Log-Sobolev-type inequalities for solutions to stationary Fokker–Planck–Kolmogorov equations

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Abstract
We prove that every probability measure \( \mu \) satisfying the stationary Fokker–Planck–Kolmogorov equation obtained by a \( \mu \)-integrable perturbation \( v \) of the drift term \( -x \) of the Ornstein–Uhlenbeck operator is absolutely continuous with respect to the corresponding Gaussian measure \( \gamma \) and for the density \( f = d\mu/d\gamma \) the integral of \( |f| \log(f + 1)|^{\alpha} \) against \( \gamma \) is estimated via \( \|v\|_{L_1(\mu)} \) for all \( \alpha < 1/4 \), which is a weakened \( L_1 \)-analog of the logarithmic Sobolev inequality. This yields that stationary measures of infinite-dimensional diffusions whose drifts are integrable perturbations of \( -x \) are absolutely continuous with respect to Gaussian measures. A generalization is obtained for equations on Riemannian manifolds.

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1 Introduction

It is known (see [12] and [13]) that a probability measure \( \mu \) on \( \mathbb{R}^d \) satisfying the stationary Fokker–Planck–Kolmogorov equation

\[
\Delta \mu - \text{div}(b \mu) = 0 \tag{1.1}
\]

in the sense of the integral identity

\[
\int_{\mathbb{R}^d} [\Delta \varphi + \langle b, \nabla \varphi \rangle] \, d\mu = 0, \quad \varphi \in C_0^\infty(\mathbb{R}^d),
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) is Borel measurable and integrable with respect to \( \mu \) on balls, possesses a density \( \varphi \) with respect to Lebesgue measure. Moreover, if \( |b| \) is locally integrable to some power \( p \) greater than \( d \) with respect to \( \mu \) or with respect to Lebesgue measure, then \( \varphi \) belongs to the Sobolev class \( W^{p,1} \) on every ball (the class \( W^{p,1}(\Omega) \) on a domain \( \Omega \) consists of

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functions belonging to $L^p(\Omega)$ along with their generalized first order derivatives). However, this is false if $p < d$ (see [13]). On the other hand, as shown in [17] (see also [13]), in the case of the global condition $|b| \in L^2(\mu)$, we have $q \in W^{1,1}(\mathbb{R}^d)$, $\sqrt{q} \in W^{2,1}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} \frac{\nabla q}{q} \, dx \leq \int_{\mathbb{R}^d} |b|^2 \, d\mu.$$  

The latter bound admits an infinite-dimensional version. To this end we write the drift $b$ in the form

$$b(x) = -x + v(x).$$

If $v = 0$, then the only solution in the class of probability measures is the standard Gaussian measure $\gamma$ with density $(2\pi)^{-d/2} \exp(-|x|^2/2)$. Hence it is natural to express $\mu$ through $\gamma$. For the corresponding density $f = d\mu/d\gamma$ one has

$$\int_{\mathbb{R}^d} \frac{\nabla f}{f} \, d\gamma \leq \int_{\mathbb{R}^d} |v|^2 \, d\mu.$$  

In this form the result extends to the infinite-dimensional case provided that $v$ takes values in the Cameron–Martin space $H$ of the Gaussian measure $\gamma$ and $|v| = |v|_H$ and $|\nabla f| = |\nabla f|_H$ are taken with respect to the Cameron–Martin norm. The logarithmic Sobolev inequality (applied to $\sqrt{f}$) yields the bound

$$\int_{\mathbb{R}^d} f |\log f| \, d\gamma \leq \int_{\mathbb{R}^d} |v|^2 \, d\mu,$$

as well as its infinite-dimensional analog, which is a constructive sufficient condition for the uniform integrability of the densities of finite-dimensional projections of solutions to infinite-dimensional equations with respect to the corresponding Gaussian measures.

It has recently been shown in [15] that these results on Sobolev differentiability of densities break down in the $L^1$-setting. It can happen that $|b| \in L^1(\mu)$, but the solution density $q$ does not belong to the Sobolev class $W^{1,1}(\mathbb{R}^d)$, i.e., $|\nabla q|$ does not belong to $L^1(\mathbb{R}^d)$, and similarly for the density $f$ the condition $|v| \in L^1(\gamma)$ does not guarantee that the function $|\nabla f|$ belongs to $L^1(\gamma)$. However, these negative results left open the important question of whether in the infinite-dimensional case the solution $\mu$ with $|v|_H \in L^1(\mu)$ is always absolutely continuous with respect to $\gamma$ as it holds in the finite-dimensional case. The main result of this paper answers positively this long-standing question. This result is based on a dimension-free finite-dimensional bound on the integral of $f |\log(f + 1)|^{\alpha}$. It is worth noting that if $f > 0$ is in the Sobolev class $W^{1,1}(\gamma)$ with respect to $\gamma$, then the measure $f \cdot \gamma$ satisfies (1.1) with $v = \nabla f / f$. It is known (see [3,22,24]) that in this very special case $f \sqrt{\log(f + 1)} \in L^1(\gamma)$.

## 2 Main results

Throughout this section we use the notation $\|v\|_{L^1(\mu)} := \|v\|_{L^1(\mu)}$.

The following theorem is our main result.

**Theorem 2.1** For every $\alpha < 1/4$, there is a number $C(\alpha)$ such that whenever $\mu = f \cdot \gamma$ is a probability measure on $\mathbb{R}^d$ satisfying (1.1) with $b(x) = -x + v(x)$, where $|v| \in L^1(\mu)$, one has

$$\int_{\mathbb{R}^d} f |\log(f + 1)|^{\alpha} \, d\gamma \leq C(\alpha) \left[ 1 + \|v\|_{L^1(\mu)} \left( \log(1 + \|v\|_{L^1(\mu)}) \right)^{\alpha} \right]. \quad (2.1)$$
It is also possible that our bound with \( \alpha < 1/4 \) can be raised up to \( 1/2 \).

The natural infinite-dimensional version of this result is considered in the next section.

The proof is based on two auxiliary results of independent interest. Let \( (T_t)_{t \geq 0} \) denote the standard Ornstein–Uhlenbeck semigroup on \( L^1(\gamma) \) defined by

\[
T_t \varphi(x) = \int_{\mathbb{R}^d} \varphi(e^{-t}x - \sqrt{1 - e^{-2t}} y) \gamma(dy).
\]

Some elementary properties of this semigroup used below can be found in [5,6], and [8].

Let \( \| \cdot \|_K \) denote the usual \( 1 \)-Kantorovich norm defined on bounded signed measures \( \sigma \) with \( \sigma(\mathbb{R}^d) = 0 \) and finite first moment by

\[
\| \sigma \|_K = \sup \left\{ \int g \, d\sigma : g \in \text{Lip}_1 \right\}.
\]

where \( \text{Lip}_1 \) is the set of all \( 1 \)-Lipschitz functions on \( \mathbb{R}^d \). It is readily seen that the supremum can be taken over the class of \( 1 \)-Lipschitz smooth compactly supported functions. This norm can be extended to the space of signed measures with finite first moment. For example, we can set

\[
\| \delta_0 \|_K = 1 \quad \text{for Dirac’s measure} \quad \delta_0 \quad \text{at zero and then let} \quad \| \sigma \|_K := \| \sigma - \sigma(\mathbb{R}^d) \delta_0 \|_K + |\sigma(\mathbb{R}^d)|.
\]

It is also possible to extend this norm by imposing the restriction \( g(0) = 0 \) when taking sup.

It is known that for every Borel probability measure \( \eta \) with finite first moment on \( \mathbb{R}^d \) there is a probability measure \( \sigma \) on \( \mathbb{R}^d \times \mathbb{R}^d \) with projections \( \eta \) and \( \gamma \) such that it minimizes the integral of \( |x - y| \) over such measures and the corresponding minimum is \( \| \eta - \gamma \|_K \). Such a measure \( \sigma \) is called a \( 1 \)-optimal transportation plan for the measures \( \eta \) and \( \gamma \). Hence the same is true for the pair of measures \( c\eta \) and \( c\gamma \) with any \( c > 0 \): their \( 1 \)-optimal transportation plan is \( c\sigma \) (its total mass is \( c \)). On this topic, see [2,7,9], and [28].

The next two lemmas are connected with properties of the Ornstein–Uhlenbeck semigroup, but not with our equation.

**Lemma 2.2** Suppose that \( g \in L^1(\gamma) \) is a nonnegative function. If the measure \( g \cdot \gamma \) has a finite first moment, i.e.,

\[
K := \| g \cdot \gamma - \| g \|_{L^1(\gamma)} \gamma \|_K
\]

is finite, then

\[
J_t(g) = \int_{\mathbb{R}^d} (T_t g) (\log(T_t g + 1))^{1/2} d\gamma
\]

\[
\leq \| g \|_{L^1(\gamma)} \left( \log(\| g \|_{L^1(\gamma)}) + 1 \right)^{1/2} + 2^{-1} K t^{-1/2} \quad \forall t \in [0, 1]. \quad (2.2)
\]

In particular, if \( g \) is a probability density with respect to \( \gamma \), we have

\[
J_t(g) = \int_{\mathbb{R}^d} (T_t g) (\log(T_t g + 1))^{1/2} d\gamma \leq \sqrt{\log 2} + 2^{-1} \| g \cdot \gamma - \gamma \|_K t^{-1/2} \quad \forall t \in [0, 1].
\]

**Proof** We employ Wang’s log-Harnack inequality for a nonnegative function \( h \in L^1(\gamma) \) established in [29,30] (see also [31,32]) in much greater generality:

\[
T_t \log h(x) \leq \log T_t h(y) + \frac{1}{2} \frac{1}{e^{2t} - 1} |x - y|^2.
\]

(2.3)

Let us take \( h = T_t g + 1 \) in Wang’s inequality. Then

\[
(T_t \log h(x))^{1/2} \leq (\log T_t h(y))^{1/2} + (4t)^{-1/2} |x - y|.
\]
Let $\sigma$ be a 1-optimal transportation plan for $g \cdot \gamma$ and $\|g\|_1 \gamma$, where we write $\|g\|_1 = \|g\|_{L^1(\gamma)}$ for simplicity in this proof. Integrating the previous bound with respect to $\sigma$ (we omit the indication of domain of integration below) and observing that
\[
\int (\log T_t h(y))^{1/2} \sigma(dx \, dy) = \|g\|_1 \int (\log T_t h(y))^{1/2} \gamma(dy)
\]
\[
\leq \|g\|_1 \left( \log \int T_t h(y) \gamma(dy) \right)^{1/2} = \|g\|_1 (\log(\|g\|_1 + 1))^{1/2},
\]
where we have used that $T_t h \geq 1$ and applied Jensen’s inequality, we arrive at the inequality
\[
\int g(x) \left( T_t \log h(x) \right)^{1/2} \gamma(dx) \leq \|g\|_1 \left( \log(\|g\|_1 + 1) \right)^{1/2} + (4t)^{-1/2} \|g \cdot \gamma - I(g)\gamma\|_K,
\]
which yields (2.2), since $T_t \left[ \left( \log h \right)^{1/2} \right] \leq (T_t \log h)^{1/2}$ and the integral of $g T_t \left[ \left( \log h \right)^{1/2} \right]$ equals the integral of $(T_t g)(\log h)^{1/2}$. \qed

It is worth noting that in the situation of this lemma the function $T_t g$ belongs to the Gaussian Sobolev class $W^{1,1}(\gamma)$, see [22, Proposition 3.5] or [8, Proposition 5.12].

The Ornstein–Uhlenbeck operator $L$ is defined by
\[
L \varphi(x) := \Delta \varphi(x) - \langle x, \nabla \varphi(x) \rangle
\]
for smooth functions $\varphi$. It can be written as
\[
L \varphi = \text{div}_\gamma \nabla \varphi,
\]
where for a smooth vector field $u$ we set
\[
\text{div}_\gamma u(x) = \text{div} u(x) - \langle x, u(x) \rangle.
\]
For smooth compactly supported functions $\varphi$ and $\psi$ and a smooth vector field $u$ we have
\[
\int_{\mathbb{R}^d} \varphi L \psi \, d\gamma = -\int_{\mathbb{R}^d} \langle \nabla \varphi, \nabla \psi \rangle \, d\gamma,
\]
\[
\int_{\mathbb{R}^d} \varphi \, \text{div}_\gamma u \, d\gamma = -\int_{\mathbb{R}^d} \langle \nabla \varphi, u \rangle \, d\gamma.
\]
These equalities extend to functions and vector fields from Gaussian Sobolev classes, which is not used below.

We shall now see that although for a general $\gamma$-integrable vector field $u$ its $\gamma$-divergence can be a singular distribution, for every $s > 0$, there is a function $T_s \text{div}_\gamma u$ in $L^1(\gamma)$ that satisfies the identity
\[
\int_{\mathbb{R}^d} \varphi \, T_s \text{div}_\gamma u \, d\gamma = -\int_{\mathbb{R}^d} \langle \nabla T_s \varphi, u \rangle \, d\gamma = -\int_{\mathbb{R}^d} e^{-s} \langle \nabla \varphi, T_s u \rangle \, d\gamma, \quad \varphi \in C_0^\infty. (2.5)
\]

**Lemma 2.3** Let $u$ be a Borel vector field on $\mathbb{R}^d$ such that $|u| \in L^1(\gamma)$. Then, for each $s > 0$, there is a function
\[
T_s \text{div}_\gamma u \in L^1(\gamma)
\]
satisfying (2.5) such that for the measure
\[
\nu_s := (T_s \text{div}_\gamma u) \cdot \gamma
\]
\[\text{ Springer} \]
we have
\[ \|v_s\| = \|T_s \text{div}_\gamma u\|_{L^1(\gamma)} \leq \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \|u\|_{L^1(\gamma)} \leq \frac{1}{\sqrt{2s}} \|u\|_{L^1(\gamma)}. \] (2.6)

In addition, \( v_s(\mathbb{R}^d) = 0 \) and
\[ \|v_s\|_K \leq e^{-s} \|u\|_{L^1(\gamma)}. \] (2.7)

This means that \( T_s \) extends to the distributional \( \gamma \)-divergences of \( \gamma \)-integrable vector fields as an operator with values in \( L^1(\gamma) \), i.e., \( T_s \text{div}_\gamma \) extends to a bounded operator from \( L^1(\gamma, \mathbb{R}^d) \) to \( L^1(\gamma) \).

**Proof** Suppose first that \( u \) is smooth with compact support. For all \( \phi \in C_0^\infty \) we have
\[ \int_{\mathbb{R}^d} T_s \text{div}_\gamma u \, \phi \, d\gamma = -\int_{\mathbb{R}^d} \langle u, \nabla T_s \phi \rangle \, d\gamma, \]
where
\[ \partial_h T_s \phi(x) = \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \int_{\mathbb{R}^d} \phi(e^{-s} x - \sqrt{1 - e^{-2s}} y)(h, y) \, \gamma(dy) \]
for all \( h \in \mathbb{R}^d \). Hence
\[ |\nabla T_s \phi(x)| \leq \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \|\phi\|_{\infty}, \]
which yields (2.6). Using that
\[ \nabla T_s \phi = e^{-s} T_s \nabla \phi, \]
we obtain (2.7). In the general case we take a sequence of smooth compactly supported vector fields \( u_j \) converging to \( u \) in \( L^1(\gamma) \). By (2.6) the sequence of smooth functions \( T_s \text{div}_\gamma u_j \) converges in \( L^1(\gamma) \). The limit is the desired function. For \( u_j \) equality (2.5) also holds for \( \phi \in C_0^\infty \), in particular, for \( \phi = 1 \), hence the integral of \( T_s \text{div}_\gamma u_j \) against \( \gamma \) vanishes. Therefore,
\[ \int_{\mathbb{R}^d} T_s \text{div}_\gamma u \, d\gamma = 0. \]
It is clear that (2.7) holds and (2.5) remains true in the limit. \( \square \)

Suppose now that a nonnegative function \( f \in L^1(\gamma) \) satisfies the equation
\[ \Delta(f \cdot \gamma) - \text{div}(f b \cdot \gamma) = 0 \]
with
\[ b(x) = -x + v(x), \]
where
\[ |v| \in L^1(\mu), \text{ i.e., } f|v| \in L^1(\gamma). \]
Using the Ornstein–Uhlenbeck operator defined by (2.4), we can rewrite the equation as
\[ \int_{\mathbb{R}^d} (L \phi + \langle v, \nabla \phi \rangle) f \, d\gamma = 0 \, \forall \phi \in C_0^\infty. \] (2.8)
Let us set 
\[ w := f \cdot v. \]
By assumption \(|w| \in L^1(\gamma)\). We have in the sense of distributions 
\[ Lf = \text{div}_\gamma w, \]
where 
\[ \text{div}_\gamma w = \text{div} w - \langle x, w \rangle. \]

For smooth \(w\) and \(f\) this would be 
\[
\int_{\mathbb{R}^d} \varphi Lf \, d\gamma = \int_{\mathbb{R}^d} \varphi \text{div}_\gamma w \, d\gamma, \quad \varphi \in C^\infty_0.
\]

It is readily seen that identity (2.8) remains valid for all \(\varphi\) from the class \(S(\mathbb{R}^d)\) of smooth rapidly decreasing functions. Since \(T_s \varphi \in S(\mathbb{R}^d)\) for all \(\varphi \in C^\infty_0\) and \(s \geq 0\), we obtain the identity
\[
\int_{\mathbb{R}^d} (LT_s \varphi + \langle v, \nabla T_s \varphi \rangle) \, f \, d\gamma = 0, \quad \forall \varphi \in C^\infty_0, \ s \geq 0.
\]
Therefore, in \(L^1(\gamma)\) we have
\[
T_t f - f = \int_0^t T_s \text{div}_\gamma w \, ds,
\]
where the last integral exists in \(L^1(\gamma)\) by Lemma 2.3. Indeed, the integrals with respect to \(\gamma\) of both sides multiplied by any \(\varphi \in C^\infty_0\) coincide, because by (2.5) and (2.9) we have
\[
\int_{\mathbb{R}^d} \varphi T_s \text{div}_\gamma w \, d\gamma = -\int_{\mathbb{R}^d} \langle \nabla T_s \varphi, w \rangle \, d\gamma = \int_{\mathbb{R}^d} LT_s \varphi \, f \, d\gamma,
\]
but the integral of the right-hand side in \(s\) over \([0, t]\) equals the integral of \(f (T_t \varphi - \varphi)\) against \(\gamma\), which is the integral of \(\varphi (T_t f - f)\) by the symmetry of \(T_t\).

**Proposition 2.4** Under the assumptions of Theorem 2.1 we have
\[
\|f \cdot \gamma - \gamma\|_K \leq \|fv\|_{L^1(\gamma)} = \|v\|_{L^1(\mu)}.
\]
In addition,
\[
\|T_t f - f\|_{L^1(\gamma)} \leq (2t)^{1/2} \|v\|_{L^1(\mu)}.
\]

**Proof** It follows by (2.10) and (2.7) that for all \(t > 0\) and \(\varphi \in C^\infty_0(\mathbb{R}^d)\) we have
\[
\left| \int_{\mathbb{R}^d} \varphi T_t f \, d\gamma - \int_{\mathbb{R}^d} \varphi f \, d\gamma \right| = \left| \int_0^t \int_{\mathbb{R}^d} \varphi T_s \text{div}_\gamma w \, ds \, d\gamma \right| \leq \|fv\|_{L^1(\gamma)} \|\nabla \varphi\|_\infty \int_0^t e^{-s} \, ds.
\]
It remains to recall that \(T_t f\) converges to 1 in \(L^1(\gamma)\) as \(t \to \infty\). The second estimate follows similarly by using (2.6). The passage to all 1-Lipschitz functions in place of smooth compactly supported ones is easily justified by Fatou’s theorem taking into account that the Gaussian measure has finite first moment. \(\square\)
The bound $\|T_t f - f\|_{L^1(\gamma)} \leq Ct^{1/2}$ can be regarded as the inclusion of $f$ in a certain fractional Besov type space with respect to $\gamma$ in the spirit of [10] and [11].

It is worth noting that this proposition yields that the norm $|x|$ is $\mu$-integrable, so $\|\mu\|_K < \infty$, which is not obvious in advance (and has not been assumed in the proof) and does not follow from the $\gamma$-integrability of the function $f$ ($\log(f + 1)^\alpha$ with $\alpha < 1/4$, unlike the case where $f$ ($\log(f + 1)^{1/2}$ is $\gamma$-integrable (which holds if $f \in W^{1,1}(\gamma)$ or at least $f \in BV(\gamma)$, but this can fail in our situation according to [15]). It is also known (see [1, Lemma 2.3]) that the bound $\|T_t f - f\|_{L^1(\gamma)} \leq Ct^{1/2}$ holds if $f \in BV(\gamma)$.

**Proof of Theorem 2.1** Let $t_n := n^{-\beta}$, where $\beta > 1$ will be picked later. Set

$$g_n := |T_{t_{n+1}} f - T_t f|,$$

$$V := \|v\|_{L^1(\mu)}.$$

For simplicity, we omit indication of $\mathbb{R}^d$ when integrating over the whole space. Let us observe that by Proposition 2.4 there is a number $C(\beta)$, depending only on $\beta$, such that

$$\int g_n \, d\gamma \leq C(\beta)n^{-(1+\beta)/2} V, \quad n \geq 1.$$  \hspace{1cm} (2.11)

In addition,

$$\int g_n (\log(g_n + 1))^{1/2} \, d\gamma \leq 2\sqrt{\log 2} + n^{\beta/2} V.$$  \hspace{1cm} (2.12)

This bound is obtained as follows:

$$\int g_n (\log(g_n + 1))^{1/2} \, d\gamma \leq \int (T_{t_{n+1}} f)(\log(T_{t_{n+1}} f + 1))^{1/2} \, d\gamma$$

$$+ \int (T_{t_n} f)(\log(T_{t_n} f + 1))^{1/2} \, d\gamma,$$

because $|f_1 - f_2|(\log(|f_1 - f_2| + 1))^{1/2}$ is dominated pointwise by the sum of the functions $f_1 (\log(f_1 + 1))^{1/2}$ and $f_2 (\log(f_2 + 1))^{1/2}$ whenever $f_1, f_2 \geq 0$. It remains to apply Lemma 2.2 and Proposition 2.4.

By Hölder’s inequality and (2.11), (2.12) we have

$$\int g_n (\log(g_n + 1))^{\alpha} \, d\gamma \leq \left[ \int g_n \, d\gamma \right]^{1-2\alpha} \left[ \int g_n (\log(g_n + 1))^{1/2} \, d\gamma \right]^{2\alpha}$$

$$\leq C(\alpha, \beta)n^{\alpha\beta - (1+\beta-2\alpha\beta)/2} V + C(\alpha, \beta)n^{-(1+\beta)(1-2\alpha)/2} V^{1-2\alpha}.$$  

If $\alpha < 1/4$ is fixed, we can find $\beta$ large enough (namely, $\beta > (2\alpha + 1)/(1 - 4\alpha)$) so that the powers obtained will be less than $-1$. Since

$$f \leq T_1 f + \sum_n g_n,$$

for obtaining the desired inequality it remains to apply the triangle inequality for the corresponding Orlicz norm. To this end, we estimate the Luxemburg norms of $g_n$. Recall (see [23]) that the Luxemburg norm is equivalent to the Orlicz norm and in our case is defined by

$$\|g\|_L = \inf \left\{ s > 0 : \int \frac{g}{s}(\log\left(\frac{g}{s} + 1\right))^\alpha \, d\gamma \leq 1 \right\}.$$
Now let us bound 
\[ \int \frac{g_n}{s} \left( \log \left( \frac{g_n}{s} + 1 \right) \right)^\alpha d\gamma \]
via \( \|g_n\|_{L^1(\gamma)} \) and the integral of \( g_n(\log(g_n + 1))^\alpha \):
\[
\int \frac{g_n}{s} \left( \log \left( \frac{g_n}{s} + 1 \right) \right)^\alpha d\gamma = \int \frac{g_n}{s} \left( \log \left( \frac{g_n}{s} + s \right) \right)^\alpha d\gamma 
\leq \int \frac{g_n}{s} \left( \log(g_n + 1) + \log(1/s + 1) \right)^\alpha d\gamma 
\leq \int \frac{g_n}{s} \left[ \left( \log(g_n + 1) \right)^\alpha + \left( \log(1/s + 1) \right)^\alpha \right] d\gamma 
\leq \frac{1}{s} \int g_n \left( \log(g_n + 1) \right)^\alpha + \frac{1}{s} \left( \log(1/s + 1) \right)^\alpha \int g_n d\gamma.
\]
Hence
\[ \|g_n\|_{L} \leq C(\alpha)n^{-\delta(1 + V)} \left( 1 + (\log(1 + V))^{\alpha} \right), \delta = \delta(\alpha) > 1. \]

Finally, we obtain convergence of the Luxemburg norms of the functions \( g_n \) and, consequently,
\[ \|f\|_{L} \leq C(\alpha)(1 + V) \left( 1 + (\log(1 + V))^{\alpha} \right). \]

It remains to bound the integral of \( f(\log(f + 1))^{\alpha} \). Let \( L := \|f\|_{L}, g := f/L \). Using the same arguments as above we have
\[
\int f(\log(f + 1))^{\alpha} d\gamma \leq L \int g(\log(g + 1))^{\alpha} d\gamma + L(\log(L + 1))^{\alpha} \int_{\mathbb{R}^d} g d\gamma 
\leq L + (\log(L + 1))^{\alpha} 
\leq C(\alpha)(1 + V) \left( 1 + (\log(1 + V))^{\alpha} \right).
\]
Thus, we have obtained the desired bound. \( \square \)

**Remark 2.5** As a corollary, we can obtain the following bound on the tail distribution of \( f \):
\[ \gamma(x: f(x) > \lambda) \leq C \frac{\log \log \lambda}{\lambda \log \log \lambda}, \lambda \geq \lambda_0. \]

Indeed, for the operator \( A_1 \) defined by
\[ A_t := \frac{1}{t} \int_0^t T_s ds, \]
Talagrand’s result [27] yields the bound
\[ \gamma(x: A_1 f(x) > \lambda) \leq C \frac{\log \log \lambda}{\lambda \log \lambda}, \lambda \geq \lambda_0. \]

Then
\[ \gamma(x: f(x) > \lambda) \leq \gamma(x: A_t f(x) > \lambda/2) + \gamma(x: |f - A_t f| > \lambda/2) \]
\[ \leq \frac{1}{\lambda t} \frac{C' \log(\log \lambda + \log t)}{\log \lambda + \log t} + t^{1/2} \frac{2}{\lambda} \|v\|_{L^1(\mu)}. \]
Taking \( t := (\log \lambda)^{-2/3} \) and assuming that \( \lambda \) is sufficiently large, we obtain the announced bound. Let us observe that a somewhat worse bound can be obtained from Lehec’s result [25] for \( T_t \) in place of \( A_t \).

**Remark 2.6** The following inequality was established in [19] for functions \( f \in W^{1,1}(\gamma) \):
\[
\| f - 1 \|_{L^1(\gamma)}^2 \leq 2\| f \cdot \gamma - \gamma \|_K \| \nabla f \|_{L^1(\gamma)}.
\]
This inequality is a generalization of the classical Hardy–Landau–Littlewood inequality for functions on the real line (see also [18] and [20]). Using the same reasoning as in [19] and the estimates obtained above one can prove that for \( f \) in Theorem 2.1 one has
\[
\| f - 1 \|_{L^1(\gamma)}^2 \leq 2\| f \cdot \gamma - \gamma \|_K \| v \|_{L^1(\mu)} \leq 2\| v \|_{L^1(\mu)}^2.
\]
A different derivation of this bound has been given by A.F. Miftakhov (see [14]).

**Remark 2.7** It is plain that the main theorem is based on two ingredients: an a priori bound and the estimates obtained above one can prove that for \( f \) in Theorem 2.1 one has
\[
\| f - 1 \|_{L^1(\gamma)}^2 \leq 2\| f \cdot \gamma - \gamma \|_K \| v \|_{L^1(\mu)} \leq 2\| v \|_{L^1(\mu)}^2.
\]
Both properties hold for a broad class of diffusion semigroups (see below). Of course, constants may be different and the Kantorovich norm must be taken with respect to the intrinsic metric. For example, in place of the standard Ornstein–Uhlenbeck semigroup we can consider the semigroup
\[
T_t^B \varphi(x) = \int \varphi\left( e^{-tB}x - \sqrt{1 - e^{-2tB}} \ y \right) \gamma_B(dy),
\]
where \( B \) is a positive definite operator on \( \mathbb{R}^d \) and \( \gamma_B \) is the centered Gaussian measure with covariance \( B^{-1} \). The measure \( \gamma_B \) is invariant for \( \{T_t^B\}_{t \geq 0} \) and satisfies the stationary equation with the drift \(-Bx\). Assume that \( B \geq \beta_1 I \), where \( \beta_1 > 0 \) is the minimal eigenvalue of \( B \). Suppose that \( \mu \) satisfies the stationary equation with \( b(x) = -Bx + v(x) \). Then \( \mu \) is absolutely continuous with respect to \( \gamma_B \) and the integral of \( \int \log(f + 1)^\mu \) for \( f = d\mu/d\gamma_B \) is finite for all \( \alpha < 1/4 \). Moreover, if \( \beta_1 \geq 1 \), then (2.1) remains valid, in the general case a constant depending on \( \beta_1 \) will appear.

Indeed, we have
\[
\nabla T_t^B \varphi = e^{-tB} T_t^B \nabla \varphi.
\]

Therefore, in the corresponding analog of Proposition 2.4 for any 1-Lipschitz function \( \varphi \) we have
\[
\int_0^\infty \int_{\mathbb{R}^d} (e^{-tB} T_t^B \nabla \varphi, u) d\gamma_B dt \leq \int_{\mathbb{R}^d} \int_0^\infty e^{-t\beta_1} |u| dt d\gamma_B = \beta_1^{-1} \| u \|_{L^1(\gamma_B)}.
\]
Hence in the bound for \(\|f \cdot \gamma_B - \gamma_B\|_K\) we have to replace \(|v|\) by \(\beta_1^{-1}|v|\). Next, for estimating \(\|T^B_tf - f\|_{L^1(\gamma_B)}\) we use the equality
\[
\partial_t T^B_t \varphi(x) = \int_{\mathbb{R}^d} \varphi(e^{-sB}x - \sqrt{1 - e^{-2sB}}y)\langle e^{-sB}(1 - e^{-2sB})^{-1/2}h, By \rangle \gamma_B(dy).
\]
The integral of \(y \mapsto |(z, B^{1/2}y)|\) with respect to \(\gamma_B\) is estimated by \(|z|\), and
\[
|e^{-sB}(1 - e^{-2sB})^{-1/2}B^{1/2}h| \leq (2s)^{-1/2}|h|.
\]
Hence \(|\nabla T^B_t \varphi| \leq (2s)^{-1/2}\|\varphi\|_{\infty}\), which gives the same estimate for \(\|T^B_t f - f\|_{L^1(\gamma_B)}\) as in Proposition 2.4. Wang’s log-Harnack inequality also holds in this case. If \(\beta_1 \geq 1\), then it holds without any change; for any \(\beta_1 > 0\) it holds with the additional factor \(e\) in front of \(|x - y|^2\).

Let us formulate an analog of Theorem 2.1 for manifolds. Let \((M, \varrho)\) be a connected complete Riemann manifold. In place of the Ornstein–Uhlenbeck operator we consider the operator
\[
L \varphi = \Delta \varphi + \langle Z, \nabla \varphi \rangle,
\]
where \(Z\) is a smooth vector field satisfying the curvature condition
\[
\text{Ric}(X, X) - \langle \nabla_X, Z \rangle \geq -K|X|^2
\]
with some number \(K \in \mathbb{R}\). In the case of the standard Ornstein–Uhlenbeck operator we have \(\text{Ric} = 0, K = -1, Z(x) = -x, \langle \nabla_X, Z \rangle = -|X|^2\). It is known (see [32, Theorem 2.3.3]) that the diffusion semigroup \(\{P_t\}\) generated by the operator \(L\) satisfies the inequality
\[
P_t(\log f)(x) \leq \log P_t f(y) + \frac{K}{2(1 - e^{-2tK})} \varrho(x, y)^2
\]
for all \(P_t\)-integrable functions \(f > 0\). It is also known (see, in particular, [32, Theorem 2.3.1]) that
\[
|\nabla P_t \varphi| \leq e^{tK} P_t|\nabla \varphi|, \quad |\nabla P_t \varphi| \leq C t^{-1/2}\|\varphi\|_{\infty}.
\]
In addition, there is a probability measure \(\mu_0\) on \(M\) satisfying the equation \(L^* \mu_0 = 0\) (see [16, Theorem 3.4]). It follows from the previous remark that the following assertion is true.

**Theorem 2.8** Suppose that a probability measure \(\mu\) on \(M\) satisfies the perturbed equation
\[
L^*_v \mu = 0,
\]
where
\[
L_v \varphi = L \varphi + \langle v, \nabla \varphi \rangle
\]
and \(v\) is a Borel vector field on \(M\) such that \(|v| \in L^1(\mu)\). Then \(\mu\) has a density \(f\) with respect to \(\mu_0\) and \(f(\log(f + 1))^\alpha \in L^1(\mu_0)\) for all \(\alpha < 1/4\). Moreover, there is a number \(C\) depending on \(\alpha\) and \(K\) such that
\[
\int_M f(\log(f + 1))^\alpha d\mu_0 \leq C(\alpha, K) \left[ 1 + \|v\|_{L^1(\mu)} \left( \log(1 + \|v\|_{L^1(\mu)}) \right)^\alpha \right].
\]
In the case of $\mathbb{R}^d$ this result applies to smooth $Z$ such that

$$\langle Z(x) - Z(y), x - y \rangle \leq -k|x - y|^2$$

for some number $k > 0$. In particular, one can take for $Z$ a negative definite linear operator.

**Remark 2.9** Since our main assumption is the integrability of $v$ with respect to the solution $\mu$ that typically is not explicitly given, it is of interest to ensure this integrability in terms of $v$ (or $b(x) = -x + v(x)$) without using $\mu$. A sufficient condition can be given by using Lyapunov functions: if $v$ is locally bounded, it suffices to have a function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ as $|x| \to +\infty$ and

$$\Delta V(x) + \langle b(x), \nabla V(x) \rangle \leq C - |v(x)|$$

with some positive constant $C$. In this case $\|v\|_{L^1(\mu)} \leq C$ (see [13, Theorem 2.3.2]). For example, if

$$\langle v(x), x \rangle \leq -C_1 < -d \quad \text{outside some ball and} \quad |v(x)| \leq C_2 \exp(|x|^2/2),$$

then one can take $V(x) = \exp(|x|^2/2)$ and conclude that there is a unique probability solution $\mu$ and $|v| \in L^1(\mu)$.

### 3 Infinite-dimensional extensions

The results of the previous section admit straightforward infinite-dimensional extensions. Let $X$ be a locally convex space, let $X^*$ be the topological dual of $X$, and let $\gamma$ be a centered Radon Gaussian measure on $X$. This means that $\gamma$ is a Borel probability measure such that for every Borel set $B$ and every $\varepsilon > 0$ there is a compact set $K \subset B$ with $\gamma(B \setminus K) < \varepsilon$, and, in addition, every functional $l \in X^*$ is a centered Gaussian random variable, i.e., is either zero almost everywhere or

$$\gamma(x : l(x) < s) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^s e^{-u^2/(2\sigma)} \, du,$$

where $\sigma = \|l\|_{L^2(\gamma)}^2$.

The Cameron–Martin space $H$ of $\gamma$ consists of all vectors with finite norm

$$|h|_H = \sup \{l(h) : l \in X^*, \|l\|_{L^2(\gamma)} \leq 1 \}.$$ 

This is a separable Hilbert space with respect to the norm $| \cdot |_H$ (called the Cameron–Martin norm), the corresponding inner product is denoted by $(h, k)_H$.

The most important example is the countable power of the standard Gaussian measure on the real line, which is defined on the space $\mathbb{R}^\infty$ of all real sequences (or on a suitable Hilbert subspace of full measure). The corresponding Cameron–Martin space is the usual space $l^2$. Moreover, by the celebrated Tsirelson theorem, every centered Radon Gaussian measure with an infinite-dimensional Cameron–Martin space is isomorphic to this particular example by means of a measurable linear mapping. Hence we can assume without loss of generality that $\gamma$ below is this countable product on $\mathbb{R}^\infty$. Given a Borel vector field

$$v : X \to H,$$

one can define solutions to the stationary equation

$$L^*_b \mu = 0, \quad b(x) = -x + v(x),$$
as follows. First, dealing with $X = \mathbb{R}^\infty$, we introduce the class $\mathcal{F}C_0$ of test functions of the form

$$\varphi(x) = \varphi_0(x_1, \ldots, x_n), \quad \varphi_0 \in C_0^\infty(\mathbb{R}^n).$$

The reader is warned that this class is not a linear space. For a general locally convex space, an analogous class consists of cylindrical functions.

Next, we define the Ornstein–Uhlenbeck operator $L$ on $\mathcal{F}C_0$ by

$$L\varphi(x) = \sum_{i=1}^\infty [\partial_{x_i}^2 \varphi(x) - x_i \partial_{x_i} \varphi(x)].$$

Obviously, for each $\varphi \in \mathcal{F}C_0$ this is a finite sum. Let

$$v = (v_i), \quad |v(x)|_H^2 = \sum_{i=1}^\infty v_i(x)^2.$$

The operator $L_b$ with $b(x) = -x + v(x)$ is defined by

$$L_b\varphi(x) = L\varphi(x) + \sum_{i=1}^\infty v_i(x) \partial_{x_i} \varphi(x).$$

Finally, if $\mu$ is a Borel probability measure on $X$ such that $v_i \in L^1(\mu)$ for all $i$ and

$$\int_X L_b \varphi \, d\mu = 0 \quad \forall \varphi \in \mathcal{F}C_0,$$

then we say that $\mu$ satisfies the equation $L_b^*\mu = 0$. Note that $L_b\varphi$ is bounded, since if $\varphi$ depends on $x_1, \ldots, x_n$, then the functions $x_i \partial_{x_i} \varphi$ are bounded.

In the case of an abstract locally convex space $X$ the definition is analogous: there are even two similar options. For a class of test functions one can use functions of the form $\varphi(l_1, \ldots, l_n)$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$, $l_i \in X^*$. Alternatively, one can fix a sequence $\{l_i\} \subset X^*$ (say, if there is a sequence separating points) and take only $l_i$ from this sequence. However, the definition of $L_b$ becomes a bit more technical (see [8, Section 4]), because it involves also $H$. Let $\{e_i\}$ be an orthonormal basis in $H$ such that there is a sequence $\{\hat{e}_j\} \subset X^*$ for which $\hat{e}_i(e_j) = \delta_{ij}$. Every functional $l \in X^*$ has a continuous restriction to $H$, hence there is a vector $\tilde{l} \in H$ with $l(u) = (\tilde{l}, u)_H$ for all $u \in H$. Note that $\tilde{h} = h$. The operator $L_b\varphi$ is defined by

$$L_b\varphi = L\varphi + \sum_{i=1}^\infty (v, e_i)_H \partial_{e_i} \varphi,$$

which for cylindrical functions as above can be written as

$$L\varphi = \sum_{j, k \leq n} (\tilde{l}_j, \tilde{l}_k)_H \partial_{x_j} \partial_{x_k} \varphi(l_1, \ldots, l_n) - \sum_{j=1}^n l_j \partial_{x_j} \varphi(l_1, \ldots, l_n).$$

If we use $l_j = \hat{e}_j$, then we arrive at the simple expression used above in the case of $\mathbb{R}^\infty$. 

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It is readily seen that if $v_n^i$ is the conditional expectation of $v_i$ with respect to the measure $\mu$ and the $\sigma$-field $B_n$ generated by $x_1, \ldots, x_n$, then the projection $\mu_n$ of $\mu$ to $\mathbb{R}^n$ satisfies the finite-dimensional equation

$$L^*_p \mu_n = 0$$

with $b^\alpha(x) = -x + v^\alpha(x)$ on $\mathbb{R}^n$, $v^\alpha = (v_1^\alpha, \ldots, v_n^\alpha)$. Actually, our infinite-dimensional equation is equivalent to this system of finite-dimensional equations for projections.

**Theorem 3.1** Let $\mu$ be a Borel probability measure on $X$ such that $|v|_H \in L^1(\mu)$ and $L^*_p \mu = 0$. Then $\mu$ is absolutely continuous with respect to $\gamma$ and for $f := d\mu/d\gamma$ we have

$$\int_X f (\log(f + 1))^{\alpha} d\gamma \leq C(\alpha) \left[ 1 + \|v|_H \right] \left( \log(1 + \|v|_H) \right)^{\alpha}.$$  

(3.1)

where $\alpha < 1/4$ and $C(\alpha)$ are the same numbers as in Theorem 2.1.

**Proof** It is readily seen that the finite-dimensional densities $f_n = d\mu_n/d\gamma_n$ regarded as functions in $L^1(\gamma)$ form a martingale with respect to $\gamma$ and the $\sigma$-fields $B_n$. By the property of conditional expectations we have

$$\|v^n\|_{L^1(\mu_n)} \leq \|v|_H\|_{L^1(\mu)}$$

By the main theorem this martingale is uniformly integrable, hence converges in the weak topology of $L^1(\gamma)$ and almost everywhere to some function $f \in L^1(\gamma)$. Then, for every $\varphi \in FC_0$, the integral of $\varphi f$ with respect to $\gamma$ equals the integral of $\varphi$ with respect to $\mu$. Hence $\mu = f \cdot \gamma$. Estimate (3.1) follows by Fatou’s theorem. \qed

Unlike the case of $\mathbb{R}^d$, the absolute continuity of $\mu$ with respect to $\gamma$ is also a substantial novelty of this theorem.

In the infinite-dimensional case it is important to distinguish between the Cameron–Martin space norm of $v$ and a weaker norm that arises if we consider $\mu$ and $\gamma$ on some continuously embedded Hilbert space $E$ of full measure. The integrability of $\|v\|_E$ does not guarantee the absolute continuity of $\gamma$ even if $v$ still takes values in the Cameron–Martin space $H$. This is why in the infinite-dimensional case we avoid writing $\|v\|_{L^1(\mu)}$ in place of $\|v|_H\|_{L^1(\mu)}$, as we did in $\mathbb{R}^d$.

**Remark 3.2** The following analog of Lemma 2.3 holds. Let $u : X \to H$ be a Borel vector field such that $|u|_H \in L^1(\gamma)$. Then, for each $s > 0$, there is a function $T_s \text{div}_\gamma u \in L^1(\gamma)$ satisfying the identity

$$\int_X \varphi T_s \text{div}_\gamma u \, d\gamma = -\int_X (D_H T_s \varphi, u)_H \, d\gamma = -\int_X e^{-s} (D_H \varphi, T_s u)_H \, d\gamma, \quad \varphi \in FC_0$$

and the bound

$$\|T_s \text{div}_\gamma u\|_{L^1(\gamma)} \leq \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \|u|_H\|_{L^1(\gamma)}.$$  

(3.2)

This means that $T_s \text{div}_\gamma$ extends to a bounded operator from $L^1(\gamma, H)$ to $L^1(\gamma)$. In addition, in the situation of Theorem 3.1 for $w = f v$ equality (2.10) holds.

For the proof we take mappings $u^j : X \to \mathbb{R}^d$ with components of class $FC_0$ such that $|u - u^j|_H \to 0$ in $L^1(\gamma)$. It follows from Lemma 2.3 that the functions $T_s \text{div}_\gamma u^j$ converge...
in $L^1(\gamma)$. The limit will be denoted by $T_\delta \text{div}_\gamma u$. Obviously, it satisfies the desired identity and inequality (3.2). Equality (2.10) for $w = f v$ follows from the finite-dimensional case applied to $w^n = f_n v^n$, because by the martingale convergence theorem we have convergence of $|v^n|_H$ to zero in $L^1(\gamma)$, which along with (3.2) enables us to pass to the limit in (2.10) for $w^n$.

It follows from the aforementioned identity that if $|u|_H \in L^p(\gamma)$ with some $p > 1$, then

$$T_\delta \text{div}_\gamma u \in L^p(\gamma),$$

since for $\varphi \in L^{p/(p-1)}(\gamma)$ one has $|D_H T_\delta \varphi|_H \in L^{p/(p-1)}(\gamma)$ and $\|D_H T_\delta \varphi\|_{L^{p/(p-1)}(\gamma)}$ is estimated through $\|\varphi\|_{L^{p/(p-1)}(\gamma)}$. Moreover, the order of integrability of $T_\delta \text{div}_\gamma u$ can be increased by writing $T_\delta = T_{\delta - \delta} T_\delta$ and using that by the hypercontractivity (see, e.g., [5]) $T_{\delta - \delta}$ takes $L^p(\gamma)$ to $L^q(\gamma)$ with $q = c^{2\delta - 2\delta} (p - 1) + 1$. Note that this does not help much for estimating $f$, because in our main situation $u = f v$, so that even if $v$ is bounded, some a priori information is needed about the integrability of $f$ (and in the general case $f$ can fail to be integrable to a power larger than 1, so increasing integrability is only possible in a logarithmic scale).

In addition, Proposition 2.4 also extends to infinite dimensions with the following modification: in the definition of the Kantorovich norm, one should take the supremum over the intersection of $\mathcal{FC}_0$ with the class $\text{Lip}_1(H)$ of Borel functions that are 1-Lipschitz along the Cameron–Martin space or over the whole class $\text{Lip}_1(H)$. By definition the class $\text{Lip}_1(H)$ consists of all Borel functions $\varphi$ for which

$$|\varphi(x + h) - \varphi(x)| \leq C|h|_H, \quad x \in X, \ h \in H.$$

The corresponding definition is this. For Borel probability measures $\mu$ and $v$ integrating all Borel functions that are Lipschitz along the Cameron–Martin space we set

$$\|\mu - v\|_{K,H} = \sup \left\{ \int_X \varphi d\mu - \int_X \varphi d\nu : \varphi \in \text{Lip}_1(H) \right\}.$$

**Proposition 3.3** Under the assumptions of Theorem 3.1 we have

$$\|f \cdot \gamma - \gamma\|_{K,H} \leq \|f v|_H\|_{L^1(\gamma)} = \|v|_H\|_{L^1(\mu)}.$$

In addition,

$$\|T_t f - f\|_{L^1(\gamma)} \leq (2t)^{-1/2} \|v|_H\|_{L^1(\mu)}.$$

**Proof** The second bound follows immediately from the finite-dimensional case as in the proof of Theorem 3.1. To obtain the first bound we need to pass from the $\mathcal{FC}_0 \cap \text{Lip}_1(H)$ to $\text{Lip}_1(H)$ in the inequality

$$\left| \int_X \varphi f d\gamma - \int_X \varphi d\gamma \right| \leq \|v|_H\|_{L^1(\mu)}.$$ 

We first observe that every function in $\text{Lip}_1(H)$ is $\gamma$-integrable (see [5, Theorem 4.5.7]). Hence by Fatou’s theorem we conclude that it is also $\mu$-integrable. Now applying Fatou’s theorem once again we conclude that the previous inequality extends to $\text{Lip}_1(H)$. 

We emphasize that in this proposition we have shown that functions from $\text{Lip}_1(H)$ are $\mu$-integrable, which is not obvious in advance. This property enables us to extend the class $\mathcal{FC}_0$, with respect to which the equation is defined, to the larger class $\mathcal{FC}_b$, in which representing
functions $\varphi_0$ are taken in the class $C^\infty_b(\mathbb{R}^n)$. The advantage of $\mathcal{FC}_b$ is that it is a linear space. However, the problem with this class in our original definition is due to the fact that no information about the integrability of

$$x_1 \partial x_1 \varphi(x_1, \ldots, x_n), \ldots, x_n \partial x_n \varphi(x_1, \ldots, x_n)$$

with respect to $\mu$ is given in advance. For $\varphi \in C^\infty_0(\mathbb{R}^n)$, such functions are bounded, hence $\mu$-integrable. In the case of an abstract locally convex space our result shows that $X^* \subset L^1(\mu)$, hence $L_b \varphi \in L^1(\mu)$ for cylindrical functions and the equation $L^*_b \mu = 0$ holds also with respect to the class $\mathcal{FC}_b$ in place of the original class $\mathcal{FC}_0$.

Suppose now that $\{w_n(t)\}$ is a sequence of independent Wiener processes, $v = (v_n)_{n=1}^\infty$ is a sequence of Borel functions on $\mathbb{R}^\infty$ such that $\sum_{n=1}^\infty |v_n(x)|^2 < \infty$, and there is a diffusion process $\xi(t) = (\xi_n(t))_{n=1}^\infty$ in $\mathbb{R}^\infty$ satisfying the perturbed Ornstein–Uhlenbeck stochastic equation

$$d\xi_n(t) = dw_n(t) - \xi_n(t)dt + v_n(\xi(t))dt.$$ 

It follows from our result that if $\xi(t)$ has a stationary measure $\mu$ for which $|v|_2 \in L^1(\mu)$, then $\mu$ is absolutely continuous with respect to the Gaussian measure that is the stationary solution for the non-perturbed linear equation. The assumption that $|v|_2 \in L^1(\mu)$ is essential and cannot be replaced by the weaker condition that the components of $v$ are $\mu$-integrable (recall that the stationary equation is meaningful with this weaker condition). Similarly, if we have the stochastic equation

$$d\xi_n(t) = dw_n(t) - \beta_n \xi_n(t)dt + v_n(\xi(t))dt$$

with some $\beta_n \geq \beta_0 > 0$ and $\mu$ is a stationary measure, then the $\mu$-integrability of $|v|_2$ ensures the absolute continuity of $\mu$ with respect to the Gaussian measure corresponding to the linear system with $v = 0$. On this direction, see, e.g., [21].

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