Frame Coherence and Sparse Signal Processing

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Abstract—The sparse signal processing literature often uses random sensing matrices to obtain performance guarantees. Unfortunately, in the real world, sensing matrices do not always come from random processes. It is therefore desirable to evaluate whether an arbitrary matrix, or frame, is suitable for sensing sparse signals. To this end, the present paper investigates two parameters that measure the coherence of a frame: worst-case and average coherence. We first provide several examples of frames that have small spectral norm, worst-case coherence, and average coherence. Next, we present a new lower bound on worst-case coherence and compare it to the Welch bound. Later, we propose an algorithm that decreases the average coherence of a frame without changing its spectral norm or worst-case coherence. Finally, we use worst-case and average coherence, as opposed to the Restricted Isometry Property, to garner near-optimal probabilistic guarantees on both sparse signal detection and reconstruction in the presence of noise. This contrasts with recent results that only guarantee noiseless signal recovery from frames that have small spectral norm, worst-case coherence, and average coherence. We first provide several examples of parameters that measure the coherence of a frame: worst-case coherence and average coherence. Next, we present a new lower bound on worst-case coherence: 

$$\nu_F := \frac{1}{N - 1} \max_{i \in \{1, \ldots, N\}} \sum_{j=1 \atop j \neq i}^{N} |\langle f_i, f_j \rangle|.$$  

Note that in addition to having zero worst-case coherence, orthonormal bases also have zero average coherence. It was established in [9] that when $\nu_F$ is sufficiently smaller than $\mu_F$, a number of guarantees can be provided for sparse signal processing. It is therefore evident from [1]–[9] that there is a pressing need for nearly tight frames with small worst-case and average coherence, especially in the area of sparse signal processing.

I. INTRODUCTION

Many classical applications, such as radar and error-correcting codes, make use of over-complete spanning systems [1]. Oftentimes, we may view an over-complete spanning system as a frame. Take $F = \{f_i\}_{i \in I}$ to be a collection of vectors in some separable Hilbert space $\mathcal{H}$. Then $F$ is a frame if there exist frame bounds $A$ and $B$ with $0 < A \leq B < \infty$ such that $A\|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B\|x\|^2$ for every $x \in \mathcal{H}$. When $A = B$, $F$ is called a tight frame. For finite-dimensional unit norm frames, where $I = \{1, \ldots, N\}$, the worst-case coherence is a useful parameter:

$$\mu_F := \max_{i,j \in \{1, \ldots, N\}} |\langle f_i, f_j \rangle|.$$  

Note that orthonormal bases are tight frames with $A = B = 1$ and have zero worst-coherence. In both ways, frames form a natural generalization of orthonormal bases.

In this paper, we only consider finite-dimensional frames. Those not familiar with frame theory can simply view a finite-dimensional frame as an $M \times N$ matrix of rank $M$ whose columns are the frame elements. With this view, the tightness condition is equivalent to having the spectral norm be as small as possible; for an $M \times N$ unit norm frame $F$, this equivalently means $\|F\|_2^2 = \frac{N}{M}$.

Throughout the literature, applications require finite-dimensional frames that are nearly tight and have small worst-case coherence [1]–[8]. Among these, a foremost application is sparse signal processing, where frames of small spectral norm and/or small worst-case coherence are commonly used to analyze sparse signals [4]–[9]. Recently, [9] introduced another notion of frame coherence called average coherence:

$$\nu_F := \frac{1}{N - 1} \max_{i \in \{1, \ldots, N\}} \sum_{j=1 \atop j \neq i}^{N} |\langle f_i, f_j \rangle|.$$  

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Definition 1. We say an $M \times N$ unit norm frame $F$ satisfies the Strong Coherence Property if

$$(SCP-1) \quad \mu_F \leq \frac{1}{164\log N} \quad \text{and} \quad (SCP-2) \quad \nu_F \leq \frac{\mu_F}{\sqrt{M}},$$

where $\mu_F$ and $\nu_F$ are given by (1) and (2), respectively.

Since average coherence is so new, there is currently no intuition as to when (SCP-2) is satisfied. As a third contribution, this paper shows how to transform a frame that satisfies (SCP-1) into another frame with the same spectral norm and worst-case coherence that additionally satisfies (SCP-2). Finally, this paper uses the Strong Coherence Property to provide new guarantees on both sparse signal detection and reconstruction in the presence of noise. These guarantees are related to those in [4], [5], [7], and we elaborate on this relationship in Section V. In the interest of space, the proofs have been omitted throughout, but they can be found in [1].

II. FRAME CONSTRUCTIONS

Many applications require nearly tight frames with small worst-case and average coherence. In this section, we give three types of frames that satisfy these conditions.

A. Normalized Gaussian frames

Construct a matrix with independent, Gaussian distributed entries that have zero mean and unit variance. By normalizing the columns, we get a matrix called a normalized Gaussian frame. This is perhaps the most widely studied type of frame in the signal processing and statistics literature.

To be clear, the term “normalized” is intended to distinguish the results presented here from results reported in earlier works, such as [9], [14], [16], which only ensure that the frame elements of Gaussian frames have unit norm in expectation. In other words, normalized Gaussian frames are frames with individual frame elements independently and uniformly distributed on the unit hypersphere in $\mathbb{R}^M$.

That said, the following theorem characterizes the spectral norm and the worst-case and average coherence of normalized Gaussian frames.

Theorem 2 (Geometry of normalized Gaussian frames). Build a real $M \times N$ frame $G$ by drawing entries independently from a Gaussian distribution of zero mean and unit variance. Next, construct a normalized Gaussian frame $F$ by taking $f_n := \frac{g_n}{\sqrt{|g_n|}}$, for every $n = 1, \ldots, N$. Provided $60\log N \leq M \leq \frac{4}{\log N}$, then the following inequalities simultaneously hold with probability exceeding $1 - 11N^{-1}$:

(i) $\mu_F \leq \frac{\sqrt{3} \log N}{\sqrt{M - \sqrt{M + \sqrt{N + 2\log N}}}} $.

(ii) $\nu_F \leq \frac{2N}{\sqrt{M - \sqrt{M + \sqrt{N + 2\log N}}}} $.

(iii) $\|F\|_2 \leq \frac{N^{1/2}}{\sqrt{M^{1/2}N^{-1}}} $.

B. Random harmonic frames

Random harmonic frames, constructed by randomly selecting rows of a discrete Fourier transform (DFT) matrix and normalizing the resulting columns, have received considerable attention lately in the compressed sensing literature [17]–[19]. However, to the best of our knowledge, there is no result in the literature that shows that random harmonic frames have small worst-case coherence. To fill this gap, the following theorem characterizes the spectral norm and the worst-case and average coherence of random harmonic frames.

Theorem 3 (Geometry of random harmonic frames). Let $U$ be an $N \times N$ non-normalized discrete Fourier transform matrix, explicitly, $U_{k\ell} := e^{2\pi i k\ell/N}$ for each $k, \ell, = 0, \ldots, N - 1$. Next, let $\{B_i\}_{i=1}^M$ be a collection of independent Bernoulli random variables with mean $\frac{M}{N}$, and take $M := \{1 : B_i = 1\}$. Finally, construct an $[M] \times N$ harmonic frame $F$ by collecting rows of $U$ which correspond to indices in $M$ and normalizing the columns. Then $F$ is a unit norm tight frame: $\|F\|_2 = \frac{N}{\sqrt{M}}$. Furthermore, provided $16\log N \leq M \leq \frac{N^2}{3}$, the following inequalities simultaneously hold with probability exceeding $1 - 4N^{-1} - N^{-2}$:

(i) $\frac{M}{3} \leq [M] \leq \frac{2}{3}M$,

(ii) $\nu_F \leq \frac{\mu_F}{\sqrt{|M|}}$,

(iii) $\mu_F \leq \frac{118(N - M)\log N}{M^2 N}$.

C. Code-based frames

Many structures in coding theory are also useful for constructing frames. Here, we build frames from a code that originally emerged with Berlekamp in [20], and found recent reincarnation with [21]. We build a $2^m \times 2^m$ frame, indexing rows by elements of $\mathbb{F}_{2^m}$ and indexing columns by $(t + 1)$-tuples of elements from $\mathbb{F}_{2^m}$. For $x \in \mathbb{F}_{2^m}$ and $\alpha \in \mathbb{F}_{2^m}$, the corresponding entry of the matrix $F$ is

$$F_{x\alpha} = \frac{1}{\sqrt{2^m}} (-1)^{\text{Tr}[\alpha, x + \sum_{i=1}^m \alpha_i z^{2^i + 1}]} ,$$

where $\text{Tr} : \mathbb{F}_{2^m} \to \mathbb{F}_2$ denotes the trace map, defined by $\text{Tr}(z) = \sum_{i=0}^{m-1} z^{2^i}$. The following theorem gives the spectral norm and the worst-case and average coherence of this frame.

Theorem 4 (Geometry of code-based frames). The $2^m \times 2^m$ frame defined by $F$ is unit norm tight, i.e., $\|F\|_2^2 = 2^m$, with worst-case coherence $\mu_F \leq \frac{1}{\sqrt{2^{2m} - 2^{m+1}}}$ and average coherence $\nu_F \leq \frac{\mu}{\sqrt{2^m}}$.

III. FUNDAMENTAL LIMITS ON WORST-CASE COHERENCE

In many applications of frames, performance is dictated by worst-case coherence. It is therefore particularly important to understand which worst-case coherence values are achievable. To this end, the following bound is commonly used in the literature:

Theorem 5 (Welch bound [1]). Every $M \times N$ unit norm frame $F$ has worst-case coherence $\mu_F \geq \frac{N - M}{M(N - 1)}$.

The Welch bound is not tight whenever $N > M^2$ [1]. For this region, the following gives a better bound:

Theorem 6 ([10], [11]). Every $M \times N$ unit norm frame $F$ has worst-case coherence $\mu_F \geq 1 - 2N^{-1/(M - 1)}$. Taking $N = \Theta(M^2)$, this lower bound goes to $1 - \frac{2}{M}$ as $M \to \infty$. 

For many applications, it does not make sense to use a
complex frame, but the bound in Theorem 6 is known to be
loose for real frames [12]. We therefore improve Theorem 6
for the case of real unit norm frames:

**Theorem 7.** Every real \( M \times N \) unit norm frame \( F \) has worst-
case coherence

\[
\mu_F \geq \cos \left[ \pi \left( \frac{M-1}{N} \right) \frac{\Gamma \left( \frac{M-1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \right].
\]  

(4)

Furthermore, taking \( N = \Theta(a^M) \), this lower bound goes to \( \cos \left( \frac{\pi}{2} \right) \) as \( M \to \infty \).

In [12], numerical results are given for \( M = 3 \), and we
compare these results to Theorems 6 and 7 in Figure 1.
Considering this figure, we note that the bound in Theorem 6 is
inferior to the maximum of the Welch bound and the bound in
Theorem 7 at least when \( M = 3 \). This illustrates the degree
to which Theorem 7 improves the bound in Theorem 6 for
real frames. In fact, since \( \cos \left( \frac{\pi}{2} \right) \geq 1 - \frac{2}{a} \) for all \( a \geq 2 \), the
bound for real frames in Theorem 7 is asymptotically better
than the bound for complex frames in Theorem 6. Moreover,
for \( M = 2 \), Theorem 7 says \( \mu \geq \cos \left( \frac{\pi}{4} \right) \), and [22] proved this bound to be tight for every \( N \geq 2 \). For \( M = 3 \), Theorem 7
can be further improved as follows:

**Theorem 8.** Every real \( 3 \times N \) unit norm frame \( F \) has worst-
case coherence \( \mu_F \geq 1 - \frac{4}{N} + \frac{1}{N^2} \).

**IV. Reducing average coherence**

In [9], average coherence is used to garner a number of
guarantees on sparse signal processing. Since average coherence
is so new to the frame theory literature, this section will
investigate how average coherence relates to worst-case coherence
and the spectral norm. We start with a definition:

**Definition 9 (Wiggling and flipping equivalent frames).** We
say the frames \( F \) and \( G \) are *wiggling equivalent* if there exists
a diagonal matrix \( D \) of unimodular entries such that \( G = FD \).

Furthermore, they are *flipping equivalent* if \( D \) is real, having
only ±1’s on the diagonal.

The terms “wiggling” and “flipping” are inspired by the
fact that individual frame elements of such equivalent frames
are related by simple unitary operations. Note that every
frame with \( N \) nonzero frame elements belongs to a flipping
equivalence class of size \( 2^N \), while being wiggling equivalent
to uncountably many frames. The importance of this type of
frame equivalence is, in part, due to the following lemma,
which characterizes the shared geometry of wiggling equivalent
frames:

**Lemma 10 (Geometry of wiggling equivalent frames).** Wiggling
equivalence preserves the norms of frame elements, the
worst-case coherence, and the spectral norm.

Now that we understand wiggling and flipping equivalence,
we are ready for the main idea behind this section. Suppose we
are given a unit norm frame with acceptable spectral norm and
worst-case coherence, but we also want the average coherence
to satisfy (SCP-2). Then by Lemma 10 all of the wiggling
equivalent frames will also have acceptable spectral norm and
worst-case coherence, and so it is reasonable to check these
frames for good average coherence. In fact, the following
theorem guarantees that at least one of the flipping equivalent
frames will have good average coherence, with only modest
requirements on the original frame’s redundancy:

**Theorem 11 (Frames with low average coherence).** Let \( F \) be
an \( M \times N \) unit norm frame with \( M < \frac{N-1}{4 \log 2N} \). Then there
exists a frame \( G \) that is flipping equivalent to \( F \) and satisfies
\( \nu_G \leq \frac{\nu_F}{\sqrt{M}} \).

While Theorem 11 guarantees the existence of a flipping
equivalent frame with good average coherence, the result does
not describe how to find it. Certainly, one could check all \( 2^N \)
frames in the flipping equivalence class, but such a procedure
is computationally slow. As an alternative, we propose a linear-
time flipping algorithm (Algorithm 1). The following theorem
guarantees that linear-time flipping will produce a frame with
good average coherence, but it requires the original frame’s
redundancy to be higher than what suffices in Theorem 11.

**Theorem 12.** Suppose \( N \geq M^2 + 3M + 3 \). Then Algorithm 1

\[
\begin{aligned}
\text{Input:} & \text{ An } M \times N \text{ unit norm frame } F \\
\text{Output:} & \text{ An } M \times N \text{ unit norm frame } G \text{ that is flipping equivalent to } F \\
& g_1 \leftarrow f_1 \quad \{ \text{Keep first frame element} \} \\
& \text{for } n = 2 \text{ to } N \text{ do} \\
& \quad \text{if } \| \sum_{i=1}^{n-1} g_i + f_n \| \leq \| \sum_{i=1}^{n-1} g_i - f_n \| \text{ then} \\
& \quad \quad g_n \leftarrow f_n \quad \{ \text{Keep frame element for shorter sum} \} \\
& \quad \text{else} \\
& \quad \quad g_n \leftarrow -f_n \quad \{ \text{Flip frame element for shorter sum} \} \\
& \text{end if} \\
& \text{end for}
\end{aligned}
\]
outputs an $M \times N$ frame $G$ that is flipping equivalent to $F$ and satisfies $\nu_G \leq \frac{\nu_F}{\sqrt{M}}$.

As an example of how linear-time flipping improves average coherence, consider the following matrix:

$$F := \frac{1}{\sqrt{5}} \begin{bmatrix} + & + & + & + & - & + & + & - & + & + & + & + & - & + & + & + & + & + & + & + & + & + & - & - & - & + & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & -&
Algorithm 2 One-Step Thresholding (OST) [9]

Input: An $M \times N$ unit norm frame $F$, a vector $y = Fx + e$, and a threshold $\lambda > 0$

Output: An estimate $\hat{K} \subseteq \{1, \ldots, N\}$ of the support of $x$ and an estimate $\hat{x} \in \mathbb{C}^N$ of $x$

1. $\hat{x} \leftarrow 0$ \text{ [Initialize]}
2. $z \leftarrow F^*y$
3. $\hat{K} \leftarrow \{n : |z_n| > \lambda\}$ \text{ [Select indices via OST]}
4. $\hat{x}_\hat{K} \leftarrow (F_{\hat{K}})^{\dagger} y$ \text{ [Reconstruct signal via least-squares]}

$t \in (0, 1)$ to denote the locations of all the entries of $x$ that, roughly speaking, lie above the noise floor $\sigma$. Finally, we use $T_\mu(t) := \{n : |x_n| > 20 \mu F \|x\| \sqrt{2 \log N}\}$ to denote the locations of entries of $x$ that, roughly speaking, lie above the self-interference floor $\mu F \|x\|$.

Theorem 15 (Reconstruction of sparse signals). Take an $M \times N$ unit norm frame $F$ which satisfies the Strong Coherence Property, pick $t \in (0, 1)$, and choose $\lambda = 2\sqrt{\sigma^2 \log N} \max\left\{\frac{\|\mu F \|\sqrt{\text{SNR}}}{\sqrt{2}}, \frac{20}{\sqrt{2}}\right\}$. Further, suppose $x \in \mathbb{C}^N$ has support $K$ drawn uniformly at random from all possible $K$-subsets of $\{1, \ldots, N\}$. Then provided

$$K \leq \frac{c \sqrt{N}}{\sigma \|F\|_2 \sqrt{\log N}},$$

Algorithm 2 produces $\hat{K}$ such that $T_\sigma(t) \cap T_\mu(t) \subseteq \hat{K} \subseteq K$ and $\hat{x}$ such that

$$\|x - \hat{x}\| \leq c_2 \sqrt{\sigma^2 |K| \log N} + c_3 \|x_K \setminus \hat{K}\|$$

with probability exceeding $1 - 10N^{-1}$. Finally, defining $T := |T_\sigma(t) \cap T_\mu(t)|$, we further have

$$\|x - \hat{x}\| \leq c_2 \sqrt{\sigma^2 |K| \log N} + c_3 \|x_{K \setminus \hat{K}}\|$$

in the same probability event. Here, $c_1 = 37e$, $c_2 = \frac{1}{2} + e^{-1-\gamma}$, and $c_3 = 1 + \frac{e}{2+e-\gamma}$ are numerical constants.

A few remarks are in order now for Theorem 15. First, if $F$ satisfies the Strong Coherence Property and $F$ is nearly tight, then OST handles sparsity that is almost linear in $M$: $K = O(M \log N)$ from [3]. Second, the $\ell_2$ error associated with the OST algorithm is the near-optimal (modulo the log factor) error of $\sqrt{\sigma^2 |K| \log N}$ plus the best $T$-term approximation error caused by the inability of the OST algorithm to recover signal entries that are smaller than $O(\mu F \|x\| \sqrt{2 \log N})$. Nevertheless, it is easy to convince oneself that such error is still near-optimal for large classes of sparse signals. Consider, for example, the case where $\mu F = O(1/\sqrt{M})$, the magnitudes of $K/2$ nonzero entries of $x$ are some $\alpha = \Omega(\sqrt{\sigma^2 \log N})$, while the magnitudes of the other $K/2$ nonzero entries are not necessarily same but scale as $O(\sqrt{\sigma^2 \log N})$. Then we have from Theorem 15 that $\|x - x_F\| = O(\sqrt{\sigma^2 |K| \log N})$, which leads to near-optimal $\ell_2$ error of $\|x - \hat{x}\| = O(\sqrt{\sigma^2 |K| \log N})$. To the best of our knowledge, this is the first result in the sparse signal processing literature that does not require RIP and still provides near-optimal reconstruction guarantees for such signals in the presence of noise, while using either random or deterministic frames, even when $K = O(M \log N)$.

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