THERMAL CONDITIONS FOR SCALAR BOSONS 
IN A CURVED SPACE TIME

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Abstract
The conditions that allow us to consider the vacuum expectation value of the energy-momentum tensor as a statistical average, at some particular temperature, are given. When the mean value of created particles is stationary, a planckian distribution for the field modes is obtained. In the massless approximation, the temperature dependence is as that corresponding to a radiation dominated Friedmann-like model.
1. Introduction

In the semiclassical approximation the vacuum expectation value (VEV) of the energy-momentum tensor (EMT) can be used as a source of the Einstein equations, in the form:

\[ G_{\mu\nu} = -8\pi G \langle T_{\mu\nu} \rangle_{\text{reg.}} \]  

(1)

This approximation is useful as an asymptotic value of more complete theories where the gravitational field is also quantized, when the time is larger than the Planck time. The mean value of \( T_{\mu\nu} \) can be obtained using the expression:

\[ \langle T_{\mu\nu} \rangle = \frac{\langle \text{out},0 | \hat{T}_{\mu\nu} | 0,\text{in} \rangle}{\langle \text{out},0 | 0,\text{in} \rangle} \]  

(2)

which was introduced in ref.[1]. We can relate eq. (2) with the VEV with respect to some particular “in” or “out” vacua (see ref.[2]), i.e.:

\[ \langle T_{\mu\nu} \rangle = \langle \text{in},0 | \hat{T}_{\mu\nu} | 0,\text{in} \rangle + i \sum_{i,j} \Lambda_{ij} T_{\mu\nu}(\phi^{*}_{\text{in},i},\phi^{*}_{\text{in},j}) \]  

\[ = \langle \text{out},0 | \hat{T}_{\mu\nu} | 0,\text{out} \rangle + i \sum_{i,j} V_{ij} T_{\mu\nu}(\phi^{*}_{\text{out},i},\phi^{*}_{\text{out},j}) \]  

(3)

where \( \{\phi_{\text{in}}\} \cup \{\phi^{*}_{\text{in}}\} \) and \( \{\phi_{\text{out}}\} \cup \{\phi^{*}_{\text{out}}\} \) are the basis of solutions defined for the Cauchy surface labeled by “in” and “out” respectively. \( \Lambda_{ij} \) and \( V_{ij} \) are functions of the Bogoliubov coefficients that determine the transformation between the two vacua. It is also useful to take into account that the expectation values can be expanded in terms of the vacuum polarization, which contains the local infinities, that are removed by the usual regularization methods (see ref.[3]) independently of the vacuum definition used, and a term which is function of the created particles, due to the interaction of the field with the curved geometry. This last term is of course null in flat space time. But in curved space it is in general infinite when it is calculated perturbatively. For instance in the particular case of fields minimally coupled to gravity, the infinity coming from the particle creation term, cannot be removed with the usual regularization methods for an arbitrary vacuum definition (see ref.[4]). However it is easy to see (ref. [5]) that when particle creation has a planckian distribution, which is lost when perturbative expansions are performed, the contribution to the EMT is finite and moreover is coincident with the standard cosmology which uses perfect fluid classical sources in the Einstein equations. A question that we can ask is; if it is reasonable to get a planckian distribution for the
mean value of created particles. The answer is affirmative because is obvious the fact that the actual background radiation of the Universe has a black body distribution with temperature approximately equal to 3°K (see ref. [6]). In another case, for example in the black holes studied by Hawking [7], a planckian distribution is predicted. Also the distribution seen by an accelerated observer (Rindler observer) is planckian like, therefore by the generalization of the equivalence principle we hope that for some Bogoliubov transformations we get a thermal spectrum. This result is shown in the present work for a scalar field minimally coupled to a Robertson-Walker metric. The expression obtained for the VEV of the EMT is compared with the one coming from the statistical average of $T_{\mu\nu}$ at a given temperature, which is obtained using Thermo Field Dynamics (TFD) theory [9]. In a previous work [10] it was proved that a field in a curved geometry presents a form similar to that of the field in a thermal bath. In TFD the interaction with the bath is half-filled by the so called “tilde” modes, which is more natural that the interaction between the classical background and the quantum field. In the present work from, the comparison mentioned before, sufficient conditions on the Bogoliubov transformation are obtained, in order to get the same functional form for VEV of the EMT and the statistical average obtained from TFD.

2. Time dependent Bogoliubov transformation

In a pioneer work [11] Parker developed a consistent method to obtain the annihilation-creation operators at each time in curved space. The operators defined at different Cauchy surfaces are related by time dependent Bogoliubov transformations. Parker found in ref.[11] the differential equation satisfied by the Bogoliubov coefficients. This equation is other form to write the field equation. In order to resolve that equation it is necessary to give some initial conditions. The Bogoliubov coefficients give us the mean value of created particles between an initial time “$t_0$” and a present time “$t$”.

In order to make the paper more comprehensive, in this section we repeat in brief the formulation given by Parker in ref. [11].

The action for a minimally coupled scalar matter field is:

$$S = \frac{1}{2} \int \sqrt{-g} d^4x (\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2)$$

(4)

and the metric is the spatially flat Robertson-Walker, given by

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2)$$

(5)
Therefore the variation of the action gives the field equation

\[(\nabla_\mu \partial^\mu + m^2)\varphi = 0\]  \hspace{1cm} (6)

(with \(\mu = 0, 1, 2, 3\)).

The metric given by eq.(5) represents an unbounded space-time, therefore in a Fourier representation of \(\varphi\) integrals appear, in order to make the calculation easier, we can use a discretization in the same form as in ref.[11], i.e. we can introduce the periodic boundary condition \(\varphi(x + nL, t) = \varphi(x, t)\), where \(n\) is a vector with integer Cartesian components and \(L\) a longitude which goes to infinity at the end of the calculation (really in our case it is not necessary because the results used do not depend on \(L\)). Then we can introduce the set of functions \(\{\phi_k(x)\} \cup \{\phi^*_k(x)\}\) defined by

\[\phi_k(x) = \frac{1}{(La(t))^{3/2}\sqrt{2W}} \exp i(kx - \int_{t_0}^t W(k, t')dt')\]  \hspace{1cm} (7)

with \(W\) an arbitrary real function of \(k = |k|\) and \(t\). The field can be expanded in the form

\[\varphi(x, t) = \sum_k [a_k(t)\phi_k(x) + a^\dagger_k(t)\phi^*_k(x)]\]  \hspace{1cm} (8)

where the operators \(a_k(t)\) and \(a_{k'}(t)\) are time dependent and satisfy the commutation relations

\[[a_k(t), a_{k'}(t)] = 0, \quad [a^\dagger_k(t), a^\dagger_{k'}(t)] = 0, \quad [a_k(t), a^\dagger_{k'}(t)] = \delta_{k,k'}\]  \hspace{1cm} (9)

The operators act on the vacuum \(|0, t>\) in the form

\[a_k(t)|0, t> = 0, \quad a^\dagger_k(t)|0, t> = |1_k, t>\]

Moreover we define the operators

\[A_k := a_k(t = t_1) \quad \text{with} \quad t \geq t_1 \geq t_0\]

\[A_k^\dagger := a^\dagger_k(t = t_1)\]  \hspace{1cm} (10)

Also we define the vacuum \(|0 >:= |0, t = t_1>\) on which the operators defined by eq.(10) act. These operators are related with the time dependent ones by the following Bogoliubov transformation.
\[ a_k(t) = \alpha(k, t)A_k + \beta(k, t)A_k^\dagger \]  

\[ a_{-k}^\dagger(t) = \alpha(k, t)A_{-k} + \beta(k, t)^*A_k \]

clearly we use the conditions

\[ \alpha(k, t_1) = 1, \quad \beta(k, t_1) = 0 \]  

We can rewrite the field \( \varphi(x, t) \) in terms of the new operators in the form

\[ \varphi(x, t) = \sum_k [A_k \Psi_k(x) + A_k^\dagger \Psi_k^*(x)] \]

with \( \Psi_k(x) \) and \( \Psi_k^*(x) \) solutions of eq. (6). We can make the separation

\[ \Psi_k(x) = h(k, t) \exp i k \cdot x \]  

with

\[
h(k, t) = \frac{1}{(La(t))^{3/2}(2W(k, t))^{1/2}}\left[\alpha(k, t)^* e^{-i \int_{t_0}^t W dt'} + \beta(k, t)^* e^{i \int_{t_0}^t W dt'}\right]
\]

Replacing eq. (14) in eq. (6) we have

\[
(\hat{\alpha} e^{i \int W dt'} + \hat{\beta} e^{-i \int W dt'}) M - \frac{\tilde{W}}{W}(\hat{\alpha} e^{i \int W dt'} + \hat{\beta} e^{-i \int W dt'})
\]

\[
+2i W(\hat{\alpha} e^{i \int W dt'} - \hat{\beta} e^{-i \int W dt'}) + \hat{\alpha} e^{i \int W dt'} + \hat{\beta} e^{-i \int W dt'} = 0
\]  

with

\[
M := -\frac{1}{2}(\tilde{W}/W) + \frac{1}{4}(\tilde{W}/W)^2 - \frac{9}{4}(\dot{a}/a)^2 - \frac{3}{2}(\dot{a}/a)^\dagger + \omega^2 - W^2
\]  

In the following we will call \( \omega^2 = k^2/a^2 + m^2 \).

From eqs (12) and (13), Parker [11] proves that the Bogoliubov coefficients satisfy the equation

\[ \dot{\beta} = -\dot{\alpha} \exp(2i \int_{t_0}^t W(k, t') dt') \]
From the functional form (14), for which the invariance of the Klein-Gordon product holds, we obtain:

\[ |\alpha|^2 - |\beta|^2 = 1 \quad (18) \]

Using eq.(11) we note that

\[ n_k := |\beta_k|^2 = <0|a_k(t)a_k(t)|0> \]

is the mean value of particles created between \( t_1 \) and \( t \). Eq. (18) allow us to write (as in ref.[11]):

\[ \alpha(k, t) = e^{-i\gamma \alpha(k, t)} \cosh \theta(k, t) \quad (19a) \]

\[ \beta(k, t) = e^{i\gamma \beta(k, t)} \sinh \theta(k, t) \quad (19b) \]

\( \gamma_\alpha, \gamma_\beta \) and \( \theta \) are functions of \( k \) and \( t \) that must satisfy the initial conditions given by eq.(12) and the field equation (15). As we can see from eq. (19 a) the number of particles is only a function of \( \theta \), i.e. \( n_k = \sinh^2 \theta \).

Replacing eqs (19) and (17) in eq. (15) and splitting in real and imaginary parts, the following system of equations is obtained:

\[ (1 + \tanh \theta \cos \Gamma)M + 2W \dot{\gamma}_\alpha = 0 \quad (20a) \]

\[ M \sin \Gamma + 2W \dot{\theta} = 0 \quad (20b) \]

with \( \Gamma := \gamma_\alpha + \gamma_\beta - 2 \int_{t_0}^{t} W dt' \)

The solutions to eqs (20) give us the set of time dependent Bogoliubov transformations compatible with the Robertson-Walker metric. The stationary particle creation is a particular case with \( \dot{\theta} = 0 \), therefore from eq. (20 b) we have:

\[ M = 0 \quad (21) \]

3. VEV of EMT as a thermodynamical object

In ref.[4] the analogy between the Bogoliubov transformation which relates states at different temperature with those at different Cauchy surfaces was shown. Here we will exploit this analogy in relation with the VEV of EMT. First of all we calculate the classical EMT in curved space-time in the usual form (see ref.[2]), i.e. as:

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \]
we obtain the expression
\[ T_{\mu\nu} = \frac{1}{2}\{\varphi_\mu, \varphi_\nu\} - \frac{1}{4}g_{\mu\nu}\{\varphi^\sigma, \varphi_\sigma\} + \frac{1}{4}m^2g_{\mu\nu}\{\varphi, \varphi\} \] (22)

where \{ , \} is the anticommutator and \( \varphi_\mu := \partial_\mu \varphi \).

We can now calculate the EMT operator replacing eq.(13) in eq.(22) obtaining

\[
\hat{T}_{\mu\nu} = \sum_{kk'} \{A_kA_{k'}D_{\mu\nu}[\Psi_k(x), \Psi_{k'}(x)] + A_kA_{k'}D_{\mu\nu}[\Psi_k(x), \Psi_{k'}^*(x)] \\
+ A_{k}A_{k'}D_{\mu\nu}[\Psi_{k}^*(x), \Psi_{k'}(x)] + A_{k}A_{k'}D_{\mu\nu}[\Psi_{k}^*(x), \Psi_{k'}^*(x)] \}
= \sum_{kk'} \{A_kA_{k'}D_{\mu\nu}[\Psi_k(x), \Psi_{k'}^*(x)] \\
+ A_{k}A_{k'}D_{\mu\nu}[\Psi_{k}^*(x), \Psi_{k'}(x)]
\}
\] (23)

with \{k \leftrightarrow k'\} indicating that the last term of eq.(23) is equal to the first changing \( k \) by \( k' \). The differential operator \( D_{\mu\nu} \) is defined by

\[
D_{\mu\nu}[\varphi, \psi] := \frac{1}{2}\partial_\mu \varphi \partial_\nu \psi - \frac{1}{4}g_{\mu\nu}\partial_\sigma \varphi \partial_\sigma \psi + \frac{1}{4}m^2g_{\mu\nu}\varphi \psi
\] (24)

We can now calculate the VEV of \( \hat{T}_{\mu\nu} \) using the vacuum \( |0\rangle \), and obtain:

\[
<0|\hat{T}_{\mu\nu}|0> = 2\sum_k Re\{D_{\mu\nu}[\Psi_k(x), \Psi_k^*(x)]\}
\] (25)

Replacing now eqs (11) and (13) we also have

\[
\Psi_k(x) = \alpha^* \phi_k(x) + \beta^* \phi_{-k}^*
\] (26)

Putting this last equation in eq. (25) yields

\[
<0|\hat{T}_{\mu\nu}|0> = 2\sum_k Re\{(1 + 2|\beta|^2)D_{\mu\nu}[\phi_k, \phi_k^*] + 2\alpha^* \beta D_{\mu\nu}[\phi_k, \phi_{-k}]
\]

\[
+ \frac{1}{2}(|\beta|^2)E_{\mu\nu}[\phi_k, \phi_k^*] + \frac{1}{2}(\alpha^* \beta)E_{\mu\nu}[\phi_k, \phi_{-k}]
\]

\[
+ \frac{1}{2}(\delta_{\mu\nu} - \frac{1}{2}g_{\mu\nu})[|\dot{\alpha}|^2 + |\dot{\beta}|^2]|\phi_k|^2 + 2\alpha^* \beta^* \phi_{-k}\phi_k]
\} (27)

with

\[
E_{\mu\nu}[\psi, \varphi] := \psi(\delta_{\mu\nu}\partial_\mu \varphi + \delta_{\nu\mu}\partial_\nu \varphi - g_{\mu\nu}\dot{\varphi})
\] (28)
For simplicity we can consider the case in which the particle creation is stationary, then $\dot{\beta} = \dot{\alpha} = 0$, therefore we have
\[
< 0|\hat{T}_{\mu\nu}|0 > = 2 \sum_k Re\{(1 + 2|\beta|^2)D_{\mu\nu}[\phi_k, \phi_k^*] + 2\alpha^*\beta D_{\mu\nu}[\phi_k, \phi_{-k}]\}
\]
(29)

As we will see the difference between the VEV given by eq. (29) and a statistical average is the “interference” term, for the modes $k$ and $-k$, of eq. (29).

A simple way to calculate the statistical mean value of some physical observable is by means of the thermo field dynamics (TFD) formulation (see ref. [12]). In this formulation the idea is to take into account the interaction between the system and the thermal bath by means of quantum fluctuation of the bath (see also ref. [9]), which are called “tilde” fields. These fields are auxiliary because they are not measurable. Those fields only act in the thermalization of the system modes. The tilde modes are associated to tilde creation annhilation operators which act on the vacuum of the thermal bath $|\tilde{0} >$. The space of states, in this formulation is extended in order to include the tilde and the states of the system, i.e.:

\[
\{|n, \tilde{n} >\} = \{|n >\} \otimes \{\tilde{n} >\}
\]

We will call the total vacuum $\{0 > := |0 > \otimes |\tilde{0} >$. Moreover we introduce a $T$ parameter in order to identify the temperature of the state, i.e. $|0, T >$ and a Bogoliubov transformation that relates the vacuum at zero temperature with the one at temperature $T$:
\[
a_k(T) = A_k \cosh \theta(k, T) - \tilde{A}_k^\dagger \sinh \theta(k, T)
\]
(30)

In order to obtain the zero temperature operators, $\theta(k, T = 0) = 0$ is necessary. Then we have the creation annhilation bosonic fields $A_k$ and $A_k^\dagger$ which operate in the form
\[
A_k |0 > = 0.
\]
\[
A_k^\dagger |0 > = |1_k >, etc.
\]

with the commutation relation
\[
[A_k, A_{k'}^\dagger] = \delta_{k,k'}
\]

The tilde operators, represent the quantum effect of the reservoir and satisfy
\[
\tilde{A}_k |0 > = 0.
\]
The thermal operators $a_k(T)$, $a^+_k(T)$, $\tilde{a}_k(T)$, $\tilde{a}^+_k(T)$, which satisfy:

$$a_k(T)\|0, T > = 0$$

$$a^+_k(T)\|0, T > = \|1_k, T >, etc.$$

$$[a_k(T), a^+_k' (T)] = \delta_{k,k}$$

and in analogous form the tilde operators. The tilde and no tilde operators are mutually commutative.

With the new Fock space we can calculate the statistical mean value of any physical observable, for example if we have the physical operator $\hat{A}$, the mean value can be obtained by means of:

$$< A > = < 0, T | \hat{A} | 0, T >$$

We will calculate now the statistical mean value of the energy momentum operator $\hat{T}_{\mu\nu}$. In order to do that we can represent the operator as a functional of the thermal modes $\phi_{kT}$ and $\phi^*_{kT}$ in a way analogous to that in curved space time:

$$\hat{T}_{\mu\nu} = \sum_{kk'} \{ a_k(T)a_{k'}(T)D_{\mu\nu}[\phi_{kT}(x), \phi_{k'T}(x)] + a_k(T)a^+_k'(T)D_{\mu\nu}[\phi_{kT}(x), \phi^*_{k'T}(x)]$$

$$+ a^+_k(T)a_{k'}(T)D_{\mu\nu}[\phi^*_{kT}(x), \phi_{k'T}(x)] + a^+_k(T)a^+_k'(T)D_{\mu\nu}[\phi^*_{kT}(x), \phi^*_{k'T}(x)]$$

$$+\{k \leftrightarrow k'\} \}$$

(31)

It is easy to prove (see ref. [13]) using the transformation given by eq. (30) and the inverse that

$$< 0|a_k^+(T)a_k(T)|0 > = < 0, T | A_k^+A_k|0, T >$$

and in analogous form with $a_k(T)a_k(T)$ and $a_k(T)a_k^+(T)$. Therefore we can do

$$< T_{\mu\nu} > = < 0|\hat{T}_{\mu\nu}[a_k(T), a_k^+(T), a_{k'}(T), a_{k'}^+(T)]|0 >$$

then the following expression is obtained

$$< 0|\hat{T}_{\mu\nu}|0 > = 2\sum_k Re\{ (1 + 2n_k) D_{\mu\nu}[\phi_{kT}, \phi_{kT}^*] \}$$

(32)
where

\[ n_k = \langle 0 | a_k^\dagger(T) a_k(T) | 0 \rangle \]

As we can see from eq.(32) the \( \phi_k \) modes will be equivalent to the thermal ones if the interference term in eq. (29) is null.

The difference with the “genuine thermality” has the same cause as that in the case of the Rindler vs Minkowski observers, analyzed in ref. [14]. It is related with the fact that \( \langle 0 | a_k(t) a_{k'}(t) | 0 \rangle \neq 0 \). In ref. [14] the thermality is restored assuming that the observation is restricted to a finite spacetime region, which is introduced mathematically by means of the wave packets (see also ref. [8]). In our case instead of using wave packets we will exploit the fact that the interference term is oscillant. First we calculate each component of that term:

\[
\text{Re}\{2\alpha^* \beta D_{00}[\phi_k, \phi_{-k}]\} = \frac{1}{2} \cosh \theta \sinh \theta |\phi|^2 \left\{ \left[ \frac{1}{4} \left( \frac{\dot{W}}{W} + 3H \right) \right]^2 + \omega^2 - W^2 \right\} \cos \Gamma - \left( \frac{\dot{W}}{W} + 3H \right) W \sin \Gamma 
\]

(33)

\[
\text{Re}\{2\alpha^* \beta D_{0j}[\phi_k, \phi_{-k}]\} = -\text{Re}\{ie^{i\gamma} \dot{\phi}_k \phi_{-k} k_j \} 
\]

(34)

\[
\text{Re}\{2\alpha^* \beta D_{jj}[\phi_k, \phi_{-k}]\} = \frac{1}{2} \cosh \theta \sinh \theta |\phi|^2 (2k_j^2 - a^2 \omega^2) \cos \Gamma 
\]

(35)

There are not contributions coming from the term given by eq. (34), because when the summation \( \sum_{k=-\infty}^{+\infty} \) is performed, a self cancellation is produced.

The terms given by eqs (33) and (35), when the limit to the continuum is taken \( (\sum_k \rightarrow (L/2\pi)^3 \int d^3k) \), are proportional to the integrals:

\[
I_{00} = \int_0^\infty k^2 dk \left\{ \left[ \frac{1}{4} \left( \frac{\dot{W}}{W} + 3H \right) \right]^2 + \omega^2 - W^2 \right\} \cos \Gamma - \left( \frac{\dot{W}}{W} + 3H \right) W \sin \Gamma \cosh \theta \sinh \theta 
\]

(36)

\[
I_{jj} = \int_0^\infty k^2 dk (2k_j^2 - a^2 \omega^2) \cos \Gamma \cosh \theta \sinh \theta 
\]

(37)

(where we used \( d^3k = 4\pi k^2 dk \), and \( \Gamma \) as in eqs (20)). The particle creation is not modified when \( \gamma \) changes, as we can see from eq. (19). Then we can propose a \( \gamma \) phase with the functional form

\[
\gamma = \mu k 
\]

(38)

If we take \( \mu \rightarrow \infty \) and introduce an ultraviolet cutoff (which is less restrictive than the wave packet and can be justified because \( k/a \) is the physical momentum associated to the \( k \) mode, as is shown in ref.[11]) in the integrals (36) and (37) we can then
apply the Riemann-Lebesgue theorem, which is as follows: “If $f$ is a real function absolutely integrable in the interval $[a, b]$, then $\lim_{\gamma \to \infty} \int_a^b f(t) \sin(\gamma t + \delta) dt = 0$ with $\delta$ a real constant”. Clearly this is also true when we have “cos” instead of “sin”. By means of this theorem we can eliminate the interference term in eq. (29), and it results equivalent to the statistical mean value given by eq. (32).

4. Thermal spectrum condition

In order to obtain thermal distribution for the $k$ modes, we need to apply an extremum condition to a thermodinamical potential. This potential will be a function of the energy and the entropy of the bosonic gas in curved space-time. The energy can be calculated by means of the metric hamiltonian $\hat{H}$ in the form

$$E = <0|\hat{H}|0>$$

where

$$\hat{H} = \int a^3 d^3x \hat{T}_{00}$$

in the discretized formulation is $\int d^3x = L^3$. For eq. (29) when the oscillatory term is dropped, we have:

$$E = \sum_k (\frac{1}{2} + n_k) \epsilon_k$$

with

$$\epsilon_k = \frac{1}{2W} \left[ \frac{1}{4} \left( \frac{\dot{W}}{W} + 3H \right)^2 + W^2 + \omega^2 \right]$$

the energy by mode. As the energy for the system studied has the form of a set of decoupled quantum oscillators, it can be considered as a bosonic gas in flat space-time, therefore we can use the expression for the entropy used in Minkowski space. In order to do that we can write the operator (see ref.[12]):

$$\hat{K} = -\sum_k \{ a_k^\dagger(t) a_k(t) \log \sinh^2 \theta - a_k(t) a_k^\dagger(t) \cosh^2 \theta \}$$

The expectation value of the operator given by eq.(41) is the entropy of a bosonic gas [15]:

$$K = <0|\hat{K}|0> = -\sum_k \{ n_k \log n_k - (1 + n_k) \log (1 + n_k) \}$$

In a way analogous to ref.[12] we will get an extremum, with respect to the Bogoliubov angle $\theta$, of the thermodinamical potential

$$\Lambda = -TK + E$$
where $T$ is a Lagrangian multiplier that can be interpreted as the temperature. Then performing the variation
\[ \delta \Lambda / \delta \theta = 0 \]
the following eq. is obtained
\[ T \log \frac{\sinh^2 \theta}{\cosh^2 \theta} + \epsilon = 0. \]

Therefore we get a Planckian spectrum
\[ n_k = \frac{1}{e^{\frac{\epsilon_k}{T}} - 1} \quad (43) \]

(white units so that $\hbar = c = k_B = 1$)

It is interesting to note that we can eliminate the interference term in the Hamiltonian, without using the phase condition given by eq. (38), by means of the criterion known as Hamiltonian diagonalization. As we can see from eq. (33) that condition is
\[ \dot{W} W = -3H \quad (44a) \]
\[ W^2 = \omega^2 \quad (44b) \]

This condition is, moreover, compatible with the dynamic equation for $W$, given by eq. (21). When at the observation time holds eq. (44), $D_{00}[\phi_k, \phi_{-k}] = 0$ is assured. Of course the condition (44) has a meaning when the calculation is “in-out”, with the parameter “t” fixed at the observation time. Then we can think of condition (44) as Cauchy data on $W$ and $\dot{W}$. So when condition (44) is satisfied the energy by mode is (as we can see from eq. (40)).
\[ \epsilon_k = \omega \quad (45) \]

In particular if we suppose that the rest energy can be dropped against the kinetic one, i.e. $k^2/a^2 >> m^2$, we can do $\epsilon_k \simeq k/a$, and obtain
\[ n_k = \frac{1}{e^{\frac{k}{T}} - 1} \quad (46) \]

From eq.(39) going to the continuum set of modes, the particle creation contribution to the energy density is given by
\[ \rho = \frac{1}{2\pi^2 a^4} \int_0^\infty k^3 n_k dk \quad (47) \]
Replacing eq.(46) in eq.(47) and performing the integral, we have

\[
\rho = \frac{1}{2\pi^2} T^4 \int_0^\infty \frac{k'^3}{e^{k'} - 1} dk' = \frac{\pi^2}{30} T^4
\]

which is the expression obtained from the standard phenomenological radiation dominated Friedmann cosmology [16].

5. Conclusions

The VEV of the EMT is different to the statistical mean value, even for the stationary particle creation. In the last case the difference is an interference term between the modes \( k \) and \( -k \), which can be eliminated by a condition on the phase of the Bogoliubov transformation. At the hamiltonian level the coincidence with the statistical value is satisfied by means of the criterion known as diagonalization of the hamiltonian condition (that for scalar field is identical to the condition of minimization of the energy \( \beta \)). By means of this condition also the mean value \(<T_{00}>\), defined by Utiyama-De Witt [1], and the VEV of the EMT are coincident (see eq.(3)).

The coincidence between the VEV and the statistical mean value, is another view of the analogy between the Bogoliubov transformation that connects the different vacua in curved space-time and the one that relates vacua to different temperature in TFD.

From the extremal conditions on the thermodynamical potential, we obtained a thermal spectrum. The fact that particle creation turns to be thermal avoids the existence of infinities coming from this effect, in the source of the Einstein equations. This is true also when the hamiltonian diagonalization conditions are used. In general those conditions produce inconsistencies with the adiabatic regularization [4], [17]. All of that suggested the utilization of the usual covariant regularization techniques, in order to eliminate the infinities coming from the vacuum polarization term, together with some thermodynamic conditions that “regularized” the creation particle term. That approach allowed us to take into account the back-reaction effect in an easy way, without using numerical calculation (see ref.[5]).

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