Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation

Martin Lüscher

Deutsches Elektronen-Synchrotron DESY
Notkestrasse 85, D-22603 Hamburg, Germany
E-mail: luscher@mail.desy.de

Abstract

It is shown that the Ginsparg-Wilson relation implies an exact symmetry of the fermion action, which may be regarded as a lattice form of an infinitesimal chiral rotation. Using this result it is straightforward to construct lattice Yukawa models with unbroken flavour and chiral symmetries and no doubling of the fermion spectrum. A contradiction with the Nielsen-Ninomiya theorem is avoided, because the chiral symmetry is realized in a different way than has been assumed when proving the theorem.

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1. A well-known problem with fermions on the lattice is that one usually ends up with breaking chiral symmetry or having more particles in the continuum limit than intended. The celebrated Nielsen-Ninomiya theorem [1] states that this is in fact unavoidable if a few plausible assumptions are made. The construction of chiral field theories on the lattice thus appears to be difficult and maybe even impossible in some cases.

Recently some intriguing results have been published by Neuberger [2,3] and by Hasenfratz, Lalena and Niedermayer [4], which suggest that chiral symmetry may be preserved in lattice QCD, at least to some extent, if the lattice Dirac operator is of a particular form. The proposed expressions for the Dirac operator have been derived in completely different ways and tend to be very complicated, but all of them satisfy a simple identity, originally due to Ginsparg and Wilson [5], which protects the quark masses from additive renormalizations [2,6] and which plays a key role in the proof of the lattice index theorem of ref. [4].

In this letter it will be shown that the lattice fermion action in fact has an exact symmetry if the Ginsparg-Wilson identity holds. The usefulness of this observation is illustrated by constructing a class of chiral Yukawa models on the lattice with unbroken chiral and flavour symmetries (and no doublers). Since the chiral transformations in these theories are not of the naively expected form, a contradiction with the Nielsen-Ninomiya theorem is avoided without having to compromise in any other ways. In particular, the flavour-singlet chiral symmetry has the expected anomaly if gauge interactions are included.

2. A particularly simple form of the Nielsen-Ninomiya theorem holds for free Dirac fermions on a euclidean lattice and it is helpful for the discussion that follows to briefly recall this. So let us consider the free field action

\[ S_F = a^4 \sum_x \overline{\psi} D \psi, \]  

(2.1)

where \( a \) denotes the lattice spacing and \( D \) the lattice Dirac operator. As usual we assume \( D \) to be invariant under translations so that

\[ D e^{ipx} u = \tilde{D}(p) e^{ipx} u \]  

(2.2)

for all constant Dirac spinors \( u \) and some complex \( 4 \times 4 \) matrix \( \tilde{D}(p) \). The theorem now states that the following properties cannot hold simultaneously (for an elegant proof see ref. [7]).

(a) \( \tilde{D}(p) \) is an analytic periodic function of the momenta \( p_\mu \) with period \( 2\pi/a \).
(b) For momenta far below the cutoff $\pi/a$, we have $\tilde{D}(p) = i\gamma_\mu p_\mu$ up to terms of order $ap^2$.

(c) $\tilde{D}(p)$ is invertible at all non-zero momenta (mod $2\pi/a$).

(d) $D$ anti-commutes with $\gamma_5$.

Property (a) is necessary if we want $D$ to be an essentially local operator, (b) and (c) ensure that the correct continuum limit is obtained and (d) guarantees that the fermion action is invariant under continuous chiral transformations $^\dagger$.

3. To escape the theorem, Ginsparg and Wilson [5] suggested many years ago to replace property (d) through the relation

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D.$$  (3.1)

A simple consequence of this equation is that the fermion propagator anti-commutes with $\gamma_5$ at non-zero distances and chiral symmetry is thus partly preserved.

Examples of free lattice Dirac operators satisfying the Ginsparg-Wilson relation can be found rather easily. A particularly simple solution is given by [2]

$$D = \frac{1}{a} \{1 - A(A^\dagger A)^{-1/2}\}, \quad A = 1 - a D_w,$$  \hspace{1cm} (3.2)

where $D_w$ denotes the standard Wilson-Dirac operator,

$$D_w = \frac{1}{2} \{\gamma_\mu (\partial^\mu + \partial_\mu) - a \partial^\mu \partial_\mu\},$$  \hspace{1cm} (3.3)

and $\partial_\mu$ and $\partial^\mu$ are the nearest-neighbour forward and backward difference operators. Because of the square root in eq. (3.2), one might assume that $D$ is a non-local operator, but this is actually not the case. Using the abbreviations $\hat{p}_\mu = (1/a) \sin(ap_\mu)$ and $\hat{p}_\mu = (2/a) \sin(ap_\mu/2)$, we have

$$a \tilde{D}(p) = 1 - \left\{1 - \frac{1}{2} a^2 \hat{p}_\mu^2 - ia\gamma_\mu \hat{p}_\mu \right\}\left\{1 + \frac{1}{2} a^4 \sum_{\mu<\nu} \hat{p}_\mu^2 \hat{p}_\nu^2 \right\}^{-1/2},$$  \hspace{1cm} (3.4)

and it is immediately clear from this formula that the conditions (a), (b) and (c) listed above are fulfilled. In particular, from the analyticity of $\tilde{D}(p)$ one infers that its Fourier transform falls off exponentially at large distances with a rate proportional $^\dagger$ The Dirac matrices are taken to be hermitean and $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$
to $1/a$. For free fermions eq. (3.2) thus provides a completely satisfactory solution
of the Ginsparg-Wilson relation.

4. We now show that eq. (3.1) implies a continuous symmetry of the fermion
action, which may be interpreted as a lattice form of chiral symmetry. No particular
assumptions need to be made here, i.e. the discussion applies to any Dirac operator
satisfying the Ginsparg-Wilson relation, including the gauge covariant operators of
refs. [2–4].

The infinitesimal variation of the fields associated with the new symmetry is

$$\delta \psi = \gamma_5 \left(1 - \frac{1}{2}a D\right) \psi, \quad \delta \bar{\psi} = \bar{\psi} \left(1 - \frac{1}{2}a D\right) \gamma_5,$$

(4.1)

where $D$ is considered to be a matrix which may be multiplied from the right with $\psi$
or from the left with $\bar{\psi}$. Flavour non-singlet chiral transformations may be defined
similarly by including a group generator in eq. (4.1). In both cases it is trivial to
check that the fermion action eq. (2.1) is invariant if the Ginsparg-Wilson identity
holds.

The flavour-singlet chiral symmetry is anomalous in the presence of gauge fields
and it now seems that we have got too much symmetry on the lattice. The paradox
is resolved by noting that the fermion integration measure is in general not invariant
under the transformation (4.1).

To work this out let us consider the theory in a finite space-time volume with
suitable boundary conditions so that the Ginsparg-Wilson identity is preserved. We
are then interested in the symmetry properties of the (unnormalized) expectation
values

$$\langle O \rangle_F = \int \prod_x d\psi(x) d\bar{\psi}(x) \mathcal{O} e^{-S_F}$$

(4.2)

of arbitrary products $\mathcal{O}$ of the fermion fields. By substituting

$$\psi \to \psi + \epsilon \delta \psi, \quad \bar{\psi} \to \bar{\psi} + \epsilon \delta \bar{\psi},$$

and expanding to first order in $\epsilon$ one obtains

$$\langle \delta \mathcal{O} \rangle_F = -a \text{tr} \left\{ \gamma_5 D \right\} \langle \mathcal{O} \rangle_F,$$

(4.3)

where the trace is to be taken over the space of all fermion fields. Evidently in the
case of free fermions, with $D$ as given above, the trace vanishes and the symmetry
is exact. The same is also true if we consider flavour non-singlet chiral rotations,
because the group generator which has to be included in the transformation law (and which then appears on the right-hand side of eq. (4.3)) is traceless.

The anomaly, \(-a \text{ tr } \{ \gamma_5 D \} \), has previously been calculated in ref. [4] and we now give a second derivation which is applicable also in those cases where the Dirac operator does not have any particular hermiticity properties. Let \( z \) be any complex number not contained in the spectrum of \( D \). A little algebra, using the Ginsparg-Wilson identity, yields

\[
a(z - D)\gamma_5(z - D) = z(2 - az)\gamma_5 - (1 - az)\{ (z - D)\gamma_5 + \gamma_5(z - D) \}, \quad (4.4)
\]

and after multiplying this equation from the right with \((z - D)^{-1}\) and taking the trace one ends up with

\[
-a \text{ tr } \{ \gamma_5 D \} = z(2 - az) \text{ tr } \{ \gamma_5 (z - D)^{-1} \}. \quad (4.5)
\]

We now divide through the factor \( z(2-a\bar{z}) \) and integrate over a small circle centred at the origin that does not encircle any spectral value of \( D \) other than 0. In particular,

\[
P_0 = \oint \frac{dz}{2\pi i} (z - D)^{-1} \quad (4.6)
\]

projects on the subspace of zero modes of \( D \) and the result

\[
-a \text{ tr } \{ \gamma_5 D \} = 2 \text{ tr } \gamma_5 P_0 = 2N_f \times \text{index}(D) \quad (4.7)
\]

is thus obtained, where \( N_f \) denotes the number of fermion flavours. Taken together eqs. (4.3) and (4.7) show that the Ward identities associated with the global flavour-singlet chiral transformations on the lattice have the correct anomaly. In view of the exact index theorem of Hasenfratz et al. [4] and the earlier work of Ginsparg and Wilson [5] this comes hardly as a surprise, but it is striking that the anomaly can be calculated with so little effort.

5. It is now relatively easy to couple fermions to scalar fields in such a way that the flavour and chiral symmetries are preserved on the lattice. Our starting point is the free fermion action

\[
S_F = a^4 \sum_x \{ \bar{\psi}D\psi - (2/a)\bar{\chi}\chi \}, \quad (5.1)
\]

where \( D \) is assumed to be a decent solution of the Ginsparg-Wilson relation such as the one discussed in section 3. The auxiliary fields \( \chi \) and \( \bar{\chi} \) will later be used to
construct chirally invariant interaction terms. For the time being we only note that they do not propagate and the physical content of the theory is hence unchanged.

As before one can show that the modified transformation

\[
\begin{align*}
\delta \psi &= \gamma_5 \left(1 - \frac{1}{2} aD\right) \psi + \gamma_5 \chi, \\
\delta \chi &= \gamma_5 \frac{1}{2} aD \psi,
\end{align*}
\]

leaves the action and the fermion integration measure invariant (gauge interactions are excluded in this section). It follows from these equations that

\[
\begin{align*}
\delta (\psi + \chi) &= \gamma_5 (\psi + \chi), \\
\delta \overline{\psi} (\overline{\psi} + \overline{\chi}) &= (\overline{\psi} + \overline{\chi}) \gamma_5,
\end{align*}
\]

and the propagator of the sum $\psi + \chi$ is hence chirally invariant in the ordinary sense. This is, incidentally, perfectly consistent with the Nielsen-Ninomiya theorem, because the Fourier transform of the propagator vanishes at some momenta and its inverse is hence singular, thus violating property (a) (cf. sect. 2).

Suppose now that $\phi$ is a complex scalar field on the lattice with the usual self-interactions. A chirally invariant Yukawa interaction term is then given by

\[
S_{\text{int}} = a^4 \sum_x g_0 (\overline{\psi} + \overline{\chi}) \left\{ \frac{1}{2} (1 - \gamma_5) \phi + \frac{1}{2} (1 + \gamma_5) \phi^* \right\} (\psi + \chi)
\]

with $g_0$ being the bare coupling constant. More complicated interactions with flavour symmetries and various multiplets of fermions can be constructed similarly. A few remarks should be added at this point to make it clear that the lattice theories defined in this way are completely sane. For simplicity attention is restricted to the phase where chiral symmetry is not spontaneously broken.

(i) In perturbation theory the one-particle irreducible diagrams are chirally invariant in the ordinary sense, because the internal fermion lines represent the propagation of $\psi + \chi$ and the vertices are manifestly invariant. Non-symmetric counterterms are hence not needed to renormalize the theory.

(ii) At sufficiently weak coupling the spectrum of fermions is exactly as expected, i.e. there are no doublers. To see this first note that there are none when the interactions are switched off. Now since the one-particle irreducible self-energy diagrams anti-commute with $\gamma_5$, they must be proportional to $\gamma_\mu P_\mu$ at small momenta. In particular, the perturbative corrections to the fermion propagator are of the form $D^{-1} B$ where $B$ is bounded and it is hence impossible that new poles arise at small couplings.
(iii) The auxiliary fields $\chi$ and $\overline{\chi}$ couple only locally and do not carry any independent physical information. Essentially these fields play the rôle of Lagrange multipliers which may integrated out if so desired although the expressions that one obtains are not particularly illuminating †.

6. The important qualitative message of this paper is that the Nielsen-Ninomiya theorem can be bypassed if we do not insist that the chiral transformations assume their canonical form on the lattice. The construction of chirally invariant lattice theories remains non-trivial, however, because the transformation laws depend on the interaction in general. This is in fact required in gauge theories as otherwise one would end up with a non-anomalous flavour-singlet chiral symmetry. An interesting observation in this connection is that the chiral rotations (5.2) become interaction dependent when the auxiliary fields are eliminated.

At this point many interesting questions have not even been touched and are left for future research. In particular, the precise conditions under which chiral symmetries can exist on the lattice remain to be uncovered and an attempt should be made to derive an identity, similar to the Ginsparg-Wilson relation, which allows one to construct exactly supersymmetric lattice theories.

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† The bilinear part of the auxiliary field action can have zero modes at large scalar fields. This does not lead to any singularities (fermion integrals are always finite), but one may prefer to avoid this complication by replacing the scalar field in eq. (5.4) through $\phi/(1 + g_0^2a^2|\phi|^2)^{1/2}$. 

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