Double integrals on a weighted projective plane and the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$

Atsuhiro Nagano

December 30, 2021

Abstract

The aim of this paper is to give an explicit extension of the classical elliptic integrals to the Hilbert modular case for $\mathbb{Q}(\sqrt{5})$. We study a family of Kummer surfaces corresponding to the Humbert surface of invariant 5 with two complex parameters. Our Kummer surface is given by a double covering of the weighted projective space $\mathbb{P}(1 : 1 : 2)$ branched along a parabola and a quintic curve. The period mapping for our family is given by double integrals of an algebraic function on chambers coming from an arrangement of a parabola and a quintic curve in $\mathbb{C}^2$.

Introduction

The aim of this paper is to give a canonical extension of the classical elliptic integrals to the Hilbert modular case for $\mathbb{Q}(\sqrt{5})$. The arrangement of 4 points on the projective line $\mathbb{P}^1(\mathbb{C})$ is deeply related to the elliptic modular functions for the principal congruence subgroup $\Gamma(2)$. The double covering of $\mathbb{P}^1(\mathbb{C})$ branched at 4 points gives an elliptic curve. The coordinate of the configuration space of 4 branch points on $\mathbb{P}^1(\mathbb{C})$ gives a modular function for $\Gamma(2)$ via the period mapping of the family of the corresponding elliptic curves.

One of the most successful extensions of the above classical story to several variables is given by K. Matsumoto, T. Sasaki and M. Yoshida [8]. They showed an interesting relation between the arrangement of 6 lines on the projective plane $\mathbb{P}^2(\mathbb{C})$ and the modular functions on a 4 dimensional bounded symmetric space of type $I$ via the period mapping of the family of $K3$ surfaces coming from the arrangement of 6 lines.

We shall give another natural extension of the classical elliptic integrals to a case of several variables. The Hilbert modular functions for real quadratic fields are very popular among modular functions of several variables. However, to the best of the author’s knowledge, to obtain simple and geometric extensions of the classical elliptic integrals to the Hilbert modular cases is a highly non-trivial problem. Although the Hilbert modular functions with level 2 structure can be obtained from the moduli of hyperelliptic curves of genus 2, they are characterized by a complicated modular equations (see Remark 1.7).

In this paper, we focus on the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. Since the real quadratic field $\mathbb{Q}(\sqrt{5})$ gives the smallest discriminant, several researchers (for example, K. B. Gundlach [2], F. Hirzebruch [4], R. Müller [10]) studied this case in detail. We shall give a simple and geometric interpretation of the Hilbert modular functions in this case. We consider the double integrals of the algebraic function $F$ in (2.2) of 2 variables on chambers surrounded by the parabola $P$ in (1.2) and the quintic curve $Q$ in (1.3) with the $(2, 5)$-cusp. These double integrals are equal to the period integrals of the Kummer surface $K(X, Y)$ in (1.1). The equation (1.1) gives a double covering of the weighted projective plane $\mathbb{P}(1 : 1 : 2)$.

Keywords: Elliptic integrals ; Hilbert modular functions ; Kummer surfaces
Mathematics Subject Classification 2010: 11F46, 14J28
Running head: Double integrals and Hilbert modular functions
branched along \( P \) and \( Q \) and the complex parameters \((X, Y)\) determine the arrangement of the branch loci. The parameters \((X, Y)\) are regarded as a pair of the Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \) via the explicit double integrals (see Remark 1.9 and Theorem 2.1). Our results are coherent with the story of the classical elliptic integrals (see Table 1). The results in this paper are used in the paper [12].

| Classical Story | Result of This Paper |
|-----------------|----------------------|
| Base Space      | \( \mathbb{P}^1(\mathbb{C}) \) | \( \mathbb{P}(1:1:2) \) |
| Branch Loci     | 4 points             | \( P \) and \( Q \) |
| Variety         | Elliptic curve       | Kummer surface \( K(X, Y) \) |
| Arrangement     | Elliptic modular function for \( \Gamma(2) \) | Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \) |

Table 1: The classical elliptic integrals and the result of this paper.

The author conjectures that we can similarly obtain simple and geometric interpretations of other Hilbert modular functions also, using suitable weighted projective planes. Our results might give a first step of such an approach to Hilbert modular functions.

1 The Kummer surface \( K(X, Y) \) and the Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \)

We consider the period mapping for the family \( K = \{ K(X, Y) \} \) of surfaces where

\[
K(X, Y) : v^2 = (u^2 - 2y^5)(u - (5y^2 - 10Xy + Y))
\]  

(1.1)

for \((X, Y) \neq (0, 0)\). The equation \(1.1\) gives a double covering of the \((y, u)\)-space branched along the parabola

\[
u = 5y^2 - 10Xy + Y
\]

(1.2)

and the quintic curve

\[
u^2 = 2y^5
\]

(1.3)

with the \((2, 5)\)-cusp \((y, u) = (0, 0)\). The parameters \((X, Y)\) define the arrangement of the divisors \( P \) and \( Q \). In this section, we see the properties of the family \( K \).

1.1 The Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \) and the K3 surface \( S(X, Y) \)

In this subsection, we survey the results of [11].

Let \( \mathcal{O} \) be the ring of integers in the real quadratic field \( \mathbb{Q}(\sqrt{5}) \). Set \( \mathbb{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \). The Hilbert modular group \( PSL(2, \mathcal{O}) \) acts on \( \mathbb{H} \times \mathbb{H} \) by

\[
 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (z_1, z_2) \mapsto \left( \frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right),
\]

for \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, \mathcal{O}) \), where \( \gamma \) means the conjugate in \( \mathbb{Q}(\sqrt{5}) \). We consider the involution \( \tau : (z_1, z_2) \mapsto (z_2, z_1) \) also.

**Definition 1.1.** If a holomorphic function \( g \) on \( \mathbb{H} \times \mathbb{H} \) satisfies the transformation law

\[
g \left( \frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right) = (\gamma z_1 + \delta) (\gamma' z_2 + \delta')^k g(z_1, z_2)
\]

for \( k \in \mathbb{Z} \), then \( g \) is called a Hilbert modular function.
for any \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PSL}(2, \mathbb{O}) \), we call \( g \) a Hilbert modular form of weight \( k \) for \( \mathbb{Q}(\sqrt{5}) \). If \( g(z_2, z_1) = g(z_1, z_2) \), \( g \) is called a symmetric modular form.

If a meromorphic function \( f \) on \( \mathbb{H} \times \mathbb{H} \) satisfies
\[
f(\begin{pmatrix} \alpha z_1 + \beta \\ \gamma z_1 + \delta \end{pmatrix}, \begin{pmatrix} \alpha' z_2 + \beta' \\ \gamma' z_2 + \delta' \end{pmatrix}) = f(z_1, z_2)
\]
for any \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PSL}(2, \mathbb{O}) \), we call \( f \) a Hilbert modular function for \( \mathbb{Q}(\sqrt{5}) \).

**Remark 1.1.** Hirzebruch [4] showed that the symmetric Hilbert modular surface \( (\mathbb{H} \times \mathbb{H})/(\text{PSL}(2, \mathbb{O}), \tau) \) is isomorphic to the weighted projective plane \( \mathbb{P}(1 : 3 : 5) = \{ (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \} \). The point \( (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0) \) gives the cusp \( (\sqrt{-1}\infty, \sqrt{-1}\infty) \) of the modular surface. Letting
\[
X = \frac{\mathfrak{B}}{\mathfrak{A}^2}, \quad Y = \frac{\mathfrak{C}}{\mathfrak{A}^5},
\]
the pair \( (X, Y) \) defines a system of affine coordinates of \( \{ \mathfrak{A} \neq 0 \} \) of \( \mathbb{P}(1 : 3 : 5) \).

**Remark 1.2.** Müller [10] gave the Hilbert modular forms \( g_2 \) \((s_6, s_{10}, s_{15}, \text{resp.})\) of weight 2 \((6, 10, 15, \text{resp.})\). They generate the ring of Hilbert modular forms for \( \mathbb{Q}(\sqrt{5}) \).

A K3 surface \( X \) is a simply connected compact complex surface with \( K_X = 0 \). The homology group \( H_2(X, \mathbb{Z}) \) has the unimodular lattice structure. Let \( \text{NS}(X) \), the Néron-Severi lattice of \( X \), denote the sublattice in \( H_2(X, \mathbb{Z}) \) generated by the divisors on \( X \). The orthogonal complement \( \text{Tr}(X) \) of \( \text{NS}(X) \) in \( H_2(X, \mathbb{Z}) \) is called the transcendental lattice of \( X \).

We set the family \( \mathcal{F} = \{ S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) | (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) - \{ (1 : 0 : 0) \} \} \) of K3 surfaces with an elliptic fibration given by the affine equation
\[
S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : z_0^2 = x_0^3 - 4y_0^2(4y_0 - 5\mathfrak{A})x_0^2 + 20\mathfrak{B}y_0^3x_0 + \mathfrak{C}y_0^4.
\]
For a generic point \( (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) \), the intersection matrix of the Néron-Severi lattice \( \text{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \) is given by \( E_8(-1) \oplus E_8(-1) \oplus \left( \begin{smallmatrix} 2 & 1 \\ 1 & -2 \end{smallmatrix} \right) \) (see [II]). Set \( \mathcal{D} = \{ \xi \in \mathbb{P}^3(\mathbb{C}) | \xi A^T \xi = 0, \xi A^T > 0 \} \), where \( A = U \oplus \left( \begin{smallmatrix} 2 & 1 \\ 1 & -2 \end{smallmatrix} \right) \) gives the transcendental lattice of \( S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \). Here, \( U \) is the parabolic lattice of rank 2. Note that \( \mathcal{D} \) is composed of 2 connected components \( \mathcal{D}_+ \) and \( \mathcal{D}_- \). We let \( (1 : 1 : -\sqrt{-1} : 0) \in \mathcal{D}_+ \). In [II], we had the multivalued period mapping \( \mathbb{P}(1 : 3 : 5) - \{ (1 : 0 : 0) \} \to \mathcal{D}_+ \) for \( \mathcal{F} \) given by
\[
\Phi : (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mapsto \left( \int_{\Gamma_1} \omega : \int_{\Gamma_2} \omega : \int_{\Gamma_3} \omega : \int_{\Gamma_4} \omega \right),
\]
where \( \omega \) is the holomorphic 2-form up to a constant factor and \( \Gamma_1, \cdots, \Gamma_4 \) are 2-cycles on \( S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \).

**Remark 1.3.** Let \( \{ \Gamma_1, \cdots, \Gamma_4 \} \) be a basis of the transcendental lattice \( A \). We can take 2-cycles \( \Gamma_1, \cdots, \Gamma_4 \) such that they satisfy \( (\Gamma_j : \Gamma_k) = \delta_{j,k} \) \((j, k = 1, \cdots, 4)\). These 2-cycles \( \Gamma_1, \cdots, \Gamma_4 \) give the period mapping \( [I, 0] \).

Note that we have a biholomorphic mapping \( j : \mathbb{H} \times \mathbb{H} \to \mathcal{D}_+ \). The multivalued mapping \( j^{-1} \circ \Phi \) on \( \{ \mathfrak{A} \neq 0 \} \) is given by
\[
(X, Y) \mapsto (z_1, z_2) = \left( \frac{\int_{\Gamma_3} \omega + \frac{1 + \sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega}, \frac{\int_{\Gamma_3} \omega + \frac{1 - \sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega} \right).
\]
Theorem 1.1. ([11]) The multivalued period mapping ([1]) gives a developing map of the Hilbert modular orbifold \( \mathbb{H} \times \mathbb{H} / \langle \text{PSL}(2, \mathbb{O}), \tau \rangle \) with the branch divisor

\[
Y(-1728X^5 + 64(5X^2 - Y)^2 + 720X^3Y - 80XY^2 + Y^3) = 0.
\]

The inverse of ([1]) gives a pair \((X(z_1, z_2), Y(z_1, z_2))\) of symmetric Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \).

Remark 1.4. The icosahedral group is deeply related to the Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \) or \([7]\). Since the divisor

\[
-1728X^5 + 64(5X^2 - Y)^2 + 720X^3Y - 80XY^2 + Y^3 = 0 
\]  

is derived from Klein’s icosahedral invariants, this relation is called Klein’s icosahedral relation.

Remark 1.5. The inverse \((X(z_1, z_2), Y(z_1, z_2))\) of ([1]) has an explicit expression in terms of the Müller’s modular forms \(g_2, g_6, s_{10}\) (see [11]).

1.2 The Kummer surface for the Humbert surface of invariant 5

In this subsection, we recall the properties of the Humbert surface of invariant 5.

Let \( \mathcal{O}_2 \) be the Siegel upper half plane of degree 2. The symplectic group \( \text{Sp}(4, \mathbb{Z}) \) acts on \( \mathcal{O}_2 \). The quotient space \( \mathcal{O}_2 / \text{Sp}(4, \mathbb{Z}) \) gives the moduli space of principally polarized Abelian surfaces. Take \( \Omega = \left( \begin{array}{cc} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{array} \right) \in \mathcal{O}_2 \). Let \( L_\Omega \) be the lattice generated by the columns of the matrix \((\Omega, I_2)\). The complex torus \( \mathbb{C} / L_\Omega \) of 2-dimension gives a principally polarized Abelian surface. We note that \( Z_\Omega \) corresponds to the Jacobian variety of a hyperelliptic curve of genus 2.

Let \( T \) be the involution of a 2-dimensional complex torus \( Z \) induced by \((z_1, z_2) \mapsto (-z_1, -z_2)\) on the universal covering \( \mathbb{C}^2 \). The minimal resolution \( \text{Kum}(Z) = \mathbb{Z}/\langle \text{id}, T \rangle \) is called the Kummer surface. \( \text{Kum}(Z) \) is a \( K^3 \) surface. Note that \( Z \) is an Abelian surface if and only if \( \text{Kum}(Z) \) is an algebraic \( K^3 \) surface.

Remark 1.6. Let \( \Omega \in \mathcal{O}_2 \) and \( Z_\Omega \) be the corresponding principally polarized Abelian surface. The Kummer surface \( \text{Kum}(Z_\Omega) \) can be given by the double covering of \( \mathbb{P}^2(\mathbb{C}) \) \( \{ (\zeta_0 : \zeta_1 : \zeta_2) \} \) whose branch divisor is given by 6 lines \( \zeta_2 = 0, \zeta_2 + 2\zeta_1 + \zeta_0 = 0, \zeta_0 = 0 \) and \( \zeta_2 + 2\lambda_2 \zeta_1 + \lambda_2^2 \zeta_0 = 0, (j \in \{1, 2, 3\}) \) with three complex parameters \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). In this paper, this Kummer surface is denoted by \( \text{Kum}(Z_\Omega) \).

An element \( \Omega = \left( \begin{array}{cc} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{array} \right) \in \mathcal{O}_2 \) is said to have a singular relation with invariant \( \Delta \) if there exist relatively prime integers \( a, b, c, d, e \in \mathbb{Z} \) such that \( a\sigma_1 + b\sigma_2 + c\sigma_3 + d(\sigma_2^2 - \sigma_1\sigma_3) + e = 0 \) and \( \Delta = b^2 - 4ac - 4de \). Set \( \mathcal{N}_5 = \{ \Omega \in \mathcal{O}_2 | \sigma \) has a singular relation with invariant \( \Delta \} \). Let \( p \) be the canonical projection \( \mathcal{O}_2 \rightarrow \mathcal{O}_2 / \text{Sp}(4, \mathbb{Z}) \). Then, the space \( \mathcal{H}_5 = p(\mathcal{N}_5) \), called the Humbert surface of invariant 5, gives the moduli space of principally polarized Abelian surfaces \( A \) such that \( \mathcal{O} \subset \text{End}(A) \).

Remark 1.7. Humbert [3] showed that \( \Omega \) has a singular relation with \( \Delta = 5 \) if and only if

\[
4(\lambda_1^2 \lambda_3 - \lambda_2^2 + \lambda_3^2(1 - \lambda_1) + \lambda_2^2 \lambda_3)(\lambda_1^2 \lambda_2 \lambda_3 - \lambda_1 \lambda_2^2 \lambda_3) \\
= (\lambda_1^2(\lambda_2 + 1) \lambda_3 - \lambda_2^2(\lambda_1 + \lambda_3) + (1 - \lambda_1) \lambda_2 \lambda_3^2 + \lambda_1(\lambda_2 - \lambda_3))^2
\]

holds (see also [3] Theorem 2.9). This relation is called Humbert’s modular equation for \( \Delta = 5 \). Let \( \mathcal{M}_{4,2} \rightarrow \mathcal{O}_2 / \text{Sp}(4, \mathbb{Z}) \) be the natural projection, where \( \mathcal{M}_{4,2} \) is the moduli space of genus two curves with level 2 structure. The equation ([1]) defines a component of the inverse image \( Q^{-1}(\mathcal{H}_5) \).

This modular equation is studied in detail by several researchers (for example, Hashimoto and Murabayashi [3]). However, since this equation ([1]) is complicated, to the best of the author’s knowledge, to study the moduli properties of the family \( \{ \text{Kum}(\lambda_1, \lambda_2, \lambda_3) \} \) corresponding to \( \mathcal{H}_5 \) is not easy.
1.3 The Shioda-Inose structure

Let $X$ be an algebraic $K3$ surface. Let $\omega$ be the unique holomorphic 2-form on $X$ up to a constant factor. If an involution $\iota : X \to X$ satisfies $\iota^* \omega = \omega$, we call $\iota$ a symplectic involution. Set $G = \langle \iota, \text{id} \rangle \subset \text{Aut}(X)$. Set $\tilde{Y} = X/G$. Letting $Y \to \tilde{Y}$ be the minimal resolution, $Y$ is a $K3$ surface. We have the rational quotient mapping $\chi : X \dashrightarrow Y$.

**Definition 1.2.** We say that a $K3$ surface $X$ admits a Shioda-Inose structure if there exists a symplectic involution $\iota \in \text{Aut}(X)$ with the rational quotient mapping $\chi : X \dashrightarrow Y$ such that $Y$ is a Kummer surface and $\chi_* \text{ induces a Hodge isometry } \text{Tr}(X)(2) \cong \text{Tr}(Y)$.

**Theorem 1.2.** (Morrison [9]) The $K3$ surface $X$ admits a Shioda-Inose structure if and only if there is an embedding $\text{E}_8(-1) \oplus \text{E}_8(-1) \rightarrow \text{NS}(X)$. A symplectic involution $\iota$ exchanging the two copies of $\text{E}_8(-1)$ induces a Shioda-Inose structure.

1.4 Kummer surface $K(X, Y)$

Due to Theorem 1.2, the $K3$ surface $S(\mathbb{A} : \mathbb{B} : \mathbb{C})$ for $(\mathbb{A} : \mathbb{B} : \mathbb{C}) \neq (1 : 0 : 0)$ admits a Shioda-Inose structure. Therefore, there exists the Kummer surface $K(\mathbb{A} : \mathbb{B} : \mathbb{C})$ and a symplectic involution $\iota$ of $S(\mathbb{A} : \mathbb{B} : \mathbb{C})$ such that the corresponding rational quotient mapping $\chi : S(\mathbb{A} : \mathbb{B} : \mathbb{C}) \dashrightarrow K(\mathbb{A} : \mathbb{B} : \mathbb{C})$ induces a Hodge isometry $\text{Tr}(S(\mathbb{A} : \mathbb{B} : \mathbb{C}))(2) \cong \text{Tr}(K(\mathbb{A} : \mathbb{B} : \mathbb{C})).$

We shall obtain an explicit defining equation of $K(\mathbb{A} : \mathbb{B} : \mathbb{C})$ by realizing the above symplectic involution $\iota$. To find such an involution, we need a special elliptic fibration on $S(\mathbb{A} : \mathbb{B} : \mathbb{C})$ given by the following lemma.

**Lemma 1.1.** The defining equation of $S(\mathbb{A} : \mathbb{B} : \mathbb{C})$ in (1.5) is birationally equivalent to

$$z_1^2 = x_1(x_1^2 + (20\mathbb{A}y_1^2 - 20\mathbb{B}y_1 + \mathbb{C})x_1 + 16y_1^5).$$

(1.10)

**Proof.** Perform the birational transformation

$$x_0 = \frac{x_1}{16y_1}, \quad y_0 = -\frac{x_1}{16y_1}, \quad z_0 = \frac{x_1z_1}{256y_1}$$

to (1.5).

![Figure 1: The singular fibres given by (1.10).](image-url)
The mapping \( \pi_1 : S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \to \mathbb{P}^1(\mathbb{C}) \) given by \((x_1, y_1, z_1) \mapsto y_1 \) defines an elliptic fibration. The fibre \( \pi_1^{-1}(0) \) (\( \pi_1^{-1}(\infty) \), resp.) is a singular fibre of \( \pi_1 \) of type \( I_{10} \) (\( III^* \), resp.). We set \( \pi_1^{-1}(0) = a_0 + a_1 + \cdots + a_4 + a'_0 + a'_1 + \cdots + a'_4 \) and \( \pi_1^{-1}(\infty) = b_0 + b_1 + c_1 + c_2 + c_3 + c'_1 + c'_2 + c'_3 \). Let \( O \) be the zero of the Mordell-Weil group. Let \( O' \) be the section of \( \pi_1 \) given by \((x_1, y_1, z_1) = (0, y_1, 0) \). Note that \( 2O' = O \) (see Figure 1).

We have the involution \( \iota \) of \( S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \) given by

\[
(x_1, y_1, z_1) \mapsto \left( \frac{16y_1^5}{x_1}, y_1, \frac{-16y_1^5}{x_1^2} \right).
\]

This is a symplectic involution. Note that \( \iota \) is a van Geemen-Sarti involution for elliptic surfaces (see \[1\]). Let \( G = (\text{id}, \iota) \). Set

\[
u_1 = x_1 + \frac{16y_1^5}{x_1}, \quad v_1 = \frac{x_1^2 - 16y_1^5}{z_1}.
\]

(1.11)

They are \( G \)-invariants. We can see that \((x_1, y_1, z_1) \mapsto (u_1, y_1, v_1) \) defines a 2 to 1 mapping.

**Theorem 1.3.** The defining equation of the Kummer surface \( K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \) is given by

\[ e^2 = (u^2 - 2y^5)(u - (5\mathfrak{A}y^2 - 10\mathfrak{B}y + \mathfrak{C})). \]

(1.12)

For generic \((\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5)\), the intersection matrix of the transcendental lattice \( \text{Tr}(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \) is given by

\[
A(2) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.
\]

Proof. By direct observations, we can check that \( \iota \) interchanges 2 copies of \( E_8(\cdot - 1) \) in \( \text{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \) (see Figure 2). Therefore, due to Theorem 1.2 the involution \( \iota \) gives the Shioda-Inose structure on \( S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \).

![Figure 2: \( E_8(\cdot - 1) \) lattices in \( \text{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \).](image)

From \[1.10\], \[1.11\], and the birational transformation

\[
u_1 = -u, \quad v_1 = \frac{\sqrt{-1}y}{u - (5\mathfrak{A}y^2 - 10\mathfrak{B}y + \mathfrak{C})}, \quad y_1 = \frac{y}{2},
\]
we can check that the defining equation of \( S((\mathfrak{A} : \mathfrak{B} : \mathfrak{C})/G \) is given by \[1.12\].

We have the intersection matrix of \( \text{Tr}(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \) since \( \iota \) gives the Shioda-Inose structure.

We have the family \( \tilde{\mathcal{K}} = \{ K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \} \) of Kummer surfaces. The projection \((y, u, v) \mapsto (y, u) \) defines the double covering \( \tilde{P} : K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \to \mathbb{P}(1 : 1 : 2) = \{(\zeta_0 : \zeta_1 : \zeta_2)\}, \) where \( y = u \tilde{a}_0 \) and \( u = \tilde{a}_0 \) on \( \{\zeta_0 \neq 0\} \). Its branch divisor is given by \( \tilde{P} \cup \tilde{Q} \), where

\[
\begin{align*}
\tilde{P} \cap \{\zeta_0 \neq 0\} &= \{(y, u)u = 5\mathfrak{A}y^2 - 10\mathfrak{B}y + \mathfrak{C}\}, \\
\tilde{Q} \cap \{\zeta_0 \neq 0\} &= \{(y, u)u^2 = 2y^5\}.
\end{align*}
\]

(1.13)
Remark 1.8. The equation \((1.12)\) gives an expression of the Kummer surface \(\text{Kum}(Z_\Omega)\) for \(\Omega \in \mathcal{H}_5\). This is different from the expression of \(K_H(\lambda_1, \lambda_2, \lambda_3)\) in Remark 1.7. Our expression has some advantages. For example, our parameter space has a simple compactification by adding the point \((\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0)\). This point is equal to the cusp of the Hilbert modular surface \((\mathbb{H} \times \mathbb{H})/(\text{PSL}(2, \mathbb{O}), \tau)\) (see Remark 1.7).

Let \(\omega_K\) be the unique holomorphic 2-form on \(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\) up to a constant factor. Set \(\chi_*(\Gamma_j) = \Delta_j\) for \(j \in \{1, 2, 3, 4\}\). We have the period mapping for \(\mathcal{K}\) given by

\[
\Phi_K : (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mapsto \left(\int_{\Delta_1} \omega_K : \int_{\Delta_2} \omega_K : \int_{\Delta_3} \omega_K : \int_{\Delta_4} \omega_K\right) \in \mathcal{D}. \tag{1.14}
\]

Since \(\chi^*(\omega_K) = \omega\) and \(\chi_*(\Gamma_j) = \Delta_j\), we have clearly the following proposition.

Proposition 1.1. It holds that

\[
\left(\int_{\Gamma_1} \omega : \cdots : \int_{\Gamma_4} \omega\right) = \left(\int_{\Delta_1} \omega_K : \cdots : \int_{\Delta_4} \omega_K\right).
\]

Remark 1.9. According to Theorem 1.1 and the above proposition, the inverse of \(j^{-1} \circ \Phi_K\) gives the pair \((X, Y)\) of Hilbert modular functions for \(\mathbb{Q}(\sqrt{5})\) via the period mapping \(\Phi_K\).

We have the projection \(\pi : K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \to \mathbb{P}^1(\mathbb{C})\) given by \((u, y, v) \mapsto y\). We have the singular fibre \(\pi^{-1}(0)\) \((\pi^{-1}(\infty), \text{resp.})\) of the elliptic surface \((K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}), \pi, \mathbb{P}^1(\mathbb{C}))\) of type \(I_5\) \((III^*, \text{resp.})\) and other five singular fibres \(\pi^{-1}(s_1), \cdots, \pi^{-1}(s_5)\) of type \(I_2\).

Proposition 1.2. The vector space \(\text{NS}(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \otimes \mathbb{Q}\) is generated by the components of singular fibres, the section \(O\) given by the zero of the Mordell-Weil group and a general fibre \(F\) of \(\pi\).

Proof. \(\text{NS}(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \otimes \mathbb{Q}\) is an 18-dimensional vector space over \(\mathbb{Q}\). Set \(\pi^{-1}(y) = \bigcup_{j=0, \cdots, r(y)} \Theta_\nu, j\), where \(\Theta_\nu, j\) is a connected component and \(\Theta_\nu, 0 \cap O \neq \emptyset\). By calculating the intersection numbers, we can check that the 18 divisors \(\Theta_{s_0, 1}, \cdots, \Theta_{s_4, 1}, \Theta_{s_1, 1}, \cdots, \Theta_{s_5, 1}, \Theta_{\infty, 1}, \cdots, \Theta_{\infty, 7}, O\) and \(F\) generate a sublattice of \(\text{NS}(K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))\) of rank 18. Hence the claim follows.

Considering the relation \((1.3)\), we have the defining equation \((1.1)\) with \((X, Y)\) parameters.

2 The double integrals of an algebraic function on chambers surrounded by a parabola and a quintic curve

In this section, we obtain an extension of the classical elliptic integrals. We shall study a single-valued branch \(g_0\to \mathcal{D}_+\) of the multivalued period mapping \(\Phi_K\) explicitly by \(g_0\) is an open set given by Figure 3 in \(\mathbb{R}^2\). By the analytic continuation of this single-valued branch, we have the multivalued period mapping \(\Phi_K\) in \((1.14)\). The arrangement of \(P\) in \((1.2)\) and \(Q\) in \((1.3)\) determines the chambers \(R_1, R_2, R_3\) and \(R_4\) in Figure 9. Theorem 2.1 gives an extension of the classical elliptic integrals to the Hilbert modular case for \(\mathbb{Q}(\sqrt{5})\).

2.1 The elliptic curve \(E(y)\)

For \(y > 0\), set \(\alpha(y) = y^2\sqrt{2y}, \beta(y) = -y^2\sqrt{2y}\) and \(p(y) = 5y^2 - 10xy + Y\) where \(\sqrt{y} > 0\). Note that these \(\alpha(y), \beta(y)\) and \(p(y)\) are real valued analytic functions for \(y \in \mathbb{R}_+\). Set

\[
E(y) : v^2 = (u - \alpha(y))(u - \beta(y))(u - p(y)), \tag{2.1}
\]

for \(y \in \mathbb{R}_+\). Of course, \(E(y)\) gives the fibre for \(y \in \mathbb{R}_+\) of the elliptic surface \((K(X, Y), \pi, \mathbb{P}^1(\mathbb{C}))\). The discriminant of the right hand side of \((2.1)\) for \(u\) has five roots in \(y\)-plane.
Let $U_0$ be the domain in $\mathbb{R}^2 = \{(X,Y)\}$ described in Figure 3. The curve in Figure 3 is Klein’s icosahedral relation in $\mathbb{L}$. If $(X,Y) \in U_0$, the five roots of the discriminant of the right hand side of (2.1) for $u$ are in $\mathbb{R}^+(\subset y-$space). So, we let $s_1 = s_1(X,Y), s_2 = s_2(X,Y), s_3 = s_3(X,Y), s_4 = s_4(X,Y)$ and $s_5 = s_5(X,Y)$ be these five roots such that $0 < s_1 < s_2 < s_3 < s_4 < s_5$.

For $(X,Y) \in U_0$ and $s_{j-1} < y < s_j$ ($j = 0, \cdots, 6$), we denote the right hand side of $E(y)$ by $(u-w_1(y))(u-w_2(y))(u-w_3(y))$, where $w_1(y) < w_2(y) < w_3(y)$ (see Table 2 and Figure 4).

| $0 < y < s_1$ | $s_1 < y < s_2$ | $s_2 < y < s_3$ | $s_3 < y < s_4$ | $s_4 < y < s_5$ | $s_5 < y$ |
|----------------|-----------------|-----------------|-----------------|-----------------|----------|
| $w_1(y)$       | $\beta(y)$     | $\beta(y)$     | $p(y)$          | $\beta(y)$     | $\beta(y)$ |
| $w_2(y)$       | $\alpha(y)$    | $p(y)$          | $\beta(y)$     | $p(y)$          | $\alpha(y)$ |
| $w_3(y)$       | $p(y)$          | $\alpha(y)$    | $\alpha(y)$    | $p(y)$          | $\alpha(y)$ |

Table 2: The correspondence between $\{w_1, w_2, w_3\}$ and $\{\alpha, \beta, p\}$.

Since the points $\alpha(y), \beta(y)$ and $p(y)$ are real-valued for $y \in \mathbb{R}^+$, the function

$$F(y, u_+) = \sqrt{(u_+ - \alpha(y))(u_+ - \beta(y))(u_+ - p(y))}$$

is single-valued on $\{(y, u_+)|y \in \mathbb{R}^+, \text{Im}(u_+) > 0\}$. Hence,

$$F(y, u) = \lim_{t \to 0} F(y, u + \sqrt{-1}t) \in \mathbb{R} \quad (2.2)$$

is single-valued for $s_{j-1} < y < s_j$ and $u \notin \{\alpha(y), \beta(y), p(y), \infty\}$ as Table 3.

Take a base point $b \in (s_2, s_3)(\subset \mathbb{R})$. We can take the basis $\{\gamma_1, \gamma_2\}$ of the homology group $H_1(\pi^{-1}(b), \mathbb{Z})$ such that $(\gamma_1 \cdot \gamma_2) = 1$ and

$$\int_{\gamma_1} \omega = 2 \int_{\beta(b)}^{p(b)} \frac{du}{\sqrt{F(b, u)}}, \quad \int_{\gamma_2} \omega = 2 \int_{\alpha(b)}^{\beta(b)} \frac{du}{\sqrt{F(b, u)}}.$$
For \( j \in \{0, 1, 2\} \) (\( \in \{3, 4, 5\} \), resp.), we put \( l_j = \{(s_j, \sqrt{-1}t)|t \geq 0\} = \{(s_j, -\sqrt{-1}t)|t \geq 0\}, \) resp.). We call \( l_j \) the cut line for \( s_j \). For \( y \in \mathbb{C} - \{l_0, \cdots, l_5\} \), take an arc \( \alpha_y \) which does not touch the cut lines \( l_j (j \in \{0, \cdots, 5\}) \) with the start (end, resp.) point \( b \) (\( y \), resp.). Let \( u \mapsto a_y(u) \) (\( 0 \leq u \leq 1 \)) be the parametric representation of \( \alpha_y \). Take a 1-cycle \( \gamma \) on \( E(b) \). For \( \gamma \in H_1(\pi^{-1}(b), \mathbb{Z}) \), we choose the 1-cycle \( \gamma_{\alpha_y}(u) \) on \( \pi^{-1}(a_y(u)) \) which depends continuously on \( u \) with \( \gamma_{\alpha_y}(0) = \gamma \). If \( \alpha'_y \) is homotopic to \( \alpha_y \) in \( \mathbb{C} - \{l_0 \cup \cdots \cup l_5\} \), we have \( \gamma_{\alpha_y}(1) = \gamma_{\alpha'_y}(1) \). So, we have a well-defined correspondence \( \mathbb{C} - \{l_0 \cup \cdots \cup l_5\} \ni y \mapsto \gamma_{\alpha_y}(1) \in H_1(\pi^{-1}(y), \mathbb{Z}) \). Then, we put

\[
\gamma = \gamma_{\alpha_y}(1) \in H_1(\pi^{-1}(y), \mathbb{Z}) \quad (y \in \mathbb{C} - \{y_0, \cdots, y_5\}).
\]

Next, let \( r_j \) (\( j = 0, 1, \cdots, 5 \)) be a closed arc on \( \mathbb{C} - \{0, s_1, \cdots, s_5\} \), starting at \( s_j \) with the positive orientation and ending at \( b \). We assume that \( r_j \) does not touch the cut line \( l_k \) if \( j \neq k \). Let \( t \mapsto u_j(t) \) (\( 0 \leq t \leq 1 \)) be the parametric representation of \( r_j \). For instance, we can take an arc \( r_1 \) as in Figure 5. We choose 1-cycles \( \gamma_1(t) \) and \( \gamma_2(t) \) on \( \pi^{-1}(u_j(t)) \) which depend continuously on \( t \) such that \( \gamma_1(0) = \gamma_1 \) and \( \gamma_2(0) = \gamma_2 \). So, we have

\[
\begin{pmatrix} \gamma_1(1) \\ \gamma_2(1) \end{pmatrix} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},
\]

where \( a_j, b_j, c_j, d_j \in \mathbb{Z} \) and \( a_jd_j - b_jc_j = 1 \). The correspondence \( r_j \mapsto M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \) gives a representation of the fundamental group \( \pi_1(\mathbb{C} - \{0, s_1, \cdots, s_5\}) \). We call the matrix \( M_j \) the monodromy matrix for \( r_j \).

![Figure 5: The points 0, s_1, \cdots, s_6, the cut lines and an arc r_1 going around s_1.](image)

**Remark 2.1.** If an arc \( r \) in the base space of an elliptic fibration goes around a singular fibre with the positive orientation, the monodromy matrix \( M_r \) is obtained by K. Kodaira (\[9\], Theorem 9.1). For example, if the singular fibre is of type \( I_0 \) (\( b > 0 \)) or \( III^* \), the monodromy matrix \( M_r \) is given by \( B^{-1}M_r^0B \), where \( M_r^0 \) is given by Table 4 and \( B \in GL(2, \mathbb{Z}) \).

**Lemma 2.1.** The monodromy matrices \( M_j \) for \( \{\gamma_1, \gamma_2\} \) are given by Table 5.

**Proof.** Let us determine matrix \( M_2 \) around \( s_2 \). The fibre \( \pi^{-1}(s_2) \) is a singular fibre of type \( I_2 \). So, the monodromy matrix \( M_2 \) is in the form

\[
M_2 = B^{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} B,
\]
### Singular Fibre Matrix

| Singular Fibre | Matrix $M^0_r$ |
|----------------|---------------|
| $I_b$          | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $III^*$        | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ |

Table 4: The matrices $M^0_r$ for the singular fibres of type $I_b$ and $III^*$.

| Type of Singular Fibre | Monodromy Matrix for $\gamma_1, \gamma_2$ |
|------------------------|------------------------------------------|
| $y_1$                  | $M_1 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ |
| $y_2$                  | $M_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ |
| $y_3$                  | $M_3 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ |
| $y_4$                  | $M_4 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ |
| $y_5$                  | $M_5 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ |
| $0$                    | $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $\infty$              | $M_\infty = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$ |

Table 5: The monodromy matrices $M_j$ ($j = 0, 1, \cdots, 5, \infty$).

where $B \in GL(2, \mathbb{Z})$. Observe that $p(y) = w_1^{(3)}(y)$ converges to $\beta(y) = w_2^{(3)}(y)$ when $y \to y_2 + 0$. So, the matrix $M_2$ fixes the 1-cycle $\gamma_1 = \gamma_1^{(3)}$. Hence, we have $B = I_2$ and $M_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. By the same argument, we obtain Table 5.

\[ \square \]

### 2.2 The transcendental lattice $\langle D_1, \cdots, D_4 \rangle$

From Table 5, we have the following relations:

\[
\begin{align*}
M_1M_2M_4M_5 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \\
M_1M_2M_5M_3 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \\
M_2^{-1}M_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
M_0M_1M_2M_0^{-1}M_3^{-1} &= \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.
\end{align*}
\tag{2.4}
\]

The transformation given by the matrix $M_1M_2M_4M_3$ fixes the 1-cycle $\gamma_2$. Let $\rho_1$ be a closed curve in $y$-plane starting from the base point $b$ and goes around $s_1, s_2, s_4$ and $s_3$ successively. Let $t \mapsto s(t)$ be a parametric representation of $\rho_1$. For $0 \leq t \leq 1$, we put the 1-cycle $\gamma^{(1)}(t)$ on the elliptic curve $\pi^{-1}(s(t))$. The 1-cycle $\gamma^{(1)}(t)$ depends continuously on $t$ and $\gamma^{(1)}(0) = \gamma^{(1)}(1) = \gamma_2$ on $\pi^{-1}(b) = \pi^{-1}(s(0)) = \pi^{-1}(s(1))$. Then, the set

\[ C_1 = \bigcup_{0 \leq t \leq 1} \gamma^{(1)}(t) \]

defines a 2-cycle on the surface $K(X, Y)$. Similarly, we have the 2-cycles $C_2, C_3$ in Figure 6 and $C_4$ in Figure 7.
Figure 6: 2-cycles $C_1, C_2, C_3$.

Figure 7: 2-cycle $C_4$. 
Lemma 2.2. The intersection matrix for \( \{C_1, C_2, C_3, C_4\} \) is given by

\[
((C_j \cdot C_k))_{j,k=1, \ldots, 4} = \begin{pmatrix}
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & -4 & -6 \\
0 & 0 & -6 & -4
\end{pmatrix}.
\]

Proof. Let \( \rho_j \) be the base arc of \( C_j \). For \( y \in \rho_j \), let \( \gamma^{(j)}(y) = C_j \cap \pi^{-1}(y) \). Suppose the base arcs \( \rho_j \) and \( \rho_k \) intersect at \( s \) points \( y_1, \ldots, y_s \) in \( y \)-plane. Then, the intersection number \( (C_j \cdot C_k) \) is given by the following formula:

\[
(C_j \cdot C_k) = \sum_{l=1}^{s} (-1)(\rho_j \cdot \rho_k)_{y_l}(\gamma^{(j)}(y_l) \cdot \gamma^{(k)}(y_l)),
\]

where \((\rho_j \cdot \rho_k)_{y_l}\) is the intersection number of the base arcs \( \rho_j \) and \( \rho_k \) at the point \( y_l \) and \((\gamma^{(j)}(y_l) \cdot \gamma^{(k)}(y_l))\) is the intersection number of 1-cycles on the elliptic curve \( \pi^{-1}(y_l) \). See Figure 6. The base arc \( \rho_1 \) and \( \rho_2 \) intersect at 2 points \( a_1 \) and \( a_2 \). We have \((\rho_1 \cdot \rho_2)_{a_1} = +1 \) and \((\rho_1 \cdot \rho_2)_{a_2} = -1 \). Then, from (2.5), we have

\[
(C_1 \cdot C_2) = (-1)(1)(-\gamma_1 - 2\gamma_2) + (-1)(-1)(-2\gamma_1 + \gamma_2) = (-1)(-2) + 0 = 2.
\]

By the same argument, the claim follows.

Due to the above lemma, the following corollary is obvious.

Corollary 2.1. Put

\[
D_1 = C_1, \quad D_2 = C_2, \quad D_3 = C_4 - C_3, \quad D_4 = C_4.
\]

Then, the intersection matrix for \( \{D_1, \ldots, D_4\} \) is given by

\[
((D_j \cdot D_k))_{j,k=1, \ldots, 4} = \begin{pmatrix}
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 4 & 2 \\
0 & 0 & 2 & -4
\end{pmatrix}.
\]

Proposition 2.1. The system \( \{D_1, D_2, D_3, D_4\} \) gives a basis of the transcendental lattice of \( K(X, Y) \) with the intersection matrix \( A(2) \).

Proof. By the above construction, 2-cycle \( D_j \) \( (j = 1, \ldots, 4) \) does not touch the singular fibres of \( (K(X, Y), \pi, \mathbb{P}^1(\mathbb{C})) \). So, from Theorem 1.1 and Proposition 1.2, the system \( \{D_1, \ldots, D_4\} \) gives a basis of \( \text{Tr}(K(X, Y)) \).

2.3 The 2 cycles \( L_1, \ldots, L_6 \)

Next, we define the 2-cycles \( L_1, \ldots, L_6 \) on \( K(X, Y) \). Let \( \delta_j \) \( (j = 1, \ldots, 6) \) be an arc in \( y \)-plane with a parametric representation \( t \mapsto q_j(t) \) \( (0 \leq t \leq 1) \) whose start point and end point is given by Table 6. Remark that we take them such that \( \delta_j \) does not touch the cut lines \( l_k \) \( (k \in \{0, \ldots, 5\}) \) if \( 0 < t < 1 \). Hence, we can put a 1-cycle \( \delta^{(j)}(q_j(t)) \) on \( \pi^{-1}(q_j(t)) \) as Table 6 with the manner in (2.8). Then, we can see that \( L_j = \bigcup_{0 \leq t \leq 1} \delta^{(j)}(q_j(t)) \) gives a 2-cycle on \( K(X, Y) \) (see Figure 8).

As we proved Lemma 2.2, we can prove the following lemma and corollary.

Lemma 2.3.

\[
((L_j \cdot C_k))_{1 \leq j \leq 6, 1 \leq k \leq 4} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 1 & 1 & -1 \\
0 & 0 & -2 & -3 \\
0 & 1 & -2 & -3 \\
0 & 0 & 2 & 0
\end{pmatrix}.
\]
Corollary 2.2.

\[
((L_j \cdot D_k))_{1 \leq j \leq 6, 1 \leq k \leq 4} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 1 & -2 & -1 \\
0 & 0 & -1 & -3 \\
0 & 1 & -1 & -3 \\
0 & 0 & -2 & 0
\end{pmatrix}.
\]  

(2.9)

Figure 8: 2-cycles \(L_1, L_2, L_3, L_4, L_5\) and \(L_6\).

Proposition 2.2. A branch of the period mapping \(\Phi_K\) in (1.14) on \(U_0\) has the following expression:

\[
\begin{aligned}
\int_{\Delta_1} \omega_K &= \int_{L_1 + L_2} \omega_K, \\
\int_{\Delta_2} \omega_K &= \int_{L_1} \omega_K = \int_{L_5 - L_4} \omega_K, \\
\int_{\Delta_3} \omega_K &= \int_{-L_4 - 3(L_6 + L_5 - L_4 - L_3 + L_2 + L_1)} \omega_K, \\
\int_{\Delta_4} \omega_K &= \int_{L_6 + L_5 - L_4 - L_3 + L_2 + L_1} \omega_K.
\end{aligned}
\]

(2.10)

Proof. According to Proposition 2.1 \(\{D_1, \cdots, D_4\}\) gives a basis of \(\text{Tr}(K(X, Y))\). Recall the construction of 2-cycles \(\Gamma_1, \cdots, \Gamma_4\) on \(S(X, Y)\) in Remark 1.3. Together with Proposition 1.1 it is sufficient to take 2-cycles \(\Delta_1, \cdots, \Delta_4 \in H_2(K(X, Y), \mathbb{Z})\) such that \((\Delta_j \cdot D_k) = \delta_{jk}\). By Corollary 2.2 we can check that the 2-cycles in the right hand side of (2.10) satisfies these properties.
2.4 The cambers \( R_1, R_2, R_3 \) and \( R_4 \)

![Diagram of cambers](image)

Figure 9: The chambers \( R_1, R_2, R_3 \) and \( R_4 \).

We set the chambers in \( \mathbb{R}^2 \) (see Figure 9):

\[
\begin{align*}
R_1 &= \{(u, y)|0 \leq y \leq s_2, w_1(y) \leq u \leq w_2(y)\}, \\
R_2 &= \{(u, y)|s_1 \leq y \leq s_4, w_2(y) \leq u \leq w_3(y)\}, \\
R_3 &= \{(u, y)|s_2 \leq y \leq s_3, w_1(y) \leq u \leq w_2(y)\}, \\
R_4 &= \{(u, y)|s_4 \leq y \leq s_5, w_2(y) \leq u \leq w_3(y)\}. 
\end{align*}
\tag{2.11}
\]

They are surrounded by the branch divisors \( P \) and \( Q \). From Table 2, we obtain Table 7.

| \( y \) | \( \frac{1}{2} \left( \int_{\gamma_1(y)} \omega_y \right) \) | \( \frac{1}{2} \left( \int_{\gamma_2(y)} \omega_y \right) \) |
|---|---|---|
| \( 0 < y < s_1 \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) |
| \( s_1 < y < s_2 \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) |
| \( s_2 < y < s_3 \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) |
| \( s_3 < y < s_4 \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) |
| \( s_4 < y < s_5 \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) |
| \( s_5 < y \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) | \( \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} + \int_{\alpha(y)}^{\beta(y)} \frac{du}{F(u, y)} \) |

Table 7: The elliptic integrals on \( E(y) \) for \( (s_{j-1}, s_j) \).

**Theorem 2.1.** A branch of the period mapping \( \Phi_K \) in \( \mathbb{C} \) on \( U_0 \) is given by the following double integrals on the chambers \( R_1, R_2, R_3 \) and \( R_4 \):

\[
\begin{align*}
\int_{\Delta_1} \omega &= 2 \int_{R_2} \frac{dudy}{F(u, y)} + 2 \int_{R_4} \frac{dudy}{F(u, y)} \\
\int_{\Delta_3} \omega &= 6 \int_{R_1} \frac{dudy}{F(u, y)} + 2 \int_{R_3} \frac{dudy}{F(u, y)} \\
\int_{\Delta_2} \omega &= 2 \int_{R_2} \frac{dudy}{F(u, y)} \\
\int_{\Delta_4} \omega &= -2 \int_{R_1} \frac{dudy}{F(u, y)}. 
\end{align*}
\tag{2.12}
\]
Proof. From Proposition (2.2), Table 6 and Table 7, we have

\[
\int_{\Delta_2} \omega_K = \int_{L_5} \omega_K - \int_{L_4} \omega_K = 2 \int_{s_1}^{s_4} \int_{\gamma_1(y)-\gamma_2(y)} \frac{dydu}{F(u,y)} - 2 \int_{s_2}^{s_3} \int_{\gamma_1(y)} \omega_K = 2 \int_{s_1}^{s_4} \int_{R_2}^{\alpha(y)} \frac{dydu}{F(u,y)} = \int_{R_2} \frac{dydu}{F(u,y)}.
\]

Similarly, we have

\[
\begin{cases}
\int_{\Delta_1} \omega_K = \int_{L_5} \omega_K - \int_{L_4} \omega_K + \int_{L_2} \omega_K = 2 \int_{R_2} \frac{dydu}{F(u,y)} + 2 \int_{s_4}^{s_5} \frac{dydu}{F(u,y)} = 2 \int_{R_2} \frac{dydu}{F(u,y)} + 2 \int_{R_4} \frac{dydu}{F(u,y)}, \\
\int_{\Delta_4} \omega_K = \int_{L_6} \omega_K + \int_{L_4} \omega_K - \int_{L_3} \omega_K + \int_{L_2} \omega_K = 2 \int_{s_1}^{s_4} \frac{dydu}{F(u,y)} + 2 \int_{s_3}^{s_4} \frac{dydu}{F(u,y)} = - \int_{R_4} \frac{dydu}{F(u,y)}, \\
\int_{\Delta_3} = - \int_{L_4} \omega_K - 3 \int_{L_3} \omega_K = 2 \int_{s_2}^{s_3} \frac{dydu}{F(u,y)} + 6 \int_{R_1} \frac{dydu}{F(u,y)} = 2 \int_{R_3} \frac{dydu}{F(u,y)} + 6 \int_{R_1} \frac{dydu}{F(u,y)}.
\end{cases}
\]

By the analytic continuation of the single-valued branch on \(U_0\) given by the integrals in (2.12), we have the multivalued period mapping \(\Phi_K\) for the family \(K\). Hence, the Hilbert modular functions for \(\mathbb{Q}(\sqrt{5})\) is deeply concerned with the arrangement of the divisors \(P\) in (1.2) and \(Q\) in (1.3). The above theorem gives a canonical extension of the classical elliptic integrals to the Hilbert modular case with the smallest discriminant.

Acknowledgment

The author would like to thank Professor Hironori Shiga for helpful advises and valuable suggestions. He is also grateful to Professor Kimio Ueno and the members of his laboratory for kind encouragements. He is grateful to the referee for careful reading and valuable comments. This work is supported by Waseda University Grant for Special Research Project 2013A - 870 and 2014B -169.

References

[1] B. van Geemen and A. Sarti, *Nikulin involutions on K3 surfaces*, Math. Z 255, 731-753, 2007.
[2] K. B. Gundlach, *Die Bestimmung der Funktionen zur Hirbertschen Modulgruppe des Zahlkörpers \(\mathbb{Q}(\sqrt{5})\)*, Math. Ann. 152, 1963, 226-256.
[3] G. Humbert, *Sur les fonctions abéliennes singulières*, Oeuvres de G. Humbert 2, pub. par les soins de Pierre Humbert et de Gaston Julia, Gauthier-Villars, 297-401, 1936.
[4] F. Hirzebruch, *The ring of Hilbert modular forms for real quadratic fields of small discriminant*, Lecture Notes in Math. 627, Springer-Verlag, 1977, 287-323.
[5] K. Hashimoto and Y. Murabayashi, *Shimura curves as intersections of Humbert equations and defining Equations of QM-curves of genus two*, Tohoku Math. J., 47 (2), 1995, 271-296.
[6] K. Kodaira, *On analytic surfaces II*, Ann. of Math., 77, 1963, 563-626.
[7] R. Kobayashi, K. Kushibiki and I. Naruki, *Polygons and Hilbert modular groups*, Tohoku Math. 41, 1989, 633-646.

[8] K. Mastumoto, T. Sasaki and M. Yoshida, *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3, 6)*, Internat. J. Math, 3, 1-164, 1992.

[9] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. 75, 105-121, 1984.

[10] R. Müller *Hilbertsche Modulformen und Modulfunktionen zu $\mathbb{Q}(\sqrt{5})$*, Arch. Math. 45, 1985, 239-251

[11] A. Nagano, *A theta expression of the Hilbert modular functions for $\sqrt{5}$ via the periods of K3 surfaces*, Kyoto J. Math., 53 (4), 2013, 815-843.

[12] A. Nagano and H. Shiga, *Modular map for the family of Kummer surfaces via K3 surfaces*, Math. Nachr., to appear, 2014.

---

Atsuhira Nagano  
Department of Mathematics  
Waseda University  
Okubo 3-4-1, Shinjuku-ku, Tokyo, 169-8555  
Japan  
(E-mail: atsuhira.nagano@gmail.com)