Inelastic interaction of nearly equal solitons for the BBM equation

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Abstract

This paper is concerned with the interaction of two solitons of nearly equal speeds for the (BBM) equation. This work is an extension of [31] addressing the same question for the quartic (gKdV) equation. We consider the (BBM) equation, for

$$\begin{align*}
\lambda &
\in
[0, 1), \\
(1 - \lambda \partial_x^2) \partial_t u + \partial_x (\partial_x^2 u - u + u^2) = 0.
\end{align*}$$

(BBM)

Solitons are solutions of the form

$$R_{\mu, x_0}(t, x) = Q_{\mu}(x - \mu t - x_0),$$

for \(\mu > -1, x_0 \in \mathbb{R}\).

For \(\mu_0 > 0\) small, let \(U(t, x)\) be the unique solution of (BBM) such that

$$\lim_{t \to -\infty} \|U(t) - Q_{-\mu_0}(\cdot + \mu_0 t) - Q_{\mu_0}(\cdot - \mu_0 t)\|_{H^1} = 0.$$

First, we prove that \(U(t)\) remains close to the sum of two solitons, for all time \(t \in \mathbb{R}\),

$$U(t, x) = Q_{\mu_1(t)}(x - y_1(t)) + Q_{\mu_2(t)}(x - y_2(t)) + \varepsilon(t) \quad \text{where} \quad \|\varepsilon(t)\| \leq \mu_0^2,$$

with \(y_1(t) - y_2(t) > 2|\ln \mu_0| + O(1)\), which means that at the main order the situation is similar to the integrable KdV case. However, we show that the collision is perfectly elastic if and only if \(\lambda = 0\) (i.e. only in the integrable case).

1 Introduction

We consider the so-called Benjamin-Bona-Mahony equation, for \(\lambda \in (0, 1),\)

$$\begin{align*}
(1 - \lambda \partial_x^2) \partial_t u + \partial_x (\partial_x^2 u - u + u^2) = 0, \quad t, x \in \mathbb{R}.
\end{align*}$$

(BBM)

We refer to Appendix C below for obtaining [BBM] from the following more standard form of the equation

$$\begin{align*}
(1 - \partial_x^2) \partial_t \mathcal{U} + \partial_x (\mathcal{U} + \mathcal{U}^2) = 0.
\end{align*}$$

(1.1)

Recall that (1.1) was originally introduced by Peregrine [40] and Benjamin, Bona and Mahony [2] as an alternate model to the standard integrable (KdV) equation, corresponding to \(\lambda = 0,\)

$$\begin{align*}
\partial_t u + \partial_x (\partial_x^2 u + u^2) = 0.
\end{align*}$$

(KdV)

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The Cauchy problem for (BBM) is globally well-posed in $H^1$ (see [2]), and any $H^1$ solution $u(t, x)$ of (BBM) satisfies for all $t \in \mathbb{R}$,

$$\int (\lambda (\partial_x u)^2 + u^2) \, dt = M(u(t)) = M(u(0))$$  \hspace{1em} (mass) \hspace{1em} (1.2)

$$\int \left( (\partial_x u)^2 + u^2 - \frac{2}{3} u^3 \right) \, dt = \mathcal{E}(u(t)) = \mathcal{E}(u(0))$$ \hspace{1em} (energy) \hspace{1em} (1.3)

It is also well-known that the (BBM) equation has soliton solutions: for $\mu > -1$, set

$$Q_\mu(x) = (1 + \mu)Q\left( \sqrt{\frac{1 + \mu}{1 + \lambda \mu}} x \right)$$

where

$$Q(x) = \frac{3}{2} \frac{1}{\cosh^2 \left( \frac{x}{2} \right)}$$ solves $Q'' + Q^2 = Q$.

Then, for any $\mu > -1$, $y \in \mathbb{R}$, $R_{\mu, y}(t, x) = Q_\mu(x - \mu t - y)$ is solution of (BBM).

1.1 Review on the collision problem for (KdV) type equations

We briefly review some results concerning the problem of collision of solitons for (KdV) type models and we refer to the introduction of [31] for more details.

First, it is very well-known that the (KdV) equation has explicit pure $N$-soliton solutions ([17], [42], [34]): for any given $c_1 > \ldots > c_N > 0$, $y^{-1}, \ldots, y^{-N} \in \mathbb{R}$, there exists an explicit multi-soliton solution $u(t, x)$ of (KdV) which satisfies

$$\lim_{t \to \pm \infty} \left\| u(t) - \sum_{j=1}^{N} c_j Q(\sqrt{c_j}(x - y_j^+ - c_j t)) \right\|_{H^1(\mathbb{R})} = 0,$$

for some $y_j^+$ (such solutions were found using the inverse scattering transform).

Stability and asymptotic stability of $N$-solitons were studied by Maddocks and Sachs [24] in $H^N$ by variational techniques and in the energy space $H^1$ by Martel, Merle and Tsai [33].

Second, recall that LeVeque [23] further investigated the behavior of the explicit 2-soliton solution $u$ above in the asymptotic $\mu = \frac{c_1 - c_2}{c_1 + c_2}$ small i.e. for nearly equal solitons. It is proved that for some explicit functions $c_j(t)$, $y_j(t)$

$$\sup_{t, x \in \mathbb{R}} \left| u(t, x) - c_1(t)Q(\sqrt{c_1(t)}(x - y_1(t))) - c_2(t)Q(\sqrt{c_2(t)}(x - y_2(t))) \right| \leq C \mu^2,$$  \hspace{1em} (1.4)

Moreover, $\min_{t \in \mathbb{R}} (y_1(t) - y_2(t)) = 2 |\ln \mu| + O(1)$, which means in particular that the minimum separation between the two solitons goes to $\infty$ as $\epsilon \to 0$. See [23] for a more precise statement.

Collision problems for (gKdV) and (BBM) have also been studied since the 60’s from both experimental and numerical points of view (see [12], [17], [46], [1], [8], [3], [18], [38], [41], [5], [33], [14]).

However, except for some integrable equations for which special explicit solutions are known, the problem of describing rigorously the collision of two solitons is mainly open.
Now, we review some recent rigorous works related to the interaction of two solitons in the nonintegrable situation for the generalized KdV equations

\[ \partial_t u + \partial_x (\partial_x^2 u + u^p) = 0, \quad t, x \in \mathbb{R}. \]  

(gKdV)

Recall that solitons of (gKdV) write

\[ R_{c,y}(t,x) = c_1^p - Q(\sqrt{c}(x - ct - y)), \text{ for } c > 0, y \in \mathbb{R} \]

where \( Q \) satisfies \( Q'' + Q^p = Q \).

Mizumachi [36] studied rigorously the interaction of two solitons of nearly equal speeds for (gKdV) for \( p = 3 \) and \( p = 4 \). For initial data \( u_0 \) close to \( Q(x) + c_1^p - Q(\sqrt{c}(x + L)) \), where \( L > 0 \) is large and \( c \), close to 1, satisfies \( c - 1 \leq e^{-\frac{L}{2}} \), Mizumachi proved the two solitons remain separated for all positive time and that eventually the corresponding solution \( u(t) \) behaves as

\[ u(t) = (c_1^+)^{p-1}Q\left(\sqrt{c_1^+(\cdot - c_1^+t - y_1^+)}\right) + (c_2^+)^{p-1}Q\left(\sqrt{c_2^+(\cdot - c_2^+t - y_2^+)}\right) + \varepsilon(t,x), \quad (1.5) \]

for large time, for some \( c_1^+ < c_2^+ \) close to 1 and \( \varepsilon \) small in some space. The analysis part in [36] relies on scattering results due to Hayashi and Naumkin [15, 16] and on the use of spaces of exponentially decaying functions (introduced in this context by Pego and Weinstein [39]).

From [36], the situation is roughly speaking similar to the one described in the integrable case by LeVeque [23]. However, two main questions were left open in this work in this regime

1. **Is the 2-soliton structure stable globally in time in the energy space \( H^1 \)?**
2. **Does there exist a pure 2-soliton in this regime?**

As in the integrable case, we call pure 2-solitons, solutions of (gKdV) satisfying

\[ u(t) - \sum_{j=1,2} (c_j^+)^{p-1}Q\left(\sqrt{c_j^+(\cdot - c_j^+t - y_j^+)}\right) \rightarrow 0 \quad \text{as } t \rightarrow \pm \infty \text{ in } H^1(\mathbb{R}). \quad (1.6) \]

Note that if (1.6) holds both at \(-\infty\) and \(+\infty\), then necessarily \( c_j^- = c_j^+ \) for \( j = 1, 2 \) (see [30], pp. 68, 69).

These two questions have been answered in a recent work by the authors [31]. Indeed, in the context of two solitons of almost equal speeds for the quartic (gKdV) equation, by constructing an approximate solution to the problem, we were able to prove first the global stability of the two soliton structure in \( H^1 \) and second, the inelastic character of the interaction. See Theorems 1 and 2 in [31].

We also point out some other recent works of the authors ([29], [30]) concerning the problem of collision of two solitons of (gKdV) for a general nonlinearity \( g(u) \) in the case where one soliton, is supposed to be large with respect to the other soliton, i.e. assuming \( 0 < c_1 \ll c_2 \). See also [32], with T. Mizumachi, extending these results to the (BBM) equation.

1.2 **Main results**

In the present paper, we extend the results of [31] to the (BBM) model.

There are two main motivations to consider these questions for the (BBM) model: first, the structure of the (BBM) equation is close to the one of the (KdV) equation but it cannot be considered as a perturbation of the (KdV) equation. Second, the present paper on (BBM) proves that our techniques extend to quadratic nonlinearity, unlike [36], based on scattering techniques critical for \( p = 3 \).
Theorem 1 (Inelastic interaction of two solitons with nearly equal speeds). Let $\lambda \in (0,1)$. There exist $C, c, \sigma, \mu_+ > 0$ such that the following holds. For $0 < \mu_0 < \mu_+$, let $U(t)$ be the unique solution of (BBM) such that

$$\lim_{t \to \infty} \|U(t) - Q_{-\mu_0}(-, \mu_0 t + \frac{1}{2}Y_0 + \ln 2) - Q_{\mu_0}(\cdot, -\mu_0 t - \frac{1}{2}Y_0 - \ln 2)\|_{H^1} = 0,$$  \hspace{1cm} (1.7)

where $Y_0 = |\ln(\mu_0^2/\alpha)|$ and $\alpha = 240/(15 + 10\lambda - \lambda^2)$. Then

(i) Global stability of 2-solitons. There exist $\mu_1(t)$, $\mu_2(t)$, $y_1(t)$, $y_2(t)$ such that,

$$w(t, x) = U(t, x) - Q_{\mu_1(t)}(x - y_1(t)) - Q_{\mu_2(t)}(x - y_2(t))$$  \hspace{1cm} (1.8)

satisfies, for all $t \in \mathbb{R}$,

$$\|w(t)\|_{H^1(\mathbb{R})} \leq C|\ln \mu_0|^3 \mu_0^2, \quad \left|\min_{t \in \mathbb{R}}(y_1(t) - y_2(t)) - Y_0\right| \leq C|\ln \mu_0|^{\sigma} \mu_0^{3/2},$$  \hspace{1cm} (1.9)

$$\sum_{j=1,2} |\mu_j(t) + (-1)^j \mu_0 \tanh(\mu_0 t)| + \sum_{j=1,2} |\dot{y}_j(t) - \mu_j(t)| \leq C|\ln \mu_0|^2 \mu_0^2.$$  \hspace{1cm} (1.10)

(ii) Asymptotics and defect. The limits $\mu_1^+ = \lim_{t \to +\infty} \mu_1$, $\mu_2^+ = \lim_{t \to +\infty} \mu_2$ exist and

$$\lim_{t \to +\infty} \|w(t)\|_{H^1(x > -(99/100)Y_0)} = 0, \quad \lim_{t \to +\infty} \|w(t)\|_{H^1(\mathbb{R})} \geq c \mu_0^3,$$  \hspace{1cm} (1.11)

$$c \mu_0^3 \leq \mu_1^+ - \mu_0 \leq C|\ln \mu_0|^{\sigma} \mu_0^4, \quad c \mu_0^3 \leq -\mu_2^+ - \mu_0 \leq C|\ln \mu_0|^{\sigma} \mu_0^4.$$  \hspace{1cm} (1.12)

It follows immediately from the lower bound (1.11) that no pure 2-soliton exists, which is a new result for the (BBM) equation in this regime.

Theorem 2 (Stability result in the energy space for (KdV) and (BBM) equations). Let $\lambda \in [0,1)$. There exists $\mu_+ > 0$, $C, \sigma > 0$, such that the following holds. Let $\bar{\mu}_0 \in \mathbb{R}$ and $\bar{Y}_0 > 0$ be such that

$$\mu_0 = \left(\bar{\mu}_0^2 + 4\alpha e^{-\bar{Y}_0}\right)^{1/2} < \mu_+.$$  \hspace{1cm} (1.13)

Let $u_0 \in H^1$ be such that

$$\|u_0 - Q_{-\bar{\mu}_0}(\cdot, -\frac{1}{2}\bar{Y}_0) - Q_{\bar{\mu}_0}(\cdot, +\frac{1}{2}\bar{Y}_0)\|_{H^1(\mathbb{R})} \leq \omega \mu_0,$$  \hspace{1cm} (1.14)

where $0 < \omega < |\ln \mu_0|^{-2}$, and let $u(t)$ be the solution of (BBM) such that $u(0) = u_0$. Then, there exist $T(t)$, $X(t)$ of class $C^1$ such that, for all $t \in \mathbb{R}$,

$$\|u(t + T(t), \cdot) + X(t) - U(t)\|_{H^1(\mathbb{R})} + |\dot{X}(t)| + \mu_0 |\dot{T}(t)| \leq C \omega \mu_0 + C|\ln \mu_0|^{\sigma} \mu_0^{3/2},$$  \hspace{1cm} (1.15)

where $U(t)$ is the solution defined in Theorem 1.

Comments on the results:

1. The (KdV) case in Theorems 1 and 2. The value $\lambda = 0$ in the BBM equation corresponds to the integrable KdV equation. In this case, estimates (1.9)–(1.10) still hold. Estimate (1.9) corresponds to (1.3) but from the proofs in the present paper, we improve the main result in [23] in this case by computing explicitly the term of size $\mu_0^2$, see Remark 3.
Note also that for $\lambda = 0$, the existence of pure 2-soliton solutions corresponds to $\mu_1^+ = \mu_0$ and $\mu_2^+ = -\mu_0$ in (1.11) and (1.12).

Moreover, Theorem 2 holds for $\lambda = 0$ and it is also a new global stability result for the \textit{(KdV)} equation in the energy space. This kind of result cannot be proved by scattering theory.

2. Except for the value of the constant $\alpha > 0$, Theorems 1 and 2 are exactly the same as for the quartic (gKdV) equation. In particular, the orders of size in $\mu_0$ in the various estimates do not depend on the power of the nonlinearity. Moreover, the function

$$Y(t) = Y_0 + 2\ln(\cosh(\sqrt{\alpha}e^{-\frac{1}{2}Y_0} t)) \text{ solution of } \ddot{Y} = 2\alpha e^{-Y}, \quad \lim_{t \to -\infty} \dot{Y}(t) = 2\mu_0, \quad \dot{Y}(0) = 0. \tag{1.16}$$

appears in both problems and has a universal character in this problem. Note that Theorems 1 and 2 can be extended to any two solitons of (1.1) of almost equal sizes using a simple scaling argument. See Appendix C.

Finally, from the present paper and [31], it is clear that the results can be extended to (gKdV) equations with general nonlinearities.

3. As in [31], the lower bounds in (1.11) and (1.12) measure the inelastic character of the collision. Moreover, the different exponents of $\mu_0$ in (1.11) and (1.12) denotes a gap in the estimates which is an open problem.

1.3 Strategy of the proofs

We describe briefly the strategy of the proofs of Theorems 1 and 2, which is the same as in [31]. We point out the analogies and the main technical differences between the (BBM) and the quartic (gKdV) case. The proof of Theorem 2 is a consequence of the proof of Theorem 1 and so we focus on the proof of Theorem 1. Let $U(t)$ be the solution of (BBM) defined in Theorem 1.

(1) The first step is the construction of an approximate solution in terms of a series in $e^{-y(t)}$ where $y(t) = y_1(t) - y_2(t)$ is the distance between the two solitons, using the exponential decay of the solitons. From Proposition 2.1, the approximate solution contains a tail of order $e^{-y(t)}$ between the two solitons, which is relevant in the description of the exact solution, see Remark 3. This tail of order $e^{-y(t)}$ is not related to inelasticity since it appears also in the integrable case $\lambda = 0$. Moreover, it does not prevent the approximate solution to be in the energy space at this order, since it is localized in space between the two solitons.

In contrast, for $\lambda \neq 0$, one cannot build an approximate solution at order $e^{-\frac{1}{2}y(t)}$ in the energy space, whereas it is possible for $\lambda = 0$. The presence of a nonzero tail at $-\infty$ in space at this order is related to nonintegrability and inelasticity.

The construction of the approximate solution for (BBM) in Section 2 is more involved that in the quartic (gKdV) case mainly because the nonlinearity is quadratic rather that quartic.

(2) After the approximate solution is constructed, we introduce the following decomposition of the solution $U(t)$:

$$U(t,x) = Q_{c_1(t)}(x - y_1(t)) + Q_{c_2(t)}(x - y_2(t)) + W(t,x) + \varepsilon(t,x),$$

where $Q_{c_1(t)}(x - y_1(t)) + Q_{c_2(t)}(x - y_2(t))$ is the modulated approximate solution and $\varepsilon(t)$ is a rest term. To prove stability of the two soliton structure, we have to control both the parameters $c_j(t)$ and $y_j(t)$ and the rest term $\varepsilon(t)$.
From the construction of the approximate solution, the parameters \( c_j(t) \) and \( y_j(t) \) have to satisfy an approximate dynamical system. Remarkably, it is exactly the same dynamical system as for the quartic (gKdV) equation (except the values of the numerical constants). This dynamical system, and the related solution \( Y(t) \) of the ODE

\[
\dot{Y} = 2\alpha e^{-Y}, \quad Y(0) = Y_0, \quad \dot{Y}(0) = 0,
\]

seem to be universal in this type of problems. The control of the dynamical system satisfied by the parameters is thus exactly the same as in [31] and we will not repeat the arguments in the present paper (see Section 4).

Concerning the control of the rest term \( \varepsilon(t) \), as in [31], we use variants of techniques developed for large time stability and asymptotic stability of solitons and multi-solitons for the (gKdV) equations in the energy space, [44], [27], [33] and [25], extended to the (BBM) case in [45], [37], [9], [10], [11] and [26]. At this point, we need some new refined arguments and the proofs are more involved than in the quartic (gKdV) case. Note that since the nonlinearity is quadratic, one cannot use scattering theory from [15], [16] as in [36].

(3) Finally, in Section 5, we prove that for \( \lambda \neq 0 \), the defect due to the interaction of two solitons is bounded from below, which implies in particular that the collision is not elastic.

Assuming for the sake of contradiction that the lower bound in (1.11) is not satisfied for any positive value of \( c \), we obtain first some symmetry properties \( (x \to -x, t \to -t) \) on the parameters \( c_j(t), y_j(t) \) at a certain order.

Second, using space decay properties of \( U(t,x) \), we obtain a gain in the control of the error term in the dynamical system satisfied by \( c_j(t), y_j(t) \). Using this refined version of the dynamical system which is not symmetric for \( \lambda \neq 0 \) (as a consequence of the tail of order \( e^{-\frac{1}{2}y(t)} \) in the approximate solution), we find a contradiction.

## 2 Construction of an approximate solution

We denote by \( \mathcal{Y} \) the set of functions \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \) such that

\[
\forall j \in \mathbb{N}, \exists C_j, \ r_j > 0, \ \forall x \in \mathbb{R}, \quad \left| f^{(j)}(x) \right| \leq C_j(1 + |x|)^r_j e^{-|x|}.
\]

**Proposition 2.1.** Let \( \lambda \in [0,1) \). There exist unique \( A_j(x), B_j(x), D_j(x), \alpha, \beta, \delta, a, b_j, d_j \) \((j = 1,2) \), \( \sigma \geq 3 \) and \( 0 < \mu_* < 1/10 \) such that for any \( 0 < \mu_0 < \mu_* \), the following hold.

(i) Properties of \( A_j, B_j, D_j \) and \( b_j \).

\[
A_j, B_j, D_j \in L^\infty(\mathbb{R}), \quad A'_j, B'_j, D'_j \in \mathcal{Y},
\]

\[
-\lim_{+\infty} A_1 = \lim_{+\infty} A_2 = \pm \theta_A, \quad \theta_A = \frac{36(5 - \lambda^2)}{15 + 10\lambda - \lambda^2}, \quad \lim_{+\infty} D_1 = \lim_{+\infty} D_2 = 0, \tag{2.1}
\]

\[
\lim_{+\infty} B_1 = \lim_{+\infty} B_2 = 0, \quad \lim_{-\infty} B_1 = -\lim_{-\infty} B_2 = -\frac{288\lambda^2}{15 + 10\lambda - \lambda^2},
\]

and \( A_j, B_j \) and \( D_j \) satisfy the orthogonality conditions of Lemmas 2.3, 2.5 and 2.6. Moreover,

\[
b_1 = b_2 \quad \text{if and only if} \quad \lambda = 0. \tag{2.2}
\]
(ii) Definition of the approximate solution. For $\Gamma = (\mu_1, \mu_2, y_1, y_2)$, define
\[
V_0(x; \Gamma) = Q\mu_1(x - y_1) + Q\mu_2(x - y_2) \\
+ e^{-(y_1 - y_2)}(A_1(x - y_1) + A_2(x - y_2)) \\
+ \theta(\mu_1 - \mu_2)xQ(x - y_1)Q(x - y_2) \\
+ (y_1 - y_2)e^{-(y_1 - y_2)}(\mu_1B_1(x - y_1) + \mu_2B_2(x - y_2)) \\
+ e^{-(y_1 - y_2)}(\mu_1D_1(x - y_1) + \mu_2D_2(x - y_2)),
\]
where
\[
\theta = \frac{(1 + \lambda)(5 - 10\lambda + \lambda^2)}{15 + 10\lambda - \lambda^2}.
\]

(iii) Equation of $V_0(x; \Gamma(t))$. Let $I$ be some time interval and $\Gamma(t) = (\mu_1(t), \mu_2(t), y_1(t), y_2(t))$ be a $C^1$ function defined on $I$ such that, for some constant $K > 1$,
\[
\forall t \in I, \quad Y_0 - 1 \leq y_1(t) - y_2(t) \leq KY_0, \quad |\mu_1(t)| \leq 2\mu_0, \quad |\mu_2(t)| \leq 2\mu_0,
\]
\[
|\mu_1(t) + \mu_2(t)| \leq Y_0^2 e^{-Y_0}, \quad |y_1(t) + y_2(t)| \leq Y_0^2 e^{-\frac{1}{2}Y_0},
\]
where
\[
Y_0 = |\ln(\mu_0^2/\alpha)| \quad \text{and} \quad \alpha = 240/(15 + 10\lambda - \lambda^2).
\]

Let
\[
V_0(t, x) = V_0(x; \Gamma(t)), \quad y(t) = y_1(t) - y_2(t).
\]

Then, on $I$, $V_0(t, x)$ solves
\[
(1 - \lambda\partial_x^2)\partial_t V_0 + \partial_x(\partial_x^2 V_0 - V_0 + V_0^2) = \bar{E}(V_0) + E_0(t, x)
\]
where
\[
\bar{E}(V_0) = \sum_{j=1,2} (\tilde{\mu}_j - \bar{M}_j)(1 - \lambda\partial_x^2)\frac{\partial V_0}{\partial \mu_j} - \sum_{j=1,2} (\bar{\mu}_j - \dot{y}_j - \bar{N}_j)(1 - \lambda\partial_x^2)\frac{\partial V_0}{\partial y_j}
\]
\[
\bar{M}_1(t) = \alpha e^{-y(t)} + \beta \mu_1(t)y(t)e^{-y(t)} + \delta \mu_1(t)e^{-y(t)},
\]
\[
\bar{M}_2(t) = -\alpha e^{-y(t)} - \beta \mu_2(t)y(t)e^{-y(t)} - \delta \mu_2(t)e^{-y(t)},
\]
\[
\bar{N}_1(t) = a e^{-y(t)} + b_1 \mu_1(t)y(t)e^{-y(t)} + d_1 \mu_1(t)e^{-y(t)},
\]
\[
\bar{N}_2(t) = a e^{-y(t)} + b_2 \mu_2(t)y(t)e^{-y(t)} + d_2 \mu_2(t)e^{-y(t)},
\]
and for some $C = C(K) > 0$,
\[
\forall t \in I, \quad \sup_{x \in \mathbb{R}} \left\{ \left(1 + e^{\frac{1}{2}(x-y_1(t))} \right) |E_0(t, x)| \right\} \leq CY_0^\sigma e^{-Y_0} e^{-y(t)}.
\]

Sections 2.1–2.5 are devoted to the proof of Proposition 2.1.

Note that the function $V_0$ is not in $L^2$ since $B_j$ have non zero limits at $-\infty$. We now introduce an $L^2$ approximation of $V_0$, using a suitable cut-off function. Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a $C^\infty$ function such that
\[
\psi' \geq 0, \quad \psi \equiv 0 \text{ on } \mathbb{R}^-, \quad \psi \equiv 1 \text{ on } [\frac{1}{2}, +\infty),
\]
As a consequence of Proposition 2.1 we obtain the following result.
Proposition 2.2 ($L^2$ approximate solution). Under the assumptions of Proposition 2.1 (i)–(iii), let
\[ V(x; \Gamma) = V_0(x; \Gamma) \psi \left( e^{-\frac{1}{2}Y_0 x} + 1 \right), \quad V(t, x) = V(x; \Gamma(t)) \]  \hspace{1cm} (2.12)

Then,

(i) Closeness to the sum of two solitons.
\[ \|V - \{Q_{\mu_1}(\cdot , - y_1) + Q_{\mu_2}(\cdot , - y_2)\}\|_{L^\infty} \leq C e^{-y}, \]  \hspace{1cm} (2.13)
\[ \|V - \{Q_{\mu_1}(\cdot , - y_1) + Q_{\mu_2}(\cdot , - y_2)\}\|_{H^1} \leq C \sqrt{y} e^{-y}. \]  \hspace{1cm} (2.14)

(ii) Equation of $V(t, x)$.
\[ (1 - \lambda \partial_x^2) \partial_t V + \partial_x (\partial_x^2 V - V + V^2) = \tilde{E}(V) + E(t, x) \]  \hspace{1cm} (2.15)

where
\[ \tilde{E}(V) = \sum_{j=1,2} (\mu_j - M_j)(1 - \lambda \partial_x^2) \frac{\partial V}{\partial \mu_j} - \sum_{j=1,2} (\mu_j - y_j - N_j)(1 - \lambda \partial_x^2) \frac{\partial V}{\partial y_j}, \]  \hspace{1cm} (2.16)

and for some $C = C(K) > 0$,
\[ \forall t \in I, \quad \sup_{x \in \mathbb{R}} \left\{ 1 + e^{\frac{1}{2}(x-y_1(t))} \right\} |E(t, x)| \leq CY_0^\sigma e^{-Y_0 e^{-y(t)}}, \]  \hspace{1cm} (2.17)
\[ \|E(t)\|_{L^2} \leq CY_0^\sigma e^{-\frac{3}{4}Y_0 e^{-y(t)}}. \]

The proof of Proposition 2.2 being very similar to the one of Proposition 2.2 in [31], it is omitted.

2.1 Preliminary expansion

We set
\[ \tilde{R}_j(t, x) = Q_{\mu_j(t)}(x - y_j(t)), \quad R_j(t, x) = Q(x - y_j(t)), \]  \hspace{1cm} (2.18)
\[ \Lambda \tilde{R}_j(t, x) = \Lambda Q_{\mu_j(t)}(x - y_j(t)), \quad \Lambda R_j(t, x) = \Lambda Q(x - y_j(t)), \]

and similarly for $\Lambda^2 R_j$, where $\Lambda Q_{\mu}, \Lambda^2 Q_{\mu}$ are defined in Claim A.2.

We introduce the notation
\[ r(t) = O_k, \quad f(t, x) = O_k, \quad \sup_{x \in \mathbb{R}} \left\{ 1 + e^{\frac{1}{2}(x-y_1(t))} \right\} \sup_{t \in I} |f(t, x)| \leq C(1 + Y_0^\sigma) e^{-kY_0}, \]  \hspace{1cm} (2.19)

Define
\[ S(v) = (1 - \lambda \partial_x^2) \partial_t v + \partial_x (\partial_x^2 v - v + v^2), \]  \hspace{1cm} (2.20)
and $M_j, N_j$ as in (2.19), for $\alpha, \beta, \delta$ and $a, b, d_j$ to be determined.

We look for an approximate solution of $S(v) = 0$ under the form $v(t, x) = v(x; \Gamma(t))$,
\[ v = \tilde{R}_1 + \tilde{R}_2 + w, \]  \hspace{1cm} (2.21)
where \( w(t, x) = w(x; \Gamma(t)) \) so that using the equation of \( Q_\mu \) (see (A.3)) and \( \frac{\partial}{\partial y_j} \tilde{R}_j = -\partial_x \tilde{R}_j \),
\[
\frac{\partial}{\partial y_j} \tilde{R}_j = -\partial_x \tilde{R}_j,
\]
\[
S(v) = \tilde{E}(v) + F + \tilde{F} + G(w) + H(w),
\]
where
\[
\tilde{E}(v) = \sum_{j=1,2} (\mu_j - M_j)(1 - \lambda \partial_x^2) \frac{\partial v}{\partial \mu_j} - \sum_{j=1,2} (\mu_j - \dot{y}_j - N_j)(1 - \lambda \partial_x^2) \frac{\partial v}{\partial y_j}
\]
\[
F = 2\partial_x \left( \tilde{R}_1 \tilde{R}_2 \right),
\]
\[
\tilde{F} = M_1(1 - \lambda \partial_x^2) \Lambda \tilde{R}_1 + M_2(1 - \lambda \partial_x^2) \Lambda \tilde{R}_2 + N_1(1 - \lambda \partial_x^2) \partial_x \tilde{R}_1 + N_2(1 - \lambda \partial_x^2) \partial_x \tilde{R}_2,
\]
and
\[
G(w) = \partial_x \left[ \partial_x^2 w - w + 2 \left( \tilde{R}_1 + \tilde{R}_2 \right) w \right] + \sum_{j=1,2} \mu_j \frac{\partial w}{\partial y_j}
\]
\[
H(w) = \partial_x \left[ w^2 \right] + \sum_{j=1,2} M_j(1 - \lambda \partial_x^2) \frac{\partial w}{\partial \mu_j} - \sum_{j=1,2} N_j(1 - \lambda \partial_x^2) \frac{\partial w}{\partial y_j}.
\]

In the rest of this section, we give preliminary expansions of \( F \) and \( \tilde{F} \).

**Lemma 2.1** (Expansion of \( F \)). **Under the assumptions of Proposition 2.1**, \( F = 2\partial_x \left( \tilde{R}_1 \tilde{R}_2 \right) = F_A + F_Q + F_B + F_D + \mathcal{O}_2, \)

where
\[
F_A = -12 e^{-y}(\partial_x R_1 + R_1) + 12 e^{-y}(-\partial_x R_2 + R_2),
\]
\[
F_Q = (1 - \lambda)(\mu_1 - \mu_2) R_1 R_2,
\]
\[
F_B = -6(1 - \lambda)\mu_1 ye^{-y}(\partial_x R_1 + R_1) + 6(1 - \lambda)\mu_2 ye^{-y}(-\partial_x R_2 + R_2)
\]
\[
F_D = e^{-y}[\mu_1 S_{F,1}(x - y_1) + \mu_2 S_{F,2}(x - y_2)],
\]
with \( S_{F,1} \in \mathcal{Y} \) and \( S_{F,2}(x) = -S_{F,1}(-x) \).

Note that the term \( F_Q \) does not exist in the quartic case (see Lemma 2.1 in [31]). The proof of Lemma 2.1 is given in Appendix A.

**Lemma 2.2** (Expansion of \( \tilde{F} \)). **Under the assumptions of Proposition 2.1**, \( \tilde{F} = \tilde{F}_A + \tilde{F}_B + \tilde{F}_D + \mathcal{O}_2, \)

where
\[
\tilde{F}_A = (1 - \lambda \partial_x^2) \left[ \alpha e^{-y} \Lambda R_1 + ace^{-y} \partial_x R_1 - ace^{-y} \Lambda R_2 + ace^{-y} \partial_x R_2 \right],
\]
\[
\tilde{F}_B = (1 - \lambda \partial_x^2) \left[ \beta \mu_1 ye^{-y} \partial_x R_1 + b_1 \mu_1 ye^{-y} \partial_x R_1 - \beta \mu_2 ye^{-y} \Lambda R_2 + b_2 \mu_2 ye^{-y} \partial_x R_2 \right]
\]
\[
\tilde{F}_D = (1 - \lambda \partial_x^2) \left[ \delta \mu_1 ye^{-y} \Lambda R_1 + \alpha \mu_1 ye^{-y} \Lambda^2 R_1 + d \mu_1 ye^{-y} \partial_x R_1 + \alpha \mu_1 ye^{-y} \partial_x \Lambda R_1 
\right.
\]
\[
\left. - \delta \mu_2 ye^{-y} \Lambda R_2 - \alpha \mu_2 ye^{-y} \Lambda^2 R_2 + d \mu_2 ye^{-y} \partial_x R_2 + \alpha \mu_2 ye^{-y} \partial_x \Lambda R_2 \right].
\]

The proof of this result is the same as the one of Lemma 2.2 in [31], thus it is omitted.
2.2 Determination of $A_j$

Lemma 2.3. Let

\[
\alpha = \frac{240}{15 + 10\lambda - \lambda^2}, \quad \theta_A = \frac{36(5 - \lambda^2)}{15 + 10\lambda - \lambda^2}. \tag{2.23}
\]

(i) There exist $a$ and $\hat{A}_1 \in \mathcal{V}$ such that $A_1 = \hat{A}_1 + \theta_A \frac{Q'}{Q}$ solves

\[
(-L A_1)' + 2\theta_A Q' + \alpha(1 - \lambda\partial^2_x)\Lambda Q + a(1 - \lambda\partial^2_x)Q' = 12(Q + Q'),
\]

\[
\int A_1(1 - \lambda\partial^2_x)Q' = \int (A_1 + \theta_A)(1 - \lambda\partial^2_x)Q = 0.
\]

(ii) Set $A_2(x) = A_1(-x)$ and

\[
w_A(t, x) = e^{-y(t)}(A_1(x - y_1(t)) + A_2(x - y_2(t))).
\]

Then,

\[
F_A + \tilde{F}_A + G(w_A) = -\frac{\theta_A}{18}(\mu_1 - \mu_2)R_1R_2
\]

\[
+ e^{-y}\left[\mu_1S_1(x - y_1) + \mu_2S_2(x - y_2)\right] + O_2
\]

where $S_1 \in \mathcal{V}$, $S_2(x) = -S_1(-x)$.

Moreover,

\[
\sum_{j=1,2} \left| \int w_A(1 - \lambda\partial^2_x)R_j \right| + \sum_{j=1,2} \left| \int w_A(1 - \lambda\partial^2_x)\partial_x R_j \right| = O_2. \tag{2.25}
\]

Proof. Proof of (i). First, we determine $\alpha$. Multiplying the equation of $A_1$ by $Q$, integrating and using $L(Q') = 0$, we obtain by (A.9) and (A.8)

\[
\alpha \int [(1 - \lambda\partial^2_x)\Lambda Q]Q = 12 \int Q^2 \quad \text{so that} \quad \alpha = \frac{240}{15 + 10\lambda - \lambda^2}. \tag{2.26}
\]

Second, we find the value of $\theta_A$. For $\theta_A$ to be chosen, set $A_1 = \theta_A \frac{Q'}{Q} + \hat{A}_1$, so that from (A.6), $\hat{A}_1$ has to satisfy

\[
(-L \hat{A}_1)' = -\alpha(1 - \lambda\partial^2_x)\Lambda Q + 12(Q + Q') - \theta_A(2Q - \frac{5}{3}Q^2) - 2\theta_A Q' - a(1 - \lambda\partial^2_x)Q'.
\]

To find $\hat{A}_1$ in $\mathcal{V}$, we need

\[
\theta_A = \frac{1}{2} \int (-\alpha(1 - \lambda\partial^2_x)\Lambda Q + 12Q) = \frac{1}{2} \left( -\frac{\alpha}{2}(1 + \lambda) + 12 \right) \int Q = \frac{36(5 - \lambda^2)}{15 + 10\lambda - \lambda^2}. \tag{2.27}
\]

For this choice of $\theta_A$, there exists $Z \in \mathcal{V}$, $\int Z(1 - \lambda\partial^2_x)Q' = 0$ such that

\[
Z' = -\alpha(1 - \lambda\partial^2_x)\Lambda Q + 12(Q + Q') - \theta_A(2Q - \frac{5}{3}Q^2) - 2\theta_A Q'.
\]
By Claim \[\text{Claim}\] there exists \(A \in \mathcal{Y}\) such that \(\int A(1 - \lambda \partial_x^2)Q' = 0\) and \(-LA = Z\). Let \(\hat{A}_1 = A - a \Lambda Q \in \mathcal{Y}\). Then, \(\hat{A}_1(1 - \lambda \partial_x^2)Q' = 0\) and \(-L\hat{A}_1 = Z - a(1 - \lambda \partial_x^2)Q\). Finally, we uniquely choose \(a\) such that \(\int (\hat{A}_1 + \theta_A)(1 - \lambda \partial_x^2)Q = 0\)

Proof of (ii). First, by the parity properties of \(Q\), \(A_2(x) = A_1(-x)\) satisfies

\[(-LA_2)' + 2\theta_A Q' - \alpha(1 - \lambda \partial_x^2)\Lambda Q + a(1 - \lambda \partial_x^2)Q' = -12(Q - Q').\]

Now, we compute \(F_A + \tilde{F}_A + G(w_A)\). Using \([A,23]\), we have

\[G(w_A) = \partial_x (\partial_x^2 w_A - w_A + 2 (R_1 + R_2) w_A)
+ 2 \partial_x ((\mu_1 \Lambda R_1 + \mu_2 \Lambda R_2) w_A) + \mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} + O_2.\]

First,

\[\partial_x (\partial_x^2 w_A - w_A + 2 (R_1 + R_2) w_A)
= e^{-y} (-LA_1 + 2\theta_A Q)' (x - y_1) + e^{-y} \partial_x (2R_1(A_2(x - y_2) - \theta_A))
+ e^{-y} (-LA_2 + 2\theta_A Q)' (x - y_2) + e^{-y} \partial_x (2R_2(A_1(x - y_1) - \theta_A)).\]

Using the estimate

\[|A_2(x - y_2) - \theta_A| \leq C(1 + |x - y_2|^\omega) e^{-(x - y_2)} \quad \text{for } x > y_2 \quad (2.28)\]

and \([A,24]\), we have

\[e^{-y} R_1(A_2(x - y_2) - \theta_A) = O_2 \quad \text{and similarly} \quad e^{-y} R_2(A_1(x - y_1) - \theta_A) = O_2.\]

Thus, using the expressions of \(F_A\) and \(\tilde{F}_A\) in Lemmas \(2.1\) and \(2.2\) and the equations of \(A_1\) and \(A_2\), we find

\[F_A + \tilde{F}_A + \partial_x (\partial_x^2 w_A - w_A + 2 (R_1 + R_2) w_A) = O_2.\]

Second, by similar arguments,

\[2 \partial_x ((\mu_1 \Lambda R_1 + \mu_2 \Lambda R_2) w_A) = 2\mu_1 e^{-y} \partial_x (\Lambda R_1(A_1(x - y_1) + \theta_A))
+ 2\mu_2 e^{-y} \partial_x (\Lambda R_2(A_2(x - y_2) + \theta_A)) + O_2.\]

Finally, we compute \(\mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2}\). We have

\[\frac{\partial w_A}{\partial y_1} = -w_A - e^{-y} A_1'(x - y_1), \quad \frac{\partial w_A}{\partial y_2} = w_A - e^{-y} A_2'(x - y_2).\]

Thus, using \([2.3]\),

\[\mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} = -(\mu_1 - \mu_2) w_A - \mu_1 e^{-y} A_1'(x - y_1) - \mu_2 e^{-y} A_2'(x - y_2)
= -\theta_A (\mu_1 - \mu_2) e^{-y} \left( \frac{\partial_x R_1}{R_1} \right) + \mu_1 e^{-y} (2\hat{A}_1 + A_1')(x - y_1) - \mu_2 e^{-y} (-2\hat{A}_2 + A_2')(x - y_2) + O_2.\]
For this term, we use Claim \( \text{A.3} \) (see Appendix A.2), i.e.

\[
e^{-y} \left( \frac{\partial_y R_1}{R_1} - \frac{\partial_x R_2}{R_2} \right) = \frac{1}{18} R_1 R_2 + \frac{1}{3} e^{-y} (R_1 + R_2) + O_{3/2}.
\]

We obtain

\[
\begin{align*}
\mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} &= -\frac{1}{18} \theta_A (\mu_1 - \mu_2) R_1 R_2 \\
- \mu_1 e^{-y} (\frac{2}{3} \theta_A Q + 2 \hat{A}_1 + A'_1)(x - y_1) - \mu_2 e^{-y} (-\frac{2}{3} \theta_A Q - 2 \hat{A}_2 + A'_2)(x - y_2) + O_2.
\end{align*}
\]

Combining these computations, we obtain

\[
\begin{align*}
F_A + \tilde{F}_A + G(w_A) &= -\frac{1}{18} \theta_A (\mu_1 - \mu_2) R_1 R_2 \\
+ \mu_1 e^{-y} \left( 2 \partial_x (\Lambda Q(A_1 + \theta_A)) - \frac{2}{3} \theta_A Q - (2 \hat{A}_1 + A'_1) \right) (x - y_1) \\
+ \mu_2 e^{-y} \left( 2 \partial_x (\Lambda Q(A_2 + \theta_A)) + \frac{2}{3} \theta_A Q + (2 \hat{A}_2 - A'_2) \right) (x - y_2),
\end{align*}
\]

so that

\[
S_1 = 2 (\Lambda Q(A_1 + \theta_A))' - \frac{2}{3} \theta_A Q - 2 \hat{A}_1 - A'_1.
\]

Using \( \text{(2.25)} \), and \( \int (A_1 + \theta_A)(1 - \lambda \partial_x^2)Q = 0 \), we have

\[
\begin{align*}
\int \ w_A(1 - \lambda \partial_x^2)R_1 &= e^{-y} \int \ [A_1(1 - \lambda \partial_x^2)Q + \theta_A(1 - \lambda \partial_x^2)Q] + O_2 = O_2,
\end{align*}
\]

and similarly for the other scalar products in \( \text{(2.25)} \). \( \square \)

### 2.3 Nonlocalized term of order \( O_{3/2} \)

**Lemma 2.4** (Approximate solution at order \( O_{3/2} \) with localized error term). Let

\[
w_Q = \theta(\mu_1 - \mu_2) x R_1 R_2, \quad \theta = 1 - \lambda - \frac{\theta_A}{18} = \frac{(1 + \lambda)(5 - 10\lambda + \lambda^2)}{15 + 10\lambda - \lambda^2}.
\]

Then

\[
G(w_Q) = \theta(\mu_1 - \mu_2) x R_1 R_2 \\
- 6 \theta \mu_1 y e^{-y} e^{-(x - y_1)} (3(R_1 - \partial_x R_1) - \partial_x (R_1^2 - R_1^2)) \\
+ 6 \theta \mu_2 y e^{-y} e^{-(x - y_2)} (3(R_2 + \partial_x R_2) + \partial_x (R_2^2 - R_2^2)) \\
+ 6 \theta \mu_1 y e^{-y} (\mu_1 \tilde{S}_1(x - y_1) + \mu_2 \tilde{S}_2(x - y_2)) + O_2,
\]

where \( \tilde{S}_1 \in Y \) and \( \tilde{S}_1(x) = -\tilde{S}_2(-x) \).

**Proof.** The proof is based on Claim \( \text{A.5} \) in Appendix A.

First, arguing as in the proof of Lemma \( \text{2.3} \), we have

\[
G(w_Q) = \partial_x (\partial^2 w_Q - w_Q + 2 (R_1 + R_2) w_Q) + O_2.
\]
Moreover, since \( x = \frac{1}{2}(x - y_1 + x - y_2) + \frac{1}{2}(y_1 + y_2) \), using (2.6), we have
\[
 w_Q = \frac{1}{2} \theta(\mu_1 - \mu_2)(x - y_1 + x - y_2)R_1R_2 + O_2.
\]
Therefore, using Claim A.5 and the asymptotics of \( Q \) from (A.14), we get
\[
 G(w_Q) = -\frac{1}{2} \theta(\mu_1 - \mu_2) \partial_x \left\{ -\partial_x^2 ((x - y_1 + x - y_2)R_1R_2) + (x - y_1 + x - y_2)R_1R_2 \right. \\
\left. - 2(R_1 + R_2)(x - y_1 + x - y_2)R_1R_2 \right\} + O_2 \\
= -\theta(\mu_1 - \mu_2)R_1R_2 \\
- 6\theta \mu_1 y e^{-y} e^{-(x-y_1)} (3(R_1 - \partial_y R_1 - \partial_x (R_1^2)) - R_1^2) \\
+ 6\theta \mu_2 y e^{-y} e^{-(x-y_2)} (3(R_2 + \partial_y R_2 + \partial_x (R_2^2)) - R_2^2) \\
+ e^{-y} \left( \mu_1 \tilde{S}_1(x - y_1) + \mu_2 \tilde{S}_2(x - y_2) \right) + O_2,
\]
where \( \tilde{S}_1 \) and \( \tilde{S}_2 \) satisfy the desired conditions.

\[ \square \]

### 2.4 Determination of \( B_j \) and \( D_j \)

**Lemma 2.5.** Let
\[
 Z(x) = 6(1 - \lambda) e^{-x} (Q - Q') + 6\theta e^{-x} (3(Q - Q' - (Q')') - Q^2)
\]
\[
 \beta = \frac{120(1 - \lambda)}{15 + 10\lambda - \lambda^2}, \quad \theta_B = \frac{144\lambda^2}{15 + 10\lambda - \lambda^2}.
\]

(i) There exist unique \( b_1 \) and \( \tilde{B}_1 \in \mathcal{Y} \) such that \( B_1 = \tilde{B}_1 + \theta_B \left( 1 + \frac{Q'}{Q} \right) \) satisfies
\[
 (-LB_1)' + \beta(1 - \lambda \partial_x^2) \Lambda Q + b_1(1 - \lambda \partial_x^2)Q' = Z
\]
\[
 \int B_1(1 - \lambda \partial_x^2)Q' = \int B_1(1 - \lambda \partial_x^2)Q = 0.
\]

(ii) There exist unique \( b_2 \) and \( \tilde{B}_2 \in \mathcal{Y} \) such that \( B_2 = \tilde{B}_2 - \theta_B \left( 1 + \frac{Q'}{Q} \right) \) satisfies
\[
 (-LB_2)' - 4\theta B Q' - \beta(1 - \lambda \partial_x^2) \Lambda Q + b_2(1 - \lambda \partial_x^2)Q' = -Z(-x)
\]
\[
 \int B_2(1 - \lambda \partial_x^2)Q' = \int (B_2 - 2\theta_B)(1 - \lambda \partial_x^2)Q = 0.
\]

Moreover,
\[
 b_2 - b_1 = 0 \iff \lambda = 0. \tag{2.29}
\]

(iii) Set
\[
 w_B(t, x) = y e^{-y(t)} (\mu_1(t)B_1(x - y_1(t)) + \mu_2(t)B_2(x - y_2(t))).
\]
Then,
\[
 F_A + \tilde{F}_A + G(w_A) + G(w_Q) + F_B + \tilde{F}_B + G(w_B)
\]
\[
 = e^{-y} \left[ \mu_1(S_1 + \tilde{S}_1)(x - y_1) + \mu_2(S_2 + \tilde{S}_2)(x - y_2) \right] + O_2, \tag{2.30}
\]

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\begin{equation}
\sum_{j=1,2} \left| \int w_B(1 - \lambda \partial_x^2) R_j \right| + \sum_{j=1,2} \left| \int w_B(1 - \lambda \partial_x^2) \partial_x R_j \right| = \mathcal{O}_{5/2}. \tag{2.31}
\end{equation}

**Proof.** We follow the strategy of the proof of Lemma 2.3. The only difference is that we now look for solutions \( B_1, B_2 \) both with limit 0 at \( +\infty \).

Proof of (i). We find the value of \( \beta \) from the equation of \( B_1 \) multiplied by \( Q \), using (A.9) and (A.18),

\[ \beta \int Q(1 - \lambda \partial_x^2) \Lambda Q = \frac{3}{10} \beta (15 + 10\lambda - \lambda^2) \]
\[ = \int ZQ = (3(1 - \lambda) + 9\theta) \int e^{-x}Q^2 - 10\theta \int e^{-x}Q^2 = 36(1 - \lambda). \]

Next, from (A.12), (A.9), (A.8), we have

\[ \int Z = 6(1 - \lambda) \int Q + 6\theta \left( 3 \int Q - 2 \int e^{-x}Q^2 \right) = 36 ((1 - \lambda) - \theta) = 2\theta_A, \]

and we find \( \theta_B \) by integrating the equation of \( B_1 \) \((2\theta_B = \int (-LB_1)'\)\)

\[ 2\theta_B = -3\beta(1 + \lambda) + 2\theta_A = \frac{288\lambda^2}{15 + 10\lambda - \lambda^2}. \]

We now obtain the existence of \( \hat{B}_1 \in \mathcal{Y} \) as in the proof of Lemma 2.3 with \( b_1 \) uniquely chosen so that \( \int B_1(1 - \lambda \partial_x^2)Q = 0 \) and \( \int B_1(1 - \lambda \partial_x^2)Q' = 0 \).

Proof of (ii). We solve the equation of \( B_2 \) exactly in the same way. We check that the values of \( \beta \) and \( \theta_B \) are suitable to solve the problem, and we obtain unique \( \hat{B}_2 \in \mathcal{Y} \) and \( b_2 \) so that \( \int B_2(1 - \lambda \partial_x^2)Q' = \int (B_2 - 2\theta_B)(1 - \lambda \partial_x^2)Q = 0 \).

We now check that \( b_1 \neq b_2 \) for \( \lambda \neq 0 \). Let \( B(x) = \hat{B}_1(x) - \hat{B}_2(-x) = B_1(x) - B_2(-x) - 2\theta_B \). Then \( B \in \mathcal{Y} \) and

\[ -LB + (b_1 - b_2)(1 - \lambda \partial_x^2)Q + 8\theta_B Q = 0, \quad \int (B + 4\theta_B)(1 - \lambda \partial_x^2)Q = 0. \]

Then, multiplying the equation of \( B \) by \( \Lambda Q \), integrating and using \( L\Lambda Q = -(1 - \lambda \partial_x^2)Q \) (see Claim A.2(ii)), we obtain

\[ \int B(1 - \lambda \partial_x^2)Q + (b_1 - b_2) \int (1 - \lambda \partial_x^2)Q\Lambda Q + 8\theta_B \int Q\Lambda Q = 0, \]

and so, by \( \int Q\Lambda Q = \frac{1}{4}(\lambda + 3) \int Q \)

\[ (b_1 - b_2) \int (1 - \lambda \partial_x^2)Q\Lambda Q = 4\theta_B \left( \int Q - 2 \int Q\Lambda Q \right) = -2\theta_B(\lambda + 1), \]

and thus, in view of the expression of \( \theta_B \), we obtain

\[ b_1 = b_2 \iff \lambda = 0. \]
Proof of (iii). We finish the proof of Lemma 2.5 as the one of Lemma 2.3. In particular, using the limits of $B_1$ and $B_2$ at $\pm \infty$

$$G(w_B) = \mu_1 y e^{-y}(-LB_1)'(x - y_1) + \mu_2 y e^{-y}(-LB_2 - 4\theta_B Q)'(x - y_2) + O_2.$$ 

This, combined with the equations of $B_1$ and $B_2$ and Lemmas 2.1, 2.2, 2.3 and A.5 proves (2.30). Note that $w_B$ is not in $L^2$ since it has a nonzero limit at $-\infty$. However, it has exponential decay as $x \to +\infty$. This allows us to prove that all rest terms are indeed of the form $O_2$ (see notation $O_2$ in (2.19)).

The control of the various scalar products is easily obtained as in Lemma 2.3 from the properties of $B_1, B_2$.

Finally, we claim without proof the following result.

**Lemma 2.6** (Definition and equation of $w_D$). Let

$$S = -S_{F,1} - S_1 - \tilde{S}_1 - (1 - \lambda \partial_x^2)(\alpha \Lambda^2 Q + a(\Lambda Q))'.$$

(i) There exist unique $\delta, \theta_D, d_1$ and $\hat{D}_1 \in \Upsilon$ such that $D_1 = \hat{D}_1 + \theta_D \left(1 + \frac{Q}{\bar{Q}}\right)$ satisfies

$$(-LD_1)' + \delta(1 - \lambda \partial_x^2)\Lambda Q + d_1(1 - \lambda \partial_x^2)Q' = S(x)$$

(ii) There exist unique $d_2$ and $\hat{D}_2 \in \Upsilon$ such that $D_2 = \hat{D}_2 - \theta_D \left(1 + \frac{Q}{\bar{Q}}\right)$ satisfies

$$(-LD_2)' - \delta(1 - \lambda \partial_x^2)\Lambda Q - 4\theta_D Q' + d_2(1 - \lambda \partial_x^2)Q' = -S(-x)$$

(iii) Set

$$w_D(t,x) = e^{-y(t)}(\mu_1(t)D_1(x - y_1(t)) + \mu_2(t)D_2(x - y_2(t))).$$

Then,

$$F_A + \tilde{F}_A + G(w_A) + G(w_Q) + F_B + \tilde{F}_B + G(w_B) + F_D + \tilde{F}_D + G(w_D) = O_2,$$

$$\sum_{j=1,2} \left| \int w_D(1 - \lambda \partial_x^2)R_j \right| + \sum_{j=1,2} \left| \int w_D(1 - \lambda \partial_x^2)\partial_x R_j \right| = O_5/2.$$

We do not need to compute $d_1 - d_2$, this is the reason why the exact expression of $S_1$ and $\tilde{S}_1$ are not needed.

**2.5 End of the proof of Proposition 2.1**

Set

$$V_0 = \tilde{R}_1 + \tilde{R}_2 + W_0, \quad W_0 = w_A + w_Q + w_B + w_D.$$ 

From the preliminary expansion (2.22), we have

$$S(V_0) = \tilde{E}(V_0) + E_0, \quad E_0 = F + \tilde{F} + G(W_0) + F(W_0).$$

In view of notation (2.19), estimate (2.10) holds true for some $\sigma > 0$ provided that $E_0 = O_2$. From Lemmas 2.3, 2.4, 2.5 and 2.6 we have $F + \tilde{F} + G(W_0) = O_2$. Thus, we only have to check that $H(W_0) = O_2$.

First, $\partial_x \left[ w_0^2 \right] = O_2$. Second, since $|M_j| + |N_j| \leq C e^{-y}$, we also obtain

$$\sum_{j=1,2} M_j \frac{\partial W_0}{\partial \mu_j} - \sum_{j=1,2} N_j \frac{\partial W_0}{\partial y_j} = O_2.$$

Thus, Proposition 2.1 is proved.
3 Preliminary long time stability arguments

3.1 Stability of the 2-soliton structure in the interaction region

We start with the decomposition of any solution of (BBM) close the approximate solution $V$ (introduced in Proposition 2.2] by modulation theory. See Appendix B for the proof.

**Lemma 3.1 (Decomposition around the approximate solution).** There exists $\omega_0 > 0$, $C > 0$, $y_0 > 0$ such that if $u(t)$ is a solution of (BBM) on some time interval $I$ satisfying for $0 < \omega < \omega_0$, $y_0 > \bar{y}_0$

$$\forall t \in I, \quad \inf_{y_1 - y_2 > y_0} \|u(t) - V(:, 0, 0, y_1, y_2)\|_{H^1} \leq \omega,$$

then there exists a unique decomposition $(\Gamma(t), \varepsilon(t))$ of $u(t)$ on $I$,

$$u(t, x) = V(x; \Gamma(t)) + \varepsilon(t, x), \quad \Gamma(t) = (\mu_1(t), \mu_2(t), y_1(t), y_2(t)) \text{ of class } C^1,$$

such that $\forall t \in I$,

$$\int \varepsilon(t, x) (1 - \lambda \partial_x^2) \bar{R}_j(t, x) dx = \int \varepsilon(t, x) (1 - \lambda \partial_x^2) \partial_x \bar{R}_j(t, x) dx = 0,$$

$$\mu_1(t) - y_2(t) > y_0 - C\omega, \quad \|\varepsilon(t)\|_{H^1} + |\mu_1(t)| + |\mu_2(t)| \leq C\omega,$$

$$\mu_2(t) - y_1(t) = \frac{1}{2} \left[\varepsilon(t) \partial_x (\partial_x^2 \varepsilon - \varepsilon + 2\nu \varepsilon + \varepsilon^2) + E(t, x) + \bar{E}(V) = 0,

where $\bar{R}_j(t, x) = Q_{\mu_j(t)}(x - y_j(t))$ and $V, E(t, x), \bar{E}(V)$ are defined in Proposition 2.2.

Moreover, assuming

$$\forall t \in I, \quad (|\mu_1(t)| + |\mu_2(t)|) y(t) \leq 1,$$

$\dot{\Gamma}(t)$ satisfies the following estimates

$$|\dot{\mu}_j - \mathcal{M}_j| \leq C \left[\|\varepsilon\|_L^2 + \nu \varepsilon \|\varepsilon\|_L^2 + \int |E| (\bar{R}_1 + \bar{R}_2)\right],$$

$$|\mu_j - y_j - \mathcal{N}_j| \leq C \left[\|\varepsilon\|_L^2 + \int |E| (\bar{R}_1 + \bar{R}_2)\right].$$

The next proposition presents almost monotonicity laws which are essential in proving long time stability results in the interaction region. They will allow us to compare the approximate solution $V(t, x)$ with exact solutions. The functional is different depending on whether $\mu_1(t) > \mu_2(t)$ or $\mu_1(t) < \mu_2(t)$.

The constant $0 < \rho < 1/32$ to be fixed later, set

$$\varphi(x) = \frac{2}{\pi} \arctan(\exp(8\rho x)), \quad \text{so that } \lim_{-\infty} \varphi = 0, \lim_{\infty} \varphi = 1,$$

$$\forall x \in \mathbb{R}, \quad \varphi(-x) = 1 - \varphi(x), \quad \varphi'(x) = \frac{8\rho}{\pi \cosh(8\rho x)},$$

$$|\varphi''(x)| \leq 8\rho |\varphi'(x)|, \quad |\varphi'''(x)| \leq (8\rho)^2 |\varphi'(x)|.$$
Proposition 3.1 (Almost monotonicity laws). For $\rho > 0$ small enough, and under the assumptions of Lemma 3.1, let

$$\mathcal{F}_+(t) = \int \left[ (\partial_x \varepsilon)^2 + \varepsilon^2 - \frac{2}{3} \left( (\varepsilon + V)^3 - V^3 - 3V^2 \varepsilon \right) \right] + \int \left[ \lambda (\partial_x \varepsilon)^2 + \varepsilon^2 \right] \Phi(t, x), \quad (3.8)$$

where $\Phi(t, x) = \mu_1(t) \varphi(x) + \mu_2(t)(1 - \varphi(x))$;

$$\mathcal{F}_-(t) = \int \left[ (\partial_x \varepsilon)^2 + \varepsilon^2 - \frac{2}{3} \left( (\varepsilon + V)^3 - V^3 - 3V^2 \varepsilon \right) \right] \Phi_1(t, x) + \int \left[ \lambda (\partial_x \varepsilon)^2 + \varepsilon^2 \right] \Phi_2(t, x), \quad (3.9)$$

where

$$\Phi_1(t, x) = \frac{\varphi(x)}{1 + \mu_1(t)^2} + \frac{1 - \varphi(x)}{(1 + \mu_2(t))^2}, \quad \Phi_2(t, x) = \frac{\mu_1(t) \varphi(x)}{(1 + \mu_1(t))^2} + \frac{\mu_2(t)(1 - \varphi(x))}{(1 + \mu_2(t))^2}. \quad (3.10)$$

There exists $C > 0$ such that

$$\|\varepsilon(t)\|^2_{H^1} \leq C \mathcal{F}_+(t), \quad \|\varepsilon(t)\|^2_{H^1} \leq C \mathcal{F}_-(t). \quad (3.11)$$

Moreover,

(i) If $t \in I$ is such that

$$\mu_1(t) \geq \mu_2(t) \quad \text{and} \quad y_2(t) \leq -\frac{1}{4} y(t), \quad y_1(t) \geq \frac{1}{4} y(t), \quad (3.12)$$

then

$$\frac{d}{dt} \mathcal{F}_+(t) \leq C \|\varepsilon\|^2_{L^2} \left[ e^{-\frac{1}{2}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2}) (e^{-2\rho y} + \|\varepsilon\|_{L^2}) \right] + C \|\varepsilon\|_{L^2} \|E\|_{L^2}. \quad (3.13)$$

(ii) If $t \in I$ is such that

$$\mu_2(t) \geq \mu_1(t) \quad \text{and} \quad y_2(t) \leq -\frac{1}{4} y(t), \quad y_1(t) \geq \frac{1}{4} y(t), \quad (3.14)$$

then

$$\frac{d}{dt} \mathcal{F}_-(t) \leq C \|\varepsilon\|^2_{L^2} \left[ e^{-\frac{1}{2}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2}) (e^{-2\rho y} + \|\varepsilon\|_{L^2}) \right] + C \|\varepsilon\|_{L^2} \|E\|_{L^2}. \quad (3.15)$$

See proof of Proposition 3.1 in Appendix 3.

Remark 1. The introduction of almost monotone variants of the energy and mass is related to Weinstein’s approach for stability of one soliton [42] and to Kato identity for the KdV equation (see [19]). These techniques have been developed in [27], [33] and then extended in [28], [9], [36] and [17].

3.2 Stability of the two soliton structure for large time

In this section, we present a stability result for the two soliton structure for large time, i.e. far away from the interaction time. The argument, similarly to the one of Propositions 3.1, is based on almost monotone variant of energy and mass. As a corollary, we obtain a sharp estimate for large negative time on the pure two solution solution considered in Theorem 1.
Proposition 3.2 (Stability for large time). For $0 < \rho < 1/32$ small enough, there exist $C > 0$ and such that for $\mu_0 > 0$ and $\omega > 0$ small enough, if $u(t)$ is an $H^1$ solution of \([\text{BBM}]\) satisfying
\[
\| u(t_0) - Q_{-\mu_0}(\cdot + \mu_0 t_0) - Q_{\mu_0}(\cdot - \mu_0 t_0) \|_{H^1(\mathbb{R})} \leq \omega \mu_0,
\]
for some $t_0 < - (\rho \mu_0)^{-1} |\log \mu_0|$, then there exist $y_1(t), y_2(t)$ and $\mu_1^+, \mu_2^+$ such that
(i) For all $t_0 \leq t \leq - (\rho \mu_0)^{-1} |\log \mu_0|,$
\[
\| u(t) - Q_{-\mu_0}(\cdot - y_1(t)) - Q_{\mu_0}(\cdot - y_2(t)) \|_{H^1(\mathbb{R})} \leq C \omega \mu_0 + C \exp (-4 \rho \mu_0 |t|),
\]
\[
y_1(t) - y_2(t) \geq \frac{3}{2} \mu_0 |t|,
\]
\[
| - \mu_0 - \dot{y}_1(t) | + | \mu_0 - \dot{y}_2(t) | \leq C \omega \mu_0 + C \exp (-4 \rho \mu_0 |t|).
\]
(ii) For all $t \leq t_0$,
\[
\| u(t) - Q_{-\mu_0}(\cdot - y_1(t)) - Q_{\mu_0}(\cdot - y_2(t)) \|_{H^1(\mathbb{R})} \leq C \omega \mu_0 + C \exp (-4 \rho \mu_0 |t_0|),
\]
\[
y_1(t) - y_2(t) \geq \frac{3}{2} \mu_0 |t|,
\]
\[
| - \mu_0 - \dot{y}_1(t) | + | \mu_0 - \dot{y}_2(t) | \leq C \omega \mu_0 + C \exp (-4 \rho \mu_0 |t_0|).
\]
(iii) Asymptotic stability.
\[
\lim_{t \to -\infty} \| u(t) - Q_{\mu_1^+}(\cdot - y_1(t)) - Q_{\mu_2^+}(\cdot - y_2(t)) \|_{H^1(\mathbb{R} < \frac{\mu_0}{4\rho} |t|)} = 0,
\]
\[
\lim_{t \to -\infty} \dot{y}_1(t) = \mu_1^+, \quad \lim_{t \to -\infty} \dot{y}_2(t) = \mu_2^+,
\]
\[
| \mu_1^+ + \mu_0 | + | \mu_2^+ - \mu_0 | \leq C \omega \mu_0 + C \exp (-4 \rho \mu_0 |t_0|).
\]

See the proof of this result in Appendix [B].

Remark 2. Using the invariance of the BBM equation by the transformation
\[
x \to -x, \quad t \to -t,
\]
\[
(3.20)
\]
it follows that a statement similar to Proposition 3.2 holds for $t_0 > (\rho \mu_0)^{-1} |\log \mu_0|.$

Corollary 3. Let $u(t)$ be the unique solution of \([\text{BBM}]\) satisfying
\[
\lim_{t \to -\infty} \| u(t) - Q_{-\mu_0}(\cdot + \mu_0 t) - Q_{\mu_0}(\cdot - \mu_0 t) \|_{H^1} = 0.
\]

Then, for all $t \leq - (\rho \mu_0)^{-1} |\log \mu_0|,$
\[
\| u(t) - Q_{-\mu_0}(\cdot + \mu_0 t) - Q_{\mu_0}(\cdot - \mu_0 t) \|_{H^1} \leq \exp (-4 \rho \mu_0 |t|).
\]

We refer to Theorem 1 in [11] for the existence and uniqueness of the solution $u(t)$.

Proof of Corollary 3 assuming Proposition 3.2. For fixed $t$, we can pass to the limit $\omega \to 0$, $t_0 \to -\infty$ in (3.17). Then, we integrate the estimates on $\dot{y}_1(t)$ and $\dot{y}_2(t)$ (see (3.17)) from $-\infty$ to $t$. \(\square\)
4 Stability of the 2-soliton structure

In this section, using the approximate solution constructed in Propositions 2.1 and 2.2 and the asymptotic arguments of Section 3, we prove the stability part of Theorem 1 and Theorem 2.

4.1 Description of the global behavior of the asymptotic 2-soliton solution

Let $0 < \rho < 1/32$ being fixed as in Propositions 3.1 and 3.2. Recall that $\sigma \geq 3$ is defined in Propositions 2.1 and 2.2.

We recall the following notation from the introduction

$$Y_0 = |\ln(\mu_0^2/\alpha)| \quad \text{or equivalently} \quad \mu_0 = \sqrt{|\alpha|} e^{-\frac{1}{2}Y_0},$$

$$Y(t) = Y_0 + 2\ln(\cosh(\mu_0t)) \text{ solution of } \ddot{Y} = 2\alpha e^{-Y}, \quad \lim_{t \to -\infty} \dot{Y}(t) = -2\mu_0, \quad \dot{Y}(0) = 0. \quad (4.2)$$

Note that $\dot{Y}(t) = 2\mu_0 \tanh(\mu_0t)$ and, for all $t \in \mathbb{R}$,

$$0 \leq Y(t) - (Y_0 + 2\mu_0|t| - 2\ln 2) < 2\exp(-2\mu_0|t|). \quad (4.3)$$

**Proposition 4.1** (Description of the 2-soliton solution in the interaction region). Let $U(t)$ be the unique solution of \eqref{BBM} such that

$$\lim_{t \to -\infty} \|U(t) - Q_{-\mu_0}(\cdot + \frac{1}{2}Y(t)) - Q_{\mu_0}(\cdot - \frac{1}{2}Y(t))\|_{H^1(\mathbb{R})} = 0. \quad (4.4)$$

Let $T > 0$ be such that $Y(T) = 400\rho^{-2}Y_0$. Then, for $\mu_0 > 0$ small enough, there exists $(\Gamma(t), \varepsilon(t)) \in C^1$ such that for all $t \in [-T, T]$,

$$U(t, x) = V(t; \Gamma(t)) + \varepsilon(t, x), \quad \Gamma(t) = (\mu_1(t), \mu_2(t), y_1(t), y_2(t)),$$

and

$$|\bar{\mu}(t)| \leq Y_0^2 e^{-Y_0}, \quad |\bar{y}(t)| \leq Y_0^4 e^{-\frac{1}{2}Y_0}, \quad (4.5)$$

$$|\mu(t) - \bar{Y}(t)| \leq CY_0^{\sigma+1} e^{-\frac{5}{4}Y_0}, \quad |y(t) - Y(t)| \leq CY_0^{\sigma+2} e^{-\frac{3}{4}Y_0}, \quad (4.6)$$

$$\|\varepsilon(t)\|_{H^1} \leq CY_0^\sigma e^{-\frac{3}{4}Y_0}, \quad (4.7)$$

where

$$\mu(t) = \mu_1(t) - \mu_2(t), \quad y(t) = y_1(t) - y_2(t),$$

$$\bar{\mu}(t) = \mu_1(t) + \mu_2(t), \quad \bar{y}(t) = y_1(t) + y_2(t). \quad (4.8)$$

Moreover, there exists $t_0$ such that

$$|t_0| \leq CY_0^\sigma e^{-\frac{3}{4}Y_0}, \quad \mu(t_0) = 0; \quad \forall t \in [-T, t_0), \quad \mu(t) < 0; \quad \forall t \in (t_0, T], \quad \mu(t) > 0. \quad (4.9)$$

The proof of Proposition 4.1 is omitted since it is exactly the same as the one of Proposition 4.1 in [31], using Sections 2 and 3.
4.2 Conclusion of the proof of the stability of the 2-soliton structure

In this section, we finish the proof of the stability part of Theorem 1.

Proof of \( (1.9) - (1.10) \) and partial proof of \( (1.11) \) and \( (1.12) \). Let \( T > 0 \) be defined as in Proposition 4.1. We prove the existence of \( \mu_j(t) \) and \( y_j(t) \) and estimates \( (1.9) - (1.10) \) separately on \( (-\infty, -T] \), \( [-T, T] \) and \( [T, +\infty) \). It is straightforward that the functions \( \mu_j(t) \) and \( y_j(t) \) can be adjusted to have \( C^1 \) regularity on \( \mathbb{R} \).

For \( t < -T \), Corollary 3 clearly implies \( (1.9) - (1.10) \).

On \( [-T, T] \), \( (1.9) - (1.10) \) are direct consequences of \( (4.5) - (4.7) \) and \( (2.14) \) (comparing in \( H^1 \) the approximate solution with the sum of two solitons).

**Remark 3.** By \( (4.7) \) and the definition of \( V \) (see \( (2.3) \) and \( (2.12) \)), for \( t \in [-T, T] \),

\[
\|U(t) - \tilde{R}_1(t) - \tilde{R}_2(t) - e^{-y(t)}(A_1(\cdot, -y_1(t)) + A_2(\cdot, -y_2(t)))\|_{H^1} \leq CY_0^\sigma e^{-\frac{\sigma}{4}Y_0},
\]

where for \( t \) close to 0, the term \( e^{-y}(A_1(x-y_1) + A_2(x-y_2)) \) is indeed relevant as a correction term in the computation of \( U(t) \). In view of the behavior at \( \pm \infty \) of the functions \( A_1 \) and \( A_2 \) (see Lemma 2.3), this term decays exponentially for \( x > y_1(t) \) and \( x < y_2(t) \) but contains a tail for \( y_2(t) < x < y_1(t) \). Note that this tail also appears in the integrable case i.e. for \( \lambda = 0 \), and thus it is not related to the lack of integrability.

Now, we consider the region \( t \geq T \). By \( (4.3) \), \( T > \frac{10}{\sqrt{5}} \rho^{-1}Y_0 e^{\frac{\sigma}{4}Y_0} > 10(\rho \mu_0)^{-1} |\ln \mu_0| \).
From \( (1.5), (1.6), (4.7) \) and \( (2.14) \) written at \( t = T \),

\[
\|U(T) - Q_{\mu_1(T)}(\cdot, -y_1(T)) - Q_{\mu_2(T)}(\cdot, -y_2(T))\| \leq CY_0^\sigma e^{-\frac{\sigma}{4}Y_0} \leq C'r_0^\sigma e^{-\frac{\sigma}{4}Y_0} \mu_0,
\]

where \( |\mu_1(T) - \mu_0| + |\mu_2(T) + \mu_0| \leq CY_0^\sigma e^{-\frac{\sigma}{4}Y_0} \mu_0 \).

Therefore, we can apply Proposition 3.2 backwards (i.e. for \( t \geq T \) – see Remark 2), with \( \omega = C'Y_0^\sigma e^{-\frac{\sigma}{4}Y_0} \). There exist \( y_1(t), y_2(t) \) and \( \mu_1^+ = \lim_{+\infty} \mu_1, \mu_2^+ = \lim_{+\infty} \mu_2 \), such that

\[
w(t) = U(t) - Q_{\mu_1(t)}(\cdot, -y_1(t)) - Q_{\mu_2(t)}(\cdot, -y_2(t))
\]

satisfies

\[
\sup_{t \in [T, +\infty)} \|w(t)\|_{H^1} \leq CY_0^\sigma e^{-\frac{\sigma}{4}Y_0}, \quad \lim_{t \to +\infty} \|w(t)\|_{H^1(x > -99/100 t)} = 0,
\]

\[
|\mu_1^+ - \mu_0| + |\mu_2^+ + \mu_0| \leq CY_0^\sigma e^{-\frac{\sigma}{4}Y_0}, \quad \lim_{+\infty} \dot{y}_j = \mu_j^+ \quad (j = 1, 2).
\]

Finally, using the conservation laws and the above asymptotics for \( w(t) \), we claim the following refined estimates on the limiting scaling parameters:

\[
0 \leq \mu_1^+ - \mu_0 \leq CY_0^{2\sigma} e^{-2Y_0}, \quad 0 \leq -\mu_2^+ - \mu_0 \leq CY_0^{2\sigma} e^{-2Y_0},
\]

which is a consequence of \( (1.11) \) and the following lemma.

**Lemma 4.1 (Monotonicity of the speeds by conservation laws).** There exists \( C > 0 \) such that

\[
\frac{1}{C} e^{Y_0} \lim_{t \to +\infty} \|w(t)\|_{H^1}^2 \leq \frac{\mu_1^+ - \mu_0}{\mu_0} - 1 \leq C e^{Y_0} \lim_{t \to +\infty} \|w(t)\|_{H^1}^2 \leq CY_0^{2\sigma} e^{-\frac{\sigma}{2}Y_0},
\]

\[
\frac{1}{C} e^{Y_0} \lim_{t \to +\infty} \|w(t)\|_{H^1}^2 \leq -\frac{\mu_2^+}{\mu_0} - 1 \leq C e^{Y_0} \lim_{t \to +\infty} \|w(t)\|_{H^1}^2 \leq CY_0^{2\sigma} e^{-\frac{\sigma}{2}Y_0}.
\]
Proof. We first write the conservation of mass and energy for \( U(t) \) (see (1.2) and (1.3)) and then pass to the limit \( t \to -\infty, t \to +\infty \), using (4.11). It follows that the limits \( \lim_{+\infty} M(w) \) and \( \lim_{+\infty} \mathcal{E}(w) \) exist and

\[
M(Q_{\mu_0}) + M(Q_{-\mu_0}) = M(Q_{\mu_1^+}) + M(Q_{\mu_2^+}) + \lim_{+\infty} M(w), \tag{4.13}
\]

\[
\mathcal{E}(Q_{\mu_0}) + \mathcal{E}(Q_{-\mu_0}) = \mathcal{E}(Q_{\mu_1^+}) + \mathcal{E}(Q_{\mu_2^+}) + \lim_{+\infty} \mathcal{E}(w). \tag{4.14}
\]

Let

\[
\nu_1 = \frac{\mathcal{E}(Q_{\mu_0}) - \mathcal{E}(Q_{\mu_1^+})}{M(Q_{\mu_1^+}) - M(Q_{\mu_0})}, \quad \nu_2 = \frac{\mathcal{E}(Q_{-\mu_0}) - \mathcal{E}(Q_{\mu_2^+})}{M(Q_{\mu_2^+}) - M(Q_{-\mu_0})},
\]

so that by (A.11) and (4.12),

\[
\left| \frac{\nu_1}{\mu_0} - 1 \right| \leq \frac{1}{4}, \quad \left| \frac{\nu_2}{-\mu_0} - 1 \right| \leq \frac{1}{4}.
\]

We combine (4.13) and (4.14) to get

\[
\lim_{+\infty} \mathcal{E}(w) = \nu_1 (M(Q_{\mu_1^+}) - M(Q_{\mu_0})) + \nu_2 (M(Q_{\mu_2^+}) - M(Q_{-\mu_0})),
\]

\[
= (\nu_1 - \nu_2) (M(Q_{\mu_1^+}) - M(Q_{\mu_0})) - \nu_2 \lim_{+\infty} M(w),
\]

\[
= (\nu_1 - \nu_2) (M(Q_{-\mu_0}) - M(Q_{\mu_2^+})) - \nu_1 \lim_{+\infty} M(w).
\]

Since \( \|w\|_{L^\infty} \leq C \|w\|_{H^1} \leq CY_0^\sigma e^{-\frac{2}{5}Y_0} \), we have

\[
\frac{1}{2} \limsup_{+\infty} \|w\|_{H^1}^2 \leq \liminf_{+\infty} \mathcal{E}(w) \leq 2 \liminf_{+\infty} \|w\|_{H^1}^2,
\]

and by \( |\nu_1| + |\nu_2| \leq \frac{1}{4} \), for \( \mu_0 \) small enough, we obtain \( \lim_{+\infty} \mathcal{E}(w) + \nu_2 \lim_{+\infty} M(w) > \frac{1}{4} \limsup_{+\infty} \|w\|_{H^1}^2, \lim_{+\infty} \mathcal{E}(w) + \nu_1 \lim_{+\infty} M(w) < 2 \limsup_{+\infty} \|w\|_{H^1}^2. \)

By \( \frac{d}{dt} Q_{\mu_{|\mu=0}} > 0 \), and so for all \( |\mu| \leq 2\mu_0, \frac{d}{dt} Q_{\mu_{|\mu'=\mu}} > C > 0 \), we get (4.13) and (4.14). \( \square \)

For future reference, we observe that the following hold for \( t \in [-T, T] \)

\[
\sup_{x \in \mathbb{R}} \{1 + e^{\frac{1}{2}(x-y_1(t))}|E(t,x)|\} \leq CY_0^\sigma e^{-Y_0} e^{-Y(t)}, \quad \|E(t)\|_{L^2} \leq CY_0^\sigma e^{-\frac{2}{5}Y_0} e^{-Y(t)}, \tag{4.15}
\]

\[
|\dot{\mu}_j - \mathcal{M}_j| \leq CY_0^\sigma e^{-\frac{2}{5}Y_0}, \quad |\mu_j - \dot{\gamma}_j - N_j| \leq CY_0^\sigma e^{-\frac{2}{5}Y_0}. \tag{4.16}
\]

### 4.3 Proof of Theorem 2

First, we claim a refined stability result around the family of asymptotic 2-soliton solutions (defined in the next claim) in the spirit of Proposition 3.2 but without the exponential error term (see (3.17)–(3.18)). The proof is given in Appendix B.

**Proposition 4.2 (Sharp stability).** Let \( U \) be defined as in Theorem 1. For \( \mu_0 > 0 \) small enough, if \( u(t) \) is a solution of (BBM) such that

\[
\|u(T_1) - U(T_1)\|_{H^1} = \omega \mu_0 \tag{4.17}
\]

for some \( T_1 \), where \( 0 < \omega < |\ln \mu_0|^{-2} \), then there exist \( t \in \mathbb{R} \mapsto (T(t), X(t)) \in \mathbb{R}^2 \) such that

\[
\forall t \in \mathbb{R}, \quad \|u(t + T(t), x + X(t)) - U(t)\|_{H^1} + |\dot{X}(t)| + e^{-\frac{2}{5}Y_0}|\dot{T}(t)| \leq C \omega \mu_0. \tag{4.18}
\]
Lemma 5.1. Let $\mu_0 \in \mathbb{R}$ and $\bar{Y}_0 > 0$ be such that

$$\mu_0 = \sqrt{\bar{\mu}_0^2 + 4\alpha e^{-\bar{Y}_0}}$$

is small enough. Let $u_0 \in H^1$ satisfy (1.14) and let $u(t)$ be the corresponding solution of (BDM). We assume that $\bar{\mu}_0 \leq 0$, the proof being the same in the case $\bar{\mu}_0 > 0$ by using the transformation $x \to -x$, $t \to -t$ and translation in space invariance.

For this value of $\mu_0$, let $U(t)$ and $Y(t)$ be defined as in Theorem 1 and Sections 4.1 and 4.2. Recall that for all $t$, $\bar{Y}^2(t) + 4\alpha e^{-Y(t)} = 4\mu_0^2$. Since $\bar{Y}_0 \geq Y_0$, there exists $\check{T}_0 < T_0$ such that $Y(\check{T}_0) = \bar{Y}_0$, so that $\check{Y}(\check{T}_0) = 2\check{\mu}_0$. We claim that for some $\lambda_1 \in \mathbb{R}$,

$$J(\check{T}_0, j(x)) = Q_{-\check{\mu}_0}(. - \check{Y}_0) - Q_{\check{\mu}_0}(. + \check{Y}_0) ||_{H^1} \leq C|\ln \mu_0|^\sigma 2^\mu_0^{3/2}. \quad (4.19)$$

Indeed, if $\check{T}_0 < -T$, then we use Corollary 3. Otherwise, by Proposition (4.1), we have

$$|y(\check{T}_0) - Y(\check{T}_0)| \leq C\ln \mu_0|^{\sigma+2} \mu_0^{3/2}, \quad \|\varepsilon(\check{T}_0)||_{H^1} \leq C\ln \mu_0|^{\sigma} \mu_0^{5/2},$$

$$\|\mu_1(\check{T}_0) + \frac{1}{2} \check{Y}(\check{T}_0)| \leq C\ln \mu_0|^{2} \mu_0^{2}, \quad \|\mu_2(\check{T}_0) - \frac{1}{2} \check{Y}(\check{T}_0)| \leq C\ln \mu_0|^{2} \mu_0^{2}.$$  

Using in addition (2.12), we get (4.19).

By (4.19) and (1.14), we obtain

$$\|u_0 - U(\check{T}_0, . + X_1)||_{H^1} \leq C\omega \mu_0 + C|\ln \mu_0|^{\sigma+2} \mu_0^{3/2},$$

and by Proposition 4.2 we obtain (1.15).

5 Nonexistence of a pure 2-soliton and interaction defect

In this section, we complete the proof of Theorem 1 by proving the lower bounds in (1.11) and (1.12).

5.1 Refined control of the translation parameters

Now, we introduce specific functionals $J_j(t)$ related to the translation parameters $y_j(t)$ to obtain a refined version of the dynamical system.

Lemma 5.1. Under the assumptions of Proposition 4.1 for $j = 1, 2$, let

$$J_j(t) = \frac{1}{\int_{\mathbb{R}} (1 - \lambda \partial_x^2) \Lambda Q} \int \varepsilon(t, x) (1 - \lambda \partial_x^2) J_j(t, x) dx,$$  

where $J_j(t, x) = \int_{-\infty}^x \Lambda \check{R}_j(t, y) dy$. Then $J_j(t)$ is well-defined and the following hold

(i) Estimates on $J_j$.

$$\forall t \in [-T, T], \quad |J_1(t)| + |J_2(t)| \leq CY_0^{\sigma+1} e^{-\frac{1}{2} \check{Y}_0}. \quad (5.2)$$

(ii) Equation of $J_j$. For $j = 1, 2$,

$$\forall t \in [-T, T], \quad \left| \frac{d}{dt} J_j(t) - (\mu_j - \dot{y}_j - N_j) \right| \leq CY_0^{\sigma+2} e^{-\frac{1}{2} \check{Y}_0}. \quad (5.3)$$
Remark 4. The constant \( \int (1 - \lambda \partial_x^2)AQ \) is not zero (see (A.9)).

Note also that \( \int A \neq 0 \) (see (A.8)), and so the functions \( J_j(x) \) are bounded but have no decay at \( +\infty \) in space. Therefore, \( J_j(t) \) is not well-defined for a general \( \varepsilon \in H^1 \). Part of the proof of Lemma 5.1 consists on obtaining decay in space for \( \varepsilon(t) \) in order to give a rigorous sense to \( J_j \).

Remark 5. Estimate (5.3) says formally that \( \mu_j - \dot{y}_j - N_j \) is of order \( O_{T/4} \), which is an decisive improvement with respect to (4.16) (gain of a factor \( e^{-\frac{1}{2}Y_0} \)).

Proof. Preliminary estimates. We work under the assumptions of Proposition 4.1, and on the interval \( [-T, T] \). First, we claim exponential decay properties of \( U(t) \) on the right \( (x > y_1(t)) \).

Claim 5.1 (Decay estimate on \( u(t) \)). There exist \( C > 1 \) and \( \rho_0 > 1 \) such that for all \( t \in [-T, T] \), for all \( X_0 > 1 \),

\[
\int_{x > x_0 + y_1(t)} (U^2 + (\partial_x U)^2)(t, x)dx \leq Ce^{-\rho_0 X_0}. \tag{5.4}
\]

Recall that the proof of Claim 5.1 is obtained by

\[
\lim_{t \to -\infty} \| U(t) \|_{H^1(x > \frac{1}{2}|t|)} = 0
\]

combined with monotonicity arguments, see e.g. [10] for the case of the \([BMB]\) equation.

Estimate of \( J_j \). Note that \( J_j \) does not belong to \( L^2 \) (see Remark 4) but satisfies

\[
\sup_{x \in \mathbb{R}} \left\{ \left( 1 + e^{-\frac{1}{2}(x-y_1(t))} \right) |J_j(t, x)| \right\} \leq C. \tag{5.5}
\]

It follows from (5.4), (5.5), and the decomposition of \( U(t) \) in Lemma 3.1 that

\[
|J_1(t)| \leq C \int_{x < y_1(t)} |\varepsilon(t, x)||1 - \lambda \partial_x^2|J_j(t, x)|dx + C \int_{y_1(t) \leq x < y_1(t) + 10\rho_0^{-1}Y_0} |\varepsilon(t, x)|dx + C \int_{x > y_1(t) + 10\rho_0^{-1}Y_0} |\varepsilon(t, x)||1 - \lambda \partial_x^2|J_j(t, x)|dx
\]

\[
\leq C(1 + Y_0)\| \varepsilon(t) \|_{L^\infty} + C \int_{x > y_1(t) + 10\rho_0^{-1}Y_0} |U(t, x)| + Ce^{-5Y_0}
\]

\[
\leq CY_0^{\sigma + 1}e^{-\frac{5}{2}Y_0}.
\]

Moreover, using \( y_1(t) - y_2(t) = y(T) \leq CY_0 \), one gets by similar arguments

\[
|J_2(t)| \leq CY_0^{\sigma + 1}e^{-\frac{5}{2}Y_0}.
\]

Equation of \( J_1 \). To prove (5.3), we make use of the equation of \( \varepsilon \) (see (3.1)), and of the special algebraic structure of the approximate solution \( V(t, x) \) introduced in Propositions 2.1 and 2.2. We have

\[
\left( \int [(1 - \lambda \partial_x^2)AQ] \frac{d}{dt} J_1(t) \right) = \int (1 - \lambda \partial_x^2)\partial(t)\varepsilon J_1 + \int \varepsilon \partial(t)(1 - \lambda \partial_x^2)J_1.
\]

First observe that

\[
\partial(t)J_1(x) = \int_{-\infty}^{x} \partial(t)\Lambda \tilde{R}_1(y)dy = \int_{-\infty}^{x} \left\{ \mu_1 \frac{\partial \Lambda \tilde{R}_1}{\partial \mu_1} + \dot{y}_1 \frac{\partial \Lambda \tilde{R}_1}{\partial y_1} \right\} (y)dy.
\]

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Thus, by $|M_j| + |N_j| \leq Ce^{-Y_0}$, $|\mu_j(t)| \leq Ce^{-\frac{1}{2}Y_0}$, (4.16) and (5.4), we have

\[
\left| \int \varepsilon \partial_t (1 - \lambda \partial_x^2) J_j \right| \leq CY_0^\sigma e^{-\frac{3}{4}Y_0}.
\] (5.6)

Next, using (3.4) and $\partial_x J_1 = \Lambda \tilde{R}_1$, we have

\[
\int (1 - \lambda \partial_x^2) \partial_t \varepsilon J_1 = \int (\partial_x^2 \varepsilon - \varepsilon + 2V\varepsilon + \varepsilon^2) \Lambda \tilde{R}_1 - \int EJ_1 + \int \tilde{E}(V) J_1.
\]

For the term $\int (\partial_x^2 \varepsilon - \varepsilon + 2V\varepsilon + \varepsilon^2) \Lambda \tilde{R}_1$, we argue as the proof of Lemma 3.1. Using $L_{\mu_j} \Lambda Q_{\mu_j} = (1 - \lambda \partial_x^2) Q_{\mu_j}$ (see (A.3), (2.13), (A.22), $\int \varepsilon (1 - \lambda \partial_x^2) \tilde{R}_j = 0$, Proposition 4.1 and the definition of $V$ (see Proposition 2.2), we obtain

\[
\left| \int (\partial_x^2 \varepsilon - \varepsilon + 2V\varepsilon + \varepsilon^2) \Lambda \tilde{R}_1 \right| \leq CY_0^2 \varepsilon^{-Y_0} \|\varepsilon\|_{L^2} + C\|\varepsilon\|_{L^2}^2 \leq CY_0^\sigma e^{-\frac{3}{2}Y_0}.
\]

By (4.15) and (5.5), we have

\[
\left| \int EJ_1 \right| \leq CY_0^\sigma e^{-2Y_0}.
\]

Next, we consider the term $\int \tilde{E}(V) J_1$. From the definition of $\tilde{E}(V)$ in (2.16), the structure of $V_0$ and $V$, see (2.1) and (2.12) (see also (1.2)), and (4.16), we have

\[
\sup_{t \in \mathbb{R}} \left\{ (1 + e^{\frac{1}{2}(x-y_1(t))}) \left| \tilde{E}(V) - \sum_{j=1,2} (\mu_j - \dot{y}_j - N_j)(1 - \lambda \partial_x^2) \partial_x \tilde{R}_j \right| \right\} \leq CY_0^\sigma e^{-2Y_0}.
\] (5.7)

Thus, by (5.5), we obtain

\[
\left| \int \tilde{E}(V) J_1 - (\mu_1 - \dot{y}_1 - N_1) \int [(1 - \lambda \partial_x^2) \partial_x \tilde{R}_1] J_1 \right| \leq CY_0^\sigma e^{-2Y_0}.
\] (5.8)

Finally, using $\int [(1 - \lambda \partial_x^2) \partial_x \tilde{R}_1] J_1 + \int [(1 - \lambda \partial_x^2) Q] \Lambda Q \leq C|\mu_1(t)| \leq Ce^{-\frac{3}{2}Y_0}$ and (4.16), we obtain (5.3).

The proof for $\frac{d}{dt} J_2$ is exactly the same.

\[\square\]

\section*{5.2 Preliminary symmetry arguments}

First, we claim the following additional information obtained on the parameters of the solution $U(t)$, under the assumptions of Proposition 4.1

\textbf{Claim 5.2.} For all $t \in [-T, T]$,

\[
|\mu_1(t) - \mu_2(-t)| \leq CY_0^\sigma e^{-\frac{3}{4}Y_0}.
\] (5.9)
Remark 6. Assuming \( U \) symmetric (see the definition of smaller \( \lambda_1 \)) let \( \mu \) be such that the following hold (5.10) and (5.11), and so (5.9)

Indeed, on the one hand, estimate \( |x_0| \leq CY_0^2 e^{-\frac{3}{4}Y_0} \) follows from (4.5) taken at time \( t = t_0/2 \) and (5.10) taken at \( t = t_0/2 \).

On the other hand, from (5.9) and (5.12), we have \( |\mu_1(t) - \mu_1(t - t_0)| \leq CY_0^{\sigma+3} e^{-\frac{3}{4}Y_0} \). Since \( \mu_1(t) \geq Ce^{-Y_0} \) for \( |t| \) close to 0, we obtain \( |t_0| \leq CY_0^{\sigma+3} e^{-\frac{3}{4}Y_0} \).

Using Section 2, the proof is similar to the one of Lemma 5.2 in [31] and it is omitted.

5.3 Lower bound on the defect

In this section, we prove the following result.

Proposition 5.1. Let \( \lambda \in (0,1) \). Under the assumptions of Proposition 4.1, for a possibly smaller \( \mu_0 > 0 \), there exists a constant \( c > 0 \) such that,

\[
\liminf_{t \to +\infty} \|w(t)\|_{H^1} \geq cY_0 e^{-\frac{3}{4}Y_0},
\]

\[
\mu_1^+ - \mu_0 \geq cY_0^2 e^{-\frac{3}{4}Y_0}, \quad -\mu_2^+ - \mu_0 \geq cY_0^2 e^{-\frac{3}{4}Y_0}.
\]
Proof. The proof is the same as the one of Proposition 5.2 in [31] but we repeat it here since the argument is the key of the nonexistence of a pure 2-soliton solution.

It suffices to prove the estimate on \( w(t) \). The estimates on the final parameters then follow from Lemma 4.1.

Let \( \epsilon > 0 \) arbitrary, and suppose for the sake of contradiction that

\[
\liminf_{t \to +\infty} \|w(t)\|_{H^1} \leq \epsilon Y_0 e^{-\frac{3}{2}Y_0}. \tag{5.14}
\]

**Step 1.** We claim that for some \( \tilde{T}(t), \tilde{X}(t) \), for all \( t \in \mathbb{R} \),

\[
\|U(-t + \tilde{T}(t), -x + \tilde{X}(t)) - U(t, x)\|_{H^1} + |\tilde{\dot{X}}(t)| + e^{-\frac{3}{2}Y_0} |\tilde{T}(t)| \leq C \epsilon Y_0 e^{-\frac{3}{2}Y_0}. \tag{5.15}
\]

**Proof of (5.15).** By Lemma 4.1 it follows that

\[
0 \leq \mu_1^+ - \mu_0 \leq C e^{2Y_0} e^{-\frac{3}{2}Y_0}, \quad 0 \leq - (\mu_2^+ + \mu_0) \leq C e^{2Y_0} e^{-\frac{3}{2}Y_0}.
\]

In particular, for all \( t \)

\[
\|Q_{\mu_1^+}(\cdot, -y_1(t)) + Q_{\mu_2^+}(\cdot, -y_2(t)) - (Q_{\mu_0}(\cdot, -y_1(t)) + Q_{-\mu_0}(\cdot, -y_2(t)))\|_{H^1} \leq C e^{2Y_0} e^{-\frac{3}{2}Y_0}.
\]

From (5.14) and the behavior of \( U \), it follows that there exist \( T_1, T_2 > T \) and \( X \) such that

\[
\|U(T_1, x) - U(-T_2, -x + X)\|_{H^1} \leq 2 \epsilon Y_0 e^{-\frac{3}{2}Y_0} + C e^{2Y_0} e^{-\frac{3}{2}Y_0} \leq 3 \epsilon Y_0 e^{-\frac{3}{2}Y_0},\tag{5.16}
\]

for \( Y_0 \) large enough. From Proposition 4.2 it follows that there exist \( \tilde{T}(t) \) and \( \tilde{X}(t) \) such that (5.15) holds.

**Step 2.** Conclusion of the proof of Proposition 5.1. Take \( 0 < t_1 < t_2 \) such that \( Y(t_1) = Y_0 + 1 \) and \( Y(t_2) = Y_0 + 2 \). Note that \( t_2 - t_1 < Ce^{2Y_0} \) since \( Y > c_0 e^{-\frac{3}{2}Y_0} \) on \([t_1, t_2] \), for some \( c_0 > 0 \).

Note that for \( t \in [-T, T] \), \( \tilde{T}(t) \) and \( \tilde{X}(t) \) are small by Proposition 4.1. Applying Lemma 5.2 at \( t_1 \) and \( t_2 \), for all \( t \in [t_1, t_2] \), we obtain (for \( Y_0 \) large enough depending on \( \epsilon \))

\[
|\mu_1(t) - \mu_2(-t + \tilde{T}(t))| + |y_1(t) + y_2(-t + \tilde{T}(t))| \leq C \epsilon Y_0 e^{-\frac{3}{2}Y_0}.
\]

By (5.15), for all \( t \in [t_1, t_2] \), we have \( |\tilde{T}(t) - \tilde{T}(t_1)| \leq C \epsilon Y_0 e^{-\frac{3}{2}Y_0}, \quad |\tilde{X}(t) - \tilde{X}(t_1)| \leq C \epsilon Y_0 e^{-Y_0} \) and thus,

\[
\forall t \in [t_1, t_2], \quad |\mu_2(-t + \tilde{T}(t_1)) - \mu_2(-t + \tilde{T}(t))| \leq C \epsilon e^{-\frac{3}{2}Y_0},
\]

\[
|y_2(-t + \tilde{T}(t_1)) - \tilde{X}(t_1) - (y_2(-t + \tilde{T}(t)) - \tilde{X}(t))| \leq C \epsilon Y_0 e^{-Y_0}
\]

Therefore, setting

\[
\nu(t) = \mu_1(t) - \mu_2(-t + \tilde{T}(t_1)), \quad z(t) = y_1(t) + y_2(-t + \tilde{T}(t_1)) - \tilde{X}(t_1),
\]

we obtain

\[
|\nu(t)| \leq C \epsilon Y_0 e^{-\frac{3}{2}Y_0}, \quad |z(t)| \leq C \epsilon Y_0 e^{-Y_0}. \tag{5.17}
\]
We claim
\[
|\mathcal{J}_1(t)| + |\mathcal{J}_2(t)| \leq CY_0^{\sigma+1}e^{-\frac{3}{4}Y_0}, \tag{5.18}
\]
\[
\left| \frac{d}{dt} \left( z(t) - \{(b_+y(t) + (b_- + d_-)e^{-y(t)}) + (\mathcal{J}_1(t) - \mathcal{J}_2(-t + \bar{T}(t))) \right| \leq CeY_0e^{-\frac{3}{4}Y_0}, \tag{5.19}
\]
where
\[ b_+ = \frac{1}{2}(b_1 - b_2), \quad d_- = \frac{1}{2}(d_1 - d_2). \]

Note that estimate (5.18) is just (5.2) from Lemma 5.1 Assuming estimate (5.19) for the moment, we complete the proof of the Proposition.

Integrating (5.19) on \([t, t_2]\), using (5.18) and \( t_2 - t < Ce^{\frac{1}{2}Y_0} \), we obtain
\[
\left| (z(t_1) - \{(b_+y(t_1) + (b_- + d_-)e^{-y(t_1)}) - (z(t_2) - \{(b_-y(t_2) + (b_- + d_-)e^{-y(t_2)}) \right| 
\leq CeY_0e^{-Y_0}. \tag{5.20}
\]

Thus, by (5.17), for \( k = 1 + \frac{d_-}{\lambda} \), \( (b_- \neq 0 \) for \( \lambda \neq 0) \),
\[
|(y(t_1) + k)e^{-y(t_1)} - (y(t_2) + k)e^{-y(t_2)}| \leq CeY_0e^{-Y_0}. \tag{5.21}
\]
But since \( Y(t_1) = Y_0 + 1 \) and \( Y(t_2) = Y_0 + 2 \), (5.21) is a contradiction for \( \epsilon \) small enough and \( Y_0 \) large enough.

Let us now prove (5.19). By (5.3) and the expression of \( N_j \) in (2.9), we have
\[
\dot{y}_1 = \mu_1 + ae^{-y} - b_1\mu_1ye^{-y} - d_1\mu_1e^{-y} = \mathcal{J}_1 + O(Y_0^{\sigma+2}e^{-\frac{3}{4}Y_0}),
\]
\[
\dot{y}_2 = \mu_2 + ae^{-y} - b_2\mu_2ye^{-y} - d_2\mu_2e^{-y} = \mathcal{J}_2 + O(Y_0^{\sigma+2}e^{-\frac{3}{4}Y_0}). \tag{5.22}
\]
Moreover, by (5.17), we check
\[
|e^{-y(t)} - e^{-y(t + \bar{T}(t))}| \leq CeY_0e^{-2Y_0}. \tag{5.23}
\]
Thus, using again (5.17), we obtain
\[
\dot{z}(t) = \dot{y}_1(t) - \dot{y}_2(-t + \bar{T}(t)) \]
\[
= \nu(t) - (b_1 - b_2)\mu_1ye^{-y} - (d_1 - d_2)\mu_1e^{-y} - (\mathcal{J}_1(t) - \mathcal{J}_2(-t + \bar{T}(t))) + O(Y_0^{\sigma+2}e^{-\frac{3}{4}Y_0}),
\]
Since \( |\mu_1 + \mu_2| \leq CY_0^{2}e^{-Y_0} \) (see (1.3)), we have \( |\mu_1 - \frac{1}{2}\mu| \leq CY_0^{2}e^{-Y_0} \) and thus, by \( |\mu - \dot{y}| \leq Ce^{-Y_0} \), we obtain \( |\mu_1e^{-y} - \frac{1}{2}\dot{y}e^{-y}| \leq CY_0^{2}e^{-2Y_0} \). Therefore,
\[
\dot{z} = \nu + b_+\dot{y}ye^{-y} + d_-\dot{y}e^{-y} - (\mathcal{J}_1(t) - \mathcal{J}_2(-t + \bar{T}(t))) + O(Y_0^{\sigma+2}e^{-\frac{3}{4}Y_0}),
\]
where \( |\nu(t)| \leq Ce^{-\frac{3}{4}Y_0} \) from (5.17). Estimate (5.19) then follows. \( \square \)
A Appendix to the construction of an approximate solution

A.1 Linearized operator, identities and asymptotics for solitons

Recall that we set
\[ Q_\mu(x) = (1 + \mu)Q\left(\sqrt{\frac{1+\mu}{1+\lambda \mu}} x\right) \quad \text{where} \quad Q(x) = \frac{3}{2} \frac{1}{\cosh^2\left(\frac{x}{2}\right)} \] (A.1)
satisfies
\[ Q'' + Q^2 = Q \quad \text{and} \quad (Q')^2 + \frac{2}{3}Q^3 = Q^2. \] (A.2)

We recall the following well-known spectral properties of \( L \) (see [45] and Lemma 2.2 from [29])

Claim A.1 (Properties of the operator \( L \)). The operator \( L \) defined in \( L^2(\mathbb{R}) \) by
\[ Lf = -f'' + f - 2Qf \]
is self-adjoint and satisfies the following properties:

(i) First eigenfunction : \( LQ^3 = -\frac{5}{4}Q^3 \);

(ii) Second eigenfunction : \( LQ' = 0 \); the kernel of \( L \) is \( \{c_1 Q', c_1 \in \mathbb{R}\} \);

(iii) For any function \( h \in L^2(\mathbb{R}) \) orthogonal to \( Q' \) for the \( L^2 \) scalar product, there exists a unique function \( f \in H^2(\mathbb{R}) \) orthogonal to \( (1 - \lambda \partial_x^2)Q' \) such that \( Lf = h \); moreover, if \( h \) is even (respectively, odd), then \( f \) is even (respectively, odd).

(iv) Suppose that \( f \in H^2(\mathbb{R}) \) is such that \( Lf \in \mathcal{Y} \). Then, \( f \in \mathcal{Y} \).

(v) There exists \( c_1 > 0 \) such that for all \( f \in H^1(\mathbb{R}) \),
\[ \int (1 - \lambda \partial_x^2)Qf = \int (1 - \lambda \partial_x^2)Q'f = 0 \quad \Rightarrow \quad (Lf, f) \geq c_1 \|f\|^2_{H^1}. \]

Claim A.2 (Preliminary computations on solitons). (i) Scaling.
\[ (1 + \lambda \mu)Q''_\mu - (1 + \mu)Q_\mu + Q^2_\mu = 0. \]

Set
\[ \Lambda Q_\mu = \left(\frac{d}{d\mu'}Q_{\mu'}\right)_{|\mu' = \mu}, \quad \Lambda^2 Q_\mu = \left(\frac{d^2}{d\mu'^2}Q_{\mu'}\right)_{|\mu' = \mu}. \]

Then,
\[ \Lambda Q = \Lambda Q_0 = Q + \frac{1}{2}(1 - \lambda)xQ', \]
\[ \Lambda^2 Q = \Lambda^2 Q_0 = \frac{3}{4}(1 - \lambda)(1 - \lambda + \lambda^2)xQ' + \frac{1}{4}(1 - \lambda)^2x^2Q - \frac{1}{4}(1 - \lambda)^2x^2Q^2. \] (A.3)
(ii) Linearized operator. Let
\[ L_\mu v = -(1 + \lambda \mu)v'' + (1 + \mu)v - 2Q_\mu v, \quad Lv = 0v = -v'' + v - 2Qv. \] (A.4)

Then,
\[ L_\mu Q_\mu = -Q_\mu^2, \quad L_\mu \Lambda Q_\mu = -(1 - \lambda \partial_x^2)Q_\mu, \quad L_\mu Q_\mu' = 0, \] (A.5)
\[ \frac{LQ'}{Q} = -\frac{5}{3}Q' + \frac{Q'}{Q}, \quad \left(\frac{LQ'}{Q}\right)' = -2Q + \frac{5}{3}Q^2, \quad \lim_{x \to \pm \infty} \frac{Q'}{Q} = \mp 1. \] (A.6)

(iii) Integral identities.
\[ \int Q = \int Q^2, \quad \int Q^3 = \frac{6}{5} \int Q^2, \quad \int Q^2 = \frac{1}{5} \int Q^2, \] (A.7)
\[ \int Q^2 = 6, \quad \int \Lambda Q = \frac{1}{2}(1 + \lambda) \int Q = 3(1 + \lambda), \] (A.8)
\[ \int [(1 - \lambda \partial_x^2)(\Lambda Q)]Q = \frac{3}{10}(15 + 10\lambda - \lambda^2), \quad \int Q\Lambda Q = \frac{3}{2}(3 + \lambda), \] (A.9)
\[ \int Q_\mu^2 = (1 + \mu)^{\frac{3}{2}}(1 + \lambda \mu)^{\frac{1}{2}} \int Q^2, \quad \int Q_\mu^3 = (1 + \mu)^{\frac{3}{2}}(1 + \lambda \mu)^{\frac{1}{2}} \int Q^3, \] (A.10)
\[ \int (Q_\mu')^2 = (1 + \mu)^{\frac{3}{2}}(1 + \lambda \mu)^{-\frac{1}{2}} \int (Q')^2, \quad -\frac{d}{d\mu} E(Q_\mu) = \mu \frac{d}{d\mu} M(Q_\mu), \quad \frac{d}{d\mu} M(Q_\mu) > 0. \] (A.11)

(iv) Pointwise identities.
\[ Q - Q' = e^x(Q + Q') = \frac{12e^{2x}}{(e^x + 1)^3}, \quad e^{-x}Q^2 = -Q^2 + 3(Q' + Q), \] (A.12)
\[ e^{-x}((\Lambda Q)' - \Lambda Q - \frac{1}{2}(1 - \lambda)Q) = -\frac{1}{2}(3 - \lambda)(Q' + Q) + \frac{1}{2}(1 - \lambda)x(Q^2 - 2(Q' + Q)). \] (A.13)

(v) Asymptotics
\[ Q(x) = 6e^{-x} - 12e^{-2x} + O(e^{-3x}) \quad \text{at } +\infty. \] (A.14)

**Proof.** (i) First, we check that \( Q_\mu(. + x_0) \) solves the following equation
\[ (1 + \lambda \mu)Q_\mu''(. + x_0) - (1 + \mu)Q_\mu(. + x_0) + Q_\mu^2(. + x_0) = 0. \] (A.15)

Indeed, we have
\[ (1 + \lambda \mu)Q_\mu''(x + x_0) = (1 + \mu)^2Q''\left(\frac{1 + \mu}{1 + \lambda \mu}(x + x_0)\right) \]
\[ = (1 + \mu)^2Q\left(\sqrt{\frac{1 + \mu}{1 + \lambda \mu}(x + x_0)}\right) - (1 + \mu)^2Q^2\left(\sqrt{\frac{1 + \mu}{1 + \lambda \mu}(x + x_0)}\right) \]
\[ = (1 + \mu)Q_\mu(x + x_0) - Q_\mu^2(x + x_0). \]
We have by direct computations
\[ \Lambda Q_\mu = Q \left( \sqrt{\frac{1 + \mu}{1 + \lambda \mu}} x \right) + \frac{1}{2} \left( 1 + \mu \right) \frac{1}{1 + \lambda \mu} \frac{1}{\sqrt{1 + \lambda \mu}} x Q' \left( \sqrt{\frac{1 + \mu}{1 + \lambda \mu}} x \right). \]

The expressions of \( \Lambda Q \) and \( \Lambda^2 Q \) then follow.

(ii) Differentiating (A.15) with respect to \( \mu \) and then with respect to \( x_0 \), we obtain
\[ (1 + \lambda \mu)(\Lambda Q_\mu)'' - (1 + \mu)\Lambda Q_\mu + 2Q_\mu \Lambda Q_\mu = -\lambda Q_\mu'' + Q_\mu, \]
\[ (1 + \lambda \mu)(Q_\mu)'' - (1 + \mu)Q_\prime + 2Q_\mu Q'_\mu = 0. \]

Let us check (A.6). First, \( \lim_{x \to \infty} \frac{Q'}{Q} = +1 \) is clear from the expression of \( Q \). Next, we have, using (A.2),
\[ \frac{Q'' - (Q')^2}{Q^2} = -\frac{1}{3} Q. \]

Thus,
\[ -L \frac{Q'}{Q} = -\frac{1}{3} Q - \frac{Q'}{Q} + 2Q' = \frac{5}{3} Q' - \frac{Q'}{Q}, \quad \left( -L \frac{Q'}{Q} \right)' = 2Q - \frac{5}{3} Q^2. \]

(iii) These identities are readily obtained from (A.2). Note for example:
\[
\int [(1 - \lambda \partial_x^2)\Lambda Q](Q + \frac{1}{2}(1 - \lambda)xQ')(Q - \lambda Q + \lambda Q^2)
= (1 - \lambda) \int Q^2 + \lambda \int Q^3 - \frac{1}{4} (1 - \lambda)^2 \int Q^2 - \frac{1}{6} (1 - \lambda) \lambda \int Q^3
= (1 - \lambda)(1 - \frac{1}{4} + \frac{\lambda}{6}) \int Q^2 + \frac{6}{5}(\lambda - \frac{\lambda^2}{6} + \frac{\lambda^2}{6}) \int Q^2 = \frac{3}{10}(15 + 10\lambda - \lambda^2).
\]

The identities on \( \int Q_\mu^2 \), \( \int (Q_\mu')^2 \) and \( \int Q_\mu^3 \) follows directly from (A.11). Now, we prove (A.11). We first observe that
\[ \frac{1}{2} \frac{d}{d\mu} E(Q_\mu) = \int (Q_\mu')^2 + \mu \Lambda Q_\mu - Q_\mu^2 \Lambda Q_\mu
= -\mu \int [\lambda (Q_\mu')^2 + Q_\mu \Lambda Q_\mu] = -\frac{1}{2} \frac{d}{d\mu} M(Q_\mu), \]
is a consequence of (A.15), multiplied by \( \Lambda Q_\mu \) and integrated over \( \mathbb{R} \).

Then, we check \( \frac{d}{d\mu} M(Q_\mu) > 0 \). For \( \mu \) small, the result is true by (A.9) and a perturbation argument. In fact, it is true for all \( \mu > -1 \) (see Weinstein [45]). Indeed, by the expressions of \( \int (\partial_x Q_\mu)^2 \) and \( \int Q_\mu^2 \), we have
\[ M(Q_\mu) = \int \lambda (Q_\mu')^2 + Q_\mu^2 = \frac{1}{5} \int Q^2(1 + \mu)^\frac{1}{2}(1 + \lambda \mu)^{-\frac{1}{2}}(5 + \lambda(1 + 6\mu)). \]

Differentiating with respect to \( \mu \), we find
\[ \frac{d}{d\mu} M(Q_\mu) = \frac{1}{10} \int Q^2(1 + \mu)^\frac{1}{2}(1 + \lambda \mu)^{-\frac{1}{2}}(15 + 10\lambda - \lambda^2 + 40\lambda \mu + 8\lambda^2 \mu + 24\lambda^2 \mu^2)
= \frac{1}{10} \int Q^2(1 + \mu)^\frac{1}{2}(1 + \lambda \mu)^{-\frac{1}{2}}(15(1 - \lambda)^2 + 8\lambda(1 + \mu)(5(1 - \lambda) + 3\lambda(1 + \mu))) > 0. \]
(iv)–(v) These identities and asymptotic properties are easily obtained from the explicit expression of \( Q \):
\[
Q(x) = \frac{6e^x}{(e^x + 1)^2} = \frac{6e^{-x}}{(e^{-x} + 1)^2}, \quad Q'(x) = \frac{6(e^x - e^{2x})}{(e^x + 1)^3} = \frac{6(e^{-2x} - e^{-x})}{(e^{-x} + 1)^3}.
\]
In particular, we observe that
\[
Q^2 = 36 \frac{e^{-2x}}{(e^{-x} + 1)^4} = 3Q' + Q \quad \text{and so} \quad e^{-x}Q^2 = -Q^2 + 3(Q' + Q).
\]
We obtain in particular
\[
10 \int e^{-x}Q^3 = 9 \int e^{-x}Q^2. \tag{A.18}
\]
Moreover,
\[
e^{-x}((AQ)' - \Lambda Q - \frac{1}{2}(1 - \lambda)Q) = \frac{1}{2}(3 - \lambda)e^{-x}(Q' - Q) + \frac{1}{2}(1 - \lambda)xe^{-x}(Q'' - Q')
\]
\[
= -\frac{1}{2}(3 - \lambda)(Q' + Q) + \frac{1}{2}(1 - \lambda)x(Q' + Q - e^{-x}Q^2).
\]

A.2 Technical claim \[A.3\]

Claim A.3.
\[
\frac{1}{18} R_1 R_2 = e^{-y} \left( \frac{\partial_x R_1}{R_1} - \frac{1}{3} R_1 - \frac{\partial_x R_2}{R_2} - \frac{1}{3} R_2 \right) + O_{3/2}. \tag{A.19}
\]

Proof. We distinguish the two regions \( x - y_2 > \frac{y}{2} \) and \( x - y_2 < \frac{y}{2} \).
- Case \( x - y_2 > \frac{y}{2} \). For \( x > y_2 + \frac{y}{2} \), we have \( R_2(t, x) = 6e^{-y(y - y_1) - y} + O(e^{-2(y - y_1) - 2y}) \) from \[A.14\] in Claim A.2. Thus, we obtain for such \( x \),
\[
R_1 R_2 - 18e^{-y} \left( \frac{\partial_x R_1}{R_1} - \frac{1}{3} R_1 - \frac{\partial_x R_2}{R_2} - \frac{1}{3} R_2 \right)
\]
\[
= -18e^{-y} \left( \frac{\partial_x R_1}{R_1} - \frac{1}{3} R_1 - \frac{1}{3} e^{-(x-y_1)} R_1 - \frac{\partial_x R_2}{R_2} - \frac{1}{3} R_2 \right) + R_1 O(e^{-2(y - y_1) - 2y}).
\]
Note that \( R_1 O(e^{-2(y - y_1) - 2y}) = O_{3/2} \).
- Case \( x - y_1 < -\frac{y}{2} \) (or equivalently \( x - y_1 < -\frac{y}{2} \)). It is treated similarly. \( \square \)
A.3 Proof of Lemma 2.1

First, we claim the following estimates.

Claim A.4. The following holds ($\omega \geq 0$)

$$\sqrt{1 + \mu_j} = 1 + \frac{1}{2} (1 - \lambda) \mu_j - \frac{1}{4} (1 - \lambda)(1 + 3\lambda) \mu_j^2 + O(e^{-\frac{1}{2}Y_0}),$$  \hspace{1cm} (A.21)

$$\tilde{R}_j(t, x) + |\partial_x \tilde{R}_j(t, x)| \leq C e^{-\langle 1-2\mu_0 \rangle |x-y_j|},$$  \hspace{1cm} (A.22)

$$\left| \tilde{R}_j(t, x) - \left\{ R_j(t, x) + \mu_j(t) \Delta R_j(t, x) + \frac{1}{2} \mu_j^2(t) \Lambda^2 R_j(t, x) \right\} \right| \leq C \mu_0^3 \left( 1 + |x - y_j(t)|^3 \right) e^{-\langle 1-2\mu_0 \rangle |x-y_j(t)|},$$  \hspace{1cm} (A.23)

$$\int (1 + |x-y_1|^{\omega} + |x-y_2(t)|^{\omega}) e^{-\langle 1-2\mu_0 \rangle |x-y_1|} e^{-\langle 1-2\mu_0 \rangle |x-y_2|} dx \leq C(1 + |y|^{1+\omega}) e^{-y}. \quad (A.25)$$

Proof. We have

$$\sqrt{1 + \mu_j} = 1 + \frac{1}{2} \mu_j - \frac{1}{8} \mu_j^2 (1 - \lambda)(1 + 3\lambda) \mu_j^2 + O(\mu_j^3)$$  \hspace{1cm} (A.26)

and (A.21) follows. Estimate (A.22) is clear from $Q(x) \leq C e^{-|x|}$ and (A.21).

Now, we prove (A.23), using the Taylor formula (in the $\mu_j$ variable):

$$\tilde{R}_j(t, x) = R_j(t, x) + \mu_j(t) \Delta R_j(t, x) + \frac{1}{2} \mu_j^2(t) \Lambda^2 R_j(t, x)$$

$$+ \frac{1}{2} \mu_j^3(t) \int_0^1 (1-s)^2 \left( \frac{\partial^3 Q_\mu}{\partial \mu^3} \right)_{\mu=s\mu_j(t)} (x-y_j(t)) ds.$$

Note that, from the proof of (A.3) and elementary computations:

$$\left| \left( \frac{\partial^3 Q_\mu}{\partial \mu^3} \right)_{\mu=s\mu_j(t)} (x) \right| \leq C(1 + |x|^{3}) e^{-\langle 1-2\mu_0 \rangle |x|};$$

(A.23) follow.

Proof of (A.25). For $y_2 < x < y_1$, we have

$$e^{-\langle 1-2\mu_0 \rangle |x-y_1|} e^{-\langle 1-2\mu_0 \rangle |x-y_2|} = e^{-\langle 1-2\mu_0 \rangle y} \leq Ce^{-y},$$

since (2.5) implies $2\mu_0 y(t) \leq C\mu_0 Y_0 \leq C'$ for $Y_0$ large enough.

For $x > y_1 > y_2$,

$$e^{-\langle 1-2\mu_0 \rangle |x-y_1|} e^{-\langle 1-2\mu_0 \rangle |x-y_2|} = e^{-\langle 1-2\mu_0 \rangle (2x-y_1-y_2)} = e^{-2(1-2\mu_0)(x-y_1)} e^{-\langle 1-2\mu_0 \rangle y(t)}$$

$$\leq Ce^{-\frac{3}{2} |x-y_1|} e^{-y(t)}.$$

Arguing similarly for the case $x < y_2 < y_1$, we prove (A.24) and (A.25).\qed
We continue the proof of Lemma 2.1. In order to expand $F$, we perform the following preliminary decomposition:

$$F = 2\tilde{R}_2\partial_x\tilde{R}_1 + 2\tilde{R}_1\partial_x\tilde{R}_2$$

Using (A.21) and (A.13), we obtain

$$= 2\tilde{R}_2 \left( \partial_x\tilde{R}_1 - \sqrt{\frac{1 + \mu_1}{1 + \lambda \mu_1}} \tilde{R}_1 \right) + 2\tilde{R}_1 \left( \partial_x\tilde{R}_2 + \sqrt{\frac{1 + \mu_2}{1 + \lambda \mu_2}} \tilde{R}_2 \right)$$

Using (A.21) and (A.23), we have

$$\partial_x\tilde{R}_1 - \sqrt{\frac{1 + \mu_1}{1 + \lambda \mu_1}} \tilde{R}_1 = \partial_x(R_1 + \mu_1\Lambda R_1) - (1 + \frac{1}{2}(1 - \lambda)\mu_1)(R_1 + \mu_1\Lambda R_1) + \tilde{\Theta}$$

Thus, by (A.22) and (A.24), we obtain

$$2\tilde{R}_2 \left( \partial_x\tilde{R}_1 - \sqrt{\frac{1 + \mu_1}{1 + \lambda \mu_1}} \tilde{R}_1 \right) = 2(R_2 + \mu_2\Lambda R_2) \left[ \partial_x R_1 - R_1 + \mu_1(\partial_x\Lambda R_1 - \Lambda R_1 - \frac{1}{2}(1 - \lambda)R_1) \right] + O_2.$$

Using (A.14), (A.3) and $x - y_2 = x - y_1 + y$, we obtain

$$2\tilde{R}_2 \left( \partial_x\tilde{R}_1 - \sqrt{\frac{1 + \mu_1}{1 + \lambda \mu_1}} \tilde{R}_1 \right) = 12e^{-y}e^{-(x - y_1)} \left[ 1 + \mu_2(1 - \frac{1}{2}(1 - \lambda)(x - y_1 + y)) \right] \times \left[ \partial_x R_1 - R_1 + \mu_1(\partial_x\Lambda R_1 - \Lambda R_1 - \frac{1}{2}(1 - \lambda)R_1) \right] + O_2.$$

Using (A.12), (A.13) and then (2.6),

$$2\tilde{R}_2 \left( \partial_x\tilde{R}_1 - \sqrt{\frac{1 + \mu_1}{1 + \lambda \mu_1}} \tilde{R}_1 \right) = -12e^{-y}(\partial_x R_1 + R_1)$$

Next, by similar computations,

$$\partial_x\tilde{R}_2 + \sqrt{\frac{1 + \mu_2}{1 + \lambda \mu_2}} \tilde{R}_2 = 12e^{-y}(-\partial_x R_2 + R_2) + 6(1 - \lambda)\mu_2 e^{-y}(-\partial_x R_2 + R_2)$$

Using (A.28), (A.29), (A.30) and then (2.6),

$$\partial_x\tilde{R}_2 + \sqrt{\frac{1 + \mu_2}{1 + \lambda \mu_2}} \tilde{R}_2 = 12e^{-y}(-\partial_x R_2 + R_2) + 6(1 - \lambda)\mu_2 e^{-y}(-\partial_x R_2 + R_2) + 6\mu_2 e^{-y}(1 - \lambda) \left[ -(\partial_x R_1 + R_1) + (x - y_2)(R_1^2 - 3(\partial_x R_1 + R_1)) \right] + O_2.$$
Finally, using Claim A.4,

$$2 \left( \frac{1 + \mu_1}{1 + \lambda \mu_1} - \frac{1 + \mu_2}{1 + \lambda \mu_2} \right) \tilde{R}_1 \tilde{R}_2 = (1 - \lambda)(\mu_1 - \mu_2)R_1 R_2 + \mathcal{O}_2.$$

### A.4 Approximate antecedent of $R_1 R_2$

**Claim A.5.** Let $x_j = x - y_j$. Then

$$\partial_x(-\partial_x^2((x_1 + x_2)R_1 R_2) + (x_1 + x_2)R_1 R_2 - 2(R_1 + R_2)(x_1 + x_2)R_1 R_2)$$

$$= 2R_1 R_2 - y(3(\partial_x R_1 - R_1) + \partial_x(R_1^2))R_2 + yR_1^2 \partial_x R_2$$

$$+ y(\partial_x R_1 + R_2) + \partial_x(R_2^2)R_1 - yR_2^2 \partial_x R_1$$

$$+ 2[R_1^2 - x_1 \partial_x R_1^2 - 3x_1(\partial_x R_1 - R_1) - 3(R_1 - \partial_x R_1)]R_2$$

$$+ 2[x_1 R_1^2 - 3(\partial_x R_1 - R_1)]\partial_x R_2$$

$$+ 2[R_2^2 - x_2 \partial_x R_2^2 + 3x_2(\partial_x R_2 + R_2) - 3(R_2 + \partial_x R_2)]R_1$$

$$+ 2[x_2 R_2^2 - 3(\partial_x R_2 + R_2)]\partial_x R_1.$$

**Proof.** For any two functions $F_1$, $F_2$, the following holds true

$$\partial_x(-\partial_x^2(F_1(x_1)F_2(x_2)) + F_1(x_1)F_2(x_2) - 2(R_1 + R_2)(F_1(x_1)F_2(x_2)))$$

$$= \partial_x(F_2(x_2)(LF_1)(x_1) + F_1(x_1)(LF_2)(x_2) - 2F_1'(x_1)F_2'(x_2) - F_1(x_1)F_2(x_2))$$

$$= (LF_1)'(x_1)F_2(x_2) + (LF_2)'(x_2)F_1(x_1)$$

$$+ (LF_1 - F_1 - 2F_1''(x_1))F_2(x_2) + (LF_2 - F_2 - 2F_2''(x_2))F_1(x_1).$$

Now, we apply formula (A.31) with $F_1 = xQ$ and $F_2 = Q$. Note that (see Claim A.2 (ii))

$$LQ = -Q^2, \quad LQ - Q - 2Q'' = -3Q + Q^2,$$

$$L(xQ) = xLQ - 2Q' = -xQ^2 - 2Q'$$

Thus, from (A.31)

$$\partial_x(-\partial_x^2(x_1 R_1 R_2) + x_1 R_1 R_2 - 2(R_1 + R_2)x_1 R_1 R_2)$$

$$= (R_1^2 - x_1 \partial_x R_1^2)R_2 + x_1 R_1^2 \partial_x R_2 - \partial_x(R_1^2)(x_1 R_1) + R_2^2(R_1 + x_1 \partial_x R_1)$$

$$- 5R_1 R_2 - 3x_1 \partial_x R_1 R_2 - 3x_1 R_1 \partial_x R_2 - 6\partial_x R_1 \partial_x R_2. \quad (A.32)$$

Similarly,

$$\partial_x(-\partial_x^2(x_2 R_1 R_2) + x_2 R_1 R_2 - 2(R_1 + R_2)x_2 R_1 R_2)$$

$$= (R_2^2 - x_2 \partial_x R_2^2)R_1 + x_2 R_2^2 \partial_x R_1 - \partial_x(R_2^2)(x_2 R_2) + R_1^2(R_2 + x_2 \partial_x R_2)$$

$$- 5R_2 R_1 - 3x_2 \partial_x R_2 R_1 - 3x_2 R_2 \partial_x R_1 - 6\partial_x R_2 \partial_x R_1. \quad (A.33)$$

Therefore, summing up,

$$\partial_x(-\partial_x^2((x_1 + x_2)R_1 R_2) + (x_1 + x_2)R_1 R_2 - 2(R_1 + R_2)(x_1 + x_2)R_1 R_2)$$

$$= (R_1^2 - x_1 \partial_x R_1^2)R_2 + x_1 R_1^2 \partial_x R_2 - \partial_x(R_1^2)(x_2 R_2) + R_2^2(R_2 + x_2 \partial_x R_2)$$

$$+ (R_2^2 - x_2 \partial_x R_2^2)R_1 + x_2 R_2^2 \partial_x R_1 - \partial_x(R_2^2)(x_1 R_1) + R_1^2(R_1 + x_1 \partial_x R_1)$$

$$- 10R_1 R_2 - 12\partial_x R_1 \partial_x R_2$$

$$- 3x_1 \partial_x R_1 R_2 - 3x_1 R_1 \partial_x R_2 - 3x_2 \partial_x R_2 R_1 - 3x_2 R_2 \partial_x R_1. \quad (A.34)$$
The terms in the last line of (A.34) are handled as follows (recall that $x_2 - x_1 = y$)

$$3x_1\partial_xR_1R_2 + 3x_1R_1\partial_xR_2 + 3x_2\partial_xR_2R_1 + 3x_2R_2\partial_xR_1$$

$$= 3x_1(\partial_xR_1 - R_1)R_2 + 3x_1R_1(\partial_xR_2 + R_2) + 3x_2(\partial_xR_2 + R_2)R_1 + 3x_2R_2(\partial_xR_1 - R_1)$$

$$= 6x_1(\partial_xR_1 - R_1)R_2 + 3y(\partial_xR_1 - R_1)R_1 + 6x_2(\partial_xR_2 + R_2)R_1 - 3y(\partial_xR_2 + R_2)R_1.$$

For the term $12\partial_xR_1\partial_xR_2$ in (A.34), we observe

$$12\partial_xR_1\partial_xR_2 = 6(\partial_xR_1 - R_1)\partial_xR_2 + 6(\partial_xR_2 + R_2)\partial_xR_1 + 6R_1\partial_xR_2 - 6R_2\partial_xR_1$$

$$= 6(\partial_xR_1 - R_1)\partial_xR_2 + 6(\partial_xR_2 + R_2)\partial_xR_1 + 12R_1R_2.$$

Thus, we obtain

$$\partial_x(-\partial_x^2((x_1 + x_2)R_1R_2) + (x_1 + x_2)R_1R_2 - 2(R_1 + R_2)(x_1 + x_2)R_1R_2)$$

$$= (R_1^2 - x_1\partial_xR_1^2)R_2 + x_1R_1^2\partial_xR_2 - \partial_x(R_1^2)(x_2R_2) + R_1^2(R_2 + x_2\partial_xR_2)$$

$$- 6x_1(\partial_xR_1 - R_1)R_2 + 6(\partial_xR_1 - R_1)\partial_xR_2 - 6(R_1 - \partial_xR_1)R_2$$

$$+ (R_2^2 - x_2\partial_xR_2^2)R_1 + x_2R_2^2\partial_xR_1 - \partial_x(R_2^2)(x_1R_1) + R_2^2(R_1 + x_1\partial_xR_1)$$

$$- 6x_2(\partial_xR_2 + R_2)R_1 + 6(\partial_xR_2 + R_2)\partial_xR_1 - 6(R_2 + \partial_xR_2)R_1$$

$$+ 3y(\partial_xR_1 + R_1)R_2 + 3y(\partial_xR_2 + R_2)R_1 + 2R_1R_2$$

$$= 2R_2^2 - 3(\partial_xR_1 - R_1) + \partial_x(R_1^2) + yR_1^2\partial_xR_2$$

$$+ y(3(\partial_xR_2 + R_2) + \partial_x(R_2^2) - yR_2^2\partial_xR_1$$

$$+ 2[R_1^2 - x_1\partial_xR_1^2 - x_2\partial_xR_2^2 - 3x_1(\partial_xR_1 - R_1) - 3(R_1 - \partial_xR_1)]R_2$$

$$+ 2x_1R_1^2 - 3(\partial_xR_1 - R_1)\partial_xR_2$$

$$+ 2x_2R_2^2 - 3(\partial_xR_2 + R_2)\partial_xR_1 + 2x_2R_2^2 - 3(\partial_xR_2 + R_2)\partial_xR_1,$$

and the proof of Claim (A.34) is complete. \qed

\section{Modulation and monotonicity arguments}

\subsection{Proof of Lemma 3.1}

Let

$$V(\omega_0, y_0) = \{u \in H^1(\mathbb{R}); \inf_{y_1 - y_2 > y_0} \|u - V(x; (0,0,y_1,y_2))\|_{H^1} \leq \omega_0\},$$

where $V(x; \Gamma)$ is defined in Proposition 2.2.

\textbf{Lemma B.1} (Time independent modulation). There exist $\omega_0$, $\bar{y}_0 > 0$ and a unique $C^1$ map $\Gamma = (\mu_1, \mu_2, y_1, y_2) : V(\omega_0, \bar{y}_0) \rightarrow (0, \infty)^2 \times \mathbb{R}^2$ such that if $u \in V(\omega, y_0)$ for $0 < \omega \leq \omega_0$, $y_0 \geq \bar{y}_0$ and

$$\varepsilon(x) = u(x) - V(x; \Gamma),$$

then, for $j = 1, 2,$

$$\int \varepsilon(1 - \lambda \partial_x^2)Q_{\mu_j}(\cdot - y_j) = \int \varepsilon(1 - \lambda \partial_x^2)Q'_{\mu_j}(\cdot - y_j) = 0$$

$$y_1 - y_2 > y_0 - C\omega, \quad \|\varepsilon\|_{H^1} + |\mu_1| + |\mu_2| \leq C\omega.$$
Proof. The proof, based on the implicit function theorem, is similar to the one of Lemma 8 in [33] (see also [11] for the BBM case), the only difference being that the modulation uses the map \((\mu_1, \mu_2, y_1, y_2) \mapsto V(x; (\mu_1, \mu_2, y_1, y_2))\) instead of the family of sums of two solitons. By the properties of \(V\) (see (2.14) and below (B.2)) and (A.9), the nondegeneracy condition is the same as in [11].

The existence, uniqueness and continuity of \(\Gamma(t)\) is a consequence of Claim B.1. The \(C^1\) regularity of \(\Gamma(t)\) is obtained by standard regularization arguments and the equation of \(\varepsilon(t)\) which is deduced easily from \(\text{BBM}\) and (2.15).

Next, we prove the estimates on \(\dot{\Gamma}(t)\), i.e. (3.5), omitting standard regularization arguments to justify the formal computations. First, we expand \(0 = \frac{d}{dt} \int \varepsilon(1 - \lambda \partial_x^2) \tilde{R}_j\). Using (3.4), we obtain \((k \neq j)\)

\[
0 = \frac{d}{dt} \int \varepsilon(1 - \lambda \partial_x^2) \tilde{R}_j = \int \varepsilon(1 - \lambda \partial_x^2) \partial_t \tilde{R}_j + \int (\partial_x^2 \varepsilon - \varepsilon + 2V \varepsilon) \partial_x \tilde{R}_j + \int \varepsilon^2 \partial_x \tilde{R}_j - \int E \tilde{R}_j - (\dot{\mu}_j - M_j) \int (1 - \lambda \partial_x^2) \frac{\partial V}{\partial \mu_j} \tilde{R}_j - (\dot{\mu}_k - M_k) \int (1 - \lambda \partial_x^2) \frac{\partial V}{\partial \mu_k} \tilde{R}_j + (\mu_j - \dot{y}_j - N_j) \int (1 - \lambda \partial_x^2) \frac{\partial V}{\partial y_j} \tilde{R}_j + (\mu_k - \dot{y}_k - N_k) \int (1 - \lambda \partial_x^2) \frac{\partial V}{\partial y_k} \tilde{R}_j.
\]

We claim the following estimates.

Claim B.1. Assuming (3.3),

\[
\left| \int \tilde{R}_1 \tilde{R}_2 \right| \leq C(y + 1)e^{-y}, \tag{B.1}
\]

\[
\begin{align*}
 j = 1, 2, & \quad \left\| \frac{\partial V}{\partial \mu_j} - \Lambda \tilde{R}_j \right\|_{H^1} + \left\| \frac{\partial V}{\partial y_j} + \partial_x \tilde{R}_j \right\|_{L^\infty} + \frac{1}{\sqrt{y}} \left\| \frac{\partial V}{\partial y_j} + \partial_x \tilde{R}_j \right\|_{H^1} \leq C e^{-y}. \tag{B.2}
\end{align*}
\]

Indeed, under assumption (3.5), (B.1) is a consequence of (A.22) and (A.25), and (B.8) is a consequence of (2.13), (2.12) and the properties of \(A_j\), \(B_j\) and \(D_j\) (see (2.11)).

By (B.2), (B.1), (2.13), \(L_{\mu_j} Q_{\mu_j} = 0\) (see Claim A.2) and \(\int (1 - \lambda \partial_x^2) \partial_x \tilde{R}_j \tilde{R}_j = 0\), we get \((j \neq k)\)

\[
0 = \dot{\mu}_j \int \varepsilon(1 - \lambda \partial_x^2) \Lambda \tilde{R}_j + (\mu_j - \dot{y}_j) \int \varepsilon(1 - \lambda \partial_x^2) \partial_x \tilde{R}_j + \|\varepsilon(t)\|_{L^2} O((y + 1)e^{-y}) + O(\|\varepsilon\|_{L^2}) - \int E \tilde{R}_j - (\dot{\mu}_j - M_j) \left( \int (1 - \lambda \partial_x^2) \Lambda \tilde{R}_j + O(e^{-\frac{1}{2}y}) \right) + (\dot{\mu}_k - M_k) O(y^2 e^{-y}) + (\mu_j - \dot{y}_j - N_j) O(e^{-y}) + (\mu_k - \dot{y}_k - N_k) O(y e^{-y}).
\]

Hence, by \(\int (1 - \lambda \partial_x^2) \Lambda \tilde{R}_j \tilde{R}_j \geq c_0 > 0\) (see (A.9)), for \(y\) large and \(\varepsilon\) small, we get

\[
|\dot{\mu}_j - M_j| \leq C \left[ \|\varepsilon\|_{L^2}^2 + ye^{-y}\|\varepsilon\|_{L^2} + \int |E \tilde{R}_j| \right] + C e^{-y}\|\dot{\mu}_j - \dot{y}_j - \dot{N}_j\| + C y^2 e^{-y}\|\mu_k - \mu_j - \dot{N}_k\| + C ye^{-y}\|\mu_j - \dot{y}_j - N_j\|.
\]

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Similarly, expanding \( 0 = \frac{d}{dt} \int \varepsilon(1 - \lambda \partial_x^2) \partial_x \tilde{R}_j \), we obtain

\[
|\mu_j - \dot{y}_j - N_j| \leq C[\|\varepsilon\|_{L^2} + \int |E \partial_x \tilde{R}_j|] \\
+ C(\|\varepsilon\|_{L^2} + e^{-\frac{t}{2}}) |\mu_j - M_j| + Ce^{-\frac{t}{2}} |\mu_k - M_k| + Cy^{-y} |\mu_k - \dot{y}_k - N_k|.
\]

Combining these estimates, for \( y \) large and \( \varepsilon \) small, (3.6) is proved.

### B.2 Proof of Proposition [3.1]

The proof of Proposition [3.1] is inspired by the proof of Proposition 3.1 in [31]. However, it is technically more involved in the BBM case. We refer to [37], [9], [11] and [32] for previous similar arguments for the (BBM) equation.

The proof of (3.11) is standard, see for example Lemma 4 in [33] and [11]. Recall that it is based on coercivity property of the operator \( L \) under orthogonality conditions, see Claim A.1(v).

We continue with the following claim:

**Claim B.2.** Let \( \varphi(x) \) be defined by (5.7). If \( a(x), b(x) \in L^2 \) are such that \( a - \lambda \partial_x^2 a = b \) then

\[
(1 - (8\rho)^2) \int a^2 \varphi' + 2\lambda \int (\partial_x a)^2 \varphi' + \lambda^2 \int (\partial_x^2 a)^2 \varphi' \leq \int b^2 \varphi'.
\]  
(B.3)

Indeed, integrating by parts and then using (3.7),

\[
\int b^2 \varphi' = \int (a - \lambda \partial_x^2 a)^2 \varphi' = \int a^2 \varphi' + 2\lambda \int (\partial_x a)^2 \varphi' - \lambda \int a^2 \varphi'' + \lambda^2 \int (\partial_x^2 a)^2 \varphi'
\]

\[
\geq (1 - (8\rho)^2) \int a^2 \varphi' + 2\lambda \int (\partial_x a)^2 \varphi' + \lambda^2 \int (\partial_x^2 a)^2 \varphi'.
\]

**• Case** \( \mu_1(t) \geq \mu_2(t) \). We first claim the following technical estimates, as consequences of (3.7), (3.12), (2.3) and (2.12).

**Claim B.3.**

\[
\|V - \tilde{R}_1 - \tilde{R}_2\|_{L^\infty} \leq Ce^{-y},
\]  
(B.4)

\[
\|V \partial_x \varphi\|_{L^\infty} \leq C(\|\mu_1\| + \|\mu_2\|)e^{-2\rho y},
\]  
(B.5)

\[
\|\Phi - \mu_j\|_{L^\infty} e^{-\frac{1}{2}|x-y_j|} \leq C(\|\mu_1\| + \|\mu_2\|)e^{-2\rho y},
\]  
(B.6)

\[
\|\Phi \partial_x V - \sum_{j=1,2} \mu_j \partial_x \tilde{R}_j\|_{L^\infty} \leq C(\|\mu_1\| + \|\mu_2\|)e^{-2\rho y} + Ce^{-y},
\]  
(B.7)

\[
\|\partial_y V - \sum_{j=1,2} \left( \mu_j \Lambda \tilde{R}_j - \dot{y}_j \partial_x \tilde{R}_j \right)\|_{L^\infty} \leq Ce^{-y}.
\]  
(B.8)
Let
\[ \Theta = \|\varepsilon\|_{L^2}^2 \left[ e^{-\frac{3}{2}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2})(e^{-2\rho y} + \|\varepsilon\|_{L^2}) \right] + \|\varepsilon\|_{L^2}^2 \|E\|_{L^2}. \]

Let us compute \( \frac{d}{dt} \mathcal{F}_+(t) \):
\begin{align*}
\frac{1}{2} \frac{d}{dt} \mathcal{F}_+(t) &= \int \partial_t \varepsilon \left( -\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^2 - V^2) + [(1 + \lambda \partial_x^2)(\varepsilon \Phi)] \right) \\
&\quad - \lambda \int (\partial_x \partial_t \varepsilon) \varepsilon \partial_x \Phi + \frac{1}{2} \int \partial_t \Phi \lambda (\partial_x \varepsilon)^2 + \varepsilon^2 \\
&\quad - \int \partial_t V ((\varepsilon + V)^2 - V^2 - 2V\varepsilon) = F_1 + F_2 + F_3 + F_4.
\end{align*}

Observe that \( \partial_x \Phi = (\mu_1 - \mu_2) \varphi' \geq 0 \) by \((3.12)\).

Using \((3.1)\) and then by direct computations and estimates, we claim the following estimates, which imply immediately \((3.13)\).

**Claim B.4.**
\begin{align*}
F_1 &\leq -\frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi - \frac{3}{8} \int \varepsilon^2 \partial_x \Phi - \int \varepsilon^2 (\mu_1 \partial_x \tilde{R}_1 + \mu_2 \partial_x \tilde{R}_2), \\
&\quad + \sum_{j=1,2} (\hat{\mu}_j - M_j) \int \varepsilon^2 \Lambda \tilde{R}_j + \sum_{j=1,2} (\mu_j - \hat{y}_j - N_j) \int \varepsilon^2 \partial_x \tilde{R}_j + C\Theta \tag{B.9} \\
F_2 &\leq \frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi + \frac{3}{8} \int \varepsilon^2 \partial_x \Phi + C\Theta, \tag{B.10} \\
F_3 &\leq C\Theta, \tag{B.11} \\
F_4 &\leq \int \varepsilon^2 (\mu_1 \partial_x \tilde{R}_1 + \mu_2 \partial_x \tilde{R}_2) \\
&\quad - \sum_{j=1,2} (\hat{\mu}_j - M_j) \int \varepsilon^2 \Lambda \tilde{R}_j - \sum_{j=1,2} (\mu_j - \hat{y}_j - N_j) \int \varepsilon^2 \partial_x \tilde{R}_j + C\Theta. \tag{B.12}
\end{align*}

Indeed,
\begin{align*}
F_1 &= -\int (-\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^2 - V^2)) (\Phi \partial_x \varepsilon + \varepsilon \partial_x \Phi) \\
&\quad - \int E(1 + \lambda \partial_x^2)^{-1} (-\partial_x^2 \varepsilon + \varepsilon + [(1 + \lambda \partial_x^2)(\varepsilon \Phi)] - ((\varepsilon + V)^2 - V^2)) \\
&\quad - \sum_{j=1,2} (\hat{\mu}_j - M_j) \int \frac{\partial V}{\partial y_j} (-\partial_x^2 \varepsilon + \varepsilon + [(1 + \lambda \partial_x^2)(\varepsilon \Phi)] - ((\varepsilon + V)^2 - V^2)) \\
&\quad + \sum_{j=1,2} (\mu_j - \hat{y}_j - N_j) \int \frac{\partial V}{\partial y_j} (-\partial_x^2 \varepsilon + \varepsilon + [(1 + \lambda \partial_x^2)(\varepsilon \Phi)] - ((\varepsilon + V)^2 - V^2)) \\
&\quad = F_{1,1} + F_{1,2} + F_{1,3} + F_{1,4}.
\end{align*}

By \((3.7)\), Claim \([3.3]\) and several integration by parts, we get
\begin{align*}
F_{1,1} &= -\frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi - \frac{1}{2} \int \varepsilon^2 \partial_x \Phi + \frac{1}{2} \int \left( 3\varepsilon^2 V + \frac{4}{3} \varepsilon^3 \right) \partial_x \Phi + \frac{1}{2} \int \varepsilon^2 \partial_x^3 \Phi - \int \varepsilon^2 \partial_x V \\
&\quad \leq \frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi - \frac{1}{2} (4\rho^2 - C\|\varepsilon\|_{H^1}) \int \varepsilon^2 \partial_x \Phi - \int \varepsilon^2 (\mu_1 \partial_x \tilde{R}_1 + \mu_2 \partial_x \tilde{R}_2) \\
&\quad + C(|\mu_1| + |\mu_2|) \|\varepsilon\|_{L^2}^2 e^{-\rho y} + C e^{-\frac{3}{2}y} \|\varepsilon\|_{L^2}^2.
\end{align*}
and
\[ |F_{1,2}| \leq C\|E\|_{L^2}\|\varepsilon\|_{L^2}. \]

For \(F_{1,3}\), we use (B.2), (A.5), (3.3) and (B.6), so that
\[
\left|[(1 - \lambda \partial_x^2)\Lambda \tilde{R}_j] \Phi - [(1 - \lambda \partial_x^2)\Lambda \tilde{R}_j]\mu_j \right|
\leq Ce^{-\frac{1}{2}|x-y_j(t)|}\|\Phi - \mu_j\| \leq C(\mu_1 + \mu_2)e^{-\frac{1}{2}|x-y_j(t)|}e^{-2\rho y},
\]
and
\[
F_{1,3} = \sum_{j=1,2} (\dot{\mu}_j - \mathcal{M}_j) \int \varepsilon^2 \Lambda \tilde{R}_j + e^{-2\rho y}(\mu_1 + \mu_2) + \|\varepsilon\|_{L^2}O(\|\varepsilon\|_{L^2}) \sum_{j=1,2} |\dot{\mu}_j - \mathcal{M}_j|. \quad (B.13)
\]

Similarly,
\[
F_{1,4} = \sum_{j=1,2} (\mu_j - \dot{y}_j - \mathcal{N}_j) \int \varepsilon^2 \partial_x \tilde{R}_j + e^{-2\rho y}(\mu_1 + \mu_2) + \|\varepsilon\|_{L^2}O(\|\varepsilon\|_{L^2}) \sum_{j=1,2} |\mu_j - \dot{y}_j - \mathcal{N}_j|. \quad (B.14)
\]

Using (3.6), (B.9) follows.

Let \(z_1, z_2\) such that \(z_1 - \lambda \partial_x^2 z_1 = \varepsilon, z_2 - \lambda \partial_x^2 z_2 = 2V\varepsilon + \varepsilon^2\). Then, using the equation of \(\varepsilon\):
\[
F_2 = \lambda \int \partial_x^2 (1 - \lambda \partial_x^2)^{-1}(\partial_x \varepsilon - \varepsilon + 2V\varepsilon + \varepsilon^2)\partial_x \Phi + \lambda \int \partial_x (1 - \lambda \partial_x^2)^{-1} \varepsilon \partial_x \Phi
\]
\[
+ \lambda \sum_{j=1,2} (\dot{\mu}_j - \mathcal{M}_j) \int \partial_x \partial_{\mu_j} \varepsilon \partial_x \Phi - \lambda \sum_{j=1,2} (\mu_j - \dot{y}_j - \mathcal{N}_j) \int \partial_x \partial_{\dot{y}_j} \varepsilon \partial_x \Phi
\]
\[
= F_{2,1} + F_{2,2} + F_{2,3} + F_{2,4}.
\]

\[
F_{2,1} = -\lambda \int \partial_x (\partial_x^2 z_1 + z_2) \partial_x \varepsilon \partial_x \Phi - \lambda \int \partial_x (\partial_x^2 z_1 + z_2) \varepsilon \partial_x \Phi - \lambda \int \partial_x^2 z_1 \partial_x \Phi
\]
\[
\leq \frac{1}{2} \int \left[ \lambda^2 (\partial_x^2 z_1)^2 + (\partial_x \varepsilon)^2 + 2\lambda^2 (\partial_x \varepsilon)^2 + \frac{1}{2} \varepsilon^2 + 4\lambda^2 (\partial_x \varepsilon)^2 + \frac{1}{4} (\partial_x \varepsilon)^2 \right] \partial_x \Phi
\]
\[
+ 2\rho \int \left[ \lambda^2 (\partial_x^2 z_1)^2 + \varepsilon^2 + 4\lambda^2 (\partial_x \varepsilon)^2 + \frac{1}{4} \varepsilon^2 \right] \partial_x \Phi,
\]
by using Cauchy Schwarz inequality and (3.7). For \(\rho\) small enough, using Claim B.2 and (B.5), we obtain
\[
|F_{2,1}| \leq \frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi + \left( \frac{1}{4} + 8\rho + C\|\varepsilon\|_{H^1} + C e^{-y} \right) \int \varepsilon^2 \partial_x \Phi + C(\mu_1 + \mu_2)\|\varepsilon\|_{L^2}^2 e^{-2\rho y}
\]
\[
\leq \frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi + \frac{3}{8} \int \varepsilon^2 \partial_x \Phi + C\Theta.
\]
The term \(F_{2,2}\) is estimated as \(F_{1,2}\) and by arguments previously used, we also obtain \(|F_{2,3}| + |F_{2,4}| \leq C\Theta\). Thus, (B.10) is proved.

Next,\[
F_3 = \frac{1}{2} \int (\dot{\mu}_1 \varphi + \dot{\mu}_2 (1 - \varphi))[\lambda (\partial_x \varepsilon)^2 + \varepsilon^2],
\]

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so that by $|M_j| \leq Ce^{-y}$ and (3.6),

$$|F_3| \leq C(e^{-y} + \sum_{j=1,2} |\tilde{\nu}_j - M_j|)\|\varepsilon\|^2_{L^2} \leq C\Theta. \quad (B.15)$$

Finally, by (B.8),

$$F_4 = -\int \partial_t V\varepsilon^2 = -\int \sum_{j=1,2} \left(\tilde{\nu}_j \Lambda \tilde{R}_j - \tilde{\nu}_j \partial_x \tilde{R}_j \varepsilon^2\right) + O(e^{-y})\|\varepsilon\|^2_{L^2}$$

$$= \sum_{j=1,2} \mu_j \int \varepsilon^2 \partial_x \tilde{R}_j - \sum_{j=1,2} (\tilde{\nu}_j - M_j) \int \varepsilon^2 \Lambda \tilde{R}_j$$

$$= \sum_{j=1,2} (\tilde{\nu}_j - \tilde{\nu}_j - N_j) \int \varepsilon^2 \partial_x \tilde{R}_j + O(e^{-y})\|\varepsilon\|^2_{L^2},$$

which proves (B.12).

- Case $\mu_1(t) \leq \mu_2(t)$. Since $\mu_2(t) \geq \mu_1(t)$ we have $\frac{1}{(1+\mu_1(t))^2} \leq \frac{1}{(1+\mu_2(t))^2}$, $\partial_x \Phi_1 \geq 0$ and $\partial_x \Phi_2 \leq 0$. Note also that by explicit computations, for $\mu_j$ small enough:

$$\left|\partial_x \Phi_2 + \frac{1}{2} \partial_x \Phi_1\right| \leq C(|\mu_1| + |\mu_2|)\partial_x \Phi_1. \quad (B.16)$$

Let us compute $\frac{d}{dt} F_\cdot(t)$:

$$\frac{1}{2} \frac{d}{dt} F_\cdot(t) = \int \partial_t \varepsilon (-\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^2 - V^2)) \Phi_1 - \int \partial_t \varepsilon \partial_x \varepsilon \partial_x \Phi_1$$

$$+ \int (1 - \lambda \partial_x^2) \partial_t \varepsilon \Phi_2 - \lambda \int \partial_x \partial_t \varepsilon \partial_x \Phi_2$$

$$+ \frac{1}{2} \int \left\{ [\partial_x^2 \varepsilon + \varepsilon^2 - \frac{2}{3} ((\varepsilon + V)^3 - V^3 - 3V^2 \varepsilon)] \partial_t \Phi_1 + \left[ \lambda (\partial_x^2 \varepsilon + \varepsilon^2) \partial_t \Phi_2 \right] \right\}$$

$$- \int \partial_t V ((\varepsilon + V)^2 - V^2 - 2V \varepsilon) \Phi_1 = G_1 + G_2 + G_3 + G_4 + G_5 + G_6.$$ 

Let $z_3$ be such that $(1 - \lambda \partial_x^2)z_3 = \varepsilon - \partial_x^2 \varepsilon$ and $z_4$ be such that $(1 - \lambda \partial_x^2)z_4 = -2V \varepsilon - \varepsilon^2$, so that by (3.4)

$$\partial_t \varepsilon = \partial_x z_3 + \partial_x z_4 - (1 - \lambda \partial_x^2)^{-1} E - \sum_{j=1,2} \left[ (\tilde{\nu}_j - M_j) \frac{\partial V}{\partial \mu_j} - (\tilde{\nu}_j - \tilde{\nu}_j - N_j) \frac{\partial V}{\partial y_j} \right]. \quad (B.17)$$

Then,

$$G_1 = \int \partial_x (z_3 + z_4)(z_3 + z_4 - \lambda \partial_x^2 (z_3 + z_4)) \Phi_1$$

$$- \int (1 - \lambda \partial_x^2)^{-1} E (-\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^2 - V^2)) \Phi_1$$

$$- \sum_{j=1,2} (\tilde{\nu}_j - M_j) \int \frac{\partial V}{\partial \mu_j} (-\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^2 - V^2)) \Phi_1$$

$$+ \sum_{j=1,2} (\tilde{\nu}_j - \tilde{\nu}_j - N_j) \int \frac{\partial V}{\partial y_j} (-\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^2 - V^2)) \Phi_1$$

$$= G_{1,1} + G_{1,2} + G_{1,3} + G_{1,4}.$$
\[ G_2 = -\int \partial_x(z_3 + z_4)\partial_x\epsilon \partial_x \Phi_1 + \int (1 - \lambda \partial_x^2)^{-1} E \partial_x \epsilon \partial_x \Phi_1 \]
\[ + \sum_{j=1,2} (\mu_j - M_j) \int \frac{\partial V}{\partial \mu_j} \partial_x \epsilon \partial_x \Phi_1 - \sum_{j=1,2} (\mu_j - y_j - N_j) \int \frac{\partial V}{\partial y_j} \partial_x \epsilon \partial_x \Phi_1 \]
\[ = G_{2,1} + G_{2,2} + G_{2,3} + G_{2,4}. \]

\[ G_3 = \int \partial_x(-\partial_x^2 \epsilon + \epsilon - ((V + \epsilon)^2 - V^2))\epsilon \Phi_2 - \int E \epsilon \Phi_2 \]
\[ - \sum_{j=1,2} (\mu_j - M_j) \int (1 - \lambda \partial_x^2)^{-1} \partial_x E \epsilon \partial_x \Phi_2 \]
\[ = G_{3,1} + G_{3,2} + G_{3,3} + G_{3,4}. \]

\[ G_4 = -\lambda \int \partial_x^2(z_3 + z_4)\epsilon \partial_x \Phi_2 + \lambda \int (1 - \lambda \partial_x^2)^{-1} \partial_x E \epsilon \partial_x \Phi_2 \]
\[ + \lambda \sum_{j=1,2} (\mu_j - M_j) \int \partial_x \frac{\partial V}{\partial \mu_j} \epsilon \partial_x \Phi_2 - \lambda \sum_{j=1,2} (\mu_j - y_j - N_j) \int \partial_x \frac{\partial V}{\partial y_j} \epsilon \partial_x \Phi_2 \]
\[ = G_{4,1} + G_{4,2} + G_{4,3} + G_{4,4}. \]

Note that the terms \( G_{1,2}, G_{2,2}, G_{3,2} \) and \( G_{4,2} \) are readily controllable by \( C \|E\|_L^2 \|\epsilon\|_L^2 \).

Now, we focus on \( G_{1,1} + G_{2,1} + G_{3,1} + G_{4,1} \). We denote by \( G_7 \) the quadratic parts of \( G_{1,1} + G_{2,1} + G_{3,1} + G_{4,1} \), i.e. the terms coming from the linear part of the equation. We have

\[ G_7 = \int \partial_x z_3(z_3 - \lambda \partial_x^2 z_3)\Phi_1 - \int \partial_x z_3 \partial_x \epsilon \partial_x \Phi_1 \]
\[ + \int \partial_x(-\partial_x^2 \epsilon + \epsilon)\epsilon \Phi_2 - \lambda \int \partial_x^2 z_3 \epsilon \partial_x \Phi_2 \]

Then, using (3.7), (3.16)

\[ G_7 = \int (z_3 - \lambda \partial_x^2 z_3)\partial_x z_3 \Phi_1 + (1 - \frac{\lambda}{2}) \int z_3 \partial_x^2 \epsilon \partial_x \Phi_1 \]
\[ + \lambda \int \partial_x^2 z_3 \epsilon \partial_x \left( \frac{1}{2} \Phi_1 - \Phi_2 \right) + (1 - \frac{\lambda}{2}) \int z_3 \partial_x \epsilon \partial_x^2 \Phi_1 + \frac{\lambda}{2} \int \partial_x z_3 \epsilon \partial_x^2 \Phi_1 \]
\[ + \int \partial_x \left( -\partial_x^2 \epsilon + \epsilon \right) \epsilon \Phi_2 \]
\[ \leq -\frac{1}{2} \int z_3 \partial_x \Phi_1 + \frac{\lambda}{2} \int (\partial_x z_3)^2 \partial_x \Phi_1 \]
\[ + (1 - \frac{\lambda}{2}) \int z_3 (\epsilon - z_3 + \lambda \partial_x^2 z_3) \partial_x \Phi_1 + 2 \int \epsilon (\epsilon - z_3 - \partial_x^2 \epsilon) \partial_x \Phi_2 \]
\[ - \frac{3}{2} \int (\partial_x \epsilon)^2 \partial_x \Phi_2 + \frac{1}{2} \int \epsilon^2 \partial_x^3 \Phi_2 - \frac{1}{2} \int \epsilon^2 \partial_x \Phi_2 + C \rho \int ((\partial_x \epsilon)^2 + \epsilon^2 + (\partial_x z_3)^2 + z_3^2) \partial_x \Phi_1 \]
Integrating by parts, using \( \text{[3.7], (B.16)} \) and then choosing \( \rho \) small enough, we find
\[
G_7 \leq \left(-\frac{3}{2} + \frac{\lambda}{2}\right) \int z_3^2 \partial_x \Phi_1 + \left(-\frac{\lambda}{2} + \frac{\lambda^2}{2}\right) \int (\partial_x z_3)^2 \partial_x \Phi_1 - \frac{1}{4} \int (\partial_x \varepsilon)^2 \partial_x \Phi_1
\]
\[
- \frac{3}{4} \int \varepsilon^2 \partial_x \Phi_1 + \left(2 - \frac{\lambda}{2}\right) \int z_3 \varepsilon \partial_x \Phi_1 + C \rho \int ((\partial_x \varepsilon)^2 + \varepsilon^2 + (\partial_x z_3)^2 + z_3^2) \partial_x \Phi_1
\]
\[
\leq -\frac{1}{12} \int \varepsilon^2 \partial_x \Phi_1 - \frac{1}{8} \int (\partial_x \varepsilon)^2 \partial_x \Phi_1.
\]

The nonlinear terms in \( G_{1,1} + G_{2,1} + G_{3,1} + G_{4,1} \) which contain \( \partial_x \Phi_1 \) or \( \partial_x \Phi_2 \) are treated by perturbation (for \( \varepsilon \) small and \( y \) large) exactly as in the previous case, using the signed rest \(-\frac{1}{12} \int \varepsilon^2 \partial_x \Phi_1 - \frac{1}{8} \int (\partial_x \varepsilon)^2 \partial_x \Phi_1 \) obtained above.

In \( G_{3,1} \), we are left with one cubic term \(-\int \varepsilon^2 \partial_x \Phi_2 \), which cannot be controlled by \( \Theta \), nor by the rest term above, since \( \Phi_2 \) does not appear with a derivative. Thus, computing the main order of this term, and estimating the rest by \( \Theta \), we obtain
\[
G_{1,1} + G_{2,1} + G_{3,1} + G_{4,1} \leq -\sum_{j=1,2} \frac{\mu_j(t)}{(1 + \mu_1(t))^2} \int \varepsilon^2 \partial_x \tilde{R}_j + C \Theta. \tag{B.18}
\]

After some computations, similarly as before, we obtain
\[
|G_{2,3}| + |G_{2,4}| + |G_{3,3}| + |G_{3,4}| + |G_{4,3}| + |G_{4,4}| \leq C \Theta,
\]
\[
G_{1,3} + G_{1,4} = \sum_{j=1,2} (\hat{\mu}_j - \mathcal{M}_j) \nu_j \int \Lambda \tilde{R}_j \varepsilon^2 + \sum_{j=1,2} (\mu_j - \hat{y}_j - N_j) \nu_j \int \partial_x \tilde{R}_j \varepsilon^2 + O(\Theta). \tag{B.19}
\]

The term \( G_5 \) is treated exactly as the term \( F_3 \) so that \( |G_5| \leq C \Theta \). Finally, using \( \text{[B.8]} \), the term \( G_6 \) writes
\[
G_6 = -\int \partial_t V \varepsilon^2 \Phi_1 = \sum_{j=1,2} \frac{\mu_j(t)}{(1 + \mu_1(t))^2} \int \varepsilon^2 \partial_x \tilde{R}_j - \sum_{j=1,2} (\hat{\mu}_j - \mathcal{M}_j) \frac{1}{(1 + \mu_1(t))^2} \int \Lambda \tilde{R}_j \varepsilon^2
\]
\[
- \sum_{j=1,2} (\mu_j - \hat{y}_j - N_j) \frac{1}{(1 + \mu_1(t))^2} \int \partial_x \tilde{R}_j \varepsilon^2 + O(\Theta).
\]
\[
\tag{B.20}
\]

In conclusion, combining \( \text{[B.18], (B.19)} \) and \( \text{[B.20]} \), we finish the proof of Proposition \( \text{3.1} \).

**B.3 Proof of Proposition \( \text{3.2} \)**

By classical arguments (based on the implicit function theorem – see Lemma \( \text{3.1} \) Lemma \( \text{B.1} \) and \( \text{[33]} \)), there exists \( \omega_1 > 0, \bar{y}_0 > 1 \) such that if
\[
\inf_{x_1 - x_2 > \bar{y}_0} \|u(t) - Q_{-\mu_0}(x - x_1) - Q_{\mu_0}(x - x_2)\|_{H^1} \leq \omega_1 \tag{B.21}
\]
then \( u(t) \) can be decomposed as follows
\[
u(t, x) = \overline{f}_1(t, x) + \overline{f}_2(t, x) + \overline{r}(t, x), \tag{B.22}
\]
where
\[
\overline{f}_1(t, x) = Q_{-\mu_0}(x - y_1(t)), \quad \overline{f}_2(t, x) = Q_{\mu_0}(x - y_2(t)) \tag{B.23}
\]
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and $y_j(t)$ are $C^1$ functions uniquely chosen so that

$$\int \bar{\tau}(t, x) \partial_x \bar{R}_j(t, x) dx = 0.$$  \hfill (B.24)

Moreover, $\|\bar{\tau}\|_{H^1} \leq C \omega_1$.

Let $\bar{y}(t) = y_1(t) - y_2(t)$. By (A.5), the functions $\bar{\tau}(t, x)$ and $y_j(t)$ satisfy the following equation

$$\begin{align*}
(1 - \lambda \partial_x^2) \partial_t \bar{\tau} + \partial_x (\partial_x^2 \bar{\tau} - \bar{\tau} + 2(\bar{R}_1 + \bar{R}_2)\bar{\tau} + \bar{\tau}^2) &= -2 \partial_x (\bar{R}_1 \bar{R}_2) \\
+ (-\mu_0 - \bar{y}_1)(1 - \lambda \partial_x^2) \partial_x \bar{R}_1 + (\mu_0 - \bar{y}_2)(1 - \lambda \partial_x^2) \partial_x \bar{R}_2,
\end{align*}$$  \hfill (B.25)

and as in Lemma 3.1, we obtain

$$\left| (-1)^j \mu_0 - \bar{y}_j \right| \leq C\|\bar{\tau}\|_{H^1} + e^{-\frac{3}{2}y},$$  \hfill (B.26)

where we have used (see (A.1) and (A.2))

$$\int \bar{R}_2(t) \bar{R}_1(t) \leq C e^{-\frac{3}{2}y(t)}. \hfill (B.27)$$

**Proof of (3.17).** For $C_s > 2$ to be chosen later, assume (3.16) and define

$$T^* = \sup \left\{ t_0 < T < - (\mu_0)^{-1} \log \mu_0 \mid \text{such that, for all } t_0 < t < T, \, u(t) \text{ satisfies (B.21)}, \right.$$  

$$\left. \|\bar{\tau}(t)\|_{H^1} \leq C_s \omega_0 + C_s e^{-4 \mu_0 |t|} \right\}.$$  

Note that for $C_s$ large enough, $T^*$ is well-defined by (3.16) and by continuity of $u(t)$ in $H^1$.

We prove that $T^* < - (\mu_0)^{-1} \log \mu_0$, for $C_s$ large enough, assuming by contradiction that $T^* < - (\mu_0)^{-1} \log \mu_0$ and working on the interval $[t_0, T^*]$.

First, we claim the following control of the scaling directions of $\bar{\tau}(t)$.

**Claim B.5.** For all $t \in [t_0, T^*]$,

$$\left| \int \bar{\tau}(t, x) \left( 1 - \lambda \partial_x^2 \right) \bar{R}_j(t, x) dx \right| \leq C \left( \mu_0^{-1} \sup_{[t_0, t]} \|\bar{\tau}\|_{H^1}^2 + \sup_{[t_0, t]} e^{-\frac{3}{2}y} + \mu_0 \omega \right).$$  \hfill (B.28)

**Proof of Claim B.5.** Indeed, (B.5) is obtained by expanding $u(t) = \bar{R}_1(t) + \bar{R}_2(t) + \bar{\tau}(t)$ in the conservation laws (1.2) and (1.3) (i.e. $M(u(t_0)) = M(u(t))$ and $\mathcal{E}(u(t_0)) = \mathcal{E}(u(t))$) using (A.5), (B.27) and (3.16):

$$\begin{align*}
M(u(t_0)) &= M(\bar{R}_1(t_0)) + M(\bar{R}_2(t_0)) + 2 \int \bar{\tau}(t_0) (1 - \lambda \partial_x^2) \bar{R}_1(t_0) \\
+ 2 \int \bar{\tau}(t_0) (1 - \lambda \partial_x^2) \bar{R}_2(t_0) + O(e^{-\frac{3}{2}y(t_0)}) + O(\|\bar{\tau}(t_0)\|_{H^1}^2) \\
&= M(u(t)) = M(\bar{R}_1(t)) + M(\bar{R}_2(t)) + 2 \int \bar{\tau}(t) (1 - \lambda \partial_x^2) \bar{R}_1(t) \\
+ 2 \int \bar{\tau}(t) (1 - \lambda \partial_x^2) \bar{R}_2(t) + O(e^{-\frac{3}{2}y(t)}) + O(\|\bar{\tau}(t)\|_{H^1}^2);
\end{align*}$$

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\[ \mathcal{E}(u(t_0)) = \mathcal{E}(\mathcal{R}_1(t_0)) + \mathcal{E}(\mathcal{R}_2(t_0)) + 2\mu_0 \int \bar{\tau}(t_0)(1 - \lambda \partial_x^2)\mathcal{R}_1(t_0) \]
\[ - 2\mu_0 \int \bar{\tau}(t_0)(1 - \lambda \partial_x^2)\mathcal{R}_2(t_0) + O(e^{-\frac{3}{4}y(t_0)}) + O(\|\bar{\tau}(t_0)\|_{H^1}^2) \]
\[ = \mathcal{E}(u(t)) = \mathcal{E}(\mathcal{R}_1(t)) + \mathcal{E}(\mathcal{R}_2(t)) + 2\mu_0 \int \bar{\tau}(t)(1 - \lambda \partial_x^2)\mathcal{R}_1(t) \]
\[ - 2\mu_0 \int \bar{\tau}(t)(1 - \lambda \partial_x^2)\mathcal{R}_2(t) + O(e^{-\frac{3}{4}y(t)}) + O(\|\bar{\tau}(t)\|_{H^1}^2) ; \]

Using \( M(\mathcal{R}_j(t_0)) = M(\mathcal{R}_j(t)) \), \( \mathcal{E}(\mathcal{R}_j(t_0)) = \mathcal{E}(\mathcal{R}_j(t)) \), and \( \|\bar{\tau}(t_0)\|_{H^1} \leq C\mu_0\omega \), we find (B.28). \( \square \)

Now, we use a functional \( \mathcal{F} \) similar to \( \mathcal{F}_- \). Let
\[ \mathcal{F}(t) = \int \left[ (\partial_x \bar{\tau})^2 + \bar{\tau}^2 - \frac{2}{3} ((\bar{\tau} + \mathcal{R}_1 + \mathcal{R}_2)^3 - (\mathcal{R}_1 + \mathcal{R}_2)^3 - 3(\mathcal{R}_1 + \mathcal{R}_2)^2 \bar{\tau}) \right] \mathcal{F}_1(x) \]
\[ + \int [\lambda(\partial_x \bar{\tau})^2 + \bar{\tau}^2] \mathcal{F}_2(x), \]
where, \( \varphi \) being defined in (3.7),
\[ \mathcal{F}_1(x) = \frac{\varphi(x)}{(1 - \mu_0)^2} \frac{1 - \varphi(x)}{(1 + \mu_0)^2}, \]
\[ \mathcal{F}_2(x) = \frac{-\mu_0 \varphi(x)}{(1 - \mu_0)^2} + \frac{\mu_0(1 - \varphi(x))}{(1 + \mu_0)^2}. \]

We perform similar (and simpler) computations as the ones of Propositions 3.1 and 3.1 (scaling parameters and \( \Phi_j \) are time independent here). We obtain, for some \( \rho > 0 \) small enough
\[ \frac{d}{dt} \mathcal{F}(t) \leq C\|\bar{\tau}\|_{L^2} \left( e^{-2\rho y}\|\bar{\tau}\|_{L^2} + e^{-\frac{3}{4}y} \right). \]

From this point, the end of the proof is the same as the one of Proposition 3.2 in [31] and it is omitted.

**Proof of (3.18).** It is completely similar.

**Proof of (3.19).** The asymptotic stability is a consequence of results in [27], [37], [9], [11] and [26].

### B.4 Proof of Proposition 4.2

For \( X_1, X_2 \in \mathbb{R} \), let \( U_{X_1,X_2} \) be the unique solution of (BMM) such that
\[ \lim_{t \to -\infty} \|U_{X_1,X_2}(t) - Q_{-\mu_0}(x + \mu_0 t - X_1) - Q_{\mu_0}(x - \mu_0 t - X_2)\|_{H^1} = 0. \] 

Then, for any \( Y_1, Y_2 \in \mathbb{R} \), one has
\[ U_{Y_1,Y_2}(t,x) = U_{X_1,X_2}(t - T_0, x - X_0) \]
where \( X_0 = \frac{1}{2}(Y_1 - X_1) + \frac{1}{2}(Y_2 - X_2), T_0 = \frac{1}{2\mu_0} (X_2 - Y_2 - (X_1 - Y_1)) \).

In particular, the map \( (X_1, X_2) \mapsto U_{X_1,X_2} \) is smooth and
\[ \frac{\partial U_{X_1,X_2}}{\partial X_j} = (-1)^j \frac{1}{2\mu_0} \frac{\partial U_{X_1,X_2}}{\partial t} - \frac{1}{2} \frac{\partial U_{X_1,X_2}}{\partial x}. \]
We assume $T_1 \in (-T, T)$, the case $|T_1| > T$ being similar, and we prove the stability result for $t \in (-\infty, T_1]$, the stability proof for $t > T_1$ following from similar arguments.

For $C_1 > 2$ to be chosen, we define

$$T^* = \inf\{t \leq T_1 ; \text{such that for all } t \leq t' \leq T_1, \inf_{X_1, X_2} \|u(t') - U_{X_1, X_2}(t')\|_{H^1} \leq C_1\omega \mu_0\}. \tag{B.34}$$

By the assumption on $u(T_1)$, $C_1 > 2$ and continuity of $u(t)$ in $H^1$, $T^* < T_1$ is well-defined. We prove that $T^* = -\infty$ by using a contradiction argument: we assume $T^* > -\infty$ and we obtain a contradiction by strictly improving the estimate of $\inf_{X_1, X_2} \|u - U_{X_1, X_2}\|_{H^1}$ on $t \in [T^*, T_1]$. By Proposition 4.1, $U_{X_1, X_2}$ is close for all time to the sum of two distant solitons. Thus, on $[T^*, T_1]$, for $\omega$ small enough, we can use modulation theory (as in Lemma 3.1) to obtain $(X_1(t), X_2(t)) \in \mathbb{R}^2$, such that

$$u(t, x) = \hat{U}(t, x) + \bar{\varepsilon}(t, x), \quad \hat{U}(t, x) = U_{X_1(t), X_2(t)}(t, x),$$

$$\int \bar{\varepsilon}(t, x)(1 - \lambda \partial_x^2) \frac{\partial \hat{U}}{\partial X_j} = 0 \quad (j = 1, 2). \tag{B.35}$$

Moreover, $\bar{\varepsilon}$ satisfies $\|\bar{\varepsilon}(t)\|_{H^1} \leq CC_1\omega \mu_0$, and

$$(1 - \lambda \partial_x^2) \partial_x \bar{\varepsilon} + \partial_x (\partial_x^2 \bar{\varepsilon} - \bar{\varepsilon} + 2\hat{U} \bar{\varepsilon} + \bar{\varepsilon}^2) + \sum_{j=1,2} \hat{X}_j (1 - \lambda \partial_x^2) \frac{\partial \hat{U}}{\partial X_j} = 0, \tag{B.36}$$

and

$$|\hat{X}_1| + |\hat{X}_2| \leq C\|\bar{\varepsilon}\|_{H^1}. \tag{B.37}$$

Note that there exists $\tilde{\mu}_j(t)$ and $\tilde{\gamma}_j(t)$ such that for all $t$:

$$\|\hat{U}(t) - \sum_{j=1,2} Q_{\tilde{\mu}_j(t)}(x - \tilde{\gamma}_j(t))\|_{H^1} \leq CY_0e^{-Y_0}. \tag{B.38}$$

Moreover, as in Proposition 4.1 there exists $t_0$ such that $\tilde{\mu}_1(t) > \tilde{\mu}_2(t)$ if $t > t_0$ and $\tilde{\mu}_1(t) < \tilde{\mu}_2(t)$ if $t < t_0$. We assume that $t_0 < T^*$.

To control $\bar{\varepsilon}(t)$ on $[t_0, T_1]$ (i.e. to prove that $T^* < t_0$), we use the functional

$$\hat{F}(t) = \int \left[ (\partial_x \bar{\varepsilon})^2 + \bar{\varepsilon}^2 - \frac{2}{3}((\bar{\varepsilon} + \hat{U})^3 - \hat{U}^3 - 3\hat{U}^2\bar{\varepsilon}) \right] \tilde{\Phi}_1 + \int \left[ \lambda (\partial_x^2 \bar{\varepsilon})^2 + \bar{\varepsilon}^2 \right] \tilde{\Phi}_2,$$

for $\tilde{\Phi}_j$ defined from $\tilde{\mu}_j(t)$ as in $(3.10)$.

We follow the same computations as in the proof of Propositions 3.1 and 3.2, except that here there is no error term $E(t, x)$, and no scaling parameter; thus we get

Claim B.6. For all $t \in [\max(T^*, t_0), T_1]$,

$$\frac{d}{dt} \hat{F}(t) \geq -C\|\bar{\varepsilon}(t)\|_{L^2}^2e^{-(\frac{1}{2}+\rho)Y_0} \quad (C > 0), \tag{B.39}$$

$$\hat{F}(t) \geq \lambda\|\bar{\varepsilon}(t)\|_{L^2}^2 + C \sum_{j=1,2} \left( \int \bar{\varepsilon}(t)(1 - \lambda \partial_x^2)Q_{\tilde{\mu}_j(t)}(x - \tilde{\gamma}_j(t)) \right)^2 \quad (\lambda > 0). \tag{B.40}$$

$$\left| \int \bar{\varepsilon}(t)(1 - \lambda \partial_x^2)Q_{\tilde{\mu}_j(t)}(x - \tilde{\gamma}_j(t)) \right| \leq C\omega \mu_0. \tag{B.41}$$
We omit the proof of Claim B.6 since it is the same as the proof of Claim B.5 in [31]. From (B.39), (B.40) and (B.41), and a continuity argument we deduce that $T^* < t_0$. Note that the estimates on $|\dot{X}|$ and $|\dot{T}|$ come from (B.37) and (B.32).

Finally, to treat the case $T^* < t_0$, i.e. to prove that $T^* = -\infty$, one uses another functional, similar to $F_+$. This completes the proof of Proposition 4.2.

C Appendix

We write the transformation from equation (1.1) to equation (BBM). Note that solitons for equation (1.1) are of the form $(c>1)$

$$R_{c,x_0}(t,x) = Q_c(x-ct-x_0) \quad \text{where} \quad Q_c(x) = (c-1)Q\left(\sqrt{\frac{c-1}{c}}x\right). \quad \text{(C.1)}$$

For $1 < c_1 < c_2$ close, let $U(t,x)$ be the unique solution of (1.1) (see [11]) such that

$$\lim_{t \to -\infty} \|U(t) - Q_{c_1}(., -c_1t - x_1) - Q_{c_2}(., -c_2t - x_2)\|_{H^1} = 0. \quad \text{(C.2)}$$

Let

$$\bar{c} = \frac{1}{2}(c_1 + c_2), \quad \lambda = \frac{c - 1}{\bar{c}}, \quad \mu_0 = \frac{c_2 - \bar{c}}{\bar{c} - 1} = \frac{\bar{c} - c_1}{\bar{c} - 1}, \quad \text{(C.3)}$$

$$x' = \lambda^{1/2} \left(x - \frac{t}{1 - \lambda}\right), \quad t' = \frac{\lambda^{3/2}t}{1 - \lambda}, \quad U(t', x') = \frac{1 - \lambda}{\lambda} U(t, x). \quad \text{(C.4)}$$

Then, $U(t,x)$ satisfies (BBM) and it is the unique solution of (BBM) such that

$$\lim_{t \to -\infty} \|U(t) - Q_{\mu_0}(., \mu_0t - y_1) - Q_{\mu_0}(., -\mu_0t - y_2)\|_{H^1} = 0,$$

where

$$y_1 = x_1\sqrt{\lambda}, \quad y_2 = x_2\sqrt{\lambda}.$$ 

Indeed, for $c > 1$, $x_0 \in \mathbb{R}$, a soliton $R_{c,x_0}(t,x) = Q_c(x-ct-x_0)$ of (1.1) transforms by (C.4) into

$$R_{c,y}(t', x') = \left(\frac{1 - \lambda}{\lambda}\right) (c-1)Q\left(\sqrt{\frac{c-1}{c}}\left(\lambda^{-\frac{1}{2}}x' + \lambda^{-\frac{3}{2}}[1 - c(1 - \lambda)] t' - x_0\right)\right). \quad \text{(C.5)}$$

Setting

$$\mu = \mu(c) = \frac{1}{\lambda}(c(1 - \lambda) - 1) = \frac{c - \bar{c}}{\bar{c} - 1}, \quad y_0 = \lambda^{\frac{3}{2}}x_0, \quad \text{(C.6)}$$

one checks

$$R_{c,y}(t', x') = (1 + \mu)Q\left(\sqrt{\frac{1 + \mu}{1 + \lambda\mu}}(x' - \mu t' - y_0)\right) = Q_\mu(x' - \mu t' - y_0).$$
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