A Least Squares Estimation of a Hybrid log-Poisson Regression and its Goodness of Fit for Optimal Loss Reserves in Insurance

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Abstract. In this article, the parameters of a hybrid log-linear model (log-Poisson) are estimated using the fuzzy least-squares (FLS) procedures (Celmíš, 987a,b; D’Urso and Gastaldi, 2000; D’Urso and Gastaldi, 2001). A goodness of fit have been derived in order to assess and compare this new model and the classical log-Poisson regression in loss reserving framework (Mack, 1991). Both the hybrid model and its goodness of fit are performed on a loss reserving data.

Keywords: fuzzy least squares, log-Poisson, goodness of fit, loss reserve, hybrid

1 Introduction

An important role of a non-life actuary is the calculation of provisions, mainly Incurred But Not Reported reserve (IBNR). Then, finding the fair value of loss reserve is a relevant topic for non-life actuaries. Indeed, insurance companies must simultaneously have enough reserves to meet their commitment to policyholders and have enough funds for their investments. Therefore several methods have been proposed in actuarial science literature to capture this fair value.

In one hand, we distinguish deterministic methods (Bornhuetter and Ferguson, 1972; Taylor, 1986; Linnemann, 1984). They provide crisp predictions for reserves. In the other hand, Taylor et al. (2003); Wüthrich and Merz (2008); Mack (1991); England and Verrall (2002) present stochastic methods. Those methods don’t give only a crisp value of the reserves but provide also their variability. But even stochastic methods have weakness.

In Straub and Swiss (1988), there are some experiences where stochastic methods can give unrealistic estimates. For example, when the claims are related to body injures, the future losses for the company will depend on the growth of the wage index that help to determine the amount of indemnity, and depends also on changes in court practices and public awareness of liability matters. Then the information is vague. Therefore the use of Fuzzy Set Theory becomes very attractive when the information is vague as in this case.

de Andrés Sánchez (2006); de Andrés-Sánchez (2007, 2012); de Andrés Sánchez (2014) present the interest of fuzzy regression models (FRM) in the calculus of loss reserves in insurance using the concept of expected value of a Fuzzy Number (FN) (de Campos Ibáñez and Muñoz, 1989). Asai (1982) is the first to develop a FRM where the coefficient are fuzzy numbers (Dubois and Prade, 1988). In the case of loss reserving, FN are easy to handle arithmetically unlike in the case of classical regression where the coefficients are random variables and are not easy to handle arithmetically. Another difference between fuzzy regression and classical regression is in dealing with errors as fuzzy variables in fuzzy regression modelling while errors are considered as random residuals in classical regression. But to integrate both fuzziness and randomness into a regression model, one should think about hybrid regression models.
Then we have developed in our previous article a hybrid log-Poisson regression inspired from the FRM (Asai, 1982) and taking into account an optimize $h$ -value in the linear program. However, the fuzzy parameters in this model are calculated through an optimization program and does not provide an explicit form of the estimated parameters.

In this article, we derive the exact form of the estimated fuzzy parameters of the hybrid log-Poisson regression by using the concept of fuzzy least-squares (Celmiš, 987a,b; D’Urso and Gastaldi, 2000; DUrso and Gastaldi, 2001). We develop a goodness of fit index to assess this new model and to compare it with the classical one. A numerical application on a loss reserving data will be provided.

The article is organized as follows: We first present some definitions and preliminaries concepts in fuzzy logic as the first section; In the second section, we derive a least squares estimation of the log-Poisson Model; The new estimation procedure for the hybrid log-Poisson regression is developed in section three; In the fourth and fifth section, a goodness of fit and the implementation algorithm of the hybrid model are respectively developed. Finally, a numerical example and a conclusion are suggested at the end of this paper.

2 Preliminaries on Fuzzy Sets

In this section, we review some concepts related to our research. That is the concept of fuzzy set, membership function, fuzzy number, weighted function of FN, fuzzy linear regression estimation according to least square approach.

2.1 Review on some Definitions and Basic Properties of Fuzzy Sets

Definition 1. (Zadeh, 1965)
Let $\Omega$ be a non empty set and $x \in \Omega$. In classical set theory, a subset $A$ of $\Omega$ can be defined by its characteristic function $\chi_A$ as a mapping from the elements of $\Omega$ to the elements of the set $\{0,1\}$,

$$\chi_A : \Omega \rightarrow \{0,1\} \tag{1}$$

This mapping may be represented as a set of ordered pairs, with exactly one ordered pair present for each element of $\Omega$. The first element of the ordered pair is an element of the set $\Omega$, and the second element is an element of the set $\{0,1\}$. The value zero is used to represent non-membership, and the value one is used to represent membership. The truth or falsity of the statement ”$x$ is in $A$” is determined by the ordered pair $(x, \chi_A(x))$. The statement is true if the second element of the ordered pair is 1, and the statement is false if it is 0.

Similarly, a fuzzy subset (also called fuzzy set) $\tilde{A}$ of a set $\Omega$ can be defined as a set of ordered pairs, each with the first element from $\Omega$, and the second element from the interval $[0,1]$, with exactly one ordered pair present for each element of $\Omega$. This defines a mapping called membership function.

Definition 2. (Zadeh, 1965)
The membership function of a fuzzy set $\tilde{A}$, denoted by $\mu_{\tilde{A}}$ is defined by

$$\mu_{\tilde{A}} : \Omega \rightarrow [0,1] \tag{2}$$

where $\mu_{\tilde{A}}$ is typically interpreted as the membership degree of element $x$ in the fuzzy set $\tilde{A}$.

The degree to which the statement ”$x$ is in $\tilde{A}$” is true is determined by finding the ordered pair $(x, \mu_{\tilde{A}}(x))$. The degree of truth of the statement is the second element of the ordered pair. A fuzzy set (Zadeh, 1965) $\tilde{A}$ on $\Omega$ can also be defined as a set of tuples:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in \Omega\}. \tag{3}$$

and could be represented by a graphic.

Definition 3. (Dubois and Prade, 1978)
Let $\Omega$ be the set of objects and $\tilde{A} \subset \Omega$. The $\alpha$-cut $A_{\alpha}$ of $\tilde{A}$ is the set defined by

$$A_{\alpha} = \{x \in \Omega, \mu_{\tilde{A}}(x) \geq \alpha\}. \tag{4}$$

2
Definition 4. (Dubois and Prade, 1988)

1. A fuzzy number $\tilde{A}$ is a fuzzy set of a universe $\Omega$ (the real line $\mathbb{R}$) such that:
   a. all its $\alpha$-cuts are convex which is equivalent to $\tilde{A}$ is convex, that is $\forall x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$, $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$;
   b. $\tilde{A}$ is normalized, that is $\exists x_0 \in \Omega$ such that $\mu_{\tilde{A}}(x_0) = 1$.
   c. $\mu_{\tilde{A}}$ is a cumulative membership function of bounded support, where $\Omega = \mathbb{R}$ and $[0, 1]$ are equipped with the natural topology.

2. A triangular fuzzy number (TFN) $\tilde{\gamma}$ is a fuzzy number denoted by $\tilde{\gamma} = (\beta^L, \alpha^c, \beta^R)$, $\beta^L, \alpha^c, \beta^R \in \mathbb{R}$, such that $\mu_{\tilde{\gamma}}(\beta^L) = \mu_{\tilde{\gamma}}(\beta^R) = 0$ and $\mu_{\tilde{\gamma}}(\alpha^c) = 1$ with $\alpha^c$ the centre of $\tilde{\gamma}$, $\beta^L$ its left spread and $\beta^R$ its right spread (Lai and Hwang, 1992).

A TFN $\tilde{\gamma}$ could be defined with its membership degree function $\mu_{\tilde{\gamma}}$ or, with its $h$-level ($\alpha$-cut ($h \in [0, 1]$) $\gamma_h$ (see Dubois and Prade (1988)), i.e

$$
\mu_{\tilde{\gamma}}(x) = \begin{cases} 
1 - \frac{\alpha^c - x}{\beta^L} & \text{if } \alpha - \beta^L < x \leq \alpha \\
1 - \frac{x - \alpha^c}{\beta^R} & \text{if } \alpha < x \leq \alpha + \beta^R \\
0 & \text{if } \text{otherwise}
\end{cases}
$$

or

$$
\tilde{\gamma}_h = [\gamma_{L_h}, \gamma_{R_h}] = [\alpha^c - \beta^L(1-h), \alpha^c + \beta^R(1-h)]
$$

- If $\alpha^c - \beta^L = \beta^R - \alpha^c$, then $\tilde{\gamma}$ define a STFN
- Otherwise $\beta^L \neq \beta^R$, then $\tilde{\gamma}$ define an ATFN (see Figure 1).

Notes and Comments. It is well known that if $\tilde{A}$ is a fuzzy number, then $\tilde{A}_h$, the $h$ level ($\alpha$-cut) of $\tilde{A}$ is a compact set of $\mathbb{R}$, for all $h \in [0, 1]$.

2.2 Review on Celmiński Least Squares Model for Fuzzy Linear Regression

In this subsection, we present the least-squares approach to estimated the fuzzy linear regression (Celmiński, 1987a,b; D’Urso and Gastaldi, 2000; D’Urso and Gastaldi, 2001) rather than the possibilistic approach (Asai, 1982; Ishibuchi and Nii, 2001).

Let us consider crisp explanatory variables $x_{ij}$ ($i = 1, \ldots, n; j = 1, \ldots, p$) describing triangular fuzzy variables $\tilde{Y}_i = (\tilde{Y}_i^L, \tilde{Y}_i^c, \tilde{Y}_i^R)$, where $\tilde{Y}_i^c$ is the centre of $\tilde{Y}_i$, $\tilde{Y}_i^L$ and $\tilde{Y}_i^R$ its left and right spreads respectively.
In matrix form, the fuzzy linear regression model between \(X\) (matrix of explanatory variables \(x_{ij}\)) and \(\tilde{Y}\) (vector of dependent variables \(\tilde{Y}_i\)) can be written as following:

\[
\begin{align*}
Y^c &= Y^c* + \varepsilon \\
Y^L &= Y^L* + \zeta \\
Y^R &= Y^R* + \eta
\end{align*}
\] (6)

where

\[
\begin{align*}
Y^c* &= X\beta \\
Y^L* &= Y^c* \theta + 1\lambda \\
Y^R* &= Y^c* \delta + 1\mu
\end{align*}
\] (7)

and

- \(X\) is a \(n \times (p + 1)\) matrix containing the vector 1 concatenated to \(p\) crisp input variables.
- \(\beta\) is a \((p + 1) \times 1\) vector of regression parameters for the regression model for \(Y^c\).
- \(Y^c\) is a \(n \times 1\) vector of the observed centres.
- \(Y^L\) is a \(n \times 1\) vector of interpolated centres.
- \(Y^L*\) is a \(n \times 1\) vector of observed left spreads.
- \(Y^R\) is a \(n \times 1\) vector of observed right spreads.
- \(Y^R*\) is a \(n \times 1\) vector of observed interpolated right spreads.
- \(\theta, \lambda, \delta, \mu\) are the regression parameters for the regression model for \(Y^L\) and \(Y^R\).
- \(1\) is a \(n \times 1\) vector of ones.
- \(\varepsilon, \zeta, \eta\) are \(n \times 1\) vector of error terms.

Remark 1. Model (6) is based on 3 sub-models. The first interpolate the centre of \(\tilde{Y}_i\). The two others are built over the first and yield the spreads.

Theorem 1. The iterative least squares estimates \(\hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\delta}\) and \(\hat{\mu}\) of the parameters \(\beta, \theta, \lambda, \delta\) and \(\mu\) in the system (6) are given by:

\[
\begin{align*}
\hat{\beta} &= \frac{1}{(1 + \theta^2 + \delta^2)} \left[ (X^T X)^{-1} X^T (Y^c + (Y^L - 1\lambda)\theta + (Y^R - 1\mu)\delta) \right] \\
\hat{\theta} &= (\beta^T X^T X \beta)^{-1} (\beta^T X^T Y^L - \beta^T X^T T 1\lambda) \\
\hat{\lambda} &= \frac{1}{n} \left( (Y^L)^T 1 - \beta^T X^T T \theta \right) \\
\hat{\delta} &= (\beta^T X^T X \beta)^{-1} (\beta^T X^T Y^R - \beta^T X^T T 1\mu) \\
\hat{\mu} &= \frac{1}{n} \left( (Y^R)^T 1 - \beta^T X^T T \delta \right)
\end{align*}
\] (8-12)

Proof. (see D’Urso (2003)).

3 A Least Squares Estimation of the log-Poisson Model

In this section, we provide a least squares estimation for the classical log-Poisson regression (Mack, 1991) in loss reserving framework.

Let Table 1 be a run-off triangle, where \(Y_{ij}\) is the total loss regarding the underwriting period \(i\) which have been paid with \(j\) periods delay and the loss amounts \(Y_{ij}\) with \(i + j = k\) have been paid in calendar year \(k \in \mathbb{N}\).

\(\{Y_{ij} : i = 1, \ldots, k; j = 1, \ldots, k\}\) are assumed to be independent and log – Poisson distributed (Mack, 1991), i.e.

\[
Y_{ij} \sim \mathcal{P}(e^{\nu_{ij}}) = \mathcal{P}(e^{\tau + \alpha_i + \gamma_j})
\] (13)
According to least squares method, we can estimate the vector of parameters (14) can be written as

\[ Y = e^{X^T\beta} \]

(15) can be written as

\[ Y = \varepsilon e^{X\beta} \]

where \( \varepsilon_{n \times 1} \) is the vector of errors terms \( \varepsilon_{ij} \) such that \( \varepsilon_{ij} \sim P(1) \) and \( \mathbb{E}(\varepsilon_{ij}) = \mathbb{V}(\varepsilon_{ij}) = 1 \).

(15) \Rightarrow \ln(Y) = \ln(\varepsilon) + X\beta \]

According to least squares method, we can estimate the vector of parameters \( \beta \) by minimizing

\[ S(\beta) = (\ln(\varepsilon))^T(\ln(\varepsilon)) \]

\[ = ((\ln(Y) - X\beta)^T ((\ln(Y) - X\beta) \]

\[ = ([\ln(Y)]^T - \beta^TX^T)(\ln(Y) - X\beta) \]

\[ = [\ln(Y)]^T[\ln(Y)] - [\ln(Y)]^T X\beta - \beta^TX^T[\ln(Y)] + \beta^T(X^TX)\beta \]

\[ = [\ln(Y)]^T[\ln(Y)] - 2[\ln(Y)]^T X\beta + \beta^T(X^TX)\beta \]
Proposition 1. Let $Y_{ij}$ be the loss amounts underwriting period $i$ which have been paid with $j$ periods delay in a certain run-off triangle.

We assume that $\{Y_{ij} : i = 1, \ldots, k; j = 1, \ldots, k-i+1\}$ are modelled by

$$Y_{ij} \sim \mathcal{P}(e^{\nu_{ij}}) \Rightarrow \nu_{ij} = \ln\mathbb{E}(Y_{ij}) \Rightarrow \mathbb{E}(Y_{ij}) = e^{X^T_i \beta} \Rightarrow \ln\mathbb{E}(Y_{ij}) = X^T_i \beta \Rightarrow \mathbb{E}(Y) = \varepsilon e^{X \beta}$$

where

- $X_{(n \times p)}$ is the matrix of explanatory variables $x_{ij}$
- $Y_{(n \times 1)}$ is the vector of observations $Y_{ij}$
- $\beta_{(p \times 1)}$ is the vector of parameters $n = \frac{1}{2}k(k+1)$ and $p = 2k - 1$.
- $\varepsilon_{(n \times 1)}$ is the vector of errors terms $\varepsilon_{ij}$ such that $\varepsilon_{ij} \sim \mathcal{P}(1)$ and $\mathbb{E}(\varepsilon_{ij}) = V(\varepsilon_{ij}) = 1$.

Then the least squares estimator of $\beta$ is given by

$$\hat{\beta}^{LS} = (X^T X)^{-1} [\ln(Y)]^T X. \quad (18)$$

Proof.

$$\min_{\beta} S(\beta) \Leftrightarrow \begin{cases} \frac{\partial S(\beta)}{\partial \beta} = 0 \\ \frac{\partial^2 S(\beta)}{\partial \beta^2} > 0 \end{cases}$$

We have

$$S(\beta) = [\ln(Y)]^T [\ln(Y)] - 2[\ln(Y)]^T X \beta + \beta^T (X^T X) \beta \Rightarrow \frac{\partial S(\beta)}{\partial \beta} = -2[\ln(Y)]^T X + 2(X^T X) \beta. \quad (20)$$

$$\Rightarrow (X^T X) \beta = [\ln(Y)]^T X \Rightarrow \beta = (X^T X)^{-1} [\ln(Y)]^T X. \quad (23)$$

But

$$\frac{\partial^2 S(\beta)}{\partial \beta^2} = 2(X^T X), \quad (24)$$

which is a semi definite positive matrix. Hence

$$\hat{\beta}^{LS} = (X^T X)^{-1} [\ln(Y)]^T X. \quad (25)$$

4 A Fuzzy Least Squares Estimation of a Hybrid Log-Poisson Regression for Loss Reserving

In this section, we present another way to estimate the parameters of the hybrid log-Poisson regression, which is the extension of the classical log-Poisson regression (Mack, 1991) in loss reserving framework.

(Mack, 1991) assumes that the incremental payments $Y_{ij}$ are log-Poisson distributed, i.e,

$$Y_{ij} \sim \mathcal{P}(e^{\nu_{ij}}) \Rightarrow \mathbb{E}(Y_{ij}) = e^{\nu_{ij}} = \varphi_{ij} \forall (i,j) \in \{i = 1, \ldots, k\} \times \{j = 1, \ldots, k-i+1\}. \quad (26)$$
We assume that uncertainty about in the run-off triangle is due both to fuzziness and randomness. We suppose then that $\hat{Y}_{ij} = (Y_{ij}^L, Y_{ij}^c, Y_{ij}^R)$ is a fuzzy Poisson random variable (Buckley, 2006), i.e.,

\[
\left[ \mathbb{E}_F(\hat{Y}_{ij}) \right]_h = \left\{ \sum_{x=0}^{+\infty} x e^{-\varphi_{ij}} \frac{(\varphi_{ij})^x}{x!} \mid \varphi_{ij} \in [\hat{Y}_{ij}]_h \right\} = \{ \varphi_{ij} \mid \varphi_{ij} \in [\hat{Y}_{ij}]_h \} = \hat{\varphi}_{ij},
\]

where $\mathbb{E}_F(\cdot)$ is the fuzzy expected value operator. So the fuzzy expected value is just the fuzzification of the crisp expected value.

The hybrid model built over the log-Poisson regression can be defined in matrix form and by using result of section 3 as follows:

\[
\begin{aligned}
\begin{cases}
\ln(Y^c) = Y^{c*} + \ln(\epsilon), \quad Y^{c*} = X\beta \\
\ln(Y^L) = Y^{L*} + \ln(\xi), \quad Y^{L*} = Y^{c*} + 1 + \lambda \\
\ln(Y^R) = Y^{R*} + \ln(\eta), \quad Y^{R*} = Y^{c*} + \delta + 1 + \mu
\end{cases}
\end{aligned}
\]

(27)

where

- $\beta = (\tau, \alpha, \gamma)^T \in \mathbb{R}^{2k-1}$ with
  \[
  \tau \in \mathbb{R}, \\
  \alpha = (\alpha_2 \ldots \alpha_k) \in \mathbb{R}^{k-1}, \\
  \gamma = (\gamma_2 \ldots \gamma_k) \in \mathbb{R}^{k-1}
\]

- $\epsilon, \xi, \eta$ are $n \times 1$ vectors of uncorrelated error terms following Poisson random variables ($\mathcal{P}(1)$) such that $\mathbb{E}(\epsilon') = \mathbb{E}(\xi') = \mathbb{E}(\eta') = 0_{n \times 1}$.

**Theorem 2.** The iterative fuzzy least squares estimators $\hat{\beta}, \hat{\theta}, \hat{\delta}, \hat{\lambda}$ and $\hat{\mu}$ of the parameters $\beta, \theta, \delta, \lambda$ and $\mu$ in model (27) are given by:

\[
\begin{aligned}
\hat{\beta} &= \frac{1}{(1 + \theta^2 + \delta^2)}(X^T X)^{-1}\left\{X^T Y^{c*} + \theta X^T (Y^{L*} - 1) + \delta X^T (Y^{R*} - 1)\right\} \\
\hat{\theta} &= [\beta^T (X^T X) \beta]^{-1} \beta^T X^T (Y^{L*} - 1) \\
\hat{\delta} &= [\beta^T (X^T X) \beta]^{-1} \beta^T X^T (Y^{R*} - 1) \\
\hat{\lambda} &= \frac{1}{n} \left(1^T Y^{L*} - \beta^T X^T 1\theta\right) \\
\hat{\mu} &= \frac{1}{n} \left(1^T Y^{R*} - \beta^T X^T 1\delta\right)
\end{aligned}
\]

(28) \hspace{2cm} (29) \hspace{2cm} (30) \hspace{2cm} (31) \hspace{2cm} (32)

where $\beta, \theta, \delta, \lambda$ and $\mu$ are the different values taken by the parameters before reaching their optimal values $\hat{\beta}, \hat{\theta}, \hat{\delta}, \hat{\lambda}$ and $\hat{\mu}$.

**Proof.** By using the fuzzy least squares method on model (27), we estimate its parameters as follows:
Denote
\[
S(\beta, \theta, \delta, \lambda, \mu) = [\varepsilon^T \varepsilon'] + [\xi^T \xi'] + [\eta^T \eta']
\]
\[
S(\beta, \theta, \delta, \lambda, \mu) = (Y^c - X\beta)^T (Y^c - X\beta) + (Y^r - X\beta\theta - 1\lambda)^T (Y^r - X\beta\theta - 1\lambda)
+ (Y^r - X\beta\delta - 1\mu)^T (Y^r - X\beta\delta - 1\mu)
\]
\[
S(\beta, \theta, \delta, \lambda, \mu) = (Y^c)^T - 2\beta^T X\beta + 2\beta^T (X^T X) + 2[Y^L]^T [Y^L] - 2[Y^L]^T X\beta
- 2[Y^L]^T 1\lambda + 2\beta^T (X^T X) + 21T X\beta\theta + n\lambda + [Y^r]^T [Y^r] - 2[Y^r]^T X\beta\delta
- 2[Y^r]^T 1\mu + 2\beta^T (X^T X) + 21T X\beta\delta + n\mu^2
\]
\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta} = -2[Y^c]^T X + 2(X^T X)\beta - 2[Y^L]^T X\beta + 2(X^T X)\beta\theta + 21T X\beta\lambda = 0
\]
\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \theta} = -2[Y^L]^T X\beta + 2\beta^T (X^T X)\beta\theta + 21T X\beta\lambda = 0
\]
\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \delta} = -2[Y^r]^T X\beta + 2\beta^T (X^T X)\beta\delta + 21T X\beta\mu = 0
\]
\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \lambda} = -2[Y^L]^T 1 + 21T X\beta\theta + 2n\lambda = 0
\]
\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu} = -2[Y^r]^T 1 + 21T X\beta\delta + 2n\mu = 0
\]
Furthermore
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta^2} = 2(X^T X)(1 + \theta^2 + \delta^2) \tag{43}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \theta \partial \beta} = -2[Y^L]^T X + 4(X^T X)\beta \theta + 21^T X \lambda \tag{44}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \delta \partial \beta} = -2[Y^R]^T X + 4(X^T X)\beta \delta + 21^T X \mu \tag{45}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \lambda \partial \beta} = 21^T X \theta, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu \partial \beta} = 21^T X \delta \tag{46}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \theta} = -2[Y^L]^T X + 4(X^T X)\beta \theta + 21^T X \lambda \tag{47}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \theta^2} = 2\beta^T (X^T X)\beta, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \delta \partial \theta} = 0 \tag{48}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \lambda \partial \theta} = 21^T X \beta, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu \partial \theta} = 0 \tag{49}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \delta} = -2[Y^R]^T X + 4(X^T X)\beta \delta + 21^T X \mu \tag{50}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \theta \partial \delta} = 2\beta^T (X^T X)\beta, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \delta^2} = 2\beta^T (X^T X)\beta \tag{51}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \lambda \partial \delta} = 21^T X \beta, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu \partial \delta} = 0 \tag{52}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \lambda} = 21^T X \theta, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \lambda} = 21^T X \beta \tag{53}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \lambda} = 0, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \lambda} = 2n \tag{54}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu \partial \lambda} = 0, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \mu} = 21^T X \delta \tag{55}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta \partial \mu} = 0, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu^2} = 21^T X \beta \tag{56}
\]
\[
\frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \lambda \partial \mu} = 0, \quad \frac{\partial^2 S(\beta, \theta, \delta, \lambda, \mu)}{\partial \mu^2} = 2n \tag{57}
\]

Let $H$ be the corresponding Hessian matrix. Then
\[
\forall u = (u_1 \ldots u_5)^T \in \mathbb{R}^5, \quad u^T H u = 2(X^T X)(1 + \theta^2)u_1^2 + 2\beta^T (X^T X)\beta(u_2^2 + u_3^2) + 2n(u_4^2 + u_5^2) > 0 \tag{58}
\]

Hence $H$ is a semi positive definite matrix and this ends the proof.

### 5 A Goodness of Fit Index for Hybrid log-Poisson Regression

In this section, we derive a goodness of fit index $\tilde{R}_F^2$ for the hybrid log-Poisson regression. This index is relevant to assess the explanatory power of the model.

In order to provide the mathematical formula of that $\tilde{R}_F^2$, let us prove some results.

**Proposition 2.** Let
\[
\begin{align*}
Y^c' &= Y^c + \varepsilon', \quad Y^c' = X \beta \\
Y^L' &= Y^L + \xi', \quad Y^L' = Y^c \theta + 1 \lambda \\
Y^R' &= Y^R + \eta', \quad Y^R' = Y^c \delta + 1 \mu
\end{align*} \tag{59}
\]
be the hybrid model built over the classical log-Poisson regression, where
\[ \beta = (\tau, \alpha, \gamma)^T \in \mathbb{R}^{2k-1} \text{ with } \]
\[ \tau \in \mathbb{R} \]
\[ \alpha = (\alpha_2 \ldots \alpha_k) \in \mathbb{R}^{k-1} \]
\[ \gamma = (\gamma_2 \ldots \gamma_k) \in \mathbb{R}^{k-1} \]

- \( \varepsilon, \xi, \eta \) are \( n \times 1 \) vectors of error terms following Poisson random variables (\( P(1) \)).

The following relationships hold:

1) \[ \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^c)^T (Y_{ij}' - Y_{ij}') = 0 \]

2) \[ \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}' - Y_{ij}') = 0 \]

3) \[ \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}' - Y_{ij}')^T Y_{ij}^L = 0 \]

Proof. 1) We have
\[ S(\beta, \theta, \delta, \lambda, \mu) = \left\{ [Y^c']^T [Y^c'] - 2 [Y^c']^T X \beta + \beta^T (X^T X) \beta \right\} + \left\{ [Y^L']^T [Y^L'] - 2 [Y^L']^T (X \theta + 1 \lambda) \right\} + \left\{ [Y^R']^T [Y^R'] - 2 [Y^R']^T (X \delta + 1 \mu) \right\} + (X \delta + 1 \mu)^T (X \delta + 1 \mu) \]
\[ \frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \beta} = 0_{2k-1} \iff X^T \left\{ (Y^c' - Y^c') + \theta (Y^L' - Y^c \theta - 1 \lambda) + \delta (Y^R' - Y^c \delta - 1 \mu) \right\} = 0_{2k-1} \]
\[ \iff X^T \left\{ (Y^c' - Y^c') + \theta (Y^L' - Y^L') + \delta (Y^R' - Y^R') \right\} = 0_{2k-1} \]
\[ \text{(60)} \]

\[ \frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \theta} = 0 \iff \beta^T (X^T X) \beta - \beta^T X^T [Y^L'] + \beta^T X^T 1 \lambda = 0 \]
\[ \iff \beta^T (X^T X) \beta = \beta^T X^T [Y^L'] - \beta^T X^T 1 \lambda \]
\[ \iff (Y^c')^T (Y^c') \theta = (Y^c')^T \left( Y^L' - 1 \lambda - Y^c \theta + Y^c \theta \right) \]
\[ \iff (Y^c')^T (Y^c') \theta = (Y^c')^T \left( Y^L' - Y^L' + Y^c \theta \right) \]
\[ \iff (Y^c')^T (Y^c') \theta = (Y^c')^T (Y^L' - Y^L') + (Y^c')^T (Y^c') \theta \]
\[ \iff (Y^c')^T (Y^L' - Y^L') = 0 \]
\[ \text{(61)} \]
\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \delta} = 0 \iff -[Y_{R'}]^T X\beta + \beta(X^T X)\beta \delta + 1^T X\beta \mu = 0
\]

\[
\iff -\beta^T X^T [Y_{R'}] + \beta(X^T X)\beta \delta + \beta^T X^T 1 \mu = 0
\]

\[
\iff \beta(X^T X)\beta \delta = \beta^T X^T [Y_{R'}] - \beta^T X^T 1 \mu
\]

\[
\iff (Y_{c'}^c)^T (Y_{c'}^c) \delta = (Y_{c'}^c)^T \left\{ Y_{R'} - 1 \mu \right\}
\]

\[
\iff (Y_{c'}^c)^T (Y_{c'}^c) \delta = (Y_{c'}^c)^T \left\{ Y_{R'} - Y_{R'}^c \right\}
\]

\[
\iff (Y_{c'}^c)^T (Y_{c'}^c) \delta = (Y_{c'}^c)^T \left\{ Y_{R'} - Y_{R'}^c + Y_{c'}^c \delta \right\}
\]

\[
\iff (Y_{c'}^c)^T (Y_{c'}^c) \delta = (Y_{c'}^c)^T \left\{ Y_{R'} - Y_{R'}^c \right\} + (Y_{c'}^c)^T (Y_{c'}^c) \delta
\]

\[
\iff (Y_{c'}^c)^T \left\{ Y_{R'} - Y_{R'}^c \right\} = 0 \quad (62)
\]

\[
\iff \beta^T X^T \left\{ (Y_{c'}^c - Y_{c'}^c) + \theta (Y_{L'}^c - Y_{L'}^c) + \delta (Y_{R'} - Y_{R'}^c) \right\} = 0
\]

\[
\iff (Y_{c'}^c)^T \left\{ (Y_{c'}^c - Y_{c'}^c) + \theta (Y_{L'}^c - Y_{L'}^c) + \delta (Y_{R'} - Y_{R'}^c) \right\} = 0 \quad (63)
\]

\[
2)
\]

\[
S(\beta, \theta, \delta, \lambda, \mu) = \left\| \begin{bmatrix} Y_{c'}^c - Y_{c'}^c \\ Y_{L'}^c - Y_{L'}^c \\ Y_{R'}^c - Y_{R'}^c \end{bmatrix} \right\|^2
\]

\[
= \left\| \begin{bmatrix} Y_{c'}^c - X\beta \\ Y_{L'}^c - X\beta \theta - 1 \lambda \\ Y_{R'}^c - X\beta \delta - 1 \mu \end{bmatrix} \right\|^2
\]

\[
= \left\| \begin{bmatrix} Y_{c'}^c \\ Y_{L'}^c - 1 \lambda \\ Y_{R'}^c - 1 \mu \end{bmatrix} - \begin{bmatrix} X \\ X\theta \\ X\delta \end{bmatrix} \beta \right\|^2
\]

\[
\min_{\beta} S(\beta, \theta, \delta, \lambda, \mu) \iff 1^T \left( \frac{Y_{c'}^c}{Y_{L'}^c} + Y_{L'}^c - 1 \lambda + Y_{R'}^c - 1 \mu \right) = 1^T \begin{bmatrix} X \\ X\theta \\ X\delta \end{bmatrix} \beta
\]

\[
\iff 1^T [Y_{c'}^c] + 1^T [Y_{L'}^c] - n\lambda - 1^T [Y_{R'}^c] - n\mu = 1^T X\beta + 1^T X\beta \theta + 1^T X\beta \delta
\]

\[
\iff 1^T [Y_{c'}^c] + 1^T [Y_{L'}^c] + 1^T [Y_{R'}^c] - n\lambda - n\mu - 1^T X\beta - 1^T X\beta \theta - 1^T X\beta \delta = 0
\]

\[
\iff 1^T ([Y_{c'}^c] - X\beta) + 1^T ([Y_{L'}^c] - X\beta \theta - 1 \lambda) + 1^T ([Y_{R'}^c] - X\beta \delta - 1 \mu) = 0
\]

\[
\iff 1^T (Y_{c'}^c - X\beta) = -1^T (Y_{L'}^c - X\beta \theta - 1 \lambda) - 1^T (Y_{R'}^c - X\beta \delta - 1 \mu) \quad (68)
\]

\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \theta} = 0 \iff -(Y_{c'}^c)^T [Y_{L'}^c] + (Y_{c'}^c)^T (Y_{c'}^c) \theta + (Y_{c'}^c)^T 1 \lambda = 0 \quad (69)
\]

\[
\frac{\partial S(\beta, \theta, \delta, \lambda, \mu)}{\partial \delta} = 0 \iff -(Y_{c'}^c)^T [Y_{R'}^c] + (Y_{c'}^c)^T (Y_{c'}^c) \delta + (Y_{c'}^c)^T 1 \mu = 0 \quad (70)
\]
\[ Y_c^* T \left( Y_L' - Y_L^* \right) = 0 \quad (71) \]
\[ Y_c^* T \left( Y_R' - Y_R^* \right) = 0 \quad (73) \]
\[ Y_c^* T \left( Y_c^* \theta - 1 \lambda \right) = 0 \quad (77) \]
\[ 1 T \left( Y_L' - Y_L^* \right) = 0 \quad (79) \]
\[ 1 T \left( Y_R' - Y_R^* \right) = 0 \quad (80) \]
\[ 1 T \left( Y_L' - Y_L^* \right) = 0 \quad (81) \]

**Definition 5.** For a set of crisp observations \( X_{n \times (p+1)} \) and by considering the hybrid log-Poisson model built over the classical one \((27)\) in loss reserving framework.

We define for the fuzzy output \( \hat{Y}_{ij} = (Y_L^{'ij}, Y_c^{'ij}, Y_R^{'ij}) \), \((i,j) \in \{1, \ldots, k\} \times \{1, \ldots, k - i + 1\} \) the following concepts:

1) **The fuzzy total sum of squares**

\[
FSST = \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} \left( Y_L^{ij} - \overline{Y_L} \right)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} \left( Y_c^{ij} - \overline{Y_c} \right)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} \left( Y_R^{ij} - \overline{Y_R} \right)^2 \quad (84)
\]
where

\[ Y_{in}^c = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} Y_{ij} \]  

\[ Y_{in}^L = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} Y_{ij}^L \]  

\[ Y_{in}^R = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} Y_{ij}^R \]  

(85)

(86)

(87)

\[ n = \frac{1}{2} k (k + 1) \]  

(88)

2) The fuzzy sum of the squares of the regression

\[ FSSR = \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^c - Y_{in}^c)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^L - Y_{in}^L)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^R - Y_{in}^R)^2 \]  

(89)

3) The fuzzy sum of the squares of errors

\[ FSSE = \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^c - Y_{in}^c)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^L - Y_{in}^L)^2 + \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (Y_{ij}^R - Y_{in}^R)^2 \]  

(90)

**Theorem 3.** Let us consider a set of crisp observations \( X_{n \times (p+1)} \) and fuzzy output \( \tilde{Y}_{ij} = (Y_{ij}^L, Y_{ij}^c, Y_{ij}^R), (i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k - i + 1\} \). By considering the hybrid log-Poisson model built over the classical one in loss reserving framework, the following relationship holds:

\[ FSST = FSSR + FSSE \]  

(91)

**Proof.**

\[ FSST = ||Y^c - 1Y_{in}^c||^2 + ||Y^L - 1Y_{in}^L||^2 + ||Y^R - 1Y_{in}^R||^2 \]  

(92)

\[ ||Y^c - 1Y_{in}^c||^2 = ||Y^c - Y^c + Y^c - 1Y_{in}^c||^2 \]  

\[ = (Y^c - Y^c)^T (Y^c - Y^c) + 2(Y^c - Y^c)^T (Y^c - 1Y_{in}^c) + (Y^c - 1Y_{in}^c)^T (Y^c - 1Y_{in}^c) \]  

But

\[ 2(Y^c - Y^c)^T (Y^c - 1Y_{in}^c) = 2(Y^c - Y^c)^T Y^c - 2(Y^c - Y^c)^T 1Y_{in}^c \]  

\[ = 0 \]  

(from proposition 2).

\[ \Rightarrow ||Y^c - 1Y_{in}^c||^2 = ||Y^c - Y^c||^2 + ||Y^c - 1Y_{in}^c||^2 \]  

(93)

\[ ||Y^L - 1Y_{in}^L||^2 = ||Y^L - Y^L + Y^L - 1Y_{in}^L||^2 \]  

\[ = (Y^L - Y^L)^T (Y^L - Y^L) + (Y^L - 1Y_{in}^L)^T (Y^L - 1Y_{in}^L) + 2(Y^L - Y^L)^T (Y^L - 1Y_{in}^L) \]  

\[ 2(Y^L - Y^L)^T (Y^L - 1Y_{in}^L) = 2(Y^L - Y^L)^T Y^L - 2(Y^L - Y^L)^T 1Y_{in}^L \]  

\[ = 0 \]  

(from proposition 2).

\[ \Rightarrow ||Y^L - 1Y_{in}^L||^2 = ||Y^L - Y^L||^2 + ||Y^L - 1Y_{in}^L||^2 \]  

(94)
\[ ||Y^R - 1Y^R_{in}||^2 = ||Y^R - Y^R + Y^R - 1Y^R_{in}||^2 \]
\[ = (Y^R - Y^R)^T (Y^R - Y^R) + (Y^R - 1Y^R_{in})^T (Y^R - 1Y^R_{in}) + 2(Y^R - Y^R)^T (Y^R - 1Y^R_{in}) \]
\[ 2(Y^R - Y^R)^T (Y^R - 1Y^R_{in}) = 2(Y^R - Y^R)^T Y^R - 2(Y^R - Y^R)^T 1Y^R_{in} \]
\[ = 0 \quad \text{(from proposition 2)} \]
\[ \Rightarrow ||Y^R - 1Y^R_{in}||^2 = ||Y^R - Y^R||^2 + ||Y^R - 1Y^R_{in}||^2 \]  \hspace{1cm} (95)

**Definition 6.** Let us consider a set of crisp observations \( X_{n 	imes (p+1)} \) and fuzzy output \( \tilde{Y}_{ij} = (Y^L_{ij}, Y^C_{ij}, Y^R_{ij}) \) in the hybrid log-Poisson regression model [27] in loss reserving framework.

We define the fuzzy goodness of fit index by:

\[ \tilde{R}_F^2 = \frac{FSSR}{FSST} = 1 - \frac{FSSE}{FSST} \]
\[ = \frac{||Y^C - 1Y^C_{in}||^2 + ||Y^L - 1Y^L_{in}||^2 + ||Y^R - 1Y^R_{in}||^2}{||Y^C - 1Y^C_{in}||^2 + ||Y^L - 1Y^L_{in}||^2 + ||Y^R - 1Y^R_{in}||^2} \]
\[ = 1 - \frac{||Y^C - Y^C||^2 + ||Y^L - Y^L||^2 + ||Y^R - Y^R||^2}{||Y^C - 1Y^C_{in}||^2 + ||Y^L - 1Y^L_{in}||^2 + ||Y^R - 1Y^R_{in}||^2} \]  \hspace{1cm} (96)

Using theorem 3, we notice that \( \tilde{R}_F^2 \in [0, 1] \).

6 Algorithm for implementation of the new model

In this section, we provide the algorithm behind our R program that will allow us to estimate the parameters of the new model and to compute the outstanding loss reserves.

1) Modelling the incremental losses \( Y_{ij} \) with the log-Poisson regression: We estimate the incremental losses \( \hat{Y}_{ij} \) through log-Poisson regression, i.e

\[ Y_{ij} \sim \mathcal{P}(e^{\nu_{ij}}), \quad \text{where} \ \nu_{ij} = \tau + \alpha_i + \gamma_j \]
\[ \Rightarrow \hat{Y}_{ij} = e^{X^T \hat{\beta}, \ (i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k - i + 1\}} \]

2) Estimation Procedure of \( Y^R_{ij}, Y^C_{ij}, Y^L_{ij} \): We assume that in the run off triangle \( Y_{ij} \) have been modelled as follows:

\[ Y_{ij} \sim \mathcal{P}(\varphi_{ij}), \ \varphi_{ij} = e^{\tau + \alpha_i + \gamma_j} \]  \hspace{1cm} (101)

we compute now the Pearson residuals

\[ \hat{r}_{ij} = \frac{Y_{ij} - \hat{Y}_{ij}}{\sqrt{\hat{\varphi}_{ij}}} \]

where \( \hat{\varphi}_{ij} = e^{\hat{\tau} + \hat{\alpha}_i + \hat{\gamma}_j} \) and \( (i, j) \in \{1, \ldots, k\} \times \{1, \ldots, k - i + 1\} \)

The adjusted residuals (England and Verrall, 1999) are computed as follows:

\[ \hat{r}'_{ij} = \sqrt{\frac{n}{n - p}} \hat{r}_{ij}, \text{where} \ n = \frac{k(k + 1)}{2}, \ p = 2k - 1 \]  \hspace{1cm} (102)

Our idea to construct the fuzzy output \( \tilde{Y}_{ij} \) is as following:
\[ Y_{ij}^c = Y_{ij} \]
\[ Y_{ij}^L = Y_{ij} - \frac{|P_{ij}|}{2} \]
\[ Y_{ij}^R = Y_{ij} + \frac{|P_{ij}|}{2} \]

3) Estimation of fuzzy parameters in the hybrid log-Poisson model:

We use the expression of \( \hat{\beta}, \hat{\delta}, \hat{\lambda}, \hat{\hat{\delta}}, \hat{\hat{\hat{\delta}}} \) given in theorem 2 to estimate the fuzzy parameters of the new model in equation (27). For that, an iterative algorithm have been written under \( R \) to estimate those parameters.

4) Estimation of the goodness of fit:

From step 3, we can get
\[
\hat{Y}^c = X\hat{\beta} \quad (103)
\]
\[
\hat{Y}^L = X\hat{\hat{\beta}} + 1\hat{\lambda} \quad (104)
\]
\[
\hat{Y}^R = X\hat{\hat{\hat{\beta}}} + 1\hat{\mu} \quad (105)
\]

Then from definition 6, the estimation of \( \hat{R}_F^2 \) is given by
\[
\hat{R}_F^2 = 1 - \frac{||Y^c - \hat{Y}^c||^2 + ||Y^L - \hat{Y}^L||^2 + ||Y^R - \hat{Y}^R||^2}{||Y^c - 1Y^c||^2 + ||Y^L - 1Y^L||^2 + ||Y^R - 1Y^R||^2} \quad (106)
\]

5) Estimation of outstandings reserves:

Et this step, we predict \( \hat{Y}_{ij} \) using the new model as follows:
\[
\hat{Y}_{ij}^c = \hat{X}_{ij}^T \hat{\beta} \Rightarrow \hat{Y}_{ij}^c = e^{\hat{X}_{ij}^T \hat{\beta}} \quad (107)
\]
\[
\hat{Y}_{ij}^L = \hat{X}_{ij}^T \hat{\hat{\beta}} + \lambda \Rightarrow \hat{Y}_{ij}^L = e^{\hat{X}_{ij}^T \hat{\hat{\beta}} + \lambda} \quad (108)
\]
\[
\hat{Y}_{ij}^R = \hat{X}_{ij}^T \hat{\hat{\hat{\beta}}} + \mu \Rightarrow \hat{Y}_{ij}^R = e^{\hat{X}_{ij}^T \hat{\hat{\hat{\beta}}} + \mu} \quad (109)
\]

where
\[
(i, j) \in \{1, \ldots, k\} \times \{k - i + 1, \ldots, k\}
\]
\[
\hat{\beta} = (\hat{\tau}, \hat{\alpha}_2, \ldots, \hat{\alpha}_k, \hat{\gamma}_2, \ldots, \hat{\gamma}_k)^T \in \mathbb{R}^{2k-1}
\]
\[
\hat{Y}_{ij} = (\hat{Y}_{ij}^L, \hat{Y}_{ij}^c, \hat{Y}_{ij}^R)
\]

Then the fuzzy total loss reserve is computed as follow:
\[
\hat{R}_{T, \text{Res}} = \sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^c \quad (110)
\]
\[
= (\sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^L, \sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^c, \sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^R) \quad (111)
\]
\[
= (R_{T, \text{Res}}^L, R_{T, \text{Res}}^c, R_{T, \text{Res}}^R) \quad (112)
\]

In this article, we use the concept of expected value of FN (de Campos Ibáñez and Muñoz, 1989) to move from the fuzzy value of total loss reserve \( \hat{R}_{T, \text{Res}} \) to the crisp value of total loss reserve \( R_{T, \text{Res}} \). Denote \( E_F \) that expected value.

The \( h \)-level of fuzzy total loss reserve is defined as following:
\[
\bar{R}_{T, \text{Res}}(h) = [h \cdot R_{T, \text{Res}}^L - (1-h) \cdot R_{T, \text{Res}}^L \cdot h \cdot R_{T, \text{Res}}^c + (1-h) \cdot R_{T, \text{Res}}^R] \quad (113)
\]

Then the expected value of FN \( \bar{R}_{T, \text{Res}} \) is defined as follows:
\[
E_F(\bar{R}_{T, \text{Res}}, \pi) = (1-\pi) \int_0^{1} (h \cdot R_{T, \text{Res}}^c - (1-h) \cdot R_{T, \text{Res}}^L) \, dh + \pi \int_0^{1} (h \cdot R_{T, \text{Res}}^L + (1-h) \cdot R_{T, \text{Res}}^R) \, dh \quad (114)
\]
where $\pi$ is the decision-maker risk aversion parameter ($0 \leq \pi \leq 1$). From (114), we have

$$\mathbb{E}_F(\hat{R}_{T,\text{Res}}, \pi) = (1 - \pi) \int_0^1 (h \cdot R^c_{T,\text{Res}} - R^L_{T,\text{Res}} + h \cdot R^R_{T,\text{Res}}) dh + \pi \int_0^1 (h \cdot R^R_{T,\text{Res}} - R^L_{T,\text{Res}} - h \cdot R^R_{T,\text{Res}}) dh$$

$$= (1 - \pi) \int_0^1 h(R^c_{T,\text{Res}} + R^L_{T,\text{Res}}) dh - (1 - \pi) \int_0^1 R^L_{T,\text{Res}} dh + \pi \int_0^1 h(R^R_{T,\text{Res}} - R^L_{T,\text{Res}}) dh$$

$$+ \pi \int_0^1 R^R_{T,\text{Res}} dh$$

$$= \frac{(1 - \pi)(R^c_{T,\text{Res}} + R^L_{T,\text{Res}})}{2} - \frac{(1 - \pi)R^L_{T,\text{Res}}}{2} + \frac{\pi(R^R_{T,\text{Res}} - R^L_{T,\text{Res}})}{2} + \frac{\pi R^R_{T,\text{Res}}}{2}$$

7 Numerical Example

In this section, we apply both the classical log-Poisson regression (Mack, 1991) and the hybrid model estimated by a fuzzy least square procedure on a real data. Let us use the numerical example from de Andrés Sánchez (2006).

| Development Year | Origin Year |
|------------------|-------------|
| $i/j$            | 0 1 2 3 4    |
| 2000             | 1120 2090 2610 2920 3130 |
| 2001             | 1030 1920 2370 2710 |
| 2002             | 1090 2140 2610 |
| 2003             | 1300 2650 |
| 2004             | 1420 |

Table 2. Numerical example from de Andrés Sánchez (2006)

According to the 1st step of the algorithm, we perform the classical log-Poisson regression on the data from de Andrés Sánchez (2006) using R software. The estimated parameters are displayed in table 3.

| $\hat{\alpha}_i$ | $\hat{\gamma}_j$ | $p$-value($\hat{\alpha}_i$) | $p$-value($\hat{\gamma}_j$) |
|------------------|------------------|----------------|----------------|
| -0.08473         | 0.66182          | $4.23 \times 10^{-10}$ | $< 2 \times 10^{-16}$ |
| 0.00587          | 0.86503          | 0.741          | $< 2 \times 10^{-16}$ |
| 0.20725          | 0.98780          | $< 2 \times 10^{-16}$ | $< 2 \times 10^{-16}$ |
| 0.26203          | 1.05240          | $4.72 \times 10^{-16}$ | $< 2 \times 10^{-16}$ |

$\hat{\tau} = 6.99639$ | $p$-value($\hat{\tau}$) $= < 2 \times 10^{-16}$ |

$R^2 = 0.9621253$ | **Total Reserve** $= 33634.89$

Table 3. Estimated parameters

From table 3 and With a threshold of 1%, we conclude that except $\hat{\alpha}_3$, the others coefficients are statistically significant. The goodness of fit of the model to the data is good, since $R^2 = 96.21\%$ and the estimation of the total loss reserve is **33634.89**

Now let us test if the model performed is adapted to a statistical perspective through a dispersion test (see table 4).

With a threshold of 1%, we do not reject the null hypothesis, i.e $p-value > 1\%$. Therefore we don’t need to perform a quasi-Poisson regression.
Let us perform the steps 2), 3) and 4) of the algorithm. The iterative algorithm to estimate parameters $\beta, \theta, \delta, \lambda, \mu$ converges after 12112 iterations and the estimation of parameters and fuzzy output are displayed in table 5 and in equation (115).

$$\hat{\theta} = 1.000429; \quad \hat{\lambda} = -0.003468438; \quad \hat{\delta} = 0.9995556; \quad \hat{\mu} = 0.003584175;$$

(115)

From these outputs, we conclude that the hybrid model is more adequate to the classical one since $\hat{R}_F^2 > R^2$.

From step 5) of our algorithm, we can predict the incremental losses as fuzzy numbers and total fuzzy loss reserve $\hat{R}_{T, Res}$. The results are given in table 6.

From the table 6 and using the expected value of fuzzy number for defuzzification purposes, we can compute the crisp value of outstanding loss reserve with the maximum decision-maker risk aversion,
\[ Y_{25}^L = 2875.014 \quad Y_{25}^c = 2875.162 \quad Y_{25}^R = 2875.293 \]
\[ Y_{34}^L = 2936.547 \quad Y_{34}^c = 2936.671 \quad Y_{34}^R = 2936.777 \]
\[ Y_{35}^L = 3131.660 \quad Y_{35}^c = 3131.716 \quad Y_{35}^R = 3131.739 \]
\[ Y_{43}^L = 3562.481 \quad Y_{43}^c = 3562.659 \quad Y_{43}^R = 3562.803 \]
\[ Y_{44}^L = 3798.981 \quad Y_{44}^c = 3799.279 \quad Y_{44}^R = 3799.538 \]
\[ Y_{52}^L = 2742.702 \quad Y_{52}^c = 2742.898 \quad Y_{52}^R = 2743.080 \]
\[ Y_{53}^L = 3355.000 \quad Y_{53}^c = 3355.077 \quad Y_{53}^R = 3355.127 \]
\[ Y_{54}^L = 3787.870 \quad Y_{54}^c = 3788.162 \quad Y_{54}^R = 3788.415 \]
\[ Y_{55}^L = 4039.332 \quad Y_{55}^c = 4039.759 \quad Y_{55}^R = 4040.141 \]
\[ R_{T,Res} = (33384.915, 33386.738, 33388.281) \]

| Table 6. Predicted values from the hybrid model |
|-----------------------------------------------|

i.e \( \pi = 1 \).

\[
\mathbb{E}_F(\hat{R}_{T,Res}, \pi = 1) = \hat{R}_{T,Res} = \frac{R_{T,Res}^c + R_{T,Res}^R}{2} = 33387.5095
\]

From those results we conclude that the new hybrid model we suggested produce best results than the classical one according to the goodness of fit.

8 Conclusion

This paper has considered the relevance of Hybrid Models in loss reserving framework, mainly when we are in presence of vague information like in medical insurance (Straub and Swiss, 1988). Those models could give best result compared to stochastic models. In our previous article, we have estimated the hybrid log-Poisson model using a linear programming problem and a numerical example have been made in view to compare that model with the classical log-Poisson regression. In this article, we have suggested a new way to estimate the parameters of the hybrid log-Poisson regression in loss reserving framework using the fuzzy least squares procedure. Furthermore we have developed a goodness of fit index to assess our model. This new model have been applied to a run-off triangle in order to estimate the outstanding loss reserve. According to the goodness of fit, the hybrid model approaches the fair value of loss reserve better than the well known log-Poisson regression model (Mack, 1991). However the weakness of that fuzzy least squares estimation of the hybrid log-Poisson regression is its computational part. Since we got an iterative estimator, the R program take some time to converge (12112 iterations).

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