1. INTRODUCTION

This introduction will give a brief summary of the physical ideas and mathematical concepts without going into the details of the physical models. Its purpose is to introduce some definitions and provide some guidance for the non-expert mathematician or physicist. In the first part of the introduction the physical motivation and scenario will be introduced, while in the second part the formalism and models will be summarized. If some terms seem unfamiliar, the reader should not be discouraged, because either the jargon used is not important or it will be defined at some later point. In the following sections more details will be given. The mathematically inclined reader who may not care about the physical applications may want to skip the first part of this introduction.

---

1 Based on lectures delivered at the (1) AMS meeting at USC, Nov. 1992, (2) Conference on Quantum Aspects of Black Holes, ITP, UC-Santa Barbara, June 1993, and (3) 25th Summer Institute, École Normale Supérieure, Paris, France, Aug. 1993. To appear in Interface Between Mathematics and Physics, Ed. S.-T. Yau .

2 Research partially supported by DOE grant No. DE-FG03-84ER-40168 and by NSF Grant No. PHY89-04035 .
I first want to make a case for studying String Theory in curved spacetime. I will mention quantum gravity and the unification of forces as the two main reasons.

It is generally agreed that quantum gravity is an obscure frontier. Every discussion of black holes or similar singularities fizzles out at the vicinity of the singularity and tries to put the blame on the lack of understanding of quantum gravity. One of the main reasons for this is that ordinary gravity is not a renormalizable quantum field theory. On the other hand, String Theory is the main known candidate for a finite theory of quantum gravity. However, in String Theory most discussions of gravity are perturbative, and in this form one cannot address issues of singularities and strong gravitational fields. Therefore, String Theory in curved spacetime must be studied in order to understand how string theory handles the issues of black holes and other singularities. Furthermore, it is important to tackle the problems of quantum gravity in a complete theory such as String Theory, since otherwise some important features may be missing in the discussion. For example, spacetimes with duality and mirror symmetry properties emerge in String Theory. It is not known whether such features are important for quantum gravity, but they are entirely missing in the traditional discussions of quantum gravity.

One of the main promises of String Theory is to explain the common origin of all forces and all matter. In the mainstream of String Theory this unification has been formulated by assuming a four dimensional flat spacetime plus additional curved compactified spacelike dimensions described by a conformal field theory, with the condition that the total theory is conformally critical. This cannot be the full story, since in this form gravity is perturbative, and one cannot discuss the effects of strong gravitational fields or singularities (such as the Big Bang) on the formation of matter during the early part of the Universe. Therefore, predictions on the unification of forces and the nature of matter such as the quark and lepton families may be incomplete or even wrong.

Therefore, I have proposed the following scenario in which I believe String Theory plays a role. I assume that during the evolution of the Universe there is a String Era during which spacetime is curved. There has been speculation that at the earliest times spacetime dissolves, there is no metric, and only topological notions survive (Witten). If this is true, at slightly later times a String Era must begin, and my discussion of String Theory would apply at those times. Once one accepts the existence of a String Era, described by a string propagating in a curved spacetime, then the mathematical consistency of the theory (i.e.
conformal invariance at the quantum level) does not necessarily require the existence of higher dimensions. Maybe there are extra dimensions beyond four, maybe not. It then becomes attractive to speculate that, in fact there are only four dimensions at all times, i.e. during the String Era and afterwards. (My discussion below is not necessarily committed to only four dimensions and it will apply to any number of dimensions from two to 26. In fact, it is perhaps embarrassing that the number of dimensions during the String Era is mathematically allowed to be less than, equal to or larger than four). I assume that during the String Era the Universe is described by a heterotic string theory (of the type given below) which provides a complete theory. It is during this era that the gauge symmetries and the matter content, i.e. quark and lepton families, are determined. When the theory makes a transition to the eventual flat spacetime of our current Universe the symmetry and matter content is expected to survive. In particular, I point out that, even in curved spacetime one can identify the low energy matter as being the lowest string states classified in chiral gauge multiplets, since these will be protected from getting a mass in either flat or curved spacetime.

Therefore, I believe that, in order to realize the prediction of String Theory for the unification of forces and matter, one must study string theory in curved spacetime during a String Era. If I am correct about this point, the earlier studies with a flat four dimensional spacetime, plus Calabi-Yau manifolds (etc.), may have been on the wrong track. As will be seen in examples given below, the gauge symmetries that emerge in purely four dimensional heterotic superstring models (in curved spacetime) contain the Standard Model or Grand Unified Model symmetries, such as SU(3)xSU(2)xU(1) or SU(5) or SO(10). In this sense there is already an encouraging sign that the curved spacetime approach is promising.

There are also some obvious problems that need more understanding than anyone can offer at the present. In particular let us consider the transition from the curved spacetime to the flat four dimensional Universe. I will assume that several phase transitions that must occur in any string theory will help make this transition. First, one knows that there has to be a phase transition that gives the dilaton a mass. Second, there has to be a phase transition that inflates a small part of the early universe to the known part of the current Universe. These phase transitions must occur even if string theory is formulated by starting with a flat four dimensional spacetime or with a curved spacetime in more than four dimensions. It is expected that the gauge forces drive these phase transitions. Such gauge forces are present in a heterotic string theory. Unfortunately, these phase transitions in string theory have not yet been understood, and I have nothing new to add to this at this
time. However, I emphasize that because of such expected phase transitions, it is perfectly plausible that even though we start from a curved spacetime that is inhomogeneous and non-isotropic, the inflated current Universe that is flat can emerge from any part of the universe of the String Era. In particular, as will be seen below, the String Era geometry has an asymptotically flat region that is described by the linear dilaton background. The final flat Universe may be the inflation of a small region of this asymptotic geometry. If there are more dimensions than four, the linear dilaton may point along the extra dimensions. If there are only four dimensions a linear dilaton would violate Poincaré invariance. However, because of inflation, the dilaton will take its values in only a small region of the initial space, thus appearing as almost a constant in the inflated universe, and therefore consistent with an apparent Poincaré invariance. In describing the final flat Universe one must use an effective low energy action after all the phase transitions have taken place. The part that remains obscure in this scenario is how to derive this final effective action and verify that indeed our kind of universe emerges.

The remarks above summarize my “religion” on the subject of unification. Having explained my point of view, I can go on with the technical aspects which may have applications whether or not the universe is purely four dimensional or not during the String Era. In order to study string theory in curved spacetime I have formulated exactly solvable models in which many questions can be more easily investigated. These models are based on non-compact Kac-Moody algebras which provide a Hamiltonian formulation of gauged Wess-Zumino-Witten models. The non-compact groups and an appropriate set of their cosets are chosen to describe spacetime with a single time coordinate. I now give a brief description of these models.

1.2. Formalism and models

More generally, there are models in string theory that are based on Wess-Zumino-Witten (WZW) models, or gauged Wess-Zumino-Witten models (GWZW), in which the quantum commutation rules can be put into the form of an affine Kac-Moody or Bardakci-Halpern algebra. One starts with a map from a Riemann surface (e.g. the sphere, parametrized by the complex coordinate $z$) to a group $G$. The map $g(z, \bar{z})$ is a matrix in a representation of $G$, usually taken as the fundamental representation. Then one constructs the “currents” that are elements in the left Lie algebra, $it^A J_A(z) = g^{-1} \partial_z g$, and the right Lie algebra, $it^A \bar{J}_A(z) = \partial_{\bar{z}} gg^{-1}$. These are holomorphic and anti-holomorphic respectively. The matrices $t^A, A = 1, 2, \ldots, \text{dim } G$, represent the Lie algebra (whose
the dimension is $\text{dim } G$) and they provide a basis for it. One can define string coordinates $X_A(z, \bar{z})$ which provide a parametrization of the group manifold $g = g(X(z, \bar{z}))$, for example $g(X) = \exp(it^A X_A)$ (later other more convenient parametrizations will be used). Then the currents are really constructed from the string coordinates $J_A = \partial_z X_A + \cdots$ and $\bar{J}_A = \partial_{\bar{z}} X_A + \cdots$, where the dots $\cdots$ are non-linear terms in $X_A$.

One expands these currents as a Laurent series in powers of $z$ (e.g. on the sphere) and identifies the expansion coefficients as the generators of the affine algebra as follows: left currents $J_A(z) = \sum_n J_{An} z^{-1-n}$, right currents $J_A(\bar{z}) = \sum_n \bar{J}_{An} \bar{z}^{-1-n}$. The canonical commutation rules that follow from the standard quantum theory take the form of the affine algebra

$$[J_{An}, J_{Bm}] = i f_{AB}^C J_{C,n+m} - k \frac{n}{2} \eta_{AB} \delta_{n+m,0}$$  (1.1)

where, in an appropriate basis, the Killing form is diagonal $\eta_{AB} = \text{diag}(+, +, \cdots, - - , \cdots)$, with +1 entries for compact generators and −1 entries for non-compact ones. The commutation rules look identical for the right moving (anti-holomorphic) currents $\bar{J}_{An}$. The constant $k$ is called the central extension, and is a parameter in the WZW action $\mathcal{L}$.

There is a quadratic form in the currents (called the Sugawara form) which has a structure similar to the quadratic Casimir operator (but is not an invariant). Actually, because of the infinite dimensional nature of the algebra there are an infinite number of such quadratic forms which, taken together, close among themselves under commutation and, form the Virasoro algebra. These are given by $L_n^G = \sum_m : J_{A,-m} J_{B,m+n} : \eta^{AB} / (-k + g)$, where $g$ is the Coxeter number for the group $G$, and the colons “:” indicate “normal ordering”. The stress tensor of the WZW model has Fourier coefficients that are precisely these Virasoro generators. The Virasoro generators are the generators of infinitesimal conformal transformations which play a fundamental role in string theory. The sum of the zero modes of the holomorphic and anti-holomorphic sectors, $L_0^G + \bar{L}_0^G$, is the Hamiltonian of the WZW model. When expressed in terms of the string coordinates $X^A$ introduced above this Hamiltonian is rather non-linear. As we shall see later the structure of these operators in terms of the string coordinates is what defines the geometry of spacetime.

The so called coset models $[11]$ of string theory correspond to gauged WZW models and they work as follows. Consider an infinite dimensional affine algebra for the group $G$, as in (1.1). Consider a subgroup $H \subset G$ and its corresponding affine algebra denoted
by \( J_{an} \), \( a = 1, 2, \ldots, \text{dim} H \), \( n \in \mathbb{Z} \). This subgroup is gauged, and the gauge currents are the \( J_{an} \). The stress tensor for the \( G/H \) coset model for string theory is constructed by subtracting the quadratic Sugawara form for \( H \) from the quadratic Sugawara form for \( G \).

This gives a new Virasoro algebra \( L_n = L_n^G - L_n^H \). An important feature of the \( L_n \) is that they are gauge invariant, i.e. they commute with the gauge currents of the subgroup, \([L_m, J_{an}] = 0\). The \( G/H \) coset scheme is very tight because of a requirement of conformal invariance that underlies string theory. Conformal invariance is successfully incorporated by demanding the closure of the Virasoro algebra mentioned above, and this is achieved by the scheme automatically. This feature makes the \( G/H \) model an exactly solvable string theory by using the representation theory of the affine algebra.

Modules of the affine algebra are used to construct the Hilbert space of the string theory. One considers the direct product of left-mover (holomorphic) and right-mover (anti-holomorphic) sectors. The string theory Hilbert space is obtained as “modular invariant” combinations of the left and right modules (modular invariance will not be explained here, however, it is guaranteed by a Lagrangian. The exact string models presented in this paper have such a Lagrangian construction in the form of gauged WZW models.). Furthermore, the string theory comes with constraints on the canonical variables. The physical states of the theory are identified as the subset of states in the Hilbert space on which the constraints vanish. One set of constraints is that currents that belong to the subalgebra of \( H \subset G \) must vanish: \( J_{an} = \bar{J}_{an} = 0 \) (on kets) for \( n \geq 1 \). This is equivalent to demanding gauge invariant physical states. Another set of constraints is that the generators of the Virasoro algebras in the holomorphic and the anti-holomorphic sectors vanish, \( L_n = \bar{L}_n = 0 \) (on kets) for \( n \geq 1 \). This is equivalent to requiring reparametrization invariant physical states.

The zero modes \( J_{a0}^G, L_0 \) do not vanish on physical states (we will append the extra letter \( G \) or \( H \) to the currents in order to emphasize that they belong to the Lie algebra of the group \( G \) or the subgroup \( H \)).

Concentrate on the ground state of the string theory, which is called the Tachyon state \( T \). It satisfies the conditions

\[
(L_0 + \bar{L}_0 - 2)T = (J_0^H + \bar{J}_0^H)T = 0, \quad J_n^G T = \bar{J}_n^G T = 0, \quad n \geq 1 \tag{1.2}
\]

\[ The \ L_n^H = \sum_m :J_{a,-m}J_{b,m+n} :\eta^{AB}/(-k + h) \text{ have a structure similar to } L_n^G \text{ above, where the metric } \eta^{ab} \text{ and Coxeter number } h \text{ are appropriate to the subgroup } H. \]
Only the zero modes $J^G_{A0}, \bar{J}^G_{A0}$ are non-zero on the Tachyon. Therefore, only the zero mode of the group element $g \in G$ is relevant for this state $T(g)$, and one can ignore the Riemann surface $(z, \bar{z})$. Note that for the subgroup $H$ the combination $J_{a0} + \bar{J}_{a0}$ must vanish. This last requirement is also a remnant of gauge invariance. It demands that the tachyon is a singlet under the action of the subgroup $H$ when it acts according to the adjoint action, i.e.

$$T(gh^{-1}) = T(g) .$$  \hspace{1cm} (1.3)

Therefore the tachyon is a function of the coset $G/H$, where $H$ acts according to the adjoint action. This coset defines the manifold for our geometry. The spacetime geometry on which the string propagates is identical to the geometry that arises through the tachyon state (this point will not be further explained here).

To find the metric and other properties of the geometry we analyze further the eigenvalue equation in (1.2) that involves the Hamiltonian $= L_0 + \bar{L}_0$, constructed from the $G/H$ coset Virasoro operators:

$$L_0 = -\frac{J_G \cdot J_G}{k-g} - \frac{J_H \cdot J_H}{k-h} , \quad \bar{L}_0 = -\frac{\bar{J}_G \cdot \bar{J}_G}{k-g} - \frac{\bar{J}_H \cdot \bar{J}_H}{k-h} . \hspace{1cm} (1.4)$$

Since only the zero modes $J^G_{A0}, \bar{J}^G_{A0}$ enter, these are just a combination of the quadratic Casimir operators of $G$ and of $H$. By writing these Casimir operators in the form of second order differential operators in the string variables $X^\mu$ (see below) we define the metric $G_{\mu\nu}(X)$ and the dilaton $\Phi(X)$

$$(L_0 + \bar{L}_0)T = \frac{-1}{e^\Phi \sqrt{-G}} \partial_\mu (e^\Phi \sqrt{-GG^{\mu\nu}} \partial_\nu T) . \hspace{1cm} (1.5)$$

By working out the left hand side of this equation through group theoretical manipulations and then comparing to the right hand side we can read off the metric defined on the manifold $G/H$. This metric and dilaton automatically solve the Einstein equations for dilaton gravity in the large $k$ limit. The Einstein equations impose conformal invariance perturbatively in lowest order in an expansion of the theory in powers of $k$. However, at finite $k$ our expressions give the conformally exact metric and dilaton.

The spectrum is given by the unitary representations of $G$ since all that is needed is to diagonalize the quadratic Casimir operators of the left and right groups in a basis that is restricted to $H$. Therefore, $T(g)$ must be a linear combination of the D-functions for the
group in the unitary representation $R$ of $G$, with a trace over the subgroup $H$ in any of the representations of $H$ (to impose a singlet of $H$). Symbolically

$$\begin{align*}T(g) &= \text{Tr}_H(D^R(g)) \quad (1.6)\end{align*}$$

Only the representations $R$ and its restrictions to $H$ that yield the eigenvalue $L_0 + \bar{L}_0 = 2$ are admitted as solutions.

Three years ago it was understood that, by appropriately choosing $G$ and $H$, one can describe strings propagating in gravitational backgrounds that correspond to curved spacetime with a single time coordinate [5]. All possible cosets that have the single time property have been found and classified [6][7]. These have also been extended by supersymmetry and classified [8]. Furthermore, by using the equivalence of the $G/H$ scheme to the gauged Wess-Zumino-Witten model (GWZW) it has been possible to connect the geometry of spacetime to the algebraic scheme [5] [9]. The first example $SL(2, \mathbb{R})/SO(1, 1)$ which was worked out explicitly was interpreted as a string propagating in the gravitational background of a black hole in two dimensions [9]. By now there are examples in three and four dimensions as well as with supersymmetry. These geometries solve the Einstein equations in dilaton gravity and have singularities that are more intricate than black holes. This kind of gravitational singularities have not arisen and have not been studied before. Furthermore, the global manifold turns out to have certain “duality” properties, by which we mean that there is a symmetry transformation that does not change the spectrum but interchanges patches of the manifold. Such manifolds and properties are of great interest for quantum gravity as well as string theory as applied to the Early Universe or to black hole physics. The fact that the models outlined above are completely solvable make them very attractive for studying the fundamental questions that arise in these physical phenomena.

As seen from the above summary, this approach involves several fields of mathematics (algebra, geometry, representation theory, differential equations, analysis) on the one hand and string theory and conformal field theory on the other. The fact that the groups $G$ of interest are non-compact means that little is known about these models at the present, and much more needs to be done.

In the remainder of this paper we will provide more details by giving a summary of the coset scheme in section 2, while examples of the singular gravitational geometries and some of their properties will be outlined in section 3.
2. THE COSET SCHEME IN STRING THEORY

2.1. An introduction to affine algebras and WZW models

Let us consider an affine algebra associated with a group $G$ which may be compact or non-compact \[3\]

\[
[J_{nA}, J_{mB}] = i f_{AB}^C J_{n+m,C} - n k \eta_{AB} \delta_{n+m}, \tag{2.1}
\]

where $\eta_{AB}$ is taken proportional to the Killing metric and is defined by $g_{AB} = f_{AC}^D f_{BD}^C g_{\eta_{AB}}$. The number $g$ is the Coxeter number, e.g. for $SU(N)$, $g = N$; $k$ is positive and larger than $g$ when $G$ is non-compact and a negative integer when $G$ is compact. Furthermore, in the non-compact case, $k$ must be a positive integer larger than $g$ if the maximal compact subgroup of $G$ is non-abelian. The generators of the affine algebra are Fourier components of a local current defined on the string worldsheet $J_A(z) = \sum_{n=-\infty}^{+\infty} J_{nA} z^{-n-1}$ ($z = \exp(\tau + i\sigma)$ and $(\tau, \sigma)$ parametrize the Euclidean continuation of the worldsheet). It may be constructed from a group element $g(z, \bar{z}) \in G$ as $t^A J_A = -i g^{-1} \partial_z g$, where $t^A$ is a basis of matrices for the Lie algebra, say in the fundamental representation. After writing an appropriate action, which is the Wess-Zumino-Witten (WZW) action, and quantizing it, one finds that the commutation rules above correspond to the canonical quantization of the model \[1\][\[2\]. Actually one finds that there is another current, $t^A \bar{J}_A(\bar{z}) = -i \partial_{\bar{z}} g g^{-1}$, whose Fourier components $\bar{J}_{nA}$ are canonical degrees of freedom independent from $J_{nA}$, and form another affine algebra with the same structure as above. The WZW action that gives rise to these structures may be written as (in the Minkowski version of the worldsheet, with $\sigma^\pm = (\tau \pm \sigma)/\sqrt{2}$):

\[
S_0(g) = \frac{k}{8\pi} \int_M d^2 \sigma \ Tr(g^{-1} \partial_+ g^{-1} \partial_- g) - \frac{k}{24\pi} \int_B Tr(g^{-1} dg^{-1} dg^{-1} dg). \tag{2.2}
\]

In order to gain a bit more insight into the meaning of these equations it is useful to consider the large $k$ limit. Large $k$ is equivalent to small $\hbar$ and hence it corresponds to the semi-classical limit of the theory (since the path integral involves the action in the form $exp(iS/\hbar)$). In this limit it is useful to rescale the generators by defining $\alpha_{nA} = J_{nA}/\sqrt{k}$ and then taking the $k \to \infty$ limit in eq.(2.1). The result is

\[
[\alpha_{nA}, \alpha_{mB}] = -n \eta_{AB} \delta_{n+m,0}. \tag{2.3}
\]

9
which is equivalent to harmonic oscillator commutation rules. A similar result holds for \( \bar{\alpha}_{nA} = \bar{J}_{nA}/\sqrt{k} \). Thus, the generators of the affine algebra may be thought of as a non-Abelian version of harmonic oscillators. This can be made more explicit as follows. With an appropriate parametrization of the group element by coordinates \( X_A \), say \( g = \exp(it^A X_A/\sqrt{k}) \), the large \( k \) limit of the action reduces to an action for the free field \( X_A(z, \bar{z}) \). The \( X_A(z, \bar{z}) \) will be interpreted as the string coordinates defined on the Euclidean worldsheet \((z, \bar{z})\), with \( z = \exp(\tau + i\sigma) \). In fact, in the large \( k \) limit, the free field can be thought of as a free string propagating in flat spacetime. For finite \( k \) the spacetime will be curved. At this point the signature of spacetime is not yet correct (i.e. one needs only one time coordinate and additional space coordinates), but we will see later how to construct a good model. The Fourier components of the free field are, in fact, the \( \alpha_{nA}, \bar{\alpha}_{nA} \) oscillators. The perturbative expansion of the non-linear theory in powers of \( 1/k \) can be worked out by starting with the free field Hilbert space defined by the Fock space of the \( \alpha_{nA}, \bar{\alpha}_{nA} \). This will correspond to an expansion around flat spacetime. However, the theory can also be solved non-perturbatively for any \( k \), in curved spacetime, by using the representation theory of the affine algebra, and this is the principal advantage of considering such a theory.

The conserved energy-momentum tensor has components \( T^G_{zz}, T^G_{\bar{z}\bar{z}}, T^G_{z\bar{z}} \) (using the letter \( G \) to emphasize that the group \( G \) is involved). After taking quantum ordering problems into account they take the form \( T^G_{zz} =: J_A(z)J_B(z) : \eta^{AB}/(-k + g) \) with a similar form for \( T^G_{\bar{z}\bar{z}} \) in terms of \( \bar{J}_A \), while \( T^G_{z\bar{z}} \) vanishes. The colon “:” is used to indicate normal ordering. Note that the form of \( T^G \) has the same structure of the quadratic Casimir operator, and it is called the Sugawara form. It has the Fourier expansion \( T^G_{zz} = \sum_n L^G_n z^{-n-2} \), where the \( L^G_n \) are given in the introduction. The \( L^G_n \) are called Virasoro generators and they could be considered to be a kind of generalization of the quadratic Casimir operator. However, unlike the usual Casimir operator, the Virasoro operators do not commute with the currents or with each other, rather

\[
[L^G_n, L^G_m] = (n - m)L^G_{n+m} + \frac{c_G}{12}n(n^2 - 1)\delta_{n+m,0}, \quad [L^G_n, J_{mA}] = -mJ_{n+m,A}. \quad (2.4)
\]

where the so called Virasoro central charge is \( c_G = k \text{dim}(G)/(k - g) \). Similarly one constructs \( \bar{L}^G_n \) from the \( \bar{J}^G_{An} \) with identical properties. The Hamiltonian of the theory is constructed from the zero mode Virasoro operators and is given by \( H = L^G_0 + \bar{L}^G_0 \).
The eigenstates of the Hamiltonian are formed by constructing the representations of the affine algebra as follows. First one defines the so-called ground states, or level zero states, \(|R >^0_G\), which correspond to the states in any unitary representation \(R\) of the group \(G\). One assumes that these states form the vacuum states for the “non-Abelian oscillators”. For \(n \geq 1\) the \(J_{-n,A}, J_{n,A}\) are considered creators and annihilators respectively, while for \(n = 0\) the \(J_{0,A}\) act via a matrix representation \(t^R_A\) appropriate for the representation \(R\)

\[
J_{0,A}|R >^0_G = i^R_A|R >^0_G, \quad J_{nA}|R >^0_G = 0, \quad n \geq 1.
\]

\(|R >^0_G\) symbolizes the basis states in the representation \(R\) of the ordinary Lie algebra of \(G\). The excited higher level states are obtained by applying integer powers \(p_i\) of the creators \(J_{-n_i,A_i}\), \(n_i \geq 1\), on the ground state, thus constructing the Hilbert space

\[
|R >^l_G = \prod_i (J_{-n_i,A_i})^{p_i}) |R >^0_G.
\]

The level is given by \(l = \sum_i n_i p_i\). This space is somewhat analogous to Fock space. In fact, for large \(k\) the states \((2.6)\) reduce to ordinary Fock space constructed from the \(\alpha_{nA}\) oscillators introduced above. Of course, the non-Abelian nature of the creators provide for a richer structure. The eigenvalue of the Hamiltonian is then obtained from

\[
L^G_0 |R >^l_G = (\frac{C(R)}{-k + g} + l)|R >^l_G.
\]

where \(C(R)\) is the eigenvalue of the usual quadratic Casimir operator for representation \(R\). A similar result is obtained for \(\bar{L}^G_0\). Thus, for any value of \(k\) the spectrum is calculated from the quadratic Casimir. Furthermore, the non-linearity of the theory has been shoved entirely into the Casimir operator through the representation content \(R\) of the state. The excitations contribute to the energy through the level, the integer \(l\), in a way that is quite analogous to excitations in a Fock space. As we shall see, the geometry of spacetime is also embedded in the Casimir operator.

2.2. Introduction to Cosets and Gauged WZW models

The action \((2.2)\) may be modified by gauging a subgroup \(H\) of \(G\). The new action takes the form \(S = S_0 + S_1\) with \([10]\)

\[
S_1(g, A) = -\frac{k}{4\pi} \int_M d^2\sigma \ Tr(A_- \partial_+ gg^{-1} - \tilde{A}_+ g^{-1} \partial_- g + A_- g \tilde{A}_+ g^{-1} - A_- A_+).
\]
where the gauge fields $A_+, A_-$ are matrices in the Lie algebra of $H$. (A generalization of this action with $A_+$ twisted relative to $A_-$ exists \cite{11}, and this is important for a phenomenon called "duality", but we will not be concerned with it in this paper.) The gauge invariance leads to constraints that require the subgroup currents $J_{n,i}, \bar{J}_{n,i}$, with $i \in H$, to vanish. In the quantum theory the constraints are applied on the states by demanding the subgroup currents with $n \geq 1$ to vanish on kets $|R>^G$, and those with $n \leq 1$ to vanish on bras, while for $n = 0$ one must take only the combination $J_{0,i} + \bar{J}_{0,i}$ to vanish. The physical states are only those states that satisfy the constraint equations.

With these constraints one is effectively removing gauge degrees of freedom that correspond to the subgroup $H$, thus dealing with a theory whose degrees of freedom are in some sense associated with the coset $G/H$. Again, it is useful to consider the large $k$ limit, for which the constraints are solved very simply: Consider the oscillators $\alpha_{An} = (\alpha_{an}, \alpha_{\mu n})$, where the first set is associated with $H$ and the second set with $G/H$, and take only the Fock space constructed from the oscillators $\alpha_{n,\mu}, \bar{\alpha}_{n,\mu}$ with $\mu \in G/H$ and ignore the subgroup oscillators (this is equivalent to a free string $X_{\mu}(\tau, \sigma)$ in flat spacetime). The removal of degrees of freedom is a consequence of the subgroup gauge invariance: one may imagine choosing a gauge in which the $H$-gauge fields "eat up" degrees of freedom from the group element $g \in G$. The gauge fields are not dynamical since they appear without derivatives; thus they are just Lagrange multipliers which, through the equations of motion, are functions of the remaining degrees of freedom. Therefore the gauged WZW model indeed depends only $G/H$ degrees of freedom. For finite $k$ the explicit dependence of the theory on the $G/H$ degrees of freedom is highly non-trivial and is different than the old $G/H$ sigma models. Furthermore, it contains the information on the geometry, as will be seen later.

The energy-momentum tensor for the gauged WZW model is constructed by subtracting the Sugawara form for the subgroup $H$ from the corresponding form for $G$ \cite{4}

$$
T^{G/H}_{zz} = \frac{J_A J_B : \eta^{AB} - J_i J_j : \eta^{ij}}{-k + g - k + h},
$$

where $g, h$ are the Coxeter numbers for $G, H$ respectively. Therefore the Virasoro generators for the gauged WZW model are given by $L^{G/H}_n = L^n_G - L^n_H$, and therefore the Hamiltonian is $H = L^{G/H}_0 + \bar{L}^{G/H}_0$.

The eigenvalues of this Hamiltonian are computed as follows: First decompose the representation $R$ into representations $r_h$ of the subgroup $H$, i.e. $|R>^G_G = \bigoplus_h |r_h>^G_H$, then compute the difference between the Casimir eigenvalues
\[ L_0^{G/H} |r_h >_{H}^{l_h} = \left\{ \frac{C^G(R)}{-k + g} - \frac{C^H(r_h)}{-k + h} + (l - l_h) \right\} |r_h >_{H}^{l_h} . \] (2.10)

In the full string theory that we will consider below we also need to impose the gauge constraints that follows from the reparametrization invariance, or conformal invariance of the string theory. This corresponds to the vanishing of the Virasoro generators \( L_n^{G/H}, \bar{L}_n^{G/H} \), for \( n \geq 1 \), on physical kets, and demanding \( L_0^{G/H} = \bar{L}_0^{G/H} \) as well. Therefore, the eigenvalue of the Hamiltonian on a physical state is \( H = 2L_0^{G/H} \). A physical state involves both sectors of the Hilbert space constructed from the \( J_{nA} \) and \( \bar{J}_{nA} \) in “modular invariant” combinations (not to be explained here).

These constraints, as well as the \( G/H \) constraints outlined above, are all automatically satisfied for the ground states \((l = 0 = l_h)\) that are gauge invariant under the subgroup \( H \). These are called the tachyonic states (although they are not necessarily tachyons in a supersymmetric theory, or in curved spacetime). Thus, a physical state of the gauged WZW model at the tachyonic level (i.e. ground state level) is automatically obtained for any state that satisfies

\[(J_{0,i} + \bar{J}_{0,i})|Tachyon, R, r_h > = 0. \] (2.11)

This is nothing but the requirement of gauge invariance with respect to the subgroup \( H \). The energy eigenvalue follows from (2.10). We will examine the states that satisfy these conditions in order to extract the spacetime geometry.

2.3. Time Coordinate

Let me review how the single time coordinate condition restricts the possible cosets \( G/H \). Let us consider a WZW model based on a non-compact group. Let us parametrize the group element by \( X^A(\tau, \sigma) \), where \( A \) is an index in the adjoint representation. The left or right moving currents take the form \( J^A = \partial X^A + \cdots \), where the dots stand for nonlinear terms in an expansion in powers of \( X \). The Fourier components of these currents \( J_n^A \) satisfy an affine algebra as in eq.(2.1), where \( k \) is the central extension and \( \eta^{AB} \) is proportional to the Killing metric. In an appropriate basis one can choose a diagonal \( \eta^{AB} = diag(1, \cdots, 1, -1, \cdots, -1) \) with +1 entries corresponding to compact generators and -1 entries to non-compact ones. For example, for \( SL(2, \mathbb{R}) \) with currents \( (J^0, J^1, J^2) \), one has the Minkowski metric in \( 2 + 1 \) dimensions: \( \eta^{AB} = diag(1, -1, -1) \).
As explained above, when $k \to \infty$ the currents behave like the free field oscillators of the flat string theory as in (2.3). Examining the signature of the oscillators as given by the Killing metric (with the extra sign in front) we interpret the free fields $X^A \sim \sum_n \frac{1}{n!} \alpha_n^A \times n + \cdots$ as time coordinates when $A$ corresponds to compact generators and as space coordinates when $A$ corresponds to non-compact generators. The signature of the coordinates are the same for finite positive $k$. This is seen by specializing the commutation rules (2.1) to $A = B$ for which the structure constant of the Lie algebra drops out. The timelike oscillators create negative norm states that ruin the unitarity of the theory. The negative norm states must be eliminated from the theory by a consistent set of constraints.

For ordinary string theory in flat spacetime the necessary constraint emerges automatically from the conformal invariance of the theory. Conformal invariance requires the vanishing of the Virasoro generators, and it can be shown that indeed these conditions remove all negative norms. This result goes under the name of the no-ghost theorem.

In a string theory one can tolerate only one time coordinate. This is because, by naive counting, the Virasoro constraints $L_n^{G/H} \sim 0$ can eliminate only the ghosts generated by the negative norm of one time-like oscillator $\alpha_0^0$, just like string theory in flat spacetime. Therefore, one must put constraints that set to zero the unwanted time-like compact generators $J_{n,i}$, except for one of them. However, first class constraints of this type must close to form an algebra. Therefore, the currents that are set equal to zero ($J^i \sim 0$ weakly on states) must form a subalgebra corresponding to a subgroup of the non-compact group $H \subset G$. The subalgebra may include compact and non-compact generators. The remaining currents $J^\mu$, $\mu = 0, 1, 2, \cdots (d - 1)$ stand in one-to-one correspondence with the coset coordinates $X^\mu$ that include just one time coordinate. Thus, one must choose a subgroup $H$ such that the coset $G/H$ has the signature of Minkowski space $\eta_{\mu\nu}$ in $d$ dimensions. As seen in the previous section this set of constraints defines an exact conformal field theory that fits the algebraic framework of the gauged WZW model. The important ingredient is that one must take an appropriate non-compact coset $G/H$. The only simple groups that give a single time coordinate were classified in [3]

\[
\begin{align*}
SO(d - 1, 2)/SO(d - 1, 1) & \quad SO(d, 1)/SO(d - 1, 1) \\
SU(n, m)/SU(n) \times SU(m) & \quad SO(n, 2)/SO(n) \\
SO(2n^*)/SU(n) & \quad Sp(2n^*)/SU(n) \\
E_6^*/SO(10) & \quad E_7^*/E_6
\end{align*}
\]
This list, which contains only simple groups, may be extended with direct products of simple groups \( G_1 \times G_2 \times \cdots \) including \( U(1) \) or \( \mathbb{R} \) factors, or their cosets, so long as the additional factors do not introduce additional time coordinates [3][4][5].

There is another way to see the same result by using a Lagrangian method at the classical level rather than the algebraic Hamiltonian argument given above at the quantum level. As discussed in the previous section, a coset theory corresponds to a gauged WZW model with the subgroup \( H \) local. Using the gauge invariance one can eat up \( \text{dim}(H) \) degrees of freedom, leaving behind \( \text{dim}(G/H) \) group parameters that contain just one timelike coordinate. Since the gauge fields are non-dynamical they can be integrated out. This leaves behind a sigma model type theory with the desired signature. The large \( k \) limit of this theory has free field quantum oscillators with a single time coordinate.

Both the Hamiltonian and Lagrangian arguments were first given by Bars and Nemeschansky [3]. The Hamiltonian approach was given more weight in [5] where several examples, including \( SL(2,\mathbb{R})/\mathbb{R} \) at \( k = 9/4 \), were investigated. It was found that this model describes a black hole in two dimensions [1][2]. With the realization that non-compact group coset methods generate singular geometries, there has been a flurry of activity to determine the geometries of higher dimensional cosets [11][12]. While these models represent only a small subset of all possible curved spacetime models described by the general sigma model, they have the advantage of being solvable in principle thanks to the algebraic formulation. Thus a lot more can be said about the spectrum, correlation functions, etc. of the quantum string theory based on these models. Furthermore, it has been realized that the special geometries described by these non-compact groups are relevant to gravitational singularities such as black holes and cosmological Big Bang. For these reasons this class of models has received considerable attention during the past couple of years. It is hoped that through such solvable models new light will be shed on unresolved gravitational issues, in string theory as well as general relativity, such as singularities, quantization and finiteness or renormalizability in curved spacetime, the question of Euclidean-Minkowski continuation, spectrum of low energy particles and excited string states in the presence of curvature, etc..

---

4 This model, although first suggested in [3] as a candidate for strings in curved spacetimes, got popularly known as the Witten Black Hole [9], since he carried out the Lagrangian argument explicitly and interpreted the sigma model metric as a black hole. It should not be very difficult to find a more appropriate name that recognizes the original discovery.
2.4. Heterotic Superstring in Curved Spacetime

The other important question for string theory is the nature and content of the low
energy matter it is supposed to predict in the form of quarks, leptons, gauge bosons, etc..
The new models have opened up the possibility of heterotic superstring theories in four
spacetime dimensions (with or without additional compactified dimensions) [13]. This is
possible because the Virasoro central charge \( c = 26 \) (or \( c = 15 \) with supersymmetry) con-
dition can be satisfied in fewer dimensions provided the space is curved. For example it
has been possible to construct consistent purely four dimensional heterotic string theories
based on non-compact current algebra cosets [8][14]. We can now ask, what are the het-
erotic models that can be constructed with the non-compact group method? Due to the
lack of space I will only give a couple of tables which are self-explanatory. For details see
the original literature. In these tables only the models in purely four dimensions are given.
The left movers are fully described in Table-1 in the form of a supersymmetric Kazama-
Suzuki coset, where \( SO(3,1)_1 \) represents four free Minkowski fermions. The right movers
are described in both tables. Table 1 contains the 4-dimensional coset for right movers
(connected with the left movers) while Table 2 gives the gauge group that is needed in
order to satisfy the \( c_R = 26 \) condition. The contribution of the gauge group to the central
charge is given by \( c_R(\text{int}) \) in Table 2.

| # | left movers with N=1 SUSY | right movers |
|---|--------------------------|--------------|
| 1 | \( SO(3,2)_{-k} \times SO(3,1)_1/ SO(3,1)_{-k+1} \) | \( SO(3,2)_{-k}/ SO(3,1)_{-k} \) |
| 2 | \( \frac{SL(2,\mathbb{R})_{-k_1} \times SL(2,\mathbb{R})_{-k_2} \times SO(3,1)_1}{SL(2,\mathbb{R})_{-k_1-k_2+2}} \times \mathbb{R} \) | \( \frac{SL(2,\mathbb{R})_{-k_1} \times SL(2,\mathbb{R})_{-k_2}}{SL(2,\mathbb{R})_{-k_1-k_2}} \times \mathbb{R} \) |
| 3 | \( SO(2,2)_{-k} \times SO(3,1)_1/ SO(2,1)_{-k+2} \) \times \mathbb{R} | \( SO(2,2)_{-k}/ SO(2,1)_{-k} \) \times \mathbb{R} |
| 4 | \( SL(2,\mathbb{R})_{-k} \times SO(3,1)_1 \times \mathbb{R} \) | \( SL(2,\mathbb{R})_{-k} \times \mathbb{R} \) |
| 5 | \( \frac{SL(2,\mathbb{R})_{-k_1} \times SL(2,\mathbb{R})_{-k_2} \times SO(3,1)_1}{\mathbb{R}^2} \) | \( \frac{SL(2,\mathbb{R})_{-k_1} \times SL(2,\mathbb{R})_{-k_2}}{\mathbb{R}^2} \) |
| 6 | \( SL(2,\mathbb{R})_{-k_1} \times SU(2)_{k_2} \times SO(3,1)_1/ \mathbb{R}^2 \) | \( SL(2,\mathbb{R})_{-k_1} \times SU(2)_{k_2}/ \mathbb{R}^2 \) |
| 7 | \( (SL(2,\mathbb{R})_{-k} \times \mathbb{R}^2 \times SO(3,1)_1)/ \mathbb{R} \) | \( (SL(2,\mathbb{R})_{-k} \times \mathbb{R}^2)/ \mathbb{R} \) |
| 8 | \( \mathbb{R}^3 \times \mathbb{R}_Q \times SO(3,1)_1 \) | \( \mathbb{R}^3 \times \mathbb{R}_Q \) |

Table 1. Current algebraic description of left movers and right movers.

In item 8, \( \mathbb{R}_Q \) corresponds to a free boson with a background charge \( Q \).
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\# & conditions for $c_L = 15$ & $c_R(int)$ & gauge group, right movers \\
\hline
1 & $k = 5$ & 11 & $(E_7)_1 \times SU(5)_1$ \\
2 & $k_1 - 2 = \frac{k_2 - 2}{2}(-1 + \sqrt{\frac{3k_2}{3k_2 - 8}})$ & $13 - \delta$ & $\delta = \frac{12}{(k_1 + k_2 - 4)(k_1 + k_2 - 2)}$ \\
3 & $k = 3$ & $11\frac{1}{2}$ & $(E_7)_1 \times SU(3)_1 \times SU(2)_2 \times U(1)_1$ \\
4 & $k = 8/3$ & 13 & $(E_8)_1 \times SO(10)_1$ \\
5 & $k_1 = \frac{8k_2 - 20}{3k_2 - 8}$, $k_1, k_2 > \frac{2}{3}$ & 13 & $(E_8)_1 \times SO(10)_1$ \\
6 & $k_1 = \frac{8k_2 + 20}{3k_2 + 8}$, $k_2 = 1, 2, 3, \ldots$ & 13 & $(E_8)_1 \times SO(10)_1$ \\
7 & $k = 8/3$ & 13 & $(E_8)_1 \times SO(10)_1$ \\
8 & $Q^2 = \frac{3}{4}$ & 13 & $(E_8)_1 \times SO(10)_1$ \\
\hline
\end{tabular}
\caption{Conditions for $c_L = 15$ and examples of symmetries that give $c_R = 26$.}
\end{table}

It is encouraging to note that the desirable low energy symmetries, including $SU(3) \times SU(2) \times U(1)$, are contained in these curved spacetime string models that have only four dimensions. Also, the grand unified gauge groups that emerge are the familiar and desirable ones. The quark and lepton states, which come in color triplets and $SU(2)$ doublets, are expected to emerge in several families. Compared to the popular approach of four flat dimensions plus compactified dimensions, the gauge groups are either the same or closely related. This gives the hope that the quark/lepton spectrum of a curved purely four dimensional heterotic superstring that describes the very early universe may be closely related to the quarks and leptons that survive to the present times.

In trying to solve the puzzles of gravitational singularities and cosmology, and those of the Standard Model with respect to the spectrum of matter (i.e. quark/lepton families) and gauge bosons, we may hope that a complete string theory in curved spacetime may guide us. For this reason I believe that it is valuable to study in great detail the models presented in Table 1. These are solvable models that should direct us toward a realistic unified theory.

Modular invariance for these models would be assured by the existence of a Lagrangian formulation. Such a Lagrangian formulation was introduced in [13]. However, it has recently been pointed out in [15] that there is a global anomaly. To cancel the anomaly they proposed to supersymmetrize both left and right movers. A simpler, and clearer version of this proposal is the following: build supersymmetric gauged WZW Lagrangian models for both right and left movers, as in [13], and then introduce additional internal fermions to make up the deficit for $c_R = 26$. This modification would change the coset
structure of the right movers: in Table 1 the left and right movers become the same, and in Table 2 the change of \( c_{\text{int}} \) modifies the internal groups slightly. Unfortunately, the authors of [15] overlooked that their proposal creates another problem: it requires the introduction of a time-like signature right-moving fermion (because of the non-compact nature of the gauge subgroup) whose negative norm states cannot be removed (since the right movers do not have overall superconformal symmetry). Therefore, this proposal is defective and cannot be used. Instead, another anomaly cancellation or modular in

Note that if the time coordinate remains flat, and supersymmetric only for left movers, the modified scheme would work without any problems. An important example of such a model was already worked out in detail in [16]. To specialize it to only 4 dimensions do as follows: on the left, take the supersymmetrized time coordinate and one space coordinate to form flat light-cone super coordinates with \( c_L = 3 \), and take the supersymmetric, left-right symmetric, \( SL(2, \mathbb{R})_k/U(1) \) (at \( k = 8/3 \) or \( c = 3k/(k-2) = 12 \)) as two more spacelike coordinates, to make altogether 4 coordinates. This gives \( c_L = 15 \). Furthermore, on the right, take 2 bosonic lightcone coordinates, plus the right-moving supersymmetric \( c = 12 \) mirror theory, and additional right moving internal coordinates with \( c_{\text{int}} = 12 \) to give \( c_R = 2+12+12 = 26 \). The 4-dimensional geometry consists of two flat lightcone coordinates plus the dual cigar/trumpet geometries of the \( SL(2, \mathbb{R})/U(1) \) model (see below). The computation of the spectrum of this 4-dimensional heterotic string model follows the path of ref. [16] with the substitution of \((k = 8/3, c = 12)\) instead of \((k = 3, c = 9)\) used there. More details will be given elsewhere.

3. GEOMETRY OF THE MANIFOLD

A gauged WZW model given by (2.2)(2.8) can be rewritten in the form of a non-linear sigma model by choosing a unitary gauge that eliminates some of the degrees of freedom from the group element, and then integrating out the non-propagating gauge fields [1][2]. The remaining degrees of freedom are identified with the string coordinates \( X^\mu(\tau, \sigma) \). The resulting action exhibits a gravitational metric \( G_{\mu\nu}(X) \) and an antisymmetric tensor \( B_{\mu\nu}(X) \) at the classical level.

\[
S_{\text{eff}} = \int d^2 \sigma (G^{\mu\nu}(X) \partial_\alpha X_\mu \partial^\alpha X_\nu + \cdots). \tag{3.1}
\]

At the one loop level there is also a dilaton \( \Phi(X) \). These fields govern the spacetime geometry of the manifold on which the string propagates. Conformal invariance at the one loop
level demands that they satisfy coupled Einstein equations. Thanks to the exact conformal properties of the gauged WZW model the Einstein equations are automatically satisfied. Therefore, any of our non-compact gauged WZW models can be viewed as generating automatically a solution of these equations. One only needs to do some straightforward algebra to extract the explicit forms of $G_{\mu\nu}, B_{\mu\nu}, \Phi$.

This algebra can be carried out by starting from the Lagrangian, such as in (2.2), and has been done for all the models in four dimensions listed in Table 1. The first case was $SL(2, \mathbb{R})/\mathbb{R}$, which was interpreted by Witten [9] as the geometry of a 2D black hole. The higher dimensional cases yield more intricate but singular geometries [11][12][13][7]. In these manifolds there are geodesically complete patches that contain singularities that hide behind a horizon, as well as patches that contain bare singularities. Although the Lagrangian method is straightforward, it has a number of drawbacks. First, it yields the geometry only in a patch that is closely connected to a particular choice of a unitary gauge. The remaining patches of the global geometry can be recovered only in other unitary gauges and may have no resemblance to the analytic form of the metric, dilaton, etc. in another unitary gauge. To overcome this problem we have introduced global coordinates [17] on the complete geometry. The global coordinates are gauge invariant. The second problem with the Lagrangian method is that it yields the semi-classical geometry up to one loop in an expansion in powers of $1/k$. However, since the gauged WZW model is conformally exact one would rather obtain the conformally exact geometry by using alternative methods. In the exact geometry some singularities are shielded by quantum effects. It turns out that the Hamiltonian method that utilizes the GKO construction solves both of these problems simultaneously and yields an exact metric and dilaton to all orders in $1/k$ [18][19][20]. More recently, the quantum effective action has been constructed exactly, at least in the zero mode sector [21] for any $G/H$ (see also [22][23] for the special case $SL(2)/\mathbb{R}$). The zero mode sector is sufficient to extract the metric, dilaton and the antisymmetric tensor that play a role in the low energy limit of string theory. From this effective action it has finally been possible to extract the general exact geometry for any coset $G/H$ [21]. However, although exact to all orders in $1/k$, the known methods with the effective action still yield the geometry in a patch instead of the global space. Therefore, in this paper we concentrate on the algebraic Hamiltonian approach, which is really the main basis for justifying the effective action.

With the Hamiltonian approach one can compute the gravitational metric and dilaton backgrounds to all orders in the quantum theory (all orders in the central extension $k$). We
have managed to obtain these quantities for bosonic, type-II supersymmetric, and heterotic string theories. It turns out that the geometry of the heterotic and type-II superstrings are obtained by deforming the geometry of the purely bosonic string by definite shifts in the exact $k$-dependence. Therefore, it is sufficient to first concentrate on the purely bosonic string. The following relations have been proven for any $G/H$ model \[18\] [24]: (i) For type-II superstrings the conformally exact metric and dilaton are identical to those of the non-supersymmetric semi-classical bosonic model. (ii) The exact expressions for the heterotic superstring are derived from their exact bosonic string counterparts by shifting the central extension $k \to 2k - h$ (except for a different shift in the overall $k$ factor). (iii) The combination $e^\Phi \sqrt{-G}$ is independent of $k$ and therefore can be computed in lowest order perturbation theory. Cases 2,5,6 in Table 1 are a bit more complicated because of the two central extensions, but the results that relate the bosonic string to superstrings are analogous. Case 6 is explicitly discussed in \[20\], and the others are just analytic continuations of this one.

The main idea is the following. For the bosonic string the conformally exact Hamiltonian is the sum of left and right Virasoro generators $L_0 + \bar{L}_0$. They may be written purely in terms of Casimir operators of $G$ and $H$ when acting on a state $T(X)$ at the tachyon level. The exact dependence on the central extension $k$ is included in this form by using the GKO formalism in terms of currents. For example for the left-movers $L_0$

\[
L_0 T = \left( \frac{\Delta_G}{k - g} - \frac{\Delta_H}{k - h} \right) T
\]

\[
\Delta_G \equiv Tr(J_G)^2, \quad \Delta_H \equiv Tr(J_H)^2,
\]

The exact quantum eigenstate $T(X) = \langle X | Tachyon \rangle$ can be analyzed in $X$-space. Then the Casimir operators become Laplacians constructed as differential operators in group parameter space (of dimension $\dim(G)$). Consider a state $T(X)$ which is a singlet under the gauge group $H$ (acting simultaneously on left and right movers)

\[
(J_H + \bar{J}_H) T = 0 .
\]

Because of the $\dim H$ conditions $T(X)$ can depend only on $d = \dim(G/H)$ parameters, $X^\mu$ (string coordinates), which are $H$-invariants constructed from group parameters (see

---

5 $g, h$ are the Coxeter numbers for the group and the subgroup. For the cases of interest in this paper $g = d - 1, h = d - 2$ for $d \geq 3$, and $g = 2, h = 0$ for $d = 2$. 

---

20
below). The fact that there are exactly \( \text{dim}(G/H) \) such independent invariants is not immediately obvious but it should become apparent to the reader by considering a few specific examples. As discussed in [14] these are in fact the coordinates that globally describe the sigma model geometry. Consequently, using the chain rule, we reduce the derivatives in (3.2) to only derivatives with respect to the \( d \) string coordinates \( X^\mu \). In this way we can write the conformally exact Hamiltonian \( L_0 + \bar{L}_0 \) as a Laplacian differential operator in the global curved space-time manifold involving only the string coordinates \( X^\mu \). By comparing to the expected general form

\[
(L_0 + \bar{L}_0)T = \frac{-1}{e^\Phi \sqrt{-G}} \partial_\mu (e^\Phi \sqrt{-GG^{\mu\nu}} \partial_\nu T)
\]

for the singlet \( T \), we read off the exact global metric and dilaton.

We have applied this program to all the models in Table 1 and obtained the exact geometry to all orders in \( 1/k \). The large \( k \) limit of our results agree with the semi-classical computations of the Lagrangian method. In the special case of two dimensions we also agree with another previous derivation of the exact metric and dilaton for the \( SL(2, \mathbb{R})/\mathbb{R} \) bosonic string [25]. We summarize here the global and conformally exact results for the metric and dilaton in the case of \( SO(d-1, 2)/SO(d-1, 1) \) for \( d=2,3,4 \) [18]. Due to the more complex expressions we refer the reader to the original literature for the remaining cases [19][20]. The group element \( g \) for \( SO(d-1, 2)/SO(d-1, 1) \) can be parametrized as a \( (d+1) \times (d+1) \) matrix in the form

\[
g = \begin{pmatrix}
1 & 0 \\
0 & \frac{1+a}{1-a} x^\alpha \beta
\end{pmatrix}
\begin{pmatrix}
 b & (b+1)x^\beta \\
-(b+1)x_\alpha & (n_\alpha^\beta - (b+1)x_\alpha x^\beta)
\end{pmatrix}
\]

where \( b = \frac{1-x^2}{1+x^2} \). The \( d \) parameters \( x_\alpha \) and \( d(d-1)/2 \) parameters \( a_\alpha \beta \) transform as a vector and an antisymmetric tensor, respectively, under the Lorentz subgroup \( H = SO(d-1, 1) \) which acts on both sides of the matrix as \( g \rightarrow hgh^{-1} \). By considering the infinitesimal left transformations \( \delta_L g = \epsilon_L g \) we can read off the generators that form an \( SO(d-1, 2) \) algebra for left transformations.

\[
J_{\alpha\beta} = \frac{1}{2}(1+a)_{\alpha\alpha'} (1+a)_{\beta\beta'} \frac{\partial}{\partial a_{\alpha'\beta'}}
\]

\[
J_{\alpha} = -\frac{1}{2}(1+x^2) (1+a)_{\alpha}^\beta \frac{\partial}{\partial x_\beta} + \frac{1}{2}(1+a)_{\alpha\alpha'} (1+a)_{\beta'\gamma} x_\gamma \frac{\partial}{\partial a_{\alpha'\beta'}}
\]

If we consider instead the infinitesimal right transformations \( \delta_R g = g \epsilon_R \) we find the following expressions for the generators of right transformations:
\[ \bar{J}_{\alpha\beta} = -\frac{1}{2}(1-a)_{\alpha\alpha'}(1-a)_{\beta\beta'} \frac{\partial}{\partial x_{\alpha'}} - x_{[\alpha} \frac{\partial}{\partial x_{\beta}]} \]

\[ \bar{J}_\alpha = \frac{1}{2}(x^2 - 1) \frac{\partial}{\partial x^\alpha} - x_{\alpha} x^\beta \frac{\partial}{\partial x^\beta} - \frac{1}{2}(1-a)_{\alpha\alpha'}(1-a)_{\gamma\beta'} x^\gamma \frac{\partial}{\partial x_{\alpha'}\beta'} . \]  

The \( \bar{J} \) currents obey the same commutation rules as \( J \) and moreover commute with each other: \([J, \bar{J}] = 0\). The quadratic Casimirs for the group and subgroup on either the left or the right are obtained by squaring these currents. For the explicit expressions see [18].

As argued above the global parametrization of the manifold is given in terms of H-invariants, i.e. Lorentz invariants in the present case. In order to obtain a diagonal metric on the manifold one must find \( d \) convenient combinations of these Lorentz invariants in \( d \) dimensions. We give here the basis that diagonalizes the semi-classical metric at large \( k \).

One of the natural invariants already occurs in the construction of the group element for every \( d \), namely \( b = \frac{1-x^2}{1+x^2} \).

3.1. Two dimensions

For \( d = 2 \) the antisymmetric tensor is Lorentz invariant \( a_{\alpha\beta} = a\epsilon_{\alpha\beta} \), and it is convenient to parametrize \( a \) as \( tanh(t) \) or \( coth(t) \). Then the global string coordinates can be taken as \( X^\mu = (t, b) \). Given all possible values for \( (a, x^\alpha) \) the ranges of the two invariants cover the entire plane \(-\infty < t, b < +\infty\). The metric is given by the line element

\[ ds^2 = 2(k-2) \left( \frac{dt^2}{4(b^2 - 1)} - \beta(b) \frac{b-1}{b+1} dt^2 \right), \quad \beta^{-1}(b) = 1 - \frac{2}{k} \frac{b-1}{b+1} . \]

For the dilaton the corresponding expression is

\[ \Phi = \ln \left( \frac{b+1}{\sqrt{\beta(b)}} \right) + \text{const} . \]

The scalar curvature for this metric is

\[ R = \frac{2k}{k-2} \frac{(k-2)b+k-4}{((k-2)b+k+2)^2} . \]

The curvature is singular at \( b = -(k+2)/(k-2) \), which is also where \( \beta(b) = \infty \). These are the properties of the exact 2d metric. The semi-classical metric is obtained by taking the large \( k \) limit, for which \( \beta = 1 \). Then the singularity is at \( b = -1 \). Following Witten this singularity is interpreted as a black hole while the horizon is at \( b = 1 \). The signature of the space is \((+-)\) or \((-+)\) depending on the region in the \((t, b)\) plane as indicated in Fig.
2 of [17]. The signature is understood by examining the semi-classical metric. To see the connection to the Kruskal coordinates used by Witten let \( b = 1 - 2uv \) and \( u^2 = e^{2t}|b-1|/2 \), \( v^2 = e^{-2t}|b-1|/2 \).

If one considers the supersymmetric version of this string model, then one can easily show that the metric is identical to the semiclassical metric, and does not get renormalized for higher orders of \( 1/k \) [18]. Therefore, in the supersymmetric case the singularity is exactly at \( b = -1 \) since \( \beta(b) = 1 \) and \( R = 2/(k - 2)(b + 1) \).

There are other 2D cosets that one may consider, such as \( SO(2, 1)/SO(2) \) and \( SO(3)/SO(2) \), that are analytic continuations of \( SO(1, 2)/SO(1, 1) \). The same procedure applied to them yields the same metrics in terms of the global coordinates given above, but in different regions of the manifold parametrized by \( b \) and \( u \), where \( u = t^2 \) is analytically continued to negative values as well. The different regions have the following interpretations (see fig.2 in [17]): \( SO(3)/SO(2) \) = a cymbal for \(-1 < b \leq 1\); \( SO(2, 1)/SO(2) \) = a cigar when \( b \geq 1 \), and a trumpet when \( b \leq -1 \); \( SO(1, 2)/SO(1, 1) \) = region outside of the horizon of a black hole when \( b \geq 1 \), region between the horizon and singularity of the black hole when \(-1 < b < 1\), bare singularity region on the other side of the black hole when \( b < -1 \).

Properties of these geometrical spaces have been discussed in various publications. In terms of the global coordinates defined here a discussion of geodesics can be found in [17]. Furthermore, these spaces have “duality” properties which correspond to a symmetry that interchanges patches of the manifold without changing its Hamiltonian. In the present case this corresponds to the operation \( b \to -b \). In terms of the original group parameters this is generated by the inversion \( x_\alpha \to x_\alpha/x^2 \). The details are found in [17] and [18]. All of these properties are shared by the higher dimensional manifolds of the following sections, but will not be discussed here due to lack of space. The interested reader can consult the above references.

In the black hole case, there are asymptotically flat regions which are displayed by the change of coordinates \( b = \pm \cosh \frac{2z_1}{\sqrt{2(k-2)}} \), \( t = \frac{z_0}{\sqrt{2k}} \). For large \( z_1 \to \pm \infty \) and any \( z_0 \) the exact metric and dilaton have the asymptotic forms

\[
\begin{align*}
    ds^2 &= dz_1^2 - dz_2^2, \\
    \Phi &= \sqrt{\frac{2}{k-2}|z_1|},
\end{align*}
\]  

(3.11)

displaying a dilaton which is asymptotically linear in the space direction, just like a Liouville field in 2d quantum gravity with a background charge. Despite the flat metric there
is no Poincaré invariance due to the linear dilaton. Note that both the region outside the horizon \((b \to +\infty)\) and the naked singularity region \((b \to -\infty)\) are asymptotically flat.

It was noted that the location of the singularity shifts as a function of \(k\). If one concentrates on the patches that represent the black hole, one concludes that the quantum effects have shielded the singularity. This was first noted in [18] and elaborated on more recently by other groups. In fact the same kind of singularity shielding phenomenon is present in the quantum effects of all other non-compact cosets as discussed and illustrated in [18]. Is this significant for black hole physics? Are black holes eliminated by quantum effects in string theory? For comparison, it is also beneficial to consider the compact \(SU(2)_k/U(1)\) at positive \(k\) (as demanded by unitarity). The geometry is given by the cymbal corresponding to the region \(-1 < b < 1\) with a classical singularity at the boundary \(b = -1\). For the quantum metric, curvature, etc. substitute \(k\) by \(-k\) in the above expressions. The singularity then moves inside the classical region and does not get shielded. On the other hand, in

In my opinion one should concentrate on the wavefunction \(T\) (rather than the quantum corrected metric) and interpret it as a probability distribution. It is then seen that the probability grows logarithmically in the vicinity of the black hole, leading to the interpretation that the particle likes to spend a lot of time there. The logarithmic behavior is square integrable with an appropriate measure given by \(\int e^\Phi \sqrt{G}|T|^2\). Depending on physical boundary conditions that are imposed, the particle may also tunnel to the geometrical patch on the other side of the black hole, a phenomenon possible in quantum mechanics. Such phenomena still need to be interpreted.

3.2. Three dimensions

For \(d = 3\) the antisymmetric tensor is equivalent to a pseudo-vector \(a_{\alpha\beta} = \epsilon_{\alpha\beta\lambda} y^\lambda\), from which we construct two convenient invariants \(v = 2/(1 + y^2)\) and \(u = -v (x \cdot y)^2/x^2\), which together with \(b\) provide a basis for the string coordinates \(X^\mu = (v, u, b)\). Given all possible values taken by \((x^\alpha, y^\alpha)\) the allowed ranges for the invariants are

\[
(+ - +) \text{ or } (- + +) \quad \{b^2 > 1 \& uv > 0\},
\]
\[
(+ + -) \quad \{b^2 < 1 \& uv < 0\}, \quad \text{except} \quad 0 < v < u + 2 < 2. \quad (3.12)
\]

The 3d conformally exact metric is given by the line element [18]
\[ ds^2 = 2(k - 2)(G_{bb} db^2 + G_{vv} dv^2 + G_{uu} du^2 + 2G_{vu} dv du) \] (3.13)

where

\[ G_{bb} = \frac{1}{4(b^2 - 1)} \]

\[ G_{vv} = -\frac{\beta(v, u, b)}{4v(v - u - 2)} \left( \frac{b + 1}{b - 1} + \frac{1}{k - 1} \frac{u + 2}{v - u - 2} \right) \] (3.14)

\[ G_{uu} = \frac{\beta(v, u, b)}{4u(v - u - 2)} \left( \frac{b - 1}{b + 1} - \frac{1}{k - 1} \frac{v - 2}{v - u - 2} \right) \]

\[ G_{vu} = \frac{1}{4(k - 1)} \frac{\beta(v, u, b)}{(v - u - 2)^2} , \]

and

\[ \beta^{-1}(v, u, b) = 1 + \frac{1}{k - 1} \frac{1}{v - u - 2} \left( \frac{b - 1}{b + 1} (u + 2) - \frac{b + 1}{b - 1} (v - 2) - \frac{2}{k - 1} \right) . \] (3.15)

The exact dilaton is

\[ \Phi = \ln \left( \frac{(b^2 - 1)(v - u - 2)}{\sqrt{\beta(v, u, b)}} \right) + \Phi_0 , \] (3.16)

In the large \( k \) limit one obtains the global version of a semi-classical metric derived in [17] using Lagrangian methods

\[ \left. \frac{ds^2}{2(k - 2)} \right|_{k \to \infty} = \frac{db^2}{4(b^2 - 1)} - \frac{1}{v - u - 2} \left( \frac{b + 1}{b - 1} \frac{dv^2}{4v} - \frac{b - 1}{b + 1} \frac{du^2}{4u} \right) \] (3.17)

The signature \((+ - +), ( - + +), ( + + -)\) depends on the region and is indicated in Fig-1 of [17]. A three-dimensional view of this metric is given in Figs-4 of [17]. The surface is where the scalar curvature blows up. This coincides with the location where the dilaton blows up in the large \( k \) limit as seen from the above expression. The space has two topological sectors denoted by the sign of a conserved “charge” \( \pm = \text{sign}(v(b + 1)) = \text{sign}(u(b - 1)) \). The sign never changes along geodesics. A more intuitive view of the space is obtained in another set of coordinates for the plus sector \((b, \lambda_+, \sigma_+)\) and the minus sector \((b, \lambda_-, \sigma_-)\), which are given by \( \lambda^2_\pm = \pm v(b + 1) \) and \( \sigma^2_\pm = \pm u(b - 1) \). Then the singularity surface is shown in Figs 3 of [17]. In the plus region the singularity surface has the topology of the double trousers with pinches in the legs. In the minus region we have the topology of two sheets that divide the space into three regions.

25
There are asymptotically flat regions that may be displayed by a change of variables to
\( b = \pm \cosh \frac{1}{\sqrt{3(k-2)}} (2z_1 - z_0) \), \( u = (\pm)^{'} \cosh \frac{1}{\sqrt{3(k-2)}} (-z_1 + 2z_0) \cosh^2 z_2 \), \( v = (\pm)^{'} \cosh \frac{1}{\sqrt{3(k-2)}} (-z_1 + 2z_0) \sinh^2 z_2 \) (here \( (\pm)^{'} \) is a set of signs independent than \( \pm \)). For large values of \( z_1 \to \pm \infty \), and finite values of \((z_0, z_2)\), the semiclassical metric and dilaton take the form
\[
ds^2 = -dz_0^2 + dz_1^2 + dz_2^2, \quad \Phi = \sqrt{\frac{6}{k-2}} |z_1 - \frac{4}{3}z_0|, \tag{3.18}
\]
showing that the dilaton is linear in a space-like direction \( z_1^{'} = \frac{5}{3}z_1 - \frac{4}{3}z_0 \) in the asymptotically flat region. Then \( z_1^{'} \) behaves just like a Liouville field, while the Lorentz transformed \( z_0^{'} = \frac{5}{3}z_0 - \frac{4}{3}z_1 \) is a time coordinate, and the diagonal metric is rewritten as
\[
ds^2 = -(dz_0^2) + (dz_1^2 + dz_2^2). \tag{3.19}
\]
The exact metric is not flat when only \(|z_1|\) is large. To display its asymptotically flat region one requires somewhat different coordinates.

3.3. Four dimensions

For \( d = 4 \) one can construct the Lorentz invariants
\[
x^2, \quad z_1 = \frac{1}{4} Tr(a^2), \quad z_2 = \frac{1}{4} Tr(a^*a), \quad z_3 = xa^2x/x^2, \tag{3.19}
\]
where \( a^* = \frac{1}{2} \epsilon_{\alpha\beta\alpha^{'}\beta^{'}} a^{\alpha^{'}\beta^{'}} \) is the dual of \( a_{\alpha\beta} \). However, the semi-classical metric is diagonal for a different set of four invariants \( X^\mu = (v, u, w, b) \) given by
\[
b = \frac{1 - x^2}{1 + x^2}, \quad u = \frac{1 + z_2^2 + 2(z_1 - z_3)}{1 - 2z_1 - z_2^2}, \quad v = \frac{1 + z_1 + \sqrt{z_1^2 + z_2^2}}{1 - z_1 - \sqrt{z_1^2 + z_2^2}}, \quad w = \frac{1 + z_1 - \sqrt{z_1^2 + z_2^2}}{1 - z_1 + \sqrt{z_1^2 + z_2^2}}. \tag{3.20}
\]
To find the ranges in which the above global coordinates take their values we consider a Lorentz frame that can cover all possibilities without loss of generality. First we notice that by Lorentz transformations the antisymmetric matrix \( a_{\alpha\beta} \) can always be transformed to a block diagonal matrix with the non-zero elements
\[
a_{01} = \tanh t \text{ or } \coth t, \quad a_{23} = \tan \phi. \tag{3.21}
\]
Then, using (3.20) one can deduce the form of the global variables: \( v = \pm \cosh 2t, \ w = \cos 2\phi, \) and \( u = \frac{1}{x^2}(w(x_0^2 - x_1^2) - v(x_2^2 + x_3^2)) \). Therefore the string variables can take values in the following regions with the signature in the \((v, u, w, b)\) basis.
$(-+++): \ b^2 > 1, \ \{ -1 < w < u < 1 < v \ \text{or} \ v < -1 < u < w < 1 \\
\text{or} \ -1 < w < 1 < u < v \}$,

$(+++): \ b^2 > 1, \ \{ -1 < w < 1 < v < u \ \text{or} \ u < v < -1 < w < 1 \}$

$(++++): \ b^2 < 1, \ \{ u < w < 1 < v \ \text{or} \ v < -1 < w < u \ \text{or} \ v < u < -1 < w < 1 \}.$

(3.22)

With this set of coordinates we compute the conformally exact dilaton and metric as before. The dilaton field is

$$\Phi = \ln \left( \frac{(b^2 - 1)(b - 1)(v - u)(w - u)}{\sqrt{\beta(b, u, v, w)}} \right) + \Phi_0. \quad (3.23)$$

and the metric is given by

$$ds^2 = 2(k - 3)(G_{bb} db^2 + G_{uu} du^2 + G_{vv} dv^2 + G_{ww} dw^2 \\
+ 2G_{uv} dudv + 2G_{uw} dudw + 2G_{vw} dv dw), \quad (3.24)$$

where

$$G_{bb} = \frac{1}{4(b^2 - 1)}$$

$$G_{uu} = \frac{\beta(b, u, v, w)}{4(u - w)(v - u)} \left( \frac{b - 1}{b - 1} - \frac{1}{k - 2} \frac{(v - w)^2}{(v - u)(u - w)} \left( 1 - \frac{1}{(k - 2) b - 1} \right) \right)$$

$$G_{vv} = -\frac{(v - w)\beta(b, u, v, w)}{4(v^2 - 1)(v - u)} \left( \frac{b + 1}{b - 1} - \frac{1}{k - 2} \frac{1}{(v - u)(u - w)} \left[ 1 - u^2 + \right] \right. \left. + \frac{b + 1}{b - 1} \right)^2(v - u)(v - w) + \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{(1 + v^2)(u + w) - 2v(1 + uw)}{v - w} \right)$$

$$G_{ww} = \frac{(v - w)\beta(b, u, v, w)}{4(1 - w^2)(u - w)} \left( \frac{b + 1}{b - 1} - \frac{1}{k - 2} \frac{1}{(v - u)(u - w)} \left[ 1 - u^2 + \right] \right. \left. + \frac{b + 1}{b - 1} \right)^2(u - w)(v - w) + \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{(1 + w^2)(u + v) - 2w(1 + uv)}{v - w} \right)$$

$$G_{uv} = \frac{\beta(b, u, v, w)}{4(k - 2)(v - u)^2} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{v - w}{u - w} \right)$$

$$G_{uw} = \frac{\beta(b, u, v, w)}{4(k - 2)(u - w)^2} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{v - w}{v - u} \right)$$

$$G_{vw} = \frac{1}{(k - 2)^2 \frac{b - 1}{4(v - u)(u - w)}},$$

27
and the function $\beta(b, u, v, w)$ is defined by

$$
\beta^{-1}(b, u, v, w) = 1 + \frac{1}{k-2} \frac{(v-w)^2}{(v-u)(w-u)} \left( \frac{b+1}{b-1} + \frac{b-1}{b+1} \frac{1-u^2}{(v-w)^2} \right)
+ \frac{1}{k-2} \left( \frac{vw+u(v+w)-3}{(v-w)^2} - \left( \frac{b+1}{b-1} \right)^2 \right)
+ \frac{2}{(k-2)^3} \frac{b+1}{b-1} \frac{vw-1}{(v-u)(u-w)}.
$$

The large $k$ limit of these expressions reduce to the semiclassical dilaton and metric that follow from the Lagrangian approach

$$
\frac{ds^2}{2(k-2)} \bigg|_{k \to \infty} = \frac{db^2}{4(b^2-1)} + \frac{b-1}{b+1} \frac{du^2}{4(v-u)(u-w)}
+ \frac{b+1}{b-1} \frac{dv^2}{4(1-w^2)(u-w)} - \frac{dw^2}{4(v^2-1)(v-u)}.
$$

We can see that the signature of the semiclassical metric for different ranges of the parameters (3.22) is precisely as required by the group parameter space which led to (3.22). However, for the exact metric $\beta(u, v, w, b)$ must remain positive to keep $-\det(G)$ positive. This implies that part of the regions in (3.22) are screened out by quantum effects for the exact geometry. This screening phenomenon is true for every dimension $d = 2, 3, 4$ and the screened regions must be interpreted in the quantum theory as tunneling or decay regions for probability amplitudes (such as the tachyon wavefunction). Under any circumstances the manifold cannot go outside of the range (3.22) dictated by the group theory.

As in the previous $d = 2, 3$ cases, we can check that our explicit expressions for the dilaton and metric give the $k$-independent combination $\sqrt{-Ge^\Phi}$. Therefore this quantity takes the same value for either the exact metric and dilaton or the semiclassical metric and dilaton. Since it is unrenormalized by quantum effects (other than one loop), it may be computed in lowest order perturbation theory. This combination appears in the d’Alembertian and is also closely related to the integration measure in the path integral. Through group theoretical arguments given in [11][13] it was possible to guess that this combination should remain unrenormalized by quantum effects. Similar to the $d = 2, 3$ cases the 4d manifold has an asymptotically flat region, but it will not be discussed here.
3.4. Particle and String Geodesics

Having global coordinates and a global geometry is not sufficient to get a feeling of the geometry, one also needs to know the behavior of the geodesics. However, for the complicated metrics that are displayed above the geodesic equation seems to be completely unmanageable. Fortunately, we have developed a procedure that relies on group theory and managed to solve for all particle geodesics. The trick is to take advantage of the fact that the global coordinates are gauge invariant under $H$-transformations. Then we may solve the equations of motion for the group element $g$ in any gauge, and use the solution to construct the $H$-invariant combinations that form the global coordinates of the geometry. In fact, there is an axial gauge in which $g$ is solved easily \[17\]. For a point particle (string shrunk to a point) it is given as a function of proper time

$$g(\tau) = e^{\alpha \tau} g_0 e^{(p-\alpha)\tau}, \quad (3.28)$$

where $g_0$ is a constant group element at initial proper time $\tau$, and $\alpha, p$ are constant matrices in the Lie algebras of $H$ and $G/H$ respectively. The equations of motion require that these constants satisfy a constraint

$$(g_0(p-\alpha)g_0^{-1})_H + \alpha = 0, \quad (3.29)$$

where the subscript $H$ implies a projection to the Lie algebra of $H$. This solution applies to any group and subgroup. As shown in \[17\] the standard geodesics equations for the geometries displayed above are automatically solved when the $H$-invariants are constructed from the solution \( (3.28) (3.29) \). In this way all light-like, space-like and time-like geodesic solutions are obtained.

With the point geodesics at hand we have discovered a number of additional interesting properties about the $d = 2, 3, 4$ manifolds \[17\] which generalize to other non-compact gauged WZW models as well. The most striking feature is that the manifolds that are pictured in the figures have many copies and the complete manifold must include all the copies. The gauge invariant coordinates (e.g. $(b, t)$ for $d = 2$) are not sufficient to fully describe the structure. There are additional discrete gauge invariants constructed from the group element $g$ that label the copies of the manifold. This can be seen easily in our examples since the gauge subgroup is just the Lorentz group and its properties are well known. In this case the invariants are Lorentz dot products constructed from a vector $x^\alpha$ and a tensor $a^{\alpha\beta}$. Let us consider the invariant $b = (1 - x^2)/(1 + x^2)$, say in the region
It is known that the time component $x^0$ could be either positive or negative and that a Lorentz transformation cannot change this sign. Therefore, the sign of $x^0$ is a discrete gauge invariant which does not show up in the metric or dilaton that characterized the manifolds discussed above. However, the model as a whole knows about this sign through the group element $g$. Such discrete invariants are present in every non-compact gauged WZW model and they label copies of the manifolds described above. We may then ask whether these copies communicate with each other? The answer is yes, they do, and this can be seen by following the behaviour of a particle geodesic. The full information about the particle geodesic is contained in the solution for $g$ in (3.28)(3.29). From this it can be verified that at the proper time that a particle touches a curvature singularity the discrete invariant switches sign and then the particle continues its journey smoothly from one copy of the manifold to the next. For example, in the 2d black hole case this happens for a time-like geodesic (i.e. massive particle) in a finite amount of proper time (on the other hand, a light-like geodesic takes an infinite amount of proper time to reach the singularity and therefore ends its journey without changing copies of the manifold).

This behavior is present in all non-compact models in this paper as well as in other models (e.g. we have verified it in the $SL(2,\mathbb{R}) \times SU(2)/\mathbb{R}^2$ model). It is reminiscent of the Reissner-Nordström black hole in which geodesics move on to other worlds. The difference is that in our case this happens at the singularity itself. When quantum corrections are included and the exact metric considered, then the singularity and the transition to other worlds no longer seem to be at the same place; at least this is the case for the 2d black hole. The spectrum of the discrete invariant depends on the group representation and therefore one expects different numbers of copies in different quantum states. The number of copies is infinite for quantum states with non-fractional quantum numbers, which is typical in unitary non-holomorphic representations of non-compact groups. When the number of copies is infinite the particle can never come back to the same world, but for a finite number of copies the particle returns to the original world by emerging from a white singularity.

So far we have discussed particle geodesics that correspond to a string collapsed to a single point. We may also investigate string geodesics in the same manifolds. That is, we are also interested in solutions for the strings moving in curved spacetime, just like one has a complete solution in flat spacetime in terms of harmonic oscillator normal modes. This problem has been solved in principle for the non-compact gauged WZW models in [11]. There the solution for the group element $g(\tau, \sigma)$ has been obtained explicitly in terms
of normal modes. This is the analog of (3.28) above. There remains to construct the appropriate dot products to form the invariants, which in turn are the solutions to the string geodesics. This last part has not yet been performed explicitly, but it is only a matter of straightforward algebra of the kind performed for the particle geodesics in [17]. This procedure gives all the solutions in curved spacetime and can answer questions of the type “what happens when a string falls into a black hole?”

3.5. Duality

Due to lack of space we have not covered other interesting topics such as duality properties of these manifolds. It was shown in [11][17] that there is a dynamical duality that generalizes the $R \to 1/R$ duality properties of conformal field theories based on tori. This is related to the axial/vector duality that is present in the 2d black hole. It was shown in [17] that the duality transformation is equivalent to an inversion in group parameter space $(x_\alpha, a_{\alpha\beta})$ given in (3.3). This inversion generates discrete jumps for the group parameter that corresponds to interchanging different patches of the geometrical manifold. For details the reader is referred to [17]. This duality property is closely related to mirror symmetry of the kind discussed for Calabi-Yau manifolds, as will be explained elsewhere. The duality symmetry mentioned here is different than the one discussed in recent months by Verlinde, Giveon, Rocek, and others.

4. CONCLUSIONS

We have only scratched the surface of the subject of non-compact gauged WZW models. We have shown that this approach is very useful for learning about strings in curved spacetime that may be relevant for the early part of the Universe. It is during this era that string theory should be relevant and it is during this era that the matter we know was formed. Therefore, in trying to solve the puzzles of the Standard Model with respect to the spectrum of matter and gauge bosons we may hope that a string theory in curved spacetime may guide us. For this reason I believe that it is valuable to study in great detail the models presented in Table 1. These are solvable models that should direct us toward a realistic unified theory.

I did not have the space to discuss a number of interesting results. Among these I want to mention the construction of the effective quantum action to all orders in $k$, at least in the zero mode sector [21]. This method generates the same exact conformal metric and
dilaton as the Hamiltonian approach, but in addition it also gives the exact antisymmetric tensor $B_{\mu\nu}(X)$ (the axion). The general results for all gauged WZW models are provided in [21].

Other important results are the computation of conformally exact quantities for supersymmetric and heterotic strings. These are obtained by a simple substitution of $k$ by a shifted value of $k$ in the semiclassical or exact results of the purely bosonic quantities. The prescription is derived in [18] and applied to a number of cases there and elsewhere [20].

There are many mathematical problems of interest. The geometries, the duality properties, the unitary representations of non-compact groups restricted to the appropriate subgroup, etc. are all problems that either have not been studied, or require a lot more research. From the point of view of physics, we are only at the beginning of our understanding of strings in curved spacetimes and many interesting results can be expected in the future.

---

6 See also [23] for the SL(2,R) case, with a partial discussion of possible non-local terms for the higher string modes. However, his treatment is not gauge invariant as noted in [21].
References

[1] E. Witten, Comm. Math. Phys. 92 (1984) 455.

[2] I. Bars, in Vertex Operators in Math and Physics, Eds. Lepowsky, Mandelstam, Singer, (Springer Verlag NY, 1985), page 373.

[3] V.G. Kac, Func. Anal. App. 1 (1967) 238; R.V. Moody, Bull. Am. Math. Soc. 73 (1967) 217.

[4] K. Bardakci and M.B. Halpern, Phys. Rev. D3 (1971) 2493; P. Goddard, A. Kent and D. Olive, Phys. Lett. 152B (1985) 88.

[5] I. Bars and D. Nemeschansky, Nucl. Phys. B348 (1991) 89.

[6] I. Bars, “Curved Spacetime Strings and Black Holes”, in Proc. XXth Int. Conf. on Diff. Geometrical Methods in Physics, eds. S. Catto and A. Rocha, Vol. 2, p. 695, (World Scientific, 1992).

[7] P. Ginsparg and F. Quevedo, Nucl. Phys. B385 (1992) 527.

[8] I. Bars, Phys. Lett 293B (1992) 315.

[9] E. Witten, Phys. Rev. D44 (1991) 314.

[10] E. Witten, Nucl. Phys. B223 (1983) 422; K. Bardakci, E. Rabinovici and B. Säring, Nucl. Phys. B299 (1988) 157; K. Gawedzki and A. Kupiainen, Phys. Lett. 215B (1988) 119; Nucl. Phys. B320 (1989) 625.

[11] I. Bars and K. Sfetsos, Mod. Phys. Lett. A7 (1992) 1091.

[12] M. Crescimanno, Mod. Phys. Lett. A7 (1992) 489; J. B. Horne and G. T. Horowitz, Nucl. Phys. B368 (1992) 444; E. S. Fradkin and V. Ya. Linetsky, Phys. Lett. 277B (1992) 73; P. Horava, Phys. Lett. 278B (1992) 101; E. Raiten, “Perturbations of a Stringy Black Hole”, Fermilab-Pub 91-338-T; D. Gershon, “Exact Solutions of Four-Dimensional Black Holes in String Theory”, TAUP-1937-91.

[13] I. Bars and K. Sfetsos, Phys. Lett. 277B (1992) 269.

[14] I. Bars, “Superstrings on Curved Spacetimes”, USC-92/HEP-B5 (hep-th/9210079), in String Quantum Gravity and Physics at the Planck Energy Scale, Ed. N. Sanchez, World Scientific (1993), page.

[15] A. Giveon, E. Rabinovici, and A.A. Tseytlin, Heterotic String Solutions and Coset Conformal Field Theories, hep-th/9304155.

[16] I. Bars, Nucl. Phys. B334 (1990) 125.

[17] I. Bars and K. Sfetsos, Phys. Rev. D46 (1992) 4495.

[18] I. Bars and K. Sfetsos, Phys. Rev. D46 (1992) 4510.

[19] K. Sfetsos, “Conformally Exact Results for $SL(2, \mathbb{R}) \otimes SO(1,1)^{d-2}/SO(1,1)$ Coset Models”, USC-92/HEP-S1 (hep-th/9206048), to appear in Nucl. Phys.
[20] I. Bars and K. Sfetsos $SL(2,\mathbb{R}) \times SU(2)/R^2$ String Model in Curved Spacetime and Exact Conformal Results”, Phys. Lett. B301 (1993) 183.

[21] I. Bars and K. Sfetsos, Phys. Rev. D48 (1993) 844.

[22] I. Bars, K. Sfetsos and A.A. Tseytlin, unpublished.

[23] A.A. Tseytlin, “Effective Action in Gauged WZW Models and Exact String Solutions”, Imperial/TP/92-93/10.

[24] A.A. Tseytlin, preprint CERN-TH.6804/93.

[25] R. Dijgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B371 (1992) 269.