Computation of the phase induced by non-newtonian gravitational potentials in atom interferometry

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Abstract

In this letter we present a computation of the phase induced by test masses of different geometry, in the framework of non-newtonian gravitation, on an ideal separated arms atom interferometer. We deduce the related limits on the non-newtonian gravitational strength in the sub-millimeter region for the potential range. These limits would be comparable with the best existing experimental limits but with the advantage of using a microscopic probe.

I. INTRODUCTION

In this first section we will recall briefly the main lines of the theoretical motivations of this kind of calculations and their interest and the present state-of-the-art of experiments in this field will be summarized. For a more detailed description of both theoretical and experimental point of view the reader is referred to [1] and [2]. The basics of atom interferometry will be then presented and, after some geometrical considerations, the signal for different kinds of potential will be obtained. Finally, we will derive the limits on the non-newtonian gravitational strength that could be obtained by an ideal experiment and we will compare the results to present limits.

In the attempt to unify the description of all known forces, two types of extensions of newtonian gravity are usually made [3]. First, one can postulate that a new force mediated by massive scalar bosons exists and that it gives rise to a Yukawa potential so that the total gravitational potential between point-like particles of mass $m_1$ and $m_2$ separated by a distance $r$ reads:

$$V_Y(r) = -\frac{Gm_1m_2}{r}(1 + \alpha_Y \exp^{-r/\lambda_Y})$$

In the preceding expression $G$ denotes the gravitational constant and the first term is the usual newtonian potential. The correction term is thus characterized by its dimensionless strength $\alpha_Y$ and its range $\lambda_Y$ which is essentially inversely proportional to the boson mass. A peculiar feature of such Yukawa potential is that the exponential factor saturates when the mutual distance $r$ tends to zero: if its strength $\alpha_Y$ is small compared to 1 then the
Yukawa correction remains always a small correction to Newtonian gravity which is itself a very weak interaction compared to the other known forces.

On the other hand, some recent theoretical developments account for such weakness supposing that there exists $n$ extra dimensions in which only gravity propagates so that the main part of it is lost for the usual 3D-space. The extra dimensions are compact so that their existence can be felt only to mutual distances smaller to a compactification radius $\lambda_n$; for distances much greater than the compactification radius then the reminder of the gravitation interaction in the extra dimensions manifests itself as a Yukawa-like potential with a range of the order of the compactification radius $\lambda_n$ and a strength $\alpha_n$ of a few units. It thus quickly decreases leaving just the weak long range usual inverse square law interaction. On the contrary, at short mutual distances, it follows from Gauss theorem in $3 + n$ spatial dimensions that the gravitational interaction $V_n(r)$ behaves as $r^{-(1+n)}$ so that it conveniently reads:

$$V_n(r) = -\frac{Gm_1m_2}{r} \alpha_n \left(\frac{\lambda_n}{r}\right)^n \text{ for } r \ll \lambda_n$$

(2)

It should be noticed that this potential does not saturates as the Yukawa potential at zero distances and "produces dramatic deviation from Newtonian gravity". The model predict a compactification radius $\lambda_2 \sim 10^{-4} - 10^{-3} m$ which make it a matter of an experimental test.

Up to now, there is experimental evidence that the inverse square law for gravitational interactions is well tested for separations $r$ between $10^{-2} m$ and $10^{15} m$ where $\alpha$ is known to be smaller than $10^{-4}$. A detailed description of the experimental works can be found in reference [2]. The best limits on $\alpha$ in the region around $\lambda \sim 10^{-4} m$ have been very recently provided by the experiment described in reference [4]. This experiment relies on a torque pendulum, the motion of which would be perturbed in a specific way by non-Newtonian potentials from a test mass. Separation between $200 \mu m$ and $10 mm$ have been achieved with a null result. The present day constraint, therefore, for $\lambda \sim 10^{-4} m$ is $\alpha \leq 100$ [4].

Up to now all the performed experiments share common features from technical constraints and scaling considerations. In particular, they are macroscopic in the sense that both the probe and test masses dimensions are in the centimeter range.

In the following, we will show that atom interferometry, that uses a microscopic probe, could explore different geometries and distances ranges. So let us now turn to the basics of atom interferometry.

**II. ATOM INTERFEROMETRY**

Atom interferometry aims at making interferences with de Broglie waves associated to the external motion of massive particles. It was born in the late 80’s when mechanical effects of light were extensively studied which lead the principal contributor, W. D. Philips, S. Chu and C. Cohen Tannoudji to Nobel price in 1997. The main difference with optical interferometry lies in the fact that the particle used (atoms or molecules) possess internal degree of freedom and mass. That make the interferometer sensitive to external fields and inertial effects we are concerned with. For an introductory review the reader is referred
to [4] and for more details to [7]. As we want to probe distance dependent potential we
will concentrate on an idealized separated arms interferometer as outlined in Fig. [1]. An
atom beam is emitted from a source S and passes through a three identical beam splitters
$G_{1,2,3}$ separated by a distance $L$. At each beam splitter, each incoming beam is coherently
divided in two parts separated by an angle $\theta$ (labels $A, O, O', B$ on Fig. [1]). Only one of
the closed paths is shown on Fig. [1] at the end of which the interference pattern is recorded
on a detector $D$. The detection scheme obviously depends on the particles used such as
alkaline ($Li, Na, Rb, Cs$) atoms or molecules, earth-alkaline metastable atoms ($Mg^*, Ca^*$),
metastable rare gas atoms ($He^*, Ar^*, Ne^*$), molecules ($I_2, Na_2, C_{60}$).

In the same way, the actual shape of the interferometer depends on the very technique
used to construct the beam splitters. The example chosen above corresponds to grating
interferometers: atoms, whose de Broglie wavelength is $\lambda_{dB}$, interacting with a modulated
perturbation of period $\Lambda$, are diffracted at a typical diffraction angle $\theta \sim \frac{\lambda_{dB}}{\Lambda}$. Such a
perturbation can be for example a material grating or a laser standing wave.

In ideal conditions, the signal recorded by the detector is then $I(\delta\phi) = I_0 \cos \delta\phi$ where
$I_0$ equals to the atom flux times the interaction time and $\delta\phi$ is the phase difference between
the two arms.

In the following, to evaluate the phase difference induced by gravitational interactions,
we will consider a semi-classical situation in which the atom velocity $v$ and mass $M$ are high
enough so that its de Broglie wavelength $\lambda_{dB} = \frac{\hbar}{Mv}$ is small compared to any relevant length
characterizing the potential $V(r, t)$ existing in between the beam splitters. This potential
will also be considered as a perturbation of the free motion of the particles which thus
fly in straight line between the source, beam splitters and detector. Then, the phase shift
accumulated by the particle with respect to the unperturbed case can be written as [8], [9]:

$$\phi = \frac{1}{\hbar} \int_{\Gamma_{cl.}} \frac{dr}{v} V(r, t(r))$$

(3)

where $\Gamma_{cl.}$ is the classical unperturbed path of the particle. The phase difference simply
reads:

$$\delta\phi = \frac{1}{\hbar} \int_{AO'_{B}} \frac{dr}{v} V(r, t(r)) - \frac{1}{\hbar} \int_{AO_{B}} \frac{dr}{v} V(r, t(r)) = \oint_{AO'_{BOA}} \frac{dr}{v} V(r, t(r))$$

(4)

To effectively compute this phase difference and thus the expected signal one has specify
the geometry of the test mass to evaluate the generated gravitation potential.

III. GEOMETRY CONSIDERATIONS AND PHASE CALCULATIONS

As explained in reference [2], one should use small size test mass to get as near as possible
and thus increase the relative sensitivity to non-newtonian potentials. Nevertheless, as we
investigate a totally different experimental technique, new possibilities are opened. Obvi-
ously, a linear ($1D$) test mass parallel to the atomic trajectory is preferable to a point-like
($0D$) one as it increases the interacting time with the probe. One the other hand the use
of a plane ($2D$) test mass is unfavourable for two reasons [10]. First, as the potentials of
interest are of short range, only a stripe of width $\sim \lambda$ parallel to the atomic trajectory will
contribute significantly to the interactions so that the 1D case give a reasonable approximation. Second, an atom near a surface is subjected to the Van der Waals potential that will screen the other ones. On the contrary, one might expect that the electrostatic image of the atom by a 1D-wire is weaker than by a 2D-plane leading to a smaller Van der Waals interaction that thus will be neglected in the following. To be exhaustive, one should go into the higher dimensions cases. The 3D case of typical size $R$ is also unfavourable because only a volume $R\lambda^2$ has to be taken into account. However, as suggested in Eq. [3], the use of time-dependent potentials (e.g. an oscillating test mass) open the way to fruitful phase sensitive detection techniques.

We thus chose for the calculations an idealized wire test mass of linear density $\mu$, of length $L$, located between the two first beam splitters at a distance $d$ from the lower partial beam and parallel to it (see Fig. [1]). Let us take $G_2$ as the $x$-axis of a reference frame and its intersection with the $z$-axis as the origin. Assuming a Yukawa extra interaction, the potential at the position $(x,z)$ is:

$$V^{1D}_Y(x,z) = \int_{-L-z}^{-z} -\frac{\alpha_Y GM}{\sqrt{(d+x)^2 + u^2}} \mu \, du \, \exp\left(-\frac{1}{\lambda_Y} \sqrt{(d+x)^2 + u^2}\right)$$

In the preceding expression, $|d+x|$ is of the order of $\lambda_Y$, thus in the sub-millimeter range, whereas $L$ is the length of the test mass which is related to the length of the apparatus which is commonly in the meter range. So, except for negligible fringe effects, the limits of the integral may be rejected to infinity as the exponential term to be integrated quickly decreases. Then, with the variable changes $v = \frac{u}{|d+x|}$ and $v = \sinh w$:

$$V^{1D}_Y(x,z) \approx -\alpha_Y GM\mu \int_{-\infty}^{+\infty} dw \, \exp\left(-\frac{|d+x|}{\lambda_Y} \cosh w\right) = -\alpha_Y GM\mu 2K_0\left(\frac{|d+x|}{\lambda_Y}\right)$$

where $K_0$ is the modified Bessel function of first kind [12].

Between the two first gratings ($z < 0$), the two partial beams can be parameterized as $x_2(z) = 0$ and $x_1(z) = e(1 + \frac{z}{L})$. So the phase delay accumulated along the lower path simply reads $\phi_2 = \frac{GMm}{\hbar v} 2\alpha_Y K_0\left(\frac{d}{\lambda_Y}\right)$, where $m = L\mu$ is simply the test mass. The phase delay accumulated on the upper path $\phi_1$ contains the mean value of the $K_0$ function which has a rather complicated analytical expression involving hypergeometric and other Bessel functions. For sake of simplicity, we will consider only the limiting case where the beam separation $e$ is much greater than the interaction range $\lambda$. Then the potential is practically negligible everywhere along the upper path and so for $\phi_1$. The phase difference is then simply $\delta\phi = \phi_2 - 0$:

$$\delta\phi^{1D}_Y = \frac{GMm}{\hbar v} 2\alpha_Y K_0\left(\frac{d}{\lambda_Y}\right) \text{ if } e \gg \lambda_Y$$

From Ref. [11] we get the following expansions: $\delta\phi^{1D}_Y \sim \sqrt{e} e^{-\frac{d}{\lambda}}$ as $\frac{d}{\lambda} \ll 1$ and $\delta\phi^{1D}_Y \sim \ln \frac{d}{\lambda}$ as $\frac{d}{\lambda} \gg 1$

The same kind of calculations can be done for the case of extra dimensions. Some difficulties arise because the law we have (Eq. [3]) is valid only for distances shorter than the compactification radius $\lambda_n$. For longer distances, the potential decreases exponentially fast
in a Yukawa fashion \[1\] and should give a negligible contribution. Let’s make the assumption that \(d \ll \lambda_n\). As done before we will suppose that \(e \gg \lambda\) to neglect the contribution of the upper path. We will get an estimate of the effect on the lower path restricting the integration range to a distance \(\lambda_n\) around the position of the particle. Under these assumptions, and neglecting fringe effects, the potential arising from \(n > 0\) extra dimensions reads:

\[
V_{nD}^1(x, z) = \int_{-\lambda_n}^{+\lambda_n} \frac{\alpha_n GM}{\sqrt{(d + x)^2 + u^2}} \mu \left( \frac{\lambda_n}{\sqrt{(d + x)^2 + u^2}} \right)^n \right. \\
= \alpha_n GM\mu \left( \frac{\lambda_n}{|d + x|} \right)^n \int_{-\frac{\lambda_n}{|d + x|}}^{\frac{\lambda_n}{|d + x|}} dv \left( 1 + v^2 \right)^{-\frac{1+n}{2}} \\
= -2\alpha_n GM\mu \left( \frac{\lambda_n}{|d + x|} \right)^n \, _2F_1 \left( \frac{1}{2} \left( n + 1 \right), \frac{1}{2} \left( 3 - \left( \frac{\lambda_n}{d + x} \right)^2 \right) \right) \tag{8}
\]

where \(2F_1\) denotes the hypergeometric function \[13\]. The phase difference then reads:

\[
\delta \phi_{nD}^1 = \frac{GMm}{\hbar v} 2\alpha_n \left( \frac{\lambda_n}{d} \right)^{n+1} \, _2F_1 \left( \frac{1}{2}, \frac{1+n}{2}, \frac{3}{2}, -\left( \frac{\lambda_n}{d} \right)^2 \right) \text{ if } e \gg \lambda_n \text{ and } n > 0 \tag{9}
\]

and, in particular,

\[
\delta \phi_{2D}^1 = \frac{GMm}{\hbar v} 2\alpha_2 \left( \frac{\lambda_2}{d} \right)^3 \left( 1 + \left( \frac{\lambda_2}{d} \right)^2 \right)^{-\frac{1}{2}} \tag{10}
\]

One can show that, assuming that \(m\) is a constant, in the region \(d \leq \lambda\) were the model holds that \(\delta \phi_{nD}^1 \sim \left( \frac{d}{\lambda} \right)^n\) when \(d \ll \lambda\).

Nevertheless, the newtonian potential \(n = 0\) has infinite range and thus must be treated separately. As the distance between the particle and the wire is much smaller than its length, one can take the infinite length of the wire limit \[14\]. So, from Gauss theorem, the gravitational field is \(g(r) = -2GM \frac{r}{r^2} = -\nabla \left( 2GM \ln \frac{r}{d} \right)\) so that the potential is null on the lower path. The potential energy is then:

\[
V_{\text{newton}}^1(x) = 2GM\mu \ln \left( 1 + \frac{x}{d} \right) \tag{11}
\]

The newtonian phase difference is then:

\[
\delta \phi_{\text{newton}}^1 = \frac{GMm}{\hbar v} 2 \ln \left( 1 + \frac{e}{d} \right) \tag{12}
\]

With all these quantitative expressions for the different phases we can now turn to numerical estimates for a reference atom interferometer.
IV. NUMERICAL RESULTS

In all cases, the phase difference is $\frac{GMm}{\hbar v}$ times a numerical factor which can be inferred from dimensional considerations. The numerical factor depends on the particular geometry used and the presumed force law. It should be noticed that this numerical factor involves dimensionless ratios as $\alpha, \lambda_d$ that can be great. Anyway, let us first give a numerical estimate of the pre-factor, taking a slowed cesium beam $v = 10 \text{ m.s}^{-1}$, $M = 2 \times 10^{-25} \text{ kg}$, and a wire $1m \times 100 \mu m \times 100 \mu m$ of density equal to 20 (gold) so that $m = 2 \times 10^{-4} \text{ kg}$. The pre-factor then amounts to some $10^{-6}$. The detection limit $\delta \phi_{\text{min}} = 10^{-3} \text{ rad}$ is commonly accepted for atom interferometry (see different contributions in [7]). It first confirms the known fact that the metrology on the regular newtonian gravity (i.e. a measurement of $G$) is practically impossible as even with $d = 10 \mu m$, $\ln(1 + \frac{d}{\lambda})$ will amount to some units so $\phi_{\text{newton}}$ (Eq.12) will be to small to be detected.

One can anyway evaluate the upper bounds on the strength of the non-newtonian interaction in the case of a hypothetical null experiment based on our calculation. The results are shown in Fig. 2 together with limits taken from Ref. [4]. They were obtained assuming that the actual wire diameter equals the distance from the wire to the atoms $d$ so that the 1D model is at the limit of acceptability. The upper part corresponds to $d = 100 \mu m$ so that, for $\lambda < d$, the wire is beyond the compactification radius for the models with extra dimensions. The curves have then been extrapolated the curve given by the Yukawa potential. $d = 10 \mu m$ is shown in the lower part for which such a problem do not arise since we have chosen the same parameters range as in Ref. [4] for sake of comparison. In the same way, the beam separation $e \sim 10^{-3} \text{ m}$ with the chosen parameter set. The limits are thus overestimated in the somehow low interest region $\lambda > 10^{-3} \text{ m}$.

While a real experiment based on this computation looks very difficult for several reason, the figure clearly shows that there is an interest in further studies of the methodology we described. For short ranges, $\lambda < 10^{-4} \text{ m}$, the method appear particularly efficient even if one uses a less sophisticated atomic source such as a supersonic beam for which $v \sim 10^3 \text{ m.s}^{-1}$. In the end, the main free parameter in the simulations is $d$ and has to be chosen with respect to the potential model of interest and its suspected range. When $d$ decreases the different potentials increase but the test mass scales typically as $d^2$ and vice versa. For the Yukawa case, an optimum is numerically, and not surprisingly, found for $d \sim \lambda$ for the typical parameters ranges we are concerned with. On the contrary for n extra dimensions the overall phase scales as $d^2 \left(\frac{\lambda}{d}\right)^n \sim d^{2-n}$. The case $n = 2$ thus exhibits a weak dependency on the choice of the wire diameter whereas one should use as small as possible a wire if $n \geq 3$. On the contrary, if the $n = 1$ case is under investigation, one should use the largest acceptable wire that is $d \sim \lambda$. These behaviors can be checked on the figure where the limits for $n = 1, 2, 3$ get worse, remain roughly unchanged or improve from $d = 100 \mu m$ (upper graph) to $d = 10 \mu m$ (lower graph).

V. CONCLUSIONS

The results presented here concerning non-newtonian gravitation are based on a completely different method than the existing ones that use macroscopic probes. Our computation show that atom interferometry could provide limits on the strength $\alpha$ of several extra
potentials comparable with the existing ones in a reasonable integration time, especially if they are of short range $\lambda < 10^{-4}m$. More, it can be easily extended to test a composition dependent interaction ("fifth-force") using, for example, as test masses two wires of different materials or two isotopes such as $^6Li$ and $^7Li$ for the probe atom. Using polarized beams spin dependent potentials could be also taken into account.

A detailed study of an actual experimental set-up is far beyond the scope of such a letter; a rapid realization of such an experiment looks anyway difficult because of some experimental critical points. In particular the short range interaction implies high collimated beams which results in low count rates and moreover to increase sensitivity challenging beam slowing techniques are necessary.

We are anyway confident that, being atom optics a fast developing field, in the near future most of the experimental difficult points will be clarified and atom interferometry will contribute to the tests of non-newtonian gravitation.

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[13] In fact, for integer $n$, $2F_1 \left(\frac{1}{2}, \frac{1+n}{2}, \frac{3}{2}, -x^2\right)$ has simple analytical expression as it can be verified integrating directly the potential by recurrence. See [12] Eq. 2.271.6 and Eq. 2.148.4:
\[
\begin{align*}
\int dx \ (1 + x^2)^{-\frac{2n+1}{2}} &= \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{x^{2k+1}}{(1+x^2)^{2k+1}} \\
\int dx \ (1 + x^2)^{-n} &= \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)...(2n-2k+1)}{(n-1)(n-2)...(n-k)(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!} \arctan x
\end{align*}
\]

[14] Here again, exact analytical expressions taking into account the finite length of the wire exist. We get $V_{\text{newton}}^{1D}(x, z) = 2GM\mu[\arcsinh \frac{L}{2d} + \arcsinh \frac{x}{d+x} - \arcsinh \frac{x}{d+x}]$. The corresponding phases can then be easily obtained from computer algebra but are so intricate that only the limiting case presented above seem to have practical interest.
FIGURES

FIG. 1. General scheme of a separated arms interferometer of overall length $2 \times L$ and maximum separation $e$. $S$ and $D$ represent respectively the source and the detector. $G_1$, $G_2$ and $G_3$ are the beam splitters that divide and recombine the atomic beam. The wire, of diameter $d$ is set parallel and at the distance $d$ of the first part of the lower arm. As an illustrative example pp. 1-83, a three gratings Mach-Zehnder interferometer of D. Pritchard’s group, MIT, uses nanofabricated gratings of period $\lambda = 200\text{nm}$ separated by $L = 0.6\text{m}$. The source is a rare gas seeded supersonic beam of sodium. Their de Broglie wavelength $\lambda_{dB} = \frac{h}{mv} \sim 16\text{pm}$. It corresponds to a diffraction angle $\theta = \frac{\lambda_{dB}}{\Lambda}$ about $80\mu\text{rad}$. The beam spacing $e \sim 50\mu\text{m}$ is big enough so that a septum can be inserted between the two arms. The signal is then recorded on an hot wire detector.

FIG. 2. Comparison of results presented in Ref. [4] (heavy lines) and detection limits of an atom interferometer assuming a Yukawa potential (thin line) or $n$ extra dimensions $n = 1$ (dash), $n = 2$ (dot) or $n = 3$ (dot dash). The vertical axis, generically labeled $|\alpha|$, represents either $\alpha_Y$ either $\alpha_{1,2,3}$ depending on the chosen scenario. Results in [4] assume a Yukawa extra potential. The upper and lower parts correspond respectively to different wire diameters and beam/wire distances $d = 100\mu\text{m}$ and $d = 10\mu\text{m}$. In the former case, the results for extra dimensions models have been extrapolated to the Yukawa model by a dotted line in the region $d \geq \lambda$. 

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