Dispersion relations and knot theory

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Abstract

I show that the crossing symmetric dispersion relation (CSDR) for 2-2 scattering leads to a fascinating connection with knot theory. In particular, the dispersive kernel can be identified naturally in terms of the generating function for the Alexander polynomials corresponding to the torus knot \((2, 2n + 1)\) arising in knot theory. In the low energy expansion, the difference between the \((n + 1)\)-th and \(n\)-th derivatives of the scattering amplitude with respect to the crossing symmetric variable can be bounded in terms of the torus \((2, 2n + 1)\)-knot invariants and the resulting bounding curve in the space of allowed S-matrices can be determined analytically in terms of the \((2, 2n + 1)\)-torus Alexander polynomial. The agreement with the pion S-matrix bootstrap is impressive. The global bounds are derived using Geometric Function Theory (GFT) techniques and shown to be identical. I discuss tree level type II string theory using the CSDR-knot connection. Finally, I correlate the \(q\)-deformed harmonic oscillator with the CSDR-knot picture.

Dedicated to the Ukranian scientists, some of whose work has been used in this paper.

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1 Introduction

In the last few years, it has been realized that combining the power of dispersion relations with crossing symmetry leads to two-sided bounds on the low energy Taylor expansion coefficients (Wilson coefficients) of 2-2 scattering amplitudes [1, 2, 3, 4, 5]. There are two complementary dispersion relations in the literature that have been used in these studies—(a) The fixed-\(t\) dispersion relation where crossing symmetry is not manifest and (b) The crossing symmetric dispersion relation (CSDR) where crossing symmetry is manifest but locality is not [6, 7, 8]. The CSDR has led to a surprising connection with an area of mathematics called Geometric Function Theory (GFT) [9, 10, 11]. One of the famous theorems in GFT is the de Branges theorem (1985) [12] which proves the famous Bieberbach conjecture (1916) concerning the Taylor expansion coefficients of univalent functions [13]. A simpler class of functions which are relevant for studying 2-2 scattering is the typically real functions introduced by Rogosinski [14] in 1932. For this class, the Bieberbach conjecture was proved in the same paper.

In this paper, we will find a very interesting connection between the CSDR and knot theory. Knot theory is a fascinating area of mathematics dealing with the properties of knots. We will review some relevant material below. It has been known for a while that there are profound and surprising connections between knot theory and areas of physics like quantum field theory and statistical physics [15, 16, 17, 18]. In the context of perturbative Feynman diagrams in QFT, a connection with knot theory has been suggested in the literature and reviewed in [19]. We will consider nonperturbative representations of 2-2 scattering via the CSDR and explore connections with knot theory. One of the main motivations is: Can a boundary curve in the space of allowed S-matrices be derived analytically? The answer will turn out to be yes and the curve in question will be shown to be related to knot polynomials.
Knot polynomials aid in characterizing knots. The oldest of these polynomials is the Alexander polynomial introduced by J. W. Alexander in 1923 [20]. This will be the main character in the present paper. I will show how this polynomial naturally appears in the CSDR. One of the simplest quantum field theories we study in an introductory course in QFT is the $\phi^3$ theory and its cousins. In the crossing symmetric variable, the Alexander polynomials for a certain class of knots, called the torus knots, make a natural appearance in the discussion of 2-2 scattering in such theories. The kernel in the CSDR resembles a linear combination of $\phi^3$ and $\phi^4$ theories with the mass parameter integrated over in a certain way. The CSDR gives a non-perturbative representation of 2-2 scattering describing, for instance, pion scattering. Some of the two-sided bounds, alluded to above, can be recast in terms of the knot invariants involving the Alexander polynomial of a $(2, 2n + 1)$ torus knot evaluated at a special value for the polynomial variable. As we will show using the CSDR, the derivatives of the scattering amplitude with respect to the crossing symmetric variable can be thought of as an average over the Alexander polynomials. Our main findings are the analytic results in eq.(3.12) and eq.(3.13) which give bounds on the derivatives of the amplitude in terms of the Alexander polynomials. In particular, eq.(3.12) gives the bounding curve in the space of S-matrices which is determined analytically in terms of the $(2, 2n + 1)$-torus Alexander polynomial while eq.(3.13) gives the global bound. The global bound will also be re-derived using the Bieberbach conjecture and shown to be identical. We use the pion S-matrix bootstrap to test these bounds and find that the agreement is quite impressive. We also discuss the low energy expansion of the 2-2 dilaton scattering at tree level in type II string theory. In particular, following [31, 32], we examine a correspondence between transcendentality and the knot polynomials. We also show that the low energy expansion is approximated remarkably well by an expression involving sum of the Alexander knot polynomials in eq.(5.5).

There are known relations between Chebyshev polynomials and the $(2, p), (3, p), (4, p)$ torus knots [21, 22, 23]. We find that linear combinations of various derivatives of the scattering amplitude can be related to the relevant Alexander polynomials and analogous bounding curves in the space of S-matrices can be determined. We will give an example of these more general bounds in eq.(3.18).

Finally, using the $q$-deformed oscillator that features in the discussion of quantum group $SU_q(2)$, we will map the amplitude to the expectation value of the $q$-deformed number operator in eq.(6.6) and relate the CSDR to an energy, averaged w.r.t. the knot parameter in eq.(6.10). In the next section we begin a review of some relevant background pertaining to Alexander polynomials.

NB: $t$ in this paper represents the knot polynomial parameter and not the Mandelstam invariant!

2 Alexander polynomials in knot theory

2.1 Important background

In this section, I will briefly review some background material on Alexander polynomials for torus knots [24]. Let me begin with a lightning review of knot polynomials in general. Knot polynomials allow us to characterise knots. Two knots are equivalent if they can be transformed into each other using a finite number of Reidemeister moves [25]. If the knot polynomials of two knots are different, then the knots are not equivalent. The converse is not true; two different knots can have the same knot polynomial. Several
famous knot polynomials are known—these include Alexander, Conway-Alexander, Jones, Kauffman and HOMFLY-PT. The oldest and perhaps the simplest of these is the Alexander polynomials. We will see that in the CSDR, Alexander polynomials make a natural appearance, so our focus will be on this. Torus knots are labeled by two co-prime integers \((p, q)\). Here the curve depicting the knot on the torus traverses \(p\) times along longitude and \(q\) times along meridian. For \((p, q)\) co-prime, the torus knot is prime; in other words, it cannot be decomposed into smaller knots, much like how we define prime numbers. In the CSDR, we will see that the torus knot \((2, 2n + 1)\) where \(n\) is a positive integer, makes an appearance.

For a \((p, q)\)-torus knot, the Alexander polynomials are given by

\[
A^{(p,q)}(t) = t^{-g} \frac{(tpq - 1)(t - 1)}{(tp - 1)(tq - 1)},
\]

where the genus-\(g\) of the knot is given by

\[
g = \frac{1}{2}(p - 1)(q - 1).
\]

In the case of interest \(p = 2, q = 2n + 1\) so that the genus is \(n\) and

\[
A^{(2,2n+1)}(t) = t^{-n} \frac{t^{2n+1} + 1}{t + 1}.
\]

Further the crossing number of the \((2, 2n + 1)\) knot is \(c = 2n + 1\) and is also a topological invariant. In figure. \(\text{Figure 1}\) we show the knots corresponding to \(n = 1, 2, 3\).

![Some torus knots](image)

(a) Trefoil (2,3)  (b) Pentafoil (2,5)  (c) Heptafoil (2,7)

Figure 1: Some torus knots. Figures generated in mathematica.

The Alexander polynomials for the \(n = 1, 2, 3\) cases are shown below:

\[
A^{(2,3)}(t) = t + \frac{1}{t} - 1,
\]

\[
A^{(2,5)}(t) = t^2 + \frac{1}{t^2} - t - \frac{1}{t} + 1,
\]

\[
A^{(2,7)}(t) = t^3 + \frac{1}{t^3} - t^2 - \frac{1}{t^2} + t + \frac{1}{t} - 1.
\]

The Alexander, Conway, Jones polynomials for the pentafoil knot are the same as the knot\(^1\) \(10_{132}\) which

\(^1\)The nomenclature means that the (prime) knot has 10 crossings and in some standard list is the 132nd knot with 10

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is distinguished by the Kauffman polynomial. The Alexander polynomials for any knot satisfy two important properties that we note:

\[ A(1) = 1, \]
\[ |A(-1)| = \text{invariant}. \]  

(2.7) \hspace{1cm} (2.8)

The quantity \(|A(-1)|\) is called the knot determinant and is a special knot invariant. For the torus knot \((2, 2n + 1)\) we have \(A^{(2, 2n+1)}(1) = 1\) and

\[ |A^{(2, 2n+1)}(-1)| = 2n + 1 = c, \]

(2.9)

which is the same as the crossing number and is the maximum absolute value for the Alexander polynomial in the interval \(\frac{1}{2}(t + \frac{1}{t}) \in [-1, 1]\).

2.2 Key observation

The key mathematical insight that enables us to correlate the CSDR with the Alexander polynomials is the relation of the latter with the Chebyshev polynomials of the second kind. This observation was made by Ukranian mathematicians in [21, 22]. Denote by \(U_n(x)\) the Chebyshev polynomials of the second kind such that \(U_0(x) = 1, U_1(x) = 2x, U_2(x) = 4x^2 - 1\). This is the same normalization used in Mathematica.

The generating function of these polynomials is given by

\[ \frac{1}{1 - 2xz + z^2} = \sum_{n=0}^{\infty} z^n U_n(x). \]  

(2.10)

This relation will be the key in relating the CSDR to the Alexander knot polynomials. The relation between the \(U_n(x)\) and \(A^{(2, 2n+1)}(t)\) is simply the following:

\[ A^{(2, 2n+1)}(t) = U_n(x) - U_{n-1}(x), \text{ where } 2x = t + \frac{1}{t}. \]  

(2.11)

Equivalently we can write

\[ U_n(x) = \sum_{k=0}^{n} A^{(2, 2k+1)}(t), \]  

(2.12)

with the above relation between \(t\) and \(x\). Generalizations of eq.(2.11) for the torus knots of the type \((3, p)\) can be found in [22]. We will touch upon these generalizations in sec.(3.3).
3 Torus knots in CSDR

3.1 Tree level $\phi^2\psi$ and Alexander polynomial

Let us begin by discussing 2-2 scattering of $\phi\phi \rightarrow \phi\phi$ in tree level $\phi^2\psi$ theory\footnote{We could have also considered the massive $\phi^3$ theory in which case we can just work with the shifted Mandelstam invariants $s = s_1 + 4/3m^2$ etc. so that $s_1 + s_2 + s_3 = 0$.}, one of the simplest examples one encounters in a first course in QFT. We will treat $\phi$ to be a massless scalar and $\psi$ to be massive scalar of mass $m$. The amplitude, up to an overall coupling constant, is given by

$$M(s, t) = \frac{1}{m^2 - s_1} + \frac{1}{m^2 - s_2} + \frac{1}{m^2 - s_3}, \quad (3.1)$$

subject to the constraint $s_1 + s_2 + s_3 = 0$. The present discussion will aid us in the next section, where we will move to the crossing symmetric dispersion relation (CSDR) introduced first by Auberson and Khuri \cite{Auberson:1977} and discussed more recently in \cite{Buchbinder:2009}. The key step is to use a different set of variables rather than the Mandelstam variables:

$$s_k = a \left( 1 - \frac{(z - z_k)^3}{z^3 - 1} \right). \quad (3.2)$$

Here $z_k = \exp(2\pi i (k - 1)/3)$ are the cube-roots of unity. One can check that $s_1 + s_2 + s_3 = 0$ and that

$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3}. \quad (3.3)$$

It is easy to check that\footnote{Using eq.(2.3), eq.(3.5) can be checked to be $\beta m^2 \tilde{z} \left( \tilde{z} - t \right) \left( \tilde{z} - \frac{1}{t} \right)$ as expected.}

$$M(\tilde{z}, a) - \frac{3}{m^2} = \frac{\tilde{\beta}}{m^2} \frac{\tilde{z}}{1 - 2\xi \tilde{z} + \tilde{z}^2} = \frac{\tilde{\beta}}{m^2} \sum_{n=0}^{\infty} U_n(\xi) \tilde{z}^{n+1} = \frac{\tilde{\beta}}{m^2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(2, 2k+1)(t) \tilde{z}^{n+1}, \quad (3.4)$$

$$= \frac{\tilde{\beta}}{m^2} \sum_{n=0}^{\infty} A(2, 2n+1)(t) \frac{\tilde{z}^{n+1}}{1 - \tilde{z}}, \quad (3.5)$$

where $2\xi = t + \frac{1}{t}$ with $\tilde{z} \equiv z^3$ and $\tilde{\beta}, \xi$ are given by

$$\tilde{\beta} = \frac{27a^2}{m^6} (3a - 2m^2), \quad 2\xi = 2 - 27\left( \frac{a}{m^2} \right)^2 + 27\left( \frac{a}{m^2} \right)^3. \quad (3.6)$$

Thus $(1 - \tilde{z})(M(\tilde{z}, a) - \frac{3}{m^2})$ can be thought of as the generating function of the $(2, 2n+1)$-Alexander knot polynomials. Now using eq.(2.11), we easily find

$$\frac{1}{\partial \tilde{z}} \left. \frac{\partial^{n+1} M(\tilde{z}, a)}{(n+1)!} - \frac{\partial^n M(\tilde{z}, a)}{n!} \right|_{\tilde{z}=0} = U_n(\xi) - U_{n-1}(\xi), \quad (3.7)$$

This relates the derivatives of the scattering amplitude to the Alexander polynomial evaluated at $t$, which is related to $\xi$ in terms of $a, m^2$. Now notice that $\xi = 1$ (equivalently $t = 1$) corresponds to $a/m^2 = 0, 1$
while $\xi = -1$ (equivalently $t = -1$) gives $a/m^2 = -1/3$ or $2/3$. Notice that if $-1 \leq \xi \leq 1$, then eq. (3.4) leads to the conclusion that there are no singularities inside the unit disc $|\tilde{z}| < 1$. This is relevant to make contact with GFT. Thus the combination of the derivatives of the amplitude evaluated at special values of the parameter $a$ is related to the knot determinant! Explicitly we have

$$|A^{(2,2n+1)}(-1)| = \left| \frac{1}{\partial_{\tilde{z}}M(\tilde{z}, a_0)} \left( \frac{n+1}{(n+1)!} \frac{\partial^n_M(\tilde{z}, a_0)}{n!} - \frac{n}{n+1} \frac{\partial^n_M(\tilde{z}, a_0)}{n!} \right) \right|_{\tilde{z}=0}, \ a_0 = -1/3 \text{ or } 2/3. \quad (3.8)$$

It is gratifying to note that there is a connection between the oldest knot polynomial and one of the simplest quantum field theories.

### 3.2 The CSDR and bounds in terms of knot invariants

The idea behind a CSDR is to write a dispersion relation treating $a$ as a parameter and fixed and the dispersive variable to be $z$. As reviewed in [7], a fully crossing symmetric amplitude then is a function of $a, z^3$. In [10], it was shown that the CSDR in [6, 7] can be written in the form of the Robertson representation of typically real functions in the context of GFT. This was then instrumental in giving two sided bounds on the (ratios of) Wilson coefficients of the amplitude. In general, in order to describe scattering of identical particles in a situation where in the complex-$s$ plane, there is a gap between the $s$-channel and $u$-channel cuts, and where the amplitude for large $|s|$ falls off faster than $|s|^2$, we have:

$$M(\tilde{z}, a) = \alpha_0 + \frac{2N}{\pi} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \frac{\tilde{z}}{\tilde{z}^2 - 2\xi \tilde{z} + 1} = \alpha_0 + \frac{2N}{\pi} \sum_{n=0}^{\infty} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi U_n(\xi) \tilde{z}^{n+1}, \quad (3.9)$$

where

$$N = \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \text{Im} M(\xi, s_1(x, a)) = \frac{\pi}{2} \partial_{\tilde{z}}M(\tilde{z}, a)|_{\tilde{z}=0}, \quad (3.10)$$

and

$$d\mu(\xi) = N^{-1} d\xi \text{Im} M(s_1(\xi, a), s_2(\xi, a)) \quad (3.11)$$

defines a probability measure with $\text{Im} M(s_1, s_2)$ denoting the $s$-channel discontinuity and $s_2 = -\frac{s_1}{2}(1 - (\frac{s_1 + 3a}{s_1 - a})^{1/2})$. More details can be found in [7]. Here $2\xi = 2 - 27(a/s_1)^2 + 27(a/s_1)^3$ as in eq. (3.6). We will choose the cut in the $s_1$ plane to begin at $s_1 = 8/3$ and this sets the normalization for us. Our focus will be on the interval $-8/9 \leq a \leq 16/9$ as in this range the amplitude is typically real [10]; this will enable us to compare with the Bieberbach conjecture. It is easy to show that $\xi_{\text{max}} = 1$ while $\xi_{\text{min}} = 1 - \frac{243a^2}{128} + \frac{729a^3}{192}$. When $a = -8/9, 16/9$, $\xi_{\text{min}} = -1$ else it is $>-1$. We will now derive an

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4In [26], such CSDRs have been generalized for external particles carrying spin, in the context of weakly coupled EFTs.
interesting inequality. Starting with eq. (3.9) we write using the triangle inequality

\[ \rho_n \equiv \left| \frac{1}{\partial \hat{z}} \mathcal{M}(\hat{z}, a) \right| = \left| \left( \frac{\partial^{n+1} \mathcal{M}(\hat{z}, a)}{(n+1)!} - \frac{\partial^n \mathcal{M}(\hat{z}, a)}{n!} \right) \right|_{\hat{z}=0} = \left| \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\mu(\xi) A^{(2,2n+1)}(t) \right|, \quad \text{where } 2\xi = t + \frac{1}{t} \]

\[ \leq \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\mu(\xi) \left| A^{(2,2n+1)}(t) \right| \leq \sup_{\xi_{\text{min}} \leq \xi \leq \xi_{\text{max}}} \left| A^{(2,2n+1)}(t) \right| \]

\[ \leq \left| A^{(2,2n+1)}(-1) \right|. \quad (3.12) \]

The last two lines give the upper bound on \( \rho_n \) in terms of knot invariants. In the last line we have used that the knot determinant maximizes \( |A^{(2,2n+1)}(t)| \) for any \( a \in [-8/9, 16/9] \). Thus the combination of the derivatives of the amplitude are bounded by the knot determinant (or equivalently the crossing number)! The equality occurs for theories like the tree-level \( \phi^3 \) theory. Near \( a \sim 0, \xi \sim 1 \) and since

\[ \left| A^{(2,2n+1)}(1) \right| = 1, \]

we find that \( \rho_n \sim 1 \) near the origin. Plots of the maximum \( \rho_1 \) and \( \rho_2 \) vs \( a \) for the pion bootstrap are shown in fig. (2). The data was obtained from the S-matrix bootstrap for pions in [27] following [28]. Since tree-level \( \phi^3 \) theory saturates the bound in eq. (3.13), this is as tight as possible; imposing locality constraints is not going to improve it. The solid blue lines in the plots are \( \sup_{\xi_{\text{min}} \leq \xi \leq \xi_{\text{max}}} \left| A^{(2,2n+1)}(t) \right| \) which can be determined analytically and defines the bounding curve. The agreement with the S-matrix data is remarkable.

**Rederiving the inequality in eq. (3.13) using GFT**

We can easily rederive the last inequality in eq. (3.13) using the Bieberbach-Rogosinski bounds for a typically real function\(^7\) which is regular inside the unit disc. Writing \( \mathcal{M}(\hat{z}, a) \) as

\[ \mathcal{M}(\hat{z}, a) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \hat{z}^n, \quad (3.14) \]

we know that it is a typically real function inside the unit disc \( |\hat{z}| < 1 \) for \( a \in [-8/9, 16/9] \) and obeys the Bieberbach-Rogosinski inequalities \(^{10}\)

\[ -\kappa_n \leq \frac{\alpha_n}{\alpha_1} \leq n, \quad n \geq 2 \]

\[ (3.15) \]

where \( \kappa_n \) is \( n \) for even \( n \) and is some number less than \( n \) for odd \( n \). The precise number is unimportant.

As an example, consider \( n = 2 \). Then the lhs of eq. (3.13) is \( |\alpha_2/\alpha_1 - 1| \). Using the lower bound on

\(^5\)The plots exhibit a curious commonality—namely that the orange, blue and green points almost line up. The S-matrices correspond to different values of Adler zeros. Further the input conditions used in [28] and those in [27] are somewhat different, the latter using scattering length constraints. In all, we consider a space of 99 S-matrices.

\(^6\)For \( a > 16/9 \), the blue line saturates to \( 2n + 1 \), while for \( a < -8/9 \) the blue line shoots off to \( \infty \). However, note that in this case, \( \text{Im} \mathcal{M} \) is not positive and hence going from the first line to the 2nd line in eq. (3.13) is not valid.

\(^7\)\( f(z) \) is typically real if \( \text{Im} f(z) \text{Im} z \geq 0 \) when \( \text{Im} z \neq 0 \). See [10] for a review of the relevant mathematics.
Figure 2: Maximum $\rho_n$ vs $a$ where $\rho_n$ is defined in eq.(3.13). The blue dots are obtained on maximizing $\rho_n$ over the S-matrices for the “upper pion river” while the orange dots are for the “lower pion river” in [27]. The green points are for the “pion lake” [28]. The red dotted line is tree level $\phi^3$ with $m=1$ while the black dotted line is one loop $\phi^4$ with $m=1$. The solid blue lines in the plots are $\sup_{\xi_{\min}\leq\xi\leq1} |A(2,2n+1)(t)|$.

$\alpha_2/\alpha_1$, which is -2, we find $|\alpha_2/\alpha_1 - 1| \leq 3$. Similarly for $n=3$ we have $|\alpha_3/\alpha_1 - \alpha_2/\alpha_1|$. Here we need the upper bound for $\alpha_3/\alpha_1$ and lower bound for $\alpha_2/\alpha_1$. Together this yields $|\alpha_3/\alpha_1 - \alpha_2/\alpha_1| \leq 5$. It is easy to check that the generalization of this argument leads to

$$\left| \frac{\alpha_n}{\alpha_1} - \frac{\alpha_{n-1}}{\alpha_1} \right| \leq 2n + 1,$$  

using eq.(3.15). The rhs is precisely the knot determinant for the torus knot $(2, 2n + 1)$. Note however, that the analysis in eq.(3.12) gives a stronger bound on $\rho_n$ in terms of the knot polynomial, which is $a$-dependent unlike the $a$-independent eq.(3.15). Writing $M(s_1, s_2) = \sum M_{pq} x^p y^q$ with $x = -(s_1 s_2 + s_1 s_3 + s_2 s_3), y = -s_1 s_2 s_3$ and defining $w_{pq} = W_{pq}/W_{10}$, we get the same two-sided bounds on $w_{01}$ as GFT [8] and somewhat stronger results for $w_{11}, w_{20}, w_{02}$ than using eq.(3.15) directly.

### 3.3 Role of other torus knots

So far, our discussion has featured the $(2, 2n + 1)$ torus knots. What is the role of the other torus knots? Let me point out here that there are known simple relations between the Chebyshev polynomials of the second kind and $(3, p), (4, p)$ torus knots [22, 23]. In principle, one can just use the orthogonality of the Chebyshev polynomials and known expressions for the $(p, q)$ Alexander polynomials to derive relations for any torus knot, but this appears to be a hard calculation to do in complete generality. It will be worthwhile to do this in the future. Let me give a specific example which serves to illustrate the potential generality of the discussion so far. It is known that the analog of eq.(2.11) for the $(3, 5)$ torus knot is

$$A^{(3,5)}(t) = U_4(x) - U_3(x) - U_2(x) + 2U_1(x) - 1.$$  

This is the same that arises from the numerical techniques of [4].
Using this it is easy to see that the analog of eq. (3.12) and eq. (3.13) will be (suppressing the arguments):

\[ \rho^{(3,5)} \equiv \left| \frac{\partial^3 M}{5!} - \frac{\partial^4 M}{4!} - \frac{\partial^5 M}{3!} + \partial^2 M - \partial M \right| \leq \sup_{\xi_{\text{min}} \leq \xi \leq \xi_{\text{max}}} A^{(3,5)}(t) \lesssim 5.91. \]  

(3.18)

Similar bounding curves can be worked out for all cases involving the \((3, p), (4, p)\) torus knots. Deriving this result using GFT appears to be nontrivial.\footnote{See \cite{29} for a discussion of certain classes of inequalities on linear combinations of the Taylor coefficients. None of the ones listed appear to capture eq. (3.18).}

4 \(A^{(2,3)}(t)\) and the S-matrix bootstrap

In this section, we will investigate the question: How close are individual S-matrices in the S-matrix bootstrap to the Alexander polynomial \(A^{(2,3)}(t) = t + 1/t - 1 = 2\xi - 1\). To quantify this, we will define the distance between two polynomials in the \(\xi\) variable as:

\[ d(p_1|p_2) = \int_{-1}^{1} d\xi (p_1(\xi) - p_2(\xi))^2. \]  

(4.1)

Using this we can compare how close \(\rho_1\) is to the Alexander polynomial for the pion S-matrices obtained from the bootstrap. From the pion bootstrap, we express \(\rho_1\) as a function of \(\xi\) where \(\xi = \xi_{\text{min}} = 1 - \frac{243a^2}{128} + \frac{729a^3}{1024}\). Then we expand up to \(O(\xi^6)\). Focusing on the “upper river” we obtain the plot in fig. (3). As is evident from the figure, most S-matrices are far from the knot polynomial. Interestingly, the S-matrix which maximizes \(\rho_1\) at \(a = 16/9\) is also the one which is nearest to the knot polynomial. All the S-matrices which have low values of \(d(p_1|p_2)\) as exhibited in fig. (3)(b) are linear in \(\xi\) and have slopes close to 2 or 1. For future work, it will be interesting to explore the connection between minimization of \(d(p_1|p_2)\) and emerging integral slope in more detail.

Figure 3: (a) \(d(p_1|p_2)\) vs \(k\) where \(k\) labels the S-matrix. (b) The \(\rho_1\) vs \(\xi\) for the \(k = 21, 26, 27, 36\) S-matrices which have low values of \(d(p_1|p_2)\). The \(k = 27\) which has the minimum value is indicated in red while the \(\phi^3\) theory is in black.
5 Type II string theory and knot polynomials

In this section, we will examine the type II tree level string amplitude for 2-2 dilaton scattering and show that the low energy expansion can be well approximated by a sum of knot polynomials. First we will recast the discussion about transcendentality in the amplitude in terms of the $\tilde{z}, a$ variables to make a statement about transcendentality and the knot polynomials. We will follow the discussions in [31, 32].

5.1 Transcendentality and knots

The amplitude, up to an overall kinematic factor of $(s_1s_2 + s_1s_3 + s_2s_3)^2$ is given by

$$\tilde{A}(s_1, s_2, s_3) = \frac{1}{s_1s_2s_3} \frac{\Gamma(1 - s_1)\Gamma(1 - s_2)\Gamma(1 - s_3)}{\Gamma(1 + s_1)\Gamma(1 + s_2)\Gamma(1 + s_3)}, \quad (5.1)$$

where $s_1 + s_2 + s_3 = 0$. In terms of the $\tilde{z}, a$ variables introduced in the main text, we have the following expansion around $\tilde{z} = 0$

$$\tilde{A} - \frac{1}{s_1s_2s_3} - 2\zeta(3) = 54a^2\tilde{z}( - \zeta(5) + a\zeta(3)^2) + 54a^2\tilde{z}^2( - 2\zeta(5) + 2a\zeta(3)^2 + 27a^2\zeta(7) + 54a^3\zeta(3)\zeta(5) + 9a^4(\zeta(9) + 2\zeta(3)^3)) + O(\tilde{z}^3). \quad (5.2)$$

Following [32], we assign the kinematic variable $a$ a transcendentality weight $-1$ and $\tilde{z}$ a weight $0$. $\zeta(n)$ has weight $n$. Then it is easy to see that each term in the above expansion has weight 3. We can discuss this a bit more generally using the CSDR in eq.(3.9). First let us focus on the powers of $a$ that appear at a particular order in $\tilde{z}$. $U_n(\xi)$ is a degree $n$ polynomial while $\xi$ itself involves degree-2 and degree-3 terms in $a$. The measure $d\xi$ gives a factor $27a^2(3a - 2s'_1)$ where we integrate over $s'_1$. Then, for instance at $\tilde{z}^2$ order, we have $n = 1$ [19] and hence the maximum degree of $a$ is 6. In a local theory this maximum degree cannot change [6, 7]. Specifically, we have all integer powers of $a$ from 2 to 6. This is precisely the pattern above. Then since we have $U_n(\xi) = \sum_{k=0}^{n} A^{(2,2k+1)}(t)$ with $t + 1/t = 2\xi$, we easily note that the highest power of $a$ for a given $n$ is associated with the top knot polynomial $A^{(2,2n+1)}(t)$. Together with the measure factor, the highest power of $a$ is $3n + 3$. With the transcendentality assignments for $a$, we conclude that at a given order in $\tilde{z}^n$, $A^{(2,2n+1)}(t)$ is associated with a maximum transcendentality weight of $3n$. Such relations between transcendentality and knot polynomials have been conjectured before in perturbative QFT–see [19]–although the details differ. For instance in [19], for perturbative $\phi^4$ theory, a correspondence between $(2,2n+1)$-torus knots and $\zeta(2n + 1)$ has been proposed as opposed to $\zeta(3n)$ we find above. For $n = 1$ they are the same but not otherwise.

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10 Recall we used $\tilde{z}^{n+1}$ in eq.(3.9).
5.2 Low energy expansion in terms of knot polynomials

Now let us examine numerically the expansion in eq.(5.1). We rewrite the rhs (up to 2 decimal places) as

\[ \tilde{z}(-55.99a^2 + 78.03a^3) + \tilde{z}^2(-111.99a^2 + 156.05a^3 + 1470.17a^4 - 3634.64a^5 + 2175.24a^6) + O(\tilde{z}^3). \] (5.3)

It can be expected that the low energy expansion should be dominated by the first massive pole (see appendix of [7] for an explicit check). So in the dispersive integral should be well approximated by the lower limit of the integral which in our normalization starts at 1. Then writing the measure factor’s contribution as

\[ \mu(a) \equiv 27a^2(3a - 2), \]

our expectation is that eq.(5.3) is going to be approximated by

\[ \mu(a) \left( \tilde{z}A^{(2,1)} + \tilde{z}^2(A^{(2,1)} + A^{(2,3)}) \right) + O(\tilde{z}^3), \] (5.4)

\[ \tilde{z}(-54a^2 + 81a^3) + \tilde{z}^2(-108a^2 + 162a^3 + 1458a^4 - 3645a^5 + 2187a^6) + O(\tilde{z}^3). \] (5.5)

The agreement with eq.(5.3) is good! In fact for higher orders in \( \tilde{z} \), the agreement becomes better as shown in fig.(4). This works mainly due to the fact that there is spin-0 dominance [2, 33, 34] and hence the contribution from the Legendre polynomial in the CSDR is trivial.

\[ \text{Figure 4: Comparison of the } \tilde{z}^{n+1} \text{ coefficients for string (red) and knot polynomial sum (blue) for } n = 2, 3. \]

6 Relation with \( SU_q(2) \)

We will now correlate the CSDR with the \( q \)-deformed oscillator [35] which features in the discussion of the quantum group \( SU_q(2) \). The \( q \)-deformed oscillator was introduced independently by Biedenharn and Macfarlane in 1989 and has been studied in great detail in the literature. The book [35] is a good reference for the material used below. The \( q \)-oscillator uses 3 generators \( b, b^\dagger, N \) satisfying

\[ bb^\dagger - qb^\dagger b = q^{-N}, [N, b] = -b, [N, b^\dagger] = b^\dagger. \]

We have \((b^\dagger)^\dagger = b, N^\dagger = N\). Here \( q \) is either a real number or a complex number with unit modulus. In the latter case, which will be our focus, we also have \( bb^\dagger - q^{-1}b^\dagger b = q^N \) so that

\[ b^\dagger b = [N]_q, \quad bb^\dagger = [N + 1]_q, \] (6.1)
where we have introduced the q-number
\[
[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}.
\]
(6.2)

When \(x\) is an integer, it can be checked that the \(q\)-numbers are just Chebyshev polynomials of the second kind \[21, 22\]. Specifically,
\[
[n]_q = U_{n-1} \left( \frac{1}{2} (q + \frac{1}{q}) \right).
\]
(6.3)

When \(q \to 1\), \([x]_q \to x\). For the \(q\)-oscillator, we can define the Hamiltonian
\[
H = w b^\dagger b = w [N]_q.
\]
(6.4)

Quite remarkably, the thermal expectation value of \(b^\dagger b\) is given by \[35\]
\[
\langle b^\dagger b \rangle_{\beta,q} = e^{\beta w} - \frac{1}{e^{2\beta w} - (q + \frac{1}{q}) e^{\beta w} + 1} = \frac{\bar{z}(1 - \bar{z})}{\bar{z}^2 - 2\xi \bar{z} + 1}, \quad \bar{z} \equiv e^{-\beta w},
\]
(6.5)

which is precisely of the form of the generating function (see eq.(3.5)) for the Alexander polynomials\[11\] in terms of the dispersive kernel with \(q \to t\). The expansion around \(\bar{z} = 0\) corresponds to \(\beta \to \infty\). Using this, we can write the map
\[
M(\bar{z}, a) = \alpha_0 + \frac{2N}{\pi Z} \int_{\xi_{min}}^{\xi_{max}} d\mu(\xi) \langle b^\dagger b \rangle_{\beta,t}, \quad Z = \frac{e^{\beta w}}{e^{\beta w} - 1},
\]
(6.6)

where \(2\xi = t + 1/t\) as before and \(Z\) is the thermal partition function for the undeformed oscillator. When \(|\xi| \leq 1\), we have \(t\) to be complex and \(|t| = 1\). This is precisely the case for \(q\) discussed above!

In other words, the amplitude is related to the \(q\)-average of the expectation value of \(a^\dagger a\) with inverse temperature set by \(-\ln \bar{z})/w\). The measure factor involves the partial wave amplitudes and hides the dynamical information. In the forward limit when \(a \to 0\), as well as for large values of the dispersive variable \(s_1^1\) we have \(t \to 1\). So in these regimes, we have the usual undeformed oscillator picture where the corresponding knot polynomial is unity. This observation helps in explaining the following feature.

Near \(a \sim 0\), all coefficients of \(\bar{z}^n\) are always negative (see fig.(4)). This is because in this region we have an approximate description in terms of the usual simple harmonic oscillator. According to eq.(6.6), the \(\bar{z}^n\) coefficients are related to the degeneracies which are positive and because of the negative sign of \(N\) in eq.(6.6) \[9\], all coefficients are negative.

We can also think of the amplitude in terms of a gas of these \(q\)-oscillators in the following way. An ideal gas of the \(q\)-oscillators has the Hamiltonian \[35\]
\[
H = \sum_i \epsilon_i [N_i]_q.
\]
(6.7)

\[11\]I could not find a reference to this simple observation in the knot theory literature.
We can write eq. (3.4) as

\[ M(\bar{z}, a) - \frac{3}{m^2} = \frac{\beta}{m^2} \sum_{n=0}^{\infty} U_n(\xi) \bar{z}^{n+1} = \frac{\beta}{m^2} \sum_{n=1}^{\infty} \epsilon_n(t)[n]_t, \quad \epsilon_n(t) = \bar{z}^n, \]  

(6.8)

where \( \beta, \xi \) are defined in eq. (3.6). Note that \( \beta \) is dependent on \( \xi \) and hence on \( t \). Thus, the amplitude is written suggestively in terms of the Hamiltonian for the gas of \( q \)-oscillators with the knot parameter \( t \) playing the role of \( q \). For the \( \phi^3 \) theory, for fixed \( a \), we had a fixed value of \( t \) as discussed previously.

For the CSDR, we have an integration over \( \xi \) as in eq. (3.9):

\[ M(\bar{z}, a) - \alpha_0 = 2 \sum_{n=1}^{\infty} \epsilon_n \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\mu(\xi)[n]_t, \quad \epsilon_n = \bar{z}^n, \]  

(6.9)

in other words, we have to consider an average over \( t \). From here we have:

\[ (1 - \bar{z}) (M(\bar{z}, a) - \alpha_0) = \frac{2N}{\pi} \sum_{n=1}^{\infty} \epsilon_n \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\mu(\xi)[2,2n-1]_t. \]  

(6.10)

Therefore, an interesting physical interpretation of the average over the sum of knot polynomials is that it yields the average energy (averaged over the knot parameter) with \( \epsilon_n = (1 - \bar{z}) \bar{z}^n \) for a gas of \( q \)-oscillators. It is tempting to conjecture that extremal theories in the S-matrix bootstrap correspond to extremizing this “average energy” in some manner. In the appendix, we give an explicit representation of the string amplitude in terms of the \( q \)-deformed oscillator.

### 7 Discussion

We end with some brief comments.

- We derived bounds on the combination of derivatives of the amplitude in terms of knot invariants. The expression in eq. (3.13) was in the form of an average over knot polynomials. We gave a physical interpretation of this quantity in terms of the \( q \)-deformed oscillator. In the future, it will be useful to develop the \( q \)-deformed oscillator picture further. An example of a question that would be interesting to answer is: In terms of the \( q \)-oscillator picture, what restrictions on the spectrum correspond to local theories? We touch upon this question in the appendix.

- Note that eq. (3.13) holds for any regular typically real function inside the unit disc which is known to respect the Robertson representation. Thus, this connects typically real functions with knot theory—a connection that has not been pointed out previously, to the best of my knowledge.

- One can ask if other knot polynomials like the Jones polynomial can be expressed in terms of the amplitude using the CSDR. It can be shown [30] that in this case, the Jones polynomials for the torus \( (2,2n+1) \) is a linear combination of derivatives including up to \( 3n+1 \) terms instead of the two terms for the Alexander case. Thus in a very tangible sense, the Jones polynomials are “more complex” from the Amplitudes perspective. We are investigating a quantification of this kind of idea in more generality.
We have not used the locality constraints anywhere in our discussion. It remains to be seen if other knot polynomials show up in the analysis when locality constraints are imposed. It will also be interesting to find a closer connection with the knot theory–perturbative Feynman diagram connection in the program of [19].

For weakly coupled theories involving a graviton exchange, where graviton loops can be ignored, the amplitude at leading order is proportional to $a^\frac{\tilde{z}}{1-\tilde{z}^2}$ which in GFT parlance is the Koebe function and extremal. We can write this as $a \sum_{n=0}^{\infty} \sum_{k=0}^{n} A^{(2k+1)}(1) \tilde{z}^{n+1}$. In this case $\rho_n = 1$ for all $n \geq 1$ with $\rho_n$ defined in eq. (3.13). The treatment of graviton loops is an interesting open problem.

As a final observation, it should be pointed out that since the expansion in $\tilde{z}$ was important in our analysis, the connection with knot polynomials would be obscure in the usual Mandelstam variables and hence in the fixed-$t$ dispersion relation.

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A String amplitude in terms of $q$-oscillator

Here I quote the formula for the type II tree level string amplitude, used in the main text, in terms of the $q$-oscillator representation. Explicitly, one can show

$$\hat{A} = \frac{1}{s_1 s_2 s_3} - 2\zeta(3) = -\frac{1}{Z} \sum_{k=1}^{\infty} \epsilon_k (b^\dagger b)_{\beta, q_k},$$

(A.1)

where $Z$ and $\beta$ are as in eq. (6.1) and $q_k$ is defined via $2\xi = 2 - 27(a/k)^2 + 27(a/k)^3 = q_k + 1/q_k$ and

$$\epsilon_k = \frac{27a^2 (3a^2 - 2)}{\pi^2 k^3 (k!)^2} (-1)^k \sin \left(\frac{k\pi}{2} (1 + \lambda_k)\right) \sin \left(\frac{k\pi}{2} (1 - \lambda_k)\right) \Gamma^2 \left[\frac{k}{2} (1 + \lambda_k)\right] \Gamma^2 \left[\frac{k}{2} (1 - \lambda_k)\right],$$

(A.2)

with $\lambda_k = \sqrt{\frac{k+3a}{k-a}}$. Between $-1/3 \leq a \leq 2/3$, which is where $q$ is complex with $|q| = 1$, it can be checked that $\epsilon_k \geq 0$. Note that there is a quantization in the $q$-deformation in terms of the level $k$. For $k \to \infty$, $q_k \to 1$ while $\epsilon_k \to 54 \frac{k^2 a - 5}{k} \zeta(5, -2a, k_0)$. When $\tilde{z} \to 0$, $k = 1$ dominates.\footnote{The locality constraints \cite{7} are equivalent to the statement that $\sum_{k=1}^{\infty} [n]_{q_k} \epsilon_k = P_n(a)$ where $P_n(a)$ is a degree-$k$ polynomial in $a$. Put differently, for a given $n$, the top knot polynomial in the description is the torus-$(2, 2n + 1)$. The simplicity of these conditions should enable a systematic study.}

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$\tilde{z} = 0$ does not contribute to the locality constraints. Also the large $k$ tail can be resummed to give $54 \frac{\zeta(5-2a, k_0)}{T_{[a]}} (1-z^2)$ where $\zeta(s, b)$ is the Hurwitz zeta function and $k_0$ is the lower $k$-cutoff. Since this is a Koebe function, the large $k$ tail saturates the Bieberbach conjecture.
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