Primitive recursive functions versus partial recursive functions: comparing the degree of undecidability

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Abstract

Consider a decision problem whose instance is a function. Its degree of undecidability, measured by the corresponding class of the arithmetic (or Kleene-Mostowski) hierarchy hierarchy, may depend on whether the instance is a partial recursive or a primitive recursive function. A similar situation happens for results like Rice Theorem (which is false for primitive recursive functions). Classical Recursion Theory deals mainly with the properties of partial recursive functions.

We study several natural decision problems related to primitive recursive functions and characterise their degree of undecidability. As an example, we show that, for primitive recursive functions, the injectivity problem is $\Pi^0_1$-complete while the surjectivity problem is $\Pi^0_2$-complete.

We compare the degree of undecidability (measured by the level in the arithmetic hierarchy) of several primitive recursive decision problems with the corresponding problems of classical Recursion Theory. For instance, the problem “does the codomain of a function have exactly one element?” is $\Pi_1$-complete for primitive recursive functions and belongs to the class $\Delta^0_2 \setminus (\Sigma^0_1 \cup \Pi^0_1)$ for partial recursive functions.

An important decision problem, “does a given primitive recursive function have at least one zero?” is studied in detail; the input and output restrictions that are necessary and sufficient for the decidability this problem – its “frontiers of decidability” – are established.

We also study a more general situation in which a primitive recursive function (the instance of the problem) is a part of an arbitrary “acyclic primitive recursive function graph”. This setting may be useful to evaluate the relevance of a given primitive recursive function as a part of a larger primitive recursive structure.

Keywords: primitive recursion; undecidability; Recursion Theory.
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1 Introduction

The primitive recursive (PR) functions form a large and important enumerable subclass of the recursive (total computable) functions. Its vastness is obvious from the fact that every total recursive function whose time complexity is bounded by a primitive recursive function is itself primitive recursive, while its importance is well expressed by the Kleene Normal Form Theorem (see [22, 4, 35]) and by the fact that every recursively enumerable set can be enumerated by a primitive recursive function.\(^2\)

The meta-mathematical and algebraic properties of PR functions have been widely studied, see for instance [22, 42, 40, 51, 43]. Special sub-classes (such as unary PR functions, see for instance [41, 45]) and hierarchies (see for instance [17, 1, 31, 33, 3, 16] and [54, Ch. VI]) of PR functions were also analysed.

Efficiency lower bounds of primitive recursive algorithms have also been studied, see [6, 7, 9, 10, 33, 34, 14, 53, 52, 29].

In this work we concentrate on decidability questions related to PR functions.

Many decision problems related to PR functions are undecidable. For instance, the question “given a PR function \(f(x)\), is there some \(x\) such that \(f(x) = 0\)” is undecidable (\(\mathbb{E}\)). In this paper we show that many other questions related to PR functions are undecidable, and classify their degree of undecidability according to the arithmetic hierarchy. Let us mention two examples of the problems studied in this work: “does a given PR function have exactly one zero?” (class \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\)), and “is a given PR function surjective (onto)?” (\(\Pi_2\)-complete).

The undecidability of some of these problems has undesirable consequences. For instance, when studying reversible computations, it would be useful to have a method for enumerating the bijective PR functions. However, although the set of PR functions is recursively enumerable, the set of bijective PR functions is not: it belongs the \(\Pi_2\)-complete class.

We also study classes of PR problems in which a PR function \(f\) is a part of a larger PR system \(S\). We obtain a simple decidability condition for the case in which

\[
S(f, \overline{x}) = h(f(g(\overline{x}))),
\]

where \(g\) and \(h\) are fixed (total) recursive functions (\(g\) may have multiple outputs), but not for the case illustrated in Figure 1 (page 4). It is interesting to compare (1) with the result of applying the Normal Form Theorem to this more general case (see page 31)

\[
S(f, \overline{x}) = h(\overline{x}, f(g(\overline{x}))).
\]

Organisation and contents of the paper.

The main concepts and results needed for the rest of the paper are presented in Section 2 (page 5).

The rest of the paper can be divided in two parts.

Part I, specific problems

In Section 3 (page 10) several natural decision problems associated with PR functions are studied and compared (in terms of the arithmetic hierarchy level) with the corresponding partial recursive
Figure 1: A PR function \( f \) may be considered an important part of a complex structure \( S \) if the following problem is undecidable: \textbf{instance:} \( f \), \textbf{question:} “\( \exists \overline{x}: S(f, \overline{x}) = 0 \)?”. We will show that an arbitrary acyclic system \( S \) containing exactly one occurrence of \( f \) can always be reduced to the normal form illustrated in Figure 8 (page 34).

problems. The list of problem includes: the existence of zeros, the equivalence of functions, “domain problems”, and problems related with the injectivity, surjectivity, and bijectivity of PR functions. For each of these problems the degree of undecidability is established in terms of their exact location in the in the arithmetic hierarchy; see Figures 2 (page 18) and 3 (page 19). The problems we study are either recursive, complete in \( \Sigma_n \) (for some \( n \geq 1 \)), complete in \( \Pi_n \) (for some \( n \geq 1 \)) or belong to the class \( \Delta_2 \setminus (\Sigma_1 \cup \Pi_1) \).

Part II, classes of problems

- Section 4 (page 24): study of the “frontiers of decidability” of the problem “does a given PR function has at least one zero?”. More precisely, we find the input and output restrictions that are necessary and sufficient for decidability; these restrictions are obtained by pre- or post-applying fixed PR functions to the PR function which is the instance of the problem.

- Section 5 (page 30): two more general settings are studied:

  - Section 5.1 (page 30): the PR function \( f \), instance of a decision problem, is located somewhere inside a larger acyclic PR system \( S \) (also denoted by \( S(f, \overline{x}) \)), see Figure 1 (page 4). In algebraic terms this means that \( S \) is obtained by \textit{composing} fixed PR functions (or, more generally, fixed recursive functions) with the instance \( f \). An example of such a system \( S \) can be seen in see Figure 7 (page 32). The class of problems, parameterised by \( S \) is: given a PR function \( f \), is there some \( \overline{x} \) such that \( S(f, \overline{x}) = 0 \)? When this problem is undecidable the PR function \( f \) can be considered an “important part” of \( S \).

  - Section 5.2 (page 34): The problem of composing the PR function \( f \) (the instance of the problem) with itself is briefly studied.

- In Section 6 (page 37) we compare the degree of undecidability of the partial recursive and primitive recursive versions of the same decision problem, see Figure 11 (page 39) for \( n \geq 2 \) assign the level \( n - 1/2 \) to the class \( \Delta_n \setminus (\Sigma_{n-1} \cup \Pi_{n-1}) \). We conjecture that there exist problems whose difference of levels in the arithmetic hierarchy is 0, 1, 3/2, 2, 5/2, 3, 7/2...
(See Figure 12, page 41). In the general case it may be difficult or impossible to determine the difference of levels because the PR level is influenced by the particularities of the PF functions.

2 Preliminaries

The notation used in this paper is straightforward. “If” denotes “if and only if” and “≡” denotes “is defined as”. The logical negation is denoted by “¬”. The complement of the set $A$ is denoted by $\overline{A}$, while the difference between the sets $A$ and $B$ is denoted by $A \setminus B$. A total function $f : A \to B$ is injective (or one-to-one) if $x \neq y$ implies $f(x) \neq f(y)$; it is surjective (or onto) if for every $y \in B$ there is at least one $x \in A$ such that $f(x) = y$; and it is bijective if it is both injective and surjective. The set $\{x : f(x) = y\}$ is denoted by $f^{-1}(y)$. For $n \geq 0$, the sequence of arguments “$x_1, x_2, \ldots, x_n$” is denoted by $\overline{x}$. Using a standard primitive recursive bijection $(x_1, x_2)$ of $\mathbb{N}^2$ in $\mathbb{N}$ we can represent a sequence $\overline{x}$ of integers $x_1, x_2, \ldots, x_n$ by a single integer $x = \langle x_1, x_2, \ldots, x_n \rangle$; the sequence $\overline{x}$ will often be denoted by $x$.

**Definition 1.** Let $\overline{x} = \langle x_1, \ldots, x_n \rangle$ and $\overline{y} = \langle y_1, \ldots, y_m \rangle$; by $\overline{y} = f(\overline{x})$ we mean that the tuple $\overline{y}$ is a function of the tuple $\overline{x}$. We say that $f$ is a multiple function or “multifunction”.

The $S_{mn}$ Theorem [22, 11, 44] will often be used without mention.

2.1 Languages, reductions and the arithmetic hierarchy

We say that the functions $f$ and $g$ are equal and write $f = g$ if for every integer $x$ either $f(x)$ and $g(x)$ are both undefined, or they are both defined and have the same value.

The terms “recursive function”, “total computable function”, and “total function” will be used interchangeably. PR denotes “primitive recursive”, ParRec denotes “partial recursive”. The ParRec function corresponding to the index $e$ is $\varphi_e$. Its domain is denoted by $W_e$. “Recursively enumerable” is abbreviated by r.e.

**Injectivity, surjectivity, bijectivity**

PR functions are total (computable) so that the definition at the beginning of Section 2 apply. We use the following definition for ParRec functions.

**Definition 2.** Let $\varphi_e$ be a partial recursive function.

$\varphi_e$ is injective if its restriction to $W_e$ (a total function) is injective.

$\varphi_e$ is surjective if its restriction to $W_e$ (a total function) is surjective.

$\varphi_e$ is bijective if it is total, injective and surjective.

Thus a ParRec functions can be injective and surjective without being bijective.

There are other reasonable definitions of bijectivity for ParRec functions. For instance, $\varphi_e$ is bijective if its restriction to $W_e$ is bijective, as a total function. The result stated in item [14, page 23] is true for both definitions.
Kleene Normal Form Theorem

The Kleene Normal Form Theorem \[44, 22, 35\] states that there is a PR predicate \(T\) and a PR function \(U\) such that for every ParRec function \(f(x)\) there is an index \(e\) satisfying \(\varphi_e(x) = f(x) = U(\mu_h: T(e, x, h))\), where (i) the predicate \(T(e, x, h)\) checks if the integer \(h\) codes a (halting) computation history of \(f(x)\) and (ii) the function \(U(h)\) extracts the final result from the computation history \(h\).

**Theorem 1** (A consequence of the Kleene Normal Form Theorem). Let \(U(e, x, t)\) be a Turing machine that executes \(t\) steps of the computation \(\varphi_e(x)\), printing \(\varphi_e(x)\) whenever \(\varphi_e(x)\) is defined and \(t\) is sufficient large. Let \(T(e, x, t)\) be a Turing machine that executes \(t\) steps of the computation \(\varphi_e(x)\), returning 0 if the computation halted in \(\leq t\) steps, and 1 otherwise (both \(U\) and \(T\) always halt). The number of steps needed for the computation of a halting computation \(\varphi_e(x)\) is thus \(\mu_t: T(e, x, t)\).

There are recursive functions \(T\) and \(U\) such that for every ParRec function \(f\) there is an index \(e\) satisfying \(\forall x: [f(x) = U(e, x, \mu_t: T(e, x, t))]\).

The function \(T(e, x, t)\) mentioned in Theorem 1 will often be used in this work:

**Definition 3.** Let \(\varphi_e(x)\) be the partial recursive function with index \(e\). Let \(T(e, x, t)\) be a Turing machine corresponding to the function

\[
T(e, x, t) = \begin{cases} 
0 & \text{if the computation of } \varphi_e(x) \text{ converges exactly at step } t \\
1 & \text{otherwise.}
\end{cases}
\]

In most cases it does not matter if we replace “converges exactly at step \(t\)” by “converges at some step \(\leq t\)”. For every integer \(e\), the function \(T_e(x, t) = T(e, x, t)\) is assumed to be primitive recursive. When we do not need to mention the index \(e\), we use the function \(F(x, t) \overset{\text{def}}{=} T_e(x, t)\).

Sometimes we will view the functions \(F\) and \(T\) as predicates, interpreting the value 1 as “false” (does not halt) and the value 0 as “true” (halts).

\[\square\]

**Arithmetic hierarchy**

The arithmetic hierarchy (see for instance \[35, 44, 22, 25, 24, 23, 19\]) is used in this paper with the purpose of measuring the degree of undecidability of PR decision problems. This is usual in Recursion Theory, see for instance its use for rewriting problems studied in \[12, 48\]. This is similar to the classification of the difficulty of a (decidable) problem in the polynomial hierarchy (PH). As we will not mention other hierarchies, the superscript 0 in the classes of the AH will be dropped. For instance, we will write \(\Pi_2\) instead of \(\Pi^0_2\). A suitable background on the AH can be found, for instance, in references \[18, \text{Chapter 7 (§29)}\], \[44, \text{Chapter 14}\] \[3, \text{Chapter 10}\] , and \[49, \text{Chapter IV}\]. In particular, for \(n \geq 1\) we have the proper inclusions \(\Delta_n \subset \Sigma_n\) and \(\Delta_n \subset \Pi_n\) (Hierarchy Theorem \[49\]).

We denote by \(\Delta\) the set \(\Sigma_0 = \Pi_0 = \Delta_0 = \Delta_1\). Thus the class of recursively enumerable, but not recursive sets is \(\Sigma_1 \setminus \Delta\).

Some classes of the AH are briefly described below.
### AH class | Decision problem
--- | ---
Δ | Decidable
Σ₁ | Semi-decidable
Σ₁ ∖ Δ | Undecidable, semi-decidable
Π₁ | Complement of a semi-decidable problem
Π₁ ∖ Δ | Undecidable, complement of a semi-decidable problem
Δ₂ = Σ₂ ∩ Π₂ | Decision problems that can be described both as “∃x ∀y : P(x, y)” and as “∀x ∃y : Q(x, y)” where P and Q are recursive predicates (x and y represent tuples of variables).

The class Δ₂ has been studied in detail, see for instance [35, Section IV.1], [36, Chapter XI], and [39]; see also [16, 50, 2, 28]. Another characterisation of Δ₂ is Shoenfield’s Limit Lemma [46]; from [35, page 373]: a set A is in Δ₂ if and only if its characteristic function is the limit of a recursive function g, i.e. cₐ(x) = \( \lim_{s \to \infty} g(x, s) \).

**Definition 4** (Many-one reduction). The language L reduces (many-one) to M, \( L \leq_m M \), if there is a recursive function \( f: \mathbb{N} \to \mathbb{N} \) such that \( x \in L \) if and only if \( f(x) \in M \).

**Property 1.** If \( L \leq_m M \) then for every integer \( n \in \mathbb{N} \): (i) \( \Gamma \leq_m M \), (ii) \( M \in \Sigma_n \implies L \in \Sigma_n \), (iii) \( M \in \Pi_n \implies L \in \Pi_n \). (For \( n = 0 \) we get: if M is recursive, then L is recursive.)

A set P is m-complete in the class \( C \) if \( P \in C \) and \( Q \leq_m P \) for every \( Q \in C \). We will often say “complete” instead of “m-complete”. A set P is m-hard in the class \( C \) if \( P \in C \) and \( Q \leq_m P \) for every \( Q \in C \).

See also the definitions and results at the beginning of Section 6, page 37. The structure of the arithmetical hierarchy classes has been widely studied, see for instance [38, 23, 15, 46, 55, 56, 27, 44, 35, 36, 49].

### 2.2 Decision problems

The decision problems related to partial recursive functions will be used for two purposes: to prove the completeness of PR problems (using Theorem 2 below), and to compare the degree of undecidability of PR decision problem with the corresponding problem of classical Recursion Theory, see Section 3.4, page 19.

**Notation.** The names of the decision problems whose instance is a PR function (the main subject of this paper) have the superscript “PR”. If the instance is a ParRec function, there is no such superscript. Examples of the former and the latter are EXACTLY-ONE-ZEROPR and FINITE-DOMAIN, respectively.

**Definition 7** (page 37) characterises the PR decision problem that corresponds to a given ParRec problem.

We list some important ParRec decision problems, see [4, 18, 35, 11].

**Problem 1.** HP, halting problem.

Instance: a Turing machine \( T \) and a word \( x \).

Question: does the computation \( T(x) \) halt?
Problem 2. SHP, self halting problem.
   Instance: a Turing machine $T$.
   Question: does the computation $T(i)$ halt, where $i$ is a Gödel index of $T$?

Problem 3. PCP, Post correspondence problem.
   Instance: two finite sets of non-null words, $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, defined over the same finite alphabet with at least two letters.
   Question: is there a finite non-null sequence $i_1, \ldots, i_k$ such that $x_{i_1}x_{i_2}\ldots x_{i_k} = y_{i_1}y_{i_2}\ldots y_{i_k}$?

Problem 4. TOTAL.
   Instance: a ParRec function $f(x)$.
   Question: Is $f(x)$ defined for every $x$?

Problem 5. FINITE-DOMAIN.
   Instance: a ParRec function $f(x)$.
   Question: is the domain of $f(x)$ finite?

Problem 6. COFINITE-DOMAIN.
   Instance: a ParRec function $f(x)$.
   Question: is the domain of $f(x)$ cofinite?

Problem 7. EQUIVALENCE.
   Instance: ParRec functions $f(x)$ and $g(x)$.
   Question: Is $f = g$ (same function)?

Theorem 2.
FINITE-DOMAIN is $\Sigma_2$-complete. TOTAL is $\Pi_2$-complete. COFINITE-DOMAIN is $\Sigma_3$-complete.
EQUIVALENCE is $\Pi_2$-complete.

Proof. For the problems FINITE-DOMAIN, TOTAL, and COFINITE-DOMAIN see for instance [10].
Let $e$ and $e'$ be the instance of EQUIVALENCE. The equality of the functions $\varphi_e$ and $\varphi_{e'}$ can be expressed as follows

\[ \forall x : (\forall t_1 : T(e, x, t_1) = T(e, x, t_1) = 1) \lor (\exists t, t' : (T(e, x, t) = T(e, x, t') = 0) \land \uparrow(e, x, t) = \uparrow(e', x, t')) \]

(see Definition[11] page[10]).
This can be expressed in the form $\forall x, t_1 \exists t, t' : \ldots$, so that the problem EQUIVALENCE is in $\Pi_2$.
To prove completeness we reduce the $\Pi_2$-complete problem TOTAL to EQUIVALENCE. Given the ParRec function $f(x)$, define the ParRec function $g(x)$ as:
1) compute $f(x)$
2) if the computation halts, output 0 (otherwise the result is undefined).
Clearly $g(x)$ is equivalent to the zero function $\varnothing(x)$ iff $f(x)$ is total. □
2.3 Models of computation

Deterministic finite automata (DFA), Turing machines (TM), and context free grammars (CFG) and examples of “models of computation”. A fundamental assumption associated with the intuitive idea of “model of computation” is the following.

Assumption 1. The instances of a “model of computation” are recursively enumerable.

For instance, we may conclude that no model of computation characterises the set of recursive (total) functions because that set is not enumerable (see for instance [37]).

The “dovetailing” technique

As an illustration of this technique consider a recursively enumerable set of Turing machines $M_1$, $M_2$, ... and suppose that we want to list the outputs of all the computations $M_i(x)$ for some fixed $x$. That is, for every $i$, if the computation $M_i(x)$ halts, its output is listed. This can be effectively done by the following method:

- Successively simulate one step of computation for each machine, in the following order:

| M: | M_1 | M_2 | M_3 | M_4 | M_5 | M_6 | M_7 | M_8 | M_9 | M_10 | M_11 | ... |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|-------|------|
| MS: | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10    | 11    |      |
| GS: | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10    | 11    |      |

where M, MS, and GS denote respectively “machine”, “machine step number” and “global step number”. During this simulation, whenever some machine $M_i$ halts, print its output. Note that every computation step of every machine is simulated: either forever or until it halts.

More generally we can for instance use the dovetailing technique to enumerate any r.e. collection of r.e. sets.

2.4 Primitive recursive functions

It is assumed that the reader is familiar with:

- the classical definition of primitive recursive, functions, see [41, 35];

- the use of register languages to characterise those functions (particularly the Loop language, see [31, 32]);

- the PR binary function "−"

\[ x - y = \begin{cases} 
  x - y & \text{if } x \geq y \\
  0 & \text{otherwise.} 
\end{cases} \]

The following property will be useful.

Property 2. Let $a$ and $b$ be arbitrary integers. Then $a = b$ if and only if $(a - b) + (b - a) = 0$. 
3 Decision problems associated with PR functions

In this section we study the decidability of some problems for which the instance is one PR function (or a pair of PR functions). First, in Section 3.1, we study several problems related to the existence of zeros of the PR function $f$ and to the equivalence of the PR functions $f$ and $g$, then, in Section 3.2, some properties related to the codomain of $f$ are studied, and finally, in Section 3.3 we prove results associated to the injectivity, surjectivity, and bijectivity of $f$. In every case it is assumed that the PR function is given by an index (or equivalently by a Loop program). The reader can find a summary of some of these results in Figure 2 (page 18).

In Section 3.4 (page 19) we also consider the case in which the instance is a ParRec function and compare the two cases. Figures 4 (page 20) and 5 (page 25) contain a summary of this comparison.

3.1 Some undecidable problems

For each of the decision problems listed below the instance consists of one or two primitive recursive functions. When the instance has the form $f(x, y)$, where $x$ is given, we can use the $S_{mn}$ Theorem to convert it to the form $g(y)$, where $g$ depends on $x$. Both these forms will be used in the sequel. All the problems are all proved to be either decidable or $\Sigma^1_n$-complete (for some $n \geq 1$) or $\Pi^1_n$-complete (for some $n \geq 1$) or members of the class $\Delta^2_2 \setminus (\Sigma^1_1 \cup \Pi^1_1)$.

Problem 8. HAS-ZEROS$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Is there some $y$ such that $f(\overline{x}, y) = 0$?
Class: $\Sigma^1_1$-complete, Theorem 3 (page 12).

Problem 9. EXACTLY-ONE-ZERO$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Does $f(\overline{x}, y)$ have exactly one zero?
Class: $\Delta^2_2 \setminus (\Sigma^1_1 \cup \Pi^1_1)$, Theorem 4 (page 12).

Problem 10. AT-LEAST-k-ZEROS$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Does $f(\overline{x}, y)$ have at least $k$ zeros? (for fixed $k \geq 1$).
Class: $\Sigma^1_1$-complete, Theorem 5 (page 13).

Problem 11. EXACTLY-k-ZEROS$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Does $f(\overline{x}, y)$ have exactly $k$ zeros? (for fixed $k \geq 2$).
Class: $\Delta^2_2 \setminus (\Sigma^1_1 \cup \Pi^1_1)$, Theorem 6 (page 13).

Many other natural decision problems could have been studied. For space reasons we did not include (i) several simple generalisations and (ii) similar problems (in terms of the degree of undecidability). An example of (i) is “does $f(x)$ have a number of zeros between $m$ and $n$?” (for fixed $m$ and $n$ with $0 < m < n$); this problem generalises problems 9 and 11. An example of (ii) is the following problem which is similar to RECURSIVE (Problem 33, page 40): “given $e$, $\exists e' : e' \in W_e$ and $W_{e'}$ is infinite?” (see [44, §14.8, Theorem XV (page 326)].
Problem 12. EQUAL-NEXT$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Is there some $y$ such that $f(\overline{x}, y) = f(\overline{x}, y + 1)$?
Class: $\Sigma_1$-complete, Theorem 7 (page 13). □

Problem 13. ZERO-FUNCTION$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Is $f(\overline{x}, y)$ the zero function $0(y)$?
Class: $\Pi_1$-complete, Theorem 8 (page 13). □

Problem 14. $\infty$-ZEROS$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Does $f(\overline{x}, y)$ have infinitely many zeros?
Class: $\Pi_2$-complete. Theorem 9 (page 13). □

Problem 15. ALMOST-ALL-ZEROS$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive function $f(\overline{x}, y)$.
Question: Is $f(\overline{x}, y) = 0$ for almost all values of $y$?
Class: $\Sigma_2$-complete, Theorem 10 (page 14). □

Problem 16. EQUAL-AT-ONE-POINT$^\text{PR}$, “functions equal at one point”.
Instance: $\overline{x}$ and the primitive recursive functions $f(\overline{x}, y)$, $g(\overline{x}, y)$.
Question: Is $f(\overline{x}, y) = g(\overline{x}, y)$ for at least one $y$?
Class: $\Sigma_1$-complete, Theorem 11 (page 14). □

Problem 17. EQUIVALENCE$^\text{PR}$.
Instance: $\overline{x}$ and the primitive recursive functions $f(\overline{x}, y)$, $g(\overline{x}, y)$.
Question: Is $f(\overline{x}, y) = g(\overline{x}, y)$ for every $y$?
Class: $\Pi_1$-complete, Theorem 12 (page 14). □

A note on polynomials and the HAS-ZEROS$^\text{PR}$ problems

For each instance $f(x)$ of the HAS-ZEROS$^\text{PR}$ problem there is a polynomial with integer coefficients $p(x_1, \ldots, x_n)$, such that $p$ has a zero iff $f(x)$ has a zero. Let us begin by quoting [13].

In an unpublished and undated manuscript from the 1930s found in Gödel’s Nachlass and reproduced in Vol. III of the Collected Works, he showed that every statement of the form “$\forall x : R(x)$” with $R$ primitive recursive [relation] is equivalent to one in the form

$$\forall x_1, \ldots, x_n \exists y_1, \ldots, y_m : [p(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0]$$

in which the variables range over natural numbers and $p$ is a polynomial with integer coefficients; it is such problems that Gödel referred to as Diophantine in the Gibbs lecture. It follows from the later work on Hilbert’s 10th problem by Martin Davis, Hilary Putnam, Julia Robinson and — in the end — Yuri Matiyasevich that, even better, one can take $m = 0$ in such a representation, when the “$=$” relation is replaced by “$\neq$”. 

The language associated with the Theorem 4.

Proof

We use the following easy to prove fact: for every \(3.1.2\) Results

Thus, given that there is an algorithm that computes \(p\) from \(f\), and relatively to the decidability of the HAS-ZEROS\(^{\text{PR}}\) problem, an arbitrary PR function is no more general than a (multiple variable) integer polynomial.

\[\Theta(x) = \begin{cases} 0 & \text{if } f(x) = 0 \\ 1 & \text{otherwise} \end{cases}\]

Clearly \(f\) has at least one zero iff the PR function \(g\) has exactly one zero. Thus the problem EXACTLY-ONE-ZERO\(^{\text{PR}}\) is not in \(\Pi_1\).
(iii) Not in $\Sigma_1$. Reduce $\neg$(HAS-ZEROS$^{PR}$) to EXACTLY-ONE-ZERO$^{PR}$. Let $f$ be the instance of $\neg$(HAS-ZEROS$^{PR}$). Define $g$ as

$$
g(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \geq 1 \text{ and } f(x-1) \neq 0 \\
0 & \text{if } x \geq 1 \text{ and } f(x-1) = 0
\end{cases}\$$

The function $g$ has exactly one zero iff the function $f$ has no zeros. It follows that EXACTLY-ONE-ZERO$^{PR}$ is not in $\Sigma_1$. □

**Theorem 5.** The AT-LEAST-k-ZEROS$^{PR}$ problem is $\Sigma_1$-complete.

**Proof.** The AT-LEAST-k-ZEROS$^{PR}$ problem is clearly semi-decidable and HAS-ZEROS$^{PR}$ (instance $f(x)$) reduces easily to AT-LEAST-k-ZEROS$^{PR}$ (instance $g(x)$) if we define for each $x \geq 0$

$$g(kx) = g(kx + 1) = \ldots = g(kx + (k-1)) = f(x)$$

so that each zero of $f$ corresponds to $k$ zeros of $g$. As the problem HAS-ZEROS$^{PR}$ is complete in $\Sigma_1$ (Theorem 3 page 12), the reduction from HAS-ZEROS$^{PR}$ to AT-LEAST-k-ZEROS$^{PR}$ proves the $\Sigma_1$-completeness of AT-LEAST-k-ZEROS$^{PR}$. □

**Theorem 6.** For every $k \geq 1$ the language associated with the EXACTLY-k-ZEROS$^{PR}$ problem is in $\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$.

**Proof.** Similar to the proof of Theorem 4 □

**Theorem 7.** The EQUAL-NEXT$^{PR}$ problem is $\Sigma_1$-complete.

**Proof.** It is obvious that EQUAL-NEXT$^{PR} \in \Sigma_1$.

The following observation suggests a reduction of HAS-ZEROS$^{PR}$ (instance $f(x)$) to EQUAL-NEXT$^{PR}$ (instance $g(x)$): $f(x)$ has at least one zero iff $g(x) \overset{\text{def}}{=} \sum_{i=0}^{x} f(i)$ has at least a value equal to the next one (note that $g(0) = 0$). As HAS-ZEROS$^{PR}$ was proved to be $\Sigma_1$-complete (Theorem 3 page 12), this reduction shows that EQUAL-NEXT$^{PR}$ is also $\Sigma_1$-complete. □

**Theorem 8.** The ZERO-FUNCTION$^{PR}$ problem is $\Pi_1$-complete.

**Proof.** The problem can be expressed as $\forall x : f(x) = 0$. Thus it belongs to $\Pi_1$.

Reduce HAS-ZEROS$^{PR}$ to $\neg$(ZERO-FUNCTION$^{PR}$). Given an instance $f$ of HAS-ZEROS$^{PR}$, define the function $g$ as

$$g(x) = \begin{cases} 
0 & \text{if } f(x) \neq 0 \\
1 & \text{if } f(x) = 0
\end{cases}\$$

The function $g$, which is clearly PR, has at least one zero iff $f$ is not the zero function. As HAS-ZEROS$^{PR}$ is $\Sigma_1$-complete (Theorem 3 page 12), this reduction proves the theorem. □

**Theorem 9.** The $\infty$-ZEROS$^{PR}$ problem is is $\Pi_2$-complete.

**Proof.** In $\Pi_2$: the statement associated with the problem can be expressed as

$$\forall m \exists x : (x \geq m) \land (f(x) = 0).$$
To prove completeness, consider $P$, the complement of the problem $\infty$-ZEROS$^{\text{PR}}$ ("finite number of zeros"). We prove the completeness of $P$ in $\Sigma_2$ using a reduction of the $\Sigma_2$-complete problem $\text{FINITE-DOMAIN}$ (Theorem 2, page 8) to $P$. Consider the function $T_e(\langle x, t \rangle) \overset{\text{def}}{=} T(e, x, t)$ (Definition 3, page 6). We assume that for each $x$ there is at most one $t$ such that $T(e, x, t) = 0$; thus, the number of zeros of $T_e$ equals the size of the domain of $\varphi_e$. The instance of the class $P$ that corresponds to the instance $\varphi_e$ of $\text{FINITE-DOMAIN}$ is defined as $T_e$. Clearly $T_e$ has a finite number of zeros iff $\varphi_e$ is in $\text{FINITE-DOMAIN}$. □

Theorem 10. The ALMOST-ALL-ZEROS$^{\text{PR}}$ problem is $\Sigma_2$-complete.

Proof. First notice that ALMOST-ALL-ZEROS$^{\text{PR}}$ is in $\Sigma_2$, because a function $f$ is in ALMOST-ALL-ZEROS$^{\text{PR}}$ iff

$$\exists x_0 \forall x : x \geq x_0 \Rightarrow f(x) = 0.$$ 

To prove completeness, use Theorem 2 (page 8) and the following characterisation of $\text{FINITE-DOMAIN}$

$$\exists a \forall x \forall t : (x \geq a \land t \geq a) \Rightarrow (\text{the computation } T(e, x, t) \text{ did not halt in time } \leq t)$$

to define a reduction of $\text{FINITE-DOMAIN}$ to the PR problem ALMOST-ALL-ZEROS$^{\text{PR}}$: given a ParRec function $\varphi_e$ (instance of $\text{FINITE-DOMAIN}$), consider the PR function $T'_e(\langle x, t \rangle) \overset{\text{def}}{=} 1 - T(e, x, t)$ (see Definition 3, page 6; assume that for each $x$ there is at most one $t$ such that $T(e, x, t) = 0$). The function $T'_e$ is in the class ALMOST-ALL-ZEROS$^{\text{PR}}$ iff $\varphi_e$ is in $\text{FINITE-DOMAIN}$. □

Theorem 11. The EQUAL-AT-ONE-POINT$^{\text{PR}}$ problem is $\Sigma_1$-complete.

Proof. Recall Definition 16, page 11. The problem EQUAL-AT-ONE-POINT$^{\text{PR}}$ is clearly in $\Sigma_1$. The HAS-ZEROS$^{\text{PR}}$ easily reduces to this problem if we fix $g(x, y) = 0$ (the zero function). The completeness follows from this reduction and the fact that HAS-ZEROS$^{\text{PR}}$ is $\Sigma_1$-complete, Theorem 3 (page 12). □

Theorem 12. The EQUIVALENCE$^{\text{PR}}$ problem is $\Pi_1$-complete.

Proof. The complement of the EQUIVALENCE$^{\text{PR}}$ problem is clearly semi-decidable. Consider the instance $\langle e, x \rangle$ of HP. Using the Kleene Normal Form, we see that the computation $\varphi_e(x)$ halts iff the corresponding PR function $T_{e,x}(t) \overset{\text{def}}{=} T(e, x, t)$ (see Theorem 1, page 6) has one zero. The PR function

$$h(t) = \begin{cases} 
0 & \text{if } T_{e,x}(t) \neq 0 \\
1 & \text{if } T_{e,x}(t) = 0 
\end{cases}$$

is not equivalent to the zero function 0(t) iff $T_{e,x}(t)$ has at least one zero. This defines a reduction $\neg\text{HP}$ to EQUIVALENCE$^{\text{PR}}$ which proves the completeness. □

3.2 Size of the codomain

We now study decision problems related to the size of the codomain of a PR function. The INFINITE-CODOMAIN$^{\text{PR}}$ problem will be used in Sections 4.2 (page 27) and 4.4 (page 28). In the following problems, the instance is the PR function $f$. 14
Problem 18. \(|\text{CODOMAIN}| = k^{\text{PR}}\), the codomain is finite with cardinality \(k\).

Question: Does the codomain of \(f\) have size \(k\)?

Class: \(\Pi_1\)-complete for \(k = 1\) (Theorem 13, page 15).

Class: \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\) for \(k \geq 2\) (Theorem 14, page 15).

Theorem 13. The language associated with the problem \(|\text{CODOMAIN}| = 1^{\text{PR}}\) is \(\Pi_1\)-complete.

Proof. A positive answer to \((|\text{CODOMAIN}| = 1^{\text{PR}})\) can be expressed as \(\forall x : (f(x) = f(0))\), so that \((|\text{CODOMAIN}| = 1^{\text{PR}})\) is in \(\Pi_1\). Reduce \((\text{ZERO-FUNCTION}^{\text{PR}})\) to \((|\text{CODOMAIN}| = 1^{\text{PR}})\), as follows. Given an instance \(f\) of \(\text{ZERO-FUNCTION}^{\text{PR}}\) define the instance \(g\) of \((|\text{CODOMAIN}| = 1^{\text{PR}})\) as

\[
\begin{align*}
g(0) &= 0 \\
g(x) &= f(x - 1) & \text{for } x \geq 1.
\end{align*}
\]

The function \(g\) has codomain with size 1 iff \(f(x) = 0\) (zero function). As \(\text{ZERO-FUNCTION}^{\text{PR}}\) is \(\Pi_1\)-complete (Theorem 8, page 13), \(|\text{CODOMAIN}| = 1^{\text{PR}}\) is also \(\Pi_1\)-complete. □

Theorem 14. For any integer \(k \geq 2\) the language associated with the problem \(|\text{CODOMAIN}| = k^{\text{PR}}\) belongs to the class \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\).

Proof. Assume \(k \geq 2\).

(i) In \(\Delta_2\): a positive answer to \(|\text{CODOMAIN}| = k^{\text{PR}}\) can be expressed as (illustrated for the case \(k = 2\))

\[
[\exists x_1, x_2 : A(x_1, x_2)] \land [\forall z_1, z_2, z_3 : B(z_1, z_2, z_3)]
\]

where

\[
\begin{align*}
A(x_1, x_2) &= f(x_1) \neq f(x_2) \\
B(z_1, z_2, z_3) &= (f(z_1) = f(z_2)) \lor (f(z_2) = f(z_3)) \lor (f(z_3) = f(z_1))
\end{align*}
\]

The question associated with the problem can thus be expressed in 2 forms:

\[
\begin{align*}
\exists x_1, x_2 \forall z_1, z_2, z_3 : A(x_1, x_2) \land B(z_1, z_2, z_3) \\
\forall z_1, z_2, z_3 \exists x_1, x_2 : A(x_1, x_2) \land B(z_1, z_2, z_3)
\end{align*}
\]

Thus \(|\text{CODOMAIN}| = k^{\text{PR}}\) belongs both to \(\Delta_2 = \Sigma_2 \cap \Pi_2\).

(ii) Not in \(\Sigma_1\): reduce \(\text{ZERO-FUNCTION}^{\text{PR}}\) to \(|\text{CODOMAIN}| = k^{\text{PR}}\): let \(f\) be an instance of \(\text{ZERO-FUNCTION}^{\text{PR}}\). Define \(g\) as

\[
g(x) = \begin{cases} 
  x & \text{for } 0 \leq x < k \\
  k \times f(x - k) & \text{for } x \geq k
\end{cases}
\]

Clearly \(|\text{cod}(g)| = k\) iff \(f(x) = 0\) (zero function). Thus \(|\text{CODOMAIN}| = k^{\text{PR}}\) is not in \(\Sigma_1\).

(iii) Not in \(\Pi_1\): reduce \((\neg \text{ZERO-FUNCTION}^{\text{PR}})\) to \(|\text{CODOMAIN}| = k^{\text{PR}}\): let \(f\) be an instance of \((\neg \text{ZERO-FUNCTION}^{\text{PR}})\). Define

\[
g(kx + i) = \begin{cases} 
  0 & \text{if } f(x) = 0 \text{ (for } i = 0, 1, \ldots, k - 1) \\
  i & \text{if } f(x) \neq 0 \text{ (for } i = 0, 1, \ldots, k - 1)\n\end{cases}
\]

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Clearly \(|\text{cod}(g)| = k\) iff \(f(x) \neq 0\) for at least a value of \(x\). Thus \(\text{CODOMAIN} = k^{\text{PR}}\) is not in \(\Pi_1\).

\[\square\]

**Problem 19.** \(\text{FINITE-CODOMAIN}^{\text{PR}}\), finite codomain.

**Question:** Is the codomain of \(f\) finite?

**Class:** \(\Sigma_2\)-complete (Theorem 15, page 16). \(\square\)

**Theorem 15.** The problem \(\text{FINITE-CODOMAIN}^{\text{PR}}\) is \(\Sigma_2\)-complete.

**Proof.** In \(\Sigma_2\): the \(\text{FINITE-CODOMAIN}^{\text{PR}}\) statement can be expressed as \(\exists m \forall x : f(x) \leq m\).

\(\Sigma_2\)-complete: the problem \(\infty\text{-ZEROS}^{\text{PR}}\) is \(\Pi_2\)-complete (Theorem 9, page 13). Reduce \(\neg(\infty\text{-ZEROS}^{\text{PR}})\) to \(\text{FINITE-CODOMAIN}^{\text{PR}}\). The PR function \(f\) has a finite number of zeros iff the PR function defined by the following program has finite codomain.

\[
\text{Function } g(n):
\]
\[
m \leftarrow 0;
\text{for } i = 0, 1, \ldots, n:
\]
\[
\text{compute } f(i);
\text{if } f(i) = 0:
\]
\[
m \leftarrow m + 1;
\]
\[
\text{return } m;
\]

\[\square\]

The problem \(\text{INFINITE-CODOMAIN}^{\text{PR}}\) is the negation of the problem \(\text{FINITE-CODOMAIN}^{\text{PR}}\). It follows from Theorem 15 that \(\text{INFINITE-CODOMAIN}^{\text{PR}}\) is \(\Pi_2\)-complete.

### 3.3 Injectivity, surjectivity, and bijectivity

The problems of deciding if a given PR function is injective, surjective, or bijective are considered in this section.

**Problem 20.** \(\text{INJECTIVE}^{\text{PR}}\), primitive recursive injectivity.

**Instance:** a PR function \(f\).

**Question:** is \(f\) an injective function? \(\square\)

**Problem 21.** \(\text{ONTO}^{\text{PR}}\), primitive recursive surjectivity.

**Instance:** a PR function \(f\).

**Question:** is \(f\) a surjective function? \(\square\)

**Problem 22.** \(\text{BIJECTIVE}^{\text{PR}}\), primitive recursive bijectivity.

**Instance:** a PR function \(f\).

**Question:** is \(f\) a bijective function? \(\square\)

#### 3.3.1 Injectivity

**Theorem 16.** The problem \(\text{INJECTIVE}^{\text{PR}}\) is \(\Pi_1\)-complete.

As a consequence of this result the injective PR functions can not be effectively enumerated and can not be characterised by a “model of computation”, see Assumption 1 (page 9).
Proof. The INJECTIVE$^{PR}$ statement can be expressed as

$$\forall m, n : (m \neq n) \Rightarrow (f(m) \neq f(n))$$

Thus INJECTIVE$^{PR}$ belongs to the class $\Pi_1$.

The HAS-ZEROS$^{PR}$ is $\Sigma_1$-complete (Theorem 3, page 12). We reduce HAS-ZEROS$^{PR}$ to $\neg$INJECTIVE$^{PR}$. Let $f$ be the instance of HAS-ZEROS$^{PR}$. Define the function $g$ as

$$\begin{cases} 
  g(0) = 0 \\
  g(n) = n & \text{if } n \geq 1 \text{ and } f(n-1) \neq 0 \\
  g(n) = 0 & \text{if } n \geq 1 \text{ and } f(n-1) = 0
\end{cases}$$

Clearly, $g$ is injective iff $f$ has no zeros. □

3.3.2 Surjectivity and bijectivity

Theorem 17. The problem ONTO$^{PR}$ is $\Pi_2$-complete.

Proof. An instance $f$ of ONTO$^{PR}$ can be expressed as $\forall y \exists x : f(x) = y$. It follows that ONTO$^{PR}$ is in the class $\Pi_2$.

To prove completeness, we reduce the $\Pi_2$-complete problem $\neg$FINITE-DOMAIN (Theorem 2, page 9) to ONTO$^{PR}$. Let $\varphi_e$ be an instance of $\neg$FINITE-DOMAIN. Consider the Turing machine $T$ in Definition 3 (page 6). Define the PR function $f$ as follows:

- $f(n)$ is the number of integers $m < n$ with $m = \langle x, t \rangle$ for which $T(e, x, t) = 0$, that is, for which the computation $\varphi_e(x)$ halts at exactly the step $t$. Clearly the codomain of the PR function $f$ is $\mathbb{N}$ iff the ParRec function $\varphi_e$ does not have finite domain. □

Theorem 18. The problem BIJECTIVE$^{PR}$ is $\Pi_2$-complete.

Proof. An instance $f$ of BIJECTIVE$^{PR}$ can be expressed as

$$\forall y, x_1, x_2 \exists x : [f(x) = y] \land [x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)].$$

Thus BIJECTIVE$^{PR}$ belongs to the class $\Pi_2$.

We reduce the $\Pi_2$-complete problem ONTO$^{PR}$ (Theorem 17 above) to BIJECTIVE$^{PR}$. Let $f$ be the instance of ONTO$^{PR}$. For convenience we will use the function $g(n) \overset{\text{def}}{=} f(\lfloor n/2 \rfloor)$ so that there are infinitely many pairs $(m, n)$ with $g(m) = g(n)$.

The function $h$, instance of BIJECTIVE$^{PR}$, is defined as

Function $h(n)$:

- Compute the values $g(0), \ldots, g(n)$;
- if $g(n) \not\in \{g(0), \ldots, g(n-1)\}$ then:
    - $h(n) = 2 \times g(n)$; // (if $g(n)$ is a new value)
- else:
    - $h(n) =$ first unused odd integer
    // (if $g(n) = g(i)$ for some $i < n$)

5By “$m = \langle x, t \rangle$” we mean: use $\langle \cdot, \cdot \rangle$, the standard bijection $\mathbb{N}^2 \to \mathbb{N}$, to extract $x$ and $t$ from $m$. 

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Figure 2: Some decision problems studied in this paper. The instance consists of one or two PR functions (two for the EQUIVALENCEPR problem). The problems are numbered (first column) for reference in Figure 3, page 19. Note that the question associated with the problems in class 8 is not “is the size of the codomain at least 2?” but “is the size of the codomain exactly equal to k?” (for some fixed $k \geq 2$).

The codomain of $h$ consists of two parts: (i) the set of even integers, having the form $2 \times a$ where $a$ belongs to the codomain of $f$, and (ii) the set of all the odd integers, $\{1, 3, 5, \ldots\}$. Clearly $h$ is always injective. If $f \in$ ONTOPR, all the even integers occur in the set (i) so that $h \in$ BIJECTIVEPR. If $f \notin$ ONTOPR, some integer $a$ is not in the codomain of $f(i)$. Thus $2a$ does not belong to the set (i), so that $h \notin$ BIJECTIVEPR. Moreover the function $h$ is PR (if $f$ is PR).

The table below illustrates the definition of the functions $g$ and $h$ from a function $f$, given as example.

| $n$ | 0 1 2 3 4 5 6 7 8 9 10 11 \ldots |
|-----|---------------------------|
| $f(n)$ | 3 2 5 5 3 40 \ldots \ldots \ldots \ldots \ldots |
| $g(n)$ | 3 3 2 2 5 5 5 5 3 3 40 40 \ldots |
| $h(n)$ | 6 1 4 3 10 5 7 9 11 13 80 15 \ldots |

Whenever the value of $g(n)$ is new (bar over the number), the value of $h(n)$ is $2 \times g(n)$. If $g(n)$ it occurred before, the successive odd integers $1, 3, 5, \ldots$ are used as values of $h(n)$. The function $h$ will always be injective. However, it will be surjective only if $f$ is also surjective.

The table in Figure 2 (page 18) summarises our undecidability results about PR functions.
Figure 3: Location in the arithmetic hierarchy of the problems mentioned in Figure 2, page 18. The problem numbers (1 to 12 in this colour) refer to the numbers in the first column of Figure 2. All the decision problems in $\Sigma_1$, $\Pi_1$, $\Sigma_2$, and $\Pi_2$ are complete in the respective class.

3.4 Primitive recursive versus partial recursive problems

It is interesting to compare the degree of undecidability of the same problem about a function $f$ in two cases: (i) $f$ if primitive recursive, (ii) $f$ is partial recursive.

**Theorem 19.** Let $A$ be a PR decision problem corresponding to the question “does a given PR function have the property $P$?” and let $X$ be any decision problem. If $X \leq_m A$, then $X \leq_m A'$, where $A'$ corresponds to the question “does a given partial recursive function have the property $P$?”.

**Proof.** Let $f$ be the function corresponding to the reduction $X \leq_m A$. The image by $f$ of a (positive or negative) instance $x$ of $X$ is a PR function, thus it is also a partial function. So, we can use the same function $f$ for the reduction $X \leq_m A'$. □

Partial recursive functions: some undecidability results

We now proof the AH classes mentioned in the last column of Figure 4, page 20. The instance of the decision problems studied in this section is thus a ParRec function.

When we mention the value $f(x)$ of a ParRec function $f$ it is implicitly assumed that this value exists, that is, that $\exists t : F(x, t)$; recall the meaning of the predicate $F(x, t)$, see Definition 3, page 8.

1. $f(0) = 0$: $\Sigma_1$-complete, easy to reduce the SHP to this problem.

2. TOTAL: $\Pi_2$-complete, see Theorem 2 (page 5).

3. HAS-ZEROS: $\Sigma_1$-complete.

4. NO-ZEROS, $f$ has no zeros: $\Pi_1$-complete.

5. EXACTLY-ONE-ZERO: $\Pi_2$-complete.

**Proof.** The corresponding statement is below (lines 1 to 5). The $\forall \exists$ sequence originates in lines (4) and (5): when we speak of the value of $f(x_1)$ we also say that $f(x_1)$ is defined; and similarly for $f(x_2)$.
| Problem                        | Primitive recursive (problem $P^{PR}$) | Partial recursive (problem $P$) |
|-------------------------------|----------------------------------------|---------------------------------|
| $f(0) = 0$                    | $\Delta$                               | $< \Sigma_1$-complete           |
| TOTAL                        | $\Delta$                               | $< \Pi_2$-complete              |
| RECURSIVE                    | $\Delta$                               | $< \Sigma_3$-complete           |
| HAS-ZEROS                    | $\Sigma_1$-complete                     | $= \Sigma_1$-complete           |
| NO-ZEROS                     | $\Pi_1$-complete                        | $= \Pi_1$-complete              |
| EXACTLY-ONE-ZERO             | $\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$ | $< \Pi_2$-complete              |
| AT-LEAST-k-ZEROS             | $\Sigma_1$-complete                     | $= \Sigma_1$-complete           |
| EXACTLY-k-ZEROS              | $\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$ | $< \Pi_2$-complete              |
| ZERO-FUNCTION                | $\Pi_1$-complete                        | $< \Pi_2$-complete              |
| $\infty$-ZEROS               | $\Pi_2$-complete                        | $= \Pi_2$-complete              |
| FINITE-DOMAIN                | $\Delta$                               | $< \Sigma_2$-complete           |
| $|\text{CODOMAIN}| = 1$    | $\Pi_1$-complete                        | $< \Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$ |
| $|\text{CODOMAIN}| = k, k \geq 2$ | $\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$ | $= \Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$ |
| EQUIVALENCE                  | $\Pi_1$-complete                        | $< \Pi_2$-complete              |
| INJECTIVE                    | $\Pi_1$-complete                        | $= \Pi_1$-complete              |
| ONTO                         | $\Pi_2$-complete                        | $= \Pi_2$-complete              |
| BIJECTIVE                    | $\Pi_2$-complete                        | $= \Pi_2$-complete              |

Figure 4: The degree of undecidability of some decision problems about one or two functions. For each problem $P$ two cases are compared: the function is primitive recursive ($P^{PR}$) and the function is partial recursive ($P$). The symbol “$=$” means that the partial recursive problem and the corresponding PR problem have the same degree of undecidability (in terms of the arithmetic hierarchy), while the symbol “$<$” means that the PR problem problem is “less undecidable” than the corresponding partial recursive problem. The RECURSIVE is defined in Example 33, page 40.
in lines (4) and (5).

(1) \( \exists x : [f(x) \text{ is defined and } f(x) = 0] \) and

(2) \( \forall x_1, x_2 : \) either

(3) \([x_1 = x_2] \) or \([f(x_1) \text{ is undefined}] \) or \([f(x_2) \text{ is undefined}] \) or

(4) \([f(x_1) \text{ defined and } f(x_1) \neq 0] \) or

(5) \([f(x_2) \text{ defined and } f(x_2) \neq 0] \).

To prove completeness we reduce the \( \Pi_2 \)-complete EQUIVALENCE (Theorem 2, page 8) to EXACTLY-ONE-ZERO. Consider an instance \( \langle f, g \rangle \) of EQUIVALENCE and define

Function \( h(n) : \)
- if \( n = 0 \) : return 0;
- if \( n \geq 1 \):
  - if for some \( x \leq n \), \( f(x) \) and \( g(x) \) are already defined at step \( \leq n \) with \( f(x) \neq g(x) \), then
    - return 0; // \( f \neq g \)
  - else
    - return 1 // \( f = g \) until now.

Clearly the function \( h(n) \) has exactly one zero iff \( f \) and \( g \) are the same (partial) function.

6. AT-LEAST-\( k \)-ZEROS: \( \Sigma_1 \)-complete.
   It is easy to show that it is in \( \Sigma_1 \) (use the “dovetailing” technique); and that HAS-ZEROS reduces easily to AT-LEAST-\( k \)-ZEROS, see the proof of Theorem 5 (page 13).

7. EXACTLY-\( k \)-ZEROS (\( k \) fixed): \( \Pi_2 \)-complete.
   Proof similar to EXACTLY-ONE-ZERO, see item 5.

8. ZERO-FUNCTION: \( \Pi_2 \)-complete.
   Proof. The corresponding question can be expressed as
   \[
   \forall x \exists t : F(x, t) \land (f(x) = 0)
   \]
   so that ZERO-FUNCTION belongs to \( \Pi_2 \).
   Consider the \( \Pi_2 \)-complete problem TOTAL, Theorem 2 (page 8). Reduce TOTAL to ZERO-FUNCTION as follows. Given an instance \( f \) of TOTAL, define
   \[
   g(n) = \begin{cases} 
   0 & \text{if the computation of } f(n) \text{ converges} \\
   \text{undefined} & \text{otherwise}
   \end{cases}
   \]
   The function \( g \) is the zero function iff \( f \) is total.

9. \( \infty \)-ZEROS: \( \Pi_2 \)-complete.
   Proof. The question associated with the problem can be expressed as
   \[
   \forall m \exists x, t : (x \geq m) \land F(x, t) \land (f(x) = 0)
   \]
   Thus \( \infty \)-ZEROS belongs to \( \Pi_2 \).
   It is not difficult to reduce \( \neg \)FINITE-DOMAIN (recall the FINITE-DOMAIN is \( \Sigma_2 \)-complete,
10. \(|\text{CODOMAIN}| = 1\): \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\).

Proof. \(|\text{CODOMAIN}| = 1\) can be expressed as the negation of

\[
\neg[\exists x, t : F(x, t)] \lor \exists x_1, x_2, t_1, t_2 : F(x_1, t_1) \land F(x_2, t_2) \land f(x_1) \neq f(x_2)
\]

Part (1) means “\(f\) it is the totally undefined function” while part (2) means “there are at least two values of \(x\) for which \(f(x)\) is defined and has different values. We can rewrite this logical statement, which is the negation of \(|\text{CODOMAIN}| = 1\), as

\[
\exists x_1, x_2, t_1, t_2 \forall x, t : [\neg F(x, t)] \lor [F(x_1, t_1) \land F(x_2, t_2) \land f(x_1) \neq f(x_2)]
\]

In this case the order of the quantifiers can be changed, see the proof of Theorem 4 (page 12). Thus \(|\text{CODOMAIN}| = 1\) (whose statement is the negation of the above) belongs to \(\Delta_2 = \Sigma_2 \cap \Pi_2\).

We now show that the problem \(|\text{CODOMAIN}| = 1\) is neither in \(\Sigma_1\) nor in \(\Pi_1\). First reduce \(\neg\text{SHP} \to (|\text{CODOMAIN}| = 1)\). Let \(n\) be the instance of \(\neg\text{SHP}\). Define

\[
f(t) = \begin{cases} 
0 & \text{if } t = 0 \\
0 & \text{if } \varphi_n(n) \text{ does not converge in } \leq t \text{ steps } (t \geq 1) \\
1 & \text{otherwise } (t \geq 1)
\end{cases}
\]

Clearly \(\varphi_n(n)\) diverges iff the codomain of \(f\) has size 1.

Now reduce \(\text{SHP} \to (|\text{CODOMAIN}| = 1)\). Let \(n\) be the instance of \(\text{SHP}\). Define

\[
f(t) = \begin{cases} 
\text{undefined} & \text{if } \varphi_n(n) \text{ does not converge in } \leq t \text{ steps} \\
0 & \text{otherwise}
\end{cases}
\]

The computation \(\varphi_n(n)\) halts iff the codomain of \(f\) has size 1. Thus \(|\text{CODOMAIN}| = 1\) belongs to \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\).

11. \(|\text{CODOMAIN}| = k, k \geq 2\): \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\).

Proof. We exemplify for the case \(k = 2\), the cases \(k \geq 3\) are similar. The the statement \(|\text{CODOMAIN}| = 2\) can be expressed as

\[
\exists x, x', t, t': F(x, t) \land F(x', t') \land (f(x) \neq f(x')) \land \forall x_1, x_2, x_3, t_1, t_2, t_3 : Q(x_1, x_2, x_3, t_1, t_2, t_3) : \text{cod}(f) \geq 2
\]

where \(Q(x_1, x_2, x_3, t_1, t_2, t_3)\) denotes

\[
\neg F(x_1, t_1) \lor \neg F(x_2, t_2) \lor \neg F(x_3, t_3) \lor (f(x_1) = f(x_2)) \lor (f(x_2) = f(x_3)) \lor (f(x_3) = f(x_1))
\]

\(7\)When \(F(x_1, t_1)\) is false (the computation of \(f(x_1)\) has not yet halted) the value of \(f(x_1)\) is irrelevant because the disjunct \(\neg F(x_1, t_1)\) is true. In more detail: if we write \(f_{t_1}(x_1)\) instead of \(f(x_1)\), the value of \(f_{t_1}(x_1)\) is arbitrary when \(F(x_1, t_1)\) is false and \(f_{t_1}(x_1) = f(x_1)\) otherwise. And similarly for \(f(x_2)\) and \(f(x_3)\).
The following problems use Definition 2, page 5.

13. ONTO: \[\Pi^2\]

12. INJECTIVE: \[\Pi^1\]-complete.

Proof. For ParRec functions the ONTO statement can be expressed as

\[Q(x_1, x_2, x_3, t_1, t_2, t_3) \land \left( \forall x_1, x_2, x_3, t_1, t_2, t_3 : [F(x, t) \land F(x', t') \land (f(x) \neq f(x'))] \right) \land \left( t_1 \neq t_2 \right) \land \left( t_2 \neq t_3 \right) \land \left( t_3 \neq t_1 \right)

(see the proof of Theorem 14, page 12). Thus \(|\text{CODOMAIN}| = 2\) belongs to \(\Sigma_2 \cap \Pi_2 = \Delta_2\).

To prove that \(|\text{CODOMAIN}| = 2\) is not in \(\Pi_1\), reduce SHP (instance \(n\)) to \(|\text{CODOMAIN}| = 2\).

Let \(n\) be the instance of SHP.

\[
f(t) = \begin{cases} 
0 & \text{if } t = 0 \\
0 & \text{if } \varphi_n([n]) \text{ did not converge in } \leq t - 1 \text{ steps (} t \geq 1 \text{)} \\
1 & \text{if } \varphi_n([n]) \text{ converges in } \leq t - 1 \text{ steps (} t \geq 1 \text{)}
\end{cases}
\]

(the case \(f(0) = 0\) is considered because the possibility of convergence for \(t = 0\)). The function \(f\) has codomain with size 2 iff \(\varphi_n([n])\).

To prove that \(|\text{CODOMAIN}| = 2\) is not in \(\Sigma_1\), reduce \(\neg\text{SHP}\) (instance \(n\)) to \(|\text{CODOMAIN}| = 2\).

Let \(n\) be the instance of \(\neg\text{SHP}\).

\[
f(t) = \begin{cases} 
0 & \text{if } t = 0 \\
1 & \text{if } t = 1 \\
0 & \text{if } \varphi_n([n]) \text{ did not converge in } \leq t - 2 \text{ steps (} t \geq 2 \text{)} \\
2 & \text{if } \varphi_n([n]) \text{ converges in } \leq t - 2 \text{ steps (} t \geq 2 \text{)}
\end{cases}
\]

The function \(f\) has codomain with size 2 iff \(\varphi_n([n])\) diverges.

The following problems use Definition 2, page 5.

12. INJECTIVE: \(\Pi_1\)-complete.

Proof. For ParRec functions the INJECTIVE statement can be expressed as

\[
\neg \left( \exists x, y, t, t' : F(x, t) \land F(y, t') \land (f(x) = f(y)) \right)
\]

(6)

Define a reduction from HAS-ZEROS to \(\neg\text{INJECTIVE}\) as follows. Let \(f\) be the instance of HAS-ZEROS. Define \(g\) as

\[
g(0) = 0 \\
g(n) = 0 & \text{if } n \geq 1, \ f(n - 1) \downarrow \text{ and } f(n - 1) = 0 \\
g(n) = \text{undefined} & \text{otherwise.}
\]

The function \(f\) has at least one 0 iff the function \(g\) is not injective. As HAS-ZEROS is \(\Sigma_1\)-complete, INJECTIVE is \(\Pi_1\)-complete.

13. ONTO: \(\Pi_2\)-complete.

Proof. For ParRec functions the ONTO statement can be expressed as

\[
\forall y \exists x, t : F(x, t) \land (f(x) = y)
\]

Thus ONTO is in \(\Pi_2\). The reduction from the \(\Pi_2\)-complete problem \(\neg\text{FINITE-DOMAIN}\) to ONTO\textsuperscript{PR} used in the
proof of Theorem 17 (page 17) is also a reduction from ¬FINITE-DOMAIN to ONTO (because a PR function is also a ParRec function).

14. BIJECTIVE (for ParRec functions): Π₂-complete.

Proof. The BIJECTIVE statement can be expressed as

\[ \forall y, x, x_1, x_2 \exists x', t \colon f(x, t) \land f(x') = y \land (x_1 \neq x_2) \Rightarrow f(x_1) \neq f(x_2). \]

Note that the conjunct “∀x∃t : f(x, t)” (total function) ensures that we can talk about f(x'), f(x_1) and f(x_2) without including any other convergence conditions. Thus BIJECTIVE is in Π₂.

Recall that the problem ONTOPR is Π₂-complete (reduction ¬FINITE-DOMAIN ≤ₘ ONTOPR, see the proof of Theorem 17, page 17) and note that the reduction ONTOPR ≤ₘ BIJECTIVEPR (proof of Theorem 18, page 17) is also a reduction ONTOPR ≤ₘ BIJECTIVE. Thus, BIJECTIVE is in Π₂ complete.

4 Frontiers of decidability: the HAS-ZEROSPR problem

This section and the next one are about classes of decision problems. In this section we consider the basic problem “does the PR function f have at least one zero?” and restrict the set of inputs and the set of outputs of f, as explained below. We will find necessary and sufficient conditions for decidability of the problem “does the restricted function have at least one zero?”.

We restrict the possible inputs of a PR function f by pre-applying to f some PR function in. The outputs of f are similarly restricted by post-applying to f some fixed PR function h. Both restrictions – the input and the output – can be applied simultaneously, see Figure 6 (page 27). Thus, we will study the decidability of a restricted form of the HAS-ZEROSPR problem, namely the existence of zeros of the function h(f(g(x))). A more general form of placing a PR function f in a fixed system consisting of PR functions is studied in Section 5.1, page 30.

In summary, we study the following classes of PR decision problems, where \( \overline{x} \) denotes the sequence of arguments \( x_1, x_2 \ldots x_n \) (\( n \geq 1 \)), g is a fixed PR multifunction (Definition 1 page 3), and h is a fixed PR function:

(a) “∃x : f(g(\overline{x})) = 0?”;
(b) “∃x : h(f(\overline{x})) = 0?”;
(c) “∃x : h(f(g(\overline{x}))) = 0?”;
(d) “∃x : h(\overline{x}, f(g(\overline{x}))) = 0?”.

The general class (d) will be studied in Section 5.1 see Theorem 24 (page 31). Consider for example the function g(x) = λx [2x^2 + 1] and the question “Given the PR function f, ∃x : f(g(x)) = f(2x^2 + 1) = 0?” This problem belongs to the Class 1 above, see Section 4.2 (page 27).

\(^8\)We use a sans serif font for fixed PR functions like g and h. Our results are also valid for fixed (total) recursive functions.
Figure 5: The AH undecidability classes of some decision problems, considered in two cases: when the instance is a *primitive recursive function* (origin of the arrow) and when it is a *partial recursive function* (tip of the arrow). \( \text{CODOMAIN}=1 \) denotes \(|\text{CODOMAIN}|=1^{\text{PR}}\), and \( f(x) \equiv 0 \) denotes \( \text{ZERO-FUNCTION}^{\text{PR}} \). Whenever the head or the tip of an arrow is in \( \Sigma_1 \) or \( \Pi_1 \) or \( \Pi_2 \), then the corresponding problem is *complete* in the corresponding class. See also Figure 4, page 20. The decision problem \textsc{RECURSIVE} (not represented here) has the \( \text{PR} \) version in \( \Delta \), while the \( \text{ParRec} \) version is complete in \( \Sigma_3 \); see Example 33, page 40.
We will see that the conditions for the decidability of (c) are simply the conjunction of the decidability conditions for (a) and for (b). In this sense, the input and the output restrictions are independent.

Why study these families of problems?

There are several reasons for studying the composition of the instance $f$ with other PR functions. For instance, many atomic predicates can be written as an equality of the form $F(x) = 0$, where $F(x)$ is obtained by composing $f$ with fixed PR functions. As an example, the question “$f(x) = c$?”, where $c$ is a constant, can be expressed as “$f(x) \neq c$” which is equivalent to “$h(f(x)) = c$” where $h(z) = \lambda z \cdot (z - c + [c - z])$. However, our main interest in this section is to look to a well known undecidable problem, namely HAS-ZEROS$^{PR}$, and restrict the inputs and outputs of the function being studied, in order to know “when the problem become decidable”. This decidability frontier is given by Theorems 20, 21, and 22 respectively in pages 27, 28 and 28.

4.1 The meaning of restricted HAS-ZEROS$^{PR}$ problems

The class of decision problems we study in this section is

Problem specification: fixed PR functions $g$ and $h$.

Instance: PR function $f$.

Question: $\exists x : h(f(g(x))) = 0$?

Recall that the unrestricted case of the problem, namely HAS-ZEROS$^{PR}$, is undecidable, Theorem 3 (page 12). If we restrict the input of $f$ by pre-applying a PR function $g$, the problem remains undecidable if and only if the set of possible inputs of $f$ (the codomain of $g$) is infinite, see Theorem 20 page 27. The output of $f$ is restricted by post-applying some other PR function $h$. As Theorem 21 (page 28) shows, the problem remains undecidable if and only if the codomain of $h$ includes 0 and some nonzero integer. The necessary and sufficient conditions for the undecidability of this problem are illustrated the diagram of Figure 6. We summarise the conclusions of this section:

1. Each of the following conditions trivially implies decidability: (i) the set of possible outputs of $g$ is finite, (ii) the output of $h$ is always (that is, for every input $g(x)$ of $f$) nonzero, (iii) the output of $h$ is always zero.

2. If neither of these conditions is satisfied, the problem is undecidable.

We can use the HAS-ZEROS-$f \cdot g^{PR}$, HAS-ZEROS-$h \cdot f^{PR}$, and HAS-ZEROS-$h \cdot f \cdot g^{PR}$ classes of problems (Theorems 20, 21 and 22 respectively) to prove the undecidability of many decision problems.

Note. In the statements of Theorems 20 (page 27), 21 (page 28), and 22 (page 28) we can replace “undecidable” by “$\Sigma_1$-complete”. This follows directly from the corresponding proofs.

□

Note. It would have been possible to prove first Theorem 22 (page 28), and then present Theorems 20 (page 27) and 21 (page 28) as corollaries. However, we prefer to prove the simpler results first. □
4.2 The HAS-ZEROS-\(g\)\(^{PR}\) class of problems

Each problem in the class is specified by the PR function \(g\).

**Problem 23.** (HAS-ZEROS-\(g\)\(^{PR}\)) Let \(g\) be a fixed PR function.

Instance: PR function \(f\).

Question: \(\exists x : f(g(x)) = 0?\)

As we will see in Theorem 20 (page 27), the decidability of the problem depends on whether the codomain of \(g\) is infinite.

We begin by proving directly the undecidability of a particular problem of this class.

**Example 1.** Let \(g(x) = 2x + 1\). Reduce HAS-ZEROS\(^{PR}\) to HAS-ZEROS-\(g\)\(^{PR}\). Let \(u\) be an instance of the HAS-ZEROS\(^{PR}\) problem. Define the function \(f\) (instance of HAS-ZEROS-\(g\)\(^{PR}\)) as \(f(x) = u([x/2])\). We have \(f(g(x)) = f(2x + 1) = u([((2x + 1)/2)] = u(x)\), so that \(u(x)\) has at least one zero iff \(f(x)\) is a positive instance of the “\(f(g(x))\)” problem for \(g(x) = 2x + 1\).

Instead of studying more particular cases we now consider a class of problems, proving an undecidability result. For this purpose we will have to look more closely to the Kleene Normal Form Theorem [44].

**Theorem 20.** The problem HAS-ZEROS-\(g\)\(^{PR}\) is undecidable if and only if the codomain of \(g\) is infinite.

**Proof.** The following is a decision procedure when the codomain of \(g\) is finite, say \(\{y_1, \ldots, y_n\}\); compute \(f(y_1), \ldots, f(y_n)\); answer YES iff any of these integers is 0, answer NO otherwise.

Suppose now that the codomain of \(g\) is infinite. It is a consequence of the Kleene Normal Form Theorem (see Theorem 11 page 14) that any ParRec function \(f(x)\) can be written as \(f(x) = U(x, \mu_t : (T(x, t)))\) where

- The PR function \(T\) “evaluates \(y\) steps” of the computation \(f(x)\). At the end, it returns 0 if the computation has already finished, and 1 otherwise. We assume that if \(T(x, t) = 0\) for some \(t\), it is also 0 for every \(t' \geq t\).

- The PR function \(U\) is similar to \(T\), but when it is “called” – if it is “called” – the value of the last argument, \(\mu_t : T(x, t)\), guarantees that the computation has halted; it then returns the result of the computation \(f(x)\).

As \(g\) has an infinite codomain, the sequence \(g(0), g(1), \ldots\) contains an increasing infinite subsequence \(g(y_0) < g(y_1) < \ldots\) (with \(y_0 < y_1 < \ldots\)), so that

\[
f(x) = U(x, \mu_t : T(x, t)) = U(x, \mu_t : T(x, g(t))) \tag{7}
\]
This holds because there are \(g(y)\)'s arbitrarily large. The arguments \(\Sigma\) can be “included” in the function \(T\); let \(T_{\Sigma}(g(y))\) be the resulting (PR) function.

The HP (unary halting problem) can now be reduced to the HAS-ZEROS-\(f\cdot g\)PR problem. Let \((f, \Sigma)\) be the instance of HP. The corresponding instance of HAS-ZEROS-\(f\cdot g\)PR is the \(T_{\Sigma}\) function, constructed from \(f\) and \(\Sigma\) as described above. The computation \(f(\Sigma)\) halts iff there is some \(y\) such that \(T_{\Sigma}(g(y)) = 0\). \(\square\)

### 4.3 The HAS-ZEROS-\(h\cdot f\)PR class of problems

Each problem is specified by the PR function \(h\).

**Problem 24.** (HAS-ZEROS-\(h\cdot f\)PR) Let \(h\) be a fixed PR function.

Instance: PR function \(f\).

Question: \(\exists \Sigma: h(f(\Sigma)) = 0\)\(\square\)

**Theorem 21.** The HAS-ZEROS-\(h\cdot f\)PR problem is undecidable if and only if \(h^{-1}(1)\) is neither \(\emptyset\) nor \(\mathbb{N}\).

**Proof.** If \(h^{-1}(1)\) is the empty set, \(h(f(\Sigma)) \neq 0\) for every \(f\) and for every \(\Sigma\), so that, for every instance \(f\), the answer to “\(\exists \Sigma: h(f(\Sigma)) = 1\)” is NO. Similarly, if \(h^{-1}(1) = \mathbb{N}\), the answer is always YES. In both cases the problem is decidable.

Suppose now that there are two integers \(a\) and \(b\) with \(h(a) = 0\) and \(h(b) \neq 0\). We reduce HAS-ZEROS\(_{\mathbb{P}}\) to HAS-ZEROS-\(h\cdot f\)PR. Let \(f\) be an instance of HAS-ZEROS\(_{\mathbb{P}}\), and define the function \(f'(\Sigma)\)

\[
f'(\Sigma) = \begin{cases} a & \text{if } f(\Sigma) = 0 \\
    b & \text{if } f(\Sigma) \neq 0 
\end{cases}
\]

The function \(f'\) is also PR, as it may be obtained from \(f\) by the additional instruction “\(\text{if } f(\Sigma) = 0 \text{ then } f'(x) = a \text{ else } f'(x) = b\)”. Clearly, \(f(\Sigma)\) has at least one zero iff \(h(f'(\Sigma))\) has at least one zero. \(\square\)

### 4.4 The HAS-ZEROS-\(h\cdot f\cdot g\)PR class of problems

Each problem is specified by the PR functions \(g\) and \(h\).

**Problem 25.** (HAS-ZEROS-\(h\cdot f\cdot g\)PR) Let \(g\) and \(h\) be fixed PR functions.

Instance: PR function \(f\).

Question: \(\exists \Sigma: h(f(g(\Sigma))) = 0\)\(\square\)

**Theorem 22.** Let \(\text{cod}(g)\) be the codomain of \(g\). The HAS-ZEROS-\(h\cdot f\cdot g\)PR problem is undecidable if and only if the following two conditions hold: \(\text{(i)}\) \(\text{cod}(g)\) is infinite, \(\text{(ii)}\) \(h^{-1}(1)\) is neither the empty set nor \(\mathbb{N}\).

**Proof.** \(g\) and \(h\) are fixed PR functions. If the codomain of \(g\) is finite, or \(h^{-1}(1) = \emptyset\), or \(h^{-1}(1) = \mathbb{N}\), the problem is clearly decidable.

Suppose that (i) and (ii) hold. Reduce the halting problem to HAS-ZEROS-\(h\cdot f\cdot g\)PR. Let \(u(x)\) be the instance of the halting problem and let \(T(x, t) = T_{x}(t)\) be the Turing machine that corresponds to the computation of \(u(x)\), see the proof of Theorem 20, page 27.
Assuming (i), the computation \( u(x) \) halts iff there is some \( t \) such that \( T_x(g(t)) = 0 \). Assuming (ii), let \( h(a) = 0 \) and \( h(b) \neq 0 \). Recall that \( T_x(t) \) is either 1 (computation not yet halted at step \( t \)) or 0 (computation already finished). Define \( T'_x(t) \) as

\[
T'_x(t) = \begin{cases} 
  a & \text{if } T_x(t) = 1 \\
  b & \text{if } T_x(t) = 0
\end{cases}
\]

Then, \( h(T'_x(t)) = 0 \) if \( T_x(t) = 1 \) and \( h(T'_x(t)) \neq 0 \) if \( T_x(t) = 0 \). Thus \( u(x) \) halts iff \( v_x(g(t)) \) has at least one zero, where \( v_x(t) = h(T'_x(t)) \). The transformation \( u(x) \to T'_x(t) \) defines the reduction \( \text{HP} \leq_m \text{HAS-ZEROS-h-f-g}^{\text{PR}} \). In summary,

\[
u(x) \text{ halts} \iff \exists t : T(x, t) = 0 \iff \exists t : T'_x(t) = 0 \\
\iff \exists t : T_x(g(t)) = 0 \iff \exists t : h(T'_x(g(t))) = 0
\]

4.5 Classifying a problem – testing the properties of \( g \) and \( h \)

Suppose that we are given a problem in one of the classes \( \text{HAS-ZEROS-f-g}^{\text{PR}} \), \( \text{HAS-ZEROS-h-f}^{\text{PR}} \) or \( \text{HAS-ZEROS-h-f-g}^{\text{PR}} \), and that we want to know if the problem is decidable or not. Perhaps not surprisingly, this “classifying problem” is itself undecidable.

– Consider first the class of problems \( \text{HAS-ZEROS-f-g}^{\text{PR}} \). In order to classify an arbitrary problem of this class, we have to solve the meta-problem, “is the codomain of a given PR function \( g \) infinite?”. However, as we have seen in Theorem 15 (page 28) this problem is undecidable, belonging to the class \( \Pi_2 \)-complete.

– Consider now the class \( \text{HAS-ZEROS-h-f-g}^{\text{PR}} \) of decision problems. The corresponding question associated with the post-function \( h \) can be expressed as the following problem

**Problem 26.** \( \text{HAS-ZEROS-AND-NONZEROS}^{\text{PR}} \)

Instance: PR function \( h \).

Question: \( \exists a, b : h(a) = 0 \) and \( h(b) \neq 0 \)?

**Theorem 23.** The problem \( \text{HAS-ZEROS-AND-NONZEROS}^{\text{PR}} \) is complete in the class \( \Sigma_1 \).

**Proof.** Membership in \( \Sigma_1 \) is obvious. Reduce the \( \Sigma_1 \)-complete problem \( \text{HAS-ZEROS}^{\text{PR}} \) to \( \text{HAS-ZEROS-AND-NONZEROS}^{\text{PR}} \). Let \( f \) be the instance of \( \text{HAS-ZEROS}^{\text{PR}} \), and let

\[
g(x) = \begin{cases} 
  1 & \text{if } x = 0 \\
  0 & \text{if } x \geq 1 \text{ and } f(x - 1) = 0 \\
  1 & \text{if } x \geq 1 \text{ and } f(x - 1) \neq 0
\end{cases}
\]

Clearly, \( f \) has at least one zero iff there are integers \( a \) and \( b \) such that \( g(a) = 0 \) and \( g(b) \neq 0 \).

Thus if the PR function \( h \) for the class \( \text{HAS-ZEROS-h-f}^{\text{PR}} \) is given by a recursive definition or by Loop program, it is not decidable whether the condition expressed in Theorem 21 (page 28) is satisfied.

– For the class of problems \( \text{HAS-ZEROS-h-f-g}^{\text{PR}} \) we get a similar “undecidability of classification”.

\[\Box\]
In conclusion, given a problem \( P \) belonging to one of the classes \( \text{HAS-ZEROS-} f \cdot g^{\text{PR}} \), \( \text{HAS-ZEROS-} h \cdot f^{\text{PR}} \), or \( \text{HAS-ZEROS-} h \cdot f \cdot g^{\text{PR}} \) (the corresponding instance is \( g \), \( h \) or \( \langle g, h \rangle \)), the classification of \( P \) as decidable or undecidable is itself an undecidable problem.

5 Generalisations of the PR decision problems

We now study two generalisations of the \( h(f(g(x))) \) problem:

- The PR function \( f \) (the instance of the decision problem) is included in an arbitrary acyclic PR system \( S \). See Section 5.1 (page 30).
- More than one occurrence of \( f \) is possible. In particular, there is the possibility of composing \( f \) with itself. See Section 5.2 (page 34).

As previously, the instance of each problem is always be a PR function \( f \), while \( g, g', g_1, g_2 \ldots, h \) denote fixed PR functions.

5.1 A primitive recursive function \( f \) in an acyclic PR structure: normal form

In Section 4.4 (page 28) we fully characterised the existence of zeros of a function with the form \( h(f(g(x))) \), where \( f \) is the instance of the problem and \( g \) and \( h \) are fixed PR functions. We consider now a more general situation in which there is a unique occurrence of \( f \) “inside” an arbitrary acyclic primitive recursive structure, see Section 5.1 (see also Figure 1) page 4. For any such structure there is an equivalent normal form, see Theorem 24 (page 31).

We begin by defining acyclic PR expressions.

**Definition 5.** An acyclic PR expression (acyclic-PR-exp) is an acyclic directed graph characterised by:

- nodes: input variables \( x_i \), inputs of the PR functions, outputs of the PR functions, and the output variable \( y \);

- edges: (\( a, b \)) where \( a \) is either an input variable or the output of a PR function, and \( b \) is either the input of a PR function or \( y \);

- the inputs of PR functions have indegree 1, the output of PR functions have positive outdegree, \( y \) has indegree 1 and outdegree 0, the input variables have indegree 0.

One PR function is \( f \). The other PR functions are fixed. There are no loops in the graph. □

A particular acyclic-PR-exp is illustrated in Figure 7, page 32.

An acyclic-PR-exp can be expanded into a single formula. In the example of Figure 7 we get

\[
m(f(x_1, q(x_1, x_2)), q(x_1, x_2), p(q(x_1, x_2), x_2)).
\]

Sub-expressions may be repeated; in this example, “\( q(x_1, x_2) \)” occurs 3 times. Another way of representing an acyclic-PR-exp is to use definitions in a system of equations. This method avoids

9Or, more generally, inside an arbitrary acyclic (total) recursive structure.
the repetition of sub-expressions. In our example (Figure 7, page 32) a possible system of equations is

\[
\begin{align*}
  z &= q(x_1, x_2) \\
  y' &= f(x_1, z) \\
  v &= p(z, x_2) \\
  y &= m(y', z, v).
\end{align*}
\]

A system of equations equivalent to an acyclic-PR-exp must satisfy the following conditions

- Every left hand side is the definition of a new variable, that is, a variable that does not occur in previous equations nor on the right hand side of this equation.
- The variable defined by the last expression is the output variable.
- Each right hand side has the form \( s(w_1, \ldots, w_n) \) where \( s \) is a function and each \( w_i \) is either an input variable or a previously defined variable.

5.1.1 The normal form of a acyclic PR expressions

Suppose that a given PR function \( f \) is placed somewhere inside a fixed acyclic PR structure (or acyclic-PR-exp) \( S \). Consider the decision problem

**Problem 27.**

Instance: A PR function \( f \)

Question: Does the function \( S(f, \overline{x}) \) have at least a zero? □

In this general case, and contrarily to the problem \( h(f(g(\overline{x}))) = 0 \) (Theorem 22, page 28) it does not seem possible to get a closed condition for the decidability of the problem.

As a step towards the analysis of this problem, we will show that such an arbitrary acyclic structure can be reduced to a normal form illustrated in Figure 8, page 34. This normalisation has nothing to do with primitive recursive functions and can be applied to an arbitrary acyclic graph of functions that includes a distinguished function \( f \).

Consider Figure 9, page 34, which represents the condition “\( \exists x : S(f, \overline{x}) = 0? \)”, where \( S \) has been reduced to the normal form (Theorem 24 below). We can see the origin of the difficulty of finding solutions of \( S(f, \overline{x}) = 0 \), even when \( S \) is in the normal form: there must be a pair \( (\overline{x}, y') \) in the set \( h^{-1}(0) \) such that, if we apply \( g \) and \( f \) in succession to \( \overline{x} \), we get that same value \( y' \). That is,

\[
\exists (\overline{x}, y') \in h^{-1}(0) : f(g(\overline{x})) = y'.
\]

We now have two interacting conditions ((1) and (2) above) for the existence of a zero, and this is the reason of the difficulty in finding a general explicit condition for the existence of zeros of \( S(f, \overline{x}) \).

The main result of this section is the following theorem.

**Theorem 24** (Normal Form Theorem). An arbitrary acyclic PR expression \( S(f, \overline{x}) = y \) containing an occurrence of \( f \) can be reduced to the form \( h(\overline{x}, f(g(\overline{x}))) \), where \( g \) is a fixed (not depending on \( f \)) PR multifunction (Definition 1, page 5) and \( h \) is a fixed PR function. The number of outputs of \( g \) is equal to the arity of \( f \).
Figure 7: An example of an *acyclic PR expression*, or “acyclic-PR-exp”. In general, the corresponding closed output expression contains multiples occurrences of the same sub-expression. In this case the closed output expression is \( y = \text{m}(f(x_1, q(x_1, x_2)), q(x_1, x_2), \text{p}(q(x_1, x_2), x_2)) \) in which the sub-expression \( q(x_1, x_2) \) occurs three times.

The structure of the normal form is illustrated in Figure 8 page 34. Before proving this result we illustrate it with an example.

**Example 2.** Consider the function of Figure 7 (page 32). Recall that \( y \) and \( y' \) denote respectively the output of the entire system \( S \) and the output of \( f \). We get

\[
\begin{align*}
  h(x_1, x_2, y') &= \text{m}(y', q(x_1, x_2), \text{p}(q(x_1, x_2), x_2)) \\
  g(x_1, x_2) &= \langle x_1, q(x_1, x_2) \rangle
\end{align*}
\]

and, as stated in Theorem 24 we have \( y = h(\text{x}, f(g(\text{x}))) \).

**Proof.** Consider an acyclic graph that corresponds to an acyclic-PR-exp. We describe an iterative algorithm that removes nodes of the graph until an acyclic-PR-exp with the form \( h(\text{x}, f(g(\text{x}))) \) is obtained, see Figure 8 (page 34). The output variable \( y \) of \( S \) must occur in a single equation (the last one) whose initial form is, say \( y = H'(\ldots); \) the final form of \( H' \) will be denoted by \( h \). Let the function \( f \) be represented by the equation \( y' = f(x') \). This form of \( f \) will not change during the execution of the algorithm.

With the exception of the input nodes \( \text{x} \), of \( H' \), and of \( f \), the nodes of the graph are primitive recursive functions of the form \( u(\text{w}) \) that can be classified in 3 types

- **INP-f**: there is at least one path from the output of \( u(\text{w}) \) to an input of \( f \).
- **OUT-f**: there is at least one path from the output \( y' \) of \( f \) to one of the inputs \( \text{w} \).
- **NEITHER**.

In Figure 7 (page 32) \( q \) is an INP-f node, \( m \) is an OUT-f node, and \( p \) is a NEITHER node.

We emphasise that \( \text{x} \) (the input nodes), \( H' \), and \( f \) belong to neither of these classes and will never be removed \((H' \) may be modified during the normalisation procedure). Notice also that as the graph is acyclic, no node can be simultaneously of the INP-f and OUT-f types.

**Step 1.** Remove all nodes \( u \) of the INP-f type that are not immediate predecessors of \( f \):

The node \( u \) is removed and “included” in all the its successors (which may be of the INP-f, OUT-for NEITHER types) (with the exception of \( f \)); no loop is created in the graph. This process may change the NEITHER nodes and the nodes in the path \( y' \) to \( y \). In both cases the reason is that the output of \( u \) may connect to inputs of those nodes.
In the end of the removal process, there is a single set of functions \( g \) between the inputs \( \bar{r} \) of the system and the inputs of \( f \). If the output of a function \( g' \) of \( g \) is also connected to an OUT-\( f \) or to a NEITHER node, \( g' \) is included in those nodes.

**Step 2.** Remove all nodes \( u \) of the OUT-\( f \) type:

In a similar way, these nodes are removed; NEITHER nodes (but not INP-\( f \)-nodes, now all included in \( g \)) can change during the process. Also, outputs of NEITHER nodes (that were inputs of the OUT-\( f \) node \( u \)) can now be inputs of \( H' \).

**Step 3.** Remove all nodes \( u \) of the NEITHER type: include them in the \( H' \) node.

In the end of the process, the final form of \( H' \) (that is, \( h \)) has \( \bar{r} \) and \( y \) as its only inputs.

□

**How to obtain the normal form from the system of equations: an example**

The normal form algorithm can be rephrased in terms of the system of equations that describes the system \( S \). Instead of giving the algorithm in detail, we illustrate it with the example of Figure 7 (page 32). The function \( f \), instance of the decision problem, is special; it plays the role of an argument of the system.

\[
\begin{align*}
\begin{cases}
  z &= q(x_1, x_2) \\
  y' &= f(x_1, z) \\
  v &= p(z, x_2) \\
  y &= m(y', z, v)
\end{cases} 
\quad \text{(1)}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  z &= q(x_1, x_2) \\
  y' &= f(x_1, z) \\
  v &= p(q(x_1, x_2), x_2) \\
  y &= m(y', q(x_1, x_2), v)
\end{cases} 
\quad \text{(2)}
\end{align*}
\]

The first system of equations corresponds directly to Figure 7. Although not done here, the first step would be to include all the inputs in the functions that have at least one input as argument (with exception of \( f \)).

**Step 1.** The definition of \( z \) depends only on the inputs \( x_1 \) and \( x_2 \) (and not on “intermediate” variables). Replace \( z \) by \( q(x_1, x_2) \) in every line after its definition, except in the line that defines \( f \); for instance, \( v = p(z, x_2) \) is replaced by \( v = p(q(x_1, x_2), x_2) \). The definition of \( z \), namely \( z = g(x_1, x_2) \) is not deleted because its output is an input of \( f \).

**Step 2.** The definition of \( v \) depends only on the inputs \( x_1 \) and \( x_2 \). Replace \( v \) by \( p(q(x_1, x_2), x_2) \) in every line after its definition; in this case

- The definition of \( y \) becomes \( y = m(y', q(x_1, x_2), p(q(x_1, x_2), x_2)) \).
- The definition of \( v \) is deleted.

**General case: a condition for the existence of a solution**

Using the normal form of \( S \), we see that
Figure 8: An arbitrary acyclic composition of one occurrence of \(f\) with fixed PR functions can be reduced to this “normal form”. The corresponding expression has the form \(y = h(x, f(g(x)))\) where \(g\) has multiple outputs.

Figure 9: (Refer to Figure 8, page 34.) The global system \(S, y = S(f, x)\), has a zero iff we have \(f(g(x)) = y'\) for some \((x, y') \in h^{-1}(0)\). For the particular case \(S(f, x) \equiv h(f(g(x)))\), that is, when there is no green arrow in the diagram, there is an explicit (“acyclic”) condition for the existence of a zero of \(S\), see Theorem 22 (page 28). But no such condition is known for the general case.

**Observation 1.** If \(S\) is a fixed acyclic PR structure (see Section 5.1.1 page 31), a primitive recursive function \(f\) is a solution of the problem “\(\exists x: S(f, x) = 0\)?” iff the following condition holds

\[
\exists x, y': [h(x, y') = 0] \land [f(g(x)) = y']
\]

Note however that, as already stated in page 31 (see also Figure 9 in page 34), this observation is not a “closed form” condition (on the functions \(g\) and \(h\)) for the existence of a solution of “\(\exists x: S(f, x) = 0\)”. Thus, it is not a satisfactory solution of the problem.

### 5.2 Composition of \(f\) with itself

#### 5.2.1 Introduction

Until now we have only studied PR decision problems in which only one occurrence of \(f\) is allowed. This is because we view \(f\), the instance of the decision problem, as a part of a larger system \(S\) – an acyclic graph that containing a node \(f\).

However, it is interesting to study systems containing more than one occurrence of \(f\), and in particular, systems in which the composition of \(f\) with itself is allowed.

**In this section we will only study a few problems of this kind.** In particular we study the decidability of the problem

“Given \(f\), does the function \(f(f(x))\) have a zero?” (8)
5.2.2 Solution of some problems in which the primitive recursive function \( f \) is composed with itself

For the class of HAS-ZEROS\(^{PR} \) problems in which \( f \) may be composed with itself, we do not know a general undecidability criterion. However, we now study a few such problems. Each of them turns out to be \( \Sigma_1 \)-complete.

The problems that we will consider are the following.

- **\( ff^{PR} \):** given \( f \), does the function \( f(f(x)) \) have at least one zero?".
- **\( (f^{(n)})^{PR} \):** given \( f \), does the function \( f(\cdots f(x)) \) have at least one zero? (\( (f^{(2)})^{PR} \) is the problem \( ff^{PR} \)).
- **\( ffZ2^{PR} \):** given \( f \), does the function \( f(f(x)) \) have two or more zeros?

We will use the concept of “graph of a function”.

**Definition 6.** The graph of a function \( f \) is the directed graph

\[(V, E) \text{ where } V = \mathbb{N} \text{ and } E = \{(i, j) : i, j \in \mathbb{N}, f(i) = j\}\]

□

**Theorem 25.** The problem \( ffZ2^{PR} \) is \( \Sigma_1 \)-complete.

**Proof.** Membership in \( \Sigma_1 \) is obvious. To prove completeness reduce HAS-ZEROS\(^{PR} \) to \( ffZ2^{PR} \).

Given an instance \( f \) of HAS-ZEROS\(^{PR} \), define the function \( g \), instance of \( ffZ2^{PR} \), as

\[
\begin{align*}
g(0) &= 0 \\
g(1) &= f(0) \\
g(2) &= f(1) \\
g(3) &= f(2) \\
&\vdots
\end{align*}
\]

As \( g(0) = g(0) = 0 \) the function \( g(g(x)) \) has at least one zero. Also, if \( f(x) = 0 \) for some \( x \), \( g(g(x)) \) has two or more zeros, because in this case we have \( g(g(x+1)) = g(f(x)) = g(0) = 0 \) (and \( x+1 \geq 1 \)).

On the other hand, if \( f(x) \) has no zeros and \( x \geq 1 \), we get

\[
g(g(x)) = g(f(x-1)) = g(y) = f(y-1) \geq 1
\]

where \( y \overset{\text{def}}{=} f(x-1) \geq 1 \). □

**Theorem 26.** The problem \( ff^{PR} \) (page 37) is \( \Sigma_1 \)-complete.

**Proof.** Notice that \( \exists x : g(g(x)) = 0 \) iff the graph of \( g \) contains a path with length 2 ending in 0.

We reduce HAS-ZEROS\(^{PR} \) to \( ff^{PR} \). Given an instance \( f \) of HAS-ZEROS\(^{PR} \), define the graph of the function \( g \), the instance of \( ff^{PR} \), as:

- The set of nodes is \( \mathbb{N} \).
Figure 10: Example of the reduction from HAS-ZEROS$^\text{PR}$ (function $f$) to $ff^\text{PR}$ (function $g$). The upper diagram (green nodes) represents part of the graph of $f$ with $f(0) = 1$, $f(1) = 1$, $f(2) = 0$, $f(3) = 4$, $f(4) = 0$, and $f(5) = 4$. The bottom diagram represents the transformed graph, or graph of the function $g$. Each node $i$ of the top diagram is “transformed” in two nodes, $2i$ (blue) and $2i + 1$ (yellow). Dotted lines connect the upper nodes 0, 1, 4, and 5 to the corresponding pairs of nodes in the bottom diagram (connections from upper nodes 2 and 3 are not represented). Each edge $(i, j)$ of $f$ (top diagram) is mapped in two edges of $g$ (bottom diagram): $(2i, 2i + 1)$ and $(2i + 1, 2j)$.

- For every pair $(i, j)$ in the graph of $f$, that is, for every $f(i) = j$ with $i \in \mathbb{N}$, there are two edges of the graph of $g$: $(2i, 2i + 1)$ and $(2i + 1, 2j)$. Notice that in the graph of $g$ every odd numbered node has indegree = outdegree = 1.

Every node $i$ of the graph of $f$ corresponds to two nodes of the graph of $g$: $2i$ and $2i + 1$. See Figure 10 (page 36) where the transformation is illustrated.

Suppose that there is a solution of HAS-ZEROS$^\text{PR}$: in the graph of $g$ the solution $f(x) = 0$ corresponds a path with length 2 ending in 0, namely $2x \rightarrow 2x + 1 \rightarrow 0$. So, $g(g(2x)) = 0$. For example $f(2) = 0$ and $g(g(4)) = g(5) = 0$.

Suppose that there is a solution of $ff^\text{PR}$: the graph of $g$ has a path of length 2 ending in 0, say $g(g(x)) = 0$; that path must have the form $2i \rightarrow 2i + 1 \rightarrow 0$, because every even node $2i$ only connects to the successor node $2i + 1$ (and never to 0). Thus, by the definition of $g$, we must have $f(i) = 0$. □

The previous proof can be easily generalised.

**Theorem 27.** For every $n \geq 1$ the problem $(f^{(n)})^\text{PR}$ is $\Sigma_1$-complete.
6 Comparing the degree of undecidability: conjectures

We compare in the general case the degree of undecidability of a ParRec problem and of the corresponding PR problem. ParRec indices will be denoted by \( e \) and PR indices by \( p \). The set of all \( p \), that is, of the indices that represent Loop programs, will be denoted by \( I_{\text{PR}} \) (the set \( I_{\text{PR}} \subset \mathbb{N} \) is recursive).

We first characterise the PR problem that corresponds to a given ParRec problem.

**Definition 7** (Problem correspondence). Let a decision problem about ParRec functions be “given \( e \in \mathbb{N} \), is St(\( \varphi_e \))?” where St(\( f \)) denotes some statement about the function \( f \). The corresponding PR decision problem is “given \( p \in I_{\text{PR}} \), is St(\( \varphi_p \))?”.

Thus, the correspondence between the two kinds of problems is simply the restriction of the index set, \( \mathbb{N} \rightarrow I_{\text{PR}} \).

**Definition 8.** If \( n \) is a positive integer, the decision problem \( P \) is located at level \( n \) of the AH if \( P \in \Sigma_n \cup \Pi_n \) but \( P \not\in \Delta_n \). If \( P \in \Sigma_n \setminus \Pi_n \), we say that \( P \) is located in \( \Sigma_n \) and similarly, if \( P \in \Pi_n \setminus \Sigma_n \), we say that \( P \) is located in \( \Pi_n \). If a decision problem \( P \) is \( \Sigma_n \)-complete or \( \Pi_n \)-complete, then \( P \) is located at level \( n \). For \( n \geq 2 \) we say that a set in \( \Delta_n \setminus (\Sigma_{n-1} \cup \Pi_{n-1}) \) is located at level \( n - \frac{1}{2} \) of the AH. If a PR decision problem is located at level \( m \) of the AH and the corresponding ParRec problem is located at level \( n \) of the AH we say that the undecidability jump (or JUMP) is \([m \rightarrow n]\). The undecidability discrepancy (or DISCREP) between those problems is \(|n - m|\).

It is a consequence of the Arithmetic Hierarchy Theorem (see for instance [35, Theorem IV.1.13]) that for every positive integer \( n \) there exist decision problems located at level \( n \) and decision problems located at level \( n + \frac{1}{2} \).

A partition of the arithmetical predicates ([18]) is

\[
\Delta_1, \text{cplt}(\Sigma_1), \text{cplt}(\Pi_1), [\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)], \text{cplt}(\Sigma_2), \text{cplt}(\Pi_2), [\Delta_3 \setminus (\Sigma_2 \cup \Pi_2)], \ldots
\]

where cplt(\( X \)) means the class of problems complete in \( X \).

Theorem [19] (page [19]) is a simple consequence of the fact that every PR index is also a ParRec index, that is, \( I_{\text{PR}} \subset \mathbb{N} \). We now show that the index restriction \( \mathbb{N} \rightarrow I_{\text{PR}} \) does not increase the AH level in which a decision problem is located.

**Theorem 28.** Let \( P \) be a ParRec decision problem and let \( P^{\text{PR}} \) be the corresponding PR version. If \( P \) and \( P^{\text{PR}} \) are located respectively at the levels \( n \) and \( m \) (possibly a half-integers) of the arithmetic hierarchy, then \( n \geq m \).

**Proof.** Suppose first that \( P \) is located in \( \Sigma_n \) or in \( \Pi_n \). Let \( P \) and \( P^{\text{PR}} \) be expressed as St(\( \varphi_e \)) and St(\( \varphi_p \)), respectively; see Definition [7] page [37]. The two statements are identical and can be written in the same Tarski-Kuratowski normal form (see for instance [14, Chapter 14]). It follows that \( P^{\text{PR}} \) is also at level \( n \) (but possibly not located at level \( n \)). The level \( m \) at which \( P^{\text{PR}} \) is located must then satisfy \( m \leq n \).

If \( P \) is located in \( \Delta_m \), the corresponding statement has the form \( \text{St}(\varphi_e) \land \text{St}'(\varphi_e) \), where one conjunct corresponds to \( \Sigma_m \) and the other to \( \Pi_m \). The problem \( P^{\text{PR}} \) also satisfies \( \text{St}(\varphi_p) \land \text{St}'(\varphi_p) \), and we may reason as above to show that \( m \leq n \).
The restriction $I_{PR} \to \mathbb{N}$ may reduce the degree of undecidability of a problem, see Figure 4 (page 20) and Figure 5 (page 25). After analysing the JUMP of a few decision problems we will conjecture that all non-negative JUMPs and every non-negative DISCREPs are possible.
| Problem                        | Function | Instance | Statement            | AH Class      | JUMP  | DISCREP |
|-------------------------------|----------|----------|----------------------|---------------|-------|---------|
| 1                             | \(\exists x : f(x) = 1\) | PR \(p\) | \(\exists x : \varphi_p(x) = 1\) | \(\Sigma_1\)-complete | \([1 \rightarrow 1]\) | 0       |
|                               | ParRec \(e\) | \(\exists x : \varphi_e(x) = 1\) | \(\Sigma_1\)-complete |               |       |         |
| 2                             | | PR \(p\) | \(|\text{cod}(\varphi_p)| = 1\) | \(\Pi_1\)-complete | \([1 \rightarrow \frac{3}{2}]\) | \(\frac{1}{2}\) |
|                               | ParRec \(e\) | \(|\text{cod}(\varphi_e)| = 1\) | \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\) |               |       |         |
| 3                             | EXACTLY-ONE-ZERO | PR \(p\) | \(\exists ! x : \varphi_p(x) = 0\) | \(\Delta_2 \setminus (\Sigma_1 \cup \Pi_1)\) | \([\frac{3}{2} \rightarrow 2]\) | \(\frac{1}{2}\) |
|                               | ParRec \(e\) | \(\exists ! x : \varphi_e(x) = 0\) | \(\Pi_2\)-complete |               |       |         |
| 4                             | HP       | PR \((p, x)\) | \(\exists t : T(p, x, t) = 0\) | recursive | \([0 \rightarrow 1]\) | 1       |
|                               | ParRec \((e, x)\) | \(\exists t : T(e, x, t) = 0\) | \(\Sigma_1\)-complete |               |       |         |
| 5                             | TOTAL    | PR \(p\) | \(\forall x \exists t : T(p, x, t) = 0\) | recursive | \([0 \rightarrow 2]\) | 2       |
|                               | ParRec \(e\) | \(\forall x \exists t : T(e, x, t) = 0\) | \(\Pi_2\)-complete |               |       |         |
| 6                             | RECURSIVE | PR \(p\) | \(W_p \) is recursive | recursive | \([0 \rightarrow 3]\) | 3       |
|                               | ParRec \(e\) | \(W_e \) is recursive | \(\Sigma_3\)-complete |               |       |         |

Figure 11: The degree of undecidability of six decision problems is compared. EOZ means EXACTLY-ONE-ZERO. For each problem two cases are considered: the instance is a partial recursive (ParRec) function and the instance is a primitive recursive (PR) function.
Some decision problems: discrepancy

**Problem 28.** “∃x : f(x) = 1?”. Line 1 of Figure 11 page 39

In detail the ParRec statement is \( \exists x : (T(e, x, t) = 0) \land (U(e, x, t_0) = 1) \), where \( t_0 \) is a value of \( t \) satisfying \( T(e, x, t_0) = 0 \) and the notation of Theorem 1 (page 6) is used. □

**Problem 29.** “|CODOMAIN| = 1?”. Line 2 of Figure 11 page 39. See item 10, page 22. □

**Problem 30.** EXACTLY-ONE-ZERO. Line 3 of Figure 11, page 39. See item 5 (page 19) and Theorem 4 (page 12). □

**Problem 31.** HP, the halting problem. Line 4 of Figure 11, page 39. We use the notation explained in Definition 3 (page 6). \( \langle e, x \rangle \) and \( \langle p, x \rangle \) denote respectively bijections \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) and \( \mathbb{I}_{PR} \times \mathbb{N} \rightarrow \mathbb{N} \). □

**Problem 32.** TOTAL problem. Line 5 of Figure 11, page 39. □

**Problem 33.** RECURSIVE problem. Line 6 of Figure 11, page 39

Let RECURSIVE be the set defined in [49, Definition 4.14 (page 21)], \( \text{RECURSIVE} = \{ e | W_e \text{ is recursive}\} \). This set is \( \Sigma_3 \)-complete, see for instance [44, §14.8, Theorem XVI (page 327)] and [49, Corollary 3.5 (page 66)]. The PR version of this problem is \( \text{RECURSIVE}^{PR} = \{ p | W_p \text{ is recursive}\} \). Once \( \forall p \in \mathbb{I}_{PR} : W_p = \mathbb{N} \), we have \( \text{RECURSIVE}^{PR} = \mathbb{I}_{PR} \), a recursive set. □

A conjecture

Looking to Figure 12 (page 41), to which other examples could easily be added, the following conjecture seems natural.

**Conjecture 1.** For every pair \( n \geq m \geq 0 \) with \( n, m \in \{0, 1, 3/2, 2, 5/2, 3, 7/2, \ldots\} \) there is a ParRec decision problem located at level \( n \) of the arithmetic hierarchy such that the corresponding PR decision problem is located at level \( m \), that is, with JUMP = \( [m \rightarrow n] \) (the case \( m > n \) is impossible by Theorem 28, page 37).

If this conjecture is true, the undecidability discrepancy between a ParRec problem and the corresponding PR can be any non-negative integer and any half-integer greater than 1.

6.1 Positive discrepancy: why?

We have seen that some decision problems have DISCREP>0. There are several reasons for this. For instance

1. **PR functions are total.**
   That is, \( \forall p \forall x \exists t : T(p, x, t) \), using the notation of Definition 3 page 3. This the reason that applies to all problems with DISCREP>0 that we studied so far.

2. **There are recursive functions that grow “too fast”.**
   There are recursive functions that are asymptotic upper bounds of every PR function. As a consequence, there are decision problems that are trivial for PR functions but undecidable for ParRec functions. One example is \( g(n) \equiv A(n, n) \) where \( A \) is the Ackermann function.
Figure 12: Levels in the arithmetic hierarchy of the PR (left) and ParRec (right) versions of 10 decision problems, including Examples 28 (page 40) to 33 (page 40) and other problems studied in this work. In the four lower decision problems the PR and ParRec levels are identical. All decision problems (in both versions) located in integer AH levels are complete in the corresponding class. (The level $\frac{1}{2}$ does not exist because $\Sigma_0 = \Pi_0 = \Delta_0 = \Delta_1$.)
as defined in [5] or [18]. The decision problem\textsuperscript{10} “given \( f, \exists n_0 \forall n \geq n_0 : f(n) < g(n) ? \)” is trivial for PR functions but undecidable for ParRec functions.

(3) There are also “small” recursive functions that are not PR.
There are recursive functions bounded by a constant that are not PR. An example is the function:

\[
\begin{align*}
    h(p) &= \begin{cases} 
        0 & \text{if } p \in I_{PR} \text{ and } \varphi_p(p) \neq 0 \\
        1 & \text{otherwise.}
    \end{cases}
\end{align*}
\]

The function \( h(p) \) has codomain \( \{0, 1\} \), is recursive, and not PR. However, the ParRec decision problem “given \( e \), is \( \varphi_e = h \)?” is undecidable.

(4) The PR index gives information about the function.
From the index \( p \) of a PR function \( \varphi_p \) we can obtain some information about the function \( \varphi_p \), for instance the PR-complexity of \( \varphi_p \), see [20, 21]. But no information about a ParRec function \( f \) can be obtained from a corresponding index (Rice Theorem, see for instance [44, 4, 11]).

We see that there are several very different reasons for a positive discrepancy. Any definition of a class of total recursive functions has as a consequence that some properties hold for all members of the class; for instance (1)-(4) hold for PR functions. These properties may be undecidable for ParRec functions but trivial for PR functions. This fact leads us to conjecture the following.

Conjecture 2. There is no algorithm that, given a decision problem about a function, computes the discrepancy between the ParRec and the PR versions of that problem.

7 Conclusions and open problems

The primitive recursive (PR) functions are an important subclass of the recursive (total computable) functions. Many interesting decision problems related to PR functions are undecidable. In this work we studied the degree of undecidability of a relatively large number of these problems. All these problems are either m-complete in the corresponding class of the arithmetic hierarchy or belong to the class \( \Delta_2 \setminus (\Sigma_1 \cup \Pi_1) \). Their degree of undecidability was compared with the corresponding decision problems associated with partial recursive (ParRec) functions, as exemplified in Figure 5, page 25.

The exact conditions for the decidability of the general decision problem “\( \exists \overline{x} : h(f(g(\overline{x}))) = 0 \)”, where \( g \) and \( h \) are fixed PR functions (\( g \) may have multiple outputs) and \( f \) is the instance of the problem (a PR function), have been established.

A more general situation was also studied: the instance of the problem, a PR function \( f \), is part of an arbitrary PR acyclic graph \( S \) (having \( f \) and fixed primitive recursive functions as nodes). Let the corresponding function be \( S(f, \overline{x}) \); the question is “\( \exists \overline{x} : S(f, \overline{x}) = 0 \)”. A normal form for these acyclic function graphs was obtained, namely \( h(\overline{x}, f(g(\overline{x}))) \); in contrast with the \( h(f(g(\overline{x}))) \) class mentioned above, we conjecture that there is no closed necessary and sufficient condition for decidability.

\textsuperscript{10} A similar function \( 2 \uparrow n \) where the Knuth superpower notation is used, see for instance [20, 20].

\textsuperscript{11} Where “\( f(n) < g(n) \)” means that \( f(n) \) is defined and its value is less than \( g(n) \).
Another generalisation was briefly studied: problems in which the instance \( f \) can be composed with itself.

Of course, many problems remain open. Some of the more interesting are:

- Clarify the relation between PR and ParRec functions, namely
  
  (i) Given any decision problem about a function, relate the degree of undecidability in two cases: when the function is PR and when the function is ParRec. In Section 6 (Conjecture 2 page 42) we conjecture that this may be not possible.

  (ii) Is every undecidability discrepancy possible? Proof or disproof the Conjecture 1 (page 40). We saw decision problems for which the undecidability discrepancy is 0, 1/2, 1, 2, and 3, see Section 5 (page 37) and Figure 12 (page 41).

  (iii) Study decision problems whose positive undecidability discrepancy is due to other reasons (besides the fact that every PR function is total); see the discussion in Section 5 page 37.

- Find closed decidability conditions for the general system \( S(f, x) \) described above.

- Study in general the decidability problems in which several occurrences of the instance \( f \) may occur; this includes expressions in which \( f \) composes with itself.

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