Tracial state space with non-compact extreme boundary

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Abstract. Let $A$ be a unital simple separable C*-algebra. If $A$ is nuclear and infinite-dimensional, it is known that strict comparison is equivalent to $\mathcal{Z}$-stability if the extreme boundary of its tracial state space is non-empty, compact and of finite covering dimension. Here we will provide the first proof of this result on the case of certain non-compact extreme boundaries. Besides, if $A$ has strict comparison of positive elements, it is known that the Cuntz semigroup of this C*-algebra is recovered functorially from the Murray-von Neumann semigroup and the tracial state space whenever the extreme boundary of the tracial state space is compact and of finite covering dimension. We will extend this result to the case of a zero-dimensional extreme boundary with finitely many cluster points.

1. Introduction

The set of traces of a C*-algebra is a very important invariant of the algebra. For a C*-algebra $A$, let $T(A)$ be its tracial state space (i.e. the set of normalized finite traces of $A$) and $\partial_e T(A)$ the extreme boundary of $T(A)$. It is known that $T(A)$ is a Choquet simplex if $A$ is unital ([18]). If $A$ is separable, then $T(A)$ is metrizable.

In this paper, we will only consider unital simple separable C*-algebras. The tracial state space of such C*-algebras can still be very complicated, such as in the Poulsen simplex, in which the extreme points are dense([16]). In fact, [2] shows that every metrizable Choquet simplex occurs as the tracial state space of some simple unital AF-algebra.

Several recent results in C*-algebras theory have been obtained under the assumption of a compact extreme boundary of the tracial state space. In 2008 A. Toms and W. Winter made the following conjecture:

Conjecture 1.1. Let $A$ be a simple unital nuclear separable C*-algebra. The following are equivalent:

(1) $A$ has finite nuclear dimension;
(2) $A$ is $\mathcal{Z}$-stable;
(3) $A$ has strict comparison of positive elements.
In 2004, M. Rørdam showed that $\mathcal{Z}$-stability implies strict comparison for unital simple exact C*-algebras ([17]). In 2010, W. Winter proved that finite nuclear dimension implies $\mathcal{Z}$-stability for unital separable simple infinite-dimensional C*-algebras ([23]). A. Toms, S. White, W. Winter ([22]), E. Kirchberg, M. Rørdam ([11]) and Y. Sato ([19]) showed that for a simple nuclear separable unital infinite-dimensional C*-algebra, strict comparison is equivalent to $\mathcal{Z}$-stability if the extreme boundary of its tracial state space is non-empty, compact and of finite covering dimension.

In the second part of this paper, we will prove the following theorem.

**Theorem 1.1.** Let $A$ be a simple nuclear separable unital infinite-dimensional C*-algebra with non-empty tracial state space. Suppose $\partial e T(A) = X$ has the tightness property and has finite covering dimension. The following conditions are equivalent:

1. $A$ is $\mathcal{Z}$-stable;
2. $A$ has strict comparison.

The tightness property in this theorem, to be introduced in the next section, will yield the outcome in the case of certain non-compact extreme boundaries, which has not been reported in previous literature. In particular, we will treat the case of the tracial state space of any recursive subhomogeneous algebra of finite topological dimension.

M. Dadarlat and A. Toms showed in [7] that for a unital simple separable C*-algebra with strict comparison of positive elements $A$, the Cuntz semigroup of $A$ is recovered functorially from the Murray-von Neumann semigroup and the tracial state space $T(A)$ whenever $\partial e T(A)$ is compact and of finite covering dimension. In this paper, we will show that this result still holds if $\partial e T(A)$ is zero dimensional and has only finitely many cluster points.

### 2. Preliminaries

#### 2.1. The Cuntz Semigroup

Let $A$ be a C*-algebra and let $\mathcal{K}$ denote the algebra of compact operators on a separable infinite-dimensional Hilbert space. Let $(A \otimes \mathcal{K})_+$ denote the set of positive elements in $A \otimes \mathcal{K}$. Given $a, b \in (A \otimes \mathcal{K})_+$, we say that $a$ is Cuntz subequivalent to $b$ (denoted $a \preceq b$) if there is a sequence $(x_n)$ in $A \otimes \mathcal{K}$ such that

$$\|x_n b x_n^* - a\| \to 0.$$ 

We say that $a$ and $b$ are Cuntz equivalent (denoted $a \sim b$) if $a \preceq b$ and $b \preceq a$. The relation $\preceq$ is clearly transitive and reflexive and $\sim$ is an equivalence relation.

We define the Cuntz semigroup to be $Cu(A) = (A \otimes \mathcal{K})_+ / \sim$, and write $\langle a \rangle$ for the equivalence class of $a \in (A \otimes \mathcal{K})_+$. $Cu(A)$ is indeed an ordered Abelian semigroup when equipped with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the partial order

$$\langle a \rangle \leq \langle b \rangle \iff a \preceq b.$$
2.2. Rank Functions. We denote by $T(A)$ the tracial state space of $A$. Given $\tau$ in $T(A)$, we define a map $d_\tau : A_+ \to \mathbb{R}^+$ by

$$d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n}).$$

$d_\tau$ extends naturally to $(A \otimes K)_+$ and we always regard this set to be its domain. This map is lower semicontinuous, and depends only on the Cuntz equivalence class of $a$. Moreover, it has the following properties:

1) if $a \prec b$, then $d_\tau(a) \leq d_\tau(b)$;

2) if $a$ and $b$ are mutually orthogonal, then $d_\tau(a + b) = d_\tau(a) + d_\tau(b)$;

3) $d_\tau((a - \varepsilon)_+) \nearrow d_\tau(a)$ as $\varepsilon \to 0$.

We then define the rank function of $a \in (A \otimes K)_+$, a map $\iota(a)$ from the tracial state space $T(A)$ to $\mathbb{R}^+$ given by the formula $\iota(a)(\tau) = d_\tau(a)$. It is easy to verify that these rank functions are lower semicontinuous, affine and nonnegative. If $A$ is simple, then they are strictly positive.

2.3. Strict Comparison and $Z$-stability. Let $A$ be a unital C*-algebra. We say that $A$ has strict comparison of positive elements if $a \prec b$ for $a, b \in (A \otimes K)_+$ whenever

$$d_\tau(a) < d_\tau(b), \forall \tau \in \{\gamma \in T(A)|d_\gamma(b) < \infty\}.$$ 

We say $A$ is $Z$-stable if $A \otimes Z \cong A$ where $Z$ is the Jiang-Su algebra (i.e. a simple unital infinite dimensional nuclear C*-algebra with Elliott invariant isomorphic to that of the complex numbers).

2.4. Choquet Simplices. We already know that the tracial state space of a unital separable C*-algebra is a metrizable Choquet simplex. For a general metrizable Choquet simplex $K$, given any point $x \in K$, there exists a unique Borel probability measure $\mu_x$ defined on the extreme boundary $\partial_x K$ such that

$$f(x) = \int_{\partial_x K} f d\mu_x$$

for any affine continuous function $f$ on $K$ ([1]).

For convenience, we denote $X = \partial_x K$ and $\partial X = \overline{X} \setminus X$. Then from above we obtain a set of Borel probability measures on $X$ for $\partial X$: $\{\mu_x : x \in \partial X\}$.

**Definition 2.1.** Recall that a set $\Gamma$ of Borel probability measures on $X$ is called tight if for every $\varepsilon > 0$ there exists a compact subset $F$ of $X$ such that

$$\mu(F) \geq 1 - \varepsilon$$

for all $\mu \in \Gamma$. We say $X$ has the tightness property if $\{\mu_x : x \in \partial X\}$ is tight.

Obviously, if $X$ has only finitely many cluster points, then it has the tightness property.
2.5. Other Notations. For convenience, we denote $\text{Aff}(K)$ the set of real-valued continuous affine functions on a compact metrizable Choquet simplex $K$, $L\text{Aff}(K)$ the set of bounded strictly positive, lower-semicontinuous affine functions on $K$, and $S\text{Aff}(K)$ the set of extended real-valued functions which can be obtained as the pointwise supremum of an increasing sequence from $L\text{Aff}(K)$.

For positive elements $a, b \in A$ we write that $a \approx b$ if there is $x \in A$ such that $x^*x = a$ and $xx^* = b$. The relation $\approx$ is an equivalence relation, and it is known that $a \approx b$ implies $a \sim b$.

3. Some Useful Results

The first part of our paper is based on the work of [7] and will be using some of its lemmas. We will first restate these lemmas as follows for the convenience of the reader.

For each $\eta > 0$ we define a continuous map $f_\eta : \mathbb{R}_+ \rightarrow [0, 1]$ by the following formula:

$$f_\eta(t) = \begin{cases} t/\eta, & 0 < t < \eta \\ 1, & t \geq \eta. \end{cases}$$

**Lemma 3.1.** (See [7, Lemma 3.1].) Let $A$ be a unital C*-algebra, with $T(A) \neq \emptyset$ and let $a \in M_k(A)$ be positive. Suppose that there are $0 < \alpha < \beta$ such that $\alpha < d_\tau(a) < \beta$ for every $\tau$ in a closed subset $Y$ of $T(A)$. Then there exists $\varepsilon > 0$ and an open neighborhood $U$ of $Y$ of $T(A)$, with the property that $\alpha < d_\tau((a - \varepsilon)_+) < \beta, \forall \tau \in U$.

**Lemma 3.2.** (See [7, Lemma 3.2].) Let $A$ be a unital separable C*-algebra with nonempty tracial state space, and let $Y \subset T(A)$ be closed. Suppose that $a \in M_k(A)$ is a positive element with the property that $\beta - \alpha < d_\tau(a) \leq \beta, \forall \tau \in Y$ for some $0 < \alpha < \beta$. Then there exists $\eta > 0$ such that $k - \beta \leq d_\tau(1_k - f_\eta(a)) < k - \beta + 2\alpha, \forall \tau \in Y$.

**Lemma 3.3.** (See [7, Lemma 4.1].) Let $A$ be a unital simple separable infinite-dimensional C*-algebra and $\tau$ a normalized trace on $A$. Let $0 < s < r$ be given. It follows that there are an open neighborhood $U$ of $\tau$ in $T(A)$ and a positive element $a$ in some $M_k(A)$ such that $s < d_\gamma(a) < r, \forall \gamma \in U$.

**Lemma 3.4.** (See [7, Lemma 4.2].) Let $A$ be a unital C*-algebra and $\tau$ a normalized trace on $A$. Let $x, y$ be positive elements in $M_k(A)$. Then $d_\tau(y^{1/2}xy^{1/2}) \geq d_\tau(x) - d_\tau(1_k - y)$, where $1_k$ denotes the unit of $M_k(A)$.

The next theorem is from Lin’s paper [12] based on work of Cuntz and Pedersen:
Theorem 3.1. (See [7, Theorem 4.3].) Let $A$ be a unital simple C*-algebra with nonempty tracial state space, and let $\tau$ be a strictly positive affine continuous function on $T(A)$. It follows that for any $\varepsilon > 0$ there is a positive element $a$ of $A$ such that $f(\tau) = \tau(a), \forall \tau \in T(A)$, and $\|a\| < \|f\| + \varepsilon$.

4. Rank Functions

If $a \in A$ is a positive element and $\tau \in T(A)$ we denote by $\nu_\tau$ the measure induced on the spectrum $\sigma(a)$ of $a$ by $\tau$. Then $d_\tau(a) = \nu_\tau((0, \infty) \cap \sigma(a))$ and more generally

$$d_\tau(f(a)) = \nu_\tau(\{t \in \sigma(a) : f(t) > 0\})$$

for all nonnegative continuous functions $f$ defined on $\sigma(a)$. (See [7], under Definition 2.1)

Lemma 4.1. Let $A$ be a unital C*-algebra, with $T(A) \neq \emptyset$ and let $a \in M_k(A)$ be positive. Suppose that $Y$ is a closed subset of $T(A)$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$, and an open neighborhood $U$ of $Y$, such that $\forall \gamma \in U$, $\exists \tau \in Y$, and

$$d_\tau(a) - \delta < d_\tau((a - \varepsilon)_+) < d_\tau(a) + \delta$$

Proof. Since $d_\tau((a - \varepsilon)_+) \nearrow d_\tau(a)$ as $\varepsilon \searrow 0$ for each $\tau$, we can fix $\varepsilon_\tau > 0$, such that $d_\gamma((a - \varepsilon_\tau)_+) > d_\tau(a) - \delta$.

Since $\gamma \mapsto d_\gamma((a - \varepsilon_\tau)_+)$ is lower semi-continuous, we can find an open neighborhood $V_\tau$ of $\tau$, such that

$$d_\gamma((a - \varepsilon_\tau)_+) > d_\tau(a) - \delta, \forall \gamma \in V_\tau.$$

The family $\{V_\tau\}_{\tau \in Y}$ is an open cover of $Y$, and so $Y \subset V_{\tau_1} \cup \ldots \cup V_{\tau_n}$ for some $\tau_1, \ldots, \tau_n \in Y$. Set $\varepsilon := \min\{\varepsilon_{\tau_1}, \ldots, \varepsilon_{\tau_n}\}$ and $V := V_{\tau_1} \cup \ldots \cup V_{\tau_n}$, so that for each $\gamma \in V$, we have $\gamma \in V_{\tau_i}$ for some $i$, and

$$d_\gamma((a - \varepsilon)_+) \geq d_\gamma((a - \varepsilon_{\tau_i})_+) > d_\tau(a) - \delta.$$

Let $\nu_\tau$ be the measure induced on $\sigma(a)$ by $\tau$, we also have

$$d_\tau((a - \varepsilon)_+) = \nu_\tau(\varepsilon, +\infty) \cap \sigma(a)) \leq \nu_\tau(\varepsilon, +\infty) \cap \sigma(a)) \leq \nu_\tau((0, +\infty) \cap \sigma(a)) \leq d_\tau(a), \forall \tau \in T(A).$$

In particular, $d_\tau((a - \varepsilon)_+) \leq \nu_\tau((0, +\infty) \cap \sigma(a)) < d_\tau(a) + \delta$ for all $\tau \in Y$. By the Portmanteau theorem, the map $\gamma \mapsto \nu_\gamma((\varepsilon, +\infty) \cap \sigma(a))$ is upper semi-continuous, and so the set $W_\tau = \{\gamma \in T(A) : \nu_\gamma((\varepsilon, +\infty) \cap \sigma(a)) < d_\tau(a) + \delta\}$ is open and contains $\tau$. Moreover, for any $\gamma \in W_\tau$, we have $d_\gamma((a - \varepsilon)_+) < d_\tau(a) + \delta$.

Set let $U_\tau = V_\tau \cap W_\tau$, $U_\tau$ is an open neighborhood of $\tau$, such that $\forall \gamma \in U_\tau$,

$$d_\tau(a) - \delta < d_\gamma((a - \varepsilon)_+) < d_\tau(a) + \delta.$$

Then $U = \bigcup_{\tau \in Y} U_\tau$ is an open neighborhood of $Y$. For any $\gamma \in U, \gamma \in U_\tau$ for some $\tau$, hence

$$d_\tau(a) - \delta < d_\gamma((a - \varepsilon)_+) < d_\tau(a) + \delta.$$

holds. \qed
The next lemma is a generalization of Lemma 4.4 of [7], which generates the "indicator rank functions".

**Lemma 4.2.** Let $A$ be a separable unital simple infinite-dimensional C*-algebra whose tracial state space $T(A)$ is nonempty and $\partial_e T(A) = X$ is a $F_\sigma$ set. It follows that for any $\delta > 0$, and compact subset $Y \subset X$, there is a nonzero positive element $a$ of $A$ with the property that

$$d_\tau(a) < \delta, \forall \tau \in Y,$$

$$d_\tau(a) = 1, \forall \tau \in X \setminus Y.$$

**Proof.** Let $\delta$ and $Y$ be given. If $X$ is compact, then this result has already been established in [7]. Assume now that $X$ is non-compact, so $Y \neq X$. Fix a decreasing sequence $\{U_n\}_{n=2}^{\infty}$ of open subsets of $X$ with the property that $Y = \bigcap_{n=2}^{\infty} U_n$ and $U_n^c \neq \emptyset$.

Since $X$ is a $F_\sigma$ set, there exists an increasing sequence $\{F_n\}_{n=2}^{\infty}$ of compact subsets of $X$ such that $X = \bigcup_{n=2}^{\infty} F_n$ and $F_n = Y \cup E_n$, where $E_n$ compact and $E_n \subset U_n^c$ for each $n$. Then by Corollary 11.15 in [8], since $F_n$ is compact, we can use Theorem 3.1 to produce a sequence of $(b_n)_{n=2}^{\infty}$ in $A_+$ with the following properties:

$$\tau(b_n) > 1 - 1/n, \forall \tau \in E_n$$

$$\tau(b_n) < \delta/2^n, \forall \tau \in Y$$

$$\|b_n\| \leq 1.$$

For any $\tau \in U_n^c$ we have

$$d_\tau((b_n - 1/n)_+) \geq \tau((b_n - 1/n)_+) \geq \tau(b_n) - 1/n > 1 - 2/n.$$ In particular $d_\tau((b_n - 1/n)_+) \neq 0$. Moreover, for any $\tau \in Y$ we have

$$d_\tau((b_n - 1/n)_+) = n \int (1/n)\chi(1/n, \infty) d\mu_\tau \leq \tau(b_n) < \delta/2^n.$$ Set $c_n := 2^{-n}(b_n - 1/n)_+$, so that $d_\tau(c_n) > 1 - 2/n$ for each $\tau \in U_n^c$, $d_\tau(c_n) < \delta/2^n$ for each $\tau \in Y$ and $\|c_n\| \leq 2^{-n}$. Now set

$$a := \sum_{n=2}^{\infty} c_n \in A_+$$

If $\tau \in Y$, then using the lower semi-continuity of $d_\tau$, we have

$$d_\tau(a) \leq \liminf_{k} d_\tau\left(\sum_{n=2}^{k} c_n\right)$$

$$\leq \liminf_{k} \sum_{n=2}^{k} d_\tau(c_n)$$

$$\leq \delta.$$
If \( \tau \in X \setminus Y \), then \( \tau \in U_k^c \) for all \( k \) sufficiently large. It follows that for these \( k \),

\[
d_\tau(a) = d_\tau\left(\sum_{n=2}^{\infty} c_n\right) \geq d_\tau(c_k) \geq 1 - 2/k
\]

We conclude that \( d_\tau(a) \geq 1 \) for each such \( \tau \). On the other hand, \( a \in A \), so \( d_\tau(a) \leq 1 \) for any \( \tau \in T(A) \). \( \square \)

For convenience, for a given set \( X \), we denote \( \partial X = X \setminus X \).

**Lemma 4.3.** Let \( A \) be a separable unital simple infinite-dimensional C*-algebra whose tracial state space \( T(A) \) is nonempty and \( \partial_e T(A) = X \) is a \( F_\sigma \) set. If there exist a compact subset \( F \subset X \) and some \( \varepsilon > 0 \) such that \( \mu_\tau(F) > \varepsilon, \forall \tau \in \partial X \). Then \( \partial X \cup F \) is a compact subset of \( T(A) \).

**Proof.** Since there exist a compact subset \( F \subset X \) and some \( \varepsilon > 0 \) such that \( \mu_\tau(F) > \varepsilon, \forall \tau \in \partial X \), then by previous lemma, there exists a nonzero positive element \( a \) of \( A \) with the property that

\[
d_\tau(a) < \varepsilon/2, \forall \tau \in F,
\]

\[
d_\tau(a) = 1, \forall \tau \in X \setminus F.
\]

Hence for each \( \tau \in \partial X \),

\[
d_\tau(a) = \int_X d_\gamma(a) d\mu_\tau = \int_F d_\gamma(a) d\mu_\tau + \int_{X \setminus F} d_\gamma(a) d\mu_\tau
\]

\[
\varepsilon/2 \cdot \mu_\tau(F) + \mu_\tau(X \setminus F) \leq 1 - \varepsilon/2
\]

Since \( d_\tau(a) \) is lower semi-continuous, \( T = \{ \tau \in T(A) | d_\tau(a) \leq 1 - \varepsilon/2 \} \) is compact. By the compactness of \( X \), \( T \cap \overline{X} = \partial X \cup F \) is compact. \( \square \)

The next lemma is a generalization of Lemma 4.5 of [7].

**Lemma 4.4.** Let \( A \) be a unital simple separable infinite dimensional C*-algebra. Suppose that \( X = \partial_e T(A) \) is a nonempty \( F_\sigma \) set. Let \( a \in M_N(A) \) be positive, and let there be given compact subset \( Y \) of \( X \) and \( \delta > 0 \). It follows that there is a positive element \( b \) of \( M_N(A) \) with the following properties:

\[
d_\tau(b) = d_\tau(a), \forall \tau \in X \setminus Y
\]

\[
d_\tau(b) \leq \delta, \forall \tau \in Y
\]

**Proof.** Use Lemma 4.2 to find a positive element \( h' \) of \( A \) satisfying

\[
d_\tau(h') < \delta/N, \forall \tau \in Y
\]

\[
d_\tau(h') = 1, \forall \tau \in X \setminus Y.
\]

Let \( h := \oplus_{i=1}^{N} h' \in M_N(A) \). We have

\[
d_\tau(h) < \delta, \forall \tau \in Y
\]

\[
d_\tau(h) = N, \forall \tau \in X \setminus Y.
\]
Let $V_1 \subset V_2 \subset V_3 \subset \ldots$ be a sequence of open subsets of $X$ such that $\bigcap_{i=1}^{\infty} V_i = U$. Trivially,

$$N - 1/2i < d_\tau(h) \leq N, \forall \tau \in \overline{V_i},$$

and so Lemma 3.4 applied for $k = \beta = N$ and $\alpha = 1/2i$ yields $\eta_i > 0$ such that

$$d_\tau(1_N - f_{\eta_i}(h)) < 1/i, \forall \tau \in \overline{V_i}.$$

To simplify notation in the remainder of the proof, relabel $f_{\eta_i}(h)$ as $h_i$. We may assume that the sequence $(\eta_i)$ is decreasing so that the sequence $(h_i)$ is increasing. Since $d_\tau(h) = d_\tau(f_{\eta_i}(h))$ for any $\tau \in T(A)$ and $\eta > 0$, it follows that

$$d_\tau(h_i) < \delta, \forall \tau \in Y$$

$$d_\tau(h_i) = N, \forall \tau \in X \setminus Y.$$

Set $a_i := a^{1/2}h_ia^{1/2}$. Since $a^{1/2}h_ia^{1/2} \approx h_i^{1/2}ah_i^{1/2} \precsim a$, we have

$$d_\tau(a_i) \leq d_\tau(a), \forall \tau \in X \setminus Y.$$

Also, since $a_i = a^{1/2}h_ia^{1/2} \precsim h_i$, we have

$$d_\tau(a_i) \leq d_\tau(h_i) < \delta, \forall \tau \in Y.$$

For our lower bound, we observe that by Lemma 3.4 we have for any $\tau \in \overline{V_i}$:

$$d_\tau(a_i) = d_\tau(h_i^{1/2}ah_i^{1/2}) \geq d_\tau(a) - d_\tau(1_N - h_i) > d_\tau(a) - 1/i.$$

Therefore we have

$$d_\tau(a_i) < \delta, \forall \tau \in Y$$

$$d_\tau(a) - 1/i < d_\tau(a_i) < d_\tau(a), \forall \tau \in \overline{V_i}.$$

Since $h_i \precsim h_{i+1}$ and $a_i := a^{1/2}h_ia^{1/2} \precsim a^{1/2}h_{i+1}a^{1/2} = a_{i+1}$. The increasing sequence $(a_i)_{i=1}^\infty$ has a supremum $b \in M_N(A)$. Since each $d_\tau$ is a supremum preserving state on $Cu(A)$, we conclude that

$$d_\tau(b) < \delta, \forall \tau \in Y$$

$$d_\tau(b) = d_\tau(a), \forall \tau \in X \setminus Y.$$

as desired. \qed

**Lemma 4.5.** Let $A$ be a unital simple separable $C^*$-algebra. Suppose that the extreme boundary $X$ of $T(A)$ is nonempty, zero dimensional and has the tightness property. Then for any compact subset $F \subset X$, and for each $0 \leq r' < r < 1$, there exists a positive element $a$ in some $M_N(A)$ with the property that

$$r' < d_\tau(a) < r, \forall \tau \in F.$$

**Proof.** Set $d := \dim(F)$, and for $i \in \{0, \ldots, 2d + 2\}$ define

$$r_i = r - \frac{(2d + 2 - i)e}{2d + 2}.$$
Fix $\tau \in F$. Use Lemma 3.3 to find, for each $k \in \{0, \ldots, d\}$, positive element $\tilde{b}_k \in M_N(A)$ and an open neighborhood $V_k$ of $\tau$ in $F$ with the property that
\[ r_{2k} < d_\gamma(\tilde{b}_k) < r_{2k+1}, \forall \gamma \in V_k. \]
Set $U_\tau = \bigcap_k V_k$, so that $U := \{U_\tau\}_{\tau \in F}$ is an open cover of $F$. By the finite-dimensionality and compactness of $F$, there are a refinement of a finite subcover of $U$, say $W = \{W_1, \ldots, W_n\}$, and a map $c : W \to \{0, 1, \ldots, d\}$ with the property that if $i \neq j$ then
\[ c(W_i) = c(W_j) \Rightarrow W_i \cap W_j = \emptyset. \]
Each $W_i$ is contained in some $U_\tau$, and so we have the positive elements $\tilde{b}_k, k \in \{0, 1, \ldots, d\}$ such that
\[ r_{2k} < d_\gamma(\tilde{b}_k) < r_{2k+1}, \forall \gamma \in W_i. \]
Set $\eta = \varepsilon/(n(2d+2))$, and use Lemma 4.4 to produce positive elements $b_k(\gamma)$ in $M_N(A)$ with the following properties:
\[ r_{2k} < d_\gamma(b_k(\gamma)) < r_{2k+1}, \forall \gamma \in W_i \]
\[ d_\gamma < \eta, \forall \gamma \in F \setminus W_i. \]
Now for each $k \in \{0, 1, \ldots, d\}$ define
\[ b_k = \bigoplus_{\{i|c(W_i) = k\}} b_k(i) \in A \otimes K. \]
The $W_i$s appearing in the sum above are mutually disjoint. Suppose that $\tau \in W_s$ and $c(W_s) = k$. We have the following bounds:
\[ d_\tau(b_k) = \sum_{\{i|c(W_i) = k\}} d_\tau(b_k(i)) = d_\tau(b_k(s)) + \sum_{\{i|c(W_i) = k, i \neq s\}} d_\tau(b_k(i)) < r_{2k+1} + n\eta \]
\[ = r_{2k+1} + \varepsilon/(2d+2) = r_{2k+2} \]
and
\[ d_\tau(b_k) > d_\tau(b_k(s)) > r_{2k}. \]
For each $k \in \{0, 1, \ldots, d\}$ we define
\[ W_k = \bigcup_{\{i|c(W_i) = k\}} W_i, \]
so that
\[ r_{2k} < d_\gamma(b_k) < r_{2k+2}, \forall \gamma \in W_k. \]
Note that $W_0, W_1, \ldots, W_d$ is a cover of $F$. To complete the proof of the theorem we proceed by induction. First observe that
since $F$ is compact and metrizable, we may find a closed subset $K_0$ of $W_0$ with the property that $K_0, W_1, ..., W_d$ is a cover of $F$. Set $c_0 = b_0$, so that
\[ r - \varepsilon = r_0 < d_\tau(c_0) < r_2, \forall \tau \in K_0. \]

Now suppose that we have found a closed set $K_k \subset W_0 \cup \ldots \cup W_k, k < d$, such that $K_k^c, W_{k+1}, ..., W_d$ covers $F$, and a positive element $c_k$ in some $M_N(A)$ with the property that
\[ r - \varepsilon = r_0 < d_\tau(c_k) < r_{2k+2}, \forall \tau \in K_k. \]

Since $F$ is compact and metrizable, we can find a closed set $K_{k+1} \subset K_k^c \cup W_{k+1}$ such that $K_{k+1}^c, W_{k+2}, ..., W_d$ covers $F$. Applying Lemma 4.6 to $c_k$ and $b_{k+1}$ we obtain a positive element $c_{k+1}$ in some $M_N(A)$ with the property that
\[ r - \varepsilon = r_0 < d_\tau(c_{k+1}) < r_{2k+4}, \forall \tau \in K_{k+1}. \]

Starting with the base case $k = 0$, applying the inductive step above $n$ times, and noting that we must have $K_d = F$, we arrive at a positive element $c_n$ in some $M_N(A)$ with the property that
\[ r - \varepsilon = r_0 < d_\tau(c_n) < r_{2d+2}, \forall \tau \in F. \]

Setting $a = c_n$ complete the proof. \hfill \Box

**Lemma 4.6.** Let $A$ be a separable unital simple infinite-dimensional C*-algebra. Suppose that the extreme boundary $X$ of its tracial state space $T(A)$ is nonempty, zero dimensional and has the tightness property. Moreover, assume that for any $0 \leq r' < r < 1$ and any compact subset $F \subset X$, there is $x$ in some $M_N(A)$ with the property that $r' < d_\tau(x) < r, \forall \tau \in F$. Suppose $Y$ is a clopen subset of $X$ and $\mu_\tau(Y) < \varepsilon, \forall \tau \in \partial X$ for some $\varepsilon > 0$. Then there exists $l \in A \otimes K$, such that
\[ |d_\tau(l) - f(\tau)| < 9\varepsilon, \forall \tau \in Y \]
\[ d_\tau(l) < 8\varepsilon, \forall \tau \in \overline{X \setminus Y}. \]

where $M \in \mathbb{N}$ is determined by $Y$ and $\varepsilon$.

**Proof.** Since $Y$ is a clopen subset and $X$ is zero dimensional, $Y$ can be written as a finite disjoint union of clopen subsets: $Y = \bigsqcup_{i=1}^n Y_i$ such that on each $Y_i$, $f$ is "almost constant": $|f(\tau_1) - f(\tau_2)| < \varepsilon/2, \forall \tau_1, \tau_2 \in Y_i$. Then we want to construct $l_1, l_2, ..., l_n$ which satisfy
\[ |d_\tau(l_i) - f(\tau)| < \varepsilon/2 + \varepsilon_i, \forall \tau \in Y_i \]
\[ d_\tau(l_i) \leq 8\varepsilon_i, \forall \tau \in \overline{X \setminus Y_i} \]

where $\varepsilon_i = \sup_{\tau \in \partial X} \{\mu_\tau(Y_i)\}$ for $i = 1, 2, ..., n$. By assumption, $\mu_\tau(Y) < \varepsilon, \forall \tau \in \partial X$. This implies that $\sum_{i=1}^n \varepsilon_i < \varepsilon$. Hence by setting $l := \oplus_{i=1}^n l_i$ we get the $l$ with the desired property.

What remains is to construct such $l_i, i = 1, 2, ..., n$. Since $Y_i$ is clopen, by Lemma...
for any $\delta > 0$, there exists $t_i \in A$ with the property $d_r(t_i) < \delta$ on $Y_i$ and $d_r(t_i) = 1$ on $X \setminus Y_i$. For $\tau \in \partial X$, $d_r(t_i) \geq 1 \cdot \mu_r(X \setminus Y_i) \geq 1 - \varepsilon_i$. So we have

$$\delta' < d_r(t_i) < \delta, \forall \tau \in Y_i$$

$$1 - \varepsilon_i \leq d_r(t_i) \leq 1, \forall \tau \in \overline{X \setminus Y_i}.$$ 

$\delta' > 0$ exists from the proof of Lemma 4.1 Since $Y_i$ and $\overline{X \setminus Y_i}$ are both compact, by Lemma 4.3 there are $\eta_1, \eta_2 > 0$, such that

$$d_r((t_i - \eta_1)_) > 1 - \varepsilon_i, \forall \tau \in \overline{X \setminus Y_i}$$

$$d_r((t_i - \eta_2)_) > \delta', \forall \tau \in Y_i.$$ 

Take $\eta = \min\{\eta_1, \eta_2\}$, then $d_r((t_i - \eta)_+) > 1 - \varepsilon_i$ on $\overline{X \setminus Y_i}$ and $d_r((t_i - \eta)_+) > \delta'$ on $Y_i$. Since $d_r((t_i - \eta)_+) = \nu_r((\eta, \infty) \cap \sigma(t_i))$, then

$$\nu_r((0, \eta] \cap \sigma(t_i)) = d_r(t_i) - \nu_r((\eta, \infty) \cap \sigma(t_i)) < 1 - (1 - \varepsilon_i) = \varepsilon_i, \forall \tau \in \overline{X \setminus Y_i}$$

$$< \delta - \delta' < \delta, \forall \tau \in Y_i.$$ 

Then

$$d_r((1 - f_\eta(t_i)) = \nu_r([0, \eta] \cap \sigma(t_i))$$

$$= \nu_r((0, \eta] \cap \sigma(t_i)) + \nu_r([0] \cap \sigma(t_i))$$

$$= \nu_r((0, \eta] \cap \sigma(t_i)) + 1 - d_r(t_i)$$

$$< \varepsilon_i + 1 - (1 - \varepsilon_i) = 2\varepsilon_i, \forall \tau \in \overline{X \setminus Y_i}.$$ 

On the other hand, $d_r((1 - f_\eta(t_i)) \geq 1 - d_r(t_i) > 1 - \delta$ on $Y_i$. Now, since we have $|f(\tau_1) - f(\tau_2)| < \varepsilon/2, \forall \tau_1, \tau_2 \in Y_i$, take $m_i = f(\tau_i)$ for some $\tau_i \in Y_i$. It suffices to find $l_i$ such that

$$|d_r(l_i) - m_i| < \varepsilon_i, \forall \tau \in Y_i$$

$$d_r(l_i) \leq 8\varepsilon_i, \forall \tau \in \overline{X \setminus Y_i}.$$ 

By our assumption, for any $r = m_i, r' = \max\{0, m_i - \varepsilon_i/2\}$, we have $x_i \in M_{N_i}(A)$ for some $N_i \in \mathbb{N}$ which satisfies

$$m_i - \varepsilon_i/2 < d_r(x_i) < m_i, \forall \tau \in Y_i.$$ 

Obviously we can choose $x_i$ such that $d_r(x_i) \leq 1, \forall \tau \in T(A)$. Then by Theorem 4.4.1 in [4] we can find $x_i' \in M_d(A)$ which is Murray-von Neumann equivalent to $x_i$. Take $\delta = \varepsilon_i/8$ and set $t_i' = \sum_{j=1}^{4} (1 - f_\eta(t_i)) \in M_d(A)$. Consider $l_i = l_i'^{1/2}x_i'l_i'^{1/2}$. Since $l_i \preceq x_i'$, we have $d_r(l_i) \leq d_r(x'_i) < m_i$ on $Y_i$. On the other hand,

$$d_r(l_i) \geq d_r(x'_i) - d_r(1 - t_i')$$

$$\geq d_r(x'_i) - 4 \cdot d_r(f_\eta(t_i))$$

$$\geq d_r(x'_i) - 4 \cdot d_r(t_i)$$

$$> m_i - \varepsilon_i/2 - 4 \cdot \varepsilon_i/8$$

$$> m_i - \varepsilon_i,$$
for any $\tau \in Y_i$. For $\tau \in X \setminus Y$, $d_\tau(t_i) \leq d_\tau(t'_i) \leq 4 \cdot d_\tau(1 - f_\gamma(t_i)) < 8\varepsilon_i$. \hfill \Box

Note that in [7], if instead of assuming the extreme boundary is compact, we restrict ourselves to a compact subset of the extreme boundary, then some of the results in [7] still hold in the new setting. In particular, from Theorem 5.2 and Lemma 4.5 in [7], we know the following lemmas are true:

**Lemma 4.7.** Let $A$ be a unital simple separable $C^*$-algebra. Suppose that $X = \partial_\varepsilon T(A)$ is a nonempty $F_\tau$ set. For each $m \in \mathbb{N}$ and any compact subset $F \subset \partial_\varepsilon T(A) = X$, there is $x \in Cu(A)$ with the property that $md_\tau(x) \leq 1 \leq (m + 1)d_\tau(x)$, $\forall \tau \in F$. It follows that for any $f \in \operatorname{Aff}(T(A))$, $\varepsilon > 0$, any compact subset $F \subset X$, there is positive $h \in A \otimes K$, such that

$$|d_\tau(h) - f(\tau)| < \varepsilon, \forall \varepsilon \in F.$$

**Lemma 4.8.** Let $A$ be a unital simple separable $C^*$-algebra with strict comparison of positive elements and at least one bounded trace. $F$ is a compact subset of the extreme boundary $X = \partial_\varepsilon T(A)$ and $a, b \in A$ is positive. Suppose that there are $0 < \alpha < \beta < \gamma \leq 1$ and open sets $U, V \subset F$ with the property that $\alpha < d_\tau(a) < \beta, \forall \tau \in U$ and $\beta < d_\tau(b) < \gamma, \forall \tau \in V$. It follows that for any closed set $K \subset U \cup V$, there is a positive element $c$ of $M_2(A)$ with the property that

$$\alpha < d_\tau(c) < \gamma, \forall \tau \in K.$$

5. Rank Functions on Zero-dimensional Extreme Boundaries

**Theorem 5.1.** Let $A$ be a unital simple separable $C^*$-algebra with strict comparison of positive elements. Suppose further that the extreme boundary $X$ of $T(A)$ is nonempty, zero dimensional and has the tightness property. It follows that $\forall f \in \operatorname{Aff}(T(A))$, $\forall \varepsilon > 0$, $\exists h \in (A \otimes K)_+$, such that

$$(*) \quad |d_\tau(h) - f(\tau)| < \varepsilon, \forall \tau \in T(A).$$

If $A$ moreover has strict comparison, then we may take $f \in \operatorname{SAff}(T(A))$, and arrange $d_\tau(h) = f(\tau)$ for each $\tau \in A$.

**Proof.** We only need to establish $(*)$ on $\partial_\varepsilon T(A) = X$.

Let $f \in \operatorname{Aff}(T(A))$ and $\varepsilon > 0$ be given. Without loss of generality assume $\|f\| = 1$. Since $f$ is uniformly continuous on $T(A)$, there exists an open neighborhood $U_1$ of $\partial X$, such that $\forall \gamma \in U_1, \exists \tau \in \partial X, |f(\tau) - f(\gamma)| < \varepsilon/6$.

Since $X$ has the tightness property, there is an open neighborhood $U_2$ of $\partial X$, such that $\mu_\tau(U_2) < \varepsilon/48, \forall \tau \in \partial X$. Let $U = U_1 \cap U_2$ and denote $F = X \setminus U$. $F$ is compact subset of $X$ and $\mu_\tau(F) > 1 - \varepsilon/6$ for all $\tau \in \partial X$.

By Lemma [4,7] there exists $h_1 \in (A \otimes K)_+$, such that $|d_\tau(h_1) - f(\tau)| < \varepsilon/6, \forall \tau \in F$. Since $d_\tau(h_1)$ is affine, $d_\tau(h_1) = \int_X d_\tau(h_1) d\mu_\tau$ for each $\tau \in \partial X$. Similar equation
holds for \( f \) since it is also affine. Then,
\[
|d_\tau(h_1) - f(\tau)| \leq |\int_F d_\gamma(h_1)d\mu_\tau - \int_F f d\mu_\tau| + |\int_{X\setminus F} d_\gamma(h_1)d\mu_\tau - \int_{X\setminus F} f d\mu_\tau|
\leq \varepsilon/6 \cdot \mu_\tau(F) + 2 \cdot \mu_\tau(X\setminus F)
\leq \varepsilon/6 + 2\varepsilon/6 = \varepsilon/2
\]
for each \( \tau \in \partial X \).

By Lemma 4.3, \( \partial X \cup F \) is compact. Then by Lemma 4.1 we obtain \( \varepsilon_1 > 0 \) and an open neighborhood \( V \) of \( \partial X \cup F \). For any \( \gamma \in V, \exists \tau \in \partial X \cup F, \) such that
\[
d_\tau(h_1) - \varepsilon/6 < d_\gamma((h_1 - \varepsilon_1)_+) < d_\tau(h_1) + \varepsilon/6
\]
and the inequality also holds for \( \gamma = \tau \in \partial X \cup F \). Denote \( h_2 := (h_1 - \varepsilon_1)_+ \). Then
\[
|d_\gamma(h_2) - f(\tau)| \leq |d_\gamma(h_2) - d_\tau(h_1)| + |d_\tau(h_1) - f(\tau)| + |f(\tau) - f(\gamma)|
\leq \varepsilon/6 + \varepsilon/3 + \varepsilon/6 = 5\varepsilon/6
\]
for all \( \gamma \in V \).

Since \( V \) is an open neighborhood of the compact subset \( \partial X \cup F \), then since \( X \) is zero-dimensional, we can replace \( V \) by a clopen set \( V' \) such that \( \partial X \cup F \subset V' \subset V \). Denote \( Y = X \setminus V' \), so \( Y \) is also clopen. Then by Lemma 4.4, there exists \( h_3 \in (A \otimes \mathcal{K})_+ \), such that
\[
d_\tau(h_3) = d_\tau(h_2), \forall \tau \in X \setminus Y
\]
\[
d_\tau(h_3) \leq \varepsilon/6, \forall \tau \in Y.
\]

So
\[
|d_\tau(h_3) - f(\tau)| < 5\varepsilon/6, \forall \tau \in X \setminus Y
\]
\[
d_\tau(h_3) \leq \varepsilon/6, \forall \tau \in Y.
\]

Then it suffices to find \( l \in (A \otimes \mathcal{K})_+ \) satisfying
\[
|d_\tau(l) - f(\tau)| < \varepsilon/3, \forall \tau \in Y
\]
\[
d_\tau(l) \leq \varepsilon/6, \forall \tau \in X \setminus Y.
\]

By setting \( h := h_3 \oplus l \), we get
\[
|d_\tau(h) - f(\tau)| \leq \varepsilon, \forall \tau \in X
\]
as desired. The construction of such \( l \) is as follows. By Lemma 4.6, we can find \( l \in (A \otimes \mathcal{K})_+ \), such that
\[
|d_\tau(l) - f(\tau)| < 9\varepsilon/48 < \varepsilon/3, \forall \tau \in Y
\]
\[
d_\tau(l) < 8\varepsilon/48 = \varepsilon/6, \forall \tau \in X \setminus Y.
\]
This finishes the proof of (*).

Now suppose that \( A \) has strict comparison. The final conclusion of the Theorem then follows from the proof of Theorem 2.5 of [10], which shows how one produces an arbitrary \( f \in S\text{Aff}(T(A)) \) by taking suprema. \( \square \)
6. Proof of the Main Result

In this section, we will prove Theorem 1.1.

First recall that $T_\infty(A)$ is the set of all traces on $A_\infty$ induced by the trace $(\tau_n)_n \mapsto \lim_{n \to \infty} \tau_n(x_n)$ on $\ell^\infty(A)$ where $(\tau_n)_n$ is a sequence in $T(A)$ and $\omega \in \beta \mathbb{N}\setminus \mathbb{N}$ is a free ultrafilter. If we choose the sequence $(\tau_n)_n$ in $Y \subset T(A)$ instead, then write $T_\infty^Y(A)$ for the collection of those traces arising in the same fashion. $T_\infty^Y(A)$ is clearly a subset of $T_\infty(A)$.

The idea of the proof of Theorem 1.1 is similar to that of Theorem 4.6 of [22], in which its Lemma 3.5 plays a key role. Note that if the result of Lemma 3.5 of [22] holds on a subset $Y$ of $T(A)$, we can replicate the arguments in Section 4 of [22] and get similar versions of Lemma 4.1, Lemma 4.2, Lemma 4.3, Proposition 4.4 and Lemma 4.5 of [22] by replacing $T_\infty(A)$ by $T_\infty^Y(A)$. Therefore the following result is true:

**Lemma 6.1.** Let $A$ be a simple separable unital nuclear nonelementary C*-algebra with $T(A) \neq \emptyset$. If for any subset $Y \subset \partial_e T(A)$, each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists some $m \in \mathbb{N}$ and cpc order zero maps $\psi(0), \ldots, \psi(m) : M_k \to A$ such that

$$\|\psi^{(i)}(x), y\| \leq \varepsilon \|x\|$$

for all $i \in \{0, \ldots, m\}$, $x \in M_k$, $y \in \mathcal{F}$ and such that for each $\tau \in Y$, there exists $i(\tau) \in \{0, \ldots, m\}$ such that $\tau(\psi^{(i(\tau))}(1_k)) > 1 - \varepsilon$. Then there exists a cpc order zero map $\phi^{(0)} : M_k \to A$ such that

$$\|\phi^{(0)}(x), y\| \leq \varepsilon \|x\|$$

for $x \in M_k$, $y \in \mathcal{F}$ and such that for each $\tau \in Y$, $\tau(\phi^{(0)}(1_k)) > 1 - \varepsilon$.

Note that Lemma 3.5 of [22] still holds if we remove the assumption of the compactness of the extreme boundary and restrict to a compact subset of it:

**Lemma 6.2.** Let $m \geq 0$, $k \geq 2$ and let $A$ be a simple separable unital nuclear nonelementary C*-algebra with $T(A) \neq \emptyset$ and $\dim(\partial_e T(A)) \leq m$. Then for any compact subset $Y \subset \partial_e T(A)$, each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists cpc order zero maps $\psi(0), \ldots, \psi(m) : M_k \to A$ such that

$$\|\psi^{(i)}(x), y\| \leq \varepsilon \|x\|$$

for all $i \in \{0, \ldots, m\}$, $x \in M_k$, $y \in \mathcal{F}$ and such that for each $\tau \in Y$, there exists $i(\tau) \in \{0, \ldots, m\}$ such that $\tau(\psi^{(i(\tau))}(1_k)) > 1 - \varepsilon$.

The next lemma follows immediately from Lemma 6.1 and Lemma 6.2:

**Lemma 6.3.** Let $m \geq 0$, $k \geq 2$ and let $A$ be a simple separable unital nuclear nonelementary C*-algebra with $T(A) \neq \emptyset$ and $\dim(\partial_e T(A)) \leq m$. Then for any compact subset $Y \subset \partial_e T(A)$, each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a cpc order zero map $\phi^{(0)} : M_k \to A$ such that

$$\|\phi^{(0)}(x), y\| \leq \varepsilon \|x\|$$
for \( x \in M_k, y \in F \) and such that for each \( \tau \in Y \), \( \tau(\phi(0)(1_k)) > 1 - \varepsilon \).

Now we can prove Theorem 6.1.

**Proof.** The implication of (1) \( \Rightarrow \) (2) has already been established in [17]. Now assume \( A \) has strict comparison. Since \( X \) has the tightness property, there is a compact subset \( Y \subset X \), such that for any \( \gamma \in X \setminus Y, \nu_\gamma(Y) > 1 - \varepsilon/2 \). Then by Lemma 6.2 there is \( \phi(0) : M_k \to A \) such that

\[
\left\| [\phi(0)(x), y] \right\| \leq \varepsilon \|x\|
\]

and \( \tau(\phi(0)(1_k)) > 1 - \varepsilon/2 \) for any \( \tau \in Y \). Define a map \( g : T(A) \to \mathbb{R} \) as follows:

\[
g(\tau) = \tau(\phi(0)(1_k)).
\]

\( g \) is affine and continuous. So for \( \gamma \in X \setminus Y \),

\[
\gamma(\phi(0)(1_k)) = g(\gamma) = \int_X g d\nu_\gamma \geq \int_Y g d\nu_\gamma \\
\geq (1 - \varepsilon/2)(1 - \varepsilon/2) > 1 - \varepsilon.
\]

Let \( U = \{ \tau \in X : \tau(\phi(0)(1_k)) > 1 - \varepsilon \} \). \( U \) is an open neighborhood of \( Y \cup (X \setminus X) \). So \( X \setminus U \) is compact. Applying Lemma 6.2 to \( X \setminus U \), we find \( \phi(1) : M_k \to A \) such that

\[
\left\| [\phi(1)(x), y] \right\| \leq \varepsilon \|x\|
\]

and \( \tau(\phi(1)(1_k)) > 1 - \varepsilon \) for any \( \tau \in X \setminus U \). Now we have two cpc order zero maps \( \phi^{(0)}, \phi^{(1)} : M_k \to A \) such that

\[
\left\| [\phi^{(i)}(x), y] \right\| \leq \varepsilon \|x\|
\]

for \( i \in \{0, 1\}, x \in M_k, y \in F \) and such that for each \( \tau \in X \), there exists \( i(\tau) \in \{0, 1\} \) such that \( \tau(\phi^{(i(\tau))}(1_k)) > 1 - \varepsilon \). By Lemma 6.2 we could reduce the number of cpc order zero maps to one. Therefore using the same argument as in Theorem 3.6 of [22], \( A \) admits uniformly tracially large cpc order zero maps \( M_k \to A_\infty \cap A' \). Then by Theorem 2.6 of [22], \( A \) is \( \mathbb{Z} \)-stable. \( \square \)

The recursive subhomogeneous algebras (RSH algebras) is an important class of C*-algebras. Recall the definition of a RSH algebra given in [15]:

**Definition 6.1.** A recursive subhomogeneous algebra is a C*-algebra given by the following recursive definition:

1. If \( X \) is a compact Hausdorff space and \( n \geq 1 \), then \( C(X, M_n) \) is a recursive subhomogeneous algebra;
2. If \( A \) is a recursive subhomogeneous algebra, \( X \) is a compact Hausdorff space, \( X^{(0)} \subset X \) is closed, \( \varphi : A \to C(X^{(0)}, M_n) \) is any unital homomorphism, and \( \rho : C(X, M_n) \to C(X^{(0)}, M_n) \) is the restriction homomorphism, then the pullback

\[
A \oplus_{C(X^{(0)}, M_n)} C(X, M_n) = \{(a, f) \in A \oplus C(X, M_n) : \varphi(a) = \rho(f)\}
\]

is a recursive subhomogeneous algebra.
PROPOSITION 6.1. Let $A$ be a simple nuclear separable unital infinite-dimensional $C^*$-algebra with non-empty tracial state space. Suppose $T(A)$ is homeomorphic to the tracial state space of some RSH algebra of finite topological dimension. The following conditions are equivalent:

1. $A$ is $Z$-stable;
2. $A$ has strict comparison.

PROOF. We only need to construct a tracially large cpc order zero map. First consider the case of a one-step RSH algebra of finite topological dimension:

$$B = C(X, M_n) \oplus C(Y, M_m) \cap C(Y, M_m) = \{(f, g) \in C(X, M_n) \oplus C(Y, M_m) : \varphi(f) = \rho(g)\}$$

where $X, Y$ are compact Hausdorff spaces of finite covering dimension, $Y(0) \subset Y$ is closed, and $\varphi : C(X, M_n) \to C(Y, M_m)$ is some unital homomorphism and $\rho : C(Y, M_m) \to C(Y, M_m)$ is the restriction homomorphism. Since $\partial_x T(B)$ and $\partial_y T(A)$ are homeomorphic, we claim that $T(A)$ is non-empty, with the tightness property and of finite covering dimension. Thus the result follows by Theorem [1.1]. Indeed, $\partial_x T(B)$ is homeomorphic to $X \cup (Y \setminus Y(0))$ whose cluster points in $X \cup Y$ all belongs to $Y(0)$. But for any point $y$ in $Y(0)$, which is the extreme point of $T(C(Y, M_m))$, $y \circ \varphi$ is a tracial state of $C(X, M_n)$. So there exists a Borel probability measure $\mu_y$ on $X$, such that $y \circ \varphi = \int_X d\mu_y$. Since $X$ is compact, this shows that $\partial_x T(B)$ has the tightness property.

We then consider the case of a two step RSH algebra of finite topological dimension:

$$C = B \oplus C(Z, M_p) \cap C(Z, M_p) = \{(b, g) \in B \oplus C(Y, M_m) : \varphi(b) = \rho_g\}$$

where $B$ is defined above, $Z$ is a compact Hausdorff space of finite covering dimension, $Z(0) \subset Z$ is closed, and $\varphi' : B \to C(Z, M_p)$ is some unital homomorphism and $\rho' : C(Z, M_p) \to C(Z, M_p)$ is the restriction homomorphism. $\partial_x T(C) = \partial_x T(A)$ is homeomorphic to $X \cup (Y \setminus Y(0)) \cup (Z \setminus Z(0))$ whose cluster points belongs to $Y(0) \cup Z(0)$. By similar argument as above, for any point $y$ in $Y(0)$, there exists a Borel probability measure $\mu_y$ on $X$, such that $y \circ \varphi = \int_X d\mu_y$. And any point $z$ in $Z(0)$, there exists a Borel probability measure $\mu_z$ on $X \cup \{Y \setminus Y(0)\}$, such that $z \circ \varphi = \int_{X \cup \{Y \setminus Y(0)\}} d\mu_z$. Then take $X$ as the compact subset in Lemma [6.3] for each finite set $F \subset C$ and $\epsilon > 0$, there exists a cpc order zero map $\phi : M_k \to C$ such that

$$\|[\phi(x), y]\| \leq \epsilon \|x\|$$

for $x \in M_k$, $y \in F$ and such that for each $\tau \in X$, $\tau(\phi(1_k)) > 1 - \epsilon/4$. Then using the same trick as in the proof of Theorem [1.1] on the cluster points in $Y(0)$, we obtain a cpc order zero map $\phi' : M_k \to C$ such that

$$\|[\phi'(x), y]\| \leq \epsilon \|x\|$$

for $x \in M_k$, $y \in F$ and such that for each $\tau \in X \cup \{Y \setminus Y(0)\}$, $\tau(\phi'(1_k)) > 1 - \epsilon/2$. Then do this again on the cluster points in $Z(0)$, we obtain a cpc order zero map $\phi'' : M_k \to C$ such that

$$\|[\phi''(x), y]\| \leq \epsilon \|x\|$$
for \( x \in M_k, y \in F \) and such that for each \( \tau \in X \cup (Y \setminus Y^{(0)}) \cup (Z \setminus Z^{(0)}) = \partial_e T(C), \)
\[ \tau(\phi_k''(1_k)) > 1 - \varepsilon. \]
And this is sufficient to yield a tracially large cpc order zero map.

Note that in the proof of the two step case, we first construct a cpc order zero map which is tracially large on the subset \( X \) of the tracial state space, then use the relation between the cluster points of \( Y \setminus Y^{(0)} \) and \( X \) to generate a cpc order zero map which is tracially large on \( X \cup (Y \setminus Y^{(0)}) \). Then again, using the relation between the cluster points of \( Z \setminus Z^{(0)} \) and \( X \cup (Y \setminus Y^{(0)}) \), we obtain a tracially large cpc order zero map. By this observation, for any RSH algebra of finite topological dimension, we do the same trick step by step, and after finitely many times we could construct a tracially large cpc order zero map. \( \Box \)

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