A note on solutions of linear systems

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Abstract

In this paper we will consider Rohde’s general form of \{1\}-inverse of a matrix $A$. The necessary and sufficient condition for consistency of a linear system $Ax = c$ will be represented. We will also be concerned with the minimal number of free parameters in Penrose’s formula $x = A^{(1)c} + (I - A^{(1)}A)y$ for obtaining the general solution of the linear system. This results will be applied for finding the general solution of various homogenous and non-
-homogenous linear systems as well as for different types of matrix equations.

Keywords: Generalized inverses, linear systems, matrix equations

1. Introduction

In this paper we consider non-homogeneous linear system in $n$ variables

$$Ax = c,$$

where $A$ is an $m \times n$ matrix over the field $\mathbb{C}$ of rank $a$ and $c$ is an $m \times 1$ matrix over $\mathbb{C}$. The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ will be denoted by $\mathbb{C}^{m \times n}$, $m, n \in \mathbb{N}$. The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ of rank $a$ will be denoted by $\mathbb{C}^{a \times n}_m$. For simplicity of notation, we will write $A_{i \rightarrow} (A_{i,j})$ for the $i^{th}$ row (the $j^{th}$ column) of the matrix $A \in \mathbb{C}^{m \times n}$.

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Any matrix $X$ satisfying the equality $AXA = A$ is called $\{1\}$-inverse of $A$ and is denoted by $A^{(1)}$. The set of all $\{1\}$-inverses of the matrix $A$ is denoted by $A^{\{1\}}$. It can be shown that $A^{\{1\}}$ is not empty. If the $n \times n$ matrix $A$ is invertible, then the equation $AXA = A$ has exactly one solution $A^{-1}$, so the only $\{1\}$-inverse of the matrix $A$ is its inverse $A^{-1}$, i.e. $A^{\{1\}} = \{A^{-1}\}$. Otherwise, $\{1\}$-inverse of the matrix $A$ is not uniquely determined. For more informations about $\{1\}$-inverses and various generalized inverses we recommend A. Ben-Israel and T. N. E. Greville [1] and S. L. Campbell and C. D. Meyer [2].

For each matrix $A \in \mathbb{C}^{m \times n}$ there are regular matrices $P \in \mathbb{C}^{n \times n}$ and $Q \in \mathbb{C}^{m \times m}$ such that

$$QAP = E_a = \begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix},$$

(2)

where $I_a$ is $a \times a$ identity matrix. It can be easily seen that every $\{1\}$-inverse of the matrix $A$ can be represented in the form

$$A^{(1)} = P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} Q$$

(3)

where $U = [u_{ij}]$, $V = [v_{ij}]$ and $W = [w_{ij}]$ are arbitrary matrices of corresponding dimensions $a \times (m - a)$, $(n - a) \times a$ and $(n - a) \times (m - a)$ with mutually independent entries, see C. Rohde [8] and V. Perić [7].

We will generalize the results of N. S. Urquhart [9]. Firstly, we explore the minimal numbers of free parameters in Penrose’s formula

$$x = A^{(1)} c + (I - A^{(1)} A) y$$

for obtaining the general solution of the system \([1]\). Then, we consider relations among the elements of $A^{(1)}$ to obtain the general solution in the form $x = A^{(1)} c$ of the system \([1]\) for $c \neq 0$. This construction has previously been used by B. Malešević and B. Radić [3] (see also [4] and [5]). At the end of this paper we will give an application of this results to the matrix equation $AXB = C$.

2. The main result

In this section we indicate how technique of an $\{1\}$-inverse may be used to obtain the necessary and sufficient condition for an existence of a general solution of a non-homogeneous linear system.
Lemma 2.1. The non-homogeneous linear system \( (1) \) has a solution if and only if the last \( m - a \) coordinates of the vector \( c' = Qc \) are zeros, where \( Q \in \mathbb{C}^{m \times m} \) is regular matrix such that \( (2) \) holds.

Proof: The proof follows immediately from Kroneker–Capelli theorem. We provide a new proof of the lemma by using the \( \{1\} \)-inverse of the system matrix \( A \). The system \( (1) \) has a solution if and only if \( c = AA^{(1)}c \), see R. Penrose [6]. Since \( A^{(1)} \) is described by the equation \( (3) \), it follows that

\[
AA^{(1)} = AP \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} Q = Q^{-1} \begin{bmatrix} I_a & U \\ 0 & 0 \end{bmatrix} Q.
\]

Hence, we have the following equivalences

\[
c = AA^{(1)}c \iff (I - AA^{(1)})c = 0 \iff \left( Q^{-1}Q - Q^{-1} \begin{bmatrix} I_a & U \\ 0 & 0 \end{bmatrix} Q \right) c = 0
\]

\[
\iff Q^{-1} \begin{bmatrix} 0 & -U \\ 0 & I_{n-a} \end{bmatrix} Qc = 0 \iff \begin{bmatrix} 0 & -U \\ 0 & I_{n-a} \end{bmatrix} c' = 0
\]

\[
c' = \begin{bmatrix} c_a' \\ c_{n-a}' \end{bmatrix} \iff \begin{bmatrix} 0 & -U \\ 0 & I_{n-a} \end{bmatrix} \begin{bmatrix} c_a' \\ c_{n-a}' \end{bmatrix} = 0 \iff \begin{bmatrix} -Uc_{n-a}' \\ c_{n-a}' \end{bmatrix} = 0
\]

\[
\iff c_{n-a}' = 0.
\]

Furthermore, we conclude \( c = AA^{(1)}c \iff c_{n-a}' = 0 \). \( \Box \)

Theorem 2.2. The vector

\[
x = A^{(1)}c + (I - A^{(1)}A)y,
\]

\( y \in \mathbb{C}^{n \times 1} \) is an arbitrary column, is the general solution of the system \( (1) \), if and only if the \( \{1\} \)-inverse \( A^{(1)} \) of the system matrix \( A \) has the form \( (3) \) for arbitrary matrices \( U \) and \( W \) and the rows of the matrix \( V(c_a' - y_{a}') + y_{(n-a)}' \)

are free parameters, where \( Qc = c' = \begin{bmatrix} c_a' \\ 0 \end{bmatrix} \) and \( P^{-1}y = y' = \begin{bmatrix} y_a' \\ y_{n-a}' \end{bmatrix} \).

Proof: Since \( \{1\} \)-inverse \( A^{(1)} \) of the matrix \( A \) has the form \( (3) \), the solution of the system \( x = A^{(1)}c + (I - A^{(1)}A)y \) can be represented in the form

\[
x = P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} Qc + \left( I - P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} QA \right) y
\]

\[
= P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} c' + \left( I - P \begin{bmatrix} I_a & U \\ V & W \end{bmatrix} QAPP^{-1} \right) y.
\]
According to Lemma 2.1 and from (2) we have
\[ x = P \left[ \begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] \left[ \begin{array}{c} c' \\ 0 \end{array} \right] + \left( I - P \left[ \begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] P^{-1} \right) y. \]

Furthermore, we obtain
\[
\begin{align*}
  x &= P \left[ \begin{array}{c} c' \\ Vc'_a \end{array} \right] + \left( I - P \left[ \begin{array}{c} I_a & U \\ \hline V & W \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] P^{-1} \right) \left[ \begin{array}{c} y_a \\ y_{n-a} \end{array} \right] \\
  &= P \left[ \begin{array}{c} c' \\ Vc'_a \end{array} \right] + \left( PP^{-1} - P \left[ \begin{array}{c} I_a & U \\ \hline V & W \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] P^{-1} \right) \left[ \begin{array}{c} y_a \\ y_{n-a} \end{array} \right] \\
  &= P \left[ \begin{array}{c} c' \\ Vc'_a \end{array} \right] + P \left( I - \left[ \begin{array}{c} I_a & U \\ \hline V & W \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \right) P^{-1} \left[ \begin{array}{c} y_a \\ y_{n-a} \end{array} \right] \\
  &= P \left[ \begin{array}{c} c' \\ Vc'_a \end{array} \right] + P \left[ \begin{array}{c} 0 \\ -V \end{array} \right] \left[ \begin{array}{c} y'_a \\ y'_{n-a} \end{array} \right],
\end{align*}
\]

where \( y' = P^{-1}y \). We now conclude
\[
\begin{align*}
  x &= P \left( \left[ \begin{array}{c} c' \\ Vc'_a \end{array} \right] + \left[ \begin{array}{c} 0 \\ -V y'_a + y'_{n-a} \end{array} \right] \right) = P \left[ \begin{array}{c} c' \\ V(c'_a - y'_a) + y'_{n-a} \end{array} \right].
\end{align*}
\]

Therefore, since matrix \( P \) is regular we deduce that \( P \left[ \begin{array}{c} c' \\ V(c'_a - y'_a) + y'_{n-a} \end{array} \right] \) is the general solution of the system (1) if and only if the rows of the matrix \( V(c'_a - y'_a) + y'_{n-a} \) are \( n - a \) free parameters. \( \square \)

**Corollary 2.3.** The vector
\[ x = (I - A^{(1)}) y, \]
\( y \in \mathbb{C}^{n \times 1} \) is an arbitrary column, is the general solution of the homogeneous linear system \( Ax = 0 \), \( A \in \mathbb{C}^{m \times n} \), if and only if the \( \{1\} \)-inverse \( A^{(1)} \) of the system matrix \( A \) has the form (3) for arbitrary matrices \( U \) and \( W \) and the rows of the matrix \( -V y'_a + y'_{(n-a)} \) are free parameters, where \( P^{-1}y = y' = \left[ \begin{array}{c} y'_a \\ y'_{n-a} \end{array} \right]. \)

**Example 2.4.** Consider the homogeneous linear system
\[
\begin{align*}
  x_1 + 2x_2 + 3x_3 &= 0 \\
  4x_1 + 5x_2 + 6x_3 &= 0.
\end{align*}
\]
The system matrix is

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.
\]

For regular matrices

\[
Q = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 0 & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{bmatrix}
\]

equality (2) holds. Rohde’s general \{1\}-inverse \(A^{(1)}\) of the system matrix \(A\) is of the form

\[
A^{(1)} = P \begin{bmatrix} 1 & 0 \\ v_{11} & v_{12} \end{bmatrix} Q
\]

According to Corollary 2.3 the general solution of the system (4) is of the form

\[
x = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v_{11} & -v_{12} & 1 \end{bmatrix} P^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},
\]

where

\[
P^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Therefore, we obtain

\[
x = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -v_{11} & -v_{12} & 1 \end{bmatrix} \begin{bmatrix} y_1 + 2y_2 + 3y_3 \\ -3y_2 - 6y_3 \\ y_3 \end{bmatrix}
= P \begin{bmatrix} 0 \\ 0 \\ -v_{11}y_1 - (2v_{11} - 3v_{12}y_2 - (3v_{11} - 6v_{12} - 1)y_3 \end{bmatrix}.
\]

If we take \(\alpha = -v_{11}y_1 - (2v_{11} - 3v_{12}y_2 - (3v_{11} - 6v_{12} - 1)y_3\) as a parameter we get the general solution

\[
x = \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 0 & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix}.
\]
Corollary 2.5. The vector 
\[ x = A^{(1)}c \]
is the general solution of the system (1), if and only if the \( \{1\} \)-inverse \( A^{(1)} \) of the system matrix \( A \) has the form (3) for arbitrary matrices \( U \) and \( W \) and the rows of the matrix \( Vc' \) are free parameters, where \( Qc = c' = \begin{bmatrix} c' \\ 0 \end{bmatrix} \).

Remark 2.6. Similar result can be found in paper B. Malešević and B. Radičić [3].

Example 2.7. Consider the non-homogeneous linear system
\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 7 \\
    4x_1 + 5x_2 + 6x_3 &= 8.
\end{align*}
\]
According to Corollary 2.5 the general solution of the system (5) is of the form
\[
x = P \begin{bmatrix} 1 & 0 \\ v_{11} & v_{12} \end{bmatrix} Q \begin{bmatrix} 7 \\ 8 \end{bmatrix} = P \begin{bmatrix} 7 \\ -20 \\ 7v_{11} - 20v_{12} \end{bmatrix}.
\]
If we take \( \alpha = 7v_{11} - 20v_{12} \) as a parameter we obtain the general solution of the system
\[
x = P \begin{bmatrix} 7 \\ -20 \\ \alpha \end{bmatrix} = P \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 0 & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -20 \\ \alpha \end{bmatrix} = \begin{bmatrix} -\frac{19}{3} + \alpha \\ \frac{20}{3} - 2\alpha \\ \alpha \end{bmatrix}.
\]
We are now concerned with the matrix equation
\[ AX = C, \]
where \( A \in \mathbb{C}^{m \times n} \), \( X \in \mathbb{C}^{n \times k} \) and \( C \in \mathbb{C}^{m \times k} \).

Lemma 2.8. The matrix equation (6) has a solution if and only if the last \( m - a \) rows of the matrix \( C' = QC \) are zeros, where \( Q \in \mathbb{C}^{m \times m} \) is regular matrix such that (2) holds.
Proof: If we write \( X = [X_{i1} \ X_{i2} \ldots \ X_{ik}] \) and \( C = \{C_{i1} \ C_{i2} \ldots \ C_{ik}\} \), then we can observe the matrix equation (6) as the system of matrix equations

\[
AX_{i1} = C_{i1} \\
AX_{i2} = C_{i2} \\
\vdots \\
AX_{ik} = C_{ik}.
\]

Each of the matrix equation \( AX_{ii} = C_{ii}, \ 1 \leq i \leq k \), by Lemma 2.1 has solution if and only if the last \( m-a \) coordinates of the vector \( C'_{ii} = QC_{ii} \) are zeros. Thus, the previous system has solution if and only if all entries of the last \( m-a \) rows of the matrix \( C' \) are zeros. \( \square \)

Theorem 2.9. The matrix

\[
X = A^{(1)}C + (I - A^{(1)}A)Y \in \mathbb{C}^{n \times k},
\]

\( Y \in \mathbb{C}^{n \times k} \) is an arbitrary matrix, is the general solution of the matrix equation (6) if and only if the \( \{1\}^{-}\text{inverse} \ A^{(1)} \) of the system matrix \( A \) has the form (3) for arbitrary matrices \( U \) and \( W \) and the entries of the matrix

\[
V(C' - Y'+ Y'_{(n-a)})
\]

are mutually independent free parameters, where \( QC = C' = \begin{bmatrix} C'_{a} \\ 0 \end{bmatrix} \) and \( P^{-1}Y = Y' = \begin{bmatrix} Y'_{a} \\ Y'_{n-a} \end{bmatrix} \).

Proof: Applying the Theorem 2.2 on the each system \( AX_{ii} = C_{ii}, \ 1 \leq i \leq k \), we obtain that

\[
X_{ii} = P \left[ \frac{C'_{a\dot{i}}}{V(C'_{a\dot{i}} - Y'_{a\dot{i}}) + Y'_{n-a\dot{i}}} \right]
\]

is the general solution of the system if and only if the rows of the matrix \( V(C'_{a\dot{i}} - Y'_{a\dot{i}}) + Y'_{n-a\dot{i}} \) are \( n-a \) free parameters. Assembling these individual solutions together we get that

\[
X = P \left[ \frac{C'_{a}}{V(C' - Y') + Y'_{n-a}} \right]
\]
is the general solution of the matrix equation \((6)\) if and only if entries of the matrix \(V(C'_a - Y'_a) + Y'_{n-a}\) are \((n-a)k\) mutually independent free parameters. □

From now on we proceed with the study of the non-homogeneous linear system of the form

\[ xB = d, \]  \quad (7)

where \(B\) is an \(n \times m\) matrix over the field \(\mathbb{C}\) of rank \(b\) and \(d\) is an \(1 \times m\) matrix over \(\mathbb{C}\). Let \(R \in \mathbb{C}^{n \times n}\) and \(S \in \mathbb{C}^{m \times m}\) be regular matrices such that

\[ RBS = E_b = \begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix}. \]  \quad (8)

An \(\{1\}\)-inverse of the matrix \(B\) can be represented in the Rohde’s form

\[ B^{(1)} = S \begin{bmatrix} I_b \\ N \\ M \\ K \end{bmatrix} R \]  \quad (9)

where \(M = [m_{ij}]\), \(N = [n_{ij}]\) and \(K = [k_{ij}]\) are arbitrary matrices of corresponding dimensions \(b \times (n-b)\), \((m-b) \times b\) and \((m-b) \times (n-b)\) with mutually independent entries.

**Lemma 2.10.** The non-homogeneous linear system \((7)\) has a solution if and only if the last \(m-b\) elements of the row \(d' = dS\) are zeros, where \(S \in \mathbb{C}^{m \times m}\) is regular matrix such that \((8)\) holds.

**Proof:** By transposing the system \((7)\) we obtain system \(B^T x^T = d^T\) and by transposing the matrix equation \((8)\) we obtain that \(S^T B^T R^T = E_b\). According to Lemma 2.1 the system \(B^T x^T = d^T\) has solution if and only if the last \(m-b\) coordinates of the vector \(S^T d^T\) are zeros, i.e. if and only if the last \(m-b\) elements of the row \(d' = dS\) are zeros. □

**Theorem 2.11.** The row

\[ x = dB^{(1)} + y(I - BB^{(1)}), \]

\(y \in \mathbb{C}^{1 \times n}\) is an arbitrary row, is the general solution of the system \((7)\), if and only if the \(\{1\}\)-inverse \(B^{(1)}\) of the system matrix \(B\) has the form \((9)\) for arbitrary matrices \(N\) and \(K\) and the columns of the matrix \((d'_b - y'_b)M + y'_{n-b}\) are free parameters, where \(dS = d' = [d'_b | 0]\) and \(yR^{-1} = y' = [y'_b | y'_{n-b}]\).
Proof: The basic idea of the proof is to transpose the system (7) and to apply the Theorem 2.2. The \( \{1\} \)-inverse of the matrix \( B \) is equal to a transpose of the \( \{1\} \)-inverse of the matrix \( B^T \). Hence, we have

\[
(B^T)^{(1)} = (B^{(1)})^T = \left( S \left[ \frac{I_b}{N} M \right] R \right)^T = R^T \left[ \frac{I_b}{M^T} N^T \right] S^T.
\]

We can now proceed analogously to the proof of the Theorem 2.2 to obtain that

\[
x^T = R^T \left[ \frac{d'_{b}^T}{M^T(d'_{b} - y'_{n-b}) + y'_{n-b}} \right]
\]

is the general solution of the system \( B^T x^T = d^T \) if and only if the rows of the matrix \( M^T(d'_{b} - y'_{n-b}) + y'_{n-b} \) are \( n - b \) free parameters. Therefore,

\[
x = [d'_{b} \ | \ (d'_{b} - y'_{n-b}) M + y'_{n-b}] R
\]

is the general solution of the system (7) if and only if the columns of the matrix \( (d'_{b} - y'_{n-b}) M + y'_{n-b} \) are \( n - b \) free parameters. □

Analogous corollaries hold for the Theorem 2.11.

We now deal with the matrix equation

\[
XB = D,
\]

where \( X \in \mathbb{C}^{k \times n} \), \( B \in \mathbb{C}^{n \times m} \) and \( D \in \mathbb{C}^{k \times m} \).

Lemma 2.12. The matrix equation (10) has a solution if and only if the last \( m - b \) columns of the matrix \( D' = DS \) are zeros, where \( S \in \mathbb{C}^{m \times m} \) is regular matrix such that (8) holds.

Theorem 2.13. The matrix

\[
X = DB^{(1)} + Y (I - BB^{(1)}) \in \mathbb{C}^{k \times n},
\]

\( Y \in \mathbb{C}^{k \times n} \) is an arbitrary matrix, is the general solution of the matrix equation (10) if and only if the \( \{1\} \)-inverse \( B^{(1)} \) of the system matrix \( B \) has the form (9) for arbitrary matrices \( N \) and \( K \) and the entries of the matrix

\[
(D'_{b} - y'_{n-b}) M + y'_{n-b}
\]

are mutually independent free parameters, where \( DS = D' = [D'_{b} \mid 0] \) and \( Y R^{-1} = Y' = [Y'_{b} \mid Y'_{n-b}] \).
3. An application

In this section we will briefly sketch properties of the general solution of the matrix equation

\[ AXB = C, \]  

(11)

where \( A \in \mathbb{C}^{m \times n} \), \( X \in \mathbb{C}^{n \times k} \), \( B \in \mathbb{C}^{k \times l} \) and \( C \in \mathbb{C}^{m \times l} \). If we denote by \( Y \) matrix product \( XB \), then the matrix equation (11) becomes

\[ AY = C. \]  

(12)

According to the Theorem 2.9 the general solution of the system (12) can be presented as a product of the matrix \( P \) and the matrix which has the first \( a = \text{rank}(A) \) rows same as the matrix \( QC \) and the elements of the last \( m - a \) rows are \((m - a)n\) mutually independent free parameters, \( P \) and \( Q \) are regular matrices such that \( QAP = E_a \). Thus, we are now turning on to the system of the form

\[ XB = D. \]  

(13)

By the Theorem 2.13 we conclude that the general solution of the system (13) can be presented as a product of the matrix which has the first \( b = \text{rank}(B) \) columns equal to the first \( b \) columns of the matrix \( DS \) and the rest of the columns have mutually independent free parameters as entries, and the matrix \( R \), for regular matrices \( R \) and \( S \) such that \( RBS = E_b \). Therefore, the general solution of the system (11) is of the form

\[ X = P \begin{bmatrix} G_{ab} & F \\ H & L \end{bmatrix} R, \]

where \( G_{ab} \) is a submatrix of the matrix \( QCS \) corresponding to the first \( a \) rows and the first \( b \) columns and the entries of the matrices \( F, H \) and \( L \) are \( nk - ab \) free parameters. We will illustrate this on the following example.

**Example 3.1.** We consider the matrix equation

\[ AXB = C, \]

where \( A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \) \text{ and } C = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix}. \) If we take \( Y = XB \), we obtain the system

\[ AY = C. \]
It is easy to check that the matrix \( A \) is of the rank \( a = 1 \) and for matrices \( Q = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \) and \( P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \) the equality \( QAP = E_a \) holds. Based on the Theorem 2.9, the equation \( AY = C \) can be rewritten in the system form

\[
AY_{i1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},
AY_{i2} = \begin{bmatrix} 2 \\ -4 \end{bmatrix},
AY_{i3} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.
\]

Combining the Theorem 2.2 with the equality

\[
\begin{bmatrix} c_1' & c_2' & c_3' \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

yields

\[
Y_{i1} = P \begin{bmatrix} 1 \\ v - vz_{11} + z_{21} \end{bmatrix},
Y_{i2} = P \begin{bmatrix} 2 \\ 2v - 2vz_{12} + z_{22} \end{bmatrix},
Y_{i3} = P \begin{bmatrix} 1 \\ v - vz_{13} + z_{23} \end{bmatrix},
\]

for an arbitrary matrix \( Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{bmatrix} \). Therefore, the general solution of the system \( AY = C \) is

\[
Y = P \begin{bmatrix} 1 & 2 & 1 \\ \alpha & \beta & \gamma \end{bmatrix}.
\]

From now on, we consider the system

\[
XB = D
\]

for

\[
D = P \begin{bmatrix} 1 & 2 & 1 \\ \alpha & \beta & \gamma \end{bmatrix} = \begin{bmatrix} 1 + 2\alpha & 2 + 2\beta & 1 + 2\gamma \\ \alpha & \beta & \gamma \end{bmatrix}.
\]
There are regular matrices \( R = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \) and \( S = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) such that \( RBS = E_b \) holds. Since the rank of the matrix \( B \) is \( b = 1 \), according to the Lemma 2.12 all entries of the last two columns of the matrix \( D' = DS \) are zeros, i.e. we have \( \gamma = \alpha, \beta = 2\alpha \). Hence, we get that the matrix \( D' \) is of the form

\[
D' = \begin{bmatrix}
1 + 2\alpha & 0 & 0 \\
\alpha & 0 & 0 
\end{bmatrix}
\]

Applying the Theorem 2.13, we obtain

\[
X = \begin{bmatrix}
1 + 2\alpha & (1 + 2\alpha - t_{11})m_{11} + t_{12} \\
\alpha & (\alpha - t_{21})m_{11} + t_{22} \\
& \gamma_{11} \\
& \gamma_{12} 
\end{bmatrix}
\begin{bmatrix}
1 + 2\alpha - t_{11}m_{11} + t_{13} \\
(1 + 2\alpha - t_{11})m_{12} + t_{13} \\
(\alpha - t_{21})m_{12} + t_{21} + t_{13} \\
(\alpha - t_{21})m_{22} + t_{22} \\
\gamma_{21} \\
\gamma_{22} 
\end{bmatrix}
\]

for an arbitrary matrix \( T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix} \). Finally, the solution of the system \( AXB = C \) is

\[
X = \begin{bmatrix}
1 + 2\alpha - \beta_1 - \beta_2 \\
\alpha - \gamma_1 - \gamma_2 \\
\gamma_1 \\
\gamma_2 
\end{bmatrix}
\]

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