ALMOST ALL LAGRANGIAN TORUS ORBITS IN $\mathbb{C}P^n$ ARE NOT HAMILTONIAN VOLUME MINIMIZING

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Abstract. All principal orbits of the standard Hamiltonian $T^n$-action on the complex projective space $\mathbb{C}P^n$ are Lagrangian tori. In this article, we prove that most of them are not volume minimizing under Hamiltonian isotopies of $\mathbb{C}P^n$ if the complex dimension $n$ is greater than two, although they are Hamiltonian minimal and Hamiltonian stable.

1. Introduction

The classical isoperimetric inequality for a simple closed curve $L$ in $\mathbb{R}^2$ (resp. the unit two-sphere $S^2$) states that

$$l(L)^2 \geq 4\pi A \quad \text{(resp. } l(L)^2 \geq 4\pi A - A^2\text{)},$$

where $l(L)$ is the length of $L$ and $A$ the area of the disc enclosed by $L$. Moreover, the equality holds if and only if $L$ is a round circle. In other words, a round circle $L = S^1$ in $\mathbb{R}^2$ (or $S^2$) has least length when we deform $L$ in such a way that the enclosed area $A$ is unchanged. Notice that without this last constraint we can easily reduce the length of $S^1$ by deforming it to the normal direction.

In papers [8] and [9], Y.-G. Oh proposed a higher dimensional analogue of such a phenomenon from the symplectic geometrical viewpoint and introduced several concepts. Let us review the settings. Let $(M, \omega, J)$ be a Kähler manifold. A submanifold $L$ of $M$ is said to be Lagrangian if $\omega|_{TL} \equiv 0$ and $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} M$. This condition is equivalent to the existence of an orthogonal decomposition

$$T_pM = T_pL \oplus J(T_pL)$$

for any $p \in L$. Throughout this article all Lagrangian submanifolds are assumed to be connected, embedded, closed and equipped with the

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induced Riemannian metric from the ambient manifold $M$. We denote by $\text{Vol}(L)$ the volume of $L$ with respect to the metric.

Notice that $\mathbb{R}^2 \cong \mathbb{C}$ and $S^2$ are one-dimensional Kähler manifolds and a simple closed curve in them is a Lagrangian submanifold. The constraint that $A$ is constant is generalised to the deformation of a Lagrangian submanifold $L$ under Hamiltonian isotopies explained below.

By definition, we have the linear isomorphism defined by

$$\Gamma(T^\perp L) \ni V \mapsto -\alpha V := \omega(V, \cdot)|_{TL} \in \Omega^1(L),$$

where $T^\perp L (\cong J(TL))$ denotes the normal bundle of $L \subset M$ and $\Omega^1(L)$ the set of all one-forms on $L$. A variational vector field $V \in \Gamma(T^\perp L)$ of $L$ is called a Hamiltonian variation if $\alpha V$ is exact. It implies that the infinitesimal deformation of $L$ with the vector field $V$ preserves the Lagrangian constraint. The following definitions are due to Oh.

**Definition 1** ([9], [8]). Let $L$ be a Lagrangian submanifold of a Kähler manifold $(M, \omega, J)$.

1. $L \subset M$ is said to be *Hamiltonian minimal* if it satisfies that

$$\left. \frac{d}{dt} \text{Vol}(L_t) \right|_{t=0} = 0$$

for any smooth deformation $\{L_t\}_{-\epsilon < t < \epsilon}$ of $L = L_0$ with a Hamiltonian variation $V = \frac{dL_t}{dt}|_{t=0}$.

2. Suppose that $L \subset M$ is Hamiltonian minimal. Then $L$ is said to be *Hamiltonian stable* if it satisfies that

$$\left. \frac{d^2}{dt^2} \text{Vol}(L_t) \right|_{t=0} \geq 0$$

for any smooth deformation $\{L_t\}_{-\epsilon < t < \epsilon}$ of $L = L_0$ with a Hamiltonian variation $V = \frac{d^2L_t}{dt^2}|_{t=0}$.

3. $L \subset M$ is said to be *Hamiltonian volume minimizing* if

$$\text{Vol}(\phi(L)) \geq \text{Vol}(L)$$

holds for any $\phi \in \text{Ham}_c(M, \omega)$, which is the set of all compactly supported Hamiltonian diffeomorphisms of $(M, \omega)$.

A diffeomorphism $\phi$ of $(M, \omega)$ is called Hamiltonian if $\phi$ is the time-one map of the flow $\{\phi^t_H\}_{0 \leq t \leq 1}, \phi^0_H = \text{id}_M$, of the (time-dependent) Hamiltonian vector field $X_H$, defined by a compactly supported Hamiltonian function $H \in C^\infty_c([0, 1] \times M)$. The isotopy $\{\phi^t_H\}_{0 \leq t \leq 1}$ is called a Hamiltonian isotopy of $M$. It is easy to see that $(\phi^t_H)^*\omega = \omega$. Note that a (time-independent) Hamiltonian vector field on $M$ gives rise to a Hamiltonian variation of a Lagrangian submanifold $L \subset M$. 
At present we know only a few non-trivial examples of Hamiltonian volume minimizing Lagrangian submanifolds except for special Lagrangian submanifolds; the real form $\mathbb{R}P^n \subset \mathbb{C}P^n$, \[8\], the product of the great circles in $S^2 \times S^2$, \[6\], and the totally geodesic Lagrangian sphere $S^{2n-1}$ in the complex hyperquadric $Q_{2n-1}(\mathbb{C})$, \[7\].

The most fundamental example of symplectic manifolds is the linear complex space $\mathbb{C}^n$ equipped with the standard symplectic structure $\omega_0 := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Its standard complex structure and $\omega_0$ define the standard Euclidean metric on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. We denote by $S^1(b) \subset \mathbb{R}^2 \cong \mathbb{C}$ the boundary of a round disc with area $b$ centred at the origin, i.e., the radius of $S^1(b)$ is $\sqrt{b/\pi}$. For positive real numbers $b_1, \ldots, b_n > 0$, the product torus (or elementary torus, see \[3\]) $T(b) = T(b_1, \ldots, b_n) := S^1(b_1) \times \cdots \times S^1(b_n) \subset \mathbb{C}^n$ is a typical example of Lagrangian submanifolds of $\mathbb{C}^n$. Here we denote $N(b) := \#\{b_1, \ldots, b_n\}$, e.g., $N(b) = 3$ for $b = (1, 2, 2, 4)$.

We can easily check, using the first variation formula (see \[9\], p. 178), that $L \subset M$ is Hamiltonian minimal if and only if the equation $\delta \alpha_H = 0$ holds on $L$, where $\delta$ and $H$ are the codifferential operator on $L$ and the mean curvature vector of $L$, respectively. Hence, $T(b) \subset \mathbb{C}^n$ is Hamiltonian minimal. Using his second variation formula \[9\], Thorem 3.4, Oh proved that the torus $T(b) \subset \mathbb{C}^n$ is a Hamiltonian stable Lagrangian submanifold (see \[9\], Theorem 4.1). Moreover, the isoperimetric inequality for closed curves in $\mathbb{R}^2$ states that $T(b_1) \subset \mathbb{C}$ is Hamiltonian volume minimizing. Based on these results, Oh proposed the following

**Conjecture 2** (Oh \[9\], p. 192). The Lagrangian torus $T(b)$ in $\mathbb{C}^n$ is Hamiltonian volume minimizing.

In a sense, Conjecture 2 is regarded as a symplectic higher dimensional analogue of the isoperimetric inequality in $\mathbb{R}^2$. Though the statement is quite natural, it turned out to be false for $n \geq 3$. Indeed, C. Viterbo \[12\], p. 419 has already pointed out that $T(1, 2, 2)$ and $T(1, 2, 3)$ are Hamiltonian isotopic based on a remarkable result by Chekanov \[3\] Theorem A, see Section 2. Namely, the second one is not Hamiltonian volume minimizing. Furthermore, in Section 2 we prove

**Corollary 3.** Let $b \in (\mathbb{R}_{>0})^n$. If $N(b) \geq 3$, then the Lagrangian torus $T(b) \subset \mathbb{C}^n$ is not Hamiltonian volume minimizing.

If $n \geq 3$, then the set $$\{b \in (\mathbb{R}_{>0})^n \mid N(b) \geq 3\}$$
is an open dense subset of $(\mathbb{R}_{>0})^n$, and hence almost all product tori in $\mathbb{C}^n$ ($n \geq 3$) are not Hamiltonian volume minimizing. Notice that $T(b)$ is represented as $\mu_0^{-1}(b_1/2\pi, \ldots, b_n/2\pi)$, where

$$
\mu_0(x_1, \ldots, x_n, y_1, \ldots, y_n) = \left(\frac{1}{2}(x_1^2 + y_1^2), \ldots, \frac{1}{2}(x_n^2 + y_n^2)\right)
$$

is the moment map $\mu_0 : \mathbb{C}^n \to (\mathbb{R}_{\geq 0})^n$ associated with the standard Hamiltonian action by the real torus $T^n \subset (\mathbb{C} \times \mathbb{C})^n$ on $\mathbb{C}^n$.

Similarly, the complex projective space $(\mathbb{C}P^n, J_{\text{std}})$ equipped with the standard Fubini-Study Kähler form $\omega_{\text{FS}}$ admits an effective Hamiltonian $T^n$-action. Each principal orbit is a flat Lagrangian torus in $\mathbb{C}P^n$ like product one in $\mathbb{C}^n$. As for its Hamiltonian minimality and Hamiltonian stability, the second author previously proved

**Proposition 4** ([10], Section 4). *Any Lagrangian torus orbit $T^n$ in $(\mathbb{C}P^n, \omega_{\text{FS}}, J_{\text{std}})$ is Hamiltonian minimal and Hamiltonian stable.*

Hence, it is worthwhile to determine whether each Lagrangian torus orbit $T^n$ is Hamiltonian volume minimizing or not. The following is the main result of the present article, which provides a negative solution for the problem (see Conjecture 1.4 in [10]).

**Theorem 5.** If $n \geq 3$, then almost all Lagrangian torus orbits in $\mathbb{C}P^n$ are not Hamiltonian volume minimizing.

The proof, which is given in Section 3, is based on a recent result of Chekanov and Schlenk [4] which gives a refinement of the Chekanov’s one mentioned above.

In general, Darboux’s theorem says that any point in a symplectic manifold $(M, \omega)$ possesses a neighbourhood which is isomorphic to a neighbourhood of the origin of $(\mathbb{C}^n, \omega_0)$. Then the Chekanov-Schlenk’s theorem ensures any symplectic manifold the existence of a pair of Lagrangian tori which are mutually Hamiltonian isotopic and not intersect. Furthermore, in the class of compact toric symplectic manifolds, we can regard the Chekanov-Schlenk’s theorem as a local model of a $T^n$-fixed point of such a manifold. Although the result is weaker than the case of $\mathbb{C}P^n$, this observation yields the following

**Theorem 6.** Let $(M, \omega, J)$ be a complex $n$-dimensional compact toric Kähler manifold. If $n \geq 3$, then there exists a toric fibre of $M$ (indeed, infinitely many) which is not Hamiltonian volume minimizing.

We prove it in Section 4. Notice that toric fibres in Theorem 6 are all Hamiltonian minimal (see Section 4).
2. Product tori in $\mathbb{C}^n$ and Chekanov-Schlenk’s Theorem

In this section, we shall consider the case of $(\mathbb{C}^n, \omega_0)$. For $a = (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n$, we use the following notations:

$$a = \min\{a_i | 1 \leq i \leq n\}, \quad \bar{a} = \max\{a_i | 1 \leq i \leq n\}, \quad |a| = \sum_{i=1}^{n} a_i,$$

$$m(a) = \#\{i | a_i = a\}, \quad \|a\| = |a| + \bar{a}, \quad \|\|a\|\| = \|a\| + \bar{a},$$

$$\Gamma(a) = \text{span}_{\mathbb{Z}}(a_1 - a, \ldots, a_n - a) \subset \mathbb{R}.$$  

For $a, a' \in (\mathbb{R}_{>0})^n$, we denote $a \simeq a'$ if

$$(a, m(a), \Gamma(a)) = (a', m(a'), \Gamma(a')),$$

and consider the set

$$\tilde{\Delta}_s := \left\{ (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n \mid \sum_{i=1}^{n} a_i < s \right\}.$$

Notice that $\mu_0^{-1}(\tilde{\Delta}_s)$ is the open ball in $\mathbb{C}^n$ with radius $\sqrt{2s}$ centred at the origin. Let $L$ and $L'$ be Lagrangian submanifolds of $(M, \omega)$. Then $L$ is said to be Hamiltonian isotopic to $L'$ if there exists $\phi \in \text{Ham}_c(M, \omega)$ such that $\phi(L) = L'$. The following result is fundamental for the arguments of this article.

**Theorem 7** (Chekanov [3]). Let $a, a' \in (\mathbb{R}_{>0})^n$. A product torus $T(a)$ of $(\mathbb{C}^n, \omega_0)$ is Hamiltonian isotopic to $T(a')$ if and only if $a \simeq a'$ holds.

**Proposition 8** (Corollary [3]). If $N(a) \geq 3$, then the product torus $\mu_0^{-1}(a) = T(2\pi a) \subset \mathbb{C}^n$ is not Hamiltonian volume minimizing.

**Proof.** For $a = (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n$, by assumption, there exist numbers $i, j \in \{1, 2, \ldots, n\}$ such that $\bar{a} < a_i < a_j$. We define a new $a'$ as

$$a' = (a'_1, \ldots, a'_n) := (a_1, \ldots, a_{j-1}, a_j - a_i + a_i, a_{j+1}, \ldots, a_n).$$

Then we have $a \simeq a'$ and $\|a\| > \|a'\|$. Since $\Pi_i a_i > \Pi_i a'_i$, Theorem 7 implies that $\mu_0^{-1}(a)$ is not Hamiltonian volume minimizing. \qed

Furthermore, the size of the support of a Hamiltonian isotopy connecting two product tori in Theorem 7 has precisely estimated as follows. This estimation is essential to treat the case of $\mathbb{C}P^n$.

**Theorem 9** (Chekanov-Schlenk [4], Theorem 1.1). For $a, a' \in (\mathbb{R}_{>0})^n$, suppose that $a \simeq a'$. Let $s$ be a positive number satisfying that $s > \max\{\|a\|, \|a'\|\}$. Then there exists a smooth Hamiltonian function $H : [0, 1] \times \mathbb{C}^n \rightarrow \mathbb{R}$ satisfying the following:
(1) \( \text{Supp}(H) \subset [0, 1] \times \mu_0^{-1}(\Delta) \).
(2) \( \phi_H(\mu_0^{-1}(a)) = \mu_0^{-1}(a') \).

3. LAGRANGIAN TORUS ORBITS IN \( \mathbb{C}P^n \)

In this section, we shall treat the case of \( \mathbb{C}P^n \) and prove the main theorem.

3.1. \( e_1 \)-action-angle coordinates. Let us consider \( \mathbb{R}^n \) and take an orthonormal basis

\[
e_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}
\]

of \( \mathbb{R}^n \) and set \( \Delta := \{(a_1, \ldots, a_n) \in ([0, \infty)^n | \sum_{i=1}^n a_i \leq 1}\}. For a notational reason, we put \( e_0 := (0, 0, \ldots, 0) \in \mathbb{R}^n \). The symplectic toric manifold corresponding to the polytope \( \Delta \) is nothing but the \( n \)-dimensional complex projective space \( (\mathbb{C}P^n, \omega_{FS}, \mu) \).

We first examine the coordinate neighbourhood given by

\[
U_0 = \{[z_0 : z_1 : \cdots : z_n] | z_0 \neq 0\} \sim \mathbb{C}^n, \quad [z_0 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right).
\]

We put

\[
r_0^i := \left|\frac{z_i}{z_0}\right|, \quad \theta_0^i := \arg \frac{z_i}{z_0}
\]

for \( i = 1, \ldots, n \). Then the moment map associated with the standard Hamiltonian \( T^n \)-action on \( (\mathbb{C}P^n, \omega_{FS}, \mu) \) is represented as

\[
\mu = \sum_{i=1}^n u_0^i e_i, \quad u_0^i := \frac{(r_0^i)^2}{1 + \sum_{j=1}^n (r_0^j)^2}.
\]

Here we introduce the coordinates defined by

\[
x_0^i := \sqrt{2u_0^i \cos \theta_0^i}, \quad y_0^i := \sqrt{2u_0^i \sin \theta_0^i}.
\]

Then, on \( U_0 \) the symplectic structure \( \omega_{FS} \) and the moment map \( \mu \) are expressed as follows:

\[
\omega_{FS}|_{U_0} = \sum_{i=1}^n du_0^i \wedge d\theta_0^i = \sum_{i=1}^n dx_0^i \wedge dy_0^i, \quad \mu|_{U_0} = \frac{1}{2} \sum_{i=1}^n ((x_0^i)^2 + (y_0^i)^2) e_i.
\]

Hence we have an isomorphism

\[(U_0, \omega_{FS}|_{U_0}, \mu|_{U_0}) \cong (\mu_0^{-1}(\Delta), \omega_0, \mu_0)\]
as Hamiltonian $T^n$-spaces. We call the coordinates \((u_0^1, \ldots, u_n^1, \theta_0^1, \ldots, \theta_n^1)\) \(e_0\)-action-angle coordinates.

Similarly, we examine the coordinate neighbourhood given by

\[ U_1 = \{ [z_0: z_1: \cdots: z_n] | z_1 \neq 0 \} \sim \mathbb{C}^n, \quad [z_0: \cdots: z_n] \mapsto \left( \frac{z_0}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1} \right). \]

(The case where \(i \geq 2\) is similar.) We put

\[ r_i^1 := \left| \frac{z_i}{z_1} \right|, \quad r_i^j := \left| \frac{z_i}{z_1} \right| (i \geq 2), \quad \theta_1^i := \arg \frac{z_0}{z_1}, \quad \theta_i^j := \arg \frac{z_i}{z_1} (i \geq 2), \]

Then we have

\[ \mu = u_i^1(e_0 - e_1) + \sum_{i=2}^n u_i^1(e_i - e_1) + e_1, \quad u_i^1 := \frac{(r_i^1)^2}{1 + \sum_{j=1}^n (r_j^1)^2} (i \geq 1). \]

We also introduce the coordinates defined by

\[ x_i^1 := \sqrt{2u_i^1 \cos \theta_i^1}, \quad y_i^1 := \sqrt{2u_i^1 \sin \theta_i^1}. \]

Then, on \(U_1\) we have

\[ \omega_{FS}|_{U_1} = \sum_{i=1}^n du_i^1 \wedge d\theta_i^1 = \sum_{i=1}^n dx_i^1 \wedge dy_i^1, \]

\[ \mu|_{U_1} = \frac{1}{2} \{(x_i^1)^2 + (y_i^1)^2\}(e_0 - e_1) + \frac{1}{2} \sum_{i=2}^n \{(x_i^1)^2 + (y_i^1)^2\}(e_i - e_1) + e_1. \]

Hence we obtain an isomorphism

\[ (U_1, \omega_{FS}|_{U_1}, \mu|_{U_1}) \cong (\mu_0^{-1}(\Delta_1), \omega_0, \mu_0) \]

as Hamiltonian $T^n$-spaces. Moreover, on \(U_0 \cap U_1\),

\[ r_i^0 = \frac{1}{r_i^1}, \quad r_j^0 = \frac{r_j^1}{r_i^1} (j \geq 2), \quad \theta_0^1 = -\theta_1^1, \quad \theta_j^0 = \theta_j^1 - \theta_1^1 (j \geq 2) \]

hold. Then we have

\[ u_0^1 = \frac{1}{1 + \sum_{j=1}^n (r_j^1)^2} = 1 - \sum_{j=1}^n u_j^1, \quad u_0^j = u_j^1 (j \geq 2) \]

and we can easily check that

\[ \mu = \sum_{j=1}^n u_j^1 e_j = u_1^1(e_0 - e_1) + \sum_{i=2}^n u_i^1(e_i - e_1) + e_1. \]

Similarly, the \(e_i\)-action coordinates \((u_i^1, \ldots, u_i^n)\) satisfies that

\[ u_i^j = u_0^j (i \neq j), \quad u_i^i = 1 - \sum_{j=1}^n u_0^j \]
on $U_0 \cap U_i$ and

$$\mu|_{U_0 \cap U_i} = \sum_{j=1}^{i} u_i^j (e_{j-1} - e_i) + \sum_{j=i+1}^{n} u_i^j (e_j - e_i) + e_i.$$ 

Hence, on $U_0 \cap U_i$ the symplectic structure $\omega_{FS}$ described as

$$\omega_{FS}|_{U_0 \cap U_i} = \sum_{j=1}^{n} du_i^j \wedge d\theta_0^j = \sum_{j=1}^{n} du_i^j \wedge d\theta_i^j.$$ 

3.2. Volume of a Lagrangian torus orbit in $\mathbb{C}P^n$. Recall that the moment map $\mu : \mathbb{C}P^n \to \Delta$ is associated with the standard Hamiltonian $T^n$-action on $(\mathbb{C}P^n, \omega_{FS})$. The volume of a $T^n$-orbit $\mu^{-1}(p)$, $p \in \text{Int}(\Delta)$, can be calculated by using Abreu’s symplectic potential (see Section 4). Let $(u_0^1, \ldots, u_0^n)$ be the $e_0$-action coordinates of $p$. Then we obtain

$$\left(\text{Vol}(\mu^{-1}(p))\right)^2 = C \left(1 - \sum_{j=1}^{n} u_0^j\right) \prod_{k=1}^{n} u_0^k,$$

where $C$ is a positive constant. As for the $e_i$-action coordinates $(u_i^1, \ldots, u_i^n)$, by the formula of the coordinate transformation examined in the previous subsection, we have the same formula

$$\left(\text{Vol}(\mu^{-1}(p))\right)^2 = C \left(1 - \sum_{j=1}^{n} u_i^j\right) \prod_{k=1}^{n} u_i^k. \tag{3.1}$$

3.3. Proof of the main theorem. Here we give a property of a moment polytope which holds only for $\mathbb{C}P^n$ among compact toric Kähler manifolds.

**Lemma 10.** Let $u_i = (u_i^1, \ldots, u_i^n)$ be the $e_i$-action coordinates of $p \in \text{Int}(\Delta)$. Then there exists a number $i$ such that $||u_i|| \leq 1$.

**Proof.** Suppose that $||u_0|| > 1$. By definition, there exists $i \in \{0, 1, \ldots, n\}$ such that $\overline{u}_0 = u_0^i$. Then we have

$$u_i = (u_i^1, \ldots, u_i^{i-1}, u_i^i, u_i^{i+1}, \ldots, u_i^n) = (u_0^1, \ldots, u_0^{i-1}, 1 - |u_0|, u_0^{i+1}, \ldots, u_0^n),$$

and hence $|u_i| = 1 - u_0^i$. Therefore,

$$||u_i|| = 1 - u_0^i + \overline{u}_i = \begin{cases} 1 - u_0^i + u_j^j \leq 1 & \text{(if } \overline{u}_i = u_0^j (i \neq j)) \\ 2 - ||u_0|| < 1 & \text{(if } \overline{u}_i = 1 - |u_0|), \end{cases}$$

which implies $||u_i|| \leq 1$. \hfill $\square$

Theorem 5 is a direct consequence of the following
Theorem 11. Let $(\mathbb{C}P^n, \omega_\text{FS})$ be the $n$-dimensional complex projective space. Let $\mu : \mathbb{C}P^n \to \Delta$ be the moment map associated with the standard Hamiltonian $T^n$-action on $\mathbb{C}P^n$. Pick a point $p \in \text{Int}(\Delta)$ and take $e_i$-action coordinates $u_i = (u_i^1, \ldots, u_i^n) \in \text{Int}(\Delta)$ of $p$ which satisfies that $\|u_i\| \leq 1$. If $N(u_i) \geq 3$, then the Lagrangian torus orbit $\mu^{-1}(p) \subset \mathbb{C}P^n$ is not Hamiltonian volume minimizing.

Remark 12. We denote by $D_n$ the set of all points in $\text{Int}(\Delta)$ which satisfy the assumption of Theorem 11. Of course, $D_n$ is open dense in $\text{Int}(\Delta)$ if $n \geq 3$.

Proof. Firstly, since $\|u_i\| < \|u_i\| \leq 1$, by virtue of Theorem 5 we can take $a := u_i$ in the proof of Proposition 5 and there exist a positive number $\varepsilon > 0$ and a smooth function $H : [0, 1] \times \mathbb{C} \to \mathbb{R}$ satisfying that

1. $\text{Supp}(H) \subset [0, 1] \times \mu_0^{-1}(\Delta_{1-\varepsilon})$,
2. $\phi_{\mu}^t(\mu_0^{-1}(u_i)) = \mu_0^{-1}(u'_i)$.

Let $p'$ be the element in $\text{Int}(\Delta)$ whose $e_i$-action coordinate is $u'_i$. Notice that we have the identification $(U_i, \omega_\text{FS}|_{U_i}, \mu|_{U_i}) \cong (\mu_0^{-1}(\Delta_i), \omega_0, \mu_0)$ as Hamiltonian $T^n$-spaces. Denoting it by $\Phi : U_i \to \mu_0^{-1}(\Delta_i) \subset \mathbb{C}^n$, we can define the following Hamiltonian function on $\mathbb{C}P^n$:

$$\hat{H}(t, x) := \begin{cases} H(t, \Phi(x)) & x \in U_i \\ 0 & x \in \mathbb{C}P^n \setminus U_i. \end{cases}$$

Then $\hat{H} \in C^\infty_c([0, 1] \times \mathbb{C}P^n)$ and we obtain $\phi_{\hat{H}}^t(\mu^{-1}(p)) = \mu^{-1}(p')$.

Secondly, let us compare their volume. By assumption, there exist numbers $a, b \in \{1, \ldots, n\}$ such that $u_i < u_i^a < u_i^b$. Then by (3.1) we obtain

$$\frac{\text{Vol}(\mu^{-1}(p))}{\text{Vol}(\mu^{-1}(p'))} - \left(\frac{\text{Vol}(\mu^{-1}(p'))}{\text{Vol}(\mu^{-1}(p))}\right)^2 = C \left(1 - \frac{1}{\prod_{k=1}^n u_i^k}\right) \left(1 - \frac{1}{\prod_{j=1}^n u_i^j}\right) \left(1 - \frac{1}{\prod_{j=1}^n u_i^j + u_i^a - u_i^b + u_i}\right)\left(1 - \frac{1}{\prod_{k=1}^n u_i^k}\right)$$

$$= C \left(1 - \frac{1}{\prod_{k=1}^n u_i^k}\right) \left(1 - \frac{1}{\prod_{j=1}^n u_i^j}\right) \left(1 - \frac{1}{\prod_{j=1}^n u_i^j + u_i^a - u_i^b + u_i}\right)\left(1 - \prod_{k=1}^n u_i^k\right)\left(1 - \prod_{j=1}^n u_i^j\right)$$

$$= C \left(1 - \frac{1}{\prod_{k=1}^n u_i^k}\right) \left(1 - \frac{1}{\prod_{j=1}^n u_i^j}\right) \left(1 - \prod_{k=1}^n u_i^k\right)\left(1 - \prod_{j=1}^n u_i^j\right)$$

$$= C \prod_{k=1}^n u_i^k \left(u_i^a - u_i\right) \left(1 - \prod_{j=1}^n u_i^j\right)$$.
Since \( \prod_{k=1}^{n} u_i^k (u_i^k - \mathbf{u}_i) > 0 \) and
\[
1 - \sum_{j=1}^{n} u_i^j - u_i^k + u_i^a - \mathbf{u}_i \geq 1 - \|\mathbf{u}_i\| + u_i^a - \mathbf{u}_i > 0
\]
hold, we conclude that \( \text{Vol}(\mu^{-1}(p)) > \text{Vol}(\mu^{-1}(p')) \). \qed

4. The case of toric Kähler manifolds

In this section, we attempt to generalise the argument of Section 3 to toric Kähler manifolds. From now on, let \((M, \omega, J)\) be a complex \(n\)-dimensional compact toric Kähler manifold, i.e., \(M\) admits an effective holomorphic action of the complex torus \((\mathbb{C} \times)^n\) such that the restriction to the real torus \(T^n\) is Hamiltonian with respect to the Kähler form \(\omega\). Its moment map is denoted by \(\mu : M \to \Delta = \mu(M) \subset \mathbb{R}^n\). We may assume, without loss of generality, that the moment polytope \(\Delta\) satisfies
\[
\Delta = \{ \mathbf{a} \in (\mathbb{R}_{\geq 0})^n \mid l_r(\mathbf{a}) := \langle \mathbf{a}, \mu_r \rangle - \lambda_r \geq 0, \lambda_r < 0, r = n+1, \ldots, d \},
\]
where each \(\mu_r\) is a primitive element of the lattice \(\mathbb{Z}^n \subset \mathbb{R}^n\) and inward-pointing normal to the \(r\)-th \((n-1)\)-dimensional face of \(\Delta\). It is known that each fibre \(\mu^{-1}(\mathbf{a})\), \(\mathbf{a} \in \text{Int}(\Delta)\), is a Lagrangian torus and Hamiltonian minimal (see, e.g., [10, Proposition 3.1]).

The point \(\mu^{-1}(0) \in M\) is a fixed point of the \((\mathbb{C} \times)^n\)-action. By the construction, there exists a toric affine neighbourhood \(U\) of \(\mu^{-1}(0)\) such that \((U, \mu^{-1}(0))\) is isomorphic to \((\mathbb{C}^n, 0)\) as \((\mathbb{C} \times)^n\)-spaces. Using this identification we can define the standard complex coordinates \((w^1, \ldots, w^n)\) on \(U\). Their polar coordinates are given by \(w^i = r^i e^{\sqrt{-1} \theta^i}\), \(i = 1, \ldots, n\).

As a set \(U\) is described as
\[
U = M \setminus \mu^{-1}(\mathcal{F}), \quad \mathcal{F} := \bigcup_{F: \text{facet of } \Delta, 0 \notin F} F.
\]

The restriction of the Kähler form \(\omega\) on \(U\) can be expressed as
\[
\omega|_U = 2\sqrt{-1} \partial \bar{\partial} \varphi,
\]
where \(\varphi\) is a real-valued function defined on \((\mathbb{R}_{\geq 0})^n\) (see [1], [5]). Then the moment map \(\mu : M \to \Delta\) is represented as
\[
\mu(p) = \left( r^1 \frac{\partial \varphi}{\partial r^1}, \ldots, r^n \frac{\partial \varphi}{\partial r^n} \right)(p) =: (u^1, \ldots, u^n)\]
Putting $x^i := \sqrt{2u^i}\cos\theta^i$ and $y^i := \sqrt{2u^i}\sin\theta^i$, a straightforward calculation yields

$$\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i = \sum_{i=1}^{n} du^i \wedge d\theta^i, \ \mu|_U = \frac{1}{2} \sum_{i=1}^{n} ((x^i)^2 + (y^i)^2) e_i$$

on $U$. Thus $(U, \omega|_U, \mu|_U)$ is isomorphic as Hamiltonian $T^n$-spaces to $(V, \omega_0|_V, \mu_0|_V)$, where $V := \mu_0^{-1}(\Delta \setminus \mathcal{F})$ and $\mu_0$ is the moment map defined in Section 1.

Now we are in a position to prove our second result (Theorem 6).

**Theorem 13.** Let $(M, \omega, J)$ be a complex $n$-dimensional compact toric Kähler manifold equipped with the moment map $\mu : M \to \Delta \subset \mathbb{R}^n$ that is specified as above. Assume that $n \geq 3$ and define a constant $s_0 > 0$ as

$$s_0 = \sup\{ s > 0 \mid \tilde{\Delta}_s \subset \Delta \}.$$

For $a \in \text{Int}(\Delta)$ with $N(a) \geq 3$, if $\|a\| < s_0$, then there exists $a' \in \text{Int}(\Delta)$ such that

$$\phi(\mu^{-1}(a)) = \mu^{-1}(a')$$

for some $\phi \in \text{Ham}(M, \omega)$. Furthermore, if $\|a\|$ is sufficiently close to 0, then in addition these Lagrangian tori $\mu^{-1}(a)$ and $\mu^{-1}(a')$ satisfy

$$\text{Vol}(\mu^{-1}(a)) > \text{Vol}(\mu^{-1}(a')).$$

In particular, the above Lagrangian torus $\mu^{-1}(a)$ is not Hamiltonian volume minimizing in $M$.

**Proof.** Given a vector $a \in \text{Int}(\Delta)$ satisfying that $N(a) \geq 3$ and $\|a\| < s_0$, according to Theorem 9 the proof of Proposition 8 enables us to take $a' \in \text{Int}(\Delta)$ and $\{\phi^t_H\}_{0 \leq t \leq 1} \subset \text{Ham}_c(\mathbb{C}^n, \omega_0)$ which satisfy

$$\phi^1_H(\mu_0^{-1}(a)) = \mu_0^{-1}(a'), \ \text{Supp}(H) \subset [0,1] \times \mu_0^{-1}(\tilde{\Delta}_{s_0})$$

and

$$\prod_{i=1}^{n} a_i > \prod_{i=1}^{n} a'_i.$$

Using the action angle coordinates $(u^1, \ldots, u^n, \theta^1, \ldots, \theta^n)$ on $U$ explained before, we identify $(U, \omega|_U, \mu|_U)$ with $(V, \omega_0|_V, \mu_0|_V)$ and extend the Hamiltonian function $H$ on $U$ to $M$ as

$$\tilde{H}(t, x) := \begin{cases} H(t, x) & , \ x \in U \\ 0 & , \ x \in M \setminus U. \end{cases}$$

Then $\tilde{H} \in C^\infty([0,1] \times M)$ and hence we obtain $\phi^1_{\tilde{H}}(\mu^{-1}(a)) = \mu^{-1}(a')$. 


In order to complete the proof of Theorem 6, we have to compare the volume of two flat tori $\mu^{-1}(a)$ and $\mu^{-1}(a')$ with respect to the induced metric from the toric Kähler manifold $(M, \omega, J)$.

In general, all $\omega$-compatible toric complex structures on $(M, \omega)$ can be parametrized by smooth functions on $\text{Int}(\Delta)$, which is shown by Abreu in [1, Section 2]. More precisely, we can choose a strictly convex function $g \in C^\infty(\text{Int}(\Delta))$ whose Hessian $\text{Hess}_x(g)$ describes the complex structure $J$ on $M$, and the determinant of $\text{Hess}_x(g)$ is given by

$$\delta(x) \prod_{r=1}^d l_r(x)$$

where $\delta \in C^\infty(\Delta)$ is a strictly positive function (see [1, Theorem 2.8]).

Then the Riemannian metric of the fibre $\mu^{-1}(a) \subset M$ of $p \in \text{int}(\Delta)$ is given by the $(n \times n)$-matrix $(\text{Hess}_x(g))^{-1}$, and hence

$$\text{Vol}(\mu^{-1}(a))^2 = (2\pi)^2 \delta(a) \prod_{i=1}^n a_i \prod_{r=n+1}^d l_r(a)$$

holds. However, in general it is difficult to compare $\text{Vol}(\mu^{-1}(a))$ with $\text{Vol}(\mu^{-1}(a'))$ from this expression. So we introduce a parameter $c \in (0, 1]$ and consider the volume of a Lagrangian torus $\mu^{-1}(ca)$. Then we obtain

$$\text{Vol}(\mu^{-1}(ca))^2 - \text{Vol}(\mu^{-1}(ca'))^2$$

$$= (2\pi \sqrt{c})^2 \delta(ca) \prod_{i=1}^n a_i \prod_{r=n+1}^d l_r(ca) - \delta(ca') \prod_{i=1}^n a'_i \prod_{r=n+1}^d l_r(ca')$$

The value at $c = 0$ of the quantity of the inside of the brackets is

$$\delta(0) \prod_{r=n+1}^d (-\lambda_r) \left( \prod_{i=1}^n a_i - \prod_{i=1}^n a'_i \right),$$

which is positive due to (4.3). Therefore, there exists a constant $c_a > 0$ such that

$$\text{Vol}(\mu^{-1}(ca)) - \text{Vol}(\mu^{-1}(ca')) > 0.$$ holds for any $c \in (0, c_a)$. Thus we complete the proof.

5. Remained open problems

Finally, let us discuss the remained part of Oh’s conjecture and add some remarks. According to Corollary 8 the unsolved part of Conjecture 2 is as follows.
Problem 14. Let $0 < a \leq b$ and $k = 1, 2, \ldots, n$. Is a product torus $T(a, \ldots, a, b, \ldots, b)_{k, n-k}$ in $\mathbb{C}^n$ Hamiltonian volume minimizing?

This problem had already been considered by Anciaux in the case where $n = 2$, and he gave a partial answer to it. He showed in [2, Main Theorem] that $T(a, a) \subset \mathbb{C}^2$ has the least volume among all Hamiltonian minimal Lagrangian tori of its Hamiltonian isotopy class. However, this result does not imply that $T(a, a)$ is Hamiltonian volume minimizing in $\mathbb{C}^2$.

Next we turn to the case of $\mathbb{C}P^n$. We proved in Theorem 5 that every Lagrangian torus that is the preimage of a point in $D_n \subset \text{Int}(\Delta)$ is not Hamiltonian volume minimizing. However, the barycentre $p_0$ of $\Delta$ is not in $D_n$. The corresponding fibre $\mu^{-1}(p_0) \subset \mathbb{C}P^n$ is a minimal Lagrangian torus and called the Clifford torus. Thus the following question raised by Oh is still open.

Problem 15 ([8], p. 516). Is the Clifford torus in $\mathbb{C}P^n$ Hamiltonian volume minimizing?

We point out that Urbano proved that the only Hamiltonian stable minimal Lagrangian torus in $\mathbb{C}P^2$ is the Clifford one (see [11, Corollary 2]).

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