Quantization of Scalar Field in the Presence of Imaginary Frequency Modes

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Abstract

Complex frequency modes occur for a scalar field near a rapidly rotating star with ergoregion but no event horizon. Such complex frequency modes must be included in the quantization of the field. As a model for this system, we have investigated a real scalar field with mass \( \mu \) in a one-dimensional square-well potential. If the depth of the potential is greater than \( \mu^2 \), then there exist imaginary frequency modes. It is possible to quantize this simple system, but the mode operators for imaginary frequencies satisfy unusual commutation relations and do not admit a Fock-like representation or a ground state. Similar properties have been discussed already by Fulking for a complex scalar field interacting with an external electrostatic potential.

We are interested in the field dynamics in the physical case where the initial state of the quantum field is specified before the complex frequency modes develop. As a model for this, we investigated a free scalar field whose “mass” is normal in the past and becomes “tachyonic” in the future. A particle detector in the far future placed in the in-vacuum state shows non-vanishing excitations related to the imaginary frequency modes as well. Implications of these results for the question of vacuum stability near rapidly rotating stars and possible applications to other fields in physics are discussed briefly.

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1 Introduction

It has for a long time been known that the ergosphere of a rotating black hole leads to superradiance\[1\]. The quantum counterpart of this classical phenomenon of wave amplification, the so-called Starobinskii-Unruh process\[2, 3\], has been studied, predating Hawking’s discovery of the black hole evaporation\[4\]. It, however, is still unresolved whether the Starobinskii-Unruh effect is primarily due to the existence of the event horizon or the ergoregion.

To resolve this question, Matacz, Davies and Ottewill\[5\] have investigated the quantum vacuum stability of a real massless scalar field near rapidly rotating stars that have an ergoregion but no event horizon. It turned out that the Starobinskii-Unruh effect is absent in this setting if one assumes only real frequency modes occur. This result may indicate that presence of the ergosphere is not sufficient for particle production. However, as mentioned by the authors in Ref.\[5\], the inclusion of complex frequency modes could change the conclusion seriously. In fact, Ashtekar and Magnon\[6\] have given a general argument based on complex structure approach indicating particle production near a star with ergoregion. In the case of rotating black holes, Whiting\[7\] has proved, for massless fields, that complex frequency modes do not occur. On the other hand, in the case of rapidly rotating stars with ergoregions, negative energy could be accumulated within the ergoregion with giving radiation of positive energy to infinity. This process indeed generates complex frequency modes\[8\]; see the appendix for details. These modes are exponentially amplified, reminiscent of a laser, form a discrete set, and may give rise to a novel form of vacuum instability after being quantized. The issue of the ergoregion instability in stars has also been studied by many authors from various points of view classically\[9\], although its quantum counterpart has not as yet been understood well. Therefore, in order to conclude whether or not the vacuum instability occurs near stars with ergoregions, one has to include these complex frequency modes as well and needs to understand their physical role in the problem. The first step will be to learn how to quantize complex frequency modes in general. In addition, these complex frequency modes are also relevant in other physical systems such as leaky optical cavities\[10\], electromagnetic waves in plasma\[11\], α-decay and the associated quantum mechanical tunnelling problems\[12\], compound nucleus theory\[13\], and wave propagation in gravitational systems\[14\].

The quantization of a charged field in some electrostatic potentials including complex frequency modes has been briefly mentioned first by Schiff, Snyder and Weinberg after they discovered the occurrence of those modes – the so-called Schiff-Snyder-Weinberg effect – when the potential becomes strong\[15\]. Schroer and Swieca\[16\] have constructed a formal quantization of a charged Klein-Gordon field with strong stationary external interactions. This quantization of field including such complex frequency modes shows that there is no Fock-like representation, e.g., no normalizable energy eigenstates, a breakdown of the vacuum postulate, and a breakdown of the particle interpretation of the quantum field theory. One specific and simple application of the above formulation was demonstrated by Schroer\[17\] for a free scalar field with a “tachyonic” mass, i.e., $\mu^2 < 0$. Here the author shows how
the inclusion of imaginary frequency modes gives a consistent quantum field theory which is relativistically causal. Fulling [18] made clear the precise relationship of the occurrence of complex frequency modes to the Klein paradox [19]. It has also been shown how particle creation near a rotating black hole can be understood in terms of the Klein paradox [18, 20], assuming complex frequency modes do not occur [21]. Similarly, the quantum instability near a rotating star, if it occurs, may be understood in terms of Schiff-Snyder-Weinberg effect.

In the present paper, we describe the general features of quantization in the presence of complex frequency modes by considering a real scalar field interacting with external scalar potential. The direct application to the system of a star with ergoregion will appear elsewhere. In Sec. 2, the quantization is carried out in the presence of imaginary frequency modes. Many peculiar changes, due to imaginary frequency modes, in the conventional formulation of second-quantization are also shown explicitly. In Sec. 3, two simple models for potentials revealing imaginary frequency modes are given. Possible implications for the issue [5] of vacuum stability near rotating stars and applications to other fields in physics are discussed in Sec. 4. In the appendix, the occurrence of complex frequency modes is reviewed for the case of rapidly rotating stars.

2 Quantization in the Presence of Imaginary Frequency Modes

As seen in the appendix, it is essential to include complex frequency modes as well as real ones in the quantization of fields near rapidly rotating stars. Instead of quantizing fields near stars with ergoregions directly, it will be much easier to consider a simpler system which contains the essential aspect of the problem. In this section, we describe a formulation of quantization for a real scalar field interacting with an external potential in Minkowski flat spacetime, assuming the potential is sufficiently negative so that imaginary frequency modes occur. Let us consider the following system in Minkowski flat spacetime:

\[ L = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + \mu^2 \phi^2 + V(x) \phi^2 \right) \]  

(1)

where \( \mu \) is the mass of the real scalar field \( \phi(x) \) and \( V(x) \) is an external scalar potential. The field equation is then

\[ \frac{\partial^2}{\partial t^2} \phi(x) + (-\nabla^2 + \mu^2 + V(x))\phi(x) = 0. \]  

(2)

If the potential \( V(x) \) is time-independent, solutions are separable in the form of

\[ \phi_{(j)}(x) = \phi_j(x)e^{-i\omega_j t} \]  

(3)

where \( \phi_j(x) \) satisfies

\[ \omega_j^2 \phi_j(x) = (-\nabla^2 + \mu^2 + V(x))\phi_j(x). \]  

(4)
This form of equation, Eq. (1), often appears for free fields in stationary curved spacetime as a radial part. For instance, see Eq. [23] in the appendix. Thus, by studying the simple system above, we are able to understand the essential behavior of fields in curved spacetime as well. Eq. (2) is also analogous to that of electromagnetic waves in an optical cavity with a position dependent dielectric constant $n^2(x)$ [22]. Before going further, let us consider the non-relativistic limit of the above equation. The energy-momentum relation for a single particle corresponding to Eq. (2) will be $E^2 = p^2 + V(x)$, or $(E - \mu)(E + \mu) = p^2 + V(x)$. For a non-relativistic particle $p \ll \mu$ in a weak potential $V \ll \mu^2$, $E \simeq \mu$ and so $E - \mu \simeq p^2/2\mu + (2\mu)^{-1}V(x)$, which is a form of Schrödinger equation with a potential $(2\mu)^{-1}V(x)$. For a strong potential $|V| \gg \mu^2$, one does not recover the usual form above and expects that $E$ becomes pure imaginary when the potential is sufficiently negative [23].

Now note that the frequency $\omega_j$ in Eq. (1) could be either real or pure imaginary depending on the value of the potential $V(x)$. That is, multiplying Eq. (1) by $\phi_j^*$ and integrating it, we find

$$\omega_j^2 = \frac{\int_{\Sigma} d^3x \left[|\nabla \phi_j|^2 + (\mu^2 + V(x))|\phi_j|^2\right] - \oint_{\partial\Sigma} \phi_j^* \nabla \phi_j \cdot d\mathbf{S}}{\int_{\Sigma} d^3x |\phi_j|^2}. \quad (5)$$

Here we assume suitable boundary behavior of the field so that the integration on the spatial boundary in Eq. (3) vanishes [24]. Now it can be easily seen in Eq. (3) that, if the potential $V(x)$ is sufficiently negative, then there could exist mode solutions such that $\omega_j^2$ would be negative and so $\omega_j$ would be pure imaginary, i.e., $\omega_j^* = -\omega_j$. In the case of free fields, e.g., $V(x) = 0$, the frequency must be real provided the boundary terms in Eq. (3) vanish. Note that even plane waves of complex frequencies with dispersion relations $\omega_j^2 = k_j^2 + \mu^2$ satisfy the KG equation. However, the field becomes singular at spatial infinity, which cannot be accepted physically in the free case.

Now let us define a Klein-Gordon inner product induced from the field equation Eq. (4). [18] Multiplying Eq. (4) by $\phi_k^*$ and subtracting the same form for $\phi_k^* \phi_j$ multiplied by $\phi_j$, we obtain

$$(\omega_j - \omega_k^*)(\omega_j + \omega_k^*) \int d^3x \phi_k^* \phi_j = -\oint_{\partial\Sigma} (\phi_k^* \nabla \phi_j - \phi_j \nabla \phi_k^*) \cdot d\mathbf{S}. \quad (6)$$

Assuming suitable boundary conditions in which the right hand side of Eq. (3) vanishes, we see

$$<\omega_j - \omega_k^*, \omega_j + \omega_k^*> = 0 \quad (7)$$

where $<\phi_k, \phi_j> = (\omega_j + \omega_k^*) \int d^3x \phi_k^* \phi_j = i \int_{t=0}^t d^3x \phi_{(k)}^* \frac{\partial}{\partial t} \phi_{(j)}$. Since $\int d^3x \phi_{(k)}^* \frac{\partial}{\partial t} \phi_{(j)}$ is time-independent, we define a Klein-Gordon inner product as follows:

$$<\phi_1, \phi_2> = i \int d^3x \phi_1^* \frac{\partial}{\partial t} \phi_2 \quad (8)$$

at any space-like Cauchy surface. This equation yields non-trivial orthogonality relations in general—the so-called “quasi-orthogonality” [18, 13]. First, notice from Eq. (3) and Eq. (4) that we have four linearly independent mode solutions

$$\phi_{(j)} = (\omega_j, \phi_j), \quad \phi_{(j)}^* = (-\omega_j^*, \phi_j^*), \quad \phi_{(j)} = (\omega_j^*, \phi_j^*), \quad \phi_{(j)}^* = (-\omega_j, \phi_j). \quad (9)$$
for given any mode solution \((\omega_j, \phi_j) \equiv \phi_j(x)e^{-i\omega_j t}\). If \(\omega_j \neq \omega_k^*\), we find from Eq. (7) that \(\phi(j)\) and \(\phi(k)\) are orthogonal:
\[
<\phi(k), \phi(j)> = 0 \quad \text{if} \quad \omega_j \neq \omega_k^*.
\]
For real \(\omega_j\), we define
\[
\epsilon_j = <\phi(j), \phi(j)> = 2\omega_j \int d^3x |\phi_j|^2.
\]
For \(\epsilon_j \neq 0\), therefore, we can normalize \(\phi_j\) so that \(\epsilon_j = \pm 1\) according to the signature of the frequency \(\omega_j\). For imaginary \(\omega_j\), however, we see from Eq. (10) that \(\phi(j)\) is null:
\[
<\phi(j), \phi(j)> = 0.
\]
But, \(\phi(j)\) and \(\phi^*(j)\) are NOT orthogonal, and
\[
\bar{\epsilon}_j = <\phi^*(j), \phi(j)> = 2\omega_j \int d^3x |\phi_j|^2
\]
can be set to \(\pm i\) according to the signature of \(\text{Im} \, \omega_j\) by normalizing \(\phi_j\). According to Eq. (11), the pairs \((\phi(j), \phi(j))\) and \((\phi^*(j), \phi^*(j))\) in Eq. (4) are not necessarily orthogonal for real \(\omega_j\), respectively, since \(\omega_j = \omega_k^*\). By linearly combining them, however, we can always find a set having the same form in which they are orthogonal. Hence we finally have the following “quasi-orthogonality” relations
\[
<\phi(j), \phi(k)> = \epsilon_j \delta(j)(k) \quad \text{for real} \quad \omega_j;
\]
\[
<\phi(j), \phi(j)> = 0, \quad <\phi^*(j), \phi(j)> = \bar{\epsilon}_j \quad \text{for imaginary} \quad \omega_j
\]
where we can normalize the field \(\phi_j\) so that \(\epsilon_j = \pm 1\) and \(\bar{\epsilon}_j = \pm i\) unless they vanish.

The total mode energy that the field mode contains classically is proportional to its norm. Thus, any imaginary frequency mode has zero total mode energy. To see this relationship it is convenient to introduce the two-component formalism[18, 19] of the Klein-Gordon equations. The canonical conjugate \(\pi(x)\) of the field \(\phi(x)\) and the two-component field \(\Phi(x)\) are defined as follows:
\[
\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}, \quad \Phi = \begin{pmatrix} \phi \\ \pi \end{pmatrix}.
\]
Then Eqs. (2) and (4) are respectively equivalent to
\[
i \frac{\partial}{\partial t} \Phi(t, x) = W \Phi(t, x), \quad W \Phi_j(x) = \omega_j \Phi_j(x)
\]
where
\[
W = \begin{pmatrix} 0 & i \\ -i(-\nabla^2 + \mu^2 + V) & 0 \end{pmatrix}, \quad \Phi_j(x) = \begin{pmatrix} \phi_j(x) \\ -i\omega_j \phi_j(x) \end{pmatrix}.
\]
Now we find from Eq. (8) that
\[
<\phi_1, \phi_2> = -\int d^3x \Phi_1^\dagger \sigma_2 \Phi_2 \equiv <\Phi_1, \Phi_2>
\]
where \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). Then we see

\[
<\Phi_{(j)}, W\Phi_{(j)}> = \int_{\Sigma} d^3 x \left[ |\pi_{(j)}|^2 + |\nabla \phi_{(j)}|^2 + (\mu^2 + V)|\phi_{(j)}|^2 \right] - \oint_{\partial \Sigma} \phi_{(j)}^* \nabla \phi_{(j)} \cdot d\mathbf{S}. \tag{17}
\]

Therefore, if the mode solution satisfies suitable boundary conditions such that the surface term above vanishes, then the total energy \( H \) of the classical field mode, which is proportional to the first term in the above equation, will be proportional to the norm of the field mode, i.e., \( \sim <\phi_{(j)}, \phi_{(j)}> \). It implies that the total energy of a real frequency mode is \( \sim \epsilon_j \omega_j \) whereas that of an imaginary frequency mode is zero. However, the energy density is not necessarily zero everywhere for imaginary frequency modes. In fact, one may easily show from Eq. (17) that the mode energy density is exponentially increasing or decreasing, depending on the signature of \( \text{Im} \omega_j \), in time and is negative in a region where the potential is strongly negative. In the case of a rapidly rotating star, the mode energy density for complex frequencies is negative in the ergoregion which is exactly canceled by the positive energy outside as shown in the appendix. This property will also be demonstrated explicitly for the case of a square-well potential in Sec. 3.3. Note also that \( W \) is Hermitian with respect to the inner product defined in Eq. (14). That is, assuming suitable boundary conditions for the solutions, \( <W\Phi_1, \Phi_2>=<\Phi_1, W\Phi_2>(\text{For } \Phi_1 = \Phi_2, \text{this relationship is always satisfied, independent of boundary conditions.}) \). Thus, the norm \( <\Phi_1(t), \Phi_2(t)> \) is time-independent and so is \( <\phi_1(t), \phi_2(t)> \) as mentioned above.

Now the general solution of Eq. (2) will be

\[
\phi(t, \mathbf{x}) = \sum_{\omega_j > 0} (a_j v_{(j)} + a_j^\dagger v_{(j)}^*) + \sum_{\text{Im } \omega_j > 0} (d_j u_{(j)} + d_j^\dagger u_{(j)}^*) + \sum_{\text{Re } \omega_j > 0} (a_j \phi_{(j)} e^{-i\omega_j t} + a_j^\dagger \phi_{(j)}^* e^{i\omega_j t} + \text{Re} \sum_{\omega_j > 0} [(d_j \phi_{(j)} + d_j^\dagger \phi_{(j)}^*) e^{-i\omega_j t} + (d_j^\dagger \phi_{(j)} + d_j \phi_{(j)}^*) e^{i\omega_j t}]. \tag{18}
\]

Note that the time dependence of mode solutions is exponentially increasing or decreasing (\( \sim e^{\pm |\omega_j| t} \)) for imaginary modes. To construct the quantum theory of this real scalar field we now interpret \( \phi(t, \mathbf{x}) \) as an operator-valued distribution with the following equal-time commutation relations

\[
[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0 \tag{19}
\]

where the canonical conjugate \( \pi(t, \mathbf{x}) = \partial \phi / \partial t \) is defined as usual. By using the following relations,

\[
a_j = <v_{(j)}, \phi(x)>, \quad d_j = -i <u_{(j)}^*, \phi(x)>, \quad d_j^\dagger = i <u_{(j)}^*, \phi(x)>, \quad \text{with } a_{-j} = a_j^\dagger, \quad d_{-j} = d_j^\dagger,
\]

we obtain commutation relations among mode operators:

\[
[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0; \quad [d_j, d_k^\dagger] = [d_j, d_k] = [d_j^\dagger, d_k^\dagger] = [d_j^\dagger, d_k] = \cdots = 0. \tag{21}
\]
Note the unusual form of commutation relations among mode operators for imaginary frequencies which result from the quasi-orthogonality properties among mode solutions. By “unusual” we mean that $d_j$ and $d_j^\dagger$ commute but $d_j$ and $d_j^\dagger$ do not. It will be shown below that this property is related to the inverted harmonic oscillator representation of mode operators for imaginary frequencies. We may redefine mode operators for imaginary frequencies in a different way that $d_j' = e^{i\pi/4}d_j$ and $d_j'' = e^{i\pi/4}d_j$. Then, $[d_j', \ d_j''] = [d_j^\dagger, \ d_j'^\dagger] = 1$, and the rescaled mode functions $u_j' = e^{-i\pi/4}u_j$ and $u_j'' = e^{i\pi/4}u_j$ in Eq. (18) will have the following orthogonality: $\langle u_j', u_j'' \rangle = -i\delta_j \rightarrow \pm 1$.

To check if these mode operators have particle interpretation let us consider the Hamiltonian operator for the Lagrangian Eq. (11).

$$H = \frac{1}{2} \int d^3x \left[ \frac{\pi^2}{2} + (\nabla \phi)^2 + (\mu^2 + V)\phi^2 \right].$$

(22)

In terms of mode operators, we find

$$\begin{align*}
H &= \frac{1}{2} \sum_{\omega_j > 0} \omega_j (a_j^\dagger a_j + a_j a_j^\dagger) \\
&\quad + \frac{1}{2} \sum_{\text{Im} \omega_j > 0} i\omega_j (d_j d_j + d_j^\dagger d_j^\dagger + d_j^\dagger d_j + d_j^\dagger d_j).
\end{align*}$$

(23)

Note $H^\dagger = H$ as expected. By linearly transforming into Hermitian operators, e.g., $a_j = (\omega_j^{1/2}Q_j + i\omega_j^{-1/2}P_j)/\sqrt{2}$ with $[Q_j, \ P_j] = i$, one can easily see that the Hamiltonian for real frequency modes has a representation of a set of attractive harmonic oscillators as usual. Thus, the energy spectrum is discrete and bounded below. By defining a vacuum state such that $a_j |0\rangle = 0$ for all $j$, one can construct a Fock space which possesses the conventional particle interpretation. For the mode operators of imaginary frequencies, let us consider following linear transformations,

$$\begin{align*}
d_j &= -\frac{1}{2}i[|\omega_j|^{1/2}q_j + |\omega_j|^{-1/2}p_j] + (|\omega_j|^{1/2}q_j - |\omega_j|^{-1/2}p_j)], \\
d_j^\dagger &= \frac{1}{2}i[|\omega_j|^{1/2}q_j - |\omega_j|^{-1/2}p_j] + (|\omega_j|^{1/2}q_j + |\omega_j|^{-1/2}p_j)].
\end{align*}$$

(24)

Here $q$ and $p$ are Hermitian operators satisfying $[q, \ p] = i$ again. Then, the Hamiltonian operator for imaginary frequency modes can be expressed by

$$\begin{align*}
\sum_{\text{Im} \omega_j > 0} H_j &= \sum_{\text{Im} \omega_j > 0} \frac{1}{2}i\omega_j (d_j d_j + d_j^\dagger d_j^\dagger + d_j^\dagger d_j^\dagger + d_j^\dagger d_j) \\
&= \sum_{\text{Im} \omega_j > 0} \frac{1}{2} (p_j^2 - |\omega_j|^2 q_j^2 + p_j^2 - |\omega_j|^2 q_j^2).
\end{align*}$$

(25)

We find that $H_j$ is a system of two decoupled repulsive harmonic oscillators with frequency $|\omega_j|$. The energy spectrum is continuous then and not bounded below. Thus, it has no ground
state. One cannot define a reasonable vacuum state for $H_j$. In fact, the Hamiltonian $H_j$ has no normalizable eigenstate. We shall show an explicit construction of energy eigenfunctions below which is non-normalizable. Suppose it has an energy eigenstate such that $H \psi_E = E \psi_E$. Notice from Eq. (21) and Eq. (23) that

$$[H, d_j] = -\omega_j d_j, \quad [H, d_j^\dagger] = -\omega_j d_j^\dagger, \quad [H, d_j] = \omega_j d_j, \quad [H, d_j^\dagger] = \omega_j d_j^\dagger.$$  \hspace{1cm} (26)

Thus, $d_j \psi_E >$ and $d_j^\dagger \psi_E >$ and $d_j \psi_E >$ and $d_j^\dagger \psi_E >$ are eigenstates with eigenvalues $(E \pm \omega_j)$, respectively, which are not real any more since $\omega_j$ is imaginary. This fact is not inconsistent with the Hermiticity of the Hamiltonian operator because energy eigenstates are not normalizable ones. All these properties described above, therefore, indicate that imaginary frequency mode operators do not possess a Fock-like space although there is a Hilbert space for them as will be shown below.

Before explicitly constructing a Hilbert space for $H_j$, one finds that it is convenient to define mode operators in a different way, which also makes the connection of imaginary mode operators to the inverted harmonic oscillator representation transparent. Let us first consider an operator $H = \frac{1}{2}(p^2 + \omega^2 q^2)$ where $p$ and $q$ are Hermitian operators satisfying $[q, p] = i$, and $\omega$ is an arbitrary complex number. Then, $H = \frac{1}{2}\omega((\omega^{1/2} q)^2 + (\omega^{-1/2} p)^2) = \frac{1}{2}\omega(a b + b a)$ where $a = (\omega^{1/2} q + i\omega^{-1/2} p)/\sqrt{2}$ and $b = (\omega^{1/2} q - i\omega^{-1/2} p)/\sqrt{2}$ satisfying $[a, b] = 1$. Note that $[H, a] = -\omega a$ and $[H, b] = \omega b$. For a real $\omega$ corresponding to an attractive harmonic oscillator, $a^\dagger = b$ and so $H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a)$ with $[a, a^\dagger] = 1$. For a pure imaginary $\omega = i|\omega|$, which corresponds to a repulsive harmonic oscillator, $a = \sqrt{i}(|\omega|^{1/2} q + |\omega|^{-1/2} p)/\sqrt{2} = i a^\dagger$ and $b = \sqrt{i}(|\omega|^{1/2} q - |\omega|^{-1/2} p)/\sqrt{2} = ib^\dagger \neq a^\dagger$. Let $\phi_j(x) = \phi_j^R(x) + i\phi_j^I(x)$ where $\phi_j^R$ and $\phi_j^I$ are real functions. Then, the field decomposition in Eq. (18) has the following form

$$\psi(t, x) = \cdots + (b_j^R \phi_j^R + c_j^R \phi_j^I)e^{-i\omega_j t} + (b_j^I \phi_j^R + c_j^I \phi_j^I)e^{i\omega_j t} + \cdots$$  \hspace{1cm} (27)

where

$$b_j^\prime = d_j + d_j^\dagger, \quad b_j^\prime = d_j + d_j^\dagger, \quad c_j^\prime = i(d_j - d_j^\dagger) = c_j^\dagger, \quad c_j^\prime = -i(d_j - d_j^\dagger) = c_j^\dagger. \hspace{1cm} (28)$$

Then, $[b_j^\prime, b_j^\prime] = [c_j^\prime, c_j^\prime] = -2i$, otherwise vanishes. By redefining $b_j^\prime = \sqrt{-2ib_j}$ and $c_j^\prime = \sqrt{-2ic_j}$ so that $[b_j, b_j] = [c_j, c_j] = 1$, the Hamiltonian operator $H_j$ in Eq. (25) becomes

$$H_j = \frac{1}{2}\omega_j(b_j b_j + b_j b_j + c_j c_j + c_j c_j).$$  \hspace{1cm} (29)

Now we can easily see that $H_j$ is equivalent to a system of two decoupled repulsive harmonic oscillators with frequency $|\omega_j|$. As a consequence, one also finds that the unusual feature of commutation relations in Eq. (21) for imaginary mode operators is indeed related to a property of the inverted harmonic oscillator system.

By defining a new set of Hermitian operators $x = \sqrt{-ib_j}, p_x = -i\partial_x = -\sqrt{-ib_j}, y = \sqrt{-ic_j},$ and $p_y = -i\partial_y = -\sqrt{-ic_j}$, one can easily find energy eigenfunctions

$$\psi_{em}(r, \varphi) = (2\pi)^{-1}r^{-i\kappa - 1}e^{im\varphi}$$  \hspace{1cm} (30)
where $\varepsilon$ is an arbitrary continuous real number and $m$ an integer. Since $H_j\psi_{em} = \frac{1}{2}\varepsilon|\omega_j|\psi_{em}$, the continuous energy eigenvalue is $E_j = \frac{1}{2}\varepsilon|\omega_j|$. These eigenfunctions are orthogonal

$$\int \psi^*_\varepsilon\psi_m r dr d\varphi = \delta(\varepsilon - \varepsilon')\delta_{mm'},$$

(31)

and are complete

$$\int_{-\infty}^{\infty} d\varepsilon \psi^*_\varepsilon(x)\psi_m(y) = \delta(x - y).$$

(32)

Note from Eq. (31) that energy eigenfunctions are not normalizable. However, one can construct normalizable wave packets from them [17]. These square integrable wave packets form a Hilbert space $H_j$. The Hilbert space for the quantum field $\phi$ is then

$$\mathcal{H} = \mathcal{H}_{\text{Re}} \otimes \prod_j \mathcal{H}_j$$

(33)

where $\mathcal{H}_{\text{Re}}$ is the usual symmetrized Fock space generated by real frequency modes and $\prod_j \mathcal{H}_j$ by imaginary frequency modes which can be either a finite number of products or an infinite number of products depending on how many imaginary frequency modes occur in the system.

### 3 Specific Models

As a simple model for the system, we investigate a real scalar field with mass $\mu$ in a one-dimensional static square-well potential, and explicitly show that a finite number of imaginary frequency modes appear as the depth of the square-well becomes deeper than $\mu^2$. In addition, we will also investigate a free scalar field in a time varying potential since we are interested in the physical case where the initial state of the quantum field is specified before the complex frequency modes develop. For instance, it will be very interesting to study the vacuum stability of quantum fields in the distribution of collapsing matter which finally settles down to a rapidly rotating star with ergoregion.

#### 3.1 Square-well potentials

Now let us apply the formulation constructed above to a specific model in two-dimensional spacetime in which the potential is

$$V(x) = \begin{cases} 
0 & \text{for } |x| > a, \\
-V_0 & \text{for } |x| < a.
\end{cases}$$

(34)

Then, the field equation Eq. (34) becomes

$$\begin{align*}
d^2\phi_j/dx^2 + (\omega_j^2 - \mu^2)\phi_j &= 0 & \text{for } |x| > a, \\
d^2\phi_j/dx^2 + (\omega_j^2 - \mu^2 + V_0)\phi_j &= 0 & \text{for } |x| < a.
\end{align*}$$

(35)
The solutions will be stationary or exponential depending on the sign of the coefficients of the second terms. As boundary conditions, we assume that $\phi_j(x)$ is not singular at $x = \pm \infty$, and that $\phi_j$ and $d\phi_j/dx$ are continuous at $x = \pm a$.25

If the frequency $\omega_j$ is complex, the argument below Eq. (3) shows that it should be pure imaginary. If $V_0 \leq \mu^2$, which also includes the case of a step potential when $V_0 < 0$, then both inside and outside solutions are exponential for imaginary frequencies. Since this class of solutions cannot satisfy the continuity of the first derivative at $x = \pm a$, there is no imaginary frequency mode solution in this case and so the quantization will end up with the Fock space $\mathcal{H}_{Re}$ only. For $V_0 > \mu^2$, however, there are three classes of mode solutions including imaginary frequency ones. Let $\omega_j^2 - \mu^2 + V_0 = k^2$. Hence, $\omega_j^2 - \mu^2 = k^2 - V_0$. Since $\omega_j$ is possibly pure imaginary only, $k^2$ must be real. If $k^2$ is negative (and so is $k^2 - V_0$ as well), there will be no solutions satisfying continuity conditions at $x = \pm a$ by the same reason above. Thus, it is sufficient to consider only real $k$ ranging from $-\infty$ to $\infty$.

(i) If $k^2 > V_0$, then $\omega_j^2 > \mu^2$ and so $\omega_j$ is real. There are two linearly independent solutions $R\phi_k(x)$ and $L\phi_k(x)$ for any continuous $k$ in this range:

$$
R\phi_k(x) = \begin{cases} 
eq \text{e}^{i\sqrt{k^2-V_0}x} + R\text{e}^{-i\sqrt{k^2-V_0}x} & \text{for } x \leq -a, \\
A_R \sin kx + B_R \cos kx & \text{for } -a \leq x \leq a, \\
T_R \text{e}^{i\sqrt{k^2-V_0}x} & \text{for } x \geq a,
\end{cases}
$$

and

$$
L\phi_k(x) = \begin{cases} 
eq \text{e}^{-i\sqrt{k^2-V_0}x} & \text{for } x \leq -a, \\
A_L \sin kx + B_L \cos kx & \text{for } -a \leq x \leq a, \\
T_L \text{e}^{i\sqrt{k^2-V_0}x} & \text{for } x \geq a.
\end{cases}
$$

The coefficients are determined by the continuity conditions at $x = \pm a$. $R\phi_k(x)$ can be regarded as a right-going wave with reflection at $x = -a$. Similarly, $L\phi_k(x)$ is a left-going wave with reflection at $x = a$.

(ii) If $V_0 - \mu^2 \leq k^2 \leq V_0$, then $0 \leq \omega_j^2 \leq \mu^2$ and so $\omega_j$ is still real, but the outside solution must be exponential since $\omega_j^2 - \mu^2 = k^2 - V_0 < 0$. There are even or odd solutions for some discrete values of $k$ in this range:

$$
eq \phi_{k_j}(x) = \begin{cases} \text{e}^{\sqrt{V_0-k_j^2}x} & \text{for } x \leq -a, \\
C_e \cos k_jx & \text{for } -a \leq x \leq a, \\
\text{e}^{-\sqrt{V_0-k_j^2}x} & \text{for } a \leq x,
\end{cases}
$$

and

$$
o \phi_{k_j}(x) = \begin{cases} \text{e}^{-\sqrt{V_0-k_j^2}x} & \text{for } x \leq -a, \\
C_o \sin k_jx & \text{for } -a \leq x \leq a, \\
\text{e}^{-\sqrt{V_0-k_j^2}x} & \text{for } a \leq x.
\end{cases}
$$

Here $k_j$ is determined by continuity conditions at $x = \pm a$ again. For even modes,

$$
k_ja \tan k_ja = \sqrt{V_0a^2 - (k_ja)^2}.
$$

(40)
Thus, depending on $\sqrt{V_0a}$, $\mu a$, and $a$, there could be no $k$ value giving a solution or a finite number of $k$’s which give mode solutions. For odd modes,

$$k_j a \cot k_j a = -\sqrt{V_0a^2 - (k_j a)^2}. \quad (41)$$

(iii) If $0 \leq k^2 \leq V_0 - \mu^2$, then $-(V_0 - \mu^2) \leq \omega_j^2 < 0$ and so $\omega_j$ is imaginary. The form of mode solutions is the same as in case (ii), i.e., Eq. (38)-(39). The equations determining the $k$’s are also the same as in Eq. (40)-(41), but the range of $ka$ is different. Since it includes zero, there always exists at least one even mode solution.

We have shown before that the total classical mode energy is zero for imaginary frequency modes. One can check this property for any imaginary frequency mode solutions. For example, the energies stored outside and inside the square-well for an even mode are

$$H_{\text{OUT}} \sim \sqrt{V_0 - k_j^2} e^{-2\sqrt{V_0 - k_j^2} a} e^{2i\omega_j t}, \quad H_{\text{IN}} \sim -k_j C_e^2 \cos k_j a \sin k_j a e^{2i\omega_j t}. \quad (42)$$

Note that both they are increasing exponentially in time. However, the total energy is

$$H = H_{\text{OUT}} + H_{\text{IN}} \sim (\sqrt{V_0 - k_j^2} - k_j \tan k_j a) e^{2i\omega_j t} \quad (43)$$

which vanishes because of Eq. (40). Note also that $H_{\text{IN}}$ is always negative for imaginary modes as expected from the form of the energy density in Eq. (22).

The field operator in Eq. (18) can be written as follows in this case:

$$\phi(t, x) = \sum_{k^2 > V_0(k > 0)} [(R_{a_k} R_{\phi_k} e^{-i\omega t} + R_{a_k}^* R_{\phi_k}^* e^{i\omega t}) + (L_{a_k} L_{\phi_k} e^{-i\omega t} + L_{a_k}^* L_{\phi_k}^* e^{i\omega t})]$$

$$+ \sum_{V_0 - k_j^2 \leq k^2 \leq V_0} [(e_{a_{k_j}} e_{\phi_{k_j}} e^{-i\omega_j t} + e_{a_{k_j}}^* e_{\phi_{k_j}}^* e^{i\omega_j t}) + (0 a_{k_j}^* 0 \phi_{k_j} e^{-i\omega_j t} + a_{k_j}^* a_{k_j}^* 0 \phi_{k_j}^* e^{i\omega_j t})]$$

$$+ \sum_{0 \leq k^2 \leq V_0 - \mu^2} [(e_{d_{k_j}} e_{\phi_{k_j}} + e_{d_{k_j}}^* e_{\phi_{k_j}}^*) e^{-i\omega_j t} + (0 d_{k_j}^* 0 \phi_{k_j} + d_{k_j}^* d_{k_j}^* 0 \phi_{k_j}^*) e^{i\omega_j t}] \quad (44)$$

where $\omega = +\sqrt{k^2 + \mu^2 - V_0}$ for real frequencies and $\omega_j = +i\sqrt{V_0 - (k_j^2 + \mu^2)}$ for imaginary frequencies. Note that $R_{\phi_k}(x) e^{-i\omega t}$ and $L_{\phi_k}(x) e^{-i\omega t}$ become purely right-going and left-going waves, i.e., $e^{ikx - i\omega t}$ and $e^{-ikx - i\omega t}$, in the limit of $V_0 \to 0$, respectively. Now the quantization of this field simply follows the general scheme described in Sec.2.

### 3.2 Time-varying mass

Instead of studying an “eternal” rotating star with ergoregion, it is more realistic to consider a dynamical rotating star system. That is, the spacetime was almost flat in the past with some matter distribution possessing non-zero total angular momentum. This matter starts to collapse and finally settles down to a rapidly rotating star having ergoregion. In this situation, we are mainly interested in the field dynamics in which the state of the quantum field in the past is specified (for instance, the in-vacuum state) before the complex frequency
modes develop. As a model for this, we shall investigate below a time-varying potential $V(t)$ such that

$$V(t) = -\frac{V_0}{2}(\tanh \rho t + 1) = \begin{cases} 0 & \text{as } t \to -\infty, \\ -V_0 & \text{as } t \to \infty. \end{cases}$$  \hspace{1cm} (45)$$

Separating $\phi(t, x) = e^{ikx}\phi_k(t)$, we find that Eq. (2) yields

$$\frac{d^2\phi_k}{dt^2} + (k^2 + \mu^2 C(t))\phi_k = 0$$  \hspace{1cm} (46)$$

where

$$C(t) = 1 + \mu^{-2}V(t) = (1 - V_0/2\mu^2) - \frac{V_0}{2\mu^2} \tanh \rho t$$

$$= \begin{cases} 1 & \text{as } t \to -\infty, \\ -\mu^{-2}(V_0 - \mu^2) & \text{as } t \to \infty. \end{cases}$$

For $V_0 > \mu^2$, the system is equivalent to a free scalar field starting with normal mass $\mu^2$ and ending up with “tachyonic” mass $-(V_0 - \mu^2) < 0$. Therefore, imaginary frequency modes occur as time evolves. For example, in the far future $t \sim \infty$, all modes satisfying $k^2 + \mu^2 < V_0$ have imaginary frequencies $\omega(k) = \pm i\sqrt{V_0 - (k^2 + \mu^2)}$. Schroer\[17\] has already studied the quantization of a scalar field with tachyonic mass, showing one of remarkable results that the tachyons propagate causally.

Eq. (46) can be solved exactly in terms of hypergeometric functions. The mode solutions which behave like plane waves with positive frequency $\omega_{\text{in}} = \sqrt{k^2 + \mu^2}$ in the remote past are

$$\phi_{\text{in}}^k(t) \sim e^{-i\omega_{\text{in}} t - i\omega_{\text{in}}/\rho \ln(2 \cosh \rho t)} F_1(1 + i\omega - /\rho, i\omega - /\rho; 1 - i\omega_{\text{in}}/\rho; (1 + \tanh \rho t)/2)$$

$$\rightarrow e^{-i\omega_{\text{in}} t} \quad \text{as } \quad t \to -\infty.$$  \hspace{1cm} (47a)$$

On the other hand, the modes which behave like plane waves with frequency $\omega_{\text{out}} = \sqrt{k^2 + \mu^2 - V_0}$ in the far future are

$$\phi_{\text{out}}^k(t) \sim e^{-i\omega_{\text{out}} t - i\omega_{\text{out}}/\rho \ln(2 \cosh \rho t)} F_1(1 + i\omega - /\rho, i\omega - /\rho; 1 + i\omega_{\text{out}}/\rho; (1 - \tanh \rho t)/2)$$

$$\rightarrow e^{-i\omega_{\text{out}} t} \quad \text{as } \quad t \to \infty.$$  \hspace{1cm} (47b)$$

where $\omega_{\pm} = (\omega_{\text{out}} \pm \omega_{\text{in}})/2$. Thus, the asymptotic behavior of $\phi_{\text{out}}^k$ for $k^2 + \mu^2 < V_0$ is that of imaginary frequency modes of $\omega_{\text{out}} = i\sqrt{V_0 - (k^2 + \mu^2)}$. These two complete bases lead to the following two equivalent expansions of the field $\phi(t, x)$

$$\phi(t, x) = \sum_k (a_k u_{\text{in}}^k + a_k^\dagger u_{\text{in}}^{\ast k}) = \sum_k (b_k u_{\text{out}}^k + b_k^\dagger u_{\text{out}}^{\ast k})$$  \hspace{1cm} (47)$$

where $u_k(t, x) = e^{ikx}\phi_k(t)$. Since the Klein-Gordon inner product is time-independent for solutions satisfying suitable boundary conditions at $x = \pm \infty$, we obtain

$$\langle u_{k'}^\text{in}, u_k^\text{in} \rangle |_{t=t} = \langle u_{k'}^\text{in}, u_k^\text{in} \rangle |_{t=-\infty} = \epsilon_k \delta_{k'k},$$  \hspace{1cm} (48)$$
and

\[
\langle u_{k'}^{\text{out}}, u_{k}^{\text{out}} \rangle \big|_{t=t} = \langle u_{0}^{\text{out}}, u_{0}^{\text{out}} \rangle \big|_{t=\infty} = \epsilon_{k} \delta_{k,k'} \\
\langle u_{-k}^{\text{out}}, u_{k}^{\text{out}} \rangle = -i, \quad \langle u_{k}^{\text{out}}, u_{-k}^{\text{out}} \rangle = 0
\]

for \( k^2 + \mu^2 > V_0 \)

Thus, the commutation relations among mode operators are

\[
[a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a_{k'}^\dagger, a_{k'}^\dagger] = 0 \\
[b_k, b_{k'}] = \delta_{kk'}, \quad [b_k, b_{k'}] = [b_{k'}^\dagger, b_{k'}^\dagger] = 0 \\
[b_k, b_{-k}] = -i, \quad [b_{k}, b_{k'}^\dagger] = 0
\]

for all \( k \), \( k^2 + \mu^2 > V_0 \), \( k^2 + \mu^2 < V_0 \).

By using the linear transformation properties of hypergeometric functions [26], we have the following Bogolubov transformations between in and out modes.

\[
u_k^{\text{in}}(t, x) = \alpha_k u_k^{\text{out}}(t, x) + \beta_k u_{-k}^{\text{out}}(t, x)
\]

where

\[
\alpha_k \sim \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(-i\omega_{\text{out}}/\rho)}{\Gamma(-i\omega_+/\rho)\Gamma(1 - i\omega_+/\rho)}, \quad \beta_k \sim \frac{\Gamma(1 - i\omega_{\text{in}}/\rho)\Gamma(i\omega_{\text{out}}/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1 + i\omega_-/\rho)}.
\]

Hence,

\[
b_k = \alpha_k a_k + \beta_k^* a_{-k}^\dagger.
\]

From Eq. (47), we obtain

\[
|\alpha_k|^2 - |\beta_k|^2 = 1, \quad \alpha_k\beta_k^* - \alpha_{-k}\beta_{-k}^* = 0 \quad \text{for} \ k^2 + \mu^2 > V_0,
\]

\[
|\alpha_k|^2 - |\beta_k|^2 = 0, \quad \alpha_k\beta_k^* - \alpha_{-k}\beta_{-k}^* = -i \quad \text{for} \ k^2 + \mu^2 < V_0.
\]

Notice that the second ‘converted’ relations result from the unusual form of the commutation relations among mode operators for imaginary frequencies.

Since the scalar field becomes free as \( t \to -\infty \), there exists a well-defined Fock representation near \( t \sim -\infty \) as usual. Let \( |0\rangle_{\text{in}} \) be a vacuum state defined as \( a_k|0\rangle_{\text{in}} = 0 \) for all \( k \). As seen before, however, there is no well-defined vacuum state in the remote future due to the presence of imaginary frequency modes. Now let us consider a “particle” detector linearly coupled to the field near \( t \sim \infty \) placed in the in-vacuum state \( |0\rangle_{\text{in}} \). The transition probability of the detector from \( |E_0\rangle \) to \( |E'\rangle \) with \( E = E' - E_0 \), in first order perturbation theory, is proportional to the response function \( \mathcal{F}(E) [27] \).

\[
\mathcal{F}(E) = \lim_{T \to \infty} \int_{T}^{T+\Delta T} dt \int_{T}^{T+\Delta T} dt' e^{-iE(t-t')\in} \langle 0|\phi(x(t))\phi(x(t'))|0\rangle_{\text{in}}.
\]

Since \( \in \langle 0|\phi(x(t))\phi(x(t'))|0\rangle_{\text{in}} = \sum_k u_k^{\text{in}}(x)u_k^{\text{in}}(x') \) by using Eq. (17), we see

\[
\mathcal{F}(E) = \lim_{T \to \infty} \sum_k | \int_{T}^{T+\Delta T} dt e^{-iEt} u_k^{\text{in}}(x) |^2.
\]
From the Bogolubov transformations Eq. (49) and the asymptotic behavior of \( u_{\text{out}}^*(x) \) near \( t \sim \infty \), we obtain finally

\[
\mathcal{F}(E) = \lim_{T \to \infty} \sum_{k^2 + \mu^2 > V_0} \frac{1}{4\pi}\Re \left[ \alpha_k \beta_k^{*} \sin \frac{E_{+} \Delta T}{E_+} \sin \frac{E_{-} \Delta T}{E_+} e^{-2i\omega(T+\Delta T/2)} \right] \\
+ \frac{1}{2} \Re \left[ \alpha_k \beta_k^{*} \sin \frac{E_{+} \Delta T}{E_+} \sin \frac{E_{-} \Delta T}{E_-} e^{-2i\omega(T+\Delta T/2)} \right] \\
+ \sum_{k^2 + \mu^2 < V_0(k>0)} \frac{1}{4\pi|\omega|} \left\{ \left| \frac{\sin \frac{E_{+} \Delta T}{E_+}}{E_+} \right|^2 \left[ (|\alpha_k|^2 + |\beta_k|^2)e^{2|\omega|(T+\Delta T/2)} \right] \right\} \\
+ (|\alpha_{-k}|^2 + |\beta_{-k}|^2)e^{-2|\omega|(T+\Delta T/2)} + \frac{1}{2} \Re \left[ \alpha_k \beta_k^{*} \sin \frac{E_{+} \Delta T}{E_+} \sin \frac{E_{-} \Delta T}{E_-} \right] \right\}
\]

where \( \omega = \omega_{\text{out}}(k) = \sqrt{k^2 + \mu^2 - V_0} \) and \( E_{\pm} = E \pm \omega_{\text{out}} \) for real frequency modes and \( \omega_{\text{out}}(k) = \pm i\sqrt{V_0 - (k^2 + \mu^2)} \) and \( E_{\pm} = E \pm i|\omega_{\text{out}}| \) for imaginary frequency modes with \( \pm |k| \) momentum, respectively.

Noticing that \( (\sin \frac{\varepsilon \Delta T}{\varepsilon})^2 / \Delta T \sim \delta(\varepsilon) \) for \( \Delta T \gg 1 \), the transition rate is given by

\[
\frac{\mathcal{F}(E)}{\Delta T} \sim \sum_{k^2 + \mu^2 > V_0} \frac{|\beta_k|^2}{4\pi\omega_{\text{out}}} \delta(E - \omega_{\text{out}}) + \sum_{k^2 + \mu^2 < V_0(k>0)} \frac{|\alpha_k|^2 + |\beta_{-k}|^2}{4\pi\omega_{\text{out}}(E^2 + |\omega|^2)} \frac{e^{2|\omega|\Delta T}}{\Delta T} e^{2|\omega|T} \tag{54}
\]

in the limit of \( T, \Delta T \to \infty \). This result shows non-vanishing excitations of the particle detector related to the imaginary frequency modes as well as the usual contributions due to the positive energy mode mixing in real frequency modes. Furthermore, the contributions related to imaginary frequency modes grow exponentially in \( T \), implying some instability in the future probably due to no boundedness below in the energy spectrum. Note also that the \( \delta \)-function in the first term implies the energy conservation, that is, at first order perturbation theory, only the real frequency field mode whose quantum energy is same as that of the particle detector \( (E = \omega_{\text{out}}) \) can excite the particle detector. For imaginary frequency modes, however, all modes contribute to the excitation possibly because the energy spectrum for any imaginary frequency mode is continuous as shown in Sec. [2].

### 4 Discussion

In this paper, we have shown why one needs to consider complex frequency modes in investigating the vacuum stability for a scalar field near stars with ergoregions, and how the quantization can be formulated in the presence of imaginary frequency modes for a real scalar field interacting with external potentials. As simple systems, we examined a one-dimensional square-well potential and a scalar field with varying mass. We found that it is possible to quantize the field in the presence of imaginary frequency modes, but the mode operators for them do not admit a Fock-like representation and so no particle interpretation. The Hamiltonian operator for imaginary frequencies are equivalent to a system of a set of
two decoupled *repulsive* harmonic oscillators. Therefore, the energy spectrum is continuous, no normalizable energy eigenstate exists, and there is no ground state. The excitation of the particle detector placed in the in-vacuum state for the field with time varying mass has contributions related to imaginary frequency modes as well. The transition rate, however, is not stationary but is exponentially increasing in time.

For the issue of vacuum stability near rapidly rotating stars with ergoregions, these results strongly indicate that there exists the quantum instability as well corresponding to the classical ergoregion instability shown in the appendix. However, its character will be very different from the conventional analysis of the quantum vacuum instability in the case of a rotating black hole because the analysis in that case relies on the existence of vacuum states natural to two asymptotic regions in the past and the future or the existence of two equivalent complete bases of the Hilbert space whereas the inclusion of imaginary frequency modes does not admit a Fock-like representation or a vacuum state. Non-vanishing $\beta$-coefficients in the Bogolubov transformations do not give a direct interpretation of particle creation for imaginary frequency mode operators, not only because the particle interpretation breaks down, but also because any energy eigenstate for imaginary frequency modes is non-normalizable. For some cases, however, one can use Unruh’s “particle” detector model to extract some useful physics as we have shown in the previous section for a scalar field with time varying mass. For general cases, it would perhaps be preferable to take the point of view that the fundamental object in quantum field theory is the field operator itself, not the “particles” defined in a preferred Fock space[28, 29]. For example, the expectation value of the energy-momentum tensor should be a meaningful quantity. However, renormalization of the energy-momentum in the presence of complex frequency modes would have to be understood first. As far as we know, this interesting issue has never been addressed. A direct application of the present work to the issue of quantum instability near a star with enoregion will appear elsewhere.

It is also an interesting issue how to construct a quantum field theory in the presence of complex frequency modes by using the algebraic approach. To obtain a quantum description of fields in this approach, one first defines the $\ast$-algebra of field operators and then constructs the Hilbert space of states by choosing an appropriate $\ast$-representation, equivalently a suitable complex structure, of this $\ast$-algebra with a set of rules for dynamics. The most difficult part in this prescription is to single out the ‘correct’ representation among all possible $\ast$-representations. In Ref. [3], Ashtekar and Magnon have shown how certain physically motivated requirements select a unique complex structure and so the ‘correct’ representation. In the presence of complex frequency mode solutions, however, it is not clear at present stage whether or not this algebraic approach is extendable. It is not only because the field grows unboundedly in time due to the imaginary part of the complex frequency so that the very construction of the $\ast$-algebra itself would break down, but also because the inner product becomes indefinite as shown in Eq. (13) so that it is unclear whether one can find a complex structure compatible with the symplectic structure. Of course, the field would not blow up in the realistic system. In other words, if we also include the dynamics of the external system which is producing the external potential, then the field will stop the exponential increase
in time when the external potential loses its energy into the field and becomes smaller than a certain critical value, for example, when $V_0 < \mu^2$ in the case of square-well potential and when the ergosphere disappears in the case of a rapidly rotating star as mentioned by the authors in Ref. [6].

There are many other fields in physics in which complex frequency modes play important roles. Generically, if a system stores some “free” energy which can be released through interactions, then some amplifications occur, revealing complex frequency modes classically. In a system of plasma, for instance, a small perturbation of electric field exponentially increases in time if the phase velocity of the perturbed field is smaller than the velocity of charged particles, and is damped in the opposite case. The energy stored in plasma is released quickly by a small perturbation, giving complex frequency modes[11]. In a tunable laser, the energy stored in dielectric material amplifies an incident light and results in the intensity increasing of the output laser beam. In a field theoretic treatment of the system, the dielectric material plays the role of a source producing an external potential and it is possible for complex frequency modes to occur under suitable conditions[20]. In the theory of linear quantum amplifiers[31], one assumes a time dependent annihilation operator, $a(t) = a(0)e^{W t/2 - i\omega t}$ with a gain factor $W$. This gain factor $W$ may be interpreted simply as coming from the imaginary part of a complex frequency mode in the second quantization scheme where one does not need to assume the non-unitary evolution of the mode operator. In addition, the repulsive harmonic oscillator representation appeared in the quantization formalism in this paper may be able to explain the model of the inverted oscillator amplifier[32] and the initial stages of the superfluorescence process modeled by Glauber and Haake[33] in a more fundamental sense. Therefore, the quantization formalism described at the present work may be useful to understand those phenomena in the context of quantum field theory.

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Appendix: Occurrence of Complex Frequency Modes

In this appendix we will basically follow Vilenkin’s demonstration showing the generation of exponentially amplified waves, which is related to the occurrence of complex frequency modes, in the background metric of rapidly rotating stars. The physical interpretation of the demonstration will be the following. An incoming radial wave in a superradiant mode will be scattered by the potential barrier near the ergosurface with a reflection coefficient bigger than unity. The transmitted wave will be trapped inside the ergoregion by passing through the center of the rotating star and by being reflected at the ergosurface repeatedly. At each reflection at the ergosurface, the transmitted waves will be amplified and escape to
infinity. Thus, the total energy of outgoing waves looks like an exponential amplification of that of the initial ingoing wave, reminiscent of laser action.

The system we consider is a massless real scalar field $\phi(x)$ satisfying the Klein-Gordon equation

$$\Box \phi = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi) = 0$$

in the spacetime of a rotating star with ergosphere but no event horizon. Thus, the system is stationary and the spacetime outside the surface of the rotating star is described by the Kerr metric. The inside of the star body is assumed to be regular. We assume that the surface of the star body is near the outside of the corresponding event horizon, if it would have existed, of the Kerr metric.

Since the Kerr metric has two Killing vector fields (i.e., $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial \varphi)^a$ in Boyer-Linquist coordinates $(t, r, \theta, \varphi)$) and Carter’s constant from $\nabla_{(a} K_{bc)} = 0$, Eq. (55) is separable and admits a complete set of solutions of the form

$$\phi(x) = e^{-i\omega t + im\varphi} U_{\omega lm}(r) S_{\omega lm}(\theta).$$

Then, Eq. (55) becomes, outside the rotating star body,

$$\frac{\partial}{\partial r} (\Delta \frac{\partial \phi}{\partial r}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \phi}{\partial \theta}) - m^2 \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \phi - 4am\omega \frac{Mr}{\Delta} \phi + \omega^2 \left[ \frac{r^2 + a^2}{\Delta} - a^2 \sin^2 \theta \right] \phi = 0$$

where $\Delta = r^2 - 2Mr + a^2$, and $M$ and $a$ are the total mass of the rotating star and the angular momentum per unit mass, respectively. $l$ and $m$ are integers satisfying $|m| \leq l$. Defining $U_{\omega lm}(r) = R_{\omega lm}(r)/(r^2 + a^2)^{1/2}$, the radial part of Eq. (57) yields

$$\frac{d^2 R_{\omega lm}}{dr^*^2} - V_{\omega lm}(r) R_{\omega lm} = 0$$

where the $r^*$ is a “generalized” tortoise coordinate defined by $dr^*/dr = (r^2 + a^2)/\Delta$. Since $\Delta(r) \to 0$ as $r$ approaches the horizon radius $r_+ = M + \sqrt{M^2 - a^2}$, one can easily see $r^* \to -\infty$ as $r$ decreases to $r_+$.

The asymptotic behavior of the effective potential $V_{\omega lm}(r)$ induced through the interaction with the gravitational field is as follows:

$$V_{\omega lm}(r) \sim \begin{cases} -\omega^2 & \text{as } r^* \to \infty \\ -(\omega - m\Omega_H)^2 & \text{as } r^* \to r_0^* \end{cases}$$

Here the $r_0^*$ corresponds to $r = r_0$ ( $\gtrsim r_+$ and so $r_0^* \sim -\infty$) at which the surface of the rotating star is located. $\Omega_H$ is defined as $\Omega_H = a/(2Mr_+)$, which is the angular velocity of the horizon of the Kerr metric. Note that, between two asymptotic regions, there exists a potential hump which grows as $l$ increases. Note also that the left asymptotic value $-(\omega - m\Omega_H)^2$ varies from 0 to $-\infty$ as $m\Omega_H$ changes. In particular, a deep potential well is produced for a big value of $m\Omega_H$ which leads to the classical superradiance mode.
The behavior of this potential will not change much when it crosses the surface of the rotating star body. However, we put a totally reflecting mirror on the surface of the star. This reflection boundary condition is justified if the scalar field does not interact with the matter of the rotating star body. That is, an ingoing spherical wave will cross the surface of the star, pass the center, and go back to the outside of the star without being changed much. We achieve this reflection boundary condition on the star surface by setting an infinite potential wall at \( r^* = r_0^* \). Therefore, any scalar field satisfying this reflection boundary condition will vanish at the reflecting mirror, i.e., \( \phi(x) = 0 \) at \( r^* = r_0^* \). In addition to this inside boundary condition, we also assume that the field does not blow up at infinity.

Now, the question arises as to whether or not there exist complex frequency modes in the geometry described above. In the case of a rotating black hole, Detweiler and Ipser have made a numerical search showing that complex frequency modes of \( \text{Im } \omega > 0 \) do not exist. After many years, Whiting finally proved this analytically for massless fields. On the other hand, in the case of an ultrarelativistic rotating star where an ergosphere appears and a reflecting mirror boundary surface is assumed, the occurrence of complex frequency modes has been shown by Vilenkin. He basically shows how the existence of the reflection boundary condition produces the exponential amplification of the waves in superradiant modes. The issue on the ergoregion instability has also been studied by many authors from various points of view.

Let us first consider the rotating black hole case. From the asymptotic form of the potential in Eq. (59), we can easily construct the linearly independent solutions \( R_{\omega lm}^\pm \) of Eq. (58) whose asymptotic forms are

\[
R_{\omega lm}^+ \sim \begin{cases} B^+ e^{-i\tilde{\omega}r^*} & \text{as } r^* \to -\infty \\ e^{-i\omega r^*} + A^+ e^{i\omega r^*} & \text{as } r^* \to \infty \end{cases}
\]

and

\[
R_{\omega lm}^- \sim \begin{cases} e^{i\tilde{\omega}r^*} + A^- e^{-i\tilde{\omega}r^*} & \text{as } r^* \to -\infty \\ B^- e^{i\omega r^*} & \text{as } r^* \to \infty \end{cases}
\]

where \( \tilde{\omega} = \omega - m\Omega_H \) and \( \omega > 0 \). As one can see, \( R_{\omega lm}^\pm \) are the waves originating at infinity and at the horizon, respectively. From Eq. (58), we obtain

\[
\frac{d}{dr^*}(R_1 \frac{dR_2}{dr^*} - R_2 \frac{dR_1}{dr^*}) = (V_2 - V_1)R_1 R_2
\]

for any two solutions \( R_{\omega l_1 m_1} \) and \( R_{\omega l_2 m_2} \). When \( V_2 = V_1 \), the Wronskian \((= R_1 R_2' - R_2 R_1')\) is constant, and then it leads to the relations

\[
1 - |A^+|^2 = \frac{\tilde{\omega}}{\omega} |B^+|^2, \quad 1 - |A^-|^2 = \frac{\omega}{\tilde{\omega}} |B^-|^2, \\
\omega B^- = \tilde{\omega} B^+, \quad A^{**} B^- = -\frac{\tilde{\omega}}{\omega} A^- B^{**}.
\]

It can be easily seen from the above equations that \( |A^+|^2 > 1 \) and \( |A^-|^2 > 1 \) for \( \tilde{\omega} < 0 \)(i.e., \( \omega < m\Omega_H \)), which are the so-called superradiant modes. This property indicates that, if an
ingoing wave packet sharply peaked in frequency in the range of superradiance is scattered off the black hole, an amplified fraction $|A^+|^2$ will be reflected back to infinity with being amplified and a fraction $|A^+|^2 - 1 = -\frac{\omega}{\bar{\omega}}|B^+|^2$ will be transmitted through the ergoregion and absorbed into the event horizon finally.

In the case of a rotating star, however, this transmitted wave becomes an outgoing wave after passing through the center of the star and will then scatter at the potential barrier in the ergoregion. The reflected fraction this time will be $|A^-|^2$ and $|A^-|^2 - 1 = -\frac{\omega}{\bar{\omega}}|B^-|^2$ for the transmission. This process will be repeated presumably until all rotational energy of the star is released to infinity. Assuming the energy of the incident wave to be $E_0$, the total energy escaped to infinity is

$$E_{\text{OUT}} = E_0(|A^+|^2 + |B^-B^+|^2 \sum_{n=0}^{\infty} |A^-|^{2n}).$$

(64)

The energy accumulated in the ergoregion equals

$$E_{\text{IN}} = E_0 \frac{\omega}{\bar{\omega}} |B^+|^2 \lim_{n \to \infty} |A^-|^{2n}.$$

(65)

For non-superradiant modes $\bar{\omega} > 0$, the above two equations are still satisfied and we find $E_{\text{IN}} = 0$ and $E_{\text{OUT}} = E_0$ since $|A^-| < 1$ and $\sum_{n=0}^{\infty} |A^-|^{2n} = (1 - |A^-|^2)^{-1}$. On the other hand, if $\bar{\omega} < 0$, i.e., superradiant modes, then $|A^-| > 1$ and we find both $E_{\text{IN}}$ and $E_{\text{OUT}}$ diverge. However, the sum is always the incident energy, $E_{\text{IN}} + E_{\text{OUT}} = E_0$. If $t_0$ is the time interval between two successive reflections at the mirror $r^* = r_0^*$, then the radiated power measured at infinity is

$$\frac{dE}{dt} \sim |A^-|^{2t/t_0} = e^{t/\tau}$$

(66)

where the “e-folding” time is $\tau = t_0/2 \ln |A^-|$. Assuming $t_0 \sim 2|r_0^*|/c$,

$$\tau \simeq |r_0^*|/\ln |A^-|.$$  

(67)

Note that $\tau \to \infty$ as $\bar{\omega} \to 0$, i.e., $|A^-| \to 1$, and so the amplification of waves becomes very weak.

This effect of exponential amplification in the presence of the reflection boundary strongly suggests that there exist unstable mode solutions of Eq. (53) which grow exponentially in time. In fact, there exists such an unstable mode solution in the presence of the reflection boundary condition which is purely outgoing at infinity like Eq. (53) up to an overall scaling. Since such a mode solution should not blow up at infinity (in order to be included in standard approach to quantization), it must satisfy the following two boundary conditions, including the one on the surface of the star,

$$R_{\omega lm}(r^* = r_0^*) = 0; \quad |R_{\omega lm}(r^* \to \infty)| < \infty.$$  

(68)

From the first condition, we find

$$e^{i\bar{\omega}r_0^*} + A^- e^{-i\bar{\omega}r_0^*} = 0.$$  

(69)
Thus, defining $A^- = |A^-| e^{i\delta}$,

$$\omega = m\Omega_H + \frac{(2n-1)\pi - \delta}{2r_0^*} + i\frac{\ln |A^-|}{-2r_0^*}. \tag{70}$$

We find that the imaginary part of the frequency $\omega$ is non-zero:

$$\omega_I = \text{Im} \omega = \ln |A^-|/(-2r_0^*). \tag{71}$$

If $|A^-| < 1$, then $\omega_I < 0$ and we see the boundary condition at infinity in Eq. (68) is not satisfied. Thus, only the “superradiant” modes ($|A^-| > 1$) as viewed from inside the ergosurface give satisfactory solutions. Since the time dependence of the unstable mode solutions are $\phi_\omega \sim e^{-i\omega t} = e^{-i\omega_R t e^{-i\omega t}}$, the radiated power to infinity will be proportional to $e^{2\omega_I t} = e^{t/\tau}$, and thus we obtain $\tau = (2\omega_I)^{-1} = -r_0^*/\ln |A^-|$, which agrees with Eq. (67).

According to Eq. (71) $\omega_I$ diverges as $|A^-| (> 1)$ increases. It, however, can be shown that $\omega_I$ is bounded. By multiplying Eq. (57) by $\phi^*$ and integrating over $r$ and $\theta$, we obtain

$$\int_{r_0}^{\infty} \int_0^\pi \sin \theta dr d\theta \left\{ \Delta |\partial \phi/\partial r|^2 + |\partial \phi/\partial \theta|^2 - \frac{m^2}{r^2 \sin^2 \theta} (\Delta - a^2 \sin^2 \theta) + \frac{4amM}{r} \omega - \frac{\omega^2}{r^2} ((r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta) \right\} \frac{r^2}{\Delta} |\phi|^2 = 0$$

since the boundary terms vanish at $r = r_0$, $\infty$ and $\theta = 0, \pi$. First notice that $\omega$ must be real for $m = 0$ in the above equation. Using $|(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta)/r^2 \geq r_+^2 + a^2 + 2Ma^2/r_+$ for $r \geq r_+$, we see

$$m\omega_R \geq 0, \quad 0 \leq |\omega_R| \leq |m|\Omega_H \tag{72}$$

and

$$\omega_I^2 \leq (1 + a^4/r_+^4)\omega_R^2 + 2mM(m + 2a\omega_R)/r_+^3. \tag{73}$$

Therefore, unstable eigenfrequencies are confined in a bounded region.

We have seen that unstable modes with complex frequencies occur in the region of superradiance for a massless scalar field when a reflection boundary condition is assumed somewhere inside the ergoregion, which can be regarded as a rapidly rotating star. In the case of rotating black holes, no such complex frequency modes exist. For a small negative energy perturbation in the ergoregion is simply absorbed by the event horizon so that the outgoing positive energy flow is not amplified any more. In contrast, for a rotating star, the negative energy is trapped inside the ergoregion giving amplified positive energy flows to infinity repeatedly, and so revealing an exponential amplification. Of course, the negative energy inside amplifies itself to conserve energy. This process leads to instability and gives unstable modes whose frequencies are complex.

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[19] The Klein paradox is a paradox in the context of a single particle theory. In a many particle theory such as the quantum field theory, it is merely a pair creation of particles when an external potential becomes strong enough to provide at least the rest mass energy of the created pair of particles. By considering a step potential enclosed in a box, Fulling clearly shows how the Klein and Schiff-Snyder-Weinberg effects appear
and how they are continuously related as the potential depth is deepened. For a brief explanation of the Klein paradox, see the p. 515 in Lectures on Quantum Mechanics by Gordon Baym, (The Benjamin/Cummings Publishing Company, Massachusetts, 1969).

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[21] This assumption has indeed been proved by Whiting\[7\].

[22] In fact, the one-dimensional Maxwell wave equation \[n^2(z)\partial_t^2 - \partial_z^2]E(z,t) = 0\[10\] can be related to Eq. (2) for massless fields in one dimension by a transformation \(E(z,t) = n^{-1/2}(z)\phi(x,t)\) and \(dx/dz = n(z)\) with the potential \(V(x) = -n^{-3/2}\partial_z^2n^{-1/2}\[14\].

[23] In a Hydrogen atom, the energy spectrum from Dirac’s equation indeed shows that the ground state energy becomes imaginary as the atomic number \(Z\) is larger than \(\alpha^{-1} \approx 137\), e.g., \(E_g = mc^2\sqrt{1 - (Z\alpha)^2}\).

[24] By suitable boundary behavior, we mean that the spatial part of solutions has a compact support, is periodic, or goes to zero sufficiently fast as one approaches spatial infinity.

[25] Since we are assuming a non-singular potential well around \(x = \pm a\), the first derivative of \(\phi_j(x)\) must be continuous.

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