Nonlinear differential equations based on nonextensive Tsallis entropy and physical applications

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Abstract

A family of nonlinear ordinary differential equations with arbitrary order is obtained by using nonextensive concepts related to the Tsallis entropy. Applications of these equations are given here. In particular, a connection between Tsallis entropy and the one-dimensional correlated anomalous diffusion equation is established. It is also developed explicitly a WKB-like method for second order equations and it is applied to solve approximately a class of equations that contains as a special case the Thomas-Fermi equation for an atom. It is expected that the present ideas can be useful in the discussion of other nonlinear contexts.

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I. INTRODUCTION

As it is well known, the equation

\[ \frac{dp}{dx} + \lambda p = 0, \]

whose general solution is

\[ p(x) = p_0 \exp(-\lambda x), \]

has many applications. In general, Eq. (1) models systems related to an extensive context (in the thermodynamical sense). Typical examples are the systems based on independent events, for instance, the radioactive decay of noninteracting nucleus. In this context, it is interesting to observe that the solution of Eq. (1) has an entropic interpretation. In fact, if the usual entropy,

\[ S = -\int_a^b p(x) \ln p(x) \, dx, \]

is maximized subject to the constraints

\[ \alpha = \int_a^b p(x) \, dx, \]

and

\[ \beta = \int_a^b x p(x) \, dx, \]

the solution of Eq. (1) for \( a \leq x \leq b \) is obtained again. The parameters \( \alpha \) and \( \beta \) are adjusted in order to obtain \( p_0 \) and \( \lambda \). If \( \alpha = 1 \), \( p(x) \) can be interpreted as a probability. However, this choice for \( \alpha \) is not necessary because all the conclusions presented here are independent of the \( \alpha \) value.

Now, it is possible to ask how Eq. (1) must be generalized in order to describe dependent events (nonextensive context). As a guide to answer this question it will be considered in this work a generalization of the entropy (3) employed by Tsallis [1] (see also Ref. [2]) in a nonextensive statistical mechanics. The above choice is motivated by the fact that
this generalized entropy (Tsallis entropy) has been applied successfully in the discussion of many situations where the nonextensivity plays an important role, for instance, Lévy superdiffusion [3] and correlated anomalous diffusion [4–7], turbulence in two-dimensional pure electron plasma [8], dynamic linear response theory [8] and Green functions [4], perturbation and variation methods for calculation of thermodynamic quantities [10], low-dimensional dissipative systems [11], simulated annealing and optimization techniques [12] and connection with quantum uncertainty [13]. In other words, the main purpose of this work is to obtain generalizations of Eq. (1) based on nonextensive Tsallis entropy, giving physical applications. More specifically, it is obtained a generalization of Eq. (1) by using Tsallis entropy (Section II), and applications to the motion of a particle in a fluid medium and chemical kinetics are presented. A further generalization for higher order nonlinear differential equations is given (Section III) and it is applied to discuss a new connection between the correlated anomalous diffusion equation and the Tsallis entropy. Moreover, it is introduced a WKB-like approximation in order to study a family of nonlinear differential equations based on Tsallis entropy (Section IV), and finally this approximated method is employed to the case of the Thomas-Fermi equation.

II. FIRST ORDER NONLINEAR DIFFERENTIAL EQUATION

The Tsallis entropy can be written as

$$S_q = -\int_{a}^{b} p(x) \frac{(1-p(x)^{q-1})}{1-q} \, dx . \quad (6)$$

Furthermore, the constraint (5) is currently substituted [1,2] by

$$\beta = \int_{a}^{b} x \, p(x)^q \, dx \quad (7)$$

and the constraint (4) remains unchanged. When the entropy (6) is maximized subject to the constraints (4) and (7), $p(x)$ can be written as

$$p(x) = p_o \left[ 1 - (1-q) \, p_o^{q-1} \, x \right]^{1/(1-q)} . \quad (8)$$
This distribution generalizes the exponential one (see Eq. (2)). In the above expressions the parameter \( q \in \mathbb{R} \) characterizes the degree of nonextensivity, in particular the entropy (3) and the constraint (4) are obtained as limiting case when \( q \to 1 \), recovering the extensive case. A remarkable fact is that Eq. (8) can be applied directly to several contexts successfully. In addition to other works already cited, the study of solar neutrinos [14] and Zipf law [15] are relevant examples of such applications.

By direct inspection it is verified that the above generalization of the exponential satisfies the nonlinear equation

\[
\frac{dp}{dx} + \lambda p^q = 0 \tag{9}
\]

subject to the initial condition \( p(0) = p_0 \). This equation is the generalization of Eq. (1) based on the nonextensive Tsallis entropy. When the notation \( \lambda_{\text{eff}} = \lambda p^{q-1} \) is employed, Eq. (8) can be interpreted as a decay with memory, thus \( q \) is in some sense related with the memory of the system.

A. Applications

Before generalizing Eq. (8) to higher orders it is illustrative to present applications for it. One of them describes the mean motion of a particle in a fluid medium without external force. In this case, the motion equation can be written as

\[
m dv/dt = -b v^q, \tag{10}
\]

where \( v \) and \( b \) are respectively the velocity and the friction coefficient of the particle relative to the medium and \( m \) is the particle mass. Here, the parameter \( q \) is related to the turbulent flow (nonextensive behavior) and for \( q = 1 \) (slow motion) the extensive behavior is recovered. Another example comes from chemical kinetics. In this case, the concentration \( C_A \) of a given species \( A \) obeys the empirical equation [16]

\[
dC_A/dt = K C_A^{\alpha} C_B^{\beta} C_C^{\gamma} \cdots, \tag{11}
\]
where $K$ is the reaction constant and $\alpha$, $\beta$, $\gamma$, \ldots, refer to the concentration of chemical species $A$, $B$, $C$, \ldots, present in the reaction. In this expression, $\alpha$, $\beta$, $\gamma$, \ldots, are respectively the order of the reaction with respect to $A$, $B$, $C$, \ldots, and the sum $\alpha + \beta + \gamma + \cdots$ is the overall order of the reaction. In some cases the concentrations $C_B$, $C_C$, ... can be considered constant, thus the above equation reduces to the form of Eq. (9) with $q = \alpha$. In this way, the parameter $q$ becomes the order of the reaction for the species $A$.

### III. FAMILY OF NONLINEAR DIFFERENTIAL EQUATIONS

The natural generalization of Eq. (1) for higher order differential equations, in the sense that $p = p_o \exp(-\lambda x)$ is a particular solution, is the family of linear differential equations of arbitrary order with constant coefficients. The $N$-order element of this family is

$$\sum_{n=0}^{N} a_n \frac{d^n p}{dx^n} = 0 \quad (12)$$

and the corresponding algebraic equation for $\lambda$ is

$$\sum_{n=0}^{N} a_n (-\lambda)^n = 0 \quad . \quad (13)$$

In view of the previous remarks, the generalization of these ideas in the Tsallis context is based on the replacement of the particular solution $p(x) = p_o \exp(-\lambda x)$ by $p(x) = p_o [1 - (1 - q)p_o^{q-1}\lambda x]^{1/(1-q)}$. Thus,

$$\sum_{n=0}^{N} a_n \frac{d^n p}{dx^n} p^{(N-n)(q-1)+1} = 0 \quad (14)$$

represents the nonlinear ordinary differential equation of constant coefficients that generalizes Eq. (12) and the corresponding generalization of Eq. (13) is

$$a_N(-\lambda)^N \prod_{j=0}^{N-2} [(j+1)q - j] + \sum_{n=1}^{N-1} a_n (-\lambda)^n \prod_{j=N-n-1}^{N-2} [(j+1)q - j] + a_0 = 0 \quad . \quad (15)$$

This equation is applicable for $N \geq 2$ and when $N = 1$ Eq. (13) must be replaced by $a_1(-\lambda) + a_0 = 0$. Without loss of generality the $N$-th term of Eq. (14), $\frac{d^n p}{dx^n}$, was considered linear in $p(x)$. As expected, Eqs. (14) and (15) reduce respectively to Eqs. (12) and (13).
and (13) in the limit $q \to 1$. Furthermore, it is important to remark that a superposition of particular solutions of Eq. (14) is not another solution of this equation, in contrast with the case of Eq. (12). For instance, the generalization of circular and hyperbolic functions based on the superposition of Eq. (8) [17] is not a solution of Eq. (14). Thus, in the following applications only particular solutions previously described will be considered.

A. Application to correlated anomalous diffusion

It was recently shown that the one-dimensional correlated anomalous diffusion equation (generalized nonlinear Fokker-Planck equation without external force),

\[
\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi^\nu}{\partial x^2} \quad (\nu \in \mathbb{R}),
\]  

(16)

has a solution related to the Tsallis entropy [4,5].

This solution is a generalization of the Gaussian one, reducing to it in the limit $q \to 1$. On the other hand, by using the generalization developed above for linear ordinary differential equations, a new relation between Tsallis entropy and correlated anomalous diffusion equation can be obtained.

The new connection is based on the substitution of

\[
\phi(x,t) = T(t) X(x)
\]

(17)

into Eq. (16). This separation of variables leads to $dT/dt = -\sigma T^\nu$ and $d^2Y/dx^2 = -\sigma Y^{1/\nu}$, where $\sigma$ is a separation variable constant and $Y = X^\nu$. Since these equations are members of the family ruled by Eq. (14), a new connection between the one-dimensional correlated anomalous diffusion equation and the Tsallis entropy is established. Furthermore, as a consequence of this general procedure, a set of particular solutions of the anomalous correlated diffusion equations is obtained, namely,

\[
\phi(x,t) = T_0 \left[ 1 - (1 - \nu) \frac{T_0^{\nu-1}}{\sigma t} \right]^{1/(1-\nu)} X_0 \left[ 1 + \left( \frac{1 - \nu}{2\nu} \right) X_0^{(1-\nu)/2} \lambda x \right]^{2/(\nu-1)},
\]

(18)
where \( T_0 \) and \( X_0 \) are constants and \( \lambda \) is the solution of the equation \((1 + \nu)\lambda^2 + 2\nu\sigma = 0\). Note that the above procedure leads to complex solutions for \( \sigma > 0 \) and it is a natural extension of the method of separation of variables applied to the usual diffusion equation \((\nu = 1)\).

The previous method can be easily applied to the generalized diffusion equation recently proposed by Tsallis and Bukman \[5\],

\[
\frac{\partial \phi^\mu}{\partial t} = \frac{\partial^2 \phi^\nu}{\partial x^2} \quad (\mu, \nu \in \mathcal{R}).
\] (19)

In fact, it is sufficient to replace \( \phi^\mu \) by \( \psi \) and \( \nu \) by \( \nu/\mu \). In general, the connection between the porous media diffusion equation and Tsallis entropy, based on separation of variables and the family of nonlinear partial differential equation (Eq. (14)), can be easily extended for other nonlinear partial differential equations.

### IV. NONLINEAR DIFFERENTIAL EQUATIONS AND WKB-LIKE APPROXIMATION

Eq. (9) can be generalized further if we allow \( \lambda \) to become a function of \( x \). In this case, the solution of Eq. (9) with \( p(0) = p_0 \) becomes

\[
p(x) = p_0 \left[ 1 - (1 - q)p_0^{q-1} \int_0^x \lambda(z)dz \right]^{1/(1-q)}.
\] (20)

In a similar way, we can allow that the constants \( a_n \) in Eq. (14) become functions of \( x \). This procedure leads to

\[
\sum_{n=0}^{N} a_n(x) \frac{d^n}{dx^n} p^{(N-n)(q-1)+1} = 0.
\] (21)

As in the linear case \((q = 1)\) there are no general solutions for these equations. Consequently, it is natural to perform approximated analyses to obtain some information about the solutions of Eq. (21). In the following discussions, among other possibilities, a generalization of WKB method for \( q \neq 1 \) is developed explicitly for the \( N = 2 \) case.
As it is well known, in the WKB method approximate solutions of equation \( \frac{d^2 p}{dx^2} = f(x)p \) can be obtained when \( f(x) \) is a slowly varying function. In this case, it is employed an auxiliary function \( g(x) \) defined by the relation \( p(x) = \exp (g(x)) \). From the previous developments, a natural generalization of these equations are respectively

\[
\frac{d^2 p}{dx^2} = f(x)p^{2q-1}
\]  

(22)

and

\[
p(x) = [1 + (1 - q)g(x)]^{1/(1-q)}.
\]  

(23)

In terms of \( g(x) \), Eq. (22) becomes

\[
[1 + (1 - q)g] \frac{d^2 g}{dx^2} + q \left( \frac{dg}{dx} \right)^2 - f = 0.
\]  

(24)

Following again the usual WKB approach, the term with \( \frac{d^2 g}{dx^2} \) is neglected in the first approximation, hence

\[
g(x) = \pm q^{-1/2} \int f(x)^{1/2} dx.
\]  

(25)

Consistently, the validity condition of this approximation is

\[
\left| \frac{d^2 g}{dx^2} \right| \approx \left| \frac{df/dx}{2q^{1/2} f^{1/2}} \right| \ll \left| \frac{1}{1 + (1 - q)g} \right|.
\]  

(26)

Since Eq. (22) is nonlinear for \( q \neq 1 \), the superposition of solutions is not a solution, therefore the present development is indicated for situations where some particular solutions can be considered as good approximations. Notice also that the solutions (25) can be improved through iterations (replacing successively improved Eq. (25) into Eq. (24)).

A. Application to Thomas-Fermi equation

To exemplify the previous development, we consider the Thomas-Fermi equation for a free atom.
\[ \frac{d^2 y}{dx^2} = x^{-1/2} y^{3/2}. \]  

(27)

In the free neutral atom case the boundary conditions are \( y(0) = 1 \) and \( y(\infty) = 0 \). By comparing Eq. (27) with Eq. (22) it is verified that \( f(x) = x^{-1/2} \) and \( q = 5/4 \). Choosing the particular solution adjustable to these conditions it becomes

\[ y(x) = \left(1 + \frac{2}{3\sqrt{5}} x^{3/4}\right)^{-4}. \]  

(28)

In this example, the validity condition (26) becomes

\[ 1 + \frac{3\sqrt{5}}{2} x^{-3/4} \ll 15. \]  

(29)

This condition indicates that the approximation is better for larger \( x \). Furthermore, Eq. (28) is in satisfactory agreement with a numerical calculation (see, for instance, Ref. [19]). In a general context, when \( f(x) \) is a smooth function, the corresponding approximate solution becomes more accurate. In particular, when \( f(x) \) is a constant this solution becomes exact.

V. CONCLUSIONS

Summing up, the nonextensive concepts based on the Tsallis entropy were employed to obtain a family of nonlinear ordinary differential equations. The first order equation of this family is a nonextensive generalization of the exponential decay equation. Moreover, by using a separation of variables procedure and the above family of equations, a connection between the correlated anomalous diffusion equation and the Tsallis entropy is obtained. In addition to this, for second order equations we presented a WKB-like approach to obtain approximated solutions. This procedure was used in the context of the Thomas-Fermi equation for a free neutral atom, and it was shown that the well known solution precisely correspond to \( q = 5/4 \). In general, the developments introduced in this work indicate that many nonlinear effects are closely related with nonextensive concepts in the Tsallis framework.
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