Are Two Given Maps Homotopic? An Algorithmic Viewpoint

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Abstract

This paper presents two algorithms. The first decides the existence of a pointed homotopy between given simplicial maps \( f, g : X \to Y \), and the second computes the group \([\Sigma X, Y]^*\) of pointed homotopy classes of maps from a suspension; in both cases, the target \( Y \) is assumed simply connected. More generally, these algorithms work relative to \( A \subseteq X \).

Keywords
Homotopy · Suspension · Polycyclic group · Algorithm

Mathematics Subject Classification
55Q05 · 55P40

1 Introduction

In this paper, we study the following problem: decide whether two given maps \( f, g : X \to Y \) are homotopic. For computational purposes, we assume that \( X \) and \( Y \) are finite simplicial complexes or, more generally, finite simplicial sets, \( f \) and \( g \) are simplicial maps, while we ask for a \textit{continuous} homotopy between them. It is well known that no algorithm for deciding this problem exists if \( Y \) is allowed to be non-simply connected; this follows easily from Novikov’s result [8] on the unsolvability of the word problem in groups. We will thus restrict our attention to the case of a simply connected \( Y \).

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Our main theorem is formulated for pointed spaces, since this is the usual situation in topology. The unpointed setup is obtained either by adjoining a disjoint basepoint to $X$ or as a special case of a more general relative setup, which is treated later in this paper. The decidability of the homotopy problem for simply connected $Y$ is completely answered by the following theorem.

**Theorem A** There is an algorithm that decides the existence of a pointed homotopy between given simplicial maps $f, g : X \to Y$, where $X$ and $Y$ are finite simplicial sets with $Y$ simply connected.

This problem is solved in [2] under an additional “stability restriction” $\dim X \leq 2 \conn Y$ relating the dimension of $X$ and the connectivity of $Y$. Under this restriction, [2] also presents an algorithm that computes $[X, Y]$, i.e., the set of homotopy classes of continuous maps from $X$ to $Y$ (in fact, an abelian group). Our next result removes the stability restriction if the domain is a suspension, generalizing the computation of homotopy groups of spaces described by Brown in [1] or, later, in [4].

**Theorem B** There is an algorithm that computes the group $[\Sigma X, Y]^*$ of pointed homotopy classes of maps from a suspension $\Sigma X$ to a simply connected $Y$, both given as finite simplicial sets.

It turns out that $[\Sigma X, Y]^*$ is a polycyclic group (a solvable group all of whose filtration quotients are cyclic) and, as such, can be effectively described by generators and relations. In addition, it will also be fully effective (to be defined in Sect. 3), allowing one, for example, to express the homotopy class of any given pointed simplicial map $f : \Sigma X \to Y$ in terms of the computed generators.

### 1.1 Possible Applications

The homotopy problem is very basic to topology and as such has many applications. We want to mention two that might be of interest to non-topologists.

It is known that oriented vector bundles of dimension $k$ over a given space $X$, or rather their isomorphism classes, are in bijection with homotopy classes of maps $X \to BSO(k)$. If a suitable simplicial model for $BSO(k)$ is constructed and vector bundles are represented by simplicial maps $X \to BSO(k)$ (or by simplicial maps $X' \to BSO(k)$ for some subdivision $X'$ of $X$), the algorithm of this paper is then able to decide whether two given vector bundles are isomorphic or, as a special case, if a given vector bundle is trivial.

For a smooth manifold $X$, homotopy classes of maps $X \to MSO(k)$ correspond to submanifolds of $X$ of codimension $k$ up to cobordism in $X$ and, thus, given appropriate simplicial representatives, our algorithm is able to decide whether two given submanifolds are cobordant in $X$ or, as a special case, if a given submanifold of $X$ bounds in $I \times X$.

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1 For non-oriented bundles, the $\mathbb{Z}/2$-equivariant version of our theorem would be needed for the target space $BO(k)$; such an extension is briefly mentioned later.
We note, however, that we are not aware of simplicial models for any of the two outlined applications. Constructing such models is an interesting topic for further research.

1.2 Relative Version

Our proof of Theorems A and B works naturally in the comma category $\mathcal{A}/\sSet$ of simplicial sets “under $\mathcal{A}$”: The objects are simplicial sets $X$ equipped with a map $\iota: \mathcal{A} \to X$ and the morphisms in this category are maps $f: X \to Y$ for which the following diagram

\[ \begin{array}{ccc}
A & \xrightarrow{\alpha} & Y \\
\downarrow \iota & & \downarrow f \\
X & & 
\end{array} \]

commutes. For spaces under $\mathcal{A}$, the appropriate notion of homotopy is that of a homotopy relative to $\mathcal{A}$. When $\iota$ is an inclusion and $Y$ is a Kan complex, the resulting set of homotopy classes will be denoted by $[X, Y]^\mathcal{A}$ (this does not mention the maps $\iota$ and $\alpha$ explicitly, but they are usually understood from the context). For general $X, Y \in \mathcal{A}/\sSet$, we define $[X, Y]^\mathcal{A}$ by first replacing, up to weak homotopy equivalence, $\iota$ by an inclusion $\mathcal{A} \hookrightarrow X^{\text{cof}}$ and $Y$ by a Kan complex $Y^{\text{fib}}$ and then setting $[X, Y]^\mathcal{A} = [X^{\text{cof}}, Y^{\text{fib}}]^\mathcal{A}$. Since the replacements are equipped with maps $X^{\text{cof}} \to X$ and $Y \to Y^{\text{fib}}$, every simplicial map $X \to Y$ under $\mathcal{A}$ defines an element of $[X, Y]^\mathcal{A}$.

However, not every element of $[X, Y]^\mathcal{A}$ is represented by a simplicial map $X \to Y$.

It is easy to define a relative suspension $\Sigma_\mathcal{A} X$ and formulate a relative version of Theorem B. In the course of the proof of the theorems, we will, however, need a further generalization coming from reinterpreting pointed maps $\Sigma X \to Y$ as maps $I \times X \to Y$ from the cylinder that are constant onto the basepoint on the “boundary” $(\partial I \times X) \cup (I \times \ast)$. In this generalization, the restriction to the boundary is not required to be constant and, obviously, a general subspace $\mathcal{A}$ is allowed instead of the basepoint $\ast$. We now proceed with a precise formulation.

For a map $f: X \to Y$ that extends the given $\alpha: \mathcal{A} \to Y$, consider

\[ (\partial I \times X) \cup (I \times \mathcal{A}) \to Y \]

\[ \downarrow \]

\[ I \times X \]

where the map at the top is $f$ on each copy of $X$ in $\partial I \times X$ and is the constant homotopy at $\alpha$ on $I \times \mathcal{A}$. For brevity, we denote the resulting set of homotopy classes of maps under the “boundary” $(\partial I \times X) \cup (I \times \mathcal{A})$ by $[I \times X, Y]_f^\mathcal{A}$.

Now we are ready to state a generalization of Theorems A and B.

**Theorem C** Let $X, Y \in \mathcal{A}/\sSet$ be given on the input, with all $A, X, Y$ finite simplicial sets and $Y$ simply connected. Then, algorithms for the following tasks exist:
C.1. Given two simplicial maps \( f, g : X \to Y \) in \( A/\sSet \), decide whether they represent the same element of \( [X, Y]^A \).

C.2. Given a simplicial map \( f : X \to Y \) in \( A/\sSet \), compute the group \( \partial f \).

Theorems A and B are obtained from Theorem C by setting \( A = * \); for the latter, we also set \( f \) constant onto the basepoint.

1.3 Equivariant and Fiberwise Setup

The theorems presented here may also be extended to

– The equivariant situation, where all spaces are equipped with a free action of a finite group \( G \) and all maps and homotopies are \( G \)-equivariant.
– The fiberwise situation, where all spaces lie over a space \( B \) and all maps and homotopies are fiberwise.

However, the correct statement of Theorem C would become rather complicated and, for this reason, we will not deal with such an extension. The extended version of Theorem C could be proved in exactly the same way as the non-extended version, using the results of [5], which treats both the equivariant and the fiberwise setup.

1.4 Notation

In the rest of the paper, the following notation will be employed: We denote the standard \( n \)-simplex by \( \Delta^n \), its \( i \)-th vertex by \( i \), its \( i \)-th face by \( d_i \Delta^n \) and its boundary by \( \partial \Delta^n \). The \( i \)-th horn in \( \Delta^n \), i.e., the simplicial subset spanned by the faces \( d_j \Delta^n \), \( j \neq i \), will be denoted \( \Lambda^n_i \). For simplicity, we will also denote \( I = \Delta^1 \). Then, \( \partial I_k \) is the obvious boundary of the \( k \)-cube, i.e., of the \( k \)-fold product \( I_k = I \times \cdots \times I \).

We will say that a map \( T \times X \to Y \) is constant at \( f : X \to Y \) if it equals the composition \( T \times X \xrightarrow{pr} X \xrightarrow{f} Y \). More generally, we use this terminology for maps defined on subspaces of \( T \times X \). Thus, e.g., \( [I \times X, Y]^f \) is the set of homotopy classes of maps \( I \times X \to Y \) whose restriction to \( (\partial I \times X) \cup (I \times A) \) is constant at \( f \).

1.5 Outline of the Proof

Let \( P_n \) be the \( n \)th stage of the Postnikov tower for \( Y \). Then, \( [X, Y]^A \cong [X, P_n]^A \) for \( n \geq \dim X \) and, thus, the problems about \( [X, Y]^A \) can be translated to \( [X, P_n]^A \), where they are solved inductively—this is the idea of the classical obstruction theory. The proofs of Theorems C.1 and C.2 are interrelated through an “exact sequence”

\[
[I \times X, P_{n-1}]_f \overset{d_f}{\longrightarrow} H^n(X, A; \pi_n) \overset{a}{\longrightarrow} [X, P_n]^A \overset{p_n^*}{\longrightarrow} [X, P_{n-1}]^A \overset{k_n^*}{\longrightarrow} H^{n+1}(X, A; \pi_n)
\]

An extension to non-free actions is a work in progress by the authors of this paper.
In fact, this is the bottom part of the long exact sequence of homotopy groups for the fibration map \((X, P_n) \to \text{map}(X, P_{n-1})\). Notably, this sequence concerns the \(\pi_0\)-sets and we want to talk about components different from the natural basepoint so that this part of the exact sequence has to be described carefully and the resulting structure is somewhat complicated—it contains (parametrized collections of) non-abelian groups, sets and actions of groups on sets.

From the computational point of view, almost nothing is decidable for \([X, P_n]^A\)—even non-emptiness of this set is undecidable, see [3]. On the other hand, the cohomology groups can be computed easily using the Smith normal form. The remaining group \([I \times X, P_{n-1}]^A_f\) is, however, non-abelian and has to be dealt with using methods different from those of the preceding papers [2] and [5].

To obtain Theorems C.1, we want to decide whether two elements \(f, g \in [X, P_n]^A\) are equal. By induction, we decide whether the images in \([X, P_{n-1}]^A\) are equal. If this is the case, the original elements differ by an action by an element of \(H^n(X, A; \pi_n)\) and we have to decide whether this element lies in the stabilizer. The exactness gives that the stabilizer is the image of \(d_f\), which can be computed from the knowledge of generators of the domain of \(d_f\)—this is achieved by Theorem C.2.

To obtain Theorem C.2, the same sequence is applied to \(I \times X\) instead of \(X\), in which case it consists of groups and group homomorphisms, and the middle term, i.e., the desired \([I \times X, P_n]^A\), is computed from the remaining terms using suitable computational tools for groups—the involved groups are all polycyclic and such tools are standard. (For reader’s convenience, we outline the relevant tools later in this paper.)

### 2 Postnikov Towers

The proof of Theorem C relies on computations in a Postnikov tower of \(Y\). Its algorithmic construction has been carried out in [4]. Here, we only give a brief summary of the main results concerned with its construction and with computations in the tower.

#### 2.1 Definition of Postnikov Tower

Let \(Y\) be a simply connected simplicial set. A (simplicial) Postnikov tower for \(Y\) is a commutative diagram

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3 In the non-relative setup, i.e., for \(A = \emptyset\), the set \([X, P_n]\) always contains the homotopy class of the constant map. The question whether this is the only homotopy class, i.e., whether \([X, P_n]\) has a single element, is a problem whose decidability is an open question.
satisfying the following conditions:

- The induced map $\varphi_n: \pi_i(Y) \to \pi_i(P_n)$ is an isomorphism for $0 \leq i \leq n$.
- $\pi_i(P_n) = 0$ for $i \geq n + 1$.
- The space $P_n$ is a principal twisted Cartesian product over $P_{n-1}$, necessarily with fiber the Eilenberg–MacLane space $K(\pi_n, n)$. Equivalently, there exists a pullback square

$$
\begin{array}{ccc}
P_n & \xrightarrow{q_n} & WK(\pi_n, n) \\
\downarrow{p_n} & & \downarrow{\delta} \\
P_{n-1} & \xrightarrow{\kappa_n} & \overline{WK}(\pi_n, n)
\end{array}
$$

(1)

identifying $P_n$ with the pullback $P_{n-1} \times_{\overline{WK}(\pi_n, n)} WK(\pi_n, n)$.

In the above, $\overline{WK}(\pi_n, n)$ is the classifying space and $WK(\pi_n, n)$ the universal principal twisted Cartesian product over it or, more precisely, their standard simplicial models with $\overline{WK}(\pi_n, n) = K(\pi_n, n+1)$ a minimal complex and $\delta$ a minimal fibration, see [9]. We will be making use of a natural isomorphism

$$\text{map}(X, WK(\pi, n)) \cong C^n(X; \pi)$$

between simplicial maps $X \to WK(\pi, n)$ and normalized cochains on $X$ with coefficients in $\pi$. It restricts to an isomorphism $\text{map}(X, K(\pi, n)) \cong Z^n(X; \pi)$. The passage between simplicial maps and cochains/cocycles is computable in both directions, see [2]; in this paper, we call a map computable if an algorithm is provided that computes the image of any given element of the domain—this involves specifying a way of encoding elements of both domain and codomain in a computer, so that they may serve as inputs and outputs of such an algorithm.

From the computational perspective, the Postnikov tower faces the following problem: The standard simplicial models for the Postnikov stages, although minimal, are generally infinite. This is solved by a somewhat technical notion of a simplicial set with effective homology that was introduced by Sergeraert et al. A detailed exposition is given in [10] and an extension to free actions of a finite group $G$ in [5]. We will not
need an explicit definition here\(^4\)—we will use directly only a small part, namely that all simplices have a specified representation in a computer. Thus, it makes sense to speak about computability of maps involving the Postnikov stages. As a special case, a simplicial map \(X \to P_n\) can be easily given by a finite amount of data—the table listing the images of all non-degenerate simplices of \(X\). The effective homology for Postnikov stages is proved in [4, Theorem 1]; here, we only state the conclusions that will be needed.

**Theorem 1** [4, Theorem 1] There is an algorithm that, given a finite simply connected simplicial set \(Y\) and an integer \(n\), constructs the first \(n\) stages of a Postnikov tower for \(Y\). The maps \(\varphi_i, k_i\) and \(p_i\) are computable.

The algorithms for these computable maps use, in an essential way, a precomputed finite amount of data describing the homotopy groups of \(Y\), the Postnikov invariants (i.e., the twisting of \(P_n\) over \(P_{n-1}\)), etc. The computation of these data is what we mean by the construction of the stages of the Postnikov tower.

From now on, we will assume that \(\iota\) is an inclusion; if this was not the case, simply replace the space \(X\) by the mapping cylinder of \(\iota\), i.e., the space \(X_{\text{cof}} = (I \times A) \cup_\iota X\).

The following Whitehead-type theorem is standard, and its fiberwise equivariant version is proved in [5, Theorem 3.3].

**Theorem 2** The map \(\varphi_n: Y \to P_n\) induces a bijection \(\varphi_{n*}: [X, Y]^A \to [X, P_n]^A\) for every \(n\)-dimensional simplicial set \(X\). \(\square\)

In the above theorem, \(P_n\) is a space under \(A\) via the composition \(\alpha_n = \varphi_n \alpha\). Since \(P_n\) is a Kan complex, the homotopy classes in \([X, P_n]^A\) are represented by simplicial maps \(X \to P_n\) under \(A\) (no replacement needed). Applying the theorem to \(I \times X\), we obtain, for \(n \geq \dim(I \times X) = 1 + \dim X\), an isomorphism \([I \times X, Y]_f^A \cong [I \times X, P_n]_{f_n}\), where \(f_n = \varphi_n \alpha\). When there is no risk of confusion, we will denote the above composites \(\alpha_n\) and \(f_n\) simply as \(\alpha\) and \(f\).

2.2 Computations with Postnikov Towers

The group structure on \([I \times X, Y]_f^A\) is defined in terms of concatenation of homotopies. In addition, homotopy lifting will be used heavily in our algorithm and, thus, we need algorithmic versions of such tasks. They will be instances of a general algorithm for lifting maps up one stage of a Postnikov tower, which could thus be thought of as an algorithmic version of the obstruction theory. In the statements below, a *diagonal* is a map indicated by the dashed arrow for which both triangles commute.

\[^4\] To give a rough idea, the effective homology of a space \(X\) consists of a computable chain homotopy equivalence \(C_\ast X \simeq C_\text{ef}^\ast X\), i.e., there are algorithms computing chain maps in both directions and algorithms computing chain homotopies; the crucial requirement is that, in each dimension, \(C_\text{ef}^\ast X\) should be a finitely generated abelian group. In this way, one may compute within this chain complex effectively and thus, using the provided chain homotopy equivalence, also within \(C_\ast X\), at least on homological level.

Of course, finite simplicial sets are easily provided with effective homology, since the identity on \(C_\ast X\) is sufficient, but many infinite simplicial sets also admit effective homology, e.g., the Eilenberg–MacLane spaces \(K(\pi, n)\). The strength of the notion of effective homology lies in the fact that many constructions with spaces can be also performed on the level of effective homology, so that, for example, a (twisted) Cartesian product of spaces with effective homology is again a space with effective homology.
Proposition 1  There is an algorithm that, given a diagram

\[
\begin{array}{ccc}
A & \rightarrow & P_n \\
\downarrow & \searrow & \downarrow p_n \\
X & \rightarrow & P_{n-1}
\end{array}
\]

with \(X\) finite decides whether a diagonal exists. If it does, it computes one.

If \(H^{n+1}(X, A; \pi_n) = 0\), a diagonal exists for any such square (the associated obstruction is necessarily zero).

Proof  Composing with the defining pullback square (1), we obtain an equivalent problem

\[
\begin{array}{ccc}
A & \rightarrow & WK(\pi_n, n) \\
\downarrow & \searrow & \downarrow \delta \\
X & \rightarrow & \overline{WK}(\pi_n, n)
\end{array}
\]

Thinking of \(c\) as a cochain in \(C^n(A; \pi_n)\), we extend it to a cochain on \(X\) by mapping all \(n\)-simplices not in \(A\) to zero. This prescribes a map \(\tilde{c} : X \rightarrow WK(\pi_n, n)\) that is a solution of the lifting-extension problem from the statement for \(z\) replaced by \(\delta \tilde{c}\). Since the lifting-extension problems and their solutions are additive, one may subtract this solution from the previous problem and obtain an equivalent problem (the equivalence is described below in more detail)

\[
\begin{array}{ccc}
A & \rightarrow & WK(\pi_n, n) \\
\downarrow & \searrow & \downarrow \delta \\
X & \rightarrow & \overline{WK}(\pi_n, n)
\end{array}
\]

A solution of this problem is a relative cochain \(c_0 \in C^n(X, A; \pi_n)\) whose coboundary is \(z_0 = z - \delta \tilde{c}\); this \(c_0\) yields a solution \(c_0 + \tilde{c}\) of the original problem. (The class \([z_0] \in H^{n+1}(X, A; \pi_n)\) is thus the sole obstruction to the existence of a lift.) Since \(C^*(X, A; \pi_n)\) is a cochain complex of finitely generated abelian groups, a computation as above (decide whether an element lies in the image of \(\delta\) and compute a preimage under \(\delta\)) is possible using the Smith normal form or using Lemma 1.

\[\square\]

2.3 Lifting and Concatenating Homotopies

The following two special cases apply even to lifting through multiple stages since the diagonals in question always exist (and, in fact, are unique up to an appropriate notion of homotopy): In both cases, the map on the left of the square, denoted here for
simplicity, \( \hat{A} \sim \hat{X} \), is a weak homotopy equivalence (in each case, \( \hat{A} \) is a deformation retract of \( \hat{X} \)) and thus the cohomology groups \( H^{k+1}(\hat{X}, \hat{A}; \pi_k) \) are zero. In the case \( m = 0 \), the Postnikov stage \( P_m \) is a point and as such may be ignored; the propositions then speak simply about extensions of maps to \( P_n \).

**Proposition 2** (Homotopy lifting/extension) Given a diagram

\[
\begin{array}{c}
(i \times X) \cup (\Delta^1 \times A) \\
\sim \\
\Delta^1 \times X \\
\longrightarrow \\
\downarrow \\
\end{array}
\]

with \( X \) finite, where \( i \in \{0, 1\} \), it is possible to compute a diagonal. In other words, one may lift/extend homotopies in Postnikov towers algorithmically.

The second special case is used to concatenate homotopies.

**Proposition 3** (Homotopy concatenation) Given a diagram

\[
\begin{array}{c}
(\Lambda^2_1 \times X) \cup (\Delta^2 \times A) \\
\sim \\
\Delta^2 \times X \\
\longrightarrow \\
\downarrow \\
\end{array}
\]

with \( X \) finite, where \( i \in \{0, 1, 2\} \), it is possible to compute a diagonal. In other words, one may concatenate homotopies in Postnikov towers algorithmically.

We will now use Proposition 3 to make the group operations in \([I \times X, P_n]_f^0\) computable. The situation is later summarized in Proposition 4, after a formal definition of a class of groups with such computational qualities.

Elements of \([I \times X, P_n]_f^0\) are represented by simplicial maps \( I \times X \to P_n \), whose restriction to \( (\partial I \times X) \cup (I \times A) \) is constant at \( f \) (in particular, when viewed as homotopies, they are relative to \( A \)).

Let \( h_2, h_0 : I \times X \to P_n \) be two such maps. Viewing each \( h_i \) as defined on \( d_1 \Delta^2 \times X \), we obtain a single map \( \Lambda^2_1 \times X \to P_n \) which, together with the map \( \Delta^2 \times A \to P_n \) that is constant at \( \alpha \), prescribes the top map in Proposition 3, in which we take \( m = 0 \) so that \( P_0 = \ast \). Let \( \Delta^2 \times X \to P_n \) be the computed diagonal map. Then, we will call its restriction to \( d_1 \Delta^2 \times X \) the concatenation of \( h_2 \) and \( h_0 \) and denote it by \( h_0 + h_2 \). Zero for the concatenation is clearly the homotopy constant at \( f \). An inverse of a homotopy (or, in fact, subtraction of homotopies) is computed similarly to the addition using \( \Lambda^2_0 \).
3 Polycyclic Groups

3.1 Semi-effective Groups

We want to axiomatize the kind of computational structure with which we equipped \([I \times X, P_n]_{f}\). It is a group, but its elements are given non-uniquely by representatives, i.e., actual simplicial maps, and the group operations are computed in terms of these representatives.

A semi-effective set is a mapping of sets \(S \rightarrow S\), denoted \(\sigma \mapsto [\sigma]\) such that elements of \(S\) have an agreed upon representation in a computer (that we will not specify concretely). We say that \(S\) is represented by \(S\) and the element \(\sigma \in S\) is a representative of \([\sigma]\). This representation of elements is generally non-unique—we may have \([\sigma] = [\tau]\) for \(\sigma \neq \tau\). A mapping \(f : S \rightarrow T\) between semi-effective sets is said to be computable, if there is provided a computable mapping \(\varphi : S \rightarrow T\) that represents \(f\), i.e., such that \(f([\sigma]) = [\varphi(\sigma)]\).

A semi-effective group, which we write additively, is a group \(G\) whose underlying set is semi-effective, represented by \(G\), and whose group operations are computable, i.e., algorithms are provided that compute \((\gamma, \delta) \mapsto \gamma + \delta, \ast \mapsto o, \gamma \mapsto -\gamma\) such that \([\gamma + \delta] = [\gamma] + [\delta], [o] = 0, [-\gamma] = -[\gamma]\).

An important example of a semi-effective group is the cohomology group \(H^n(X, A; \pi, n) \cong [X, K(\pi, n)]A\). It is represented by the set \(Z^n(X, A; \pi, n)\) of relative cocycles or equivalently by the set map \((X, A), (K(\pi, n), 0))\) of simplicial maps \(X \rightarrow K(\pi, n)\) that are zero on \(A\). In this case, much more is true—see Example 1 below.

Returning to the homotopy concatenation, we have already obtained the following result.

Proposition 4 The set \([I \times X, P_n]_{f}\) is a semi-effective group represented by the set of simplicial maps \(I \times X \rightarrow P_n\) whose restriction to \((\partial I \times X) \cup (I \times A)\) is constant at \(f\). \(\square\)

A semi-effective collection of groups, \(G = (G_\sigma)_{\sigma \in S}\), is a parametrized version of a semi-effective group, where a parametrization is a computable mapping \(\varphi : G \rightarrow S\); each fiber \(G_\sigma = \varphi^{-1}(\sigma)\) represents a semi-effective group \(G_\sigma\) and the operations are given, uniformly in \(\sigma \in S\), by computable mappings \(G \times_S G \rightarrow G, S \rightarrow G\) and \(G \rightarrow G\). Using this notion, the above proposition can be reformulated in the following way: \([I \times X, P_n]_{f}\) is a semi-effective collection of groups parametrized by simplicial maps \(f : X \rightarrow Y\).

3.2 Fully Effective Groups

Semi-effective groups and collections of groups provide a “local” computational structure. Without additional knowledge, one is not able to solve global problems, such as the word problem in the group or computing the order of a given element.

Such tasks can be easily solved when the group is represented in some special form, for example, as a product of cyclic groups or as a suitable generalization to polycyclic...
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groups; see [11], references therein or the documentation for the computational algebra systems such as GAP [6] and Magma [7] that have algorithms for many similar tasks implemented.

Since we insist on elements of \([X, P_n]\) to be represented by simplicial maps \(X \to P_n\) rather than a tuple of integers, we define a fully effective group to be a semi-effective group together with a computable isomorphism with a group in a special form as above (e.g., a product of cyclic groups), allowing one to transport any problem, such as computing the order of an element, to the relevant group in the special from, to solve it there and to transport the result back to the original group.

3.3 Fully Effective Abelian Groups

First, we recall from [2] some simple observations regarding computational aspects of abelian groups.

**Definition 1** A semi-effective abelian group \(G\) is said to be *fully effective* if there is given an isomorphism \(\mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_r \cong G\), computable together with its inverse. In detail, denoting by \(g_i \in G\) the element corresponding to the canonical generator of \(\mathbb{Z}/q_i\), this consists of

- An algorithm that outputs a finite list of elements \(g_1, \ldots, g_r \in G\) (given by representatives) and their orders \(q_1, \ldots, q_r \in \{1, 2, \ldots\} \cup \{0\}\) (where \(q_i = 0\) gives \(\mathbb{Z}/q_i = \mathbb{Z}\)),

- An algorithm that, given \(\gamma \in G\), computes integers \(z_1, \ldots, z_r\) so that \([\gamma] = z_1 g_1 + \cdots + z_r g_r\); each coefficient \(z_i\) is unique within \(\mathbb{Z}/q_i\).

We allow \(q_i = 1\) to simplify our arguments later; it is clearly possible to obtain a reduced list with all \(q_i \neq 1\) by throwing out from the list the generators \(g_i\) with \(q_i = 1\).

**Example 1** As explained, \(H^n(X, A; \pi)\) is semi-effective when represented by either relative cocycles or simplicial maps \(X \to K(\pi, n)\) that are zero on \(A\). It is also fully effective, since one may compute the isomorphism type of the cohomology groups using a Smith normal form algorithm applied to the differentials in the cochain complex (see, e.g., [12]).

**Lemma 1** (Kernel and Cokernel, [2, Lemmas 2.2 and 2.3]) Let \(f: G \to H\) be a computable homomorphism of fully effective abelian groups. Then, both \(\ker f\) and \(\coker f\) can be represented as fully effective abelian groups. More generally, the computation of \(\coker f\) only requires \(H\) fully effective abelian, a list of generators of \(G\) (not necessarily abelian) and \(f\) computable.

Another useful construction is [2, Lemma 2.4] that shows that the class of fully effective abelian groups is closed under extensions. We will not use this result; instead, we will need its generalization to the case of polycyclic groups, namely Proposition 6.

3.4 Fully Effective Polycyclic Groups

The group \([I \times X, Y]\) is generally non-abelian, and we will thus need to extend some of the machinery from abelian groups to a wider class of groups, called polycyclic.
According to Theorem 2, \([I \times X, Y]_f^3 \cong [I \times X, P_n]_f^3\) and the latter is a semi-effective group by Proposition 4. In this way, the original group \([I \times X, Y]_f^3\) is also semi-effective, but not represented by maps \(I \times X \to Y\). Of course, this is also in accordance with the fact that \(Y\) is not assumed to be a Kan complex.

**Definition 2** A group \(G\) is called *polycyclic* if it has a subnormal series with cyclic factors. In detail, there exists a sequence of subgroups

\[
G = G_r \geq \cdots \geq G_0 = 0
\]

such that:

- \(G_{i-1}\) is a normal subgroup of \(G_i\) for \(i = 1, \ldots, r\),
- \(G_i/G_{i-1}\) is a cyclic group for \(i = 1, \ldots, r\).

**Example 2** Every finitely generated abelian group is polycyclic: When \(G \cong \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_r\) with the corresponding generators \(g_1, \ldots, g_r\), the filtration is given by the subgroups \(G_i\) generated by \(g_1, \ldots, g_i\).

Suppose that elements \(g_i \in G_i\) have been chosen in such a way that their images in \(G_i/G_{i-1}\) are generators of these cyclic groups; clearly, such a choice is possible. Denoting by \(q_i\) the order of \(G_i/G_{i-1}\) \(\cong \mathbb{Z}/q_i\),

\[
\mathbb{Z}/q_1 \times \cdots \times \mathbb{Z}/q_r \to G
\]

\[(z_1, \ldots, z_r) \mapsto z_1g_1 + \cdots + z_r g_r
\]

is easily seen to be bijective: Given \(g \in G\), consider its image \(z_r \in G_r/G_{r-1} \cong \mathbb{Z}/q_r\). Then \(g - z_r g_r \in G_{r-1}\) and we continue in the same manner to show that \(g - z_r g_r - \cdots - z_1 g_1 \in G_0 = 0\), i.e., \(g = z_1g_1 + \cdots + z_r g_r\) in a unique way. In particular, \(G\) is generated by \(g_1, \ldots, g_r\). At the same time, the *word problem* in \(G\), i.e., the problem of deciding whether two given words in the generators \(g_i\) are equal, can be translated to \(\mathbb{Z}/q_1 \times \cdots \times \mathbb{Z}/q_r\) and easily solved there. This leads to the following definition:

**Definition 3** We say that a semi-effective group \(G\), represented by a set \(G\), is *fully effective polycyclic* if it is polycyclic with subnormal series (2) and a bijection \(\mathbb{Z}/q_1 \times \cdots \times \mathbb{Z}/q_r \cong G\) as above is computable together with its inverse. In detail, this consists of

- An algorithm that outputs a finite list of elements \(g_1 \in G_1, \ldots, g_r \in G_r\) (given by representatives) and the orders \(q_1, \ldots, q_r \in \{1, 2, \ldots\} \cup \{0\}\) of \(G_i/G_{i-1}\) (where \(q_i = 0\) gives \(\mathbb{Z}/q_i = \mathbb{Z}\)),
- An algorithm that, given \(\gamma \in G\), computes integers \(z_1, \ldots, z_r\) so that \([\gamma] = z_1g_1 + \cdots + z_r g_r\); each coefficient \(z_i\) is unique within \(\mathbb{Z}/q_i\).

As explained just prior to the definition, the algorithm in the second point is equivalent to the computability of the projections \(p_i : G_i \to G_i/G_{i-1} \cong \mathbb{Z}/q_i\).
Remark 1 In fact, it is even possible to specify (the isomorphism type of) the whole group by a finite amount of data: This consists of the conjugates \( g_i + g_j - g_i \in G_{i-1} \) for \( i > j \) and the multiples \( q_i g_i \in G_{i-1} \).

Proposition 5 Let \( G \) be a fully effective polycyclic group, \( H \) a fully effective abelian group and \( f : G \to H \) a computable homomorphism. Then, it is possible to compute \( K = \ker f \) as a fully effective polycyclic group.

Proof We will proceed by induction with respect to the length \( r \) of the subnormal series for \( G \). We denote \( K_i = \ker f|_{G_i} = G_i \cap K \). In the following diagram, every row is a short exact sequence and so are the solid columns.

\[
\begin{array}{ccccccccc}
0 & \to & K_{r-1} & \to & K_r & \to & K_r / K_{r-1} & \to & 0 \\
0 & \to & G_{r-1} & \to & G_r & \to & G_r / G_{r-1} & \to & 0 \\
0 & \to & f(G_{r-1}) & \to & f(G_r) & \to & f(G_r) / f(G_{r-1}) & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

It is easy to see that the dashed column is then also exact (e.g., by the snake lemma). By induction, \( K_{r-1} \) is fully effective polycyclic. By Lemma 1, it is possible to compute \( \ker f' \cong K_r / K_{r-1} \); say that it is generated by \( t_r \in G_r / G_{r-1} \cong \mathbb{Z}/q_r \). This means that \( f(t_r g_r) \in f(G_{r-1}) \) and thus, from the knowledge of the generators of \( G_{r-1} \), it is possible to compute some \( h \in G_{r-1} \) with \( f(t_r g_r) = f(h) \). Finally, \( -h + t_r g_r \in K_r \) is the required element mapping to the generator \( t_r \in K_r / K_{r-1} \subseteq G_r / G_{r-1} \). The projection \( K_r \to K_r / K_{r-1} \cong \mathbb{Z}/(q_r t_r^{-1}) \) is the composition

\[
K_r \xrightarrow{c} G_r \xrightarrow{f} G_r / G_{r-1} \cong \mathbb{Z}/q_r \xrightarrow{t_r^{-1} \times} \mathbb{Z}/(q_r t_r^{-1})
\]

(the multiplication by \( t_r^{-1} \) is defined on the image of \( K_r \)) and is thus computable. \( \square \)

Finally, we show that fully effective polycyclic groups are closed under extensions. We state a more general version of an exact sequence for pointed sets, in which “\( \ker d_n \)” is interpreted as the equivalence relation associated with the mapping \( d_n \):

Definition 4 A semi-effective exact sequence of pointed sets is an exact sequence

\[
\cdots \longrightarrow G_{n+1} \xrightarrow{d_{n+1}} G_n \xrightarrow{d_n} G_{n-1} \xrightarrow{d_{n-1}} G_{n-2} \longrightarrow \cdots
\]
of semi-effective pointed sets and computable pointed maps such that the induced maps

\[ d_n : G_n / \ker d_n \xrightarrow{\sim} \im d_n \]

have computable inverses, called sections. Since \( G_n / \ker d_n \) is represented by \( \mathcal{G}_{n-1} \), this amounts to computable partial mappings \( \rho_{n-1} : \mathcal{G}_{n-1} \rightarrow \mathcal{G}_n \), defined on representatives of \( \im d_n \), such that \( d_n[\rho_{n-1}(\gamma)] = [\gamma] \). In general, it may happen that \([\gamma] = [\gamma']\), while \([\rho_{n-1}(\gamma)] \neq [\rho_{n-1}(\gamma')]\).

When the sequence in question is bounded from either side, we ask the condition for all inner maps. (This is the case in our major example.)

A semi-effective exact sequence (of groups) has all \( G_n \) semi-effective groups and all \( d_n \) homomorphisms of groups (there is no condition on sections \( \rho_{n-1} \)). In this case, one may freely pass between images and kernels and, thus, one may write \( d_n : \coker d_{n+1} \rightarrow \ker d_{n-1} \).

**Proposition 6** Given a semi-effective short exact sequence of groups

\[ 0 \rightarrow K \xrightarrow{f} G \xrightarrow{g} H \rightarrow 0 \]

with \( K, H \) fully effective polycyclic, it is possible to equip \( G \) with a structure of a fully effective polycyclic group.

**Proof** We denote the inverses \( \rho \) and \( \sigma \). Since \( f \) is injective, \( \rho \) induces a well-defined mapping (retraction) \( r : f(K) \rightarrow K \). Let \( H = H_s \geq \cdots \geq H_0 = 0 \) and \( K = K_t \geq \cdots \geq K_0 = 0 \) be subnormal series. Then, we have the following subnormal series

\[ G = g^{-1}(H_s) \geq \cdots \geq g^{-1}(H_0) = f(K_t) \geq \cdots \geq f(K_0) = 0 \]

for \( G \) with filtration quotients either \( H_i / H_{i-1} \) or \( K_j / K_{j-1} \), the corresponding projections

\[ g^{-1}(H_i) \xrightarrow{g} H_i \rightarrow H_i / H_{i-1}, \]

\[ f(K_j) \xrightarrow{f} K_j \rightarrow K_j / K_{j-1}, \]

and generators either \([\sigma(\eta_i)]\), where \( \eta_i \) represents the generator \( h_i \in H_i \), or \( f(k_j) \), where \( k_j \in K_j \) is the generator. \( \square \)

**4 Proof of Theorem C**

We fix a map \( f : X \rightarrow Y \) under \( A \) as in the statement. We introduce the following abbreviations

\[ K_n = K(\pi_n, n), \quad W K_n = W K(\pi_n, n), \quad WK_n = WK(\pi_n, n), \quad \mathcal{G}_{n,k} = [I^k \times X, P_n]_f, \]

\[ \text{Springer} \]
the last being the set of homotopy classes relative to \((\partial I^k \times X) \cup (I^k \times A)\) of maps, whose restriction to this subspace is constant at \(f\).

First, we state a number of claims. They all depend on two nonnegative integers \(n\) and \(k\); for (gen) and (poly), we require \(k \geq 1\).

(\text{gen})_{n,k} \quad \text{It is possible to compute generators of} \ G_{n,k}.

(htpy)_{n,k} \quad \text{For a given} \ g_n : I^k \times X \to P_n, \text{it is possible to compute a homotopy} \ f_n \sim g_n, \text{relative to} \ (\partial I^k \times X) \cup (I^k \times A), \text{from the map constant at} \ f_n \text{to} \ g_n \text{\(a \ “\text{nullhomotopy}”\)} \text{or decide that such a homotopy does not exist.}

(poly)_{n,k} \quad \text{It is possible to equip} \ G_{n,k} \text{with a structure of a fully effective polycyclic group.}

By (gen)_n, we will understand that (gen)_{n,k} holds for all \(k\) and similarly for other claims. We will now stick to (gen)_n, etc., and return to the refined claims later.

By Theorem 2, in order to prove Theorem C.1, it is enough to prove (htpy)_{n,0} for \(n = \dim X\) and, in order to prove Theorem C.2, it is enough to prove (poly)_{n,1} for \(n = 1 + \dim X\). We prove the claims inductively in the following way:

(\begin{tikzcd}
(\text{gen})_{n-1} & (\text{gen})_n \\
(\text{htpy})_{n-1} & (\text{htpy})_n \\
(\text{poly})_{n-1} & (\text{poly})_n
\end{tikzcd})

To show the claimed implications, we use the following theorem, that will be proved later:

\textbf{Theorem 3} \quad \text{Let} \ f : X \to Y \text{be a map in } A/\text{ssSet}. \text{Then, there is an exact sequence of groups and pointed sets}

\[ [I \times X, P_{n-1}]^A_f \xrightarrow{d_f} H^n(X, A; \pi_n) \xrightarrow{a} [X, P_n]^A \xrightarrow{p_{n*}} [X, P_{n-1}]^A \xrightarrow{k_{n*}} H^{n+1}(X, A; \pi_n) \]

where the basepoints of all sets of homotopy classes of maps to Postnikov stages are represented either by \(f\) or the constant map at \(f\). Assuming (htpy)_{n-1,0}, it is a semi-effective exact sequence.

Applying Theorem 3 to the pair \((I^k \times X, (\partial I^k \times X) \cup (I^k \times A))\) in place of \((X, A)\), we get an exact sequence

\[ \mathbb{G}_{n-1,k+1} \xrightarrow{d_f} H^{n-k}(X, A; \pi_n) \xrightarrow{a} \mathbb{G}_{n,k} \xrightarrow{p_{n*}} \mathbb{G}_{n-1,k} \xrightarrow{k_{n*}} H^{n+1-k}(X, A; \pi_n) \]

(3)

that is semi-effective under (htpy)_{n-1,k}. Also, by the last paragraph of the proof of Theorem 3, it is an exact sequence of groups.

\textbf{Proof of} (poly)_n \Rightarrow (gen)_n \text{ This is clear.}

\textbf{Proof of} (gen)_{n-1} + (htpy)_{n-1} \Rightarrow (htpy)_n \text{ We denote by} f_i \text{and} g_i \text{the projections of} f \text{and} g \text{to the} \ i \text{-th stage of the Postnikov tower. First, using (htpy)_{n-1}, we compute}

\[ \begin{tikzcd}
\vdots \\
\vdots \\
\vdots
\end{tikzcd} \]
a homotopy \( h': f_{n-1} \sim g_{n-1} \) or decide that \( h' \) does not exist. Next, we lift \( h' \) using Proposition 2 to a homotopy \( \tilde{h}' : f'_n \sim g_n \). Since \( p_n f'_n = p_n f_n \), we may write \( f'_n = f_n + \zeta_n \) for a unique map \( \zeta_n : I^k \times X \to K_n \), zero on the “boundary.” We use \( \text{(gen)}_{n-1} \) to decide whether \( [\zeta_n] \in \text{im} \, d_f \) and to further compute \( h'' \) with \( d_f[h''] = [\zeta_n] \). Using Proposition 7, it is possible to find a lift \( \tilde{h}'' \) that starts at \( f_n \) and finishes at \( f''_n \); it is computed using Proposition 1. Thus, the concatenation \( h = \tilde{h}' + \tilde{h}'' \), computed by Proposition 3, is a homotopy from \( f_n \) to \( g_n \). If any of \( h' \), \( h'' \) fails to exist, the maps \( f_n \), \( g_n \) are not homotopic.

(Proof of (htpy)\(_{n-1} + \text{(poly)}_{n-1} \) \( \Rightarrow \) \( \text{(poly)}_n \)) The exact sequence (3) induces a short exact sequence

\[
0 \longrightarrow \text{coker } d_f \longrightarrow \mathbb{G}_{n,k} \xrightarrow{p_{n\ast}} \text{ker } k_{n\ast} \longrightarrow 0
\]

that is still semi-effective—this is easily seen by viewing the sections as maps from the kernel to the cokernel. The first term of the short exact sequence is fully effective abelian by Lemma 1, and the last term is fully effective polycyclic by Proposition 5; both claims use \( \text{(poly)}_{n-1} \). The proof now follows from Proposition 6.

**Remark 2** In more detail, we have proved implications

\[
\text{(gen)}_{n-1,k+1} + \text{(htpy)}_{n-1,k} \Rightarrow \text{(htpy)}_{n,k}
\]

\[
\text{(gen)}_{n-1,k+1} + \text{(htpy)}_{n-1,k} + \text{(poly)}_{n-1,k} \Rightarrow \text{(poly)}_{n,k}
\]

(for coker \( d_f \), generators of \( \mathbb{G}_{n-1,k+1} \) are sufficient).

There is a further implication \( \text{(gen)}_{n-1,k} \Rightarrow \text{(gen)}_{n,k} \) that works only for \( k \geq 2 \): One may compute generators of \( \mathbb{G}_{n,k} \) from those of \( \text{ker } k_{n\ast} \) and coker \( d_f \); clearly, coker \( d_f \) is generated by the images of generators of \( H^{n-k}(X, A; \pi_n) \) and it is not too difficult to compute a set of generators of \( \text{ker } k_{n\ast} \) from a set of generators of \( \mathbb{G}_{n-1,k} \) (denoting these generators \( g_{\alpha} \in \mathbb{G}_{n-1,k} \), compute relations between the \( k_{n\ast}(g_{\alpha}) \in H^{n+1-k}(X, A; \pi_n) \); viewing these as certain integral combinations of the \( k_{n\ast}(g_{\alpha}) \) being zero, the corresponding combinations of the \( g_{\alpha} \) generate \( \text{ker } k_{n\ast} \); in the non-abelian case \( k = 1 \), this does not work, since each relation gives rise to infinitely many combinations of the \( g_{\alpha} \), varying in the ordering of summands).

To summarize, for \( n = \text{dim } X \), it is possible to organize the algorithm of Theorem C.2, i.e., the claim \( \text{(poly)}_{n+1,1} \), in such a way that, in its course, we only use \( \text{(gen)}_{m,2} \), for \( m \leq n \), and \( \text{(poly)}_{m,1} \), for \( m \leq n+1 \). Similarly, to obtain Theorem C.2, i.e., the claim \( \text{(htpy)}_{n,0} \), we invoke \( \text{(poly)}_{n-1,1} \) and, thus, we only use \( \text{(gen)}_{m,2} \), for \( m \leq n-2 \), and \( \text{(poly)}_{m,1} \), for \( m \leq n-1 \).

### 5 Proof of Theorem 3

First, we introduce a general “exact sequence” that relates the sets of homotopy classes of maps to consecutive stages of a Postnikov tower and that does not depend on the choices of basepoints.
5.1 Exact Sequences

Our (semi-effective) exact sequence will take the following form

\[ G \xrightarrow{d} H \xrightarrow{s} D \xrightarrow{t} E \xrightarrow{f} F \]

where \( D \) and \( E \) are semi-effective sets, \( F \) a semi-effective pointed set with basepoint \([o] \in F\), \( H \) a semi-effective group, and \( G \) a semi-effective collection of groups \( G_\varepsilon \) indexed by \( \varepsilon \in \mathcal{E} \). The maps \( s \) and \( t \) are computable maps of sets, represented by \( \sigma \) and \( \tau \), the arrow at \( D \) denotes a computable action of \( H \) on \( D \), and \( d \) is a computable collection of group homomorphisms \( d_\delta : G_{\sigma(\delta)} \rightarrow H \) indexed by \( \delta \in \mathcal{D} \).

**Definition 5** We say that the above sequence is exact if

- \( \text{im } s = t^{-1}([o]) \),
- \( s(d) = s(d') \) if and only if \( d, d' \) lie in the same orbit of the \( H \)-action, i.e., \( d' = d + h \) for some \( h \in H \), and
- the stabilizer of \([\delta] \in D\) is exactly the image of \( d_\delta \).

We may construct out of this sequence an ordinary exact sequence of semi-effective pointed sets and computable maps (it is not semi-effective—generally, the sections are not computable) in the following way: choose a basepoint \( \delta \in \mathcal{D} \) and then consider

\[ G_{\sigma(\delta)} \xrightarrow{d_\delta} H \xrightarrow{a} D \xrightarrow{s} E \xrightarrow{f} F \]

with \( a(h) = [\delta] + h \), the action of \( H \) on the fixed element \([\delta]\). It is easily seen to be really exact, where \( G_{\sigma(\delta)} \) and \( H \) are equipped with the respective zeros as basepoints, \( D \) with basepoint \([\delta]\), \( E \) with basepoint \([\sigma(\delta)]\) and \( F \) with the given element \([o] \in F\).

5.2 Exact Sequence Relating Consecutive Stages

By composing \( \alpha : A \rightarrow Y \) with various maps in the Postnikov tower of \( Y \), we make all \( P_n, P_{n-1}, WK_n \) and \( \overline{WK}_n \) into spaces under \( A \). Further, \( K_n \) is considered as a space under \( A \) via the constant map onto the zero of \( K_n \). For the purpose of the description of the exact sequence, we will denote maps \( X \rightarrow P_{n-1} \) by \( \ell_{n-1}, \ell'_{n-1} \), etc. and maps \( X \rightarrow P_n \) by \( \ell_n, \ell'_n \), etc. Our main exact sequence is

\[ [I \times X, P_{n-1}] \xrightarrow{\delta} [X, K_n]^A \bigcup [X, P_n]^A \xrightarrow{\beta} [X, P_{n-1}]^A \xrightarrow{\gamma} [X, \overline{WK}_n]^A, \]

where the first term is a collection of groups, indexed by maps \( \ell_{n-1} : X \rightarrow P_{n-1} \) under \( A \), with the corresponding group \([I \times X, P_{n-1}]_{\ell_{n-1}}^\delta \) as in Proposition 4, i.e., the group of homotopy classes of maps whose restriction to \((\partial I \times X) \cup (I \times A)\) is constant at \( \ell_{n-1} \). The element \([o] \in [X, \overline{WK}_n]^A \) is the only homotopy class in the image of \( \delta_{ns} : [X, WK_n]^A \rightarrow [X, \overline{WK}_n]^A \). (Since \( WK_n \) is contractible, there is a unique homotopy class \( X \rightarrow WK_n \), obtained by extending the given \( A \rightarrow WK_n \) arbitrarily, see, e.g., the proof of Proposition 1.)
The maps $p_n*$ and $k_{n*}$ are induced by $p_n$ and $k_n$, respectively. The action is also induced by the action of $K_n$ on $P_n$. Both maps and the action are clearly computable. It remains to describe the homomorphisms $d_{\ell_n} : [I \times X, P_{n-1}]^3_{p_n(\ell_n)} \to [X, K_n]^A$.

Starting with a homotopy $h : I \times X \to P_{n-1}$ as above, lift it to a homotopy $\tilde{h} : I \times X \to P_n$ so that its restriction to $(0 \times X) \cup (I \times A)$ is constant at the given map $\ell_n$. The restriction of $\tilde{h}$ to $1 \times X$ is then of the form $\ell_n + \zeta$ for a unique $\zeta : X \to K_n$ and we set $d_{\ell_n}[h] = [\zeta]$; a well-defined map according to Proposition 7 below. Each map $d_{\ell_n}$ is computable by Proposition 2. It is also a group homomorphism, since a concatenation of homotopies is computed, as in Proposition 3, using the lift in $\ell$, starting at $\tilde{h}$ relative to $A$, implying $\zeta \sim \zeta'$; thus, $d_{\ell_n}$ is well defined.

For the second part, concatenating the homotopy $\tilde{h} : \ell_n \sim \ell_n + \zeta$, with the homotopy $\ell_n + \zeta \sim \ell_n + \zeta'$, we obtain $\tilde{h}' : \ell_n \sim \ell_n + \zeta'$. If the concatenation of homotopies is computed, as in Proposition 3, using the lift in $\ell$, then this concatenation will also be a lift of $h$; here $s^1 : \Delta^2 \to \Delta^1$ is the map sending the non-degenerate 2-simplex of $\Delta^2$ to the $s_1$-degeneracy of the non-degenerate 1-simplex of $\Delta^1$.

5.3 Proof of Exactness

The exactness at $[X, P_{n-1}]^A$ means that $\ell_{n-1} : X \to P_{n-1}$ lifts to $P_n$ if and only if the composition $k_{n}(\ell_{n-1})$ factors (up to homotopy) through $WK_n$ (the basepoint of $[X, \overline{WK}_n]^A$ is the unique such homotopy class) and is thus clear.

The exactness at $[X, P_n]^A$ means that, given two maps $\ell_n, \ell'_n : X \to P_n$, their projections to $P_{n-1}$, denoted $\ell_{n-1} = p_n\ell_n, \ell'_{n-1} = p_n\ell'_n$, are homotopic if and only if $\ell_n + \zeta \sim \ell'_n$ for some $\zeta : X \to K_n$. By lifting a homotopy $h : \ell_{n-1} \sim \ell'_{n-1}$ to a homotopy $\tilde{h} : \ell_n \sim \ell'_n$, we may replace $\ell'_n$ by a homotopic map $\ell''_n$ in such a way that $\ell_{n-1} = \ell''_{n-1}$ and then the result is clear.
To prove exactness at \([X, K_n]^A\), we observe that every homotopy \(h : \ell_n \sim \ell_n + \zeta\) is a lift of its projection \(p_n h\); therefore, the image of \(d_{\ell_n}\) consists exactly of homotopy classes of maps \(\zeta : X \to K_n\) such that \(\ell_n \sim \ell_n + \zeta\) and this is exactly the claimed exactness.

5.4 Resulting Exact Sequence of Pointed Sets

Let \(f : X \to Y\) be a map in \(A/\text{sSet}\). Applying the general construction to (4) yields an exact sequence of pointed sets

\[
[I \times X, P_{n-1}]_f^a \xrightarrow{df} [X, K_n]^A \xrightarrow{a} [X, P_n]^A \xrightarrow{p_{n*}} [X, P_{n-1}]^A \xrightarrow{k_{n*}} [X, WK_n]^A.
\]

Section of \(p_{n*}\) is computed as in the first paragraph of the proof of exactness using Proposition 1. A section of \(a\) is computed as in the second paragraph of the proof of exactness but with \(\ell_{g-1}\) equal to the basepoint \(f_{n-1}\); we use \((htpy)_{n-1}\) to compute \(h\) and then lift it to \(\tilde{h}\) using Proposition 2. Next, \(\ell''\) is obtained by restriction and the representative \(\zeta\) is obtained as the “difference” \(\ell'' - \ell_n\).

To finish the proof of Theorem 3, we need to identify the second and the last term with cohomology groups. For the second term, we invoke the computable isomorphism \([X, K_n]^A \cong H^n(X, A; \pi_n)\). We recall that \(WK_n\) is made into a space under \(A\) via \(k_n f_{n-1}\), which is generally nonzero, but has an extension \(k_n f_{n-1} : X \to WK_n\). Thus, we get an isomorphism

\[
[X, WK_n]^A \cong H^{n+1}(X, A; \pi_n), \quad [g] \mapsto [g - k_n f_{n-1}]
\]

where \(g - k_n f_{n-1} : X \to WK_n\) is a map that is zero on \(A\) and as such can be thought of as a relative cocycle. The inverse map is obtained by adding \(k_n f_{n-1}\) and, thus, both directions are computable.

5.5 Case of Groups

It remains to show that (3) consists of group homomorphisms for \(k \geq 1\). This has been proved for \(d_f\) with respect to addition in \(K_n\), and the remaining maps are clearly homomorphisms with respect to concatenation of homotopies. By Eckmann–Hilton argument, the various group structures coincide.

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