A modified two-sided approximation method for a four-point Vallée-Poussin type problem

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A MODIFIED TWO-SIDED APPROXIMATION METHOD FOR A FOUR-POINT VALLÉE–POUSSIN TYPE PROBLEM

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Abstract. We develop a modified two-sided approximation method for a four-point boundary value problem of the Vallée–Poussin type for a system of non-linear differential equations of fourth order with argument deviations.

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1. INTRODUCTION

There are many works dealing with constructive methods for approximate integration of boundary value problems for ordinary differential equations, which allow one to obtain a direct algorithm to error estimation (see, e.g., [4, 10, 11] and references therein). These methods include the two-sided methods, which give provide a possibility to construct approximate solutions and, on every step of iteration, obtain a posteriori error estimates of the successive approximations. Numerous research papers are devoted to the construction of new modifications of two-sided methods aimed at the study of various boundary value problems for ordinary differential equations (see, e.g., [1–3, 9].

This paper is devoted to the investigation of a four-point boundary-value problem of the Vallée–Poussin type for a system of non-linear differential equations with argument deviation by using a suitable version of the two-sided method generalising the works [5, 6].

2. PROBLEM SETTINGS, DEFINITIONS AND NOTATIONS

Let us consider the following problem of Vallée-Poussin’s type: to find a solution $Y = (y_i)_{i=1}^n$ of the system of differential equations

$$Y^{(4)}(x) = F(x, Y(x), (J_A Y)(x), (J_{J} Y)(x)), \quad x \in [0, \ell], \quad (2.1)$$

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which satisfies the conditions

\[ Y(0) = A_1, \quad Y(\ell/3) = A_2, \quad Y(2\ell/3) = A_3, \quad Y(\ell) = A_4, \]

and

\[ Y(x) = \begin{cases} \Phi(x) & \text{if } x \in [\lambda_0, 0], \\ \Psi(x) & \text{if } x \in [\ell, \theta_0], \end{cases} \]

where \( F: [0, \ell] \times \mathbb{R}^3 \rightarrow \mathbb{R}^n \), the vector-functions \( A = (\lambda_i)_{i=1}^n \) and \( \Theta = (\theta_i)_{i=1}^n \) from \( C([0, \ell], \mathbb{R}^n) \) are such that \( \lambda_i(x) \leq x, \theta_i(x) \geq x \) for all \( x \in [0, \ell], i = 1, \ldots, n \),

\[ \lambda_0 := \min \{ \lambda_i(x) \mid x \in [0, \ell], i = 1, \ldots, n \}, \quad \theta_0 := \max \{ \theta_i(x) \mid x \in [0, \ell], i = 1, \ldots, n \}, \]

and \( A_s = (a_{i,s})_{i=1}^n \in \mathbb{R}^n \) for \( s = \frac{1}{4}, \frac{1}{2}, \) and \( \Phi \in C([\lambda_0, 0], \mathbb{R}^n), \Psi \in C([\ell, \theta_0], \mathbb{R}^n) \) are given initial vector-functions satisfying the conditions

\[ \Phi(0) = A_1, \quad \Psi(\ell) = A_4. \]

The operator \( \mathcal{J}_F : C([\lambda_0, \theta_0], \mathbb{R}^n) \rightarrow C([0, \ell], \mathbb{R}^n) \) appearing in (2.1) is defined by the formula

\[ \mathcal{J}_F Y(x) := \left( y_i(y_i(x)) \right)_{i=1}^n, \quad x \in [0, \ell], \]

for any \( \Gamma = (y_i)_{i=1}^n \in C([0, \ell], \mathbb{R}^n) \) and \( Y = (y_i)_{i=1}^n \in C([\lambda_0, \theta_0], \mathbb{R}^n). \)

3. ASSUMPTIONS

In the sequel, let us suppose that the right-hand side \( F: [0, \ell] \times \mathcal{D}^3 \rightarrow \mathbb{R}^n \), \( \mathcal{D} \subseteq \mathbb{R}^n \), of the equation (2.1) belongs to the class \( \mathcal{M}_D([0, \ell]) \), where \( \mathcal{M}_D([0, \ell]) \) denotes the set of the vector-functions \( F \) satisfying the following conditions:

1. \( F \in C([0, \ell] \times \mathcal{D}^3, \mathbb{R}^n); \)
2. there exists a vector-function \( H \in C([0, \ell] \times \mathcal{D}^6, \mathbb{R}^n) \) such that:
   - (a) the equality
     \[ H(x, U, U) = F(x, U) \]
     holds for all \( x \in [0, \ell] \) and \( U \in \mathcal{D}^3; \)
   - (b) the inequality
     \[ H(x, P_1(x), (\mathcal{J}_A P_1)(x), (\mathcal{J}_\Theta P_1)(x), Q_2(x), (\mathcal{J}_A Q_2)(x), (\mathcal{J}_\Theta Q_2)(x)) \]
     \[ \geq H(x, Q_1(x), (\mathcal{J}_A Q_1)(x), (\mathcal{J}_\Theta Q_1)(x), P_2(x), (\mathcal{J}_A P_2)(x), (\mathcal{J}_\Theta P_2)(x)) \]
     (3.1)

is satisfied for all \( x \in [0, \ell] \) and every vector-functions \( P_k, Q_k : [\lambda_0, \theta_0] \rightarrow \mathbb{R}^n, k = 1, 2 \), whose restrictions on \([0, \ell]\) belong to \( C^4([0, \ell], \mathbb{R}^n) \), such that

\[ P_k(x), Q_k(x) \in \mathcal{D} \quad \text{for all } x \in [\lambda_0, \theta_0], k = 1, 2, \]

\[ P_k(x) \leq Q_k(x) \quad \text{for } x \in [0, \ell/3] \cup [2\ell/3, \ell], k = 1, 2, \]

\[ P_k(x) \geq Q_k(x) \quad \text{for } x \in [\ell/3, 2\ell/3], k = 1, 2, \]

\[ P_k^{(4)}(x) \geq Q_k^{(4)}(x) \quad \text{for } x \in [0, \ell], k = 1, 2. \]

\(^*\text{C([0, \ell], \mathbb{R}^n)} \text{ is the usual Banach space of continuous vector-functions from [0, \ell] to } \mathbb{R}^n.\)
(c) the vector-function \(H\) satisfies the Lipschitz condition with a non-negative matrix \(K = (k_{ij})_{i,j=1}^n\), i.e.,

\[
|H(x, P_{00}, P_{01}, P_{02}, Q_{00}, Q_{01}, Q_{02}) - H(x, P_{00}, P_{01}, P_{02}, Q_{00}, Q_{01}, Q_{02})| 
\leq K \left( \sum_{s=0}^{2} |P_{1s} - P_{0s}| + |Q_{1s} - Q_{0s}| \right),
\]

for all \(P_{0s}, P_{1s}, Q_{0s}, Q_{1s}\) from \(\mathcal{D}\), \(s = 0, 1\), and all \(x \in [0, \ell]\).

In (3.1), (3.2), and all similar relations below, the inequalities between vectors and the absolute value sign are understood component-wise.

4. Preliminary Considerations

Due to the fact that the corresponding linearised homogeneous boundary value problem has only the trivial solution on \([0, \ell]\), the solution \(Y\) of problem (2.1)–(2.3) can be represented in the form

\[
Y(x) = \begin{cases} 
\Phi(x) & \text{for } x \in [\lambda_0, 0], \\
\Omega(x) - (TF(\cdot, Y(\cdot), (J_A Y)(\cdot), (J_B Y)(\cdot))(x) & \text{for } x \in [0, \ell], \\
\Psi(x) & \text{for } x \in [\ell, \theta_0],
\end{cases}
\]

where the vector-function \(\Omega(x) = (\omega_i(x))_{i=1}^n\) has the components

\[
\omega_i(x) = a_{i1} + \frac{243}{4\ell^6} \begin{vmatrix} x & 0 & x^2 & x^3 \\
a_{i2} - a_{i1} & \frac{\ell^2}{\ell} & \frac{\ell^3}{\ell^2} & \frac{\ell^4}{\ell^3} \\
\frac{\ell}{\ell} & \frac{\ell}{\ell} & \frac{\ell}{\ell} & \frac{\ell}{\ell} \end{vmatrix}, \quad x \in [0, \ell],
\]

the operator \(T: C([0, \ell], \mathbb{R}^n) \rightarrow C([0, \ell], \mathbb{R}^n)\) for any \(Z \in C([0, \ell], \mathbb{R}^n)\) is defined by the formula

\[
(TZ)(x) = \frac{81}{8\ell^6} \int_0^\ell \mathcal{G}(x, \xi) Z(\xi) d\xi, \quad x \in [0, \ell],
\]

and \(\mathcal{G}\) is the Green function [7, 8] of the problem given by the relations

\[
\mathcal{G}_1(x, \xi) = \begin{cases} 
R_{11}(x, \xi), & 0 \leq x \leq \frac{\ell}{2}, \\
R_{12}(x, \xi), & \frac{\ell}{2} \leq x \leq \frac{\ell}{3}, \\
R_{13}(x, \xi), & \frac{\ell}{3} \leq x \leq \ell,
\end{cases} \quad \mathcal{G}_1(x, \xi) = \begin{cases} 
R_{11}(x, \xi), & 0 \leq x \leq \frac{\ell}{2}, \\
R_{12}(x, \xi), & \frac{\ell}{2} \leq x \leq \frac{\ell}{3}, \\
R_{13}(x, \xi), & \frac{\ell}{3} \leq x \leq \ell,
\end{cases}
\]

\[
\mathcal{G}_2(x, \xi) = \begin{cases} 
R_{21}(x, \xi), & 0 \leq x \leq \frac{\ell}{4}, \\
R_{22}(x, \xi), & \frac{\ell}{4} \leq x \leq \frac{\ell}{3}, \\
R_{23}(x, \xi), & \frac{\ell}{3} \leq x \leq \frac{\ell}{2}, \\
R_{24}(x, \xi), & \frac{\ell}{2} \leq x \leq \ell,
\end{cases} \quad \mathcal{G}_3(x, \xi) = \begin{cases} 
R_{31}(x, \xi), & 0 \leq x \leq \frac{\ell}{4}, \\
R_{32}(x, \xi), & \frac{\ell}{4} \leq x \leq \frac{\ell}{3}, \\
R_{33}(x, \xi), & \frac{\ell}{3} \leq x \leq \frac{\ell}{2}, \\
R_{34}(x, \xi), & \frac{\ell}{2} \leq x \leq \ell.
\end{cases}
\]
\[ R_{k1}(x, \xi) = \begin{bmatrix} x & (x-\xi)^3 & x^2 & x^3 \\ \frac{t}{3} & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \\ \frac{2t}{3} & \frac{2t^2}{9} & \frac{4t^3}{27} & \frac{2t^3}{8t^3} \\ (l-\xi)^3 & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \end{bmatrix}, \quad R_{k4}(x, \xi) = \begin{bmatrix} x & 0 & x^2 & x^3 \\ \frac{t}{3} & 0 & \frac{t^2}{9} & \frac{t^3}{27} \\ \frac{2t}{3} & 0 & \frac{4t^2}{9} & \frac{8t^3}{27} \\ (l-\xi)^3 & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \end{bmatrix} \]

for \( k = 1, 3, \)

\[ R_{12}(x, \xi) = \begin{bmatrix} x & 0 & x^2 & x^3 \\ \frac{t}{3} & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \\ \frac{2t}{3} & \frac{2t^2}{9} & \frac{4t^3}{27} & \frac{2t^3}{8t^3} \\ (l-\xi)^3 & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \end{bmatrix}, \quad R_{33}(x, \xi) = \begin{bmatrix} x & (x-\xi)^3 & x^2 & x^3 \\ \frac{t}{3} & 0 & \frac{t^2}{9} & \frac{t^3}{27} \\ \frac{2t}{3} & 0 & \frac{4t^2}{9} & \frac{8t^3}{27} \\ (l-\xi)^3 & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \end{bmatrix} \]

and

\[ R_{22}(x, \xi) = R_{32}(x, \xi) = \begin{bmatrix} x & (x-\xi)^3 & x^2 & x^3 \\ \frac{t}{3} & 0 & \frac{t^2}{9} & \frac{t^3}{27} \\ \frac{2t}{3} & \frac{2t^2}{9} & \frac{4t^3}{27} & \frac{2t^3}{8t^3} \\ (l-\xi)^3 & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \end{bmatrix}, \]

\[ R_{13}(x, \xi) = R_{23}(x, \xi) = \begin{bmatrix} x & 0 & x^2 & x^3 \\ \frac{t}{3} & 0 & \frac{t^2}{9} & \frac{t^3}{27} \\ \frac{2t}{3} & \frac{2t^2}{9} & \frac{4t^3}{27} & \frac{2t^3}{8t^3} \\ (l-\xi)^3 & \frac{t^2}{9} & \frac{t^3}{27} & \frac{t^3}{8t^3} \end{bmatrix}. \]

It is easy to see that
\[ \mathcal{J}_1(x, \xi) \geq 0, \quad \mathcal{J}_2(x, \xi) \leq 0, \quad \mathcal{J}_3(x, \xi) \geq 0 \quad \text{for } (x, \xi) \in [0, t] \times [0, t]. \quad (4.2) \]

**Definition.** Vector-functions \( Z_0, V_0 : \left[ \lambda_0, \theta_0 \right] \to \mathbb{D} \) whose restrictions on \([0, t]\) belong to the space \( C^3([0, t], \mathbb{R}^n) \) are called comparison functions of problem (2.1)–(2.3) if they satisfy the boundary conditions (2.2), the initial condition (2.3), and the inequalities\( Z_0(x) \leq V_0(x) \) for \( x \in [0, t/3] \cup [2t/3, t] \),\n\( Z_0(x) \geq V_0(x) \) for \( x \in [t/3, 2t/3] \). \quad (4.3)

**Notation.** For any vector-functions \( P, Q : \left[ \lambda_0, \theta_0 \right] \to \mathbb{R}^n \) we set
\( \langle P, Q \rangle = \left\{ u \in \mathbb{R}^n \mid \min \{ P(x), Q(x) \} \leq u \leq \max \{ P(x), Q(x) \} \right\} \) for some \( x \in [\lambda_0, \theta_0] \),
where the operations “min” and “max” for vectors are understood component-wise.
5. CONSTRUCTION OF THE ALTERNATIVE TWO-SIDED METHOD FOR PROBLEM (2.1)–(2.3)

Let us construct the successive approximations \( \{Z_p\}_{p=1}^{\infty} \) and \( \{V_p\}_{p=1}^{\infty} \) of a solution of problem (2.1)–(2.3) according to the formulae

\[
Z_{p+1}(x) = \begin{cases} 
\Phi(x) & \text{for } x \in [\lambda_0, 0], \\
\Omega(x) - (TF_p)(x) & \text{for } x \in [0, \ell], \\
\Psi(x) & \text{for } x \in [\ell, \theta_0],
\end{cases}
\]

\[
V_{p+1}(x) = \begin{cases} 
\Phi(x) & \text{for } x \in [\lambda_0, 0], \\
\Omega(x) - (TF_p)(x) & \text{for } x \in [0, \ell], \\
\Psi(x) & \text{for } x \in [\ell, \theta_0],
\end{cases}
\]

(5.1)

where

\[
F^p(x) = H(x, Z_p(x), (J_A Z_p)(x), (J_\theta Z_p)(x), V_p(x), (J_A V_p)(x), (J_\theta V_p)(x)),
\]

\[
F_p(x) = H(x, V_p(x), (J_A V_p)(x), (J_\theta V_p)(x), Z_p(x), (J_A Z_p)(x), (J_\theta Z_p)(x))
\]

for all \( x \in [0, \ell] \), and the zero approximations \( Z_0 \) and \( V_0 \) are comparison functions of problem (2.1)–(2.3) satisfying the conditions

\[
\alpha_0(x) := Z_0^{(4)}(x) - F_0(x) \geq 0,
\]

\[
\beta_0(x) := V_0^{(4)}(x) - F^0(x) \leq 0
\]

(5.2)

for all \( x \in [0, \ell] \).

The iteration process (5.1) can be represented in the form

\[
Z_{p+1}(x) - Z_p(x) = (T\alpha_p)(x), \quad V_{p+1}(x) - V_p(x) = (T\beta_p)(x), \quad x \in [0, \ell],
\]

(5.3)

where

\[
\alpha_p(x) := Z_p^{(4)}(x) - F_p(x), \quad \beta_p(x) := V_p^{(4)}(x) - F_p(x), \quad x \in [0, \ell], \quad p \in \mathbb{N}.
\]

(5.4)

Hence, from (5.3) and (5.4), for any \( p \in \mathbb{N} \cup \{0\} \), we obtain

\[
\alpha_{p+1}(x) = F_p(x) - F_{p+1}(x), \quad \beta_{p+1}(x) = F_p(x) - F^p_{p+1}(x), \quad x \in [0, \ell],
\]

(5.5)

\[
Z_p(x) - Z_{p+2}(x) = -T(\alpha_p + \alpha_{p+1})(x), \quad x \in [0, \ell],
\]

(5.6)

\[
V_p(x) - V_{p+2}(x) = -T(\beta_p + \beta_{p+1})(x), \quad x \in [0, \ell],
\]

(5.7)

and

\[
\alpha_{p+1}(x) + \alpha_{p+2}(x) = F_p(x) - F_{p+2}(x), \quad x \in [0, \ell],
\]

\[
\beta_{p+1}(x) + \beta_{p+2}(x) = F_p(x) - F^{p+2}(x), \quad x \in [0, \ell].
\]

(5.8)

Taking into account conditions (4.2), (5.2), and (5.3) with \( p = 0 \), we can see that

\[
Z_1(x) - Z_0(x) \geq 0, \quad V_1(x) - V_0(x) \leq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell],
\]

\[
Z_1(x) - Z_0(x) \leq 0, \quad V_1(x) - V_0(x) \geq 0, \quad x \in [\ell/3, 2\ell/3].
\]
Thus, if \( Z_1(x), V_1(x) \in \mathcal{D} \) for all \( x \in [\lambda_0, \theta_0] \), then from (5.5) with \( p = 0 \), by virtue of (5.2), (5.8), and (3.1), we obtain \( \alpha_1(x) \leq 0, \beta_1(x) \geq 0 \) for all \( x \in [0, \ell] \). Therefore, from (4.2) and (5.3) with \( p = 1 \) we get

\[
\begin{align*}
Z_2(x) - Z_1(x) & \leq 0, \quad V_2(x) - V_1(x) \geq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\
Z_2(x) - Z_1(x) & \geq 0, \quad V_2(x) - V_1(x) \leq 0, \quad x \in [\ell/3, 2\ell/3].
\end{align*}
\]

(5.9)

Assume, in addition, that

\[
\alpha_0(x) + \alpha_1(x) \geq 0, \quad \beta_0(x) + \beta_1(x) \leq 0, \quad x \in [0, \ell].
\]

(5.10)

Then from (5.6) with \( p = 0 \) we obtain

\[
\begin{align*}
Z_0(x) - Z_2(x) & \leq 0, \quad V_0(x) - V_2(x) \geq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\
Z_0(x) - Z_2(x) & \geq 0, \quad V_0(x) - V_2(x) \leq 0, \quad x \in [\ell/3, 2\ell/3],
\end{align*}
\]

(5.11)

and thus (5.9) and (5.11) result in

\[
Z_0(x) \leq Z_2(x) \leq Z_1(x), \quad V_1(x) \leq V_2(x) \leq V_0(x),
\]

for \( x \in [0, \ell/3] \cup [2\ell/3, \ell] \). (5.12)

and

\[
Z_1(x) \leq Z_2(x) \leq Z_0(x), \quad V_0(x) \leq V_2(x) \leq V_1(x) \quad \text{for} \quad x \in [\ell/3, 2\ell/3].
\]

(5.13)

Therefore, we have proved that if \( \langle Z_0, Z_1 \rangle \subseteq \mathcal{D}, \langle V_1, V_0 \rangle \subseteq \mathcal{D} \), and conditions (5.10) hold, then the values \( Z_2(x) \) and \( V_2(x) \) of the next approximations which are obtained according to (5.1) also belong to the set \( \mathcal{D} \).

From (3.1), (5.10), (5.12), (5.13), and (5.3), (5.5), (5.7) with \( p = 2, 1, 0 \), we get

\[
\begin{align*}
\alpha_2(x) & \geq 0, \quad \beta_2(x) \leq 0, \quad x \in [0, \ell], \\
Z_3(x) - Z_2(x) & \geq 0, \quad V_3(x) - V_2(x) \leq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\
Z_3(x) - Z_2(x) & \leq 0, \quad V_3(x) - V_2(x) \geq 0, \quad x \in [\ell/3, 2\ell/3],
\end{align*}
\]

and

\[
\alpha_1(x) + \alpha_2(x) \leq 0, \quad \beta_1(x) + \beta_2(x) \geq 0, \quad x \in [0, \ell].
\]

Hence, from (5.6) with \( p = 1 \) we obtain

\[
\begin{align*}
Z_1(x) - Z_3(x) & \geq 0, \quad V_1(x) - V_3(x) \leq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\
Z_1(x) - Z_3(x) & \leq 0, \quad V_1(x) - V_3(x) \geq 0, \quad x \in [\ell/3, 2\ell/3].
\end{align*}
\]

Consequently,

\[
\begin{align*}
Z_0(x) \leq Z_2(x) \leq Z_3(x) \leq Z_1(x), \quad V_1(x) \leq V_3(x) \leq V_2(x) \leq V_0(x), & \quad \text{for} \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\
Z_1(x) \leq Z_3(x) \leq Z_2(x) \leq Z_0(x), \quad V_0(x) \leq V_2(x) \leq V_3(x) \leq V_1(x), & \quad \text{for} \quad x \in [\ell/3, 2\ell/3],
\end{align*}
\]

and thus \( Z_3(x), V_3(x) \in \mathcal{D} \) for all \( x \in [\lambda_0, \theta_0] \).
Using the method of the mathematical induction we can show that if \(\{Z_0, Z_1\} \subseteq \mathcal{D}, \{V_1, V_0\} \subseteq \mathcal{D}\), and conditions (5.10) hold, then the sequences \(\{Z_p\}_{p=1}^{\infty}\) and \(\{V_p\}_{p=1}^{\infty}\), which are constructed according to (5.1), satisfy the inequalities

\[
Z_{2p}(x) \leq Z_{2p+2}(x) \leq Z_{2p+3}(x) \leq Z_{2p+1}(x),
\]

\[
V_{2p}(x) \leq V_{2p+2}(x) \leq V_{2p+3}(x) \leq V_{2p+1}(x)
\]

for \(x \in [0, \ell/3] \cup [2\ell/3, \ell]\), \(p = 0, 1, 2, \ldots\), and

\[
Z_{2p+1}(x) \leq Z_{2p+3}(x) \leq Z_{2p+2}(x) \leq Z_{2p}(x),
\]

\[
V_{2p}(x) \leq V_{2p+2}(x) \leq V_{2p+3}(x) \leq V_{2p+1}(x)
\]

for \(x \in [\ell/3, 2\ell/3]\), \(p = 0, 1, 2, \ldots\).

Let us now find a sufficient condition for the uniform, on \(\mathcal{I}_0\), convergence of the sequences \(\{Z_p\}_{p=1}^{\infty}\) and \(\{V_p\}_{p=1}^{\infty}\) to the unique solution of the boundary value problem (2.1)–(2.3).

For any vector \(P = (p_i)_{i=1}^{n} \in \mathbb{R}^n\), we set

\[
\|P\| := \max_{i=1,n} |p_i|.
\]

Let us also put

\[
W_p(x) := Z_p(x) - V_p(x), \quad x \in [\lambda_0, \theta_0], \quad p = 0, 1, 2, \ldots,
\]

\[
\epsilon := \max_{x \in [0, \ell]} \left\{ \|Z_0(x) - Z_1(x)\|, \|V_0(x) - V_1(x)\|, \|W_0(x)\| \right\},
\]

and

\[
d := \max_{x \in [0, \ell]} \int_0^\ell |g(x, \xi)| d\xi = \frac{4\ell^{10}}{3^7}.
\]

Then using (5.3), (5.5), we can prove by induction the error estimate

\[
\max_{x \in [0, \ell]} \left\{ \|Z_{p+1}(x) - Z_p(x)\|, \|V_{p+1}(x) - V_p(x)\| \right\}
\]

\[
\leq \epsilon \left( \frac{81}{8\ell^6} d \|K\| \right)^p = \epsilon \left( \frac{\ell^4}{9} \|K\| \right)^p (5.14)
\]

valid for all \(p \in \mathbb{N}\), where \(K\) is the matrix appearing in the Lipschitz condition (3.2) and \(\|K\| = \max_{i=1,n} \left\{ \sum_{j=1}^{n} k_{ij} \right\}\).

If \(\|K\|\) satisfies the inequality

\[
\|K\| < \frac{9}{\ell^4}, \quad (5.15)
\]

then it follows from estimate (5.14) that the approximations \(\{Z_p\}_{p=1}^{\infty}\) and \(\{V_p\}_{p=1}^{\infty}\) converge, respectively, to certain limits \(Y_*\) and \(Y^*\) uniformly on \([\lambda_0, \theta_0]\).
Let us show that $Y(x) \equiv Y^*(x)$. From (5.1) we have

$$W_{p+1}(x) = \begin{cases} 0 & \text{for } x \in [\lambda_0, 0], \\ (T(F^p - F_p))(x) & \text{for } x \in [0, \ell], \\ 0 & \text{for } x \in [\ell, \theta_0]. \end{cases}$$

It is easy to show that the estimate

$$\max_{x \in [0, \ell]} \|W_p(x)\| \leq \xi \left( \frac{81}{8\ell^6} d e \|K\| \right)^p = \xi \left( \frac{\ell^4}{9} \|K\| \right)^p \tag{5.16}$$

is true for $p \in \mathbb{N}$. If condition (5.15) holds, then $\lim_{p \to \infty} W_p(x) = 0$ uniformly on $[0, \ell]$, and thus

$$Y(x) = Y^*(x) =: Y(x), \quad x \in [\lambda_0, \theta_0].$$

Passing in equalities (5.1) to the limit as $p \to \infty$, we obtain the equality

$$Y(x) = \begin{cases} \Phi(x) & \text{for } x \in [\lambda_0, 0], \\ \Omega(x) - (T\tilde{H})(x) & \text{for } x \in [0, \ell], \\ \Psi(x) & \text{for } x \in [\ell, \theta_0], \end{cases}$$

where

$$\tilde{H}(x) := H(x, Y(x), (\mathcal{J}_A Y)(x), (\mathcal{J}_\Theta Y)(x), Y(x), (\mathcal{J}_A Y)(x), (\mathcal{J}_\Theta Y)(x))$$

$$= F(x, Y(x), (\mathcal{J}_A Y)(x), (\mathcal{J}_\Theta Y)(x)), \quad x \in [0, \ell],$$

i.e., $Y$ is a solution of problem (2.1)–(2.3).

The uniqueness of the solution $Y$ under the condition (5.15) can be easily proved by using the Lipschitz condition (3.2).

Consequently, we have proved the following

**Theorem.** Let $F \in \mathcal{M}_D([0, \ell])$ and $Z_0, V_0$ be comparison functions of problem (2.1)–(2.3) satisfying conditions (5.2). In addition, let the first approximations $Z_1$ and $V_1$ constructed according to formulæ (5.1) be such that $(Z_0, Z_1) \subseteq D$, $(V_1, V_0) \subseteq D$, and conditions (5.10) hold. Assume also that condition (5.15) is satisfied.

Then the sequences of approximations $\{Z_p\}_{p=1}^\infty$ and $\{V_p\}_{p=1}^\infty$ constructed according to (5.1) converge uniformly on $[\lambda_0, \theta_0]$ to the unique solution $Y$ of problem (2.1)–(2.3) and, moreover,

$$Z_{2p}(x) \leq Z_{2p+2}(x) \leq Y(x) \leq Z_{2p+3}(x) \leq Z_{2p+1}(x),$$

$$V_{2p+1}(x) \leq V_{2p+3}(x) \leq Y(x) \leq V_{2p+2}(x) \leq V_{2p}(x)$$

for $x \in [0, \ell/3] \cup [2\ell/3, \ell]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \leq Z_{2p+3}(x) \leq Y(x) \leq Z_{2p+2}(x) \leq Z_{2p}(x),$$

for $x \in [\ell/3, 2\ell/3]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \leq Z_{2p+3}(x) \leq Y(x) \leq Z_{2p+2}(x) \leq Z_{2p}(x),$$

for $x \in [\ell/3, 2\ell/3]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \leq Z_{2p+3}(x) \leq Y(x) \leq Z_{2p+2}(x) \leq Z_{2p}(x).$$
A TWO-SIDED APPROXIMATION METHOD FOR A VALLÉE–POUSSIN TYPE PROBLEM

\[ V_{2p}(x) \leq V_{2p+2}(x) \leq Y(x) \leq V_{2p+3}(x) \leq V_{2p+1}(x) \]
for \( x \in [\ell/3, 2\ell/3] \), \( p = 0, 1, 2, \ldots \).

\textbf{Remark.} If the domain \( D \) is “large” enough, then there exist comparison functions \( Z_0, V_0 \) of problem (2.1)–(2.3) satisfying conditions (5.2).

Indeed, let \( U: [\lambda_0, \theta_0] \to \mathbb{R}^n \) be an arbitrary vector-function which satisfies the boundary conditions (2.2) and the initial condition (2.3) and is such that \( U|_{[0, \ell]} \in C^4([0, \ell], \mathbb{R}^n) \) and \( U(x) \in D \) for all \( x \in [\lambda_0, \theta_0] \). Then we set
\[
\alpha(x) := U^{(4)}(x) - F(x, U(x), (f_A U)(x), (f_B U)(x)) , \quad x \in [0, \ell].
\]
It is clear that the problems
\[
\eta^{(4)} = |\alpha(x)|, \\
\eta(0) = 0, \quad \eta(\ell/3) = 0, \quad \eta(2\ell/3) = 0, \quad \eta(\ell) = 0
\]
and
\[
q^{(4)} = -|\alpha(x)|, \\
q(0) = 0, \quad q(\ell/3) = 0, \quad q(2\ell/3) = 0, \quad q(\ell) = 0
\]
have unique solutions \( \eta \) and \( q \), respectively. Relations (4.1) and (4.2) yield
\[
\eta(x) \leq 0, \quad q(x) \geq 0 , \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\
\eta(x) \geq 0, \quad q(x) \leq 0 , \quad x \in [\ell/3, 2\ell/3].
\]

Now we put
\[
Z_0(x) = U(x) + \eta(x), \quad V_0(x) = U(x) + q(x), \quad x \in [0, \ell], \\
Z_0(x) = U(x), \quad V_0(x) = U(x), \quad x \in [\lambda_0, 0] \cup [\ell, \theta_0].
\]
It is easy to see that \( Z_0 \) and \( V_0 \) satisfy the boundary conditions (2.2), the initial condition (2.3), and inequalities (4.3). If \( Z_0(x), V_0(x) \in D \) for all \( x \in [\lambda_0, \theta_0] \), then \( Z_0, V_0 \) are comparison functions of problem (2.1)–(2.3) and, using (5.17), (5.18) and assumptions (2a) and (2b) of Section 3, we get
\[
Z_0^{(4)}(x) - F_0(x) = U^{(4)}(x) + |\alpha(x)| - F_0(x) = \\
= \alpha(x) + |\alpha(x)| + F(x, U(x), (f_A U)(x), (f_B U)(x)) - F_0(x) \geq 0
\]
and
\[
V_0^{(4)}(x) - F_0^0(x) = U^{(4)}(x) - |\alpha(x)| - F_0^0(x) = \\
= \alpha(x) - |\alpha(x)| + F(x, U(x), (f_A U)(x), (f_B U)(x)) - F_0^0(x) \leq 0
\]
for all \( x \in [0, \ell] \). Consequently, \( Z_0 \) and \( V_0 \) also satisfy conditions (5.2).
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