Meson-meson potential extracted from the tetraquark system

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(Dated: January 1, 2021)

We extract the potential between two heavy mesons from the corresponding tetraquark system. By expanding the tetraquark state in terms of the two independent meson-meson states and taking the sum over all the Casimir scaling potentials between two heavy quarks, we obtain the potential for the tetraquark state. Using the lattice QCD simulated mixing angle between the two color-singlet states and subtracting the inner potentials of the two mesons, we derive the meson-meson potential controlled by the mixing angle.

Among the studies of exotic hadrons which can not be explained as normal mesons and baryons, there are many theoretical works focusing on heavy tetraquark systems \(QQ\bar{Q}Q\) \((Q = c, b)\) \([1,22]\). Recently, a narrow structure around 6.9 GeV, named \(X(6900)\), is observed by the LHCb Collaboration at colliding energy \(\sqrt{s} = 7, 8, 13\) TeV \([23]\). This is the first candidate of fully-heavy tetraquarks observed in experiment.

The molecular picture is an often used mechanism to understand the properties of multi-quark states, for instance the phenomenon that some of the tetraquark states locate just below the threshold of the corresponding two mesons \([9,14,16,17,19–22,31]\). As an approximation to calculate the total potential of the tetraquark state, we consider only the one-gluon-exchange potential between any two quarks, and the total potential is a sum over all such calculations \([5,16,17,19–22,31]\). As an approximation to calculate the total potential of the tetraquark state, we consider only the one-gluon-exchange potential between any two quarks, and the total potential is a sum over all such Casimir scaling potentials \([32]\). For a tetraquark system, considering all the possible combinations, there are two independent meson-meson states, \(|m_1\rangle = |(Q_1\bar{Q}_3)_{1_1},(Q_2\bar{Q}_4)_{1_1}\rangle\) and \(|m_2\rangle = |(Q_1\bar{Q}_4)_{1_1},(Q_2\bar{Q}_3)_{1_1}\rangle\). If one takes only one meson-meson configuration, there will be no interaction between the two mesons at one-gluon-exchange level, see Ref.\([30]\) and the discussion below. In this paper, we consider a general tetraquark system which can be expressed as a combination of the two independent meson-meson states with a mixing angle, calculate the potential for the tetraquark state as a sum over all the Casimir scaling potentials between two heavy quarks, and then extract the meson-meson potential by subtracting the inner potentials of the two mesons. We will focus on the effect of the mixing angle on the meson-meson potential.

From the decomposition in color space, the two independent states \(|s_1\rangle = \|(QQ)\bar{s},(Q\bar{Q})\bar{s}\rangle\) and \(|s_2\rangle = \|(QQ)\bar{s},(Q\bar{Q})\bar{s}\rangle\) form a complete and orthogonal set of color-singlet states. The two meson-meson states can be explicitly expressed in terms of them,

\[
|m_1\rangle = \sqrt{1/3}|s_1\rangle + \sqrt{2/3}|s_2\rangle,

|m_2\rangle = -\sqrt{1/3}|s_1\rangle + \sqrt{2/3}|s_2\rangle. \tag{1}
\]

Note that, different from the two color-singlet states \(|s_1\rangle\) and \(|s_2\rangle\), the two meson-meson states are normalized but not orthogonal to each other. A general tetraquark state can be expanded as a linear combination of the two meson-meson states \([30]\),

\[
|QQ\bar{Q}Q\rangle = \sin \Theta |s_1\rangle + \cos \Theta |s_2\rangle = \sqrt{3/4} \left[ (\cos \Theta/\sqrt{2} + \sin \Theta) |m_1\rangle + (\cos \Theta/\sqrt{2} - \sin \Theta) |m_2\rangle \right], \tag{2}
\]

where \(\Theta\) represents the mixing angle between the two color-singlet states. It is easy to check that, the general tetraquark state \(|QQ\bar{Q}Q\rangle\) is reduced to the meson-meson state \(|m_1\rangle\) at the mixing angle \(\Theta = \Theta_0\) with \(\tan \Theta_0 = 1/\sqrt{2}\) and the other meson-meson state \(|m_2\rangle\) at \(\Theta = -\Theta_0\).

An often used method to investigate multi-heavy-quark systems is the potential model which largely simplifies the calculation \([5,16,17,19,22,31]\). As an approximation to calculate the total potential of the tetraquark state, we consider only the one-gluon-exchange potential between any two quarks, and the total potential is a sum over all such Casimir scaling potentials \([32]\),

\[
V = \sum_{i<j}^4 \frac{(QQ\bar{Q}Q)\lambda_i^a \otimes \lambda_j^a (QQ\bar{Q}Q)}{-16/3} V_c(|r_{ij}|), \tag{3}
\]

where the matrix \(\lambda_i^a\) is defined as \(2T^a\) for quark \(i\) and \(-2(T^a)^*\) for anti-quark \(i\) with \(T^a\) being the Gell-Mann matrices, \(|r_{ij}| = |r_j - r_i|\) is the distance between the two quarks \(i\) and \(j\), and \(V_c(r)\) is the static Cornell potential

\[
V_c(r) = -\frac{\alpha}{r} + \sigma r. \tag{4}
\]

The two parameters \(\alpha\) and \(\sigma\) can be fixed by fitting the charmonium masses in vacuum \([31]\).
Since the Cornell potential depends only on the distance between the two interacting quarks, the motion of the tetraquark system which is controlled by the four-body Schrödinger equation [22] can be divided into a center-of-mass part and a relative part, and the total potential controls the relative motion. In the center-of-mass frame, see Figure 1, we take \( r_{13} = r_3 - r_1, r_{24} = r_4 - r_2 \) and the distance between the two sub-systems \((Q_1 \bar{Q}_3)\) and \((Q_2 \bar{Q}_4)\)

\[
\mathbf{r} = \frac{M_2 \mathbf{r}_2 + M_4 \mathbf{r}_4}{M_2 + M_4} - \frac{M_1 \mathbf{r}_1 + M_3 \mathbf{r}_3}{M_1 + M_3},
\]

as the three independent coordinate vectors, where \(M_i\) is the quark mass for \(i = 1, 2\) and anti-quark mass for \(i = 3, 4\).

The other coordinates can be expressed in terms of them,

\[
\begin{align*}
\mathbf{r}_{12} &= \mathbf{r} + \frac{M_3 \mathbf{r}_{13}}{M_1 + M_3} - \frac{M_4 \mathbf{r}_{24}}{M_2 + M_4}, \\
\mathbf{r}_{14} &= \mathbf{r} + \frac{M_3 \mathbf{r}_{13}}{M_1 + M_3} + \frac{M_2 \mathbf{r}_{24}}{M_2 + M_4}, \\
\mathbf{r}_{32} &= \mathbf{r} - \frac{M_1 \mathbf{r}_{13}}{M_1 + M_3} - \frac{M_4 \mathbf{r}_{24}}{M_2 + M_4}, \\
\mathbf{r}_{34} &= \mathbf{r} - \frac{M_1 \mathbf{r}_{13}}{M_1 + M_3} + \frac{M_2 \mathbf{r}_{24}}{M_2 + M_4}.
\end{align*}
\]

Calculating directly the matrix elements in (3) leads to the total static potential \(V(\mathbf{r}, r_{13}, r_{24})\) for the tetraquark state,

\[
V = \frac{1}{8} (1 - 3 \cos(2\Theta)) \left[ V_1(r_{12}) + V_2(r_{34}) \right] + \frac{1}{16} \left( 7 + 3 \cos(2\Theta) + 6\sqrt{2} \sin(2\Theta) \right) \left[ V_1(r_{13}) + V_2(r_{24}) \right] + \frac{1}{16} \left( 7 + 3 \cos(2\Theta) - 6\sqrt{2} \sin(2\Theta) \right) \left[ V_1(r_{14}) + V_2(r_{32}) \right].
\]

It is clear to see that, at the specific mixing angle \(\Theta = \Theta_0\) or \(\Theta = -\Theta_0\) the tetraquarks state is reduced to a meson-meson state, the total potential contains only the inner potentials of the two mesons, \(V = V_1(r_{13}) + V_2(r_{24})\) at \(\Theta = \Theta_0\) and \(V = V_1(r_{14}) + V_2(r_{32})\) at \(\Theta = -\Theta_0\), and the interaction between the two mesons totally disappear.

The mixing angle \(\Theta\) depends in general case on the three relative coordinates \(\Theta(\mathbf{r}, r_{13}, r_{24})\). To extract the meson-meson potential from the above tetraquark potential as a function of the distance \(r\) between the two mesons \((Q_1 \bar{Q}_3)\) and \((Q_2 \bar{Q}_4)\) at fixed meson sizes \(r_{13}\) and \(r_{24}\), we neglect the dependence on the azimuth angles of \(r\) for \(\Theta\) and \(V\).

This means that, the meson-meson interaction in the following calculation is taken to be radial symmetric. This is true when \(r\) approaches to infinity but only an approximation at finite \(r\). Further more, we keep the azimuth-angle dependence of \(r_{13}\) and \(r_{24}\) for the potential \(V\) but neglect this dependence for the mixing angle \(\Theta\), namely we take \(V(r, r_{13}, r_{24})\) and \(\Theta(r, r_{13}, r_{24})\). These two assumptions are also taken into account in the lattice QCD simulation for the mixing angle \(\Theta\).

With the definition of

\[
\begin{align*}
r_{13} &= r_{13}(\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1), \\
r_{24} &= r_{24}(\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2)
\end{align*}
\]
for the azimuth angles of $r_{13}$ and $r_{24}$, we calculate the averaged potential over the angles

$$\overline{V}(r, r_{13}, r_{24}) = \frac{1}{16\pi^2} \int d\Omega_1 d\Omega_2 V(r, r_{13}, r_{24})$$  \hspace{1cm} (9)$$

with $\Omega_1 = (\theta_1, \phi_1)$ and $\Omega_2 = (\theta_2, \phi_2)$. After a straightforward but tedious calculation, we obtain

$$\overline{V} = \frac{1}{8} (1 - 3 \cos(2\Theta)) [V(r, r_{13}, r_{24}|M^3_{13}, M^2_{24}) + V(r, r_{13}, r_{24}|M^1_{13}, M^4_{24})] + \frac{1}{16} \left(7 + 3 \cos(2\Theta) + 6\sqrt{2}\sin(2\Theta)\right) [V_c(r_{13}) + V_c(r_{24})] + \frac{1}{16} \left(7 + 3 \cos(2\Theta) - 6\sqrt{2}\sin(2\Theta)\right) [V(r, r_{13}, r_{24}|M^3_{13}, M^2_{24}) + V(r, r_{13}, r_{24}|M^1_{13}, M^4_{24})]$$  \hspace{1cm} (10)$$

with

$$V(r, r_{13}, r_{24}|a_1, a_2) = -\frac{\alpha}{r} + \sigma_1 \frac{r_1^2 + r_2^2}{2} + \left\{ \begin{array}{ll}
\frac{r_1(r_1 - r_2)^2}{4a_1a_2r_1r_2} - \sigma_1 \frac{(r_1 - r_2)^4}{24a_1a_2r_1r_2r^2}, & r < r_-
\frac{r_1(r_1 - r_2)^2}{4a_1a_2r_1r_2} - \sigma_1 \frac{(r_1 - r_2)^4}{24a_1a_2r_1r_2r^2}, & r_- \leq r < r_+
0, & r \geq r_+
\end{array} \right.\hspace{1cm} (11)$$

$M^i_{jk} = M_i/(M_j + M_k)$ and $r_{\pm} = |a_1r_{13} \pm a_2r_{24}|$.

By subtracting the inner potentials of the two mesons from the total potential of the tetraquark system, we finally obtain the averaged meson-meson potential $V_{mm}(r, r_{13}, r_{24})$,

$$V_{mm} = V(r, r_{13}, r_{24}) - V_c(r_{13}) - V_c(r_{24}),$$  \hspace{1cm} (12)$$

where $r_{13}$ and $r_{24}$ are the averaged radii of the two mesons.

The key factor to control the tetraquark potential $V$ and the meson-meson potential $V_{mm}$ is now the mixing angle $\Theta(r, r_{13}, r_{24})$. It is difficult to be self-consistently determined in the frame of potential models. We take here the lattice QCD simulated $\Theta$ [30] as a function of $r$ at fixed meson sizes $r_{13} = r_{24} = r_m$, see Figure 2. In the limit of $r \to \infty$ but with finite $r_{13}$ and $r_{24}$, $\Theta$ approaches to $\Theta_0$, and the tetraquark system becomes two free mesons ($Q_1Q_3$) and ($Q_2Q_4$) with potentials

$$\overline{V}(r \to \infty) = V_c(r_{13}) + V_c(r_{24}),$$
$$V_{mm}(r \to \infty) = 0.$$  \hspace{1cm} (13)$$

In the other limit of $r_{13}, r_{24} \to \infty$ but with finite $r$, $\Theta$ approaches to $-\Theta_0$, and the tetraquark system becomes again two free mesons ($Q_1Q_3$) and ($Q_2Q_4$). If we consider a fully-heavy tetraquark system $cc\bar{c}\bar{c}$ or $bb\bar{b}\bar{b}$, there is $M^i_{jk} = 1/2$ and $r_{13} = r_{24}$, the averaged and meson-meson potentials in this limit become

$$\overline{V}(r_{13} \to \infty) = 2\sigma r,$$
$$V_{mm}(r_{13} \to \infty) = 2(\sigma r - V_c(r_{13})).$$  \hspace{1cm} (14)$$

The case of $r \to 0$ but with finite $r_{13}$ and $r_{24}$ is close to the second limit, see Figure 2.

In general case with $0 < r < \infty$, the lattice QCD simulated $\Theta$ [30] at $r_{13} = r_{24} = r_m$ can be well fitted by

$$\Theta(r|r_m) = \frac{\Theta(r|r_m)}{\Theta_0} = \frac{\cos^2 x \sin [\lambda_1 (r - r_m)] + \sin^2 x \sin [\lambda_2 (r - r_m)]}{\cos^2 x \cosh [\lambda_1 (r - r_m)] + \sin^2 x \cosh [\lambda_2 (r - r_m)]},$$  \hspace{1cm} (15)$$

with three parameters $\lambda_1, \lambda_2$ and $x$. The lattice data and the fitted lines with different values of the parameters are shown in Figure 2.

Employing the lattice simulated mixing angle $\Theta(r)$ with different meson size $r_m/\alpha = 3, 4, \cdots, 11$, we calculated the $J/\psi - J/\psi$ potential as a function of the distance $r$ between the two $J/\psi$s. The result is shown in Figure 3 and the corresponding parameters are taken to be $\alpha = 0.5$ and $\sigma = 0.17$ (GeV)$^2$ [31]. Note that, for tetraquark systems $cc\bar{c}\bar{c}$ and $bb\bar{b}\bar{b}$, the mass parameter is $M^i_{jk} = M_i/(M_j + M_k) = 1/2$, and the potential is independent of the value of the
heavy quark mass. Of course, the Schrödinger equation or the wave function of the system in this case depends still on the mass value. Considering the $J/\psi$ size $r_{J/\psi}$ as two times the $J/\psi$ radius $\sim 0.8$ fm, the $J/\psi - J/\psi$ potential is close to the rightmost line with $r_m/a = 11$ in Figure 3. The potential is attractive at small $r$ which bounds the two mesons together, then becomes continuously repulsive around the meson size $r_m$, and finally approaches to zero when the distance is large enough. The numerical result here is similar to a recent calculation where the colorless Pomeron exchange produces the attractive force [33].

In summary, we extracted the meson-meson potential from the quark structure of the corresponding tetraquark system. The two mesons are not treated as color singlet particles, there is color exchange between them, and only the whole tetraquark system is in color singlet state. The color structure of the meson-meson interaction is controlled by the mixing angle between the two independent color singlet states, its behavior as a function of the distance between the two mesons is taken from the lattice QCD simulations.

Acknowledgement: We thank Prof. Lianyi He for helpful discussions in the beginning of this work. The work is supported by the NSFC under grant No. 11890712.

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