A Generalization of the notion of a $P$-space to proximity spaces

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Abstract

In this note, we shall generalize the notion of a $P$-space to proximity spaces and investigate the basic properties of $P$-proximities. We therefore define a $P$-proximity to be a proximity where if $A_n \prec B$ for all $n \in \mathbb{N}$, then $\bigcup_n A_n \prec B$. It turns out that the class of $P$-proximities is equivalent to the class of $\sigma$-algebras. Furthermore, the $P$-proximity coreflection of a proximity space is the $\sigma$-algebra of proximally Baire sets.

1 Introduction

We begin by reviewing basic facts on proximity spaces without proofs. All our preliminary information on proximity spaces can be found in [2]. In this paper, we shall assume all proximity spaces are separated and all topologies are completely regular. If $\delta$ is a relation, then we shall write $\overline{\delta}$ for the negation of the relation $\delta$. In other words, we have $R \overline{\delta} S$ if and only if we do not have $R \delta S$.

A proximity space is a pair $(X, \delta)$ where $X$ is a set and $\delta$ is a relation on $P(X)$ that satisfies the following axioms.

1. $A \delta B$ implies $B \delta A$.
2. $(A \cup B) \delta C$ if and only if $A \delta C$ or $B \delta C$.
3. If $A \delta B$, then $A \neq \emptyset$ and $B \neq \emptyset$.
4. If $A \overline{\delta} B$, then there is a set $E$ such that $A \overline{\delta} E$ and $E \overline{\delta} B$.

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5. If \( A \cap B \neq \emptyset \), then \( A \delta B \).

A proximity space is *separated* if and only if \( \{x\} \delta \{y\} \) implies \( x = y \).

Intuitively, \( A \delta B \) whenever the set \( A \) touches the set \( B \) in some sense. Therefore a proximity space is a set with a notion of whether two sets are infinitely close to each other.

If \( (X, \delta) \) is a proximity space, then let \( \prec \) be the binary relation on the power set \( P(X) \) where \( A \prec B \) if and only if \( A \delta B^c \). The relation \( \prec \) satisfies the following.

1. \( X \prec X \).
2. If \( A \prec B \), then \( A \subseteq B \).
3. If \( A \subseteq B, C \subseteq D, B \prec C \), then \( A \prec D \).
4. If \( A \prec B_i \) for \( 1 \leq i \leq n \), then \( A \prec \bigcap_{i=1}^n B_i \).
5. If \( A \prec B \), then \( B^c \prec A^c \).
6. If \( A \prec B \), then there is some \( C \) with \( A \prec C \prec B \).

If \( \prec \) satisfies 1–6 and if we define \( \delta \) by letting \( A \delta B \) if and only if \( A \prec B^c \), then \((X, \delta)\) is a proximity space.

If \( (X, \delta) \) is a proximity space, then we put a topology on \( X \) by letting \( A = \{x| x \delta A\} \). A set \( U \subseteq X \) is open if and only if \( \{x\} \prec U \) whenever \( x \in U \).

**Example 1.** If \( X \) is a set, then \( \delta \) is the discrete proximity if \( A \delta B \) whenever \( A \cap B = \emptyset \).

**Example 2.** Let \( X \) be a completely regular space. Let \( A \delta B \) if there is a function \( f : X \to [0, 1] \) with \( f(A) \subseteq \{0\}, f(B) \subseteq \{1\} \). Then \((X, \delta)\) is a proximity space that induces the topology on \( X \). It is well known that a topology \( X \) is induced by some proximity if and only if \( X \) is completely regular.

**Example 3.** Compact spaces have a unique proximity where \( A \delta B \) if and only if \( \overline{A} \cap \overline{B} \neq \emptyset \). Furthermore, if \((X, \delta)\) is a compact proximity space, and \((Y, \rho)\) is a proximity space, then a map \( f : X \to Y \) is continuous if and only if \( f \) is a proximity map.

If \((X, \delta)\) and \((Y, \rho)\) are proximity spaces, then a function \( f : X \to Y \) a *proximity map* if \( f(A) \rho f(B) \) whenever \( A \delta B \). It can easily be shown that \( f \) is a proximity map if and only if \( f^{-1}(C) \delta f^{-1}(D) \) whenever \( C \rho D \). Furthermore, \( f \) is a proximity map if and only if \( f^{-1}(C) \prec f^{-1}(D) \) whenever \( C \prec D \). Every proximity map is continuous.

If \((X, \delta)\) is a proximity space and \( Y \subseteq X \), then define a relation \( \delta_Y \) on \( P(Y) \) by letting \( A \delta_Y B \) if and only if \( A \delta B \). Then \( \delta_Y \) is a proximity on \( Y \) that induces the subspace topology on \( Y \) called the induced proximity.
If \((X, \delta)\) is a proximity space, then \(A \delta B\) if and only if there is a proximity map \(g : X \to [0, 1]\) with \(g(A) \subseteq \{0\}, g(B) \subseteq \{1\}\).

If \((X, \delta)\) is a proximity space, then there is a unique compactification \(C\) of \(X\) where \(A \delta B\) if and only if \((\text{cl}_C A) \cap (\text{cl}_C B) \neq \emptyset\). This compactification is called the *Smirnov compactification* of \(X\) and the proximity \(\delta\) is the proximity induced by the unique proximity on the compact space \(C\). If \(X\) is a completely regular space, then the proximity spaces that induce the topology on \(X\) are in a one-to-one correspondence with the compactifications of \(X\). If \((X, \delta), (Y, \rho)\) are proximity spaces and \(C\) is the Smirnov compactification of \(X\) and \(D\) is the Smirnov compactification of \(Y\), then every proximity map \(f : X \to Y\) has a unique extension to a continuous map from \(C\) to \(D\).

An algebra of sets \((X, \mathcal{M})\) is reduced if and only if whenever \(x, y \in X\) are distinct then there is an \(R \in \mathcal{M}\) with \(x \in R, y \in R^c\). We assume that all algebras of sets are reduced. If \(X\) is a space, then a *zero set* is a set of the form \(f^{-1}(0)\) where \(f : X \to \mathbb{R}\). The union of finitely many zero sets is a zero set, and the intersection of countably many zero sets is a zero set. A *\(P\)-space* is completely regular space where every \(G_\delta\)-set is open. It is well known and it can easily be shown that a completely regular space is a \(P\)-space if and only if every zero set is open. If \(X\) is a completely regular space, then the *\(P\)-space coreflection* \((X)_{\aleph_1}\) is the space with underlying set \(X\) and where the \(G_\delta\)-sets in \(X\) form a basis for the topology on \((X)_{\aleph_1}\).

## 2 \(P\)-proximities

A separated proximity space \((X, \delta)\) is a *\(P\)-proximity space* if whenever \(A_n \subseteq X\) for \(n \in \mathbb{N}\) and \(B \subseteq X\) and \(\bigcup_{n=0}^\infty A_n \delta B\), then \(A_n \delta B\) for some \(n\). In other words, \(X\) is a \(P\)-proximity space if and only if whenever \(A_n \prec B\) for each natural number \(n\), then \(\bigcup_n A_n \prec B\). Equivalently, \(X\) is a \(P\)-proximity if and only if whenever \(A \prec B_n\) for all \(n\), then \(A \prec \bigcap_n B_n\).

A proximity space \((X, \delta)\) is said to be *zero-dimensional* if and only if whenever \(A \delta B\), then there is a \(C \subseteq X\) with \(A \delta C, B \delta C^c, C \delta C^c\). In other words, \((X, \delta)\) is zero-dimensional if and only if whenever \(A \prec B\) there is a \(C\) with \(A \prec C\prec C \prec B\). If \((X, \delta)\) is a proximity space, then let \(\mathcal{M}_\delta = \{R \subseteq X|R \delta R^c\} = \{R \subseteq X|R \prec R\}\). If \((X, \mathcal{M})\) is an algebra of sets, then let \(\delta_\mathcal{M}\) be the relation on \(P(X)\) where \(U \delta_\mathcal{M} V\) if and only if there is some \(R \in \mathcal{M}\) with \(U \subseteq R, V \subseteq R^c\). The constructions \(\delta_\mathcal{M}\) and \(\mathcal{M}_\delta\) form a duality between zero-dimensional proximity spaces and algebras of sets. We shall therefore regard algebras of sets as the same as zero-dimensional
proximities.

**Theorem 4.** 1. If \((X, \delta)\) is a proximity space, then \((X, \mathcal{M}_\delta)\) is an algebra of sets.

2. If \((X, \mathcal{M})\) is an algebra of sets, then \((X, \delta_{\mathcal{M}})\) is a zero-dimensional proximity space.

3. If \(\delta\) is a zero-dimensional proximity, then \(\delta_{\mathcal{M}_\delta} = \delta\).

4. If \((X, \mathcal{M})\) is an algebra of sets, then \(\mathcal{M} = \mathcal{M}_{\delta_{\mathcal{M}}}\).

5. If \((X, \delta)\) is a zero-dimensional proximity space, then \(\mathcal{M}_{\delta}\) is a basis for the topology on \(X\).

**Proof.** See [1].

**Theorem 5.** Let \((X, \mathcal{M}), (Y, \mathcal{N})\) be algebras of sets. Then a mapping \(f : X \to Y\) is a proximal map if and only if \(f^{-1}(U) \in \mathcal{M}\) for each \(U \in \mathcal{N}\).

**Proof.** \(\to\) Assume that \(f\) is a proximal mapping. For each \(R \in \mathcal{N}\) we have \(R < R\), so \(f^{-1}(R) < f^{-1}(R)\), and thus \(f^{-1}(R) \in \mathcal{M}\).

\(\leftarrow\) Let \(U, V \subseteq Y\), and let \(U < V\). Then there is an \(R \in \mathcal{N}\) with \(U \subseteq R \subseteq V\). Therefore \(f^{-1}(R) \in \mathcal{M}\), and \(f^{-1}(U) \subseteq f^{-1}(R) \subseteq f^{-1}(V)\). Therefore \(f^{-1}(U) < f^{-1}(V)\), so \(f\) is a proximity map.

**Theorem 6.** Every \(P\)-proximity space is zero-dimensional.

**Proof.** Let \(X\) be a \(P\)-proximity space. Assume \(A < B\). Then there is a sequence \((C_n)_{n \in \mathbb{N}}\) of subsets of \(X\) where \(A = C_0 < C_1 < ... < C_n < ... < B\). Therefore let \(C = \bigcup_{n \in \mathbb{N}} C_n\). Since \(X\) is a \(P\)-proximity space, we have \(A < C < B\). Furthermore, since \(C_n < C\) for all \(n\), and \(X\) is a \(P\)-proximity space, we have \(C = \bigcup_n C_n < C\). Therefore \(X\) is a zero-dimensional proximity space.

**Theorem 7.** An algebra of sets \((X, \mathcal{M})\) is a \(\sigma\)-algebra if and only if \(\delta_{\mathcal{M}}\) is a \(P\)-proximity space.

**Proof.** \(\to\) Assume \((X, \mathcal{M})\) is a \(\sigma\)-algebra. Assume that \(A_n < B\) for all \(n\). Then for each \(n\) there is an \(C_n \in \mathcal{M}\) with \(A_n \subseteq C_n \subseteq B\). Therefore \(\bigcup_n A_n \subseteq \bigcup_n C_n \subseteq B\), so \(\bigcup_n A_n < B\).

\(\leftarrow\) Assume that \(\delta_{\mathcal{M}}\) is a \(P\)-proximity space, and let \(R_n \in \mathcal{M}\) for each natural number \(n\). Then \(R_n < R_n \subseteq \bigcup_n R_n\), so \(\bigcup_n R_n < \bigcup_n R_n\), so \(\bigcup_n R_n \in \mathcal{M}\). Therefore \((X, \mathcal{M})\) is a \(\sigma\)-algebra.
In view of the above theorem, the class of σ-algebras is equivalent to the class of P-proximities.

Now given a proximity space \((X, \delta)\), we shall characterize the smallest σ-algebra \((X, \mathcal{M})\) where the identity function from \((X, \mathcal{M})\) to \((X, \delta)\) is a proximity map, but we must first generalize the notion of a zero set to proximity spaces. Let \((X, \delta)\) be a proximity space. Then a \textit{proximally zero set} is a set of the form \(f^{-1}(0)\) where \(f : X \to [0, 1]\) is proximally continuous. If \(C\) is the Smirnov compactification of \(X\), then \(f\) has a unique extension to a continuous function \(\hat{f} : C \to [0, 1]\). Hence \(f^{-1}(0) = \hat{f}^{-1}(0) \cap X\). Therefore the proximally zero sets on a proximity space are precisely the sets of the form \(X \cap Z\) where \(Z \subseteq C\) is a zero set. As a consequence, the intersection of countably many proximally zero sets is a proximally zero set, and the union of finitely many proximally zero sets is a proximally zero set.

The σ-algebra of \textit{proximally Baire sets} on a proximity space \((X, \delta)\) is the smallest σ-algebra containing the proximally zero sets. If \((X, \delta)\) is a proximity space with Smirnov compactification \(C\), and \(\mathcal{M}\) is the Baire σ-algebra on \(C\), then \(\{R \cap X | R \in \mathcal{M}\}\) is the σ-algebra of proximally Baire sets on \(X\).

**Remark.** Every proximally zero set is a zero set, but in general there are zero sets that are not proximally zero sets. For example, let \(A\) be an uncountable discrete space, and let \(\delta\) be the proximity induced by the one point compactification \(A \cup \{\infty\}\). It is well known that for normal spaces the closed \(G_\delta\)-sets are precisely the zero sets, so it suffices to characterize the closed \(G_\delta\)-subsets of \(A \cup \{\infty\}\). Let \(R \subseteq A \cup \{\infty\}\) be a closed \(G_\delta\)-set. If \(\infty \not\in R\), then \(R\) is finite. If \(\infty \in R\), then \(R = \bigcap_n U_n\) for some sequence of open sets \(U_n \subseteq A \cup \{\infty\}\), but each \(U_n\) is co-finite, so \(R\) is co-countable.

Therefore each zero set in \(A \cup \{\infty\}\) is either co-finite or co-countable. Therefore every proximally zero set in \(A\) is either co-finite or co-countable. We conclude that not every zero set in \(A\) a proximally zero set in \(A\).

**Theorem 8.** Let \((X, \delta)\) be a proximity space. Then a set \(Z \subseteq X\) is a proximally zero set if and only if there is a sequence \((Z_n)_{n \in \mathbb{N}}\) with \(Z = \bigcap_n Z_n\) and where \(\ldots Z_n < Z_{n-1} < \ldots < Z_1\).

**Proof.** → If \(Z \subseteq X\) is a proximally zero set, then there is a proximity map \(f : X \to [0, 1]\) with \(Z = f^{-1}(0)\). For all \(n \geq 1\), we have \([0, \frac{1}{n}] < [0, \frac{1}{n+1}]\), so \(f^{-1}([0, \frac{1}{n+1}]) < f^{-1}([0, \frac{1}{n}])\), and \(Z = f^{-1}(0) = \bigcap_n f^{-1}([0, \frac{1}{n}])\).

← Suppose that \((Z_n)_{n \in \mathbb{N}}\) is such a sequence. Then for all \(n\) we have \(Z_{n+1} \subseteq Z_n\), so there is a continuous function \(f_n : X \to [0, 1]\) with \(Z_{n+1} \subseteq \)
Corollary 9. A proximity space \((X, \delta)\) is a \(P\)-proximity space if and only if \(\mathcal{M}_\delta\) contains each proximally zero set.

Proof. \(\Rightarrow\) Let \((X, \delta)\) be a \(P\)-proximity space. If \(Z\) is a zero set, then there is a sequence \((Z_n)_{n \in \mathbb{N}}\) with \(Z = \bigcap_{n \in \mathbb{N}} Z_n\) and where \(Z_{n+1} \prec Z_n\) for all \(n\). Therefore \(Z \subset Z_n\) for all \(n\), so \(Z \subset \bigcap_{n \in \mathbb{N}} Z_m = Z\), so \(Z \in \mathcal{M}_\delta\).

\(\Leftarrow\) Assume \(\mathcal{M}_\delta\) contains each proximally zero set. Assume \(A \prec B_n\) for all \(n \geq 0\). For \(n \geq 0\) there is a proximally zero set \(Z_n\) with \(A \subseteq Z_n \subseteq B_n\), so \(\bigcap_n Z_n\) is a proximally zero set, so \(A \subseteq \bigcap_n Z_n \prec \bigcap_n Z_n \subseteq \bigcap_n B_n\).

Let \(X\) be a completely regular space with a compactification \(\mathcal{C}\). Then \(X\) is locally compact if and only if \(X\) is open in \(\mathcal{C}\), so \(X\) is locally compact if and only if \(\mathcal{C} \setminus X\) is closed in \(\mathcal{C}\). Furthermore, \(X\) is \(\sigma\)-compact if and only if \(\mathcal{C} \setminus X\) is a \(G_\delta\)-set in \(\mathcal{C}\). Therefore \(X\) is locally compact and \(\sigma\)-compact if and only if \(\mathcal{C} \setminus X\) is a closed \(G_\delta\)-set in \(\mathcal{C}\). Since \(\mathcal{C}\) is normal, the closed \(G_\delta\)-sets in \(\mathcal{C}\) are precisely the zero sets. Therefore \(X\) is locally compact and \(\sigma\)-compact if and only if \(\mathcal{C} \setminus X\) is a zero set in \(\mathcal{C}\).

Theorem 10. Let \((X, \delta)\) be a proximity space, and assume that \(X\) is locally compact and \(\sigma\)-compact. Then \(Z \subseteq X\) is a proximally zero set if and only if \(Z\) is a zero set.

Proof. Clearly every proximally zero set is a zero set. It suffices to show that every zero set is a proximally zero set. If \((X, \delta)\) is a proximity space, then assume that the proximity \(\delta\) is inherited from a compactification \(\mathcal{C}\) of \(X\). If \(Z\) is a zero set then there is a function \(f : X \to [0, 1]\) with \(Z = f^{-1}(0)\). Furthermore, since \(X\) is locally compact and \(\sigma\)-compact, there is a function \(g : \mathcal{C} \to [0, 1]\) where \(g(x) > 0\) for \(x \in X\) and \(g(c) = 0\) for \(c \in \mathcal{C} \setminus X\). Therefore let \(h : \mathcal{C} \to [0, 1]\) be the mapping where \(h(x) = f(x)g(x)\) whenever \(x \in X\), and \(h(c) = 0\) for each \(c \in \mathcal{C} \setminus X\). It is easy to see that \(h\) is continuous. Therefore if \(\hat{h}\) is the restriction of \(h\) to \(X\), then \(\hat{h}\) is a proximal map with \(\hat{h}^{-1}(0) = f^{-1}(0) = Z\). Therefore \(Z\) is a proximally zero set.

In particular, for a locally compact \(\sigma\)-compact proximity space, the proximally Baire sets coincide with the Baire sets.

Remark. We may have \(O\delta E\) even when \(O\) and \(E\) are disjoint proximally zero sets. Furthermore, it is possible that \(C\delta C^c\) even though \(C\) and \(C^c\)
are proximally zero sets. Give \( Z \) the proximity induced by the one-point compactification. Let \( C \) be the collection of even integers. Then \( C^c \) is the collection of all odd integers. Since \( C \) and \( C^c \) are both closed sets we have \( C, C^c \) be disjoint proximally zero sets. On the other hand, \( \text{cl}_{Z \cup \{\infty\}}(C) \cap \text{cl}_{Z \cup \{\infty\}}(C^c) = \{\infty\} \), so \( C \delta C^c \).

**Definition 11.** If \( (X, \delta) \) is a proximity space, then let \( (X, \delta)_{\aleph_1} = (X, (\delta)_{\aleph_1}) \) be the proximity space equivalent to the \( \sigma \)-algebra of proximally Baire sets on \( X \).

**Theorem 12.** Let \( (X, \mathcal{N}) \) be a \( \sigma \)-algebra, and let \( (Y, \delta) \) be a proximity space. Then a map \( f : (X, \mathcal{N}) \to (Y, \delta)_{\aleph_1} \) is proximally continuous if and only if \( f \) is a proximal mapping from \( (X, \mathcal{N}) \) to \( (Y, \delta)_{\aleph_1} \).

*Proof.* → Assume that \( f : (X, \mathcal{N}) \to (Y, \delta) \) is a proximity map. If \( C \subseteq Y \) is a proximally zero set, then there is a proximal map \( g : Y \to [0, 1] \) with \( C = g^{-1}(0) \). Therefore \( g \circ f \) is a proximity map as well, \( f^{-1}(C) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0) \) is a proximally zero set, and since \( (X, \mathcal{N}) \) is a \( P \)-proximity space, we have \( f^{-1}(C) \in \mathcal{N} \) for each proximally zero set \( C \). Therefore \( f^{-1}(R) \in \mathcal{N} \) for each proximally Baire set \( R \).

← Now assume that \( f : (X, \mathcal{N}) \to (Y, \delta)_{\aleph_1} \) is a proximal mapping. Then let \( C, D \subseteq Y \) be sets with \( C \delta D \). Then there is a proximal zero set \( Z \subseteq Y \) with \( C \subseteq Z, D \subseteq Z^c \). Therefore \( f^{-1}(Z) \in \mathcal{N} \), so \( f^{-1}(C) \subseteq f^{-1}(Z) \), and \( f^{-1}(D) \subseteq f^{-1}(Z^c) \), thus \( f^{-1}(C) \delta f^{-1}(D) \).

In particular, if \( (X, \mathcal{M}) \) is a \( \sigma \)-algebra, and \( \delta \) is any proximity on \( \mathbb{R} \) that induces the Euclidean topology, then \( f : X \to \mathbb{R} \) is measurable if and only if \( f \) is a proximal mapping.

**Theorem 13.** If \( (X, \delta) \) is a proximity space, then the topology on \( (X, \delta)_{\aleph_1} \) is the topology on the \( P \)-space coreflection of the topology on \( X \).

*Proof.* The proof is left to the reader.

**Lemma 14.** Let \( f : (X, \delta) \to (Y, \delta) \) be a function. Then \( f \) is a proximity map if and only if whenever \( g : (Y, \delta) \to [0, 1] \) is a proximity map, then \( g \circ f : (X, \delta) \to [0, 1] \) is a proximity map.

*Proof.* → If \( f \) is a proximity map, then clearly any composition \( g \circ f \) must be a proximity map.

← Now assume that each composition \( g \circ f \) is a proximity map. Assume that \( A, B \subseteq Y \) are sets with \( A \delta B \). Then there is a proximity map
Theorem 15. Let \((X, \delta)\) be a proximity space. Then the following are equivalent.

1. \((X, \delta)\) is a \(P\)-proximity space.

2. If \(f_n : (X, \delta) \to [0, 1]\) is a proximity map for each \(n \in \mathbb{N}\), and \(f_n \to f\) pointwise (here we do not assume \(f\) is continuous), then \(f : (X, \delta) \to [0, 1]\) is also a proximity map.

3. For each proximity space \((Y, \rho)\), if \(f_n : (X, \delta) \to (Y, \rho)\) is a proximity map for each \(n \in \mathbb{N}\), and \(f_n \to f\) pointwise, then \(f : (X, \delta) \to (Y, \rho)\) is also a proximity map.

Proof. 1 \(\to\) 2 The map \(f\) is a proximity map if and only if \(f\) is a measurable function on the \(\sigma\)-algebra \((X, \mathcal{M}_\delta)\). The implication follows since measurable functions are closed under pointwise convergence.

2 \(\to\) 3 Assume that \((Y, \rho)\) is any proximity space, and assume that \(f_n \to f\) pointwise and each \(f_n\) is a proximity map from \(X\) to \(Y\). Then let \(g : (Y, \rho) \to [0, 1]\) be a proximity map. Then \(g \circ f_n \to g \circ f\) pointwise, and since each \(g \circ f_n : (X, \mathcal{M}) \to [0, 1]\) is a proximity map, we have \(g \circ f\) also be a proximity map. Therefore \(f\) is a proximity map by lemma 14.

3 \(\to\) 2 This is trivial.

2 \(\to\) 1 Assume \(Z \subseteq X\) is a proximally zero set. Then there is a proximal mapping \(f : X \to [0, 1]\) with \(Z = f^{-1}(1)\). On the other hand, we have \(f^n \to \chi_Z\) pointwise where \(\chi_Z\) denotes the characteristic function, so \(\chi_Z\) is a proximal mapping. Therefore \(Z \delta Z^c\), so \(Z \in \mathcal{M}_\delta\). We conclude that \(X\) is a \(P\)-proximity space.

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Theorem 16. Let \(X\) be a completely regular space. Then \(X\) is a \(P\)-space if and only if whenever \(f_n : X \to [0, 1]\) is continuous for all \(n\), and \(f_n \to f\) pointwise, then \(f\) is continuous.
Proof. → Assume $X$ is a $P$-space. Then for each continuous $f_n : X \to [0,1]$ and $x \in X$ there is an open neighborhood $U_n$ of $x$ where $f_n(U_n) = f_n(x)$. Therefore if $U = \bigcap_n U_n$, then $U$ is an open neighborhood of $x$. Furthermore, if $f_n \to f$ pointwise, then $f(U) = f(x)$. Therefore $f$ is continuous at each point $x \in X$.

← Assume $Z \subseteq X$ is a zero set. Then let $f : X \to [0,1]$ be a continuous function where $f^{-1}(1) = Z$. Then $f^n \to \chi_Z$ pointwise. Therefore since $\chi_Z$ is continuous we have $Z$ be open. Therefore $X$ is a $P$-space.

Corollary 17. If $X$ is completely regular and $\delta$ is the proximity induced by the Stone-Cech compactification of $X$, then $X$ is a $P$-space if and only if $(X, \delta)$ is a $P$-proximity space.

Proof. The proximal mappings from $X$ to $[0,1]$ are precisely the continuous functions from $X$ to $[0,1]$. Therefore by theorems 15 and 16 we have $X$ be a $P$-space if and only if $(X, \delta)$ is a $P$-proximity space.

3 Conclusions and applications

We conclude this paper by demonstrating that it is sometimes better to consider $\sigma$-algebras as $P$-proximities since proximity spaces are often easier to work with than $\sigma$-algebras.

If $(X, \mathcal{M})$ is a $\sigma$-algebra, then let $L^\infty(X, \mathcal{M})$ denote the collection of all bounded measurable functions from $X$ to $\mathbb{C}$. Clearly $L^\infty(X, \mathcal{M})$ is a Banach-algebra and even a $C^*$-algebra. If $Y$ is a compact space, then let $C(Y)$ be the Banach-algebra which consists of all continuous functions from $Y$ to $\mathbb{C}$. Let $\mathcal{C}$ be the Smirnov compactification of $(X, \mathcal{M})$. Then $L^\infty(X, \mathcal{M})$ is isomorphic as a Banach-algebra to $C(\mathcal{C})$. One can easily show that $\mathcal{C}$ is the collection of all ultrafilters on the Boolean algebra $\mathcal{M}$. The maximal ideal space of $L^\infty(X, \mathcal{M})$ is therefore homeomorphic to the collection of all ultrafilters on $\mathcal{M}$. Furthermore, from these facts one can easily show that if $(X, \mathcal{M}, \mu)$ is a measure space, then the maximal ideal space of $L^\infty(\mu)$ is homeomorphic to the collection of all ultrafilters on the quotient Boolean algebra $\mathcal{M}/\{R \in \mathcal{M}|\mu(R) = 0\}$.

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