TALBOT EFFECT FOR THE SCHRÖDINGER EQUATION

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Abstract. We study fractal properties of the solution to the periodic Schrödinger equation with singular potentials and bounded variation initial data. The solution turns out to have fractal behavior at irrational times, whereas it resembles the initial data in regularity at rational times. This extends the results of Oskolkov [18] and Rodnianski [20] on the free Schrödinger equation to general cases having potentials. For the purpose, we employ the Bourgain space and obtain smoothing estimates for the Duhamel part involving singular potentials.

1. Introduction

When a quantum particle is in a box with reflecting boundaries, the graph of the probability density in spacetime forms certain pattern called quantum carpet. This interesting phenomenon was experimentally observed in classical optics firstly by H. F. Talbot [22] in 1836. He discovered that when an incident plane wave passes through a periodic grating, the image of the diffracted wave is refocused and recovers the initial grating with certain periodicity. Later, the time period to recover the initial grating pattern, which is called Talbot distance $d_T$, was calculated by Rayleigh [19] as $d_T = a^2/\lambda$, where $a$ is the spacing of the grating and $\lambda$ is the wavelength of the incident wave.

Until recently, there have been intensive mathematical studies to understand this Talbot effect ([2, 3, 6, 7, 9, 10, 15, 17, 18, 20, 24]). In [3], Berry and Klein mathematically justified the Talbot effect by using the paraxial propagator defined by a Gauss sum involving the free Schrödinger evolution to model the wave function due to an evenly spaced diffraction grating. They obtained a striking dichotomy between “rational” and “irrational” times showing that at every $t \in d_T\mathbb{Q}$ a finite copies of the grating pattern reappear, whereas at every $t \notin d_T\mathbb{Q}$ the wave function exhibits fractal nowhere differentiable profile. Meanwhile, many authors considered the periodic Schrödinger equation

$$\begin{align*}
\begin{cases}
\imath \partial_t u + \partial_{xx} u = 0, & x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \ t \in \mathbb{R} \\
u(0, x) = f(x),
\end{cases}
\end{align*}
$$

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to study implication of algebraic properties of time in the wave function. Under the assumption that the initial $f$ is of bounded variation, Oskolkov [18] proved that the wave function $u(t, x)$ is a continuous but nowhere differentiable function of $x$ if $t \notin \pi \mathbb{Q}$, while it is necessarily discontinuous if $t \in \pi \mathbb{Q}$ whenever $f$ contains discontinuities. Later Kapitanski and Rodnianski [15] showed better regularity of $u(t, \cdot)$ for $t \notin \pi \mathbb{Q}$.

For every $t \in \pi \mathbb{Q}$, Taylor [24] found that $u(t, \cdot)$ is a finite linear sum of translates of the delta functions with the coefficients being Gauss sums if $f$ is the delta function. As another manifestation of the Talbot effect, for a step function $f$, Berry [2] calculated the fractal dimension $1$ of the temporal, spatial and diagonal restrictions of the graph of the density function $|u(t, x)|^2$. Based on the calculations he conjectured that the graphs of $\text{Re} u(t, \cdot)$, $\text{Im} u(t, \cdot)$ and $|u(t, \cdot)|^2$ have fractal dimension $D = n + 1/2$ at almost all irrational times $t$. Rodnianski [20] proved that for every bounded variation $f$ which is not in $\bigcup_{\epsilon > 0} H_2^{s_0 + \epsilon}$, the graphs of the real and imaginary parts of the solution $e^{it\partial_x} f$ have fractal dimension $D = 3/2$ at almost all irrational times. This rigorously justifies Berry’s conjecture in one dimension other than the statement for the density $|e^{it\partial_x} f|^2$. Recently, Chousionis, Erdo\'gan and Tzirakis [7] extended this result by considering initial data in larger class $\text{BV} \setminus \bigcup_{\epsilon > 0} H_2^{s_0 + \epsilon}$, $1/2 \leq s_0 < 3/4$, and also settled Berry’s conjecture in one dimension proving that the dimension of the graph of $|e^{it\partial_x} f|^2$ is $3/2$ at almost all $t \notin \pi \mathbb{Q}$ whenever $f$ is a step function having jumps only at rational points.

In a series of recent papers [7, 9, 10], Erdo\'gan with his collaborators greatly developed the mathematical theory of the Talbot effect. In particular, a nonlinear Schr"odinger equation was studied in [9, 10]. In their argument the key ingredient was to obtain smoothing estimate in the Bourgain spaces for the nonlinear part, which, combined with the known results on the linear part ([18, 15, 20]), gives the quantization and dimension results for the cubic NLS.

Adapting this framework we aim to investigate the Talbot effect for the general Schrödinger equation with potentials:

$$
\begin{aligned}
&\{iu_t + u_{xx} = V(x)u, \quad x \in T, \ t \in \mathbb{R}, \\
&u(0, x) = f(x).
\end{aligned}
$$

To facilitate the statements of our results we introduce some notations. We denote by $\text{BV}$ the space of functions of bounded variation in $T$. For a $2\pi$-periodic function $f$ we define its Fourier coefficient by $\hat{f}(k) = \frac{1}{2\pi} \int_T e^{-ikx} f(x)dx$ for any $r \in \mathbb{R}$. For every $s \geq 0$ we denote by $H^s$ the Sobolev space on $T = \mathbb{R}/2\pi \mathbb{Z}$ equipped with the norm

$$
\|f\|_{H^s} := \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{f}(k)|^2 \right)^{1/2}
$$

The fractal dimension, also called upper Minkowski dimension, of a set $S \subset \mathbb{R}^n$ is defined by

$$
\limsup_{\epsilon \to 0} \frac{\log(N(S, \epsilon))}{\log(1/\epsilon)},
$$

where $N(S, \epsilon)$ is the minimum number of balls of radius $\epsilon$ required to cover $S$. 

\[1\]
where $\langle k \rangle = (1 + |k|^2)^{1/2}$. We also use the notation $H^{s+} := \bigcup_{\epsilon > 0} H^{s+\epsilon}$ and $H^+ := H^{0+}$. Furthermore, throughout this paper, if a Banach space of functions $B^s$ is decreasing (in the sense of the set inclusion) with respect to a regularity index $s \in \mathbb{R}$, then we use the following handy notation

$$B^s := \bigcup_{\epsilon > 0} B^{s+\epsilon}, \quad B^s := \bigcap_{\epsilon > 0} B^{s-\epsilon}. \tag{1.4}$$

Examples of $B^s$ include the Sobolev space $H^s$, the Besov space $B^s_{p,q}$ and the Hölder space $C^s$. Our first result is on the dichotomy in regularity at rational and irrational times.

**Theorem 1.1.** Let $u$ be a solution to (1.2) with $f \in BV$ and $V \in H^+$. If $t \notin \pi\mathbb{Q}$ then $u(t,x)$ is a continuous function of $x$. If $t \in \pi\mathbb{Q}$ and $f$ has at least one discontinuity on $T$, then $u(t,x)$ is a bounded function and necessarily contains (at most countable) discontinuities. If $f$ is continuous, then $u(t,x)$ is jointly continuous on $\mathbb{R} \times T$.

We also compute the fractal dimension of the graph of the solution, in terms of regularity of potentials and initial data, at irrational time slices.

**Theorem 1.2.** Let $u$ be a solution to (1.2) with $f \in BV$ and $V \in H^+$. Suppose that

$$\sigma_0 := \sup\{\sigma \in \mathbb{R}: f \in H^\sigma\} < 3/4.$$

If we set

$$r_0 := \sup\{r \in \mathbb{R}: V \in H^r\},$$

then, for almost all $t \in \mathbb{R} \setminus \pi\mathbb{Q}$,

1. the upper Minkowski dimension of the graphs of $\text{Re}u(t,\cdot)$, $\text{Im}u(t,\cdot)$ and $|u(t,\cdot)|^2$ is less than or equal to $\max\{\frac{3}{2}, \frac{3}{2} - \sigma_0 - r_0\}$;

2. the upper Minkowski dimension of the graphs of $\text{Re}u(t,\cdot)$ and $\text{Im}u(t,\cdot)$ are greater than or equal to $\frac{5}{2} - 2\sigma_0$ provided that $\sigma_0 - \frac{1}{2} < r_0$.

**Remark 1.3.** In the above theorems, the space $H^+$ of potentials contains a number of interesting functions. It is a classical result due to Haslam-Jones [14] that the Fourier coefficients of the unbounded function

$$V(x) = \frac{\phi(x)}{|x|^\nu(\log |x|/\kappa)^a}, \quad -\pi \leq x < \pi, \tag{1.5}$$

where $\phi \in BV$, $0 < \nu < 1$, $\kappa > \pi$ and $a \in \mathbb{R}$, satisfy

$$|\hat{V}(0)| \lesssim 1 \quad \text{and} \quad \hat{V}(k) = O(|k|^{\nu-1}(\log |k|)^{-a}), \quad k \neq 0.$$

Thus, for every $a \in \mathbb{R}$ and $\nu < \frac{1}{2}$, the unbounded potential (1.5) is in $H^+$. On the other hand, since $C^\alpha \hookrightarrow H^s$ for $0 < s < \alpha \leq 1$ (10 Theorem 1.13), we see that the space $H^+$ includes all classes of Hölder continuous periodic functions.

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2Since $f \in BV$ it follows that $\sigma_0 \geq 1/2$. 

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Organization. In Section 2, we state the crucial smoothing estimate for the Duhamel part of the initial value problem (1.2) (Proposition 2.1) as well as the known results on the free Schrödinger evolution that we need. Then we prove Theorems 1.1 and 1.2. In the remaining sections, we focus on proving the smoothing estimate. In Section 3, we first obtain bilinear estimates in the Bourgain space for the potential part $V u$ in (1.2). Based upon this, in Section 4, we establish well-posedness for (1.2) and then prove the smoothing estimate.

Notation. In inequalities, we employ the letter $C$ to denote a positive constant which may change at each occurrence. For $A, B > 0$ we write $A \lesssim B$ if $A \leq CB$ with some constant $C > 0$. We also use the notation $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Talbot effect for the Schrödinger equation

In this section, we prove Theorems 1.1 and 1.2. Let us first rearrange the equation in (1.2) as follows:

$$iu_t + (\partial_{xx} - \hat{V}(0))u = R(V,u),$$

where $R(V,u)$ is the function on $\mathbb{R} \times \mathbb{T}$ defined by

$$R(V,u)(t,x) = (V(x) - \hat{V}(0))u(t,x).$$

Then, by Duhamel’s formula, the solution to the initial value problem (1.2) can be equivalently written as

$$u(t,x) = e^{it(\partial_{xx} - \hat{V}(0))}f(x) - i \int_0^t e^{i(t-t')(\partial_{xx} - \hat{V}(0))}R(V,u)(t',x)dt'.$$

We shall prove that the Duhamel term

$$\mathcal{P}(t,x) := -i \int_0^t e^{i(t-t')(\partial_{xx} - \hat{V}(0))}R(V,u)(t',x)dt'$$

is in fact continuous on $\mathbb{R} \times \mathbb{T}$. For the purpose we obtain the following smoothing property for $\mathcal{P}(t,\cdot)$, which is the key ingredient in this paper. We provide the proof of the property in Section 4.2.

Proposition 2.1. Suppose that $V \in H^r$ and $f \in H^s$ for $r \geq 0$ and $0 < s < r + 1$. If $a \leq r$ and $a < \min\{1 + r - s, 1/2\}$, then $\mathcal{P}(t,\cdot) \in H^{s+a}$ for every $t \in \mathbb{R}$, and is continuous in the time variable $t$.

For the evolution $e^{it(\partial_{xx} - \hat{V}(0))}$, we make use of the following known result due to Oskolkov [18, Proposition 14 and page 390]:

Theorem 2.2. Suppose that $g \in BV$.

(i) If $t \notin \pi \mathbb{Q}$, then $e^{it\partial_{xx}}g$ is a continuous function of $x$. If $g$ has at least one discontinuity on $\mathbb{T}$ and $t \in \pi \mathbb{Q}$, then $e^{it\partial_{xx}}g$ necessarily contains discontinuities.

(ii) If $g$ is continuous, then $e^{it\partial_{xx}}g$ is jointly continuous in temporal and spatial variables.
Remark 2.3. The quantization results of Berry and Klein [3] and Taylor [24] state that if $t \in \pi \mathbb{Q}$ then $e^{it\partial_x x}g$ is a linear sum of finitely many translates of $g$ (see [12, Theorem 2.14]). Hence, in Theorem 2.2 (i), $e^{it\partial_x x}g \in BV$ whenever $t \in \pi \mathbb{Q}$, so the possible discontinuities in this case are at most countable.

Proof of Theorem 1.2. By the assumption $V \in H^+$ it is possible to pick a sufficiently small number $\alpha \in (0, \frac{1}{2})$ such that $V \in H^{2\alpha}$. Since $f \in BV$ it follows that $f \in H^{\frac{1}{2}-}$ and in particular, $f \in H^{\frac{1}{2}-\alpha}$. Thus, making use of Proposition 2.1 (with $r = 2\alpha$, $s = \frac{1}{2} - \alpha$ and $\alpha = 2\alpha$), we conclude that $P(t,x) \in C_t H^\frac{1}{2}+\alpha$. Moreover, from the Sobolev embedding

$$H^s \hookrightarrow C^{s-\frac{1}{2}} \quad \text{for} \quad s > 1/2,$$  \hspace{1cm} (2.2)

it follows that

$$\mathcal{P}(t,x) \in C_t C^\alpha_x. \hspace{1cm} (2.3)$$

For $t \notin \pi \mathbb{Q}$, Theorem 2.2 (i) shows that

$$e^{-it(\partial_x x-\hat{V}(0))}f = e^{it\hat{V}(0)}e^{-it\partial_x x}f \hspace{1cm} \text{(2.4)}$$

is continuous. Hence, combined with (2.3), it follows from (2.1) that

$$u(t,\cdot) = e^{-it(\partial_x x-\hat{V}(0))}f(\cdot) + \mathcal{P}(\cdot,\cdot) \hspace{1cm} \text{(2.5)}$$

is also continuous.

If $t \in \pi \mathbb{Q}$ and $f$ is discontinuous, then it follows from Theorem 2.2 (i) and Remark 2.3 that the evolution (2.4) is of bounded variation and contains (at most countable) discontinuity. Thus, (2.3) tells us that the solution (2.5) is a discontinuous bounded function with at most countable discontinuity.

If $f$ is continuous, then the function (2.4) is also continuous by Theorem 2.2 (ii). Hence, combining with 2.3 we see that the solution (2.5) is jointly continuous on $\mathbb{R} \times T$. \hfill \Box

Now we prove Theorem 1.2. Let us first recall the Besov space and its properties that we need. Let $\phi \in C_0^\infty([-2, -1/2] \cup [1/2, 2])$ be such that $\sum_{j \in \mathbb{Z}} \phi(2^{-j}t) = 1$ for $t \in \mathbb{R} \setminus \{0\}$, and let $\phi_0(t) := 1 - \sum_{j \geq 1} \phi(2^{-j}t)$. We denote by $P_j$ the projections defined by

$$P_0 f(x) := \sum_{k \in \mathbb{Z}} \phi_0(k) \hat{f}(k) e^{ikx}, \quad P_j f(x) := \sum_{k \in \mathbb{Z}} \phi(2^{j-k}) \hat{f}(k) e^{ikx}, \quad j \geq 1.$$

For $1 \leq p, q \leq \infty$ and $s \geq 0$, the inhomogeneous Besov space $B^s_{p,q}$ on the periodic domain $T$ is a Banach space of functions equipped with the norm

$$||f||_{B^s_{p,q}} := \left\{ \begin{array}{ll}
||P_0 f||_{L^p(T)} + \left( \sum_{j \geq 1} (2^{sj} ||P_j f||_{L^p(T)})^q \right)^{1/q} & \text{if} \quad q < \infty, \\
\sup_{j \geq 0} 2^{sj} ||P_j f||_{L^p(T)} & \text{if} \quad q = \infty.
\end{array} \right. \hspace{1cm} (2.6)$$

It is well-known\footnote{See, for example, [21] (Remark 4 on p. 164 and Theorem (i), (v) on pp. 168-169).} that $B^s_{2,2} = H^s$ for every $s$ and $B^s_{\infty,\infty} = C^\alpha$ for $\alpha \in (0, \infty) \setminus \mathbb{N}$. By complex interpolation between Besov spaces (see [11, Theorem 6.4.5]) and Hölder’s...
inequality, we have, for \( s_1 \neq s_2 \),
\[
(B^{s_1}_{1,\infty}, B^{s_2}_{\infty,\infty})_{\frac{1}{2}} = B^{s_1 + \varepsilon}_{2,\infty} \hookrightarrow B^{s_2}_{2,\infty} = H^{s_2 - \varepsilon}.
\] (2.7)

We also make use of the following theorems of Chousionis–Erdo˘gan–Tzirakis \cite{7} and Deliu–Jawerth \cite{8}.

**Theorem 2.4** (\cite{7}). Let \( \frac{1}{2} \leq \sigma_0 < \frac{3}{4} \) and suppose \( g \in BV \setminus H^{\sigma_0^+} \). Then, for almost all \( t \in \mathbb{R} \setminus \pi \mathbb{Q} \), we have \( e^{it\partial_x} g \in C^{1/2} \setminus B^{2\sigma_0 - 1/4}_1 \).

**Theorem 2.5** (\cite{8}). Let \( 0 < s < 1 \). Assume that \( f: \mathbb{T} \rightarrow \mathbb{R} \) is continuous and \( f \notin B^{s}_{1,\infty} \). Then the upper Minkowski dimension of the graph of \( f \) is at least \( 2 - s \).

**Proof of Theorem 2.5** Let us first prove the part (1). By the definition of \( \sigma_0 \) and \( r_0 \) it is clear that
\[
f \in H^{\sigma_0^+} \quad \text{and} \quad V \in H^{r_0^+}.
\]
By applying Proposition 2.1 (with \( s = \sigma_0 - \epsilon \) and \( r = r_0 - 2\epsilon \) for an infinitesimal \( 0 < \epsilon \ll 1 \)) we have
\[
\mathcal{P}(t, \cdot) \in H^{\sigma_0^+ + \epsilon}, \quad \forall t \in \mathbb{R},
\] (2.8)
whenever
\[
a < \min\{r_0, 1 + r_0 - \sigma_0, 1/2\} = \min\{r_0, 1/2\} =: a_0.
\]
Hence, by the Sobolev embedding (2.2) we have
\[
\mathcal{P}(t, \cdot) \in C^{\sigma_0^+ + a - \frac{1}{2}}, \quad \forall t \in \mathbb{R},
\] (2.9)
for every \( a < a_0 \). On the other hand, by Theorem 2.4 we have
\[
e^{it\partial_x} f \in C^{1/2} \quad \text{for almost all} \quad t \in \mathbb{R} \setminus \pi \mathbb{Q}.
\] (2.10)
Hence, it follows from (2.5), (2.9) and (2.10) that
\[
u(t, \cdot) \in C^{|\min(\frac{1}{2}, \sigma_0 + a - \frac{1}{2})|} \quad \text{for almost all} \quad t \in \mathbb{R} \setminus \pi \mathbb{Q}
\] (2.11)
for every \( a < a_0 \). It is obvious that the same statement is still valid for \( \text{Re}u(t, \cdot), \text{Im}u(t, \cdot) \) and \( |u(t, \cdot)|^2 \) in place of \( u(t, \cdot) \) in (2.11). We now use the following classical result on the upper Minkowski dimension for Hölder continuous functions (for a proof we refer the reader to [13 Corollary 11.2]):

**Lemma 2.6.** Let \( 0 \leq \alpha \leq 1 \). If a function \( f: \mathbb{T} \rightarrow \mathbb{R} \) is \( C^\alpha \), then the upper Minkowski dimension of the graph of \( f \) is at most \( 2 - \alpha \).

Indeed, by Lemma 2.6, the upper Minkowski dimension of the graphs of \( \text{Re}u(t, \cdot), \text{Im}u(t, \cdot) \) and \( |u(t, \cdot)|^2 \) is at most
\[
2 - \min\{1/2, \sigma_0 + a_0 - 1/2\} = \max\{3/2, 5/2 - \sigma_0 - a_0\}
\]
for almost all \( t \in \mathbb{R} \setminus \pi \mathbb{Q} \). We notice further that if \( a_0 = 1/2 \), then \( 5/2 - \sigma_0 - a_0 = 2 - \sigma_0 \in (5/4, 3/2] \), so the maximum is equal to \( 3/2 \). Therefore, we obtain (1).

Let us now prove the part (2). Since \( f \notin H^{\sigma_0^+} \), we see that for almost every \( t \), neither \( \text{Re}e^{it\partial_x} f \) nor \( \text{Im}e^{it\partial_x} f \) belong to \( H^{\sigma_0^+} \) (see \cite{7} Lemma 3.2). It follows from (2.10) and the embedding \( B^{1}_{1,\infty} \cap B^{2}_{\infty,\infty} \subset H^{1/2} \) (see (2.7)) that, for almost
all \( t \in \mathbb{R} \setminus 2\pi \mathbb{Q} \), both the real and the imaginary parts of \( e^{it\partial_{xx}} f \) do not belong to \( B^{2\sigma_0-\frac{1}{2}}_{1,\infty} \). On the other hand, by (2.8) we see \( \mathcal{P}(t,\cdot) \in B^{\sigma_0+\min\{r_0,\frac{1}{2}\}}_{1,\infty} \) for all \( t \in \mathbb{R} \). Combining these we have

\[
\hat{u}(t,\cdot) = \text{Re}(e^{it\partial_{xx} \hat{V}(0)} f) + i \text{Im}(e^{it\partial_{xx} \hat{V}(0)} f) + \mathcal{P}(t,\cdot)
\]

\( \notin B^{\sigma_0-\frac{1}{2}}_{1,\infty} \) for a.e. \( t \notin B^{\sigma_0-\frac{1}{2}}_{1,\infty} \) for a.e. \( t \notin B^{\sigma_0+\min\{r_0,\frac{1}{2}\}}_{1,\infty} \) for all \( t \).

From this we conclude that if \( 2\sigma_0 - \frac{1}{2} < \sigma_0 + \min\{r_0,\frac{1}{2}\} \), that is, either \( r_0 \geq \frac{1}{2} \) or \( \sigma_0 - \frac{1}{2} < r_0 < \frac{1}{2} \), then for almost all \( t \notin 2\pi \mathbb{Q} \) neither the real nor the imaginary parts of \( u(t,\cdot) \) belong to \( B^{2\sigma_0-\frac{1}{2}}_{1,\infty} \). By Theorem 2.5 we conclude that the graphs of real and imaginary parts of \( u(t,\cdot) \) have Minkowski dimension \( \geq 2 - (2\sigma_0 - \frac{1}{2}) = \frac{5}{2} - 2\sigma_0 \).

### 3. Bilinear estimate in the Bourgain space

In this section, we obtain bilinear estimates for \( R(V,u) \) in the Bourgain space, which are essential in proving Proposition 2.1.

#### 3.1. The Bourgain space

For \( s, b \in \mathbb{R} \), we denote by \( X^{s,b} \) the closure of the set of Schwartz functions \( \mathcal{S}(\mathbb{R}; C^\infty(\mathbb{T})) \) under the norm

\[
\|u\|_{X^{s,b}} := \| (k)^s (\tau + k^2)^b \hat{u}(\tau, k) \|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{Z})},
\]

where \( \hat{u} \) denotes the space-time Fourier transform of \( u \) defined by

\[
\hat{u}(\tau, k) = \int_{\mathbb{R}} e^{-it\tau} u(t, k) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{Z}} e^{-i(\tau t + kx)} u(t, x) dx dt, \quad (\tau, k) \in \mathbb{R} \times \mathbb{Z}.
\]

The \( X^{s,b} \)-space is also called the Bourgain space or dispersive Sobolev space. We also define, for a closed interval \( I \subset \mathbb{R} \), the restricted space \( X^{s,b}_I \) as the Banach space of functions on \( I \times \mathbb{T} \) equipped with the norm

\[
\|u\|_{X^{s,b}_I} = \inf \|w\|_{X^{s,b}} : w|_{I \times \mathbb{T}} = u.
\]

We notice that \( X^{s,b}_\mathbb{R} = X^{s,b} \).

The following are some basic properties of the Bourgain space that we need to prove the smoothing estimate (Proposition 2.1). The proofs of the properties, which we omit, can be obtained by routine adaptation of those of Corollary 2.10, Lemma 2.11 and Proposition 2.12 in the spatially periodic setting and time translation argument.

#### Lemma 3.1

Let \( s \in \mathbb{R}, b > \frac{1}{2} \) and \( I \subset \mathbb{R} \) a closed interval. Then \( X^{s,b}_I \hookrightarrow C(I; H^s) \) and

\[
\sup_{t \in I} \|u(t,\cdot)\|_{H^s} \leq C \|u\|_{X^{s,b}_I},
\]

where \( C \) is a constant depending only on \( b \).

#### Lemma 3.2

Let \( s \in \mathbb{R}, -\frac{1}{2} < b < b' < \frac{1}{2} \) and \( I \) a closed interval of length \( \delta \). Then

\[
\|u\|_{X^{s,b}_I} \leq C \delta^{b'-b} \|u\|_{X^{s,b'}_I},
\]

where the constant \( C \) depends only on \( b \) and \( b' \).
Lemma 3.3. Let $s \in \mathbb{R}$, $\frac{1}{2} < b \leq 1$ and $I = [t_0, t_0 + \delta]$ for $t_0 \in \mathbb{R}$ and $0 < \delta \leq 1$. Then, for $t \in I$, we have
\[
\|e^{i(t-t_0)\partial_x} f\|_{X^{s,b}_I} \leq C\|f\|_{H^s},
\]
\[
\left\| \int_{t_0}^t e^{i(t-t')\partial_x} F(t', \cdot) dt' \right\|_{X^{s,b}_I} \leq C\|F\|_{X^{s,b-1}_I},
\]
where $C$ depends only on $b$.

3.2. Bilinear estimate for $R(V, u)$. In this section we estimate $R(V, u) = (V - \tilde{V}(0))u$ in the $X^{s,b}$-space.

Proposition 3.4. Let $r \geq 0$ and $0 < s < 1 + r$, and suppose that
\[ a \leq r, \quad a < 1 + r - s \quad \text{and} \quad a < 1/2. \]

Then, for every interval $I$,
\[
\|R(V, u)\|_{X_I^{s+a,b'-1}} \lesssim \|V\|_{H^r} \|u\|_{X^{s,b}_I} \tag{3.1}
\]
provided that $b, b' \in (\frac{1}{2}, \frac{1}{2} + \epsilon)$ for an $\epsilon > 0$ small enough.

We first recall from [11, Lemma 3.3] the following simple lemma which is used several times in proving the proposition.

Lemma 3.5. Let us define, for $\beta \geq 0$,
\[
\phi_\beta(k) := \sum_{|n| \leq |k|} \frac{1}{(n!)^\beta} \approx \begin{cases} 1, & \beta > 1, \\ \log(1 + \langle k \rangle), & \beta = 1, \\ \langle k \rangle^{1-\beta}, & \beta < 1. \end{cases} \tag{3.2}
\]

If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$, then
\[
\sum_n \frac{1}{(n-k_1)^\beta (n-k_2)^\gamma} \approx \int_0^\infty \frac{1}{(\tau-k_1)^\beta (\tau-k_2)^\gamma} d\tau \lesssim \frac{\phi_\beta(k_1-k_2)}{\langle k_1-k_2 \rangle^\gamma}.
\]

Proof of Proposition 3.4. First, let us prove (3.1) by replacing the restricted spaces $X_I^{s+a,b'-1}$ and $X_I^{s,b}$ with $X^{s+a,b'-1}$ and $X^{s,b}$, respectively. We write
\[
\|R(V, u)\|_{X_I^{s+a,b'-1}}^2 \tag{3.3}
\]
\[
= \left| \int \sum_{k \in \mathbb{Z}^+} \left| \sum_{l \in \mathbb{Z} \setminus \{k\}} \langle k \rangle^{s+a} \langle r + k^2 \rangle^{b'-1} \tilde{V}(k-l) \tilde{u}(r,l) \right|^2 d\tau \right|
\]
\[
= \left| \int \sum_{k \in \mathbb{Z}^+} \left| \sum_{l \in \mathbb{Z} \setminus \{k\}} M(k,l,r) \langle k-l \rangle^r \tilde{V}(k-l) \langle l \rangle^s \langle r + l^2 \rangle^b \tilde{u}(r,l) \right|^2 d\tau \right|
\]
where
\[
M(k,l,r) := \langle k \rangle^{s+a} \langle r + k^2 \rangle^{b'-1} \langle l \rangle^{-s} \langle k-l \rangle^{-r} \langle r + l^2 \rangle^{-b}.
\]
By the Cauchy–Schwarz inequality and Young’s inequality for the convolution, \[ (3.3) \]
is bounded by
\[
\sup_{\tau, k} \sum_{l \in \mathbb{Z}} M(k, l, \tau)^2 \int \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (k - m)^{2r} |\tilde{V}(k - m)|^2 (m)^{2s} (\tau + m^2)^{2b} |\tilde{u}(\tau, m)|^2 \, d\tau 
\leq \sup_{\tau, k} \sum_{l \in \mathbb{Z}} M(k, l, \tau)^2 \|V\|^2_{H^r} \|u\|^2_{\dot{H}^{s,b}}.
\]
Hence it remains to prove that
\[
\sup_{\tau, k} \sum_{l \in \mathbb{Z}} M(k, l, \tau)^2 < \infty.
\]

From the assumption, it is clear that \(2b \geq 2 - 2b'\). Hence, by the easy inequality \(\langle \tau + m \rangle \langle \tau + n \rangle \geq \frac{1}{2} (n - m)\), we have
\[
\sum_{l \in \mathbb{Z} \setminus \{k\}} M(k, l, \tau)^2 \leq \sum_{l \in \mathbb{Z} \setminus \{k\}} \frac{\langle k \rangle^{2a + 2a} \langle l \rangle^{-2s} \langle k - l \rangle^{-2r}}{(\tau + k^2)^{2 - 2b'} (\tau + l^2)^{2 - 2b'}} 
\leq \frac{1}{\langle k \rangle^{2a - 2r}} \sum_{l \in \mathbb{Z} \setminus \{\pm k\}} \frac{\langle k \rangle^{2a + 2a} \langle l \rangle^{-2s} \langle k - l \rangle^{-2r}}{(l^2 - k^2)^{2 - 2b'}} + \langle k \rangle^{2a - 2r}.
\]
(3.4)

Since \(a \leq r\) it is obvious that \(\langle k \rangle^{2a - 2r} \leq 1\), so we need only to show the summation in (3.4) is bounded uniformly in \(k \in \mathbb{Z}\). For the purpose, we break the set of summation indices \(\mathbb{Z} \setminus \{\pm k\}\) and separately consider the contribution of
\[
A_k := \{l \in \mathbb{Z}: l \neq \pm k, |k - l| \geq |k|/2\} \quad \text{and} \quad B_k := \{l \in \mathbb{Z}: l \neq \pm k, |l| > |k|/2\}.
\]
It is obvious that \(A_k \cup B_k = \mathbb{Z} \setminus \{\pm k\}\).

Let us first prove the boundedness of the summation in (3.4) over the set \(A_k\). If \(l \in A_k\) then \(\langle k - l \rangle \gtrsim \langle k \rangle\). It is also clear from the assumptions on \(s\) and \(b'\) that \(2 - 2b' + 2s > 1\). Hence application of Lemma [3.5] yields
\[
\sum_{l \in A_k} \frac{\langle k \rangle^{2s + 2a} \langle l \rangle^{-2s} \langle k - l \rangle^{-2r}}{(l + k)^{2 - 2b'} (l - k)^{2 - 2b'}} \lesssim \frac{1}{\langle k \rangle^{2s + 2a - 2r + 2b'} - 2} \sum_{l \in A_k} \frac{1}{(l + k)^{2 - 2b'} \langle l \rangle^{2s}} 
\lesssim \frac{\langle k \rangle^{2s + 2a - 2r + 2b'} - 2}{\langle k \rangle^{\min(2s, 2 - 2b')}} \phi_{2a}(k).
\]
(3.5)

In the case \(s \geq \frac{1}{2}, \max\{2s, 2 - 2b'\} = 2s\) since \(b' > \frac{1}{2}\). It follows from (3.2) that
\[
\phi_{2s}(k) = \langle k \rangle^{2s + 2a - 2r + 2b' - 4} \phi_{2s}(k) \lesssim \langle k \rangle^{2s + 2s + 2a - 2r + 2b' - 4} \log(1 + \langle k \rangle).
\]
Since \(b' < \frac{1}{2} + \epsilon\), the last quantity is again bounded by
\[
\langle k \rangle^{2s + 2a - 2r - 2 + 4r} \log(1 + \langle k \rangle),
\]
which is bounded uniformly in \(k\) since \(a < 1 + r - s\) and \(\epsilon\) is small enough.
In the other case \(0 < s < \frac{1}{2}\), we have \(\max\{2s, 2 - 2b'\} = 2 - 2b'\) when \(\epsilon\) is sufficiently small. Thus, we get from (3.2) that
\[
\langle k \rangle^{2a - 2r + 2b' - 2} \phi_{2 - 2b'}(k) \leq \langle k \rangle^{2a - 2r + 4b' - 3} \leq \langle k \rangle^{2a - 2r - 1 + 4\epsilon},
\]
which is uniformly bounded since \(a < \frac{1}{2}\) and \(\epsilon\) is small enough.

We now show the boundedness of the summation in (3.4) over the set \(B_k\). We note that \(|k| \geq |l|\) on \(B_k\). Since \(r \geq 0\) and \(b' - \frac{1}{2} < \epsilon\ll 1\) we see that \(2(2 - 2b') + 2r > 1\). Hence, application of Lemma 3.5 gives
\[
\sum_{l \in B_k} \frac{\langle k \rangle^{2a + 2b' - 2s} \langle k - l \rangle^{-2r}}{(l + k)^{2 - 2b'} (l - k)^{2 - 2b'}} \lesssim \langle k \rangle^{2a} \sum_{l \in B_k} \frac{1}{(l + k)^{2 - 2b'} (l - k)^{2 - 2b' + 2r}} \lesssim \langle k \rangle^{2a + 2b' - 2} \phi_{2 - 2b' + 2r}(k).
\]
By the estimate (3.2) we have
\[
\langle k \rangle^{2a + 4b' - 3} \quad \text{if} \quad r = 0,
\]
\[
\langle k \rangle^{2a + 2b' - 2} \quad \text{if} \quad r > 0,
\]
both of which are uniformly bounded since \(a < \frac{1}{2}\) and \(\epsilon\) is sufficiently small.

We now prove (3.1). Let \(\chi_I \in C_0^\infty(\mathbb{R})\) be such that \(\chi_I = 1\) on \(I\). By the definition of the restricted Bourgain space and the estimate what we have obtained, we see that
\[
\|R(V, u)\|_{X^{s+a,\nu-1}} \leq \|\chi_I R(V, u)\|_{X^{s+a,\nu-1}} \lesssim \|V\|_{H^r} \|\chi_I u\|_{X^{s,b}}.
\]
Taking infimum over all functions \(w\) such that \(w|_{I \times \mathbb{R}} = u\) on the right side, we get the estimate (3.1). □

4. LOCAL WELL-POSEDNESS AND SMOOTHING ESTIMATE

In this section, we make use of Proposition 3.4 and employ the contraction mapping principle to establish local well-posedness in the Sobolev spaces for the initial value problem (1.2). We then prove Proposition 2.1.

Let \(u\) be the solution to the equation (1.2) and set \(v(t, x) := e^{it\hat{V}(0)} u(t, x)\). Since
\[
iv_t + v_{xx} = e^{it\hat{V}(0)} (iu_t - \hat{V}(0) u + u_{xx}) = e^{it\hat{V}(0)} (V - \hat{V}(0)) u = (V - \hat{V}(0)) v = R(V, v),
\]
the initial value problem (1.2) is equivalent to the following:
\[
\begin{cases}
iv_t + v_{xx} = R(V, v), \\
v(0, x) = f(x).
\end{cases}
\]
Let us recall the definition of \(\mathcal{P}(t, x)\) from (2.1) and note that
\[
v(t, x) - e^{it\partial_x} f = e^{it\hat{V}(0)} (u(t, x) - e^{it\partial_x - \hat{V}(0)} f) = e^{it\hat{V}(0)} \mathcal{P}(t, x).
\]
Since \(\|\mathcal{P}(t, x)\|_{H^s} = \|e^{it\hat{V}(0)} \mathcal{P}(t, x)\|_{H^s}\), in order to prove Proposition 2.1 it is enough to prove the smoothing estimate for \(e^{it\hat{V}(0)} \mathcal{P}(t, x)\) instead of \(\mathcal{P}(t, x)\). Therefore, in this section, we may and shall consider the equation (4.1) instead of (1.2).
Local well-posedness. We now prove the initial value problem (4.1) is locally well-posed in $H^s$. We use the notation $X^{s,b}_δ := X^{s,b}_{[0,δ]}$.

**Theorem 4.1.** Let $V ∈ H^r$ for $r ≥ 0$ and suppose $0 < s < 1 + r$ and $\frac{1}{2} < b < \frac{1}{2} + \epsilon$ for some $\epsilon > 0$ small enough. For every $f ∈ H^s$, there exists a time $δ > 0$ and an open ball $B$ in $H^s$ containing $f$, and a subset $X$ of $X^{s,b}_δ$, such that for each $g ∈ B$ there exists a unique solution $v_g ∈ X$ for the integral equation

$$v_g(t,x) = e^{it\partial_{xx}} g - i \int_0^t e^{i(t-t')\partial_{xx}} R(V,v_g)(t')dt'.$$

(4.3)

Furthermore, the mapping $B \ni g \mapsto v_g ∈ X$ is Lipschitz continuous with the estimate

$$\|v_g\|_{X^{s,b}_δ} \lesssim \|g\|_{H^s}.$$ (4.4)

**Remark 4.2.** By Lemma 3.1 we have $X^{s,b}_δ \hookrightarrow C([0,δ],H^s)$. Thus, the theorem establishes the local well-posedness of (4.1) (hence (1.1)) in $H^s$, in the sense of [23, Definition 3.4].

**Remark 4.3.** In fact, as can be seen in the proof (see (4.6) below), the small time $δ$ is independent of the initial data $f ∈ H^s$. Since the initial value problem (4.1) is invariant under time translations, we can patch the solution (4.3) along $t ∈ \mathbb{R}$ by repeatedly applying Theorem 4.1 to obtain the global solution $v ∈ C(\mathbb{R},H^s)$ to (4.1). Also, the estimate (4.4) combined with Lemma 3.1 implies the following global bound

$$\|v(t)\|_{H^s} \lesssim e^{\epsilon t}\|f\|_{H^s}, \quad \forall t ∈ \mathbb{R}.$$ (4.5)

**Proof of Theorem 4.1.** For $f ∈ H^s$ we set

$$B := \{g ∈ H^s : \|g - f\|_{H^s} < 1\},$$

and for each $g ∈ B$ we define

$$\Gamma_g(v)(t,x) := e^{it\partial_{xx}} g - i \int_0^t e^{i(t-t')\partial_{xx}} R(V,v)(t')dt'.$$

We aim to show that, for some small time $0 < δ < 1$ and $K > 0$ to be chosen later, the mapping $\Gamma_g$ is a contraction on the set

$$X := \{w ∈ X^{s,b}_δ : \|w\|_{X^{s,b}_δ} ≤ K \max\{1,\|f\|_{H^s}\}\}$$

whenever $g ∈ B$.

Let us first show that the mapping $\Gamma_g : X → X$ is well-defined. Lemmas 3.2 and 3.3 imply that if $g ∈ B$ and $w ∈ X$, then

$$\|\Gamma_g(w)\|_{X^{s,b}_δ} \leq \|e^{it\partial_{xx}} g\|_{X^{s,b}_δ} + \|\int_0^t e^{i(t-t')\partial_{xx}} R(V,w)(t')dt'\|_{X^{s,b}_δ} \leq C\|g\|_{H^s} + C\|R(V,w)\|_{X^{s,b-1}_δ} \leq C(1 + \|f\|_{H^s}) + Cδ^{b'-b}\|R(V,w)\|_{X^{s,b-1}_δ}$$
then it follows that

\[ \text{if we set } 0 < \delta < 1 \text{ and } \frac{1}{2} < b < b' < 1. \]

We then apply Proposition 3.4 (with \( a = 0 \)) to see that

\[ \| \Gamma_g(w) \|_{X^{s,b}_\delta} \leq C(1 + \| f \|_{H^\varepsilon} + C\delta^{b'-b}\| V \|_{H^\varepsilon}\| w \|_{X^{s,b}_\delta}, \]

provided that \( \frac{1}{2} < b < b' < \frac{1}{2} + \epsilon < 1 \) for \( \epsilon > 0 \) small enough. Since \( w \in X \) we get

\[ \| \Gamma_g(w) \|_{X^{s,b}_\delta} \leq C_0(2 + K\delta^{b'-b}\| V \|_{H^\varepsilon}) \max\{1, \| f \|_{H^\varepsilon}\}. \]

If we set \( K = 3C_0 \) and take \( 0 < \delta < 1 \) so small that

\[ \delta^{b'-b} < 1/(1 + K\| V \|_{H^\varepsilon}), \]

then it follows that

\[ \| \Gamma_g(w) \|_{X^{s,b}_\delta} \leq K\| f \|_{H^\varepsilon}. \]

Therefore, \( \Gamma_g(X) \subset X \) for \( g \in B \).

Secondly, let us prove that the map \( \Gamma_g : X \to X \) \((g \in B)\) is a contraction. In a similar manner, by Lemmas 3.2 and 3.3 and Proposition 3.4 we have

\[ \| \Gamma_g(w_1) - \Gamma_g(w_2) \|_{X^{s,b}_\delta} \leq C\delta^{b'-b}\| R(V, w_1 - w_2) \|_{X^{s,b'-1}_\delta} \]

\[ \leq C\delta^{b'-b}\| V \|_{H^\varepsilon}\| w_1 - w_2 \|_{X^{s,b}_\delta} \leq \frac{1}{3}\| w_1 - w_2 \|_{X^{s,b}_\delta}. \]

Therefore, by applying the contraction mapping principle, it follows that there exists a unique \( v_g \in X \) solving the equation \( \Gamma_g(v_g) = v_g \).

The estimate (4.4) also follows from (4.7) since \( \| f - g \|_{H^\varepsilon} < 1 \). The continuity of the map \( g \mapsto v_g \) also follows from utilizing Lemmas 3.2 and 3.3 and Proposition 3.4 as in the above. Indeed, by (4.3) we see that for \( g, h \in B \)

\[ \| v_g - v_h \|_{X^{s,b}_\delta} \]

\[ \leq \| e^{it\partial_x^2} (g - h) \|_{X^{s,b}_\delta} + \| \int_0^t e^{i(t-t')\partial_x} \left( R(V, v_g)(t') - R(V, v_h)(t') \right) dt' \|_{X^{s,b}_\delta} \]

\[ \leq C\| g - h \|_{H^\varepsilon} + C\delta^{b'-b}\| V \|_{H^\varepsilon}\| v_g - v_h \|_{X^{s,b}_\delta} \]

\[ \leq C\| g - h \|_{H^\varepsilon} + \frac{1}{3}\| v_g - v_h \|_{X^{s,b}_\delta}, \]

from which it follows that \( \| v_g - v_h \|_{X^{s,b}_\delta} \lesssim \| g - h \|_{H^\varepsilon}. \)

\[ \square \]

4.2. Smoothing estimate: Proof of Proposition 2.1 In this final section, we prove Proposition 2.1.

Let \( f \) and \( V \) be given as in the statement of Proposition 2.1. It is enough to prove that for every \( t_0 \geq 0 \) there exists a constant \( C > 0 \) such that

\[ \| P(t_0, x) \|_{H^{s+m}} \leq C\| V \|_{H^\varepsilon}\| f \|_{H^\varepsilon}. \]

This gives regularity gain for \( P(t, \cdot) \) compared to the global estimate (4.5) for the solution \( v \). In order to prove (4.8), the bilinear estimate (Proposition 3.4) as well as the local well-posedness (Theorem 4.1) is crucial.

Let us invoke the \( \delta > 0 \) in Theorem 4.1 and pick \( m \in \mathbb{N} \) such that \((m-1)\delta \leq t_0 < m\delta \). We also set

\[ v_j(x) = v(j\delta, x) \text{ and } I_j = [\delta j, \delta(j + 1)]. \]
for $j \in \{0, 1, 2, \cdots, m-1\}$. Applying Theorem 4.1 with the initial data $v_j$, it is possible to write the solution $v$ as

$$v(t) = e^{i(t-\delta_j)\partial_{xx}}v_j - i \int_{\delta_j}^t e^{i(t-t')\partial_{xx}} R(V, v)(t') dt', \quad t \in I_j.$$ 

By Lemmas 3.1 and 3.3 we have, for $t \in I_j$,

$$\|v(t) - e^{i(t-\delta_j)\partial_{xx}}v_j\|_{H^{s+a}} \leq \left\| \int_{\delta_j}^t e^{i(t-t')\partial_{xx}} R(V, v)(t') dt' \right\|_{X^{s+a,b}_{I_j}} \lesssim \|R(V, v)\|_{X^{s+a,b-1}_{I_j}}$$

whenever $\frac{1}{2} < b \leq 1$. Let us choose $b$ sufficiently close to $\frac{1}{2}$, and then apply Proposition 3.4 (with $b' = b$) to see that

$$\|R(V, v)\|_{X^{s+a,b-1}_{I_j}} \lesssim \|V\|_{H^r} \|v\|_{X^{s,b}_{I_j}}.$$ 

Hence, by the estimate (4.5), we conclude that

$$\|v(t) - e^{i(t-\delta_j)\partial_{xx}}v_j\|_{H^{s+a}} \leq C e^{\delta_s} \|V\|_{H^r} \|f\|_{H^s}, \quad t \in I_j, \quad (4.10)$$

for $j \in \{0, 1, 2, \cdots, m-1\}$. We recall (4.2) and write

$$e^{it_0 \hat{V}(0)} P(t_0, x) = v(t_0) - e^{it_0 \partial_{xx}} f$$

$$= v(t_0) - e^{i(t_0 - \delta(m-1))\partial_{xx}} v_{m-1} + \sum_{j=1}^{m-1} e^{i(t_0 - \delta_j)\partial_{xx}} (v_j - e^{i\delta\partial_{xx}} v_{j-1}).$$

Application of the estimate (4.10) gives

$$\|P(t_0, \cdot)\|_{H^{s+a}}$$

$$\leq \|v(t_0) - e^{i(t_0 - \delta(m-1))\partial_{xx}} v_{m-1}\|_{H^{s+a}} + \sum_{j=1}^{m-1} \|v_j - e^{i\delta\partial_{xx}} v_{j-1}\|_{H^{s+a}}$$

$$\leq C m e^{\delta_s} \|V\|_{H^r} \|f\|_{H^s}.$$ 

Therefore, we obtain the desired estimate (4.8).

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