COUNTING CURVES ON TORIC SURFACES
TROPICAL GEOMETRY & THE FOCK SPACE

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ABSTRACT. We study the stationary descendant Gromov–Witten theory of toric surfaces by combining and extending a range of techniques – tropical curves, floor diagrams, and Fock spaces. A correspondence theorem is established between tropical curves and descendant invariants on toric surfaces using maximal toric degenerations. An intermediate degeneration is then shown to give rise to floor diagrams, giving a geometric interpretation of this well-known bookkeeping tool in tropical geometry. In the process, we extend floor diagram techniques to include descendants in arbitrary genus. These floor diagrams are then used to connect tropical curve counting to the algebra of operators on the bosonic Fock space, and are shown to coincide with the Feynman diagrams of appropriate operators. This extends work of a number of researchers, including Block–Gottsche, Cooper–Pandharipande, and Block–Gathmann–Markwig.

1. INTRODUCTION

1.1. Overview. The scope of this manuscript is to explore the relationships between the following enumerative and combinatorial geometric theories of surfaces, studied by a number of researchers in the last decade:

(1) decorated floor diagram counting;
(2) logarithmic and relative Gromov–Witten theory of Hirzebruch surfaces;
(3) tropical descendant Gromov–Witten theory of Hirzebruch surfaces;
(4) matrix elements of operators on a bosonic Fock space.

Floor diagrams are loop free graphs on a linearly ordered set of vertices, further endowed with vertex, edge, and half-edge decorations as specified in Definition 4.1. Each floor diagram is counted with a weight, coming from context in which it arises. Floor diagrams capture the combinatorial essence of the other three theories, in the sense that the simplest way to exhibit the above equivalences is through a weight preserving bijection between floor diagrams and specific ways to organize the enumeration in the other theories.

Gromov–Witten theory studies the intersection theory on moduli spaces of maps from pointed curves to a target surface. We are concerned with two distinct flavours of this theory – the relative and logarithmic invariants – which impose tangency conditions along certain boundary divisors, as in Definition 2.3 and Definition 2.4. These moduli spaces admit a virtual fundamental class, and zero dimensional cycles are constructed by capping with the virtual class two types of cycles: point conditions, corresponding to requiring a point on the curve to map to a specified point on the surface; and descendant insertions, which are Euler classes of certain tautological line bundles on the moduli space, associated to each marked point. The word stationary refers to the fact that descendant insertions are always coupled with point conditions. In this work, we specify special tangency orders to the 0 and ∞ sections of Hirzebruch surfaces, taking inspiration from the geometry of double Hurwitz numbers. In the logarithmic case, we specify transverse contact
along the torus invariant fibers. By using a degeneration of the relative geometry to a chain of Hirzebruch surfaces, in Theorem 4.9, the equivalence of the relative invariants with floor diagram counts is established. The relationship to logarithmic invariants is more subtle and passes through the tropical equivalence described below.

**Tropical Gromov–Witten theory** of surfaces consists of the study of piecewise linear, balanced maps from tropical curves into \( \mathbb{R}^2 \), see Definition 3.3. One obtains a finite count by imposing point conditions (i.e. specifying the image of a contracted marked end on the plane), and tropical descendant conditions. The descendant conditions constrain the valency of the vertex adjacent to a marked end. Each map is counted with a weight that arises as an intersection number on a certain moduli space of logarithmic stable maps. In good cases, these weights can be further spread out as products of combinatorial factors over the vertices. The directions and multiplicities of the infinite ends define a Newton fan, which determines at the same time a toric surface, a curve class on it, and prescribed tangencies along the toric divisors, offering a natural candidate for a correspondence between the logarithmic and tropical theories.

The logarithmic theory is shown to coincide with the tropical count in Theorem 3.9, using the recently established decomposition formula for logarithmic Gromov–Witten invariants [2]. The correspondence between the tropical count and the floor diagram count is established by a combinatorial argument. Specifically, after specializing the tropical point conditions, the contributing curves take a very special form, and become floor decomposed, meaning that certain subgraphs of the tropical curves may be contracted to give rise to a floor diagram. The floor decomposition yields a nontrivial result for the logarithmic invariants – namely, that the multiplicity of a floor decomposed tropical curve can be obtained in terms of the multiplicities associated to its vertices. A general such statement for logarithmic invariants is unknown, even for toric surfaces.

The **bosonic Fock space** is a countably infinite dimensional vector space with a basis indexed by ordered pairs of partitions of positive integers. It has an action of a Heisenberg algebra of operators, generated by two families of operators \( a_s, b_s \) parameterized by the integers. The distinguished basis vectors can naturally be identified with tangency conditions along the \( 0 \) and \( \infty \) sections of a Hirzebruch surface. In Definition 6.1, we construct a family of linear operators \( M_l \) on the Fock space which are naturally associated to stationary descendant insertions. To each (relative or logarithmic) Gromov–Witten invariant then corresponds a matrix element for an operator obtained as an appropriate composition of the \( M_l \)'s above. The equality between a the Gromov–Witten invariant and the corresponding matrix element goes through a comparison with the floor diagrams count: by Wick’s theorem a matrix element can be naturally evaluated as a weighted sum over Feynman graphs (see Definition 6.4). In Theorem 6.3 we exhibit a weight preserving bijection between the Feynman graphs for a given matrix element, and the floor diagrams for the corresponding Gromov–Witten invariant.

1.2. **Context and Motivation.** This work provides an extension and unification of several previous lines of investigation on the subject. Correspondence theorems between tropical curve counts and primary Gromov–Witten invariants of surfaces – those with only point conditions and no descendant insertions – were established by Mikhalkin, Nishinou–Siebert, and Gathmann–Markwig in [36, 38, 21]; the tropical descendant invariants in genus 0 was first investigated by Markwig–Rau [34], and correspondence theorems were established independently, using different techniques, by A. Gross [25] and by Mandel–Ruddat [32]. Tropical descendants have also arisen in aspects of the SYZ conjecture [26, 40].
Cooper and Pandharipande pioneered a Fock space approach to the Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ by using degeneration techniques [19]. Block and Götsche generalized their work to a broader class of surfaces (the $h$-transverse surfaces, see for instance [5] and [11]), and to refined curve counts, via quantum commutators on the Fock space side [8]. Both cases deal only with primary invariants. Block and Götsche assign an operator on the Fock space to point insertions, and observe the connection between floor diagrams and Feynman graphs. We generalize their operator to a family of operators, one for each descendant insertion, and notice that the operators can be written with summands naturally corresponding to the possible sizes of the floor (see Definition 6.1) containing a particular descendant insertion. In the primary case, there were only floors of size 0 (elevators) or 1 (floors), and hence the operator had two terms.

Section 4 contains a brief summary of how floor diagrams came to be employed for these types of enumerative problems (Subsection 4.2). This discussion follows our definition of floor diagrams (Definition 4.1), to explain and motivate some of the minor combinatorial tweaks we made in order to adapt to the current geometric context.

It is at this point a well understood philosophy that correspondence theorems between classical and tropical enumerative invariants are based on the fact that tropical curves encode the combinatorics of possible degenerations of the classical objects. The decomposition formula for the Gromov–Witten invariants of simple normal crossings degenerations allows us to equip tropical curves with a virtual multiplicity, and state the correspondence theorem between the virtual counts of tropical and algebraic curves [2].

An appealing feature of the generality provided by the logarithmic setup is that it establishes a formula from which one can witness the collapsing of geometric inputs in different settings to give rise to a purely combinatorial theory. In genus 0, the descendant contributions collapse into closed combinatorial formulas. Conceptually, this is because the intersection theory of the space of genus 0 logarithmic maps is essentially captured by the intersection theory on a particular toric variety, see [41]. Without descendants but still in higher genus, there is a different collapsing – on a surface, one can degenerate in such a way that all the algebraic inputs are 1 up to multiplicity – the multiplicity can be detected combinatorially, leading to Mikahilkin’s formula (3.4).

A drawback of the logarithmic approach to this enumerative problem is that there is not yet a formula expressing the virtual multiplicity of a tropical curve in terms of vertex multiplicities, although such a formula is expected to exist. In lieu of it, there are two options. The first is to change our geometric setup to the older relative maps geometry. The second is to prove a vertex multiplicity formula for special choices of configurations of points. We do this by using
tropical arguments to limit the types of tropical curves that can contribute to horizontally stretched descendant constraints. In both cases, the floor diagram connects the invariants to the Fock space.

Restricting our attention to the study of invariants of Hirzebruch surfaces is a stylistic choice, as we strived to write a paper that communicates the various connections we explore, rather than making the most general statements possible. Results of Section 3 could as well be formulated for any toric surface, results of Sections 4, 5, and 6 for any toric surface dual to a h-transverse lattice polygon, see [8, Section 2.3].

This paper is a sequel to the authors’ work in [14], in which the relationship between tropical curves, Fock spaces, and degeneration techniques was studied for target curves, combining Okounkov and Pandharipande’s seminal work in [39], with the tropical perspective on the enumerative geometry of target curves [12, 13, 16, 15]. We refer the reader to [14] for a more detailed discussion of the history of the target curve case.

The paper is organized as follows. In Section 2 we present some basic facts about the geometry of Hirzebruch surfaces, and introduce logarithmic and relative stationary descendant Gromov–Witten invariants. Section 3 introduces the tropical theory of descendant stationary invariants of Hirzebruch surfaces, and proves the correspondence theorem with the logarithmic theory. In Section 4 we define our version of decorated floor diagrams, explain the connection with the previous notions in the literature, and then compare floor diagram counts with the relative theory, as an application of the degeneration formula. In section 5, we develop a vertex multiplicity formula for floor decomposed tropical maps. We then provide a correspondence theorem relating the count of floor diagrams with the tropical theory, using a combinatorial argument and keeping track of the local vertex multiplicities. Section 6 provides a brief and hopefully friendly introduction to the Fock space, and then proves the equivalence between floor diagram counts and matrix elements for specific operators in the Fock space.

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2. Relative and logarithmic descendants

We study two closely related algebro-geometric curve counting theories attached to a Hirzebruch surface – the relative and logarithmic Gromov–Witten invariants with stationary descendents.

For $k \geq 0$, the Hirzebruch surface $F_k$ is defined to be the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$; it is a smooth projective toric surface. The 1-skeleton of its fan $\Sigma_k$ is given by the four vectors $e_1, \pm e_2, -e_1 + ke_2$. The 2-dimensional cones are spanned by the consecutive rays in the natural counterclockwise ordering. The zero section $B$, the infinity section $E$, and the fiber $F$ have intersections

$$B^2 = k, \quad E^2 = -k, \quad BF = EF = 1, \quad 	ext{and} \quad F^2 = BE = 0.$$  

The Picard group of $F_k$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ generated by the classes of $B$ and $F$. In particular, we have $E = B - kF$. A curve in $F_k$ has **bidegree** $(a, b)$ if its class is $aB + bF$. The polygon depicted in Figure 2 defines $F_k$ as a projective toric surface polarized by an $(a, b)$ curve.

![Figure 2](image-url)

**Figure 2.** The polygon defining the Hirzebruch surface $F_k$ as a toric surface embedded in projective space with hyperplane section the class of a curve of bidegree $(a, b)$. The vertical sides corresponds to the sections $B$ (left) and $E$ (right).

We study the virtual enumerative invariants of curves in Hirzebruch surfaces that have prescribed special contact orders with the zero and infinity sections, and generic intersection with the invariant fibers. This numerical data is encoded in terms of the Newton fan.

**Definition 2.1.** A **Newton fan** is a sequence $\delta = (v_1, \ldots, v_k)$ of vectors $v_i \in \mathbb{Z}^2$ satisfying

$$\sum_{i=1}^{k} v_i = 0.$$  

If $v_i = (v_{i1}, v_{i2})$, then the positive integer $w_i = \gcd(v_{i1}, v_{i2})$ (resp. the vector $\frac{1}{w_i}v_i$) is called the expansion factor (resp. the primitive direction) of $v_i$. We use the notation

$$\delta = (v_1^{m_1}, \ldots, v_k^{m_k})$$  

to indicate that the vector $v_i$ appears $m_i$ times in $\delta$. 

We study the virtual enumerative invariants of curves in Hirzebruch surfaces that have prescribed special contact orders with the zero and infinity sections, and generic intersection with the invariant fibers. This numerical data is encoded in terms of the Newton fan.
For a Newton fan \( \delta \), one can construct a polarized toric surface, identified by the dual polygon \( \Pi_\delta \) in \( \mathbb{R}^2 \), in the following way: for each primitive integer direction \( (\alpha, \beta) \) in \( \delta \), we consider the vector \( w(-\beta, \alpha) \), where \( w \) is the sum of the expansion factors of all vectors in \( \delta \) with primitive integer direction \( (\alpha, \beta) \). Up to translation, \( \Pi_\delta \) is the unique (convex, positively oriented) polygon whose oriented edges are exactly the vectors \( w(-\beta, \alpha) \).

**Notation 2.2.** *(Discrete data)* Fix a Hirzebruch surface \( F_k \). The following discrete conditions govern the enumerative geometric problems we study throughout the paper:

- A positive integer \( n \);
- Non-negative integers \( g, a, k_1, \ldots, k_n, n_1, n_2 \);
- A vector \( \phi = (\phi_1, \ldots, \phi_n) \in (\mathbb{Z} \setminus 0)^{n_1} \);
- A vector \( \mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{Z} \setminus 0)^{n_2} \);
- We assume that \( \phi \) and \( \mu \) are non-decreasing sequences.
- We denote by \( (\phi^+, \mu^+) \) the positive entries of \( (\phi, \mu) \), and by \( (\phi^-, \mu^-) \) the negative ones.

Further, the following two equations must be satisfied:

1. \[ \sum_{i=1}^{n_1} \phi_i + \sum_{i=1}^{n_2} \mu_i + ka = 0; \]
2. \[ n_2 + 2a + g - 1 = n + \sum_{j=1}^{n} k_j. \]

### 2.1. Logarithmic invariants.

The first enumerative geometric problem we introduce is stationary, descendant, logarithmic Gromov–Witten invariants of \( F_k \), which morally count curves in \( F_k \) with prescribed tangency conditions along the boundary, and satisfying some further geometric constraints, called **descendant insertions** (see Section 2.1), at a number of fixed points in the interior of the surface. In this context, \( g \) is the arithmetic genus of the curves being counted, \( n \) is the number of ordinary marked points on the curves, and the \( k_i \) are the degrees of the descendant insertions at each point. The sequences \( (\phi, \mu) \) identify a curve class in \( H_2(F_k, \mathbb{Z}) \), as well as the required tangency with the toric boundary, as we now explain.

The tuple \( (\phi, \mu) \) determines the curve class

\[ \beta = aB + \left( \sum_{\phi_i \in \phi^+} \phi_i + \sum_{\mu_i \in \mu^+} \mu_i \right) F. \]

The compatibility condition (1) ensures that \( \beta \) is an effective, integral curve class in \( H_2(F_k, \mathbb{Z}) \).

The Newton fan

\[ \delta(\phi, \mu) := \{(0, -1)^a, (k, 1)^a, \varphi_1 \cdot (1, 0), \ldots, \varphi_{n_1} \cdot (1, 0), \mu_1 \cdot (1, 0), \ldots, \mu_{n_2} \cdot (1, 0)\}. \]

encodes contact orders a curve may have with the toric boundary of \( F_k \). Such a curve is necessarily of class \( \beta \).

We count curves with contact orders \( |\varphi_i| \) for \( \varphi_i < 0 \) (resp. \( \varphi_i > 0 \)) with the zero (resp. infinity) section at fixed points, and contact orders \( |\mu_i| \) for \( \mu_i < 0 \) (resp. \( \mu_i > 0 \)) with the zero (resp. infinity) section at arbitrary points.
Logarithmic stable maps and logarithmic Gromov–Witten invariants were developed in [1, 17, 28]. There is a moduli stack with logarithmic structure

$$\overline{M}^\log_{g,n_1+n_2}(F_k, \delta(\phi, \mu)),$$

such that, ignoring the marked points for simplicity, a map from a logarithmic scheme $S$ to the moduli space is equivalent to a diagram

$$\mathcal{C} \xrightarrow{f} F_k,$$

where $\mathcal{C}$ is a family of connected marked genus $g$ nodal logarithmic (?) curves and $f$ is a map of logarithmic schemes, whose underlying map is stable in the usual sense.

Concerning the discrete data, contact orders with the toric boundary are specified by the Newton fan $\delta(\phi, \mu)$. We choose to mark the points of contact with the zero and infinity sections, and to not mark the points of contact with the torus-invariant fibers, where the behavior requested is transverse.

This moduli space is a virtually smooth Deligne-Mumford stack and it carries a virtual fundamental class denoted by $[1]^{\log}$ in degree $(g-1) + 2a + n + n_1 + n_2$. For each of the first $n$ marked points, which carry trivial contact orders, there are evaluation morphisms

$$ev_i : \overline{M}^\log_{g,n_1+n_2}(F_k, \delta(\phi, \mu)) \to F_k.$$

The points marking the contact points with the zero and infinity sections give rise to evaluation morphisms

$$\hat{ev}_i : \overline{M}^\log_{g,n_1+n_2}(F_k, \delta(\phi, \mu)) \to \mathbb{P}^1.$$

Here, the target $\mathbb{P}^1$ is the the zero section $B$ for negative entries of $\phi$ or $\mu$, and the infinity section $E$ for positive entries.

For each of the first $n$ marks (with trivial contact order) there is a cotangent line bundle, whose first Chern class is denoted $\psi_i$.

**Definition 2.3.** Fix a Hirzebruch surface $F_k$ and discrete data as in Notation 2.2.

The stationary descendant log Gromov–Witten invariant is defined as the following intersection number on $\overline{M}^\log_{g,n_1+n_2}(F_k, \delta(\phi, \mu))$:

$$\langle (\phi^-, \mu^-) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\phi^+, \mu^+) \rangle^\log_g = \int_{[1]^{\log}} \prod_{j=1}^n \psi_j^{k_j} ev_j^*(|pt|) \prod_{i=n+1}^{n+n_1} \hat{ev}_i^*(|pt|).$$  \hspace{1cm} (5)

Condition (2) comes from equating the expected dimension of the moduli space with the codimension of the intersection cycle, and hence it is a necessary condition for Equation (5) to be non-zero.

2.2. Relative invariants. Closely related to the logarithmic invariants studied above are relative Gromov–Witten invariants. Let $F_k$ continue to denote the Hirzebruch surface, now considered as a pair $(F_k, B + E)$, the divisor consisting of the disjoint union of the zero and infinity sections. Fix discrete data as in Notation 2.2. There is a moduli space

$$\overline{M}^{rel}_{g,n_1+n_2}(F_k, \delta(\phi, \mu)).$$
where each parameterizing families of maps to expansions
\[ \mathcal{C} \rightarrow S_1 \cup \cdots \cup S_m \rightarrow \mathbb{F}_k, \]
where each \( S_i \) is a copy of the Hirzebruch surface \( \mathbb{F}_k \), where the zero section of \( S_i \) is glued to the infinity section of \( S_{i+1} \). As before, the curve \( \mathcal{C} \) carries \( n + n_1 + n_2 \) markings, and the contact orders at fixed points of the zero section of \( S_1 \) (resp. infinity section of \( S_m \)) are given by \( |\varphi_i| \) for \( \varphi_i < 0 \) (resp. \( \varphi_i > 0 \)), and at arbitrary points of the zero section of \( S_1 \) (resp. infinity section of \( S_m \)) are specified to be \( |\mu_i| \) for \( \mu_i < 0 \) (resp. \( \mu_i > 0 \)). See [24, 31] for additional details on maps to expansions.

Both the relative and logarithmic invariants are virtual counts for the same enumerative problem. The main difference between the logarithmic and relative setups is that contact orders are not prescribed with the torus invariant fibers in the latter. The two theories are expected to be close related by a degeneration formula for logarithmic Gromov–Witten invariants.

There is once again a virtual fundamental class in the homology of \( \overline{M}_{g,n+n_1+n_2}^{\text{rel}}(\mathbb{F}_k \delta(\varphi, \mu)) \), in degree \( (g-1) + 2a + n_1 + n_2 + n \). The spaces of relative stable maps come equipped with evaluation morphisms. The following definition is analogous to Definition 2.3.

**Definition 2.4.** The stationary descendant relative Gromov–Witten invariant is defined by

\[
\langle \phi^-, \mu^- \rangle \tau_{k_1}(\text{pt}) \cdots \tau_{k_n}(\text{pt}) \langle \phi^+, \mu^+ \rangle^\text{rel} = \int_{[1]^\text{rel}} \prod_{j=1}^{n} \psi_j^{k_j} \left[ \text{ev}_1([\text{pt}]) \right] \prod_{i=n+1}^{n+n_1} \tilde{\text{ev}}_i([\text{pt}]).
\]

### 3. Tropical descendants

#### 3.1. Tropical preliminaries

An (abstract) tropical curve is a connected metric graph \( \Gamma \) with unbounded rays or “ends” and a genus function \( g : \Gamma \rightarrow \mathbb{N} \) which is nonzero only at finitely many points. Locally around a point \( p \), \( \Gamma \) is homeomorphic to a star with \( r \) half-rays. The number \( r \) is called the valence of the point \( p \) and denoted by \( \text{val}(p) \). We require that there are only finitely many points with \( \text{val}(p) \neq 2 \). A finite set of points containing (but not necessarily equal to the set of) all points of nonzero genus and valence larger than 2 may be chosen; its elements are called vertices. By abuse of notation, the underlying graph with this vertex set is also denoted by \( \Gamma \). Correspondingly, we can speak about edges and flags of \( \Gamma \). A flag is a tuple \((V, e)\) of a vertex \( V \) and an edge \( e \) with \( V \in \partial e \). It can be thought of as an element in the tangent space of \( \Gamma \) at \( V \), i.e. as a germ of an edge leaving \( V \), or as a half-edge (the half of \( e \) that is attached to \( V \)). Edges which are not ends have a finite length and are called bounded edges.

A marked tropical curve is a tropical curve such that some of its ends are labeled. An isomorphism of a tropical curve is a homeomorphism respecting the metric, the markings of ends, and the genus function. The genus of a tropical curve is the first Betti number \( b^1(\Gamma) \) plus the genera of all vertices. A curve of genus 0 is called rational.

The combinatorial type of a tropical curve is obtained by dropping the information on the metric.

Let \( \Sigma \) be a polyhedral decomposition of \( \mathbb{R}^2 \).

**Definition 3.1.** A tropical stable map to \( \Sigma \) is a tuple \((\Gamma, f)\) where \( \Gamma \) is a marked abstract tropical curve and \( f : \Gamma \rightarrow \Sigma \) is a piecewise integer-affine map of polyhedral complexes satisfying:
On each edge \( e \) of \( \Gamma \), \( t \) is of the form
\[
\begin{aligned}
t &\mapsto a + t \cdot v \\
&\text{with } v \in \mathbb{Z}^2,
\end{aligned}
\]
where we parametrize \( e \) as an interval of size the length \( l(e) \) of \( e \). The vector \( v \), called the \textit{direction}, arising in this equation is defined up to sign, depending on the starting vertex of the parametrization of the edge. We will sometimes speak of the direction of a flag \( v(V,e) \). If \( e \) is an end we use the notation \( v(e) \) for the direction of its unique flag.

- The \textbf{balancing condition} holds at \textit{every} vertex, i.e.
\[
\sum_{e \in \partial V} v(V,e) = 0.
\]
- The \textbf{stability condition} holds, \textit{i.e.} for \textit{every} 2-valent vertex \( v \) of \( \Gamma \), the \textit{star} \( v \) is not contained in the relative interior of any single cone of \( \Sigma \).

For an edge with direction \( v = (v_1,v_2) \in \mathbb{Z}^2 \), we call \( w = \gcd(v_1,v_2) \) the \textit{expansion factor} and \( \frac{1}{w} \cdot v \) the \textit{primitive direction} of \( e \).

An isomorphism of tropical stable maps is an isomorphism of the underlying tropical curves respecting the map. The \textit{degree} of a tropical stable map is the Newton fan given as the multiset of directions of its ends. The \textit{combinatorial type} of a tropical stable map is the data obtained when dropping the metric of the underlying graph. More explicitly, it consists of the data of a finite graph \( \Gamma \), and (1) for each vertex \( v \) of \( \Gamma \), the cone \( \sigma_v \) of \( \Sigma \) to which this vertex maps, and (2) for each edge \( e \) of \( \Gamma \), the expansion factor and primitive direction of \( e \).

Note that in practice, the precise polyhedral decomposition plays a limited role, and we will often drop this from the discussion, simply referring to the maps by the notation \([\Gamma \rightarrow \mathbb{R}^2]\).

**Convention 3.2.** We consider tropical stable maps to Hirzebruch surfaces, \textit{i.e.} the degree is a Newton fan dual to the polygons of Figure 2. Furthermore, we require the vertical and diagonal ends to be non-marked and of expansion factor 1. The horizontal ends can have any expansion factor, and are marked.

In what follows, we fix conditions for tropical stable maps — the degree, the genus, point conditions, high valency (descendant) conditions, and end conditions — and then count tropical stable maps satisfying the conditions, with multiplicity. We consider degrees containing integer multiples of \((1,0)\). An end whose direction vector is a multiple of \((1,0)\) is mapped to a line segment of the form \(\{(a,b) + t \cdot (\pm 1,0)\} \), where \((a,b) \in \mathbb{R}^2\). The unique \( b \) appearing here is the \textit{y-coordinate} of the respective end. Our end conditions fix some of the \( y \)-coordinates of ends.

**Definition 3.3.** Fix discrete invariants as in Notation 2.2. Let
\[
\Delta = \delta(\Phi,\mu) \cup \{0^n\}
\]
identify a degree for tropical stable maps. Fix \( n \) points \( p_1,\ldots,p_n \in \mathbb{R}^2 \) in general position, and two sets \( E_0 \) and \( E_\infty \) of pairwise distinct real numbers together with bijections \( E_0 \rightarrow \{\varphi_i | \varphi_i < 0\} \) (resp. \( E_\infty \rightarrow \{\varphi_i | \varphi_i > 0\} \)).

The \textbf{tropical descendant Gromov–Witten invariant}
\[
\langle (\Phi^-,\mu^-) | \tau_{k_1}(p_1) \cdots \tau_{k_n}(p_n) (\Phi^+,\mu^+) \rangle^\text{trop}_g
\]
is the weighted number of marked tropical stable maps \((\Gamma, \Gamma)\) of degree \( \Delta \) and genus \( g \) satisfying:
- For \( j = 1,\ldots,n \), the marked end \( j \) is contracted to the point \( p_j \in \mathbb{R}^2 \).
- The end \( j \) is adjacent to a vertex \( V \) in \( \Gamma \) of valence \( \text{val}(V) = k_j + 3 - g(V) \).
• $E_0$ and $E_\infty$ are the $y$-coordinates of ends marked by the set $\Phi$.

Each such tropical stable map is counted with multiplicity $\frac{1}{\text{Aut}(f)}^m(\Gamma, f)$, to be defined in Definition 3.6.

3.2. Superabundance and rigid curves. The set of tropical stable maps of a fixed combinatorial type can be parametrized by a polyhedral cone in a real vector space; in the present generality and notation, a proof is recorded in [42], though the main ideas can be found in [22, 23, 37]. The expected dimension of the cone associated to the type of a map $(\Gamma, f)$ is

$$\#\{\text{ends}\} + b_1(\Gamma) - 1 - \sum V_i \cdot (\text{val}(V) - 3) = \#\{\text{bounded edges}\} - 2b_1(\Gamma),$$

see for instance [42, Section 2.2]. When a combinatorial type has this expected dimension, it is said to be non-superabundant. In superabundant cases, there may be nontrivial families even when the expected dimension is zero, so we introduce the notion of rigidity to reduce to a finite combinatorial count.

Definition 3.4. Choose general points $p_1, \ldots, p_n \in \mathbb{R}^2$, a degree, genus, incidence, and descendant constraints defining a tropical descendant Gromov–Witten invariant. Let $(\Gamma, f)$ be a tropical stable map satisfying these chosen constraints. The map $(\Gamma, f)$ is said to be rigid if $(\Gamma, f)$ is not contained in any nontrivial family of tropical curves having the same combinatorial type.

The following result follows from a simple adaptation of the proof of [36, Lemma 4.20].

Lemma 3.5. Let $(\Gamma, f)$ be a rigid stable map satisfying the conditions of Definition 3.3. Then every connected component of $\Gamma$ minus the marked ends is rational and contains exactly one non-fixed end.

3.3. The virtual multiplicity of a rigid tropical curve. Let

$$f : \Gamma \rightarrow \mathbb{R}^2$$

be a rigid tropical stable map contributing to a tropical descendant Gromov–Witten invariant. Assume that all vertices of $\Gamma$ map to integer points in $\mathbb{R}^2$. Note that since edge expansion factors are rational, this is always possible after a translation and dilation. We define the multiplicity of $(\Gamma, f)$. This is done in two steps. Let $\mathbb{F}_k^\dagger$ denote the product $\mathbb{F}_k \times \text{Spec}(\mathbb{N} \rightarrow \mathbb{C})$, where the latter is the standard logarithmic point. Consider a minimal logarithmic stable map

$$\begin{array}{ccc}
C & \rightarrow & \mathbb{F}_k^\dagger \\
\downarrow & & \downarrow \\
S & \rightarrow & \text{Spec}(\mathbb{N} \rightarrow \mathbb{C}).
\end{array}$$

A logarithmic curve over a standard logarithmic point $(\Gamma)$ determines a tropical curve $\Gamma_C$. Specifically, we take the underlying graph of $\Gamma_C$ to be the dual graph of $C$. Attached to each node $q$ of $C$ is a deformation parameter $\delta_q \in \mathbb{N}$. We take the length of the corresponding edge $e_q$ to be the value $\delta_q$. As explained in [2, Section 2], by dualizing the maps of monoids underlying the map of logarithmic schemes $C \rightarrow \mathbb{F}_k^\dagger$, we obtain a tropical map

$$\Gamma_C \rightarrow \mathbb{R}^2.$$

Fix $(\Gamma, f)$ a tropical stable map. Consider logarithmic stable maps $C \rightarrow \mathbb{F}_k^\dagger$, equipped with an edge contraction from the tropicalization $\Gamma_C \rightarrow \mathbb{R}^2$ to $(\Gamma, f)$. That is, the logarithmic maps are at least as degenerate as dictated the fixed tropical map $(\Gamma, f)$. However, the marking by $(\Gamma, f)$ is an
additional rigidifying datum. In [2, Section 4], it is shown that such maps are parameterized by an algebraic stack with a finite map

\[ \overline{M}_{g, n_1+n_2}^{\log}(\mathbb{F}^\dagger_k, \delta_{\phi, \mu}) \]

We may now define the virtual multiplicity of \((\Gamma, f)\). For simplicity, we only treat the case without fixed points on the boundary. It is a notational exercise to extend this to the general case. Given a standard logarithmic point

\[ p^\dagger : \text{Spec}(\mathbb{N} \to \mathbb{C}) \to \mathbb{F}^\dagger_k, \]

there is an associated tropical point in \(\mathbb{R}^2\) as follows. Let \(U_\sigma\) be the affine invariant open to which the underlying scheme theoretic point maps. Let \(S_\sigma\) be the associated dual cone of characters that extend to regular functions on \(U_\sigma\). By the definition of a logarithmic morphism, there is an induced map

\[ S_\sigma \to \mathbb{N}, \]

which gives an integral point \(p \in \mathbb{R}^2\). We refer to \(p^\dagger\) as a logarithmic lifting of \(p\).

Fix discrete data. Choose general points \(p_1, \ldots, p_n \in \mathbb{R}^2\), and let \((\Gamma, f)\) be a rigid tropical stable map passing through the points \(p_i\) contributing to a tropical descendant Gromov–Witten invariant. Choose a logarithmic lifting \(p^\dagger = (p_1^\dagger, \ldots, p_n^\dagger)\) of \((p_1, \ldots, p_n)\). Evaluation at the \(n\) markings gives rise to a morphism

\[ \text{ev} : \overline{M}_{(\Gamma, f)} \to (\mathbb{F}^\dagger_k)^n. \]

By taking a fiber product with the point \(p \to (\mathbb{F}^\dagger_k)^n\), we obtain a new moduli space \(\overline{M}_{(\Gamma, f)}(p^\dagger)\) of logarithmic maps with given tropical type passing through the points. As explained in [2, Section 6.3.2], this moduli space comes equipped with a virtual fundamental class.

**Definition 3.6.** For a rigid tropical curve \((\Gamma, f)\) contributing to a tropical descendant Gromov–Witten invariant, define its **virtual multiplicity** to be

\[ m_{(\Gamma, f)} := \int_{[\overline{M}_{(\Gamma, f)}(p^\dagger)]^\text{vir}} \prod_{j=1}^n \psi_j^{k_j}. \]

A non-rigid tropical curve \((\Gamma, f)\) is defined to have multiplicity 0.

A requisite for the above definition is the fact that the multiplicity defined above is independent of the choice of logarithmic lifting. This follows from the logarithmic deformation invariance of the virtual fundamental class [28, Theorem 0.3]. This completes the definition of the tropical descendant Gromov–Witten invariant.

### 3.4. When does the virtual multiplicity collapse?

The multiplicity in Definition 3.6 is difficult to compute in practice, which limits the utility of the logarithmic decomposition. However, in Section 5, we observe that after specializing the point conditions to horizontally stretched position (leading to floor decomposed tropical maps), it is indeed possible to write the multiplicity \(m_{(\Gamma, f)}\) in terms of local multiplicities attached to vertices, yielding a *degeneration* formula, as opposed to only a decomposition. We point out two previous instances in which this was already known [34, 36].
Figure 3. A tropical stable map to $\mathbb{F}_1$ contributing to $\langle((−2), (−2, −1))|\tau_0(p_1)\ldots\tau_0(p_8)|((1), (1))\rangle_0^\text{trop}$ as in Example 3.7.

(1) **If all $\psi$-powers are 0**, i.e. $k_1 = \ldots = k_n = 0$: the valency condition implies that the vertex adjacent to end $i$ is trivalent and of genus 0. Since the end $i$ is contracted, the image of a neigbourhood of this vertex just looks like an edge passing through $p_i$. We thus count plane tropical curves passing through the points (and possibly with some fixed $y$-coordinates for the ends). An example can be found in Example 3.7, see Figure 3. They are counted with multiplicity equal to the product of the normalized areas of the triangles in the dual subdivision (notice that all vertices are trivalent and of genus 0 for dimension reasons). In case of fixed $y$-coordinates, the product above has to be multiplied in addition with $\prod_e \frac{1}{w(e)}$, where the product goes over all fixed ends $e$ and $w(e)$ denotes their expansion factor [21]. That all local Gromov–Witten invariants are 1 follows from the correspondence theorem proved in [38, 41].

(2) **If the genus $g = 0$**: in [34], tropical, rational, stationary, descendant invariants are studied. The appropriate tropical maps are counted with multiplicity equal to the product of the normalized areas of the triangles (dual to non-marked vertices) in the dual subdivision as above, with a factor of $\prod_e \frac{1}{w(e)}$ for fixed ends, see [7]. The correspondence theorem for such invariants is proved in the papers [25, 41] and using different methods in [32].

**Example 3.7.** We show two examples. The point conditions $p_1 \in \mathbb{R}^2$ are chosen to be in horizontally stretched position, see [20, Definition 3.1].

(1) Let $k = 1$, $(\phi) = (−2, 1)$, $(\mu) = (−2, −1, 1)$. Then $\sum \varphi_i + \sum \mu_i + 3 \cdot 1 = 0$, so $a = 3$. Let $g = 0$, $n = 8$, and $k_1 = \ldots = k_8 = 0$. Since $n_2 = 3$ and $3 + 2 \cdot 3 − 1 = 8$, this choice satisfies the condition of Definition 3.3. Figure 3 shows the image of a tropical stable map contributing to $\langle((−2), (−2, −1))|\tau_0(p_1)\ldots\tau_0(p_8)|((1), (1))\rangle_0^\text{trop}$ with multiplicity 72 (see Remark 3.4 (1)). The Figure reflects the image of the map, decorated by some data of the parametrization — for that reason, the picture indicates a crossing instead of a 4-valent vertex. We draw the fixed $y$-coordinates as points at the end of an end. Expansion factors bigger one are written next to the edges, so that the direction is visible from the picture.

(2) As before, let $k = 1$, $(\phi) = (−2, 1)$, $(\mu) = (−2, −1, 1)$, $a = 3$ and $g = 0$. Let $n = 4$ and $k_1 = 0$, $k_2 = 1$, $k_3 = 3$ and $k_4 = 0$. Then $3 + 2 \cdot 3 − 1 = 4 + 1 + 3$, so the condition of Definition 3.3 is satisfied for this choice. Figure 4 shows a tropical stable map contributing to $\langle((−2), (−2, −1))|\tau_0(p_1)\tau_1(p_2)\tau_3(p_3)\tau_0(p_4)|((1), (1))\rangle_0^\text{trop}$ with multiplicity 4 (see Remark 3.4 (2) above).

**Remark 3.8.** The image $f(\Gamma) \subset \mathbb{R}^2$ of a tropical stable map is a tropical plane curve as considered e.g. in [36, 43]. We assume that the reader is familiar with basic concepts concerning tropical
plane curves, in particular their duality to subdivisions of the Newton polygon. In our situation, the image of any tropical stable map contributing to the count above is dual to a subdivision of the polygon attached to the Newton fan $\delta_{(\phi, \mu)}$, which defines the Hirzebruch surface $F_k$ as a projective toric surface with hyperplane section the class of a curve of bidegree $(a, \sum_{i} \varphi_i > 0 \varphi_i + \sum_{i} \mu_i > 0 \mu_i)$. Figure 5 shows the dual Newton subdivisions of the images of the stable maps of Example 3.7.

**Theorem 3.9** (Correspondence theorem). Fix a Hirzebruch surface $F_k$ and discrete data as in Notation 2.2. The tropical stationary descendant log Gromov–Witten invariant coincides with its algebro-geometric counterparts, i.e. we have

$$\langle (\phi^-, \mu^-) | \tau_{k_1} (pt) \ldots \tau_{k_n} (pt) | (\phi^+, \mu^+) \rangle_{g} = \langle (\phi^-, \mu^-) | \tau_{k_1} (pt) \ldots \tau_{k_n} (pt) | (\phi^+, \mu^+) \rangle_{g}^{\text{trop}}$$

**Proof.** The proof is a consequence of the decomposition formula for logarithmic Gromov–Witten invariants, due to Abramovich, Chen, Gross, and Siebert [2]. We explain the geometric setup, and how to deduce the multiplicity above from the formulation in loc. cit. We assume that there are no fixed boundary conditions to lower the burden of the notation; the general case is no more complicated.

Consider the moduli space $\mathcal{M}_{\log}^{g,n+n_1+n_2}(F_k, \delta_{(\phi, \mu)})$, and on it, the descendant cycle class given by $\psi_1^{k_1} \ldots \psi_n^{k_n}$. We compute the invariant by degenerating the point conditions, and cutting down the virtual class to

$$\psi_1^{k_1} \ldots \psi_n^{k_n} \cap [1]^{\log}.$$

Working over $\text{Spec} (\mathbb{C}[[t]])$, choose points $p_1, \ldots, p_n \in T \subset F_k$, whose tropicalizations $p_1^{\text{trop}}, \ldots, p_n^{\text{trop}}$ are in general position in $\mathbb{R}^2$. Since the tropical moduli space with the prescribed discrete data has only finitely many cones, it follows that there are finitely many rigid tropical stable maps meeting the stationary constraints. Suppose $(\Gamma, f)$ is a tropical stable map with an end $p_i$ incident to a vertex $V$. Since a point $p_i$ must support the descendant class $\psi_i^{k_i}$, a dimension argument
forces that the valency of \( V \) is \( k_j + 3 - g(V) \). In other words, the tropical curves contributing to the count are precisely the ones outlined in Definition 3.3.

Enumerate the finitely many tropical stable maps \( (\Gamma_1, f_1), \ldots, (\Gamma_r, f_r) \) contributing to the invariant

\[
\langle (\phi_-, \mu^-)\tau_{k_1}(p_1^\text{trop}) \ldots \tau_{k_n}(p_n^\text{trop}) | (\phi^+, \mu^+) \rangle^\text{trop}_{g}.
\]

Choose a polyhedral decomposition \( \mathcal{P} \) of \( \mathbb{R}^2 \) such that every tropical stable map \( f_i^\text{trop} \) factors through the one-skeleton of \( \mathcal{P} \) and that the fan of unbounded directions of \( \mathcal{P} \) (i.e. the recession fan) is the fan \( \Sigma_k \). Note that \( \mathcal{P} \) can always be chosen to be a common refinement of the images of \( f_i^\text{trop} \). The contact order conditions on the tropical maps ensure that the recession fan is \( \Sigma_k \).

The polyhedral decomposition \( \mathcal{P} \) determines a toric degeneration \( \mathcal{X} \) of \( F_k \), over \( \text{Spec}(\mathbb{C}[\left[ t \right]]) \), see [29]. By the deformation invariance property of logarithmic Gromov–Witten invariants, we may compute \( \langle (\phi_-, \mu^-)\tau_{k_1}(p_1) \ldots \tau_{k_n}(p_n) | (\phi^+, \mu^+) \rangle^\text{log}_{g} \) on the central fiber of this degeneration, as

\[
\text{ev}^*(p) \cap \psi_1^{k_1} \ldots \psi_n^{k_n} \cap [1]^\text{log},
\]

where

\[
\text{ev} : \mathcal{M}_{g,n+n_1+n_2}(\mathcal{X}, \delta(\phi, \mu)) \rightarrow \mathcal{X}^n,
\]

is the product of all evaluation morphisms, and \( p \) is the specialization of the point \( (p_1, \ldots, p_n) \) chosen above.

By applying the decomposition formula for logarithmic Gromov–Witten invariants for point conditions [2, Theorem 6.3.9], this invariant can be written as a sum of the invariants associated to each tropical curve in the manner described. \( \Box \)

4. FLOOR DIAGRAMS VIA THE RELATIVE THEORY

Floor diagrams are connected, via combinatorial manipulations, both to relative descendant Gromov–Witten invariants and tropical descendant Gromov–Witten invariants. This section deals with the former. Floor diagrams naturally organize the computation of a relative descendant via the degeneration formula. A brief discussion of the connection between floor diagrams appearing here and in previous work is found in Section 4.2.

4.1. Floor diagrams. The enumerative geometry studied in this section is the relative descendant Gromov–Witten theory of \( F_k \), as outlined in Section 2.2

**Definition 4.1.** Let \( D \) be a loop-free connected graph on a linearly ordered vertex set. \( D \) has two types of edges: compact edges, composed of two flags (or half-edges), adjacent to different vertices, and unbounded edges, also called ends, with only one flag. \( D \) is called a floor diagram for \( F_k \) of degree \( (\phi, \mu) \) if:

1. Three non-negative integers are assigned to each vertex \( V \): \( g_V \) (called the genus of \( V \)), \( s_V \) (called the size of \( V \)) and \( k_V \) (called the \( \psi \)-power of \( V \)).
2. Each flag may be decorated with a thickening. We require that for each compact edge precisely one of its two half-edges is thickened.
3. At each vertex \( V \), \( k_V + 2 - 2s_V - g_V \) adjacent half-edges are thickened.
4. Each edge \( e \) comes with an expansion factor \( w(e) \in \mathbb{N}_{>0} \).
5. At each vertex \( V \), the signed sum of expansion factors of the adjacent edges (where we use negative signs for edges pointing to the left and positive signs for edges pointing to the right) equals \(-k \cdot s_V\).
Figure 6. An example of a floor diagram. The genus at all vertices is 0.

(6) The sequence of expansion factors of non-thick ends (where we use negative signs for the ends pointing to the left and positive signs for the ends pointing to the right) is \((\Phi)\), and the sequence of expansion factors of thickened ends (with the analogous sign convention) is \((\mu)\).

(7) The ends of the graph are marked by the parts of \((\phi, \mu)\).

The genus of a floor diagram is defined to be the first Betti number of the graph plus the sum of the genera at all vertices.

Example 4.2. Figure 6 shows a floor diagram for \(F_1\) of degree \((-2, 1), (-2, -1, 1)\) and genus 0.

Definition 4.3. Given a floor diagram for \(F_k\), let \(V\) be a vertex of genus \(g_V\), size \(s_V\) and with \(\psi\)-power \(k_V\). Let \((\phi_V, \mu_V)\) denote the expansion factors of the flags adjacent to \(V\); the first sequence encodes the normal half edges, the second the thickened ones. We define the multiplicity \(\text{mult}(V)\) of \(V\) to be the one-point stationary relative descendant invariant

\[
\text{mult}(V) = \langle (\phi_V^-, \mu_V^-) | \tau_{k_V}(\text{pt}) | (\phi_V^+, \mu_V^+) \rangle^{\text{rel}}_{g_V}.
\]

Definition 4.4 (Floor multiplicity for relative geometries). Fix discrete data as in Notation 2.2. We define:

\[
\langle (\phi_V^-, \mu_V^-) | \tau_{k_1}(\text{pt}) \cdots \tau_{k_n}(\text{pt}) | (\phi_V^+, \mu_V^+) \rangle^{\text{floor}}_{g}
\]

to be the weighted count of floor diagrams \(D\) for \(F_k\) of degree \((\phi, \mu)\) and genus \(g\), with \(n\) vertices with \(\psi\)-powers \(k_1, \ldots, k_n\), such that \(a\) equals the sum of all sizes of vertices, \(a = \sum_{V=1}^{n} s_V\).

Each floor diagram is counted with multiplicity

\[
\text{mult}(D) = \prod_{e \in \text{C.E.}} w(e) \cdot \prod_{V} \text{mult}(V),
\]

where the second product is over the set C.E. of compact edges and \(w(e)\) denotes their expansion factors; the third product ranges over all vertices \(V\) and \(\text{mult}(V)\) denotes their multiplicities as in Definition 4.3.

4.2. Motivation and relation to other work. For readers who are familiar with floor diagrams and their relation to tropical curves in \(\mathbb{R}^2\), we insert a section discussing the special aspects of the definition we use here, and their relation to common definitions in the literature.

Floor diagrams were introduced for counts of curves in \(\mathbb{P}^2\) by Brugallé-Mikhalkin [10], and further investigated by Fomin-Mikhalkin [20], leading to new results about node polynomials. The results were generalized to other toric surfaces, including Hirzebruch surfaces, in [5].

The main observation is that by picking horizontally stretched point conditions, the images of tropical stable maps contributing to a Gromov–Witten invariant become floor decomposed: this means that the dual subdivision of the Newton polygon is sliced (i.e. a refinement of a subdivision of the trapezoid by parallel vertical lines — see Figure 7). Floor diagrams are then obtained by shrinking
Figure 7. The subdivision of a floor decomposed tropical curve refines the sliced Newton polygon. Each strip corresponds to a floor, and the integral width of the strip is called the size of the floor.

Figure 8. The floors in the tropical stable map of Example 3.7(1), and the corresponding floor diagram.

each floor (i.e. a part of the plane tropical curve which is dual to a (Minkowski summand of a) slice in the Newton polygon) to a white vertex. Each floor contains precisely one marked point. Further marked points lie on horizontal edges which connect floors, the so-called elevators, and are represented with black vertices. Fixed horizontal ends are given a (double circled) vertex, while other horizontal ends are shrunk so that the diagram has no unbounded edges.

Example 4.5. In Figure 8 we revisit the tropical stable map observed in the first part of Example 3.7. The floors are circled by dashed lines. On the right-hand side we have the corresponding floor diagram. Following the convention in [7], fixed ends terminate with a double circle, and other ends are contracted to the corresponding black vertex.

For rational stationary descendant Gromov–Witten invariants, the floor diagram technique was studied by Block, Gathmann and the third author [7] (the diagrams are called ψ-floor diagrams). There are two main difference with respect to the primary case:

• descendant insertions force us to consider floor decomposed curves with floors of size larger than one. The size of a floor thus becomes part of the data of a floor diagram: each floor vertex comes with two numbers, the ψ-power $k_i$ of the corresponding marked point, and the size of the floor;
• marked points may now be supported at a vertex of the tropical stable map, and horizontal edges incident to such vertex are fixed by the point condition. This condition is encoded by thickening the corresponding half-edges in the floor diagram.

Example 4.6. Figure 9 illustrates the second part of Example 3.7. Some half-edges are thickened, indicating that the corresponding edge in the tropical curve leading to this diagram is adjacent to the marked point in the floor.

---

1The bizarre nomenclature makes intuitive sense if everything is rotated by 90°.
The language in [7] was modeled after the work by Fomin–Mikhalkin [20], which was motivated by a computational approach aiming at new results about node polynomials. Our current motivation to study floor diagrams comes from their connections to degeneration techniques and Fock space formalisms to enumerative geometry. Hence our definitions introduce the following modifications with respect to [20, 7]:

1. The distinction between floors and marked points on elevators is not needed anymore (to the contrary, it only complicates the combinatorics and clouds the connection to the Fock space). We do away with bi-colored vertices by considering marked points on elevators as floors of size zero. The adjacent half-edges have to be thickened, since they are adjacent to the marked point.
2. We thicken ends that correspond to a tangency at a non-fixed point, have unthickened ends for tangency to a fixed point (rather than marking the end with a double circle) and remove the vertices at the end of these edges.
3. We draw all elevator edges adjacent to marked points, as that allows us to record the complete tangency data for the invariant we are trying to compute (in the convention of [20, 7], obvious continuations of edges in the tropical curve are dropped in the floor diagram).

As an example, the floor diagram of Figure 9 becomes with our conventions the diagram in Figure 6.

4.3. Floor diagrams and degeneration. As seen in the previous section, tropical curves naturally arise by counting curves in maximal degenerations, while the correspondence from relative descendants to floor diagrams follows from the simpler “accordion” degeneration. This was originally observed and discussed in [9, 6]. We work in this section with Jun Li’s degeneration formula for relative stable maps. We recall Li’s theorem, stated in the specific geometric context that is of interest to us [31].

**Theorem 4.7.** Let \( \mathcal{X}_t \) be a flat family of surfaces such that the general fiber is a smooth Hirzebruch surface \( \mathbb{F}_k \) and the central fiber is the union of two surfaces \( S_1 \cup_D S_2 \) both isomorphic to \( \mathbb{F}_k \), meeting transversely along the divisor \( D = E_{S_1} = B_{S_2} \). Fix a two part partition of the set \([n + m]\): without loss of generality we may choose \([1, \ldots, n) \cup (n + 1, \ldots, n + m]\). Set discrete invariants \( g, n, k_1, \ldots, k_n, (\Phi, \mu) \) as in Notation 2.2.
Then:

$$\langle (\Phi^-, \mu^-) | \tau_{k_1}(pt), \ldots, \tau_{k_{n+m}}(pt) | (\Phi^+, \mu^+) \rangle_{\text{rel}^*}^{g_1} = \sum_{(\lambda_i \eta_j)} \frac{\prod \lambda_i \eta_j}{|\text{Aut}(\Lambda)||\text{Aut}(\eta)|},$$

$$\langle (\Phi^-, \mu^-) | \tau_{k_1}(pt), \ldots, \tau_{k_n}(pt) | (\lambda, \eta) \rangle_{\text{rel}^*}^{g_1} ((-\eta, -\lambda) | \tau_{k_{n+1}}(pt), \ldots, \tau_{k_{n+m}}(pt) | (\Phi^+, \mu^+)),$$

where $\Lambda = \lambda_1, \ldots, \lambda_r, \eta = \eta_1, \ldots, \eta_s$ are an $r$-tuple and an $s$-tuple of positive integers and the sum is over all discrete data $(g_1, g_2, (\eta, \Lambda))$ such that:

1. $\langle (\Phi^-, \lambda), (\mu^-, \eta) \rangle$ (resp. $\langle (-\eta, \Phi^+), (-\lambda, \mu^+) \rangle$) determines an effective curve class $a_1 B_{S_1} + b_1 F_{S_1}$ (resp. $a_2 B_{S_2} + b_2 F_{S_2}$) in $H_2(F_k, \mathbb{Z})$ with $a_1, a_2 \geq 0$, $a_1 + a_2 = a$, $b_1 = a_2 k + b$, $b_2 = b$;
2. $g = g_1 + g_2 + r + s - 1$.

Remark 4.8. The following details are important in parsing Equation (7):

1. The formula is organized as a sum over the gluing data $(\Lambda, \eta)$. Each term in the summand is however weighted by a factor of $\frac{1}{|\text{Aut}(\Lambda)||\text{Aut}(\eta)|}$, which corrects the overcounting coming from different labelings of points that give rise to the same gluing. More geometrically, one may think that in Equation (7) the sum is over the distinct topological types of maps (where the points that get glued are unlabeled), and the multiplicity of each summand omits the above factor.
2. The switching of the roles of $\Lambda$ and $\eta$ on the two sides of the product comes from the Kunneth decomposition of the class of the diagonal in $\mathbb{P}^1 \cong D$.

To realize the hypotheses of the theorem, one may start from a trivial family $\mathcal{X}_1 \times \Lambda^1 \to \Lambda^1$ together with $n + m$ non-intersecting sections $s_\nu$, the first $n$ staying away from $E$, the last $m$ meeting but not tangent to $E$ at $t = 0$; one obtains $\mathcal{X}_1$ by blowing up $E \times \{0\}$ and considering the proper transforms of the sections. This construction may be iterated a finite number of times, and Theorem 4.7 applies with the appropriate bookkeeping. This is what gives rise to the correspondence with the floor diagram count, as we make explicit in the next theorem.

Theorem 4.9. Fix a Hirzebruch surface $\mathbb{P}_k$ and discrete data as in Notation 2.2. The descendant log Gromov–Witten invariant coincides with the weighted count of floor diagrams from Definition 4.4

$$\langle (\Phi^-, \mu^-) | \tau_{k_1}(pt), \ldots, \tau_{k_n}(pt) | (\Phi^+, \mu^+) \rangle_{\text{floor}}^g = \langle (\Phi^-, \mu^-) | \tau_{k_1}(pt), \ldots, \tau_{k_n}(pt) | (\Phi^+, \mu^+) \rangle_{\text{rel}^*}^g.$$

Proof. As with many proofs based on iterated applications of the degeneration formula, a completely explicit and accurate bookkeeping would be extremely cumbersome and cloud the actual simplicity of the argument. We choose therefore to carefully outline the construction, and omit the bookkeeping.

Iterate the construction from the previous paragraph $n - 1$ times, each time separating exactly one section from all others. In the end one obtains a family $\mathcal{X}_1$ such that the general fiber is a smooth Hirzebruch surface $\mathbb{P}_k$ and the central fiber is the union of $n$ surfaces $S_0 = S_1 \cup D_1, S_2 \cup D_2, \ldots, S_{n-1} \cup D_{n-1}, S_n$, where all surfaces $S_i$ are isomorphic to $\mathbb{F}_k$ and $S_i$ and $S_{i+1}$ meet transversely along the divisor $D_1 = E_{S_1} = B_{S_{i+1}}$. For $V = 1, \ldots, n$, the section $s_V$ is obtained as the proper transform of the original section.

Applying the appropriately iterated version of Theorem 4.7, the stationary descendant invariant is expressed as a sum over the topological types of maps from nodal curves to the central fiber,
weighted by the appropriate (disconnected) relative Gromov–Witten invariants. Since each $S_i$ contains exactly one marked point, the disconnected maps to $S_i$ have one connected component hosting the marked point; by dimension reasons, the other components consist of rational curves mapping with degree $dF$ (multiple of the class of a fiber), and in fact mapping as a $d$-fold cover of a fiber, fully ramified at the points of contact with $D_{i-1}$ and $D_i$. Further, the relative conditions at the boundary must have one fixed point on one side, and a moving point on the other. The contribution of any such component to the disconnected invariant is $1/d$.

For every summand in the degeneration formula, consider the dual graph of the source curve, label each edge with the ramification order of the corresponding point, and thicken half edges corresponding to moving boundary point conditions. For every two-valent vertex adjacent to two flags of opposite thickening, contract the vertex and the two neighboring flags. We claim (and leave the verification to the patient reader) that the object thus obtained is a floor diagram for the stationary descendant invariant we are trying to compute, and further that this construction establishes a bijection between the summands in the degeneration formula and the floor diagrams described in Definition 4.4.

The proof is concluded by showing that each floor diagram is counted with the same multiplicity. The degeneration formula assigns the same vertex and compact edge multiplicities to the dual graphs of maps as the floor diagram enumerative count. The proof is then concluded by noticing that the operation of removing a two-valent vertex (which contributes with multiplicity $1/d$) and its two adjacent flags does not alter the multiplicity of the graph: for each such vertex removed we lose a compact edge of weight $d$, which contributes a factor of $d$ to the multiplicity of the graph.

Since the proof of the correspondence is based on a bijection between dual graphs of maps and floor diagrams that preserves connectedness, one immediately obtains the following corollary.

**Corollary 4.10.** The version of Theorem 4.9 for connected invariants also holds:

\begin{equation}
\langle (\phi^-, \mu^-) | \tau_{k_1}(pt), \ldots, \tau_{k_n}(pt) | (\phi^+, \mu^+) \rangle^\text{floor} = \langle (\phi^-, \mu^-) | \tau_{k_1}(pt), \ldots, \tau_{k_n}(pt) | (\phi^+, \mu^+) \rangle^\text{rel}.
\end{equation}

5. **Floor diagrams via the tropical theory**

In this section, we examine the effect of horizontally stretched constraints on tropical maps. Tropical maps meeting such constraints are floor decomposed, and better behaved than for general conditions. This allows us to prove a degeneration formula for the logarithmic descendants, expressing them as a product over vertices. We use this to present a direct weighted bijection between counts of tropical stable maps and floor diagrams.

If we consider the enumerative problem with $k_i = 0$ for all $i$, then the tropical descendant invariant considered above is nothing but a count of tropical plane curves satisfying point conditions. For such a count, it is well-known, see for instance [36], that all tropical stable maps $(\Gamma, f)$ that contribute with nonzero multiplicity have $g$ cycles which are visible in the image $f(\Gamma)$ and have only trivalent vertices. It then follows that their spaces of deformations have the expected dimension, because the $g$ visible cycles impose $2g$ linearly independent conditions in the orthant parametrizing all lengths on bounded edges. Hence superabundancy is not present for tropical plane curve counts.

The primary tool in this section is a specialization of the tropical points into special configuration.
Definition 5.1. A set $p_1, \ldots, p_n \in \mathbb{R}^2$, with $p_i = (x_i, y_i)$ is said to be in **horizontally stretched position** if

- The $x$-coordinates increase, i.e. $x_i < x_{i+1}$ for all $i$.
- The $y$-coordinates increase, i.e. $y_i < y_{i+1}$ for all $i$.
- The $x$ coordinates are much larger than the $y$-coordinates, $\min_{i \neq j} |x_i - x_j| \gg \max_{i \neq j} |y_i - y_j|$.

As observed by Brugallé, Fomin, and Mikhalkin, the choice of distinguished stretching direction determines a distinguished decomposition of a tropical map.

Definition 5.2. Let $\Gamma \to \mathbb{R}^2$ be a tropical stable map. An **elevator edge** is an edge in $\Gamma$ whose image has edge direction parallel to $(1, 0)$. A **floor** of a tropical map $\Gamma \to \mathbb{R}^2$ is a connected component in $\Gamma$ of the complement of all elevator edges.

Note that if a descendant lies on an elevator edge, the corresponding vertex is a floor supporting the marked point. The following proposition shows the usefulness of the notions above.

Proposition 5.3. Let $\Gamma \to \mathbb{R}^2$ be a rigid tropical stable map meeting horizontally stretched stationary descendant constraints. Each floor of $\Gamma$ meets and is fixed by exactly one stationary descendant constraint.

Proof. The proof follows from identical arguments as in [20].

A tropical stable map meeting the horizontally stretched descendant constraints will be called **floor decomposed**.

5.1. **Excluding superabundant maps.** When descendant insertions are allowed, even if all cycles are visible in the image, they do not need to impose linearly independent conditions. The existence of superabundant tropical stable maps satisfying the conditions implies the existence of rigid tropical stable maps with ”additional overvalency”, e.g. a 4-valent vertex which is not adjacent to a marked point (see Example 3.10 in [22]). In what follows, we exclude such behavior for the case of floor-decomposed tropical stable maps.

Lemma 5.4. Let $(\Gamma, f)$ be a rigid floor decomposed tropical stable map, satisfying horizontally stretched conditions. Assume $g'$ independent cycles are visible in the image $f(\Gamma) \subset \mathbb{R}^2$, then these cycles impose $2g'$ linearly independent conditions, i.e. the space of deformations of $(\Gamma, f)$ is of codimension $2g'$ in the orthant parametrizing all lengths on bounded edges of $\Gamma$.

Proof. The space of deformations is cut out by $2g'$ equations in $\mathbb{R}^b$, where $b$ denotes the number of bounded edges. We show that the $2g'$ equations are independent. Since $(\Gamma, f)$ is rigid, there cannot be cycles contained in a floor (each floor is fixed by only one point condition). The $g'$ cycles thus have to be between two floors each, and thus each involve at least two elevator edges. We can put an arbitrary order on the pairs of floors, and then on the cycles involving the same two floors as imposed by the maximal $y$-coordinates of the elevator edges. We aim at producing an upper triangular matrix for the $2g'$ equations, i.e. we order the cycles (each in charge of two rows of the matrix), and the bounded edges (each in charge of one column) in such a way that we produce a diagonal with nonzero entries, and with zeros below it. For cycles involving the same two floors, we start with the cycle involving an elevator with maximal $y$-coordinate. For the columns, we pick this elevator edge, and second a column of an edge in the leftmost floor of the cycle which is not needed by any other cycle. The second elevator edge of the cycle is possibly part of another cycle,
if yes, we pick this cycle next. In terms of columns, we pick the elevator edge shared by the first two cycles, plus an edge in the floor which is not used by any other cycle. Continuing like this, we obtain a diagonal with nonzero entries, and zeros below it. For cycles involving another pair of floors, we repeat the procedure. With this choices, the coefficients of the linear equations cutting out the space of deformations of \((\Gamma, f)\) from the orthant parametrizing all lengths of bounded edges produce an upper triangular matrix; this implies the deformation space is of codimension \(2g^*\) as expected.

**Proposition 5.5.** If a floor decomposed \((\Gamma, f)\) contributes with non-zero multiplicity to a descendant Gromov–Witten invariant, then all cycles of \(\Gamma\) are visible in \(f(\Gamma)\). That is, no cycles are mapped to a line segment or contracted to a point.

*Proof.* Since we assume that \((\Gamma, f)\) contributes with non-zero multiplicity, it has to be rigid. If a cycle of \(\Gamma\) was contracted to a point, then \((\Gamma, f)\) would not be rigid because the lengths of edges of the contracted cycle can be varied without changing the image \(f(\Gamma)\). We could vary \((\Gamma, f)\) in an at least one-dimensional family still meeting the point and \(y\)-coordinate conditions.

Assume a cycle of \(\Gamma\) is mapped to a line segment \(S\). Let \(V\) be a vertex of the cycle mapping to an endpoint of \(S\). We may assume that no marking is incident to the vertex \(V\): by the genericity of the incidence conditions only one side of the flat cycle supports a marked point, and since we ruled out completely contracted cycles, the segment \(S\) has two distinct endpoints.

Now we argue that \(\Gamma\) does not contribute to the logarithmic invariant. If \(V\) is trivalent, it follows by the balancing condition that all three edges are mapped to the same line. The lengths of the three edges may be varied in such a way that the image \(f(V)\) moves along the line, but \(f(\Gamma)\) remains unchanged (see Figure 10). The resulting tropical stable maps would still satisfy all the conditions. Thus \((\Gamma, f)\) is not rigid.

The local picture in the remaining case is exemplified by the vertex \(V\) in Figure 11. Let \(W\) be the edge on the other side of the flat cycle. If neither \(V\) nor \(W\) carry a marked point, then the invariant vanishes because the (virtual) dimension of the associated moduli space of maps meeting the constraints is positive. Indeed, by moving one of the edges of the flat cycle away from the other, we obtain a one-dimensional family of maps meeting the constraints, and since the cycle spans \(\mathbb{R}^2\) in each of these deformations, the resulting deformations of the logarithmic map are unobstructed and there is a virtually positive dimensional family of logarithmic maps meeting the conditions.

Assume that the vertex \(W\) carries the marked point \(p\), and therefore the unique point fixing the floor to which \(W\) belongs is supported on \(W\). Let \(k\) be the power of the \(\psi\)-class at the marked point \(p\). We claim that \(m_{(\Gamma, f)}\) vanishes.

Assume first the flat cycle is horizontal. Consider the union of all non-elevator edges of \(\Gamma\) that are connected to \(W\), i.e., the floor of \(W\). This floor can be projected onto the \(y\)-axis, giving rise to a cover of \(\mathbb{R}\). Let \(C_W\) be a curve dual to the floor of \(W\) and \(C_W \to \mathbb{F}_k\) the associated logarithmic map. By translating \(W\) to the origin, we may assume that \(C_W\) maps to the dense torus orbit in \(\mathbb{F}_k\). The projection from \(\mathbb{R}^2\) to \(\mathbb{R}\) onto the \(y\)-axis induces a projection \(\mathbb{F}_k \to \mathbb{P}^1\), and hence a logarithmic map \(C_W \to \mathbb{P}^1\). Let \(M_W(\mathbb{F}_k)\) and \(M_W(\mathbb{P}^1)\) be the associated moduli space of logarithmic maps.

We may interpret the point condition attached to \(W\) as the intersection two conditions: that \(p\) passes through a fiber of this projection \(\mathbb{F}_k \to \mathbb{P}^1\), and that \(p\) passes through the closure of a subtorus that is transverse to this fiber. We may remove the subtorus condition by studying logarithmic maps to \(\mathbb{F}_k\) up to the translation action of the fiber torus. Let \(M_W(\mathbb{F}_k)^-\) denote this space of logarithmic maps up to the fiberwise torus action c.f. [35].
Tropical maps with a trivalent vertex at the end of a hidden cycle are not rigid.

A rigid superabundant tropical stable map of multiplicity zero.

In the space $M_W(\mathbb{P}_k)^\sim$, the stationary constraint at $W$ translates to $W$ intersecting a fixed fiber of the projection. The edges that are incident to $W$ that are a part of the flat cycle have degree 0 in the projection. Consider the map of moduli spaces $M_W(\mathbb{P}_k)^\sim \to M_W(\mathbb{P}^1)$. The logarithmic tangent bundle of $\mathbb{P}_k$ is naturally isomorphic to the trivial bundle $\mathbb{N} \otimes \mathbb{C}$ where $\mathbb{N}$ is the cocharacter lattice. The projection $\mathbb{P}_k \to \mathbb{P}^1$ thus induces a corresponding quotient of the logarithmic tangent bundle of $\mathbb{P}_k$. It follows that the induced map on moduli spaces carries a logarithmic relative perfect obstruction theory of expected dimension $-g$. Since the valency of $W$ in the projection decreases by at least 2, the locus of maps to $\mathbb{P}^1$ that factor through the projection has virtual dimension at most $k-1$. Upon imposing the stationary constraint $ev_p^*([pt])\psi^k$ the virtual dimension is negative. The projection formula now guarantees the requisite vanishing.

If the flat cycle is not horizontal, the argument above may be adapted as follows. As before, translate $W$ to the origin in $\mathbb{R}^2$. Let $\rho$ be the ray spanned by the image of the flat cycle. Make a toric modification $X_\rho$ of the surface $\mathbb{P}_k$ such that the projection morphism

$$\mathbb{R}^2 \to \mathbb{R}^2/\langle \rho \rangle = \mathbb{R}$$

is a morphism of fans, where $\mathbb{R}$ is given the structure of the fan of $\mathbb{P}^1$. The virtual dimension of the moduli space of maps is unchanged under this modification by [4]. The argument above may now be repeated – in the induced projection $X_\rho \to \mathbb{P}^1$, the fact that the valency at the vertex $W$ drops by 2 implies, via the projection formula, that the invariant vanishes.

**Example 5.6.** Figure 11 shows an example of a rigid superabundant tropical stable map which has multiplicity zero. The figure is supposed to reflect both the image of the stable map and the parametrizing abstract graph — we draw two edges close together if their images in $\mathbb{R}^2$ coincide.

Combining Lemma 5.4 and Proposition 5.5, we deduce the following non-trivial fact:
**Corollary 5.7.** For horizontally stretched point conditions leading to floor-decomposed tropical stable maps, any \((\Gamma, f)\) that contributes with non-zero multiplicity to a tropical descendant Gromov–Witten invariant is non-superabundant.

*Proof.* From Proposition 5.5 we can conclude that all cycles of \(\Gamma\) are visible in the image \(f(\Gamma)\). From Lemma 5.4 we can conclude that they form independent conditions. It follows that the space of deformations of \((\Gamma, f)\) is of the expected dimension, and hence \((\Gamma, f)\) is not superabundant. \(\square\)

**Remark 5.8.** For tropical stable maps to \(\mathbb{R}^n\) with \(n \geq 3\), there is no analogous statement known, i.e. it is not known whether there is a configuration of points such that all tropical stable maps (of non-zero multiplicity) satisfying the conditions are not superabundant, or even if there is a configuration of points forcing all cycles to be visible, which is a much weaker condition. It would be interesting to study the effect of floor stretched conditions in higher dimensions.

**Lemma 5.9.** Let \((\Gamma, f)\) be a floor-decomposed tropical stable map contributing to a tropical descendant Gromov–Witten invariant with non-zero multiplicity. Then every vertex of \(\Gamma\) which is not adjacent to a marked end is trivalent and of genus 0.

*Proof.* Let \((\Gamma, f)\) be a floor-decomposed tropical stable map with non-zero multiplicity. Assume the marked points with \(\mathfrak{i}\)-conditions \(k_1, ..., k_n\) are at vertices of genus \(g_1, ..., g_n\), and accordingly, of valence \(k_i + 3 - g_i\).

By Corollary 5.7, \((\Gamma, f)\) is not superabundant. The number of edges in the graph \(\Gamma\) is

\[ n + n_1 + n_2 + 2a - 3 + 3(g - g_1 - \ldots - g_n) - \sum V (\text{val}(V) - 3), \]

which follows from an Euler characteristic computation. The space of deformations of \((\Gamma, f)\) has dimension:

\[ 2 + \#(\text{edges}) - 2 \cdot \#(\text{visible cycles}) = n + n_1 + n_2 + 2a - 1 + (g - g_1 - \ldots - g_n) - \sum V (\text{val}(V) - 3) = n + n_1 + n + \sum_i (k_i - g_i) - \sum V (\text{val}(V) - 3) \]

by the requirement on the conditions. Since \(\sum_v (\text{val}(V) - 3) = \sum_i (k_i - g_i) + \sum_{V'} (\text{val}(V') - 3)\) (where now the sum goes over all vertices \(V'\) which are not adjacent to one of the marked ends \(i\)) by the valency conditions, and since the \(y\)-coordinates of \(n_1\) ends are fixed and \(n\) generic point conditions are satisfied, the dimension has to be at least \(n_1 + 2n\), which can only be satisfied if any vertex besides the ones adjacent to the marked ends, is trivalent and of genus 0. \(\square\)

### 5.2. Deformations of rigid floor decomposed maps.

Let \((\Gamma, f)\) be a floor decomposed, rigid tropical stable map contributing to a stationary descendant invariant. We wish to express the virtual multiplicity \(m_{(\Gamma, f)}\) attached to such a floor decomposed curve in terms of its vertices. Specifically, each vertex \(V\) of \(\Gamma\) gives rise to discrete data for a logarithmic stable map. Distributing the stationary conditions, as well as boundary conditions along outgoing edges at \(V\), we obtain a candidate for local multiplicities which should comprise \(m_{(\Gamma, f)}\). In order to carry out this strategy, we need to understand the non-rigid tropical maps nearby \((\Gamma, f)\).

**Proposition 5.10.** Let \((\Gamma, f)\) be a rigid tropical map as above and let \((\Gamma', f')\) be a deformation of it, meeting the same stationary descendant constraints. Then, the image of \(f\) coincides with the image of \(f'\).
Proof. By the lemmas in the previous section, we see that \((\Gamma, f)\) must be trivalent and of genus 0 away from its marked ends. Thus, the only deformations of \((\Gamma, f)\) occur locally near a vertex of high valency or genus. We will show that locally near such a vertex, no deformation that changes the image of \(f\) can meet the stationary descendant conditions.

Assume that \(\Gamma\) has a single vertex \(V\) of genus \(g\), supporting a marked point, and having valency \(r\). Let \(k\) be the power of the descendant attached to the marking. Then we have the equality 
\[
    r - 3 + g = k.
\]

The image of \(f\) is dual to a Newton polygon \(\Delta_f\) with at most \(r\) sides and at least \(g\) interior lattice points. The deformations of \((\Gamma, f)\) that change the image correspond to a subdivision \(\Delta'_f\) of the Newton polygon \(\Delta_f\). Recall that the 2-dimensional polygons contained in the subdivision \(\Delta'_f\) are dual to the vertices in the deformed tropical (image) curve. Moreover, the genus of each such vertex is equal to the number of interior lattice points. In order to meet the descendant condition, we must produce a subdivision \(\Delta'_f\) such that for some vertex \(V'\) dual to a polygon in the subdivision, we achieve the equality
\[
    \text{val}(V') - 3 + g(V') = k.
\]

The vertices of the polygons in \(\Delta'_f\) can include the interior lattice points of \(\Delta_f\). We first deal with the case when there are no parallelograms in the subdivision. Then any interior point that is used produces a visible cycle in the image. Assume \(g_1\) interior points are used in constructing \(\Delta'_f\). In the resulting tropical curve, any vertex \(V'\) can have genus at most \(g - g_1\). A polygon in the subdivision can have at most \(g_1 + 1\) edges that are not edges of \(\Delta'_f\), i.e. ”new edges” that appear in the subdivision. Furthermore, any polygon in the subdivision can have at most \(r - 2\) boundary edges of \(\Delta_f\). Thus, we see that
\[
    (r - 2 + g_1 + 1) - 3 + (g - g_1) < k
\]
and the deformation cannot meet the descendant condition.

Now assume that there is a parallelogram in \(\Delta'_f\). Here, it is not necessarily the case that the genus of the deformed curve is smaller. However, the parallelogram is dual to two edges of the tropical curve crossing, whose dual edges must be adjacent to different polygons in the subdivision. Thus, a vertex can have at most \(r - 3\) edges dual to boundary edges of \(\Delta_f\), and we again have the inequality
\[
    (r - 3 + g_1 + 1) - 3 + (g - g_1 + 1) < k.
\]
Thus, we see that any parallelogram in \(\Delta'_f\) reduces the valency, while polytopes in \(\Delta'_f\) other than parallelograms reduce the genus. In all cases, it is impossible that the deformed curve continues to meet the descendant condition. Thus, the only deformations of \((\Gamma, f)\) must leave the image unchanged, and the proposition follows. \(\square\)

5.3. Tweaking the logarithmic moduli space. The consequence of Proposition 5.10 is that all tropical stable maps – not just rigid ones – that contribute to a stationary descendant Gromov–Witten invariant have the same image as a rigid map. Thus, there are only finitely many images of tropical curves that we need consider in the enumerative problem. This allows us to use an elegant idea developed by Gross–Pandharipande–Siebert [27] to relate the logarithmic stationary descendants of the Hirzebruch surface \(\mathbb{F}_k\) to the stationary descendants of an open geometry obtained by first degenerating \(\mathbb{F}_k\) to accommodate all tropical curves, and then deleting its codimension 2 strata. The resulting moduli space will not be complete, but retains sufficient properness to support the stationary descendants because of the results of the previous section. This in turn will give us access to the degeneration formula.
Fix the numerical data defining a tropical stationary descendant Gromov–Witten invariant. Choose floor decomposed points, and let \((\Gamma_1, f_1), \ldots, (\Gamma_s, f_s)\) be the rigid tropical curves meeting these conditions. Let \(\mathcal{P}\) be a polyhedral decomposition of \(\mathbb{R}^2\) such that for each \(i\), the image

\[ f_i : \Gamma_i \to \mathbb{R}^2 \]

is contained in the 1-skeleton of \(\mathcal{P}\).

**Proposition 5.11.** Let \(f' : \Gamma' \to \mathbb{R}^2\) be any tropical stable map meeting the given stationary descendant constraints as above. Then \(f'\) factors through the one skeleton of \(\mathcal{P}\).

**Proof.** This is a restatement of Proposition 5.10 in the previous section. \(\square\)

Let \(\mathcal{X}\) be the special fiber of the toric degeneration associated to \(\mathcal{P}\), and let \(\mathcal{X}^\circ\) denote the complement of the codimension 2 strata in the degeneration. We consider \(\mathcal{X}^\circ\) as a logarithmic scheme over \(\text{Spec}(\mathbb{N} \to \mathbb{C})\) equipped with its divisorial logarithmic structure.

Let \((\Gamma, f)\) be rigid and let \(\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ)\) denote the space of logarithmic stable maps to \(\mathcal{X}^\circ\) equipped with a marking by \((\Gamma, f)\) in the sense of Section 3.3. Choose logarithmic lifts \(p_1^\dagger = (p_1^\dagger, \ldots, p_n^\dagger)\) of the points \((p_1, \ldots, p_n) \in \mathbb{R}^2\). Let \(\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger)\) be the moduli space of logarithmic stable maps passing through these points.

This space admits a virtual fundamental class \([\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger)]_{\text{vir}}\) in Borel–Moore homology. Moreover, the stationary descendant logarithmic invariant is rationally equivalent to a class supported on \(\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger)\).

We view the space \(\mathcal{X}^\circ\) as a logarithmically smooth scheme with a rank 1 logarithmic structure, since the higher rank loci are the (at least) threefold intersections of components which have been removed. As a consequence, we may replace the logarithmic stable maps moduli space \(\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger)\) with the space \(\overline{M}_{(\Gamma, f)}^\exp(\mathcal{X}^\circ, p^\dagger)\) of relative maps to logarithmic expanded degenerations of \(\mathcal{X}^\circ\), equipped with a marking of the source by \((\Gamma, f)\), see [30]. Since it mainly serves as a technical tool, we only briefly mention how to work with the space.

The invariant curves in \(\mathcal{X}^\circ\) give it the structure of a logarithmic scheme with divisorial log structure over \(\text{pt}^\dagger = \text{Spec}(\mathbb{N} \to \mathbb{C})\). As such, one may consider maps from logarithmic curves \(C/S \to \mathcal{X}^\circ/\text{pt}^\dagger\) that meet the strata transversely. This space may be compactified by considering maps to expanded degenerations \(\mathcal{X}^\circ \to \mathcal{X}^\circ\) obtained by logarithmic blowings up of the invariant curves, with an appropriate transversality condition just as in the relative geometry. The resulting space is once again virtually smooth with a virtual class \([\overline{M}_{(\Gamma, f)}^\exp(\mathcal{X}^\circ, p^\dagger)]_{\text{vir}}\) in the expected dimension.

By forgetting the expansion, stabilizing the map as necessary, and pushing forward the logarithmic structure, we obtain a morphism

\[ \mu : \overline{M}_{(\Gamma, f)}^\exp(\mathcal{X}^\circ, p^\dagger) \to \overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger). \]

The following result of Abramovich–Marcus–Wise allows us to work with the expanded theory without changing the enumerative problem [3].

**Proposition 5.12.** There is an equality of classes

\[ \mu_*[\overline{M}_{(\Gamma, f)}^\exp(\mathcal{X}^\circ, p^\dagger)]_{\text{vir}} = [\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger)]_{\text{vir}} \in H^*_{BM}(\overline{M}_{(\Gamma, f)}(\mathcal{X}^\circ, p^\dagger))_\mathbb{Q}. \]
We define the descendant logarithmic Gromov–Witten invariant, as a cycle, attached to the expanded moduli space in the same manner as before by capping the virtual class of the space of maps through points with descendants. The moduli space of maps to $\mathcal{X}^o$ includes into the moduli space of maps to $\mathcal{X}$ by composition. By the arguments in previous section, the difference
\[
\left( \prod_{j=1}^{n} \psi_j^{k_j} \cap [\overline{\mathcal{M}}(\Gamma,f)(\mathcal{X}^o,p^\dagger)]^{vir} \right)
\]
is equal to the stationary descendant invariant in question. Indeed, any logarithmic map meeting the stationary descendant constraints has to have a tropicalization that factors through the one skeleton of $\mathcal{P}$, and thus lie in the open target. Thus, the invariant coming from the expanded moduli space has a well-defined degree.

5.4. **Vertex multiplicities.** The virtual class of $\overline{\mathcal{M}}^{exp}(\mathcal{X}^o,p^\dagger)$ satisfies a degeneration formula analogous to the one used in the previous section, proved by Q. Chen [18]. This allows us to write the virtual multiplicity $m_{(\Gamma,f)}$ in terms of the vertices of $(\Gamma,f)$.

Let $(\Gamma,f)$ be a rigid tropical stable map as above, contributing to a stationary descendant Gromov–Witten invariant. Orient the edges of $\Gamma$ minus the marked ends in each component towards the unique non-fixed end.

**From oriented edges to boundary incidence conditions.** Locally around each vertex $V$ of $\Gamma$, the directions of the adjacent flags define a Newton fan $\delta_V$. We let $\delta_\phi$ be the subset given by all entries of $\delta_V$ corresponding to edges which are oriented towards $V$, and $\delta_\mu$ consist of the vectors in $\delta_V$ oriented away from $V$.

Let $\overline{\mathcal{M}}_V$ is the moduli space of maps to the open surface $X^o_V$ determined by the local picture near $V$, such that if an edge is incoming, we consider maps that pass through a pre-determined point of the corresponding boundary curve.

More precisely, at a vertex $V$ there is a corresponding moduli space of maps $f : C_V \to X_V$, where the surface $X_V$ is determined by the Newton fan $\delta_V$ and the genus, marked points, and contact orders are given by the star of $V$ in $(\Gamma,f)$. We consider the open toric surface $X^o_V$ obtained by deleting the torus fixed point. Moreover, let $\overline{\mathcal{M}}_V$ be the moduli space of relative maps to expansions of $X^o_V$, with the boundary incidences specified by the orientation as follows. If an edge is oriented towards $V$, we consider maps passing through a fixed point of the corresponding boundary curve.

This moduli space of maps admits a virtual fundamental class $[\overline{\mathcal{M}}_V]^{vir}$.

**Definition 5.13.** Define the local vertex multiplicity at $V$ to be

$$\operatorname{mult}_V(\Gamma,f) = \langle \tau_{k_i}(pt) \rangle_{\delta_\phi \cup \delta_\mu, g_V} := \int [\overline{\mathcal{M}}_V]^{vir} \psi_i^{k_i} \cdot \ev^*([pt]).$$

if the marked end $i$ is adjacent to $V$ and

$$\operatorname{mult}_V(\Gamma,f) = \langle \rangle_{\delta_\phi \cup \delta_\mu, g_V} := \deg([\overline{\mathcal{M}}_V]^{vir})$$

otherwise. Here $g_V$ denotes the genus of $\Gamma$ at $V$.

Since we require the marked ends to meet distinct points, there cannot be more than one end adjacent to a vertex $V$. 
Note that the arguments in the previous section guarantee that the possible degenerations of the local map near \( V \) that satisfy the valency and incidence conditions are contained in the a priori non-compact space of maps \( \overline{M}_V \).

Given an edge \( e \) of a rigid tropical curve \((\Gamma, f)\), we let \( w(e) \) denote the expansion factor along the edge.

**Remark 5.14.** The only possibly non-vanishing local vertex multiplicities happen when the virtual dimension of the moduli space of logarithmic stable maps equals 0 in the case of an unmarked vertex, and \( k_i + 2 \) for a vertex adjacent to the \( i \)-th mark. Let \( V \) denote a vertex whose star gives the Newton fan \( \delta \). Let \( \delta_\phi \cup \delta_\mu = \delta \) be an arbitrary two-part partition of \( \delta \), and let \( M_V \) the moduli space of logarithmic stable maps identified by this data. The virtual dimension is:

\[
\text{virdim}(M_V) = g - 1 + \text{val}(v) - \ell(\phi)
\]

If \( V \) is an unmarked vertex, using \( \text{val}(V) = \ell(\phi) + \ell(\mu) \), it follows that for the virtual dimension of \( M_V \) to equal 0,

\[
\ell(\mu) = 1 - g.
\]

We showed in Lemma 3.5 that unmarked vertices are rational, and therefore \( \ell(\mu) = 1 \).

If \( v \) is adjacent to the \( i \)-th marked leg, recall that \( \text{val}(v) = k_i + 3 - g \). Therefore, for the virtual dimension of \( M_v \) to be \( k_i + 2 \) it must be that \( \ell(\phi) = 0 \).

**Proposition 5.15.** The virtual multiplicity \( m_{(\Gamma, f)} \) can be written in terms of the local vertex multiplicities of \((\Gamma, f)\). Specifically,

\[
m_{(\Gamma, f)} = \frac{1}{|\text{Aut}(f)|} \cdot \prod_{e: \text{C.E.}} w(e) \cdot \prod_V \text{mult}_V(\Gamma, f).
\]

where C.E. stands for compact edge.

**Proof.** The proof is a standard application of the degeneration formula [18], essentially identical to the proof already outlined in Theorem 4.9. As noted above, the orientation chosen is the only way in which to obtain a nonzero invariant, since the virtual dimension at each vertex must be zero to obtain a nonzero contribution in the degeneration formula. The product of edge expansion factors comes directly from the statement of the formula, while the automorphism factor arises from passing to a rigidified moduli space where the maps can be uniquely decomposed into their constituent components. We leave the bookkeeping to the reader. \( \square \)

**Remark 5.16.** Assume a floor-decomposed rigid tropical stable map \((\Gamma, f)\) contributes with non-zero multiplicity to a tropical descendant invariant \( \langle [\phi^-] [\phi^+] \cdot \tau_{k_1}(p_1) \cdots \tau_{k_n}(p_n) | [\phi^+] [\phi^-] \rangle_{\text{trop}} \) as in Definition 3.3. Let \( V_1, \ldots, V_l \) be the vertices that do not support a marked point. By Lemma 5.9, these vertices are trivalent. Then the factor

\[
\prod_{e: \text{C.E}} w(e) \cdot \prod_{i=1}^{l} \text{mult}_{V_i}(\Gamma, f)
\]

appearing in the multiplicity \( m_{(\Gamma, f)} \) is equal to the product of all (normalized) areas of triangles dual to the trivalent non-marked vertices in the dual subdivision, divided by the weights of fixed ends. Indeed, the Gromov–Witten invariant at trivalent vertices is 1, and the gluing factors and edge weights together contribute the product of areas above, as follows from [23, 34, 33].
5.5. Logarithmic floor multiplicity. We keep the definition of a floor diagram used in the previous section but we change the multiplicity from a relative invariant to the corresponding logarithmic invariant.

Warning 5.17. In what follows, to avoid overburdening the notation, we repurpose the symbols from the previous section, replacing the relative multiplicities with their logarithmic multiplicities.

Definition 5.18. Given a floor diagram for \( \mathbb{F}_k \), let \( V \) be a vertex of genus \( g_V \), size \( s_V \) and with \( \psi \)-power \( k_V \). Let \( (\Phi_V, \mu_V) \) denote the expansion factors of the flags adjacent to \( V \); the first sequence encodes the normal half edges, the second the thickened ones. We define the logarithmic multiplicity \( \text{mult}(V) \) of \( V \) to be the one-point stationary logarithmic descendant invariant

\[
\text{mult}(V) = \langle (\Phi_V, \mu_V) | \tau_{k_V}(pt) | (\Phi_V, \mu_V) \rangle_{g_V}^{\text{log}}.
\]

Definition 5.19 (Floor multiplicity for logarithmic geometries). Fix discrete data as in Notation 2.2. We define:

\[
\langle (\Phi_V, \mu_V) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\Phi_V, \mu_V) \rangle_{g}^{\text{floor}}
\]

to be the weighted count of floor diagrams \( D \) for \( \mathbb{F}_k \) of degree \( (\Phi, \mu) \) and genus \( g \), with \( n \) vertices with \( \psi \)-powers \( k_1, \ldots, k_n \), such that \( a \) equals the sum of all sizes of vertices, \( a = \sum_{V=1}^{s} s_V \).

Each floor diagram is counted with multiplicity

\[
\text{mult}(D) = \prod_{e \in \text{C.E.}} w(e) \cdot \prod_{V} \text{mult}(V),
\]

where the second product is over the set \( \text{C.E.} \) of compact edges and \( w(e) \) denotes their expansion factors; the third product ranges over all vertices \( V \) and \( \text{mult}(V) \) denotes their multiplicities as in Definition 5.18.

Theorem 5.20. Fixing all discrete invariants as in Notation 2.2, the weighted count of floor diagrams equals the tropical descendant log Gromov–Witten invariant, i.e. we have

\[
(\Phi_V, \mu_V) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\Phi_V, \mu_V) \rangle_{g}^{\text{floor}} = \langle (\Phi_V, \mu_V) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\Phi_V, \mu_V) \rangle_{g}^{\text{trop}}.
\]

Proof. The proof of this theorem is in two parts. Construction 5.21 associates a floor diagram to a floor decomposed tropical stable map contributing to \( \langle (\Phi_V, \mu_V) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\Phi_V, \mu_V) \rangle_{g}^{\text{trop}} \). By Proposition 5.22, the weighted number of tropical stable maps yielding the same floor diagram \( D \) under this procedure equals the multiplicity \( \text{mult}(D) \) from Definition 5.19. \( \square \)

5.6. Constructing floor diagrams from tropical curves. Let \( (\Gamma, f) \) be a rigid floor decomposed tropical stable map contributing to

\[
(\Phi_V, \mu_V) | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | (\Phi_V, \mu_V) \rangle_{g}^{\text{trop}}.
\]

Because of the horizontally stretched point conditions, each marked point is either on a horizontal edge (resp. elevator edge) of \( f(\Gamma) \) (i.e. an edge of primitive direction \((1, 0)\)) or on a part dual to a slice in the Newton subdivision. On each part dual to a slice, there is exactly one marked point. Consider the preimage in \( \Gamma \) under \( f \) of a part dual to a slice, this is a subgraph that we call \( \Gamma' \). Assume the slice in the Newton polygon has width \( s > 0 \) (i.e. in plane coordinates, it is a slice between the lines \( x = i \) and \( x = i + s \) for some \( i \)). Since the image of \( \Gamma' \) is fixed by exactly one point (and conditions on the coordinates of its horizontal edges), \( \Gamma' \) consists of only rational connected components. Furthermore, all but one of these components is just one edge which is mapped horizontally. This connected component (which contains \( s \) ends of direction \((0, -1)\) and \( s \) ends of direction \((k, 1)\)) is called a floor of size \( s \). We refer to other connected components as horizontal edges passing through the floor. For an example, see Figures 8 and 9.
Construction 5.21. Let \((\Gamma, f)\) be a non-superabundant floor decomposed tropical stable map contributing to \(\langle (\Phi^-_i, \mu^-_i) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_i, \mu^+_i) \rangle_{\text{trop}}\).

We associate a floor diagram \(D\) contributing to \(\langle (\Phi^-_i, \mu^-_i) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_i, \mu^+_i) \rangle_{\text{floor}}\) to \((\Gamma, f)\) by contracting each floor to a vertex; also marked points adjacent to only horizontal edges are considered vertices. The vertices are equipped with:

- the \(\psi\)-power \(k_i\) of the adjacent marked point \(i\),
- the size \(s_i\) (i.e. the width) of the dual slice of the Newton polygon for vertices corresponding to a floor; \(s_i = 0\) for marked points on elevators,
- the genus \(g_i\) of the vertex adjacent to the marked end \(i\) in the tropical curve.

We thicken flags if they come from half-edges of \(f(\Gamma)\) which are adjacent to a marked point.

Proof. We show that Construction 5.21 yields a floor diagram of the right degree and genus. Because of the horizontally stretched point conditions, we obtain a graph \(D\) on a linearly ordered vertex set.

The balancing condition satisfied by \((\Gamma, f)\) implies that the signed sum of expansion factors of edges adjacent to vertex \(i\) of the floor diagram equals \(-k \cdot s_i\).

By Lemma 3.5, removing from the subgraph underlying a floor of size \(s_i\) the marked end \(i\) together with its end vertex yields connected components each containing at most one of the \(2s_i\) ends of direction \((0, -1)\) resp. \((k, 1)\). It follows that the valence of the vertex adjacent to the \(i\)-th mark is \(2s_i\) plus one (for the marked end itself) plus the number of adjacent horizontal edges. The latter correspond to the thick flags in the floor diagram \(D\). Thus at vertex \(i\) of \(D\), \((k_i - g_i + 3) - 1 = 2s_i\) edges are thickened, as required. Furthermore, each horizontal edge of \(\Gamma\) must be fixed, either by a condition on the \(y\)-coordinates of ends, or by a marked point. It cannot be fixed more than once because of the genericity of the conditions. It follows that every edge of the associated floor diagram \(D\) has precisely one thickened flag, as required. Since all floors of \((\Gamma, f)\) are rational, the genus of \(D\) is \(g\). Obviously, the degree of \(D\) is \((\Phi, \mu)\). Thus \(D\) is a floor diagram contributing to \(\langle (\Phi^-_i, \mu^-_i) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_i, \mu^+_i) \rangle_{\text{floor}}\).

\[\square\]

Proposition 5.22. Let \(D\) be a floor diagram contributing to

\[\langle (\Phi^-_i, \mu^-_i) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_i, \mu^+_i) \rangle_{\text{floor}}\]

The weighted number of tropical stable maps contributing to

\[\langle (\Phi^-_i, \mu^-_i) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_i, \mu^+_i) \rangle_{\text{trop}}\]

that yield \(D\) under the procedure described in Construction 5.21 equals \(\text{mult}(D)\).

Proof. Let \((\Gamma, f)\) be a tropical stable map that yields \(D\) using Construction 5.21.

Notice first that if \((\Gamma, f)\) contributes with multiplicity 0 (as e.g. the one in Figure 11), then also \(D\) contributes with multiplicity 0: the floor containing the vertex of multiplicity 0 also has multiplicity 0. Vice versa, if \(D\) has a floor of multiplicity 0, any tropical stable map producing \(D\) must have a vertex of multiplicity 0.

So we can assume now that \((\Gamma, f)\) is of non-zero multiplicity. In particular it is rigid, and not superabundant, and all its non-marked vertices are trivalent by the above. Using our convention
of marking horizontal ends, it follows also that \((\Gamma, f)\) has no nontrivial automorphisms, since it cannot have edges which are not distinguishable.

Following Definition 5.13 and using Remark 5.16, \((\Gamma, f)\) contributes a product of

1. areas of triangles dual to non-marked vertices and factors \(\frac{1}{w}\) for the weights of fixed ends, and
2. local vertex multiplicities \(\text{mult}_V(\Gamma, f)\).

Every compact edge \(e\) of \(D\) of weight \(w(e)\) comes from a bounded edge \(e'\) of \(\Gamma\) of weight \(w(e)\). Since \(e\) has precisely one non-thickened flag, \(e'\) is adjacent to precisely one trivalent vertex \(V\) not adjacent to a marked point (see Lemma 5.9). Denote by \(e''\) an edge in the floor which is adjacent to \(V\). Every non-horizontal edge in a floor is of direction \((0, 1) + c \cdot (1, 0)\) for some \(c\) (by the balancing condition, the fact that the floor contains no cycles, and since we can connect every edge to an end of direction \((0, -1)\)), and so the area of the triangle dual to \(V\) (formed by the duals of \(e'\) and \(e''\)) is \(w(e)\).

A non-fixed end of \(\Gamma\) has to be adjacent to a marked point by rigidity, so it is not adjacent to a trivalent vertex as above. A fixed end of \(\Gamma\) is adjacent to a trivalent vertex whose dual triangle has area \(w(e)\) by the above.

Altogether we can see that the first item above — the product over all areas of triangles dual to non-marked vertices in the dual subdivision of \((\Gamma, f)\) divided by factors \(w\) for fixed ends — equals the product of weights of the compact edges of \(D\).

We cut \((\Gamma, f)\) into floors. Each floor \((\Gamma', f')\) can be viewed as a tropical stable map contributing to the count

\[\langle (\Phi^-_V, \mu^-_V) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_V, \mu^+_V) \rangle_{g_V}\]

which gives the multiplicity of the floor viewed as a vertex \(V\) of \(D\). As such, the floor contributes its tropical multiplicity, which is again a product as above.

Let \(v\) be a vertex of \(D\). By Theorem 3.9, \(\text{mult}(v)\) equals the weighted sum of all floors \((\Gamma', f')\) of some \((\Gamma, f)\) that map to \(v\) under Construction 5.21. In this weighted count, each summand contributes with its tropical multiplicity as above. Since every end of \(\Gamma'\) which is not adjacent to the marked point in \(\Gamma'\) has to be fixed by rigidity, the only contribution we have for the whole floor is the local vertex multiplicity \(\text{mult}_V(\Gamma', f')\) of the vertex \(V\) of \(\Gamma'\) adjacent to the marked point. Thus, \(\text{mult}(V)\) equals the weighted sum over all floors that can possibly be inserted, each counted with the factor \(\text{mult}_V(\Gamma', f')\) where \(V\) is the vertex adjacent to the marked point.

Since we can freely combine floors by gluing them to elevator edges as imposed by \(D\), \(\text{mult}(D)\) equals the weighted count of all tropical stable maps contributing to the invariant

\[\langle (\Phi^-_V, \mu^-_V) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\Phi^+_V, \mu^+_V) \rangle_{\text{trop}}\]

and yielding \(D\) under the procedure described in Construction 5.21, where each tropical stable map is counted with a product of weights for the compact edges of \(D\) times \(\text{mult}_V(\Gamma', f')\) where \(V\) is the vertex adjacent to the marked point. We have seen above that the product of weights for the compact edges of \(D\) equals the product of the areas of triangles dual to non-marked edges, divided by the weights of the fixed ends. Thus \(\text{mult}(D)\) equals the weighted count of all tropical stable maps yielding \(D\), each weighted with its tropical multiplicity. The statement follows.

\(\square\)
6. FLOOR DIAGRAMS VIA THE OPERATOR THEORY

In this section we build on work of Cooper and Pandharipande [19] and Block and Götsche [8] and express relative descendant Gromov–Witten invariants of Hirzebruch surfaces as matrix elements for an operator on a Fock space. The results of this section continue to hold if one replaces the stationary logarithmic descendants of the previous section. We begin the section by reviewing the formalism of Fock spaces in our context.

Let $\mathcal{H}$ denote the algebra presented with generators $a_n, b_n$ for $n \in \mathbb{Z}$ satisfying the commutator relations

$$\{a_n, a_m\} = 0, \quad \{b_n, b_m\} = 0, \quad \{a_n, b_m\} = n \cdot \delta_{n,-m},$$

where $\delta_{n,-m}$ is the Kronecker symbol. We let $a_0 = b_0 = 0$.

The Fock space $F$ is the vector space generated by letting the generators $a_n, b_n$ for $n < 0$ act freely (as linear operators) on the so-called vacuum vector $v_\emptyset$. We define $a_n \cdot v_\emptyset = b_n \cdot v_\emptyset = 0$ for $n > 0$. For a pair of partitions $\Phi = (\phi_1, \ldots, \phi_n)$ and $\mu = (\mu_1, \ldots, \mu_m)$, we denote

$$v_{\Phi,\mu} = \frac{1}{|\text{Aut}(\Phi)| \cdot |\text{Aut}(\mu)|} a_{-\phi_1} \cdot \ldots \cdot a_{-\phi_n} \cdot b_{-\mu_1} \cdot \ldots \cdot b_{-\mu_m} \cdot v_\emptyset.$$

The vectors $\{v_{\Phi,\mu}\}$ indexed by pairs of partitions $\Phi, \mu$ form a basis for $F$. We define an inner product on $F$ by declaring $\langle v_\emptyset | v_\emptyset \rangle = 1$ and $a_n$ to be the adjoint of $a_{-n}$, $b_n$ of $b_{-n}$. The structure constants for the inner product in the two-partition basis are:

$$\langle v_{\Phi,\mu} | v_{\Phi',\mu'} \rangle = \prod_{i=1}^n \phi_i \cdot \prod_{i=1}^m \mu_i \cdot \frac{1}{|\text{Aut}(\Phi)|} \cdot \frac{1}{|\text{Aut}(\mu)|} \cdot \delta_{\Phi,\mu'} \cdot \delta_{\mu,\mu'}.$$

Following standard conventions, for $\alpha, \beta \in F$ and an operator $A \in \mathcal{H}$, we write $\langle \alpha | A | \beta \rangle$ for $\langle \alpha | A | \beta \rangle$. Such expressions are referred to as matrix elements. We write $\langle A \rangle$ for $\langle v_\emptyset | A | v_\emptyset \rangle$; such a value is called a vacuum expectation.

We also introduce normal ordering of operators in $\mathcal{H}$. If $c_i, i = 1, \ldots, n$ are operators in $\mathcal{H}$, then the normally ordered product $\prod_{i=1}^n c_i$ reorders the $c_i$ so that any $c_i$ with $i > 0$ occurs after the $c_j$ with $j < 0$. For example, we have $\hat{a}_2 \hat{b}_2 \hat{a}_2 \hat{a}_2 := b_2 a_1 a_2 a_2$.

As before, we fix $k \in \mathbb{N}$ to identify a Hirzebruch surface $F_k$.

**Definition 6.1.** Let $m \in \mathbb{N}_{>0}$, $l$, $s$ and $g \in \mathbb{N}$ be given. Let $z \in (\mathbb{Z} \setminus \{0\})^m$ satisfy $\sum_{i=1}^m z_i = -k \cdot s$. Denote $\mu = (z_1, \ldots, z_{l+2-2s-g})$ and $\Phi = (z_{l+2-2s-g+1}, \ldots, z_m)$, and let superscripts $\pm$ denote the subsets of positive (resp. negative) entries.

Define

$$\hat{a}_n = \begin{cases} u a_n & \text{if } n < 0 \\ a_n & \text{if } n > 0 \end{cases} \quad \text{and} \quad \hat{b}_n = \begin{cases} u b_n & \text{if } n < 0 \\ b_n & \text{if } n > 0 \end{cases}.$$  

We define the following series of operators in $\mathcal{H}[t, u]$, indexed by $l \in \mathbb{N}$:

$$M_l = \sum_{g \in \mathbb{N}} u g^{-1} \sum_{s \in \mathbb{N}} t^s \sum_{m \in \mathbb{N}_{>0}} \sum_{z \in \mathbb{Z}^m} \langle (\Phi^-, \mu^-) | \tau_l (p t) (\Phi^+, \mu^+) \rangle_g^{\text{rel}} \cdot \hat{b}_{z_1} \cdot \ldots \cdot \hat{b}_{z_{l+2-2s-g}} \cdot \hat{a}_{z_{l+2-2s-g+1}} \cdot \ldots \cdot \hat{a}_{z_m}.$$
where the fourth sum is taken over all \( z \) satisfying \( \sum z_i = -k \cdot s \) (where \( s \) is the index of the second sum), and where the one-point Gromov–Witten invariant \( \langle (\Phi^-_m, \mu^-_m) | \tau_0(pt)(\Phi^+_n, \mu^+_n) \rangle_{g} \) depends on the indices \( l, g \) and \( z \) as above.

**Remark 6.2.** Consider the operator \( M_0 \). It has only two summands for \( s, s = 0 \) and \( s = 1 \), since \( 2 - 2s - g < 0 \) for \( s > 1 \). If \( s = 0 \), the curve class in the Gromov–Witten invariant \( \langle (\Phi^-_m, \mu^-_m) | \tau_0(pt)(\Phi^+_n, \mu^+_n) \rangle_{g} \) is a multiple of the class of a fiber. This implies that the moduli space of maps is non-empty only if \( g = 0 \) and \( m = 2 \). The invariant \( \langle \mu^-_m | \tau_0(pt) | \mu^+_m \rangle_{g} \), for \( \mu = (d, -d) \) is readily seen to be \( 1 \): there is a unique map of degree \( d \) from a rational curve to the fiber identified by the point condition, fully ramified at 0 and \( \infty \) (the intersections of the sections with the given fiber). Such a map has no automorphisms because we have marked one point on the rational curve.

If \( s = 1 \), we must have \( g = 0 \) and no \( b \) factors. The invariants \( \langle \Phi^-_m | \tau_0(pt) | \Phi^+_m \rangle_{g} \) are all \( 1 \) by the genus 0 correspondence theorem and a tropical computation, see [38, 41].

So we have

\[
M_0 = \sum_{z_1 + z_2 = 0} b_{z_1} \cdot b_{z_2} + \sum_{\Phi \in (\mathbb{Z} \setminus \{0\})^m} t \cdot u |\Phi^-|^{-1} a_{z_1} \cdots a_{z_m},
\]

where the second sum goes over all \( z \in (\mathbb{Z} \setminus \{0\})^m \) satisfying \( \sum z_i = -k \). Here the normal ordering is unnecessary since the \( a_i \) commute amongst themselves, as do the \( b_i \). Since the genus can be computed from the Euler characteristic of the underlying Feynman graphs, the variable \( u \) is superficial in this scenario. Setting \( u = 1 \), we obtain the operator \( H_k(t) \) defined in [8], Theorem 1.1. Our family of operators \( M_k \) generalizes the operator of Block-Götsche to one operator for each power of descendant insertions.

**Theorem 6.3.** With discrete data fixed as in Notation 2.2, the disconnected relative descendant Gromov–Witten invariant \( \langle (\Phi^-_m, \mu^-_m) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt)(\Phi^+_n, \mu^+_n) \rangle_{g} \) equals the matrix element

\[
(15) \quad \left| \frac{\text{Aut}(\mu)}{|\text{Aut}(\Phi)|} \prod |\mu_i| \prod |\Phi_i| \right| \nu_{\mu}^{-1} \cdot \Phi^- \left| \text{Coeff}_{t^u u h} M_{k_1} \left( \prod_{i=1}^{n} M_{k_i} \right) \nu_{\mu}^+ \cdot \Phi^+ \right|,
\]

where the operators \( M_{k_i} \) are as defined in Definition 6.1, and for a series of operators \( M \in \mathcal{H}[t,u] \) \( \text{Coeff}_{t^u h} M \) denotes the \( t^h u^h \)-coefficient.

**Important detail.** Notice the order of the partitions is switched on the two sides of Equation (15), thus the \( \mu_i \) entries are associated to \( a \) variables and vice-versa.

Before we start a formal proof of Theorem 6.3, we make a relevant definition and recall an important tool for the proof.

After translating the matrix element in Equation (15) to a vacuum expectation, we compute it as the weighted sum over **Feynman graphs** associated to each monomial contributing to the expectation. This can be viewed as a variant of Wick’s theorem [44] and is proved in Proposition 5.2 of [8]. Generalizing the situation in [8], the Feynman graphs in question are essentially floor diagrams and Theorem 6.3 follows because of a natural weighted bijection of Feynman graphs and floor diagrams.

**Definition 6.4.** Let \( P = m_+ \cdot m_1 \cdots m_n \cdot m_- \) be a product of monomials in the variables \( a_s \) or \( b_s \), such that:
We associate graphs to $P$ called **Feynman graphs** for $P$, via the following algorithm.

**Step 1: local pieces.** To any monomial $m_i$, associate a star graph with vertex denoted $v_i$: for each factor $a_s$ appearing in $m_i$, draw a (non-thickened) edge germ of weight $|s|$ which is directed to the left if $s < 0$ and to the right if $s > 0$. For each factor $b_s$, draw a thickened edge germ of weight $|s|$ which is directed to the left if $s < 0$ and to the right if $s > 0$.

To the special monomials $m_+, m_-$ associate a collection of disconnected, marked edge germs of weight equal to the absolute value of the index of each operator appearing in the monomials. Thicken the germs corresponding to the operators $b_s$.

**Step 2: Feynman fragment.** We call the Feynman fragment associated to $P$ the disconnected graph obtained by linearly ordering the union of all the local pieces: first come the edge germs relative to $m_+$, then vertices $v_i$ (ordered according to their index $i$, and finally the edge germs corresponding to $m_-$.

**Step 3: filling the gaps.** A Feynman graph completing the Feynman fragment is any (marked, weighted, ordered) graph obtained by promoting edge germs to half edges, and gluing pairs of half edges until there is none left. A pair of half edges may be glued if:

- one is directed to the right and the other to the left, and the vertex adjacent to the germ directed to the right is smaller than the one adjacent to the germ directed to the left,
- the two edge germs have the same weight, and
- one edge germ is thickened and one is not.

**Example 6.5.** Let $P$ be the product

$$P = (b_2 \cdot a_1 \cdot a_2) \cdot (b_{-2} \cdot b_2) \cdot (a_{-2} \cdot b_{-1} \cdot a_2) \cdot (b_{-2} \cdot a_{-2} \cdot a_1 \cdot a_1) \cdot (b_{-1} \cdot b_1) \cdot (b_{-1} \cdot a_{-1}),$$

where the factors $m_i$ are separated by parentheses. Following Definition 6.4, a Feynman graph for $P$ is any graph completing the Feynman fragment depicted in Figure 12. In Figure 13, the dotted lines suggest a way to complete the fragment to a Feynman graph for $P$. After removing all external half edges, we recognize the floor diagram depicted in Figure 6.
Proposition 6.6 (Wick’s Theorem, see Proposition 5.2 of [8]). The vacuum expectation \( \langle P \rangle \) for a product \( P \) as in Definition 6.4 equals the weighted sum of all Feynman graphs for \( P \), where each Feynman graph is weighted by the product of weights of all edges (interior edges and ends).

A detailed proof of this proposition may be found in [8]. Here we provide an intuitive and informal description of the mechanism that underlies the proof, as we feel this will be more beneficial to a reader who is not already an expert on these techniques.

Proof. In the product \( P \), we take the right most factor \( a_i \) or \( b_i \) with \( i > 0 \), and try to move it to the right. To simplify notations, let us assume that this right most factor is \( a_i \) for some \( i > 0 \). If this factor \( a_i \) reaches the very right in a contribution we produce in this way (i.e. ends up being the right most factor of a contributing term), then we obtain zero since by definition \( a_i \cdot v_0 = b_i \cdot v_0 = 0 \) for \( i > 0 \). The commutator relations produce several contributing terms for \( \langle P \rangle \) when moving \( a_i \) to the right. We can make \( a_i \) jump over any \( a_j \), or \( b_k \) with \( k \neq -i \). If \( a_i \) is the left neighbour of \( b_{-i} \) however, the commutator relation replaces \( a_i b_{-i} \) by \( b_{-i} a_i + i \). That is, we get two summands, one in which we manage to move \( a_i \) further to the right, and one where we cancel this factor together with its neighbour \( b_{-i} \).

With both summands, we continue moving the right most factor with positive index right. For the summand in which we cancel \( a_i \) together with a factor of \( b_{-i} \) appearing right of \( a_i \) in \( P \), we add to the Feynman fragment of \( P \) by drawing an edge connecting the germ corresponding to \( a_i \) and the germ corresponding to \( b_{-i} \).

By following this procedure we draw all Feynman graphs completing the Feynman fragment for \( P \). Each Feynman graph corresponds to a way to group the factors of \( P \). Each Feynman graph corresponds to a way to group the factors of \( P \) in pairs \( \{a_i, b_{-i}\} \) corresponding to edges completing the corresponding marked edge germs. Each such pair produces a contribution of \( i \) because of the commutator relations, so altogether each Feynman graph should be counted with weight equal to the product of its edge weights to produce \( \langle P \rangle \).

Proof of Theorem 6.3: First we express the matrix element in Equation (15) as a vacuum expectation:

\[
\frac{|\text{Aut}(\mu)||\text{Aut}(\phi)|}{\prod |\mu_i| \prod |\varphi_i|} \left\langle v_{\mu^{-},\phi^{-}} |M| v_{\mu^{+},\phi^{+}} \right\rangle = \frac{|\text{Aut}(\mu)||\text{Aut}(\phi)|}{\prod |\mu_i| \prod |\varphi_i|} \frac{1}{|\text{Aut}(\phi^{+})||\text{Aut}(\mu^{+})||\text{Aut}(\phi^{-})||\text{Aut}(\mu^{-})|} \left\langle v_{0} \prod_{\mu_i \in \mu^{-}} a_{|\mu|} \prod_{\varphi_i \in \phi^{-}} b_{|\varphi|} M \prod_{\mu_i \in \mu^{+}} a_{-\mu_i} \prod_{\varphi_i \in \phi^{+}} b_{-\varphi_i} \right\rangle \left\langle v_{0} \right\rangle
\]

(16)

By Theorem 4.9, the left-hand side in Equation (16) equals an appropriate count of floor diagrams. By Proposition 6.6, each term contributing to the right-hand side can be expressed in terms of a weighted count of suitable Feynman diagrams. We show that the floor diagrams contributing to the left-hand side are essentially equal to the Feynman graphs contributing to the right, and that they are counted with the same weight on both sides.

Expand the left-hand side so that it becomes a sum of vacuum expectations, where each summand is of the form \( w_P \cdot P \) such that \( w_P \) is a number and \( P = m_+ \cdots m_- \) a monomial as described...
in Definition 6.4. For each summand, 
\[ m_+ = \prod_{\mu_i \in \mathcal{M}^-} a_{| \mu_i |} \cdot \prod_{\phi_i \in \mathcal{F}^-} b_{| \phi_i |} \quad \text{and} \quad m_- = \prod_{\mu_i \in \mathcal{M}^+} a_{-| \mu_i |} \cdot \prod_{\phi_i \in \mathcal{F}^+} b_{-| \phi_i |}. \]

A factor \( m_i \) for \( i = 1, \ldots, n \) comes from a summand of \( M_{k_i} \), i.e. is of the form 
\[ ((\mathcal{F}^-, \mathcal{M}^-)|\tau_{k_i}(pt)|(\mathcal{F}^+, \mathcal{M}^+)) \cdot b_{z_1} \cdot \ldots \cdot b_{z_{k_i+2-2s_i-g_i-1}} \cdot \hat{a}_{z_{k_i+2-2s_i-g_i+1}} \cdot \ldots \cdot \hat{a}_{z_m}; \]
where \( s_i \) is encoded in the power of \( t \) and \( g_i \) in the power of \( u \).

Enrich the Feynman fragment for \( P \) by adding three numbers to each vertex \( i \), namely the \( \psi \)-power \( k_i \) (imposed by the operator \( M_{k_i} \) of which the factor corresponding to vertex \( i \) is taken), the size \( s_i \) (imposed by the power of \( t \)) and the genus \( g_i \) (imposed by the power of \( u \)). Any Feynman diagram completing this Feynman fragment is by definition a weighted loop-free graph with ends on the linearly ordered vertex set \( v_1, \ldots, v_n \). After removing all external half edges, the conditions (1), (2) and (3) we impose in the definition of a floor diagram (Definition 4.1) are satisfied. By definition of the operator \( M_1 \) (see Definition 6.1), the signed sum of weights of edges adjacent to a vertex equals \(-k \cdot s_i\), so condition (5) is satisfied. By definition of the operator \( M_L \), in each factor \( m_i \), exactly \( k_i + 2 - 2s_i - g_i \) factors are \( b \)-operators and thus correspond to thickened edge germs, so condition (4) is satisfied.

Since we take the \( t^a \) coefficient of the product \( M_{k_1} \cdot \ldots \cdot M_{k_n} \) for the operator in Equation (15), we obtain floor diagrams satisfying \( a = \sum s_i \). The degree \( \langle \delta, \mu \rangle \) is determined by the boundary conditions. To see that the floor diagram is of the right genus, notice that the variable \( u \) is in charge of genus. Let us build a Feynman graph from the left to the right, starting with the left ends, and adding in vertex after vertex from 1 to \( n \), taking the change in genus into account in each step. The genus of the graph consisting of \( \ell(\mathcal{F}^-) + \ell(\mathcal{M}^-) \) left ends (at first disconnected) has genus \(-\ell(\mathcal{F}^-) - \ell(\mathcal{M}^-) + 1\). For the vertex \( i \) of local genus \( g_i \), by definition of the operator \( M_L \), we get a contribution of \( u^{g_i-1} \), and we get as many additional factors of \( u \) as the vertex has incoming edges (by the \( \hat{a}_i \) resp. \( b_i \) convention). Since \( h \) incoming edges potentially close up \( h - 1 \) cycles, the vertex \( i \) increases the genus by \( g_i + h_i - 1 \), where \( h_i \) denotes the number of incoming edges. By taking the \( u^{g + \ell(\mathcal{F}^-) + \ell(\mathcal{M}^-) - 1} \) coefficient in total, we thus obtain floor diagrams of genus \( g \).

Each Feynman graph for \( P \) can thus be viewed (after removing external half edges) as a floor diagram contributing to the left-right-hand-hand side, and vice versa, each floor diagram gives a Feynman graph.

It remains to show that a Feynman graph and the corresponding floor diagram contribute to Equation (15) with the same multiplicity. For the right-hand side, note that a Feynman graph contributes with the product of the weight of all of its edges times the coefficient \( w_P \) of the product \( P \) in the expansion of the product of the \( M_1 \)-operators. Dividing by the factor \( \frac{1}{| \mathcal{M}_v |} \cdot \frac{1}{| \mathcal{F}_v |} \) (see the right-hand side of Equation (16)), we see that we are giving the Feynman graph weight equal to the product of the weights of its internal edges times the factor \( w_P = \prod_{v=1}^{n} \langle (\mathcal{F}_v^-, \mathcal{M}_v^-)|\tau_{k_v}(pt)|(\mathcal{F}_v^+, \mathcal{M}_v^+) \rangle \cdot g_v \). This is precisely the weight of the corresponding floor diagram in Equation (4.4).

\[ \square \]

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