ON WINTGEN IDEAL SURFACES

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Abstract

Wintgen proved in [29] that the Gauss curvature $K$ and the normal curvature $K^D$ of a surface in the Euclidean 4-space $E^4$ satisfy

$$K + |K^D| \leq H^2,$$

where $H^2$ is the squared mean curvature. A surface $M$ in $E^4$ is called a Wintgen ideal surface if it satisfies the equality case of the inequality identically. Wintgen ideal surfaces in $E^4$ form an important family of surfaces; namely, surfaces with circular ellipse of curvature. In this paper, we provide a brief survey on some old and recent results on Wintgen ideal surfaces and more generally Wintgen ideal submanifolds in definite and indefinite real space forms.

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1 Introduction

For surfaces $M$ in a Euclidean 3-space $E^3$, the Euler inequality

$$K \leq H^2,$$

(1)

whereby $K$ is the intrinsic Gauss curvature of $M$ and $H^2$ is the extrinsic squared mean curvature of $M$ in $E^3$, at once follows from the fact that

$$K = k_1k_2, \quad H = \frac{1}{2}(k_1 + k_2),$$

whereby $k_1$ and $k_2$ denote the principal curvatures of $M$ in $E^3$. And, obviously, $K = H^2$ everywhere on $M$ if and only if the surface $M$ is totally...
umbilical in $E^3$, i.e. $k_1 = k_2$ at all points of $M$, or still, by a theorem of Meusnier, if and only if $M$ is a part of a plane $E^2$ or of a round sphere $S^2$ in $E^3$.

Consider an isometric immersion $\psi : M \to \tilde{M}^4$ of a surface $M$ into a Riemannian 4-manifold $\tilde{M}^4$, the ellipse of curvature at a point $p$ of $M$ is defined as

$$E_p = \{h(X,X) \mid X \in T_pM, \|X\| = 1\},$$

where $h$ is the second fundamental form of $M$ in $\tilde{M}^4$.

In 1979, P. Wintgen [29] proved a basic relationship between the intrinsic Gauss curvature $K$, the extrinsic normal curvature $K^D$, and squared mean curvature $H^2$ of any surface $M$ in a Euclidean 4-space $E^4$; namely,

$$K + |K^D| \leq H^2,$$

with the equality holding if and only if the curvature ellipse is a circle. Wintgen’s inequality was generalized to surfaces in 4-dimensional real space forms in [20]. A similar inequality holds for surfaces in pseudo-Euclidean 4-space $E^4_2$ with neutral metric [7, 9].

Following L. Verstraelen et al. [14, 26], we call a surface $M$ in $E^4$ Wintgen ideal if it satisfies the equality case of Wintgen’s inequality identically. Obviously, Wintgen ideal surfaces in $E^4$ are exactly superminimal surfaces.

In this article, we provide a brief survey on some old and some recent results on Wintgen ideal surfaces; and more generally, Wintgen ideal submanifolds in definite and indefinite real space forms. Some related results are also presented in this paper.

2 Some known results on superminimal surfaces

2.1 $R$-surfaces

A surface $\psi : M \to \tilde{M}^4$ is superminimal if and only if, at each point $p \in M$, the ellipse of curvature $E_p$ is a circle with center at the origin $o$ (see [15]). Simple examples of superminimal surfaces in the Euclidean 4-space $E^4$ are $R$-surfaces, i.e., graphs of holomorphic functions:

$$\{(z, f(z)) : z \in U\},$$

where $U \subset \mathbb{C} \cong \mathbb{R}^2$ is an open subset of the complex plane and $f$ is a holomorphic function.
When the ambient space $\hat{M}^4$ is a space of constant curvature, O. Borůvka proved in 1928 that the family of superminimal immersions $\psi : M \to \hat{M}^4$ depends (locally) on two holomorphic functions.

### 2.2 Isoclinic surfaces

For an oriented plane $E$ in $\mathbb{E}^4$, let $E^\perp$ denote the orthogonal complement with the orientation given by the condition

$$E \oplus E^\perp = \mathbb{E}^4.$$  

Two oriented planes $E, F$ are called oriented-isoclinic if either

(a) $E = F^\perp$ (as oriented planes) or

(b) the projection $pr_F : E \to F$ is a non-trivial, conformal map preserving the orientations.

Consider an oriented surface $\psi : M \to \hat{M}^4$. If $\gamma$ is a curve in $\hat{M}^4$, denote by $\tau_\gamma$ the parallel displacement along $\gamma$ in the tangent bundle $T\hat{M}^4$. The surface $M^2$ is called a negatively oriented-isoclinic surface if, for every curve $\gamma$ in $M$ from $x$ to $y$, the planes $\tau_{\psi_0\gamma}(T_{\psi(x)}M)$ and $T_{\psi(y)}M$ are negatively oriented isoclinic planes in $T_{\psi(y)}\hat{M}^4$.

S. Kwietniowski proved in his 1902 dissertation at Zürich that a surface in $\mathbb{E}^4$ is superminimal if and only if it is negatively oriented-isoclinic. Th. Friedrich in 1997 extended this result for surface in $\hat{M}^4$.

### 2.3 Representation

In 1982, R. Bryant studied a superminimal immersion of a Riemann surface $M$ into $S^4$ by lifting it to $CP^3$, via the twistor map

$$\pi : CP^3 \to S^4$$

of Penrose. The lift is a holomorphic curve, of the same degree as that of the immersion, which is horizontal with respect to the twistorial fibration; moreover, the lift is a holomorphic curve in $CP^3$ satisfying the differential equation

$$z_0d\bar{z}_1 - z_1dz_0 + z_2dz_3 - z_3dz_2 = 0.$$  

Setting  

$$z_0 = 1, \quad z_1 + z_2z_3 = f, \quad z_2 = g,$$  

the differential equation is satisfied.
one can solve for \(z_1, z_2, z_3\) in terms of the meromorphic functions \(f\) and \(g\), which serves as a kind of Weierstrass representation. Via this, R. Bryant showed the existence of a superminimal immersion from any compact Riemann surface \(M\) into the 4-sphere \(S^4\).

M. Dajczer and R. Tojeiro established in [13] a representation formula for superminimal surfaces in \(E^4\) in terms of pairs \((g, h)\) of conjugate minimal surfaces in \(E^4\).

### 2.4 Twister space

On an oriented Riemannian 4-manifold \(\tilde{M}^4\), there exists an \(S^2\)-bundle \(Z\), called the **twistor space** of \(\tilde{M}^4\), whose fiber over any point \(x \in \tilde{M}^4\) consists of all almost complex structures on \(T_x\tilde{M}^4\) that are compatible with the metric and the orientation. It is known that there exists a one-parameter family of metrics \(g^t\) on \(Z\), making the projection

\[ Z \to \tilde{M}^4 \]  

(6)

into a Riemannian submersion with totally geodesic fibers.

Th. Friedrich proved in 1984 that superminimal surfaces are characterized by the property that the lift into the twistor space is holomorphic and horizontal.

### 2.5 Central sphere congruence

The central sphere congruence of a surface in Euclidean space is the family of 2-dimensional spheres that are tangent to the surface and have the same mean curvature vector as the surface at the point of tangency.

In 1991, B. Rouxel [27] proved the following results:

**Theorem 2.1.** If the ellipse of curvature of a surface in \(E^4\) is a circle, then the surface of centers of the harmonic spheres is a minimal surface of \(E^4\).

**Theorem 2.2.** If \(M\) is a surface of \(E^4\) with circular ellipse of curvature and if the harmonic spheres of \(M\) have a common fixed point, then \(M\) is a conformal transform of a superminimal surface of \(E^4\).

**Theorem 2.3.** The surface of centers of such sphere congruence is a minimal surface.
2.6 Ramification divisor

Let $M$ be a compact Riemann surface of genus $g$ and let $\phi : M \to \mathbb{C}P^1$ be a holomorphic map of degree $d$. A point $x \in M$ is a ramification point of $\phi$ if $d\phi(x) = 0$, and its image $\phi(x) \in \mathbb{C}P^1$ is called a branch point of $\phi$.

By the Riemann-Hurwitz Theorem the number of branch points of $\phi$ (counting multiplicities) is $2g + 2d - 2$.

The ramification divisor of $\phi$ is the formal sum

$$
\sum_i a_i p_i,
$$

where $p_i$ is a ramification point of $\phi$ with multiplicity $a_i$, and where the sum is taken over all ramification points of $\phi$. Let $\text{Ram}(\phi)$ denote the ramification divisor of $\phi$.

If we put

$$
f_1 = \frac{z_1}{z_0}, \quad f_2 = \frac{z_3}{z_2},
$$

then $f_1$ and $f_2$ are known of degree $d$ satisfying $\text{ram}(f_1) = \text{ram}(f_2)$, where $\text{ram}(f)$ is the ramification divisor of the meromorphic function $f$.

This provides a method for constructing the moduli space $M_d(M)$ of horizontal holomorphic curves of degree $d$ for a Riemann surface $M$ in $S^4$.

For $M = S^2$, B. Loo proved [24] that the moduli space $M_d(M)$ is connected and it has dimension $2d + 4$.

2.7 Riemann surfaces of higher genera

By applying algebraic geometry, Chi and Mo studied in [11] the moduli space over superminimal surfaces of higher genera. In particular, they proved the following 5 results:

**Theorem 2.4.** Let $M$ be a Riemann surface of genus $g \geq 1$. Then all the branched superminimal immersions of degree $d < 5$ from $M$ into $S^4$ are totally geodesic.

**Theorem 2.5.** Let $M$ be a Riemann surface of genus $g \geq 1$. Then $M$ admits a non-totally geodesic branched superminimal immersions of degree 6 into $S^4$ if and only if $M$ is a hyper-elliptic surface, i.e., it is an elliptic fibration over an elliptic curve.

**Theorem 2.6.** Let $M$ be a hyper-elliptic surface of genus $g > 3$. Then non-totally geodesic branched superminimal immersions of degree 6 from $M$ into
$S^4$ are the pullback of non-totally geodesic branched superminimal spheres of degree 3 via the branched double covering of $M$ onto $CP^1$.

**Theorem 2.7.** Let $M$ be a Riemann surface of genus $g \geq 2$ ($g = 1$, respectively). If $d > 5g + 4$, $(d \geq 6$, respectively), then there is a non-totally geodesic branched superminimal immersion of degree $d$ from $M$ into $S^4$. The immersion is generically one-to-one.

**Theorem 2.8.** Let $M$ be a Riemann surface of genus $g \geq 1$. If the degree $d$ of a superminimal immersion of $M$ in $S^4$ satisfies $d \geq 2g - 1$, then the dimension of the moduli space $M_d(M)$ is between $2d - 4g + 4$ and $2d - g + 4$, where the upper bound is achieved by the totally geodesic component.

### 3 Wintgen’s inequality

We recall the following result of P. Wintgen [29].

**Theorem 3.1.** Let $M$ be a surface in Euclidean 4-space $E^4$. Then we have
\[
H^2 \geq K + |K^D|
\]
at every point in $M$. Moreover, we have

(i) If $K^D \geq 0$ holds at a point $p \in M$, then the equality sign of (9) holds at $p$ if and only if, with respect to some suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$ at $p$, the shape operator at $p$ satisfies
\[
A_{e_3} = \begin{pmatrix} \mu + 2\gamma & 0 \\ 0 & \mu \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.
\]

(ii) If $K^D < 0$ holds at $p \in M$, then the equality sign of (9) holds at $p$ if and only if, with respect to some suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$ at $p$, the shape operator at $p$ satisfies
\[
A_{e_3} = \begin{pmatrix} \mu - 2\gamma & 0 \\ 0 & \mu \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.
\]

### 4 Wintgen ideal surfaces in $E^4$

In this and the next sections we present some recent results on Wintgen ideal surfaces.
Proposition 4.1. Let $M$ be a Wintgen ideal surface in $E^4$. Then $M$ has constant mean curvature and constant Gauss curvature if and only if $M$ is totally umbilical.

The following results classifies Wintgen ideal surfaces in $E^4$ with equal Gauss and normal curvatures.

Theorem 4.1. Let $\psi : M \to E^4$ be a Wintgen ideal surface in $E^4$. Then $|K| = |K^D|$ holds identically if and only if one of the following four cases occurs:

1. $M$ is an open portion of a totally geodesic plane in $E^4$.
2. $M$ is a complex curve lying fully in $C^2$, where $C^2$ is the Euclidean 4-space $E^4$ endowed with some orthogonal almost complex structure.
3. Up to dilations and rigid motions on the Euclidean 4-space $E^4$, $M$ is an open portion of the Whitney sphere defined by
   \[
   \psi(u, v) = \frac{\sin u}{1 + \cos^2 u} \left( \sin v, \cos v, \cos u \sin v, \cos u \cos v \right).
   \]
4. Up to dilations and rigid motions of the Euclidean 4-space $E^4$, $M$ is a surface with $K = K^D = \frac{1}{2}H^2$ defined by
   \[
   \psi(x, y) = \frac{2\sqrt{y}}{5} \cos x \cos \left( \frac{x}{2} \right) \cos(y) \cos \left( \frac{1}{2} \tanh^{-1} \left( \tan \frac{x}{2} \right) \right) \\
   \times \left( \tan \left( \frac{1}{2} \tanh^{-1} \left( \tan \frac{x}{2} \right) \right) (2 - \tan(y)) + \tan \left( \frac{x}{2} \right) (1 + 2 \tan(y)) \right),
   \]
   where
   \[
   \tan \left( \frac{1}{2} \tanh^{-1} \left( \tan \frac{x}{2} \right) \right) (1 + 2 \tan(y)) - \tan \left( \frac{x}{2} \right) (2 - \tan(y)),
   \]
   \[
   \tan \left( \frac{x}{2} \right) \tan \left( \frac{1}{2} \tanh^{-1} \left( \tan \frac{x}{2} \right) \right) (1 + 2 \tan(y)) + \tan(y) - 2,
   \]
   \[
   \tan \left( \frac{x}{2} \right) \tan \left( \frac{1}{2} \tanh^{-1} \left( \tan \frac{x}{2} \right) \right) (\tan(y) - 2) - 2 \tan(y) - 1.
   \]

According to I. Castro [3], up to rigid motions and dilations of $C^2$ the Whitney sphere is the only compact orientable Lagrangian superminimal surface in $C^2$.

Remark 4.1. In order to prove Theorem 4.1 we have solved the following
fourth order differential equation:

\[ p^{(4)}(x) - 2(tan x)p''(x) + \left( 1 + \frac{5}{8} sec^2 x \right) p''(x) + \left( \frac{5}{8} sec^2 x - 2 \right) (tan x)p'(x) + \frac{185}{256} (sec^4 x)p(x) = 0. \]  

(12)

to obtain the following exact solutions:

\[ p(x) = \sqrt{\cos x} \left\{ (c_1 \cos \left( \frac{x}{2} \right) + c_2 \sin \left( \frac{x}{2} \right)) \cos \left( \frac{1}{2} \tanh^{-1} \left( \tan \left( \frac{x}{2} \right) \right) \right) \\
+ \left( c_3 \cos \left( \frac{x}{2} \right) + c_4 \sin \left( \frac{x}{2} \right) \right) \sin \left( \frac{1}{2} \tanh^{-1} \left( \tan \left( \frac{x}{2} \right) \right) \right) \right\} \]  

(13)

5  Wintgen ideal surfaces in \( \mathbb{E}_2^4 \)

For space-like oriented surfaces in a 4-dimensional indefinite real space form \( R_2^4(c) \) with neutral metric, one has the following Wintgen type inequality (cf. [7, 9, 10]).

**Theorem 5.1.** Let \( M \) be an oriented space-like surface in a 4-dimensional indefinite space form \( R_2^4(c) \) of constant sectional curvature \( c \) and with index two. Then we have

\[ K + K^D \geq \langle H, H \rangle + c \]  

(14)

at every point.

The equality sign of (14) holds at a point \( p \in M \) if and only if, with respect to some suitable orthonormal frame \( \{ e_1, e_2, e_3, e_4 \} \), the shape operator at \( p \) satisfies

\[ A_{e_3} = \begin{pmatrix} \mu + 2\gamma & 0 \\ 0 & \mu \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}. \]  

(15)

As in surfaces in 4-dimensional real space forms, we call a surface in \( R_2^4(c) \) Wintgen ideal if it satisfies the equality case of (14) identically.

**Theorem 5.2.** Let \( M \) be a Wintgen ideal surface in a neutral pseudo-Euclidean 4-space \( \mathbb{E}_2^4 \). Then \( M \) satisfies \( |K| = |K^D| \) identically if and only if, up to dilations and rigid motions, \( M \) is one of the following three types of surfaces:
(i) A space-like complex curve in $C^2_1$, where $C^2_1$ denotes $E^4_2$ endowed with some orthogonal complex structure;

(ii) An open portion of a non-minimal surface defined by

$$\sec^2 x \left( \sin x \sinh y, \sqrt{2 - \sin^2 x} \cosh y, \sin x \cosh y, \sqrt{2 - \sin^2 x} \sinh y \right);$$

(iii) An open portion of a non-minimal surface defined by

$$\frac{\cosh x}{6\sqrt{2}y} \left( 6\sqrt{2}\sqrt{2+(1-2 \tanh x)\sqrt{1+\tanh x}} + y^2 \sqrt{2+\sqrt{1+\tanh x}}, ight.$$ 

$$6\sqrt{2}\sqrt{2+(2 \tanh x-1)\sqrt{1+\tanh x}} + y^2 \sqrt{2+\sqrt{1+\tanh x}},$$

$$6\sqrt{2}\sqrt{2+(1-2 \tanh x)\sqrt{1+\tanh x}} - y^2 \sqrt{2+\sqrt{1+\tanh x}},$$

$$-y^2 \sqrt{2+\sqrt{1+\tanh x}} \left( \sqrt{2} \cosh x \sqrt{1+\tanh x} - e^x \right) \bigg).$$

6 Surfaces with null normal curvature in $E^4_2$.

The following theorem of Chen and Suceava from [10] classifies surfaces with null normal curvature in $E^4_2$.

**Theorem 6.1.** Let $M$ be a space-like surface in the pseudo-Euclidean 4-space $E^4_2$. If $M$ has constant mean and Gauss curvatures and null normal curvature, then $M$ is congruent to an open part of one of the following six types of surfaces:

1. A totally geodesic plane in $E^4_2$ defined by $(0, 0, x, y)$;
2. A totally umbilical hyperbolic plane $H^2(-\frac{1}{a^2}) \subset E^3_1 \subset E^4_2$ given by

$$\left( 0, a \cosh u, a \sinh u \cos v, a \sinh u \sin v \right),$$

where $a$ is a positive number;

3. A flat surface in $E^4_2$ defined by

$$\frac{1}{\sqrt{2m}} \left( \cosh(\sqrt{2}mx), \cosh(\sqrt{2}my), \sinh(\sqrt{2}mx), \sinh(\sqrt{2}my) \right),$$
where \( m \) is a positive number;

(4) A flat surface in \( \mathbb{E}_2^4 \) defined by
\[
\left( 0, \frac{1}{a} \cosh(ax), \frac{1}{a} \sinh(ax), y \right),
\]
where \( a \) is a positive number;

(5) A flat surface in \( \mathbb{E}_2^4 \) defined by
\[
\left( \frac{\cosh(\sqrt{2}x)}{\sqrt{2m}}, \frac{\cosh(\sqrt{2}y)}{\sqrt{2m(2m-r)}}, \frac{\sinh(\sqrt{2}x)}{\sqrt{2m}}, \frac{\sinh(\sqrt{2}y)}{\sqrt{2m(2m-r)}} \right),
\]
where \( m \) and \( r \) are positive numbers satisfying \( 2m > r > 0 \);

(6) A surface of negative curvature \(-b^2\) in \( \mathbb{E}_2^4 \) defined by
\[
\left( \frac{1}{b} \cosh(bx) \cosh(by), \int_0^y \cosh(by) \sinh \left( \frac{4\sqrt{m^2-b^2}}{b} \tan^{-1} \left( \frac{\tanh \frac{by}{2}}{2} \right) \right) dy, \right.
\]
\[
\left. \frac{1}{b} \sinh(bx) \cosh(by), \int_0^y \cosh(by) \cosh \left( \frac{4\sqrt{m^2-b^2}}{b} \tan^{-1} \left( \frac{\tanh \frac{by}{2}}{2} \right) \right) dy \right),
\]
where \( b \) and \( m \) are real numbers satisfying \( 0 < b < m \).

7 Spacelike minimal surfaces with constant Gauss curvature.

From the equation of Gauss, we have

**Lemma 7.1.** Let \( M \) be a space-like minimal surface in \( \mathbb{R}_2^4(c) \). Then \( K \geq c \). In particular, if \( K = c \) holds identically, then \( M \) is totally geodesic.

For space-like minimal surfaces in \( \mathbb{R}_2^4(c) \), Theorem 1 of [28] implies that \( M \) has constant Gauss curvature if and only if it has constant normal curvature.

We recall the following result of M. Sasaki from [28].

**Theorem 7.1.** Let \( M \) be a space-like minimal surface in \( \mathbb{R}_2^4(c) \). If \( M \) has constant Gauss curvature, then either

1. \( K = c \) and \( M \) is a totally geodesic surface in \( \mathbb{R}_2^4(c) \);
2. \( c < 0 \), \( K = 0 \) and \( M \) is congruent to an open part of the minimal surface defined by
\[
\frac{1}{\sqrt{2}} \left( \cosh u, \cosh v, 0, \sinh u, \sinh v \right),
\]
or

(3) \( c < 0, K = c/3 \) and \( M \) is isotropic.

Let \( \mathbb{R}^2 \) be a plane with coordinates \( s, t \). Consider a map \( B : \mathbb{R}^2 \to \mathbb{E}_3^5 \) given by

\[
B(s, t) = \left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2t}{\sqrt{3}}}, t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2t}{\sqrt{3}}}, \right.
\]

\[
\quad \quad \quad \quad \quad \left. \frac{1}{2} + \frac{t^2}{2} e^{\frac{2t}{\sqrt{3}}}, t + \left( \frac{t^3}{3} + \frac{t}{4} \right) e^{\frac{2t}{\sqrt{3}}}, \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{1}{8} + \frac{t^4}{18} \right) e^{\frac{2t}{\sqrt{3}}} \right).
\]

The first author proved in [5] that \( B \) defines a full isometric parallel immersion

\[
\psi_B : H^2_{\left(-\frac{1}{3}\right)} \to H^2_{\left(-1\right)}
\]

of the hyperbolic plane \( H^2_{\left(-\frac{1}{3}\right)} \) of curvature \( -\frac{1}{3} \) into \( H^2_{\left(-1\right)} \).

The following result was also obtained in [5].

**Theorem 7.2.** Let \( \psi : M \to H^2_{\left(-1\right)} \) be a parallel full immersion of a space-like surface \( M \) into \( H^2_{\left(-1\right)} \). Then \( M \) is minimal in \( H^2_{\left(-1\right)} \) if and only if \( M \) is congruent to an open part of the surface defined by

\[
\left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2t}{\sqrt{3}}}, t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2t}{\sqrt{3}}}, \right.
\]

\[
\quad \quad \quad \quad \quad \left. \frac{1}{2} + \frac{t^2}{2} e^{\frac{2t}{\sqrt{3}}}, t + \left( \frac{t^3}{3} + \frac{t}{4} \right) e^{\frac{2t}{\sqrt{3}}}, \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{1}{8} + \frac{t^4}{18} \right) e^{\frac{2t}{\sqrt{3}}} \right).
\]

Combining Theorem 7.1 and Theorem 7.2, we obtain the following.

**Theorem 7.3** Let \( M \) be a non-totally geodesic space-like minimal surface in \( H^2_{\left(-1\right)} \). If \( M \) has constant Gauss curvature \( K \), then either

(1) \( K = 0 \) and \( M \) is congruent to an open part of the surface defined by

\[
\frac{1}{\sqrt{2}} \left( \cosh u, \cosh v, 0, \sinh u, \sinh v \right),
\]

or
(2) $K = -\frac{1}{3}$ and $M$ is is congruent to an open part of the surface defined by

\[
\left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}} t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}} \right),
\]

\[
\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}} t + \left( \frac{t^3}{3} + \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}},
\]

\[
\sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{1}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}}.
\]

8 Wintgen ideal surfaces satisfying $K^D = -2K$.

We need the following existence result from \[10\].

**Theorem 8.1.** Let $c$ be a real number and $\gamma$ with $3\gamma^2 > -c$ be a positive solution of the second order partial differential equation

\[
\frac{\partial}{\partial x} \left( \frac{(3\gamma \sqrt{c + 3\gamma^2} - c)(6\gamma + 2\sqrt{3c + 9\gamma^2})\sqrt{3}\gamma x}{2\gamma(c + 3\gamma^2)} \right) - \frac{\partial}{\partial y} \left( \frac{(3\gamma \sqrt{c + 3\gamma^2} - c)\gamma y}{2\gamma(c + 3\gamma^2)(6\gamma + 2\sqrt{3c + 9\gamma^2})^{\frac{3}{2}}} \right) = \gamma \sqrt{c + 3\gamma^2}
\]

defined on a simply-connected domain $D \subset \mathbb{R}^2$. Then $M_\gamma = (D, g_\gamma)$ with the metric

\[
g_\gamma = \frac{\sqrt{c + 3\gamma^2}}{\gamma(6\gamma + 2\sqrt{3c + 9\gamma^2})^{\frac{3}{2}}} \left( dx^2 + (6\gamma + 2\sqrt{3c + 9\gamma^2})^{\frac{3}{2}} dy^2 \right)
\]

admits a non-minimal Wintgen ideal immersion $\psi_\gamma : M_\gamma \rightarrow R^4_2(c)$ into a complete simply-connected indefinite space form $R^4_2(c)$ satisfying $K^D = 2K$ identically.

The following result from \[10\] classifies Wintgen ideal surfaces in $R^4_2(c)$ satisfying $K^D = 2K$.

**Theorem 8.2.** Let $M$ be a Wintgen ideal surface in a complete simply-connected indefinite space form $R^4_2(c)$ with $c = 1, 0$ or $-1$. If $M$ satisfies $K^D = 2K$ identically, then one of following three cases occurs:

1. $c = 0$ and $M$ is a totally geodesic surface in $\mathbb{E}^4_2$.
(2) $c = -1$ and $M$ is a minimal surface in $H^4_2(-1)$ congruent to an open part of $\psi : H^2(-\frac{1}{3}) \to H^2_2(-1) \subset \mathbb{E}^3_5$ defined by

$$
\begin{align*}
&\left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}}, t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}} \right) \\
&\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left( \frac{t^3}{3} + \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}}, \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{1}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}} \
&\right);
\end{align*}
$$

(3) $M$ is a non-minimal surface in $R^4_2(c)$ which is congruent to an open part of $\psi_\gamma : \gamma \to R^4_2(c)$ associated with a positive solution $\gamma$ of the partial differential equation (18) as described in Theorem 8.1.

9 An application to minimal surfaces in $H^4_2(-1)$.

A function $f$ on a space-like surface $M$ is called logarithm-harmonic, if $\Delta (\ln f) = 0$ holds identically on $M$, where $\Delta (\ln f) := *d \ast (\ln f)$ is the Laplacian of $\ln f$ and $*$ is the Hodge star operator. A function $f$ on $M$ is called subharmonic if $\Delta f \geq 0$ holds everywhere on $M$.

In this section we present some results from [7].

Theorem 9.1. Let $\psi : M \to H^4_2(-1)$ be a non-totally geodesic, minimal immersion of a space-like surface $M$ into $H^4_2(-1)$. Then

$$K + K_D \geq -1$$

(20)

holds identically on $M$.

If $K + 1$ is logarithm-harmonic, then the equality sign of (20) holds identically if and only if $\psi : M \to H^4_2(-1)$ is congruent to an open portion of the immersion $\psi_\phi : H^2(-\frac{1}{3}) \to H^2_2(-1)$ which is induced from the map $\phi : \mathbb{R}^2 \to \mathbb{E}^3_5$ defined by

$$
\phi(s, t) = \left( \sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{7}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}}, t + \left( \frac{t^3}{3} - \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}} \right) \\
\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left( \frac{t^3}{3} + \frac{t}{4} \right) e^{\frac{2s}{\sqrt{3}}} \\
\sinh \left( \frac{2s}{\sqrt{3}} \right) - \frac{t^2}{3} - \left( \frac{1}{8} + \frac{t^4}{18} \right) e^{\frac{2s}{\sqrt{3}}} \right).
$$

(21)
Corollary 9.1. Let \( \psi : M \to H^4_2(-1) \) be a minimal immersion of a space-like surface \( M \) of constant Gauss curvature into \( H^4_2(-1) \). Then the equality sign of (20) holds identically if and only if one of the following two statements holds.

1. \( K = -1, K^D = 0 \), and \( \psi \) is totally geodesic.
2. \( K^D = 2K = -\frac{2}{3} \) and \( \psi \) is congruent to an open part of the minimal surface \( \psi_\phi : H^2_2(-\frac{1}{3}) \to H^4_2(-1) \) induced from (21).

Proposition 9.1. Let \( \psi : M \to E^4_2 \) be a minimal immersion of a space-like surface \( M \) into the pseudo-Euclidean 4-space \( E^4_2 \). Then

\[
K \geq -K^D
\]  
holds identically on \( M \).

If \( M \) has constant Gauss curvature, then the equality sign of (22) holds identically if and only if \( M \) is a totally geodesic surface.

Proposition 9.2. Let \( \psi : M \to E^4_2 \) be a minimal immersion of a space-like surface \( M \) into \( E^4_2 \). We have

1. If the equality sign of (20) holds identically, then \( K \) is a non-logarithm-harmonic function.
2. If \( M \) contains no totally geodesic points and the equality sign of (22) holds identically on \( M \), then \( \ln K \) is subharmonic.

Proposition 9.3. Let \( \psi : M \to S^4_2(1) \) be a minimal immersion of a space-like surface \( M \) into the neutral pseudo-sphere \( S^4_2(1) \). Then

\[
K + K^D \geq 1
\]  
holds identically on \( M \).

If \( M \) has constant Gauss curvature, then the equality sign of (23) holds identically if and only if \( M \) is a totally geodesic surface.

Moreover, we have the following result from [7].

Proposition 9.4. Let \( \psi : M \to S^4_2(1) \) be a minimal immersion of a space-like surface \( M \) into \( S^4_2(1) \). We have

1. If the equality sign of (23) holds identically, then \( K - 1 \) is non-logarithm-harmonic.
2. If \( M \) contains no totally geodesic points and if the equality case of (23) holds, then \( \ln(K - 1) \) is subharmonic.
10 Wintgen ideal submanifolds are Chen submanifolds

Consider a submanifold $M^n$ of a real space form $\tilde{M}^{n+m}(\epsilon)$, the normalized normal scalar curvature $\rho^\perp$ is defined as

$$\rho^\perp = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n;1\leq r<s\leq m} \langle R^\perp(e_i,e_j)\xi_r,\xi_s\rangle^2},$$

where $R^\perp$ is the normal connection of $M^n$, and $\{e_1,\ldots,e_n\}$ and $\{\xi_1,\ldots,\xi_m\}$ are the orthonormal frames of the tangent and normal bundles of $M^n$, respectively.

in 1999, De Smet, Dillen, Vrancken and Verstraelen proved in [12] the Wintgen inequality

$$\rho \leq H^2 - \rho^\perp + c \quad (24)$$

for all submanifolds $M^n$ of codimension 2 in all real space forms $\tilde{M}^{n+2}(c)$, where $\rho$ is the normalized scalar curvature defined by

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} \langle R(e_i,e_j)e_j,e_i\rangle \quad (25)$$

and $R$ is the Riemann curvature tensor of $M^n$.

The Wintgen inequality (24) was conjectured by De Smet, Dillen, Vrancken and Verstraelen to hold for all submanifolds in all real space forms in the same paper [12], known as DDVV conjecture.

Recently, Z. Lu [25] and J. Ge and Z. Tang [17], settled this conjecture independently in general. A submanifold $M^n$ of a real space form $\tilde{M}^m(c)$ is called a Wintgen ideal submanifold if it satisfies the equality case of (24) identically.

An $n$-dimensional submanifold $M$ of a Riemannian manifold is called a Chen submanifold if

$$\sum_{i,j} \langle h(e_i,e_j),\vec{H}\rangle h(e_i,e_j) \quad (26)$$

is parallel to the mean curvature vector $\vec{H}$, where $h$ is the second fundamental form and $\{e_i\}$ is an orthonormal frame of the submanifold $M$ (cf. [19]).
The following theorem was proved by S. Decu, M. Petrović-Torgašev and L. Verstraelen in [14] which provides a very simple relationship between Wintgen ideal submanifolds and Chen submanifolds for submanifolds in real space forms.

**Theorem 10.1.** Every Wintgen ideal submanifold of arbitrary dimension and codimension in a real space form is a Chen submanifold.

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