THE $S^1$-EQUIVARIANT COHOMOLOGY OF SPACES OF LONG EXACT SEQUENCES

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Abstract. Let $S$ denote the graded polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. We interpret a chain complex of free $S$-modules having finite length homology modules as an $S^1$-equivariant map $\mathbb{C}^m \setminus \{0\} \to X$, where $X$ is a moduli space of exact sequences. By computing the cohomology of such spaces $X$ we obtain obstructions to such maps, including a slight generalization of the Herzog-Kühl equations.

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1. Introduction

Let $X(c_1, \ldots, c_n)$ denote the space of long exact sequences

$$0 \longrightarrow k^{c_1} \longrightarrow k^{c_2} \longrightarrow \cdots \longrightarrow k^{c_3} \longrightarrow 0$$

which we handle by fixing bases for each term and identifying a sequence with the $n-1$-tuple of matrices representing the differentials. The principal results we obtain are a calculation of the cohomology of this space, and a description of the $E_{\infty}$-page of a spectral sequence converging to the $S^1$-equivariant cohomology of $X$.

The eventual applications we have in mind are to the conjecture of Buchsbaum-Eisenbud & Horrocks and the conjecture of Carlsson in homological algebra. The
first of these conjectures, in a special case, claims that if $M$ is a nonzero finite-length module over the graded polynomial ring $S = \mathbb{C}[x_1, \ldots, x_m]$, and if

$$
0 \longrightarrow F_a \longrightarrow F_{a-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M
$$

is a free resolution, then $\text{rank } F_i \geq \binom{m}{i}$, or more informally, that the smallest resolution is the Koszul resolution.

Decomposing the free modules into their graded parts: $F_i = \bigoplus_{j=0}^{\infty} S(j)^{\beta_{i,j}}$, we obtain a matrix $\beta_{i,j}$ of integers, referred to as the Betti table of the resolution.

Recently, in [Erm09], some progress was made on the conjecture by means of the striking theory of Boij-Soderberg [BS08], [ES09], that states that the Betti table of a resolution of such a module is constrained to be a rational linear combination of certain simpler tables. The starting point for the study of Betti tables are the following formulas, which go by the name of the Herzog-Kühl equations.

$$
\sum_{i=1}^{n} \sum_{j \in \mathbb{Z}} (-1)^j \beta_{i,j} j^s = 0
$$

where $s$ ranges over the integers $\{0, \ldots, n\}$.

Parallel to Buchsbaum-Eisenbud & Horrocks conjecture is a conjecture of Carlson, [Car87], which is more general in its applicability but less specific in its claims. According to that conjecture, if $M$ is a differential-graded free module over $S$, and if $M$ has nonzero finite-length homology, then it is claimed that $\text{rank } M \geq 2^m$.

The simplest examples of differential-graded modules are the total spaces of resolutions: $\bigoplus_{i=0}^{a} F_i$. Occupying an intermediate position between resolutions on the one hand, and the full generality of $dgm$s on the other, there are the chain complexes of graded free modules

$$
\Theta : 0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_a \longrightarrow 0
$$

having finite length homology.

The differential of $\Theta$ can be written as a matrix $D : \bigoplus_{i} F_i \rightarrow \bigoplus_{i} F_i$, taking entries in the ring $S$. The graded nature of $D$ and $\bigoplus_{i} F_i$ ensures that the polynomials appearing in $D$ are homogeneous, although they may be of varying degrees. We view such a matrix as a map $\mathbb{C}^{m} \rightarrow \text{Mat}_{N \times N}(\mathbb{C})$, for the appropriate value of $N$. A basic result, c.f. [Car86], is that the homology of $\Theta$ is an $S$-module which is supported only at the ideal $(x_1, \ldots, x_m)$. In this case, this implies that evaluating $\Theta$ at any point $x \in \mathbb{C}^{m} \setminus \{0\}$ yields a long exact sequence. We have a map

$$
D : \mathbb{C}^{m} \setminus \{0\} \longrightarrow X
$$

where $X$ is the moduli space of long exact sequences considered above. Since the entries in $D$ are homogeneous, this map is compatible with a $\mathbb{C}^{*} \simeq S^{1}$-action given by scalar multiplication. By considering equivariant cohomology with respect to this $\mathbb{C}^{*}$-action, we are able to obtain obstructions to the existence of $D$. These obstructions include, most notably, the Herzog-Kühl equations, generalized to this case. There are also some other obstructions in a similar vein, which, if the Buchsbaum-Eisenbud-Horrocks conjecture holds, are conjecturally vacuous in the case of resolution, but may still have relevance in the more general case.

The usual derivation of the Herzog-Kühl equations for resolutions is by consideration of the Hilbert polynomial of the module $M$ being resolved. The Hilbert polynomial, understood here as defined on $K_0(\mathbb{P}^{n-1})$ corresponds via a Riemann-Roch theorem to the the Chern polynomial of $M$ in $CH^*(\mathbb{P}^{n-1})$, [Eis95, p. 490].
It should come as no surprise that the obstructions we obtain in theorem 10 are exactly the Chern classes of $M$ in the case of a resolution. It is instructive to see quite how little is required for the derivation of these obstructions, since they rely only on the $\mathbb{C}^*$-equivariance of the map $\mathbb{C}^m \setminus \{0\} \to X$; no other property of homogeneous polynomials is used.

All equivariant homology and cohomology will be Borel equivariant homology and cohomology with respect to the group $S^1 \subset \mathbb{C}^*$ of unit complex numbers, unless otherwise stated. We abbreviate $\text{Gl}(n, \mathbb{C})$ to $\text{Gl}(n)$ throughout.

This paper will be recast in the language of equivariant motivic cohomology in future work, thus extending the result to the case of arbitrary characteristic. Since motivic cohomology theory is bigraded, having a ‘weight’ grading in addition to the usual one. This grading facilitates a number of arguments about differentials in spectral sequences, which are forced to vanish for reasons of grading. As a consequence, we leave in sketch form a handful of arguments concerning spectral sequence, mostly to the effect that certain classes are transgressive. All such arguments can be carried out without recourse to algebraic theories, generally by ungainly arguments considering the action of cohomology operations on the spectral sequence.

2. Graded Vector Spaces

We work exclusively over $\mathbb{C}$. We understand a graded vector space $V$ to be a vector space equipped with a decomposition $V = \bigoplus_{i \in \mathbb{Z}} V_i$. The subspace $V_i$ is the $i$-th graded part of $V$. Graded vector spaces can be realized as representations of $S^1$, by letting $S^1$ act on $V_i$ by $z \circ v = z^i v$. When we consider a basis of a graded vector space, we will generally require that such a basis consist of homogeneous elements.

Suppose $V$, $W$ are finite dimensional graded vector spaces, then there are two ways in which we may give $\text{Hom}_{\mathbb{C}}(V, W)$ an $S^1$-action.

1. The left action, so that $(z \circ T)v = z \circ (Tv)$.
2. The left-right action so that: $(z \circ T)(z \circ v) = z \circ (Tv)$.

Both actions will occur in the sequel. Since the action of $S^1$ on a graded vector space maps bases to bases, it is easy to see that $z \circ A$ has the same rank as $A$ in either case. If we fix graded bases $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ of $V, W$, and if these elements lie in gradings $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_m\}$ respectively, then a given map $V \to W$ can be written in matrix form, $A$. The two actions of $z \in S^1 \subset \mathbb{C}^*$ on $A$ are both given by matrix multiplication

$$ z \cdot A = \begin{pmatrix} z^{w_1} & z^{w_2} & \cdots & z^{w_m} \\ \end{pmatrix} A \quad \text{in the case of action on the left} $$

and

$$ z \cdot A = \begin{pmatrix} z^{w_1} & z^{w_2} & \cdots & z^{w_m} \\ \end{pmatrix} A \begin{pmatrix} z^{-v_1} & z^{-v_2} & \cdots & z^{-v_m} \\ \end{pmatrix} \quad \text{in the case of action on both the left and right} $$
Given a finite-dimensional graded vector space \( V \), it has a dual space \( \text{Hom}_\mathbb{C}(V, \mathbb{C}) \). This can be given an \( S^1 \) action, and consequently a grading, in either of the two ways above, but in the case of vector spaces over \( \mathbb{C} \) there is also a third construction. We continue in our choice of basis \( \{x_i\} \) of \( V \), and we can define a sesquilinear form by \( \langle x_i, x_j \rangle = \delta_{ij} \) and extending to all of \( V \); this gives an explicit isomorphism \( V \cong \hat{V} \). We make \( \hat{V} \) into a graded vector space by means of this isomorphism. It is not hard to see that the grading on \( \hat{V} \) we obtain in this way does not depend on the chosen graded basis. Henceforth, whenever we take the dual of a graded vector space, we shall give it this grading.

We continue to employ the same bases \( \{x_i\} \) and \( \{y_i\} \) for the graded vector spaces \( V, W \) as before, and deliberately confuse transformations and matrices. If \( A \in \text{Hom}(V,W) \), then \( A \) has a Hermitian conjugate dual, \( A' \), which is a map of vector spaces \( A' \in \text{Hom}(\hat{W}, \hat{V}) \). With this notation, we can assert the following.

**Proposition 1.** The map \( \text{Hom}(V,W) \to \text{Hom}(\hat{W}, \hat{V}) \) given by \( A \mapsto A' \) is \( S^1 \)-equivariant when both spaces are given the left-right action.

**Proof.** This is most easily seen by arguing explicitly with matrices. Write \( A = (a_{i,j}) \). With respect to the left-right action on \( A' \), we have
\[
(z \circ A')(y_i) = (z \circ A')(z^{-w_i}(z \circ y_i)) = z^{-w_i}(z \circ A')(z \circ y_i) =
\]
\[
z^{-w_i} \left[ z \circ \sum_{j=1}^{n} a_{j,i} x_j \right] = z^{-w_i} \sum_{j=1}^{n} a_{j,i} z^{v_j} x_j
\]
as a result, we have
\[
z \circ A' = \begin{pmatrix}
    z^{v_1} \\
z^{v_2} \\
\vdots \\
z^{v_n}
\end{pmatrix} \begin{pmatrix}
    z^{-w_1} \\
z^{-w_2} \\
\vdots \\
z^{-w_n}
\end{pmatrix} = \begin{pmatrix}
    z^{v_1} \\
z^{v_2} \\
\vdots \\
z^{v_n}
\end{pmatrix} A' \begin{pmatrix}
    z^{-w_1} \\
z^{-w_2} \\
\vdots \\
z^{-w_n}
\end{pmatrix} = (z \circ A)'
\]
as required. \( \square \)

3. The Homology and Cohomology of \( \text{GL}(n) \)

We take coefficients in a commutative ring \( R \). In all applications \( R \) shall be \( \mathbb{Z}, \mathbb{Z}/p \) or \( \mathbb{Q} \). The cohomology of \( \text{GL}(n) \) is an exterior algebra \( H^*(\text{GL}(n); R) = \Lambda_R(\alpha_1, \ldots, \alpha_n) \) with \( |\alpha_i| = 2i - 1 \). The elements \( \alpha_i \) are primitive, in the sense of Hopf algebra. The homology is given a ring structure by the Pontrjagin product, which turns out to be an exterior algebra again \( H_*(\text{GL}(n); R) = \Lambda_R(\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \).

Indeed one has \( \hat{\beta} = \beta \hat{\gamma} \) with respect to these two product structures.

There is a map \( \text{GL}(n) \times \text{GL}(m) \to \text{GL}(n + m) \) by matrix direct-sum. The induced map on homology is
\[
\Lambda_R(\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \otimes \Lambda_R(\hat{\alpha}'_1, \ldots, \hat{\alpha}'_m) \to \Lambda_R(\hat{\alpha}''_1, \ldots, \hat{\alpha}''_{n+m})
\]
\[
\hat{\alpha}_i \otimes \hat{\alpha}'_j \mapsto \hat{\alpha}''_i \hat{\alpha}''_j
\]

There are two noteworthy involutions \( \text{GL}(n) \to \text{GL}(n) \). They are \( \iota : a \mapsto a^{-1} \) and \( c : a \mapsto a' \). If we restrict to \( U(n) \cong \text{GL}(n) \), they coincide. As a result,
the automorphisms they induce on homology and cohomology coincide. From the sequence of maps
\[ \text{Gl}(n) \xrightarrow{\Delta} \text{Gl}(n) \times \text{Gl}(n) \xrightarrow{\iota \times \text{id}} \text{Gl}(n) \times \text{Gl}(n) \xrightarrow{\mu} \text{Gl}(n) \]
we obtain a diagram on homology groups
\[ H_*(\text{Gl}(n); R) \xrightarrow{\Delta_*} H_*(\text{Gl}(n) \times \text{Gl}(n); R) \xrightarrow{\iota_* \otimes \text{id}} H_*(\text{Gl}(n) \times \text{Gl}(n); R) \rightarrow H_*(\text{Gl}(n); R) \]
which, when applied to \( \hat{\alpha}_i \) gives
\[ \hat{\alpha}_i \rightarrow \hat{\alpha}_i \otimes 1 + 1 \otimes \hat{\alpha}_i \rightarrow \iota_*(\hat{\alpha}_i) \otimes 1 + 1 \otimes \hat{\alpha}_i \rightarrow c_*(\hat{\alpha}_i) + \hat{\alpha}_i \]
since the composition is \( \text{Gl}(n) \rightarrow \{I_n\} \subset \text{Gl}(n) \), we must have \( c_*(\hat{\alpha}_i) = \iota_*(\hat{\alpha}_i) = -\hat{\alpha}_i \).

4. The Equivariant Cohomology of \( \text{Gl}(n) \)

Given a space \( X \) with a \( G \) action, we call the Borel equivariant cohomology \( H^*(EG \times G X; R) \) the equivariant cohomology. There exists a fiber sequence \( X \rightarrow EG \times G X \rightarrow BG \), which, under hypotheses which will always be satisfied in our applications, gives a Serre spectral sequence
\[ E_{p,q}^2 = H^p(BG; H^q(X); R) \Rightarrow H^{p+q}(EG \times G X; R). \]
In general we shall not compute equivariant cohomology in full, but shall content ourselves to describe the \( E_2 \)-page of this spectral sequence and the differentials.

4.1. The Left Action. Let \( \vec{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n \) be a set of \( n \) weights. We begin by considering the following left \( S^1 \)-action on \( \text{Gl}(n) \).

\[ \phi: S^1 \times \text{Gl}(n) \rightarrow \text{Gl}(n), \quad z \cdot A = \begin{pmatrix} z^{w_1} & & \\ & \ddots & \\ & & z^{w_n} \end{pmatrix} A \]

We consider initially the cohomology Serre spectral sequence arising from the fibration \( ES^1 \times S^1 \text{Gl}(n) \rightarrow BS^1 \cong \mathbb{C}P^\infty \), with fiber \( \text{Gl}(n) \). We write \( T \) for the maximal torus of \( U(n) \subset \text{Gl}(n) \). The inclusion \( U(n) \rightarrow \text{Gl}(n) \) is a homotopy equivalence. There is a sequence of group homomorphisms \( S^1 \xrightarrow{f} T \rightarrow U(n) \rightarrow \text{Gl}(n) \), where the map \( f \) is given by \( z \mapsto \text{diag} z^{w_1}, z^{w_2}, \ldots, z^{w_n} \). We obtain in this way a compatible sequence of group actions on \( \text{Gl}(n) \).

\[ S^1 \times \text{Gl}(n) \xrightarrow{\pi} T \times \text{Gl}(n) \rightarrow \text{Gl}(n) \times \text{Gl}(n) \]
\[ \text{Gl}(n) \xrightarrow{\pi} \text{Gl}(n) \rightarrow \text{Gl}(n) \]
and consequently maps of fibrations, each with fiber \( \text{Gl}(n) \),

\[
\begin{array}{c}
ES^1 \times_{S^1} \text{Gl}(n) \\
\downarrow \\
BS^1
\end{array}
\xrightarrow{id} \begin{array}{c}
ET \\
\downarrow \\
BT
\end{array}
\xrightarrow{\phi} \begin{array}{c}
\text{Gl}(n) \\
\downarrow \\
B \text{Gl}(n)
\end{array}
\]

The cohomology of \( \text{Gl}(n) \) is \( H^*(\text{Gl}(n); R) \cong \Lambda_R(\alpha_1, \ldots, \alpha_n) \), with \( |\alpha_i| = 2i - 1 \).

The cohomology of \( B \text{Gl}(n) \), \( BT \) and \( BS^1 \) are

\[
H^*(B \text{Gl}(n); R) \cong R[c_1, \ldots, c_n], \quad H^*(BT; R) \cong R[t_1, \ldots, t_n], \quad H^*(BS^1; R) \cong R[t]
\]

where \( |c_i| = 2i \), and \( |t_i| = |t| = 1 \). The map induced by the inclusion \( T \to \text{Gl}(n) \) on the cohomology of classifying spaces is \( \psi : \mathbb{Z}[c_1, \ldots, c_n] \to \mathbb{Z}[t_1, \ldots, t_n] \) sending \( c_i \) to \( \sigma_i(t_1, \ldots, t_n) \), the \( i \)-th elementary symmetric function in the \( t_j \).

The map on the cohomology of classifying spaces induced by \( S^1 \to T \) is given by \( \theta : \mathbb{Z}[t_1, \ldots, t_n] \to \mathbb{Z}[t] \), sending \( t_i \) to \( w_it \).

We remark that all cohomology groups we encounter are free \( R \)-modules, as a consequence there is never any nontrivial behaviour in either Künneth or Universal Coefficient formulas.

The Serre spectral sequence arising from the \( \text{Gl}(n) \)-fiber bundle \( E \text{Gl}(n) \times_{\text{Gl}(n)} \text{Gl}(n) \to B \text{Gl}(n) \) is simply the standard fibration \( E \text{Gl}(n) \to B \text{Gl}(n) \). The action of the differentials in this spectral sequence is summarily described by

\[
d_2(\alpha_i) = c_i, \quad d_j(\alpha_i) = 0 \quad \text{for} \ i \neq j
\]

all other differentials can be deduced easily from these by using the product structure.

By comparison with this spectral sequence, the differentials in the Serre spectral sequence for the fibration \( ES^1 \times_{S^1} \text{Gl}(n) \to BS^1 \) can be computed. There is a comparison map, which on the \( E_2 \)-page is

\[
(1) \quad \text{id} \otimes (\theta \psi)^* : H^*(\text{Gl}(n); R) \otimes H^*(B \text{Gl}(n); R) \to H^*(\text{Gl}(n); R) \otimes H^*(BS^1; R)
\]

In the latter spectral sequence, the differentials are now seen to be described by

\[
(2) \quad d_2(\alpha_i) = \sigma_i(\bar{w})t^i \quad \text{mod} \ \sigma_1(\bar{w})t^i, \sigma_2(\bar{w})t^i, \ldots, \sigma_{i-1}(\bar{w})t^i, \quad d_j(\alpha_i) = 0 \quad \text{for} \ i \neq j
\]

As with the former sequence, the differentials on all other elements may be deduced from those given.

4.2. **Action on both Left & Right.** We now consider an action of \( S^1 \) on \( \text{Gl}(n) \) both on the left and on the right. If \( \vec{u}, \vec{v} \in \mathbb{Z}^n \) are two \( n \)-tuples of integers, one defines an action of \( S^1 \) on \( \text{Gl}(n) \) by

\[
z \cdot A = \begin{pmatrix}
z^{u_1} & z^{u_2} & \cdots & z^{u_n} \\
z^{-v_1} & z^{v_2} & \ddots & \\
& \ddots & \ddots & z^{-v_n}
\end{pmatrix} A
\]

The argument in this section being rather long, if routine, we state it here for convenience of reference.
Proposition 2. For the $S^1$ action given above, the Serre spectral sequence of the fiber sequence $\text{GL}(n) \to E \times_{S^1} \text{GL}(n) \to BS^1$ has $E_2$-page

$$H^*(\text{GL}(n); R) \otimes H^*(BS^1; R) \cong \Lambda_R(\alpha_1, \ldots, \alpha_n) \otimes R[\theta].$$

The differentials in this sequence are described summarily by

$$d_2(\alpha_i) = [\sigma_i(\vec{u}) - \sigma_i(\vec{v})] \theta^i \pmod{\sigma_1(\vec{u}) - \sigma_1(\vec{v}), \ldots, \sigma_{i-1}(\vec{u}) - \sigma_{i-1}(\vec{v})}$$

where $i$ is any integer between 1 and $n$; the differentials can be deduced on all other elements by means of the product structure.

A portion of this spectral sequence is illustrated below

\[\begin{array}{cccccc}
\alpha_3 & 0 & \alpha_3 \theta & 0 & \alpha_3 \theta^2 \\
\alpha_1 \alpha_2 & 0 & \alpha_1 \alpha_2 \theta & 0 & \alpha_1 \alpha_2 \theta^2 \\
\alpha_2 & 0 & \alpha_2 \theta & 0 & \alpha_2 \theta^2 \\
0 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & \alpha_1 \theta & 0 & \alpha_1 \theta^2 \\
1 & 0 & \theta & 0 & \theta^2
\end{array}\]

The differentials denoted by dotted arrows can be deduced from those denoted by solid arrows. A differential between illustrated groups that is not marked by an arrow must be 0.

There is a group homomorphism $f : S^1 \to \text{GL}(n) \times \text{GL}(n)$ given by

$$f(z) = \begin{pmatrix}
  z^{w_1} & & \\
  z^{w_2} & \ddots & \\
  & \ddots & \ddots & \ddots & \\
  & & \ddots & \ddots & z^{v_n}
\end{pmatrix},
\begin{pmatrix}
  z^{v_1} & & \\
  z^{v_2} & \ddots & \\
  & \ddots & \ddots & \ddots & \\
  & & \ddots & \ddots & z^{v_n}
\end{pmatrix}$$

Let $\text{GL}(n) \times \text{GL}(n)$ act on $\text{GL}(n)$ by $(g, h) \cdot A = gAh^{-1}$. We write the action map as $\alpha : (\text{GL}(n) \times \text{GL}(n)) \times \text{GL}(n) \to \text{GL}(n)$. There is a commutative diagram

$$\begin{array}{ccc}
S^1 \times \text{GL}(n) & \xrightarrow{f} & (\text{GL}(n) \times \text{GL}(n)) \times \text{GL}(n) \\
\text{GL}(n) \times \text{GL}(n) & \xrightarrow{\alpha} & \text{GL}(n)
\end{array}$$

We concentrate for now on the action $\alpha$, and in particular, on the Serre spectral sequence associated with the fibration $E((\text{GL}(n) \times \text{GL}(n)) \times_{\text{GL}(n) \times \text{GL}(n)} \text{GL}(n)) \to B(\text{GL}(n) \times \text{GL}(n))$. 
In the first place we know that \( B(\text{Gl}(n) \times \text{Gl}(n)) \simeq B \text{Gl}(n) \times B \text{Gl}(n) \), and the cohomology \( H^*(B(\text{Gl}(n) \times \text{Gl}(n)); R) \) is \( R[c_1, c_1', c_2, c_2', \ldots, c_n, c_n'] \). The homomorphisms \( \tau_1, \tau_2 : \text{Gl}(n) \to \text{Gl}(n) \times \text{Gl}(n) \) sending \( a \mapsto (a, I_n) \) and \( a \mapsto (I_n, a) \) respectively, induce evaluations of \( R[c_1, c_1', c_2, c_2', \ldots, c_n, c_n'] \) at \( (c_1', \ldots, c_n') = 0 \) and \( (c_1, \ldots, c_n) = 0 \) respectively.

There is a diagram

\[
\begin{array}{ccc}
\text{Gl}(n) \times \text{Gl}(n) & \xrightarrow{\tau_1 \cdot \text{id}} & (\text{Gl}(n) \times \text{Gl}(n)) \times \text{Gl}(n) \\
\downarrow & & \downarrow \\
\text{Gl}(n) & \xrightarrow{\text{id} \otimes c_1'} & \text{Gl}(n) \\
\end{array}
\]

where the action on the left is the usual action of \( \text{Gl}(n) \) on itself on the left. There is a map of Serre spectral sequences, which on the \( E_2 \)-page is

\[
H^*(\text{Gl}(n); R) \otimes H^*(B(\text{Gl}(n) \times \text{Gl}(n)); R) \xrightarrow{\text{id} \otimes c_1'} H^*(E(\text{Gl}(n) \times \text{Gl}(n)) \times \text{Gl}_n \times \text{Gl}_n \text{Gl}(n); R)
\]

the lower sequence of which has already been described in equations (1) and (2). By comparing the sequences, we know the generators \( \alpha_i \) of \( H^*(\text{Gl}(n); \mathbb{Z}) \) are transgressive in the upper sequence. General considerations imply that the differentials in the upper sequence satisfy

\[
d_{2i}(\alpha_i) = p_i(c_1, c_1', \ldots, c_i, c_i') \pmod{p_1, \ldots, p_{i-1}}
\]

where \( p_i(c_1, c_1', \ldots, c_i, c_i') \) is a homogeneous polynomial of degree \( 2i \). By comparison with the lower sequence, we know that after evaluating at \( c_1' = c_2' = \cdots = c_n' = 0 \), this polynomial becomes precisely \( c_i \). By degree-counting, we know that

\[
p_i = c_i + a c_i' + q_i(c_1, c_1', c_2, c_2', \ldots, c_{i-1}, c_{i-1}') \pmod{p_1, \ldots, p_{i-1}}
\]

There is an obvious involution, \( \tau \), on \( \text{Gl}(n) \times \text{Gl}(n) \) interchanging the two factors, and an involution, \( \sigma \), on \( \text{Gl}(n) \), sending \( A \) to \( A^{-1} \). These are compatible in the sense that the following commutes

\[
\begin{array}{ccc}
(\text{Gl}(n) \times \text{Gl}(n)) \times \text{Gl}(n) & \xrightarrow{\tau \cdot \sigma} & (\text{Gl}(n) \times \text{Gl}(n)) \times \text{Gl}(n) \\
\downarrow & & \downarrow \\
\text{Gl}(n) & \xrightarrow{\sigma} & \text{Gl}(n) \\
\end{array}
\]

The vertical action maps are \( \alpha \) both cases (this is simply the equation \( hA^{-1}g^{-1} = (gAh^{-1})^{-1} \)). There is a resulting map of Serre spectral sequences associated with these group actions, it is in this case an automorphism, which on the \( E_2 \)-page

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1 The two projection homomorphisms \( \text{Gl}(n) \times \text{Gl}(n) \to \text{Gl}(n) \) induce the inclusions \( R[c_1, \cdots, c_n], R[c_1', \cdots, c_n'] \subset R[c_1, c_1', \cdots, c_n, c_n'] \), but we shall not use this fact in the sequel.
manifests as

\[ H^*(\text{Gl}(n); R) \otimes H^*(B(\text{Gl}(n) \times \text{Gl}(n)); R) \longrightarrow H^*(\text{Gl}(n); R) \otimes H^*(B(\text{Gl}(n) \times \text{Gl}(n)); R) \]

\[ \alpha_i \otimes 1 - \alpha_i \otimes 1 \\
1 \otimes c_i \longmapsto 1 \otimes c_i' \\
1 \otimes c_i' \longmapsto 1 \otimes c_i \]

From this and linearity of the differentials, one sees that

\[ p_i(c_1, c_1', \ldots, c_i, c_i') = d_2(i) = -d_2(-\alpha_i) = -p_i(c_i', c_1, \ldots, c_i, c_i) \quad (\text{mod } p_1, \ldots, p_{i-1}). \]

Comparing this with equation (3), we see that \( a = -1 \) and furthermore that \( q_i \) also satisfies

\[ q_i(c_i', c_1, c_2, \ldots, c_{i-1}, c_{i-1}) = -q_i(c_1, c_1', c_2, \ldots, c_{i-1}, c_{i-1}) \quad (\text{mod } p_1, \ldots, p_{i-1}) \]

One now sees by a simple induction that the following holds

**Lemma 3.** In the notation of the previous paragraphs, for all \( i \leq n \)

\[ d_2(i) = c_i - c_i' \quad (\text{mod } c_1 - c_1', c_2 - c_2', \ldots, c_{i-1} - c_{i-1}'). \]

**Proof.** This is evident for \( i = 1 \). Suppose the result holds for \( i - 1 \). Then, from equation (3), we have

\[ d_2(i) = c_1 - c_1' + q_i(c_1, c_1', c_2, c_2', \ldots, c_{i-1}, c_{i-1}) \quad (\text{mod } c_1 - c_1', c_2 - c_2', \ldots, c_{i-1} - c_{i-1}). \]

It is a basic fact that

\[ q_i(c_1, c_1', c_2, c_2', \ldots, c_{i-1}, c_{i-1}) = q_i(c_1', c_1', c_2, c_2', \ldots, c_{i-1}, c_{i-1}) \]

\[ (\text{mod } c_1 - c_1', c_2 - c_2', \ldots, c_{i-1} - c_{i-1}) \]

but by comparison with equation (4), we see that the left hand side above is 0. \( \Box \)

We now return to consideration of the action of \( S^1 \) on \( \text{Gl}(n) \) on the left & right. Our basic tool is the homomorphism \( S^1 \rightarrow \text{Gl}(n) \times \text{Gl}(n) \). This induces a homomorphism on the cohomology of classifying spaces

\[ R[c_1, c_1', c_2, c_2', \ldots, c_n, c_n'] = H^*(B(\text{Gl}(n) \times B \text{Gl}(n)); R) \rightarrow \mathbb{Z}[t] = H^*(BS^1; R) \]

which is given by \( c_i \rightarrow \sigma_i(\bar{w})t^i \) and \( c_i' \rightarrow \sigma_i(\bar{v})t^i \).

The group homomorphism being compatible with the action on \( \text{Gl}(n) \), we have a map of Serre spectral sequences, on the \( E_2 \)-page this manifests as

\[ H^*(\text{Gl}(n); R) \otimes H^*(B(\text{Gl}(n) \times \text{Gl}(n)); R) \longrightarrow H^*(\text{Gl}(n); R) \otimes H^*(BS^1; R) \]

\[ \left( \begin{array}{cc} \alpha_i \otimes 1 \\ 1 \otimes c_i \\ 1 \otimes c_i' \end{array} \right) \longmapsto \left( \begin{array}{cc} \alpha_i \otimes 1 \\ 1 \otimes \sigma_i(\bar{w})t^i \\ 1 \otimes \sigma_i(\bar{v})t^i \end{array} \right) \]

the differentials in the second sequence can be determined from this comparison, since we know that

\[ d_2(i) = \sigma_i(\bar{v})t^i - \sigma_i(\bar{v})t^i \quad (\text{mod } \sigma_1(\bar{v}) - \sigma_1(\bar{w}), \sigma_2(\bar{v}) - \sigma_2(\bar{w}), \ldots, \sigma_{i-1}(\bar{v}) - \sigma_{i-1}(\bar{w})) \]

as claimed in proposition 2.
5. The Equivariant Cohomology of Stiefel Manifolds

We shall consider the unreduced Stiefel manifolds, to wit, the spaces \( W(n, m) \) of sequences of \( m \) linearly independent vectors in \( \mathbb{C}^n \). One has \( W(n, n) = \text{Gl}(n) \), of course. One can view \( W(n, m) \) either as the space of surjective maps \( \mathbb{C}^n \to \mathbb{C}^m \) or as the space of injective maps \( \mathbb{C}^m \to \mathbb{C}^n \). We adopt the former interpretation for preference. Given \( m' \leq n - m \), and a surjective map \( A : \mathbb{C}^n \to \mathbb{C}^{m+m'} \), we can compose with the evident projection \( \mathbb{C}^{m+m'} \to \mathbb{C}^m \), and so obtain a surjective map \( A' : \mathbb{C}^n \to \mathbb{C}^m \). We therefore have a map \( \pi : W(n, m + m') \to W(n, m) \). In coordinates, this map is straightforward

\[
\begin{pmatrix}
    a_{1,1} & \cdots & a_{m,1} & a_{m+1,1} & \cdots & a_{m+m',1} \\
    \vdots & & \vdots & \vdots & & \vdots \\
    a_{1,n} & \cdots & a_{m,n} & a_{m+1,n} & \cdots & a_{m+m',n}
\end{pmatrix} \mapsto
\begin{pmatrix}
    a_{1,1} & \cdots & a_{m,1} \\
    \vdots & & \vdots \\
    a_{1,n} & \cdots & a_{m,n}
\end{pmatrix}
\]

We now summarise what we will need of the non-equivariant cohomology of such Stiefel manifolds.

**Proposition 4.** The cohomology of the Stiefel manifold \( W(n, m) \) is given by

\[
H^*(W(n, m); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(\alpha_{n-m+1}, \alpha_{n-m+1}, \ldots, \alpha_n).
\]

For \( n \geq m + m' \), there exists a projection map \( \pi : W(n, m + m') \to W(n, m) \) (as above) by taking the first \( m \) of \( m' \) vectors. On cohomology, such a map induces an inclusion

\[
H^*(W(n, m); R) \cong \Lambda_R(\alpha_{n-m+1}, \alpha_{n-m+1}, \ldots, \alpha_n) \xrightarrow{\alpha_i} \Lambda_R(\alpha_{n-m+1}, \alpha_{n-m'+1}, \ldots, \alpha_n)
\]

If the vector spaces are given a grading, the space \( W(n, m) \) is subject to a left-right action. More precisely, if \( U, V \) are graded vector spaces of dimensions \( n \) and \( m \) respectively, admitting bases of homogeneous elements in degrees \( (a_1, \ldots, a_n) \in \mathbb{Z}^n \) and \( (v_1, v_2, \ldots, v_m) \in \mathbb{Z}^m \), and if \( A : V \to W \) is a surjective map, then we can write \( A \) with respect to the given bases, and the left-right action of \( S^1 \) of \( A \) is given by

\[
z \cdot A = \begin{pmatrix}
    z^{w_1} \\
    z^{w_2} \\
    \vdots \\
    z^{w_m}
\end{pmatrix} A \begin{pmatrix}
    z^{-v_1} \\
    z^{-v_2} \\
    \vdots \\
    z^{-v_m}
\end{pmatrix}
\]

Given graded vector spaces \( V, V' \) of dimensions \( m, m' \) with \( m + m' \leq n \), and a surjective map \( A : U \to V \oplus V' \), we can take the composite \( A : U \to V \oplus V' \to V \). This gives an \( S^1 \)-equivariant version of the map \( \pi \) in the obvious manner.

**Theorem 5.** Suppose \( U, V \) are \( n, m \)-dimensional graded vector spaces, respectively, with admitting bases of homogeneous elements in degrees \( \vec{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n \) and \( \vec{v} = (v_1, v_2, \ldots, v_m) \in \mathbb{Z}^m \). Let \( W(n, m) \to ES^1 \times_{S^1} W(n, m) \to BS^1 \) be the fiber sequence associated with the induced \( S^1 \)-action on \( W(n, m) \), the space of surjective maps \( U \to V \). Let \( s_1, \ldots, s_{n-m} \) be integers defined (recursively) by the relations

\[
\sigma_i(\vec{u}) = \sum_{j=1}^{i} \sigma_j(\vec{v}) s_{i-j}.
\]
and let $s_i = 0$ for $i > n - m$.

The Serre spectral sequence associated with the given fibration has $E_2$-page

$$H^*(W(n, m); Z) \otimes H^*(BS^1; Z) = \Lambda_Z(\alpha_{n-m+1}, \ldots, \alpha_n) \otimes Z[\theta]$$

Let $n - m + 1 \leq k \leq n$ and suppose that

$$d_{2j}(\alpha_j) = 0 \quad \text{for} \quad n - m + 1 \leq i \leq k$$

then we have, in the given spectral sequence

$$(5) \quad d_{2k}(\alpha_k) = [\sigma_k(\bar{v}) - \sum_{j=1}^{k} \sigma_j(\bar{v})s_{k-j}] \theta^k$$

Proof. Strictly speaking, the proof proceeds by induction on $\ell = k - (n - m + 1)$, although most of the difficulty is already evident in the case $\ell = 0$. The arguments for the case $\ell = 0$ and for the induction step are very similar; we shall give both in parallel as much as possible.

Given the form of the spectral sequence, and the fact that all $\alpha_i$ are transgressive, one certainly has $d_{2k}(\alpha_k) = C\theta^k$. The argument in the proof will be to reduce to $\mathbb{Z}/p$-coefficients, then verify equation (5) for infinitely many primes $p$. The case of $\mathbb{Z}$ coefficients then follows.

Consider the integer polynomial

$$f(x) = x^{n-m+1} - s_1x^{n-m} + s_2x^{n-m-1} + \cdots + (-1)^{n-m-1}s_{n-m-1}x + (-1)^{n-m}s_{n-m}$$

By the a corollary of the Frobenius density theorem, this polynomial splits over $\mathbb{Z}/p$ for infinitely many primes $p$. Let $\mathcal{P}$ denote the set of all such primes. We will first establish equation (5) for all $p \in \mathcal{P}$.

To show the problem reduces well to prime coefficients, we need only observe that the natural map induced by $\mathbb{Z} \to \mathbb{Z}/p$ commutes with differentials, and since all the $\mathbb{Z}$-modules under consideration are free, on the $E_2$-page it takes the form of a map

$$H^p(BS^1; H^q(W(n, m); Z)) \to H^p(BS^1; H^q(W(n, m); \mathbb{Z}/p))$$

which is always surjective.

Working modulo a particular prime, $p \in \mathcal{P}$, we write $d_{2k}(\alpha_k) = C\theta^k$, by abuse of notation, where all the terms are understood as the reduction mod $p$ of their integral analogues. By the choice of the prime $p$, we can find $\bar{v}' = (v'_1, \ldots, v'_{n-m})$, roots of the polynomial $f(x)$ in $\mathbb{Z}/p$. We chose particular integer representatives for the $v'_i$, denoting them by $v'_i$, again by abuse of notation. We note that this choice of $v'_i$ has been made so that $\sigma_i(\bar{v}') \equiv s_i \pmod{p}$. We work exclusively with $\mathbb{Z}/p$ coefficients from now on. We therefore have $\sigma_i(\bar{v}') = s_i$, which is key to the whole argument.

Let $V'$ be a graded vector space with homogeneous basis elements in degrees $\bar{v}' = (v'_1, \ldots, v'_{n-m})$. Let $\pi$ denote the equivariant projection map $\pi : \text{Gl}(n) \to W(n, m)$ we obtain by considering the former as surjective maps $U \to V \oplus V'$, and the latter as surjective maps $U \to V$. This map induces a map of Serre spectral sequences.

For the purposes of computing with the $S^1$-action on $\text{Gl}(n)$, it will be convenient to define $\bar{v}''$ as the concatenation of $\bar{v}'$ and $\bar{v}'$, that is $(v_1, \ldots, v_m, v'_1, \ldots, v'_{n-m})$. The spectral sequence arising from the $S^1$-action on $\text{Gl}(n)$ has $E_2$ page

$$H^*(BS^1; \mathbb{Z}/p) \otimes H^*(\text{Gl}(n); \mathbb{Z}/p) = \mathbb{Z}/p[\theta] \otimes \Lambda_{\mathbb{Z}/p}(\alpha_1, \ldots, \alpha_n)$$
the first nonvanishing differential is given by $d_j(\alpha_j) = [\sigma_j(\vec{u}) - \sigma_j(\vec{v})]^{\partial^i}$, where $j$ is the least positive integer such that $\sigma_j(\vec{u}) - \sigma_j(\vec{v}) \neq 0$.

Suppose for the sake of contradiction that this $j \leq n - m$.

$$d_j(\alpha_j) = \sigma_j(\vec{u}) - \sigma_j(\vec{v}) = \sigma_j(\vec{u}) - \sum_{i=1}^{j} \sigma_i(\vec{v})\sigma_{j-i}(\vec{v}) = \sigma_j(\vec{u}) - \sum_{i=1}^{j} \sigma_i(\vec{v})s_{j-i}$$

The term on the RHS is 0, by the definition of $s_{j-i}$, a contradiction.

If we are in the case where $k > n - m + 1$, that is, not the base case for the purpose of induction, we suppose for the sake of contradiction that $j < k$. Again we have

$$d_j(\alpha_j) = \sigma_j(\vec{u}) - \sigma_j(\vec{v}) = \sigma_j(\vec{u}) - \sum_{i=1}^{j} \sigma_i(\vec{v})\sigma_{j-i}(\vec{v}) = \sigma_j(\vec{u}) - \sum_{i=1}^{j} \sigma_i(\vec{v})s_{j-i}$$

but here we know that, by applying the result to compute $d_j(\alpha_j)$, that we have $d_j(\alpha_j) = \sigma_j(\vec{u}) - \sum_{i=1}^{j} \sigma_i(\vec{v})s_{j-i}$. Since $d_j(\alpha_j)$ was assumed to be 0 for $j$ in the range $n - m < j < k$, we have a contradiction.

We return to considering both cases. The purpose of the inductive argument is to be able to assert that $d_j(\alpha_j) = 0$ for $j < k$ in the spectral sequence

$$H^*(BS^1; \mathbb{Z}/p) \otimes H^*(\text{Gl}(n); \mathbb{Z}/p) \Rightarrow H^*(ES^1 \times_{S^1} \text{Gl}(n,m); \mathbb{Z}/p).$$

This allows us to compute the differential $d_{2k}(\alpha_k)$ in this spectral sequence. It is

$$d_{2k}(\alpha_k) = [\sigma_k(\vec{u}) - \sigma_k(\vec{v})]^{\partial^k} = [\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})\sigma_{k-j}(\vec{v})] = [\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}]$$

We again consider the cohomology of $ES^1 \times_{S^1} \text{W}(n,m)$. From the comparison map

$$\xymatrix{ H^*(BS^1; \mathbb{Z}/p) \otimes H^*(\text{W}(n,m); \mathbb{Z}/p) \ar[r]^{\text{id} \otimes \pi^*} \ar[d]_{\text{id} \otimes \pi} & H^*(ES^1 \times_{S^1} \text{W}(n,m); \mathbb{Z}/p) \ar[d] \ar[r] & H^*(ES^1 \times_{S^1} \text{Gl}(n); \mathbb{Z}/p) \ar[d] }$$

we obtain a commutative square

$$\xymatrix{ \alpha_k \ar[r]^{d_{2k}} & C^{\partial^k} \ar[d] \ar[r] & \\
\alpha_k \ar[r]^{d_{2k}} & C^{\partial^k} \ar[d] \ar[r] & \\
[\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}]^{\partial^k} \ar[r] & C - \left[\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}\right]^{\partial^k} \ar[r] & \\
}$$

which proves that $C = [\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}]$ in $\mathbb{Z}/p$, or equivalently that we have

$$p \mid \left(C - \left[\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}\right]\right)$$

Since there are infinitely many $p \in \mathcal{P}$, and this relation holds for them all, it follows that

$$C = [\sigma_k(\vec{u}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}]$$
as required.

Unfortunately this method of proof establishes only the first non-zero differential of the form $d_2k(\alpha_k) = C\theta^k$, we cannot push it further to describe the subsequent differentials. We conjecture that the pattern established in the theorem continues, that the differential takes the form

$$d_2k(\alpha_k) = [\sigma_k(\vec{v}) - \sum_{j=1}^{k} \sigma_j(\vec{v})s_{k-j}]\theta^k$$

modulo the appropriate indeterminacy for all $k$.

6. The General Rothenberg-Steenrod Spectral Sequence

This section is devoted to a construction of a spectral sequence which is well-known to homotopy theorists, but seldom published in the form we need, we establish it here for want of a good reference. It seems to be most properly called a spectral sequence of Eilenberg-Moore-Rothenberg-Steenrod type, after [RS65]; we shall abbreviate and refer to it as the Rothenberg-Steenrod spectral sequence.

We shall be working with simplicial spaces in this section, where appropriate (e.g. for purposes of computing singular homology) we shall tacitly replace such a simplicial space by its geometric realization.

Let $G$ be a topological group. Let $X$, $Y$ be pointed spaces on which $G$ acts on the right and on the left respectively. As in [May75], one defines $B(X,G,Y)$ as the simplicial space whose $n$-simplices are $Z_n(X,G,Y) = X \times G \times \cdots \times G \times Y$. The face maps in this simplicial space are given by combining successive pairs of spaces in $Z_n$ via the action maps $X \times G \to X$, $G \times Y \to Y$ and the multiplication map $G \times G \to G$.

As in [Seg68], for any simplicial space $A$, one has spectral sequences (in loc. cit. only the cohomology case is handled and that for semisimplicial spaces, but the case of homology is similar and the necessary modifications for simplicial spaces not difficult)

$$E^1_{p,q} = H_q(A_p; R) \Rightarrow H_{p+q}(A; R)$$

$$E^{p,q}_1 = H^q(A_p; R) \Rightarrow H^{p+q}(A; R)$$

which are natural in $A$. In our particular case, we have spectral sequences

$$E^1_{p,q} = H_q(Z_p(X,G,Y); R) \Rightarrow H_{p+q}(B(X,G,Y); R)$$

$$E^1_{p,q} = H^q(Z_p(X,G,Y); R) \Rightarrow H^{p+q}(B(X,G,Y); R)$$

It is proved in loc. cit. that the $d_1$-differential in the spectral sequences $d_1 : H^q(Z_p(X,G,Y); R) \to H^p(Z_{p+1}(X,G,Y); R)$ is the alternating sum of the maps induced on homology by the face maps of the simplicial space. The analogous result holds in homology.

We make the simplifying assumption that $H_*(G; R)$, $H_*(X; R)$ and $H_*(Y; R)$ are projective $R$-modules, so that a K"unneth isomorphism obtains

$$H_*(Z_*(X,G,Y); R) = H_*(X; R) \otimes_R H_*(G; R) \otimes_R H_*(Y; R)$$
and one also has universal coefficient isomorphisms for $X, Y$ and $G$

$$H^*(X; R) \cong \text{Hom}(H_*(X; R), R)$$

and similarly for $Y$ and $G$.

In this case, a homology class can be represented as a sum of terms of the form

$$\xi \otimes \gamma_1 \otimes \ldots \otimes \gamma_q \otimes \eta,$$

and one has

$$d_1(\xi \otimes \gamma_1 \otimes \ldots \otimes \gamma_q \otimes \eta) = \xi \gamma_1 \otimes \gamma_2 \otimes \ldots \otimes \gamma_q \otimes \eta + \sum_{i=1}^{q} (-1)^i \xi \otimes \gamma_1 \otimes \ldots \otimes \gamma_{i-1} \otimes \gamma_i \gamma_{i+1} \otimes \gamma_{i+2} \otimes \ldots \otimes \gamma_q \otimes \eta$$

The differential in the case of cohomology is dual to the differential described above.

We now concentrate on the case of a single space, $Y$, on which $G$ acts on the left.

**Proposition 6.** There is a convergent spectral sequence

$$E_2^{p,q} = \text{Ext}^p_{H_* (G; R)}(H_* (Y; R), R) \Rightarrow H^{p+q}(B(\text{pt}, G, Y); R)$$

which is natural in both $Y$ and $G$, in the sense that if $\phi : G \to G'$ is a group homomorphism, and $f : Y \to Y'$ is a map from a $G$-space to a $G'$-space so that the following diagram commutes

$$
\begin{array}{ccc}
G \times Y & \longrightarrow & G' \times Y' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y'
\end{array}
$$

then there is a map of spectral sequences

$$E_2^{p,q} = \text{Ext}^p_{H_* (G'; R)}(H_* (Y'; R), R) \longrightarrow H^{p+q}(B(\text{pt}, G', Y'); R)$$

$$E_2^{p,q} = \text{Ext}^p_{H_* (G; R)}(H_* (Y; R), R) \longrightarrow H^{p+q}(B(\text{pt}, G, Y); R)$$

**Proof.** The sequence is exactly that described above, for the cohomology of $B(\text{pt}, G, Y)$. The key is the identification of the $E_2$-page.

For convenience, we denote the $R$-algebra $H_* (G; R)$ by $S$. We know that the $E_2$ page is the homology of a cocomplex

(6) $$
\begin{array}{ccc}
H^*(Z_p (\text{pt}, G, Y); R) & \longrightarrow & H^*(Z_{p+1} (\text{pt}, G, Y); R)
\end{array}
$$

We have

$$H^*(Z_p; R) = \text{Hom}_R(S^p \otimes_R H_*(Y; R), R)$$

$$\cong \text{Hom}_S(S \otimes_R S^p \otimes_R H_*(Y; R), R) \cong \text{Hom}_S(H_*(Z_p(G, G, Y); R), R)$$

The group $H_*(Z_p(G, G, Y); R)$ is the $p$-th term in the $E_1$-page of the spectral sequence associated with $B(G, G, Y)$. A routine but messy argument regarding differentials allows us to identify the cocomplex in (6) with the result of applying the
functor \( \text{Hom}_S(\cdot, R) \) to the complex

\[
\begin{array}{c}
S^{\otimes p+2} \otimes_R H_* (Y; R) \cong H_* (Z_{p+1} (G, G, Y); R) \\
S^{\otimes p+1} \otimes_R H_* (Y; R) \cong H_* (Z_p (G, G, Y); R)
\end{array}
\]

This complex is exactly the algebraic bar resolution of \( H_* (Y; R) \) as an \( S \)-module over the ring \( R \), which is, as the name suggests is a resolution of \( H_* (Y; R) \), \cite{Wei94}. It follows that the \( E_2 \)-page of the spectral sequence, viz. the homology of \( S \), is exactly as claimed.

The convergence of the spectral sequence goes by the book, \cite{Boa99}, since it is concentrated in one quadrant of the plane.

We remark on the naturality of the sequence. In the first place, given a map \( \phi : B \rightarrow C \) with \( \phi = \text{id} \), then we have an induced \( G \)-action on \( Y \) and a map \( B(\text{pt}, G, Y) \rightarrow B(\text{pt}, G, Y) \). In both cases we obtain maps of spectral sequences, and composing these maps yields exactly the naturality claimed in the proposition.

We remark that the maps on \( E_2 \)-pages obtained in this way are exactly the maps

\[
\text{Ext}^p_{H_* (G; R)} (H_* (Y'; R), R) \longrightarrow \text{Ext}^p_{H_* (G; R)} (H_* (Y; R), R)
\]

one obtains by functoriality of \( \text{Ext} \) with respect to the maps \( H_* (Y; R) \rightarrow H_* (Y'; R) \) and \( H_* (G; R) \rightarrow H_* (G'; R) \).

One case of particular importance is the following

**Corollary 6.1.** Suppose \( X \) is a space with a free \( G \)-action on the left, then there is a spectral sequence

\[
E^{p,q}_2 = \text{Ext}^p_{H_* (G; R)} (H_* (X; R), R) \Longrightarrow H^{p+q} (X/G; R)
\]

**Proof.** Under the assumption of a free \( G \)-action, there is a weak equivalence \( X/G \simeq B(\text{pt}, X, G) \), and the spectral sequence follows. \( \square \)

**7. The Cohomology of Spaces of Long Exact Sequences**

**7.1. Presentation as Homogeneous Spaces.** We consider long exact sequences of graded \( \mathbb{C} \)-vector spaces, for instance:

\[
0 \longrightarrow A_1 \xrightarrow{d_1} B_1 \xrightarrow{e_1} A_2 \xrightarrow{d_2} \cdots \xrightarrow{e_{n-1}} A_n \xrightarrow{d_n} B_n \longrightarrow 0
\]

since there are \( 2n \) terms to this sequence, we designate this as the even case, the odd case will be that where the last term is \( A_n \). We shall treat mainly the even case in what follows, the odd case is generally much the same and we shall try to spare the reader by proving each result only once.

Our initial treatment does not involve the grading, and is equally true of the ungraded case. Later, the grading will play a meaningful role.

We define \( a_i = \dim_k A_i \) and \( b_i = \dim_k B_i \). When the grading is not important, we denote the space of such sequences by \( X (a_1, \ldots, a_n, b_1, \ldots, b_n) \).

The spaces \( A_i, B_i \) can each be decomposed into graded parts. Write \( \mathbb{C} (n) \) for the vector space \( \mathbb{C} \), placed in degree \( n \). We write \( A_i \cong \oplus_{j=1}^{a_i} \mathbb{C} (v_{i,j}) \), with \( v_{i,j} \leq v_{i,j+1} \).
The integers $v_{i,j}$ encapsulate all the grading information of the $A_i$. We make an equivalent definition of integers $w_{i,j}$ for the $B_i$. We will occasionally write

$$X(v_{1,1}, \ldots, v_{1,a_1}; v_{2,1}, \ldots, v_{2,a_2}; \ldots; v_{n,1}, \ldots, v_{n,a_n};$$

$$w_{1,1}, \ldots, w_{1,b_1}; w_{2,1}, \ldots, w_{2,b_2}; \ldots; w_{n,1}, \ldots, w_{n,b_n})$$

in place of $X(a_1, \ldots, a_n, b_1, \ldots, b_n)$ when the grading is important.

In the interests of concreteness, we fix a basis of homogeneous elements for each $A_i$ and each $B_i$, and equip each space with a complex inner-product with respect to which the given basis is orthonormal.

The space of long exact sequences is now identified with the space of matrices

$$G = \text{Gl}(A_1) \times \text{Gl}(B_1) \times \text{Gl}(A_2) \times \text{Gl}(B_2) \times \cdots \times \text{Gl}(A_n) \times \text{Gl}(B_n)$$

has a left-action on the space of such sequences by

$$(a_1, b_1, \ldots, a_n, b_n) \cdot (d_1, e_1, d_2, e_2, \ldots, d_n, e_n)$$

$$= (b_1 d_1, a_1^{-1} d_2 e_1 b_1^{-1}, \ldots, a_n e_{n-1}^{-1} b_{n-1}^{-1} b_n d_n a_n^{-1})$$

This left-action is readily seen to be transitive. We will describe the space $X$ as a set of left cosets of $G$, and use this description to calculate the cohomology of $X$.

To do this, we fix notation

We write the homology of $\text{Gl}(A_i)$ as $\Lambda_R(\hat{\alpha}_{i,1}, \ldots, \hat{\alpha}_{i,a_i})$ and that of $\text{Gl}(B_i)$ as $\Lambda_R(\hat{\beta}_{i,1}, \ldots, \hat{\beta}_{i,b_i})$. We then have

$$H_s(G; R) = \prod_{i=1}^n \Lambda_R(\hat{\alpha}_{i,1}, \ldots, \hat{\alpha}_{i,a_i}) \times \prod_{i=1}^n \Lambda_R(\hat{\beta}_{i,1}, \ldots, \hat{\beta}_{i,b_i}).$$

At times it is more convenient to distinguish the two families of spaces $A_i$ and $B_i$, and at other times it is more convenient to view them as being of a kind. We will occasionally therefore use the notation

$$\alpha_{i,j} = \gamma_{2i-1,j}, \quad \beta_{i,j} = \gamma_{2i,j}, \quad a_i = e_{2i-1}, \quad b_i = e_{2j}.$$

In considering $X$ as a homogeneous $G$-space, the stabilizer of a point can be computed without too much difficulty. Denote by $\{x_{i,j}\}$ the $j$-th element in our basis for $A_i$, and by $\{y_{i,j}\}$ the $j$-th element in our basis for $B_i$, and by $r_i, s_i$ the ranks of $d_i, e_i$ respectively. Consider the sequence $z_0$ for which

$$d_i(x_{i,j}) = y_{i,j}, \quad \text{for } 1 \leq j \leq r_i, \quad d_i(x_{i,j}) = 0, \quad \text{for } j > r_i$$

$$e_i(y_{i,b_i-j}) = x_{i+1,a_i+1-j}, \quad \text{for } 1 \leq j \leq s_i, \quad e_i(y_{i,b_i-j}) = 0 \quad \text{for } j > s_i$$

We take this sequence as an origin for the homogeneous space $X$ whenever we require such a point.

Supposing $(a_1, b_1, \ldots, a_n, b_n)$ fixes this sequence, then we have, in particular, the equations $b_i d_i a_i^{-1} = d_i$ and $a_{i+1} e_i b_i^{-1}$. In coordinates

$$b_i \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix} a_i^{-1} = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}, \quad a_{i+1} \begin{pmatrix} I_{s_i} & 0 \\ 0 & 0 \end{pmatrix} b_i^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_s \end{pmatrix}$$

From these equations it follows that the matrices $a_i$ and $b_i$ decompose as

$$b_i = \begin{pmatrix} f_i & * \\ 0 & g_i \end{pmatrix}, \quad a_i = \begin{pmatrix} f_i & * \\ 0 & g_{i-1} \end{pmatrix}$$
We see that the stabiliser of an arbitrary element is therefore a group having the homotopy type of

\[ K = \text{Gl}(r_1) \times \text{Gl}(s_1) \times \text{Gl}(r_2) \times \text{Gl}(s_2) \times \cdots \times \text{Gl}(s_{n-1}) \times \text{Gl}(r_n) \]

We write the homology of this space as

\[ H_\ast(K; R) = \prod_{i=1}^{2n} \Lambda_R(\hat{\tau}_{i,1}, \ldots, \hat{\tau}_{i,i}) \times \prod_{i=1}^{n-1} \Lambda_R(\hat{\sigma}_{i,1}, \hat{\sigma}_{i,2}, \ldots, \hat{\sigma}_{i,s_i}) \]

but again, in keeping with our dual understanding of \( A_i, B_i \) we will write

\[ \hat{\rho}_{i,j} = \hat{\tau}_{2i-1,j}, \quad \hat{\sigma}_{i,j} = \hat{\tau}_{2i,j}, \quad r_i = t_{2i-1}, \quad s_i = t_{2i} \]

we shall also use the convention \( t_0 = 0 \), so that for all \( c_i \) there is a corresponding \( t_{i-1} \).

7.2. The Non-Equivariant Cohomology. The inclusion of \( K \) in \( G \) induces the following map on homology

\[ H_\ast(K; R) \xrightarrow{\iota_*} H_\ast(G; R) \]

\[ \prod_{i=1}^{2n} \Lambda_R(\hat{\tau}_{i,1}, \ldots, \hat{\tau}_{i,i}) \xrightarrow{\iota_*} \prod_{i=1}^{2n} \Lambda_R(\hat{\gamma}_{i,1}, \ldots, \hat{\gamma}_{i,c_i}) \]

which is a map of Hopf algebras, in particular of rings. By a change of coordinates, replacing \( \hat{\gamma}_{i,j} \) by \( \iota_* (\hat{\tau}_{i,j}) \) in our presentation of \( H_\ast(G; R) \) \([10]\), we see that \( H_\ast(G; R) \) is itself an exterior algebra over \( H_\ast(K; R) \). We find a set \( \hat{N} \) so that \( H_\ast(G; R) = \Lambda_{H_\ast(K; R)}(\hat{N}) \).

Let \( i \) be an integer satisfying \( 1 \leq i \leq 2n \). Suppose \( j \) is an integer satisfying \( t_{i-1} < j \leq c_i \). We define \( \ell \) to be the least integer \( i \leq \ell \) such that \( t_\ell < j \)

\[ \kappa_{i,j} = \sum_{k=i}^\ell (-1)^{k-1} \gamma_{k,j}. \]

We denote the set of all \( \kappa_{i,j} \) by \( N(X) \), or by \( N \) when the dependence on \( X \) is clear. By \( \hat{N} \), we mean the set of duals of \( N \), taken with respect to the evident basis of \( H_\ast(G; R) = \Lambda_R(\{\gamma_{i,j}\}) \).
A pictorial description of the $\kappa_{i,j}$ is perhaps of use. Take for example the case where $(c_1, c_2, c_3, c_4, c_5) = (1, 4, 5, 4, 3, 1)$ and $(t_1, t_2, t_3, t_4, t_5) = (1, 3, 2, 2, 1)$. Pictorially we denote this as

\[(15) \quad \gamma_{3,5} \quad \gamma_{2,4} \quad \gamma_{3,4} \quad \gamma_{4,4} \quad \gamma_{2,3} \quad \gamma_{3,3} \quad \gamma_{4,3} \quad \gamma_{5,3} \quad \gamma_{2,2} \quad \gamma_{3,2} \quad \gamma_{4,2} \quad \gamma_{5,3} \quad \gamma_{1,1} \quad \gamma_{2,1} \quad \gamma_{3,1} \quad \gamma_{4,1} \quad \gamma_{5,1} \quad \gamma_{6,1} \]

the horizontal lines being numbered by the $t_i$. In this case one has

$\kappa_{1,1} = \gamma_{1,1} + \gamma_{2,1} + \gamma_{3,1} + \gamma_{4,1} + \gamma_{5,1} + \gamma_{6,1}$

$\kappa_{2,2} = \gamma_{2,2} + \gamma_{3,2} + \gamma_{4,2} + \gamma_{5,2}$

$\kappa_{2,3} = \gamma_{2,3} + \gamma_{3,3}$

$\kappa_{i,j} = \gamma_{i,j}$ if $(i, j)$ is any of $(4, 3), (5, 3), (2, 4), (3, 4), (4, 4)$ or $(5, 3)$

**Proposition 7.** In the notation of this section, the cohomology of

\[X = X(a_1, \ldots, a_n, b_1, \ldots, b_n)\]

is

\[\operatorname{Hom}_{H_*(K;R)}(H_*(G; R), R)\]

The map $G \to G/K = X$ induces the evident injective map

\[H^*(X; R) \cong \operatorname{Hom}_{H_*(K;R)}(H_*(G; R), R) \to \operatorname{Hom}_{R}(H_*(G; R), R) \cong H^*(G; R)\]

and we can identify $H^*(X; R) = \Lambda_R(N) \subset H^*(G; R)$.

**Proof.** Since $X$ is the homogeneous space of cosets of $G$ by $K$, we have a spectral sequence as in corollary 6.1 which on the $E_2$-page is $\operatorname{Ext}^{p,q}_{H_*(K;R)}(H_*(G; R), R)$. The $H_*(K; R)$-module structure of $H_*(G; R)$ is given by $K R$-module structure of $H_*(G; R)$, in particular it is a free $H_*(K; R)$-module. The $E_2$-page is concentrated in the column $p = 0$, and it collapses thereafter. We therefore have $H^*(X; R) = \operatorname{Hom}_{H_*(K;R)}(H_*(G; R), R)$, which is the first assertion.

We can calculate the comparison map $H_*(G/K; R) \to H_*(G; R)$ by taking $G$ to be $G/\{e\}$, the trivial quotient, and using the naturality of the Rothenberg-Steenrod spectral sequence. The map is evidently an injection for algebraic reasons.

To prove the last assertion, that $H^*(X; R) = \Lambda_R(N) \subset H^*(G; R)$, we first remark that since the $\kappa_{i,j}$ are $R$-independent and satisfy $\kappa_{i,j}^2 = 0$, the subalgebra of $H^*(G; R)$ they generate is exactly $\Lambda_R(N)$.

Secondly, it is easily verified that the $\kappa_{i,j}$ are in fact $H_*(K; R)$-linear, so we certainly have an inclusion $\Lambda_R(N) \subset H^*(X; R)$. To prove the equality, we first
observe that it suffices to deal with the case $R = \mathbb{Z}$, since the case of general $R$ can be obtained by base change.

Consider therefore the short exact sequence

$$0 \rightarrow \Lambda_\mathbb{Z}(N) \rightarrow H^*(X; \mathbb{Z}) \rightarrow Q \rightarrow 0$$

It suffices to show $Q = 0$. Since $Q$ is a finitely generated abelian group, we need only show $Q \otimes \mathbb{Q} = 0$ and $Q \otimes \mathbb{Z}/p = 0$ for all $p$. After changing base to any field, $k$, we have

$$0 \rightarrow \Lambda_k(N) \rightarrow H^*(X; k) \rightarrow Q \otimes \mathbb{Z}/k \rightarrow 0$$

We show that $Q \otimes \mathbb{Z}/k$ is $0$ by counting dimensions. In the first place we have $\dim_k \Lambda_k(N) = 2^{|N|}$. In the second we have, writing $S$ for $H_\mathcal{S}(K; k)$ for brevity,

$$\dim_k(H^*(X; k)) = \dim_k \hom_S(H_\mathcal{S}(G; k), k) = \dim_k \hom_S(\Lambda_S(\hat{N}), k) = \dim_k \hom_S(S^{2^{|N|}}, k) = 2^{|\hat{N}|}$$

It follows that $Q \otimes \mathbb{Z}/k = 0$, as claimed. \qed

7.3. The $S^1$-action. There is an $S^1$-action on $G = \operatorname{Gl}(A_1) \times \operatorname{Gl}(B_1) \times \cdots \times \operatorname{Gl}(A_n) \times \operatorname{Gl}(B_n)$, since these vector spaces are graded and we can give $\operatorname{Gl}(A_i)$ an action of $S^1$ on the left, following section 2. This gives rise to an action of $S^1$ on $G/K = X$, which is of central importance. To give explicit formulas, we take $\xi \in X$ to be of given by $g_\xi X$, where $\xi_0$ is as defined in equation (11), and $g$ is determined only up to indeterminacy by $K$. For the sake of concreteness, take $d_1 : A_1 \rightarrow B_1$, one of the differentials in the sequence $x$. We can write this as $b_1 d_0^0 d_1^{-1} = d_1$, where $a_1, b_1$ are terms in $g$. The $S^1$ action on $d_1$ is therefore given by

$$z \cdot d_1 = (z \cdot b_1) d_0^0 (z \cdot a_1)^{-1} = \begin{pmatrix} z^{u_1} & & \\
 & z^{u_2} & \\
 & & \ddots \\
 & & & z^{u_n} a_1 \end{pmatrix} d_1 \begin{pmatrix} z^{-v_1} & & \\
 & z^{-v_2} & \\
 & & \ddots \\
 & & & z^{-v_n a_1} \end{pmatrix}$$

viz. it is an action on the left & right as described in section 2.

8. Comparison Maps of Spaces of Exact Sequences

8.1. Comparison By Folding. We begin with a sequence of graded complex vector spaces

$$0 \rightarrow A_1 \stackrel{d_1}{\rightarrow} B_1 \stackrel{e_1}{\rightarrow} A_2 \stackrel{d_2}{\rightarrow} \cdots \stackrel{e_{n-1}}{\rightarrow} B_{n-1} \stackrel{e_n}{\rightarrow} A_n \stackrel{d_n}{\rightarrow} B_n \rightarrow 0$$

We continue to assume that each vector space is equipped with an inner product and an orthonormal basis of homogeneous elements, this makes it possible to identify $\hat{A}_i$ and $A_i$ graded vector spaces, as in section 2 and so obtain a complex

$$0 \rightarrow B_1 \stackrel{e_1 \oplus \hat{d}_1}{\rightarrow} A_1 \oplus A_2 \stackrel{0 \oplus d_2}{\rightarrow} B_2 \rightarrow \cdots \rightarrow A_n \stackrel{d_n}{\rightarrow} B_n \rightarrow 0$$

The map $\hat{d}_1$ is injective into $A_1 \cong \hat{A}_1$. If $e_1(b) = 0$, then $b \in \operatorname{im} d_1$, by exactness, and we can write $b = d_1a$. We have $\hat{d}_1 d_1 a \neq 0$, by nondegeneracy. It follows that $\ker(e_1) \cap \ker(\hat{d}_1) = \{0\}$, and so $e_{n-1} \oplus \hat{d}_n$ is an injective map. The complex is
consequently an exact sequence in \( X(a_1, \ldots, a_n, b_1, \ldots, b_{n-1} + b_n) \), and we have a map

\[
\psi : X = X(a_1, \ldots, a_n, b_1, \ldots, b_n) \to X(a_1, \ldots, a_n, b_1, \ldots, b_{n-1} + b_n) = X'
\]

From the graded point of view, the latter space above is

\[
\begin{align*}
X(\mathbf{v}_1, \ldots, \mathbf{v}_{1,a_1}, \mathbf{v}_{2,a_2}, \ldots; \mathbf{v}_{n,1}, \ldots, \mathbf{v}_{n,a_n}; \mathbf{w}_{1,1}, \ldots, \mathbf{w}_{1,b_1}; \mathbf{w}_{2,1}, \ldots, \mathbf{a}_{2,b_2}; \ldots; \\
\mathbf{w}_{n-1,1}, \ldots, \mathbf{w}_{n-1,b_{n-1}}; \mathbf{w}_{n,1}, \ldots, \mathbf{w}_{n,b_n})
\end{align*}
\]

and the map \( \psi \) is \( S^1 \)-equivariant map by the same argument used in proposition 1.

There is a map \( \phi : G = \text{Gl}(A_1, B_1, \ldots, B_{n-1}, A_n, B_n) \to G' = \text{Gl}(A_1, B_1, \ldots, B_{n-1} \oplus \tilde{B}_n, A_n) \), which restricts to the composition \( \text{Gl}(B_{n-1}) \times \text{Gl}(B_n) \to \text{Gl}(B_{n-1}) \times \text{Gl}(\tilde{B}_n) \to \text{Gl}(B_{n-1} \oplus \tilde{B}_n) \).

This map lifts \( \psi \)

\[
\begin{array}{ccc}
K & \longrightarrow & G \\
\downarrow & & \downarrow \psi \\
K' & \longrightarrow & G'
\end{array}
\]

the image of the isotropy subgroup \( K \) in \( G' \) lies in \( K' \), so one has a map of group actions on spaces

\[
\begin{array}{ccc}
G \times K & \longrightarrow & G' \times K' \\
\downarrow & & \downarrow \\
G & \longrightarrow & G'
\end{array}
\]

by naturality in proposition 6.1 we have a comparison map of spectral sequences, which allows us to compute the map \( \psi^* \) on cohomology.

**Proposition 8.** We adopt the notation of this section so far. Writing

\[
H^*(X; \mathbb{Z}) = \Lambda_{\mathbb{Z}}(N(X)) \quad \text{and} \quad H^*(X'; \mathbb{Z}) = \Lambda_{\mathbb{Z}}(N(X'))
\]

where \( N(X) \) is as defined after equation (14). We write \( N = N(X) \) and \( N' = N(X') \). The map

\[
\psi^* : \Lambda_{\mathbb{Z}}(N') \to \Lambda_{\mathbb{Z}}(N)
\]

is defined by

\[
\kappa_{1,j} \mapsto \kappa_{1,j} + \kappa_{2,j} + \kappa_{3,j}, \quad \kappa_{i,j} \mapsto \kappa_{i+1,j} \quad \text{for} \ i \geq 2
\]

where we employ the convention that \( \kappa_{i,j} = 0 \) if it is not otherwise defined.

We remark that, because of exactness, \( \kappa_{1,j} \) and \( \kappa_{2,j} \) are never simultaneously defined.

**Proof.** The maps on homology induced by \( G \to G' \) and \( K \to K' \) are easily computed, c.f. section 3, since these maps restrict to matrix direct-sum. We can then appeal to naturality in corollary 6.1 to obtain the result. The details are not particularly enlightening, and we suppress them. \( \square \)
8.2. The Comparison With $W(b_1, a_1)$. In order to compute the $S^1$-equivariant cohomology of $X$, at least to the extent of computing the first nonzero differential in the Serre spectral sequence associated with $X \to ES^1 \times S^1 X \to BS^1$

$$E_2 = R[\theta] \otimes \Lambda_R(\{\kappa_{i,j}\}) \Longrightarrow H^*(ES^1 \times S^1 X; R)$$

we use a comparison with a Stiefel manifold $W(b_1, a_1)$. This comparison is straightforward, one takes the sequence ($\mathbf{10}$), and forgets all but the differential $A_1 \xrightarrow{d_1} B_1$. This is represented by a $b_1 \times a_1$ matrix of rank $a_1$. The map $X \to W(b_1, a_1)$ is evidently $S^1$-equivariant.

By a straightforward comparison argument, using naturality in corollary ($\mathbf{11}$) we obtain

**Proposition 9.** The induced map on cohomology arising from the projection $X \to W(b_1, a_1)$ is

$$\Lambda_R(\alpha_{b_1-a_1+1}, \alpha_{b_2-a_2+2}, \ldots, \alpha_{b_1}) \cong H^*(W(b_1, a_1); R) \xrightarrow{\alpha_j} H^*(X; R) \cong \Lambda_R(N)$$

$$\alpha_j \longmapsto \kappa_{2,j}$$

This comparison, in tandem with the folding comparison, gives a general procedure for computing many of the differentials in the Serre spectral sequence of the fibration $X \to ES^1_{S^1} X \to BS^1$. We say a class $\kappa_{i,j}$ supports a nonzero differential if $d_{2j}(\kappa_{i,j}) \neq 0$ in the aforementioned spectral sequence. The general outline of the method is to use comparison with $W(b_1, a_1)$ to compute the differentials supported by $\kappa_{2,b_1-a_1+i}$ for $1 \leq i$, at least as far as the first such class to support a nonzero differential. Then, one applies comparison by folding in order to extend the reach of this method to differentials supported by $\kappa_{2,j} + \kappa_{3,j}$ for certain $j$, and so forth. The difficulty in deducing all the differentials currently lies in theorem ($\mathbf{13}$) in that the method for computing the differentials in the sequence associated with a Stiefel manifold does not work past the first nonvanishing differential. Since this is the impediment, however, we may always compute differentials $d_{2j}(\kappa_{i,j})$ if no $\kappa_{i,j'}$ for $j' < j$ supports a nonzero differential.

9. Herzog-Kühl Equations

We return to considering chain complexes over the ring $\mathbb{C}[x_1, \ldots, x_m] = S$. We treat the case of even length, the case of odd length being similar. We begin with a chain complex of graded free $S$-modules

$$\Theta : 0 \longrightarrow F_1 \xrightarrow{D_1} G_1 \xrightarrow{E_1} \cdots \xrightarrow{D_n} G_n \longrightarrow 0$$

having artinian homology. We denote the rank of $D_j, E_j$ by $r_j, s_j$ respectively.

For reasons of expediency we fix homogeneous bases of each of the free $S$-modules appearing in the above resolution. We denote by $v_{j,1}, \ldots, v_{j,a_j}$ the degrees of the basis of $F_{2j-1}$, and by $w_{j,1}, \ldots, w_{j,b_j}$ the degrees of a basis of $F_{2j}$. With respect to these bases, each differential in the resolution takes the form of a matrix over $S$. Since the resolution is graded, it follows that the entries in these matrices are themselves homogeneous polynomials of (possibly) varying degrees.

The main result of this section is the following.
**Theorem 10 (Herzog-Kühl Equations).** Let $\Theta$, $v_{j,k}$ and $w_{j,k}$ be as defined above (a similar result holds for a complex of odd length). Let $q \leq n$ and consider $r_q = \text{rank} \, D_q$ (a similar result holds with the roles of $F_j$, $G_j$ reversed). Let $\bar v_q$, $\bar w_q$ denote the vectors of integers $(v_{j,k})$, $(w_{j,k})$ for $j \leq q$. Let $u_1, \ldots, u_q$ be integers defined recursively by

$$
\sigma_i(\bar w_q) = \sum_{j=1}^i \sigma_j(\bar v_q)u_{i-j}
$$

for $1 \leq i \leq r_q$. Let $u_j = 0$ for $j > q$. Then we have

$$
\sigma_i(\bar w_q) = \sum_{j=1}^i \sigma_j(\bar v_q)u_{i-j}
$$

for $r_q + 1 \leq i \leq m - 1$.

**Proof.** It follows from the exactness result of [Car86] that evaluation at all prime ideals of $S$ other than the irrelevant ideal yields a long exact sequence of graded vector spaces

$$
0 \longrightarrow \mathbb{C}^{a_1} \longrightarrow \mathbb{C}^{b_1} \longrightarrow \cdots \longrightarrow \mathbb{C}^{b_n} \longrightarrow 0
$$

in particular, the matrices representing the differentials yields a map

$$
f : \mathbb{C}^m \setminus 0 \rightarrow X(a_1, \ldots, a_n; b_1, \ldots, b_n) = X
$$
given by homogeneous polynomial functions. The homogeneity of the polynomials implies that this map $f$ is $S^1$ equivariant, where $S^1 \subset \mathbb{C}^*$ acts on $\mathbb{C}^m \setminus 0$ in the obvious way and on $X$ according to the grading on the various $F_{2j-1} \cong S^{a_i}$, $F_{2j} \cong S^{b_j}$. Properly $X$ is to be understood as

$$
X = X(v_1,1; \ldots, v_{1,a_1}; v_2,1; \ldots, v_{2,a_2}; \ldots; v_{n,1}; \ldots, v_{n,a_n}; w_1,1; \ldots, w_{1,b_1}; w_2,1; \ldots, w_{2,b_2}; \ldots; w_{n,1}; \ldots, w_{n,b_n})
$$

but we will try to avoid this notation as far as possible.

Since we understand the equivariant cohomology of $X$ reasonably well, we shall try to obtain obstructions to $S^1$-equivariant maps $\mathbb{C}^m \setminus 0 \rightarrow X$.

We consider the equivariant composition:

$$
\mathbb{C}^m - 0 \longrightarrow X \longrightarrow \text{pt}
$$

There is a composition of maps of Serre spectral sequences arising from this, each fibration having base $BS^1$ and the fibres varying. The Serre spectral sequence for $\mathbb{C}^m \setminus 0$ has $E_2$-page:

$$
H^*(\mathbb{C}^m \setminus 0; R) \otimes_R H^*(BS^1; R) \cong \frac{R[\alpha, \theta]}{(\alpha^2)}, \quad |\alpha| = (0, 2m - 1), |\theta| = (2, 0)
$$

The sole nontrivial differential is $d_{2m}(\alpha) = \theta^m$.

The spectral sequence computing the equivariant cohomology of $\text{pt}$ is even more straightforward, which is precisely $H^*(BS^1; R) = R[\theta]$ and is in an immediate state of collapse.

The point is that the comparison map $R[\theta] \cong H^*_{S^1}(\text{pt}; R) \rightarrow H^*_{S^1}(\mathbb{C}^m \setminus 0; R) \cong R[\theta]/(\theta^{2m})$ is induced by comparison maps of spectral sequences. In particular, we obtain, on every page subsequent to the $E_2$-page, for $j < m$ and all $\ell > 1$

$$
R \cong E^{2j,0}_\ell(ES^1 \times S^1 \text{pt}) \cong E^{2j,0}_\ell(ES^1 \times S^1 \mathbb{C}^m \setminus 0) \cong R
$$
Since this map must factor through $\mathbb{C}^m \setminus 0 \to X$, taking $R = \mathbb{Q}$, we see that the spectral sequence computing $H^*_S(X; \mathbb{Q})$ must have $E^{2j,0}_r \cong \mathbb{Q}$. The differentials $d_2(\kappa_{1,j})$ must all vanish for $j < m$. We now apply comparison theorems to deduce the nature of these differentials.

We first apply the comparison-by-folding map, proposition $\square$ 2q-times. We have a comparison

$$X(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \to$$

$$X(a_1 + a_2 + \cdots + a_q, a_{q+1}, \ldots, a_n; b_1 + b_2 + \cdots + b_q, b_{q+1}, \ldots, b_n)$$

To this we apply a comparison with a Stiefel manifold

$$X(a_1 + a_2 + \cdots + a_q, a_{q+1}, \ldots, a_n; b_1 + b_2 + \cdots + b_q, b_{q+1}, \ldots, b_n) \to$$

$$W(b_1 + \cdots + b_q, a_1 + \cdots + a_q)$$

Write $b = \sum_{i=1}^q b_i$, $a = \sum_{i=1}^q a_i$. We point out that by an exactness argument, we have $b - a = r_q$. The map induced on cohomology by the composite of the two comparison maps is $\alpha_k \in \Lambda^q(\alpha_{b-a+1}, \ldots, \alpha_b)$ to $\sum_{i=1}^{2q} \kappa_{i,k}$. Since the differentials in the Serre spectral sequence supported by the classes $\kappa_{i,k}$ vanish when $k < m$, and since the comparison maps are $S^1$-equivariant, it follows that the differentials supported by the classes $\alpha_k$, where $k < m$, in the cohomology of $W(b, a)$ similarly vanish. The $S^1$-action on $W(b, a)$ which makes the comparison maps equivariant is given by the weights $(v_{1,1}, v_{1,2}, \ldots, v_{q,a_q}; w_{1,1}, w_{1,2}, \ldots, w_{q,b_q})$. The numerical formulas of the proposition now follow by considering theorem $\square$.

The numerical conditions of this theorem may be restated in the following form. Given $(v_{j,k})$ and $(w_{j,k})$ as in the theorem, there exist complex numbers $v_{q+1,1}, \ldots, v_{q+1,r_q}$ (unique up to permutation) so that, taking $\vec{v}_q$ to be the vector of all $v_{j,k}$ and $\vec{v}_{q+1,k}$, we have

$$\sigma_i(\vec{v}_q) = \sigma_i(\vec{w}_q)$$

The $u_j$ of the theorem are the elementary symmetric functions $\sigma_j(\{v_{1,1}^{q+1}, \ldots, v_{q+1,r_q}\})$. The theory of symmetric polynomials now allow us to restate the theorem as follows

**Corollary 10.1.** Let $\Theta$, $v_{j,k}$ and $w_{j,k}$ be as defined above (a similar result holds for a complex of odd length). Let $q \leq n$ and consider $r_q = \text{rank} D_q$ (a similar result holds with the roles of $F_j$, $G_j$ reversed). Let $\vec{v}_q$, $\vec{w}_q$ denote the vectors of integers $(v_{j,k}), (w_{j,k})$ for $j \leq q$. Let $v'_{q+1,1}, \ldots, v'_{q+1,r_q}$ be complex numbers defined by the system of equations

$$\sum_{j=1}^q \sum_{k=1}^{b_j} w_{j,k}^i = \sum_{j=1}^q \sum_{k=1}^{a_j} u_{j,k}^i + \sum_{k=1}^{r_q} (v'_{q+1,k})^i$$

for $1 \leq i \leq r_q$. Then we have

$$\sum_{j=1}^q \sum_{k=1}^{b_j} w_{j,k}^i = \sum_{j=1}^q \sum_{k=1}^{a_j} u_{j,k}^i + \sum_{k=1}^{r_q} (v'_{q+1,k})^i$$

for $r_q + 1 \leq i \leq m - 1$.

The classical Herzog-K"uhl equations correspond to the case where $q = n$, in that case $r_q = 0$ and there are no integers $u_j$ (and therefore no $v'_{q+1,j}$) to be considered. In this case we simply have the following
Corollary 10.2. Let $\Theta$, $v_{j,k}$ and $w_{j,k}$ be as defined above (a similar result holds for a complex of odd length). We have

$$\sum_{j=1}^{n} \sum_{k=1}^{b_j} w_{j,k} = \sum_{j=1}^{n} \sum_{k=1}^{a_j} v_{j,k}$$

for $0 \leq i \leq n$.

In cases where $r_q \geq m$, of course, all the above statements are vacuous. When the complex under consideration is a resolution, the strong Buchsbaum-Eisenbud conjecture says that the rank $r_q \geq (m-1)/(2q-1)$. Since this generally exceeds $m-1$, the more intricate relations above are conjectured not to matter for resolutions. For arbitrary complexes, however, the author does not know of a conjecture or result that eliminates them from consideration.

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