Faddeev-Jackiw and Canonical Analysis of the Jackiw-Teitelboim model written as a BF theory

J. Manuel-Cabrera

División Académica de Ciencias Básicas, Universidad Juárez Autónoma de Tabasco,
Km 1 Carretera Cunduacán-Jalpa, Apartado Postal 24, 86690 Cunduacán, Tabasco, México

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The Jackiw-Teitelboim model written as a BF theory in two dimensions is analyzed by using the Dirac's and Faddeev-Jackiw formalism. The analysis consists in finding the full structure of the constraints, the gauge transformations, the counting of degrees of freedom and the generalized Faddeev-Jackiw brackets. The Poincaré symmetry and the diffeomorphisms are found. Further, we show that the Faddeev-Jackiw and Dirac’s brackets coincide to each other.

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I. INTRODUCTION

The interest of studying lower dimensional models have been of enormous use in several branches of physics and specifically lower dimensional gravity theories have proven highly instructive in the understanding of the quantum gravity theories. Furthermore, some mathematical complexities found in four dimensions can be avoided, and certain basic physical phenomena can be easily demonstrated. Quite recently, D. Grummier, J. Hartong, S. Prohazka and J. Salzer has been studied the AdS/CFT correspondence for a class of 2D theories of gravity called two-dimensional dilaton gravity. Furthermore, this approach has been useful to study a proposed duality among the Sanchdev-Ye-Kitaev and the AdS$_2$ gravity in the shape of the Jackiw-Teitelboim [JT] model. On the other hand, S. Josephine argued in that it is possible to find a general solution for correlators of external boundary operators in black holes states of JT model. As we have remarked before, the models in lower dimensional are important since they help to generate new ideas, and to stimulate new insights into their dimensional counterparts.

The aim of this paper is to investigate the canonical Dirac formalism and the Faddeev-Jackiw method of 2D gravity proposed by JT written in the BF formulation in 1+1 dimensions introduced by K. Isler and C.A.Kamimura. It is important to comment that there exist analysis of the BF action developed in a smaller phase space through Dirac’s algorithm reported in and the Hamilton-Jacobi formalism has been studied in; however, in the analysis of Dirac’s formalism was developed in a smaller phase space and the complete structure of the constraints on

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1. This means that only those variables that occur in the action with temporal derivative are considered as dynamical.
the full phase space was not reported. The implementation of Dirac’s approach in the full phase space, it is important in the classification of the constraints in first and second class, in some cases, it is complicated and is not a trivial task [14, 15], therefore, it is necessary to use another framework that could give us a complete description of singular systems. In this context, the Faddeev-Jackiw [FJ] [16] formalism with the Barcelos Neto-Wotzasek extension [BW] [17, 18] is an alternative scheme to the Dirac approach. The combination of [FJ] and [BW] is sometimes called in the literature as modified FJ formalism, or symplectic approach. We can find diverse applications, developments and the equivalence between Dirac method and the original [FJ] in [17–23]. Recently, the [FJ] symplectic formalism has been employed to study the Bonzom-Livine action [24], topologically massive gravity [TMG] [25], and the abelian analog version of TMG theory at the chiral point [26] and also in four-dimensional General Relativity (GR) and to GR extensions [27].

The advantages in using the [FJ] approach come from the fact that it is not necessary to do the classification of the constraints in first and second class as carried out in Dirac algorithm since all the constraints of the theory are on the same footing. Additionally, all the relevant information of the theory can be obtained through an invertible symplectic matrix where the entries of this inverse matrix correspond to the [FJ] generalized brackets and coincide with the Dirac brackets [20, 21]. Furthermore, Montani-Wotzasek [MW] [19], presented the procedure of dealing with the gauge symmetry transformation over all the configuration space.

In this paper, we consider the analysis of JT theory in the BF formulation in the context of Dirac’s formalism and [FJ] symplectic approach. If we want to compare both scheme, it is mandatory to work in Dirac’s formalism with the full configuration space [28]. This is an important difference from the analysis given by [11, 12] where this fact was not considered.

The organization of this paper is as follows. In the next sections, we briefly discuss of two-dimensional gravity proposed by [JT] model and the BF formulation. In Section IV, we show how to arrive at the complete structure of the constraints, and we find the Dirac brackets. In Section V we obtain the generator of the general gauge transformations with Castellani method. In Section VI, we develop a complete analysis of the BF model in the [FJ] symplectic formalism, and we find the full constraints, the symplectic matrix. In Section VII, we obtained the gauge symmetries of the theory with the Montani-Wotzasek [MW] formalism [19]. We work with the configuration space field as symplectic variables with the purpose of reproducing all the Dirac results, as we will see the generalized [FJ] brackets and Dirac’s brackets coincide to each other. In Section VIII finally, we present the results.

II. TWO DIMENSIONAL GRAVITY OF JACKIW-TEITELBOIM MODEL

Gravity in 1+1 dimensions, cannot be formulated on the Einstein-Hilbert action \( \int d^2x \sqrt{-g} R \), which is a surface term, therefore, the Einstein’s tensor \( G_{\mu \nu} \) vanishes identically in two-dimensions, for that reason, it is necessary to invent a model. A 2D gravity model has been suggested by Jackiw and Teitelboim [6, 8] and the connection with BF formulation was analysed by K. Isler and C.A.
Kamimura [9], who also showed that JT model can be written as a gauge theory.

As we have mentioned earlier, Jackiw and Teitelboim [JT] 2 developed a model simple but nontrivial of General Relativity, where include, in addition to the space-time the metric $g_{\mu \nu}$, a scalar field $\Psi(x)$ [7, 8].

The [JT] model is described down the next action

$$S_{JT}[g_{\mu \nu}, \Psi] = \frac{1}{2} \int d^2 x \sqrt{-g} \Psi (R - 2\Lambda). \quad (1)$$

The Euler-Lagrange equations for [JT] model are given by

$$\frac{\delta S_{JT}[g_{\mu \nu}, \Psi]}{\delta \Psi} : R - 2\Lambda = 0, \quad (2)$$

$$\frac{\delta S_{JT}[g_{\mu \nu}, \Psi]}{\delta g_{\mu \nu}} : \nabla_\mu \nabla_\nu \Psi + \Lambda g_{\mu \nu} \Psi = 0. \quad (3)$$

The first equations of motion refer to Einstein’s equation (3), the parameter $\Lambda$ playing the role of the cosmological constant and the second is the equation of motion for the scalar field [30].

III. JACKIW-TEITELBOIM MODEL IN THE BF FORMULATION

In $n$-dimensional spacetime, BF theory with gauge group $G$ involves two fields: a $G$ connection $A$, and a $g$-valued ($n$-2)-form $E$. In the absence of matter, the lagrangian is simply

$$L = Tr(E \wedge F). \quad (4)$$

The BF theory of two dimensional gravity consist on the gauge connection 1-form $A$ and a scalar field $B$, also called background field ($B$-field), whose action is given by

$$S_{BF}[B, A] = \int_M Tr(B \wedge F), \quad (5)$$

the trace is taken on the adjoint representation of $G$, and $F = dA + A \wedge A$ is the curvature of $A$. The corresponding Euler-Lagrange equations are $F = 0$ and $D_A B = 0$, where the first equation simply say that the connections $A$ is flat, and the second equation say that $B$ is covariant constant-its covariant exterior derivative $D_A B$ vanishes.

It is well-know that geometric dynamics of [JT] model can be written as a BF theory in two dimensions [9], which is a generalization of Witten’s work in three dimensions on Chern-Simons theory [32]. The gauge group $G$ of two dimensional gravity is given by de Sitter or anti-de Sitter in

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2 The generalized of [JT] model through a function $f(\mu_i, R)$, where $f$ is a suitable polynomial function of $\mu_i$ and $R$ has been proposed by Lichtzeir and Odintsov [29]
Riemannian or Lorentzian space-time. Hence, the (A)dS algebra of the generators will satisfy the following commutation relations

\[ [J, P_I] = \epsilon_{IJ} P_J, \quad [P_I, P_J] = \epsilon_{IJ} \Lambda J, \]  

(6)

where the operators \( P_I \) and \( J \) are the translation generators and the Lorentz boost generator.

If we define generators of the (A)dS algebra as \( \{ T_i \} = \{ T_0, T_1, T_2 \} = \{ P_0, P_1, J \} \), the Lie algebra can be written as

\[ [T_i, T_j] = f_{ijk} T_k = \Lambda \epsilon_{ijl} \eta^{lk} T_k, \]  

(7)

and the metric \( \eta_{ij} \) on the Lie algebra can be expressed as

\[ \eta_{ij} = \text{diag}(\Lambda \sigma, \Lambda, 1). \]  

(8)

Nonetheless, the presence of a non vanishing cosmological constant \( \Lambda \) is necessary to ensure the non degeneracy of the Killing metric and to build a consistent gauge theory \([9][31]\).

IV. HAMILTONIAN DYNAMICS OF THE JACKIW-TEITELBOIM MODEL IN THE BF FORMULATION

The "BF" action (5) can be written in the form \([9]\)

\[ S_{BF}[B, A] = \int \text{Tr}(B \wedge F) \quad \rightarrow \quad S_{BF}[\phi, A] = \int \text{Tr}(\phi \wedge F). \]  

(9)

The "B-field" \( (B \rightarrow \phi = \phi^i T_i =: \phi^j P_j + \Psi J) \) of the theory is a 0-form and \( F = F^i T_i = F^I P_I + F^2 J \) is the Yang-Mills curvature, where \( F^I = de^I + \omega^I J \wedge e^J \) and \( F^2 = d\omega + \frac{4}{\epsilon} \epsilon_{IJK} e^I \wedge e^J \) represent the torsion and curvature 2-form of the zweibein field in the first order formalism\([6]\).

The action (9) can be now rewritten as

\[ S_{BF}[\phi, A] = \int \text{Tr}(\phi \wedge F) = \int \text{Tr}(T_i T_j) \int \phi^i \wedge F^j = \int \phi^j \wedge F^j = \frac{1}{2} \int d^2 x \epsilon^{\mu\nu} \eta_{ij} \phi^i F^j_{\mu\nu}. \]  

(10)

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\( \text{The antisymmetric tensor } \epsilon_{IJ} \text{ is defined by } \epsilon_{01} = 1. \) The indices \( I, J, \ldots = 0, 1 \) are lowered by the flat metric \( \eta_{IJ} = \text{diag}(\sigma, 1) \) or its inverse \( \eta^{IJ} \) where \( \sigma = \pm 1 \) for Riemannian, resp. Lorentzian theory. For \( \sigma = -1 \) or 1, is the Lie algebra of \( SO(3) \) or \( SO(1, 2) \).\[3][13]\)

\( \text{The completely antisymmetric tensor } \epsilon_{ijl} \text{ is defined by } \epsilon_{012} = 1. \)

\( \text{The notation } "\text{Tr}" \text{ represents a non-degenerate invariant bilinear form on the Lie algebra and } \text{Tr}(T_i T_j) = (T_i, T_j) = \eta_{ij} \text{ plays the role of a metric on the Lie algebra.} \)

\( \text{The values of the spacetime of the indices } \mu, \nu, \ldots \text{ are label by } t, x. \) The antisymmetric tensor \( \epsilon^{\mu\nu} \) is defined by \( \epsilon^{12} = 1. \)
The variation of the action (10) leads to the equations

\[ \frac{\delta S_{BF}[\phi, A]}{\delta \phi_i} : F^i = 0, \quad \frac{\delta S_{BF}[A, \phi]}{\delta A^i} : D\phi_i = 0, \]

(11)

the equations (11) for two dimensional BF gravity are equivalent to equations [JT] model [9].

Therefore, in order to carry out the Hamiltonian analysis, we assume that space-time has the topological structure \( M = \Sigma \times \mathbb{R} \), where \( \Sigma \) is a 1-dimensional manifold, representing the "space" and \( \mathbb{R} \) represents an evolution parameter. By performing the 1+1 decomposition, we can write the action (10) as

\[ S_{BF}[A, \phi] = \int d^2x (\phi_i \partial_t A^i_x + A^i_x D_x \phi_i). \]

(12)

The definition of the momenta \((\Pi^x_i, \Pi^t_i, \Pi^\phi_i)\) canonically conjugate to the configuration variables \((A^i_x, A^i_t, \phi^i)\) is given by

\[ \Pi^x_i = \frac{\delta L}{\delta \dot{A}^i_x}, \quad \Pi^t_i = \frac{\delta L}{\delta \dot{A}^i_t}, \quad \Pi^\phi_i = \frac{\delta L}{\delta \dot{\phi}^i}. \]

(13)

The matrix elements of the Hessian is given by

\[ H_{LM} = \frac{\partial^2 L}{\partial Q^i_L \partial Q^j_M} = 0. \]

(14)

Note that rank of the Hessian is zero, thus, we expect 9 primary constraints. From the definition of the momenta (13) we identify the following 9 primary constraints

\[ \Phi^x_i := \Pi^x_i - \phi_i \approx 0, \quad \Phi^t_i := \Pi^t_i \approx 0, \quad \Phi^\phi_i := \Pi^\phi_i \approx 0. \]

(15)

The canonical Hamiltonian is given by

\[ H_c = -\int dx A^i_t D_x \phi_i, \]

(16)

and the corresponding primary Hamiltonian \( H_P \)

\[ H_P = H_c + \int dx \left[ \lambda^x_i \Phi^x_i + \lambda^t_i \Phi^t_i + \lambda^\phi_i \Phi^\phi_i \right], \]

(17)

where \((\lambda^x_i, \lambda^t_i, \lambda^\phi_i)\) are the corresponding Lagrange multipliers associated of these constraints \((\Phi^x_i, \Phi^t_i, \Phi^\phi_i)\). The fundamental Poisson brackets of the theory are determined by the commutation relations

\[ \{A^i_j(x, \Pi^x_j(y))\} = \delta^i_j \delta(y-x), \]

\[ \{\phi^i(x, \Pi^\phi_j(y))\} = \delta^i_j \delta(y-x). \]

(18)

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7 The component fields are given by \( \phi_i = (\varphi_0, \varphi_1, \Psi) \), \( A^i_x = (e^0_x, e^1_x, \omega_x) \), \( A^i_t = (e^0_t, e^1_t, \omega_t) \). \( D_x \phi_i = \partial_x \phi_i + f_{ij}^k A^k_x \phi_k \).

8 Where \( Q_L \) label the sets of variables \( Q_L = \{A^i_x, A^i_t, \phi^i\} \).
The next steps is to observer if there are more constraints, so that, we calculate the following $9 \times 9$ matrix whose entries are the Poisson brackets among the constraints \([15]\)

\[
\{ \Phi^x_i(x), \Phi^x_j(y) \} = 0, \\
\{ \Phi^x_i(x), \Phi^\phi_j(y) \} = -\eta_{ij}\delta(x - y),
\]

and expressed in matrix form, namely,

\[
W = \{ \Phi^A(x), \Phi^B(y) \} = \begin{pmatrix}
0 & 0 & -\eta_{ij}\delta(x - y) \\
0 & 0 & 0 \\
\eta_{ij}\delta(x - y) & 0 & 0
\end{pmatrix},
\]

where, \(\Phi^A = (\Phi^x_i, \Phi^t_i, \Phi^\phi_i)\). It is easy to see that this matrix has rank=6 and 3 null-vectors. By using these 3 null-vectors and the evolution of \(\Phi^t_i\) produces the following 3 secondary constraints

\[
\dot{\Phi}^t_i = \{ \Phi^t_i(x), H_P \} \approx 0 \quad \Rightarrow \quad \beta_i := D_x\phi_i \approx 0,
\]

and consistency conditions of \(\Phi^x_i\) and \(\phi_i\) leads to 6 Lagrangian multipliers

\[
\dot{\Phi}^x_i = \{ \Phi^x_i(x), H_P \} \approx 0 \quad \Rightarrow \quad \lambda_i^x + f_{ijk}A_k^i \phi_k \approx 0,
\]

\[
\dot{\Phi}^\phi_i = \{ \Phi^\phi_i(x), H_P \} \approx 0 \quad \Rightarrow \quad \lambda_i^\phi - D_xA^i_1 \approx 0.
\]

Consistency conditions of the secondary constraints leads to no new constraints. Having found all constraints, we need to identify from the primary and secondary constraints which corresponds to first and second class. In order to classify the full set of constraints, we have to calculate the rank and the null-vectors of the $12 \times 12$ matrix whose entries will be the Poisson brackets between primary and secondary constraints, this is

\[
\{ \Phi^x_i(x), \Phi^\phi_j(y) \} = -\eta_{ij}\delta(x - y), \\
\{ \Phi^x_i(x), \beta_j(y) \} = f_{ijk}\phi_k\delta(x - y), \\
\{ \Phi^\phi_i(x), \beta_j(y) \} = -\eta_{ij}\partial_y(x - y) - f_{ijl}A^l_y\delta(x - y).
\]

This matrix has a vanishing determinant. After a long calculation, we found that this matrix has a rank=6 and 6 null vectors, thus, the theory presents a set of 6 first class constraints and 6 second class constraints. The structure of the first class constraints is obtained by means the null vectors, where, the null vectors of the matrix \([23]\) are given by

\[
V^1_i = (0, \delta_i^j, 0, 0)\delta(x - y), \\
V^2_i = (0, \delta_i^j, 0, -f_{ijl}\phi_k, \delta_i^l)\delta(x - y).
\]
In order to identify the following 6 first class constraints, we used the contraction of the null vectors \( \mathbf{24} \) with the constraints \( \mathbf{15} \) and \( \mathbf{21} \)
\[
\gamma_i = D_x\chi^x_i + D_x\phi_i - f_{ij}^k\phi_k\chi^\phi_j \approx 0,
\gamma_i^t = \Phi_t^i = \Pi_i^t \approx 0,
\]
and the following 6 second class constraints
\[
\chi^x_i = \Phi^x_i = \Pi^x_i - \phi_i \approx 0,
\chi^\phi_i \approx 0.
\]

At this point, it is worth noting that these constraints \( \mathbf{26} \) have not been reported in the full phase space. As was pointed out at the introduction, it is mandatory to know the correct structure of the constraints on the full phase space in order to get complete information of the fundamental gauge transformation and the Dirac brackets. As we well know, the structure of constraints are related to gauge symmetries, besides, they have an important role an important role on the formulation canonical approaches of quantization \[15\].

We now give the complete algebra among the constraints \( \mathbf{25} \) and \( \mathbf{26} \)
\[
\{\gamma_i(x), \gamma_j(y)\} = f_{ij}^k \gamma_k \delta(x - y) \approx 0,
\{\gamma_i(x), \gamma_j^t(y)\} = 0,
\{\gamma_i^t(x), \gamma_j^t(y)\} = 0,
\{\gamma_i(x), \chi_j^\phi(y)\} = f_{ij}^k \chi_k^\phi \delta(x - y) \approx 0,
\{\gamma_i(x), \chi_j^y(y)\} = f_{ij}^k \chi_k^y \delta(x - y) \approx 0,
\{\chi_i^\phi(x), \chi_j^y(y)\} = \eta_{ij} \delta(x - y),
\]
where we can observe that the algebra of constraints \( \mathbf{27} \) is closed and is the local version of the Lie algebra of the group A(dS). These constraints generate the A(dS) gauge transformation. Additionally, with all the information obtained until now, we can construct the Dirac brackets. For this aim, we shall use the matrix whose elements are only the Poisson brackets among second class constraints, namely \( C_{\alpha\beta}(x, y) = \{\zeta^\alpha(x), \zeta^\beta(y)\} \), given by
\[
[C_{\alpha\beta}(x, y)]_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \eta_{ij} \delta(x - y),
\]
the inverse matrix \([C_{\alpha\beta}(x, y)]^{ij}\) is given by

\[^9\text{Where } f_{ij}^k = \Lambda e_{ij}^k.\]
The Dirac brackets among two functionals $A$, $B$ is defined as

$$\{A(x), B(y)\}_D = \{A(x), B(y)\}_P - \int dudv \{A(x), \zeta^\alpha(u)\} C^{-1}_{\alpha\beta}(u,v) \{\zeta^\beta(v), B(y)\},$$  \tag{30}$$

where $\{A(x), B(y)\}_P$ is the usual Poisson bracket between the functionals $A$, $B$ and $\zeta^\alpha = (\chi_i^\phi, \chi_i^\psi)$ is the set of second class constraints. By using (29) and (30), yields the following Dirac’s brackets of the theory

$$\{A^i(x), \phi_j(y)\}_D = \delta^i_j \delta(x-y),$$  \tag{31}$$

$$\{A_i^i(x), \Pi_j^i(y)\}_D = \delta^i_j \delta(x-y),$$  \tag{32}$$

$$\{A_i^i(x), \Pi_j^j(y)\}_D = \delta^i_j \delta(x-y),$$  \tag{33}$$

we can observer that in BF model the fields $A_i^i$ and $\phi_i$ they are non-commutative.

We calculate the Dirac brackets among the first and second class constraints, and we have found that non trivial part of the Dirac Brackets is given by

$$\{\gamma_i(x), \gamma_j(y)\}_D = f_{ij}^k (\chi_i^\phi + f_{k}^{mn} \chi_m \chi_n^\phi) \delta(x-y).$$  \tag{34}$$

According to the Dirac formalism, the Dirac brackets among first class constraints must be square of second class constraints and linear of first class constraints \textsuperscript{10}. Additionally, the Dirac Brackets amongst the second class constraints $\{\zeta^\alpha(x), \zeta^\beta(y)\}_D = 0$, and with all other quantities turn out be zero \textsuperscript{15}.

On the other hand, the identification complete of the constraints and Lagrange multipliers will allow us to identify the extended action. By using the first class constraints (25), the second class class constraints (26), and the multipliers Lagrange multipliers (22) we find that the extended action takes the form

$$S_E[\phi^i, \Pi_i^\phi, A_i^i, \Pi_i^\mu, \lambda_i, u_i^i, u_i^\phi] = \int d^2x (\Pi_i^\phi \dot{\phi}^i + \Pi_i^\mu \dot{A}_i^i - \mathcal{H} - \lambda^i \gamma_i - \lambda^i \gamma_i^t - u^i \chi_i^\phi - u^\phi \chi^i_{\phi}),$$  \tag{35}$$

and

$$\mathcal{H} = -A_i^i \gamma_i = -A_i^i (D_x \chi_i^x + D_x \phi_i - f_{i}^{jk} \phi_k \chi_j^\phi),$$  \tag{36}$$

\textsuperscript{10} Quadratic terms in (32) may be present since the square of second class constraint is a first class one \textsuperscript{14, 33}.\]
where $\lambda^i, \lambda^t_i, u^i_x, u^t_{\phi}$, are the Lagrange multipliers that enforce the first and second class constraints. We are to observable, by considering the second class constraints as strong equation, that the Hamiltonian (36) is reduced to the usual expression found in the literature [11, 12], which is defined on a reduced phase space context. From the extend action we can identify the extend Hamiltonian, which is given by

$$H_E = \int dx(H + \lambda^i \gamma_i + \lambda^t_i \gamma^t_i),$$

(37)

thus, the extended Hamiltonian is a linear combination of first-class constraints as expected.

The equations of motion obtained from the extend Hamiltonian and brackets Dirac are expressed by

$$\dot{\phi}_i = \{\phi, H_E\}_D = f_{ij}^k(\lambda^j - A^j_i)\phi_k,$$
$$\dot{A}^i_x = \{A^i_x, H_E\}_D = D_x(A^i_x - \lambda^i),$$
$$\dot{A}^t_i = \{A^t_i, H_E\}_D = \lambda^t_i,$$
$$\dot{\Pi}^x_i = \{\Pi^x_i, H_E\}_D = f_{ijk}(\lambda^j - A^j_i)\Pi^j_k,$$
$$\dot{\Pi}^t_i = \{\Pi^t_i, H_E\}_D = \gamma_i.$$ (38)

V. GAUGE GENERATOR

We will calculate the Fundamental Gauge Transformation (FGT) defined on the full phase space. The construction of the FGT is based on the Castellani method and the gauge generators are given by first class [34]. According to the Castellani method, the gauge generator is given by

$$G = \int \sum \left[D_t \tau^i \gamma^t_i + \epsilon^i \gamma_i\right].$$ (39)

By using the gauge generator, we obtain the following gauge transformation on the phase space

$$\delta_0 \phi_i = f_{ij}^k \epsilon^j \phi_k,$$
$$\delta_0 A^i_x = -D_x \epsilon^i,$$
$$\delta_0 A^t_i = D_t \epsilon^i,$$
$$\delta_0 \Pi^x_i = f_{ij}^k \epsilon^j \chi^x_k,$$
$$\delta_0 \Pi^t_i = f_{ij}^k \epsilon^j \chi^t_k + f_{ij}^k \epsilon^j \phi_k,$$
$$\delta_0 \Pi^t_i = f_{ij}^k \epsilon^j \gamma^t_j.$$ (40)

We can see that FGT of BF model are given by (40) and do not correspond to diffeomorphisms. Nevertheless, it is well known that a theory with background independence is diffeomorphisms
covariant, and this symmetry can be obtained from the FGT. Hence, the diffeomorphisms must be found by redefining the gauge parameters as $\tau^i = -\varepsilon^i = v^\rho A^i_\rho$, and where $v$ is a vector field

\begin{align*}
\delta_0 \phi_i & = -f_{ij}^k v^\rho A^j_\rho \phi_k, \\
\delta_0 A^i_\mu & = D_\mu (v^\rho A^i_\rho),
\end{align*}

and the gauge transformation of the fields takes the following form\(^{11}\)

\begin{align*}
\phi'_i & \rightarrow \phi_i + \Sigma_\nu \phi_i - v^\rho D_\rho \phi_i, \\
A'^i_\mu & \rightarrow A^i_\mu + \Sigma_\nu A^i_\mu + v^\rho F^i_{\rho\nu}.
\end{align*}

The expression for diffeomorphisms are obtained (on shell) from the FGT as an internal symmetry of the theory. For other hand, the symmetries obtained in (40), are related with Poincaré transformation. We can redefine the gauge parameters as $\tau^i = -\varepsilon^i = \theta^i + v^\rho A^i_\rho$\(^{35}\)

\begin{align*}
\delta_0 A^i_\mu & = D_\mu \theta^i + \partial_\mu v^\rho A^i_\rho + v^\rho \partial_\mu A^i_\mu + v^\rho F^i_{\rho\mu} = \delta_{\text{PGT}} A^i_\mu + v^\rho F^i_{\rho\mu}, \\
\delta_0 \phi_i & = -f_{ij}^k \theta^j \phi_k + v^\rho \partial_\rho \phi_i - v^\rho D_\rho \phi_i = \delta_{\text{PGT}} \phi_i - v^\rho D_\rho \phi_i,
\end{align*}

where

\begin{align*}
\delta_{\text{PGT}} A^i_\mu & = D_\mu \theta^i + \partial_\mu v^\rho A^i_\rho + v^\rho \partial_\rho A^i_\mu, \\
\delta_{\text{PGT}} \phi_i & = -f_{ij}^k \theta^j \phi_k + v^\rho \partial_\rho \phi_i.
\end{align*}

We can see that the gauge symmetries (43) take back to the Poincaré symmetries up to terms proportional to the equations of motion (11).

**VI. FADDEV-JACKIW SYMPLECTIC ANALYSIS FOR BF THEORY**

In this section, we focus now on the FJ method. In order to perform this aim, we observe that Lagrangian density of the action (12) can be written by a first-order Lagrangian like

\[ L(\xi, \dot{\xi}) = \dot{\xi}^{(0)a} K_a^{(0)}(\xi) - V^{(0)}(\xi), \quad a = 1, 2, 3. \]

Using (45), we may rewrite (12)

\[ L = \dot{\xi}^{(0)a} K_a^{(0)}(\xi) - V^{(0)}(\xi), \quad a = 1, 2, 3. \]

\(^{11}\) $\mathcal{L}$ represents the Lie derivative.
\[ \mathcal{L}^{(0)} = \phi_i \partial_t A_i^x + A_i^x D_x \phi_i = \phi_i \partial_t A_i^x - V^{(0)}, \]  

(46)

where the superscript \(^{(0)}\) means initial Lagrangian and \(V^{(0)} = -A_i^x D_x \phi_i\) is called the symplectic potential.

In the [FJ] method, the Euler-Lagrange equations of motion are

\[ f^{(0)}_{ab}(x,y) \dot{\xi}^{(0)a}(y) = \frac{\partial V^{(0)}(y)}{\partial \xi^{(0)a}(x)}, \]  

(47)

\[ f^{(0)}_{ab}(x,y) = \frac{\delta K^{(0)}_a(y)}{\delta \xi^{(0)a}(x)} - \frac{\delta K^{(0)}_b(y)}{\delta \xi^{(0)a}(x)}, \]  

(48)

where \(f^{(0)}_{ab}\) is the symplectic matrix and with \(\xi^{(0)a}\) representing a set of symplectic variables, \(K^{(0)}_a\) is called the canonical 1-form. From expression (46) we can identify the coefficients \(K^{(0)}_a(x) = \{K^{(0)}_1, K^{(0)}_2, K^{(0)}_3\} = \{0, \phi_i, 0\}\). In order to obtain all the Dirac results of previous section, we will use the configuration space \(\xi^{(0)a}(x) = \{\xi^{(0)1}, \xi^{(0)2}, \xi^{(0)3}\} = \{\phi_i, A_i^x, A_i^t\}\) as symplectic variables.

It is important to comment, that in [FJ] framework we are free to choose the symplectic variables, we can choose the configuration variables or the phase space variables.

The symplectic matrix is given by

\[
\begin{pmatrix}
\phi_j & A_j^x & A_j^t \\
\phi_i & 0 & \delta_i^j \\
A_i^x & -\delta_i^j & 0 \\
A_i^t & 0 & 0
\end{pmatrix}
\delta(x - y). 
\]  

(49)

The symplectic matrix \(f^{(0)}_{ab}\) represents a \([9 \times 9]\) singular matrix. In [FJ] scheme, this implies that the theory has constraints. In order to obtain these constraints, we calculate the zero modes of the symplectic matrix in this case we have a zero mode, and is given by \((v^{(0)a})^T_1 = (0, 0, v^{A_i^t})\), where \(v^{A_i^t}\) is an arbitrary function. By multiplying the two sides of (45) by the zero modes \((v^{(0)a})^T_1\), the left-hand side of (45) is

\[
\int dx (v^{(0)a})^T_1(x) f^{(0)}_{ab}(x,y) = 0.
\]  

(50)

Then, the right-hand side from (45) we can get the primary constraint as

\[
\Omega^{(0)}_i = \int dx (v^{(0)a})^T_1(x) \frac{\delta}{\delta \xi^{(0)a}(x)} \int dy V^{(0)}(y),
\]

\[
= -\int dx v^{A_i^t}(x) \delta_i^j D_x \phi_j,
\]

\[
= -v^{A_i^t} \delta_i^j D_x \phi_j = 0,
\]  

(51)

since \(v^{A_i^t}\) is an arbitrary function, we obtain the following constraint

\[
\Omega^{(0)}_i = \delta_i^j D_x \phi_j = 0,
\]  

(52)
this constraint is the secondary constraint found by means Dirac’s method in above section. In order to
determine whether there are more constraints, we calculate the following \[ F_{cb}^{(1)} \dot{\xi}^b = Z_c(\xi), \] (53)
where
\[ Z_c(\xi) = \left( \begin{array}{c} \frac{\partial V^{(0)}(\xi)}{\partial \xi^a} \\ \frac{\partial V^{(0)}(\xi)}{\partial \xi^b} \end{array} \right), \] (54)
and
\[ F_{cb}^{(1)}(x, y) = \left( \begin{array}{c} f^{(0)}_{ab} \\ \frac{\partial \Omega^{(0)}(0)}{\partial \xi^b} \end{array} \right) = \left( \begin{array}{ccc} \phi_j & A^j_y & A^j_i \\ 0 & \delta^j_i & 0 \\ A^j_y & -\delta^j_i & 0 \\ \frac{\partial \Omega^{(0)}(0)}{\partial \xi^b} & \delta^j_i \partial_x - f^j_{ik} k A^k_x & f^j_{ik} \phi_k & 0 \end{array} \right) \delta(x - y). \] (55)

We can observe that the matrix (55) is not a square matrix, nevertheless, note that this matrix has
an independent mode given by \((v^{(1)})^T = (-f^j_{ik} k \phi_k \delta(x - y), \delta^j_i \partial_x \delta(x - y), -f^j_{ik} k A^k_k \delta(x - y), v^A_k, \delta^j_i \delta(x - y)),\) this mode is used in order to obtain further constraints. By means of the following expression
\[ (v^{(1)})^T Z_c = 0, \] (56)
where \(c = 1,\) we obtain that (56) is an identity, thus, leads to no new constraints for the theory
under study.

According to the [FJ] symplectic, we will write a new Lagrangian, this is done by means of the
\(A_i^t = \dot{\lambda}^i\) Lagrange multiplier associated to that constraint \(\Omega^{(0)}_t,\) therefore, we can write the next
symplectic Lagrangian
\[ \mathcal{L}^{(1)} = \phi_i A_i^t + \Omega^{(0)}_t \dot{\lambda}^i - V^{(1)}, \] (57)
where \(V^{(1)} = V^{(0)} \big|_{\Omega^{(0)}_t = 0} = 0,\) the symplectic potential vanish reflecting the general covariance
of the theory, just like it is present in General Relativity. From the first-order Lagrangian (53),
we can identify the next symplectic variables \(\xi^{(1)a}(x) = \{\phi_i, A^i_x, \lambda^i\}\) and the new coefficients of
1-forms \(K^{(1)}_a(x) = \{0, \phi_i, \Omega^{(0)}_a\}.\) Therefore, having considered this new information, we can obtain
the following symplectic matrix
where rows and columns follow the order \(\phi_i, A_x^i, \lambda^i.\) The symplectic matrix \(f_{ab}^{(1)}\) represents a \([9 \times 9]\)
singular matrix. However, as discussed above there are not more constraints; the noninvertibility of
\( f^{(1)}_{ab}(x, y) = \begin{bmatrix} \phi_j & A^j_y & \lambda^j & \sigma_j \\ \phi_i & 0 & \delta^i_j & \delta^i_j \delta_y + f_{jk} A^k_y = \delta(x - y). \end{bmatrix} \) (58)

implies that there is a gauge symmetry. If we want to invert the symplectic matrix, we choose the following gauge fixing

\[ A^j_i(x) = 0, \] (59)

according to the [FJ] symplectic formalism, we have to introduce the gauge fixing as constraint by mean of lagrange multiplier \( \sigma_i \). Now, introducing this new information into (57), leads to new symplectic Lagrangian

\[ L^{(2)} = \phi_i \dot{A}^i_x + (\Omega^{(0)}_i + \sigma_i) \dot{\lambda}^i, \] (60)

thus, we identify the following set of symplectic variables \( \xi^{(2)} \alpha (x) = \{\phi_i, A^j_x, \lambda^i, \sigma_i\} \) and the symplectic 1-forms \( K^{(2)}_a(x) = \{0, \phi_i, \Omega^{(0)}_i + \sigma_i, 0\} \). Furthermore, by using these symplectic variables we find that the symplectic matrix is given by

\[ f^{(2)}_{ab}(x, y) = \begin{bmatrix} \phi_j & A^j_y & \lambda^j & \sigma_j \\ \phi_i & 0 & \delta^i_j & \delta^i_j \delta_y + f_{jk} A^k_y = \delta(x - y). \end{bmatrix} \] (61)

The symplectic matrix \( f^{(2)}_{ab} \) represents a \([12 \times 12]\) nonsingular matrix. After a long calculation, the inverse is given by

\[ [f^{(2)}_{ab}(x, y)]^{-1} = \begin{bmatrix} 0 & -\delta^i_j & 0 & -f_{ji}^k \phi_k \\ \delta^i_j & 0 & 0 & -\delta^i_j \delta_y + f_{jk} A^k_y \\ 0 & 0 & 0 & \delta^i_j \\ -f_{ji}^k \phi_k & \delta^i_j \delta_x + f_{jk} A^k_x & -\delta^i_j & 0 \end{bmatrix} \] (62)

Therefore, from (62) it is possible to identify the following [FJ] generalized brackets by means of

\[ \{\xi^{(2)}_{ia}(x), \xi^{(2)}_{ja}(y)\}_{FJ} = [f^{(2)}_{ij}(x, y)]^{-1}, \] (63)
thus, the following brackets are identified
\[
\{A^i_x(x), \phi_j(y)\}_{FJ} = \delta^i_j \delta(x-y),
\]
\[
\{\phi_i(x), \sigma_j(y)\}_{FJ} = -f^{i, k}_{j, l} \phi_k \delta(x-y),
\]
\[
\{A^i_x(x), \sigma_j(y)\}_{FJ} = -\delta^i_j \partial_y \delta(x-y) - f^{i, k}_{j, l} A^k_y \delta(x-y),
\]
\[
\{\lambda^i(x), \sigma_j(y)\}_{FJ} = \delta^i_j \delta(x-y).
\] (64)

It is important to comment, that the generalized [FJ] brackets obtained from (62) agreed with the brackets Dirac (31). In fact, if we make a redefinition of the fields introducing the momenta given by
\[
\Pi^x_i = \phi_i,
\]
\[
\Pi^\phi_i = 0,
\] (65)

the generalized [FJ] brackets (64) take the form
\[
\{A^i_x(x), \phi_j(y)\}_{FJ} = \delta^i_j \delta(x-y),
\]
\[
\{A^i_x(x), \Pi^x_j(y)\}_{FJ} = \delta^i_j \delta(x-y),
\] (66)

where we can observe that (66) coincide with the full Dirac’s brackets found in (31, 33) if we choose to impose the gauge $A^i_t = 0$ in the Dirac’s method.

As we have discussed earlier, in [FJ] approach it is not necessary classify the constraints in first class or second class, since all the constraints are at the same footing. Therefore, we can perform the counting of physical degrees of freedom in the following form; there are 6 dynamical variables ($\phi_i, A^i_x$) and 6 constraints ($\Omega^{(0)}_i, A^i_t$), therefore, the theory lacks of physical degrees of freedom.

**VII. GAUGE GENERATOR IN THE MW METHOD**

Finally, we have calculated the gauge transformations of the theory, for this aim we calculate the mode of the matrix (58), this mode is given by\textsuperscript{12}
\[
[W^{(1)a}]^T = (f^{i, k}_{j, l} \phi_k, \eta^\mu \partial_x + f^{i, k}_{j, l} A^k_x, -\eta^\nu \delta(x-y)).
\] (67)

It can be seen that the zero-mode ($W^{(1)a})^T$ is the generator of the infinitesimal gauge symmetry on the constraint surface of the action (58) and the infinitesimal gauge transformation of fields in compact notation is given by $\delta_G \xi^{(1)a} = \int dx [W^{(1)a}]^T \varepsilon$ and "$\varepsilon$" is a set of infinitesimal arbitrary parameters.

\textsuperscript{12} In this context \( T \) means matrix transposition.
From the above, we can see that the gauge transformation is therefore given by
\[ \delta_G \xi^{(1)a} = (\delta_G \phi_i, \delta_G A^i_x, \delta_G \lambda^i) = \int dx (f^i_{jk} \phi_k, \eta^{ij} \partial_x + f^{ij}_k A^k_x, -\eta^{ij}) \delta(x - y) \varepsilon_j, \]

or, more explicitly
\[ \delta_G \phi_i(x) = f^i_{jk} \phi_k \varepsilon^j, \quad \delta_G A^i_x(x) = -D_x \varepsilon^i, \quad \delta_G \lambda^i(x) = -\varepsilon^i. \]

Now, coming back to the original variables\(^\text{13}\)
\[ \delta_G \phi_i(x) = f^i_{jk} \phi_k \varepsilon^j, \quad \delta_G A^i_x(x) = -D_x \varepsilon^i, \quad \delta_G A^i_t(x) = -\partial_t \varepsilon^i = \partial_t \tau^i. \]

In this manner, by using the [FJ] symplectic framework we have reproduced the first two components of gauge transformations reported in Dirac’s method but the last component fails to do so.

Montani and Wotzasek [MW]\(^\text{19}\) developed one scheme of dealing with restrictions in the FJ method. Besides, they modified the FJ approach for obtained the symmetry transformations of action over all the configuration space, contrary to transformation (70) generated by [FJ] formalism that only holds on the constraint surface. According to the MW method one should write the functional variation of the corresponding Lagrangians\(^\text{14}\) to zero. Therefore,
\[ \frac{\delta \mathcal{L}^{(1)}}{\delta \xi^{(1)a}} = f^i_{ab} \dot{\xi}^{(1)b} - \frac{\partial V^{(1)}}{\partial \xi^{(1)a}}. \]

So, according to MW, on multiplying (71) by the zero-mode (67), we have
\[ [W^{(1)a}^T \frac{\delta \mathcal{L}^{(1)}}{\delta \xi^{(1)a}} = [W^{(1)a}^T \left[ f^i_{ab} \dot{\xi}^{(1)b} - \frac{\partial V^{(1)}}{\partial \xi^{(1)a}} \right], \]

from (72), we obtained the following equations\(^\text{15}\)
\[ (f^i_{jk} \phi_k, \eta^{ij} \partial_x + f^{ij}_k A^k_x, -\eta^{ij}) \left[ \begin{array}{c} \frac{\delta \xi^{(1)}}{\delta \phi_i} \\ \frac{\delta \xi^{(1)}}{\delta A^i_x} \\ \frac{\delta \xi^{(1)}}{\delta \lambda^i} \end{array} \right] = \left[ \begin{array}{ccc} 0 & \delta^i_j & D^i_{jy} \\ -\delta^i_j & 0 & -f^{ij}_k \phi_k \\ -D^i_{ix} & -f^{ij}_k A^k_x & 0 \end{array} \right] \left[ \begin{array}{c} \dot{\phi}_j \\ \dot{A}^i_x \\ \dot{\lambda}^i \end{array} \right], \]

from eq.(73), one finds the GT over all the configuration space. Using this equations, the GT can be written as

\(^\text{13}\) \(\partial_t \delta \lambda^i = \delta A^i_t\).

\(^\text{14}\) In term of matrix form \[ \left[ \frac{\delta \mathcal{L}^{(1)}}{\delta \xi^{(1)a}} \right]^T = \left( \frac{\delta \mathcal{L}^{(1)}}{\delta \phi_i}, \frac{\delta \mathcal{L}^{(1)}}{\delta A^i_x}, \frac{\delta \mathcal{L}^{(1)}}{\delta \lambda^i} \right). \]

\(^\text{15}\) Where \(D^i_{jy} = \partial_j \phi_y + f^{jk}_i A^k_y\) and \(D^i_{ix} = \delta^i_x \partial_x + f^{ik}_j A^k_x\).
\[ f^{ik} \phi_k \frac{\delta L^{(1)}}{\delta \phi_i} + (\eta^{il} \partial_x + f^{li} k A^k_x) \frac{\delta L^{(1)}}{\delta A^i_x} - \eta^{il} \frac{\delta L^{(1)}}{\delta \lambda^i} = f^{ik} \phi_k (A^i_x - D_x \dot{\lambda}^i) + (\eta^{il} \partial_x + f^{li} k A^k_x)(-\dot{\phi}_i - f_{lpq} \phi^p \dot{\lambda}^q) + (-\eta^{il})(-\dot{\Omega}^{(0)}_l). \]

Now, coming back to the original variables

\[
\begin{align*}
\dot{\lambda}^i & \rightarrow A^i_t, \\
\frac{\delta L^{(1)}}{\delta \lambda^i} & \rightarrow \frac{\delta L^{(1)}}{\delta A^i_t}, \\
\frac{\delta L^{(1)}}{\delta \phi_i} & \rightarrow \frac{\delta L^{(0)}}{\delta \phi_i}, \\
\frac{\delta L^{(1)}}{\delta A^i_x} & \rightarrow \frac{\delta L^{(0)}}{\delta A^i_x},
\end{align*}
\]

and implementing the equations of motion for the gauge fields

\[ \frac{\delta L^{(0)}}{A^i_t} = \Omega^{(0)}_i, \]

we obtain\(^{16}\)

\[ (-1)\eta^{il} D_t \varepsilon_l \frac{\partial L^{(0)}}{\partial A^i_t} + (-1)\eta^{il} D_x \varepsilon_l \left( \frac{\partial L^{(0)}}{\partial A^i_t} - \partial_i \frac{\partial L^{(0)}}{\partial \dot{A}^i_x} \right) + f^{ik} \phi_k \varepsilon_l \frac{\partial L^{(0)}}{\partial \phi_i} = 0. \]

Then, we can find that the gauge field transformations are given by

\[ \delta G A^i_t(x) = -D_t \varepsilon^i = D_t \tau^i, \quad \delta G A^i_x(x) = -D_x \varepsilon^i, \quad \delta G \phi_i(x) = f_{ij} k \phi_k \varepsilon^j. \]

This means that the transformations will be a gauge symmetry over all the configuration space. We found the same transformations as obtained by using Dirac approach.

\section*{VIII. CONCLUSIONS AND PROSPECTS}

In this work, we analyse the BF model of JT theory from point of view of the Dirac formalism and FJ symplectic method for constrained systems. In Dirac formalism, we find the first and the second-class constraints, and we can see that \( S_{BF} \) action for gravity in two dimensions is devoid of degrees of freedom, this is clearly in concordance with the fact this theory contain 18 canonical variables \( (\phi, A^i_\mu, \Pi^{\phi}_1, \Pi^{\phi}_2) \), 6 first class constraints \( (\gamma_i, \gamma^i_\lambda) \) and 6 second class constraints \( (\chi^\phi_i, \chi^\tau_i) \), as

\(^{16}\) \( \eta^{il} D_t = \eta^{il} \partial_t + f^{i_1 i_2} A^\alpha_{i_1} \).
a result, one gets zero degrees of freedom $18 - 2(6) - 6 = 0$, consequently, the theory is topological.

Furthermore, in this approach we have obtained the fundamental gauge structure as well the algebra between the constraints, and it has been shown that the set of constraints form a closed algebra (AdS) \( (27) \). Besides, by defining the gauge parameters, diffeomorphisms and Poincaré symmetries can be obtained from the fundamental gauge symmetry. On the other hand, considering the second class constraints as strong equality, the results is reduced to the usual expression found in the literature \[11, 12\], which is defined on a reduced phase space context. These results obtained can be compared with those calculated by means of the Hamilton-Jacobi \[13\] formalism, where, we have no distinction between the first and the second-class constraints.

In the \([FJ]\) symplectic approach we have a set of the constraints of the theory are at the same footing and generally leads to a less number of constraints that the Dirac formalism, and this fact allows that the \([FJ]\) symplectic method is more convenient to perform. Moreover, we have showed that the generalized \([FJ]\) brackets and the Dirac’s ones coincide to each other. Besides, we have obtained that the number of physical degrees of freedom is the same as the one obtained from the Dirac formalism. On the other hand, was obtained the gauge symmetry over all the configuration space by using the \([MW]\) algorithm. In this manner, we have reproduced all relevant Dirac’s results by working with \([FJ]\) symplectic, in particular we can see that \([FJ]\) symplectic method is more economical when it is compared with the Dirac formalism.

We finish this paper with some comments, as discussed above, in the \([FJ]\) symplectic framework it is not necessary to classify the constraints in second class or first class as in Dirac’s method is done, consequently, the algebraic operations that involving constraints analysis are shortened. This fact allows that the \([FJ]\) symplectic method is more convenient to develop. In this sense, we can carry out the analysis to other models of 2D gravity. The action \((78)\) is an alternative model reproducing Einstein’s equations with a cosmological constant and dynamical torsion \[36, 37\]

$$
S[e_\mu, \omega_\mu] = \int d^2x \left( \frac{1}{16\alpha} R^{I,J}_{\mu\nu} R^I_{\mu\nu} - \frac{1}{8\beta} T^I_{\mu\nu} T^I_{\mu\nu} - \Lambda \right), \quad (78)
$$

at the same time, the action \((78)\) contains solutions with constant curvature and zero torsion, it also includes several other 2D gravity models \[7–9\] and this is of particular interest for investigations of the quantum structure of gravity. The Hamiltonian analysis of the model \((78)\) has been developed in \[38\] and its canonical quantization in \[39\]. On the other hand, we can find in the literature \[37\] that the model \((78)\) in the region $e = \det(e^I_\mu) \neq 0$ can be written as a gauge theory based on the quadratic extension of the Poincaré algebra and can be rewritten as

$$
\tilde{S}[e_\mu, \omega_\mu, \varphi, \varphi^I] = \int d^2x \left[ \frac{1}{2} \delta^{\mu\nu} (\varphi R_{\mu\nu} + \varphi_1 T^I_{\mu\nu}) - e(\alpha \varphi^2 + \beta \varphi_1 \varphi_1^I + \Lambda) \right]. \quad \text{(79)}
$$

The Hamiltonian analysis on the full phase space of the action \((79)\) has not been reported and the complete structure of the constraints it is unknown. As previously discussed, in some cases, to implement the Dirac algorithm is large and tedious task, hence, it is necessary to use alternative
formulations that could give us a complete canonical description of the theory, in this sense, we will utilize the [FJ] symplectic formalism with the purpose of studying the action \( \mathcal{S} \) in \([10]\).

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