu = \frac{1}{\sqrt{V} + 1}\), then \(\nu\) is a global defining function for \(S^{n-1}\) in \(\Sigma\) and

\[
u^2 h = \frac{1}{s^2(\sqrt{V} + 1)^2}(ds^2 + (1 - \frac{s^2}{4})^2d\sigma_0 + \tau s^n + o(s^n))
\]

where

\[
s^2(\sqrt{V} + 1)^2 = (\sqrt{s^2V + s})^2 = s^2V + 2s\sqrt{s^2V} + s^2 = 1 + 2s + O(s^2).
\]
So

\[(3.4) \quad u^2 h = (1 + 2s + O(s^2))ds^2 + (1 + 2s + O(s^2))d\sigma_0 + O(s^2).\]

Thus \((\Sigma, u^2 h)\) is a compact manifold with the standard \((n-1)\)-sphere as its boundary and the second fundamental form for \(\partial \Sigma\) in \(\Sigma\) is \(d\sigma_0\) (i.e. the boundary is totally umbilical). In the light of (2.30) one may compute the scalar curvature for \(u^2 h\) as follows:

\[(3.5) \quad R[u^2 h] = u^{-\frac{n+2}{2}}(-\frac{4(n-1)}{n-2}\Delta u^{-\frac{n-2}{2}} + R[h]u^{-\frac{n-2}{2}}) = n(n-1)(V - |\nabla \sqrt{V}|^2 - 1),\]

which goes to zero as \(s \to 0\) by (3.3). We observe the following lemma which will allow us to apply the Strong maximum principle to conclude that \(R[u^2 h] \geq 0\). Namely,

**Lemma 3.1.**

\[(3.6) \quad -\Delta(V - |\nabla \sqrt{V}|^2 - 1) = 2|\nabla^2 \sqrt{V} - \sqrt{V} h|^2 - \frac{\nabla \sqrt{V}}{\sqrt{V}} \cdot \nabla(V - |\nabla \sqrt{V}|^2 - 1).\]

**Proof.** We simply compute

\[-\Delta V = 2(-nV - |\nabla \sqrt{V}|^2)\]

and

\[\Delta|\nabla \sqrt{V}|^2 = 2(||\nabla^2 \sqrt{V}|^2 + (\sqrt{V})_{ij} \nabla V |_{ij}) \]

\[= 2(||\nabla^2 \sqrt{V}|^2 + (\sqrt{V})^{-1}(\nabla V)^{ij} \nabla \sqrt{V} | (\sqrt{V})_{ij} (\sqrt{V})_{ij}).\]

Therefore

\[(3.7) \quad -\Delta(V - |\nabla \sqrt{V}|^2 - 1) = 2|\nabla^2 \sqrt{V} - \sqrt{V} h|^2 - \frac{(\sqrt{V})_{ij}}{\sqrt{V}}(2(\sqrt{V})(\sqrt{V})_{ij} - 2(\sqrt{V})_{ij} (\sqrt{V})_{ij}) \]

\[= 2|\nabla^2 \sqrt{V} - \sqrt{V} h|^2 - \frac{\nabla \sqrt{V}}{\sqrt{V}} \cdot \nabla(V - |\nabla \sqrt{V}|^2 - 1).\]
**Theorem 3.2.** Suppose that \((\Sigma, h, \sqrt{V})\) is a static solution to the vacuum Einstein equations with negative cosmological constant, i.e. \((\Sigma, h, \sqrt{V})\) comes from Theorem 2.8 in the previous section. Then \((\Sigma, h, \sqrt{V}) = (B^n, (\frac{2}{1-|x|^2})^2|dx|^2, \frac{1+|x|^2}{1-|x|^2})\) for some choice of coordinate in dimension between 3 and 7.

**Proof.** We consider the defining function \(u = \frac{1}{\sqrt{V+1}}\) and the compact manifold \((\Sigma, u^2h)\). From (3.4) we know that \(\partial \Sigma = S^{n-1}\) and \(u^2h|_{\partial \Sigma} = d\sigma_0\). Moreover, from (3.4), we know that the standard round \(S^{n-1}\) is the boundary of \((\Sigma, u^2h)\) and has the second fundamental form \(d\sigma_0\) (i.e. it is totally umbilical). On the other hand, by (3.5) and (3.3), the scalar curvature \(R[u^2h]\) goes to zero as \(s \to 0\). Using the above Lemma 3.1 and the strong maximum principle, we therefore conclude that \(R[u^2h] \geq 0\).

Now we appeal to the recent work of Miao [Mi] (see also works of Shi and Tam [ST]). We apply the work in [Mi] to the manifold \((M, g)\) where \(\Omega = \Sigma\) and \(g_\pm = u^2h\), and \(M \setminus \Omega = R^n \setminus B^n\) and \(g_+\) is the Euclidean metric. By Corollary 5.1 in [Mi], for example, we then conclude that \(R[u^2h] = 0\) for \(3 \leq n \leq 7\).

Back to \((\Sigma, h, \sqrt{V})\), in the light of Lemma 3.1 again, we observe that

\[
\nabla^2 \sqrt{V} = \sqrt{V} h.
\]

Similar to what was proved in [Ob], we prove that (3.8) implies that \((\Sigma, h)\) is isometric to the standard hyperbolic space form and \(V = \sqrt{1+r^2}\) for some choice of coordinates in the following lemma. Then the proof of theorem is complete.

**Lemma 3.3.** Suppose that \((M^n, g)\) is a complete Riemannian manifold. And suppose that there is a positive function \(\phi\) such that

\[
\nabla^2 \phi = \phi g. \tag{3.9}
\]

Then \((M^n, g) = (R^n, g_H)\) and \(\phi = c\sqrt{1+r^2}\) for some choice of coordinates.

**Proof.** First one observes that, \(\phi\) has one and only one global minimum point \(p_0\) on \(M\). Due to the homogeneity of (3.9), one may assume that \(\phi(p_0) = 1\). Let us consider a geodesic \(\gamma(s)\) emanating from \(p_0\) and parameterized with its length \(s\). Then, along this geodesic, for \(\phi(\gamma(s))\), we have

\[
\begin{aligned}
\phi'' - \phi &= 0 \\
\phi(0) &= 1 \\
\phi'(0) &= 0.
\end{aligned} \tag{3.10}
\]
Thus $\phi(s) = \cosh s$. Now take an orthonormal base $X^0 = \frac{\partial}{\partial s}, X^1, X^2, \ldots X^{n-1}$ at $p_0$ and parallel translate them along $\gamma(s)$. We want to calculate $d(\exp_{p_0})(sX^0)(X^k)$. That is, we compute the Jacobi field $Y^k(s)$ along $\gamma(s)$ such that

\[
\begin{cases}
\nabla_{X^0} \nabla_{X^0} Y^k + R(Y^k, X^0)X^0 = 0 \\
Y^k(0) = 0 \\
\nabla_{X^0} Y^k(0) = X^k(0).
\end{cases}
\]

Let $Y^k(s) = \sum f_i(s)X^i(s)$. Then (3.11) becomes

\[
\begin{cases}
f_i''X^i + f_iR_{0i0j}X^j = 0 \\
f_i = 0 \\
f_i'(0) = \delta_{ik}.
\end{cases}
\]

Notice that, by (3.9) and Ricci identity,

\[
\phi_{a,cb} - \phi_{a,cb} = \phi_dR_{dabc} = \sinh sR_{0abc} = \phi_c\delta_{ab} - \phi_b\delta_{ac},
\]

which gives us

\[
R_{0i0j} = \frac{1}{\sinh s}(\phi_j\delta_{i0} - \phi_0\delta_{ij}) = -\delta_{ij}.
\]

Plugging into (3.12), we have

\[
\begin{cases}
f_i'' - f_i = 0 \\
f_i(0) = 0 \\
f_i'(0) = \delta_{ik}.
\end{cases}
\]

Thus $Y^k(s) = \sinh sX^k(s)$. To show that $(M, g)$ is a hyperbolic space form, we use the exponential map $\exp_{p_0}$ which takes the tangent space $T_{p_0}M$ onto $M$ in the light of completeness. Clearly this gives a nice global coordinate chart. Next we want to calculate the metric $g$ under these coordinates. Let us use spherical coordinates for $T_{p_0}M$, that is, $(s, v) \in [0, \infty) \times S^{n-1}$ and $\exp_{p_0}(sv) \in M$. By the above calculations of Jacobi fields, we immediately have

\[
g = ds^2 + (\sinh s)^2 \, d\sigma_0,
\]

which is the hyperbolic metric. So $(M, g)$ is a hyperbolic space form. Finally let us point out that, if we denote $r = \sinh s$, then $\phi = \cosh s = \sqrt{1 + r^2}$.
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