FRAMED MOTIVIC $\Gamma$-SPACES

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In memory of Vladimir Voevodsky

ABSTRACT. We combine several mini miracles to achieve an elementary understanding of infinite loop spaces and very effective spectra in the algebro-geometric setting of motivic homotopy theory. Our approach combines $\Gamma$-spaces and Voevodsky’s framed correspondences into the concept of framed motivic $\Gamma$-spaces; these are continuous or enriched functors of two variables that take values in framed motivic spaces. We craft proofs of our main results by imposing further axioms on framed motivic $\Gamma$-spaces such as a Segal condition for simplicial Nisnevich sheaves, cancellation, $\mathbb{A}^1$- and $\sigma$-invariance, Nisnevich excision, Suslin contractibility, and grouplikeness. This adds to the discussion in the literature on coexisting points of view on the $\mathbb{A}^1$-homotopy theory of algebraic varieties.

1. INTRODUCTION

The category $\Gamma$ of correspondences or multivalued functions on finite sets is of fundamental importance in topology [29]. Following Boardman and Vogt [6], Segal’s work on $\Gamma$-spaces gives convenient models for $E_\infty$ spaces — spaces with multiplications that are unital, associative, and commutative up to higher coherent homotopies — and for infinite loop spaces. Segal applied his ideas to prove the celebrated Barratt-Priddy-Quillen theorem identifying the group completion of the disjoint union $\bigsqcup_n B\Sigma_n$ of classifying spaces of symmetric groups with the infinite loop space of $\mathbb{S}$ — the topological sphere. Soon afterwards, Bousfield and Friedlander carried out their homotopical identification of connective spectra and $\Gamma$-spaces — an early striking success in the development of stable homotopy theory [7]. Moreover, $\Gamma$-spaces have the advantage that they are simple to define, as well as being intrinsically tied to $K$-theory and topological Hochschild homology [10].

In this paper we introduce the concept of framed motivic $\Gamma$-spaces together with a few axioms. The main purpose of our set-up is to advance our practical understanding of infinite loop spaces along with new viewpoints on connective and very effective spectra in the algebro-geometric setting of $\mathbb{A}^1$-homotopy theory [26], [32]. Voevodsky envisioned this new direction of development in his work on framed correspondences in motivic homotopy theory [34].

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Working over a field \( k \), our approach combines Segal’s category \( \Gamma \) with Voevodsky’s symmetric monoidal category \( \text{Sm}/k_+ \) of framed correspondences of level zero \([34]\) — a slight enlargement of \( \text{Sm}/k \), the category of smooth separated schemes of finite type over \( \text{Spec}(k) \).

Recall from \([19]\) that a framed motivic space is a pointed simplicial Nisnevich sheaf on the category of framed correspondences \( \text{Fr}_+(k) \). As noted in \( \S 2 \), \( \text{Sm}/k_+ \), the opposite category \( \Gamma^{\text{op}} \) and \( \text{Sm}/k_+ \), taking values in framed motivic spaces
\[
\mathcal{X}: \Gamma^{\text{op}} \otimes \text{Sm}/k_+ \longrightarrow \mathcal{M}^{\text{fr}}
\]
and call them framed motivic \( \Gamma \)-spaces (see Definition 2.5). We should note that there is a canonically induced faithful functor
\[
\mathcal{M}^{\text{fr}} \longrightarrow \mathcal{M}
\]
obtained from the composite
\[
\text{Sm}/k \longrightarrow \text{Sm}/k_+ \longrightarrow \text{Fr}_+(k).
\]
The definition of framed correspondences invented in \([34]\) uses an algebro-geometric analogue of a framing on the stable normal bundle of a manifold. We shall review additional background on (1), (2), and (3) in \( \S 2 \).

In our quest to carry over Segal’s programme for \( \Gamma \)-spaces to \( \mathbb{A}^1 \)-homotopy theory we begin by formulating some homotopical axioms for framed motivic \( \Gamma \)-spaces. These axioms concern both of the “variables” \( \Gamma^{\text{op}} \) and \( \text{Sm}/k_+ \) in (1). Informally speaking, the pointed finite sets accounts for the \( S^1 \)-suspension whereas the framed correspondences accounts for the \( \mathbb{G}_m \)-suspension in stable motivic homotopy theory. We may and will view \( \Gamma^{\text{op}} \) as the full subcategory of pointed finite sets with objects \( n_+ = \{0, \ldots, n\} \) pointed at 0 for every integer \( n \geq 0 \).

Every \( \Gamma \)-space gives rise to a simplicial functor and hence an associated \( S^1 \)-spectrum; for details, see \([10, \text{Chapter 2}] \). Similarly in the motivic setting, see (9), we show that every \( U \in \text{Sm}/k_+ \) and \( \mathcal{X} \) as in (1) give rise to a presheaf of \( S^1 \)-spectra \( \mathcal{X}(\mathbb{S}, U) \). We refer to \([24]\) for a comprehensive introduction to the homotopical algebra of such presheaves. In Axioms 1.1 below we employ the notions of local equivalences for simplicial presheaves \([24, \text{Chapter 4}] \) and stable local equivalences for presheaves of \( S^1 \)-spectra \([24, \text{Chapter 10}] \).

For \( n \geq 0 \) and every finitely generated field extension \( K/k \) we write \( \hat{\Delta}_K^n \) for the semilocalization of the standard algebraic \( n \)-simplex
\[
\Delta_K^n = \text{Spec}(K[x_0, \ldots, x_n]/(x_0 + \cdots + x_n - 1))
\]
with closed points the vertices \( v_0, \ldots, v_n \in \Delta_K^n \) — see \([32, \text{§3}] \) for the colimit preserving realization functor from simplicial sets to Nisnevich sheaves. We recall that \( v_i \) is the closed subscheme of \( \Delta_K^n \).
defined by \( x_j = 0 \) for \( j \neq i, \ 0 \leq i \leq n \). Following [31, §2] we write \( \hat{\Delta}_K^* \) for the corresponding cosimplicial semilocal scheme.

We are ready to introduce the main objects of study in this paper.

1.1. Axioms. A framed motivic \( \Gamma \)-space \( \mathcal{X} \) is called special if (1)-(5) holds:

1. We have \( \mathcal{X}(0_+, U) = \ast = \mathcal{X}(n_+, \emptyset) \) for all \( n \geq 0 \) and \( U \in \mathrm{Sm}/k_+ \), while for all \( n \geq 1 \) and nonempty \( U \in \mathrm{Sm}/k_+ \) the naturally induced morphism
\[
\mathcal{X}(n_+, U) \rightarrow \mathcal{X}(1_+, U) \times \cdots \times \mathcal{X}(1_+, U)
\]
is a local equivalence of pointed motivic spaces.

2. For all \( n \geq 0 \) and \( U \in \mathrm{Sm}/k_+ \) the framed presheaf of stable homotopy groups
\[
V \mapsto \pi^*_n \mathcal{X}(S, U)(V)
\]
is \( \mathbb{A}^1 \)-invariant, additive and \( \sigma \)-stable (see Remark 1.2).

3. (Cancellation) Let \( G \) denote the cone of the 1-section \( \mathrm{Spec}(k) \rightarrow \mathbb{G}_m \) in \( \Delta^\mathrm{op} \mathrm{Sm}/k_+ \). For all \( n \geq 0 \) and \( U \in \mathrm{Sm}/k_+ \) there is a canonical stable local equivalence
\[
\mathcal{X}(S, G^{\wedge n} \times U) \rightarrow \mathrm{Hom}(G, \mathcal{X}(S, G^{\wedge n+1} \times U)).
\]

4. (\( \mathbb{A}^1 \)-invariance) For all \( U \in \mathrm{Sm}/k_+ \) there is a naturally induced stable local equivalence
\[
\mathcal{X}(S, U \times \mathbb{A}^1) \rightarrow \mathcal{X}(S, U).
\]

5. (Nisnevich excision) For every elementary Nisnevich square in \( \mathrm{Sm}/k \)
\[
\begin{array}{ccc}
U' & \rightarrow & X' \\
\downarrow & & \downarrow \\
U & \rightarrow & X
\end{array}
\]
there is a homotopy cartesian square in the stable local model structure:
\[
\begin{array}{ccc}
\mathcal{X}(S, U') & \rightarrow & \mathcal{X}(S, X') \\
\downarrow & \downarrow & \downarrow \\
\mathcal{X}(S, U) & \rightarrow & \mathcal{X}(S, X)
\end{array}
\]

Moreover, a special framed motivic \( \Gamma \)-space \( \mathcal{X} \) is called very effective if (6) holds and very special if (7) holds.

6. (Suslin contractibility) For all \( U \in \mathrm{Sm}/k_+ \) and any finitely generated field extension \( K/k \), the geometric realization of the simplicial \( S^1 \)-spectrum
\[
\mathcal{X}(S, G \times U)(\hat{\Delta}_K^*)
\]
is contractible.
(7) (Grouplikeness) For all \( U \in \text{Sm}/k_+ \) the Nisnevich sheaf \( \pi^{\text{nis}}_0 \mathcal{X}(1_+, U) \) associated to the presheaf
\[
\pi_0 \mathcal{X}(1_+, U)(V)
\]
of connected components on \( \text{Sm}/k \) takes values in abelian groups.

1.2. Remark. The reader will recognize axioms (1) and (7) as sheaf versions of special and very special Segal \( \Gamma \)-spaces, respectively \[7, 29\]. Axiom (2) makes use of the assumption that \( X \) is a framed motivic \( \Gamma \)-space. A framed presheaf \( F \) is \( \sigma \)-stable if \( F(\sigma V) = \text{id} F(V) \) for all \( V \in \text{Sm}/k \). Here the level 1 explicit framed correspondence \((\{0\} \times V, \mathbb{A}^1 \times V, \text{pr}_{\mathbb{A}^1}, \text{pr}_V) \in \text{Fr}_1(V, V)\) defines a map \( \sigma_V : V \rightarrow V \) in \( \text{Fr}_+ (k) \); see \[19, \S 2\]. \( \mathcal{F} \) is radditive if \( \mathcal{F}(\emptyset) = * \) and \( \mathcal{F}(X_1 \sqcup X_2) = \mathcal{F}(X_1) \times \mathcal{F}(X_2) \) for all \( X_1, X_2 \in \text{Sm}/k \). In (3), \( \mathbb{G} \) is a simplicial object in \( \text{Sm}/k_+ \) with smash product \( \mathbb{G} \wedge^n \) formed in \( \Delta^{op}\text{Sm}/k_+ \) \[19, \text{Notation 8.1}\]. Axioms (2), (3), (4), and (5) are concerned with presheaves of \( S^1 \)-spectra as in \[24, \text{Part IV}\]. Axiom (6) traces back to Suslin’s work on rationally contractible presheaves in \[31\]; see also \[4\] and \[20\].

1.3. Example. An example of a quintessential special framed motivic \( \Gamma \)-space is given by
\[
(n_+, U) \in \Gamma^{op} \boxtimes \text{Sm}/k_+ \mapsto C_+ \text{Fr}(-, n_+ \otimes U) \in \mathcal{M}^{\text{fr}}.
\]
Here Fr refers to stable framed correspondences and \( C_+ \text{Fr}(-, X') \) to the simplicial framed functor \( X \mapsto \text{Fr}(X \times \Delta^n, X') \) — see \[19 \] and \[34\]. By \( K \otimes U \), where \( K \in \Gamma^{op} \) and \( U \in \text{Sm}/k \), we mean the coproduct of copies of \( U \) indexed by the non-based elements in \( K \).

The evaluation functor in (15) associates to every framed motivic \( \Gamma \)-space \( \mathcal{X} \) an object in the category of framed motivic spectra in the sense of \[20, \text{Definition 2.1}\]
\[
\mathcal{X}_{S^1, \mathbb{G}} \in \text{Sp}_{S^1, \mathbb{G}}^{\text{fr}}(k).
\]
Recall that the triangulated category of framed bispectra \( \text{SH}_{\text{nis}}^{\text{fr}}(k) \) whose objects are those of \( \text{Sp}_{S^1, \mathbb{G}}^{\text{fr}}(k) \) is equivalent to the stable motivic homotopy category \( \text{SH}(k) \) via the identity with quasi-inverse the big framed motive functor \[20, \text{Theorem 2.2}\]. The big framed motive functor is closely related to Example 1.3 — for details we refer to \[19, \text{Section 12}\].

For the purposes of this paper it is not necessary to discuss model structures on framed motivic \( \Gamma \)-spaces. Our next definition is inspired by Segal’s homotopy category of \( \Gamma \)-spaces \[29\].

1.4. Definition. The homotopy category of framed motivic \( \Gamma \)-spaces
\[
\mathcal{H}_{\Gamma, \mathbb{G}}(k)
\]
has objects special framed motivic \( \Gamma \)-spaces and with morphisms given by
\[
\mathcal{H}_{\Gamma, \mathbb{G}}(k)(\mathcal{X}, \mathcal{Y}) := \text{SH}_{\text{nis}}^{\text{fr}}(k)(\mathcal{X}_{S^1, \mathbb{G}}, \mathcal{Y}_{S^1, \mathbb{G}}).
\]
In §3 we discuss how $H_{\Gamma,Fr}^r(k)$ relates to the unstable pointed motivic homotopy category $H(k)$ and to connective motivic spectra $SH(k)_{\geq 0}$ via the commutative — up to equivalence of functors — diagram of adjunctions:

$$
\begin{array}{ccc}
H(k) & \xrightarrow{\Sigma^\infty} & SH(k)_{\geq 0} \\
\downarrow_{\Gamma M_{fr}} & & \downarrow_{\Gamma M_{fr}} \\
H_{\Gamma,Fr}^r(k) & \xrightarrow{\Omega S^1, G} & \mathcal{X} \\
\end{array}
$$

Here $\mathcal{X} \in H_{\Gamma,Fr}^r(k)$ is mapped to its underlying motivic space $\mathcal{X}(1_{+}, pt) \in H(k)$ and to its framed motivic spectrum $\mathcal{X}_{S^1, G} \in SH(k)_{\geq 0}$ under the equivalence between $SH_{fr}^r(k)$ and $SH(k)$ in [20]. We refer to Remark 3.2 for the definition of $\Gamma M_{fr}$ — a version of the big framed motive functor introduced in [19, Section 12].

1.5. **Theorem.** For every infinite perfect field $k$ there is an equivalence of categories

$$
H_{\Gamma,Fr}^r(k) \xrightarrow{\simeq} SH(k)_{\geq 0}
$$

Its quasi-inverse $SH(k)_{\geq 0} \xrightarrow{\simeq} H_{\Gamma,Fr}^r(k)$ takes $\mathcal{E} \in SH(k)_{\geq 0}$ to an explicitly constructed framed motivic $\Gamma$-space $\Gamma M_{fr}^{\mathcal{E}} \in H_{\Gamma,Fr}^r(k)$.

Let $SH^{eff}(k)$ be the full subcategory of $SH(k)$ that is generated under homotopy colimits and extensions by motivic $\mathbb{P}^1$-suspension spectra of smooth schemes. This category is of interest since it gives rise to the very effective slice filtration introduced in [30]. We note $SH^{eff}(k)$ is contained in the triangulated category $SH(k)_{\geq 0}$ — generated under homotopy colimits and extensions by motivic $\mathbb{P}^1$-suspension spectra $\Sigma^{p,q} U_+$, where $p \geq q$, and $U \in Sm/k$.

We shall study $SH^{eff}(k)$ from the point of view of framed motivic $\Gamma$-spaces.

1.6. **Definition.** The homotopy category of very effective framed motivic $\Gamma$-spaces

$$
H^{veffr}_{\Gamma,Fr}(k)
$$

is the full subcategory of $H_{\Gamma,Fr}^r(k)$ comprised of very effective special framed motivic $\Gamma$-spaces.

We show that Axiom (6) on Suslin contractibility of special framed motivic $\Gamma$-spaces captures precisely the difference between $SH^{veff}(k)$ and $SH(k)_{\geq 0}$.

1.7. **Theorem.** For every infinite perfect field $k$ there is an equivalence of categories:

$$
H^{veffr}_{\Gamma,Fr}(k) \xrightarrow{\simeq} SH^{veff}(k)
$$

Finally, we employ Axiom (7) in our recognition principle for infinite motivic loop spaces.
1.8. **Theorem.** For every infinite perfect field \( k \) and every \( \mathcal{E} \in \text{SH}(k) \) there exists a very special framed motivic \( \Gamma \)-space \( \Gamma M^E_{fr} \) and a local equivalence of pointed motivic spaces:

\[
\Gamma M^E_{fr}(1_+, \text{pt}) \simeq \Omega^\infty_S \Omega^\infty_{G_E} \mathcal{E}
\]

(7)

Moreover, if \( \mathcal{X} \) is a very special framed motivic \( \Gamma \)-space then \( \mathcal{X}(1_+, \text{pt}) \) is an infinite motivic loop space.

**Guide to the paper.** For the convenience of the reader we begin §2 by reviewing background on enriched categories, with the aim of introducing framed motivic \( \Gamma \)-spaces. As prime examples we discuss the motivic sphere spectrum \( S \), algebraic cobordism \( \text{MGL} \), motivic cohomology \( \text{MZ} \), and Milnor-Witt motivic cohomology \( \tilde{\text{MZ}} \). Our main results, Theorems 1.5, 1.7, and 1.8 are shown in §3. Finally, in §4 we record some novel homotopical properties of framed motivic \( \Gamma \)-spaces.

**Notation.** Throughout the paper we employ the following notation.

\[\begin{align*}
 k, \text{pt} & \quad \text{infinite perfect field of exponential characteristic } e, \spec(k) \\
 \text{Sm}/k & \quad \text{smooth separated schemes of finite type} \\
 \text{Sm}/k_+ & \quad \text{framed correspondences of level zero} \\
 \text{Shv}_*(\text{Sm}/k) & \quad \text{closed symmetric monoidal category of pointed Nisnevich sheaves} \\
 \mathcal{M} & = \Delta^{op}\text{Shv}_*(\text{Sm}/k) \quad \text{pointed motivic spaces, a.k.a. pointed simplicial Nisnevich sheaves} \\
 \text{Fr}_+(k) & \quad \text{the category of framed correspondences} \\
 \text{Pre}^{fr}_+(k) & \quad \text{framed presheaves, a.k.a. presheaves of sets on Fr}_+(k) \\
 i: \text{Sm}/k \rightarrow \text{Fr}_+(k) & \quad \text{the composite functor Sm}/k \rightarrow \text{Sm}/k_+ \rightarrow \text{Fr}_+(k) \\
 S^{s,t}, \Omega^{s,t}, \Sigma^{s,t} & \quad \text{motivic } (s,t)\text{-sphere, loop space, and suspension} \\
 S_+ & \quad \text{pointed simplicial sets}
\end{align*}\]

Our standard convention for motivic spheres is that \( S^1 \simeq \mathbb{P}^1 \simeq T \) and \( S^{1,1} \simeq \mathbb{A}^1 \setminus \{0\} \) as in [26].

**Relations to other works.** Our approach in this paper is a homage to Segal’s work on categories and cohomology theories [29]. Along the same line we use minimal machinery to achieve concrete models for infinite motivic loop spaces and motivic spectra with prescribed properties. Based on Voevodsky’s notes [34], the machinery of framed motives is developed in [19]. As an application, explicit computations of infinite motivic loop spaces are given as follows: \( \Omega^\infty_S \Sigma^\infty_{\text{gp}}A, A \in \mathcal{M} \), is locally equivalent to the space \( C_+\text{Fr}(A^c)^{\text{gp}} \) (‘gp’ for group completion), where \( A^c \) is a projective cofibrant replacement of \( A \) — see [19, Section 10]. Based on [1, 19, 18, 17], a motivic recognition principle for infinite motivic loop spaces using the language of infinity categories is given in [14].

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2. Framed Motivic $\Gamma$-Spaces

We refer to [5] and [13] for the projective motivic model structure on the closed symmetric monoidal category of pointed motivic spaces $\mathcal{M}$. This model structure is combinatorial, proper, simplicial, symmetric monoidal, and weakly finitely generated. Let $\Delta[\bullet]$ be the standard cosimplicial simplicial set $n \mapsto \Delta[n]$. If there is no likelihood of confusion, we sometimes regard it as a cosimplicial smooth scheme, where each $\Delta[n]$ is regarded as the disjoint union $\bigsqcup_{\Delta[n]} \text{pt}$. The simplicial function object between pointed motivic spaces $A$ and $B$ is given by

$${}\mathcal{S}_\bullet(A, B) = \text{Hom}_{\mathcal{M}}(A \land \Delta[\bullet]_+, B) = \text{Hom}_{\mathcal{M}}(A, B(\Delta[\bullet] \times -)).$$

For every $U \in \text{Sm}/k$ the Yoneda lemma identifies $\mathcal{S}_\bullet(U_+, A)$ with the pointed simplicial set of sections $A(U)$.

Recall $A \in \mathcal{M}$ is finitely presentable if the functor $\text{Hom}_{\mathcal{M}}(A, -)$ preserves directed colimits. For example, the representable pointed motivic space $U_+$ is finitely presentable for every $k$-smooth scheme $U \in \text{Sm}/k$.

A collection $\mathcal{C}$ of finitely presentable pointed motivic spaces can be enriched in $\mathcal{M}$ by means of the $\mathcal{M}$-enriched Hom-functor

$$[A, B](X) := \text{Hom}_{\mathcal{M}}(A, B)(X) = \mathcal{S}_\bullet(A \land X_+, B) = \text{Hom}_{\mathcal{M}}(A \land \Delta[\bullet]_+, B(\Delta[\bullet] \times -)) = \text{Hom}_{\mathcal{M}}(A, B(\Delta[\bullet] \times -)), \quad A, B \in \mathcal{C}, X \in \text{Sm}/k. \quad (8)$$

The enriched composition in $\mathcal{C}$ is inherited from the enriched composition in $\mathcal{M}$. We write $[\mathcal{C}, \mathcal{M}]$ for the category of $\mathcal{M}$-enriched covariant functors from $\mathcal{C}$ to $\mathcal{M}$, and refer to [12, Section 4] for its projective model structure — the weak equivalences and fibrations are defined pointwise.

Voevodsky [34] defined the morphisms in $\text{Sm}/k_+$ by setting

$$\text{Sm}/k_+(X, Y) := \text{Hom}_{\text{Shv}^+(\text{Sm}/k)}(X_+, Y_+), \quad X, Y \in \text{Sm}/k.$$ 

In case $X$ is connected we have $\text{Sm}/k_+(X, Y) = \text{Hom}_{\text{Sm}/k}(X, Y)_+$ by [34, Example 2.1].

2.1. Lemma. With the notation above we have identifications of constant simplicial sets

$$[U_+, V_+](X) = \text{Hom}_{\mathcal{M}}((U \times X)_+, V_+) = \text{Sm}/k_+(U \times X, V),$$

where $U, V, X \in \text{Sm}/k$.

Proof. By definition we have

$$[U_+, V_+](X) = \text{Hom}_{\mathcal{M}}(U_+, V_+)(X) = \text{Hom}_{\mathcal{M}}((U \times X)_+, V_+).$$

It is evident that $\text{Hom}_{\mathcal{M}}((U \times X)_+, V_+) = \text{Sm}/k_+(U \times X, V)$. \qed
2.2. **Remark.** In fact $\text{Sm}/k_+(-, V)$, $V \in \text{Sm}/k$, is the Nisnevich sheaf associated to the presheaf $U \mapsto \text{Hom}_{\text{Sm}/k}(U, V) \sqcup \text{pt}$.

Our first example is Segal’s category $\Gamma^{\text{op}}$ of pointed finite sets and pointed maps.

2.3. **Example.** As in [19, Section 5] we view $\Gamma^{\text{op}}$ as a full subcategory of $\mathcal{M}$ by sending $K \in \Gamma^{\text{op}}$ to $(\sqcup_{K \times \ast \text{pt}}) = \sqcup_{K}$ — the coproduct is indexed by the non-based elements in $K$. This turns $\Gamma^{\text{op}}$ into a symmetric monoidal $\mathcal{M}$-category. Hence $[\Gamma^{\text{op}}, \mathcal{M}]$ is a closed symmetric monoidal category by [9].

We claim that $[\Gamma^{\text{op}}, \mathcal{M}]$ can be identified with the category $\Gamma \mathcal{M}$ of covariant functors from $\Gamma^{\text{op}}$ to $\mathcal{M}$ sending $0_\ast$ to the basepoint $\ast$ of $\mathcal{M}$ — in this case $\mathcal{C} = \{\sqcup_{K \times \ast \text{pt}} \mid K \in \Gamma^{\text{op}}\}$: An $\mathcal{M}$-enriched functor $\mathcal{X} \in [\Gamma^{\text{op}}, \mathcal{M}]$ sends $K \in \Gamma^{\text{op}}$ to $\mathcal{X}(\sqcup_{K \times \ast \text{pt}}) \in \mathcal{M}$ and for $K, L \in \Gamma^{\text{op}}$ there is a morphism

$$\alpha_{K,L} : \mathcal{X}(\sqcup_{K \times \ast \text{pt}}) \to \mathcal{X}(\sqcup_{L \times \ast \text{pt}}).$$

Here the motivic space $[\sqcup_{K \times \ast \text{pt}}, \sqcup_{L \times \ast \text{pt}}]$ is given by

$$U \mapsto \Gamma^{\text{op}}(\sqcup_{K \times n(U)_+ \times \{\ast, +\}} \text{pt}, \sqcup_{L \times \ast \text{pt}}),$$

where $n(U)$ is the number of connected components of $U \in \text{Sm}/k$ and $n(U)_+ = \{0, 1, \ldots, n(U)\}$. Since $\mathcal{X}$ takes values in simplicial sheaves it follows that

$$\mathcal{X}(K)(U) = \mathcal{X}(K)(U_1) \times \cdots \times \mathcal{X}(K)(U_{n(U)}),$$

and consequently we have

$$\alpha_{K,L}(U) = \alpha_{K,L}(U_1) \times \cdots \times \alpha_{K,L}(U_{n(U)}).$$

To a morphism $f : K \to L$ in $\Gamma^{\text{op}}$ we associate the morphism $\mathcal{X}(\sqcup_{K \times \ast \text{pt}}) \to \mathcal{X}(\sqcup_{L \times \ast \text{pt}})$ with $U$-sections

$$\alpha_{K,L}(U_1)(f) \times \cdots \times \alpha_{K,L}(U_{n(U)})(f).$$

Clearly, this yields the identification of $[\Gamma^{\text{op}}, \mathcal{M}]$ with pointed functors from $\Gamma^{\text{op}}$ to $\mathcal{M}$.

We are passing to the definition of the category of framed motivic spaces $\mathcal{M}^{\text{fr}}$ and to its natural enrichment over $\mathcal{M}$. Let $\text{Fr}_+(k)$ be the category of framed correspondences as in [19, Section 2]. Let $\text{Pr}^{\text{fr}}(k)$ be the category of framed presheaves, that is the category of presheaves of sets on $\text{Fr}_+(k)$. Let $i : \text{Sm}/k \to \text{Sm}/k_+ \to \text{Fr}_+(k)$ be the composite functor. Recall from [19, Section 2] that a framed Nisnevich sheaf on $\text{Sm}/k$ is a framed presheaf such that its restriction to $\text{Sm}/k$ via the functor $i$ is a Nisnevich sheaf. Let $\text{Shv}^{\text{fr}}(k)$ denote the category of pointed framed Nisnevich sheaves. The morphisms in this category are just morphisms of pointed framed presheaves. The category of framed motivic spaces $\mathcal{M}^{\text{fr}}$ is the category of simplicial objects in $\text{Shv}^{\text{fr}}(k)$. There is a canonically induced faithful functor $\iota : \mathcal{M}^{\text{fr}} \to \mathcal{M}$ obtained from the composite $i : \text{Sm}/k \to \text{Sm}/k_+ \to \text{Fr}_+(k)$.

Following [34, Section 6] there is a natural pairing $\text{Sm}/k_+ \times \text{Fr}_+(k) \to \text{Fr}_+(k)$ taking $(X,Y)$ to $X \times Y$ and $(f, \alpha)$ to $f \times \alpha$. In what follows this pairing will be used systematically without referring to it. We also use it in the natural enrichment of $\mathcal{M}^{\text{fr}}$ over $\mathcal{M}$.
First, we can associate a framed Nisnevich sheaf $\mathcal{F}(X \times -)$ to every framed Nisnevich sheaf $\mathcal{F}$ and every $X \in \text{Sm}/k_+$. In detail, given $\alpha \in \text{Fr}_+(U',U)$ put $\alpha^* : \mathcal{F}(X \times U) \rightarrow \mathcal{F}(X \times U')$ to be $(\text{id}_X \times \alpha)^*$. If $\mathcal{F}$ is a pointed framed Nisnevich sheaf then the framed Nisnevich sheaf $\mathcal{F}(X \times -)$ is pointed also.

Second, every morphism $f : X' \rightarrow X$ in $\text{Sm}/k_+$ induces a morphism of framed sheaves $f^* : \mathcal{F}(X \times -) \rightarrow \mathcal{F}(X' \times -)$. Namely, if $U \in \text{Fr}_+(k)$ one sets $f^* : \mathcal{F}(X \times U) \rightarrow \mathcal{F}(X' \times U)$ to be $(f \times \text{id}_U)^*$. If $\mathcal{F}$ is a pointed framed Nisnevich sheaf, then the morphism of framed sheaves $f^* : \mathcal{F}(X \times -) \rightarrow \mathcal{F}(X' \times -)$ is a morphism of pointed framed Nisnevich sheaves.

Finally, similarly to (8), $\mathcal{M}^{fr}$ is naturally enriched over $\mathcal{M}$. Namely,

$$\mathcal{M}(A,B)(X) := \text{Hom}_{\mathcal{M}}(A,B(\Delta(\bullet) \times -)), \quad A,B \in \mathcal{M}^{fr}, X \in \text{Sm}/k.$$ 

The enriched composition in $\mathcal{M}^{fr}$ is inherited from the enriched composition in $\mathcal{M}$.

Our second example is Voevodsky’s category of framed correspondences of level zero.

2.4. Example. We enrich $\text{Sm}/k_+$ in $\mathcal{M}$ by setting

$$[U,V] := \text{Hom}_{\mathcal{M}}(U_+,V_+), \quad U,V \in \text{Sm}/k_+.$$ 

This turns $\text{Sm}/k_+$ into a symmetric monoidal $\mathcal{M}$-category with tensor products $U \times V \in \text{Sm}/k$. It follows that $[\text{Sm}/k_+,\mathcal{M}]$ is a symmetric monoidal $\mathcal{M}$-category [9]. Framed correspondences of level zero form the underlying category of the $\mathcal{M}$-category $\text{Sm}/k_+$. According to Lemma 2.1 the pointed motivic space $[U,V]$ has $Y$-sections the constant simplicial set

$$[U,V](Y) = \text{Hom}_{\mathcal{M}}((U \times Y)_+,V_+) = \text{Sm}/k_+(U \times Y,V).$$

Owing to the $\mathcal{M}$-enrichment every $\mathcal{X} \in [\text{Sm}/k_+,\mathcal{M}]$ gives rise to a morphism

$$[U,V] \rightarrow \text{Hom}_{\mathcal{M}}(\mathcal{X}(U),\mathcal{X}(V)).$$

On $Y$-sections we obtain a morphism from $[U,V](Y)$ to

$$\text{Hom}_{\mathcal{M}}(\mathcal{X}(U),\mathcal{X}(V))(Y) = S_{\bullet}(\mathcal{X}(U) \wedge Y_+,\mathcal{X}(V)) =$$

$$S_{\bullet}(\mathcal{X}(U),\mathcal{X}(V)(Y \times -)) = \text{Hom}_{\mathcal{M}}(\mathcal{X}(U) \wedge \Delta(\bullet)_+,\mathcal{X}(V)(Y \times -)).$$

The monoidal product $\Gamma^{op} \boxtimes \text{Sm}/k_+$ is the $\mathcal{M}$-category with objects $\text{Ob}\Gamma^{op} \times \text{Ob}\text{Sm}/k_+$ and

$$[(K,A),(L,B)] = [K,L] \times [A,B].$$

Note that $\Gamma^{op} \boxtimes \text{Sm}/k_+$ is a symmetric monoidal $\mathcal{M}$-category.

2.5. Definition. (1) A motivic $\Gamma$-space is an $\mathcal{M}$-enriched functor $\mathcal{X} : \Gamma^{op} \boxtimes \text{Sm}/k_+ \rightarrow \mathcal{M}$.

(2) A framed motivic $\Gamma$-space is an $\mathcal{M}$-enriched functor $\mathcal{X} : \Gamma^{op} \boxtimes \text{Sm}/k_+ \rightarrow \mathcal{M}^{fr}$.

2.6. Remark. Let $\Gamma^{op} \times \text{Sm}/k_+$ denote the underlying category of the $\mathcal{M}$-category $\Gamma^{op} \boxtimes \text{Sm}/k_+$. Every motivic $\Gamma$-space $\mathcal{X} : \Gamma^{op} \boxtimes \text{Sm}/k_+ \rightarrow \mathcal{M}$ gives rise to a functor $\mathcal{X} : \Gamma^{op} \times \text{Sm}/k_+ \rightarrow \mathcal{M}$ denoted by the same letter.

Unravelling the previous definition, a framed motivic $\Gamma$-space is equivalent to giving the following data:
\(\triangleright\) an \(\mathcal{M}\)-functor \(\mathcal{X} : \Gamma^\text{op} \boxtimes \text{Sm}/k_+ \to \mathcal{M}\);

\(\triangleright\) a functor \(\mathcal{X} : \Gamma^\text{op} \times \text{Sm}/k_+ \to \mathcal{M}\);

\(\triangleright\) the induced functor \(\mathcal{X} : \Gamma^\text{op} \times \text{Sm}/k_+ \to \mathcal{M}\) equals the composite functor \(\Gamma^\text{op} \times \text{Sm}/k_+ \xrightarrow{\mathcal{X}'} \mathcal{M}^\text{fr} \to \mathcal{M}\) such that the canonical morphism

\[ [U,V](Y) \to \text{Hom}_{\mathcal{M}}(\mathcal{X}(K,U), \mathcal{X}(K,V)(Y \times -)) \]

factors through \(\text{Hom}_{\mathcal{M}}(\mathcal{X}'(K,U), \mathcal{X}'(K,V)(Y \times -))\) for all \(K \in \Gamma^\text{op}, U,V,Y \in \text{Sm}/k_+\).

2.7. Evaluation Functors. Every motivic \(\Gamma\)-space \(\mathcal{X} \in [\Gamma^\text{op} \boxtimes \text{Sm}/k_+, \mathcal{M}]\) and \(U \in \text{Sm}/k_+\) gives rise to an enriched functor \(\mathcal{X}(U) \in [\Gamma^\text{op}, \mathcal{M}]\). In Example 2.3 we identified \(\mathcal{X}(U)\) with the datum of a pointed functor from \(\Gamma^\text{op}\) to \(\mathcal{M}\). Following [10, Example 2.1.2.1] by the sphere spectrum we mean the inclusion \(S : \Gamma^\text{op} \hookrightarrow S_*\). By taking the left Kan extension along the sphere spectrum \(S : \Gamma^\text{op} \hookrightarrow S_*\) we obtain the evaluation functor with values in motivic \(S^1\)-spectra

\[ \text{ev}_{S^1} : [\Gamma^\text{op}, \mathcal{M}] \to \text{Sp}_{S^1}(k) \]  

\(\mathcal{X}(U) \to \mathcal{X}(S,U) = (\mathcal{X}(S^0)(U), \mathcal{X}(S^1)(U), \mathcal{X}(S^2)(U), \ldots)\).

We refer to \(\mathcal{X}(S,pt)\) as the underlying motivic \(S^1\)-spectrum of \(\mathcal{X}\).

On the other hand, for \(K \in \Gamma^\text{op}\) we obtain an enriched functor \(\mathcal{X}(K) \in [\text{Sm}/k_+, \mathcal{M}]\) — see Example 2.4. Moreover, for \(U,V \in \text{Sm}/k_+\) there are natural morphisms in \(\mathcal{M}\)

\[ V_+ \to [U,U \times V] \to \text{Hom}_{\mathcal{M}}(\mathcal{X}(K)(U), \mathcal{X}(K)(U \times V)) \]

By adjunction we obtain morphisms

\[ \mathcal{X}(K)(U) \land V_+ \to \mathcal{X}(K)(U \times V) \quad \text{and} \quad \mathcal{X}(K)(U) \to \text{Hom}_{\mathcal{M}}(V_+, \mathcal{X}(K)(U \times V)). \]

(10)

The simplices of \(G \in \Delta^\text{op}\text{Sm}/k_+\) consist of finite disjoint unions \(G_{m,\leq n}^{\leq \infty}\) of copies of the multiplicative group scheme \(G_m\) and \(pt\). Namely, the simplices are \(G_m, G_m \sqcup pt, G_m \sqcup pt \sqcup pt, \ldots\) (we also refer the reader to [19, Notation 8.1]). As a special case of (10) we have

\[ \mathcal{X}(K)(U) \land (G_{m,\leq n}^{\leq \infty})_+ \to \mathcal{X}(K)(U \times G_{m,\leq n}^{\leq \infty}) \quad \text{and} \quad \mathcal{X}(K)(U) \to \text{Hom}_{\mathcal{M}}((G_{m,\leq n}^{\leq \infty})_+, \mathcal{X}(K)(U \times G_{m,\leq n}^{\leq \infty})). \]

(11)

For the smash powers of \(G\) we define the morphisms

\[ \mathcal{X}(K)(G^{\leq n}) \land G_+ \to \mathcal{X}(K)(G^{\leq n+1}) \quad \text{and} \quad \mathcal{X}(K)(G^{\leq n}) \to \text{Hom}_{\mathcal{M}}(G_+, \mathcal{X}(K)(G^{\leq n+1})). \]

(12)

to be the geometric realization of

\[ \{ \mathcal{X}(K)((G^{\leq n})_t) \land (G_+)_t \to \mathcal{X}(K)((G^{\leq n+1})_t) \} \]

\[ l \mapsto \{ \mathcal{X}(K)((G^{\leq n})_t) \land (G_+)_t \to \mathcal{X}(K)((G^{\leq n+1})_t) \} \]
and

\[ I \mapsto \{ X(K)((G^m)_I) \to \text{Hom}_\mathcal{M}((G_+)_I, X(K)((G^m+1)_I)) \} \]

obtained from (11). Due to (12) we obtain the evaluation functor with values in motivic G-spectra

\[ \text{ev}_G : [\text{Sm}/k_+\text{, }\mathcal{M}] \to \text{Sp}_G(k) \]

(13)

\[ X \mapsto X(\mathcal{X}(K)(\text{pt}), X(\mathcal{X}(K)(G)), X(\mathcal{X}(G^2), \ldots) ). \]

We refer to [11, Chapter 3, Section 2.3] for a discussion of the category \( \text{Sp}_{S^{1},G}(k) \) of motivic \( (S^1, G) \)-bispectra. Its associated homotopy category is equivalent to \( \text{SH}(k) \). Combining (9) and (13) we obtain the evaluation functor:

\[ \text{ev}_{S^{1},G} : [\Gamma^{op} \boxtimes \text{Sm}/k_+, \mathcal{M}] \to \text{Sp}_{S^{1},G}(k) \]

(14)

\[ \mathcal{X} \mapsto \mathcal{X}_{S^{1},G} = \text{ev}_{S^{1},G}(\mathcal{X}). \]

More precisely, for \( i, j \geq 0 \) we have

\[ \text{ev}_{S^{1},G}(\mathcal{X})_{i,j} = X(\mathcal{X}_{S^{1},G}(S^i, G^j)) \in \mathcal{M}. \]

The evident structure maps turn \( \mathcal{X}_{S^{1},G} \) into a motivic \( (S^1, G) \)-bispectrum.

In turn, let \( [\Gamma^{op} \boxtimes \text{Sm}/k_+, \mathcal{M}^{fr}] \) denote the category of \( \mathcal{M} \)-enriched functors from \( \Gamma^{op} \boxtimes \text{Sm}/k_+ \) to \( \mathcal{M}^{fr} \). Its objects are the framed motivic \( \Gamma \)-spaces following Definition 2.5. If \( \mathcal{X} \) is a framed motivic \( \Gamma \)-space then the structure morphisms

\[ X(S^i, G^j) \to \text{Hom}(S^1, X(S^i+1, G^j)) \]

\[ X(S^i, G^j) \to \text{Hom}(G_+, X(S^i+1, G^j+1)) \]

are morphisms in \( \mathcal{M}^{fr} \). Therefore \( \mathcal{X}_{S^{1},G} \in \text{Sp}^{fr}_{S^{1},G}(k) \) is a framed motivic \( (S^1, G) \)-bispectrum in the sense of [20, Definition 2.1]. Similarly to (14) we obtain the evaluation functor:

\[ \text{ev}^{fr}_{S^{1},G} : [\Gamma^{op} \boxtimes \text{Sm}/k_+, \mathcal{M}^{fr}] \to \text{Sp}^{fr}_{S^{1},G}(k) \]

(15)

\[ \mathcal{X} \mapsto \mathcal{X}_{S^{1},G} = \text{ev}^{fr}_{S^{1},G}(\mathcal{X}). \]

2.8. Example. For every \( X \in \text{Sm}/k \) we can form the motivic \( \Gamma \)-space with sections

\[ (K, U) \mapsto \text{Sm}/k_+(-, K \otimes (X \times U)). \]

Its evaluation is the suspension bispectrum \( \Sigma_{G} \Sigma_{G} X_+ \) of \( X \). Similarly, we can form the special framed motivic \( \Gamma \)-space \( \text{Hom}(X, C_{\text{Fr}}) \) with sections

\[ (K, U) \mapsto C_{\text{Fr}}(-, K \otimes (X \times U)). \]

Its underlying motivic \( S^1 \)-spectrum \( \text{Hom}(X, C_{\text{Fr}})(\mathbb{S}, \text{pt}) \) is the framed motive of \( X \) [19].

There is a natural morphism of motivic \( \Gamma \)-spaces

\[ \text{Sm}/k_+(-, - \otimes (X \times -)) \to C_{\text{Fr}}(-, - \otimes (X \times -)). \]

(16)
By [19, Theorem 11.1] the evaluation functor in (14) takes the morphism in (16) to a stable motivic equivalence. In particular, the special framed motivic $\Gamma$-space $\text{Hom}(pt, C, \text{Fr})$ is a model for the motivic sphere $1$.

By linearization we obtain the special framed motivic $\Gamma$-space $\text{Hom}(X, C, \mathbb{Z}F)$ with sections

$$(K, U) \mapsto C, \mathbb{Z}F((-K \otimes (X \times U))).$$

The underlying motivic $S^1$-spectrum $\text{Hom}(X, C, \mathbb{Z}F)(\mathbb{S}, pt)$ is the linear framed motive of $X$ [19].

2.9. Example. Let $\mathcal{E}$ be a motivic symmetric Thom $T$- or $T^2$-spectrum with a bounding constant $d \leq 1$ and contractible alternating group action in the sense of [16, Section 1] — the main examples are algebraic cobordism $\text{MGL}$ [32] and the $T^2$-spectra $\text{MSL}$, $\text{MSp}$ in [27] (in all of these cases $d = 1$). Under these assumptions there exists a special framed motivic $\Gamma$-space $\text{Hom}(X, C, \text{Fr}^\mathcal{E})$ with sections

$$(K, U) \mapsto C, \text{Fr}^\mathcal{E}((-K \otimes (X \times U))).$$

The evaluation $\text{ev}_{\mathcal{E}} : (\text{Hom}(X, C, \text{Fr}^\mathcal{E}))$ agrees with $\mathcal{E} \wedge X_+$ by the proof of [16, Theorem 9.13]. Moreover, $\text{Hom}(X, C, \text{Fr}^\mathcal{E})(\mathbb{S}, pt)$ is the $\mathcal{E}$-framed motive of $X$ in the sense of [16, Section 9].

Likewise, we obtain the special framed motivic $\Gamma$-space $\text{Hom}(X, C, \mathbb{Z}F^\mathcal{E})$, whose underlying motivic $S^1$-spectrum is the linear $\mathcal{E}$-framed motive of $X$ defined in [16, Section 9].

2.10. Example. Suppose that $\mathcal{A}$ is a strict category of Voevodsky correspondences in the sense of [15, Definition 2.3] and there exists a functor $\text{Fr}^\mathcal{A}(k) \rightarrow \mathcal{A}$ which is the identity map on objects. Examples include finite Milnor-Witt correspondences $\text{Cor}$ [8], finite correspondences $\text{Cor}$ [33], and $K^0_0$-correspondences [36]. We define $C, \mathcal{A}$ to be the very special framed motivic $\Gamma$-space with sections the Suslin complex of the Nisnevich sheaf $\mathcal{A}(-, K \otimes U)^{\text{nisl}}$ — sectionwise we have

$$(K, U) \mapsto C, \mathcal{A}((-K \otimes (X \times U)).$$

Note that $\text{Hom}(X, C, \mathcal{A})(\mathbb{S}, pt)$ is the $\mathcal{A}$-motive of $X$ defined in [15, Section 2], where $\text{Hom}(X, C, \mathcal{A})$ stands for the very special framed motivic $\Gamma$-space with sections $(K, U) \mapsto C, \mathcal{A}((-K \otimes (X \times U)).$

2.11. Remark. The motivic $\Gamma$-spaces in Examples 2.8, 2.9, and 2.10 share the common trait of factoring through the functor $\otimes: \text{Gr}^\text{op} \otimes \text{Sm}/k_+ \rightarrow \text{Sm}/k_+$.

3. SPECIAL FRAMED MOTIVIC $\Gamma$-SPACES AND INFINITE MOTIVIC LOOP SPACES

Let $\mathcal{E}$ be a motivic $(S^1, G)$-bispectrum. Using the $n$th weight motivic $S^1$-spectrum $\mathcal{E}(n)$ of $\mathcal{E}$—

defined by $\mathcal{E}(n) = \mathcal{E}_{i, n}$ — we write $\mathcal{E} = (\mathcal{E}(0), \mathcal{E}(1), \ldots)$. For integers $p, n \in \mathbb{Z}$ let $\pi_{p,n}^\mathcal{E}$ be the Nisnevich sheaf on $\text{Sm}/k$ associated to the presheaf

$$U \mapsto \text{SH}(k)(U_+ \wedge S^{p-n} \wedge G^{\wedge n}, \mathcal{E}).$$

Recall that $\mathcal{E}$ is connective if $\pi_{p,n}^\mathcal{E} = 0$ for all $p < n$. Similarly, a motivic $S^1$-spectrum $\mathcal{E} \in \text{Sp}_{S^1}(k)$ is connective if $\pi_{n}^\mathcal{E}$ is connective if $\pi_{n}^\mathcal{E} = 0$ for all $n < 0$. For a Nisnevich sheaf $F$ of abelian groups on $\text{Sm}/k$, let $F_{-1}$ denote the Nisnevich sheaf given by $U \mapsto \ker(1^*: F(U \times \mathbb{G}_m) \rightarrow F(U))$. 
3.1. **Lemma.** A framed motivic $(S^1,\mathbb{G})$-bispectrum $\mathcal{E} = (\mathcal{E}(0), \mathcal{E}(1), \ldots)$ in the sense of [20, Section 2] is connective if and only if $\mathcal{E}(n)$ is a connective motivic $S^1$-spectrum for every $n \geq 0$.

**Proof.** Without loss of generality we may assume that the underlying motivic bispectrum $\mathcal{E}$ is fibrant (we use here [20, Lemma 2.6]). Writing $|−|$ for the absolute value we have $\pi^{\mathcal{E}}_{p,n} = \pi^{\mathcal{E}}_{p,n}(\mathcal{E}(n))$ if $n \leq 0$, while $\pi^{\mathcal{E}}_{p,n} = \pi^{\mathcal{E}}_{p,n}(\mathcal{E}(0))$ if $n > 0$. Here $\pi^{\mathcal{E}}_{p,n}$ denotes the Nisnevich sheaf associated to $\mathcal{E}$. The proof of the sublemma in [19, Section 12] shows that

$$\pi^{\mathcal{E}}_{p,n}(\mathcal{E}(0)) = \pi^{\mathcal{E}}_{p,n}(\mathcal{E}(0)) - n.$$ 

If $\mathcal{E}$ is connective then $\pi^{\mathcal{E}}_{p,n}(\mathcal{E}(n)) = 0$ for all $n \leq 0$ and $p < n$. In particular, for all $s > 0$ and $n \leq 0$, the sheaf $\pi^{\mathcal{E}}_{n}(\mathcal{E}(n))$ is trivial. The converse implication is evident. □

Recall that $Sp_{S^1}(k)$ is naturally enriched in $\mathcal{M}$ — see the proof of [22, Theorem 6.3]. In fact, for $\mathcal{E}, \mathcal{F} \in Sp_{S^1}(k)$ one defines $\mathcal{M}(\mathcal{E}, \mathcal{F})$ as the equalizer of the diagram

$$\prod_n \mathcal{M}(\mathcal{E}_n, \mathcal{F}_n) \longrightarrow \prod_n \mathcal{M}(\mathcal{E}_n, \mathcal{F}_n).$$

(17)

Here we employ the morphism $\mathcal{M}(\mathcal{E}_n, \mathcal{F}_n) \longrightarrow \mathcal{M}(\mathcal{E}_n, \mathcal{F}_n)$ induced by the adjoint of the structure maps of $\mathcal{F}$, and the canonically induced morphism

$$\mathcal{M}(\mathcal{E}_{n+1}, \mathcal{F}_{n+1}) \rightarrow \mathcal{M}(\mathcal{E}_n \wedge S^1, \mathcal{F}_{n+1}) \cong \mathcal{M}(\mathcal{E}_n, \mathcal{F}_n).$$

We shall refer to $Sp_{S^1}([Sm/k_+, \mathcal{M}])$ as the category of spectral functors — see [20, Section 5]. The objects are $S^1$-spectra in the closed symmetric monoidal $\mathcal{M}$-category $[Sm/k_+, \mathcal{M}]$ introduced in Example 2.4. Similarly to (13), see [20, Section 5, (3)], there exists an evaluation functor

$$ev \mathcal{G} : Sp_{S^1}([Sm/k_+, \mathcal{M}]) \rightarrow Sp_{S^1}(k).$$

We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** For $\mathcal{X} \in H_{\mathcal{G}}^f(k)$ and $n \geq 0$, see Definition 1.4, the geometric realization functor furnishes the associated $\mathcal{M}$-enriched functor

$$\mathcal{X}(\mathcal{G}^\wedge n) : = |l \rightarrow \mathcal{X} (\mathcal{G}^\wedge n)| \in [\Gamma_{\mathcal{G}}^\mathcal{M}, \mathcal{M}].$$

Owing to Example 2.3 this is a pointed functor from $\Gamma_{\mathcal{G}}^\mathcal{M}$ to $\mathcal{M}$. Applying the functor $ev_{\mathcal{G}}$ in (9) yields the motivic $S^1$-spectrum $ev_{\mathcal{G}}(\mathcal{X}(\mathcal{G}^\wedge n)) = \mathcal{X}(\mathcal{G}^\wedge n)$. By [20, Lemma 2.5] $\mathcal{X}(\mathcal{G}^\wedge n)$ is $\mathcal{A}^1$-local. Moreover, $\mathcal{X}(\mathcal{S}, \mathcal{G}^\wedge n)$ is sectionwise connective because on every section it is the $S^1$-spectrum associated to a $\Gamma$-space. It follows that $\mathcal{X}(\mathcal{S}, \mathcal{G}^\wedge n)$ is a connective motivic $S^1$-spectrum for every $n \geq 0$. For the evaluation functor $ev_{\mathcal{S}, \mathcal{G}}$ in (14) we have

$$ev_{\mathcal{S}, \mathcal{G}}(\mathcal{X})(n) = \mathcal{X}(\mathcal{S}, \mathcal{G}^\wedge n).$$

Combined with Lemma 3.1 we conclude that $ev_{\mathcal{S}, \mathcal{G}}(\mathcal{X}) \in SH(k)_{\geq 0}$. Therefore we obtain the induced evaluation functor

$$ev_{\mathcal{S}, \mathcal{G}} : H_{\mathcal{G}}^f(k) \rightarrow SH(k)_{\geq 0}.$$
By the construction of $H_{Fr}^G(k)$ — see Definition 1.4 — the functor $ev_{S^1}^G$ in (18) is fully faithful. It remains to show essential surjectivity — this is the most interesting part of the proof.

Suppose $\mathcal{E}$ is a cofibrant and fibrant symmetric motivic $(S^1, G)$-bispectrum. Then there exists a framed spectral functor $\mathcal{M}^G_{Fr}$ in the sense of [20, Definition 6.1] such that $ev_{G}(\mathcal{M}^G_{Fr})$ is naturally isomorphic to $\mathcal{E}$ in $SH(k)$ — see [20, Section 6]; in fact, $\mathcal{M}^G_{Fr}$ enables the equivalence between $SH(k)$ and framed spectral functors in [20, Theorem 6.3, Definition 6.5].

We briefly recall the construction of $\mathcal{M}^G_{Fr}$ since it is important for the details of this proof. The motivic spaces $C_i Fr(\mathcal{E}_{i,j})$ conspire into a motivic $(S^1, G)$-bispectrum $C_i Fr(\mathcal{E})$. For $n \geq 0$ we let $R^n_G C_i Fr(\mathcal{E})$ denote $\text{Hom}(G^\wedge n, C_i Fr(\mathcal{E}[n]))$, where $\mathcal{E}[n]$ is the $n$th shift of $\mathcal{E}$ in the $G$-direction. In weight $i \geq 0$ we have the motivic $S^1$-spectrum

$$R^n_G C_i Fr(\mathcal{E})(i) = \text{Hom}(G^\wedge n, C_i Fr(\mathcal{E}(n+i))).$$

There is a canonical morphism of motivic $(S^1, G)$-bispectra

$$R^n_G C_i Fr(\mathcal{E}) \to R^{n+1}_G C_i Fr(\mathcal{E}),$$

and we set

$$R^n_G C_i Fr(\mathcal{E}) := \text{colim}(C_i Fr(\mathcal{E}) \to R^1_G C_i Fr(\mathcal{E}) \to R^2_G C_i Fr(\mathcal{E}) \to \cdots).$$

Owing to [20, Claim 2, Section 6] there are stable motivic equivalences

$$\mathcal{E} \to C_i Fr(\mathcal{E}) \to R^n_G C_i Fr(\mathcal{E}).$$

For $n \geq 0$ we define the spectral functor $G^\wedge C_i Fr(\mathcal{E})[n]$ sectionwise by

$$U \mapsto \text{Hom}(G^\wedge n, C_i Fr(\mathcal{E}(n+U))).$$

By construction there is a natural morphism of spectral functors

$$G^\wedge C_i Fr(\mathcal{E})[n] \to G^\wedge C_i Fr(\mathcal{E})[n+1],$$

and we set

$$\mathcal{M}^G_{Fr} := \text{colim}(G^\wedge C_i Fr(\mathcal{E})[0] \to G^\wedge C_i Fr(\mathcal{E})[1] \to \cdots).$$

By [20, Lemma 6.6] there is a morphism of motivic $(S^1, G)$-bispectra

$$ev_{G}(\mathcal{M}^G_{Fr}) \to R^n_G C_i Fr(\mathcal{E}).$$

In every weight, (19) is a stable local equivalence of motivic $S^1$-spectra due to [20, Lemma 6.7]. This implies the zigzag of stable motivic equivalences

$$\mathcal{E} \to R^n_G C_i Fr(\mathcal{E}) \leftrightarrow ev_{G}(\mathcal{M}^G_{Fr}),$$

and therefore an isomorphism in $SH(k)$

$$ev_{G}(\mathcal{M}^G_{Fr}) \cong \mathcal{E}. \quad (20)$$
For $U \in \text{Sm}/k_+$ the motivic $S^1$-spectrum $\mathcal{M}^\infty_{fr}(U)$ is not necessarily a sectionwise $\Omega$-spectrum. However, the said property holds for the framed spectral functor $\mathcal{M}^\infty_{fr}$ with sections

$$U \mapsto \Theta^\infty_{S^1}(\mathcal{M}^\infty_{fr}(U)).$$

Here $\Theta^\infty_{S^1}$ is the motivic $S^1$-stabilization functor defined in [22, Definition 4.2]. By construction there is a canonical morphism

$$\mathcal{M}^\infty_{fr}(U) \rightarrow \mathcal{M}^\infty_{fr}(U).$$

We note that (21) is a sectionwise stable equivalence of motivic $S^1$-spectra.

Next we use (17) to define the motivic $\Gamma$-space $\Gamma \mathcal{M}^\infty_{fr}$ by setting

$$\Gamma \mathcal{M}^\infty_{fr}(n_+, U) := \mathcal{M}(S^{\times n}, \mathcal{M}^\infty_{fr}(U)), \quad n \geq 0, U \in \text{Sm}/k.$$

Here the $S^1$-spectrum $S^{\times n} := S \times \cdots \times S$ is regarded as a constant motivic $S^1$-spectrum. For all $U, V \in \text{Sm}/k_+$ and the adjunction $(\text{ev}_{S^1}, \Phi)$ between $\Gamma$-spaces and spectra in [7, Section 5] we have

$$\Gamma \mathcal{M}^\infty_{fr}(n_+, U)(V) = \Phi(\mathcal{M}^\infty_{fr}(U)(V))(n_+) = S_*(S^{\times n}, \mathcal{M}^\infty_{fr}(U)(V)).$$

This expression determines the values of $\Phi$ at the $S^1$-spectrum $\mathcal{M}^\infty_{fr}(U)(V)$. Moreover, the counit $\text{ev}_{S^1} \circ \Phi \rightarrow \text{id}$ induces a morphism of spectral functors

$$\text{ev}_{S^1}(\Gamma \mathcal{M}^\infty_{fr}) \rightarrow \mathcal{M}^\infty_{fr}.$$  \hfill (23)

By construction, $\Gamma \mathcal{M}^\infty_{fr}$ is a framed motivic $\Gamma$-space in the sense of Definition 2.5. Moreover, in weight $n \geq 0$, (21) induces a sectionwise stable equivalence of motivic $S^1$-spectra

$$\text{ev}_G(\mathcal{M}^\infty_{fr})(n) \rightarrow \text{ev}_G(\mathcal{M}^\infty_{fr})(n).$$

In combination with (20) we deduce an isomorphism in $\text{SH}(k)$

$$\text{ev}_G(\mathcal{M}^\infty_{fr}) \cong \mathcal{E}.$$  \hfill (24)

We will show that $\Gamma \mathcal{M}^\infty_{fr}$ satisfies (1)-(4) in Axioms 1.1 and also (5) provided $\mathcal{E} \in \text{SH}(k)_{\geq 0}$.

Clearly we have $\Gamma \mathcal{M}^\infty_{fr}(0_+, U) = * = \Gamma \mathcal{M}^\infty_{fr}(n_+, \emptyset)$ for all $U \in \text{Sm}/k_+$ and $n \geq 0$. Moreover, the canonical sectionwise stable equivalence of cofibrant motivic $S^1$-spectra

$$S \vee \cdots \vee S \rightarrow S \times \cdots \times S$$

induces — via (17) and (22) — the sectionwise equivalence of motivic spaces

$$\Gamma \mathcal{M}^\infty_{fr}(n_+, U) = \mathcal{M}(S^{\times n}, \mathcal{M}^\infty_{fr}(U)) \rightarrow \mathcal{M}(S^{\times n}, \mathcal{M}^\infty_{fr}(U)) \cong \Gamma \mathcal{M}^\infty_{fr}(1_+, U)^{\times n}.$$  

This establishes Axiom (1).

For $U \in \text{Sm}/k_+$ the presheaf of stable homotopy groups $\pi_* \text{ev}_{S^1}(\Gamma \mathcal{M}^\infty_{fr}(U))$ is isomorphic to $\pi_* \mathcal{M}^\infty_{fr}(U)$ if $n \geq 0$ and trivial if $n < 0$ — this follows as in [7, Theorem 5.1]. By (21) there is an isomorphism of presheaves between $\pi_* \mathcal{M}^\infty_{fr}(U)$ and $\pi_* (\mathcal{M}^\infty_{fr}(U))$. Since the former is framed in addition to being $\mathbb{A}^1$- and $\sigma$-invariant, the same holds for $\pi_* \text{ev}_{S^1}(\Gamma \mathcal{M}^\infty_{fr}(U))$. This shows that Axiom (2) holds.
Axioms (3) and (4) hold because $M_{fr}^\phi$ is a framed spectral functor and the presheaves of stable homotopy groups $\pi_n ev_S(\Gamma M_{fr}^\phi(U))$ of the connective $A^1$-local motivic $S^1$-spectrum $ev_S(\Gamma M_{fr}^\phi(U))$ are isomorphic to $\pi_n(M_{fr}^\phi(U))$ for all $n \geq 0$ and $U \in Sm/k_+$.

Axiom (5) holds if we assume $E \in SH(k)_{\geq 0}$. Indeed, the proof of [20, Theorem 6.3] shows $E \wedge U_+ \in SH(k)_{\geq 0}$ is isomorphic to $ev_S(M_{fr}^\phi(\cdot \times U))$ for all $U \in Sm/k_+. Here M_{fr}^\phi(\cdot \times U)$ is the framed spectral functor with sections

$$X \mapsto M_{fr}^\phi(X \times U).$$

By Lemma 3.1 the $A^1$-local motivic $S^1$-spectrum $M_{fr}^\phi(U)$ is connective. Indeed, $M_{fr}^\phi(U)$ is the zeroth weight of the framed bispectrum $ev_S(M_{fr}^\phi(\cdot \times U))$ whose weights are $A^1$-local by [20, Lemma 2.6]. Thus for all $U \in Sm/k_+$ the morphism (23) yields a stable local equivalence of connective motivic $S^1$-spectra

$$\Gamma M_{fr}(S, U) \rightarrow M_{fr}^\phi(U).$$

We conclude $\Gamma M_{fr}(S, \cdot)$ is a framed spectral functor and the framed motivic $\Gamma$-space $\Gamma M_{fr}^\phi$ satisfies Nisnevich excision as in Axiom (5). This completes the proof.

3.2. Remark. The proof of Theorem 1.5 shows that a quasi-inverse functor $\Gamma M_{fr}$ to the equivalence $ev_{S^1,G} : H_{fr}^{G}(\cdot) \rightarrow SH(k)_{\geq 0}$ is given as follows: For $E \in SH(k)_{\geq 0}$ take a functorial cofibrant and fibrant replacement $E'$ in the stable model structure on symmetric motivic $(S^1,G)$-bispectra. Then map $E$ to the framed motivic $\Gamma$-space $\Gamma M_{fr}^\phi$.

With Theorem 1.5 in hand we can prove Theorem 1.7.

Proof of Theorem 1.7. Following [3, Section 3, p. 1131], [30, Section 5] we have

$$SH^{eff}(k) = SH(k)_{\geq 0} \cap SH^{eff}(k),$$

where $SH^{eff}(k)$ is the full subcategory of $SH(k)$ comprised of effective bispectra. For $X \in H_{fr}^{G}(k)$ the evaluation $X^{S^1,G}$ is contained in $SH(k)_{\geq 0}$ due to Theorem 1.5. By Axiom (6) the $S^1$-spectrum

$$|X^{S^1,G}(\cdot \times U)(\Delta^*_K/k)|$$

is stably contractible for any finitely generated field extension $K/k$ and $U \in Sm/k$. It follows that

$$|X^{S^1,G}(\cdot \times U)(\Delta^*_K/k)|$$

is stably contractible for every $n > 0$. We deduce that $X^{S^1,G} \in SH^{eff}(k)$ and thus $X^{S^1,G} \in SH^{eff}(k)$ by reference to [4, Theorem 4.4] and [20, Definition 3.5, Theorem 3.6].

We have shown the restriction of the equivalence $ev_{S^1,G} : H_{fr}^{G}(\cdot) \rightarrow SH(k)_{\geq 0}$ in Theorem 1.5 to the full subcategory $H_{fr}^{eff}(k)$ takes values in $SH^{eff}(k)$. It remains to show that it is essentially surjective.

Suppose $E$ is a very effective cofibrant and fibrant symmetric motivic $(S^1,G)$-bispectrum. By Theorem 1.5 there exists a framed motivic $\Gamma$-space $\Gamma M_{fr}^\phi$ and an isomorphism between $ev_{S^1,G}(\Gamma M_{fr}^\phi)$
and $\mathcal{E}$ in $\text{SH}(k)_{\geq 0}$. Moreover, the proof of Theorem 1.5 shows that for every $U \in \text{Sm}/k_{+}$ there is an isomorphism in $\text{SH}(k)_{\geq 0}$ between $\mathcal{E} \wedge U_{+}$ and $\text{ev}_{S^{1},G}^{\text{fr}}(\Gamma M_{\text{fr}}^{\mathcal{E}}(- \times U))$. Here $\Gamma M_{\text{fr}}^{\mathcal{E}}(- \times U)$ is the framed motivic $\Gamma$-space with sections

$$(n_{+},X) \mapsto \Gamma M_{\text{fr}}^{\mathcal{E}}(n_{+},X \times U).$$

Recall that $\text{SH}^{\text{fr}}(k)$ is closed under the smash product in $\text{SH}(k)$ by [30, Lemma 5.6]. In particular we have $\mathcal{E} \wedge U_{+} \in \text{SH}^{\text{fr}}(k)$. To conclude the $S^{1}$-spectrum

$$\Gamma M_{\text{fr}}^{\mathcal{E}}(\Delta_{k/\mathbb{K}}^{\mathcal{E}})$$

is stably contractible we appeal to [20, Theorem 3.6]. It follows that the framed motivic $\Gamma$-space $\Gamma M_{\text{fr}}^{\mathcal{E}}$ is effective, and hence $\mathcal{E}$ is isomorphic to $\text{ev}_{S^{1},G}^{\text{fr}}(\Gamma M_{\text{fr}}^{\mathcal{E}})$ in $\text{SH}^{\text{fr}}(k)$. $\square$

Suppose $\mathcal{E}$ is a motivic $(S^{1},G)$-bispectrum with a motivic fibrant replacement $\mathcal{E}^{f}$. We will write $\Omega_{S^{1}}^{\infty} \Omega_{G}^{\infty} \mathcal{E}$ for the pointed motivic space $\mathcal{E}_{0,0}^{f}$.

**3.3. Definition.** A pointed motivic space $A$ is an infinite motivic loop space if there exists a motivic $(S^{1},G)$-bispectrum $\mathcal{E}$ and a local equivalence $A \simeq \Omega_{S^{1}}^{\infty} \Omega_{G}^{\infty} \mathcal{E}$.

**3.4. Lemma.** Suppose $\mathcal{X}^{f}$ is a very special framed motivic $\Gamma$-space. Then the bispectrum $\mathcal{X}_{S^{1},G}^{f}$ obtained from $\mathcal{X}_{S^{1},G}^{f}$ by taking levelwise local fibrant replacements is motivically fibrant. $\square$

**Proof.** This follows from [20, Lemma 2.6] since the $S^{1}$-spectrum associated with a very special $\Gamma$-space is an $\Omega$-spectrum after taking levelwise fibrant replacements [10, Corollary 2.2.1.7]. $\square$

The above brings us to the proof of Theorem 1.8.

**Proof of Theorem 1.8.** Without loss of generality we may assume $\mathcal{E} \in \text{SH}(k)_{\geq 0}$. Indeed, it follows from [2, p. 374] that for any $\mathcal{E}$ the connective cover $\tau_{\geq 0} \mathcal{E} \rightarrow \mathcal{E}$ yields a sectionwise equivalence

$$\Omega_{S^{1}}^{\infty} \Omega_{G}^{\infty} (\tau_{\geq 0} \mathcal{E}) \rightarrow \Omega_{S^{1}}^{\infty} \Omega_{G}^{\infty} (\mathcal{E}).$$

Now every $\mathcal{E} \in \text{SH}(k)_{\geq 0}$ is isomorphic to $\text{ev}_{S^{1},G}^{\text{fr}}(\Gamma M_{\text{fr}}^{\mathcal{E}})$ for some special framed motivic $\Gamma$-space $\Gamma M_{\text{fr}}^{\mathcal{E}}$ — see the proof of Theorem 1.5. For $n \geq 0$ and $U, V \in \text{Sm}/k_{+}$, item (21) yields

$$\Gamma M_{\text{fr}}^{\mathcal{E}}(n_{+},U)(V) = \Phi(M_{\text{fr}}^{\mathcal{E}}(U)(V))(n_{+}) = S_{*}(S^{n},M_{\text{fr}}^{\mathcal{E}}(U)(V)).$$

Here $M_{\text{fr}}^{\mathcal{E}}(U)(V)$ is the $\Omega$-spectrum $\Theta_{S^{1}}^{\mathcal{E}} \cdot M_{\mathcal{E}}^{\mathcal{E}}(U)(V)$ introduced in the proof of Theorem 1.5. It follows that $\Gamma M_{\text{fr}}^{\mathcal{E}}(1_{+},U)(V)$ is the zero space $M_{\mathcal{E}}^{\mathcal{E}}(U)(V)_{0}$ of the $\Omega$-spectrum $M_{\mathcal{E}}^{\mathcal{E}}(U)(V)$. Thus $\pi_{0}M_{\mathcal{E}}^{\mathcal{E}}(U)(V)_{0}$ is an abelian group, and $\pi_{0}^{\text{Nis}} M_{\text{fr}}^{\mathcal{E}}(1_{+},U)$ is a Nisnevich sheaf of abelian groups. This shows that $\Gamma M_{\text{fr}}^{\mathcal{E}}$ is a very special framed motivic $\Gamma$-space — see Axiom (7).

By appeal to Lemma 3.4 the bispectrum $\text{ev}_{S^{1},G}^{\text{fr}}(\Gamma M_{\text{fr}}^{\mathcal{E}})^{f}$ obtained by taking levelwise local fibrant replacements is motivically fibrant. Hence there exists a sectionwise equivalence of pointed motivic spaces

$$\Gamma M_{\text{fr}}^{\mathcal{E}}(1_{+},pt)^{f} \simeq \text{ev}_{S^{1},G}^{\text{fr}}(\Gamma M_{\text{fr}}^{\mathcal{E}})_{0,0}^{f} \simeq \Omega_{S^{1}}^{\infty} \Omega_{G}^{\infty} \mathcal{E}.$$
Now suppose $\mathcal{X}$ is a very special framed motivic $\Gamma$-space. By Lemma 3.4, $\mathcal{X}^f_{S^1,G}$ is motivically fibrant and we deduce

$$\mathcal{X}^f_{(1,+),\text{pt}} = ev_{S^1,G}(\mathcal{X})^f_{0,0} \simeq \Omega^\infty_{S^1,G} \mathcal{X}^f_{S^1,G}.$$  

Since $\mathcal{X}^f_{(1,+),\text{pt}}$ is locally equivalent to $\mathcal{X}^f_{(1,+),\text{pt}}$, then it follows that $\mathcal{X}^f_{(1,+),\text{pt}}$ is an infinite motivic loop space in the sense of Definition 3.3. □

3.5. Remark. Every special framed motivic $\Gamma$-space $\mathcal{X} : \Gamma^{op} \boxtimes \text{Sm}/k_+ \to \mathcal{M}$ has a canonically associated very special framed motivic $\Gamma$-space with sections $$(n_+,U) \mapsto \Omega^\infty_{S^1} \text{Ex}^\infty \mathcal{X}^f(S^1 \wedge n_+,U).$$

In this expression, Kan’s fibrant replacement functor $\text{Ex}^\infty$ is applied sectionwise in $S^\bullet$. 

We finish this section by discussing the diagram (4) of adjoint functors from the introduction:

$$
\begin{array}{ccc}
\mathbf{H}(k) & \xrightarrow{\Sigma^\infty_{S^1,G}} & \mathbf{SH}(k)_{\geq 0} \\
\downarrow & & \downarrow \\
\mathbf{H}^\Gamma_{\text{Fr}}(k) & \leftarrow & \Gamma_{\text{Fr}}
\end{array}
$$

The functor $u : \mathbf{H}^\Gamma_{\text{Fr}}(k) \to \mathbf{H}(k)$ sends a framed motivic $\Gamma$-space $\mathcal{X}$ to its underlying motivic space $\mathcal{X}^f_{(1,+),\text{pt}}$. Moreover, $C_*\text{Fr}$ sends a motivic space $A$ to $C_*(A^c \otimes -)$ — the projective cofibrant replacement $A^c$ of $A$ is a filtered colimit of simplicial smooth schemes from $\Delta^{op}\text{Sm}/k_+$. According to [19, Section 11] $C_*\text{Fr}$ is a functor from $\mathbf{H}(k)$ to $\mathbf{H}^\Gamma_{\text{Fr}}(k)$.

The composite functor $ev_{S^1,G} \circ C_*\text{Fr}$ is equivalent to $\Sigma^\infty_{S^1,G}$ due to [19, Section 11]. Theorem 1.8 implies that $u \circ \Gamma_{\text{Fr}}$ is equivalent to $\Omega^\infty_{S^1,G}$. Thus the adjoint pair $(\Sigma^\infty_{S^1,G}, \Omega^\infty_{S^1,G})$ is equivalent to $(ev_{S^1,G} \circ C_*\text{Fr}, u \circ \Gamma_{\text{Fr}})$. Since $(ev_{S^1,G}, \Gamma_{\text{Fr}})$ is an adjoint equivalence by Theorem 1.5, it follows that $(C_*\text{Fr}, u)$ is a pair of adjoint functors.

3.6. Corollary. The diagram of adjoint functors (4) commutes up to equivalence of functors.

4. Further properties of motivic $\Gamma$-spaces

Let $\mathcal{X} : \Gamma^{op} \boxtimes \text{Sm}/k_+ \to \mathcal{M}_{\text{Fr}}$ be a framed motivic $\Gamma$-space. One has an enriched functor

$$\mathcal{X}^f_{(1,+),\text{pt}} : \text{Sm}/k_+ \to \mathcal{M}^\text{fr}, \quad U \mapsto \mathcal{X}^f_{(1,+),U}.$$ 

For all $U,V \in \text{Sm}/k_+$ we have the elementary Nisnevich square:

$$
\begin{array}{ccc}
\emptyset & \to & V \\
\downarrow & & \downarrow \\
U & \to & U \sqcup V
\end{array}
$$
If $\mathcal{X}$ is (very) special in the sense of Axioms 1.1, then Axioms (1) and (5) imply the stable local equivalence
$$\mathcal{X}(S, U) \vee \mathcal{X}(S, V) \longrightarrow \mathcal{X}(S, U \sqcup V).$$
(25)

On the other hand, the sectionwise stable equivalence
$$\mathcal{X}(S, U) \vee \mathcal{X}(S, V) \longrightarrow \mathcal{X}(S, U) \times \mathcal{X}(S, V)$$
factors as
$$\mathcal{X}(S, U) \vee \mathcal{X}(S, V) \longrightarrow \mathcal{X}(S, U \sqcup V) \longrightarrow \mathcal{X}(S, U) \times \mathcal{X}(S, V).$$

It follows that the rightmost morphism is a local stable equivalence. This shows the morphism of motivic spaces
$$\mathcal{X}(1_+, U \sqcup V) \longrightarrow \mathcal{X}(1_+, U) \times \mathcal{X}(1_+, V)$$
is a local equivalence, and likewise for
$$\mathcal{X}(1_+, n_+ \otimes U) \longrightarrow \mathcal{X}(1_+, U) \times \cdots \times \mathcal{X}(1_+, U), \quad n \geq 1.$$}%

Here we write $n_+ \otimes U := U \sqcup \cdots \sqcup U \in \text{Sm}/k_+$. Axiom (1) ensures that $\mathcal{X}(1_+, 0_+ \otimes U) = *$ since by definition $0_+ \otimes U := \emptyset$. Moreover, if $\mathcal{X}$ is very special then the Nisnevich sheaf $\pi_0^{\text{nis}} \mathcal{X}(1_+, U)$ takes values in abelian groups due to Axiom (7). We record these observations in the next lemma.

4.1. Lemma. For any very special framed motivic $\Gamma$-space $\mathcal{X}$ and $U \in \text{Sm}/k_+$ the functor
$$n_+ \mapsto \mathcal{X}(1_+, n_+ \otimes U)$$
is locally a very special $\Gamma$-space.

Let us fix a cofibrant replacement functor $A \longrightarrow A^c$ in the projective motivic model structure on $\mathcal{M}$ in the sense of [5, Section 3], [13] — $A^c$ is a sequential colimit of simplicial schemes in $\Delta^{\text{op}} \text{Sm}/k_+$. For a motivic $\Gamma$-space $\mathcal{X}$, we define the functor $\mathcal{X}(1_+, -) : \mathcal{M} \longrightarrow \mathcal{M}$ by setting
$$\mathcal{X}(1_+, A) := \text{colim}_{(\Delta[n] \times U)_+} \mathcal{X}(1_+, \Delta[n]_+ \otimes U), \quad A \in \mathcal{M}.$$%

Here we identify the pointed motivic space $A$ with colim_{(\Delta[n] \times U)_+} A(\Delta[n] \times U)_+.

A key property of $\Gamma$-spaces says that if $f : K \longrightarrow L$ is an equivalence in $S_\bullet$, then so is $F(f) : F(K) \longrightarrow F(L)$ for every $F : \Gamma^{\text{op}} \longrightarrow S_\bullet$ — see [7, Proposition 4.9], [10, Lemma 2.2.1.3]. The following result is a motivic counterpart of this property.

4.2. Theorem. For any very special framed motivic $\Gamma$-space $\mathcal{X}$ the functor
$$\mathcal{X}(1_+, -) : \mathcal{M} \longrightarrow \mathcal{M}, \quad A \mapsto \mathcal{X}(1_+, A^c)$$
takes motivic equivalences to local equivalences of motivic spaces. Hence if $\mathcal{X}$ is a special framed motivic $\Gamma$-space, then the functor
$$\mathcal{X}(S, -) : \mathcal{M} \longrightarrow \text{Sp}_n(\mathbb{S}), \quad A \mapsto \mathcal{X}(S, A^c)$$
takes motivic equivalences to stable local equivalences of motivic $S^1$-spectra.
Our proof of Theorem 4.2 is inspired by Voevodsky’s theory of left derived radditive functors as in [35, Theorem 4.19] — the basic notions we will need in this paper are recalled below. In this context, we note that the category Sm/k_+ has finite coproducts.

Recall that a morphism \( e : A \to X \) in a category \( C \) is a coprojection if it is isomorphic to the canonical morphism \( A \to A \sqcup Y \) for some \( Y \) [35, Section 2]. A morphism \( f : A \to X \) in \( \Delta^{op}C \) is a termwise coprojection if for all \( i \geq 0 \) the morphism \( f_i : A_i \to X_i \) is a coprojection. As observed in [35, Section 2] a morphism \( f : B \to A \) and an object \( X \) conspire into the pushout:

\[
\begin{array}{c}
B \\
\downarrow \\
A
\end{array} \quad \begin{array}{c}
e_B \\
\downarrow \\
e_A
\end{array} \quad \begin{array}{c}
\to B \sqcup X \\
\downarrow \\
A \sqcup X
\end{array}
\]

It follows that there exist pushouts for all pairs of morphisms \( (e, f) \) with \( e \) a coprojection whenever \( C \) is a category with finite coproducts — and likewise for pairs of morphisms \( (e, f) \) in \( \Delta^{op}C \), where \( e \) is a termwise coprojection. Following [35, Section 2] a square in \( \Delta^{op}C \) is called an elementary pushout square if it is isomorphic to the pushout square for a pair of morphisms \( (e, f) \), where \( e \) is a termwise coprojection.

If \( C \) has finite coproducts, then for any commutative square \( Q \) of the form

\[
\begin{array}{c}
B \\
\downarrow \\
A
\end{array} \quad \begin{array}{c}
X \\
\downarrow \\
Y
\end{array}
\]

we define the object \( K_Q \) by the elementary pushout square:

\[
\begin{array}{c}
B \sqcup B \\
\downarrow \\
A \sqcup Y
\end{array} \quad \begin{array}{c}
\to B \otimes \Delta[1] \\
\downarrow \\
K_Q
\end{array}
\]

There is a canonically induced morphism \( p_Q : K_Q \to X \). An important example is the cylinder \( \text{cyl}(f) \) of a morphism \( f : X \to X' \); in terms of the construction above, this is the object associated to the square:

\[
\begin{array}{c}
X \\
\downarrow \\
X
\end{array} \quad \begin{array}{c}
f \\
\downarrow \\
f
\end{array} \quad \begin{array}{c}
X' \\
\downarrow \\
X'
\end{array}
\]

By [35, Lemma 2.9] the natural morphisms \( X' \to \text{cyl}(f) \) and \( \text{cyl}(f) \to X' \) are mutually inverse homotopy equivalences.
4.3. **Lemma.** Suppose $X$ is a special framed motivic $\Gamma$-space. Then $X(S, -)$ takes elementary pushout squares in $\Delta^\op \Sm/k_+$ to homotopy pushout squares in the stable local model structure on motivic $S^1$-spectra.

**Proof.** Consider the pushout square in $\Delta^\op \Sm/k_+$ with horizontal coprojections:

\[
\begin{array}{ccc}
B & \xrightarrow{e_B} & B \sqcup X \\
\downarrow & & \downarrow \\
A & \xrightarrow{e_A} & A \sqcup X
\end{array}
\]

The associated square of spectra

\[
\begin{array}{ccc}
X(S, B) & \xrightarrow{} & X(S, B \sqcup X) \\
\downarrow & & \downarrow \\
X(S, A) & \xrightarrow{} & X(S, A \sqcup X)
\end{array}
\]

is a homotopy pushout because by (25) it is stably locally equivalent to the pushout square:

\[
\begin{array}{ccc}
X(S, B) & \xrightarrow{} & X(S, B) \vee X(S, X) \\
\downarrow & & \downarrow \\
X(S, A) & \xrightarrow{} & X(S, A) \vee X(S, X)
\end{array}
\]

By definition, an elementary pushout square is isomorphic to the pushout square of morphisms $(e, f)$, where $e$ is a termwise coprojection. It remains to observe that the geometric realization of a simplicial homotopy pushout square of spectra is a homotopy pushout. \hfill \square

4.4. **Corollary.** Suppose $X$ is a special framed motivic $\Gamma$-space and

\[
\begin{array}{ccc}
C & \xrightarrow{e} & D \\
\downarrow{f} & & \downarrow \\
C' & \xrightarrow{e'} & D'
\end{array}
\]

is an elementary pushout square in $\Delta^\op \Sm/k_+$ of morphisms $(e, f)$, where $e$ is a termwise coprojection. If $X(S, e)$ is a stable local equivalence of spectra, then so is $X(S, e')$.

**Proof of Theorem 4.2.** Let $Q$ denote an elementary Nisnevich square in $\Sm/k$:

\[
\begin{array}{ccc}
U' & \xrightarrow{} & X' \\
\downarrow & & \downarrow \\
U & \xrightarrow{} & X
\end{array}
\]
By applying the cylinder construction and forming pushouts in \( \mathcal{M} \) we obtain the commutative diagram:

\[
\begin{array}{ccc}
U' + & \xrightarrow{\text{cyl}(U' + \to X')} & X' + \\
\downarrow & & \downarrow \\
U + & \xrightarrow{\text{cyl}(U' + \to X')} & U + \sqcup X + \\
\end{array}
\]

Note that \( U' + \to \text{cyl}(U' + \to X') \) is a termwise coprojection and a projective cofibration between projective cofibrant objects of \( \mathcal{M} \). Thus \( s(Q) := \text{cyl}(U' + \to X') \sqcup U + \) is projective cofibrant [21, Corollary 1.1.11] and \( U + \to s(Q) \) is a termwise coprojection. Likewise, applying the cylinder construction to \( s(Q) \to X + \) and setting \( t(Q) := \text{cyl}(s(Q) \to X +) \) we get a projective cofibration

\[
\text{cyl}(Q) : s(Q) \longrightarrow t(Q).
\]

Here \( \text{cyl}(Q) \) is a termwise coprojection and a local equivalence in \( \mathcal{M} \).

In the following we let \( J_{\text{mot}} = J_{\text{proj}} \cup J_{\text{nis}} \cup J_{\mathbb{A}^1} \) where

\[
J_{\text{proj}} = \{ \Delta[n]_+ \wedge U_+ \to \Delta[n]_+ \wedge U_+ \mid U \in \text{Sm}/k, n > 0, 0 \leq r \leq n \},
\]

\[
J_{\text{nis}} = \{ \Delta[n]_+ \wedge s(Q) \coprod_{\partial \Delta[n]_+ \wedge s(Q)} \partial \Delta[n]_+ \wedge t(Q) \to \Delta[n]_+ \wedge t(Q) \mid Q \text{ is an elementary Nisnevich square} \},
\]

\[
J_{\mathbb{A}^1} = \{ \Delta[n]_+ \wedge U \times \mathbb{A}^1_+ \coprod_{\partial \Delta[n]_+ \wedge U \times \mathbb{A}^1_+} \partial \Delta[n]_+ \wedge \text{cyl}(U \times \mathbb{A}^1_+ \to U_+) \to \Delta[n]_+ \wedge \text{cyl}(U \times \mathbb{A}^1_+ \to U_+) \mid U \in \text{Sm}/k \}.
\]

We note that every map in \( J_{\text{mot}} \) is a termwise coprojection. According to [13, Lemma 2.15] a morphism is a fibration with fibrant codomain in the projective motivic model structure if and only if it has the right lifting property with respect to \( J_{\text{mot}} \).

Arguing as in [7, Proposition 4.9] the functor \( \mathcal{X}(1_+, -) \) maps members of \( J_{\text{proj}} \) to local equivalences. We note that \( \mathcal{X}(1_+, -) \) preserves naive simplicial homotopies — if \( A \) is a pointed motivic space then \( \mathcal{X}(1_+, \Delta[1]_+ \otimes A^c) \) is a cylinder object for \( \mathcal{X}(1_+, A^c) \). Axiom (4) implies there is a canonically induced local equivalence

\[
\mathcal{X}(1_+, U \times \mathbb{A}^1_+) \to \mathcal{X}(1_+, \text{cyl}(U \times \mathbb{A}^1_+ \to U)).
\]

Axiom (5) implies the same holds for \( \mathcal{X}(1_+, \text{cyl}(Q)) \).
To show that $\mathcal{X}(1_+, -)$ maps members of $J_{\text{nis}}$ to local equivalences, let us start with a cofibration of simplicial sets $K \hookrightarrow L$ and the induced commutative diagram:

\[
\begin{array}{ccc}
K \vee s(Q) & \longrightarrow & L \vee s(Q) \\
\downarrow a_0 & & \downarrow a_1 \\
K \vee t(Q) & \longrightarrow & L \vee t(Q)
\end{array}
\]

Applying Lemma 4.1 to $a_0 = K \vee \text{cyl}(Q)$ implies the induced morphism $\mathcal{X}(1_+, a_0)$ is a local equivalence. The same applies to $a_2 = L \vee \text{cyl}(Q)$ and $\mathcal{X}(1_+, a_2)$. Since $\mathcal{X}$ is very special, Corollary 4.4 shows $\mathcal{X}(1_+, a_1)$ is a local equivalence. Thus $\mathcal{X}(1_+, a_3)$ is a local equivalence and our claim for $J_{\text{nis}}$ follows. Likewise, $\mathcal{X}(1_+, -)$ maps members of $J_{\text{mot}}$ to local equivalences.

So far we have established that $\mathcal{X}(1_+, -)$ takes members of $J_{\text{mot}}$ to local equivalences. For every motivic equivalence $f : A \longrightarrow B$ the induced morphism $f^\ast : A^\ast \longrightarrow B^\ast$ is also a motivic equivalence. It remains to show the canonical morphism

\[
\mathcal{X}(1_+, f^\ast) : \mathcal{X}(1_+, A^\ast) \longrightarrow \mathcal{X}(1_+, B^\ast)
\]

is a local equivalence. To that end we apply the small object argument [21, Theorem 2.1.14].

To begin we note that all the morphisms in $J_{\text{mot}}$ have finitely presentable (co)domains. For every pointed motivic space $A \in \mathcal{M}$, let $\alpha : A \longrightarrow \mathcal{L}A$ be the transfinite composition of the $\mathcal{K}_0$-sequence

\[
A = E^0 \xrightarrow{a_0} E^1 \xrightarrow{a_1} E^2 \xrightarrow{a_2} \ldots
\]

constructed as follows: For $n \geq 0$ we let $S_n$ denote the set of all commutative squares

\[
\begin{array}{ccc}
C & \longrightarrow & E^n \\
g \downarrow & & \downarrow \\
D & \longrightarrow & *
\end{array}
\]

where $g \in J_{\text{mot}}$ and form the pushout:

\[
\begin{array}{ccc}
\bigsqcup_{S_n} C & \longrightarrow & E^n \\
\downarrow \sqcup g & & \downarrow a_0 \\
\bigsqcup_{S_n} D & \longrightarrow & E^{n+1}
\end{array}
\]

This construction is plainly functorial in $A$. By definition, $\alpha$ is a trivial motivic cofibration in $\mathcal{M}$ belonging to $J_{\text{mot-cell}}$ [21, Definition 2.1.9].
We claim the horizontal morphisms in the commutative diagram
\[
\begin{array}{ccc}
\mathcal{X}(1_+, A^c) & \longrightarrow & \mathcal{X}(1_+, L(A^c)) \\
\downarrow & & \downarrow \\
\mathcal{X}(1_+, f^c) & \longrightarrow & \mathcal{X}(1_+, L(f^c))
\end{array}
\]
are local equivalences: Corollary 4.4 shows $\mathcal{X}(1_+, -)$ maps the cobase change of a member of $J_{\text{mot}}$ to a local equivalence — here we use the assumption that $\mathcal{X}$ is very special. Local equivalences are closed under filtered colimits and $\mathcal{X}(1_+, -)$ preserves filtered colimits, so the same holds for members of $J_{\text{mot-cell}}$. Since $L(A^c)$ and $L(f^c)$ are cofibrant and fibrant, $L(f^c)$ is a homotopy equivalence. As noted above $\mathcal{X}(1_+, -)$ preserves naive simplicial homotopies and therefore $\mathcal{X}(1_+, L(f^c))$ is a homotopy equivalence. Thus $\mathcal{X}(1_+, f^c)$ is a local equivalence.

Let $MZ$ be the motivic ring spectrum representing integral motivic cohomology in the sense of Suslin-Voevodsky [32]. Up to inversion of the exponential characteristic $e$ of the base field $k$, the highly structured category of $MZ$-modules is equivalent to Voevodsky’s derived category of motives — see [28, Theorem 58] and also [23, Theorem 5.8]. A crucial part of the proof shows that for every $U \in \text{Sm}/k$ the natural assembly morphism
\[
MZ \wedge U_+ \longrightarrow MZ \circ (- \wedge U_+)
\]
is an isomorphism in $\text{SH}(k)[1/e]$. For a $\Gamma$-space $F : \Gamma^{\text{op}} \longrightarrow S_*$ the corresponding statement says that the morphism
\[
ev_{S^1}(F) \wedge K \longrightarrow \ev_{S^1}(F(- \wedge K))
\]
is a stable equivalence for every pointed simplicial set $K \in S_*$ — see [7, Lemma 4.1]. We show a similar property for special framed motivic $\Gamma$-spaces.

4.5. Theorem. Suppose $k$ is an infinite perfect field of exponential characteristic $e$. Let $U \in \text{Sm}/k$ be such that $U_+$ is strongly dualizable in $\text{SH}(k)$, e.g., $U$ is a smooth projective algebraic variety. For every special framed motivic $\Gamma$-space $\mathcal{X}$ the natural morphism of bispectra
\[
ev_{S^1,G}(\mathcal{X}) \wedge U_+ = ev_G(\mathcal{X}(S,-)) \wedge U_+ \longrightarrow ev_G(\mathcal{X}(S,- \otimes U)) = ev_{S^1,G}(\mathcal{X}(- \otimes U))
\]
is a stable motivic equivalence. Moreover, for every pointed motivic space $A \in \mathcal{M}$ the natural morphism of bispectra
\[
ev_{S^1,G}(\mathcal{X}) \wedge A^c \longrightarrow ev_{S^1,G}(\mathcal{X}(- \otimes A^c))
\]
is an isomorphism in $\text{SH}(k)[1/e]$.

Proof. Without loss of generality we may assume that $\mathcal{X}$ is very special — see Remark 3.5. We view $\mathcal{X}(1_+, -)$ as an $\mathcal{M}$-enriched functor from $\text{Sm}/k_+$ to $\mathcal{M}$.

Recall from §2 the $\mathcal{M}$-category of finitely presentable motivic spaces $f.\mathcal{M}$. Via an enriched left Kan extension functor the inclusion of $\mathcal{M}$-categories $\iota : \text{Sm}/k_+ \rightarrow f.\mathcal{M}$ yields the functor
\[
\Upsilon : [\text{Sm}/k_+, \mathcal{M}] \longrightarrow [f.\mathcal{M}, \mathcal{M}].
\]
By expressing $\mathcal{V} \in [\text{Sm}/k_+ \mathcal{M}]$ as a coend

$$\mathcal{V} = \int_{U \in \text{Sm}/k_+} \mathcal{V}(U) \sma [U, -],$$

we obtain

$$\Upsilon(\mathcal{V}) = \int_{U \in \text{Sm}/k_+} \mathcal{V}(U) \sma \iota(U), -].$$

By construction, $\Upsilon(\mathcal{V})(V) = \mathcal{V}(V)$ for all $V \in \Delta^\op \text{Sm}/k_+$. More generally, $\Upsilon(\mathcal{V})(A^c) = \mathcal{V}(A^c)$ for every pointed motivic space $A \in \mathcal{M}$.

Theorem 4.2 implies that $\Upsilon(\mathcal{X}(1_+, -))$ maps motivic weak equivalences of projective cofibrant motivic spaces to local equivalences. Owing to [28, Corollary 56] the $G$-evaluation of the assembly morphism

$$\Upsilon(\mathcal{X}(1_+, - \otimes S)) \sma U_+ \longrightarrow \Upsilon(\mathcal{X}(1_+, - \otimes S \otimes U))$$

is a stable motivic equivalence between motivic $(S^1, G)$-bispectra if $U_+$ is strongly dualizable in $\text{SH}(k)$. Here $\mathcal{X}(1_+, - \otimes S \otimes U)$ is the evaluation at the sphere $S$ of the $\Gamma$-space of Lemma 4.1.

Since $\Upsilon(\mathcal{X}(1_+, V)) = \mathcal{X}(1_+, V)$ for all $V \in \Delta^\op \text{Sm}/k_+$ the same holds for the $G$-evaluation of the morphism

$$\mathcal{X}(1_+, - \otimes S) \sma U_+ \longrightarrow \mathcal{X}(1_+, - \otimes S \otimes U).$$

We denote by $\mathcal{X}(S^n, -), n > 0$, the very special framed motivic $\Gamma$-space with sections

$$(k_+, U) \longmapsto \mathcal{X}(S^n \sma k_+, U).$$

Replacing $\mathcal{X}$ with $\mathcal{X}(S^n, -)$, we deduce the stable motivic equivalence of motivic $(S^1, G)$-bispectra

$$\text{ev}_G(\mathcal{X}(S^n, - \otimes S)) \sma U_+ \longrightarrow \text{ev}_G(\mathcal{X}(S^n, - \otimes S \otimes U)). \quad (28)$$

Combining (28) with [7, Lemma 4.1] we obtain the stable motivic equivalences of motivic $(S^1, S^1, G)$-trispectra

$$\text{ev}_G(\mathcal{X}(S, - \otimes S)) \sma U_+ \longrightarrow \text{ev}_G(\mathcal{X}(S, - \otimes S \otimes U))$$

and

$$\text{ev}_G(\mathcal{X}(S, -)) \sma U_+ \sma S \longrightarrow \text{ev}_G(\mathcal{X}(S, - \otimes U)) \sma S.$$

For the cofibrant replacements of $\text{ev}_G(\mathcal{X}(S, -)) \sma U_+$ and $\text{ev}_G(\mathcal{X}(S, - \otimes U))$ in $\text{Sp}_{S^1, G}(k)$ we find a stable motivic equivalence between cofibrant motivic $(S^1, S^1, G)$-trispectra

$$(\text{ev}_G(\mathcal{X}(S, -)) \sma U_+)^c \sma S \longrightarrow \text{ev}_G(\mathcal{X}(S, - \otimes U))^c \sma S.$$

Since $- \sma S^1$ is a Quillen auto-equivalence on $\text{Sp}_{S^1, G}(k)$ we deduce the stable motivic equivalence

$$(\text{ev}_G(\mathcal{X}(S, -)) \sma U_+)^c \longrightarrow \text{ev}_G(\mathcal{X}(S, - \otimes U))^c$$

between cofibrant motivic $(S^1, G)$-bispectra — see also [22, Theorem 5.1]. Therefore (26) is a stable motivic equivalence.
Recall that \( U_+ \) is strongly dualisable in \( \text{SH}(k)[1/e] \) for every \( U \in \text{Sm}/k \) — see [25, Appendix B]. The previous arguments show that (26) is an \( e^{-1} \)-stable motivic equivalence — note that [28, Corollary 56] concerns the stable motivic model structure on motivic functors, but it readily extends to the \( e^{-1} \)-stable model structure.

Finally, when \( A \in \mathcal{M} \), recall that \( A^e \) is a sequential colimit of simplicial schemes from \( \Delta^0 \text{Sm}/k_+ \). Since the geometric realization functor preserves \( e^{-1} \)-stable motivic equivalences we conclude (27) is an isomorphism in \( \text{SH}(k)[1/e] \).

\[\square\]

REFERENCES

[1] A. Ananyevskiy, G. Garkusha, I. Panin, Cancellation theorem for framed motives of algebraic varieties, Adv. Math. 383 (2021), article 107681.
[2] A. Ananyevskiy, M. Levine, I. Panin, Witt sheaves and the \( \eta \)-inverted sphere spectrum, J. Topology 10(2) (2017), 370-385.
[3] T. Bachmann. The generalized slices of hermitian \( K \)-theory. J. Topology 10(4) (2017), 1124-1144.
[4] T. Bachmann, J. Fasel, On the effectivity of spectra representing motivic cohomology theories, preprint arXiv:1710.00594v3.
[5] B. Blander, Local projective model structures on simplicial presheaves, K-Theory 24(3) (2001), 283-301.
[6] J. M. Boardman, R. M. Vogt, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, 1971.
[7] A. K. Bousfield, E. M. Friedlander, Homotopy theory of \( \Gamma \)-spaces, spectra, and bisimplicial sets, in Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Mathematics, Vol. 658, Springer-Verlag, 1978, pp. 80-130.
[8] B. Calmès, J. Fasel, The category of finite \( MW \)-correspondences, preprint arXiv:1412.2989v2.
[9] B. Day, On closed categories of functors, In Reports of the Midwest Category Seminar, IV, pp. 1-38. Springer, Berlin, 1970.
[10] B. I. Dundas, T. Goodwillie, R. McCarthy, The local structure of algebraic \( K \)-theory. Algebra and Applications, Vol. 18. Springer-Verlag, 2013.
[11] B. I. Dundas, M. Levine, P. A. Østvær, O. Røndigs, V. Voevodsky, Motivic homotopy theory. Lectures from the Summer School held in Nordfjordeid, August 2002. Universitext. Springer-Verlag, 2007.
[12] B. I. Dundas, O. Røndigs, P. A. Østvær, Enriched functors and stable homotopy theory, Doc. Math. 8 (2003), 409-488.
[13] B. I. Dundas, O. Røndigs, P. A. Østvær, Motivic functors, Doc. Math. 8 (2003), 489-525.
[14] E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, M. Yakerson, Motivic infinite loop spaces, Cambridge J. Math. 9(2) (2021), 431-549.
[15] G. Garkusha, Reconstructing rational stable motivic homotopy theory, Compos. Math. 155(7) (2019), 1424-1443.
[16] G. Garkusha, A. Neshitov, Fibrant resolutions for motivic Thom spectra, preprint arXiv:1804.07621.
[17] G. Garkusha, A. Neshitov, I. Panin, Framed motives of relative motivic spheres, Trans. Amer. Math. Soc. 374(7) (2021), 5131-5161.
[18] G. Garkusha, I. Panin, Homotopy invariant presheaves with framed transfers, Cambridge J. Math. 8(1) (2020), 1-94.
[19] G. Garkusha, I. Panin, Framed motives of algebraic varieties (after V. Voevodsky), J. Amer. Math. Soc. 34(1) (2021), 261-313.
[20] G. Garkusha, I. Panin, The triangulated categories of framed bispectra and framed motives, preprint arXiv:1809.08006.
[21] M. Hovey, Model categories, American Mathematical Society, Providence, RI, 1999.
[22] M. Hovey, Spectra and symmetric spectra in general model categories, J. Pure Appl. Algebra 165(1) (2001), 63-127.

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[23] M. Hoyois, S. Kelly, P. A. Østvær, The motivic Steenrod algebra in positive characteristic, J. Eur. Math. Soc. 19, 3813-3849, 2017.
[24] J. F. Jardine, Local homotopy theory. Springer Monographs in Mathematics. Springer, 2015.
[25] M. Levine, Y. Yang, G. Zhao, J. Riou, Algebraic elliptic cohomology theory and flops, I, Math. Ann. 375 (2019), 1823-1855.
[26] F. Morel, V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, Publ. Math. IHES 90 (1999), 45-143.
[27] I. Panin, C. Walter, On the algebraic cobordism spectra $MSL$ and $MSP$, preprint arXiv:1011.0651.
[28] O. Röndigs, P. A. Østvær, Modules over motivic cohomology, Adv. Math. 219 (2008), 689-727.
[29] G. Segal, Categories and cohomology theories, Topology 13 (1974), 293-312.
[30] M. Spitzweck, P. A. Østvær, Motivic twisted $K$-theory, Algebr. Geom. Topol. 12 (2012), 565-599.
[31] A. Suslin, On the Grayson spectral sequence. Tr. Mat. Inst. Steklova 241 (2003), Teor. Chisel, Algebra i Gebr. Geom., 218-253; translation in Proc. Steklov Inst. Math. 2003, no. 2(241), 202-237.
[32] V. Voevodsky, $\mathbb{A}^1$-homotopy theory, Doc. Math., Extra Vol. ICM 1998(1), 417-442.
[33] V. Voevodsky, Triangulated category of motives over a field, in Cycles, Transfers and Motivic Homology Theories, Ann. Math. Studies, Princeton Univ. Press, 2000.
[34] V. Voevodsky, Notes on framed correspondences, www.math.ias.edu/vladimir/publications, unpublished, 2001.
[35] V. Voevodsky, Simplicial radditive functors, J. of K-Theory 5 (2010), 201-244.
[36] M. E. Walker, Motivic cohomology and the $K$-theory of automorphisms, PhD Thesis, University of Illinois at Urbana-Champaign, 1996.

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