LOCAL WELL-POSEDNESS FOR THE MODIFIED KDV EQUATION IN ALMOST CRITICAL $\hat{H}^r$-SPACES

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Abstract. We study the Cauchy problem for the modified KdV equation
\[ u_t + u_{xxx} + (u^3)_x = 0, \quad u(0) = u_0 \]
for data $u_0$ in the space $\hat{H}^s_x$ defined by the norm
\[ \|u_0\|_{\hat{H}^s_x} := \|\langle \xi \rangle^s \hat{u}_0\|_{L^r_x}. \]
Local well-posedness of this problem is established in the parameter range
$2 \geq r > 1$, $s \geq 1 - \frac{1}{2r}$, so the case $(s, r) = (0, 1)$, which is critical in view
of scaling considerations, is almost reached. To show this result, we use an
appropriate variant of the Fourier restriction norm method as well as bi- and
trilinear estimates for solutions of the Airy equation.

1. Introduction and main result

In this paper we study the local well-posedness (LWP) of the Cauchy problem
for the modified KdV equation
\[ u_t + u_{xxx} + (u^3)_x = 0, \quad u(0) = u_0, \quad x \in \mathbb{R}. \]
As long as data $u_0$ in the classical Sobolev spaces $H^s_x$ are considered, this problem is
known to be well-posed for $s \geq \frac{1}{4}$ and ill-posed (in the $C^0$ - uniform sense) for $s < \frac{1}{4}$.
Both, the positive and the negative result, were shown by Kenig, Ponce, and the
second author, see [KPV93, Theorem 2.4] and [KPV01, Theorem 1.3], respectively.
The situation remains the same, when the defocusing modified KdV equation, i. e.
(1) with a negative sign in front of the nonlinearity, is considered. In this case the proof
of the well-posedness result remains identically valid, while the ill-posedness
result here is due to Christ, Colliander and Tao, cf. [CCT03, Theorem 4]. In both
cases the standard scaling argument suggests LWP for $s > -\frac{1}{4}$, so - on the $H^s_x$-scale
- there is a considerable gap of $\frac{1}{2}$ derivatives between the scaling prediction and
the optimal LWP result.

This gap could be closed partially by the first author in [G04], where data in the
spaces $H^s_x$ are considered, which are defined by the norms
\[ \|u_0\|_{\hat{H}^s_x} := \|\langle \xi \rangle^s \hat{u}_0\|_{L^r_x}, \]
where $\hat{u}_0$ denotes the Fourier transform of $u_0$, $\langle \xi \rangle = (1 + \xi^2)^{\frac{1}{4}}$ and $\frac{1}{r} + \frac{1}{r'} = 1$.
The choice of these norms was motivated by earlier work of Cazenave, Vilela and
the second author on nonlinear Schrödinger equations, see [CVV01], yet another
alternative class of data spaces has been considered in [VV01].

The main result in [G04] was LWP for the v parameter range \(2 \geq r > \frac{4}{3}\),
\(s \geq s(r) = \frac{1}{2} - \frac{1}{2r}\), which coincides for \(r = 2\) with the optimal result on the \(H^s_{\infty}\)
scale. The proof used an appropriate variant of Bourgain’s Fourier restriction norm
method, cf. [G93]. Especially the function spaces \(X^r_{s,b}\), defined by

\[
\|f\|_{X^r_{s,b}} := \left( \int d\xi d\tau (\xi)^{sr'} (\tau - \xi^3)^{br'} |\hat{f}(\xi, \tau)|^{r'} \right)^{\frac{1}{r'}}
\]

were utilised, as well as the time restriction norm spaces

\[
X^r_{s,b}(\delta) := \{ f = \hat{f} \in \mathbb{F} : \hat{f} \in X^r_{s,b} \}
\]

with norm

\[
\|f\|_{X^r_{s,b}(\delta)} := \inf \{ \|\hat{f}\|_{X^r_{s,b}} : \hat{f}|_{[\delta, \delta] \times \mathbb{R}} = f \}.
\]

A key estimate in [G04] was the following Airy-version of the Fefferman-Stein-
estimate (cf. [F70] and [G04, Corollary 3.6])

\[
\|e^{-t\partial_x^3} u_0\|_{L^3_t L^6_x} \leq c \|I - \frac{x}{x'} u_0\|_{L^r_x}, \quad r > \frac{4}{3},
\]

Here and below \(I (J)\) denotes the Riesz (Bessel) potential operator of order \(-1\) and
\(\mathcal{L}_x = H^s_x\). This estimate fails to be true for \(r \leq \frac{4}{3}\), which explains the restriction
\(r > \frac{4}{3}\) in [G04].

It is the aim of the present paper to show, how this difficulty can be overcome
by using bi- and trilinear estimates for solutions of the Airy equation (instead of linear and bilinear ones). This allows us to extend the LWP result for the v parameter range \(2 \geq r > 1, s \geq s(r)\). More precisely, the following theorem is the main result of this paper.

**Theorem 1.** Let \(2 \geq r > 1, s \geq s(r) = \frac{1}{2} - \frac{1}{2r}\) and \(u_0 \in H^s_x\). Then there exist \(b > \frac{1}{r}, \delta = \delta(\|u_0\|_{H^s_x}) > 0\) and a unique solution \(u \in X^r_{s,b}(\delta)\) of (2). This solution is persistent and the flow map \(S : u_0 \mapsto u, H^s_x \to X^r_{s,b}(\delta_0)\) is locally Lipschitz continuous for any \(\delta_0 \in (0, \delta)\).

Theorem 1 is sharp in the sense that, for given \(r \in (1, 2]\), we have ill-posedness in the \(C^0\)-uniform sense for \(\frac{1}{2} - 1 < s < s(r)\). This can be seen by using the counterexample from [KPV01], as it was discussed in [G04] section 5). Combined with scaling considerations - observe that \(H^s_x\) scales like \(H^s_{\infty}\), if \(s - \frac{1}{r} = \sigma - \frac{1}{r}\) - this shows, that the case \((s, r) = (0, 1)\) remains critical in our setting and that our result covers the whole subcritical range. Unfortunately, our argument breaks down - even for small data - in the critical case, and we must leave this as an open problem. Notice, however, that for specific data

\[
u_0 = a \delta + \mu p.v. \frac{1}{x}, \quad (a, \mu \text{ small})
\]

of critical regularity the existence of global solutions of (1) was shown in [PV05].

**Theorem 2.** Let \(2 \geq r > 1\) and \(s \geq s(r) = \frac{1}{2} - \frac{1}{2r}\). Then for all \(b' < 0\) and \(b > \frac{1}{r}\) the estimate

\[
\|\partial_x (\prod_{i=1}^{3} u_i)\|_{X^r_{s,b'}} \leq c \prod_{i=1}^{3} \|u_i\|_{X^r_{s,b}}
\]

holds true.
respectively. We use $\delta$ of Mathematics at the UPV in Bilbao for its kind hospitality during his visit.

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(Below we will always write $\hat{F}$ in the space variable the symbol needed. Certain bi- and trilinear expressions involving these solutions will be estimated in the spaces $H^s_{\vec{r}}(L^t_x)$ and $L^r_{x,t} := L^r_x(L^t_x)$, where

$$\|f\|_{L^r_x(L^t_x)} := \left( \int \left( \int |\hat{f}(\xi, \tau)|^{p'} d\tau \right)^{\frac{p}{p'}} d\xi \right)^{\frac{1}{p'}}, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{p'} = 1.$$  

(Below we will always write $p'$, $q'$ etc. to indicate conjugate Hölder exponents, $\hat{f}$ or $\hat{F}f$ denote the Fourier transform of $f$, while for the partial Fourier transform in the space variable the symbol $F_x$ is used.) We begin with the following bilinear estimate, which we state and prove in a slightly more general version than actually needed.

Remarks:

i) (On the lifespan of local solutions) Using \cite{G05} Lemma 5.2, we have for $u_1$, $u_2$, $u_3$ supported in $[-\delta, \delta] \times \mathbb{R}$ ($0 < \delta \leq 1$) the estimate

$$\|\partial_x (\prod_{i=1}^3 u_i)\|_{X^r_{s,b,-1}} \leq c\delta^{1-\frac{1}{p} - \frac{1}{q}} \prod_{i=1}^3 \|u_i\|_{X^r_{s,b}},$$

provided $2 \geq r > 1, s \geq s(r), b > \frac{1}{2}, \varepsilon > 0$. Inserting this estimate, especially the specific power of $\delta$, into the proof of the local result, we obtain a lifespan of size $\delta \sim \|u_0\|_{H^s_x}^{\frac{2}{p-1} - \varepsilon' r''}$. For $r = 2$ this coincides - up to $\varepsilon'$ - with the result in \cite{KPV93} (see also \cite{PLP99} Theorem 1.1).

ii) Concerning related results for the one-dimensional cubic NLS and DNLS equations we refer to \cite{G05}.

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2. Bi- and trilinear Airy estimates

Throughout this section we consider solutions $u(t) = e^{-t\partial_x^2}u_0$, $v(t) = e^{-t\partial_x^2}v_0$ and $w(t) = e^{-t\partial_x^2}w_0$ of the Airy equation with data $u_0$, $v_0$ and $w_0$, respectively. Certain bi- and trilinear expressions involving these solutions will be estimated in the spaces $H^s_{\vec{r}}(L^t_x)$ and $L^r_{x,t} := L^r_x(L^t_x)$, where

$$\|f\|_{L^r_x(L^t_x)} := \left( \int \left( \int |\hat{f}(\xi, \tau)|^{p'} d\tau \right)^{\frac{p}{p'}} d\xi \right)^{\frac{1}{p'}}, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{p'} = 1.$$  

Let $I^*$ denote the Riesz potential of order $-s$ and let $I^s_{\vec{r}}(f, g)$ be defined by its Fourier transform (in the space variable):

$$F_x I^s_{\vec{r}}(f, g)(\xi) := \int_{\mathbb{R}} d\xi_1 |\xi - \xi_1|^s F_x f(\xi_1) F_x g(\xi_2),$$

where $\int_*$ is shorthand for $\int_{\xi_1 + \xi_2 = \xi}$. Then we have

$$\|I^s_{\vec{r}}(u, v)\|_{L^r_x(L^t_x)} \leq c\|u_0\|_{L^r_x} \|v_0\|_{L^r_x},$$

provided $1 \leq q \leq r_{1,2} \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r_{1}} + \frac{1}{r_{2}}$.

Proof: Taking the Fourier transform first in space and then in time we obtain

$$F_x F^* I^s_{\vec{r}}(u, v)(\xi, t) = c\|\| \int_{\mathbb{R}} d\xi_1 |\xi - \xi_1|^s F_x u_0(\xi_1) F_x v_0(\xi_2)$$

and

$$F^* F I^s_{\vec{r}}(u, v)(\tau, \xi) = c\|\| \int_{\mathbb{R}} d\xi_1 |\xi - \xi_1|^s \delta(\tau - \xi_1) F_x u_0(\xi_1) F_x v_0(\xi_2).$$

respectively. We use $\delta(g(x)) = \sum_n \frac{1}{|g(x_n)|} \delta(x - x_n)$, where the sum is taken over all simple zeros of $g$, which in our case is

$$g(\xi_1) = \tau - \xi_1^3 - \xi_2^3 = \tau - \xi^3 + 3\xi_1(\xi - \xi_1)$$


with the zeros
\[ \xi_{1}^{\pm} = \frac{\xi \pm y}{2}, \quad y := 2 \sqrt{\frac{r}{3\xi}} \frac{\xi^2}{12}. \]
and the derivative
\[ g'(\xi_{1}^{\pm}) = 3\xi(\xi - 2\xi_{1}^{\pm}) = \mp 3\xi y. \]
Hence
\begin{align*}
\frac{1}{p} I^{\frac{1}{p}} \left( I_{-}^{\frac{1}{p}} (u, v) (\xi, \tau) \right) & = c|\xi|^{-\frac{1}{p}} y^{-\frac{2}{p}} \left( F_{x} u_{0}(\xi + y) F_{x} v_{0}(\xi - y) + F_{x} u_{0}(\xi - y) F_{x} v_{0}(\xi + y) \right). \\
\end{align*}
Using \( d\tau = 3|\xi|ydy \), we see that the \( L_2^r \) - norm of the first contribution equals
\[ \left( \int dy |F_{x} u_{0}(\xi + y) F_{x} v_{0}(\xi - y)|^{p'} \right)^{\frac{1}{p'}} = c \left( |F_{x} u_{0}|^{p'} * |F_{x} v_{0}|^{p'} (\xi) \right)^{\frac{1}{p'}}. \]
Now Young's inequality is applied to see that
\[ \left( \int \xi (|F_{x} u_{0}|^{p'} * |F_{x} v_{0}|^{p'} (\xi)) \right)^{\frac{1}{p'}} \leq c\|u_{0}\|_{L_2^{2\beta}} \|v_{0}\|_{L_2^{2\beta}} \]
(cf. the proof of \cite{Go5} Lemma 1), which is the desired bound. Finally we observe that the second contribution in \( \Box \) can be treated in precisely the same manner with \( r_{1} \) and \( r_{2} \) interchanged.

Arguing similarly as in the proof of Lemma 2.1 in \cite{Go4} we obtain:

**Corollary 1.** For \( p, q, r_{1,2} \) as in the previous lemma and \( b_{i} > \frac{1}{r_{i}} \) the estimate
\[ \|I_{-}^{\frac{1}{p}} \left( u_{1}, u_{2} \right)\|_{L_{2}^{2}(L_{2}^{r})} \leq c\|u_{1}\|_{X_{0, b_{1}}^{r_{1}}} \|u_{2}\|_{X_{0, b_{2}}^{r_{2}}} \]
is valid.

The next step is to dualize the preceding corollary. For that purpose we recall the bilinear operator \( I_{+}^{p} \), defined by
\[ F_{x} I_{+}^{p} (f, g)(\xi) := \int_{\xi_{1} + \xi_{2} = \xi} d\xi_{1} |\xi + \xi_{2}|^{p} F_{x} f(\xi_{1}) F_{x} g(\xi_{2}), \]
and the linear operators
\[ M_{u} v := I_{+}^{p} (u, v) \quad \text{and} \quad N_{u} w := I_{+}^{p} (w, \overline{u}), \]
which are formally adjoint w. r. t. the inner product on \( L_{2}^{2} (\tau) \) (cf. \cite{Go4} p.3299)). With this notation, Corollary \( \Box \) expresses the boundedness of
\[ I_{+}^{p} M_{u} \rightarrow X_{0, b_{2}}^{r_{2}} \rightarrow L_{2}^{2}(L_{2}^{r}) \]
with operator norm \( \leq c\|u_{1}\|_{X_{0, b_{1}}^{r_{1}}}. \) By duality, under the additional hypothesis \( 1 < p, q, r_{1,2} < \infty \), it follows that
\[ N_{u_{1}} I_{+}^{p} : L_{2}^{2}(L_{2}^{r}) \rightarrow X_{0, -b_{2}}^{r_{2}} \]
is bounded with the same norm. Thus we obtain the following estimate:

**Corollary 2.** Let \( 1 < q \leq r_{1,2} \leq p < \infty \), \( \frac{1}{q} + \frac{1}{r_{2}} = \frac{1}{r_{1}} + \frac{1}{r_{2}} \) and \( b_{i} > \frac{1}{r_{i}} \). Then
\begin{align*}
\|I_{+}^{p} (I_{-}^{q} u_{2}, \overline{u_{1}})\|_{X_{0, -b_{2}}^{r_{2}}} & \leq c\|u_{1}\|_{X_{0, b_{1}}^{r_{1}}} \|u_{2}\|_{L_{2}^{2}(L_{2}^{r})}. \\
\end{align*}
Remark: Since the phase function $\phi(\xi) = \xi^3$ is odd, we have $\|u_1\|_{X^r_{s,b}} = \|\overline{u_1}\|_{X^r_{s,b}}$, and we may replace $\overline{u_1}$ by $u_1$ in the left hand side of (5).

The special case in (5), where $p = q = r_1, 2$, will be sufficient for our purposes. In this case, (5) can be written as

\[ \| \mathcal{I}^\dagger (I^\dagger u_2, u_1) \|_{X^r_{0,b}, Y} \leq c\|u_1\|_{X^r_{0,b}, Y} \|u_2\|_{L^r_{x,t}, Y}, \]

provided $1 < r < \infty$, $b' < -\frac{1}{p}$. Combining this with the trivial endpoint of the Hausdorff-Young inequality, i.e.,

\[ \|u_2 u_1\|_{L^r_{x,t}, Y} \leq c\|u_1\|_{L^r_{x,t}, Y} \|u_2\|_{L^r_{x,t}, Y}, \]

we obtain by elementary Hölder estimates

\[ \| \mathcal{I}^\dagger (I^\dagger u_2, u_1) \|_{X^r_{0,b}, Y} \leq c\|u_1\|_{X^{r'}_{0,-b}, Y} \|u_2\|_{L^r_{x,t}, Y}, \]

where $0 \leq \frac{1}{p'} \leq \frac{1}{p}$ and $b < -\frac{1}{p'}$. In this form actually we shall make use of Corollary 2.

Now we turn to the trilinear estimates. Again we take the Fourier transform first in $x$ and then in $t$ to obtain

\[ \mathcal{F}_x(uvw)(\xi, t) = c \int d\xi_1 d\xi_2 e^{i(\xi_1^3 + \xi_2^3 + \xi_1^3)\xi} \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2) \mathcal{F}_x w_0(\xi_3), \]

where now $\int_x = \int_{\xi_1 + \xi_2 + \xi_3 = \xi}$ and

\[ \mathcal{F}(uvw)(\xi, \tau) = c \int d\xi_1 d\xi_2 \delta(\xi_1^3 + \xi_2^3 + \xi_3^3 - \tau) \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2) \mathcal{F}_x w_0(\xi_3). \]

Now the argument of $\delta$, that is

\[ g(\xi_2) = 3(\xi - \xi_1)\xi_2^2 - 3(\xi - \xi_1)^2\xi_2 - 3\xi_1(\xi - \xi_1) + \xi^3 - \tau \]

has exactly two zeros

\[ \xi_2^\pm = \frac{\xi - \xi_1}{2} \pm \sqrt{\frac{(\xi + \xi_1)^2}{4} + \frac{\tau - \xi_1^3}{3(\xi - \xi_1)}} = \frac{\xi - \xi_1}{2} \pm y, \]

with

\[ |g'(\xi_2^\pm)| = 6|\xi - \xi_1| \sqrt{\frac{(\xi + \xi_1)^2}{4} + \frac{\tau - \xi_1^3}{3(\xi - \xi_1)}} = 6|\xi - \xi_1|. \]

Using $\delta(g(\xi_2)) = \sum_{g(x_0)} \frac{\delta(x_0 - x_0)}{|g'(x_0)|}$, where the sum is taken over all simple zeros of $g$, we see that

\[ \mathcal{F}(uvw)(\xi, \tau) = c(K_+(\xi, \tau) + K_-(\xi, \tau)), \]

where

\[ K_\pm(\xi, \tau) = \int d\xi_1 \frac{1}{|\xi - \xi_1|} \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_1, \tau) \pm y) \mathcal{F}_x w_0(\xi_1, \tau), \]

with $y$ as defined in (8).

In order to estimate $\|uvw\|_{L^r_{x,t}} = \|\mathcal{F}(uvw)\|_{L^r_{x,t}, Y}$, we distinguish between three cases depending on the relative size of the frequencies $\xi_1, \xi_2$ and $\xi_3$:

i) $|\xi_1| \sim |\xi_2| \gg |\xi_3|$, 

ii) $|\xi_2 - \xi_1| \geq |\xi_2 + \xi_3|$, 

iii) $1 \leq |\xi_2 - \xi_3| \leq |\xi_2 + \xi_3|$. 

To treat the first case we define the trilinear operator $T$ by

$$F_x T(f, g, h) := \int f(\xi_1, \xi_2) \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2) \mathcal{F}_x h(\xi_3) \, d\xi_1 d\xi_2 d\xi_3$$

where again $\int = \int_{\xi_1 + \xi_2 + \xi_3 = \xi}$. In this case we have:

**Lemma 2.** Let $1 \leq r \leq 2$ and $s_1 > \frac{1}{2r} - \frac{1}{r}$, $s_2 \geq \frac{1}{2r}$. Then

$$\|T(u, v, w)\|_{L^r_x} \leq c \|u\|_{H^r_x} \|v\|_{H^r_x} \|w\|_{H^r_x}. \tag{12}$$

Proof: By the above computation we have

$$F T(u, v, w)(\xi, \tau) = c(K^+(\xi, \tau) + K^-(\xi, \tau)),$$

with

$$K^\pm(\xi, \tau) = \int_{A_{\pm}} \frac{1}{|\xi - \xi'|} F_x u(\xi_1) F_x v(\xi_2) F_x w(\xi_3) \, d\xi_1 d\xi_2 d\xi_3,$$

where $A_{\pm} = \{|\xi_1| \sim |\xi_2| \pm \xi_3\}$ and $y$ is defined by (8). Since in $A_{\pm}$ the inequality $|\xi_1| \leq c|\xi_2| + y$ holds, we get the upper bound

$$K^\pm(\xi, \tau),$$

leading to

$$\|T(u, v, w)\|_{L^r_x} \leq c\|J^{-1} u\|_{L^r_x} \|J^{-1} v\|_{L^r_x} \|w\|_{L^r_x}. \tag{13}$$

By symmetry between the first two factors and multilinear interpolation we obtain

$$\|T(u, v, w)\|_{L^r_x} \leq c\|J^{-1} u\|_{L^r_x} \|J^{-1} v\|_{L^r_x} \|w\|_{L^r_x}. \tag{14}$$

On the other hand side we have

$$\|uvw\|_{L^r_x} \leq c\|u\|_{L^r_x(L^1_x)} \|v\|_{L^r_x(L^1_x)} \|w\|_{L^r_x(L^\infty_x)}, \tag{15}$$

with

$$\|w\|_{L^r_x(L^\infty_x)} \leq c\|J^w u\|_{L^r_x} \tag{16}$$

which is the maximal function estimate from [SS7, Thm. 3]. Concerning the first two factors we interpolate between the sharp version of Kato’s smoothing effect, i.e. $\|Ju\|_{L^r_x(L^1_x)} = c\|u\|_{L^r_x}$, see [KPV91, Thm. 4.1], and (10) to obtain

$$\|J^w u\|_{L^r_x(L^1_x)} \leq c\|u\|_{L^r_x} \tag{17}$$

such that

$$\|T(u, v, w)\|_{L^r_x} \leq c\|J^{-1} u\|_{L^r_x} \|J^{-1} v\|_{L^r_x} \|w\|_{L^r_x}. \tag{18}$$

Using multilinear interpolation again, now between (19) and (18), we finally see that, for $1 \leq r \leq 2$,

$$\|T(u, v, w)\|_{L^r_x} \leq c\|J^{-1} u\|_{L^r_x} \|J^{-1} v\|_{L^r_x} \|w\|_{L^r_x}, \tag{19}$$

where in the last step we have used the Sobolev type embedding $\tilde{H}^r_x \subset \tilde{H}^s_x$, which holds true for $s = \frac{1}{r} > \sigma - \frac{1}{p}$, $r \leq 2$.

\[\square\]

**Corollary 3.** For $r, s_1, 2$ as in the previous lemma and $b > \frac{1}{r}$ the estimate

$$\|T(u_1, u_2, u_3)\|_{L^r_x} \leq c\|u_1\|_{X^r_{s_1, b}} \|u_2\|_{X^r_{s_1, b}} \|u_3\|_{X^r_{s_2, b}} \tag{20}$$

holds true.
Next we introduce \( T_{\geq} \) \( (T_{\leq}) \) by
\[
F_{x} T_{\geq} (f, g, h) := \int d\xi_1 d\xi_2 F_x f(\xi_1) F_x g(\xi_2) F_x h(\xi_3) \chi_{\{\xi_2 - \xi_3 \leq |\xi_2 + \xi_3|\}},
\]
and
\[
F_{x} T_{\leq} (f, g, h) := \int d\xi_1 d\xi_2 F_x f(\xi_1) F_x g(\xi_2) F_x h(\xi_3) \chi_{\{1 \leq |\xi_2 - \xi_3| \leq |\xi_2 + \xi_3|\}}.
\]

**Lemma 3.** Let \( 1 < p_1 < p < p_0 < \infty, p < p_0', \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} \) and \( -\frac{1}{p_1} < 1 + \frac{1}{p} \). Then the estimate
\[
|T_{\geq}(u, v, w)|_{L^p_x} \leq c\|u_0\|_{L^{p_0}_x} \|I^- \frac{\partial}{\partial y} v_0\|_{L^p_x} \|I^- \frac{\partial}{\partial y} w_0\|_{L^p_x}
\]
is valid.

Proof: For the Fourier transform of \( T_{\geq}(u, v, w) \) in both variables we obtain
\[
\mathcal{F} T_{\geq}(u, v, w)(\xi, \tau) = c(K_{\geq}^+ (\xi, \tau) + K_{\leq}^- (\xi, \tau)),
\]
where
\[
K_{\geq}^+ (\xi, \tau) = \int \frac{d\xi_1}{|\xi - \xi_1|} F_x u_0(\xi_1) F_x v_0(\frac{\xi - \xi_1}{2} + y) F_x w_0(\frac{\xi - \xi_1}{2} + y),
\]
with \( y \) as in \([3]\) again. By symmetry we may restrict ourselves to the estimation of \( K_{\geq}^+ \). Using \( \frac{\xi - \xi_1}{2} \leq y \) and Hölder’s inequality, we see that
\[
K_{\geq}^+ (\xi, \tau) \leq c \left( \int \frac{d\xi_1}{|\xi - \xi_1|^{(1-\theta)p}} \right)^{\frac{1}{\theta}} \times \ldots
\]
\[
\ldots \times \left( \int \frac{d\xi_1}{|\xi - \xi_1|^{(1-\theta)p}} F_x I^- \frac{\partial}{\partial y} v_0(\frac{\xi - \xi_1}{2} + y) F_x I^- \frac{\partial}{\partial y} w_0(\frac{\xi - \xi_1}{2} + y)^{\theta'} \right)^{\frac{1}{\theta'}}
\]
where \( \theta = \frac{3}{p'} - \frac{1}{p} \in (0, 1) \) by our assumptions. Taking the \( L^{p'}_x \)-norm of both sides and using \( d\tau = 6|\xi - \xi_1|ydy \) we arrive at
\[
\|\mathcal{F} T_{\geq}(u, v, w)(\xi, \tau)\|_{L^{p'}_x} \leq c(\|F_x u_0\|^p \|\xi^{(\theta-1)p}) \eta \times \ldots
\]
\[
\ldots \times \left( \int \frac{d\xi_1}{|\xi - \xi_1|^{(1-\theta)p}} F_x I^- \frac{\partial}{\partial y} v_0(\frac{\xi - \xi_1}{2} + y) F_x I^- \frac{\partial}{\partial y} w_0(\frac{\xi - \xi_1}{2} + y)^{\theta'} \right)^{\frac{1}{\theta'}}.
\]
Changing variables \( z_\pm := \frac{\xi - \xi_1}{2} \pm y \) we see that the second factor equals
\[
\left( \int \frac{dz_+ + dz_-}{|z_+ + z_-|^{(1-\theta)p}} |F_x I^- \frac{\partial}{\partial y} v_0(z_+) F_x I^- \frac{\partial}{\partial y} w_0(z_-)|^{\theta'} \right)^{\frac{1}{\theta'}} \leq c \|I^- \frac{\partial}{\partial y} v_0\|_{L^{p'}_x} \|I^- \frac{\partial}{\partial y} w_0\|_{L^{p'}_x},
\]
by the Hardy-Littlewood-Sobolev-inequality, requiring \( \theta \) to be chosen as above and \( 1 < \theta' < 2 \), which follows from our assumptions. It remains to estimate the \( L^{p'}_x \)-norm of the first factor, that is
\[
\|F_x u_0\|^p \|\xi^{(\theta-1)p} \|^\frac{1}{\theta'} \leq c(\|F_x u_0\|^p \|\xi^{(\theta-1)p} \|_{L^{p'}_x} \|\xi^{(\theta-1)p} \|^\frac{1}{\theta'} \)
\]
\[
\leq c\|u_0\|_{L^{p_0}_x},
\]
where the HLS inequality was used again. For its application we need
\[
0 < (1 - \theta)p < 1; \quad 1 < \frac{p_0}{p} < \frac{1}{1 - (1 - \theta)p} \quad \text{and} \quad \theta = \frac{1}{p_0},
\]
which follows from the assumptions, too. \( \square \)
Corollary 4. For $1 < r < 2$ there exist $s_{0,1} \geq 0$ with $s_{0} + 2s_{1} = \frac{1}{r}$, such that
\begin{equation}
\|T_{2}^{\leq}(u, v, w)\|_{L_{t}^{r_{1}}} \leq c\|I^{-s_{0}}u_{0}\|_{L_{x}^{r}}\|I^{-s_{1}}v_{0}\|_{L_{x}^{r}}\|I^{-s_{1}}w_{0}\|_{L_{x}^{r}}.
\end{equation}
In addition, for $b > \frac{1}{r}$ we have
\[\|T_{2}^{\leq}(u, u_{2}, u_{3})\|_{L_{t}^{r_{1}}} \leq c\|I^{-s_{0}}u_{1}\|_{X_{0,b}^{s_{1}}}\|I^{-s_{1}}u_{2}\|_{X_{0,b}^{s_{1}}}\|I^{-s_{1}}u_{3}\|_{X_{0,b}^{s_{1}}}.
\]

Proof of (12): Using Hölder’s inequality and the Airy-version of the Fefferman-Stein-estimate, that is
\begin{equation}
\|u\|_{L_{t}^{q_{0}}} \leq c\|I^{\frac{1}{r_{0}}u_{0}\|_{L_{x}^{r_{0}}}}, \quad q > \frac{4}{3},
\end{equation}
see [G04], Corollary 3.6, we get for
\begin{equation}
\frac{4}{3} < q_{0} < 2 < q_{1}, \quad \text{with} \quad \frac{1}{3} = \frac{1}{q_{0}} + \frac{1}{4} = \frac{1}{q_{1}}.
\end{equation}
that
\begin{equation}
\|T_{2}(u, v, w)\|_{L_{t}^{r_{1}}} \leq \|uvw\|_{L_{t}^{r_{1}}} \leq c\|I^{-\frac{1}{30}}u_{0}\|_{L_{x}^{r_{0}}}\|I^{-\frac{2}{3_{0}}v_{0}}\|_{L_{x}^{r_{1}}}\|I^{-\frac{1}{3_{0}}w_{0}}\|_{L_{x}^{r_{1}}}.
\end{equation}
Multilinear interpolation of (15) with Lemma 3 yields (12), provided $p, p_{0}, p_{1}; q_{0}, q_{1}$, defined by the interpolation conditions
\begin{equation}
\frac{1}{r} = \frac{1 - \theta}{p} + \frac{\theta}{2} = \frac{1 - \theta}{p_{0}} + \frac{\theta}{q_{1}} = \frac{1 - \theta}{p_{0}} + \frac{\theta}{q_{1}},
\end{equation}
fulfill the assumptions of Lemma 3 and (13), respectively, which can be guaranteed by choosing $\theta$ sufficiently small. Now $s_{0,1}$ are obtained from
\[s_{0} = \frac{1 - \theta}{3q_{0}} \quad \text{and} \quad s_{1} = \frac{1 - \theta}{2p} + \frac{\theta}{3q_{1}},\]
which gives
\[s_{0} + 2s_{1} = \frac{1 - \theta}{p} + \frac{\theta}{3}(\frac{1}{q_{0}} + \frac{2}{q_{1}}) = \frac{1}{r},\]
as desired.

\begin{flushright}
\text{\&}
\end{flushright}

Remark: By (13), Corollary 4 still holds true for $r \geq 2$ (with $s_{0} = s_{1} = \frac{1}{4}$).

Lemma 4. Let $1 \leq r \leq \rho \leq \infty$. Then
\begin{equation}
\|T_{2}^{\leq}(u, v, w)\|_{L_{t}^{r_{1}}} \leq c\|u_{0}\|_{L_{x}^{r_{0}}}\|I^{-\frac{1}{r_{0}}v_{0}}\|_{L_{x}^{r_{1}}}\|I^{-\frac{1}{r_{0}}w_{0}}\|_{L_{x}^{r_{1}}}.
\end{equation}
Proof: We have
\begin{equation}
\mathcal{F}T_{2}^{\leq}(u, v, w)(\xi, \tau) = c(K_{+}^{\leq}(\xi, \tau) + K_{-}^{\leq}(\xi, \tau)),
\end{equation}
where
\[K_{\pm}^{\leq}(\xi, \tau) = \int_{\{1 \leq y \leq |\xi - \xi_{1}|\}} \frac{d\xi_{1}}{|\xi - \xi_{1}|}|y|\mathcal{F}u_{0}(\xi_{1})\mathcal{F}v_{0}\left(\frac{\xi - \xi_{1}}{2} \pm y\right)\mathcal{F}w_{0}\left(\frac{\xi - \xi_{1}}{2} \mp y\right)
\]
with $y$ as defined in (8). By symmetry between $v$ and $w$ it suffices to treat $K_{+}^{\leq}$, which we decompose dyadically with respect to $y$ to obtain the upper bound:
\begin{align*}
&\left|c\int_{\{1 \leq y \leq |\xi - \xi_{1}|, y \sim 2^{j}\}} \frac{d\xi_{1}}{|\xi - \xi_{1}|}|y|\mathcal{F}u_{0}(\xi_{1})\mathcal{F}v_{0}\left(\frac{\xi - \xi_{1}}{2} \pm y\right)\mathcal{F}w_{0}\left(\frac{\xi - \xi_{1}}{2} \mp y\right)\right| \\
\leq &\left|c\sum_{j=0}^{\infty} 2^{-j} \int_{\{y \sim 2^{j}\}} \frac{d\xi_{1}}{|\xi - \xi_{1}|}|y|\mathcal{F}u_{0}(\xi_{1})\mathcal{F}v_{0}\left(\frac{\xi - \xi_{1}}{2} \pm y\right)\mathcal{F}w_{0}\left(\frac{\xi - \xi_{1}}{2} \mp y\right)\right| \\
\leq &\left|c\sum_{j=0}^{\infty} 2^{-j}\|u_{0}\|_{L_{x}^{r_{0}}}\|v_{0}\|_{L_{x}^{r_{1}}}\|w_{0}\|_{L_{x}^{r_{1}}}\right|.
\end{align*}
where \( \lambda(y \sim 2^j) \) denotes the Lebesgue measure of \( \{ \xi_1 : y(\xi_1) \sim 2^j \} \), which is bounded by \( c2^j \). Hence, for any \( p > 1 \),

\[
\| K^+_{\leq} \|_{\mathcal{L}^r_{\xi,\tau}} \leq c \sum_{j=0}^{\infty} 2^{-\frac{\beta}{p}} \| u_0 \|_{\mathcal{L}^r_{\xi}} \| I^{-\frac{\beta}{2}} v_0 \|_{\mathcal{L}^r_{\xi}} \| I^{-\frac{\beta}{2}} w_0 \|_{\mathcal{L}^r_{\xi}} 
\]

\[
(16)
\]

On the other hand, by integration with respect first to \( d\tau = 6y(\xi - \xi_1)dy \), to \( d\xi \) and finally to \( d\xi_1 \), we see that

\[
\| K^+_{\leq} \|_{\mathcal{L}^r_{\xi,\tau}} \leq c \| u_0 \|_{\mathcal{L}^r_{\xi}} \| v_0 \|_{\mathcal{L}^r_{\xi}} \| w_0 \|_{\mathcal{L}^r_{\xi}}.
\]

(17)

Now multilinear interpolation between (16) and (17) leads to

\[
\| K^+_{\leq} \|_{\mathcal{L}^r_{\xi,\tau}} \leq c \| u_0 \|_{\mathcal{L}^r_{\xi}} \| I^{-\frac{\beta}{2}} v_0 \|_{\mathcal{L}^r_{\xi}} \| I^{-\frac{\beta}{2}} w_0 \|_{\mathcal{L}^r_{\xi}},
\]

which gives the desired result.

\[\square\]

**Corollary 5.** Let \( 1 \leq r < p \leq \infty, \beta > \frac{1}{p}, b > \frac{1}{r} \) and \( \epsilon > 0 \). Then

\[
\| T_{\leq}(u_1, u_2, u_3) \|_{\mathcal{L}^r_{\xi,\tau}} \leq c \| u_1 \|_{X_{0,b}^p} \| I^{-\frac{\beta}{2}} u_2 \|_{X_{0,b}^r} \| I^{-\frac{\beta}{2}} u_3 \|_{X_{0,b}^r}
\]

and

\[
\| T_{\leq}(u_1, u_2, u_3) \|_{\mathcal{L}^r_{\xi,\tau}} \leq c \| u_1 \|_{X_{\sigma,b}^p} \| I^{-\frac{\beta}{2}} u_2 \|_{X_{\sigma,b}^r} \| I^{-\frac{\beta}{2}} u_3 \|_{X_{\sigma,b}^r}
\]

are valid.

### 3. Proof of Theorem 2

Without loss of generality we may assume that \( s = s(r) \). Then we rewrite the left hand side of (13) as

\[
\mathcal{L}^r_{\xi,\tau} \sim \{ |f(\xi) - \xi| \} \int dv \prod_{i=1}^{3} \mathcal{L}^r_{\xi,\tau},
\]

where \( dv = d\xi_1 d\xi_2 d\tau_1 d\tau_2 \) and \( \sum_{i=1}^{3} (\xi_i, \tau_i) = (\xi, \tau) \).

In the sequel, we shall use the following notation:

- \( \xi_{\text{max}}, \xi_{\text{med}}, \xi_{\text{min}} \) are defined by \( |\xi_{\text{max}}| \geq |\xi_{\text{med}}| \geq |\xi_{\text{min}}| \),
- \( p \) denotes the projection on low frequencies, i.e. \( p f(\xi) = \chi_{|\xi| \lesssim 1} \hat{f}(\xi) \),
- \( f \gtrsim g \) is shorthand for \( \| f \| \leq c \| g \| \),
- for the mixed weights coming from the \( X_{\sigma,b}^{r,s} \) norms we shall write \( \sigma_0 := \tau - \xi^3 \) and \( \sigma_i := \tau_i - \xi^3, 1 \leq i \leq 3 \), respectively,
- the Fourier multiplier associated with these weights is denoted by \( \Lambda^b := \mathcal{F}^{-1} (\tau - \xi^3)^b \mathcal{F} \),
- for a real number \( x \) we write \( x \pm \varepsilon \) to denote \( x \pm \varepsilon \) for arbitrarily small \( \varepsilon > 0 \), \( \infty \) stands for an arbitrarily large real number.

---

\(^1\)To see this, we write \( \{ \xi_1 : y(\xi_1) \sim 2^j \} = S_1 \cup S_2 \), where in \( S_1 \) we assume that \( |\xi - \xi_1| \lesssim 2^j \), \( |\xi + \xi_1| \lesssim 2^j \) or \( |\xi - 3\xi_1| \lesssim 2^j \). Then \( S_1 \) consists of a finite number of intervals of total length bounded by \( c2^j \). For \( S_2 \) we have \( |\xi - \xi_1| \gg 2^j, |\xi + \xi_1| \gg 2^j \) and \( |\xi - 3\xi_1| \gg 2^j \), implying that

\[
\left| \frac{dy}{d\xi_1} \right| = \frac{1}{2y(\xi - \xi_1)} \left| \frac{\xi + \xi_1)(\xi - 3\xi_1)}{4} + y^2 \right| \gtrsim \frac{|\xi + \xi_1| |\xi - 3\xi_1|}{y|\xi - \xi_1|} \gtrsim 1,
\]

which gives

\[
\lambda(S_2) = \int_{S_2} d\xi_1 \leq \int \frac{d\xi_1}{dy} \chi_{(y \sim 2^j)} dy \leq c2^j.
\]
Apart from the trivial region where $|\xi_{\text{max}}| \leq 1$, whose contribution can be estimated by

\[
\left\| \prod_{i=1}^{3} pu_{i} \right\| \leq c \left\| \prod_{i=1}^{3} pu_{i} \right\| \leq c \sum_{i=1}^{3} \left\| pu_{i} \right\| X_{s, b}^{r} \leq c \prod_{i=1}^{3} \left\| u_{i} \right\| X_{s, b}^{r},
\]

we consider three cases:

1. The nonresonant case, where $|\xi_{\text{max}}| \gg |\xi_{\text{med}}|$, $|\xi_{\text{max}}| \sim |\xi_{\text{med}}| \gg |\xi_{\text{min}}|$ and, finally, $|\xi_{\text{max}}| \sim |\xi_{\text{min}}|$.

2. In the nonresonant case we assume without loss of generality that $|\xi_{1}| \geq |\xi_{2}| \geq |\xi_{3}|$. Then we have for this region

\[
J^{s} \partial_{x}(u_{1}u_{2}u_{3}) \leq \partial_{x}(J^{-s}u_{1}J^{-s}u_{2}J^{-s}u_{3}) \leq \left(I_{+}^{\frac{1}{2}} + I_{+}^{\frac{1}{2}}(J^{s}u_{1}, J^{s}u_{2})J^{1-s-\frac{2}{3}}u_{3}ight) \leq \left(I_{+}^{0} + I_{+}^{\frac{1}{2}}(J^{s}u_{1}, J^{s}u_{2})J^{1-s-\frac{2}{3}}u_{3} \right).
\]

Now the dual version \([7]\) of the bilinear estimate is applied to obtain

\[
\left\| I_{+}^{0} + I_{+}^{\frac{1}{2}}(J^{s}u_{1}, J^{s}u_{2}) \right\| X_{s, b}^{r} \leq c \left\| I_{+}^{0} + I_{+}^{\frac{1}{2}}(J^{s}u_{1}, J^{s}u_{2}) \right\| X_{s, b}^{r} \leq c \prod_{i=1}^{3} \left\| u_{i} \right\| X_{s, b}^{r},
\]

where in the last step we have used the bilinear estimate itself (Corollary \([1]\)) for the first and Sobolev-type embeddings for the second factor.

2. In the semiresonant case we assume again $|\xi_{1}| \geq |\xi_{2}| \geq |\xi_{3}|$ and consider two subcases: If, in addition, $|\xi_{1} + \xi_{2}| \leq 1$ (so that $(\xi) \leq c\langle \xi \rangle$), we can argue as in case

\[
J^{s} \partial_{x}(u_{1}u_{2}u_{3}) \leq \partial_{x}(J^{-s}u_{1}J^{-s}u_{2}J^{-s}u_{3}) \leq \left(I_{+}^{0} + I_{+}^{\frac{1}{2}}(J^{s}u_{1}, J^{s}u_{2})J^{1-s-\frac{2}{3}}u_{3} \right)
\]

which can be treated as above by applying \([7]\), Sobolev-type embeddings and Corollary \([1]\). On the other hand, if $|\xi_{1} + \xi_{2}| \geq 1$, we have

\[
|\sigma_{0} - \sigma_{1} - \sigma_{2} - \sigma_{3}| = 3|\xi_{1} + \xi_{2}||\xi_{2} + \xi_{3}||\xi_{3} + \xi_{1}| \geq \langle \xi_{1} \rangle \langle \xi_{2} \rangle,
\]

and hence, for any $\varepsilon > 0$,

\[
\langle \xi_{1} \rangle^{\varepsilon} \langle \xi_{2} \rangle^{\varepsilon} \leq c \prod_{i=0}^{3} \langle \sigma_{i} \rangle^{\varepsilon}.
\]

So, in this subcase, we have the upper bound

\[
\left\| T(J^{0}u_{1}, J^{0}u_{2}, J^{0}u_{3}) \right\| X_{s, b}^{r} \leq c \prod_{i=1}^{3} \left\| u_{i} \right\| X_{s, b}^{r}
\]

by Corollary \([3]\).

3. In the resonant case we distinguish several subcases:

3.1: At least for one pair $(i, j)$ we have $|\xi_{i} - \xi_{j}| \geq |\xi_{i} + \xi_{j}|$. 

Here we may assume by symmetry that $|\xi_2 - \xi_3| \geq |\xi_2 + \xi_3|$. Then we have for nonnegative $s_0,1$ with $s_0 + 2s_1 = \frac{1}{r}$

$$\partial_x J^s(u_1 u_2 u_3) \preceq T_{2^s}(J^{s+s_0} u_1, J^{s+s_1} u_2, J^{s+s_1} u_3),$$

so that Corollary 4 leads to the desired bound.

3.2: $|\xi_1 - \xi_2| \leq |\xi_1 + \xi_2|$, $|\xi_2 - \xi_3| \leq |\xi_2 + \xi_3|$ and $|\xi_3 - \xi_1| \leq |\xi_3 + \xi_1|$, so that all the $\xi_i$ have the same sign, which implies

$$|\xi_i|^3 \sim |\xi_2|^3 \sim |\xi_3|^3 \leq \prod_{i=0}^3 (\sigma_i).$$

3.2.1: At least one of the $|\xi_i - \xi_j| \geq 1$.

By symmetry we may assume that $|\xi_2 - \xi_3| \geq 1$. Gaining a $\langle \xi \rangle^\varepsilon$ from the $\sigma_i$’s we obtain as an upper bound for this subcase

$$\|T_{\xi}(J^{\varepsilon-} \text{A}^0+u_1, J^{\varepsilon+} \text{A}^0+u_2, J^{\varepsilon+} \text{A}^0+u_3)\|_{L_{r,t}^\infty} \leq c \prod_{i=1}^3 \|u_i\|_{X_{s,b}^r},$$

where we have used the second part of Corollary 5.

3.2.2: $|\xi_i - \xi_j| \leq 1$ for all $1 \leq i \neq j \leq 3$.

Again, we can gain a $\langle \xi \rangle^\varepsilon$ from the $\sigma_i$’s. Now, writing

$$f_{\varepsilon}(\xi, \tau) = \langle \xi \rangle^\varepsilon (\tau - \xi_3) \mathcal{F} u_{1}(\xi, \tau), \quad 1 \leq i \leq 3,$$

such that $\|f_{\varepsilon}\|_{L_{r,t}^\infty} = \|u_i\|_{X_{s,b}^r}$, it suffices to show

$$\|\langle \xi \rangle^\varepsilon|\xi| \int_A d\nu \prod_{i=1}^3 \langle \xi_i \rangle^{-\varepsilon} \langle \tau_i - \xi_i \rangle^{\frac{1}{2} - \varepsilon} f_{\varepsilon}(\xi, \tau)\|_{L_{r,t}^\infty} \leq c \prod_{i=1}^3 \|f_{\varepsilon}\|_{L_{r,t}^\infty},$$

where in $A$ all the differences $|\xi_k - \xi_j|$, $1 \leq k \neq j \leq 3$, are bounded by 1 and $|\xi| \sim |\xi_i| \sim |\xi_j|$ for all $1 \leq i \leq 3$. By Hölder’s inequality and Fubini’s Theorem the proof of (18) is reduced to show that

$$\sup_{\xi\tau} \langle \xi \rangle^{1-2s-} \left( \int_A d\nu \prod_{i=1}^3 \langle \tau_i - \xi_i \rangle^{-1-} \right)^\frac{1}{2} < \infty.$$

Using [GTV97] Lemma 4.2 twice, we see that

$$\int_A d\nu \prod_{i=1}^3 |\tau_i - \xi_i|^{-1-} \leq c \int_{A'} d\xi_1 d\xi_2 (\tau - \xi^3 + 3(\xi_1 + \xi_2)(\xi - \xi_1)(\xi - \xi_2))^{-1-},$$

where $A'$ is simply the projection of $A$ onto $\mathbb{R}^2$. We decompose

$$A' = A_0 \cup A_1 \cup \bigcup_{0 \leq k,j \leq \ln \langle \xi \rangle} A_{kj},$$

where in $A_0$ ($A_1$) we have that $|\xi_1 + \xi_2 - \frac{2\xi}{3}| \leq \frac{100}{\langle \xi \rangle}$ ($|\xi_1 + \xi_3 - \frac{2\xi}{3}| \leq \frac{100}{\langle \xi \rangle}$), so that the contributions of these subregions are bounded by $\frac{c}{\langle \xi \rangle}$, while in $A_{kj}$ it should hold that $|\xi_1 + \xi_2 - \frac{2\xi}{3}| \sim 2^{-k}$ and $|\xi_1 + \xi_3 - \frac{2\xi}{3}| \sim 2^{-j}$. By symmetry we may assume $k \leq j$. To estimate the integral over $A_{kj}$, we introduce new variables $x_1 := \xi_1 + \xi_2 - \frac{2\xi}{3}$ and $x_2 := \xi_1 - \xi_2$, such that

$$|x_1| \sim 2^{-k} \quad \text{and} \quad |x_2| = |\xi_1 + \xi_2 - \frac{2\xi}{3}| + 2(\xi_1 + \xi_3 - \frac{2\xi}{3}) \preceq 2^{-k}.$$
Then
\[
\int_{A_{kj}} d\xi_1 d\xi_2 (\tau - \xi_3^3 + 3(\xi_1 + \xi_2)(\xi - \xi_1)(\xi - \xi_2))^{-1} - 1 \leq \int_{|x_1| \approx 2^{-k}} dx_1 \int_{|x_2| \approx 2^{-k}} dx_2 (\tau - \xi_3^3 + 3(x_1 + \frac{2\xi}{3})(\frac{x_1}{2} - \frac{2\xi}{3}))^{-1}.
\]
Substituting \( z := (x_1 + \frac{2\xi}{3})(\frac{x_1}{2} - \frac{2\xi}{3}) \), so that
\[
\left| \frac{dz}{dx_1} \right| = \left| \frac{3x_1^2 - x_2^2}{4} - x_1 \xi \right| \sim |x_1 \xi| \sim |\xi|^{2-k},
\]
we see that the latter is bounded by
\[
\int_{|x_2| \leq 2^{-k}} dx_2 2^k \left| \frac{dz}{|\xi|} \right| (\tau - \xi^3 + 3z)^{-1} \leq \frac{c}{|\xi|}.
\]
Finally, summing up over \( j \) and \( k \), we have
\[
\int_{A'} d\xi_1 d\xi_2 (\tau - \xi_3^3 + 3(\xi_1 + \xi_2)(\xi - \xi_1)(\xi - \xi_2))^{-1} \leq c \left( \frac{(\ln |\xi|)^2}{|\xi|} \right) \leq c|\xi|^{-1+},
\]
which gives (19).

□

References

[B93] Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, GAFA 3 (1993), 107 - 156 and 209 - 262

[CVV01] Cazenave, T., Vega, L., Vilela, M. C.: A note on the nonlinear Schrödinger equation in weak \( L^p \) spaces, Communications in contemporary Mathematics, Vol.3, No. 1 (2001), 154-162

[CCT03] Christ, M., Colliander, J., Tao, T.: Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. 125 (2003), no. 6, 235 - 1293

[F70] Fefferman, C.: Inequalities for strongly singular convolution operators, Acta Math. 124 (1970), 9 - 36

[FLP99] Fonseca, G., Linares, F., Ponce, G.: Global well-posedness for the modified Korteweg-de Vries equation, Comm. PDE, 24 (1999), 683 - 705

[GTV97] Ginibre, J., Tsutsumi, Y., Velo, G.: On the Cauchy Problem for the Zakharov System, J. of Functional Analysis 151 (1997), 384 - 436

[G04] Grünrock, A.: An improved local well-posedness result for the modified KdV equation, IMRN 2004, No. 61, 3287 - 3308

[G05] Grünrock, A.: Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, IMRN 2005, No.41, 2525 - 2558

[KPV91] Kenig, C., Ponce, G., Vega, L.: Oscillatory Integrals and Regularity of dispersive equations, Indiana University Math. J., 40, 1991, 33 - 69

[KPV99] Kenig, C., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, CPAM 46 (1993), 527 - 620

[KPV01] Kenig, C., Ponce, G., Vega, L.: On the illposedness of some canonical dispersive equations, Duke Math. J. 106 (2001), 617 - 633

[Perelman, G., Vega, L.: Self-similar planar curves related to modified Korteweg-de Vries equation, preprint

[S87] Sjölin, P.: Regularity of solutions to the Schrödinger equation, Duke Math. J. 55 (1987), no. 3, 699-715

[VV01] Vargas, A., Vega, L.: Global wellposedness for 1D nonlinear Schrödinger equation for data with an infinite \( L^2 \) norm, J. Math. Pures Appl. 80, 10(2001), 1029-1044