Limits of Weierstrass points in regular smoothings of curves with two components

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Abstract. In [3] D. Eisenbud and J. Harris posed the following question: What are the limits of Weierstrass points in families of curves degenerating to stable curves not of compact type? We answer their question for one-dimensional families of smooth curves degenerating to stable curves with just two components meeting at points in general position. In this note we treat only those families whose total space is regular. Nevertheless, we announce here our most general answer, to be presented in detail in [5].

1. Regularly smoothable linear systems

Let $C$ be a connected, projective, nodal curve defined over an algebraically closed field $k$. Let $C_1, \ldots, C_n$ be its irreducible components. Let $B := \text{Spec}(k[[t]])$; let $o$ denote its special point and $\eta$ its generic point. A projective and flat map $\pi: S \to B$ is said to be a smoothing of $C$ if the generic fiber $S_\eta$ is smooth and the special fiber $S_o$ is isomorphic to $C$. In addition, if $S$ is regular then $\pi$ is called a regular smoothing.

Assume from now on that $n = 2$. Let $\Delta$ be the reduced Weil divisor with support $|\Delta| = C_1 \cap C_2$ and $\delta := \deg \Delta$. For $i = 1, 2$ let $g_i$ denote the arithmetic genus and $\omega_i$ the dualizing sheaf of $C_i$. Then $g := g_1 + g_2 + \delta - 1$ is the arithmetic genus of $C$. Let $\omega$ denote the dualizing sheaf of $C$.

To avoid exceptional cases, assume from now on that $C$ is semi-stable, that is, assume that $\delta > 1$ or $g_1g_2 > 0$. Let $\ell_i := [g_{3-i}/\delta]$ and $m_i := \ell_i \delta - g_{3-i}$ for $i = 1, 2$. If $\ell_1\ell_2 \neq 0$ set $\lambda_i := \ell_i/\gcd(\ell_1, \ell_2)$ for $i = 1, 2$. Let $L_{i, j} := \omega_j((1 + (-1)^{i-j}\ell_i)\Delta)$ for all $i, j \in \{1, 2\}$.

If $\pi$ is a regular smoothing of $C$, let $\omega_\pi$ be its (relative) dualizing sheaf. Put $L_{\pi, i} := \omega_\pi(-\ell_iC_i)$ and $L_{\pi, i}|_C$ for $i = 1, 2$. Note that $L_{\pi, i}|_C \simeq L_{i, j}$ for all $i, j \in \{1, 2\}$. Once we fix isomorphisms we obtain restriction maps

$$\rho_{\pi, i, j}: H^0(L_{\pi, i}) \to H^0(L_{i, j})$$
for all $i, j \in \{1, 2\}$. Let $V_{\pi,i} := \text{Im}(\rho_{\pi,i,i})$ for $i = 1, 2$.

**Lemma 1.** For $i = 1, 2$ consider the following condition:

\[(1.i) \quad h^0(\omega_{3-i}(-n\Delta)) = \max(g_{3-i} - n\delta, 0) \text{ for every } n \geq 0.\]

Let $\pi$ be a regular smoothing of $C$. If (1.i) holds, then $h^0(L_{\pi,i}) = g$ and $\rho_{\pi,i,j}$ is injective for $j = i$ and non-zero for $j = 3 - i$.

**Proof.** The natural exact sequence,

$$0 \to L_{\pi,i}|C_i(-\Delta) \to L_{\pi,i} \to L_{\pi,i}|C_{3-i} \to 0,$$

induces an exact sequence on global sections,

\[(2) \quad 0 \to H^0(L_{i,i}(-\Delta)) \to H^0(L_{\pi,i}) \xrightarrow{\rho_{\pi,i,3-i}} H^0(L_{i,3-i}).\]

Since (2) is exact,

$$h^0(L_{\pi,i}) \leq h^0(L_{i,i}(-\Delta)) + h^0(L_{i,3-i}).$$

Now, $h^0(L_{i,i}(-\Delta)) + h^0(L_{i,3-i}) = g$ by Riemann-Roch and (1.i). So $h^0(L_{\pi,i}) \leq g$. But $L_{\pi,i} = \omega_{\pi}(-\ell_iC_i)|C_i$, hence $h^0(L_{\pi,i}) \geq g$ by semi-continuity. So $h^0(L_{\pi,i}) = g$.

Now, $g > g + m_i - \delta$, hence $h^0(L_{\pi,i}) > h^0(L_{i,i}(-\Delta))$ if $\ell_i > 0$. If $\ell_i = 0$, then $g_{3-i} = 0$; since $C$ is semi-stable, $\delta > 1$ and hence $g > g_i$. It follows that $h^0(L_{\pi,i}) > h^0(L_{i,i}(-\Delta))$ as well. In any case, $\rho_{\pi,i,3-i} \neq 0$ by the exactness of (2).

Finally, $h^0(L_{i,3-i}(-\Delta)) = 0$ by (1.i), because $g_{3-i} \leq \ell_i\delta$. So $\rho_{\pi,i,i}$ is 1–1. The proof is complete.

**Theorem 2.** For $i = 1, 2$ let $L_i$ be an invertible sheaf on $C$.

(a) If $\ell_i \neq 0$, then there is a regular smoothing $\pi$ of $C$ such that $L_{\pi,i} \cong L_i$ if and only if $L_i|C_j \cong L_{i,j}$ for $j = 1, 2$.

(b) If $\ell_1\ell_2 \neq 0$, then there is a regular smoothing $\pi$ of $C$ such that $L_{\pi,i} \cong L_i$ for $i = 1, 2$ if and only if $L_i|C_j \cong L_{i,j}$ for all $i, j \in \{1, 2\}$ and $L_1^\lambda L_2^\lambda \cong \omega^{\lambda_1+\lambda_2}$.

**Proof.** Let $\mathbb{P}$ denote the Picard group of $C$ and $\mathbb{T} \subseteq \mathbb{P}$ the subgroup of sheaves whose restrictions to $C_1$ and $C_2$ are trivial. Since $C$ is nodal, $\mathbb{T}$ is a torus of dimension $\delta - 1$. Let $\mathbb{E} \subseteq \mathbb{P}$ denote the set of invertible sheaves $N$ on $C$ such that $N|C_i \cong \mathcal{O}_{C_i}((-1)^i\Delta)$ for $i = 1, 2$. Note that $\mathbb{T}\mathbb{E} \subseteq \mathbb{E}$.

Let’s prove (a). If $\pi$ is a regular smoothing of $C$ then $L_{\pi,i}|C_j \cong L_{i,j}$ for $j = 1, 2$, as observed before. Conversely, suppose that $L_i|C_j \cong L_{i,j}$ for $j = 1, 2$. Let $N \in \mathbb{E}$ and $M := L_i\omega^{-1}N(-1)^{i+1}\ell_i$. Then $M \in \mathbb{T}$. Since $\mathbb{T}$ is a torus and $\ell_i \neq 0$, there is $Q \in \mathbb{T}$ such that $M = Q(-1)^i\ell_i$. Then $L_i = \omega(QN)(-1)^i\ell_i$. Since $QN \in \mathbb{E}$, by ([6],
Prop. 3.16) there is a regular smoothing \( \pi: S \to B \) of \( C \) such that \( \mathcal{O}_S(C_1)|_C \cong QN \). Then \( L_{\pi,i} \cong L_i \), completing the proof of (a).

Let’s prove (b). If \( \pi \) is a regular smoothing of \( C \) then \( L_{\pi,i}|_{C_j} \cong L_{i,j} \) for all \( i,j \in \{1,2\} \) and

\[
L_{\pi,1}^2 L_{\pi,2}^2 \cong \omega_{\pi}^{\lambda_1 + \lambda_2} (-\lambda_2 \ell_1 C_1 - \lambda_1 \ell_2 C_2)|_C.
\]

Since \( \lambda_2 \ell_1 \) and \( \lambda_1 \ell_2 \) are equal and \( C_1 + C_2 \equiv 0 \), we have \( L_{\pi,1}^2 L_{\pi,2}^2 \cong \omega^{\lambda_1 + \lambda_2} \).

Conversely, suppose that \( L_i|_{C_j} \cong L_{i,j} \) for all \( i,j \in \{1,2\} \) and \( L_{\pi,1}^2 L_{\pi,2}^2 \cong \omega^{\lambda_1 + \lambda_2} \).

Let \( N \in \mathbb{E} \). For \( i = 1,2 \) set \( M_i \) := \( L_i \omega^{-1} N(-1)^{i+1} \ell_i \). Then \( M_1 \) and \( M_2 \) are in \( \mathcal{T} \) and \( M_1 \lambda_2 M_2 \lambda_1 \cong \mathcal{O}_C \). Since \( \gcd(\lambda_1,\lambda_2) = 1 \) and \( \mathcal{T} \) is a torus, there is \( Q \in \mathcal{T} \) such that \( M_i \cong Q(-1)^{i} \ell_i \) for \( i = 1,2 \). Then \( L_i \cong \omega((QN)^{-1} \ell_i \) for \( i = 1,2 \). Since \( QN \in \mathbb{E} \), by ([6], Prop. 3.16) there is a regular smoothing \( \pi: S \to B \) of \( C \) such that \( \mathcal{O}_S(C_1)|_C \cong QN \). Then \( L_{\pi,i} \cong L_i \) for \( i = 1,2 \). The proof is complete.

Note that \( \ell_i \leq g_{3-i} \) and so \( L_{i,i} \cong \omega_i((1 + g_{3-i})\Delta) \) for \( i = 1,2 \). Let \( G := G_1 \times G_2 \) where

\[
G_i := \text{Grass}_g(H^0(\omega_i((1 + g_{3-i})\Delta))) \text{ for } i = 1,2.
\]

Suppose (1.1). We say that \( V_i \in G_i \) is **regularly smoothable** if there is a regular smoothing \( \pi \) of \( C \) such that \( V_i = V_{\pi,i} \). Let \( \tilde{V}_i \subseteq G_i \) be the subset of regularly smoothable subspaces.

Suppose (1.1–2). We say that \( \nu = (V_1,V_2) \in G \) is **regularly smoothable** if there is a regular smoothing \( \pi \) of \( C \) such that \( V_i = V_{\pi,i} \) for \( i = 1,2 \), and denote \( \nu = \nu_\pi \). Let \( \tilde{V} \subseteq G \) denote the subset of regularly smoothable pairs. Of course, \( \tilde{V} \subseteq \tilde{V}_1 \times \tilde{V}_2 \).

**Theorem 3.** For \( i = 1,2 \) consider the following condition:

\[
(3.i) \quad h^0(\omega_{3-i}( -\ell_i \Delta + I)) = 0 \text{ for every effective divisor } I < \Delta \text{ with } \deg I = m_i.
\]

(a) If (3.i) holds, then \( \tilde{V}_i \) is locally closed in \( G_i \) and isomorphic to a torus of dimension \( \delta - 1 \), unless \( \delta|g_{3-i} \); in the exceptional case, \( \tilde{V}_i = \{H^0(L_{i,i})\} \).

(b) If (3.1–2) hold and \( \delta \not| \gcd(g_1,g_2) \), then \( \tilde{V} \) is closed in \( \tilde{V}_1 \times \tilde{V}_2 \) and isomorphic to a torus of dimension \( \delta - 1 \).

**Proof.** Fix isomorphisms \( \zeta_{i,j}: L_{i,j}|_{\Delta} \cong \mathcal{O}_{\Delta} \), and let \( e_{i,j}: H^0(L_{i,j}) \to H^0(\mathcal{O}_{\Delta}) \) be the induced maps for all \( i,j \in \{1,2\} \). For \( i = 1,2 \) let \( L_i \) be the invertible sheaf on \( C \) obtained by identifying \( L_{i,1} \) and \( L_{i,2} \) along \( \Delta \) by means of \( \zeta_{i,1} \) and \( \zeta_{i,2} \). If \( \ell_1 \ell_2 \neq 0 \) choose the \( \zeta_{i,j} \) such that \( L_{1}^{\lambda_2} L_{2}^{\lambda_1} \cong \omega^{\lambda_1 + \lambda_2} \).

Assume that (3.i) holds. Then (1.i) holds as well. It follows from Lemma 1 and Riemann-Roch that \( \dim V_{\pi,i} = g \) and \( \text{codim}(V_{\pi,i}, H^0(L_{i,i})) = m_i \) for every regular smoothing \( \pi \) of \( C \).
If \( \delta | g_{3-i} \) then \( m_i = 0 \), and so \( \tilde{V}_i = \{ H^0(L_{i,i}) \} \). Suppose now that \( \delta \not| g_{3-i} \). So \( \ell_i > 0 \), and thus \( e_{i,i} \) is surjective by Riemann-Roch. Let \( G_i := \text{Grass}_{\delta - m_i}(H^0(O_\Delta)) \).

Taking inverse images by \( e_{i,i} \) gives us a closed embedding \( \mu_i : G_i \to \mathbb{G}_i \). Let \( W_i \) be the image of \( e_{i,3-i} \). By (1.i), \( \dim W_i = \delta - m_i \), so \( W_i \in G_i \). Let \( T := H^0(O_\Delta) \), and consider its natural action on \( H^0(O_\Delta) \). Let \( O_i \) denote the orbit of \( W_i \) under the induced action of \( T \) on \( G_i \). So \( O_i \subseteq G_i \) is locally closed. Now, by (3.i), all the Plücker coordinates of \( W_i \) in \( G_i \) are non-zero. In addition, \( W_i \not= H^0(O_\Delta) \) because \( \delta \not| g_{3-i} \). It follows that the orbit map \( \psi : T \to \mathcal{O}_i \) factors through an isomorphism \( T/k^* \cong \mathcal{O}_i \). So \( \mathcal{O}_i \) is isomorphic to a torus of dimension \( \delta - 1 \). Since \( \ell_i \not= 0 \), it follows from Theorem 2 that \( \tilde{V}_i = \mu_i(\mathcal{O}_i) \). Since \( \mu_i \) is an embedding, \( \tilde{V}_i \) is locally closed in \( \mathcal{G}_i \) and isomorphic to a torus of dimension \( \delta - 1 \). So (a) is proved.

Assume now that (3.1–2) hold, and let’s prove (b). If \( \delta | g_{3-i} \) then \( \tilde{V}_i = \{ H^0(L_{i,i}) \} \) by (a). Hence \( \tilde{V} = \tilde{V}_1 \times \tilde{V}_2 \) and \( \tilde{V} \cong \tilde{V}_i \), showing (b). Suppose now that \( \delta \not| g_i \) for \( i = 1, 2 \). Then \( \ell_1 \ell_2 \not= 0 \). Consider the subgroups \( D := \langle (t_1, t_2) \in T \times T \mid t_1^{\lambda_2} = t_2^{\lambda_1} \rangle \) and \( Z := D \cap (k^* \times k^*) \). Since \( \lambda_1 \) and \( \lambda_2 \) are coprime, \( D \) is a subtorus of dimension \( \delta \) of \( T \times T \), and \( Z \) is a one-dimensional subtorus of \( D \). Let \( \mathcal{O} \) be the orbit of \( (W_1, W_2) \) under the induced action of \( D \) on the product \( G_1 \times G_2 \). Then \( \mathcal{O} \subseteq \mathcal{O}_1 \times \mathcal{O}_2 \), and the orbit map \( \psi : D \to \mathcal{O} \) is the restriction to \( D \) of \( \psi_1 \times \psi_2 \). Since \( \psi_i \) factors through an isomorphism \( T/k^* \cong \mathcal{O}_i \) for \( i = 1, 2 \), then \( \psi \) factors through an isomorphism \( D/Z \cong \mathcal{O} \). So \( \mathcal{O} \) is closed in \( \mathcal{O}_1 \times \mathcal{O}_2 \) and isomorphic to a torus of dimension \( \delta - 1 \). Since \( \ell_1 \ell_2 \not= 0 \) and \( L_1^{\lambda_2} L_2^{\lambda_1} \cong \omega^{\lambda_1 + \lambda_2} \), it follows from Theorem 2 that \( \tilde{V} = (\mu_1 \times \mu_2)(\mathcal{O}) \). So \( \tilde{V} \) is closed in \( \tilde{V}_1 \times \tilde{V}_2 \) and isomorphic to a torus of dimension \( \delta - 1 \). The proof is complete.

If \( \pi : S \to B \) is a smoothing of \( C \), let \( W_\pi := \overline{W}_\eta \cap C \), where \( W_\eta \subseteq S \) is the Weierstrass subscheme of the generic fiber \( S_\eta \) of \( \pi \). Call the associated Weil divisor \( [W_\pi] \) a limit Weierstrass divisor. For each pair \( \nu = (V_1, V_2) \in \mathbb{G} \), let

\[
W_\nu := W_{\nu,1} + W_{\nu,2} + g(\delta - 2)\Delta,
\]

where \( W_{\nu,i} \) is the ramification divisor of the linear system \( (V_i, \omega_i((1 + g_{3-i})\Delta)) \) for \( i = 1, 2 \).

**Theorem 4.** If \( k \supseteq \mathbb{Q} \) and (1.1–2) hold, then \( [W_\pi] = W_\nu \), for each regular smoothing \( \pi \) of \( C \).

**Proof.** For \( i = 1, 2 \), since (1.i) holds, Lemma 1 says that \( L_{\pi,i} \) is the extension associated to \( C_i \) of the canonical sheaf on the generic fiber of \( \pi \) (see [4], p. 26). By ([4], Thm. 7),

\[
(W_\pi) = \overline{W}_{\pi,1} + \overline{W}_{\pi,2} + g(g - 1 - \ell_1 - \ell_2)\Delta,
\]
where $W_{\pi,i}$ is the ramification divisor of the linear system $(V_{\pi,i}, L_{i,i})$ for $i = 1, 2$. Now, $L_{i,i} = \omega_i((1 + \ell_i)\Delta)$, hence $W_{\pi,i} = W_{\pi,i} - g(g - i - \ell_i)\Delta$ for $i = 1, 2$. Substituting in (4) we obtain $[W_\pi] = W_{\nu_\pi}$.

**Corollary 5.** Assume that $k \supseteq \mathbb{Q}$ and $\delta | \gcd(g_1, g_2)$. If $h^0(\omega_i(-g_i/\delta)\Delta)) = 0$ for $i = 1, 2$, then

$$[W_\pi] = W_1 + W_2 + (g^2 - \frac{g(g + 1)}{\delta})\Delta,$$

for every regular smoothing $\pi$ of $C$, where $W_i$ is the ramification divisor of the complete system $|\omega_i((1 + g_{3-i}/\delta)\Delta)|$ for $i = 1, 2$.

Proof. Since $\delta|(g_1, g_2)$, we have $\ell_i = g_{3-i}/\delta$ for $i = 1, 2$. Apply (4) to finish the proof.

The above corollary was stated in the case $\delta = 1$ in ([3], Cor. 4.3).

**2. Smoothable linear systems**

We present now the main results of our forthcoming [5]. From now on assume that the condition,

$$(5.\ i)\ h^0(\omega_{3-i}(-D)) = 0 \quad \text{for every effective divisor } D \text{ on } C_{3-i}$$

holds for $i = 1, 2$. Condition (5.\ i) is stronger than (3.\ i), but still is of general type. In [5] we construct a closed subvariety $V \subseteq G$ such that $\{W_\nu | \nu \in V\}$ is the set of limit Weierstrass divisors on $C$. This construction is described below.

First of all, every smoothing of $C$ corresponds to a regular smoothing of a certain semi-stable model of $C$. For each $\Delta$-tuple $\mu$ of positive integers consider the semi-stable model $C_\mu$ of $C$ obtained by splitting the branches of $C$ at each $p \in \Delta$ and connecting them by a chain of $\mu_p - 1$ smooth rational curves. Let $\Upsilon_\mu$ denote the collection of irreducible components of $C_\mu$.

Let $\tilde{\pi}: S \to B$ be a regular smoothing of $C_\mu$ and $\omega_{\tilde{\pi}}$ its (relative) dualizing sheaf. For each $\Upsilon_\mu$-tuple $\lambda$ of integers let

$$L_{\tilde{\pi},\lambda} := \omega_{\tilde{\pi}}(\sum_{E \in \Upsilon_\mu} \lambda_E E) \quad \text{and} \quad L_{\tilde{\pi},\lambda} := L_{\tilde{\pi},\lambda}|_{C_\mu}.$$

For each $E \in \Upsilon_\mu$, consider the restriction map $\rho_{\tilde{\pi},\lambda,E}: H^0(L_{\tilde{\pi},\lambda}) \to H^0(L_{\tilde{\pi},\lambda}|_E)$. Let $i = 1, 2$. By ([4], Thm. 1), for each regular smoothing $\tilde{\pi}$ of $C_\mu$, there is a unique $\Upsilon_\mu$-tuple of integers $\lambda_i$ with $\lambda_i|_{C_i} = 0$ such that $\rho_{\tilde{\pi},\lambda_i,E}$ is injective for $E = C_i$ and non-zero for $E \in \Upsilon_\mu - C_i$. In [5] we prove that $\lambda_i$ depends only on $\mu$, and that $h^0(L_{\tilde{\pi},\lambda_i}) = g$ for every regular smoothing $\tilde{\pi}$ of $C_\mu$. In fact, we
give a completely numerical recipe to compute $\lambda_i$ in terms of $\mu$, and show that $\lambda_{i,E} \geq 0$ for each $E \in \Upsilon_\mu$ and $\lambda_{i,E} \leq g_{3-i}$ for each $E \in \Upsilon_\mu$ intersecting $C_i$. So $L_{\tilde{\pi},\lambda_i}|_{C_i} \subseteq \omega((1 + g_{3-i})\Delta)$.

A pair $\nu = (V_1, V_2) \in G$ is said to be $\mu$-smoothable if there is a regular smoothing $\tilde{\pi}$ of $C_\mu$ such that $V_i = \text{Im}(\rho_{\tilde{\pi},\lambda_i,C_i})$ for $i = 1, 2$. We denote $\nu = \nu_{\tilde{\pi}}$. Let $\mathbb{V}_\mu \subseteq G$ be the set of $\mu$-smoothable pairs. In [5] we describe $\mathbb{V}_\mu$ in a way similar to the way $\tilde{V}$ is described in Theorem 2. As in Theorem 3, we prove that $\mathbb{V}_\mu$ is locally closed in $G$ and isomorphic to a torus, and compute its dimension in terms of $\mu$.

Let $U \subseteq \mathbb{P}^\Delta_{\mathbb{Q}}$ be the open subset parameterizing classes of $\Delta$-tuples of positive rational numbers, and give it the analytic topology. Changing the base of a regular smoothing $\tilde{\pi}$ of $C_\mu$ doesn’t change $L_{\tilde{\pi},\mu}$. Hence $\mathbb{V}_\mu = \mathbb{V}_{t\mu}$ for every positive integer $t$. Define $\mathbb{V}_\mu$ for each $\mu \in U$ in the natural way.

Let

$$\mathbb{V} := \bigcup_{\mu \in U} \mathbb{V}_\mu \subseteq G.$$

Given $\mu, \mu' \in U$, we show in [5] that $\mathbb{V}_\mu$ and $\mathbb{V}_{\mu'}$ either are equal or don’t intersect. We show also that $\mathbb{V}$ is the union of finitely many $\mathbb{V}_\mu$, and prove the following theorem:

**Theorem 6.** Assume (5.1–2). Then $\mathbb{V}$ is projective, connected and of pure dimension $\delta - 1$.

To prove the theorem we show first a remarkable relation between the topologies of $U$ and $\mathbb{V}$. By studying boundary points of tori orbits in Grassmannians we prove in [5] that, for each $\mu \in U$,

$$\overline{\mathbb{V}}_\mu = \bigcup_{\mu' \in U_\mu} \mathbb{V}_{\mu'},$$

where $U_\mu \subseteq U$ is any sufficiently small (analytic) open neighborhood of $\mu$.

We show also that $\mathbb{V}$ is irreducible if and only if $g_1 = g_2 = 1$, and compute the number of irreducible components of $\mathbb{V}$ if $\delta = 2$; it is $g - \gcd(g_1 + 1, g_2 + 1)$.

Finally, let $\nu = (V_1, V_2) \in \mathbb{V}$. By definition, there are a semi-stable model $C_\mu$ of $C$ and a regular smoothing $\tilde{\pi}$ of $C_\mu$ such that $\nu = \nu_{\tilde{\pi}}$. Let $\pi$ be the smoothing of $C$ induced by $\tilde{\pi}$ and $[W_\pi]$ the limit Weierstrass divisor on $C$. In [5] we prove that $[W_\pi] = W_\nu$ if $k \supseteq \mathbb{Q}$.

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