SOME PARTICULAR SELF-INTERACTING DIFFUSIONS: ERGODIC BEHAVIOR AND ALMOST SURE CONVERGENCE

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Abstract. This paper deals with some self-interacting diffusions \((X_t, t \geq 0)\) living on \(\mathbb{R}^d\). These diffusions are solutions to stochastic differential equations:

\[
dX_t = dB_t - g(t)\nabla V(X_t - \mu_t)dt,
\]

where \(\mu_t\) is the mean of the empirical measure of the process \(X_t\), \(V\) is an asymptotically strictly convex potential and \(g\) is a given function. We study the ergodic behavior of \(X_t\) and prove that it is strongly related to \(g\). Actually, we show that \(X_t\) is ergodic (in the limit-quotient sense) if and only if \(\mu_t\) converges a.s. We also give some conditions (on \(g\) and \(V\)) for the almost sure convergence of \(X_t\).

1. Introduction

Processes with path-interaction have been an intensive research area since the seminal work of Norris, Rogers and Williams \[13\]. More precisely, self-interacting diffusions have been first introduced by Durrett and Rogers \[7\] under the name of Brownian polymers. They proposed a model for the shape of a growing polymer. Denoting by \(X_t\) the location of the end of the polymer at time \(t\), \(X_t\) satisfies a Stochastic Differential Equation (SDE) with a drift term depending on its own occupation measure (in dimension 1, we define it through the local time of \(X_t\)). One is then interested in rescaling \(X_t\). This process has been studied by different authors (see \[5, 6, 9, 12, 15\]). They show in particular in the self-attracting case, that \(X_t\) converges a.s. Later, an other model of polymers has been proposed by Benaïm, Ledoux and Raimond \[2\]. They have studied, in the compact case, self-interacting diffusions depending on the empirical measure. When the process is living on a compact Riemannian manifold, they have proved that the asymptotic behavior of the empirical measure can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. Benaïm and Raimond \[3\] went further in this study and in particular, they gave sufficient conditions for the a.s. convergence of the empirical measure. Very recently, Raimond \[16\] has generalized the previous work: he has studied the asymptotic properties of a process \(X_t\), living on a Riemannian compact manifold \(M\), solution to the SDE

\[
dX_t = dB_t - g(t)\nabla V \ast \mu_t(X_t)dt,
\]

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with $V * \mu_t(x) = \frac{1}{t} \int_0^t V(x, X_s) ds$, $\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$ and $g(t) = a \log(1 + t)$ (or $|g(t)| \leq a \log(t)$ and $g'(t) = O(t^{-\gamma})$ with $0 < \gamma \leq 1$). He has proved that, else if $g$ is constant, the approximation of $\mu_t$ by a deterministic flow is not true. He has more particularly investigated the example $M = S^n$ and $V(x, y) = -\cos d(x, y)$ (where $d$ is the geodesic distance on $S^n$) and proved that a.s. $\mu_t$ converges weakly towards a Dirac measure. For an overview on reinforced processes, we refer the reader to Pemantle’s survey [14].

In the present paper, we are concerned with some self-interacting processes living on $\mathbb{R}^d$. Consider a smooth potential $V : \mathbb{R}^d \to \mathbb{R}_+$ and an application $g : \mathbb{R}_+ \to \mathbb{R}_+^*$. Our goal is to study the ergodic behavior of the self-interacting diffusion $X$ solution to

\begin{equation}
\begin{cases}
\quad dX_t = dB_t - g(t) \nabla V(X_t - \bar{\mu}_t) dt \\
\quad X_0 = x
\end{cases}
\end{equation}

where $B$ is a standard Brownian motion and $\bar{\mu}_t$ denotes the empirical mean of the process $X$:

\begin{equation}
\bar{\mu}_t = \frac{1}{r + t} \left( r \bar{\mu} + \int_0^t X_s ds \right), \quad \bar{\mu}_0 = \bar{\mu}.
\end{equation}

Here $\mu$ is an initial (given) probability measure on $\mathbb{R}^d$, $\bar{\mu}$ denotes the mean of $\mu$ and $r > 0$ is an initial weight (it permits to consider any initial probability measure).

First, note that for a quadratic interaction potential $V$, the process satisfying (1.2) is exactly of the form of (1.1) and in both cases, the occupation measure is penalized by $g(t)$. Afterwards, a natural generalization of this process is the class of self-interacting diffusions discussed here. The interesting point is that we manage to study precisely the asymptotic behavior of $X$ and prove a convergence criterion. Moreover, this model could be used to modelize the behavior of social insects, as the ants trails. Indeed, ants mark their paths with the trails pheromones. Certain ants lay down an initial trail of pheromones as they return to the nest with food. This trail attracts other ants and serves as a guide. As long as the food source remains, the pheromone trail will be continually renewed. Despite the quick evaporation, the path is reinforced and so, the ants manage to gradually find the best route. In this (simplified) model, the function $g$ is the speed of evaporation and $X$ denotes the trail.

In order to study the behavior of $X$, solution to (1.2), it is natural to introduce the process $Y$, defined by

\begin{equation}
Y_t = X_t - \bar{\mu}_t.
\end{equation}

It appears that $(Y_t, t \geq 0)$ is the solution to the SDE

\begin{equation}
\begin{cases}
\quad dY_t = dB_t - g(t) \nabla V(Y_t) dt - Y_t \frac{dt}{r + t}, \quad Y_0 = x - \bar{\mu} \\
\quad d\bar{\mu}_t = Y_t \frac{dt}{r + t}.
\end{cases}
\end{equation}

The study of $Y$ is obviously easier than the study of $X$, because $Y$ is a (non-homogeneous) Markov process. Indeed, we will prove that $Y$ converges a.s. and satisfies the pointwise ergodic theorem. Because of that, the behavior of $X$ could seem a bit easy at first glance. But, it really shows unexpected behaviors and in particular, it does not satisfy the
pointwise ergodic theorem in general, (because \( \mu_t \) does not converge, except for functions \( g \) going fast to infinity). This explains how difficult is the study of more general self-interacting diffusions in non-compact spaces (see Kurtzmann [10]), driven by the equation

\[
dX_t = dB_t - \int_{\mathbb{R}^d} \nabla V(X_t, x) d\mu_t(x) dt.
\]

In this paper, we give a description of the asymptotic behavior of both \( \mu_t \) and \( X_t \). For simplicity, we suppose that the potential \( V \) does not admit any degenerate critical point. Of course, this assumption will be weakened in the following. First, we state the ergodic result:

**Theorem 1.1.**

1. The process \( Y_t \) satisfies the pointwise ergodic theorem: with probability 1, the empirical measure of \( Y \) converges weakly to a random measure, and what is more, this last measure is a convex combination of Dirac measures taken in the critical points of \( V \).
2. The process \( X_t \) satisfies the pointwise ergodic theorem if and only if \( \mu_t \) converges almost surely.

Remark, that a necessary condition for the convergence of \( \mu_t \) is that \( V \) admits a unique minimum in 0. We will prove this result in §5.1.

The second and main result of this paper is the following description of the asymptotic behavior of \( X_t \), shown in §6.2:

**Theorem 1.2.** Suppose \( \sqrt{g(t)^{-1} \log G(t)} = O(h(t)^{-1}) \), where \( G \) is a primitive of \( g \) and \( \int_0^\infty \frac{ds}{(1+s)h(s)} < \infty \).

1. Then the process \( Y_t \) converges almost surely to \( Y_\infty \), where \( Y_\infty \) belongs to the set of the local minima of \( V \). Moreover, for all local minimum \( m \) of \( V \), one has \( \mathbb{P}(Y_\infty = m) > 0 \).
2. On one hand, on the set \( \{Y_\infty = 0\} \), we have that both \( X_t \) and \( \mu_t \) converge almost surely to \( \bar{\mu}_\infty := \bar{\mu} + \int_0^\infty \frac{dY_s}{r+s} \). On the other hand, on the set \( \{Y_\infty \neq 0\} \), we get that \( \lim_{t \to \infty} X_t / \log t = Y_\infty \).

The remainder of the paper is organized in the following way. In Section 2, we motivate our study by the basic case \( V \) quadratic, for which we have an explicit expression of \( X \) and \( Y \) (in terms of Brownian martingales). Afterwards, we introduce the notations and hypotheses. Section 4 deals with the description of the behavior of \( Y \) around the local extrema of \( V \). Then, we study in Section 5 the ergodic behavior of \( Y \) and give conditions for the almost sure convergence of \( Y \). Finally, Section 6 is divided in two parts. The first one is devoted to the proof of the main results, whereas the second one deals with conditions for the almost sure convergence of \( X \) (depending on \( g \)).

### 2. A MOTIVATING EXAMPLE: THE QUADRATIC CASE

Let \( X \) be the solution of the SDE

\[
\begin{align*}
\left\{ \begin{array}{l}
dX_t = dB_t - g(t) \nabla V * \mu_t(X_t) dt \\
X_0 = x
\end{array} \right.
\end{align*}
\]  

(2.1)
where $V \ast \mu_t(x) := \int V(x - y)\mu_t(dy)$ and $\mu_t$ is the empirical measure of the process, namely

$$\mu_t = \frac{r}{r + t}\mu + \frac{1}{r + t} \int_0^t \delta x_s ds.$$  

We consider $V(x) = \frac{1}{2}(x, cx)$, where $c$ is a symmetric positive definite matrix. We suppose that $g$ is non-decreasing and denote $G(t) := \int_0^t g(s)ds$. 

**Remark 2.1.** Without any loss of generality, we suppose that $d = 1$, because the method is exactly the same when $d \geq 1$. The only thing to do is to diagonalize the matrix $c$ and to remember that for an orthogonal matrix $U$, the process $(U \cdot B_s, s \geq 0)$ is also a Brownian motion. So, we consider

$$\nabla V \ast \mu_t(x) = cx - \frac{1}{r + t} \int_0^t cX_s ds - \frac{r}{r + t}c\mu.$$  

2.1. **Explicit expression of $X$.** When the interaction function is quadratic, $X$ is the sum of a Brownian martingale and a deterministic part. Moreover, one can prove the convergence of the empirical measure with the expression of $X_t$ and $\bar{\mu}_t$. 

**Proposition 2.2.** If $X$ is the solution to (2.1), then we have

$$Y_t := X_t - \bar{\mu}_t = \frac{1}{r + t}e^{-cG(t)} \left( \int_0^t (r + s)e^{cG(s)}dB_s + r(x - \bar{\mu}) \right).$$

**Proof.** The process $Y$ satisfies

$$dY_t = dB_t - \left( cg(t) + \frac{1}{r + t} \right) Y_t dt, \quad Y_0 = x - \bar{\mu}. \tag{2.2}$$

Our strategy to express $Y$ in terms of a Brownian martingale is to consider the modification of $Y$, defined by $U_t := (r + t)e^{cG(t)}Y_t$. Then Itô’s formula implies

$$dU_t = (r + t)e^{cG(t)}dB_t, \quad U_0 = r(x - \bar{\mu}).$$

**Corollary 2.3.** The solution to the SDE (2.1) is given by

$$X_t = x + rc(\bar{\mu} - x) F(t) + \int_0^t \left[ 1 - (r + s)e^{cG(s)}(F(t) - F(s)) \right] dB_s.$$

Moreover

$$\bar{\mu}_t := \frac{1}{r + t} \int_0^t X_s ds + \frac{r}{r + t} \bar{\mu}$$

$$= x + r(\bar{\mu} - x) \left( \frac{1}{r + t}e^{-cG(t)} + cF(t) \right)$$

$$+ \int_0^t \left[ 1 - (r + s)e^{cG(s)} \left( F(t) - F(s) + \frac{1}{r + t} \frac{e^{-cG(t)}}{r + s} \right) \right] dB_s$$

where $F(t) = \int_0^t e^{cG(s)}g(s) \frac{ds}{r + s}$. 


Proof. We already know the expression of $Y_t := X_t - \bar{\mu}_t$. We easily get $d\bar{\mu}_t = \frac{Y_t}{r+t} dt$, and so, by Fubini’s theorem for stochastic integrals, we have
\[
\bar{\mu}_t = \int_0^t (r+s) e^{cG(s)} (H(t) - H(s)) dB_s + r(x - \bar{\mu}) H(t) + \bar{\mu}
\]
with $H(t) := \int_0^t \frac{e^{-cG(u)}}{(r+u)^2} du$. As $X_t = Y_t + \bar{\mu}_t$, the latter result implies
\[
X_t = \int_0^t (r+s) e^{cG(s)} \left( \frac{e^{-cG(t)}}{r+t} + H(t) - H(s) \right) dB_s + \int_0^t \frac{e^{-cG(t)}}{r+t} + H(t) r(x - \bar{\mu}) dB_s + \bar{\mu}.
\]
Integrating by parts, we get $H(t) - H(s) = \frac{e^{-cG(s)}}{r+s} - \frac{e^{-cG(t)}}{r+t} - c \int_s^t g(u) e^{-cG(u)} \frac{du}{r+u}$ and the result follows.

**Remark 2.4.** According to the expression of $X$, we find that $(X_t, t \geq 0) \overset{d}{=} (-X_t, t \geq 0)$ if and only if $\bar{\mu} = x = 0$.

2.2. Ergodic result. We begin to prove the pointwise ergodic theorem for the following non-homogeneous (Gauss-)Markov process.

**Lemma 2.5.** Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, such that $\lim_{t \to \infty} a(t) > 0$, $A(t) := \int_0^t a(s) ds$ increases to $A(\infty) = \infty$ and for $V(t) := e^{-2A(t)} \int_0^t e^{2A(s)} ds$, $V(\infty) < \infty$. Consider the process defined by
\[
dZ_t = -a(t) Z_t dt + dB_t, \quad Z_0 = z.
\]
Then, denoting by $\gamma$ the centered Gaussian measure with variance $V(\infty)$ by an abuse of notations, let $\mathcal{N}(0,0) = \delta_0$, we have for all continuous bounded function $\varphi$
\[
\frac{1}{t} \int_0^t \varphi(Z_s) ds \overset{a.s.}{\underset{t \to \infty}{\longrightarrow}} \int \varphi(z) \gamma(dz).
\]

**Proof.** It is enough prove the result for the Fourier transform. First, note that we can give an explicit expression of this process, that is
\[
Z_t = e^{-A(t)} \left( \int_0^t e^{A(s)} dB_s + z \right).
\]
Let $\mathcal{F}_s := \sigma(B_u, 0 \leq u \leq s)$. It is obvious that, knowing $\mathcal{F}_s$, $Z_t$ has a Gaussian law with mean $m(s,t) := e^{-(A(t)-A(s))} Z_s$ and variance $V(s,t) := e^{-2A(t)} \int_s^t e^{2A(u)} du$. Fix $t \in \mathbb{R}_+, u \in \mathbb{R}$ and define the martingale $M_{s,u}^t := \mathbb{E} \left( e^{iuZ_t} | \mathcal{F}_s \right) = \exp \left\{ iuZ_s m(s,t) - \frac{u^2}{2} V(s,t) \right\}$. Applying Itô’s formula to $s \mapsto M_{s,u}^t$, we find that $dM_{s,u}^t = iue^{-A(t)-A(s)} M_{s,u}^t dB_s$. So,
\[
e^{iuZ_t} = \mathbb{E} e^{iuZ_t} + \int_0^t iue^{-A(t)-A(s)} M_{s,u}^t dB_s.
\]
Then, applying Fubini’s theorem for stochastic integrals (see [7] p.175), we easily get

\[(2.3) \quad \int_0^t e^{iuZ_s}ds = \int_0^t \mathbb{E}e^{iuZ_s}ds + \int_0^t dB_s \int_s^t iue^{-(A(r) - A(s))}M_r^r,udr\]

But, as \(Z\) is Gaussian with variance \(V(0,t)\), it converges in distribution to a Gaussian variable of law \(\gamma = \mathcal{N}(0, V(\infty))\). Because of Cesàro’s result, we find

\[\frac{1}{t} \int_0^t \mathbb{E}[e^{iuZ_s}]ds \to e^{-u^2V(\infty)^2/2}.\]

It only remains to find an equivalent to the stochastic part of \((2.3)\). Denote \(N_{s,t}^{u}(v) := \int_s^t iue^{A(v) - A(r)}M_r^r,udr\). First, on the set \(\{\int_0^\infty < N_{s,t}^{u}(s) > ds < \infty\} \) the stochastic part of \((2.3)\) converges a.s. to a finite variable and so, it is of the order of \(o(t)\). Indeed, we decompose it as

\[(2.4) \quad \int_0^t N_{s,\infty}^{u}(s)dB_s - \int_0^t N_{t,\infty}^{u}(s)dB_s\]

On the set \(\{\int_0^\infty < N_{s,t}^{u}(s) > ds = \infty\}\), the law of large numbers for martingales implies a.s.

\[\int_0^t dB_s \int_s^\infty iue^{-(A(r) - A(s))}M_r^r,udr = o\left(\int_0^t \int_s^\infty iue^{-(A(r) - A(s))}M_r^r,udr ds\right).\]

Indeed, we find the rough upper bound by using the initial definition of \(M_{s,\infty}^{r,\infty}\):

\[|N_{s,t}^{u}(s)| \leq |u| \int_s^t e^{A(s) - A(r)}dr = |u|e^{A(s)}(K_t - K_s)\]

where \(K_t := \int_0^t e^{-A(r)}dr\). As the asymptotic development of \(K\) is \(K_t = K_{\infty} - \frac{e^{-A(t)}}{a(t)} + o\left(\frac{e^{-A(t)}}{a(t)}\right)\), we find

\[\int_0^t e^{2A(s)}(K_t - K_s)^2ds \leq 2(K_t - K_{\infty})^2 \int_0^t e^{2A(s)}ds + 2 \int_0^t e^{2A(s)}(K_{\infty} - K_s)^2ds\]

So, the previous integral is of the order of \(O(t)\). Denote \(S_t := \int_0^t N_{t,\infty}^{u}(s)dB_s\), we have

\[\frac{1}{t^2} \mathbb{E}(S_t^2) = \frac{1}{t^2} \int_0^t \mathbb{E}(N_{t,\infty}^{u}(s))^2ds \leq \frac{|u|^2}{t^2} \int_0^t e^{2A(s)}ds(K_{\infty} - F(t))^2 = o(t^{-2}).\]

By Borel-Cantelli, we find that \(\frac{S_t}{n}\) converges almost surely to 0. To conclude, it remains to write

\[\frac{1}{t} \int_0^t e^{iuZ_s}ds = \frac{[t]}{t} \int_0^t e^{iuZ_s}ds + \frac{1}{t} \int_{[t]} e^{iuZ_s}ds,\]

where \([t]\) denotes the bigger integer less than \(t\). \(\square\)
Lemma 2.6. Suppose that $g'(t)/g^2(t)$ converges to 0 and that $\lim g(t) > 0$. Then the random variable $\overline{\mu}_t = \frac{1}{r+t} \int_0^t X_s ds + \frac{r}{r+t} \overline{\mu}$ converges almost surely.

Proof. Introduce $H(t) := \int_0^t \frac{e^{-cG(u)}}{(r+u)^2} du$. First, decompose the process $\overline{\mu}_t = \overline{\mu}_t^1 + \overline{\mu}_t^2 + \overline{\mu}_t^3$ where

\[
\overline{\mu}_t^1 = \overline{\mu} + r(x - \overline{\mu})H(t);
\]
\[
\overline{\mu}_t^2 = (H(t) - H(\infty)) \int_0^t (r + s)e^{cG(s)} dB_s;
\]
\[
\overline{\mu}_t^3 = \int_0^t (r + s)e^{cG(s)}(H(\infty) - H(s)) dB_s.
\]

Obviously, the deterministic part $\overline{\mu}_t^1$ converges because of the convergence of $H(t) = \frac{1}{r} - cF(t) - e^{-cG(t)}$. We need the following development of $H$:

\[
(2.5) \quad H(t) = H(\infty) - \frac{1}{c(g(t)(r+t)^2} e^{-cG(t)} + o \left( \frac{e^{-cG(t)}}{t^2 g(t)} \right).
\]

The deterministic term of $\overline{\mu}_t^2$ is equivalent to $\frac{1}{c(g(t)(r+t)^2} e^{-cG(t)}$. Remark, that the quadratic variation of the stochastic term in $\overline{\mu}_t^2$ equals $\int_0^t (r + s)^2 e^{2cG(s)} ds = O \left( \frac{tr e^{2cG(t)}}{g(t)} \right)$ because $g'(t)/g^2(t)$ tends to 0. By the law of the iterated logarithm ([11] Theorem 3), we finally get $\overline{\mu}_t^2 \stackrel{a.s.}{\longrightarrow} 0$.

Last, the term $\overline{\mu}_t$ is a local martingale, and actually a $L^2$-bounded-martingale. Thus $\overline{\mu}_t^3$ converges a.s. to $\overline{\mu}_\infty^3$.

We conclude that $\overline{\mu}_t \stackrel{a.s.}{\longrightarrow} \overline{\mu}_\infty$ with $\overline{\mu}_\infty = \overline{\mu} + H(\infty)r(x - \overline{\mu}) + \overline{\mu}_\infty^3$. □

Theorem 2.7. Suppose that $g'(t)/g^2(t)$ converges to 0 and that $\lim g(t) > 0$. Then, with probability 1, the empirical measure $\mu_t$ converges weakly to a random measure $\mu_\infty$, and the previous limit $\overline{\mu}_\infty$ is the mean of $\mu_\infty$.

Proof. We first point out that the deterministic part of $X_t$ converges, because of the formula (2.3).

Decompose the process $X$ into three parts: $X_t = \overline{\mu}_\infty + \phi(t)U_t + o(1)$ where

\[
\overline{\mu}_\infty := x + cr(\overline{\mu} - x)F(\infty) + \int_0^\infty \left[ 1 - (r+s)ce^{cG(s)}(F(\infty) - F(s)) \right] dB_s
\]
\[
U_t := \frac{e^{-cG(t)}}{r+t} \int_0^t (r+s)e^{cG(s)} dB_s
\]
\[
\phi(t) := c(r+t)(F(\infty) - F(t)) e^{cG(t)}
\]

Again, we prove the result for the Fourier transform. We have the following:

\[
\frac{1}{t} \int_0^t e^{iuX_s} ds = \frac{e^{iu(\overline{\mu}_\infty + o(1))}}{t} \int_0^t e^{iu(\phi(t)U_t)} ds.
\]
As shown in Lemma 2.6, the random variable $\mathfrak{m}_\infty$ is well-defined. By Lemma 2.5, the random variable $\phi(t)U_t$ satisfies the pointwise ergodic theorem. So, we get the ergodic result: $\frac{1}{t}\int_0^t e^{iu\phi(s)U_s}ds$ converges a.s. So, the Fourier transform of $\mu_t$ converges a.s. □

Corollary 2.8. Suppose that $g'(t)/g^2(t)$ converges to 0 and that $g$ converges to $0 < g(\infty) < \infty$. Then the limit $\mu_\infty$ is a Gaussian measure with a random mean: $\mu_\infty = \mathcal{N}(\mu_\infty, \frac{1}{2g(\infty)c})$.

2.3. Asymptotic behavior of $X$. In the preceding subsection, we have shown that $X$ satisfies the pointwise ergodic theorem. We prove here that, depending on $g$, this process exhibits different behaviors: $X$ converges either almost surely, or in probability (and not a.s.), or it diverges.

First, we describe roughly the asymptotic behavior of $X$.

Proposition 2.9. Suppose that $g'(t)/g^2(t)$ converges to 0 and that $g$ converges to $0 < g(\infty) < \infty$. Then we get

$$P\left(\limsup_{t \to \infty} X_t = +\infty \right) = P\left(\liminf_{t \to \infty} X_t = -\infty \right) = 1.$$ 

Proof. We start to prove that the measure $\mu_\infty$ is diffusive. Let $A$ be a non negligible subset of $\mathbb{R}$. We have the asymptotic equivalence

$$\int_0^t \delta_{X_s}(A)ds \sim tl$$

where $l$ is a positive constant depending on $A$. So, $\int_0^\infty \delta_{X_s}(A)ds = \infty$ a.s. It then implies that for all $K > 0$, $\int_0^\infty \delta_{X_s}([K, \infty])ds = \infty$ a.s. and so

$$P\left(\bigcap_{K \geq 1} \left\{ \int_0^\infty 1_{X_s \geq K} ds = \infty \right\} \right) = 1.$$ 

We conclude that $P(\limsup_{t \to \infty} X_t = +\infty) = 1$. The proof is exactly the same for $\liminf_{t \to \infty} X_t$. □

Proposition 2.10. Suppose that $g'(t)/g^2(t)$ converges to 0 and that $\lim_{t \to \infty} g(t) = \infty$. Then $X_t$ converges in probability to a random variable $X_\infty$ and a.s. $\mu_t$ converges weakly to $\delta_{X_\infty}$.

Proof. As $Y$ is a Gaussian process and $E(Y_t^2) = O(g(t)^{-1})$, we find that $Y$ converges in $L^2$ and so in probability to 0. Decomposing $X$ as $X_t = Y_t + \int_0^t Y_s \frac{ds}{r+s}$, we get that the sequence $E|X_t|^2, t \geq 0$ is Cauchy and thus converges. As a consequence, $X$ converges to $X_\infty$ in $L^2$. We then easily get that $\mu_t$ converges toward $\delta_{X_\infty}$ in probability. By Theorem 2.7 a.s. $\mu_t$ converges weakly and we conclude by uniqueness of the limit. □

Proposition 2.11. Suppose that $g'(t)/g^2(t)$ converges to 0 and that $g(t)^{-1}\log G(t)$ is positive and bounded on $\mathbb{R}_+$. Then there exists $M > 0$ such that

$$P(\limsup_{t \to \infty} |Y_t| \leq M) = 1.$$
Proof. We can rewrite $Y$ as a Brownian (local) martingale: $Y_t = \frac{1}{f(t)} \left( Y_0 + \int_0^t f(s) dB_s \right)$ where $f(t) := (r + t)e^{cG(t)}$. We point out the following asymptotic result

$$\int_0^t (r + s)^2 e^{2cG(s)} ds = O \left( (r + t)^2 e^{2cG(t)} \right).$$

By the law of iterated logarithm, there exists $M > 0$ such that a.s. $\limsup_{t \to \infty} |Y_t| \leq M$. \hfill $\square$

**Corollary 2.12.** Suppose that $g'(t)/g^2(t)$ converges to 0 and that $g(t)^{-1} \log G(t)$ is bounded for all $t \geq 0$. Then the process $X_t$ is bounded a.s., converges in probability to $X_\infty = \overline{\mu}_\infty$ and a.s. $\mu_t$ converges weakly to $\delta_{X_\infty}$.

**Proof.** We know that $X_t = Y_t + \overline{\mu}_t$. As $Y$ is a.s. bounded and $\overline{\mu}_t$ converges a.s., $X$ is also a.s. bounded. Moreover, $Y$ is Gaussian and thus converges (in law) to a centered Gaussian variable. This last variable being bounded, $Y$ converges in probability to 0. But $Y$ does not converge a.s. to 0 because of the law of the iterated logarithm. As a consequence, $X$ converges in probability to $X_\infty = \overline{\mu}_\infty$. We conclude by uniqueness of the limit that a.s. $\mu_t$ converges weakly to $\delta_{X_\infty}$. \hfill $\square$

**Proposition 2.13.** Suppose that $g'(t)/g^2(t)$ converges to 0 and that $\lim_{t \to \infty} g(t)^{-1} \log G(t) = 0$. Then the following hold:

1. The process $Y_t := X_t - \overline{\mu}_t$ converges to 0 a.s.
2. The processes $X_t$ and $\overline{\mu}_t$ converge to $\overline{\mu}_\infty$ a.s. and a.s. $\mu_t$ converges weakly to $\delta_{\overline{\mu}_\infty}$.

**Proof.** We only need to prove that $Y_t := X_t - \overline{\mu}_t$ converges a.s. to 0. We have already seen that $Y_t = \frac{e^{-cG(t)}}{r + t} \int_0^t (r + s)e^{cG(s)} dB_s + r(x - \overline{\mu}) \frac{e^{-cG(t)}}{r + t} =: Y^1_t + Y^2_t$. The deterministic part of $Y_t$, namely $Y^2_t$, converges obviously to 0. Then, the law of the iterated logarithm implies that $Y_t$ converges a.s. to 0. As a.s. $\mu_t$ converges to a random measure $\mu_\infty$, by uniqueness of the limit, we conclude that $\mu_\infty = \delta_{\overline{\mu}_\infty}$. \hfill $\square$

3. Notation, hypotheses and existence

Let $G$ be the function $G(t) = \int_0^t g(s) ds$ and $G^{-1}$ its generalized inverse: $G^{-1}(t) := \inf\{u \geq 0; G(u) \geq t\}$. In the sequel, $(\cdot, \cdot)$ stands for the Euclidian scalar product. We also denote by $P(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$.

In the sequel, the assumptions on the potential $V: \mathbb{R}^d \to \mathbb{R}_+$ are:

1. (regularity and positivity) $V \in \mathcal{C}^2(\mathbb{R}^d)$ and $V \geq 0$;
2. (convexity) $V = W + \chi$ where $\chi$ is a compactly supported function such that $\nabla \chi$ is Lipschitz (with the constant $\tilde{C} > 0$) and there exists $c > 0$ such that $\nabla^2 W(x) \geq cI d$;
3. (growth) there exists $a > 0$ such that for all $x \in \mathbb{R}^d$, we have

$$\Delta V(x) \leq a(1 + V(x)) \text{ and } \lim_{|x| \to \infty} \frac{|\nabla V(x)|^2}{V(x)} = \infty.$$
Remark 3.1. 1) The convexity condition implies that $V$ is strictly uniformly convex out of a compact set $K$.

2) By the growth condition (3.1), $|\nabla V|^2 - \Delta V$ is bounded by below.

We also assume that $V$ has a finite number of critical points. Let $\text{Max} = \{M_1, \ldots, M_p\}$ be the set of saddle points and local maxima of $V$ and $\text{Min} = \{m_1, \ldots, m_n\}$ be the set of the local minima of $V$. We assume that $\forall i, \forall \xi \in \mathbb{R}^d, (\nabla^2 V(m_i) \xi, \xi) > 0$ and for all $M_i$, $\nabla^2 V(M_i)$ admits a negative eigenvalue.

For the moment, we also suppose that $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is non-decreasing and $g \in C^1(\mathbb{R}_+)$. We will give the precise assumptions on $g$ in the statement of the results. Remark, that if $\lim_{t \rightarrow \infty} g(t) = \infty$, then for all $T > 0$, we have that $G^{-1}(t + T) - G^{-1}(t) \rightarrow 0$.

The main goal of this paper is to study the asymptotic behavior of $\mu_t$ and $X$. First, we show that $X$ satisfies the pointwise ergodic theorem, that is

Definition 3.2. The process $X$ satisfies the pointwise ergodic theorem if there exists a measure $\mu_\infty$ such that a.s. $\mu_t := \frac{1}{t-r} \left( r \mu + \int_r^t \delta_{X_s} ds \right) \rightarrow \mu_\infty$ for the weak convergence of measures: for all continuous bounded function $f$, $\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow{a.s.} \int f d\mu_\infty$.

Let us begin to prove that the SDE studied admits a unique global strong solution.

Proposition 3.3. For any $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $r > 0$, there exists a unique global strong solution $(X_t, t \geq 0)$ of (1.2).

Proof. The local existence and uniqueness of the SDE studied in this paper is standard (see [17] theorem 11.2). We just need to prove here that $Y$, hence $X$ (because $X_t := Y_t + \int_0^t Y_s \frac{ds}{s_{t+r}^+}$), does not explode in a finite time.

We apply Itô’s formula to the function $x \mapsto V(x)$:

$$
\begin{align*}
\,dV(Y_t) &= (\nabla V(Y_t), dB_t) \\
&\quad + \left( \frac{1}{2} \Delta V(Y_t) - g(t)|\nabla V(Y_t)|^2 - \frac{1}{r+t}(\nabla V(Y_t), Y_t) \right) \,dt.
\end{align*}
$$

Let us introduce the sequence of stopping times

$$\tau_n = \inf\{t \geq 0; V(Y_t) + \int_0^t g(s)|\nabla V(Y_s)|^2 \,ds > n\}.$$

We note that $\int_0^{t \wedge \tau_n} (\nabla V(Y_s), dB_s)$ is a true martingale. The growth condition (3.1) implies that $(\nabla V(y), y) \xrightarrow{|y| \rightarrow +\infty} +\infty$ and that $|\nabla V|^2 - \Delta V$ is bounded from below. So, there exists $C$ such that

$$
\mathbb{E}V(Y_{t \wedge \tau_n}) \leq \mathbb{E}(V(Y_0)) + Ct. \quad \square
$$

4. Study of the process $Y$

We study the process $Y$, which is the solution to the following SDE

$$(4.1) \quad \,dY_t = dB_t - \left( g(t)\nabla V(Y_t) + \frac{Y_t}{r+t} \right) \,dt; \quad Y_0 = x - \mathbb{E}r.$$
More precisely, we study the behavior of $Y$ around the critical points of $V$. We show in particular for each local minimum of $V$, that $Y$ stays close to it with positive probability; whereas this probability vanishes for an unstable critical point.

4.1. The process $Y_t$ gets close to the critical points of $V$.

**Proposition 4.1.** Suppose that $\lim g(t) > 1/2$, and for all $T > 0$, $G^{-1}(t+T) - G^{-1}(t) \to 0$. Then, almost surely, the process $Y$ gets as close as we want to the set $\text{Min} \cup \text{Max}$: for all $t > 0$, $T^\varepsilon_t := \inf\{s \geq t; d(Y_s, \text{Min} \cup \text{Max}) < \varepsilon\} < \infty$ a.s.

**Proof.** Let $\varepsilon > 0$. Applying Itô’s formula to $x \mapsto V(x)$, we obtain

$$\text{d}V(Y_t) = (\nabla V(Y_t), dB_t) - D(t,Y_t)\text{d}t.$$  

where we have introduced

$$D(t,y) = g(t)|\nabla V(y)|^2 + \frac{1}{r+t}(y, \nabla V(y)) - \frac{1}{2}\Delta V(y).$$

We denote by $\text{Min} \cup \text{Max}$ the set of the critical points of $V$. Then it follows from the growth condition that on the set $\{z; d(z, \text{Min} \cup \text{Max}) > \varepsilon\}$ and for $t \geq 0$, the functions $y \mapsto \frac{1}{r+t}(y, \nabla V(y)) + \frac{1}{2}||\nabla V(y)|^2 - \Delta V(y)||$ and $y \mapsto (g(t) - \frac{1}{r+t})|\nabla V(y)|^2$ are bounded from below. Moreover, the application $y \mapsto (g(t) - \frac{1}{r+t})|\nabla V(y)|^2$, is positive for $t$ large enough. So, there exists $t_0 = t_0(\varepsilon)$ such that: $\forall t > t_0$, $\forall y \in \{z; d(z, \text{Min} \cup \text{Max}) > \varepsilon\}$ we have

$$g(t)|\nabla V(y)|^2 + \frac{1}{r+t}(y, \nabla V(y)) - \frac{1}{2}\Delta V(y) \geq \frac{g(t)}{2}|\nabla V(y)|^2 > 0.$$ 

Let us introduce the stopping time $T^\varepsilon_t = \inf\{s \geq t; d(Y_s, \text{Min} \cup \text{Max}) < \varepsilon\}$. We want to prove that, for all $t > t_0$, $\mathbb{P}(T^\varepsilon_t < +\infty) = 1$. Then, it follows from (4.2) and (4.4) that, for $t > t_0$, the following processes are two super-martingales:

$$\left(\frac{\text{d}}{\text{d}u} \int_0^s \frac{g(u)|\nabla V(Y_u)|^2}{2}\text{d}u, s \geq t \right)$$

As they are nonnegative, they converge a.s. as $s \to \infty$. So, the process $\left(\int_0^{s \wedge T^\varepsilon_t} \frac{g(u)|\nabla V(Y_u)|^2}{2}\text{d}u, s \geq t \right)$ also converges a.s. On the set $\{T^\varepsilon_t = +\infty\}$, we have

$$|\nabla V(Y_{s \wedge T^\varepsilon_t})|^2 \xrightarrow{s \to \infty} 0.$$ 

Thus $Y_{s \wedge T^\varepsilon_t}$ gets close to $\text{Min} \cup \text{Max}$ and there is a contradiction. Finally, $\mathbb{P}(T^\varepsilon_t < +\infty) = 1$ for all $t \geq t_0$. For $t \leq t_0$ we remark that $t \mapsto T^\varepsilon_t$ is increasing. 

**Corollary 4.2.** Suppose that $\lim g(t) > 1/2$ and for all $T > 0$, $G^{-1}(t+T) - G^{-1}(t) \to 0$. Then, almost surely, the process $Y$ gets close to the set $\text{Min} \cup \text{Max}$ infinitely often: for all $\varepsilon > 0$, there exists a sequence of stopping times $(T_n)_{n \geq 1}$ such that $T_n$ tends to infinity and

$$\forall n \geq 1, \mathbb{P}(T_n < +\infty) = 1 \text{ and } d(Y_{T_n}, \text{Min} \cup \text{Max}) < \varepsilon.$$ 

**Proof.** Let $T^\varepsilon_n = \inf\{s > n; d(Y_s, \text{Min} \cup \text{Max}) < \varepsilon\} < \infty$ a.s. We conclude by choosing $T_n = T^\varepsilon_n$ in the proof of Proposition 4.1. 

□
4.2. Case of a stable critical point: local minimum. We will prove that if the process $Y$ is close to a local minimum $m$, then the probability that the set $\{Y_s; s \geq 0\}$ is included in a neighborhood of $m$ is positive. Indeed, a second-order Taylor expansion permits us to compare $(y - m, \nabla V(y))$ and $|y - m|^2$ and we use a comparison theorem for the associate SDE.

Let $m$ be a local minimum of $V$ such that $\nabla^2 V(m) > 0$. Taylor’s formula implies that there exists $a > 0$ and $\varepsilon_0 > 0$ such that for all $|y - m| \leq \varepsilon_0$ we have $(y - m, \nabla V(y)) \geq a|y - m|^2$.

**Proposition 4.3.** Suppose that $\lim g(t) > 1/2$, $\frac{g'(t)}{g(t)}$ converges to 0 when $t$ tends to infinity. Assume also that $\sqrt{g(t)^{-1} \log G(t)}$ is bounded on $\mathbb{R}_+$. Let $\varepsilon_0 > \varepsilon > 0$. Then, there exists $T_0 > 0$ such that, for all $T > T_0$, we have on the event $\{|Y_T - m| < \varepsilon\}$, that $\mathbb{P}(\forall s \geq T; |Y_s - m| < \varepsilon) > 0$. Moreover, on the event $\{|Y_T - m| < \varepsilon\}$, we have

$$|Y_{t+T} - m| = O\left(\sqrt{g(t + T)^{-1} \log G(t + T)}\right) \text{ a.s.}$$

**Proof.** We will show at the end of the proof that $\mathbb{P}(\forall s \geq T; |Y_s - m| < \varepsilon) > 0$. Suppose for the moment that this event has a positive probability to occur. Without any loss of generality, we can suppose that $m = 0$. Consider the time-changed process $\tilde{Y}_t := Y_{t+T}$.

1) Suppose that $d = 1$. Let $\varepsilon > 0$. As $V''(m) > 0$, there exists $a > 0$ such that for all $0 \leq y \leq \varepsilon$

$$V'(y) \geq ay. \tag{4.5}$$

Let us introduce the non-negative process $U$ solution to the SDE

$$dU_t = dB_t^T - ag(t + T)U_t dt + dL_t, \quad U_0 = \tilde{Y}_0 \geq 0, \tag{4.6}$$

where $L$ corresponds to the local time of $U$ in $0$.

**Step 1:** the equation (4.6) admits a unique solution $U_t$, which is nonnegative. Let $Z$ be the process defined by $Z_t = e^{aG(t+T)}U_t$. By definition of $U$, we easily obtain that

$$dZ_t = e^{aG(t+T)}dB_t^T + e^{aG(t+T)}dL_t.$$ 

Let $\alpha(t)$ be the function such that $\int_0^{\alpha(t)} e^{2aG(s+T)}ds = t$ and define

$$A_t := \int_0^t e^{aG(s+T)}dL_s.$$ 

If we consider the process $Z_{\alpha(t)} = Z_{\alpha(0)} + W_t + A_{\alpha(t)}$ (where $W_t = \int_0^{\alpha(t)} e^{aG(s+T)}dB_s^T$), then $A_t$ increases if and only if $L_t$ increases and so, $A_{\alpha(t)}$ increases if and only if $Z_{\alpha(t)} - Z_{\alpha(0)}$ vanishes. This means that $A_{\alpha(t)}$ is the local time at zero of $W$. Skorokhod’s lemma (see [8]) then entails that the process $Z_{\alpha(t)}$ is uniquely defined by $Z_{\alpha(t)} = W_t^+$ where $W_t^+$ is the reflected Brownian motion associated to $W$. So, there exists a unique (strong) solution

$$U_t = U_0 + e^{-aG(t+T)}W_{\alpha^{-1}(t)}^+ \tag{4.7}.$$

**Step 2:** by the law of the iterated logarithm, there exists a constant $C > 0$ such that

$$U_t \leq Ce^{-aG(t+T)}\sqrt{\alpha^{-1}(t) \log(\log(\alpha^{-1}(t)))} \text{ a.s.}$$

As $\alpha^{-1}(t) = \int_0^t e^{2aG(s+T)}ds$, we get that a.s. $U_t = O\left(\sqrt{g(t + T)^{-1} \log G(t + T)}\right)$. 


Step 3: $\tilde{Y}_t \leq U_t$ a.s. by a martingale comparison theorem. Indeed, let $l$ be a function of class $C^2$ such that:
\[
\begin{aligned}
&\forall x > 0, \ l(x) > 0 \text{ and } l'(x) > 0, \\
&\forall x \leq 0, \ l(x) = 0
\end{aligned}
\]

According to Itô's formula, we have
\[
\begin{align*}
l(\tilde{Y}_t - U_t) &= \ - \int_0^t l'(\tilde{Y}_s - U_s) \left( g(s + T)V'(\tilde{Y}_s) - g(s + T)aU_s + \frac{\tilde{Y}_s}{r + s + T} \right) ds \\
&\quad - \int_0^t l' (\tilde{Y}_s - U_s) dL_s.
\end{align*}
\]

As $U$ is nonnegative, $\tilde{Y}$ is positive on the event $\{\tilde{Y}_s > U_s\}$, and so by the bound (4.5), we find that $g(s + T)V'(\tilde{Y}_s) - ag(s + T)U_s \geq ag(s + T)(\tilde{Y}_s - U_s)$. We then have $l(\tilde{Y}_t - U_t) \leq 0$ a.s. and this leads to
\[
(4.8) \quad \tilde{Y}_t \leq U_t \text{ a.s.}
\]

Using the same argument on $[-\varepsilon, 0]$ we get the lower bound $V_t \leq \tilde{Y}_t$, where $V_t$ is a nonpositive process.

Finally, the processes $V$ and $U$ satisfy, by the law of the iterated logarithm
\[
\limsup_{t \to \infty} U_t = - \liminf_{t \to +\infty} V_t = \lim_{t \to \infty} \sqrt{g(t + T)^{-1} \log G(t + T)}.
\]

2) Suppose that $d \geq 2$. Define $\tau := \inf\{t > 0; \tilde{Y}_t = 0\}$. Itô's formula implies
\[
(4.9) \quad d|\tilde{Y}_{t \wedge \tau}| = dW_{t \wedge \tau} - g(t \wedge \tau + T) \left( \frac{\tilde{Y}_{t \wedge \tau}}{|\tilde{Y}_{t \wedge \tau}|}, \nabla V(\tilde{Y}_{t \wedge \tau}) \right) dt \\
- \frac{|\tilde{Y}_{t \wedge \tau}|}{r + t \wedge \tau + T} dt + \frac{d - 1}{2|\tilde{Y}_{t \wedge \tau}|} dt
\]

where $W_t = \int_0^t \left( \frac{\tilde{\pi}}{|\tilde{\pi}|}, dB_s^T \right)$ is a standard Brownian motion. The condition $\nabla^2 V(0) > 0$ implies that there exists $a > 0$ such that
\[
(4.10) \quad \forall |y| \leq \varepsilon, \ (y, \nabla V(y)) \geq a|y|^2.
\]

Let us introduce the $(d - 1)$-dimensional Bessel process $R$. Consider the time-changed process $U_t := e^{-aG(t+T)} R_{e^{2aG(t+T)} ds}$, which is the nonnegative (strong) solution to
\[
(4.11) \quad dU_t = d\beta_t^T - ag(t + T)U_t dt + \frac{d - 1}{2U_t} dt,
\]

where $\beta_t$ is a Brownian motion. On the event $\{\forall s \geq T; |Y_s| < \varepsilon\}$, we apply the previous comparison theorem and we obtain that $|\tilde{Y}_t| \leq U_t$. On the other hand, $R_t$ is the radial part of a $d$-dimensional Brownian motion. Similarly to the one dimensional case, the law
of the iterated logarithm implies that a.s. \( R_t = O(\sqrt{(t+T) \log \log(t+T)}) \), and so a.s. 

\[
U_t = O \left( \sqrt{g(t + T)^{-1} \log G(t + T)} \right)
\]

It remains to prove that \( \mathbb{P}(\forall s \geq T; |Y_s - m| < \varepsilon) > 0 \). Let \( \tau_T := \inf\{s > T; |Y_s - m| > \varepsilon\} \). For all \( T < t < \tau_T \), we have a.s. \( |Y_t - m| \leq U_t + V_t \). As \( U_t = e^{-aG(t+T)}W_{\alpha-1}(t) \) and

(4.12) \[
\limsup_{t \to \infty} (|U_t| + |V_t|) \sqrt{\frac{g(t + T)}{\log G(t + T)}} \leq 1.
\]

So, for \( T \) large enough, we have \( \mathbb{P}\left( \sup_{s \geq T}(U_s + V_s) < \varepsilon \right) > 0 \) (because \( \sqrt{\frac{\log G(t)}{g(t)}} \) is bounded) and finally \( \mathbb{P}(\tau_T = \infty) > 0 \). \( \square \)

**Corollary 4.4.** Suppose that \( \lim g(t) > 1/2 \), and that \( \frac{g(t)}{g(0)^{2t}} \) and \( \frac{\log G(t)}{g(t)} \) converge to 0 when \( t \) tends to infinity. Then, there exists \( T_0 > 0 \) such that for all \( T > T_0 \), the process \( Y_t - m \) converges almost surely to 0 on the event \( \{\forall s \geq T; |Y_s - m| < \varepsilon\} \).

**Proof.** We follow the previous proof and recall that \( |Y_t - m| \leq U_t + V_t \). We conclude with (4.12). \( \square \)

### 4.3. Case of an unstable critical point.

#### 4.3.1. Case of a local maximum.

Let \( M \) be a local maximum of \( V \). As \( \Delta V(M) < 0 \), \( \varepsilon_1 := \sup\{\varepsilon; \forall|y| < \varepsilon, \Delta V(M + y) < 0\} \) exists and is finite.

**Proposition 4.5.** Suppose that \( \lim g(t) > 1/2 \). Let \( 0 < \varepsilon < \varepsilon_1 \), \( M \) a local maximum of \( V \) and \( T := \inf\{t > 0; |Y_t - M| < \varepsilon\} \). Then

\[
\mathbb{P}(\forall s \geq T; |Y_s - M| < \varepsilon) = 0.
\]

**Proof.** For the sake of simplicity, we restrict our attention to the case \( M = 0 \), because the method is exactly the same when \( M \neq 0 \). Note, that \( T < \infty \) a.s. by Proposition 4.4. By Itô’s formula:

\[
dV(Y_{t+T}) = (\nabla V(Y_{t+T}), dB_{t+T}) - D(t+T,Y_{t+T})dt
\]

where \( D(t, y) \) is defined by (4.3). On the event \( A := \{\forall s \geq T; |Y_s| < \varepsilon\} \) we obtain the bound

\[
D(t+T, Y_{t+T}) = g(t+T)\|\nabla V(Y_{t+T})\|^2 + \frac{(Y_{t+T}, \nabla V(Y_{t+T}))}{r + t + T} - \frac{1}{2} \Delta V(Y_{t+T}) \geq \frac{C_1}{r + t + T} + C_2
\]

where \( C_1 = \inf\{(y, \nabla V(y)); |y| < \varepsilon\} \) and \( C_2 = -\frac{1}{2} \sup\{\Delta V(y); |y| < \varepsilon\} > 0 \). We thus find for \( t \) large enough that \( D(t+T, Y_{t+T}) \geq C > 0 \) and so

(4.13) \[
\mathbb{E}(V(Y_{t+T})1_A) \leq \mathbb{E}(V(Y_T)1_A) - Ct\mathbb{P}(A) + o(t).
\]

Finally, this last inequality is impossible since \( V \) is nonnegative. So, we conclude that \( \mathbb{P}(A) = 0 \). \( \square \)
Remark 4.6. If $M \neq 0$ we have an additional term $M \log(t + T)$ and the proof is exactly the same.

4.3.2. Case of a saddle point. Let $M$ be a saddle point of $V$. First, remark that, if $\Delta V(M) < 0$, then we can follow the proof of Proposition 4.5 to conclude. Nevertheless, we give here a general proof.

Let $e$ be an unstable direction (that is $\partial^2_{ee} V(M) < 0$) associate to the saddle point $M$ and $P_e : \mathbb{R}^d \mapsto \mathbb{R}$ the projection on $\mathbb{R} e$. We know that such a direction exists (because $\nabla^2 V$ admits a negative eigenvalue in $M$). As $\partial^2_{ee} V(M) < 0$, $\varepsilon_2 := \sup \{\varepsilon; \forall |y| < \varepsilon, \partial^2_{ee} V(M + y) < 0 \text{ and } (\partial_e V(P_e(y)), \partial_e V(y)) > 0\}$ exists and is finite.

Proposition 4.7. Suppose that $\lim g(t) > 1/2$. Let $0 < \varepsilon < \varepsilon_2$, $M$ a saddle point of $V$ and $T$ a positive stopping time such that $|Y_T - M| < \varepsilon$. Then

$$\mathbb{P}(\forall s \geq T; |Y_s - M| < \varepsilon) = 0.$$

Proof. Suppose for simplicity that $M = 0$. Itô’s formula applied to the function $x \mapsto V(P_e(x))$ implies that

$$dV(P_e(Y_{t+T})) = dM_{t+T} - \tilde{D}(t + T, Y_{t+T}) dt,$$

where $M_t := \int_0^t (\partial_e V(P_e(Y_s)), P_e(dB_s))$ is a local martingale and

$$\tilde{D}(t, Y_t) := g(t) (\partial_e V(P_e(Y_t)), \partial_e V(Y_t)) + \frac{1}{r + t} (\partial_e V(P_e(Y_t)), P_e(Y_t)) - \frac{1}{2} \partial^2_{ee} V(P_e(Y_t)).$$

On the event $A := \{\forall s \geq T; |Y_s| < \varepsilon\}$ we get the bound

$$\tilde{D}(t + T, Y_{t+T}) \geq \frac{C_3}{r + t + T} + C_4,$$

with $C_3 := \inf \{(P_e(y), \partial_e V(P_e(y))); |y| < \varepsilon\}$ and $C_4 := -\frac{1}{2} \sup \{\partial^2_{ee} V(P_e(y)); |y| < \varepsilon\} > 0$. Thus, for $t$ large enough, there exists $C > 0$ such that $\tilde{D}(t + T, Y_{t+T}) \geq C > 0$ and so

$$\mathbb{E}(V(P_e(Y_{t+T})) \mathbb{I}_A) \leq \mathbb{E}(V(P_e(Y_{T})) \mathbb{I}_A) - Ct\mathbb{P}(A) + o(t).$$

Finally, as $V$ is a nonnegative function, we conclude that $\mathbb{P}(A) = 0$. \hfill \Box

5. Asymptotic behavior of $Y$

5.1. Pointwise ergodic theorem. The aim of this paragraph is to prove that $Y$ satisfies the pointwise ergodic theorem. We begin to show that $Y$ is bounded in $L^2$.

Lemma 5.1. Suppose that $\lim g(t) = \infty$ and $\lim \frac{g'(t)}{g(t)^2} = 0$. Then, the process $Y$ is $L^2$-bounded.

Proof. We will prove a stronger result and show that $\mathbb{E}V(Y_t)$ is bounded. Itô’s formula implies

$$dV(Y_t) = (\nabla V(Y_t), dB_t) - g(t)|\nabla V(Y_t)|^2 dt - \frac{(Y_t, \nabla V(Y_t))}{r + t} dt + \frac{1}{2} \Delta V(Y_t) dt.$$
As $W$ is strictly convex everywhere, with a constant of convexity $C$, and $\chi$ is a compactly supported function, the map $y \mapsto (y, \nabla \chi(y))$ is bounded (by $M > 0$). For all $n \in \mathbb{N}$, define the stopping time $\tau_n = \inf \{ t ; |Y_t| > n \}$. Then, we get by localization (and because $g$ is non-decreasing):

$$ \mathbb{E}|Y_{t\wedge \tau_n}|^2 \leq \mathbb{E}V(Y_0) + Mt < \infty. $$

Let $n$ go to infinity and use Fatou’s lemma to get, for all $t \geq 0$, that $V(Y_t) \in L^1$. By the growth hypothesis (3.1), there exists some $\alpha > 0$ such that $-g(t)V(x) + aV(x) \leq -\alpha g(t)V(x)$. So, the following holds

$$ \frac{d}{dt}\mathbb{E}V(Y_t) \leq -\frac{C}{r + t}\mathbb{E}|Y_t|^2 + a - \alpha g(t)\mathbb{E}V(Y_t). $$

Now, we solve this inequality by solving $\dot{u} = a - \alpha g(t)u$ and then $\mathbb{E}V(Y_t) = O(1)$. \hfill \Box

In order to obtain the ergodic result for $Y$, we introduce a dynamical system $\phi$ for which $Y$ is an asymptotic pseudotrajectory in probability (see [1] for more details on this notion), that is

**Definition 5.2.** The process $Y$ is an asymptotic pseudotrajectory (in probability) for the flow $\phi$ if $\forall T, \alpha > 0$, we have

$$ \lim_{t \to \infty} \mathbb{P}\left( \sup_{0 \leq h \leq T} |Y_{t+h} - \phi_h(Y_t)| \geq \alpha \right) = 0. \tag{5.1} $$

Let us consider the time-changed process $Y_{G^{-1}(t)}$. It satisfies for all $h \geq 0$

$$ Y_{G^{-1}(t+h)} - Y_{G^{-1}(t)} = B_{G^{-1}(t+h)} - B_{G^{-1}(t)} - \int_0^h \nabla V(Y_{G^{-1}(t+s)}) ds $$

$$ - \int_0^h Y_{G^{-1}(t+s)} \frac{ds}{\kappa(t+s)}, $$

where we have defined $\kappa(t) := (r + G^{-1}(t))g(G^{-1}(t)).$

**Proposition 5.3.** Suppose that $\lim g(t) = \infty$, and that $\frac{g(t)}{g(t)^2}$ vanishes when $t$ tends to infinity. Let $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be the flow generated by

$$ \frac{d}{dt}\phi_t(x) = -\nabla V(\phi_t(x)); \quad \phi_0(x) = x. \tag{5.2} $$

Then $Y$ is an asymptotic pseudotrajectory for $\phi$: for all $T > 0$ and $\alpha > 0$,

$$ \lim_{t \to \infty} \mathbb{P}\left( \sup_{0 \leq h \leq T} |Y_{G^{-1}(t+h)} - \phi_h(Y_{G^{-1}(t)})| \geq \alpha \right) = 0. $$

**Proof.** A simple computation yields to, with $\tilde{Y}_t = Y_{G^{-1}(t)}$ and $\tilde{B}_t = B_{G^{-1}(t)}$

$$ \tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t) = \tilde{B}_{t+h} - \tilde{B}_t + \int_0^h \left( \nabla V(\phi_s(\tilde{Y}_t)) - \nabla V(\tilde{Y}_{t+s}) \right) ds $$

$$ - \int_0^h \tilde{Y}_{t+s} \frac{ds}{\kappa(t+s)}. $$
Let us apply Itô’s formula to $h \mapsto e^{-2\hat{c}h} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2$. As $V = W + \chi$, where $(\xi, \nabla^2 W(x)\xi) \geq C|\xi|^2$ and $\nabla \chi$ is $\tilde{C}$-Lipschitz, we have

$$
\frac{1}{2} \ d(e^{-2\hat{c}h} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2) = e^{-2\hat{c}h}\left(\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t), d\tilde{B}_{t+h}\right) + e^{-2\hat{c}h}\frac{1}{\kappa(t+h)} \left(\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t), \tilde{Y}_{t+h}\right) \ dh + e^{-2\hat{c}h}\left(\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t), \nabla V(\phi_h(\tilde{Y}_t)) - \nabla V(\tilde{Y}_{t+h})\right) \ dh + \frac{1}{2}e^{-2\hat{c}h} d < \tilde{Y}_t. - \phi(\tilde{Y}_t) >_h - \tilde{C}e^{-2\hat{c}h}|\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2 dh.
$$

So, we have the following upper bound:

$$
\frac{1}{2} \sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2 \leq \sup_{0 \leq h \leq T} e^{2\hat{c}h} \int_0^h e^{-2\hat{c}s}(\tilde{Y}_{t+s} - \phi_s(\tilde{Y}_t), d\tilde{B}_{t+s}) + \frac{e^{2\hat{c}T}}{2} \int_0^T \frac{1}{g(G^{-1}(t+s))} ds + \sup_{0 \leq h \leq T} e^{2\hat{c}h} \int_0^h e^{-2\hat{c}s} \frac{(\tilde{Y}_{t+s} - \phi_s(\tilde{Y}_t), \tilde{Y}_{t+s})}{\kappa(t+s)} ds.
$$

By BDG’s inequality for the local martingale $\int_0^h e^{-2\hat{c}s}(\tilde{Y}_{t+s} - \phi_s(\tilde{Y}_t), d\tilde{B}_{t+s})$ and a rough upper bound for its quadratic variation, we deduce the existence of a positive constant $C_2$ such that:

$$
E\left(\sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2\right) \leq e^{2\hat{c}T}(G^{-1}(t+T) - G^{-1}(t)) + C_2 e^{4\hat{c}T}(G^{-1}(t+T) - G^{-1}(t)) \left[ E\left(\sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_T(\tilde{Y}_t)|^2\right)\right]^{1/2} + e^{2\hat{c}T} E\left(\sup_{0 \leq h \leq T} \int_0^h \frac{(\tilde{Y}_{t+s} - \phi_s(\tilde{Y}_t), \tilde{Y}_{t+s}) ds}{\kappa(t+s)}\right).
$$

We now need to estimate the last mean of the latter inequality. We have:

$$
\int_0^h \frac{(\tilde{Y}_{t+s} - \phi_s(\tilde{Y}_t), \tilde{Y}_{t+s})}{\kappa(t+s)} ds \leq \frac{1}{2} \int_0^h \frac{|\tilde{Y}_{t+s} - \phi_s(\tilde{Y}_t)|^2}{\kappa(t+s)} ds + \frac{1}{2} \int_0^h \frac{|\tilde{Y}_{t+s}|^2}{\kappa(t+s)} ds.
$$
Because $\kappa$ is non-decreasing and by Lemma 5.1, we get the bounds:

$$
E \left( \sup_{0 \leq h \leq T} \int_0^h \frac{\tilde{Y}_{t+s} - \phi_h(\tilde{Y}_t)}{\kappa(t+s)} ds \right) \leq \frac{MT}{\kappa(t)};
$$

$$
E \left( \sup_{0 \leq h \leq T} \int_0^h \frac{\tilde{Y}_{t+s} - \phi_h(\tilde{Y}_t)}{\kappa(t+s)} ds \right) \leq \frac{T}{\kappa(t)} E \left( \sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2 \right).
$$

But by hypothesis on $g$, $(G^{-1}(t+T) - G^{-1}(t))$ and $\kappa(t)^{-1}$ converge to 0 when $t$ increases to the infinity. So, we obtain for $t$ large enough:

$$
E \left( \sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2 \right) \leq 2e^{4\bar{C}T}(G^{-1}(t+T) - G^{-1}(t)) + 2Me^{2\bar{C}T} \frac{T}{\kappa(t)}.
$$

To conclude, we just need to use Markov’s inequality to conclude:

$$
P \left( \sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)| \geq \alpha \right) \leq \frac{1}{\alpha^2} E \left( \sup_{0 \leq h \leq T} |\tilde{Y}_{t+h} - \phi_h(\tilde{Y}_t)|^2 \right). \quad \square
$$

**Lemma 5.4.** Suppose that for all $T > 0$, $G^{-1}(t+T) - G^{-1}(t)$ vanishes when $t$ tends to infinity. Let $(\mu_t^{G^{-1}}, t \geq 0)$ be the family defined by $\mu_t^{G^{-1}} = \frac{1}{t} \int_0^t \delta_{Y_{G^{-1}(s)}}(Y_t) ds$. Then, $(\mu_t^{G^{-1}}, t \geq 0)$ is a tight family of measures.

**Proof.** It is enough to show that a.s. $\varphi(t) := \int_0^t V(Y_{G^{-1}(s)}) ds = O(t)$. Indeed, letting $A > 0$ and $K$ being a compact set such that $\forall x \in K^c$ we have $V(x) \geq A$, then we get $\mu_t^{G^{-1}}(V) \geq A \mu_t^{G^{-1}}(K^c)$.

From the growth assumption (3.1), there exists $a > 0$ and for all $\varepsilon > 0$, there exists $k_\varepsilon$ such that

$$
\Delta V \leq a(1 + V) \quad \text{and} \quad V \leq k_\varepsilon + \varepsilon \|
abla V\|^2.
$$

It then easily implies that

$$
\varphi(t) \leq k_\varepsilon t + \varepsilon \int_0^t \|\nabla V(Y_{G^{-1}(s)})\|^2 ds
$$

and $\int_0^t \Delta V(Y_{G^{-1}(s)}) ds \leq at + a\varphi(t)$. Applying Itô’s formula to the process $t \mapsto V(Y_{G^{-1}(t)})$, we obtain

$$
V(Y_{G^{-1}(t)}) - V(Y_{G^{-1}(0)}) = \int_{G^{-1}(0)}^{G^{-1}(t)} \langle \nabla V(Y_s), dB_s \rangle - \int_0^t \|\nabla V(Y_{G^{-1}(s)})\|^2 ds
$$

$$
- \int_0^t \frac{\langle Y_{G^{-1}(s)}, \nabla V(Y_{G^{-1}(s)}) \rangle ds}{(r + G^{-1}(s)) g(G^{-1}(s))}
$$

$$
+ \frac{1}{2} \int_0^t \Delta V(Y_{G^{-1}(s)}) \frac{ds}{g \circ G^{-1}(s)}.
$$

Let us consider the martingale part of the equality. On the right hand, on the set $\{\int_{G^{-1}(0)}^\infty \|\nabla V(Y_s)\|^2 ds < \infty\}$, this (local) martingale is bounded in $L^2$ and thus converges.
So, the limit-measure in the ergodic theorem for $t$ large enough, a.s.

$$\int_{G^{-1}(0)}^{G^{-1}(t)} (\nabla V(Y_s), dB_s) \leq \frac{1}{2} \int_0^t |\nabla V(Y_{G^{-1}(s)})|^2 ds.$$ 

Itô's formula (5.4) implies for $t$ large enough

$$\int_0^t |\nabla V(Y_{G^{-1}(s)})|^2 ds \leq \int_0^t \Delta V(Y_{G^{-1}(s)}) ds - 2V(Y_{G^{-1}(t)}) + 2V(Y_{G^{-1}(0)})$$

$$- 2 \int_0^t \frac{(Y_{G^{-1}(s)}, \nabla V(Y_{G^{-1}(s)}))}{(r + G^{-1}(s))g(G^{-1}(s))} ds$$

$$\leq \frac{a(t + \varphi(t))}{g \circ G^{-1}(t)} + 2V(Y_{G^{-1}(0)})$$

$$+ O \left( \int_0^t \frac{1}{(r + G^{-1}(s))g \circ G^{-1}(s)} ds \right).$$

So, we have a.s. $\int_0^t |\nabla V(Y_{G^{-1}(s)})|^2 ds = O(t) + a\varphi(t)$. Putting this result in (5.3) and choosing $\varepsilon$ small enough, we conclude that $\varphi(t) = O(t)$ a.s. \hfill $\Box$

**Remark 5.5.** Assuming that $g$ increases to the infinity is enough to ensure the convergence of $G^{-1}(t + T) - G^{-1}(t)$ to 0.

**Theorem 5.6.** Suppose that $\lim g(t) = \infty$, and that $\frac{\varphi(t)}{g(t)}$ vanishes when $t$ tends to infinity. Then the process $Y$ satisfies the pointwise ergodic theorem. More precisely, there exist some deterministic constants $a_i \geq 0$, such that $\sum a_i = 1$ and $\mu_t^Y$ converges (for the weak convergence of measures) toward $\sum_{1 \leq i \leq n} a_i \delta_{m_i}$.

**Proof.** Consider the time-changed process $Y_{G^{-1}(t)}$. By Benaim & Schreiber [4], Proposition 5.3 implies that the limit points of the empirical measure of $Y_{G^{-1}(t)}$ are included in the set of all the “invariant measures” for $\frac{d}{dt} \phi_t(x) = -\nabla V(\phi_t(x))$ with the initial condition $\phi_0(x) = x$. All these invariant measures are included in $\text{Vect}\{\delta_{m_1}, \ldots, \delta_{m_n}, \delta_{M_1}, \ldots, \delta_{M_p}\}$. So, if we have only a local minimum, we are done and we have proved the result. Else, let $\mu_t^{G^{-1}} = \frac{1}{t} \int_0^t \delta_{Y_{G^{-1}(s)}} ds$. Lemma 5.4 asserts that $(\mu_t^{G^{-1}}, t \geq 0)$ is a tight family of measures. Moreover, the ergodic theorem is actually true for any continuous function $\varphi$ such that $|\varphi| \leq \kappa(1 + |x|)$ with $\kappa > 0$. Let us now prove the uniqueness of the limiting probability measure. Indeed, one easily shows that $\mu_t$ is a Cauchy sequence in $L^1$. There exists $M > 0$ such that for any $s > 0$,

$$|\mathbb{E} \mu_{t+s} - \mathbb{E} \mu_t| \leq \frac{s}{t(t+s)} \int_0^t \mathbb{E} |X_u| du + \frac{1}{t+s} \int_t^{t+s} \mathbb{E} |X_u| du \leq M \frac{s}{t+s}.$$ 

So, the limit-measure in the ergodic theorem for $Y_{G^{-1}}$ is $\sum_{i=1}^n a_i \delta_{m_i} + \sum_{i=1}^p b_i \delta_{M_i}$ (where $a_i, b_i$ are nonnegative constants such that $\sum (a_i + b_i) = 1$) and the last step is to show
that the same result holds for $Y$.

For all continuous bounded function $\psi$ and $t > s$, we have (by an integration by parts)

$$
\int_s^t \psi(Y_u) du = \frac{G(t)}{g(t)} \mu^{G^{-1}}_{G(t)} \psi - \frac{G(s)}{g(s)} \mu^{G^{-1}}_{G(s)} \psi + \int_s^t \frac{g'(u)G(u)}{g^2(u)} \mu^{G^{-1}}_{G(u)} \psi du
$$

$$
= (t - s) \mu^{G^{-1}}_{G(t)} \psi + \frac{G(s)}{g(s)} \left( \mu^{G^{-1}}_{G(t)} \psi - \mu^{G^{-1}}_{G(s)} \psi \right)
$$

$$
+ \int_s^t \frac{g'(u)G(u)}{g^2(u)} \left( \mu^{G^{-1}}_{G(u)} \psi - \mu^{G^{-1}}_{G(s)} \psi \right) du.
$$

As $\mu^{G^{-1}}_{G(t)} \psi$ converges a.s., we deduce that

$$
\mu_i \psi = o(1) + \mu^{G^{-1}}_{G(t)} \psi + \frac{1}{t} \int_s^t \frac{g'(u)G(u)}{g^2(u)} \left( \mu^{G^{-1}}_{G(u)} \psi - \mu^{G^{-1}}_{G(s)} \psi \right) du.
$$

We conclude because $\frac{1}{t} \int_s^t \frac{g'(u)G(u)}{g^2(u)} du$ is bounded. It only remains to show that $b_i = 0$ for all $i$. Propositions 4.5 and 4.7 imply that for an unstable critical point, there exists a direction $j$ such that for all $\varepsilon > 0$, $P(\forall s \geq T, |Y_s^{(j)} - M^{(j)}| \leq \varepsilon) = 0$. Consider a nonnegative continuous function $f$, which support is a small ball (of radius $\alpha > 0$) around $M$: $f$ vanishes in all critical points except $M$ and $f(M) = 1$. Then, we have a.s. $\int_0^t 1_{\{|Y_s^{(j)} - M^{(j)}| \leq \alpha\}} ds = o(t)$ and the ergodic result still holds: $\frac{1}{t} \int_0^t f(Y_s) ds$ converges almost surely to $b$. So, by definition of $f$, we conclude that $b = 0$.

5.2. **Almost sure convergence.** We will prove that $Y$ converges a.s. towards a minimum. Let $0 < \varepsilon < \varepsilon_0$ and $T > T_0$ be as in Section 4. Let $m$ be a local minimum of $V$ such that $|Y_T - m| < \varepsilon$.

**Lemma 5.7.** If $\lim_{t \to \infty} g(t)^{-1} \log G(t) = 0$, then for all $c > 0$, we get $\int_0^\infty e^{-c g(t)} dt < +\infty$.

**Proof.** For all $\varepsilon > 0$, there exists $t$ large enough, such that for all $s \geq t$, we have $g(s)/\log G(s) \geq \varepsilon^{-1}$. Moreover, we know that there exists a positive constant $a$ such that for $t$ large enough $g(t) \geq a$ and then $G(t) \geq at$. As a consequence, we get $g(t) \geq \varepsilon^{-1} \log(at)$. We now conclude that $\int_{t_0}^\infty e^{-(c \log(at))/\varepsilon} dt < \infty$ (choose for instance $\varepsilon = c/2$).

**Proposition 5.8.** Suppose that $\lim_{t \to \infty} \frac{g(t)}{g(\varepsilon) g(t)} = 0$. If $g(t)^{-1} \log G(t)$ converges to $0$, then $Y_t$ converges a.s. and for all $i$, we have $P \left( \lim_{t \to \infty} Y_t = m_i \right) > 0$ and $P \left( \lim_{t \to \infty} Y_t = M_i \right) = 0$.

**Proof.** We begin to prove that $Y$ converges a.s. by using a result of Benaïm (1 Proposition 4.6). It asserts that if $F(x) = -\nabla V(x)$ is a continuous globally integrable vector field, and if for all $c > 0$, we have $\int_0^\infty e^{-cgG^{-1}(t)} dt < +\infty$ and $P(\sup_i |Y_t| < \infty) = 1$, then $Y$ is almost surely an asymptotic pseudotrajectory for the flow induced by $F$. Actually, the first and last conditions are fulfilled under our hypothesis. Moreover, as $G^{-1}$ is a nondecreasing function, the (finite) integral $\int_0^\infty e^{-cg(t)} dt$ is a upper bound for the preceding integral. As
a consequence, \( Y \) is an a.s. asymptotic pseudotrajectory for the flow \( \Phi \) defined by \( \Sigma \), and so, the flow restricted to the set of the limit points of \( Y \) does not admit any other attractor than the set of limit points. Finally, \( Y \) converges a.s. and the limit points of \( Y \) are included into the set \( \{ x\left. ; \nabla V(x) = 0 \right. \} \).

If \( Y \) converges to \( Y_\infty \), then the limit-process \( Y_\infty \) is not a local maximum \( M_i \) because of Proposition 4.5. On the event \( \{ \forall s \geq T; \left| Y_s - m_i \right| < \varepsilon \} \), which has a positive probability to occur by Proposition 4.3, we have a.s. that \( V_t \leq Y_{t+T} - m_i \leq U_t \). As \( \lim_{t \to \infty} U_t / \sqrt{\log G(t)} = 1 \) a.s., we get \( U_t \xrightarrow{a.s.} 0 \). The same holds for \( V_t \).

**Corollary 5.9.** Suppose that \( \lim g(t)^{-1} \log G(t) = 0 \). If \( V \) is a strictly uniformly convex function everywhere (with a unique minimum \( m \)), then \( Y_t \xrightarrow{a.s.} m \).

**Remark 5.10.** If there exists a local minimum \( m \neq 0 \), then \( \mathbb{P}(\overline{\mu}_t \text{ converges}) < 1 \). On the other hand, if 0 is a local minimum, then \( \overline{\mu}_t \) converges if and only if \( \int_0^\infty Y_t \frac{d\gamma}{r+s} < \infty \) a.s.

### 6. Behavior of \( X \) in the case of a general potential

#### 6.1. Ergodicity of \( X \)

At that stage, we have proved that \( Y \) satisfies the pointwise ergodic theorem. The main question of this paper is to know whether \( X \) also satisfies the pointwise ergodic theorem or not. Remark, that the process \( \overline{\mu}_t \) converges a.s. if and only if \( \int_0^t Y_s \frac{d\gamma}{r+s} \) converges. In particular, if \( Y_t \xrightarrow{a.s.} 0 \) polynomially fast then \( \overline{\mu}_t \) converges a.s.

A necessary condition for the almost sure convergence of \( Y \) to 0 is to consider a potential \( V \) with 0 as unique minimum (for instance \( V \) is symmetric and strictly convex).

**Proposition 6.1.** Suppose that \( \lim g(t) = \infty \), and that \( \frac{g'(t)}{g(t)^2} \) vanishes when \( t \) tends to infinity. Then, the measure \( \mu_t \) converges weakly if and only if \( \overline{\mu}_t \) converges a.s.

**Proof.** We recall that \( X_t = Y_t + \overline{\mu}_t \). We have shown in Section 5 under our assumptions, that \( Y \) always satisfies the pointwise ergodic theorem. Consider the Fourier transform of \( \mu_t \). We have for all \( u \in \mathbb{R}^d \):

\[
\frac{1}{t} \int_0^t e^{i(u,X_s)} ds = \frac{e^{i(u,\overline{\mu}_\infty)}}{t} \int_0^t e^{i(u,Y_s)} ds + \frac{1}{t} \int_0^t e^{i(u,Y_s)} \left( e^{i(u,\overline{\mu}_s)} - e^{i(u,\overline{\mu}_\infty)} \right) ds.
\]

The first right member converges a.s. to \( e^{i(u,\overline{\mu}_\infty)} \int e^{i(u,y)} \gamma(dy) \). For the second right member, we use Cesàro’s result to prove that it converges a.s. to 0 if and only if \( \overline{\mu}_t \) converges a.s. So, \( X \) satisfies the pointwise ergodic theorem.

#### 6.2. Almost sure convergence

In order to study the asymptotic behavior of \( (X_t, t \geq 0) \), we will consider the process \( Y \) defined by \( Y_t = X_t - \overline{\mu}_t \).

**Theorem 6.2.** Assume that \( \lim g(t) = \infty \) and \( g'(t)/g^2(t) \) goes to zero as \( t \) tends to infinity. Suppose also that \( \sqrt{g(t)^{-1} \log G(t)} = O(h(t)^{-1}) \), with \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \int_0^\infty \frac{ds}{(1+s)h(s)} < +\infty \). One of the following holds:
(1) If $0$ is the unique local minimum of $V$ then
\[ P \left( \lim_{t \to \infty} X_t = \mu + \int_0^\infty Y_s \frac{ds}{r + s} \right) = P \left( \lim_{t \to \infty} \mu_t = \mu + \int_0^\infty Y_s \frac{ds}{r + s} \right) = 1; \]

(2) If $0$ is a local minimum of $V$ and there exists other local minima, then on the event \( \{ Y_t \overset{a.s.}{\to} 0 \} \), we have that $X_t$ converges and else it diverges. More precisely, we get
\[ P \left( \lim_{t \to \infty} X_t = \mu + \int_0^\infty Y_s \frac{ds}{r + s} \right) + P(\lim_{t \to \infty} |X_t| = \infty) = 1 \]
and
\[ 1 > P \left( \lim_{t \to \infty} \mu_t = \mu + \int_0^\infty Y_s \frac{ds}{r + s} \right) = P \left( \lim_{t \to \infty} \mu_t = \mu + \int_0^\infty Y_s \frac{ds}{r + s} \right) > 0; \]

(3) If $0$ is not a local minimum of $V$, then
\[ P \left( \lim_{t \to \infty} |X_t| = \infty \right) = P \left( \lim_{t \to \infty} |\mu_t| = \infty \right) = 1. \]

Moreover, on the set \( \{ Y_\infty \neq 0 \} \), $X_t/\log t$ converges to $Y_\infty$.

Proof. Denote $m = (m_1, \ldots, m_d)$. Recall, that $Y_t = X_t - \mu_t$. First, suppose that $m = 0$. By Proposition 5.8, $Y_t$ has a positive probability to converge toward 0. On this event, Proposition 4.3 implies that the integral \( \int_0^T \frac{Y_s}{r + s} ds \) converges (because of the law of the iterated logarithm). So, $\mu_t$ converges toward this integral and the result follows for $X_t$.

On the other hand, if $m \neq 0$, then $P(Y_t \to m) > 0$ and so the $j$th-coordinate of $\mu_t$ converges to $\text{sgn}(m_j)\infty$. So, the direction $j$ is unstable and $X_t$ does not converge a.s. What is more, on the set \( \{ Y_\infty \neq 0 \} \), we have
\[ \left| \frac{\mu_t}{\log t} - Y_\infty \right| \leq \frac{1}{\log t} \int_0^t \frac{|Y_s - Y_\infty|}{r + s} ds \leq \frac{1}{\log t} \int_0^t \frac{\log G(s)}{g(s)} \frac{ds}{r + s}. \]
The latter upper bound tends to 0 by the law of the iterated logarithm (Proposition 4.3). As $\frac{X_t}{\log t} = \frac{\mu_t}{\log t} + \frac{Y_t}{\log t}$, the result follows. \[ \square \]

Remark 6.3. Any polynomial $h$ satisfies the required condition. In particular, one can choose $g(t) = t^\alpha (\log(1 + t))^{\beta}$ with $\alpha > 0$ or $\alpha = 0$ and $\beta > 2$.

If $g$ does not satisfy the latter conditions, then it can happen that $\mu_t$ does not converge a.s. and so, $X$ does not converge. This is a sufficient condition for the a.s. convergence of $X$.

Conclusion
We have obtained the following. The process $X$ converges a.s. if and only if $g$ is greater than a polynomial (and increases to infinity) and $V$ admits 0 as unique local minimum. Otherwise, we will prove in a forthcoming paper that:
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• if $g$ is such that $\lim \frac{\log G(t)}{g(t)} = k > 2\text{osc} \chi$, then $X$ converges in distribution (to a random variable concentrated on the global minima of $V$) if and only if $\sum a_i m_i = 0$ (where the constants $a_i$ are defined by Theorem 5.6);

• if $\lim g(t) = 1$, then $X$ converges in probability if and only if $\int_{\mathbb{R}^d} x e^{-2V(x)} dx = 0$.

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