Propagation of quantum gravity-modified gravitational waves on a classical FLRW spacetime

Angel Garcia-Chung,1,2,∗ James B. Mertens,3,4,5, † Saeed Rastgoo,4,‡ Yaser Tavakoli,6,7,§ and Paulo Vargas Moniz8,¶

1Departamento de Física, Universidad Autónoma Metropolitana - Iztapalapa, San Rafael Atlíxco 186, Ciudad de México 09340, México
2Universidad Panamericana, Tecoyotitla 366, Col. Ex Hacienda Guadalupe Chimalistac, C.P. 01050 Ciudad de México, México
3Department of Physics and McDonnell Center for the Space Sciences, Washington University, St. Louis, MO 63130, USA
4Department of Physics and Astronomy, York University 4700 Keele Street, Toronto, Ontario M3J 1P3 Canada
5Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
6Department of Physics, University of Guilan, Namjoo Blvd., 41335-1914 Rasht, Iran
7School of Astronomy, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5531, Tehran, Iran
8Departamento de Física, Centro de Matemática e Aplicações: CMA-UBI, Universidade da Beira Interior, 6200 Covilhã, Portugal

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The linearized Einstein field equations provide a low energy wave equation for propagation of gravitational fields which may be originated from a high energy source. Motivated by loop quantum gravity (LQG), we propose the polymer quantization scheme to derive the effective propagation of such waves on a classical FLRW spacetime. To overcome the challenge of polymer quantizing a time-dependent Hamiltonian, we rewrite such a Hamiltonian in a time-independent manner in the extended phase space, polymerize it, and then transform it back to the usual phase space. This way we obtain a time-dependent polymer Hamiltonian for the gravitational waves. We then derive the effective equations of motion and show that i) the form of the waves are modified ii) the speed of the waves depend on their frequencies iii) the quantum effects are amplified by the distance/time the waves travel.

I. INTRODUCTION

Recent observations of Gravitational Waves (GWs) and rapid increase in the sensitivity of GW observatories has opened up a great opportunity in connecting theory and phenomenology with experiment in many areas of physics and astronomy. Particularly, precision cosmology, black hole physics and quantum gravity can benefit hugely from this development.

∗ alechung@xanum.uam.mx
† jmertens@wustl.edu
‡ srastgoo@yorku.ca
§ yaser.tavakoli@guilan.ac.ir
¶ pmoniz@ubi.pt
Such observations have also the potential of guiding us towards the correct theory of quantum gravity, among other ways, by revealing the information about the deep structure of spacetime encoded in such waves. Although these effects might be extremely small, the distances these waves travel can act as amplifiers of such quantum effects, making them observable in our current or near future experiments.

There have been numerous studies connecting theories of quantum gravity with potential observation regarding the structure of quantum spacetime. Particularly, in Loop Quantum Gravity (LQG) [1], there have been studies to understand the consequence of nonperturbative quantization in propagation of Gamma Ray Bursts (GRBs), other matter fields and GWs on cosmological or black holes spacetimes (for some examples see [2–27] and the references within).

In this work we consider GWs as effective perturbation propagating on a classical FLRW cosmological spacetime. The effective form of such waves is derived by applying the techniques of polymer quantization [28–32] to the classical perturbations. Such a quantization is a representation of the classical algebra on a Hilbert space that is unitarily inequivalent to the usual Schroödinger’s representation. In it, operators are regularized and are written in a certain exponential form. In such theories, the infinitesimal generator corresponding to some of the operators do not exist on the Hilbert space. As a consequence, the conjugate variables to those operators only admit finite transformations. Thus the dynamics of the theory leads to discretization of the spectrum of the conjugate operators (For more details and some examples of polymer quantization applied to particles and path integral formulation of black holes see [30, 31, 33]). Polymer quantizations usually come in two “polarizations”. In one polarization one or several configuration variables are regularized, and in the other polarization one or more of the momenta are polymerized.

Since the Hamiltonian of our model is time-dependent, we apply a certain method to overcome the challenge of polymerizing such a time-dependent systems. We first write the Hamiltonian of the system in a time-independent form in the extended phase space, polymerize such a time-independent Hamiltonian, and transform it back to the usual phase space, yielding a polymerized time-dependent Hamiltonian. In fact we derive two effective Hamiltonians, each corresponding to one of the polarizations of the polymer quantization. Using these modified Hamiltonians, we study the effective equations of motion of polymerized GWs and show that i) the form of the waves are modified ii) the speed of the waves depend on their frequencies iii) the quantum effects are amplified by the distance/time the waves travel.

This paper is organized as follows: in Sec. II, we derive the classical Hamiltonian of perturbation on an FLRW classical background. In Sec. III, this time-dependent Hamiltonian is turned into a polymer effective time-dependent Hamiltonian by applying a certain method that is inspired by an approach used to deal with time-dependent Harmonic oscillators. We derive two Hamiltonians each corresponding to one of the polarizations of the polymer quantization. In Sec. IV we derive the equations of motions corresponding to each effective polymer Hamiltonian, and solve them both perturbatively and numerically in order to explore deviations from the classical behavior. Finally, in Sec. V, we present our concluding remarks and comment about future outlook and projects.


II. HAMILTONIAN FORMALISM FOR GWS

GWs are the result of weak field approximation to Einstein’s field equations. On a curved spacetime, we fix the (unperturbed) background as a 4-manifold $M = T^3 \times \mathbb{R}$, with a spatial three-torus topology, equipped with coordinates $x^j \in (0, \ell)$, and a temporal coordinate $x^0 \in \mathbb{R}$. We then consider a small (metric) perturbation to this background and study the GWs generated by this perturbation.

Hence, given the (unperturbed) Einstein-Hilbert gravitational action

$$S_{\text{grav}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \mathcal{R},$$

(2.1)

the starting point of writing the Hamiltonian of the GWs, is the general perturbed metric

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu},$$

(2.2)

where $\hat{g}_{\mu\nu}$ is the unperturbed background metric, while $h_{\mu\nu}$ denotes a small perturbation concerning $\hat{g}_{\mu\nu}$. Moreover,

$$h^{\mu\nu} = \hat{g}^{\mu\sigma} \hat{g}^{\nu\tau} h_{\sigma\tau}.$$  

(2.3)

In order to reduce the number of terms in the linearized Einstein field equations, it is convenient to express the Einstein tensor in terms of the trace-reversed metric perturbation

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} h,$$

(2.4)

where, $h = h^{\mu}_{\mu} = \eta^{\mu\nu} h_{\mu\nu}$ with $\eta^{\mu\nu}$ being the Minkowski spacetime metric. Thereby, the linearized Einstein field equation in terms of $\bar{h}_{\mu\nu}$ can be expressed as a wave equation, in the Lorentz gauge

$$\nabla_\mu \bar{h}^{\mu\nu} = 0.$$  

(2.5)

Indeed, in this gauge, the metric perturbation looks like a transverse wave. By imposing an additional (synchronous) transverse-traceless gauge, where

$$\bar{h} = 0, \quad \bar{h}_{0\mu} = 0, \quad \text{and} \quad \nabla_i \bar{h}^{ij} = 0,$$

(2.6)

we consider only spatial, transverse, traceless perturbations. In the latter case, the metric perturbations $h_{ij}$ correspond at present time to GWs propagating on the unperturbed spacetime background. A wave traveling along, say, the $x^3$ direction, can be separated into two polarization scalar modes $h_+(x)$ and $h_\times(x)$ as

$$h_{ij}(x) = h_+(x) e_+^{ij} + h_\times(x) e_\times^{ij},$$

(2.7)

where

$$e_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e_\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(2.8)

\footnote{To avoid a discussion of boundary conditions on fields (generated by perturbation), we will assume that the spatial 3-manifold is $T^3$.}
Let us now consider the GWs propagating in a homogeneous, isotropic universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = -N^2(x^0) d(x^0)^2 + a^2(x^0) dx^2,$$

where \(x^0\) is an arbitrary time coordinate, \(N(x^0)\) is the lapse function which depends on the choice of \(x^0\), and \(dx^2 = \sum_i dx^i)^2\) is a unit three-sphere. To study the linearized Einstein equations, to be comparable with the Minkowski spacetime, it is more convenient to work with a conformally (perturbed) flat metric:

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} = a^2(\eta_{\mu\nu} + \tilde{h}_{\mu\nu}) \quad \text{(2.10)}$$

Here, the conformal metric perturbation \(\tilde{h}_{ij}\), for a wave traveling along the \(x^3\) direction, is related to the physical metric perturbation (2.7) by the scale factor as

$$\tilde{h}_{ij}(x) := a^{-2}h_{ij}(x) \quad \text{(2.11)}$$

The metric perturbation produces a perturbation to the action (2.1). At second order in linear perturbation, for a traceless-traverse gauge, we get the perturbed action as

$$\delta S^{(2)}_{\text{grav}} = \frac{1}{4\kappa^2} \int d^4x \sqrt{-\hat{g}} \hat{\Box} \hat{h}^{ij}. \quad \text{(2.12)}$$

This represents the action governing the GWs propagating on the unperturbed background \(\hat{g}_{\mu\nu}\) in the \(x^3\)-direction.

Let us, for convenience, introduce the new scalars \(\tilde{h}_{\pm}(x)\) by

$$\tilde{h}_{ij}(x) := \sqrt{2\kappa} \left[ \tilde{h}_+(x)e^+_ij + \tilde{h}_x(x)e^x_ij \right] \quad \text{(2.13)}$$

where

$$\tilde{h}_+(x) = \frac{a^{-2}}{\sqrt{2\kappa}} h_+(x) \quad \text{and} \quad \tilde{h}_x(x) = \frac{a^{-2}}{\sqrt{2\kappa}} h_x(x). \quad \text{(2.14)}$$

By substitution the Eqs. (2.7) and (2.13) into the perturbed action (2.12), the perturbed Lagrangian density at second order in linear perturbations becomes

$$\mathcal{L}_{\tilde{h}} = \frac{1}{2} \sum_{\lambda=+,-,x} \tilde{h}_\lambda \tilde{\Box} \tilde{h}_\lambda + \mathcal{O}(\tilde{h}_\lambda^2). \quad \text{(2.15)}$$

The effective action of the independent polarization modes, provided by the Lagrangian density (2.15), is that of two massless scalar fields. Thus, the equation of motion for the (scalar) perturbation \(\tilde{h}_\lambda(x)\), with a fixed \(\lambda\), is given by the familiar Klein-Gordon equation

$$\tilde{\Box} \tilde{h}_\lambda(x) = 0 \quad \text{(2.16)}$$

Our aim henceforth, will be to study the quantum theory of scalar perturbations \(\tilde{h}_\lambda(x)\)–satisfying the Klein-Gordon equation (2.16)–propagating on the cosmological spacetime (2.9).

The canonically conjugate pair for the field \(\tilde{h}_\lambda(x)\) consists of \((\tilde{h}_\lambda, \tilde{\pi}_\lambda)\) on a \(x^0 = \text{const.}\) slice. As usual we would like to write the field \(\tilde{h}_\lambda(x)\) in terms of its Fourier modes. However, we are not \textit{a priori} assuming Lorentz invariance, and in fact, we will be considering its
possible violations. Hence, we do not perform a four-dimensional Fourier transform on \( h_\lambda(x) \), rather we only consider such a transformation over spatial coordinates for \( h_\lambda(x) \) and its conjugate momentum \( \pi_\lambda(x) \). The classical solutions of the equation of motion (2.16) can be expanded in Fourier modes as

\[
\begin{align*}
\tilde{h}_\lambda(x^0, \mathbf{x}) &= \frac{1}{\ell^{3/2}} \sum_{\mathbf{k} \in \mathcal{L}} h_{\lambda, \mathbf{k}}(x^0) e^{i \mathbf{k} \cdot \mathbf{x}}, \\
\tilde{\pi}_\lambda(x^0, \mathbf{x}) &= \frac{1}{\ell^{3/2}} \sum_{\mathbf{k} \in \mathcal{L}} \Pi_{\lambda, \mathbf{k}}(x^0) e^{i \mathbf{k} \cdot \mathbf{x}},
\end{align*}
\]

where the wave vector \( \mathbf{k} \in (2\pi \mathbb{Z}/\ell)^3 \) spans a three-dimensional lattice\(^2 \mathcal{L} \) [6]. The Fourier coefficients are canonically conjugate satisfying the commutation relations \( \{h_{\lambda, \mathbf{k}}, \Pi_{\lambda, \mathbf{k'}}\} = \delta_{\mathbf{k}, -\mathbf{k'}}. \) Moreover, the reality conditions on the field \( h_\lambda(x^0, \mathbf{x}) \) imply that \( h_{\lambda, \mathbf{k}} = (h_{\lambda, -\mathbf{k}})^* \) and \( \Pi_{\lambda, \mathbf{k}} = (\Pi_{\lambda, -\mathbf{k}})^* \) are satisfied for each mode.

From the Lagrangian (2.15), we can write the (time-dependent) Hamiltonian of the perturbation field propagating on the background \( (M, g_{\mu\nu}) \). In terms of the conjugated pairs \((\tilde{h}_\lambda, \tilde{\pi}_\lambda)\), by using Eqs. (2.17), the Hamiltonian of the GW is obtained as

\[
H(x^0) = \sum_{\lambda = +, \times} \frac{N(x^0)}{2a^3(x^0)} \int d^3x \left[ (\tilde{\pi}_\lambda)^2 + a^4(x^0)(\partial_i \tilde{h}_\lambda)^2 \right] = \sum_{\lambda = +, \times} \frac{N(x^0)}{2a^3(x^0)} \sum_{\mathbf{k}} \left[ (\Pi_{\lambda, \mathbf{k}})^* \Pi_{\lambda, \mathbf{k}} + k^2 a^4(x^0) (h_{\lambda, \mathbf{k}})^* h_{\lambda, \mathbf{k}} \right],
\]

where \( k = |\mathbf{k}|. \)

Following the above reality conditions for the perturbation field \( h_\sigma(t, \mathbf{x}) \), it turns out that not all modes \( h_{\lambda, \mathbf{k}}(t) \) of the GWs are independent. In other words, when decomposing each field mode \( h_{\lambda, \mathbf{k}}(t) \) and its conjugate momentum \( \Pi_{\lambda, \mathbf{k}}(t) \) as

\[
\begin{align*}
\mathbf{b}_{\sigma, \mathbf{k}} &:= \frac{1}{\sqrt{2}} \left( \mathbf{b}_{\sigma, \mathbf{k}}^{(1)} + i \mathbf{b}_{\sigma, \mathbf{k}}^{(2)} \right), \\
\Pi_{\sigma, \mathbf{k}} &:= \frac{1}{\sqrt{2}} \left( \Pi_{\sigma, \mathbf{k}}^{(1)} + i \Pi_{\sigma, \mathbf{k}}^{(2)} \right),
\end{align*}
\]

the reality conditions imply that

\[
\begin{align*}
\mathbf{b}_{\sigma, -\mathbf{k}} &= \mathbf{b}_{\sigma, \mathbf{k}}^{(1)}, & \mathbf{b}_{\sigma, -\mathbf{k}}^{(2)} &= -\mathbf{b}_{\sigma, \mathbf{k}}^{(2)}, \\
\Pi_{\sigma, -\mathbf{k}} &= \Pi_{\sigma, \mathbf{k}}^{(1)}, & \Pi_{\sigma, -\mathbf{k}}^{(2)} &= -\Pi_{\sigma, \mathbf{k}}^{(2)}.
\end{align*}
\]

For each \( \mathbf{k} = (k_1, k_2, k_3) \), relation above enables us to split the lattice \( \mathcal{L} \) into positive and negative sectors [6]

\[
\begin{align*}
\mathcal{L}_+ &= \{ \mathbf{k} : k_3 > 0 \} \cup \{ \mathbf{k} : k_3 = 0, k_2 > 0 \} \cup \{ \mathbf{k} : k_3 = k_2 = 0, k_1 > 0 \}, \\
\mathcal{L}_- &= \{ \mathbf{k} : k_3 < 0 \} \cup \{ \mathbf{k} : k_3 = 0, k_2 < 0 \} \cup \{ \mathbf{k} : k_3 = k_2 = 0, k_1 < 0 \} \\
&= \{ \mathbf{k} : -\mathbf{k} \in \mathcal{L}_+ \},
\end{align*}
\]

\(^2\) Because of the periodicity of the torus \( T^3 \), equipped with coordinates \( x^j \in (0, \ell) \), the allowed Fourier components are those with the wavevectors in the reciprocal space of \( T^3 \). We therefore consider an elementary cell \( \mathcal{V} \) by fixing a fiducial (spatial) flat metric \( \gamma_{ij} \) and denote by \( V_o = \ell^3 \) the volume of the \( \mathcal{V} \) in this geometry. (For simplicity, we assume that this cell is cubical with respect to \( \gamma_{ij} \).) Then, all integrations in the Fourier expansion will be restricted to this volume.
respectively. This decomposition of $L$ enables further to decompose any summation over $k \in L$ into its positive and negative parts. Then, by defining new variables $A_{\lambda,k}$ and $E_{\lambda,k}$,

$$A_{\lambda,k} := \begin{cases} h^{(1)}_{\lambda,k}, & \text{for } k \in L_+ \\ h^{(2)}_{\lambda,-k}, & \text{for } k \in L_- \end{cases} \quad (2.24a)$$

and

$$E_{\lambda,k} := \begin{cases} \Pi^{(1)}_{\lambda,k}, & \text{for } k \in L_+ \\ \Pi^{(2)}_{\lambda,-k}, & \text{for } k \in L_- \end{cases} \quad (2.24b)$$

which are canonically conjugate

$$\{A_{\lambda,k}, E_{\lambda',k'}\} = \delta_{kk'} \delta_{\lambda\lambda'}. \quad (2.25)$$

Now, we can reexpress the Hamiltonian (2.18) as

$$H(x^0) = \frac{N}{2a^3} \sum_{\lambda=\pm} \sum_{k \in L} \left[ E_{\lambda,k}^2 + k^2 a^4 A_{\lambda,k}^2 \right] =: \sum_{\lambda=\pm} \sum_{k \in L} H_{\lambda,k}(x^0). \quad (2.26)$$

Eq. (2.26) represents the Hamiltonian of a set of decoupled harmonic oscillators defined by conjugate pairs $(A_{\lambda,k}, E_{\lambda,k})$ associated with any $k$ mode for a fixed polarization $\lambda$, satisfying the relation (2.25).

At this point, we choose the harmonic time gauge where $N(x^0 = \tau) = a^3(\tau)$ to get rid of the factor $a^{-3}$ in front of (2.26). Hence, the Hamiltonian of the perturbations (for the fixed mode $k$ and polarization $\lambda$) over the FLRW background in harmonic times becomes

$$H_{\lambda,k}(\tau) = \frac{1}{2} \left[ E_{\lambda,k}^2 + k^2 a^4 A_{\lambda,k}^2 \right]. \quad (2.27)$$

This Hamiltonian, Eq. (2.27), however, resembles an oscillator with time-dependent frequency so that analyzing and particularly in quantum theory, finding its effective counterpart is very complicated. In the next section we will show how we bypass this problem and obtain an effective Hamiltonian.

### III. POLYMER QUANTIZATION AND THE EFFECTIVE HAMILTONIAN

As mentioned in the previous section, the Hamiltonian (2.26) is a time-dependent one which makes finding its effective counterpart complicated. In order to circumvent this issue, we will lift the system to the extended phase space (EPS), find the effective counterpart there, and bring that effective Hamiltonian to the usual phase space where the effective Hamiltonian is now time-dependent. A schematics of our method can be seen in Fig. 1. The steps are written below or close to the arrows in parenthesis, hence “to EPS” is step (1) etc. In the following sections we detail this procedure. Section IIIA is devoted to steps (1) and (2), section IIIB discusses step (3) and in section IIIC we will follow steps (4) and (5).

#### A. Obtaining time-independent classical Hamiltonian

Let us consider a time-dependent harmonic oscillator

$$S = \int \left\{ \frac{dq}{dt} \right\} dt, \quad (3.1)$$
where the time-dependent Hamiltonian is of the form

\[ H(t) = \frac{1}{2m} p^2 + \frac{1}{2} m \omega(t)^2 q^2. \quad (3.2) \]

We can now move to the extended phase space (step (1) in Fig. 1) in which time \( t \) is now one of the configuration variables whose conjugate is denoted by \( p_t \). Hence, the system is now described by the coordinates \((q, t, p, p_t)\). In accordance with Dirac’s formalism, the system is now described by the extended action

\[ S = \int \left\{ p \frac{dq}{d\tau} + p_t \frac{dt}{d\tau} - \lambda \phi \right\} d\tau, \quad (3.3) \]

where

\[ \phi = p_t + H(t) \approx 0, \quad (3.4) \]

is a first class constraint ensuring the compatibility of the two actions in extended and usual phase space on the constrained surface, and \( \lambda \) is a Lagrange multiplier.

In the next step (step 2 in Fig. 1), we perform (the inverse of) a canonical transformation

\[ Q = \frac{1}{\rho(t)} q, \quad (3.5) \]

\[ T = \int \frac{1}{\rho^2(t)} dt, \quad (3.6) \]

\[ P = \rho(t) p - m \dot{\rho}(t) q, \quad (3.7) \]

\[ P_T = \rho^2(t) p_t + \rho(t) \dot{\rho}(t) q p - \frac{m}{2} q^2 \left[ \dot{\rho}(t) + \frac{W^2}{\rho^2(t)} - \omega^2(t) \rho^2(t) \right], \quad (3.8) \]

in order to transform the \( H \) appearing in the first class constraint \( \phi \) into a time-independent one. Here, \( W \) is the time-independent frequency of the time-independent system as we will see in (3.11), and \( \rho \) is an auxiliary variables to be determined by the specific properties of the system, more precisely by \( \omega \) and \( W \). Such a canonical transformation turns the action (3.3) into

\[ S = \int \left\{ P \frac{dQ}{d\tau} + P_T \frac{dT}{d\tau} - \lambda \phi \right\} d\tau, \quad (3.9) \]
where, the first class constraint now reads
\[ \tilde{\phi} = \rho^2(T) [P_T + K] \approx 0, \] (3.10)
and the corresponding Hamiltonian \( K \) appearing in it is
\[ K = \frac{1}{2m} P^2 + \frac{1}{2} m W^2 Q^2. \] (3.11)
Moreover, the auxiliary equation used to fix \( \rho(t) \) becomes
\[ \ddot{\rho}(t) + \omega^2(t) \rho(t) = \frac{W^2}{\rho^3(t)}. \] (3.12)

Now one can polymer quantize the time-independent Hamiltonian (3.11) as usual, find its effective counterpart, and then apply the canonical transformations (3.5)-(3.8) to obtain its associated extended action similar to (3.3), and from there read-off the time-dependent Hamiltonian in the usual (non-extended) phase space. These are steps 3-5 in Fig. 1. These steps will be detailed in the following subsections. Before continuing, notice that in our paper the following correspondence holds
\[ A_{\sigma,k} \rightarrow Q \quad \text{and} \quad E_{\sigma,k} \rightarrow P. \] (3.13)

B. Polymer quantization and effective time-independent Hamiltonian

Let us consider a time-independent Hamiltonian of the form (3.11) where the Poisson algebra of the canonical variables is given by
\[ \{Q, P\} = 1, \] (3.14)
with other Poisson brackets being zero. Such a Poisson bracket allows us to construct the Weyl algebra \( \mathcal{W} \) whose generators \( \hat{W}(a, b) \) satisfy the Weyl algebra multiplication
\[ \hat{W}(a_1, b_1) \hat{W}(a_2, b_2) = e^{i \frac{\pi}{\hbar}(a_1 b_2 - b_1 a_2)} \hat{W}(a_1 + a_2, b_1 + b_2), \] (3.15)
where \( a_i \)'s and \( b_i \)'s (with \( i = 1, 2 \)) are parameters associated to the group member \( \hat{W} \). An example is the standard or Schrödinger representation, where the Weyl algebra \( \mathcal{W} \) generators can be written as the formal exponential
\[ \hat{W}(a, b) = e^{\frac{i}{\hbar}(a \hat{Q} - b \hat{P})}. \] (3.16)
If the infinitesimal generators \( \hat{Q}, \hat{P} \) are both well-defined on the Hilbert space, then the Weyl algebra can be essentially reduced to \( [\hat{Q}, \hat{P}] = 1 \) of the Schrödinger representation. However, we would like to perform a different quantization of our classical system, known as the “polymer representation”. As we will see, in this type of quantization motivated by loop quantum gravity, usually at least one of the infinitesimal generators \( \hat{Q} \) or \( \hat{P} \) are not well-defined on the Hilbert space due to the lack of weak continuity of the operators (see below). This makes the polymer representation and Hilbert space unitarily inequivalent to the standard Schrödinger representation, and hence is expected to yield distinctly different physical results.
Polymer representation usually comes in two “polarizations”: either \( \hat{Q} \) is not well-defined but \( \hat{P} \) is, or vice versa. In the polarization where \( \hat{Q} \) is not well-defined, the spectrum of its conjugate variable \( \hat{P} \) becomes discrete. This is basically because there is no \( \hat{Q} \) to generate infinitesimal transformations in \( \hat{P} \). The inverse of this statement are valid for the case where \( \hat{P} \) is not well-defined. We will consider both cases in what follows. However, note that in LQG, the connection is holonomized/polymerized and the triad is discretized. Now, in our notation \( Q \) corresponds to \( A_{\sigma,k} \) which itself corresponds to the metric perturbations, see (2.24a). Hence a polarization where \( \hat{P} \) or \( E_{\sigma,k} \) is polymerized which results in \( \hat{Q} \) or \( A_{\sigma,k} \) becoming discrete is more in line with LQG.

1. Case (i): Polymer \( \hat{P} \), Discrete \( \hat{Q} \)

In this case, the polymer Hilbert space is of the form
\[
\mathcal{H}_{\text{poly}}^{(p)} = L^2(\mathbb{R}, dP_{\text{Bohr}}) \ni \Psi(P) = \sum_{\{Q_i\}} \Psi_{Q_i} e^{\frac{i}{\hbar}Q_i P},
\]
(3.17)
where \( \{Q_i\} \), thought of as a graph, are discrete values corresponding to \( Q \) and the inner product is
\[
\langle \Psi(P) | \Phi(P) \rangle = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \Psi(P)^* \Phi(P) dP.
\]
(3.18)
The representation of the Weyl algebra generators on \( \mathcal{H}_{\text{poly}}^{(p)} \) is given by
\[
\hat{W}(a,b)\Psi(P) = e^{\frac{\hbar}{2}ab} e^{\frac{i}{\hbar}bP} \Psi(P + a).
\]
(3.19)
In this scheme, the operator \( \hat{W}(0, b) \) is not weakly continuous
\[
\langle e^{\frac{i}{\hbar}Q_j P} | \hat{W}(0, b) | e^{\frac{i}{\hbar}Q_j P} \rangle = \delta_{b,0},
\]
(3.20)
and consequently, it violates the Stone-von Neumann theorem requirements for this representation to be unitarily equivalent to the standard (Schrödinger representation of) quantum mechanics. As a result, we cannot obtain an infinitesimal generator for the operator \( \hat{W}(0, b) \), which, in the standard Schrödinger representation corresponds with \( \hat{P} \). For this reason, in polymer quantum mechanics, we are forced to introduced a combination of Weyl generators that mimics the term \( \hat{P}^2 \) in the quantum Hamiltonian. In order to introduce such a combination, the so called polymer scale is needed. This scale, denoted by \( \mu \) mimics the role of the Planck length in Loop Quantum Gravity. While this is a free parameter of the theory that should be fixed by experiment, it should be is small enough as to provide a good agreement with the experiments in the standard quantum mechanics (\( \mu/l_0 \sim 10^{-7} \), where \( l_0 \) is the proper length scale of the standard quantum harmonic oscillator). Therefore, this polymer scale admits an upper bound. One way to put a bound on the value of this scale is via the comparison of predicted theoretical effects of polymer quantum mechanics on the propagation of a GW and the experimental observations. This is part of the motivation for the present work.

Let us then consider a polymer scale \( \mu \) with a fixed, albeit unknown, value. Using \( \mu \), the standard combination of Weyl generators to provide the analog of \( \hat{P}^2 \) is given by
\[
\hat{P}_{\text{poly}}^2 = \frac{\hbar^2}{\mu^2} \left[ 2\hat{1} - \hat{W}(0, \mu) - \hat{W}(0, -\mu) \right].
\]
(3.21)
As a result, the action of this operator is
\[
\hat{P}_{\text{poly}}^2 \Psi(P) = \left[ \frac{2\hbar}{\mu} \sin \left( \frac{\mu P}{2\hbar} \right) \right]^2 \Psi(P). \quad (3.22)
\]

It can be checked [34] that in the semi-classical limit, this operator yields the following expression for the quadratic term \( P^2 \) in the Hamiltonian
\[
P_{\text{eff}}^2 = \left[ \frac{2\hbar}{\mu} \sin \left( \frac{\mu P}{2\hbar} \right) \right]^2, \quad (3.23)
\]
and using this result, the effective Hamiltonian for polymer quantized harmonic oscillator is of the form
\[
K_{\text{eff}}^{(q)} = \frac{1}{2m} \left[ \frac{2\hbar}{\nu} \sin \left( \frac{\nu Q}{2\hbar} \right) \right]^2 + \frac{mW^2}{2}Q^2. \quad (3.24)
\]

2. Case (ii): Polymer \( Q \), Discrete \( P \)

In this case we can follow on the same lines of the case (i). The Hilbert space is now given by
\[
\mathcal{H}_{\text{poly}}^{(q)} = L^2 \left( \mathbb{R}, dQ_{\text{Bohr}} \right) \ni \Psi(Q) = \sum_{\{P_j\}} \Psi_{P_j} e^{i\frac{\hbar}{\nu}P_jQ}, \quad (3.25)
\]
and the inner product is
\[
\langle \Psi(Q) | \Phi(Q) \rangle = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \Psi^*(Q)\Phi(Q) dQ. \quad (3.26)
\]

The representation for the Weyl generator in this Hilbert space is
\[
\hat{W}(a, b) \Psi(Q) = e^{-\frac{\hbar}{\nu}ab} e^{-\frac{\hbar}{\nu}aQ} \Psi(Q + b). \quad (3.27)
\]
Note that the polymer scale in this case has units of \( P \) and thus we will use a different notation, \( \nu \), for the polymer scale in this case. In this polarization the operator \( \hat{Q} \) is not well-defined and hence the term \( \hat{Q}^2 \) in the Hamiltonian is to be expressed using a combination of Weyl generators. The combination is similar to the one considered for the case (i)
\[
\hat{Q}_{\text{poly}}^2 = \frac{\hbar^2}{\nu^2} \left[ 2\hat{1} - \hat{W}(\nu, 0) - \hat{W}(-\nu, 0) \right], \quad (3.28)
\]
and it can be checked that the action of this operator is
\[
\hat{Q}_{\text{poly}}^2 \Psi(Q) = \left[ \frac{2\hbar}{\nu} \sin \left( \frac{\nu Q}{2\hbar} \right) \right]^2 \Psi(Q) \quad (3.29)
\]
Similarly, the effective correction to the potential of the harmonic oscillator is then given by
\[
Q_{\text{eff}}^2 = \left[ \frac{2\hbar}{\nu} \sin \left( \frac{\nu Q}{2\hbar} \right) \right]^2, \quad (3.30)
\]
and the effective Hamiltonian in this case turns out to be
\[
K_{\text{eff}}^{(q)} = \frac{1}{2m} P^2 + \frac{mW^2}{2} \left[ \frac{2\hbar}{\nu} \sin \left( \frac{\nu Q}{2\hbar} \right) \right]^2. \quad (3.31)
\]
C. Polymer time-dependent effective Hamiltonian

After obtaining $\tilde{\phi}$, Eq. (3.10) from step 2, we can fix it using $\tilde{\phi} = 0$ and $dT/d\tau = \tilde{T}$, to obtain the Hamiltonian (3.11). In step 3, this time-independent Hamiltonian is polymerized as discussed in section IIIIB and from that an effective Hamiltonian is derived either in the form of (3.24) or (3.31), depending on the polarization. This time-independent effective polymer Hamiltonian is then replaced back into one of the following extended phase space actions

$$S^{(p)} = \int \left\{ P \frac{dQ}{d\tau} + P_T \frac{dT}{d\tau} - \tilde{\lambda}(\tilde{\phi}(P)) \right\} d\tau, \quad (3.32)$$

$$S^{(Q)} = \int \left\{ P \frac{dQ}{d\tau} + P_T \frac{dT}{d\tau} - \tilde{\lambda}(\tilde{\phi}(Q)) \right\} d\tau. \quad (3.33)$$

based on the polarization used. In the next step, 4, we perform the canonical transformations (3.5)-(3.8) on the above action, and particularly $\tilde{\phi}$, to obtain $\tilde{\phi}(p)$ or $\tilde{\phi}(q)$ as a function of extended phase space $q, t, p, p_t$. It is worth noting that the canonical transformation introduces a boundary term, which at classical level does not affect the Lagrangian equations of motion.

Finally, in step 5, we solve the constraint set $\tilde{\phi} \approx 0$ to obtain the time dependent Hamiltonian in usual phase space of $(q, p)$. This one obtains the effective time-dependent polymer Hamiltonians

$$H_{\text{eff}}^{(p)} = \frac{2\hbar^2}{m \mu \rho^2} \sin^2 \left( \frac{\mu (\rho p - m \rho q)}{2\hbar} \right) + \frac{\hat{p}q \rho + \frac{mq^2}{2} \left[ \omega^2 - \frac{p^2}{\rho^2} \right]}{\rho}, \quad (3.34)$$

$$H_{\text{eff}}^{(q)} = \frac{p^2}{2m} + \frac{2m\hbar^2}{\nu^2} (\rho \tilde{\rho} + \omega^2 \rho^2) \sin^2 \left( \frac{\nu q}{2\hbar \rho} \right) - \frac{mq^2 \rho}{2\rho}. \quad (3.35)$$

The effective equations of motion corresponding to $H_{\text{eff}}^{(p)}$ are

$$\frac{dq}{dt} = \{ q, H_{\text{eff}}^{(p)} \} = \frac{1}{m \rho \mu} \sin \left( \frac{\mu (\rho p - m \rho q)}{h} \right) + \frac{\hat{p}(t)}{\rho(t)} q, \quad (3.36)$$

$$\frac{dp}{dt} = \{ p, H_{\text{eff}}^{(p)} \} = \frac{\hat{p} \hbar}{\rho^2 \mu} \sin \left( \frac{\mu (\rho p - m \rho q)}{h} \right) + \frac{mq^2 \hat{p}}{\rho^2} - m\omega^2 q - \frac{\hat{p}(t)}{\rho(t)} p, \quad (3.37)$$

and the ones corresponding to $H_{\text{eff}}^{(q)}$ are

$$\frac{dq}{dt} = \{ q, H_{\text{eff}}^{(q)} \} = \frac{p}{m}, \quad (3.38)$$

$$\frac{dp}{dt} = \{ p, H_{\text{eff}}^{(q)} \} = -m \hbar (\tilde{p} + \rho \omega^2) \frac{\sin \left( \frac{\nu q}{h \rho} \right)}{\nu} + \frac{mq \tilde{p}}{\rho}. \quad (3.39)$$

IV. EFFECTIVE EQUATIONS OF MOTION AND PHENOMENOLOGY

The correspondence between the generic analysis of previous section and our specific model is expressed as

$$q \rightarrow A_{\sigma,k}, \quad p \rightarrow E_{\sigma,k}, \quad W^2 = |k|^2, \quad \omega^2 = |k|^2 \alpha^4, \quad m = 1. \quad (4.1, 4.2)$$
Using these, we study two polarization of our mode in what follows.

A. Polymer $\mathcal{E}$, Discrete $A$

By applying (4.1)-(4.2) to (3.34) we obtain the effective polymer Hamiltonian with polymer $E_{\sigma,k}$ as

$$H_{\text{eff}}^{(E)} = \sum_{\lambda=+,-} \sum_{k \in \mathcal{L}} \left\{ \frac{2}{\mu^2 \rho^2} \sin^2 \left( \frac{\mu (\rho E_{\sigma,k} - \dot{\rho} A_{\sigma,k})}{2} \right) + \frac{\dot{\rho} A_{\sigma,k} E_{\sigma,k}}{\rho} + \frac{A_{\sigma,k}^2}{2} \left[ \omega^2 - \frac{\dot{\rho}^2}{\rho^2} \right] \right\},$$

where we have set $\hbar = 1$. The corresponding equations of motion read

$$\frac{dA_{\sigma,k}}{dt} = \frac{1}{\rho} \sin (\mu (\rho E_{\sigma,k} - \dot{\rho} A_{\sigma,k})) + \frac{\dot{\rho}}{\rho} A_{\sigma,k},$$

$$\frac{dE_{\sigma,k}}{dt} = \frac{\dot{\rho}}{\rho^2} \sin (\mu (\rho E_{\sigma,k} - \dot{\rho} A_{\sigma,k})) + \frac{\dot{\rho}^2}{\mu} A_{\sigma,k} - \omega^2 A_{\sigma,k} - \frac{\dot{\rho}^2}{\rho} E_{\sigma,k}.$$ (4.3, 4.4)

These equations are nonlinear in both $A_{\sigma,k}$ and $E_{\sigma,k}$, and their $\mu \to 0$ limit matches the classical equations of motion as expected.

B. Polymer $A$, Discrete $\mathcal{E}$

In this case and by using (4.1)-(4.2) to (3.35) we obtain an effective polymer Hamiltonian with polymer $A_{\sigma,k}$ as

$$H_{\text{eff}}^{(A)} = \sum_{\lambda=+,-} \sum_{k \in \mathcal{L}} \left\{ \frac{E_{\sigma,k}^2}{2} + \frac{2}{\nu^2} (\rho \ddot{\rho} + \omega^2 \rho^2) \sin^2 \left( \frac{\nu A_{\sigma,k}}{2 \rho} \right) - \frac{A_{\sigma,k}^2}{2 \rho} \right\}.$$ (4.5)

The equations of motion in this case are

$$\frac{dA_{\sigma,k}}{dt} = E_{\sigma,k},$$

$$\frac{dE_{\sigma,k}(t)}{dt} = -\ddot{\rho} + \frac{\rho \omega^2}{\nu} \sin \left( \frac{\nu A_{\sigma,k}}{\rho} \right) + \frac{\dot{\rho}}{\rho} A_{\sigma,k},$$ (4.6, 4.7)

which are now nonlinear only in both $A_{\sigma,k}$ while their $\nu \to 0$ limit also matches the classical equations of motion.

C. Perturbative and nonperturbative numerical solutions

We can solve Eqs. (4.4)-(4.5) and (4.7)-(4.8) for specific field-space configurations, both perturbatively, and numerically in order to compute exact solutions that can be compared to perturbative calculations. We will begin by looking solutions with a time-independent background, for which $\rho = 1$ and $\dot{\rho} = \ddot{\rho} = 0$. After gaining some insight in this setting, we
examine the solutions for $\rho$ required to study behavior in a time-dependent background. We can obtain solutions in this case by applying the transformation given in Eqs. (3.5)-(3.7) to the time-independent solution, or by directly solving the time-dependent equations of motion.

The essential parameters we would like to vary include the mode amplitude at some initial time $t_I$, $A_I \equiv A(t = t_I)$, momentum $E_I \equiv E(t = t_I)$, frequency $\omega$, and $\mu$ (or $\nu$). We can reduce this parameter space by considering the physical behavior of the system, and making note of several re-scalings the equations of motion are invariant under. We first note that the equations of motion are highly analogous to the case of a physical pendulum, and will similarly result in periodic behavior, albeit with a different period. We can therefore seek solutions with an initial amplitude $A_I = 0$ without loss of generality. We then note that the equations of motion and auxiliary equation are invariant under a re-scaling of the frequency,

$$t \rightarrow kt \quad \mu \rightarrow k\mu \quad \nu \rightarrow \nu \quad A \rightarrow A \quad E \rightarrow E/k,$$  

and so it suffices to obtain solutions for a single frequency.

The equations of motion are also invariant under a re-scaling of the initial momentum $E_I$,

$$t \rightarrow t \quad \mu \rightarrow E_I \mu \quad \nu \rightarrow E_I \nu \quad A \rightarrow A/E_I \quad E \rightarrow E/E_I,$$  

and so we can take the initial momentum to be $E_I = 1$ in numerical solutions, although we will leave this factor in later analytic expressions. The parameters $\mu$ and $\nu$ then determine the “smallness” of oscillations. We note that the equations are invariant under a similar re-scaling of $\mu$ and $\nu$, and so we could equivalently choose to vary $E_I$; the important thing is to vary one of these quantities, which will determine how “small” the oscillations are. In either case, we have reduced the parameter space to a simple one in which we can vary only $\mu$ and $\nu$.

For the time-independent EOMs, the solution for each wavevector is in fact identical to that of an ordinary physical pendulum for polymer $A$. While solutions to this equation, and the corresponding one for polymer $E$ are periodic, due to the nonlinear structure both higher harmonics may be excited and a frequency shift develops. Both of these corrections are of order $O(\nu^2)$ (or $\mu^2$). The frequency shift is not readily found using a standard perturbative approach, where the solution appears to contain a secular, growing term. However, this term can be eliminated by also expanding in a small perturbation of the frequency– this is the essence of the Poincaré–Lindstedt method, which we employ here to obtain an approximate analytic solution.

For polymer $A$, again fixing the phase so that $A_I = 0$, the solution we obtain is given by

$$A(t) \simeq E_I \sin \left[ (1 - (E_I \nu)^2/16)kt \right] - \frac{E_I^3 \nu^2}{48} \sin^3 \left[ (1 - (E_I \nu)^2/16)kt \right],$$  

while for polymer $E$ the solution is

$$A(t) \simeq E_I \sin \left[ (1 - (E_I k\mu)^2/16)kt \right] - \frac{E_I^3 k^2 \mu^2}{16} \sin^2 \left[ (1 - (E_I k\mu)^2/16)kt \right] \cos \left[ (1 - (E_I k\mu)^2/16)kt \right].$$  

(4.12)
Figure 2. Time evolution of $A$ with $A_I = 0$, $E_I = 1$, and $k = 1$ for two different choices of $\mu = \nu$ in the case of a time-independent background spacetime, i.e., $\rho$ = constant. The solutions are shown at early times, and the axis is broken to show the behavior at a much later time. Solutions can be mapped to different choices of $k$ and $E_I$ using the rescalings in Eq. (4.9) and Eq. (4.10), while changing $A_I = 0$ can be viewed as a phase shift.

These solutions can be seen to contain a frequency shift of order $\nu^2$ or $\mu^2$, and a cubic correction term. The frequency shifts in both cases are nearly identical; this is because while the role of conjugate variables has been interchanged in the solutions, the form has remained unchanged. The second, cubic term can also be re-written, and thought of, as an introduction of higher harmonics using angle identities. In observations, the frequency shift may be rather more important to account for than the excited harmonics, as the frequency shift can manifest as a phase shift that has considerable time to develop as a wave traverses cosmological distances. In Fig. 2 we demonstrate this, comparing the perturbative solution to the exact and classical ones for the time-independent case.

We can also analyze the above perturbative solutions and obtain some insight into the speed of propagation of the waves. For that, we note that the dominant contributions to equations (4.20) and (4.21) can be written as

$$A(t) \simeq E_I \sin \left[ \left( 1 - \left( \frac{E_I \nu}{4} \right)^2 \right) kt \right], \quad (4.13)$$

$$A(t) \simeq E_I \sin \left[ \left( 1 - \left( \frac{E_I k \mu}{4} \right)^2 \right) kt \right]. \quad (4.14)$$

Comparing with the classical solution where we identify $ka^2 = \omega_c$, with $\omega_c$ being the classical
angular speed, we notice that up to first order the polymer angular speeds are

\[ \omega^{(A)} \simeq \omega_c \left[ 1 - \left( \frac{E I \nu}{4} \right)^2 \right], \quad (4.15) \]

\[ \omega^{(E)} \simeq \omega_c \left[ 1 - k^2 \left( \frac{E I \mu}{4} \right)^2 \right]. \quad (4.16) \]

Although these are perturbative and approximate and even though we have neglected higher harmonics in (4.20) and (4.21), the above two equations reveal a curious phenomenon. Noting that \( \omega_c = k a^2 \) and the group velocity being

\[ v = \frac{d \omega_{\text{poly}}}{d (k a^2)} \]

with \( \omega_{\text{poly}} \) being either \( \omega^{(A)} \) or \( \omega^{(E)} \), we obtain

\[ v^{(A)} \simeq 1 - \left( \frac{E I \nu}{4} \right)^2, \quad (4.18) \]

\[ v^{(E)} \simeq 1 - k^2 \left( \frac{E I \mu}{4} \right)^2. \quad (4.19) \]

where \( v^{(A)} \) and \( v^{(E)} \) are velocities of the effective waves in case of polymer \( A \) and polymer \( E \) respectively. One can see from (4.18) that in polymer \( A \) case, the group velocity of the waves are slower than the speed of light by a factor \( \left( \frac{E I \nu}{4} \right)^2 \) that does not depend on the frequency of the waves, but is dependent on the initial momentum \( E I \) of the waves and the polymer parameter in this case \( \nu \). Hence all the waves in this case move slower than the speed of light and this effect is amplified if the wave has a larger initial momentum \( E I \). For polymer \( E \) case in which we are more interested, we can see from (4.19) that such a lower-than-the-speed-of-light propagation also happens for the waves, and it also depends on the initial momentum \( E I \) of the waves and the polymer parameter \( \mu \) due to the factor \( k^2 \left( \frac{E I \mu}{4} \right)^2 \).

However, in this case, there is an important difference: the deviation from the speed of light also depends on the modes \( k \). Hence, waves with larger \( k \) (i.e., larger energies), have a lower speed compare to the ones with smaller \( k \) and are more affected by the quantum structure of spacetime. Also, notice that this case leads to the violation of Lorentz symmetry as can be seen by squaring both sides if (4.16). Of course, due to the sheer smallness of the expected values of \( \mu \) and \( \nu \), and the appearance of their squares in the above expressions, these effects are very small, but a highly energetic phenomenon with a large \( E I \) may help amplifying it to an extent that future observatories can detect it.

For the case of a time-dependent background, we can obtain a solution one of two ways: directly integrating the EOMs, or using the canonical transformation in Eqs. (3.5)-(3.8). In either case, we will need to obtain a solution for \( \rho \) by solving Eq. (3.12). In general, this choice determines whether the mode amplitude will be purely decaying, or will contain oscillatory behavior. Here we will seek purely growing solutions for \( \rho \), choosing initial conditions such that oscillatory behavior is minimized; in our case simply choosing \( \rho = 1 \) and \( \dot{\rho} = 0 \) is sufficient. Choosing a different initial amplitude for \( \rho \) is in any case equivalent to re-scaling of the scale factor \( a \), polymer scale, momentum, and time coordinate.
Figure 3. Evolution of the auxiliary variable $\rho(t)$. The full numerical non-oscillatory solution is shown in solid red, an approximate power law solution is shown in dashed blue, and a solution with initial conditions that result in oscillatory behavior is shown in light grey.

For the case of a time-dependent background, the solutions can be obtained by transforming the ones with the time-independent background,

\begin{equation}
A(t) \simeq \mathcal{E}_I \rho \sin \left[ (1 - (\mathcal{E}_I \nu)^2/16)kT(t) \right] - \frac{\mathcal{E}_I^3 \nu^2}{48} \rho \sin^3 \left[ (1 - (\mathcal{E}_I \nu)^2/16)kT(t) \right], \tag{4.20}
\end{equation}

\begin{equation}
A(t) \simeq \mathcal{E}_I \rho \sin \left[ (1 - (\mathcal{E}_I k \mu)^2/16)kT(t) \right] - \frac{\mathcal{E}_I^3 k^2 \mu^2}{16} \rho \sin^2 \left[ (1 - (\mathcal{E}_I k \mu)^2/16)kT(t) \right] \cos \left[ (1 - (\mathcal{E}_I k \mu)^2/16)kT(t) \right], \tag{4.21}
\end{equation}

where

\begin{equation}
T(t) = \int_{t_i}^t dt' \frac{1}{\rho(t')}^2 \tag{4.22}
\end{equation}

For GWs emitted at a time much greater than the characteristic wave timescale, i.e. $t_I \gg k^{-1}$, and for non-oscillatory solutions, the second-derivative term is small, and solutions to the auxiliary equations are well-approximated by a simple power law, $\rho = 1/a$. In Fig. 3 we show the behavior of $\rho$ for several sets of initial conditions, and for a universe with a cosmological constant with $w = -1$, $a \propto t^{1/3}$, and $t_I = 10^3$ (in units of $k^{-1}$). In subsequent plots we will use initial conditions which do not result in oscillatory behavior.

From the canonical transformation, Eqs. (3.5)-(3.7), or rather their inverse, we see that the time-dependent waveform amplitude will pick up an overall factor of $\rho$ relative to the time-independent one, the time coordinate will be altered, and the momentum will be similar rescaled but will also pick up an additional factor proportional to the wave amplitude. Due to the monotonically decreasing nature of $\rho$, and the smallness of its derivative, this additional factor will be a strongly sub-dominant contribution. In Fig. 4 we show the final solution for the field $A(t)$ for this time-dependent background. Somewhat counter-intuitively, the frequency is seen to increase at later times; more commonly the frequency is considered to decrease (redshift) with cosmological expansion. This is due to the choice of harmonic slicing we have made, with $N = a^3$ instead of a more commonly used $N = 1$ (synchronous) or $N = a$ (comoving) time coordinate.
Figure 4. The time evolution of $\mathcal{A}$ (as in Fig. 2) for two different choices of $\mu = \nu$, for the case of a time-dependent background, i.e., $\rho(t)$ as described in the text. The axis is broken to show the behavior at a later time.

V. DISCUSSION AND CONCLUSION

In this work we have studied a certain effective form of GWs, considered as quantized perturbations propagating over a classical FLRW spacetime, in order to derive observational signatures to be compared with the results of experiments held by GW observatories. We have considered the Hamiltonian of classical gravitational perturbations, a time-dependent Hamiltonian, and have applied the techniques of polymer quantization to it. Polymer quantization is a nonperturbative method of quantization, inspired by LQG, in which some of the operators are regularized and written in a certain exponential form. Since such a quantization is unitarily inequivalent to the standard Schrödinger representation, one expects to obtain physically distinct result compared to it. Polymer quantization usually comes in two polarizations. In one polarization the configuration variables are regularized (or polymerized) and their momenta are discretized. In the other polarization momenta are polymerized and the configuration variables are discretized. We consider both of such polarizations for the polymer quantization of the aforementioned time-dependent Hamiltonian: one in which the perturbations are polymerized and their momenta are discrete, and another one where the momenta of the perturbations are polymerized and hence the perturbations themselves are descretized. Of course the latter case is of more interest to us.

Since the classical Hamiltonian we obtain is time-dependent and to overcome the challenge of polymer quantizing such a time-dependent system, we use a method that is used to deal with the same issue in time-dependent harmonic oscillators [35]. We first write such
a Hamiltonian in a time-independent format in the extended phase space by applying a
certain canonical transformation, polymer quantize it, apply the inverse of such a canonical
transformation to make it time-dependent again, and finally transform it back into the
standard phase space. This way, we obtained a time-dependent polymer quantized effective
non-perturbative Hamiltonian. We then derived and studied the corresponding effective fully
nonperturbative equations of motion numerically. We also derived a perturbative analytical
expression for the solutions and analyzed them to obtain further insight into the behavior
of such waves.

As a result, we found that

i) The form of the waves are modified. More precisely, there is a phase shift with respect
to the classical case. Furthermore, small amplitude harmonics are excited.

ii) The speed of the waves turns out to be smaller than the speed of light. In a perturba-
tive analysis we see that for the time-independent background (the same qualitative
behavior is seen numerically for the time-dependent case):

(a) In the case where the gravitational perturbations are polymerized and their con-
jugate momenta are discretized, the wave speeds are \( v^{(A)}_\nu \simeq 1 - (\frac{E^I_\nu}{4})^2 \). Hence the
factor, \( (\frac{E^I_\nu}{4})^2 \), by which the speed of waves differ from the speed of light depends
on the polymer scale \( \nu \) and the initial wave momentum \( E^I_\nu \), and this is the same
for all the waves regardless of their wave vectors or frequencies. Of course this
factor is very small due to expected small value of polymer parameter, in this
case, \( \nu \).

(b) In the case where the momentum of the perturbations are polymerized and the
gravitational perturbations themselves discretized, which is the case of more in-
terest to us, the wave speeds are \( v^{(E)}_\mu \simeq 1 - k^2 (\frac{E^I_\mu}{4})^2 \). Hence in this case, the
factor, \( k^2 (\frac{E^I_\mu}{4})^2 \), by which the wave speed is smaller than the speed of light not
only depends on the polymer scale \( \mu \) and the initial momentum of the perturba-
tions \( E^I_\mu \), but now it also depends on the wave vector \( k \) or equivalently frequency of
the waves. Thus the higher energy waves show more deviation from the classical
behavior compared to the low energy waves.

iii) The modifications to the classical behavior due to quantum effects are amplified by
the distance/time the waves travel: the corrections to the frequency shift become of
order unity when \( E^I_\mu \mu^2 k^3 D_s \) or \( E^I_\nu \nu^2 k D_s \) are of order unity for a distance \( D_s \) traveled.

We plan to obtain a more robust constraint on \( \mu \) and \( \nu \) in future works, where we will
apply these results to initial data known from real GWs, and compare the numerical results
of applying our method to waves with such initial values with the observed results of GW
observatories, particularly those of LIGO. Furthermore, we will proceed to apply our method
to the case where both the background spacetime and the perturbations are effective.

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Appendix A: Friedmann equations in harmonic slicing

In a majority of cosmological literature, the slicing condition used either coincides with a synchronous time and corresponding lapse $N = 1$, or conformal time with $N = a$. In this work we choose harmonic slicing with $N = a^3$, which results in a modified behavior for the evolution of the scale factor. The coupled Einstein-fluid equations for a homogeneous, isotropic universe in a 3+1 language are given by

$$\partial_t \ln \gamma = -2NK$$
$$\partial_t K = \frac{N}{3}K^2 + 4\pi N(\rho_m + 3P)$$
$$\partial_t (a^3 \rho_m) = 0.$$  \hspace{1cm} \text{(A1)} \hspace{1cm} \text{(A2)} \hspace{1cm} \text{(A3)}

for spatial metric determinant $\gamma = a^6$, trace of the extrinsic curvature $K$, and ADM density and pressure $\rho_m$ and $P$. Assuming an equation of state $P = w\rho_m$ to close the system, and choosing the lapse $N = a^3$, this system has solutions of the form

$$a(t) = \left(\frac{t}{t_I}\right)^{\frac{1}{3} \frac{1}{1-w}}.$$  \hspace{1cm} \text{(A4)}

For equations of state $w = -1, 0, 1/3$ (cosmological constant, dust, radiation), the scale factor shows power-law growth. We are eventually interested in solving Eq. 3.12 for a given choice of $a$; in general, eg. in a universe with multiple components, we will need to solve for $\rho$ numerically.

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