LONG TIME EXISTENCE OF SMOOTH SOLUTION FOR THE POROUS MEDIUM EQUATION IN A BOUNDED DOMAIN

SUNGHOON KIM

ABSTRACT. In this paper, we are going to show the long time existence of the smooth solution for the porous medium equations in a smooth bounded domain:

\[ \begin{cases} u_t = \Delta u^m & \text{in } \Omega \times [0, \infty) \\ u(x, 0) = u_0 > 0 & \text{in } \Omega \\ u(x, t) = 0 & \text{for } x \in \partial \Omega \end{cases} \]  

(0.1)

where \( m > 1 \) is the permeability. The proof is based on the short time existence of \( C^{2, \gamma}_s \)-smooth solution, the global \( C^1_s \)-estimate, the Hölder estimate of divergence type degenerate equation with measurable coefficients and \( C^1_s \)-estimate of mixed type equation with Lipschitz coefficients.

1. INTRODUCTION

We consider in this paper the initial value problem for the Porous Medium Equation (PME)

\[ \begin{cases} u_t = \Delta u^m & \text{in } \Omega \\ u(x, 0) = u_0 > 0 & \text{in } \Omega \\ u(x, t) = 0 & \text{for } x \in \partial \Omega \end{cases} \]  

(1.1)

posed in a bounded domain \( \Omega \) with the range of exponents \( m > 1 \), with initial data \( u_0 \) nonnegative, integrable and compactly supported.

In \[6\], Kim and Lee dealt with the short time existence of solution to PME in a bounded domain. More precisely, the main result in their paper is that, if \( u \) is a solution to (1.1) and \( f = u^m \), then, in some regularity conditions on the initial data \( f_0 = u_0^m \) and its first and second derivatives, the solution of the degenerate equation

\[ f_t = m f^{m-1} \Delta f = m f^\alpha \Delta f, \quad \left( f = u^m, \quad \alpha = 1 - \frac{1}{m} \right) \]  

(1.2)

exists on a short time interval \([0, T]\) and \( f \in C^{2, \gamma}_s(\Omega) \) on \([0, T]\), i.e., the solution and its first and second derivative are Hölder continuous with respect to a suitable Riemannian metric \( s \) under which the distance between two points \( x_1 \) and \( x_2 \) in \( \Omega \) is equivalent to the function

\[ \frac{|x_1 - x_2|}{d(x_1)^\alpha + d(x_2)^\alpha + \sum_{i=1}^{n-1} |x_1 - x_2|^\alpha} \]

with \( d = d(x) \) denoting the distance to the boundary of \( \Omega \).

In this work, we will show that, under the same assumptions, the solution \( f \) of (1.2) exists on the long time \([0, \infty)\) and it is also of class \( C^{2, \gamma}_s(\Omega) \), i.e., assuming that the initial value \( f_0 \) is strictly positive in the interior of a domain \( \Omega \subset \mathbb{R}^n \), with \( f_0 = 0 \) on the boundary, and denoting by \( d \) the distance to the boundary of \( \Omega \), we will obtain the following result.

**Theorem 1.1.** If the functions \( f_0, Df_0 \) and \( d^2D^2f_0 \), restricted to the compact domain \( \Omega \), extended continuously up to the boundary of \( \Omega \), with extensions which are Hölder continuous on \( \Omega \) of class \( C^\gamma(\Omega) \), for some \( \gamma > 0 \) and \( Df_0 \neq 0 \).
along $\partial \Omega$, then the initial value problem

$$
\begin{align*}
  f_i = \frac{1}{m} f^\alpha \Delta f, & \quad (x, t) \in \Omega \times (0, \infty), \\
  f(x, 0) = f^0(x), & \quad x \in \Omega \\
  f(x, t) = 0, & \quad x \in \partial \Omega \times [0, \infty)
\end{align*}
$$

(1.3)

admits a solution $f$ which is $C^{2,\gamma}_\Omega$-smooth up to the boundary, when $0 < t < \infty$.

As in the Section 4 in [6], the coordinate change, $(z = f(x', x_n, t) \rightarrow x_n = h(x', z, t))$, converts domain

$$
\Omega \in \mathbb{R}^n \quad \Rightarrow \quad D \in \mathbb{R}^{n-1} \times \mathbb{R}^+
$$

and the equation (1.2) to

$$
(1.4) \quad h_t = z^a \left[ \Delta_x h + \left( -\frac{1 + |\nabla_x h|^2}{h_z} \right) \right]_x, \quad (\alpha = 1 - \frac{1}{m}).
$$

In addition, $h_{z_i}$, $(i = 1, \cdots, n - 1)$ satisfies

$$
(1.5) \quad w_l = z^a \nabla_k \left( a^{kl} \nabla_l w \right), \quad (k, l = 1, \cdots, n)
$$

where

$$
da^{mm} = \frac{1 + |\nabla_x h|^2}{h_z^2}, \quad a^{kn} = -\frac{2h_k}{h_z}, \quad a^{nk} = 0 \quad \text{and} \quad a^{k'l'} = \delta_{kl}, \quad (k', l' = 1, \cdots, n - 1).
$$

Since the solution $f$ of (1.3) is strongly related to the solution $h$ of (1.4), by the Theorem 1.1 in [6], the solution $h$ exists on a short time interval. Let $(0, T)$ be the maximal interval of existence for $C^{2,\gamma}_\Omega$-solution $h$ and suppose that $T < \infty$. Then, solution does not belong to the space $C^{2,\gamma}_\Omega$ at time $t = T$ anymore. However, the existence theory gave $h \in W^{1,2}$. This implies that the coefficients of the equation (1.4) are only measurable and bounded. Thus, if it could be shown that $h \in C^{1,\gamma}_\Omega$, then the coefficients in (1.4) would belong to $C^{1,\gamma}_\Omega$.

Then, Schauder estimate in [6] provides $C^{2,\gamma}_\Omega$ regularity on $h$ at $t = T$ and we can get an extended interval $[0, T')$, $(T < T')$ in which $f \in C^{2,\gamma}_\Omega$. This contradicts the maximality of $T$. Therefore, $T$ must be infinity and the Theorem 1.1 follows. Hence, the missing step for the regularity problem to be solved is

$$
h \in W^{1,2} \Rightarrow h \in C^{1,\gamma}_\Omega, \quad (0 < \gamma < 1).
$$

In this paper, we show the long time existence of solution $f = u^m$ by solving the missing step (1.6).

The paper is divided into five parts: In Part 1 (Section 2) we review the metric $ds$ which controls the diffusion and state theorem for the short time existence of $f = u^m$ in [6]. In Part 2 (Section 3) we deal with global estimates, including Gradient estimate and Non-degeneracy. In part 3 and 4 (Section 4 and 5) we establish the Hölder estimate for the solution of degenerated parabolic equation with divergence type and $C^{1,\gamma}_\Omega$ estimate for the solution of mixed equation with Lipschitz coefficients. Finally, in Section 6, we prove the long time existence of the solution $f$ of (1.2) which is in $C^{2,\gamma}_\Omega(\Omega)$.

2. Preliminaries

The diffusion in (1.4) is governed by the Riemannian metric $ds$ where

$$
\text{ds}^2 = \frac{dx_1^2 + \cdots + dx_n^2}{2x_n^2}.
$$

The distance between two points $x^1 = (x_1^1, \cdots, x_n^1)$ and $x^2 = (x_1^2, \cdots, x_n^2)$ in this metric is a function $s[x^1, x^2]$, which is equivalent to the function

$$
s_\gamma \left[ x^1, x^2 \right] = \frac{\sum_{i=1}^n |x_i^1 - x_i^2|}{|x_1^1|^{\gamma} + |x_2^1|^{\gamma} + \sum_{i=1}^{n-1} |x_i^1 - x_i^2|^{\gamma}}
$$

in the sense that

$$
s \leq Cs \quad \text{and} \quad \frac{s}{\gamma} \leq Cs
$$
for some constant $C > 0$. For the parabolic problem we use the parabolic distance

$$s \left[ (x^1, t_1), (x^2, t_2) \right] = s \left[ x^1, x^2 \right] + \sqrt{|t_1 - t_2|}.$$ 

In terms of this distance, we can define Hölder semi-norm and norm of continuous function $g$ on a compact subset $A$ of the half-space $\{(x_1, \cdots, x_n, t) : x_n \geq 0\}$:

$$\|g\|_{H^s(A)} = \sup_{P_1, P_2 \in A} \frac{g(P_1) - g(P_2)}{|P_1 - P_2|^s},$$

$$\|g\|_{C^s(A)} = \|g\|_{C^s(A)} + \|g\|_{H^s(A)}^s.$$ 

With these norms, the space $C^{2, \gamma}_s(A)$ is the Banach space of all such functions with norm:

$$\|g\|_{C^{2, \gamma}_s(A)} = \|g\|_{C^s(A)} + \sum_{i=1}^n \|g_{x_i}\|_{C^s(A)} + \|g\|_{H^s(A)} + \frac{\|g\|_{C^s(A)}}{\sup_{P \in A} |g(P)|}.$$ 

Imitating the case where the operators are defined on the half-space $\{(x_1, \cdots, x_n, t) : x_n \geq 0\}$ we can define the distance function $s$ in $\Omega$. In the interior of $\Omega$ the distance will be equivalent to the standard Euclidean distance, while around any point $x_0 \in \partial \Omega$, $s$ is defined as the pull back of the distance on the half space $\{(x_1, \cdots, x_n, t) : x_n \geq 0\}$ through a map $\varphi^* : ((x_1, \cdots, x_n, t) : x_n \geq 0) \rightarrow \Omega$ that straightens the boundary of $\Omega$ near $x_0$.

It can be easily shown that the distance between two points $P_1$ and $P_2$ in $\Omega$ is equivalent to the function

$$s(P_1, P_2) = \frac{|P_1 - P_2|}{d(P_1) + d(P_2)^s + \sum_{i=1}^n |P_1 - P_2|^s}$$

with $d = d(P)$ denoting the distance to the boundary of $\Omega$. The parabolic distance in the metric is equivalent to the function

$$s \left[ (x^1, t_1), (x^2, t_2) \right] = s \left[ x^1, x^2 \right] + \sqrt{|t_1 - t_2|}.$$ 

Suppose that $A$ is a subset of $\Omega \times [0, \infty)$. As above, we denote by $C^s_{\gamma}(A)$ the space of Hölder continuous functions on $A$ with respect to the metric $s$ and by $C^{2, \gamma}_s(A)$ the space of all functions $w$ on $A$ such that $w$, $w_t$, $w_i$ and $d^i w_{ij}$, $i, j \in \{1, \cdots, n\}$ and with $d$ denoting the distance function to the boundary of $\Omega$, extend continuously up to the boundary of $A$ and the extensions are Hölder continuous on $A$ of class $C^s_{\gamma}(A)$. Then, they are both Banach spaces under the norm $\|w\|_{C^s_{\gamma}(A)}$ and

$$\|w\|_{C^{2, \gamma}_s(A)} = \|w\|_{C^s(A)} + \sum_{i=1}^n \|w_{x_i}\|_{C^s(A)} + \|w\|_{C^s(A)} + \sum_{1 \leq i \leq m} \|d^i w_{x_i}\|_{C^s(A)}.$$ 

Under this metric, it is known that the problem (1.3) has the $C^{2, \gamma}_s$ solution for a short time interval $(0, T)$. We now state the short time existence for (1.3).

**Theorem 2.1 (Theorem 1.1 in [6]).** Let $d$ be the distance to the boundary $\Omega$. If the functions $f^0$, $D^0$ and $D^2 f^0$, restricted to the compact domain $\Omega$, extended continuously up to the boundary of $\Omega$, with extensions which are Hölder continuous on $\Omega$ of class $C^s_{\gamma}(\Omega)$, for some $\gamma > 0$ and $D f_0 \neq 0$ along $\partial \Omega$, then there exists a number $T > 0$ for which the initial value problem

$$\begin{cases}
  f_t = mf^2 \Delta f, & (x, t) \in \Omega \times (0, \infty), \\
  f(x, 0) = f^0(x), & x \in \Omega \\
  f(x, t) = 0, & x \in \partial \Omega \times [0, \infty)
\end{cases}$$

admits a solution $f$ which is $C^{2, \gamma}_s$-smooth up to the boundary, when $0 < t < T$. 


3. Global $C^1$-estimate

This section is devoted to prove some properties of the solution $f$ to (1.3), including global estimates, gradient estimate on the boundary, etc. To get them, we construct sub and super solutions to the problem (1.3). Now, we first deal with the $L^\infty$ estimate of $f$.

**Lemma 3.1** ($L^\infty$-estimate). Let $d$ be the distance to the boundary of $\partial \Omega$ and let $f$ be the solution of (1.3) with initial data $f^0$ such that

\[ f^0, Df^0, d^a D^2f^0 : \text{ continuous up to the boundary } \partial \Omega. \]

There are constants $c > 0$ and $C \leq \infty$ such that

1. \[ \|f\|_{L^\infty(\Omega)} \leq \min \left\{ \frac{C}{(1 + t)^{1/\alpha}}, \|f_0\|_{L^\infty(\Omega)} \right\}, \quad (\alpha = 1 - \frac{1}{m}) \]

2. There is a ball $B_{\delta_0} \in \Omega$ such that

\[ \inf_{B_{\delta_0}} f(x,t) \geq \frac{c}{(1 + t)^{1/\alpha}}. \]

**Proof.** i) We take a ball $B_R = B_R(0)$ of radius $R$ strictly containing $\Omega$, especially we assume that $\Omega \subset B_{\frac{R}{2}}$, and consider the function $z(x,t)$ defined in $B_R \times (0, \infty)$ by

\[ z(x,t) = A \left( R^2 - |x|^2 \right) \left( 1 + t \right)^{\frac{1}{m}} \]

for suitable constant $A$ to be chosen presently. Since $z$ is positive in $B_R \times (0, \infty)$, we have

\[ f(x,t) = 0 < z(x,t) \quad \text{on } \partial \Omega \times (0, \infty). \]

In addition, if we choose $A$ larger than $\frac{\|f_0\|_{L^\infty(\Omega)}}{R^2}$, we get

\[ f_0(x) \leq z(x,0) \quad \text{in } \Omega. \]

Finally, we will obtain the inequality $me^{\alpha} \Delta z - z_t \leq 0$ in $\Omega$ whenever

\[ A > \left( \frac{R^{2(1-\alpha)}}{2mn\alpha} \right)^{\frac{1}{m}}. \]

With this choice, the comparison principle implies that

\[ f(x,t) \leq z(x,t) = \frac{A(R^2 - |x|^2)}{(1 + t)^{\frac{1}{m}}} \leq \frac{AR^2}{(1 + t)^{\frac{1}{m}}} = \frac{C}{(1 + t)^{\frac{1}{m}}} \quad \text{in } \Omega \times (0, \infty). \]

On the other hand, one can easily check $f(x,t) \leq \|f_0\|_{L^\infty(\Omega)}$ by comparison principle. Hence the proof of (i) is finished.

ii) We first suppose that the initial data $u_0 = f_0^\frac{1}{m}$ satisfies the following condition,

\[ (m-1)\Delta u_0^m \leq -u_0 \quad \forall x \in \Omega. \]

Then, the initial data $u_0$ is controlled from below by the solution $g(x)$ of

\[ \begin{cases} (m-1)\Delta g^m + g = 0 & \text{in } \Omega \\ g = 0 & \text{on } \partial \Omega, \end{cases} \]

i.e.,

\[ u_0(x) \geq g(x). \]

In addition, the function

\[ \frac{g(x)}{(1 + t)^{\frac{1}{m}}} \]
is also a solution of porous medium equation (1.1) with the initial data \( g \). Hence, by the comparison principle, we have

\[ u(x, t) \geq \frac{g(x)}{(1 + t)^{\frac{1}{m}}}. \]

Therefore, for any ball \( B_{\delta_0} \in \Omega \), we have

\[ \inf_{x \in B_{\delta_0}} f(x, t) \geq \inf_{x \in B_{\delta_0}} g(x) \frac{1}{(1 + t)^{\frac{1}{m}}}. \]

Next, we denote by \( \Omega_{u_0} \) the set

\[ \Omega_{u_0} = \{ x \in \Omega : (m - 1)\Delta u_0^m > -u_0 \} \]

and we assume that \( \Omega_{u_0} \neq \emptyset \). Let’s define the function \( \overline{u}_0 \) such that

\[ \overline{u}_0 = u_0 \quad \text{on} \quad \Omega_{u_0} \]

and

\[ (m - 1)\Delta \overline{u}_0^m > -\overline{u}_0 \quad \text{on} \quad \Omega. \]

Then, by comparison principle for elliptic equation, \( \overline{u}_0 \leq u_0 \) in \( \Omega \). We also let \( \overline{u} \) be the solution of (1.1) with the initial data \( u_0 \) being replaced by \( \overline{u}_0 \). Then, by comparison principle for parabolic equation, we have

\[ u \geq \overline{u} \quad \text{in} \quad \Omega. \]

On the other hand, by the Problem 8.1(i) in [9], \( \overline{u} \) satisfies

\[ \overline{u}_t \geq -\overline{u} \frac{(m - 1)(1 + t)}{\overline{u}_0}. \]

Thus, by the Gronwell’s inequality, we have

\[ \overline{u}(x, t) \geq \frac{\overline{u}_0(x)}{(1 + t)^{\frac{m}{m-1}}}. \]

Since \( \Omega_{u_0} \) is open set, there exists a ball \( B_{\delta_0} \in \Omega_{u_0} \). Hence

\[ u(x, t) \geq \overline{u}(x, t) \geq \frac{\overline{u}_0(x)}{(1 + t)^{\frac{m}{m-1}}} = \frac{u_0(x)}{(1 + t)^{\frac{m}{m-1}}} \quad \text{on} \quad B_{\delta_0}. \]

Therefore

\[ \inf_{x \in B_{\delta_0}} f(x, t) = \inf_{x \in B_{\delta_0}} u^m(x, t) \geq \frac{\inf_{x \in B_{\delta_0}} u_0^m(x)}{(1 + t)^{\frac{m}{m-1}}} = \frac{\inf_{x \in B_{\delta_0}} f_0(x)}{(1 + t)^{\frac{m}{m-1}}}, \]

which implies the conclusion. \( \square \)

Next property is the gradient estimate which will play an important role to show \( C^{1,\gamma}_{\alpha} \) continuity of solution.

**Lemma 3.2** (Gradient estimate). Let \( d \) be the distance to the boundary of \( \partial \Omega \) and let \( f \) be the solution of (1.3) with initial data \( f^0 \) such that

\[ f^0, \ Df^0, \ d^\alpha D^2 f^0 : \quad \text{continuous up to the boundary} \ \partial \Omega. \]

There are uniform constant \( 0 < C_\alpha < \infty \) such that

\[ |||f|||_{L^\infty(\Omega)} < C_\alpha \frac{|||f^0|||_{C^{1,\gamma}(\Omega)}}{(1 + t)^{1/\alpha}} \]

\[ (3.1) \]
Then, by the comparison principle, we get
\[ |\nabla f(x, t)| < C_1 \quad \forall x \in \Omega, \quad 0 \leq t < t_0. \]  

Thus, (3.1) holds for a short time.

Let \( \psi \) be the solution of
\[
\begin{cases}
\Delta \psi + \frac{1}{m\tau} \psi^{\frac{5}{4}} = 0 & \text{in } \Omega \\
\psi(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since \( 0 < |\nabla \psi| < \infty \) on \( \partial \Omega \), we can select constants \( 0 < c_1 < 1 < c_2 < \infty \) such that
\[
c_1 \psi(x) \leq f_0(x) \leq c_2 \psi(x), \quad \text{in } \Omega.
\]

Then, by the comparison principle, we get
\[
\frac{c_1 \psi(x)}{(1 + t)\frac{5}{4}} \leq f(x, t) \leq \frac{c_2 \psi(x)}{(1 + t)\frac{5}{4}}, \quad \forall x \in \Omega.
\]

Since \( \|\psi(x)\|_{C^1(\partial \Omega)} < \infty \), we also have
\[
|\nabla f(x, t)| \leq \frac{c_2 \|\psi(x)\|_{C^1(\partial \Omega)}}{(1 + t)\frac{5}{4}} \leq \frac{c_2 \|f_0(x)\|_{C^1(\Omega)}}{c_1(1 + t)\frac{5}{4}} < \infty, \quad \forall 0 \leq t < \infty, x \in \partial \Omega.
\]

Denoting by \( \Omega_{\alpha} \), for \( \alpha > 0 \), the set
\[ \Omega_{\alpha} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \alpha \}. \]

Then, by (3.3), there exist constants \( c_0 > 0 \) and \( c_3 > 0 \) such that
\[
|\nabla f(x, t)| \leq \frac{c_3 \|f_0(x)\|_{C^1(\Omega)}}{(1 + t)\frac{5}{4}} < \infty, \quad \forall 0 \leq t < \infty, x \in \Omega \setminus \Omega_{\alpha}.
\]

To finish the proof, let us define the family of rescaled functions
\[
f_k(x, t) = k^\frac{5}{4} f(x, kt)
\]
in \( \Omega_{c_0} \), with parameter \( k > 0 \). Then, they are again solutions of
\[
f_t = m f^{\alpha} \Delta f.
\]

Since \( f > 0 \) in \( \Omega_{c_0} \), the coefficient \( m f^{\alpha} \) is bounded above and below. Hence the equation (3.5) becomes uniformly parabolic and the solutions are smooth with derivatives locally bounded in terms of the bounds for \( f \). We conclude that there exist a constant \( c_4 > 0 \) such that
\[
|\nabla f_k(x, 1)| < c_4 \quad x \in \Omega_{c_0}.
\]

This means that
\[
|\nabla f(x, k)| < \frac{c_4}{k^{\frac{5}{4}}} \quad x \in \Omega_{c_0}.
\]

For some constant \( c_5 > 0 \), putting \( t = k \) and \( c_4 = c_5 \|f_0\|_{C^1(\Omega)} \) in inequality (3.6). Then
\[
|\nabla f(x, t)| \leq \frac{c_5 \|f_0\|_{C^1(\Omega)}}{t^{\frac{5}{4}}} \quad x \in \Omega_{c_0},
\]

Hence, we get
\[
|\nabla f(x, t)| \leq \frac{c_6 \|f_0\|_{C^1(\Omega)}}{(1 + t)^{\frac{5}{4}}}, \quad \forall t \geq t_0 \quad x \in \Omega_{c_0}
\]

where \( c_6 = c_5 \left(1 + \frac{1}{n} \right)^{\frac{5}{4}} \). By (3.2), (3.4) and (3.3), (3.1) holds for all \( t > 0 \).

Finally, in this section, we will show the non degeneracy of solution \( f \) to (1.3) near the boundary. This estimate guarantees the uniformly ellipticity of coefficients \( a^{ij} \) in (1.5).
Lemma 3.3 (Non-degeneracy estimate). Let $d$ be the distance to the boundary of $\partial \Omega$ and let $f$ be the solution of (1.3) with initial data $f^0$ such that
\[ f^0, \; Df^0, \; d^2D^2f^0 : \text{ continuous up to the boundary } \partial \Omega. \]
There are uniform constant $0 < c_1 < \infty$ such that
\[ \frac{c_0}{(1 + t)^{1+\frac{\alpha}{2}}} < \| \nabla f \|_{L^\infty(\partial \Omega)} \]
Proof. By the Lemma 3.1(ii), there exists a ball $B_{b_0} \subset \Omega$ such that
\[ \inf_{x \in B_{b_0}} f(x,t) \geq \frac{c_0}{(1 + t)^{1+\frac{\alpha}{2}}} . \]
For fixed $t > 0$, let $v$ be the solution of the problem
\[
\begin{aligned}
\begin{cases}
\Delta v(x,t) = 0 & \text{in } \Omega \backslash B_{b_0} \\
v(x,t) = c_0 & \text{on } \partial B_{b_0} \\
v(x,t) = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]
Then, the function $V(x,t) = \frac{v(x,t)}{(1 + t)^{\frac{\alpha}{2}}}$ satisfies
\[ V^\alpha \Delta V - V_1 = \frac{v}{\alpha(1 + t)^{1+\frac{\alpha}{2}}} \geq 0. \]
In addition, we have
\[ V(x,t) = \frac{v(x,t)}{(1 + t)^{\frac{\alpha}{2}}} = \frac{c_0}{(1 + t)^{\frac{\alpha}{2}}} \leq f(x,t) \quad \text{on } \partial B_{b_0} \]
and
\[ V(x,t) = \frac{v(x,t)}{(1 + t)^{\frac{\alpha}{2}}} = 0 = f(x,t) \quad \text{on } \partial \Omega. \]
By the comparison principle, we have
\[ f(x,t) \geq V(x,t) \quad \text{in } \Omega \backslash B_{b_0} . \]
By the Hopf’s inequality for the harmonic equation, there exists some constant $c_1 > 0$ such that
\[ \frac{\partial v}{\partial \nu} \leq -c_1 < 0 \quad \text{on } \partial \Omega \]
for the outer normal direction $\nu$ to $\partial \Omega$. Therefore,
\[ \| \nabla f \|_{L^\infty(\partial \Omega)} \geq \| \nabla V \|_{L^\infty(\partial \Omega)} = \frac{\| \nabla v \|_{L^\infty(\partial \Omega)}}{(1 + t)^{\frac{\alpha}{2}}} \geq \frac{c_1}{(1 + t)^{\frac{\alpha}{2}}} \]
and lemma follows. \qed

4. Hölder estimate I

In the previous section, we obtained the global $C^1$ regularity of solution $f$ to the problem (1.3). By the relation between $f$ and $h$, we can say the same story on the solution $h$ of (1.4). Hence, we can have basic informations for coefficients $a^{ij}$ and solution $w$ in (1.5), including boundedness of solution and coefficients. With this basic properties, we devote this section and next one for solving the missing step, (1.6), for the regularity theory.

Let $H$ be the half space $\{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0 \}$. We are going to show Hölder estimate on a solution $w$ of the equation
\[ w_t = x_n^a \nabla_i \left( a^{ij} \nabla_j w \right) + x_n^ag \quad \text{in } H \]
for a forcing term $g$. Assume that the coefficients $a^{ij}(x,t)$ are measurable functions and satisfy
\[ \lambda |\xi|^2 \leq a^{ij}(x,t) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad (i, j = 1, \cdots, n). \]

\[ a_n \]
In addition, we suppose that the forcing term \( g \) satisfies

\[
(4.3) \quad |g(x)| \leq C|w(x)|
\]

for some constant \( C < \infty \).

For the H"older estimates of the solution \( w \) to \((1.1)\), we need the following two inequalities. The first one is a weighted version of Sobolev’s inequality. Let \( C_0^\infty(H) \) be the space of restriction of functions in \( C_0^\infty(\mathbb{R}^n) \) to \( H \).

**Lemma 4.1** (See Theorem 4.2.2 in [8]). Let \( 1 \leq p < q < \infty \) and we assume that \( \sigma \leq 1 \) satisfies

\[
\frac{1}{q} - \frac{1}{n} = \frac{1}{p}.
\]

Then

\[
\left( \int_{H} x_n^{q\sigma} |u^s| dx \right)^{\frac{1}{q}} \leq c \left( \int_{H} x_n^{(q+\sigma)\sigma} |\nabla u|^q dx \right)^{\frac{1}{q}}
\]

for the closure of \( C_0^\infty(H) \).

The next inequality is the parabolic version of Sobolev’s inequality with weight:

**Lemma 4.2.** Let \( Q = B \times (a, b) \) is cylinder in \( H \times (0,1) \) and \( f \in C_0^\infty(Q) \). Then there are constant \( C > 0 \) such that if \( l_1 = \frac{n+2-2\alpha}{n-\alpha} \) we have

\[
\left( \int_{a}^{b} \int_{B} x_n^{-a} |f|^{2l_1} dx dt \right)^{\frac{1}{l_1}} \leq C \left[ \sup_{(a,b)} \int_{B} x_n^{-a} |f|^2 dx + \int_{a}^{b} \int_{B} |\nabla f|^2 dx dt \right].
\]

**Proof.** By the H"older inequality,

\[
\left( \int_{B} x_n^{-a} |f|^{2l_1} dx \right)^{\frac{1}{l_1}} \leq C_1 \left( \int_{B} x_n^{-a} |f|^2 dx \right)^{\frac{1}{2l_1}} \left( \int_{B} x_n^{-a} |f|^{2l_1} dx \right)^{\frac{1}{2}}.
\]

for some constant \( C_1 > 0 \). By the Lemma 4.1 it follows that

\[
\left( \int_{B} x_n^{-a} |f|^{2l_1} dx \right)^{\frac{1}{l_1}} \leq C_2 \left( \int_{B} x_n^{-a} |f|^2 dx \right)^{1-\frac{1}{l_1}} \left( \int_{B} |\nabla f|^2 dx \right)^{\frac{1}{l_1}}.
\]

for some constant \( C_2 > 0 \). Now, taking the \( l_1 \) power and integrating in \( (a,b) \), we get

\[
\left( \int_{a}^{b} \int_{B} x_n^{-a} |f|^{2l_1} dx dt \right)^{\frac{1}{l_1}} \leq C_3 \left[ \sup_{(a,b)} \int_{B} x_n^{-a} |f|^2 dx + \int_{a}^{b} \int_{B} |\nabla f|^2 dx dt \right]
\]

for some constant \( C_3 > 0 \) and the lemma follows. \( \square \)

Define the balls \( B_r \) and \( B^+_r \) of radius \( r \) around \( x = x_0 \) to be the sets

\[
B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \quad \text{and} \quad B^+_r(x_0) = B_r(x_0) \cap \{x_n > 0\}.
\]

We let \( B_r \) be the ball around the point \( x = 0 \). We define the round cubes \( Q_r \) of radius \( r \) around \( (x,t) = (0,1) \) to be the sets

\[
Q_r = B_r \times (1 - r^{2-\alpha}, 1).
\]

We also define the general round cubes:

\[
Q_r(x,t) = Q_r + (x,t) \quad \text{and} \quad Q^+_r(x,t) = Q_r(x,t) \cap \{x_n > 0\}.
\]

Let us give the Harnack inequality first.
**Lemma 4.3** (Harnack’s Inequality). Let $w$ be a solution of equation (4.1) defined in $B^+ \times (0,1)$ with conditions (4.2) and (4.3). Let $|w| \leq M$ on $Q^+_1$ and $\hat{h} = w + M + 1 \geq 1$. Then,

$$
\max_{Q^+_1(0,—\hat{t})} \hat{h} \leq C \min_{Q^+_1(0,—\hat{t})} \hat{h}
$$

**Proof.** Let $Q^+ = B^+ \times (s_2,s_1) \in H \times [0,\infty)$ be an round cube in $Q^+_1$ and we take $\phi(x,t) = \eta^2(x,t)\hat{h}(x,t)$ as test function, where $\eta \in C^\infty(Q^+)$ with $\eta = 0$ on $\partial_p Q^+ \cap \{x_n > 0\}$. Since $w$ is solution of (4.1) in $Q^+_1$, $\hat{h}$ satisfies

$$
\frac{\partial \hat{h}}{\partial t} = \nabla \cdot (a^{ij} \nabla \hat{h}) + g \quad \text{in } H.
$$

Hence, multiplying (4.4) by the test function $\phi(x,t)$, we arrive at

$$
\frac{4\Lambda |\beta|}{|1+\delta|^2} \int_{Q^+} |\nabla \phi|^2 \, dx \, dt + \frac{1}{|1+\delta|^2} \left( \sup_{|\xi| \leq \delta, \xi \in B} \int_{Q^+} x_n^{-\alpha} \eta^2 \hat{h}^{1+\delta} \, dx \, dt \right)
\leq \frac{8\Lambda (1+3|\delta|)}{|1+\delta|^2} \int_{Q^+} \hat{h}^{1+\delta} |\nabla \eta|^2 \, dx \, dt + \frac{8\Lambda (1+2|\beta|)}{|1+\delta|^2} \int_{Q^+} \hat{h}^{1+\delta} |\nabla \eta|^2 \, dx \, dt
\leq 2 \int_{Q^+} \eta^2 \hat{h}^{1+\delta} |\xi|^2 \, dx \, dt + \frac{4}{|1+\delta|^2} \int_{Q^+} x_n^{-\alpha} \hat{h}^{1+\delta} \eta |\xi| \, dx \, dt
$$

for $\delta > 0$ or $\delta < -1$. Use of Young’s inequality yields

$$
\sup_{|\xi| \leq \delta, \xi \in B} \left( \int_{Q^+} x_n^{-\alpha} \eta^2 \hat{h}^{1+\delta} \, dx \, dt \right) \leq C_1 \left( \int_{Q^+} (|\nabla \eta|^2 + x_n^{-\alpha} |\eta| l_1) \left( \hat{h}^{1+\delta} \right)^2 \, dx \, dt + \int_{Q^+} \eta^2 \hat{h}^{1+\delta} |\xi|^2 \, dx \, dt \right)
$$

for some constant $C_1 > 0$.

We first consider the case $\delta > 0$. Let $r_1$ and $r_2$ be such that $\frac{3}{8} \leq r_2 \leq r_1 \leq \left( \frac{1}{2} \right)^{\frac{1}{\alpha}}$, choosing $\eta$ in such a way that $\eta(x,t) = 1$ in $Q^+_n \left( 0, -\frac{3}{8} \right)$, $\eta(x,t) = 0$ on $\partial_p Q^+_n \left( 0, -\frac{3}{8} \right) \cap H$, $0 \leq \eta \leq 1$ in $Q^+_n \left( 0, -\frac{3}{8} \right)$, $|\nabla \eta| \leq \frac{C_2}{r_1-r_2}$ and $|\eta| \leq \frac{C_2}{(r_1-r_2)^{\alpha-1}}$ for some constant $C_2 > 0$. Then, for $Q^+ = Q^+_n \left( \frac{3}{8}, \frac{5}{8} \right) = B^+ \left( \frac{3}{8}, \frac{5}{8} \right) \times \left( \frac{1}{1, \frac{5}{8}} \right)$,

$$
\int_{Q^+} (|\nabla \eta|^2 + x_n^{-\alpha} |\eta| l_1) \left( \hat{h}^{1+\delta} \right)^2 \, dx \, dt \leq \frac{C_3}{|r_1-r_2|^2} \int_{Q^+_n \left( 0, -\frac{3}{8} \right)} x_n^{-\alpha} \left( \hat{h}^{1+\delta} \right)^2 \, dx \, dt
$$

for some constant $C_3 > 0$. On the other hand, by (4.3), the second term of the right hand side in (4.6) is changed to

$$
\int_{Q^+} \eta^2 \hat{h}^{1+\delta} |\xi| \, dx \, dt \leq C_4 \int_{Q^+} \eta^2 \hat{h}^{1+\delta} \, dx \, dt \leq C_4 \int_{Q^+_n \left( 0, -\frac{3}{8} \right)} \hat{h}^{1+\delta} \, dx \, dt
$$

for some constants $C_4 > 0$. Applying (4.6), (4.7) to (4.5), we can get

$$
\sup_{\frac{3}{8} \leq \xi \leq \frac{5}{8}} \int_{B^+} x_n^{-\alpha} \eta^2 \hat{h}^{1+\delta} \, dx \, dt + \int_{Q^+} |\nabla \left( \eta \hat{h}^{1+\delta} \right)|^2 \, dx \, dt
\leq \frac{C_5}{|r_1-r_2|^2} \int_{Q^+_n \left( 0, -\frac{3}{8} \right)} x_n^{-\alpha} \left( \hat{h}^{1+\delta} \right)^2 \, dx \, dt
$$

for some constants $C_5 > 0$. Applying (4.6), (4.7) to (4.5), we can get
for some constant $C_5 > 0$. By the Lemma 3.2, there exists $l_1 > 1$ such that

$$
\left( \int_{Q^*} x_n^{-a} \left( \frac{\hat{h}}{x_n^{\alpha}} \right)^{2l_1} dx dt \right)^{\frac{1}{l_1}} 
\leq C_6 \left[ \sup_{\frac{1}{r_2}\leq \frac{1}{r_1}} \int_{B^{*}} x_n^{-a} \hat{h}^{1+a} (, t) \ dx + \int_{Q^*} \left| \nabla \left( \frac{\hat{h}}{x_n^{\alpha}} \right) \right|^2 dx dt \right]
$$

(4.9)

for some constant $C_6 > 0$. Then, by (4.8) and (4.9),

$$
\left( \int_{Q^*_1(0, -\frac{1}{2})} x_n^{-a} \left( \frac{\hat{h}}{x_n^{\alpha}} \right)^{2l_1} dx dt \right)^{\frac{1}{l_1}} \leq \frac{C_7}{|r_1 - r_2|^{2 - a}} \int_{Q^*_1(0, -\frac{1}{2})} x_n^{-a} \left( \frac{\hat{h}}{x_n^{\alpha}} \right)^2 dx dt
$$

for some constant $C_7 > 0$. Taking the $\frac{1}{1 + \delta}$th root on each side of (4.10), we have

$$\|\hat{h}\|_{L^{1+\delta}(Q^*_1(0, -\frac{1}{2}), x_n^a)} \leq \frac{C_8}{|r_1 - r_2|^{1 + \delta}} \|\hat{h}\|_{L^{1+\delta}(Q^*_1(0, -\frac{1}{2}), x_n^a)}$$

for some constant $C_8 > 0$. Define $\gamma_1 = p_0 > 0, \gamma_i = l_1 \gamma_{i-1}$ and $r_i = \frac{1}{8} + \left( \frac{1}{8} \right)^{1 + \delta} - \frac{1}{8}$. Then, the previous inequality becomes

$$\|\hat{h}\|_{L^{1+\delta}(Q^*_i(0, -\frac{1}{2}), x_n^a)} \leq C_8 \left( \frac{8}{9} \right) \|\hat{h}\|_{L^{1+\delta}(Q^*_i(0, -\frac{1}{2}), x_n^a)}$$

Therefore, iteration yields

$$\max_{Q^*_i(0, -\frac{1}{2})} \hat{h} \leq C_9 \|\hat{h}\|_{L^{1+\delta}(B^{*} \times (\frac{1}{8}, 1), x_n^a)} \quad (C_9 > 0).$$

For the other case $\delta < -1$, we also take $\phi(x, t) = \tau_2(x, t) \hat{h}^{1+\alpha}(x, t)$ as test function, choosing $\eta$ in such a way that $\eta(x, t) = 1$ in $Q^*_t \cap H, 0 \leq \eta \leq 1$ in $Q^*_t, |\nabla \eta| \leq \frac{C_{10}}{|r_1 - r_2|^{\frac{1}{1 + \delta}}}$ and $|\eta| \leq \frac{C_{11}}{|r_1 - r_2|^{\frac{1}{1 + \delta}}}$ for some constant $C_{10} > 0$ and $\frac{1}{8} \leq r_2 \leq r_1 \leq \left( \frac{8}{9} \right)^{\frac{1}{1 + \delta}}$. Then, by the similar computation above, we have

$$\|\hat{h}\|_{L^{1+\delta}(Q^*_2, x_n^a)} \geq \frac{C_{11}}{|r_1 - r_2|^{1 + \delta}} \|\hat{h}\|_{L^{1+\delta}(Q^*_1, x_n^a)}$$

for some constant $C_{11} > 0$. Define $\gamma_0 = -p_0 > 0, \gamma_i = l_1 \gamma_{i-1}$ and $r_i = \frac{1}{8} + \left( \frac{1}{8} \right)^{\frac{1}{1 + \delta}} - \frac{1}{8}$. Then, the previous inequality becomes

$$\|\hat{h}\|_{L^{1+\delta}(Q^*_i, x_n^a)} \geq C_{11} \left( \frac{8}{9} \right) \|\hat{h}\|_{L^{1+\delta}(Q^*_i, x_n^a)}.$$ 

Therefore, iteration yields

$$\min_{Q^*_i} \hat{h} \geq C_{12} \|\hat{h}\|_{L^{1+\delta}(B^{*} \times (\frac{1}{8}, t), x_n^a)}, \quad (C_{12} > 0).$$

(4.12)

To get the harnack’s inequality for $\hat{h}$, we finally examine the case $\delta = -1$. Let $0 < \rho \leq \left( \frac{1}{8} \right)^{\frac{1}{1 + \delta}}$. We take $\phi(x, t) = \tau_2(x, t) \hat{h}^{-1}(x, t)$ as test function, where $\eta \in C^\infty_0 (B^{*}_{1}, [0, 1]), \eta(x, t) = 1$ in $Q^*_t, \text{supp}(\eta) \subset Q^*_{\frac{1}{2} - \frac{1}{1 + \delta}} \subset Q^*_t$ and $|\nabla \eta| \leq \frac{C_{13} \rho}{|r_1 - r_2|^{\frac{1}{1 + \delta}}}$ and $|\eta| \leq \frac{C_{14} \rho}{|r_1 - r_2|^{\frac{1}{1 + \delta}}}$ for some constant $C_{13} > 0$. Then, since $w$ satisfies (4.11) in $Q^*_t$, we can get

$$\left( \int_{Q^*_t} \left| \nabla \log \hat{h}^2 \right| dx dt \right)^{\frac{1}{2}} \leq C_{14}$$

(4.13)
for some constant $C_{14} > 0$. Let $U = \log \hat{h}$ and $p = \frac{2(n-1)}{n-2}$. Then, by the Lemma 4.1 there exists constant $C_{15}, C_{16} > 0$ such that

$$\frac{1}{|x_n(B^+_p)|} \int_{Q^+_p} x_n^{-\alpha} |U - U_{Q^+_p}| \, dx \, dt \leq \left( \int_{B^+_p} x_n^{-\alpha} |U - U_{Q^+_p}|^p \, dx \right)^{\frac{1}{p}} \int_{1 - \rho^{2-a}}^1 \left( \int_{B^+_p} |\nabla U|^2 \, dx \right)^{\frac{1}{2}} \, dt$$

(4.13)

$$\leq C_{15} \rho^{-2} \int_{1 - \rho^{2-a}}^1 \left( \int_{B^+_p} |\nabla U|^2 \, dx \right)^{\frac{1}{2}} \, dt$$

$$\leq C_{16} \rho^{-2} + \rho^{-\frac{3}{x}} = C_{16} \rho^{-2} \leq C_{16}.$$

The parabolic version of John and Nirenberg Lemma for BMO (see [14]) yields that there exist two positive constants $P_0$ and $C_{17}$ such that

$$\left[ \frac{1}{x_n(B^+_p)} \left( \frac{h}{r} \right)^{\frac{1}{\alpha}} \int_{B^+_p} x_n^{-\alpha} \, dx \right]^\frac{1}{\alpha} \int_{1 - \rho^{2-a}}^1 \int_{B^+_p} x_n^{-\alpha} e^{\rho U} \, dx \, dt$$

(4.14)

$$\leq C_{17}.$$

Then combining this with (4.11) and (4.12), we get

$$\max_{Q^+_p(\frac{b-\rho}{2})} h \leq C_{18} \min_{Q^+_p} \hat{h}$$

for some constant $C_{18} > 0$ and lemma follows.

We can now state the first main result of our paper.

**Theorem 4.4.** Let $w$ be a solution of equation (4.1) defined in $B^+_1 \times (0, 1)$ with conditions (4.2) and (4.3). Suppose that

$$\max_{Q^+_1} |w| \leq C_{|w|} < \infty.$$

Then, $w$ is locally Hölder continuous in $B^+_1 \times (0, 1)$ and

$$\|w\|_{C^s(Q^+_1)} \leq C \left( \|w\|_{L^\infty(Q^+_1)} + 1 \right), \quad (s < 1).$$

**Proof.** For $\bar{h} = w + C_{|w|}$, let $m_r = \inf_{Q^+_r} \bar{h}$, $M_r = \sup_{Q^+_r} \bar{h}$. Then $\bar{h} - m_r + 1, M_r - \bar{h} + 1 \geq 1$ and satisfy the equation (4.4) in $Q_r$. Applying the Harnack inequality (Lemma 4.3) to those equations, we get

$$M_r - M^*_r + 1 = \inf_{Q^+_r} (M_r - \bar{h} + 1) \geq \frac{1}{C} \sup_{Q^+_r} (M_r - \bar{h} + 1) = \frac{1}{C} (M_r - m^*_r + 1)$$

and

$$m^*_r - m_r + 1 = \inf_{Q^+_r} (\bar{h} - m_r + 1) \geq \frac{1}{C} \sup_{Q^+_r} (\bar{h} - m_r + 1) = \frac{1}{C} (M^*_r - m_r + 1).$$

Hence,

$$M^*_r - m^*_r \leq \left( \frac{C - 1}{C + 1} \right) (M_r - m_r) + \frac{2(C - 1)}{C + 1}.$$
Let $\text{osc}(r) = M_r - m_r$. Then

$$\text{osc}\left(\frac{r}{\delta}\right) \leq \left(\frac{C - 1}{C + 1}\right) \text{osc}(r) + \frac{2(C - 1)}{C + 1}.$$ 

By an elementary iteration, we get

$$\text{osc}(r) \leq C'r' \left[\text{osc}(1) + 1\right]$$

for some constant $C' > 0$ depending on $C$, i.e., $\overline{h}$ is Hölder at $(x,t) = (0,1)$. Since $\overline{h} = w + C|w|$, we also have Hölder continuity of $w$ at $(x,t) = (0,1)$. Finally, by translating and dilating, we have

$$\|w\|_{C^s(Q_t)} \leq C'' \left(\|w\|_{C^s(Q_{t_0})} + 1\right), \quad (s < 1)$$

and lemma follows.

\[ \square \]

5. Hölder estimates II

Through the previous section, we have established that the gradient of $h$ in (1.4) with respect to $x_i$ ($i = 1, \cdots, n - 1$), is Hölder continuous near the boundary $\partial \Omega$. However, one can easily check that $h_z$ is not a solution of (1.5). Hence, solving the missing step (1.6) is not completed at this moment. Thus, we will devote this section for the Hölder regularity for $h_z$.

Let $w$ be a solution of the problem

$$\begin{cases} 
  w_t = x_n^a \left(a^{ij} w_{ij}\right) & \text{in } H \\
  w(x, t) = 0 & \text{on } x_n = 0 \\
  w(x, 0) = w_0(x) & \text{in } H
\end{cases}$$

with initial value $w_0(x) \in C^{0,1}(H)$. Assume that the coefficients $a^{ij}(x,t)$ are measurable functions and satisfy

$$\lambda |\xi|^2 \leq a^{ij}(x,t)\xi_i\xi_j \leq \Lambda |\xi|^2, \quad (i, j = 1, \cdots, n)$$

for some constants $0 < \lambda \leq \Lambda < \infty$.

To get the Hölder regularity for $h_z$, we will first construct an important, for our purpose, barrier function.

**Lemma 5.1.** Let $Q = B^+ \times (0, 1)$ and $P_i(x') = (x', r) \in H$ for $0 < r < 1$ and let $K = \{x \in B^+_r : |x_i - \left(P_i(0)\right)| < \frac{\rho}{r}, \; i = 1, \cdots, n\}$. Then, there exists a solution $f > 0$ and time $T(r) > 0$ such that

$$\begin{cases} 
  x_n^a \left(a^{ij} f_{ij}\right) - f_t = 0 & \text{in } Q \\
  f = 0 & \text{on } \partial Q \\
  |\nabla f| \geq c(T) > 0 & \text{in } B^+_r \cap \{x_n = 0\}, \; t \geq T
\end{cases}$$

Moreover, at $t = 0$, $f$ satisfies

$$\text{supp}(f(\cdot, 0)) \subset K \quad \text{and} \quad 0 \leq f(\cdot, 0) \leq m_0 1_k < \infty$$

for a constant $m_0 > 0$.

**Proof.** In this proof, we will use a modification of the technique of [7] to prove the lemma. For each $x' \in \mathbb{R}^{n-1}$ such that $|x'| < \frac{\rho}{10}$, let $P_i = P_i(x')$ and consider

$$g(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-P_i|^2}{4t}}$$

and $\tilde{g}(x, t) = e^{-Mi}g(x, t + \tau_0)$ for $\tau_0 > 0$ we can choose later. By a direct computation, we can get

$$\tilde{g}_t(x, t) = e^{-Mi}g(x, t + \tau_0) \left[-M - \frac{n}{2(t + \tau_0)} + \frac{\beta |x-P_i|^2}{(t + \tau_0)^2}\right]$$

and

$$\tilde{g}_{ij} = e^{-Mi}g(x, t + \tau_0) \left[\frac{4\beta^2 (x_i - P_{i\tau_0})(x_j - P_{j\tau_0})}{(t + \tau_0)^2}\right], \quad (i \neq j \text{ and } 1 \leq i, j \leq n)$$
Thus, the condition (5.6) holds for subsolution of (5.1). We are going to choose proper constants \( t \geq t_0 \) and sufficiently small constant \( \beta \) such that \( \tau_0 > 0 \). Since \( \tilde{g} \) takes maximum value of two subsolutions to (5.1) at each point in \( H \), it is also a subsolution of (5.1). We are going to choose proper constants \( \beta, \tau_0 \) and \( \epsilon_0 \) so that (5.6) holds for sufficiently large constant \( \tau_0 \) and \( \beta \).

Now, we define the function \( \tilde{g}_{\epsilon_0} \) by

\[
\tilde{g}_{\epsilon_0} = \max\{0, \tilde{g} - \epsilon_0\}
\]

for small \( \epsilon_0 > 0 \). Since \( \tilde{g}_{\epsilon_0} \) takes maximum value of two subsolutions to (5.1) at each point in \( H \), it is also a subsolution of (5.1). We are going to choose proper constants \( \beta, \tau_0 \) and \( \epsilon_0 \) so that (5.6) holds for sufficiently large constant \( \tau_0 \) and \( \beta \).

For (5.4), we take sufficiently small constant \( \tau_0 \) and large one \( \beta >> 1 \) so that

\[
\frac{1}{4\pi \tau_0^2} e^{-\frac{4r^2}{\tau_0^2}} < \epsilon_0 \quad \text{and} \quad \epsilon_0 < \frac{1}{4\pi \tau_0^2} < m_0 + \epsilon_0.
\]

Then,

\[
\tilde{g}(\cdot, 0) < \epsilon_0 \quad \text{on} \quad B_r^+ \setminus K \quad \text{and} \quad \epsilon_0 < \tilde{g}(P_r, 0) \leq m_0 + \epsilon_0.
\]

Thus, the condition (5.6) holds for sufficiently large constant \( \tau_0 \) and \( \beta \).

For (5.7), we will focus on \( \text{supp}\{\tilde{g}_{\epsilon_0}(\cdot, t)\} \). Since \( \tilde{g}_{\epsilon_0}(x, t) \) is radially symmetric about the point \( P_r \) for each \( t \geq 0 \), \( \epsilon_0 \)-level set of \( \tilde{g}_{\epsilon_0}(\cdot, t) \) is a ball centered at \( P_r \). Let’s denote by \( R(t) \) the radius of the \( \epsilon_0 \)-level set of \( \tilde{g}_{\epsilon_0}(\cdot, t) \).

Then, one can easily check that there exists time \( t = t' \) such that \( R(t) \) is increasing on \( (0, t') \) and decreasing on \( (t', \infty) \). Hence, at time \( t' > 0 \), \( R(t) \) satisfies

\[
\frac{d[R(t)]}{dt} = 0 \quad \text{at} \quad t = t'.
\]

To show that there exists a time \( T > 0 \) such that

\[
\text{supp}\{\tilde{g}_{\epsilon_0}(\cdot, T)\} = B_r^+(P_r),
\]

we will show that \( R(t') \geq \frac{1}{2} \) with the following equation

\[
e^{-\frac{4r^2}{\epsilon_0 \tau_0}} = \epsilon_0 [4\pi (t + \tau_0)]^\frac{n}{2}.
\]

Differentiating on both sides of (5.9) with respect to \( t \), we can get

\[
R(t')^2 = \frac{(t' + \tau_0)^2}{\beta} \left[ M + \frac{\epsilon_0 n}{2(t' + \tau_0)} \right] \geq \frac{\tau_0^2 M}{\beta} \quad \text{at} \quad t' \geq t'.
\]

By (5.5), the last term of (5.10) is bounded below by \( 4r^2 \). Hence, we can get

\[
R(t') \geq 2r.
\]

This immediately implies that there exists a time \( 0 < T \leq t' \) such that

\[
R(T) = \frac{r}{4}.
\]
Then, the second term on the right hand side of (5.11) is positive. Hence, since (5.7) is true, one can easily check that there exists a constant \( c = c(T) \) such that
\[
\left| \nabla \tilde{g}_{c_0}(P_0, T) \right| \geq c(T) > 0.
\]

Next, we define function \( G \) by
\[
G(x, t) = \epsilon_1 \left( 1 - \frac{4|x - P_+|^2}{r^2} \right) \tilde{g}(x, t)
\]
\[
= \frac{\epsilon_1}{(4\pi(t + \tau_0))^2} \left( 1 - \frac{4|x - P_+|^2}{r^2} \right) e^{-Mrt} \frac{[|P_x|^2]}{r_{0}} \quad \text{in} \quad B_\frac{1}{4}(P_+) \times [T, 1).
\]
for some constant \( \epsilon_1 > 0 \). Note that if we choose \( \epsilon_1 \) sufficiently small, then
\[
G(x, T) \leq \tilde{g}_c(x, T), \quad \text{in} \quad B_\frac{1}{4}(P_+)
\]
since, for some constants \( c_1 \) and \( c_2 \),
\[
|\nabla \tilde{g}_c(x, T)| \geq c_1 > 0 \quad \text{and} \quad |\nabla G| \leq c_2 < \infty \quad \text{on} \quad \partial B_\frac{1}{4}(P_+).\]

On the other hand, by a direct computation, we can get
\[
G_t(x, t) = G \left[ M - \frac{n}{2(t + \tau_0)} + \frac{\beta |x - P_+|^2}{(t + \tau_0)^2} \right]
\]
and
\[
G_{ij} = G \left[ \frac{4\beta^2 (x_i - P_{+,i})(x_j - P_{+,j})}{(t + \tau_0)^2} \right] \\
+ \epsilon_1 \tilde{g} \left[ \frac{8\beta (x_i - P_{+,i})(x_j - P_{+,j})}{r^2(t + \tau_0)} \right], \quad (i \neq j \text{ and } 1 \leq i, j \leq n)
\]
and
\[
G_{ii} = G \left[ -\frac{2\beta}{t + \tau_0} + \frac{4\beta^2 (x_i - P_{+,i})^2}{(t + \tau_0)^2} \right] + \epsilon_1 \tilde{g} \left[ \frac{8\beta |x_i - P_{+,i}|^2}{r^2(t + \tau_0)} - \frac{8}{r^2} \right], \quad (1 \leq i \leq n).
\]

Then
\[
\chi_n (a^{ij} G_{ij}) - G_t \geq G \left[ \frac{M}{2} - \frac{2\beta r^2}{T + \tau_0} - \frac{r^2}{16(T + \tau_0)^2} \right] \\
+ \epsilon_1 \tilde{g} \left[ \frac{M}{2} \left( 1 - \frac{4|x - P_+|^2}{r^2} \right) + \frac{8\Lambda \beta x_n^2 |x - P_+|^2}{r^2(1 + \tau_0)} - \frac{8\Lambda x_n^2}{r^2} \right]
\]
in \( B_\frac{1}{4}(P_+) \times [T, 1) \). Let
\[
\beta \geq \frac{64\Lambda(1 + \tau_0)}{\lambda r^2} \quad \text{and} \quad M \geq \frac{256\Lambda}{15^2}.
\]

Then, the second term on the right hand side of (5.11) is positive. Hence,
\[
\chi_n (a^{ij} G_{ij}) - G_t \geq 0 \quad \text{in} \quad B_\frac{1}{4}(P_+) \times [T, 1)
\]
if
\[
M \geq \max \left\{ \frac{4\beta \Lambda r^2}{\tau_0} + \frac{8\beta r^2}{\tau_0^2} - \frac{256\Lambda}{15^2} \right\}.
\]
Now we let \( f \) be a solution of
\[
\begin{align*}
\frac{\partial}{\partial t} f_i (x, t) &- d_i f_i = 0 & (x, t) \in Q \\
\frac{\partial}{\partial x_j} f_i (x, t) &- (f_i)_{x_j} = 0 & (x, t) \in B^+_r \times (0, 1) \\
f_i (x, t) &- f_0 (x, 0) = m_0 & x \in B^+_r, \text{ for } i = 1, \ldots, n.
\end{align*}
\]
(5.14)

Then, for sufficiently small \( \varepsilon > 0 \) and \( M \) in (5.13) and \( \beta \) satisfying (5.8) and (5.12), we have
\[
\lim_{\varepsilon \to 0} \int_{B^+_r} f \, dx dt = \int_{B^+_r} \tilde{g}_e \, dx dt,
\]
and
\[
\int_{B^+_r} f \, dx dt \leq \beta \int_{B^+_r} g \, dx dt
\]
by the comparison principle. Since \( \varepsilon' \in \mathbb{R}^{n-1} \) is an arbitrary point with \( |\varepsilon'| \leq \frac{\varepsilon}{10} \), by the definition of functions \( \tilde{g}_e \) and \( G \), the solution \( f \) has nontrivial gradients on \( \partial B^+_r \cap \{x_n = 0\} \), i.e., the third inequality of (5.3) holds.

To complete the proof of lemma, we are now going to show the existence of solution \( f \) to (5.3) with initial condition (5.4). We begin by constructing a sequence of approximate domain so as to avoid the degeneracy of the equation. We may simply put
\[
B^{+}_{r,n} = B^+_r \cap \left\{ x_n > \frac{r}{n} \right\}.
\]

We now solve the problem
\[
\begin{align*}
\frac{\partial}{\partial t} f_i (x, t) - (f_i)_{x_j} &- (f_n)_{x_j} = 0 & (x, t) \in B^+_r \times (0, 1) \\
f_i (x, t) &- f_0 (x, 0) = m_0 & x \in B^+_r, \text{ for } i = 1, \ldots, n.
\end{align*}
\]
(5.15)

The maximum principle, which holds for classical solutions, implies that
\[
0 \leq f_n \leq m_0 \quad \text{in } B^+_r \times (0, 1).
\]

Moreover, again by the maximum principle
\[
f_n \leq f_{n+1} \quad \text{in } B^+_r \times (0, 1), \quad \forall n \geq 4.
\]

Hence we can define the function
\[
f(x, t) = \lim_{n \to \infty} f_n (x, t), \quad \text{in } Q
\]
as a monotone limit of bounded non-negative functions. On the other hand, we also have
\[
f_n (x, 0) \leq \frac{4m_0}{r_0} x_n \quad \text{in } B^+_r.
\]

Hence, \( 0 \leq f \leq Mx_n \) for some constant \( M > 0 \) and \( f \) is continuous up to the boundary. Since \( f_n \) is a classical solution of (5.15), \( f \) clearly satisfies (5.3) and (5.4). Hence, we complete the proof. \( \square \)

With this barrier function, we are going to show the second main result for our paper: \( C^{1, \gamma} \) regularity of solution \( w \) to (5.1).

**Lemma 5.2.** Let \( w \) be a solution of (5.1). If \( w \) is in \( C^{0,1} (H) \), with respect to space variable \( x \), then, it is also in \( C^{1, \gamma} \), \( 0 < \gamma < 1 \) near the plane \( \{x_n = 0\} \), i.e., for each \( x_0 \in \{x_n = 0\} \) and \( T > 0 \), there is a constant \( L \) which is depending on \( x_0 \) such that
\[
[w(x, T) - Lx_n] \leq C_0 |x - x_0|^{1+\gamma}, \quad \text{in } B^+_T (x_0)
\]
(5.16)

for some constant \( C_0 > 0 \)

**Proof.** For each \( T > 0 \), let \( r = r(T) \) be given in Lemma 5.1 for \( l = 1 - (16)^{\gamma-2} \). We denote by
\[
M_0 = \max_k w_0 \quad \text{and} \quad m_0 = \min_k w_0
\]
for the set \( K = \{ x \in B^+_r : |x_i - P^*_x| < \frac{\kappa}{r}, \ P^*_x = (0, \cdots, 0, \frac{\kappa}{r}) \in \mathbb{R}^n \text{ and } i = 1, \cdots, n \} \). Then, by the Lemma 5.1 there exists a solution \( f \) of
\[
\begin{align*}
\begin{cases}
  f_i = x^0_i \left( d^j f_{ij} \right) & \text{in } B^+_r \\
  f = 0 & \text{on } \partial B^+_r 
\end{cases}
\]
with the conditions \( w(x, 0) \geq f(x, 0) \) in \( B^+_r \) and
\[
|\nabla f(x, t)| \geq c_0 > 0, \quad \forall t \geq 1T, \ x \in \partial B^+_r \cap \{ x_n = 0 \}.
\]
Hence,
\[
|\nabla w(x, t)| \geq |\nabla f(x, t)| \geq c_0 > 0, \quad \forall t \geq 1T, \ x \in \partial B^+_r \cap \{ x_n = 0 \}
\]
and this implies that there exists a constant \( A > 0 \) such that
\[
w(x, t) \geq Ax_n \quad \text{in } B^+_r \times [1T, T].
\]
In addition, there also exists a constant \( B > 0 \) such that
\[
w(x, t) \leq Bx_n \quad \text{in } B^+_r \times [1T, T]
\]
since \( w_0 \in C^{0,1}(H) \) and the function \( Bx_n \) is a solution of (5.1). Now we are going to show (5.16) with scaling properties. To get them, we define the new function by
\[
\tilde{w}(x', t') = 16w(x, t) \quad \left( x' = 16x, \quad t' = (16)^{2-n} (t - 1T) \right).
\]
Then, the function \( \tilde{w} \) is also a solution of (5.1) with initial data \( \tilde{w}(\cdot, 1T) \). By (5.17) and (5.18), we also obtain
\[
Ax_n' \leq \tilde{w}(x', t') \leq Bx_n', \quad \text{in } B^+_r \times [0, T].
\]
Let’s assume that \( A_0 \) and \( B_0 \) are the optimal constants for (5.19). If
\[
\left\{ x' \in B^+_r : \tilde{w}(x', 0) \geq \left( \frac{A_0 + B_0}{2} \right) x_n' \right\} \geq \frac{1}{2} |B^+_r|,
\]
then there exist constants \( \delta_1 > 0 \) and \( m_1 > 0 \) such that
\[
w(x', 0) - A_0 x_n' \geq m_1, \quad \text{in } K.
\]
This is because \( \tilde{w}(x', 0) - A_0 x_n' \) is Lipschitz continuous with respect to space variable. On the other hand, if
\[
\left\{ x' \in B^+_r : \tilde{w}(x', 0) \leq \left( \frac{A_0 + B_0}{2} \right) x_n' \right\} \geq \frac{1}{2} |B^+_r|,
\]
then there exist constants \( \delta'_1 > 0 \) and \( m'_1 > 0 \) such that
\[
B_0 x_n' - \tilde{w}(x', 0) \geq m'_1, \quad \text{in } K.
\]
Note that the constants \( m_1 \) and \( m'_1 \) are only obtained by \( |A_0 - B_0| \) and Lipschitz continuity. We first assume that
\[
\left\{ x' \in B^+_r : \tilde{w}(x', 0) \geq \left( \frac{A_0 + B_0}{2} \right) x_n' \right\} \geq \frac{1}{2} |B^+_r|.
\]
Then, by similar argument as above, there is a function \( f' \) such that
\[
\tilde{w}(x', t') - A_0 x_n' \geq f'(x', 0) \quad \text{in } B^+_r \times (0, T)
\]
and
\[
|\nabla f'(x', t')| \geq c_0 > 0, \quad \forall t' \geq 1T, \ x' \in \partial B^+_r \cap \{ x_n' = 0 \}.
\]
Note that \( m_1 \) in (5.20) is only depending on \( |B - A| \) and Lipschitz continuity of \( \tilde{w} \). Hence, the constant \( c_0 \) is proportion to \( |A_0 - B_0| \). Thus, there exists a constant \( 0 < \kappa < \frac{1}{16} \) such that
\[
\tilde{w}(x', t') - B_0 x_n' \geq \kappa (B_0 - A_0) x_n' \quad \text{in } B^+_r \times [1T, T].
\]
Similarly, if
\[
\left\{ x' \in B^+_r : \tilde{w}(x', 0) \leq \left( \frac{A_0 + B_0}{2} \right) x_n' \right\} \geq \frac{1}{2} |B^+_r|,
\]
then, we also have
\[ B_0 x'_n - w'(x', t') \geq \kappa (B_0 - A_0) x'_n \quad \text{in } B^+_T \times [T, T]. \]

Therefore, we get \( B_1 \geq A_1 > 0 \) such that \( 0 < A_0 \leq A_1 \leq B_1 \leq B_0 < \infty, B_1 - A_1 \leq (1 - \kappa)(B_0 - A_0) \)
and
\[
A_1 x'_n \leq w'(x', t') \leq B_1 x'_n, \quad \text{in } B^+_T \times [T, T].
\]

Hence, from (5.21),
\[
A_1 x_n \leq w(x, t) \leq B_1 x_n, \quad \text{in } B^+_T \times [T + (1 - l)T, T].
\]

By iteration arguments, we conclude that there exist series \(|A_k|\) and \(|B_k|\) such that \( 0 < A_0 \leq A_1 \leq \cdots \leq A_k \leq A_{k+1} \leq \cdots \leq B_k \leq \cdots \leq B_0 < \infty, B_{k+1} - A_{k+1} \leq (1 - \kappa)(B_k - A_k) \)
and
\[
A_k x_n \leq w(x, t) \leq B_k x_n, \quad \text{in } B^+_T \times [(1 - (1 - l)^{k+1})T, T].
\]

This will imply
\[
\frac{|w(x, t) - C x_n|}{x_n} \leq B_k - A_k \leq (1 - \kappa)^k (B_0 - A_0), \quad \text{in } B^+_T \times [(1 - (1 - l)^{k+1})T, T]
\]
for \( C = \lim_{k \to \infty} A_k = \lim_{k \to \infty} B_k \). Now we get (5.22) when \( x \in B^+_T \). Hence, by the property of logarithm function, we get
\[
\frac{|w(x, T) - C x_n|}{x_n} \leq \left( \frac{16|x|}{r} \right)^{\gamma - \log_{10} (1 - \kappa)}, \quad \text{in } B^+_T.
\]

Since \( 0 < \kappa < \frac{15}{16} \), we have \( 0 < - \log_{10} (1 - \kappa) < 1 \). Hence (5.16) holds for constant \( \gamma = - \log(1 - \kappa) \) and the lemma follows.

From the previous Lemma 5.2 we can cover the remaining part of missing step for the long time existence.

**Corollary 5.3.** Under the hypotheses of Lemma 5.2 we also have
\[
|w_2(x) - L| \leq C_1 |x - x_0|'^r, \quad \text{in } B^+_T (x_0)
\]
for some constant \( C_1 > 0 \).

**Proof.** Let \( x_0 \) be a point on \( \{x_n = 0\} \) and let \( x^s = (x', s) \) be a point in \( B^+_T (x_0) \). Then,
\[
|w_2(x^s) - La| = \lim_{b \to a} \frac{|w(x) - w(x^b)|}{b - a}.
\]

By the previous Lemma (Lemma 5.2), we can get
\[
|w(x^b) - La| = C_1 |x^b - x_0|'^r + \text{h.o.t. of } |x^a - x_0|
\]
and
\[
|w(x^b) - Lb| = C_1 |x^b - x_0|'^r + \text{h.o.t. of } |x^b - x_0|
\]
for some constant \( C_1 > 0 \) if \( x^s \) is sufficiently close to \( x^b \). Here, \( \text{h.o.t.} \) means ‘higher order terms’. Then, by (5.24), (5.24) and (5.25), we obtain
\[
|w_2(x^s) - L| \leq \lim_{b \to a} \frac{|w(x) - Lb - (w(x^s) - La)|}{b - a} \leq C_1 |x^a - x_0|'^r + \text{h.o.t. of } |x^a - x_0|.
\]

Since \( x^a \) is chosen arbitrary in \( B^+_T (x_0) \), we can get a desired result.
6. Proof of Main Theorem

6.1. Local coordinate change. To motivate the proof of Main Theorem, we will first compute the transformation of the equation

\[ f_t = mf^\alpha \triangle f \]

when one exchanges dependent and independent variables near the boundary. This change of coordinates converts the boundary into a flat boundary. More precisely, assume that the function \( f \) belong to the space \( C^{2+\alpha}_t(\Omega \times [0, T]) \). Let \( X_0 = (x^0, t_0) = (x_0^0, x^0_n, t_0) \) at the boundary \( \partial \Omega \times (0, T) \). We can assume (by rotating the coordinates) that \( f_{x_i}(X_0) = \cdots = f_{x_n}(X_0) = 0 \), and \( f_n(X_0) > 0 \). It follows from the Implicit Function Theorem that if the number \( \eta \) is sufficiently small, we can solve the equation \( z = f(x, t) \) with respect to \( x_n \) around the point \( X_0 \). This yields to a map \( x_n = h(x', z, t) \) defined in a small box \( B_\eta^+(x^0, 0, t_0) \). We wish to compute the evolution equation of \( h \) from the one of \( f \). To compute the evolution of \( h \), we use the identities

\[
\begin{align*}
 f_{x_n} &= \frac{1}{h_z}, & f_{x_i} &= -\frac{h_i}{h_z}, & f_t &= -\frac{h_t}{h_z}, & f_{x_n x_n} &= -\frac{1}{h_z^2} h_{zz}, \\
 f_{x_i x_n} &= -\frac{1}{h_z^2} (h_i h_{zz} - 2h_i h_{z} h_{z} + h_{x_i x_n}) & (i = 1, \cdots, n - 1).
\end{align*}
\]

This follows immediately from above computations, the equation

\[ f_t - mf^\alpha \triangle f = 0 \]

transforms into the equation

\[ M(h) = h_t - z^a \left[ \triangle_x h + \left( -\frac{1 + |\nabla_x h|^2}{h_z} \right) \right] = 0. \]

Set

\[ a^{mn} = \frac{1 + |\nabla_x h|^2}{h_z^2}, \quad a^{ik} = -\frac{2h_i}{h_z} \quad \text{and} \quad a^{k'\ell} = \delta_{k'\ell} \]

for \( k, l' = 1, \cdots, n - 1 \). Then, \( h \) is a solution of the equation

\[ L[w] = w_t - z^a \left( h^{ik} w_{ik} \right), \quad (k, l = 1, \cdots, n). \]

The operator \( M \) defined above becomes degenerate when \( z = 0 \). We can also easily compute its first derivatives with respect to \( x_i \) (\( i = 1, \cdots, n - 1 \)):

\[ L[w : x_i] = w_t - z^a \left( \frac{1 + |\nabla_x h|^2}{h_z^2} w_z - \sum_{j=1}^{n-1} \frac{2h_j}{h_z} w_{x_j} \right) + z^a \nabla_{x'} (\nabla_x w) \]

Set

\[ a^{mn} = \frac{1 + |\nabla_x h|^2}{h_z^2}, \quad a^{kn} = -\frac{2h_k}{h_z} \quad \text{and} \quad a^{ik} = 0 \quad \text{and} \quad a^{k'\ell} = \delta_{k'\ell} \]

for \( k, l' = 1, \cdots, n - 1 \). Then, \( h_{x_i} \) is a solution of the equation

\[ L[w : x_i] = w_t - z^a \nabla_{x_i} \left( h^{ij} \nabla_{x_j} w \right), \quad (k, l = 1, \cdots, n). \]

We finish with the proof of Theorem 1.1.

The Proof of Theorem 1.1 By the Theorem 2.1 there exists a solution \( f \) of \( (1.3) \) which is \( C^{2+\gamma}_t \)-smooth up to the boundary, when \( 0 < t < \tau \) for some constant \( \tau > 0 \). Let \( (0, \tau_0) \) be the maximal interval of existence for \( C^{2+\gamma}_t \)-solution \( f \) of \( (1.3) \) and suppose that \( \tau_0 < \infty \). By scaling time around \( t = \tau_0 \), we can assume that

\[ \tau_0 = 1. \]

We will show the solution \( f \) is in \( C^{2+\gamma}_t(\Omega) \) at \( t = 1 \). Then, by applying Theorem 2.1 again, we can extend \( C^{2+\gamma}_t \)-smoothness of \( f \) to \( t = 1 + \tau_1 \) for some constant \( \tau_1 > 0 \). By maximality of \( \tau_0 = 1 \), contradiction arise.
Hence, $\tau_0 = \infty$ and theorem follows.

Since $\Omega$ is compact domain, we can express $\Omega$ as the finite union

$$\Omega = \Omega_0 \cup \bigcup_{l \geq 1} \Omega_l$$

of compact domains in such a way that

$$\text{dist}(\Omega_0, \partial \Omega) \geq \frac{\rho}{2} > 0$$

and for all $l \geq 1$

$$\Omega_l = B_\rho(x_l) \cap \Omega$$

with $B_\rho(x_l)$ denoting the ball center at $x_l \in \partial \Omega$ of radius $\rho > 0$. The number $\rho > 0$ will be determined later. The equation (1.2), when restricted on the interior domain $\Omega_0$, is nondegenerate. Therefore, the classical Schauder theory for linear parabolic equations implies that

$$f \in C_2^{\gamma_s}(\Omega_0) \quad \text{at } t = 1.$$ 

Hence, we are going to concentrate our attention on the domains $\Omega_l$ ($l \geq 1$), close to the boundary of $\Omega$. Under the local coordinate change above, we can change the function $f$ defined on $\Omega_l$ into the function $h$ defined on $\mathcal{D}_l \in H$. In addition,

$$h \in C_2^{\gamma_s}(\mathcal{D}_l) \quad (0 \leq t < 1).$$

By the maximality of $\tau_0 = 1$, $h$ is no longer in $C_2^{\gamma_s}(\mathcal{D}_l)$ at time $t = 1$. However the existence theory gave $h \in W^{1,2}(\mathcal{D}_l)$ (See chapter 5 in [9] for the details). By the Lemmas 3.1, 3.2 and 3.3, the coefficients of $L$ in (6.1) and (6.2) satisfy the hypotheses in the corollary 5.3 and the Lemma 4.4 respectively. Then, the Lemma 4.4 and the corollary 5.3 tell us that

$$\nabla h \in C_2^{\gamma_s}(\mathcal{Q}_2^{+}(x_l, 1))$$

where $\gamma = \frac{2\nu}{2-\alpha}$ and $r < 1$.

Hence, the coefficients of the equation (6.1) satisfy the assumption of Lemma 3.2 in [6]. By Lemma 3.2 in [6],

$$h \in C_2^{\gamma_s}(\mathcal{Q}_2^{+}(x_l, 1)).$$

Let’s choose $\rho > 0$ so small that $\mathcal{D}_l \in B_{\frac{\rho}{2}}(x_l)$. Then

$$h \in C_2^{\gamma_s}(\mathcal{D}_l) \quad \text{at } t = 1.$$ 

This immediately implies

$$f \in C_2^{\gamma_s}(\Omega_l) \quad \text{at } t = 1.$$ 

Therefore, by (6.3), (6.4) and (6.6), we get

$$f \in C_2^{\gamma_s}(\Omega) \quad \text{at } t = 1$$

and theorem follows.

\[\square\]

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Sungchoon Kim: Department of Mathematics and PMI (Pohang Mathematics Institute), Pohang University of Science and Technology (POSTECH), Hyoja-Dong San 31, Nam-gu, Pohang 790-784, South Korea
E-mail address: math.s.kim@postech.ac.kr