Herbert Stahl’s proof of the BMV conjecture

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Abstract. This paper contains a simplified version of Stahl’s proof of a conjecture of Bessis, Moussa and Villani on the trace of matrices $A + tB$ with Hermitian $A$ and $B$.

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This paper presents a simplified version of Herbert Stahl’s proof of the BMV conjecture [1]. The proof preserves all the main ideas of Stahl; the simplification consists in technical details.

Theorem. Let $A$ and $B$ be two $n \times n$ Hermitian matrices. Then the function

$$f(t) = \text{Tr} \exp(A - tB)$$ (1)

has a representation

$$f(t) = \int_{b_1}^{b_n} e^{-ts} d\mu(s),$$ (2)

where $\mu$ is a nonnegative measure and $b_1$ and $b_n$ are the smallest and the largest eigenvalues of $B$.

If $B$ is positive semi-definite, it follows that $(-1)^n f^{(n)}(t) \geq 0$, $t \in \mathbb{R}$. Such functions are called absolutely monotone. The result was conjectured in [2]. Two equivalent statements for positive semi-definite matrices $B$ are [3]:

a) the polynomial $t \mapsto \text{Tr}(A + Bt)^p$, $p \in \mathbb{N}$, has all non-negative coefficients, and

b) the function $t \mapsto \text{Tr}(A + tB)^{-p}$, $p > 0$, is absolutely monotone.

Before the work of Stahl, this theorem was known for $2 \times 2$ matrices. A survey of the previous attempts to prove it is contained in [4]. The proof of Stahl, which is explained in these notes, is completely elementary: all the required tools were available in the middle of the 19th century.

Without loss of generality, one can assume that $B$ is a diagonal matrix with eigenvalues $b_n > b_{n-1} > \cdots > b_1 > 0$. This is achieved by simultaneous conjugacy of $A$ and $B$, adding a scalar to $B$, and approximating the resulting $B$ by a matrix whose eigenvalues are distinct.

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Eigenvalues $\lambda$ of $A - tB$ are determined by the equation

$$\det(\lambda I - A + tB) = 0.$$  

This determinant is a polynomial in two variables $t$, $\lambda$. Taking $t$ out of the determinant, and denoting $y = \lambda/t$, $x = 1/t$, we obtain a polynomial equation of the form

$$0 = \det(yI + B - xA) = \prod_{j=1}^{n} (y + b_j - xa_{j,j}) + O(x^2),$$

where $O(x^2)$ is a polynomial divisible by $x^2$.

This implies that there are $n$ holomorphic branches of the multivalued implicit function $\lambda(t)$ in a neighborhood of infinity which satisfy

$$\lambda_j(t) = -b_j t + a_{j,j} + O\left(\frac{1}{t}\right), \quad t \to \infty,$$

and all the $\lambda_j$ are real on the real line. Moreover, each of these branches has an analytic continuation in a region containing the real line, according to Rellich’s theorem ([5], Theorem 12.4). The algebraic function $\lambda(t)$ is defined on a Riemann surface $S$ with $n$ sheets spread over the Riemann sphere. This Riemann surface is not necessarily connected. It has $n$ unramified sheets over a region that contains the real line and a neighborhood of infinity.

**Special case:** suppose that $A$ is also diagonal, then the $O(1/t)$ terms in (3) can be omitted, and we obtain

$$f(t) = \sum_{j=1}^{n} e^{a_{j,j}} e^{-b_j t} = \int_{0}^{\infty} e^{-st} \sum_{j=1}^{n} e^{a_{j,j} \delta_{b_j}(s)} ds.$$  

(4)

Thus $\mu$ is a discrete measure with positive atoms at the eigenvalues of $B$.

In the general case, the discrete component of $\mu$ is the same, and the continuous component is a positive function on $(b_1, b_n)$. Stahl found the following explicit expression for the density.

**Proposition.** The measure

$$d\mu(s) = \left(\sum_{j=1}^{n} e^{a_{j,j} \delta_{b_j}(s)} + w(s)\right) ds,$$

(5)

where

$$w(s) = \frac{1}{2\pi i} \sum_{j:b_j<s} \int_{C} e^{\lambda_j(\zeta) + s\zeta} d\zeta,$$  

(6)

satisfies (1) and (2). Here $C$ is any circle centred at the origin, of sufficiently large radius, described counterclockwise.

Stahl writes that it was nontrivial to guess (5), (6) and gives no explanations. So we include a heuristic argument which could be used to guess this formula. The inversion formula for the Laplace transform gives the density in the form

$$\frac{1}{2\pi i} \int_{L} f(\zeta) e^{s\zeta} d\zeta,$$
where $L$ is a vertical line sufficiently far to the right. For $|\zeta|$ large enough, the expression under the integral equals

$$e^{s\zeta} f(\zeta) = \sum_{j=1}^{n} e^{\lambda_j(\zeta) + s\zeta}.$$  

As $\lambda_j(\zeta) = -b_j\zeta + \cdots$, the summands for which $b_j > s$ are exponentially decreasing in the right half-plane, therefore, for these summands, the line $L$ can be shifted to the right, and all these summands vanish. The rest of the summands decrease exponentially to the left, and for them, the contour can be bent to the left to obtain a circle $C$.

Of course one can give a rigorous justification of these arguments, but once the formula is guessed, it is easy to verify it directly, and we reproduce Stahl’s argument.

**Lemma 1.** For every $s$, we have

$$\sum_{j=1}^{n} \int_{C} e^{\lambda_j(\zeta) + s\zeta} d\zeta = 0.$$  

**Proof.** Indeed, this is an integral of an entire function over a closed contour.

It follows that the density $w$ defined by (6) is zero for $s > b_n$, and it is evidently zero for $s < b_1$.

**Proof of Proposition 1.** We compute the Laplace transform of the density $w$ defined by (6):

$$\int_{0}^{\infty} e^{-st} w(s) \, ds = \sum_{k=1}^{n-1} \int_{b_k}^{b_{k+1}} e^{-ts} w(s) \, ds =: \sum_{k=1}^{n-1} I_k(t).$$

We fix $t > 0$ and deform the contour $C$ in (6) so that the positive ray is outside $C$. This is possible to do because all $\lambda_j$ are holomorphic in a region containing the real line and $C$. Thus $t$ is outside the deformed contour $C'$. According to (6), we have

$$I_k(t) = \int_{b_k}^{b_{k+1}} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta) + s(\zeta - t)} \, d\zeta \, ds.$$

Changing the order of integration and the order of summation, we obtain

$$\sum_{k=1}^{n-1} I_k(t) = \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta + s(-t))} \, ds \, d\zeta$$

$$= \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)} (e^{b_n(\zeta - t)} - e^{b_j(\zeta - t)}) \frac{d\zeta}{\zeta - t}.$$  

The last expression is transformed using Cauchy’s formula and the fact that $t$ is outside $C'$. We have

$$\sum_{j=1}^{n} \int_{C'} e^{\lambda_j(\zeta)} e^{b_n(\zeta - t)} \frac{d\zeta}{\zeta - t} = 0,$$
similarly to Lemma 1, so

\[ \sum_{k=1}^{n-1} I_k(t) = -\sum_{j=1}^{n} \frac{1}{2\pi i} \int_{C_j} e^{\lambda_j(\zeta) + b_j(\zeta-t)} \frac{d\zeta}{\zeta-t}. \]

Using (3) we write

\[ \lambda_j(\zeta) = -b_j\zeta + a_{j,j} + r_j(\zeta), \]

where \( r_j(\infty) = 0 \), and apply Cauchy’s formula again. We obtain for every \( j \):

\[ -\frac{1}{2\pi i} \int_{C_j} e^{\lambda_j(\zeta) + b_j(\zeta-t)} \frac{d\zeta}{\zeta-t} = -\frac{e^{-b_j t + a_{j,j}}}{2\pi i} \int_{C_j} e^{r_j(\zeta)} \frac{d\zeta}{\zeta-t} = e^{-b_j t + a_{j,j}} (e^{r_j(t)} - 1) = e^{\lambda_j(t)} - e^{-b_j t + a_{j,j}}. \]

Adding these expressions for \( j = 1 \ldots n \) and comparing with (5) and the second equation in (4), we obtain Proposition 1.

It remains to prove that (6) is nonnegative for every \( s \). Let us fix \( s \) and \( k \) so that \( b_k < s < b_{k+1} \). The idea of Stahl is to replace the contour of integration in (6) by an ingeniously chosen homologous contour, on which the integral is nonnegative simply because the integrand is nonnegative.

We recall that \( S \) is a (possibly disconnected) Riemann surface spread over the \( \zeta \)-sphere. We denote a generic point of \( S \) by \( p \), and let \( \pi: S \to \mathbb{C} \) be the projection onto the \( \zeta \)-sphere. Then \( \lambda \) is a meromorphic function on \( S \) all of whose poles are simple and lay over \( \zeta = \infty \).

The asymptotic expressions (3) imply that there exists \( R > 0 \) such that for all \( j \leq k \) the functions

\[ \lambda_j(\zeta) + s\zeta = (s - b_j)\zeta + \cdots \]  

(7)

are holomorphic for \( |\zeta| > R \), real on the real line, and have strictly positive derivatives for \( \zeta > R \) and \( \zeta < -R \), while for \( j > k \) they have strictly negative derivatives. By increasing \( R \) if necessary, we achieve that for \( |\zeta| > R/2 \) and \( j \leq k \), we have that \( \text{Im}(\lambda_j(\zeta) + s\zeta) \) has the same sign as \( \text{Im}\zeta \). And for \( |\zeta| > R/2 \) and \( j > k \), \( \text{Im}(\lambda_j(\zeta) + s\zeta) \) has the opposite sign from \( \text{Im}\zeta \).

The surface \( S \) has an anti-conformal involution, induced by complex conjugation. The set of fixed points of this involution consists of \( n \) curves, \( \pi \)-preimages of the real line. These curves break \( S \) into two halves \( S^+ \) and \( S^- \) which are mapped onto each other by the involution. The projections of these halves are the upper and lower half-planes.

We set \( C = \{ \zeta : |\zeta| = R \} \) in (6), where \( R \) is as previously chosen.

Consider the open sets

\[ D^+ := \{ p \in S : |\pi(p)| < R, \text{Im}\pi(p) > 0, \text{Im}(\lambda(p) + s\pi(p)) > 0 \}, \]

\[ D^- := \{ p \in S : |\pi(p)| < R, \text{Im}\pi(p) < 0, \text{Im}(\lambda(p) + s\pi(p)) < 0 \} \]

and

\[ D = \text{int}(D^+ \cup D^-). \]

The set \( \{ p \in S : |\pi(p)| = R \} \) consists of \( n \) disjoint circles \( C_j \subset S \) which we label according to the branches of \( \lambda_j \) in (7), so that \( \lambda = \lambda_j \) on \( C_j \). According to the
paragraph after (7), the circles \( C_j \) with \( j \leq k \) belong to \( \partial D \) while the \( C_j \) with \( j > k \) are disjoint from \( D \).

Let \( D_1 \) be a component\(^1\) of \( D \) whose boundary contains some circles \( C_j \). We are going to prove that

\[
\sum_{j: C_j \subset \partial D_1} \int_{C_j} e^{\lambda(p) + s\pi(p)} \, d\pi(p) > 0,
\]

where the circles are oriented counterclockwise, which agrees with their orientation as part of \( \partial D \). Adding these inequalities over all components of \( D \) will prove the theorem. Indeed, each circle \( C_j \) with \( j \leq k \) belongs to the boundary of exactly one component of \( D \), and circles \( C_j \) with \( j > k \) do not belong to the boundary of \( D \).

Each component \( D_1 \) of \( D \) is a Riemann surface of finite type, whose boundary consists of several curves parametrized by circles. This parametrization is piecewise smooth, but may be neither smooth nor injective. We call these curves the boundary curves of \( D_1 \). Our choice of \( R \) guarantees that the part of the boundary of \( D_1 \) that projects into \( |\zeta| < R \) is exactly the chain on which integration is performed in (8).

Consider the rest of the boundary \( \partial D_1 \) which projects into \( |\zeta| < R \).

**Lemma 2.** No boundary curve of \( D \) over \( \{ \zeta: |\zeta| < R \} \) can project into the open upper or lower half-plane.

**Proof.** Indeed suppose that \( \gamma \) is a boundary curve whose projection does not intersect the real axis. It is oriented in the standard way, so that \( D \) is on the left. Suppose without loss of generality that \( \gamma \) projects to the upper half-plane. Let \( g(p) = \lambda(p) + s\pi(p) \). As \( \text{Im} \, g > 0 \) in \( D^+ \) and \( \text{Im} \, g = 0 \) on \( \gamma \), we conclude that the normal derivative of \( \text{Im} \, g \) has constant sign on \( \gamma \). Then by the Cauchy-Riemann equations, the tangential derivative of \( \text{Re} \, g \) along \( \gamma \) is of constant sign, which is impossible because \( \gamma \) is a closed curve, and \( \text{Re} \, g \) is single-valued on \( \gamma \).

Thus every boundary curve of \( D_1 \) intersects the real line. Let \( \gamma \) be a boundary curve of \( D_1 \) which projects into \( \{ \zeta: |\zeta| < R \} \). By Lemma 2, \( \gamma \) is mapped into itself by the involution, so it consists of two symmetric pieces: one piece \( \gamma^+ \) projects into the upper half-plane, another \( \gamma^- \) into the lower half-plane. At all endpoints \( p \) of \( \gamma^+ \) or \( \gamma^- \) we have \( \text{Im} \, \pi(p) = 0 \). We have

\[
e^g = e^{\text{Re} \, g + i \text{Im} \, g} = e^{\text{Re} \, g} (\cos(\text{Im} \, g) + i \sin(\text{Im} \, g)).
\]

Since \( \text{Im} \, g = 0 \) on \( \gamma \) and \( \text{Re} \, g \) is increasing, we conclude that \( \varphi(t) := e^{g(\gamma(t))} \) is a real increasing function of the natural parameter \( t \) on \( \gamma^+ \). Thus

\[
\frac{1}{2\pi i} \int_{\gamma^+} e^{g(p)} \, d\pi(p) = \frac{1}{2\pi i} \left( \int_{\gamma^+} + \int_{\gamma^-} \right) \varphi(t)(d\xi(t) + i \, d\eta(t))
= \frac{1}{\pi} \int_{\gamma^+} \varphi(t) \, d\eta(t) = -\frac{1}{\pi} \int_{\gamma^+} \eta(t) \, d\varphi(t) < 0,
\]

where we have integrated by parts using \( \eta(t) = 0 \) at the endpoints of \( \gamma^+ \).

\(^1\)One can prove using the maximum principle that every component of \( D \) has some \( C_j \) on the boundary, but we are not using this fact.
As the integral of the holomorphic 1-form over the boundary equals zero,
\[ \int_{\partial D_1} e^g \, d\pi = 0, \]
by Cauchy’s theorem, the contribution to the integral from the part of \( \partial D_1 \) which projects into \( \{ \zeta : |\zeta| < R \} \) is the negative of the contribution of the part of \( \partial D_1 \) over \( C \). This completes the proof of (8) and of the theorem under consideration.

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