Present status of nuclear cluster physics (Theory)

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Abstract. Present status is discussed, from the theoretical viewpoint, for some selected topics of nuclear cluster physics. What is stressed here is that many of basic concepts of nuclear cluster physics since the days of K. Wildermuth are now examined and revived in new lights. An important example is the duality character of the ground-state wave function which was well known already in 1950’s and has been recently revived in order to explain observed large magnitudes of E0 transitions between cluster states and the ground state in many nuclei. The other topics we discuss are (1) spatial localization of clusters v.s. THSR wave function, (2) coexistence of cluster states with mean-field-type states, and (3) spatial localization of clusters and ab initio calculations of cluster states.

1. Introduction

According to the request of the organizers, I discuss present status of nuclear cluster physics from the theoretical side by selecting some topics. What is stressed in my talk is the fact that many of basic concepts of nuclear cluster physics since the days of K. Wildermuth are now examined and revived in new lights with fruitful results.

An important example is the concept of the duality of the shell-model wave function of the ground state which means that the ground-state wave function can be rewritten in the form of cluster-model wave function. The duality character of the ground-state wave function which was well known already in 1950’s was recently revived in order to explain observed large magnitudes of E0 transitions between cluster states and the ground state in many nuclei.

Other topics we selected in addition to the duality of the shell model wave function of the ground state are (1) spatial localization of clusters v.s. THSR wave function, (2) coexistence of cluster states with mean-field-type states, and (3) spatial localization of clusters and ab initio calculations of cluster states.

2. The duality of the shell model wave function of the ground state

2.1. Recognition of the duality of the ground state wave function in 1950’s

When Wildermuth started his microscopic cluster model study of nuclear structure in 1958 [1], the shell model had already been well established because it was proposed about 10 years ago by Mayer and Jensen. Wildermuth’s cluster model was constructed based on the duality property of the shell-model wave function of the ground state which means that the ground-state wave function can be rewritten in the form of cluster-model wave function. A typical example is the duality of the doubly-closed-shell wave function of the $^{16}$O ground state which is equivalent to
$^{12}$C + α cluster-model wave function;
\[
\det |(0s)^4(0p)^{12}| = c_L A \left[ R_{(4-L)/2,L}(rC-\alpha, 3\nu) |Y_L(rC-\alpha)\phi_L^{(12C)}| J=0 \phi(\alpha) \right] g(X_G, 16\nu),
\]
where $R_{\nu L}(r)$ is radial H.O. function, $g(r, \gamma) = (2\gamma/\pi)^{3/4} \exp(-\gamma r^2)$ and $X_G$ is the total center-of-mass coordinate. The duality character of ground state shell-model wave functions was known by many people who studied microscopic cluster-model wave functions in 1950’s. For example, Perring and Skyrme wrote at the beginning of the abstract of their paper [2] “It is shown that it is possible to write down α-particle wave functions for the ground states of $^8$Be, $^{12}$C, and $^{16}$O, which become, when antisymmetrized, identical with shell-model wave functions.”

Widermuth considered that the formation of cluster states is due to the activation of the clustering degree of freedom embedded in the ground state due to its duality nature. Namely he considered that, while the ground state has the heaviest overlap of clusters, excited cluster-states are formed by relaxing the overlap of clusters. He assigned cluster structures to many excited states in light nuclei [3]. For example, in $^{16}$O he assigned $^{12}$C($0^+$) + α structure to $0^+$(6.06 MeV), $2^+$(6.91 MeV), $4^+$(10.36 MeV), $1^-$ (9.58 MeV), and $3^{-}$ (11.62 MeV) and $^{12}$C($2^+$) + α structure to $2^+$(9.84 MeV). Widermuth’s assignments are mostly supported by the present nuclear structure study. But, at the days around 1960, the assignments of cluster structure to excited states were not regarded as convincing. Therefore this situation even meant that the duality could lead to the unnessicity of the cluster model, because its wave function can be rewritten by the shell-model wave function.

A convincing signature of cluster structure which is not easily describable by shell model or by deformed mean-field model is “spatial localization of clusters”. Spatial localization of clusters has been one of the most important concepts of nuclear clustering. Before discussing spatial localization of clusters in the next section, we discuss recent finding that the duality character of the ground state is well reflected in the observed large magnitudes of electric monopole (E0) transitions between cluster states and the ground state.

2.2. Experimental proof of the duality

In spite of large difference of structure between cluster states and the ground state, observed E0 transitions between them are recently pointed out to be strong and comparable with the single-nucleon strength [4]. Single-nucleon strength of E0 transition is roughly given by $M(E0) \sim \langle u_f | r^2 | u_i \rangle \sim (3/5) R^2$ with $R$ being nuclear radius which is about 5.4 fm$^2$ for $R = 3$ fm. E0 transition of the Hoyle state in $^{12}$C and those of many cluster states in $^{16}$O are good examples. The observed values of them are 5.4 fm$^2$ for $0_1^+ \leftrightarrow 0_2^+$ in $^{12}$C, and 3.6 fm$^2$, 4.0 fm$^2$, 3.3 fm$^2$, for $0_1^- \leftrightarrow 0_2^+, 0_1^+ \leftrightarrow 0_3^+$, $0_1^+ \leftrightarrow 0_5^+$, respectively. From the shell-model viewpoint, this looks very contradictory because cluster states are described by superposition of many-particle many-hole configurations. However, the cluster model can explain easily and naturally why E0 transitions are of the use of the duality character of the ground-state wave function. This fact is explained as follows in the case of $^{16}$O: Since $0_2^+$ and $0_3^+$ states are dominantly described by the wave functions $A\{\chi_L(r)|Y_L(r)\phi_L^{(12C)}\}_0\phi(\alpha)$ with $L = 0$ and $L = 2$ respectively, the expression of the ground state as $A\{R_{(4-L)/2,L}(r)|Y_L(r)\phi_L^{(12C)}\}_0\phi(\alpha)$ by its duality property teaches us that the E0 transitions between $0_2^+$ and $0_3^+$ and the ground state are nothing but the transitions of relative motion between $\chi_L(r)$ and $R_{(4-L)/2,L}(r)$. Namely, the E0 transitions are caused by a single degree of freedom (relative motion), which is the reason why observed $M(E0)$ are comparable to single-particle value. It is to be noticed that when the E0 transition operator $O^{(16)}$ is expressed as $O^{(16)} = O^{(12C)} + O(\alpha) + [(1/2)(12 \times 4)/16] r^2$, only the relative-coordinate operator $[(1/2)(12 \times 4)/16] r^2$ makes non-zero contribution in the matrix element $\langle 0_1^+ | O(16) | 0_2^+ or 0_3^+ \rangle$. 

2.3. Bayman-Bohr theorem of the duality

Bayman and Bohr gave a mathematically neat description of the duality by the aid of $SU(3)$ symmetry in 1958/59 [5]. With this theorem, Eq. (1) is expressed more precisely as

$$|(0s)^4(0p)^{12}(\lambda, \mu) = (0, 0)) = C_0 A\left\{\left[R_{(4,0)}(r_{C-\alpha}, 3\nu)\phi_{(0,4)}^{(12)C}\right](0,0)\phi(\alpha)\right\}g(X_G, 16\nu).$$ (2)

The H.O. function $R_{nL}(r, \gamma)Y_{LM}(\hat{r})$ has $SU(3)$ symmetry $(2n + L, 0)$ and hence is written as $R_{(2n+L,0)L,M}(r, \gamma)$, while $\phi_{LM}^{(12)C}$ has $SU(3)$ symmetry $(0,4)$ and is written as $\phi_{(0,4)L,M}^{(12)C}$. In the equality of Eq.(2), we note the conservation of the $SU(3)$ symmetry $(\lambda, \mu) = (0, 0)$ in both left and right sides. Therefore we have $A\left\{[R_{(4,0)}(r_{C-\alpha}, 3\nu)\phi_{(0,4)}^{(12)C}]_{(\lambda,\mu)}\phi(\alpha)\right\} = 0$ for $(\lambda, \mu) \neq (0, 0)$. Ref. [5] was published soon after the proposal of the SU(3) shell model by Elliott in 1958 [6].

Thanks to the Bayman-Bohr theorem, we can judge easily whether the ground-state wave function $\psi$ has duality character or not. When $\psi$ is dominantly described by a single SU(3) shell-model wave function, it has duality character. In heavy nuclei, since the Elliott’s SU(3) symmetry is largely lost in the ground state, we can say that $\psi$ does not have duality character. But this does not mean the denial of the possible existence of excited states whose wave functions have duality character.

3. Spatial localization of clusters v.s. THSR wave function

An important convincing proof of the spatial localization of clusters is the phenomenon of the inversion doublet rotational bands with small energy gap between plus and minus parity bands. Since the inversion doublet comes from the parity-violating intrinsic state, the inversion doublet bands with non-small alpha-decay widths should have spatial localization of clusters. Observed typical examples of inversion-doublet bands are in $^{16}O$ and $^{20}Ne$ [7]. In $^{16}O$, the plus-parity band is $K^\pi = 0^+$ band upon the first excited state with $J^\pi = 0^+$ at $E_x = 6.05$ MeV, and the minus-parity band is $K^\pi = 0^-$ band upon the $1^-$ state at $E_x = 9.58$ MeV which is slightly above the $^{12}C + \alpha$ threshold. The $K^\pi = 0^-$ band states have large $\alpha$-decay reduced widths comparable to the Wigner-limit value. Hence in accordance with Wildermuth, spatially-localized cluster structure of $^{12}C + \alpha$ is assigned to this inversion-doublet. In the case of $^{20}Ne$, the plus-parity band is the ground-state band and the minus-parity band is $K^\pi = 0^-$ band upon the $1^-$ state at $E_x = 5.80$ MeV which is slightly above the $^{16}O + \alpha$ threshold. To this inversion-doublet, spatially-localized cluster structure of $^{16}O + \alpha$ is assigned.

A very surprising fact is that the RGM/GCM wave functions of the inversion-doublet bands of $^{20}Ne$ were found, recently, to be almost the same as single THSR (Tohsaki-Horiuchi-Schuck-Roepke) wave functions [8]:

$$|\langle\Phi(\text{single THSR})|\Phi(^{16}O + \alpha \text{ RGM})\rangle|^2 \approx 100\% \quad \text{for inversion doublet band states.} \quad (3)$$

As is seen in the fact that the THSR wave function was originally introduced for describing $\alpha$-codensate-like states [9], it expresses non-localized motion of clusters. Therefore this relation looks inconsistent with the idea of the parity-violating deformation due to the spatially localized $^{16}O + \alpha$ clustering. This seeming inconsistency is resolved by considering the inter-cluster Pauli repulsion due to the antisymmetrization (Fermi statistics of nucleons) [10]. The inter-cluster Pauli repulsion prevents two clusters to come close to each other and hence gives rise to the spatial localization of $^{16}O$ and $\alpha$ clusters. In Fig. 1, we show the nucleon-density distribution of the $^{16}O + \alpha$ THSR wave function before projection where we clearly see the spatial localization of $^{16}O$ and $\alpha$ clusters. We may say that dynamics prefers non-localized clustering while kinematics (Pauli repulsion) makes the system look like localized clustering.
4. Coexistence of cluster states with mean-field-type states

Coexistence of cluster states with mean-field-type states is a natural consequence of the duality nature of the ground state, as is shown in Fig. 2. However, as mentioned before, at the days around 1960 when Wildermuth proposed his microscopic cluster model, the assignments of cluster structure to excited states were not regarded as convincing. Different from the days of Wildermuth, the assignments of cluster structure to many excited states are now regarded as convincing in many nuclei. Signatures or evidences of cluster structure have been pursued in many ways. We now see coexistence of cluster states with mean-field-type states in a wide region of nuclear chart including unstable nuclei.

In several nuclei, the number of cluster states is larger than that of mean-field-type states up to rather high excitation energy, say, 15 MeV. A typical example is the \(^{16}\text{O}\) nucleus which is a double-closed-shell nucleus implying a fundamental nucleus for the shell model.

The coexistence feature is now seen up to pf-shell region. In this region, cluster states include molecular resonances in addition to alpha-cluster states and mean-field-type states include superdeformed states in addition to normal deformed states. In Fig. 3 AMD reproduction is shown of the coexistence feature of cluster states with mean-field-type states in \(^{44}\text{Ti}\) [11].

Formation of molecular resonance states is due to the duality property of usually not the ground state but excited mean-field-type states, which is in contrast to alpha-cluster states formed through the duality of the ground state. For example, in \(^{28}\text{Si}\), \(^{16}\text{O} + ^{12}\text{C}\) molecular-resonance states are considered to be formed on the basis of the duality property of the excited rotational band with prolate deformation upon the \(0^+\) state at 6.69 MeV [12]. This band has, as its dominant component of the intrinsic state, \(|(sd)^{12}[4^3](\lambda, \mu) = (12, 0)\rangle\) which has the duality character that it is equivalent to \(^{16}\text{O} + ^{12}\text{C}\) cluster-model wave function. Long time ago the \(^{16}\text{O} + ^{12}\text{C}\) Brink-GCM calculation reported that the calculation reproduced both \(^{16}\text{O} + ^{12}\text{C}\) resonance states and the prolate band states upon the \(0^+\) state at 6.69 MeV [13]. Another example is the case of \(^{16}\text{O} + ^{16}\text{O}\) molecular resonance states which are considered to be formed on the basis of the superdeformed states in \(^{32}\text{S}\) [14, 15]. Ref. [15] reports that the AMD-GCM calculation along the energy curve with respect to the quadrupole deformation gives rise to \(^{16}\text{O} + ^{16}\text{O}\) molecular resonance states. The dominant component of the superdeformed-state wave function of this state is \(|(sd)^{16}[4^3](\lambda, \mu) = (12, 0)\rangle\).
function is \(|(sd)^{12}(pf)^{4[44]}(\lambda, \mu) = (24, 0)\rangle\) which has the duality character that it is equivalent to \(^{16}\text{O} + ^{16}\text{O}\) cluster-model wave function. Fig. 4 gives a schematical picture of the two-step formation of molecular state on the basis of the duality of the mean-field-type excited state.

In neutron-rich nuclei, clustering is largely governed by valence-neutron structures around clustered core. We have molecular orbits spread in the whole nucleus, atomic orbits around single clusters, and others. Fig. 5 shows AMD study of such variety of nuclear structures in \(^{22}\text{Ne}\) where the core part \((N = Z = 10)\) of the ground band states has dominantly \((8, 0)\) \(SU(3)\) symmetry which has the duality of \(^{16}\text{O} + \alpha\) clustering [16].

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**Figure 3.** Reproduction by AMD of the coexistence of cluster states with mean-field-type states in \(^{44}\text{Ti}\). \(^{40}\text{Ca} + \alpha\) cluster states are shown by bold lines.

**Figure 4.** Two-step formation of molecular state due to the duality of the mean-field-type excited state.

**Figure 5.** Coexistence of cluster states and mean-field-type states in neutron-rich \(^{22}\text{Ne}\) studied with AMD.
5. Spatial localization of clusters and ab initio calculations of cluster states

Convincing evidences of clustering are now abundantly given by ab initio calculations of nuclear structure. A well-known example is the quantum Monte Carlo calculation of $^8$Be by Wiringer et al. [17] which showed spatially-localized $2\alpha$ clusters which are separated from each other by about 2 fm.

Spatial localization of clusters requires very high $\hbar\omega$ for the reproduction of cluster states in shell model. If the $\hbar\omega$ space is not wide enough, cluster states can not be sufficiently bound. This fact tells us that an ab initio shell model calculation gives us a good method to discriminate cluster states from mean-field-type states. For example, no-core shell-model calculations have not succeeded until now to reproduce the low excitation energy of the Hoyle state of $^{12}\text{C}$ [18]. Therefore, no-core shell-model calculations indicate very clearly that the Hoyle state is largely different from the mean-field-type state. This theoretical method of discriminating cluster states from mean-field-type states is very convincing and was not existent in the days of Wildermuth.

Quite recently a similar study was reported in Ref. [19] where $^8\text{Be}$ was investigated by TOSM (tensor optimized shell model). Different from no-core shell model, TOSM includes very high $\hbar\omega$ space, but on the other hand it adopts only $0p-0h$ and $2p-2h$ configurations. Thus, for the well-developed $2\alpha$ structure of the $^8\text{Be}$ ground band states, TOSM gives insufficient description.

Ab initio approaches have a powerful merit which is absent in other approaches to nuclear clustering. In ab initio approaches clustering can be judged not only by density distribution but also by tensor-force effects. A typical example is given in Ref. [20] where it is shown that the probabilities of $S$, $P$, and $D$ states of the second $0^+$ state of $^4\text{He}$ are very close to those of $^3\text{He}$ and $t$. This result strongly supports the $3N+N$ cluster structure of the second $0^+$ state.

It is of course desirable that we have supports to the $\alpha$-condensate model of the Hoyle state by ab initio approaches. However, at present there is no fully converged ab initio description of the Hoyle state. We can even say that the very frontier of ab initio study of nuclear structure is the study of the Hoyle state. Recent result by the quantum Monte Carlo approach given in Ref. [21] is maybe most advanced result. When we compare the density distribution $r^2\rho(r)$ of the Hoyle state between the quantum Monte Carlo calculation and the $3\alpha$ THSR wave function [22],

![Figure 6. Reproduction of $^{8}\text{Be}$ levels by TOSM. The calculated energy position of the first $T=1\ 2^+$ state is shifted to be the same as the observed energy position. Although the mean-field-type states are well reproduced by TOSM, the calculated levels of the ground band are underbound compared with the observed levels, which is due to the insufficiency of TOSM for describing $2\alpha$ cluster states.](image-url)
we see good correspondence between two density distributions, although the quantum Monte Carlo calculation has not yet fully converged for the Hoyle state [23].

6. Summary

(i) Many of basic concepts of nuclear cluster physics since the days of K. Wildermuth are now examined and revived in new lights.

(ii) The concept of the duality of the shell-model wave function of the ground state was the starting point of Wildermuth’s cluster model.

(iii) Widermuth considered that the formation of cluster states is due to the activation of the clustering degree of freedom embedded in the ground state due to its duality nature. But, at the days around 1960, the assignments of cluster structure to excited states were not regarded as convincing.

(iv) Recently it has been discussed that the duality character of the ground state explains well and naturally observed large magnitudes of $E0$ transitions between cluster states and the ground state.

(v) Spatial localization of clusters which is a convincing signature of cluster structure has recently been discussed that it comes from the inter-cluster Pauli repulsion in nonlocalized cluster dynamics.

(vi) Abundance of cluster states is seen in a wide region of nuclear chart including unstable nuclei and up to high excitation energy. Molecular resonance states are discussed through the duality of excited mean-field states, which is in contrast to alpha-cluster states formed through the duality of the ground state.

(vii) Ab initio calculations of nuclear structure are presenting convincing evidences of clustering which include spatial localization, characteristic tensor-force contribution, and gas-like clustering.

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