Generalizing a Unified Model of Dark Matter, Dark Energy, and Inflation with Non Canonical Kinetic Term

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We study a unification model for dark energy, dark matter, and inflation with a single scalar field with non canonical kinetic term. In this model the kinetic term of the Lagrangian accounts for the dark matter and dark energy, and at early epochs a quadratic potential accounts for slow roll inflation. The present work is an extension to the work by Bose and Majumdar [Phys. Rev. D 79, 103517 (2009)] with a more general kinetic term that was proposed by Chimento in Phys. Rev. D 69, 123517 (2004). We demonstrate that the model is viable at the background and linear perturbation levels.

I. INTRODUCTION

In some recent works, the possibility to unify the description of dark matter, dark energy, and inflation it has been studied. For example, in the work by Liddle and Ureña-Lopez [1] is studied through a single scalar field that in the early Universe can account for the inflaton field and at late times oscillates around a nonzero minimum of the potential to yield the dark matter and dark energy components the unification. The key issue there is that the reheating epoch must reduce the field density enough to be subdominant during the radiation epoch, but not completely, for it to account for the dark matter, something that proved to be nontrivial to achieve in standard reheating schemes [2]. Further works tried to solve the issue by considering plasma masses [3, 4], a brief thermal inflation period [5], and braneworld cosmological equations of motion [6]. Another unification model was proposed in [7], where a phantom scalar field accomplishes for a $\omega < -1$ period in inflation and in the late time Universe. In [8] unification is achieved from the compactification of the six dimensional supergravity Salam-Sezgin model, where an analysis is included on the gravitational waves power spectrum.

On the other hand, the idea that a modified kinetic term in the Lagrangian could produce accelerated expansion was first proposed in the context of inflation [9] and later in dark energy [10]. This work was then generalized in what now we call k-essence [11, 12]. This kind of model has been used in several works to unify dark energy and dark matter [13, 14].

In the present work, we study the proposal by Bose and Majumdar [17] in which a scalar field with non canonical kinetic term accounts for the three phenomena. This field has a quadratic potential that drives inflation and at late times behaves almost as a purely kinetic scalar field that accounts for both dark matter and dark energy. We generalize this work by introducing a bigger class of kinetic terms [13] in which the proposal of Ref. [17] is a particular case. Then, we fit the parameters using the values of the density components in the Universe at the present time. Finally, we study the behavior of the field density at early times and conclude that it is possible to obtain inflation through a potential almost in the same way as in usual inflation models.

This work is organized as follows. In Sec. II we review how a purely kinetic Lagrangian can yield effective dark matter and dark energy densities. In Sec. III we analyze the cosmological constraints to be fulfilled by the free parameters and in Sec. IIIA we add an extra constant to our Lagrangian to build the general theory. In Sec. IIIB we analyze particular cases and show their viability. Then, in Sec. IV we study the effects on linear perturbations during the matter dominated era. In Sec. V we analyze the behavior of the field during inflation and consider a quadratic potential in slow roll. Finally, in Sec. VI we conclude.

II. PURELY KINETIC LAGRANGIANS

To start our approach, we will be working with a scalar field having a purely kinetic Lagrangian,

$$\mathcal{L} = F(X),$$

with $X = -\frac{1}{2} \phi_{\mu\nu} \phi^{\mu\nu}$ and signature $(-,+,+,+)$. The canonical kinetic term is recovered for $F(X) = X$.

These Lagrangians are invariant with respect to the symmetry transformation

$$\phi \mapsto \phi + \phi_0.$$  \hspace{1cm} (2)

Applying the Noether theorem, there is a conserved current $J^{\mu} = F_X \partial^{\mu} \phi$, where $F_X$ is the derivative of the

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Lagrangian with respect to $X$. It satisfies $J^{\mu \nu} = 0$, that can be transformed to

$$\sqrt{-g} J^{\mu \nu} = 0.$$  \hspace{1cm} (3)

Substituting the determinant of the FRW metric, $g = -a^6$, “$a$” being the scale factor, and using the homogeneity property to get rid of the spatial derivatives, one obtains the expression

$$a^6 F_X^2 X = \kappa,$$  \hspace{1cm} (4)

with $\kappa$, an integration constant.

Another way to get this result is through the equations of motion of these fields for a cosmological background \cite{[13],[14],[18]}:

$$3M_{pl}^2 H^2 = \rho,$$  \hspace{1cm} (5)

$$\rho = -3H(\rho + P),$$  \hspace{1cm} (6)

where $P = F$ is the pressure, $\rho = 2XF_X - F$ the energy density, $H = \dot{a}/a$ the Hubble parameter, and $M_{pl}^2 \equiv 1/(8\pi G)$.

Once we have specified the functional form of $F(X)$, it is possible to use the above simplification to obtain $X$ in terms of the scale factor and then other dynamic quantities such as the energy density and the sound velocity. In this work, we begin with a Lagrangian proposed by Chimento in \cite{[13]},

$$F(X) = \frac{1}{2\alpha - 1} \left( (AX)^{\alpha} - 2\alpha a_0 \sqrt{AX} \right),$$  \hspace{1cm} (7)

where the factors $A$, $\alpha$, and $a_0$ are constants chosen in that particular arrangement for the solution of the energy density to have a simple form, as shown below. Here $A$ has dimensions of $[A] = E^{4/\alpha - 4}$, $a_0$ of $[a_0] = E^{1 - 2/\alpha}$, where $E$ stands for energy and $\alpha$ is a dimensionless parameter. As the density can be expressed in terms of $X$ as $\rho = (AX)^\alpha$ using \cite{[14]}, we can obtain the explicit equation of state

$$P = \frac{1}{2\alpha - 1} \left( \rho - 2\alpha a_0 \rho^{1/2\alpha} \right).$$  \hspace{1cm} (8)

Also, using Eq. \cite{[14]}, it is possible to get the energy density as a function of the scale factor

$$\rho = \left[ a_0 + \frac{c_0}{a^3} \right]^{n},$$  \hspace{1cm} (9)

with $n = 2\alpha/(2\alpha - 1)$ and $c_0$ an integration constant. This expression for the density has the property that can be adjusted to behave as the dark densities of the $\Lambda$CDM model, as we will show it in detail in the next section.

### III. DARK ENERGY AND DARK MATTER

There are several interesting properties of the solution \cite{[14]}. It can be seen that when $a \gg \sqrt{c_0/a_0}$, the density tends to the constant value $a_0^n$, and at the same time the pressure

$$P = \frac{1}{(2\alpha - 1)} \left( (\alpha_0 + \frac{c_0}{a^3})^{n} - 2\alpha a_0 \left( \alpha_0 + \frac{c_0}{a^3} \right)^{n/2\alpha} \right)$$  \hspace{1cm} (10)

tends to $-\alpha_0^2$, which implies that the solution approaches to a cosmological constant. These kinds of solutions are called freezing cosmological models \cite{[19]}. The squared adiabatic sound speed for this model is \cite{[20]}

$$c_s^2 = \frac{P}{\rho X} = \frac{1}{(2\alpha - 1)} \frac{1}{\alpha a_0^2/c_0 + 1},$$  \hspace{1cm} (11)

which tends to zero for $a \gg \sqrt{c_0/a_0}$. It is important for the theories that intend to reproduce dark matter to have a small sound speed to allow the structure formation, as we will see in detail in Sec. \cite{[IV]}.

Expanding the solution \cite{[11]}, one has that

$$\rho = \sum_{k=0}^{n} \left( \frac{n}{k} \right) \alpha_0^{n-k} \left( \frac{c_0}{a^3} \right)^k,$$  \hspace{1cm} (12)

which is a finite series for $n \in \mathbb{N}$. The first terms are

$$\rho = \alpha_0^n + \frac{nc_0 \alpha_0^{n-1}}{\rho_0} + \frac{n(n-1)c_0^2 \alpha_0^{n-2}}{2\rho_0^2} + ..., $$  \hspace{1cm} (13)

where the first term can be identified with the cosmological constant and the second with the dark matter density. This can be written as

$$\frac{\alpha_0^n}{a_0^3} = \rho_{dc0},$$  \hspace{1cm} (14)

$$\frac{nc_0 \alpha_0^{n-1}}{a_0^3} = \rho_{dm0} \approx \frac{\rho_{dc0}}{3},$$  \hspace{1cm} (15)

where $\rho_{dc0}$ and $\rho_{dm0}$ are the current values for dark energy and dark matter, respectively.

It has been shown that some purely kinetic models can be rewritten as generalized Chaplygin gas \cite{[21],[22]}, and in Ref. \cite{[23]}, it is shown that these models when used to unify DM and DE need to be indistinguishable from the $\Lambda$CDM model in order to be viable. This also has been proven for the affine adiabatic models in \cite{[16],[24],[25]}. In our model, one can write $P = P(\rho)$, as shown in Eq. \cite{[8]}, making it similar, but not equal, to a generalized Chaplygin gas, since our equation of state has an extra term, then the above constraints do not apply here. In our case, we demand the $\rho_0$ term and subsequent in the expansion \cite{[13]} to be small compared to the relevant terms since nucleosynthesis. In this way we avoid a different expansion history that would spoil the observed abundances of the light elements and posterior dynamics. This condition can be written as $\rho_0(a_{nuc}) \ll \rho_{dc0}(a_{nuc})$. Substituting their values, we get

$$\frac{n(n-1)c_0^2 \alpha_0^{n-2}}{2a_{nuc}^6} \ll (\rho_c)(a_{nuc})^4,$$ \hspace{1cm} (16)
which implies
\[ z_{\text{nuc}}^2 \ll \frac{2 (\rho_r)_0 a_0^6}{n(n-1)c_\alpha^2 a_0^{n-2}}. \] (17)

Using the conditions (14) and (15), we can obtain
\[ z_{\text{nuc}}^2 \ll \frac{\rho_0}{\rho_{de0}} -36\alpha, \] (18)
and by using the present values of the densities and the redshift at nucleosynthesis \( z_{\text{nuc}} \sim 10^{10} \), the condition becomes
\[ \alpha \gg 10^{21}. \] (19)

As the parameter \( \alpha \) comes from the Lagrangian \( \mathcal{L} \), the condition (19) states that the kinetic term should include the \( X \) term to a huge exponent, that is a quite unnatural election. In fact what is happening is that the exponent in (9) is approaching to one, \( n = 1 + \mathcal{O}(10^{-21}) \), in a very fine tuning manner. Let us try to avoid this as follows.

### A. Extra term

In the papers by Bose and Majumdar [17, 26], it is considered an additional constant to the Lagrangian \( \mathcal{L} \) that permits to avoid the condition (19). In fact, they choose \( \alpha = 1 \) from the beginning and still accomplish the observational constraints equivalent to (14), (15), and (17). Let us adopt this view but without fixing the value of \( \alpha \). The new Lagrangian becomes
\[ F(X) = \frac{1}{(2\alpha - 1)} [(AX)^\alpha - 2\alpha_0 \sqrt{AX}] + M. \] (20)

The new energy density in terms of the scale factor is
\[ \rho = \alpha_0^\alpha - M + \frac{nc_0\alpha_0^{n-1}}{a^3} + \sum_{k=2}^n \binom{n}{k} \alpha_0^{n-k} \left( \frac{c_0}{a^3} \right)^k. \] (21)

Hereafter, we will suppose that \( n \in \mathbb{N} \) therefore being the series finite. The conditions (14) and (15) are now described by
\[ \alpha_0^\alpha - M = \rho_{de0}, \] (22)
\[ \frac{nc_0\alpha_0^{n-1}}{a^3} \approx \frac{\rho_{de0}}{3}. \] (23)

The equivalent to (17) corresponds to demand the extra terms in the energy density (21) to be small enough during the known evolution of the Universe. Defining
\[ \rho_{3k} \equiv \binom{n}{k} \alpha_0^{n-k} \left( \frac{c_0}{a^3} \right)^k, \] (24)
it is enough to require those terms to be small in the nucleosynthesis epoch, i.e., \( \rho_{3k}(a_{\text{nuc}}) \ll \rho_r(a_{\text{nuc}}) \). Substituting the values and using the condition (23), we get
\[ \binom{n}{k} \alpha_0^{n-k} \ll 3^k n^{k} \rho_{r0} \left( \frac{\alpha_0}{\rho_{de0}} \right)^{k-1} \rho_{de0} \] (25)
for \( k = 2, 3, \ldots n \). This constraint provides a condition for \( \alpha_0 \), one for each \( k \), namely
\[ \alpha_0^\alpha \gg \rho_{de0} \left[ \binom{n}{k} \alpha_0^{3k-4} \left( \frac{\alpha_0}{\rho_{de0}} \right)^{k-1} \rho_{de0} \right]^{1/(k-1)}. \] (26)

Comparing the right-hand side of the above expressions for \( n \) constant and varying \( k \) the most restrictive condition occurs when \( k = n \). Thus, it is only necessary to ensure that \( \alpha_0 \) satisfies the inequality for that case:
\[ \alpha_0^\alpha \gg \rho_{de0} \left[ \alpha_0^{3n-4} \left( \frac{\alpha_0}{\rho_{de0}} \right)^{n} \right]^{1/(n-1)}/\rho_{r0}. \] (27)

The last expression tells us that \( \alpha_0^\alpha \) has a magnitude bigger than \( 10^{10} \) times the dark energy density, but from the condition (22), the difference between this constant and \( M \) accounts for the dark energy. This means that both constants are almost equal, with the difference needed to be fine tuned in order to accomplish the constraints. On the other side, one advantage of the model is that the vacuum density \( M \) can be big enough to be considered the one produced from quantum field theory considerations. This density just needs to fulfill an expression similar to (24):
\[ M \gg \rho_{de0} \left[ \alpha_0^{3n-4} \left( \frac{\alpha_0}{\rho_{de0}} \right)^{n} \right]^{1/(n-1)}/\rho_{r0}. \] (28)

In order to avoid super-Planckian density values, we demand \( M < M_{\text{pl}}^2 \sim 10^{122} \rho_{de0} \). As the condition (28) implies at most that \( M \) shall be bigger than \( 10^{29} \) times the dark energy density, we still have 93 orders of magnitude in which \( M \) can take values. Even more because the condition (28) for a particular \( n \) is usually less restrictive than the \( 10^{29} \) value, as we will see in the next section for specific models.

The sound speed of the model is specified in (11) and is negligible once \( a \gg \sqrt{\alpha_0/\alpha_0} \), and given the constraints (23) and (24), this happens at very early times for every \( n \).

### B. Particular cases

As we mentioned in the previous subsection, the case for \( n = 2, \alpha = 1 \) was studied in the papers [17, 26]. With the assumption of \( A \sim 1 \), being this parameter dimensionless only in this particular case, they found the parameter \( \alpha_0 \) to lie in the range
\[ 10^{-48} M_{\text{pl}}^2 \leq 2\alpha_0 \leq 10^{-40} M_{\text{pl}}^2, \] (29)
where the upper bound stems from the inflation model they used. Assuming \( \alpha_0 \) to have the maximum value, the speed of sound at the beginning of matter radiation equality turns out to be \( c_s(t_{eq}) \simeq 4.1 \times 10^{-32} \) and smaller thereafter.
dark matter (UDM). The perturbation theory for UDM has been studied in Refs. 16, 27, in particular, for the purely kinetic model of Scherrer 14:
\[
F(X) = F_m + F_2(X - X_m)^2,
\]
where the constant \(F_m\) corresponds to the negative of the dark energy density and the quadratic term accomplishes for a matter like component producing an energy density of the form
\[
\rho = -F_m + 4F_2X_m^2(\alpha/a_1)^{-3},
\]
for \(X\) sufficiently close to the function minimum \(X_m\), where \(\epsilon = (X - X_m)/X_m\).

Our Lagrangian, given by Eq. (20), has a minimum since the negative term \(\sqrt{AX}\) decreases as \(X\) increases for small \(X\), and the term \((AX)\alpha\) increases and dominates for large \(X\). The cosmological dynamics brings the system to the minimum, which is asymptotically reached at \(a = \infty\). However, one can see that the minimum is almost attained long before the matter domination. To see this, one can evaluate the deviation parameter \(\epsilon\), that for our model takes the form \(\epsilon \approx \rho_3/\alpha\Omega_0^2\). The bound \(23\) can be used to obtain that
\[
\epsilon \ll \frac{2(n-1)\rho_3}{n\rho_{de0}} \left(\frac{3n}{2}\Omega_{nuc0}\rho_{de0}\right)^{1/(n-1)}
\]
or
\[
\epsilon \ll \frac{(z+1)^{3}}{(n-1)(3n)^{1/(n-1)}(30n+36)/(n-1)}.
\]
For the cases \(n = 2\) and \(n = 3\), the deviation at the beginning of matter domination \(z_{eq}\) is smaller than \(10^{-13}\) and \(10^{-16}\), respectively, and this deviation becomes even smaller with the cosmological evolution. Then, we can approximate our model by one of the previous type, Eq. (31), by considering a Taylor expansion around the minimum \(X_m = \alpha_0^{2/(2a-1)}/A\) as
\[
F(X) \approx M - \alpha_0^n + \frac{1}{4} A^2 \alpha_0^{(2a-4)/(2a-1)} (X - X_m)^2,
\]
which is valid long before the beginning of the matter domination, and then the considerations on the perturbation theory for our model are the same as for Scherrer’s model.

In the paper by Giannakis and Hu 27, the linear perturbation theory is developed for Scherrer model and the transfer function \(T(k)\) calculated. The deviation from usual cold dark matter (CDM) model can be quantified by \(T_Q(k)\) defined as \(T(k) = T_Q(k)/T_{CDM}(k)\). The numerical calculation for this function can be fitted by the function
\[
T_Q(k) \approx \frac{3j_1(x)}{x} \left[1 + (x/3.4)^2\right]^{1/(\beta+1)}
\]
with
\[
x = \left(\frac{k\eta_0}{7.74}\right), \quad \beta = 0.21 \left[\frac{c_0}{10^{10}} \left(\frac{\Omega_{m0} h^2}{0.14}\right)^{3/2}\right]^{0.12},
\]
where $\eta_*$ is the conformal time evaluated at $a_* = 14\epsilon_0^{1/3}$. For $\epsilon_0 \to 0$, the transfer function becomes more and more similar to the usual from CDM, i.e., $T_Q(k) \to 1$. In Ref. [27], it is concluded that its value today has to be smaller than $10^{-16}$ in order to yield a satisfactory CMB power spectrum. In our model, we can use the condition [26], that for the $n = 2$ and $n = 3$ cases states that the present value of the parameter $\epsilon$ is smaller than $10^{-23}$ and $10^{-26}$, respectively, accomplishing a correct fit to the CMB power spectrum. Figure 2 shows the modifications to the transfer function for these cases, where $T_Q$ deviates significantly from 1 only for small scales, which are outside the range of validity of the linear perturbation theory. For larger $n$, the effects on $T_Q(k)$ are important for even larger wave numbers.

V. INFLATION AND EARLY UNIVERSE

Now we proceed to analyze the conditions under which inflation could occur if the above solution is valid at early times. Let us see how two physical scales of the Universe are compared. A typical length at the beginning of inflation $\lambda_i = H_i^{-1}$ will evolve as the Universe expands as

$$\lambda_i(a) = \lambda_i \left( \frac{a}{a_i} \right).$$

This length should be expanded by a sufficient factor during inflation so that at the present time it could become bigger than the length of our visible Universe $\lambda_0 = H_0^{-1}$. The way to achieve this is by the exponential expansion that during inflation makes $a_0/a_i = e^{N} a_0/a_f$, where $a_f$ is the scale factor at the end of inflation. Usually, an expansion of $N \approx 60$ makes the ratio

$$\frac{\lambda_i(a_0)}{\lambda_0} = \frac{H_i^{-1}}{H_0^{-1}} e^{N} \left( \frac{a_0}{a_f} \right),$$

bigger than one. If the ratio is smaller than 1, it would mean that different parts of our Universe came from causally disconnected regions, and we could not explain the isotropy measured, for example, in the CMB. This is called the horizon problem.

To calculate the ratio, we can use the definition $\Omega_3 \equiv \rho_3/3M^2 H^2$ to replace the Hubble factors in the expression. We obtain

$$\frac{\lambda_i(a_0)}{\lambda_0} = \sqrt{\frac{\Omega_3 \rho_3}{\Omega_3 \rho_{30}}},$$

but as $\rho_3$ is proportional to $a^{-3}$

$$\frac{\lambda_i(a_0)}{\lambda_0} = \sqrt{\frac{\Omega_3}{\Omega_{30}} \left( \frac{a_0}{a_i} \right)}.$$  

In the limiting case where the amount of inflation is just enough to resolve the horizon problem, this ratio is equal to one, in this case implying that

$$\Omega_3 = \Omega_{30} \frac{a_0}{a_i}.$$  

But as $\Omega_{30} \approx 0.27$ and $a_0/a_i$ is typically of the order of $10^{51}$, the density parameter at the beginning of inflation needs to be huge.

The reason for this result is that we are demanding the dark matter density to be present before inflation and that the expansion during this epoch does not dilute it too much. For that to happen as we see, the initial density needs to be huge. To solve this problem, in the next subsection we will add a term to the Lagrangian in order to modify the behavior of the density at early times; essentially what we need is this term to be present only after inflation has end.

We could be worried about the other terms in the expansion [24]. We know these terms should not be relevant at nucleosynthesis and thereafter, and it would be desirable to make them arbitrarily small during the inflation epoch with the choice of a sufficiently large $M$ parameter. This has the effect of setting the domination of these terms back in the preinflationary era and then being subdominant at the beginning of inflation. To achieve this one can constrain the constants $a_0$ and $c_0$. However, this cannot be done for the $\rho_3$ term, because its value is already determined by the observational constraints, and as we explained, it will not be small during inflation. Thus, a new ingredient has to be added to the theory.

A. Inflation driven by a potential field

To avoid the important problems mentioned at the end of the previous subsection, we need to break the validity of the solution [21] during inflation. One way to achieve it is to add an interaction term, coupled to the inflaton
that could make it to decay into the k-essence field during reheating or later, and thus making \((21)\) valid only after this period. On the other side, the solution that we explore here is to make the \(\phi\) field the only one responsible for the whole dynamics, including inflation. Therefore, a potential term in the Lagrangian is necessary.

We consider the new Lagrangian

\[
\mathcal{L}(\phi, X) = F(X) - V(\phi),
\]
with \(F\) from \((24)\) and \(V\) a quadratic potential

\[
V(\phi) = \frac{1}{2} m^2 \phi^2.
\]

This term that will be important at early times will make the solution to the energy density not blow up at \(a \to 0\), as it happens with \((21)\). The crucial ingredient will be considering the slow roll (sr) approximation at which \(X|_{sr} \ll V|_{sr}\).

The equations of motion are now

\[
3M_{pl}^2 H^2 = 2X F_X - F + V, \tag{45}
\]

\[
(F_X \dot{\phi})' + 3H F_X \dot{\phi} + V' = 0, \tag{46}
\]

where \(V' = dV/d\phi\). We suppose that for the early Universe, most of the energy density is in the potential term, so that we can use the slow roll approximation, in which the terms in \(X\) can be neglected. After inflation, the energy density will be transferred to the kinetic term, so we can treat the field as purely kinetic and proceed with the analysis made in the previous sections.

During slow roll, the equations of motion transform to

\[
3M_{pl}^2 H^2 = V(\phi), \tag{47}
\]

\[
3H F_X \dot{\phi} + V'(\phi) = 0, \tag{48}
\]

and we expect them to be valid during the inflation period, with the end of its validity corresponding to the end of inflation.

Using the previous equations, one obtains the value

\[
F_X = -V' M_{pl}/\dot{\phi} \frac{3V}{V'} ,
\]

that for the case of the potential \((44)\) and \(\dot{\phi} < 0\) can be expressed as

\[
F_X = \frac{m M_{pl}}{\sqrt{3X}}.
\]

It is interesting to note that from this equation, the value of \(X\) will be totally defined once we have specified an expression for \(F(X)\), and this value will be approximately constant during the period of validity of the slow roll approximation, that is, \(X = X_{sr}\) for all the slow roll inflation period. This property is characteristic of the quadratic potential chosen in \((44)\); for other potentials, the right-hand side of Eq. \((49)\) will depend on \(\phi\), and the parameter \(X_{sr}\) no longer will be constant.

The slow roll parameters have in this case the form \((17)\)

\[
\epsilon = \frac{M_{pl}^2}{2F_X} \left(\frac{V'}{V}\right)^2, \tag{50}
\]

\[
\eta = \frac{M_{pl}^2}{F_X} \frac{V''}{V}. \tag{51}
\]

The slow roll approximation is valid as long as these parameters are small compared to one. The only difference between these expressions and the ones with a canonical kinetic term is the \(F_X\) factor, which is however constant during slow roll because it depends only on \(X_{sr} \approx\) const., as above explained.

It is considered that the end of inflation happens when \(\epsilon \approx 1\) which for the potential \((44)\) corresponds to

\[
\phi_i^2 = \frac{2M_{pl}^2}{F_X}.
\]

If we suppose that the parameter \(\eta\) also becomes one at this time we will have with the expression \((51)\) and the value of the potential that \(2M_{pl}^2/\phi_i^2 F_X^2 \approx 1\). Substituting the value of the field at the end of inflation \((52)\) leads us to \(F_X(X_{sr} = 1)\). Thanks to that the expressions for the slow roll parameters \((50)\) \((51)\) and the equations of motion \((44)\) \((45)\) become the same as the canonical ones, so the subsequent analysis can be carried out with the standard slow roll approximation.

The number of e-folds obtained is then

\[
N = \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{F_X}{M_{pl}^2} \frac{V'}{V} dt \approx \frac{\phi_i^2 - \phi_f^2}{4M_{pl}^2}, \tag{53}
\]

where \(\phi_i\) and \(\phi_f\) are the field values at the beginning and at the end of inflation, respectively. Assuming that \(N \approx 60\) and using \((52)\), we obtain

\[
\phi_i = 15.5 M_{pl}.
\]

We can obtain the value of the parameter \(m\) using the measured amplitude of temperature perturbations in the CMB \((28)\) \((29)\), \(\delta = 2 \times 10^{-5}\), and as in \((30)\) \((31)\), we get \(m = 7 \times 10^{-6} M_{pl} (n - 1)^{1/4}\).

The values for the slow roll parameters at the beginning of inflation can also be obtained by using \((54)\) to give \(\epsilon_i = 8.3 \times 10^{-3}\) and \(\eta_i = 8.3 \times 10^{-3}\). This justifies the use of the slow roll approximation. Now we are in the position to calculate the spectral index \(n_s = 1 - 0.3\sqrt{n}/(n - 1)\) and the tensor to scalar ratio \(r = 0.15\sqrt{n}/(n - 1)\), in accordance with the typical values in standard, chaotic inflation \((30)\) but with a correction due to the noncanonical kinetic term \((32)\).

The energy scale during inflation can also be determined with the use of Eqs. \((52)\) \((53)\) and the value of \(m\). The potential at the beginning and end of inflation will be, respectively,

\[
V_i = 5.9 \times 10^{-9} M_{pl}^4 \sqrt{n - 1},
\]

\[
V_f = 4.9 \times 10^{-11} M_{pl}^4 \sqrt{n - 1}, \tag{55}
\]

that implies that the energy scale of the potential at the beginning and end of inflation is safely low compared to the Plank scale.

From the derivative of the second of the equations of motion \((48)\) one gets \(V'' = -3H F_X\), where \(H\) can be obtained from the second of the Friedmann equations.
\( \dot{H} = \frac{(\rho + P) - 2M_{pl}^2}{2M_{pl}^2} \) with \( \rho + P = 2XF_X \). In this case we arrive at \( V'' = 3F_X^2 X/M_{pl}^2 \). This expression can be substituted into (51) to find \( \eta = 3X/V \). At the end of inflation as we supposed that \( \eta \) is of order 1, we can find the value of the kinetic parameter \( X \) as being one third of the value of the potential at that moment. This is reasonable because we think about the end of inflation as the moment at which the potential stops its domination and gets the same order of magnitude than the kinetic term. From (54), we find

\[
X_{sr} \sim 1.6 \times 10^{-11} M_{pl}^4 \sqrt{n - 1}.
\]

To obtain the value of \( X_{sr} \), we did not need the particular expression for the kinetic term (20), but we have a new equation that has to be satisfied, named \( F_X(X_{sr}) = 1 \). This will be accomplished by a suitable choice of the parameter \( A \) that is the only one in the Lagrangian that we have not constrained yet. The equation is as follows

\[
\frac{n}{2} \left( A^\alpha X_{sr}^{\alpha - 1} - a_0 \sqrt{\frac{A}{X_{sr}}} \right) = 1,
\]

which cannot be exactly solved for \( A \), but we can make the assumption that the term containing \( a_0 \) is small and then obtain \( A^\alpha = 2X_{sr}^{1 - \alpha}/n \). For the case \( n = 2 \), for example, we get \( A = 1 \), and the neglected term in (57) is around \( 10^{-37} \) showing that our approximation is valid. For \( n = 3 \), the parameter \( A \) is \( 10^{-4} M_{pl}^4 \) and the neglected order \( 10^{-28} \).

### B. End of inflation

After the end of inflation, the kinetic term will start to dominate and the field will behave as purely kinetic with the characteristics analyzed in the previous sections. Part of the energy density in the field shall be transformed into radiation density that ultimately will give rise to the electromagnetic radiation and baryonic matter present in the Universe today. This transformation needs to be incomplete in order to let the kinetic field behave as dark matter and dark energy. One possibility is through gravitational particle production. Standard calculations (33, 54) indicate that this process yields an energy density at the end of inflation given by \( \rho_{rf} \sim 0.01gH_f^2 \), where the subindex refers to the end of inflation and \( g \) is the number of degrees of freedom of the particles produced that we suppose to be \( \sim 100 \). Using (55), we can obtain \( \rho_{rf} \sim 10^{-21} M_{pl}^4 \).

If the potential energy at the end of inflation (55) transforms into kinetic energy and starts to behave as (21), it will initially decay as \( a^{-3\eta} \) while the radiation density decays only as \( a^{-4} \). Then, after an expansion of \( 25/(3n - 4) \) e-folds, these densities will cross and the radiation will start to dominate. Thus, the domination of the radiation term starts sooner for bigger \( n \). For example, for \( n = 2 \), the Universe needs to expand \( 10^3 \) times (12 e-folds) after the end on inflation, but for the model with \( n = 4 \), the Universe has to expand only 23 times (3 e-folds). After that, the model begins with the usual radiation dominated early Universe. We can also calculate the density of the Universe at the moment when this crossing occurs, which is given by

\[
\rho_c \sim V_f \left( \frac{a_0}{a_c} \right)^{3n}.
\]

For example, for \( n = 2 \), the energy will be \( 10^{30} \) GeV\(^4\); for \( n = 3 \), one has \( \rho_c \sim 10^{43} \) GeV\(^4\); and for bigger \( n \), this energy density will be bigger. In all the cases, the radiation dominated Universe starts well before the nucleosynthesis epoch which occurs at energies around 1 MeV.

### VI. CONCLUSIONS

In this work, we have generalized a particular type of solution that unifies dark energy and dark matter with only one purely kinetic scalar field. We obtained the conditions on the parameters of the Lagrangian to allow a realistic cosmological evolution. These conditions are equivalent to demand that the model be indistinguishable from \( \Lambda \)CDM at least since nucleosynthesis. First, we noted that without the introduction of the vacuum density \( M_f \), the parameter \( \alpha \) to which the \( X \) term is raised in the Lagrangian had to have a value bigger than \( 10^{21} \), something pretty unnatural. Once we added the \( M \) parameter, we were in the position to choose the \( \alpha \) parameter. An interesting feature of this \( M \) term is that it is the only one necessary to cancel all the remaining \( \rho_{sk} \) terms in (21), and at the same time, it sets the proper cosmological constant today.

Even when the introduction of the vacuum density \( M \) could be considered a drawback of our model because it is equivalent to put by hand a cosmological constant, unlike the case of introducing directly the dark energy term, in this case, the constant has a bigger magnitude and is easier to reconcile with the vacuum energy obtained from quantum field theory. For example, in the \( n = 3 \) case, we set \( M \) from around \( (10^{-24} M_{pl})^4 \) to \( M_{pl}^4 \). In fact, the bigger the value of the constant term, the better this model reproduces the \( \Lambda \)CDM model. Once we have chosen the value of the constant \( M \), we can calculate the other parameters in the Lagrangian in order to satisfy the constraints.

Having demonstrated the validity of the generalized Eq. (20) as a unified dark matter and dark energy model for the background dynamics, we proceeded to compute its influence in the linearly perturbed Universe. In particular, we demonstrated that long before the matter domination era, the system is near its minimum, \( \epsilon \ll 1 \), and following Ref. (24), we found that transfer function of this model does not significantly deviate from the CDM model, emulating a perturbed, standard cosmological scenario.
The behavior of our solution at early times also gave us a clue that this model should be modified in order to let the inflationary epoch to occur. A possible solution to this scenario that had been studied in [17] is to add a potential term in the Lagrangian in order to make this field also the responsible for inflation. We analyzed this possibility in our models and obtained the corresponding values of the remaining free parameter in the kinetic term and the new free parameter in the potential term. Also, it is interesting that many parameters can be fixed without specifying the explicit expression for $F_X$, but only the value of $F_X$ during inflation, which, for the quadratic potential in the slow roll regime is constant. It is possible to introduce other potential terms to drive inflation following an equivalent analysis to the one stated here.

The present work shows that the dynamics of inflation, dark energy, and dark matter can be unified in a theory with a standard inflationary potential and a nontrivial kinetic term. Most of the ideas present in this work were already stated in [17], and we proved that they can be extended to the more general Lagrangian (43) with (20) for $n = 2, 3, 4 \cdots$.

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