SYZYGIES OF A TOWER OF COMPACT LOCAL HERMITIAN SYMMETRIC SPACES OF FINITE TYPE

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Abstract. Let $X$ be a $n$ dimensional compact local Hermitian symmetric space of non-compact type and $L = \mathcal{O}(K_X) \otimes \mathcal{O}(qM)$ be an adjoint line bundle. Let $c > 0$ be a constant. Assume the curvature of $M$ is $\geq c\omega$, where $\omega$ is the kähler form of $X$, and $X$’s injectivity radius has a lower bound $\tau > \sqrt{2e}$, where $e$ is the Euler number. In this article, we prove that if $q > \frac{2}{(p+1)}n$, then $L$ enjoys Property $N_p$. Applying this result to a tower of compact local Hermitian symmetric spaces $\cdots \rightarrow X_{s+1} \rightarrow X_s \rightarrow \cdots \rightarrow X_0 = X$, we prove that $2K_s$ has Properties $N_p$ for $s \gg 0$ and fixed $p$. Based on the same technique, we show a criterion of projective normality of algebraic curves and a division theorem with small power difference.

1. Introduction

1.1. Background And Main Results. Let $X$ be a compact local Hermitian symmetric space of non-compact type, namely $X$ can be written as $G/H$, where $G$ is a semi-simple Lie group of non-compact type and $H$ is a maximal compact subgroup. Under this natural setting, we consider a tower of manifolds $X_s = \Gamma_s \backslash G/H$ such that $\Gamma_{s+1} < \Gamma_s$ is a normal subgroup, which associates a sequence of finite maps

$$\cdots \rightarrow X_{s+1} \rightarrow X_s \rightarrow \cdots \rightarrow X_0 = X.$$ 

In [8] and [9], S.-K. Yeung shows that for $s \gg 0$ the canonical bundle $K_s$ is very ample and can separate the $k$-th jet. Inspired by the result of very ampleness, we investigate the properties of higher normality, namely Property $N_p$. In particular, Property $N_0$ corresponds to projective normality. We expect that for enough high tower covering, the higher normality should also be satisfied. In particular, if $X = B^n/\Gamma$ is a ball quotient with the injectivity radius $\rho_X$ and $L$ is numerically equivalent to $qK_X$, in [10] J.-M. Hwang and W.-K. To show that $K_X \otimes L$ enjoys Property $N_p$ if $p$ and $q$ satisfy

$$p + 1 < \frac{2(n+1)}{n} \sinh^2 \left( \frac{\rho_X}{4} \right).$$

For fixed $q \geq 2$ and $p$, in the tower of $X$, $qK_s$ has Property $N_p$ for $s \gg 0$. In this article we want to generalize this result to local symmetric Hermitian spaces of non-compact type and show that $qK_s$ has Property $N_p$ for fixed $q \geq 2$, $p$ and $s \gg 0$. Our main theorem is

**Theorem 1.1.** Let $X$ be a compact kähler manifold and $\omega$ be the kähler metric. Let the curvature form $R_M$ of the holomorphic line bundle $M$ satisfy $R_M \geq c\omega, c > 0$. Fix an integer $q \geq 1$ and let $\tau$ be the injectivity radius of the manifold $X$. Then, if
(a) $\tau$ is bounded from below: $\tau > \sqrt{2e}$, where $e$ is the Euler number and

$$q \geq \frac{1}{c} \cdot \sqrt{\frac{2}{e} (p + 1)n} > \frac{2e}{c\tau} \cdot (p + 1)n,$$

(b) or

$$q > \frac{2e}{c} \left(\frac{1}{\tau} + \frac{1}{\tau^2}\right) \cdot (p + 1)n,$$

$K_X + qM$ satisfies Property $N_p$.

If the tower of covering is enough high, the injectivity radius will approach infinity. Thus, we have the following direct implication.

**Corollary 1.2.** Let $X$ be a compact local Hermitian symmetric space of non-compact type, and $L = qK_X$, $q \geq 2$. Let $\{X_s\}$ be a tower of covering of $X$. Then, for fixed $p$, there exists $s_0$ such that $L_s = qK_{X_s}$ satisfies Property $N_p$ for every $s \geq s_0$.

Suggested by cohomological criterion of Property $N_0$, it has the format of division theorem. Let $L = K_X + M$. Property $N_0$ is equivalent to the surjectivity of the map

$$\beta_k : H^0(X, L) \otimes H^0(X, L^\otimes k) \rightarrow H^0(X, L^\otimes (k+1))$$

for every $k \geq 1$. If we intend to apply Skoda’s division theorem [7] to $\beta_k$, $k$ has to be large. In this article, we remove this constrain by introducing the injectivity radius into the estimate.

**Corollary 1.3** (Division Theorems with small power difference). Let $P_\tau \subset \mathbb{C}^n$ be a polydisc with side length $\tau$ with respect to a Kähler metric $\omega$. Let the curvature form $R_M$ of the holomorphic line bundle $M$ satisfy $R_M \geq c\omega$, $c \geq 0$. If

(a) $\tau$ is bounded from below: $\tau > \sqrt{2e}$, where $e$ is the Euler number

(b) or

$$q > \frac{2e}{c} \left(\frac{1}{\tau} + \frac{1}{\tau^2}\right) \cdot (p + 1)n,$$

the map $\beta_k : H^0(P_\tau, L) \otimes H^0(P_\tau, L^\otimes k) \rightarrow H^0(P_\tau, L^\otimes (k+1))$ is surjective.

By using similar techniques, we can prove a theorem of projective normality of algebraic curves.

**Theorem 1.4.** Let $X$ be a Riemann surface and $\omega$ be a kähler metric. Let the curvature form $R_M$ of the holomorphic line bundle $M$ satisfy $R_M \geq c\omega$, $c \geq 0$. Let $\tau$ be the injectivity radius of the manifold $X$. If $\tau > \sqrt{2e}$ and $c > \frac{2e}{\tau}$, then $K_X + M$ satisfies Property $N_0$.

**Remark 1.** For conveniency, we will introduce two small positive constants $\epsilon$ and $\epsilon'$ in doing later estimates, so the conditions in Theorem 1.4 can be phrased in terms of $\epsilon$ and $\epsilon'$:

(a) $\tau$ is bounded from below: $\tau \geq \sqrt{e(2 + \epsilon')}$, where $e$ is the Euler number and

$$q \geq \frac{1}{c} \cdot \sqrt{\frac{2 + \epsilon'}{e} (p + 1)(n + \epsilon)} \geq \frac{e(2 + \epsilon')}{c\tau} \cdot ((p + 1)n + \epsilon),$$

(b) or

$$q \geq \frac{e(2 + \epsilon')}{c} \left(\frac{1}{\tau} + \frac{1}{\tau^2}\right) \cdot ((p + 1)n + \epsilon),$$

and so are the conditions in Corollary 1.3 and Theorem 1.4.
1.2. Contents. This article is structured as follows: in Section 2 we review the definition and equivalent cohomological characterization of Property \( N_p \). In Section 3, we use the extension theorem to prove the main Theorem 1.1. In Section 4, we use the techniques developed in Section 3 to show the projective normality of algebraic curves (Theorem 1.4) and division theorem with small power difference (Corollary 1.3).

Acknowledgements

We want to specially thank professor Sai-Kee Yeung for useful discussion and generous advice on this paper.

2. \( N_p \) Properties

2.1. Setting. Let \( X \) be an irreducible projective variety, and \( L \) be a very ample line bundle on \( X \) defining an embedding

\[
\phi_L : X \rightarrow \mathbb{P} = \mathbb{P} H^0(X, L).
\]

Consider the graded ring \( R_L = R(X, L) = \bigoplus H^0(X, L^m) \) determined by \( L \), and write \( S = \text{Sym} H^0(X, L) \) for the homogeneous coordinate ring of \( \mathbb{P} \). Then \( R_L \) admits a free resolution \( E_\bullet \):

\[
\cdots \rightarrow \bigoplus_j S(-a_{2,j}) \rightarrow \bigoplus_j S(-a_{1,j}) \rightarrow S \bigoplus \bigoplus_j S(-a_{0,j}) \rightarrow R_L \rightarrow 0.
\]

We hope the resolution for the first \( p \) terms in \( E_\bullet \) are as simple as possible. For example, every two adjacent grades are just different by 1, i.e.

**Definition 2.1** (Property \( N_p \), [5], Definition 1.8.50). The embedding line bundle \( L \) satisfies Properties \( N_p \) if \( E_0 = S \), and

\[
a_{i,j} = i + 1 \quad \text{for all} \quad j
\]

whenever \( 1 \leq i \leq p \).

There is an alternative way to characterize Property \( N_p \) by cohomologies. Let \( X \) be a projective variety and \( L \) be a line bundle generated by global sections. Then, there exists a natural exact sequence:

\[
0 \rightarrow M_L \rightarrow H^0(L) \otimes \mathcal{O}_X \overset{i}{\rightarrow} L \rightarrow 0,
\]

where \( M_L \) is the kernel of \( i \), and (1) naturally induces a Koszul complex:

\[
0 \rightarrow \bigwedge^{p+1} M_L \rightarrow \bigwedge^{p+1} H^0(L) \otimes \mathcal{O}_X \rightarrow \bigwedge^p M_L \otimes L \rightarrow 0,
\]

where the map is described by

\[
f_{i_1, \ldots, i_{p+1}} e^{i_1} \wedge \cdots \wedge e^{i_{p+1}} \mapsto \sum (-1)^k f_{i_1, \ldots, i_{p+1}} s_{i_k} e^{i_1} \wedge \cdots \wedge \hat{e}^{i_k} \cdots \wedge e^{i_{p+1}}.
\]

2.2. Cohomological Criterion of Property \( N_p \). Under the preceding setting, the criterion of Property \( N_p \) is as follows.

**Lemma 2.2** ([1], Lemma 1.6). Assume that \( L \) is very ample, and that \( H^1(X, L^k) = 0 \) for all \( k \geq 1 \). Then \( L \) satisfies Property \( N_p \) iff

\[
H^1(X, \bigwedge^a M_L \otimes L^b) = 0, \quad \forall a \leq p + 1 \text{ and } b \geq 1.
\]
In characteristic zero case, the wedge product is a direct summand of the tensor product. Therefore, \( L \) will have Property \( P \) if the following condition holds:

\[
H^1(X, \bigotimes^a M_L \otimes L^b) = 0, \quad \forall a \leq p + 1 \text{ and } b \geq 1.
\]

In general, it is hard to deal with \( M_L \) directly, so an improved version of vanishing condition is needed. The idea is to consider vanishing of cohomology groups on the product of \( X \) rather than \( X \) itself.

**Proposition 2.3** ([4], Lemma 1.5). Let \( L \) be an ample line bundle on a projective manifold \( X \) with \( H^1(X, L^k) = 0 \) for all \( k \geq 1 \). Then for an integral \( \ell \geq 2 \), \( L \) satisfies Property \( N_\ell-2 \) if for all integers \( m \) and \( b \) satisfying \( 2 \leq m \leq \ell \) and \( b \geq 1 \),

\[
H^1(X^m, q_1^* L^b \otimes \cdots \otimes q_m^* L \otimes \mathcal{I}_\Sigma) = 0,
\]

where \( \Sigma = D_{1,m} \cup D_{2,m} \cup \cdots \cup D_{m-1,m} \) is the union of pairwise diagonals in \( X \times \cdots \times X \).

This proposition has a direct implication.

**Theorem 2.4** ([4], Theorem 1.6). Let \( X \) be a projective variety and let \( L \) be an ample line bundle on \( X \). Then for every positive integer \( p_0 \), there exists a number \( n_0 \) such that \( L^n \) has property \( N_{p_0} \) for every \( n \geq n_0 \).

Remark that in this theorem, \( L \) has to be raised to enough high power without upper bound estimate, but in our Theorem 1.1 the required power of the line bundle is effective and explicit.

### 3. Proof of The Main Theorem

In this section we will break the proof of Theorem 1.1 into several steps. Basically, we will follow the framework proposed in [4]. Nevertheless, instead of applying Kodaira Vanishing theorem, we will use extension theorems in order to obtain the effective power of \( L = qK_X \). Throughout this section, we will assume

\[
H^1(X, L^b) = 0
\]

for all \( b \geq 1 \). Remark that the assumption of Theorem 1.1 satisfies this condition since \( R_M > 0 \).

3.1. \( p = 0 \) case. According to the argument in [4], it is sufficient to justify the conditions in the following lemma to show Property \( N_0 \).

**Lemma 3.1** (Lemma 1.1, [4]). Denote \( \mathcal{I}_D \) the ideal sheaf of the diagonal embedding of \( X \) in \( X^2 = X \times X \). Then

(a) \( H^0(X, M_L \otimes L^b) = H^0(X^2, \mathcal{I}_D \otimes L \otimes L^b) \),

(b) \( H^1(X, \mathcal{I}_D \otimes L \otimes L^b) = 0 \Rightarrow H^1(X, M_L \otimes L^b) = 0 \),

for every \( b \geq 1 \).

The main ingredient of the the proof in [4] is Kodaira-Viehweg vanishing theorem. Instead of applying the vanishing theorem, we prove the vanishing by extending the sections in the cohomology groups. Let

\[
V = H^0(X, L).
\]
Consider exact sequence (1) and tensor it with $L^{\otimes (b+1)}$, which induces a long exact sequence

$$0 \longrightarrow H^0(M_L \otimes L^{\otimes (b+1)}) \longrightarrow V \otimes H^0(L^{\otimes (b+1)}) \longrightarrow H^0(L^{\otimes (b+2)})$$

$$\longrightarrow H^1(M_L \otimes L^{\otimes (b+1)}) \longrightarrow H^1(L^{\otimes (b+1)}) = 0.$$  

(5)

The last term is vanishing because of the assumption (3). Thus, if we can show the map $\iota$ is surjective, which implies $H^1(X, M_L \otimes L^{\otimes (b+1)}) = 0$. Then the original arguments in [4] follow and we are done.

3.1.1. Setting for Applying Extension Theorem. Before we proceed the extension theorem, let us treat the exact sequence (5) as the cohomology groups of line bundle on $X \times X$. Consider the exact sequence on $X \times X$:

$$0 \longrightarrow L \otimes L^{b+1} \otimes \mathcal{J}_D \longrightarrow L \otimes L^{b+1} \longrightarrow L^{b+2} \otimes \mathcal{O}_D \longrightarrow 0,$$

where $D \longrightarrow X \times X$ is the diagonal embedding of $X$. Then, there are natural isomorphisms:

$$V \otimes H^0(X, L^{\otimes (b+1)}) \cong H^0(X \times X, \pi_1^* L \otimes \pi_2^* L^{\otimes (b+1)})$$

$$H^0(X, L^{\otimes (b+2)}) \cong H^0(D, \pi_1^* L \otimes \pi_2^* L^{\otimes (b+1)}).$$

**Notation 3.2.** Let $L, M$ be line bundles on $X$. We will denote $L \boxtimes M$ the line bundle $\pi_1^* L \otimes \pi_2^* M$ on $X \times X$ for short later.

Hence the extension problem of the map $\iota$ becomes a problem of extending sections of $L \boxtimes L^{\otimes (b+1)}$ on the diagonal $D \subset X \times X$. Let $\dim X = n$. Since the codimension of $D \subset X \times X$ is greater than 1 if $\dim X \geq 2$, we need to blow up $D$ on $X \times X$ to fix this issue. Let $\alpha : Y = Bl_D X \times X \longrightarrow X \times X$ be the blowup, and we have the following diagram

$$\begin{array}{ccc}
E & \subset & Y \\
\| & & \| \\
D & \subset & X \times X
\end{array}$$

where $E$ is the exceptional divisor. Then, we turn to consider the extension problem on $Y$:

$$H^0(Y, \alpha^* L \boxtimes L^{\otimes (b+1)}) \longrightarrow H^0(E, \alpha^* L \boxtimes L^{\otimes (b+1)}) .$$

$$H^0(X \times X, L \boxtimes L^{\otimes (b+1)}) \longrightarrow H^0(D, L \boxtimes L^{\otimes (b+1)})$$

By the Ohsawa-Takegoshi theorem ([6] theorem 1.1), we need to justify the curvature condition:

$$\sqrt{-1} \Theta(\alpha^* L \boxtimes L^{\otimes (b+1)}) + \text{Ric}_Y \geq (1 + \epsilon)\sqrt{-1} \Theta(E)$$

$$\Longleftrightarrow \sqrt{-1} \Theta(\alpha^* L \boxtimes L^{\otimes (b+1)}) \geq \sqrt{-1} \Theta(\alpha^* K_X \boxtimes K_X) + (n + \epsilon)\sqrt{-1} \Theta(E)$$

where $0 < \epsilon \ll 1$. Here we utilize the blowup formula

$$\text{Ric}_Y = -K_Y = -(\alpha^* K_{X \times X} + (\text{codim } D - 1)E)$$
and codim $D = 2n - n = n$. In particular, if $L = (q + 1)K_X$ and $q \geq 1$, the above curvature condition becomes
\[
\sqrt{-1}\Theta(\alpha^*K_X \boxtimes K_X) + \sqrt{-1}\Theta(\alpha^*(p_1^*K_X^\otimes(q-1) + p_2^*K_X^\otimes((q+1)b+(q-1)))) \geq (n+\epsilon)\sqrt{-1}\Theta(E).
\]
Since $\sqrt{-1}\Theta(K_X) > 0$, we only need to require
\[
q\sqrt{-1}\Theta(\alpha^*K_X \boxtimes K_X) \geq (n + \epsilon)\sqrt{-1}\Theta(E). \tag{6}
\]
Note that $X \times X$ is also a Hermitian symmetric space of non-compact type because $\sqrt{-1}\Theta(K_{X \times X}) > 0$ and bounded below by using the Hermitian-Einstein metric. Let us examine the bundle $O_Y(E)$ closely. Choose an appropriate hypersurface $H$ on $X \times X$ such that $O_Y(E)$ is trivial on $U = X \times X - H$, namely, the transition function $h_{\alpha\beta}$ on the intersection of open sets $U_\alpha \cap U_\beta$ is 1. Let $U_D = (E - H)|_D \subset (D - H)$ is a local open set. After blowing up, the exceptional divisor $E \to D$ is a $\mathbb{P}^{n-1}$ projective bundle and
\[
E|_{U_D} \cong D \times \mathbb{P}^{n-1}.
\]
Let $x = (x_1, x_2) \in D \subset X \times X$, take $\Omega = B_\tau(x_1) \times B_\tau(x_2) \subset X \times X$, so that in $\Omega$, $X \times X$ can be seen as flat. Take an open set $U' \subset U$ if necessary so that $E|_{U'} \cong D \times \mathbb{C}^{n-1} \subset Y|_{U'}$. Let $\Omega' = \Omega \cap U'$ Then, on $\Omega'$, we are able to choose local coordinates
\[
z_1, \ldots, z_n
\]
for $D|_{U'}$, 
\[
w_1, \ldots, w_{n-1}
\]
for the exceptional direction, and extend the set to 
\[
w_1, \ldots, w_{n-1}, z_1, \ldots, z_n, z_{n+1},
\]
which is the coordinate system of $Y|_{U'}$. In particular we can choose $z_{n+1}$ so that 
\[
E = \{z_{n+1} = 0\}.
\]

3.1.2. Metric of $O_Y(E)$. By the standard technique in proving the Kodaira vanishing theorem, we take a two-open sets covering to cover $Y|_{U'}$. Denote
\[
P_\epsilon = \{x = (w, z) \in Y|_{U'}; |z_{n+1}| < \epsilon\},
\]
and then consider
\[
V_1 = P_{2\epsilon} \text{ and } V_2 = Y|_{U'} - P_\epsilon.
\]
Fix $0 < \epsilon' \ll 1$ a small constant, and we need a technical lemma to construct a special cut-off function.

Lemma 3.3. There exists a cut-off function $\chi$ such that
\[
\chi(t) = 1, t \leq \frac{\tau}{2}, \quad \chi(t) = 0, t \geq \tau
\]
\[
-\frac{2 + \epsilon'}{\tau^2} \leq \chi'(t) \leq 0, \quad |\chi''(t)| \leq \frac{4(2 + \epsilon')}{\tau^2}. \tag{7}
\]

Proof. This technical cut-off function is constructible. We refer the details to the proof of Theorem 1 in [8]. \qed
Let us consider the partition of unity functions associated with \( \chi \):
\[
\rho_1(z) = \rho(z) = e^{-\chi(\sigma)} \quad \text{and} \quad \rho_2(z) = 1 - \rho(z),
\]
where \( \sigma = |z_{n+1}| \Rightarrow 0 \leq \rho_1, \rho_2 \leq 1. \)

By construction, it is easy to see
\[
\frac{1}{e} \leq e^{-\chi} \leq 1 \quad \text{and} \quad |\chi'(\sigma)| \leq \frac{(2 + \epsilon')}{\epsilon^2},
\]
where \( \epsilon \) is the Euler number.

Now we are ready to construct a metric on \( O_Y(E) \). Let
\[
\begin{cases}
  h_1 = e^{-\varphi} = 1 + |w|^2 := 1 + |w_1|^2 + \cdots + |w_{n-1}|^2 \geq 1 \quad \text{and} \\
  h_2 = 1.
\end{cases}
\]

Note that \( h_1 \) is a natural metric of \( O(-1) \). Then, we define
\[
\begin{align*}
  h &= \rho_1 h_1 + \rho_2 h_2 = \rho h_1 + (1 - \rho)h_2 \\
  &= \rho(h_1 - 1) + 1 = \rho(e^{-\varphi} - 1) + 1.
\end{align*}
\]

This metric is well defined. Since on \( U' \subset U \), \( O(E)|_{U'} \) is trivial. Thus, the transition function is 1, which allows us to manipulate the metric freely without worrying about the transition laws. Note that the curvature induced by \( h \) has signs:
\[
\sqrt{-1} \Theta(\mathcal{O}_{Y,x}(E)) = \begin{cases}
  0 & \text{on } Y_x - P_{2\epsilon}, \\
  \text{bounded} & \text{on } P_{2\epsilon} - P_{\epsilon}, \\
  \leq 0 & \text{on } P_{\epsilon} (= 0 \text{ along radius direction}), \\
  < 0 & \text{on } E.
\end{cases}
\]

3.1.3. Curvature of \( \mathcal{O}_Y(E) \). Let us further explore the curvature of \( \mathcal{O}_Y(E) \) with respect to the metric \( h \) defined in (10). We have
\[
\partial_k \partial_{\ell} \log h = \mathcal{O} \left( \frac{(\rho(e^{-\varphi} - 1) + 1)(\partial_k \partial_{\ell} \rho(e^{-\varphi} - 1) + \partial_k \rho \partial_{\ell} e^{-\varphi} + \partial_{\ell} \rho \partial_k e^{-\varphi} + \rho \partial_k \partial_{\ell} e^{-\varphi})}{(\rho(e^{-\varphi} - 1) + 1)^2} \right)
\]
\[
= \frac{1}{(\rho(e^{-\varphi} - 1) + 1)^2} \left( (\rho(e^{-\varphi} - 1) + 1)(\partial_k \partial_{\ell} \rho(e^{-\varphi} - 1) + \rho \partial_k \partial_{\ell} e^{-\varphi}) \right)
\]
\[
+ (\rho(e^{-\varphi} - 1) + 1)(\partial_k \rho \partial_{\ell} e^{-\varphi} + \partial_{\ell} \rho \partial_k e^{-\varphi})
\]
\[
- (\partial_k \rho(e^{-\varphi} - 1) + \rho \partial_k e^{-\varphi})(\partial_{\ell} \rho(e^{-\varphi} - 1) + \rho \partial_{\ell} e^{-\varphi})
\]
Therefore, \( \partial_k \partial_\ell \log h \) is
\[
\frac{1}{\rho(e^{-\varphi} - 1) + 1} \left( \partial_k \partial_\ell \rho(e^{-\varphi} - 1) + \rho \partial_k \partial_\ell e^{-\varphi} \right) + \frac{1}{(\rho(e^{-\varphi} - 1) + 1)^2} \left( \partial_\ell \rho \partial_k e^{-\varphi} + \partial_k \rho \partial_\ell e^{-\varphi} - \partial_\ell \rho \partial_k \rho(e^{-\varphi} - 1)^2 - \rho^2 \partial_k e^{-\varphi} \partial_\ell e^{-\varphi} \right). \]
\( (11) \)

Recalling that
\[ -\partial_k \partial_\ell \log \rho = \frac{\partial_k \partial_\ell \rho}{\rho} + \frac{\partial_\ell \rho \partial_k \rho}{\rho^2}, \]
we aim to identify such shapes in \( (11) \). Matching up the terms and introducing a tangent vector \( v \), we have
\[
-\partial_k \partial_\ell \log h \, v^k \bar{v}^\ell = -\frac{\partial_k \partial_\ell \rho \rho}{\rho} v^k \bar{v}^\ell A + \frac{\partial_k \rho \partial_\ell \rho \rho^2}{\rho^2} v^k \bar{v}^\ell A^2 - \frac{\partial_k \partial_\ell e^{-\varphi}}{e^{-\varphi}} v^k \bar{v}^\ell B + \frac{\partial_k e^{-\varphi} \partial_\ell e^{-\varphi}}{(e^{-\varphi})^2} v^k \bar{v}^\ell B^2 - \frac{1}{(\rho(e^{-\varphi} - 1) + 1)^2} \left( \partial_\ell \rho \partial_k e^{-\varphi} + \partial_k \rho \partial_\ell e^{-\varphi} \right) v^k \bar{v}^\ell, \]
\( (12) \)
where
\[
A = \frac{\rho(e^{-\varphi} - 1)}{\rho(e^{-\varphi} - 1) + 1}, \quad B = \frac{\rho e^{-\varphi}}{\rho(e^{-\varphi} - 1) + 1}.
\]

By \( (9) \), it is easy to see
\[ A \leq 1, \quad \text{and} \quad B = \frac{\rho e^{-\varphi}}{\rho e^{-\varphi} + (1 - \rho)} \leq 1. \]

Let us further investigate the term \( \frac{\partial_\ell \rho \partial_k \rho \rho^2}{\rho^2} v^k \bar{v}^\ell \) in \( (12) \). Compute
\[
\frac{\partial_\ell \rho \partial_k \rho \rho^2}{\rho^2} v^k \bar{v}^\ell = \partial_\ell \chi v^k \cdot \partial_k \bar{\chi} \bar{v}^\ell = |\partial_\ell \chi v^k|^2 \geq 0,
\]
which implies
\[
-\partial_k \partial_\ell \log h \, v^k \bar{v}^\ell \leq \left( -\frac{\partial_k \partial_\ell \rho}{\rho} + \frac{\partial_\ell \rho \partial_k \rho}{\rho^2} \right) v^k \bar{v}^\ell A + \left( -\frac{\partial_k \partial_\ell e^{-\varphi}}{e^{-\varphi}} + \frac{\partial_\ell e^{-\varphi} \partial_k e^{-\varphi}}{(e^{-\varphi})^2} \right) v^k \bar{v}^\ell B - \frac{1}{(\rho(e^{-\varphi} - 1) + 1)^2} \left( \partial_\ell \rho \partial_k e^{-\varphi} + \partial_k \rho \partial_\ell e^{-\varphi} \right) v^k \bar{v}^\ell \leq \partial_\ell \partial_\ell v^k \bar{v}^\ell + \partial_k \partial_\ell \varphi v^k \bar{v}^\ell + \frac{2}{(\rho e^{-\varphi})^2} |\partial_\ell \partial_k e^{-\varphi} v^k \bar{v}^\ell|.
\]
\( (13) \)

- Regarding the first term in \( (13) \), we will take care of it by multiplying \( e^{-\chi} \) to the metric of \( M \) (cf. \( (10) \)).
- Regarding the second term in \( (13) \), recall \( \rho = e^{-\chi(\sigma)} \) and compute
\[
\frac{2}{(\rho e^{-\varphi})^2} |\partial_\ell \rho \partial_k e^{-\varphi} v^k \bar{v}^\ell| = \frac{2}{\rho e^{-\varphi}} |\partial_\ell \chi v^k \partial_k \bar{\chi} \bar{v}^\ell| = \frac{1}{\rho e^{-\varphi}} |\bar{\chi} v^k \partial_k \bar{\chi} \bar{v}^\ell| = \frac{1}{\rho e^{-\varphi}} |\partial_k \sigma v^k \partial_\ell \bar{v}^\ell| \leq e^{-\varphi} \left( \frac{1}{e^{-\varphi}} |\partial_k \sigma v^k|^2 + |\partial_\ell \bar{v}^\ell|^2 \right),
\]
\( (14) \)
where \( \sigma = |z_{n+1}|^2 \). Note that here we estimate \( 1/\rho = e^x \leq e \) because \( 0 \leq x \leq 1 \).

3.1.4. Estimates of (d) and (e) in (14). Regarding term (e), we aim to combine the estimate of term (e) = \( \frac{\sum_k |w_k|^2}{e} \) and the negativity of \( \partial_k \partial_{\bar{k}} v^k \bar{v}^\ell \) in (13). By using the explicit expression \( \varphi = -\log(1 + |w|^2) \) and \( |w|^2 = \sum_k |w_k|^2 \), we can compare these two terms. Recall

\[
-\partial_k \partial_{\bar{k}} \log(1 + |w|^2) = -\frac{(1 + |w|^2)\delta_k - w_k \bar{w}_k}{(1 + |w|^2)^2},
\]

which implies

\[
\partial_k \partial_{\bar{k}} v^k \bar{v}^\ell = -\frac{(1 + |w|^2)|v|^2 + w_k \bar{w}_k v^k \bar{v}^\ell}{(1 + |w|^2)^2}
= -\frac{|v|^2}{(1 + |w|^2)^2} - \frac{|w|^2|v|^2}{(1 + |w|^2)^2} + \frac{|w_k \bar{v}^\ell|^2}{(1 + |w|^2)^2}
\leq -|v|^2.
\]

On the other hand,

\[
\frac{1}{e^{-\varphi}} |\partial_{\bar{k}} \varphi \partial^k \bar{v}^\ell|^2 = \frac{1}{1 + |w|^2} \left| \frac{w_k \bar{v}^\ell}{1 + |w|^2} \right|^2 \leq \frac{1}{(1 + |w|^2)^3} |w|^2 |v|^2 \leq \frac{|v|^2}{(1 + |w|^2)^2} \leq |v|^2.
\]

Thus, by using (17), we require the numerical condition

\[
e|\chi| \leq e \cdot \frac{2 + \epsilon'}{\tau^2} \leq 1 \iff \tau \geq \sqrt{e(2 + \epsilon')}.
\]

If this is not the case, namely, the injectivity radius is small, then we need the positivity of \( M \) to take over the positivity of \( |\partial_{\bar{k}} \varphi \partial^k \bar{v}^\ell|^2 \). Again, by choosing the normal coordinates plus \( \sqrt{-1} \Theta(M) \geq c w \), we have \( \sqrt{-1} \Theta(M) k^k \bar{v}^\ell \geq c |v|^2 \). Thus, we require the numerical condition

\[
qc \geq (n + \epsilon) e \left( \frac{2 + \epsilon'}{\tau^2} \right),
\]

which corresponds to the \( 1/\tau^2 \) term in condition (b) in Theorem 11.

Let us examine term (d). By introducing the estimate of term (e), we obtain the estimate:

\[
-\partial_k \partial_{\bar{k}} \log h v^k \bar{v}^\ell \leq \partial_k \partial_{\bar{k}} \chi v^k \bar{v}^\ell + \frac{e|\chi|}{e^{-\varphi}} |\partial_k \sigma v^k|^2
= \partial_k \partial_{\bar{k}} \chi v^k \bar{v}^\ell + \frac{e}{1 + |w|^2} |\chi| \cdot |(\partial_k z_{n+1}) v^k|^2
\leq \partial_k \partial_{\bar{k}} \chi v^k \bar{v}^\ell + \frac{e}{\tau^2} \cdot \tau |v|^2.
\]

Recall (11), we require

\[
qc \sqrt{-1} \Theta(\alpha^* M \boxtimes M) \geq (n + \epsilon) \sqrt{-1} \Theta(E),
\]

namely, we need

\[
qc \geq (n + \epsilon) \cdot \frac{e}{\tau^2},
\]

which is the condition (a) in Theorem 11 in the case of \( p = 0 \).
3.1.5. **Extension Theorem.** Let $e^{-\varphi_M}$ be the smooth metric of $M$ such that $\partial_k \partial_{\bar{k}} e^{-\varphi_M} \geq c|v|^2$, and equip $M$ with the metric

$$e^{-\varphi_M + \bar{\varphi}}.$$ (16)

Let $L = K_X + qM$. By the construction,

$$\partial_k \partial_{\bar{k}} (q \varphi_M + \chi) v^k \bar{v}^\ell \geq (n + \epsilon) e^\frac{(2 + \epsilon')}{\tau} |v|^2 + \partial_k \partial_{\bar{k}} \chi v^k \bar{v}^\ell \geq -\partial_k \partial_{\bar{k}} \log h v^k \bar{v}^\ell.

Then, a section $f \in H^0(D, (K_X \otimes qM) \boxtimes (K_X \otimes qM)^{\otimes (b + 1)})$ on the diagonal $D$ satisfying the $L^2$ condition

$$\int_D \|f\|^2 dV_D = \int_U |f|^2 e^{-q(b + 2)(\varphi_M + \bar{\varphi})} dV_U < \infty.$$

can be extended to $\tilde{f}$ on $\Omega \times \Omega$ with $L^2$ estimates, and then $\tilde{f}$ can be extended to $X \times X$ and be a section in $H^0(X \times X, L \boxtimes L^{\otimes (b + 1)})$. Since the metric $e^{-(\varphi_M + \bar{\varphi})}$ is smooth, every section in $H^0(D, L \boxtimes L^{\otimes (b + 1)})$ is extendible. Thus, the map $\iota$ is surjective as desired and $K_X + qM$ has Property $N_0$. In particular, if $M = K_X$, $\chi = 1$, for enough high tower $X_\chi$, the injectivity radius will be sufficient large. Hence, $2K_X$ will enjoy Property $N_0$ for $s \gg 0$.

3.2. **$p = 1$ case.** By Lemma 2.22, Property $N_1$ is equivalent to $H^1(X, \otimes^a M_L \otimes L^{\otimes b}) = 0$ for $2 \geq a \geq 0$, $b \geq 1$ which implies the original cohomological condition $H^1(X, \otimes^a M_L \otimes L^{\otimes b}) = 0$ for $2 \geq a \geq 0$, $b \geq 1$.

3.2.1. **Setting for Applying Extension Theorems.** We consider the exact sequence

$$0 \longrightarrow M_L^{\otimes 2} \otimes L^{\otimes (b + 1)} \longrightarrow V \otimes M_L \otimes L^{\otimes (b + 1)} \overset{\iota}{\longrightarrow} M_L \otimes L^{\otimes (b + 2)} \longrightarrow 0,$$

and its induced long exact sequence

$$0 \longrightarrow H^0(M_L^{\otimes 2} \otimes L^{\otimes (b + 1)}) \longrightarrow V \otimes H^0(M_L \otimes L^{\otimes (b + 1)}) \overset{\iota}{\longrightarrow} H^0(M_L \otimes L^{\otimes (b + 2)})$$

$$\longrightarrow H^1(M_L^{\otimes 2} \otimes L^{\otimes (b + 1)}) \longrightarrow H^1(M_L \otimes L^{\otimes (b + 1)}) = 0.$$

Note that the last vanishing $H^1(M_L \otimes L^{\otimes (b + 1)}) = 0$ is by the previous step, i.e. Lemma 3.1(b). Our aim is to show $H^1(M_L^{\otimes 2} \otimes L^{\otimes (b + 1)}) = 0$. Similar to the proof of Lemma 3.1 it is sufficient to show the following lemma.

**Lemma 3.4** (Lemma 1.3, [4]). *Assume that $H^1(M_L \otimes L^{\otimes (b + 1)}) = 0$. Let $\Sigma^{(3)} = D_{1,3} \cup D_{2,3}$, where $D_{1,3}$ is the diagonal embedding of $X$ in $X_1 \times X_2 \subset X \times X \times X$. Denote $\mathcal{F}_{\Sigma^{(3)}}$ the ideal sheaf of $\Sigma^{(3)}$. Then

(a) $H^0(X, M_L^{\otimes 2} \otimes L^{\otimes (b + 1)}) = H^0(X^3, \mathcal{F}_{\Sigma^{(3)}} \otimes L \boxtimes L \boxtimes L^{\otimes (b + 1)}),$

(b) $H^1(X^3, \mathcal{F}_{\Sigma^{(3)}} \otimes L \boxtimes L \boxtimes L^{\otimes (b + 1)}) = 0 \Rightarrow H^1(X, M_L^{\otimes 2} \otimes L^{\otimes (b + 1)}) = 0,$

for every $b \geq 1$.

The proof is similar to the proof of lemma 3.1. Instead of showing $H^1(X^3, \mathcal{F}_{\Sigma^{(3)}} \otimes L \boxtimes L \boxtimes L^{\otimes (b + 1)}) = 0$ directly, we proceed the proof by using extension theorem.
Recall $V = H^0(X, L)$ and the canonical isomorphisms:

$$V \otimes H^0(X, M_L \otimes L^\otimes(b+1)) \cong V \otimes H^0(X^2, \mathcal{J}_D \otimes L \boxtimes L^\otimes(b+1))$$

$$\cong H^0(X^3, \mathcal{J}_{D, 2} \otimes L \boxtimes L^\otimes(b+1))$$

$$H^0(X, M_L \otimes L^\otimes(b+2)) \cong H^0(X^2, \mathcal{J}_D \otimes L \boxtimes L^\otimes(b+2))$$

$$\cong H^0(D_{1,3}, \mathcal{J}_{D, 2} \otimes L \boxtimes L^\otimes(b+1)),$$

where $D_{i,j} \cong X^2 \subset X \times X \times X$ is the partial diagonal embedding defined by $D_{i,j} = \{(x_1, x_2, x_3) \in X^3 \mid x_i = x_j\}$. Then, the vanishing of $H^1(X^3, \mathcal{J}_{\Sigma(3)} \otimes L \boxtimes L \boxtimes L^\otimes(b+1))$ is equivalent to the surjectivity of the restriction map

$$H^0(X^3, \mathcal{J}_{D, 2} \otimes L \boxtimes L^\otimes(b+1)) \longrightarrow H^0(D_{1,3}, \mathcal{J}_{D, 2} \otimes L \boxtimes L^\otimes(b+2)). \quad (17)$$

3.2.2. Extension Theorem. In order to apply the Ohsawa-Takegoshi theorem (Theorem 1.1), we increase the dimension of the extension center by blowup. Let $\alpha : Y = Bl_{D_{1,3}}X^3 \longrightarrow X^3$ be the blowup, and we have the following diagram:

$$\begin{array}{ccc}
E & \subset & Y \\
\downarrow & & \downarrow \alpha \\
D_{1,3} & \subset & X^3
\end{array}$$

where $E$ is the exceptional divisor. Then, we turn to consider the extension problem on $Y$:

$$H^0(Y, \alpha^* \mathcal{J}_{D, 2} \otimes L \boxtimes L \boxtimes L^\otimes(b+1)) \longrightarrow H^0(E, \alpha^* \mathcal{J}_{D, 2} \otimes L \boxtimes L^\otimes(b+2)).$$

$$H^0(X^3, \mathcal{J}_{D, 2} \otimes L \boxtimes L \boxtimes L^\otimes(b+1)) \longrightarrow H^0(D_{1,3}, \mathcal{J}_{D, 2} \otimes L \boxtimes L^\otimes(b+2)).$$

Note that the multiplier ideal sheaf $\mathcal{J}_{D, 2}$ associated to a singular weight function which takes $\infty$ along $D_{2,3}$. By the same justification as the $p = 0$ case, we calculate the curvature conditions of the bundles $L \boxtimes L \boxtimes L^\otimes(b+1)$, and require

$$qc \geq (2n + \epsilon) \cdot \frac{2 + \epsilon'}{\tau},$$

to obtain the desired curvature estimate:

$$q\sqrt{-1} \Theta(\alpha^* K_{X_x} \boxtimes K_{X_x} \boxtimes K_{X_x}) \geq (n + \epsilon)\sqrt{-1} \Theta(E).$$

The coefficient 2 of $n$ is coming from the blow up formula

$$\text{Ric}_Y = -K_Y = -(\alpha^* K_{X \times X} + (\text{codim } D_{1,3} - 1)E)$$

and $\text{codim } D_{1,3} = 3n - n = 2n$. Thus, the morphism in (17) is surjective and $K_X + qM$ has Property $N_1$. In particular, if $M = K_X, q = 1$, for enough high tower $X$, the injectivity radius will be sufficient large. Hence, $2K_{X_X}$ will enjoy Property $N_1$ for $s \gg 0$. 
3.3. **General case of** \( p \). Following the previous arguments in the \( p = 0 \) and \( p = 1 \) cases, we proceed the mathematical induction on \( p \). Similar to Lemma 3.1 and Lemma 3.4, we assume the following statements.

**Inductive Hypothesis.** Let the partial diagonal embedding of \( X \) be
\[
D_{i,j} = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j\},
\]
and denote \( \Sigma^{(p)} = D_{1,2} \cup D_{2,3} \cup \cdots \cup D_{p-1,p} \). Then
\[
\begin{align*}
(a) \quad & H^0(X, M_L^{\otimes (p-1)} \otimes L^{\otimes (b+1)}) = H^0(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}), \\
(b) \quad & H^1(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}) = 0 \Rightarrow H^1(X, M_L^{\otimes (p-1)} \otimes L^{\otimes (b+1)}) = 0,
\end{align*}
\]
for every \( b \geq 1 \). We take the singular weight function of \( \mathcal{J}_{\Sigma^{(p)}} \) as follows. Locally, \( \mathcal{J}_{\Sigma^{(p)}} \) is choosen as
\[
e^{-\varphi} = e^{-\varphi_{1,p}} \cdots e^{-\varphi_{p-1,p}},
\]
where \( e^{-\varphi_{i,p}} = \frac{1}{|f_{i,p}|} \) where \( f_{i,p} \) is the local defining equation of \( D_{i,p} \). By the construction, \( \varphi \) is a plurisubharmonic function. Then, we can prove the general lemma by using extension theorem.

**Lemma 3.5** (Lemma 1.5, [4]). Assume that \( L \) has Property \( N_{p-1} \). Denote \( \mathcal{J}_{\Sigma^{(p)}} \) the ideal sheaf of \( \Sigma^{(p)} \). Then
\[
\begin{align*}
(a) \quad & H^0(X, M_L^{\otimes (p-1)} \otimes L^{\otimes (b+1)}) = H^0(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}), \\
(b) \quad & H^1(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}) = 0 \Rightarrow H^1(X, M_L^{\otimes (p-1)} \otimes L^{\otimes (b+1)}) = 0.
\end{align*}
\]
Recall \( V = H^0(X,L) \) and the natural isomorphisms:
\[
V \otimes H^0(X, M_L^{\otimes (p-1)} \otimes L^{\otimes (b+1)}) \cong V \otimes H^0(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}) \\
\cong H^0(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}),
\]
\[
H^0(X, M_L^{\otimes (p-1)} \otimes L^{\otimes (b+2)}) \cong H^0(X^p, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+2)}) \\
\cong H^0(D_{1,p}, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}),
\]
it is sufficient to solve the extension problem:
\[
H^0(X^{p+1}, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}) \rightarrow H^0(D_{1,p}, \mathcal{J}_{\Sigma^{(p)}} \otimes L \boxtimes \cdots \boxtimes L \boxtimes L^{\otimes (b+1)}). \tag{18}
\]
Let be the \( \alpha : Y = Bl_{D_{1,p}} X^{p+1} \rightarrow X^{p+1} \) blowup and \( E \) be the exceptional divisor. Then, we apply the same argument as before. We require
\[
qc \geq ((p+1)n + \epsilon) \cdot \epsilon \left( \frac{2 + \epsilon}{\tau} \right)
\]
to obtain the desired curvature estimate:
\[
\sqrt{-1} \Theta(\alpha^* K_{X_s} \boxtimes \cdots \boxtimes K_{X_s}) \geq (n + \epsilon) \sqrt{-1} \Theta(E)
\]
Thus, the morphism in (18) is surjective as desired and \( K_X + qM \) has Property \( N_p \). In particular, if \( M = K_X, q = 1 \), for enough high tower \( X_s \), the injectivity radius will be sufficient large. Hence, \( 2K_{X_s} \) will enjoy Property \( N_p \) for \( s \gg 0 \) and we complete the proof.
4. Proof of Corollary 1.3 and Theorem 1.4

4.1. Normality of Riemann Surfaces. In this section, we will use the same framework and techniques to handle projective normality of Riemann surfaces. It is well known that Property $N_0$ is equivalent to projective normality, namely, if $L$ is projective normal if

$$Sym^k H^0(X, L) \to H^0(X, L^\otimes k)$$

is surjective (cf. [2], Introduction). Particularly, if one can show that

$$\beta_k : H^0(X, L) \otimes H^0(X, L^{\otimes (k-1)}) \to H^0(X, L^\otimes k)$$

is surjective for every $k$, then $L$ is projective normal. In our case, $L$ is an adjoint bundle, i.e. $L = K_X + M$. For sufficient large $k$, one can show that $\beta_k$ is surjective by Skoda’s division theorem. Thus, the difficulty lies in the case when $k$ is small, especially, when $k = 2$. The key is the natural isomorphisms we utilized in Section 3.1.4:

$$V \otimes H^0(X, L^{\otimes (k-1)}) \cong H^0(X \times X, \pi_1^* L \otimes \pi_2^* L^{\otimes (k-1)})$$

and the blowup diagram:

$$H^0(Y, \alpha^* L \boxtimes L^{\otimes (k-1)}) \xrightarrow{res} H^0(E, \alpha^* L \boxtimes L^{\otimes (k-1)}) ,$$

where $\alpha : Y = Bl_D X \times X$ is the blow-up along the diagonal. Then, we intend to apply the technique of extension similar to Section 3.1.2 to show that the restriction map

$$\beta_k : H^0(Y, \alpha^* L \boxtimes L^{\otimes (k-1)}) \to H^0(E, \alpha^* L \boxtimes L^{\otimes (k-1)})$$

is surjective.

Recall the curvature estimate of $O_Y(E)$ (13) and (14):

$$-\partial_k \partial_\bar{\ell} \log h v^k \bar{v}^\ell \leq \partial_k \partial_\bar{\ell} X v^k \bar{v}^\ell + \partial_k \partial_\bar{\ell} \varphi v^k \bar{v}^\ell + \frac{2}{(pe-v)^2} |\partial_k \rho \partial_\bar{\ell} e^{-\varphi} v^k \bar{v}^\ell|$$

$$\leq \partial_k \partial_\bar{\ell} X v^k \bar{v}^\ell + \partial_k \partial_\bar{\ell} \varphi v^k \bar{v}^\ell + \frac{e |\chi|}{e^{-\varphi}} (|\partial_k \sigma v^k|^2 + |\partial_\bar{\ell} \varphi \bar{v}^\ell|^2).$$ (19)

When the injectivity radius $\tau$ is large, such as

$$\tau \geq \sqrt{e(2 + e')},$$

we can apply the estimate in Section 3.1.4 and use the negativity of $O(-1)$ to control the term (e). For term (d), we aim to find a global coordinate to construct a metric to take over it. Since $X$ is a Riemann surface, the universal cover $\tilde{X}$ is $\mathbb{P}^1, \mathbb{C}$ or a disc $B(0, 1)$. By removing the branch points and the branch cuts, the fundamental domain $\Omega \subset \tilde{X}$ is biholomorphic to an open set $U \subset X$. By Riemann’s theorem, $U$ is further biholomorphic to a disc. Thus, we have a global coordinate $z$ which
enables us to construct an appropriate plurisubharmonic function with compact support to dominate term (d). Recall the estimate of the last term in (15):

\[
\frac{e^{\chi'}}{e^{-\nu}}|\partial_k \sigma v^k|^2 \leq \frac{e(2 + \epsilon')}{\tau^2}, \quad \tau |v|^2 = \frac{e(2 + \epsilon')}{\tau} |v|^2.
\]

We aim to construct a plurisubharmonic function \( \eta_\tau \) such that

\[
\partial_z \partial_{\bar{z}} \eta_\tau \geq \frac{e(2 + \epsilon')}{\tau}.
\]

Consider

\[
\eta_\tau = \begin{cases} 
  c_1 \tau |z|^2 & z < \tau \\
  c_1 \tau & z = \tau 
\end{cases}
\]

where \( c_1 \) is a constant to be determined. Then,

\[
\partial_z \partial_{\bar{z}} \eta_\tau = c_1 \tau \cdot \frac{1}{\tau^2} \partial_w \partial_{\bar{w}} |w|^2 \quad \text{(where } w = \frac{z}{\tau})
\]

\[
= \frac{c_1}{\tau}, \quad \text{on } |w| < 1.
\]

Thus, we can take

\[
f_\tau = e^{-\eta_\tau}, \quad \text{and } c_1 = e(2 + \epsilon') \quad (20)
\]

so that \( f_\tau \) is globally defined on \( X \) because the extended values on the branch points and branch cuts are 1 by taking the limit, and the curvature of the weight function \( f_\tau \) is

\[
-\partial \bar{\partial} \log f_\tau = \partial \bar{\partial} \eta_\tau \geq \frac{e(2 + \epsilon')}{\tau}.
\]

Therefore, we equip \( L \) with the metric

\[
e^{-(\phi M + \frac{1}{4} + \eta_\tau)},
\]

which has the desired curvature estimate

\[
q \sqrt{-1} \Theta(\alpha^* M \boxtimes M) \geq (1 + \epsilon) \sqrt{-1} \Theta(E).
\]

Hence, every section of \( H^0(E, \alpha^* L \boxtimes L^{\otimes (k-1)}) \) is extendible, and we finish the proof of Theorem 1.4.

Remark 2. In applying the extension theorem, we require a smooth weight function so that every section on the extension center is \( L^2 \) finite. Here we skip a standard technical detail. In order to make the weight function smooth, we need a family of smoothifiers to smooth out the corner of \( f_\tau \) at \( z = \tau, 0 \), i.e. \( w = 1, 0 \). By taking limit, we can still obtain the desired estimates and extend the section.

4.2. Division Theorem with Small Power Difference. The key step in proving Theorem 1.4 is to find a global coordinate. The setting of Corollary 1.3, the division theorem with small power difference, assures the existence of such coordinate, so by introducing the function \( f_\tau \) constructed in Section 4.1 and the estimates in Section 3.1.2 and Section 3.1.4, Corollary 1.3 follows.
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