Charm quark mass dependence of the electromagnetic dipole operator contribution to $\bar{B} \rightarrow X_s \gamma$ at $O(\alpha_s^2)$

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Abstract

We extend existing calculations of the electromagnetic dipole operator contribution to the total decay rate and the photon energy spectrum of the decay $\bar{B} \rightarrow X_s \gamma$ at $O(\alpha_s^2)$ by working out the exact dependence on the charm quark mass.
1 Introduction

As a flavor changing neutral current process the inclusive decay $\bar{B} \to X_s \gamma$ is loop-induced and therefore highly sensitive to new degrees of freedom beyond the Standard Model. To tap the full potential of this decay channel in deriving constraints on the parameter space of new physics models both the experiments and the Standard Model calculations should be known as accurately as possible.

On the experimental side, the latest measurements by Belle and BABAR are reported in [1, 2], and the world average performed by the Heavy Flavor Averaging Group [3] for $E_\gamma > 1.6$ GeV reads

$$\text{Br}(\bar{B} \to X_s \gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4},$$

(1.1)

where the errors are statistical, systematical, due to the extrapolation to the common lower-cut in the photon energy, and due to the $\bar{B} \to X_d \gamma$ contamination, respectively.

In order to compete with the given experimental accuracy the theoretical prediction of the $\bar{B} \to X_s \gamma$ branching ratio has to be known at the next-to-next-to-leading order (NNLO) level. There have been great efforts of several groups within the last few years to achieve this goal. The three-loop dipole operator matching was found in [4], the three-loop mixing of the four-quark operators in [5], and the three-loop mixing of the dipole operators was calculated in [6]. Furthermore, the four-loop mixing of the four-quark operators into the dipole operators was calculated in [7]. The two-loop matrix elements of the electromagnetic dipole operator together with the corresponding bremsstrahlung terms can be found in [8–11]. The three-loop matrix elements of the four-quark operators were found in [12] within the so-called large-$\beta_0$ approximation, and a calculation that goes beyond this approximation by employing an interpolation in the charm quark mass $m_c$ was presented in [13]. The combination of all these individual contributions culminated in a first estimate of the $\bar{B} \to X_s \gamma$ branching ratio at $O(\alpha_s^2)$ [14]. For $E_\gamma > 1.6$ GeV it reads

$$\text{Br}(\bar{B} \to X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4}.$$  

(1.2)

Here, we should mention that there are several perturbative and non-perturbative effects that have not been considered when deriving this estimate. Some of these are already available in the literature: the four-loop mixing of $O_1, \ldots, O_6$ into $O_8$ [7]; the bremsstrahlung contributions of the $(O_2, O_2)$, $(O_2, O_7)$ and $(O_7, O_8)$-interferences at $O(\alpha_s^2 \beta_0)$ [15]; those $(O_2, O_2)$ and $(O_2, O_7)$ contributions which are due to the renormalization of $m_c$ [16] (written as expansions in $m_c/m_b$); photon energy cut-off related effects [17–21]; and estimates for the $O(\alpha_s \Lambda_{QCD}/m_b)$ corrections [22]. Other effects are unknown at the moment, like the complete virtual- and bremsstrahlung contributions to the $(O_7, O_8)$- and $(O_8, O_8)$-interferences at $O(\alpha_s^2)[1]$ the exact $m_c$-dependence of various matrix elements beyond the large $\beta_0$-approximation, in order to improve (or even remove) the uncertainty due to the interpolation in $m_c$ [13]; and the emission process of photons from four-quark operators at tree-level. The individual contributions listed above are all expected to remain within the uncertainty given in (1.2), nevertheless they should be taken into account in future updates.

In the present paper we extend the calculations of the $(O_7, O_7)$-interference contribution performed in [8–11] to include the charm quark mass at its physical value. Since the results

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1 In (1.2) the virtual corrections to the $(O_7, O_8)$-interference at $O(\alpha_s^2 \beta_0)$ were taken into account.
given there were presented for $N_H$ heavy quarks with masses equal to $m_b$ and $N_L$ light quarks with masses equal to zero, exact results for the $(O_7, O_7)$ contribution are only available for two extreme cases. Either we can set $N_H = 2$ and $N_L = 3$, that is the charm quark mass is equal to $m_b$, or we can set $N_H = 1$ and $N_L = 4$, that is the charm quark is considered to be massless. The $N_H$ heavy quarks are of course kinematically not allowed to appear in the final state, and hence the first possibility corresponds qualitatively to the experimental situation since events with charmed hadrons are not included on the experimental side. However, since $m_c \approx m_b / 4$ in reality, one could choose as well the second possibility, but in this case the inclusion of contributions from the $c \bar{c}$ production is required on the theoretical side in order to get rid of infrared divergences. As argued in [10] we expect the true result to lie somewhere in between. To check this statement, we calculate the exact charm quark mass dependence of the $(O_7, O_7)$-interference contribution to the photon energy spectrum $d\Gamma(b \to X_s^{\text{partonic}} \gamma)/dE_\gamma$ and the total decay width $\Gamma(b \to X_s^{\text{partonic}} \gamma)$, excluding charm quarks in the final state. The impact of the exact $m_c$-dependence on the branching ratio will be taken into account together with other new contributions in a forthcoming analysis.

The organization of this paper is as follows. In section 2 we describe briefly the calculation of the relevant Feynman diagrams and present our final results for the photon energy spectrum and the total decay width. Furthermore, we comment on the numerical importance of our result on the branching ratio $\text{Br}(\bar{B} \to X_s \gamma)$ in this section.

## 2 Results for the charm quark contribution

Within the low-energy effective theory the partonic $b \to X_s \gamma$ decay rate can be written as

$$\Gamma(b \to X_s^{\text{partonic}} \gamma)_{E_\gamma > E_0} = \frac{G_F^2 \alpha_{\text{em}} m_b^3}{32 \pi^4} \left| V_{tb} V_{ts}^* \right|^2 \sum_{i,j} C^\text{eff}_i(\mu) C^\text{eff}_j(\mu) G_{ij}(E_0, \mu),$$

where $m_b$ and $\overline{m}_b(\mu)$ denote the pole and the running $\overline{MS}$ mass of the $b$ quark, respectively, $C^\text{eff}_i(\mu)$ the effective Wilson coefficients at the low-energy scale, and $E_0$ the energy cut in the photon spectrum. As already anticipated in the introduction, we will focus on the function $G_{77}(E_0, \mu)$ corresponding to the self-interference of the electromagnetic dipole operator

$$O_7 = \frac{e}{16\pi^2} \overline{m}_b(\mu) (\bar{s} \sigma^{\mu\nu} P_R b) F_{\mu\nu}. \tag{2.2}$$

More precisely, we extend the calculations performed in [8–11] to include the effects of a massive charm quark. To this end we calculate the cuts of the $b$ quark self-energies displayed in figure 1 with two or three particles in the intermediate state. We do not have to calculate cuts with four particles in the intermediate state since such cuts would run through the charm quark bubble and events involving charmed hadrons in the final state are not included on the experimental side.

The reduction of the 2-particle-cut of the first Feynman diagram visualized in figure 1 to a set of a few so-called master integrals is done by means of the systematic Laporta algorithm [23] based on the integration-by-part technique first proposed in [24, 25]. Since the reduction procedure applied to the present problem has already been described in great

\[^2\text{For an automatized implementation of this algorithm see, e.g., [26].}\]
Figure 1: 2- and 3-particle-cuts of the irreducible $b$ quark selfenergy diagrams with a massive charm quark bubble contributing to the $b \to s\gamma$ (blue dashed line) and $b \to s\gamma g$ (orange dashed lines) transitions at $O(\alpha_s^2)$. Thick lines denote $b$ quarks, thin lines $s$ quarks, wiggly lines photons and curly lines gluons. Left-right reflected diagrams are not shown.

detail in [10,11], we refrain from repeating it here once more. We just remark that after the reduction there remain only three two-loop integrals which do not factorize into a product of one-loop integrals. To solve these integrals, we first introduce Feynman parameters in the standard way and perform the loop-integrations. At this level, the Feynman parameter integrals contain denominators of the form

$$\frac{1}{(m_c^2 P_1 + m_b^2 P_2)\alpha},$$  

where $P_1$ and $P_2$ are polynomials in the Feynman parameters. Next, applying the Mellin-Barnes representation [27]

$$\frac{1}{(x+y)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_C ds \frac{x^s}{2\pi i y^{s+\alpha}} \Gamma(-s)\Gamma(\alpha + s), \quad x = m_c^2 P_1, \quad y = m_b^2 P_2,$$  

where the integration contour $C$ runs from $-i\infty$ to $+i\infty$ such that it separates the poles generated by the two $\Gamma$ functions, proves very useful because then the integration over the Feynman parameters becomes trivial. (For explicit examples of this method, see e.g., [28]). Finally, we close the integration contour $C$ by a half-circle with infinite radius at either side and sum up the enclosed residues. In this way we managed to obtain solutions for the three non-trivial two-loop integrals valid for arbitrary values of $m_c$.

The remaining 2- and 3-particle-cuts which all contain cut-diagrams with external wave function corrections are taken into account by proper insertions of $s$ quark and gluon wave function renormalization constants into the corresponding 2- and 3-particle-cuts of irreducible two-loop $b$ quark self-energy diagrams. A collection of the relevant renormalization constants necessary to render our amplitudes finite can be found in appendix A.

Having briefly described the technical details of our calculation, we turn our attention to the results for the function $G_{77}(E_0, \mu)$ which we write as an integral over the (rescaled) photon energy spectrum:

$$G_{77}(E_0, \mu) = \int_{z_0}^{1} dz \frac{dG_{77}(z, \mu)}{dz}, \quad z = \frac{2E_0}{m_b}, \quad z_0 = \frac{2E_0}{m_b}.$$  

(2.5)
In NNLO approximation the photon energy spectrum can be decomposed as follows:\footnote{In the notation of [11] we have $\hat{H}^{(1)} = 4 H^{(1)}$ and $\hat{H}^{(2,i)} = 16 H^{(2,i)}$ for $i = a, na, NL, NH.$}

$$\frac{dG_{\gamma\gamma}(z, \mu)}{dz} = \delta(1 - z) + \frac{\alpha_s(\mu)}{4\pi} C_F \hat{H}^{(1)}(z, \mu) + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 C_F \hat{H}^{(2)}(z, \mu) + O(\alpha_s^3), \tag{2.6}$$

where

$$\hat{H}^{(2)}(z, \mu) = C_F \hat{H}^{(2,a)}(z, \mu) + C_A \hat{H}^{(2,na)}(z, \mu)
+ T_R N_L \hat{H}^{(2,NL)}(z, \mu) + T_R N_H \hat{H}^{(2,NH)}(z, \mu) + T_R N_V \hat{H}^{(2,NV)}(z, \mu). \tag{2.7}$$

Here, $N_L, N_H$ and $N_V$ denote the number of light ($m_q = 0$), heavy ($m_q = m_b$), and virtual ($m_q = m_c$) quark flavors, that is the total number of quark flavors is $N_F = N_L + N_H + N_V$. Furthermore, $\alpha_s(\mu)$ is the running coupling constant in the $\overline{MS}$ scheme, and the numerical values of the color factors are given by $C_F = 4/3$, $C_A = 3$, and $T_R = 1/2$.

The functions $\hat{H}^{(1)}$ and $\hat{H}^{(2,i)}$ ($i = a, na, NL, NH$) appearing in (2.6) and (2.7) receive contributions from the $b \to s\gamma$, $b \to s\gamma q\bar{q}$, $b \to s\gamma gg$ and $b \to s\gamma q\bar{q}$ ($q \in \{u, d, s\}$, $m_q = 0$) transitions. They can be found in [11]. On the other hand, the function $\hat{H}^{(2,NV)}$ is completely new. It originates from the $b \to s\gamma$ and $b \to s\gamma g$ transitions with massive charm quark bubbles, see figure II. Our result for this function reads

$$\hat{H}^{(2,NV)}(z, \mu) = \left(\frac{124}{27} \pi^2 + \frac{32}{9} (7 + \pi^2) L_\mu + \frac{16}{3} L^2_\mu + \left(\frac{614}{27} - \frac{8}{9} \pi^2\right) \ln \rho + f(\rho)\right) \delta(1 - z)
+ \frac{16}{3} (2 L_\mu - \ln \rho) \left[\frac{\ln(1 - z)}{1 - z}\right]_+ + \frac{28}{3} (2 L_\mu - \ln \rho) \left[\frac{1}{1 - z}\right]_+
- \frac{8}{3} \left(7 + z - 2 z^2 - 2 (1 + z) \ln(1 - z)\right) L_\mu
+ \frac{4}{3} \left(7 + z - 2 z^2\right) \ln \rho - \frac{8}{3} (1 + z) \ln(1 - z) \ln \rho, \tag{2.8}$$

where

$$f(\rho) = -\frac{\pi^2}{9} \left(162 \sqrt{\rho} + 70 \rho^{3/2} - 36 \rho^2\right) + \frac{32}{9} \rho \ln \rho + \frac{2}{9} \left(25 + 27 \rho^2\right) \ln^2 \rho + \frac{4}{9} \ln^3 \rho
- \frac{4}{9} (31 + 27 \rho^2) \ln(1 - \rho) \ln \rho - \frac{4}{9} \sqrt{\rho} (81 + 35 \rho) \text{artanh}(\sqrt{\rho}) \ln \rho
- \frac{2}{9} \left(62 + 81 \sqrt{\rho} + 35 \rho^{3/2} + 54 \rho^3\right) \text{Li}_2(\rho) + \frac{8}{3} \ln \rho \text{Li}_2(\rho)
+ \frac{8}{9} \sqrt{\rho} (81 + 35 \rho) \text{Li}_2(\sqrt{\rho}) - \frac{16}{3} \text{Li}_3(\rho) + \frac{5578}{81} + \frac{172}{9} \rho. \tag{2.9}$$

Here,

$$\rho = \frac{m_c^2}{m_b^2}, \quad L_\mu = \ln \left(\frac{\mu}{m_b}\right), \tag{2.10}$$
Li_3(z) = \int_0^z dx \, Li_2(x)/x, \text{ and } [\ldots]_+, \text{ are plus-distributions defined in the standard way. Note that our result (2.8) holds for } \rho \in (0, \infty). \text{ The individual 2- and 3-particle-cuts contributing to } \hat{H}^{(2,NV)} \text{ can be found in appendix B.}

In the remainder of this section we comment on the numerical importance of the new contribution given in (2.8). To this end we consider the function

\[ G_{77}(0, \mu) = 1 + \frac{\alpha_s(\mu)}{4\pi} \left\{ \frac{64}{9} \pi^2 - \frac{16}{9} \pi^2 - \frac{16}{3} L_{\mu} \right\} + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left\{ (3.55556 N_F - 44.4446) L_{\mu}^2 + (21.6168 N_F - 334.803) L_{\mu} + 37.8172 N_L + h(\rho) N_V - 2.16077 N_H - 519.250 \right\}, \]  
(2.11)

which follows from (2.5) when setting \( E_0 = 0 \) and performing the integration over \( z \). The function \( h(\rho) \) which incorporates the \( m_c \)-dependence is given by

\[ h(\rho) = \frac{248}{81} \pi^2 + \left( \frac{1972}{81} - \frac{16}{27} \pi^2 \right) \ln \rho + \frac{2}{3} f(\rho). \]  
(2.12)

Equation (2.11) generalizes our result presented in [10] to include \( N_V \) massive charm quarks.

Denoting the coefficient of \( (\alpha_s(\mu)/(4\pi))^2 \) in (2.11) by \( X_2(N_H, N_L, N_V) \), we find (for \( \mu = m_b \) and \( m_c/m_b = 0.26 \))

\[ X_2(1, 4, 0) = -370.142, \quad X_2(1, 3, 1) = -386.638, \quad X_2(2, 3, 0) = -410.120, \]  
(2.13)

that is the result with the physical charm quark mass is almost equal to the average of the two approximations. From these numbers we conclude that the branching ratio \( \text{Br}(\bar{B} \to X_s\gamma) \) will stay within the errors quoted in (1.2) when implementing the exact \( m_c \)-dependence of the \( (O_7, O_7) \)-contribution. Therefore we do not repeat the interpolation performed in [13] which would be necessary to properly implement our new result since it also estimates the \( m_c \)-dependence of the \( (O_7, O_7) \)-contribution. We postpone this until more progress towards an improved evaluation of the branching ratio at NNLO has been achieved.

Before closing this section we mention that a normalization with \( m_b^5 \) rather than \( \overline{m}_b^2(\mu)m_b^3 \) is sometimes used in the definition of (2.11). The relation between the \( \overline{\text{MS}} \) mass \( \overline{m}_b(\mu) \) and the on-shell mass \( m_b \) (including the exact dependence on \( \rho \)) necessary to convert between both normalizations can be found in appendix A.

Acknowledgements

We would like to thank M. Misiak and M. Steinhauser for helpful discussions. H. M. A. is partially supported by the ANSEF N 05-PS-hepth-0825-338 program. T. E. and C. G. are supported by the Swiss National Foundation as well as EC-Contract MRTN-CT-2006-035482 (FLAVIAnet).
A Renormalization constants

The renormalization schemes applied in this work are exactly the same as used in [10], and the relevant renormalization constants for $N_V = 0$ can be found in appendix A of that reference. In order to obtain the renormalization constants for $N_V \neq 0$ one has to proceed as follows: (i) set $N_F$, the total number of quarks appearing there, equal to $N_H + N_L + N_V$; (ii) add additional contributions $\delta Z_3^{os}$, $\delta Z_{2s}^{os}$ and $\delta Z_{2b}^{os}$ to the gluon, s quark and b quark wave function renormalization constants, respectively. These additional contributions are proportional to $N_V$, and their explicit expressions read

$$\delta Z_3^{os} = -\frac{4}{3} T_R N_V \Gamma(\epsilon) e^{\gamma \epsilon} \left( \frac{\mu}{m_b} \right)^{2\epsilon} \rho^{-\epsilon} \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s^2),$$

$$\delta Z_{2s}^{os} = C_F T_R N_V \frac{2(1 + \epsilon)(3 - 2\epsilon) \Gamma(\epsilon)^2 e^{2\gamma \epsilon}}{(1 - \epsilon)(2 - \epsilon)(1 + 2\epsilon)(3 + 2\epsilon)} \left( \frac{\mu}{m_b} \right)^{4\epsilon} \rho^{-2\epsilon} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 + O(\alpha_s^3),$$

$$\delta Z_{2b}^{os} = C_F T_R N_V \left\{ \frac{1}{\epsilon} (1 + 8 L_\mu - 4 \ln \rho) + \left( \frac{44}{3} - 16 \ln \rho \right) L_\mu + 24 L_\mu^2 + \frac{443}{18} + 28 \rho + \frac{\pi^2}{3} (5 - 18\sqrt{\rho} - 30 \rho^{3/2} + 12 \rho^2) + \frac{8}{3} (2 + 3 \rho) \ln \rho + 2 (2 + 3 \rho^2) \ln^2 \rho - 4 (1 + 3 \rho^2) \ln(1 - \rho) \ln \rho - 4 \rho \arctan(\sqrt{\rho}) \ln \rho + 8 \sqrt{\rho} (3 + 5 \rho) \operatorname{Li}_2(\sqrt{\rho}) - 2 (2 + 3 \sqrt{\rho} + 5 \rho^{3/2} + 6 \rho^2) \operatorname{Li}_2(\rho) \right\} \rho^{-2\epsilon} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 + O(\alpha_s^3),$$

(A.1)

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant and $d = 4 - 2\epsilon$ the space-time dimension. While the first two expressions in (A.1) can be obtained in a straightforward way, the calculation of the analytic expression for $\delta Z_{2b}^{os}$ is more involved. We checked numerically that our result is in agreement with the one given in [29]. We remark that the results in (A.1) are independent of the gauge parameter appearing in the gluon propagator.

Finally, we give the relation between the pole-mass $m_b$ and the $\overline{\text{MS}}$ mass $\overline{m}_b(\mu)$ up to two-loops including the contribution of the $N_V$ quarks,

$$\frac{m_b(\mu)}{m_b} = 1 - C_F \left( 4 + 6 L_\mu \right) \frac{\alpha_s(\mu)}{4\pi}$$

$$+ C_F \left\{ C_F \left( \frac{7}{8} + 8\pi^2 \ln 2 - 5\pi^2 - 12 \zeta_3 + 21 L_\mu + 18 L_\mu^2 \right) - C_A \left( \frac{1111}{24} + 4\pi^2 \ln 2 - \frac{4}{3} \pi^2 - 6 \zeta_3 + \frac{185}{3} L_\mu + 22 L_\mu^2 \right) + T_R N_H \left( \frac{143}{6} + \frac{8}{3} \pi^2 + \frac{52}{3} L_\mu + 8 L_\mu^2 \right) \right\},$$

(A.1)
\[ + T_R N_L \left( \frac{71}{6} + \frac{4}{3} \pi^2 + \frac{52}{3} L_\mu + 8 L_\mu^2 \right) \]

\[ + T_R N_V \left( \frac{52}{3} L_\mu + 8 L_\mu^2 + \frac{71}{6} + 12 \rho + \frac{4}{3} \pi^2 (1 - 3 \sqrt{\rho} - 3 \rho^{3/2} + \rho^2) \right. \]

\[ + 4 \rho \ln \rho + 2 \rho^2 \ln^2 \rho - 4 \left( 1 + \rho^2 \right) \ln(1 - \rho) 
\ln \rho \]

\[ - 8 \sqrt{\rho} (1 + \rho) \text{artanh}(\sqrt{\rho}) \ln \rho + 16 \sqrt{\rho} (1 + \rho) \text{Li}_2(\sqrt{\rho}) \]

\[ - 4 \left( 1 + \sqrt{\rho} + \rho^{3/2} + \rho^2 \right) \text{Li}_2(\rho) \right) \right] \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 + O(\alpha_s^3), \quad (A.2) \]

where \( \zeta_3 \) is the Riemann zeta-function. We recalculated the term proportional to \( N_V \) and our finding agrees with \([30]\) (bearing in mind that a factor of 4/3 is missing in front of the function \( \Delta(M_i/M) \) appearing in equation (17) of that reference, as also observed in \([31]\)). The remaining terms in \( (A.2) \) have been taken from \([30]\).

### B Individual cut contributions to \( \widehat{H}^{(2,NV)} \)

The function \( \widehat{H}^{(2,NV)} \) introduced in \([271]\) receives contributions from the 2-particle-cut of the first diagram given in figure 1, as well as contributions from 2- and 3-particle-cuts where at least one renormalization constant proportional to \( N_V \) is present. Denoting these contributions by \( \widehat{H}_2^{(2,NV,\text{bare})} \), \( \widehat{H}_2^{(2,NV,\text{ct})} \), and \( \widehat{H}_3^{(2,NV,\text{ct})} \), respectively, we have

\[ \widehat{H}^{(2,NV)}(z, \mu) = \widehat{H}_2^{(2,NV,\text{bare})}(z, \mu) + \widehat{H}_2^{(2,NV,\text{ct})}(z, \mu) + \widehat{H}_3^{(2,NV,\text{ct})}(z, \mu), \quad (B.1) \]

with the individual contributions given by

\[ \widehat{H}_2^{(2,NV,\text{bare})}(z, \mu) = \frac{8}{3} \frac{1}{\epsilon^3} + \left( \frac{20}{3} + 16 L_\mu - \frac{8}{3} \ln \rho \right) \frac{1}{\epsilon^2} \]

\[ + \left( \frac{188}{9} - \frac{2}{9} \pi^2 + [40 - 16 \ln \rho] L_\mu + 48 L_\mu^2 - \frac{16}{3} \ln \rho + \frac{4}{3} \ln^2 \rho \right) \frac{1}{\epsilon} \]

\[ + \frac{7612}{81} - \frac{80}{9} \rho + \pi^2 \left( \frac{73}{27} - 12 \sqrt{\rho} + \frac{20}{9} \rho^{3/2} \right) - 8 \zeta_3 \]

\[ + \left( \frac{376}{3} - \frac{4}{3} \pi^2 - 32 \ln \rho + 8 \ln^2 \rho \right) L_\mu + [120 - 48 \ln \rho] L_\mu^2 + 96 L_\mu^3 \]

\[ + \left( \frac{10}{9} \pi^2 + \frac{8}{27} [7 - 15 \rho] \right) \ln \rho + \frac{56}{9} \ln^2 \rho - \frac{88}{9} \ln \rho \ln(1 - \rho) \]

\[ - \frac{8}{9} \sqrt{\rho} (27 - 5 \rho) \text{artanh}(\sqrt{\rho}) \ln \rho - \frac{4}{9} \left( 22 + 27 \sqrt{\rho} - 5 \rho^{3/2} \right) \text{Li}_2(\rho) \]

\[ + \frac{8}{3} \ln \rho \text{Li}_2(\rho) + \frac{16}{3} \sqrt{\rho} (27 - 5 \rho) \text{Li}_2(\sqrt{\rho}) - \frac{16}{3} \text{Li}_3(\rho) \right) \delta(1 - z), \quad (B.2) \]
\( \hat{H}^{(2,NV,ct)}_2(z, \mu) = \left\{ -\frac{8}{3} \frac{1}{\epsilon^3} - (5 + 8 L_\mu) \frac{4}{3} \frac{1}{\epsilon^2} - \left( \frac{188}{9} - \frac{4}{9} \pi^2 + \frac{64}{3} L_\mu + \frac{64}{3} L^2_\mu + 4 \ln \rho \right) \frac{1}{\epsilon} \right. \\
- \frac{226}{9} + 28 \rho + \pi^2 \left( \frac{8}{3} - 6 \sqrt{\rho} - 10 \rho^{3/2} + 4 \rho^2 \right) + \frac{64}{9} \zeta_3 \\
- \left( \frac{520}{9} - \frac{16}{9} \pi^2 + 24 \ln \rho \right) L_\mu - \frac{64}{3} L^2_\mu - \frac{256}{9} L^3_\mu - \left( \frac{2}{3} - 8 \rho \right) \ln \rho \\
+ (4 + 6 \rho^2) \ln^2 \rho - 4 \left( 1 + 3 \rho^2 \right) \ln \rho \ln(1 - \rho) \\
- 4 \sqrt{\rho} (3 + 5 \rho) \text{artanh}(\sqrt{\rho}) \ln \rho - 2 \left( 2 + 3 \sqrt{\rho} + 5 \rho^{3/2} + 6 \rho^2 \right) \text{Li}_2(\rho) \\
+ 8 \sqrt{\rho} (3 + 5 \rho) \text{Li}_2(\sqrt{\rho}) \right\} \delta(1 - z), \\
(B.3) \\
\hat{H}^{(2,NV,ct)}_3(z, \mu) = \left\{ -2 L_\mu + \ln \rho \frac{8}{3} \frac{1}{\epsilon^2} \right. \\
- \left( \frac{2}{9} \pi^2 + \left( \frac{56}{3} - 16 \ln \rho \right) L_\mu + \frac{80}{3} L^2_\mu - \frac{28}{3} \ln \rho + \frac{4}{3} \ln^2 \rho \right) \frac{1}{\epsilon} \\
- \frac{7}{9} \pi^2 - \left( \frac{128}{3} - \frac{28}{9} \pi^2 - 56 \ln \rho + 8 \ln^2 \rho \right) L_\mu - \left( \frac{280}{3} - 48 \ln \rho \right) L^2_\mu \\
- \frac{608}{9} L^3_\mu + \left( \frac{64}{3} - 2 \pi^2 \right) \ln \rho - \frac{14}{3} \ln^2 \rho + \frac{4}{9} \ln^3 \rho + \frac{8}{9} \zeta_3 \right\} \delta(1 - z) \\
+ \frac{16}{3} \left( 2 L_\mu - \ln \rho \right) \left[ \frac{\ln(1 - z)}{1 - z} \right]_{+} + \frac{28}{3} \left( 2 L_\mu - \ln \rho \right) \left[ \frac{1}{1 - z} \right]_{+} \\
- \frac{8}{3} \left( 7 + z - 2 z^2 - 2 (1 + z) \ln(1 - z) \right) L_\mu \\
+ \frac{4}{3} \left( 7 + z - 2 z^2 \right) \ln \rho - \frac{8}{3} \left( 1 + z \right) \ln(1 - z) \ln \rho. \\
(B.4) \\
The three contributions given above are by themselves independent of the gauge parameter entering the gluon propagator.

References

[1] P. Koppenburg et al. [Belle Collaboration], Phys. Rev. Lett. 93 (2004) 061803 [arXiv:hep-ex/0403004].

[2] B. Aubert et al. [BaBar Collaboration], [arXiv:hep-ex/0607071].

B. Aubert et al. [BABAR Collaboration], Phys. Rev. D 72 (2005) 052004 [arXiv:hep-ex/0508004].

[3] E. Barberio et al. [Heavy Flavor Averaging Group], [arXiv:hep-ex/0603003].
[4] M. Misiak and M. Steinhauser, Nucl. Phys. B 683 (2004) 277 [arXiv:hep-ph/0401041].
[5] M. Gorbahn and U. Haisch, Nucl. Phys. B 713 (2005) 291 [arXiv:hep-ph/0411071].
[6] M. Gorbahn, U. Haisch and M. Misiak, Phys. Rev. Lett. 95 (2005) 102004 [arXiv:hep-ph/0504194].
[7] M. Czakon, U. Haisch and M. Misiak, [arXiv:hep-ph/0612329].
[8] I. Blokland, A. Czarnecki, M. Misiak, M. Slusarczyk and F. Tkachov, Phys. Rev. D 72 (2005) 033014 [arXiv:hep-ph/0506055].
[9] K. Melnikov and A. Mitov, Phys. Lett. B 620 (2005) 69 [arXiv:hep-ph/0505097].
[10] H. M. Asatrian, A. Hovhannisyan, V. Poghosyan, T. Ewerth, C. Greub and T. Hurth, Nucl. Phys. B 749 (2006) 325 [arXiv:hep-ph/0605009].
[11] H. M. Asatrian, T. Ewerth, A. Ferroglia, P. Gambino and C. Greub, [arXiv:hep-ph/0607316].
[12] K. Bieri, C. Greub and M. Steinhauser, Phys. Rev. D 67 (2003) 114019 [arXiv:hep-ph/0302051].
[13] M. Misiak and M. Steinhauser, [arXiv:hep-ph/0609241].
[14] M. Misiak et al., [arXiv:hep-ph/0609232].
[15] Z. Ligeti, M. E. Luke, A. V. Manohar and M. B. Wise, Phys. Rev. D 60 (1999) 034019 [arXiv:hep-ph/9903305].
[16] H. M. Asatrian, C. Greub, A. Hovhannisyan, T. Hurth and V. Poghosyan, Phys. Lett. B 619 (2005) 322 [arXiv:hep-ph/0505068].
[17] M. Neubert, Eur. Phys. J. C 40 (2005) 165 [arXiv:hep-ph/0408179].
[18] T. Becher and M. Neubert, Phys. Lett. B 633 (2006) 739 [arXiv:hep-ph/0512208].
[19] T. Becher and M. Neubert, Phys. Lett. B 637 (2006) 251 [arXiv:hep-ph/0603140].
[20] T. Becher and M. Neubert, [arXiv:hep-ph/0610067].
[21] J. R. Andersen and E. Gardi, [arXiv:hep-ph/0609250].
[22] S. J. Lee, M. Neubert and G. Paz, [arXiv:hep-ph/0609224].
[23] S. Laporta, Int. J. Mod. Phys. A 15 (2000) 5087 [arXiv:hep-ph/0102033].
[24] F. V. Tkachov, Phys. Lett. B 100 (1981) 65.
[25] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B 192 (1981) 159.
[26] C. Anastasiou and A. Lazopoulos, JHEP 0407 (2004) 046 [arXiv:hep-ph/0404258].
[27] E. E. Boos and A. I. Davydychev, Theor. Math. Phys. 89 (1991) 1052 [Teor. Mat. Fiz. 89 (1991) 56].
[28] C. Greub, T. Hurth and D. Wyler, Phys. Rev. D 54 (1996) 3350 [arXiv:hep-ph/9603404].

[29] D. J. Broadhurst, N. Gray and K. Schilcher, Z. Phys. C 52 (1991) 111.

[30] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C 48 (1990) 673.

[31] K. G. Chetyrkin and M. Steinhauser, Nucl. Phys. B 573 (2000) 617 [arXiv:hep-ph/9911434].