Research Article

Asymmetric Truncated Hankel Operators: Rank One, Matrix Representation

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Asymmetric truncated Hankel operators are the natural generalization of truncated Hankel operators. In this paper, we determine all rank one operators of this class. We explore these operators on finite-dimensional model spaces, in particular, their matrix representation. We also give their matrix representation and the one for asymmetric truncated Toeplitz operators in the case of model spaces associated to interpolating Blaschke products.

1. Introduction

Let $H^2$ be the standard Hardy space of the unit disc $\mathbb{D}$ identified with the subspace of the boundary functions of its functions in $L^2(\mathbb{T})$.

A function in $H^\infty(\mathbb{D})$ is inner if it is unimodular on the unit circle $\mathbb{T}$. To each inner function $\alpha$ we associate a model space, $K_\alpha = H^2 \ominus \alpha H^2$. The model space is a reproducing kernel Hilbert space with reproducing kernel, $k_\alpha(z) = (1 - \frac{\alpha(\bar{\lambda})\overline{\alpha(z)}}{\lambda - z})$, for $z, \lambda \in \mathbb{D}$.

The inner function $\alpha$ has an angular derivative in the sense of Carathéodory (ADC) at a point $\eta \in \mathbb{T}$ if and only if every $f$ in $K_\alpha$ has nontangential limit at $\eta$. In particular, $k_\alpha^\eta(z) \in K_\alpha$.

If $\alpha$ is an inner function, then $\alpha^\eta = \overline{\alpha(z)}$ is also an inner function and the associated model space is noted $K_{\alpha^\eta}$.

To each model space, we associate a natural conjugation $C_\alpha$ such that $C_\alpha K_\alpha = K_{\alpha^\eta}$, given by $C_\alpha f(z) = \overline{\alpha(z)f(z)}$, for $z \in \mathbb{T}$. The image of the kernel function for $C_\alpha$, called conjugate kernel function, $\tilde{k}_\alpha^\eta(z)$, is given by $\tilde{k}_\alpha^\eta(z) = (\overline{\alpha(z)} - \alpha(\bar{\lambda}))/(z - \lambda)$, $z, \lambda \in \mathbb{D}$. If $\alpha$ has an ADC at $\eta \in \mathbb{T}$,

$$
\tilde{k}_\alpha^\eta(z) = \frac{\alpha(z) - \alpha(\bar{\eta})}{z - \eta} = \frac{\alpha(\eta) - \overline{\alpha(z)}}{\eta - \bar{z}} = \alpha(\eta)\overline{\alpha(z)}\tilde{k}_\alpha^\eta(z).
$$

The model spaces can also be defined as the only invariant subspaces of the backward shift operator $S^*$ on $H^2$. Denote by $S_\alpha(S^\alpha)$ the compression of the shift operator (resp., backward shift) to the model space $K_\alpha$.

The only finite-dimensional model spaces are the one associated to finite Blaschke products, and its dimension is the same as the multiplicity of the associated Blaschke product. Since a finite Blaschke product is analytic on a neighborhood of the unit disc, it has an ADC everywhere on $\mathbb{T}$ and $k_\alpha^\eta \in K_\alpha$, for every $\eta \in \mathbb{T}$. In this case, let $m$ denote the multiplicity of a finite Blaschke product $\alpha$. If we arbitrarily choose a collection $\{\lambda_i, i = 1, \cdots, m\}$ of distinct points in $\mathbb{D}$, $\{k_\alpha^\eta, i = 1, 2, \cdots, m\}$ forms a basis for $K_\alpha$.

In infinite-dimensional case the kernel functions $(k_\alpha^\eta)_{\eta \in \mathbb{T}}$ form a Riesz basis if and only if $\alpha$ is an interpolating Blaschke product or equivalently, $(\lambda_i)_{i \in \mathbb{N}}$ is a uniformly separated Blaschke sequence that satisfies $\inf_{k \in \mathbb{N}} \prod_{k \leq k' \leq k''}(\lambda_k - \lambda_{k''})/(1 - \lambda_k \lambda_{k''}) > 0$.

Asymmetric truncated Toeplitz and Hankel operators were first introduced in [1, 2], respectively. For $\varphi \in L^2(\mathbb{T})$, let $\alpha$ and $\beta$ be two inner functions, the asymmetric truncated Toeplitz operator $A^\varphi_{\alpha \beta}$ and the asymmetric truncated Hankel operator $B^\varphi_{\alpha \beta}$ are defined on $K_\alpha^\infty = K_\alpha \cap H^\infty$ by
\[ A^αβ_ψ : K^∞_α \rightarrow K^∞_β, \quad (2) \]

\[ f \mapsto A^αβ_ψ(f) = P_β(ϕf), \quad (3) \]

\[ B^αβ_ψ : K^∞_α \rightarrow K^∞_β, \quad (4) \]

\[ f \mapsto B^αβ_ψ(f) = P_β(I - P)(ϕf), \quad (5) \]

where \( P \) and \( P_β \) denote the orthogonal projection on \( H^2 \) and \( K_β \), respectively, and \( I \) is the flip operator from \( H^2_β \) onto \( H^2_0 = \mathbb{C}H^2 \) defined on \( T \) by \( Jf(z) = \bar{zf}(\bar{z}) \). We denote the set of bounded asymmetric truncated Toeplitz operators (ATTO) (bounded asymmetric truncated Hankel operators (ATHO)) by \( \mathfrak{S}(α, β) \). 

In [5], Sarason introduced truncated Toeplitz operators and showed that all rank one TTOs are of the form \( c\, k_α^c \otimes \bar{k}_α^c \) or \( c\, k_β^c \otimes \bar{k}_α^c \), where \( c \in \mathbb{C} \) and \( λ \in \mathbb{D} \) or in \( \mathbb{T} \) such that \( α \) has an ADC at \( λ \). Similar results were obtained in [4] for truncated Hankel operators \((\bar{k}_λ^c \otimes k_λ^c) \, k_α^c \otimes k_β^c \). Surprisingly, this is not always true in the case of asymmetric truncated Toeplitz operators.

**Theorem 1** (see [5, 6]). Let \( α \) and \( β \) be two inner functions.

1. Let \( ω \in \mathbb{D} \) or \( ω \in \mathbb{T} \) such that \( α \) and \( β \) have an ADC at \( ω \)

\[ k_ω^β \otimes \bar{k}_ω^α = A^αβ_ω \in \mathfrak{S}(α, β) \] and \( k_ω^β \otimes \bar{k}_ω^α = A^αβ_ω \in \mathfrak{S}(α, β) \). \quad (6)

2. The only rank one asymmetric truncated Toeplitz operators in \( \mathfrak{S}(α, β) \) are nonzero scalar multiples of \( k_0^α \otimes \bar{k}_0^α \) and \( k_0^β \otimes \bar{k}_0^α \) where \( α \in \mathbb{D} \) or \( ω \in \mathbb{T} \) such that \( α \) and \( β \) have an ADC at \( ω \) if and only if \( \{m \geq 2 \} \) or \( \{m > 1 \text{ and } n > 1 \} \), where \( m, n \in \{0, \infty\} \) are the dimensions of \( K_α \) and \( K_β \), respectively

In [7], Cima et al. raised the question of which linear transformations on finite-dimensional model spaces are truncated Toeplitz operators, and they proved that the matrix representation of TTOs with respect to kernel basis (conjugate kernel basis, Clark and modified Clark bases) is entirely determined by the entries of the main diagonal and first row. In [8, 9], Lanucha obtained similar results for TTOs acting on model spaces associated to interpolating Blaschke products and for THOs. In [6], Jurak and Lanucha generalized these results to asymmetric truncated Toeplitz operators on finite-dimensional model spaces.

This paper determines all rank one asymmetric truncated Hankel operators and generalizes the results about matrix representation to ATTOs and ATTOs on special model spaces. In Section 2, we cite some results from [3, 10–12]. We precise all rank one ATTOs in Section 3. In Section 4, we explore the ATTOs on finite-dimensional spaces, we calculate the dimension of \( \mathfrak{S}(α, β) \) and exhibit a basis for it, and we give the matrix representation of ATTOs in finite dimensional model spaces associated to Blaschke products each with distinct zeros. Section 5 is dedicated to the matrix representation of ATTOs and ATTOs on special infinite-dimensional model spaces, in particular, we study the action of the unitary operator \( V_{ξ} = \sqrt{dξ_1 \circ τ_ξ} \), where \( τ_ξ(z) \) is disc automorphism.

**2. Preliminaries**

Like for truncated Toeplitz and Hankel operators, the symbol of an ATTO and an ATTO is not unique, in fact, we have the following theorem.

**Theorem 2** (see [10, 12]). Let \( α \) and \( β \) be inner functions. For \( ϕ \in L^2(\mathbb{T}) \), we have \( A^αβ_ϕ = 0 \) if and only if \( ϕ \in αH^2 + βH^2 \) and \( B^αβ_ϕ = 0 \) if and only if \( ϕ \in H^2 + αβH^2 \). Where \( β^α(\bar{z}) = \bar{β}(z) \).

We will use the following technical lemma.

**Lemma 3** (see [3]). Let \( α \) be inner, for \( λ \in \mathbb{D} \), we have

\[ S_αk^α_λ = \bar{λ}k^α_λ - α(λ)k^α_0, \quad \text{and} \quad S_αk^α_λ = \bar{λ}k^α_λ - α(λ)k^α_0. \quad (7) \]

For \( λ \in \mathbb{D} \setminus \{0\} \), we have

\[ S_αk^α_λ = \frac{1}{λ}k^α_λ - \frac{1}{λ}k^α_0. \quad (8) \]

Clark, in [11], proved that the only unitary rank one perturbations of the compressed shift operator are

\[ U^α_λ = S_α + \frac{α(0) + λ}{1 - |α(0)|^2} k^α_0 \otimes \bar{k}^α_0, \quad \text{for} \quad λ \in \mathbb{T}, \quad (9) \]

and that the point spectrum of \( U^α_λ \) is the set of the solutions of the equation \( α(η) = α(0) + λ(1 + α(0)λ) \) at which \( α \) has an ADC, denote them by \( η_m \). The corresponding eigenvectors are the normalized boundary kernels \( v^α_η \) corresponding to the points \( η_m \). Whenever the point spectrum of \( U^α_λ \) is pure, the family of eigenvectors forms a basis for \( K_α \) called the Clark basis.

The modified Clark basis \( (e^α_η)_m \) satisfies \( C_αe^α_η = e^α_η \) and is given by \( e^α_η = η_m^αv^α_η \), where \( η_m^α = \exp(-i(\arg η_m - \arg λ)) \).

One of the most important results about TTOs and THOs is their characterization in terms of compressed shift operator and operators of rank at most 2. Recently, the authors in [10] proved similar characterizations for both ATTO and ATTO.

**Theorem 4** (see [10]). Let \( A \) be a bounded linear operator from \( K_α \) to \( K_β \). Then
(1) \( A \in \mathfrak{T}(\alpha, \beta) \) if and only if
\[
A - S_\beta A S_\alpha^* = \psi \otimes k_\alpha^\beta + k_0^\beta \otimes \chi,
\]
for some \( \psi \in K_\alpha, \chi \in K_\beta \) if and only if
\[
A - U_\beta^\alpha A \left( U_\beta^\alpha \right)^* = \psi \otimes k_\alpha^\beta + k_0^\beta \otimes \chi,
\]
for some \( \psi \in K_\alpha, \chi \in K_\beta \) and some \( \lambda_\alpha, \lambda_\beta \) in \( \mathbb{T} \).

(2) \( B \in \mathfrak{S}(\alpha, \beta) \) if and only if
\[
B - S_\beta B S_\alpha^* = \psi \otimes k_\alpha^\beta + k_0^\beta \otimes \chi,
\]
for some \( \psi \in K_\alpha, \chi \in K_\beta \) and some \( \lambda_\alpha, \lambda_\beta \) in \( \mathbb{T} \).

Where \( U_\beta^\alpha, U_\alpha^\beta \) are Clark operators (9).

In the same paper [10], we also have the following theorem.

**Theorem 5** (see [10]). Let \( \alpha \) and \( \beta \) be two inner functions and \( \varphi \in L^2(\mathbb{T}) \). Then

(1) \( A \in \mathfrak{T}(\alpha, \beta) \) if and only if \( C_\beta A C_\alpha \in \mathfrak{T}(\alpha, \beta) \). In addition, \( C_\beta A^\alpha \varphi C_\alpha = A_\lambda^\alpha \beta \varphi \)

(2) \( B \in \mathfrak{S}(\alpha, \beta) \) if and only if \( C_\beta B C_\alpha \in \mathfrak{S}(\alpha, \beta) \). In addition, \( C_\beta B_\beta^\alpha C_\alpha = B_\lambda^\alpha \beta \varphi \)

(3) \( C_\beta A^\alpha \beta f^\# = B_\lambda^\alpha \beta \varphi^\# \), where \( f^\# \) is a conjugation on \( L^2(\mathbb{T}) \)

defined by \( f^\# = f(\overline{\lambda}) \)

3. Asymmetric Truncated Hankel Operators of Rank One

In this section, we describe all rank one asymmetric truncated Hankel operators through the results for asymmetric truncated Toeplitz operators. In what follows, let \( \alpha \) and \( \beta \) denote two arbitrary inner functions.

The following proposition gives some rank one asymmetric truncated Hankel operators.

**Proposition 6.** We have

(1) For every \( \lambda \in \mathbb{D} \), the operators \( k_\lambda^\beta \otimes k_\lambda^\alpha = B_\lambda^\beta \alpha \) and \( k_\lambda^\beta \otimes k_\lambda^\alpha = B_\lambda^\beta \alpha \) belong to \( \mathfrak{S}(\alpha, \beta) \)

(2) If \( \alpha \) and \( \beta \) have an ADC at \( \eta \) and \( \overline{\eta} \), respectively, the operators \( k_\eta^\beta \otimes k_\eta^\alpha = B_\eta^\beta \alpha \) and \( k_\eta^\beta \otimes k_\eta^\alpha = B_\eta^\beta \alpha \) belong to \( \mathfrak{S}(\alpha, \beta) \)

Proof. From Theorem 5, we have
\[
C_\beta A^\alpha \beta f^\# = B_\lambda^\alpha \beta \varphi^\#,
\]
or
\[
C_\beta A^\alpha \beta f^\# = B_\lambda^\alpha \beta \varphi^\#
\]

Choose an arbitrary \( \lambda \in \mathbb{D} \) or \( \lambda \in \mathbb{T} \) such that \( \alpha^\# \) and \( \beta^\# \) have an ADC at \( \lambda \), by Theorem 1, we have

\[
C_\beta \left( k_\lambda^\beta \otimes k_\lambda^\alpha \right) f^\# = C_\beta A^\alpha \beta \alpha^\# = k_\lambda^\beta \otimes k_\lambda^\alpha,
\]
\[
C_\beta \left( k_\lambda^\beta \otimes k_\lambda^\alpha \right) f^\# = C_\beta A^\alpha \beta \alpha^\# = k_\lambda^\beta \otimes k_\lambda^\alpha.
\]

For \( \lambda \in \mathbb{D} \), the operators \( k_\lambda^\beta \otimes k_\lambda^\alpha \) and \( k_\lambda^\beta \otimes k_\lambda^\alpha \) belong to \( \mathfrak{S}(\alpha, \beta) \). This is also true for \( \lambda \in \mathbb{T} \), since \( \alpha^\# \) has an ADC at \( \lambda \), if and only if \( \alpha \) has an ADC at \( \lambda \).

Now, we give the main theorem of this section.

**Theorem 7.** All rank one operators in \( \mathfrak{S}(\alpha, \beta) \) are the non-zero scalar multiples of \( k_\Lambda^\beta \otimes k_\Lambda^\alpha \) and \( k_\Lambda^\beta \otimes k_\Lambda^\alpha \), where \( \lambda \in \mathbb{D} \) or \( \lambda \in \mathbb{T} \) such that \( \alpha \) and \( \beta \) have an ADC at \( \lambda \) and \( \overline{\lambda} \), respectively, if and only if \( \{mn \leq 2\} \) or \( \{m > 1 \text{ and } n > 1\} \), where \( m, n \in \mathbb{N} \cup \{+\infty\} \) are the dimensions of \( K_\alpha \) and \( K_\beta \), respectively.

Proof. Every rank one operator in \( \mathfrak{S}(\alpha, \beta) \) is of the form \( f \otimes g \) for \( f \in K_\beta \) and \( g \in K_\alpha \). Since \( \mathfrak{S}(\alpha, \beta) = C_\beta \mathfrak{T}(\alpha^\#, \beta^\#) \) ([10]), we can find \( f' \in K_\beta \) and \( g' \in K_\alpha \) such that \( f' \otimes g' = C_\beta f' \otimes g'^\# \). For \( f' \otimes g' \) is rank one in \( \mathfrak{T}(\alpha^\#, \beta) \), by Theorem 1, there exist \( c \in \mathbb{C} \) and \( \lambda \in \mathbb{D} \) or \( \lambda \in \mathbb{T} \) (\( \alpha^\#, \beta \) have an ADC at \( \lambda \)) such that \( f' \otimes g' = c k_\lambda^\beta \otimes k_\lambda^\alpha \) or \( f' \otimes g' = c k_\lambda^\beta \otimes k_\lambda^\alpha \) if and only if \( \{mn \leq 2\} \) or \( \{m>1 \text{ and } n>1\} \). Finally, \( f \otimes g = c C_\beta (k_\lambda^\beta \otimes k_\lambda^\alpha)^\# = c k_\lambda^\beta \otimes k_\lambda^\alpha \) or \( f \otimes g = c C_\beta (k_\lambda^\beta \otimes k_\lambda^\alpha)^\# = c k_\lambda^\beta \otimes k_\lambda^\alpha \) if and only if \( \{mn \leq 2\} \) or \( \{m>1 \text{ and } n>1\} \).
4. Asymmetric Truncated Hankel Operators in Finite-Dimensional Model Spaces

In this section, we suppose that both \( \alpha \) and \( \beta \) are finite Blaschke products of respective multiplicities \( m \) and \( n \).

4.1. Dimension and Basis of \( \mathcal{H}(\alpha, \beta) \)

Theorem 8. Let \( K_\alpha \) and \( K_\beta \) have dimensions \( m \) and \( n \), respectively, then the dimension of \( \mathcal{H}(\alpha, \beta) \) equals \( m + n - 1 \).

Proof. Using Theorem 2, we can write

\[
\mathcal{H}(\alpha, \beta) = \left\{ B^{\alpha, \beta}_\lambda : \chi \in K_{\alpha^{m+n}} \right\},
\]

where \( \dim K_{\alpha^{m+n}} = m + n \). Since the constants are in \( K_{\alpha^{m+n}} \) and \( B_c^{\alpha, \beta} = 0 \) for all \( c \in \mathbb{C} \),

\[
\dim \mathcal{H}(\alpha, \beta) = \dim K_{\alpha^{m+n}} - 1 = m + n - 1.
\]

\( \square \)

Remark 9. From Theorem 4, \( C_c \mathcal{H}(\alpha^{m+n}, \beta) = \mathcal{H}(\alpha, \beta) \). Since \( C_\beta \) and \( J^\# \) preserve the dimensions, \( \dim \mathcal{H}(\alpha, \beta) = \dim \mathcal{H}(\alpha^{m+n}, \beta) = m + n - 1 \).

Theorem 10. Let \( \alpha, \beta \) be two finite Blaschke products of respective multiplicities \( m \) and \( n \). Then for any \( m + n - 1 \) distinct points from \( \mathbb{D} \), denoted by \( \{ \lambda_j \}_{j=1}^{m+n-1} \),

1. \( \{ k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j} : i = 1, \ldots, m + n - 1 \} \) is a basis of \( \mathcal{H}(\alpha, \beta) \)
2. \( \{ k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j} : i = 1, \ldots, m + n - 1 \} \) is also a basis of \( \mathcal{H}(\alpha, \beta) \).

Proof. We only prove (1), the other case follows by application of the conjugation of the coputations. From [6], a basis of \( \mathcal{H}(\alpha^{m+n}, \beta) \) is \( \{ k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j} : i = 1, \ldots, m + n - 1 \} \). Using the proof of Proposition 6, \( \{ C_{\beta}(k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j}) J^\# = k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j} : i = 1, \ldots, m + n - 1 \} \) is a basis of \( \mathcal{H}(\alpha, \beta) \).

\( \square \)

Remark 11. We can prove the result directly as in [6] for ATTOs.

4.2. Matrix Representation of ATHO on Finite-Dimensional Spaces

In this subsection, we give the matrix representation of an ATHO acting on two finite dimensional model spaces, with respect to kernel and conjugate kernel bases, Clark and modified Clark bases. In all this section, suppose that the inner function \( \alpha \) has distinct zeros \( (a_i)_{j=1}^m \) and \( \beta \) has distinct zeros \( (b_j)_{j=1}^n \).

4.2.1. Matrix Representation with respect to the Kernel Bases and Conjugate Kernel Bases. Choose \( m + n - 1 \) points \( \{ \lambda_j \}_{j=1}^{m+n-1} \) in \( \mathbb{D} \) distinct from \( (b_j)_{j=1}^n \) then \( \{ k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j} : i = 1, \ldots, m + n - 1 \} \) is a basis for \( \mathcal{H}(\alpha, \beta) \). Since the zeros of the considered inner functions are distinct, \( \{ k^\beta_{\lambda_j} \otimes k^\alpha_{\lambda_j} : i = 1, \ldots, m \} \) and \( \{ k^\beta_{\lambda_j} : i = 1, \ldots, n \} \) are bases of \( K_\alpha \) and \( K_\beta \) (same for the conjugate kernel functions). We denote the matrix representation of a bounded operator \( B \) with respect to the abovementioned kernel bases (conjugate kernel bases) by \( \mathcal{H}(\alpha, \beta) \).

\[
s_{ij} = \frac{1}{\beta^j(b_j)} \sum_{q=1}^{m+n-1} d_{ij} k^\beta_{\lambda_q} \otimes k^\alpha_{\lambda_q}. \]

We have

\[
B^{\alpha, \beta}_\lambda = \left( \sum_{q=1}^{m+n-1} d_{ij} k^\beta_{\lambda_q} \otimes k^\alpha_{\lambda_q} \right) k^\beta_{\lambda_j} = \sum_{q=1}^{m+n-1} d_{ij} \lambda_j - b_j \lambda_q.
\]

Now replace (20), we have

\[
s_{ij} = \frac{1}{\beta^j(b_j)} \sum_{q=1}^{m+n-1} d_{ij} \lambda_j - b_j \lambda_q - b_j \lambda_q - b_j.
\]

We can write

\[
\sum_{q=1}^{m+n-1} d_{ij} \lambda_j - b_j \lambda_q = \left( \frac{1}{\lambda_j - b_j} - \frac{1}{\lambda_j} \right) \frac{1}{\lambda_j - b_j}.
\]

By adding and subtracting \( 1 - b_j/\lambda_q - b_j \) and \( 1 - a_i/1 - \lambda_q a_i \), we find

\[
s_{ij} = \frac{\beta^j(b_j)(1 - a_i/1 - b_j)}{\beta^j(b_j)(1 - a_i/1 - b_j)}.
\]
Conversely, suppose that the matrix representation of the bounded transformation $B$ satisfies (21). From the proof of the first implication, we know that the subspace of the matrices satisfying (21) is a subspace of $\mathcal{S}(\alpha, \beta)$, and its dimension is obviously $m + n - 1$. But $\dim \mathcal{S}(\alpha, \beta) = m + n - 1$, we have the equality. \hfill $\Box$

The following theorem establishes the matrix representation with respect to conjugate kernel bases.

**Theorem 13.** Let $B$ be a bounded linear transformation from $K_{\alpha}$ to $K_{\beta}$. $B \in \mathcal{S}(\alpha, \beta)$ if and only if for $1 \leq i \leq n$ and $1 \leq j \leq m$, \[ p_{ij} = \frac{\beta'(b_j) - \beta'(b_j)(1 - a_j b_i) p_{ij} + \beta'(b_j)(1 - a_j b_i) p_{ij}}{\beta'(b_i)(1 - a_i b_j)}. \] \hfill (26)

**Proof.** We have \[ p_{ij} = \frac{1}{\beta'(b_i)} \left( C_{\beta, \alpha} a_{\alpha} b_{\beta} k_{\beta} b_{\beta} \right) = \frac{1}{\beta'(b_i)} \left( B_{\alpha} a_{\alpha} b_{\beta} k_{\beta} b_{\beta} \right) \]

by Theorem 12, it satisfies (21) and we have \[ p_{ij} = \frac{\beta'(b_j)(1 - a_j b_i) p_{ij} - \beta'(b_j)(1 - a_j b_i) p_{ij} + \beta'(b_j)(1 - a_j b_i) p_{ij} \beta'(b_i)(1 - a_i b_j)}{\beta'(b_i)(1 - a_i b_j)} \]

where $s_{ij}$ is the entry in the $i$th row and $j$th column of the matrix representing $B_{\alpha} a_{\beta} b_{\beta} \phi$. \hfill (27)

Therefore \[ p_{ij} = s_{ij}, \] \hfill (28)

4.2.2. Matrix Representation with respect to Clark and Modified Clark Bases. Let $a$ be $m$-finite Blaschke product and $\beta$ be $n$-finite Blaschke product. Choose arbitrary $\lambda_1, \lambda_2 \in \mathbb{T}$, then the following equations \[ \alpha(\eta) = \frac{\alpha(0) + \lambda_1}{1 + \alpha(0) \lambda_1} = \alpha_{\lambda_1} \text{ and } \beta(\eta) = \frac{\beta(0) + \lambda_2}{1 + \beta(0) \lambda_2} = \beta_{\lambda_2}, \] \hfill (30)

have exactly $m$ and $n$ distinct solutions, respectively, denoted by $(\eta)^m_i$ and $(\zeta)^n_i$. The families of normalized eigenvectors $\nu^\beta = \|k_{\eta}^\alpha\|^{-1} k_{\eta}^\alpha$ and $\nu^\beta = \|k_{\zeta}^\beta\|^{-1} k_{\zeta}^\beta$ form orthonormal bases for $K_{\alpha}$ and $K_{\beta}$, respectively. We also have \[ \|k_{\eta}^\alpha\| = \sqrt{\alpha' (\eta)} \text{ and } \|k_{\zeta}^\beta\| = \sqrt{\beta' (\zeta)}. \] \hfill (31)

Reorder the sequences $(\eta)^m_i$ and $(\zeta)^n_i$ such that $l$ be the maximal integer such that $\eta_i = \zeta_i$ for any $i$. Denote by $(t_{l,i})$, for $1 \leq i \leq n$ and $1 \leq j \leq m$, the matrix representation of any bounded transformation from $K_{\alpha}$ into $K_{\beta}$ with respect to Clark bases, and by $(u_{l,i})$, for $1 \leq i \leq n$ and $1 \leq j \leq m$, the matrix representation with respect to the modified Clark bases, we have \[ u_{l,i} = \langle B_{\alpha} a_{\zeta}, \nu^\beta \rangle = \omega^\beta a_{\zeta} \langle B_{\alpha} a_{\zeta}, \nu^\beta \rangle = \omega^\beta a_{\zeta}, t_{l,i}. \] \hfill (32)

**Theorem 14.** A bounded linear transformation $B$ from $K_{\alpha}$ to $K_{\beta}$ belongs to $\mathcal{S}(\alpha, \beta)$ if and only if

1. For $l \geq 1$ with $i \leq l$ and $i=j$ \[ t_{l,i} = \sqrt{\beta' (\zeta)} \left( \eta_j - \zeta_j \right)_i t_{l,i} \]

2. For $l = 0$ \[ t_{l,i} = \sqrt{\beta' (\zeta)} \left( \eta_j - \zeta_j \right)_i t_{l,i} \]

Proof. We proceed as in the proof of Theorem 12. \hfill $\Box$
We deduce the following theorem from Theorem 14 and formula (32).

**Theorem 15.** A bounded linear transformation $B$ from $K_\alpha$ to $K_\beta$ belongs to $\mathcal{S}(\alpha, \beta)$ if and only if for $1 \leq i \leq n$ and $1 \leq j \leq m$, except that $i = j \leq l$.

1. For $l = 1$ with $i = l$ and $i = j$

$$
\begin{align*}
\omega^\alpha \sqrt[\omega^\alpha]{|B'(\xi_i)|} (\eta_j - \xi_i) \\
\omega^\beta \sqrt[\omega^\beta]{|B'(\xi_i)|} (\eta_j - \xi_i) = u_{i,j}
\end{align*}
$$

(36)

For $l \geq 1$ with $i > l$,

$$
\begin{align*}
\omega^\alpha \sqrt[\omega^\alpha]{|B'(\xi_i)|} (\eta_j - \xi_i) \\
\omega^\beta \sqrt[\omega^\beta]{|B'(\xi_i)|} (\eta_j - \xi_i) = u_{i,j}
\end{align*}
$$

(37)

2. For $l = 0$

$$
\begin{align*}
\omega^\alpha \sqrt[\omega^\alpha]{|B'(\xi_i)|} (\eta_j - \xi_i) \\
\omega^\beta \sqrt[\omega^\beta]{|B'(\xi_i)|} (\eta_j - \xi_i) \\
\omega^\alpha \sqrt[\omega^\alpha]{|B'(\xi_i)|} (\eta_j - \xi_i) \\
\omega^\beta \sqrt[\omega^\beta]{|B'(\xi_i)|} (\eta_j - \xi_i) = u_{i,j}
\end{align*}
$$

(38)

5. **Matrix Representation in Infinite-Dimensional Case**

In all this section, unless mentioned, we will suppose that $\alpha$ and $\beta$ are two interpolating Blaschke sequences with respective zeros $(a_m)_{m \geq 1}$ and $(b_n)_{n \geq 1}$, then the kernel functions $(k_\alpha^a)_{a \neq 1}$ and conjugate kernel functions $(\bar{k}_\beta^b)_{b \neq 1}$ form two Riesz bases for $K_\alpha$ (same for $K_\beta$) (see [8,9]). For all sequence of complex numbers $(f_m)_{m \geq 1}$ such that $\sum_{m=1}^\infty |f_m|^2 (1 - |a_m|^2) < \infty$, the unique solution $f$ in $K_\alpha$ of the interpolation problem $f(a_m) = f_m$ is given by

$$
f = \sum_{i=1}^\infty \tilde{f}(a_i) k_\alpha^a = \sum_{i=1}^\infty \tilde{f}(a_i) \bar{k}_\beta^b.
$$

(39)

We will also keep the previous notations of the matrix representations of an operator with respect to the different bases $(s_{i,j}, p_{i,j}, t_{i,j},$ and $u_{i,j})$.

**5.1. Matrix Representation of ATHOs.** Denote by $(a_{1m})_{m \geq 1} = (b_{1n})_{n \geq 1}$ the subsequence of common elements between $(a_m)_{m}$ and $(b_n)_{n}$ ordered such that $a_i = b_i$ and that $1 = l_1 \in \{l \leq l_1 \}$, We will prove that the above matrix representations are also true in the case of model spaces associated to interpolating Blaschke products.

**Theorem 16.** Let $B$ be a bounded linear transformation from $K_\alpha$ to $K_\beta$. Then, $B \in \mathcal{S}(\alpha, \beta)$ if and only if for any $i, j$

$$
\begin{align*}
\bar{\beta}(b_i) (1 - a_i b_i)s_{i,j} = \bar{\beta}(b_i) (1 - a_i b_i)s_{i,j} + \bar{\beta}(b_i) (1 - a_i b_i)s_{i,j},
\end{align*}
$$

(40)

**Proof.** Suppose that $B = B^{\alpha \beta}_\varphi$ and we will prove that it satisfies (40). As in the proof of Theorem 8, for any $\varphi$ in $L^2(T)$, there is a $\phi \in K_{\alpha^{\beta^{\varphi}}}$ such that $B^{\alpha \beta}_\varphi = B^{\alpha \beta}_\phi$, where $K_{\alpha^{\beta^{\varphi}}} = K_\alpha \otimes K_\beta$.

Then, there are $\chi \in K_\alpha$ and $\psi \in K_{\beta^{\varphi}}$ such that $B^{\alpha \beta}_\varphi = \chi^{\alpha \beta}_\varphi$.

Let $\beta^{\alpha \beta}_\chi$ be an interpolating Blaschke product with zeros $(b_m)_{m \geq 1}$. For the bases $(k_\alpha^a)_{a \neq 1}$ and $(\bar{k}_\beta^b)_{b \neq 1}$, we have

$$
\chi = \sum_{m=1}^\infty C_m k_\alpha^a = \sum_{n=1}^\infty \bar{d}_n \bar{k}_\beta^a.
$$

(41)

Replacing in $B^{\alpha \beta}_\varphi$,

$$
B^{\alpha \beta}_\varphi = \sum_{m=1}^\infty c_m B^{\alpha \beta}_\varphi + \sum_{n=1}^\infty d_n \chi^{\alpha \beta}_\varphi.
$$

(42)

By (6), we have

$$
B^{\alpha \beta}_\varphi = \sum_{m=1}^\infty c_m k_\alpha^a \otimes k_\beta^a + \sum_{n=1}^\infty d_n \chi^{\alpha \beta}_\varphi.
$$

(43)

So

$$
B^{\alpha \beta}_\varphi = \sum_{m=1}^\infty c_m a_m k_\alpha^a \otimes k_\beta^a + \sum_{n=1}^\infty \bar{d}_n \bar{k}_\beta^b.
$$

(44)
Replacing in (20), we get
\[
s_{ij} = \frac{1}{\beta'(b_i)} \left( \sum_{m=1}^{\infty} c_m a_m \ \beta'\alpha_m - \sum_{n=1}^{\infty} d_n a_n \ k_n(a_i) \right).
\] (45)

When \((l_k)_k\) is empty, we have
\[
s_{ij} = \frac{1}{\beta'(b_i)} \left( \sum_{m=1}^{\infty} c_m a_m \ \beta'(a_m) - \sum_{n=1}^{\infty} d_n a_n \ \beta'(b_n) \right).
\] (46)

When \((l_k)_k\) is not empty, we have
\[
\tilde{k}_n(a_m) = \begin{cases} 0 & \text{if } i \in (l_k)_k, m \neq i, \\ \beta'\alpha_m & \text{if } i \in (l_k)_k, m = i. \end{cases}
\] (47)

\[
\tilde{k}_n(a_m) = \begin{cases} 0 & \text{if } j \in (l_k)_k, n \neq j, \\ \alpha'(a_i) & \text{if } j \in (l_k)_k, n = j, \end{cases}
\] (48)

Therefore, we have 4 cases, \(j \in (l_k)_k\) and \(i \in (l_k)_k\), \(j \in (l_k)_k\) and \(i \in (l_k)_k\), \(j \in (l_k)_k\) and \(i \in (l_k)_k\), and \(i \in (l_k)_k\) and \(j \in (l_k)_k\). In all these cases, we decompose, add, and subtract as in the finite-dimensional case. Using the fact that \(a_1 = b_1\), we obtain
\[
s_{ij} = \frac{\beta'(b_i)(1 - a_i b_i)s_{1j} - \beta'(b_i)(1 - a_i b_i)s_{1j} + \beta'(b_i)(1 - a_i b_i)s_{1j}}{\beta'(b_i)(1 - a_i b_i)}.
\] (49)

Conversely, we proceed as in [9] for truncated Hankel operators but using the generalized characterization in Theorem 4.

As for Theorem 10, we also generalize the matrix representation of the ATHOs with respect to conjugate kernel bases.

**Theorem 17.** Let \(B\) be a bounded linear transformation from \(K_a\) to \(K_b\). We have \(B \in \mathcal{S}(\alpha, \beta)\) if and only if for any \(i, j \geq 1\), we have
\[
P_{ij} = \frac{\beta'(b_i)(1 - a_i b_i)P_{ij} - \beta'(b_i)(1 - a_i b_i)P_{ij} + \beta'(b_i)(1 - a_i b_i)P_{ij}}{\beta'(b_i)(1 - a_i b_i)}.
\] (50)

Suppose that \(\alpha\) and \(\beta\) are inner functions such that \(K_a\) and \(K_\beta\) have Clark bases \((v^{\alpha}_{\eta_1})\) and \((v^{\beta}_{\eta_1})\), where \((\eta_1)\) and \((\zeta_1)\) are sequences of eigenvalues for some \(U^{\alpha}_{_1}\) and some \(U^{\beta}_{_2}\) satisfying the equations (30). We have the relations \(\sqrt{\alpha'(\eta_1)} = ||k_{\eta_1}||\) and \(\sqrt{\beta'(\zeta_1)} = ||k_{\zeta_1}||\). Reorder these sequences such that \(\eta_1 = \zeta_1\), for any \(k \geq 1\) and that \(1 = l_1 \in (l_k)_k\). We have the following theorem.

**Theorem 18.** Let \(B\) be a bounded linear transformation from \(K_a\) to \(K_\beta\). \(B \in \mathcal{S}(\alpha, \beta)\) if and only if

1. **When \((l_k)_k \neq 1\) is empty, for \(i \in (l_k)_k\) or \(j \in (l_k)_k\) and \(j \neq i\)**

\[
t_{ij} = \frac{\sqrt{\beta'(\zeta_1)} (\eta_j - \zeta_j)}{\sqrt{\beta'(\zeta_1)}} t_{ij} + \frac{\sqrt{\alpha'(\eta_1)} (\eta_j - \zeta_j)}{\sqrt{\alpha'(\eta_1)}} t_{ij}.
\] (51)

2. **When \((l_k)_k = 1\) is empty, for any \(i, j \geq 1\)**

\[
t_{ij} = \frac{\sqrt{\beta'(\zeta_1)} (\eta_j - \zeta_j)}{\sqrt{\beta'(\zeta_1)}} t_{ij} + \frac{\sqrt{\alpha'(\eta_1)} (\eta_j - \zeta_j)}{\sqrt{\alpha'(\eta_1)}} t_{ij}.
\] (52)

\[
t_{ij} = \frac{\sqrt{\beta'(\zeta_1)} (\eta_j - \zeta_j)}{\sqrt{\beta'(\zeta_1)}} t_{ij} - \frac{\sqrt{\alpha'(\eta_1)} (\eta_j - \zeta_j)}{\sqrt{\alpha'(\eta_1)}} t_{ij}.
\] (53)
Proof. The proof is similar to the one of Theorem 14, but we use formula (44). For the converse implication, we proceed as in Theorem 21, except using the following characterization of ATOs in Theorem 4, there exist \( \chi \in K_a \) and \( \psi \in K_\beta \),

\[
B = \left( U^\beta \right)^* B \left( U^\alpha \right)^* = \psi \otimes k_0^\beta + k_0^\alpha \otimes \chi. \tag{54}
\]

\( \square \)

As for Theorem 15, we deduce the matrix representation with respect to modified Clark bases.

**Theorem 19.** A bounded linear transformation \( B \) from \( K_a \) to \( K_\beta \) belongs to \( S(\alpha, \beta) \) if and only if

1. When \( (l_k)_{k \geq 1} \) is not empty, for \( \{ i \in (l_k) \text{ and } j \in (l_k) \} \) or \( \{ i \in (l_k) \text{ and } j \in (l_k) \text{ and } j \neq i \} \)

\[
u_{ij} = \frac{\omega^i}{\omega^j} \frac{\sqrt{\beta'(\xi_i)}}{\sqrt{\beta'(\xi_j)}} \frac{\sqrt{\eta_i - \xi_i}}{\sqrt{\eta_j - \xi_j}} u_{ij},
\]

\[
+ \frac{\omega^a}{\omega^\beta} \frac{\sqrt{|\alpha'(\eta_i)|}}{\sqrt{|\beta'(\xi_j)|}} \frac{1}{\sqrt{\eta_i - \xi_i}} u_{ij},
\]

(55)

For \( \{ i \in (l_k) \} \text{ or } \{ j \in (l_k) \} \) or \( \{ i \in (l_k) \text{ and } j \in (l_k) \} \),

\[
u_{ij} = \frac{\omega^i}{\omega^j} \frac{\sqrt{\beta'(\xi_i)}}{\sqrt{\beta'(\xi_j)}} \frac{\sqrt{\eta_i - \xi_i}}{\sqrt{\eta_j - \xi_j}} u_{ij},
\]

\[+ \frac{\omega^a}{\omega^\beta} \frac{\sqrt{|\alpha'(\eta_i)|}}{\sqrt{|\beta'(\xi_j)|}} \frac{1}{\sqrt{\eta_i - \xi_i}} u_{ij}.
\]

(56)

2. When \( (l_k)_{k \geq 1} \) is empty

\[
u_{ij} = \frac{\omega^i}{\omega^j} \frac{\sqrt{\beta'(\xi_i)}}{\sqrt{\beta'(\xi_j)}} \frac{\sqrt{\eta_i - \xi_i}}{\sqrt{\eta_j - \xi_j}} u_{ij},
\]

\[+ \frac{\omega^a}{\omega^\beta} \frac{\sqrt{|\alpha'(\eta_i)|}}{\sqrt{|\beta'(\xi_j)|}} \frac{1}{\sqrt{\eta_i - \xi_i}} u_{ij},
\]

\[+ \frac{\omega^a}{\omega^\beta} \frac{\sqrt{|\alpha'(\eta_i)|}}{\sqrt{|\beta'(\xi_j)|}} \frac{1}{\sqrt{\eta_i - \xi_i}} u_{ij},
\]

(57)

5.2. Matrix Representation of ATOs. To get the matrix representation of ATOs on interpolating model spaces, we will work as in the proof of Theorem 2.2 in [8]. We first need to explore the action of \( V_{\xi_{\omega}} \) on \( \mathcal{S}(\alpha, \beta) \), where \( V_{\xi_{\omega}} \) acts from \( K_a \) into \( K_{\alpha_{\omega_{\xi_{\omega}}}} \) and is defined by \( V_{\xi_{\omega}} = \sqrt{\tau_{\xi_{\omega}}^{-1}(f \circ \tau_{\xi_{\omega}})} \).

**Proposition 20.** Let \( \alpha \) and \( \beta \) be two inner functions, \( \phi, \psi \in L^2 \) and \( A^\alpha_{\psi_{\xi_{\omega}}} = \mathcal{S}(\alpha, \beta) \). Then for some \( c \in \mathbb{D} \) and \( \xi \in \mathbb{T} \),

\[
V_{\xi_{\omega}} A^\alpha_{\psi_{\xi_{\omega}}},\psi_{\xi_{\omega}} = A^\alpha_{\psi_{\xi_{\omega}}},\psi_{\xi_{\omega}}. \tag{58}
\]

Proof. For \( f \in K_a, \phi, \psi \in L^2, \xi \in \mathbb{T} \) and \( a \in \mathbb{D} \), we have

\[
V_{\xi_{\omega}} [P_{\beta}(\phi f)](z) = \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[
= \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(z)f(\tau_{\xi_{\omega}}(z))}
\]

\[
= \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(z)f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi.
\]

(59)

Let \( \chi = \tau_{\xi_{\omega}}(\omega) \). Since \( \tau_{\xi_{\omega}}(z) = \xi(|z|^2 - 1)/((1 - cz)^2) \), we have

\[
V_{\xi_{\omega}} [P_{\beta}(\phi f)] = \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi = \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi = \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi.
\]

(60)

Since

\[
\sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi = \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi = \sqrt{\tau_{\xi_{\omega}}^{-1} \phi(\tau_{\xi_{\omega}}(z))f(\tau_{\xi_{\omega}}(z))}
\]

\[\cdot \int_{\mathbb{T}} \frac{1 - \beta(\tau_{\xi_{\omega}}(z))}{1 - \tau_{\xi_{\omega}}(z)} d\chi.
\]

(61)
In conclusion,

\[ V_{\xi_{e}} A_{\varphi} = A_{\varphi_{\xi_{e}}} V_{\xi_{e}}. \] (63)

Applying \( V_{\xi_{e}}^{-1} = V_{\xi_{e}}^{*} \), we get the result. \( \square \)

As before, the formula (20) is also true for ATTOs. We will prove that the matrix characterizations of ATTOs on finite-dimensional model spaces obtained in [6] are also true in the infinite case. Denote by \( (a_{k}, b_{k}) \) the subsequence of common zeros between \( \alpha \) and \( \beta \) ordered such that \( a_{k} = b_{k} \), for any \( k \geq 1 \) and that \( 1 = l_{1} \in (l_{k})_{k} \) when \( (l_{k})_{k} \) is not empty.

**Theorem 21.** A bounded linear transformation \( A \) from \( K_{\alpha} \) to \( K_{\beta} \) belongs to \( \mathfrak{S}(\alpha, \beta) \) if and only if its matrix representation with respect to the kernel bases satisfies

(1) When \( (l_{k})_{k>1} \) is not empty, for \( \{i \in (l_{k})_{k} \text{ and } j \in (l_{k})_{k} \} \) or \( \{i \in (l_{k})_{k} \text{ and } j \in (l_{k})_{k} \text{ and } j \neq i \} \)

\[ s_{i,j} = \frac{\bar{\beta}'(b_{j}) \left( \bar{\alpha}_{j} - \bar{b}_{j} \right) s_{i,j} + \bar{\beta}'(b_{j}) \left( \bar{\alpha}_{j} - \bar{b}_{j} \right) s_{i,j} \}{\bar{\beta}'(b_{j}) \left( \bar{\alpha}_{j} - \bar{b}_{j} \right)} \] (64)

For \( \{i \in (l_{k})_{k} \text{ and } j \in (l_{k})_{k} \} \) or \( \{i \in (l_{k})_{k} \text{ and } j \in (l_{k})_{k} \} \),

\[ s_{i,j} = \frac{\bar{\beta}'(b_{i}) \left( \bar{\alpha}_{i} - \bar{b}_{i} \right) s_{i,j} + \bar{\beta}'(b_{i}) \left( \bar{\alpha}_{i} - \bar{b}_{i} \right) s_{i,j} \}{\bar{\beta}'(b_{i}) \left( \bar{\alpha}_{i} - \bar{b}_{i} \right)} \] (65)

(2) When \( (l_{k})_{k>1} \) is empty

\[ s_{i,j} = \frac{\bar{\beta}'(b_{j}) \left( \bar{\alpha}_{j} - \bar{b}_{j} \right) s_{i,j} + \bar{\beta}'(b_{j}) \left( \bar{\alpha}_{j} - \bar{b}_{j} \right) s_{i,j} \}{\bar{\beta}'(b_{j}) \left( \bar{\alpha}_{j} - \bar{b}_{j} \right)} \] (66)

**Proof.** The proof of necessity is the same as in [6] for ATTOs acting on finite-dimensional model spaces. Conversely, consider any bounded linear transformation from \( K_{\alpha} \) into \( K_{\beta} \) whose matrix representation satisfies Theorem 21. To show that \( A \in \mathfrak{S}(\alpha, \beta) \), or equivalently to show that \( A \) satisfies the characterization in Theorem 4, we will find a \( \chi \in K_{\alpha} \) and a \( \psi \in K_{\beta} \) such that

\[ A - S_{\beta}A_{\alpha}^{*} = \psi \otimes k_{\alpha}^{0} + k_{\beta}^{0} \otimes \chi. \] (67)

This is equivalent to

\[ \langle Ak_{a}, k_{b}^{0} \rangle - \langle S_{\beta}A_{\alpha}^{*}k_{a}^{0}, k_{b}^{0} \rangle = \langle \psi \otimes k_{\alpha}^{0} \rangle \hat{k}_{b}^{0} + \langle k_{\beta}^{0} \rangle \chi \hat{k}_{b}^{0}, \] (68)

for any \( i, j \geq 1 \). Using the relations in Lemma 3, when \( b_{i} \neq 0 \), for any \( i \geq 1 \), we have for every \( i, j \geq 1 \)

\[ \left( 1 - \frac{\bar{\alpha}_{j}}{\bar{b}_{j}} \right) \langle Ak_{a}, k_{b}^{0} \rangle - \frac{\bar{\alpha}_{j}}{\bar{b}_{j}} \langle k_{a}^{0}, A^{*}k_{b}^{0} \rangle = \frac{\bar{\beta}(0)}{\bar{b}_{j}} \chi(a_{j}). \] (69)

or

\[ \left( 1 - \frac{\bar{\alpha}_{j}}{\bar{b}_{j}} \right) \langle k_{a}^{0}, A^{*}k_{b}^{0} \rangle - \frac{\bar{\beta}(0)}{\bar{b}_{j}} \chi(a_{j}). \] (70)

for any \( i, j \geq 1 \).

When \( (l_{k})_{k} \) is not empty, suppose that \( \beta(0) \neq 0 \). Since the matrix representation of \( A \) satisfies the formulas in Theorem 21, the above system is equivalent to

\[
\begin{align*}
\left( 1 - \frac{\bar{\alpha}_{j}}{\bar{b}_{j}} \right) k_{a}^{0} & + \frac{\bar{\alpha}_{j}}{\bar{b}_{j}} \langle k_{a}^{0}, A^{*}k_{b}^{0} \rangle = \frac{\bar{\beta}(0)}{\bar{b}_{j}} \chi(a_{j}), & j \geq 1; \\
\langle k_{a}^{0}, A^{*}k_{b}^{0} \rangle & = \frac{\bar{\beta}(0)}{\bar{b}_{j}} \chi(a_{j}), & k \geq 1; \\
\left( 1 - \frac{\bar{\alpha}_{i}}{\bar{b}_{i}} \right) k_{a}^{0} & + \frac{\bar{\alpha}_{i}}{\bar{b}_{i}} \langle k_{a}^{0}, A^{*}k_{b}^{0} \rangle = \frac{\bar{\beta}(0)}{\bar{b}_{i}} \chi(a_{i}), & i \geq 1, i \in (l_{k})_{k}.
\end{align*}
\] (71)
Set an arbitrary \( \tilde{\psi}(b_i) \), then the solution of the system is

\[
\begin{align*}
\tilde{\psi}(b_i) &= -\langle k_{\alpha}^a, A^* \chi^\beta \rangle - \frac{b_i(0)}{b_i} \chi(a_i) ; \\
\tilde{\psi}(b_i) &= \left(1 - \frac{\alpha_i}{b_i}\right) \bar{\beta}^i (b_i) S_{1, j} + \frac{\alpha_i}{b_i} \langle k_{\alpha}^a, A^* \chi^\beta \rangle + \frac{b_i(0)}{b_i} \tilde{\chi}(a_i) .
\end{align*}
\]

To show that the solutions of the system \( \chi \) and \( \psi \) are in \( K_\alpha \) and \( K_\beta \), respectively, it suffices to prove that \( \chi \psi \) are the unique solutions in \( K_\alpha \) and \( K_\beta \) of the interpolation problems corresponding to \( (a_m)_{m \geq 1} \) and \( (b_n)_{n \geq 1} \) (39). In fact, since \( A^* \chi^\beta \), \( A^* \chi^\beta \), \( k_{\alpha}^a \in K_\beta \), and \( (a_j)_{j \geq 1}, (b_j)_{j \geq 1} \) are Blaschke sequences, we have

\[
\sum_{j=1}^{\infty} |\chi(a_j)|^2 \left(1 - |a_j|^2 \right) \leq \sum_{j=1}^{\infty} \left| \psi(b_j) \right|^2 \left(1 - |b_j|^2 \right) \leq \sum_{j=1}^{\infty} \left(1 - |a_j|^2 \right) + C \sum_{j=1}^{\infty} \left(1 - |a_j|^2 \right) + C \sum_{j=1}^{\infty} \left(1 - |a_j|^2 \right) \leq \infty.
\]

If \( b(0) = 0, \beta(c) \neq 0 \) for some \( c \in \mathbb{D} \). Let \( \xi = 1 \), by Proposition 20, \( A \in \mathcal{A} (\alpha, \beta) \) if and only if \( V_\alpha V^*_{\alpha} \in \mathcal{G}(\alpha \circ \tau, \beta \circ \tau) \). Since \( \beta \circ \tau(0) = \beta(c) \neq 0 \), we need to show that the matrix representation of \( V_\alpha V^*_{\alpha}, (w_{ij}) \) with respect to the kernel bases \( (k_{\alpha}^\beta \tau_{(a_i)})_{m} \) and \( (k_{\alpha}^\beta \tau_{(b_j)})_{n} \) satisfies the relations in Theorem 21. In fact, as in [8], we have

\[
V_\alpha V^*_{\alpha} \tau_{(a_i)}(z) = \frac{1 - \bar{\tau} \tau}{|1 - \bar{z}|^2} k_{\alpha}^a(z), \quad V_\alpha V^*_{\alpha} \tau_{(b_j)}(z) = \frac{1 - \bar{\tau} \tau}{|1 - \bar{z}|^2} k_{\beta}(z).
\]

\[
\omega_{ij} = \frac{1}{(\beta \circ \tau)(b_j)} \left| V_\alpha V^*_{\alpha} \tau_{(a_i)}(z) \right|^2 \left| \tau_{(b_j)}(z) \right|^2 = \frac{1 - \bar{\tau} \tau}{|1 - \bar{z}|^2} \left| k_{\alpha}^a(z) \right|^2 \left| k_{\beta}(z) \right|^2
\]

for any \( i, j \geq 1 \). We also have

\[
(\beta \circ \tau)'(\tau_{(b_j)}) = \frac{\beta'(b_j)}{\tau_{(b_j)}}, \quad (\tau_{(a_i)}) = \frac{c - z}{1 - cz} \quad \text{and} \quad r_{(z)} = \frac{|c|^2 - 1}{(1 - cz)^2},
\]

using these formulas we can prove that every \( \omega_{ij} \) satisfies first relation in Theorem 21,

\[
\begin{align*}
(\beta \circ \tau)'(\tau_{(a_i)}) \omega_{ij} + (\beta \circ \tau)'(\tau_{(b_j)}) \omega_{ij} &= \frac{1 - \bar{\tau} \tau}{1 - \bar{z}} \omega_{ij}.
\end{align*}
\]
If \((l_k)_k\) is empty, then the equation (70) becomes with the assumption \(\beta(0) \neq 0\),
\[
\begin{align*}
\left(1 - \frac{\pi}{b_1}\right)\beta'(b_1) s_{i,1} + \frac{\pi}{b_1} \langle k_\alpha^e, A^* k_\beta^e \rangle = \psi(b_1) - \frac{\beta(0)}{b_1} \chi(a_i), & \quad j \geq 1; \\
\left(1 - \frac{\pi}{b_j}\right)\beta'(b_j) s_{j,1} + \frac{\pi}{b_j} \langle k_\alpha^e, A^* k_\beta^e \rangle = \psi(b_j) - \frac{\beta(0)}{b_j} \chi(a_i), & \quad i \geq 2.
\end{align*}
\]
(79)

If we set an arbitrary \(\bar{\psi}(b_i)\), then, the solutions are
\[
\begin{align*}
\chi(a_i) = \frac{b_1}{\beta(0)} \psi(b_1) - \left(1 - \frac{\pi}{b_1}\right)\beta'(b_1) s_{i,1} + \frac{\pi}{b_1} \langle k_\alpha^e, A^* k_\beta^e \rangle, & \quad j \geq 1; \\
\psi(b_i) = \left(1 - \frac{\pi}{b_j}\right)\beta'(b_j) s_{j,1} + \frac{\pi}{b_j} \langle k_\alpha^e, A^* k_\beta^e \rangle + \frac{\beta(0)}{b_j} \chi(a_i), & \quad i \geq 2.
\end{align*}
\]
(80)

It remains to check that \(\chi \in K_\alpha\) and \(\psi \in K_\beta\) by showing that they are solutions of the corresponding interpolation problems. The case \(\beta(0) = 0\) can be treated in the same way as in the previous case.

To get the matrix representation of ATTOs with respect to conjugate kernel bases \((k_\alpha^e)_m\) and \((k_\beta^e)_n\), we proceed as in Theorem 13 and use the fact that \(C_\beta AC_\alpha = A_\alpha^e\beta\) from the Theorem 5.

**Theorem 22.** A bounded linear transformation belongs to \(\mathcal{X}(\alpha, \beta)\) if and only if

1. When \((l_k)_{k \geq 1}\) is not empty, for \(i \in (l_k)_k\) or \(j \in (l_k)_k\) and \(j \neq i\)
   \[
   p_{i,j} = \frac{\beta'(b_j)(a_j - b_i)p_{i,j} + \beta'(b_j)(a_j - b_i)p_{j,i}}{\beta'(b_i)(a_j - b_i)},
   \]
   (81)

   For \(i \in (l_k)_k\) and \(j \in (l_k)_k\) or \(i \in (l_k)_k\) and \(j \in (l_k)_k\),
   \[
   p_{i,j} = \frac{\beta'(b_j)(a_j - b_i)p_{i,j} + \beta'(b_j)(a_j - b_i)p_{j,i}}{\beta'(b_j)(a_j - b_i)},
   \]
   (82)

2. When \((l_k)_{k \geq 1}\) is empty
   \[
   p_{i,j} = \frac{\beta'(b_j)(a_j - b_i)p_{i,j} - \beta'(b_j)(a_j - b_i)p_{j,i} + \beta'(b_j)(a_j - b_i)p_{i,j}}{\beta'(b_j)(a_j - b_i)},
   \]
   (83)

Suppose that \(\alpha\) and \(\beta\) are inner functions such that the spaces \(K_\alpha\) and \(K_\beta\) have Clark bases, \((v_{\eta_i})_i\) and \((v_{\zeta_i})_i\). Denote the subsequence of common elements of \((\eta_i)_i\) and \((\zeta_i)_i\) by \((\eta_i)_k = (\zeta_i)_k\) ordered such that \(\eta_i = \zeta_i\), for any \(k \geq 1\) and that \(1 = l_1 \in (l_k)_k\).

**Theorem 23.** A bounded linear transformation belongs to \(\mathcal{X}(\alpha, \beta)\) if and only if

1. When \((l_k)_{k \geq 1}\) is not empty, for \(i \in (l_k)_k\) and \(j \in (l_k)_k\) or \(i \in (l_k)_k\) and \(j \in (l_k)_k\) and \(j \neq i\)
   \[
   t_{i,j} = \frac{\sqrt{|\beta'(|\zeta_i|)(1 - \zeta_i)\eta_i|}}{\sqrt{|\beta'(|\zeta_i|)(1 - \zeta_i)\eta_i|}} t_{i,j} + \frac{\sqrt{|\alpha'(\eta_j)(1 - \zeta_i)\eta_i|}}{\sqrt{|\alpha'(\eta_j)(1 - \zeta_i)\eta_i|}} t_{j,i},
   \]
   (84)

For \(i \in (l_k)_k\) and \(j \in (l_k)_k\) or \(i \in (l_k)_k\) and \(j \in (l_k)_k\),
\[
\begin{align*}
& t_{i,j} = \frac{\sqrt{|\beta'(|\zeta_i|)(1 - \zeta_i)\eta_i|}}{\sqrt{|\beta'(|\zeta_i|)(1 - \zeta_i)\eta_i|}} t_{i,j} + \frac{\sqrt{|\alpha'(\eta_j)(1 - \zeta_i)\eta_i|}}{\sqrt{|\alpha'(\eta_j)(1 - \zeta_i)\eta_i|}} t_{j,i}.
\end{align*}
\]
(85)

2. When \((l_k)_{k \geq 1}\) is empty
   \[
   t_{i,j} = \frac{\sqrt{|\beta'(|\zeta_i|)(1 - \zeta_i)\eta_i|}}{\sqrt{|\beta'(|\zeta_i|)(1 - \zeta_i)\eta_i|}} t_{i,j} - \frac{\sqrt{|\alpha'(\eta_j)(1 - \zeta_i)\eta_i|}}{\sqrt{|\alpha'(\eta_j)(1 - \zeta_i)\eta_i|}} t_{j,i}.
   \]
   (86)

**Proof.** The proof is the same as in Theorem 21, except using the equivalent characterization from Theorem 4 instead.

We also deduce the matrix representation with respect to the modified Clark bases.

**Theorem 24.** A bounded linear transformation belongs to \(\mathcal{X}(\alpha, \beta)\) if and only if
(1) When \((l_k)_{k=1}^\infty\) is not empty, for \(\{i \in (l_k)_k\) and \(j \in (l_k)_k\) \(\) or \(i \in (l_k)_k\) and \(j \in (l_k)_k\) and \(j \neq i\)

\[
\begin{align*}
    u_{i,j} &= \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j} \\
    &+ \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j} \\
    &+ \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j}.
\end{align*}
\]

(87)

For \(\{i \in (l_k)_k\) and \(j \in (l_k)_k\) or \(i \in (l_k)_k\) and \(j \in (l_k)_k\)

\[
\begin{align*}
    u_{i,j} &= \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j} \\
    &+ \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j}.
\end{align*}
\]

(88)

(2) When \((l_k)_{k=1}^\infty\) is empty

\[
\begin{align*}
    u_{i,j} &= \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j} \\
    &- \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j} \\
    &+ \frac{\omega_i^\beta}{\omega_j^\beta} \sqrt{\beta^\prime(\zeta_i)} \left(1 - \zeta_i \pi_j \right) u_{i,j}.
\end{align*}
\]

(89)

Remark 25. Note that since \(\mathbb{J}^\# a_j = k_n^\#\) and \(\mathbb{J}^\# \tilde{k}_j = \tilde{k}_n^\#\), we have

\[
\begin{align*}
    s_{i,j} &= \frac{1}{\beta^\prime(\zeta_i)} \left( Bk_j^\# \tilde{k}_i^\# \right) \frac{1}{\beta^\prime(\zeta_i)} \left( CgJ^\# k_n^\# \tilde{k}_n^\# \right) \\
    &= \frac{1}{\beta^\prime(\zeta_i)} \left( J^\# BCa_j^\# \tilde{k}_i^\# \right) \frac{1}{\beta^\prime(\zeta_i)} \left( CgJ^\# k_n^\# \tilde{k}_n^\# \right).
\end{align*}
\]

(90)

Similarly, we also can obtain the matrix representation of an ATTO with respect to kernel and conjugate kernel bases, and to conjugate kernel and kernel bases via the passage formulas.

Data Availability

There is no underlying data in this paper, and all of its research is the derivation of basic theory.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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