A Rigorous Analysis of the Superconducting Phase of an Electron - Phonon System

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**Introduction**

In these notes we give a very rough sketch of nonperturbative methods of many body quantum field theory that are powerful enough to rigorously control weak coupling instabilities in condensed matter physics, for example, the Cooper instability in an electron-phonon system.

We restrict our attention to a $d \geq 2$ dimensional model for $\ell = 0$ superconductivity. Precisely, to the model formally characterized by the Euclidean action

\[ A(\psi, \psi) = \int dk \left( i k_0 - e(k) \right) \bar{\psi}_k \psi_k + \frac{\lambda}{2} \int \prod_{i=1}^4 dk_i \left( 2\pi \right)^{d+1} \delta(k_{1+k_2-k_3-k_4}) \bar{\psi}_{k_1} \psi_{k_3} \langle k_1, k_2 | V | k_3, k_4 \rangle \bar{\psi}_{k_2} \psi_{k_4} \]

where $e(k) = \frac{k^2}{2m} - \mu$, $\lambda > 0$. In these expressions, the electron fields are vectors $\psi_k = (\psi_{k,\uparrow}, \psi_{k,\downarrow})$ and $\bar{\psi}_k = (\bar{\psi}_{k,\uparrow}, \bar{\psi}_{k,\downarrow})$ whose components $\psi_{k,\sigma}$, $\bar{\psi}_{k,\sigma}$ are generators of an infinite dimensional Grassmann algebra over $\mathbb{C}$. That is, the fields anticommute with each other. The generating functional for the associated connected, amputated Euclidean Green's functions is

\[ S(\phi, \bar{\phi}) = \log \frac{1}{Z} \int e^{-A(\psi + \phi, \bar{\psi} + \bar{\phi})} \prod_{k,\sigma} d\psi_{k,\sigma} d\bar{\psi}_{k,\sigma} \]

It is assumed that the reduced interaction $\langle s', -s' | V | t', -t' \rangle$ is attractive and dominant in the zero angular momentum sector. Here, $k' = (0, \frac{k}{|k|} k_F)$ is the projection of $k = (k_0, k) \in \mathbb{R}^{d+1}$ onto the Fermi surface. In more detail, expanding in spherical harmonics,

\[ -\lambda \langle s', -s' | V | t', -t' \rangle = \sum_{\ell \geq 0} \lambda_{\ell(0)} \pi_{\ell}(s', t') \]

our assumption becomes $\lambda_{0(0)} > 0$ and $\lambda_{0(0)} \gg \lambda_{\ell(0)}$, $\ell \geq 1$.

We now precisely formulate one of our goals. For each $L > 0$, let $d\mu_L$ be the Grassmann Gaussian measure over the torus $\mathbb{R}^{d+1}/L \mathbb{Z}^{d+1}$ whose covariance is the multiplication operator

\[ \frac{1}{i k_0 - e(k)} \left( 1 - \delta_{k_0,0} \delta_{e(k),0} \right) \]

on $\ell^2(\frac{2\pi}{L} \mathbb{Z}^{d+1})$. For convenience, let

\[ \int dk f(k) = \left( \frac{2\pi}{L} \right)^{d+1} \sum_{k \in \frac{2\pi}{L} \mathbb{Z}^{d+1}} f(k) \]

The finite volume action $A_{L,r}(\psi, \bar{\psi})$ is given by

\[ A_{L,r}(\psi, \bar{\psi}) = A(\psi, \bar{\psi}) + \delta \mu(\lambda, \mu; L, r) \int \bar{\psi}_k \psi_k - r \int \bar{\psi}_{k\uparrow} \psi_{-k\downarrow} + \psi_{-k\downarrow} \psi_{k\uparrow} \]
where there is a counterterm for the chemical potential and a small external field that ultimately selects a pure phase. The corresponding finite volume generating functional is

$$S_{L,r}(\phi, \bar{\phi}) = \log \frac{1}{Z} \int e^{-A_{L,r}(\psi, \bar{\psi})} \prod_{k,\sigma} d\psi_{k,\sigma} d\bar{\psi}_{k,\sigma}$$

We are strongly convinced that the tools are at hand to give a completely rigorous proof of the

**Theorem.** Let $d = 2, 3$ and let $\langle k_1, k_2|V|k_3, k_4 \rangle$ be a sufficiently regular, real function on $\mathbb{R}^{4(d+1)}$ satisfying

$$\langle Rk_1, Rk_2|V|Rk_3, Rk_4 \rangle = \langle k_1, k_2|V|k_3, k_4 \rangle$$
$$\langle Tk_1, Tk_2|V|Tk_3, Tk_4 \rangle = \langle k_1, k_2|V|k_3, k_4 \rangle$$
$$\langle k_1, k_2|V|k_3, k_4 \rangle = \langle -k_3, k_2|V| -k_1, k_4 \rangle$$
$$= \langle k_1, -k_4|V|k_3, -k_2 \rangle$$

for all $R \in O(d, \mathbb{R})$ where $Rk = (k_0, Rk)$, and $Tk = (-k_0, k)$. Let

$$-\lambda \langle s', -s'|V|t', -t' \rangle = \sum_{\ell \geq 0} \lambda_\ell(0) \pi_\ell(s', t')$$

be the expansion of the rotation invariant reduced interaction in spherical harmonics. Fix $\varepsilon > 0$. Let $\lambda > 0$ and $\varepsilon' > 0$ be sufficiently small. If $\lambda_0(0) > 0$ and $\varepsilon'\lambda_0(0) > \lambda_\ell(0)$, $\ell \geq 1$ , then the limit

$$S(\phi, \bar{\phi}) = \lim_{r \downarrow 0} \lim_{L \uparrow \infty} S_{L,r}(\phi, \bar{\phi})$$

exists and has the following properties:

(i) (U(1) Symmetry Breaking) There is a $\Delta > 0$ with $\Delta = \text{const} e^{-\text{const} \frac{\lambda}{\varepsilon}}$ such that

$$\langle \psi_{k'\uparrow} \psi_{-p\downarrow} \rangle = \langle \bar{\psi}_{-k'\downarrow} \bar{\psi}_{p\uparrow} \rangle = -\frac{(2\pi)^{d+1}}{\Delta} \delta(k' - p)$$

(ii) The $2n$ point functions of $S(\phi, \bar{\phi})$ with $n$ odd decay exponentially at a rate at least $(1 - \varepsilon)\Delta$.

(iii) (Goldstone Boson) The $2n$ point functions of $S(\phi, \bar{\phi})$ with $n$ odd decay at least polynomially. In particular, there are constants $c_1, c_2 > 0$ such that

$$\int ds dt dp \langle \bar{\psi}_{s+q\uparrow} \bar{\psi}_{s\downarrow} + \psi_{s+q\downarrow} \psi_{s\uparrow}; \bar{\psi}_{t-p\downarrow} \bar{\psi}_{t\uparrow} + \psi_{t+p\downarrow} \psi_{t\uparrow} \rangle = -\frac{1}{c_1 q_0^2 + c_2 q^2} + O(1)$$
In other words, there is a channel in the four point function, due to the $U(1)$ Goldstone boson, that does not decay exponentially fast.

For simplicity, we have stated a fragment of a more complete theorem.

In order to systematically investigate the long range behavior of correlation functions at low temperature, it is natural to use a renormalization group analysis ([FT2],[FT3]) near the Fermi surface. This entails slicing the free propagator around it singularity on the Fermi sphere. The renormalization group generates an effective slice-dependent interaction.

To analyze the ultraviolet and infrared behavior of a relativistic Euclidean field theory, one defines a momentum $k$ to be of scale $j$ if $|k| \approx 2^j$. Here, 2 is just a fixed constant that determines the scale units. As $j \to \infty$, the momentum $k$ approaches the ultraviolet end of the model. As $j \to -\infty$, $k$ approaches the infrared end of the model.

In non-relativistic solid state physics the natural scales consist of finer and finer shells around the Fermi surface. For each negative integer $j = 0, -1, -2, \ldots$ the $j$-th slice contains all momenta in a shell of thickness $2^j$ a distance $2^j$ from the singular locus

$$\left\{ k \in \mathbb{R}^{d+1} \mid k_0 = 0, \ |k| = \sqrt{2m\mu} \right\}$$

The propagator for the $j$-th slice is

$$C^j(\xi_1, \xi_2) = \delta_{\sigma_1, \sigma_2} \int dk \frac{e^{i(k_0\xi_1 - k_0\xi_2)}}{i k_0 - e(k)} 1_j(k_0^2 + e(k)^2)$$

(1)

where $1_j(k_0^2 + e(k)^2)$ is the characteristic function for the set $2^j \leq |i k_0 - e(k)| < 2^{j+1}$.

For simplicity, we have introduced a sharp partition of unity even though a smooth one is required for a complete, technically correct analysis [FT3,II.1]. Summing over $j \leq 0$, we obtain the full infrared propagator $C(\xi_1, \xi_2) = \sum_{j \leq 0} C_j(\xi_1, \xi_2)$. The full Schwinger functions are obtained by assigning each line of each Feynman diagram a scale $j$ and then summing over all such assignments.

Each single scale propagator (1) is supported in momentum space on a $d+1$ dimensional manifold with boundaries. The natural coordinates for this manifold are $k_0$, $\eta = e(k)$ and $k' = \sqrt{2m\mu} \frac{k}{|k|}$. In these coordinates the shell is $\left\{ k \mid 2^j \leq \sqrt{k_0^2 + \eta^2} \leq \text{const} \, 2^j \right\}$ and is topologically $S^{d-1} \times S^1 \times [0,1]$. However the first factor, the Fermi sphere $S^{d-1}$, should be viewed as having a macroscopic radius of order 1 while the remaining factors $S^1 \times [0,1]$ should be viewed as having a small diameter of order $2^j$ at scale $j$. 

The fact that this manifold has two length scales, 1 and $2^j$, of radically different size reflects the basic anisotropy between frequency $k_0$ and momentum $k$. It implies, in contrast to the field theory case, that the behavior of $C^j(\xi_1,\xi_2)$ at large $\xi_1 - \xi_2$ cannot be simply characterized as ‘decay at rate $2^{-j}$’. Rather, $C^j$ looks like

$$|C^j(\xi_1,\xi_2)| \leq \text{const} \cdot 2^j \left[1 + |x_1 - x_2| \right]^{(1-d)/2} \left[1 + 2^j |\xi_1 - \xi_2| \right]^{-N}$$

when a smooth cutoff function is used.

These shells induce an infrared renormalization group flow that acts in the ladder approximation on the running coupling constants $\lambda_{\ell}(j)$, $\ell \geq 0$, associated to the quartic local part at scale $j \leq 0$, by [FT3, I.85]

$$\lambda_{\ell}(j-1) = \lambda_{\ell}(j) + \beta(j) \lambda_{\ell}(j)^2$$

where $\beta(j) > 0$ and $\lim_{j \to -\infty} \beta(j) = \beta > 0$. In this approximation, our assumption $\lambda_0(0) > 0$ and $\lambda_0(0) > \lambda_{\ell}(0)$, $\ell \geq 1$, implies $\lim_{j \to -\infty} \lambda_0(j) = \infty$ and that (see, [FT3]) to all orders in the full flow $\lambda_0(j)$ dominates $\lambda_n(j)$, $n \geq 1$. In more detail, $\lambda_0(j)$ grows slowly as $j$ goes down to $\delta + \text{const}$ where, $\delta = -\left[\frac{1}{4}\right]$ is the symmetry breaking scale, and then quickly takes off to infinity. The divergence of a flow generated by a “Fermi surface” to a nontrivial fixed point is typical of many symmetry breaking or mass generation phenomena in condensed matter physics.

This renormalization group analysis reveals three distinct energy regimes. Fix $a \gg 2$ and let $\Delta \approx 2^{\delta}$ be the BCS gap. In the first regime at scales $j$, for which $2^j > a\Delta$, the effective coupling constant $\lambda_0(j)$ can be used as a small parameter. Symmetry breaking takes place in the second regime where $\frac{1}{a}\Delta < 2^j < a\Delta$. In the third regime, $2^j < \frac{1}{a}\Delta$, the physics of the Goldstone boson dominates. As explained above, the effective coupling constant is not small in the latter two regimes.

The “First Regime”:

In this section, we concentrate on the problem of summing high orders of perturbation theory and, in particular, discuss the results of [FMRT3]. To do so we first consider an artificial model which retains the essential difficulties we are interested in, but has no renormalization problems. The toy world consists of

- $d + 1$ dimensional Euclidean space-time
- four types of fermions, denoted $\psi_\uparrow$, $\psi_\downarrow$, $\bar{\psi}_\uparrow$ and $\bar{\psi}_\downarrow$, that play the roles of spin up and spin down electrons and positrons/holes
- momenta “morally” in the range $M^j \leq |p| \leq M^{j+1}$ with $M > 1$ being some fixed constant. This is typical of one slice of a field theory. In a many body model we would have $M^j \leq |p_0| + |p - k_F| \leq M^{j+1}$. In a realistic model we would have to sum over $j$ using the renormalization group.

We say “morally” because momentum is never actually going to appear in the toy world. Instead we are going to mimic the assumed momentum range by two space-time properties of the model. First, because the momentum space of our toy world has volume $M^{j(d+1)}$ the Pauli exclusion principle says that there can be at most one $\psi_\uparrow$, for example, in any region of volume $M^{-j(d+1)}$ in position space. Thus we define the fields of our model to be

$$\left\{ \psi_\uparrow(x), \psi_\downarrow(x), \bar{\psi}_\uparrow(x), \bar{\psi}_\downarrow(x) \mid x \in W := M^{-j} \mathbb{Z}^{d+1} \right\}$$

They are the generators of a Grassmann algebra. Thus

$$\bar{\psi}_\alpha(x) \bar{\psi}_\beta(y) = -\bar{\psi}_\beta(y) \bar{\psi}_\alpha(x)$$

and in particular

$$\left( \bar{\psi}_\alpha(x) \right)^2 = 0$$

The second concerns the propagator. That is, the free two point Euclidean Green’s function. The interacting two point Euclidean Green’s function is

$$S_2(x, x') = \frac{\int \bar{\psi}_\uparrow(x) \bar{\psi}_\uparrow(x') e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C}$$

where the interaction

$$V = \frac{1}{2} \sum_{y \in W} M^{-j(d+1)} \bar{\psi}_\uparrow(y) \bar{\psi}_\downarrow(y) \psi_\downarrow(y) \psi_\uparrow(y)$$

and

$$d\mu_C = \exp \left\{ \sum_{z,z' \in W} \bar{\psi}_\sigma(z') C^{-1}(z', z) \psi_\sigma(z) \right\} \prod_{z \in W} d\psi_\sigma(z) d\bar{\psi}_\sigma(z)$$

is the Grassmann Gaussian measure with covariance $C$, to be specified shortly.
Here are all the properties of Grassmann Gaussian measures that we are going to use. The symbol $\int \cdot d\mu_C$ is a linear functional that assigns a complex number to every polynomial in the fields and that obeys

1. $\int \psi_\sigma(x) \bar{\psi}_\sigma(y) d\mu_C = \delta_{\sigma,\sigma'} C(x, y)$

2. $\int \psi_\sigma(x) F(\psi, \bar{\psi}) d\mu_C = \sum_{y \in M^{-j} \mathbb{Z}^{d+1}} C(x, y) \int \frac{\delta}{\delta \psi_\sigma(y)} F(\psi, \bar{\psi}) d\mu_C$

$\int F(\psi, \bar{\psi}) \bar{\psi}_\sigma(y) d\mu_C = \sum_{x \in M^{-j} \mathbb{Z}^{d+1}} \int F(\psi, \bar{\psi}) \frac{\delta}{\delta \psi_\sigma(x)} d\mu_C C(x, y)$

$= \sum_{x \in M^{-j} \mathbb{Z}^{d+1}} C(x, y) \int \frac{\delta}{\delta \psi_\sigma(x)} F(\psi, \bar{\psi}) d\mu_C$

Except for signs the left and right derivatives $\frac{\delta}{\delta \psi_\sigma(y)} F(\psi, \bar{\psi})$ and $F(\psi, \bar{\psi}) \frac{\delta}{\delta \psi_\sigma(y)}$ behave like ordinary derivatives.

We assume, as the second characteristic of our momentum range, that the covariance $C$ decays at a rate typical of a smooth function whose Fourier transform has support in a neighbourhood of $|p| = M^j$. Precisely,

$$|C(x, y)| \leq \text{const} M^{(d+1)j/2} e^{-M^j |x-y|}.$$ 

The coefficient $M^{(d+1)j/2}$ is chosen to give power counting typical of a strictly renormalizable field theory. The position space behaviour of the many-Fermion propagator is somewhat more complicated than this. However, by decomposing the Fermi surface into a union of $M^{-(d-1)j}$ “rectangles” of side $M^j$, one can think of the many-Fermion field at scale $j$ as a sum $\psi^{(j)} = \sum_\alpha \psi^{(j,\alpha)}$ of $M^{-(d-1)j}$ independent fields with each “coloured” field having a covariance that obeys $|C(x, y)| \leq \text{const} M^{dj} e^{-M^j |x-y|}$. See [FMRT3].

**Theorem.** Let $S_{2,n}(x, x')$ be the coefficient of $\lambda^n$ in the formal power series expansion of $S_2(x, x')$. That is, $S_2(x, x') = \sum_{n=0}^{\infty} S_{2,n}(x, x') \lambda^n$. There exists a constant $R$, independent of $j, x, x'$, such that

$$\sup_x \sum_{x'} |S_{2,n}(x, x')| \leq K_j R^n.$$ 

In other words $S_2$ is analytic in $|\lambda| < \frac{1}{R}$. In other words, the sum of all connected Feynman diagrams converges for all $|\lambda| < \frac{1}{R}$. 
Proof: We first describe the logic of the proof. Denote by $S_2(x,x';\Lambda)$ the two point function of the model gotten by restricting the world to a finite subset of $\Lambda$ of $W$. It is easy to see [FMRT3], by Gram’s or Hadamard’s inequality, that both the numerator and denominator of $S_2(x,x';\Lambda)$ are entire functions of $\lambda$. The denominator $\int e^{-\lambda V}d\mu_C$ can have many $\Lambda$ dependent zeros. But when $\lambda = 0$, the denominator is one so that $S_2(x,x';\Lambda)$ is meromorphic on all of $\mathbb{C}$ and analytic at zero. We shall develop a formal power series expansion for $S_2(x,x';\Lambda)$ with the property that for every $N$

$$S_2(x,x';\Lambda) = \sum_{n=0}^{N} S_{2,n}(x,x';\Lambda)\lambda^n + O(\lambda^{N+1}).$$

A priori we do not claim that the tail $O(\lambda^{N+1})$ is uniform in $\Lambda$. Nevertheless, since $S_2(x,x';\Lambda)$ is analytic at zero we must have

$$S_2(x,x';\Lambda) = \sum_{n=0}^{\infty} S_{2,n}(x,x';\Lambda)\lambda^n$$

in some, possibly $\Lambda$ dependent, neighborhood of zero. We remark in passing that $S_{2,n}(x,x';\Lambda)$ must be the sum of all connected Feynman diagrams of order $n$ with 2 external legs, since we have an asymptotic expansion.

The heart of the proof is to show that there exists a constant $R$, independent of $\Lambda, j$ and a constant $K_j$ independent of $\Lambda$ such that

$$\sup_x \sum_{x'} |S_{2,n}(x,x';\Lambda)| \leq K_j R^n \quad (**)$$

As a consequence, equation (*) applies for all $|\lambda| < R^{-1}$. Any zeroes of the denominator that appear in this disk must be cancelled by zeroes of the numerator. It shall also be clear from the proof of (**) that the limits $S_{2,n}(x,x') = \lim_{\Lambda \to W} S_{2,n}(x,x';\Lambda)$ exist. This will prove, by the Lebesgue dominated convergence theorem, that

$$S_2(x,x') = \lim_{\Lambda \to W} S_2(x,x';\Lambda) = \sum_{n=0}^{\infty} S_{2,n}(x,x')\lambda^n$$

for all $|\lambda| < R^{-1}$ and that the coefficients $S_{2,n}(x,x')$ obey the bound (**).

We now describe the expansion used. To emphasize that everything is uniform in $\Lambda$, we suppress $\Lambda$. The first step is to use integration by parts (Property 2) to turn the $\psi_\uparrow(x)$ of the two point function into a covariance:

$$S_2(x,x') = \frac{\int \psi_\uparrow(x)\bar{\psi}_\uparrow(x')e^{-\lambda V}d\mu_C}{\int e^{-\lambda V}d\mu_C}$$

$$= C(x,x') + \frac{\sum_y \lambda C(x,y)\int \bar{\psi}_\uparrow(x')\frac{\delta V}{\delta \psi(y)}e^{-\lambda V}d\mu_C}{\int e^{-\lambda V}d\mu_C}$$
The first term is the trivial Feynman diagram giving the free value of $S_2$. For the second, apply integration by parts again to turn the $\bar{\psi}_↑(x')$ into another propagator.

$$S_2(x, x') = C(x, x') - \frac{\sum_{y, y'} \lambda^2 C(x, y) C(y', x') \int \left[ \frac{\delta}{\delta \bar{\psi}_↑(y')} \frac{\delta}{\delta \psi_↑(y)} V \right] e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C} - \frac{\sum_{y, y'} \lambda^2 C(x, y) C(y', x') \int \frac{\delta V}{\delta \bar{\psi}_↑(y)} \frac{\delta V}{\delta \psi_↑(y')} e^{-\lambda V} d\mu_C}{\int e^{-\lambda V} d\mu_C}$$

In each step select any $\psi$ downstairs and use integration by parts to turn it into one end of a propagator. When a term has no fields downstairs, the $\int e^{-\lambda V} d\mu_C$ in the numerator exactly cancels that in the denominator, leaving a Feynman diagram. This was how the trivial diagram $C$ arose. Leave such terms alone. Upon completion of the expansion, we have $S_2(x, x')$ expressed as the sum of all connected two point Feynman diagrams.

To illustrate the principal difficulty in estimating $S_2$ consider the following $n^{th}$ order term that arises in the midst of the expansion:

$$\frac{\lambda^n}{2^n} \sum_{y_1, \ldots, y_n \in W} M^{-j(d+1)n} C(x, y_1) C(y_1, y_2) \cdots C(y_n, x') \int \prod_{m=1}^n \bar{\psi}_↑(y_m) \psi_↑(y_m) e^{-\lambda V} d\mu_C$$

The functional integral

$$\int \prod_{m=1}^n \bar{\psi}_↑(y_m) \bar{\psi}_↑(y_m) e^{-\lambda V} d\mu_C$$

$$= - \sum_{z \in W} C(z, y_1) \int \frac{\delta}{\delta \bar{\psi}_↑(z)} \left[ \psi_↑(y_1) \prod_{m=2}^n \bar{\psi}_↑(y_m) \psi_↑(y_m) e^{-\lambda V} \right] d\mu_C$$

$$= - \sum_{i=1}^n C(y_i, y_1) \int \bar{\psi}_↑(y_1) \cdots \bar{\psi}_↑(y_i) \cdots \bar{\psi}_↑(y_n) \psi_↑(y_n) e^{-\lambda V} d\mu_C + O(\lambda)$$

We did a single integration by parts to get rid of $\bar{\psi}_↑(y_1)$ and ended up with $n$ terms of order $\lambda^n$. If we perform $n - 1$ further integrations by parts to get rid of $\bar{\psi}_↑(y_2), \cdots, \bar{\psi}_↑(y_2)$ we will generate $n!$ diagrams of order $\lambda^n$. Naive bounds on these $n!$ terms will fail to produce an acceptable bound on $S_{2,n}$.

Fortunately, the Pauli exclusion principle saves us. Note first that, if $y_m = y_{m'}$ for any $m \neq m'$, then $\psi_↑(y_m) \psi_↑(y_m') = -\psi_↑(y_m) \psi_↑(y_m)$ so that $\psi_↑(y_m) \psi_↑(y_m') = 0$ and hence
Let \( \psi_\downarrow(y_1) \prod_{m=2}^n \bar{\psi}_\downarrow(y_m) \psi_\downarrow(y_m) = 0 \). Let \( A_i = \int \psi_\downarrow(y_1) \cdots \bar{\psi}_\downarrow(y_\ell) \cdots \bar{\psi}_\downarrow(y_n) \psi_\downarrow(y_n) e^{-\lambda V} d\mu_G \). Then we may bound

\[
\sum_{i=1}^n |C(y_i, y_1) A_i| \leq \max_{1 \leq i \leq n} \left| e^{M^j|y_1-y_i|/2} C(y_i, y_1) A_i \right| \sum_{i=1}^n e^{-M^j|y_1-y_i|/2}
\]

\[
\leq \max_{1 \leq i \leq n} \left| M^{j(d+1)/2} e^{-M^j|y_1-y_i|/2} A_i \right| \sum_{y \in M \mathbb{Z}^d+1} e^{-M^j|y_1-y|/2}
\]

\[
= \max_{1 \leq i \leq n} \left| M^{j(d+1)/2} e^{-M^j|y_1-y_i|/2} A_i \right| \sum_{x \in \mathbb{Z}^d+1} e^{-|x|/2}
\]

\[
= \mathcal{E} \max_{1 \leq i \leq n} \left| M^{j(d+1)/2} e^{-M^j|y_1-y_i|/2} A_i \right| \quad (**) \]

where \( \mathcal{E} = \sum_{x \in \mathbb{Z}^d+1} e^{-|x|/2} < \infty \). The crucial consequence of the Pauli exclusion principle, that the \( y_i \)'s all are different, was used in going from line one to line two. Think of \( M^{j(d+1)/2} e^{-M^j|y_1-y_i|/2} \) as a propagator (replacing \( C(y_1, y_i) \)) for a line in a graph. This propagator joins a vertex at \( y_1 \) to a vertex at \( y_i \). The fields \( \bar{\psi} \) downstairs in the functional integral \( A_i \) are external legs for the graph.

Proceed by induction. In each step of the induction we integrate by parts once and apply the above bounding procedure. We start the \( k \)th step in the induction process with a maximum over \( k-1 \) indices like the \( i \) in (**) and we end the step with \( k \) such indices. We think of each such index as specifying the target vertex of a propagator. At the end of the expansion we find

\[
|S_{2,n}(x, x')| \leq \frac{1}{2^n} \max_G \sum_{y_1, \cdots, y_n} M^{-j(d+1)n} \prod_{\ell \in G} \left[ (\mathcal{E} + 1) M^{j(d+1)/2} e^{-M^j|y_{i_{\ell}}-y_{f_{\ell}}|/2} \right]
\]

The maximum is over all connected Feynman diagrams with two one-legged vertices, labeled \( x, x' \) and \( n \) four-legged vertices labeled \( y_1, \cdots, y_n \). The labels of the two vertices at the ends of line \( \ell \) are denoted \( i_\ell \) and \( f_\ell \). The reason for the +1 in \( \mathcal{E} + 1 \) is that each functional derivative arising from an application of the integration by parts formulae can act on the exponent as well as on interaction vertices downstairs. In preparation for bounding the graph \( G \), select a spanning tree \( T \) for \( G \). A spanning tree is a subgraph \( T \subset G \) which has no loops and contains all the vertices of \( G \). Bound all factors \( e^{-M^j|y_{i_{\ell}}-y_{f_{\ell}}|/2} \) that are associated with lines \( \ell \in G \setminus T \) by one. Then apply

\[
\sum_{y \in W} e^{-M^j|y-y'|/2} \leq \mathcal{E}
\]
to each vertex of $G$ starting with those farthest from $x$ in the partial ordering of $T$. The result is

$$\sum_{x'} |S_{2,n}(x, x')| \leq \frac{1}{2^n} \max_{G} \sum_{y_1, \ldots, y_n, x'} M^{-j(d+1)n} \prod_{\ell \in G} \left[ (\mathcal{E} + 1) M^{j(d+1)/2} e^{-M^j |y_{i\ell} - y_{f\ell}|/2} \right]$$

$$\leq \frac{1}{2^n} M^{-j(d+1)n} \max_{G} (\mathcal{E} + 1) |G|)^{G|j(d+1)/2} \sum_{y_1, \ldots, y_n, x'} \prod_{\ell \in T} \left[ e^{-M^j |y_{i\ell} - y_{f\ell}|/2} \right]$$

$$\leq \frac{1}{2^n} M^{-j(d+1)n} \max_{G} (\mathcal{E} + 1) |G|^{G|j(d+1)/2} \mathcal{E}^{n+1}$$

As we are currently considering an $n^{th}$ order diagram contributing to the two point function

$$|G| = \frac{2 + 4n}{2} = 2n + 1$$

and the final bound is

$$\sum_{x'} |S_{2,n}(x, x')| \leq \frac{1}{2^n} M^{-j(d+1)n} (\mathcal{E} + 1)^{2n+1} M^{(2n+1)j(d+1)/2} \mathcal{E}^{n+1}$$

$$\leq \frac{(\mathcal{E} + 1)^{3n+2}}{2^n} M^{j(d+1)/2},$$

which proves the Theorem with $R = \frac{1}{2} (\mathcal{E} + 1)^3$ and $K_j = (\mathcal{E} + 1)^2 M^{j(d+1)/2}$.

More generally, for a $p$ point function, $|G| = \frac{p+4n}{2} = 2n + p/2$, the number of sums controlled by the tree decay is $n + p - 1$ and

$$\sum_{x_2, \cdots, x_p} |S_{p,n}(x_1, \cdots, x_p)| \leq \frac{1}{2^n} M^{-j(d+1)n} (\mathcal{E} + 1)^{2n+p/2} M^{(2n+p/2)j(d+1)/2} \mathcal{E}^{n+p-1}$$

$$\leq \frac{(\mathcal{E} + 1)^{3n+3p/2-1}}{2^n} M^{pj(d+1)/4}.$$

The preceding techniques for summing perturbation theory do not apply directly to many fermions systems. The assumption $M^j \leq |p| \leq M^{j+1}$ is an over simplification. The presence of the Fermi surface forces $M^j \leq |p_0| + |p| - k_F | \leq M^{j+1}$ and this makes it much more difficult to implement the Pauli exclusion principle quantitatively.

To explain the difficulty, observe that the shell in momentum space about the Fermi surface has volume $M^{2j}$, while the position space volume of a “dual” box is $M^{-(d+1)j}$. The Pauli exclusion principle now permits a dual box to contain, in contrast to the “toy” system
discussed above, \(O(M^{-(d-1)j})\) electrons. For \(d = 1\) there is no true Fermi surface and there are \(O(1)\) electrons in a dual box. As the dimension grows the Pauli principle becomes progressively weaker.

There are three naive ways to force the volume in phase space to be independent of the scale \(j\). One either makes the dual boxes smaller, or decomposes the shell into sufficiently small sectors, or both. In each case, the number of electrons in such a constrained region would be of order one, achieving duality in the sense of the exclusion principle. The first alternative, however, violates duality in the sense of decay of the propagator.

Let us decompose the shell \(||k| - k_F| \approx M^j\) about the Fermi surface into \(M^{-(d-1)j}\) sectors of side \(M^j\) by another partition of unity. The free propagator at scale \(j\) in sector \(\Sigma\) is given by

\[
C_{j,\Sigma}(\xi_1, \xi_2) = \delta_{\sigma_1, \sigma_2} \int d\xi \frac{e^{i(k_0 - e(k)^2)}}{i k_0 - e(k)} 1_j(k_0^2 + e(k)^2) S_\Sigma(k)
\]

where \(S_\Sigma\) is supported on the sector \(\Sigma\). Of course, \(C_j = \sum_\Sigma C_{j,\Sigma}\). There is a corresponding decomposition of the fields. Observe that there are at most two spin one half fields at scale \(j\) in a position space box of side \(M^{-j}\) and momenta in the support of \(1_j(k_0^2 + e(k)^2) S_\Sigma(k)\), while there are \(O(M^{-(d-1)j})\) fields with momenta in the whole shell. That is, sectors enforce the full Pauli exclusion principle, while the whole shell allows a scale dependent accumulation of fields, reminiscent of Bosons.

It is easy to derive standard momentum space power counting for an individual graph using sectors. As usual, one selects a spanning tree for the graph. To each line not in the tree there is a corresponding momentum loop obtained by joining its ends through a path in the tree. This construction produces a complete set of independent loops. Ignoring unimportant constants, each propagator is bounded by its supremum \(M^{-j}\). The volume of integration for each loop is now \(M^{(d+1)j}\). A priori, there is one sector sum with \(M^{-(d-1)j}\) terms for each line. But, by conservation of momentum, there is only one sector sum per loop and one obtains the usual \(M^{\frac{4}{3}(4-E)j}\) where \(E\) is the number of external lines.

In the course of a non-perturbative construction, estimates cannot be made graph by graph because there are too many of them. Rather, as in the proof of the theorem, the perturbation series must be blocked and the blocks estimated as units. The blocks are estimated using the exclusion principle to implement strong cancellations between the roughly \(n!^2\) graphs of order \(n\). However, once the series is blocked, momentum loops can’t be defined and the argument leading to the power counting estimate cannot be made. Conservation of momentum has to be implemented at each vertex rather than through loops. Loosely
speaking, the Fermi surface makes it hard to fit the Pauli exclusion principle and conservation of momentum together.

Specializing to two dimensions, one can show (using the observation that four planar vectors of equal length whose sum is zero form a parallelogram, see, [FMRT3]) that the number of active sector 4-tuples at a vertex is of order $|j|M^{-2j}$. The factor $M^{-2j}$ is natural since a parallelogram is determined by two of its sides. The logarithmic factor arises from the degenerate situation in which all four vectors are roughly collinear. One can combine [FMRT3] the methods used to prove the theorem with the decomposition of the propagator into sectors to obtain a rigorous nonperturbative analysis in two dimensions of the full many electron system down to the scale $a\Delta$, that is throughout the first regime.

In three dimensions, the parallelogram is hinged and the logarithm $|j|$ jumps to the power $M^{-\frac{1}{2}j}$. This power makes the problem of controlling the first regime in three dimensions much more difficult. New ideas are required that we will not discuss here.

Anderson [A] has conjectured the failure of Fermi liquid theory in two dimensional interacting electron systems even at weak coupling. He argues that an orthogonality catastrophe leads to Luttinger liquid behavior roughly characterized by the continuity of the number density $n_k$ across the Fermi surface. Our two dimensional expansion can also be used to construct weakly coupled electron systems that in contrast to Anderson’s conjecture are rigorously Fermi liquids (see, [FKLT]) in the sense that $n_k$ jumps down discontinuously at the Fermi surface.

The “Second Regime”:

We now make an Ansatz that can be rigorously justified. We suppose that the effective vertex at scale $\delta_+ = \delta + \text{const}$ is the BCS interaction for Cooper pairs

$$V_{\text{eff}} = -2\lambda_0(\delta_+)\int [q] < \text{const} \Delta dq dt ds \bar{\psi}_{t+\frac{q}{2}, \uparrow} \psi_{t-\frac{q}{2}, \downarrow} \bar{\psi}_{-s+\frac{q}{2}, \downarrow} \psi_{s+\frac{q}{2}, \uparrow}$$

Here, $0 < \lambda_0(\delta') = O(1)$, $\Delta \approx M^\delta$ the symmetry breaking energy (the BCS gap), and all the fields are at scale $\delta_+$ so that the integrals are implicitly constrained by $|e(\pm t + \frac{q}{2})|, |e(\pm s + \frac{q}{2})|$, $|t_0|, |s_0| < \text{const} \Delta$. Approximating $\pm k + \frac{q}{2}$ by $\pm k$, using the notation $V = M^{-(d+1)\delta}$ and introducing sectors at scale $\delta_+$, the effective vertex becomes

$$\sum_{\Sigma_1, \Sigma_2} \frac{-2\lambda_0(\delta_+)}{V} \int ds_1 ds_2 \bar{\psi}_{\Sigma_1, \uparrow(s_1)} \psi_{-\Sigma_1, \downarrow(s_1)} \bar{\psi}_{-\Sigma_2, \downarrow(s_2)} \psi_{\Sigma_2, \uparrow(s_2)}$$
We denote by $-\Sigma$ the sector antipodal to $\Sigma$. Note that the sums run individually over $(\text{const } \Delta)^{-(d-1)}$ sectors.

Our vertex has the structure typical of an $N(= (\text{const } \Delta)^{-(d-1)})$-component vector model. Pictorially, sectors ("colors") $\Sigma_1$ and $-\Sigma_1$ enter at the left of an interaction squiggle and the sectors $\Sigma_2$ and $-\Sigma_2$ at the right. Thus, the sector is conserved up to a flip as it flows along particle lines in a graph. We now check, by rough power counting that the effective coupling constant of the vertex is $1/N$.

To estimate the size of the vertex, observe that the momentum space free propagator at scale $\delta_+$ in sector $\Sigma$ is

$$\frac{1_{\delta_+} \left( k_0^2 + e(k)^2 \right) S_\Sigma(k)}{ik_0 - e(k)} \approx \frac{1_{\delta_+} \left( k_0^2 + e(k)^2 \right) S_\Sigma(k)}{\Delta}$$

Fixing $\Sigma_1$ and $\Sigma_2$ there is one $d+1$ dimensional momentum integration and two propagators per vertex. Thus, the power counting weight of the vertex is

$$\int dk \frac{1_{\delta_+} \left( k_0^2 + e(k)^2 \right) S_\Sigma(k)}{\Delta^2} \approx \frac{\Delta^{d+1}}{\Delta^2} = \Delta^{d-1} = O(1/N)$$

Recall that diagramatically, the Goldstone boson propagator is the sum of all one particle irreducible four legged graphs. The simplest subseries is the geometric series of bubbles. It follows from the last two paragraphs that the bubble is $O(1)$ in $1/N$ and all other four legged graphs are of higher order. It follows from these remarks that the full Goldstone boson propagator can be expanded in powers of $1/N$ around the explicit sum of all bubbles. The large number of components can be used to rigorously control the nonperturbative effects of pairing and the associated $U(1)$ Goldstone boson that appear in the transition between the first and second regimes.

**The “Third Regime”:**

In the third regime the electrons have already paired and the number symmetry is broken. The physics is dominated by the Golstone boson that mediates the interaction between pairs of Cooper pairs. The Goldstone boson will appear as a singularity in the four point function. That is,

$$\int ds dt dp \left\langle \bar{\psi}_s - q_\uparrow \bar{\psi}_s ^\downarrow + \psi_{s+q\downarrow} \psi_s \uparrow \bar{\psi}_{t - p\uparrow} - \bar{\psi}_{t - p\downarrow} + \psi_{t - p \downarrow} \psi_{t \uparrow} \right\rangle = -\frac{1}{c_1 q_0^2 c_2 q^2} + O(1)$$
To rigorously control the symmetry breaking we study the effective interaction
\[ V_{\text{eff}} = -2g^2 \int ds \ dt \ dq \ \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \bar{\psi}_\downarrow(-t+\frac{\xi}{2}) \psi_\downarrow(-s+\frac{\xi}{2}) \psi_\uparrow(s+\frac{\xi}{2}) \]

\[ = -2g^2 \int dp \ dq \left( \int dt \ \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \bar{\psi}_\downarrow(-t+\frac{\xi}{2}) \right) B(p,-q) \left( \int ds \ \psi_\downarrow(-s+\frac{\xi}{2}) \psi_\uparrow(s+\frac{\xi}{2}) \right) \]

with \( B(p,q) = (2\pi)^{d+1} \delta(p+q) \). Note that, by antisymmetry,
\[ \int ds \ dt \ dq \ \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \bar{\psi}_\downarrow(-t+\frac{\xi}{2}) \psi_\downarrow(-s+\frac{\xi}{2}) \psi_\uparrow(s+\frac{\xi}{2}) = 0 \]

Let \( (\gamma_1, \gamma_2) \) be a \( \mathbb{V}^2 \) valued Gaussian variable with the real, even covariance
\[ \langle \gamma_i(p) \gamma_j(q) \rangle = \delta_{i,j} B(p,q) \]

Observe that the position space covariance \( \langle \gamma_i(\xi_1) \gamma_j(\xi_2) \rangle = \delta_{i,j} \delta(\xi_1 - \xi_2) \) is also real. Thus we can choose \( \gamma_i(\xi) \) to be real valued. Set
\[ \Gamma(\xi) = \gamma_1(\xi) - i\gamma_2(\xi) \]

We have
\[ e^{-V_{\text{eff}}} = \int e^g \int dt dq (\Gamma(q) \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \bar{\psi}_\downarrow(-t+\frac{\xi}{2}) + \bar{\Gamma}(q) \psi_\downarrow(-s+\frac{\xi}{2}) \psi_\uparrow(s+\frac{\xi}{2})) d\mu(\gamma_1, \gamma_2) \]

since for all functions \( X(q) \) and \( Y(q) \)
\[ \int e \int dq (X(q)\Gamma(q) + Y(q)\bar{\Gamma}(q)) d\mu(\gamma_1, \gamma_2) = \int e \int dq (X(q)+Y(-q)) \gamma_1(q)-i(X(q)-Y(-q)) \gamma_2(q) d\mu(\gamma_1, \gamma_2) \]

\[ = e^\frac{1}{2} \int dp dq (X(p)+Y(-p)) B(p,q) (X(q)+Y(-q)) e^{-\frac{1}{2} \int dp dq (X(p)-Y(-p)) B(p,q) (X(q)-Y(-q))} \]

\[ = e^2 \int dp dq X(p)B(p,q)Y(-q) \]

Changing variables and combining terms,
\[ g \int dt dq \ (\Gamma(q) \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \bar{\psi}_\downarrow(-t+\frac{\xi}{2}) + \bar{\Gamma}(q) \psi_\downarrow(-s+\frac{\xi}{2}) \psi_\uparrow(s+\frac{\xi}{2})) \]

\[ = g \int dt dq \ (\Gamma(q) \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \bar{\psi}_\downarrow(-t+\frac{\xi}{2}) + \bar{\Gamma}(q) \psi_\downarrow(-t+\frac{\xi}{2}) \psi_\uparrow(t+\frac{\xi}{2})) \]

\[ = g \int dq dt \ \bar{\psi}_\uparrow(t+\frac{\xi}{2}) \psi_\downarrow(-t+\frac{\xi}{2}) \begin{pmatrix} 0 & \Gamma(q) \\ \Gamma(-q) & 0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow(t+\frac{\xi}{2}) \\ \psi_\downarrow(-t+\frac{\xi}{2}) \end{pmatrix} \]

\[ = g \int dq dt \ \bar{\Psi}(t+\frac{\xi}{2}) \begin{pmatrix} 0 & \Gamma(q) \\ \Gamma(-q) & 0 \end{pmatrix} \Psi(t+\frac{\xi}{2}) = g \int d\xi \ \bar{\Psi}(\xi) \begin{pmatrix} 0 & \Gamma(\xi) \\ \Gamma(-\xi) & 0 \end{pmatrix} \Psi(\xi) \]
where

\[ \Psi(k) = \begin{pmatrix} \Psi_1^1(k) \\ \Psi_2^2(k) \end{pmatrix} = \begin{pmatrix} \psi_{k\uparrow}^* \\ \psi_{-k\downarrow} \end{pmatrix} \]

are Nambu fields. For convenience set

\[ \gamma = \gamma_1 \sigma^1 + \gamma_2 \sigma^2 = \begin{pmatrix} 0 & \Gamma(\xi) \\ \Gamma(\xi) & 0 \end{pmatrix} \]

Then,

\[ e^{-V_{\text{eff}}} = \int \exp \left( g \int d\xi \, \bar{\Psi}(\xi) \gamma(\xi) \Psi(\xi) \right) d\mu(\gamma) \]

Performing the Fermionic integration

\[ \int e^{-V_{\text{eff}}} d\mu(\Psi, \bar{\Psi}) = \int \int \exp \left( g \int d\xi \, \bar{\Psi}(\xi) \gamma(\xi) \Psi(\xi) \right) d\mu(\gamma_1, \gamma_2) d\mu(\Psi, \bar{\Psi}) \]

we obtain (the exponential of) an effective interaction for the intermediate boson field \( \gamma \). Here, \( \gamma \) is a multiplication operator in position space acting on \( \mathbb{R}^{d+1} \)-valued functions and \( C \) is the multiplication operator in momentum space given by

\[ C(p) = -\rho(p) \frac{ip_0 + e(p) \sigma^3}{p_0^2 + e(p)^2} \]

where \( \rho(p) \) is the characteristic function of the set \( \{ p \in \mathbb{R}^{d+1} \mid p_0^2 + e(p)^2 < 1 \} \). Thus \( \rho(p) \) imposes an ultraviolet, but no infrared, cutoff on the Fermions.

To analyze the effective model we introduce (to be technically precise, already in the second regime) a new scale that resolves the delta function covariance defining the bosonic Gaussian measure \( d\mu(\gamma) \). This decomposition induces another renormalization group flow.

The determinant \( \det(\mathbb{1} - g C\gamma) \) is a complicated function of \( \gamma \). To get some feeling for it we do a mean field computation. We consider constant \( \gamma \)'s and introduce the periodized Fermionic covariance \( P_j(\xi) \),

\[ P_j(\xi) = \sum_c C(\xi - c) \]

The sum runs over the lattice \( M^{-j} \mathbb{Z}^{d+1} \) so that \( P_j \) is periodic on a large box \( \Lambda \) of side \( M^{-j} \).

It is not hard to show that:

\[ \log \det(\mathbb{1} - g P_j \gamma) = |\Lambda| \sum_p \frac{1}{|\Lambda|} \log \left( 1 + \frac{g^2 \gamma_1^2 \rho(p)}{p_0^2 + e(p)^2} \right) \]

where the sum is over \( p \) in \( 2\pi M^j \mathbb{Z}^{d+1} \) and where with abuse of notation \( \gamma^2 = (\gamma_1^2 + \gamma_2^2) \mathbb{1}_2 \) is identified with \( (\gamma_1^2 + \gamma_2^2) \).
Remark. Notice that $\sum_p \frac{1}{|\Lambda|} \to \int_{\mathbb{R}^{d+1}} dp$ as $j \to -\infty$. On the other hand, the volume prefactor $|\Lambda| = M^{-(d+1)j}$ tends to infinity as usual.

Formally,

$$\det(1 - g C \gamma) d\mu(\gamma) = e^{\log \det(1 - g C \gamma)} e^{-\frac{1}{2} \int d\gamma \gamma B^{-1} \gamma} \prod_{\xi \in \mathbb{R}^{d+1}} d\gamma(\xi)$$

$$= e^{-\frac{1}{2} \left( \int d\gamma(\xi)^2 - \log \det(1 - g C \gamma) \right)} \prod_{\xi \in \mathbb{R}^{d+1}} d\gamma(\xi)$$

Thus, the full effective potential in a box $\Lambda$ of side $M^{-j}$ evaluated at the constant field configuration $\gamma$ is

$$M^{-(d+1)j} \left( \frac{1}{2} \gamma^2 - \sum_p \frac{1}{|\Lambda|} \log \left( 1 + g^2 \frac{\rho(p)}{p_0 + c(p)^2} \gamma^2 \right) \right)$$

We want to show that its graph is a Bordeaux wine bottle (also referred to as a Mexican hat) and determine its dimensions.

To do this it is convenient to replace the sum $\sum_p \frac{1}{|\Lambda|}$ by an integral and study the mean field effective potential per unit volume

$$\mathcal{E}(r) = \frac{1}{2} r^2 - \int dp \log \left( 1 + g^2 \frac{\rho(p)}{p_0 + c(p)^2} r^2 \right)$$

where $r = \sqrt{\gamma_1^2 + \gamma_2^2}$. In terms of the variable $s = r^2$ (but, by abuse of notation, retaining the name $\mathcal{E}$)

$$\mathcal{E}(s) = \frac{1}{2} s - \int dp \log \left( 1 + g^2 \frac{\rho(p)}{p_0 + c(p)^2} s \right)$$

$$\frac{d\mathcal{E}}{ds}(s) = \frac{1}{2} - \int dp \frac{g^2 \rho(p)}{p_0^2 + c(p)^2 + g^2 s}$$

$$\frac{d^2\mathcal{E}}{ds^2}(s) = \int dp \frac{g^4 \rho(p)}{(p_0^2 + c(p)^2 + s)^2}$$

Hence $\mathcal{E}(s)$ is continuous on $[0, \infty)$, is zero at $s = 0$ and grows like $s/2$ at $s = \infty$. The first derivative diverges logarithmically to $-\infty$ at $s = 0$ and converges to $1/2$ at $s = \infty$. The second derivative is always positive. Thus $\mathcal{E}(s)$ has a unique critical point $s_*$ and this critical point is a global minimum.
Integrating over the angular variables, changing variables to $\eta = e(p)$ and then using polar coordinates to replace $p_0$ and $\eta$

$$\mathcal{E}(s) = \frac{1}{2}s - \frac{|S^{d-1}|}{(2\pi)^{d-1}} \int dp_0 \, d|p| |p|^{d-1} \log \left( 1 + g^2 \frac{\rho(p)}{p_0 + e(p)} s \right)$$

$$= \frac{1}{2}s - mk_F^d |S^{d-1}| \frac{|p_0|}{(2\pi)^d} \int d\eta \left( 1 + \frac{2m}{k_F^d} \eta \right)^{d-2} \log \left( 1 + g^2 \frac{\rho(p)}{p_0 + e(p)} s \right)$$

$$= \frac{1}{2}s - mk_F^d |S^{d-1}| \frac{|p_0|}{(2\pi)^d} \int_0^1 dR R \left( 1 + o(R^2) \right) \log \left( 1 + \frac{g^2 s}{R^2} \right)$$

When $d = 2$ the $O(R^2)$ term is absent. When $d > 2$ we used oddness to show that the $O(\eta)$ term vanishes.

For $d = 2$ we can explicitly evaluate the integral to show that the graph of the effective potential is a Bordeaux wine bottle whose absolute minimum is at $\eta = g|\gamma|_s \approx \exp \left\{ -\frac{\pi}{mg^2} \right\}$ and has depth approximately $\frac{m}{4\pi} (g|\gamma|_s)^2$ and curvature at the minimum approximately $\frac{m}{\pi} g^2$. The picture in dimensions $d > 2$ is similar. Note that the depth, $M^{-(d+1)j} g|\gamma|_s$, of the effective potential $M^{-(d+1)j} \left( \frac{1}{2} \gamma^2 - \sum_p \frac{1}{|\Lambda|} \log \left( 1 + g^2 \frac{\rho(p)}{p_0 + e(p)} \gamma^2 \right) \right)$ in the whole box is enormous due to the volume factor $M^{-(d+1)j}$. It is deep enough to break the symmetry of the whole model.

Symmetry breaking forces the value of $\gamma$ to be concentrated near some point $\Delta/g = \frac{1}{g}(\Delta_1 \sigma_1 + \Delta_2 \sigma_2)$ with $|\Delta| = \sqrt{\Delta_1^2 + \Delta_2^2} = g|\gamma|_s$. The phase is determined by a boundary condition. Then it is natural to shift $\gamma$ by $\Delta/g$ and define the radial and tangential components

$$\gamma = \frac{1}{2\Delta} \text{tr} (\gamma \Delta) \Delta + \frac{1}{2\Delta} \text{tr} (\gamma \Delta^#) \Delta^#$$

$$= \Delta/g + \gamma_{\text{rad}} \Delta/|\Delta| + \gamma_{\text{tan}} \Delta^#/|\Delta|$$

where $|\Delta| = \sqrt{\Delta^2}$. While $\gamma_{\text{rad}}$ and $\gamma_{\text{tan}}$ are globally defined they can only be interpreted as radial and tangential components in a small neighbourhood of $\gamma = \Delta/g$. In the new variables the measure

$$e^{-\mathcal{V}_{\text{eff}}} \, d\mu(\Psi, \bar{\Psi}) = \int e^g \int d\xi \, \Psi(\xi) \gamma(\xi) \Psi(\xi) \, d\mu(\gamma) \, d\mu(\Psi, \bar{\Psi})$$

$$= \text{const} \int e^g \int d\xi \, \bar{\Psi}_s \Psi + \int d\xi \, \bar{\Psi} \Delta \Psi \, e^{-|\Delta|/g} \int d\xi \, \gamma_{\text{rad}} \, d\mu(\gamma_s) \, d\mu(\Psi, \bar{\Psi})$$

$$= \text{const} \int e^g \int d\xi \, \bar{\Psi}_s \Psi \, e^{-|\Delta|/g} \int d\xi \, \gamma_{\text{rad}} \, d\mu(\gamma_s) \, d\mu(\Delta(\Psi, \bar{\Psi}))$$

where $\gamma_s = \gamma_{\text{rad}} \Delta/|\Delta| + \gamma_{\text{tan}} \Delta^#/|\Delta|$ is the shifted field and $d\mu_{\Delta}$ is the Grassmann-Gaussian measure with covariance

$$C_{\Delta} = \frac{\rho(k)}{ik_0 - e(k)\sigma^3 - \Delta \rho(k)}$$
Expanding the integral
\[
\int e^g \int d\xi \bar{\Psi} \gamma_s \Psi d\mu_{\Delta} (\Psi, \Psi) = \det (1 - gC_{\Delta} \gamma_s)
\]
in powers of \( g \) generates vertices whose naive power counting is nonrenormalizable. We derive [FMRT4] quantitative Ward identities that force the model to be superrenormalizable.

It is not straightforward to rigorously exploit the structure of the effective potential and to nonperturbatively implement the Ward identities. Roughly, at each bosonic scale the Goldstone boson field is divided into a slowly varying “background” and a quickly varying fluctuation. Phase space is decomposed into “large” and “small” field regions relative to the scale dependent background fields. In small field regions the Goldstone boson field is trapped “near” the bottom of the Bordeaux wine bottle and its gradient is “small”. Here, “near” and “small” quantitatively reflect the bosonic scale. In large field regions, the Goldstone boson field is “far” from the bottom and its gradient is “large”. We expand, renormalize and apply Ward identities in small field regions to extract the momentum space singularity symptomatic of the long range nature of the Goldstone boson. We use probabilistic estimates in large field regions to show that they are strongly suppressed.

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