Two characterizations of simple circulant tournaments

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July 15, 2015

Abstract

The acyclic disconnection $\omega(D)$ (resp. the directed triangle free disconnection $\omega_3(D)$) of a digraph $D$ is defined as the maximum possible number of connected components of the underlying graph of $D \setminus A(D^*)$ where $D^*$ is an acyclic (resp. a directed triangle free) subdigraph of $D$. We say that $\pi$ is an externally acyclic (resp. a $\mathcal{C}_3$-free) partition if the set of external arcs of $D$ (an arc $(u, v)$ of $D$ is said to be external if $u$ and $v$ belong to different equivalence classes of $\pi$) induces an acyclic subdigraph (resp. a $\mathcal{C}_3$-free subdigraph) of $D$. A digraph $D$ is said to be $\omega$-keen (resp. $\omega_3$-keen) if there exists an optimal externally acyclic (resp. $\mathcal{C}_3$-free) partition $\pi$ of $V(D)$ having exactly one equivalence class of cardinality one and no externally acyclic (resp. $\mathcal{C}_3$-free) partition leaves more than one such a class. In this paper, we generalize some previous results and solve some problems posed by V. Neumann-Lara (The acyclic disconnection of a digraph, Discrete Math. 197/198 (1999), 617-632). Let $\mathcal{C}_{2n+1}(J)$ be a circulant tournament. We prove that $\mathcal{C}_{2n+1}(J)$ is $\omega$-keen and $\omega_3$-keen, respectively, and $\omega(\mathcal{C}_{2n+1}(J)) = \omega_3(\mathcal{C}_{2n+1}(J)) = 2$ for every $\mathcal{C}_{2n+1}(J)$. Finally, it is showed that $\omega_3(\mathcal{C}_{2n+1}(J)) = 2$, $\mathcal{C}_{2n+1}(J)$ is simple and $J$ is aperiodic are equivalent propositions. The proofs of the results are essentially based on classic results for abelian groups of additive number theory.

Keywords: circulant tournament; acyclic disconnection; tight tournament; simple (or prime) tournament; addition theorems

Math. Subj. Class.: 05C20, 11P70

1 Introduction

The acyclic disconnection $\omega(D)$ of a digraph $D$ is defined in [11] as the maximum possible number of connected components of the underlying graph of $D \setminus A(D^*)$ where $A(D^*)$ denotes the arc set of an acyclic subdigraph $D^*$ of $D$ (that is, $D^*$ contains no directed cycle). The so called directed triangle-free disconnection (in brief, the $\mathcal{C}_3$-free disconnection) $\omega_3(D)$ of $D$ (a closed definition introduced in the same paper) is the maximum possible number of connected components of the underlying graph of $D \setminus A(D^*)$ where $D^*$ is a directed triangle-free subdigraph of $D$ (that is, $D^*$

*Research supported by CONACYT Project CB-2012-01 178910.
contains no $\vec{C}_3$. Equivalently, $\mathcal{W}(D)$ can be defined as the maximum number of colors in a coloring of the vertices of $D$ such that no cycle is properly colored (in a proper coloring, consecutive vertices of the directed cycle receive different colors). Similarly, the $\vec{C}_3$-free disconnection $\mathcal{W}_3(D)$ of $D$ is the maximum number of colors in a coloring of the vertices of $D$ such that no 3-cycle $\vec{C}_3$ is 3-colored. A small value of both parameters implies a large cyclic complexity of the digraph $D$. Apart from the first and seminal results for these parameters in general digraphs that were developed in [11], the acyclic and the $\vec{C}_3$-free disconnection have mainly been studied for tournaments, specially, for circulant tournaments (see for instance the already mentioned paper, [9] and [8]) and for some classes of (non-circulant) regular tournaments in [9].

The following definitions were introduced in [11]. Let $k \in \mathbb{N}$. Consider a partition $\pi = V_1 | V_2 | \cdots | V_k$ of the vertex set of a digraph $D$ into nonempty subsets $V_j$ for $1 \leq j \leq k$. An equivalence class $V_j$ of $\pi$ is singular if $V_j$ is a singleton. An arc $(u, v) \in A(D)$ is said to be external if $u$ and $v$ belong to different equivalence classes of $\pi$. We say that $\pi$ is an externally acyclic (resp. a $\vec{C}_3$-free) partition if the set of external arcs of $D$ induces an acyclic subdigraph (resp. a $\vec{C}_3$-free subdigraph) of $D$. An externally acyclic (resp. $\vec{C}_3$-free) partition $\pi = V_1 | \cdots | V_k$ of $V(D)$ of a digraph $D$ is optimal if $\mathcal{W}(D) = k$ (resp. $\mathcal{W}_3(D) = k$). A digraph $D$ is said to be $\mathcal{W}$-keen (resp. $\mathcal{W}_3$-keen) if every optimal externally acyclic (resp. $\vec{C}_3$-free) partition $\pi$ of $V(D)$ has exactly one singular equivalence class.

In [8], it was proved that every prime order circulant tournament $\vec{C}_p(J)$ with $p \geq 3$ (see the definition in the next section) is $\mathcal{W}_3$-keen, $\mathcal{W}$-keen and $\mathcal{W}(\vec{C}_p(J)) = \mathcal{W}_3(\vec{C}_p(J)) = 2$.

In this paper, we generalize the aforementioned results and solve the following problems posed in [11]:

(i) On p. 623, end of Section 3, it is said: "We do not know whether every circulant tournament is $\mathcal{W}$-keen (resp. $\mathcal{W}_3$-keen)." We prove that every circulant tournament is $\mathcal{W}$-keen and $\mathcal{W}_3$-keen, see Theorems [13] and [14], respectively.

(ii) Problem 6.6.1. Characterize the circulant (or regular) tournaments $T$ for which $\mathcal{W}(T) = 2$ (resp. $\mathcal{W}_3(T) = 2$). Are they the same? We prove that $\mathcal{W}(\vec{C}_{2n+1}(J)) = \mathcal{W}_3(\vec{C}_{2n+1}(J)) = 2$

for every simple circulant tournaments $\vec{C}_{2n+1}(J)$ (a consequence of Theorem [19] and Corollary [22]).

(iii) Problem 6.6.3: $\mathcal{W}(\vec{C}_{2n+1}(J)) = \mathcal{W}_3(\vec{C}_{2n+1}(J))$ for every $\vec{C}_{2n+1}(J)$ circulant tournament (Corollary [22]). This problem remains open when $T$ is regular in general.

We give an interesting characterization of compositions of circulant tournaments in terms of an additive combinatorial property of their symbol sets (Theorem [4] and in terms of their $\vec{C}_3$-free disconnection (Theorem [19] a fact conjectured by V. Neumann-Lara in a personal communication). The main results of these paper are summarized in Corollary [21]. The proofs of the theorems are essentially based on classic results for abelian groups of additive number theory. Additive (or combinatorial as P. Erdős used to call it) number theory could be described as the study of the discrete properties of subsets of abelian groups in general. In this work, we focus on applying these tools to the group of residues modulo a positive integer $n$ in order to solve three coloring and structural problems for tournaments.
2 Preliminary results

A tournament $T$ on $n$ vertices is an orientation of the complete graph $K_n$. For every tournament $T$, we have that $\overrightarrow{\omega}_3(T) \geq 2$ (\cite{11}, Remark 4.1(i)). We say that $T$ is tight if $\overrightarrow{\omega}_3(T) = 2$.

**Proposition 1** (\cite{11}, Corollary 2.6). If $T$ is a regular tournament, then every externally $\overrightarrow{C}_3$-free partition of $T$ has at most one singular equivalence class.

**Remark 2.** (i) As a consequence of this proposition, if a regular tournament $T$ has an optimal externally $\overrightarrow{C}_3$-free partition with a unique singular equivalence class, then $T$ is $\overrightarrow{\omega}_3$-keen.

(ii) In particular, if $\overrightarrow{\omega}_3(T) = 2$ (that is, $T$ is tight), where $T$ is regular, then $T$ is $\overrightarrow{\omega}_3$-keen.

Throughout this paper, we deal with regular tournaments which are well-known to have odd order. We denote by $\mathbb{Z}_{2m+1}$ the group of residues modulo $2m+1$, where $m$ is a positive integer. Let $J \subseteq \mathbb{Z}_{2m+1} \setminus \{0\}$ such that $|J| = m$ and $|\{j,-j\} \cap J| = 1$ for every $j \in J$ (or equivalently, $J = -J \cup \{0\}$). A circulant tournament $T = \overrightarrow{C}_{2m+1}(J)$ (or a rotational tournament $RT_{2m+1}(J)$, $m \in \mathbb{N}$, see \cite{12}) with symbol set $J$ is defined by

\[
V(\overrightarrow{C}_{2m+1}(J)) = \mathbb{Z}_{2m+1} \text{ and } \\
A(\overrightarrow{C}_{2m+1}(J)) = \{(i,j) : i, j \in \mathbb{Z}_{2m+1} \text{ and } j-i \in J\}.
\]

Notice that from the definition, $0 \notin J + J$ and $|J| = m$. It is well-known that a circulant tournament is regular and vertex-transitive, that is, for every pair of vertices $u, v \in V(\overrightarrow{C}_{2m+1}(J))$, there exists an automorphism of $\overrightarrow{C}_{2m+1}(J)$ that maps $u$ to $v$. This property is constantly used in proofs as it shows that circulant tournaments are highly symmetric structures.

The tournament $\overrightarrow{C}_{2m+1}(1,2,\ldots,m)$ is called the cyclic tournament. It is easy to check that for all $m$ there is only one cyclic tournament up to isomorphism.

Let $D$ and $F$ be digraphs and $F_v$ a family of mutually disjoint isomorphic copies of $F$ for all $v \in V(D)$. The composition (or lexicographic product, see \cite{6}) $D[F]$ of the digraphs $D$ and $F$ is defined by

\[
V(D[F]) = \bigcup_{v \in V(D)} V(F_v) \text{ and } \\
A(D[F]) = \bigcup_{v \in V(D)} A(F_v) \cup \{(i,j) : i \in V(F_v), j \in V(F_w) \text{ and } (v,w) \in A(D)\}.
\]

The composition of digraphs is not commutative, but it is associative (an easy proof left to the reader). A digraph $D$ is said to be simple (or prime, see \cite{6}, p. 116) if it is not isomorphic to a composition of digraphs (see \cite{10}).

**Proposition 3** (\cite{11}, Neumann-Lara). If $D$ is a $\overrightarrow{\omega}_3$-keen (resp. $\overrightarrow{\omega}$-keen) digraph and $F$ is a digraph, then

\[
\overrightarrow{\omega}_3(D[F]) = \overrightarrow{\omega}_3(D) + \overrightarrow{\omega}_3(F) - 1
\]

(respectively, \[
\overrightarrow{\omega}(D[F]) = \overrightarrow{\omega}(D) + \overrightarrow{\omega}(F) - 1.
\]

Moreover, if $D$ and $F$ are $\overrightarrow{\omega}_3$-keen (resp. $\overrightarrow{\omega}$-keen) digraphs, then $D[F]$ is also $\overrightarrow{\omega}_3$-keen (resp. $\overrightarrow{\omega}$-keen).
We define the interval of positive integers $[1, k] = \{1, 2, ..., k\}$ ($k \geq 1$).

**Proposition 4** ([11], Neumann-Lara). Let $\overrightarrow{C}_{2n+1}(J)$ and $\overrightarrow{C}_{2n+1}(K)$ be circulant tournaments. Then the composition $\overrightarrow{C}_{2n+1}(J) \overrightarrow{C}_{2n+1}(K)$ is a circulant tournament isomorphic to $\overrightarrow{C}_{(2m+1)(2n+1)}(L)$, where the symbol set

$$L = (2m+1)K \cup (J + (2m+1)[1, 2n+1])$$

Given nonempty subsets $A, B \subseteq \mathbb{Z}_n$, we define

$$A + B = \{a + b : a \in A, b \in B\}, \quad cA = \{ca : a \in A, c \in \mathbb{Z}_n\},$$

$$-A = \{-a : a \in A\} \text{ and } A - B = A + (-B).$$

If $A$ and $B$ are sets, we denote the set difference by $A \setminus B$. A set $\emptyset \neq A \subseteq \mathbb{Z}_n$ is an arithmetic progression if there exist $a, d \in \mathbb{Z}_n$, $d \neq 0$ such that

$$A = \{a + id : 0 \leq i \leq |A| - 1\}.$$

Let $A$, $B$ and $C$ nonempty subsets of $\mathbb{Z}_n$. As a simple consequence of the above definitions we have (see [13])

$$(A + B) \cap C = \emptyset \Leftrightarrow A \cap (B - C) = \emptyset. \quad (1)$$

Let $G$ be an abelian group written additively and $\emptyset \neq C \subseteq G$. Following [5] (and [4]), we define the period of the set $C$ as

$$H(C) = \{g \in G : C + g = C\}.$$

Then $C + H(C) = C$ and $H(C)$ is a subgroup of $G$. We say that $C$ is periodic if $H(C) \neq \{0\}$ and aperiodic if $H(C) = \{0\}$. Observe that $C$ is periodic if and only if $C$ is the union of $H(C)$-cosets The set $C$ is called quasi-periodic if there exists $H$, a nontrivial subgroup of $G$, such that $C = C' \cup C''$ ($C'$ and $C''$ could be empty) and $C' \cap C'' = \emptyset$, where

(i) $C'$ is periodic and

(ii) $C''$ is properly contained in an $H$-coset, that is, $C'' \subset c + H$, where $c \in C''$.

In this setting, notice that if $C$ is periodic, then it is quasi-periodic. It is straightforward to check that an arithmetic progression of length at least 3 is an aperiodic set.

**Theorem 5** ([7] Kneser). Let $G$ be a non-trivial abelian group and $A$ and $B$ nonempty finite subsets of $G$. Then $H = H(A + B)$ satisfies that

$$|A + B| \geq |A + H| + |B + H| - |H| \geq |A| + |B| - |H|$$

Moreover, $A + B$ is periodic if $|A + B| \leq |A| + |B| - 2$.

The following result is a consequence of more general theorems proved in [5] by Kemperman, see also [4] for a thorough study of the so called Kemperman Structure Theorem.

**Theorem 6** ([5] Kemperman). Let $G$ be an abelian finite group of order $n$ and $A, B$ subsets of $G$ such that $|A| \geq 2$, $|B| \geq 2$ and $|A + B| = |A| + |B| - 1$. Then either $A + B$ is an arithmetic progression or $A + B$ is quasi-periodic with period $H$, a nontrivial subgroup of $G$. 

4
3 Compositions of circulant tournaments and quasi-periodic symbol sets

Let $m$ be a positive integer. Observe that $\gcd(m, 2m + 1) = 1$ and as a consequence, if $J$ is the set symbol of a circulant tournament, then $J$ is not periodic. Therefore, $J$ is aperiodic or quasi-periodic. Using Theorems 5 and 6 and the definition of the symbol set $J$ (recall that $0 \notin J$ for the second inequality), we have that

$$2m - 1 = 2 |J| - 1 \leq |J + J| \leq 2 |J| = 2m.$$  

The following theorem gives a characterization of compositions of circulant tournaments in terms of the quasi-periodicity of its symbol set.

**Theorem 7.** Let $\tilde{C}_{2p+1}(L)$ be a circulant tournament. $\tilde{C}_{2p+1}(L)$ is a composition of circulant tournaments if and only if $L$ is quasi-periodic.

**Proof.** Suppose that $\tilde{C}_{2p+1}(L)$ is isomorphic to $\tilde{C}_{2m+1}(J) \left[ \mathcal{C}_{2n+1}(K) \right]$. According to Proposition 4,

$$L = (2m + 1)K \cup (J + (2m + 1)[1, 2n + 1])$$

and $2p + 1 = (2n + 1)(2n + 1)$. First, notice that

$$H = (2m + 1)[1, 2n + 1] = \{(2m + 1)i \mod (2p + 1) : i = 0, 1, \ldots, 2n\}$$

$$= \{0, 2m + 1, 4m + 2, \ldots, 2n(2m + 1)\}$$

is a subgroup of $\mathbb{Z}_{2p+1}$ of order $2n + 1$ and every nontrivial subgroup of the cyclic group $\mathbb{Z}_{2p+1}$ is of this form if $2n + 1$ is a divisor of $2p + 1$. We have that $J \cap H = \emptyset$, since $J \subseteq \mathbb{Z}_{2m+1} \setminus \{0\}$. Therefore, $J + H$ is a union of $H$-cosets not containing the subgroup $H$ ($0 \notin J$). Similarly, $K \subseteq \mathbb{Z}_{2n+1} \setminus \{0\}$ and consequently $(2m + 1)K \subseteq H$ (recall that all products are taken modulo $2p + 1$). We have that $(2m + 1)K \cap (J + H) = \emptyset$. Let $L = C' \cup C''$, where $C' = J + H$ and $C'' = (2m + 1)K$. We conclude that $L$ is quasi-periodic.

Conversely, suppose that the symbol set $L$ of cardinality $p$ is quasi-periodic. By the definition of a quasi-periodic set, there exists $H$, a nontrivial subgroup of $\mathbb{Z}_{2p+1}$, such that $L = C' \cup C''$ and $C' \cap C'' = \emptyset$, where $C' + H = C'$ and $C'' \subseteq c + H$ with $c \in C''$. Observe that $H \cap C' = \emptyset$ because $0 \notin L$. Then

$$C' = \{j_i + H : 0 \neq j_i \in G\}.$$ 

Let $J = \{j_1, j_2, \ldots, j_s\}$, where $s$ is the number of cosets contained in $C'$. Therefore $C' = J + H$. Observe that if $j_k \in J$ where $k \in [1, s]$, then $-j_k \notin J$ (otherwise, $j_k \in L$ and $-j_k \in L$, a contradiction). Moreover, $j_k + H \subseteq L$ if and only if $-j_k + H \not\subseteq L$ by the definition of the symbol set $L$. Observe that in the quotient group $\mathbb{Z}_{2p+1}/H$, we have that $-j_k + H = -j_k + H$. Consequently, $C'' \subseteq c + H = H$. Let

$$C'' = \{(2m + 1)k_\alpha : k_\alpha \in [1, 2n + 1]\}.$$ 

We define $K = \{k_1, k_2, \ldots, k_t\}$. Once again, if $k_\alpha \in K$, then $-k_\alpha \notin K$ for every $\alpha \in [1, t]$. Observe also that $|C''| = t$. Therefore, $s(2n + 1) + t = p$. Since $p = m(2n + 1) + n$, where $m$, $n$, $s$ and $t$ are positive integers, and $2n + 1 > |n - t|$ (here $|x|$ stands for the absolute value of the integer $x$), we have that

$$s(2n + 1) + t = m(2n + 1) + n \Leftrightarrow (s - m)(2n + 1) = n - t \Leftrightarrow s = m \text{ and } n = t.$$
Hence $L = (2m + 1)K \cup (J + H)$. By Proposition 4 we conclude that $\overrightarrow{C}_{2p+1}(L)$ is isomorphic to $\overrightarrow{C}_{2m+1}(J + 1)$. 

Using Theorem 6 we have the following consequences.

**Corollary 8.** Let $\overrightarrow{C}_{2m+1}(J)$ be a circulant tournament such that $J$ is not an arithmetic progression. Then $|J + J| = 2|J| + 1$ if and only if $J$ is quasi-periodic. Moreover, if $J$ is an arithmetic progression (resp. quasi-periodic), then $J + J$ is an arithmetic progression (resp. quasi-periodic).

**Corollary 9.** Let $\overrightarrow{C}_{2m+1}(J)$ be a simple circulant tournament. Then $J$ and $J + J$ are aperiodic.

4 Simple circulant tournaments are $\omega_3$-keen and tight

Let $\overrightarrow{C}_{2n+1}(J)$ be a circulant tournament. We use the following results proved in 8 for circulant tournaments of prime order. The following first lemma remains valid for circulant tournaments of any odd order (see Remark 4 of the already mentioned paper). The second one is also true for general circulant tournaments since the proof only uses the additive properties of the ring $\mathbb{Z}_{2n+1}$.

**Lemma 10** (8, Lemma 1). Let $\pi = A \mid B \mid C$ be an externally $\overrightarrow{C}_3$-free partition of $\overrightarrow{C}_{2n+1}(J)$. Then

$$(((A + J) \cap B) + J) \cap (A - J) = \emptyset.$$  

**Lemma 11** (8, Lemma 2). Let $A$, $C$ and $J$ be nonempty subsets of $\mathbb{Z}_{2n+1}$ such that $A \cap C = \emptyset$ and $J = -J \cup \{0\}$. Then

$$(C \cap (A - J)) - J = (C - J) \cap (A - J).$$

Suppose that $\pi = A \mid B \mid C$ is an externally $\overrightarrow{C}_3$-free partition of $\overrightarrow{C}_{2n+1}(J)$. Then by Lemma 10 and using (1) we have

$$(((A + J) \cap B) + J) \cap A = \emptyset$$

$\Leftrightarrow (((A + J) \cap B) + J) \cap (A - J) = \emptyset$

$\Leftrightarrow ((A + J) \cap B) + J \cap (C \cap (A - J)) = \emptyset$

$\Leftrightarrow ((A + J) \cap B) \cap ((C \cap (A - J)) - J) = \emptyset.$

By Lemma 11 the last equality is equivalent to

$$(A + J) \cap (C - J) \cap (A - J) = \emptyset$$

$\Leftrightarrow (A + J) \cap B \cap (C - J) \subseteq A - J - J,$

where the complement is relative to $\mathbb{Z}_{2n+1}$. So we have proved the following

**Lemma 12.** Let $\pi = A \mid B \mid C$ be an externally $\overrightarrow{C}_3$-free partition of $\overrightarrow{C}_{2n+1}(J)$. Then

$$(((A + J) \cap B) + J) \cap A = \emptyset$$

$\Leftrightarrow (A + J) \cap B \cap (C - J) \subseteq A - J - J.$
We will follow the procedure provided in [8]. First, we show the following proposition, whose proof is made in the same way as did in Proposition 5 of [8], using Lemmas [10] [11] and [12] stated before. In fact, the proof we give here is shorter than that given in our previous paper.

**Proposition 13.** Let \( n \) be a positive integer and \( \overrightarrow{C}_{2n+1}(J) \) be a circulant tournament, where \( J \) is neither an arithmetic progression nor a quasi-periodic set. Then \( \overrightarrow{C}_{2n+1}(J) \) is \( \overrightarrow{w}_3 \)-keen.

**Proof.** Let us suppose that \( \overrightarrow{w}_3(\overrightarrow{C}_{2n+1}(J)) = k \geq 2 \). If \( k = 2 \), then by Remark [2](ii), \( \overrightarrow{C}_{2n+1}(J) \) is \( \overrightarrow{w}_3 \)-keen. Therefore, we consider the case when \( k \geq 3 \). By Proposition [4] the tournament \( \overrightarrow{C}_{2n+1}(J) \) has at most one singular equivalence class in every externally \( \overrightarrow{C}_3 \)-free partition. Suppose that there exists an optimal externally \( \overrightarrow{C}_3 \)-free partition of \( V(\overrightarrow{C}_{2n+1}(J)) \) without singular equivalence classes. Let \( \pi = V_1 \mid V_2 \mid \cdots \mid V_k \) (\( k \geq 3 \)) be this optimal externally \( \overrightarrow{C}_3 \)-free partition of \( \mathbb{Z}_p \) such that \( 2 \leq |V_1| \leq |V_2| \leq \cdots \leq |V_k| \) and define the sets

\[
A = V_1, \quad B = \bigcup_{i=1}^{m} V_{j_i}, \quad \text{and} \quad C = \bigcup_{i=m}^{k} V_{j_i},
\]

where \( 1 \leq m \leq k - 1 \) and the set family \( \{ V_{j_i} : i = 2, 3, \ldots, k \} \) is a permutation of \( \{ V_j : j = 2, 3, \ldots, k \} \). Without loss of generality, we can assume that \( 2 \leq |A| \leq |B| \) and \( |A| \leq |C| \). Since \( \pi \) is an externally \( \overrightarrow{C}_3 \)-free partition, so is \( \pi' = A \mid B \mid C \). Then by Lemma [10]

\[
(((A + J) \cap B) \cap (C + J)) \cap A = \emptyset
\]

which is equivalent to

\[
(A + J) \cap B \cap (C - J) \subseteq A - J - J
\]

by Lemma [12] (recall that the complement is relative to \( \mathbb{Z}_{2n+1} \)). Since \( J \) is neither an arithmetic progression nor a quasi-periodic set by assumption, \( J \) is aperiodic and hence \( J + J \) is aperiodic too in virtue of Corollary [9]. Hence, \( |J + J| = 2n \) and applying Theorem [8] we have that

\[
2n + 1 \geq |A + (-J - J)| = |A - J - J| \geq |A| + |-(J + J)| = |A| + 2n.
\]

This implies that \( |A| \leq 1 \), a contradiction to the assumption that \( |A| \geq 2 \).

Notice that cyclic circulant tournaments \( \overrightarrow{C}_{2n+1}(1, 2, \ldots, n) \) (the case when \( J \) is an arithmetic progression) are tight (Theorem 4.4(i) of [11]) and therefore, \( \overrightarrow{w}_3 \)-keen by Remark [2](ii)). Applying Proposition [13] to the case when \( J \) is aperiodic, we have the following

**Corollary 14.** Simple circulant tournaments \( \overrightarrow{C}_{2n+1}(J) \) are \( \overrightarrow{w}_3 \)-keen.

On the other hand, if \( J \) is quasi-periodic, then Theorem [7] implies that \( \overrightarrow{C}_{2n+1}(J) \) is a composition of circulant tournaments, say \( T_1[T_2] \) (\( T_1 \) and \( T_2 \) not necessarily distinct). If \( T_1 \) is not simple, then \( T_1 = T_3[T_4] \). By Proposition [4] the tournaments \( T_3 \) and \( T_4 \) are circulant. By the associativity of the composition,

\[
T_1[T_2] = (T_3[T_4])[T_2] = T_3[T_4[T_2]].
\]

Analogously, if \( T_2 \) is not simple, then there exist circulant tournaments \( T_5 \) and \( T_6 \) such that

\[
T_2 = T_5[T_6] \quad \text{and} \quad T_1[T_2] = T_1[T_5[T_6]].
\]
If both $T_1$ and $T_2$ are not simple, then we have a composition of type
\[ T_1[T_2] = (T_3[T_4])[T_5[T_6]] = T_3[T_4][T_5[T_6]]. \]
Observe that for $3 \leq j \leq 6$, the tournaments $T_j$ are not necessarily distinct. The procedure can be repeated until the tournament $\vec{C}_{2n+1}(J)$ is isomorphic to the nested composition
\[ T_1[T_2[\cdots T_{k-1}[T_k]\cdots]], \]
where $T_i$ is simple for every $1 \leq i \leq k$. We have proved the following proposition.

**Proposition 15.** Let $\vec{C}_{2n+1}(J)$ be a composition of circulant tournaments. Then there exist $T_1, T_2, \ldots, T_k$ simple circulant tournaments such that $\vec{C}_{2n+1}(J)$ is isomorphic to
\[ T_1[T_2[\cdots T_{k-1}[T_k]\cdots]]. \]

We notice that the proposition stated before is a very special case of a decomposition of a tournament (digraph) into simple (prime) tournaments (digraphs) in the case of the lexicographic product. The problem for general graphs is studied in Chapter 6 of [6] by Klavžar and Imrich and for digraphs in [2] by Dürrer and Imrich.

Proposition 15, Corollary 14 and the repeated application of Proposition 3 yield the following consequence:

**Corollary 16.** Compositions of circulant tournaments are $\omega_3$-keen.

Putting together Proposition 13 and Corollaries 14 and 16 we obtain the following result.

**Theorem 17.** Every circulant tournament is $\omega_3$-keen.

In a similar way one can prove that

**Theorem 18.** Every circulant tournament is $\omega$-keen.

Theorems 17 and 18 affirmatively answer the problem posed in [11] (p. 623, end of Section 3) already mentioned in the Introduction and generalize Theorem 3 of [8].

The next result is proved in a similar (not exact) way as Theorem 4 of [8]. The proof also uses a well-known result due to B. Alspach [1]: every arc of a regular tournament is contained in a directed cyclic triangle.

**Theorem 19.** Every simple circulant tournament $\vec{C}_{2n+1}(J)$ is tight.

**Proof.** For a contradiction, let us suppose that $\vec{C}_{2n+1}(J)$ is simple and not tight. Then there exists an externally $\vec{C}_3$-free partition of $V(\vec{C}_{2n+1}(J))$ with three equivalence classes. By Corollary 13, one of such classes is singular and since the automorphism group of $\vec{C}_{2n+1}(J)$ is vertex-transitive, we can assume without loss of generality that the externally $\vec{C}_3$-free partition of $V(\vec{C}_{2n+1}(J))$ is $\pi = \{0\} \cup B \cup C$, where $|B| + |C| = 2n$. Without loss of generality, we suppose that $|B| \leq |C|$. Taking $A = \{0\}$ in Lemma 12 we have that
\[ J_B \cap (C - J) \subseteq \overline{J - J} - J = \overline{(J + J)}, \]
where $J_B = J \cap B$ (similarly, $-J_B = (-J) \cap B$, $J_c = J \cap C$ and $-J_C = (-J) \cap C$). Since $0 \notin J + J$ and $J$ is aperiodic by Corollary 9, it follows that

$$|J + J| = 2|J| = 2n.$$  

Therefore, $0 \notin -(J + J)$ and

$$|-(J + J)| = 2|-J| = 2|J| = 2n.$$  

From this, $-(J + J) = \{0\}$ and consequently $J_B \cap (C - J) \subseteq \{0\}$. Observe that if $J_B \cap (C - J) = \{0\}$, then $0 \in J$ and $0 \in B$, which is a contradiction. Therefore, $J_B \cap (C - J) = \emptyset$. By 1, this equality is equivalent to

$$(J_B + J) \cap C = \emptyset \iff J_B + J \subseteq \overline{C} = B \cup \{0\}.$$  

Note that $0 \notin J_B + J \subseteq J + J$ and so,

$$J_B + J \subseteq B = J_B \cup (-J_B).$$  

Then

$$|J_B| + |J| \leq |J_B + J| \leq |J_B| + |-J_B| \leq n,$$

where for the inequality on the left side we use that $\overrightarrow{C}_p(J)$ is simple, Theorem 6 and Corollary 9, and for that on the right side, we use that $J_B \cap (-J_B) = \emptyset$, $|B| + |C| = 2n$ and $|B| \leq |C|$. It follows from the last inequality that $|J| = |-J_B| = n$.

Since $-J_B \subseteq -J$ and $|J| = |-J| = n$, we have that $|J_B| = |J| = n$ and so,

$$|J_B| = n \iff -J_C = \emptyset.$$  

We conclude that $-J \subseteq B$ and $C \subseteq J$. Then, the tournament $\overrightarrow{C}_{2n+1}(J)$ contains the arcs $0 \rightarrow j$ for all $j \in J_C = C$. By the already mentioned result by Alspach (see 11), there exists a directed cyclic triangle

$$
\begin{array}{c}
J \\
\cap C \ni j & \rightarrow & j' \in -J \subseteq B.
\end{array}
$$

We have obtained a contradiction, $\pi = \{0\} | B | C$ is an externally $\overrightarrow{C}_3$-free partition of $V(\overrightarrow{C}_{2n+1}(J))$.

\begin{corollary}[	extit{8}, Theorem 4] Let $p$ be an odd prime number. Then $\overrightarrow{C}_p(J)$ is tight.
\end{corollary}

Using Proposition 3 and Theorems 7 and 10, we can state the following characterization of simple circulant tournaments (and therefore, of compositions of circulant tournaments).

\begin{corollary}
Let $\overrightarrow{C}_{2n+1}(J)$ be a circulant tournament. The following conditions are equivalent:

(i) $\overrightarrow{C}_{2n+1}(J)$ is tight.

(ii) $\overrightarrow{C}_{2n+1}(J)$ is simple.

(iii) $J$ is aperiodic.
\end{corollary}

From the definitions of the acyclic and the $\overrightarrow{C}_3$-free disconnection, we have that $2 \leq \overrightarrow{J}(T) \leq \overrightarrow{J}_3(T)$ for every tournament $T$. Using again Theorems 12 and 13 combined with Proposition 9, we obtain the following corollary which solves Problem 6.6.3 of [11].

\begin{corollary}
$\overrightarrow{J}(\overrightarrow{C}_{2n+1}(J)) = \overrightarrow{J}_3(\overrightarrow{C}_{2n+1}(J))$ for every positive integer $n$.
\end{corollary}
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