CONTROLLABILITY OF THE LAGUERRE AND THE JACOBI EQUATIONS.

DIOMEDES BARCENAS\(^{(1)}\), HUGO LEIVA\(^{(1)}\), YAMILTEL QUINTANA\(^{(2)}\) AND WILFREDO URBINA\(^{(1)}\)

Abstract. In this paper we study the controllability of the controlled Laguerre equation and the controlled Jacobi equation. For each case, we found conditions which guarantee when such systems are approximately controllable on the interval \([0, t_1]\). Moreover, we show that these systems can never be exactly controllable.

Key words and phrases. Laguerre equation, Jacobi equation, controllability, compact semi-group.

2001 Mathematics Subject Classification. Primary 93B05. Secondary 93C25.

1. Introduction.

The study of orthogonal polynomials which are eigenfunctions of a differential operator have a long history. In 1929 S. Bochner \([4]\) posed the problem of determining all families of orthogonal polynomials in \(\mathbb{R}\) that are eigenfunctions of some arbitrary but fixed second-order differential operators. In that article, he proved that this property characterizes the so-called classical orthogonal polynomials, linked with the names of Hermite, Laguerre and Jacobi (this last family containing as particular cases the Legendre, Tchebychev and Gegenbauer polynomials). Later H.L. Krall and O. Frink \([16]\) considered the Bessel polynomials, that are also orthogonal polynomials that satisfies a second order equation, but their orthogonality measure does not have support is \(\mathbb{R}\) but on the unit circle of the complex plane. The general problem, for a differential operator of any order was posed by H. L. Krall \([14]\) in 1938, he proved that the differential operator has to be of even order and, in \([15]\), he obtained a complete classification for the case of an operator of order four (see \([5], [14], [15]\) and \([19]\) for a more detailed references and further developments). There have been recent developments in the direction of connecting the study of orthogonal polynomials with modern problems related to Harmonic Analysis and PDE’s, see for instance \([9], [12], [24]\).

Date: October, 2006.

\((1)\) Research partially supported by ULA and FONACIT#G-97000668
\((2)\) Research partially supported by DID-USB under Grant DI-CB-015-04.
On the other hand, it is well known that many differential equations can be solved using the separation variable method, obtaining solutions in terms of a orthogonal expansion. Nevertheless, is an absolute merit of C. Sturm and J. Liouville in the 1830s, the knowledge of the existence of such solutions - long before the advent of Hilbert spaces Theory in the XX-th century-. Their results were precursors of the Operator Theory, but from our present viewpoint can be more naturally obtained as consequences of the spectral Theorem for compact hermitian operators (the reader is referred to [26] for the proof of this statement).

With respect to recent developments in controllability of evolution equations of fluid mechanics and controllability of the wave and heat equations via numerical approximation schemes, we refer to [13] and [27], respectively.

Following the point of view of connecting the study of diverse aspects of Orthogonal Polynomials Theory with PDE’s, in this paper we are going to study:

(1) The controllability of controlled Laguerre equation

\[ z_t - \sum_{i=1}^{d} \left( x_i \frac{\partial^2 z}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial z}{\partial x_i} \right) + \sum_{n=0}^{\infty} \sum_{|\nu|=n} u_{n}(t) \langle b, l_{\nu}^\alpha \rangle_{\mu_\alpha}, \quad t > 0, \quad x \in \mathbb{R}^d, \]

where \{l_{\nu}^\alpha\} are the normalized Laguerre polynomials of type \( \alpha \) in \( d \) variables which are orthogonal polynomials with respect to the the Gamma measure in \( \mathbb{R}^d_+ \), \( \mu_\alpha(x) = \prod_{i=1}^{d} \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + 1)} x_{\alpha_i} \), \( b \in L^2(\mathbb{R}^d_+, \mu_\alpha) \) and the control \( u \in L^2(0, t_1; l^2) \), where with \( l^2 \) the Hilbert space complex square sumable sequences, that for convenience, it will be written as

\[ l^2 = \left\{ U = \{U_{\nu}\}_{|\nu|=n} : U_{\nu} \in \mathbb{C}, \sum_{n=0}^{\infty} \sum_{|\nu|=n} |U_{\nu}|^2 < \infty \right\}, \]

with the inner product and norm defined as

\[ \langle U, V \rangle_{l^2} = \sum_{n=0}^{\infty} \sum_{|\nu|=n} U_{\nu} \overline{V_{\nu}}, \quad \|U\|^2_{l^2} = \sum_{n=0}^{\infty} \sum_{|\nu|=n} |U_{\nu}|^2, \quad U, V \in l^2. \]

We will prove the following statement: If for all \( \nu = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathbb{N}_0^d \)

\[ \langle b, l_{\nu}^\alpha \rangle_{\mu_\alpha} = \int_{\mathbb{R}^d_+} b(x) l_{\nu}^\alpha(x) \mu_\alpha(dx) \neq 0, \]

then the system is approximately controllable on \([0, t_1]\). Moreover, the system can never be exactly controllable.
In particular, we consider the Laguerre equation in one variable with a single control
\[ z_t = xz_{xx} + (\alpha + 1 - x)z_x + b(x)u \quad t \geq 0, \quad x \in \mathbb{R}_+, \]
where \( b \in L^2(\mathbb{R}_+, \mu) \) and the control \( u \) belong to \( L^2(0, t_1; \mathbb{R}_+) \). This system is approximately controllable if and only if
\[
\int_{\mathbb{R}_+} b(x)\ell^\alpha(x)x^{-\alpha}e^\nu dx \neq 0, \quad \nu = 0, 1, 2, \ldots.
\]

(2) The controllability of controlled Jacobi equation

\[
(1.2) \quad z_t = \sum_{i=1}^d \left[ (1-x_i^2)\frac{\partial^2 z}{\partial x_i^2} + (\beta - \alpha - (\alpha_i + \beta_i + 2) x_i)\frac{\partial z}{\partial x_i} \right] + \sum_{n=0}^\infty \sum_{|\nu|=n} u_\nu(t)\langle b, p^{\alpha,\beta}_\nu \rangle_{\mu,\alpha,\beta} p^{\alpha,\beta}_\nu,
\]

\( t > 0, \quad x \in [-1, 1]^d \) where \( \{p^{\alpha,\beta}_\nu\} \) are the normalized Jacobi polynomials of type \( \alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}_+^d, \alpha_i, \beta_i > -1 \), in \( d \) variables, which are orthogonal polynomials with respect to the Jacobi measure in \([-1, 1]^d\mu_{\alpha,\beta}(x) = \prod_{i=1}^d(1-x_i)^{\alpha_i}(1+x_i)^{\beta_i} \ dx, \ b \in L^2([-1, 1]^d, \mu_{\alpha,\beta}) \) and the control \( u \in L^2(0, t_1; L^2) \).

Analogous to the previous case, we will prove that if for all \( \nu = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathbb{N}_0^d \)
\[
\langle b, p^{\alpha,\beta}_\nu \rangle_{\mu,\alpha,\beta} = \int_{[-1,1]^d} b(x)p^{\alpha,\beta}_\nu(x)\mu_{\alpha,\beta}(dx) \neq 0,
\]
then the system is approximately controllable on \([0, t_1]\); but, it can never be exactly controllable.

Also, in particular, for \( \alpha, \beta > -1 \) we consider the Jacobi equation in one variable with a single control
\[
z_t = (1-x^2)z_{xx} + (\beta - \alpha - (\alpha + \beta + 2) x)z_x + b(x)u, \quad t \geq 0, \quad x \in [-1, 1],
\]
where \( b \in L^2([-1, 1], \mu_{\alpha,\beta}) \) and the control \( u \) belong to \( L^2(0, t_1; [-1, 1]) \). This system is approximately controllable if and only if
\[
\int_{[-1,1]} b(x)p^{\alpha,\beta}_\nu(1-x)^{-\alpha}(1+x)^{-\beta} dx \neq 0, \quad \nu = 0, 1, 2, 3, \ldots.
\]
The Laguerre differential operator,

\[
(1.3) \quad L^\alpha = -\sum_{i=1}^d \left[ x_i\frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i} \right]
\]
and the Jacobi differential operator,

\[
(1.4) \quad L^{\alpha,\beta} = -\sum_{i=1}^d \left[ (1-x_i^2)\frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i)\frac{\partial}{\partial x_i} \right]
\]
are well-known operators in the theory Orthogonal Polynomials, in Probability Theory, in Quantum Mechanics and in Differential Geometry (see [12], [18], [19], [20], [23]).

With the results of this paper, we complete the study of controllability problem for the operators associated to classical orthogonal polynomials. In a previous paper [3] it was considered the case of Ornstein-Uhlenbeck operator, and as far as we know, these controlled equations have not been studied until now. Also we obtain results, as in [22], on approximate controllability for some higher dimensional systems associated to a Sturm-Liouville operators of the form

$$\mathcal{L} = \frac{1}{\rho(x)} \sum_{i,j=1}^d \partial_{x^i} (a_{ij}(x)\partial_{x^j}),$$

where $x \in \mathbb{R}^d$, $\rho : \mathbb{R}^d \to \mathbb{R}$ is a constant function and $A(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ is a constant matrix. It remains open the study of the general case. The arguments used in this paper can be extended to this more general setting.

Two important tools which allow to improve and complete the study of controllability problem for the operator associated to classical orthogonal polynomials were used in [3] and come from [2] (Theorem 3.3) and [9] (Theorem A.3.22).

The outline of the paper is the following. Section 2 is dedicated to preliminary results. Section 3 we present main results of the paper, the controllability of the controlled Laguerre equation (1.1) and the controllability of the controlled Jacobi equation (1.2).

2. PRELIMINARY RESULTS.

In this section we shall choose the spaces where our problems will be set and we shall present some results that are needed in the next section. Also, we will give the definition of exact and approximate controllability.

To deal with polynomials in several variables we use the standard multi-index notation. A multi-index is denoted by $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$, where $\mathbb{N}_0$ is the set of non negative integers numbers. For $\nu \in \mathbb{N}_0^d$ we denote by $\nu! = \prod_{i=1}^d \nu_i!$, $|\nu| = \sum_{i=1}^d \nu_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_1^{\nu_1} \cdots \partial_d^{\nu_d}$.

Then the normalized Laguerre polynomials of type $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, $\alpha_i > -1$, and order $\nu$ in $d$ variables is given by the tensor product

$$l_\nu^\alpha(x) = \frac{\sqrt{\nu!}}{\sqrt{\Gamma(\alpha + \nu + 1)}} \prod_{i=1}^d (-1)^{\alpha_i} x_i^{-\alpha_i} e^{x_i} \partial_i^{\nu_i} (x_i^\nu e^{x_i} x_i^\nu e^{-x_i}).$$

It is well known, that the Laguerre polynomials are eigenfunctions of the Laguerre operator $\mathcal{L}^\alpha$,

$$\mathcal{L}^\alpha l_\nu^\alpha(x) = -|\nu| l_\nu^\alpha(x).$$
Given a function \( f \in L^2(\mathbb{R}_+^d, \mu_\alpha) \) its \( \nu \)-Fourier-Laguerre coefficient is defined by

\[
\langle f, l^\alpha_{\nu} \rangle_{\mu_\alpha} = \int_{\mathbb{R}_+^d} f(x) l^\alpha_{\nu}(x) \mu_\alpha(dx),
\]

Let \( C^\alpha_n \) be the closed subspace of \( L^2(\mathbb{R}_+^d, \mu_\alpha) \) generated by \( \{l^\alpha_{\nu} : |\nu| = n\} \), \( C^\alpha_n \) is a finite dimensional subspace of dimension \( \binom{n+d-1}{n} \). By the orthogonality of the Laguerre polynomials with respect to \( \mu_\alpha \) it is easy to see that \( \{C^\alpha_n\} \) is a orthogonal decomposition of \( L^2(\mathbb{R}_+^d, \mu_\alpha) \),

\[
L^2(\mathbb{R}_+^d, \mu_\alpha) = \bigoplus_{n=0}^\infty C^\alpha_n,
\]

which is called the Wiener-Laguerre chaos.

The orthogonal projection \( P^\alpha_n \) of \( L^2(\mathbb{R}_+^d, \mu_\alpha) \) onto \( C^\alpha_n \) is given by

\[
P^\alpha_n f = \sum_{|\alpha|=n} \langle f, l^\alpha_{\nu} \rangle_{\mu_\alpha} l^\alpha_{\nu}, \quad f \in L^2(\mathbb{R}_+^d, \mu_\alpha),
\]

and for a given \( f \in L^2(\mathbb{R}_+^d, \mu_\alpha) \) its Laguerre expansion is given by \( f = \sum_n P^\alpha_n f \).

Using this notation one can prove the following espectral decomposition of \( \mathcal{L}^\alpha \)

\[
\mathcal{L}^\alpha f = \sum_{n=0}^\infty (-n) P^\alpha_n f, \quad f \in L^2(\mathbb{R}_+^d, \mu_\alpha),
\]

and its domain \( D(\mathcal{L}^\alpha) \) is

\[
D(\mathcal{L}^\alpha) = \left\{ f \in L^2(\mathbb{R}_+^d, \mu_\alpha) : \sum_{n=0}^\infty n^2 \|P^\alpha_n f\|_{2,\mu_\alpha} < \infty \right\}.
\]

Let \( Z = L^2(\mathbb{R}_+^d, \mu_\alpha) \) and \( l^2 \) be the Hilbert space of complex square summable sequences. Now, suppose that \( b \) is a fixed element of \( Z \) and consider the linear and bounded operator \( B : l^2 \to Z \) defined by

\[
BU = \sum_{n=0}^\infty \sum_{|\nu|=n} U_{\nu}(b, l^\alpha_{\nu}) l^\alpha_{\nu}.
\]

Then, the system (1.1) can be written as follows

\[
z' = \mathcal{L}^\alpha z + Bu, \quad t > 0.
\]

By a similar way, the normalized Jacobi polynomials of type \( \alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}_+^d, \alpha_i, \beta_i > -1 \), of order \( \nu \) in \( d \) variables is given by the tensor product
Consider the set,

\[ f \]

and given a function \( f \) with \( \nu \)-Fourier-Jacobi coefficient is defined by

\[
\langle f, p^{\alpha,\beta}_\nu \rangle_{\mu,\alpha,\beta} = \int_{[-1,1]^d} f(x) p^{\alpha,\beta}_\nu(x) \mu_\alpha(dx).
\]

As the eigenvalues of the Jacobi operator are not linear in \( n \), following [1] we are going to consider a alternative decomposition, in order to obtain an espectral decomposition of \( \mathcal{L}^{\alpha,\beta} f \) for any \( f \in L^2([-1,1]^d, \mu_{\alpha,\beta}) \) in terms of the orthogonal projections.

For fixed \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d), \beta = (\beta_1, \beta_2, \ldots, \beta_d) \), in \( \mathbb{R}^d \) such that \( \alpha_i, \beta_i > -\frac{1}{2} \) let us consider the set,

\[
R^{\alpha,\beta} = \left\{ r \in \mathbb{R}^+ : \text{there exists } (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}_0^d, \text{ with } r = \sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1) \right\}.
\]

\( R^{\alpha,\beta} \) is a numerable subset of \( \mathbb{R}^+ \), we can write an enumeration of \( R^{\alpha,\beta} \) as \( \{r_n\}_{n=0}^\infty \) with \( 0 = r_0 < r_1 < \ldots \). Let

\[
A^{\alpha,\beta}_n = \left\{ \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}_0^d : \sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1) = r_n \right\}.
\]

Notice that \( A^{\alpha,\beta}_0 = \{(0, \ldots, 0)\} \) and that if \( \kappa \in A^{\alpha,\beta}_n \) then \( \sum_{i=1}^d \kappa_i (\kappa_i + \alpha_i + \beta_i + 1) = r_n \).

Let \( C^{\alpha,\beta}_n \) denote the closed subspace of \( L^2([-1,1]^d, \mu_{\alpha,\beta}) \) generated by the linear combinations of \( \{p^{\alpha,\beta}_\kappa : \kappa \in A^{\alpha,\beta}_n\} \). By the orthogonality of the Jacobi polynomials with respect to \( \mu_{\alpha,\beta} \) and the density of the polynomials, it is not difficult to see that \( \{C^{\alpha,\beta}_n\} \) is an orthogonal decomposition of \( L^2([-1,1]^d, \mu_{\alpha,\beta}) \), that is

\[
L^2([-1,1]^d, \mu_{\alpha,\beta}) = \bigoplus_{n=0}^\infty C^{\alpha,\beta}_n.
\]

We call (2.9) a modified Wiener–Jacobi decomposition.
The orthogonal projection $P_n^{\alpha,\beta}$ of $L^2([-1, 1]^d, \mu_{\alpha,\beta})$ onto $C_n^{\alpha,\beta}$ is given by

$$P_n^{\alpha,\beta} f = \sum_{\nu \in \Lambda_n^{\alpha,\beta}} \langle f, p_{\nu}^{\alpha,\beta} \rangle_{\mu_{\alpha,\beta}} p_{\nu}^{\alpha,\beta}, \quad f \in L^2([-1, 1]^d, \mu_{\alpha,\beta}),$$

and for a given $f \in L^2([-1, 1]^d, \mu_{\alpha,\beta})$ its Jacobi expansion is then given by

$$f = \sum_{n=0}^{\infty} P_n^{\alpha,\beta} f.$$ 

Therefore $\{ P_n^{\alpha,\beta} \}_{n \geq 0}$ is a complete system of orthogonal projections in $L^2([-1, 1]^d, \mu_{\alpha,\beta})$.

Using this notation one can prove the following spectral decomposition of the operator $L^{\alpha,\beta}$

$$L^{\alpha,\beta} = \sum_{n=0}^{\infty} (-r_n) P_n^{\alpha,\beta} f,$$

$f \in L^2([-1, 1]^d, \mu_{\alpha,\beta})$, and its domain $D(L^{\alpha,\beta})$ is given by

$$D(L^{\alpha,\beta}) = \left\{ f \in L^2([-1, 1]^d, \mu_{\alpha,\beta}) : \sum_{n=0}^{\infty} (r_n)^2 \| P_n^{\alpha,\beta} f \|_{2, \mu_{\alpha,\beta}} < \infty \right\}.$$ 

Let $W = L^2([-1, 1]^d, \mu_{\alpha,\beta})$ and $l^2$ be the Hilbert space of complex square summable sequences. Again, suppose that $b$ is a fixed element of $W$ and consider the linear and bounded operator $\tilde{B} : l^2 \to W$ defined by

$$\tilde{B}U = \sum_{n=0}^{\infty} \sum_{|\nu|=n} U_{\nu} \langle b, p_{\nu}^{\alpha,\beta} \rangle_{\mu_{\alpha,\beta}} p_{\nu}^{\alpha,\beta}.$$ 

Then, the system (1.2) can be written as follows

$$w' = L^{\alpha,\beta} w + \tilde{B}u, \quad t > 0,$$

**Theorem 2.1.** The operators $L^\alpha$ and $L^{\alpha,\beta}$ are the infinitesimal generators of analytic semigroups $\{ T^\alpha(t) \}_{t \geq 0}$ and $\{ T^{\alpha,\beta}(t) \}_{t \geq 0}$, respectively. They are given as

$$T^\alpha(t) z = \sum_{n=0}^{\infty} e^{-nt} P_n^{\alpha} z, \quad z \in Z, \quad t \geq 0,$$ 

and $T^{\alpha,\beta}(t)$ as

$$T^{\alpha,\beta}(t) z = \sum_{n=0}^{\infty} e^{-nt} P_n^{\alpha,\beta} z, \quad z \in Z, \quad t \geq 0.$$
where \( \{ P_\alpha^n \}_{n \geq 0} \) is a complete orthogonal projections in the Hilbert space \( Z \) given by
\[
P_\alpha^n z = \sum_{|\nu| = n} \langle z, l_\nu^\alpha \rangle \mu_\nu \mu_\alpha l_\nu^\alpha, \quad n \geq 0, \quad z \in Z,
\]
and
\[
E_{\alpha}^{\beta} = \sum_{n \geq 0} \frac{P_\alpha^n}{n!},
\]
where \( \{ P_\alpha^{\alpha,\beta}_n \}_{n \geq 0} \) is a complete orthogonal projections in the Hilbert space \( W \) given by
\[
P_\alpha^{\alpha,\beta}_n w = \sum_{\nu \in A_{\alpha,\beta}^n} \langle w, p_\nu^{\alpha,\beta} \rangle \mu_{\alpha,\beta} p_\nu^{\alpha,\beta}, \quad n \geq 0, \quad w \in W.
\]

**Lemma 2.1.** The semigroups given by (2.12) and (2.13) are compact for \( t > 0 \).

**Proof.** Since \( T_\alpha(t) \) is given by
\[
T_\alpha(t) z = \sum_{n=0}^{\infty} e^{-nt} P_\alpha^n z, \quad t > 0,
\]
we can consider the following sequence of compact operators
\[
T_\alpha^k(t) z = \sum_{n=0}^{k} e^{-nt} P_\alpha^n z, \quad t > 0.
\]

It is easy to see that the sequence of compact operators \( \{ T_\alpha^n(t) \} \) converges uniformly to \( T_\alpha(t) \) for all \( t > 0 \).

Analogously, \( T^{\alpha,\beta}(t) \) is given by
\[
T^{\alpha,\beta}(t) w = \sum_{n=0}^{\infty} e^{-r_n t} P^{\alpha,\beta}_n w, \quad t > 0,
\]
so that, we can consider the following sequence of compact operators
\[
T^{\alpha,\beta}_k(t) w = \sum_{n=0}^{k} e^{-r_n t} P^{\alpha,\beta}_n w, \quad t > 0.
\]

and again it is easy to see that the sequence of compact operators \( \{ T^{\alpha,\beta}_k(t) \} \) converges uniformly to \( T^{\alpha,\beta}(t) \) for all \( t > 0 \).

Then, from part e) of Theorem A.3.22 of [9] we conclude the compactness of the semigroups \( T^{\alpha}(t) \) and \( T^{\alpha,\beta}(t) \), respectively. \( \square \)

Now, we shall give the definitions of exact and approximate controllability in terms of system (2.7) and (2.11). In spite of this definitions can be given for more general evolutions equations, we concentrated our attention to the cases of our interest.
For all $z_0 \in Z$, $w_0 \in W$ and given controls $u \in L^2(0,t_1;l^2)$ and $\tilde{u} \in L^2(0,t_1;l^2)$ the equations (2.7) and (2.11) have a unique mild solution given -in each case- by

\begin{equation}
    z(t) = T^\alpha(t)z_0 + \int_0^t T^\alpha(t-s)Bu(s)ds, \quad 0 \leq t \leq t_1.
\end{equation}

\begin{equation}
    w(t) = T^{\alpha,\beta}(t)w_0 + \int_0^t T^{\alpha,\beta}(t-s)\tilde{B}\tilde{u}(s)ds, \quad 0 \leq t \leq t_1.
\end{equation}

**Definition 2.1.** (Exact Controllability).

We shall say that the system (2.7) (respectively, (2.11)) is exactly controllable on $[0,t_1]$, $t_1 > 0$, if for all $z_0,z_1 \in Z$ (respectively, $w_0,w_1 \in W$) there exists a control $u \in L^2(0,t_1;l^2)$ (respectively, $\tilde{u} \in L^2(0,t_1;l^2)$) such that the solution $z(t)$ of (2.7) corresponding to $u$ (respectively, the solution $w(t)$ of (2.11) corresponding to $\tilde{u}$), that verifies $z(t_1) = z_1$ (respectively, $w(t_1) = w_1$).

Consider the following bounded linear operators

\begin{equation}
    G : L^2(0,t_1;l^2) \to Z, \quad Gu = \int_0^{t_1} T^\alpha(t_1-s)Bu(s)ds,
\end{equation}

\begin{equation}
    \tilde{G} : L^2(0,t_1;l^2) \to W, \quad \tilde{G}\tilde{u} = \int_0^{t_1} T^{\alpha,\beta}(t_1-s)\tilde{B}\tilde{u}(s)ds.
\end{equation}

Then, the following Proposition is a characterization of the exact controllability of the sytems (2.7) and (2.11).

**Proposition 2.1.**

i) The system (2.7) is exactly controllable on $[0,t_1]$ if and only if, the operator $G$ is surjective, that is to say

\[ GL^2(0,t_1;l^2) = GL^2 = \text{Range}(G) = Z. \]

ii) The system (2.11) is exactly controllable on $[0,t_1]$ if and only if, the operator $\tilde{G}$ is surjective, that is to say

\[ \tilde{G}L^2(0,t_1;l^2) = \tilde{G}L^2 = \text{Range}(\tilde{G}) = W. \]

**Definition 2.2.** We say that (2.7) (respectively, (2.11)) is approximately controllable in $[0,t_1]$ if for all $z_0,z_1 \in Z$ (respectively, $w_0,w_1 \in W$) and $\epsilon > 0$, there exists a control $u \in L^2(0,t_1;l^2)$ (respectively, $\tilde{u} \in L^2(0,t_1;l^2)$) such that the solution $z(t)$ given by (2.7) (respectively, the solution $w(t)$ given by (2.11)) satisfies

\[ ||z(t_1) - z_1|| \leq \epsilon, \quad (\text{respectively}, \quad ||w(t_1) - w_1|| \leq \epsilon). \]
Via duality, the following Theorem allows to give a characterization of the approximate controllability for our systems. Such characterization holds in general and the reader is referred to [9] for the details of its proof.

**Theorem 2.2.**

i) The system (2.7) is approximately controllable on \([0, t_1]\) if and only if
\[
B^* (T^\alpha)^* (t) z = 0, \quad \forall t \in [0, t_1], \text{ implies } z = 0.
\]

ii) The system (2.11) is approximately controllable on \([0, t_1]\) if and only if
\[
\tilde{B}^* (T^{\alpha,\beta})^* (t) w = 0, \quad \forall t \in [0, t_1], \text{ implies } w = 0.
\]

### 3. Controllability of the controlled Laguerre equation and the controlled Jacobi equation.

In this section we shall prove the main results of the paper.

**Theorem 3.1.**

i) If for all \(n \in \mathbb{N}_0\) and \(|\nu| = n\) we have
\[
\langle b, l^\alpha_\nu \rangle_{\mu_\alpha} = \int_{\mathbb{R}^d} b(x) l^\alpha_\nu (x) \mu_\alpha (dx) \neq 0,
\]
then the system (2.7) is approximately controllable on \([0, t_1]\), but never exactly controllable.

ii) If for all \(n \in \mathbb{N}_0\) and \(|\nu| = n\) we have
\[
\langle b, p^{\alpha,\beta}_\nu \rangle_{\mu_{\alpha,\beta}} = \int_{[-1,1]^d} b(x) p^{\alpha,\beta}_\nu (x) \mu_{\alpha,\beta} (dx) \neq 0,
\]
then the system (2.11) is approximately controllable on \([0, t_1]\), but never exactly controllable.

**Remark 3.1.** Notice that it is sufficient to prove the first part of the Theorem, since the proof depends of relation between the adjoint operator of \(B\) (respectively, \(\tilde{B}\)) and the adjoint operator of \(T^\alpha(t)\) (respectively, \(T^{\alpha,\beta}(t)\)) given by the Theorem 2.2.
Proof. Suppose condition (3.20). Next, we compute $B^* : Z \to l^2$. In fact,

$$\langle BU, z \rangle_{\mu_n} = \left\langle \sum_{n=0}^{\infty} \sum_{|\nu|=n} U_\nu \langle b, l^\alpha_\nu \rangle_{\mu_n} l^\alpha_\nu, z \right\rangle_{Z,Z} = \sum_{n=0}^{\infty} \sum_{|\nu|=n} U_\nu \langle b, l^\alpha_\nu \rangle_{\mu_n} \langle z, l^\alpha_\nu \rangle_{Z,Z} = \left\langle U, \{\{b, l^\alpha_\nu \rangle_{\mu_n} \langle z, l^\alpha_\nu \rangle\}_{|\nu|=n}\}_{n \geq 0} \right\rangle_{l^2,l^2}.$$ 

Therefore,

$$B^* z = \{\{b, l^\alpha_\nu \rangle_{\mu_n} \langle z, l^\alpha_\nu \rangle\}_{|\nu|=n}\}_{n \geq 0} = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \langle b, l^\alpha_\nu \rangle_{\mu_n} \langle z, l^\alpha_\nu \rangle e_\nu,$$

where $\{e_\nu\}_{|\nu|=n} \}_{n \geq 0}$ is the canonical basis of $l^2$.

On the other hand,

$$(T^\alpha)^*(t) z = \sum_{n=0}^{\infty} e^{-nt} P^\alpha_n z, \quad z \in Z, \quad t \geq 0.$$

Then,

$$B^* (T^\alpha)^*(t) z = \left\{\{b, l^\alpha_\nu \rangle_{\mu_n} \langle (T^\alpha)^*(t) z, l^\alpha_\nu \rangle\}_{|\nu|=n}\}_{n \geq 0}.$$ 

According with the part (i) of Theorem 2.2 the system (2.7) is approximately controllable on $[0, t_1]$ if and only if

(3.22) $\langle b, l^\alpha_\nu \rangle_{\mu_n} \langle (T^\alpha)^*(t) z, l^\alpha_\nu \rangle = 0, \quad \forall t \in [0, t_1], \quad |\nu| = n, \quad n = 0, 2, \ldots, \infty, \quad \Rightarrow z = 0.$

Since $\langle b, l^\alpha_\nu \rangle_{\mu_n} \neq 0$ for $|\nu| = n, \quad n \geq 0$, then condition (3.22) is equivalent to

(3.23) $\langle (T^\alpha)^*(t) z, l^\alpha_\nu \rangle = 0, \quad \forall t \in [0, t_1], \quad |\nu| = n, \quad n \geq 0, \quad \Rightarrow z = 0.$

Now, we shall check condition (3.23):

$$\langle (T^\alpha)^*(t) z, l^\alpha_\nu \rangle = \sum_{m=0}^{\infty} e^{-nt} \langle P_m z, l^\alpha_\nu l^\alpha_\nu \rangle = 0, \quad |\nu| = n, \quad n = 0, 1, 2, \ldots, \infty; \quad t \in [0, t_1].$$

Applying Lemma 3.14 from [10], pag. 62 (see also Lemma 3.1 of [3]), we conclude that

$$\langle P^\alpha_m z, l^\alpha_\nu \rangle = 0, \quad |\nu| = n, \quad m, n = 0, 1, 2, \ldots, \infty.$$
\[ \sum_{|\nu|=m} \langle z, l^\alpha_\nu \rangle \langle l^\alpha_\nu, l^\alpha_\nu \rangle = 0, \quad |\nu| = n, \quad m, n = 0, 1, 2, \ldots, \infty. \]

i.e.,

\[ \langle z, l^\alpha_\nu \rangle = 0, \quad |\nu| = n, \quad n = 0, 1, 2, \ldots, \infty. \]

Since \( \{l^\alpha_\nu \}_\nu \) is a complete orthonormal basis of \( Z \), we conclude that \( z = 0 \).

On the other hand, from Lemma 2.1 we know that \( T^\alpha(t) \) is compact for \( t > 0 \), then applying Theorem 3.3 from [2] we conclude that the system \((2.7)\) is not exactly controllable on any interval \([0, t_1]\). This last fact and the remark 3.1 finish the proof. \( \Box \)

Since an important ingredient in the above proof is Theorem 3.3 from [2], for completeness of this work we shall include here its proof -adapted to our context-

In fact, from Proposition 2.1 it is enough to prove that the operator

\[ G : L^2(0, t_1; l^2) \to Z, \quad Gu = \int_0^{t_1} T^\alpha(t_1 - s)Bu(s)ds \]

satisfies

\[ \text{Range}(G) \neq Z. \]

In order to do that, we shall prove that the operator \( G \) is compact. For all \( \delta > 0 \) small enough the operator \( G \) can be written as follows

\[ G = G_\delta + S_\delta, \quad G_\delta, S_\delta \in L(L^2(0, t_1; l^2, Z), \]

where

\[ G_\delta u = \int_0^{t_1-\delta} T^\alpha(t_1 - s)Bu(s)ds \quad \text{and} \quad S_\delta u = \int_{t_1-\delta}^{t_1} T^\alpha(t_1 - s)Bu(s)ds. \]

**Claim 1.** The operator \( G_\delta \) is compact. In fact,

\[ G_\delta u = \int_0^{t_1-\delta} T^\alpha(\delta)T^\alpha(t_1 - \delta - s)Bu(s)ds \]

\[ = T^\alpha(\delta) \int_0^{t_1-\delta} T^\alpha(t_1 - \delta - s)Bu(s)ds \]

\[ = T^\alpha(\delta)H_\delta u. \]

Since \( T^\alpha(\delta) \) is compact and \( H_\delta \in L(L^2(0, t_1; l^2), Z) \), then \( G_\delta \) is compact.
Claim 2. For $\epsilon > 0$ there exists $\delta > 0$ such that $\|S_\delta\| < \epsilon$. In fact,

$$
\|S_\delta u\| \leq \int_{t_1 - \delta}^{t_1} \|T^\alpha(t_1 - s)\| B \|u(s)\| ds \leq \int_{t_1 - \delta}^{t_1} M \|B\| \|u(s)\| ds,
$$

where

$$
M = \sup_{0 \leq s \leq t \leq t_1} \|T^\alpha(t - s)\|.
$$

Applying Hölder’s inequality we obtain

$$
\|S_\delta u\| \leq M \|B\| \delta \|u\|_{L^2}.
$$

Therefore, $\|S_\delta\| < \epsilon$ if $\delta < \frac{\epsilon}{M \|B\|}$.

Hence, for all natural number $n$ the exists $\delta_n > 0$ such that

$$
\|G - G_{\delta_n}\| = \|S_{\delta_n}\| < \frac{1}{n}, \quad n = 1, 2, 3, \ldots.
$$

So that, the sequence of compact operators $\{G_{\delta_n}\}$ converges uniformly to $G$. Then applying part e) of Theorem A.3.22 from [9] we obtain that $G$ is compact. Finally, from part g) of the same Theorem we obtain that $\text{Range}(G) \neq Z$.

As special cases of Theorem 3.1 we consider

Example 3.1.

a) The Laguerre equation in one variable with a single control

$$
(3.24) \quad z_t = x z_{xx} + (\alpha + 1 - x) z_x + b(x) u \quad t \geq 0, \quad x \in \mathbb{R}_+,
$$

where $b \in L^2(\mathbb{R}_+, \mu_\alpha)$ and the control $u$ belong to $L^2(0, t_1; \mathbb{R}_+)$. The equation (3.24) is approximately controllable if and only if

$$
\int_{\mathbb{R}_+} b(x) t^\alpha_\nu(x) x^{-\alpha} e^x dx \neq 0, \quad \nu = 0, 1, 2, \ldots.
$$

In particular, if $\alpha = \frac{n}{2} - 1$ then the equation (3.24) is associated to the Cox-Ingersoll-Ross (CIR) processes with a single control and therefore the controlled CIR can never be exactly controllable on $[0, t_1]$.

b) The Jacobi equation in one variable with a single control

$$
(3.25) \quad z_t = (1 - x^2) z_{xx} + ((\beta - \alpha - (\alpha + \beta + 2) x) z_x + b(x) u \quad t \geq 0, \quad x \in [-1, 1],
$$

where $b \in L^2([-1, 1], \mu_{\alpha, \beta})$ and the control $u$ belong to $L^2(0, t_1; [-1, 1])$. 
The equation (3.25) is approximately controllable if and only if

\[ \int_{[-1,1]} b(x)p_{\alpha,\beta}^\nu (1-x)^{-\alpha} (1+x)^{-\beta} \, dx \neq 0, \nu = 0, 1, 2, \ldots. \]

Remark 3.2. Notice that in each case, the approximated controllability is totally determined by the non-orthogonality of the function \( b \in L^2(\mathbb{R}_+, \mu_\alpha) \) (respectively, \( b \in L^2([-1,1], \mu_{\alpha,\beta}) \)) and the Laguerre (respectively, Jacobi) polynomials and it is independent of choice of control \( u \).

Finally, we will make some comments about the controllability of general Sturm-Liouville equations. From a general point of view our arguments require of the following ingredients:

1. A measure space \((\Omega, \Sigma, \mu)\), where \(\Omega \subseteq \mathbb{C}^d\) and \(\mu\) is a Borel measure defined on \(\Omega\).
2. An differential operator Sturm-Liouville type \(L\), whose eigenfunctions \(\{y_n\}_{n \geq 0}\) form a complete orthogonal system in \(L^2(\Omega, d\mu)\) with complex eigenvalues \(\{\lambda_n\}_{n \geq 0}\) such that \(\Re(\lambda_n) \to \infty\) as \(n \to \infty\).
3. A sequence of orthogonal projections \(\{P_n\}_{n \geq 0}\) associated to the complete orthogonal system \(\{y_n\}_{n \geq 0}\).
4. The Hilbert space of complex square sumable sequences \(l^2\).

With these ingredients the semigroup of operators \(\{T_t\}_{t \geq 0}\) given by

\[ T_t f = \sum_{n \geq 0} e^{-\lambda_n t} P_n f \]

is a strongly continuous semigroup of compact operators, having infinitesimal generator,

\[ \mathcal{L} = \sum_{n \geq 0} (-\lambda_n) P_n f, \]

with domain

\[ D(\mathcal{L}) = \left\{ f \in L^2(\Omega, d\mu) : \sum_{n \geq 0} \|\lambda_n P_n f\|_{L^2(\Omega, d\mu)}^2 < \infty \right\}. \]

Then for \(b \in L^2(\Omega, d\mu)\) fixed, we consider the linear and bounded operator

\[ B : l^2 \to L^2(\Omega, d\mu) \]

defined by

\[ BU = \sum_{n \geq 0} U_n \langle b, y_n \rangle_{L^2(\Omega, d\mu)} y_n. \]

Then, the controlled equation associated to the Sturm-Liouville differential operator \(\mathcal{L}\),

\[ z'(t) = \mathcal{L}z(t) + Bu(t), \quad t \geq 0 \]

is approximately controllable on \([0, t_1]\), if and only if,

\[ P_n b \neq 0, \quad \text{for all } n \geq 0. \]
The special case $d = 1$ and $\Omega$ be the unit circle of the complex plane, the support of the orthogonality measure $\mu$ for the Besell polynomials $\{B_n\}_{n \geq 0}$, which are eigenfunctions of the differential operator

\begin{equation}
(3.26) \quad \mathcal{L} = x^2 \frac{d^2}{dx^2} + (2x + 2) \frac{d}{dx},
\end{equation}

with eigenvalue $n(n+1)$, $n \geq 0$. Since these polynomials constitute a complete orthogonal system in $L^2(\Omega, d\mu)$, then, if we consider the Bessel equation with a single control

\begin{equation}
(3.27) \quad z_t = x^2 z_{xx} + (2x + 2) z_x + b(x) u \quad t \geq 0, \quad x \in \Omega,
\end{equation}

where $b \in L^2(\Omega, d\mu)$ and the control $u$ belong to $L^2(0, t_1; \Omega)$, we have that (3.27) is approximately controllable if and only if

$$
\langle b, B_n \rangle_{L^2(\Omega, d\mu)} \neq 0, \quad \text{for all } n \geq 0.
$$

**References**

[1] C. BALEDRAMA, AND W. URBINA, *Fractional Integration and Fractional Differentiation for d-dimensional Jacobi Expansions*. Sent for publication (2006). arXiv: math.AP/0608639.

[2] D.BARCENAS, H. LEIVA AND Z. SIVOLI, *A Broad Class of Evolution Equations are Approximately Controllable, but Never Exactly Controllable*. IMA J. Math. Control Inform. 22, no. 3 (2005), 310–320.

[3] D.BARCENAS, H. LEIVA AND W. URBINA, *Controllability of the Ornstein-Uhlenbeck Equation*. IMA J. Math. Control Inform. 23 no. 1, (2006), 1–9.

[4] S. BOCHNER, *Uber Sturm-Liouville Polynomsysteme*. Math. Z. 29 (1929), 730–736.

[5] S. BOCHNER, *Sturm-Liouville and heat equations whose eigenfunctions are ultraspherical polynomials or associated Bessel functions*. Collected Papers of Salomon Bochner, ed. R. C. Gunning, AMS (1991).

[6] J.C. COX, J.E.Jr. INGERSOLL AND S.A. ROSS, *A Theory of the term structure of interest rates*. Econometrica. 53 (1985), 385-407.

[7] C. CROETSCH, *Elements of applicable Functional Analysis*. Marcel Dekker, New York (1980). Lecture Notes in Control and Information Sciences, vol. 8. Springer Verlag, Berlin (1978).

[8] R.F. CURTAIN, A.J. PRITCHARD, *Infinite Dimensional Linear Systems*. Lecture Notes in Control and Information Sciences, 8. Springer Verlag, Berlin (1978).

[9] R.F. CURTAIN, H.J. ZWART, *An Introduction to Infinite Dimensional Linear Systems Theory*. Text in Applied Mathematics, 21. Springer Verlag, New York (1995).

[10] H. O. FATTORINI, *Some Remarks on Complete Controllability of Linear Systems*. SIAM J. Control 4 (1966), 686–694.

[11] H. O. FATTORINI, *On Complete Controllability of linear Systems*. J. Diff. Eqs. 3 (1967), 391–402.

[12] P. GRACZYK, J. J. LOEB, I., LOPEZ, A., NOWAK, W. URBINA, *Higher order Riesz Transforms, Fractional Derivatives and Sobolev spaces for Laguerre expansions*, J. Math. Pures Appl. (9) 84 (2005) no. 3, 375–405.

[13] O. Y. IMANUVILOV, *Controllability of evolution equations of fluid dynamics*. Proceedings of International Congress of Mathematicians, vol. III, Eur. Math. Soc. (2006), 1321–1338.

[14] H.L. KRALL, *Certain differential equations for the Tchebycheff polynomials*. Duke Math. J. 4 (1938), 705–718.
[15] H.L. KRALL, On orthogonal polynomials satisfying a certain fourth order differential equation, The Pennsylvania State College Studies, no. 6 (1940).
[16] H.L. KRALL, O. FRINK. A new class of orthogonal polynomials: The Bessel polynomials. Trans. Amer. Math. Soc. 65, (1949), 100–115.
[17] T. W. KÖRNER, Fourier Analysis. Cambridge University Press, Cambridge. (1993).
[18] P. A. MEYER, Quelques resultats analytiques sur le semigroupe d’Ornstein-Uhlenbeck en dimension infinie. Lectures Notes in Contr. and Inform. Sci. Springer-Verlag. 49 (1983), 201–214.
[19] L. MIRANIAN, On classical orthogonal polynomials and differential operators. J. Phys. A: Math. Gen. 38 (2005), 6379–6383.
[20] B. MUCKENHOUPT, Poisson Integrals for Hermite and Laguerre expansion. Trans. Amer. Math. Soc. 139 (1969) 231–242.
[21] A. NAYLOR, G. SELL, Linear Operator Theory in Engeeniring and Science. Holt-Rinehart-Winston, New York (1971).
[22] D. L. RUSSELL, Controllability and Stabilizability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions. SIAM Rev. 20 No. 4 (1978), 636–739.
[23] G. SZEGÖ, Orthogonal polynomials, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc. Providence, R. I., 1959.
[24] J. L. TORREA, Algunas observaciones sobre el semigrupo de Laguerre. MARGARITA MATHEMATICA. Editors: L. Español and J. L. Varona, Servicio de Publicaciones, Universidad de La Rioja, Logroño, Spain, (2001).
[25] R. TRIGGIANI, Extensions of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators. SIAM J. Control Optimization 14 No. 2 (1976), 313–338.
[26] N. YOUNG, An introduction to Hilbert spaces. Cambridge University Press, hardback edition. Reprinted (1995).
[27] E. ZUAZUA, Control and numerical approximation of the wave and heat equations. Proceedings of International Congress of Mathematicians, vol. III, Eur. Math. Soc. (2006), 1389–1417.
Wilfredo Urbina
Departamento de Matemáticas Facultad de Ciencias
Universidad Central de Venezuela, Caracas, VENEZUELA

and Department of Mathematics and Statistics,
University of New Mexico, Albuquerque, New Mexico, 8713, USA

E-mail address: barcenas@ula.ve, hleiva@ula.ve, yquintana@usb.ve, wurbina@euler.ciens.ucv.ve