ON COBWEB POSETS’ MOST RELEVANT CODINGS (***)

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SUMMARY:
One considers here acyclic digraphs named KoDAGs (****) which represent the outmost general chains of di-bi-cliques denoting thus the outmost general chains of binary relations. Because of this fact KoDAGs start to become an outstanding concept of nowadays investigation. We propose here examples of codings of KoDAGs looked upon as infinite hyper-boxes as well as chains of rectangular hyper-boxes in $\mathbb{N}_\infty$. Neither of KoDAGs’ codings considered here is a poset isomorphism with $\Pi = (P, \leq)$. Nevertheless every example of coding supplies a new view on possible investigation of KoDAGs properties. The codes proposed here down are by now recognized as most relevant codings for practical purposes including visualization. More than that. Employing quite arbitrary sequences $F = \{n_F\}_{n \geq 0}$ infinitely many new representations of natural numbers called an $F$-base or base-$F$ number system representations are introduced. These constitute mixed radix-type numeral systems. $F$- base non-standard positional numeral systems in which the numerical base varies from position to position have picturesque interpretation due to KoDAGs graphs and their correspondent posets which in turn are endowed on their own with combinatorial interpretation of uniquely assigned to KoDAGs $F$ – nomial coefficients. The base-$F$ number systems are used for KoDAGs’ coding and are interpreted as chain coordinatization in KoDAGs pictures as well as systems of infinite number of boxes’ sequences of $F$-varying containers capacity of subsequent boxes. Needless to say how crucial is this base-$F$ number system for KoDAGs - hence - consequently for arbitrary chains of binary relations. New $F$-based numeral systems are umbral base-$F$ number systems in a sense to be explained in what follows.

(****) - for the history of this name see The Internet Gian-Carlo Polish Seminar
Subject 1 oDAGs and KoDAGs in Company - here ⇒ http://ii.uwb.edu.pl/akk/sem/sem_rota.htm.

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1 Introduction. Cobweb poset definition.

Let any natural numbers valued sequence $F$-sequence be chosen. (Sometimes value zero might be admitted as for example $F_0 = 0$ in the case of $F$ being Fibonacci sequence). Then $F$ designates all structures considered here. Here a directed acyclic graph, also called DAG, is a directed graph with no directed cycles. Note that posets $\Pi = \langle P, \leq \rangle$ defined below are not lattices except for trivial case.

1.1 Plane $N_0 \times N$ definition of $\Pi = \langle P, \leq \rangle$ [13] [20, 19, 18, 27]

Cobweb poset $\Pi = \langle P, \leq \rangle$ is defined via its Hasse digraph as follows:

$$P = \bigcup_{s \geq 0} \Phi_s, \quad \Phi = \{ (j, s) : 1 \leq j \leq s_F \}$$

$s$-level, $j$ indicates position of a point in $s$-th level

$$(j, s) \leq (k, t) \iff s < t \lor (j = k \land s = t).$$

Any cobweb poset and any cobweb subposet graph $P_k = \bigcup_{s \geq 0} \Phi_s$ is a DAG [20] [19] [18].

More than that - all these posets as well as any cobweb layer $\langle \Phi_k \to \Phi_n \rangle = \bigcup_{s \geq k} \Phi_s$ are faithfully represented via orderable DAGs i.e. oDAGs [29, 7, 30] and since recently these have being called KoDAGs [2, 3, 7, 8, 9, 10].

While cobweb layer $\langle \Phi_k \to \Phi_n \rangle$, $n \geq k$ is an $(n - k + 1)$-level digraph oDAG and cobweb subposet $P_n$ is an $(n + 1)$-level digraph specifically $\langle \Phi_k \to \Phi_{k+1} \rangle$ is just a complete bipartite digraph

$$(1) \quad \langle \Phi_k \to \Phi_{k+1} \rangle = K_{F_k,F_{k+1}}, \quad k = 0, 1, 2, ...$$

1.2 Chain of di-bi-cliques’ definition of $\Pi = \langle P, \leq \rangle$ [20]

Identification [11] might be considered as source of the definition of $F_0$ rooted $F$-cobweb graph $P(F) = P$. The usual convention is to establish $F_0 = 1$.

Namely: Cobweb poset $\Pi = \langle P, \leq \rangle$ is defined via its Hasse digraph as a chain of complete binary relations each one link of the chain being represented by its di-bi-clique graph i.e. complete bipartite digraph $K_{F_k,F_{k+1}}$; $k = 0, 1, 2, ...$ where $(j, s) \leq (k, t) \iff s < t \lor j = k \land s = t$.

Obviously we thus establish natural and obvious bijection uniquely coding complete relations’ chain via its KoDAG.

Obviously any chain of binary relations is obtainable from the cobweb poset chain of complete relations via deleting arcs in di-bicliques of the complete relations chain as of course any relation $R_k$ as a subset of $\Phi_k \times \Phi_{k+1}$ is represented by a bipartite digraph $D_k$. Recall then that a ”complete relation” $C_k$ is identified by definition with its di-biclique graph $K_{F_k,F_{k+1}}$. A digraph $D_k$ is a sub-digraph of $C_k$. 
In brief then [20] cobweb posets’ and thus KoDAGs’s defining di-bicliques are links of any complete relations’ chain.

For more on that see [20] from where we quote

DAGs considered as a generalization of trees have a lot of applications in computer science, bioinformatics, physics and many natural activities of humanity and nature. For example in information categorization systems, such as folders in a computer or in Serializability Theory of Transaction Processing Systems and many others. Here we introduce specific DAGs as generalization of trees being inspired by algorithm of the Fibonacci tree growth. For any given natural numbers valued sequence the graded (layered) cobweb posets’ DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated complete bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition - a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chain of complete relations via deleting arcs (arrows) in di-bicliques. Let us underline it again: any chain of relations is obtainable from the cobweb poset chain of complete relations via deleting arcs in di-bicliques of the complete relations chain. For that to see note that any relation \( R_k \) as a subset of \( A_k \times A_{k+1} \) is represented by a one-direction bipartite digraph \( D_k \). A "complete relation" \( C_k \) by definition is identified with its one direction di-biclique graph \( d-B_k \). Any \( R_k \) is a subset of \( C_k \). Correspondingly one direction digraph \( D_k \) is a subgraph of an one direction digraph of \( d-B_k \). The one direction digraph of \( d-B_k \) is called since now on the di-biclique i.e. by definition - a complete bipartite one direction digraph. Another words: cobweb poset defining di-bicliques are links of a complete relations’ chain. (end of quote)

Here come some examples of KoDAGs (Fig. 1 - 4).

Notation used in above figure an figures to follow comes from [20, 19, 18] and [27] where

\[ \sigma P_m = C_m[F;\sigma < F_1,F_2,...,F_m>] \]

is an equipotent sub-poset obtained from \( P_m \) with help of a permutation \( \sigma \) of the sequence \( < F_1,F_2,...,F_m> \) encoding \( m \) layers of \( P_m \) thus obtaining the equinumerous sub-poset \( \sigma P_m \) with the sequence \( < F_1,F_2,...,F_m> \) encoding now \( m \) layers of \( \sigma P_m \). Then \( P_m = C_m[F;< F_1,F_2,...,F_m>] \).

Further readings: [13, 20, 19, 18, 25, 2, 3, 4, 7, 8, 9, 10, 27].

Motto for further investigations:

KoDAGs represent the outmost general chains of di-bicliques denoting thus the outmost general chains of binary relations. Because that fact KoDAGs start to become an outstanding concept of nowadays investigations.
2 Cobweb posets’ coding via $N^\infty$ lattice boxes and more

**Basic:** $N^\infty = N \times N \times N \times \ldots = \times_{s \geq 0} N_s$, $N_s \equiv N$. Neither of codings introduced in this section is the posets’ isomorphism with $\langle \Pi, \leq \rangle$.

Recall that posets $\Pi = \langle P, \leq \rangle$ defined below are not lattices except for trivial case.

At the same time for the sake of Kwaśniewski combinatorial interpretation of cobweb poset’s inherited $F$-nomial coefficients $[20, 11, 12, 22, 23, 19, 18, 5, 16, 15, 14, 6]$ - maximal chains of layers $\langle \Phi_k \rightarrow \Phi_n \rangle$ were and are looked upon by Kwaśniewski expressis verbis as the set points of the set

$$\{ \text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle \} \equiv C_{\text{max}}(\Phi_k \rightarrow \Phi_n) \equiv C_{\text{max}}^{k,n}. $$

On the other hand - Dziemiańczuk [3,2,1] - while investing tiling problem posed in $[20, 19, 18]$ had observed in practice of computer calculations an usefulness of his so called geometric coding of the posets $\Pi = \langle P, \leq \rangle$. In the course of further joint searching his concept clarified to be of much more than of technical rele-
2.1 Geometric $N^\infty$ coding of $\Pi$ via the set $V$ of maximal chains of $\Pi$  

The poset $K$ is to be defined as $K = \langle V, \leq \rangle$, where $V \subset N^\infty$ is selected to be the set of points from $N^\infty$ forming the infinite discrete $F$-hyper-box (discrete rectangular hyper-box designated by $F$-sequence)... or in everyday parlance just hyper-box i.e. $V = [0_F] \times [1_F] \times [2_F] \times \ldots$ where $[0_F]$ might correspond as in Fibonacci sequence case to an "empty root" and just by convention has one point $\{\emptyset\}$.

$$V \equiv \{ \chi = (c_0, c_1, c_2, \ldots) : c_n \in \{0, 1, \ldots, n_F - 1\}, n \geq 0 \} \equiv \{ \chi = (\phi_0, \phi_1, \phi_2, \ldots) : \phi_n \in \Phi_n, n \geq 0 \}$$

or in plane $N_0 \times N$ presentation

$$V \Leftrightarrow \{ \chi = (c_0, c_1, c_2, \ldots) : 1 \leq c_n \leq n_F, n \geq 0 \}$$

with the point $\chi = (c_0, c_1, c_2, \ldots), \chi \in V$ interpreted as a maximal-chain in $\Pi$. $V$ denotes the family of maximal chains of $\Pi$.

**Example:** for $\leq_{\Pi} \leq_V$ where for $\chi_1 = (c_0, c_1, c_2, \ldots), \chi_2 = (d_0, d_1, d_2, \ldots)$ we define
\[ \chi_1 \leq_V \chi_2 \iff (\exists n \in N \ c_n < d_n \land \forall m > n \ c_m \leq d_m) \]

le voila a chain of coded \( \Pi \) poset maximal chains 

\((0,0,0,\ldots) \leq_V (0,0,1,0,0,\ldots) \leq_V (0,0,0,1,0,\ldots) \ldots \leq_V \ldots \)

The above is an example of a chain in \( K \). It is characteristic example as \( \leq_V \) is a total order.

Consider now \( V^* = \{(c_0,c_1,\ldots,c_m,0,0,\ldots) \in V \} \). It appears to be a chain of chains.

**Lemma 1** \( \leq_V \) is total order in \( V^* \) hence \( (V^*,\leq_V) \) is a chain.

**Proof.** Indeed. Let \( \chi_1 \leq_V \chi_2, \chi_1 \neq \chi_2 \).

Then either

\[ \chi_1 \leq_V \chi_2 \iff (\exists n \in N : c_n < d_n \land \forall m > n : c_m \leq d_m) \]

or (note that \( \chi_1 \neq \chi_2 \) is true)

\[ \neg (\chi_1 \leq_V \chi_2) \iff \neg (\exists n \in N : c_n < d_n \land \exists n \in N , \forall m > n : c_m \leq d_m) \]

\[ \iff \forall n \in N : (c_n < d_n) \lor \neg (\exists n \in N , \forall m > n : c_m \leq d_m) \]

\[ \iff (\forall n \in N : d_n \leq c_n) \lor \neg (\exists n \in N , \forall m > n : c_m \leq d_m) \]

\[ \iff (\forall n \in N : d_n \leq c_n) \land (\exists n \in N , \forall m > n : c_m \leq d_m) \lor \neg (\exists n \in N , \forall m > n : c_m \leq d_m) \]

\[ \iff \chi_2 \leq_V \chi_1 \lor [\forall n \in N : d_n \leq c_n \land (\exists n \in N , \forall m > n : c_m \leq d_m)] \]

\[ \iff \chi_2 \leq_V \chi_1 \lor [\forall m \in N : d_m \leq c_m \land (\exists n \in N , \forall m > n : c_m \leq d_m)] \]

\[ \iff \chi_2 \leq_V \chi_1 \lor \chi_2 \leq_V \chi_1 \lor \chi_2 \leq_V \chi_1 \lor \chi_2 \leq_V \chi_1 \]

\[ \iff \chi_2 \leq_V \chi_1 \]

Let \( \chi_1 \neq \chi_2 \). Then either \( \chi_1 \leq_V \chi_2 \) or \( \chi_2 \leq_V \chi_1 \]

### 2.2 Natural Partial Order Relations \( \leq_\Pi \) defining coding poset \( K \equiv \langle V, \leq_\Pi \rangle \)

**2.2.1** \( \leq_V, \Gamma \equiv \langle V, \leq_V \rangle \)

For the reasons to be apparent soon (see also Section 4 pictures) the codings to follow shall be called discrete geometric.

A partial order relation \( \leq_\Pi = \leq_V \) already introduced and used for the sake of computer calculations with maximal chains’ counting by Dziemiańczuk [http://www.dejaview.cad.pl/cobwebposets.html] has been defined as follows

\[ \chi_1 \leq_V \chi_2 \iff (\exists n \in N \ c_n < d_n \land \forall N \exists m > n \ c_m \leq d_m) \lor \forall n \ c_n = d_n \]

As shown in Lemma the poset \( \Gamma \equiv \langle V, \leq_V \rangle \) is a chain. \( \Gamma \) denotes then a new linearly ordered poset comprising the family of maximal chains of \( \Pi \) as points of \( V \). The source of this kind of point of view is inherited from combinatorial interpretation [20] of \( F \)-nomial coefficients [11]–[19], [4,5,6]. Indeed. Recall [20] [19] [18] that maximal chains of layers \( (\Phi_k \rightarrow \Phi_n) \) are set points of the set
\{\text{maximal chains in } (\Phi_k \to \Phi_n) \}\equiv C_{\text{max}} (\langle \Phi_k \to \Phi_n \rangle) \equiv C_{\text{max}}^{k,n}.

Denoting with \( V_{k,n} \subseteq V \) the discrete finite rectangular \( F \)-hyper-box or \((k,n)-F\)-hyper-box or in everyday parlance just \((k,n)\)-box

\[ V_{k,n} \equiv [kF] \times [(k+1)F] \times ... \times [nF] \]

we identify the two just by agreement according to the natural identification:

\[ C_{\text{max}}^{k,n} \equiv V_{k,n}. \]

Here down as in above we shall use [see: Appendix, Section 5] the so called upside down notation: \( F_k \equiv kF \).

\[ \subseteq, B \equiv \langle V, \subseteq \rangle \]

A partial ordered relation \( \leq_{\Pi} = \subseteq \) which sets the pace with the identification KoDAGs treated as chains of discrete \( F \)- hyper-boxes from \( N^\infty \) is obvious and of primary importance - including practical (tilings) and visualization purposes.

The natural partial order \( \subseteq \) forced upon discrete boxes \( C_{\text{max}}^{k,n} \) might be additionally and extra introduced also in \( \Pi = \langle P, \leq \rangle \) and then it is just set inclusion. For

\[ B \equiv \langle V, \subseteq \rangle = L_{\text{box}}(V) \subset L(V) \equiv 2^V \]

this would mean correspondingly the discrete hyper-box into discrete hyper-box inclusion with the welcomed scenario of an infinite Boolean lattice \( L(V) \equiv 2^V \) or chains of finite Boolean lattices \( L(V_{k,n}) \equiv 2^{V_{k,n}} \).

The \( \leq_V \) choice is of course different from \( \subseteq \) though both are relatives in a sense. The chain poset \( \Gamma \) and the sub Lattice \( B \) of Boolean Lattice are already complementarily useful - both.

More than that. One is not forbidden - not at all - to consider simultaneously two partial orders i.e. the structure \( \langle V, \leq_{\Pi}, \subseteq \rangle \) in general and specifically one of them being linear order \( \langle V, \leq_V, \subseteq \rangle \).

Summing up:

**Mantra: self-quotation:**

\textit{KoDAGs represent the outmost general chains of di-bi-cliques denoting thus the outmost general chains of binary relations. Because of that fact KoDAGs start to become an outstanding concept of nowadays investigations.}

**Mantra: Refrazed = coded:**

\textit{KoDAGs in } \langle V, \leq_{\Pi} \rangle \text{ coding are "just chains" of discrete hyper-boxes thus coding the outmost general chains of binary relations. Because of that fact this KoDAGs in this coding start to become an outstandingly natural concept of nowadays investigations.}
- Hyper-boxes?
- Yes. For example chains of *di-bi-cliques hyper-boxes*

\[ V_{k,k+1} = [k_F] \times [(k+1)F] \Rightarrow C_{\text{max}}^{k,k+1}. \]

Here down as in above we shall use [see: Appendix, Section 5] the so called upside down notation: \( F_k \equiv k_F \).

**2.2.3 \( \leq_{\Phi}, \Phi = \times_{s \geq 0}\Phi_s \) coding (\( \Phi \) is infinite Cartesian product of levels \( \Phi_s \))**

\( \leq_{\Pi} = \leq_{\Phi} \). Consider a sequence of posets \( \langle \Phi_s, \leq_{\Phi_s} \rangle \) where \( \Phi_s \) stays for set of vertices on \( s \)-th level i.e. \( \Phi_s = \{ (j, s) : 1 \leq j \leq s_F \} \) and linear order between vertices on this level i.e.

\[ (j, s) \leq_{\Phi_s} (k, s) \iff j \leq k, \quad j, k \in [s_F] \equiv \{ x; x \leq s_F \}. \]

If so then the poset \( \Lambda = (\times_{s \geq 0}\Phi_s, \leq_{\Phi}) \) is no more cobweb poset i.e. it is not isomorphic to \( \langle P, \leq \rangle \). This is obvious already from the fact that the sine qua non feature of \( \langle P, \leq \rangle \) is that \( \Phi_s \) are independent sets and \( \langle P_n, \leq \rangle \) is an \( n \)-level graded poset; \( n \in N \cup \{0\} \).

**Specifications of \( \leq_{\Phi} \).**

There are three of the possible "WIKI-common" partial orders on the Cartesian product of two totally ordered sets:

1. Lexicographical order: \( (a, b) \leq (c, d) \) if and only if \( a < c \) or \( (a = c \land b \leq d) \). This is a total order.

2. \( (a, b) \leq (c, d) \) if and only if \( a \leq c \) and \( b \leq d \) (the product order). This is a partial order.

3. \( (a, b) \leq (c, d) \) if and only if \( (a < c \land b < d) \) or \( (a = c \land b = d) \) (the reflexive closure of the direct product of the corresponding strict total orders). This results also a partial order.

As for the \( \leq_{\Phi} \) we have at our disposal unbounded multitude of combining these three choices at the start for example for di-bi-cliques and then subsequently imposing various orders on the resulted products of posets - freely, step by step using the induction for example.

**Conviction.**

Even more? We are not limited only to this possibility...? [Hypothesis?]

We leave this for further investigation. Here we content ourselves with an example to be used for visualizations.

**Example:**

Consider a poset \( \Lambda = (\times_{s \in N \cup \{0\}}\Phi_s, \leq_{\Phi}) \), \( s \in N \cup \{0\} \) with Cartesian product of vertices’ sets \( \Phi = \times_{s \in N \cup \{0\}} = \{ (\phi_0, \phi_1, ...) : \phi_n \in \Phi_n \} \) and the partial order \( \leq_{\Phi} \) defined as follows

\[ \chi_1 \leq_{\Phi} \chi_2 \iff \forall s \in N \cup \{0\} \; \epsilon_s \leq_{\Phi_s} d_s \]
where $\chi_1 = (c_0, c_1, c_2, \ldots), \chi_2 = (d_0, d_1, d_2, \ldots) \in \Phi = \times_{s \in \mathbb{N} \cup \{0\}} \Phi_s$

We obtain a lattice with set points representing maximal chains in $\Pi$, with partial order relation $\leq_\Phi$. The "level by level" Cartesian product poset $\Lambda$ is a lattice as obviously for any two elements from $\Phi$ there exists a supremum (join; $\lor$) and an infimum (meet; $\land$).

**Now note.** The above choice of $\leq_\Phi$ means that both $\chi_1$ and $\chi_2$ are elements of an infinite "vertical" strip $S_{\chi_1, \chi_2} = \{\chi \in \Phi : \chi_1 \land \chi_2 \leq \chi \leq \chi_1 \lor \chi_2\}$ which is also a hyper-box $S_{\chi_1, \chi_2} \subseteq \Phi$. Thus again discrete hyper-box into discrete hyper-box inclusion is inevitably around.

**Illustration**

An illustration of $\Lambda = \langle \Phi, \leq_\Phi \rangle \equiv \langle \times \Phi_s, \leq_\Phi \rangle$ lattice points = maximal chains in $\Pi$-disposal is supplied by the figure of the coded di-bi-clique layer $V_{3,4} \subset \Phi$, where $F = \text{Natural numbers}$ (Fig. 5).

![Figure 5: Hasse diagram of lattice $\langle V_{3,4}, \leq_\Phi \rangle$](image)

### 3 How does it work

**Recall.** Here down as in above we shall use (see: Appendix, Section 5) the so called upside down notation: $F_k \equiv k_F$. $V$ is a code of $\Pi$ and $V_{k,n}$ is a code of the layer $\langle \Phi_k \to \Phi_n \rangle$. **Obviously:** any cobweb poset $\Pi$ designated by the sequence $F$ has its own representation $V$ **independently of the partial order chosen** i.e. $K \equiv \langle V, \leq_\Pi \rangle$. Here comes an illustration (Fig. 6).

As a matter of illustration we show soon using pictures **how** any cobweb poset $\Pi$ can be represented as a hyper-box $V$ in discrete subspace of $\mathbb{N}^\infty$, so that any maximal-chain of $\Pi$ poset becomes a point of $V$. More than that.

Employing quite arbitrary sequences $F \equiv \{n_F\}_{n \geq 0}$ infinitely many new representations of natural numbers called an $F$-base or base-$F$ number system representations are introduced. These are used for coding and interpreted as chain coordinatization in KoDAGs as well as systems of infinite number of boxes’ sequences of $F$-varying containers capacity of subsequent boxes. Needless to say how crucial this base-$F$ number system for KoDAGs - hence - consequently for arbitrary chains of binary relations. New positional-type number $F$-based systems are umbral base-$F$ number systems in a sense to explained in what
follows.

Let us introduce and recall the basic notions and notation.

Let \( k, n \in \mathbb{N} \cup \{0\} \) unless other stated. For \( k = n \) we deal then with "empty boxes".

Denoting with \( V_{k,n} \in V \) the discrete finite rectangular \( F \)-hyper-box or \((k,n)\)-hyper-box or in everyday parlance just \((k,n)\)-box

\[
V_{k,n} \equiv [k_F] \times [(k+1)_F] \times ... \times [n_F]
\]

we identify the following two just by agreement according to the \( F \)-natural identification:

\[
C_{\text{max}}^{k,n} \equiv V_{k,n}
\]

**Note.** Any point \( \chi \in V_{1,m} \) from finite hyper-box is represented by point from infinite hyper-box \( V \) according to the identification

\[
\chi = (c_0, c_1, ..., c_m) \equiv (c_0, c_1, ..., c_m, 0, 0, ...) \in V
\]

**Definition 1** \((\lambda)\) We define now \( F \)-coding map \( \lambda : V^* \rightarrow \mathbb{N} \cup \{0\} \) where \( V^* = \{(c_0, c_1, ..., c_m, 0, 0, ...) \in V\} \) simply as

\[
\lambda((c_0, c_1, ..., c_m)) = c_0 \cdot 1 + c_1 \cdot k_F^1 + c_2 \cdot k_F^2 + ... c_m \cdot k_F^m
\]

where, in Kwański upside down notation (see Section 5)

\[
k_F^s \equiv k_F(k+1)_F...(k+s-1)_F, \quad 0_F^s \equiv 1\text{ hence } 1_F^s \equiv F(1+1)_F...(1+s-1)_F \equiv s_F!
\]

hence for \( k > 0 \)

\[
(\lambda) \quad \lambda((c_0, c_1, ..., c_m)) = c_0 \cdot 1 + \sum_{s=1}^m c_s \cdot \frac{(k+s-1)_F!}{(k-1)_F!}
\]

and consequently for \( V_{1,m} \equiv V_m \):

\[
\lambda(\chi) = c_0 \cdot 1 + \sum_{s=1}^m c_s \cdot s_F!
\]

In not Kwański upside down notation.
\[ k_F^n \equiv k_F(k_F + 1)(k_F + s - 1) \quad k_F^0 \equiv 1 \]

hence

\[
(\lambda \text{ not}) \quad \lambda((c_0, c_1, ..., c_m)) = c_1 \cdot 1 + \sum_{s=1}^{m} c_s \cdot \frac{(k_F + s - 1)!}{(k_F - 1)!}
\]

and consequently

\[
\lambda(\chi) = c_0 \cdot 1 + \sum_{s=1}^{m} c_s \cdot s!
\]

For \( F \neq N \), this \( \lambda \) map defining formulas in not Kwaśniewski upside down notation above appear not "\( F \)-natural" i.e. not \( F \)-designated structures-consistent. The structures in mind comprise KoDAGs, their incidence algebras, combinatorial interpretation, tiling problem solutions and so on.

Let \( F \neq N \). Then of course

\[
c_0 \cdot 1 + \sum_{s=1}^{m} c_s \cdot \frac{(k + s - 1)_F!}{(k - 1)_F!} \neq c_0 \cdot 1 + \sum_{s=1}^{m} c_s \cdot \frac{(k_F + s - 1)!}{(k_F - 1)!}
\]

It is the Kwaśniewski upside down notation (see Section 6) which appears natural for the important statements to hold and vague "umbral" sequence based positional systems to work. This is this very Kwaśniewski upside down notation to be used here down.

For arbitrary admissible \( F \) the \( (\lambda) \) definition formulas in upside down notation appear "\( F \)-natural" i.e. \( F \)-designated structures-consistent. The structures in mind comprise KoDAGs, their incidence algebras, combinatorial interpretation, tiling problem solutions and so on.

Recall now the notation and the content of the lemma to be used.

**Lemma 2** \( \leq_V \) is total order in \( V^* \) hence \( \langle V^*, \leq_V \rangle \) is a chain.

**Lemma 3** Let \( k > 0 \). If \( \chi_1 = (k_F - 1, (k+1)_F - 1, ..., (k+m-1)_F - 1, 0) \in V_{k,n} \) and \( \chi_2 = (0, 0, ..., 1) \in V_{k,n} \) where \( n = k + m \in N \) then

\[
(2) \quad \lambda(\chi_1) = \lambda(\chi_2) - 1
\]
Proof: by induction.

1. For \( m = 1 \) we have \( \lambda((k_F - 1, 0)) = \lambda((0, 1)) - 1 \), obvious.

2. If (2) is true for \( m > 1 \) i.e.

\[
\sum_{s=0}^{m-1} [(k + s)_F - 1] \cdot k_F^s = k_F^m - 1
\]

then (2) is true for \( (m + 1) \) i.e.

\[
\sum_{s=0}^{m} [(k + s)_F - 1] \cdot k_F^s = k_F^{m+1} - 1.
\]

Indeed. Just check.

\[
\sum_{s=0}^{m-1} [(k + s)_F - 1] \cdot k_F^s + [(k + m)_F - 1] \cdot k_F^m = (k + m)_F \cdot k_F^m - 1,
\]

\[
\sum_{s=0}^{m-1} [(k + s)_F - 1] \cdot k_F^s - k_F^m + 1 + (k + m)_F \cdot k_F^m = (k + m)_F \cdot k_F^m.
\]

Then according to the induction assumption \( 0 + (k + m)_F \cdot k_F^m = (k + m)_F \cdot k_F^m \).

\[\blacksquare\]

Comment. This lemma proves a \textit{F-coding numeral system property} via \( \lambda \) such as that of the binary numeral system or base-2 number system as example shows: 01112 = 10002 - 1.

Hence the purpose aimed convention \( s \in \{0, 1, ..., (k + m)_F - 1\} \) for hyper-box and other definitions in order to exhibit the rule: 11...10 = 00...01 while moving one step up to the next level of any given KoDAG.

Lemma 4 For any hyper-box \( V_{k,n} \) its \( F \)-coding, labeling function \( \lambda \) is an injection from \( V_{k,n} \) to \( N \cup \{0\} \) i.e. \( \lambda(\chi_1) = \lambda(\chi_2) \Rightarrow \chi_1 = \chi_2 \) for any \( \chi_1, \chi_2 \in V_{k,n} \).

Proof a contrario

Let us assume \( \lambda(\chi_1) = \lambda(\chi_2) \land \chi_1 \neq \chi_2 \). Then (1) \( \sum_{s=0}^{m} c_s \cdot k_F^s = \sum_{s=0}^{m} d_s \cdot k_F^s \land (2) \exists s : c_s \neq d_s \). Then \( \sum_{s=0}^{m} (c_s - d_s) \cdot k_F^s = 0 \) and from (2) we infer that there exists one or more indices such that \( (c_s - d_s) \neq 0 \). Let \( \Omega = \{0, 1, ..., m\} \) and \( I = \{s \in \Omega : (c_s - d_s) \neq 0\} \). Then \( \sum_{s \in \Omega \setminus I} (c_s - d_s) \cdot k_F^s = 0 \). Therefore we need to consider only \( \sum_{s \in I} (c_s - d_s) \cdot k_F^s = 0 \) case. Let us identify the maximal number \( r \) from the set \( I \) i.e. \( r = \max\{s \in I\} \). Then (3) \( \sum_{s \in I \setminus \{r\}} (c_s - d_s) \cdot k_F^s + (c_r - d_r) \cdot k_F^r = 0 \). According to Lemma 3 we know that the sum of all \( \delta_s \cdot k_F^s \) where \( \delta_s = (c_s - d_s), s \in \{0, 1, ..., r - 1\} \) is smaller than \( 1 \cdot k_F^r \) therefore the sum (3) is not equal to 0 as seen from

\[
\sum_{s \in I \setminus \{r\}} (c_s - d_s) \cdot k_F^s \leq \lambda(\chi_3) < \lambda(\chi_4) \leq |(c_r - d_r)| \cdot k_F^r
\]

\[
\Rightarrow \sum_{s \in I \setminus \{r\}} (c_s - d_s) \cdot k_F^s + (c_r - d_r) \cdot k_F^r \neq 0
\]

where \( \chi_3 = (k_F - 1, (k + 1)_F - 1, ..., (k + r - 1)_F - 1, 0), \chi_4 = (0, ..., 0, 1) \in V_{k,k+r} \) contrary to the assumption. \( \blacksquare \)

Conclusion. As a conclusion from the above statements the following is true.
Theorem 1. Chains $\langle V^*, \leq_V \rangle$ and $\langle N, \leq \rangle$ are isomorphic.

Proof. It is enough to prove that $\lambda$ an order preserving bijection of $V^*$ and $N$. We already know from Lemma 4 that $\lambda$ is an injection. At first we prove that for any two points $\chi_1, \chi_2 \in V_{k,n}$ where $k, n \in N \cup \{0\}$.

$\chi_1 \leq_V \chi_2 \Leftrightarrow \lambda(\chi_1) \leq \lambda(\chi_2)$

Let $\chi_1 = (c_0, c_1, \ldots, c_m)$, $\chi_2 = (d_0, d_1, \ldots, d_m)$. The case $\chi_1 = \chi_2$ is obvious, therefore let us consider $\chi_1 \neq \chi_2$.

1. $\chi_1 \leq_V \chi_2 \Rightarrow \lambda(\chi_1) \leq \lambda(\chi_2)$

From definition of $\leq_V$ relation, we have that (1) $\exists 0 \leq j \leq m (c_j < d_j)$ and (2) $\forall t > j (c_t \leq d_t)$ therefore according to Lemma 3 and (1) we infer

$$\sum_{s=0}^{j} c_s k_F^s + \sum_{s > j}^{m} c_s k_F^s < d_j k_F^j + \sum_{s > j}^{m} d_s k_F^s$$

Hence $\lambda$ is monomorphism.

2. The $\lambda$ is monomorphism is surjection. This is being proved by supplying an appropriate algorithm for retrieving $\chi_\alpha$ from its $F$-coding natural number $\alpha$.

Algorithm 1

Input: $\alpha \in N \cup \{0\}$, Output: $\chi_\alpha \in V$ such that $\lambda(\chi_\alpha) = \alpha$

Step 1. Find the smallest number $m \in N$ such that $\alpha < k_F^{m+1}$, and set $r_m = \alpha$

Step 2. Identify $c_m$ and $r_{m-1}$ from $r_m = c_m \cdot k_F^m + r_{m-1}$ equation so that $r_{m-1}$ stays for the the remainder after appropriate dividing i.e. $r_{m-1} = r_m \mod k_F^m$

then $r_{m-1} = c_{m-1} \cdot k_F^{m-1} + r_{m-2}$ and so on ... $r_1 = c_1 \cdot k_F + r_0$ until $r_0 = c_0$ or using the while instruction $F$-code algorithm reads:

function lambda($\alpha \in N \cup \{0\}$) : $\chi_\alpha \in V$

begin
set $m$ such that $k_F^m \leq \alpha < k_F^{m+1}$
$r_m = \alpha;$
while ($m > 0$)
begin
$c_m = r_m \mod k_F^m;$
$r_{m-1} = r_m - c_m \cdot k_F^m;$
$m = m - 1;$
end
$c_0 = r_0;$
end

This is then just multiple application of Euclidean algorithm and the uniqueness is obvious.
The result:
From Step 1 we have \( \alpha = r_m \). From Step 2 \( \alpha = c_m \cdot k_F^m + r_{m-1} \) and so on until \( r_0 = c_0 \)

\[
\alpha = c_m \cdot k_F^m + c_{m-1} \cdot k_F^{m-1} + \cdots + c_1 \cdot k_F + c_0 = \lambda(\chi_\alpha) \equiv (c_m c_{m-1} \cdots c_1 c_0)_F \]

Name
This representation of zero or any natural number \( \alpha \) we shall call an \textbf{-base-F} or \textbf{base-F} number system representation of \( \alpha \).

Needless to say how crucial is this base-F number system for KoDAGs - hence - consequently for arbitrary chain \( s \) of binary relations.

Examples:
Consider now the hyper-box \( V_{1,n} \), \( n \in N \).

1. For constant sequence \( F = \{ p \} \) \( p \neq 0 \), where \( p \in N \) the points of the hyper-box \( V(F) \) are \( F \)-coded with the above used standard numeral system with the \( p \) base or base-\( p \) number system i.e.

\[
\lambda ((c_0, c_1, \ldots, c_m)) = (c_m, c_{m-1}, \ldots, c_1, c_0)_p = \sum_{s=0}^{m} c_s \cdot p^s \ i.e.
\]

\[
\lambda ((c_0, c_1, \ldots, c_m)) = (c_m c_{m-1} \cdots c_1 c_0)_F .
\]

Note the habit-indispensable order of \( c_i \) variables \textbf{inversion} in \( F \)-representation.

The case \( p = 2 \) is the case of binary system; for example \( \chi = (1, 0, 1, 1) \equiv (1101)_2 \equiv (\lambda(\chi))_2 = 11 \). Here then \( F \)-coding \( \lambda \) is a way to convert \( F \)-base representation to binary numeral base \( \lambda((c_0, c_1, \ldots, c_m)) = (c_m c_{m-1} \cdots c_1 c_0)_2 \).

In general it may be considered as a mean to convert \( F \)-base representation of a natural number to any other standard base-\( d \) representation. For example with \( p = 10 \) choice we arrive at quite ancient decimal system \( \lambda ((c_0, c_1, \ldots, c_m)) = (c_m c_{m-1} \cdots c_1 c_0)_F \).

2. For non constant sequences like \( F \)=Fibonacci sequence next weight coefficients are in general different equal to \( s_p \) i.e.

\[
\lambda((c_0, c_1, \ldots, c_m)) = \sum_{s=0}^{m} c_s \cdot k_F^s \ i.e. \ 
\lambda((c_0, c_1, \ldots, c_m)) = (c_m c_{m-1} \cdots c_1 c_0)_F .
\]

Recall: the representation \( \lambda((c_0, c_1, \ldots, c_m)) = (c_m c_{m-1} \cdots c_1 c_0)_F \) of zero or any natural number \( \alpha \) we call the \textbf{-base-F} or \textbf{base-F} number system representation of \( \alpha \).

Let \( F \)=Fibonacci sequence. Next weight coefficients are different starting from \( s = 2 \). There \( \lambda(0, 0, 0, 0, 0, \ldots) = 0 \), \( \lambda(0, 0, 1, 0, 0, 0, \ldots) = 1 \cdot 2_p! = 1 \), \( \lambda(0, 0, 0, 1, 0, 0, \ldots) = 1 \cdot 2_p! = 1 \cdot 2_p = 2 \), \( \lambda(0, 0, 1, 0, 0, \ldots) = 1 \cdot 2_p! + 1 \cdot 3_p = 3 \), \( \lambda(0, 0, 0, 2, 0, 0, \ldots) = 2 \cdot 3_p! = 4 \), \( \lambda(0, 0, 1, 2, 0, 0, \ldots) = 1 \cdot 2_p + 2 \cdot 3_p! = 5 \) and so on. For example \( 32 = \lambda(0, 0, 0, 2, 0, 1, \ldots) = 2 \cdot 3_p! + 5_p! = 3_p! + 5_p! \cdot 4 \cdot 3_p! \), \( 24 = \lambda(0, 0, 0, 0, 0, 1, \ldots) = 4 \cdot 4_p! \), \( 29 = \lambda(0, 0, 1, 2, 4, 0, \ldots) \), ...

The way to add one ["ball"] to the store is unique. How it goes?
\( 6 \leftarrow 5 + 1 = \lambda(0, 0, 1, 2, 0, 0, \ldots) + \lambda(0, 0, 0, 0, 0, 0, \ldots) = \lambda(0, 0, 0, 0, 1, 0, \ldots) \)
\( 7 \leftarrow 6 + 1 = \lambda(0, 0, 0, 0, 1, 0, \ldots) + \lambda(0, 0, 0, 0, 0, 0, \ldots) = \lambda(0, 0, 0, 1, 1, 0, \ldots) \).

The way to add one ["ball"] to the store is unique if one insists to place additional balls one by one.
\[ 7 ← 5 + 2 = λ(0, 0, 1, 2, 0, 0...) + λ(0, 0, 0, 1, 0, 0...) = λ(0, 0, 1, 2, 0, 0...) + λ(0, 0, 1, 0, 0, 0...) + λ(0, 0, 0, 0, 1, 0, 0...) + λ(0, 0, 1, 0, 0, 0...) = λ(0, 0, 1, 1, 0, 0...). \] This rule is just the \( F \)-base algorithm of adding:

\[
\begin{align*}
(0,0,1,2,0,0...) \\
+ (0,0,1,0,0,0...) \\
\hline
= (0,0,0,0,1,0...) 
\end{align*}
\]

The way to add ball or ball containers or just store to the store is unique. How it goes?

We now visualize this "phenomenon" with boxes, containers and balls, [using containers and thus making exact a glimpse’s idea with balls only instead of containers suggested by W. Bajguz - also the participant of our join Gian Carlo Rota Polish Seminar [http://ii.uwb.edu.pl/akk/sem/sem_rota.htm].

Let \( \alpha \in N \).

1. **For constant sequence** \( F \equiv \{p\}_{n \geq 0} \) imagine infinite number of boxes’ sequence of the \( (p - 1) \) containers capacity - each box, as Fig. 8 shows. Almost all boxes are empty. So containers contain containers. And ultimately - containers in containers which are in containers and so on - store balls.

   Figure 8: Containers’ sequence interpretation of the p-base numeral system

   The rule of the base-\( F \) (i.e. here base-\( p \)) number system is the following: "adding \( p \)-th container to \( s \)-th box causes the surplus and this whole surfeit is passed over as the one container to the next \( (s+1) \)-th box" leaving the \( s \)-th box void of containers.

   Delivering \( \alpha \) balls one by one into 0-th box and applying The rule one arrives at the final distribution of containers of containers etc. with balls. Distribution is then just Russian "Babushkas in Babushka" sequence. Naturally application of this incrassating "Babushka in Babushka" rule i.e. The rule ends up with no more than \( (p - 1) \) containers per box distribution (Figure 8).

   In zero-th box there are \( c_0 \) balls \( \bullet \ldots \bullet \). In s-th box \( s > 0 \), there are \( c_s \) containers \( \square \ldots \square \).

2. **For non constant sequence** \( F \equiv \{p\}_{n \geq 0} \)

   The \( F \)-rule of any base-\( F \) number system in which the numerical base varies from position to position is the following:
"Adding one \( s_F \)-th ball to the \( s \)-th box causes the surplus and this whole surfeit is passed over as the one container to the next \((s+1)\)-th box leaving the \( s \)-th box void of containers."

Another words: the \( s_F \) containers overflow in the \( s \)-th box is passed over as an additional one container into the \((s + 1)\)-th box void of containers.

Naturally application of this incrassating different in size "Babushka in Babushka" rule i.e. The F-rule ends up with no more than \((s_F - 1)\) containers per \( s \)-th box distribution (Figure 9).

\[
\begin{array}{cccccc}
\bullet \bullet \bullet & | & \bullet \bullet \bullet \bullet \bullet & | & \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet & | & 0 \quad 0 \\
1_F - 1 & | & 2_F - 1 & | & \ldots & | & (m+1)_F - 1 & | & (m+2)_F - 1 & | & (m+3)_F - 1
\end{array}
\]

Figure 9: Sequence of containers’ interpretation of the F-base numeral system

In zero-th box there are \( c_0 \) balls \( \bullet \bullet \bullet \). In \( s \)-th box \( s > 0 \), there are \( c_s \) containers \( \square \square \square \). Recall: \( c = (c_0, c_1, ..., c_m, 0, 0, ...) \). Observe: the number of containers in the \( s \)-th box equals to the \( s \)-th coordinate of the point \( \chi \in V(F) \)

**Summing up:**

\[
\alpha = c_m \cdot k_F^m + c_{m-1} \cdot k_F^{m-1} + \ldots + c_1 \cdot k_F + c_0 = \lambda(\chi_\alpha)
\]

\[
\lambda((c_0, c_1, ..., c_m)) \equiv (c_m c_{m-1} ... c_1 c_0)_F
\]

This representation \( \lambda((c_0, c_1, ..., c_m)) \equiv (c_m c_{m-1} ... c_1 c_0)_F \) of zero or any natural number \( \alpha \) we shall call an \textit{F-base} or \textit{base-F} number system representation of \( \alpha \).

Needless to say how crucial is this base-F number system for KoDAGs - hence consequently for arbitrary chain \( s \) of binary relations.

New positional-type number F-based systems are umbral base-F number systems in a following sense explained symbolically in a pictogram sequence way

\[
\sum_{s=0}^{m} c_s \cdot a^s \rightarrow \text{[umbral view]} \rightarrow \sum_{s=0}^{m} c_s \cdot a_s \rightarrow
\]

\[
\rightarrow \text{[example and upside down notation]} \rightarrow \sum_{s=0}^{m} c_s \cdot k_F^s \rightarrow
\]

\[
\rightarrow \text{[example’s case } k = 1, \text{akk upside – down notation]} \rightarrow \sum_{s=0}^{m} c_s \cdot s_F!
\]
The base-$F$ number system is to be afterwards and soon compared with using the Fibonacci numbers to represent whole numbers [http://ii.uwb.edu.pl/akk/suprasl/akk.htm]. We mean by this the Fibonacci base system. Recall: Zeckendorf proved in 1972 [31] that: each representation of a number $n$ as a sum of distinct Fibonacci numbers, is unique but where no two consecutive Fibonacci numbers are used (and there is only one column headed "1"). Another words, a number $α$ might be written as a sum of nonconsecutive Fibonacci numbers $α = \sum_{s=0}^{m} c_s F_s$ where $c_s$ are 0 or 1 and $c_s \cdot c_{s+1} = 0$

4 Application. Cobweb Hyper-Box Tillig Phenomenon.

For Kwaśniewski tiling problem solution see [3, 4].

Here we refraze the tile notion in Dziemiańczuk geometric-coded setting in order to accomplish illustrative pictures’ delivery.

Geometric code description of the cobweb tile.

The set $τ_m$ of $λ$ points $π ∈ V_{k,n}$ and a permutation $σ$ of the set \{1\text{\_}F, 2\text{\_}F, ..., m\text{\_}F\} such that

\[ τ_m ≡ \{π = (c_1, ..., c_m) : c_s ∈ [(σ \cdot s)\text{\_}F], s = 1, 2, ..., m\}\]

where $λ = m\text{\_}F!, m = n − k + 1$ is called a cobweb tile.

Here come some pictures of cobweb hyper-box tiles below (Fig. 10, 11).

Figure 10: Picture of all Natural numbers’ tiles $τ_2, τ_3$. 
Definition of \textit{cobweb tiling} in the geometric code.
A set $T_{k,n}$ of tiles $\tau_s$ from finite cobweb hyper-box $V_{k,n}$ such that

$$T_{k,n} = \{ \tau_s \in V_{k,n} : \tau_i \cap \tau_j = \emptyset, i \neq j \land \bigcup_{i=1}^{m_F} \tau_i = V_{k,n} \}$$

is called cobweb tiling.

**Notation.** Let $T(V_{k,n})$ denotes the family of all tilings $T_{k,n}$ of finite cobweb hyper box $V_{k,n}$.

**Information.** The compact formula for the number $\{ n \}_F$ of all tilings of the layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ i.e. $\{ n \}_F = |T(V_{k,n})|$ is still not known \[20, 18\]. See more in \[29, 8, 4\].

To this end by now we supply some pictures of the cobweb hyper-box tilings with (Fig. 12 - 15).

**To be continued.**

**Summing up.**

Cobweb hyper-box representation of cobweb poset in infinite discrete space as a rectangular hyper box invented as a result of cobweb tilings' visualization [http://www.dejaview.cad.pl/cobwebposets.html] already shows up efficient and promising for future investigations. Further readings: \[20, 19, 18, 8, 3, 4, 29\].
5 Appendix - Kwaśniewski upside down notation

The Kwaśniewski upside-down notation $n_F = F_n$ has been used for mnemonic reasons - as in the case of Gaussian numbers in finite geometries and the so called "quantum groups"; (see [10,11], [18-20] and references therein). It has been used then consequently in all relevant papers by Ewa Krot, by Ewa Krot-Sieniawska and Dziemiańczuk and Bajguz.

Precise formulation of the Upside Down Principle may be found in recent (20 Feb 2009) The Internet Gian Carlo Rota Polish Seminar affiliated article [32]. There one also finds the recent formulation of Kwaśniewski combinatorial interpretation of the $F$-nomial coefficients (consult Appendix 2. for more on that).

Given any sequence $\{F_n\}_{n \geq 0}$ of nonzero reals ($F_0 = 0$ being sometimes acceptable as $0! = F_0! = 1$). one defines its corresponding binomial-like $F$-nomial coefficients as follows.

**Definition 2.**

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_F^n}{k_F!}, \quad n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_1 F;$$

$$0_F! = 1; \quad n_F^k = n_F(n-1)_F \ldots (n-k+1)_F.$$

Kwaśniewski had made above an analogy driven identifications in the spirit of Ward’s Calculus of sequences [27]. Identification $n_F \equiv F_n$ is the notation used in extended Fibonomial Calculus case [10,11-15,4,5,6] being also there inspiring as $n_F$ mimics $n_q$ established notation for Gaussian integers exploited in much elaborated family of various applications including quantum physics (see [10,11] and references therein).

**Now compare:** [10,11]

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \rightarrow \binom{n}{k}_q = \frac{n_q!}{k_q!(n-k)_q!} \rightarrow \binom{n}{k}_\psi = \frac{n_\psi!}{k_\psi!(n-k)_\psi!}$$

$\binom{n}{k} \rightarrow \binom{n}{k}_q \rightarrow \binom{n}{k}_\psi \ldots$ incidence-like coefficients, connection constants see: [25,26] and references therein; **here and there throughout upside down notation is mnemonic source of associations**

$$\binom{n}{k} \rightarrow \binom{n}{k}_q \rightarrow \binom{n}{k}_\psi \ldots$$

$\psi$ denotes an extension of

$$\left\{ \frac{1}{n!} \right\}_{n \geq 0}$$

sequence to quite arbitrary one ("admissible") and the specific choices are for example: Fibonomialy-extended ($F_n$, $n \geq 0$ -Fibonacci sequence) or Gauss $q$-extended
\[\{\psi_n\}_{n \geq 0} = \left\{ \frac{1}{F_n!} \right\}_{n \geq 0}, \quad \{\psi_n\}_{n \geq 0} = \left\{ \frac{1}{n_q!} \right\}_{n \geq 0},\]

admissible sequences of extended umbral operator calculus - see more below. With such an extension we may \(\psi\)-mnemonic repeat with exactly the same simplicity and beauty much of what was done by Rota years ago. Thus via practisizing we get used to write down these extensions in mnemonic upside down notation [1-25]:

\[n_\psi \equiv \psi_n, \quad x_\psi \equiv \psi(x) \equiv \psi_x, \quad n_\psi! = n_\psi(n - 1)_\psi!, \quad 0_\psi = 1\]

\[x_k^\psi = x_\psi(x-1)_\psi...(x-k+1)_\psi \equiv \psi(\psi(x-1)...\psi(x-k+1))\]

You may consult for further development and use of this notation [10,11,22] and references therein. As for references - the papers of main references are: [10,11,22].

The Kwaśniewski upside-down notation imposes associations with more general schemes notions as Whitney numbers aside of incidence coefficients [18]

**Summarizing.** While identifying general properties of such \(\psi\)-extensions in all their connotations mentioned above the merit consists indeed in notation i.e. here - in writing objects of these extensions in mnemonic convenient **upside down notation**.

(3) \[\frac{\psi(n-1)}{\psi_n} \equiv n_\psi, \quad n_\psi! = n_\psi(n - 1)_\psi!, \quad n > 0, \quad x_\psi \equiv \frac{\psi(x-1)}{\psi(x)}\]

(4) \[x_k^\psi = x_\psi(x-1)_\psi(x-2)_\psi...(x-k+1)_\psi\]

(5) \[x_\psi(x-1)_\psi...(x-k+1)_\psi = \frac{\psi(x-1)_\psi(x-2)_\psi...(x-k)_\psi}{\psi(x)_\psi(x-1)_\psi...(x-k+1)_\psi}\]

If one writes the above in the form \(x_\psi \equiv \frac{\psi(x-1)}{\psi(x)} \equiv \Phi(x) \equiv \Phi_x \equiv x_\Phi\), one sees that the name upside down notation is legitimate.

**The KoDAG Enterprise Information.** For most recent (20 Feb 2009) developments on upside down notation efficiency see [32] and

*This is The Definition which is the right answer to the leitmotiv questions* i.e. The Internet Gian-Carlo Polish Seminar article,

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http://ii.uwb.edu.pl/akk/sem/sem_rota.htm

6 Appendix - Cobweb posets and KoDAGs’ ponderables of Kwaśniewski relevant recent productions. Excerpts from [32].
Definition 3 Let \( n \in \mathbb{N} \cup \{0\} \cup \{\infty\} \). Let \( r, s \in \mathbb{N} \cup \{0\} \). Let \( \Pi_n \) be the graded partial ordered set (poset) i.e. \( \Pi_n = (\Phi_n, \leq) = (\bigcup_{k=0}^{n} \Phi_k, \leq) \) and \( (\Phi_k)_{k=0}^{n} \) constitutes ordered partition of \( \Pi_n \). A graded poset \( \Pi_n \) with finite set of minimal elements is called cobweb poset iff
\[
\forall x, y \in \Phi \text{ i.e. } x \in \Phi_r \text{ and } y \in \Phi_s \text{ } r \neq s \Rightarrow x \leq y \text{ or } y \leq x,
\]
\( \Pi_\infty \equiv \Pi \).

Note. By definition of \( \Pi \) being graded its levels \( \Phi_r \in \{\Phi_k\}_{k=0}^{\infty} \) are independence sets and of course partial order \( \leq \) up there in Definition 6.1. might be replaced by \( < \).

The Definition is the reason for calling Hasse digraph \( D = (\Phi, \leq) \) of the poset (\( \Phi, \leq \)) a KoDAG as in Professor Kazimierz Kuratowski native language one word Komplet means complete ensemble - see more in [23] and for the history of this name see: The Internet Gian-Carlo Polish Seminar Subject 1. oDAGs and KoDAGs in Company (Dec. 2008).

Definition 4 Let \( F = (k_F)_{k=0}^{n} \) be an arbitrary natural numbers valued sequence, where \( n \in \mathbb{N} \cup \{0\} \cup \{\infty\} \). We say that the cobweb poset \( \Pi = (\Phi, \leq) \) is denominated (encoded=labelled) by \( F \) iff \( |\Phi_k| = k_F \) for \( k = 0, 1, \ldots, n \).

Observation 1 Let \( n = k + m \). The number of subposets equipotent to subposet \( P_m \) rooted at any fixed point at the level labeled by \( F_k \) and ending at the \( n \)-th level labeled by \( F_n \) is equal to
\[
\binom{n}{m}_F = \binom{n}{k}_F = \frac{n!_F}{k!_F} \frac{k!_F}{(n-k)!_F} = \frac{n!_F}{k!_F} \frac{n-k)_F!}{k!_F} \frac{(n-1)_F!}{k!_F} \ldots \frac{(1)_F!}{k!_F} = \frac{n!_F}{k!_F} \frac{k!_F}{k!_F}.
\]

Equivalently - now in a bit more mature 2009 year the answer is given simultaneously viewing layers as biunivoquely representing maximal chains sets. Let us make it formal.

Such recent equivalent formulation of this combinatorial interpretation is to be found in [33] from where we quote it here down (see also [32]).

Let \( \{F_n\}_{n \geq 0} \) be a natural numbers valued sequence with \( F_0 = 1 \) (or \( F_0! \equiv 0! \) being exceptional as in case of Fibonacci numbers). Any such sequence uniquely designates both \( F \)-nomial coefficients of an \( F \)-extended umbral calculus as well as \( F \)-cobweb poset introduced by this author (see :the source [19] from 2005 and earlier references therein). If these \( F \)-nomial coefficients are natural numbers or zero then we call the sequence \( F \) - the \( F \)-cobweb admissible sequence.

Definition 5 Let any \( F \)-cobweb admissible sequence be given then \( F \)-nomial coefficients are defined as follows
\[
\binom{n}{k}_F = \frac{n!_F}{k!_F(n-k)!_F} = \frac{n!_F \cdot (n-1)_F \cdot \ldots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \ldots \cdot k_F} = \frac{n!_F}{k!_F} \frac{k!_F}{k!_F} = \frac{n!_F}{k!_F} \frac{k!_F}{k!_F} = \frac{n!_F}{k!_F} \frac{k!_F}{k!_F} = \frac{n!_F}{k!_F} \frac{k!_F}{k!_F}.
\]
while \( n, k \in \mathbb{N} \) and \( 0_F! = n_F! = 1 \).
Definition 6. \( C_{\text{max}}(P_n) \equiv \{ c = < x_0, x_1, ..., x_n >, x_s \in \Phi_s, s = 0, ..., n \} \) i.e. \( C_{\text{max}}(P_n) \) is the set of all maximal chains of \( P_n \).

Definition 7. Let

\[
C_{\text{max}}(\Phi_k \rightarrow \Phi_n) \equiv \{ c = < x_k, x_{k+1}, ..., x_n >, x_s \in \Phi_s, s = k, ..., n \}.
\]

Then the \( C(\Phi_k \rightarrow \Phi_n) \) set of Hasse sub-diagram corresponding maximal chains defines biunivocally the layer \( (\Phi_k \rightarrow \Phi_n) = \bigcup_{s=k}^{n} \Phi_s \) as the set of maximal chains’ nodes and vice versa - for these graded DAGs (KoDAGs included).

The equivalent to those from [17,19] formulation of combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) is the following.

Theorem [33] (Kwaśniewski) For \( F \)-cobweb admissible sequences \( F \)-nomial coefficient \( \binom{n}{k}_F \) is the cardinality of the family of equipotent to \( C_{\text{max}}(P_n) \) mutually disjoint maximal chains sets, all together partitioning the set of maximal chains \( C_{\text{max}}(\Phi_{k+1} \rightarrow \Phi_n) \) of the layer \( (\Phi_{k+1} \rightarrow \Phi_n) \), where \( m = n - k \).

For February 2009 readings on further progress in combinatorial interpretation and application of the partial order sets named cobweb posets and their’s corresponding encoding Hasse diagrams KoDAGs see [32-37] and references therein. For active presentation of cobweb posets see [38].

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