Abstract

In this paper we collect results concerning the operator-norm convergent Trotter product formula for two semigroups \( \{ e^{-tA} \}_{t \geq 0} \) and \( \{ e^{-tB} \}_{t \geq 0} \), with densely defined generators \( A \) and \( B \) in a Banach space. Although the strong convergence in Banach space for contraction semigroups is known since the seminal paper by Trotter (1959), which after more than three decades was extended to convergence in the operator-norm topology in Hilbert spaces by Rogava (1993), the operator-norm convergence in a Banach space was established only in (2001). For the first time this result was established under hypothesis that one of the involved into the product formula contraction semigroups, e.g. \( \{ e^{-tA} \}_{t \geq 0} \), is holomorphic together with certain conditions of smallness on generators \( B \) and \( B^* \) with respect to generators \( A \) and \( A^* \). Note that in spite of a quite strong assumptions on operators \( A \) and \( B \) the proof of the operator-norm convergent Trotter product formula on a Banach space is (unexpectedly) involved and technical.

To elucidate the question of how far these conditions are from optimal ones we show an Example of the operator-norm convergent Trotter product formula for two semigroups \( \{ e^{-tA} \}_{t \geq 0} \) and \( \{ e^{-tB} \}_{t \geq 0} \) on a Banach space, where hypothesis on operator \( A \) is relaxed to condition that \( A \) is generator of a contraction semigroup.

1 Preliminaries

1.1 Bounded semigroups on \( X \)

For what follows the properties of holomorphic (contraction) semigroups on a Banach space \( X \) are essential. Therefore, we start by a suitable for our aim recall of
details concerning the bounded, holomorphic semigroups, and fractional powers of their generators. We begin with definitions and properties to introduce certain notations adapted in this section for semigroups on \( \mathcal{X} \).

**Definition 1.1.** We would remind that a family \( \{U(t)\}_{t \geq 0} \) of bounded linear operators on a Banach space \( \mathcal{X} \) is called a one-parameter strongly continuous semigroup if it satisfies the conditions:

(i) \( U(0) = 1 \),

(ii) \( U(s + t) = U(s)U(t) \) for all \( s, t \geq 0 \),

(iii) \( \lim_{t \to +0} U(t)x = x \) for all \( x \in \mathcal{X} \).

We recall some immediate consequences of this definition:

- There are constants \( C_A \geq 1 \) and \( \gamma_A \in \mathbb{R} \), depending on the generator of the semigroup, such that \( \|U(t)\| \leq C_A e^{\gamma_A t} \) for all \( t \geq 0 \).
- \( t \mapsto U(t) \) is a strongly continuous function from \( [0, +\infty) \) onto the algebra \( \mathcal{L}(\mathcal{X}) \) of bounded linear operators on \( \mathcal{X} \).
- There exists a closed densely defined linear operator \( A \) on \( \mathcal{X} \) with domain \( \text{dom}(A) \), called the generator of the semigroup, such that \( \lim_{t \to +0} (U(t)x - x)/t = -Ax \) for any \( x \in \text{dom}(A) \), that is, by convention \( U(t) := e^{-tA} \).
- The resolvent of the generator satisfies the estimate \( \|R_A(-\lambda)\| = \|(A + \lambda I)^{-1}\| \leq C_A/(\text{Re}(\lambda) - \gamma_A) \) for all \( \lambda \) with \( \text{Re}(\lambda) > \gamma_A \), thus the open half plane with \( \text{Re}(\zeta) < -\gamma_A \) is contained into the resolvent set of \( A \), which is defined as \( \rho(A) = \{z \in \mathbb{C} : \|R_A(z)\| < +\infty\} \).
- If \( \gamma_A \leq 0 \), \( U(t) \) is called a bounded semigroup (otherwise, \( U(t) \) is called a quasi-bounded semigroup of type \( \gamma_A > 0 \)). For any strongly continuous semigroup, we can construct a bounded semigroup by adding some constant \( \eta \geq \gamma_A \) to its generator. Let \( \tilde{A} = A + \eta I \), then for the semigroup \( \tilde{U}(t) \) generated by \( \tilde{A} \), one has \( \|\tilde{U}(t)\| \leq C_A, t \geq 0 \), and the open half-plane \( \text{Re}(\lambda) < \eta - \gamma_A \) is included into the resolvent set \( \rho(\tilde{A}) \) of \( \tilde{A} \). So it is not restrictive to suppose that the considered semigroup \( U(t) \) is bounded and that the set \( \{z \in \mathbb{C} : \text{Re}(z) \leq 0\} \subseteq \rho(A) \).
- If \( \|U(t)\| \leq 1, t \geq 0 \), the semigroup is called a contraction semigroup. We comment that the method of the preceding remark does not permit to construct a contraction semigroup from a bounded semigroup in general, since it can not change the value of the constant \( C_A \).

Below we need a characterization of generators of these contraction semigroups. First we recall that the space of linear bounded functionals \( \mathcal{X}^* = \mathcal{L}(\mathcal{X}, \mathbb{C}) \) is a dual of the Banach space \( \mathcal{X} \) and that \( \mathcal{X}^* \) is itself a Banach space. Recall that a linear operator \( A \) in \( \mathcal{X} \) is accretive if for all pairs \( (u, \phi) \in \text{dom}(A) \times \mathcal{X}^* \) with \( \|u\|_X = 1 \), \( \|\phi\|_{\mathcal{X}^*} = 1 \), \( (u, \phi) = 1 \), one has \( \text{Re}(Au, \phi) \geq 0 \). We also add that a densely defined in \( \mathcal{X} \) accretive operator \( A \) is generator of contraction semigroup if the range of \( \lambda I + A \) is \( \mathcal{X} \) for some \( \lambda > 0 \).

Now we prove a series of estimates indispensable throughout this paper.

**Lemma 1.2.** Let \( U(t) \) be a bounded semigroup with boundedly invertible generator \( A \), then for all \( t \geq 0 \), and for any \( n \in \mathbb{N} \), we have:
Preliminaries

\[
\begin{align*}
(U(t) - \sum_{k=0}^{n} \frac{(-tA)^k}{k!}) A^{-n-1} &= -\int_0^t d\tau \left( U(\tau) - \sum_{k=0}^{n-1} \frac{(-\tau A)^k}{k!} \right) A^{-n}, \quad (1.1) \\
\left\| \left( U(t) - \sum_{k=0}^{n} \frac{(-tA)^k}{k!} \right) A^{-n-1} \right\| &\leq C_A \frac{\rho^{n+1}}{(n+1)!}. \quad (1.2)
\end{align*}
\]

Proof. We proceed by induction, and we first prove that

\[
(U(t) - 1)x = -\int_0^t d\tau U(\tau) A x, \quad x \in \text{dom}(A). \quad (1.3)
\]

Note that for any \( \epsilon > 0 \) the semigroup properties yields the representation

\[
\int_0^t ds U(s) U(\epsilon) - \frac{1}{\epsilon} = \int_0^t ds U(s + \epsilon) - \frac{U(s)}{\epsilon} = \int_t^{t+\epsilon} ds \frac{U(s)}{\epsilon} - \int_0^t ds \frac{U(s)}{\epsilon} = (U(t) - 1) \frac{1}{\epsilon} \int_0^t ds U(s).
\]

Moreover, one also gets:

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\epsilon ds U(s) x = x, \quad x \in \mathbb{R},
\]

\[
\lim_{\epsilon \to 0} \frac{U(\epsilon) - 1}{\epsilon} x = -A x, \quad x \in \text{dom}(A).
\]

This proves (1.3), and since \( A \) is boundedly invertible, we obtain (1.1) for \( n = 0 \). Furthermore, since \( U(t) \) is bounded by \( C_A \), we obtain the estimate (1.2) for \( n = 0 \).

Suppose that (1.1) and (1.2) are true for some \( n \), then a straightforward calculation leads to (1.1) for \( n + 1 \). Hence, using the representation (1.1) and (1.2) for \( n \) to estimate the integrand, we obtain (1.2) for \( n + 1 \), which completes the proof by induction. \( \square \)

Similarly, we obtain a representation for a restricted development of \((1 + A)^{-1}\).

**Lemma 1.3.** Let \( A \) be as in Lemma 1.2 Then for any \( n \geq 0 \):

\[
(1 + A)^{-1} A^{-n-1} = \left( \sum_{k=0}^{n} (-A)^k \right) A^{-n-1} + (-1)^{n+1}(1 + A)^{-1}. \quad (1.4)
\]

Proof. For \( n = 0 \), the representation (1.4) follows from the resolvent formula:

\[
(1 + A)^{-1} - A^{-1} = -(1 + A)^{-1} A^{-1}. \quad (1.5)
\]

Suppose that (1.4) holds for an integer \( n > 1 \), then:
\((1 + A)^{-1}A^{-n-2} = \sum_{k=0}^{n}(-A)^k A^{-n-2} + (-1)^{n+1}(1 + A)^{-1}A^{-1}\). \(\text{(1.6)}\)

Applying \(\text{(1.5)}\) to the last term of \(\text{(1.6)}\) we get the representation \(\text{(1.4)}\) for \(n+1\), and thus for any \(n\) by induction. \(\square\)

**Lemma 1.4.** If \(U(t)\) is a bounded semigroup with boundedly invertible generator \(A\) then:

\[\left\| \frac{1}{t} \left( (1 + tA)^{-1} - U(t) \right) A^{-2} \right\| \leq 3C_A/2, \quad t > 0.\] \(\text{(1.7)}\)

**Proof.** By Lemma 1.2 one gets

\[\left\| (U(t) - (1 + tA)) \frac{1}{t^2} A^{-2} \right\| \leq \frac{C_A}{2}.\] \(\text{(1.8)}\)

On the other hand by Lemma 1.5 we have

\[\left\| ((1 + tA)^{-1} - (1 + tA)) \frac{1}{t^2} A^{-2} \right\| = \left\| (1 + tA)^{-1} \right\| \leq C_A.\] \(\text{(1.9)}\)

Here the last estimate follows from \((1 + tA)^{-1} = (1/t)R_A(-1/t)\) and \(\|R_A(-\lambda)\| \leq C_A/\lambda + \delta, \delta \geq 0\), which is valid for bounded semigroups with boundedly invertible generators. Hence \(\text{(1.7)}\) follows from \(\text{(1.8)}\) and \(\text{(1.9)}\). \(\square\)

### 1.2 Holomorphic contraction semigroups on \(X\)

Now let \(U: z \mapsto U(z)\) be a family of operators with \(z\) taking their values in the sector of the complex plane:

\[S_\theta = \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \theta\}\] \(\text{(1.10)}\)

where \(0 < \theta \leq \pi/2\).

**Definition 1.5.** Recall that the family of operators \(\{U(z)\}_{z \in S_\theta}\) is a bounded holomorphic semigroup of semi-angle \(\theta \in (0, \pi/2]\) on a Banach space \(X\) if it satisfies the following conditions:

(i) If \(0 < \epsilon < \theta\), then \(\|U(z)\| \leq M_\epsilon\) for all \(z \in S_{\theta-\epsilon}\) and some \(M_\epsilon < \infty\).

(ii) \(U(z_1)U(z_2) = U(z_1 + z_2)\) for all \(z_1, z_2 \in S_\theta\).

(iii) \(U: z \mapsto U(z)\) is analytic function of \(z \in S_\theta\).

(iv) If \(x \in X\) and \(0 < \epsilon < \theta\), then \(\lim_{z \to 0} U(z)x = x\) provided \(z \in S_{\theta-\epsilon}\).

Let \(\sigma(A) = \mathbb{C} \setminus \rho(A)\) denote the spectrum of \(A\). It can be used for the following characterisation of the holomorphic semigroup generators, \([\text{Kato95}], \text{Chapter IX}\):

**Proposition 1.6.** Operator \(A\) in a Banach space \(X\) is the generator of a bounded holomorphic semigroup of semi-angle \(\theta \in (0, \pi/2]\) if and only if \(A\) is a closed operator with a dense domain \(\text{dom}(A)\) such that:
Preliminaries

Similarly to strongly continuous semigroups, a family called a quasi-bounded if there exists a constant $\beta > 0$ such that $\|U(z)\| = \beta e^{-\theta |z|}$ for $z \in S_{\theta}$.

Proposition 1.7. If $U(z)$ is a bounded holomorphic semigroup of semi-angle $\theta$ with generator $A$, then for all $z \in S_{\theta}$ and $n \in \mathbb{N}$ one has $U(z)X \subseteq \text{dom}(A^n)$. Moreover, there are positive constants $C_A, C_A(n)$ such that for $t > 0$:

$$\left\| \frac{dU(t)}{dt} \right\| = \|AU(t)\| \leq \frac{C_A}{t}$$

and

$$\left\| \frac{d^n U(t)}{dt^n} \right\| = \|A^n U(t)\| \leq \frac{C_A(n)}{t^n}.$$

(1.11)

Let $0 < \theta < \pi/2$. Then estimates (1.11) are valid for complex argument $z \in S_{\theta}$ with constants depending on $\theta$.

Remark 1.8. Similarly to strongly continuous semigroups, a family $U(z)$, $z \in S_{\theta}$ is called a quasi-bounded holomorphic semigroup of semi-angle $\theta$ if there exists a constant $\beta > 0$ such that restriction of $[e^{-i\xi} U(z)]_{\xi \in S_{\theta}}$ to $[0, \pi]$ is a bounded $C_0$-semigroup.

The class of semigroups that we consider here is restricted to holomorphic contraction semigroups.

To this aim we recall definition of this notion below, [Kato95], Chapter IX.

Definition 1.9. We say that $\{U_A(z)\}_{z \in S_{\theta}}$ is a holomorphic contraction semigroup with generator $A \in \mathcal{H}_c(\theta, 0)$, if its restriction $\{U_A(t)\}_{t \geq 0}$ to $[0, \pi]$ is a contraction $C_0$-semigroup, that is $\mathcal{H}_c(\theta, 0) := \mathcal{H}(\theta, 0) \cap \mathcal{G}(1, 0)$.

Note that this class of semigroups is not empty and corresponding generators have the following properties:

(i) Let $\{U_A(t)\}_{t \geq 0}$, be a contraction semigroup with generator $A \in \mathcal{G}(1, 0)$ in a Banach space $X$, such that $U_A(t)X \subseteq \text{dom}(A)$ for $t > 0$. If $\|AU(t)\| \leq M_1 t^{-1}$ for some $M_1 > 0$ and all $t > 0$, then there exists $\theta = \arcsin (eM_1)^{-1} (< \pi/2)$ such that $U_A(t)$ may be analytically continued to contraction holomorphic semigroup of semi-angle $\theta$.

(ii) Let $A$ be a sectorial operator in a Hilbert space $\mathcal{H}$, i.e. its numerical range $W = \{Au, u \in \text{dom}(A) \text{ and } \|u\| = 1\} \subseteq S_{\pi/2-\theta}$ for $0 < \theta \leq \pi/2$. If the operator $A$ is closed, then it is generator of the holomorphic contraction semigroup of semi-angle $\theta$.

(iii) Let $A$ be a generator of holomorphic semigroup on a Banach space $X$. If $A$ is accretive, then $A$ generates a holomorphic contraction semigroup.

(iv) If $A$ is the generator of a strongly continuous group $\{U_A(t)\}_{t \in \mathbb{R}}$ of contractions $\|U_A(t)\| \leq 1$, then $\pm A \in \mathcal{G}(1, 0)$, and $A^2 \in \mathcal{H}_c(\pi/2, 0)$, [EN00], Corollary II.4.9.
1.3 Fractional powers of generators

We scrutinize in this section some properties of fractional powers of the generators for bounded semigroups in a Banach space, see, e.g., [Yos80], Chapter IX.

Recall that fractional power $A^\alpha$, $0 < \alpha < 1$, of generator of a bounded $C_0$-semigroup $U(t)$ ($\|U(t)\| \leq C_A$) can be expressed by the integral (when it is well-defined):

$$A^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty d\lambda \lambda^{-\alpha - 1} (U(\lambda) - I) x, \quad x \in \text{dom}(A), \quad (1.12)$$

where $\Gamma(\cdot)$ is the Gamma-function and $A^\alpha$ is chosen to be positive for $\lambda > 0$. Since for any $x \in \text{dom}(A)$ and $0 < \alpha \leq 1$ the integral (1.12) is convergent, $\text{dom}(A) \subseteq \text{dom}(A^\alpha)$. We set $A^0 := I$ and define $A^\alpha = A^{\alpha-[\alpha]} A^{[\alpha]}$ for any $\alpha > 0$, where $[\alpha]$ denotes the integer part of $\alpha$.

**Proposition 1.10.** For each $\alpha \in [0, 1]$, there exists a constant $C_{A,\alpha}$ depending only on $C_A$ and $\alpha$, such that for all $\mu > 0$,

$$\|A^\alpha (A + \mu I)^{-1}\| \leq \frac{C_{A,\alpha}}{\mu^{1-\alpha}}, \quad (1.13)$$

**Proof.** For $\alpha = 0$ or $\alpha = 1$, the result follows directly from the estimate of the resolvent. Let $0 < \alpha < 1$ and $x \in X$. Note that $\text{ran}(A + \mu I)^{-1} = \text{dom}(A) \subseteq \text{dom}(A^\alpha)$. Then

$$A^\alpha (A + \mu I)^{-1} x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty d\lambda \lambda^{-\alpha - 1} (U(\lambda) - I)(A + \mu I)^{-1} x. \quad (1.14)$$

We divide the integral (1.14) into two parts: $0 < \lambda \leq \mu^{-1}$ and $\lambda > \mu^{-1}$, and we use the representation (1.3):

$$A^\alpha (A + \mu I)^{-1} x = \frac{1}{\Gamma(-\alpha)} \int_0^{\mu^{-1}} d\lambda \lambda^{-\alpha - 1} \int_0^1 dt (-U(t))(I - \mu(A + \mu I)^{-1}) x$$

$$+ \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} d\lambda \lambda^{-\alpha - 1} (U(\lambda) - I)(A + \mu I)^{-1} x.$$

Now by the estimate of the resolvent $\|(A + \mu I)^{-1}\| \leq C_A/\mu$ for all $\mu > 0$ one obtains:

$$\|A^\alpha (A + \mu I)^{-1} x\| \leq \frac{C_A (1 + C_A)}{\Gamma(-\alpha)} \left( \int_0^{\mu^{-1}} d\lambda \lambda^{-\alpha} + \frac{1}{\mu} \int_{\mu^{-1}}^{\infty} d\lambda \lambda^{-\alpha - 1} \right) \|x\|$$

$$\leq \frac{C_A (1 + C_A) \mu^{1-\alpha}}{\alpha (1-\alpha) \Gamma(-\alpha)} \|x\|.$$

Setting $C_{A,\alpha} := C_A (1 + C_A)/(\alpha (1-\alpha) \Gamma(-\alpha))$ we obtain the estimate (1.13). \qed
Next we recall the following well-known property of the semigroup generator $A$:

**Lemma 1.11.** \( \text{dom}((A + \delta I)^\alpha) = \text{dom}(A^\alpha) \) for all \( \delta > 0 \) and \( 0 < \alpha < 1 \).

**Theorem 1.12.** Let \( U_A(t) \) be a bounded holomorphic semigroup with generator \( A \), then for any real \( \alpha > 0 \), we have

\[
\sup_{t > 0} \| t^\alpha A^\alpha U_A(t) \| = M_\alpha < \infty. \tag{1.15}
\]

**Proof.** Let \( 0 < \alpha < 1 \). By \( \text{dom}(A) \subseteq \text{dom}(A^\alpha) \) one gets \( \text{dom}(A^\alpha U_A(t)) = \mathbb{X} \). Hence by (1.12) we have

\[
A^\alpha U_A(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty d\lambda \lambda^{-\alpha-1} (U_A(t + \lambda) - U_A(t)). \tag{1.16}
\]

Now we split the integral (1.16) in two parts: \( 0 < \lambda \leq t \) and \( \lambda > t \), and we use the estimate of the derivative of the holomorphic semigroup (see Proposition 1.7) to obtain

\[
\|U_A(t + \lambda) - U_A(t)\| \leq \lambda \sup_{t \leq \tau \leq t + \lambda} \| \partial_\tau U_A(\tau) \| \leq \lambda \frac{C'_A}{t}. \tag{1.17}
\]

This leads to the estimate

\[
\|A^\alpha U_A(t)\| \leq \frac{1}{\Gamma(-\alpha)} \left( \int_0^t d\lambda \lambda^{-\alpha} \frac{C'_A}{t} + \int_t^\infty d\lambda 2C_A \lambda^{-\alpha-1} \right) \\
\leq \frac{t^{-\alpha}}{\Gamma(-\alpha)} \left( \frac{C'_A}{1 - \alpha} + \frac{2C_A}{\alpha} \right).
\]

Therefore one obtains (1.15) for \( 0 < \alpha < 1 \) by setting \( M_\alpha := \Gamma(-\alpha)^{-1}(C'_A/(1 - \alpha) + 2C_A/\alpha) \).

For integer powers \( \alpha \), (1.15) follows directly from Proposition 1.7. Notice that by Proposition 1.7 \( \text{ran}(U_A(t)) \subseteq \text{dom}(A^\alpha) \) for \( t > 0 \). Then result (1.15) follows for any non integer \( \alpha > 1 \), from: the observation that \( \text{dom}(A^\alpha) = A^{\alpha-1}[A^\alpha] = \text{dom}(A^{[\alpha]}) \), the representation (1.16), and the estimate (1.11) of derivatives of order \([\alpha] + 1\). \( \square \)

## 2 Operator-norm Trotter product formula

### 2.1 Perturbation of holomorphic contraction semigroups

In this section we prove the operator-norm convergence of the Trotter product formula in Banach space \( \mathbb{X} \). Note that convergence of the abstract version of this formula in the strong operator topology is known since \( \text{Trot59} \). The proof goes via estimate of the rate of convergence in the case of small perturbations of the holomorphic contraction semigroups, cf. \( \text{CZ01} \).
We require that operator \( A \) generates a holomorphic contraction semigroup, and that perturbation \( B \) satisfies the following hypothesis:

(H1) \( A \) is generator of a contraction semigroup on \( X \).

(H2) There is a real \( \alpha \in [0, 1) \) such that \( \text{dom}(A^\alpha) \subseteq \text{dom}(B) \) and that \( \text{dom}(A^*) \subseteq \text{dom}(B^*) \) adjoint operator in the dual space \( X^* \).

Notice that we can suppose the operator \( A \) boundedly invertible. If it is not the case, one considers \( A + \eta \) for some \( \eta > 0 \). Then by Lemma 2.1 we have \( \text{dom}(A + \eta \mathbb{1}) = \text{dom}(A^\alpha) \subseteq \text{dom}(B) \).

**Remark 2.1.** We note that assumption (H2) implies that \( \eta \) is relatively bounded with respect to \( A \) and the relative bound in (2.2) can be made infinitesimally small for the large enough shift parameter \( \eta > 0 \).

Since the operators \( A^\alpha \) and \( B \) are closed, the inclusions in (H2) are equivalent to \( A^\alpha \)-boundedness of \( B \) and the \( A^\alpha \)-boundedness of \( B^* \). In particular, \( \|BA^{-\alpha}\| \leq d \) and \( \|BA^{-\alpha}A^{-\alpha}\| \leq d' \) for some \( d, d' > 0 \). Therefore, for any \( x \in \text{dom}(A) \subseteq \text{dom}(B) \), we have the estimate

\[
\|Bx\| \leq \frac{C_\alpha d}{\eta^{1-\alpha}} \|Ax\| + \eta^\alpha C_\alpha d \|x\| \tag{2.2}
\]

and the relative bound in (2.2) can be made infinitesimally small for the large enough shift parameter \( \eta > 0 \).

For this class of (small) perturbations of holomorphic contraction semigroup with generator \( A \in \mathcal{H}(\theta, 0) \) (Definition 1.9) one gets the following result:

**Lemma 2.2.** Let \( \{e^{-\alpha t}\}_{\alpha \in S^1} \) be a holomorphic contraction semigroup of semi-angle \( \theta \) on \( X \) and perturbation \( B \) satisfy the hypothesis (H1) and (H2). Then the algebraic sum \( A + B \) of operators defined on \( \text{dom}(A + B) = \text{dom}(A) \) is also a generator of holomorphic contraction semigroup with the same semi-angle, that is, \( (A + B) \in \mathcal{H}(\theta, 0) \).

**Proof.** To this end we verify conditions of Proposition 1.6. Let \( \epsilon \in (0, \theta) \). Then by (2.2) we obtain inequality

\[
\|B(A + z\mathbb{1})^{-1}\| \leq \frac{C_\alpha \|BA^{-\alpha}\|}{\eta^{1-\alpha}} \|A(A + z\mathbb{1})^{-1}\| + \eta^\alpha C_\alpha \|BA^{-\alpha}\| \|A(A + z\mathbb{1})^{-1}\|,
\]

for \( |\arg(z)| < \theta + \pi/2 - \epsilon \). Seeing that \( \|(z I + A)^{-1}\| \leq N_\epsilon |z|^{-1} \) for \( N_\epsilon > 0 \) and \( z \in S_{\theta+\pi/2-\epsilon} \), this inequality leads to

\[
\|B(A + z\mathbb{1})^{-1}\| \leq \frac{C_\alpha \|BA^{-\alpha}\|}{\eta^{1-\alpha}} (1 + N_\epsilon) + \eta^\alpha C_\alpha \|BA^{-\alpha}\| \frac{N_\epsilon}{|z|}, \tag{2.3}
\]
for $z \in S_{\theta + \pi/2 - \epsilon}$. Therefore, the Neumann series for $(A + B + z I)^{-1}$ converges if the right hand side of $\overline{\mathcal{C}_3}$ is smaller than 1. Since we can choose $\eta$ and $z \in S_{\theta + \pi/2 - \epsilon}$ such that the right-hand side of the estimate $\overline{\mathcal{C}_3}$ becomes smaller than 1, we obtain:

$$\| (A + B + z I)^{-1} \| \leq \frac{M}{|z - \gamma|}.$$  

Here $M$ and $\gamma$ are some positive constants. Then by Proposition 1.6 we conclude that operator $(A + B) \in \mathcal{H}(\theta, \gamma)$, that is, it generates a quasi-bounded holomorphic semigroup $\{U_{A + B}(z)\}_{z \in \mathbb{D}_S}$.

On the other hand, the conditions of lemma imply that $A$ and $B$ are accretive, thus operator $A + B$ is also accretive. Since for $\lambda < 0$, and $|\lambda|$ sufficiently large ($|\lambda| > \gamma$), the point $\lambda$ is in the resolvent set $\rho(A + B)$, we conclude that $(A + B) \in \mathcal{H}(1, 0)$ generates a contraction semigroup.

Since by $(A + B) \in \mathcal{H}(\theta, \gamma)$ this semigroup is also holomorphic, one finally obtains the assertion. □

The proof of the main theorem of this section involves three technical lemmata. For the two of them we need only that $B(B^*)$ are $A(A^*)$-bounded in the Kato sense, i.e., there are positive constants $a$ and $b$ such that:

1. $x \in \text{dom}(A) \subseteq \text{dom}(B)$, $\| Bx \| \leq a \| Ax \| + b \| x \|$,
2. $\phi \in \text{dom}(A^*) \subseteq \text{dom}(B^*)$, $\| B^* \phi \| \leq a \| A^* \phi \| + b \| \phi \|$.

If $A$ is boundedly invertible, then we can put $b = 0$ with the relative bound $a + b |A^{-1}|$ instead of $a$.

**Lemma 2.3.** Let boundedly invertible $A$ and operator $B$ be generators of bounded semigroups. Let $B$ and $B^*$ verify (2.4), (2.5) and suppose that operator $H = (A + B)$ with $\text{dom}(H) = \text{dom}(A)$ is the boundedly invertible generator of a bounded semigroup. Then there exists constant $L_1$ such that for all $\tau \geq 0$:

$$\left\| A^{-1} \left( e^{-\tau B} e^{-\tau A} - e^{-\tau(A+B)} \right) \right\| \leq L_1 \tau, \tag{2.6}$$

$$\left\| \left( e^{-\tau B} e^{-\tau A} - e^{-\tau(A+B)} \right) A^{-1} \right\| \leq L_1 \tau. \tag{2.7}$$

**Proof.** By virtue of the identity

$$A^{-1} \left( e^{-\tau B} e^{-\tau A} - e^{-\tau(A+B)} \right) = A^{-1} \left( e^{-\tau B} - 1 \right) e^{-\tau A} + A^{-1} \left( e^{-\tau A} - 1 \right) + A^{-1} HH^{-1} \left( 1 - e^{-\tau H} \right)$$

and by Lemma 1.2 we get (2.6):
Let $A$, $B$ and $H = A + B$ be the same as in Lemma 2.4. Then there exists a constant $L_2 > 0$ such that for all $\tau \geq 0$:

\[
\begin{align*}
\left\| A^{-1} \left( e^{-\tau B} e^{-\tau A} - e^{-\tau(A+B)} \right) A^{-1} \right\| &\leq L_2 \tau^2, \quad (2.8) \\
\left\| A^{-1} \left( e^{-\tau A} e^{-\tau B} - e^{-\tau(A+B)} \right) A^{-1} \right\| &\leq L_2 \tau^2. \quad (2.9)
\end{align*}
\]

**Proof.** By virtue of

\[
e^{-\tau B} e^{-\tau A} - e^{-\tau H} = (\mathbb{1} - e^{-\tau B})(\mathbb{1} - e^{-\tau A}) + \left( e^{-\tau A} - (\mathbb{1} + \tau A)^{-1} \right) + \left( e^{-\tau B} - (\mathbb{1} + \tau B)^{-1} \right) + (\mathbb{1} + \tau H)^{-1} - e^{-\tau H}
\]

and by identity

\[
A^{-1} \left( \tau H(\mathbb{1} + \tau H)^{-1} - \tau A(\mathbb{1} + \tau A)^{-1} - \tau B(\mathbb{1} + \tau B)^{-1} \right) A^{-1}
\]

we obtain the representation

\[
A^{-1} \left( e^{-\tau B} e^{-\tau A} - e^{-\tau H} \right) A^{-1} =
\]

\[
A^{-1}(\mathbb{1} - e^{-\tau B})(\mathbb{1} - e^{-\tau A}) A^{-1} + \left( e^{-\tau A} - (\mathbb{1} + \tau A)^{-1} \right) A^{-2} + A^{-1} \left( e^{-\tau B} - (\mathbb{1} + \tau B)^{-1} \right) A^{-1} + A^{-1} H \left( (\mathbb{1} + \tau H)^{-1} - e^{-\tau H} \right) H^{-2} H^{-1} A^{-1} + \tau^2 (\mathbb{1} + \tau A)^{-1} + \tau^2 A^{-1} B(\mathbb{1} + \tau B)^{-1} BA^{-1} - \tau^2 A^{-1} H(\mathbb{1} + \tau H)^{-1} H A^{-1}.
\]

This presentation yields the following estimate:
Operator-norm Trotter product formula

\[
\frac{1}{\tau^2} \| A^{-1}(e^{-\tau A} - e^{-\tau B})A^{-1} \| \leq \frac{1}{\tau} \| A^{-1}B \| \int_0^\tau ds \ e^{-\tau B} \frac{1}{\tau} \| (1 - e^{-\tau A})A^{-1} \| + \frac{1}{\tau^2} \| (e^{-\tau A} - (1 + \tau A)^{-1})A^{-2} \| + \frac{1}{\tau^2} \| A^{-1}B \| \int_0^\tau ds \int_0^s ds_1 ds_2 e^{-s_2 B} - \tau^2 (1 + \tau B)^{-1} \| BA^{-1} \| + \frac{1}{\tau^2} \| A^{-1}H \| \| (1 + \tau H)^{-1} - e^{-\tau H} \| \| HA^{-1} \| + 1 + \| A^{-1}B \| \| BA^{-1} \| + \| A^{-1}H \| \| HA^{-1} \|
\]

Now Lemmata 1.2 and 1.4 together with (2.4), (2.5) imply (2.8), where we can take \( L_2 = \alpha' C A C_B + 3 C_A/2 + 3 C_B a^2/2 + 3 C_H(1 + a')^2/2 + 1 + a'^2 + (1 + a')^2 \) with \( a' = a + b \| A^{-1} \| \). Similarly one obtains (2.9).

\[ \square \]

Note that for the proof of the third lemma (Lemmata 2.4) we do need conditions (H2), as well as requirement that semigroups are contractive.

**Lemma 2.5.** Let \( A \) be a boundedly invertible generator of holomorphic contraction semigroup. If \( B \) is generator of a contraction semigroup and there exists \( \alpha \in (0, 1) \) such that \( \text{dom}(A^\alpha) \subseteq \text{dom}(B) \), then for any \( k \geq 1 \) and \( \tau > 0 \) one gets the estimates

\[
\left\| \left( e^{-\tau B}e^{-\tau A} \right)^k \right\| \leq \frac{L_3}{\tau^\alpha} + \frac{C_A}{k\tau}, \quad \text{if } \alpha > 0, \quad (2.10)
\]

\[
\left\| \left( e^{-\tau B}e^{-\tau A} \right)^k \right\| \leq \tilde{L}_3(1 + \ln k) + \frac{C_A}{k\tau}, \quad \text{if } \alpha = 0. \quad (2.11)
\]

**Proof.** We start with the following chain of estimates:

\[
\left\| \left( e^{-\tau B}e^{-\tau A} \right)^k \right\| \leq \left\| \left( e^{-\tau B}e^{-\tau A} \right)^k - e^{-k\tau A} \right\| + \left\| e^{-k\tau A} \right\|
\]

\[
\leq \sum_{j=0}^{k-1} \left\| \left( e^{-\tau B}e^{-\tau A} \right)^{k-1-j} \left( e^{-\tau B} - I \right) e^{-\tau A} \right\| + \left\| e^{-k\tau A} \right\|
\]

\[
\leq \sum_{j=0}^{k-1} \int_0^\tau ds \ e^{-s B} A^{-1} \left\| A^\alpha e^{-j+1}\tau A \right\| + \left\| e^{-k\tau A} \right\|
\]

Notice that the second inequality is in particular due to contractions of \( e^{-\tau A} \) and \( e^{-\tau B} \); and to equation (1.13) of Lemma 1.2. From the hypothesis \( \text{dom}(A^\alpha) \subseteq \text{dom}(B) \) we deduce that \( \| BA^{-1} \| \leq d \), see Remark 2.1. By Propositions 1.7 and Theorem 1.12 we get respectively:

\[
\left\| e^{-k\tau A} \right\| \leq \frac{C_A}{k\tau} \text{ and } \left\| A^\alpha e^{-j+1}\tau A \right\| \leq \frac{M_\alpha}{(j+1)\tau^{1+\alpha}}.
\]
Therefore, we conclude that:

\[
\left\| \left( e^{-\tau B} e^{-\tau A}\right)^k A \right\| \leq \frac{M_\alpha A}{\tau^\alpha} \sum_{j=0}^{k-1} \frac{1}{(j+1)^{1+\alpha}} + \frac{C'_A}{k\tau}.
\]

Since \( \alpha > 0 \), this gives the announced result (2.10) with

\[
L_3 = d M_\alpha \sum_{j=1}^{\infty} (1/j)^{1+\alpha},
\]

and (2.11) for \( \alpha = 0 \) with \( \tilde{L}_3 = \|B\| C'_A \). □

Note that since \( \text{dom}(A^\alpha) \subseteq \text{dom}(B) \) implies \( \text{dom}(A^{\alpha'}) \subseteq \text{dom}(B) \) for \( \alpha' \geq \alpha \), the estimate (2.10) is valid in fact for any \( \alpha' \geq \alpha \).

### 2.2 Convergence rate

**Theorem 2.6.** Let \( \{e^{-tA}\}_{t \in S} \) be a holomorphic contraction semigroup, that is, \( A \in \mathcal{H}_c(S,0) \). Let \( B \) be generator of a contraction semigroup. If there exists \( \alpha \in [0,1) \) such that \( \text{dom}(A^\alpha) \subseteq \text{dom}(B) \) and \( \text{dom}(A^*) \subseteq \text{dom}(B^*) \), then there are constants \( M_1, M_2, \tilde{M}_2, \eta > 0 \), such that for any \( t \geq 0 \) and \( n > 1 \) one gets estimates

\[
\left\| \left( e^{-tB/n} e^{-tA/n}\right)^\alpha - e^{-t(A+B)} \right\| \leq e^{\eta t} (M_1 + M_2 t^\alpha) \frac{\ln n}{n^{1+\alpha}}, \quad \alpha > 0, \quad (2.12)
\]

\[
\left\| \left( e^{-tB/n} e^{-tA/n}\right)^\alpha - e^{-t(A+B)} \right\| \leq e^{\eta t} (M_1 + \tilde{M}_2 t) \frac{(\ln n)^2}{n}, \quad \alpha = 0. \quad (2.13)
\]

**Proof.** Since \( B \) satisfies hypothesis (H1) and (H2), by Lemma 2.2 the operator \( H = (A + B) \) is generator of a holomorphic contraction semigroup. If operator \( A \) has no bounded inverse, let \( \tilde{A} := A + \eta \) and \( \tilde{H} := \tilde{A} + B \) for some \( \eta > 0 \) (see Remark 2.1). Then both operators are boundedly invertible. As we indicated above, these changes of generators do not modify the domain inclusions. If we want to obtain \( \|B\tilde{A}^{-1}\| < 1 \) then by the estimate (2.11) we have to choose a sufficiently large shift parameter \( \eta > 0 \). This gives us the estimate \( \|A\bar{H}^{-1}\| = \|(1 + B\tilde{A}^{-1})\| \leq 1/(1 - \alpha) \) where we set \( a = \|B\tilde{A}^{-1}\| \).

Now we put \( \tau := t/n, \bar{U}(t) := e^{-t\tilde{H}}, \) and \( \bar{T}(\tau) := e^{-\tau B} e^{-\tau A}. \) To estimate the left-hand side of (2.12) we use

\[
\left( e^{-tB/n} e^{-tA/n}\right)^\alpha - e^{-t(A+B)} = (\bar{T}^\alpha(\tau) - \bar{U}^\alpha(\tau)) e^{\eta t}, \quad (2.14)
\]

and telescopic identity:
\[
\hat{T}(\tau)^n - \hat{U}(\tau)^n = \sum_{m=0}^{n-1} \hat{T}(\tau)^{n-m-1}(\hat{T}(\tau) - \hat{U}(\tau))\hat{U}(\tau)^m \\
= \hat{T}(\tau)^{n-1}\tilde{A}\tilde{A}^{-1}(\hat{T}(\tau) - \hat{U}(\tau)) + (\hat{T}(\tau) - \hat{U}(\tau))\tilde{A}\tilde{A}^{-1}\tilde{A}\tilde{H}^{-1}\tilde{H}\hat{U}(\tau)^{n-1} + \sum_{m=1}^{n-2} \hat{T}(\tau)^{n-m-1}\tilde{A}\tilde{A}^{-1}(\hat{T}(\tau) - \hat{U}(\tau))\tilde{A}\tilde{H}^{-1}\tilde{H}\hat{U}(\tau)^m,
\]

which implies
\[
\left\|\hat{T}(\tau)^n - \hat{U}(\tau)^n\right\| \leq \left\|\hat{T}(\tau)^{n-1}\tilde{A}\right\|\left\|\tilde{A}^{-1}(\hat{T}(\tau) - \hat{U}(\tau))\right\| + \left\|(\hat{T}(\tau) - \hat{U}(\tau))\tilde{A}\tilde{A}^{-1}\right\|\left\|\tilde{A}\tilde{H}^{-1}\tilde{H}\hat{U}(\tau)^{n-1}\right\| + \sum_{m=1}^{n-2} \left\|\hat{T}(\tau)^{n-m-1}\tilde{A}\right\|\left\|\tilde{A}^{-1}(\hat{T}(\tau) - \hat{U}(\tau))\tilde{A}\tilde{H}^{-1}\tilde{H}\hat{U}(\tau)^m\right\|.
\]

Hence by Lemmata 2.3 and 2.5 (it is at this point that we use the hypothesis of contraction), and by Proposition 1.17, we obtain the estimate:
\[
\left\|\hat{T}(\tau)^n - \hat{U}(\tau)^n\right\| \leq \left(\frac{L_3}{r^n} + \frac{C_A}{n-1} \right) L_1 \tau + \frac{L_1}{1-a} \frac{C_H'}{n-1} \\
+ \sum_{m=1}^{n-2} \left( L_3 \tau^{1-a} + \frac{C_A}{n-m-1} \right) L_1 \frac{C_H'}{1-a} \\
\leq L_3 L_1 \frac{\tau^{1-a}}{n-1} + \frac{L_1}{1-a} \left( C_A + \frac{C_H'}{n-1} \right) + \frac{L_2}{1-a} \frac{C_H'}{n-1} \sum_{m=1}^{n-2} \frac{1}{n-m-1} + \frac{1}{m} \\
+ \frac{L_2 C_H' C_A}{1-a} \sum_{m=1}^{n-2} \frac{1}{n-m-1} \frac{1}{m} \\
\leq L_3 L_1 \frac{\tau^{1-a}}{n-1} + \frac{L_1}{1-a} \left( C_A + \frac{C_H'}{1-a} \right) \\
+ 2 L_3 L_2 C_H' C_A \frac{\ln n}{1-a} + 4 \frac{L_2 C_H' C_A}{1-a} \ln n.
\]

Here we used that:
\[
\sum_{m=1}^{n} \frac{1}{(n-m)m} = 2 \sum_{m=1}^{n-1} \frac{1}{m} \leq 2 (1 + \ln(n-1)) \leq 4 \frac{\ln n}{n}.
\]

The estimate (2.15) and (2.14) imply the announced result (2.12) for \( \alpha > 0 \), with \( M_1 = 4 L_1 \left( C_A + \frac{C_H'}{1-a} \right) + 4 \frac{L_2 C_H' C_A}{1-a} \) and \( M_2 = 2 L_3 L_1 + 2 \frac{L_3 L_2 C_H' C_A}{1-a} \).

In a similar way one gets also estimate for \( \alpha = 0 \):
\[\|T(t)^n - U(t)^n\| \leq L_3(1 + \ln(n - 1)) \left( \frac{L_1 t}{n} + \frac{L_1 C_A}{a n - 1} + \frac{L_1 C_H'}{1 - a} \right) + \sum_{m=1}^{n-2} \left( \frac{L_1 t}{n} \left( 1 + \ln(n - m - 1) \right) + \frac{C_A'}{n - m - 1} \right) \frac{L_2 C_H'}{1 - a} \]

This estimate together with (2.14) yield (2.13) for \(\tilde{H}\) for any \(t\). \(\Box\)

**Theorem 2.7.** Let \(\{e^{-\lambda t}\}_{t \in \mathbb{R}}\) be a holomorphic contraction semigroup, that is, \(A \in \mathcal{L}(\mathcal{H}(0, 0)). \) Let \(B\) be a generator of a contraction semigroup, and there exists \(\alpha \in [0, 1]\) such that \(\text{dom}((A^\alpha)'') \subseteq \text{dom}(B')\) and \(\text{dom}(A) \subseteq \text{dom}(B)\). If in addition \(\text{dom}(A^\alpha) \subseteq \text{dom}(B')\) (for the case, when the space \(\mathcal{H}\) is not reflexive), then there are constants \(M_3, M_4, \tilde{M}_4, \eta > 0\), such that for any \(t \geq 0\) and \(n > 2\):

\[\left\| \left( e^{-\alpha t/n} e^{-B/n} \right)^n - e^{-\alpha t + B} \right\| \leq e^{\eta n} (M_3 + M_4 t^{1 - \alpha}) \frac{\ln n}{n^{1/2}}, \quad \alpha > 0, \quad (2.16)\]

\[\left\| \left( e^{-\alpha t/n} e^{-B/n} \right)^n - e^{-\alpha t + B} \right\| \leq e^{\eta n} (M_3 + \tilde{M}_4 t) \frac{2 (\ln n)^2}{n}, \quad \alpha = 0, \quad (2.17)\]

for any \(t \geq 0\) and \(n > 2\).

**Proof.** Let \(\bar{F}(t) := e^{-\lambda t} e^{-tB}\). Then by the same arguments as in the proof of Theorem 2.6, one obtains:

\[\bar{U}(t)^n - \bar{F}(t)^n = \sum_{m=0}^{n-1} \bar{U}(t)^{n-m-1}(\bar{U}(t) - \bar{F}(t))\bar{F}(t)^m\]

\[= \bar{U}(t)^{n-1} \bar{A}^{n-1} \bar{A}^{-1}(\bar{U}(t) - \bar{F}(t))\]

\[+ (\bar{U}(t) - \bar{F}(t)) \bar{A}^{-1} \bar{A} \bar{F}(t)^{n-1}\]

\[+ \sum_{m=1}^{n-2} \bar{U}(t)^{n-m-1} \bar{H} \bar{H}^{-1} \bar{A}^{-1} (\bar{U}(t) - \bar{F}(t)) \bar{A}^{-1} \bar{A} \bar{F}(t)^m.\]

Notice that the Lemmata 2.3 and 2.4 hold for \(\bar{F}(t)\). By a simple modification of Lemma 2.5, one uses \(\|\bar{A}^{-1} B\| = \|B'(\bar{A}^{-1})\| < \infty\), we find that

\[\left\| \bar{A} \left( e^{-\lambda t} e^{-tB} \right)^k \right\| \leq \frac{L_4}{t^\alpha} + \frac{C_A'}{k t}, \quad \alpha > 0,\]

\[\left\| \bar{A} \left( e^{-\lambda t} e^{-tB} \right)^k \right\| \leq L_4(1 + \ln k) + \frac{C_A'}{k t}, \quad \alpha = 0.\]

These ingredients ensure the estimates (2.16) and (2.17). \(\Box\)

**Corollary 2.8.** Under the same conditions as in Theorem 2.6, we have the operator-norm convergence of the symmetrised Trotter formula, i.e., there exists \(M_5, M_6, \tilde{M}_6, \eta > 0\), such that for any \(t \geq 0\) and \(n > 2\):
\[
\left\| (e^{-tA/2n} - e^{-tB/2n} - e^{-t(A+B)/2n})^n \right\| \leq e^{\eta t} (M_5 + M_6 n^{1-\alpha}) \frac{\ln n}{n^{1-\alpha}}, \quad \text{for } 0 < \alpha < 1, \quad (2.18)
\]

\[
\left\| (e^{-tA/2n} - e^{-tB/2n} - e^{-t(A+B)/2n})^n \right\| \leq e^{\eta t} (M_5 + M_6 n^{1-\alpha}) \frac{2 (\ln n)^2}{n}, \quad \text{for } \alpha = 0. \quad (2.19)
\]

**Proof.** Since Lemmata 2.3, 2.4, and 2.5 can be easily extended to the symmetrized product \( e^{-tA/2n} e^{-tB/2n} e^{-tA/2n} \), the proof of the Theorem 2.6 carries through verbatim to obtain (2.18) and (2.19).

**Remark 2.9.** Seeing that in Theorems 2.6, 2.7 and in Corollary 2.8 the perturbation \( B \) of dominating operator \( A \) is either infinitesimally \( A \)-small, or simply bounded, the corresponding results in Banach space \( X \) are weaker than those in Hilbert space \( H \), see [Rog93], [NZ98], [NZ99], and [Zag20]. Recall that in [Rog93], [NZ98] the perturbation \( B \) is Kato-small with respect to operator \( A \) for relative bound \( b < 1 \). The fractional condition (H2) in a Hilbert space \( H \) was introduced in [IT97].

Note that in the both cases: \( X \) and \( H \), the dominating operator \( A \) is supposed to be generator of a holomorphic semigroup.

### 3 Example

Resuming Remark 2.9, the question arises: whether the Trotter product formula converges in the operator-norm topology if the condition on dominating generator \( A \in \mathcal{H}(1, 0) \) is relaxed to hypothesis that \( A \in \mathcal{G}(1, 0) \), i.e., it is generator of a contraction (but not holomorphic!) semigroup and \( B \) is a bounded generator?

The aim of this section is to give an answer to this question using example of a certain class of generators and semigroups. It turns out that appropriate for this purpose is the class of generators of evolution semigroups.

#### 3.1 Evolution semigroups and Trotter product formula

To proceed further we need some key notions from the evolution semigroups theory and in particular the notion of solution operator.

A strongly continuous mapping \( U(\cdot, \cdot) : \Delta \to L(X) \), where domain \( \Delta : = \{(t, s) : 0 < s \leq t \leq T\} \) and \( L(X) \) is the set of bounded operators on separable Banach space \( X \), is called a solution operator if the conditions

(i) \( \sup_{(t, s) \in \Delta} \|U(t, s)\|_{L(X)} < \infty, \)

(ii) \( U(t, s) = U(t, r)U(r, s), \quad 0 < s \leq r \leq t \leq T, \)
More precisely, operator induced by generator $A$ are satisfied. Let us consider the Banach space $L^p(I, \mathcal{X})$ for $I := [0, T]$ and $p \in [1, \infty)$. The operator $\mathcal{K}$ is an evolution generator of the evolution semigroup $\{U(t) := e^{-t\mathcal{K}}\}_{t \geq 0}$ if there is a solution operator such that the Howland-Evans-Neidhardt representation, see [How74], [Ev76], [Nei79] and [Nei81]:

\[
(e^{-t\mathcal{K}}f)(t) = U(t, t-\tau)\chi_I(t-\tau)f(t-\tau), \quad f \in L^p(I, \mathcal{X}),
\]

holds for a.a. $t \in I$ and $\tau \geq 0$. Seeing that on account of (3.1) the semigroup $\{e^{-\tau\mathcal{K}}\}_{\tau \geq 0}$ is nilpotent: $e^{-\tau\mathcal{K}}f = 0$ for $\tau \geq T$, the evolution generator $\mathcal{K}$ can never be generator of a holomorphic semigroup.

A simple example of an evolution generator is the differentiation operator, cf. [NSZ20]:

\[
(D_0f)(t) := \partial_t f(t), \quad f \in \text{dom}(D_0) := \{f \in W^{1,p}(I, \mathcal{X}) : f(0) = 0\},
\]

where $W^{1,p}(I, \mathcal{X})$ is the Sobolev space of order $(1, p)$ of Bochner $p$-integrable functions. Then by (3.2) one obviously gets the contraction shift semigroup:

\[
(e^{-\tau D_0}f)(t) = \chi_I(t-\tau)f(t-\tau), \quad f \in L^p(I, \mathcal{X}),
\]

for a.a. $t \in I$ and $\tau \geq 0$. Hence, (3.1) implies that the corresponding solution operator of the non-holomorphic evolution semigroup $\{e^{-\tau D_0}\}_{\tau \geq 0}$ is given by $U_{D_0}(t, s) = 1$, for all $(t, s) \in A$.

Below we consider the operator $\mathcal{K}_0 := D_0 + \mathcal{A}$, where $\mathcal{A}$ is the multiplication operator induced by generator $A$ of a holomorphic contraction semigroup on $\mathcal{X}$. More precisely

\[
(\mathcal{A}f)(t) := Af(t), \quad \text{and } (e^{-\tau\mathcal{A}}f)(t) = e^{-\tau A}f(t), \quad f \in \text{dom}(\mathcal{A}) := \{f \in L^p(I, \mathcal{X}) : Af \in L^p(I, \mathcal{X})\}.
\]

Then the perturbation of the shift semigroup (3.3) by $\mathcal{A}$ corresponds to the semigroup with generator $\mathcal{K}_0$. One easily checks that $\mathcal{K}_0$ is an evolution generator of a contraction semigroup on $L^p(I, \mathcal{X})$, that is never holomorphic [NSZ20]. Indeed, since the generators $D_0$ and $\mathcal{A}$ commute, the representation (3.1) for evolution semigroup $\{e^{-\tau\mathcal{K}_0}\}_{\tau \geq 0}$ takes the form:

\[
(e^{-\tau\mathcal{K}_0}f)(t) = e^{-\tau A}\chi_I(t-\tau)f(t-\tau), \quad f \in L^p(I, \mathcal{X}),
\]

for a.a. $t \in I = [0, T]$ and $\tau \geq 0$. Then by (3.1) the solution operator $U_0(t, s) = e^{-(t-s)A}$. Therefore, $e^{-\tau\mathcal{K}_0}f = 0$ for $\tau \geq T$, that is, semigroup $\{e^{-\tau\mathcal{K}_0}\}_{\tau \geq 0}$ is nilpotent.

Furthermore, if $\{B(t)\}_{t \in I}$ is a strongly measurable family of generators of contraction semigroups on $\mathcal{X}$, that is, $B(\cdot) : I \rightarrow \mathcal{G}(1,0)$, then the induced multiplication operator $\mathcal{B}$:
\[(Bf)(t) := B(t)f(t),\quad f \in \text{dom}(B) := \left\{ f \in L^p(I, \mathbb{X}) : f(t) \in \text{dom}(B(t)) \text{ for a.a. } t \in I \right\},\]

is a generator of a contraction semigroup on \(L^p(I, \mathbb{X})\).

In the next Subsection 3.2 we consider perturbation of generator \(\mathcal{K}_0\) by multiplication operator \(B\) (3.4). Thereupon we construct by means of the Trotter product formula approach a corresponding perturbed semigroup.

**Remark 3.1.** We conclude by remarks concerning some *notations* and *definitions* that we use below throughout Section 3.

1. For characterisation the rate of convergence we use, so-called, *Landau’s symbols*:

   \[g(n) = O(f(n)) \iff \limsup_{n \to \infty} \left| \frac{g(n)}{f(n)} \right| < \infty,\]

   \[g(n) = o(f(n)) \iff \limsup_{n \to \infty} \left| \frac{g(n)}{f(n)} \right| = 0,\]

   \[g(n) = \Theta(f(n)) \iff 0 < \liminf_{n \to \infty} \left| \frac{g(n)}{f(n)} \right| \leq \limsup_{n \to \infty} \left| \frac{g(n)}{f(n)} \right| < \infty,\]

   \[g(n) = \omega(f(n)) \iff \limsup_{n \to \infty} \left| \frac{g(n)}{f(n)} \right| = \infty.\]

2. We also use the notation \(C^{0,\beta}(I) := \{ f : I \to \mathbb{C} : \text{there exists some } K > 0 \text{ such that } |f(x) - f(y)| \leq K |x - y|^{\beta}, \text{ for any } x, y \in I \text{ and } \beta \in (0, 1) \}\).

3. Below we consider the Banach space \(L^p(I, \mathbb{X})\) for \(I := [0, T]\) and \(p \in [1, \infty)\).

### 3.2 Trotter product formula

We reminisce (cf. Subsection 3.1) that semigroup \(\{U(t)\}_{t \geq 0}\), on the Banach space \(L^p(I, \mathbb{X})\) is called the *evolution semigroup* if there is a *solution operator*: \(\{U(t, s)\}_{(t, s) \in I}\), such that representation (3.1) holds.

Let \(\mathcal{K}_0\) be the generator of an evolution semigroup \(\{U_0(t)\}_{t \geq 0}\) and let \(B\) be a multiplication operator induced by a measurable family \(\{B(t)\}_{t \in I}\) of generators of contraction semigroups. Note that in this case the multiplication operator \(B\) (3.4) is a generator of a contraction semigroup \((e^{-\tau B}f)(t) = e^{-\tau B(t)}f(t)\), on the Banach space \(L^p(I, \mathbb{X})\). Since \(\{U_0(t)\}_{t \geq 0}\) is an evolution semigroup, then by definition (3.1) there is a propagator \((U_0(t, s))_{(t, s) \in I}\) such that the representation

\[ (U_0(t)\tau f)(t) = U_0(t, t-\tau)\chi_I(t, t-\tau)f(t-\tau), \quad f \in L^p(I, \mathbb{X}), \]

is valid for a.a. \(t \in I\) and \(\tau \geq 0\). Then we define \(\tau_n := (t-s)/n\), for \(n \in \mathbb{N}\), and
\( G_j(t, s; n) := U_0(s + j \tau_n, s + (j - 1) \tau_n) e^{\tau_n B(t + (j - 1) \tau_n)},\ (t, s) \in \mathcal{A}, \)

where \( j \in \{1, 2, \ldots, n\}, n \in \mathbb{N}, (t, s) \in \mathcal{A}, \) and we set

\[
V_n(t, s) := \prod_{j=0}^{n} G_j(t, s; n), \quad n \in \mathbb{N}, \quad (t, s) \in \mathcal{A}.
\]

That is, the product is increasingly ordered in \( j \) from the right to the left. Then a straightforward computation shows that the representation

\[
\left( \left( e^{\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n f \right)(t) = V_n(t, t - \tau) \chi_I(t - \tau)f(t - \tau),
\]

\( f \in L^p(I, \mathcal{X}), \) holds for each \( \tau \geq 0 \) and a.a. \( t \in \mathcal{I}. \)

**Theorem 3.2.** Let \( \mathcal{K} \) and \( \mathcal{K}_0 \) be generators of evolution semigroups on the Banach space \( L^p(I, \mathcal{X}) \) for some \( p \in [1, \infty). \) Further, let \( \{B(t) \in \mathcal{B}(1, 0)\}_{t \in I} \) be a strongly measurable family of generators of contraction semigroups on \( \mathcal{X}. \) Then

\[
\sup_{t \in [0, \infty)} \left\| e^{-\tau \mathcal{K}} - \left( e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n \right\|_{L^p(I, \mathcal{X})}
= \text{ess sup}_{(t, s) \in I} \| U(t, s) - V_n(t, s) \|_{L^p(\mathcal{X})}, \quad n \in \mathbb{N}.
\]

**Proof.** Let \( \{L(t)\}_{t \geq 0} \) be the left-shift semigroup on the Banach space \( L^p(I, \mathcal{X}): \)

\[
(L(t)f)(t) = \chi_I(t + \tau)f(t + \tau), \quad f \in L^p(I, \mathcal{X}).
\]

Using that we get

\[
\left( L(t) \left( e^{-\tau \mathcal{K}} - \left( e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n \right) \right)(t)
= \| U(t + \tau, t) - V_n(t + \tau, t) \|_{L^p(\mathcal{X})} \chi_I(t + \tau)f(t),
\]

for \( \tau \geq 0 \) and a.a. \( t \in I. \)

It turns out that for each \( n \in \mathbb{N} \) the operator \( L(t) \left( e^{-\tau \mathcal{K}} - \left( e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n \right) \) is a multiplication operator induced by \( \{(U(t + \tau, t) - V_n(t + \tau, t)) \chi_I(t + \tau)\}_{t \in I}. \) As a consequence,

\[
\left\| L(t) \left( e^{-\tau \mathcal{K}} - \left( e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n \right) \right\|_{L^p(I, \mathcal{X})}
= \text{ess sup}_{t \in I} \| U(t + \tau, t) - V_n(t + \tau, t) \|_{L^p(\mathcal{X})} \chi_I(t + \tau),
\]

for each \( \tau \geq 0. \) Note that one has

\[
\sup_{\tau \geq 0} \left\| L(t) \left( e^{-\tau \mathcal{K}} - \left( e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n \right) \right\|_{L^p(I, \mathcal{X})}
= \text{ess sup}_{\tau \geq 0} \left\| L(t) \left( e^{-\tau \mathcal{K}} - \left( e^{-\tau \mathcal{K}_0/n} e^{-\tau \mathbf{B}/n} \right)^n \right) \right\|_{L^p(I, \mathcal{X})}.
\]
This is based on the fact that if $F(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathbb{X})$ is strongly continuous, then
\[
\sup_{\tau \geq 0} \|F(\tau)\|_{\mathcal{L}(\mathbb{X})} = \text{ess sup}_{\tau \geq 0} \|F(\tau)\|_{\mathcal{L}(\mathbb{X})}.
\]
Hence, we find
\[
\begin{align*}
\sup_{\tau \geq 0} \left\| L(\tau) \left( e^{-\tau K} - \left( e^{-\tau K_0/n} e^{-\tau B/n} \right)^n \right) \right\|_{\mathcal{L}(\mathcal{H})} \\
= \text{ess sup} \sup_{\tau \geq 0} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{L}(\mathbb{X})} \chi_{\widetilde{T}}(t + \tau) = 0.
\end{align*}
\]
Further, if $\Phi(\cdot, \cdot) : \mathbb{R}_+ \times I \to \mathcal{L}(\mathbb{X})$ is a strongly measurable function, then
\[
\text{ess sup} \|\Phi(\tau, t)\|_{\mathcal{L}(\mathbb{X})} = \text{ess sup} \|\Phi(\tau, t)\|_{\mathcal{L}(\mathbb{X})}.
\]
Then, taking into account two last equalities, one obtains
\[
\begin{align*}
\sup_{\tau \geq 0} \left\| L(\tau) \left( e^{-\tau K} - \left( e^{-\tau K_0/n} e^{-\tau B/n} \right)^n \right) \right\|_{\mathcal{L}(\mathbb{X})} \\
= \text{ess sup} \sup_{\tau \geq 0} \|U(t + \tau, t) - V_n(t + \tau, t)\|_{\mathcal{L}(\mathbb{X})} \chi_{\widetilde{T}}(t + \tau) \\
= \text{ess sup} \|U(t, s) - V_n(t, s)\|_{\mathcal{L}(\mathbb{X})} ,
\end{align*}
\]
that proves (3.6). \qed

We study bounded perturbations of the evolution generator $\mathcal{K}_0 = D_0$ (3.2). To this aim we consider $I = [0, 1], \mathbb{X} = \mathbb{C}$ and we denote by $L^p(I)$ the Banach space $L^p(I, \mathbb{C})$.

For $t \in I$, let $q : t \mapsto q(t) \in L^\infty(I)$. Then, $q$ induces a bounded multiplication operator $\mathcal{B} = Q$ on the Banach space $L^p(I)$:
\[
(Qf)(t) = q(t)f(t), \quad f \in L^p(I).
\]
For simplicity we assume that $q \geq 0$. Then $Q$ generates on $L^p(I)$ a contraction semigroup $\{e^{-\tau Q}\}_{\tau \geq 0}$. Since generator $Q$ is bounded, the closed operator $\mathcal{A} := D_0 + Q$, with domain $\text{dom}(\mathcal{A}) = \text{dom}(D_0)$, is generator of a $C_0$-semigroup on $L^p(I)$. By the Trotter product formula in the strong operator topology it follows immediately that
\[
\lim_{n \to \infty} \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n f = e^{-\tau (D_0 + Q)} f, \quad f \in L^p(I),
\]
uniformly in $\tau \in [0, T]$ on bounded time intervals.

Then we define on Banach space $\mathbb{X} = \mathbb{C}$ a family of bounded operators $\{V(t)\}_{t \in I}$ by
\[
V(t) := e^{-\int_0^t ds q(s), \quad t \in I}.
\]
Note that for almost every $t \in I$ these operators are positive. Then $V^{-1}(t)$ exists and it has the form
\[
V^{-1}(t) = e^{\int_0^t ds q(s), \quad t \in I}.
\]
The operator families \( \{V(t)\}_{t \in I} \) and \( \{V^{-1}(t)\}_{t \in I} \) induce two bounded multiplication operators \( V \) and \( V^{-1} \) on \( L^p(I) \), respectively. Then invertibility implies that
\[ V V^{-1} = V^{-1} V = 1 \mid_{L^p(I)}. \]
Using the operator \( V \) one easily verifies that \( D_0 + Q \) is similar to \( D_0 \), that is, one has
\[ V^{-1}(D_0 + Q) V = D_0 \quad \text{or} \quad D_0 + Q = V D_0 V^{-1}. \]

Hence, the semigroup generated on \( L^p(I) \) by \( D_0 + Q \) gets the explicit form:
\[
\left( e^{-\tau(D_0+Q)} \right) (t) = \left( V e^{-\tau D_0} V^{-1} f \right) (t) = e^{-\int_{t}^{\tau} dy q(y)} f(t - \tau) \chi(t - \tau). \tag{3.8}
\]

Since by (3.1) the solution operator \( U(t, s) \) that corresponds to evolution semigroup (3.8) is defined by equation
\[
\left( e^{-\tau D_0} \right) f(t) = U(t, t - \tau) f(t - \tau) \chi(t - \tau),
\]
we deduce that it is equal to \( U(t, s) = e^{-\int_{t}^{s} dy q(y)} \).

Now we study the corresponding Trotter product formula. For a fixed \( \tau \geq 0 \) and \( n \in \mathbb{N} \), we define approximates \( \{V_n\}_{n \geq 1} \) by
\[
\left( \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right) (t) =: V_n(t, t - \tau) \chi(t - \tau) f(t - \tau).
\]

Then by straightforward calculations, which are similar to (3.5), one finds that approximants have the following explicit form:
\[
V_n(t, s) = e^{-\tau_n \sum_{k=0}^{n-1} q(s + k \tau_n)}, \quad (t, s) \in A, \quad \tau_n = (t - s)/n, \quad n \in \mathbb{N}.
\]

**Theorem 3.3.** Let \( q \in L^\infty(I) \) be non-negative. Then
\[
\sup_{\tau \geq 0} \left\| e^{-\tau D_0 + Q} - \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L^p(I)} = \Theta \left( \text{ess sup}_{(t, s) \in A} \int_{s}^{t} dy q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right), \quad n \in \mathbb{N},
\]
as \( n \to \infty \), where \( \Theta \) is the Landau symbol defined in Remark 3.1 see Subsection 3.1

**Proof.** First, by Theorem 3.2 and by \( U(t, s) = e^{-\int_{t}^{s} dy q(y)} \) we obtain
\[
\sup_{\tau \geq 0} \left\| e^{-\tau (D_0 + Q)} - \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L^p(I)} = \text{ess sup}_{(t, s) \in A} \left| e^{-\int_{t}^{s} dy q(y)} - e^{-\tau_n \sum_{k=0}^{n-1} q(s + k \tau_n)} \right|. \tag{3.9}
\]
Then, using the inequality
e^{-\max(x,y)} |x - y| \leq |e^{-x} - e^{-y}| \leq |x - y|, \quad 0 \leq x, y,
for $0 \leq s < t \leq 1$ one finds the estimates
\[ e^{\leq \|q\|_{L^\infty}} R_n(t, s; q) \leq \left| e^{-\int_s^t dy q(y)} - e^{-\tau_n \sum_{k=0}^{n-1} q(s+k \tau_n)} \right| \leq R_n(t, s; q), \]
where
\[ R_n(t, s, q) := \left| \int_s^t dy q(y) - \tau_n \sum_{k=0}^{n-1} q(s+k \tau_n) \right|, \quad (t, s) \in \Delta, \quad n \in \mathbb{N}. \quad (3.10) \]
Hence, for the left-hand side of (3.9) we get the estimate
\[ e^{\leq \|q\|_{L^\infty}} R_n(q) \leq \sup_{\tau \geq 0} \left\| e^{-\tau (D_0+Q)} - \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L^p(I)} \leq R_n(q), \]
where $R_n(q) := \text{ess sup}_{(t, s) \in \Delta} R_n(t, s; q)$, $n \in \mathbb{N}$. These estimates together with definition of $\Theta$ prove the assertion. \(\square\)

Note that by virtue of (3.10) and Theorem 3.3 the operator-norm convergence rate of the Trotter product formula for the pair $\{D_0, Q\}$ coincides with the convergence rate of the integral Darboux-Riemann sum approximation of the Lebesgue integral.

### 3.3 Rate of convergence

First we consider the case of a real Hölder-continuous function $q \in C^{0,\beta}(I)$.

**Theorem 3.4.** If $q \in C^{0,\beta}(I)$ is non-negative, then
\[ \sup_{\tau \geq 0} \left\| e^{-\tau (D_0+Q)} - \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L^p(I)} = O(1/n^\beta), \]
as $n \to \infty$.

**Proof.** One gets for $\tau_n = (t - s)/n$ and $n \in \mathbb{N}$:
\[
\int_s^t dy q(y) - \tau_n \sum_{k=0}^{n-1} q(s+k \tau_n) \\
= \sum_{k=0}^{n-1} \int_{k \tau_n}^{(k+1) \tau_n} dy \left( q(s+y) - q(s+k \tau_n) \right),
\]
which yields the estimate
\[ \left| \int_s^t dy \ q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right| \]
\[ \leq \sum_{k=0}^{n-1} \int_{k \tau_n}^{(k+1) \tau_n} dy \ |q(s + y) - q(s + k \tau_n)| . \]

Since \( q \in C^{0,\beta}(I) \), there is a constant \( L_\beta > 0 \) such that for \( y \in [k \tau_n, (k+1) \tau_n] \) one has
\[ |q(s + y) - q(s + k \tau_n)| \leq L_\beta |y - k \tau_n|^{\beta} \leq L_\beta \frac{(t - s)^\beta}{n^\beta} . \]

Hence, we find
\[ \left| \int_s^t dy \ q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right| \leq L_\beta \frac{(t - s)^{1+\beta}}{n^{\beta}} \leq L_\beta \frac{1}{n^{\beta}} , \]
which proves (cf. Remark 3.1)
\[ \text{ess sup}_{(t,s) \in \Delta} \left| \int_s^t dy \ q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right| = O \left( \frac{1}{n^{\beta}} \right) . \]

Applying now Theorem 3.3 one completes the proof. □

The next natural question is: what happens, when function \( q \) is only continuous?

**Theorem 3.5.** If \( q : I \to \mathbb{C} \) is continuous and non-negative, then
\[ \left\| e^{-\tau_n D_0 + Q} - \left( e^{-\tau_n D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L(U(I))} = o(1) , \quad (3.11) \]
as \( n \to \infty \).

**Proof.** Seeing that \( q \) is continuous, for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for \( y, x \in I \) and \( |y - x| < \delta \) we have \( |q(y) - q(x)| < \varepsilon \). Therefore, if \( 1/n < \delta \), then for \( y \in (k \tau_n, (k+1) \tau_n) \) we have
\[ |q(s + y) - q(s + k \tau_n)| < \varepsilon, \quad (t, s) \in \Delta . \]

Hence,
\[ \left| \int_s^t dy \ q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right| \leq \varepsilon(t - s) \leq \varepsilon , \]
which yields (cf. Remark 3.1)
\[ \text{ess sup}_{(t,s) \in \Delta} \left| \int_s^t dy \ q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right| = o(1) . \]

Now it remains only to apply Theorem 3.3. □
Here it is worth to note that for general continuous function $q$ one can say nothing about the convergence rate. Indeed, it can be shown that in (3.11) the convergence to zero can be arbitrary slow (3.12) for a bounded perturbation $Q$. This is drastically different to the case, when dominating generator corresponds to a holomorphic semigroup and perturbation operator is bounded, cf. (2.17), (2.19) for $\alpha = 0$, or to the case of unbounded perturbation, when $0 < \alpha < 1$, see (2.16), (2.18).

**Theorem 3.6.** Let $\delta_n > 0$ be a sequence with $\delta_n \to 0$ as $n \to \infty$. Then there exists a continuous function $q : I = [0, 1] \to \mathbb{R}$, such that

$$\sup_{\tau \geq 0} \left\| e^{-\tau D_0 + \tau Q} - \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L(L^p(I))} = \omega(\delta_n),$$

(3.12)
as $n \to \infty$, where $\omega$ is the Landau symbol defined in Remark 3.1, see Subsection 3.1.

**Proof.** Taking into account the Walsh-Sewell theorem ([WaSe37], Theorem 6), we find that for any sequence $\{\delta_n\}_{n \in \mathbb{N}}$, $\delta_n > 0$ satisfying $\lim_{n \to \infty} \delta_n = 0$ there exists a continuous function $f : [0, 2\pi] \to \mathbb{R}$, such that

$$\left| \int_0^{2\pi} dx f(x) - \frac{2\pi}{n} \sum_{k=1}^n f(2k\pi/n) \right| = \omega(\delta_n),$$
as $n \to \infty$. Setting $q(y) := f(2\pi(1 - y))$ for $y \in [0, 1]$, we get a continuous function $q : [0, 1] \to \mathbb{R}$, such that

$$\left| \int_0^1 dy q(y) - \frac{1}{n} \sum_{k=0}^{n-1} q(k/n) \right| = \omega(\delta_n).$$

Given that function $q$ is continuous, we find

$$\text{ess sup}_{(t,s) \in \Delta} \left( \int_s^t dy q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right) \geq \left| \int_0^1 dy q(y) - \frac{1}{n} \sum_{k=0}^{n-1} q(k/n) \right|,$$

which yields

$$\text{ess sup}_{(t,s) \in \Delta} \left( \int_s^t dy q(y) - \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n) \right) = \omega(\delta_n).$$

Applying now Theorem 3.3 we prove (3.12). \qed

Our final comment concerns the case, when function $q : [0, 1] \to \mathbb{R}$ is only measurable. Then it can happen that the Trotter product formula for that pair $(D_0, Q)$ does not converge in the operator-norm topology.
Theorem 3.7. There is a non-negative function \( q \in L^\infty([0, 1]) \) such that

\[
\limsup_{n \to \infty} \sup_{\tau \geq 0} \left\| e^{-\tau D_0 + Q} - \left( e^{-\tau D_0/n} e^{-\tau Q/n} \right)^n \right\|_{L^p(I)} > 0 .
\] (3.13)

Proof. Let us introduce open intervals:

\[
\Delta_{0,n} := (0, \frac{1}{2^{2n-2}}),
\]

\[
\Delta_{k,n} := (t_{k,n} - \frac{1}{2^{2n-2}}, t_{k,n} + \frac{1}{2^{2n-2}}), \quad k = 1, 2, \ldots, 2^n - 1,
\]

\[
\Delta_{2^n,n} := (1 - \frac{1}{2^{2n-2}}, 1),
\]

\[n \in \mathbb{N},\]

where \( t_{k,n} = \frac{k}{2^n}, k = 0, \ldots, n, n \in \mathbb{N}.\)

Notice that \( t_{0,n} = 0 \) and \( t_{2^n,n} = 1.\) One easily checks that the intervals \( \Delta_{k,n}, k = 0, \ldots, 2^n, \) are mutually disjoint. We introduce the open sets

\[
O_n = \bigcup_{k=0}^{2^n} \Delta_{k,n} \subseteq I, \quad n \in \mathbb{N}.
\]

and

\[
O = \bigcup_{n \in \mathbb{N}} O_n \subseteq I.
\]

Then it is clear that

\[
|O_n| = \frac{1}{2^{n+1}}, \quad n \in \mathbb{N}, \quad \text{and} \quad |O| \leq \frac{1}{2}.
\]

Therefore, the Lebesgue measure of the closed set \( C := I \setminus O \subseteq I \) can be estimated by

\[
|C| \geq \frac{1}{2}.
\]

Using the characteristic function \( \chi_C(\cdot) \) of the set \( C \) we define

\[
q(t) := \chi_C(t), \quad t \in I.
\]

The function \( q \) is measurable and it satisfies \( 0 \leq q(t) \leq 1, t \in I.\)

Let \( \varepsilon \in (0, 1). \) We choose \( s \in (0, \varepsilon) \) and \( t \in (1 - \varepsilon, 1) \) and we set

\[
\xi_{k,n}(t, s) := s + k \frac{t - s}{2^n}, \quad k = 0, \ldots, 2^n - 1, \quad n \in \mathbb{N}, \quad (t, s) \in \Delta.
\]

Note that \( \xi_{k,n}(t, s) \in (0, 1), k = 0, \ldots, 2^n - 1, n \in \mathbb{N}.\) Moreover, we have

\[
t_{k,n} - \xi_{k,n}(t, s) = k \frac{1}{2^n} - s - k \frac{t - s}{2^n} = k \frac{1 - t + s}{2^n} - s,
\]

which leads to the estimate
\[ |t_{k,n} - \xi_{k,n}(t, s)| \leq \varepsilon \left( 1 + k/2^{n-1} \right), \quad k = 0, \ldots, 2^n - 1, \quad n \in \mathbb{N}. \]

Hence
\[ |t_{k,n} - \xi_{k,n}(t, s)| \leq 3\varepsilon, \quad k = 0, \ldots, 2^n - 1, \quad n \in \mathbb{N}. \]

Let \( \varepsilon_n := 1/(3 \cdot 2^{n+2}) \) for \( n \in \mathbb{N} \). Then we get that \( \xi_{k,n}(t, s) \in \mathcal{A}_{k,n} \) for \( k = 0, \ldots, 2^n - 1, n \in \mathbb{N}, s \in (0, \varepsilon_n) \) and for \( t \in (1 - \varepsilon_n, 1) \).

Now let
\[ S_n(t, s; q) := \tau_n \sum_{k=0}^{n-1} q(s + k \tau_n), \quad (t, s) \in \mathcal{A}, \quad \tau_n = (t-s)/n, \quad n \in \mathbb{N}. \]

We consider
\[ S_{2^n}(t, s; q) = \frac{t-s}{2^n} \sum_{k=0}^{2^n-1} q(s + k 2^{-n}) = \frac{t-s}{2^n} \sum_{k=0}^{2^n-1} q(\xi_{k,n}(t, s)), \]

\( n \in \mathbb{N}, (t, s) \in \mathcal{A} \). If \( s \in (0, \varepsilon_n) \) and \( t \in (1 - \varepsilon_n, 1) \), then \( S_{2^n}(t, s; q) = 0, n \in \mathbb{N} \) and
\[ \left| \int_s^t dy q(y) - S_{2^n}(t, s; q) \right| = \int_s^t dy q(y), \quad n \in \mathbb{N}, \]

for \( s \in (0, \varepsilon_n) \) and \( t \in (1 - \varepsilon_n, 1) \). In particular, this yields
\[ \text{ess sup}_{(t,s) \in \mathcal{A}} \left| \int_s^t dy q(y) - S_{2^n}(t, s; q) \right| \geq \text{ess sup}_{(t,s) \in \mathcal{A}} \int_s^t dy q(y) \geq \int_I dy \chi_C(y) \geq \frac{1}{2}. \]

Hence, we obtain
\[ \limsup_{n \to \infty} \text{ess sup}_{(t,s) \in \mathcal{A}} \left| \int_s^t dy q(y) - S_{2^n}(t, s; q) \right| \geq \frac{1}{2}, \]

and applying Theorem 3.3 we finish the prove of (3.13).

**Remark 3.8.** We note that Theorem 3.7 does not exclude the convergence of the Trotter product formula for the pair \( \{D_0, Q\} \) in the strong operator topology. We would remind that the same kind of dichotomy is known for the Trotter product formula on a Hilbert space, see the Hiroshi Tamura example in [Tam00], Theorem B. By virtue of (3.7) and (3.13), Theorem 3.7 yields an example of this dichotomy on a Banach space.

Again, there is a drastic difference between the origin of these conclusions in a Hilbert space ([Tam00], Theorem B) for **bounded** perturbation of the holomorphic semigroup and in a Banach space (Theorem 3.7) for **bounded** perturbation of a (non-holomorphic) **contractive** semigroup.
4 Notes

Notes to Section 1. Characterisation of holomorphic contraction semigroups at the end of Subsection 1.2 (iv), is due to Corollary II.4.9 [EN00]. For the proof of Lemma 1.11 see, for example [Tan75], Lemma 2.3.5.

Notes to Section 2. Here we extend to the operator-norm convergence of the product formula on a Banach space for perturbation B with a relative zero $A$-bound for holomorphic semigroup $\{e^{-tA}\}_{t \geq 0}$ some of the Trotter-Chernoﬀ results, cf. [Trot59], [But20], [Zag20]. This shows that hypothesis of self-adjointness in the case of a Hilbert space [IT97] has only a technical importance.

On the other hand the operator-norm topology is "natural" for holomorphic $C_0$-semigroups, which may lead one to think that it is an essential hypothesis for the operator-norm convergence of the Trotter product formula. In Section 3 we showed that this hypothesis is also technical, but we have to assume contraction of semigroup $\{e^{-tA}\}_{t \geq 0}$. We would like to remark that demand of contraction is not as super­fluous as one could suppose. For demonstration we address the reader to instructive example by Trotter [Trot59], where it is called "the norm condition".

This section contains a revision of result [CZ01], Section 3, where the operator-norm convergence of the Trotter product formula on a Banach space $X$ has been proven (up to our knowledge) for the first time. For a survey of similar results in this direction see [NSZ18b].

Notes to Section 3. In contrast to Section 2, where operator-norm convergence holds true if the dominating operator $A \in \mathcal{H}(\theta, 0)$ generates a holomorphic contraction semigroup and operator $B$ is a $A$-infinitesimally small generator of a contraction semigroup (in particular, if $B$ is a bounded operator), we present Example that this is also possible if condition on generator $A$ is relaxed. The conditions are [NSZ18a]:

1. Operator $A = K_0$ generates a contractive (not holomorphic!) semigroup.
2. $B = B$ is a bounded operator.

There it is also demonstrated that the operator-norm convergence generally fails (even for bounded operators $B$) if unbounded $K_0$ is not a holomorphic generator and that operator-norm convergence of the Trotter product formula can be arbitrary slow, cf. Subsection 3.3. This is again very different to the holomorphic case: $A \in \mathcal{H}(\theta, 0)$ (cf. Subsection 2.2), where the rate of the operator-norm convergence is of the order $O((\ln n)\alpha/n)$ for any bounded perturbation $B (\alpha = 0)$, see Theorems 2.6, 2.7 and Corollary 2.8.
References

But20. Y. A. Butko, The method of Chernoff approximation, pp 19–46. In: J. Banasiak et al. (eds.), Semigroups of Operators – Theory and Applications, SOTA 2018, Springer Proceedings in Mathematics and Statistics, vol. 325. Springer, Berlin 2020.

CZ01. V. Cachia and V. A. Zagrebnov, Operator-norm convergence of the Trotter product formula for holomorphic semigroups, J. Oper. Theory 46 (2001), 199–213.

EN00. K. J. Engel and R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Springer-Verlag, Berlin, 2000.

Ev76. D. E. Evans. Time dependent perturbations and scattering of strongly continuous groups on Banach spaces. Math. Ann., 221(3):275–290, 1976.

How74. J. S. Howland. Stationary scattering theory for time-dependent Hamiltonians, Math. Ann. 207 (1974) 315–335.

IT97. T. Ichinose and Hideo Tamura, Error estimate in operator norm for Trotter-Kato product formula, Integral Equations Oper. Theory 27 (1997), 195–207.

Kato95. T. Kato. Perturbation Theory for Linear Operators. (Corrected Printing of the Second Edition 1980) Springer-Verlag, Berlin, Heidelberg, 1995.

Nei79. H. Neidhardt. Integration of Evolutionsgleichungen mit Hilfe von Evolutionshalbgruppen. Dissertation, AdW der DDR, Berlin, 1979.

Nei81. H. Neidhardt, On abstract linear evolution equations. I. Math. Nachr., 103:283–298, 1981.

NZ98. H. Neidhardt and V. A. Zagrebnov, On error estimates for the Trotter-Kato product formula, Lett. Math. Phys. 44 (1998), 169–186.

NZ99. H. Neidhardt and V. A. Zagrebnov, Fractional powers of self-adjoint operators and Trotter-Kato product formula, Integral Equations Oper. Theory 35 (1999), 209–231.

NSZ18a. H. Neidhardt, A. Stephan and V. A. Zagrebnov, Remarks on the operator-norm convergence of the Trotter product formula, Integral Equations Oper. Theory 90:15 (2018), pp.1–14.

NSZ18b. H. Neidhardt, A. Stephan and V. A. Zagrebnov, Operator-norm convergence of the Trotter product formula on Hilbert and Banach spaces: a short survey, in: Current Research in Nonlinear Analysis. Springer Optimization and Its Applications vol. 135, Springer, Berlin 2018, pp. 229–247.

NSZ20. H. Neidhardt, A. Stephan, and V. A. Zagrebnov. Convergence rate estimates for Trotter product approximations of solution operators for non-autonomous Cauchy problems, RIMS Kyoto Univ. 56 (2020), 83–135.

Rog93. Dzh. L. Rogava, Error bounds for Trotter-type formulas for self-adjoint operators, Funct. Anal. Appl. 27 (1993), 217–219.

Tam00. Hiroshi Tamura, A remark on operator-norm convergence of Trotter-Kato product formula, Integral Equations Oper. Theory 37 (2000), 350–356.

Tan75. H. Tanabe. Equations of evolution, Tokyo, Iwanami, 1975 (in Japanese). English translation: Pitman Advanced Publishing Program, London, 1979.

Trot59. H. F. Trotter, On the products of semigroups of operators, Proc. Amer. Math. Soc. 10 (1959), 545–551.

WaSe37. J. L. Walsh and W. E. Sewell, Note on degree of approximation to an integral by Riemann sums, The American Mathematical Monthly 44 (1937), 155–160.

Yos80. K. Yosida, Functional Analysis, Springer-Verlag, Berlin, 1980.

Zag20. V. A. Zagrebnov, Notes on the Chernoff product formula, J. Funct. Anal. 279 (2020), 108696–24pp.