Anomalous Band Structure in Odd-Odd Nuclei
with the Quadrupole-Quadrupole Interaction

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Abstract

We perform shell model calculations in odd-odd nuclei using a quadrupole-
quadrupole interaction with single-particle splittings chosen so as to obtain
the $SU(3)$ results. Elliott had shown that such an interaction gives rotational
bands for which the energies go as $I(I + 1)$. This certainly is true for even-
even and for odd-even or even-odd nuclei with $K \neq 1/2$. We have looked
at odd-odd nuclei e.g. $^{22}\text{Na}$ and found somewhat different behaviour. In
$^{22}\text{Na}$ the $I = 1^+_1 \ T = 0$ and $I = 0^+_1 \ T = 1$ states are degenerate, and a
rotational band built on the $I = 0^+_1 \ T = 1$ state behaves in a normal fashion.
For the $I = 1^+_1 \ T = 0$ band however, we find that the energy is given by
$E(I) - E(1^+_1) = AI(I + 1)$. This differs from the ‘normal’ behaviour which
would be $E(I) - E(1^+_1) = AI(I + 1) - 2A$.

I. INTRODUCTION

In the rotational model the formula for the energy of a state in a rotational band with
total angular momentum $I$ is given by [1]

$$E_I = E_0 + \frac{\hbar^2}{2J} \left[ I(I + 1) + \delta_{K,1/2}a(-1)^{I+1/2}(I + 1/2) \right]$$

where $a$ is the decoupling parameter given by $a = -\langle K = 1/2 \mid J_+ \mid K = 1/2 \rangle$ and where if $|K\rangle = \sum_j C_{j,k} \phi_{j,k}$ then $|\bar{K}\rangle = \sum_j C_{j,k} (-1)^{j+k} \phi_{j,-k}$. 

For even-even nuclei, and for odd-even and even-odd nuclei with $K \neq 1/2$, one gets the familiar $I(I+1)$ spectrum \[^3\].

It is generally thought that the Elliott $SU(3)$ model also gives an $I(I+1)$ spectrum. This has been discussed most explicitly in the context of even-even nuclei. The $SU(3)$ results also give the more complex $K = 1/2$ behaviour where the decoupling parameter $a$ has a value corresponding to that obtained from an asymptotic Nilsson wave function. This will be discussed briefly in section II. But the main thrust of this work will be to show that for odd-odd nuclei one obtains in certain cases deviations from the above formula.

We have performed shell model calculations with all possible configurations in a given major shell using the interaction $\sum_{i<j} Q(i) \cdot Q(j)$ where, in order to get Elliott’s $SU(3)$ results we must also add single-particle splittings, e.g. in the $1s-0d$ shell we have $\epsilon_{0d} - \epsilon_{1s} = 18 \bar{\chi}$ and in the $1p-0f$ shell we have $\epsilon_{0f} - \epsilon_{1p} = 30 \bar{\chi}$, where $\bar{\chi} = \frac{50^+}{32\pi}$ with $b$ the oscillator length parameter ($b^2 = \frac{\hbar^2}{m\omega}$).

As has been previously noted \[^3\][^4], we use the $\vec{r}$-space $Q \cdot Q$ interaction rather than the mixed $\vec{r}$ and $\vec{p}$-space one. With such an interaction $2/3$ of the above single-particle splitting comes from the $i = j$ part of $Q \cdot Q$ and $1/3$ from the interaction of the valence particle with the core.

\section{II. A BRIEF LOOK AT $K = 1/2$ BANDS}

Let us be specific and discuss $^{19}F$ and $^{43}Sc$. We consider in each case three valence nucleons beyond a closed shell. In $^{19}F$ the particles are in the $1s-0d$ shell, whereas in $^{43}Sc$ they are in the $1p-0f$ shell. The energy levels of the lowest bands are given in Table I for the two cases. The results for the two nuclei are striking but different. In $^{19}F$, the lowest state is a $I = 1/2^+$ singlet, and at higher energies we get degenerate pairs $(3/2^+, 5/2^+)$, $(7/2^+, 9/2^+)$, $(11/2^+, 13/2^+)$. In $^{43}Sc$ the ground state is degenerate, and the degenerate pairs are $(1/2^+, 3/2^+)$, $(5/2^+, 7/2^+)$, ..., $(17/2^+, 19/2^+)$. If we look at the rotational formula, we find that these results are consistent with a decoupling parameter $a = +1$ for $^{19}F$ and $a = -1$ for $^{43}Sc$. It is easy to show that these are precisely the results one obtains with asymptotic Nilsson wave functions. In both cases the odd particle will be in a $\Lambda = 0$ $\Sigma = 1/2$ state in the asymptotic limit. From the definition of $\vec{K}$, the state $|\Lambda = 0 \bar{\Sigma} = 1/2\rangle$ can be shown to be equal to $-(-1)^\pi |\Lambda = 0 \Sigma = -1/2\rangle$ where $\pi$ is (+) for an even-parity major shell and (−) for an odd-parity one. Hence:

$$a = (-1)^\pi (\Sigma = +1/2 \mid J_+ \mid \Sigma = -1/2) = (-1)^\pi$$
It has long ago been noted by Bohr and Mottelson [1] that \( a = +1 \) corresponds to weak coupling of the odd particle to \( I = 0^+, 2^+, 4^+, ... \) states, whereas \( a = -1 \) corresponds to weak coupling to \( I = 1, 3, 5, ... \) states. It should be emphasized that the results in Table I are not the realistic ones—they represent the asymptotic extremes.

At any rate, we have shown that the \( Q \cdot Q \) interaction gives the same results for these two \( K = 1/2 \) bands as does the rotational formula with asymptotic Nilsson wave functions.

### III. ODD-ODD NUCLEI E.G. \(^{22}NA\)

#### A. The Energy Spectra

In table II we show a fairly detailed list of energy levels for the odd-odd nucleus \(^{22}Na\) obtained with the \( Q \cdot Q \) interaction. We show \( T = 0 \) and \( T = 1 \) states in separate columns. We have underlined \( T = 0 \) and \( T = 1 \) rotational bands, and will now discuss them in more detail. We use the same parameters as in \(^{19}F\) just to bring out some similarities. If one is interested in a best fit, one should of course have an \( A \) dependence in \( \chi \).

Note that the ground state consists of two degenerate states, one with \( I = 1^+ T = 0 \) and the other with \( I = 0^+ T = 1 \). Both states have \( L = 0 \) and the simple spin-independent interaction gives the same energy for \( S = 0 \) and \( S = 1 \). Let us first look at the \( T = 1 \) states. The ground state is \( I = 0^+ \). The \( 2^+ \) state is at 1.588 and is doubly degenerate. If we follow the underlined states we have a \( 2^+ \) at 1.588 MeV, \( 3^+ \) at 3.177 MeV, \( 4^+ \) at 5.293 MeV, \( 5^+ \) at 7.941 MeV until we reach \( 10^+ \) at 29.117 MeV.

The energy levels of \( I = 2^+, 3^+, ..., 9^+, 10^+ \) are given by

\[
E^*(I) = E(I) - E(1^+_1) = AI(I + 1) \quad \text{where} \quad A = \frac{\hbar^2}{2\mathcal{J}} = E(2^+)/6
\]

At first sight there would appear to be nothing wrong. But remember that \( E^*(I) \) is the energy of a state of angular momentum \( I \) for which the \( I = 1^+ \) state has been set to zero energy. If we put \( I = 1 \) into the above formula we would get \( E^*(1) = 2A \).
To put it in a better way, the rotational formula at the beginning of this paper (Eq. (1)) would yield
\[ E(I) - E(1^+) = AI(I + 1) - 2A \]
However, the results that we obtain are
\[
E^*(I) = E(I) - E(1^+) = AI(I + 1) \quad I \neq 1 \\
= 0 \quad I = 1
\]
Thus, for the case of \( T = 0 \) states in odd-odd nuclei we get a difference between the rotational formula and the \( SU(3) \) limit.

### B. The \( B(E2) \) Values for \( T = 0 \rightarrow T = 0 \) Transitions

To clarify the structure of these bands, we performed calculations of \( B(E2) \) values for various \( T = 0 \rightarrow T = 0 \) transitions up to \( I = 4 \). They are shown in Tables III and IV, where we introduced a small spin-orbit splitting in order to remove the degeneracies as our shell model code does not handle transitions involving degenerate states very well. Note that with bare \( E2 \) charges \( e_p = 1, \ e_n = 0 \) we obtain \( B(E2 : 1^+_1 \rightarrow 2^+_1 = 0) = 34.9 \ e^2 fm^4 \). This is quite large, and in our opinion justifies treating the \( I = 1^+ \) state as a member of the band. Actually, if we used the usual effective charges \( e_p = 1.5, \ e_n = 0.5 \), the \( B(E2) \) value would increase four-fold (i.e. to about 140 \( e^2 fm^4 \)). Note also that the cross-over transition \( I = 1^+_1 \rightarrow I = 3^+_2 \) at 3.2 \( MeV \) is zero. This is consistent with the \( I^+_1 \) state being \( L = 0 \ S = 1 \) and the \( I = 3^+_2 \) state being \( L = 3 \ S = 1 \). One cannot connect from \( L = 0 \) to \( L = 3 \) via the \( E2 \) operator. There is some strength to a lower \( 3^+ \) state which is not a member of the rotational band \( (B(E2) = 6.15 \ e^2 fm^4) \). That \( 3^+ \) state must be \( L = 2 \ S = 1 \).

Our work suggests that the rotational model formula requires an additional term for odd-odd nuclei in order to be consistent with the \( SU(3) \) results [2]. We gain further insight by examining the degeneracies associated with the \( T = 0 \) underlined states of Table II, i.e. those with energy \( AI(I + 1) \). The even \( I \) states up to \( I = 8 \) are doubly degenerate whereas the others are singlets. This suggests that there are two bands for which the states with the same \( I \) values are degenerate. One band is a \( K = 2 \) band with all values of \( I \) from 2 to 10, and there is nothing anomalous about it. The other band consists of states of angular momentum 1,2,4,6 and 8. For the latter band, the orbital angular momentum of the states are 0,2,4,6 and 8 respectively, and they all have \( S = 1 \). Their energies can be fit to the formula \( E^*(I) = BL(L + 1) \) rather than \( AI(I + 1) \).
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REFERENCES

[1] A. Bohr and B. Mottelson, *Nuclear Structure*, Vol. II (W.A. Benjamin Inc., Reading, Massachusetts, 1975)

[2] J.P. Elliott, Proc. Royal Soc. **A 245** 128(1958); **A245** 562(1958).

[3] M.S. Fayache, L. Zamick and Y.Y. Sharon, Phys. Rev. **C 55**, 1575(1997).

[4] E. Moya de Guerra, P. Sarriguren and L. Zamick, Phys. Rev. **C 56**, 863(1997).
TABLE I. Energy Levels (in MeV) of Excited States Corresponding to the $K = 1/2$ Ground State Bands in $^{19}F$ and $^{43}Sc$ with the $-\chi Q \cdot Q$ Interaction.

| $^{19}F$ | $^{43}Sc$ |
|----------|----------|
| $I^\pi$  | $E^*$    | $I^\pi$  | $E^*$    |
| $(1/2)^+$ | 0        | $(1/2)^-$ | 0        |
| $(3/2)^+$ | 1.588    | $(3/2)^-$ | 0        |
| $(5/2)^+$ | 1.588    | $(5/2)^-$ | 0.679    |
| $(7/2)^+$ | 5.295    | $(7/2)^-$ | 0.679    |
| $(9/2)^+$ | 5.295    | $(9/2)^-$ | 1.900    |
| $(11/2)^+$ | 11.118  | $(11/2)^-$ | 1.900 |
| $(13/2)^+$ | 11.118  | $(13/2)^-$ | 3.664 |
|         |          | $(15/2)^-$ | 3.664 |
|         |          | $(17/2)^-$ | 5.971 |
|         |          | $(19/2)^-$ | 5.971 |

$^a$For $^{19}F$ we use $\chi = 0.1841$ ($\bar{\chi} = 0.0882$)

$^b$For $^{43}Sc$ we use $\chi = 0.0294$ ($\bar{\chi} = 0.0218$)
TABLE II. The Energy Levels (in $MeV$) of $^{22}Na$ Calculated with the $-\chi Q \cdot Q$ Interaction$^a$

| $I^\pi$ | $T = 0$ States | $T = 1$ States |
|--------|----------------|----------------|
| 0$^+$  | 8.999          | 0.000          |
|        | 12.176         | 2.647          |
|        | 12.176         | 8.999          |
|        | 13.235         | 9.000          |
|        | 16.410         | 12.176         |
| 1$^+$  | 0.000          | 2.647          |
|        | 1.588          | 8.999          |
|        | 1.588          | 8.999          |
|        | 2.647          | 10.059         |
|        | 9.000          | 10.059         |
| 2$^+$  | 1.588          | 1.588          |
|        | 1.588          | 1.588          |
|        | 3.176          | 2.647          |
|        | 8.999          | 5.294          |
|        | 10.059         | 9.000          |
| 3$^+$  | 1.588          | 3.176          |
|        | 1.588          | 5.293          |
|        | 3.177          | 10.058         |
|        | 5.294          | 10.058         |
|        | 5.294          | 11.646         |
| 4$^+$  | 3.176          | 5.293          |
|        | 5.293          | 5.293          |
|        | 5.293          | 5.293          |
|        | 7.941          | 10.059         |
|        | 11.647         | 11.647         |
| 5$^+$  | 5.294          | 7.941          |
|        | 5.294          | 10.059         |
|        | 7.941          | 13.763         |
|   | 10.059 | 13.763 |
|---|-------|-------|
| 11.118 | 13.763 |

| 6+ | 7.941 | 10.059 |
|----|------|-------|
| 11.117 | 11.117 |
| 11.117 | 11.118 |
| 14.824 | 16.412 |
| 16.411 | 16.412 |

| 7+ | 11.117 | 14.823 |
|----|-------|-------|
| 11.117 | 16.940 |
| 14.823 | 19.587 |
| 16.941 | 19.587 |
| 19.058 | 19.587 |

| 8+ | 14.823 | 16.941 |
|----|-------|-------|
| 19.058 | 19.058 |
| 19.059 | 19.059 |
| 22.763 | 22.767 |
| 23.292 | 23.293 |

| 9+ | 19.058 | 23.822 |
|----|-------|-------|
| 19.058 | 25.939 |
| 23.822 | 26.470 |
| 25.940 | 27.527 |
| 26.469 | 27.527 |

| 10+ | 23.823 | 25.942 |
|----|-------|-------|
| 29.117 | 29.117 |
| 30.705 | 30.706 |
| 32.293 | 32.294 |
| 33.881 | 32.294 |

\(^a\)In this table and in the following tables, the same value of \( \chi \) (and of \( \bar{\chi} \)) was used for \(^{22}\text{Na}\) as for \(^{19}\text{F}\).
TABLE III. Calculated $B(E2)$ from the Ground State in $^{22}\text{Na}$ with the $-\chi Q \cdot Q$ Interaction.

$I = 1^+_1 \ T = 0 \rightarrow I = 2^+ \ T = 0$

| $E^*(I = 2^+, T = 0)$ | $B(E2) \ (e^2 fm^4)$ |
|------------------------|----------------------|
| 1.591                  | 34.89                |
| 1.598                  | 4.31                 |
| 3.199                  | 0.00                 |
| 8.995                  | 0.00                 |
| 10.054                 | 0.00                 |

$I = 1^+_1 \ T = 0 \rightarrow I = 3^+ \ T = 0$

| $E^*(I = 3^+, T = 0)$ | $B(E2) \ (e^2 fm^4)$ |
|------------------------|----------------------|
| 1.570                  | 6.15                 |
| 1.586                  | 48.74                |
| 3.183                  | 0.00                 |
| 5.306                  | 0.00                 |
| 5.320                  | 0.00                 |
### TABLE IV. Calculated $B(E2)$ Between Excited States in $^{22}$Na with the $-\chi Q \cdot Q$ Interaction.

For $I = 2^+ T = 0 \rightarrow I = 3^+ T = 0$

| $E^*(I = 2^+, T = 0)$ | $E^*(I = 3^+, T = 0)$ | $B(E2) \ (e^2f m^4)$ |
|------------------------|------------------------|----------------------|
| 1.591 | 1.570 | 0.00 |
| 1.591 | 1.586 | 13.73 |
| 1.591 | 3.183 | 3.18 |
| 1.598 | 1.570 | 13.5 |
| 1.598 | 1.586 | 0.00 |
| 1.598 | 3.183 | 26.57 |
| 3.199 | 1.570 | 1.44 |
| 3.199 | 1.586 | 0.20 |
| 3.199 | 3.183 | 0.00 |

For $I = 2^+ T = 0 \rightarrow I = 4^+ T = 0$

| $E^*(I = 2^+, T = 0)$ | $E^*(I = 4^+, T = 0)$ | $B(E2) \ (e^2f m^4)$ |
|------------------------|------------------------|----------------------|
| 1.591 | 3.161 | 1.41 |
| 1.591 | 5.296 | 30.60 |
| 1.591 | 5.299 | 12.14 |
| 1.598 | 3.161 | 11.03 |
| 1.598 | 5.296 | 1.14 |
| 1.598 | 5.299 | 25.00 |
| 3.199 | 3.161 | 0.00 |
| 3.199 | 5.296 | 0.59 |
| 3.199 | 5.299 | 5.83 |

For $I = 3^+ T = 0 \rightarrow I = 4^+ T = 0$

| $E^*(I = 3^+, T = 0)$ | $E^*(I = 4^+, T = 0)$ | $B(E2) \ (e^2f m^4)$ |
|------------------------|------------------------|----------------------|
| 1.570 | 3.161 | 40.10 |
| 1.570 | 5.296 | 0.15 |
| 1.570 | 5.299 | 2.68 |
\begin{align*}
I = 3^+ \ T = 0 & \rightarrow I = 5^+ \ T = 0 \\
E^*(I = 3^+, T = 0) & \quad E^*(I = 5^+, T = 0) \quad B(E2) \ (e^2 fm^4) \\
1.570 & \quad 5.278 & \quad 20.31 \\
1.570 & \quad 5.289 & \quad 0.36 \\
1.570 & \quad 7.945 & \quad 0.00 \\
1.586 & \quad 5.278 & \quad 8.74 \\
1.586 & \quad 5.289 & \quad 36.00 \\
1.586 & \quad 7.945 & \quad 0.00 \\
3.183 & \quad 5.278 & \quad 4.53 \\
3.183 & \quad 5.289 & \quad 0.20 \\
3.183 & \quad 7.945 & \quad 25.09 
\end{align*}