Last passage isometries for the directed landscape

Duncan Dauvergne

Abstract
Consider the restriction of the directed landscape $L(x, s; y, t)$ to a set of the form $\{x_1, \ldots, x_k\} \times \{s_0\} \times \mathbb{R} \times \{t_0\}$. We show that on any such set, the directed landscape is given by a last passage problem across $k$ locally Brownian functions. The $k$ functions in this last passage isometry are built from certain marginals of the extended directed landscape. As applications of this construction, we show that the Airy difference profile is locally absolutely continuous with respect to Brownian local time, that the KPZ fixed point started from two narrow wedges has a Brownian-Bessel decomposition around its cusp point, and that the directed landscape is a function of its geodesic shapes.

Keywords Last passage percolation · Directed landscape · KPZ universality · Airy sheet · Brownian local time

Mathematics Subject Classification 60K35

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1 Introduction

The KPZ (Kardar–Parisi–Zhang) universality class is a large collection of 2-dimensional random metrics and 1-dimensional models of random interface growth. Models in this class are expected to exhibit the same behaviour under scaling. The past twenty-five years have seen a period of intense and fruitful research on this class, with progress propelled by the discovery of a handful of exactly solvable models, including tasep, last passage percolation, directed polymers in a random environment, and the KPZ equation itself. The solvable structure in these models has allowed researchers to identify universal limiting objects that should arise as scaling limits of all KPZ models, e.g. Tracy-Widom distributions, Airy processes, the KPZ fixed point, the Airy sheet, the directed landscape. See the books and expository articles [8, 31, 32, 36] and references therein for background on the KPZ universality class and related areas.

This article studies the directed landscape, which is the full scaling limit of random metric models in the KPZ universality class. Our goal in this paper is to study and highlight the connection between the directed landscape and a particularly important prelimiting model, known as Brownian last passage percolation. We start by introducing last passage percolation in a line environment.

Let $f$ be a sequence of continuous functions $f_i : \mathbb{R} \to \mathbb{R}, i \in \mathbb{Z}$. For a nonincreasing cadlag function $\pi$ from $[x, y]$ to the integer interval $[m, n] := \{m, \ldots, n\}$, henceforth a path from $(x, n)$ to $(y, m)$, define the length of $\pi$ with respect to the environment $f$ by

$$\|\pi\|_f = \sum_{i=m}^{n} f_i(\pi_i) - f_i(\pi_{i+1}).$$  \hspace{1cm} (1)

Here $\pi_i = \inf\{t \in [x, y] : \pi(t) < i\}$ is the time when $\pi_i$ jumps off of line $i$, and if this set is empty we set $\pi_i = y$. We will visualize the space $\mathbb{R} \times \mathbb{Z}$ so that the $\mathbb{Z}$-coordinate decreases as we move up the page. This means that paths move up and to the right across the page, see Fig. 1. For $x \leq y \in \mathbb{R}$ and $m \leq n \in \mathbb{Z}$, define the last passage value across the environment $f$ by

$$f[ (x, n) \to (y, m) ] = \sup_\pi \|\pi\|_f,$$ \hspace{1cm} (2)

where the supremum is over all paths from $(x, n)$ to $(y, m)$. A path $\pi$ that achieves this supremum is a geodesic. When the environment is a collection of independent two-sided Brownian motions $B = \{B_i : i \in \mathbb{Z}\}$, this model is known as Brownian last passage percolation.

Brownian last passage percolation is best thought of as a natural model of a random metric on the (semi-discrete) plane. The only differences between Brownian last passage percolation and a true metric are that distances (i.e. last passage values) are only defined between points that come in a certain order and distances are defined by maximizing, rather than minimizing, path length. Nonetheless, Brownian last passage percolation is expected to have the same limiting behaviour as more natural random metric models in the KPZ class.
The payoff that we get from studying Brownian last passage percolation is that it has an exactly solvable structure. As a result of this structure, we can extract very precise asymptotic information. Indeed, in [11] the Brownian last passage percolation \((x, n; y, m) \mapsto B[(x, n) \to (y, m)]\) was shown to converge under scaling to a four-parameter scaling limit: the directed landscape \(L\). More precisely, letting \((x, s)_n = (s + 2x/n^{1/3}, s - \lfloor sn \rfloor)\) be the translation between limiting and pre-limiting locations, [11, Theorem 1.5] gives that there is a coupling of Brownian last passage percolation and the directed landscape such that

\[
B_n[(x, s)_n \to (y, t)_n] = 2(t - s)\sqrt{n} + 2(y - x)n^{1/6} + n^{-1/6}(L + o_n)(x, s; y, t).
\]

Here each \(B_n\) is a sequence of independent Brownian motions, and the parameter space for all the above functions is

\[
\mathbb{R}^4_+ = \{u = (p; q) = (x, s; y, t) \in \mathbb{R}^4 : s < t\}.
\]

The directed landscape \(L\) is a random continuous function from \(\mathbb{R}^4_+ \to \mathbb{R}\) and the error terms \(o_n\) almost surely satisfies \(\sup_K |o_n| \to 0\) for every compact set \(K \subset \mathbb{R}^4_+\). More recently, \(L\) was shown to be the scaling limit of other integrable models of last passage percolation [13].

Just like with Brownian last passage percolation, the value \(L(p; q) = L(x, s; y, t)\) is best thought of as a distance between two points \(p\) and \(q\) in the space-time plane. Here \(x, y\) are spatial coordinates and \(s, t\) are time coordinates. We cannot move backwards or instantaneously in time, so \(L(x, s; y, t)\) is not defined for \(s \geq t\). Unlike with an ordinary metric, \(L\) is not symmetric and may take negative values. As in last passage percolation, it also satisfies the triangle inequality backwards:

\[
L(p; r) \geq L(p; q) + L(q; r) \quad \text{for all } (p; r), (p; q), (q; r) \in \mathbb{R}^4_+.
\]

\(^1\) Note that time coordinates are in increasing order in \(\mathbb{R}^4_+\) whereas in the prelimit the lines \(n, m\) come in decreasing order. This switch in ordering allows for a more natural indexing in the limit. The last passage indexing is chosen to better accommodate the description of last passage across the Airy line ensemble, see Sect. 3.1.
Certain marginals of $\mathcal{L}$ are well-known. For example, for every fixed $x, s, y, t$ we have

$$\mathcal{L}(x, s; y, t) \overset{d}{=} (t - s)^{1/3} T - \frac{(x - y)^2}{t - s} \quad \text{and} \quad \mathcal{L}(x, 0; \cdot, 1) \overset{d}{=} \mathcal{A}(\cdot) - (x - \cdot)^2$$

(3)

where $T$ is a Tracy–Widom GUE random variable and $\mathcal{A}$ is an Airy$_2$ process. It may be useful for the reader to keep in mind the first equality above when thinking about the general shape of $\mathcal{L}$. In particular, the parabolic term $\frac{(x - y)^2}{t - s}$ is what we get if we apply KPZ limiting scaling to the standard Euclidean metric.

A priori, there is no reason to expect that $\mathcal{L}$ retains characteristics particular to any one of its prelimits. However, one might guess that Brownian last passage percolation bears a stronger connection with $\mathcal{L}$ than other models since $\mathcal{L}$ is known to have locally Brownian behaviour as we vary $x$ and $y$. Quite surprisingly, certain marginals of the directed landscape can essentially be expressed as Brownian last passage problems!

For this theorem and throughout the paper we say that $B$ is a $k$-dimensional Brownian motion of variance $\alpha$ if $B = \sqrt{\alpha} B'$, where $B'$ is a standard $k$-dimensional Brownian motion. We also write $X \ll Y$ for two random variables $X, Y$ if the law of $X$ is absolutely continuous with respect to the law of $Y$.

**Theorem 1.1** Let $s < t \in \mathbb{R}, b > 0$ and $x_1 < \cdots < x_k \in \mathbb{R}$. Let $\mathcal{L}$ denote the directed landscape and let $B$ be a collection of $k$ independent Brownian motions of variance 2. Consider the random continuous functions $f_\mathcal{L}, f_B : [1, k] \times [-b, b] \to \mathbb{R}$ given by

$$f_\mathcal{L}(i, y) = \mathcal{L}(x_i, s; y, t) - \mathcal{L}(x_i, s; -b, t)$$

$$f_B(i, y) = B[(-b - 1, i) \to (y, 1)] - B[(-b - 1, i) \to (-b, 1)].$$

Then $f_\mathcal{L} \ll f_B$.

The recentering by $\mathcal{L}(x_i, s; -b, t)$ and $B[(-b - 1, i) \to (-b, 1)]$ is necessary to deal with the fact that the last passage values $B[(-b - 1, i) \to (y, 1)]$ are increasing in $i$, but the corresponding landscape values are not. Note that the $k = 1$ case of Theorem 1.1 is just local absolute continuity of the Airy$_2$ process with respect to Brownian motion; this was first shown in [9]. Finally, the function $f_\mathcal{L}$ has an implicit dependence on $s, t$—it only depends on these times via a straightforward scaling relation, via scale and translation invariance properties of $\mathcal{L}$.

Theorem 1.1 follows from a more refined structural theorem—Theorem 1.2—that expresses values of the form $\mathcal{L}(x_i, s; y, t), i \in [1, k], y \in \mathbb{R}$ as a last passage problem across $k$ locally Brownian functions. To state that theorem and explain how Theorem 1.1 arises, we need to introduce a few more notions.

### 1.1 The Airy sheet, multi-point last passage, and the RSK isometry

The fundamental building block in the directed landscape is the **Airy sheet** $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$ given by $\mathcal{S}(x, y) := \mathcal{L}(x, 0; y, 1)$. The rescaled object $\mathcal{S}_t(x, y) := s\mathcal{S}(x/s^2, y/s^2)$
is called an **Airy sheet of scale** \( s \). The directed landscape is built from independent Airy sheets of different scales in an analogous way to how Brownian motion is built from independent normal distributions. More precisely, the directed landscape is the unique (in law) random continuous function \( \mathcal{L} : \mathbb{R}_+^4 \rightarrow \mathbb{R} \) satisfying the metric composition law

\[
\mathcal{L}(x, r; y, t) = \max_{z \in \mathbb{R}} \mathcal{L}(x, r; z, s) + \mathcal{L}(z, s; y, t) \quad \forall (x, r; y, t) \in \mathbb{R}_+^4, \ s \in (r, t),
\]

and with the property that \( \mathcal{L}(\cdot, t_i; \cdot, t_i + s_i^3) \) are independent Airy sheets of scale \( s_i \) for any set of disjoint time intervals \( (t_i, t_i + s_i^3) \).

The Airy sheet was constructed by understanding how a continuous version of the Robinson-Schensted-Knuth (RSK) correspondence interacts with random inputs.

Let \( C^n_0 \) be the space of \( n \)-tuples of continuous functions \( f = (f_1, \ldots, f_n), f_i : [0, \infty) \rightarrow \mathbb{R} \) with \( f(0) = 0 \). The **continuous RSK correspondence** is a map \( W : C^n_0 \rightarrow C^n_0 \). It can be presented in purely geometric terms by looking at **multi-point last passage percolation**. Consider an environment \( f = (f_i, i \in \mathbb{Z}) \) and vectors \( p = (p_1, \ldots, p_k), q = (q_1, \ldots, q_k) \), where \( p_i = (x_i, n_i), q_i = (y_i, m_i) \in \mathbb{R} \times \mathbb{Z} \) and \( x_i \leq x_{i+1}, n_i \leq n_{i+1} \) and \( y_i \leq y_{i+1}, m_i \leq m_{i+1} \) for all \( i \). Let

\[
f[p \rightarrow q] = \sup_\pi \sum_{i=1}^k \| \pi \|_f
\]

where the supremum is over all disjoint \( k \)-tuples of paths \( \pi = (\pi_1, \ldots, \pi_k) \), where each \( \pi_i \) goes from \( (x_i, n_i) \) to \( (y_i, m_i) \), and \( \pi_i(z) < \pi_{i+1}(z) \) for \( z \in (x_i, y_i) \cap (x_{i+1}, y_{i+1}) \). We call a \( k \)-tuple that achieves this supremum a (disjoint) **optimizer**. As we have not imposed any relationship between \( p \) and \( q \) it is possible that no disjoint \( k \)-tuples exist. In this case we set \( f[p \rightarrow q] = -\infty \). Having \( f[p \rightarrow q] > -\infty \) forces certain inequalities to hold relating the \( x_i, y_i, n_i \) and \( m_i \).

Now, for a point \( p \in \mathbb{R} \times \mathbb{Z} \), we write \( p^k \) for the vector consisting of \( k \) copies of a point \( p \). For \( f \in C^n_0 \), define \( Wf \in C^n_0 \) by

\[
\sum_{i=1}^k (Wf)_i(y) = f[(0, n)^k \rightarrow (y, 1)^k]
\]

for \( k \in [1, n] \), \( t \in [0, \infty) \). Moving forward, we will write \( Wf_i \) rather than \( (Wf)_i \) to simplify notation. Remarkably, the RSK map \( W \) is an **isometry** between the upper and lower boundaries of \( f \). In other words, it preserves multi-point last passage values between clusters of points on line \( n \) and line \( 1 \). For \( x \in \mathbb{R}^k_\leq := \{ x \in \mathbb{R}^k : x_1 \leq \cdots \leq x_k \} \), we let \( (x, m) \) denote the \( k \)-tuple of points \( (x_1, m), \ldots, (x_k, m) \). For any vectors \( (x, n), (y, 1) \) with \( 0 \leq x_1 \), we have

\[
f[(x, n) \rightarrow (y, 1)] = Wf[(x, n) \rightarrow (y, 1)].
\]
The isometry (6) was shown in [11, Proposition 4.1], though closely related formulas had been previously observed by Biane, Bougerol, and O’Connell [5] and Noumi and Yamada [25].

The RSK correspondence behaves well with certain random inputs. In particular, if $B \in C_0^{\infty}$ is a sequence of independent Brownian motions, then $WB$ is a sequence of nonintersecting Brownian motions, see [26, 28]. In the KPZ scaling limit, $WB$ converges to the parabolic Airy line ensemble $W$, constructed by Prähofer and Spohn [30], and realized as a sequence of infinity many nonintersecting, locally Brownian functions by Corwin and Hammond [9].

The Airy sheet was built in [11] by understanding how the isometry (6) passes to the KPZ limit for single points. As the precise way in which this happens is rather technical, we postpone it until Definition 3.2. In [14], this analysis was extended to multiple points to construct an extended Airy sheet and an extended directed landscape. The extended directed landscape can alternately be described using the usual directed landscape $L$ once we define path lengths in $L$.

In $L$, a path from $(x, s)$ to $(y, t)$ is a continuous function $\pi : [s, t] \to \mathbb{R}$ with $\pi(s) = x$ and $\pi(t) = y$, with length

$$\|\pi\|_L = \inf_{k \in \mathbb{N}} \inf_{s = t_0 < t_1 < \ldots < t_k = t} \sum_{i=1}^{k} L(\pi(t_{i-1}), t_{i-1}; \pi(t_i), t_i).$$

(7)

This is analogous to defining the length of a curve in Euclidean space by piecewise linear approximation. A path $\pi$ is a geodesic if $\|\pi\|_L$ is maximal among all paths with the same start and end points. Equivalently, a geodesic is any path $\pi$ with $\|\pi\|_L = L(\pi(s), s; \pi(t), t)$. Almost surely, geodesics exist between every pair of points $p = (x, s), q = (y, t)$ with $s < t$ and for any fixed pair of points $p, q$ almost surely there is a unique geodesic from $p$ to $q$, see [11, Section 12].

Now, for $x, y \in \mathbb{R}^k, s < t$ let

$$L(x, s; y, t) = \sup_{\pi} \sum_{i=1}^{k} \|\pi_i\|_L,$$

(8)

where the supremum is over all $k$-tuples of paths $\pi_i : [s, t] \to \mathbb{R}$ with $\pi_i(s) = x_i, \pi_i(t) = y_i$ and $\pi_i(r) < \pi_{i+1}(r)$ for all $r \in (s, t)$. This extension of $L$ is called the extended directed landscape, abbreviated as extended landscape. The extended landscape is continuous on its domain, and satisfies a limiting RSK isometry. Define an infinite collection of functions $WL_i : \mathbb{R} \to \mathbb{R}, i \in \mathbb{N}$ by

$$\sum_{i=1}^{k} WL_i(y) = L(0^k, 0; y^k, 1).$$

(9)

Then $WL$ is a parabolic Airy line ensemble and for any $x, y \in \mathbb{R}^k$, we have

$$L(x, 0; y, 1) = WL[x \to y].$$

(10)
Of course, one needs to make sense of the right side of (10). As this is somewhat technical and is not used moving forward in the introduction, we leave it until Theorem 3.3 for details.

### 1.2 Other isometries

In addition to $W_f$, for a given $f \in \mathcal{C}_0^n$ there are other interesting environments $g \in \mathcal{C}_0^n$ that satisfy the isometric property (6). Our main theorems in this paper come from exploring these other isometries in the limit $L$.

One of the standard methods for constructing $W_f$ is via iterating two-line versions of the RSK map, known as Pitman transforms. This method originates in [5], see also [10] for a more expository description. For $n \in \mathbb{N}$, an adjacent transposition $\sigma_i := (i, i + 1) \in S_n$ and $f \in \mathcal{C}_0^n$, define

$$W_{\sigma_i} f = (f_1, \ldots, f_{i-1}, W(f_i, f_{i+1}), f_{i+2}, \ldots, f_n).$$

For more general permutations $\tau \in S_n$, we define

$$W_\tau = W_{\sigma_{i_1}} \cdots W_{\sigma_{i_k}}$$

(11)

where $\sigma_{i_1} \cdots \sigma_{i_k} = \tau$ is a reduced decomposition of $\tau$, i.e. a minimal length decomposition of $\tau$ as a product of adjacent transpositions. The right-hand side of (11) is the same for any reduced decomposition, see the discussion preceding Proposition 2.8 in [5]. The $n$-line map $W$ in (5) is equal to $W_{\text{rev}_n}$, where $\text{rev}_n = n \cdots 1$ is the reverse permutation, see [5] or Section 3 in [10].

As with $W_{\text{rev}_n}$, all of the environments $W_\tau f$ turn out to satisfy (6), see Proposition 2.4. Moreover, these environments can also be described in terms of certain multi-point last passage values in analogy with (5). Also, if $B$ is a collection of independent Brownian motions, then while $W_\tau B$ does not have the tractable non-intersecting structure of $WB$, its paths are still locally absolutely continuous with respect to Brownian motion, as is the case with $WB$. See Sect. 2 for more details.

This story has an analogue in the directed landscape. The environments we construct will no longer satisfy (10) for all possible $x, y$, but rather only for $x$ with entries in a particular finite set $\{x_1, \ldots, x_k\}$. On the other hand, these environments are in one sense significantly simpler than $W_L$: they will consist of only $k$ lines. As in the finite setting, while these isometric environments no longer have the integrable or nonintersecting structure of $W_L$, they are still locally Brownian and their lines can be described via certain marginals of the extended directed landscape, similarly to (9).

To set up our main theorem, we need a notion of last passage from $-\infty$. Let $f = (f_1, \ldots, f_k), f_i : \mathbb{R} \to \mathbb{R}$ be an environment of continuous functions. For $I = \{I_1 < \cdots < I_\ell\} \subset [1, k]$ and an $\ell$-tuple $p \in (\mathbb{R} \times [1, k])^\ell$, define

$$f[(-\infty, I) \to p] := \lim_{z \to -\infty} f[(z, I) \to p] + \sum_{i \in I} f_i(z).$$

(12)
Note that the right-hand side above is monotone increasing in \( z \), so the limit necessarily exists. In (12) and throughout we write \((z, I)\) for the \(|I|\)-tuple \((z, I_1), \ldots, (z, I_{|I|})\). This type of last passage can equivalently be defined using a supremum over disjoint paths, see Remark 3.8.

**Theorem 1.2**  For \( x \in \mathbb{R}^k \) and a set \( I \subset [1, k] \), write \( x^I \) for the vector in \( \mathbb{R}^{|I|} \) consisting only of coordinates \( x_i \) of \( x \) with \( i \in I \). Define a sequence of \( k \) functions \( W_x \mathcal{L} = \{ W_x \mathcal{L}_i : \mathbb{R} \to \mathbb{R} , i \in [1, k] \} \) by the formula

\[
\sum_{i=1}^\ell W_x \mathcal{L}_i(y) = \mathcal{L}(x^{[1, \ell]}, 0; y^\ell, 1), \quad \text{for } \ell \in [1, k], y \in \mathbb{R}. \tag{13}
\]

1. (Asymptotics and stability) Almost surely, for any \( x \in \mathbb{R}^k \) with \( x_1 < \cdots < x_k \) we have the following. For any \( i \in [1, k] \), we have

\[
\lim_{z \to \pm \infty} \frac{W_x \mathcal{L}_i(z) + z^2}{z} = 2x_i. \tag{14}
\]

In particular, this implies that for \( y_0 \in \mathbb{R} \), there exists a random \( Z_0(y_0) \in (-\infty, y_0) \cap \mathbb{Z} \) such that

\[
W_x \mathcal{L}|(-\infty, I) \to (y, 1)] = W_x \mathcal{L}|(z, I) \to (y, 1)] + \sum_{i \in I} W_x \mathcal{L}_i(z) \tag{15}
\]

for all \( I \subset [1, k] \), \( z \leq Z_0 \) and \( y \in \mathbb{R}^{|I|} \) with \( y_1 \geq y_0 \).

2. (Locally Brownian) Fix \( x \in \mathbb{R}^k \) with \( x_1 < \cdots < x_k \). Conditional on \( W_x \mathcal{L}(0) \) the function \( W_x \mathcal{L}(\cdot) - W_x \mathcal{L}(0) \) is locally absolutely continuous with respect to a \( k \)-dimensional Brownian motion \( B \) of variance 2. In other words, if we let \( B, \mathcal{L} \) be independent then on any interval \([a, b], \) we have \( (W_x \mathcal{L}(0), W_x \mathcal{L}|_{[a, b]} - W_x \mathcal{L}(0)) \ll (W_x \mathcal{L}(0), B|_{[a, b]}).\)

3. (Isometry) Almost surely, for any \( x \in \mathbb{R}^k \) with \( x_1 < \cdots < x_k \), any \( I \subset [1, k] \), and any \( y \in \mathbb{R}^{|I|} \), we have

\[
W_x \mathcal{L}|(-\infty, I) \to (y, 1)] = \mathcal{L}(x^I, 0; y, 1). \]

**Remark 1.3**

1. Theorem 1.2 allows us to represent complicated marginals of the directed landscape and the Airy sheet in terms of last passage in a finite environment of locally Brownian continuous functions. In particular, this allows us to show that almost sure properties of Brownian last passage percolation across finitely many lines hold in \( \mathcal{L} \). We discuss a few consequences of this in Sect. 1.3.

2. Theorem 1.2 also has a version when \( x \) is allowed to have repeated entries. This is quite a technical extension that can be derived from the distinct entry case by an approximation argument. As this extension is not required in our applications later on and the ideas are not central to the paper, we leave its statement and proof to the appendix. Note that the case where \( x_i = 0 \) for all \( i \) is implicit in Proposition 5.9 in
In this special case, \( W_x \mathcal{L} \) returns the top \( k \) lines of \( \mathcal{W}_L \). This case corresponds to the usual RSK isometry.

3. Even though we have fixed the times in Theorem 1.2 equal to 0 and 1, the result also holds for arbitrary times \( s < t \) by invariance properties of \( \mathcal{L} \).

4. We believe the construction in Theorem 1.2 should be useful for studying Busemann functions for the extended landscape. Consider the limiting field

\[
B_{\mathcal{L}}(x, y) = \lim_{t \to \infty} \mathcal{L}(tx, -t; 0, y) - \mathcal{L}(tx, -t; 0, 0^k),
\]

which should give the Busemann function in a direction \( x \in \mathbb{R}^k \leq \) between locations \( (0, y) \) and \( (0, 0^k) \). Taking a limit of the construction in Theorem 1.2 should yield insight into the nature of \( B_{\mathcal{L}} \). Indeed, for any \( x \in \mathbb{R}^k \leq \) with \( x_1 < \cdots < x_k \), defining \( W_x^\infty \mathcal{L} = \{ W_x \mathcal{L}_i : \mathbb{R} \to \mathbb{R}, i \in [1, k] \} \) by

\[
\sum_{i=1}^\ell W_x^\infty \mathcal{L}_i(y) := B_{\mathcal{L}}(x^{[1, \ell]}, y^\ell),
\]

for any \( I \subset [1, k] \) and \( y \in \mathbb{R}^{\ell} \leq \), we should have the equality

\[
B_{\mathcal{L}}(x^I, y) = W_x^\infty \mathcal{L}([-\infty, I) \to (y, 1]).
\]

Moreover, each of the lines \( W_x^\infty \mathcal{L}_i \) should be an independent, variance 2 Brownian motion with drift \( 2x_i \). We do not attempt to make this limiting picture rigorous, or to justify the existence of the limit in (16).

Busemann functions have been studied in detail in last passage models, yielding remarkable insights about the geometry of infinite geodesics and other last passage structures, e.g. see [7, 18, 21, 34]. At the level of single points, a Busemann function structure closely related to (17) was shown for exponential last passage percolation by Fan and Seppäläinen [15], building on work of Ferrari and Martin [16] studying stationary measures in multi-type tasep. In [15], the analogue of the equality (17) is distributional. The lines corresponding to \( W_x^\infty \mathcal{L}_i \) are constructed via certain ‘multiclass processes’ which, a priori, are unrelated to multi-point last passage values.

1.3 Consequences

In addition to Theorem 1.1, we present three other fairly straightforward consequences of Theorem 1.2. First, fix \( x_1 < x_2 \) and define the Airy difference profile

\[
A^{x_1, x_2}(y) = \mathcal{L}(x_2, 0; y, 1) - \mathcal{L}(x_1, 0; y, 1).
\]

The function \( A^{x_1, x_2} \) is a continuous increasing function, and hence is the cumulative distribution function of a random measure \( \mu_{x_1, x_2} \). Moreover, basic symmetries of \( \mathcal{L} \)
imply that the process $A^{x_1,x_2} - \mathbb{E}A^{x_1,x_2}$ is stationary and by the first equality in (3) we have

$$\mathbb{E}A^{x_1,x_2}(y) = -x_2^2 - x_1^2 + 2y(x_2 - x_1),$$

so on average, $A^{x_1,x_2}$ increases linearly. However, this is not the case for individual realizations of $A^{x_1,x_2}$; in fact, $\mu_{x_1,x_2}$ is supported on a lower dimensional set.

In [3], Basu, Ganguly and Hammond showed that the support of $\mu_{x_1,x_2}$ almost surely has Hausdorff dimension $1/2$, and in [4], Bates, Ganguly and Hammond proved that this support coincides with a set of exceptional events for geodesics in $L$. Moving beyond a Hausdorff dimension estimate, Ganguly and Hegde [17] showed that $A^{x_1,x_2}$ can be represented as a ‘patchwork quilt’ of objects that are locally absolutely continuous with respect to the running maximum of a Brownian motion. The techniques used in [17] have similarities with our methods, i.e. they also use properties of iterated Pitman transforms to study $A^{x_1,x_2}$.

As a consequence of Theorem 1.2 in the special case when $x = (x_1, x_2)$, we recover all of these previous results and more. In particular, we show that $A^{x_1,x_2}$ is locally absolutely continuous with respect to the running maximum of a Brownian motion without any need for a patchwork quilt.

**Theorem 1.4** Fix $x = (x_1, x_2)$ with $x_1 < x_2$.

1. For all $y \in \mathbb{R}$,

$$A^{x_1,x_2}(y) = \sup_{z \leq y} W_x \mathcal{L}_2(z) - W_x \mathcal{L}_1(z) = \sup_{z \leq y} \mathcal{L}((x_1, x_2), 0; z^2, 1) - 2\mathcal{L}(x_1, 0; z, 1).$$

2. For any compact set $[a, b]$, the law of $A^{x_1,x_2}|_{[a,b]}$ is absolutely continuous with respect to the law of the running maximum $M$ of a Brownian motion $B$ on $[a-1, b]$ of variance $4$, i.e. $M(x) = \sup_{a-1 \leq y \leq x} B(y)$.

3. The support of $\mu_{x_1,x_2}$ is the set of $y \in \mathbb{R}$ where

$$\mathcal{L}((x_1, x_2), 0; y^2, 1) = \mathcal{L}(x_1, 0; y, 1) + \mathcal{L}(x_2, 0; y, 1).$$

Equivalently, this is the of set of $y \in \mathbb{R}$ where there exist geodesics $\pi_1$ from $(x_1, 0) \rightarrow (y, 1)$ and $\pi_2$ from $(x_2, 0) \rightarrow (y, 1)$ such that $\pi_1(r) < \pi_2(r)$ for all $r \in [0, 1)$.

The disjoint geodesic characterization of $\text{supp}(\mu_{x_1,x_2})$ in Theorem 1.4.3 was shown in [3]. We have included it above to highlight how it can alternately be obtained as an immediate consequence of our main theorem and one of the main results of [14].

As a consequence of Theorem 1.4, we can show that $\mathcal{L}$ can be reconstructed from only the shapes of its geodesics, without any information about their lengths.

**Theorem 1.5** Let $Q^4_+ = \mathbb{R}^4_+ \cap Q^4$. For every $u = (p, q) \in Q^4_+$, let $\pi\mathcal{L}(u)$ be the almost surely unique $\mathcal{L}$-geodesic from $p$ to $q$, and for $s < t \in Q$ let $\mathcal{F}_{s,t}$ be the $\sigma$-algebra generated by $\{\pi\mathcal{L}(x, s; y, t) : (x, y) \in Q^2\}$ and all null sets. Then $\mathcal{L}(\cdot, s; \cdot, t) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $\mathcal{F}_{s,t}$-measurable.
In particular, we can almost surely reconstruct the entire directed landscape $L$ using only the information from $\pi_L(u), u \in \mathbb{Q}_1^4$.

We thank Bálint Virág for pointing out the key step in the proof of Theorem 1.5 and the reference [35].

Our final consequence concerns the KPZ fixed point, constructed by Matetski, Quastel, and Remenik [22]. The KPZ fixed point is a Markov process $h_t, t \in [0, \infty)$ taking values in the space of upper semicontinuous functions $h: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ with a growth bound at $\pm \infty$. It is the scaling limit of random growth models in the KPZ universality class. By results of [13, 24], it is related to the directed landscape $L$ by the following formula. Letting $h_0: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ denote the initial condition of the KPZ fixed point, we can write

$$h_t(x) = \max_{x \in \mathbb{R}} h_0(x) + L(x, 0; y, t). \quad (18)$$

In [19], Hammond (see also [6, 20]) demonstrated how for fixed $t$, $h_t$ can be decomposed as a patchwork quilt made of Brownian pieces. More precisely, he showed that under mild assumptions on the initial condition, $h_t$ has the following description:

- There exists an ordered (finite or countable) random sequence $\ldots A_{-1} < A_0 < A_1 < \ldots$ which is finite when restricted to any compact set, an (unordered) random sequence $p_i, i \in \mathbb{Z}$, and a sequence of random continuous functions $Y_i : \mathbb{R} \to \mathbb{R}, i \in \mathbb{Z}$ such that

$$h_t(x) = \sum_{i \in \mathbb{Z}} 1(x \in [A_i, A_{i+1}]) [Y_i(x) + p_i]. \quad (19)$$

- Each of the functions $Y_i$ is locally absolutely continuous with respect to a Brownian motion of variance 2.

In this description, each patch $Y_i(x) + p_i = L(y_i, 0; x, t) + h_0(y_i)$ for some strictly increasing sequence $y_i$.

More recently, Sarkar and Virág [33] showed that in fact the KPZ fixed point started from any initial condition is locally absolutely continuous with respect to Brownian motion. From this point of view, the cusp points $A_i$ are not seen. However, we still expect interesting behaviour at these points. Using Theorem 1.2, we can identify what is actually happening at these cusp points. For simplicity, we restrict ourselves to the easiest nontrivial case when the initial condition $h_0$ is combination of two narrow wedge initial conditions. In this case there is exactly one cusp, and the KPZ fixed point has the following basic structure.

**Fact 1.6** Let $h_t$ denote the KPZ fixed point at time $t$ started from an initial condition $h_0$ which is equal to $-\infty$ except at two points $p_1 < p_2$ where $h_0(p_1) = a_1, h_0(p_2) = a_2$.

\footnote{Since we first started working with the directed landscape, Bálint Virág and I have long been interested in these sort of reconstruction questions for $L$. There are many interesting ones. One we particularly like (which would significantly strengthen Theorem 1.5) is whether a landscape value $L(p; q)$ can be reconstructed from only the shape of the geodesic from $p$ to $q$.}
for some $a_1, a_2 \in \mathbb{R}$. Then there exists $A \in \mathbb{R}$ such that

$$h_t(y) = \lfloor \mathcal{L}(p_1, 0; y, 1) + a_1 \rfloor I(y < A) + \lfloor \mathcal{L}(p_2, 0; x, 1) + a_2 \rfloor I(y \geq A).$$

We are concerned with understanding the joint law of $(A, h_t)$.

**Theorem 1.7** Let $X \in \mathbb{R}$ be any random variable with a positive Lebesgue density everywhere, let $B : \mathbb{R} \to \mathbb{R}$ be a two-sided standard Brownian motion, let $R : \mathbb{R} \to \mathbb{R}$ be a two-sided Bessel-3 process. That is, $R = \|Y\|_2$, where $Y$ is a 3-dimensional standard Brownian motion. Suppose that all 3 objects are independent. Then with $h_t$ as in the setting of Fact 1.6, for any $t > 0$ and any compact interval $I \subset \mathbb{R}$, we have

$$(A, h_t(\cdot)|I - h_t(A)) \ll (X, [B + R](-X + \cdot)|I).$$

Theorem 1.7 shows that conditional on its cusp location $A$, $h_t(A + \cdot) - h_t(A)$ is locally absolutely continuous with respect to a two-sided Brownian-Bessel process. If we start from a more general initial condition, then there will be many cusp points as in (19). If we condition on all of these cusp points, then similar ideas could be used to show that the process $h_t$ is locally absolutely continuous with respect to a Brownian motion started at the cusp closest to 0, plus a string of independent Bessel-3 bridges running between all cusp points. For brevity, we do not pursue this technical extension here.

## 2 Properties of last passage percolation

### 2.1 Basics

Recall the definition of multi-point last passage from (4). Multi-point last passage percolation satisfies the following useful **metric composition law**. Its proof is immediate from the definitions.

**Proposition 2.1** Let $f = (f_i, i \in \mathbb{Z})$ be a sequence of continuous functions and let $j \in \mathbb{Z}$. For any endpoints $p, q$ where each $p_i \in \mathbb{R} \times \{j, j + 1, j + 2, j + 1\}$ and each $q_i \in \mathbb{R} \times \{j, j − 1, \ldots\}$ we have that

$$f[\mathbf{p} \to \mathbf{q}] = \max_{z \in \mathbb{R}^k_{\leq}} f[\mathbf{p} \to (z, j + 1)] + f[(z, j) \to \mathbf{q}].$$

Similarly, for any $x \in \mathbb{R}$ and endpoints $p, q$ with $p_i \in (-\infty, x] \times \mathbb{Z}, q_i \in (x, \infty) \times \mathbb{Z}$ we have that

$$f[\mathbf{p} \to \mathbf{q}] = \max_{I \in \mathbb{Z}^k_{\leq}} f[\mathbf{p} \to (x, I)] + f[(x, I) \to \mathbf{q}].$$

Here and throughout $\mathbb{Z}^k_{\leq} = \{I \in \mathbb{Z}^k : I_1 \leq I_2 \cdots \leq I_k\}$.

We will also use the following facts about optimizers. First, suppose that $\pi, \pi'$ are paths from $(x, n)$ to $(y, m)$ and $(x', m')$ to $(y', m')$, respectively. We write $\pi \leq \pi'$ and say that $\pi'$ is **to the right** of $\pi$ if:
An example of Proposition 2.2(ii) with $k = 3, k' = 1, s = 2$. Here the red paths give the optimizer $\pi$ from $(0, I)$ to $(y, 1)$ and the purple path is the geodesic $\pi'$ from $(0, I_3)$ to $(y_3, 1)$. The proposition guarantees that $\pi_3 \geq \pi'$. For $z \in [x, y] \cap [x', y']$, we have $\pi(z) \leq \pi'(z)$.

For $k$-tuples of paths $\pi, \pi'$, we write $\pi \leq \pi'$ if $\pi_i \leq \pi'_i$ for all $i \in [1, k]$. Then we have the following, from [14]. The first part is Lemma 2.2 from that paper, and part (ii) is a slight variant of Lemma 2.3, whose proof goes through verbatim.

**Proposition 2.2** Let $f = (f_i, i \in \mathbb{Z})$ be a sequence of continuous functions, and let $(p, q)$ be a pair of $k$-tuples such that there is at least one disjoint $k$-tuple (of paths) from $p$ to $q$.

(i) There always exists an optimizer $\pi$ in $f$ from $p$ to $q$ such that for any optimizer $\tau$ from $p$ to $q$, we have $\tau \leq \pi$. We call $\pi$ the rightmost optimizer from $p$ to $q$.

(ii) For this part, suppose that $p = (x, I)$ for some $x \in \mathbb{R}$, $I \subset \mathbb{N}_\leq^k$ and $q = (y, 1)$ for some $y \in \mathbb{R}_\leq^k$. Also let $k' \in \mathbb{N}$ and assume $(p', q') = (x, I', y', 1)$ is a $k'$-tuple of endpoints such that there is at least one disjoint $k'$-tuple from $p'$ to $q'$. Finally let $s \in [0, 1, \ldots, k - 1]$ be such that $k - s \leq k'$ and suppose that for all $i \in [1 + s, k]$ we have $I_i \geq I'_{i-s}$ and $y_i \geq y'_{i-s}$. Then if $\pi, \pi'$ are the rightmost optimizers from $p$ to $q$ and $p'$ to $q'$, we have $\pi_i \geq \pi'_i$ for all $i \in [1 + s, k]$.

Part (ii) above holds in greater generality than stated (i.e. we could take endpoints in a more general form) but the version above suffices for our purposes here. Note that the concrete case when $s = 0$ and $k = k'$ is easiest to understand. In this case, part (ii) amounts to a simple monotonicity of rightmost optimizers as we shift endpoints up/down or left/right. See Fig. 2 for an example.

### 2.2 Pitman transforms

Recall from the introduction that $\mathcal{C}_0^n$ is the space of $n$-tuples of continuous functions $f = (f_1, \ldots, f_n), f_i : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$. A fruitful way of studying last passage percolation across environments in $\mathcal{C}_0^n$ is by sorting the environments two lines at a time using a series of two-line Pitman transforms, an approach introduced in...
The Pitman transform is simply the map $W : \mathcal{C}_0^2 \rightarrow \mathcal{C}_0^2$ from (5) for 2 lines. The following proposition records important properties of this map.

**Proposition 2.3** Let $W : \mathcal{C}_0^2 \rightarrow \mathcal{C}_0^2$ and $f \in \mathcal{C}_0^2$. We have

(i) $Wf_2 \leq f_2 \leq Wf_1$ and $Wf_2 \leq f_1 \leq Wf_1$.

(ii) For any endpoints $(x, 2), (y, 1)$ with $0 \leq x_1$, we have the **isometry**

$$Wf[(x, 2) \rightarrow (y, 1)] = f[(x, 2) \rightarrow (y, 1)].$$

Item (i) is immediate from the definition, and (ii) is [11, Lemma 4.3]. As discussed in Sect. 1.2, we can apply Pitman transforms two lines at a time to understand last passage percolation in $\mathcal{C}_0^\infty$ to produce a series of environments $W_\tau f$ indexed by permutations $\tau \in S_n$.

In [11], the isometry in Proposition 2.3(ii) was extended to yield an isometry for $W = W_{\text{rev}}$ by using the metric composition law, Proposition 2.1. We will also extend Proposition 2.3(ii) to get an isometric property for general environments $W_\tau f$; the proof is essentially identical to the proof of [11, Proposition 4.1].

**Proposition 2.4** Let $\tau \in S_n$ be a permutation and suppose that for some interval $[a, b] \subset [1, n]$, $\tau$ is the identity on $[a, b]^c$. Then for any endpoints $p, q$ with $p_i \in [0, \infty) \times [b, n]$ and $q_i \in [0, \infty) \times [1, a]$, we have

$$W_\tau f[p \rightarrow q] = f[p \rightarrow q].$$

**Proof** By (11), it suffices to prove the proposition when $\tau$ is an adjacent transposition $\sigma_j$ and $[a, b] = \{j, j + 1\}$. First assume $j \neq 1, n - 1$. For any endpoints $p, q$ of size $k$ with $p_i \in [0, \infty) \times [j + 1, n]$ and $q_i \in [0, \infty) \times [1, j]$ for all $i$, Proposition 2.1 ensures that

$$f[p \rightarrow q] = \max_{z, z' \in \mathbb{R}_+^k} f[p \rightarrow (z, j + 2)]$$

$$+ f[(z, j + 1) \rightarrow (z', j)] + f[(z', j - 1) \rightarrow q].$$

Under the maximum in (20), the first and third terms are unchanged when we apply $W_{\sigma_j}$ since $W_{\sigma_j} f_k = f_k$ for $k \neq j + 1, j$. The middle term is unchanged by Proposition 2.3(ii). Hence the left side of (20) is unchanged when we apply $W_{\sigma_j}$. The cases when $j = 1, n - 1$ are similar, except there will be fewer terms on the right side of (20).

Next, for $j \leq i$ define the permutation $\tau_{i,j} = \sigma_j \cdots \sigma_{i-1}$. We use the convention that $\tau_{i,i} = \text{id}_n$. The maps $W_{\tau_{i,j}}$ are related to last passage by the following lemma, Lemma 3.10 from [10].

**Lemma 2.5** Let $j \leq i \in [1, n]$ and $f \in \mathcal{C}_0^{\infty}$. For all $y \geq 0$, we have

$$f[(0, i) \rightarrow (y, j)] = W_{\tau_{i,j}} f_j(y).$$

$$\square$$ Springer
The next proposition builds on Lemma 2.5. Recall that for a finite set \( J = \{ j_1 < \cdots < j_k \} \subset \mathbb{Z} \) and \( x \in \mathbb{R} \), we use the shorthand \((x, J) = ((x, j_1), \ldots, (x, j_k))\).

**Proposition 2.6** Let \( I = \{ I_1 < \cdots < I_k \} \subset \llbracket 1, n \rrbracket \), \( f \in C_n^0 \), set \( \tau_I := \tau_{I_k} \cdots \tau_{I_1, 1} \), and define \( m : I \to \llbracket 1, n \rrbracket \) by \( m(I_j) = j \) for all \( j \). For any nonempty subset \( J \subset I \) and any vector \( y \in \mathbb{R}_{\leq 0}^{|J|} \) with \( 0 \leq y_{1, \ell} \), we have

\[
W_{\tau_I} f \left[(0, m(J)) \to (y, 1)\right] = f \left[(0, J) \to (y, 1)\right].
\]

We can also explicitly describe the functions \( W_{\tau_I} f_1, \ldots, W_{\tau_I} f_k \) as follows:

\[
\sum_{i=1}^{\ell} W_{\tau_I} f_i(y) = f \left[(0, I^\ell) \to (y^\ell, 1)\right],
\]

for all \( 0 \leq y, \ell \in \llbracket 1, k \rrbracket \), where \( I^\ell = \{ I_1, \ldots, I_\ell \} \).

See Fig. 3 for a concrete example of Proposition 2.6 and an outline of some of the main proof ideas.

**Proof** For each \( \ell \in \llbracket 0, k \rrbracket \), define a map \( m_\ell : I \to \llbracket 1, n \rrbracket \) by setting \( m_\ell(I_j) = j \) for \( j \leq \ell \), and \( m_\ell(I_j) = I_j \) otherwise. Let \( \tau_\ell = \tau_{I_{\ell}} \cdots \tau_{I_1, 1} \). We will inductively prove the stronger claim that for every \( \ell \in \llbracket 0, k \rrbracket \), for any \( J, y \) as in the proposition, we have

\[
W_{\tau_\ell} f \left[(0, m(\ell(J))) \to (y, 1)\right] = f \left[(0, J) \to (y, 1)\right].
\]

---

**Fig. 3** An example of Proposition 2.6 with \( I = \{2, 4\} \), \( n = 4 \). In each of the five diagrams, the indicated last passage values are equal and in this simple example, all equalities can be verified by inspection. In the proof, the equalities (a) = (b) and (c) = (d) follow from Proposition 2.4 and the equalities (b) = (c) and (d) = (e) follow from Lemma 2.5.
The base case when \( \ell = 0 \) is trivially true. Now suppose that the claim holds at \( \ell - 1 \). It is enough to show that for any \( g \in \mathcal{C}^n_0 \) and \( J, y \) as in the proposition, we have

\[
W_{\tau_{J}, \ell} g[(0, m_\ell(J)) \to (y, 1)] = g[(0, m_{\ell-1}(J)) \to (y, 1)].
\] (24)

Indeed, (24) implies (23) by taking \( g = W_{\tau_{J-1}} f \) and applying the inductive hypothesis. First, by Proposition 2.4, (24) holds with \( m_\ell(J) \) replaced by \( m_{\ell-1}(J) \). This immediately implies (24) if \( I_\ell \notin J \), so from now on we may assume \( I_\ell \in J \). Now,

\[
W_{\tau_{J}, \ell} g[(0, m_\ell(J)) \to (y, 1)] \leq W_{\tau_{J}, \ell} g[(0, m_{\ell-1}(J)) \to (y, 1)],
\] (25)

since all disjoint \( k \)-tuples from \( (0, m_\ell(J)) \) to \( (y, 1) \) are also disjoint \( k \)-tuples from \( (0, m_{\ell-1}(J)) \) because \( m_\ell \leq m_{\ell-1} \). For the opposite inequality, consider any disjoint \( k \)-tuple \( \pi \) from \( (0, m_{\ell-1}(J)) \) to \( (y, 1) \). Let \( \pi_\ell \) be the path starting at \( (0, I_\ell) \), and let \( y = \sup \{ t \geq 0 : \pi(t) \geq \ell \} \), where we take \( y = 0 \) if this set is empty. Then

\[
\|\pi_\ell[0,y]\|_{W_{\tau_{J}, \ell} g} \leq W_{\tau_{J}, \ell} g[(I_\ell, 0) \to (y, \ell)] = g[(I_\ell, 0) \to (y, \ell)] = W_{\tau_{J}, \ell} g(y).
\]

The first equality follows from the isometry in Proposition 2.4, and the second equality follows from Lemma 2.5. Therefore if we define a new path \( \rho = \ell \land \pi_\ell \), we have

\[
\|\pi_\ell\|_{W_{\tau_{J}, \ell} g} \leq \|\rho\|_{W_{\tau_{J}, \ell} g}.
\]

Moreover, replacing the path \( \pi_\ell \) with \( \rho \) in the \( k \)-tuple \( \pi \) yields a new disjoint \( k \)-tuple \( \pi_* \) from \( (0, m_\ell(J)) \) to \( (y, 1) \) since all paths starting above \( \pi_\ell \) started at lines in \([1, \ell - 1]\). This gives the opposite inequality in (25), yielding (24).

**Remark 2.7** Moving forward, we will also want to apply Pitman transforms to more general environments \( f \), opened up at times other than 0. Let \( f = (f_1, \ldots, f_n) \), \( f_i : \mathbb{R} \to \mathbb{R} \). Letting \( T_a f(x) = f(x + a) - f(a) \), for \( \tau \in S_n \), define

\[
W_{a, \tau} f(x) = W_{\tau} T_a f(x - a)
\] (26)

so that \( W_{a, \tau} f_i : [a, \infty) \to \mathbb{R} \) for all \( i \in [1, n] \).

### 2.3 Brownian motion and the Pitman transform

When \( B \in \mathcal{C}^2_0 \) is a Brownian motion, Pitman’s 2M − X theorem [29] identifies the law of \( WB \).

**Theorem 2.8** (Theorem 1.3, [29]). Let \( B = (B_1, B_2) \in \mathcal{C}^2_0 \) be two independent standard Brownian motions. Then \( (WB_1 + WB_2, WB_1 - WB_2) \overset{d}{=} (\sqrt{2}B, \sqrt{2}R) \), where \( R \) is a Bessel-3 process, \( B \) is a standard Brownian motion, and the two objects are independent.
Theorem 2.8 implies that for any interval \([a, b] \subset (0, \infty)\), we have \(W B|_{[a, b]} \ll B|_{[a, b]}\). We will need a version of this result for iterated Pitman transforms \(W_\sigma\). This will require the following strengthening of this absolute continuity observation.

**Proposition 2.9** Let \(B \in C^n_0\) be a sequence of \(n\) independent Brownian motions. For any \(\tau \in S_n\) and any \([a, b] \subset (0, \infty)\), we have \(W_\tau B|_{[a, b]} \ll B|_{[a, b]}\).

We use the following lemma to help with the inductive step.

**Lemma 2.10** (Lemma 4.5, [17]). Let \(X \in C^2_0\) be a random function such that for every \(0 < \epsilon < T\), we have \(X|_{[\epsilon, T]} \ll B|_{[\epsilon, T]}\) where \(B \in C_0^2\) is a standard Brownian motion. Suppose also that a.s.

\[
\max_{x \in [0, \epsilon]} X_2(x) - X_1(x) > 0 \tag{27}
\]

for all \(\epsilon > 0\). Then for every \(0 < \epsilon < T\), we have \(WX|_{[\epsilon, T]} \ll B|_{[\epsilon, T]}\).

Note that in [17], Ganguly and Hegde prove a version of Proposition 2.9 for particular permutations \(\sigma\). Just as in our case, their key input is Lemma 2.10.

**Proof** We prove the proposition by induction on \(n\). The \(n = 2\) base case follows from Theorem 2.8, as discussed above. Now suppose the lemma holds at \(n - 1\), and consider \(\tau \in S_n\). Let \(m = \tau(n)\). Using the notation \(\tau_{i,j}\) introduced prior to Proposition 2.6, we can write \(\tau = \tau_{n,m,\rho}\) for some \(\rho \in S_n\) with \(\rho(n) = n\). By the inductive hypothesis and the fact that \((W_\rho B)_n = B_n\), the proposition holds if \(m = n\). We now induct backwards on \(m \in [1, n]\). Let \(m \leq n - 1\) and suppose the proposition holds for \(m + 1\). By Lemma 2.10, the proposition also holds if the set of \(x\) where

\[
W_{\tau_{n,m+1,\rho}} B_{m+1}(x) - W_{\tau_{n,m+1,\rho}} B_m(x) > 0 \tag{28}
\]

has a limit point at 0 a.s. Now, \(W_{\tau_{n,m+1,\rho}} B_m = W_\rho B_m\), and

\[
W_\rho B_m(x) \leq W_\rho B[0, n - 1) \rightarrow (x, 1)] = B[(0, n - 1) \rightarrow (x, 1)],
\]

where the equality follows from Proposition 2.4. Moreover, repeated applications of Proposition 2.3(i) imply that \(W_{\tau_{n,m+1,\rho}} B_{m+1}(x) \geq W_\rho B_n(x) = B_n(x)\). Therefore to show (28), we just need to show that the set of \(x\) where

\[
B_n(x) - B[(0, n - 1) \rightarrow (x, 1)] > 0 \tag{29}
\]

has a limit point at 0 a.s. Let \(I_m\) denote the indicator of the event in (29) for \(x = 1/2^m\). By Blumenthal’s 0 – 1 law, the process \(\{I_m, m \in \mathbb{N}\}\) is stationary and ergodic. Therefore by the ergodic theorem, (29) occurs a.s. for infinitely many \(x = 1/2^m, m \in \mathbb{N}\) as long as \(\mathbb{E} I_m > 0\). The fact that \(\mathbb{E} I_m > 0\) follows since \(B_n(x), B[(0, n - 1) \rightarrow (x, 1)]\) are independent and \(B_n(x)\) has an unbounded upper tail.

In order to prove Theorem 1.7, we will also need to understand how two-line Brownian last passage percolation evolves from a vertical initial condition.
Proposition 2.11 Let $B' \in \mathcal{C}_0^2$ and suppose $B' \ll B$, where $B \in \mathcal{C}_0^2$ is a standard Brownian motion. Let $a_1 > a_2$, and define $M_i : [0, \infty) \to \mathbb{R}$ by $M_i(y) = a_i + B'[0, i) \to (y, 1)]$. Set $H = \max(M_1, M_2)$ and define

$$
\tau = \inf\{t \in (0, \infty) : B'_2(t) - B'_1(t) = a_1 - a_2\}.
$$

Then a.s.

$$
H(x) = 1(x < \tau)M_1(x) + 1(x \geq \tau)M_2(x).
$$

Next, let $X$ be any random variable on $\mathbb{R}$ with positive Lebesgue density everywhere, let $B^*$ be a two-sided standard Brownian motion, let $R$ be a two-sided Bessel-3 process, and suppose that all three objects are independent. Then

$$
(\tau, \sqrt{2}(H - H(\tau))) \ll (X, [B^* + R](-X + \cdot)|_{[0, \infty)}).
$$

When $B = B'$, a more precise description of the law of $(\tau, \sqrt{2}(H - H(\tau)))$ falls out of the proof. To prove Proposition 2.11 we need a lemma.

Lemma 2.12 Let $B : [0, t] \to \mathbb{R}$ be a standard Brownian motion, let $R$ be a Bessel-3 process on $[0, t]$ started at 0 and conditioned to end at a location $a$, and let $R'$ be an unconditioned Bessel-3 process on $[0, t]$. Suppose all objects are independent. Then $(B + R)|_{[0, t]} \ll (B + R')|_{[0, t]}$.

Proof First, the processes $(B + R')|_{[0, t/2]}$ and $(B + R)|_{[0, t/2]}$ are mutually absolutely continuous since $R$, $R'$ are mutually absolutely continuous when restricted to $[0, t/2]$. Let $\mathcal{F}$ be the $\sigma$-algebra generated by $(B + R)|_{[0, t]}$ and let $\mathcal{F}'$ be the $\sigma$-algebra generated by $(B + R')|_{[0, t]}$. Let $\mu$ denote the conditional distribution of $R_{[t/2,t]} - R(t/2)$ given $\mathcal{F}$ and let $\mu'$ denote the conditional distribution of $R'_{[t/2,t]} - R'(t/2)$ given $\mathcal{F}'$. Since the increment $B|_{[t/2,t]} - B(t/2)$ is independent of $\mathcal{F}$, $\mathcal{F}'$, $R$, $R'$, to prove the lemma it is enough to show that a.s. $\mu \ll \mu'$. The laws of $R(t/2)$, $R'(t/2)$ given $\mathcal{F}$, $\mathcal{F}'$ respectively are a.s. both mutually absolutely continuous with respect to Lebesgue measure on $[0, \infty)$. A Bessel process defined on $[t/2, t]$ started from a point $X \in (0, \infty)$ whose law is mutually absolutely continuous with respect to Lebesgue measure is mutually absolutely continuous with respect to the law $v$ of $B|_{[t/2,t]} - B(t/2)$, where $B$ is a standard Brownian motion. Therefore a.s. $v \ll \mu'$. On the other hand, a Bessel bridge on $[t/2, t]$ between points $X, Y \in (0, \infty)$ where $X - Y$ has a Lebesgue density is absolutely continuous with respect to Brownian motion, so a.s. $\mu \ll \mu'$. □

Proof of Proposition 2.11 Because the proposition claims only absolute continuity, since $B' \ll B$ it is enough to prove it when $(B'_1, B'_2)$ is replaced by $(B_1, B_2)$. Let $a = a_1 - a_2$, $C_1 = B_1 - B_2$, $C_2 = B_1 + B_2$. We have that

$$
\tau = \inf\{t \in (0, \infty) : C_1(t) = -a\}.
$$
Hence \( \tau \ll X \) since \( C_1 \) is a Brownian motion, and \( \tau \) is a stopping time with respect to \( C_1 \)'s filtration. In particular, given \( \tau \), the three processes

\[
\frac{C_2(\tau - x) - C_2(\tau)}{\sqrt{2}}, \quad x \in [0, \tau]; \quad B_1(\tau + x) - B_1(\tau), B_2(\tau + x) - B_2(\tau),
\]

\[x \in [0, \infty)\tag{31}\]

are all independent standard Brownian motions. Moreover, given \( \tau \) the process

\[
\frac{C_1(\tau - x) - C_1(\tau)}{\sqrt{2}}, \quad x \in [0, \tau]
\]

is a standard Brownian motion conditioned to stay positive and go from \((0, 0)\) to \((\tau, a)\), independent of all the processes in (31). In other words, it is a Bessel-3 process on \([0, \tau]\) conditioned to end at \(a\). Therefore by Lemma 2.12,

\[
(\tau, 2[M_1(\tau - x) - M_1(\tau)]) = (\tau, [C_1 + C_2](\tau - x) - [C_1 + C_2](\tau)) \ll (\tau, \sqrt{2}[B^* + R](-x)), \tag{32}
\]

where the absolute continuity is as functions on \([0, \tau]\). Here \(B^*\) is a standard two-sided Brownian motion and \(R\) is an unconditioned two-sided Bessel 3-process, independent of each other and \(\tau\). Now, for \(y \leq \tau\) we have the following calculation:

\[
M_2(y) = a_2 + B[(0, 2) \to (y, 1)]
\]

\[= a_2 + B_1(1) - B_2(0) + \sup_{x \in [0,y]} B_2(x) - B_1(x)\]

\[= a_2 + B_1(1) - B_2(0) + \sup_{x \in [\tau,y]} B_2(x) - B_1(x)\]

\[= a_2 + B[(\tau, 2) \to (y, 1)] + B_2(\tau) - B_2(0).\]

The third equality above follows since \(\tau\) is the first time when \(B_2 - B_1\) equals \(a\), and the remain equalities are by definition. Therefore

\[
M_2(y) - M_2(\tau) = B_2[(\tau, 2) \to (y, 1)]).
\]

The observation from (31) and Theorem 2.8 imply that given \(\tau\) and \(C_1, C_2\) on \([0, \tau]\), we have

\[
\sqrt{2}B_2[(\tau, 2) \to (\tau + \cdot, 1)] \overset{d}{=} B^* + R
\]

as functions on \([0, \infty)\). Combining this with (32) completes the proof. \(\square\)
3 Isometries in the limit

In this section, we build up to the proof of Theorem 1.2. Section 3.1 deals with the substantial necessary background on the directed landscape, the Airy sheet, and related objects and Sect. 3.2 proves the theorem.

3.1 Preliminaries on universal KPZ limits

In this section, we gather the necessary background about the Airy line ensemble, the Airy sheet, the directed landscape, and extended versions of these objects.

First, the parabolic Airy line ensemble $\mathcal{W} = (\mathcal{W}_i, i \in \mathbb{N})$ is a random sequence of continuous functions $\mathcal{W}_i : \mathbb{R} \to \mathbb{R}$. Its lines satisfy $\mathcal{W}_1 > \mathcal{W}_2 > \ldots$ a.s. It is defined through a determinantal formula, see [9] for details. The main fact about the Airy line ensemble that we need is the following. For this next proposition, we let $\mathcal{F}_{a,b,k}$ denote the $\sigma$-algebra generated by

$$\mathcal{W}_i(t), \quad (i, t) \notin [1, k] \times (a, b).$$

**Proposition 3.1** Fix $k \in \mathbb{N}, a < b$ and let $B$ be a $k$-dimensional Brownian motion of variance 2 started from $B(a) = 0$. Then conditionally on $\mathcal{F}_{a,b,k}$, for any $c \in (a, b)$ we have

$$(\mathcal{W}_1|_{[a,c]} - \mathcal{W}_1(a), \ldots, \mathcal{W}_k|_{[a,c]} - \mathcal{W}_k(a)) \ll (B_1, \ldots, B_k)|_{[a,c]}.$$ (33)

Other absolute continuity statements for $\mathcal{W}$ have appeared earlier (e.g. [9, Proposition 4.1]), but these are not phrased to give the conditional absolute continuity we need, so we include a proof.

**Proof** The statement is almost immediate from the Brownian Gibbs property for $\mathcal{W}$, shown in [9, Theorem 3.1]. The Brownian Gibbs property says that for any $a < b$ and $k \in \mathbb{N}$, conditional on $\mathcal{F}_{a,b,k}$, the Airy lines $\mathcal{W}_1 > \cdots > \mathcal{W}_k$ restricted to the interval $[a, b]$ are given by $k$ independent Brownian bridges $B'_1, \ldots, B'_k$ of variance 2 between $B'_i(a) = \mathcal{W}_i(a)$ and $B'_i(b) = \mathcal{W}_i(b)$ for $i \in [1, k]$, conditioned so that $B'_1(x) > \cdots > B'_k(x) > \mathcal{W}_{k+1}(x)$ for all $x \in (a, b)$.

Therefore the distribution $(\mathcal{W}_1|_{[a,b]} - \mathcal{W}_1(a), \ldots, \mathcal{W}_k|_{[a,b]} - \mathcal{W}_k(a))$ is absolutely continuous with respect to that of $k$ independent Brownian bridges started at 0 and ending at certain $\mathcal{F}_{k,a,b}$-measurable endpoints. Since a Brownian bridge on $[a, b]$ is absolutely continuous with respect to Brownian motion on any interval $[a, c]$ with $c < b$, the result follows. \(\square\)

**Definition 3.2** Let $\mathcal{W}$ be a parabolic Airy line ensemble, and let $\tilde{\mathcal{W}}(x) = \mathcal{W}(-x)$. For $(x, y, z) \in \mathbb{Q}^+ \times \mathbb{Q}^2$ let

$$S'(x, y, z) = \lim_{k \to \infty} \mathcal{W}(-\sqrt{k/(2x)}, k) \to (y, 1) - \mathcal{W}(-\sqrt{k/(2x)}, k) \to (z, 1)],$$

$$S'(-x, y, z) = \lim_{k \to \infty} \tilde{\mathcal{W}}(-\sqrt{k/(2x)}, k) \to (-y, 1) - \tilde{\mathcal{W}}(-\sqrt{k/(2x)}, k) \to (-z, 1)].$$ (34)
Fig. 4 An illustration of $S(x, y, z)$ in the definition of the Airy sheet. This quantity is a limit of differences of last passage values from $(-\sqrt{k/(2x)}, k)$ in $W$. As we take $k$ to infinity, the portions of the last passage paths illustrated in red and blue above stabilize, and only the tail (illustrated in purple) changes for large $k$. This implies that the limit in (34) exists and allows us to define $S(x, y) - S(x, z) := S(x, y, z)$. Equation (35) extracts single Airy sheet values from these difference by ergodicity. Note that if we increase $x$, the tail will move from left to right.

For $(x, y) \in \mathbb{Q} \setminus \{0\} \times \mathbb{Q}$, define

$$S(x, y) = \mathbb{E}A_1(0) + \lim_{n \to \infty} \frac{1}{2n} \sum_{z = -n}^{n} S'(x, y, z) - (z - x)^2,$$

The Airy sheet defined from $W$ is the unique continuous extension of $S$ to $\mathbb{R}^2$. See Fig. 4 for an illustration of Definition 3.2. It is not at all clear that the limits in Definition 3.2 exist, or that the resulting function has a continuous extension to $\mathbb{R}^2$. However, a.s. these properties do hold and so the construction above is well-defined. This is shown in [13, Section 1.11]. While Definition 3.2 is somewhat involved, one should think of it as capturing the idea that

$$\text{“}S(x, y) = W[(x, \infty) \to (y, 1)]\text{“},$$

which can be thought of as the single-point limit of Proposition 2.4. The more complex Definition 3.2 is required to rigorously make sense of the right hand side of (36). The exact expression $(-\sqrt{k/(2x)}, k)$ for the asymptotic direction representing $(x, \infty)$ is not particularly intuitive and follows from calculations. See [11, 13] for more background.

The extended landscape is built from the extended Airy sheet. To describe this precisely, we need a notion of multi-point last passage percolation across $W$. For $x \in [0, \infty)$, $y \in \mathbb{R}$, and $n \in \mathbb{N}$, a nonincreasing cadlag function $\pi : (-\infty, y] \to \{n, n + 1, \ldots\}$ is a parabolic path from $x$ to $(y, n)$ if

$$\lim_{z \to -\infty} \frac{\pi(z)}{2z^2} = x.$$
This definition guarantees that for $x > 0$, if $z_k$ is the largest time when $\pi(z) \leq k$, then $z_k = \sqrt{k/(2x + o(1))}$, matching the asymptotic direction from (34). Note that in [14], this definition was only used with $n = 1$. For us, it will be convenient to also work with other parabolic paths. All lemmas from [14] for parabolic paths to line 1 go through for parabolic paths to a general line $n$, since our definition ensures that any parabolic path to line $n$ is also a parabolic path to line 1.

For a parabolic Airy line ensemble $\mathcal{W}$ with corresponding Airy sheet $S$ as in Definition 3.2, and any parabolic path starting at $x \geq 0$, define its path length by

$$\|\pi\|_{\mathcal{W}} = S(x, y) + \lim_{z \to -\infty} (\|\pi|_{[z, y]}\|_{\mathcal{W}} - \mathcal{W}([z, \pi(z)]) \to (y, 1)).$$

(38)

Note that we use $(y, 1)$ on the right hand side even if the parabolic path ends at a line $n \neq 1$. The limit above always exists and is nonpositive, see [14, Lemma 5.1]. The idea of definition (38) is that the best possible path length for $\pi$ should be the last passage value in (36). This accounts for the first term in (38). For fixed $z$, the second term in (38) measures the discrepancy between the length of $\pi$ and the optimal length on the compact interval $[z, y]$; taking $z \to -\infty$ then gives the discrepancy between $\|\pi\|_{\mathcal{W}}$ and $S(x, y)$.

For $x \in \mathbb{R}_k$ with $x_1 \geq 0$ and a $k$-tuple $p = (p_1, \ldots, p_k)$ with $p_i = (y_i, n_i) \in \mathbb{R} \times \mathbb{N}$ and $y_1 \leq \cdots \leq y_k, n_1 \leq \cdots \leq n_k$, we can then define the multi-point last passage value

$$\mathcal{W}[x \to p] = \sup_{\pi} \sum_{i=1}^{k} \|\pi_i\|_{\mathcal{W}},$$

(39)

where the supremum is over disjoint $k$-tuples of parabolic paths $\pi = (\pi_1, \ldots, \pi_k)$ from $x_i$ to $p_i$ satisfying $\pi_i(t) > \pi_j(t)$ for all $i < j, t < y_i$. A $k$-tuple achieving (39) is a (disjoint) optimizer from $x$ to $p$, or a geodesic from $x$ to $(y, n)$ when $k = 1$. In [14, Theorem 1.3], equation (39) was used along with a translation invariance property to characterize the extended Airy sheet. Thanks to the recently established landscape symmetry

$$\mathcal{L}(x, s; y, t) \overset{d}{=} \mathcal{L}(-x, s; -y, t)$$

(40)

shown in [13, Proposition 1.23] this characterization can be strengthened. The equality (40) is as random continuous functions on $\mathbb{R}_4^4$. Theorem 3.3 Let $\mathcal{L}$ be a directed landscape. Similarly to (9), define

$$\sum_{i=1}^{k} W_x \mathcal{L}_i(y) = \mathcal{L}(x^k, 0; (y - x)^k, 1).$$

(41)

Then for all $x \in \mathbb{R}$, the process $W_x \mathcal{L}$ is a parabolic Airy line ensemble and all the parabolic Airy line ensembles $W_x \mathcal{L}$ can be expressed in terms of the $x = 0$ ensemble
Indeed, a.s. we have that
\[\sum_{i=1}^k W_x L_i(y) = W_0 L[x^k \to ((y - x)^k, 1)], \quad \text{for all } x \geq 0, \quad \text{and}\]
\[\sum_{i=1}^k W_x L_i(y) = \tilde{W}_0 L[(-x)^k \to ((x - y)^k, 1)], \quad \text{for all } x \leq 0.\]

Here similarly to Definition 3.2, \(\tilde{W}_0 L(y) := W_0 L(-y).\) Moreover, define the extended Airy sheet \(S(x, y) = L(x, 0; y, 1).\) This is a real-valued continuous function with domain \(\mathcal{X} := \bigcup_{k \geq 1} \mathbb{R}^k_\leq \times \mathbb{R}^k_\leq.\) A.s. \(S\) satisfies
\[S(x, y) = W_z L[x - z^k \to (y - z^k, 1)], \quad x_1 \geq z,\]
\[S(x, y) = \tilde{W}_z L[z^k - x \to (z^k - y, 1)], \quad x_k \leq z\]
for all \((x, y) \in \mathcal{X}, z \in \mathbb{Q}\) satisfying the above constraints. In particular, combining (42) and (43) implies that \(S\) is a function of \(W_0 L\) and that \(S|_{\mathbb{R}^2}\) and \(W_0 L\) satisfy the same relationship as \(S, W\) in Definition 3.2.

The equations in (43) should hold not just for \(z \in \mathbb{Q}\), but simultaneously for all \(z\). We do not pursue this technical extension here as it is only tangential to the paper. Note that to make sense of the points \(-x, -y \in \mathbb{R}^k_\leq\) in (43) we need to reverse the order of the coordinates in \(x, y\). From now on we ignore this minor point. While we encourage the reader to simply think of Theorem 3.3 as a black box result from [14] strengthened with the input (40), we include a bookkeeping proof below in case the reader decides to delve into the literature and wants to locate precise statements.

**Proof** The first equality in (43) for \(x = 0\) and \(z = 0\) is one of the main results of [14]. More precisely, it follows immediately by combining Equation (7), Theorem 1.3, and Theorem 1.6 from that paper. This combination also immediately implies (42) for \(x \geq 0\). Now, we have the symmetry
\[L(x, s; y, t) \stackrel{d}{=} L(x + r, s; y + r, t),\]
where the equality is as random functions in \(\mathbb{R}^4_\uparrow\), see [11, Lemma 10.2.2]. This implies that the first equality in (43) also holds a.s. for any \(z \in \mathbb{Q}\). The symmetry (40) then implies all the ‘tilde’ equalities in both (42) and (43). \(\square\)

To understand the extended Airy sheet, we study geodesics, path length and disjoint optimizers in \(W\). We record the following basic properties of these objects, from [14].

**Proposition 3.4** Let \(W\) be an Airy line ensemble. We have the following properties.

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(i) (Lemma 4.2(i), [14]) For any fixed \((x, y) \in [0, \infty) \times \mathbb{R}\), there exists a unique geodesic in \(\mathcal{W}\) from \(x\) to \((y, 1)\) a.s. We call this geodesic \(\pi\{x, y\}\).

(ii) (Lemma 4.2(iv), [14]) For any fixed \(0 \leq x < x'\) and \(y \in \mathbb{R}\), we have

\[
\lim_{r \to \infty} \mathbb{P}(\pi\{x, y\}(z) < \pi\{x', y + r\}(z) \text{ for all } z < y) = 1.
\]

(iii) (Lemma 4.3, [14]) The following statement holds a.s. Let \(\pi_1, \pi_2\) be any two parabolic paths across \(\mathcal{W}\) from any point \(x\) to any points \((z_1, n_1), (z_2, n_2)\) respectively, such that for some \(z_0 < z_1 \wedge z_2\), we have \(\pi_1(y) = \pi_2(y)\) for any \(y \leq z_0\). Then

\[
\|\pi_1\|_{\mathcal{W}} - \|\pi_1|_{[z_0, z_1]}\|_{\mathcal{W}} = \|\pi_2\|_{\mathcal{W}} - \|\pi_2|_{[z_0, z_2]}\|_{\mathcal{W}}.
\]

Here note that \(\|\pi\|_{[a,b]}\|_{\mathcal{W}}\) is a usual path length in the environment \(\mathcal{W}\) defined with a sum of increments as in (1), not an infinite path length.

(iv) (Proposition 4.5(i), [14]) Let \(x \in \mathbb{R}_+^k\) with \(x_1 \geq 0\), and let \(y \in \mathbb{R}_+^k\). Suppose that \(\pi\) is a disjoint optimizer from \(x\) to \((y, 1)\). Then \(\pi\) is locally optimal.

(v) (Proposition 4.5(ii), [14]) For parts (v), (vi), let \(x \in \mathbb{R}_+^k\) with \(0 < x_1 < \cdots < x_k\), and let \(y \in \mathbb{R}_+^k\). A.s. there is a unique optimizer \(\pi = (\pi_1, \ldots, \pi_k)\) from \(x\) to \((y, 1)\) in \(\mathcal{W}\). Moreover, letting \(\pi\{x_i, 0\}\) be the a.s. unique geodesic in \(\mathcal{W}\) from \(x_i\) to \((0, 1)\), there exists a random \(Y \in \mathbb{R}\) such that \(\pi\{x_i, 0\}(t) = \pi_i(t)\) for all \(t \leq Y\), \(i \in [1, k]\).

(vi) (Proposition 4.5(iii), [14]) A.s. the only \(k\)-tuple \(\pi\) from \(x\) to \((y, 1)\) in \(\mathcal{W}\) which is locally optimal is the unique optimizer from \(x\) to \((y, 1)\).

(vii) (Proposition 5.8, [14]) A.s. for every \(x, y \in \mathbb{R}_+^k\) with \(x_1 \geq 0\) there is at least one optimizer in \(\mathcal{W}\) from \(x\) to \((y, 1)\).

Note that in [14], part (iv) above is stated only for \(0 < x_1 < \cdots < x_k\) as it is part of a broader proposition. However, the proof in that paper only uses that \(x_1 \geq 0\).

We also require a crude bound on the extended Airy sheet, and a simple fact about equality of Airy sheet values.

**Lemma 3.5** (Part of Lemma 7.7, [14]). For every \(\eta > 0\), there exists an \(R > 0\) such that

\[
\left| S(x, y) + \|x - y\|_2^2 \right| \leq R(1 + \|x\|_1 + \|y\|_1)^\eta
\]

for all \((x, y) \in \mathfrak{X}\).

**Lemma 3.6** (Lemma 7.10, [14]). Let \(S\) be the extended Airy sheet. Then a.s. for any \(x^{(1)} \leq x^{(2)} \leq x^{(3)} \leq x^{(4)} \in \mathbb{R}_+^k\), and \(y^{(1)} \leq y^{(2)} \leq y^{(3)} \leq y^{(4)} \in \mathbb{R}_+^k\), if

\[
S((x^{(2)}, x^{(3)}), (y^{(2)}, y^{(3)})) = S(x^{(2)}, y^{(2)}) + S(x^{(3)}, y^{(3)}),
\]

then

\[
S((x^{(1)}, x^{(4)}), (y^{(1)}, y^{(4)})) = S(x^{(1)}, y^{(1)}) + S(x^{(4)}, y^{(4)}).
\]
Finally, we use the following relationship between extended landscape values and geodesic disjointness.

**Proposition 3.7** (Corollary 1.11, [14]). A.s. the following holds. For every \((x, s; y, t) \in \mathcal{X}_t\), we have

\[
\mathcal{L}(x, s; y, t) = \sum_{i=1}^k \mathcal{L}(x_i, s; y_i, t)
\]

if and only if there exist \(\mathcal{L}\)-geodesics \(\pi_1, \ldots, \pi_k\) where \(\pi_i\) goes from \((x_i, s)\) to \((y_i, t)\), satisfying \(\pi_i(r) < \pi_{i+1}(r)\) for all \(i \in [1, k-1]\) and \(r \in (s, t)\).

### 3.2 Proof of Theorem 1.2

Before proceeding with the proof, we digress slightly to give a path-based definition of (12) that may help the reader with intuition.

**Remark 3.8** Let \(f = (f_1, \ldots, f_k), f_i : \mathbb{R} \to \mathbb{R}\). Let \([m, n] \subset [1, k]\) and consider a nonincreasing cadlag function \(\pi : (-\infty, y] \to [m, n]\), and assume that \(\pi\) stabilizes at line \(n\) as \(z \to -\infty\). We call \(\pi\) a path from \((-\infty, n)\) to \((y, m)\). Matching with (1), define the path length

\[
\|\pi\|_f = \sum_{i=m}^n [f_i(\pi_i) - f_i(\pi_{i+1})].
\]

In the above sum, for \(i \leq n\) we let \(\pi_i = \inf\{t \in (-\infty, y] : \pi(t) < i\}\) be the time when \(\pi_i\) jumps off of line \(i\). Also, for all \(i\) set \(\pi_i = y\) if \(\pi(t) \geq i\). In the above sum, the term \(f_n(\pi_{n+1})\) needs to be defined separately; we simply set it to 0. Then for \(I = (I_1, \ldots, I_\ell)\) and \(p = ((x_1, m_1), \ldots, (x_\ell, m_\ell))\) we can write

\[
f[(-\infty, I) \to p] = \sup_{\pi} \sum_{i=1}^k \|\pi_i\|_f,
\]

where the supremum is over all \(k\)-tuples \(\pi = (\pi_1, \ldots, \pi_k)\), where \(\pi_j\) is a path from \((-\infty, I_j)\) to \(p_i = (y_i, m_i)\), and for all \(i\) we have \(\pi_i < \pi_{i+1}\) on \((-\infty, x_i)\). The equality between the right sides of (45) and (12) follows from comparing definitions.

**Proof of Theorem 1.2, part 1.** The asymptotics in (14) are immediate from Lemma 3.5. The stability in (15) then follows from (14) and the fact that \(x_1 < x_2 < \cdots < x_k\). Figure 5 in the appendix may be helpful for visualizing the implication (14) \(\implies\) (15).

To prove Theorem 1.2.2, we need a few lemmas. The first lemma contains versions of basic facts about last passage percolation in the context of parabolic paths.
Lemma 3.9  A.s. the following statements hold. For all \( x, y \in \mathbb{R}^k \leq \) with \( x_1 \geq 0 \) and every \( z \leq y_1 \) we have

\[
\mathcal{W}[x \to (y, 1)] = \sup_{I \in \mathbb{N}^k_{\leq}} \mathcal{W}[x \to (z, I)] + \mathcal{W}[(z, I) \to (y, 1)].
\] (46)

Moreover, this supremum is always achieved by some \( I \), and letting \( \sigma \) be any optimizer from \( x \) to \( (y, 1) \) we have

\[
\mathcal{W}[x \to (y, 1)] = \mathcal{W}[x \to (z, \sigma(z))] + \mathcal{W}[(z, \sigma(z)) \to (y, 1)],
\] (47)

and \( \sigma|_{(-\infty, z]} \) is an optimizer from \( x \) to \( (z, \sigma(z)) \).

In the statement above, the first term under the supremum is a last passage value involving parabolic paths whereas the second term is simply a regular last passage value.

**Proof**  We work on the almost sure event specified by Proposition 3.4(iii) and (vii). Fix \( I \in \mathbb{N}^k_{\leq} \) and let \( \epsilon > 0 \). First, note that from the definition

\[
\mathcal{W}[x \to (z, I)] \leq \mathcal{W}[x \to (z^k, 1)] \leq \sum_{i=1}^{k} S(x_i, z) < \infty.
\]

Therefore \( \mathcal{W}[x \to (z, I)] \) is necessarily finite and so we can find a disjoint \( k \)-tuple \( \pi \) of parabolic paths from \( x \) to \( (z, I) \) with

\[
\mathcal{W}[x \to (z, I)] \leq \epsilon + \sum_{i=1}^{k} \|\pi_i\|_{\mathcal{W}}.
\]

Also, let \( \tau \) be a disjoint optimizer from \( (z, I) \) to \( (y, 1) \). Let \( \sigma \) be the concatenation of \( \pi \) and \( \tau \), i.e. \( \sigma_i \) is a parabolic path from \( x_i \) to \( (y_i, 1) \) and \( \sigma_i = \pi \) on \( (-\infty, z] \) and \( \tau \) on \( (z, y_i] \). Then \( \sigma \) is a disjoint \( k \)-tuple from \( x \) to \( (y, 1) \) and by Proposition 3.4(iii),

\[
\sum_{i=1}^{k} \|\sigma_i\|_{\mathcal{W}} = \sum_{i=1}^{k} \|\pi_i\|_{\mathcal{W}} + \sum_{i=1}^{k} \|\tau_i\|_{\mathcal{W}}.
\] (48)

The left-hand side of (48) is bounded above by the left hand side of (46) and the right-hand side is bounded below by the right hand side of (46) minus \( \epsilon \). This holds for all \( I, \epsilon \) so \((\text{LHS}) \geq (\text{RHS})\) in (46).

For the other direction, by Proposition 3.4(vii) there exists a disjoint optimizer \( \sigma \) from \( x \) to \( (y, 1) \). Splitting this optimizer at time \( z \) yields disjoint \( k \)-tuples \( \pi = \sigma|_{(-\infty, z]} \) from \( x \) to \( (z, \sigma(z)) \) and \( \tau = \sigma|_{(z, y]} \) from \( (z, \sigma(z)) \) to \( (y, 1) \). Again, Proposition 3.4(iii)
ensures that (48) holds, giving that
\[
\mathcal{W}[\mathbf{x} \to (\mathbf{y}, 1)] = \sum_{i=1}^{k} \|\pi_i\|_{\mathcal{W}} + \sum_{i=1}^{k} \|\tau_i\|_{\mathcal{W}}
\leq \mathcal{W}[\mathbf{x} \to (\mathbf{z}, \sigma(\mathbf{z}))] + \mathcal{W}[\mathbf{(z}, \sigma(\mathbf{z})) \to (\mathbf{y}, 1)].
\]

This completes the proof of (46) and also implies all claims in the ‘Moreover’ statement.

For the next lemma and throughout the section, we let \(\mathcal{S}, \mathcal{L}, \mathcal{W} := W_0 \mathcal{L}\) be coupled as in Theorem 3.3.

**Lemma 3.10** Let \(\mathbf{x} \in \mathbb{R}^k_\leq\) satisfy \(0 < x_1 < \cdots < x_k\), let \([a, b] \subset \mathbb{R}\), and let \(\pi = (\pi_1, \ldots, \pi_k)\), where each \(\pi_i\) is the (a.s. unique) \(\mathcal{W}\)-geodesic from \(x_i\) to 0. Then there exists a random integer \(Y < a\) such that for all \(\mathbf{z} \leq Y, \mathbf{I} \subset [1, k]\) and \(\mathbf{y} \in [a, b]^{\mathbf{I}_\leq}\), we have
\[
\mathcal{S}(\mathbf{x}^I, \mathbf{y}) = \mathcal{W}[(\mathbf{z}, [\pi(z)]^I) \to (\mathbf{y}, 1)] + \sum_{i \in \mathcal{I}} \mathcal{W}[\mathbf{x}_i \to (\mathbf{z}, \pi_i(z))] \tag{49}
\]

Here recall that \(\mathbf{x}^I\) denotes the vector in \(\mathbb{R}^{|\mathcal{I}|}\) whose coordinates are \(x_i, i \in \mathcal{I}\).

**Proof** First, since there are only finitely many choices for \(\mathcal{I}\), it suffices to prove that for each \(\mathcal{I}\) there exists \(Y = Y(\mathcal{I})\) such that (49) holds whenever \(\mathbf{z} \leq Y, \mathbf{y} \in [a, b]^{\mathbf{I}_\leq}\).

Without loss of generality, we may then assume \(\mathcal{I} = [1, k]\).

By Proposition 3.4(v), the a.s. unique optimizers \(\tau, \tau'\) from \(\mathbf{x}\) to \(a^k\) and \(\mathbf{x}\) to \(b^k\) satisfy \(\tau(z) = \tau'(z) = \pi(z)\) for all \(\mathbf{z} \leq Y\) for some random integer \(Y < a\). Next, for any \(\mathbf{y} \in [a, b]_{\leq}^k\) and \(\mathbf{z} \leq Y\), let \(\rho_{\mathbf{z}, \mathbf{y}}\) be the rightmost optimizer from \((\mathbf{z}, \pi(z))\) to \((\mathbf{y}, 1)\) in \(\mathcal{W}\). By the monotonicity in Proposition 2.2(ii), we have \(\tau \leq \rho_{\mathbf{z}, \mathbf{y}} \leq \tau'\) and so
\[
\tau|_{\mathbf{z}, \mathbf{y}} = \rho_{\mathbf{z}, \mathbf{y}}|_{\mathbf{z}, \mathbf{y}} = \tau'|_{\mathbf{z}, \mathbf{y}}. \tag{50}
\]

Therefore \(\rho_{\mathbf{z}, \mathbf{y}}|_{\mathbf{z}, \mathbf{z}', \infty} = \rho_{\mathbf{z}', \mathbf{y}}|_{\mathbf{z}, \mathbf{z}', \infty}\) for all \(\mathbf{z}, \mathbf{z}' \leq Y\). Hence, there is a \(k\)-tuple \(\rho\) from \(\mathbf{x}\) to \(\mathbf{y}\) such that \(\rho(t) = \rho_{\mathbf{z}, \mathbf{y}}(t)\) for all \(\mathbf{z} \leq Y, t \geq z\). By construction \(\rho\) is a disjoint \(k\)-tuple from \(\mathbf{x}\) to \(\mathbf{y}\) which is locally optimal since each \(\rho_{\mathbf{z}, \mathbf{y}}\) is an optimizer. Therefore by Proposition 3.4(vi), \(\rho\) is a.s. an optimizer from \(\mathbf{x} \to \mathbf{y}\), and for \(\mathbf{z} \leq Y\) we have
\[
\mathcal{S}(\mathbf{x}, \mathbf{y}) = \mathcal{W}[\mathbf{x} \to (\mathbf{y}, 1)] = \sum_{i=1}^{k} \|\rho_i|_{(-\infty, \mathbf{z})}\|_{\mathcal{W}} + \mathcal{W}[(\mathbf{z}, \rho(\mathbf{z})) \to (\mathbf{y}, 1)]
\leq \sum_{i=1}^{k} \mathcal{W}[\mathbf{x}_i \to (\mathbf{z}, \pi_i(z))] + \mathcal{W}[(\mathbf{z}, \pi(\mathbf{z})) \to (\mathbf{y}, 1)].
\]

Here the first equality uses the ‘Moreover’ claim in Lemma 3.9 applied to the optimizer \(\rho\). The second equality uses that \(\pi = \rho\) on \((-\infty, Y)\), and that each \(\pi_i|_{(-\infty, \mathbf{z})}\) is a
geodesic from \(x_i\) to \((z, \pi(z))\); this follows from the ‘Moreover’ claim in Lemma 3.9 applied to the geodesics \(\pi_i\).

The next lemma gives a simple measurability property for path length; it is a variant of the stronger [14, Lemma 4.3].

**Lemma 3.11** Fix \(a \in \mathbb{R}\), let \(\Pi\) be the set of parabolic paths from \(x\) to \(y\) for some \(y \leq a, x \geq 0\) (recall the definition of a parabolic path from (37)), and let \(\mathcal{W}\) be the parabolic Airy line ensemble. Let \(\mathcal{F}_a\) be the \(\sigma\)-algebra generated by all null sets and by \(\mathcal{W}|_{(-\infty, a]}\), and let \(F : \Pi \rightarrow \mathbb{R} \cup \{-\infty\}\) be the function recording path length in \(\mathcal{W}\). Then \(F\) is \(\mathcal{F}_a\)-measurable.

In particular, for any \(x \in \mathbb{R}_k\) with \(x_1 \geq 0\) and any \(p = (p_1, \ldots, p_k)\) with each \(p_i \in (-\infty, a] \times \mathbb{N}\), the last passage value \(\mathcal{W}|_{x \rightarrow p}\) is \(\mathcal{F}_a\)-measurable.

**Proof** By looking at the definition of parabolic path length in (38), we see that it is enough to prove that \(S(\cdot, a)|_{[0, \infty)}\) is \(\mathcal{F}_a\)-measurable. By continuity of \(S\) it suffices to prove that \(S(x, a)\) is \(\mathcal{F}_a\)-measurable for all \(x \in \mathbb{Q}^+\). Going back to Definition 3.2, we have that \(S'(x, n, a)\) is \(\mathcal{F}_a\)-measurable for all \(n \leq a\). Therefore it is enough to show that a.s.

\[
S(x, a) = \mathbb{E}\mathcal{W}_1(0) + \lim_{n \rightarrow \infty} \frac{1}{a + n} \sum_{m \in [-n, a] \cap \mathbb{Z}} S'(x, a, m) - (m - a)^2.
\]

By Definition 3.2, \(S'(x, a, m) = S(x, a) - S(x, m)\) so the above is equivalent to the equation

\[
\mathbb{E}\mathcal{W}_1(0) = \lim_{n \rightarrow \infty} \frac{1}{a + n} \sum_{m \in [-n, a] \cap \mathbb{Z}} S(x, m) + (m - x)^2,
\]

which follows from the fact that \(f(m) = S(x, m) + (m - x)^2\) is a stationary Airy process with pointwise mean \(\mathbb{E}\mathcal{W}_1(0)\), and this process is ergodic, see equation (5.15) in [30].

Finally, we will use a fact about locally Brownian functions. We omit the straightforward proof.

**Lemma 3.12** Fix \(n \in \mathbb{N}\) and \(z < -n\). Let \(X\) be a random vector in \(\mathbb{R}_k\), and let \(W : [z, n] \rightarrow \mathbb{R}_k\) be any random continuous function such that conditional on \(X\) we have

\[
W|_{[-n,n]} \ll B|_{[-n,n]},
\]

where \(B\) is a standard \(k\)-dimensional Brownian motion started from \(B(z) = 0\). Then conditional on both \(X\) and \(W(0)\) we have

\[
W|_{[-n,n]} - W(0) \ll B'|_{[-n,n]},
\]

where \(B'\) is a standard \(k\)-dimensional Brownian motion started from \(B'(0) = 0\).
Proof of Theorem 1.2, part 2  First, by translation invariance of $L$ (Eq. 44), we may assume $x_1 > 0$. Now, as in Lemma 3.10, let $\pi = (\pi_1, \ldots, \pi_k)$, where $\pi_i$ is the unique $\mathcal{W}$-geodesic from $x_i$ to 0. Fix $n \in \mathbb{N}$, and let $Y_n$ be the random integer specified by Lemma 3.10 for the interval $[-n, n]$. For all $\ell \in [1, k]$ and $y \in [-n, n]$ we have

$$
\sum_{i=1}^\ell W_x L_i(y) = S(x^{[1,\ell]}, y^\ell) = \mathcal{W}[(Y_n, [\pi(Y_n)]^{[1,\ell]}) \rightarrow (y^\ell, 1)]
+ \sum_{i=1}^\ell \mathcal{W}[x_i \rightarrow (Y_n, \pi_i(Y_n))].
$$

(51)

Now, using the notation $\tau_I$, $W_{z, \tau_I}$ from Proposition 2.6 and Remark 2.7, for $z \in \mathbb{Z}$ and $I = \{I_1 < \cdots < I_k\} \subset \mathbb{N}$, we set

$$
\mathcal{W}^{l,z} := W_{z, \tau_I}(W_1, W_2, \ldots, W_k).
$$

We also let $X^{l,z} \in \mathbb{R}^k$ denote the vector with $X_i^{l,z} = \mathcal{W}[x_i \rightarrow (z, I_i)]$. By Proposition 2.6 and (51) we then have

$$
W_x L_i(y) = \mathcal{W}_i^{\pi(Y_n), Y_n}(y) + X_i^{\pi(Y_n), Y_n} \quad \text{for all } i \in [1, k], y \in [-n, n].
$$

(52)

There are only countably many choices for $Y_n$, $\pi(Y_n)$. Therefore to complete the proof it suffices to show that for every fixed $n \in \mathbb{N}$, $I = \{I_1 < \cdots < I_k\} \subset \mathbb{N}$ and $z \in \mathbb{Z}$ with $z < -n$, conditional on $\mathcal{W}^{l,z}(0) + X^{l,z}$, the function

$$
[\mathcal{W}^{l,z} + X^{l,z}]|_{[-n,n]} - [\mathcal{W}^{l,z} + X^{l,z}](0) = \mathcal{W}^{l,z}|_{[-n,n]} - \mathcal{W}^{l,z}(0)
$$

is a.s. absolutely continuous with respect to $B'|_{[-n,n]}$ where $B'$ is an independent $k$-dimensional Brownian motion of variance 2 with $B'(0) = 0$. Our aim will be to appeal to Lemma 3.12.

By Lemma 3.11, $X^{l,z}$ is $\mathcal{F}_z$-measurable. Therefore by Proposition 3.1, conditional on $X^{l,z}$, on the interval $[z, n]$ the sequence

$$(\mathcal{W}_1 - \mathcal{W}_1(z), \ldots, \mathcal{W}_k - \mathcal{W}_k(z))$$

is absolutely continuous with respect to an $I_k$-dimensional Brownian motion $B$ of variance 2 with $B(z) = 0$. Hence by Proposition 2.9, conditional on $X^{l,z}$ we have that $\mathcal{W}^{l,z}|_{[-n,n]} \ll B|_{[-n,n]}$, where $B$ is an $I_k$-dimensional Brownian motion of variance 2 started from $B(z) = 0$. Lemma 3.12 then yields the result. \[\square\]

We split the proof of Theorem 1.2.3 up using two lemmas.
Lemma 3.13 Fix $x \in \mathbb{R}^k$ with $x_1 < \cdots < x_k$. A.s. there exists a random vector $y \in [0, \infty)^k$ such that for every $J \subset [1, k]$, we have

$$S(x^J, y^J) = \sum_{i \in J} S(x_i, y_i), \quad \text{and} \quad W_{x\mathcal{L}}[x^J \to y^J] = \sum_{i \in J} W_{x\mathcal{L}}[x_i \to y_i].$$

(53)

Also, for any $y \in \mathbb{R}$ a.s. there exists $R > 0$ such that for $r \geq R, \ell \in [1, k - 1]$ we have

$$S(x, ((y - r)^\ell, y^{k-\ell})) = S(x^{[1, \ell]}, (y - r)^\ell) + S(x^{[\ell+1, k]}, y^{k-\ell}).$$

(54)

Proof Let $Z_0(0)$ be as in Theorem 1.2.1, and let $\tau_1$ be the rightmost geodesic in $W_{x\mathcal{L}}$ from $(Z_0(0), i)$ to $(y_1, 1)$. Also, let $\pi_1$ be the a.s. unique geodesic in $\mathcal{W}$ from $x_i$ to $y$. To prove (53), it is enough to show that we can find random points $0 = y_1 < \cdots < y_k \in \mathbb{Z}$ such that $\tau_i(y_i)(z) < \tau_{i+1}(y_{i+1})(z)$ and $\pi_i(y_i)(z) < \pi_{i+1}(y_{i+1})(z)$ for all $z < y_i, i \in [1, k - 1]$. We do this by induction on $\ell \in [1, k]$. The base case $\ell = 1$ is trivially true. Suppose now that we have constructed $y_1, \ldots, y_\ell$. Then we let $y_{\ell+1}$ be the first integer such that $\tau_\ell(y_\ell)(z) < \tau_{\ell+1}(y_{\ell+1})(z)$ and $\pi_\ell(y_\ell)(z) < \pi_{\ell+1}(y_{\ell+1})(z)$ for all $z < y_\ell$. We need to justify that such a $y_{\ell+1}$ exists. This uses the asymptotics in (14) to establish the inequality for $\tau_\ell, \tau_{\ell+1}$, and the disjointness estimate in Proposition 3.4(ii) applied to the points $x_\ell < x_{\ell+1}$ and $y_\ell$ to establish the inequality for $\pi_\ell, \pi_{\ell+1}$. Note that Proposition 3.4(ii) only applies to fixed triples, whereas we are using a random point $y_\ell$. This does not cause any issues since $y_\ell$ can only take on countably many values.

We move to (54). By translation invariance of $\mathcal{L}$, (44), we may assume $y = 0$. Moreover, by the flip symmetry (40) it is enough to show that there exists $R > 0$ such that for all $r \geq R$ and $\ell \in [1, k - 1]$ we have

$$S(x, (0^\ell, r^{k-\ell})) = S(x^{[1, \ell]}, 0^\ell) + S(x^{[\ell+1, k]}, r^{k-\ell}).$$

(55)

Letting $y = (y_1, \ldots, y_k)$ be as constructed above, (53) implies that

$$S(x, y) = S(x^{[1, \ell]}, y^{[1, \ell]}) + S(x^{[\ell+1, k]}, y^{[\ell+1, k]}).$$

(56)

Now, letting $R = y_k$ and using that $y_1 = 0$, (55) follows from (56) and Lemma 3.6. \hfill \Box

Lemma 3.14 A.s. for every $J \subset [1, k]$ and $y \in \mathbb{R}^{|J|}$, we have

$$S(x^J, 0^{\lfloor J \rfloor}) - S(x^J, y) = W_{x\mathcal{L}}((-\infty, J) \to (0^{\lfloor J \rfloor}, 1]) - W_{x\mathcal{L}}((-\infty, J) \to (y, 1]).$$

(57)

In the proof, we use notation and results contained within the proof of Theorem 1.2.2.

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Both sides of (57) are continuous in y, so it suffices to prove that the equality holds a.s. for fixed $J = \{j_1 < \cdots < j_\ell\}$ and $y \in \mathbb{R}^\ell_\leq$. Let $[-m, m]$ be an interval containing all $y_i$, let $R > 0$ be as in Lemma 3.13 with $y = -m$, and let $Z_0(-m) \leq -m$ be as in Theorem 1.2.1. Set

$$N = \max(\lceil R \rceil, -Z_0(-m)).$$

Letting $Y_n, n \in \mathbb{N}$ and $\pi$ be as in the proof of Theorem 1.2.2, by Lemma 3.10, the left side of (57) equals

$$\mathcal{W}([Y_N, \lceil \pi(y) \rceil] \to (0^\ell, 1]) - \mathcal{W}([Y_N, \lceil \pi(y) \rceil] \to (y, 1]).$$

Using the notation $\mathcal{W}^{j, z}$ from Theorem 1.2.2, by Proposition 2.6 this equals

$$\mathcal{W}^{\pi(Y_N), Y_N}[(Y_N, J) \to (0^\ell, 1)] - \mathcal{W}^{\pi(Y_N), Y_N}[(Y_N, J) \to (y, 1)]. \quad (58)$$

We claim that (58) equals

$$\mathcal{W}^{\pi(Y_N), Y_N}[(\neg N, J) \to (0^\ell, 1)] - \mathcal{W}^{\pi(Y_N), Y_N}[(\neg N, J) \to (y, 1)]. \quad (59)$$

For this, it is enough to show that rightmost optimizers in $\mathcal{W}^{\pi(Y_N), Y_N}$ from $(Y_N, J)$ to $(0^\ell, 1)$ and $(y, 1)$ simply have constant paths following the lines in $J$ up to time $-N$. For this, by the monotonicity in Proposition 2.2(ii), it suffices to prove the same claim for the rightmost optimizer $\pi = (\pi_1, \ldots, \pi_\ell)$ from $(Y_N, J)$ to $((-m)^\ell, 1)$. Again using Proposition 2.2(ii), we have $\pi_i \geq \tau^i$, where $\tau^i = (\tau^i_1, \ldots, \tau^i_{k-j_i+1})$ is the rightmost optimizer in $\mathcal{W}^{\pi(Y_N), Y_N}$ from $(Y_N, [j_i, k])$ to $((-m)^{k-j_i+1}, 1)$. The paths in $\tau^i$ follow the lines $j_i, \ldots, k$ up to time $-N$ if and only if

$$\mathcal{W}^{\pi(Y_N), Y_N}[(Y_N, [1, j_i-1]) \to ((-N)^{j_i-1}, 1)] + \mathcal{W}^{\pi(Y_N), Y_N}[(Y_N, [j_i, k]) \to ((-N)^{k-j_i}, 1)].$$

By Proposition 2.6, this equality is equivalent to an equality of last passage values in $\mathcal{W}$,

$$\mathcal{W}([Y_N, \lceil \pi(y) \rceil]^{[1, j_i-1]} \to ((-N)^{j_i-1}, 1]) + \mathcal{W}([Y_N, \lceil \pi(y) \rceil]^{[j_i, k]} \to ((-N)^{k-j_i}, 1]).$$

which by Lemma 3.10, is equivalent to an equality for extended Airy sheet values:

$$S(x[^{1, j_i-1}], (-N)^{j_i-1}) + S(x[^{j_i, k}], (-m)^{j_i-1}) = S(x[^{1, k}], ((-N)^{j_i-1}, (-m)^{j_i-1})).$$

This equality follows from (54) since $N \geq R$. This completes the proof that (58)=(59).
Now, increments of $W(Y_N), Y_N$ and $W_XL$ are the same on the interval $[-N, N]$ by (52), and since $-N \leq Z_0(-m)$, the right side of (57) is unchanged if we replace $-\infty$ with $-N$. Therefore (59) equals the right side of (57), completing the proof.

Proof of Theorem 1.2, part 3 By Lemma 3.14, for every $J \subset [1, k]$, there exists a constant $\alpha(J)$ such that

$$S(x^J, y) = W_XL[(-\infty, J) \to (y, 1)] + \alpha(J)$$

for all $y$. We just need to show that $\alpha(J) = 0$ for all $J$. First, (53) in Lemma 3.13 implies that

$$\alpha(J) = \sum_{\ell \in J} \alpha(\ell), \quad \text{and} \quad \alpha([1, \ell]) = \alpha([1, \ell - 1]) + \alpha(\ell)$$

for all $J, \ell$, so it is enough to show that $\alpha([1, \ell]) = 0$ for all $\ell \in [1, k]$. This follows from the definitions (68) and (12), which together imply that

$$S(x^{[1, \ell]}, 0^\ell) = \sum_{i=1}^\ell W_XL_i(0) = W_XL[(-\infty, [1, \ell]) \to (0^\ell, 1)].$$

\[\Box\]

4 Consequences

In this section we prove all the remaining theorems. We start with Theorems 1.4, 1.5, and 1.7, as Theorem 1.1 is a bit more involved.

Proof of Theorem 1.4 Part 1 is immediate from parts 1 and 2 of Theorem 1.2 and the definition (13) of $W_XL$. For part 2, using part 1 and the asymptotics in Theorem 1.2.1 there exists a random integer $N < a - 1$ such that for all $y \in [a, b]$ we can write

$$A^{x_1, x_2}(y) = \max_{N \leq z \leq y} W_XL_2(z) - W_XL_1(z).$$

Therefore we just need to show that for all $n \in -N, n < a$, that the process $\max_{n \leq z \leq y} W_XL_2(z) - W_XL_1(z), y \in [a, b]$ is absolutely continuous with respect to the running maximum of a Brownian motion $B$ of variance 4 on $[a, b]$, started at time $a - 1$. This follows from local absolute continuity of $W_XL_2 - W_XL_1$ with respect to Brownian motion, Theorem 1.2.2. Theorem 1.2.2 also implies that the set of points $y$ where

$$W_XL_2(y) - W_XL_1(y) = \sup_{z \leq y} W_XL_2(z) - W_XL_1(z) = A^{x_1, x_2}(y)$$

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is exactly the support of the associated measure $\mu_{x_1,x_2}$. In other words, there cannot exist a point $y$ where

$$W_x \mathcal{L}_2(y) - W_x \mathcal{L}_1(y) = \sup_{z \leq y'} W_x \mathcal{L}_2(z) - W_x \mathcal{L}_1(z)$$

for all $y'$ in some open neighbourhood containing $y$, since a.s. no such $y$ exists for Brownian motion. Therefore by part 1 of Theorem 1.4, the support of $\mu_{x_1,x_2}$ is the set of times where

$$\mathcal{L}((x_1, x_2), 0; z^2, 1) = \mathcal{L}(x_1, 0; z, 1) + \mathcal{L}(x_2, 0; z, 1).$$

By Proposition 3.7, this is the same as the second set described in Theorem 1.4.3. \Box

**Proof of Theorem 1.5** By shift and scale invariance of $\mathcal{L}$, we can set $s = 0$, $t = 1$. For every $x_1, x_2, y_1, y_2 \in \mathbb{Q}$, the $\sigma$-algebra $\mathcal{F}_{0,1}$ contains the information about whether or not the geodesics from $(x_1, 0)$ to $(y_1, 1)$ and $(x_2, 0)$ to $(y_2, 1)$ are disjoint. Using Proposition 3.7, this is exactly the set where

$$\mathcal{L}((x_1, x_2), 0; (y_1, y_2), 1) = \mathcal{L}(x_1, 0; y_1, 1) + \mathcal{L}(x_2, 0; y_2, 1).$$

By continuity of the extended landscape $\mathcal{L}$ and Theorem 1.4.3, this gives us access to all the supports $S_{x_1,x_2}$ of the measures $\mu_{x_1,x_2}$ with CDFs

$$A^{x_1,x_2}(z) = \mathcal{L}(x_2, 0; z, 1) - \mathcal{L}(x_1, 0; z, 1)$$

for $x_1 < x_2 \in \mathbb{Q}$. Now, fix $x_1 < x_2 \in \mathbb{Q}$ and a compact rational interval $[a, b]$. By Theorem 1.4.1 and the asymptotics in (14), a.s. for all large enough $n \in \mathbb{N}$ we have

$$A^{x_1,x_2}(z) = \max_{-n \leq y \leq z} W_x \mathcal{L}_2(y) - W_x \mathcal{L}_1(y) \tag{60}$$

for all $z \in [a, b]$. For each $n$, let $S_{x_1,x_2}^n \subset [-n, \infty)$ be the support of the measure corresponding to the CDF $\max_{-n \leq y \leq z} W_x \mathcal{L}_2(y) - W_x \mathcal{L}_1(y)$. Since $W_x \mathcal{L}_2 - W_x \mathcal{L}_1$ is absolutely continuous with respect to Brownian motion on $[-n, b]$ by Theorem 1.2.3, by Theorem 1 in [35] (see also Section 6.4 in [23]), there is an explicit function $H$ such that a.s.

$$H(S_{x_1,x_2}^n \cap [a, b]) = \max_{-n \leq y \leq b} [W_x \mathcal{L}_2(y) - W_x \mathcal{L}_1(y)] - \max_{-n \leq y \leq a} [W_x \mathcal{L}_2(y) - W_x \mathcal{L}_1(y)].$$

(The exact nature of the function $H$ is not important; as an aside, it is type of generalized Hausdorff measure). By (60), for all large enough $n$ the right hand side above equals $A^{x_1,x_2}(b) - A^{x_1,x_2}(a)$ and $S_{x_1,x_2}^n \cap [a, b] = S_{x_1,x_2} \cap [a, b]$. Therefore a.s.

$$H(S_{x_1,x_2} \cap [a, b]) = A^{x_1,x_2}(b) - A^{x_1,x_2}(a).$$
This holds simultaneously a.s. for all rationals \( a < b, x_1 < x_2 \). Next, a.s. we have
\[
A^{x_1, x_2}(a) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A^{x_1, x_2}(a) - A^{x_1, x_2}(i) + \mathbb{E} A^{x_1, x_2}(i), \quad \text{and}
\]
\[
\mathcal{L}(x_2, 0; a, 1) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A^{x_2-i, x_2}(a) + \mathbb{E} \mathcal{L}(x_2 - i, 0; a, 1)
\]
This follows from ergodicity of the stationary Airy processes \( \mathcal{L}(:, 0; a, 1) + \mathbb{E} \mathcal{L}(:, 0; a, 1) \) and \( \mathcal{L}(x_i, 0; :, 1) + \mathbb{E} \mathcal{L}(x_i, 0; :, 1) \), see equation (5.15) in [30]. Putting all this together gives that each of the points \( \mathcal{L}(x_2, 0; a, 1) \), \( x_2, a \in \mathbb{Q} \) is \( \mathcal{F}_{0,1} \)-measurable. Continuity of \( \mathcal{L} \) then gives the result.

\[\square\]

**Proof of Fact 1.6 and Theorem 1.7** By scale invariance of \( \mathcal{L} \), we may assume \( t = 1 \). Let \( p = (p_1, p_2) \). By Theorem 1.2, we have
\[
h_i(y) = \max (\mathcal{L}(p_1, 0; y, 1) + a_1, \mathcal{L}(p_2, 0; y, 1) + a_2),
\]
\[
\mathcal{L}(p_1, 0; y, 1) = W_p \mathcal{L}_1(y),
\]
\[
\mathcal{L}(p_2, 0; y, 1) = W_p \mathcal{L}_1(y) + \sup_{z \leq y} W_p \mathcal{L}_2(z) - W_p \mathcal{L}_1(z). \quad (61)
\]
From this representation, it is clear that an \( A \in [−\infty, \infty] \) satisfying the conditions of Fact 1.6 exists. The fact that \( A \neq \pm \infty \) follows from the asymptotics in Theorem 1.2.1. Next, fix \( n \in \mathbb{N} \), and define \( W_n \mathcal{L} : [−n, \infty) \to \mathbb{R}^2 \) by
\[
W_n \mathcal{L}_p(x) = \begin{cases} W_p \mathcal{L}(x) - W_p \mathcal{L}(-n), & x \in [−n, n] \\ W_p \mathcal{L}(n) + B(x - n), & x \geq n \end{cases}
\]
where \( B \) is an independent 2-dimensional Brownian motion of variance 2 with \( B(0) = 0 \). Consider the process \( h_n^p = \max(M_1^n, M_2^n) : [−n, \infty) \to \mathbb{R} \), where
\[
M_1^n(x) = W_n \mathcal{L}([-n, -1) \to (x, 1)] + \max(a_1 + W_p \mathcal{L}_1(-n), a_2 + W_p \mathcal{L}_2(-n) + 1),
\]
\[
M_2^n(x) = W_n \mathcal{L}([-n, 2) \to (x, 1)] + a_2 + W_p \mathcal{L}_2(-n). \quad (62)
\]
By Theorem 1.2.2 and the spatial stationarity (44) for \( \mathcal{L} \), conditional on \( W_p \mathcal{L}(-n) \) the process \( W_p \mathcal{L} : [−n, \infty) \to \mathbb{R} \) is absolutely continuous with respect to a 2-dimensional Brownian motion of variance 2 on \( [−n, \infty) \) started from \( B(-n) = 0 \). Adding the independent Brownian motion to the end of \( W_n \mathcal{L} \) ensures that we can make the comparison on all of \( [−n, \infty) \), rather than just on \( [−n, n] \).

Therefore conditional on \( W_p \mathcal{L}(-n) \), the processes \( M_1^n, M_2^n \) satisfy the conditions of Proposition 2.11. Here we have used that
\[
\max(a_1 + W_p \mathcal{L}_1(-n), a_2 + W_p \mathcal{L}_2(-n) + 1) > a_2 + W_p \mathcal{L}_2(-n).
\]
Therefore letting $\tau_n$ denote the maximal $x \in (-n, \infty)$ such that $h^n_i(x) = M^n_i(x)$, by Proposition 2.11 we have that

$$
(\tau_n, h^n_i|_{(-n, \infty)} - h^n_i(\tau_n)) \ll (X, [B + R](-X + \cdot)|_{(-n, \infty)}),
$$

where $B, R, X$ are as in the statement of Theorem 1.7. Finally, comparing (61) and (62) and using the asymptotics in Theorem 1.2.1, on any interval $I$, there exists a random $N \in \mathbb{N}$ such that

$$
(\tau_N, h^N_i|_I - h^N_i(\tau_N)) = (A, h_I|_I - h_I(A)),
$$

yielding the result. \(
\square
\)

For Theorem 1.1, we first need an analogue of Proposition 2.4 for side-to-side, rather than bottom-to-top, last passage values. We will need a notion of discrete last passage percolation. Consider an $m \times n$ array $G = G_{i,j}, i \in [1, m], j \in [1, n]$. For $k \in \mathbb{N}$ and vectors $I, J \in [1, n]^k$ define the last passage value

$$
G[(1, I) \rightarrow (m, J)] = \sup_{\pi_1, \ldots, \pi_k} \sum_{i=1}^k \sum_{v \in \pi} G_v.
$$

Here the supremum is over all $k$-tuples of disjoint lattice paths $\pi_1, \ldots, \pi_k$, where $\pi_i$ starts at $(1, I_i)$, ends at $(n, J_i)$ and only moves up and to the right in the coordinate system of Fig. 1, i.e. all of its steps are in $\{(0, -1), (1, 0)\}$. If no such disjoint paths exist, we set $G[(1, I) \rightarrow (n, J)] = -\infty$. For points $(1, i), (n, j)$ we write

$$
G[(1, i)^k \rightarrow (m, j)^k]
$$

for the $k$-point last passage value from $(i, I)$ to $(n, J)$, where $I = (i + k - 1, \ldots, i + 1, i), J = (j, j - 1, \ldots, j - k + 1)$. These definitions are analogous to the definitions of semi-discrete last passage introduced in Section 1. Moreover, discrete last passage percolation also satisfies an isometry theorem, see Section 8 of [10] and references therein to precursor theorems. For an $m \times n$ array $G$ with $m \geq n$, define an $n \times n$ array $WG$ by the system of equations

$$
G[(1, n)^{(n+1-k)} \rightarrow (m, \ell)^{(n+1-k)}] = \sum_{i=k}^n \sum_{j=\ell}^n WG_{i,j}
$$

for $\ell, k \in [1, n]$. Then we have the side-to-side isometry

$$
G[(1, I) \rightarrow (m, J)] = WG[(1, I) \rightarrow (n, J)]
$$

(63)

for any $I, J$.  

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Now suppose $f$ is a sequence of $n$ continuous functions from $[0, t] \to \mathbb{R}$. For all $m \in \mathbb{N}$, let $f^m$ be the $m \times n$ array where $f^m_{i,j} = f^m_{j}(ti/m) - f^m_{j}(t(i - 1)/m)$. Then since $f$ is continuous it is easy to check that as $m \to \infty$, for any $I, J$ we have

$$f^m[(1, I) \to (m, J)] - f[(0, I) \to (t, J)] \to 0$$

and for any $i, j$ we have

$$f^m[(1, i)^k \to (m, j)^k] - f[(0, i)^k \to (t, j)^k] \to 0.$$  

Combining these facts with (63) immediately gives a side-to-side isometry for continuous functions.

**Proposition 4.1** Let $f = (f_i : \mathbb{R} \to \mathbb{R}, i \in [1, n])$ be a sequence of continuous functions, let $t > 0$, and define an $n \times n$ array $W^t f$ given by

$$f[(0, n)^{n+1-k} \to (t, \ell)^{n+1-k}) = \sum_{i=k}^{n} \sum_{j=\ell}^{n} W^t f_{i,j}.$$  

Then for any $I, J \in [1, n]^k \leq k$ we have

$$f[(0, I) \to (t, J)] = W^t f[(1, I) \to (n, J)].$$

We use this proposition to prove the following lemma.

**Lemma 4.2** For a sequence of $n$ continuous functions $f$ and $t > 0$ define $g_{t,f} : \{(i, j) \in [1, n] : i \geq j\} \to \mathbb{R}$ by

$$g_{t,f}(i, j) = f[(0, i) \to (t, j)].$$

Then for any $0 < s, t$, if $B$ is a sequence of independent standard Brownian motions, then $g_{t,B}$ and $g_{s,B}$ are mutually absolutely continuous.

**Proof** By Proposition 4.1, it is enough to show that $W^t B$ and $W^s B$ are mutually absolutely continuous for all $t, s$. The array $W^t B$ contains exactly the same information as the triangular array $X^t = \{X^t_{i,j}, i \leq j \in [1, n]\}$ given by the rule

$$\sum_{i=1}^{k} X^t_{i,j} = f[(1, n)^k \to (t, n-j + 1)^k].$$

To complete the proof, it enough to note that for any $t$, the law of $X^t$ is mutually absolutely continuous with respect to Lebesgue measure on the set of all arrays $x = x_{i,j}$ satisfying the interlacing inequalities

$$x_{i,j} \geq x_{i,j-1}, \quad x_{i,j-1} \geq x_{i+1,j}$$  

(64)
for all \(i,j\) (i.e. Lebesgue measure on Gelfand-Tsetlin patterns). This is well-known. Indeed, \(X_t^1/\sqrt{t}\) has the same law as the GUE minors eigenvalue process \(\lambda\), see p. 3691 in [27]. The top row of GUE eigenvalues in this process has an explicit Lebesgue density (e.g. see [1, Theorem 2.5.2]) and can hence be seen to be mutually absolutely continuous with respect to Lebesgue measure on the set where \(\lambda_{1,n} \geq \cdots \geq \lambda_{n,n}\). Conditional on this top row, the law of the remaining rows is uniform on the compact set satisfying the inequalities in (64), see [2].

**Proof of Theorem 1.1** By combining parts 1 and 3 of Theorem 1.2, we get that there is a random integer \(N \leq -b - 1\) such that for all \(i \in \{1, \ldots, k\}\) and \(y \in [-b, b]\) we have

\[\mathcal{L}(x_i, 0; y, 1) - \mathcal{L}(x_i, 0; -b, 1) = W_x \mathcal{L}((N, i) \rightarrow (-b, 1)) - W_x \mathcal{L}((N, i) \rightarrow (y, 1)).\]

By Theorem 1.2.2, for every integer \(n \leq -b - 1\), the process

\[W_x \mathcal{L}((n, i) \rightarrow (-b, 1)) - W_x \mathcal{L}((n, i) \rightarrow (y, 1)), i \in \{1, \ldots, k\}, y \in [-b, b]\]

is absolutely continuous with respect to the same last passage process with \(W_x \mathcal{L}\) replaced by \(k\) independent Brownian motions \(B\):

\[B((n, i) \rightarrow (-b, 1)) - B((n, i) \rightarrow (y, 1)), i \in \{1, \ldots, k\}, y \in [-b, b]. \tag{65}\]

Therefore to complete the proof, we just need to check that for all integers \(n \geq -b - 1\), the process in (65) is absolutely continuous with respect to the same process with \(n\) replaced by \(-b - 1\). We will check the stronger claim that the processes

\[B((a, i) \rightarrow (y, 1)), i \in \{1, \ldots, k\}, y \in [-b, b] \tag{66}\]

are all mutually absolutely continuous for \(a < -b\). Indeed, the laws of the random vectors

\[(B((a, i) \rightarrow (-b, j)), j \leq i \in [1, k]) \tag{67}\]

are mutually absolutely continuous for all \(a < -b\) by Lemma 4.2. Therefore to complete the proof, it is enough to show that conditional on the vector in (67), the distribution in (66) does not depend on \(a\). Under this conditioning the function \(B|_{[-b, b]}\) is still just a collection of \(n\) independent Brownian motions, and by the metric composition law (Proposition 2.1) at the vertical line \(-b\) \(\times\) \(\mathbb{Z}\), for \(i \in [1, k]\), \(y \in [-b, b]\) we have

\[B((a, i) \rightarrow (y, 1)) = \max_{j \in [1, i]} B((a, i) \rightarrow (-b, j)) + B((-b, j) \rightarrow (y, 1)).\]

The right hand side is a function of (67) and \(B|_{[-b, b]}\), completing the proof.

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
A Appendix: Theorem 1.2 with repeated endpoints

In this appendix, we extend Theorem 1.2 to the case of repeated endpoints. To state the theorem, for \( x \in \mathbb{R}_\leq^k \), let \( P([1, k], x) \) denote the set of all subsets \( I \) of \([1, k]\) such if \( i \in I \) and \( i > 1 \), then either \( i - 1 \in I \) or else \( x_{i-1} < x_i \).

**Theorem A.1** For every \( x \in \mathbb{R}_\leq^k \), recall from the introduction that we define \( W_x \mathcal{L} = \{ W_x \mathcal{L}_i : \mathbb{R} \rightarrow \mathbb{R}, i \in [1, k] \} \) by the formula

\[
\sum_{i=1}^{\ell} W_x \mathcal{L}_i(y) = \mathcal{L}(x^{[1,\ell]}, 0, y^{\ell}, 1), \quad \text{for } \ell \in [1, k], y \in \mathbb{R}. \tag{68}
\]

1. (Asymptotics and stability) A.s. for any \( x \in \mathbb{R}_\leq^k \) and \( i \in [1, k] \) we have

\[
\lim_{z \to \pm \infty} \frac{W_x \mathcal{L}_i(z) + z^2}{z} = 2x_i. \tag{69}
\]

In particular, this implies that for \( y_0 \in \mathbb{R} \), there exists a random \( Z_0(y_0) \in (-\infty, y_0) \cap \mathbb{Z} \) such that

\[
W_x \mathcal{L}((-\infty, I) \to (y, 1)) = W_x \mathcal{L}((z, I) \to (y, 1)) + \sum_{i \in I} W_x \mathcal{L}_i(z) \tag{70}
\]

for all \( I \in P([1, k], x) \), \( z \leq Z_0 \) and \( y \in \mathbb{R}^{|I|}_\leq \) with \( y_1 \geq y_0 \).

2. (Locally Brownian) Let \( x \in \mathbb{R}_\leq^k \). Conditional on \( W_x \mathcal{L}(0) \) the function \( W_x \mathcal{L}(-) - W_x \mathcal{L}(0) \) is locally absolutely continuous with respect to a \( k \)-dimensional Brownian motion \( B \) of variance 2. In other words, if we let \( B, \mathcal{L} \) be independent then on any interval \([a, b]\), we have \( (W_x \mathcal{L}(0), W_x \mathcal{L}_{[a,b]} - W_x \mathcal{L}(0)) \ll (W_x \mathcal{L}(0), B_{[a,b]}) \).

3. (Isometry) A.s. for any \( x \in \mathbb{R}_\leq^k, I \in P([1, k], x) \) and \( y \in \mathbb{R}^{\vert I \vert \leq} \), we have

\[
W_x \mathcal{L}((-\infty, I) \to (y, 1)) = \mathcal{L}(x^I, 0; y, 1). \tag{71}
\]

In practice, the restriction to \( P([1, k], x) \) does not lose us any generality since by construction, for any \( I \subset [1, k] \) we can find \( I' \in P([1, k], x) \) with \( x^I = x^{I'} \). In particular, \( P([1, k], x) \) contains all subsets of \([1, k]\) when \( x \) has no repeated entries.

We will use an approximation argument to move from Theorem 1.2 to the more general statement in Theorem A.1. For parts 1 and 3 this is quite straightforward.

**Proof of Theorem A.1, parts 1 and 3** As in the \( x_1 < \cdots < x_k \) case, the asymptotics in (69) follow from Lemma 3.5. Again, these asymptotics imply (70). Note that here, it is important that we restrict to starting index sets \( I \in P([1, k], x) \), see Fig. 5.

For part 3, let \( x \in \mathbb{R}_\leq^k, I \in P([1, k], x), y \in \mathbb{R}_\leq^k \) and let \( x^n \) be any sequence converging to \( x \), where \( x_1^n < \cdots < x_k^n \). By Theorem 1.2, Eq. (71) holds for all \( x^n \) with \( I, y \) as given. Moreover, the right-hand side of (71) is continuous as \( x^n \to x \) by
Fig. 5 A example of the implication (14) \( \Rightarrow \) (15), where only the asymptotic slopes of lines are shown. In the example \( I = \{2, 4\}, x_1 < x_2 = x_3 < x_4 \). As we take the starting points down to \(-\infty\), each optimizer path will spend all but \( O(1) \) amount of time on its starting line, which has maximal slope among available lines. In the example, this would not hold if we started a path on line 3 without starting a path on line 2, since then said path would have two equal slope functions to choose from. This issue is the reason for restricting the set \( I \) to \( P(\llbracket 1, k \rrbracket, x) \).

Continuity of the extended landscape. Continuity of the left-hand side will also follow if we can find a value of \( z \) such that for every \( x^n \) (and also for \( x \)) Eq. (70) holds:

\[
W_{x^n} L[(\infty, I) \rightarrow (y, 1)] = W_{x^n} L[(z, I) \rightarrow (y, 1)] + \sum_{i \in I} W_{x^n} L_i(z).
\]

Again this follows from the uniform control over the extended Airy sheet in Lemma 3.5.

Part 2 requires a great deal more care. The basic issue is this. If \( X_n \) is sequence of random functions (defined on \([0, 1]\), say) that are all absolutely continuous with respect to Brownian motion and converge uniformly a.s. to a limit \( X \), then \( X \) need not be absolutely continuous with respect to a Brownian motion \( B \). Because of this we need to look beyond uniform convergence statements. Our goal will be to prove the following lemma, which will quickly imply Theorem 1.2.2 in the general case via the case when \( x_1 < \cdots < x_k \).

**Lemma A.2** Let \( x \in \mathbb{R}_+^k \). Define the map \( f_x : \llbracket 1, k \rrbracket \rightarrow \llbracket 1, k \rrbracket \) by

\[
f_x(i) = \#\{j \in [i + 1, k] : x_j = x_i\}.
\]

For every \( n \in \mathbb{N} \) define \( x^n \) by letting \( x^n_i = x_i - f_x(i)/n \). For large enough \( n \), this construction guarantees that \( x^n_1 < x^n_2 < \cdots < x^n_k \).

Fix an interval \([a, b]\) \( \subset \mathbb{R} \) and let \( C^k_{a,b} \) be the space of \( k \)-tuples of continuous functions from \([a, b]\) to \( \mathbb{R} \) with the uniform topology. Let \( \mu \) be the law on \( C^k_{a,b} \) of \( W_x L \)
and let $\mu^n$ be the law on $C^k_{a,b}$ of $W_x^n$. Then

$$\mu \ll \sum_{n=1}^{\infty} \mu^n.$$ 

**Proof of Theorem A.1.2 given Lemma A.2.** Fix an interval $[a, b]$ containing 0, and let $\mu$, $\mu^n$ be as in Lemma A.2. By the Lebesgue decomposition theorem, for every $n$ we can write

$$\mu^n = \nu^n + \rho^n,$$

where $\nu^n$, $\rho^n$ are positive measures, $\rho^n$ and $\mu$ are mutually singular, and $\nu^n \ll \mu$. Let $J \subset \mathbb{N}$ be the set of all $n$ with $\nu^n(\Omega) > 0$. By Lemma A.2, this set is nonempty and $\mu \ll \sum_{n \in J} \nu^n$. Finally, for every $n \in J$ we can choose $\sigma^n \in (0, 1]$ so that $\nu^n = \sigma^n \tilde{\nu}^n$, where the $\tilde{\nu}^n$ are probability measures. We write $\tilde{W}_x^a \mathcal{L}_{[a,b]}$ for a sample from $\tilde{\nu}^n$.

Now, let $A$ be a Borel subset of $\mathbb{R}^k \times C^k_{a,b}$, and suppose that

$$\mathbb{P}((W_x \mathcal{L}(0), W_x \mathcal{L}_{[a,b]} - W_x \mathcal{L}(0)) \in A) > 0.$$ 

Then for some $n \in J$ we have $\mathbb{P}((\tilde{W}_x^a \mathcal{L}(0), \tilde{W}_x^a \mathcal{L}_{[a,b]} - \tilde{W}_x^a \mathcal{L}(0)) \in A) > 0$, and so by Theorem 1.2.2, we have that $\mathbb{P}((\tilde{W}_x^a \mathcal{L}(0), \tilde{B}_{[a,b]}(0)) \in A) > 0$, where $\tilde{B}$ is a Brownian motion on $[a, b]$, independent of all other objects. Finally, since $\tilde{\nu}^n \ll \mu$ we have

$$(\tilde{W}_x^a \mathcal{L}(0), \tilde{B}_{[a,b]}(0)) \ll (W_x \mathcal{L}(0), B_{[a,b]}(0))$$

and so $\mathbb{P}((W_x \mathcal{L}(0), B_{[a,b]}(0)) \in A) > 0$, as desired. \qed

To set up the proof of Lemma A.2, we require two lemmas about last passage percolation. The first compares multi-point last passage values to $(x, 1)$ and $(x^n, 1)$ in a Brownian environment.

**Lemma A.3** Let $x \in \mathbb{R}^k_+$ with $x_1 > 0$, and let $I \in \mathbb{N}^k$ be such that there is at least one disjoint $k$-tuple from $(0, I)$ to $(x, 1)$. For every $i \in \llbracket 1, k \rrbracket$, define

$$g_x(i) = \#\{j \in \llbracket 1, I \rrbracket : x_j = x_i\}$$

and recall the notation $x^n_i = x_i - f_x(i)/n$ of Lemma A.2. Now let $B = (B_i : i \in \mathbb{N})$ be a sequence of independent Brownian motions. Then for all large enough $n$, we can write

$$B[(0, I) \to (x, 1)] = \gamma_{n,1}^B(B) + \sum_{i=1}^{k} \alpha_i^n(B), \quad (72)$$

and

$$B[(0, I) \to (x^n, 1)] = \gamma_{n,1}^B(B) + \sum_{i=1}^{k} \beta_i^n(B), \quad (73)$$
Fig. 6 An example of optimizers for $B[(0, I) \to (x, 1)]$ and $B[(0, I) \to (x^n, 1)]$. The red parts of the optimizer paths (which yield $\gamma^{n, I}(B)$) typically coincide and stay in $S_n$. The remaining (blue and green) parts of the paths can be well understood and yield $\alpha^n(B)$, $\beta^n(B)$

where

$$\alpha^n_i(B) := B_{g_x(i)}(x_i) - B_{g_x(i)}(x^n_i - 1/n),$$

$$\beta^n_i(B) := B[(x^n_i - 1/n, g_x(i)) \to (x^n_i, 1)]$$

and $\gamma^{n, I}(B)$ is the maximum weight in $B$ of $k$ paths $\pi = (\pi_1, \ldots, \pi_k)$ where

- Each $\pi_i$ goes from $(0, I_i)$ to $(x^n_i - 1/n, g_x(i))$.
- We have $\pi_i < \pi_j$ on $(0, x^n_i - 1/n]$ for all $i < j$.
- The graphs of all $\pi_i$ stay in the region

$$S_n := \{(x, m) \notin \bigcup_{i=1}^k [x^n_i - 1/n, x_i + 1/\sqrt{n}] \times \{g_x(i)\}\}.$$ 

Moreover, as long as $n$ is large enough so that $x_i - k/\sqrt{n} > x_{i-1}$ whenever $x_i \neq x_{i-1}$, we have that \((LHS) \geq (RHS)\) in (72), (73).

Lemma A.3 has a somewhat technical statement. It is best understood through a picture, see Fig. 6.

**Proof** First, note that for $n$ satisfying $x_i - k/n > x_{i-1}$ whenever $x_i \neq x_{i-1}$ we can concatenate a $k$-tuple of paths satisfying the three bullet points in the definition of $\gamma^{n, I}$ with a $k$-tuple paths $\tau$ where either

- each $\tau_i$ is defined on $[x^n_i - 1/n, x_i]$ and equals $g_x(i)$ on that interval, or
- each $\tau_j$ is defined on $[x^n_j - 1/n, x^n_j]$ and is a geodesic from $(x^n_j - 1/n, g_x(i))$ to $(x^n_i, 1)$ on that interval

to get disjoint $k$-tuples from $(0, I)$ to $(x, 1)$ and $(x^n, 1)$, respectively. This shows that \((LHS) \geq (RHS)\) in (72), (73).

Now, formulas (72) and (73) amount to the claim that in leftmost optimizers $\pi$, $\pi^n$ from $(0, I)$ to $(x, 1)$, $(x^n, 1)$, each of the constituent paths $\pi_i$, $\pi^n_i$ goes through the point $(x^n_i - 1/n, g_x(i))$ and avoids the region $S_n$ prior to the time $x^n_i - 1/n$. Taking into account the ordering constraints on $\pi$, $\pi^n$ and the fact that all paths are nonincreasing, this holds if
\[ \pi_i(x^n_i - 1/n) = \pi_i^n(x^n_i - 1/n) = g_X(i) \text{ for all } i \in [1, k]. \]

- For every \( i \in [1, k - 1] \) such that \( x_i < x_{i+1} \), we have \( \pi_{i+1}(x_i + 1/\sqrt{n}) \geq \pi_{i+1}(x_i + 1/\sqrt{n}) \geq g_X(i) + 1. \)

Now, note that by the monotonicity in Proposition 2.2(ii), \( \pi_{i+1} \geq \pi_{i+1}^n \) for all \( i, n \). Moreover, the ordering on the paths forces \( \pi_i^n(x_i - 1/n) \geq g_X(i) \) for all \( i, n \). Therefore to show that the two bullets above hold for large enough \( n \) it is enough to show that

1. There exists \( \epsilon > 0 \) such that \( \pi_i(t) = g_X(i) \) for all \( t \in [x_i - \epsilon, x_i] \), for all \( i \in [1, k] \).
2. For all large enough \( n \), for every \( i \in [1, k - 1] \) such that \( x_i < x_{i+1} \), we have \( \pi_{i+1}(x_i + 1/\sqrt{n}) \geq g_X(i) + 1. \)

The two points have similar proofs. We start with point 1. The only way this fails is if there is an index \( i^* \) such that \( \pi_{i^*}(t) > g_X(i^*) \) for all \( t < x_i^* \). Suppose that this is indeed the case, and that \( i^* \) is the smallest index where this occurs. Let \( m \geq g_X(i^*) + 1 = \lim_{t \rightarrow x_i^*} \pi_{i^*}(t) \). Then for some \( \delta > 0 \) we have that

- \( \{ \pi_i(t) : t \in [x_i - \delta, x_i], i < i^* \} \subset [1, g_X(i^*) - 1] \}. \) This uses that \( i^* \) was chosen minimally.
- \( \{ \pi_i(t) : t \in [x_i - \delta, x_i], i > i^* \} \subset \{ m + 1, \ldots \} \).
- \( \pi_{i^*}(t) = m \) for \( t \in [x_i - \delta, x_i] \).

In particular, if we replace \( \pi_{i^*}[x_i - \delta, x_i] \) with the last passage geodesic from \( (x_i - \delta, m) \) to \( (x_i, m - 1) \) then we still have a disjoint \( k \)-tuple, and so by the optimality of \( \pi \) we have

\[
B[(x_i - \delta, m) \rightarrow (x_i, m - 1)] \leq \| \pi_{i^*}[x_i - \delta, x_i] \|_B = B_m(x_i) - B_m(x_i - \delta). \quad (74)
\]

On the other hand, as a function of \( \delta \) the difference of the left- and right-hand sides above is the running maximum of a Brownian motion, and hence is a.s. positive for all \( \delta > 0 \). Since there are only finitely many choices of \( m, x_i \), this is a contradiction.

Now suppose that the claim 2 does not hold. Since \( \pi^n \geq \pi^{n-1} \) for all \( n \) by Proposition 2.2(ii), as \( n \rightarrow \infty \pi^n \) converges to a limit \( \pi \) in the Skorokhod topology on cadlag paths. The ordering conditions are preserved in the limit so \( \pi \) is a disjoint \( k \)-tuple from \( (0, I) \) to \( (x, 1) \). Moreover, path length is continuous in this topology and multi-point last passage is a continuous function of the endpoints so \( \pi \) is an optimizer from \( (0, I) \) to \( (x, 1) \). Since we have assumed that claim 2 fails, there is some path \( \pi_{i+1} \) with \( x_i < x_{i+1} \) and \( \pi_{i+1}(x_i) \leq g_X(i) \). The ordering constraints on \( \pi \) also imply that \( \lim_{t \rightarrow x_i} \pi_{i+1}(t) \geq g_X(i) + 1 \), and so \( \pi \) jumps at time \( x_i \).

Let \( \delta > 0 \) be the largest index such that \( \pi_{x_i} \) jumps at time \( x_i \), and choose a rational \( \delta > 0 \) small enough so that \( x_i + \delta < x_{i+1} \) and none of paths \( \pi_{i+1}, \ldots, \pi_k \) jump in the interval \( [x_i, x_i + \delta] \). We can guarantee the existence of such a \( \delta \) since \( i^* \) was chosen maximally. Let

\[
j_1 := \lim_{t \rightarrow x_i} \pi_{i^*}(t) < j_2 := \pi_{i^*}(x_i) = \pi_{i^*}(\delta). \]

Since none of the paths \( \pi_{i+1}, \ldots, \pi_k \) jump in the interval \( [x_i, x_i + \delta] \), the only one of the paths of \( \pi \) that enters the interval \( [j_1, j_2] \) in the time period \( (x_i, x_i + \delta) \) is \( \pi_{i^*} \).
Therefore $\tau^*_t|_{[x_i, x_i+\delta]}$ is a geodesic from $(x_i, j_1)$ to $(x_i+\delta, j_2)$, and so

$$B[(x_i, j_2-1) \to (x_i+\delta, j_2)] \leq B[(x_i, j_1) \to (x_i+\delta, j_2)] = B_{j_2}(x_i+\delta) - B_{j_2}(x_i).$$

As with (74), the difference between the left- and right-hand sides above is the maximum of a Brownian motion on $[0, \delta]$, and hence is a.s. positive. Since there are only countably many choices of $j_2, x_i$ and $\delta$, this is a contradiction. \qed

**Lemma A.4** For every $z < b \in \mathbb{R}$ and $k \in \mathbb{N}$, a.s. there exists $m \in \mathbb{N}$ such that the following holds. For every $x \in \mathbb{R}_k$ with $0 \leq x_1 \leq x_k \leq b$, every $y \in \mathbb{R}_k^+$ with $z \leq y_1 \leq y_k \leq b$ and every optimizer $\pi$ from $x$ to $(y, 1)$ in $\mathcal{W}$, we have $\pi_k(z) \leq m$.

**Proof** Consider the a.s. unique optimizer $\tau$ from $\tilde{x} := (b+1, \ldots, b_k)$ to $(b^k, 1)$ (existence and uniqueness of $\tau$ follows from Proposition 3.4(v)), and let $m = \tau_k(z)$. Let $\pi$ be an arbitrary optimizer from $x$ to $(y, 1)$ for some $x, y$ satisfying the criteria of the lemma. The asymptotics in the definition of parabolic paths ensure that for $w \leq z$ with $w$ sufficiently large and negative, we have $\pi_i(w) \leq \tau_i(w)$ for all $i$. Moreover, $\pi|_{[w, \infty)}$ is an optimizer from $(w, \pi(w))$ to $(y, 1)$ by Proposition 3.4(iv) and $\tau|_{[w, \infty)}$ is the only optimizer from $(w, \tau(w))$ to $(b^k, 1)$ by Proposition 3.4(vi). Since $\pi(w) \leq \tau(w)$ and $y \leq b^k$, the monotonicity in Proposition 2.2(ii) guarantees that $\pi_k(z) \leq \tau_k(z) = m$. \qed

Finally, we will need a two-point bound on the deviations of the Airy line ensemble.

**Lemma A.5** (Lemma 6.1, [12]) There are constants $c, d > 0$ such that for every $t \in \mathbb{R}, k \in \mathbb{N}, s \in (0, 1), a > 0$, the $k$th line $\mathcal{W}_k$ in the parabolic Airy line ensemble satisfies

$$\mathbb{P}(|\mathcal{W}_k(t) + t^2 - \mathcal{W}_k(t+s) - (t+s)^2| > a\sqrt{s}) \leq e^{ck-da^2}.$$ 

**Proof of Lemma A.2** Fix $x \in \mathbb{R}_k^+$. By translation invariance (Equation (44)) it suffices to prove the lemma for $a \geq 0$. First, we have the following symmetries for the (single-point) directed landscape, from [11, Lemma 10.2]:

$$\mathcal{L}(x, s; y, t) \overset{d}{=} \mathcal{L}(-x, -s; -y, -t), \quad \mathcal{L}(x, s; y, t) \overset{d}{=} \mathcal{L}(x, s+1; y, t+1).$$

Here the equality in distribution is joint in $x, s, y, t$. These symmetries and the one in (40) apply also to the extended landscape since it is a measurable function of the usual directed landscape. Combining these gives that

$$\mathcal{L}(z, 0; y, 1) \overset{d}{=} \mathcal{L}(y, 0; z, 1),$$
jointly in all \( y, z \). Therefore

\[
\sum_{i=1}^{\ell} W_{x}L_i(y) \overset{d}{=} \mathcal{W}[y^{\ell} \rightarrow (z^{[1,\ell]}, 1)] = \sup_{I \in \mathbb{N}^\ell \leq} \mathcal{W}[y^{\ell} \rightarrow (x_1 - 1, I)] + \mathcal{W}[(x_1 - 1, I) \rightarrow (z^{[1,\ell]}, 1)].
\]  

(75)

The equality in distribution is joint in all \( z \in \mathbb{R}^k \subseteq \) with \( z_1 \geq x_1 - 1 \), \( y \geq 0 \), \( \ell \in [1, k] \) and uses Theorem 3.3. The equality on the second line uses Lemma 3.9. For the remainder of the proof, we work in a coupling where the above equality in distribution holds a.s.

By the monotonicity established in Lemma A.4, for every \( \epsilon > 0 \) there exists \( m \in \mathbb{N} \) such that with probability at least \( 1 - \epsilon \) the expression (75) equals

\[
\max_{I \in [1, m]^{\ell} \leq} \mathcal{W}[y^{\ell} \rightarrow (x_1 - 1, I)] + \mathcal{W}[(x_1 - 1, I) \rightarrow (z^{[1,\ell]}, 1)]
\]  

(76)

for all \( y \in [a, b] \), \( z \in \mathbb{R}^k \subseteq \) with \( z_1 \leq x_1 - 1 \leq z_k \leq x_k \) and \( \ell \in [1, k] \). Now, recall from Proposition 2.9 that for any \( m \in \mathbb{N} \) the law of \((\mathcal{W}_1 - \mathcal{W}_1(x_1 - 1), \ldots, \mathcal{W}_m - \mathcal{W}_m(x_1 - 1))\) on \([x_1 - 1, x_k] \) is absolutely continuous with respect to the law of \( m \) independent Brownian motions. Therefore by Lemma A.3, with probability at least \( 1 - \epsilon \), for all large enough \( n \) we have

\[
\sum_{i=1}^{\ell} W_{x}L_i(y) = \max_{I \in [1, m]^{\ell} \leq} \mathcal{W}[y^{\ell} \rightarrow (x_1 - 1, I)] + \gamma^{n,I}(\mathcal{W}) + \sum_{i=1}^{k} \alpha_i^n(\mathcal{W}),
\]

\[
\sum_{i=1}^{\ell} W_{x^n}L_i(y) = \max_{I \in [1, m]^{\ell} \leq} \mathcal{W}[y^{\ell} \rightarrow (x_1 - 1, I)] + \gamma^{n,I}(\mathcal{W}) + \sum_{i=1}^{k} \beta_i^n(\mathcal{W})
\]  

(77)

for all \( y \in [a, b] \), \( \ell \in [1, k] \). Here the \( \alpha^n, \beta^n, \gamma^{n,I} \) functions are as in Lemma A.3. Also, using the ‘Moreover’ in Lemma A.3 there exists a deterministic \( n_0 \) such that for all \( n \geq n_0 \), \( y \in [a, b], \ell \in [1, k] \) we have

\[
\sum_{i=1}^{\ell} W_{x}L_i(y) \geq \sup_{I \in \mathbb{N}^\ell \leq} \mathcal{W}[y^{\ell} \rightarrow (x_1 - 1, I)] + \gamma^{n,I}(\mathcal{W}) + \sum_{i=1}^{k} \alpha_i^n(\mathcal{W}),
\]

\[
\sum_{i=1}^{\ell} W_{x^n}L_i(y) \geq \sup_{I \in \mathbb{N}^\ell \leq} \mathcal{W}[y^{\ell} \rightarrow (x_1 - 1, I)] + \gamma^{n,I}(\mathcal{W}) + \sum_{i=1}^{k} \beta_i^n(\mathcal{W}).
\]  

(78)

Combining (77) and (78) and taking \( \epsilon \to 0 \) implies that a.s. the two sides of the two equations in (78) are equal for all large enough \( n \). Next, let \( S_n \) be as in Lemma A.3 and
let $C_n$ be the space of continuous functions from $S_n$ to $\mathbb{R}$ with the uniform-on-compact topology. Noting that $\gamma_{n, 1}(\mathcal{V})$ only depends on $\mathcal{V}|_{S_n}$, as does $\mathcal{W}|_{y^\ell} \rightarrow (x_1 - 1, \ell)$ (by Lemma 3.11), we have

$$W_x \mathcal{L}|_{[a, b]} = f^n(\mathcal{V}|_{S_n}, \alpha^n(\mathcal{V})) \quad \text{and} \quad W_x \mathcal{L}|_{[a, b]} = f^n(\mathcal{V}|_{S_n}, \beta^n(\mathcal{V}))$$

(79)

with probability tending to 1 as $n \rightarrow \infty$ for a sequence of functions $f_n : C_n \times \mathbb{R}^k \rightarrow C^n_{a, b}$. Our goal in the remainder of the proof is to show that for every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$\lim \sup \sup_{n \rightarrow \infty} \left\{ \frac{\mathbb{P}(\{\mathcal{V}|_{S_n}, \alpha_n(\mathcal{V}) \in S\})}{\mathbb{P}(\{\mathcal{V}|_{S_n}, \beta_n(\mathcal{V}) \in S\})} : S \subset C_n \times \mathbb{R}^k, \mathbb{P}(\{\mathcal{V}|_{S_n}, \alpha_n(\mathcal{V}) \in S\}) \geq \epsilon \right\}$$

$$\leq c_\epsilon.$$ 

(80)

Here the supremum is over all Borel subsets $S$ of $C_n \times \mathbb{R}^k$. Let us first explain why (79) and (80) imply the result. Let $A$ be a Borel subset of $C^n_{a, b}$ with $\mathbb{P}(W_x \mathcal{L}|_{[a, b]} \in A) = \epsilon > 0$. Then by (79), for all large enough $n$ we have $\mathbb{P}(\{\mathcal{V}|_{S_n}, \alpha_n(\mathcal{V}) \in f_n^{-1}(A)\}) \geq \epsilon/2$, and hence by (80), for all large enough $n$ we have $\mathbb{P}(\{\mathcal{V}|_{S_n}, \beta_n(\mathcal{V}) \in f_n^{-1}(A)\}) \geq \epsilon/(4c_\epsilon/2)$. Using (79) again gives that $\mathbb{P}(W_x \mathcal{L}|_{[a, b]} \in A) \geq \epsilon/(8c_\epsilon/2) > 0$ for all large enough $n$, as desired.

To prove (80) we first consider $\alpha^{n_0}(B)$, $\beta^{n_0}(B)$, where $B$ is a sequence of independent Brownian motions and $n_0$ is chosen to be large enough so that none of the sets $[x_i^{n_0} - 1/n, x_i + 1/\sqrt{n_0}] \times \{g_\epsilon(i)\}$ overlap. Without loss of generality, assume we can take $n_0 = 1$.

The coordinates of $\alpha^1(B)$ are independent, as are those of $\beta^1(B)$, and have explicit Lebesgue densities which are bounded above and below on every compact set: the coordinates of $\alpha^1(B)$ are normals of different variances and (as mentioned in the proof of Lemma 4.2) the coordinates of $\beta^1(B)$ are top $m \times m$ GUE eigenvalues for different choices of $m \in [1, k]$ and hence have densities given in [1, Theorem 2.5.2]). Therefore for every $\epsilon > 0$,

$$\sup \left\{ \frac{\mathbb{P}(\alpha^1(B) \in S)}{\mathbb{P}(\beta^1(B) \in S)} : S \subset C_n \times \mathbb{R}^k, \mathbb{P}(\beta^1(B) \in S) \geq \epsilon \right\} < \infty.$$ 

Now, to complete the proof of (80), it is enough to show that

$$d_{TV}(\mathbb{P}(\sqrt{n} \beta^n(\mathcal{V}) \in \cdot \mid \mathcal{V}|_{S_n}), \mathbb{P}(\beta^1(B) \in \cdot)) \rightarrow 0$$

(81)

as $n \rightarrow \infty$, and similarly for $\alpha^n$. Here the left-hand side above is a total variation distance between the random measure $\mathbb{P}(\sqrt{n} \beta^n(\mathcal{V}) \in \cdot \mid \mathcal{V}|_{S_n})$ and the deterministic measure $\mathbb{P}(\beta^1(B) \in \cdot)$. Equation (81) will follow from the Brownian Gibbs property for $\mathcal{V}$ (see the proof of Proposition 3.1).

Indeed, given $\mathcal{V}|_{S_n}$, define $\hat{\mathcal{V}}$ by connecting up the missing segments of $\mathcal{V}|_{S_n}$ with independent Brownian bridges, and let $N_n$ be the event that $\hat{\mathcal{V}}_1 > \hat{\mathcal{V}}_2 > \ldots$. Then
the Brownian Gibbs property states that \( \mathcal{W} \) is equal in distribution to \( \tilde{\mathcal{W}} \) conditioned on the event \( N_n \). Since \( \mathcal{W}_1 > \mathcal{W}_2 > \ldots \) a.s. and the missing segments of \( \mathcal{W}|_{S_n} \) all have length at most \( k/\sqrt{n} \), which tends to 0 with \( n \), \( \mathbb{P}(N_n \mid \mathcal{W}|_{S_n}) \to 1 \) in probability as \( n \to \infty \). Therefore it suffices to prove (81) with \( \sqrt{n} \beta_n(\tilde{\mathcal{W}}) \) in place of \( \sqrt{n} \beta_n(\mathcal{W}) \).

We do not have that \( \mathbb{P}(\sqrt{n} \beta_n(\tilde{\mathcal{W}}) \in \cdot \mid \mathcal{W}|_{S_n}) = \mathbb{P}(\beta^1(B) \in \cdot) \) since in \( \sqrt{n} \beta_n(\tilde{\mathcal{W}}) \) the underlying lines are Brownian bridges with a slope coming from endpoints in \( \mathcal{W}|_{S_n} \) rather than Brownian motions. However, both the slopes and the bridge pinning become increasingly negligible as \( n \to \infty \) since we are sampling the Brownian bridges on intervals of the form \( [x^n_j - 1/n, x^n_j + 1/\sqrt{n}] \) which have length \( O(\sqrt{n}) \), whereas the function \( \beta_n \) only looks at time intervals of length \( 1/n \). Indeed, by Brownian scaling and translation invariance, conditional on \( \mathcal{W}|_{S_n} \) we have that

\[
\sqrt{n} \beta_n(\tilde{\mathcal{W}}) \xrightarrow{d} \beta^1(\tilde{B} + n^{-1/4}Z^n)
\]

where \( \tilde{B} \) is a sequence of functions where each \( \tilde{B}_i \) is pinned to be equal to 0 at exactly at the times \( x^n_j - 1, x^n_j + \sqrt{n} \) for all \( j \) with \( g_x(j) = i \) and connected up by Brownian pieces that are independent of each other and \( \mathcal{W}|_{S_n} \), and \( Z^n : \mathbb{R} \to \mathbb{R}^k \) is a \( \sigma(\mathcal{W}|_{S_n}) \)-measurable sequence of piecewise linear functions. By Lemma A.5, \( \max_{i \in [1,k], u \in \mathbb{R}} |dZ^n_i(u)|/du \) is tight, and so (81) holds with \( \sqrt{n} \beta_n(\tilde{\mathcal{W}}) \) in place of \( \sqrt{n} \beta_n(\mathcal{W}) \). The proof for \( \alpha_n^* \) is similar. \( \square \)

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