PERTURBATIONS OF GEODESIC FLOWS BY RECURRENT DYNAMICS

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ABSTRACT. We consider a geodesic flow on a compact manifold endowed with a Riemannian (or Finsler, or Lorentz) metric satisfying some generic, explicit conditions. We couple the geodesic flow with a time-dependent potential, driven by an external flow on some other compact manifold. If the external flow satisfies some very general recurrence condition, and the potential satisfies some explicit conditions that are also very general, we show that the coupled system has orbits whose energy grows at a linear rate with respect to time. This growth rate is optimal. We also show the existence of symbolic dynamics.

The existence of orbits whose energy grows unboundedly in time is related to Arnold’s diffusion problem.

A particular case of this phenomenon is obtained when the external dynamical system is quasi-periodic, of rationally independent frequency vector, not necessarily Diophantine, thus extending [19]. We also recover ‘Mather’s acceleration theorem’ for time-periodic perturbations [51, 16, 34].

Since the general class of perturbations that we consider in this paper are not characterized by a frequency of motion, we are not able to use KAM or Aubry-Mather or averaging theory. We devise a new mechanism to construct orbits whose energy evolves in a prescribed fashion. This mechanism intertwines the inner dynamics restricted to some normally hyperbolic manifold with the outer dynamics corresponding to two distinct choices of homoclinic excursions to that manifold. As this mechanism uses very coarse information on the dynamics, it is potentially of wider applicability.

1. Introduction

1.1. The Hamiltonian instability problem. The phenomenon of instability in nearly integrable Hamiltonian systems was discovered in [3], which described a geometric mechanism based on whiskered tori and verified it in an example. Conceptually, this mechanism showed how to increase the energy of a conservative mechanical system by using small perturbations. Many subsequent developments in studying the Arnold mechanism, including incorporating variational methods, can be found in the literature, e.g., in [12, 6, 29]. Due to its importance to practical applications, the instability phenomenon has also been studied by physicists using heuristic and numerical methods, e.g., in [11, 67, 45, 63].

A remarkable revival of interest in the problem was stimulated by the unpublished work [51], which led to several new approaches – variational, geometric, or mixed – e.g., in [51, 52, 53, 8, 9, 10, 22, 40, 41, 42].

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The geometric approach of [16, 34, 35] does not only use whiskered tori but also normally hyperbolic invariant manifolds (NHIMs) whose stable and unstable manifolds intersect. (The method of [34] bypasses whiskered tori completely.) The mechanism is based on interleaving homoclinic excursions (conveniently quantified by the scattering map) with the dynamics on the NHIM.

In the mechanisms discussed in the above papers [16, 34, 35], there are two time scales. The homoclinic excursions are fast while the dynamics in the NHIM has slow components. If one assumes that the perturbations are periodic (or quasi-periodic) the inner dynamics can be controlled using the classical averaging method.

The main idea of this paper is that we can show Hamiltonian instability by relying almost exclusively on homoclinic excursions, through carefully choosing the times when to jump onto the homoclinics. To apply this novel mechanism, we do not need to assume much on the form of the perturbations. As it turns out, by jumping onto the homoclinics at about the same place, we obtain effects similar to averaging. The gain of energy is obtained by timing the place of the jumps and by taking advantage of the fact that there are several homoclinics available for jumping. To concatenate these excursions, we take advantage of the method of ‘correctly aligned windows’ which is more flexible than other shadowing techniques, and, at the same time, is precise enough to keep track of time. Two of the features of the method of correctly aligned windows that are crucial for us are that it works in systems with only partial hyperbolicity, and that it allows to concatenate segments of trajectories.

To illustrate the approach (we hope that there are several other applications forthcoming) we study the model considered in [51], consisting of a geodesic flow perturbed by a time-dependent potential. Since we do not need to control the inner dynamics much we do not need averaging and we do not need to assume that the external flow is periodic or quasi-periodic. Our results also show instability in the models considered in [16, 34, 35] but remove several hypotheses and also give optimal diffusion time. Of course, the orbits produced are very different from those in the above papers. Note that the models considered in this paper do not easily admit a quantitative averaging theory in the style of the classical one (good expositions of classical averaging theory are in [48,4]. We do not know if the ergodic averaging theory [2] is enough to obtain geometric results.

Of course, we do not know whether the variational methods can be applied to general time dependent systems. Some of the geometric features of minimizers well known for periodic perturbations have counter-examples for quasi-periodically forced systems [46]. Of course, variational methods work under the assumption that the system is convex (i.e., its Hessian matrix is positive definite), but this is not required by our method. In this paper, we do not assume that the metric defining the geodesic flow is Riemannian, and the arguments apply just as well to Finsler or Lorentz metrics.

1.2. Perturbed geodesic flow model. The explicit model that we treat in this paper is the following. We consider an (unperturbed) geodesic flow on the unit tangent bundle of a compact manifold has a hyperbolic periodic orbit whose stable and unstable manifolds intersect transversally. Via a potential, we couple the dynamics of the geodesic flow with that of an external flow on some other compact manifold. We assume that the external flow has a non-trivial uniformly recurrent point. We show that, if the coupling between the geodesic flow and the external
flow satisfies some non-degeneracy conditions, which we formulate explicitly, then the resulting system possesses some orbits along which the energy grows to infinity linearly in time; such growth rate is optimal. Also, we show that there exist orbits whose energy follows a prescribed energy path.

We point out that the explicit assumptions on the geodesic flow, on the external dynamics, and on the coupling, are very general. They hold generically and can be checked in concrete examples by a finite precision calculation. Also, the construction of orbits whose energy grows to infinity at an optimal rate, or of orbits whose energy behaves in some prescribed way, is explicit, a fact which can be potentially exploited in concrete applications, such as in celestial mechanics.

A particular class of systems that we refer to in this paper are quasi-periodic perturbations of the geodesic flow, which have been considered in the literature [19]. Nevertheless, the mechanisms we use in this paper are very different from those of the previous papers, and the produced orbits are different as well. This is relevant for applications, when one is not only interested in establishing the presence of the ‘diffusion’ phenomenon, but also in exploring various diffusion mechanisms which can feature different characteristics [63].

An intuition to keep in mind is that the unperturbed geodesic flow describes the dynamics of a mechanical system with a Maupertuis metric. Its energy is conserved. The external flow, which is governed by its own intrinsic dynamics, exerts a time-dependent perturbation on the mechanical system. For the perturbed system, the energy conservation law does not hold in general. If the external flow moves with some frequency, it seems conceivable that by choosing trajectories of the geodesic flow that are in resonance with that frequency one can obtain unbounded energy growth for the perturbed geodesic flow.

The striking conclusion of this paper is that one can achieve unbounded energy growth even when the external flow has no frequency of motion. The only assumption we use is recurrence. We can paraphrase this situation:

\[ \text{A little recurrence goes a long way.} \]

1.3. Contents of the paper. The set up and main results appear in Section 2. The proofs of the main results are based on a new mechanism combining geometric methods, perturbation theory, and topological techniques. Since all the constructions are rather explicit, we anticipate that this can be used in concrete applications, such as in celestial mechanics.

Section 3 describes the geometric method that is used in the paper. The perturbed system is first expressed as a slow and small perturbation of an integrable Hamiltonian, through some rescaling of the coordinates and of the time. Due to the assumptions on the geodesic flow, the unperturbed system possesses a normally hyperbolic invariant manifold whose stable and unstable manifolds intersect transversally, at various places. The transverse homoclinic intersections are used to define different ‘scattering maps’. A scattering map is defined on a subset of the normally hyperbolic invariant manifold, and encodes information on the homoclinic trajectories. For all sufficiently small perturbations – which in our case amounts to all sufficiently large energies – the normally hyperbolic invariant manifold from the unperturbed system survives to the perturbed system, and its stable and stable manifolds keep intersecting transversally. This allows us to continue the scattering
maps from the unperturbed problem to the perturbed one. The effect of a scattering map on the energy of the system can be computed very explicitly using the fact that the energy is a slow variable.

In Section 4 we show that, by interspersing the ‘outer’ dynamics, given by a scattering map, with the ‘inner’ dynamics, given by the restriction of the flow to the normally hyperbolic invariant manifold, we can arrange that the energy changes by arbitrarily large quantities, and, in particular, grows to infinity.

At the first stage, we construct some elementary building blocks for the two-map dynamics, each block consisting of one applications of a scattering map followed by a segment of a trajectory of the inner dynamics. We can compute explicitly the change of energy along such an elementary building block.

At the second stage, we show that, under appropriate conditions, we can construct sequences of elementary building blocks whose energy experiences the desired changes. We also show that these effects do not happen with one scattering map, but that is crucial to use at least two scattering maps. In particular, we show that we can arrange the choices of the scattering maps and of the corresponding blocks so that we can consistently grow the energy of the system, at an optimal rate.

These sequences of blocks determine pseudo-orbits, which are concatenation of orbits of the inner dynamics and of orbits of the outer dynamics. They do not immediately yield true orbits for the perturbed geodesic flow.

To prove the existence of true orbits, in Section 5 we apply the topological method of correctly aligned windows. Around the pseudo-orbits we construct windows that are correctly aligned, i.e., sequences of multi-dimensional rectangles that successively cross one another in a non-trivial way under the inner or the outer dynamics. A topological version of the Shadowing Lemma implies the existence of true orbits near those pseudo-orbits. We note that the main difficulty to carry the shadowing argument is that the underlying dynamical system is not hyperbolic, as there are neutral directions along the normally hyperbolic invariant manifold. Hence the customary hyperbolic shadowing methods do not apply.

The orbits constructed in our paper are very different from the orbits previously considered in the literature. The orbits we consider stay very little time near the unperturbed periodic orbits between successive homoclinic excursions. This has several technical consequences: we do not need to rely on the KAM theorem, on Aubry-Mather theory, or on averaging, in order to control the behavior of the inner dynamics (the dynamics near the closed orbits between homoclinic excursions). Avoiding the use of averaging theory allows us to consider very general perturbations. Also, we do not need to study the geometric details on how the inner dynamics is affected by the perturbation. On the other hand, we do have to rely more heavily on the scattering map, and to develop some sophisticated shadowing mechanisms that are not based on hyperbolicity but rather on topological tools.

1.4. Some other related works. Results related to the ones considered in this paper were proved in \cite{58, 59}, using the method of the separatrix map and the idea of the anti-integrable limit. See also \cite{7}. The existence of orbits whose energy grows in time at different rates, depending on the type of perturbation, was proved in \cite{22, 30, 65} for general Hamiltonian systems that slowly depend on time. Their method is quite different: under some assumptions, they identify in the frozen system two families of periodic orbits that give rise, in the time-dependent system, to orbits that jump from near one family to near the other in a manner that yields unbounded
growth of energy. A general discussion on the speed of diffusion in a priori unstable Hamiltonian systems appears in [20]; in particular, this paper considers diffusion in the planar elliptic restricted three-body problem. The diffusion phenomenon in the case of the spatial circular restricted three-body problem is considered in [17, 18].

2. Set-up and main results

2.1. The geodesic flow. In this subsection we review some well known facts on the geodesic flow, and we set up the notation.

Let $M$ be an $n$-dimensional $C^r$-smooth compact manifold and $g$ be a $C^r$ Riemannian metric on $M$, where $r \geq r_0$. (Some non-optimal values of $r_0$ are discussed in Subsection 5.7 at the end of the proofs of the main results.) The geodesic flow $\xi : TM \times \mathbb{R} \to TM$ is the flow on the tangent bundle $TM$ (of dimension $2n$), defined by

$$\xi_t(x, v) = (\xi_{x,v}(t), d\xi_{x,v}/dt(t)),$$

where $\xi_{x,v}$ is the unique geodesic with $\xi_{x,v}(0) = x$ and $d\xi_{x,v}/dt(0) = v$. To a geodesic curve $\xi$ on $M$ corresponds a trajectory of the geodesic flow given by

$$t \mapsto (\xi(t), (d\xi/dt)(t)).$$

In this paper we do not distinguish between a geodesic curve and the corresponding trajectory of the geodesic flow. Also, we do not distinguish between a parametrized curve and its image.

Since the speed along a geodesic is constant, every geodesic can be reparametrized as a unit speed geodesic. This way we reduce the study of the geodesic flow to its restriction to the unit tangent bundle $T^1M$ (of dimension $2n - 1$).

Using the standard identification between $TM$ and $T^*M$ given by $g$, we can interpret the geodesic flow as the Hamiltonian flow for the Hamiltonian $H_0 : T^*M \to \mathbb{R}$ given by

$$H_0(x, y) = \frac{1}{2} g_x(y, y).$$

On $T^*M$ we consider the standard symplectic form $\omega = dy \wedge dx$.

Since $H_0$ is independent of time, it is a conserved quantity. The energy surfaces $\Sigma_E = \{H_0 = E\}$ are invariant under $\xi_t$. We will denote by $\xi_{E,t}$ the flow restricted to the energy surface $\Sigma_E$. The restriction $\xi_{1/2,t}$ of $\xi_t$ to the energy manifold $\Sigma_{1/2} = \{H_0 = 1/2\}$ corresponds to the geodesic flow on $T^1M$.

It is clear that the flow restrictions to different energy surfaces are equivalent to one another and can be obtained just by a rescaling of time and momentum. The mapping $D_{\sqrt{2E}}(x, y) = (x, \sqrt{2E}y)$ – dilation along the fibers of $T^*M$, which is a well defined operation, because the fibers are linear spaces – gives a diffeomorphism between $\Sigma_{1/2}$ and $\Sigma_E$. Furthermore

$$\xi_{E,t} \circ D_{\sqrt{2E}} = D_{\sqrt{2E}} \circ \xi_{1/2,t},$$

$$\xi_t \circ D_{\sqrt{2E}} = D_{\sqrt{2E}} \circ \xi_{\sqrt{2E}t}.$$  

Explicitly, the trajectory of the geodesic flow $\xi_E = (\xi_E^x, \xi_E^y)$ at energy $E$ is related to the trajectory of the geodesic flow $\xi_{1/2} = (\xi_{1/2}^x, \xi_{1/2}^y)$ at energy $1/2$ by the following formula:

$$(\xi_E^x(t), \xi_E^y(t)) = (\xi_{1/2}^x(\sqrt{2E} \cdot t), \sqrt{2E} \cdot \xi_{1/2}^y(\sqrt{2E} \cdot t)).$$
Below we will regard the geodesic flow as an unperturbed dynamical system, to which we will apply an external perturbation.

2.2. Assumptions on the geodesic flow. We make the following assumptions on the geodesic flow.

A1. We assume that there exists a closed, unit speed geodesic $\lambda$, which is a hyperbolic periodic orbit for the geodesic flow on $T^1M$.

A2. We assume that there exists a unit speed geodesic $\gamma$ which is a transverse homoclinic orbit to $\lambda$ for the geodesic flow on $T^1M$.

Note that condition A1 means that $\lambda$ has stable and unstable manifolds $W^s(\lambda)$ and $W^u(\lambda)$, of dimension $n$, in $T^1M$. The fact that a geodesic $\xi$ is unit speed means that $g(\frac{d\xi}{dt},\frac{d\xi}{dt}) = 1$. Without loss of generality, we can assume that the period of the geodesic $\lambda$ from assumption A1 is 1.

The transversality condition A2 means that $T_zW^s(\lambda) + T_zW^u(\lambda) = T_z(T^1M)$, for all $z \in \gamma$.

We note that there are many manifolds for which the conditions A1 and A2 are satisfied for an abundance of Riemannian metrics; a partial review of the existing results is provided in [19]. As an example, any surface of genus 2 or higher, with any $C^{2+\delta}$ metric, $\delta > 0$, has hyperbolic geodesics with transverse homoclinic connections [43]. A very general result was obtained in [15], showing that on any closed manifold $M$ with $\dim(M) \geq 2$ the set of $C^\infty$ Riemannian metrics whose geodesic flow contains a non-trivial hyperbolic basic set is $C^2$-open and $C^\infty$-dense. This implies that the hypothesis A1 and A2 from above hold for a $C^2$-open and $C^\infty$-dense set of Riemannian metrics.

Note that the energy $H_0$ along both the hyperbolic orbit $\lambda$ in A1 and the homoclinic orbit $\gamma$ in A2 equals $1/2$, but similar orbits exist in all energy surfaces because of (2.3).

2.3. Coupling the geodesic flow with an external dynamical system. Next we will describe a class of perturbations of the geodesic flow for which we will show the existence of orbits with unbounded growth of energy over time, as well as of symbolic dynamics.

We first describe an external dynamical system which we couple with the geodesic flow through a time dependent potential. Let $X : N \to TN$ be a $C^1$-smooth vector field on a compact manifold $N$. Let $\chi_t$ be the flow on $N$ associated to $X$. Let $\theta_0 \in N$ and let $\chi_t(\theta_0)$ be an integral curve to $X$ with the initial condition $\theta_0 \in N$.

Let

$$V = \{ V : M \times N \to \mathbb{R} \mid V \text{ is a } C^r \text{ - differentiable in } q \text{ and } C^1 \text{-differentiable in } \theta \}.$$ We think of $V \in V$ as a potential depending on the parameter $\theta$ evolving in $N$. For every $V \in V$ we consider a parameter-dependent, time-dependent Hamiltonian $H_{\theta_0} : T^*M \times \mathbb{R} \to \mathbb{R}$ given by

$$(2.4) \quad H_{\theta_0}(x,y,t) = H_0(x,y) + V(x,\chi_t(\theta_0)).$$

2.4. Assumptions on the external dynamical system. We now describe some conditions on the external dynamical system.

Given $(N,\chi)$, a point $\theta_0 \in N$ is said to be uniformly recurrent (or syndetically recurrent) if for every open neighborhood $U$ of $\theta_0$ there exists $T > 0$ such that,
every interval \((a, b)\) with \(b - a > T\) contains a time \(t\) with \(\chi_t(\theta_0) \in U\). That is, a uniformly recurrent point is one which is recurrent with ‘bounded return times’.

The flow \(\chi\) on \(N\) is said to be minimal if every orbit is dense in \(N\).

Since \(N\) is compact, if the flow in minimal then for every open set \(O \subseteq N\) there exists \(T \geq 0\), depending on \(O\), such that \(\bigcup_{t \in [0, T]} \chi_t(O) = N\).

Since \(N\) is compact, then there always exists a point \(\theta_0 \in N\) that is uniformly recurrent for \((N, \chi)\). This follows from the Poincaré Recurrence Theorem.

Moreover, if \((N, \chi)\) is minimal then every point \(\theta \in N\) is uniformly recurrent.

Conversely, if \(\theta_0 \in N\) is uniformly recurrent then its orbit closure is a minimal set.

If every point \(\theta \in N\) is uniformly recurrent then \(N\) is the disjoint union of its minimal subsystems, in which case \(N\) is called semi-simple. See [33].

The following two alternative conditions will be used in the statements of the main results.

\(
A3. \text{ We assume that the flow } \chi_t \text{ has a uniformly recurrent point } \theta_0 \in N \text{ with } X(\theta_0) \neq 0. \\
A3'. \text{ We assume that the flow } \chi_t \text{ is minimal on } N.
\)

2.5. **Statement of the results.** The following result provides the existence of trajectories with unbounded growth of energy for the system \((2.4)\).

**Theorem 2.1.** Let \(g\) be a Riemannian metric on \(M\) satisfying the conditions \(A1, A2\), and let \((N, \chi)\) be an external dynamical system satisfying \(A3\). Then there exist \(\theta_0 \in N\), and \(V'\), \(C^r\)-open and \(C^r\)-dense in \(V\) with respect to the \(C^r\)-topology, \(r \geq r_0\), such that, for every \(V \in V'\), the system \((2.4)\) has a solution with \(H_{\theta_0}(x(t), y(t), t) \geq A t + B\), for some \(A, B \in \mathbb{R}\) with \(A > 0\), and for all \(t\) sufficiently large.

For example, the condition \(A3'\) in Theorem 2.1 is automatically satisfied if \(X(\theta) \neq 0\) for all \(\theta\). Note that in this case the only requirement on the flow \(\chi\) on \(N\) is that it does not have any fixed points. This condition is necessary since we want, as we shall see in the argument in Subsection 4.4, that the flow line \(\chi_t(\theta_0)\) leaves some neighborhood of the point \(\theta_0\) before it returns again to that neighborhood.

**Theorem 2.2.** Let \(g\) be a Riemannian metric on \(M\) satisfying the conditions \(A1, A2\), and let \((N, \chi)\) be an external dynamical system satisfying \(A3'\). Then there exists a set \(V'\), open and dense in \(V\) with respect to the \(C^r\)-topology, \(r \geq r_0\), such that, for every \(V \in V'\), and every \(\theta_0 \in N\), the system \((2.4)\) has a solution for which the energy \(H_{\theta_0}(x(t), y(t), t)\) grows linearly to infinity as \(t\) tends to infinity, i.e., \(H_{\theta_0}(x(t), y(t), t) \geq A t + B\) for some \(A, B \in \mathbb{R}\) with \(A > 0\), and for all \(t\) sufficiently large.

The hypothesis \(A3\) in Theorem 2.1 requires one to choose the parameter \(\theta_0\) to be a non-trivial uniformly recurrent point, and leads an unstable trajectory corresponding to that particular choice, while the hypothesis \(A3'\) in Theorem 2.2 allows the parameter \(\theta_0\) to be arbitrary.

We also note that Theorem 2.2 remains valid under the weaker assumption that the flow \(\chi_t\) on \(N\) is semi-simple.

The linear growth rate \(H_{\theta_0}(x(t), y(t), t) \approx t\) in Theorem 2.1 and Theorem 2.2 is optimal. Indeed, the energy \(H_{\theta_0}(x(t), y(t), t)\) cannot grow in time faster than
linearly, as we can easily show. By (2.4), we have that
\[
\frac{d}{dt} H_{\theta_0}(x(t), y(t), t) = \frac{\partial V}{\partial x}(x(t), \chi_t(\theta_0)) X'(\chi_t(\theta_0)),
\]
which is bounded due to the compactness of \( M \) and \( N \).

In Subsection 4.2, we provide an explicit condition \( A4 \) that ensures \( V \in V' \), as in Theorem 2.1 and Theorem 2.2. We emphasize here that this condition amounts to an explicit computation (4.14) that is verifiable in concrete systems. The potentials \( V \) satisfying this condition form a \( C^2 \)-open and \( C^\infty \)-dense set in \( V \). The condition \( A4 \) depends on a hyperbolic closed geodesic and on a pair of geometrically distinct homoclinic orbits associated to it. A generic geodesic flow has infinitely many homoclinic orbits to the same hyperbolic closed geodesic. Of course, for the main results we only need to verify the condition for just a single pair of homoclinic orbits.

Besides orbits whose energy grows unboundedly in time, there also exist orbits whose energy makes chaotic excursions, i.e., they follow any prescribed energy path. In other words, the system admits symbolic dynamics.

**Theorem 2.3.** Assume the conditions of Theorem 2.1 and a potential \( V \in V' \), with the set \( V' \) as in the statement of that theorem. Let \( E_\star \) be sufficiently large. There exists \( C > 0 \), depending on the potential \( V \), such that, for any function \( E : [0, \infty) \to [E_\star, \infty) \), such that \( |\mathcal{E}'| \leq C \), there exist \( \theta_0 \in N \), a time reparametrization \( T : [0, \infty) \to [0, \infty) \), and a solution \((x(t), y(t))\) of the system (2.4), and a constant \( D > 0 \) such that
\[
|H_{\theta_0}(x(T(t)), y(T(t))) - \mathcal{E}(t)| \leq D \mathcal{E}(t)^{-1/2}.
\]

### 2.6. Some examples of applications

The conditions for the flow on \( N \) in Theorem 2.1 are very general. What makes the examples presented below more surprising is that the flow on \( N \) can have a very simple orbit structure. The results include, as a particular case, quasi-periodic forcing, in which case we have the presence of the KAM and Nekhoroshev phenomena, which prevent, or delay, the onset of linear growth of energy, for a positive measure set of orbits. This is why we single out some examples to showcase Theorem 2.2.

For systems with a rich orbit structure (e.g., horseshoes giving rise to symbolic dynamics), there are simpler arguments that show that one can get instability. Roughly, if the forcing system has essentially arbitrary orbits, we can choose initial conditions in the forcing that lead to orbits that always increase the energy. Hence, for systems with complicated orbit structure there are many results of instability. A representative paper of this line of reasoning is [57]. Of course, even in the case when simpler mechanisms apply, the orbits constructed here are different, since we rely on the homoclinic excursions and not on riding the external forcing.

#### 2.6.1. Perturbation of the geodesic flow by a quasi-periodic potential

We consider the particular case when \( N = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \), and the flow on \( N \) is a linear flow of rationally independent frequency vector \( \nu \), i.e., \( \nu \cdot k \neq 0 \) for all \( k \in \mathbb{Z}^d \). Such a flow can be written as \( \chi_t(\theta_0) = \theta_0 + \nu \cdot t \), for any initial point \( \theta_0 \in \mathbb{T}^d \); for simplicity, we let \( \theta_0 = 0 \). The flow \( \chi_t \) is minimal on \( N = \mathbb{T}^d \).

The corresponding perturbed dynamical system is described by the time-dependent Hamiltonian \( H : T^* M \times \mathbb{R} \to \mathbb{R} \) given by
\[
H(x, y, t) = H_0(x, y) + V(x, \nu t).
\]
As a consequence of Theorem 2.1, we obtain that, for a generic set of \( C^r \)-smooth potentials \( V : M \times \mathbb{T}^d \to \mathbb{R} \), the system (2.5) has solutions \((x(t), y(t), t)\) for which the energy \( H(x(t), y(t), t) \) grows linearly to infinity as \( t \) tends to infinity. Note that the frequency \( \nu \) is not required to be Diophantine, as in [19]. In addition to recovering the results from [19] under weaker conditions, we also obtain the existence of orbits whose energy grows at an optimal speed. Again, we point out that the orbits constructed in this paper are very different from those in [19], where the orbits stay long times near the closed geodesics.

When \( d = 1 \), the flow \( \chi_t \) describes a periodic motion on \( \mathbb{T}^1 \), thus we obtain, as a particular case, the geodesic flow perturbed by a generic, periodic potential considered in, e.g., [51, 7, 16, 39].

2.6.2. Flows on Lie groups. In this section, we describe some other examples of external dynamical systems \((N, \chi_t)\) that can be used in Theorem 2.2. These examples are mild enough so that the mechanisms in [57] do not apply but nevertheless there is no averaging theory for them.

Note that we can interpret the example in Subsection 2.6.1 as a flow on a Lie group generated by a left-invariant vector field. Given a compact Lie group \( N \), recall that a left translation on \( N \) is a map \( L_g : N \to N \) given by \( L_g(\theta) = g\theta \) for some \( g \in N \). A vector field \( X \) on \( N \) is called left-invariant if \( X \) is invariant with respect to all left-translations, i.e. \( (L_g)_* (X) = X \) for all \( g \in N \). Let \( \chi_t \) be the flow of \( X \) on \( N \). It is well known that each integral curve of \( \chi_t \) is homeomorphic to the integral curve through the identity, that is, \( \chi_t(\theta) = \theta \cdot \chi_t(e) \). The flow \( \chi_t \) is minimal if and only if \( N \) is Abelian. However, every compact Abelian Lie group is a torus. Thus, our example in Subsection 2.6.1 has a natural interpretation as a left-invariant flow on a compact, Abelian, Lie group.

2.6.3. Horocycle flows. Let \( P \) be a compact connected manifold with a Riemannian metric of negative curvature, and \( \xi_t \) be the geodesic flow restricted to the unit tangent bundle \( T^1 P \). The horocyle flow is the unit speed flow on \( T^1 P \) whose orbits (referred as horocycles) are the strong stable manifolds \( \{W^{ss}(z)\} \) of the geodesic flow

\[
W^{ss}(z) = \{ z' \in T^1 M \mid \lim_{t \to \infty} d(\xi_t(z'), \xi_t(z)) = 0 \}.
\]

As a particular case, assume that \( P \) is a surface of constant negative curvature. The universal covering space is the Poincaré upper half-plane \( \mathbb{H} \) with the Poincaré metric denoted \( ds \). The geodesics in \( \mathbb{H} \) are vertical lines and circles orthogonal to the real axis, and the horocycles are horizontal lines and circles tangent to the boundary. The orientation-preserving isometries of \( \mathbb{H} \) are the linear fractional transformations, i.e., the elements of

\[
PSL(2, \mathbb{R}) = \{ z \mapsto \frac{az + b}{cz + d} \mid ad - bc = 1 \}.
\]

We can identify \( P \) as \( PSL(2, \mathbb{R})/\Gamma \) where \( \Gamma \) is a discrete co-compact subgroup\(^1\) of \( PSL(2, \mathbb{R}) \). The geodesic flow is given by

\[
\xi_t(z) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} z \Gamma
\]

\(^1\)\( \Gamma \) is co-compact if \( PSL(2, \mathbb{R})/\Gamma \) is compact.
and the horocycle flow
\[ \chi_t(z) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} z \Gamma. \]

When the manifold \( P \) has (variable) negative curvature, the universal Riemannian covering surface is the upper half-plane with metric \( f^2 ds \) where \( f \) is a non-zero function. Then \( P \) can be regarded as the quotient of this space via a discrete co-compact group of isometries.

The geodesic flow on the unit tangent bundle is Anosov. Hence the unit tangent bundle is foliated by stable and unstable manifolds. When \( P \) is a surface, the stable and unstable manifolds are 1-dimensional. The horocycle flow is the motion at unit speed along the stable/unstable manifolds.

Hedlund [36] proved that the horocycle flow on a compact surface of constant negative curvature is minimal. Furstenberg [32] showed that it is uniquely ergodic (i.e., it admits a unique ergodic measure). B. Marcus proved the same result for compact surfaces of variable negative curvature [50]. Results on the minimality of the horocycle flow in higher dimensions were obtained by Eberlein, see e.g. [25]. We note that for surfaces the horocycle foliation is \( C^2 - \varepsilon \) [31] but in general not \( C^2 \). This is enough for our result as argued in Subsection 5.7.

Hence, an interesting class of examples of external dynamical systems \((N, \chi)\) that can be used in Theorem 2.1 comes from horocycle flows on compact connected manifolds of negative curvature. We remark that horocycle flows do not determine perturbations of a fixed frequency (as in the quasi-periodic case).

2.6.4. Flows on homogeneous spaces. Let \( G \) be a Lie group, \( \Gamma \) a discrete subgroup, \( g : \mathbb{R} \to \Gamma \) a one-parameter subgroup, and \( \chi : \mathbb{R} \times G/\Gamma \to G/\Gamma \) be the \( G \)-induced flow, given by \( \chi_t(z\Gamma) = (g(t)z)\Gamma \). \( G/\Gamma \) with the \( G \)-induced flow is a homogeneous space. An element \( g \in G \) is unipotent if the adjoint \(^2\) \( \text{Ad}(g) \) of \( g \) is a unipotent matrix, i.e., 1 is its only eigenvalue. If \( \chi_t \) is unipotent for each \( t \), then the flow \( \chi \) is called a unipotent flow.

Ratner classification theory [60] asserts that, in the case when \( \Gamma \) is co-compact, and the flow \( \chi \) is unipotent, if \( \chi \) is uniquely ergodic then it is minimal. If \( \text{Vol}(G/\Gamma) < \infty \), where \( \text{Vol} \) denotes the Haar measure induced by \( \chi \), then the properties of minimality and unique ergodicity are equivalent.

Hence, another interesting class of examples of external dynamical systems \((N, \chi)\) that can be used in Theorem 2.1 are unipotent flows on homogeneous spaces of finite volume which are uniquely ergodic. Some specific examples appear in, e.g., [62, 54, 66].

2.6.5. Celestial mechanics and astrodynamics. We consider the Kepler problem, described by the Hamiltonian \( H(y, x) = \frac{1}{2}|y|^2 - \frac{1}{|x|} \). The solutions are conic sections or collision orbits. The regularized Keplerian motions for \( H < 0 \) can be lifted to trajectories of the geodesic flow on \( S^2 \). Let us consider this problem as a model for the motion of a satellite around the Earth. Following [14], the set of \( C^\infty \) Riemannian metrics on \( S^2 \) whose geodesic flow contains a non-trivial hyperbolic basic set is open and dense in the \( C^2 \) topology. Thus, we can choose a Riemannian metric that approximates the standard \( S^2 \)-metric and for which there exists a hyperbolic closed

\(^2\) The adjoint \( \text{Ad}(g) \) of \( g \) is the derivative of the map \( \Psi_g : G \to G, z \mapsto gzg^{-1} \) at \( e \), i.e., \( \text{Ad}(g) = (d\Psi_g)_e : T_eG \to T_eG \), where \( d \) is the differential and \( T_eG \) is the tangent space at the origin \( e \).
geodesic with transverse homoclinic connection, satisfying conditions $A1$, $A2$. A closed geodesic will correspond to a closed orbit around the Earth. The motion of Moon and Sun can be regarded as a quasi-periodic forcing, as in $A3$. It seems possible that the method used to prove Theorem 2.2, which is constructive, can be adapted to this example in order to design explicit maneuvers on how to move, in specific ways, the satellite from one closed orbit to another around the Earth, or to increase the size and shape of a satellite orbit. The fact that an approximation to the metric on $S^2$ is used in this argument, will result in some errors that can be corrected by low energy maneuvers. Satellite trajectory repositioning is very useful in astrodynamics, see, e.g., [56]. A paper of related interest is [1].

3. Geometric method

In this section we review briefly the geometric method to study perturbations of the geodesic flow developed in [16]. This method is based on the theory of normally hyperbolic invariant manifolds and on the scattering map. We will take the general setup from [16] but we will make substantial modifications.

We consider the parameter-dependent Hamiltonian $H : T^*M \times N \to \mathbb{R}$ given by $H(x, y, \theta) = H_0(x, y) + V(x, \theta)$. The Hamilton equations and the parameter evolution equation are

$$
\begin{align*}
\frac{dx}{dt} &= \frac{\partial H_0}{\partial y}, \\
\frac{dy}{dt} &= -\frac{\partial H_0}{\partial x} - \frac{\partial V}{\partial x}, \\
\frac{d\theta}{dt} &= X(\theta).
\end{align*}
$$

We write the corresponding flow $\psi : T^*M \times N \times \mathbb{R} \to T^*M \times N$ as

$$
\psi_t(x, y, \theta) = (\xi_t(x, y, \theta), \chi_t(\theta)),
$$

where $\chi_t$ is the flow defined by the vector field $X$ on $N$.

3.1. Normal hyperbolicity. First we consider the unperturbed system, which is described by the Hamiltonian $H_0$ given by (2.1). Each energy manifold $\Sigma_E = \{(x, y) \mid H_0(x, y) = E\}$ is invariant under the geodesic flow $\xi_t$ (now viewed as the Hamiltonian flow of $H_0$). We will denote by $\xi_{E,t} = (\xi^x_{E,t}, \xi^y_{E,t})$ a trajectory of the geodesic flow lying on $\Sigma_E$. Due to the rescaling property of the geodesic flow, the assumptions $A1$ and $A2$ imply that for each energy level $E$ there exists a periodic orbit $\lambda_E$ that is hyperbolic in $\Sigma_E$, and a transverse homoclinic orbit $\gamma_E$ to $\lambda_E$ in $\Sigma_E$.

We consider a sufficiently large initial energy level $E_* \geq 0$ (to be specified later in the argument), and we define the 2-dimensional cylinder $\Lambda_0 = E_{\geq E_*} \lambda_E$ in $T^*M$. Note that $\Lambda_0$ is a manifold with boundary but the flow is tangent to the boundary. This is a normally hyperbolic invariant manifold for the Hamiltonian flow on $T^*M$, whose stable and unstable manifolds are given by $W^s(\Lambda_0) = E_{\geq E_*} W^s(\lambda_E)$ and $W^u(\Lambda_0) = E_{\geq E_*} W^u(\lambda_E)$, respectively. The stable and unstable manifolds of $\Lambda_0$ intersect transversally along the 2-dimensional homoclinic manifold $\Gamma_0 = E_{\geq E_*} \gamma_E$ in $T^*M$. 
3.2. **Scaled coordinates.** We now rescale the coordinates \((x, y)\) and the time \(t\) so that, for high energies, the flow corresponding to the rescaled Hamiltonian is a small and slow perturbation of the geodesic flow. For \(E_*\) sufficiently large we introduce a new parameter \(\varepsilon = 1/\sqrt{E_*}\); we note that \(E_* \to \infty\) if and only if \(\varepsilon \to 0\).

The rescaled coordinates are \((q, p)\) defined by \(q = x, p = \varepsilon y\), and the rescaled time \(s\) is given by \(s = t/\varepsilon\). The variable \(\theta\) remains unchanged. The parameter-dependent Hamiltonian in these new variables is

\[
H_\varepsilon(q, p, \theta) = H_0(q, p) + \varepsilon^2 V(q, \theta) = \varepsilon^2 H(x, y, \theta).
\]

The corresponding Hamilton equations and the parameter evolution equation are

\[
\begin{align*}
\frac{dq}{ds} &= \frac{\partial H_0}{\partial p}, \\
\frac{dp}{ds} &= -\frac{\partial H_0}{\partial q} - \varepsilon^2 \frac{\partial V}{\partial q}, \\
\frac{d\theta}{ds} &= \varepsilon X(\theta).
\end{align*}
\]

The corresponding flow \(\psi_\varepsilon^s\) is of the form \(\psi_\varepsilon^s = (\xi^s(q, p, \theta), \chi^s(\theta))\).

When \(\varepsilon \to 0\), the flow \(\chi^s_{\varepsilon}\) approaches the constant flow \(\text{id}_\theta\) on \(N\), and the flow \(\xi^s_{\varepsilon}\) approaches the flow \(\xi_s\) in the \(C^{r-1}\) topology. The limiting flow \(\psi_s = (\xi_s, \text{id}_\theta)\) on the extended phase space \(T^* M \times N\) has a normally hyperbolic invariant manifold \(\tilde{\Lambda}_0 = \bigcup_{E \geq E_*} \lambda_E \times N\), since the exponential expansion rates of \(\xi_s\) are larger than those of \(\text{id}_\theta\). The stable and unstable manifolds of \(\tilde{\Lambda}_0\) are given by \(W^s(\tilde{\Lambda}_0) = \bigcup_{E \geq E_*} W^s(\lambda_E) \times N\) and \(W^u(\tilde{\Lambda}_0) = \bigcup_{E \geq E_*} W^u(\lambda_E) \times N\), respectively. Obviously, \(W^u(\tilde{\Lambda}_0)\) and \(W^s(\tilde{\Lambda}_0)\) intersect transversally along the \((2 + d)\)-dimensional homoclinic manifold \(\tilde{\Gamma}_0 = \bigcup_{E \geq E_*} \gamma_E \times N\) in \(T^* M \times N\).

Now we refer to the theory of normal hyperbolicity (see [38, 27, 5]) that shows that the invariant manifolds, which were identified in the limiting system, survive as locally invariant manifolds for the perturbed system. A very explicit proof of this can be found in [19]. The locally invariant manifolds are exactly invariant for a modified system constructed explicitly in [19]. The modifications are supported on \(E \leq E_0\). When we refer to stable and unstable manifolds, we mean the stable and unstable manifolds of the extended system. Since we will be considering orbits whose energy stays large enough (in particular in regions where the extended system agrees with the original system, the orbits we construct will also be orbits of the original system.

**Remark 3.1.** When the system is differentiable enough and the perturbation is periodic or Diophantine, it is easy to remark [16] that the invariant manifold will contain KAM tori that act as boundaries, so that the manifold is in fact invariant. Of course, the argument above does not require any differentiability and works for general perturbations.

Since \(\varepsilon\) enters both in the size of the perturbation parameter and in the time reparametrization, in order to apply the standard normally hyperbolicity theory one can rewrite the perturbed Hamiltonian as a two parameter problem, with one parameter for the size of the perturbation and the other for the time change, and
prove the persistence of the normally hyperbolic, locally invariant manifold from the unperturbed case to the perturbed case for all small enough sizes of the perturbation and uniformly in the time change parameter (see [16, 19]). In the end, it follows that, for all sufficiently small ε (and, implicitly, for all sufficiently large $E_\varepsilon$) the manifold $\Lambda_0$ can be continued to a normally hyperbolic locally invariant manifold $\Lambda_\varepsilon = \Lambda_\varepsilon \times N$ for the flow $\psi_\varepsilon^t$.

Moreover, the manifold $\Lambda_\varepsilon$ depends smoothly on the parameter $\varepsilon$, in the sense that there exists a $C^{r-1}$-smooth parametrization $\tilde{k}_\varepsilon : \tilde{\Lambda}_0 \to \Lambda_\varepsilon$ of the type $\tilde{k}_\varepsilon = (k_\varepsilon, \text{id}_\theta)$, which depends $C^{r-2}$-smoothly on the parameter, such that $k_\varepsilon(\Lambda_0) = \Lambda_\varepsilon$ (and, in particular, $k_\varepsilon(\Lambda_0) = \Lambda_\varepsilon$).

In addition, there exist stable and unstable manifolds $W^u(\Lambda_\varepsilon)$ and $W^s(\Lambda_\varepsilon)$ that vary $C^{r-1}$-smoothly with $\varepsilon$. Similarly, there exist $C^{r-1}$-smooth parametrizations $\tilde{k}_\varepsilon^u : W^u(\Lambda_0) \to W^u(\Lambda_\varepsilon)$, $\tilde{k}_\varepsilon^s : W^s(\Lambda_0) \to W^s(\Lambda_\varepsilon)$ of the local stable and unstable manifolds of $\Lambda_\varepsilon$, respectively, which agree with $\tilde{k}_\varepsilon$ on $\Lambda_0$.

Since transversality is an open condition, then $W^u(\Lambda_\varepsilon)$ and $W^s(\Lambda_\varepsilon)$ intersect transversally along a locally unique homoclinic manifold $\tilde{\Gamma}_\varepsilon = \Gamma_\varepsilon \times N$, for all $\varepsilon$ sufficiently small (and so for all sufficiently large $E_\varepsilon$).

3.3. Action-angle coordinates. For the unperturbed system, on the normally hyperbolic invariant manifold $\Lambda_0$ we can put a system of action-angle coordinates $(J, \phi)$, where the action coordinate is $J = \sqrt{2E}$, and the angle coordinate $\phi \in T^1$ is symplectically conjugate with $J$, i.e., $dJ \wedge d\phi = (dy \wedge dx)|_{\Lambda_0}$. The unperturbed Hamiltonian is integrable on $\Lambda_0$, and it takes the following form in the action-angle coordinates

$$H_0(\phi, J) = \frac{1}{2} J^2. \tag{3.5}$$

The unperturbed Hamiltonian flow on $\Lambda_0$ takes the form $J(t) = J, \phi(t) = \phi_0 + Jt$. Thus $\Lambda_0$ is foliated by invariant tori $T_E = \{(J, \phi) \mid J = \sqrt{2E}, \phi \in T^1\}$ corresponding to each energy level $H_0 = E \geq E_\varepsilon$.

For the perturbed system, the action-angle coordinate system $(J, \phi)$ on $\Lambda_0$ can be continued via $k_\varepsilon$ to an action-angle coordinate system $(J_\varepsilon, \phi_\varepsilon)$ on $\Lambda_\varepsilon$. In these coordinates, the perturbed Hamiltonian function restricted to $\Lambda_\varepsilon$ takes the form

$$H_\varepsilon(\phi_\varepsilon, J_\varepsilon, \theta) = \frac{1}{2} J_\varepsilon^2 + \varepsilon^2 V(\phi_\varepsilon, J_\varepsilon, \theta), \tag{3.6}$$

where $V(\phi_\varepsilon, J_\varepsilon, \theta)$ is obtained by expressing $V(\phi, J, \theta)$ in these coordinates. The Hamilton equations and the parameter evolution equation are:

$$\frac{d\phi_\varepsilon}{ds} = J_\varepsilon + \varepsilon^2 \frac{\partial V}{\partial J_\varepsilon}$$

$$\frac{dJ_\varepsilon}{ds} = -\varepsilon^2 \frac{\partial V}{\partial \phi_\varepsilon}$$

$$\frac{d\theta}{ds} = \varepsilon X(\theta). \tag{3.7}$$

Note that, in general, the foliation of $\Lambda_0$ by invariant tori $T_E$ in the unperturbed case does not survive to the perturbed case. For a perturbation driven by a general external flow $\chi$ on $N$ we cannot apply the KAM theorem since the perturbation affecting the tori is not necessarily periodic/quasiperiodic. For the same reason, we
cannot apply averaging theory as in [34], since we cannot obtain level sets of the action that remain almost-invariant for sufficiently long time.

3.4. The scattering map. A key tool to study the dynamics of a normally hyperbolic invariant manifold with a transverse homoclinic manifold is the scattering map [16]. The role of the scattering map is to relate the past asymptotic trajectory of an orbit in the homoclinic manifold to its future asymptotic trajectory. An extended study of the scattering map and its geometric properties can be found in [21].

Here we briefly recall the construction of the scattering map. We consider a flow on some manifold $P$. Let $\Lambda$ be a normally hyperbolic invariant manifold for the flow, with the unstable and stable manifolds $W^u(\Lambda)$ and $W^s(\Lambda)$ intersecting transversally along a homoclinic manifold $\Gamma$. This means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $z \in \Gamma$, we have

$$T_z \Gamma = T_z W^u(\Lambda) \cap T_z W^s(\Lambda).$$

(3.8)

By the normal hyperbolicity of $\Lambda$, $W^u(\Lambda)$ is foliated by 1-dimensional fibers $W^u(z)$, $z \in \Lambda$, and $W^w(\Lambda)$ is foliated by 1-dimensional fibers $W^w(z)$, $z \in \Lambda$. For each $z \in W^u(\Lambda)$ there exists a unique $z^- \in \Lambda$ such that $z \in W^u(z^-)$, and for each $z \in W^s(\Lambda)$ there exists a unique $z^+ \in \Lambda$ such that $z \in W^s(z^+)$. We define the wave maps

$$\Omega^+ : W^s(\Lambda) \to \Lambda, \quad \Omega^+(z) = z^+,$$

$$\Omega^- : W^u(\Lambda) \to \Lambda, \quad \Omega^-(z) = z^-.$$

(3.9)

These maps are differentiable.

To define the scattering map, we make the additional assumption that for each $z \in \Gamma$ we have

$$T_z W^s(\Lambda) = T_z W^s(z^+) \oplus T_z (\Gamma),$$

$$T_z W^u(\Lambda) = T_z W^u(z^-) \oplus T_z (\Gamma).$$

(3.10)

By restricting $\Gamma$ to some open subset of it, if necessary, we can ensure that the restrictions of $\Omega^\pm$ to $\Gamma$ are diffeomorphisms. We define the scattering map associated to $\Gamma$ to be the diffeomorphism $S = \Omega^+ \circ (\Omega^-)^{-1}$ from $U^- := \Omega^-(\Gamma)$ in $\Lambda$ to $U^+ := \Omega^+(\Gamma)$ in $\Lambda$.

As pointed out in [21], the scattering map can be defined in an analogous way in the case of time-dependent systems. In this case, one considers a flow associated to a skew-product vector field $(Y(z, \theta), X(\theta))$ on a product manifold $P \times N$, where the skew product vector field is assumed to be close to an autonomous vector field, i.e.,

$$\|Y(z, \theta) - Y_0(z)\|_{C^r} \ll 1$$

for some vector field $Y_0(z)$ on $N$.

Assume that there exists a normally hyperbolic invariant manifold $\Lambda_0$ in $P$ for the flow of $Y_0(z)$. Let $S_0$ be the scattering map associated to a homoclinic channel $\Gamma_0$. Assuming that the exponential rates of the flow on $N$ are smaller than the exponential rates for the flow on $P$, it follows that $\Lambda_0 \times N$ is normally hyperbolic for the product flow of $(Y_0(z), X(\theta))$, that $\Gamma_0 \times N$ is a homoclinic channel, and the corresponding scattering map $\tilde{S}_0$ is a product of the form $\tilde{S}_0(z, \theta) = (S_0(z), \theta)$. 
By the theory of normal hyperbolicity, there exists a normally hyperbolic invariant manifold close to $\Lambda \times N$ in $P \times N$ for the flow of $(Y(z, \theta), X(\theta))$, provided $Y(z, \theta)$ is sufficiently close to $Y_0(z)$, in the sense described above. There also exist a homoclinic channel close to $\Gamma \times N$. The corresponding scattering map takes the skew product form $\tilde{S}(z, \theta) = (S(z, \theta), \theta)$.

3.5. The scattering map for the unperturbed geodesic flow. In the case of the unperturbed geodesic flow, described by the Hamiltonian $H_0$, the condition (3.8) is satisfied due to the assumption A2, and the condition (3.10) follows from the fact that $\Lambda_0$ is foliated by the invariant tori $T_\varepsilon$. The scattering map preserves each of these invariant tori and it only changes the phase $\phi$ along each torus by an amount $a$ that is independent of the torus. The scattering map $S_0: U^-_0 \rightarrow U^+_0$ is expressed relative to the $(J, \phi)$-coordinates on $\Lambda_0$ by

$$(3.11) \quad S_0(J^-, \phi^-) = (J^+, \phi^+ - \phi^- + a),$$

where $J^- = J^+$ and the phase shift $a = \phi^+ - \phi^-$ is independent of the point $z^- = z^-(\phi^-, J^-)$ in $U^-_0$, as it only depends on the homoclinic manifold $\Gamma_0$.

Remarkably, in this case the scattering map can be globally defined as a continuous map on the whole of $\Lambda_0$, as it has no monodromy (see [21]). The continuation, however, is rather subtle because when the base point moves along a non-contractible closed curve in $\Lambda$ the point of homoclinic intersection changes. This causes, as observed already in [16] that after perturbations, this global definition may be impossible. When we continue the scattering map along a non-contractible closed curve we may end up with a different map.

3.6. The scattering map for the perturbed geodesic flow. We now describe the scattering map for the system (3.4). We stress that we use the notation $^\sim$ to denote the variables in the extended system. That is, $^\sim$ refers to adding the extra variable $\theta \in N$.

It is important to note that, if the dynamics in $N$ has a small growth rate – which happens in our case for small enough $\varepsilon$ because the dynamics is given by the vector field $\varepsilon X$ – then we can apply the theory of scattering map for non-autonomous systems recalled in Subsection 3.4.

As described in Subsection 3.2, $\bar{\Lambda}_\varepsilon = \Lambda_\varepsilon \times N$ is a normally hyperbolic invariant manifold for the extended dynamics given by $\psi_\varepsilon$. Furthermore, $W^s(\bar{\Lambda}_\varepsilon) = W^s(\Lambda_\varepsilon) \times N$ is the stable manifold of $\bar{\Lambda}_\varepsilon$ under the extended dynamics. We also have that $W^s(z, \theta) = W^s(z) \times \{\theta\}$. Analogously for the unstable manifold.

Hence, if $\Gamma_\varepsilon$ is a homoclinic manifold satisfying (3.8) and (3.10), we see that $\tilde{\Gamma}_\varepsilon = \Gamma_\varepsilon \times N$ will also satisfy (3.8) and (3.10) in the extended system. The wave maps (3.9) associated to $\tilde{\Gamma}_\varepsilon$ in the extended system can be written as

$$\tilde{\Omega}_{\varepsilon}^\pm(z, \theta) = (\Omega_{\varepsilon}^\pm(z, \theta), \theta).$$

Therefore, if we can associate a scattering map to $\Gamma_\varepsilon$, we can also associate a scattering map to $\tilde{\Gamma}_\varepsilon$ and we have

$$\tilde{S}_\varepsilon(z, \theta) = (S_\varepsilon(z, \theta), \theta).$$

Note that the scattering map is the identity in the $N$ component. For $\theta$ fixed, let $S_{\varepsilon, \theta}$ denote the map given by $S_{\varepsilon, \theta}(z) = S_\varepsilon(z, \theta)$. 
We can think of the scattering map either as the mapping \( \tilde{S}_\varepsilon \), or as a family of mappings \( S_{\varepsilon,\theta} \), indexed by the parameter \( \theta \). As proved in [21], the mappings \( \tilde{S}_\varepsilon \) and \( S_{\varepsilon,\theta} \) are smooth and depend smoothly on parameters.

It is proved in [21], that \( S_{\varepsilon,\theta} \) is symplectic as a mapping on a domain in \( \Lambda_\varepsilon \) if the flow is a time dependent symplectic flow. (It is also proved in [21] that \( \Lambda_\varepsilon \) is a symplectic manifold.)

In our situation, one of the consequences of the smooth dependence on parameters of \( \tilde{\Lambda}_\varepsilon \) and of its stable and unstable manifolds is that we can find regions \( U_{0}^\pm \subset \Lambda_0 \) which are independent of \( \varepsilon \), for \( 0 < \varepsilon \leq \varepsilon_0 \ll 1 \), such that \( \tilde{k}_\varepsilon(U_{0}^\pm \times N) \subseteq \tilde{U}_{\varepsilon}^\pm \), where \( \tilde{k}_\varepsilon \) is the parametrization of \( \tilde{\Lambda}_\varepsilon \) described in Subsection 3.2.

Via the parametrization \( \tilde{k}_\varepsilon \), we can consider the scattering map \( S_{\varepsilon,\theta} \) as defined from \( U_{0}^- \) to \( U_{0}^+ \), for \( \theta \in N \). That is, we can consider the scattering map as being defined between domains that are of product type and are independent of \( \varepsilon \) (of size of order \( O(1) \)).

4. Elementary building blocks for the dynamics

In this section we construct some elementary building blocks of the dynamics. Each building block is a pseudo-orbit determined by one application of the scattering map followed by the application of the inner dynamics for some time. The repeated construction of such elementary building blocks will produce a two-dynamics pseudo-orbit which intersperses the scattering map dynamics with the inner dynamics.

In Section 5.3, given a sequence of elementary building blocks, we will construct a sequence of windows which are correctly aligned by the dynamics. Afterwards, we will use the shadowing property of correctly aligned windows from Subsection 5.1 to conclude the existence of a true orbit following the sequence of elementary building blocks.

4.1. The effect of the scattering map on the scaled energy. The goal of this section is to compute the change of the energy \( H_\varepsilon \) by one of the scattering maps when \( \varepsilon \in (0,\varepsilon_0) \), for \( \varepsilon_0 \) sufficiently small. Our goal will be to obtain estimates uniform in \( \varepsilon \). The main observation is that the energy is a slow variable.

4.1.1. Preliminaries. We will be working with the scaled flow (3.4) in a fixed bounded range of scaled energies, which we choose to be \( H_\varepsilon \in [1,2] \). This determines a compact subset \( H_{-1}[1,2] \cap \tilde{\Lambda}_\varepsilon \) in \( \tilde{\Lambda}_\varepsilon \). Of course, obtaining estimates for all \( \varepsilon \in (0,\varepsilon_0) \) corresponds to letting the physical variables \((x,y)\) take values in a non-compact domain.

When we will say that some error term is bounded by a constant (or by \( O(\varepsilon^n) \)) it will mean uniformly in that compact set.

Denote \( \psi_\varepsilon^s(\tilde{z}_\varepsilon) \equiv (\xi_\varepsilon^s(q(\tilde{z}_\varepsilon)),\xi_\varepsilon^s(p(\tilde{z}_\varepsilon)),\chi_\varepsilon^s(\theta)) \) the trajectory of the scaled Hamilton equations (3.4) with initial condition \( \tilde{z}_\varepsilon = (\tilde{z}_\varepsilon,\theta) \in T^*M \times N \).

We have:

\[
\frac{d}{ds}H_\varepsilon(\psi_\varepsilon^s(\tilde{z}_\varepsilon)) = \varepsilon^3\nabla_\theta V(\xi_\varepsilon^s(q(\tilde{z}_\varepsilon)),\chi_\varepsilon^s(\theta)) \cdot X(\chi_\varepsilon^s(\theta))
\equiv \varepsilon^3D_X V(\xi_\varepsilon^s(q(\tilde{z}_\varepsilon)),\chi_\varepsilon^s(\theta)).
\]
Here we regard $\frac{d}{ds}H_\varepsilon$ as a functional acting on the solution curves of the Hamiltonian equations (3.4) in the extended phase space.

We will compute the leading term in $\varepsilon$ of the change in energy by a scattering map in the perturbed equation (see Subsection 3.6).

In the rest of the subsection, we will consider a fixed homoclinic intersection $\tilde{\Gamma}_\varepsilon$, and we will denote by $\tilde{S}_\varepsilon$ the corresponding scattering map.

The scattering map is defined from some domain to some range in $\tilde{\Lambda}_\varepsilon$. Since the manifold $\tilde{\Lambda}_\varepsilon$ depends on $\varepsilon$, it is convenient to reduce the scattering map to some manifold which is independent of $\varepsilon$. We use that the manifold $\tilde{\Lambda}_\varepsilon$ is diffeomorphic to the manifold $\Lambda_0$ via the parametrization $\tilde{k}_\varepsilon$ mentioned in Subsection 3.2. It is important to note that $\tilde{k}_\varepsilon$ is $\varepsilon$-close to the identity relative to the $C^r$-topology for $\varepsilon$ small. We will use this fact below, when we compute the leading term in the expansion with respect to $\varepsilon$ of the change of energy by the scattering map; we will be able to approximate $\tilde{k}_\varepsilon$ by the identity and incur only error terms which are subdominant. Moreover, $\tilde{k}_\varepsilon$ can be chosen so that it is symplectic [19].

As $\Lambda_0$ is foliated by geodesics $\lambda_E$, $E \geq E_\ast$, we can parametrize $\Lambda_0$ by the map $(E,s) \mapsto \lambda_E(s)$, where $E \geq E_\ast$ and the time $s$ is considered mod $(1/\sqrt{2E})$, i.e. $\lambda_E(s) = \lambda_E(s')$ if $s' - s \in (1/\sqrt{2E}) \cdot \mathbb{Z}$. Hence, we can parametrize $\Lambda_\varepsilon$ by $(E,s) \mapsto \tilde{k}_\varepsilon(\lambda_E(s))$.

Thus, each points $z_\varepsilon \in \Lambda_\varepsilon$ can be written as $z_\varepsilon = k_\varepsilon(\lambda_E(s))$, for some unique $E$ and $s$. Note that $E$ and $s$ depend on both the point $z_\varepsilon \in \Lambda_\varepsilon$ and on the perturbation parameter $\varepsilon$. Therefore, each point $\tilde{z}_\varepsilon \in \tilde{\Lambda}_\varepsilon$, with $\tilde{z}_\varepsilon = (z_\varepsilon,\theta)$ can be written as $\tilde{z}_\varepsilon = \tilde{k}_\varepsilon(\lambda_E(s))$, where $\tilde{\lambda}_E(s) = (\lambda_E(s),\theta)$.

Via the parametrization $\tilde{k}_\varepsilon$, instead of $\tilde{S}_\varepsilon$ we consider the reduction $\tilde{S}_\varepsilon = \tilde{k}_\varepsilon^{-1} \circ \tilde{S}_\varepsilon \circ \tilde{k}_\varepsilon : \tilde{k}_\varepsilon^{-1}(\tilde{U}_\varepsilon^-) \to \tilde{k}_\varepsilon^{-1}(\tilde{U}_\varepsilon^+)$; note that the domain and codomain of $\tilde{S}_\varepsilon$ are subsets of $\tilde{\Lambda}_0$. We note that when $\varepsilon \to 0$ the scattering map $\tilde{S}_\varepsilon$ approaches the unperturbed scattering map $S_0$ in the $C^1$ topology.

Since on $\Lambda_0$ we consider two coordinates systems, $(E,s)$ and $(J,\phi)$, we would like to make explicit the unperturbed scattering map in both coordinates. Given $S_0(z^-) = z^+$, in the action-angle coordinates, if $z^- = (J^-,\phi^-)$, $z^+ = (J^+,\phi^+)$, then $J^- = J^+$ and $\phi^- + a = \phi^+$, and in the energy-time coordinates, if $z^- = (E^-,s^-)$, $E^+ = (E^+,s^+)$, then $E^- = E^+$ and $s^- + a/\sqrt{2E} = s^+$.

### 4.1.2. The effect of the scattering map on the scaled energy

The first goal of this subsection is getting quantitative estimates on the change of scaled energy achieved by the scattering map:

$$\Delta(\tilde{z}_\varepsilon^-) := H_\varepsilon(\tilde{z}_\varepsilon^+) - H_\varepsilon(\tilde{z}_\varepsilon^-) = H_\varepsilon(\tilde{S}_\varepsilon(\tilde{z}_\varepsilon^-)) - H_\varepsilon(\tilde{z}_\varepsilon^-).$$

We will write each point $\tilde{z}_\varepsilon^-$ as $(k_\varepsilon(E,s),\theta)$, for some $E,s,\theta$. We will express the leading term of the expansion of $\Delta(\tilde{z}_\varepsilon^-)$ with respect to $\varepsilon$ in terms of the unperturbed system, and specifically in terms of the variables $(E,s,\theta)$.

**Proposition 4.1.** Let $\tilde{z}_\varepsilon^- = (k_\varepsilon(\lambda_E(s)),\theta) \in \tilde{U}_\varepsilon^-$ and let $\tilde{z}_\varepsilon^+ = \tilde{S}_\varepsilon(\tilde{z}_\varepsilon^-)$. The change of the scaled energy $H_\varepsilon$ by the scattering map from $\tilde{z}_\varepsilon^-$ to $\tilde{z}_\varepsilon^+$ is given by

$$\Delta(\tilde{z}_\varepsilon^-) = \varepsilon^3 \Delta_1(E,s,\theta) + O(|\varepsilon| \ln \varepsilon^4),$$

where $\Delta_1(E,s,\theta)$ is the leading term in the change of energy by the unperturbed scattering map.
where the leading term $\Delta_1$ in the expansion with respect to $\varepsilon$ is given by

$$\Delta_1(E, s, \theta) = \lim_{T\to \pm\infty} \int_{T_-}^{T_+} (DXV)(\gamma_E^q(\sigma), \theta) d\sigma$$

$$- \int_{0}^{T_+} (DXV)(\lambda_1^q(\sigma + s + a/\sqrt{2E}), \theta) d\sigma$$

$$- \int_{T_-}^{0} (DXV)(\lambda_2^q(\sigma + s), \theta) d\sigma.$$  \hspace{1cm} (4.4)

**Proof.** Let $\tilde{z}_\varepsilon = (\Omega^-)^{-1}(\tilde{z}_\varepsilon^-) = (\Omega^+)^{-1}(\tilde{z}_\varepsilon^+) \in \tilde{\Gamma}_\varepsilon$. We start by noting that the orbit starting in $\tilde{z}_\varepsilon$ is asymptotic in the future to the orbit of $\tilde{z}_\varepsilon^+$ and in the past to the orbit of $\tilde{z}_\varepsilon^-$. We can write

$$\Delta(\tilde{z}_\varepsilon^-) = \lim_{T\to \pm\infty} \left[ H_{\varepsilon}(\psi_{T_+}(\tilde{z}_\varepsilon^-)) - H_{\varepsilon}(\psi_{T_-}(\tilde{z}_\varepsilon^-)) \right.$$  

$$\left. - H_{\varepsilon}(\psi_{T_+}(\tilde{z}_\varepsilon^+)) + H_{\varepsilon}(\tilde{z}_\varepsilon^+) \right.$$  

$$\left. + H_{\varepsilon}(\psi_{T_-}(\tilde{z}_\varepsilon^-)) - H_{\varepsilon}(\tilde{z}_\varepsilon^-) \right].$$  \hspace{1cm} (4.5)

By the Fundamental Theorem of Calculus, we have

$$\Delta(\tilde{z}_\varepsilon^-) = \lim_{T\to \pm\infty} \left[ \int_{T_-}^{T_+} \left( \frac{d}{d\sigma} H_{\varepsilon}(\psi_\sigma(\tilde{z}_\varepsilon^-)) \right) d\sigma \right.$$  

$$\left. - \int_{0}^{T_+} \frac{d}{d\sigma} H_{\varepsilon}(\psi_\sigma(\tilde{z}_\varepsilon^-)) d\sigma \right.$$  

$$\left. - \int_{T_-}^{0} \frac{dH_{\varepsilon}}{d\sigma}(\psi_\sigma(\tilde{z}_\varepsilon^-)) d\sigma \right].$$  \hspace{1cm} (4.6)

By the exponential convergence of the integrands in the integrals (4.6) converge exponentially fast. So do their derivatives of low order with respect to the initial conditions. It is shown in [21] that there is exponential convergence of the integrand in (4.6) for the derivatives of an order lower than the ratio of the Lyapunov exponents in the stable and unstable directions and the Lyapunov exponents tangent to the manifold. In our case, since the directions tangent to the manifold have zero exponent for $\varepsilon = 0$, one can get that the number of derivatives of the integrand that converge exponentially fast is arbitrarily large for $\varepsilon$ small. In the arguments presented in this paper, we will need just a moderate number of derivatives.

Our next goal is to prune (4.6) to extract a convenient expression for the leading term.

By the exponential convergence of the integrands, we see that for appropriately chosen constant $K > 0$, if we take $T_+ = -T_- = K|\ln(\varepsilon)|$, the integrals differ from
the limit by not more than $O(\varepsilon^4)$. Therefore,

$$
\Delta(\tilde{z}^-) = \int_{-K\|\ln(\varepsilon)\|}^{K\|\ln(\varepsilon)\|} \left( \frac{d}{d\sigma} H_\varepsilon \right) (\psi_\sigma^\varepsilon(\tilde{z}_\varepsilon^-)) d\sigma \\
- \int_{0}^{K\|\ln(\varepsilon)\|} \left( \frac{d}{d\sigma} H_\varepsilon \right) (\psi_\sigma^\varepsilon(\tilde{z}_\varepsilon^+)) d\sigma \\
- \int_{-K\|\ln(\varepsilon)\|}^{0} \left( \frac{d}{d\sigma} H_\varepsilon \right) (\psi_\sigma^\varepsilon(\tilde{z}_\varepsilon^-)) d\sigma + O(\varepsilon^4).
$$

(4.7)

We now express $\Delta(\tilde{z}^-)$ in terms of the orbits of the unperturbed flow. By the smooth dependence on parameters of solutions of ordinary differential equations, for $|\sigma| \leq K\|\ln(\varepsilon)\|$, we have that the orbits of the perturbed system are $O(\varepsilon\|\ln(\varepsilon)\|)$-close to the corresponding orbits of the unperturbed flow. Notice that, because $\varepsilon\|\ln(\varepsilon)\|$ is small, the separation between the orbits is still growing linearly and has not yet started to grow exponentially fast with time [64, estimate (3.5.4)].

More precisely, the orbits $\xi^\varepsilon(\tilde{z}_\varepsilon^\pm)$ are $O(\varepsilon\|\ln(\varepsilon)\|)$-close to the orbit $(\lambda_E, \chi^\varepsilon)$ where $\lambda_E$ is a closed geodesic, and the orbit $\xi^\varepsilon(\tilde{z}_\varepsilon^-)$ is $O(\varepsilon\|\ln(\varepsilon)\|)$-close to $(\gamma_E, \chi_E)$, where $\gamma_E$ is a homoclinic orbit to $\lambda_E$ for the geodesic flow. Also, in these intervals of time, the variable $\theta$ changes only by $O(\varepsilon\|\ln(\varepsilon)\|)$. Thus, for $|\sigma| \leq K\|\ln(\varepsilon)\|$, we have

$$
d(\psi_\sigma^\varepsilon(\tilde{z}_\varepsilon^-), \tilde{\gamma}_E(\sigma)) \leq C\varepsilon\|\ln(\varepsilon)\|, \\
d(\psi_\sigma^\varepsilon(\tilde{z}_\varepsilon^-), \tilde{\lambda}_E(\sigma + s)) \leq C\varepsilon\|\ln(\varepsilon)\|, \\
d(\psi_\sigma^\varepsilon(\tilde{z}_\varepsilon^-), \tilde{\lambda}_E(\sigma + s + a/\sqrt{2E})) \leq C\varepsilon\|\ln(\varepsilon)\|,
$$

for some constant $C > 0$, where $\tilde{\gamma}_E(\sigma) = (\gamma_E(\sigma), \theta)$ and $\tilde{\lambda}_E(\sigma) = (\lambda_E(\sigma), \theta)$.

Therefore, substituting in the integral in (4.7) the orbits of the geodesic flow instead of the orbits of the perturbed flow and keeping $\theta$ constant, we incur an error $O(\varepsilon^4\|\ln(\varepsilon)\|)$. Using also the notation $D_XV$ introduced in (4.1), we obtain:

$$
\Delta(\tilde{z}^-) = \int_{-K\|\ln(\varepsilon)\|}^{K\|\ln(\varepsilon)\|} \left( \frac{d}{d\sigma} H_\varepsilon \right) (\tilde{\gamma}_E(\sigma)) d\sigma \\
- \int_{0}^{K\|\ln(\varepsilon)\|} \left( \frac{d}{d\sigma} H_\varepsilon \right) (\tilde{\lambda}_E(\sigma + s + a/\sqrt{2E})) d\sigma \\
- \int_{-K\|\ln(\varepsilon)\|}^{0} \left( \frac{d}{d\sigma} H_\varepsilon \right) (\tilde{\lambda}_E(\sigma + s)) d\sigma + O(\varepsilon^4\|\ln(\varepsilon)\|)
$$

(4.8)

$$
= \varepsilon^3 \int_{-K\|\ln(\varepsilon)\|}^{K\|\ln(\varepsilon)\|} (D_XV) (\gamma_E^q(\sigma), \theta) d\sigma \\
- \varepsilon^3 \int_{0}^{K\|\ln(\varepsilon)\|} (D_XV) (\lambda_E^q(\sigma + s + a/\sqrt{2E}), \theta) d\sigma \\
- \varepsilon^3 \int_{-K\|\ln(\varepsilon)\|}^{0} (D_XV) (\lambda_E^q(\sigma + s), \theta) d\sigma + O(\varepsilon^4\|\ln(\varepsilon)\|),
$$

where $\lambda_E^q, \gamma_E^q$ denote the $q$-components of $\lambda_E, \gamma_E$, respectively.
Finally, because of the exponentially fast convergence we can change the integral over a $O(\ln(\varepsilon))$ time interval as $t \to \infty$ incurring an error $O(\varepsilon^4)$, so we obtain:

\[
\Delta(\tilde{z}_\varepsilon) = \varepsilon^3 \lim_{T_+ \to \infty} \left[ \int_{T_-}^{T_+} (D_X V)(\gamma_E^q(\sigma), \theta) d\sigma 
- \int_0^{T_+} (D_X V)(\lambda_E^q(\sigma + s + a/\sqrt{2E}), \theta) d\sigma 
- \int_{T_-}^0 (D_X V)(\lambda_E^q(\sigma + s), \theta) d\sigma \right] 
+ O(\varepsilon^4|\ln(\varepsilon)|) 
= \varepsilon^3 \left[ \lim_{T_+ \to \infty} \int_0^{T_+} (D_X V)(\gamma_E^q(\sigma), \theta) - (D_X V)(\lambda_E^q(\sigma + s + a/\sqrt{2E}), \theta) d\sigma 
+ \lim_{T_- \to -\infty} \int_{T_-}^0 (D_X V)(\gamma_E^q(\sigma), \theta) - (D_X V)(\lambda_E^q(\sigma + s), \theta) d\sigma \right] 
+ O(\varepsilon^4|\ln(\varepsilon)|) 
= \varepsilon^3 \Delta_1(E, s, \theta) + O(\varepsilon^4|\ln(\varepsilon)|).
\]

In the last line we have just defined $\Delta_1 = \Delta_1(E, s, \theta)$ as the leading term of $\Delta = \Delta(\tilde{z}_\varepsilon)$, where $\tilde{z}_\varepsilon = (k(\lambda_E(s)), \theta) \in \tilde{U}_\varepsilon$.

Notice that in the above argument we used the exponential convergence of the integrands to justify the change of the limits, which is often done in Melnikov theory.

It is important to realize that the expression for the leading term $\Delta_1$ is in terms of the unperturbed trajectories and depends only on the perturbing potential. It can be considered as a global Melnikov function. In contrast with many standard treatments in which the Melnikov function is only defined for periodic or quasi-periodic orbits, (4.9) is well defined for all orbits in the domain of the scattering map independently of what is their dynamics. Also, note that the function $\Delta_1$ can be viewed as an analogue of what was called the reduced Poincaré function in [16, 19].

4.1.3. The effect of the scattering map on the action-angle coordinates. Using action angle coordinates, we write the geodesic $\lambda_E(s)$ as $\lambda_{J^2/2}(s)$ and the homoclinic $\gamma_E(s)$ as $\gamma_{J^2/2}(s)$, where $E = J^2/2$. By the rescaling property of the geodesic flow (2.3) we have that $\lambda_{J^2/2}(s) = \lambda_1(Js)$ and $\gamma_{J^2/2}(s) = \gamma_1(Js)$. Note that for an energy $E = 1$ the corresponding action is $J = \sqrt{2}$. By the change of variable formula we obtain the following
\[ \Delta_1(J, \phi, \theta) = \lim_{T_+ \to \infty} \int_0^{T_+} [(D_X V)(\gamma^q_{j/2}(\sigma), \theta) - (D_X V)(\lambda^q_{j/2}(\sigma + \phi + a), \theta)]d\sigma \]

\[ + \lim_{T_- \to -\infty} \int_{T_-}^0 [(D_X V)(\gamma^q_{j/2}(\sigma), \theta) - (D_X V)(\lambda^q_{j/2}(\sigma + \phi), \theta)]d\sigma \]

\[ = \lim_{T_+ \to \infty} \int_0^{T_+} [(D_X V)(\gamma^q(J\sigma), \theta) - (D_X V)(\lambda^q(J(\sigma + \phi + a)), \theta)]d\sigma \]

\[ + \lim_{T_- \to -\infty} \int_{T_-}^0 [(D_X V)(\gamma^q(J\sigma), \theta) - (D_X V)(\lambda^q(J(\sigma + \phi)), \theta)]d\sigma \]

\[ = \frac{1}{J}\Delta_1(\sqrt{2}, J\phi, \theta). \]

Thus, we have the following rescaling property of \( \Delta_1 \) relative to action-angle coordinates:

\[ (4.10) \quad \Delta_1(J, \phi, \theta) = \frac{1}{J}\Delta_1(\sqrt{2}, J\phi, \theta). \]

4.2. Change of energy over an elementary building block of a pseudo-orbit. The goal of this section is to compute the change of energy over an elementary building block of a pseudo-orbit, which consists in applying the scattering map followed by applying the inner dynamics for some prescribed time. We will also formulate later some conditions that ensure that the effect on the energy is non-trivial. Of course, this requires some choices (e.g., the time we decide to follow the inner dynamics), and, given the choices made, some non-degeneracy assumptions on the perturbations (e.g., a perturbation that vanishes identically will not produce any effect).

We first compute the energy change over an elementary building block obtained by starting at a point \( \tilde{z}^- = (z^-, \theta) \), applying the scattering map, and then applying the inner dynamics for some time. We specify the time for which we apply the inner dynamics implicitly by requiring that the change of angle coordinate along the pseudo-orbit starting from \( \tilde{z}^- \) is some fixed number \( L \), chosen sufficiently large, to be specified later. We denote such an elementary building block by \( B(\tilde{z}^-) \).

As specified before, we focus on a bounded energy range \( H^\ast \in [1, 2] \). From Subsection 4.1, the leading term in the energy change \( \Delta_1 \) depends on the effect of the unperturbed scattering potential on the unperturbed trajectories. Let \( \tilde{z}^- = (k_{z}(\lambda_E(s)), \theta) \), with \( E \) in the energy range. Assume that the angle-action coordinates of \( \lambda_E(s) \in \Lambda_0 \) are \( (J, \phi) \), with \( J = \sqrt{2E} \).

By \( \ref{zm1} \), the effect of the unperturbed scattering map on \( \Lambda_0 \), taking a point \( z^- \) to \( z^+ \), is to increase the angle coordinate \( \phi \) by \( a \). The scaled time \( s \) to follow the inner dynamics starting from \( z^+ \) and ending at a point of angle coordinate equal to \( L \) is \( (L - a)/\sqrt{2E} \). Hence, in terms of the unperturbed system, one follows the geodesic flow trajectory \( \lambda_E(s) \) for the time interval \( s \in [a/\sqrt{2E}, L/\sqrt{2E}] \).

In the perturbed system, we choose to follow the inner dynamics for the same time interval, which is independent of \( \varepsilon \). Since the energy is a slow variable and we are considering only scaled times of order 1, the change of energy during the time spent along the inner dynamics can be computed, with a very small error, using the fundamental theorem of calculus.
This implies that the change of energy along an orbit segment starting at some point \( \tilde{z}^* = (z^*, \theta^*) \), where \( \theta^* = \chi_{a/\sqrt{2E}}(\theta) \), and following it for a time interval \( s \in [a/\sqrt{2E}, L/\sqrt{2E}] \) is

\[
\int_{a/\sqrt{2E}}^{L/\sqrt{2E}} \frac{d}{dt} H_\varepsilon(\xi_\sigma(z^+_s), \chi_\sigma(\theta)) d\sigma = \varepsilon^3 \int_{a/\sqrt{2E}}^{L/\sqrt{2E}} (D_X V)(\lambda^q_E(\sigma), \theta)d\sigma + O(\varepsilon^4 |\ln \varepsilon|).
\]

**Proposition 4.2.** Let \( \tilde{z}_c^- = (k_c(\lambda_E(s), \theta), \tilde{S}_c(\tilde{z}_c^-) = \tilde{z}_c^+ \). Consider an elementary building block consisting of one application of the scattering map \( \tilde{S}_c(\tilde{z}_c^-) = \tilde{z}_c^+ \), and a trajectory segment with initial point \( \tilde{z}_c^+ \) following the inner dynamics for a time interval \( s \in [a/\sqrt{2E}, L/\sqrt{2E}] \). The change \( G(\tilde{z}_c^-) \) of scaled energy \( H_\varepsilon \) over the building block is of the form

\[
G(\tilde{z}_c^-) = \varepsilon^3 G_1(E, p, s) + O(\varepsilon^4 |\ln \varepsilon|),
\]

where \( G_1 \), the leading term of \( G \), is given by

\[
G_1(E, p, s) = \lim_{T_+ \to \infty} \lim_{T_- \to -\infty} \left[ \int_{T_-}^{T_+} (D_X V)(\gamma^q_E(s), \theta) d\sigma - \int_{0}^{T_+} (D_X V)(\lambda^q_E(\sigma + s + a/\sqrt{2E}), \theta) d\sigma - \int_{T_-}^{0} (D_X V)(\lambda^q_E(\sigma + s), \theta) d\sigma + \int_{a/\sqrt{2E}}^{L/\sqrt{2E}} (D_X V)(\lambda^q_E(\sigma), \theta) d\sigma \right].
\]

**Proof.** By the fundamental theorem of calculus, the change of energy along an orbit segment starting at \( \tilde{z}_c^+ \) and following it for a time interval \( s \in [a/\sqrt{2E}, L/\sqrt{2E}] \) is

\[
\int_{a/\sqrt{2E}}^{L/\sqrt{2E}} \frac{d}{dt} H_\varepsilon(\xi_\sigma(z^+_s), \chi_\sigma(\theta)) ds = \varepsilon^3 \int_{a/\sqrt{2E}}^{L/\sqrt{2E}} (D_X V)(\lambda^q_E(s), \theta) ds + O(\varepsilon^4 |\ln \varepsilon|).
\]

Combining with \([4.9]\), we see that, over an elementary building block, the energy has changed by

\[
G(\tilde{z}_c^-) = \Delta(\tilde{z}_c^-) + \int_{a/\sqrt{2E}}^{L/\sqrt{2E}} \frac{d}{dt} H_\varepsilon(\xi_\sigma(z^+_s), \chi_\sigma(\theta)) ds = \varepsilon^3 \left[ \Delta_1(\tilde{z}_c^-) + \int_{a/\sqrt{2E}}^{L/\sqrt{2E}} (D_X V)(\lambda^q_E(s), \theta) ds \right] + O(\varepsilon^4 |\ln \varepsilon|).
\]

In conclusion,

\[
G(\tilde{z}_c^-) = \varepsilon^3 G_1(E, s, \theta) + O(\varepsilon^4 |\ln \varepsilon|).
\]
Similarly to (4.10), we have the following rescaling property of $G_1$ relative to action-angle coordinates

$$(4.13) \quad G_1(J, \phi, \theta) = \frac{1}{J} G_1(\sqrt{2}, J\phi, \theta).$$

We make two important remarks.

Remark 4.3. For the scattering map $\tilde{S}_L(\tilde{z}_-^\gamma) = \tilde{z}_+^\gamma$, there is no trajectory of the system asymptotic to $\tilde{z}_-^\gamma$ in the past and to $\tilde{z}_+^\gamma$ in the future. Rather, $\psi_{T_+}(\tilde{z}_\epsilon)$ will approach $\psi_{T_-}(\tilde{z}_\epsilon^-)$ as $T_- \to -\infty$, and $\psi_{T_+}(\tilde{z}_\epsilon)$ will approach $\psi_{T_-}(\tilde{z}_\epsilon^+)$ as $T_+ \to +\infty$. Here $\tilde{z}_\epsilon = (\tilde{\Omega}^-)^{-1}(\tilde{z}_-^\epsilon) = (\tilde{\Omega}^+)^{-1}(\tilde{z}_+^\epsilon)$. For $-T_- = T_+ = K |\ln(\epsilon)|$, the change of energy $\Delta(\tilde{z}_\epsilon^-)$ along the homoclinic trajectory from $\psi_{T_-}(\tilde{z}_\epsilon^-)$ to $\psi_{T_+}(\tilde{z}_\epsilon^+)$ is the same as in (4.9), up to an error term which is subdominant.

Remark 4.4. Similarly, if instead of fixing the angle shift to a constant value $L$, we allow to chose, for each elementary building block, a value of $L$ which is constant plus an $O(\epsilon)$-term, the change of energy $G(\tilde{z}_\epsilon^-)$ is the same as in (4.11), up to an error term which is subdominant.

In Section 5.3 we will show that there is a trajectory of the dynamics that follows closely the pseudo-orbit consisting of the segment of the homoclinic trajectory from $\psi_{T_-}(\tilde{z}_\epsilon^-)$ to $\psi_{T_+}(\tilde{z}_\epsilon)$, followed by a segment of the trajectory of the inner flow $(\psi^t_s)_{|\tilde{\Lambda}_s}$ with initial point $\psi_{T_+}(\tilde{z}_\epsilon^+)$.

Based on these remarks, we will keep in mind that to an elementary building block we can associate the following objects:

- One application of the scattering map $\tilde{S}_L(\tilde{z}_-^\gamma) = \tilde{z}_+^\gamma$, plus one segment of the trajectory of the inner flow $(\psi^t_s)_{|\tilde{\Lambda}_s}$ with initial point $\tilde{z}_+^\gamma$;
- A pseudo-orbit, consisting of a segment of a homoclinic orbit from $\psi_{T_-}(\tilde{z}_\epsilon^-)$ to $\psi_{T_+}(\tilde{z}_\epsilon)$, followed by a segment of the trajectory of the inner flow $(\psi^t_s)_{|\tilde{\Lambda}_s}$ with initial point $\psi_{T_+}(\tilde{z}_\epsilon^+)$;
- A true orbit, that follows closely the pseudo-orbit described above.

The change of energy along either one of these objects is given by the estimate in Proposition 1.2 up to a subdominant error term.

4.3. Generic set of potentials. In this section we specify the set of potentials $\mathcal{V}'$ claimed in Theorem 2.4. The potential $V \in \mathcal{V}'$ are required to satisfy a condition that ensures that one can achieve consistent energy growth by applying the scattering map followed by the inner dynamics as in Proposition 1.2.

A key observation is that, since there exist a closed hyperbolic geodesic $\gamma_E$ and a corresponding transverse homoclinic orbit $\gamma_E$, by the Birkhoff-Smale Homoclinic Orbit Theorem (see e.g., [44]) there exist, in fact, at least two geometrically distinct homoclinic orbits to the same geodesic. The existence of at least two homoclinic orbits can also be argued via variational methods. We shall denote a pair of such homoclinic orbits by $\gamma^1_E$, $\gamma^2_E$, and the associated scattering maps $\tilde{S}_L^1$, $\tilde{S}_L^2$. We denote the corresponding leading terms $G_1$ from Proposition 1.2 by $G^1_1$, $G^2_1$, respectively. The main idea is that, under some generic condition on the potential $V$, utilizing one of the homoclinic orbits to grow energy will be more advantageous than utilizing the other, so one can select which one of the two homoclinic orbits to use in a way to ensure a net gain of energy over time.
To express the generic condition on the potentials \( V \in \mathcal{V} \), we write \( G_1^1, G_2^1 \) in action-angle coordinates. Indeed, the energy growth (4.11) along an elementary building block depends only on the initial point \( \tilde{z}_c = (k_c(\gamma_{1/2}(\phi)), \theta) \in \Lambda_\gamma \) of the block, and the leading term \( G_1 \) in (4.12) depends on the corresponding angle-action coordinates and parameter value \((J, \phi, \theta)\) corresponding to \( \tilde{z}_c \).

We define \( \mathcal{V} \) to be the set of potentials \( V \) for which the following non-degeneracy condition holds:

A4. Fix \( J_0 = \sqrt{2} \) so the corresponding energy level of \( H_0 \) is \( E_0 = 1 \). Fix \( \theta_0 \in \mathbb{N} \) a non-trivial uniformly recurrent point. We assume that for geodesic flow at this energy level there exist two geometrically different homoclinic trajectories \( \gamma_{E_0}^1, \gamma_{E_0}^2 \) to the same closed geodesic \( \lambda_{E_0} \), and a common domain \( \tilde{U} \) for the corresponding scattering maps \( \tilde{S}_c^1, \tilde{S}_c^2 \) such that:

\[
(4.14) \quad \sup_{\phi_1} G_1^1(J_0, \phi_1, \theta_0) \neq \sup_{\phi_2} G_2^1(J_0, \phi_2, \theta_0),
\]

where \( G_1^1, G_2^1 \) are the leading terms of the energy gain, given by (4.12).

In the above, it is understood that the angles \( \phi_1, \phi_2 \) are restricted to some closed intervals where \((\lambda_{J_0^2/2}(\phi_1), \theta_0) \in \tilde{U}, (\lambda_{J_0^2/2}(\phi_2), \theta_0) \in \tilde{U} \), where \( k_c(\tilde{U}) \) is the domain \((\tilde{U}_c^1)^- \) of \( \tilde{S}_c^1 \) and also in the domain \((\tilde{U}_c^2)^- \) of \( \tilde{S}_c^2 \). Of course, a domain \( \tilde{U} \) as in condition A4 is not unique. The condition requires only the existence of at least one domain \( \tilde{U} \) on which (4.14) holds.

We notice that because we assumed the closed geodesics in the unit tangent bundle have transverse homoclinic connections for the geodesic flow in the unit tangent bundle, we can define the projections along each of the points on the orbit. Similarly, we can lift for any value of the energy by the scaling invariance.

Therefore, we can always define locally two scattering maps and we can continue them. The only obstruction to define a scattering map in an arbitrary domain is that the local continuation along a closed loop may have some monodromy. Each of the scattering maps can be defined on any domain that does not contain essential circles, i.e., non-contractible loops of the cylinder. Hence, the assumption in A4 that the two scattering maps have a common domain is satisfied automatically.

Condition A4 is a condition on \( V \) along trajectories of the unperturbed system, as shown by (4.12). This is an explicit condition, that can be verified for a given \( V \) along a given closed geodesic and a given pair of homoclinic orbits to that closed geodesic.

Assuming condition A4, suppose, without loss of generality, that

\[
(4.15) \quad \sup_{\phi_1} G_1^1(J_0, \phi_1, \theta_0) > \sup_{\phi_2} G_2^1(J_0, \phi_2, \theta_0).
\]

Then there exists \( \delta > 0 \), \( \phi_* \) and an open neighborhood \( \mathcal{P} \subseteq \mathbb{N} \) of \( \theta_0 \) such that

\[
G_1^1(J_0, \phi_*, \theta) - G_2^1(J_0, \phi, \theta) \geq 2\delta
\]

for all \( \theta \in \mathcal{P} \) and all \( \phi \) with \( k_c(\lambda_{E_0}^1(\phi), \theta_0) \) is in the domain of \( \tilde{S}_c^2 \). Moreover, we can choose the neighborhood \( \mathcal{P} \) to be a flow box for the flow \( \chi_\Sigma^c \) on \( \Sigma \), i.e., a homeomorphic copy in \( N \) of a \( d \)-dimensional open rectangle, of the form \( \{ \chi_{\Sigma}^c(\theta) | \theta \in \Sigma, s \in (-\rho, \rho) \} \), where \( \Sigma \) is a \((d-1)\)-dimensional open disk transverse to the flow \( \chi_\Sigma^c \), and \( \rho > 0 \) is sufficiently small so that the flow \( \chi_{\Sigma}^c \) is transverse to each surface \( \{ \chi_{s_0}^c(\theta) | \theta \in \Sigma \} \) for each \( s_0 \in (-\rho, \rho) \).
The condition (A4) is formulated in terms of the geodesic and homoclinic orbits at some fixed energy level $E_0 = 1$, corresponding to $J_0 = \sqrt{2}$. By the rescaling property of the geodesic flow, and by the corresponding rescaling property (4.13), it follows that
\begin{equation}
G_1^1(J, \frac{1}{J} \phi_*, \theta) - G_1^2(J, \phi', \theta) \geq \frac{2\delta}{J},
\end{equation}
for all $\theta \in \mathcal{P}$ and all $\phi' = \frac{1}{J} \phi$ with $\phi$ restricted as before. Since we restrict to the interval $H_\varepsilon \in [1, 2]$, we have $\sqrt{2} \leq J \leq 2$ hence
\begin{equation}
G_1^1(J, \frac{1}{J} \phi_*, \theta) - G_1^2(J, \phi', \theta) \geq \delta,
\end{equation}
for all $\theta \in \mathcal{P}$ and all $\phi'$ as before.

**Lemma 4.5.** Given a geodesic flow, a closed geodesic $\lambda$ and two geometrically distinct homoclinic orbits $\gamma_{E_0}^1$, $\gamma_{E_0}^2$, at the energy level $E_0 = 1$. Then, the set $\mathcal{V}'$ of potentials $V$ that satisfy assumption (A4) is $C^0$ open and $C^\infty$ dense in the set of all the potentials.

**Proof.** We assume by contradiction that, for some domain $\mathcal{U}$, the condition (A4) does not hold.

Let $\sup_{\phi_0} G_1^1(J_0, \phi_1, \theta_0) = G_1^1(J_0, \phi_1^*, \theta_0)$ for some $\phi_1^*$, and $\sup_{\phi_2} G_1^2(J_0, \phi_2, \theta_0) = G_1^2(J_0, \phi_2^*, \theta_0)$ for some $\phi_2^*$. Then we have
\begin{equation}
G_1^1(J_0, \phi_1^*, \theta_0) = G_1^2(J_0, \phi_2^*, \theta_0).
\end{equation}

Since for fixed $(J_0, \phi_1^*)$ and $(J_0, \phi_2^*)$ the functions $G_1^1(J_0, \phi_1^*, \theta_0)$ and $G_1^2(J_0, \phi_2^*, \theta_0)$, respectively, considered as functionals of $V$, are continuous when the set $\mathcal{V}$ of potentials is given the $C^0$ topology, it is clear that (4.17) defines a $C^0$-closed set (intersection of closed sets) and, therefore the condition (A4) holds in a $C^0$-open set of potentials.

To complete the proof of Lemma 4.5 it suffices to show that, given a potential $V$ that satisfies (4.17), there is an arbitrarily $C^\infty$-small perturbation of $V$ which does not satisfy (4.17).

The construction is very clear. We note that $G_1^j, j \in \{1, 2\}$ is a sum of integrals over several trajectory segments of the geodesic flow: some are segments of closed geodesics in $\Lambda_0$, which are recurrent, and one of them is a segment of a homoclinic trajectory.

Because the two homoclinic orbits $\gamma_{E_0}^1, \gamma_{E_0}^2$ are geometrically different, we can find a $s_0 \in \mathbb{R}$ and a small enough ball $B \subset T^*M$ centered around $\gamma_{E_0}^j(s_0)$ in such a way that $B \cap \lambda_{E_0} = B \cap \gamma_{E_0}^2 = \emptyset$. Moreover, we can choose the ball $B$ such that there exists a small interval $I$ around $s_0$ such that $\gamma^1(\mathbb{R}) \cap B = \gamma^1(I)$. Choose $B'$ a small ball in $N$ centered around $\theta_0$.

Now we choose a function $W : T^*M \times N \to \mathbb{R}$ with support in $B \times B'$ so that $\nabla_\theta W(\gamma_{E_0}^j(s), \theta_0) \cdot X(\theta_0) \geq \rho > 0$ for all $s \in I$ and $\theta \in B'$, and for some $\rho > 0$. It follows from (4.14) that perturbing $V$ to $V + W$ yields $G_1^1(J_0, \phi_1^*, \theta_0) \neq G_1^2(J_0, \phi_2^*, \theta_0)$. \hfill $\Box$

4.4. **Gain of energy along sequences of elementary building blocks.** We will estimate the gain of the scaled energy $H_\varepsilon$ in time, along some suitably chosen sequences of elementary building blocks, for potentials $V \in \mathcal{V}'$. We will account for both the scaled time $s$ and the physical time $t$. 
Assume that $V$ satisfies condition (A4) from the previous section. Then for given any $J_0$ with $\|J_0\| = \sqrt{2}$ there exist $\phi_*$ with $k_\varepsilon(\lambda_{E_0}(\phi_*), \theta_0) \in (\bar{U}_\varepsilon)^{-}$ and a flow box $\mathcal{P} \subseteq \mathcal{N}$, of size $O(1)$, with the property that, if $\theta \in \mathcal{P}$ and $k_\varepsilon(\lambda_{E_0}(\phi), \theta_0) \in (\bar{U}_\varepsilon)^{-}$, then

$$G_1^{1}(J_0, \phi_*, \theta) - G_1^{2}(J_0, \phi, \theta) \geq 2\delta.$$ 

Due to the rescaling property of the geodesic flow, it follows that for any $J \in [\sqrt{2}, 2]$ (corresponding to the energy range $E \in [1, 2]$ fixed in Subsection 4.2), there exists $\phi_*(J) = \phi/J_*$ such that

$$(4.18) \quad G_1^{1}(J, \phi_*(J), \theta) - G_1^{2}(J, \phi, \theta) \geq \delta,$$

for all $\theta \in \mathcal{P}$ and all $\phi$ in the appropriate domain.

The flow $\chi_s^\varepsilon$ is slow, and so is its time-1 map. It takes a time $O(1/\varepsilon)$ to travel a distance $O(1)$. The flow box $\mathcal{P}$ was chosen of the form $\{\chi_s^\varepsilon(\theta) \mid \theta \in \Sigma, s \in (-\rho, \rho)\}$ with $\Sigma$ as in Subsection 4.3.

Assume that $\theta_0 \in \mathcal{N}$ is a uniformly recurrent point with $X(\theta_0) \neq 0$. Choosing the neighborhood $\mathcal{P}$ small enough ensures that the trajectory of $\theta_0$ will successively leave $\mathcal{P}$ and return to $\mathcal{P}$. The lengths of the time intervals when the trajectory of $\theta_0$ moves through $\mathcal{P}$ are uniformly bounded above and below, and, because of the uniform recurrence hypothesis A3, so are the lengths of the time intervals when the trajectory of $\theta_0$ moves through $\mathcal{N} \setminus \mathcal{P}$. More precisely, there exist $0 < \tau_0 < \tau'_0$, independent of $\varepsilon$, such that the trajectory of $\theta_0$ spends a scaled time between $\tau_0/\varepsilon$ and $\tau'_0/\varepsilon$ in $\mathcal{P}$, and there exist $0 < \tau_1 < \tau'_1$, independent of $\varepsilon$, such that the trajectory of $\theta_0$ spends a scaled time between $\tau_1/\varepsilon$ and $\tau'_1/\varepsilon$ in $\mathcal{N} \setminus \mathcal{P}$ between successive returns to $\mathcal{P}$.

Now we compute the growth of energy during a range of scaled time $\Delta s = 1/\varepsilon^2$. We follow a sequence of elementary building blocks of the type $B^1(J, \phi, \theta)$, $B^2(J, \phi, \theta)$, where the superscripts correspond to the two choices of homoclinic orbits/scattering maps $\gamma_1^J$, $\gamma_2^J$, respectively, where the successions of blocks is chosen as follows. When $\theta \in \mathcal{P}$ we use blocks of the type $B^1(J, \phi_*(J), \theta)$, where $\phi_*(J)$ is defined as before. When $\theta \notin \text{cl}(\mathcal{P})$ we use blocks of the type $B^2(J, \phi_0, \theta)$ for some $\phi_0$ fixed. Thus, the sequence is composed of strings of the type $B^1(J, \phi_*(J), \theta)$, alternating with strings of the type $B^2(J, \phi_0, \theta)$; the proportion of time when we switch from $B^1(J, \phi_*(J), \theta)$ to $B^2(J, \phi_0, \theta)$ or vice versa is $O(\varepsilon)$.

We compute the growth of energy along such a sequence of building block spanning a range of scaled time of $1/\varepsilon^2$. The initial condition for the dynamics on $N$ is the point $\theta_0$ which is assumed to be uniformly recurrent. The $J$-coordinate along the pseudo-orbit takes the successive values $J(n)$, and the $\theta$-coordinate along the pseudo-orbit takes the successive values $\theta(n)$; the $\phi$-coordinate is maintained fixed $\phi = \phi_0$.

We have:
\[ \Delta H_\varepsilon = \varepsilon^3 | \sum_{n \in [0, 1/\varepsilon^2]} \sum_{\theta(n) \in \mathcal{P}, \theta((n+1)\varepsilon) \not\in \mathcal{P}} G_1^n(J(n), \phi_n(J(n)), \theta(n)) \]

\[ + \sum_{n \in [0, 1/\varepsilon^2]} \sum_{\theta(n) \in \mathcal{P}, \theta((n+1)\varepsilon) \not\in \mathcal{P}} G_2^n(J(n), \phi_n, \theta(n)) \]

\[ + \sum_{n \in [0, 1/\varepsilon^2]} \sum_{\theta(n) \in \mathcal{P}, \theta((n+1)\varepsilon) \not\in \mathcal{P}} G_2^n(J(n), \phi_n, \theta(n)) \]

\[ + O(\varepsilon^2 | \ln(\varepsilon)|), \]

where the above error term is due to the accumulation error term of \(O(\varepsilon^4 | \ln(\varepsilon)|)\) from (4.12) over 1/\(\varepsilon^2\) time steps. The terms corresponding to the times \(n\) when \(\theta(n) \in \mathcal{P}, \theta((n+1)\varepsilon) \not\in \mathcal{P}\) and \(\theta(n) \not\in \mathcal{P}, \theta((n+1)\varepsilon) \in \mathcal{P}\) are \(O(\varepsilon^4 \cdot 1/\varepsilon^2) = O(1/\varepsilon)\), and since \(G_1^n, G_2^n\) are bounded, they contribute to a combined error term \(O(\varepsilon^3 \cdot 1/\varepsilon) = O(\varepsilon^2)\), which is subdominant.

Thus we can write

\[ \Delta H_\varepsilon \geq \varepsilon^3 | \sum_{n \in [0, 1/\varepsilon^2]} \sum_{\theta(n) \in \mathcal{P}, \theta((n+1)\varepsilon) \in \mathcal{P}} G_1^n(J(n), \phi_n, \theta(n)) \]

\[ + \sum_{n \in [0, 1/\varepsilon^2]} \sum_{\theta(n) \in \mathcal{P}, \theta((n+1)\varepsilon) \not\in \mathcal{P}} G_2^n(J(n), \phi_n, \theta(n)) \]

\[ + O(\varepsilon^2 | \ln(\varepsilon)|). \]

Now we rearrange the summation above to estimate the total gain of energy while \(\theta \in \mathcal{P}\):

\[ \Delta H_\varepsilon \geq \varepsilon^3 \sum_{n \in [0, 1/\varepsilon^2]} |G_1^n(J(n), \phi_n, \theta(n)) - G_2^n(J(n), \phi_n, \theta(n))| \]

\[ + \varepsilon^3 \sum_{n \in [0, 1/\varepsilon^2]} G_2^n(J(n), \phi_n, \theta(n)) \]

\[ + O(\varepsilon^2 | \ln(\varepsilon)|) \]

\[ \geq \varepsilon^3 \frac{\tau_0}{2 \varepsilon^2} + \varepsilon^3 \sum_{n \in [0, 1/\varepsilon^2]} G_2^n(J(n), \phi_n, \theta(n)) + O(\varepsilon^2 | \ln(\varepsilon)|) \]

\[ = \varepsilon \tau_0 \delta + \varepsilon^3 \sum_{n \in [0, 1/\varepsilon^2]} G_2^n(J(n), \phi_n, \theta(n)) + O(\varepsilon^2 | \ln(\varepsilon)|). \]

Now we treat the remaining summation from above as a Riemann sum of mesh 1 and we approximate it by a Riemann integral. Since \(J(\sigma)\) changes by at most \(O(\varepsilon^2)\) and \(\theta(\sigma)\) changes by at most \(O(\varepsilon)\) over each interval \([n, n+1]\), we have \(\max_{\sigma \in [n, n+1]} |\frac{d}{d\sigma} G_1(J(\sigma), \phi, \theta(\sigma))| = O(\varepsilon)\) for all \(n\). Hence the error in approximating the integral on a subdivision by a term of the Riemann sum is less than \(O(\varepsilon)\), and since there are 1/\(\varepsilon^2\) terms in the Riemann sum, the error in approximating the integral by the Riemann sum is \(O(1/\varepsilon)\). Taking into account the \(\varepsilon^3\)-leading
factor we obtain

\[
\varepsilon^3 \sum_{n \in [0,1/\varepsilon^2]} G^2_n(J(n), \phi_0, \theta(n)) \leq \varepsilon^3 \int_0^{1/\varepsilon^2} G^2_0(J(\sigma), \phi_0, \theta(\sigma))d\sigma + O(\varepsilon^{-1}/\varepsilon)
\]

\[
\leq \varepsilon^3[A^2_1(J(1/\varepsilon^2), \phi_0, \theta(1/\varepsilon^2)) - A^2_1(J(0), \phi_0, \theta(0))] + O(\varepsilon^2) = O(\varepsilon^2),
\]

where \(G^2_1 = D_X A^2_1\), where the antiderivative \(A^2_1\) of \(G^2_1\) is bounded by the compactness of \((\Lambda_0 \times N) \cap H^{-1}_0[1,2]\). (Here we used the fact that \(\frac{d}{dt} A^2_1(J(t), \phi_0(t)) = \frac{\partial^2}{\partial t^2} J + D_X A^2_1 = D_X A^2_1 + O(\varepsilon^2) = G^2_1 + O(\varepsilon^2)\).)

From (4.21) we conclude that, under the non-degeneracy assumption \(A4\), the gain of energy following a string of elementary building blocks of the type \(B^1(J, \phi_0, \theta)\), \(B^2(J, \phi_0, \theta)\), over a range of scaled time \(1/\varepsilon^2\) satisfies

\[
\Delta H_\varepsilon \geq \varepsilon \theta_0 + O(\varepsilon^2 |\log(\varepsilon)|).
\]

Note that during this time interval we do not leave the scaled energy interval \(E \in [1,2]\) fixed at the beginning of the argument.

Thus, during a time period of \(1/\varepsilon^2\), moving along the pseudo-orbits corresponding to the elementary building blocks of the type \(B^1(J, \phi_0, \theta)\), \(B^2(J, \phi_0, \theta)\), in the specified order, corresponds to a physical energy growth of \(O(\varepsilon)\). Since \(H_\varepsilon = \varepsilon^2 H_0\) and \(\Delta s = \Delta t / \varepsilon\), we obtain a physical energy growth of \(\Delta H = O(1/\varepsilon^2)\) during a physical time interval \(\Delta t = O(1/\varepsilon^2)\), that is, a linear growth rate of the physical energy in physical time.

**Remark 4.6.** If we construct a sequence of elementary building blocks corresponding to a single homoclinic \(\gamma^j_{T_0}, j \in \{1,2\}\), for a time of \(O(1/\varepsilon)\), and if the variable \(\phi_0\) in the construction is fixed as above, a calculation as in (4.22) shows that the change of scaled energy is \(O(\varepsilon^3)\). This implies that \(G^1_1\) cannot be always positive or always negative for all this time. Along any sequence of elementary building blocks for which the external flow \(\chi^\varepsilon\) returns to a small neighborhood of its initial point \(\theta_0\), there will always be regions in \(\Lambda_0\) where \(G^1_1\) is positive as well as regions where \(G^1_1\) is negative. By the same reason, we cannot have \(G^1_1 > G^2_1\) for all time. Hence, besides a flow box \(P \subseteq N\) such that \(G^1_1(J_0, \phi_0, \theta) - G^2_1(J_0, \phi_0, \theta) > 0\) for all \(\theta \in P\), there should also exist another flow box \(P'\) such that \(G^1_1(J_0, \phi_0, \theta) - G^1_1(J_0, \phi_0, \theta) > 0\) for all \(\theta \in P'\).

**4.5. Sequences of elementary building blocks achieving unbounded growth of energy.** The above construction of pseudo-orbits can be continued for a time \(O(1/\varepsilon^3)\) to achieve an energy growth corresponding to the whole interval \(H_\varepsilon \in [1,2]\). Since \(\varepsilon = 1/\sqrt{E_*}\), the corresponding growth of physical energy is \(H = \frac{1}{\varepsilon^3} H_\varepsilon \in [E_*, 2E_*]\). To grow the physical energy to infinity, we repeat the procedure, reinitializing the process starting with \(\varepsilon = 1/\sqrt{2E_*}\). For this new value of the small parameter \(\varepsilon\) in (4.3), growing the physical energy \(H \in [2E_*, 4E_*]\) amounts to growing the scaled energy \(H_\varepsilon \in [1,2]\). So all the estimates made in this subsection remain valid and carry forward. Thus, this construction of pseudo-orbits featuring energy growth can be repeated indefinitely.

In Section [5] we show the existence of true orbits ‘shadowing’ the pseudo-orbits constructed in this section.

**4.6. Sequences of elementary building blocks achieving symbolic dynamics.** In order to construct elementary building blocks along which the energy follows
a prescribed path \( \mathcal{E} : [0, \infty) \to \mathbb{R} \), we alternate elementary building blocks leading to energy growth with blocks leading to energy loss. Choosing the proportions of the energy growth and of the energy loss allows us to control the energy change. In particular, we can obtain rates close to zero by alternating blocks which gain energy with blocks which loose energy. The change of energy along an elementary building block is not more that \( \varepsilon^3 \) in the scaled variables, which corresponds to a change of energy of \( E^{-1/2} \) in the physical variables. In this way, we can follow the prescribed energy path \( \mathcal{E} \) up to \( E^{-1/2} \). Then, once the sequence of elementary building blocks is constructed, in Section 5.3 we will construct a sequence of correctly aligned windows along this sequence of block, and apply the shadowing Theorem 5.3 and obtain an orbit that follows these windows.

**Remark 4.7.** We remark that in the present mechanism we do not achieve small rates of growth by staying near a KAM torus, as in [16, 19], rather the orbits we construct are performing homoclinic excursions most of the time. Thus, these orbits are very different from the previously constructed orbits.

## 5. Existence of orbits following sequences of elementary building blocks

In this section, we show that we can concatenate infinitely many elementary building blocks as above, and that there exist a true orbit that follows the pseudo-orbit underlying those blocks, thus achieving infinite energy growth. Since our system is not hyperbolic, the classical shadowing lemma for hyperbolic systems, saying that any pseudo-orbit can be ‘shadowed’ by a true orbit, does not apply. We will show that, nevertheless, the pseudo-orbits constructed in the previous section can be approximated by a true orbit. For this, we use a topological argument based on correctly aligned windows. This argument is constructive and robust, so it allows us to also estimate the energy growth rate along the resulting orbit.

### 5.1. Topological method

In this section we review briefly the topological method of correctly aligned windows, following [68, 34, 35]. Earlier versions of the method go back to [13, 24, 23].

A window is a triple consisting of a mapping, a set, and a partition of the boundary of that set: the mapping is a homeomorphism from a multi-dimensional rectangle in some Euclidean space to a manifold, the set is the image of the multi-dimensional rectangle through the homeomorphism, and the partition divides the boundary of the set into an exit set and an entry set, which play a dynamical role.

**Definition 5.1.** A \((n_1,n_2)\)-window in an \(m\)-dimensional manifold \(M\), where \(n_1 + n_2 = m\), is an ensemble \((W,W^{\text{exit}},W^{\text{entry}},c)\) consisting of:

1. a homeomorphism \(c : \text{dom}(c) \to \text{im}(c)\), where \(\text{dom}(c)\) is an open neighborhood of \([0,1]^{n_1} \times [0,1]^{n_2} \subseteq \mathbb{R}^m\), and \(\text{im}(c)\) is an open set in \(M\),
2. a homeomorphic copy \(W := c([0,1]^{n_1} \times [0,1]^{n_2}) \subseteq \text{im}(c)\) of \([0,1]^{n_1} \times [0,1]^{n_2}\) in \(M\),
3. an ‘exit set’ \(W^{\text{exit}} := c(\partial[0,1]^{n_1} \times [0,1]^{n_2})\)
   and an ‘entry set’ \(W^{\text{entry}} := c([0,1]^{n_1} \times \partial[0,1]^{n_2})\).
We adopt the following notation: \( W_c = c^{-1}(W) = [0,1]^{n_1} \times [0,1]^{n_2} \), \( (W^{\text{exit}})_c = c^{-1}(W^{\text{exit}}) = \partial[0,1]^{n_1} \times [0,1]^{n_2} \), and \( (W^{\text{entry}})_c = c^{-1}(W^{\text{entry}}) = [0,1]^{n_1} \times \partial[0,1]^{n_2} \). When the coordinate system \( c \) is evident from context, we suppress the subscript \( c \) from the notation.

Informally, two windows are correctly aligned under some map, provided that the image of the first window under the map crosses the second window all the way through and across its exit set. Below we present a version of the definition of correct alignment that is sufficient for the purpose of this paper. More details can be found in [63]. Given two windows \( (W_1, W_1^{\text{exit}}, W_1^{\text{entry}}, c_1) \) and \( (W_2, W_2^{\text{exit}}, W_2^{\text{entry}}, c_2) \) and a continuous map \( f : M \to M \) with \( f(\text{im}(c_1)) \subseteq \text{im}(c_2) \), we will denote \( f_{c_1, c_2} = c_2^{-1} \circ f \circ c_1 \).

**Definition 5.2.** The window \( W_1 \) is correctly aligned with the window \( W_2 \) under \( f \) if the following conditions are satisfied:

1. In the case when \( n_1, n_2 \neq 0 \), the conditions are:
   1.i) There exists a continuous homotopy \( h : [0,1] \times (W_1)_{c_1} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \)
   with
   \[
   h_0 = f_{c_1, c_2},
   h([0,1], (W_1^{\text{exit}})_{c_1}) \cap (W_2)_{c_2} = \emptyset,
   h([0,1], (W_1)_{c_1}) \cap (W_2^{\text{entry}})_{c_2} = \emptyset,
   \]
   1.ii) There exists a linear map \( A : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \), such that
   1.ii.a) \( h_1(x,y) = (Ax,0) \) for \( x \in [0,1]^{n_1} \) and \( y \in [0,1]^{n_2} \),
   1.ii.b) \( A(\partial[0,1]^{n_1}) \subseteq \mathbb{R}^{n_1} \setminus [0,1]^{n_1} \).

2. In the case when \( n_2 = 0 \), the conditions are:
   2.i) \( (W_1^{\text{exit}})_{c_1} = (\partial W_1)_{c_1}, (W_2^{\text{exit}})_{c_2} = (\partial W_2)_{c_2} \),
   2.ii) \( (W_2)_{c_2} \subseteq \text{int}(f_{c_1, c_2}((W_1)_{c_1})) \),

3. In the case when \( n_1 = 0 \), the conditions are:
   3.i) \( (W_1^{\text{exit}})_{c_1} = \emptyset, (W_2^{\text{exit}})_{c_2} = \emptyset \),
   3.ii) \( \text{int}(W_2)_{c_2} \supseteq f_{c_1, c_2}((W_1)_{c_1}) \).

The correct alignment of windows is robust, in the sense that if two windows are correctly aligned under a map, then they remain correctly aligned under a sufficiently small \( C^0 \)-perturbation of the map. This allows to verify the correct alignment of long, finite sequences of windows, by breaking them into shorter, finite sequences of windows whose correct alignment can be easily controlled by perturbative arguments. This property will not be used in this paper.

Also, the correct alignment satisfies a natural product property. Given two windows and a map, if each window can be written as a product of window components, and if the components of the first window are correctly aligned with the corresponding components of the second window under the appropriate components of the map, then the first window is correctly aligned with the second window under the given map. We refer to [34] for details.

The following result can be thought of as a topological version of the Shadowing Lemma. Note that this result does not assume that the system is hyperbolic.

**Theorem 5.3 ([63]).** Assume that \( \{W_i\}_{i \in \mathbb{Z}} \) is a bi-infinite sequence of \((n_1, n_2)\)-windows in \( M \), and \( \{f_i\}_{i \in \mathbb{Z}} \) are continuous maps on \( M \). If \( W_i \) is correctly aligned with \( W_{i+1} \) under \( f_i \) for every \( i \in \mathbb{Z} \), then there exists a point \( p \in W_0 \) such that
\[
(f_i \circ \ldots \circ f_0)(p) \in W_{i+1},
\]
Assume now that \( \{W_i\}_{i \in \{0,\ldots,d\}} \) is a finite sequence of \((n_1,n_2)\)-windows in \( M \), and \( \{f_i\}_{i \in \{0,\ldots,d\}} \) are continuous maps on \( M \). If \( W_i \) is correctly aligned with \( W_{i+1} \) under \( f_i \) for every \( i = 0,\ldots,d-1 \), and \( W_d \) is correctly aligned with \( W_0 \) under \( f_d \), then there exists a point \( p \in W_0 \) such that 

\[
(f_d \circ \ldots \circ f_0)(p) = p.
\]

A sequence of windows \( \{W_i\} \) as above will be referred to as a sequence of correctly aligned windows. Note that the verification of the correct alignment of the sequence amounts to verifying correct alignment relations between successive pairs \( W_i, W_{i+1} \). A consequence of this fact, which is important for our applications, is that the concatenation of finite sequences of correctly aligned windows is a finite sequence of correctly aligned windows. That is, if \( W_0,\ldots,W_k \) forms a sequence of correctly aligned windows, and \( W_k,\ldots,W_l \) forms a sequence of correctly aligned windows, then \( W_0,\ldots,W_l \) also forms a sequence of correctly aligned windows (for simplification, we omit here to specify the mappings under which the correct alignment is realized).

In the context of this paper, the maps \( f_i \) from Theorem 5.3 will be different powers of the time-1 map associated to the flow.

We emphasize again the difference between hyperbolic dynamics and correct alignment of windows. To assert that an orbit is hyperbolic one needs to examine the expansion and contraction rates of the derivative of the map along the whole orbit. The concatenation of hyperbolic segments could fail to be hyperbolic (if the stable and unstable directions do not match). In contrast, to assert that a sequence of windows is correctly aligned, one only needs to verify that the image of one window under the map is correctly aligned with the next window in the sequence. Also, concatenations of finite sequences of correctly aligned windows are correctly aligned.

### 5.2. Reduction to a discrete dynamical system

We reduce the perturbed geodesic flow to a discrete dynamical system by considering the time-1 map of the flow \( \psi^\varepsilon \) – where the time refers to the rescaled time \( s \) in (3.3) – which we denote \( F_{\varepsilon} \). Also, we denote by \( \chi_{\varepsilon}^l \) the time-1 map (relative to the rescaled time \( s \)) associated to the flow \( \chi^l \) on \( N \). Since \( s = t/\varepsilon \), the time-1 map relative to the rescaled time \( s \) is the time-\( \varepsilon \) map relative to the physical time \( t \).

The distinguished geometric objects for the perturbed geodesic flow, described in the earlier sections, give rise to similar objects for the discrete dynamical system. We start by listing these objects and summarizing their properties.

**B1.** The map \( F_{\varepsilon} : T^*M \times N \to T^*M \times N \) is a \( C^{r-1} \)-diffeomorphism.

**B2.** The manifold \( \Lambda_\varepsilon = \Lambda_\varepsilon \times N \subseteq T^*M \times N \) is a normally hyperbolic invariant manifold for \( F_{\varepsilon} \), of dimension \((d+2)\); this manifold has stable and unstable manifolds \( W^s(\Lambda_\varepsilon) \) and \( W^u(\Lambda_\varepsilon) \), of dimension \((d+n+1)\).

**B3.** There exist exponential rates \( 0 < \lambda_\varepsilon < \lambda_+ < \lambda_1 < 1 < \mu_1 < \mu_\varepsilon < \mu_+ \) such that \( \lambda_\varepsilon < \|DF_{\varepsilon}|_{E^s_{\varepsilon,\varepsilon}}\| < \lambda_+ \), \( \mu_\varepsilon < \|DF_{\varepsilon}|_{E^u_{\varepsilon,\varepsilon}}\| < \mu_+ \), \( \lambda_1 < \|DF_{\varepsilon}|_{E^s_{\varepsilon,\varepsilon}}\| < \mu_1 \), where \( E^s_{\varepsilon,\varepsilon} \) and \( E^u_{\varepsilon,\varepsilon} \) are the stable and unstable bundles in the decomposition \( T_{\varepsilon}(T^*M \times N) = T_{\varepsilon}\Lambda_\varepsilon \oplus E^s_{\varepsilon,\varepsilon} \oplus E^u_{\varepsilon,\varepsilon} \). The above exponential rates can be chosen independently of \( \varepsilon \).
B4. The stable and unstable manifolds $W^s(\Lambda_\varepsilon)$ and $W^u(\Lambda_\varepsilon)$ have a transverse intersection along a $(d + 2)$-dimensional manifold $\tilde{\Gamma}_\varepsilon$.

B5. For the discrete dynamical system defined by $F_\varepsilon$ there exists two scattering maps $\tilde{S}_2^j : \tilde{U}_2^{j-} \to \tilde{U}_2^{j+}$ associated to two homoclinic channels $\tilde{\Gamma}_j^2$, $j = 1, 2$, satisfying Assumption (A4). The set $\tilde{U}_2^{j-}$ is of size $O(1)$ in the sense that there exist open sets $U_2^{j-} \subseteq \Lambda_0$ such that $\tilde{k}_j(U_2^{j-} \times N) \subseteq \tilde{U}_2^{j-}$ for $j = 1, 2$, and for all $\varepsilon$ sufficiently small, where $\tilde{k}_j : \Lambda_0 = \Lambda_0 \times N \to \Lambda_{\varepsilon}$ is the parametrization of $\Lambda_{\varepsilon}$ from Subsection 3.2.

B6. Consider the action-angle coordinates $(J_\varepsilon, \phi_\varepsilon)$ on $\Lambda_{\varepsilon}$. The restriction of $F_\varepsilon$ to $\Lambda_{\varepsilon}$ has the form

$$F_\varepsilon(J_\varepsilon, \phi_\varepsilon, \theta) = (J_\varepsilon + O(\varepsilon^2), \phi_\varepsilon + J_\varepsilon + O(\varepsilon^2), \chi_\varepsilon(\theta)).$$

There exists $\tau > 0$, independent of $\varepsilon$, such that

$$\frac{\partial (\pi_{\phi_\varepsilon} \circ F_\varepsilon_{\mid \Lambda_{\varepsilon}})}{\partial J_\varepsilon}(J_\varepsilon, \phi_\varepsilon) > \tau,$$

for all $(J_\varepsilon, \phi_\varepsilon, \theta) \in \Lambda_{\varepsilon}$. In particular $F_{\varepsilon_{\mid \Lambda_{\varepsilon}}}$ is an integrable twist map in the variables $(J_\varepsilon, \phi_\varepsilon)$, up to order $O(\varepsilon^2)$, with the twist coefficient lower bounded by $\tau$.

B7. Each scattering map $\tilde{S}_2^j : \tilde{U}_2^{j-} \to \tilde{U}_2^{j+}$, associated to the homoclinic channel $\tilde{\Gamma}_j$, $j \in \{1, 2\}$, is of the form

$$\begin{align*}
\tilde{S}_2^j(J_\varepsilon, \phi_\varepsilon, \theta^-) &= (J_\varepsilon^+, \phi_\varepsilon^+, \theta^+), \\
\phi_\varepsilon^+ &= \phi_\varepsilon^+ + a + O(\varepsilon^2), \\
|J_\varepsilon^+ - J_\varepsilon^-| &= O(\varepsilon^2), \\
\theta^+ &= \theta^-.
\end{align*}$$

B7. There exists a sequence of elementary building blocks, of the type $B^j(J, \phi, \theta)$, $j = 1, 2$, as in Subsection 4.3, along which the scaled energy $H_\varepsilon$ grows by $O(\varepsilon)$ in a scaled time interval $\Delta s = O(1/\varepsilon^2)$. Each building block consists of one application of one of the scattering maps, $\tilde{S}_2^j$, $j = 1, 2$, succeeded by an orbit of the inner dynamics followed for a scaled time of $O(1)$. The succession of elementary building blocks is chosen as in Subsection 6.4.

The general strategy to prove the Theorems 2.1 [2.2] [2.3] is the following. We will choose a sequence of elementary building blocks as before, and will fix a two-dynamics pseudo-orbit determined by it. We will prove that there exists a true orbit that follows this pseudo-orbit. Also, we will estimate the time it takes to such an orbit to perform the trip.

As explained in Subsection 4.2 an elementary building block consists of a segment of a homoclinic orbit $\psi_\varepsilon^s(z_\varepsilon)$, $s \in [T_{-}, T_{+}]$, followed by a trajectory segment $(\psi_\varepsilon^s)_{\Lambda_\varepsilon}(z_\varepsilon^+)$ of the flow $\psi_\varepsilon^s$ restricted to $\Lambda_{\varepsilon}$, during a time interval of order $O(1)$. Consider a sequence of $(n_1 + 1)$ successive elementary building blocks. For each one of them, we denote by $\tilde{z}_i^s(s)$ the homoclinic segment of the elementary building block, and by $\tilde{\lambda}_i(s)$ the trajectory segment of the flow $\psi_\varepsilon^s$ restricted to $\Lambda_{\varepsilon}$ corresponding to that building block, where $i = 0, \ldots, n_1$. Also, we denote by $z_{i-1}^s$, the corresponding homoclinic point on $\tilde{z}_i^s$, with $i \in \{1, \ldots, n_1\}$. We allow the considered homoclinic orbit segments to correspond to different homoclinic manifolds $\tilde{\Gamma}_\varepsilon$. 


with \( j = 1, 2 \), but we do not make this distinction in the notation (for typographical reasons), since all estimates are uniform.

Each trajectory segment that is a part of an elementary building block can be written as \( \tilde{\gamma}_i^j(s) = (\gamma_i^j(s), \theta(s)) \), and \( \tilde{\lambda}_i^j(s) = (\lambda_i^j(s), \theta(s)) \), respectively. Since a trajectory segment \( \tilde{\lambda}_i^j(s) \) is followed for a time \( O(1) \), the action \( J_\varepsilon \)-coordinate along \( \lambda_i^j(s) \) stays constant up to an \( O(\varepsilon^2) \) error. Thus, to the trajectory segment \( \lambda_i^j(s) \) we can associate a level set \( \{ J_\varepsilon = J_{\lambda i}^j \} \) of the \( J_\varepsilon \)-coordinate in \( \Lambda_\varepsilon \), which is almost invariant up to \( O(\varepsilon^2) \).

For each \( i \in \{ 0, \ldots, n_1 \} \), denote \( \mathcal{L}_i^j = \lambda_i^j \times N \); this is a \((1 + d)\)-dimensional manifold in \( T^*M \times N \). The stable and unstable manifolds of \( \mathcal{L}_i^j \) are \((n + d)\)-dimensional manifolds \( W^s(\mathcal{L}_i^j) = \bigcup_{z \in \mathcal{L}_i^j} W^s(z) \) and \( W^u(\mathcal{L}_i^j) = \bigcup_{z \in \mathcal{L}_i^j} W^u(z) \), respectively, where the stable and unstable fibers of points are well defined due to the normal hyperbolicity of \( \Lambda_\varepsilon \). Note that these sets are not, in general, invariant, as their fibers are not invariant; we have \( F_\varepsilon(W^s(z)) \subseteq W^s(F_\varepsilon(z)) \) and \( F_\varepsilon^{-1}(W^u(z)) \subseteq W^u(F_\varepsilon^{-1}(z)) \). By construction, we have that for each \( i \in \{ 1, \ldots, n_1 \} \), \( W^u(\mathcal{L}_{i-1}^j) \) intersects transversally with \( W^s(\mathcal{L}_i^j) \) along \( \{ z_{i-1}^j \} \times N \).

To show the existence of an orbit that follows the pseudo-orbit determined by the given sequence of elementary building blocks, we will use the stable and unstable manifolds to build a chain of sets \( \mathcal{L}_i^j \), with \( i = 0, \ldots, n_1 \), around which we will construct windows that are correctly aligned. The construction will be done according to the following sequence of steps, which we first describe informally below.

At the first step, for each \( i = 1, \ldots, n_1 \), we construct a pair of windows \( W_{i-1}^- \) and \( W_{i-1}^+ \) about the heteroclinic intersection \( \{ z_{i-1}^j \} \times \tilde{\Gamma}_i^j \) and \( W_{i-1}^- \times \tilde{\Gamma}_i^j \), such that \( W_{i-1}^- \) is correctly aligned with \( W_{i-1}^+ \) under the identity mapping. Here the homoclinic intersection \( \tilde{\Gamma}_i^j \) is one of the two homoclinic intersections defined by the distinguished homoclinic orbits \( \gamma_i^j \), \( j \in \{ 1, 2 \} \), described in Assumption (A4).

At the second step we iterate \( W_{i-1}^- \) forward in time along \( W^s(\mathcal{L}_i^j) \) until its image is contained in some conveniently chosen neighborhood of \( \mathcal{L}_i^j \) in \( T^*M \), and we construct a new window \( W_i^- \) near \( \mathcal{L}_i^j \), such that \( W_i^- \) is correctly aligned with \( W_i^+ \) under some positive iterate of \( F_\varepsilon \). Also, we iterate \( W_{i-1}^- \) backwards in time along \( W^u(\mathcal{L}_{i-1}^j) \) until its image is contained in some conveniently chosen neighborhood of \( \mathcal{L}_{i-1}^j \), and we construct a new window \( W^-_{i-1} \) about \( \mathcal{L}_{i-1}^j \), such that \( W^-_{i-1} \) is correctly aligned with \( W^+_i \) under some positive iterate of \( F_\varepsilon \). At this step, the neighborhoods of \( \mathcal{L}_i^j \) and \( \mathcal{L}_{i-1}^j \) are chosen so that the system of coordinates from Subsection 5.3.1 near the normally hyperbolic invariant manifold \( \Lambda_\varepsilon \), is well defined in those neighborhoods. The \( \Lambda_\varepsilon \)-component of the window \( W_{i-1}^- \) is close to the level set \( J_\varepsilon = J_{\lambda_{i-1}}^j \) of the action coordinate \( J_\varepsilon \), and the \( \Lambda_\varepsilon \)-component of the window \( W_i^- \) is close to the level set \( J_\varepsilon = J_{\lambda_{i+1}}^j \) of the action coordinate \( J_\varepsilon \) on \( \Lambda_\varepsilon \).

At the third step we align the window \( W_i^- \), corresponding to the heteroclinic connection \( W^u(\mathcal{L}_{i-1}^j) \cap W^s(\mathcal{L}_i^j) \), to the window \( W_i^- \), corresponding to the heteroclinic connection \( W^u(\mathcal{L}_i^j) \cap W^s(\mathcal{L}_{i+1}^j) \). The \( \Lambda_\varepsilon \)-components of these two windows are close to the same level set \( J_\varepsilon = J_{\lambda_i}^j \) of the action coordinate. The construction at the third step concatenates the sequence of correctly aligned windows constructed about one heteroclinic connection with the sequence of correctly aligned windows constructed about the next heteroclinic connection.

At the fourth step we concatenate short sequences of correctly aligned windows constructed as above, obtaining a long sequence of correctly aligned windows that
follows the pseudo-orbit underlying the sequence of elementary building blocks that achieves the desired energy growth.

The conclusion is that, once the windows have been constructed, the shadowing result Theorem 5.3 will provide the existence of true orbits that visit the windows in the prescribed order, hence these orbits will visit the prescribed level sets of the averaged action.

The construction of windows is similar to that in [34]. Therefore we will describe most steps of the construction succinctly. One step that is quite different is the third step, in which we align two windows $\hat{W}_i$ and $\tilde{W}_i$ about $L_{\epsilon_i}$ under some iterate of the map $F_{\epsilon}$. The difference is that in [34] the map is a twist map, while in our case, the map is a twist map in one component and a time-1 map of some general flow in the other component. We will explain this step of the construction in a greater detail.

Now we explain how the above strategy is used to prove the statements from Theorems 2.1, 2.2, 2.3.

To prove that there exist trajectories along which the energy grows unboundedly we proceed as follow. We fix a potential $V \in V'$, i.e., satisfying condition $A_4$. Under the hypotheses of Theorem 2.1 we choose an initial condition $(J^0, \phi_0, \theta_0)$, where $\theta_0$ is chosen to be a non-trivial uniformly recurrent point, $\phi_0$ is fixed as in condition $A_4$. Under the hypotheses of Theorem 2.2 we choose the same initial condition with an arbitrary $\theta_0$. Starting with this initial condition, we construct a finite sequence of elementary building blocks as in Subsection 4.2, with the underlying pseudo-orbit $\tilde{\gamma}^0_0, \tilde{\lambda}^0_0, \ldots, \tilde{\gamma}^0_{n_1}, \tilde{\lambda}^0_{n_1}$, for some $n_1 = O(1/\epsilon^3)$, along which the scaled energy grows from $E = 1$ to $E = 2$ in a scaled time $O(1/\epsilon^3)$.

This corresponds to a growth of the physical energy by $O(1/\epsilon^2)$ in a physical time $O(1/\epsilon^2)$. This physical energy growth is linear in the physical time. If the initial energy level of the physical energy is $E_*=E_*$, the physical energy at the end of this sequence of elementary building blocks is $2E_*$. The windowing construction from above above provides a finite sequence of correctly aligned windows, whose $\Lambda_\epsilon$-components are close to these level sets. Any shadowing orbit obtained from Theorem 5.3 will result in an growth of the scaled energy of $O(1)$ in a scaled time $O(1/\epsilon^3)$.

Note that, as remarked before, segments of correctly aligned windows can be concatenated. Since the constructions are uniform for all $\epsilon < \epsilon_0$, we can construct infinitely many segments that make the energy grow to $\infty$, and concatenate them.

More precisely, to obtain orbits whose scaled energy grows to infinity we proceed as in Subsection 4.5. We reset $\epsilon$ to $\epsilon = 1/\sqrt{2E_*}$ and we repeat the construction of a sequence of elementary building blocks whose scaled energy grows from $E = 1$ to $E = 2$ in a scaled time $O(1/\epsilon^3)$. The energy growth rate is still linear, and at least as large as in the previous step. The physical energy grows from $2E_*$ to $4E_*$. Then we concatenate the segment of correctly aligned windows that grows the physical energy from $E_*$ to $2E_*$, with the segment that grows the physical energy from $2E_*$ to $4E_*$. The concatenation of these segments of correctly aligned windows constructed is also a segment of correctly aligned windows. This construction of segments of elementary building blocks and corresponding segments of correctly aligned windows can be repeated indefinitely. The shadowing orbit that visits the resulting infinite sequence of correctly aligned windows yields an infinite physical
energy growth at a linear rate with respect to the physical time. In this way we obtain the statement on the existence of orbits with unbounded energy growth from Theorems 2.1 and 2.2.

To obtain an orbit whose scaled energy follows a prescribed energy path, we note that a path $E : [0, \infty) \rightarrow \mathbb{R}$ determines, via time-discretization, a sequence $J_\epsilon$-action level sets $\{ \sqrt{2 \mathcal{E}(k)} \}_{k \in \mathbb{N}}$ in $\Lambda_\epsilon$. As in Subsection 4.6 we construct a sequence of elementary building blocks whose corresponding $J_\epsilon$-values visit, in the prescribed order, the values $\sqrt{2 \mathcal{E}(k)}$, $k \in \mathbb{Z}$. Since, in general, one cannot move from one value $\sqrt{2 \mathcal{E}(k)}$ to the next $\sqrt{2 \mathcal{E}(k+1)}$ via a single elementary building block, the construction yields a sequence of action level sets $\{ J_\epsilon^i \}_{i \in \mathbb{Z}}$, such that successive elementary building blocks correspond to successive value of $J_\epsilon^i$, and there exists a subsequence $\{ i_k \}_{k \in \mathbb{N}}$ such that $J_\epsilon^{i_k} = \sqrt{2 \mathcal{E}(k)}$. Thus, we are in the same situation as described above, and we can proceed in the same way.

5.3. Construction of windows. In this subsection we will work out the details of the construction of correctly aligned windows described earlier.

We consider a finite sequence of elementary building blocks as in Subsection 4.2, with the corresponding pseudo-orbit

$$\gamma_{\epsilon}^0, \tilde{\lambda}_{\epsilon}^0, \ldots, \gamma_{\epsilon}^{n_1}, \tilde{\lambda}_{\epsilon}^{n_1},$$

for some $n_1 = O(1/\epsilon^3)$. The corresponding action $J$ level sets corresponding to the curves $\lambda_i^\epsilon$ in $\Lambda_\epsilon$, $i = 0, \ldots, n_1$, are

$$J_\epsilon^0, J_\epsilon^1, \ldots, J_\epsilon^{n_1}.$$

5.3.1. A system of coordinates near a normally hyperbolic invariant manifold. The construction of windows requires a coordinate systems relative to which the windows can be defined.

We now describe a general construction of a system of coordinates near a normally hyperbolic invariant manifold. Given a normally hyperbolic invariant manifold $\Lambda \subseteq M$ for $F : M \rightarrow M$, the bundle $N_x = E_x^s \oplus E_x^u$, $x \in \Lambda$, gives a concrete realization of the normal bundle $N_\Lambda$. That is, $N_x$ is the complementary subspace to $T_x \Lambda$, i.e., $T_x M = T_x \Lambda \oplus N_x$, for all $x \in \Lambda$.

For any point $p$ in a sufficiently small neighborhood a neighborhood $\mathcal{N}(\Lambda)$ of $\Lambda$ in $M$, we can find unique $x \in \Lambda$, $s \in E_x^s$, and $u \in E_x^u$, with $u, s$ small, such that $p = \exp_x(s+u)$. Therefore, it is natural to use $(x, s, u)$ as a system of coordinates in $\mathcal{N}(\Lambda)$. We denote $h(x, s, u) = \exp_x(s+u)$, where exp : $TM \rightarrow M$ is the exponential map.

In general, this system of coordinates is as smooth as the bundles $E^s, E^u$, whose regularity is limited by the regularity of the map $F$ and by the ratios of the exponential rates.

In our case, because the exponential rate of $F_\epsilon$ on the manifold $\tilde{\Lambda}_\epsilon$ is close to $0$ for $\epsilon$ small enough, we have that the corresponding maps $h_{\epsilon}, h_{\epsilon}^{-1}$ are $C^r$. We also note that if we express $F_\epsilon$ in terms of the coordinate mapping, we have, for $F_\epsilon \in C^2$,

$$h_{\epsilon}^{-1} \circ F_\epsilon \circ h_{\epsilon}(x, s, u) = (F_\epsilon(x), DF_\epsilon(x)s, DF_\epsilon(x)u) + O(\delta^2),$$

where $\delta > 0$ is the size of the neighborhood $\mathcal{N}(\Lambda_\epsilon)$. Furthermore

$$Dh_{\epsilon}^{-1} \circ F_\epsilon \circ h_{\epsilon}^{-1}(x, s, u) = DF_\epsilon(x) + O(\delta).$$
If we express \( F_\varepsilon \) relative this system of coordinates, \( \tilde{F}_\varepsilon = h_\varepsilon^{-1} \circ F_\varepsilon \circ h_\varepsilon \) is close to the tangent map

\[
TF_\varepsilon(x, s, u) = (F_\varepsilon(x), DF_\varepsilon(x)s, DF_\varepsilon(x)u).
\]

Indeed, we have

\[
\|\tilde{F}_\varepsilon - TF_\varepsilon\|_{C^0} \leq C\delta^2,
\]

\[
\|\tilde{F}_\varepsilon - TF_\varepsilon\|_{C^1} \leq C\delta,
\]

for some \( C > 0 \), where \( \delta \) is the size of the neighborhood \( \mathcal{N}(\Lambda_\varepsilon) \). The constants \( \delta \) and \( C \) can be chosen independent of \( \varepsilon \).

In particular, by considering a sufficiently small neighborhood \( \mathcal{N}(\Lambda_\varepsilon) \) we can ensure that the map \( F_\varepsilon \) is contracting the \( s \)-components, expanding in the \( u \)-components, and more or less neutral in \( x \).

Noting that we can describe the manifold \( \tilde{\Lambda}_\varepsilon \) via the coordinates \((J, \phi, \theta)\), the above construction provides us with a \( C^1 \)-smooth coordinate system \((J, \phi, \theta, s, u)\) in a neighborhood \( \mathcal{N}(\Lambda_\varepsilon) \) of \( \tilde{\Lambda}_\varepsilon \). Relative to this coordinate system the map \( F_\varepsilon \) can be approximated by a skew product of a map acting in the center directions of \( \tilde{\Lambda}_\varepsilon \) and a map acting in the hyperbolic directions. To simplify notation, we will not make explicit the dependence on \( \varepsilon \) of the coordinate systems. Also, from this point on we will denote \( F_\varepsilon \) by \( F \).

We will use this system of coordinates to construct windows near \( \Lambda_\varepsilon \), and, in particular, about the sets \( \mathcal{L}_i \).

We will also need to construct windows about the heteroclinic intersections \( (\{z_{i-1,i}^\varepsilon\} \times N) \cap \tilde{\Gamma}_\varepsilon \) with \( W^s(\mathcal{L}_i^\varepsilon) \). For this, we propagate the above coordinate system from a neighborhood of the set \( \mathcal{L}_i^{\varepsilon -1} \) in \( \tilde{\Lambda}_\varepsilon \) along the unstable manifold \( W^u(\mathcal{L}_i^{\varepsilon -1}) \), giving rise to a \( C^1 \)-smooth coordinate system \((J^-, \phi^-, \theta, s^-, u^-)\) in a neighborhood of \( (\{z_{i-1,i}^\varepsilon\} \times N) \cap \tilde{\Gamma}_\varepsilon \). Also, we propagate the same coordinate system from a neighborhood of the leaf \( \mathcal{L}_i^\varepsilon \) along the stable manifold \( W^s(\mathcal{L}_i^\varepsilon) \) to a neighborhood of the heteroclinic intersection \( (\{z_{i-1,i}^\varepsilon\} \times N) \cap \tilde{\Gamma}_\varepsilon \), producing a \( C^1 \)-smooth coordinate system \((J^+, \phi^+, \theta, s^+, u^+)\) in the neighborhood of \( (\{z_{i-1,i}^\varepsilon\} \times N) \cap \tilde{\Gamma}_\varepsilon \).

The two coordinate systems differ one from the other by order \( O(1) \), that is, if \( \Phi \) denotes the coordinate change from one system to another then \( C_3^{-1} \leq \|D\Phi\| \leq C_3 \) uniformly in a compact neighborhood of \( \tilde{\Gamma}_\varepsilon \), for some \( C_3 > 1 \). Since we obtain two coordinate systems around \( (\{z_{i-1,i}^\varepsilon\} \times N) \cap \tilde{\Gamma}_\varepsilon \), we can construct windows and verify their correct alignment in either coordinate system. The sizes of the components of a window in one coordinate system will differ by the sizes in the other coordinate system by some multiplicative constants that are independent of \( \varepsilon \) and, due to compactness, they can be chosen to be the same for all heteroclinic intersections.

5.3.2. Choice of constants. We choose some constants that will be used throughout the proof. Let \( a^\pm \) be defined as follows

\[
a^\pm = \sup_{z \in H^{-1}[1, 2] \cap \tilde{\Gamma}_\varepsilon} d(z^\pm, z),
\]

where the distance is measured along the stable (unstable) fiber through \( x \), respectively. Since \( H^{-1}[1, 2] \cap \tilde{\Gamma}_\varepsilon, H^{-1}[1, 2] \cap \tilde{\Lambda}_\varepsilon \) are compact, then \( 0 < a^\pm < \infty \).
First we choose some constant $\varepsilon_1 > 0$, independent of $\varepsilon$, and then we choose the positive constants $\alpha^-, \alpha^+, \hat{\alpha}, \hat{\alpha}$ and $\beta^-, \beta^+, \hat{\beta}, \hat{\beta}$, independent of $\varepsilon$, such that
\begin{align}
3\varepsilon_1 < \alpha^- &= \hat{\alpha} = \alpha < C_3^{-1} \alpha^+, \\
3\varepsilon_1 < \beta^+ &= \hat{\beta} = \beta < C_3^{-1} \beta^-.
\end{align}

Second, we choose positive integers $N, M$ sufficiently large so that
\begin{align}
\lambda_N^N(a^+ + \alpha) &< 2\varepsilon_1, \quad \hat{\beta} + \varepsilon_1 < \mu_N^N \beta^+, \quad \mu_M^{-M}(\beta^- + a^-) < 2\varepsilon_1, \\
F^N(\tilde{\Gamma}_\varepsilon) &\subseteq N(\tilde{\Lambda}_\varepsilon), \quad F^{-M}(\tilde{\Gamma}_\varepsilon) \subseteq N(\tilde{\Lambda}_\varepsilon),
\end{align}
where $N(\tilde{\Lambda}_\varepsilon)$ is the neighborhood of $\tilde{\Lambda}_\varepsilon$ where the coordinate system described in Subsection 5.3.1 is defined. Note that there exist finite $N, M$ as in (5.3) due to the definition of $\tilde{\Gamma}_\varepsilon$. Since the dynamical system (5.3) tends to a product system when $\varepsilon \to 0$, then we can choose $N, M$ to be independent of $\varepsilon$ for all $\varepsilon$ sufficiently small.

Third, we choose a positive integer $K$, satisfying the following condition.
\begin{align}
2\hat{\gamma} + \hat{\gamma} + 3\varepsilon_1 &< K\tau\delta,
\end{align}
where $\tau$ is the twist constant from (B6). Such a finite $K$ exists, for all small enough $\varepsilon$, since the inner map $F_i$ is a twist map in the $(J, \phi)$-variables.

Fourth, we choose the positive constants $\delta^-, \delta^+, \delta, \gamma^-, \gamma^+, \hat{\gamma}, \hat{\gamma}$, such that they satisfy the following inequalities
\begin{align}
\hat{\delta} < \delta + \varepsilon_1 < \delta^+ &< C_3^{-1} \gamma^- < \gamma^- < \gamma^- + M\tau\delta^- + \varepsilon_1 < \hat{\gamma}, \\
\hat{\delta} < \delta + \varepsilon_1 < \delta < \delta^+ &< C_3^0 \delta^- < \gamma^+ < \gamma^+ + N\tau\delta^+ + \varepsilon_1 < \hat{\gamma}.
\end{align}
Additionally, we want these constants to be sufficiently small. We will explain later in the argument how small should they be, but we emphasize here that the smallness condition is independent of $\varepsilon$ and can be made precise from the beginning of the argument.

Fifth, we make $\varepsilon$ even smaller, if necessary, as described below. Note that the estimates on the inner map $F_i |\tilde{\Lambda}$ and on the outer map $S_\varepsilon$ involve some error terms of orders $O(\varepsilon^2)$, as in B6 and B7. Due to the compactness of $H^{-1}[1, 2] \cap \tilde{\Lambda}_\varepsilon$ and $H^{-1}[1, 2] \cap \tilde{\Gamma}_\varepsilon$, these error terms can be bounded from above by $C_4\varepsilon^2$, for some constant $C_4 > 0$ independent of $\varepsilon$. Now we choose $\varepsilon$ sufficiently small so that
\begin{align}
NC_4\varepsilon^2 &< \varepsilon_1, \quad MC_4\varepsilon^2 < \varepsilon_1, \quad KC_4\varepsilon^2 < \varepsilon_1.
\end{align}
Thus, when we will estimate the error terms when iterating the inner map or the outer map up to max{$M, N, K$} times, we will be able to conclude that the error terms are always less than $\varepsilon_1$.

From now on $\varepsilon$ is sufficiently small and fixed.

5.3.3. Step 1. Let us consider the heteroclinic intersection $W^u(L_{i-1}^-) \cap W^s(L_i^+)$ at $\{z_{i-1}^\varepsilon\} \times N$. As seen before, the normal hyperbolicity implies that there exist $z_{i-1}^\varepsilon, z_i^\varepsilon \in \tilde{\Gamma}_i^\varepsilon$ such that $z_{i-1}^\varepsilon \in W^u(z_{i-1}^\varepsilon) \cap W^s(z_i^\varepsilon)$. Let $a_{i-1}^-$ be the distance along $W^u(L_{i-1}^-)$, and let $a_i^+$ be the distance along $W^s(L_i^+)$, we have that $a_i^+ < a^+$ and $a_i^- < a^-$.

At this step we construct a pair of windows $W_{i-1}^-, W_i^+$ about $\{z_{i-1}^\varepsilon\} \times \tilde{\Gamma}_\varepsilon$, such that $W_{i-1}^-$ is correctly aligned with $W_i^+$ under the identity mapping.
We define the window \( W_{i-1}^- \) in the coordinates \((s^-, u^-, \phi^-, J^-, \theta)\) of the type
\[
W_{i-1}^- = (R^{e-}_{i-1} \times R^{u-}_{i-1}) \times (R^{\phi-}_{i-1} \times R^{J-}_{i-1}) \times R^0_{i-1},
\]
where \( R^e \) denotes a rectangle in the coordinate \( c \). We will think of \( W_{i-1}^- \) as a product of three window components: \( R^{e-}_{i-1} \times R^{u-}_{i-1} \), corresponding to the hyperbolic coordinates, \( R^{\phi-}_{i-1} \times R^{J-}_{i-1} \), corresponding to the angle-action coordinates, and \( R^0_{i-1} \) corresponding to the external system on \( N \).

We define the exit set \( (W^-)_{i-1}^{\text{exit}} \) of \( W_{i-1}^- \) by
\[
(W^-)_{i-1}^{\text{exit}} = \left( (R^{e-}_{i-1} \times \partial R^{u-}_{i-1}) \times (R^{\phi-}_{i-1} \times R^{J-}_{i-1}) \times R^0_{i-1} \right)
\]
\[
\cup \left( (R^{e-}_{i-1} \times R^{u-}_{i-1}) \times (\partial R^{\phi-}_{i-1} \times R^{J-}_{i-1}) \times R^0_{i-1} \right),
\]
\[
\cup \left( (R^{e-}_{i-1} \times R^{u-}_{i-1}) \times (R^{\phi-}_{i-1} \times \partial R^{J-}_{i-1}) \times \partial R^0_{i-1} \right).
\]

This means that the exit directions of \( W_{i-1}^- \) correspond to the unstable direction \( u^- \) of the hyperbolic window \( R^{e-}_{i-1} \times R^{u-}_{i-1} \), to the angle direction \( \phi^- \) of the action-angle window \( R^{\phi-}_{i-1} \times R^{J-}_{i-1} \), and to all directions of the external-state window \( R^0_{i-1} \). This definition of the exit set follows the construction of product of windows described in [34].

Similarly, we define the window \( W_i^+ \) in the coordinates \((s^+, u^+, \phi^+, J^+, \theta)\) to be given by the product
\[
W_i^+ = (R^{e+}_{i} \times R^{u+}_{i}) \times (R^{\phi+}_{i} \times R^{J+}_{i}) \times R^0_{i},
\]
and its exit set \( (W^+)_{i}^{\text{exit}} \) by
\[
(W^+)_{i}^{\text{exit}} = \left( (R^{e+}_{i} \times \partial R^{u+}_{i}) \times (R^{\phi+}_{i} \times R^{J+}_{i}) \times R^0_{i} \right)
\]
\[
\cup \left( (R^{e+}_{i} \times R^{u+}_{i}) \times (\partial R^{\phi+}_{i} \times R^{J+}_{i}) \times R^0_{i} \right),
\]
\[
\cup \left( (R^{e+}_{i} \times R^{u+}_{i}) \times (R^{\phi+}_{i} \times \partial R^{J+}_{i}) \times \partial R^0_{i} \right).
\]

The exit directions of \( W_i^+ \) correspond to the direction \( u^+ \) of \( R^{e+}_{i} \times R^{u+}_{i} \), to the direction \( J^+ \) of \( R^{\phi+}_{i} \times R^{J+}_{i} \), and to all directions of \( R^0_{i} \). Note that the only difference in the exit directions of \( W_i^+ \) from \( W_{i-1}^- \) is the switching from the direction \( \phi^- \) to the direction \( J^+ \) in the angle-action components. Since we are not using yet any dynamics in constructing these windows, this switching in the exit direction of the action-angle components may seem arbitrary. The reason for this switching will become apparent later when we align windows by the inner map: when an action-angle component is iterated under the inner dynamics until it crosses another action-angle component, the twist property of the inner map causes the first window to have its action direction stretched across the second window along its angle direction. Therefore we will have a switching of the exit directions due to the alignment under the inner map. By having another switching of the exit direction at the heteroclinic intersection, we ensure the consistency of the correct alignment in the two-step process. This will be explained in greater detail at Step 3.

We set the sizes of the rectangular components of the windows as follows:
\[
\| R^{e-}_{i-1} \| = \alpha^-, \quad \| R^{u-}_{i-1} \| = \beta^-
\]
\[
\| R^{\phi-}_{i-1} \| = \gamma^-, \quad \| R^{J-}_{i-1} \| = \delta^-.
\]
Above, by \( \| R^c \| \) we mean the diameter of the rectangle \( R^c \) of the projection onto the coordinate \( c \).

We want to ensure that \( W^-_{i-1} \) is correctly aligned with \( W^+_i \) under the identity mapping. The two windows are defined in two different coordinate systems, so we need to use the coordinate change \( \Phi \) to compare them relative to the same coordinate system. We therefore require that \( \Phi(R^-_{i-1}) \subseteq \text{int}(R^+_i) \), \( \Phi(R^u_{i-1}) \supseteq R^u_i \), \( \int(\Phi(R^0_{i-1})) \supseteq R^I_i \), and \( \Phi(R^\phi_{i-1}) \subseteq \text{int}(R^\phi_i) \). These conditions imply that \( W^-_{i-1} \) is correctly aligned with \( W^+_i \) under the identity mapping. These conditions are consistent with the size specifications in (5.10) and (5.11), due to the choice of constants in (5.1), (5.2), (5.3), and (5.4).

5.3.4. Step 2. At this step we take a forward iterate of the window \( W^+_i \) along the stable manifold \( W^*(\mathcal{L}^+_i) \), and we align its image with a window \( W_i \) near \( \Lambda \). Similarly, we take a backwards iterate of the window \( W^-_{i-1} \) along the unstable manifold \( W^u(\mathcal{L}^-_{i-1}) \), and we align its image with a window \( W^-_i \) near \( \Lambda \).

First, we consider the positive iterate \( F^N(W^+_i) \) of \( W^+_i \), with \( N \) as in Subsection 5.3.2. Since the window \( W^+_i \) has been defined in the coordinate system \( (s^+, u^+, \phi^+, J^+, \theta) \) near \( \Gamma \), which was obtained by propagating the system near \( \Lambda \) along the stable manifold, and \( N \) as in (5.4), the image \( F^N(W^+_i) \) is naturally defined as a multi-dimensional rectangle in the coordinate system \( (s, u, \phi, J, \theta) \) defined in \( \mathcal{N}(\mathcal{L}^+_i) \). The dynamics in these coordinates is the skew product of the dynamics in the center directions with the dynamics in the hyperbolic directions.

The twist condition satisfied by \( F \) in the \((\phi, J)\) variables, described in (B6), determines a sheering of the \( \phi \)-direction by a quantity of \( \tau J \) per iterate along each level set of the action variable \( J \), modulo an error up to \( \varepsilon \): also, the \( J \)-coordinate is preserved up to \( \varepsilon \). Since \( F^N(W^+_i) \) is in \( \mathcal{N}(\Lambda) \), there exists a rectangle \( \hat{R}^\phi_i \times \hat{R}^J_i \) in the \((\phi, J)\) coordinate such that the \((\phi, J)\) component of \( W^+_i \) is correctly aligned under \( F^N \) with \( \hat{R}^\phi_i \times \hat{R}^J_i \). Since the size of the \((\phi, J)\) component of \( W^+_i \) is \( \gamma^- \times \delta^- \), we can choose \( \hat{R}^\phi_i \times \hat{R}^J_i \) to be of size \( \| \hat{R}^\phi_i \| = \gamma^- + N \tau \delta^- + \varepsilon \), and \( \| \hat{R}^J_i \| = \delta \). These inequalities are justified by (5.6) and (5.7).

In the \( \theta \)-direction the map \( F \) acts as the time-\( \varepsilon \) map \( \chi \) of the flow \( \chi \). To achieve correct alignment of the windows in the \( \theta \)-variable, it is sufficient to choose \( \hat{R}^\theta_i \) a topological rectangle in the interior of \( \chi^N(\hat{R}^\theta_i) \).

The distance between the point \( F^N(\varepsilon_{i-1,1}) \) in \( F^N(W^+_i) \) and \( F^N(\varepsilon_{i-1,1}) \), measured along the stable manifold \( W^*(\mathcal{L}^+_i) \), is less than \( \lambda^- \alpha^+ \).

The size of the projection of \( F^N(W^+_i) \) onto the \( s \)-coordinate is at most \( \lambda^N \alpha^+ + \varepsilon \). The size of the \( u \)-component of \( F^N(W^+_i) \) is at least \( \mu^N \beta^+ - \varepsilon \). By the choice of the coordinates in Subsection 5.3.1, the hyperbolic directions of \( F^N(\hat{R}^s_i \times \hat{R}^u_i) \) coincide with the hyperbolic directions in \( \mathcal{N}(\Lambda) \). Since by (5.1) and (5.3), we have \( \hat{\beta} + \varepsilon < \mu^N \beta^+ \), then we can construct a rectangle \( \hat{R}^s_i \times \hat{R}^u_i \) in the hyperbolic variables, of size \( \hat{\alpha} \times \hat{\beta} \), such that \( \hat{R}^s_i \times \hat{R}^u_i \) is correctly aligned with \( R^s_i \times R^u_i \) under \( F^N \), relative to the \( \theta \) hyperbolic variables.

\[
\| R^s_i \| = \alpha^+, \| R^u_i \| = \beta^+, \\
\| R^\phi_i \| = \gamma^+, \| R^J_i \| = \delta^+.
\]
Through these choices, we construct a new window $\tilde{W}_i$ about $z_{i-1,i}^+,\tilde{s}$, given by

$$\tilde{W}_i = (\tilde{R}_i^s \times \tilde{R}_i^u) \times (\tilde{R}_i^\phi \times \tilde{R}_i^J) \times \tilde{R}_i^\theta,$$

in the coordinates $(s, u, \phi, J, \theta)$, such that $F^N(W_i^+)$ is correctly aligned with $\tilde{W}_i$ under the identity mapping, or, equivalently, $W_i^+$ is correctly aligned with $\tilde{W}_i$ under $F^N$. Its exit set $(\tilde{W})_i^{\text{exit}}$ is given by

$$(\tilde{W})_i^{\text{exit}} = (\tilde{R}_i^s \times \partial \tilde{R}_i^u) \times (\tilde{R}_i^\phi \times \tilde{R}_i^J) \times \tilde{R}_i^\theta$$

$$\cup (\tilde{R}_i^s \times \tilde{R}_i^u) \times (\tilde{R}_i^\phi \times \partial \tilde{R}_i^J) \times \tilde{R}_i^\theta$$

$$\cup (\tilde{R}_i^s \times \tilde{R}_i^u) \times (\tilde{R}_i^\phi \times \tilde{R}_i^J) \times \partial \tilde{R}_i^\theta.$$

We note that the correct alignment of $W_i^+$ with $\tilde{W}_i$ under $F^N$ follows from the correct alignment of the window components, according to Definition [34], and the product property from [34]. The time it takes to achieve this alignment is $N$, which is independent of $\varepsilon$, and so is of order $O(1)$.

In a similar fashion, we construct a new window $\tilde{W}_{i-1}$ near $L_{i-1}$ such that $F^M(W_{i-1})$ is correctly aligned with $W_{i-1}$ under the identity mapping, or, equivalently, $\tilde{W}_{i-1}$ is correctly aligned with $W_{i-1}$ under $F^M$, for some $M$ as in Subsection 5.3.2. For this purpose, we take a negative iterate $F^{-M}(W_{i-1}^-)$ of $W_{i-1}$ and we construct a new window $\tilde{W}_{i-1}$ about $z_{i-1,i}^-$, of the following type

$$\tilde{W}_{i-1} = \tilde{R}_{i-1}^s \times \tilde{R}_{i-1}^u \times \tilde{R}_{i-1}^\phi \times \tilde{R}_{i-1}^J \times \tilde{R}_{i-1}^\theta,$$

in the coordinates $(s, u, \phi, J, \theta)$, with the exit set given by

$$(\tilde{W})_{i-1}^{\text{exit}} = \tilde{R}_{i-1}^s \times \partial \tilde{R}_{i-1}^u \times \tilde{R}_{i-1}^\phi \times \tilde{R}_{i-1}^J \times \tilde{R}_{i-1}^\theta$$

$$\cup \tilde{R}_{i-1}^s \times \tilde{R}_{i-1}^u \times \partial \tilde{R}_{i-1}^\phi \times \tilde{R}_{i-1}^J \times \tilde{R}_{i-1}^\theta$$

$$\cup \tilde{R}_{i-1}^s \times \tilde{R}_{i-1}^u \times \tilde{R}_{i-1}^\phi \times \tilde{R}_{i-1}^J \times \partial \tilde{R}_{i-1}^\theta.$$

We choose the size of the window components to be $\|\tilde{R}_{i-1}^s\| = \tilde{\alpha}$, $\|\tilde{R}_{i-1}^u\| = \tilde{\beta}$, $\|\tilde{R}_{i-1}^\phi\| = \tilde{\gamma}$, and $\|\tilde{R}_{i-1}^J\| = \tilde{\delta}$. By the choice of the coordinates in Subsection 5.3.1, we can choose $\tilde{R}_{i-1}^s \times \tilde{R}_{i-1}^u$ so that it is correctly aligned with $R_{i-1}^s \times R_{i-1}^u$ under $F^M$. By condition B6 we can choose $\tilde{R}_{i-1}^\phi \times \tilde{R}_{i-1}^J$ so that it is correctly aligned with $R_{i-1}^\phi \times R_{i-1}^J$ under $F^M$. In the $\theta$-variable we choose $\tilde{R}_{i-1}^\theta$ so that $R_{i-1}^\theta \subseteq \text{int}(\chi^M(\tilde{R}_{i-1}^\theta)).$

We also require that $(\tilde{R}_{i-1}^\phi \times \tilde{R}_{i-1}^J) \times \tilde{R}_{i-1}^\theta$ is contained, via the parametrization $k$ described in Subsection 5.3.2 in $U^- \times N \subseteq k^{-1}(U^-)$.

Now we verify that the choice of the sizes of the window components is compatible with these correct alignment relations. Using [6.1], [6.2], [6.4], we can ensure that $\tilde{\beta} > \mu^- M (\beta^- + a^-) + \varepsilon_1$ and $\tilde{\alpha} + \varepsilon_1 < \lambda^- M a^-$. By [6.6] and [6.7] we can ensure that $\delta + \varepsilon_1 < \delta^-$ and $\tilde{\gamma} > \gamma^- + M \tau \delta^- + \varepsilon_1$, where the term $M \tau \delta^-$ represents the effect of the twist map on $R_{i-1}^\phi \times R_{i-1}^J$ under $M$ negative iterates.

By the product property of correctly aligned windows, we obtain that $F^M(\tilde{W}_{i-1})$ is correctly aligned with $W_{i-1}$ under the identity mapping, or, equivalently, $\tilde{W}_{i-1}$ is correctly aligned with $W_{i-1}$ under $F^M$.

We note that to achieve the correct alignment of the windows in Step 2 we do not use the Lambda Lemma as in [49] [28] [34]. The role of the Lambda Lemma in
this paper is taken by the choice of coordinates coordinates near $\tilde{\Lambda}$ from Subsection 5.3.3 that are extended along the stable and unstable manifolds to produce convenient coordinate systems about $\Gamma_\varepsilon$.

5.4. Step 3. We consider the heteroclinic intersection $W^u(L_{t-1}) \cap W^s(L_t^i)$, and the heteroclinic intersection $W^u(L_t^i) \cap W^s(L_{t+1}^i)$.

By applying the Step 2 for each heteroclinic connection we obtain a pair of windows $\tilde{W}_i$ and $W_i$, both located about the set $L_t^i$.

At this step we want to get $\tilde{W}_i$ correctly aligned with $\tilde{W}_i$ under some iterate $F^K$. We pointed out earlier that the dynamics in the coordinates $(\phi, J, \theta, u, s)$ is the skew product of the dynamics in the center directions and the dynamics in the hyperbolic directions.

The first task is to ensure the correct alignment of the rectangle $\hat{R}_i^u \times \hat{R}_i^s$, of exit set $\partial \hat{R}_i^u \times \hat{R}_i^s$, with the rectangle $\hat{R}_i^u \times \hat{R}_i^s$, of exit set $\partial \hat{R}_i^u \times \hat{R}_i^s$. These rectangles are defined in the same coordinate system on $\tilde{N}(\Lambda_z)$. The correct alignment reduces to ensuring the following inclusions $\text{int}(F^K(\hat{R}_i^u)) \supseteq \hat{R}_i^u$ and $F^K_i(\hat{R}_i^s) \subseteq \text{int}(\hat{R}_i^s)$. Since the unstable directions get uniformly expanded and the stable directions get uniformly contracted, it is sufficient for the correct alignment to have $\hat{R}_i^u \times \hat{R}_i^s$ and $\hat{R}_i^u \times \hat{R}_i^s$ of the same size $\hat{\alpha} \times \hat{\beta} = \hat{\alpha} \times \hat{\beta}$; see (5.1) and (5.2).

The second task is to correctly align $(\hat{R}_i^u \times \hat{R}_i^s) \times \hat{R}_i^0$ with $(\hat{R}_i^u \times \hat{R}_i^s) \times \hat{R}_i^0$. The exit set of $\hat{R}_i^u \times \hat{R}_i^s$ is given by $\hat{R}_i^u \times \partial \hat{R}_i^s$, while the exit set of $\hat{R}_i^u \times \hat{R}_i^s$ is given by $\partial \hat{R}_i^u \times \hat{R}_i^s$. Under $F^K$, the rectangle $\hat{R}_i^u \times \hat{R}_i^s$ is transformed into a topological rectangle whose exit set components – ‘top’ edge and ‘bottom’ edge – are shifted apart one from the other by at least $K\tau \hat{\delta} - \varepsilon_1$ in the $\phi$-direction, and they are separated by a distance of at least $\hat{\delta} - \varepsilon_1$ in the $J$-direction. In order for $F^K(\hat{R}_i^u \times \hat{R}_i^s)$ to be correctly aligned with $\hat{R}_i^u \times \hat{R}_i^s$ under the identity, the image $F^K(\hat{R}_i^u \times \hat{R}_i^s)$ should stretch across $\hat{R}_i^0 \times \hat{R}_i^0$ in the direction of $\phi$ and its exit set components should come out through the exit set components – left edge and right edge – of $\hat{R}_i^u \times \hat{R}_i^s$. To ensure the stretching all the way of the first window across the second, we need that $(K\tau \| \hat{R}_i^0 \| - \varepsilon_1) - 2(\| \hat{R}_i^0 \| + \varepsilon_1)$, representing the shearing in the $\phi$ direction of $F^K(\hat{R}_i^u \times \hat{R}_i^s)$ minus the size of the top and the bottom edge, should be bigger than $\| \hat{R}_i^0 \|$, representing the ‘width’ of $\hat{R}_i^u \times \hat{R}_i^s$ in the $\phi$-direction. Also, to ensure that the image of the first rectangle does not meet the entry set of the second rectangle, we need that the quantity $\| \hat{R}_i^0 \| + \varepsilon_1$, representing an upper bound of the ‘height’ of $F^K(\hat{R}_i^u \times \hat{R}_i^s)$ in the $J$-direction, should be smaller than the quantity $\| \hat{R}_i^0 \|$, representing the height of $\hat{R}_i^u \times \hat{R}_i^s$ in the $J$-direction. By construction $\| \hat{R}_i^0 \| = \hat{\gamma}_i$, $\| \hat{R}_i^0 \| = \hat{\delta}$, $\| \hat{R}_i^0 \| = \hat{\gamma}_i$, and $\| \hat{R}_i^0 \| = \hat{\delta}$. By (5.5) we have that $K\hat{\delta} > 2\hat{\gamma}_i + \hat{\gamma}_i + 3\varepsilon_1$, and by (5.7) we have $\hat{\delta} > \hat{\delta} + \varepsilon_1$.

We also need to ensure the correct alignment of the $d$-dimensional rectangle $\hat{R}_i^0$, of exit set $\partial \hat{R}_i^0$, with the $d$-dimensional rectangle $\hat{R}_i^0$, of exit set $\hat{R}_i^0$. The correct alignment condition for windows that have the exit sets consisting of their whole boundary means that the image of the first window contains the second window in its interior. Thus, to make $\hat{R}_i^0$ correctly aligned with $\hat{R}_i^0$ under $F^K$, we need to choose $\hat{R}_i^0$ so that $\hat{R}_i^0 \supset \text{int}(\chi^K_i(\hat{R}_i^0))$.

By the product property of correctly aligned windows, the outcome of this step is that $\tilde{W}_i$ is correctly aligned with $\tilde{W}_i$ under $F^K$. 

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5.4.1. **Step 4.** To construct a long sequence of correctly aligned windows that visits all the sets \( \{ \mathcal{L}_i^\varepsilon \}_{i=0,...,n_1} \), we start with constructing a pair of correctly aligned windows \( W_0^- \) and \( W_1^+ \), by the heteroclinic intersection \( \{ z_{0,1} \} \times N \) of \( W^n(\mathcal{L}_0^\varepsilon) \) and \( W^s(\mathcal{L}_1^\varepsilon) \), as in Step 1. Then, as in Step 2, we construct the window \( \hat{W}_1 \) by \( \mathcal{L}_0^\varepsilon \) and \( W_1 \) by \( \mathcal{L}_1^\varepsilon \). At this initial step, for the uniformity of the notation, we set \( W_0 = \hat{W}_1 \). Then, we continue the construction recursively, following Step 1, Step 2, and Step 3, until we arrive with a window \( \hat{W}_n \), by \( \mathcal{L}_{n-1}^\varepsilon \). Hence, we obtain the sequence of correctly aligned windows

\[
W_0, W_0, W_1, W_1, \ldots, \hat{W}_{n-1}, W_{n-1}, W_n, \hat{W}_n
\]

starting from \( \mathcal{L}_0^\varepsilon \) and ending by \( \mathcal{L}_{n-1}^\varepsilon \). We apply the Shadowing Lemma type of result Theorem 5.3 and conclude that there is an orbit \( \{ y_i \}_{i=0,...,n_1} \) with \( y_i \in \hat{W}_i \) for all \( i \), and \( y_{i+1} = F_{R^2 M} N (y_i) \) for \( i = 0, \ldots, n_1 - 1 \).

5.5. **Proof of Theorem 2.1 and of Theorem 2.2** For Theorem 2.1 the initial condition \( \theta_0 \) is chosen to be a non-trivial uniformly recurrent point. For Theorem 2.2 the initial condition \( \theta_0 \) can be any point in \( N \). Then the construction from the previous section is performed. The outcome is an orbit \( \{ y_i \}_{i=0,...,n_1} \) is \( O(1) \) close to the pseudo-orbit underlying the sequence of elementary building blocks constructed in Subsection 4.3. We can estimate the time it takes of an orbit \( y_i \) to move from a window \( \hat{W}_i \) to the window \( W_n \). The pseudo-orbit from Subsection 4.3 achieves a growth of scaled energy \( \Delta H_\varepsilon = O(\varepsilon) \) in a scaled time \( \Delta \tau = O(1/\varepsilon^2) \). Each pseudo-orbit corresponding to a single elementary building block gives rise to a sequence of windows that are correctly aligned. The time required by the correct alignment of windows, and the corresponding time it takes to the orbit to move along those windows, is \( O(1) \), with the constants being the same for all windows in the sequence. Thus, the scaled time it takes the orbit \( \{ y_i \}_{i=0,...,n_1} \) to travel from \( y_0 \) to \( y_{n_1} \) is \( O(1/\varepsilon^2) \), equal to the time along the sequence of elementary building blocks multiplied by some constant independent of \( \varepsilon \), and hence on the energy. The gain of physical energy along this orbit is \( \Delta H = O(1/\varepsilon) \), and the physical time that is spent is \( \Delta t = O(1/\varepsilon^2) \). Thus the change in the physical energy is proportional to the time along the orbit, i.e. \( \Delta t \approx \Delta H \approx O(1/\varepsilon) \), as claimed.

Since concatenations of correctly aligned windows are still correctly aligned, the above construction of correctly aligned windows can be continued for a time \( O(1) \) to achieve a \( O(1) \) change of the physical energy that covers the interval \( H = [E_\varepsilon, 2E_\varepsilon] \), where \( E_\varepsilon \) was the choice of initial energy. To grow the physical energy to infinity, we re-initialize the process staring with \( \varepsilon = 1/\sqrt{2E_\varepsilon} \), and we construct a sequence of correctly aligned windows as before. Again, the fact that concatenations of correctly aligned windows are still correctly aligned yields a longer sequence of correctly aligned windows. The construction of correctly aligned windows can be repeated inductively, yielding a sequence of correctly aligned windows, and a corresponding shadowing orbit whose energy grows to infinity in time. The estimate on the time does not get any worse since moving up along the infinite chains corresponds to higher energy levels where the speed of diffusion is only growing faster. Thus the energy growth is linear with respect to time.

**Remark 5.4.** In the above construction we should remark that the windows are of size \( O(1) \), while the distance from \( \mathcal{L}_i^\varepsilon \) to \( \mathcal{L}_{i+1}^\varepsilon \) is only of order \( O(\varepsilon^3) \). Thus, for an orbit \( \{ y_i \}_{i=0,...,n_1} \) that goes from \( \hat{W}_i \) about \( \mathcal{L}_i^\varepsilon \) to \( \hat{W}_{i+1} \) about \( \mathcal{L}_{i+1}^\varepsilon \) the net energy
change is undetermined. However, the method of correctly aligned windows does detect an orbit whose physical energy changes by $O(1)$, from $E = E_*$ to $E = 2E_*$, within a scaled time $O(1)$. Hence, the topological argument is not capable to detect the detailed changes of the energy along the orbit, but only the significant changes.

We could get more control on the energy levels visited by choosing the windows smaller but then we would get worse estimates on the time. This phenomenon resembles the ‘energy-time’ uncertainty principle of Heisenberg [37].

5.6. Proof of Theorem 2.3. The function $E$ represents a prescribed energy path with upper bounded derivative; this condition is necessary as the energy of the perturbed system cannot grow faster than linearly. The function $T$ represents a parametrization of the time. The argument is the same as for Theorem 2.1 provided that we choose the infinite sequence of level sets $\{J^i\}_{i \in \mathbb{N}}$ to follow the energy path $E$, as described in Subsection 5.2.

5.7. Regularity. We note that if we assume that the metric and the potential are $C^r$, we conclude that the flow is $C^{r-1}$ and so is the map $F$. The theory of normal hyperbolicity requires at least $C^1$-differentiability.

Since we are using a derivative with respect to parameters and even estimate the remainders, we would like that $r - 1 \geq 2$.

We note that the $C^1$-differentiability of the flow $\chi$ on the external manifold $N$ is sufficient for the argument, since the estimates do not involve any derivatives along the solution curves of $\chi$.

Hence, in Section 2 it suffices to take $r_0 = 3$.

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