SOURCES OF HIGH LEVERAGE
IN LINEAR REGRESSION MODEL

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Abstract. Some reasons for high leverage are analytically investigated by decomposing leverage into meaningful components. The results in this work can be used for remedial action as a next step of data analysis.

AMS Mathematics Subject Classification : 62J20.
Key words and phrases : leverage, linear regression, outlier.

1. Introduction

Inclusion of high leverage points in regression data can mislead our conclusion and cause some other statistical problems. Regression outliers may not be identified by looking at the least squares residuals when the outliers are high leverage points because high leverage points tend to have very small residuals as the least squares fit is pulled too much in the direction of these outlying points ([1], [4], [6]). Gunst [2] and Mason and Gunst [5] argued that if a high leverage point possesses extreme values on some regressors, then it can induce a collinearity among regressors. Mason and Gunst [5] showed that collinearity can be increased without bound by increasing the leverage of a point. A high leverage point can hide or create collinearity ([1], [3], [7]).

Chatterjee and Hadi [1] gave some conditions for high leverage. However, no attempt has been made to uncover sources of high leverage. In this work we will make an analytic investigation of reasons for high leverage. To this end two decompositions of leverage into meaningful components are derived. For these derivations some preliminary results are also obtained. The results in this work can be used for remedial action as a next step of data analysis.
2. Preliminaries

Consider a multiple linear regression model defined by
\[ y = \beta_0 1_n + X \beta + \varepsilon, \]
where \( y \) is an \( n \times 1 \) vector of observations on a response variable, \( X = (x_1, ..., x_n)^T \) is an \( n \times p \) matrix of measurements on \( p \) regressors, \( 1_n \) is the \( n \times 1 \) vector of all elements equal to one, \( \beta_0 \) and \( \beta = (\beta_1, ..., \beta_p)^T \) are unknown regression coefficients, and \( \varepsilon \) is an \( n \times 1 \) vector of unobservable random errors.

Let \( Z = (1_n X) \). Then the hat matrix is \( H = Z(Z^T Z)^{-1}Z^T \). We have the following identity ([1])
\[ H = \frac{1}{n} 1_n 1_n^T + \tilde{X}(\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T , \]
where \( \tilde{X} = (I_n - (1/n) 1_n 1_n^T)X \) is the centered data matrix and \( I_n \) is the identity matrix of order \( n \). The leverage of the \( r \)-th observation is the \( r \)-th diagonal element \( h_{rr} \) of \( H \) given by
\[ h_{rr} = \frac{1}{n}(1 + D_r^2) \]
where \( D_r^2 = n(x_r - \bar{x})^T (\tilde{X}^T \tilde{X})^{-1} (x_r - \bar{x}) \) is the squared Mahalanobis distance from \( x_r \) to the data mean \( \bar{x} = (1/n) \sum_{j=1}^n x_j \). Eq. (1) implies that sources of high leverage can be investigated by figuring out reasons for large \( D_r^2 \).

2.1. Regression of a regressor on the other regressors

The \( i \)-th column of \( \tilde{X} \) is written as \( \tilde{x}_i \). Let \( A \) be the lower triangular matrix with positive diagonal elements such that \( (1/n) \tilde{X}^T \tilde{X} = AA^T \). That is, \( A \) is the Cholesky root of the data covariance matrix. We put \( B^T = A^{-1} \) and denote the last column of \( B \) by \( b = (b_1, ..., b_p)^T \). We partition
\[ A = \begin{bmatrix} A_1 & 0 \\ a_1^T \\ a_{pp} \end{bmatrix} \]
such that \( A_1 \) is the leading principal submatrix, having order \( p - 1 \), of \( A \). If we partition \( \tilde{X} = (\tilde{X}_1 \tilde{x}_p) \), then the least squares estimator of the vector of regression coefficients for the regression of \( \tilde{x}_p \) on the other regressors is easily computed as
\[ (\tilde{X}_1^T \tilde{X}_1)^{-1} \tilde{X}_1^T \tilde{x}_p = (A_1 A_1^T)^{-1} A_1 a_1 = - \frac{1}{b_{pp}} (b_{1p}, ..., b_{p-1,p})^T. \]
Sources of High Leverage

The \( r \)-th residual from the regression of the \( p \)-th regressor on the other regressors becomes

\[
\frac{1}{b_{pp}} b^T (x_r - \bar{x}).
\]

The error sum of squares is computed as \( \text{SSE} = nb_{pp}^{-2} \). Hence the \( r \)-th standardized residual is \( b^T (x_r - \bar{x}) \) multiplied by a constant.

In general, similar results can be obtained for the regression of the \( i \)-th (\( 1 \leq i \leq p \)) regressor on the remaining regressors by appropriately permuting regressors.

### 2.2. Multiple correlation coefficient

Let \( r_i^2 \) be the squared multiple correlation coefficient of the \( i \)-th regressor with the remaining regressors and \( s_i^2 = (1/n) \sum_{r=1}^{n} (x_{ri} - \bar{x}_i)^2 \) be the data variance of the \( i \)-th regressor, where \( x_{ri} \) is the \( i \)-th element of \( x_r \) and \( \bar{x}_i \) is the \( i \)-th element of \( \bar{x} \). Then the squared multiple correlation coefficient of the last regressor \( \tilde{x}_p \) with the other regressors is easily computed as

\[
1 - r_p^2 = (s_p^2 b_{pp}^2)^{-1}.
\]

Further we have

\[
1 - r_p^2 = (s_p^2 b_{pp}^2)^{-1}
\]

since \( s_p^2 = (1/n) \tilde{x}_p^T \tilde{x}_p = a_1^T a_1 + a_{pp}^2 \).

In general, we can compute \( r_i^2 \) (\( 1 \leq i \leq p \)) by appropriately permuting regressors.

### 3. Some reasons for high leverage

In this section we provide two decompositions of leverage, using the results in Section 2, that can explain some reasons for high leverage.

#### 3.1. Decomposition I

We denote a generic vector of \( p \) regressor variables by \( x \). Let \( x_{(i)} \) be a vector obtained by interchanging the \( i \)-th and \( p \)-th regressors of \( x \) and \( P_{(i)} \) be the associated permutation matrix such that \( x_{(i)} = P_{(i)} x \). Let \( \tilde{X}_{(i)} = \tilde{X} P_{(i)} \). It is
understood that \( x_{(p)} = x \), \( \tilde{X}_{(p)} = \tilde{X} \) and \( P_{(p)} = I_p \). Let \( A_{(i)} \) be the Cholesky root of the data covariance matrix for \( x_{(i)} \), that is, \( (1/n)\tilde{X}^T_{(i)}\tilde{X} = A_{(i)}A_{(i)}^T \). This can be expressed as \( \{ (1/n)\tilde{X}^T_{(i)}\tilde{X} \} P_{(i)} b_{(i)} = P_{(i)} a_{(i)} \) (\( i = 1, \ldots, p \)), where \( a_{(i)} \) and \( b_{(i)} \) are the last columns of \( A_{(i)} \) and \( B_{(i)} \), respectively. Collection of these \( p \) equations into a matrix form gives the inverse of the data covariance matrix as

\[
\left( \frac{1}{n} \tilde{X}^T \tilde{X} \right)^{-1} = [b_{(1)pp} P_{(1)} b_{(1)}, \ldots, b_{(p)pp} P_{(p)} b_{(p)}],
\]

where \( b_{(i)pp} \) is the last element of \( b_{(i)} \). Inserting (5) into \( D_r^2 \) with use of (4) yields the following decomposition of \( D_r^2 \)

\[
D_r^2 = \sum_{i=1}^{p} (1 - r_{ii}^2)^{-1/2} [b_{(i)pp} P_{(i)}(x_{ri} - \bar{x})] \left( \frac{x_{ri} - \bar{x}_i}{s_i} \right).
\]

Note that \( A_{(p)} = A \), \( B_{(p)} = B \), \( b_{(pp)} = b_{pp} \) and \( b_{(p)} = b \).

Together with (11), eq. (13) reveals some sources of high leverage. The \( i \)-th term of (13) shows that the contribution of the \( i \)-th term to the \( r \)-th leverage \( h_{rr} \) depends on three components. The first component \( (1 - r_{ii}^2)^{-1/2} \) will be large whenever there is a high relationship between the \( i \)-th regressor and the set of remaining regressors. In this case the contribution of \( (1 - r_{ii}^2)^{-1/2} \) to the \( r \)-th leverage \( h_{rr} \) will be effective. The second component \( b_{(i)pp} P_{(i)}(x_{ri} - \bar{x}) \) is the standardized residual from the regression of the \( i \)-th regressor on the remaining regressors in the light of (13), and it will be large in its absolute value whenever the value of the \( i \)-th regressor \( x_{ri} \) is far from the hyperplane formed by the remaining regressors, that is, whenever \( x_{ri} \) is an outlier for the regression of the \( i \)-th regressor on the remaining regressors. The third component \( (x_{ri} - \bar{x}_i)/s_i \) indicates a marginally standardized deviation and it will be large in its absolute value whenever \( x_{ri} \) is a marginally outlier.

### 3.2. Decomposition II

If we partition the data covariance matrix \( (1/n)\tilde{X}^T \tilde{X} = AA^T \) according to the partition of \( A \) in (2), then the inverse of the resulting partitioned matrix is computed as

\[
\left( \frac{1}{n} \tilde{X}^T \tilde{X} \right)^{-1} = \begin{bmatrix}
(A_1 A_1^T)^{-1} + b_1 b_1^T & b_{pp} b_1 \\
 b_{pp} b_1^T & b_{pp}^2
\end{bmatrix},
\]

where \( b_1 = (b_{1p}, \ldots, b_{p-1,p})^T \). Note that \( A_1 A_1^T \) is the data covariance matrix for the first \( p - 1 \) regressor variables. Let \( x_{r(-p)} = (x_{r1}, \ldots, x_{rp-1})^T \) and \( \bar{x}_{(-p)} = \)
$(\bar{x}_1, \ldots, \bar{x}_{p-1})^T$. We write the squared Mahalanobis distance based on the first $p-1$ regressor variables as $D^2_{r(-p)} = (x_r(-p) - \bar{x}(-p))^T (A_1 A_1^T)^{-1} (x_r(-p) - \bar{x}(-p))$. Then a little computation with use of (7) gives the following decomposition of $D^2_r$

$$D^2_r = D^2_{r(-p)} + [b^T(x_r - \bar{x})]^2. \tag{8}$$

In view of this decomposition, the removal of the $p$-th regressor from the regression model decreases the value of $h_{rr}$ by the second term of (8) divided by $n$ when the remaining regressors are still kept in the model.

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