Ballot Theorems for the Two-Dimensional Discrete Gaussian Free Field

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Abstract
We provide uniform bounds and asymptotics for the probability that a two-dimensional discrete Gaussian free field on an annulus-like domain and with Dirichlet boundary conditions, stays negative as the ratio of the radii of the outer and the inner boundary tends to infinity. Such estimates are often needed in the study of extreme values of the discrete Gaussian free field on planar domains.

Keywords Gaussian free field · Ballot theorems · Extreme value theory · Log correlated fields

1 Introduction and Statement of the Main Results

1.1 Setup

In this work we provide estimates for the probability that a two-dimensional discrete Gaussian free field (DGFF) on a domain which is a suitable generalization of an annulus, stays negative. We give both asymptotics and bounds for such probabilities in terms of the assumed Dirichlet boundary conditions, as the ratio of the radii of the outer and inner boundary tends to infinity. These results should be seen as the two-dimensional analogs of the classical ballot-type theorems, in which the probability that one-dimensional random walk (or bridge) with prescribed starting and ending positions stays negative for a given number of steps. We shall therefore keep referring to such probabilities and results as being of ballot-type, even in the context of the DGFF.
As in other approximately logarithmically correlated fields (such as branching Brownian motion or branching random walks), such ballot estimates prove crucial for the study of extreme value phenomena. Indeed, they have been used ad hoc in [3, 6] to study the microscopic structure of extreme values of the DGFF, and they are relied on in the analysis in many forthcoming works in the area [4, 17, 18]. The purpose of this manuscript is therefore to derive estimates for such ballot probabilities in a general and unified setting that could be used as a black box in a wide range of applications.

To describe the setup precisely, let us recall that the two-dimensional DGFF on a discrete finite domain $\emptyset \subseteq D \subset \mathbb{Z}^2$ with Dirichlet boundary conditions $w \in \mathbb{R}^{\partial D}$ is the Gaussian field $h^{D,w} = (h^{D,w}(x) : x \in \partial D)$ with mean and covariances given for $x, y \in \partial D$ by
\[
\mathbb{E} h^{D,w}(x) = w(x), \quad \text{Cov}(h^{D,w}(x), h^{D,w}(y)) = G_D(x, y).
\] (1.1)

Above, $w$ is a harmonic extension of $w$ to $\mathbb{Z}^2$ and $G_D$ is the discrete Green function associated with a planar simple random walk killed upon exit from $D$. Here and after, $\overline{D} := D \cup \partial D$ and $\partial D$ is the outer boundary of $D$, namely the set
\[
\partial D := \{ x \in D^C : |x - y| = 1 \text{ for some } y \in D \},
\] (1.2)

with $| \cdot |$ always representing the Euclidean norm. Also, whenever $f$ is a real function from a finite non-empty subset of $\mathbb{Z}^2$, we denote by $\overline{f}$ its unique bounded harmonic extension to the whole discrete plane. We also set $h^{D} := h^{D,0}$ (i.e. zero boundary conditions) and formally set $h^\emptyset := 0$.

It follows from the Gibbs-Markov description of the law of $h^{D,w}$, that for any other non-empty domain $D'$ with $\overline{D'} \subset D$ and any $w' \in \mathbb{R}^{\partial D'}$, conditioned on $\{h^{D,w}(x) = w'(x) : x \in \partial D'\}$, the law of $h^{D,w}$ on $\overline{D'}$ is that of the DGFF $h^{D',w'}$. The definition of the DGFF extends naturally to an infinite domain which is not the whole discrete plane, but then the harmonic extension (and therefore the field itself) may not be unique. When $D = \mathbb{Z}^2$, the field does not exist, but for notational reasons we shall nevertheless denote by $h \equiv h^{\mathbb{Z}^2}$ a field that exists only under the formal conditional probability:
\[
\mathbb{P}(h \in \cdot | h_{\partial D'} = w'), \quad \text{for } \emptyset \subseteq D' \subseteq \mathbb{Z}^2, \ w' \in \mathbb{R}^{\partial D'}, \ \text{and in which case its law on } \overline{D'} \text{ is that of } h^{D',w'} \text{ as above.}
\] (1.3)

Henceforth $f_A$ will stand for the restriction of a function $f$ to a subset of its domain $A$.

To get an asymptotic setup, we will consider discrete approximations of scaled versions of a given domain $W \subset \mathbb{R}^2$, which we define for $\ell \geq 0$ as
\[
W_\ell = \{ x \in \mathbb{Z}^2 : d(e^{-\ell}x, W^c) > e^{-\ell}/2 \}.
\] (1.4)

Above $d$ is the usual distance from a point to a set under the Euclidean norm. We shall sometimes loosely refer to such $W_\ell$ as a (discrete) set or domain of (exponential) “scale” $\ell$.

We note that the scale $\ell$ need not be an integer.

We also define $W^+$, $W^-$ and $W^\pm$ as the interior of $W$, the interior of $W^c$ and their union $W^+ \cup W^-$. For $\eta \geq 0$, let
\[
W^\eta := \{ x \in W : d(x, W^c) > \eta \},
\] (1.5)

Precedence is fixed so that $\partial U_{\ell}^{\eta} := \partial(((U^\eta)^c)_{\ell})$. We shall also loosely call $W^\eta$ or $W^\eta_{\ell}$ the “bulk” of $W$, resp. of $W_\ell$. For $\epsilon > 0$, we let $\mathcal{D}_\epsilon$ denote the collection of all open subsets $W \subset \mathbb{R}^2$ whose topological boundary $\partial W$ consists of a finite number of connected components each of which has an Euclidean diameter at least $\epsilon$, and that satisfy
\[
B(0, \epsilon) \subset W \subset B(0, \epsilon^{-1}).
\] (1.6)
Above and in the sequel $B(x, r)$ is the open ball of radius $r$ about $x$ and $B \equiv B(0, 1)$. We also set $\mathcal{D} := \cup_{\varepsilon > 0} \mathcal{D}_\varepsilon$.

We will study the DGFF on domains of the form $U_n \cap V_k^-$, where $U, V \in \mathcal{D}$ with $U, V^-$ being connected and $n, k \geq 0$. Thanks to (1.6) this can be thought of as a generalized version of the annulus $B_n \cap B_k^-$. It is well known [16] that under zero boundary conditions, the maximum of the DGFF on domain of scale $\ell$, reaches height $m_\ell + O(1)$ with high probability, where

$$m_\ell := 2\sqrt{g} \ell - \frac{3}{4} \sqrt{g} \log^+ \ell, \quad g := 2/\pi.$$  \hfill (1.7)

It is therefore natural to consider the boundary conditions $-m_n + u$ on $\partial U_n$ and $-m_k + v$ on $\partial V_k^-$, where $u \in \mathbb{R}^{\partial U_n}$ and $v \in \mathbb{R}^{\partial V_k^-}$. Indeed, when scales are exponential, the harmonic extension inside $U_n \cap V_k^-$ is roughly the linear interpolation between the values on the boundary. Therefore such boundary conditions will push down the mean of the field on the annulus at scale $\ell$ about the origin by precisely the height of the field maximum there on first order.

Ballot estimates and assumptions will often be expressed in terms of the (unique bounded) harmonic extensions $\bar{u}$, $\bar{v}$ of $u$, $v$ respectively. For bounds and conditions it will often be sufficient to reduce the boundary data to the value of $\bar{u}$ and $\bar{v}$ at test points, together with their oscillation in the bulk. We will typically take 0 and $\infty$ as the test points for $\bar{u}$ and $\bar{v}$ respectively, and to this end define $f(\infty) := \lim_{|x| \to \infty} f(x)$. We note that $f(\infty)$ always exists if $f : \mathbb{Z}^2 \to \mathbb{R}$ is the bounded harmonic extension of a function on a finite domain (this is a consequence of Proposition 6.6.1 of [19]). The oscillations of $\bar{u}$ and $\bar{v}$ in the bulk are given by $\text{osc } \bar{u}_n$ and $\text{osc } \bar{v}_k$, where $\bar{u}_n$ and $\bar{v}_k$ stand for the restrictions of $\bar{u}$ and $\bar{v}$ to $U_n^-$ and $V_k^-$ respectively, and $\text{osc } f := \sup_{x, y \in D} |f(x) - f(y)|$, for $f : D \mapsto \mathbb{R}$. We shall also write $\text{osc}_{D'} f$ for $\text{osc } f_{D'}$. Functions which are used outside their domain of definition implicitly take their value there to be zero and indicators $1_A$ are assumed to be defined only on $A$.

Lastly, dependence of constants and asymptotic bounds on parameters are specified as subscripts. When present, such subscripts represent all the parameters on which the quantity depends. For example we shall write $c_\epsilon$ for a constant that depends only on $\epsilon$ and $a_n = o_\epsilon(b_n)$, if there exists a sequence $c_n$ which depends only on $\epsilon$ such that $|a_n/b_n| \leq c_n \to 0$. As usual, constants change from one use to another and are always assumed to be positive and finite.

### 1.2 Main Results

We start with uniform asymptotics for the ballot probability.

**Theorem 1.1** Fix $\epsilon \in (0, 1)$, $\eta, \zeta \in [0, \epsilon^{-1}]$. Suppose that $U, V \in \mathcal{D}_\varepsilon$ such that $U, V^-$ are connected and $B(0, \epsilon) \subset U^\circ$. For all $k, n \geq 0$, there exist $L_n = L_{n, \eta, U} : \mathbb{R}^{\partial U_n} \to (0, \infty)$, $R_k = R_{k, \eta, V} : \mathbb{R}^{\partial V_k^-} \to (0, \infty)$ such that as $n - k \to \infty$,

$$\mathbb{P}(h_{U_n^\eta \cap V_k^-} \leq 0 \mid h_{\partial U_n} = -m_n + u, h_{\partial V_k^-} = -m_k + v) = (2 + o_\epsilon(1)) \frac{L_n(u) R_k(v)}{g(n - k)},$$  \hfill (1.8)

for all $u \in \mathbb{R}^{\partial U_n}$ and $v \in \mathbb{R}^{\partial V_k^-}$ satisfying

$$\max \left\{ \bar{u}(0), \bar{v}(\infty), \text{osc } \bar{u}_n, \text{osc } \bar{v}_k \right\} \leq \epsilon^{-1}, \quad (1 + \bar{u}(0)^-)(1 + \bar{v}(\infty)^-) \leq (n - k)^{1-\epsilon}.$$  \hfill (1.9)
The reader should compare the statement in the theorem to that in the usual ballot theorem for a one-dimensional random walk, say with standard Gaussian steps. Indeed, letting \( (S_t)_{t \geq 0} \) denote such a walk, it can be shown, e.g., using the reflection principle for Brownian Motion, that as \( n - k \to \infty \),

\[
P(S_{k,n} \leq 0 : S_k = -u, S_n = -v) = (2 + o(1)) \frac{F(u)F(v)}{n - k},
\]

uniformly in \( u, v \leq \epsilon^{-1} \) satisfying \((1 + u^-)(1 + v^-) \leq (n - k)^{1-\epsilon}\), where \( P \) is the underlying probability measure and \( \epsilon > 0 \). Moreover \( F(w) \) can be explicitly constructed via

\[
F(w) := \lim_{r \to \infty} E(S_{r^-}; S_{(0,r]} \leq 0 \mid S_0 = w); \quad w \in \mathbb{R},
\]

and obeys

\[
F(w) > 0 \text{ for all } w \leq \epsilon^{-1} \quad \text{and} \quad F(w) = (1 + o(1))w^- \text{ as } w \to -\infty. \tag{1.12}
\]

In analogy to (1.11) we have the following explicit construction for the functionals \( \mathcal{L}_n \) and \( \mathcal{R}_k \) from Theorem 1.1.

**Proposition 1.2** Fix \( \epsilon \in (0, 1), \eta, \xi \in [0, \epsilon^{-1}] \). There exists \((r_n)_{n \geq 0} \) satisfying \( r_n \to \infty \) when \( n \to \infty \) such that the following holds: For \( U, V \) as in Theorem 1.1 and all \( k, n \geq 0 \), the functions \( \mathcal{L}_n \) and \( \mathcal{R}_k \) can be defined via

\[
\mathcal{L}_n(u) := E\left( \frac{h_{\partial B_n \cap \partial B_n}(0) + m_n - r_n}{h_{\partial B_n}(0) - h_{\partial B_n} - m_n + u}; \quad h_{\partial B_n}(0) = \overline{u}(0) \right) \tag{1.13}
\]

\[
\mathcal{R}_k(v) := \lim_{n \to \infty} E\left( \frac{h_{\partial B_n \cap \partial B_n}(\infty) + m_k + r_n}{h_{\partial B_n}(\infty) - h_{\partial B_n} - m_k + v}; \quad h_{\partial B_n}(\infty) = \overline{v}(0) \right) \tag{1.14}
\]

for \( u \in \mathbb{R}^{\partial U_n}, v \in \mathbb{R}^{\partial V_k^-} \).

From now on, we shall assume that \( \mathcal{L}_n \) and \( \mathcal{R}_k \) are defined as in Proposition 1.2.

While explicit constructions of the functionals \( \mathcal{L}_n \) and \( \mathcal{R}_k \) are illustrative, they are of less use in computations involving the right-hand side of (1.8). The purpose of the next several propositions is therefore to provide bounds and asymptotics for them. For what follows, we let \( \mathbb{H}(W) \) be the space of bounded harmonic functions on an open non-empty domain \( W \subset \mathbb{R}^2 \), which we equip with the supremum norm.

The first proposition shows that, viewed as functionals of the harmonic extension of their argument, \( \mathcal{L}_n \) and \( \mathcal{R}_k \) admit an infinite version after proper scaling.

**Proposition 1.3** Fix \( \epsilon \in (0, 1), \xi, \eta \in [0, \epsilon^{-1}] \) and let \( U, V \) be as in Theorem 1.1. There exist \( \mathcal{L} = \mathcal{L}_{\xi, \eta} : \mathbb{H}(U^n) \to (0, \infty) \) and \( \mathcal{R} = \mathcal{R}_{\xi, \eta} : \mathbb{H}(V^{-\xi}) \to (0, \infty) \) such that,

\[
\mathcal{L}_n(u_n) \overset{n \to \infty}{\longrightarrow} \mathcal{L}(\widehat{u}_\infty) \quad \text{whenever} \quad \max_{x \in U_n} \left| \overline{u}_n(x) - \widehat{u}_\infty(e^{-n}x) \right| \overset{n \to \infty}{\longrightarrow} 0,
\]

\[
\mathcal{R}_k(v_k) \overset{k \to \infty}{\longrightarrow} \mathcal{R}(\widehat{v}_\infty) \quad \text{whenever} \quad \max_{x \in V_k^{-\xi}} \left| \overline{v}_k(x) - \widehat{v}_\infty(e^{-k}x) \right| \overset{k \to \infty}{\longrightarrow} 0. \tag{1.15}
\]

Above \( u_n \in \mathbb{R}^{\partial U_n}, v_k \in \mathbb{R}^{\partial V_k^-} \) for all \( k, n \geq 0 \) and \( \widehat{u}_\infty \in \mathbb{H}(U^n), \widehat{v}_\infty \in \mathbb{H}(V^{-\xi}) \).

Henceforth, we denote by \( \mathbb{H}(U, \eta) \) the subspace of those \( \widehat{u} \in \mathbb{H}(U^n) \) for which there exists an approximating sequence \( u_n \in \mathbb{R}^{\partial U_n} \) with \( \max_{x \in U_n} \left| \overline{u}_n(x) - \widehat{u}(e^{-n}x) \right| \overset{n \to \infty}{\longrightarrow} 0 \). Similarly,
by $\mathbb{H}(V^-, \zeta)$ we denote the subspace of those $\widehat{\nu} \in \mathbb{H}(V^-, \zeta)$ which admit an approximating sequence $v_k \in \mathbb{R}^{\partial V^-}$ with max $x \in V^- \left| \overline{v_k}(x) - \widehat{\nu}(e^{-k}x) \right| \xrightarrow{k \to \infty} 0$. By Proposition 1.3, $\mathcal{L}(\widehat{u})$ and $\mathcal{R}(\widehat{v})$ are well-defined for $\widehat{u} \in \mathbb{H}(U, \eta)$ and $\widehat{v} \in \mathbb{H}(V^-, \zeta)$.

Next, in analogy to (1.12) we have the following two propositions:

**Proposition 1.4** Fix $\epsilon \in (0, 1)$, $\eta, \zeta \in [0, \epsilon^{-1}]$ and let $U, V$ be as in Theorem 1.1. There exists $c = c_\epsilon < \infty$ such that for all $k \geq 0$, $n \geq c$,

$$
\mathcal{L}_n(u) = (1 + o_\epsilon(1))\overline{u}(0)^- \quad \text{as } \overline{u}(0)^- \to \infty \quad \text{with } \text{osc } \overline{u}_n \leq \epsilon^{-1},
$$

$$
\mathcal{R}_k(v) = (1 + o_\epsilon(1))\overline{v}(\infty)^- \quad \text{as } \overline{v}(\infty)^- \to \infty \quad \text{with } \text{osc } \overline{v}_\zeta \leq \epsilon^{-1},
$$

and also

$$
\mathcal{L}(\widehat{u}) = (1 + o_\epsilon(1))\widehat{u}(0)^- \quad \text{as } \widehat{u}(0)^- \to \infty \quad \text{with } \text{osc } \widehat{u}_n \leq \epsilon^{-1},
$$

$$
\mathcal{R}(\widehat{v}) = (1 + o_\epsilon(1))\widehat{v}(\infty)^- \quad \text{as } \widehat{v}(\infty)^- \to \infty \quad \text{with } \text{osc } \widehat{v}_\zeta \leq \epsilon^{-1}.
$$

**Proposition 1.5** Fix $\epsilon \in (0, 1)$, $\eta, \zeta \in [0, \epsilon^{-1}]$ and let $U, V$ be as in Theorem 1.1. There exists $c = c_\epsilon > 0$ such that for all $k \geq 0$, $n \geq c$ and all $u \in \mathbb{R}^{\partial U_n}$ and $v \in \mathbb{R}^{\partial V^-}$ with max $\{\overline{u}_n(0), \overline{v}_\zeta(\infty), \text{osc } \overline{u}_n, \text{osc } \overline{v}_\zeta\} \leq \epsilon^{-1}$,

$$
\mathcal{L}_n(u) > c, \quad \mathcal{R}_k(v) > c.
$$

Also, for all $\widehat{u} \in \mathbb{H}(U, \eta)$ and $\widehat{v} \in \mathbb{H}(V^-, \zeta)$ with max $\{\widehat{u}(0), \widehat{v}(\infty), \text{osc } \widehat{u}, \text{osc } \widehat{v}\} \leq \epsilon^{-1}$, we have

$$
\mathcal{L}(\widehat{u}) > c, \quad \mathcal{R}(\widehat{v}) > c.
$$

Next, we address the continuity of $\mathcal{L}_n$ and $\mathcal{R}_k$ in the domain, the values assigned to the field on the boundary, and $\eta, \zeta$. To measure distances between domains, if $W, W'$ are open non-empty and bounded subsets of $\mathbb{R}^2$, we let $d_{\mathbb{H}}(W, W')$ be the Hausdorff distance of their complements inside any closed and bounded subset of $\mathbb{R}^2$ which includes both $W$ and $W'$. The reader can easily verify that the above definition is proper and that $d_{\mathbb{H}}$ is indeed a metric on the space of open, non-empty and bounded subsets of $\mathbb{R}^2$ and in particular on $\mathcal{D}$.

**Proposition 1.6** Fix $\epsilon \in (0, 1)$. There exist $c = c_\epsilon < \infty$, $\rho = \rho_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(t) \downarrow 0$ as $t \downarrow 0$ such that the following holds: For all $n > c$, all $U, U' \in \mathcal{D}_\epsilon$ such that $U, U'$ are connected, all $\eta, \eta' \in [0, \epsilon^{-1}]$ such that $B(0, \epsilon) \subset U^\eta \cap U'^{\eta'}$, and all $u \in \mathbb{R}^{\partial U_n}, u' \in \mathbb{R}^{\partial U'_n}$ such that max $\{\overline{u}_n(0), \overline{u}'_n(0), \text{osc } \overline{u}_n, \text{osc } \overline{u}'_n\} \leq \epsilon^{-1}$, we have

$$
\left| \frac{\mathcal{L}_{n, n, U}(u)}{\mathcal{L}_{n, n', U}(u')} - 1 \right| \leq \rho \left( d_{\mathbb{H}}(U, U') + \text{Leb}(U^\eta \Delta U'^\eta') + \|\overline{u} - \overline{u}'\|_{L_\infty(U^\eta \cap U'^\eta')} + n^{-1} \right).
$$

(1.20)

Similarly, for all $k \geq 0$, all $V, V' \in \mathcal{D}_\epsilon$ such that $V^-, V'^-$ are connected, all $\zeta, \zeta' \in [0, \epsilon^{-1}]$ and all $v \in \mathbb{R}^{\partial V^-}, v' \in \mathbb{R}^{\partial V'^-}$ such that max $\{\overline{v}_\zeta(\infty), \overline{v}'_{\zeta'}(\infty), \text{osc } \overline{v}_\zeta, \text{osc } \overline{v}'_{\zeta'}\} \leq \epsilon^{-1}$, we have

$$
\left| \frac{\mathcal{R}_{k, \zeta, V}(v)}{\mathcal{R}_{k, \zeta', V'}(v')} - 1 \right| \leq \rho \left( d_{\mathbb{H}}(V^-, V'^-), \Delta V^\zeta \Delta V'^{-\zeta'} + \|\overline{v} - \overline{v}'\|_{L_\infty(V^\zeta \cap V'^{-\zeta'})} + k^{-1} \right).
$$

(1.21)

The above holds also for $\mathcal{L}$ and $\mathcal{R}$ in place of $\mathcal{L}_n$ and $\mathcal{R}_k$, with $u, u', v, v', \overline{u}, \overline{u}', \overline{v}, \overline{v}'$ and $\overline{u}_n, \overline{u}'_n, \overline{v}_\zeta, \overline{v}'_{\zeta'}$ all replaced by $\widehat{u}, \widehat{u}', \widehat{v}, \widehat{v}'$ which are now in $\mathbb{H}(U, \eta), \mathbb{H}(U', \eta'), \mathbb{H}(V^-, \zeta)$.
Remark. Large and that (1.9) holds. We also remark that all proofs pass through when the coefficients are replaced with arbitrary constants strictly larger than 1.

Lastly, we turn to bounds on the ballot probability. The following lower bound is a simple consequence of Theorem 1.1, Propositions 1.4, 1.5 and monotonicity.

**Corollary 1.7** Under the conditions of Theorem 1.1, there exist \( c = c_\epsilon > 0 \) and \( C = C_\epsilon < \infty \), such that

\[
P\left( h_{U^n_k} \cap V^{-,\xi} \leq 0 \bigg| h_{\partial U^n_k} = -m_n + u, h_{\partial V^{-,\xi}} = -m_k + v \right) > c \frac{(\overline{u}(0) + 1)(\overline{v}(\infty) + 1)}{n - k}
\]

(1.22)

for all \( n - k \geq C \).

While a similar upper bound can be derived in the same way, a stronger result without any restrictions on \( u \) and \( v \) is given by the following theorem. For what follows, we abbreviate:

\[
u_* = \overline{u}(0) - 2osc \overline{u}_n, \quad v_* = \overline{v}(\infty) - 2osc \overline{v}_\xi.
\]

**Theorem 1.8** Fix \( \epsilon \in (0, 1), \eta, \xi \in [0, \epsilon^{-1}] \) and let \( U, V \) be as in Theorem 1.1. Then there exists \( C = C_\epsilon < \infty \) such that the following holds: for all \( 0 \leq k < n \) and all \( u \in \mathbb{R}^{\partial U^n_k} \) and \( v \in \mathbb{R}^{V^{-,\xi}}, \)

\[
P\left( h_{U^n_k} \cap V^{-,\xi} \leq 0 \bigg| h_{\partial U^n_k} = -m_n + u, h_{\partial V^{-,\xi}} = -m_k + v \right) \leq C \frac{(u_* + 1)(v_* + 1)}{n - k},
\]

(1.24)

If, moreover, \( u_*^{-} \leq (n - k)^{1-\epsilon}, \quad v_*^{-} \leq (n - k)^{1-\epsilon}, \quad n - k \geq C \), then we have the stronger bound

\[
P\left( h_{U^n_k} \cap V^{-,\xi} \leq 0 \bigg| h_{\partial U^n_k} = -m_n + u, h_{\partial V^{-,\xi}} = -m_k + v \right) \leq C \frac{(u_*^{-} + e^{-(u_*^{-})^{2-\epsilon}})(v_*^{-} + e^{-(v_*^{-})^{2-\epsilon}})}{n - k}.
\]

(1.25)

In light of Theorem 1.1 and Proposition 1.4, we remark that the constants in (1.22) and (1.24) can be chosen arbitrarily close to \( \frac{1}{2} / g \) under the restriction that \( u_*^{-}, v_*^{-} \) and \( n - k \) are sufficiently large and that (1.9) holds. We also remark that all proofs pass through when the coefficients 2 in (1.23) are replaced with arbitrary constants strictly larger than 1.

**Remarks.**

1. **Improved upper bound for positive boundary conditions.** In Theorem 1.8, the upper bound (1.25) is stronger than (1.24) when positive and large values are assumed on either the outer or the inner (or both) “part” of the boundary of the underlying domain. The almost-Gaussian decay is the cost of a shift down of the field close to the corresponding part of the boundary, to make room for its maximum there to still be negative, despite the positive boundary conditions. We take this cost only into account when the boundary conditions on the opposite scale are not too low, as otherwise the field may be sufficiently tilted down for this cost to be lower.

2. **Discretization.** One can of course choose different methods for discretizations than the one we used in (1.4). In particular, one can replace \( e^{-\ell}/2 \) with any multiple, greater or equal to \( 1/2 \), of \( e^{-\ell} \) (as was done in [5–7]) with all the statements still holding. The choice of \( 1/2 \) ensures that each connected component of the boundary of \( W \in \mathcal{D} \) is noticeable.
also in \( W_\ell \) for large \( \ell \). Indeed, as each connected component of \( \partial W \) has positive diameter, it crosses for large enough \( \ell \) an edge of \( e^{-\ell}Z^2 \), thus there exists a vertex \( x \in Z^2 \) with \( d(e^{-\ell}x, \partial W) \leq e^{-\ell}/2 \). Then (1.4) implies \( x \notin W_\ell \), while we would have \( x \in W_\ell \) if \( d(e^{-\ell}x, W^c) > e^{-\ell}/2 \).

At the same time, the choice of \( 1/2 \) in (1.4) allows, for \( \ell = 0 \), to have discrete domains in which only a few isolated points are excluded from the bulk, as for instance \( (B(0, 1/4))_0^\perp = Z^2 \setminus \{0\} \). More generally, as in [6] one can take as \( W_\ell \) any set satisfying

\[
\{ x \in Z^2 : d(e^{-\ell}x, W^c) > \lambda \ell \} \subset W_\ell \subset \{ x \in Z^2 : d(e^{-\ell}x, \partial W) > e^{-\ell} / 2 \},
\]

for any fixed sequence \( (\lambda \ell) \) satisfying \( \lim_{\ell \to \infty} \lambda \ell = 0 \) and \( \lambda \ell \geq e^{-\ell} / 2 \). All the statements in the present subsection still hold and uniformly with respect to the particular choice of the discretized versions of \( U \) and \( V^- \), but one must then restrict oneself to the case \( \eta, \zeta > \epsilon \). The continuity of the functionals \( L_n \) and \( R_k \) in the underlying domain, which is uniform in \( n \) and \( k \) respectively, can be also used to compare different discretization methods.

3. **Continuity in the underlying domain.** Continuity in the underlying domain is handled in Proposition 1.6. The \( n^{-1} \) and \( k^{-1} \) terms in (1.20) and (1.21) are the result of the discretization. Indeed, two continuous domains can be arbitrarily close to each other, with their respective scaled-up discretizations differing by at least one point. These terms can be left out if \( U = U' \) or \( V = V' \), respectively.

The distance \( d_{H^c} \), as defined above the proposition, can be reformulated as

\[
d_{H^c}(W, W') = \inf \{ \eta > 0 : W^\eta \subset W' \text{ and } W'\eta \subset W \}. \tag{1.27}
\]

The condition in the infimum guarantees that the Green Functions: \( G_W \) and \( G_{W'} \) are close to one another in the bulk of \( W^\eta \cap W'\eta \subset W \cap W' \), which in turn ensures that the corresponding DGFFs are stochastically not too far apart on this set. This shows that the topology induced by \( d_{H^c} \) is a rather natural choice for the question of continuity in \( U \) and \( V^- \) of the functionals \( L_n \) and \( R_k \), resp.

We remark, that using the Hausdorff distance for the set itself and not its complement would not have been a good choice. Even if we ignore the fact that the Hausdorff function is only a metric when one takes compact sets, using it to measure distances would have rendered the distance between a disk and a disk with a slit to be zero. This is clearly not desirable as the DGFFs on the discretized scale-ups of these two domains are very different from each other and therefore so are the corresponding \( L_n \) or \( R_k \) functionals.

### 1.3 Proof Outline

Let us explain the strategy behind the proof of the ballot asymptotics (Theorem 1.1) and ballot upper bound (first part of Theorem 1.8), which constitute the main contribution of this manuscript. To study the ballot probability (and associated expectations such as those appearing in the definition of \( L_n \) and \( R_k \)) we recast the event

\[
\left\{ h_{U^\eta \cap V'_\xi} \leq 0 \right\}
\]

as

\[
\bigcap_{\ell=1}^{T} \{ S_\ell + D_\ell \leq 0 \} \tag{1.29}
\]
for some processes \((S_\ell)^T_{\ell=0}, (D_\ell)^T_{\ell=1}\) which are defined in terms of the field \(h\) and some additional independent randomness. We show that under \(P(\cdot | h_{\partial U_n} = -m_n + u, h_{\partial V_n} = -m_k + v)\), for \(u,v\) satisfying the conditions in Theorem 1.1, these processes satisfy the following three assumptions:

\(\text{(A1)}\) \((S_\ell)^T_{\ell=0}\) is a non-homogeneous random walk with Gaussian steps of \(O(1)\) variance, conditioned to start from \(\gamma(0)\) at time 0 and end at \(\gamma(\infty)\) at time \(T\). Moreover \(T = n - k + O(1)\).

\(\text{(A2)}\) For all \(\ell\), given \(S_\ell\), the random variables \((S_{\ell'}, D_{\ell'})_{\ell' \leq \ell}\) are conditionally independent of \((S_{\ell'})_{\ell' > \ell}\).

\(\text{(A3)}\) \((D_\ell)^T_{\ell=1}\) are stretched exponentially tight around their mean and satisfy \(|\mathbb{E}D_\ell| = O(\min(\ell, T - \ell)^{1/2-\epsilon})\) for all \(\ell\) and some \(\epsilon > 0\).

More formal versions of these assumptions appear in Sect. 2.2 as (A1) – (A3).

We shall refer to the \(D_\ell\)-s as decorations and to the pair of processes \(((S_\ell)^T_{\ell=0}, (D_\ell)^T_{\ell=1})\), as a decorated (non-homogeneous) random walk (DRW).

To carry out the reduction to the DRW, we use a new version of the concentric decomposition, which was originally introduced in [6], that we call inward concentric decomposition. Assuming for illustrative purposes that \(U = V = B\) and that \(\eta = \zeta = 0\), the idea is to condition on the values of the field at the boundaries of (discrete) balls of radii \(e^{n-\ell}\) for \(\ell = 1, 2, \ldots, n - k - 1\). Thanks to the Gibbs-Markov property of the DGFF, the field \(h_{B_n \cap B_k^c}\) (with zero boundary conditions) can then be written as a sum of a binding field \(\varphi\), namely the conditional expectation of \(h_{B_n \cap B_k^c}\) given its values on \(\bigcup_{\ell=1}^{n-k-1} \partial B_{n-\ell}\), plus independent DGFFs \(h_{A_\ell}^{A_{\ell-1}}\)

where \(A_\ell\) is the annulus \(B_{n-\ell+1} \cap B_{n-\ell}\).

Performing the conditioning in successive order from \(\ell = 1\) to \(n - k - 1\), we construct a process \((S'_\ell)_{\ell=1}^{n-k}\) as a function of \(\varphi\) plus some additional independent randomness. This process is defined so that it has the law of a random walk with Gaussian steps, conditioned to start and end at 0. More importantly, \(S'_\ell\) approximates \(\varphi\) in the bulk of \(A_\ell\), up to an additive error which has a uniform stretched exponential tail. If instead of zero, one takes boundary conditions \(-m_n + u\) on \(\partial B_n\) and \(-m_k + v\) on \(\partial B_k^c\), then the added mean \((-m_n 1_{\partial B_n} + u - m_k 1_{\partial B_k^c} + v)\) is equal on \(A_\ell\) to

\[
\frac{(n - \ell)(-m_n + \gamma(0)) + (\ell - k)(-m_k + \gamma(\infty))}{n - k} + O(1)
\]

whenever the conditions of Theorem 1.1 are satisfied. This follows from the fact that for a well-behaved harmonic function \(f\) on \(B_n \cap B_k^c\), \(\log |x| \to f(x)\) is approximately linear.

Combining the above and using also the stretched exponential tightness of the maximum of the DGFF around its mean, it follows that under \(P(\cdot | h_{\partial B_n} = u, h_{\partial B_k^c} = v)\) the event in (1.28) can be written as

\[
\bigcap_{\ell=1}^{n-k} \left\{ \max_{A_\ell} \left( \varphi + (-m_n 1_{\partial B_n} + u - m_k 1_{\partial B_k^c} + v) \frac{\partial B_n \cup \partial B_k^c}{\partial B_n \cup \partial B_k^c} + h_\ell \right) \leq 0 \right\}
\]

\[
\bigcap_{\ell=1}^{n-k} \left\{ S'_\ell + \frac{(n - \ell)u(0) + (\ell - k)\gamma(\infty)}{n - k} + D_\ell \leq 0 \right\},
\]
where \((D_{\ell})_{\ell=1}^T\) are stretched exponentially tight after centering by \(O(\log^+((\ell-k) \wedge (n-\ell)))\). Setting \(S_{\ell} := S_{\ell}^* + \frac{(n-\ell)w(0)+((\ell-k)T}\right)\), we then obtain that the pair \(((S_{\ell})_{\ell=0}^T, (D_{\ell})_{\ell=1}^T)\) satisfies Assumptions (A1)–(A1) above and that the equality between events (1.28) and (1.29) holds.

This converts the original task to that of deriving asymptotics and an upper bound for the probability of the DRW ballot event (1.29). To this end, we can use the tightness of the decorations to reduce the problem to that in which the decoration process \((D_{\ell})_{\ell=1}^T\) is replaced by a deterministic curve or barrier: \((Y_{\ell})_{\ell=1}^T\), with \(Y_{\ell} = O(\min(\ell, T-\ell)^{1/2-\epsilon})\) for some \(\epsilon > 0\). Upper and lower bounds for such barrier probabilities are then (essentially) readily available (for example in the work of Bramson [9] on barrier estimates for BM, from which such results can be derived, or [12]). These in turn are sufficient to derived the desired estimates.

A difficulty is that because of the dependence between the walk \((S_{\ell})_{\ell=0}^T\) and the decorations \((D_{\ell})_{\ell=1}^T\), as captured by Assumption (A2), the reduction to such a deterministic barrier problem can only be done when the endpoint of the walk \(S_T\) is sufficiently low, that is if \(\bar{v}(\infty) < -T^\epsilon\) for some \(\epsilon > 0\). In other words, using the reduction we can only prove weak versions of Theorem 1.1 and (the first part of) Theorem 1.8, where one imposes the additional constraint that \(\bar{v}(\infty) < -(n-k)^\epsilon\).

To remove this constraint we proceed as follows. First, we define a new DRW \((S_0^0)_{\ell=0}^T, (D_0^0)_{\ell=1}^T\) which (under the conditions of Theorem 1.1) satisfies Assumptions (A1)–(A1) as before, only that the starting and end points are now reversed: \(S_0^0 = \bar{v}(\infty)\) and \(S_0^0 = \bar{v}(0)\), and with the equivalence of events (1.28) and (1.29) (with \(S_0^0, D_0^0\) in place of \(S_{\ell}^*\) and \(D_{\ell}\) still holding. This is done by employing an outward concentric decomposition, whereby the order of conditioning is reversed, such that \(S_0^0 + D_0^0\) now corresponds to the value of the field on the annulus \(B_{k+\ell} \cap B_{k+\ell-1}^-\). Proceeding as before, this gives weak version of Theorems 1.1 and Theorem 1.8 (first part), where now the additional assumption is that \(\bar{v}(0) < -(n-k)^\epsilon\).

Now, given \(U_n\) and \(V_k^-\), we pick an intermediate scale, say, \(\ell := (n+k)/2\) and condition on the values of \(h\) on the \(\partial B_{\ell}^-\). Thanks to the Gibbs-Markov property, whenever \(n-k\) is large, we can write

\[
\mathbb{P}(h_{U_n^0 \cap V_k^{-}\ell} \leq 0 \mid h_{\partial U_n^0} = -m_n + u, h_{\partial V_k^{-}\ell} = -m_k + v) \approx \int_{w \leq m_\ell} \mathbb{P}(h_{U_n^0 \cap B_{\ell}^-} \leq 0 \mid h_{\partial U_n} = -m_n + u, h_{\partial B_{\ell}^-} = -m_\ell + w) \times \mathbb{P}(h_{B_{\ell}^\infty \cap V_k^{-}\ell} \leq 0 \mid h_{\partial B_{\ell}^-} = -m_\ell + w, h_{\partial V_k^{-}\ell} = -m_k + v) \times \mathbb{P}(h_{B_{\ell}^\infty} + m_\ell \in dw \mid h_{\partial U_n} = -m_n + u, h_{\partial V_k^{-}\ell} = -m_k + v).
\]  

(1.32)

Observing that the first two terms in the integrand are precisely ballot probabilities of the form treated in this paper, we can use the weak ballot upper bound together with discrete harmonic analysis to show that for any \(\delta > 0\), the integral can be restricted to \(w \in \mathbb{R}^{\partial B_{\ell}^\pm}\) satisfying \(\bar{w}(0)^- \in [(n-\ell)^\delta, (n-\ell)^{1-\delta}], \bar{w}(\infty)^- \in [(-\ell-k)^\delta, (-\ell-k)^{1-\delta}]\) and \(\text{osc}_{B_{\ell}^\pm, \bar{w}(\infty)} \bar{w} = O(1)\) at the cost of a negligible error. The fact that \(h_{\partial B_{\ell}^\pm} \leq -(n-k)^\delta = -(-\ell-k)^\delta = -((n+k)/2)^\delta\) in the bulk with high probability conditional on (1.28) is the DGFF analog of the entropic repulsion of a random walk bridge conditioned to stay negative.

We can now apply the weak version of Theorem 1.1 to estimate the first two terms in the integrand (first with \(U_n^0 \cap B_{\ell}^{-}\delta\) and \(B_{\ell}^\infty \cap V_k^{-}\ell\) in place of \(U_n^0 \cap B_{\ell}^-\) and \(B_{\ell} \cap V_k^{-}\ell\), respectively, and then taking \(\delta \to 0\). This together with known asymptotics for the third term in the integrand is sufficient to obtain the full version of Theorem 1.1. The method,
which we refer to as **stitching** (because we “stitch” the two ballot estimates together), can be used in a similar way to obtain the first part of Theorem 1.8 as well.

### 1.4 Additional Background and Context

The results in this manuscript should be seen as part of the theory of extreme values for logarithmically or hierarchically correlated fields. Examples for such fields are abundant and include, to name a few, branching Brownian motion (BBM), the branching random walk (BRW), the two-dimensional continuous or discrete Gaussian free field (CGFF/DGFF) and even the square root of the local times of a planar random walk (RW) or Brownian motion (BM) at all points in the underlying domain. Central to the study of these fields and the models in which they are used, are statistical properties of their extreme values. As such, considerable effort has been devoted to their study.

In the case of the DGFF, which is the focus of this work, such study can be traced back at least to the work [8], where first order asymptotics for the height of the maximum were derived in order to quantify the effect of conditioning the field to be positive in the bulk. Sharp asymptotics, tightness and finally convergence in law for the centered maximum, were proved more than a decade later in the sequence of works [10, 11, 16]. The treatment of all extreme values, namely all vertices at which the height of the field is within $O(1)$ from $m_n$ - the typical height of its maximum, followed soon after [5–7] with the derivation of a limit in law for the full extremal process of the field.

A common feature of the extreme order statistics of such fields, is the divergence with the size of the system of the first moment of the number of extreme values. This phenomenon, which is a consequence of the strong correlations at short range, makes any analysis that relies on bare moment computation fail. A common way to overcome this obstacle is the use of **truncation**. That is, an additional restriction imposed on the extreme values, which effectively removes an atypical scenario that results in an atypically large number of extreme values to appear.

In the case of the DGFF, a truncation strategy which was introduced in [6], is a restriction to the global event that the field stays below $m_n + v$ at all vertices, where $v \in \mathbb{R}$. Tightness of the maximum around $m_n$ implies that this is indeed a typical event so that for large $v$, intersecting it with an $O(1)$ probability event is harmless. At the same time, imposing this restriction on the global maximum, prevents the unlikely scenario that the entire field is globally shifted upwards, and as a result caps the number of extreme values.

This truncation has proved useful in several other works on this topic. In [4], it was used in order to prove tightness for the number of **near-extreme values** - values whose height is within order $\sqrt{n}$ of $m_n$, and then to derive the weak limit for their joint law, which factorizes into the critical Liouville Quantum Gravity measure and the Rayleigh distribution. Truncation was also used in [17] (forthcoming) where finer properties of extremal level sets are studied, in analogy to a similar work on BBM [13]. Here the truncation allows for the necessary control on the first moment of the number of extreme values with a certain local structure. A truncation is also used in [18] where the growth of the infinite volume pinned field is studied.

The restriction that the maximum stays below $m_n + v$ is included by definition in many other events of interest. Such an event appears, for example, when asymptotics are derived for the joint law of the height and position of the global maximum, with [10] or without ([6], Theorem 2.5) spatial scaling. It is also a restriction imposed when studying the local structure around the global maximum, in the derivation of the cluster law, namely the distribution of the relative heights at vertices near an extreme value [17].
Such past and foreseen examples seem to justify the need for a theory by which one can estimate the probability of events that include such typical bounds on the maximum. Since the Gibbs-Markov property is often employed in any analysis of the DGFF, a useful theory would include estimates which hold also conditionally, when values on a boundary of a subset \( D' \) of the underlying domain of the field \( D \) are specified. Under the conditioning, the probability of such events breaks into a product of two terms, in which the restriction on the maximum is imposed once for a DGFF on a “ball-like” domain \( D' \) and once for a DGFF on an “annulus-like domain” \( D \setminus \overline{D'} \).

Ball-like domains have been studied in [10]. In [3, 6], ballot theorems for the DGFF are given in [6, Propositions 5.1 and 5.2] and [3, Lemma 11.13]. There, the inner domain consists of a single vertex and the boundary values on the outer domain are constant and restricted to a compact interval. Moreover, by conditioning on the value of the field at a single vertex inside its domain of definition and using the Gibbs-Markov property, any DGFF on a ball-like domain can be turned into a DGFF on an annulus-like domain (with the inner radius being 1). Therefore, estimates for the probability that the maximum stays below \( m_n + v \) for general annulus-like domains and with prescribed boundary conditions are both necessary and sufficient. We center the boundary conditions such that the expected maximum is at the order of the boundary values, and by shifting the field by \( -m_n - v \), we arrive precisely at the ballot probabilities which appear in Sect. 1.2.

Let us now demonstrate the use of the theory by estimating the second moment of the number of extreme values in \( U^n_\epsilon \) at height above \( m_n + u \) for a DGFF on \( U_n \), namely the size of

\[
\{ x \in U^n_\epsilon : h^{U_n}(x) \geq m_n + u \}, \tag{1.33}
\]

where \( \epsilon > 0, u \in \mathbb{R} \) and \( U \in \mathcal{D}_\epsilon \). To get a non-diverging quantity, we introduce the truncation event:

\[
\{ h^{U_n}_{U_n} \leq m_n + v \}, \tag{1.34}
\]

where \( v > u \) (eventually to be taken to \( \infty \), making the probability of the event above tend to 1, by the tightness of the maximum of \( h^{U_n}_{U_n} \) around \( m_n \)). Then the second moment of the size of (1.33) on the event in (1.34) is the sum over all \( x, y \in U^n_\epsilon \) of

\[
\mathbb{P}\left( h^{U_n}_{U_n} \leq m_n + v, h^{U_n}_{U_n}(x) > m_n + u, h^{U_n}_{U_n}(y) > m_n + u \right). \tag{1.35}
\]

To estimate the probability above, we let \( k := \log(|x - y|) - C \) with the \( C \) chosen so that \( (x + B_k) \cap (y + B_k) = \emptyset, (x + B_k) \cup (y + B_k) \subseteq U^n_{\epsilon/2} \) and let \( W \) be such that \( W_k = (x + B_k) \cup (y + B_k) \). Conditioning in addition on \( h^{U_n}_{\partial W_k} \), the last probability is equal to

\[
\int \mathbb{P}\left( h^{U_n}_{U_n} \leq m_n + v \bigg| h^{U_n}_{U_n}(x) = m_n + s, h^{U_n}_{U_n}(y) = m_n + t, h^{U_n}_{\partial W_k} = m_n - m_k + v + w \right) \times \mathbb{P}\left( h^{U_n}_{U_n}(x) - m_n \in ds, h^{U_n}_{U_n}(y) - m_n \in dt, h^{U_n}_{\partial W_k} - m_n + m_k - v \in dw \right), \tag{1.36}
\]

where the integral is over all \( s > u, t > u \) and \( w \in \mathbb{R}_{\partial W_k} \). Shifting height and space and using the Gibbs-Markov property, the first term in the integrand breaks into the product of three ballot probabilities:

\[
\mathbb{P}\left( h^{U_n}_{U_n \cap W_k'} \leq 0 \bigg| h_{\partial U'} = -m_n - v, h_{\partial W_k'} = -m_k + w \right).
\]
\[
\mathbb{P}\left( h_{B_k^\setminus \{0\} \land k} \leq 0 \middle| h_{\partial B_k} = -m_k + w''', h(0) = s - v \right) \\
\mathbb{P}\left( h_{B_k^\setminus \{0\} \land k} \leq 0 \middle| h_{\partial B_k} = -m_k + w''', h(0) = t - v \right),
\]

(1.37)

where \( U', W' \) are proper shifts of \( U \) and \( W \) and \( w'', w''' \) are properly translated restrictions of \( w \). The theory in Sect. 1.2 can then be applied to all three terms to yield estimates which depend on \( n, k, s, t \) and \( w \). When \( k \to \infty \) or when bounds are only required, Proposition 1.3 or Proposition 1.4 can be used to turn the dependence on \( w \) to the dependence on \( \overline{w} \). Known bounds and asymptotics for the joint law of \( \overline{h_{Un}}_{\partial W_k^\pm}, h_{Un}(x) \) and \( h_{Un}(y) \) can then be applied to estimate the integral in (1.36).

Lastly, let us remark that while a limit in law for the centered maximum was derived for general Gaussian fields with log-correlations [15], it is not clear to us how to extend the results to such fields. Indeed, the notion of boundary conditions is central to all the statements in Sect. 1.2 and, lacking a Gibbsian structure, it is not clear how to recover them in the case of general fields. Moreover, even if one could make sense of boundary conditions, the method of proof here still relies in an essential way on both the Markovian structure of the field and the specific form of its covariances.

**Paper Outline**

The remainder of the paper is organized as follows. In Sect. 2 we show how to reduce the DGFF ballot event to a corresponding ballot event involving the DRW. Two different DRWs are defined, using both the inward and the outward decompositions. This reduction is then used in Sect. 3, in conjunction with ballot theory for the DRW, to derive a weak upper bound on the DGFF ballot probability, which is then used to prove Theorem 1.8 via stitching. Sect. 4 includes the proof of Theorem 1.1. As in the upper bound, the proof is based on weak statements for the asymptotics of the DGFF ballot probability, which are stated in this section and used, but not proved. The proofs of these weak statements together with the proofs of all the remaining main results are given in Sect. 5.

This manuscript has three appendices. Appendix A includes statements concerning extreme value theory for the DGFF as well as general discrete harmonic analysis results. Appendix B includes estimates for the harmonic extension of DGFF on a subset of its domain. Appendix C includes ballot theory for the DRW. While needed for the proofs in the paper, all of these results are derived using standard arguments and computations, and as such found their place in an appendix. We remark that the results in Appendix B can be of independent use.

**2 Reduction to a Decorated Random Walk Ballot Event**

In this section we show how to express the ballot event on the left-hand side of (1.8) in terms of a corresponding ballot event for a suitable one-dimensional non-homogeneous random walk with decorations (DRW). The latter event can then be handled by generalized versions of the usual ballot estimates for random walks, which are given in Appendix C and used in the sections to follow to prove the main results in this manuscript.

The section begins with some general notation which will be used throughout the paper (Sect. 2.1). We then formally define the DRW process (Sect. 2.2) and introduce the inward concentric decomposition (Sect. 2.3). The decomposition is then used to perform the reduction...
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from the DGFF ballot event to the DRW ballot event (Sect. 2.4). The reduction is summarized in Theorem 2.3. The proof of this theorem, in which the conditions of the DRW are verified, comes next (Sect. 2.5). Finally (Sect. 2.6), we introduce another reduction to a DRW, which is based on an outward concentric decomposition and results in a DRW of which the starting and end point are reversed. The reduction is summarized in Theorem 2.10. As the construction and proofs are analogous to the inward case, we only outline the required changes.

2.1 General Notation

In this subsection we introduce some notation and make some observations that will be used throughout the remaining part of the paper, as will be the definitions from Sect. 1.1.

We define the collections of domains

\[ \Omega^\epsilon = \{ U \in \mathcal{D}_\epsilon : U \text{ is connected}, \epsilon B \subset U^\eta \} \]

\[ \Omega_\epsilon = \{ V \in \mathcal{D}_\epsilon : V^- \text{ is connected} \} \]

for \( \epsilon \in (0, 1) \), \( \eta \geq 0 \). These definitions and (1.6) imply the useful properties that

\[ U^\eta_n \supset B_l \quad \text{whenever} \quad U \in \Omega^\epsilon, \ n - l > -\log \epsilon, \]

\[ V^{-\eta}_k \supset B_l^- \quad \text{whenever} \quad V \in \Omega_\epsilon, \ l - k > \log(\zeta + \epsilon^{-1}). \]

We also note that in all statements in the paper, the assumptions only become more general when \( \epsilon \) is chosen smaller (at the expense of less sharp constants and rates of convergence). For instance, we have \( \Omega^\epsilon \subset \Omega_\epsilon \), if \( \epsilon' \leq \epsilon \).

We will use the shorthand notation

\[ \mathbb{P} \left( \cdot \mid h_{\partial U_n} = -m_n + u, h_{\partial V^\eta_k} = -m_k + v \right) = \mathbb{P}^{U, n, u}_{V, k, v} = \mathbb{P}^{n, u}_{k, v}. \]

The expectation, variance and covariance associated with this probability measure will also be denoted by \( \mathbb{E}^{U, n, u}_{V, k, v} \) or \( \mathbb{E}^{n, u}_{k, v}, \var{V}^n_k \) and \( \text{Cov}^n_k \), respectively. Also, the integer \( T_{n-k} := \lfloor n - k \rfloor + \lfloor \log \epsilon \rfloor - \lfloor \log(\zeta + \epsilon^{-1}) \rfloor \) will be used in (2.12) below to describe number of scales in the annulus-like domain \( U^\eta_n \cap V^{-\eta}_k \).

The following property of our discretized domains will be useful:

**Remark 2.1** For \( W \in \mathcal{D} \), we consider the domain \( W^\pm = W^+ \cup W^- \) and we show in this remark that the boundary of its discretization \( W^\pm_l \) is given by

\[ \partial W^\pm_l = \partial W_l \cup \partial W^-_l \quad \text{for all} \ l \geq 0. \]

In particular, for sufficiently large \( l \), there are vertices “between” \( W^+_l \) and \( W^-_l \) such that these sets are not path-connected (cf. Remark 2 on p. 8). To prove (2.4), it suffices by the definition of the outer boundary to show that \( W_l \cap \partial W^-_l = \emptyset \) and \( W^-_l \cap \partial W_l = \emptyset \). The first statement follows as \( x \in W_l \) implies by (1.4) that \( |e^{-l} x - z| > e^{-l}/2 \) for all \( z \in \partial W \), hence \( |e^{-l} y - e^{-l}x| > e^{-l} \) for all \( y \in W^-_l \). The second statement follows analogously.

As a consequence of (2.4) and the definition of the outer boundary, \( \partial W^\pm_l \setminus \partial W_l \) is not path-connected to \( W_l \). Hence, specifying boundary conditions on \( \partial W^\pm_l \) instead of \( \partial W_l \) does not alter the DGFF on \( W_l \):

\[ \mathbb{P}(h \in \cdot \mid h_{\partial W_l} = w_{\partial W_l}) = \mathbb{P}(h \in \cdot \mid h_{\partial W^\pm_l} = w) \quad \text{on} \ W_l \]

for any \( w \in \mathbb{R}^{\partial W^\pm_l} \). Analogously, specifying boundary conditions on \( \partial W^\pm_l \) instead of \( \partial W^-_l \) does not alter the DGFF on \( U_n \cap W^-_l \). Also, as a consequence of the above, \( W_l \) and \( W^-_l \) are
not path-connected, and the sets \( W_1, \partial W^\pm_1 \) and \( W_1^- \) are disjoint. If \( W_1 \subset U_n \) and \( W_1^- \subset V_k^- \), this allows to apply the Gibbs-Markov property (Lemma \( B.10 \)) to decompose the DGFF \( h \) on a domain of the form \( U_n \cap V_k^- \) into the sum of independent DGFFs on the disjoint subsets \( U_n \cap W_1^- \) and \( W_1 \cap V_k^- \), and the binding field \( \overline{h}_0 W^\pm_1 \).

Let \(( S_t )_{t \geq 0}^\infty \) under a probability measure \( P_x \) with associated expectation \( E_x \) denote simple random walk on \( \mathbb{Z}^2 \) started at \( x \in \mathbb{Z}^2 \). For \( A \subsetneq \mathbb{Z}^2 \), we denote the first exit time of \( S \) from \( A \) by \( \tau_A := \inf \{ t \in \mathbb{N} : S_t \not\in A \} \), and the discrete Poisson kernel on \( A \) by \( \Pi_A( x, \cdot ) := P_x( S_{\tau_A} = \cdot ) \). We recall that \( G_A(x, y) = \sum_{t=0}^\infty P_x(S_t = y, i < \tau_A) \), \( x, y \in \mathbb{Z}^2 \) denotes the Green kernel of \( S \) on \( A \), and \( a(x) = \sum_{i=0}^\infty (P_x(S_i = x) - P_0(S_i = x)) \) the potential kernel of \( S \) on \( \mathbb{Z}^2 \).

By Theorem 4.4.4 of [19], there exists a constant \( c_0 > 0 \) such that

\[
a(x) = g \log |x| + c_0 + O( |x|^{-2} ) , \quad x \in \mathbb{Z}^2 . \tag{2.6}
\]

We reserve the notation \( c_0 \) for this constant throughout the paper. By Proposition 4.6.2 of [19], for finite \( A \),

\[
G_A(x, y) = \sum_{z \in \partial A} \Pi_A(x, y)a(y - z) - a(x - y) . \tag{2.7}
\]

We will often use the representation

\[
\overline{f}_A(x) = \sum_{z \in \partial A^c} f(z) P_x(S_{\tau A^c} = z) , \quad x \in \mathbb{Z}^2 \tag{2.8}
\]

for the bounded harmonic extension of a function \( f : A \to \mathbb{R} \), for \( A \subset \mathbb{Z}^2 \) finite and non-empty.

Furthermore, we write \( \wedge_{n,k} := k \wedge (n - k) \) and \( \wedge_{n,l,k} := (n - l) \wedge (l - k) \), \( g := 2/\pi \) and \( \alpha := 2/\sqrt{g} \). We also set \( \varphi^{A,B} := \mathbb{E}(h^A | h^{A \setminus B}) \) a.s. By Lemma \( B.10 \), \( \varphi^{A,B} := \overline{h}_0^{A \setminus B} \) a.s. for \( B \subset A \subset \mathbb{Z}^2 \).

### 2.2 The Decorated Non-homogeneous Random Walk Process

We now formally define the DRW process. Let \( T \in \mathbb{N} \cup \{ \infty \} \), \( a, b \in \mathbb{R} \) and \( (\sigma_k^2 : k = 1, \ldots, T) \) be a sequence of positive real numbers. Set \( s_k := \sum_{t=1}^k \sigma_t^2 \) and consider the collections of random variables \(( S_k )_{k=0}^T \) and \(( D_k )_{k=1}^T \) which we assume to be defined on the same probability space, equipped with a probability measure to be denoted by \( \mathbb{P} \) (with expectation, covariance and variance given by \( \mathbb{E}, \mathbb{Cov} \) and \( \mathbb{V}ar \)).

We shall impose the following three assumptions which depend on a parameter \( \delta > 0 \) (this parameter will usually determine the value of constants and rates of convergence in the theorems we prove for this walk).

(A1) The process \(( S_k )_{k=0}^T \) is Gaussian with means and covariances given by

\[
\mathbb{E} S_k = \frac{b s_k + a (s_T - s_k)}{s_T} , \quad \mathbb{Cov}(S_k, S_m) = \frac{s_k (s_T - s_m)}{s_T} : \quad 0 \leq k \leq m \leq T . \tag{2.9}
\]

Moreover, \( \sigma_k^2 \in (\delta, \delta^{-1}) \) for all \( k = 1, \ldots, T \).

(A2) For all \( m = 1, \ldots, T - 1 \), given \( S_m \) the two collections below are conditionally independent of each other:

\[
(S_k, D_k : k = 1, \ldots, m) , \quad (S_\ell : \ell = m + 1, \ldots, T) . \tag{2.10}
\]
(A3) For all $k = 1, \ldots, T, t > 0$,
\[
\mathbb{P}(|D_k| > \delta^{-1} \wedge_{T,k}^{1/2-\delta} + t) \leq \delta^{-1} e^{-t^\delta}.
\]  
(2.11)

If $T = \infty$, we take $a$ and $s_k$ to be the meaning of the respective right-hand sides in Assumption (A1). The reader will recognize that the law of $(S_k)_{k=0}^T$ under $\mathbb{P}$ is that of a random walk whose $k$-th step is $\mathcal{N}(0, \sigma_k^2)$, starting from $a$ at time $0$ and conditioned to be at $b$ at time $T$, if $T < \infty$, or otherwise unrestricted (in which case $b$ is irrelevant). We shall sometimes explicitly indicate the boundary conditions of such walk by writing $\mathbb{P}_{0,a}$ in place of $\mathbb{P}$ (and using a similar convention for variances and covariances). Notice that the $D_k$-s, which we refer to as “decorations”, may indeed depend on the walk, as long as they satisfy the Markovian-type structure given in Assumption (A2). We shall refer to the pair $((S_k)_{k=0}^T, (D_k)_{k=1}^T)$ as a DRW process.

2.3 Inward Concentric Decomposition

Next, we introduce the inward concentric decomposition. For $\epsilon \in (0, 1), \eta, \zeta \geq 0, 0 \leq k < n,$ and $U \in 1_{\epsilon}^\nu, V \in \mathbb{N}_\epsilon$, we now define two collections of concentric sets that resemble balls or complements of balls, using which we will construct collections of annulus-like sets that we then pack into the domain $U_n \cap V_k^-$ (Figure 1 depicts the inward concentric decomposition). Throughout the paper, we always define $T_{n-k}$ as a function of $n - k$ by
\[
T_{n-k} = [n - k] + [\log \epsilon] - [\log(\zeta + \epsilon^{-1})],
\]  
(2.12)
and we often abbreviate $T = T_{n-k}$. We assume that $T_{n-k} \geq 1$, then (2.2) ensures that the concentric ball-like sets
\[
\Delta_1 = U_n, \quad \Delta_0 = U_n^{\eta}, \quad \Delta_p = B_{n+\lfloor \log \epsilon \rfloor - p} \quad \text{for} \quad p = 1, \ldots, T - 1, \quad \Delta_T = (V_k^{\zeta})^c,
\]  
\[
\Delta_{T+1} = (V_k^-)^c
\]
are nested, $\Delta_1 \supset \ldots \supset \Delta_{T+1}$. Furthermore, we define the sets
\[
\Delta'_1 = (U_n)^c, \quad \Delta'_0 = U_n^{\eta^c}, \quad \Delta'_p = B_{n+\lfloor \log \epsilon \rfloor - p} \quad \text{for} \quad p = 1, \ldots, T - 1, \quad \Delta'_T = V_k^{\zeta^c}.
\]  
(2.13)

For $p = 0, \ldots, T$, let $h_p$ be a DGFF on $A_p := \Delta'_p \cap \Delta_{p-1}$ with boundary values zero. Moreover, we define $J_p := A_p' \cup (\Delta_p \cap V_k^-)$ for $p = 0, \ldots, T - 1, J_T = A_T'$, and let $\varphi_p$ be distributed as $(h'_p)_{\mathbb{Z}^2 \setminus J_p}$ where $h'_p$ is a DGFF on $\Delta_{p-1} \cap V_k^-$ with boundary values 0 (and, according to our notational conventions, equal to zero on $\mathbb{Z}^2 \setminus (\Delta_{p-1} \cap V_k^-)$). We note that $(h'_p)_{\mathbb{Z}^2 \setminus J_p}$ is the binding field from $\Delta_{p-1} \cap V_k^-$ to $J_{p}$, as defined in the Gibbs-Markov decomposition (Lemma B.10). We assume that the random fields $\varphi_0, \ldots, \varphi_T, h_0, \ldots, h_T$ are independent.

We will also use the notation $\varphi_{p,q} := \sum_{j=p}^q \varphi_j$ where $0 \leq p \leq q \leq T$. For $p < q$, we set $\varphi_{p,q} = 0$. Note that for $q \leq T - 1, \varphi_{p,q}$ is distributed as $\varphi_{\Delta_{p-1} \cap V_k^- \cup A_p' \cup \ldots \cup A_q' \cup V_k^-}$, and that $\varphi_{p,T}$ is distributed as $\varphi_{\Delta_{p-1} \cap V_k^- \cup A_p' \cup \ldots \cup A_T' \cap V_k^-}$, which can be seen by applying the Gibbs-Markov property (Lemma B.10) successively to the subsets $A_p', \ldots, A_q'$.

In the next proposition, we decompose the DGFF on $U_n \cap V_k^-$ on each of the annuli $A_p := \Delta_{p-1} \setminus \Delta_p$ with $p = 0, \ldots, T + 1$ in terms of the fields $\varphi_0, \ldots, \varphi_{p \wedge T}$ and $h_{p \wedge T} 1_{q \leq T}$. We note that $A_p \supset A_p'$ for $p = 0, \ldots, T$ and that $(A_p)_{p=0}^{T+1}$ forms a disjoint covering of $U_n \cap V_k^-$. 

\[\text{Springer}\]
Fig. 1 Inward concentric decomposition. The outermost boundary in each subfigure belongs to \( U_n \), the innermost one to \( V_k^- \). (a) The discrete sets \( \Delta_p \) (marked in yellow for one \( p \)) are similar to balls, and the discrete sets \( \Delta'_p \) (marked in green for one \( p \)) are similar to complements of balls. The enumerating index \( p = -1, \ldots, T + 1 \) runs from the outer domain \( \Delta_{-1} = U_n, \Delta_0 = U_n^{\eta} \) to the inner domain \( \Delta_{T+1} = (V_k^-)^c, \Delta'_0 = V_k^- \). (b) The intersections \( A'_p := \Delta_{p-1} \cap \Delta'_p \) (marked in light blue) and the slightly larger set differences \( A_p := \Delta_{p-1} \setminus \Delta_p \) (which comprises of the regions marked in light and in dark blue) are annulus-like sets. (c) \( J_p \) is the union of \( A'_p \) and \( \Delta_p \cap V_k^- \) (marked in purple).

**Proposition 2.2** (Inward concentric decomposition) Assume that \( T = T_{n-k} \geq 1 \). There exists a coupling of \( h_{U_n \cap V_k^-} \) and \((\varphi_p, h_p)^T_{p=0}\) such that

\[
h_{U_n \cap V_k^-} (x) = \sum_{p=0}^{T} (\varphi_p(x) + h_p(x)) = \sum_{p=0}^{q{T}} \varphi_p(x) + h_q(x) 1_{q \leq T} \tag{2.15}
\]

for \( x \in A_q, q = 0, \ldots, T + 1 \).

Note that the binding field \( \varphi_p \) is zero outside \( \Delta_{p-1} \cap V_k^- \), and \( h_p \) is zero outside \( A_p \). Also, if \( \eta = 0 \), then \( A_0 = \emptyset \) and \( \varphi_0 = 0 \).

**Proof of Proposition 2.2** By the Gibbs-Markov property (Lemma B.10), \( h_{U_n \cap V_k^-} \) is distributed as \( \varphi_0 + h'_0 \) where \( h'_0 \) is a DGFF on \( A'_0 \) with boundary values zero that is independent of \( \varphi_0 \). The assertion follows by iterating this step, next applying the Gibbs-Markov property to \( h'_0 \) and \( A'_1 \) and so on.

\( \square \)

### 2.4 The Reduction

We now provide the correspondence between the DGFF and the decorated random walk bridge, based on the inward concentric decomposition. An analogous correspondence based on the outward concentric decomposition is given in Sect. 2.6.

In Sects. 2.4 and 2.5, we assume \( n - k \) to be sufficiently large such that \( T = T_{n-k} \geq 1 \), and that \( h \) under \( \mathbb{P}^{h,\eta,\mu}_{k,v} \) is coupled to \((\varphi_p, h_p)^T_{p=1}\) from Proposition 2.2 such that

\[
h = \sum_{p=0}^{T} (\varphi_p + h_p) + (-m_n 1_{\partial U_n} + u - m_k 1_{\partial V_k^-} + v) 1_{\partial U_n \cup \partial V_k^-}. \tag{2.16}
\]
In each of the sets $A_p$ for $p = 1, \ldots, T$, we want to approximate $h$ by the position of a random walk bridge. To this aim, we define the harmonic average

$$X_p = \sum_{z \in \partial \Delta_p} \varphi_p(z) \Pi_{\Delta_p}(0, z) = (\varphi_p)_{\partial \Delta_p}(0) = (h_p^*)_{\partial \Delta_p}(0)$$

(2.17)

whose variance we denote by $\sigma_p^2 := \text{Var} X_p$, and where we recall the definition of the Poisson kernel $\Pi_{\Delta_p}$ from Sect. 2.1. We write $s_{p,q} = \sum_{i=p}^{q} \alpha_i^2$. Now we couple $(X_1, \ldots, X_T)$ with the process

$$\tilde{S}_t := (s_{1,T} - t) \int_0^t \frac{dW_\tau}{s_{1,T} - \tau}, \quad t \in [0, s_{1,T}]$$

(2.18)

by choosing the Brownian motion $(W_\tau, \tau \in [0, s_{1,T}])$ as follows: we define $W_0 = 0$, $W_{s_{1,p}} = X_1 + \ldots + X_p$ for $p = 1, \ldots, T$, and for $\tau \in [s_{1,p-1}, s_{1,p}]$, we let

$$W_\tau = X_{p-1} + \frac{s_{1,p-1}}{\sigma_p^2} (X_p - X_{p-1}) + \sqrt{\frac{s_{1,p-1}}{\sigma_p^2}} B^{(p)}(\tau - s_{1,p-1})/\sigma_p^2,$$

(2.19)

where $B^{(p)}$ is a standard Brownian bridge from $0$ to $0$ of length $1$, independent of everything else. As

$$\text{Cov}(\tilde{S}_t, \tilde{S}_{t'}) = (s_{1,T} - t)(s_{1,T} - t') \int_0^{t'} \frac{d\tau}{(s_{1,T} - \tau)^2} = \frac{t(s_{1,T} - t')}{s_{1,T}}$$

(2.20)

for $0 \leq t < t' \leq s_{1,T}$, the centered Gaussian process $(\tilde{S}_t, t \in [0, s_{1,T}])$ is a Brownian bridge of length $s_{1,T}$ from $0$ to $0$. Hence,

$$S'_p := \tilde{S}_{s_{1,p}}, \quad p = 1, \ldots, T, \quad S'_0 = 0$$

(2.21)

is distributed as a random walk with centered Gaussian steps having variances $\sigma_p^2$, $p = 1, \ldots, T$, starting at $0$ and conditioned on hitting $0$ after $T$ steps.

We approximate the harmonic extension of the boundary values $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k^-}$ by

$$\beta_p = \frac{s_{1+p,T}}{s_{1,T}} \overline{u}(0) + \frac{s_{1,p}}{s_{1,T}} \overline{v}(\infty), \quad p = 0, \ldots, T.$$  

(2.22)

Also recall the independent DGFF $h_p$. By the Gibbs-Markov property,

$$h_p = h - \varphi_{0,p} - (-m_n 1_{\partial U_n} + u - m_k 1_{\partial V_k^-} + v) \frac{\partial}{\partial U_n \cup \partial V_k^-}$$

on $A_p$.

(2.23)

Moreover, we define

$$\gamma(y) = (-m_n 1_{\partial U_n} + u - m_k 1_{\partial V_k^-} + v) \frac{\partial}{\partial U_n \cup \partial V_k^-}(y) + m_{n-p} - \beta_p$$

for $y \in A_p$.

(2.24)

Finally, we set

$$D_p = \max_{y \in A_p} \left\{ h_p(y) - m_{n-p} + \varphi_{0,p}(y) - S'_p + \gamma(y) \right\},$$

(2.25)

and

$$S_p = S'_p + \beta_p.$$  

(2.26)

With the definitions above, we can now state the following theorem, which shows that the processes just defined form a suitable DRW and relates the DGFF ballot event with that of the DRW. The proof of this theorem is the subject of the next subsection. Notice that (2.28) follows directly and immediately from the definitions above.
Theorem 2.3 Let \( \epsilon \in (0, 1) \). Then there exists \( \delta \in (0, 1/3) \) such that for all \( \eta, \zeta \in [0, \epsilon^{-1}] \), \( U \in \mathcal{U}_n^0, V \in \mathcal{V}_k, 0 \leq n < k \) with \( T_{n-k} \geq 1, u \in \mathbb{R}^\partial U_u, v \in \mathbb{R}^\partial V_k \) that satisfy
\[
\max \{ \| \bar{u}(0), \bar{v}(\infty), \text{osc} \bar{u}_{\eta}, \text{osc} \bar{v}_{\zeta} \} \leq \epsilon^{-1}, \quad |\bar{u}(0) - \bar{v}(\infty)| \leq \epsilon^{-1}(n - k), \quad (2.27)
\]
we have that \((S_i)_{i=0}^T, (D_i)_{i=1}^T \) satisfies Assumptions (A1) – (A3) with \( \alpha = \bar{u}(0) \) and \( b = \bar{v}(\infty) \) under the identification \( P = P_{0,\bar{u}(0)}^T, \bar{v}(\infty) \equiv \mathbb{P}_{k,v}^n \). Moreover,
\[
\{ S_p + D_p \leq 0 \} = \{ h_{A_p} \leq 0 \} \text{ for } p = 1, \ldots, T, \quad \text{and}
\]
\[
\bigcap_{p=1}^T \{ S_p + D_p \leq 0 \} = \left\{ h_{U_1^p \cap V_k^{-\zeta}} \leq 0 \right\}. \quad (2.28)
\]

2.5 Verification of the DRW Conditions: Proof of Theorem 2.3

In this subsection we prove Theorem 2.3. We begin, by verifying the assumption on the dependency structure of the decorated random walk.

Lemma 2.4 The random variables \((S_p)_{p=0}^T, (D_p)_{p=1}^T \) satisfy (A2) under the identification \( P = P_{0,\bar{u}(0)}^T, \bar{v}(\infty) \equiv \mathbb{P}_{k,v}^n \).

Proof As \( S_p - S'_p = \beta_p \) is deterministic, it suffices to show the assertion for \((S'_p)_{p=0}^T, (D_p)_{p=1}^T \). For \( p = 1, \ldots, T - 1 \), let \( P_p \) be the \( \sigma \)-algebra generated by \((W_t, t \leq s_{1,p}), \varphi_1, \ldots, \varphi_p, h_1, \ldots, h_p \). By the Gibbs-Markov property and the definition of \( W_t \), the process \((W_{s_{1,p}+t} - W_{s_{1,p}}, t \in [0, s_{1,T} - s_{1,p}] \) is independent of \( P_p \). While \( D_1, \ldots, D_p \) and \( S'_1, \ldots, S'_p \) are \( P_p \)-measurable, the following representation shows that \((S'_j - S'_p)_{j=p+1}^T \) is measurable with respect to \((W_{s_{1,p}+t} - W_{s_{1,p}}, t \in [0, s_{1,T} - s_{1,p}] \) and \( S'_p \) and hence conditionally independent of \( P_p \) given \( S'_p \):
\[
S'_p = (s_{1,T} - s_{1,j}) \int_{s_{1,p}}^{s_{1,T}} \frac{dW_t}{s_{1,T} - \tau} - (s_{1,j} - s_{1,p}) \int_0^{s_{1,p}} \frac{dW_t}{s_{1,T} - \tau}
\]
\[
= (s_{1,T} - s_{1,j}) \int_{s_{1,p}}^{s_{1,T}} \frac{dW_t}{s_{1,T} - \tau} - s_{1,j} - s_{1,p} \int_{s_{1,p}}^{s_{1,T}} S'_p, \quad (2.29)
\]
where we used the definition of \((S'_p) \) which is given by (2.21) and (2.18).

As the binding field is harmonic, we have
\[
\varphi_p(y) = \sum_{z \in \partial A_p \cup \partial A_p} \varphi_p(z) \prod_{A_p \cup A_p \cap V_k}(y, z), \quad y \in \mathbb{Z}^2 \quad (2.30)
\]
for \( p = 0, \ldots, T \).

Next we show bounds for the variance \( \sigma_p^2 = \text{Var}_k \mathcal{X}_p \).

Lemma 2.5 Let \( \epsilon \in (0, 1) \). There exists \( C = C_\epsilon < \infty \) such that \( C^{-1} < \text{Var}_k \mathcal{X}_p < C \) and
\[
|s_{p,q} - g(p, q + 1)| \leq C(1 + \log(1 + q - p)) \quad (2.31)
\]
for all \( \eta, \zeta \in [0, \epsilon^{-1}] \), \( 0 \leq k < n \) with \( T_{n-k} \geq 1, p, q \in \mathbb{N} \) with \( p \leq q \leq T \), and all \( U \in \mathcal{U}_n^0, V \in \mathcal{V}_k \). Moreover, for \( p \in \mathbb{N} \) fixed, \( \text{Var}_k \mathcal{X}_p \) converges as \( n - k \to \infty \) uniformly in \( U \in \mathcal{U}_n^0, V \in \mathcal{V}_k \). We also have \( \text{Var}_k \mathcal{X}_0 < C \).
Proof For $p \geq 1$, the bound for $\text{Var}_k^p \mathcal{X}_p$ and the convergence of $\text{Var}_k^p \mathcal{X}_p$ follow from Lemma B.6 where we choose $j = 1$, and we choose $U, W$ and $n$ such that $U_n$ becomes our $\Delta_p - 1$, and $W_{n-1} = \Delta_p$. Also $\text{Var}_k^p \mathcal{X}_0$ is bounded by a constant as

$$\text{Var}_k^p \left( h_{\partial U_n}(0) \right) \left( h_{(2e^{-1}B)_n} \right) = 0, h_{\partial V_k} = 0 \right) = \text{Var}_k^p \mathcal{X}_0 + \text{Var} \left( \varphi(2e^{-1}B)_n \cap V_k. U_n \cap V_k \right)_{\partial U_n}(0),$$

(2.32)

by the Gibbs-Markov property, and the left-hand side is bounded by a constant by the same argument as for $\mathcal{X}_p$ with $p \geq 1$.

For assertion (2.31), let $j \in \{p, \ldots, q\}$. By the Gibbs-Markov property (Lemma B.10) applied at $\partial V_k^-$,

$$\text{Var}_k^p \mathcal{X}_j = \text{Var} \left( h_{\partial \Delta_j}^{-1}(0) \right) - \text{Var} \left( \varphi(2e^{-1}B)_n \cap V_k^{-1} \right)_{\partial \Delta_j}(0).$$

(2.33)

The second variance on the right-hand side is of order $(T - j + 2)^{-1}$ by Lemma B.13.

Moreover, by successively applying the Gibbs-Markov property at $\partial \Delta_p, \ldots, \partial \Delta_q$,

$$\sum_{j=p}^{q} \text{Var} \left( h_{\partial \Delta_j}^{-1}(0) \right) = \text{Var} \left( \varphi(2e^{-1}B)_n \cap V_k^{-1} \right)_{\partial \Delta_q}(0).$$

(2.34)

By Lemma B.7, $\left| \text{Var} \left( h_{\partial \Delta_j}^{-1}(0) \right) - g(q - p + 1) \right|$ is uniformly bounded by a constant. We now also sum the second term on the right-hand side of (2.33) over $j = p, \ldots, q$, then we obtain

$$|s_{p,q} - g(q - p + 1)| \leq C \left( 1 + \log(1 + T - p) - \log(1 + T - q) \right)$$

(2.35)

Assertion (2.31) now follows from subadditivity of the logarithm.

To compare $\mathcal{S}_p$ with $\varphi_{0,p}$ on $\mathcal{F}_p$, we also define

$$\mathcal{Y}_p = \sum_{j=1}^{p} \frac{s_{p,T}}{s_{j,T}} \mathcal{X}_j$$

(2.36)

for $p = 1, \ldots, T$. The centered Gaussian random variables $\mathcal{Y}_p$ and $\mathcal{S}_p'$ can be compared as follows.

Lemma 2.6 Let $\varepsilon \in (0, 1)$. Then there exists $C = C_\varepsilon < \infty$ such that

$$\text{Var}_k^p(\mathcal{S}_p' - \mathcal{Y}_p) \leq C(n - k - p + 1)^{-1}$$

(2.37)

for all $\eta, \xi \in [0, \varepsilon^{-1}], 0 \leq n < k$ with $T_{n-k} \geq 1, U \in \mathcal{U}_k^n, V \in \mathcal{V}_\varepsilon, p = 1, \ldots, T$.

Proof If $p = T$, then we use that $\mathcal{S}_p' = 0$ a.s. and we have

$$\text{Var}_k^p(\mathcal{S}_p' - \mathcal{Y}_p) = \sum_{j=1}^{T} \frac{s_{j,T}^2}{s_{j,T}^2} \leq C_\varepsilon$$

(2.38)

by Lemma 2.5. In the following, we can thus assume $p \leq T - 1$. From the definition of $\mathcal{S}_p'$ and $\mathcal{Y}_p$, we obtain

$$\text{Var}_k^p(\mathcal{S}_p' - \mathcal{Y}_p) = \sum_{j=1}^{p} \int_{s_{j-1,T}}^{s_{j,T}} \left( \frac{s_{p+1,T} - s_{p,T}}{s_{j,T}} \right)^2 \frac{dt}{s_{j,T}^2}.$$  

(2.39)
The expression in the brackets on the right-hand side can be bounded as follows:

\[
- \frac{\sigma_p^2}{s_j,T} \leq \frac{S_{p+1,T}}{s_{1,T} - t} - \frac{S_p,T}{s_{j+1,T}} - \frac{S_{p+1,T}}{s_{j+1,T}} + \sigma_j^2 \leq - \frac{\sigma_p^2}{s_{j+1,T}} + s_{p+1,T} \frac{\sigma_j^2}{s_j,T s_{j+1,T}}
\]

for \( t \in [s_1, j-1, s_{1,j}] \). Using also Lemma 2.5, we obtain that

\[
\forall \text{Var}_k'(S_p' - \mathcal{Y}_p) \leq C_\epsilon \sum_{j=1}^{p} (T - j + 2)^{-2} \leq C_\epsilon (T - p + 1)^{-1},
\]

and the assertion follows. \( \square \)

We now come to the tails of the decorations. First we consider the part of \( \mathcal{D}_p \) which does not depend on the boundary values \( u \) and \( v \), namely \( \overline{\mathcal{D}}_p := \max_{y \in A_p} \{ \varphi_{0,p}(y) - S'_p + h_p(y) - m_{n-p} \} \).

**Lemma 2.7** For each \( \epsilon \in (0, 1) \), there exists \( \delta \in (0, \frac{1}{2}) \) such that the bounds in (A3) hold with \( (S'_p)_{T = 0}^T = (\overline{\mathcal{D}}_p)_{T = 0}^T \) in place of \( (S_p)_{T = 0}^T = (\mathcal{D}_p)_{T = 0}^T \) there, with \( a = \overline{u}(0), b = \overline{\nu}(\infty) \), for all \( \eta, \zeta \in [0, 1] \), \( U \in \mathcal{U}_\eta, V \in \mathcal{U}_\zeta, 0 \leq k < n \) with \( T_{n-k} \geq 1 \) and \( u \in \mathbb{R}^\partial U_n, v \in \mathbb{R}^\partial V_k \), under the identification \( P = P_{\mathbf{0}, \mathbf{1}(0)} \equiv P_{n,u}^{(0)} \).

**Proof** To avoid the roughness of \( \varphi_{0,p} \) near \( \partial A'_p \), we lump the binding fields \( \varphi_p \) and \( \varphi_{p-1} \) together with \( h_p \) by defining

\[
\tilde{h}_p(y) = h_p(y) + \varphi_p(y) + \varphi_{p-1}(y) - m_{n-p}, \quad y \in A_p
\]

and

\[
\mathcal{D}'_p(y) = \varphi_{0,p-2}(y) - \mathcal{Y}_p, \quad y \in A_p,
\]

for \( p = 1, \ldots, T \) (we recall that \( \varphi_{0,-1} = 0 \) by definition). Then, for \( t > 0 \) and \( p = 1, \ldots, T \), a union bound gives

\[
\mathbb{P}_{k,v}^n(\overline{\mathcal{D}}_p > t) \leq \mathbb{P}_{k,v}^n(\max_{A_p} \tilde{h}_p > t/3) + \mathbb{P}_{k,v}^n(\max_{A_p} \mathcal{D}'_p > t/3) + \mathbb{P}_{k,v}^n(|\mathcal{Y}_p - S'_p| > t/3)
\]

and

\[
\mathbb{P}_{k,v}(\overline{\mathcal{D}}_p > t) \leq \mathbb{P}_{k,v}[\max_{A_p} \tilde{h}_p < -t/3] + \mathbb{P}_{k,v}[\min_{A_p} \mathcal{D}'_p < -t/3] + \mathbb{P}_{k,v}[|\mathcal{Y}_p - S'_p| > t/3].
\]

By the Gibbs-Markov property (Lemma B.10) and by (2.42), the restriction of \( \tilde{h}_p + m_{n-p} \) to \( A_p \) is distributed as \( h^{\Delta_p - 2 \cap V_k} \). Hence, again by the Gibbs-Markov property, we can represent the restriction of the DGFF \( h^{\Delta_p - 2 \cap V_k} \) to \( A_p \) as the sum of the independent fields \( \tilde{h}_p + m_{n-p} \) and the binding field \( \varphi^{\Delta_p - 2, \Delta_p - 2 \cap V_k} \). As \( \varphi^{\Delta_p - 2, \Delta_p - 2 \cap V_k} \) is with probability 1/2 positive at the maximizer of \( \tilde{h}_p \), it follows that

\[
\frac{1}{2} \mathbb{P}_{k,v}^n(\max_{A_p} \tilde{h}_p > t/3) \leq \mathbb{P}(\max_{A_p} h^{\Delta_p - 2} - m_{n-p} > t/3).
\]
By (2.17) and (2.30), the first sum on the right-hand side is bounded by variance is bounded by a constant by Lemma 2.5. For all
so as to obtain that the triple sum in (2.48) is bounded by a constant times

Again by Lemma 2.5, and the other terms on the right-hand side of (2.47) are bounded by a constant by Lemma 2.5. Using (2.49), the analogous lower bound, and Lemma A.6, we bound the absolute differences in (2.48) by the left-hand side of

and so as to obtain that the triple sum in (2.48) is bounded by a constant times

By (2.17) and (2.30), the first sum on the right-hand side is bounded by

and the other terms on the right-hand side of (2.47) are bounded by a constant by Lemma 2.5. Again by Lemma 2.5,

Using (2.49), the analogous lower bound, and Lemma A.6, we bound the absolute differences in (2.48) by the left-hand side of

so as to obtain that the triple sum in (2.48) is bounded by a constant times \( \sum_{j=1}^{p-2} (p-j)^{-2+1/10} \text{Var}_k \mathcal{X}_j \). By Lemma 2.5, it follows that \( \text{Var}_k^n D'_p(y) \) bounded by a constant. Hence, the Borell-TIS inequality (see e.g. Theorem 2.1.1 in [2]) shows that

for all \( t > 0, U \in \Sigma^n_k, V \in \Omega^n_k, 0 \leq k < n, p = 1, \ldots, T \).

Now we show that \( \mathbb{E}^{n,u}_{k,v} \max_{A_p} D'_p \) is bounded by a constant. From Definitions (2.36) and (2.43), we have \( \max_{y \in A_p} D'_p(y) = \mathcal{Y}_1 = \mathcal{X}_1 \) which is a centered Gaussian whose variance is bounded by a constant by Lemma 2.5. For \( p \geq 2 \), we have the following bound for the (squared) intrinsic metric: For \( x, y \in A_p, \)

The right-hand side is bounded from above by an exponential tail which follows from extreme value theory for the DGFF (Lemma A.1). By Lemma A.2, the first probability on the right-hand side of (2.45) is bounded by a constant times \( e^{-t^{2-\epsilon}} \). The third summand on the right-hand side of (2.44) and (2.45) has uniformly Gaussian tails by Lemma 2.6.

To bound the second summand on the right-hand side of (2.44) and of (2.45), we plug (2.36) into (2.43), so as to obtain for \( y \in A_p \) that

\[
\text{Var}_k^n D'_p(y) = \sum_{j=1}^{p-2} \text{Var}_k^n \left( \phi_j(y) - \frac{s_{p,T}}{s_{j,T}} \right) + 1_{|p|=2} \text{Var}_k^n \phi_0(y) \\
+ \text{Var}_k^n \left( \frac{s_{p,T}}{s_{p-1,T}} \right) \mathcal{X}_{p-1} + \text{Var}_k^n \mathcal{X}_p.
\]
\[
\times \left[ \Pi_{\Delta_j \cap V_k^c}(x, z') - \Pi_{\Delta_j \cap V_k^c}(y, z') \right]
\times \mathbb{E}\left[ h^{\Delta_j \cap V_k^c}(z) h^{\Delta_j \cap V_k^c}(z') \right]
\]
(2.52)

where we used the definition of \( \varphi_{0,p-2} \), the Gibbs-Markov property, and (2.30). Applying Theorem 6.3.8 of [19] as in the proof of Lemma B.8 in Sect. B.3, we see that the differences of the Poisson kernels in the last display are bounded (in absolute value) by a constant times \( e^{-2(n-j)} |x-y| \Pi_{\Delta_j \cap V_k^c}(x, z) \) and \( e^{-2(n-j)} |x-y| \Pi_{\Delta_j \cap V_k^c}(x, z') \), respectively. Hence, (2.52) is further bounded from above by a constant times

\[
\sum_{j=0}^{p-2} \frac{|x-y|^2}{e^{2(n-j)}} \sum_{z, z' \in \partial \Delta_j} \Pi_{\Delta_j}(x, z) \Pi_{\Delta_j}(y, z) \mathbb{E}\left[ h^{\Delta_j \cap V_k^c}(z) h^{\Delta_j \cap V_k^c}(z') \right] \leq C e^{\frac{|x-y|^2}{e^{2(n-p)}}}.
\]
(2.53)

where we used that on the left-hand side, the covariance is nonnegative, and the sum over \( z, z' \) is bounded by a constant as it is equal to \( \sigma(x, y) \) in Proposition B.3.

By Fernique majorization (see e.g. Theorem 4.1 in [1]) with the uniform probability measure on \( A_p \) as the majorizing measure, it follows that the second terms on the right-hand sides of (2.44) and (2.45) are uniformly upper bounded by a Gaussian tail, which completes the verification of (A3).

Next we treat the full decoration process \( (S_p)_{p=1}^T \).

**Lemma 2.8** For each \( \epsilon \in (0, 1) \), there exists \( \delta \in (0, \frac{1}{2}) \) such that \( (S_p)_{p=1}^T \) satisfies the bounds in (A3) with \( a = \overline{\mu}(0), b = \overline{v}(\infty) \) for all \( \eta, \xi \in [0, \epsilon^{-1}], U \in \mathcal{U}_0, V \in \mathcal{V}_\epsilon, 0 \leq k < n \) with \( T_{n-k} \geq 1 \) and \( u \in \mathbb{R}^{\delta \overline{V}_k}, v \in \mathbb{R}^{\delta \overline{V}_k} \) that satisfy (2.27), under the identification \( P = P_{0, \overline{\mu}(0)} \equiv P_{n,k,v}^\infty \).

**Proof** Thanks to Lemma 2.7, it will turn out sufficient to bound \( \gamma \) on \( A_p \). By (2.24), we have \( \gamma(y) = \mathbb{E}_{n-k, u} h(y) + m_{n-p} - \beta_p \) for \( y \in A_p, p = 1, \ldots, T \). We now use the Definition (2.22) of \( \beta_p \) and linearity in the following estimate:

\[
\left| \beta_p - \frac{(T-p)\overline{\mu}(0) + p\overline{v}(\infty)}{T} \right| = \left| \frac{s_{1,p}}{s_{1,T}} - \frac{p}{T} \right| \left| \overline{v}(\infty) - \overline{\mu}(0) \right| \leq C_\epsilon \left| \overline{v}(\infty) - \overline{\mu}(0) \right| \frac{\log p}{T}
\]
(2.54)

where we also used that

\[
\frac{s_{1,p}}{s_{1,T}} \leq \frac{p + C_\epsilon (1 + \log p)}{T} \leq \frac{p}{T} + C_\epsilon \log p + C_\epsilon \frac{p}{T^2} \log T
\]
(2.55)

by Lemma 2.5. Analogously, we also have

\[
\left| \beta_p - \frac{(T-p)\overline{\mu}(0) + p\overline{v}(\infty)}{T} \right| = \left| \frac{s_{1+p,T}}{s_{1,T}} - \frac{T-p}{T} \right| \left| \overline{v}(\infty) - \overline{\mu}(0) \right| \leq C_\epsilon \left| \overline{v}(\infty) - \overline{\mu}(0) \right| \frac{\log(1 + T - p)}{T},
\]
(2.56)

so that

\[
\left| \beta_p - \frac{(T-p)\overline{\mu}(0) + p\overline{v}(\infty)}{T} \right| \leq C_\epsilon \log\left(1 + \Delta_{T,p}\right).
\]
(2.57)
Furthermore, by combining Lemmas B.14 and A.5(i) (as in (B.43) in the proof of Proposition B.2) we obtain
\[
\left| \mathbb{E}^{n,u}_{k,v} h(y) - \frac{T - p}{T} \mu(0) - \frac{p}{T} \overline{v}(\infty) - \overline{m}_{n-p} \right| \leq 2 \text{osc} \, \mu + 2 \text{osc} \, \overline{v} + C_{\epsilon} + C_{\epsilon} \frac{|\overline{v}(0)| + |\overline{v}(\infty)|}{n - k}
\]
(2.58)
as \(n - k - T\) and \(|n - p - \log |y|\) are bounded by a constant. Combining the above estimates and bounding \(|m_{n-p} - \overline{m}_{n-p}| \leq \log \wedge T, p\) by Lemma B.1, we obtain that
\[
|\gamma (y)| \leq 2 \text{osc} \, \mu + 2 \text{osc} \, \overline{v} + \delta^{-1} + \delta^{-1} \wedge 1/2 - \delta
\]
(2.59)
for sufficiently small \(\delta\). The assertion now follows from Lemma 2.7 as
\[
\mathbb{P}^{n,u}_{k,v}(|\mathcal{D}_{p}| > t + \delta^{-1} + \delta^{-1} \wedge 1/2 - \delta) \leq \mathbb{P}^{n,u}_{k,v}(|\mathcal{D}_{p}| > t)
\]
(2.60)
for sufficiently small \(\delta > 0\).

Combining the above, we obtain:

**Proof of Theorem 2.3** The correspondence (2.28) follows from the definitions in Sect. 2.4. Assumption (A1) is verified by (2.20), (2.21), (2.22) and (2.26). For \(\delta > 0\) sufficiently small, depending only on \(\epsilon\), Lemma 2.5 shows that \(\sigma_{k} \in (\delta, \delta^{-1}) \) for \(k = 1, \ldots, T\). Assumptions (A2) and (A3) are verified by Lemmas 2.4 and 2.8.

### 2.6 Outward Concentric Decomposition and Corresponding DRW

In this subsection we present another reduction from the DGFF ballot event to that involving a DRW, only that the DRW is defined using an outward concentric decomposition instead of the inward concentric decomposition which was used before. Here the conditioning is performed in the opposite order, so that scales are increasing. This results in a DRW \((\mathcal{D}_{i}^{p})_{i=0}^{T}, (\mathcal{D}_{i}^{p})_{i=1}^{T}\) in which the starting and ending point are reversed: \(\mathcal{S}_{0}^{p} = \overline{v}(\infty), \mathcal{S}_{0}^{p} = \overline{v}(0)\). The reduction is summarized in Theorem 2.10 which is the analog of Theorem 2.3.

In analogy to Sect. 2.3, we assume that \(T_{n-k} \geq 1\) and define the concentric sets
\[
\Delta_{n-1}^{0} = V_{k}^{-}, \quad \Delta_{0}^{0} = V_{k}^{-}, \quad \Delta_{p} = B_{k+|\log (\epsilon^{-1} + \zeta)| + p}^{1}, \quad \Delta_{T}^{0} = (U_{n})^{C}, \quad \Delta_{T+1}^{0} = (U_{n})^{C}
\]
(2.61)
which are nested, \(\Delta_{n-1} \supset \cdots \supset \Delta_{T+1}\). Furthermore, we define the sets
\[
\Delta_{n-1}^{0} = (V_{k}^{-})^{C}, \quad \Delta_{0}^{0} = (V_{k}^{-})^{C}, \quad \Delta_{p}^{0} = B_{k+|\log (\epsilon^{-1} + \zeta)| + p}^{1}, \quad \Delta_{T}^{0} = U_{n}^{C}.
\]
(2.62)

For \(p = 0, \ldots, T\), let \(h_{p}^{0}\) be a DGFF on \(A_{p}^{0} := \Delta_{T}^{0} \cap \Delta_{p-1}^{0}\) with boundary values zero. Moreover, we define \(J_{p}^{0} := A_{p}^{0} \cup (\Delta_{p}^{0} \cap U_{n})\) for \(p = 0, \ldots, T - 1, J_{T}^{0} = A_{T}^{0}\). Also, let \(\varphi_{p}^{0}\) be distributed as \((h_{p}^{0})_{\mathbb{Z}^{2} \setminus J_{p}^{0}}\) where \(h_{p}^{0}\) is a DGFF on \(\Delta_{T}^{0} \cap U_{n}\) with boundary values zero (and, according to our notational conventions, equal to zero in \(\mathbb{Z}^{2} \setminus (\Delta_{p-1}^{0} \cap U_{n})\)). We note that \((h_{p}^{0})_{\mathbb{Z}^{2} \setminus J_{p}^{0}}\) is the binding field from \(\Delta_{T}^{0} \cap U_{n}\) to \(J_{p}^{0}\), as defined in the Gibbs-Markov decomposition (Lemma B.10). We assume that the random fields \(\varphi_{0}^{0}, \ldots, \varphi_{T}^{0}, h_{0}^{0}, \ldots, h_{T}^{0}\) are independent.

With the annuli \(A_{p}^{0} := \Delta_{p-1}^{0} \setminus \Delta_{p}^{0}, p = 0, \ldots, T + 1\), which form a disjoint covering of \(U_{n} \cap V_{k}^{1}\) and satisfy \(A_{p}^{0} \supset A_{p}^{0}\) for \(p \leq T\), we have the following analog of Proposition 2.2:
Proposition 2.9 (Outward concentric decomposition) Assume that $T = T_{n-k} \geq 1$. There exists a coupling of $h_{U_n \cap V_k^c}$ and $(\varphi_p^o, h_p^o)^T_{p=0}$ such that

$$h_{U_n \cap V_k^c}(x) = \sum_{p=0}^{T} (\varphi_p^o(x) + h_p^o(x)) = \sum_{p=0}^{q \wedge T} \varphi_p^o(x) + h_q^o(x) 1_{q \leq T}$$

(2.63)

for $x \in A_q^o$, $q = 0, \ldots, T + 1$.

**Proof** We proceed as in the proof of Proposition 2.2 by applying the Gibbs-Markov property first to $h_0^o$ and $A_0^o$.

We represent the DGFF on $U_n \cap V_k^c$ analogously to Sect. 2.4, now building on the outward concentric decomposition. We assume that $h$ under $\mathbb{P}_{k,v}^{n,u}$ is coupled to $(\varphi_p^o, h_p^o)^T_{p=1}$ from Proposition 2.9 such that

$$h = \sum_{p=0}^{T+1} (\varphi_p^o + h_p^o) + ( - m_n 1_{\partial U_n} + u - m_k 1_{\partial V_k} + v ) 1_{\partial U_n \cup \partial V_k}.$$  

(2.64)

We define the harmonic average

$$\mathcal{X}_p^o = \sum_{z \in \partial A_p^o \cup \partial \Delta_p^o} \varphi_p^o(z) \Pi_{\Delta_p^o}(\infty, z) = (\varphi_p^o)_{\partial \Delta_p^o}(\infty)$$

(2.65)

and denote its variance by $\sigma_{p,2}^o := \text{Var} \mathcal{X}_p^o$. Writing $s_{p,q}^o = \sum_{i=p}^{q} \sigma_{i,2}^o$, we set

$$\beta_p^o = \frac{s_{1+p,T}^o}{s_{1,T}^o} \overline{v}(\infty) + \frac{s_{1}^o}{s_{1,T}^o} \overline{u}(0),$$

(2.66)

and we define $(S_p^o)_{p=0}^T$ from $(\mathcal{X}_p^o)^T_{p=1}$ and independent Brownian bridges in the same way as $(S_p')_{p=0}^T$ was defined from $(\mathcal{X}_p')^T_{p=1}$ in (2.18) and (2.21). Moreover, we define

$$\gamma^o(y) = ( - m_n 1_{\partial U_n} + u - m_k 1_{\partial V_k} + v ) 1_{\partial U_n \cup \partial V_k}(y) + m_k + \beta_p^o$$

(2.67)

for $y \in A_p^o$.

We set

$$D_p^o = \max_{y \in A_p^o} \left\{ h_p^o(y) - m_k + \varphi_0^o(y) - S_p^o + \gamma^o(y) \right\}$$

(2.68)

and $S_p^o = S_p^o + \beta_p^o$. Then, in analogy to Theorem 2.3, we have:

**Theorem 2.10** Let $\epsilon \in (0, 1)$. Then there exists $\delta \in (0, 1/3)$ such that for all $\eta, \xi \in [0, \epsilon^{-1}]$, $U \in \mathcal{U}_\eta$, $V \in \mathcal{V}_\xi$, $0 \leq n < k$ with $T_{n-k} \geq 1$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k}$ that satisfy (2.27), we have that $(S_p^o)_{p=0}^T, (D_p^o)_{p=1}^T$ satisfies Assumptions (A1) – (A3) with $\delta, a = \overline{v}(\infty)$ and $b = \overline{u}(0)$ under the identification $P = P_{k,v}^{n,u}(\overline{v}(0)) = \mathbb{P}_{k,v}^{n,u}$. Moreover,

$$\{ S_p^o + D_p^o \leq 0 \} = \{ h_{A_p^o} \leq 0 \} \text{ for } p = 1, \ldots, T, \text{ and } \bigcap_{p=1}^{T} \{ S_p^o + D_p^o \leq 0 \} = \{ h_{U_n \cap V_k^c} \leq 0 \}.$$  

(2.69)
When a DGFF $h^D$ on an infinite domain $D \subseteq \mathbb{Z}^2$ with boundary values $u \in \mathbb{R}^{\partial D}$ arises in the setting of the outward concentric decomposition, we define it as the Gaussian field on $\overline{D}$ with mean and covariance given by (1.1), where $\overline{w}$ denotes the bounded harmonic extension of $w$ to $\mathbb{Z}^2$.

Proof of Theorem 2.10 The correspondence (2.69) again follows from the definitions above. Assumptions (A1) – (A3) are verified along the lines of Sect. 2.3 in the same way as for Theorem 2.3. We always use the objects defined in this subsection, e.g. $A_p$ in place of $A^o_p$. Equation (2.33) now reads

$$\text{Var}_k \mathcal{X}_j^o = \text{Var} h^\Delta_j \partial \Delta_j^o(\infty) - \text{Var} ((h^\Delta_j)_{\partial \Delta_j^o(\infty)}, 0) = (2.70)$$

where the second variance is bounded by (B.25) by a constant times $(T - j + 2)^{-1}$. We also interchange the use of Lemmas A.6 and A.7, and we now use (A.2) and (A.7) in Lemmas A.1 and A.2, respectively. In the analog of Lemma 2.5, it suffices to assume that $n - k \to \infty$ and $k \to \infty$ for the convergence of $\text{Var} \mathcal{X}_j^o$ as this will only be used in Lemma 5.19 below. □

3 The Ballot Upper Bound

In this section we prove Theorem 1.8. Following the strategy which was outlined in Sect. 1.3, we first prove a weak version of the upper bound (Sect. 3.1), which is a direct consequence of a similar bound for the DRW ballot probability. We then show (Sect. 3.2) that, conditional on the balloting event, the harmonic extension of the values of the field at an intermediate scale is entropically repelled in the bulk with high probability and otherwise well behaved. This is then used (Sect. 3.3) to prove the full version of the theorem via “stitching” as explained in the proof outline.

3.1 A Weak Upper Bound

As a first consequence of the reduction to the DRW (Theorems 2.3 and 2.10), one obtains the following weaker version of Theorem 1.8, where the boundary values on either the inner or outer domains are sufficiently low. This follows from a corresponding weak upper bound on the ballot event for the DRW (Theorem C.1). We recall that $u_*$ and $v_*$ were defined in (1.23) in terms of $u, v, \eta, \zeta$. Also, we recall from Sect. 2.1 that $B = B(0, 1)$ denotes the unit ball, $S$ under $P_\chi$ simple random walk started at $x \in \mathbb{Z}^2$ with first hitting time $\tau^A$ of a set $A$.

Proposition 3.1 Let $\epsilon \in (0, 1)$. There exist $C = C_\epsilon < \infty$ and $\theta = \theta_\epsilon \in (0, \epsilon/3)$, such that for all $\eta, \zeta \in [0, \epsilon^{-1}]$, $0 \leq k < n$, $U \in \cup_k^\eta$, $V \in \cup_k^\zeta$, and all $u \in \mathbb{R}^\partial U_k$, $v \in \mathbb{R}^\partial V_k$ with $\max\{u^\epsilon, v^\epsilon\} \geq (n - k)^\epsilon$, we have

$$\mathbb{P}_{k, v}^{\eta, u}(h_{(\eta B)_n \cap (\theta^{-1} B^-)_k} \leq 0) \leq C \frac{(u^\epsilon + 1)(v^\epsilon + 1)}{n - k}. \quad (3.1)$$

Remark 3.2 We note that $U \in \cup_k^\eta$, $V \in \cup_k^\zeta$, $\eta \in [0, \epsilon^{-1}]$ and $\theta < \epsilon/3$ imply that $(\eta B)_n \cap (\theta^{-1} B^-)_k \subset U_k^\eta \cap V_k^{-\zeta}$. Therefore, the right-hand side in (3.1) is also an upper bound for $\mathbb{P}_{k, v}^{\eta, u}(h_{U_k^\eta \cap V_k^{-\zeta}} \leq 0)$ under the conditions of the proposition. Also, as we show in the proof below, the following alternate formulations of Proposition 3.1 hold under the same
assumptions but without the restriction on \( \max\{u_*, v_*\}^2 \):
\[
\mathbb{P}^{n, u}_k (h_{(\theta B)_n \cap (\theta^{-1} B^-)_k} \leq 0) \leq C \frac{(u_* + 1)(v_* + (n - k)^\epsilon)}{n - k}, \tag{3.2}
\]
\[
\mathbb{P}^{n, v}_k (h_{(\theta B)_n \cap (\theta^{-1} B^-)_k} \leq 0) \leq C \frac{(u_* + (n - k)^\epsilon)(v_* + 1)}{n - k}. \tag{3.3}
\]

The following lemma conveniently reduces the general case to that involving constant boundary conditions. It will be used in various places in the sequel, including in the proof of Proposition 3.1.

**Lemma 3.3** Let \( \epsilon \in (0, 1) \). There exist \( C = C_\epsilon < \infty \) and \( \theta = \theta_\epsilon \in (0, \epsilon/3) \) such that for all \( \eta, \zeta \in [0, \epsilon^{-1}] \), \( 0 \leq k \leq n - C, \ U \in \mathcal{U}_k, \ V \in \mathcal{V}_k, \ u \in \mathbb{R}^{\partial U_k}, \ v \in \mathbb{R}^{\partial V_k} \), we have
\[
\mathbb{P}^{U, n, u}_k (h_{(\theta B)_n \cap (\theta^{-1} B^-)_k} \leq 0) \leq \mathbb{P}^{U, n, u}_k (h_{(\theta B)_n \cap (\theta^{-1} B^-)_k} \leq 0). \tag{3.4}
\]

**Proof** From the definition of the DGFF, we have that \( \theta \) under \( \mathbb{P}^{n, u}_k \) is distributed as \( h + \mathbb{P}^{\partial U_{n} \cup \partial V_{k}} + \mathbb{P}^{\partial U_{n} \cup \partial V_{k}} \) under \( \mathbb{P}^{n,0}_k \). By Lemma B.14,
\[
u_{\partial U_{n} \cup \partial V_{k}} (x) \geq P_{\lambda} (\tau V_{\epsilon - 1} > \tau U_{\epsilon}) \mathbb{P}_{\lambda} (x) - \text{osc}_{\lambda} \mathbb{P}_{\lambda}
\]
for \( x \in U_{\epsilon} \cap V_{\epsilon} \), and (B.33) in Lemma B.15 yields \( \theta = \theta_\epsilon < \epsilon/3 \) such that for \( x \in (\theta B)_n \cap (\theta^{-1} B^-)_k \), we can further bound
\[
u_{\partial U_{n} \cup \partial V_{k}} (x) \geq P_{\lambda} (\tau U_{\epsilon} \leq \tau V_{\epsilon}) \mathbb{P}_{\lambda} (x) - 2\text{osc}_{\lambda} \mathbb{P}_{\lambda} = (u_* 1_{\partial U_{n}})_{\partial U_{n} \cap V_{k}}. \tag{3.6}
\]
Analogously, using (B.34) in place of (B.33), we obtain
\[
u_{\partial U_{n} \cup \partial V_{k}} (x) \geq (v_* 1_{\partial V_{k}})_{\partial U_{n} \cap V_{k}} \tag{3.7}
\]
for \( x \in (\theta B)_n \cap (\theta^{-1} B^-)_k \) and sufficiently small \( \theta \). Using monotonicity of \( \mathbb{P}^{n,0}_k \) and that \( h + (u_* 1_{\partial U_{n}})_{\partial U_{n} \cup V_{k}} + (v_* 1_{V_{k}})_{\partial U_{n} \cup V_{k}} \) under \( \mathbb{P}^{n,0}_k \) is distributed as \( h \) under \( \mathbb{P}^{n, u}_k \), yields the assertion.

**Remark 3.4** Lemma 3.3 and Theorem 1.8 remain valid and the proofs pass through when \( u_* \) and \( v_* \) are, instead of (1.23), defined as
\[
u_{u_*} \equiv \nu (0) - (1 + \epsilon) \text{osc} \nu, \quad v_* \equiv \nu (\infty) - (1 + \epsilon) \text{osc} \nu \tag{3.8}
\]
for arbitrarily small \( \epsilon > 0 \).

**Proof of Proposition 3.1** By Lemma 3.3 and Remark 3.2, there exists \( \theta < \epsilon/3 \) with
\[
\mathbb{P}^{n, u}_k (h_{U_{n} \cap V_{k}} \leq 0) \leq \mathbb{P}^{n, u}_k (h_{(\theta B)_n \cap (\theta^{-1} B^-)_k} \leq 0), \tag{3.9}
\]
whenever \( n - k \) is sufficiently large, which we can assume. We can also assume w.l.o.g. that \( \max\{u_*, v_*\} \leq 0 \).

In case \( |u_* - v_*| \leq n - k \) and \( -v_* > (n - k)^\epsilon \), Theorem 2.3 is applicable with \( a = u_* \), \( b = v_* \), so that (3.1) and (3.2) follow from the upper bound in Theorem C.1 applied to
\[
\mathbb{P}^{n, u}_k (h_{(\theta B)_n \cap (\theta^{-1} B^-)_k} \leq 0), \tag{3.10}
\]

Here we choose \( p_* \) such that \( \Delta_{p_*-1} \setminus \Delta_{T-p_*} \subset (\theta B)_n \cap (\theta^{-1} B^-)_k \) and assume that \( n-k \) is sufficiently large such that \( T \geq 2p_* + 1 \).

In case \( |u_* - v_*| > n-k \) and \( -v_* > (n-k)^\epsilon \), the right-hand sides in (3.1) and (3.2) are bounded from below by \( C \), so that the assertion follows in this case as we may assume w.l.o.g. that \( C \geq 1 \).

The corresponding cases with \( -u_* > n - k \), as well as (3.3), are handled in the same way by using Theorem 2.10 in place of Theorem 2.3.

3.2 Entropic Repulsion at an Intermediate Scale

In order to derive Theorem 1.8 we will condition on the values of the DGFF \( h \) at the boundary of a centered ball at an intermediate scale and use the weak ballot estimate from the previous subsection on each of the resulting inner and outer domains. For the latter to be in effect, we need to control the harmonic extension of the values of the boundary of the intermediate ball. In particular, we need to show that it is typically (entropically) repelled below 0 and that its oscillations are not too large.

To this end, we set for \( l, M \geq 0, \epsilon > 0 \),

\[
E_{l,M,\epsilon} := \left\{ w \in \mathbb{R}^{\partial B^\pm_k} : -\wedge_{n,l,k}^{1-\epsilon} \leq \overline{w}(0) \leq -\wedge_{n,l,k}^{\epsilon}, \ \text{osc}_{B^\pm_k \cap \overline{w}} \leq M \right\}, \tag{3.11}
\]

where we recall that \( \wedge_{n,l,k} := (n-l) \wedge (l-k) \). We then show,

**Proposition 3.5** Let \( \epsilon \in (0, \frac{1}{10}] \) and \( c = c_{\epsilon} > 0 \) such that for all \( M \geq 0, \eta, \xi \in [0, e^{-1}], U \ni \mathcal{U}_\eta, V \ni \mathcal{V}_\xi, \) all \( 0 \leq k < l < n \) with \( l \in k + (n-k)[\epsilon, 1-\epsilon] \), and all \( u \in \mathbb{R}^{\partial U_\eta}, v \in \mathbb{R}^{\partial V_\xi} \) with \( (|\mathcal{U}_\eta|, |\mathcal{V}(\infty)|, \text{osc}_{U_\eta \cap \overline{w}}, \text{osc}_{V_\xi \cap \overline{w}}) \leq \wedge_{n,l,k}^{\epsilon} \), and \( \theta = \theta_{\epsilon} \in (0, \epsilon/3) \) as in Lemma 3.3, we have

\[
\mathbb{P}_{V,k,v}^U(h_{(\partial B)_n \cap (\partial^{-1} B^-)_k}) \leq 0, h_{\partial B^\pm_k} + m_1 \notin E_{l,M,\epsilon} \leq C \frac{(u_* + 1)(u_* - 1)}{n-k} \left( -\wedge_{n,l,k}^{3/2+4\epsilon} + e^{-cM^2} \right). \tag{3.12}
\]

In order to prove the proposition, we decompose the complement of the set \( E_{l,M,\epsilon} \) as \( E_{l,M,\epsilon}^c = E_1 \cup E_2 \cup E_3 \cup E_4 \), where:

\[
E_1 = \left\{ w \in \mathbb{R}^{\partial B^\pm_k} : |\overline{w}(0)| > \wedge_{n,l,k}^{1-\epsilon} \right\}, \quad E_2 = \left\{ w \in \mathbb{R}^{\partial B^\pm_k} : -\wedge_{n,l,k}^{\epsilon} \leq \overline{w}(0) \leq \wedge_{n,l,k}^{2\epsilon} \right\},
\]

\[
E_3 = \left\{ w \in \mathbb{R}^{\partial B^\pm_k} : \text{osc}_{B^\pm_k \cap \overline{w}} > M \right\} \cap E_1 \cap E_2, \quad E_4 = \left\{ w \in \mathbb{R}^{\partial B^\pm_k} : \wedge_{n,l,k}^{2\epsilon} \leq \overline{w}(0) \leq \wedge_{n,l,k}^{1-\epsilon} \right\} \cap E_3^c,
\]

and treat each of the \( E_i \)'s separately. \( E_1 \) is handled by,

**Lemma 3.6** Let \( \epsilon \in (0, \frac{1}{10}] \) and \( \eta, \xi \in [0, e^{-1}] \). Then there exist \( C = C_{\epsilon} < \infty \) and \( c = c_{\epsilon} > 0 \) such that

\[
\mathbb{P}_{k,v}^U(h_{\partial B^\pm_k} + m_1 \in E_1) \leq C e^{-c \wedge_{n,l,k}^{1-2\epsilon}} \tag{3.14}
\]

and

\[
\mathbb{P}_{k,v}^U((h_{\partial B^\pm_k} + m_1)^- + \text{osc}_{B^\pm_k \cap \overline{w}} h_{\partial B^\pm_k} + m_1 \in E_1) \leq C e^{-c \wedge_{n,l,k}^{1-2\epsilon}} \tag{3.15}
\]
for all $U \in \mathcal{U}^1_{\nu}, V \in \mathcal{V}_\epsilon$, all $k, l, n \geq 0$ with $\partial B_l \subset U^{\nu,\epsilon}_n$ and $\partial B_l^+ \subset V_k^{-,\epsilon}$, and all $u \in \mathbb{R}^d U_n$, $v \in \mathbb{R}^d V_k$ with max $|\bar{u}(0)|, |ar{v}(\infty)|$, osc $U^{\nu,\epsilon}_n \bar{u}$, osc $V_k^{-,\epsilon} \bar{v} \leq 1^{-1,2\epsilon}$.

**Proof** The random variable $h_{\partial B_l^+}^-(0) + m_l$ under $P_{n,u}^{n,u}_{k,v}$ is Gaussian. By Lemma B.1, and Propositions B.2 and B.3, its mean is bounded (in absolute value) by $C_\epsilon + C_{\epsilon, \wedge_{n,l,k}^{1-2\epsilon}}$ and its variance is bounded from above by $g \wedge_{n,l,k} + C_\epsilon$. Hence, the left-hand side of (3.14) is bounded by a constant times

$$\wedge_{n,l,k}^{-1/2} \int_{|r| \geq \wedge_{n,l,k}^{-1}} \epsilon e^{-c_r^2 r_{n,l,k}^2} dr = C_\epsilon \int_{|r| \geq \wedge_{n,l,k}^{-1/2}} e^{-c_r^2 r_{n,l,k}^2} dr \leq C_\epsilon e^{-c_r^2 \wedge_{n,l,k}^{1-2\epsilon}}$$

(3.16)

where the equality follows by changing the variable $r \wedge_{n,l,k}^{-1/2}$ to $r$. The left-hand side of (3.15) is bounded by a constant times

$$\wedge_{n,l,k}^{-1/2} \int_{|r| \geq \wedge_{n,l,k}^{-1}} e^{-c_r^2 r_{n,l,k}^2} \sum_{a=1}^{\infty} \mathbb{P}_{n,u}^{n,u}_{k,v} \left( \text{osc}_{B_l^{+,-}} h_{\partial B_l^+} \geq a - 1 \right) \int_{|r| \geq \wedge_{n,l,k}^{-1/2}} e^{-c_r^2 r_{n,l,k}^2} dr$$

\leq C_\epsilon \wedge_{n,l,k}^{-1/2} \int_{|r| \geq \wedge_{n,l,k}^{-1}} e^{-c_r^2 r_{n,l,k}^2} \sum_{a=1}^{\infty} e^{-c_r \left( \frac{|r|}{\wedge_{n,l,k}} \right)^2} (r^- + a) dr \leq C_\epsilon e^{-c_r \wedge_{n,l,k}^{1-2\epsilon}}

(3.17)

Next we treat $E_2$.

**Lemma 3.7** Let $\epsilon \in (0, \frac{1}{10})$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then there exists $C = C_\epsilon < \infty$ such that

$$\mathbb{P}_{n,u}^{n,u}_{k,v} \left( \left( \left( h_{\partial B_l^+}^-(0) + m_l \right)^- + \text{osc}_{B_l^{+,-}} h_{\partial B_l^+} + \wedge_{n,l,k}^\epsilon \right)^2 ; h_{\partial B_l^+} + m_l \in E_2 \right) \leq C \wedge_{n,l,k}^{-1/2+4\epsilon}$$

(3.18)

for all $U \in \mathcal{U}^1_{\nu}, V \in \mathcal{V}_\epsilon$, all $0 \leq k < l < n$ with $l \in (n-k)[\epsilon, 1-\epsilon]$, $\partial B_l \subset U^{\nu,\epsilon}_n$ and $\partial B_l^+ \subset V_k^{-,\epsilon}$, and all $u \in \mathbb{R}^d U_n$, $v \in \mathbb{R}^d V_k^{-,\epsilon}$ with max $|\bar{u}(0)|, |ar{v}(\infty)|$, osc $U^{\nu,\epsilon}_n \bar{u}$, osc $V_k^{-,\epsilon} \bar{v} \leq 1^{-1,2\epsilon}$.

**Proof** The left-hand side of (3.14) is bounded from above by

$$\int_{r \in \left[ -\wedge_{n,l,k}^{-\epsilon}, \wedge_{n,l,k}^{2\epsilon} \right]} \mathbb{P}_{n,u}^{n,u}_{k,v} \left( h_{\partial B_l^+}^-(0) + m_l \in dr \right)$$

\times \sum_{a=0}^{\infty} \mathbb{P}_{n,u}^{n,u}_{k,v} \left( \text{osc}_{B_l^{+,-}} h_{\partial B_l^+} \geq a \right) \left[ \int_{r \in \left[ -\wedge_{n,l,k}^{-\epsilon}, \wedge_{n,l,k}^{2\epsilon} \right]} e^{-c_r a^2 \wedge_{n,l,k}^\epsilon + r^- + a} dr \right]^2

(3.19)

By Proposition B.3, we bound the Gaussian density by a constant times $\wedge_{n,l,k}^{-1/2}$, and we plug in the conditional tail probability from Proposition B.4. Then the previous expression is bounded from above by a constant times

$$\int_{r \in \left[ -\wedge_{n,l,k}^{-\epsilon}, \wedge_{n,l,k}^{2\epsilon} \right]} \sum_{a=0}^{\infty} e^{-c_r a^2 \wedge_{n,l,k}^\epsilon + r^- + a} \wedge_{n,l,k}^{-1/2} dr .$$

(3.20)

Bounding $r^-$ by $\wedge_{n,l,k}^\epsilon$ yields the assertion.

□
For $E_3$ we have,

**Lemma 3.8** Let $\epsilon \in (0, \frac{1}{10})$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then there exist $C = C_\epsilon < \infty$, $c = c_\epsilon > 0$ such that

$$
\mathbb{P}^n_{k, v} \left[ \left( \hat{h}_{\partial B_1^\epsilon}^{\pm}(0) + m_l \right)^- + \text{osc}_{B_1^nx}^{e, \epsilon} \hat{h}_{\partial B_1^\epsilon}^{\pm} + \land_{n, l, k} \left( \hat{h}_{\partial B_1^\epsilon}^{\pm}(0) + m_l \right) \right] \leq C e^{-cM^2} \left( 1 + \left( \frac{n - l}{n - k} \right)^2 + \left( \frac{l - k}{n - k} \right)^2 \right)
$$

(3.21)

and

$$
\mathbb{P}^n_{k, v} \left[ \left( \hat{h}_{\partial B_1^\epsilon}^{\pm}(0) + m_l \right)^- + \text{osc}_{B_1^nx}^{e, \epsilon} \hat{h}_{\partial B_1^\epsilon}^{\pm} + 1 \right] \leq \land_{n, l, k, \epsilon} \geq M \left( \hat{h}_{\partial B_1^\epsilon}^{\pm}(0) + m_l \right) \leq C e^{-cM^2} \left( 1 + \left( \frac{n - l}{n - k} \right)^2 + \left( \frac{l - k}{n - k} \right)^2 \right)
$$

(3.22)

for all $M \geq 0$, $U \in \mathcal{U}_\epsilon$, $V \in \mathcal{V}_\epsilon$ all $0 \leq k < l < n$ with $\partial B_l \subset U_n^{\eta \vee \epsilon}$ and $\partial B_l \subset V_{k}^{\neg \zeta \vee \epsilon}$, and all $u \in \mathbb{R}^d U_n$, $v \in \mathbb{R}^d V_{k}$ with $\max(\|\mathbb{P}(0)\|, \|\mathbb{P}(\infty)\|, \text{osc}_{U_n}^{e}, \text{osc}_{V_{k}}^{\neg \zeta}) \leq \epsilon^{-1} \land_{n, l, k}$.

**Proof** The left-hand side in assertion (3.21) is bounded from above by a constant times

$$
\sum_{a = [1, M]}^\infty \int_{\land_{n, l, k} \leq |r| \leq \land_{n, l, k}^{1-\epsilon}} \mathbb{P}^n_{k, v} \left[ \left( \hat{h}_{\partial B_1^\epsilon}(0) + m_l \right) \right] \mid a \mid \left( \hat{h}_{\partial B_1^\epsilon}(0) + m_l \right) \left( 1 + r^2 \right) \left( 1 + a^2 \right).
$$

(3.23)

The second probability in the integrand is bounded from above by $e^{-2c_{\epsilon}a^2}$ for a constant $c_{\epsilon} > 0$ by Proposition B.4, so that (3.23) is further bounded from above by a constant times

$$
e^{-cM^2} \left( 1 + \mathbb{E}^n_{k, v} \left[ \left( \hat{h}_{\partial B_1^\epsilon}(0) + m_l \right)^2 \right] \right).
$$

(3.24)

Bounding the second moment by Lemma B.1 and Propositions B.2 and B.3 yields (3.21). The proof of (3.22) is analogous.  

Finally for $E_4$, we have

**Lemma 3.9** Let $\epsilon \in (0, \frac{1}{10})$, and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then there exists $C = C_\epsilon < \infty$ such that

$$
\mathbb{P}^n_{k, v} \left[ \exp \left\{ - \left( \left( \hat{h}_{\partial B_1^\epsilon}(0) + m_l \right)^{-2-\epsilon} \right) ^{2-\epsilon} \right\} \right] \land_{n, l, k} \left( \hat{h}_{\partial B_1^\epsilon}(0) + m_l \right) \in E_4 \leq C e^{-\land_{n, l, k}^{1-\epsilon}}
$$

(3.25)

for all $M, U, V, k, l, n, u, v$ as in Lemma 3.8.

**Proof** The left-hand side of (3.25) is bounded from above by a constant times

$$
\int_{r \in \land_{n, l, k}^{1-\epsilon}} \mathbb{P}^n_{k, v} \left[ \left( \hat{h}_{\partial B_1^\epsilon}(0) + m_l \right) \right] dr.
$$
By Proposition B.3, we bound the Gaussian density by a constant, and we plug in the bound from Proposition B.4 for the second probability in the integrand so that the previous expression is bounded by a constant times

$$\int_{r \in [0, 2\beta]} \sum_{a=0}^{\infty} e^{-c a^2} e^{-r + 2a^2} dr .$$

(3.27)

As a last ingredient in the proof of Proposition 3.5, we need an upper bound on the probability that the DGFF stays negative in the bulk despite positive boundary values. This will be used also later for the stronger statement in Theorem 1.8.

Lemma 3.10 Let \( \epsilon \in (0, \frac{1}{10}) \) and \( \eta, \zeta \in [0, \epsilon^{-1}] \). Then there exists \( C = C_{\epsilon} < \infty \) such that, for \( \theta \) as in Lemma 3.3, we have

$$\mathbb{P}^{n, u}(h_{(\theta B) n} \cap (\theta^{-1}B^-)_{\kappa} \leq 0) \leq C \exp \left\{ - (u_\kappa^*)^{2-\epsilon} 1_{u_\kappa \geq -(n-k)} \right\}$$

(3.28)

and

$$\mathbb{P}^{n, u}(h_{(\theta B) n} \cap (\theta^{-1}B^-)_{\kappa} \leq 0) \leq C \exp \left\{ - (v_\kappa^*)^{2-\epsilon} 1_{v_\kappa \geq -(n-k)} \right\}$$

(3.29)

for all \( U \in \Omega^U \), \( V \in \Omega^V \), \( 0 \leq k < n - C \), and all \( u \in \mathbb{R}^{\partial U_n}, v \in \mathbb{R}^{\partial V_n} \).

Proof We show (3.28), the proof of (3.29) is analogous. By Lemma 3.3, \( \mathbb{P}^{n, u}_k(h_{(\theta B) n} \cap (\theta^{-1}B^-)_{\kappa} \leq 0) \leq \mathbb{P}^{n, u}_k(h_{(\theta B) n} \cap \mathbb{R}^{\partial V_n} \leq 0) \) for all sufficiently large \( n - k \). Moreover, \( h \) under \( \mathbb{P}^{n, u}_k \) is distributed as the field

$$h_{U_n \cap V_n^\kappa}(x) + (-m_n + u_\kappa) P_x (\tau_{U_n} \leq \tau_{V_n}^\kappa) + (-m_k + v_\kappa \wedge 0) P_x (\tau_{U_n} > \tau_{V_n}^\kappa), \quad x \in U_n \cap V_n^\kappa$$

(3.30)

by definition of the DGFF and the representation (2.8) for the harmonic extension. W.l.o.g. we assume that \( u_\kappa \geq 0 \) and \( v_\kappa \leq 0 \). For \( x \in A := (\theta B) n \cap (\frac{\kappa}{2} B) n \), we estimate

$$(-m_n + u_\kappa) P_x (\tau_{U_n} \leq \tau_{V_n}^\kappa) + (-m_k + v_\kappa) P_x (\tau_{U_n} > \tau_{V_n}^\kappa)$$

$$= -m_n + (1 - P_x (\tau_{U_n} > \tau_{V_n}^\kappa)) u_\kappa + P_x (\tau_{U_n} > \tau_{V_n}^\kappa) (m_n - m_k + v_\kappa)$$

$$\geq -m_n + c_\epsilon u_\kappa - C_\epsilon + C_\epsilon (n - k)^{-1} v_\kappa,$$

(3.31)

for some \( c_\epsilon > 0 \) and \( C_\epsilon < \infty \), where we used for the inequality the definition (1.7) of \( m_n, m_k \), that \( P_x (\tau_{U_n} > \tau_{V_n}^\kappa) \leq C_\epsilon (n - k)^{-1} \) by Lemma A.5(i), in particular that \( 1 - P_x (\tau_{U_n} > \tau_{V_n}^\kappa) \geq c_\epsilon \) for sufficiently large \( n - k \). Then,

$$\mathbb{P}^{n, u}_k(h_A \leq 0) \leq \mathbb{P}(h_{A \cap V_n^\kappa}^\kappa - m_n + c_\epsilon u_\kappa - C_\epsilon + C_\epsilon (n - k)^{-1} v_\kappa \leq 0).$$

(3.32)

The right-hand side is bounded by Lemma A.2 for sufficiently large \( n - k \), and the left-hand side bounds \( \mathbb{P}^{n, u}_k(h_{(\theta B) n} \cap (\theta^{-1}B^-)_{\kappa} \leq 0) \), so that (3.28) follows.

We can now prove Proposition 3.5.
Proof of Proposition 3.5} We assume w.l.o.g. that \( n - k \) is sufficiently large (depending only on \( \epsilon \)) such that the assumptions of Lemmas 3.6 – 3.10 are satisfied in their applications below. We can choose \( C \) sufficiently large such that the right-hand side of (3.12) is larger than 1 when \( n - k \) is not as large.

By a union bound,

\[
\mathbb{P}_{k,v}^{n,u}(h_{(\partial B)^{\sigma_{\beta}}}) \leq 0, h_{\partial B^{\pm}} + m_l \notin E_l, M, \epsilon
\]

\[
\leq \mathbb{P}_{k,v}^{n,u}(h_{\partial B^{\pm}}) + m_l \in E_l) + \sum_{i=2}^{4} \mathbb{P}_{k,v}^{n,u}(h_{(\partial B)^{\sigma_{\beta}}}) \leq 0, h_{\partial B^{\pm}} + m_l \in E_i).
\]

(3.33)

The first term on the right-hand side is, by Lemma 3.6 and the assumptions on \( u, v, l \), bounded to be of smaller order than the right-hand side of (3.12). Using the Gibbs-Markov property (Lemma B.10) and Remark 2.1, we bound each of the summands with \( i = 2, 3, 4 \) in (3.33) by

\[
\mathbb{P}_{V,k,u}^{U,n,u}(h_{(\partial B)^{\sigma_{\beta}}}) \leq 0, h_{B_l^{\sigma_{\beta}}} \leq 0, h_{\partial B^{\pm}} + m_l \in E_i)
\]

\[
= \int_{E_i} \mathbb{P}_{V,k,u}^{U,n,u}(h_{(\partial B)^{\sigma_{\beta}}}) \leq 0) \mathbb{P}_{V,k,v}^{U,n,u}(h_{B_l^{\sigma_{\beta}}}) \leq 0) \mathbb{P}_{V,k,v}^{U,n,u}(h_{\partial B^{\pm}} + m_l \in dw).
\]

(3.34)

We bound the factors in the integrand on the right-hand side by Proposition 3.1 and Remark 3.2 so that the integral is bounded by a constant times

\[
\frac{(u^+ + 1)(v^+ + 1)}{(n - l)(l - k)} \int_{E_i} \left( \frac{w(0)^{+} + \text{osc}_{B_l^{\pm, \epsilon}} w + (l - k)^{2\epsilon}}{w(0)^{+} + \text{osc}_{B_l^{\pm, \epsilon}} w + (n - l)^{2\epsilon}} \right) \mathbb{P}_{V,k,v}^{U,n,u}(h_{\partial B^{\pm}} + m_l \in dw).
\]

(3.35)

For \( i = 2 \), the integral in the last display is bounded by \( C_{\epsilon, l, k}^{-1/2 + 4\epsilon} \) by the assumption on \( l \) and Lemma 3.7. For \( i = 3 \), the integral in (3.35) is bounded by a constant times the right-hand side of (3.21) by the first part of Lemma 3.8. To bound the summand with \( i = 4 \) on the right-hand side of (3.33), we bound the first factor in the integrand in (3.34) by 1 and the second factor \( \mathbb{P}_{V,k,v}^{U,n,u}(h_{B_l^{\sigma_{\beta}}} \leq 0) \) by (3.28), so as to obtain

\[
\mathbb{P}_{k,v}^{n,u}(h_{(\partial B)^{\sigma_{\beta}}}) \leq 0, h_{\partial B_{l}^{\pm}} + m_l \in E_4)
\]

\[
\leq C_{\epsilon} \int_{E_4} e^{-\epsilon \left( \frac{w(0)^{+} - \text{osc}_{B_l^{\pm, \epsilon}} w}{w(0)^{+} + \text{osc}_{B_l^{\pm, \epsilon}} w} \right)^{2\epsilon}} \mathbb{P}_{V,k,v}^{U,n,u}(h_{\partial B^{\pm}} + m_l \in dw)
\]

(3.36)

which is bounded by a constant times \( \epsilon^{-\epsilon_{n,l,k}} \) by Lemma 3.9. Hence, the right-hand side of (3.33) is bounded by the right-hand side of (3.12).

\[\square\]

### 3.3 Stitching: Proof of Theorem 1.8

From the previous subsection, we have an upper bound on the probability that the DGFF takes unlikely values at the boundary of an intermediated scaled ball, so that we can apply the weak upper bound from Proposition 3.1 at the two subdomains that are separated by this boundary. We are therefore ready for,
Proof of Theorem 1.8 As noted in Sect. 2.1, we can assume w.l.o.g. that $\epsilon \in (0, 1/10)$. We first show the weaker statement, namely (1.24). We set $l = k + \frac{1}{2}(n - k)$ and take $n - k$ sufficiently large such that Lemma 3.3 is applicable, and such that $\partial B_l \subset (\theta B)_n$ and $\partial B_l^c \subset (\theta^{-1} B^-)_k$, where we take $\theta \in (0, \epsilon/3)$ from Lemma 3.3. (We can always choose $C$ sufficiently large such that the right-hand side of (1.24) is larger than 1 if $n - k$ is not as large.) By Lemma 3.3, it suffices to bound $P_{k,v_*}^n(h(\theta B)_n \cap (\theta^{-1} B^-)_k) \leq 0)$ by the right-hand side of (1.24). W.l.o.g. we assume that $\max\{u_*, v_*\} \leq 0$. We can also suppose that $\min\{u_*, v_*\} \geq -(n - k)^{\epsilon}$, as in the converse case, the assertion follows directly from Proposition 3.1.

Clearly,

$$P_{V,k,v_*}^n(h(\theta B)_n \cap (\theta^{-1} B^-)_k) \leq 0) \leq P_{V,k,v_*}^n(h(\theta B)_n \cap (\theta^{-1} B^-)_k \leq 0, h_{\partial B_l^c} \leq 0) \leq 0 \quad (3.37)$$

Then, by conditioning on the value of $h$ on $\partial B_l^c$ and using Remark 2.1, we write the probability on the right-hand side of (3.37) as

$$\int P_{B,l,w}^n(h(\theta B)_n \cap (\theta^{-1} B^-)_k, h_{\partial B_l^c} + m_l \in dw) \quad (3.38)$$

As the constant boundary values $u_*$ on $\partial U_*$ and $v_*$ on $\partial V_{-k}$ also have zero oscillation, we can apply Proposition 3.5 to bound $P_{V,k,v_*}^n(h(\theta B)_n \cap (\theta^{-1} B^-)_k) \leq 0, h_{\partial B_l^c} + m_l \not\in E_{l,1,}\epsilon)$ by the right-hand side of (1.24).

It therefore suffices to bound the integral in (3.38) restricted to $w \in E_{l,1,}\epsilon$. We bound each factor in the integrand there by Proposition 3.1 and obtain as a bound for the integral a constant times

$$\frac{(u_* + 1)(v_* + 1)}{(n - l)(l - k)} E_{k,v_*}^n\left[\left(\overline{h_{\partial B_l^c}}(0) + m_l\right)^{-} + \text{osc}_{B_l^c} \overline{h_{\partial B_l^c}} + 1\right]$$

$$\times \left(\overline{h_{\partial B_l^c}}(0) + m_l\right)^{-} + \text{osc}_{B_l^c} \overline{h_{\partial B_l^c}} + 1; h_{\partial B_l^c} + m_l \in E_{l,1,}\epsilon\right]. \quad (3.39)$$

By definition of $E_{l,1,}\epsilon$, both oscillation terms in the last display are bounded by 1. Hence, the expectation there is bounded by $2P_{k,v_*}^n\left(\left(\overline{h_{\partial B_l^c}}(0) + m_l\right)^{-}\right)^2 + 8$, where the second moment in turn is bounded by $\text{Var}_k^\epsilon\left(\overline{h_{\partial B_l^c}}(0) + m_l\right)^2$. From Lemma B.1 and Propositions B.2 and B.3, it now follows that

$$P_{k,v_*}^n(h(\theta B)_n \cap (\theta^{-1} B^-)_k) \leq 0 \leq C \epsilon^{(u_* + 1)(v_* + 1)} (n - k), \quad (3.40)$$

which yields assertion (1.24).

Next, we show the stronger statement in the theorem, namely (1.25), and suppose that $u_* > 0$ or $v_* > 0$ (as otherwise (1.25) identifies with (1.24)). We first consider the case that $0 < u_* \leq (n - k)^\epsilon$ and $(n - k)^{-\epsilon} \leq v_* \leq 0$. We now set $l = n - u_*^{1/2^{(n,j,k)}}$. As $\epsilon \in (0, 1/10)$ by assumption, we have that $\frac{3}{2}(n,l,k) \leq (n - k)^{-\epsilon}$. We also assume that $n - k$ is sufficiently large such that $n - l \leq l - k$, $\partial B_l \subset (\theta B)_n$, $\partial B_l^c \subset (\theta^{-1} B^-)_k$, the assumptions of Lemmas 3.3 and 3.10 are satisfied, and we take $\theta$ again from Lemma 3.3. If $n - k$ is not as large, then $u_*$ is bounded by a constant that depends only on $\epsilon$, in which case we can make the right-hand side of (1.25) larger than 1 by choosing $C$ sufficiently large.

We again apply Lemma 3.3 to bound the left-hand side of (1.25) by (3.38). In (3.38), we now bound the second factor in the integrand by (3.40) (with $l$ in place of $n$, etc). We bound
the first factor as follows: if $|\overline{w}(0)| \leq \frac{4}{3} \wedge n,l,k$ and $\text{osc}_{B_l^+} \overline{w} \leq \frac{1}{3} \wedge n,l,k$, we bound it by

$$\mathbb{P}^{U,K,\overline{w}}_{B_l,k} \left( h(\overline{\theta} B_l) \cap B_l^+ \leq \epsilon \right) \leq C \epsilon \exp^{-u_{\epsilon}^2}$$

by (3.28), and otherwise we bound it by 1. This gives

$$\mathbb{P}^{n,u_{\epsilon}}_{k,v_{\epsilon}} \left( h(\overline{\theta} B_l) \cap (\theta^- B^-) \leq 0 \right) \leq C e^{u_{\epsilon} + 1} \frac{l - k}{l - 2k}$$

$$= \mathbb{E}^{n,u_{\epsilon}}_{k,v_{\epsilon}} \left( h_{\text{max}} + 1 ; \left| h_{\partial B_l^+}(0) + m_l \right| \leq \frac{1}{3} \wedge n,l,k, \text{osc}_{B_l^+} \overline{h}_{\partial B_l^+} \leq \frac{1}{3} \wedge n,l,k \right) \exp^{-u_{\epsilon}^2}$$

$$+ \mathbb{E}^{n,u_{\epsilon}}_{k,v_{\epsilon}} \left( h_{\text{max}} + 1 ; \left| h_{\partial B_l^+}(0) + m_l \right| \leq \frac{1}{3} \wedge n,l,k \right)$$

$$+ \mathbb{E}^{n,u_{\epsilon}}_{k,v_{\epsilon}} \left( h_{\text{max}} + 1 ; \left| h_{\partial B_l^+}(0) + m_l \right| \leq \frac{1}{3} \wedge n,l,k, \text{osc}_{B_l^+} \overline{h}_{\partial B_l^+} \geq \frac{1}{3} \wedge n,l,k \right) \right),$$

(3.42)

where we write

$$h_{\text{max}} := \left( h_{\partial B_l^+}(0) + m_l \right) - 2 \text{osc}_{B_l^+} \overline{h}_{\partial B_l^+}.$$ 

(3.43)

The first expectation in the curly brackets on the right-hand side of (3.42) is bounded by

$$n - l + 1 \leq u_{\epsilon}^2 + 1,$$

The second expectation there is bounded by a constant times $\exp^{-u_{\epsilon}^2}$ by Lemma 3.6. The expectation in the fourth line in (3.42) is bounded by Lemma 3.8 by a constant times $(1 + u_{\epsilon}^2) \exp^{-u_{\epsilon}^2}$. Plugging these bounds into (3.42) yields the assertion when we replace $\epsilon$ with $\epsilon/5$.

For the case that $u_{\epsilon} > (n - k) \epsilon$ and $-(n - k) \epsilon \leq v_{\epsilon} \leq 0$, we estimate

$$\mathbb{P}^{n,u_{\epsilon}}_{k,v_{\epsilon}} \left( h(\overline{\theta} B_l) \cap (\theta^- B^-) \leq 0 \right) \leq C e^{-(u_{\epsilon}^{+2})^{2 - \epsilon} / 2},$$

(3.44)

by Lemma 3.10 (with $\epsilon/2$ in place of $\epsilon$), which is less than the right-hand side of (1.25) in this case.

The case $v_{\epsilon} > 0$, $u_{\epsilon} \leq 0$ is analogous. If $u_{\epsilon} \geq v_{\epsilon} > 0$, we use monotonicity in the boundary conditions and the previously handled case ($u_{\epsilon} > 0$, $v_{\epsilon} \leq 0$) in the estimate

$$\mathbb{P}^{n,u_{\epsilon}}_{k,v_{\epsilon}} \left( h(\overline{\theta} B_l) \cap (\theta^- B^-) \leq 0 \right) \leq \mathbb{P}^{n,u_{\epsilon}}_{k,0} \left( h(\overline{\theta} B_l) \cap (\theta^- B^-) \leq 0 \right) \leq C \epsilon \exp^{-u_{\epsilon}^2}$$

$$\leq C \epsilon \frac{\exp^{-u_{\epsilon}^2} / 2 \exp^{-u_{\epsilon}^2 / 2}}{n - k} \frac{2 \epsilon}{n - k}.$$  

(3.45)

which is further bounded by the right-hand side in the assertion when we replace there $\epsilon$ with $2\epsilon$, and assume in particular $C \geq 1$ so that we also have $u_{\epsilon} > 0$ and $v_{\epsilon} > 0$. If $v_{\epsilon} > u_{\epsilon} > 0$, we argue as in (3.45) but instead of $v_{\epsilon}$ we now replace $u_{\epsilon}$ with $0$. \hfill \Box

4 The Ballot Asymptotics

As in the upper bound, the reduction to the DRW (Theorems 2.3 and 2.10), permits us to derive ballot asymptotics for the DGFIF from the corresponding asymptotics of the DRW. For the latter, asymptotics are given by Theorem C.3, albeit in a “weak” form, namely when one end point of the walk is required to be sufficiently negative. In addition, work is required to convert the asymptotic statement in Theorem C.3 to a form which is similar to that in Theorem 1.1. Accordingly, the desired weak versions of Theorem 1.1 are first stated (Sect. 4.1) with their proofs relegated to the next section. We then use these weak statements to prove Theorem 1.1 in its full generality (Sect. 4.2).
4.1 Weak Ballot Asymptotics

Let $L_n = L_{n,h,U}$ and $R_k = R_{k,\zeta,v}$ be defined as in Proposition 1.2.

**Proposition 4.1** Let $\epsilon \in (0, 1)$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then

$$P_{k,v}^{n,u} \left( h_{U_k^n \cap V_{k}^{-\zeta}} \leq 0 \right) = \frac{(2 + o_\epsilon(1)) \frac{L_n(u)\vartheta(\infty)^-}{g(n-k)}}{g(n-k)}$$

as $n - k \to \infty$, for all $U \in \Psi_{e}^{n}$, $V \in \Psi_{e}$, $0 \leq k < n$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k}$ satisfying (1.9) and $\vartheta(\infty)^- \geq (n-k)^{\epsilon}$.

**Proposition 4.2** Let $\epsilon \in (0, 1)$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then,

$$P_{k,v}^{n,u} \left( h_{U_k^n \cap V_{k}^{-\zeta}} \leq 0 \right) = \frac{(2 + o_\epsilon(1)) \frac{R_k(v)\vartheta(0)^-}{g(n-k)}}{g(n-k)}$$

as $n - k \to \infty$, for all $U \in \Psi_{e}^{n}$, $V \in \Psi_{e}$, $0 \leq k < n$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k}$ satisfying (1.9) and $\vartheta(0)^- \geq (n-k)^{\epsilon}$.

4.2 Stitching: Proof of Theorem 1.1

As in the proof of Theorem 1.8, the proof of Theorem 1.1 in its full generality goes by conditioning on the value of the DGFF on the boundary of a centered ball $B_l$ at an intermediate scale. Using the entropic repulsion statement, Proposition 3.5, one can restrict attention to the case when the corresponding harmonic extension in the bulk is sufficiently repelled below 0 and its oscillation therein is not too large. This then permits us to use the weak ballot asymptotic statements in the previous sub-section, as long as the restriction to stay negative is imposed away from the boundary of the ball $B_l$. To make sure that the asymptotics for the ballot probability does not increase by leaving out the region $B_{l-1}^{-} \cap B_{l+1}$ close to this boundary, we shall also need the following proposition, which bounds the DGFF in that region under the restriction that the DGFF is repelled from zero and behaves regularly at $\partial B_{l-2}^{-}$ and at $\partial B_{l+2}^{\pm}$. The restriction is expressed in terms of the set $E_{l,M} \in \Psi_{e}$ which was defined in (3.11).

**Proposition 4.3** Let $\epsilon \in \left(0, \frac{1}{10}\right)$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then there exist $C = C_\epsilon < \infty$ and $c = c_\epsilon > 0$ such that

$$P_{k,v}^{n,u} \left( \max_{B_{l+1} \cap B_{l-1}^{-}} h > 0, h_{\partial B_{l-2}^{\pm}} + m_{l-2} \in E_{l-2,M,\epsilon}, h_{\partial B_{l+2}^{\pm}} + m_{l+2} \in E_{l+2,M,\epsilon} \right) \leq C \epsilon^{-c} \quad \epsilon \in (0, 1)$$

for all $M \geq 0$, $U \in \Psi_{e}^{n}$, $V \in \Psi_{e}$ all $k, l, n \geq 0$ with $\partial B_{l+2} \subset U_{n}^{l}$ and $\partial B_{l-2}^{-} \subset V_{k-l}^{-\zeta}$, and for all $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k}$.

**Proof** By Lemma B.14, under $P_{k,v}^{n,u}$, for $x \in B_{l+1} \cap B_{l-1}^{-},$

$$\frac{h_{\partial B_{l-2}^{-} \cup \partial B_{l+2}^{\pm}}(x)}{\vartheta_{\partial B_{l-2}^{-} \cup \partial B_{l+2}^{\pm}}(x)} = \left( \frac{h_{\partial B_{l-2}^{-}}}{\vartheta_{\partial B_{l-2}^{-}}}(x) + \frac{h_{\partial B_{l+2}^{\pm}}}{\vartheta_{\partial B_{l+2}^{\pm}}}(x) \right) \leq P_x \left( \tau_{B_{l+2}^{\pm}} \leq \tau_{B_{l-2}} \right) \frac{h_{\partial B_{l-2}^{-}}}{\vartheta_{\partial B_{l-2}^{-}}}(x) + P_x \left( \tau_{B_{l+2}^{\pm}} \leq \tau_{B_{l-2}} \right) \frac{h_{\partial B_{l+2}^{\pm}}}{\vartheta_{\partial B_{l+2}^{\pm}}}(x) \leq P_x \left( \tau_{B_{l-2}} \leq \tau_{B_{l+2}^{\pm}} \right) \frac{h_{\partial B_{l-2}^{-}}}{\vartheta_{\partial B_{l-2}^{-}}}(x) + P_x \left( \tau_{B_{l+2}^{\pm}} \leq \tau_{B_{l-2}} \right) \frac{h_{\partial B_{l+2}^{\pm}}}{\vartheta_{\partial B_{l+2}^{\pm}}}(x).$$
Hence, on the event $E$ that $h_{\partial B_i^+} + m_{l-2} \in E_{l-2,M,\varepsilon}$ and $h_{\partial B_i^+} + m_{l+2} \in E_{l+2,M,\varepsilon}$, we have

$$\overline{h_{\partial B_i^+}}(x) + m_{l-2} \leq -\min\left\{ \wedge_{n,l-2,k}^\varepsilon, \wedge_{n,l+2,k}^\varepsilon \right\} + 2M \quad (4.5)$$

under $\mathbb{P}_{k,v}^{n,u}$. Now the Gibbs-Markov property (Lemma B.10) yields

$$\mathbb{P}_{k,v}^{n,u}\left( \max_{B_{i-1}^+ \cap B_{i+1}} h > 0, E \right) \leq \mathbb{P}\left( \max_{B_{i-1}^+ \cap B_{i+1}} h_{B_i^+ \cap B_{i+1}^+} > m_{l-2} + \min\left\{ \wedge_{n,l-2,k}^\varepsilon, \wedge_{n,l+2,k}^\varepsilon \right\} - 2M \right) \leq C e^{-c \wedge_{n,l,k}^\varepsilon + 2M} \quad (4.6)$$

for constants $C = C_\varepsilon < \infty, c = c_\varepsilon > 0$ from extreme value theory of the DGFF (Lemma A.1).

In the next proof, we also rely on Propositions 1.4 and 1.5 which we prove independently in Sect. 5.

**Proof of Theorem 1.1 and Proposition 1.2.** We assume w.l.o.g. that $\varepsilon \in (0, 1/15)$. It suffices to consider the case that $\max[\overline{h}(0)^-, \overline{h}(\infty)^-] \leq (n - k)^\varepsilon$ as the converse case is already dealt with in Propositions 4.1 and 4.2.

For $0 \leq k \leq n$, we set $l := k + \frac{1}{2}(n - k)$ and we abbreviate $t := \frac{1}{2}(n - k) = \wedge_{n,l,k}$. For sufficiently large $n - k$ such that $\partial B_{l+2} \subset U_{n}^u$ and $\partial B_{l-2} \subset V_{n}^{-\varepsilon}$, we have, as in (3.38),

$$\mathbb{P}_{B_l^+}^{U, n,u}(h_{U_{n}^u \cap V_{n}^{-\varepsilon}} \leq 0) \leq \mathbb{P}_{B_l^+}^{U, n,u}(h_{U_{n}^u \cap B_{l}^+} \leq 0, h_{B_{l}^+ \cap V_{n}^{-\varepsilon}} \leq 0) = \int \mathbb{P}_{B_l^+}^{U, n,u}(h_{U_{n}^u \cap B_{l}^+} \leq 0) \mathbb{P}_{V_{n}^u,k,v}(h_{B_{l}^+ \cap V_{n}^{-\varepsilon}} \leq 0) \mathbb{P}_{V_{n}^u,k,v}(h_{\partial B_{l}^+} + m_l \in dw). \quad (4.7)$$

Let $M \in (1, \infty)$. We first evaluate the integral on the right-hand side of (4.7) restricted to $w \in E_{l,M,3\varepsilon}$. In this range of integration, for the first factor in the integrand, we plug in the uniform asymptotics from Proposition 4.1, and for the second one, we use Proposition 4.2. Then, as $n - k \to \infty$, the restricted integral is asymptotically equivalent to

$$\frac{L_n(u)}{g^2(n-l)(l-k)} \mathbb{P}_{k,v}^{n,u}(h_{\partial B_{l}^+}(0) + m_l)^- (h_{\partial B_{l}^+}(\infty) + m_l)^-; h_{\partial B_{l}^+} + m_l \in E_{l,M,3\varepsilon}). \quad (4.8)$$

On $E_{l,M,3\varepsilon}$, we have $\text{osc}_{B_{l}^+} h_{\partial B_{l}^+} \leq M$ and $(h_{\partial B_{l}^+}(0) + m_l)^- \geq C_\varepsilon$, and hence

$$(h_{\partial B_{l}^+}(0) + m_l)^- (h_{\partial B_{l}^+}(\infty) + m_l)^- = \left[(h_{\partial B_{l}^+}(0) + m_l)^- \right]^2 (1 + O(M t^{-3\varepsilon})). \quad (4.9)$$

Setting

$$\mu(0; u, v) := \mathbb{E}_{k,v}^{n,u} h_{\partial B_{l}^+}(0), \quad (4.10)$$

we also have

$$|\mu(0; u, v) + m_l| \leq C_\varepsilon (1 + t^\varepsilon) \quad (4.11)$$

by Lemma B.1, Proposition B.2 and the assumptions on $u$ and $v$. Hence, (4.9) is equal to $\left[(h_{\partial B_{l}^+}(0) - \mu(0; u, v))^- \right]^2 (1 + O(t^{-\varepsilon} + M t^{-3\varepsilon}))$ so that the expectation in (4.8) equals

$$\mathbb{E}_{k,v}^{n,u}(((h_{\partial B_{l}^+}(0) - \mu(0; u, v))^-)^2; h_{\partial B_{l}^+} + m_l \in E_{l,M,3\varepsilon}) (1 + O(M t^{-\varepsilon})). \quad (4.12)$$
We claim that

\[ E_{k,u}^{n,u}\left(\left(h_{\partial B_{l}^{+}}^{0}(0) - \mu(0; u, v)\right)^{2}; h_{\partial B_{l}^{+}} + m_{l} \notin E_{l+M,3\epsilon}\right) = O(t^{6\epsilon} + t\epsilon^{-cM^{2}}) \].

(4.13)

The sum of the expectations in (4.12) and (4.13) equals \( \frac{1}{2}\text{Var}_{k}^{n}h_{\partial B_{l}^{+}}(0) \). By Proposition B.3 and as \( n - l = l - k = \frac{1}{2}(n - k) = t \), we have

\[ \text{Var}_{k}^{n}h_{\partial B_{l}^{+}}(0) = g\frac{(n - l)(l - k)}{n - k} + O(1) = \frac{1}{4}g(n - k)(1 + O(t^{-1})) . \]

(4.14)

Using the claim (4.13), we thus obtain that the expectation in (4.8) equals

\[ \frac{1}{8}g(n - k)(1 + O(Mt^{-\epsilon} + t^{6\epsilon - 1} + e^{-cM^{2}})) . \]

(4.15)

Moreover, the restriction of the integral on the right-hand side of (4.7) to \( w \notin E_{l+M,3\epsilon} \) is, by Proposition 3.5, bounded from above by a constant times \( (e^{-cM^{2}} + (n - k)^{-1}) \) times the right-hand side of (1.8), as \( L_{u}(u) \geq c_{e}(1 + \bar{u}(v)^{-1}) \) and \( R_{k}(v) \geq c_{e}(1 + v(\infty)^{-1}) \) by Propositions 1.4 and 1.5 (which are proved independently in Sect. 5). Altogether it then follows that the right-hand side of (4.7) is asymptotically equivalent to the right-hand side of (1.8) as \( n - k \to \infty \) followed by \( M \to \infty \).

We now show the claim (4.13). Using the decomposition (3.13) (with \( 3\epsilon \) in place of \( \epsilon \)), we have to bound

\[ E_{k,u}^{n,u}\left(\left(h_{\partial B_{l}^{+}}^{0}(0) - \mu(0; u, v)\right)^{2}; h_{\partial B_{l}^{+}} + m_{l} \in E_{l}\right) \]

for \( i = 1, \ldots, 4 \). For \( i = 1 \), this expectation is bounded by

\[ E_{k,u}^{n,u}\left(\left(h_{\partial B_{l}^{+}}^{0}(0) - \mu(0; u, v)\right)^{4}\left(h_{\partial B_{l}^{+}} + m_{l} \in E_{l}\right) \right) \]

by the Cauchy-Schwarz inequality. The centered fourth moment of the Gaussian random variable \( h_{\partial B_{l}^{+}}(0) \), whose variance is given by Proposition B.3, is bounded by a constant times \( t^{2} \), and by Lemma 3.6, the probability is bounded by a constant times \( e^{-t^{1-6\epsilon}} \). For \( i = 2, (4.16) \) is bounded by a constant times \( t^{6\epsilon} + t^{2\epsilon} \) by definition of \( E_{2} \) and (4.11). For \( i = 3, (4.16) \) is bounded by a constant times \( e^{-cM^{2}}(1 + t + t^{2\epsilon}) \) by Lemma 3.8, (4.11) and the assumptions on \( u, v \). On \( E_{4} \), we have \( h_{\partial B_{l}^{+}}(0) + m_{l} \geq t^{6\epsilon} \). Hence, using also (4.11), the expression in (4.16) for \( i = 4 \) is equal to zero whenever \( t^{6\epsilon} - t^{2\epsilon} \geq 0 \). This shows the claim (4.13).

It remains to bound the difference between the right and left sides of (4.7) which is equal to

\[ \mathbb{P}_{k,v}^{n,u}\left(h_{U_{n}^{\epsilon}\cap B_{l}^{-\epsilon}} \leq 0, h_{B_{l}^{+}\cap V_{k}^{-\epsilon}} \leq 0 \max_{Z^{2} \backslash B_{l}^{+\epsilon}} h > 0 \right) . \]

(4.18)

This probability is bounded from above by

\[ \mathbb{P}_{k,v}^{n,u}\left(h_{U_{l}^{\epsilon}\cap V_{k}^{-\epsilon}} \leq 0, h_{\partial B_{l}^{+\epsilon}} + m_{l-2} \notin E_{l-2,M,\epsilon} \right) + \mathbb{P}_{k,v}^{n,u}\left(h_{U_{l}^{\epsilon}\cap V_{k}^{-\epsilon}} \leq 0, h_{\partial B_{l}^{+\epsilon}} + m_{l+2} \notin E_{l+2,M,\epsilon} \right) \]

\[ + \mathbb{P}_{k,v}\left(\max_{Z^{2} \backslash B_{l}^{+\epsilon}} h > 0, h_{\partial B_{l}^{+\epsilon}} + m_{l-2} \in E_{l-2,M,\epsilon}, h_{\partial B_{l}^{+\epsilon}} + m_{l+2} \in E_{l+2,M,\epsilon} \right) \]

(4.19)
by a union bound. Applying Proposition 3.5 to the first two probabilities and Proposition 4.3 to the third one shows that (4.18) is bounded by a constant times
\[
\left(\frac{\bar{n}(0)^{-1} + 1}{n-k}\right)\left(\frac{\bar{n}(\infty)^{-1} + 1}{n-k}\right)
\left(\frac{3/2+4e^2}{n,k} + e^{-cM^2} + e^{-c\epsilon(n-k)^{\epsilon} + C\epsilon M}\right),
\] (4.20)
where we also used (1.9). The expression in (4.20) is further bounded by a constant times
\[
\frac{\mathcal{L}_n(u)\mathcal{R}_k(v)}{n-k}
\left(\frac{3/2+4e^2}{n,k} + e^{-cM^2} + e^{-c\epsilon(n-k)^{\epsilon} + C\epsilon M}\right)
\] (4.21)
by Propositions 1.4 and 1.5 and hence of smaller order than the right-hand side of (1.8) as \(n \to \infty\) followed by \(M \to \infty\). This completes the proof.

\[\square\]

5 The Ballot Functionals

Comparison statements for the ballot functionals for the DRW and the DGFF (from Proposition 1.2) are stated (Sect. 5.1) and proved (Sect. 5.3). These are then used (Sect. 5.2) to prove Propositions 4.1, 4.2, and 1.3 – 1.6. The presentation focuses on the functional \(\mathcal{L}_n\). The functional \(\mathcal{R}_k\) is handled analogously (Sect. 5.4).

5.1 Auxiliary Results

In the setting of the inward concentric decomposition (Sect. 2.4) and using the correspondence from Theorem 2.3, we define
\[
\ell_{T,r}(\bar{u}(0), \bar{v}(\infty)) = \mathbb{E}^{U,n,u}_{V,k,v}\left(\mathcal{S}_r^-; \max_{i=1}^r \left(\mathcal{S}_i + \mathcal{D}_i\right) \leq 0\right)
\] (5.1)
in accordance with (C.3) in Appendix C, and recalling from (2.12) that \(T := T_{n-k} := [n-k] + [\log \epsilon] - [\log(\epsilon^{-1} + \zeta)]\). The functional \(\ell_{T,r}(\bar{u}(0), \bar{v}(\infty))\) differs from \(\mathcal{L}_n(u)\) from Proposition 1.2 in several ways: it is defined in terms of the decorated random walk instead of the DGFF, it depends on the inner domain \(V_r^-\) and the boundary values \(v\), and the sequence \(r_n\) from Proposition 1.2 is replaced by an additional parameter \(r\). To compare these functionals, we define further functionals in which only one of the aforementioned features changes. We begin with the functional
\[
\mathcal{L}^r_{n,k}(u,v) := \mathcal{L}_{n-k}^r(U,n,u,k,V(u,v)) := \mathbb{E}^{U,n,u}_{V,k,v}(h_{\partial B_{n-r}}(0) + m_{n-r})^-; h_{U^n_{\partial B_{n-r}}} \leq 0
\] (5.2)
in which \(\mathcal{S}_r^-\) is replaced with \(h_{\partial B_{n-r}}(0) + m_{n-r}\), and the truncation event is reformulated in terms of the DGFF.

We note that the law of the decorated random walk \((\mathcal{S}_p, \mathcal{D}_p)\) and the functionals \(\ell_{T,r}(\bar{u}(0), \bar{v}(\infty))\), \(\mathcal{L}^r_{n,k}(u,v)\) depend on \(U, V, n, k, u,v, \eta, \zeta\). Therefore, assumptions on these quantities are part of the lemmas below.

The functionals \(\ell_{T,r}(\bar{u}(0), \bar{v}(\infty))\) and \(\mathcal{L}^r_{n,k}(u,v)\) can be compared as follows:

Lemma 5.1 Let \(\epsilon \in (0,1)\). Then,
\[
\lim_{r \to \infty} \lim_{n \to \infty} (1 + \bar{u}(0)^{-\epsilon})^{-\epsilon} \left|\ell_{T_{n-k},r + [\log \epsilon]}(\bar{u}(0), \bar{v}(\infty)) - \mathcal{L}^r_{n,k}(u,v)\right| = 0
\] (5.3)
uniformly in \(\eta, \zeta \in [0, \epsilon^{-1})\), \(U \in \mathcal{U}^n, V \in \mathcal{V}_n, u \in \mathbb{R}^{\partial U_n}, v \in \mathbb{R}^{\partial V_r}\) satisfying
\[
\max \left\{\bar{u}(0), \bar{v}(\infty), \text{osc} \bar{u}_n, \text{osc} \bar{v}_n\right\} \leq \epsilon^{-1}, \quad \|\bar{u}(0) - \bar{v}(\infty)\| \leq (n-k)^{1-\epsilon}.
\] (5.4)
We note that condition (5.4) follows from condition (1.9) (with \( \epsilon/2 \) in place of \( \epsilon \)) and implies condition (2.27), which are used in the ballot asymptotics (Theorem 1.1) and in the correspondence between the DGFF and the DRW (Theorems 2.3 and 2.10), respectively. The following lemma allows to vary the parameter \( \eta \) in \( \mathcal{L}_{n,k}^r \). For a subset \( W \subset U \), we define

\[
\mathcal{L}_{n,w,U,k,V}^r(u,v) = \mathbb{E}^{U,n,u}_{W,k,v} \left( \frac{1}{\eta \partial B_{n-r}} (0) + m_{n-r} \right); \quad h_{w,n} \setminus B_{n-r} \leq 0 \),
\]

which generalizes (5.2) and satisfies \( \mathcal{L}_{n,w,U,k,V}^r(u,v) = \mathcal{L}_{n,w,U,k,V}^r(u,v) \). We call \( W \) the test set as it occurs in the restriction event in (5.5) and, when scaled by \( n \), corresponds to the set on which the DGFF has to be negative in the ballot event. The more general test set \( W \) in place of \( U \) will be used in the proof of Lemma 5.4 below.

**Lemma 5.2** Let \( \epsilon \in (0,1) \) and \( \eta, \zeta \in [0, \epsilon^{-1}] \). There exists \( C = C_\epsilon < \infty \) such that

\[
\left| \frac{\mathcal{L}_{n,w,U,k,V}^r(u,v)}{\mathcal{L}_{n,\omega,U,k,V}^r(u,v)} - 1 \right| \leq C\sqrt{\text{Leb}(W \Delta W')} \tag{5.6}
\]

for all \( U \in \mathfrak{U}_e \), \( V \in \mathfrak{V}_e \), Borel measurable \( W, W' \) with \( \epsilon^2 B \subset W \), \( W' \subset U^n \), \( n, k, r \geq 0 \) with \( \partial B_{n-r} \subset (\epsilon^2 B)_n \cap V_k^{-\zeta} \), and \( u \in \mathbb{R}^{\partial U_n}, v \in \mathbb{R}^{\partial V_k} \) that satisfy \( \max \{ \overline{u}(0), \text{osc} \overline{u}, \eta, \overline{v}(\infty), \text{osc} \overline{v} \} < \epsilon^{-1} \).

To further approach the functional \( \mathcal{L}_n \) from Proposition 1.2, we also set \( \mathcal{L}_{n}^r(u) = \mathcal{L}_{n,\eta,U,0,B}^r(u, \overline{u}(0)). \) To compare \( \mathcal{L}_n^r(u) \) with \( \mathcal{L}_{n,k}^r(u,v) \), we need the following statement:

**Lemma 5.3** Let \( \eta \in (0,1), \epsilon \in (0, \frac{1}{10}) \). Then there exists \( C = C_{\eta, \epsilon} < \infty \) such that for all \( \eta, \zeta \in [0, \epsilon^{-1}] \), \( U \in \mathfrak{U}_e \), \( V \in \mathfrak{V}_e \), \( 0 < k < n \) with \( \partial B_{n-r} \subset U_n^{\eta \vee \epsilon} \) and \( \partial B_{n-r-1}^{-} \subset V_k^{-\zeta} \), and all \( u \in \mathbb{R}^{\partial U_n}, v \in \mathbb{R}^{\partial V_k} \) satisfying (5.4), we have

\[
\left| \mathcal{L}_{n,\eta,U,0,B}^r(u, \overline{u}(0)) - \mathcal{L}_{n,\eta,U,k,V}^r(u,v) \right| \leq C(n - k)^{-\epsilon/4} \tag{5.7}
\]

With the next lemma, in which the functional \( \mathcal{L}_{n,k}^r(u,v) \) is considered for fixed \( r \), we prepare the proof of Proposition 1.6.

**Lemma 5.4** Let \( \epsilon \in (0,1), \eta \in [0, \epsilon^{-1}] \) and \( q \in \mathbb{N} \). For \( n \geq 0 \), assume that \( u \in \mathbb{R}^{\partial U_n} \) satisfies

\[
\lim_{n \to \infty} \max_{x \in U_n^q} \left| \overline{u}(x) - \widehat{u}_\infty(e^{-n}x) \right| = 0 \tag{5.9}
\]

for some \( \widehat{u}_\infty \in \mathbb{H}\mathcal{L}(\partial U_n) \). In the definition of \( (S_{p, \eta}, \mathcal{D}_{\eta})_{p=1}^q \), also set \( V = B, k = 0, v = \overline{u}(0), \zeta = 0 \). Then \( (S_{p, \eta}, \mathcal{D}_{\eta})_{p=1}^q \) converges in distribution as \( n - k \to \infty \) to some limit \( (S_{p, \eta}^{(\infty)}, \mathcal{D}_{\eta}^{(\infty)})_{p=1}^q \) whose distribution is absolutely continuous with respect to Lebesgue measure and depends only on \( U, \widehat{u}_\infty \) and \( \eta \).
5.2 Proofs of Weak Ballot Asymptotics and Properties of $L_n$

In place of the parameter $r$ in $\mathcal{L}_{r,n,U,1,0}(u, \overline{u}(0))$, we will put a sequence $(r_n)$ which translates the double limit $n \to \infty$ followed by $r \to \infty$ in the decorated random walk asymptotics (Theorem C.3), and other statements, to the single limit $n \to \infty$ in Theorem 1.1. To this aim, $r_n$ has to tend to infinity sufficiently slowly such that several statements which are formulated for fixed $r$ remain valid when $r$ is replaced with $r_n$ or a smaller value, as stated in (i) – (vi) below.

As in Proposition 1.2, we define

$$L_n(u) = L_{n,n,U}(u) = L_{r_n,U,0,0}(u, \overline{u}(0)), \quad u \in \mathbb{R}^{\partial U}.$$  \hspace{1cm} (5.10)

Throughout the paper, in particular in Proposition 1.2, we choose $(r_n)_{n \geq 0}$ as a collection of integers such that $r_n \in [0, n^{\epsilon/2}]$ and $r_n \to \infty$ as $n \to \infty$ slowly enough, depending only on given $\epsilon \in (0, 1)$, such that the following properties (i)–(vi) hold. Here, we take $\delta$ as half the supremum of the $\delta$ that we can obtain from both Theorems 2.3 and 2.10 for our given $\epsilon$. The conditions involving Lemmas 5.16, 5.18 and 5.21 can be ignored for the moment and will only be needed in Sect. 5.4.

(i) Lemmas 5.1 and 5.16 hold when we replace $r$ with any element of $[r, r_{n-k}]$.

(ii) Lemmas 5.3, 5.18 and 5.21 hold with $C = C_\epsilon$ when we replace $r$ with any element of $[r, r_{n-k}]$.

(iii) We have $r_n \leq n^{\delta/4}$.

(iv) Proposition C.4 holds when we replace $r$ with any element of $[r, r_{T'}]$.

(v) With $T(r)$ from Proposition C.5, we have $T(r_n) \leq T_n$.

(vi) The constants $C$ and $c$ in Lemma B.12 can be chosen as depending only on $\epsilon$ (not on $r$), under the restriction that $t \geq 1$ and $r \leq r_{n-k}$.

We now prove Proposition 4.1 and the assertions on $L_n$ and $L$ in Propositions 1.3–1.6. The proofs of Proposition 4.2 and of the assertions on $R_k$ and $R$ are outlined in Sect. 5.4.

We need a lower bound on the ballot probability:

**Lemma 5.6** Let $\epsilon \in (0, 1)$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then there exist $c = c_\epsilon > 0$, $C = C_\epsilon$ such that

$$\mathbb{P}_{n,k,v}(h_{U_{n,k}^{\eta} \cap V_k^{1-\zeta}} \leq 0) \geq c \frac{(1 + \overline{\pi}(0)^{-\epsilon})\mathbb{V}(\infty)^{-\epsilon}}{n - k}. \hspace{1cm} (5.11)$$

for all $U \in \mathcal{U}_{\eta}^{n}$, $V \in \mathcal{V}_\eta$, $0 \leq k < n$ with $n - k > C$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k}$ satisfying (1.9) and $\mathbb{V}(\infty)^{-\epsilon} \geq (n - k)^{\epsilon}$.

**Proof** By Theorems 2.3 and C.2, there exists $a_0 \in (-\infty, 0)$, depending only on $\epsilon$, such that the assertion holds if $\overline{u}(0) < a_0$. To infer the general case, we use FKG (see e.g. Sect. 5.4 of [3]) in

$$\mathbb{P}_{n,k,v}(h_{U_{n,k}^{\eta} \cap V_k^{1-\zeta}} \leq 0) \geq \mathbb{P}_{n,k,v}(h_{U_{n,k}^{\eta} \setminus (\epsilon B)_{n-1}} \leq 0) \mathbb{P}_{n,k,v}(h_{(\epsilon B)_{n-1} \cap V_k^{1-\zeta}} \leq 0). \hspace{1cm} (5.12)$$

To bound the first factor, we decompose the set $U_{n,k}^{\eta} \setminus (\epsilon B)_{n-1}$ by intersecting, for given $\ell \in \mathbb{N}$, with discretizations of the annuli

$$W_{n,j,\ell} := (e^{-\epsilon^{2}})^{(j-1)/\ell} e^{-1} B \setminus (e^{-\epsilon^{2}})^{(j+1)/\ell} e^{-1} B, \quad j = 1, \ldots, \ell. \hspace{1cm} (5.13)$$

For sufficiently large $n$, depending only on $\epsilon$ and $\ell$, we have

$$W_{n,j,\ell} \supset ((e^{-\epsilon^{2}})^{(j-1)/\ell} e^{-1} B)_{n} \setminus ((e^{-\epsilon^{2}})^{(j+1)/\ell} e^{-1} B)_{n}. \hspace{1cm} (5.14)$$
for all $j = 1, \ldots, \ell$, by definition of the discretization in (1.4). As also $U \subset \epsilon^{-1} B$, it then follows that

$$U_1^\eta \cap (\epsilon B)_{n-1} \subset U_0^\eta \cap \bigcup_{j=1}^{\ell} W^{(j, \ell)}_n,$$

and hence the FKG inequality yields

$$\mathbb{P}^n_{k,v}(h_{U_1^\eta \setminus (\epsilon B)_{n-1}} \leq 0) \geq \prod_{j=1}^{\ell} \mathbb{P}^n_{k,v}(h_{W^{(j, \ell)}_n \cap U_0^\eta} \leq 0).$$

(5.15)

For $\ell$ sufficiently large, depending only on $\epsilon$, we have

$$\mathbb{P}^n_{k,v}(h_{W^{(j, \ell)}_n \cap U_0^\eta} \leq 0) = 1 - \mathbb{P}^n_{k,v}(\max_{W^{(j, \ell)}_n \cap U_0^\eta} h > 0) \geq \frac{1}{2}$$

(5.16)

for all $j = 1, \ldots, \ell$ by Lemma A.4 as the Lebesgue mass of $W^{(j, \ell)}_n \cap U_0^\eta$ then becomes arbitrarily small. Hence, the first probability on the right-hand side of (5.12) is bounded from below by $2 - \epsilon > 0$.

To bound the second factor on the right-hand side of (5.12), let $g : U_n \cap V_k^{-} \to \mathbb{R}$ be harmonic with boundary values $u$ on $\partial U_n$ and $v$ on $\partial V_k^{-}$. By Lemma B.14 and Assumption (1.9),

$$g \leq 2 \epsilon^{-1} \text{ on } U_0^\eta \cap V_k^{-}.\text{ Hence, for } \lambda > 0,$$

$$\mathbb{P}^n_{k,v}(h_{W^{(j, \ell)}_n \cap U_0^\eta} \leq 0) = \mathbb{P}^n_{k,0}(h_{(\epsilon B)_{n-1} \cap V_k^{-}} + g \leq 0) \geq \mathbb{P}^n_{k,2\epsilon^{-1}}(h_{(\epsilon B)_{n-1} \cap V_k^{-}} \leq 0)$$

(5.17)

$$\geq \mathbb{P}^n_{k,2\epsilon^{-1}}(h_{(\epsilon B)_{n-1} \cap V_k^{-}} \leq 0, h_{\partial(\epsilon B)_n}(0) + m_n \in (a_0 - 1, a_0), \text{osc}(\epsilon B)_{n-1} h_{\partial(\epsilon B)_n} \leq \lambda)$$

(5.18)

$$\geq \mathbb{P}^n_{k,2\epsilon^{-1}}(h_{\partial(\epsilon B)_n}(0) + m_n \in (a_0 - 1, a_0), \text{osc}(\epsilon B)_{n-1} h_{\partial(\epsilon B)_n} \leq \lambda)$$

$$\times \inf \left\{ \mathbb{P}^n_{B,2\epsilon^{-1}}(h_{(\epsilon B)_{n-1} \cap V_k^{-}} \leq 0) : w \in \mathbb{R}^{\partial(\epsilon B)_n}, \overline{w}(0) \in (a_0 - 1, a_0), \text{osc}(\epsilon B)_n w \leq \lambda \right\},$$

where we used the Gibbs-Markov property in the last step. By Lemma B.1 and Propositions B.2 and B.3,

$$\mathbb{P}^n_{k,2\epsilon^{-1}}(h_{(\epsilon B)_n}(0) + m_n \in (a_0 - 1, a_0)) \geq c \epsilon.$$  

(5.19)

Moreover, by Proposition B.4, and for sufficiently large $\lambda$, depending only on $\epsilon$,

$$\mathbb{P}^n_{k,2\epsilon^{-1}}(\text{osc}(\epsilon B)_n h_{\partial(\epsilon B)_n} \leq \lambda | h_{\partial(\epsilon B)_n}(0) + m_n = r) \geq 1 - C \epsilon e^{-C \lambda} \geq \frac{1}{2}$$

(5.20)

for all $r \in (a_0 - 1, a_0)$. Hence, the probability in the third line of (5.18) is bounded from below by a constant. As noted in the beginning of the proof, the infimum in the last line of (5.18) is bounded from below as in the right-hand side of (5.11). Combining the above bounds yields the assertion. \( \square \)

We are now ready to prove the weak ballot asymptotics using the corresponding statement for decorated random walk (Theorem C.3), the correspondence to the DGFF (Theorem 2.3) and the comparison between the ballot functionals for the decorated random walk and the DGFF (Lemma 5.1).
Proof of Proposition 4.1} From Theorem C.3, condition (iii) in the choice of $r_n$, and Theorem 2.3, we have
\[
\mathbb{P}_{n, k}^n \left( h_{U_n^k \cap V_k^{-\xi}} \leq 0 \right) = \mathbb{P}_{n, k}^n \left( \max_{i=1}^{T_{n+k}} (S_i + D_i) \leq 0 \right) \\
= 2 \ell_{T_{n+k}, r_{n+k} + \lfloor \log \epsilon \rfloor} (\bar{u}(0), \bar{v}(\infty)) / \bar{v}(\infty) - a(1) \left( 1 + \bar{u}(0) \right) / n - k \tag{5.21}
\]
as $n - k \to \infty$, uniformly as in the assertion. We now convert the $\ell$ functional on the right-hand side to the $L$ functional in the statement of the proposition. To this aim, we write
\[
\left| \ell_{T_{n+k}, r_{n+k} + \lfloor \log \epsilon \rfloor} (\bar{u}(0), \bar{v}(\infty)) / \bar{v}(\infty) - L_n (u) / g(n - k) \right| \\
\leq \left| \ell_{T_{n+k}, r_{n+k} + \lfloor \log \epsilon \rfloor} (\bar{u}(0), \bar{v}(\infty)) - L_{n+k} (u, v) \right| + \left| L_{n+k} (u, v) - L_n (u) \right| \left( 1 + \bar{u}(0) \right) / n - k, \tag{5.22}
\]
The first ratio on the right-hand of (5.22) is $a(1)$ as $n - k \to \infty$ by Lemmas 5.1 and 5.3 and the choice of $r_n$. Hence,
\[
\mathbb{P}_{n, k}^n \left( h_{U_n^k \cap V_k^{-\xi}} \leq 0 \right) = 2 L_n (u) / g(n - k) + a(1) \left( 1 + \bar{u}(0) \right) / n - k \\
= (2 + a(1)) L_n (u) / g(n - k), \tag{5.23}
\]
where the first equality follows from (5.21) and (5.22), and the second equality from Lemma 5.6. $\square$

By combining a convergence result for the $\ell_{T_{n}, r_n}$ functional for decorated random walk (Proposition C.7), the correspondence to the DGFF (Theorem 2.3), and the comparison between the $\ell_{T_{n}, r_n}$ and the $L_n$ functional (Lemma 5.1), we obtain:

Proof of Proposition 1.3 (for $L_n$) With $\ell_{T_{n}, r_n} (\bar{u}(0), \bar{u}(0))$ defined by (5.1) for $V = B, k = 0, \xi = 0, v = \bar{u}(0)$, we write
\[
L_n (u) = L_{n, 0} (u, \bar{u}(0)) \\
= \left( L_{n, 0} (u, \bar{u}(0)) - \ell_{T_{n}, r_n + \lfloor \log \epsilon \rfloor} (\bar{u}(0), \bar{u}(0)) \right) + \ell_{T_{n}, r_n + \lfloor \log \epsilon \rfloor} (\bar{u}(0), \bar{u}(0)). \tag{5.24}
\]
The expression in brackets on the right-hand side converges to zero as $n \to \infty$ by Lemma 5.1 and as $\bar{u}(0)$ is bounded by the assumption on the convergence of $u_n$. The last term on the right-hand side of (5.24) converges to a limit $\mathcal{L}(\hat{u}_\infty)$ by Proposition C.7 which we apply as follows: for any sequence $(n_i)_{i=1}^\infty$ with $n_i \to \infty$, we set $a_i = \bar{u}_n (0), b_i = \bar{u}_n (\infty), T_i = T_{n_i}, S_i = S_{n_i}, D_i = D_{n_i}$, and we define $\mathcal{L}(S_i), \mathcal{L}(D_i)$ as in (2.18), (2.21) and (2.25) using $U_n, n = n_i, u = u_n (0), v = \bar{u}(0)$ and $\xi = 0$. Then, Lemma 5.5 yields the limit in distribution $(S_i, D_i)$ of $(S_i, D_i)$, verifies Assumption (C.10) and the assumption on stochastic absolute continuity, and shows that
\[
\mathcal{L}(\hat{u}) := \lim_{r \to \infty} \mathbb{E} \left( S_r (\infty) + D_r \leq 0 \right) \tag{5.25}
\]
depends only on $U, \tilde{u}_\infty$ and $\eta$. Assumptions ($A1$) – ($A3$) are again verified by Theorem 2.3. □

In the next proof, we derive asymptotics for the $\mathcal{L}_n$ functional in its argument again from the corresponding statement for the decorated random walk functional (Propositions C.5). Then we infer the asymptotics for the $\mathcal{L}$ functional using Proposition 3.1. 

**Proof of Proposition 1.4 (for $\mathcal{L}_n$, $\mathcal{L}$)**

With $\ell_{\mathbb{T}} n, r_n, L(0), L(0))$ defined by (5.1) for $V = B, k = 0, \zeta = 0, v = \tilde{u}(0)$, we write

$$
\frac{\mathcal{L}^{r_n}_{\mathbb{T}}(u, \tilde{u}(0))}{\tilde{u}(0)^{-}} = \frac{\mathcal{L}^{r_n}_{\mathbb{T}}(u, \tilde{u}(0)) - \ell_{\mathbb{T}} n, r_n, \log r_n(\tilde{u}(0), \tilde{u}(0))}{\tilde{u}(0)^{-}} + \frac{\ell_{\mathbb{T}} n, r_n, \log r_n(\tilde{u}(0), \tilde{u}(0))}{\tilde{u}(0)^{-}}.
$$

Using also the choice of $r_T$, we obtain that the first term on the right-hand side is at most $C_\epsilon (1 + \tilde{u}(0))^{-1}$ by Lemma 5.1, thus the assertion on $\mathcal{L}_n$ follows from Proposition C.5.

To show the assertion on $\mathcal{L}$, let $\epsilon' > 0$. Using the assertion on $\mathcal{L}_n$, we find $M < \infty$ with the following property: for each $U \in \mathfrak{U}_L^0$, $\tilde{u}(0) \in \mathfrak{H}(U, \eta) \tilde{u}(0) > M$, there exist $n \geq c_\epsilon, u_n \in \mathbb{R}^{\partial U_n}$ such that on the right-hand side of

$$
\frac{\mathcal{L}(\tilde{u})}{\tilde{u}(0)^{-}} = \frac{\mathcal{L}(\tilde{u}) - \mathcal{L}(u_n)}{\tilde{u}(0)^{-}} + \frac{\mathcal{L}(u_n)}{\tilde{u}(0)^{-}} / \tilde{u}(0)^{-},
$$

the first summand is bounded by $\epsilon'$ by Proposition 1.3, and both factors in the second summand are in $(1 - \epsilon', 1 + \epsilon')$. This shows the assertion on $\mathcal{L}$. □

The lower bound for the $\mathcal{L}_n$ functional is shown in the next proof using the lower bound for the ballot probability (Lemma 5.6) and the weak ballot asymptotics (Proposition 4.1). 

**Proof of Proposition 1.5 (for $\mathcal{L}_n$, $\mathcal{L}$)**

To show the assertion on $\mathcal{L}_n$, we first consider $u \in \mathbb{R}^{\partial U_n}$ and $v \in \mathbb{R}^{\partial B_0}$ that satisfy (1.9) and $\tilde{u}(0)^{-} \geq n^\epsilon$ with $\zeta = 0$. Then Proposition 4.1 and Lemma 5.6 (where we set $V = B$ and $k = 0$) together show that

$$
\mathcal{L}_n(u) > c_\epsilon (1 + \tilde{u}(0)^{-})
$$

for sufficiently large $n$, depending only on $\epsilon$. As the left-hand side in (5.28) does not depend on $v$, the assertion follows for all $u \in \mathbb{R}^{\partial U_n}$ with $\text{osc} \tilde{u}_\eta < \epsilon^{-1}$ and $\tilde{u}(0) \in (-n^{-1-\epsilon}, \epsilon^{-1})$. If $\tilde{u}(0) \leq -n^{-1-\epsilon}$ and $\text{osc} \tilde{u}_\eta < \epsilon^{-1}$, we have $\mathcal{L}_n(u) > c_\epsilon$ for sufficiently large $n$ by Proposition 4.1. Hence, $\mathcal{L}_n(u) > c_\epsilon$ for all $U, n, u$ as in the assertion for some constant $c = c_\epsilon > 0$.

To show the assertion on $\mathcal{L}$, we find for every $U \in \mathfrak{U}_L^0$, $\tilde{u} \in \mathfrak{H}(U, \eta)$ by Proposition 1.3 some $n \geq c_\epsilon, u_n \in \mathbb{R}^{\partial U_n}$ with $|\mathcal{L}(\tilde{u}) - \mathcal{L}(u_n)| < c/2$ and $\tilde{u}_n(0) \vee \text{osc} \tilde{u}_\eta \leq 2\epsilon^{-1}$. Using the assertion on $\mathcal{L}_n$, we then obtain $\mathcal{L}(\tilde{u}) > c/2$. □

Finally, we show the continuity of the ballot functional $\mathcal{L}_n$ in the domain, the boundary values and the test set using the corresponding statement for $\mathcal{L}_{n, 0}$ (Lemma 5.4). Here the comparison between $\mathcal{L}_n$ and $\mathcal{L}_n = \mathcal{L}_n^{\mathbb{T}}$ goes via the corresponding comparison for the decorated random walk functionals. The continuity statement for $\mathcal{L}$ is inferred using Proposition 1.3.

**Proof of Proposition 1.6 (for $\mathcal{L}_n$, $\mathcal{L}$)**

We recall the definition $\mathcal{L}_n(u) = \mathcal{L}_{n, \eta, U, 0, B}(u, \tilde{u}(0))$ from (5.10). We write

$$
\begin{align*}
|\mathcal{L}_{n, \eta, U, 0, B}(u, \tilde{u}(0)) - \mathcal{L}_{n, \eta, U, 0, B}(u, \tilde{u}(0))| &\leq |\mathcal{L}_{n, \eta, U, 0, B}(u, \tilde{u}(0)) - \ell_{\mathbb{T}} n, r_n, \log r_n(\tilde{u}(0), \tilde{u}(0))| \\
+ |\ell_{\mathbb{T}} n, r_n, \log r_n(\tilde{u}(0), \tilde{u}(0)) - \ell_{\mathbb{T}} n, r_n, \log r_n(\tilde{u}(0), \tilde{u}(0))| &\leq |\ell_{\mathbb{T}} n, r_n, \log r_n(\tilde{u}(0), \tilde{u}(0))|
\end{align*}
$$

(5.29)
where we define $L_{n, r_n}^r \left( \overline{u}(0), \overline{u}(0) \right)$ by (5.1) for $V = B$, $k = 0$, $\zeta = 0$, $v = \overline{u}(0)$. By Lemma 5.1, Proposition C.4, and by choice of $r_n$, all terms on the right-hand side of (5.29) are $\alpha(n) (1 + \overline{u}(0)^{-1})$ as $n \to \infty$ followed by $r \to \infty$. Let $\epsilon' \in (0, 1/2)$. Using Propositions 1.4 and 1.5 and the above, we obtain

$$
(1 - \epsilon') L_{n, \eta, U, 0, B}^r(u, \overline{u}(0)) \leq L_{n, \eta, U, 0, B}^r(u, \overline{u}(0)) \leq (1 + \epsilon') L_{n, \eta, U, 0, B}^r(u, \overline{u}(0))
$$

(5.30)

for all sufficiently large $n$, $r$ (and all $U, \eta, u$ as in the assertion). In particular, using Propositions 1.4 and 1.5, we obtain a constant $\tilde{c} = \epsilon_{r, \epsilon} > 0$ such that

$$
\tilde{c}(1 + \overline{u}(0)^{-1}) \leq L_{n, \eta, U, 0, B}^r(u, \overline{u}(0)) \leq \tilde{c}^{-1}(1 + \overline{u}(0)^{-1})
$$

(5.31)

and analogously also

$$
\tilde{c}(1 + u'(0)^{-1}) \leq L_{n, \eta, U, 0, B}^r(u', \overline{u'}(0)) \leq \tilde{c}^{-1}(1 + u'(0)^{-1})
$$

(5.32)

for sufficiently large $n$ and all $U, u, \eta$ as in the assertion. Hence, by Lemma 5.4, there exists $c = \epsilon_{r, \epsilon} \in (0, 1)$ such that

$$
(1 - \epsilon' Y) L_{n, \eta, U', 0, B}^r(u', \overline{u'}(0)) \leq L_{n, \eta, U, 0, B}^r(u, \overline{u}(0)) \leq (1 + \epsilon' Y) L_{n, \eta, U', 0, B}^r(u', \overline{u'}(0))
$$

(5.33)

where

$$
Y := 1 + \tilde{c}^{-1} + \tilde{c}^{-2} \frac{1 + \overline{u}(0)^{-1}}{1 + u'(0)^{-1}},
$$

(5.34)

for all $U, U', u, u', \eta, \eta'$ as in the assertion with

$$
d_{\mathbb{E}^c}(U, U') + \sqrt{\text{Leb}(U' \Delta U', \eta^c)} + \| \overline{u} - \overline{u'} \|_{U_u^c \cap U_{u'}^c} + n^{-1} \leq c,
$$

(5.35)

for which we note that $Y$ is bounded by a constant (that depends on $\epsilon, r$). From (5.33), (5.30), and the argument leading to (5.30) albeit now for $L_{n, \eta, U', 0, B}^r(u', \overline{u'}(0))$, we obtain

$$
(1 - \epsilon')^2 (1 - \epsilon' Y) L_{n, \eta, U', 0, B}^r(u', \overline{u'}(0)) \leq L_{n, \eta, U, 0, B}^r(u, \overline{u}(0))
$$

$$
\leq (1 + \epsilon')^2 (1 + \epsilon' Y) L_{n, \eta, U', 0, B}^r(u', \overline{u'}(0))
$$

(5.36)

whenever $n$ is sufficiently large and (5.35) holds. As the choice of $r$ depended only on $\epsilon$ and $\epsilon'$, assertion (1.20) follows.

To show also the assertion on $L_{\eta, U}$ and $L_{\eta', U'}$, we find, for given $\widehat{u}, \widehat{u}', U, U'$ as in the assertion, $n, u_n \in \mathbb{R}^3 U_n, u'_n \in \mathbb{R}^3 U'_n$ such that

$$
(1 - \epsilon') L_{\eta, U}^r(\widehat{u}) \leq L_{n, \eta, U, 0, B}^r(u, \overline{u}(0)) \leq (1 + \epsilon') L_{\eta, U}^r(\widehat{u}')
$$

(5.37)

and the analogous statement with $\eta', U', \widehat{u}', u'_n$ hold by Proposition 1.3. From the second inequality in (5.36) with $u_n, u'_n$ in place of $u, u'$, we then obtain

$$
(1 - \epsilon')^2 (1 + \epsilon' Y) L_{n, \eta', U', 0, B}^r(u', \overline{u'}(0)) \leq L_{n, \eta, U', 0, B}^r(u', \overline{u'}(0))
$$

(5.38)

and the analogous bound in the other direction for which we use the first inequality in (5.36). This implies the assertion on $L_{\eta, U}, L_{\eta', U'}$. □
5.3 Proofs of Auxiliary Results

First we show that the distribution of the maximum of the DGFF on a subset of the domain is absolutely continuous with respect to Lebesgue measure. We need this property to prove Lemmas 5.3 and 5.4. Henceforth, we denote \( a := 2/\sqrt{g} = \sqrt{2\pi} \).

**Lemma 5.7** Let \( r \geq 0, \epsilon \in (0, 1), \eta, \zeta \in [0, \epsilon^{-1}] \). There exists \( C = C_{r, \epsilon} < \infty \) such that

\[
\mathbb{P}_{V, k, v}^{U_n, u}(\max_{x \in U_n} h \in [-\lambda, \lambda]) \leq C\lambda(1 + \lambda)^2 e^{a(\lambda + \pi(0))}(1 + \tilde{u}(0)^{-}) \tag{5.39}
\]

for all \( \lambda \geq 0, U \in \mathcal{U}_n^1, V \in \mathcal{U}_n, n, k \geq 0 \) with \( \partial B_{n-r} \subset U_n^{\eta \vee \epsilon}, \partial B_{n-r-1} \subset V_k^{\zeta \vee \epsilon} \), and all \( u \in \mathbb{R}^{dU_n}, v \in \mathbb{R}^{dV_k} \) satisfying (2.27).

In the proof of Lemma 5.7, we apply a generalization of Theorem 1.8 which bounds the probability that the DGFF stays positive on more general domains than \( U_n^\eta \cap V_k^{\zeta} \). We recall the definitions of \( u_{\eta}, v_{\eta} \) from (1.23) (which use \( u, \eta \) and \( v, \zeta \)), as well as the annulus-like sets \( A_p \) from Sect. 2.3.

**Lemma 5.8** Let \( \epsilon \in (0, 1), \eta, \zeta \in [0, \epsilon^{-1}], 0 \leq k < n, U \in \mathcal{U}_n^0, V \in \mathcal{U}_n, u \in \mathbb{R}^{dU_n}, v \in \mathbb{R}^{dV_k}, \) and \( D \subset U_n^\eta \cap V_k^{\zeta} \) such that for each \( p = 1, \ldots, T \), the set \( D \cap A_p \) contains a discrete ball of radius \( \epsilon e^{n-p} \). Then there exists \( C = C_{\epsilon} < \infty \) such that

\[
\mathbb{P}_{k, v}^{n, u}(h_D \leq 0) \leq C\frac{(u_{\zeta} + 1)(v_{\eta} + 1)}{n - k} \tag{5.40}
\]

**Proof** This follows from the proof of Theorem 1.8 with the following modification: we redefine the decoration \( D_p \) from (2.25) as

\[
D_p = \max_{y \in A_p \cap D} \left\{ h_p(y) - m_{n-p} + \varphi_{0, p}(y) - S_{p'} + \gamma(y) \right\} \tag{5.41}
\]

which yields the correspondence

\[
\{h_{A_p \cap D} \leq 0\} = \{S_p + D_p \leq 0\} \text{ for } p = 1, \ldots, T \tag{5.42}
\]

in place of (2.28), and the first summand on the right-hand side of (2.45) becomes

\[
\mathbb{P}_{k, v}^{n, u}(\max_{A_p \cap D} h_p < -t/3) \tag{5.43}
\]

This probability is bounded in the same way by Lemma A.2, using the assumption that \( D \cap A_p \) contains a discrete ball of radius \( \epsilon e^{n-p} \). \( \square \)

**Proof of Lemma 5.7** Let \( g : U_n \cap V_k^- \to \mathbb{R} \) be the harmonic function with boundary values \( u \) on \( \partial U_n \) and \( v \) on \( \partial V_k^- \). Then, by definition of the DGFF, the superposition principle for the harmonic extension, and by a union bound, we have

\[
\mathbb{P}_{V, k, v}^{U_n, u}(\max_{x \in U_n \setminus B_{n-r}} h \in [-\lambda, \lambda]) \leq \sum_{x \in U_n^\eta \setminus B_{n-r-1}} \mathbb{P}_{V, k, 0}^{U_n, u}(h(x) + g \in [-\lambda, \lambda], \max_{y \in U_n^\eta \setminus B_{n-r}} h(y) + g(y) \leq h(x) + g(x)) \tag{5.44}
\]

For each vertex \( x \) over which the sum on the right-hand side is taken, we consider its vicinities

\[
A := \{y \in U_n \cap B_{n-r-1} : \ |x - y| < d(x, (U_n)^\xi) - \frac{1}{2} \}
\]
Furthermore, using that
\[ A' := \{ y \in A \cap U_n^\eta \setminus B_{n-r} : |x - y| \leq \frac{1}{2}d(x, (U_n)^c) \}. \tag{5.45} \]
We will also need the logarithmic distance between \( e^{-n}x \) and the complement of \( U \), which is defined by
\[ p := -\log d(e^{-n}x, U^c). \tag{5.46} \]
As \( x \in U_n^\eta \), we note from (1.4) that \( p \leq n + \log 2 \). The probability in the second line of (5.43) can only increase when we take the maximum only over \( A' \) which is a subset of \( U_n^\eta \setminus B_{n-r} \).

At \( \partial A \), we now apply the Gibbs-Markov property,
\[
\mathbb{P}_{V,k,0}^{U,n,0}(h(x) + g(x) \in [-\lambda, \lambda], \max_{y \in A'} h(y) + g(y) \leq h(x) + g(x)) \leq \int_{w \in \mathbb{R}^{\partial A}} \mathbb{P}_{V,k,0}^{U,n,0}(h_{\partial A} + m_n \in dw) \mathbb{P}(h(x) + g(x) \in [-\lambda, \lambda], h_{\partial A} = -m_n + w) \times \mathbb{P}(\max_{y \in A'} h(y) + g(y) \leq \lambda \mid h_{\partial A} = -m_n + w, h(x) + g(x) = -\lambda), \tag{5.47}
\]
in the third line we also used monotonicity of the ballot event and of the DGFF. Using Assumption (2.27) and Lemmas A.5(i) and B.14, we obtain
\[
\max_{x \in U_n^\eta \setminus B_{n-r}} \left| g(x) - \bar{\Omega}(0) \right| \leq C_{r,\varepsilon}. \tag{5.48}
\]
The ballot probability in the third line of (5.47) equals
\[
\mathbb{P}(\max_{y \in A' - x} h(y) + g(x) + y \leq \lambda \mid h_{\partial A - x} = -m_n - m_n - m_n - n + w', h(0) + g(x) = -\lambda), \tag{5.49}
\]
where \( w'(y) := w(x + y) \). From (1.4), we note that \( A - x = \tilde{A}_{n-p} \) for
\[
\tilde{A} = B \cap e^p(e^{-r}B - e^{-n}x), \tag{5.50}
\]
and that \( \tilde{A} \in \mathcal{L}_n^{\delta} \) for some \( \delta > 0 \). As also \( A' - x \subset \tilde{A}_{n-p} \) and as \( A' \) contains half of the ball around \( x \) with radius \( e^{-p}d(x, (U_n)^c) \), we can apply Lemma 5.8 with \( D = A' - x \), and \( \tilde{A}, \frac{1}{10}B \) in place of \( U, V \), and \( n - p, 0, \delta \) in place of \( n, k, \eta \). This bounds the probability in (5.49) from above by a constant times \((1 + \lambda)^2(1 + \bar{\Omega}(0)^{-})(1 + m_n - m_n + m_n + \bar{\Omega}(0)^{-} + \text{osc}_{B_{n-r}}(\bar{\Omega}))/(n - p)

We use the shorthand \( \mathbb{E}^w := \mathbb{P}(\cdot \mid h_{\partial A} = -m_n + w) \) and denote by \( \mathbb{E}^w \) and \( \text{Var}^w \) the expectation resp. the variance under \( \mathbb{P}^w \). As a consequence of (2.6) and (2.7) we have
\[
g(n - p) - C_{r,\varepsilon} \leq \text{Var}^w h(x) = G_A(x, x) \leq g(n - p) + C_{r,\varepsilon}. \tag{5.51}
\]
Moreover, let \( \mu(x) = m_n + \mathbb{E}^w h(x) + g(x) \). Then, as \( h \) under \( \mathbb{P}^w \) is Gaussian,
\[
\mathbb{P}^w(h(x) + g(x) \in [-\lambda, \lambda]) \leq C_{r,\varepsilon}(n - p)^{-1/2}\lambda \sup_{s \in [-\lambda, \lambda]} e^{-\frac{(s - \mu(x) + m_n - m_n + m_n - p)^2}{2g(n - p) + C_{r,\varepsilon}}}. \tag{5.52}
\]
Furthermore, using that \( m_n - p = 2\sqrt{g(n - p)} - \frac{3}{8}\sqrt{g} \log^+(n - p), \alpha = 2/\sqrt{g} \) and binomial expansion (as in (5.13) of [4]) yields
\[
e^{-\frac{(s - \mu(x) + m_n - m_n + m_n - p)^2}{2g(n - p) + C_{r,\varepsilon}}} \leq C_{r,\varepsilon}(n - p)^{3/2}e^{-2(n - p) - \alpha(s - \mu(x)) - \alpha(m_n - m_n - p)}. \tag{5.53}
\]
From (5.48) and as $E^w h(x)$ is the harmonic extension at $x$ of the boundary values $w$ by the definition of the DGFF, we also have
\[ g(x) + E^w h(x) \leq -m + \max_{A'} |\overline{w}| + \overline{u}(0) + C_{r,e}. \tag{5.54} \]
Moreover, we note that
\[ -\alpha(m_n - m_{n-p}) \leq -4p + \frac{3}{2} \log^+(n) - \frac{3}{2} \log^+(n-p) \leq -\frac{5}{2} p + \text{const.} \tag{5.55} \]
Thus we obtain from (5.52) and (5.53) that
\[ \mathbb{P}^w \left( h(x) + g(x) \in [-\lambda, \lambda] \right) \leq C_{r,e} (n - p) \lambda e^{-2n + \alpha \left( \lambda + \max_{A'} |\overline{w}| + \overline{u}(0) \right) - p/2}, \tag{5.56} \]
and that the right-hand side of (5.47) is bounded from above by
\[ C_{r,e} \int_{t \geq 0} \mathbb{P}^U_{V,n,0} \left( \max_{A'} |\overline{h}_{\partial A} + m_n| \in t \right) \lambda (1 + \lambda)^2 e^{-2n + \alpha \left( \lambda + \overline{u}(0) \right) + t} (1 + \overline{u}(0)^{-})(1 + t), \tag{5.57} \]
where we also used the bound on (5.49), that $\overline{w}(0) + \text{osc}_{A'} \overline{w} \leq 3 \max_{A'} |\overline{w}|$, and that $(1 + m_n - m_{n-p}) e^{-p/2}$ is bounded by a constant. We write $\tilde{\mu}(x) = m_n + \mathbb{E}_{V,k,0} U^n_{V,n} \overline{h}_{\partial A}(x)$. By Lemma B.1 and Proposition B.2, we have $|\tilde{\mu}(x)| \leq C_{r,e}$. Claiming that
\[ \mathbb{P} \left( \max_{A'} |\overline{h}_{\partial A} + m_n| > t \right) \leq C_{r,e} e^{-c_r e t^2}, \tag{5.58} \]
for $t \geq 0$, we now bound (5.57) by a constant times
\[ \lambda (1 + \lambda)^2 e^{-2n + \alpha \left( \lambda + \overline{u}(0) \right) + t} (1 + \overline{u}(0)^{-}). \tag{5.59} \]
Summing over $x \in U^n_{\partial} \setminus B_{n-r}$ and using also (5.43), the assertion then follows as $|\{x \in U^n_{\partial} \setminus B_{n-r}\}| \leq e^{2n}$.

It remains to show the claim (5.58). To this aim, we use Fernique majorization (see e.g. Theorem 4.1 in [1]) with the uniform probability measure on $A'$ as the majorization measure, and the Borell-TIS inequality (see e.g. Theorem 2.1.1 in [2]). For the application of the Borell-TIS inequality, we bound $\text{Var}_{U_{V,k} h_{\partial A}(y)}$ uniformly in $y \in A'$. Applying the representation (2.8) twice and using that $G_{U_n \cap V_{k}^-} \leq G_{U_n}$, we have
\[ \text{Var}_{U_{V,k} \overline{h}_{\partial A}}(y) = \sum_{w,w' \in \partial A} \Pi_{\partial A}(y, w) \Pi_{\partial A}(y, w') G_{U_n \cap V_{k}^-}(w, w') \leq \text{Var} \overline{h}_{\partial A} U_n(y). \tag{5.60} \]
By the Gibbs-Markov property (Lemma B.10), we have
\[ \text{Var} \overline{h}_{\partial A} U_n(y) = \text{Var} \overline{h}_{\partial A} U_n(y) - \text{Var} \overline{h}_{\partial A}(y), \tag{5.61} \]
and both variances on the right-hand side differ from $g \log d(y, (U_n)^c)$ by an additive constant by Lemma 3.2 of [4]. Hence, $\text{Var}_{U_{V,k} \overline{h}_{\partial A}}(y)$ is bounded by a constant for all $y \in A'$. For the application of Fernique’s inequality, we moreover bound the squared intrinsic metric $\mathbb{E} \left( (\overline{h}_{\partial A} U_n(y) - \overline{h}_{\partial A} U_n(z))^2 \right), y, z \in A'$, by a constant times $|y - z|^2 / d(x, (U_n)^c)^2$. This follows analogously to the proof of Lemma B.8 in which we now use also the bound for $\text{Var}_{U_{V,k} \overline{h}_{\partial A}}(y)$. \hfill \Box

The next lemma bounds the probability that $h$ is non-positive outside of an intermediated scaled disk. This estimate will be used in the proof of Lemma 5.1 below.
Lemma 5.9  Let $\epsilon \in (0, 1)$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. There exists $C = C_\epsilon < \infty$ such that
\begin{equation}
\mathbb{P}^{n,u}_{k,v}\left(h^{U_{n}\setminus B_{n-r}} \leq 0\right) \leq C(1 + \bar{\eta}(0)^{-})^2 r^{-1/2},
\end{equation}
for all $U \in \Omega^0_n$, $V \in \Omega_e$, $n \geq 0$, $r \in (0, (n-k)^{+})$ such that $\partial B_{n-r} \subset U_{n}^{\vee e}$ and $\partial B_{n-r-1}^{-} \subset V_{k}^{-}\vee e$, and all $u \in \mathbb{R}^{\partial U_{n}}$, $v \in \mathbb{R}^{\partial V_{k}^{-}}$ satisfying (5.4).

**Proof**  As $h$ under $\mathbb{P}^{n,u}_{k,v}$ is distributed as $h + \mathbb{E}^{n,u}_{k,v} h - \mathbb{E}^{n,0}_{k,0} h$ under $\mathbb{P}^{n,0}_{k,0}$, and as
\begin{equation}
\mathbb{E}^{n,u}_{k,v} h_{\partial B_{n-r}}(0) - \mathbb{E}^{n,0}_{k,0} h_{\partial B_{n-r}}(0) \leq C_\epsilon + \bar{\eta}(0)^{-},
\end{equation}
by Proposition B.2 and (5.4), we have
\begin{equation}
\mathbb{P}^{n,u}_{k,v}\left(h^{U_{n}\setminus B_{n-r}} \leq 0\right) \leq \mathbb{P}^{n,0}_{k,0}\left(h_{U_{n}\setminus B_{n-r}} - C_\epsilon - \bar{\eta}(0)^{-} \leq 0\right).
\end{equation}
By the Gibbs-Markov property, the right-hand side of (5.64) equals
\begin{equation}
\int_{w \in \mathbb{R}^{\partial B_{n-r-1}}} \mathbb{P}^{n,0}_{k,0}\left(h_{U_{n}\setminus B_{n-r-1}} + m_{n-r-1} \in dw\right) \mathbb{P}^{U_{n},0}_{B_{n-r-1},w}\left(h_{U_{n}\setminus B_{n-r}} - C_\epsilon - \bar{\eta}(0)^{-} \leq 0\right),
\end{equation}
where the last probability is bounded by a constant times
\begin{equation}
r^{-1}(1 + \bar{\eta}(0)^{-})(1 + \bar{\eta}(0)^{-} + \bar{\eta}(\infty)^{-} + \text{osc}(B_{n-r}) \bar{\eta}) \cdot
\end{equation}
by Theorem 1.8. Hence, (5.65) is bounded by
\begin{equation}
\int_{t \in \mathbb{R}} \mathbb{P}^{n,0}_{k,0}\left(h_{\partial B_{n-r-1}}(0) + m_{n-r-1} \in dt\right)
\times \sum_{a=1}^{\infty} \mathbb{P}^{n,0}_{k,0}\left(\text{osc}(B_{n-r}) h_{\partial B_{n-r-1}} \geq a \left|h_{\partial B_{n-r-1}}(0) + m_{n-r-1} = t\right| r^{-1}(1 + \bar{\eta}(0)^{-})^2(1 + t^{-})(1 + a)\right).
\end{equation}
The conditional probability in (5.67) is bounded by $e^{-c_\epsilon a + C_\epsilon |t|/r}$ by Proposition B.4. Furthermore, $h_{\partial B_{n-r-1}}(0)$ is a Gaussian with $s^2 := \text{Var}_k[h_{\partial B_{n-r-1}}(0)] \leq gr + C_\epsilon$ by Proposition B.3, and $|\mathbb{E}^{n,0}_{k,0} h_{\partial B_{n-r-1}}(0) + m_{n-r-1}| \leq C_\epsilon (1 + \log r)$ by Lemma B.1 and Proposition B.2, hence (5.67) is bounded by a constant times
\begin{equation}
\int_{t \in \mathbb{R}} dt \left|s^{-1/2}e^{-c_\epsilon t^2/2} \sum_{a=1}^{\infty} e^{-c_\epsilon a + C_\epsilon |t|/r} r^{-1}(1 + \bar{\eta}(0)^{-})^2(1 + t^{-})(1 + a)\right|.
\end{equation}
Changing variables such that $ts^{-1/2}$ becomes $t$, and using (5.64) yields (5.62). \hfill \Box

We are now ready to prove the remaining statements from Sect. 5.1. The proof of the comparison statement between the ballot functionals $\mathcal{U}_{n,k}$ for the DGFF and $\ell_{T,\tau}$ for the DRW uses various definitions and bounds in the context of the concentric decomposition:

**Proof of Lemma 5.1** We go back to the notation that was introduced in Sects. 2.3 and 2.4. In particular, we keep assuming that the coupling of $h^{U_{n},V_{k}^{\vee e}}$, $h$ and $(\varphi_p, h_p)^{p=1}$ from (2.15) and (2.16) holds $\mathbb{P}^{n,u}_{k,v}$-a.s. Set $\tau = r + [\log \epsilon]$. Then $B_{n-r} = \Delta_{\tau}$ for $\tau = 1, \ldots, T - 1$ by (2.13), and by Theorem 2.3 also
\begin{equation}
\left\{ \max_{i=1}^{\tau} (S_i' + \beta_i + D_i) \leq 0 \right\} = \left\{ h_{U_{n}\setminus B_{n-r}} \leq 0 \right\}.
\end{equation}
Using also Definitions (5.1) and (5.2), we obtain for any \( z \in A_{\gamma r} \) that

\[
\left| C_{n,k}^\epsilon (u,v) - \ell_{T, \gamma r} (\pi, \overline{\gamma}) \right|
\leq \mathbb{E}_{k,v} \left( (S_{\gamma}^\epsilon - \gamma_{\overline{\gamma}}) \right) + \mathbb{E}_{k,v} \left( |\gamma_{\overline{\gamma}} - \varphi_{0, \gamma r}(z)| ; h_{U_n^\gamma \setminus B_{n-r}} \leq 0 \right)
\leq \mathbb{E}_{k,v} \left( |\varphi_{0, \gamma r}(z) - h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (0) | \right) + \left| \mathbb{E}_{k,v} \left( h_{U_n^\gamma \setminus B_{n-r}} (0) | h_{U_n^\gamma \setminus B_{n-r}} \leq 0 \right) \right|
\]

(5.70)

where, for \( x \in \partial B_{n-r} \subset A_r \),

\[
\gamma(x) = m_{n-r} - \beta_r + (-m_r 1_{\partial U_n} + u - m_k 1_{\partial V_k^\gamma} + v)_{\partial U_n \cup \partial V_k^\gamma}(x)
\]

(5.71)

is defined as in (2.24) and we have \( |\mathbb{E}_{\partial B_{n-r}} (0) | \leq C_{\epsilon, \delta} (1 + r^\delta) \) as in (2.59) for an arbitrarily small constant \( \delta > 0 \). The first term on the right-hand side of (5.70) is bounded from above by \( \mathbb{E}_{k,d} \left( (S_{\gamma}^\epsilon - \gamma_{\overline{\gamma}})^2 \right)^{1/2} \) by the Jensen inequality. This expression is further bounded by a constant times \((n - k - \gamma r)^{-1/2}\) by Lemma 2.6 and as \( S_{\gamma}^\epsilon, \gamma_{\overline{\gamma}} \) are centered Gaussians. The probability in the second line of (5.70) is bounded by its \( \epsilon/4\)-th power, which in turn is bounded by a constant times \((1 + \overline{\pi}(0))^{-1/2} r^{-\epsilon/8}\) by Lemma 5.9.

The third term on the right-hand side of (5.70) is, by the Jensen inequality, bounded from above by the square root of

\[
\mathbb{E}_{k,v} \left( (\varphi_{0, \gamma r}(z) - h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (0)) \right) \leq \sum_{w, w' \in \Delta_{\gamma r}} |\Pi_{\Delta_{\gamma r}} (z, w) - \Pi_{\Delta_{\gamma r}} (0, w)|
\]

\[
\times |\Pi_{\Delta_{\gamma r}} (z, w') - \Pi_{\Delta_{\gamma r}} (0, w')| \mathbb{E}_{k,v} \left( h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (w) h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (w') \right),
\]

(5.72)

for the last equality we used that \( \varphi_{0, \gamma r}(z) = h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (0) \) and the analogous representation of the harmonic extension \( h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (0) \) in terms of simple random walk from (2.8). By Lemma A.6 with \( j = \gamma r \) and \( p = 2r \),

\[
|\Pi_{\Delta_{\gamma r}} (z, w) - \Pi_{\Delta_{\gamma r}} (0, w)| \leq C_{\epsilon} \left( \frac{r}{n - k - r} + r^{-1} \right) \Pi_{\Delta_{\gamma r}} (0, w)
\]

(5.73)

in (5.72). Hence, (5.72) is bounded by

\[
C_{\epsilon} \left( \frac{r}{n - k - r} + r^{-1} \right)^2 \mathbb{E}_{k,v} \left( h_{U_n^\gamma \setminus V_k^\gamma \setminus \partial B_{n-r}} (0) \right),
\]

(5.74)

where the variance is bounded by \( C_{\epsilon} r \) by Proposition B.3.

The second term on the right-hand side of (5.70) is bounded by

\[
\mathbb{E}_{k,v} \left( (\gamma_{\overline{\gamma}} - \varphi_{0, \gamma r}(z))^2 \right)^{1/2} \mathbb{E}_{k,v} \left( h_{U_n^\gamma \setminus B_{n-r}} \leq 0 \right)^{1/2}
\]

(5.75)

by the Cauchy-Schwarz inequality. By the above, it suffices to show that the second moment in this expression is bounded by some \( C_{\epsilon} < \infty \). By (2.36), (2.17), (2.30), and the definition of the binding field \( \varphi_j \) from Lemma B.10, we can set

\[
\gamma_{\overline{\gamma}} = \sum_{j=1}^{\gamma r} \sum_{y \in \Delta_j} \frac{S_{\gamma r, T}}{S_{j, T}} \Pi_{\Delta_j} (0, y) h_{\Delta_j - 1 \setminus V_k^\gamma} (y),
\]

(5.76)
where the DGFFs \( h^{\Delta_j \cap V^-} \), \( j = 1, \ldots, \tilde{r} \) are independent. Likewise, using the definition of \( \varphi_0, \tilde{r} \) and again (2.30), we have

\[
\varphi_0, \tilde{r}(z) = \varphi_0(z) + \sum_{j=1}^{\tilde{r}} \sum_{y \in \partial \Delta_j} \Pi_{\Delta_j \cap V^-}(z, y) h^{\Delta_j \cap V^-}(y) \quad (5.77)
\]

with the same realizations of \( h^{\Delta_j \cap V^-} \) as in (5.76), which are independent of \( \varphi_0 \). As in (2.48) and (2.50) with \( p = 2\tilde{r} \), we have

\[
\left| \Pi_{\Delta_j \cap V^-}(z, y) - \frac{\mathbf{S}_{\Delta_j}(z, y)}{\mathbf{S}_{\Delta_j}(0, y)} \Pi_{\Delta_j}(0, y) \right| \leq C \epsilon \log \frac{r}{r} \Pi_{\Delta_j}(0, y). \quad (5.78)
\]

By subtracting (5.76) from (5.77) and then using (5.78), we obtain

\[
\mathbb{E}^{n, u}_{k, v}( (\varphi_{\tilde{r}} - \varphi_0, \tilde{r}(z))^2 ) \leq \mathbb{V} \mathbb{A}^{n}_{k, v}( \varphi_0(z) + C \epsilon \left( \log \frac{r}{r} \right)^2 \sum_{j=1}^{\tilde{r}} \mathbb{V} \mathbb{A}_{h^{\Delta_j \cap V^-}}(\partial \Delta_j)(0) ) . \quad (5.79)
\]

which is bounded by a constant as \( \mathbb{V} \mathbb{A}_{n, v}^{n}(\varphi_0(z)) \) and the other variances on the right-hand side are bounded by a constant by Lemma 2.5 and Proposition B.3, respectively. This yields the assertion.

The comparison between DGFF ballot functionals with different test sets is derived from a bound for the DGFF to reach a high value on a small set (Lemma A.4) in the next proof, which also uses the FKG inequality (for which we refer to e.g. Sect. 5.4 of [3]).

**Proof of Lemma 5.2** We can assume that \( W \subset W' \); if this is not the case, we compare both \( \mathcal{L}^r_{n, W, U, k, V}(u, v) \) and \( \mathcal{L}^r_{n, W', U, k, V}(u, v) \) to \( \mathcal{L}^r_{n, W \cap W', U, k, V}(u, v) \), using that \( W, W' \supset W \cap W' \).

By the definition of \( \mathcal{L}^r_{n, W, U, k, V}(u, v) \) in (5.5), monotonicity and linearity of the expectation therein, and by FKG, we have

\[
0 \leq \mathcal{L}^r_{n, W, U, k, V}(u, v) - \mathcal{L}^r_{n, W', U, k, V}(u, v) = \mathbb{E}^{n, u}_{V, k, v}( \mathbb{H}_{B_{n-r}}(0) - m_{n-r}; h_{W \setminus B_{n-r}} \leq 0, \max_{x \in W \setminus B_{n}} h(x) > 0 ) \\
\leq \max_{x \in W \setminus B_{n}} h(x) > 0 \\
\leq \max_{x \in W \setminus B_{n}} h(x) > 0
\]

(5.80) and we bound the probability on the right-hand side using Lemma A.4. This yields the assertion.

A key for the proof of Lemma 5.3, and for the proof of the continuity statement in Lemma 5.4, is the following Lemma 5.10, in which we compare \( \mathcal{L}^r_{n, W, U, k, V}(u, v) \) with the functional

\[
\mathcal{L}^r_{n, W, U, k, V}(u, v, t) := \mathbb{E}^{n, u}_{V, k, v}( (h_{\partial B_{n-r}}(0) + t + m_{n-r})^-; h_{W \setminus B_{n-r}} + t \leq 0 ), \quad (5.81)
\]

where \( t \in \mathbb{R} \) and all other parameters are as in (5.5). The functional \( \mathcal{L}^r_{n, W, U, k, V}(u, v, t) \) contains the additional parameter \( t \) which lifts the DGFF by a constant value.

**Lemma 5.10** Let \( r \geq 0, \epsilon \in (0, 1), \eta, \zeta \in [0, e^{-1}) \). There exists \( C = C_{r, \epsilon} < \infty \) such that for all \( U \in \mathcal{U}_\epsilon, V \in \mathcal{V}_\epsilon \), \( 0 \leq k < n \) with \( \partial B_{n-r} \subset U_\eta \) and \( \partial B_{n-r-1}^- \subset V_k \cap \epsilon \), all \( u \in \mathbb{R}^{\partial U_n}, v \in \mathbb{R}^{\partial V_k} \) satisfying (2.27), and all \( t \geq 0 \), we have
\[ \mathcal{L}^r_{n,k}(u,v,t) \leq \mathcal{L}^r_{n,k}(u,v) \leq \mathcal{L}^r_{n,k}(u,v) + Ct^{1/2}e^{at} \] (5.82)

and

\[ \mathcal{L}^r_{n,k}(u,v,-t) - Ct^{1/2}e^{at} \leq \mathcal{L}^r_{n,k}(u,v) \leq \mathcal{L}^r_{n,k}(u,v,-t) \] (5.83)

**Proof** The first inequality in (5.82) follows by monotonicity in \( t \) of the expectation in (5.81). For the second inequality in (5.82), we note that

\[
\mathbb{E}_{k,v}^n \left( (\overline{h_{B_{n-r}}} + m_{n-r})^-; h_{U_c \setminus B_{n-r}} + t \leq 0 \right) \\
\geq \mathbb{E}_{k,v}^n \left( (\overline{h_{B_{n-r}}} + m_{n-r})^-; h_{U_c \setminus B_{n-r}} \leq 0 \right) \\
- \mathbb{E}_{k,v}^n \left( (\overline{h_{B_{n-r}}} + m_{n-r})^-; \max_{U_c \setminus B_{n-r}} h \in (-t, 0) \right) - t. \] (5.84)

Therefore, by Cauchy-Schwarz,

\[
\mathcal{L}^r_{n,k}(u,v) - \mathcal{L}^r_{n,k}(u,v,t) \leq t + \left( \mathbb{E}_{V,k,v}^U (\overline{h_{B_{n-r}}} + m_{n-r})^2 \right)^{1/2} \\
\times \mathbb{E}_{V,k,v}^U \left( \max_{U_c \setminus B_{n-r}} h \in (-t, 0) \right)^{1/2}. \] (5.85)

The probability on the right-hand side is bounded by \( C_{r,e}e^{a\pi(0)}(1 + \overline{u}(0))re^{2at} \) by Lemma 5.7. The expectation on the right-hand side of (5.85) is bounded by \( C_{r,e}(1 + \overline{u}(0)^2) \) by Propositions B.2, B.3, Lemma B.1 and Assumption (2.27). Hence, (5.85) is bounded by \( t + C_{r,e}t^{1/2}e^{at} \) which yields assertion (5.82) when we consider large and small \( t \) separately. The proof of (5.83) is analogous. \( \square \)

Next we apply Lemma 5.10 in conjunction with a Gibbs-Markov decomposition of the DGFF corresponding to the different inner domains in Lemma 5.3:

**Proof of Lemma 5.3** We show (5.7) with \( \mathcal{L}^r_{n,\eta,U,0,e_B}(u, \overline{u}(0)) \) in place of \( \mathcal{L}^r_{n,\eta,U,0,B}(u, \overline{u}(0)) \). Then the assertion of the lemma follows as

\[
\left| \mathcal{L}^r_{n,\eta,U,0,B}(u, \overline{u}(0)) - \mathcal{L}^r_{n,\eta,U,k,V}(u,v) \right| \\
\leq \left| \mathcal{L}^r_{n,\eta,U,0,B}(u, \overline{u}(0)) - \mathcal{L}^r_{n,\eta,U,0,e_B}(u, \overline{u}(0)) \right| \\
+ \left| \mathcal{L}^r_{n,\eta,U,0,e_B}(u, \overline{u}(0)) - \mathcal{L}^r_{n,\eta,U,k,V}(u,v) \right|. \] (5.86)

By Definition (2.1) of \( \mathcal{G}_\epsilon \), we have \((\epsilon B^-)_0 \supset V^-_k \). To account for the boundary values, let \( g \) denote the harmonic function on \( U_n \cap V^-_k \) with boundary values \(-m_n + u \) on \( \partial U_n \) and \(-m_n + v \) on \( \partial V^-_k \), and let \( \tilde{g} \) denote the harmonic function on \( U_n \cap (\epsilon B^-)_0 \) with boundary values \(-m_n + u \) on \( \partial U_n \) and \( \overline{u}(0) \) on \( \partial (\epsilon B^-)_0 \). Then, by the Gibbs-Markov property, \((h - \tilde{g})_{U_n \cap V^-_k} \) under \( \mathbb{P}_{e_B,0,\overline{u}(0)}^{U_n} \) is distributed as the sum of the independent fields \( h - g \) under \( \mathbb{P}_{V,k,v}^{U} \) and \( q^{U_n \cap (\epsilon B^-)_0,U_n \cap V^-_k} \), where the latter is the usual binding field. Using also Definitions (5.2) and (5.81), in particular the monotonicity in \( t \), we obtain

\( \heartsuit \) Springer
\[ L_{n, \eta, U, 0, \epsilon B}^r (u, \bar{\eta}(0)) \geq \int \mathbb{P}\left( \max_{U_n^r \setminus B_{n-r}} \left| \varphi_{\Sigma_U \cap (\epsilon B^-)_0} U_n^r \cap V_k^- + \tilde{g} - g \right| \in dt \right) L_{n, \eta, U, k, V}^r (u, v, t) \]  

(5.87)

and

\[ L_{n, \eta, U, 0, \epsilon B}^r (u, \bar{\eta}(0)) \leq \int \mathbb{P}\left( \max_{U_n^r \setminus B_{n-r}} \left| \varphi_{\Sigma_U \cap (\epsilon B^-)_0} U_n^r \cap V_k^- + \tilde{g} - g \right| \in dt \right) L_{n, \eta, U, k, V}^r (u, v, t) \]  

(5.88)

In light of Assumption (5.4), Lemma 5.11 below gives

\[ \max_{U_n^r \setminus B_{n-r}} |\tilde{g} - g| \leq r (n - k)^{-\epsilon} \]  

(5.89)

Applying Lemma 10 in (5.87) and (5.88), we obtain

\[ |L_{n, \eta, U, 0, \epsilon B}^r (u, \bar{\eta}(0)) - L_{n, \eta, U, k, V}^r (u, v)| \leq C_{r, \epsilon} \left( \mathbb{E}\left( \max_{U_n^r \setminus B_{n-r}} \left| \varphi_{\Sigma_U \cap (\epsilon B^-)_0} U_n^r \cap V_k^- + \tilde{g} - g \right| \right) \right)^{1/2} \]

\[ + \sum_{t=1}^{\infty} \mathbb{P}\left( \max_{U_n^r \setminus B_{n-r}} \left| \varphi_{\Sigma_U \cap (\epsilon B^-)_0} U_n^r \cap V_k^- + \tilde{g} - g \right| > t \right) C_{r, \epsilon} e^{C_{r, \epsilon} t} , \]  

(5.90)

where we also used Jensen’s inequality in the expectation on the right-hand side. From this, (5.89) and Lemma B.12 (with \( V = \epsilon B \)), we obtain (5.7) with \( \epsilon B \) in place of \( B \), as required.

\( \square \)

The following bound on the difference of the harmonic extensions of boundary values was used in the above proof.

**Lemma 5.11** Let \( \epsilon \in (0, 1) \). Then,

\[ \max_{U_n^r \setminus B_{n-r}} |g - \tilde{g}| \leq (n - k)^{-\epsilon} r \]  

(5.91)

for all \( \eta, \zeta \in [0, \epsilon^{-1}] \), \( U \subseteq \Sigma_{\epsilon} \), \( V \subseteq \Sigma_{\epsilon} \), \( 0 \leq k \leq n \), \( u \in \mathbb{R}^{\partial U} \), \( v \in \mathbb{R}^{\partial V} \) satisfying (5.4), all \( r \geq 0 \) with \( \partial B_{n-r} \subseteq U_{n_{r+\epsilon}}^{-} \) and \( \partial B_{n-r-1}^{-} \subseteq V_{k-\epsilon}^{-} \), and for \( g, \tilde{g} \) defined as in the proof of Lemma 5.3.

**Proof** Let \( g^0 \) be defined as in \( g \) although with \( \bar{\eta}(0) \) in place of \( v \). Then, by harmonicity, \( g = g^0 + (v - \bar{\eta}(0)) \frac{1}{\partial V_k^-} \partial U_{n_{r+\epsilon}} \). By (2.8), Lemmas B.14, B.15, A.5(i) and Assumption (5.4),

\[ \max_{U_n \setminus B_{n-r}} \frac{(v - \bar{\eta}(0))}{\partial V_k^-} \partial U_{n_{r+\epsilon}} \]

\[ \leq \max_{x \in U_n \setminus B_{n-r}} P_x (\tau_{V_k^-} < \tau_{U_n}) \left| \Sigma (\infty) - \bar{\eta}(0) \right| + C_{\epsilon} e^{-(n-r-k) \text{osc} \Sigma_{\epsilon}} \leq C_{\epsilon} (n - k)^{-\epsilon} r \]

(5.92)

It therefore suffices to prove the lemma with \( g^0 \) in place of \( g \).

As \( g^0 - \tilde{g} \) is harmonic on \( U_n \cap V_k^- \), and as \( \tilde{g} \) is harmonic on \( U_n \cap (\epsilon B^-)_0 \), we have

\[ \max_{U_n \setminus B_{n-r}} |g^0 - \tilde{g}| \leq \max_{x \in U_n \setminus B_{n-r}} \sum_{y \in \partial V_k^-} P_x (S_{\tau_{U_n \cap (\epsilon B)^{-}}_0} = y) \]

\[ \times | - m_k + \bar{\eta}(0) - \sum z \in \partial U_n P_y (S_{\tau_{U_n \cap (\epsilon B)^{-}}_0} = z)(-m_n + u(z)) - P_y (\tau_{(\epsilon B)^{-}}_0 \leq \tau_{U_n}) \bar{\eta}(0) | \]
The restriction of \( V_k \) to \( \tau U_n \) and (5.81), we obtain that
\[
\left| \mu_{\partial U_n \cup \partial (e^{-B})_y}(y) - P_y(\tau U_n < \tau (e^{-B})_y)(0) \right| \leq \text{osc} \eta_n. 
\]
Hence, using also Lemma A.5(i), we can further bound the expression on the right-hand side of (5.93) by
\[
\left( r + C \epsilon \right)^{\frac{1}{2}} \left\lfloor \frac{1}{n} m_n - m_k \right\rfloor ,
\]
where we also used that \( m_n \) is of order \( n \). As \( \frac{k}{n} m_n = \tilde{m}_{n,k,0} \) by (B.3), Lemma B.1 shows that
\[
The absolute difference in (5.94) is bounded by a constant times \( \log(n - k) \), which yields the assertion.
\]
Also in the next proof, we apply Lemma 5.10 in a Gibbs-Markov decomposition corresponding to the different domains and boundary values in Lemma 5.4. To account for the different test sets therein, we use Lemma 5.2.

**Proof of Lemma 5.4** To account for the boundary values, let \( g \) be harmonic on \( U_n \cap B_0^- \) with boundary values \( -m_n + u \) on \( \partial U_n \) and \( \mu(0) \) on \( \partial B_0^- \). Let \( \tilde{g} \) be harmonic on \( U_n \cap B_0^- \) with boundary values \( -m_n + u' \) on \( \partial U_n \) and \( \tilde{u}(0) \) on \( \partial B_0^- \). Furthermore, we define \( \tilde{u} \) as the restriction of \( \tilde{g} \) to \( \partial (U \cap U')_n \). By the Gibbs-Markov property, \( (h - g)_{U_n \cap B_0^-} \) under \( \mathbb{P}^{U_n,u} \) is distributed as the sum of the following two independent fields: \( (h - \tilde{g})_{(U \cap U')_n \cap B_0^-} \) under \( \mathbb{P}^{U \cap (U')_n, \mu(0)} \), and the binding field \( \psi_{U_n \cap B_0^-, (U \cap U')_n \cap B_0^-} \). Using this decomposition in (5.2) and (5.81), we obtain that
\[
\mathcal{L}_{r,n,U,0,B}(u, \mu(0)) \leq \int \mathcal{L}_{r,n,U \cap U',0,B}^{r}(\tilde{u}, \tilde{u}'(0), -t) \mathbb{P} \left( \max_{(U \cap U')_n \cap B_{n-r}} \left| \psi_{U_n \cap B_0^-, (U \cap U')_n \cap B_0^-} \right| + \left| \tilde{g} - g \right| \right) \leq \mathbb{E} \psi(M) ,
\]
where we set \( \psi(t) = C_{r,\epsilon} t^{1/2} e^{\alpha t} \) and
\[
M := \max_{(U \cap U')_n \cap B_{n-r}} \left| \psi_{U_n \cap B_0^-, (U \cap U')_n \cap B_0^-} \right| + \max_{(U \cap U')_n \cap B_{n-r}} \left| \tilde{g} - g \right|.
\]
From Lemmas B.11 and 5.12 below, we then obtain for each \( \epsilon' > 0 \) that \( \mathbb{E} \psi(M) < \epsilon' \) whenever \( d_{\text{HJ}}(U', U') \| \tilde{u} - \tilde{u} \|_{L^\infty(U \cap U')_n} \) and \( n^{-1} \) are sufficiently small (only depending on \( r, \epsilon, \epsilon' \)).

We note that the change from the test set \( U_n \) to the smaller test set \( U_{n'} \cup U''_n \) in (5.95) further increases the right-hand side as the restriction event in the definition of the \( \mathcal{L}_{r,n,U \cap U',0,B}^{r}(\tilde{u}, \tilde{u}'(0)) \) functional then becomes smaller. For the opposite direction, we first apply Lemma 5.2 in
\[
(1 + C_\epsilon \sqrt{\text{Leb}(U_{n'} \cup U''_n)}) \mathcal{L}_{r,n,U,0,B}^{r}(u, \mu(0)) \geq \mathcal{L}_{r,n,U \cap U',0,B}^{r}(u, \mu(0)),
\]
then we can obtain a lower bound similar to (5.96) by estimating analogously to (5.95) that
\[
L^r_{n,U\cap U',0,B}(u, \overline{u}(0)) = \int L^r_{n,U\cap U',0,B}(\overline{u}, \overline{u}'(0), \tau) \max_{(U\cap U')_n \setminus \Lambda_n} \left\{ |g_{U\cap U'}(\overline{u}, (U\cap U')_n \cap B_0)| + |\overline{u} - g| \right\} \, dr.
\]
and we obtain
\[
L^r_{n,U\cap U',0,B}(\overline{u}, \overline{u}'(0)) - L^r_{n,\eta,0,B}(u, \overline{u}(0)) \leq \mathbb{E}\psi(M) + L^r_{n,\eta,0,B}(u, \overline{u}(0))C_e \sqrt{\text{Leb}(U^{\eta \Delta U'})}.
\]
(5.100)
The assertion now follows from the analogous estimates for \(L^r_{n,\eta',0,B}u, \overline{u}'(0))\) and the triangle inequality
\[
|L^r_{n,\eta',0,B}(u, \overline{u}'(0)) - L^r_{n,\eta,0,B}(u, \overline{u}(0))| \leq |L^r_{n,\eta,0,B}(u', \overline{u}'(0)) - L^r_{n,\eta',0,B}(u', \overline{u}'(0))| + |\overline{u} - \overline{u}'(0)|.
\]
(5.101)

It remains to prove the bound on the difference of harmonic extensions of the boundary values when the outer domain varies slightly.

**Lemma 5.12** Let \(r \geq 0, \epsilon \in (0, 1), \eta, \eta' \in [0, \epsilon^{-1}], n \geq 0, \) and let \(U, U', u, u', g\) and \(\tilde{g}\) be defined as in the above proof of Lemma 5.4. Then
\[
\max_{(U\cap U')_n \setminus \Lambda_n} |\tilde{g} - g| \leq \|\overline{u} - \overline{u}'\|_{L_{\infty}(U\cap U')} + o_{\epsilon, r}(1),
\]
(5.102)
where \(o_{\epsilon, r}(1)\) tends to zero as \(n \to \infty\) and \(d_{\mathbb{H}}(U, U') \to 0.\)

**Proof** For \(x \in U^n_0 \cap U'^n_0 \setminus \Lambda_n\), we write
\[
g(x) - \tilde{g}(x) = m_n \left( P_x(\tau^U_n > \tau^B_n) - P_x(\tau^U_{n'} > \tau^B_{n'}) \right) + \left( \overline{u}_{\partial U_n \cup \partial B_0}(x) - \overline{u}(x) \right) + \left( \overline{u}(x) - \overline{u}'(x) \right) - \left( \overline{u}_{\partial U'_{n'} \cup \partial B_0}(x) - \overline{u}'(x) \right)
\]
(5.103)
using the representation (2.8). The second term in the last line of (5.103) is bounded by \(\|\overline{u} - \overline{u}'\|_{L_{\infty}(U\cap U')}\). The first and the third difference in the last line of (5.103) are bounded by \(C_en^{-1}\) by Lemmas B.14, A.5(i) and the assumptions.

We now come to the first term on the right-hand side of (5.103). Let \(\delta \in (0, \epsilon/2), \delta' \in (d_{\mathbb{H}}(U, U'), \delta), \) then we have \(U^{\delta'} \subset U'.\) First, we consider \(x \in U^{\delta'} \setminus U', \) then we obtain by monotonicity and Lemma A.5(ii) that
\[
m_n \left( P_{\epsilon^x}(\tau^U_n > \tau^B_n) - P_{\epsilon^x}(\tau^U_{n'} > \tau^B_{n'}) \right)
\]
\[
\leq m_n \left( P_{\epsilon^x}(\tau^U_n > \tau^B_n) - P_{\epsilon^x}(\tau^U_{n'} > \tau^B_{n'}) \right)
\]
\[
\leq 2\sqrt{g} (\int_{U^{\delta'}} \Pi_U(x, dz) \log |z| - \int_{U^{\delta'}} \Pi_{U^{\delta'}}(x, dz) \log |z|) + o_{\epsilon, r, \delta}(1),
\]
(5.104)
where \( o_{\varepsilon, \eta}(1) \to 0 \) as \( n \to \infty \) and the difference in the last line tends to zero as \( \delta' \downarrow 0 \) with rate depending only on \( \varepsilon, \delta \) by Lemma A.10, and we have the analogous lower bound.

Finally, for \( x \in (U_n^\eta \cap U_n^{\eta'}) \setminus U_n^{\delta + \delta'} \), we bound from above the first term on the right-hand side of (5.103) by leaving out the second probability in the difference, that is, by \( m_n P_x(\tau_{\mathcal{B}^0_n} > \tau_{\mathcal{B}^0_n}) \). The maximum over such \( x \) of this probability is bounded from above by the maximum over \( (U^{\delta} \setminus U^{2\delta})_n \) by monotonicity of the probability measure \( P_{[e^n, x]} \). Thus we bound

\[
\begin{align*}
\max_{U_n^\eta \cap U_n^{\eta'} \setminus U_n^{\delta + \delta'}} m_n P_x(\tau_{\mathcal{B}^0_n} > \tau_{\mathcal{B}^0_n}) & \leq \max_{x \in U^{\delta} \setminus U^{2\delta}} m_n P_{[e^n, x]}(\tau_{\mathcal{B}^0_n} > \tau_{\mathcal{B}^0_n}) \\
& \leq 2\sqrt{\beta} \sup_{x \in U^{\delta} \setminus U^{2\delta}} \int_{\partial U} \Pi_U(x, dz) \log |z| - \log |x| + o_{\varepsilon, \delta}(1) \quad (5.105)
\end{align*}
\]

where \( o_{\varepsilon, \delta}(1) \to 0 \) as \( n \to \infty \) by Lemma A.5(ii). By Lemma A.11, the last line in (5.105) converges to zero as \( \delta' \downarrow 0 \) uniformly for fixed \( \varepsilon \). The lower bound for the first term on the right-hand side of (5.103) is analogous, and the assertion follows.

To prove the convergence of the random walk steps and the decorations, we use again at the definitions and bounds for the concentric decomposition in Sects. 2.3–2.5:

**Proof of Lemma 5.5** First we recall from (2.26) that \( S_p = S_p' + \beta_p \). From the definition of \( \beta_p \) in (2.22) and as the quantities \( s_{1,p}, s_{1,T}/T \) therein converge by Lemma 2.5, we have

\[
\beta_p = \frac{s_{1,T} - s_{1,p}}{s_{1,T}} \mathbb{P}(0) + \frac{s_{1,p}}{s_{1,T}} \mathbb{P}(\infty) = \tilde{u}_\infty(0) + o_{\varepsilon, q}(1) \quad (5.106)
\]

as \( n - k \to \infty \) for all \( p = 1, \ldots, q \).

We consider the centered and spatially scaled harmonic extension \( \phi_p \) of the DGFF at \( \partial A'_p \). This will be defined as a random field on the continuum domain \( \tilde{\mathcal{A}}'_p \), where \( \varepsilon' \in (0, 1) \) is arbitrary and specifies the bulk as in (1.5), and we set \( \tilde{\mathcal{A}}_1 = U_n^\eta \cap e^{-1 + [\log \varepsilon']} B^- \) and \( \tilde{\mathcal{A}}_p = e^{-p + [\log \varepsilon]} B \cap e^{-p + [\log \varepsilon]} B^- \) for \( p \geq 2 \). Then, \( (\tilde{\mathcal{A}}'_p)_n \subset A'_p \) for \( p \geq 1 \) and \( n \) sufficiently large. We now define

\[
\phi_p(y) := \frac{h_{\partial \tilde{A}'_p}([e^n y]) + m_{n-p}}{\sqrt{p} \mathbb{P}(0)} \quad (5.107)
\]

under \( \mathbb{P}^{\mu, \eta}_{0, \mathbb{P}}(0) \) with the coupling from (2.16) which we always assume.

Next, we show that \( (S'_p + \beta_p, \phi_p)_{p=1}^q \) converges in distribution as \( n-k \to \infty \). Using (2.16) and applying the Gibbs-Markov property at \( A'_p \), we obtain

\[
\frac{h_{\partial \tilde{A}'_p}}{\sqrt{p} \mathbb{P}(0)} = \sum_{j=0}^p \varphi_j + g_n \quad (5.108)
\]

where we define \( g_n \) as the harmonic function on \( U_n \cap B^-_0 \) with \( g_n = -m_n + u \) on \( \partial U_n \), \( g_n = \mathbb{P}(0) \) on \( \partial B^-_0 \).

We consider the Gaussian process \( (\chi'_p, \phi_p)_{p=1}^q \) as \( \varphi_p \), \( p = 0, \ldots, T \) are independent, Lemmas 2.5 and 5.13 below imply the convergence of the covariances of the process \( (\chi'_p, \phi_p)_{p=1}^q \) as \( n-k \to \infty \). Moreover, \( g_n([e^n y]) - m_{n-p} \) converges as \( n \to \infty \) uniformly in \( y \in \tilde{\mathcal{A}}'_p \) by the assumptions on \( u, v \), Lemma 5.14 below, continuity of \( \Pi_U(\cdot, dz) \), and as \( m_n - m_{n-p} \to 2\sqrt{p} \) as \( n \to \infty \). As moreover \( \chi'_p \) and \( \varphi_j \) are centered, it follows that the finite-dimensional distributions of the process \( (\chi'_p, \phi_p)_{p=1}^q \) converge.
process \((\mathcal{X}_p, \phi_p)^q_{p=1}\) also converges in distribution when the state space is endowed with the supremum norm as the argument from the proof of Lemma 4.4 and Sect. 6.5 of [3] passes through in our setting.

From the definition of \(S'_p\) in (2.18), (2.19) and (2.21), we observe that \(S'_p\) can be expressed as a continuous function of \(s_{1,1}, \ldots, s_{1,q}, (\mathcal{X}'_1, \ldots, \mathcal{X}'_q)\) and of the independent Brownian bridges \((B'(1), \ldots, B'(q))\). As also \((s_{1,T} - t)/(s_{1,T} - \tau) \to 1\) uniformly in \(\tau, t \in [0, s_{1,q}]\) as \(n \to \infty\), it now follows from (2.18) and (2.21) that \((S'_p, \beta_p, \phi_p)^q_{p=1}\) converges in distribution to a limit process which we denote by \((S^{(\infty)}, \phi^{(\infty)})\) under a probability measure \(\mathbb{P}^{(\infty)}\).

Next, we show the joint convergence of \(S'_p\) and \(S'_p + \beta_p + D_p\). As in (2.28) and by the Gibbs-Markov property, we have

\[
S'_p + \beta_p + D_p = \max_{y \in A_p} \left\{ \overline{h_0 \mathcal{A}_p}(y) + h_p(y) \right\} = \max_{y \in A_p} h(y). \tag{5.109}
\]

We will control the maximum in (5.109) using estimates on the spatial fluctuation of the binding field \(\phi_p\) and on the clustering property of the extremal process \(\xi_p\) of the DGFF \(h_p\).

Defining the cluster radii in terms of a sequence \(c_n > 0\) with \(c_n = o(e^n), c_n \to \infty\) as \(n \to \infty\), the extremal process \(\xi_p\) is defined (in accordance with [6]) as the simple point measure on \(\tilde{A}_p' \times (\mathbb{R} \cup \{\infty\})\) with atoms in those \((x, t) \in \tilde{A}_p' \times \mathbb{R}\) for which

\[
x \in e^{-n}\mathbb{Z}^2, \quad h_p(e^n x) = \max_{y \in \mathbb{Z}^2: y - e^n x \leq c_n} h_p(y) = m_{n-p} + t. \tag{5.110}
\]

Here we endow \(\tilde{A}_p' \times (\mathbb{R} \cup \{\infty\})\) with the product topology, where the topology on \(\mathbb{R} \cup \{\infty\}\) is such that sets of the form \((a, \infty), a \in \mathbb{R}\) are compact. By Theorem 2.1 of [6] and Theorem 2.1 of [7], \(\xi_p\) converges vaguely in distribution as \(n \to \infty\) to a Cox process with intensity \(Z_p(dx) \otimes e^{-\alpha t} dt\) where \(\alpha = 2/\sqrt{\pi}\) and \(Z_p\) is a random measure with a.s. positive and finite mass and no atoms.

While existence a point \((x, t)\) of \(\xi_p\) directly implies that \(S_p + \beta'_p + D_p > t + \phi_p(x)\) by (5.107), (5.109) and (5.110), we can draw, for the converse direction, a correspondence between the vertex at which \(h\) assumes its maximum and a point of \(\xi_p\) only with high probability, that is, on the complement of some error events which we now define:

First, let

\[
E_{p,n} := \left\{ \sup_{x, y \in \tilde{A}_p', |y - x| \leq 2 e^{-n} c_n} |\phi_p(x) - \phi_p(y)| \geq \epsilon' \right\} \tag{5.111}
\]

be the event that the binding field fluctuates non-negligibly over twice the cluster size. As \(E_p\) is uniformly continuous and by the argument from Sect. 6.5 of [3], \(\phi^{(\infty)}\) is a.s. uniformly continuous on \(\tilde{A}_p'\), and it follows that \(\lim_{n \to \infty} \mathbb{P}^{n,u}_{0,\overline{\mathbb{P}}(0)}(E_{p,n}) = 0\).

Second, for \(\lambda \in \mathbb{R}\), let

\[
H_{\lambda,n} := \bigcup_{p=1}^q \left\{ \max_{A_p \setminus (\tilde{A}_p')} h > \lambda \right\} \tag{5.112}
\]

be the event that \(h\) exceeds \(\lambda\) outside the bulk of the \(A_p\). Writing \(\tilde{A}_p'^- := \{y \in U: d(y, \tilde{A}_p) < \epsilon'\}\), we have \((\tilde{A}_p'^-) \supset \tilde{A}_p\) for \(p \leq q\) and sufficiently large \(n\). Thus Lemma A.4 yields
\[ \mathbb{P}^{n,u}_{0,\pi(0)}(H_{\lambda,n}) \leq C_{\epsilon,\lambda} \sqrt{\text{Leb} \left( \bigcup_{p=1}^{q} (\tilde{A}_p \setminus \hat{A}_p) \right)} , \]  
(5.113)

which tends to zero as \( \epsilon' \to 0 \).

Third, let \( F_{p,n} \) be defined as the event that there exists \( y \in U_n' \cap B_0^{-} \) with \( c_n/2 < |y - \arg \max (\tilde{A}_p')_n| h \) and \( h(y) + \epsilon' \geq \max (\tilde{A}_p')_n h \). By Lemma A.3, we have

\[ \lim_{n \to \infty} \mathbb{P}^{n,u}_{0,\pi(0)}(F_{p,n}) = 0. \]

On \( E_{p,n}^c \cap F_{p,n}^c \), there exists in a \( c_n/2 \)-neighborhood of \( \tilde{x} := \arg \max (\tilde{A}_p')_n h \) a local maximum of \( h_p \) that corresponds to a point of \( \hat{\xi}_p \). Indeed, for all \( y \in (\tilde{A}_p')_n \) with \( c_n/2 < |y - \tilde{x}| < 2c_n \), we then have

\[ h_p(y) = h(y) - \tilde{h}_{\partial A_p'}(y) \leq h(\tilde{x}) - \epsilon' - \tilde{h}_{\partial A_p'}(\tilde{x}) ≤ h_p(\tilde{x}) \]  
(5.114)

and hence there exists \( \tilde{y} \in (\tilde{A}_p')_n \) with \( |\tilde{x} - \tilde{y}| \leq c_n/2 \) such that \( h_p(\tilde{y}) = \max_{z:|z-\tilde{y}| \leq c_n} h_p(z) \).

For \( \lambda_p \in \mathbb{R} \), we moreover have on \( E_{p,n}^c \cap F_{p,n}^c \cap H_{\lambda,n}^c \), that \( \{S_p' + \beta_p + D_p > \lambda_p\} = \{h(\tilde{x}) > \lambda_p\} \) by (5.109) and by definition of \( \tilde{x}, H_{\lambda,n} \). On the intersection of these four events and if also \( \phi_p = w_p \) for a continuous function \( w_p \) on \( \tilde{A}_p' \), we also have

\[ h_p(\tilde{y}) \geq h_p(\tilde{x}) = h(\tilde{x}) + m_n - p w_p(e^{-n}\tilde{x}) > \lambda_p + m_n - p w_p(e^{-n}\tilde{y}) - \epsilon' \]  
(5.115)

by (5.107), and this implies \( \xi_p(e^{-n}\tilde{y}, (\lambda_p - p w_p(e^{-n}\tilde{y}) - \epsilon'), \infty)) > 0 \).

Now we are ready to show the convergence in distribution asserted in the lemma. Let \( \kappa_p, \lambda_p \in \mathbb{R} \) for \( p = 1, \ldots, q \), and let \( \lambda := \min_{p=1}^{q} \lambda_p \). Then, by (5.107), (5.109), the definitions of \( \phi_p \) and \( \xi_p \) and by (5.115),

\[ \left\{ S_p' + \beta_p + D_p > \lambda_p \right\} \subset \left\{ \max_{y \in \tilde{A}_p'} (\phi_p(y) + h_p(|e^{-n}y|)) \right\} \cup H_{\lambda,n} \]

\[ \subset \left\{ \exists (x, t) : \xi_p(x, t) > 0, t + \phi_p(x) > \lambda_p \right\} \cup E_{p,n} \cup F_{p,n} \cup H_{\lambda,n} \]  
(5.116)

Using the independence of \( \xi_p \), we obtain:

\[ \mathbb{P}^{n,u}_{0,\pi(0)}(S_1' + \beta_1 > \kappa_1, \ldots, S_q' + \beta_q > \kappa_q, S_1' + \beta_1 + D_1 > \lambda_1, \ldots, S_q' + \beta_q + D_q > \lambda_q) \]

\[ \leq \int \mathbb{P}^{n,u}_{0,\pi(0)}(S_1' + \beta_1 \in dt_1, \ldots, S_q' + \beta_q \in dt_q, \phi_p \in dw_p, p = 1, \ldots, q) \prod_{p=1}^{q} 1_{\{t_p > \kappa_p\}} \]

\[ \times \mathbb{P}^{n,u}_{0,\pi(0)}(\xi_p(x, t) \in \tilde{A}_p' \times \mathbb{R} : t + w_p(x) > \lambda_p > 0) \]

\[ + \sum_{p=1}^{q} \mathbb{P}^{n,u}_{0,\pi(0)}(E_{p,n}) + \sum_{p=1}^{q} \mathbb{P}^{n,u}_{0,\pi(0)}(F_{p,n}) + \mathbb{P}^{n,u}_{0,\pi(0)}(H_{\lambda,n}) . \]  
(5.117)

The integral on the right-hand side of (5.117) is bounded from above by

\[ \int \mathbb{P}^{(\infty)}(S_1' + \beta_1 \in dt_1, \ldots, S_q' + \beta_q \in dt_q, \phi_p(\infty) \in dw_p, p = 1, \ldots, q) \prod_{p=1}^{q} 1_{\{t_p + \epsilon > \kappa_p\}} \]

\[ \times \mathbb{P}^{n,u}_{0,\pi(0)}(\xi_p(x, t) \in \tilde{A}_p' \times \mathbb{R} : t + w_p(x) + 2\epsilon > \lambda_p > 0) \]

\[ + \sum_{p=1}^{q} \mathbb{P}^{(\infty)}(\left| S_p' + \beta_p - S_p(\infty) \right| \vee \sup_{x \in \tilde{A}_p'} |\phi_p(x) - \phi_p(\infty)(x)| > \epsilon') , \]  
(5.118)
where we assume w.l.o.g. a coupling under \( P^{(\infty)} \) in which \( (S_p^t + \beta_p, \phi_p)^q_{p=1} \) converges to \( (S_p^{(\infty)}, \phi_p^{(\infty)})^q_{p=1} \) in probability. The integral in (5.118) converges as \( n \to \infty \) followed by \( \epsilon' \to 0 \) by dominated convergence. By the above, the other summands on the right-hand sides of (5.117) and (5.118) vanish as \( n \to \infty \) followed by \( \epsilon' \to 0 \). The limit of (5.118) is \((0, 1)\)-valued and continuous in the \( \lambda_p, \kappa_p \) as \( (S_p^{(\infty)}, \phi_p^{(\infty)}) \) is Gaussian, and as the second component of the intensity measure of the limit of \( \xi_p \) is absolutely continuous with respect to Lebesgue measure. A lower bound analogous to (5.117) follows by the same argument (in which we now do not require Lemmas A.3 and A.4), this yields the assertion. 

It remains to show the following two lemmas. For \( W \in \mathcal{O} \), we denote the continuum Green kernel on \( W \) by

\[
G_W(x, y) = g \int \Pi_W(x, dz) \log |y - z| - g \log |x - y| ,
\]

where \( \Pi_W \) denotes the continuum Poisson kernel, i.e. \( \Pi_W(x, \cdot) \) is the exit distribution from the domain \( W \) of Brownian motion started at \( x \).

**Lemma 5.13** Let \( \epsilon \in (0, 1), \eta \in [0, \epsilon^{-1}], q \in \mathbb{N} \). For \( U \in \mathcal{U}^g_\epsilon, p = 1, \ldots, q, \tilde{A}_p \) defined as in the proof of Lemma 5.5, \( V = B, k = \xi = 0, \) and \( x, y \in \tilde{A}_q \), we have, as \( n \to \infty \), that

\[
\text{Cov}_0^n(\varphi_p([e^a x]), \varphi_p([e^a y])) = G_{\tilde{\Delta}_{p-1}}(x, y) - G_{\tilde{\Delta}_{p}}(x, y) + o_{\epsilon, q}(1)
\]

and

\[
\text{Cov}_0^n(\varphi_p([e^a x]), \lambda'_p) = g \int_{z, z' \in \partial \tilde{\Delta}_p} \Pi_{\tilde{\Delta}_{p}}(x, dz) \Pi_{\tilde{\Delta}_p}(0, dz') G_{\tilde{\Delta}_{p-1}}(z, z') + o_{\epsilon, q}(1),
\]

where \( \tilde{\Delta}_0 = U^n, \tilde{\Delta}_p = e^{-p+[\log \epsilon]} B \).

**Proof** By the Gibbs-Markov property and the definition of \( \varphi_p \), the left-hand side in (5.120) equals

\[
G_{\Delta_{p-1}}([e^a x], [e^a y]) - (G_{\Delta_{p-1}} - G_{\Delta_{p-1} \cap B_0})([e^a x], [e^a y]) - G_{A_{p} \cup \Delta_{p}}([e^a x], [e^a y]) + (G_{A_{p} \cup \Delta_{p}} - G_{A_{p} \cup \Delta_{p} \cap B_0})([e^a x], [e^a y]) .
\]

The differences in brackets are by the Gibbs-Markov property equal to the left-hand side in (B.24) (for the first one with \( \Delta_{p-1} \) in place of \( U_n \), and the second one is nonzero only for \( p < q \) when we take \( \Delta_{p} \) in place of \( U_n \)), hence it is bounded by \( q^2 (n - q)^{-1} \) by Lemma B.13. The remaining expression is equal to

\[
g \int_{\partial \tilde{\Delta}_{p-1}} \Pi_{\tilde{\Delta}_{p-1}}(x, dz) \log |z - y| - g \int_{\partial \tilde{\Delta}_p} \Pi_{\tilde{\Delta}_{p}}(x, dz) \log |z - y| + o_{\epsilon, q}(1)
\]

by (2.6), (2.7) and Lemma A.9.

To show (5.121), we use the representation (2.30) and Definition (2.17) in

\[
\text{Cov}_0^n(\varphi_p([e^a x]), \lambda'_p) = \sum_{z \in \partial A_p, z' \in \partial \Delta_p} \Pi_{A_{p} \cup \Delta_{p} \cap B_0}([e^a x], z) \Pi_{\Delta_{p}}(0, z') G_{\Delta_{p-1} \cap B_0}(z, z') .
\]
By Lemma B.13 and the Gibbs-Markov property, we have \( G_{\Delta_{p-1} \cap B_0^-}(z, z') = G_{\Delta_{p-1}}(z, z') + O_{\epsilon, q}(1/(n - k)) \) which we plug into (5.124). For \( p < q \) we also use that
\[
\Pi_{\Delta_{p} \cap (\partial B)_n}(\cdot, z) \leq \Pi_{\Delta_{p} \cap B_0^-}^-(\cdot, z) \leq \Pi_{\Delta_{p}}(\cdot, z)
\] (5.125)
for \( z \in \Delta_{p} \). Then we use Lemma A.9 and finally let \( \delta \to 0 \). This yields the assertion. \( \Box \)

**Lemma 5.14** Let \( \epsilon \in (0, 1) \), \( \eta \in [0, \epsilon^{-1}] \). For all \( U \in \mathcal{U}_n \), \( n > 0 \), \( \hat{u}_\infty \in \mathbb{H}_{\infty}(U^n) \), \( u \in \mathbb{R}_{\partial U_n} \), let \( g_n = -m_n + u \) on \( \partial U_n \), \( g_n = \overline{u}(0) \) on \( \partial B_0^- \), and let \( g_n \) be harmonic on \( U_n \cap B_0^\circ \). Then, as \( n \to \infty \),
\[
\max_{x \in (U^n \cap \epsilon B)_n} \left| g_n(x) + m_n - \hat{u}_\infty(e^{-n}x) - 2\sqrt{g} \log |e^{-n}x| \right|
\]
\[
+ 2\sqrt{g} \int_{\partial U} \Pi_U(e^{-n}x, \, dz) \log |z| \leq \max_{x \in (U^n \cap \epsilon B)_n} \left| \overline{u}(x) - \hat{u}_\infty(e^{-n}x) \right| + o_{\epsilon}(1). \tag{5.126}
\]

**Proof** We write
\[
g_n(x) = (-m_n \mathbb{1}_{\partial U_n} \mathbb{1}_{U_n \cup \partial B_0^-}(x) + \overline{u}_{\partial U_n \cup \partial B_0^-}(x) + \overline{u}(0) \mathbb{1}_{U_n \cup \partial B_0^-}(x). \tag{5.127}
\]

For \( x \in (U^n \cap \epsilon B)_n \), the first term on the right-hand side of (5.127) equals
\[
-m_n + m_n P_A(\tau_V - \tau_{U_n})
\]
\[
= -m_n + 2\sqrt{g} \left( \int_{\partial U} \Pi_U(e^{-n}x, \, dz) \log |z| - \log |e^{-n}x| \right) + o_{\epsilon}(1) \tag{5.128}
\]
where the equality holds by Lemma A.5(ii). The third term on the right-hand side of (5.127) equals \( P_A(\tau_{B_0^-} - \tau_{U_n})\overline{u}(0) \), which is bounded by \( C_{\epsilon} n^{-\epsilon} \) by Lemma A.5(i) and the assumptions. For the second term on the right-hand side of (5.127), we estimate
\[
\left| \overline{u}_{\partial U_n \cup \partial B_0^-}(x) - \hat{u}_\infty(e^{-n}x) \right| \leq \left| \overline{u}_{\partial U_n \cup \partial B_0^-}(x) - \overline{u}(x) \right| + \left| \overline{u}(x) - \hat{u}_\infty(e^{-n}x) \right|, \tag{5.129}
\]
where the first term on the right-hand side is bounded by \( |\overline{u}(0)| P_A(\tau_{B_0^-} \leq \tau_{U_n}) + C_{\epsilon} e^{-n} \log \overline{u}_\eta \) by Lemmas B.14 and B.15, which in turn is bounded by \( C_{\epsilon} n^{-\epsilon} \) by (1.9) and Lemma A.5(i). This shows the assertion. \( \Box \)

### 5.4 The Functional \( \mathcal{R}_k \)

To prove the results on the \( \mathcal{R}_k \) and \( \mathcal{R} \) functionals from Sect. 1.2 and the weak ballot asymptotics (Proposition 4.2), we proceed to a large part along the lines of the proofs for the \( \mathcal{L}_n \) and \( \mathcal{L} \) functionals from Sects. 5.1–5.3. The only change is that the functional \( \mathcal{R}_k \) in Proposition 1.2 is defined already in terms of a limit as \( n \to \infty \) which is handled in Lemma 5.15 below.

Analogously to (5.2), we define the functional
\[
\mathcal{R}_{k,n}^r(v, u) := \mathcal{R}_{k,\xi, V, n, U}^r(v, u) := \mathbb{E}_{V, k, v}^{U, n, u} \left( \overline{h}_{\partial B_{k+r}^-}(\infty) + m_{k+r} \right); h_{V, k, v} \leq 0 \tag{5.130}
\]
in which the restriction to negative values and the harmonic average are next to the inner boundary at scale \( k \). To obtain a functional that is independent of \( n \) and to allow for arbitrarily large values of \( k \), we take the limit as \( n \to \infty \) in (5.130) in the definition

\[
\mathcal{R}_k(v) := \mathcal{R}_{k,\xi}(v) := \lim_{n \to \infty} \mathcal{R}_{k,\xi,v,n,B}^n(v, 0)
\]

(5.131)

which is in accordance with Proposition 1.2, and where existence of the limit is shown in Lemma 5.15 below. We use the setting from Sect. 2.6 and also define the functional of the decorated random walk

\[
\ell_{T_{n-k},r}(\overline{v}(\infty), \overline{u}(0)) = \mathbb{E}_{V,k,v}^U \left( (S_r^o)^i : \max_{i=1}^r (S_r^i + D_r^o) \leq 0 \right)
\]

(5.132)

in accordance with (C.3) and using the correspondence given by Theorem 2.10. Besides existence of the limiting functional \( \mathcal{R}_k \) in (5.131), the following lemma also shows that the limit is uniform and coincides with the limit of the decorated random walk functional.

**Lemma 5.15** Let \( \epsilon \in (0, 1), \ \xi, \eta \in [0, \epsilon^{-1}], \ V \in \mathcal{D}_\epsilon, \ n, k \geq 0, \ v \in \mathbb{R}^{\partial V_k^-} \) with \( \text{osc}_v \overline{v} \leq \epsilon^{-1} \) and \(- (n - k)^{-1} - \leq \overline{v}(\infty) \leq \epsilon^{-1} \). Then the limit \( \mathcal{R}_k(v) \) in (5.131) exists and satisfies

\[
\mathcal{R}_k(v) = (1 + o_\epsilon(1))\mathcal{R}_{k,\xi,v,n,B}(v, 0) + o_\epsilon(1)(1 + \overline{v}(\infty))^{-1}
\]

(5.133)
as \( n \to k \to \infty \), and

\[
\mathcal{R}_k(v) = \lim_{n \to k} \ell_{T_{n-k},r}(\overline{v}(\infty), \overline{u}(0)) \]

(5.134)

where \( \ell_{T_{n-k},r}(\overline{v}(\infty), \overline{u}(0)) \) is defined as in (5.132) with \( U = B, \ u = \overline{v}(\infty), \ \zeta = 0 \).

In the proofs in this subsection, we use analogs of the auxiliary lemmas from Sect. 5.1 which are proved along the lines of Sect. 5.3, using the outward concentric decomposition:

**Lemma 5.16** Let \( \epsilon \in (0, 1) \). Then,

\[
\lim_{r \to \infty} \lim_{n \to k \to \infty} \left( 1 + \overline{v}(\infty)^{-1} - \epsilon \right) \ell_{T_{n-k},r-\log(e^{-1}+\zeta)}(\overline{v}(\infty), \overline{u}(0)) = 0
\]

(5.135)

uniformly in \( \eta, \zeta \in [0, \epsilon^{-1}], \ U \in \mathcal{D}_\eta, \ V \in \mathcal{D}_\epsilon, \ u \in \mathbb{R}^{\partial U_n}, \ v \in \mathbb{R}^{\partial V_k^-} \) satisfying (5.4).

**Proof** This is proved analogously to Lemma 5.1. \( \square \)

Analogously to (5.5), we generalize the functional \( \mathcal{R}_{k,\xi,v,n,u} \) to a general test set \( W \) in place of \( V^{-\zeta} \) in which the DGFF is restricted to stay negative at scale \( k \):

\[
\mathcal{R}_{k,W,v,n,u}^r := \mathbb{E}_{V,k,v}^U \left( h_{\partial B_{k+r}^-}(\infty) + m_{k+r} \right)^{-1} \mathbb{E}_{W_{k+r}^-}(\overline{u}(0)) \leq 0 \).
\]

(5.136)

Then we have in particular that \( \mathcal{R}_{k,W,v,n,u}^r = \mathcal{R}_{k,\xi,v,n,u}^r \) for \( W = V^{-\zeta} \). We obtain the following analog of Lemma 5.2 which also allows to vary the parameter \( \zeta \) in \( \mathcal{R}_{k,\xi,v,n,u} \).

**Lemma 5.17** Let \( \epsilon \in (0, 1) \) and \( \eta, \zeta \in [0, \epsilon^{-1}] \). There exists \( C = C_\epsilon < \infty \) such that

\[
\left| \frac{\mathcal{R}_{k,w,v,n,u}^r(v, u)}{\mathcal{R}_{k,W,v,n,u}^r(v, u)} - 1 \right| \leq C \sqrt{\text{Leb}(W \Delta W')} \]

(5.137)

for all \( U \in \mathcal{D}_\epsilon, \ V \in \mathcal{D}_\eta, \ Borel measurable \ W, W' \) with \( e^{-2}B^- \subset W, W' \subset V^{-\zeta}, \ n, k, r \geq 0 \) with \( \partial B_{k+r} \subset U_n \cap (e^{-2}B^-)_k \), and \( u \in \mathbb{R}^{\partial U_n}, \ v \in \mathbb{R}^{\partial V_k^-} \) that satisfy

\[
\max\{\overline{u}(0), \text{osc}\overline{u}, \overline{v}(\infty), \text{osc}\overline{v} \} < \epsilon^{-1}
\]
This is proved analogously to Lemma 5.2.

The next lemma allows to compare the $R_{k,n}^r$ functionals when the outer domains and the outer boundary values differ:

**Lemma 5.18** Let $r \geq 0$ and $\epsilon \in (0, \frac{1}{10})$. Then there exists $C = C_{r, \epsilon} < \infty$ such that for all $\zeta, \eta \in [0, \epsilon^{-1}]$, $U \in \Omega^n_\epsilon$, $V \in \Omega_\epsilon$, $0 < k < n$ with $\partial B_{k+r+1} \subset U^{\eta \vee \epsilon}$ and $\partial B_{k+r} \subset V^{-\zeta \vee \epsilon}$, and all $u \in R^{\partial U_n}$, $v \in R^{\partial V_k}$ satisfying (5.4), we have

$$\left|R_{k,\zeta,V,n,U}(v, \pi(\infty)) - R_{k,\zeta,V,n,U}(v, u)\right| \leq C(n-k)^{-\epsilon/4}. \quad (5.138)$$

**Proof** This is proved analogously to Lemma 5.3, we use Lemma 5.21 below in place of Lemma 5.10.

The following lemma shows convergence of the first random walk steps and decorations as $n-k \to \infty$. Here $k$ may stay fixed, this case will be used to prove existence of the $R_{k,n}^r$ functional from (5.130). The case that $k$ tends to infinity with $n-k$ will be applied in the proof of Proposition 1.3.

**Lemma 5.19** Let $\epsilon \in (0, 1)$, $\zeta \in [0, \epsilon^{-1}]$ and $q \in N$. For $n, k \geq 0$, assume that $v \in R^{\partial V_k}$ satisfies

$$\lim_{k \to \infty} \max_{x \in V_k^{-\zeta}} \left|\pi(x) - \hat{\pi}(\epsilon^{-k} x)\right| = 0 \quad (5.139)$$

for some $\hat{\pi}_\infty \in H_\infty(V^{-\zeta})$, and let $U = B$, $u = \hat{\pi}(\infty) \in R^{\partial U_n}$, $\eta = 0$. Then $(S^p_{\epsilon, p}, S^o_{p} + D^o_{p,q} p=1)$ converges in distribution as $n-k \to \infty$ to some limit $(S^p_{\epsilon, \hat{\pi}(\infty)}, S^o_{\hat{\pi}(\infty)} + D^o_{\hat{\pi}(\infty),q} p=1)$ whose distribution is absolutely continuous with respect to Lebesgue measure and depends only on $V$, $\hat{\pi}_\infty$ and $\zeta$.

**Proof** We first consider the case that $n-k \to \infty$ while $k \geq 0$ stays fixed. In this case Assumption (5.139) is not needed and the assertion follows as $P_{V,k,v}(h_{\Delta_q} \in \cdot)$ converges weakly to $P(h_{\Delta_q} \in \cdot | h_{\partial V_k} = v)$, and as $(S^o_{\epsilon,j}, D^o_{\epsilon,j} j=1)$ is a continuous functional of $h_{\Delta_q}$ by the definitions in Sect. 2.6. The case when also $k \to \infty$ is treated analogously to Lemma 5.5.

Also, the following lower bound is proved analogously to Lemma 5.6:

**Lemma 5.20** Let $\epsilon \in (0, 1)$ and $\eta, \zeta \in [0, \epsilon^{-1}]$. Then there exists $c = c_\epsilon > 0$ such that

$$P_{k,v}(h_{U^{\eta \vee \epsilon}_k} \cap V_k^{-\zeta} \leq 0) \geq c \frac{(1 + \pi(\infty) - \bar{\pi}(0))^{-}}{n-k} \quad (5.140)$$

for all $U \in \Omega^n_\epsilon$, $V \in \Omega_\epsilon$, $0 \leq k < n$ with $T_{n-k} \geq 1$, $u \in R^{\partial U_n}$, $v \in R^{\partial V_k}$ satisfying (1.9) and $\bar{\pi}(0)^- \geq (n-k)^{\epsilon}$.

We will also refer to the following lemma in which the $R_{k,n}^r$ functionals are compared when the DGFF is lifted by a constant value $t$:

**Lemma 5.21** Let $\epsilon \in (0, 1)$, $r \geq 0$, $\zeta, \eta \in [0, \epsilon^{-1}]$. There exists $C = C_{r, \epsilon} < \infty$ such that for all $V \in \Omega^n_\epsilon$, $U \in \Omega^n_\epsilon$, $0 \leq k < n$ with $\partial B_{k+r} \subset V_k^{-\zeta \vee \epsilon}$ and $\partial B_{k+r} \subset U_k^{\eta \vee \epsilon}$, all $u \in R^{\partial U_n}$, $v \in R^{\partial V_k}$ satisfying (2.27), and all $t \in R$, we have

$$|R_{k,n}^r(v, u, t) - R_{k,n}^r(v, u)| \leq C|t|^{1/2} e^{\alpha |t|}. \quad (5.141)$$
Proof This is proved analogously to Lemma 5.10.

We are now ready to give the proofs of Lemma 5.15, Proposition 4.2, and the remaining parts for Propositions 1.3 – 1.6.

Proof of Lemma 5.15 We obtain existence of the limit on the right-hand side of (5.134) from Proposition C.7 which we apply as follows: for any sequence \((n^{(i)})\) with \(n^{(i)} \to \infty\), we set
\[
a^{(i)} = \overline{v}(\infty), b^{(i)} = \overline{v}(\infty), T^{(i)} = T_{n^{(i)} - k}, r^{(i)} = r_{n^{(i)} - k} - [\log(e^{-1} + \gamma)], T^{(\infty)} = r_{\infty} = \infty, a^{(\infty)} = b^{(\infty)} = \overline{v}(\infty),\]
and in place of \((S_{j}^{(\infty)}), (D_{j}^{(\infty)})\), we use \((S_{j}^{0}), (D_{j}^{0})\) as defined in Sect. 2.6 for \(u = \overline{v}(\infty), U = B\) and \(\eta = 0\). Lemma 5.19 yields that \((S_{j}^{(\infty)}, D_{j}^{(\infty)})\) has a limit in distribution \((S_{j}^{(\infty)}, D_{j}^{(\infty)})\), that Assumption (C.10) and the assumption on stochastic absolute continuity are satisfied, and that \(R_{k}\) depends only on \(V, v, \gamma\) and \(\gamma\). Assumptions (A1) – (A3) are verified by Theorem 2.10.

Next we show the equality in (5.134). By Lemma 5.16 and the choice of \((r_{n})\), we have
\[
\lim_{n \to \infty} \left( R_{n-k}^{(r_{n-k})}(v, \overline{v}(\infty)) - \ell_{T_{n-k} - [\log(e^{-1} + \gamma)]}(\overline{v}(\infty), \overline{v}(\infty)) \right) = 0. \quad (5.142)
\]
Moreover, Lemma 5.18 shows that
\[
\lim_{n \to \infty} \left( R_{n-k}^{(r_{n-k})}(v, \overline{v}(\infty)) - R_{n-k}^{(r_{n-k})}(v, 0) \right) = 0, \quad (5.143)
\]
Combining (5.142) and (5.143) then yields that the limits in (5.131) and (5.134) coincide.

To show assertion (5.133), we first show the Cauchy property for \(R_{n-k}^{(r_{n-k})}(v, 0)\) as \(n - k \to \infty\) for fixed \(k \geq 0\). We note that \(h\) under \(\mathbb{P}_{V, k, u}^{B,n,0}\) is distributed as \(h_{B^{n} \cap V_{k}^{-}} + g_{n}\), where \(g_{n}\) is the harmonic function on \(B_{n} \cap V_{k}^{-}\) – \((5.142)\)

For \(n' \geq n\), by the Gibbs-Markov property, \(h - g_{n'}\) under \(\mathbb{P}_{V, k, u}^{B,n',0}\) is distributed as the sum of independent fields \(h - g_{n} + \varphi_{B_{n} \cap V_{k}^{-}, B_{n} \cap V_{k}^{-}}\) under \(\mathbb{P}_{V, k, u}^{B,n,0}\). We set
\[
\varphi^{*} = \max_{V_{k}^{-} \setminus B_{k+r}} |\varphi_{B_{n} \cap V_{k}^{-}, B_{n} \cap V_{k}^{-}}|, \quad g^{*} = \max_{V_{k}^{-} \setminus B_{k+r}} |g_{n} - g_{n'}| \quad (5.144)
\]
Then, with
\[
R_{n-k}^{(r_{n-k})}(v, 0, t) := \mathbb{E}_{V, k, u}^{B,n,0}(h_{B_{k+r}}^{-}\overline{V}_{k}^{-}(\infty) + m_{k+r} + t) - h_{V_{k}^{-}\overline{V}_{k}^{-}}(\infty) - t \leq 0 \quad (5.145)
\]
for \(t \in \mathbb{R}\), we obtain analogously to (5.87) and (5.88) that
\[
R_{n-k}^{(r_{n-k})}(v, 0, 0) \geq \int \mathbb{P}(\varphi^{*} + g^{*} \in dt) R_{n-k}^{(r_{n-k})}(v, 0, t) \quad (5.146)
\]
and
\[
R_{n-k}^{(r_{n-k})}(v, 0, 0) \leq \int \mathbb{P}(\varphi^{*} + g^{*} \in dt) R_{n-k}^{(r_{n-k})}(v, 0, -t) \quad (5.147)
\]
By Lemma 5.21 and the choice of \((r_{n})\), we have
\[
|R_{n-k}^{(r_{n-k})}(v, 0, t) - R_{n-k}^{(r_{n-k})}(v, 0)| \leq C_{\epsilon} |t|^{1/2} e^{\alpha|t|} \quad (5.148)
\]
for \(t \in \mathbb{R}\), uniformly in \(r \leq r_{n-k}\). Lemma B.12 and the choice of \((r_{n})\) give
\[
\mathbb{E}\varphi^{*} \leq C_{\epsilon}(n - k)^{-1/4}, \quad \mathbb{P}(\varphi^{*} > s) \leq C_{\epsilon} e^{-\gamma_{s}(n-k)s^{2}} \quad (5.149)
\]
for \( s \geq 0 \), uniformly in \( r \leq r_{n-k} \). Furthermore, it can be shown analogously to Lemma 5.11 that
\[
g^* \leq C_{\varepsilon} r(n-k)^{-\varepsilon}. \tag{5.150}
\]
Plugging these estimates into (5.146) and (5.147), we obtain
\[
|\mathcal{R}_{k,n'}^r(v,0) - \mathcal{R}_{k,n}^r(v,0)| \leq C_{\varepsilon} \left( \mathbb{E}\varphi^* + g^* \right)^{1/2} + \sum_{s=1}^{\infty} \mathbb{P}(\varphi^* > s - g^*) C_{\varepsilon} e^{\alpha s} = o_{\varepsilon}(1) \tag{5.151}
\]
as \( n-k \to \infty \), uniformly in \( r \leq r_{n-k} \). In particular,
\[
\mathcal{R}_{k,n'}^r(v,0) - \mathcal{R}_{k,n}^r(v,0) = o_{\varepsilon}(1) \tag{5.152}
\]
as \( n-k \to \infty \), where we recall that \( n' \geq n \). By Lemmas 5.16, 5.18 and the choice of \((r_n)\),
\[
\mathcal{R}_{k,n'}^r(v,0) - \ell_{r_{n-k}'-r_{n-k}} - [\log(e^{-1} + \zeta)](\overline{v}(\infty), \overline{v}(\infty)) = o_{\varepsilon}(1)(1 + \overline{v}(\infty)^{-}) \tag{5.153}
\]
as \( n-k \to \infty \). By Proposition C.4,
\[
\ell_{r_{n-k}'-r_{n-k}} - [\log(e^{-1} + \zeta)](\overline{v}(\infty), \overline{v}(\infty)) - \ell_{r_{n-k}'-r_{n-k}} - [\log(e^{-1} + \zeta)](\overline{v}(\infty), \overline{v}(\infty)) = o_{\varepsilon}(1)(1 + \overline{v}(\infty)^{-}) \tag{5.154}
\]
as \( n-k \to \infty \). Using again (5.153) shows that
\[
\mathcal{R}_{k,n'}^r(v,0) - \mathcal{R}_{k,n}^r(v,0) = o_{\varepsilon}(1)(1 + \overline{v}(\infty)^{-}) \tag{5.155}
\]
as \( n-k \to \infty \), as desired. This shows
\[
\mathcal{R}_k(v) = \mathcal{R}_{k,\zeta,V,n,B}^r(v,0) + o_{\varepsilon}(1)(1 + \overline{v}(\infty)^{-}) \tag{5.156}
\]
and hence (5.133). \( \square \)

**Proof of Proposition 4.2** By Lemmas 5.15 and 5.20, it suffices to prove that
\[
\mathbb{P}_{k,i}^{n,u}(h_{U_{n}^u \cap V_{k}^{-}}(0) \leq 0) = (2 + o_{\varepsilon}(1)) \frac{\mathcal{R}_{k,\zeta,V,n,B}^r(v,0)\overline{u}(0)^{-}}{g(n-k)} \tag{5.157}
\]
The asymptotics (5.157) follows analogously to the proof of Proposition 4.1, where we also use Lemma 5.20. \( \square \)

**Proof of Proposition 1.3 (for \( \mathcal{R}_k \))**

With \( \ell_{r_{n-k}'-r_{n-k}}(\overline{v}(\infty), \overline{u}(\infty)) \) defined by (5.132) for \( v = v_k, U = B, u = \overline{v}(\infty), \) we have
\[
\lim_{k \to \infty} \mathcal{R}_k(v_k) = \lim_{k \to \infty} \lim_{n-k \to \infty} \ell_{r_{n-k}'-r_{n-k}}(\overline{v}(\infty), \overline{u}(\infty)) = \lim_{n \to \infty} \ell_{r_n,\zeta}(\overline{v}(\infty), \overline{u}(\infty)) \tag{5.158}
\]
by (5.134) if the limit on the right-hand side exists for all \((k_n)_{n \geq 0}\) that tend to infinity sufficiently slowly as \( n \to \infty \). To show existence of the limit on the right-hand side, we apply Proposition C.7 as follows: for any sequence \((n^{(i)}, k^{(i)})_{i=1}^{\infty}\) with \( n^{(i)} - k^{(i)} \to \infty \), we set \( a^{(i)} = \overline{u}(\infty), b^{(i)} = \overline{v}(\infty), T^{(i)} = T_{n^{(i)}-k^{(i)}}, r^{(i)} = r_{n^{(i)}-k^{(i)}}, \) and in place of \((S^{(i)}), (D^{(i)}))\), we use \((S_0), (D_0)\) as defined in Sect. 2.6 for \( k = k^{(i)}, v = v^{(i)}_k, u = \overline{v}(\infty), U = B \) and \( \eta = 0 \). Lemma 5.19
yields that \((S_j^{(i)}, D_j^{(i)})\) has a limit in distribution \((S_j^{(\infty)}, D_j^{(\infty)})\), that Assumption (C.10) and the assumption on stochastic absolute continuity are satisfied, and that \(R_k\) depends only on \(V, k, v\) and \(\zeta\). Assumptions (A1) – (A3) are verified by Theorem 2.10.

Proof of Proposition 1.4 (for \(R_k, R\))
The assertion on \(R_k\) follows from (5.134) and Proposition C.5 with \(T = \infty\). The assertion on \(R\) follows by Proposition 1.3 as in (5.27).

Proof of Proposition 1.5 (for \(R_k, R\))
The assertion on \(R_k\) follows analogously to \(L\) from Lemma 5.20 and Proposition 4.2. The assertion on \(R\) follows from Proposition 1.3 as before for \(L\).

Proof of Proposition 1.6 (for \(R_k, R\))
For any \(\epsilon' > 0\), we find, by Lemma 5.15 and Proposition 1.5, \(n \geq k\) such that

\[(1 - \epsilon') R_{V, n, B}(v, 0) \leq R_k(v) \leq (1 + \epsilon') R_{V, n, B}(v, 0) \tag{5.159}\]

which we then use for all \(V, k, v, \zeta\) as in the assertion in place of (5.30) and proceed as in the proof for \(L_k\), using the analog of Lemma 5.4 to show the assertion on \(R_k\). The assertion on \(R\) follows analogously to \(L\) using Proposition 1.3.

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Availability of data
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendices
In the Appendices A and B, we use the notions introduced in Sects. 1.1 and 2.1.

A General Facts
In this appendix we collect several results from the literature that we use frequently or in a modified way.

A.1 Extreme Value Theory for the DGFF
We need an upper bound for the tail of the DGFF maximum.

Lemma A.1 Let \(\epsilon \in (0, 1)\). There exist constants \(C = C_\epsilon < \infty\) and \(c = c_\epsilon > 0\) such that for all \(U \in \mathcal{U}_k^0\), \(V \in \mathcal{V}_\epsilon\), \(k, n \geq 0\) and \(t > 0\),

\[\mathbb{P}\left( \max_{U_{\epsilon}} h_{U_n} > m_n + t \right) \leq Ce^{-ct} \tag{A.1}\]

and

\[\mathbb{P}\left( \max_{(2e^{-1}B)k} h_{V_k} > m_n + t \right) \leq Ce^{-ct} \tag{A.2} \]

Proof Assertion (A.1) follows e.g. from Lemma 8.3 of [3] and the Gibbs-Markov decomposition (Lemma B.10). For assertion (A.2), we decompose \(h_{V_k}^-\) as \(h_{V_k}^- \cap (4e^{-1}B)k + \)
\( \varphi_{k}^{-} \cap (4e^{-1}B_{k}) \) by Gibbs-Markov and estimate
\[
\mathbb{P}\left( \max_{(2e^{-1}B_{k})} h_{V_{k}}^{V_{k}} > m_{n} + t \right) 
\leq \mathbb{P}\left( \max_{(2e^{-1}B_{k})} h_{V_{k}^{-} \cap (4e^{-1}B_{k})} > m_{n} + t/2 \right) + \mathbb{P}\left( \max_{(2e^{-1}B_{k})} \varphi_{V_{k}^{-} \cap (4e^{-1}B_{k})} > t/2 \right).
\] (A.3)

The second summand on the right-hand side of the last display is bounded from above by \( C_{e} e^{-c_{e}t^{2}} \) by Lemma B.12 (where we let \( n \to \infty \)). For the first summand, we use Gibbs-Markov in
\[
\frac{1}{2} \mathbb{P}\left( \max_{(2e^{-1}B_{k})} h_{V_{k}^{-} \cap (4e^{-1}B_{k})} > m_{n} + t/2 \right) \leq \mathbb{P}\left( \max_{(2e^{-1}B_{k})} h_{(4e^{-1}B_{k})} > m_{n} + t/2 \right),
\] (A.4)

where the factor \( \frac{1}{2} \) stands for the probability that the binding field \( \varphi_{(4e^{-1}B_{k}), (4e^{-1}B_{k}) \cap V_{k}^{-}} \) is positive at \( \arg \max_{(2e^{-1}B_{k})} h_{V_{k}^{-} \cap (4e^{-1}B_{k})} \). Then we bound the right-hand side of (A.4) by (A.1).

We also need an upper bound for the left tail. To this aim, we build on the proof of Theorem 1.1 of Ding [14] where (in particular) double exponential bounds for the left tail of the global maximum of the DGFF are shown for moderately small values. In the following lemma, we consider the maximum in a subdomain, and we state the bound for the full left tail by interpolating with the Borell-TIS bound.

**Lemma A.2** Let \( \varepsilon \in (0, 1) \). There exists \( C = C_{\varepsilon} < \infty \) such that for all domains \( U, V \in \mathcal{D}_{\varepsilon}, x \in U^{\varepsilon} \cap \varepsilon^{2}B, 0 \leq k \leq n \) with \( B(x, \varepsilon/2)_{n} \subset V_{k}^{\varepsilon} \), and for all \( t > 0 \), we have
\[ \mathbb{P}\left( \max_{B(x, \varepsilon/2)_{n}} h_{U_{n}}^{U_{n} - m_{n} \leq -t} \right) \leq C_{\varepsilon} e^{-2^{2-\varepsilon}}, \] (A.5)
\[ \mathbb{P}\left( \max_{B(x, \varepsilon/2)_{n}} h_{U_{n} \cap V_{k}^{\varepsilon} - m_{n} \leq -t} \right) \leq C_{\varepsilon} e^{-2^{2-\varepsilon}}, \] (A.6)

and for all \( y \in V^{\varepsilon} \cap \varepsilon^{2}B \) with \( B(y, \varepsilon/2)_{n} \subset U_{n}^{\varepsilon} \), we have
\[ \mathbb{P}\left( \max_{B(y, \varepsilon/2)_{n}} h_{U_{n} \cap V_{k}^{\varepsilon} - m_{k} \leq -t} \right) \leq C_{\varepsilon} e^{-2^{2-\varepsilon}}. \] (A.7)

**Proof** We first consider a square domain \( E \subset \mathbb{R}^{2} \) of side length \( \varepsilon/2 \). Let \( E' \subset E \) be the square domain of side length \( \varepsilon/4 \) with the same center point as \( E \). We show that
\[ \mathbb{P}\left( \max_{E'_{n}} h_{E'_{n}} - m_{n} \leq -t \right) \leq C_{\varepsilon} e^{-t^{2-\varepsilon}} \] (A.8)
for all \( n \geq 0 \) and \( t > 0 \).

Let \( \lambda = t^{\varepsilon} \). First we consider the case that \( \lambda \leq n^{2/3} \) (or equivalently \( t \leq n^{2/3} \)). For \( c_{1}, c_{2}, c_{3} > 0 \), let \( r = \exp[n - c_{1} \lambda + c_{2}], \ell = \exp[n - c_{1} \lambda/3 + c_{2}/3] \) and \( m = \lfloor (2\ell)^{-1} e^{\alpha} \rfloor \). For \( i = 1, \ldots, m \), let \( C_{i} \subset E'_{n} \) be disjoint balls with radius \( r \), and let \( B_{i} \) be the box of side-length \( r/8 \) that is centered in the same point as \( C_{i} \). We write \( C := \bigcup_{i=1}^{m} C_{i} \) and
\[ \chi := \arg \max_{z \in \bigcup_{i=1}^{m} B_{i}} h_{C}^{C}(z). \] (A.9)
Then, as in Sect. 2.4 of Ding [14], the constants \( c_{1}, c_{2}, c_{3} > 0 \) can be chosen such that
\[ \mathbb{P}(h_{C}^{C}(\chi) \leq m_{n} - \lambda) \leq \exp\left(-c_{3} e^{c_{1}/3}\right), \] (A.10)
which can be seen as a consequence of (14) in [14].

Let \( t_n > 0 \) be such that

\[
e^{-t_n^2} = \exp\left(-c_3 e^{c_1 n^{2/3}/3}\right).
\]

(A.11)

For \( t \in \left[\frac{2}{n^2}, t_n\right] \), we define \( C_i, B_i, \) and \( m \) as for \( \lambda = n^{2/3} \), and we use monotonicity in \( t \) to bound the probability on the left-hand side of (A.10) from above by \( \exp\left(-c_3 e^{c_1 n^{2/3}/3}\right) \). This shows assertion (A.5). For (A.6), we argue analogously to derive (A.5), we assume that \( x \) is the center point of \( E \). Then, by inclusion of events and the Gibbs-Markov property,

\[
\mathbb{P}\left(\max_{B(x, \epsilon/2)} h_{E_n} - m_n \leq -t\right) \leq \mathbb{P}\left(\max_{E_n} h_{E_n} - m_n \leq -t\right)
\]

\[
\leq \mathbb{P}\left(\max_{E_n} h_{E_n} - m_n \leq -t/2\right) + \mathbb{P}\left(\min_{E_n} \varphi_{B_n - E_n} \leq -t/2\right).
\]

(A.14)

The second summand on the right-hand side has a uniformly Gaussian tail, which follows from e.g. Lemma 4.4 of [3]. This shows assertion (A.5). For (A.6), we argue analogously using the binding field \( \varphi_{U_n \cap V_{-\epsilon}} - E_n \) for which the same tail bound holds as the intrinsic metric and the variance that are used in the Fernique and Borell-TIS estimates in Lemma 4.4 of [3] can only be smaller than for \( \varphi_{U_n - E_n} \). For (A.7), we assume that \( y \) is the center point of \( E \) and argue again as in (A.14), where the tail bound for \( \varphi_{U_n \cap V_{-\epsilon}} - E_n \) now follows from Lemma B.12.

\( \square \)

We also need the following statement on the geometry of local extrema which is a variant of Theorem 9.2 of [3].

**Lemma A.3** Let \( \epsilon \in (0, 1) \), \( \eta > 0 \), and let \((c_n)_{n \geq 0}\) satisfy \( c_n = o(e^n) \) and \( c_n \to \infty \) as \( n \to \infty \). Let \( U \in \varOmega_0, A \) open with \( eB \subset A \subset U^\eta, n > 0, u \in \mathbb{R}^{2B_{un}} \) satisfy (5.9) for some \( \tilde{u}_\infty \in \mathbb{H}_\infty(U^\eta) \). We choose \( V = B \). Then we have \( \lim_{n \to \infty} \varphi^{n}_{0, \tilde{u}(0)}(F_\epsilon) = 0 \), where \( F_\epsilon \) denotes the event that there exists \( y \in U_n^\eta \cap B_0 \) with \( c_n/2 < |y - \arg \max_{A_n} h| < 2c_n \) and \( h(y) + \epsilon \geq \max_{A_n} h \).
Proof Let $\tilde{F}$ denote the event that there exist $x, y \in A_n$ with $c_n/2 < |x - y| < 2c_n$ and $h^{U_n}(x) \wedge h^{U_n}(y) \geq m_n - \varepsilon$ for some $\varepsilon > 0$. From Theorem 9.2 of [3], we have $\lim_{n \to \infty} P(F) = 0$. We now relate $\tilde{F}$, $h^{U_n}$ with $F_n$ and $h$ under $\mathbb{P}^{n,u}_{0,0}(0)$. By definition of the DGFF, $h$ under $\mathbb{P}^{n,u}_{0,0}(0)$ is distributed as $h^{U_n \cap B_0} + \mathbb{E}^{n,u}_{0,0}(0) h$. By the Gibbs-Markov property, $h^{U_n}$ is distributed as the sum of the independent fields $\varphi^{U_n, U_n \cap B_0}$ and $h^{U_n \cap B_0}$. Moreover, by a union bound,
\begin{align}
P^{n,u}_{0,0}(F) & \leq P(\tilde{F}) + P(\max_{h^{U_n}} m_n - n \leq -t + 3\varepsilon) + P(\max_{h^{U_n}} |\varphi^{U_n, U_n \cap B_0}| > \varepsilon) \\
& \quad + 1_{\sup_{y' \in A_n, |x - y'| \leq 2c_n} |\mathbb{E}^{n,u}_{0,0}(0) h(x) - \mathbb{E}^{n,u}_{0,0}(0) h(y)| > \varepsilon}.
\end{align}
(A.15)

Let $\varepsilon' > 0$. By Lemma A.2, we find $t > 0$ such that $P(\max_{h^{U_n}} m_n - n \leq -t + 3\varepsilon) \leq \varepsilon'$ for sufficiently large $n$. By Lemma B.12, we have $P(\max_{h^{U_n}} |\varphi^{U_n, U_n \cap B_0}| > \varepsilon/2) \leq \varepsilon'$ for sufficiently large $n$. As $\mathbb{E}^{n,u}_{0,0}(0) h = (-m_n1_{\partial U_n} + u)_{\partial U_n \cup \partial B_0} + (\mathbb{P}(0)1_{\partial B_0})_{\partial U_n \cup \partial B_0}$, Lemma 5.14 yields the convergence of $x \mapsto m_n + \mathbb{E}^{n,u}_{0,0}(0) h(\{e^n x\})$ to some $\tilde{u}_\infty$ in $P_\infty(U^n \cap \varepsilon B)$. The indicator variable on the right-hand side of (A.15) is equal to $\varepsilon/2$ for $n$ sufficiently large as $\tilde{u}_\infty$ is harmonic and thus uniformly continuous on $A$. Hence, the right-hand side in (A.15) is bounded by $3\varepsilon'$ for $n$ sufficiently large which shows the assertion as $\varepsilon' > 0$ was arbitrary. \hfill $\square$

The next lemma bounds the probability that the DGFF $h$ exceeds zero on a test set near the boundary of its domain. Here we do not have entropic repulsion as in Proposition 4.3 and therefore the probability only becomes small when the test set is small or the boundary values are low. This is a version of Lemma B.12 of [6] and Lemma 3.8 of [10] in our setting under $\mathbb{P}^{n,u}_{0,0}(0)$.

Lemma A.4 Let $\varepsilon \in (0, 1)$ and $\eta, \zeta \in [0, e^{-1}]$. There exists $C = C_\varepsilon < \infty$ such that for all $U \in \mathcal{U}_n$, $V \in \mathcal{V}_\varepsilon$, $0 \leq k < n$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k}$ with $\text{osc} \, \mu_{\eta} \leq (n - k)^{1-\varepsilon}$, $\text{osc} \, \nu_{\zeta} \leq (n - k)^{1-\varepsilon}$, we have
\begin{align}
P^{n,u}_{0,k,v}(\max_{x \in W_n} h(x) > 0) & \leq C \sqrt{\text{Leb}(W)}(1 + |M|)e^{\alpha M} \\
(A.16)
\end{align}
and
\begin{align}
P^{n,u}_{0,k,v}(\max_{x \in W_k} h(x) > 0) & \leq C \sqrt{\text{Leb}(W)}(1 + |M|)e^{\alpha M} \\
(A.17)
\end{align}
for all measurable $W, W' \in \mathbb{R}^2$ with $W_n \subset U_n^0 \cap (\varepsilon^{-1}B_n) \cap V_k^{-\zeta} \wedge W_n^k \subset V_k^{-\zeta} \cap (\varepsilon^{-2}B)_k \cap U_n^0$, where we write $M = (\mu_{\eta}(0) + \text{osc} \, \mu_{\eta} + \text{osc} \, \nu_{\zeta}) \vee (\nu_{\zeta}(\infty) + \text{osc} \, \nu_{\zeta} + \text{osc} \, \mu_{\eta})$.

We note that the Lebesgue measure on the right-hand sides of (A.16) and (A.17) is proportional to the number of discrete points in $W_n$ as the discretization (1.4) implies for each $x \in W_n$ that the ball of radius $e^{-n/2}$ around $e^{-n} x$ is contained in $W$.

Proof of Lemma A.4 By Lemma B.14, we have that
\begin{align}
\max_{W_n} \{( -m_n1_{\partial U_n} + u)_{\partial U_n \cup \partial V_k} \leq -m_n + M + C_\varepsilon. \\
(A.18)
\end{align}

Then the left-hand side of (A.16) is bounded from above by the left-hand side of
\begin{align}
P \left( \max_{W_n} h^{U_n \cap V_k} > m_n - M - C_\varepsilon \right) & \leq C_\varepsilon \sqrt{\text{Leb}(W)}(1 + |M|)e^{\alpha M} \\
(A.19)
\end{align}
The last inequality follows from Lemma B.12 of [6], and as $|W_n|e^{-2n}$ is bounded by a constant times Leb$(W)$. The latter boundedness follows from the discretization (1.4) which
ensures that the points of $e^{-n}W_n \subset \mathbb{R}^2$ are surrounded by disjoint balls of radius $e^{-n}/2$ that are contained in $W$, each having area $\pi e^{-2n}/4$.

To show (A.17), we assume w.l.o.g. that $2\varepsilon^{-2}B \subset U^\eta$ (otherwise we can argue as for (A.16)). Let $g$ be the harmonic function on $U_n \cap V_k^-$ with boundary values $u$ on $\partial U_n$ and $v$ on $\partial V_k^-$. Again from Lemma B.14, we obtain $g \leq M + C_\varepsilon$ on $U_n^\eta \cap V_k^-$. By the Gibbs-Markov property and monotonicity in the boundary values,

$$P_{\varepsilon \cdot}^{n,\eta} \left( \max_{x \in W_n^\eta} h(x) > 0 \right) = P_{\varepsilon \cdot}^{n,0} \left( \max_{x \in W_n^\eta} h(x) + g(x) > 0 \right)$$

$$= \int P_{\varepsilon \cdot}^{n,0} \left( h_{\partial(2\varepsilon^{-2}B)_k} + m_k \in du \right) P_{\varepsilon \cdot}^{2\varepsilon^{-2}B_{k,0}} \left( \max_{x \in W_n^\eta} h(x) + M > 0 \right)$$

$$\leq \int P_{\varepsilon \cdot}^{n,0} \left( h_{\partial(2\varepsilon^{-2}B)_k} + m_k \in dr \right) \sum_{a=1}^\infty P_{\varepsilon \cdot}^{n,0} \left( \text{osc}_{(\varepsilon^{-2}B)_k} h \geq a \right) h_{\partial(2\varepsilon^{-2}B)_k} (0) + m_k = r \right)$$

$$\times P_{\varepsilon \cdot}^{2\varepsilon^{-2}B_{k,0}} \left( \max_{x \in W_n^\eta} h(x) + M > 0 \right). \quad (A.20)$$

Using Propositions B.3 and B.4 for the first two probabilities in the last expression and arguing as for (A.16) for the third probability, we further bound (A.20) by

$$C_\varepsilon \int e^{-c_\varepsilon r^2} \sum_{a=1}^\infty e^{-(\alpha+1)a + (\alpha+1)|r|} \sqrt{\text{Leb}(W^\eta)} (1 + a + |r|)(1 + |M|)e^{\alpha(a+|r|)}e^{aM} \quad (A.21)$$

which implies (A.17).

\[\Box\]

### A.2 Discrete Harmonic Analysis

First we need an estimate for the probability that simple random walk $S$ on $U_n \cap V_k^-$ started in a point $x$ at scale $l$ reaches $U_n$ before $V_k^-$. This is essentially the ruin probability for simple random walk on $(k, \ldots, n)$. The following lemma is a variant of Proposition 6.4.1 of [19], we consider more general sets $U, V$ instead of balls at the cost of a larger error. We recall that $\Pi_U$ denotes the continuum Poisson kernel, i.e. $\Pi_U(x, \cdot)$ is the exit distribution from the domain $U$ of Brownian motion started at $x$.

**Lemma A.5** Let $\varepsilon \in (0, 1)$. Then there exists $C = C_\varepsilon < \infty$ such that for all $U \in \Upsilon^0, V \in \Upsilon_\varepsilon, 0 \leq k < n$, the following holds:

(i) For all $x \in U_n \cap V_k^-$, we have

$$P_x(S_{U_n \cap V_k^-} \in \partial U_n) - \frac{\log |x| - k}{n - k} \leq C(n - k)^{-1}. \quad (A.22)$$

(ii) For all $x \in U^\varepsilon \cap \varepsilon B^-$, we have

$$P_{\varepsilon x}^\varepsilon(S_{U_n \cap V_k^-} \in \partial V_k^-) = \frac{\int_{\partial U} \Pi_U(x, dz) \log |z| - \log |x| + o_\varepsilon(1)}{n - k} \quad (A.23)$$

as $n - k \to \infty$.

(iii) For all $x \in V^{-\varepsilon} \cap \varepsilon^{-1}B$, we have

$$P_{\varepsilon x}^{\varepsilon}(S_{U_n \cap V_k^-} \in \partial U_n) = \frac{\log |x| - \int_{\partial V^-} \Pi_{V^-}(x, dz) \log |z| + o_\varepsilon(1)}{n - k} \quad (A.24)$$

whenever $n - k \to \infty$ followed by $k \to \infty$. 

Proof As in the proof of Proposition 6.4.1 of [19], we use that \(a(S_{\tau_{U_n\cap V_k^-}})\) is a bounded martingale under \(P_x\) for any \(x \in U_n \cap V_k^-\). To show assertion (i), let \(q := P_x(\tau_{V_k^-} < \tau_{U_n})\). By optional sampling, using that \(\tau_{U_n}\) is stochastically bounded by a geometric random variable, we have

\[
\alpha(x) = E_x a(S_{\tau_{U_n\cap V_k^-}}) = (1 - q) E_x a(S_{\tau_{U_n}}) \mid \tau_{U_n} \leq \tau_{V_k^-} + q E_x a(S_{\tau_{V_k^-}}) \mid \tau_{V_k^-} < \tau_{U_n}.\]

For assertion (i), it suffices to estimate by (2.6) the conditional expectations on the right-hand side of the last display by \(n + O_\epsilon(1)\) and \(k + O_\epsilon(1)\), respectively, and the left-hand side by \(\log |x| + O_\epsilon(1)\), and to solve for \(1 - q\).

To show assertion (ii), we set \(q := P_{[e^n x]}(\tau_{V_k^-} < \tau_{U_n})\), and obtain from (A.25) with \([e^n x]\) in place of \(x\) that

\[
E_{[e^n x]} a(S_{\tau_{U_n\cap V_k^-}}) = (1 - q) gn + (1 - q) g E_{[e^n x]} \log |e^{-n} S_{\tau_{U_n}}| + q g(k + O(1)) + o_\epsilon(1) \text{ (A.26)}
\]

for \(x \in U^\epsilon \cap eB^-\), where we now also used (2.6), that

\[
E_{[e^n x]} a(S_{\tau_{U_n\cap V_k^-}}) = (1 - q) g n + (1 - q) g E_{[e^n x]} \log |e^{-n} S_{\tau_{U_n}}| \mid \tau_{U_n} \leq \tau_{V_k^-} + o_\epsilon(1) + q (g k + O(1)) \text{ (A.27)}
\]

by (2.6), and that

\[
E_{[e^n x]} \log |e^{-n} S_{\tau_{U_n}}| - q C_\epsilon \leq E_{[e^n x]} \log |e^{-n} S_{\tau_{U_n}}| \mid \tau_{U_n} \leq \tau_{V_k^-} \text{ (A.28)}
\]

where \(q = o_\epsilon(1)\) by (i), and \(|e^{-n} S_{\tau_{U_n}}| \in [\epsilon, \epsilon^{-1}]\) by definition of \(V_{\epsilon}^0\). By Lemma A.9 below, \(E_{[e^n x]} \log |e^{-n} S_{\tau_{U_n}}| = \int_{\bar{U}} \Pi_U(0, d\eta) \log |\eta| + o_\epsilon(1) \text{ (A.29)}\)

and (ii) follows by plugging (A.29) into (A.26) and solving for \(q\).

Assertion (iii) follows analogously.

In the next lemma, we work in the setting of Sect. 2.3. We compare the discrete Poisson kernel on the annulus-like set \(\Delta_j \cap V_k^-\) and the ball-like set \(\Delta_j\). For the latter, we multiply the discrete Poisson kernel with the probability that the random walk that underlies the discrete Poisson kernel first reaches a part of the boundary of the annulus-like set that is also contained in the boundary of the ball-like set.

Lemma A.6 Let \(\epsilon \in (0, 1)\) and \(\eta, \zeta \in [0, \epsilon^{-1}]\). There exists a constant \(C = C_\epsilon < \infty\) such that

\[
|\Pi_{\Delta_j \cap V_k^-}(x, z) - \frac{T + 1 - p}{T + 1 - j} \Pi_{\Delta_j}(0, z)| \leq C \Pi_{\Delta_j}(0, z) \left(\frac{1}{T + 1 - j} + (p - j) e^{-(p - j)}\right)
\]

for all \(U \in \Omega^0, V \in \Omega, 0 \leq k < n, and all j = 1, \ldots, T - 2, p = j + 2, \ldots, T, x \in A_p,\) and \(z \in \partial \Delta_j\).

Proof We apply Lemma A.5(i) and Proposition 6.4.5 of [19] \((C_n, A, \text{and } C_{2m}\) there correspond to our \(\Delta_j, \Delta_j \cap V_k^-, \text{and } \Delta_{p - 1,}\) respectively).
We now give a modification of the previous lemma in the setting of the outward concentric decomposition from Sect. 2.6.

**Lemma A.7** Let $\epsilon \in (0, 1)$ and $\zeta, \eta \in [0, \epsilon^{-1}]$. There exists a constant $C = C_\epsilon < \infty$ such that

$$\left| \Pi_{\Delta_j^0 \cap U_n}(x, z) - \frac{T + 1 - p}{T + 1 - j} \Pi_{\Delta_j^0}(\infty, z) \right| \leq C \Pi_{\Delta_j^0}(\infty, z) \left( \frac{1}{T + 1 - j} + (p - j)e^{-(p - j)} \right)$$

(A.31)

for all $U \in \Omega^0_k, V \in \Omega_\epsilon, 0 \leq k < n$, and all $j = 1, \ldots, T - 2$, $p = j + 2, \ldots, T$, $x \in A_p^0$, and $z \in \partial \Delta_j^0$.

**Proof** By Lemma A.5(i),

$$\left| \Pi_{\Delta_j^0 \cap U_n}(x, \partial \Delta_j^0) - \frac{T + 1 - p}{T + 1 - j} \right| \leq \frac{C_\epsilon}{T + 1 - j}$$

(A.32)

for $x \in A_p^0$. Moreover, by Lemma A.8 below, we have

$$\Pi_{\Delta_j^0}(x, z) = \Pi_{\Delta_j^0}(\infty, z) \left[ 1 + O(e^{-(p - j)}) \right]$$

(A.33)

for $x \in A_p^0$. Using this in place of (6.23) in [19], we then proceed as in the proof of Proposition 6.4.5 of [19] to show that

$$P_x(\tau_{\partial \Delta_j^0 \cap U_n} = \infty) = \Pi_{\Delta_j^0}(\infty, z) \left[ 1 + O((p - j)e^{-(p - j)}) \right].$$

(A.34)

Combined with (A.32) this yields the assertion. □

**Lemma A.8** Let $\epsilon \in (0, 1)$. There exists a constant $C = C_\epsilon < \infty$ such that for all $k \geq 0$, $l \geq k + \epsilon$, $x \in B_{-}^l$, $z \in \partial B_{-}^k$, we have

$$\left| \Pi_{B_{-}^k}(x, z) - \Pi_{B_{-}^k}(\infty, z) \right| \leq C e^{-(l - k)}\Pi_{B_{-}^k}(\infty, z).$$

(A.35)

**Proof** By optional stopping, it suffices to estimate the absolute difference $|\Pi_{B_{-}^k}(x, z) - \Pi_{B_{-}^k}(y, z)|$ for $x, y \in \partial B_{-}^l$ (and $z \in \partial B_{-}^k$). By Lemma 6.3.6 of [19],

$$\Pi_{B_{-}^k}(x, z) = \sum_{w \in B_{-}^{l+\epsilon/2}} G_{B_{-}^k}(x, w) P_w \left( \tilde{x}_{\epsilon} \cap B_{-}^{l+\epsilon/2} = z \right),$$

(A.36)

where $\tilde{x}_A = \inf \{i \geq 1 : S_i \notin A \}$ for $A \subset \mathbb{Z}^2$. By (2.6) and (2.7), for $x, y, w$ as before,

$$G_{B_{-}^k}(x, w) - G_{B_{-}^k}(y, w) = g \left\{ \log |y - w| - \log |x - w| \right\} + \sum_{w' \in \partial B_{-}^k} \Pi_{B_{-}^k}(w, w') \left\{ \log |w' - x| - \log |w' - y| \right\} + O(e^{-2l}).$$

(A.37)

For our $x, y \in \partial B_{-}^l$ and $w \in \partial (B_{-}^k \setminus B_{-}^{l+\epsilon/2})$, we obtain, as $\partial B_{-}^l$ is spherically symmetric up to lattice effects that are negligible in comparison to the range of possible $w_1$, that

$$\log |y - w| - \log |x - w| \leq \log \frac{e_{l} + e_{k}}{e_{l} - e_{k}} \leq C_\epsilon e^{k-l}.$$
From (A.36), (A.37) and (A.38), we obtain
\[
|\Pi_{B_k^-}(y, z) - \Pi_{B_k^-}(x, z)| \leq C e^{-k} \sum_{w \in B_{k+\epsilon/2}} P_w(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} = z). \tag{A.39}
\]

Summing over the undirected paths that start in \( w \in B_{k+\epsilon/2}^- \) and thereby are in \( B_k^- \setminus B_{k+\epsilon/2}^- \) until they hit \( z \in \partial B_k^- \), we obtain
\[
P_w(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} = z) = P_z(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} = w) \tag{A.40}
\]
from the symmetry of the transition kernel of simple random walk. Hence, (A.39) is further bounded from above by the left-hand side of
\[
C e^{-k} P_z(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} \in \partial(B_{k+\epsilon/2}^-)^c) \leq C e^{-(l-k)} \Pi_{B_k^-}(y, z). \tag{A.41}
\]
For the second inequality in (A.41), we used a straightforward modification of Lemma 6.3.7 of [19]. As in Lemma 6.3.4 of [19], optional stopping, dominated convergence and (2.6) yield
\[
gk + O(e^{-k}) = a(z) = E_z a(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}}) = P_z(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} \in \partial B_{k+\epsilon/2}^-) g(k + \epsilon/2) + \left(1 - P_z(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} \in \partial B_{k+\epsilon/2}^-) \right) gk \tag{A.42}
\]
where the error term \( O(e^{-k}) \) comes from the discretization and lattice effects in the approximation of the potential kernel. Solving for \( P_z(S_{\frac{r}{\epsilon}B_k^- \setminus B_{k+\epsilon/2}} \in \partial B_{k+\epsilon/2}^-) \) then yields the first inequality in (A.41). Now the assertion follows from (A.40) and (A.41).

We also use the following strengthening of Lemma A.1 of [7]:

\begin{lemma}
Let \( \epsilon \in (0, 1) \). Then,
\[
\sum_{z \in \partial D_{\epsilon}^\pm} \Pi_{D_{\epsilon}^\pm}([e^n x], z) f(e^{-n} z) \xrightarrow{n \to \infty} \int_{\partial D} \Pi_{D^\pm}(x, dz) f(z) \tag{A.43}
\]
uniformly in \( D \in \mathcal{D}_\epsilon \), \( x \in D^\pm \epsilon \) and in the bounded continuous functions \( f : \mathbb{R}^2 \to \mathbb{R} \) with \( \| f \|_\infty \leq e^{-1} \) and modulus of continuity bounded by \( e^{-1} \).
\end{lemma}

\begin{proof}
For \( x \in D^\epsilon \), this can be gleaned from the proof of Lemma A.1 of [7] (which uses Skorohod coupling with a Brownian path).

To show the assertion for \( x \in D^{-\epsilon} \), we fix \( \epsilon' > 0 \) and assume w.l.o.g. that \( \| f \|_\infty \leq 1 \). Letting \( g(w) = \int_{\partial D} \Pi_{D^-}(w, dz) f(z) \) for \( w \in e^{-1}B^- \), we find by e.g. [20, Theorem 3.46] and a standard approximation argument some \( p > -\log \epsilon \) such that
\[
\left| \int_{e^{-1}B^-} \left[ \Pi_{e^{-1}B^-}(x, dz) - \Pi_{e^{-1}B^-}(\infty, dz) \right] g(z) \right| < \epsilon' \tag{A.44}
\]
for all \( x \in e^p B^- \). By definition of \( g \) and the strong Markov property of the Brownian motion underlying the Poisson kernel, it follows that
\[
\left| \int_{\partial D} \left[ \Pi_{D^-}(x, dz) - \Pi_{D^-}(\infty, dz) \right] f(z) \right| < \epsilon' \tag{A.45}
\]
for all \( x \in e^p B^- \).
\end{proof}
As before, we glean from the proof of Lemma A.1 of [7] that (A.43) holds uniformly in $D \in \mathcal{D}_\epsilon$ and $x \in D^{-,\epsilon} \cap e^{\theta+1}B$ (to apply the Skorohod coupling, we use that the hitting time of $D^-$ for planar Brownian motion is tight in its starting point $x \in D^{-,\epsilon} \cap e^{\theta+1}B$, which can be seen by replacing $D$ with its subdomain $\epsilon B$ so that tightness follows by monotonicity and spherical symmetry from the finiteness of the hitting time for a single starting point $x \in \partial e^{\theta+1}B$).

To include also $x \in e^{\theta}B_-$, it suffices by (A.45) to note that for fixed $x' \in B_{n+p} \cap B_{n+p+1}$, the difference $\sum_{z \in \partial D^\pm_\epsilon} \Pi_\partial D^\pm_\epsilon (x, z) f(e^{-n}z) - \sum_{z \in \partial D^\pm_\epsilon} \Pi_\partial D^\pm_\epsilon (x', z) f(e^{-n}z)$ is bounded harmonic in $x \in B_{n+p}$. Then, by the maximum principle for harmonic functions, the difference attains its absolute maximum on $B_{n+p+1} \cap B_{n+p}$, which is, by (A.45) and the assertion for $x \in e^{\theta+1}B$, bounded by $2\epsilon'$ for sufficiently large $n$, depending only on $\epsilon$ and $f$. □

Also the following statement can be gleaned from the proof of Lemma A.2 of [7]:

**Lemma A.10** Let $\epsilon \in (0, 1)$. Then,

\[
\int_{\partial D^{\pm,\delta}} \Pi_{D^{\pm,\delta}}(x, dz) f(z) \rightarrow \int_{\partial D} \Pi_{D^{\pm}}(x, dz) f(z) \tag{A.46}
\]

uniformly in $D \in \mathcal{D}_\epsilon$, $x \in D^{\pm,\epsilon}$ and in the bounded continuous functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\|f\|_\infty \leq \epsilon^{-1}$ and modulus of continuity bounded by $\epsilon^{-1}$.

We also need the following statement:

**Lemma A.11** Let $\epsilon \in (0, 1)$. Then,

\[
\sup_{x \in D^\pm \setminus D^{\pm,\delta}} \left| \int_{\partial D^\pm} \Pi_{D^\pm}(x, dz) [f(z) - f(x)] \right| \rightarrow 0 \tag{A.47}
\]

uniformly in $D \in \mathcal{D}_\epsilon$ and in the bounded continuous functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\|f\|_\infty \leq \epsilon^{-1}$ and modulus of continuity bounded by $\epsilon^{-1}$.

**Proof** For arbitrary $\epsilon' \in (0, \epsilon)$, let $\delta' \in (0, \epsilon/2)$ such that $|y - y'| < \delta'$ implies $|f(y) - f(y')| < \epsilon'$. For $\delta > 0$ and $x \in D^\pm \setminus D^{\pm,\delta}$, we consider concentric annuli around $x$ with exponentially decreasing radii, $A_j = x + \delta'e^{-j+1}B \setminus \delta'e^{-j}B$, $j \in \mathbb{N}$. By the assumption that each connected component of $\partial D^\pm$ has diameter at least $\epsilon$ and by an elementary geometric consideration, the number of those $j \in \mathbb{N}$ for which $\partial D^\pm$ crosses $A_j$ (i.e. $A_j \setminus \partial D^\pm$ has genus 0) is bounded from below by $j_0 := \lfloor \log \delta / \log \delta' \rfloor - 1$. By self-similarity, the probability that a standard Brownian motion $W$ started at $x$ completes a circle inside the annulus $A_j$ after hitting its middle line $x + \delta'e^{-j-1/2}B$ is bounded from below by some $p \in (0, 1)$ uniformly in $j$. On this event, $W$ hits $\partial D^\pm \cap (x + \delta'B)$. Using the strong Markov property at each such middle line, we thus obtain $(1 - p)^{j_0}$ as a geometric bound for the probability that $W$ does not hit $\partial D^\pm$ within in $x + \delta'B$. This bound becomes smaller than $\epsilon'^{-2}$ for sufficiently small $\delta > 0$, depending only on $\epsilon$ and $\epsilon'$.

Using the definition of the Poisson kernel $\Pi_{\partial D^\pm}(x, dz)$, we then bound the integral in (A.47) by $\epsilon'^{-2}\|f\|_\infty + \sup_{y \in x + \delta'B} |f(y) - f(x)|$, which is bounded by $2\epsilon'$, as required. □
B Estimates for the Harmonic Extensions of the DGFF

B.1 Harmonic Extensions at Intermediate Scales

In many applications, the ballot estimates are used on a subset of the domain of definition of the field, with either the inner or outer part of the boundary conditions being on an “intermediate scale”. To be more precise, if $U_n \cap V_k^-$ is the original domain on which $h$ is considered, with $U, V \in \mathcal{D}$ and $n, k \geq 0$, then one would like to apply the ballot theorems on $U_n \cap W_l^- \cap W_l^-$ or $W_l \cap V_k^-$, where $W \in \mathcal{D}$ and $l \geq 0$ is some “intermediate” scale satisfying $k + O(1) \leq l \leq n + O(1)$.

Thanks to the Gibbs-Markov property and as $U_n \cap W_l^-$ and $W_l \cap V_k^-$ are not path-connected, conditional on the values of $h$ on $\partial W_l^\pm$, the restrictions of $h$ to the latter sub-domains form independent fields, with each distributed like a DGFF with boundary conditions given by the original ones on $\partial U_n$ and $\partial V_k^-$ together with the values on $\partial W_l^\pm$ we condition $h$ to take. We may then use the ballot estimates in both sub-domains, but for these to be useful, we must have good control over the harmonic extension of the values of $h$ on $\partial W_l^\pm$, as many of the conditions, bounds and asymptotics in the previous subsection are expressed in terms of the latter.

The purpose of this subsection is therefore to study $\overline{h_{\partial W_l^\pm}}$ – the unique bounded harmonic extension of the values of $h$ on $\partial W_l^\pm$ to the entire plane, where $h$ is the DGFF on $U_n \cap V_k^-$ with boundary conditions given as before by $u$ on $\partial U_n$ and $v$ on $\partial V_k^-$. As the harmonic extension is a linear function of the field, it is also Gaussian and hence its law is completely determined by its mean and covariance, which we denote by

$$\mu(x; u, v) := \mu_{n,l,k}(x; u, v) = \mathbb{E}(\overline{h_{\partial W_l^\pm}}(x) \mid h_{\partial U_n} = -m_n + u, h_{\partial V_k^-} = -m_k + v)$$

(B.1)

and

$$\sigma(x, y) = \sigma_{n,l,k}(x, y) := \text{Cov}(\overline{h_{\partial W_l^\pm}}(x), \overline{h_{\partial W_l^\pm}}(y) \mid h_{\partial U_n} = 0, h_{\partial V_k^-} = 0).$$

(B.2)

For what follows, we let $\widehat{m}_l = \widehat{m}_{n,l,k}$ be the linear interpolation at scale $l$ between $m_k$ and $m_n$, that is

$$\widehat{m}_l = \widehat{m}_{n,l,k} := \frac{(l-k)m_n + (n-l)m_k}{n-k}; \quad 0 \leq k < l < n. \quad (B.3)$$

Setting $\wedge_{n,l} = l \wedge (n-l)$ and $\wedge_{n,l,k} = (l-k) \wedge (n-l)$, the following is well known and shows that the difference between $\widehat{m}_{n,l,k}$ and $m_l$ is of logarithmic order.

**Lemma B.1** For all $0 \leq k < l < n < \infty$,

$$-2\sqrt{g} \leq \widehat{m}_l - m_l \leq \frac{3}{2} \sqrt{g} \log^+ \wedge_{n,l,k} + \frac{3}{2} \sqrt{g}. \quad (B.4)$$

The proofs for this and the next subsection are given in Sect. B.3. The following three propositions provide bounds on the mean, covariances and (conditional) oscillations of $\overline{h_{\partial W_l^\pm}}$.

**Proposition B.2** Fix $\epsilon \in (0, 1)$, $\eta, \zeta \in [0, \epsilon^{-1}]$ and let $U, V \in \mathcal{D}_\epsilon$ such that $U$ and $V$ are connected. Let also $k, l, n \geq 0$ with $k < n$ and $W \in \mathcal{D}_\epsilon$ such that $\partial W_l \subset \overline{U_n^\eta}$ and $\partial W_l^- \subset \overline{V_k^-}$. Then, for all $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k^-}$ and $x \in \mathbb{Z}^2 \cup \{\infty\}$,

$$\left| \mu(x; u, v) - \left( -\widehat{m}_l + \frac{(l-k)\overline{u}(0) + (n-l)\overline{v}(\infty)}{n-k} \right) \right|$$
\[ \leq 2 \text{osc } \mu\eta + 2 \text{osc } \nu\zeta + c \frac{|\mu(0)| + |\nu(\infty)|}{n-k} + C. \quad (B.5) \]

Above \( C = C_e < \infty. \)

**Proposition B.3** Let \( \epsilon \in (0, 1). \) There exists \( C = C_\epsilon < \infty \) such that for all \( U, V \in \mathcal{D}_\epsilon \) for which \( U \) and \( V \) are connected, all \( 0 \leq k < n \) and \( W \in \mathcal{D}_\epsilon \) with \( \partial W_1 \subset U_1^\epsilon \) and \( \partial W_{-1} \subset V_{-1}^\epsilon, \) and for all \( x, y \in W_1^\epsilon \cup \{ \infty \}, \) we have
\[
|\sigma(x, y) - g\frac{(n-l)(l-k)}{n-k}| \leq C. \quad (B.6)
\]

The next proposition bounds the oscillation of \( h_{\partial W_i}^\pm \) in the bulk.

**Proposition B.4** Let \( \epsilon \in (0, 1), \eta, \zeta \in [0, \epsilon^{-1}]. \) There exists \( C = C_\epsilon < \infty \) and \( c = c_\epsilon > 0 \) such that for all \( U, V \in \mathcal{D}_\epsilon \) for which \( U \) and \( V \) are connected, all \( 0 \leq k < l \leq n \) and \( W \in \mathcal{D}_\epsilon \) with \( \partial W_1 \subset U_1^\epsilon \) and \( \partial W_{-1} \subset V_{-1}^\epsilon, \) all \( u \in \mathbb{R}^\partial U_n, v \in \mathbb{R}^\partial V_1^\epsilon, \) and for all \( x \in W_l^\epsilon \cup \{ \infty \}, w \in \mathbb{R}, t \geq 0, \) we have
\[
\mathbb{P}(\text{osc}_{W_l^\epsilon, e} h_{\partial W_1^\pm} > t \big| h_{\partial U_n} = -m_n + u, h_{\partial V_1^\epsilon} = -m_k + v, h_{\partial W_1^\pm}(x) = -m_l + w) \\
\leq C \exp \left( -c((t - C\mu)^+)^2 \right), \quad (B.7)
\]

where
\[
\mu = \mu_{n,l,k,\eta}(u, v, w) := \frac{|w| + \text{osc } \mu\eta + \text{osc } \nu\zeta + |\mu(0)|}{(n-l) \wedge (l-k)} + \frac{|\nu(0)|}{n-l} + \frac{|\nu(\infty)|}{l-k}. \quad (B.8)
\]

We note the following one-sided variant of Proposition B.4 which will be of use in [4].

**Proposition B.5** Let \( \epsilon \in (0, 1), \eta, \zeta \in [0, \epsilon^{-1}]. \) There exist \( C = C_\epsilon < \infty \) and \( c = c_\epsilon > 0 \) such that for all connected \( U \in \mathcal{D}_\epsilon, \) all \( W \in \mathcal{D}_\epsilon, \) and all \( t > 0, \) all \( x \in W_l^\epsilon \cup \{ \infty \}, \) and all boundary conditions \( u \in \mathbb{R}^\partial U_n, \) we have
\[
\mathbb{P}(\text{osc}_{W_l^\epsilon, e} h_{\partial W_1^\pm} > t \big| h_{\partial U_n} = -m_n + u, h_{\partial V_1^\epsilon} = -m_k + v, h_{\partial W_1^\pm}(x) = -m_l + w) \\
\leq C \exp \left( -c((t - C\mu)^+)^2 \right), \quad (B.9)
\]

where
\[
\mu = \mu(u, w) := \frac{|w| + |\mu(0)| + \text{osc } \mu\eta}{n-l}. \quad (B.10)
\]

For the proof of Proposition 1.3, we also need an asymptotic estimate for the covariance.

**Lemma B.6** Let \( \epsilon \in (0, 1) \) and \( j \in \mathbb{N}. \) Uniformly in \( U, V, W \in \mathcal{D}_\epsilon \) with \( W \subset e^j U^\epsilon \) and for which \( U \) and \( V \) are connected, and in \( x, y \in W_l^\epsilon \cup \{ \infty \}, \) we have
\[
\lim_{n-k \to \infty} \sigma_{n-j,k,\epsilon}([e^n-x], [e^n-y]) \\
= g \int_{w, w' \in \partial W} \prod_{x \in \partial U}(x, dw) \prod_{y \in \partial U}(y, dw') \left( \int_{z \in \partial (e^j U)} \Pi_{e^j U}(w, dz) \log |z - w'| - \log |w - w'| \right). \quad (B.11)
\]

Uniformly in \( U, V, W \in \mathcal{D}_\epsilon \) with \( e^j W \subset V_{-\epsilon} \) and for which \( U, V \) are connected, and in \( x, y \in W_l^\epsilon \cup \{ \infty \}, \) we have
\[
\lim_{n-k \to \infty} \sigma_{n-k+j,\epsilon}([e^{k+j}-x], [e^{k+j}-y])
\]
Lemma B.7 Let $\epsilon \in (0, 1)$. There exists $C = C_\epsilon < \infty$ such that for all $U, V, W \in D_\epsilon$, $0 \leq k \leq l \leq n$ with $\partial W_l \subset U_n^\epsilon$ and $\partial W_l^- \subset V_k^\epsilon$, and for all $x, y \in W_l^\epsilon \cup \{\infty\}$, we have
\[
\text{Cov}(h_{\partial W_l}(x), h_{\partial W_l}(y)) \big| h_{\partial U_n} = 0 \big) - g(n - l) \leq C, \quad \text{(B.13)}
\]
\[
\text{Cov}(h_{\partial W_l}(x), h_{\partial W_l}(y)) \big| h_{\partial V_k}^- = 0 \big) - g(l - k) \leq C. \quad \text{(B.14)}
\]
We also bound the intrinsic metric of the centered DGFF.

Lemma B.8 Let $\epsilon \in (0, 1)$. Then, there exists $C = C_\epsilon < \infty$ such that for all $U, V, W \in D_\epsilon$ and $0 \leq k \leq l \leq n$ with $\partial W_l \subset U_n^\epsilon$ and $\partial W_l^- \subset V_k^\epsilon$, all $x, y$ that are in the same connected component of $W_l^\epsilon$, we have
\[
\mathbb{E}\left[\left(\frac{h_{U_n \cap V_k^-}^\epsilon(x) - h_{U_n \cap V_k^-}^\epsilon(y)}{\partial W_l}(y)\right)^2\right] \leq C \frac{|x - y|^2}{\epsilon^4}. \quad \text{(B.15)}
\]

Remark B.9 As the proofs show, in all statements in Sect. B.1, it suffices to assume instead of $W \in D_\epsilon$, that $W$ satisfies the conditions for $W \in D_\epsilon$ with (1.6) replaced by $W \subset \epsilon^{-1}B$ and $\partial W \subset \epsilon B^-$. 

### B.2 Harmonic Extensions as Binding Fields

This subsection contains estimates on the harmonic extension of the DGFF on $U_n \cap V_k^\epsilon$ to a subdomain of $U_n \cap V_k^-$. A harmonic extension on such a domain occurs as the binding field in the Gibbs-Markov decomposition, namely $\varphi$ in following lemma:

Lemma B.10 (Gibbs-Markov property) Let $h$ be a DGFF on a nonempty set $A \subset \mathbb{Z}^2$, let $B \subset A$ be finite, and $\varphi = \mathbb{E}(h \mid h_{A \setminus B})$. Then $h - \varphi$ is a DGFF on $B$ with zero boundary conditions, and $\varphi$ is independent of $h - \varphi$ and satisfies $\varphi = h_{A \setminus B}$ a.s.

**Proof** See e.g. Lemma 3.1 in [3].

The estimates from this subsection will be used in the proofs in Sects. 2 and 3. The first of these estimates shows that the binding field vanishes in the bulk when the Hausdorff distance of the complement of the domains goes to zero. It contains a discrete version of Lemma 3.7 in [7].

Lemma B.11 Let $\epsilon \in (0, 1)$, $r \geq 0$. There exist $C = C_{\epsilon, r} < \infty$ and $\rho = \rho_{\epsilon, r} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(s) \downarrow 0$ as $s \downarrow 0$ such that
\[
\mathbb{E}\left(\max_{(U \cap \bar{B}_{n-r}^\epsilon) \setminus B_{n-r}} \frac{|h_{U_n \cap \bar{B}_{n-r}}^\epsilon(a(U \cap \bar{B}_{n-r}^\epsilon))|}{|a(U \cap \bar{B}_{n-r}^\epsilon)|\cup \partial B_0^\epsilon}\right) \leq \rho(d_{\mathbb{H}}(U, \tilde{U})) + o_{\epsilon, r}(1) \quad \text{(B.16)}
\]
\[
\mathbb{P}\left(\max_{(U \cap \bar{B}_{n-r}^\epsilon) \setminus B_{n-r}} \frac{|h_{U_n \cap \bar{B}_{n-r}}^\epsilon(a(U \cap \bar{B}_{n-r}^\epsilon))|}{|a(U \cap \bar{B}_{n-r}^\epsilon)|\cup \partial B_0^\epsilon} > t\right) \leq C \exp\left\{- \frac{t^2}{\rho(d_{\mathbb{H}}(U, \tilde{U})) + o_{\epsilon, r}(1)}\right\} \quad \text{(B.17)}
\]
and
\[
\mathbb{E}\left( \max_{(V^- \cap \tilde{V}^-)^c \setminus B^+_{k+r}} |h_{B_k \cap \tilde{V}_k^-} - h_{B_k \cup \partial(V^- \cap \tilde{V}^-)_k}| \right) \leq \rho(d_{\mathbb{H}^+}(V^-, \tilde{V}^-)) + o_{\epsilon, r}(1) \tag{B.18}
\]
\[
\mathbb{P}\left( \max_{(V^- \cap \tilde{V}^-)^c \setminus B^+_{k+r}} |h_{B_k \cap \tilde{V}_k^-} - h_{B_k \cup \partial(V^- \cap \tilde{V}^-)_k}| > t \right) \leq C \exp\left\{ - \frac{t^2}{\rho(d_{\mathbb{H}^+}(V^-, \tilde{V}^-)) + o_{\epsilon, r}(1)} \right\} \tag{B.19}
\]

where \( o_{\epsilon, r}(1) \) tends to zero as \( n \to \infty \) in (B.16), (B.17), and as \( n - k \to \infty \), followed by \( k \to \infty \) in (B.18) and (B.19), and the statements hold for all \( t > 0, U, \tilde{U}, V, \tilde{V} \in \mathcal{D}_\epsilon \) such that \( U, \tilde{U}, V, \tilde{V}^- \) are connected, \( 0 \leq k < n < \infty \).

The next lemma shows that the binding field also becomes small in a region that is many scales away from where the domains differ.

**Lemma B.12** Let \( \epsilon \in (0, 1) \), \( r \geq 0 \). There exist \( C = C_{r, \epsilon} < \infty \), \( c = c_{r, \epsilon} > 0 \) such that
\[
\mathbb{E}\left( \max_{U \cap B_{n-r}} |h_{U \cap B_{n-r}} - U_{\partial B_{n-r}} \cup V_k^-| \right) \leq C \frac{\sqrt{n} k^{1/2}}{n - k} \tag{B.20}
\]
\[
\mathbb{P}\left( \max_{U \cap B_{n-r}} |h_{U \cap B_{n-r}} - U_{\partial B_{n-r}} \cup V_k^-| > t \right) \leq C e^{-c(n-k)t^2} \tag{B.21}
\]

and
\[
\mathbb{E}\left( \max_{V_k^- \cap B_{k+r}} |h_{V_k^- \cap B_{k+r}} - V_k^- \cup \partial V_k^-| \right) \leq C \frac{\sqrt{n} k^{1/2}}{n - k} \tag{B.22}
\]
\[
\mathbb{P}\left( \max_{V_k^- \cap B_{k+r}} |h_{V_k^- \cap B_{k+r}} - V_k^- \cup \partial V_k^-| > t \right) \leq C e^{-c(n-k)t^2} \tag{B.23}
\]

for all \( t > 0, 0 \leq k < n \leq \tilde{n} < \infty \), and \( U, \tilde{U}, V, \tilde{V} \in \mathcal{D}_\epsilon \) such that \( U, \tilde{U}, V^- \) and \( \tilde{V}^- \) are connected, \( \tilde{U} \supset U, \tilde{V} \subset V \), and where we may replace \( \tilde{V}_0^- \) with \( \mathbb{Z}^2 \).

We also note the following estimate for the covariance.

**Lemma B.13** Let \( \epsilon \in (0, 1) \). Then, there exists \( C = C_\epsilon < \infty \) such that
\[
\text{Cov}\left(h_{U \cap B_{n-r}}(x), h_{U \cap B_{n-r}}(y)\right) \leq C \left( \frac{n - \log(|x| \wedge |y|)}{n - k} \right)^2 \tag{B.24}
\]

and
\[
\text{Cov}\left(h_{V_k^- \cap B_{k+r}}(x), h_{V_k^- \cap B_{k+r}}(y)\right) \leq C \left( \frac{\log(|x| \lor |y|) - k}{n - k} \right)^2 \tag{B.25}
\]

for all \( U, V \in \mathcal{D}_\epsilon \) such that \( U \) and \( V^- \) are connected, \( 0 \leq k < n < \infty \), and all \( x, y \in U_h \cap V_k^- \).

### B.3 Proofs

We now prove the statements from Sects. B.1 and B.2.
B.3.1 Mean, Covariance, and Intrinsic Metric

Proof of Lemma B.1 We write $e_l = \begin{pmatrix} 3 \sqrt{g} \end{pmatrix}^{-1} (\hat{m}_l - m_l)$. By definition of $\hat{m}_l$ and $m_l$,

$$
e_l = \log^+ l - \frac{l - k}{n - k} \log^+ n - \frac{n - l}{n - k} \log^+ k \quad \frac{d}{dl} e_l = \frac{1_{[l>1]}}{l} - \frac{\log^+ n - \log^+ k}{n - k},
$$

(B.26)

for all $0 \leq k < l < n$, and $l \neq 1$ for the derivative. Note that $e_k = e_n = 0$ and $-(k \vee 1)^{-1} \leq \frac{d}{dl} e_l \leq l^{-1} 1_{[l\geq 1]}$. Hence, if $l \in [n - 2, n]$ or $l \in [k, k + 2]$, then $|e_l| \leq 2$.

If $l \in [k + 1, n - 2]$, then $|e_l| \leq 2 + \log(l - k)$ where we also used subadditivity of the logarithm. For $n \geq 1 \vee (2k)$, we have $\frac{d}{dl} e_l \geq -2n^{-1} \log n$. Hence, if $k \leq n/2$ and $l \in [k + 1, n - 2]$, we also have $|e_l| \leq 2 + (n - l)2n^{-1} \log n \leq 2 + 2 \log(n - l)$, again by subadditivity of the logarithm for the last inequality.

Furthermore, $e_l \leq 2$ if $k \geq \max\{1, n/2\}$. By concavity of the logarithm, $e_l \geq 0$ if $l \geq 1$. □

In the following lemma, we compare harmonic averages of the boundary conditions. According to our notational conventions, e.g. $u_{\partial U \cup \partial V_k^-}$ is obtained by harmonically extending the boundary conditions $u$ on $\partial U_n$ and 0 on $\partial V_k^- \setminus \partial U_n$.

Lemma B.14 Let $\epsilon \in (0, 1)$. Then

$$
|P_x(\tau^u \leq \tau^{v_k^-}) \bar{u}(x) - \bar{u}_{\partial U_n \cup \partial V_k^-}(x)| \leq P_x(\tau^{v_k^-} < \tau^u) \text{osc}_{U} \bar{u} \tag{27}
$$

and

$$
|P_x(\tau^{v_k^-} < \tau^u) \bar{u}(x) - \bar{u}_{\partial U_n \cup \partial V_k^-}(x)| \leq P_x(\tau^u < \tau^{v_k^-}) \text{osc}_{V} \bar{u} \tag{28}
$$

for all $U, V \in \mathcal{D}_e$, $0 \leq k < n < \infty$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k^-}$, $x \in U_n \cap V_k^-$, $U' \supseteq \{x\} \cup \partial V_k^-$ and $V' \supseteq \{x\} \cup \partial U_n$.

Proof We show (B.27), the proof of (B.28) is analogous. Let $0 \leq k \leq n$. Decomposing at the exit time from $V_k^-$ yields

$$
E_x u(S_{\tau^u}) = E_x(u(S_{\tau^u \cap V_k^-}; S_{\tau^u \cap V_k^-} \in \partial U_n) + P_x(\tau^{V_k^-} < \tau^u) \sum_{y \in \partial V_k^-} P_x(S_{\tau^u} = y | \tau^{V_k^-} < \tau^u) E_y u(S_{\tau^u}) . \tag{29}
$$

Inserting $E_x u(S_{\epsilon \tau^u}) = \bar{u}(x) + (\bar{u}(y) - \bar{u}(x))$ and rearranging terms, we obtain

$$
|E_x(u(S_{\tau^u \cap V_k^-}; S_{\tau^u \cap V_k^-} \in \partial U_n) - P_x(\tau^u \leq \tau^{V_k^-}) \bar{u}(x)| \leq P_x(\tau^{V_k^-} < \tau^u) \max_{y \in \partial V_k^-} |\bar{u}(x) - \bar{u}(y)| . \tag{30}
$$

The expectation in the last display equals $\bar{u}_{\partial U_n \cup \partial V_k^-}$. □

To further bound the oscillation terms in Lemma B.14, we have the following lemma:

Lemma B.15 Let $\epsilon \in (0, 1)$. There exist $C = C_\epsilon < \infty$ and $\theta = \theta_\epsilon < \epsilon/3$ such that the following holds: For all $\eta, \zeta \in [0, \epsilon^{-1}]$, $U \in \mathcal{U}_\theta$, $V \in \mathcal{V}_\epsilon$, $0 \leq k < n < \infty$, $u \in \mathbb{R}^{\partial U_n}$, $v \in \mathbb{R}^{\partial V_k^-}$, $x \in U_n^\eta \cap V_k^- \zeta$, we have

$$
\text{osc}_{x | \partial U \cup \partial V_k^-} \bar{u} \leq C e^{-n} |x| \text{osc } \bar{u}_\eta \tag{31}
$$
and
\[ \text{osc}_{|x| \in \partial U_n} v \leq C \frac{e^k}{|x|} \text{osc}_{|x|} v. \] (B.32)

If \( x \in (\theta B)_n \cap (\theta^{-1} B^-)_k \) and \( n - k > C \), then we also have
\[ \text{osc}_{|x| \in \partial V^-_k} u \leq \epsilon P_x(\tau_{V_n} < \tau_{V^-_k}) \text{osc}_{\eta} \bar{u}. \] (B.33)
and
\[ \text{osc}_{|x| \in \partial U_n} \bar{u} \leq \epsilon P_x(\tau_{V^-_k} < \tau_{U_n}) \text{osc}_{\eta} \bar{u}. \] (B.34)

We note that (B.33) and (B.34) allow us to invert the inequality signs in the hitting probabilities on the right-hand sides of (B.27) and (B.28) as needed in the proof of Lemma 3.3.

**Proof of Lemma B.15** We first show assertions (B.31) and (B.33). If \( x \in U_n^\theta \setminus (\frac{\epsilon}{2} B)_n \), then we have \( e^{-n} |x| \geq \frac{\epsilon}{2} \), and there is nothing to show in (B.31) as \( \{x\} \cup \partial V^-_k \subset U_n^\theta \). We can thus assume that \( x \in (\frac{\epsilon}{2} B)_n \cap V^-_k \). From the representation (2.8), the definition of the oscillation and the strong Markov property of the random walk underlying the definition of the Poisson kernel, we obtain
\[
\text{osc}_{|x| \in \partial V^-_k} \bar{u} \leq 2 \max_{y \in \partial V^-_k} \left| \sum_{w \in \partial U_n} \left[ \Pi_{U_n}(y, w) - \Pi_{U_n}(x, w) \right] [u(w) - \bar{u}(x)] \right|
\]
\[
\leq 2 \max_{y \in \partial V^-_k} \left| \sum_{w' \in \partial (\epsilon B)_n} \left[ \Pi_{(\epsilon B)_n}(y, w') - \Pi_{(\epsilon B)_n}(x, w') \right] \sum_{w \in \partial U_n} \Pi_{U_n}(w', w) [u(w) - \bar{u}(x)] \right|
\]
\[
\leq 2 \max_{y \in \partial V^-_k} \sum_{w' \in \partial (\epsilon B)_n} \left| \Pi_{(\epsilon B)_n}(y, w') - \Pi_{(\epsilon B)_n}(x, w') \right| \left| \bar{u}(w') - \bar{u}(x) \right|. \] (B.35)

Applying Theorem 6.3.8 of [19] along a shortest path \((x_t)_{t=0}^l\) from \( x_0 := x \) to \( x_l := y \) in \((\epsilon B)_n\), we obtain that
\[ \left| \Pi_{(\epsilon B)_n}(y, w') - \Pi_{(\epsilon B)_n}(x, w') \right| \leq C \epsilon e^{-n} \sum_{i=1}^{l} \Pi_{(\epsilon B)_n}(x_i, w'), \] (B.36)

hence we can bound the right-hand side of (B.35) from above by
\[ C \epsilon \epsilon e^{-n} \max_{w' \in (\epsilon B)_n} \left| \bar{u}(w') - \bar{u}(x) \right| \leq C \epsilon \epsilon e^{-n} \text{osc}_{\eta} \bar{u}. \] (B.37)

which shows (B.31) as \( l \) is the length of a shortest path from \( x \in V^-_k \) to \( y \in \partial V^-_k \) in \((\epsilon B)_n\) whose length is bounded by a constant times \(|x|\). Moreover,
\[ C \epsilon \epsilon e^{-n} \leq C \epsilon \frac{|x|}{e^n} = C \epsilon \frac{|x|e^{-k}}{e^{n-k}} \leq 2 \frac{\log |x| - k}{n - k} \leq \epsilon P_x(\tau_{V_n} < \tau_{V^-_k}) \] (B.38)
for \( x \in (\theta B)_n \cap (\theta^{-1} B^-)_k \) and \( n - k > C \), where we choose \( \theta \) sufficiently small and \( C \) sufficiently large such that last and second last inequalities hold by a straightforward computation and Lemma A.5(i), respectively. This yields assertion (B.33).

To show assertion (B.32), we argue analogously but use Lemma A.8 in place of (B.36) in
\[ \left| \Pi_{(-1 B^-)_k}(y, w) - \Pi_{(-1 B^-)_k}(x, w') \right| \leq C \epsilon \frac{e^k}{|x|} \Pi_{(-1 B^-)_k}(\infty, w). \] (B.39)
For (B.34), we then also have
\[ C \varepsilon^{e^k/|x|} \leq \frac{\log e^n}{\log e^{n-k}} \leq \varepsilon P_x \left( V^< \sim U^< \right), \tag{B.40} \]
again by Lemma A.5(i) and for \( x \in (\theta B)_n \cap (\theta^{-1} B^-)_k \) for \( \theta \) sufficiently small and \( n - k \) sufficiently large. \( \square \)

**Proof of Proposition B.2** By linearity of the expectation, we have
\[ \mu(x; u, v) = \sum_{y \in \partial W^\pm_i} \Pi_{\mathbb{Z}^2 \setminus W^\pm_i}(x, y) E_k^{n, u, v}(y), \tag{B.41} \]
where
\[ E_k^{n, u, v}(y) = -m_n + (m_n - m_k) P_w \left( V^< \sim U^< \right) + \overline{h}_{\partial U_n \cup \partial V^-_k}(y) + \overline{v}_{\partial U_n \cup \partial V^-_k}(y) \tag{B.42} \]
by definition of the DGFF and the representation (2.8). By (B.42), as \(| \log |y| - l | \leq C_\varepsilon \) for \( y \in \partial W^\pm_i \), and by Lemmas B.14 and A.5(i),
\[ \left| E_k^{n, u, v}(y) - \frac{l - k}{n - k} \overline{\pi}(0) - \frac{n - l}{n - k} \overline{\pi}(\infty) - \overline{\pi}_l \right| \leq 2 \text{osc} \overline{\pi}_l + 2 \text{osc} \overline{\pi}_\varepsilon + C_\varepsilon + C_\varepsilon \frac{\overline{\pi}(0) + \overline{\pi}(\infty)}{n - k} \tag{B.43} \]
and the assertion follows from (B.41). \( \square \)

**Proof of Proposition B.3** Representing the harmonic extension in terms of the Poisson kernel and inserting the covariance of the DGFF, we obtain
\[ \text{Cov}_{k} \left( \overline{h}_{\partial W^\pm_i}(x), \overline{h}_{\partial W^\pm_i}(y) \right) = \sum_{w, w' \in \partial W^\pm_i} \Pi_{\mathbb{Z}^2 \setminus \partial W^\pm_i}(x, w) \Pi_{\mathbb{Z}^2 \setminus \partial W^\pm_i}(y, w') G_{U_n \cap V^-_k}(w, w') \tag{B.44} \]
for \( x, y \in \mathbb{Z}^2 \cup \{ \infty \} \). Here, by (2.7),
\[ G_{U_n \cap V^-_k}(w, w') = \sum_{z \in \partial U_n} \Pi_{U_n \cap V^-_k}(w, z) (a(z - w') - l) + \sum_{z \in \partial V^-_k} \Pi_{U_n \cap V^-_k}(w, z) (a(z - w') - l) - (a(w - w') - l) \tag{B.45} \]
for \( w, w' \in \partial W^\pm_i \). By Lemma A.5(i) and (2.6), the first sum on the right-hand side of (B.45) equals
\[ g \frac{l - k + O_\varepsilon(1)}{n - k} \left( \log \left( e^n + O_\varepsilon(e^l) \right) - l \right). \tag{B.46} \]
Using also the analogous estimate for the second sum in (B.45), we obtain
\[ G_{U_n \cap V^-_k}(w, w') = g \frac{(l - k)(n - l)}{n - k} - (a(w - w') - l) + O_\varepsilon(1) \tag{B.47} \]
for \( w, w' \in \partial W^\pm_i \).

We now plug (B.47) into (B.44). Then it remains to show that the second term on the right-hand side of (B.47), under the sum over \( w, w' \) from (B.44), remains bounded. To this

\( \square \)
aim, we let $\delta > 0$, and decompose this sum into two parts: first we consider $w, w'$ that are further than $\delta e^l$ apart from each other: uniformly in $W \in \mathcal{D}_x$ and $x', y' \in W_{\pm}^{\pm, \epsilon} \cup \{\infty\}$, we have

$$
\lim_{l-k \to \infty} \sum_{w, w' \in \partial W^\pm_l \cap \mathcal{D}_x; |w-w'| \geq \delta e^l} \Pi_{\mathbb{R}^2 \backslash \partial W^\pm_l}(e^l x', w) \Pi_{\mathbb{R}^2 \backslash \partial W^\pm_l}(e^l y', w')(a(w - w') - l)
$$

$$
= g \int_{w, w' \in \partial W : |w-w'| \geq \delta} \Pi_{\mathbb{R}^2 \backslash \partial W}(x', dw) \Pi_{\mathbb{R}^2 \backslash \partial W}(y', dw') \log |w-w'|
$$

(B.48)

by (2.6) and by Lemma A.9. Hence, if $l-k$ is larger than a constant, the sum on the left-hand side of (B.48) stays bounded, and this is also the case when $l$ is smaller than a constant by (2.6) and the definition of $\mathcal{D}_x$. The sum over $w, w' \in \partial W^\pm_l$ with $|w-w'| < \delta e^l$ that is complementary to (B.48) can be made arbitrarily small uniformly also in $l-k$ by choosing $\delta > 0$ small enough by Lemma B.17 below.

**Proof of Lemma B.7** Assertion (B.13) follows by the technique from the proof of Proposition B.3. For (B.14), we obtain from the representation (B.44) and monotone convergence that the covariance in the assertion is the limit as $n \to \infty$ of the covariance in Proposition B.3.

**Proof of Lemma B.8** First we show assertion (B.15) when $x, y \in W_{j}^{\pm, \epsilon} \cap (2\epsilon^{-1}B)_j$ and $x$ is adjacent to $y$. Then we have

$$
\mathbb{E}\left[\left(\overline{h_{U_n} \cap V_k^{-} \partial W^+_l}(x) - \overline{h_{U_n} \cap V_k^{-} \partial W^+_l}(y)\right)^2\right] = \sum_{w \in \partial W^+_l} \left(\Pi_{W^+_l}(x, w) - \Pi_{W^+_l}(y, w)\right)
$$

$$
\times \sum_{w' \in \partial W^+_l} \left(\Pi_{W^+_l}(x, w') - \Pi_{W^+_l}(y, w')\right) E\left(h_{U_n \cap V_k^{-}}(w)h_{U_n \cap V_k^{-}}(w')\right).
$$

(B.49)

We set $p = \max\{j : (2\epsilon^{-1}B)_j \subset \Delta_{j+1}\}$. By applying the Gibbs-Markov property successively at $\partial \Delta_1, \ldots, \partial \Delta_{p-1}$ as in Proposition 2.2, and inserting the covariance kernel $G_{\Delta_{j-1} \cap V_k^{-}}$ of a DGFF on $\Delta_{j-1} \cap V_k^{-}$, we obtain

$$
\mathbb{E}\left(h_{U_n \cap V_k^{-}}(w)h_{U_n \cap V_k^{-}}(w')\right) = \sum_{j=0}^{p} \mathbb{E}\left(\varphi_{\Delta_{j-1} \cap V_k^{-}, \Delta_{j} \cap V_k^{-}}(w)\varphi_{\Delta_{j-1} \cap V_k^{-}, \Delta_{j} \cap V_k^{-}}(w')\right)
$$

$$
= \sum_{j=0}^{p} \sum_{z, z' \in \partial \Delta_j} \Pi_{\Delta_j \cap V_k^{-}}(w, z) \Pi_{\Delta_j \cap V_k^{-}}(w', z') G_{\Delta_{j-1} \cap V_k^{-}}(z, z')
$$

(B.50)

for $w, w' \in \partial W^+_l$. The expression on the right-hand side can only increase when we replace the discrete Poisson kernels $\Pi_{\Delta_j \cap V_k^{-}}$ by $\Pi_{\Delta_j}$ as $G_{\Delta_{j-1} \cap V_k^{-}}(z, z')$ is nonnegative and vanishes for $z$ or $z'$ in $\partial V_k^{-}$. Moreover, we have $\sum_{w \in \partial W^+_l} \Pi_{W^+_l}(x, w) \Pi_{\Delta_j}(w, z) = \Pi_{\Delta_j}(x, z)$ for our $x \in W_{j}^{\pm, \epsilon} \cap (2\epsilon^{-1}B)_j$, which follows from the strong Markov property of the random walk underlying the definition of the discrete Poisson kernel. Thus, we obtain from (B.49)
and (B.50) that
\[
\mathbb{E}\left[\left(\frac{h^{U_n \cap V_k^-}}{\partial W_i^\pm}(x) - \frac{h^{U_n \cap V_k^-}}{\partial W_i^\pm}(y)\right)^2\right] \leq \sum_{j=0}^{p} \sum_{z,z' \in \partial \Delta_j} |\Pi_{\Delta_j}(x, z) - \Pi_{\Delta_j}(y, z)| \times |\Pi_{\Delta_j}(x, z') - \Pi_{\Delta_j}(y, z')| G_{\Delta_j \cap V_k^-}(z, z').
\] (B.51)

By Theorem 6.3.8 of [19], the absolute differences of the Poisson kernels in the previous expression are bounded by \(C e^{-\varepsilon(n - j)\Pi_{\Delta_j}(x, z)}\) and \(C e^{-(n - j)\Pi_{\Delta_j}(x, z')}\), respectively. Hence, the right-hand side of (B.51) is bounded by a constant times
\[
\sum_{j=0}^{p} e^{-2(n - j)\varepsilon|\omega(\Delta_j)|} \leq C e^{-2(n - p)} \leq C e^{-2l},
\] (B.52)

where we used that the variances in the last display are bounded by a constant by Proposition B.3. This shows (B.15) for adjacent \(x, y \in W_{l^\pm, \varepsilon} \cap (2e^{-1}B)_l\).

For adjacent \(x, y \in W_{l^\pm, \varepsilon} \cap (e^{-1}B^-)_l\), we note from the maximum principle for harmonic functions that \(\frac{h^{U_n \cap V_k^-}}{\partial W_i^\pm}(x) - \frac{h^{U_n \cap V_k^-}}{\partial W_i^\pm}(y)\) attains its maximum over those \(x, y\) in \(\partial(eB^-)_l\), hence the bound given by (B.51) and (B.52) also holds for all adjacent \(x, y \in W_{l^\pm, \varepsilon}\).

For \(x, y\) as in the assertion, there exists by Lemma B.16 (with \(\varepsilon/4\) in place of \(\varepsilon\)) a path \(x = x_0, x_1, \ldots, x_r = y\) of length \(r \leq C_\varepsilon|x - y|\) within a connected component of \(W_{l^\pm, \varepsilon/2}\), hence
\[
\mathbb{E}\left[\left(\frac{h^{U_n \cap V_k^-}}{\partial W_i^\pm}(x_i) - \frac{h^{U_n \cap V_k^-}}{\partial W_i^\pm}(x_{i-1})\right)^2\right] \leq C_\varepsilon e^{-l}
\] (B.53)

by the assertion for adjacent vertices which we already proved (now with \(\varepsilon/2\) in place of \(\varepsilon\)), and by the triangle inequality for the intrinsic metric. \(\square\)

**Lemma B.16** Let \(\varepsilon \in (0, 1)\). There exists \(C = C_\varepsilon < \infty\) such that for all \(W \in \mathcal{D}_\varepsilon, l \geq 0\), any connected component \(\Gamma\) of \(W_{l^\pm, \varepsilon}\), and all \(x, y \in \{z \in \Gamma : d(x, \Gamma^c) \geq \varepsilon l\}\), there exists a path from \(x\) to \(y\) in \(\Gamma\) of length at most \(C|\varepsilon - y|\).

**Proof** We define \(\Gamma' = \{z \in \Gamma : d(x, \Gamma^c) \geq \varepsilon l\} \cap (3e^{-1}B)_l\). The intersection with \((3e^{-1}B)_l\) ensures that \(e^{-l}\Gamma'\) has bounded diameter. We first bound the maximal length of a shortest path in \(\Gamma\) between two vertices of \(\Gamma'\). To this aim, we observe that the \(\frac{\varepsilon l}{2}\)-balls around points in \(\{z \in \Gamma : d(x, \Gamma^c) \geq \frac{\varepsilon l}{2}\} \cap (3e^{-1}B)_l\) form a finite covering \(C\) of \(\Gamma'\) whose cardinality is bounded by a constant that depends only on \(\varepsilon\), and that \(\cup C \subset \Gamma\). Hence, between any two vertices of \(\Gamma'\), there exists a path that crosses each ball in \(C\) only once and whose length is bounded by a constant times \(e^l\).

If a shortest path in \(\mathbb{Z}^2\) from \(x\) to \(y\) as in the assertion lies in \(\Gamma\), then the assertion holds.

If no shortest path in \(\mathbb{Z}^2\) between \(x, y \in \Gamma'\) lies in \(\Gamma\), then we have \(|x - y| \geq \varepsilon l\) as \(\varepsilon l\)-ball around \(x\) is contained in \(\Gamma\), and the assertion follows as a shortest path has length at most a constant times \(e^l\) by the above.

If \(x, y \in (e^{-1}B^-)_l\), then there always exists a path in \((e^{-1}B^-)_l\) of length \(3|x - y|\) between \(x\) and \(y\), and \((e^{-1}B^-)_l\) is contained in a connected component of \(W_{l^\pm}\) by definition of \(\mathcal{D}_\varepsilon\).

If \(x \notin (3e^{-1}B)_l\) and \(y \in \Gamma' \setminus (e^{-1}B^-)_l\), then a shortest path from \(x\) to \(y\) must go through some \(z \in \partial(2e^{-1}B)_l\). The length of the subpath from \(z\) to \(y\) is bounded by a constant times \(e^l\).
by the above, and the length of the subpath from \( x \) to \( z \) is bounded by \( 3|x-z| \). As moreover

\[ |x-y| \geq 2e^{-1}e^j \]

in this case, and \( |x-y| + 4e^{-1}e^j \geq |x-z| \), the assertion follows also in this case.

\[ \square \]

**Proof of Lemma B.6** The proof is similar to Proposition B.3. We discuss assertion (B.11), the proof of (B.12) is analogous. First we represent the covariance under consideration as

\[ \lim_{n \to \infty} \sum_{w,w' \in \partial W^\pm_j} \Pi_{Z^2 \backslash \partial W} (x,w) \Pi_{Z^2 \backslash \partial W} (y,w') \sum_{z \in \partial U_n} \Pi_{U_n} (w,z) (a(z-w') - l) \]

\[ = g \int_{w,w' \in \partial W: |w-w'| > \delta} \Pi_{Z^2 \backslash \partial W} (x,dw) \Pi_{Z^2 \backslash \partial W} (y,dw') \int_{z \in \partial (e^j U)} \Pi_{e^j U}(w,dz) \log |z-w'| \]

(B.54)

uniformly as in the assertion. The difference of the left-hand side in the previous display and of the first term on the right-hand side of (B.45), as well as the second term there, vanish as \( n-k \to \infty \) followed by \( \delta \to 0 \) by (2.6) and Lemmas A.5(i), B.17. The third term on the right-hand side of (B.45) is treated analogously.

\[ \square \]

It remains to prove the following lemma. We recall the constant \( c_0 \) from (2.6).

**Lemma B.17** Let \( \epsilon \in (0,1) \). Then

\[ \lim_{\delta \to 0} \lim_{l \to \infty} \sum_{w,w' \in \partial W^\pm_j: |w-w'| < 3e^j} \Pi_{Z^2 \backslash \partial W}^\pm (\lfloor xe^j \rfloor, w) \Pi_{Z^2 \backslash \partial W}^\pm (\lfloor ye^j \rfloor, w') (a(w-w') - l) = 0 \]

(B.55)

uniformly in \( W \in D_\epsilon, x, y \in W^\pm, \{ \infty \} \).

**Proof** By plugging in the estimate (2.6) for the potential kernel, we bound the sum in (B.55) in absolute value by

\[ \sum_{j=1}^{j^*} \left( l - j + 1 + c_0 + C_\epsilon e^{-2j} \right) \pi_j \]

(B.56)

where \( j^* := [l + \log \delta] \) and

\[ \pi_j = \sum_{w,w' \in \partial W^\pm_j: e^{j-l} \leq |w-w'| < e^j} \Pi_{Z^2 \backslash \partial W}^\pm (\lfloor xe^j \rfloor, w) \Pi_{Z^2 \backslash \partial W}^\pm (\lfloor ye^j \rfloor, w') \quad \text{for } j \geq 2, \]

(B.57)

\[ \pi_1 = \sum_{w,w' \in \partial W^\pm_j: |w-w'| < e^1} \Pi_{Z^2 \backslash \partial W}^\pm (\lfloor xe^j \rfloor, w) \Pi_{Z^2 \backslash \partial W}^\pm (\lfloor ye^j \rfloor, w') \].

(B.58)

The assertion of the lemma is implied by the claim that \( \pi_j \leq C e^{-c(l-j)} \) for \( j \leq j^* \) and constants \( c, C \in (0,\infty) \) that depend only on \( \epsilon \). To prove the claim, we first observe that

\[ \pi_j \leq \max_{z \in W^\pm, \delta} P_z \left( d(S_{\partial W^+\delta}, w) \leq e^j \right). \]

(B.59)
For \( i \geq 2 \) we consider the annuli \( \hat{A}_i := (w + B(e^i)) \setminus (w + B(e^{-i})) \) where \( B(r) \subset \mathbb{Z}^2 \) denotes the Euclidean ball of radius \( r \) around 0. Let \( E_i \) be the event that the path of a simple random walk started in \( \partial B(e_i^{1/2}) \) separates the inner and the outer boundary of \( \hat{A}_i \) before it hits \( \partial \hat{A}_i \). The probability of \( E_i \) is uniformly in \( i \geq 2 \) bounded from below by a constant \( p > 0 \) that depends only on \( \epsilon \), this follows from Donsker’s invariance principle for sufficiently large \( i \) and as the minimum over the smaller annuli is finite. Let \( i^* = [\log(\epsilon/2) + 1] \). Within any annulus \( \hat{A}_i \) for \( i = 2, \ldots, i^* \), every closed path that separates the inner and the outer boundary of \( \hat{A}_i \) has a nonempty intersection with \( \partial W_i^{\pm} \) as every connected component of \( \partial W \) has Euclidean diameter at least \( \epsilon \) and as we use the discretization (1.4). As the starting point \( z \) of \( S \) has distance at least \( e^{i-1} - 1 \) from \( w \), the event in the probability on the right-hand side of (B.59) can only occur on \( \bigcap_{i=0}^{i^*} E_i^c \). Using successively the strong Markov property at \( \partial B(e_i^{-1/2}) \) for \( i = j + 2, \ldots, i^* \), we thus obtain the geometric bound \( \pi_j \leq (1 - p)^{i^*-j-1} \). The claim that \( \pi_j \leq C(1 - p)^{j-2} \) for all \( j \leq i^* \) follows as \( l - i^* \) is bounded by a constant.

### B.3.2 Fluctuation

**Proof of Proposition B.4** The oscillation is defined as the absolute maximum of the difference field. To center this Gaussian field, we bound its conditional expectation by

\[
\left| \mathbb{E}_{k,v}^{n,u} \left( \bar{h}_{\partial W_i}^{\pm}(y) - \bar{h}_{\partial W_i}^{\pm}(z) \big| \bar{h}_{\partial W_i}^{\pm}(x) = -m_l + w \right) \right|
\]

\[
\leq \frac{\text{Cov}_k^{n,u}(\bar{h}_{\partial W_i}^{\pm}(x), \bar{h}_{\partial W_i}^{\pm}(y) - \bar{h}_{\partial W_i}^{\pm}(z))}{\text{Var}_k^{n,u}(\bar{h}_{\partial W_i}^{\pm}(x))} \times \left[ |w| + |m_l + \hat{m}_l| + |\mu(0)|(l-k) + |\mu(\infty)|(n-l) \right] \frac{1}{n-k}
\]

\[+ 2 \text{osc}_W \mu + C\varepsilon(1 + |\mu(0)| + |\mu(\infty)|)(n-k)^{-1}\]

(B.60)

for \( y, z \in W_i^{\pm,e} \), this bound follows from the usual expression for Gaussian conditional expectations where we plugged in Proposition B.2 for \( \mathbb{E}_{k,v}^{n,u} \bar{h}_{\partial W_i}^{\pm}(x) \).

First we consider the oscillation in each connected component \( \Gamma \) of \( W_i^{e} \) separately. By Proposition B.3, the covariance in the numerator on the right-hand side of (B.60) is bounded in absolute value by a constant, and the variance in the numerator is bounded from below by a constant times \( (n-k) \). Using also Lemma B.1, we further bound (B.60) by a constant times \( \mu \) for \( y, z \in \Gamma \). Now we apply Fernique majorization for the oscillation (Theorem 6.14 of [3] with \( R = \infty \)) to the centered Gaussian field \( \bar{h}_{\partial W_i}^{\pm}(y) - \mathbb{E}_{k,v}^{n,u}(\bar{h}_{\partial W_i}^{\pm}(y) | \bar{h}_{\partial W_i}^{\pm}(x) = -m_l + w), y \in \Gamma \) with the uniform probability measure on \( \Gamma \) as the majorizing measure, so as to see that \( \mathbb{E}_{k,v}^{n,u}(\text{osc}_W \bar{h}_{\partial W_i}^{\pm} | \bar{h}_{\partial W_i}^{\pm}(x) = -m_l + w) \) is bounded by a constant. To this aim, we recall from theory of the multivariate Gaussian distribution that the intrinsic metric (which can be seen as a variance) under the conditioning on \( \bar{h}_{\partial W_i}^{\pm}(x) \) is bounded by the unconditional intrinsic metric, which in turn is bounded by Lemma B.8. To bound the conditional tail probability (i.e. to show (B.7) but still with \( \Gamma \) instead of \( W_i^{\pm,e} \), we use the Borell-TIS inequality, now using Lemma B.8 as a bound for the variance of the difference field (and for its conditional variance which can only be smaller than the unconditional variance).

For the connected component \( W_i^{e,-} \), we apply Fernique majorization and the Borell-TIS inequality as above but for \( y, z \in W_i^{e,-} \) with \( |y|, |z| \leq e^{l+2} \text{diam} W_i \) and note that \( \bar{h}_{\partial W_i}^{\pm}(y) - \bar{h}_{\partial W_i}^{\pm}(z) \) assumes its maximum (and its minumum) over \( (W_i^{e,-})^2 \cup \{\infty\} \) at some
(y, z) with \(|y|, |z| \leq e^{d+2} \text{diam} W\), which follows from the maximum principle for bounded harmonic functions, applied to y and z separately.

It remains to control the oscillations between the connected components of \(W_I^{\pm,\epsilon} \cup \{\infty\}\).

The number of these connected components is equal to the number of connected components of \(W_I^{\pm,\epsilon} \cap (e^{-1} B)_I\), which in turn is bounded by \(16e^{-4}\) as \(|(e^{-1} B)_I| \leq 4\epsilon^{-2} e^{2l}\), and as \(W_I^{\pm,\epsilon} \cap (e^{-1} B)_I\) contains balls of radius \(\epsilon e^l\) that are centered in each of the connected components of \(W_I^{\pm,\epsilon} \cap (e^{-1} B)_I\) and do not intersect. Hence, it suffices to show that \(\mathbb{E}^{n,u}_{k,v}(\hat{h}_{\partial W}^\pm(y) − \hat{h}_{\partial W}^\pm(z)) > t + \mu |\hat{h}_{\partial W}^\pm(x)| = -m_l + w\) has uniformly Gaussian tails for \(y, z \in W_I^{\pm,\epsilon} \cup \{\infty\}\). To this aim, it suffices by (B.60) to bound

\[
\text{Var}_k^u\left(\hat{h}_{\partial W}^\pm(y) − \hat{h}_{\partial W}^\pm(z) \bigg| \hat{h}_{\partial W}^\pm(x) = -m_l + w\right) \leq \text{Var}_k^u\left(\hat{h}_{\partial W}^\pm(y) − \hat{h}_{\partial W}^\pm(z) \bigg| \hat{h}_{\partial W}^\pm(x) \right) + 2 \text{Cov}_k^u\left(\hat{h}_{\partial W}^\pm(y), \hat{h}_{\partial W}^\pm(z) \bigg| \hat{h}_{\partial W}^\pm(x) \right),
\]

which in turn is bounded by a constant Proposition B.3.

\[\square\]

**Proof of Proposition B.5** This is proved analogously to Proposition B.4. Let \(\mathbb{E}^{n,u}, \text{Var}^n\) and \(\text{Cov}^n\) denote the expectation, variance and covariance associated with \(\mathbb{E}^{n,u}\). For \(y, z \in W_I^{\pm,\epsilon}\), we consider the Gaussian conditional expectation

\[
\left| \mathbb{E}^{n,u}(\hat{h}_{\partial W}^\pm(y) − \hat{h}_{\partial W}^\pm(z) \bigg| \hat{h}_{\partial W}^\pm(x) = -m_l + w) \right| \leq \frac{|\text{Cov}^u(\hat{h}_{\partial W}^\pm(x), \hat{h}_{\partial W}^\pm(y) − \hat{h}_{\partial W}^\pm(z)) |}{\text{Var}^u(\hat{h}_{\partial W}^\pm(z))} \leq \frac{C \epsilon^{2d}}{\epsilon^{2d}} \epsilon^2\]

where we have

\[\mathbb{E}^{n,u}\hat{h}_{\partial W}^\pm(x) = -m_n + \hat{\mu}(x). \]

**Lemma B.7** bounds the covariance in the numerator in (B.62) by a constant and the variance in the denominator by a constant times \(n − l\). We use (B.62) to center the difference field that appears in the definition of the oscillation. In each connected component \(\Gamma\) of \(W_I \cap (2e^{-1} B)_I\), we then apply the Fernique and Borell-TIS inequalities as before, using the estimate

\[\mathbb{E}\left[\left(\bar{h}_{\partial W}^{U_n}(y') − \bar{h}_{\partial W}^{U_n}(z')\right)^2\right] \leq C \epsilon |y' − z'|^2 \]

for \(y', z' \in \Gamma\), which is a straightforward analog of Lemma B.8. The oscillation between different connected components is estimated analogously to (B.61) using Lemma B.7.

\[\square\]

**B.3.3 Proofs of Binding Field Estimates**

**Proof of Lemma B.11** We show assertions (B.16) and (B.17), the proofs of the other assertions are analogous. We write \(\varphi = \bar{h}_{\partial W}^{U_n}(\partial U \cap \bar{U})\). Let \(\eta \in \left(\text{diam}(U, \bar{U}), \epsilon/2\right)\), where we assume w.l.o.g. that \(\text{diam}(U, \bar{U}) < \epsilon/2\) and \(n \geq r − \log \epsilon\). Then we have \(U \cap \bar{U} \supset B^{\eta,\epsilon}\) and \(B^{\eta,\epsilon}_n \subset \bar{B}^{\eta,\epsilon}_{n−r}\). For \(x \in (U \cap \bar{U}) \setminus \bar{B}^{\eta,\epsilon}_{n−r}\), we bound by the Gibbs-Markov property (Lemma B.10) and monotonicity of the Green function in its domain

\[
\text{Var}_n(x) \leq \text{Var}_n(U_n(\partial U \cap \bar{U})), \quad \text{Var}_n(U_n(x, x) − G(U \cap \bar{U}))(x, x) \leq G(U_n(x, x) − G(U_n)(x, x))
\]

\[
\leq g \sum_{z \in \partial U_n} \Pi_{U_n}(x, z) \log |e^{-n}(x − z)| − g \sum_{z \in \partial U_n^\eta} \Pi_{U_n^\eta}(x, z) \log |e^{-n}(x − z)| + o_{\epsilon, \eta}(1)
\]
where we used (2.6), (2.7) and Lemma A.9. The difference of the integrals in the last line can be bounded by $\rho(\eta)$ by Lemma A.10.

For $x, y \in (U \cap \tilde{U})_n^\epsilon \cap B_{n-r}^-$, we also bound the (squared) intrinsic metric

$$\mathbb{E}\left((\varphi(x) - \varphi(y))^2\right) \leq \mathbb{E}\left((\varphi_{U_n \cap B_0^-, \tilde{U}_n^\epsilon \cap B_0^-}(x) - \varphi_{U_n \cap B_0^-, \tilde{U}_n^\epsilon \cap B_0^-}(y))^2\right).$$

We use (2.6), (2.7) and Lemma A.9. The difference of the integrals in the last line can be bounded by $\rho(\eta)$.

For $x, y$ are adjacent, then the absolute differences in the last display are bounded by $C_{\epsilon, r} e^{-n} \Pi_{U_n}(x, z)$ and $C_{\epsilon, r} e^{-n} \Pi_{U_n^\epsilon}(x, z')$, respectively, by Theorem 6.3.8 of [19], so that we obtain

$$\mathbb{E}\left((\varphi(x) - \varphi(y))^2\right) \leq C_{\epsilon, m} e^{-2n} \mathbb{V} \ar \varphi_{U_n \cap B_0^-, \tilde{U}_n^\epsilon \cap B_0^-}(x) \leq C_{\epsilon, r} e^{-2n} \rho(\eta),$$

where we used the bound for the variance from before. Analogously to the proof of Lemma B.8, we also obtain

$$\mathbb{E}\left((\varphi(x) - \varphi(y))^2\right) \leq C_{\epsilon, m} |x - y|^2 e^{-2n} \mathbb{V} \ar \varphi_{U_n \cap B_0^-, \tilde{U}_n^\epsilon \cap B_0^-}(x) \leq C_{\epsilon, r} |x - y|^2 e^{-2n} \rho(\eta)$$

for all $x, y \in (U \cap \tilde{U})_n^\epsilon \cap B_{n-r}^-$. Using (B.68), we can now apply the Fernique inequality as in the proof of Proposition B.4 which yields assertion (B.16). From this, the bound for the variance, and the Borell-TIS inequality, we then obtain (B.17). \Box

**Proof of Lemma B.12** We show (B.20) and (B.21), the proof of (B.22) and (B.23) is analogous. W.l.o.g. we assume that $B_{n-r-1}^- \subset V_k^{-\epsilon}$. For $x, y \in U_n^\epsilon \setminus B_{n-r}^-$, we estimate the (squared) intrinsic metric

$$\mathbb{E}\left((h_{U_n \cap V_k^-}(x, z) - h_{U_n \cap V_k^-}(y, z))^2\right) = \sum_{z, z' \in \partial V_k^-} [\Pi_{U_n \cap V_k^-}(x, z) - \Pi_{U_n \cap V_k^-}(y, z')] \times [\Pi_{U_n \cap V_k^-}(z, z') - \Pi_{U_n \cap V_k^-}(y, z')] \mathbb{E}\left[h_{U_n \cap V_k^-}(z) h_{U_n \cap V_k^-}(z')\right].$$

Restricting to the case that $x, y$ are adjacent, we obtain from Theorem 6.3.8 of [19] that

$$\left|\Pi_{U_n \cap V_k^-}(x, z) - \Pi_{U_n \cap V_k^-}(y, z)\right| \leq C_{\epsilon, r} e^{-n} \Pi_{U_n \cap V_k^-}(x, z)$$

where $z \in \partial V_k^-$. We define $\Delta^\zeta$ from $U, V, n, k$ and $\eta = \zeta = 0$ as in Sect. 2.6 and assume furthermore that $B_{n-r}^- \subset \Delta^\zeta$. Then, by the Markov property of the random walk underlying the definition of the discrete Poisson kernel, we estimate

$$\Pi_{U_n \cap V_k^-}(x, z) = \sum_{w \in \partial \Delta^\zeta} \Pi_{U_n \cap \Delta^\zeta}(x, w) \Pi_{U_n \cap V_k^-}(w, z)$$
\[ \leq \frac{C_{r,\varepsilon}}{n-k} \sum_{w \in \partial \Delta_i^0} \Pi_{\Delta_i^0} (\infty, w) \Pi_{V_k^-} (w, z) = \frac{C_{r,\varepsilon}}{n-k} \Pi_{V_k^-} (\infty, z), \] (B.71)

for the inequality, we used Lemma A.7 for \( \Pi_{U_n \cap \Delta_i^0} \), and in \( \Pi_{U_n \cap V_k^-} \), we used that \( z \in \partial V_k^- \) and left out the absorbing boundary at \( \partial U_n \). Plugging these estimates into (B.69) yields

\[
\mathbb{E} \left[ \left( h_{U_n \cap \overline{V}_0^-}^{-1} \partial U_n \cup \partial V_k^- (x) - h_{U_n \cap \overline{V}_0^-}^{-1} \partial U_n \cap \partial V_k^- (y) \right)^2 \right] \\
\leq C_{\varepsilon} e^{-2n(n-k)^{-2}} \sum_{z, z' \in \partial V_k^-} \Pi_{\overline{V}_0^-} (\infty, z) \Pi_{\overline{V}_0^-} (\infty, z') G_{U_n \cap \overline{V}_0^-} (z, z') \\
\leq C_{\varepsilon} e^{-2(n-k)^{-2}} \wedge_{n,k,0},
\] (B.72)

where we used Proposition B.3 in the last inequality (resp. Lemma B.7 if we replace \( \overline{V}_0^- \) with \( \mathbb{Z}^2 \)). Analogously to the proof of Lemma B.8 we also obtain

\[
\mathbb{E} \left[ \left( h_{U_n \cap \overline{V}_0^-}^{-1} \partial U_n \cup \partial V_k^- (x) - h_{U_n \cap \overline{V}_0^-}^{-1} \partial U_n \cap \partial V_k^- (y) \right)^2 \right] \leq C_{\varepsilon} |x - y|^2 e^{-2n(n-k)^{-2}} \wedge_{n,k,0}
\] (B.73)

for all \( x, y \in U_n^\varepsilon \setminus B_{n-r} \). We now apply the Fernique inequality as in the proof of Proposition B.4 which yields (B.20) when we replace in the maximum there \( U_n \) with \( U_n^\varepsilon \), which does not decrease the maximum by harmonicity of the binding field. Assertion (B.21) follows from the Borell-TIS inequality by using (B.20) and Lemma B.13.

**Proof of Lemma B.13** We show (B.24), the proof of (B.25) is analogous. By harmonicity, we write the covariance in (B.24) as

\[
\sum_{z, z' \in \partial V_k^-} \Pi_{U_n \cap V_k^-} (x, z) \Pi_{U_n \cap V_k^-} (y, z') \text{Cov} \left( h_{U_n}(z), h_{U_n}(z') \right).
\] (B.74)

We define \( T = |n - k| + [\log \varepsilon] - [\log (\varepsilon^{-1} + \zeta)] \) and \( \Delta_i^0 \) in terms of \( U, V, n, k, \) and \( \eta = \zeta = \varepsilon \) as in Sect. 2.6. W.l.o.g. we assume that \( x, y \in \Delta_i^0 \). For \( z \in \partial V_k^- \),

\[
\Pi_{U_n \cap V_k^-} (x, z) \leq \sum_{w \in \partial \Delta_i^0} \Pi_{\Delta_i^0 \cap U_n} (x, w) \Pi_{\Delta_i^0} (w, z),
\] (B.75)

where we used the strong Markov property of the random walk that underlies the definition of the discrete Poisson kernel, and we removed an absorbing boundary. By Lemma A.7,

\[
\Pi_{\Delta_i^0 \cap U_n} (x, w) \leq C_{\varepsilon} \frac{T + 1 - p}{T} \Pi_{\Delta_i^0} (\infty, w)
\] (B.76)

for all \( w \in \partial \Delta_i^0 \) when \( x \in A_i^0 \), corresponding to \( k + p - 1 - \log \varepsilon \leq \log |x| \leq k + p - \log \varepsilon \).

Inserting this back into (B.75), we further bound (B.74) by

\[
C_{\varepsilon} \frac{(T + 1 - p)^2}{T^2} \text{Var} \left( h_{U_n \cap \overline{V}_k^-}(\infty) \right).
\] (B.77)

The variance in the last display is bounded by \( g T + C_{\varepsilon} \) by Lemma B.7.

\[
\square
\]
C Ballot Theorems for Decorated Random Walks

C.1 Statements

The first result concerning such walks is a uniform ballot-type upper bound.

**Theorem C.1** (Upper bound) Fix $\delta \in (0, 1/3)$. There exists $C = C_\delta < \infty$ such that

$$
\mathbb{P}
\left(
\max_{k=1}^{\lfloor T - \delta^{-1}\rfloor} (S_k + D_k) \leq 0
\right)
\leq C \frac{(a^- + 1)(b^- + 1)}{T},
$$

for all $T \in \mathbb{N}$, $a \in \mathbb{R}$, $b < -T^\delta$ and all $(S_k)_{k=0}^T$ satisfying (A1), (A2) and (A3).

The inequality holds without any restriction on $b$, if we replace $(b^- + 1)$ by $(b^- + T^{\delta})$ on the r.h.s.

An analogous lower bound is given by the following theorem.

**Theorem C.2** (Lower bound) Fix $\delta \in (0, 1/3)$. There exist $C = C_\delta > 0$ and $a_0 = a_{0, \delta} > -\infty$, such that

$$
\mathbb{P}
\left(
\max_{k=1}^{\lfloor T \rfloor} (S_k + D_k) \leq 0
\right)
\geq C \frac{(a^- + 1)b^-}{T},
$$

for all $T \in \mathbb{N}$, $a \leq a_0$, $b < -T^\delta$, $(a^- + 1)b^- \leq T^{1-\delta}$ and all $(S_k)_{k=0}^T$, $(D_k)_{k=1}^T$ satisfying Assumptions (A1), (A2) and (A3).

We now turn to asymptotics. For $1 \leq r \leq T \leq \infty$ and $r < \infty$, set

$$
\ell_{T, r}(a, b) := \mathbb{E}
\left(
S_{r}; \max_{k=1}^{\lfloor T \rfloor} (S_k + D_k) \leq 0
\right)
$$

and observe that this quantity does not depend on $b$ when $T = \infty$. Then,

**Theorem C.3** (Asymptotics) Fix $\delta \in (0, 1/3)$ and let $\ell_{T, r}(a, b)$ be defined via

$$
\mathbb{P}
\left(
\max_{k=1}^{\lfloor T \rfloor} (S_k + D_k) \leq 0
\right) = 2 \frac{\ell_{T, r}(a, b)b^-}{s_T} + e_{T, r}(a, b).
$$

Then,

$$
\frac{T}{(a^- + 1)b^-} e_{T, r}(a, b) \to 0 \text{ as } r \to \infty,
$$

uniformly in $r^{4/\delta} \leq T \leq \infty$, $a \leq \delta^{-1}$, $b < -T^\delta$, $(a^- + 1)b^- \leq T^{1-\delta}$ and all $(S_k)_{k=0}^T$, $(D_k)_{k=1}^T$ satisfying Assumptions (A1), (A2) and (A3).

Next we state some properties of $\ell_{T, r}(a, b)$. The first property shows that $\ell_{T, r}(a, b)$ is Cauchy in $r$ and consequently that one can extend the definition of $\ell_{T, r}(a, b)$ to the case when $r = \infty$.

**Proposition C.4** Fix $\delta \in (0, 1/3)$. Then,

$$
\lim_{r, r' \to \infty} \lim_{T' \to \infty} \sup_{T \in [T', \infty]} \left| \frac{\ell_{T, r}(a, b) - \ell_{T, r'}(a, b)}{(a^- + 1)} \right| = 0,
$$

(\text{C.6})
The limits above hold uniformly in \( a \leq \delta^{-1} \), \( b < \delta^{-1} \), \(|b - a|/(a^{-1} + 1) \leq T^{1-\delta}\) and all \((S_k^T)_{k=0}^T, (D_k^T)_{k=1}^∞\) satisfying Assumptions (A1), (A2) and (A3). In particular, the limit

\[
\ell_{∞,∞}(a, b) := \lim_{r → ∞} \ell_{∞,r}(a, b),
\]

holds uniformly in all \( a \leq \delta^{-1} \), \( b < \delta^{-1} \) and all \((S_k^∞)_{k=0}^∞, (D_k^∞)_{k=1}^∞\) satisfying Assumptions (A1), (A2) and (A3).

We again remark that by definition \( \ell_{∞,∞}(a, b) \) does not depend on \( b \), but we leave the dependence on this parameter in the notation to allow a uniform treatment of all cases. Next we treat the asymptotics of \( \ell_{T,r}(a, b) \) in \( a^{-} \) for all \( T \) and \( r \).

**Proposition C.5** Fix \( \delta \in (0, 1/3) \). Then for all \( 1 \leq r \leq ∞ \),

\[
\lim_{a→−∞} \frac{\ell_{T,r}(a, b)}{a^{-}} = 1,
\]

uniformly in \( r \leq T ≤ ∞ \), \( b < \delta^{-1} \), \(|b - a|r/T \leq (a^{-1} + 1)^{1-δ} \) and all \((S_k^T)_{k=0}^T, (D_k^T)_{k=1}^∞\) satisfying Assumptions (A1), (A2) and (A3). Furthermore, there exists \((T(r))_{r=1}^∞ ∈ N^∞\) such that the above convergence is also uniform in \( r ≥ 1 \), provided that \( T(r) ≤ T ≤ ∞ \).

We now turn to the question of continuity of \( \ell_{T,r}(a, b) \) in the quantities through which it is defined. The first statement is a simple consequence of the Cauchy-Schwarz Inequality.

**Lemma C.6** Fix \( \delta \in (0, 1/3) \). Then, for some \( C = C_δ < ∞ \),

\[
|E(\bar{S}_r; \max_k(S_k + D_k) ≤ λ) − E(\bar{S}_r; \max_k(S_k + D_k) ≤ −λ)| ≤ C(a^{-1} + 1)r \left(\max_k P(S_k + D_k ∈ (−λ, λ)]\right)^{1/2},
\]

for all \( 1 \leq r \leq T ≤ ∞ \), \( b < \delta^{-1} \), \(|b - a|r/T ≤ (a^{-1} + 1)^{1-δ} \) and all \((S_k^T)_{k=0}^T, (D_k^T)_{k=1}^∞\) satisfying Assumptions (A1), (A2) and (A3).

In the second continuity statement we treat the setting of a triangular array. We therefore assume that for each \( i ∈ N ∪ \{∞\} \), there are defined \( a^{(i)}, b^{(i)} ∈ R \), \( 1 ≤ r^{(i)} ≤ T^{(i)} ≤ ∞ \) and random variables \((S_k^{(i)})_{k=0}^{T^{(i)}}, (D_k^{(i)})_{k=1}^{T^{(i)}}\) taking the roles of \( a, b, T \) and \((S_k^T)_{k=0}^T, (D_k^T)_{k=1}^∞\) from before. We also denote by \( P^{(i)} \), \( E^{(i)} \) the underlying probability measure and expectation and let \( \ell^{(i)}(r^{(i)})(a^{(i)}), b^{(i)}) \) be defined as in (C.3) only with respect to \((S_k^{(i)})_{k=0}^{T^{(i)}}, (D_k^{(i)})_{k=1}^{T^{(i)}}\).

**Proposition C.7** (Triangular array) Fix \( \delta \in (0, 1/3) \) and suppose that for all \( 1 \leq i ≤ ∞ \), \((S_k^{(i)})_{k=0}^{T^{(i)}}, (D_k^{(i)})_{k=1}^{T^{(i)}}\) satisfy Assumptions (A1), (A2) and (A3) with \( a^{(i)}, b^{(i)} \) and \( T^{(i)} \).

Suppose also that in the sense of weak convergence of finite dimensional distributions we have

\[
\left((S_k^{(i)})_{k=0}^{T^{(i)}}, (D_k^{(i)})_{k=1}^{T^{(i)}}, T^{(i)}, r^{(i)}, a^{(i)}, b^{(i)}\right) \quad \xrightarrow{i→∞} \quad \left((S_k^{(∞)})_{k=0}^{T^{(∞)}}, (D_k^{(∞)})_{k=1}^{T^{(∞)}}, T^{(∞)}, r^{(∞)}, a^{(∞)}, b^{(∞)}\right),
\]

and that 0 is a stochastic continuity point of the law of \( S_k^{(∞)} + D_k^{(∞)} \) under \( P^{(∞)} \), for all \( k = 1, \ldots, r^{(∞)} \). Lastly assume also that if \( r^{(∞)} = ∞ \) then \( r^{(i)} ≤ r^{(i)} \) and that \( a^{(i)} < \delta^{-1} \),
\[ b^{(i)} < \delta^{-1}, |b^{(i)} - a^{(i)}|/(a^{(i)} - 1) \leq (T^{(i)})^{1-\delta} \text{ for all } i \text{ large enough. Then,} \]
\[
\lim_{i \to \infty} \ell^{(i)}_{r^{(i)}}(a^{(i)}) = \ell^{(\infty)}_{r^{(\infty)}}(a^{(\infty)}) . \quad (C.11)
\]

### C.2 Proofs

A key step in the proofs is a reduction to the case where there are no decorations. This will come at the cost of adding/subtracting a deterministic curve from the barrier. We shall then rely on the following barrier probability estimates, which involve only the process \((S_k)_k\) and \((\bar{S}_k)_k\).

#### Proposition C.8

Fix \( \delta \in (0, 1/3) \). There exist non-increasing functions \( \bar{f}, f : \mathbb{R} \to (0, \infty) \) satisfying \( \bar{f}(a) \sim f(a) \sim a^{-\delta} \text{ as } a \to -\infty \text{ such that for all } T < \infty, a, b, a', b' \in \mathbb{R} \text{ with } |a' - b'| < T^{1/2-\delta} \) and \((S_k)_k\) satisfying Assumption (A1),

\[
P\left( \max_{k=0}^T (S_k - \delta^{-1} \wedge_{T,k} 1/2 - a' \mathbb{1}_{[0,T/2]}(k) - b' \mathbb{1}_{[T/2,T]}(k)) \leq 0 \right) \leq (2 + o(1)) \frac{\bar{f}(a - a') \bar{f}(b - b')}{s_T} . \quad (C.12)
\]

If in addition \( (a - a')^{-1} + 1) \leq T^{1-\delta} \) then also,

\[
P\left( \max_{k=0}^T (S_k + \delta^{-1} \wedge_{T,k} 1/2 - a' \mathbb{1}_{[0,T/2]}(k) - b' \mathbb{1}_{[T/2,T]}(k)) \leq 0 \right) \leq (2 + o(1)) \frac{f(a - a')f(b - b')}{s_T} . \quad (C.13)
\]

Both \( o(1) \) terms above depend only on \( \delta \) and tend to 0 as \( T \to \infty. \)

To carry through the reduction to the case where there are no decorations, we need to control the growth of the latter as well as the growth of the steps of the walk. Setting \( \xi_k := S_k - \bar{S}_k-1 \), we then introduce the “control variable”

\[
R := \min \left\{ \tau \geq 1 : \max_{k=1}^T \left( (|D_k| - \delta^{-1} \wedge_{T,k} 1/2 - \delta) \vee |\xi_k| - (k \vee \tau)^{\delta/2} \right) \leq 0 \right\} . \quad (C.14)
\]

The first lemma shows that \( R \) cannot be large, even conditional on the ballot event.

#### Lemma C.9

Fix \( \delta \in (0, 1/3) \). There exists \( C = C_\delta < \infty \) such that for all \( 1 \leq r < T < \infty \) and \( a \leq \delta^{-1}, b \leq -\delta^{\delta} \).

\[
P\left( \max_{k=1}^{[T-\delta^{-1}]} (S_k + D_k) \leq 0, R \geq r \right) \leq C \frac{(a^{-1} + 1)(b^{-1} + 1)}{T} e^{-r^2/\delta} . \quad (C.15)
\]

**Proof** Without loss of generality we can assume that \( T \) is large enough. For any \( 1 \leq r < T/2 \), setting \( r' := [r \vee \delta^{-1}] \), the event

\[
\left\{ \max_{k=1}^{[T-\delta^{-1}]} (S_k + D_k) \leq 0, R = r \right\} \cap \left\{ S_T = b \right\} \quad (C.16)
\]

is included in the intersection of
\[ \exists k \in \{1, \ldots, r - 1\}: \left( |D_k| - \delta^{-1} \wedge_{T,k}^{1/2-\delta} \right) \lor |\xi_k| > (r - 1)^{\delta/2} \] \quad \{S_{r'} \geq a - r'^2\} \quad (C.17)

and

\[ \left\{ \max_{k=r'}^T \left( S_k - 2\delta^{-1}k^{1/2-\delta}1_{r',T/2}(k) - (\delta^{-1}(T - k)^{1/2-\delta} + T^{\delta/2})1_{T/2,T}(k) \right) \leq 0 \right\}. \quad (C.18) \]

Above we have used that (C.16) implies that \( S_k + D_k \leq b + \delta^{-2} + (1 + \delta^{-1})T^{\delta/2} \leq 0 \) for all \( k = [T - \delta^{-1}], \ldots, T \), whenever \( T \) is large enough, in light of the restrictions on \( b \).

By Assumptions (A1) and (A3) and the union bound, the probability of the left event in (C.17) is bounded above by \( Ce^{-r^2/3} \). On the other hand, we observe that

\[ 2\delta^{-1}k^{1/2-\delta}1_{r',T/2}(k) + (\delta^{-1}(T - k)^{1/2-\delta} + T^{\delta/2})1_{T/2,T}(k) \]

is at most

\[ 2\delta^{-1}(r'^{1/2-\delta} + (k - r'^{1/2-\delta})1_{1,(T-r')/2}(k - r') + (2\delta^{-1}((T - r') - (k - r'))^{1/2-\delta} + T^{\delta/2})1_{(T-r')/2,T-r'}(k - r'). \quad (C.19) \]

Setting \( T' := T - r' \) and using stochastic monotonicity, conditional on the second event in (C.17), the \( P \)-probability of (C.18) is at most the \( P^{T',b}_{0,a-r'^2} \) probability of

\[ \left\{ \max_{k=0}^T \left( S_k' - 2\delta^{-1} \wedge_{T',k}^{1/2-\delta} - 2\delta^{-1}r'^{1/2-\delta}1_{0,T'/2}(k) - T^{\delta/2}1_{T'/2,T'}(k) \right) \leq 0 \right\}. \quad (C.20) \]

where the law of \( (S_k')_{k=0}^T \) under \( P^{T',b}_{0,a-r'^2} \) is that of \( (S_{k+r})_{k=0}^T \) under \( P(\cdot | S_{r'} = a - r'^2) \). Thanks to Assumption (A2) and Proposition C.8 the last probability is at most

\[ C \frac{(a - r'^2 - r'^{1/2-\delta})^-(b - T^{\delta/2})^-}{(T - r')} \leq C \frac{(a^- + 1)(b^- + 1)}{T} (r^2 + 1). \quad (C.21) \]

whenever \( b^- \geq T^\delta \) and \( a \leq \delta^{-1} \).

Invoking the union bound and the product rule and combining the above estimates, we get that the probability of (C.16) is bounded above by

\[ C \frac{(a^- + 1)(b^- + 1)}{T} e^{-r^2/4}. \quad (C.22) \]

The same bound applies also when \( r \geq T/2 \), in which case we only take the first event in (C.17) as the one including (C.16). Summing the right-hand side in the last display from \( r \) to \( \infty \) we obtain (C.15).

As immediate consequences, we get proofs for Theorem C.1 and Theorem C.2.

**Proof of Theorem C.1** Using \( r = 1 \) in Lemma C.9 give the desired bound when \( b \leq -T^\delta \). If \( b \) is not that low, we let \( S_k' := S_k - (b + T^\delta)k/T \) for all \( k = 0, \ldots, T \). Then \( (S_k')_{k=0}^T, (D_k')_{k=1}^T \) satisfies Assumptions (A1), (A2) and (A3) with the same parameters, except that \( b \) is now \(-T^{-\delta}\). Therefore the same argument as in the first part of the proof gives \( C(a^- + 1)(T^{\delta} + 1)/T \) as an upper bound on the left-hand side in (C.1) only with \( S_k' \) in place of \( S_k \). Since \( S_k \geq S_k' \) for all \( k \), the original left-hand side is even smaller. It remains to observe that the upper bound \( C'(a^- + 1)(b^- + T^\delta)/T \) applies in both cases.

\( \Box \)
Proof of Theorem C.2 For any $1 \leq r \leq T$, the desired probability is bounded below by
\[
P \left( \max_{k=1}^{T} \left( S_k + \frac{\delta^{-1}}{1-\delta} + (k \vee r)^{\delta/2} \right) \leq 0, \ R \leq r \right) \geq \mathbb{P} \left( \max_{k=1}^{T} \left( S_k + 2\delta^{-1} \frac{1}{1-\delta} + r^{\delta/2} 1_{[1,T/2]}(k) + T^{\delta/2} 1_{[T/2,T]}(k) \right) \leq 0 \right) - \mathbb{P} \left( \max_{k=1}^{T} S_k \leq 0, \ R > r \right) . \tag{C.23}
\]
Then from Proposition C.8, for any $r$ the first term on the right-hand side is at least $C(a^- - r^{\delta/2})(b^- - T^{\delta/2})/T \geq C_0a^- b^- / T$ for some $C_0 = C_{0,\delta} > 0$ whenever $a^- > 2r^{\delta/2}$ and under the restrictions in the theorem.

At the same time, by assuming $D_k = 0$ for all $k$, it follows from Lemma C.9 that the second term on the right-hand side of (C.23) can be made at most $(C_0/2)a^- b^- / T$ by choosing $r$ large enough. Combined this gives the desired lower bound for all $a < a_0$ for some $a_0 = a_{0,\delta} > -\infty$. \hfill \Box

Our next task is to derive asymptotics, but to this end, we shall first need several preliminary results. The first one shows that under the ballot event the random walk is repelled. In all lemmas in the remaining part of this section, we assume (A1), (A2) and (A3).

**Lemma C.10** Fix $\delta \in (0, 1/3)$. There exists $C = C_\delta < \infty$ such that for all $0 \leq r \leq T/2$, $a \leq \delta^{-1}$ and $b \leq -T^\delta$,
\[
\mathbb{P} \left( \max_{k=1}^{T} (S_k + D_k) \leq 0, \ S_r > -r^{1/2-\delta/2} \right) \leq C \frac{(a^- + 1)(b^- + 1)}{T} r^{-\delta/2} . \tag{C.24}
\]

**Proof** For $1 \leq r' \leq r \leq T/2$, the event in the statement of the lemma intersected with $\{ R \leq r' \}$ is included in
\[
\left\{ \max_{k=1}^{T} (S_k + D_k) \leq 0 \right\} \cap \left\{ S_r > -r^{1/2-\delta/2} \right\} \cap \left\{ \max_{k=r}^{T} (S_k - 2\delta^{-1} k^{1/2-\delta} 1_{[r,T/2]}(k) - (\delta^{-1} (T - k)^{1/2-\delta} + T^{\delta/2}) 1_{[T/2,T]}(k)) \leq 0 \right\} . \tag{C.25}
\]

As in Lemma C.9, we bound $2\delta^{-1} k^{1/2-\delta} 1_{[r,T/2]}(k) + (\delta^{-1} (T - k)^{1/2-\delta} + T^{\delta/2}) 1_{[T/2,T]}(k)$ by
\[
2\delta^{-1} \frac{1}{1-\delta} 1_{[r-k,-r]} + 2\delta^{-1} r^{1/2-\delta} 1_{[0,(T-r)/(T-r)]}(k - r) + T^{\delta/2} 1_{[r,T-r)]}(k - r) , \tag{C.26}
\]
and $(k \vee r')^{1/2-\delta}$ by $1_{[r-k,B]} 1_{[0,r/2]}(k) + r^{1/2-\delta} 1_{[r/2,B]}(k)$. We then use Assumption (A1) and Proposition C.8 to bound the probability of (C.25) by
\[
C \int_{-r^{1/2-\delta}}^{-r^{1/2-\delta}} \frac{(a^- - r^{1/2-\delta})^2 (b^- - T^{\delta/2})}{s_r (s_T - s_r)} \mathbb{P}(S_r \in dw) . \tag{C.27}
\]

When $a \leq \delta^{-1}$, $b \leq -T^\delta$, the last fraction can be bounded by $C(a^- + 1)(b^- + 1)r' r^-T^{-1}$ in the domain of integration and therefore this bound applies to the entire integral as well. Notice that we have used that $s_r \in (k\delta, k\delta^{-1})$. Choosing $r' = r^{\delta/2}$, invoking Lemma C.9 for the case when $\{ R > r' \}$ and finally using the union bound then completes the proof. \hfill \Box

The next three lemmas provide needed bounds on the mean of the walk at step $m$.  

\[\text{Springer}\]
Lemma C.11 Fix $\delta \in (0, 1/2)$. There exists $C = C_\delta < \infty$ such that for all $0 \leq r \leq T$, $a \leq \delta^{-1}$ and $b \leq -T^\delta$ satisfying $(a^- + 1)b^- \leq T^{1-\delta}$,

$$
\mathbb{E}(S_r^-; \max_{k=1}^r (S_k + D_k) \leq 0, \, S_r \notin \left[a - r^2, -r^{1/2-\delta}/2\right]) \leq C(a^- + 1)r^{-\delta/2}. \quad (C.28)
$$

Proof We bound the expectation restricted to the events $\{S_r > -r^{1/2-\delta}/2\}$ and $\{S_r < a - r^2\}$ separately. In the first range, the expectation is bounded above by

$$
r^{1/2-\delta/2} \times \mathbb{P}\left(\max_{k=1}^r (S_k + D_k) \leq 0, \, S_r > -r^{1/2-\delta}/2\right). \quad (C.29)
$$

By Assumption (A3) and the union bound, the probability that there exists $k \in (0, r]$ such that $-D_k \geq \delta^{-1}k^{1/2-\delta} + r^{\delta/4}$ is at most $e^{-r^{\delta^2}/8}$. On the complement of the probability of the event in (C.29) is at most

$$
\int_{-r^{1/2-\delta}/2}^{2\delta^{-1}r^{1/2-\delta}} \mathbb{P}_{0,a}^w\left(\max_{k=1}^r (S_k - \delta^{-1}k^{1/2-\delta} - r^{\delta/4}) \leq 0\right) \mathbb{P}(S_r \in dw). \quad (C.30)
$$

Upper bounding $k^{1/2-\delta} \leq r_k^{1/2-\delta} + r^{1/2-\delta}1_{[r/2, r]}(k)$, we may use Proposition C.8 to bound the first probability in the integral by

$$
C\left((a - r^{\delta/4})^- + 1\right)(w - 2a^{-1}r^{1/2-\delta}^- + 1) \leq C(a^- + 1)r^{\delta/4}r^{1/2-\delta/2}/r \leq C(a^- + 1)r^{-1/2-\delta/4}. \quad (C.31)
$$

Since this is also an upper bound on the entire integral, plugging this in (C.29) gives $C(a^- + 1)r^{-3\delta/4}$ as an upper bound for the expectation in the first range.

For the restriction to the event $\{S_r < a - r^2\}$, we bound the expectation in (C.28) by $\mathbb{E}(S_r^-; \, S_r \leq a - r^2)$ and then use Cauchy-Schwarz. This gives the bound

$$
\left(\mathbb{E}(T_{0,a}^b, S_r^2)\mathbb{E}_{0,a}^{T,b}(S_r \leq a - r^2)\right)^{1/2}. \quad (C.32)
$$

Under the restrictions on $a, b$ and $T$ we have, $|\mathbb{E}(S_r - a)| = |b - a|r/T \leq 2r$ and $\text{Var} S_r \leq Cr$. Therefore, the second moment in (C.32) is at most $C((a^- + 1)^2 + r^2)$ while the probability there is at most $Ce^{-Cr^3}$. Combined, (C.32) is at most $C(a^- + 1)e^{-r^2}$, while the entire expectation in (C.28) obeys the desired bound. \hfill \Box

Lemma C.12 Fix $\delta \in (0, 1/2)$. There exists $C = C_\delta$ such that for all $0 \leq r \leq T/2$, $a \leq \delta^{-1}$ and $b \in \mathbb{R}$ such that $|b - a|r/T \leq \delta^{-1}(a^- + 1)^{1-\delta}$,

$$
\mathbb{E}\left(S_r^-; \max_{k=1}^r (S_k - \delta^{-1}k^{1/2-\delta}) \leq 0\right) \leq C(a^- + 1) \quad (C.33)
$$

Proof Thanks the conditions in the lemma and Assumption (A1),

$$
\left|\mathbb{E}(S_r - a)\right| = \frac{S_r}{s_T} |b - a| \leq C(a^- + 1)^{1-\delta}, \quad \text{Var} S_r = \frac{S_r(s_T - s_r)}{s_T} \leq Cr \quad (C.34)
$$

Now without the restricting event, the expectation in (C.33) is at most

$$
\mathbb{E}\left((S_r - \mathbb{E}S_r)^-\right) + \left(\mathbb{E}S_r\right)^{1/2} \leq \left(\text{Var} S_r\right)^{1/2} + (a^- + (\mathbb{E}(S_r - a))^-) \leq C(r^{1/2} + a^- + 1). \quad (C.35)
$$

Therefore, if $a^- \geq \sqrt{r}$ this shows the desired claim immediately.
In the converse case, we write the expectation in \((C.33)\) as

\[
\int_{-\infty}^{0} w^{-\mathbf{P}_{0,a}^{r,w}} \left( \max_{k=1}^{r} (S_{k} - \delta^{-1} k^{1/2-\delta}) \leq 0 \right) \mathbf{P}(S_{r} \in dw). \tag{C.36}
\]

The first probability in the integrand only increases if we replace \(k^{1/2-\delta}\) by \(\wedge_{r,k}^{1/2-\delta} + r^{1/2-\delta} 1_{[r/2,r]}(k)\). But then we can use the upper bound in Proposition C.8 and Assumption (A1) to bound this probability by \(C(a^{-} + 1)(w^{-} + r^{1/2-\delta})/r\). Using this in the last integral gives the bound

\[
C a^{-} + \frac{1}{r} \left( E S_{r}^{-2} + r^{1/2-\delta} E S_{r}^{-} \right) \leq C(a^{-} + 1) \frac{\text{Var} S_{r} + (a + |E(S_{r} - a)|)^{2} + r^{1/2-\delta} E S_{r}^{-}}{r}. \tag{C.37}
\]

Using the bounds in \((C.34)\) and \((C.35)\) proves \((C.33)\) in the case that \(a^{-} < \sqrt{r}\) as well. \(\square\)

Next, we strengthen the previous lemma by adding also the decorations.

**Lemma C.13** Fix \(\delta \in (0, 1/3)\). There exists \(C = C_{\delta}\) such that for all \(0 \leq r \leq T^{1/3}, a \leq \delta^{-1}\) and \(b \in \mathbb{R}\) such that \(|b - a| r / T \leq \delta^{-1}(a^{-} + 1)^{1-\delta}\),

\[
\mathbf{E}(S_{r}^{-}; \max_{k=1}^{r} (S_{k} + D_{k}) \leq 0) \leq C(a^{-} + 1) \tag{C.38}
\]

**Proof** The proof is similar to that of Lemma C.9 and therefore allow ourselves to be brief. Analogously to \(R\), we define a control variable:

\[
L := \min \left\{ \ell \geq 0 : \max_{k=1}^{r} \left( (|D_{k}| - \delta^{-1} k^{1/2-\delta}) \vee |\xi_{k}| - (k \vee \ell)^{\delta/2} \right) \leq 0 \right\}, \tag{C.39}
\]

and observe that by the union bound, it is clearly enough to show that for all \(\ell \geq 0\),

\[
\mathbf{E}(S_{r}^{-}; \max_{k=1}^{r} (S_{k} + D_{k}) \leq 0, L = \ell) < C(a^{-} + 1)e^{-\ell^{2}/4}. \tag{C.40}
\]

When \(\ell < r\), the last expectation is at most

\[
C e^{-\ell^{2}/3} \mathbf{E}_{0,a-\ell^{2}}^{T, b} \left( S_{r}^{-}; \max_{k=1}^{r} \left( S_{k}' - 2\delta^{-1} (\ell^{1/2-\delta} + k^{1/2-\delta}) \right) \leq 0 \right) \leq C e^{-\ell^{2}/3} \mathbf{E}_{0,a'}^{T', b'} \left( S_{r}^{-}; \max_{k=1}^{r} \left( S_{k}'' - 2\delta^{-1} k^{1/2-\delta} \right) \leq 0 \right), \tag{C.41}
\]

where \(T' := T - \ell, r' := r - \ell, a' := a - \ell^{2} - 2\delta^{-1} \ell^{1/2-\delta}, b' := b - 2\delta^{-1} \ell^{1/2-\delta}\) and \((S_{k}')_{k=0}^{T'}\) under \(P_{0,a-\ell^{2}}^{T', b'}\) and \((S_{k}'')_{k=0}^{T'}\) under \(P_{0,a'}^{T', b'}\) have the same laws as \((S_{k+k} - S_{k})_{k=0}^{T'}\) and \((S_{k+k} - S_{k} - 2\delta^{-1} \ell^{1/2-\delta} T)_{k=0}^{T'}\) under \(P(-|S_{T} = a - \ell^{2})\) respectively.

Thanks to Lemma C.12, the last expectation is at most \(C(a'^{-} + 1) \leq C(a^{-} + 1)(1 + \ell^{2})\) since

\[
|b' - a'| r'/T' \leq C(|b - a| + \ell^{2}) r / T \leq C(a^{-} + 1)^{1-\delta} \leq C'(a^{-} + 1)^{1-\delta}. \tag{C.42}
\]

This shows that \((C.40)\) indeed holds. When \(\ell \geq r\), we remove the ballot event from the expectation in \((C.40)\) and then use the definition of \(L\) to bound it by

\[
C(a^{-} + \ell^{2}) \mathbf{P}(L = \ell) \leq C'(a^{-} + \ell^{2}) e^{-\ell^{2}/3} \leq C''(a^{-} + 1)e^{-\ell^{2}/4}. \tag{C.43}
\]

\(\square\)
**Proof of Theorem C.3** Let $1 \leq r \leq T^{\delta/4}$. We will show the desired statement by upper and lower bounding the left-hand side of (C.4) separately. For the upper bound, we can appeal to Lemma C.9 and Lemma C.10 to consider instead of the left-hand side of (C.4) the probability of the event

$$
\left\{ \max_{k=1}^{T} (S_k + D_k) \leq 0, \quad R \leq r, \quad S_r \leq -r^{1/2-\delta/2} \right\}. 
$$

(C.44)

which is further included in

$$
\left\{ \max_{k=1}^{r} (S_k + D_k) \leq 0 \right\} \cap \left\{ a - r^2 \leq S_r \leq -r^{1/2-\delta/2} \right\}
\cap \left\{ \max_{k=r}^{T} (S_k - 2\delta^{-1}k^{1/2-\delta}1_{[r,T/2]}(k) - (\delta^{-1}(T - k)^{1/2-\delta} + T^{\delta/2})1_{[T/2,T]}(k)) \leq 0 \right\}. 
$$

(C.45)

As before, we bound $2\delta^{-1}k^{1/2-\delta}1_{[r,T/2]}(k) + (\delta^{-1}(T - k)^{1/2-\delta} + T^{\delta/2})1_{[T/2,T]}(k)$ by

$$
(2\delta^{-1}r^{1/2-\delta} + 2\delta^{-1}(k - r)^{1/2-\delta})1_{[0,(T-r)/2]}(k - r)
+ (2\delta^{-1}((T - r) - (k - r))^{1/2-\delta} + T^{\delta/2})1_{[(T-r)/2,T-r]}(k - r). 
$$

(C.46)

and then use Assumption (A2) and Proposition C.8 to upper bound the probability of (C.45) by

$$
(2 + o(1)) \int_{a-r^2}^{-r^{1/2-\delta/2}} P_{0,a}^{uw} \left( \max_{k=1}^{r} (S_k + D_k) \leq 0 \right) \frac{(w - 2\delta^{-1}r^{1/2-\delta}) - (b - T^{\delta/2})}{(s_T - s_r)} P(S_r \in dw),
$$

(C.47)

where the $o(1)$ depends only on $r$ and goes to 0 when $r \to \infty$. To get the $(2 + o(1))$ multiplying the integral, we have also used that both factors in the numerator above tend to $\infty$ as $r \to \infty$ uniformly in the range of integration.

In the range of integration, the fraction in the integrand is at most $(1+o(1))w-b^-/s_T$, with the $o(1)$ as before. Integrating over all $w \in \mathbb{R}$ then gives $(1+o(1))2\ell_{T,r}(a,b)(a)b^-/s_T$ as an upper bound. Lemma C.13 can then be used to turn the multiplicative error into an additive one satisfying (C.5) uniformly as desired.

Turning to the lower bound, we now consider the event

$$
\left\{ \max_{k=1}^{r} (S_k + D_k) \leq 0 \right\} 
\cap \left\{ \max_{k=r}^{T} (S_k + 2\delta^{-1}k^{1/2-\delta}1_{[r,T/2]}(k) + (\delta^{-1}(T - k)^{1/2-\delta} + T^{\delta/2})1_{[T/2,T]}(k)) \leq 0 \right\}. 
$$

(C.48)

By Lemma C.9 with $(D_k)_{k=1}^{T}$ replaced by $(D_k 1_{(0,r)})_{k=1}^{T}$, we may further intersect the above event with $\{ R \leq r \}$ at the cost of decreasing its probability by at most a quantity satisfying the same as $e_{T,r}(a, b)$ in (C.5). Since the new event is now a subset of the event in the statement of theorem, for the sake of the lower bound, it suffices to bound the probability of (C.48) from below.

Using Assumption (A2), Proposition C.8 and the bound in (C.46), we proceed as in the upper bound to bound this probability from below by...
where the $o(1)$ depends only on $r$ and tends to 0 as $r \to \infty$. The lower bound in Proposition C.8 is in force (with $\delta/3$), since the numerator above is at most $C(a - 1)b^{-r^2} \leq T^{1 - \delta/3}$ for $r$ large enough, thanks to the assumptions in the theorem.

Proceeding as in the upper bound, the last display is at least

$$
(2 - o(1)) \frac{b^{-}}{ST}\mathbb{E}
\left(S_{r}^{-}; \max_{k=1}^{r}(S_{k} + D_{k}) \leq 0, S_{r} \in [a - r^{2}, -r^{2} - \delta/2]\right).
\tag{C.50}
$$

Invoking Lemma C.11 and then Lemma C.13 we remove the additional restriction on $S_{r}$ and turn the multiplicative error into an additive one satisfying (C.5) uniformly as desired. \hfill \Box

Next we turn to the proofs of Propositions C.4 and C.5. The following lemma is key.

**Lemma C.14** Fix $\delta \in (0, 1/3)$. For all $r \geq 1$,

$$
\lim_{a \to -\infty} \frac{\ell_{T,r}(a, b)}{a^{-}} = 1
\tag{C.51}
$$

uniformly in $T \in [r, \infty]$, $b \in \mathbb{R}$, $|b - a|r/T \leq (a^{-} + 1)^{1-\delta}$ and all $(S_{k})_{k=0}^{T}$, $(D_{k})_{k=1}^{T}$ satisfying Assumptions (A1), (A2) and (A3). Furthermore,

$$
\lim_{r \to \infty} \lim_{T \to \infty} \sup_{T \in [T', \infty]} \left| \frac{\ell_{T,r}(a, b) - \ell_{T',r}(a, b)}{a^{-} + 1}\right| = 0,
\tag{C.52}
$$

uniformly in $a \leq \delta^{-1}$, $b < \delta^{-1}$, $|b-a|/(a^{-} + 1) \leq T^{1-\delta}$ and all $(S_{k})_{k=0}^{T}$, $(D_{k})_{k=1}^{T}$ satisfying Assumptions (A1), (A2) and (A3).

**Proof** Starting with the first statement, for fixed $r \geq 1$, if $|b - a|r/T \leq (a^{-} + 1)^{1-\delta}$ and $T \in [r, \infty]$ then as in (C.34) for all $k \leq r$,

$$
\left| E(S_{k} - a) \right| \leq C(a^{-} + 1)^{1-\delta}, \quad \text{Var} S_{k} \leq C'r.
\tag{C.53}
$$

If in addition $a^{-} > r^{1/2 + \delta}$ then also,

$$
\mathbb{P}(S_{k} + D_{k} > 0) \leq \mathbb{P}(D_{k} > a^{-}/2) + \mathbb{P}(S_{k} - a > a^{-}/2) \leq Ce^{-(a^{-})^2}.
\tag{C.54}
$$

Using then the Cauchy-Schwarz inequality and the Union Bound gives for $r$ large enough,

$$
\mathbb{E}(\lfloor S_{r} \rfloor; \max_{k=1}^{r}(S_{k} + D_{k}) > 0) \leq \left(\mathbb{E} S_{r}^{2} \sum_{k=1}^{r} \mathbb{P}(S_{k} + D_{k} > 0)\right)^{1/2}
\leq C \left((r + a^{2})re^{-(a^{-})^{2}}\right)^{1/2} \leq e^{-(a^{-})^{2}/3}.
\tag{C.55}
$$

We define $\tilde{\ell}_{T,r}(a, b) = E(\lfloor -S_{r} \rfloor; \max_{k=1}^{r}(S_{k} + D_{k}) \leq 0)$, then the left-hand side in (C.55) bounds $|E(\lfloor -S_{r} \rfloor) - \tilde{\ell}_{T,r}(a, b)|$. Using also the first inequality in (C.53) with $k = r$, we get

$$
\left| \frac{\tilde{\ell}_{T,r}(a, b)}{a^{-}} - 1 \right| \leq C(a^{-})^{-\delta}. \tag{C.56}
$$
Moreover,
\[
|\ell_{T,r}(a, b) - \ell_{T,r}(a, b)| = E(S_r^+: r \max_{k=1}^r (S_k + D_k) \leq 0) \leq \left( E((S_r - a)^2) \right)^{1/2} + a^+
\leq C(a^- + 1)^{1-\delta} + \sqrt{C'r} + a^+ = o(a^-). \tag{C.57}
\]

Since also the right-hand side of (C.56) goes to 0 as \(a \to -\infty\), this shows the first part of the lemma.

Turning to the proof of (C.52), we first treat the case \(T < \infty\). If we assume that \(|b - a|/(a^- + 1) \leq T^{1-\delta}\) and \(T > r^{2/\delta}\) then as in (C.53) we get \(|E(S_k - a)| \leq C(a^- + 1)T^{-\delta/2}\) for all \(k \leq r\). Repeating the argument before, we get
\[
\left| \frac{\ell_{T,r}(a, b)}{a^-} - 1 \right| \leq Cr^{-\delta/2}, \tag{C.58}
\]
for all \(a^- > r^{1/2+\delta}\). This shows that (C.52) hold uniformly whenever \(|b - a|/(a^- + 1) \leq T^{1-\delta}\), \(a^- > (r \vee r')^{1/2+\delta}\) and for any choice of sequences \((S_k)_{k=0}^T, (D_k)_{k=1}^T\) satisfying Assumptions (A1), (A2) and (A3).

Now let \(\delta' \in (0, \delta)\). If we assume that \(a \leq \delta'^{-1}, b < -T^{\delta'}\) and \((a^- + 1)b^- \leq T^{1-\delta'}\), then it follows from Theorem C.3, applied once with \(r\) and once with \(r'\) (and with all other arguments the same), that the left-hand side in (C.52) is at most
\[
\lim_{r, r' \to \infty} \lim_{T \to \infty} C\left( \frac{T|e_{T,r}(a, b)|}{(a^- + 1)b^-} + \frac{T|e_{T,r'}(a, b')|}{(a^- + 1)b^-} \right) = 0. \tag{C.59}
\]
This shows that (C.52) holds uniformly in \(a \leq \delta'^{-1}, b < -T^{\delta'}\), \((a^- + 1)b^- \leq T^{1-\delta'}\) and all \((S_k)_{k=0}^T, (D_k)_{k=1}^T\) satisfying Assumptions (A1), (A2) and (A3).

We now we claim that the union of the last two domains of uniformity includes the one in the statement of the lemma with the additional restriction that \(b < -T^\delta\). Indeed, if \(a \leq \delta^{-1}, b < -T^\delta\), \(|b - a|/(a^- + 1) \leq T^{1-\delta}\) but \((a^- + 1)b^- > T^{1-\delta'}\), then \((a^- + 1)^\delta(1 + T^{1-\delta}) > T^{1-\delta'}\), which implies that \(a^- > T^{(\delta-\delta')/2}\) and hence for \(T\) large enough, depending only on \(r, r'\), also that \(a^- > (r \vee r')^{1/2+\delta}\).

Next to replace the condition \(b < -T^\delta\) with \(b < \delta^{-1}\), we consider the processes \((S_k')_{k=0}^T\) and \((D_k')_{k=1}^T\), defined by setting \(S_k' := S_k - 2kT^\delta - 1\) and \(D_k' := D_k + 2kT^\delta - 1\) for \(k \leq r\). Then the new processes satisfy the same assumptions as before with \(a' := a, b' := b - 2T^\delta\) provided \(T\) is large enough. Moreover, if \(b < \delta^{-1}\), then for \(T\) large enough also \(b' < -T^\delta\) and \(|b' - a'|/(a^- + 1) \leq C'T^\delta\). Therefore (C.52) holds uniformly as in the conditions of the lemma, if we replace \(\ell_{T,r}(a, b)\) by \(\ell'_{T,r}(a', b')\), which we defined in terms of \(S_k'\) and \(D_k'\) in place of \(S_k\) and \(D_k\) in the obvious way. But then, \(|\ell_{T,r}(a, b) - \ell'_{T,r}(a', b')| \leq 2r^\delta\) which goes to 0 in the stated limits uniformly in \(a\) and \(b\).

Finally, to treat the case \(T = \infty\), we again define new processes \((S_k')_{k=0}^{\infty}\) and \((D_k')_{k=1}^{\infty}\) for some finite \(T'\) to be chosen depending on \(r\). To do this, we let \((V_s^r : s \in [0, s_T])\) be a standard Brownian Motion coupled with \((S_k)_{k=0}^{\infty}\) such that \(S_k = W_{s_k} + a\) for all \(k = 0, \ldots, T'\). We then define
\[
W_s' := (s_{T'} - s) \int_0^s \frac{dW_t}{s_T' - t}, \quad s \in [0, s_{T'}], \tag{C.60}
\]
and finally set \(S_k' := a + W_{s_k}'\) for \(k = 0, \ldots, T'\) and \(D_k' := D_k + (S_k' - S_k)_1\} \) for all \(k = 1, \ldots, T'\).
Now it is a standard fact that $W_s'$ is a Brownian Bridge on $[0, s'_T]$, so that $S'_k$ is a random walk as in Assumption (A1) with $a' = a$, $b' = a$, $T'$ in place of $a$, $b$, $T$. At the same time,

$$S_k - S'_k = W_s - W_{s'} = \int_0^{s_k} \frac{s_k - t}{s_{T'} - t} dW_t,$$  \hspace{1cm} (C.61)

is a centered Gaussian with variance bounded by $s_k^2/(s_{T'} - s_k) \leq C r^3/(T' - r)$ for all $k \leq r$. It follows from the Union Bound that for any fixed $r$, if $T'$ is large enough, then Assumption (A3) will hold for $(S'_k, D'_k)_{k=0}^{T'}$. Since the remaining Assumption (A2) holds as well and since $a'$, $b'$, $T'$ fall into the domain of uniformity, we have (C.52), again with $\ell_{T', r}(a', b')$ in place of $\ell_{T, r}(a, b)$. Lastly we bound the difference between the two quantities by $(E|S_r - S'_r|^2)^{1/2}$, which tends to 0 as $T' \to \infty$ followed by $r \to \infty$, uniformly in $a$. \hfill \Box

**Proof of Proposition C.4** The first part of the proposition is precisely (C.52). The second part is an immediate consequence. \hfill \Box

**Proof of Proposition C.5** The first part of the proposition with $r < \infty$ follows immediately from the first part of Lemma C.14. To get the desired uniformity and the case $r = \infty$, we use (C.52) from the same lemma. This implies the existence of $r_\epsilon > 0$ for all $\epsilon \in (0, 1)$ and $T_r \geq r$ for all $r > 0$ such that

$$\left| \frac{\ell_{T, r}(a, b) - \ell_{T, r_\epsilon}(a, b)}{a^- + 1} \right| \leq \epsilon,$$  \hspace{1cm} (C.62)

for all $\epsilon \in (0, 1)$, $r \in [r_\epsilon, \infty)$, $T \in [T_r, \infty]$ and $a, b$ satisfying the conditions in the proposition.

Thanks to (C.51), for any $\epsilon > 0$, we may also find $a_\epsilon < 0$ such that $|\ell_{T, r}(a, b)/a^- - 1| \leq \epsilon$, for all $a < a_\epsilon, r \leq r_\epsilon$, $T \in [r, \infty]$ and $b$ satisfying the conditions in the proposition. It follows from the Triangle Inequality that $|\ell_{T, r}(a, b)/a^- - 1| \leq 3\epsilon$ for all $a < a_\epsilon$, $r \in [1, \infty)$, $T \in [T_r, \infty]$ and all $b$ in its allowed range. Taking $r \to \infty$ then shows the same bound for $\ell_{T, \infty}(a, b)$. \hfill \Box

We now turn to the proofs of the continuity statements. They are not difficult, given what we have proved so far.

**Proof of Lemma C.6** Writing the difference in the expectations as $E(S_r^-; \max_{k=1}^r (S_k + D_k) \in (-\lambda, \lambda])$ the latter is upper bounded thanks to the Cauchy-Schwarz Inequality and the Union Bound by

$$\left( E S_r^2 \sum_{k=1}^r P(S_k + D_k \in (-\lambda, \lambda]) \right)^{1/2}.$$  \hspace{1cm} (C.63)

As in the proof of Lemma C.14, the second moment is bounded above by $C(r + a^2) \leq C' r (a^- + 1)^2$. The sum, on the other hand, is bounded by $r$ times the supremum of all terms. \hfill \Box

**Proof of Proposition C.7** If $r^{(\infty)} < \infty$, then we may assume without loss of generality that $r^{(i)} \equiv r$ for all $i \geq 1$ and some $r < \infty$. Then thanks to the assumption of stochastic continuity, we know that $S^{(i)}_r - 1_{\max_{k=1}^r (S^{(i)}_k + D^{(i)}_k) \leq \lambda}$ is $P^{(i)}$-almost surely continuous in $(S^{(i)}_k)_{k=0}^r$, $(D^{(i)}_k)_{k=1}^r$ when $i = \infty$ and bounded by $S^{(i)}_r$ for all $i \leq \infty$. The latter converges in mean as $i \to \infty$ to $S^{(\infty)}_r$, under the assumption of (C.10) as the underlying random variables are
Gaussian. (C.11) therefore follows from of (C.10) in light of the Dominated Convergence Theorem.

When \( r(\infty) = \infty \), we use the third part of Lemma C.14 to approximate \( \ell_i(a(i)) \) up to an arbitrarily small error for all \( i \leq \infty \) large enough, by choosing \( r \) large enough and using the assumptions on \( a(i), b(i) \) and \( T(i) \). The result then follows by the first part of the proposition together with a standard three \( \epsilon \)-s argument. \( \square \)

Finally, we give,

**Proof of Proposition C.8** The case when \( a' = b' = 0 \) is folklore and can be shown, for instance, via a straightforward modification of the proofs of Proposition 1.1 and Proposition 1.2 in the supplementary material to [13] (also available as a standalone manuscript in [12]). To reduce the general case to that when \( a' = b' = 0 \), one can simply tilt both probabilities in the statement of the proposition by subtracting \( a' + (b' - a')s_k/s_T \) from the value of \( S_k \). From the assumptions in the proposition and since \( \sigma_k \in (\delta, \delta^{-1}) \), we have

\[
\left| b' - a' \right| \frac{s_k}{s_T} \leq \delta^{-2} T^{-1/2 - \delta} k^{1/2 - \delta}
\]

(C.64)

for all \( k \leq T/2 \). A similar bound holds for \( \left| b' - a' \right| (s_T - s_{T-k})/s_T \).

Consequently, the first and second probabilities in the statement of the proposition are bounded above and below respectively by

\[
P_{(S_k \pm \delta^{-1} \wedge \delta^{-2} T^{-1/2 - \delta} k^{1/2 - \delta}) \leq 0} \leq P_{(S_k \pm \delta^{-2} T^{-1/2 - \delta} k^{1/2 - \delta}) \leq 0}.
\]

(C.65)

Applying now the result in the first case with \((\delta^{-1} + \epsilon^{-2})^{-1}\) in place of \(\delta\), the desired statement follows. \( \square \)

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