Hellinger distance in approximating Lévy driven SDEs and application to asymptotic equivalence of statistical experiments

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Abstract. In this paper, we get some convergence rates in total variation distance in approximating discretized paths of Lévy driven stochastic differential equations, assuming that the driving process is locally stable. The particular case of the Euler approximation is studied. Our results are based on sharp local estimates in Hellinger distance obtained using Malliavin calculus for jump processes. As an immediate consequence of the pathwise convergence in total variation, we deduce the asymptotic equivalence in Le Cam sense of the experiment based on high-frequency observations of the SDE and its approximation.

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1 Introduction

On a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we consider the process \((X_t)_{t \in [0,1]}\) solution of the stochastic equation

\[
X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s- )dL_s,
\]

where \(L\) is a pure jump locally stable Lévy process. Pure jump driven stochastic equations are widely used to model dynamic phenomena appearing in many fields such as insurance and finance and approximation of such processes attracts...
many challenging problems. A large part of the literature is devoted to the study of weak convergence at terminal date \( E_g(X_T) - E_g(\bar{X}_T) \) (we assume in this paper that \( T = 1 \)), where \( \bar{X} \) is a numerical scheme. Let us mention some results obtained in approximating \( \text{Lévy driven stochastic equations} \) by the simplest and widely used Euler scheme. The weak order 1 for equations with smooth coefficients and for smooth functions \( g \) is obtained in Protter and Talay \([18]\) and some extensions to Hölder coefficients are studied in Mikulevičius and Zhang \([16]\) and Mikulevičius \([14]\). Expansions of the density are considered in Konakov and Menozzi \([10]\). Turning to pathwise approximation, convergence rates in law for the error process are obtained by Jacod \([6]\) and some strong convergence results have been established in Mikulevičius and Xu \([15]\). To overcome the difficulties related to the simulation of the small jumps of \( L \), more sophisticated schemes have been considered. We quote among others the works of Rubenthaler \([19]\) and Kohatsu-Higa and Tankov \([8]\).

In this paper, we consider a different control of the accuracy of approximation and we focus on high-frequency pathwise approximation of \((1.1)\) in total variation distance. Actually, convergence in total variation implies an asymptotic equivalence in Le Cam sense of corresponding experiments and permits to derive asymptotic properties (such as efficiency) by means of the simplest experiment. We mention the works by Milstein and Nussbaum \([17]\), Genon-Catalot and Larédo \([5]\), Mariucci \([13]\) for the study of asymptotic equivalence of diffusion processes and Euler approximations, in a non parametric setting.

We now precise the schemes considered in the present work. To deal with small values of the Blumenthal-Getoor index of \( L \) (characterizing the jump activity), we not only consider the Euler approximation of \((1.1)\) but also a scheme with better drift approximation. Introducing the time discretization \((t_i)_{0 \leq i \leq n}\) with \( t_i = i/n \), we approximate the process \((X_t)_{t \in [0,1]}\) by \((\bar{X}_t)_{t \in [0,1]}\) defined by \( \bar{X}_0 = x_0 \) and for \( t \in [t_{i-1}, t_i], \ 1 \leq i \leq n \)

\[
\bar{X}_t = \xi_{t-t_{i-1}}(\bar{X}_{t_{i-1}}) + a(\bar{X}_{t_{i-1}})(L_t - L_{t_{i-1}}),
\]

(1.2)

where \((\xi_t(x))_{t \geq 0}\) solves the ordinary equation

\[
\xi_t(x) = x + \int_0^t b(\xi_s(x))ds.
\]

(1.3)

Approximating \( \xi \) by

\[
\tilde{\xi}_t(x) = x + b(x)t,
\]

we obtain the Euler approximation \((\tilde{X}_t)_{t \in [0,1]}\) defined by \( \tilde{X}_0 = x_0 \) and for \( t \in [t_{i-1}, t_i], \ 1 \leq i \leq n \)

\[
\tilde{X}_t = \tilde{X}_{t_{i-1}} + b(\tilde{X}_{t_{i-1}})(t - t_{i-1}) + a(\tilde{X}_{t_{i-1}})(L_t - L_{t_{i-1}}).
\]

(1.4)

Our aim is to study the rate of convergence of \((\bar{X}_t)_{0 \leq i \leq n}\) or \((\tilde{X}_t)_{0 \leq i \leq n}\) to \((X_t)_{0 \leq i \leq n}\) in total variation distance. Let us present briefly our results. For the scheme \((1.2)\), we obtain some rates of convergence, depending on the jump activity index \( \alpha \in (0, 2) \). Essentially the rate of convergence is of order \( 1/n^{1/\alpha - 1/2} \).
if $\alpha > 1$ and $1/n^{1/2-\varepsilon}$ if $\alpha \leq 1$. If the scale coefficient $a$ is constant, we obtain in some cases the rate $1/\sqrt{n}$ for any value of $\alpha$. For the Euler scheme, the results are similar if $\alpha \geq 1$ but are working less well if $\alpha < 1$, and we have no rate at all if $\alpha \leq 2/3$. This means that for small value of $\alpha$ an approximation of (1.3) with higher order than the Euler one is required. To get these results, our methodology consists in estimating the local Hellinger distance at time $1/n$ and to conclude by tensorisation. Using Malliavin calculus for jump processes, we can bound the Hellinger distance by the $L^2$-norm of a Malliavin weight. The difficult part is next to identify a sharp rate of convergence for this weight. This is done by remarking some judicious compensations between the rescaled jumps.

The paper is organized as follows. Section 2 introduces the notation and some preliminary results. Bounds for the local Hellinger distance are given in Section 3. The main results are presented in Section 4. Section 5 contains the technical part of the paper involving Malliavin calculus and the proof of the local estimates of Section 3.

## 2 Preliminary results and notation

We first recall some properties of total variation and Hellinger distance (see Strasser [20]). Let $P$ and $Q$ be two probability measures on $(\Omega, A)$ dominated by $\nu$, the total variation distance between $P$ and $Q$ on $(\Omega, A)$ is defined by

$$d_{TV}(P, Q) = \sup_{A \in A} |P(A) - Q(A)| = \frac{1}{2} \int |\frac{dP}{d\nu} - \frac{dQ}{d\nu}| d\nu.$$

The total variation distance can be estimated by using the Hellinger distance $H(P, Q)$ defined by

$$H^2(P, Q) = \left( \int \left( \sqrt{\frac{dP}{d\nu}} - \sqrt{\frac{dQ}{d\nu}} \right)^2 d\nu \right) = 2 \left( 1 - \int \sqrt{\frac{dP}{d\nu}} \sqrt{\frac{dQ}{d\nu}} d\nu \right).$$

and we have

$$\frac{1}{2} H^2(P, Q) \leq d_{TV}(P, Q) \leq H(P, Q).$$

If $P$, respectively $Q$, is the distribution of a random variable $X$, respectively $Y$, we also use the notation $d_{TV}(X, Y)$ for $d_{TV}(P, Q)$ and $H(X, Y)$ for $H(P, Q)$. The Hellinger distance has interesting properties, in particular for product measures

$$H^2(\otimes_{i=1}^nP, \otimes_{i=1}^nQ_i) \leq \sum_{i=1}^n H^2(P, Q_i).$$

We extend this property in the next proposition to the distribution of Markov chains.

Let $(X_i)_{i \geq 0}$ and $(Y_i)_{i \geq 0}$ be two homogenous Markov chains on $\mathbb{R}$ with transition density $p$ and $q$ with respect to the Lebesgue measure. We define the
conditional Hellinger distance between $X_1$ and $Y_1$ given $X_0 = Y_0 = x$ by

$$H^2_x(p, q) = \int \left( \sqrt{p(x, y)} - \sqrt{q(x, y)} \right)^2 dy.$$ 

We denote by $P^n$, respectively $Q^n$, the distribution of $(X_i)_{1 \leq i \leq n}$ given $X_0 = x_0$, respectively $(Y_i)_{1 \leq i \leq n}$ given $Y_0 = x_0$ (the two Markov chains have the same initial value), then we can bound $H(P^n, Q^n)$ with $H_x(p, q)$.

**Proposition 2.1.** With the previous notation, we have

$$H^2(P^n, Q^n) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} H^2_{X_{i-1}}(p, q) + \mathbb{E} H^2_{Y_{i-1}}(p, q) \leq n \sup_{x \in \mathbb{R}} H^2_x(p, q).$$

**Proof.** We have from (2.1)

$$H^2(P^n, Q^n) = 2 \left( 1 - \int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} p(x_{i-1}, x_i) \prod_{i=1}^{n} q(x_{i-1}, x_i) \right)^{1/2} dx_1 \ldots dx_n \right).$$

But

$$\int_{\mathbb{R}} \sqrt{p(x_{n-1}, x_n)} q(x_{n-1}, x_n) dx_n = 1 - \frac{1}{2} H^2_{x_{n-1}}(p, q),$$

consequently

$$H^2(P^n, Q^n) = H^2(P^{n-1}, Q^{n-1}) + \int_{\mathbb{R}^{n-1}} \left( \prod_{i=1}^{n-1} p(x_{i-1}, x_i) \prod_{i=1}^{n-1} q(x_{i-1}, x_i) \right)^{1/2} H^2_{x_{n-1}}(p, q) dx_1 \ldots dx_{n-1},$$

and from the inequality $\sqrt{ab} \leq \frac{1}{2}(a + b)$, this gives

$$H^2(P^n, Q^n) \leq H^2(P^{n-1}, Q^{n-1}) + \frac{1}{2} (\mathbb{E} H^2_{X_{n-1}}(p, q) + \mathbb{E} H^2_{Y_{n-1}}(p, q)).$$

We deduce then the first inequality in Proposition 2.1 by induction, the second inequality is immediate. \qed

The result of Proposition 2.1 motivates the study of the Hellinger distance between $X_{1/n}$ and $\overline{X}_{1/n}$ given $X_0 = \overline{X}_0 = x$ (respectively $\overline{X}_{1/n}$) to bound $d_{TV}(X_{1/n}, \overline{X}_{1/n}) (\text{respectively } d_{TV}(X_{1/n}, \overline{X}_{1/n})).$ Before stating our main results, let us explain briefly our approach.

We will use the Malliavin calculus developed in [1] and [2] and follow the methodology proposed in [3] with some modifications. This requires some regularity assumptions on the coefficients $a$ and $b$. To simplify the presentation, we assume that $a$ and $b$ are real functions satisfying the following regularity conditions. In the sequel, we use the notation $||f||_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ for $f$ bounded. We make the following assumptions.
HR: the functions $a$ and $b$ are $C^3$ with bounded derivatives and $a$ is lower bounded
\[ \forall x \in \mathbb{R}, \quad 0 < \underline{a} \leq a(x). \]

The Lévy process $L$ admits the decomposition
\[ L_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} z \hat{\mu}(ds, dz), \]
with $\hat{\mu} = \mu - \overline{\mu}$, where $\mu$ is a Poisson random measure and $\overline{\mu}(dt, dz) = dt F(dz)$ its compensator. We assume that $L$ satisfies assumption A (i) and either (ii) or (iii).

A: (i) $(L_t)_{t \geq 0}$ is a Lévy process with triplet $(0, 0, F)$ with
\[ F(dz) = \frac{g(z)}{|z|^\alpha + 1} 1_{\mathbb{R}\setminus\{0\}}(z)dz, \quad \alpha \in (0, 2), \]
where $g: \mathbb{R} \mapsto \mathbb{R}$ is a continuous symmetric non negative bounded function with $g(0) = c_0 > 0$.

(ii) We assume that $g$ is differentiable on $\{|z| > 0\}$ and $g'/g$ is bounded on $\{|z| > 0\}$.

(iii) We assume that $g$ is supported on $\{|z| \leq \frac{1}{2||a'||_\infty}\}$ and differentiable with $g'$ bounded on $\{0 < |z| \leq \frac{1}{2||a'||_\infty}\}$ and that
\[ \int_{\mathbb{R}} \left| \frac{g'(z)}{g(z)} \right|^p g(z)dz < \infty, \quad \forall p \geq 1. \]

In the sequel we use the notation
- A0: A (i) and (ii),
- A1: A (i) and (iii).

Let us make some comments on these assumptions. We remark that A0 is satisfied by a large class of processes, in particular $\alpha$-stable processes ($g = c_0$) or tempered stable processes ($g(z) = c_0 e^{-\lambda|z|}$, $\lambda > 0$). On the other hand, assumption A1 is very restrictive. Actually, the restriction on the support of $g$ implies the non-degeneracy assumption (Assumption (SC) p.14 in [1]) that can be written in our framework
\[ \forall x, z, \quad |1 + a'(x)z| \geq \xi > 0. \quad (SC) \]

This condition permits to apply Theorem 5.2 in Section 5 (integrability of the inverse of $U_{r,n;K}^{K,n,r}$). Assumption A1 is required to deal with a non constant scale function $a$ ($||a'||_\infty > 0$). Conversely, if $a$ is constant, then the non-degeneracy assumption (SC) is satisfied and we get our results assuming the weaker assumption A0.

Since Malliavin calculus requires integrability properties for the driving process $L$, to deal with assumption A0, we introduce a truncation function in order to suppress the jumps larger than a constant $K$ (the truncation is useless under
In a second step we will make \( K \) tend to infinity. Note that contrarily to [3], a localization around zero is not sufficient. So we consider the truncated Lévy process \((L^K_t)_{t \geq 0}\) with Lévy measure \( F^K \) defined by

\[
F^K(dz) = \tau_K(z)F(dz),
\]

where \( F \) is the Lévy measure of \( L \) and \( \tau_K \) is a smooth truncation function such that \( \tau_K \) is supported on \( \{|x| \leq K\} \) and equal to 1 on \( \{|x| \leq K/2\} \).

We associate to \( L^K \) the truncated process that solves

\[
X^K_t = x_0 + \int_0^t b(X^K_s)ds + \int_0^t a(X^K_s)dL^K_s, \quad t \in [0, 1],
\]

and its discretization defined by \( \overline{X}^K_0 = x_0 \) and (with \( \xi \) defined in (1.3))

\[
\overline{X}^K_t = \xi_{t-t_{i-1}}(\overline{X}^K_{t_{i-1}}) + a(\overline{X}^K_{t_{i-1}})(L^K_t - L^K_{t_{i-1}}), \quad t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n.
\]

Thanks to the truncation \( \tau_K \), \( \mathbb{E}|L^K_t|^p < \infty \), for any \( p \geq 1 \), we can apply the Malliavin calculus on Poisson space introduced in [1].

Now under \( HR \) and \( A0 \) or \( A1 \), the random variables \( X^K_t \) and \( \overline{X}^K_t \) admit a density for \( t > 0 \) (see [2]). Note that under \( A1 \), \( X = X^K \) and \( X = \overline{X}^K \) for \( K \) large enough. Let \( p^{K}_{1/n} \), respectively \( \overline{p}^{K}_{1/n} \), be the transition density of the Markov chain \((X^K_{i/n})_{i \geq 0}\), respectively \((\overline{X}^K_{i/n})_{i \geq 0}\). From Proposition 2.1, we have

\[
d_{TV}((X^K_{i/n}), (\overline{X}^K_{i/n})) \leq \left( \frac{1}{2} \sum_{i=1}^{n} \left( \mathbb{E} H^2(X^K, p^{K}_{1/n}, \overline{p}^{K}_{1/n}) + \mathbb{E} H^2(\overline{X}^K, p^{K}_{1/n}, \overline{p}^{K}_{1/n}) \right) \right)^{1/2}.
\]

Consequently to bound the total variation distance between \((X^K_{i/n})_{0 \leq i \leq n}\) and \((\overline{X}^K_{i/n})_{0 \leq i \leq n}\) it is sufficient to control \( H_2(p^{K}_{1/n}, \overline{p}^{K}_{1/n}) \) in terms of \( n \), \( K \) and \( x \).

Bounds for \( H_2(p^{K}_{1/n}, \overline{p}^{K}_{1/n}) \) are presented in the next section. They are obtained by connecting \( H_2(p^{K}_{1/n}, \overline{p}^{K}_{1/n}) \) to the \( L^2 \)-norm of a Malliavin weight. This technical part of the paper is postponed to Section 5.

Of course, the methodology is exactly the same if we replace the scheme \( X \) by the Euler scheme \( \tilde{X} \). In that case we consider the truncated Euler scheme defined by \( \tilde{X}^K_0 = x_0 \) and for \( t \in [t_{i-1}, t_i], 1 \leq i \leq n, \)

\[
\tilde{X}^K_t = \tilde{X}^K_{t_{i-1}} + b(\tilde{X}^K_{t_{i-1}})(t - t_{i-1}) + a(\tilde{X}^K_{t_{i-1}})(L^K_t - L^K_{t_{i-1}}).
\]

We denote by \( \tilde{p}^{K}_{1/n} \) the transition density of the Markov chain \((\tilde{X}^K_{i/n})_{i \geq 0}\).

Throughout the paper, \( C(a, b, \alpha) \) denotes a constant, independent of \( n, K \) but depending on \( a, b, \alpha \), whose value may change from line to line. We write simply \( C \) if \( C(a, b, \alpha) \) does not depend on \( a, b, \alpha \).
3 Estimates for the local Hellinger distance

We state in this section our main results concerning the rate of convergence in approximating $X_{1/n}^K$ solution of (2.2) starting from $x$, by $X_{1/n}^K$ or $\tilde{X}_{1/n}^K$ that solve respectively (2.3) or (2.5) with initial value $x$. In what follows, the constant $C(a, b, \alpha)$ does not depend on $x$.

Before stating our results, we precise the assumptions on the auxiliary truncation $\tau_K$. Let $\tau$ be a symmetric $C^1$ function such that $0 \leq \tau(x) \leq 1$, $\tau(x) = 1$ if $|x| \leq 1/2$ and $\tau(x) = 0$ if $|x| \geq 1$. We assume moreover that

$$\forall p \geq 1, \quad \int |\tau'(z)|^p \tau(z)dz < \infty. \quad (3.1)$$

For $K \geq 2$, we define $\tau_K$ by $\tau_K(x) = \tau(x/K)$.

We first assume that $a$ is constant. In that case, our methodology does not require additional non-degeneracy assumptions on the Lévy measure and we assume $A0$. We first focus on the discretization scheme defined by (2.3).

**Theorem 3.1.** We assume $A0$ and HR with a constant, then we have

(i) \[ \sup_x H^2_x(p_{1/n}^K, \tilde{p}_{1/n}^K) \leq \frac{C(a, b, \alpha)}{n^2} \left(1 + \frac{K^{2-\alpha}}{n}\right), \]

where $C(a, b, \alpha)$ has exponential growth in $||b'||_\infty$ and polynomial growth in $||b''||_\infty$, $1/a$, $1/\alpha$ and $1/(\alpha-2)$.

(ii) Moreover, if $g$ satisfies $\int |z|g(z)dz < \infty$, then the bound does not depend on the truncation $K$ \[ \sup_x H^2_x(p_{1/n}^K, \tilde{p}_{1/n}^K) \leq \frac{C(a, b, \alpha)}{n^2} \left(1 + \frac{K^{2-\alpha}}{n^3}\right). \]

(iii) In the stable case ($g = c_0$), (i) can be improved

\[ \sup_x H^2_x(p_{1/n}^K, \tilde{p}_{1/n}^K) \leq \frac{C(a, b, \alpha)}{n^2} \left(1 + \frac{K^{2-\alpha}}{n^3}\right). \]

We now study the local Hellinger distance $H_x(p_{1/n}^K, \tilde{p}_{1/n}^K)$, where $\tilde{p}_{1/n}^K$ is the density of the Euler scheme $\tilde{X}_{1/n}^K$ defined by (2.5).

**Theorem 3.2.** We assume $A0$ and HR with a constant, then we have

\[ H^2_x(p_{1/n}^K, \tilde{p}_{1/n}^K) \leq \frac{C(a, b, \alpha)}{n^2} \left(1 + \frac{K^{2-\alpha}}{n} + |b(x)|^2 \frac{n^{2/\alpha}}{n^2}\right). \]

Moreover, if $g$ satisfies $\int |z|g(z)dz < \infty$, then the bound does not depend on the truncation $K$

\[ H^2_x(p_{1/n}^K, \tilde{p}_{1/n}^K) \leq \frac{C(a, b, \alpha)}{n^2} \left(1 + |b(x)|^2 \frac{n^{2/\alpha}}{n^2}\right). \]
In the general case \((a \text{ non constant})\), we need strong restrictions on the support of the Lévy measure \(F\) and assume \(A_{1}\). So we have \(X^K = X\) and \(\overline{X}^K = \overline{X}\) for \(K\) large enough and we omit the dependence on \(K\).

**Theorem 3.3.** We assume \(A_{1}\) and HR with \(||a'||\infty > 0\), then we have

\[
H^2_\varepsilon(p_{1/n}, \overline{p}_{1/n}) \leq \begin{cases} 
C(a, b, \alpha)(1 + |x|^2)^{-\frac{1}{2\alpha}} & \text{if } \alpha > 1, \\
C(a, b, \alpha)(1 + |x|^2)^{-\frac{1}{\alpha}} & \text{if } \alpha \leq 1, \forall \varepsilon > 0,
\end{cases}
\]

where \(C(a, b, \alpha)\) has exponential growth in \(||b'||\infty\) and polynomial growth in \(||b''||\infty, ||a'||\infty, 1/||a'||\infty, b(0), a(0), 1/a, 1/\alpha\) and \(1/(\alpha - 2)\).

(ii) For the Euler scheme \((1.4)\), we obtain for \(\alpha > 1/2\)

\[
H^2_\varepsilon(p_{1/n}, \tilde{p}_{1/n}) \leq \begin{cases} 
C(a, b, \alpha)(1 + |x|^2)^{-\frac{1}{2\alpha}} & \text{if } \alpha > 1, \\
C(a, b, \alpha)(1 + |x|^2)^{-\frac{1}{\alpha}} & \text{if } \alpha = 1, \forall \varepsilon > 0, \\
C(a, b, \alpha)(1 + |x|^2)^{-\frac{1}{\alpha+2}} & \text{if } 1/2 < \alpha < 1.
\end{cases}
\]

**Remark 3.1.** In the Brownian case \((\alpha = 2)\), we obtain the rate of convergence \(1/n\) for the square of the Hellinger distance between \(X_{1/n}\) and its Euler approximation \(\tilde{X}_{1/n}\). This (probably sharp) rate does not permit to obtain a path control of the total variation distance between the stochastic equation and the Euler scheme. This is why we focus in this paper on pure jump processes. To obtain pathwise convergence in the Brownian case, one has to consider a discretization scheme with finer step as in Konakov and al. \([9]\).

The proof of these three theorems is given in Sections 5.4 and 5.5.

4 Pathwise total variation distance and application to asymptotic equivalence of experiments

The local behavior of the Hellinger distance established in Section 3 permits to obtain some pathwise rates of convergence in total variation. As in the previous section, we distinguish between the cases \(a \text{ constant}\) (where the rate of convergence is better) or \(a \text{ non constant}\) and we study rate of convergence for the total variation distance between \((X_{i/n})_{0\leq i\leq n}\) and \((\overline{X}_{i/n})_{0\leq i\leq n}\) (respectively \((\overline{X}_{i/n})_{0\leq i\leq n}\) defined by \((1.1)\) and \((1.2)\) (respectively \((1.4)\)).

**Theorem 4.1.** We assume \(A_{0}\) and HR with a constant.

(i) Then we have

\[
d_{TV}((X_{i/n})_{0\leq i\leq n}, (\overline{X}_{i/n})_{0\leq i\leq n}) \leq C(a, b, \alpha) \max\left(\frac{1}{\sqrt{n}}, \frac{1}{n^{2a/(\alpha+2)}}\right),
\]

where \(C(a, b, \alpha)\) has exponential growth in \(||b'||\infty\) and polynomial growth in \(||b''||\infty, 1/a, a, 1/\alpha\) and \(1/(\alpha - 2)\).
(ii) Moreover if \( g \) satisfies the following integrability condition

\[
\int_{\mathbb{R}} |z| g(z) \, dz < \infty,
\]

then for any \( \alpha \in (0, 2) \), we have the better bound

\[
d_{TV}((X_n)_{0 \leq i \leq n}, (\overline{X}_n)_{0 \leq i \leq n}) \leq C(a, b, \alpha)/\sqrt{n}.
\]

(iii) In the stable case \((g = c_0)\), we obtain

\[
d_{TV}((X_n)_{0 \leq i \leq n}, (\overline{X}_n)_{0 \leq i \leq n}) \leq C(a, b, \alpha) \max(\frac{1}{\sqrt{n}}, \frac{1}{n^{\alpha/(\alpha+2)}}).
\]

**Remark 4.1.** We observe that without integrability assumptions on \( g \), the rate of convergence vanishes if \( \alpha \) goes to zero. Moreover we have \( \max(\frac{1}{\sqrt{n}}, \frac{1}{n^{\alpha/(\alpha+2)}}) = \frac{1}{\sqrt{n}} \) if \( \alpha \geq 2/3 \). In the stable case, the rate \( \frac{1}{\sqrt{n}} \) is obtained if \( \alpha \geq 2/7 \).

**Proof.** We first establish a relationship between \( d_{TV}((X_i/n)_{0 \leq i \leq n}, (\overline{X}_i/n)_{0 \leq i \leq n}) \) and \( d_{TV}((X^K_{i/n})_{0 \leq i \leq n}, (\overline{X}^K_{i/n})_{0 \leq i \leq n}) \). On the same probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) we consider the Lévy process \((L_t)_{t \geq 0}\) with Lévy measure \( F \) and the truncated Lévy process \((L^K_t)_{t \geq 0}\) with Lévy measure \( F^K \) defined by

\[
F^K(dz) = \tau_K(z) F(dz).
\]

We recall (see Section 4.1 in [3]) that this can be done by setting \( L_t = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}(ds, dz) \), respectively \( L^K_t = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}^K(ds, dz) \), where \( \tilde{\mu} \), respectively \( \tilde{\mu}^K \), are the compensated Poisson random measures associated respectively to

\[
\mu(A) = \int_{[0,1]} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t, z) \mu^*(dt, dz, du), \quad A \subset [0,1] \times \mathbb{R}
\]

\[
\mu^K(A) = \int_{[0,1]} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t, z) 1_{\{u \leq \tau_K(z)\}} \mu^*(dt, dz, du), \quad A \subset [0,1] \times \mathbb{R},
\]

for \( \mu^* \) a Poisson random measure on \([0,1] \times \mathbb{R} \times [0,1]\) with compensator \( \mathcal{P}^*(dt, dz, du) = dt F(dz) du \). By construction, the measures \( \mu \) and \( \mu^K \) coincide on the event

\[
\Omega_K = \{ \omega \in \Omega; \mu^*([0,1] \times \{ z \in \mathbb{R}; |z| \geq K/2 \} \times [0,1]) = 0 \}.
\]

(4.1)

Since \( \mu^*([0,1] \times \{ z \in \mathbb{R}; |z| \geq K/2 \} \times [0,1]) \) has a Poisson distribution with parameter

\[
\lambda_K = \int_{|z| \geq K/2} g(z) / |z|^\alpha \, dz \leq C/(\alpha K^\alpha),
\]

we deduce that

\[
\mathbb{P}(\Omega_K^c) \leq C/(\alpha K^\alpha).
\]

(4.2)
We observe that \((X_t, \overline{X}_t, L_t)_{t \in [0,1]} = (X^K_t, \overline{X}^K_t, L^K_t)_{t \in [0,1]} \) on \(\Omega_K\) and so we deduce
\[
d_{TV}((X^n_k)_{0 \leq k \leq n}, (\overline{X}^n_k)_{0 \leq k \leq n}) \leq d_{TV}((X^K_k)_{0 \leq k \leq n}, (\overline{X}^K_k)_{0 \leq k \leq n}) + C/(\alpha K^n). \tag{4.3}
\]

(i) Combining (4.3), (2.4) with Theorem 3.1 (i) we have
\[
d_{TV}((X^n_k)_{0 \leq k \leq n}, (\overline{X}^n_k)_{0 \leq k \leq n}) \leq C(a, b, \alpha) \left( \frac{1}{\sqrt{n}} + \frac{K^{1-\alpha/2} + C(\alpha)}{n^{3/2}} \right).
\]

Choosing \(K = n^{2/(\alpha+2)}\), we deduce
\[
\frac{K^{1-\alpha/2}}{n} = \frac{1}{n^{2\alpha/(\alpha+2)}} = \frac{1}{K^{\alpha}}.
\]

this gives the first part of the result.

(ii) Now with the integrability assumption on \(g\), we have
\[
d_{TV}((X^n_k)_{0 \leq k \leq n}, (\overline{X}^n_k)_{0 \leq k \leq n}) \leq C(a, b, \alpha) \left( \frac{1}{\sqrt{n}} + \frac{K^{1-\alpha/2} + C(\alpha)}{n^{3/2}} \right).
\]

and we conclude choosing \(K = n^{1/(2\alpha)}\).

(iii) In the stable case, we have
\[
d_{TV}((X^n_k)_{0 \leq k \leq n}, (\overline{X}^n_k)_{0 \leq k \leq n}) \leq C(a, b, \alpha) \left( \frac{1}{\sqrt{n}} + \frac{K^{1-\alpha/2} + C(\alpha)}{n^{3/2}} \right).
\]

We conclude with \(K = n^{4/(\alpha+2)}\).

Considering now the Euler scheme given by (1.4), we obtain the following rate of convergence in total variation distance between \((X^n_k)_{0 \leq k \leq n}\) and \((\tilde{X}^n_k)_{0 \leq k \leq n}\). We remark that we have no rate at all if \(\alpha \leq 2/3\).

**Proposition 4.1.** We assume A0, HR with a constant and \(\alpha > 2/3\). Let \((\tilde{X}^n_k)_{0 \leq k \leq n}\) be the Euler scheme defined by (1.4), then we have

(i) \[
d_{TV}((X^n_k)_{0 \leq k \leq n}, (\tilde{X}^n_k)_{0 \leq k \leq n}) \leq C(a, b, \alpha) \max \left( \frac{1}{\sqrt{n}}, \frac{1}{n^{3/2+2}} \right).
\]

(ii) Moreover, with the additional assumption on \(g\)
\[
\int_{\mathbb{R}} |z| g(z) dz < \infty,
\]

then
\[
d_{TV}((X^n_k)_{0 \leq k \leq n}, (\tilde{X}^n_k)_{0 \leq k \leq n}) \leq \begin{cases} 
C(a, b, \alpha) \frac{1}{\sqrt{n}}, & \text{if } \alpha \geq 1, \\
C(a, b, \alpha) \frac{1}{n^{3/2}}, & \text{if } \frac{2}{3} < \alpha < 1.
\end{cases}
\]
Proof. (i) From (2.4) and Theorem 3.2 (i) we have
\[
d_{TV}((X^K_n)_{0 \leq i \leq n}, (\tilde{X}^K_n)_{0 \leq i \leq n}) \leq \frac{C(a, b, \alpha)}{\sqrt{n}} \left( 1 + \frac{K^{2-\alpha}}{n} \right) \left( \sup_{t \in [0,1]} \mathbb{E}|X_t^K|^2 + \sup_{t \in [0,1]} \mathbb{E}|\tilde{X}_t^K|^2 \right)^{1/2}.
\]

Standard computations give
\[
\sup_{t \in [0,1]} \mathbb{E}|X_t^K|^2 \leq C(a, b, \alpha)K^{2-\alpha}, \quad \sup_{t \in [0,1]} \mathbb{E}|\tilde{X}_t^K|^2 \leq C(a, b, \alpha)K^{2-\alpha}.
\]

So we obtain
\[
d_{TV}((X^K_n)_{0 \leq i \leq n}, (\tilde{X}^K_n)_{0 \leq i \leq n}) \leq \frac{C(a, b, \alpha)}{\sqrt{n}} \left( 1 + \frac{K^{1-\alpha/2}n^{1/\alpha}}{n} \right) + \frac{C(a)K^n}{K^{\alpha}}.
\]

Now proceeding as in the beginning of the proof of Theorem 4.1, we see that (4.3) holds, replacing \(X_t\) by \(\tilde{X}_t\), and we deduce
\[
d_{TV}((X^K_n)_{0 \leq i \leq n}, (\tilde{X}^K_n)_{0 \leq i \leq n}) \leq \frac{C(a, b, \alpha)}{\sqrt{n}} \left( 1 + \frac{K^{1-\alpha/2}n^{1/\alpha}}{n} \right) + \frac{C(a)}{K^\alpha}.
\]

Choosing \(K = n^{(3\alpha-2)/(\alpha(2+\alpha))}\) gives the first result.

(ii) Now with the integrability assumption on \(g\), the \(L^2\)-norm of \((X^K_t)\) and \((\tilde{X}^K_t)\) does not depend on \(K\) and we have
\[
\sup_{t \in [0,1]} \mathbb{E}|X_t^K|^2 \leq C(a, b, \alpha), \quad \sup_{t \in [0,1]} \mathbb{E}|\tilde{X}_t^K|^2 \leq C(a, b, \alpha).
\]

So it yields
\[
d_{TV}((X^K_n)_{0 \leq i \leq n}, (\tilde{X}^K_n)_{0 \leq i \leq n}) \leq \frac{C(a, b, \alpha)}{\sqrt{n}} \left( 1 + \frac{n^{1/\alpha}}{n} \right) + \frac{C(a)}{K^\alpha}.
\]

With \(K = n^{1/(2\alpha)}\) we deduce
\[
d_{TV}((X^K_n)_{0 \leq i \leq n}, (\tilde{X}^K_n)_{0 \leq i \leq n}) \leq C(a, b) \max(\frac{1}{\sqrt{n}}, \frac{1}{n^{(3\alpha-2)/(12\alpha)}}).
\]

Remark 4.2. We can apply our methodology if the Lévy process \(L\) is a Brownian Motion. In that case the Malliavin calculus is more standard and we compute easily the Malliavin weight of Section 5. Assuming HR and a constant, we obtain the rate of convergence \(1/\sqrt{n}\) in total variation distance between \((X^K_n)_{0 \leq i \leq n}\) and \((\tilde{X}^K_n)_{0 \leq i \leq n}\).

We now study the convergence rate in total variation distance for a general scale coefficient \(a\), assuming \(A1\). We observe that in the Brownian case \(\alpha = 2\), we do not have convergence.
Theorem 4.2. We assume A1 and HR with $||a'||_\infty > 0$.

(i) Then we have

$$d_{TV}((X^0_{\frac{1}{n}})_{0 \leq i \leq n}, (\tilde{X}^0_{\frac{1}{n}})_{0 \leq i \leq n}) \leq \begin{cases} C(a,b,\alpha) \frac{1}{n^{\alpha/2}}, & \text{if } \alpha > 1, \\ C(a,b,\alpha) \frac{1}{n^{1/2}}, & \text{if } \alpha \leq 1, \forall \varepsilon > 0. \end{cases}$$

where $C(a,b,\alpha)$ has exponential growth in $||b'||_\infty$ and polynomial growth in $||b''||_\infty$, $||a'||_\infty$, $||a''||_\infty$, $1/||a'||_\infty$, $b(0)$, $a(0)$, $1/\alpha$, $1/\alpha$ and $1/(\alpha - 2)$.

(ii) For the Euler scheme (1.4), we obtain if $\alpha > 2/3$

$$d_{TV}((X^0_{\frac{1}{n}})_{0 \leq i \leq n}, (\tilde{X}^0_{\frac{1}{n}})_{0 \leq i \leq n}) \leq \begin{cases} C(a,b,\alpha) \frac{1}{n^{\alpha/2}}, & \text{if } \alpha > 1, \\ C(a,b,\alpha) \frac{1}{n^{1/2}}, & \text{if } \alpha = 1, \forall \varepsilon > 0, \\ C(a,b,\alpha) \frac{1}{n^{2/3}}, & \text{if } 2/3 < \alpha < 1. \end{cases}$$

Proof. Under A1, $g$ is a truncation function and the result is an immediate consequence of (2.4) and Theorem 3.3 observing that for any $p \geq 1$

$$\sup_{\tau \in [0,1]} E|X^0_{\tau}|^p \leq C(a,b,\alpha), \quad \sup_{\tau \in [0,1]} E|\tilde{X}^0_{\tau}|^p \leq C(a,b,\alpha), \quad \sup_{\tau \in [0,1]} E|\tilde{X}^0_{\tau}|^p \leq C(a,b,\alpha).$$

The result of Theorems 4.1 and 4.2 has interesting consequences in statistics. Indeed, it permits to control the Le Cam deficiency distance $\Delta$ between the experiment based on the discretely observed SDE solution of (1.1) and the experiment based on the discretization scheme (1.2). We refer to Le Cam [1] and Le Cam and Yang [12] for the definition and properties of $\Delta$.

Assume that $b$ and $a$ depend on unknown parameters $\theta$ and $\sigma$ and that we are interested in estimating the three parameters $\beta = (\theta, \sigma, \alpha)$ assuming that $\beta \in \Theta \times K_0 \times K_1$ where $\Theta$ is a compact subset of $\mathbb{R}$, $K_0$ a compact subset of $\mathbb{R}$ and $K_1$ a compact subset of $(0,2)$. Let $\mathcal{E}^n = (\mathbb{R}^n, B(\mathbb{R}^n), (\mathcal{F}^n, \beta)_{\beta \in \Theta \times K_0 \times K_1})$ be the experiment based on the observations $(X^0_{\frac{1}{n}})_{0 \leq i \leq n}$ given by (1.1) and let $\tilde{\mathcal{E}}^n = (\mathbb{R}^n, B(\mathbb{R}^n), (\tilde{\mathcal{F}}^n, \beta)_{\beta \in \Theta \times K_0 \times K_1})$ be the experiment based on the observations $(\tilde{X}^0_{\frac{1}{n}})_{0 \leq i \leq n}$ given by (1.2) (respectively $(\tilde{X}^0_{\frac{1}{n}})_{0 \leq i \leq n}$ given by (1.4)). We denote by $\Delta(\mathcal{E}^n, \tilde{\mathcal{E}}^n)$ (respectively $\Delta(\mathcal{E}^n, \tilde{\mathcal{E}}^n)$) the Le Cam distance between these two experiments. From the previous results, we deduce that this distance goes to zero with $n$ (the two experiments are asymptotically equivalent).

Corollary 4.1. We assume either ($\ast$) or (\ast\ast):

($\ast$) A0, HR with a constant, $b = b(., \theta)$ with $\theta \in \Theta$ a compact subset of $\mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}, \theta \in \Theta} |b'(x, \theta)| \leq C, \quad \sup_{x \in \mathbb{R}, \theta \in \Theta} |b''(x, \theta)| \leq C.$$
(**) A1, HR and $b = b(., \theta)$ with $\theta \in \Theta$ a compact subset of $\mathbb{R}$, $a = a(x, \sigma)$ with $\sigma \in K_0$ a compact subset of $\mathbb{R}$ such that

$$
sup_{x \in \mathbb{R}, \theta \in \Theta} |b'(x, \theta)| \leq C, \quad sup_{x \in \mathbb{R}, \theta \in \Theta} |b''(x, \theta)| \leq C, \quad sup_{\theta \in \Theta} |b(0, \theta)| \leq C,$$

$$
sup_{x \in \mathbb{R}, \sigma \in K_0} |a'(x, \sigma)| \leq C, \quad \inf_{\sigma \in K_0} ||a'(., \sigma)||_{\infty} > 0,$$

$$
sup_{x \in \mathbb{R}, \sigma \in K_0} |a''(x, \sigma)| \leq C, \quad sup_{\sigma \in K_0} |a(0, \sigma)| \leq C,$$

$$\forall x \in \mathbb{R}, \forall \sigma \in K_0, \quad a(x, \sigma) \geq \underline{a} > 0.$$

Then

$$\lim_{n \to \infty} \Delta(E^n, \tilde{E}^n) = 0,$$

and if $K_1$ is a compact subset of $(2/3, 2)$ \(\lim_{n \to \infty} \Delta(E^n, \tilde{E}^n) = 0\).

Proof. Since the Le Cam distance is bounded by the total variation distance

$$\Delta(E^n, \tilde{E}^n) \leq sup_{\beta \in \Theta \times K_0 \times K_1} d_{TV}((X^{\beta}_{0 \leq i \leq n}, (\overline{X}^{\beta}_{0 \leq i \leq n})),$$

the first part of Corollary 4.1 is an immediate consequence of Theorem 4.1 (case (*)) and Theorem 4.2 (case (**)) , observing that

$$\sup_{\beta \in \Theta \times K_0 \times K_1} C(a, b, \alpha) \leq C$$

and

$$\lim_{n} sup_{a \in K_1} \frac{1}{n^{2a/(\alpha+2)}} = 0, \quad \lim_{n} sup_{a \in K_1} \frac{1}{n^{1/(\alpha-1/2)}} = 0.$$

The second part comes from Proposition 4.1 and Theorem 4.2 since

$$\lim_{n} sup_{a \in K_1} \frac{1}{n^{(3a-2)/(\alpha+2)}} = 0, \quad \lim_{n} sup_{a \in K_1} \frac{1}{n^{3/2-1/\alpha}} = 0.$$

The main interest of Corollary 4.1 is that statistical inference in experiment $E^n$ inherits the same asymptotic properties as in experiment $\tilde{E}^n$. Efficiency in $E^n$ is still an open problem for a general scale coefficient $a$ (assuming $a$ constant, the LAMN property for $(\theta, a)$ has been established in [4] assuming additionally that $(L_t)$ is a truncated stable process). The main difficulty comes from the fact that the likelihood function is not explicit. But since $E^n$ and $\tilde{E}^n$ are asymptotically equivalent, it is sufficient to study asymptotic efficiency in the simplest experiment $\tilde{E}^n$ where the likelihood function has an explicit expression in term of the density of the driving Lévy process.
5 Local Hellinger distance and Malliavin calculus

This section is devoted to the proof of Theorems 3.1, 3.2 and 3.3. Our methodology consists in writing the Hellinger distance as the expectation of a Malliavin weight and to control this weight. We define Malliavin calculus with respect to the truncated Lévy process \((L^K_t)\) specified in Section 2, recalling that if A1 holds the additional truncation is useless.

5.1 Interpolation and rescaling

The first step consists in introducing a rescaled interpolation between the processes \((X^K_t)_{0 \leq t \leq 1/n}\) and \((\tilde{X}^K_t)_{0 \leq t \leq 1/n}\) (or \((\tilde{X}^K_t)_{0 \leq t \leq 1/n}\)) starting from \(x\), defined in Section 2.

Let us define \(Y^{K,n,r}_t\) for \(0 \leq r \leq 1\) and \(0 \leq t \leq 1\) by

\[
Y^{K,n,r}_t = x + \frac{1}{n} \int_0^t \left[ r(b(Y^{K,n,r}_s) + (1 - r)b(x)) \right] ds + \frac{1}{n^{1/\alpha}} \int_0^t \left[ ra(Y^{K,n,r}_s) + (1 - r)a(x) \right] dL^{K,n}_s
\]

with

\[
\xi^n_t(x) = x + \frac{1}{n} \int_0^t b(\xi^n_s(x)) ds,
\]

and where \((L^{K,n}_t)_{t \in [0,1]}\) is a Lévy process admitting the decomposition

\[
L^{K,n}_t = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}^{K,n}(dt, dz), \quad t \in [0,1],
\]

where \(\tilde{\mu}^{K,n}\) is a compensated Poisson random measure, \(\tilde{\mu}^{K,n} = \mu^{K,n} - \tilde{\nu}^{K,n}\), with compensator \(\tilde{\nu}^{K,n}(dt, dz) = dt \frac{\nu(z/n^{1/\alpha})}{\nu(\{0\})} K(z/n^{1/\alpha}) 1_{\mathbb{R} \setminus \{0\}}(z) dz\).

By construction, the process \((L^{K,n}_t)_{t \in [0,1]}\) is equal in law to the rescaled truncated process \((n^{1/\alpha} L^{K,n}_{t/n})_{t \in [0,1]}\). Moreover if \(r = 0\), \(Y_{1,n}^{K,n,0}\) has the distribution of \(\tilde{X}^{K}_{1/n}\) starting from \(x\), and if \(r = 1\), \(Y_{1,n}^{K,n,1}\) has the distribution of \(X^{K}_{1/n}\) starting from \(x\), so we have \(H_x(p^{K,n}_{1/n}, \tilde{p}^{K,n}_{1/n}) = H_x(Y^{K,n,1}_{1,n}, Y^{K,n,0}_{1,n})\).

For the Euler scheme, to study the Hellinger distance \(H_x(p^{K,n}_{1/n}, \tilde{p}^{K,n}_{1/n})\), we proceed as previously, replacing the interpolation \(Y^{K,n,r}_t\) by \(\tilde{Y}^{K,n,r}_t\) with

\[
\tilde{Y}^{K,n,r}_t = x + \frac{1}{n} \int_0^t [rb(\tilde{Y}^{K,n,r}_s) + (1 - r)b(x)] ds + \frac{1}{n^{1/\alpha}} \int_0^t [ra(\tilde{Y}^{K,n,r}_s) + (1 - r)a(x)] dL^{K,n}_s.
\]
We check easily that $\tilde{Y}_{1}^{K,n,1}$ has the distribution of $X_{1/n}^{K}$ starting from $x$ and $\tilde{Y}_{1}^{K,n,0}$ the distribution of $X_{1/n}^{K}$ starting from $x$.

To simplify the notation, we set

\begin{align}
 b(r, y, t) &= rb(y) + (1 - r)b(\xi^n_t(x)) \quad (5.5) \\
 \tilde{b}(r, y) &= rb(y) + (1 - r)b(x) \quad (5.6) \\
 a(r, y) &= ra(y) + (1 - r)a(x), \quad (5.7)
\end{align}

so we have

\begin{align}
 dY_{t}^{K,n,r} &= \frac{1}{n}b(r, Y_{t}^{K,n,r}, t)dt + \frac{1}{n^{1/\alpha}}a(r, Y_{t}^{K,n,r})dL_{t}^{K,n}, \\
 d\tilde{Y}_{t}^{K,n,r} &= \frac{1}{n}\tilde{b}(r, \tilde{Y}_{t}^{K,n,r})dt + \frac{1}{n^{1/\alpha}}a(r, \tilde{Y}_{t}^{K,n,r})dL_{t}^{K,n}.
\end{align}

5.2 Integration by Part

For the reader convenience, we recall some results on Malliavin calculus for jump processes, before stating our main results. We follow [3] Section 4.2 and also refer to [1] for a complete presentation. We will work on the Poisson space associated to the measure $\mu^{K,n}$ defining the process $(L_{t}^{K,n})$ assuming that $n$ is fixed. By construction, the support of $\mu^{K,n}$ is contained in $[0, 1/2] \times E_{n}$, where

\[E_{n} = \{z \in \mathbb{R}; |z| < Kn^{1/\alpha}\}.\]

We recall that the measure $\mu^{K,n}$ has compensator

\[\overline{\mu}^{K,n}(dt, dz) = dt \frac{g(z/n^{1/\alpha})}{|z|^{\alpha+1}} \tau_{K}(z/n^{1/\alpha})1_{\mathbb{R}\setminus\{0\}}(z)dz := dtF_{K,n}(z)dz. \quad (5.8)\]

We define the Malliavin operators $L$ and $\Gamma$ (we omit here the dependence in $n$ and $K$) and their basic properties (see Bichteler, Gravereaux, Jacod, [1] Chapter IV, sections 8-9-10). For a test function $f : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ (f is measurable, $C^2$ with respect to the second variable, with bounded derivatives, and $f \in \cap_{p \geq 1}L^p(dtF_{K,n}(z)dz)$), we set $\mu^{K,n}(f) = \int_{0}^{1} \int_{\mathbb{R}} f(t, z)\mu^{K,n}(dt, dz)$. As auxiliary function, we consider $\rho : \mathbb{R} \mapsto [0, \infty)$ such that $\rho$ is symmetric, two times differentiable and such that $\rho(z) = z^4$ if $z \in [0, 1/2]$ and $\rho(z) = z^2$ if $z \geq 1$. Thanks to the truncation $\tau_{K}$, we check that $\rho$, $\rho'$ and $\rho^\frac{F_{K,n}}{F_{K,n}}$ belong to $\cap_{p \geq 1}L^p(F_{K,n}(z)dz)$. We also observe that at this stage the truncation is useless if we have for any $p \geq 1$

\[\int_{\mathbb{R}} |z|^p g(z)dz < \infty.\]

This assumption is satisfied for the tempered stable process. But to include the stable process in our study, we need to introduce the truncation function.

With the previous notation, we define the Malliavin operator $L$, on a simple functional $\mu^{K,n}(f)$ as follows
$$L(\mu_{K,n}^n(f)) = \frac{1}{2} \mu_{K,n}^n \left( \rho' f' + \rho \frac{F'_{K,n}}{F_{K,n}} f' + \rho f'' \right),$$

where $f'$ and $f''$ are the derivatives with respect to the second variable. This definition permits to construct a linear operator on a space $D \subset \cap_{p \geq 1} L^p$ which is self-adjoint:

$$\forall \Phi, \Psi \in D, \quad \mathbb{E} \Phi L \Psi = \mathbb{E} L \Phi \Psi.$$ We associate to $L$, the symmetric bilinear operator $\Gamma$:

$$\Gamma(\Phi, \Psi) = L(\Phi \Psi) - \Phi L \Psi - \Psi L \Phi.$$

If $f$ and $h$ are two test functions, we have:

$$\Gamma(\mu_{K,n}^n(f), \mu_{K,n}^n(h)) = \mu_{K,n}^n(\rho f' h').$$

The operators $L$ and $\Gamma$ satisfy the chain rule property:

$$LG(\Phi) = G'(\Phi) L \Phi + \frac{1}{2} G''(\Phi) \Gamma(\Phi, \Phi),$$

$$\Gamma(G(\Phi), \Psi) = G'(\Phi) \Gamma(\Phi, \Psi).$$

These operators permit to establish the following integration by parts formula (see [1] Theorem 8-10 p.103).

**Theorem 5.1.** Let $\Phi$ and $\Psi$ be random variables in $D$, and $f$ be a bounded function with bounded derivatives up to order two. If $\Gamma(\Phi, \Phi)$ is invertible and $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p \geq 1} L^p$, we have

$$\mathbb{E} f'\Phi(\Psi) = \mathbb{E} f(\Phi) \mathcal{H}_\Phi(\Psi),$$

with

$$\mathcal{H}_\Phi(\Psi) = \Psi \frac{\Gamma(\Phi, \Gamma(\Phi, \Phi))}{\Gamma^2(\Phi, \Phi)} - 2 \Psi \frac{L \Phi}{\Gamma(\Phi, \Phi)} \frac{\Gamma(\Phi, \Psi)}{\Gamma(\Phi, \Phi)}.$$ (5.10)

We apply now the result of Theorem 5.1 to the random variable $Y_{t_1}^{K,n,r}$ observing that under $A_0$ (or $A_1$) and $HR$, $(Y_{t_1}^{K,n,r})_{t \in [0,1]} \in D, \forall r \in [0,1]$ and then the following Malliavin operators are well defined (see Section 10 in [1]). Let us introduce some more notation. For $0 \leq t \leq 1$, we set

$$\Gamma(Y_{t_1}^{K,n,r}, Y_{t_1}^{K,n,r}) = U_{t}^{k,n,r}$$

$$L(Y_{t_1}^{K,n,r}) = \mathbb{E}_{t_1}^{K,n,r}$$

and for the vector $V_{t_1}^{K,n,r} = (Y_{t_1}^{K,n,r}, \partial_r Y_{t_1}^{K,n,r}, U_{t}^{K,n,r})^T$, we denote by $W_{t}^{K,n,r} = (W_{t_1}^{K,n,r,(i,j)})_{1 \leq i,j \leq 3}$ the matrix $\Gamma(V_{t_1}^{K,n,r}, V_{t_1}^{K,n,r})$ such that

$$U_{t}^{K,n,r} = W_{t}^{K,n,r,(1,1)}$$

$$\Gamma(Y_{t_1}^{K,n,r}, \partial_r Y_{t_1}^{K,n,r}) = W_{t}^{K,n,r,(2,1)}$$

$$\Gamma(Y_{t_1}^{K,n,r}, \Gamma(Y_{t_1}^{K,n,r}, Y_{t_1}^{K,n,r})) = W_{t}^{K,n,r,(3,1)}.$$ (5.13)
We also introduce the derivative of \( Y^{K,n,r} \) with respect to \( r \), denoted by \( \partial_r Y^{K,n,r} \) and solving the equation
\[
d\partial_r Y^{K,n,r}_t = \frac{1}{n} \partial_y b(r, Y^{K,n,r}_t, t) \partial_r Y^{K,n,r}_t dt + \frac{1}{n^{1/2}} \partial_y a(r, Y^{K,n,r}_t) \partial_r Y^{K,n,r}_t dL_t^{K,n} + \frac{1}{n} \partial_r b(r, Y^{K,n,r}_t, t) dt + \frac{1}{n^{1/2}} \partial_y a(r, Y^{K,n,r}_- t) dL_t^{K,n},
\]
(5.15)
with \( \partial_r Y^{K,n,r}_0 = 0 \) and
\[
\partial_r b(r, y, t) = b(y) - b(\xi^0_r(x)), \quad \partial_y b(r, y, t) = rb'(y), \quad \partial_y a(r, y) = a(y) - a(x), \quad \partial_y a(r, y) = ra'(y).
\]

With this notation, we establish the following bound for \( H^2(p^{K,n}_1, \overline{p}^{K}_1) \). It is obvious that the same bound holds for \( H^2(p^{K,n}_1, \overline{p}^{K}_1) \), replacing the process \( Y^{K,n,r} \) by \( \overline{Y}^{K,n,r} \), but to shorten the presentation we only state the result for \( Y^{K,n,r} \).

**Theorem 5.2.** We assume \( \text{HR, A0 or A1} \) and that for any \( r \in [0, 1] \), \( U^{K,n,r}_1 \) is invertible and \( (U^{K,n,r}_1)^{-1} \in \cap_{p \geq 1} \mathbb{L}^p \). Then we have
\[
H^2(p^{K}_1, \overline{p}^{K}_1) = H^2(\{1^{K,n,1} \}, \{1^{K,n,0} \}) \leq \sup_{r \in [0, 1]} \mathbb{E}_x[\mathcal{H} \partial_r Y^{K,n,r}_1(\partial_r Y^{K,n,r}_1)^2],
\]
where
\[
\mathcal{H} \partial_r Y^{K,n,r}_1(\partial_r Y^{K,n,r}_1) = \frac{\partial_r Y^{K,n,r}_1}{U^{K,n,r}_1}, \quad \frac{\partial_r Y^{K,n,r}_1}{U^{K,n,r}_1} - 2\partial_r Y^{K,n,r}_1 \frac{\|1^{K,n,r}_1\|_{U^{K,n,r}_1}}{U^{K,n,r}_1} - \frac{W^{K,n,r}_1}{U^{K,n,r}_1}.
\]
(5.16)

**Proof.** We first observe that under \( \text{A0 or A1, HR} \) and assuming \( U^{K,n,r}_1 \) invertible with \( (U^{K,n,r}_1)^{-1} \in \cap_{p \geq 1} \mathbb{L}^p \), \( \forall r \in [0, 1] \), the random variable \( Y^{K,n,r}_1 \) (starting from \( x \)) admits a density for any \( r \in [0, 1] \). Moreover this density is differentiable with respect to \( r \). We denote by \( q^{K,n,r} \) this density and by \( \partial_r q^{K,n,r} \) its derivative with respect to \( r \). We have
\[
H^2(p^{K}_1, \overline{p}^{K}_1) = \int_{\mathbb{R}} \left( \sqrt{q^{K,n,1}(y) - q^{K,n,0}(y)} \right)^2 dy
\]
\[
= \frac{1}{4} \int_{\mathbb{R}} \left( \int_0^1 \frac{\partial_r q^{K,n,r}(y)}{\sqrt{q^{K,n,r}(y)}} dr \right)^2 dy
\]
\[
\leq \frac{1}{4} \int_0^1 \mathbb{E}_x \left( \frac{\partial_r q^{K,n,r}}{q^{K,n,r}} (Y^{K,n,r}_1) \right)^2 dr.
\]
Using the integration by part formula, we obtain a representation for \( \frac{\partial_r q^{K,n,r}}{q^{K,n,r}} \).
Let \( f \) be a smooth function, by differentiating \( r \mapsto E[f(Y^K_{1,n,r})] \), we obtain

\[
\int f(u) \partial_r q^{K,n,r}(u) du = E_f(Y^K_{1,n,t}) \partial_r Y^K_{1,n,t}
\]

\[
= E_f(Y^K_{1,n,r}) H_{1,n,r}(\partial_r Y^K_{1,n,r})
\]

\[
= E_f(Y^K_{1,n,r}) \mathbb{E}[H_{1,n,r}(\partial_r Y^K_{1,n,r})|Y^K_{1,n,r}]
\]

\[
= \int f(u) \mathbb{E}[H_{1,n,r}(\partial_r Y^K_{1,n,r})|Y^K_{1,n,r} = u] q^{K,n,r}(u) du.
\]

This gives the representation

\[
\frac{\partial_r q^{K,n,r}}{q^{K,n,r}}(y) = \mathbb{E}_x[H_{1,n,r}(\partial_r Y^K_{1,n,r})|Y^K_{1,n,r} = y],
\]

and we deduce the bound

\[
H^2_x(p^{K}_{1/n}, p^{K}_{1/n}) \leq \sup_{r \in [0,1]} \mathbb{E}_x[H_{1,n,r}(\partial_r Y^K_{1,n,r})^2].
\]

The computation of the weight \( H_{1,n,r}(\partial_r Y^K_{1,n,r}) \) is derived in the next section.

### 5.3 Computation of \( U^K_{1,n,r} \), \( \mathbb{L}^K_{1,n,r} \) and \( W^K_{1,n,r} \)

We derive here the stochastic equations satisfied by versions of processes \((U^K_{t,n,r})_{t \in [0,1]}\), \((\mathbb{L}^K_{t,n,r})_{t \in [0,1]}\) and \((W^K_{t,n,r})_{t \in [0,1]}\), assuming HR and A0 or A1. Using the result of Theorem 10-3 in [1] (we omit the details), we obtain the following equations. These equations are solved in the next sections.

We first check that \((U^K_{t,n,r})\) and \((\mathbb{L}^K_{t,n,r})\) solve respectively

\[
U^K_{t,n,r} = \mathbb{E}_0 \int_0^t \partial_y b(r, Y^K_{s,n,r}, s) U^K_{s,n,r} ds + \frac{1}{n} \int_0^t \partial_y a(r, Y^K_{s,n,r}) U^K_{s,n,r} z \tilde{\mu}^K_{n}(ds, dz) 
\]

\[
+ \frac{1}{2n} \int_0^t \int_0^t \partial_y^2 a(r, Y^K_{s,n,r}) U^K_{s,n,r} (\rho(z) + \rho(z) \mathbb{E}_z[F^K_{K,n}(z)]) \tilde{\mu}^K_{n}(ds, dz).
\]

\[
\mathbb{L}^K_{t,n,r} = \int_0^t \partial_y b(r, Y^K_{s,n,r}, s) \mathbb{L}^K_{s,n,r} ds + \frac{1}{n} \int_0^t \partial_y a(r, Y^K_{s,n,r}) \mathbb{L}^K_{s,n,r} z \tilde{\mu}^K_{n}(ds, dz) 
\]

\[
+ \frac{1}{2n} \int_0^t \int_0^t \partial_y^2 a(r, Y^K_{s,n,r}) U^K_{s,n,r} (\rho(z) + \rho(z) \mathbb{E}_z[F^K_{K,n}(z)]) \tilde{\mu}^K_{n}(ds, dz).
\]
with $B^{K,n,r}(.,.,.,t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $A^{K,n,r} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ (precised below) and $V_0^{K,n,r} = (x,0,0)^T$.

\[
B^{K,n,r,1}(v_1, v_2, v_3, t) = \frac{1}{n} b(r, v_1, t),
\]

\[
B^{K,n,r,2}(v_1, v_2, v_3, t) = \frac{1}{n} (\partial_y b(r, v_1, t)v_2 + \partial_r b(r, v_1, t)),
\]

\[
B^{K,n,r,3}(v_1, v_2, v_3, t) = \frac{2}{n} \partial_y b(r, v_1, t)v_3 + \frac{1}{n^{2/\alpha}}(\partial_y a(r, v_1))^2 v_3 \int_\mathbb{R} z^2 F_{K,n}(z) dz
\]

\[
\quad + \frac{1}{n^{2/\alpha}} a(r, v_1)^2 \int_\mathbb{R} \rho(z) F_{K,n}(z) dz,
\]

\[
A^{K,n,r}(v_1, v_2, v_3, z) = \frac{1}{n^{1/\alpha}} \left( \begin{array}{c}
\frac{a(r, v_1)z}{2\partial_y a(r, v_1)v_3 z + \frac{1}{n^{2/\alpha}}(\partial_y a(r, v_1))^2 v_3 z^2 + \frac{1}{n^{1/\alpha}} a(r, v_1)^2 \rho(z)}
\end{array} \right).
\]

We use the notation

\[
D_v B^{K,n,r}(v, t) = \begin{pmatrix}
\partial_{v_1} B^{K,n,r,1}(v, t) & \partial_{v_2} B^{K,n,r,1}(v, t) & \partial_{v_3} B^{K,n,r,1}(v, t) \\
\partial_{v_1} B^{K,n,r,2}(v, t) & \partial_{v_2} B^{K,n,r,2}(v, t) & \partial_{v_3} B^{K,n,r,2}(v, t) \\
\partial_{v_1} B^{K,n,r,3}(v, t) & \partial_{v_2} B^{K,n,r,3}(v, t) & \partial_{v_3} B^{K,n,r,3}(v, t)
\end{pmatrix},
\]

we obtain

\[
D_v B^{K,n,r}(v, t) = \begin{pmatrix}
\frac{1}{n} b''(v_1) & 0 & 0 \\
\frac{1}{n}[rb''(v_1)v_2 + b''(v_1)] & \frac{1}{n} b'(v_1) & 0 \\
0 & 0 & \partial_{v_3} B^{K,n,r,3}(v, t)
\end{pmatrix}
\]

with

\[
\partial_{v_1} B^{K,n,r,3}(v, t) = 2n rb''(v_1)v_3 + \frac{2}{n^{2/\alpha}} r^2 (a' a')(v_1)v_3 \int_\mathbb{R} z^2 F_{K,n}(z) dz
\]

\[
\quad + \frac{2}{n^{2/\alpha}} ra(r, v_1)a'(v_1) \int_\mathbb{R} \rho(z) F_{K,n}(z) dz,
\]

\[
\partial_{v_3} B^{K,n,r,3}(v, t) = 2n rb'(v_1) + \frac{1}{n^{2/\alpha}} r^2 a'(v_1)^2 \int_\mathbb{R} z^2 F_{K,n}(z) dz.
\]

Defining analogously the matrix $D_v A^{K,n,r}(v, z)$ and the vector $D_v A^{K,n,r}$, we have

\[
D_v A^{K,n,r}(v, t) = \begin{pmatrix}
\frac{1}{n^{1/\alpha}} [ra''(v_1)v_2 + a''(v_1)z] & 0 & 0 \\
\frac{1}{n^{1/\alpha}} ra'(v_1)z & \frac{1}{n^{1/\alpha}} ra'(v_1)z & 0 \\
0 & 0 & \partial_{v_3} A^{K,n,r,3}(v, t)
\end{pmatrix}
\]

with

\[
\partial_{v_1} A^{K,n,r,3}(v, t) = 2n rb''(v_1)v_3 z + \frac{2}{n^{2/\alpha}} r^2 (a' a'')(v_1)v_3 z^2 + \frac{2}{n^{2/\alpha}} ra(r, v_1)a'(v_1) \rho(z)
\]

\[
\partial_{v_3} A^{K,n,r,3}(v, t) = \frac{2}{n^{2/\alpha}} ra'(v_1)z + \frac{1}{n^{2/\alpha}} r^2 a'(v_1)^2 z^2.
\]
\[ D_z A^{K,n,r}(v,t) = \frac{1}{n^{1/\alpha}} \begin{pmatrix} a(r, v_1) \\ ra'(v_1)v_2 + (a(v_1) - a(x)) \\ 2ra'(v_1)v_3 + \frac{2}{n^{2/\alpha}} a'(v_1)^2 v_3z + \frac{1}{n^{3/\alpha}} a(r, v_1)^2 \rho'(z) \end{pmatrix}. \]

With this notation, the matrix \( W_t^{K,n,r} \) solves

\[ W_t^{K,n,r} = \int_0^t \left[ W_s^{K,n,r} D_v B^{K,n,r}(V_s^{K,n,r}, s) + D_v B^{K,n,r}(V_s^{K,n,r}, s)(W_s^{K,n,r})^T \right] ds \]

\[ + \int_0^t \int_{\mathbb{R}} \left[ W_{s-}^{K,n,r} D_v A^{K,n,r}(Y_{s-}^{K,n,r}, z) + D_v A^{K,n,r}(Y_{s-}^{K,n,r}, z)(W_{s-}^{K,n,r})^T \right] \hat{\mu}^{K,n}(ds, dz) \]

\[ + \int_0^t \int_{\mathbb{R}} D_v A^{K,n,r}(V_{s-}^{K,n,r}, z)W_{s-}^{K,n,r} D_v A^{K,n,r}(V_{s-}^{K,n,r}, z)^T \hat{\mu}^{K,n}(ds, dz) \]

\[ + \int_0^t \int_{\mathbb{R}} D_v A^{K,n,r}(V_{s-}^{K,n,r}, z)D_v A^{K,n,r}(V_{s-}^{K,n,r}, z)^T \rho(z) \mu^{K,n}(ds, dz). \]

From this, we extract directly the equations for \( W_{K,n,r}(2,1) = \Gamma(Y_{K,n,r}, \partial_s Y_{K,n,r}) \) and \( W_{K,n,r}(3,1) = \Gamma(Y_{K,n,r}, \Gamma(Y_{K,n,r}, Y_{K,n,r})). \)

\[ W_t^{K,n,r,(2,1)} = \frac{2}{n} \int_0^t \rho' (Y_{s-}^{K,n,r}) W_{s-}^{K,n,r,(2,1)} ds \]

\[ + \frac{2}{n^{1/\alpha}} \int_0^t \int_{\mathbb{R}} ra'(Y_{s-}^{K,n,r})w_{s-}^{K,n,r,(2,1)} \tilde{\mu}^{K,n}(ds, dz) \]

\[ + \frac{1}{n^{2/\alpha}} \int_0^t \int_{\mathbb{R}} 2a'(Y_{s-}^{K,n,r})^2 w_{s-}^{K,n,r,(2,1)} \tilde{\mu}^{K,n}(ds, dz) \]

\[ + \frac{1}{n} \int_0^t \left[ (rb''(Y_{s-}^{K,n,r})\partial_s Y_{s-}^{K,n,r} + a'(Y_{s-}^{K,n,r}))U_{s-}^{K,n,r} \right] ds \]

\[ + \frac{1}{n^{1/\alpha}} \int_0^t \int_{\mathbb{R}} (ra''(Y_{s-}^{K,n,r})\partial_s Y_{s-}^{K,n,r} + a'(Y_{s-}^{K,n,r}))U_{s-}^{K,n,r} \tilde{\mu}^{K,n}(ds, dz) \]

\[ + \frac{1}{n^{2/\alpha}} \int_0^t \int_{\mathbb{R}} ra'(Y_{s-}^{K,n,r})\partial_s Y_{s-}^{K,n,r} + a'(Y_{s-}^{K,n,r}))U_{s-}^{K,n,r} \hat{\mu}^{K,n}(ds, dz) \]

\[ + \frac{1}{n^{3/\alpha}} \int_0^t \int_{\mathbb{R}} a(r, Y_{s-}^{K,n,r})ra'(Y_{s-}^{K,n,r})\partial_s Y_{s-}^{K,n,r} + a(Y_{s-}^{K,n,r}) - a(x)p(z) \mu^{K,n}(ds, dz). \]
\[ W_{1}^{K,n,r,(3,1)} = \frac{3}{n} \int_{0}^{t} r b'(Y_{s}^{K,n,r}) W_{s}^{K,n,r,(3,1)} ds \]

\[ + \frac{3}{n^{1/\alpha}} \int_{0}^{t} \int_{R} r a'(Y_{s}^{K,n,r}) W_{s}^{K,n,r,(3,1)} z \tilde{\mu}^{K,n}(ds, dz) \]

\[ + \frac{3}{n^{2/\alpha}} \int_{0}^{t} \int_{R} r^2 a'(Y_{s}^{K,n,r})^2 W_{s}^{K,n,r,(3,1)} z^2 \mu^{K,n}(ds, dz) \]

\[ + \frac{2}{n} \int_{0}^{t} \int_{R} r b''(Y_{s}^{K,n,r})(U_{s}^{K,n,r})^2 ds + \frac{2}{n^{1/\alpha}} \int_{0}^{t} \int_{R} r a''(Y_{s}^{K,n,r})(U_{s}^{K,n,r})^2 z \tilde{\mu}^{K,n}(ds, dz) \]

\[ + \frac{1}{n^{2/\alpha}} \int_{0}^{t} \int_{R} r a'(Y_{s}^{K,n,r}) \left( 2r a''(Y_{s}^{K,n,r}) U_{s}^{K,n,r} z + \frac{2}{n^{1/\alpha}} r^2 a'(Y_{s}^{K,n,r})^2 U_{s}^{K,n,r} z^2 \right) \]

\[ + \frac{2}{n^{1/\alpha}} r a(r, Y_{s}^{K,n,r}) a'(Y_{s}^{K,n,r}) \rho(z) U_{s}^{K,n,r} z \tilde{\mu}^{K,n}(ds, dz) \]

\[ + \frac{1}{n^{2/\alpha}} \int_{0}^{t} \int_{R} a(r, Y_{s}^{K,n,r}) \left( 2r a'(Y_{s}^{K,n,r}) U_{s}^{K,n,r} + \frac{2}{n^{1/\alpha}} r^2 a'(Y_{s}^{K,n,r})^2 U_{s}^{K,n,r} z \right) \]

\[ + \frac{1}{n^{1/\alpha}} a(r, Y_{s}^{K,n,r}) \rho'(z) \mu^{K,n}(ds, dz). \]

### 5.4 Proof of Theorems 3.1 and 3.2 (a constant and A0)

Assuming \( a \) constant, the interpolation \( Y^{K,n,r} \) between (2.2) and (2.3) solves the equation

\[ Y_{t}^{K,n,r} = x + \frac{1}{n} \int_{0}^{t} [rb(Y_{s}^{K,n,r}) + (1 - r)b(\xi(x))] ds + \frac{1}{n^{1/\alpha}} a L_{t}^{K,n} \]

with \( \xi(x) \) defined by (5.2) and \( L_{t}^{K,n} \) by (5.3).

**Proof of Theorem 3.1.** To apply Theorem 5.2, we have to check that \( U_{1}^{K,n,r} \) is invertible and \( (U_{1}^{K,n,r})^{-1} \in \mathcal{L}_{\infty}^{p} \).

We start by solving the equations (5.15), (5.17), (5.18), (5.19), (5.20) defining respectively \( \partial_{r} Y_{1}^{K,n,r}, U_{1}^{K,n,r}, L_{1}^{K,n,r}, W_{1}^{K,n,r,(2,1)} \) and \( W_{1}^{K,n,r,(3,1)} \). This is done easily since \( a \) is constant. We define \( Z_{t}^{K,n,r} \) by

\[ Z_{t}^{K,n,r} = e^{\frac{\dot{t}}{n} b'(Y_{s}^{K,n,r}) ds}. \]

Then we obtain the following explicit expressions.

\[ \partial_{r} Y_{1}^{K,n,r} = \frac{Z_{1}^{K,n,r}}{n} \int_{0}^{1} (Z_{s}^{K,n,r})^{-1} [b(Y_{s}^{K,n,r}) - b(\xi(x))] ds \]
\[ U_{1}^{K,n,r} = a^{2} \frac{(Z_{1}^{K,n,r})^{2}}{n^{2/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} (Z_{s-}^{K,n,r})^{-2} \rho(z) \mu^{K,n}(ds,dz) \tag{5.24} \]

\[ L_{1}^{K,n,r} = \frac{(Z_{1}^{K,n,r})^{2}}{2n} \int_{0}^{1} (Z_{s}^{K,n,r})^{-1} r b''(Y_{s}^{K,n,r}) U_{s-}^{K,n,r} ds + a Z_{1}^{K,n,r} \frac{1}{2n^{1/\alpha}} \int_{0}^{1} (Z_{s-}^{K,n,r})^{-1} (\rho'(z) + \rho(z) \frac{F_{K,n}''(z)}{F_{K,n}(z)}) \mu^{K,n}(ds,dz) \tag{5.25} \]

\[ W_{1}^{K,n,r,(2,1)} = \frac{(Z_{1}^{K,n,r})^{2}}{n} \int_{0}^{1} (Z_{s}^{K,n,r})^{-2} U_{s}^{K,n,r} \left[ rb''(Y_{s}^{K,n,r}) \partial_{r} Y_{s}^{K,n,r} + b'(Y_{s}^{K,n,r}) \right] ds \tag{5.26} \]

\[ W_{1}^{K,n,r,(3,1)} = \frac{2r(Z_{1}^{K,n,r})^{3}}{n} \int_{0}^{1} (Z_{s}^{K,n,r})^{-3} r^{3} U_{s}^{K,n,r} b''(Y_{s}^{K,n,r}) ds + a^{3} \frac{(Z_{1}^{K,n,r})^{3}}{n^{3/\alpha}} \int_{0}^{1} (Z_{s-}^{K,n,r})^{-3} \rho'(z) \rho(z) \mu^{K,n}(ds,dz). \tag{5.27} \]

We obviously have the bounds

\[ \sup_{t \leq 1} |Z_{t}^{K,n,r}| \leq C(b), \quad \sup_{t \leq 1} |(Z_{t}^{K,n,r})^{-1}| \leq C(b). \tag{5.28} \]

This implies that

\[ \sup_{t \leq 1} |U_{t}^{K,n,r}| \leq \frac{a^{2}}{n^{2/\alpha}} C(b) \mu^{K,n}(\rho), \tag{5.29} \]

\[ \frac{1}{|U_{1}^{K,n,r}|} \leq C(b) \frac{n^{2/\alpha}}{\mu^{K,n}(\rho)}. \tag{5.30} \]

With the definition of \( \rho \), we can then check that for any \( p \geq 1 \)

\[ \mathbb{E} \left( \frac{1}{|\mu^{K,n}(\rho)|^{p}} \right) \leq C. \]

The proof follows the same line as in [3] section 4.2 equation (4.25) and we omit it. Consequently \( U_{t}^{K,n,r} \) is invertible and \( (U_{t}^{K,n,r})^{-1} \in \cap_{p \geq 1} L^{p} \). From Theorem 5.2 it is now sufficient to bound \( \mathbb{E}[\mathcal{H}_{Y_{t}^{K,n,r}}(\partial_{r} Y_{1}^{K,n,r})^{2}] \)

where

\[ \mathcal{H}_{Y_{t}^{K,n,r}}(\partial_{r} Y_{1}^{K,n,r}) = \frac{\partial_{r} Y_{1}^{K,n,r} W_{1}^{K,n,r,(3,1)}}{U_{1}^{K,n,r}} - \frac{\partial_{r} Y_{1}^{K,n,r} Y_{1}^{K,n,r}}{U_{1}^{K,n,r}} - 2 \partial_{r} Y_{1}^{K,n,r} \frac{Y_{1}^{K,n,r}}{U_{1}^{K,n,r}} - \frac{W_{1}^{K,n,r,(2,1)}}{U_{1}^{K,n,r}}. \]

We study the \( L^{2} \)-norm of each term. We first deduce from Gronwall’s Lemma,

\[ \sup_{t \leq 1} |Y_{t}^{K,n,r} - \xi_{t}^{n}(x)| \leq ae \frac{||b'||_{\infty}/n}{n^{1/\alpha}} \sup_{s \leq 1} |L_{s}^{K,n}| \leq C(a,b) \frac{1}{n^{1/\alpha}} \sup_{s \leq 1} |L_{s}^{K,n}|. \tag{5.31} \]
Combining this with (5.28), (5.29) and (5.30), we obtain the intermediate bounds

\[ |\partial_r Y_{1}^{K,n,r}| \leq \frac{C(a,b)}{n} \left( \frac{1}{n^{1/\alpha}} \sup_{t \in [0,1]} |L_{t}^{K,n}| + \frac{1}{n} |L_{t}^{K,n}| \right), \]

\[ |\varepsilon K,n,r| \leq \frac{C(a,b)}{n} \left( \frac{\mu K,n(\rho)}{n^{2/\alpha}} + \frac{\mu K,n(|\rho' + \rho F_{K,n}|)}{n^{1/\alpha}} \right), \]

\[ |W_{1}^{K,n,r,(2,1)}| \leq \frac{C(a,b)}{n} \left( \frac{\mu K,n(\rho)}{n^{2/\alpha}} \right) \]

\[ |W_{1}^{K,n,r,(3,1)}| \leq \frac{C(a,b)}{n} \left( \frac{\mu K,n(|\rho' + \rho F_{K,n}|)}{n^{1/\alpha}} \right). \]

With this background, we control each term in \( H_{Y_{1}^{K,n,r}}(\partial_r Y_{1}^{K,n,r}) \)

\[ \left| \partial Y_{1}^{K,n,r} \frac{W_{1}^{K,n,r}(3,1)}{U_{1}^{K,n,r}} \right| \leq \frac{C(a,b)}{n} \left( \frac{\sup_{t \in [0,1]} |L_{t}^{K,n}|}{n^{1/\alpha}} + \frac{1}{n} |L_{t}^{K,n}| \frac{\mu K,n(\rho)}{\mu K,n(\rho)} \right), \]

\[ \left| \partial Y_{1}^{K,n,r} \frac{W_{1}^{K,n,r}(2,1)}{U_{1}^{K,n,r}} \right| \leq \frac{C(a,b)}{n} \left( \frac{\sup_{t \in [0,1]} |L_{t}^{K,n}|}{n^{1/\alpha}} + \frac{1}{n} |L_{t}^{K,n}| \frac{\mu K,n(\rho)}{\mu K,n(\rho)} \right). \]

This permits to deduce that

\[ |H_{Y_{1}^{K,n,r}}(\partial_r Y_{1}^{K,n,r})| \leq \frac{C(a,b)}{n} (1 + \frac{1}{n^{1/\alpha}} + T_{1} + T_{2} + T_{3}), \]

with

\[ T_{1} = \sup_{t \in [0,1]} |L_{t}^{K,n}| \frac{\mu K,n(\rho)}{n^{1/\alpha}}, \quad T_{2} = \frac{\sup_{t \in [0,1]} |L_{t}^{K,n}| |\mu K,n(\rho')} {\mu K,n(\rho)}, \]

\[ T_{3} = \frac{\sup_{t \in [0,1]} |L_{t}^{K,n}| |\mu K,n(\rho') + \rho F_{K,n}|}{\mu K,n(\rho)}. \]

We first study the \( L^{2}\)-norm of \( T_{1} \). Since \( L_{t}^{K,n} = \int_{0}^{t} \int_{\mathbb{R}} z \tilde{K}^{n}(ds,dz) \), we have immediately using the definition of the compensator (5.8)

\[ \mathbb{E} \left| \sup_{t \in [0,1]} |L_{t}^{K,n}| \right|^{2} \leq \frac{C}{n^{2/\alpha}} \left( \int_{0}^{K^{1/\alpha}} z^{2} g \left( \frac{z}{n^{1/\alpha}} \right) \frac{1}{z^{\alpha+1}} dz \right). \]

Since \( g \) is bounded, we deduce after some calculus

\[ \mathbb{E} T_{1}^{2} = \mathbb{E} \left| \sup_{t \in [0,1]} |L_{t}^{K,n}| \right|^{2} \leq C(\alpha) K^{2-\alpha} / n. \]
Now if $g$ satisfies the additional assumption $\int_R |z|g(z)dz < \infty$, then

$$\mathbb{E}T_1^2 = \mathbb{E} \left[ \sup_{t \in [0,1]} \left| \frac{L_{t}^{K,n}}{n^{1/\alpha}} \right|^2 \right] \leq C(\alpha)/n, \quad (5.34)$$

with $C(\alpha)$ independent of $K$.

Turning to $T_2$, we decompose $L_{t}^{K,n}$ (using the symmetry of $F_{K,n}$) into the small jump part and the large jump part as

$$L_{t}^{K,n} = \int_0^t \int_{0<|z|<1} z\mu_{K,n}(ds, dz) + \int_0^t \int_{|z|\geq 1} z\mu_{K,n}(ds, dz).$$

Since the small jump part is bounded in $L^p$, for any $p \geq 1$, by a constant independent of $K$, we focus on the large jump part and study the worst term in $T_2$

$$\frac{\int_0^t \int_{|z|\geq 1} z\mu_{K,n}(ds, dz)\mu_{K,n}(\{\rho|1_{|z|\geq 1}\})}{\mu_{K,n}(\{\rho_1|1_{|z|\geq 1}\})^2} \leq C \mu_{K,n}(\{|z| \geq 1\})^{1/2}.$$

Then observing that $\mu_{K,n}(\{|z| \geq 1\})$ has a Poisson distribution with parameter $\lambda_{K,n} \leq C(\alpha)$, we obtain

$$\mathbb{E}T_2^2 \leq C(\alpha).$$

For the last term $T_3$, the definition of $F_{K,n}$ gives for $z \neq 0$

$$|\rho(z)F'_{K,n}(z)| \leq C\left( \frac{\rho(z)}{|z|} + \frac{\rho(z)}{n^{1/\alpha}} \frac{g'(\frac{z}{n^{1/\alpha}})}{g'\left(\frac{z}{n^{1/\alpha}}\right)} + \frac{\rho(z)}{n^{1/\alpha}} \frac{T'_{K}(\frac{z}{n^{1/\alpha}})}{T_{K}(\frac{z}{n^{1/\alpha}})} \right).$$

Consequently $T_3$ can be split into three terms, $T_3 \leq T_{3,1} + T_{3,2} + T_{3,3}$ with

$$T_{3,1} = \sup_{t \in [0,1]} \frac{|L_{t}^{K,n}| \mu_{K,n}(\{\rho'\} + |\rho/z|)}{\mu_{K,n}(\rho)},$$

$$T_{3,2} = \frac{1}{n^{1/\alpha}} \sup_{t \in [0,1]} \frac{|L_{t}^{K,n}| \mu_{K,n}(\rho g'(\frac{z}{n^{1/\alpha}}))}{\mu_{K,n}(\rho)},$$

$$T_{3,3} = \frac{1}{n^{1/\alpha}} \sup_{t \in [0,1]} \frac{|L_{t}^{K,n}| \mu_{K,n}(\rho \frac{T'_{K}(\frac{z}{n^{1/\alpha}})}{T_{K}(\frac{z}{n^{1/\alpha}})})}{\mu_{K,n}(\rho)}.$$

For $T_{3,1}$, we obtain by distinguishing between the small jump part and the large jump part (as for $T_2$)

$$\mathbb{E}(T_{3,1})^2 \leq C(\alpha).$$
Since \( g'/g \) is bounded, we deduce for \( T_{3,2} \)

\[
\mathbb{E}(T_{3,2})^2 \leq C \mathbb{E} \left[ \left| \sup_{t \in [0,1]} |L_t^{K,n}| \right|^{2} \right],
\]

and we conclude using (5.33) or (5.34). Remark that \( T_{3,2} = 0 \) in the stable case \( g = c_0 \).

Finally, considering \( T_{3,3} \), we first remark that by definition of \( \tau_K \)

\[
T_{3,3} \leq \frac{1}{n^{1/\alpha}} \sup_{t \in [0,1]} |L_t^{K,n}| \mu^{K,n} \left( 1_{\{Kn^{1/\alpha}/2 \leq |z| \leq Kn^{1/\alpha}\}} \frac{\tau'_K}{\tau_K} \left( \frac{z}{n^{1/\alpha}} \right) \right).
\]

From Burkholder inequality (see Lemma 2.5, inequality 2.1.37 in [7]),

\[
\mathbb{E} \left| \sup_{t \in [0,1]} |L_t^{K,n}| \right|^4 \leq C(\alpha) \frac{K^{4-\alpha}}{n},
\]

and using a change of variables and assumption (3.1)

\[
\mathbb{E} \mu^{K,n} \left( 1_{\{Kn^{1/\alpha}/2 \leq |z| \leq Kn^{1/\alpha}\}} \frac{\tau'_K}{\tau_K} \left( \frac{z}{n^{1/\alpha}} \right) \right)^4 \leq C(\alpha) \frac{1}{n K^{4+\alpha}}.
\]

This permits to deduce from Cauchy Schwarz inequality

\[
\mathbb{E}(T_{3,3})^2 \leq C(\alpha) \frac{1}{n K^{\alpha}} \leq C(\alpha)/n.
\]

To summarize, we have established in case (i) (the worst term comes from \( T_{3,2} \))

\[
\mathbb{E}_x |\mathcal{H}_{Y_{1,K,n,r}(\partial_r Y_{1,K,n,r})}|^2 \leq \frac{C(a,b,\alpha)}{n^2} (1 + \frac{K^{2-\alpha}}{n}),
\]

and if we have additionally \( \int_{\mathbb{R}} |z|g(z)dz < \infty \) (case (ii)), then

\[
\mathbb{E}_x |\mathcal{H}_{Y_{1,K,n,r}(\partial_r Y_{1,K,n,r})}|^2 \leq \frac{C(a,b,\alpha)}{n^2}.
\]

In the stable case (iii), \( T_{3,2} = 0 \) and the worst term is \( T_1/n \)

\[
\mathbb{E}_x |\mathcal{H}_{Y_{1,K,n,r}(\partial_r Y_{1,K,n,r})}|^2 \leq \frac{C(a,b,\alpha)}{n^2} (1 + \frac{K^{2-\alpha}}{n^3}).
\]

To simplify the presentation, we have not expressed explicitly the dependence of \( C(a,b,\alpha) \) in \( a, \alpha \) and the derivatives of \( b \), but it is not difficult to check that we have

\[
C(a,b,\alpha) \leq C \left( \frac{\|b'\|_{\infty}}{\|b'\|_{\infty}} + \frac{\|b''\|_{\infty}}{\alpha p_1 + \frac{1}{\alpha p_1}} + \frac{1}{\alpha p_1} + \frac{1}{(2 - \alpha) p_2} \right),
\]

with \( p_i \geq 1 \), for \( 1 \leq i \leq 5 \).

The proof of Theorem 3.1 is finished. \qed
Proof of Theorem 3.2. The proof follows the same lines as the one of Theorem 3.1 and we only indicate the main changes observing that (5.4) is obtained replacing \( b(\xi^n_s(x)) \) in (5.1) by \( b(x) \). We first deduce from Gronwall’s Lemma,

\[
\sup_{t \leq 1} |\hat{Y}^{K,n,r}_t - x| \leq C(a,b) \left( \frac{|b(x)|}{n} + \frac{1}{n^{1/\alpha}} \sup_{s \leq 1} |L_s^{K,n}| \right).
\]

(5.35)

This yields

\[
|\partial_t \hat{Y}^{K,n,r}_1| \leq C(a,b) \left( \frac{|b(x)|}{n} + \frac{1}{n^{1/\alpha}} \sup_{t \in [0,1]} |L_t^{K,n}| \right).
\]

Consequently, comparing to (5.32), we have the additional term \( \frac{|b(x)|}{n} \), so we deduce the bound

\[
|\mathcal{H}^{K,n,r}(\partial_t \hat{Y}^{K,n,r}_1)| \leq C(a,b) \left( 1 + \frac{1}{n} T_1 + T_2 + T_3 + \frac{|b(x)|}{n^2} + |b(x)| \frac{n^{1/\alpha}}{n} \left( \frac{\mu_{K,n}(\rho)}{\mu_{K,n}(\rho')} + \frac{\mu_{K,n}(\rho' + \rho_{\xi^n_s(x)})}{\mu_{K,n}(\rho)} \right) \right).
\]

We show easily that \( \frac{\mu_{K,n}(\rho')}{\mu_{K,n}(\rho)} \) and \( \frac{\mu_{K,n}(\rho' + \rho_{\xi^n_s(x)})}{\mu_{K,n}(\rho)} \) are bounded in \( L^2 \) and with the previous study of the terms \( T_1, T_2, T_3 \) we obtain the result of Theorem 3.2.

\( \square \)

5.5 Proof of Theorem 3.3 (a non constant and A1)

Since \( g \) is compactly supported, \( X_{1/n} \) and \( \overline{X}_{1/n} \) have moments of all order and the additional truncation \( \tau_K \) is useless (or \( K = \infty \)). So from now on, the interpolation \( Y^{n,r} \) and the Malliavin operators do not depend on \( K \).

To solve equations (5.15), (5.17), (5.18), (5.19), (5.20) (defining \( \partial_t Y^{n,r}_1, U^{n,r}_1, L^{n,r}_1, W^{n,r}_{1,1}, W^{n,r}_{1,1} \) and \( A_1 \)), we introduce \( (Z^{n,r}_t) \) that solves the linear equation

\[
Z^{n,r}_t = 1 + \frac{1}{n} \int_0^t rb'(Y^{n,r}_{s-})Z^{n,r}_s ds + \frac{1}{n^{1/\alpha}} \int_0^t \int_{\mathbb{R}} ra'(Y^{n,r}_{s-})Z^{n,r}_{s-} z \tilde{\mu}^n(ds,dz).
\]

(5.36)

Under A1, \( Z^{n,r}_t \) is invertible and from Itô’s formula, we check that

\[
\partial_t Y^{n,r}_t = Z^{n,r}_t \int_0^t (Z^{n,r}_{s-})^{-1} \left( b(Y^{n,r}_s) - b(\xi^n_s(x)) \right) ds
\]

(5.37)

\[
+ \frac{Z^{n,r}_t}{n^{1/\alpha}} \int_0^t \int_{\mathbb{R}} (Z^{n,r}_{s-})^{-1} (a(Y^{n,r}_s) - a(x)) z \tilde{\mu}^n(ds,dz)
\]

\[
- \frac{Z^{n,r}_t}{n^{1/\alpha}} \int_0^t \int_{\mathbb{R}} (Z^{n,r}_{s-})^{-1} \left( \frac{a(Y^{n,r}_s) - a(x)}{1 + \frac{ra'(Y^{n,r}_{s-})^2}{n^{1/\alpha}}} \right) ra'(Y^{n,r}_{s-}) z^2 \tilde{\mu}^n(ds,dz),
\]

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From Burkholder inequality.

We also use the notation

\[ \tau_{n,r} \]

dependence on \( n \). The solution for \( W \) is more complicated, we just explicit the structure of the solution for \( W^{n,r}(2,1) \) and \( W^{n,r}(3,1) \), where \( P^{n,0}, P^{n,1}, P^{n,2} \) are obtained from (5.19) and (5.20) respectively.

\[ W^{n,r}(2,1) = \left( Z_{s}^{n,r} \right)^{2} \int_{0}^{t} \left( Z_{s}^{n,r} \right)^{-2} \left( P^{n,0}_{s} + \int_{R} \frac{P^{n,1}_{s}(z)}{1 + \left( \frac{ra'(Y_{s}^{n,r})z}{n^{1/\alpha}} \right)^{2}} \mu^{n}(ds, dz) \right) ds \\
+ \int_{R} P^{n,2}_{s-}(z) \mu^{n}(ds, dz) - \int_{R} P^{n,2}_{s-}(z) \left( 1 - \frac{1}{1 + \left( \frac{ra'(Y_{s}^{n,r})z}{n^{1/\alpha}} \right)^{2}} \right) \mu^{n}(ds, dz) \right), \]

\[ W^{n,r}(3,1) = \left( Z_{s}^{n,r} \right)^{3} \int_{0}^{t} \left( Z_{s}^{n,r} \right)^{-3} \left( P^{n,0}_{s} + \int_{R} \frac{P^{n,1}_{s}(z)}{1 + \left( \frac{ra'(Y_{s}^{n,r})z}{n^{1/\alpha}} \right)^{3}} \mu^{n}(ds, dz) \right) ds \\
+ \int_{R} P^{n,2}_{s-}(z) \mu^{n}(ds, dz) - \int_{R} P^{n,2}_{s-}(z) \left( 1 - \frac{1}{1 + \left( \frac{ra'(Y_{s}^{n,r})z}{n^{1/\alpha}} \right)^{3}} \right) \mu^{n}(ds, dz) \right). \]

To identify the rate of convergence in the previous expressions and to simplify the study, we introduce some integrable processes \( (P_{t})_{t \in [0,1]} \) (we omit the dependence on \( n \)), whose expressions change from line to line, but such that

\[ \forall n \geq 1, \forall r \in [0,1], \quad \mathbb{E}_{x} \sup_{s \in [0,1]} |P_{s}|^{p} \leq C(a, b, \alpha)(1 + |x|^{p}), \quad \forall p \geq 1. \]

We also use the notation

\[ M_{t} = \int_{0}^{t} P_{s-} dE_{s}^{n}, \quad R_{t} = \int_{0}^{t} \int_{R} |z| 1_{\{|z| > 1\}} \mu^{n}(ds, dz), \quad t \in [0,1]. \]

From Burkholder inequality,

\[ \mathbb{E}_{x} \sup_{t \in [0,1]} |M_{t}|^{p} \leq C(a, b, \alpha)(1 + |x|^{p}), \]

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that is $M_t/n^{1/\alpha} = P_t$. Moreover using $|z|/n^{1/\alpha} \leq 1/(2||a'||_{\infty})$, we also have $R_t/n^{1/\alpha} = P_t$. In the following, we distinguish between the small jump part and the large jump part of $M_t$

$$M_t^{S,J} = \int_0^t \int_{\mathbb{R}} P_s - z 1_{\{|z| \leq 1\}} \mu^n(ds, dz), \quad M_t^{L,J} = \int_0^t \int_{\mathbb{R}} P_s - z 1_{\{|z| > 1\}} \mu^n(ds, dz),$$

where we used the symmetry of the compensator for the second expression. We check that $M_t^{S,J} = P_t$ and that $|M_t^{L,J}| \leq P_t R_t$.

We now give some relatively simple expressions or bounds for the variables $\partial_t Y_t^{n,r}$, $U_t^{n,r}$, $\mathbb{L}^{n,r}$, $W_1^{n,r,2,(1)}$, $W_1^{n,r,3,(1)}$. We first remark that from $\textbf{A1}$, $\mu^n$ has support in $\{|z| \leq n^{1/\alpha} \frac{1}{2||a'||_{\infty}}\}$ and we have for any $y$ and any $z$ such that $|z| \leq n^{1/\alpha} \frac{1}{2||a'||_{\infty}}$

$$\frac{2}{3} \leq \frac{1}{1 + r a'(y) \frac{n^{2/\alpha}}{n^{1/\alpha}}} \leq 2.$$ 

Moreover standard arguments give $Z_t^{n,r} = P_t$ and $(Z_t^{n,r})^{-1} = P_t$. This permits to deduce

$$\forall t \in [0, 1], \quad 0 \leq U_t^{n,r} \leq P_t \frac{\mu^n(\rho)}{n^{2/\alpha}}, \quad (5.43)$$

$$0 \leq \frac{1}{U_t^{n,r}} \leq P_t \frac{n^{2/\alpha}}{\mu^n(\rho)} \quad (5.44)$$

So as in Section 5.4, we check that $1/U_1^{n,r} \in \cap_{p \geq 1} L^p$. We also observe that

$$\forall t \in [0, 1], \quad Y_t^{n,r} - x = \frac{P_t}{n} + \frac{M_t}{n^{1/\alpha}}, \quad (5.45)$$

and from Gronwall’s inequality, we have

$$\forall t \in [0, 1], \quad |Y_t^{n,r} - \xi_t^n(x)| \leq C(b) \frac{\sup_{t \in [0, 1]} |M_t|}{n^{1/\alpha}}. \quad (5.46)$$

The next lemma summarizes our results, having in mind that we want to identify the rate of convergence of $\partial_t Y_t^{n,r} W_1^{n,r,2,(1)}/(U_1^{n,r})^2$, $\partial_t Y_t^{n,r} \mathbb{L}^{n,r}/U_1^{n,r}$ and $W_1^{n,r,2,(1)}/U_1^{n,r}$, where $U_1^{n,r}$ is approximately $\mu^n(\rho)/n^{2/\alpha}$.

**Lemma 5.1.** With $R_t = \int_0^t \int_{\mathbb{R}} |1_{\{|z| \leq 1\}}| \mu^n(ds, dz)$, we have the bounds

1. 

$$\sup_{t \in [0, 1]} |\partial_t Y_t^{n,r}| \leq \frac{P_1}{n^{1+1/\alpha}} (1 + R_1) + \frac{P_1}{n^{2/\alpha}} \left( 1 + R_1 + \int_0^1 \int_{\mathbb{R}} R_s - |z| 1_{\{|z| > 1\}} \mu^n(ds, dz) \right),$$

2. 

$$|L_1^{n,r}| \leq \frac{P_1}{n^{1+2/\alpha}} \mu^n(\rho) + \frac{P_1}{n^{2/\alpha}} (1 + \mu^n(\rho)) + \frac{P_1}{n^{1+1/\alpha}} \mu^n(|\rho'| + \rho \frac{F_n}{P_n}).$$
3. \[ |W^{n,r,(2,1)}_1| \leq \frac{P_1}{n^{1+2/\alpha}} \mu^n(\rho) + \frac{P_1}{n^{2/\alpha}} \mu^n(\rho) \sup_{t} |\partial_r Y^{n,r}_t| + \frac{P_1}{n^{3/\alpha}} \mu^n(\rho) + R_1 + R_t \int_0^1 \int_\mathbb{R} R_s|z|1_{\{|z|>1\}} \mu^n(ds,dz), \]

4. \[ |W^{n,r,(3,1)}_1| \leq \frac{P_1}{n^{1+4/\alpha}} \mu^n(\rho)^2 + \frac{P_1}{n^{4/\alpha}} (1 + \mu^n(\rho)^2) + \frac{P_1}{n^{3/\alpha}} \mu^n(|\rho'|). \]

**Proof.** 1. Using equation (5.37) with (5.45) and (5.46), \( \forall t \in [0,1] \)
\[ |\partial_r Y^{n,r}_t| \leq \frac{P_1}{n} \sup_t |M_t| + \frac{P_1}{n} \frac{1}{n^{1/\alpha}} \int_0^t \int_\mathbb{R} \frac{z^2}{n^{1/\alpha}} \mu^n(ds,dz) \]
\[ + \frac{P_t}{n^{2/\alpha}} \int_0^t \int_\mathbb{R} P_s - M_s - z\tilde{\mu}^n(ds,dz) + \frac{P_t}{n^{2/\alpha}} \int_0^t \int_\mathbb{R} P_s - M_s - z\tilde{\mu}^n(ds,dz). \]

In this expression to identify a sharp rate of convergence, we distinguish between the small jumps and the large jumps for each integral. Remark that \( |z|/n^{1/\alpha} \)

is bounded, the first two terms on the right-hand side of the inequality are bounded by
\[ \frac{P_t}{n^{1+1/\alpha}} (1 + R_1). \]

Moreover the last term satisfies
\[ \frac{P_t}{n^{2/\alpha}} |\int_0^t \int_\mathbb{R} P_s - M_s - z\tilde{\mu}^n(ds,dz)| \leq \frac{P_t}{n^{2/\alpha}} (1 + R_1 + \int_0^t \int_\mathbb{R} R_s|z|1_{\{|z|>1\}} \mu^n(ds,dz)). \]

Considering \( \int_0^t \int_\mathbb{R} P_s - M_s - z\tilde{\mu}^n(ds,dz) = \int_0^t M_s - dM_s \)

we split into four integrals (small jumps and large jumps of \( M \))
\[ \int_0^t \int_\mathbb{R} P_s - M_s - z\tilde{\mu}^n(ds,dz) = I_1^1 + I_1^2 + I_1^3 + I_1^4 \]

with \( I_1^1 = \int_0^t M_{S,J}^L dM_{S,J}^L = P_t, \)
\[ |I_1^2| = |\int_0^t M_{S,J}^L dM_{S,J}^L| \leq P_t \int_0^t \int_\mathbb{R} R_s|z|1_{\{|z|>1\}} \mu^n(ds,dz), \]
\[ |I_1^3| = |\int_0^t M_{S,J}^L dM_{S,J}^L| \leq P_t R_t, \]
\[ |I_1^4| = \int_0^t M_{S,J}^L dM_{S,J}^L \leq P_t R_t, \]

For \( I^4 \), observing that \( |M_{S,J}^L, M_{S,J}^L| = 0 \), we deduce from Itô’s formula that
\[ \int_0^t M_{S,J}^L dM_{S,J}^L = M_{S,J}^L M_{S,J}^L - \int_0^t M_{S,J}^L dM_{S,J}^L \]

and then \( |I_1^4| \leq P_t R_t \). Putting together these inequalities, we finally deduce the first result.
2. On a similar way, using equation (5.39) we obtain

$$
|W_{1}^{n,r}| \leq \frac{P_1}{n^{1+2/\alpha}} \mu^n(\rho) + \frac{P_1}{n^{1/\alpha}} \int_{0}^{1} \left( |\int_{\mathbb{R}} P_{s} U_s - z\tilde{\mu}^n(ds,dz)| + \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - \frac{z^2}{n^{1/\alpha}} \mu^n(ds,dz) \right).
$$

We check easily

$$
\frac{P_1}{n^{1/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - \frac{z^2}{n^{1/\alpha}} \mu^n(ds,dz) \leq \frac{P_1}{n^{2/\alpha}} \mu^n(\rho).
$$

To bound $|\int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - z\tilde{\mu}^n(ds,dz)|$, we introduce the process $Q_t = \int_{0}^{t} P_{s} - \rho(z) \mu^n(ds,dz)$ and its decomposition

$$
Q_t^{S,J} = \int_{0}^{t} P_{s} - \rho(z) 1_{\{|z| \leq 1\}} \mu^n(ds,dz) = P_t,
$$

$$
|Q_t^{L,J}| = |\int_{0}^{t} P_{s} - \rho(z) 1_{\{|z| > 1\}} \mu^n(ds,dz)| \leq P_t \mu^n(\rho).
$$

So we have $U_t = \int_{0}^{t} \int_{\mathbb{R}} P_{s} U_s - z\tilde{\mu}^n(ds,dz) = \frac{P_1}{n^{2/\alpha}} \int_{0}^{1} Q_s - dM_s$. We conclude by splitting $\int_{0}^{t} Q_s - dM_s$ into the small jumps and large jumps of $Q$ and $M$, with Itô’s formula for $\int_{0}^{t} Q_s - dM_s$ (as for $I^1$ in 1.), that

$$
\frac{P_1}{n^{1/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - z\tilde{\mu}^n(ds,dz) \leq \frac{P_1}{n^{2/\alpha}} (1 + \mu^n(\rho)).
$$

3. We turn to $W_{1}^{n,r,(2,1)}$. From (5.40) and (5.19), we have

$$
|W_{1}^{n,r,(2,1)}| \leq \frac{P_1}{n^{1+2/\alpha}} \mu^n(\rho) + \frac{P_1}{n^{1/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - \frac{z^2}{n^{1/\alpha}} \mu^n(ds,dz) + \frac{P_1}{n^{3/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - z\tilde{\mu}^n(ds,dz) + \frac{P_1}{n^{2/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} |\partial_s Y_{s}^{n,r} + P_{s} - [Y_{s}^{n,r} - x]| \rho(z) \mu^n(ds,dz),
$$

where we also used for some terms that $\partial_s Y_{s}^{n,r} = P_t$ (this can be deduced from 1.). We see easily that $P_1 \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - \frac{z^2}{n^{1/\alpha}} \mu^n(ds,dz) \leq P_1 \mu^n(\rho)$, but this does not permit to control $W_{1}^{n,r,(2,1)}/U_{1}^{n,r}$. So we write once again $U_t = \frac{P_1}{n^{2/\alpha}} Q_t$ with $Q$ defined above. Using $\rho(z) = z^2$ if $|z| > 1$, we have $|Q_t^{L,J}| \leq P_t R_1 R_t$. Consequently we obtain

$$
\frac{P_1}{n^{1/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} P_{s} U_s - \frac{z^2}{n^{1/\alpha}} \mu^n(ds,dz) \leq \frac{P_1}{n^{3/\alpha}} \mu^n(\rho) + R_1 + \frac{R_1}{n^{1/\alpha}} \int_{0}^{1} \int_{\mathbb{R}} R_{s} - z|1_{\{|z| > 1\}} \mu^n(ds,dz)|.
$$
The same inequality holds for \( \frac{d}{dt} \int_0^t P_s U_{s-} z \tilde{\mu}_n^\alpha (ds, dz) = \frac{d}{dt} \int_0^t Q_{s-} dM_s \) by decomposing into the small jumps and large jumps of \( Q \) and \( M \), as already done previously. Finally, considering the last term, we have

\[
\frac{P_1}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} P_{s-} |\partial_t Y_{s-}^n|^\alpha \mu^n (ds, dz) \leq \frac{P_1}{n^{2/\alpha}} \mu^n (\rho) \sup_t |\partial_t Y_t^n|,
\]

and from (5.45)

\[
\frac{P_1}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} P_{s-} |Y_{s-}^n - x|^\alpha \mu^n (ds, dz) \leq \frac{P_1}{n^{1+2/\alpha}} \mu^n (\rho) + \frac{P_1}{n^{3/\alpha}} |\mu^n (\rho)| + R_1 + R_1 \int_0^1 \int_{\mathbb{R}} R_{s-} |z| 1_{\{|z| > 1\}} \mu^n (ds, dz)\].

This completes the proof of 3.

4. Using (5.41) and (5.20)

\[
|W_{t, n}^{r, (3, 1)}| \leq \frac{P_1}{n^{1+4/\alpha}} \mu^n (\rho)^2 + \frac{P_1}{n^{1/\alpha}} \int_0^1 \int_{\mathbb{R}} P_{s-} U_{s-}^2 z^2 \mu^n (ds, dz) + \frac{P_1}{n^{3/\alpha}} |\mu^n (\rho)| + \frac{P_1}{n^{2/\alpha}} |\mu^n (\rho)|.
\]

We have

\[
\frac{P_1}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} P_{s-} U_{s-} \mu^n (ds, dz) \leq \frac{P_1}{n^{2/\alpha}} \mu^n (\rho)^2,
\]

\[
\frac{P_1}{n^{1/\alpha}} \int_0^1 \int_{\mathbb{R}} P_{s-} U_{s-}^2 \mu^n (ds, dz) \leq \frac{P_1}{n^{3/\alpha}} \mu^n (\rho)^2.
\]

Turning to the integral with respect to \( \tilde{\mu}_n \), \( J = \int_0^1 \int_{\mathbb{R}} P_{s-} U_{s-}^2 \tilde{\mu}_n^\alpha (ds, dz) \), we have the representation (recalling that \( U_t = \frac{1}{n^{\alpha/\alpha}} Q_t \))

\[
J = \frac{1}{n^{4/\alpha}} \int_0^1 (Q_{s-})^2 dM_s
\]

and analyzing each term in the decomposition of \( J \) between the large and small jumps of \( Q \) and \( M \), we obtain

\[
\frac{P_1}{n^{1/\alpha}} \int_0^1 \int_{\mathbb{R}} P_{s-} U_{s-}^2 \tilde{\mu}_n^\alpha (ds, dz) \leq \frac{P_1}{n^{4/\alpha}} (1 + \mu^n (\rho)^2).
\]

The proof of lemma 5.1 is finished. \( \square \)

Lemma 5.1 combined with (5.44) permits to obtain simple bounds for the Malliavin weight \( \mathcal{H}_{Y_{t, n}^{r, (p, q)}}(\partial_t Y_{t, n}^{r, (p, q)}) : \)

\[
|\partial_t Y_{t, n}^{r, (p, q)}| \leq P_1 |\partial_t Y_{t, n}^{r, (p, q)}| + P_1 n^{1/\alpha} |\partial_t Y_{t, n}^{r, (p, q)}| \frac{\mu^n (\rho) + \rho \rho'}{\mu^n (\rho)}, \quad (5.47)
\]

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\[
\left| \frac{W_{n,r}^{n,r,2(1)}}{U_{1}^{n,r}} \right| \leq P_{1} \left( \frac{1}{n} + \sup_{t} \left| \partial_{r} Y_{t}^{n,r} \right| \right) + \frac{P_{1}}{n^{1/\alpha}} \left( 1 + \frac{R_{1}}{\mu^{n}(\rho)} + \frac{R_{1} \int_{0}^{1} \int_{\mathbb{R}} R_{s_{1}}|z|1_{\{|z|>1\}} \mu^{n}(ds,dz)}{\mu^{n}(\rho)} \right), 
\]

(5.48)

\[
\left| \frac{\partial_{r} Y_{1}^{n,r} W_{n,r,3(1)}}{(U_{1}^{n,r})^{2}} \right| \leq P_{1} |\partial_{r} Y_{1}^{n,r}| + P_{1} n^{1/\alpha} |\partial_{r} Y_{1}^{n,r}| \frac{\mu^{n}(\rho')}{\mu^{n}(\rho)^2}. 
\]

(5.49)

It remains to evaluate the $L^2$-norm of these three terms. For this purpose, we establish an intermediate result.

**Lemma 5.2.** We recall that $R_{t} = \int_{0}^{t} \int_{\mathbb{R}} |z|1_{\{|z|>1\}} \mu^{n}(ds,dz)$. We have $\forall \varepsilon > 0$

(a)

\[
\mathbb{E}_{x} \left( P_{1} \int_{0}^{1} \int_{\mathbb{R}} R_{s_{1}}|z|1_{\{|z|>1\}} \mu^{n}(ds,dz) \right)^{2} \leq C(a,b,\alpha)(1 + |x|^{2})\frac{n^{4/\alpha}}{n^{2-\varepsilon}},
\]

(b)

\[
\mathbb{E}_{x} \left( P_{1} \frac{R_{1} \int_{0}^{1} \int_{\mathbb{R}} R_{s_{1}}|z|1_{\{|z|>1\}} \mu^{n}(ds,dz)}{\mu^{n}(\rho)} \right)^{2} \leq \begin{cases} 
C(a,b,\alpha)(1 + |x|^{2}), & \text{if } \alpha > 1, \\
C(a,b,\alpha)(1 + |x|^{2})\frac{n^{2/\alpha}}{n^{2}}, & \text{if } \alpha \leq 1.
\end{cases}
\]

**Proof.** We first recall that $\int_{0}^{t} \int_{\mathbb{R}} f(z)1_{\{|z|>1\}} \mu^{n}(ds,dz) = \sum_{i=1}^{N_{t}} f(Z_{i})$, where $(N_{t})$ is a Poisson process with intensity $\lambda_{n} = \int_{\mathbb{R}} F_{n}(z)1_{\{|z|>1\}} dz$ such that $\lambda_{n} \leq C(\alpha)$ and $(Z_{i})_{i \geq 1}$ are i.i.d. variables with density $\frac{F_{n}(z)1_{\{|z|>1\}}}{\lambda_{n}} dz$.

(a) We have

\[
|P_{1} \int_{0}^{1} \int_{\mathbb{R}} R_{s_{1}}|z|1_{\{|z|>1\}} \mu^{n}(ds,dz)| \leq P_{1} \sum_{i=1}^{N_{t}} |Z_{i}| \sum_{j=1}^{i-1} |Z_{j}| \leq P_{1} \sum_{i \neq j} |Z_{i}||Z_{j}|.
\]

So, we obtain from Hölder’s inequality for any $p > 1$

\[
\mathbb{E}_{x}(P_{1} \int_{0}^{1} \int_{\mathbb{R}} R_{s_{1}}|z|1_{\{|z|>1\}} \mu^{n}(ds,dz))^{2} \leq C(a,b,\alpha)(1 + |x|^{2})[\mathbb{E}(\sum_{i \neq j} |Z_{i}||Z_{j}|)^{2p}]^{\frac{1}{p}}.
\]

But we easily check that

\[
\mathbb{E}(\sum_{i \neq j} |Z_{i}||Z_{j}|)^{2p} \leq \mathbb{E}(N_{1}^{4p})[\mathbb{E}|Z_{i}|^{2p}]^{2},
\]

and that (the constant depends on $a$ through the truncation)

\[
\mathbb{E}|Z_{i}|^{2p} \leq C(a,\alpha)\frac{n^{2p/\alpha}}{n}.
\]

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This leads to
\[
\left[ \mathbb{E} \left( \sum_{i \neq j} |Z_i||Z_j| \right)^{2p} \right]^\frac{1}{p} \leq C(a, \alpha) \frac{n^{4/\alpha}}{n^{2/p}},
\]
and (a) is proved.

(b) Observing that \( \mu^n(\rho) \geq \mu_n(\rho \mathbb{1}_{\{|z| > 1\}}) \) and proceeding as in (a)
\[
|P_1 \int_0^1 \int_{|z| > 1} \frac{R_{a-} |z| \mathbb{1}_{\{|z| > 1\}} \mu^n(ds, dz)}{\mu^n(\rho)}| \leq P_1 \frac{\sum_{i=1}^{N_1} |Z_i| \sum_{i=1}^{N_1} |Z_i| \sum_{j=1}^{i-1} |Z_j|}{\sum_{i=1}^{N_1} |Z_i|^2}.
\]
But using successively Cauchy Schwarz inequality and \(|Z_i||Z_j| \leq \frac{1}{2}(|Z_i|^2 + |Z_j|^2)\)
\[
\left( \frac{\sum_{i=1}^{N_1} |Z_i| \sum_{i=1}^{N_1} |Z_i| \sum_{j=1}^{i-1} |Z_j|}{\sum_{i=1}^{N_1} |Z_i|^2} \right)^2 \leq N_1 \frac{\sum_{i \neq j} |Z_i||Z_j|}{\sum_{i=1}^{N_1} |Z_i|^2} \leq N_1^2 \sum_{i \neq j} |Z_i||Z_j|.
\]
Now for any \( p > 1 \) we have
\[
\mathbb{E} \left( \sum_{i \neq j} |Z_i||Z_j| \right)^p \leq \mathbb{E} \left( N_1^{2p} \right)^\frac{1}{2} \mathbb{E} \left( |Z_i|^p \right)^2 \leq C(a, \alpha)(1 + \frac{n^{2p/\alpha}}{n})^2.
\]
If \( \alpha > 1 \), choosing \( 1 < p < \alpha \) gives \( \mathbb{E} \left( \sum_{i \neq j} |Z_i||Z_j| \right)^p \leq C(a, \alpha) \) and we obtain the first part of (b) from Hölder’s inequality.
If \( \alpha \leq 1 \) then \( \mathbb{E} \left( \sum_{i \neq j} |Z_i||Z_j| \right)^p \leq C(a, \alpha) \frac{n^{2p/\alpha}}{n^{2/p}} \) and finally Hölder’s inequality gives \( \forall p > 1 \)
\[
\mathbb{E}_x \left( P_1 \int_0^1 \int_{|z| > 1} \frac{R_{a-} |z| \mathbb{1}_{\{|z| > 1\}} \mu^n(ds, dz) R_1}{\mu^n(\rho)} \right)^2 \leq C(a, b, \alpha)(1 + |x|^2) \frac{n^{2/\alpha}}{n^{2/p}}.
\]
\( \square \)

From Lemma 5.2 (a) and Lemma 5.1, we obtain immediately
\[
\mathbb{E}_x \sup_t |\partial_t Y_{t}^{n,r}|^2 \leq C(a, b, \alpha)(1 + |x|^2)(\frac{1}{n^2} + \frac{1}{n^{2/\alpha}} + \frac{1}{n^{2-\varepsilon}}),
\]
(5.50)
Consequently combining (5.50), (5.48), Lemma 5.2 (b) and observing that \( R_1 / \mu^n(\rho) \leq 1 \), we have
\[
\mathbb{E}_x \left| \frac{W_{n,r}^{(2,1)}}{\hat{U}_{1,n}^{n,r}} \right|^2 \leq C(a, b, \alpha)(1 + |x|^2)(\frac{1}{n^2} + \frac{1}{n^{2/\alpha}} + \frac{1}{n^{2-\varepsilon}}).
\]
To control the \( L^2 \)-norm of \( \frac{\partial_t Y_{t}^{n,r} W_{n,r}^{(3,1)}}{\hat{U}_{1,n}^{n,r}} \), in view of (5.49) and (5.50) it remains to bound
\[
n^{1/\alpha} |\partial_t Y_{t}^{n,r}| \frac{\mu^n(\rho |p|)}{\mu^n(\rho)^2}.
\]
We check that $\frac{\mu^n(|\rho'|)}{\mu^n(\rho)}(1 + R_1) \leq P_1$, and using $\mu^n(\rho) = \mu^n(|\rho'|1_{|z|\leq 1}) + \mu^n(|\rho'|1_{|z|> 1})$ with $\mu^n(|\rho'|1_{|z|> 1}) \leq 2R_1\mu^n(\rho)$, it yields

$$\frac{\mu^n(|\rho'|)}{\mu^n(\rho)} \int_0^1 \int_{\mathbb{R}} s - |z|1_{|z|> 1} \mu^n(ds, dz) \leq P_1 + P_1 \frac{R_1 \int_0^1 \int_{\mathbb{R}} s - |z|1_{|z|> 1} \mu^n(ds, dz)}{\mu^n(\rho)}.$$  

So from Lemma 5.1 we have

$$n^{1/\alpha}|\partial r Y_{1}^{\alpha,n,\tau}| \frac{\mu^n(|\rho'|)}{\mu^n(\rho)} \leq P_1 \left( \frac{1}{n} + \frac{1}{n^{1/\alpha}} \right) + P_1 \frac{R_1 \int_0^1 \int_{\mathbb{R}} s - |z|1_{|z|> 1} \mu^n(ds, dz)}{\mu^n(\rho)}.$$  

and consequently from (5.49), (5.50) and Lemma 5.2 we conclude

$$E_x \left| \partial r Y_{1}^{\alpha,n,\tau}| W_{1}^{\alpha,n,\tau}(3,1) \right|^2 \leq C(a, b, \alpha)(1 + |x|^2)(\frac{1}{n^2} + \frac{1}{n^{2/\alpha}} + \frac{1}{n^{2-\tau}}).$$  

For the last term $\frac{\partial Y_{1}^{\alpha,n,\tau}| W_{1}^{\alpha,n,\tau}}{U_{1}^{\alpha,n,\tau}}$, in view of (5.47) and (5.50) it remains to study

$$P_1 n^{1/\alpha}|\partial r Y_{1}^{\alpha,n,\tau}| \frac{\mu^n(|\rho'| + \frac{F'}{F_n})}{\mu^n(\rho)}.$$  

where $\frac{F'}{F_n}(z) = \frac{1}{z} + \frac{1}{n^{1/\alpha}} \frac{g'(\frac{z}{n^{1/\alpha}})}{g(\frac{z}{n^{1/\alpha}})}.$ For any $p \geq 1$, we have using A1

$$E \int_0^1 \int_{\mathbb{R}} \left| g'(\frac{z}{n^{1/\alpha}}) \right|^p 1_{|z|> 1} \mu^n(ds, dz) = 2 \int_{\mathbb{R}} \left| g'(\frac{z}{n^{1/\alpha}}) \right|^p g(\frac{z}{n^{1/\alpha}}) \frac{1}{z^{\alpha+1}} dz $$

$$= \frac{2}{n} \int_{1/n^{1/\alpha}}^{|x|^\infty} \left| g'(u) \right|^p g(u) \frac{1}{u^{\alpha+1}} du $$

$$\leq C \int_{1/n^{1/\alpha}}^1 \frac{1}{u^{\alpha+1}} du + \int \left| g'(u) \right|^p g(u) du $$

$$\leq C(\alpha).$$  

So it yields, introducing $1_{|z|\leq 1}$ and $1_{|z|> 1}$

$$\mu^n(|\rho'| + \frac{F'}{F_n}) \leq P_1(1 + R_1).$$  

Next, Lemma 5.1 and the previous bound give

$$P_1 n^{1/\alpha}|\partial r Y_{1}^{\alpha,n,\tau}| \frac{\mu^n(|\rho'| + \frac{F'}{F_n})}{\mu^n(\rho)} \leq P_1 \left( \frac{1}{n} + \frac{1}{n^{1/\alpha}} \right) + P_1 \frac{R_1 \int_0^1 \int_{\mathbb{R}} s - |z|1_{|z|> 1} \mu^n(ds, dz)}{\mu^n(\rho)},$$  

and we conclude with (5.47), (5.50) and Lemma 5.2

$$E_x \left| \partial r Y_{1}^{\alpha,n,\tau}| W_{1}^{\alpha,n,\tau} \right|^2 \leq C(a, b, \alpha)(1 + |x|^2)(\frac{1}{n^2} + \frac{1}{n^{2/\alpha}} + \frac{1}{n^{2-\tau}}).$$  

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Collecting all these results, we finally have proved, ∀ε > 0
\[ \mathbb{E}_x |H_{Y_{1,k,n,r}}^K(\partial_r \tilde{Y}_{n,r}^K)|^2 \leq C(a, b, \alpha) (1 + |x|^2) \left( \frac{1}{n^2} + \frac{1}{n^{2/\alpha}} + \frac{1}{n^{2-\varepsilon}} \right). \]

We can easily see that the constant \( C(a, b, \alpha) \) has exponential growth in \( ||b'||\infty \) and polynomial growth in \( ||b''||\infty, ||a'||\infty, ||a''||\infty, 1/||a'||\infty, b(0), a(0), 1/\alpha \) and \( 1/(\alpha - 2) \).

To complete the proof of Theorem 3.3, we consider the Euler approximation. The proof follows the same lines but the bound for \( \partial_r \tilde{Y}_{n,r}^K \) has the additional term \( b(x)/n^2 \). So the first item in Lemma 5.1 is replaced by
\[ \sup_{t \in [0, 1]} |\partial_r \tilde{Y}_{n,r}^K| \leq \frac{P_1}{n^2} + \frac{P_1}{n^{1+1/\alpha}} (1 + R_1) \]
\[ + \frac{P_1}{n^{2/\alpha}} \left( 1 + R_1 + \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} R_s - |z|1(|z|>1) \mu^n(ds, dz) \right). \]

Since we have to control not only \( \sup_t |\partial_r \tilde{Y}_{n,r}^K| \) but also \( n^{1/\alpha} \sup_t |\partial_r \tilde{Y}_{n,r}^K| \), we have the extra term \( n^{1/\alpha}/n^2 \) and finally
\[ \mathbb{E}_x |H_{Y_{1,k,n,r}}^K(\partial_r \tilde{Y}_{n,r}^K)|^2 \leq C(a, b, \alpha) (1 + |x|^2) \left( \frac{n^{2/\alpha}}{n^4} + \frac{1}{n^2} + \frac{1}{n^{2/\alpha}} + \frac{1}{n^{2-\varepsilon}} \right). \]

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