AN INDEX UPPER BOUND FOR NON-ORIENTABLE MINIMAL SURFACES IN THE $n$–DIMENSIONAL EUCLIDEAN SPACE.

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Abstract. In the present paper we obtain an upper bound on the Morse index of a complete (possibly branched) immersed non-orientable minimal surface in the $n$–dimensional Euclidean space. It is an analog of the upper bound of Ejiri and Micale for orientable surfaces. The obtained upper bound enables us to compute the index of the Alarcón-Forstnerič-López Möbius band in the 4-dimensional Euclidean space. According to our computation it is equal to one. In Appendix we discuss inequalities on the index of a closed non-orientable minimal surface in a general Riemannian manifold.

1. Introduction

A minimal submanifold $u : N \rightarrow (M, h)$ of a Riemannian manifold $(M, h)$ is a critical point of the volume functional

$$V[u] = \int_N dv_{u^*h}.$$ 

A minimal submanifold is not necessarily a point of minimum of the volume functional. The (Morse) index of a minimal submanifold measures how far our manifold is from to be a point of minimum of the volume functional. Roughly speaking, the index of a minimal submanifold is the maximal number of linearly independent infinitesimal variations of it which decrease its volume. If our minimal submanifold is a point of minimum then this number is 0. In this case we say that our minimal submanifold is stable.

In the present paper we are concentrated on minimal surfaces (i.e. submanifolds of dimension 2) of the Euclidean space $\mathbb{E}^n$. Moreover, we will also require that minimal surfaces are complete. It follows from the maximum principle applied to the coordinate functions of a minimal surface - they are harmonic with respect to the interior Laplacian on the surface - that complete minimal surfaces in $\mathbb{E}^n$ can not be compact. The classification of non-compact surfaces is a difficult task. The index allows us to classify minimal surfaces in $\mathbb{E}^n$. There are also approaches to the classification of minimal surfaces in $\mathbb{E}^n$ by the ends and the total curvature of the surface. These approaches are rather complementary to each other than opposite.

However, the computation of indices of minimal surfaces in $\mathbb{E}^n$ is not easy. Trivially, any plane has index 0, i.e. it is stable. It was proved by Fisher-Colbrie-Schoen [FCS80], do Carmo-Peng [dCP79] and Pogorelov [Pog81] independently that the plane is the only complete orientable stable minimal surface in $\mathbb{E}^3$. It is known
that the index of the catenoid in $E^3$ is one. Moreover, the catenoid is the only embedded complete minimal surface in $E^3$ of index one (see [CT88]). Without requiring embeddedness Enneper’s surface is another example of a complete minimal surface in $E^3$ of index one. It is interesting that there are no embedded complete minimal surfaces in $E^3$ of index 2 (see [CM16]). Recently, Chodosh and Maximo have proved in the paper [CM18] that there are no embedded complete orientable minimal surfaces of index 3 neither. If we remove the embeddedness assumption then the Chen-Gackstatter surface, the Richmond surface and the Jorge-Meeks surface with 3 ends are examples of minimal surfaces of index 3 (see for example [MR91, Tuz91]).

For more information and conjectures about the index of minimal surfaces in $E^3$ see for example [CM16].

The first example of a non-orientable minimal surface in $E^3$ was Henneberg’s surface [Hen75]. Topologically this surface is a Möbius band with two branched points, i.e. this surface is not regular. Meeks in [MI81] constructed the first example of a properly immersed minimal Möbius band. The Meeks Möbius band which we denote by $\mathbb{M}^M$ is the only immersed complete non-orientable surface of total Gauss curvature $-6\pi$ in $E^3$. Later, López in [Lop93] constructed a properly immersed non-orientable surface of genus one with one puncture (here the genus of a non-orientable surface is understood as the genus of its orientable cover), i.e. a punctured Klein bottle. The López Klein bottle which is denoted be $\mathbb{K}^L$ is the only immersed complete non-orientable surface of total curvature $-8\pi$ in $E^3$. De Oliveira in [DO86] obtains the Weierstrass representation of non-orientable minimal surfaces and proves that the total curvature of a non-orientable complete regular minimal surface in $E^n$ is at most $-4\pi$. In this paper we are also interested in properly embedded complete non-orientable minimal surfaces in $E^n$. By the topological reasons no such surfaces exist in $E^3$. However, there are no topological obstruction to the existence of these surfaces in $E^4$. To the best of our knowledge the first example of a properly embedded complete non-orientable minimal surface in $E^4$ was constructed in the paper [AFL20]. This embedding is given by the formula

$$u(\rho, \varphi) = \left( \sin \varphi, - \cos \varphi, - \frac{1}{2} \left( \frac{1}{\rho^2} + \rho^2 \right) \sin 2\varphi, \frac{1}{2} \left( \frac{1}{\rho^2} + \rho^2 \right) \cos 2\varphi \right),$$

where $(\rho, \varphi) \in (0, +\infty) \times \mathbb{R}$. This embedding is invariant with respect to the map $(\rho, \varphi) \mapsto (1/\rho, \pi + \varphi)$ and therefore $u$ determines a proper minimal embedding of the Möbius band into $E^4$, that we call the Alarcón-Forstnerič-López Möbius band and denote it by $\mathbb{M}^AFL$. As it was computed in [AFL21, Section 2.8.11] the total curvature of $\mathbb{M}^AFL$ is finite and is equal to $-4\pi$. It is the unique complete regular non-orientable minimal surface in $E^4$ of total curvature $-4\pi$ as it follows from [DO86, Theorem 2.2].

Not much is known about the indices of non-orientable surfaces in $E^n$. Ros in [Ros06] proved that there are no stable complete immersed non-orientable surfaces (see also [Ros92]). However, Henneberg’s surface which is not regular is stable as it was shown in [Cho90].
The classical result of Fisher-Colbrie in [FC85] that the finiteness of the total curvature implies the finiteness of the index holds for non-orientable surfaces as it follows from [Nay90, Theorem 1] [Eji91, Theorem B], [CT94] by passing to the orientable cover. The first explicit upper bound for the index of a non-orientable surface was given in [CT88, Theorem 2] in the case of $\mathbb{E}^4$:

$$\text{Ind}(\Sigma) \leq 12.72 \left( \frac{1}{\pi} \int_{\Sigma} (-K_g) dv_g \right),$$

where $g$ is the induced metric on $\Sigma$ and $K_g$ is its Gauss curvature. Recently, Chodosh and Maximo have proved in the paper [CM18] the following explicit index estimate for non-orientable immersed complete minimal surfaces $\Sigma$ in $\mathbb{E}^3$:

$$\frac{1}{3} + \frac{1}{6\pi} \int_{\Sigma} (-K_g) dv_g \leq \text{Ind}(\Sigma) \leq -6 + \frac{3}{\pi} \int_{\Sigma} (-K_g) dv_g,$$

where the lower bound actually follows from another estimate obtained by the authors in the same paper

$$\text{Ind}(\Sigma) \geq \frac{1}{3} \left( \gamma + 2 \sum_{i=1}^{r} (d_i + 1) - 4 \right),$$

for a complete immersed non-orientable minimal surface $\Sigma$ of genus $\gamma$ with ends $E_1, \ldots, E_r$ of multiplicities $d_i, i = 1, r$. Analizing their estimates Chodosh and Maximo also proved that there are no complete immersed non-orientable minimal surfaces of index 1 in $\mathbb{E}^3$. The upper bound in (1.1) follows from the Ejiri-Micallef estimate on the index of a complete (possibly branched) immersed orientable minimal surface in $\mathbb{E}^n$ of finite total curvature (see [EM08, Theorem 1.2]):

$$\text{Ind}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K_g) dv_g + 2\gamma - 2,$$

We reprove estimate (1.3) in the case of non-orientable surfaces and show that a similar inequality holds true in the case of complete non-orientable minimal surfaces of finite total curvature in $\mathbb{E}^n$. Namely, we prove the following theorem

**Theorem 1.1.** Let $\Sigma$ be a (possibly branched) immersed complete non-orientable minimal surface of genus $\gamma$ of finite total curvature in $\mathbb{E}^n$. Then

$$\text{Ind}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K_g) dv_g + \gamma - 1,$$

where $g$ is the induced metric on $\Sigma$ and $K_g$ is its Gauss curvature.

1.1. **Discussion.** The main application of Theorem 1.1 is the following result

**Theorem 1.2.** The index of the Alarcón-Forstnerič-López Möbius band is one.

In order to prove this theorem we first show that $\text{Ind}(MB_{AFL}) \geq 1$ using the same approach as in the paper [Med21]. The straightforward application of Theorem 1.1 implies that

$$\text{Ind}(MB_{AFL}) \leq \frac{1}{4\pi} 4\pi - 1 = 3,$$
since the total curvature of $MB_{AFL}$ is $-4\pi$ (see [AFL20, Section 2.8.11]). However, as we show in Section 5 in the case of the Alarcón-Forstnerič-López Möbius band Theorem 1.1 can be improved. With this improvement one gets that $\text{Ind}(MB_{AFL}) \leq 1$. Note that as it was shown by Chodosh and Maximo in [CM18, Theorem 1.8] there are no complete immersed non-orientable minimal surfaces of index 1 in $E^3$. Theorem 1.2 shows that complete immersed non-orientable minimal surfaces of index 1 exist in $E^4$. We suppose that the Alarcón-Forstnerič-López Möbius band plays the same role as the catenoid in $E^3$. Therefore, the following conjecture seems plausible

**Conjecture 1.3.** The Alarcón-Forstnerič-López Möbius band is the only complete embedded non-orientable surface of index one in $E^4$.

For more applications see Subsection 6.2 of Appendix. We would like to end this section with the following remark

**Remark 1.1.** To the best of our knowledge currently it is not known whether complete immersed stable non-orientable surfaces exist in $E^n$ where $n \geq 4$. If they do not exist then the result that $\text{Ind}(MB_{AFL}) \geq 1$ would follow immediately from it.

1.2. **Plan of the paper.** The paper is organized in the following way. We start with a necessary background described in Section 2. Section 3 contains some technical results that we use in the consequent sections. Section 4 is principle. Here we prove Theorem 1.1 in a slightly more general way as well as its improvement for non-orientable surfaces admitting a global normal field under a certain assumption. In Section 5 we prove that the index of the Alarcón-Forstnerič-López Möbius band equals one. Finally, in Section 6 we consider an Ejiri-Micallef type index estimates for closed non-orientable minimal surfaces in a general Riemannian manifold and more applications of Theorem 1.1.

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2. **Preliminaries**

In this paper we use the following notation

- $E^n$ is the Euclidean space;
- $\Sigma$ is a complete minimal surface in $E^n$ given by the immersion $u: \Sigma \to E^n$;
- $\cdot \cdot$ denotes the standard Euclidean dot-product;
- $\langle \cdot, \cdot \rangle$ and $g$ denote the scalar product and the metric induced on $\Sigma$;
- $\Gamma(N\Sigma), \Gamma(T\Sigma)$ denote the sections of the normal bundle $N\Sigma$ and the tangent bundle $T\Sigma$ over $\Sigma$ respectively;
- $\mathcal{T}\Sigma$ and $\Omega\Sigma$ denote the sheafs of the sections of the bundles $T\Sigma$ and $\Lambda\Sigma$ respectively;
- for any vector $v \in E^n$ $v^\perp$ denotes the projection onto $\Gamma(N\Sigma)$ and $v^\top$ is the projection onto $\Gamma(T\Sigma)$;
\( \nabla^\perp \) is the connection in \( N\Sigma \) and \( \nabla^\top \) is the connection in \( T\Sigma \); the covariant derivative on \( \mathbb{E}^n \) is denoted by \( \nabla \).

Let us also recall the main definitions and facts that we use in the paper:

- the Laplacian on the normal bundle is defined by
  \[
  \Delta^\perp X = \sum_{i=1}^{2} \left( \nabla_{e_i}^\perp \nabla_{e_i}^\perp X - \nabla_{(\nabla_{e_i}^\perp)}^\perp X \right), \quad \forall X \in \Gamma(N\Sigma),
  \]
  where \( e_1, e_2 \) is a local orthonormal basis in \( \Gamma(T\Sigma) \);
- the second fundamental form of \( \Sigma \) is defined as
  \[
  B(X, Y) = (\nabla_X Y)^\perp, \quad \forall X, Y \in \Gamma(T\Sigma),
  \]
  particularly, \( b_{ij} = B(e_i, e_j) \);
- the Simons operator on \( X \in \Gamma(N\Sigma) \) is given by the formula
  \[
  B(X) = 2 \sum_{i,j=1}^{2} (b_{ij} \cdot X)b_{ij};
  \]
- the Jacobi operator on \( X \in \Gamma(N\Sigma) \) is defined as
  \[
  L(X) = \Delta^\perp X + B(X);
  \]
- the second variation of the area of \( \Sigma \) towards the direction \( X \in \Gamma(N\Sigma) \) with compact support in a geodesic ball \( B_R(p) \) centred at a point \( p \in \Sigma \) is the following quadratic form:
  \[
  S(X, X) = -\int_{B_R(p)} \langle L(X), X \rangle dv_g;
  \]
- the (Morse) index \( \text{Ind}(B_R(p)) \) of \( B_R(p) \) is the maximal dimension of a vector subspace \( V \subset \Gamma(NB_R(p)) \) on which \( S \) is negative-definite; the (Morse) index \( \text{Ind}(\Sigma) \) of \( \Sigma \) is defined as \( \lim_{R \to \infty} \text{Ind}(B_R(p)) \);
- the nullity \( \text{Nul}(B_R(p)) \) is the maximal dimension of a vector subspace \( V \subset \Gamma(NB_R(p)) \) on which \( S \) vanishes; \( \text{Nul}(\Sigma) = \lim_{R \to \infty} \text{Nul}(B_R(p)) \);
- the energy functional of the immersion \( u \) is defined as
  \[
  E[u] = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 dv_g;
  \]
- the second variation of the energy of \( \Sigma \) towards the direction \( X \in \mathbb{E}^n \) with compact support in a geodesic ball \( B_R(p) \) centred at a point \( p \in \Sigma \) is the following quadratic form:
  \[
  S_E(X, X) = \int_{B_R(p)} |\nabla X|^2 dv_g;
  \]
- the energy (Morse) index \( \text{Ind}_E(B_R(p)) \) is the maximal dimension of a vector subspace \( V \subset \Gamma(TE^n) \) on which \( S_E \) is negative-definite; also \( \text{Ind}_E = \lim_{R \to \infty} \text{Ind}_E(B_R(p)) \); notice that since \( S_E \) is non-negative definite then one has \( \text{Ind}_E(\Sigma) = 0 \);
- the maximal dimension of a vector subspace \( V \subset \Gamma(TE^n) \) on which \( S_E \) vanishes is called the nullity of the energy functional and is denoted by \( \text{Nul}_E(B_R(p)) \); \( \text{Nul}_E = \lim_{R \to \infty} \text{Nul}_E(B_R(p)) \); notice that \( \text{Nul}_E(\Sigma) = n \) since one can take the constant fields \( v_1, \ldots, v_n \), the standard basis in \( \mathbb{R}^n \).
• the definitions of \( \text{Ind}(\Sigma) \), \( \text{Nul}(\Sigma) \) and \( \text{Ind}_E(\Sigma) \), \( \text{Nul}_E(\Sigma) \) are correct (see for example [FC85]).

We will also consider (possibly branched) minimal immersions of a closed surface \( \Sigma \) into a Riemannian manifold \((M, h)\). The indices \( \text{Ind}(\Sigma) \), \( \text{Nul}(\Sigma) \) and \( \text{Ind}_E(\Sigma) \), \( \text{Nul}_E(\Sigma) \) are defined in a similar way. Moreover, in this case it is not necessary to take exhaustions of \( \Sigma \) by geodesic balls.

3. Technical results

In this section we collect some technical results that we use in the consequent sections. Many of these results are similar to the results in the eponymous section in the paper [Med21]. Therefore, we do not discuss their proofs if they are absolutely similar. The original idea of the results of this section is borrowed from the paper [KW18] (see also [KNPS21, Section 6] for the non-orientable case).

It is well-known that any surface \( \Sigma \) endowed with a Riemannian metric \( g \) admits isothermal local coordinates \((x, y)\) in which \( g \) takes the form \( g = e^{2\omega}(dx^2 + dy^2) \), where \( \omega \in C^\infty(\Sigma) \). Moreover, one can introduce the local complex coordinate \( z = x + iy \) such that \( g = e^{2\omega}|dz|^2 \). If \( E \) is either the normal or the tangent bundle then for any local sections \( X, Y \) of \( E \otimes \mathbb{R} \mathbb{C} \) one can use the Hermitian scalar product \( X \cdot \bar{Y} \), where \( \bar{Y} \) is conjugate to \( Y \). Particularly, \( |X|^2 = X \cdot \bar{X} \). Note also that a minimal immersion \( u: \Sigma \to \mathbb{R}^n \) is always conformal and harmonic.

Claim 1. One has

\[
\begin{align*}
|u_x|^2 &= u_x \cdot u_x = u_y \cdot u_y = e^{2\omega} \quad \text{and} \quad |u_z|^2 = u_z \cdot u_z = \frac{1}{2} e^{2\omega}; \\
u_z \cdot u_z &= u_z \cdot u_z = 0 \quad \text{and} \quad u_{zz} = 0.
\end{align*}
\]

Consider \( u_{zz} \) which is a local section of \( \mathbb{N} \Sigma \otimes \mathbb{R} \mathbb{C} \) and denote it by \( \Omega \). In the following claims we use the notation \( \nabla_z^\perp := \nabla_{z/\partial z} \) and \( \nabla_z^\top := \nabla_{\bar{z}/\partial z} \) and similarly \( \nabla_z^\perp \) and \( \nabla_z^\top \).

Claim 2. One has

\[
\begin{align*}
u_{zz} &= 2\omega_z u_z + \Omega, \\
X_z &= \nabla_z^\perp X - 2e^{-2\omega}(X \cdot \Omega)u_z, \\
X_{\bar{z}} &= \nabla_{\bar{z}}^\perp X - 2e^{-2\omega}(X \cdot \bar{\Omega})u_z,
\end{align*}
\]

for any local section \( X \) of \( \mathbb{N} \Sigma \otimes \mathbb{R} \mathbb{C} \).

Claim 3. The following identities hold

\[
\begin{align*}
\nabla_z^\perp \Omega &= 0, \\
\nabla_z^\perp \nabla_{z}^\perp X - \nabla_{\bar{z}}^\perp \nabla_{z}^\perp X &= 2e^{-2\omega} \left( (X \cdot \Omega) \bar{\Omega} - (X \cdot \bar{\Omega}) \Omega \right),
\end{align*}
\]

for any local section \( X \) of \( \mathbb{N} \Sigma \otimes \mathbb{R} \mathbb{C} \).

Claim 4. The Laplacian on the normal bundle takes the form

\[
\Delta_z X = 2e^{-2\omega} \left( \nabla_z^\perp \nabla_z^\perp X + \nabla_{\bar{z}}^\perp \nabla_{\bar{z}}^\perp X \right),
\]
for any local section $X$ of $N\Sigma \otimes_\mathbb{R} \mathbb{C}$.

Since $\Omega$ is a local section of $N\Sigma \otimes_\mathbb{R} \mathbb{C}$ then we introduce the local sections $\Omega_1, \Omega_2$ of $N\Sigma$ such that $\Omega = \Omega_1 + i\Omega_2$.

**Claim 5.** One has $\Omega_1 = \frac{1}{2}e^{2\varphi}b_{11}$ and $\Omega_2 = -\frac{1}{2}e^{2\varphi}b_{12}$.

### 4. Index upper bounds

This section is devoted to the proof of Theorem 1.1. This theorem is an analog of the Ejiri-Micallef inequality (1.3) for non-orientable surfaces. We start this section with a short description of the Ejiri-Micallef method. Theorem 1.1 is proved in Subsection 4.2. In the end of the section (see Subsection 4.3) we discuss some improvement of Theorem 1.1.

#### 4.1. Ejiri-Micallef method

Suppose that we are given a (possibly branched) minimally immersed closed orientable surface $\Sigma$ in a Riemannian manifold $(M, h)$. We denote this immersion by $u$. Let $g = u^*h$ be the induced metric on $\Sigma$. Assume that there exists a normal field $\xi \in \Gamma(N\Sigma)$ such that the second variation of the volume functional of $\Sigma$ is non-positive. In other words the deformation of $\Sigma$ along the field $\xi$ does not increase its volume. This deformation gives rise to a family of immersions $u_\tau: \Sigma \to (M, h)$ for small values of the parameter $\tau$. If the immersions $u_\tau$ are conformal then as it is well-known the second variations of the volume and the energy functionals coincide. Of course in general $u_\tau$ does not have to be conformal. The idea of Ejiri and Micallef is in a reparametrization of the family of maps $u_\tau$ in order to make it conformal. They solve this problem in the following way. Given $\xi \in \Gamma(N\Sigma)$ we wish to find $X_\xi \in \Gamma(T\Sigma)$ such that the map $\xi \mapsto X_\xi$ is linear and the family of immersions of $\Sigma$ corresponding the field $\xi + X_\xi$ is a family of conformal maps. One then has $S_E(\xi + X_\xi, \xi + X_\xi) = S(\xi, \xi)$. All the fields $\xi$ for which this strategy can be realized form a vector space $U$. Then $\text{Ind}_E(\Sigma) + \text{Nul}_E(\Sigma) \geq \dim U$. Ejiri and Micallef show that this strategy can be realized if the following condition is satisfied

$$
D^{1,0}X_{\xi}^0 = -(\nabla^{1,0}\xi)^\top,
$$

where $X_{\xi}^{0,1}$ is the antiholomorphic part of the field $X_\xi$, $D^{1,0} = \nabla_z^\top \otimes dz$, $\nabla^{1,0} = \nabla_z \otimes dz$ and the superscript $\top$ denotes the projection onto $T\Sigma \otimes \mathbb{C}$. Further by the Fredholm alternative equation (4.1) can be solved if and only if $(\nabla^{1,0}\xi)^\top$ is $L^2$—orthogonal to $\text{Ker}(D^{1,0})^\ast$. One can show that $\text{Ker}(D^{1,0})^\ast = H^0(T^{0,1}\Sigma \otimes \Lambda^{1,0}\Sigma)$, the space of holomorphic sections of the bundle $T^{0,1}\Sigma \otimes \Lambda^{1,0}\Sigma$. Note that using the hermitian metric this space can be identified with the space $H^0((T^{1,0}\Sigma)^\ast \otimes \Lambda^{1,0}\Sigma)$ of holomorphic section of the bundle $(T^{1,0}\Sigma)^\ast \otimes \Lambda^{1,0}\Sigma$. Then the Fredholm alternative reads

$$
\int_\Sigma \langle (\nabla^{1,0}\xi)^\top, W \rangle_g dv_g = 0, \forall W \in H^0((T^{1,0}\Sigma)^\ast \otimes \Lambda^{1,0}\Sigma).
$$

The Fredholm alternative then implies that $\dim U \geq \text{Ind}(\Sigma) + \text{Nul}(\Sigma) - \dim H^0((T^{1,0}\Sigma)^\ast \otimes \Lambda^{1,0}\Sigma)$. Hence,

$$
\text{Ind}(\Sigma) + \text{Nul}(\Sigma) \leq \text{Ind}_E(\Sigma) + \text{Nul}_E(\Sigma) + \dim H^0((T^{1,0}\Sigma)^\ast \otimes \Lambda^{1,0}\Sigma).
$$
Remark 4.1. This approach was also used in the paper [FS16] for the critical catenoid and the critical Möbius band. See also papers [Lim17, Med21] for further developments of the Ejiri-Micallef method in the free boundary case.

4.2. Proof of Theorem 1.1. Let us formulate Theorem 1.1 in a slightly more general way.

**Theorem 4.1.** Let \( \Sigma \) be a (possibly branched) immersed complete non-orientable minimal surface of genus \( \gamma \) of finite total curvature in \( \mathbb{E}^n \). Then

\[
\text{Ind}(\Sigma) + \text{Nul}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K_g)dv_g + \gamma - 1,
\]

where \( g \) is the induced metric on \( \Sigma \) and \( K_g \) is its Gauss curvature.

**Proof.** Pass to the orientable cover \( \tilde{\Sigma} \) and consider \( \iota \), the fixed point free involution changing the orientation. According to the classical results of Hubert and Osserman \( \tilde{\Sigma} \) is conformally equivalent to a compact Riemann surface \( \tilde{\Sigma}_c \) with finitely many punctures and the Gauss maps extents holomorphically to the punctures. The tangent and the normal bundles \( T\Sigma \) and \( N\Sigma \) extend to the ramified bundles \( T\tilde{\Sigma}_c \) and \( N\tilde{\Sigma}_c \) i.e. \( T\tilde{\Sigma}_c|_\Sigma = T\Sigma \) and \( N\tilde{\Sigma}_c|_\Sigma = N\Sigma \). Moreover, the quadratic forms of the second variation of the volume and the energy functionals extend to sections of the bundles \( N\tilde{\Sigma}_c \) and \( \tilde{\Sigma}_c \times \mathbb{R}^n \) respectively. It is proved in [FC85, Corollary 2] and in [Nay90, Remark 1] that \( \text{Ind}(\tilde{\Sigma}) = \text{Ind}(\tilde{\Sigma}_c) \) and \( \text{Nul}(\tilde{\Sigma}) = \text{Nul}(\tilde{\Sigma}_c) \). Taking quotient by \( \iota \) we get that the same statements hold for the tangent and normal bundles over \( \Sigma \) and that \( \text{Ind}(\Sigma) = \text{Ind}(\Sigma_c) \) and \( \text{Nul}(\Sigma) = \text{Nul}(\Sigma_c) \) where \( \Sigma_c = \tilde{\Sigma}_c/\iota \). Consider the space of \( \iota \)-invariant holomorphic sections of the bundle \( (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c \) that we denote by \( H^0_\iota((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) \). As in the proof of [EM08, Theorem 1.1] (see Subsection 4.1) one can show that

\[
\text{Ind}(\Sigma) + \text{Nul}(\Sigma) = \text{Ind}(\Sigma_c) + \text{Nul}(\Sigma_c) \leq \text{Ind}_E(\Sigma_c) + \text{Nul}_E(\Sigma_c) + h^0_\iota,
\]

where \( h^0_\iota = \dim_\mathbb{R} H^0_\iota((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) \). One also obviously gets that

\[
\text{Ind}_E(\Sigma_c) \leq \text{Ind}_E(\tilde{\Sigma}_c) = \text{Ind}_E(\tilde{\Sigma}) = 0,
\]

since any (immersed in our case) complete orientable minimal surface in \( \mathbb{E}^n \) is energetically stable, i.e. the corresponding quadratic form of the second variation is non-negative definite on \( \Gamma T\mathbb{E}^n \) (see Section 2). Hence \( \text{Ind}_E(\Sigma) = 0 \). One also has that \( \text{Nul}_E(\Sigma) = \text{Nul}_E(\tilde{\Sigma}) = n \) (see Section 2). Hence,

\[
(4.3) \quad \text{Ind}(\Sigma) + \text{Nul}(\Sigma) \leq h^0_\iota + n.
\]

Now the question is to estimate \( h^0_\iota \). Let \( \tilde{s} \in H^0_\iota((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) \), i.e. \( \tilde{s}(\iota(x)) = \tilde{s}(x), \forall x \in \tilde{\Sigma} \). Suppose that \( \tilde{s} \) is non-trivial. Then any section of \( H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) \) has form \( f\tilde{s} \), where \( f \) is a meromorphic function on \( \tilde{\Sigma}_c \). Any function \( f \) on \( \tilde{\Sigma}_c \) can be
decomposed into the sum of a $\iota$--invariant function $f_i$ and a $\iota$--antiinvariant function $f_a$. Indeed,

$$f(x) = \frac{f(x) + f(\iota(x))}{2} + \frac{f(x) - f(\iota(x))}{2}.$$ 

Then we take $f_i(x) = \frac{f(x) + f(\iota(x))}{2}$ and $f_a(x) = \frac{f(x) - f(\iota(x))}{2}$. Therefore, any section of $H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c)$ can be decomposed into the sum of a $\iota$--invariant section and $\iota$--antiinvariant section. Let us denote the space of $\iota$--antiinvariant sections of $H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c)$ as $H^0_a((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c)$. Then one has

$$H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) = H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) \oplus H^0_a((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c).$$

Let $\dim_{\mathbb{R}} H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) = 2h^0$, $\dim_{\mathbb{R}} H^0_a((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c) = h^0_a$. Then we have

$$2h^0 = h^0_i + h^0_a.$$

By the Riemann-Roch Theorem one has

$$\dim_{\mathbb{C}} H^0(T^{1,0}\tilde{\Sigma}_c) - \frac{1}{2}(h^0_i + h^0_a) = \deg(T^{1,0}\tilde{\Sigma}_c) + 1 - \gamma.$$

Obviously, $\dim_{\mathbb{C}} H^0(T^{1,0}\tilde{\Sigma}_c) = 0$ since $c_1(T^{1,0}\tilde{\Sigma}_c) = \frac{1}{2\pi}\int_{\tilde{\Sigma}} K_g dv_g < 0$, where $\tilde{g}$ is the metric induced on $\tilde{\Sigma}_c$ and $K_\tilde{g}$ is its Gauss curvature. Then

$$\frac{1}{2}(h^0_i + h^0_a) = \frac{1}{2\pi}\int_{\tilde{\Sigma}} (-K_\tilde{g}) dv_\tilde{g} + \gamma - 1.$$

Therefore, we have

$$h^0_i + h^0_a = \frac{1}{\pi}\int_{\tilde{\Sigma}} (-K_\tilde{g}) dv_\tilde{g} + 2\gamma - 2. \quad (4.4)$$

In order to get the second equality on $h^0_i$ and $h^0_a$ we use the Lefschetz Fixed Point Theorem. Since the involution $\iota: \tilde{\Sigma}_c \to \tilde{\Sigma}_c$ has no fixed points then the Lefschetz Fixed Point Theorem implies

$$0 = \text{trace } \iota^*|_{H^0(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c)} - \text{trace } \iota^*|_{H^1(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c)}, \quad (4.5)$$

where $\iota^*|_{H^0(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c)}$ is the induced action of $\iota$ on the sheaf cohomology group $H^0(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c)$. Choosing the basis in $H^0(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c) \cong H^0((T^{1,0}\tilde{\Sigma}_c)^* \otimes \Lambda^{1,0}\tilde{\Sigma}_c)$ consisting of $h^0_i$ $\iota$--invariant sections and $h^0_a$ $\iota$--antiinvariant sections one may easily see that

$$\text{trace } \iota^*|_{H^0(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c)} = h^0_i - h^0_a.$$

Indeed, the $\iota$--invariant sections are those who have eigenvalue 1 with respect to the action $\iota^*$ and the $\iota$--antiinvariant sections are those who have eigenvalue -1 with respect to the action $\iota^*$. Further, by the Serre duality one has

$$H^1(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c) \cong H^0(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^*).$$
Further we see that $H^0(\tilde{\Sigma}_c, T^{1,0}\tilde{\Sigma}_c) = H^0(T^{1,0}\tilde{\Sigma}_c) = 0$ as we have already discussed. Hence, $H^1(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c) = 0$ and

$$\text{trace} t^*|_{H^1(\tilde{\Sigma}_c, (T^{1,0}\tilde{\Sigma}_c)^* \otimes \Omega^{1,0}\tilde{\Sigma}_c)} = 0.$$ 

Then (4.5) takes the form

$$h^0_i - h^0_a = 0.$$ 

Equalities (4.6) and (4.4) then imply

$$h^0_i = \frac{1}{2\pi} \int_{\tilde{\Sigma}} (-K_\tilde{g}) dv_\tilde{g} + \gamma - 1.$$ 

Substituting the latter into (4.3) we get

$$\text{Ind}(\Sigma) + \text{Nul}(\Sigma) \leq \frac{1}{2\pi} \int_{\tilde{\Sigma}} (-K_\tilde{g}) dv_\tilde{g} + \gamma - 1 + n = \frac{1}{\pi} \int_{\Sigma} (-K_g) dv_g + \gamma + n - 1.$$ 

□

4.3. Improvement of Theorem 1.1. It turns out that the estimate in Theorem 4.1 can be improved in the case when the non-orientable minimal surface admits a global normal vector field under certain condition. This result is analogous to second part of [EM08, Theorem 1.2].

**Theorem 4.2.** Let $\Sigma$ be an immersed complete non-orientable minimal surface of genus $\gamma$ of finite total curvature in $\mathbb{E}^n$, which admits a global section $\nu$ of the normal bundle such that $\nu \cdot \Omega \neq 0$. Then

$$\text{Ind}(\Sigma) + \text{Nul}(\Sigma) \leq \frac{1}{\pi} \int_{\Sigma} (-K_g) dv_g + \gamma + n - 2.$$ 

**Proof.** The proof of this theorem follows the same steps as the proof of the second part of [EM08, Theorem 1.2]. We start with the following lemma

**Lemma 4.3.** Let $\Sigma$ be an immersed complete non-orientable minimal surface in $\mathbb{E}^n$, which admits a global section of the normal bundle. Let $\tilde{\Sigma}$ be the orientable cover of $\Sigma$ and $\tilde{\nu}$ a smooth unit normal vector field on $\tilde{\Sigma}$ which is invariant under the involution $\iota$ changing the orientation. Then $(\nabla \tilde{\nu})^\top \in H^0_t(\tilde{\Sigma}, T^{0,1}\tilde{\Sigma} \otimes \Lambda^{1,0}\tilde{\Sigma}).$

**Proof.** Let $u: \Sigma \to \mathbb{E}^n$ be the minimal immersion of $\Sigma$ and $\tilde{u}: \tilde{\Sigma} \to \mathbb{E}^n$ the minimal immersion of $\tilde{\Sigma}$. One has

$$(\nabla_{\tilde{z}} \tilde{\nu})^\top = \frac{\nabla_{\tilde{z}} \tilde{\nu} \cdot \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z + \frac{\nabla_{\tilde{z}} \tilde{\nu} \cdot \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z = -\frac{\tilde{\nu} \cdot \tilde{u}_{zz} \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z - \frac{\tilde{\nu} \cdot \tilde{u}_{zz} \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z = -\frac{\tilde{\nu} \cdot \tilde{u}_{zz} \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z,$$

since $\tilde{u}_{zz} = 0$ by minimality of the immersion $\tilde{u}$. Hence $(\nabla \tilde{\nu})^\top \in \Gamma_t(T^{0,1}\tilde{\Sigma} \otimes \Lambda^{1,0}\tilde{\Sigma})$. In order to show that $(\nabla \tilde{\nu})^\top$ is holomorphic one computes that $\nabla_{\tilde{z}} (\nabla_{\tilde{z}} \tilde{\nu})^\top = 0$. Indeed,

$$\nabla_{\tilde{z}} (\nabla_{\tilde{z}} \tilde{\nu})^\top = -\nabla_{\tilde{z}} \left( \frac{\tilde{\nu} \cdot \tilde{u}_{zz} \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z \right) = \nabla_{\tilde{z}} \left( \frac{\nabla_{\tilde{z}} \tilde{\nu} \cdot \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z \right) = \nabla_{\tilde{z}} \left( \frac{(\nabla_{\tilde{z}} \tilde{\nu})^\top \cdot \tilde{u}_z}{|\tilde{u}_z|^2} \tilde{u}_z \right) = 0,$$

since $(\nabla_{\tilde{z}} \tilde{\nu})^\top \cdot \tilde{u}_z = 0$. □
Now let $\mathcal{M}_i$ denote the vector space of $\iota$–invariant meromorphic functions on $\tilde{\Sigma}_c$. We then see that

$$H^0_\iota(\tilde{\Sigma}_c, T^{0,1}\tilde{\Sigma}_c \otimes \Lambda^{1,0}\tilde{\Sigma}_c) = \{ f(\nabla\tilde{\nu})^\top \mid f \in \mathcal{M}_i, (f) + ((\nabla\tilde{\nu})^\top) \geq 0 \},$$

where $(f)$ and $((\nabla\tilde{\nu})^\top)$ denote the divisors of $f$ and $(\nabla\tilde{\nu})^\top$ respectively. We define the vector space

$$\mathcal{M}_{L,\iota} := \{ f \in \mathcal{M}_i \mid (f) + ((\nabla\tilde{\nu})^\top) \geq 0 \},$$

where $L = T^{0,1}\tilde{\Sigma}_c \otimes \Lambda^{1,0}\tilde{\Sigma}_c$. Then for any section of the form $\varphi \nu$, where $\varphi \in C^\infty(\tilde{\Sigma})$ is $\iota$–invariant, the Fredholm alternative 4.2 reads

$$\int_{\tilde{\Sigma}} \langle (\nabla\varphi\tilde{\nu})^\top, f(\nabla\tilde{\nu})^\top \rangle_{\tilde{g}} dv_{\tilde{g}} = 0, \forall f \in \mathcal{M}_{L,\iota}. \quad (4.7)$$

Note that $\langle (\nabla\tilde{\nu})^\top, (\nabla\tilde{\nu})^\top \rangle_{\tilde{g}} = |\tilde{\nu} \cdot \tilde{\Omega}|^2$, where $\tilde{\Omega}$ is the lift of $\Omega$, i.e. $\tilde{\Omega} = \tilde{u}_\perp^{\perp}$. Then we get

$$\int_{\tilde{\Sigma}} \varphi f|\tilde{\nu} \cdot \tilde{\Omega}|^2 dv_{\tilde{g}} = 0, \forall f \in \mathcal{M}_{L,\iota}.$$

Since $|\tilde{\nu} \cdot \tilde{\Omega}| \neq 0$ (otherwise $|\nu \cdot \Omega| = 0$ which is impossible by the hypothesis of the theorem) and the constant functions belong to $\mathcal{M}_{L,\iota}$ one gets that real codimension of $\iota$–invariant functions $\varphi$ satisfying (4.7) is $h^0_\iota - 1 = \frac{1}{\pi} \int_{\Sigma}(−K_\iota)dv_\iota + \gamma - 2. \quad \Box$

5. Index bounds for the Alarcón-Forstnerič-López Möbius band

In this section we show that the index of the Alarcón-Forstnerič-López Möbius band is one.

First of all we reparametrize the Alarcón-Forstnerič-López Möbius band in a suitable way. Recall that the original parametrization is given by the formula

$$u(\rho, \varphi) =$$

$$= \left( \left( \frac{1}{\rho} - \rho \right) \sin \varphi, -\left( \frac{1}{\rho} - \rho \right) \cos \varphi, -\frac{1}{2} \left( \frac{1}{\rho^2} + \rho^2 \right) \sin 2\varphi, \frac{1}{2} \left( \frac{1}{\rho^2} + \rho^2 \right) \cos 2\varphi \right),$$

where $(\rho, \varphi) \in (0, +\infty) \times \mathbb{R}$. This embedding is invariant with respect to the map $(\rho, \varphi) \mapsto (1/\rho, \pi + \varphi)$. Now let $t := e^\rho$ and $\theta := -\varphi$. Then we get the following parametrization

$$u(t, \theta) = (2 \sinh t \sin \theta, 2 \sinh t \cos \theta, \cosh 2t \sin 2\theta, \cosh 2t \cos 2\theta), \quad (5.1)$$

where $(t, \theta) \in (-\infty, +\infty) \times \mathbb{R}$. Note that in these coordinates the embedding is invariant under the map $(t, \theta) \mapsto (-t, \pi + \theta)$.

Remark 5.1. The parametrization (5.1) is the same (up to changing the order of coordinates) as the parametrization of the critical Möbius band (see [FS16, Proposition 7.1]) which makes the coordinates $(t, \theta)$ more suitable for the further computations since many of them are similar to the computations for the critical Möbius band in the paper [Med21].
Proposition 5.1. The index of the Alarcón-Forstnerič-López Möbius band is at least one.

Proof. Pass to the orientable cover of the Alarcón-Forstnerič-López Möbius band that we denote as $\overline{\operatorname{MB}_{AFL}}$. It is defined as the immersion $\tilde{u}: \overline{\operatorname{MB}_{AFL}} \to \mathbb{E}^4$. Topologically, $\overline{\operatorname{MB}_{AFL}}$ is a cylinder $(-\infty, +\infty) \times S^1$. Introduce the complex coordinate $z = t + i\theta, (t, \theta) \in (-\infty, +\infty) \times S^1$, which is global on $\overline{\operatorname{MB}_{AFL}}$. Hence, the fields $\partial/\partial z, \tilde{u}_z, \tilde{u}_\bar{z}$ are also globally defined on $\overline{\operatorname{MB}_{AFL}}$. Then Claim 2 implies that $\Omega$ is a global section of $N\overline{\operatorname{MB}_{AFL}} \otimes \mathbb{C}$.

Lemma 5.2. Let $\overline{\operatorname{MB}_{AFL}}$ be the orientable cover of the Alarcón-Forstnerič-López Möbius band then the function $\Omega \cdot \Omega$ on it is a positive real constant.

Proof. First, we prove that the function $(\Omega \cdot \Omega)$ is well-defined. Consider the quartic Hopf differential $\mathcal{H} = (\Omega \cdot \Omega)dz^4$ and compute it on the global field $\partial/\partial \theta$:

$$\mathcal{H}(\partial/\partial \theta) = (\Omega \cdot \Omega)dz(\partial/\partial \theta)^4 = (\Omega \cdot \Omega)(-i)^4 = \Omega \cdot \Omega.$$ 

Hence, $\Omega \cdot \Omega$ is globally well-defined. Further, using the computations in [Med21, Example 5.1] one has that $\Omega_1 \cdot \Omega_2 = 0$ and then $\Omega \cdot \Omega = |\Omega_1|^2 - |\Omega_2|^2$. Moreover, by Claim 5 $\Omega_1 = \frac{1}{2}e^{2\phi}b_{11} = \frac{1}{2}u_i^1$ and $\Omega_2 = -\frac{1}{2}e^{2\phi}b_{12} = -\frac{1}{2}u_\bar{\phi}^1$. Hence, $\Omega \cdot \Omega = \frac{1}{4}(|u_i^1|^2 - |u_\bar{\phi}^1|^2) = 3$ as it follows from [Med21, Example 5.1].

Lemma 5.3. Let $\overline{\operatorname{MB}_{AFL}}$ be the orientable cover of the Alarcón-Forstnerič-López Möbius band. Then one has

- $\Omega_1 \cdot \Omega_2 = 0$ and $\Omega \cdot \Omega = |\Omega_1|^2 - |\Omega_2|^2$;
- $\Delta^4 \Omega = 4e^{-4\phi}((\Omega \cdot \Omega)\bar{\Omega} - (\Omega \cdot \bar{\Omega})\Omega)$;
- $\Delta^4 \Omega_1 = -8e^{-4\phi}|\Omega_2|^2\Omega_1$ and $\Delta^4 \Omega_2 = -8e^{-4\phi}|\Omega_1|^2\Omega_2$;
- $B(\Omega_j) = 8e^{-4\phi}|\Omega_j|^2\Omega_j, j = 1, 2$.

Proof. The proof repeats the proof of [Med21, Claim 6].

Let $\varphi \in C^\infty_0(\mathbb{B}_{2R}(p))$ be a smooth positive function with compact support in the ball $\mathbb{B}_{2R}(p) \subset \overline{\operatorname{MB}_{AFL}}$ centred at some point $p$. Choose $\varphi$ such that $\varphi = 1$ on $\mathbb{B}_R(p) \subset \mathbb{B}_{2R}(p)$. Consider the fields $\Omega_1' := \varphi \Omega_1$ and $\Omega_2' := \varphi \Omega_2$. Then one sees that

$$S(\Omega_1, \Omega_1) = \lim_{R \to \infty} S(\Omega_1', \Omega_1').$$

It follows from Lemma 5.3 that one has

$$L(\Omega_1) = 8e^{-4\phi}(|\Omega_1|^2 - |\Omega_2|^2)\Omega_1.$$

By Lemma 5.2 $\Omega \cdot \Omega$ is a positive real constant. Without loss of generality one can assume that $\Omega \cdot \Omega = 1$. We have

$$L(\Omega_1) = -8e^{-4\phi}\Omega_1$$

and hence

$$S(\Omega_1, \Omega_1) = -\int_{\overline{\operatorname{MB}_{AFL}}} L(\Omega_1) \cdot \Omega_1 dv_{\bar{g}} = -8 \int_{\overline{\operatorname{MB}_{AFL}}} e^{-4\phi}|\Omega_1|^2 dv_{\bar{g}},$$
which is strictly negative since $|\Omega_2|^2 = 1 + |\Omega_2|^2$ is strictly positive.

Finally, noticing that $\Omega_1$ is invariant under the involution changing the orientation
we get that the field $\Omega_1$ descends to $\mathcal{MB}_{AFL}$ as a global normal field and the second
variation of the volume is negative on it. Then $\text{Ind}(\mathcal{MB}_{AFL}) \geq 1$.

\[ \square \]

It remains to show that $\text{Ind}(\mathcal{MB}_{AFL}) \leq 1$. In order to obtain this inequality we
first estimate the nullity of $\mathcal{MB}_{AFL}$.

**Proposition 5.4.** The nullity of the Alarcón-Forstnerič-López Möbius band is at least
five.

**Proof.** The normal bundle to $\tilde{\mathcal{MB}}_{AFL}$ is trivial since $\tilde{\mathcal{MB}}_{AFL}$ is orientable (see e.g. [Fra07]).

As in the end of the proof of Proposition 5.1 without loss of generality one can assume $|\Omega_1|^2 - |\Omega_2|^2 = 1$. Moreover, $\Omega_1 \cdot \Omega_2 = 0$. Then there exist global unit normal
fields $N_1, N_2$ and a function $\mu$ such that $\Omega_1 = \cosh \mu N_1, \Omega_2 = \sinh \mu N_2$. Indeed, since
$|\Omega_1| \geq 1$ we can set $N_1 = \frac{\Omega_1}{|\Omega_1|}$. Then the field $N_2$ is defined as a unit field such that
the orthogonal frame $\tilde{u}_t, \tilde{u}_\theta, N_1, N_2$ is positive oriented at every point $p \in \tilde{\mathcal{MB}}_{AFL}$.

Following the proof of [KW18, Theorem 3.1 (2)] we introduce a complex structure $J$ on $N\tilde{\mathcal{MB}}_{AFL}$ as

$$ JN_1 = N_2, \quad JN_2 = -N_1, $$

which is integrable by the Newlander-Nirenberg Theorem. Note also that $\nabla J = J\nabla$ and hence $\nabla^\perp J = J\nabla^\perp$. Moreover $L(J\Omega_2) = 0$. Indeed,

$$ \Delta^\perp (J\Omega_2) = J\Delta^\perp (\Omega_2) = -8e^{-4\omega}|\Omega_1|^2 J\Omega_2 = 8e^{-4\omega} \sinh \mu \cosh^2 \mu N_2 $$

and

$$ B(J\Omega_2) = \sum_{i,j=1}^2 (b_{ij} \cdot J\Omega_1)b_{ij} = 8e^{-4\omega} ((\Omega_1 \cdot J\Omega_2)\Omega_1 + (\Omega_2 \cdot J\Omega_2)\Omega_2) = $$

$$ = -8e^{-4\omega} \sinh \mu \cosh^2 \mu N_2. $$

We have used Claims 5 and Lemma 5.3 in both computations. Thus, $S(J\Omega_2, J\Omega_2) = 0$, i.e. $J\Omega_2$ is a Jacobi field. Moreover, there are 4 linearly independent Jacobi fields $v_1^+, \ldots, v_4^+$, where $v_i, i = 1, 4$ is the standard basis in $\mathbb{R}^4$. We need to show that the fields $J\Omega_2$ and $v_i^+, i = 1, 4$ are linearly independent. Suppose that

$$ J\Omega_2 + v_i^+ = 0 \text{ on } \tilde{\mathcal{MB}}_{AFL} $$

for some non-zero vector $v \in \mathbb{R}^4$. This particularly implies that $v \cdot N_2 = 0$ for any
point $p \in \tilde{\mathcal{MB}}_{AFL}$. Since $\Omega_2 = \sinh \mu N_2$ we then get $v \cdot \Omega_2 = 0$ which the same as $v \cdot \tilde{u}_\theta = 0$. Of course this identity is not trivial at the points where $\mu \neq 0$. These
points correspond to the points where $\tilde{u}_\theta \neq 0$. For example the last inequality holds
for the points \((t, \theta)\) from the set \(\{(0,0), (0, \frac{\pi}{2}), (1, 0), (1, \frac{\pi}{4})\}\). Using the formulae
\[
\tilde{u}_\iota = (2 \cosh t \cos \theta, 2 \cosh t \sin \theta, 2 \sinh 2t \cos 2\theta, 2 \sinh 2t \sin 2\theta), \\
\tilde{u}_\theta = (-2 \sinh t \sin \theta, 2 \sinh 2t \cos 2\theta, 2 \cosh 2t \cos 2\theta), \\
\tilde{u}_{\iota\theta} = (2 \sinh t \cos \theta, 2 \sinh 2t \sin \theta, 4 \cosh 2t \cos 2\theta, 4 \cosh 2t \sin 2\theta), \\
\tilde{u}_{t\theta} = (-2 \cosh t \sin \theta, 2 \cosh 2t \sin \theta, -4 \sinh 2t \sin 2\theta, 4 \sinh 2t \cos 2\theta),
\]
and computing \(\tilde{u}_{\iota\theta}^\perp\) at the points \(\{(0,0), (0, \frac{\pi}{2}), (1, 0), (1, \frac{\pi}{4})\}\) we obtain that the identity \(v \cdot \tilde{u}_{\iota\theta}^\perp = 0\) holds if and only if the vector \(v\) is zero. Contradiction. \(\square\)

**Corollary 5.5.** The index of the Alarcón-Forstnerič-López Möbius band is one.

**Proof.** Notice that for the Alarcón-Forstnerič-López Möbius band one can define a global normal vector field \(\nu\) in the following way. As in the end of the proof of Proposition 5.1 one can assume that \(|\Omega_1|^2 - |\Omega_2|^2 = 1\). Hence \(|\Omega_1| \geq 1\) and we will define \(\nu\) as the projection of the unit normal vector field \(\frac{\Omega_1}{|\Omega_1|}\) under the quotient by \(\iota\). For this normal field one actually has that \(\nu \cdot \Omega = \Omega_1 \neq 0\) since \(|\Omega_1| \geq 1\). Then the Alarcón-Forstnerič-López Möbius band satisfies Theorem 4.2. We get
\[
\text{Ind}(\mathcal{MB}_{AFL}) + \text{Nul}(\mathcal{MB}_{AFL}) \leq 6.
\]
On the other hand by Proposition 5.4 we know that \(\text{Nul}(\mathcal{MB}_{AFL}) \geq 5\). This implies that \(\text{Ind}(\mathcal{MB}_{AFL}) \leq 1\). But by Proposition 5.1 \(\text{Ind}(\mathcal{MB}_{AFL}) \geq 1\). \(\square\)

### 6. Appendix

6.1. **Ejiri-Micallef type inequalities for non-orientable surfaces.** In this section we formulate a result analogous to the Ejiri-Micallef inequalities [EM08, Theorem 1.1] for non-orientable surfaces. This result was used in [KNPS21, Section 6] in the case when \(\Sigma\) is the Klein bottle \(\tilde{\tau}_{3,1}\) (the bipolar Lawson surface) in the round sphere \(S^4\).

**Theorem 6.1.** Let \(\Sigma\) be a (possibly branched) immersed closed non-orientable minimal surface of genus \(\gamma\) in a Riemannian manifold \((M, g)\). Then
\[
\text{Ind}_E(\Sigma) \leq \text{Ind}(\Sigma) \leq \text{Ind}_E(\Sigma) + r,
\]
where, if \(b\) is the number of branch points of the immersion counted with multiplicity, then
\[
r = \begin{cases} 
3\gamma - 3 - b, & \text{if } b \leq \gamma - 2, \\
2\gamma - 1 - b, & \text{if } \gamma - 1 \leq b \leq 2\gamma - 2, \\
0, & \text{if } b \geq 2\gamma - 1.
\end{cases}
\]
Particularly, if \(\gamma = 0\), i.e. \(\Sigma\) is \(\mathbb{RP}^2\) then \(r = 0\). If \(\gamma = 1\), i.e. \(\Sigma\) is \(\text{KL}\) then \(r = 1\) if \(b = 0\) and \(r = 0\) if \(b > 0\).
Proof. The proof is analogous to the proof of Theorem 4.1. First we show that
\[ \text{Ind } \Sigma \leq \text{Ind } E \Sigma + h^0, \]
where \( h^0 = \dim_{\R} H^0(\ell \circ \Sigma) \) is the dimension of the space of the \( \ell \)-invariant sections of the bundle \((T^1,0 \Sigma)^{*} \otimes \Lambda^1,0 \Sigma)\) and \( \tilde{\Sigma} \) is the orientable cover of \( \Sigma \). The lower bound \( \text{Ind } E \Sigma \leq \text{Ind } (\Sigma) \) is trivial. Next using the Riemann-Roch Theorem, the Lefschetz Fixed Point Theorem and the Serre duality we show that \( h^0 = h^0 \), where 2\( h^0 \geq \dim_{\R} H^0(\Sigma, (T^1,0 \Sigma)^{*} \otimes \Lambda^1,0 \Sigma) \). As it was computed in the proof of [EM08, Theorem 1.1] by the Riemann-Roch Theorem
\[ h^0 = 3\gamma - 3 - \tilde{b} + \dim_{\C} H^0(T^1,0 \tilde{\Sigma}), \]
where \( \tilde{b} = 2b \) is the number of branched points on \( \tilde{\Sigma} \). If \( \tilde{b} \leq 2\gamma - 3 \) then \( c_1(T^1,0 \tilde{\Sigma}) < 0 \) hence \( \dim_{\C} H^0(T^1,0 \tilde{\Sigma}) = 0 \) and \( h^0 = 3\gamma - 3 - \tilde{b} \). If \( \tilde{b} \geq 4\gamma - 3 \) then \( c_1((T^1,0 \Sigma)^{*} \otimes \Lambda^1,0 \Sigma) < 0 \) hence \( h^0 = 0 \). If \( 2\gamma - 2 \leq \tilde{b} \leq 4\gamma - 4 \) then \( 0 \leq c_1((T^1,0 \Sigma)^{*} \otimes \Lambda^1,0 \Sigma) \leq 2\gamma - 2 \) and by Clifford’s theorem \( h^0 \leq \left\lfloor \frac{4\gamma - 2 - \tilde{b}}{2} \right\rfloor \). \( \square \)

6.2. Other applications of Theorem 1.1. The Meeks Möbius band and the López Klein bottle are properly immersed in \( \E^3 \). Therefore, by the Chodosh-Maximo result their index is at least 2. On the other hand since the total curvature of the Meeks Möbius band is \(-6\pi\) and of the López Klein bottle is \(-8\pi\) we get

**Corollary 6.2.** Let \( \text{MB}_M \) be the Meeks Möbius band and \( \text{KL}_L \) be the López Klein bottle. Then
\[ 2 \leq \text{Ind}(\text{MB}_M) \leq 5, \quad 2 \leq \text{Ind}(\text{KL}_L) \leq 8. \]

Moreover, estimate (1.4) enables us to give an upper bound for the total curvature of non-orientable minimal surfaces in \( \E^3 \). As it was shown in [MI81] the total curvature of complete immersed non-orientable minimal surfaces in \( \E^3 \) is at most \(-6\pi\). Combining estimate (1.2) with our estimate we get
\[ \frac{1}{3}(\gamma + 2 \sum_{i=1}^{r} (d_i + 1) - 4) \leq \text{Ind } \Sigma \leq \frac{1}{\pi} \int_{\Sigma} (-K_g) dv_g + \gamma - 1, \]
which implies
\[ \int_{\Sigma} K_g dv_g \leq \frac{\pi}{3} \left( 1 + \gamma - 2 \sum_{i=1}^{r} (d_i + 1) \right). \]

Of course this estimate makes sense for surfaces of high number of ends or lower genus. This estimate improves the Meeks result in the case of
\[ \sum_{i=1}^{r} (d_i + 1) \geq \frac{\gamma + 19}{2}, \]
as it follows from
\[ \frac{\pi}{3} \left( 1 + \gamma - 2 \sum_{i=1}^{r} (d_i + 1) \right) \leq -6\pi. \]
Note that by the result [CM18, Lemma 2.1] \[ \sum_{i=1}^{r}(d_i + 1) \geq \frac{\gamma + 19}{2}. \] Therefore, we get

**Corollary 6.3.** Let \( \Sigma \) be a complete immersed non-orientable minimal surface of genus \( \gamma \) in \( \mathbb{E}^3 \). Let \( E_1, \ldots, E_r \) be the ends of \( \Sigma \) with multiplicities \( d_i, i = 1, \ldots, r \). Suppose that \[ \sum_{i=1}^{r}(d_i + 1) \geq \frac{\gamma + 19}{2}. \] Then

\[ \int_{\Sigma} K_g \, dv_g \leq \frac{\pi}{3} \left( 1 + \gamma - 2 \sum_{i=1}^{r}(d_i + 1) \right). \]

We end this section with the following remark

**Remark 6.1.** If there exist complete non-orientable surfaces of genus 0 and of total curvature \(-\pi\) in \( \mathbb{E}^n \) then they must be stable as it follows from (1.4). These non-orientable surfaces must have branch points as it follows from the result of Meeks in \( \mathbb{E}^3 \) [MI81] and from the result of de Oliveira in \( \mathbb{E}^n, n \geq 4 \) [DO86, Proposition 1.4].

**References**

[AFL20] A. Alarcón, F. Forstnerič, and F. J. López. *New complex analytic methods in the study of non-orientable minimal surfaces in \( \mathbb{R}^n \)*, volume 1283. American Mathematical Soc., 2020.

[AFL21] A. Alarcón, F. Forstnerič, and F. J. López. *Minimal surfaces from a complex analytic viewpoint*. Springer, 2021.

[Cho90] J. Choe. Index, vision number and stability of complete minimal surfaces. *Archive for Rational Mechanics and Analysis*, 109(3):195–212, 1990.

[CM16] O. Chodosh and D. Maximo. On the topology and index of minimal surfaces. *Journal of Differential Geometry*, 104(3):399–418, 2016.

[CM18] O. Chodosh and D. Maximo. On the topology and index of minimal surfaces ii. *arXiv preprint arXiv:1808.06572*, 2018.

[CT88] S.-Y. Cheng and J. Tysk. An index characterization of the catenoid and index bounds for minimal surfaces in \( \mathbb{R}^4 \). *Pacific Journal of Mathematics*, 134(2):251–260, 1988.

[CT94] S.-Y. Cheng and J. Tysk. Schrödinger operators and index bounds for minimal submanifolds. *The Rocky Mountain Journal of Mathematics*, 24(3):977–996, 1994.

[dCP79] M. do Carmo and C.-K. Peng. Stable complete minimal surfaces in \( \mathbb{R}^3 \) are planes. *Bulletin (New Series) of the American Mathematical Society*, 1(6):903–906, 1979.

[DO86] M. E. G. G. De Oliveira. Some new examples of nonorientable minimal surfaces. *Proceedings of the American Mathematical Society*, 98(4):629–636, 1986.

[Eji91] N. Ejiri. Two applications of the unit normal bundle of a minimal surface in \( \mathbb{R}^n \). *Pacific Journal of Mathematics*, 147(2):291–300, 1991.

[EM08] N. Ejiri and M. Micallef. Comparison between second variation of area and second variation of energy of a minimal surface. *Advances in Calculus of Variations*, 1(3):223–239, 2008.

[FC85] D. Fischer-Colbrie. On complete minimal surfaces with finite morse index in three manifolds. *Inventiones mathematicae*, 82(1):121–132, 1985.

[FCS80] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Communications on Pure and Applied Mathematics*, 33(2):199–211, 1980.

[Fra07] A. Fraser. Index estimates for minimal surfaces and \( k \)-convexity. *Proceedings of the American Mathematical Society*, 135(11):3733–3744, 2007.

[FS16] A. Fraser and R. Schoen. Sharp eigenvalue bounds and minimal surfaces in the ball. *Inventiones mathematicae*, 203(3):823–890, 2016.
[Hen75] L. Henneberg. *Ueber solche Minimalflächen: welche eine vorgeschriebene ebene Curve zur geodätischen Linie haben...*, volume 9. Druck von Zürcher und Furrer, 1875.

[KNPS21] M. Karpukhin, M. Nahon, I. Polterovich, and D. Stern. Stability of isoperimetric inequalities for Laplace eigenvalues on surfaces. *arXiv preprint arXiv:2106.15043*, 2021.

[KW18] R. Kusner and P. Wang. On the index of minimal 2-tori in the 4-sphere. *arXiv preprint arXiv:1803.01615*, 2018.

[Lim17] V. Lima. Bounds for the Morse index of free boundary minimal surfaces. *arXiv preprint arXiv:1710.10971*, 2017.

[KW18] R. Kusner and P. Wang. On the index of minimal 2-tori in the 4-sphere. *arXiv preprint arXiv:1803.01615*, 2018.

[Med21] V. Medvedev. On the index of the critical Möbius band in $\mathbb{B}^4$. *arXiv preprint arXiv:2112.04883*, 2021.

[MIL81] W. H. Meeks III. The classification of complete minimal surfaces in $\mathbb{R}^3$ with total curvature greater than $-8\pi$. *Duke Mathematical Journal*, 48(3):523–535, 1981.

[MR91] S. Montiel and A. Ros. Schrödinger operators associated to a holomorphic map. In *Global differential geometry and global analysis*, pages 147–174. Springer, 1991.

[Nay90] S. Nayatani. On the morse index of complete minimal surfaces in euclidean space. *Osaka Journal of Mathematics*, 27(2):441–451, 1990.

[Pog81] A. V. Pogorelov. On the stability of minimal surfaces. In *Doklady Akademii Nauk*, volume 260, pages 293–295. Russian Academy of Sciences, 1981.

[Ros92] M. Ross. Complete nonorientable minimal surfaces in $\mathbb{R}^3$. *Commentarii mathematici Helvetici*, 67(1):64–76, 1992.

[Ros06] A. Ros. One-sided complete stable minimal surfaces. *Journal of Differential Geometry*, 74(1):69–92, 2006.

[Tuz91] A. A. Tuzhilin. Morse-type indices of of two-dimensional minimal surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 55(3):581–607, 1991.

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