Accuracy of Gaussian approximation in nonparametric Bernstein – von Mises Theorem*

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Abstract

The prominent Bernstein – von Mises (BvM) result claims that the posterior distribution after centering by the efficient estimator and standardizing by the square root of the total Fisher information is nearly standard normal. In particular, the prior completely washes out from the asymptotic posterior distribution. This fact is fundamental and justifies the Bayes approach from the frequentist viewpoint. In the nonparametric setup the situation changes dramatically and the impact of prior becomes essential even for the contraction of the posterior; see van der Vaart and van Zanten (2008), Bontemps (2011), Castillo and Nickl (2013, 2014) for different models like Gaussian regression or i.i.d. model in different weak topologies. This paper offers another non-asymptotic approach to studying the behavior of the posterior for a special but rather popular and useful class of statistical models and for Gaussian priors. First we derive tight finite sample bounds on posterior contraction in terms of the so-called effective dimension of the parameter space. Our main results describe the accuracy of Gaussian approximation of the posterior. In particular, we show that restricting to the class of all centrally symmetric credible sets around the penalized maximum likelihood estimator (pMLE) allows to get Gaussian approximation up to order $n^{-1}$. We also show that the posterior distribution mimics well the distribution of the pMLE and reduce the question of reliability of credible sets to

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consistency of the pMLE-based confidence sets. The obtained results are specified for nonparametric log-density estimation and generalized regression.

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Bernstein – von Mises (BvM) Theorem is one of most prominent results in statistical inference. It claims that the posterior measure is asymptotically normal with the mean close to the maximum likelihood estimator (MLE) and the variance close to the variance of the MLE. This explains why this result is often considered as the Bayesian counterpart of the frequentist Fisher Theorem about asymptotic normality of the MLE. The BvM result provides a theoretical background for different Bayesian procedures. In particular, one can use Bayesian computations for evaluation of the MLE and its variance. Also one can build elliptic credible sets using the first two moments of the posterior. The main questions to address by studying the behavior of a nonparametric Bayes procedure are

- concentration: find possibly small concentration sets of the posterior distribution;
- asymptotic normality or any other asymptotic approximation of the posterior;
- covering: whether one can use credible sets as frequentist confidence sets.

The classical versions of the BvM Theorem claim that the posterior concentrates in a root-n vicinity of the true parameter, after proper centering and scaling it is root-n standard normal, and credible sets can be well used in place of classical confidence sets. However, these results require a fixed finite dimensional parameter set, correct model specification, and large samples. We refer to van der Vaart and Wellner (1996); van der Vaart (1998) for a detailed historical overview.

Any extension of the BvM approach to the case of a large or infinite dimensional parameter space appears to be very involved, in particular, more involved than the expansions of the maximum likelihood estimate. The first problem is related to the posterior concentration. Such a result requires to bound the integral of the likelihood process in the complement of the local vicinity and this is a hard task in the nonparametric setup. The second problem is due to fact that a standard Gaussian measure on $\mathbb{R}^\infty$ is only defined
in a weak sense. In particular, it does not concentrate on any $\ell_2$ ball in $\mathbb{R}^\infty$. This makes it difficult to study the total variation distance between the posterior and the Gaussian law. We refer to Castillo and Nickl (2013, 2014), and Ghosal and van der Vaart (2017) for a more discussion. One more crucial issue is an inconsistency problem: in some situations, the BvM-based Bayesian credible sets do not contain the true parameter with the probability close to one; cf. Cox (1993); Freedman (1999), or Kleijn and van der Vaart (2006, 2012). The results of this paper help to address this phenomenon in a rather general situation. It appears that the posterior in the case of a Gaussian prior is nearly normal but its first two moments mimic the penalized maximum likelihood estimator (pMLE) with the quadratic penalization coming from the prior distribution. It is well known that the penalization yields some estimation bias. If the squared bias exceeds the variance of the penalized MLE, the Bayesian credible sets become unreliable. A number of studies explain how an empirical or hierarchical Bayes approach can be used for building “honest” confidence sets; see e.g. Knapik et al. (2016); Nickl and Szabó (2016); Sniekers and van der Vaart (2015). We indicate below how the approach advocated in this paper can be combined with the existing results on empirical or full Bayesian model selection.

The main results of this paper provide at a detailed description of the properties of the posterior distribution for Gaussian priors in a high-dimensional or nonparametric setups. In particular, we establish a nonasymptotic upper bound on concentration and on the error of Gaussian approximation for the posterior in total variation distance in terms of efficient dimension of the problem. We also show that the latter bound can be dramatically improved if we restrict ourselves to the class of centrally symmetric sets around pMLE. We consider a rather general setup, however, impose two important conditions. The first condition requires that the stochastic part of the log-likelihood is linear in the target parameter, while the second one is about concavity of the expected log-likelihood. These two conditions are automatically fulfilled in a number of popular models like Gaussian, Poissonian, Binary or Generalized Linear regression, log-density estimation, linear diffusion, etc. Under these assumptions we manage to state and prove our results in concise and avoid the empirical process machinery. A forthcoming paper Spokoiny (2019) explains how the approach and the results can be extended to much more general setups including nonlinear (generalized) regression and nonlinear inverse problems with noisy observations, Bayesian deconvolution, error-in-operator and nonparametric instrumental variable problems, nonparametric diffusion, etc.

The main contributions of the paper are:
• sharp finite sample bounds for concentration of pMLE and of posterior distribution;
• Gaussian approximation of the posterior with an explicit error term for the total variation distance and for the class of centrally symmetric sets around pMLE;
• systematic use of an effective dimension in place of the total parameter dimension;
• addressing the question of prior impact in Bayesian inference and of frequentist validity of Bayesian credible sets;
• addressing the question of posterior contraction: under a bound on the bias the contraction result can be stated in term of effective dimension;
• the whole approach is “coordinate free”, we do not use any spectral decomposition and/or any basis representation for the target parameter and penalization;
• specification of the results to log-density estimation and generalized regression.

The paper is structured as follows. Section 2 describes our setup, presents the main conditions and states the main results about the properties of the pMLE and of the posterior. Properties of the pMLE $\tilde{\theta}_G$ are described in Section 2.2. Section 2.3 presents the result about posterior concentration. The main result of the paper is Theorem 2.5 of Section 2.4 and its Corollary 2.6 about quality of Gaussian approximation of the posterior. Section 2.5 discusses the important question of prior impact. We also provide sufficient conditions which enable us to use credible sets as frequentist confidence sets: this question can be reduced to a simpler question of the bias-variance trade-off in penalized estimation using the recent results on Gaussian comparison; see Theorem A.1. We also address the issue of uniform Gaussian approximation of the posterior for a family of priors in Section 2.6. This uniform approximation allows to reduce the problem of Bayesian model selection to the well studied Gaussian case. All announced results are stated for a Gaussian prior, however, most of them can be extended in a straightforward way to a general prior with a log-concave density; see again Section 2.6. Section 3.1 comments how the result can be applied to the case of the Bayesian nonparametric log-density estimation, while Section 3.2 discusses generalized regression estimation. The proofs and some useful auxiliary facts are collected in the Appendix.

2 Nonparametric BvM Theorem

This section discusses the BvM result for a rather general model with a high-dimensional or infinite dimensional parameter set for a Gaussian prior. Compared to existing literature, our results provide finite sample bounds on posterior concentration and on accu-
Accuracy of Gaussian approximation in BvM Theorem

Moreover, we show that the quality of Gaussian approximation can be gradually improved up to order $n^{-1}$ if we only consider credible sets which are centrally symmetric around the pMLE.

First we specify our setup. Let $Y$ denote the observed data and $P$ mean their distribution. A general parametric assumption (PA) means that $P$ belongs to infinite-dimensional family $(P_{\theta}, \theta \in \Theta \subseteq \mathbb{R}^p)$ with $p \leq \infty$ dominated by a measure $\mu_0$. This family yields the log-likelihood function $L(\theta) = L(Y, \theta) \overset{\text{def}}{=} \log \frac{dP_{\theta}}{d\mu_0}(Y)$. The PA can be misspecified, so, in general, $L(\theta)$ is a quasi log-likelihood. The classical maximum likelihood principle suggests to estimate $\theta$ by maximizing the function $L(\theta)$:

$$\tilde{\theta} \overset{\text{def}}{=} \arg\max_{\theta \in \Theta} L(\theta). \quad (2.1)$$

If $P \not\in (P_{\theta})$, then the estimate $\tilde{\theta}$ from (2.1) is still meaningful and it appears to be an estimate of the value $\theta^*$ defined by maximizing the expected value of $L(\theta)$:

$$\theta^* \overset{\text{def}}{=} \arg\max_{\theta \in \Theta} E L(\theta).$$

Here $\theta^*$ is the true value in the parametric situation and can be viewed as the parameter of the best parametric fit in the general case. In the Bayes setup, the parameter $\vartheta$ is a random element following a prior measure $\Pi$ on the parameter set $\Theta$. The posterior describes the conditional distribution of $\vartheta$ given $Y$ obtained by normalization of the product $\exp\{L(\theta)\} \Pi(d\theta)$. This relation is usually written as

$$\vartheta \mid Y \propto \exp\{L(\theta)\} \Pi(d\theta).$$

Below we focus on the case of a Gaussian prior. Without loss of generality, a Gaussian prior $\Pi(\theta)$ will be assumed to be centered at zero. By $G^{-2}$ we denote its covariance matrix, so that, $\Pi \sim \mathcal{N}(0, G^{-2})$. The main question studied below is to understand under which conditions on the prior covariance $G^{-2}$ and the model, the BvM-type result holds and what is the error term in the BvM approximation. For a Gaussian likelihood, the posterior is Gaussian as well and its properties can be studied directly; see e.g. Bontemps (2011); Leahu (2011). For the case when the log-likelihood function is not quadratic in $\theta$, the study is more involved. The posterior is obtained by normalizing the product density $\exp\{L_G(\theta)\}$ with

$$L_G(\theta) = L(\theta) - \|G\theta\|^2 / 2.$$ 

This expression arises in penalized maximum likelihood estimation, one can treat the prior term $\|G\theta\|^2 / 2$ as roughness penalty. Therefore, we expect the same effect of using
the Gaussian prior as in the penalized MLE case: it improves the concentration properties but can introduce some bias in estimation. Define

$$\tilde{\theta}_G = \arg\max_{\theta \in \Theta} L_G(\theta), \quad \theta^*_G = \arg\max_{\theta \in \Theta} \mathbb{E} L_G(\theta).$$

Also define

$$F(\theta) \overset{\text{def}}{=} -\nabla^2 \mathbb{E} L(\theta),$$

$$F_G(\theta) \overset{\text{def}}{=} -\nabla^2 \mathbb{E} L_G(\theta) = -\nabla^2 \mathbb{E} L(\theta) + G^2 = F(\theta) + G^2,$$

and let $D_G$ be a symmetric matrix with $D^2_G = F_G(\theta^*_G)$. First we state the required conditions.

### 2.1 Conditions

This section collects the conditions which are systematically used in the text. We mainly require that the stochastic part of the log-likelihood process $L(\theta)$ is linear in $\theta$, while its expectation is a smooth concave function of $\theta$. We also implicitly assume that the parameter set $\Theta$ is an open subset of $\mathbb{R}^p$ where $p$ is typically equal to $\infty$. The model and complexity reduction will be done via the the prior structure, so that only the effective dimension shows up in all the results. Below we assume without explicit mentioning that all the presented conditions are fulfilled.

1. **(L)** The set $\Theta$ is open and convex in $\mathbb{R}^p$. The function $\mathbb{E} L(\theta)$ is concave in $\theta$. 
2. **(E)** The stochastic component $\zeta(\theta) = L(\theta) - \mathbb{E} L(\theta)$ of the process $L(\theta)$ is linear in $\theta$. We denote by $\nabla \zeta$ its gradient: $\nabla \zeta \equiv \nabla \zeta(\theta)$. 

We also require some exponential moment of $\nabla \zeta$.

3. **(EV)** There exist a positive symmetric matrix $V$, and constants $g > 0$, $\nu_0 \geq 1$ such that $\text{Var}(\nabla \zeta) \leq V^2$ and

$$\sup_{u \in \mathbb{R}^p} \log \mathbb{E} \exp \left\{ \lambda \frac{\langle u, \nabla \zeta \rangle}{\| V u \|} \right\} \leq \frac{\nu_0^2 \lambda^2}{2}, \quad |\lambda| \leq g.$$ 

Condition **(EV)** basically requires that the gradient $\nabla \zeta$ of the stochastic component $\zeta(\theta)$ has finite exponential moments. Then one can use the fact that existence of the exponential moment $\mathbb{E} e^{\lambda \xi}$ for a centered random variable $\xi$ and some fixed $\lambda_0$ implies that the moment generating function $f_\xi(\lambda) \overset{\text{def}}{=} \log \mathbb{E} e^{\lambda \xi}$ is analytic in $\lambda \in (0, \lambda_0)$ with
$f_\xi(0) = f'_\xi(0) = 0$ and hence, it can be well majorated by a quadratic function in a smaller interval $[0, \lambda_1]$ for $\lambda_1 < \lambda_0$; see Golubev and Spokoiny (2009). In fact this condition is only used to establish the deviation bounds for the norm $\|D_G^{-1}\nabla \xi\|$; see (2.5). One can directly operate with the quantiles of the corresponding distribution.

The signal-to-noise condition relates the matrices $V^2$ and $F_G = F_G(\theta_G^*)$; see (2.2).

$(V|G)$ Define

$$B_{V|G} \overset{\text{def}}{=} F^{-1/2} G V^2 F^{-1/2}$$

with $V^2$ from $(EV)$. There are fixed constants $\lambda_{V|G}$ and $p_{V|G}$ such that

$$\text{tr} B_{V|G} \leq p_{V|G}, \quad \|B_{V|G}\| \leq \lambda_{V|G}.$$

These constants enter in the definition of the upper quantile function $z(B_{V|G}, x)$ for $\|D_G^{-1}\nabla \xi\|$; see Theorem A.2 or Theorem A.5 below.

Apart the basic conditions $(L)$, $(E)$, $(EV)$, $(V|G)$, we need some local properties of the expected log-likelihood $IEL(\theta)$. Let $\Theta^o$ be a local set. It is required that this set contains the concentration set $A_G(\tau_G)$ of the estimate $\tilde{\theta}_G$; see Theorem 2.1 below.

The next condition is closely related to $(V|G)$ and it defines the notion of local effective dimension $p_G(\theta)$ at the point $\theta \in \Theta^o$.

$(D|G)$ It holds for a fixed constant $p^*_G$

$$p_G(\theta) \overset{\text{def}}{=} \text{tr}\{F(\theta)F_G^{-1}(\theta)\} \leq p^*_G, \quad \theta \in \Theta^o.$$

In fact, we assume that each value $p_G(\theta)$ is not large while the full dimension $p$ can be infinite.

Usually the values $p_{V|G}$ and $p_G(\theta)$ are of the same order and even close to each other; see Section 3.

Finally, we state a condition on the local smoothness properties of the expected log-likelihood $IEL(\theta)$. In particular, we require that this function is four times differentiable. Define for each $\theta \in \Theta^o$, and any $\mathbf{u} \in \mathbb{R}^p$, the directional derivative

$$\delta_m(\theta, \mathbf{u}) \overset{\text{def}}{=} \frac{1}{m!} \frac{d^m}{dt^m} IEL(\theta + t\mathbf{u}) \bigg|_{t=0}, \quad m = 3, 4. \quad (2.3)$$

Clearly the value $\delta_m(\theta, \mathbf{u})$ is proportional to $\|\mathbf{u}\|^m$. 

The functions $\delta_3(\theta, u)$ and $\delta_4(\theta, u)$ are well defined and uniformly bounded for all $\theta \in \Theta^o$ and all $u \in U^o$ for specific sets $\Theta^0$ and $U^0$ in $\mathbb{R}^p$.

A particular choice of $\Theta^0$ and $U^0$ will be specified later.

### 2.2 Properties of the pMLE $\tilde{\theta}_G$

This section briefly reviews some properties of the penalized MLE $\tilde{\theta}_G = \arg\max L_G(\theta)$.

Our results are based on conditions $(\mathcal{L})$ through $(\mathcal{L}_0)$ even if not mentioned explicitly. In particular, we systematically use that the stochastic term in the log-likelihood only linearly depends on $\theta$ and that the expected log-likelihood is concave in $\theta$. We state two results, the first one claims a kind of local concentration of the penalized MLE $\tilde{\theta}_G$, while the second one describes some useful expansions for the estimator $\tilde{\theta}_G$ and for the fitted log-likelihood $L_G(\tilde{\theta}_G)$. The presented results substantially improve similar statements in Spokoiny (2017).

Let $\theta^*_G = \arg\inf_{\theta} \mathbb{E}L_G(\theta)$ and $D^2_G = \mathbb{F}((\theta^*_G) + G^2$. Below we show that the penalized MLE $\tilde{\theta}_G$ concentrates with a high probability in the elliptic set

$$
A_G(r) \overset{\text{def}}{=} \{ \theta : \|D_G(\theta - \theta^*_G)\| \leq r \} \tag{2.4}
$$

under a proper choice of $r$.

As the stochastic component of $L_G(\theta)$ is linear in $\theta$, the gradient $\nabla \zeta = \nabla \{L_G(\theta) - \mathbb{E}L_G(\theta)\}$ does not depend on $\theta$. Under condition $(EV)$, there exists a random set $\Omega(x)$ with $\mathbb{P}(\Omega(x)) \geq 1 - C e^{-x}$ such that on this set

$$
\|D_G^{-1} \nabla \zeta\| \leq z(B_{V|G}, x), \tag{2.5}
$$

where $B_{V|G} = D^{-1}_GV^2D^{-1}_G$ and $z(B_{V|G}, x)$ is given by (A.3); see Theorem A.5. It is worth mentioning that this deviation bound is the only place in the proof where the stochastic nature of the log-likelihood $L(\theta)$ is accounted for. In the rest, we only use the condition $(E)$ about linearity the stochastic component $\zeta(\theta)$ in $\theta$.

With $\delta_3(\theta, u)$ from (2.3), define for each $r > 0$

$$
\delta_{3,G}(r) \overset{\text{def}}{=} \sup_{\theta : \|D_G(\theta - \theta^*_G)\| \leq r} \sup_{u : \|D_Gu\| \leq r} |\delta_3(\theta, u)|. \tag{2.6}
$$

This value is finite under $(\mathcal{L}_0)$ provided that $\{ \theta : \|D_G(\theta - \theta^*_G)\| \leq r \} \subseteq \Theta^0$ and $\{ u : \|D_Gu\| \leq r \} \subseteq U^o$.

Our first result claims that the penalized MLE $\tilde{\theta}_G$ belongs with a high probability to the vicinity $A_G(r)$ from (2.4) with $r \leq 2z(B_{V|G}, x)$. 

Theorem 2.1. Let (2.5) hold on a random set $\Omega(x)$ with $\mathbb{P}(\Omega(x)) \geq 1 - e^{-x}$. Let also $r_G$ be such that $\Theta^0$ contain the set $A_G(r_G) \overset{\text{def}}{=} \{ \theta : \| D_G(\theta - \theta^*_G) \| \leq r_G \}$,

\[
\frac{3\delta_3G(r_G)}{r_G^2} \leq \rho \leq \frac{1}{2},
\]

(1 - \rho)r_G \geq z(B_{V|G}, x).

Then on $\Omega(x)$, the estimate $\tilde{\theta}_G$ belongs to this set as well, that is,

\[
\| D_G(\tilde{\theta}_G - \theta^*_G) \| \leq r_G.
\]

Remark 2.1. In typical situations $\delta_3G(r_G) \asymp r_G^3 n^{-1/2}$ while $z^2(B_{V|G}, x) \asymp p_{V|G} = \text{tr}(V^2 D_G^{-2})$; see Section 3. Therefore, $\delta_3G(r_G) r_G^{-2} \asymp r_G n^{-1/2}$, and conditions (2.7) require only that the value $p_{V|G}$ is smaller than the sample size $n$, i.e. $p_{V|G} \ll n$.

Due to the concentration result of Theorem 2.1, the estimate $\tilde{\theta}_G$ lies with a dominating probability in a local vicinity of the point $\theta^*_G$. Now one can use a quadratic approximation for the penalized log-likelihood process $L_G(\theta)$ to establish an expansion for the penalized MLE $\tilde{\theta}_G$ and for the excess $L_G(\tilde{\theta}_G) - L_G(\theta^*_G)$.

Theorem 2.2. Under the conditions of Theorem 2.1, it holds on $\Omega(x)$

\[
\| D_G(\tilde{\theta}_G - \theta^*_G) - D_G^{-1} \nabla \zeta \| \leq 4 \delta_3G(r_G),
\]

(2.8)

\[
\left| L_G(\tilde{\theta}_G) - L_G(\theta^*_G) - \frac{1}{2} D_G^{-1} \nabla \zeta \right| \leq \delta_3G(r_G),
\]

(2.9)

and also, for any $\theta \in A_G(r_G)$,

\[
\left| L_G(\tilde{\theta}_G) - L_G(\theta) - \frac{1}{2} D_G(\tilde{\theta}_G - \theta) \right| \leq \delta_3G(r_G),
\]

(2.10)

where the random matrix $\tilde{D}_G^2 = F_G(\tilde{\theta}_G)$ fulfills on $\Omega(x)$ for some universal constant $C

\[
\| D_G^{-1}(\tilde{D}_G^2 - D_G^2) D_G^{-1} \| \leq C r_G^{-2} \delta_3G(r_G).
\]

(2.11)

Similarly to Theorem 2.1, the results of Theorem 2.2 are meaningful if $p_{V|G}$ is significantly smaller than $n$. Using the CLT for the standardized score $V^{-1} \nabla \zeta$ and (2.8), one can easily prove asymptotic normality of $\tilde{\theta}_G - \theta^*_G$.

Now we bound the estimation bias $\mathbb{E} L_G(\theta^*_G) - \mathbb{E} L_G(\theta^*)$ and $\theta^*_G - \theta^*$ induced by the penalization $\| G\theta \|^2$. The bias is not critical if the true value $\theta^*$ is “smooth”, that is, $\| G\theta^* \|^2$ is not too big.
Theorem 2.3. It holds

$$\mathbb{E}L_G(\theta^*_G) - \mathbb{E}L_G(\theta^*) \leq \frac{1}{2} \|G\theta^*\|^2.$$  

If, in addition, \(\|G\theta^*\|^2 \leq \frac{r_b^2}{2}\) for some \(r_b\) such that \(\delta_{3,G}(r_b)/r_b^2 \leq 1/2\), then

$$\left| \mathbb{E}L_G(\theta^*_G) - \mathbb{E}L_G(\theta^*) - \|D_G(\theta^*_G - \theta^*)\|^2/2 \right| \leq \delta_{3,G}(r_b),$$

and for \(D^2 = \mathbb{F}(\theta^*_G)\)

$$\|D_G(\theta^* - \theta^*_G) - D_G^{-1}G^2\theta^*\|^2 \leq \|D^{-1}D_G^2(\theta^* - \theta^*_G) - D^{-1}G^2\theta^*\|^2 \leq 4\delta_{3,G}(r_b). \quad (2.12)$$

Moreover, for any linear mapping \(Q\) in \(\mathbb{R}^p\) with \(\sqrt{Q^2} \leq D^{-1}D_G^2\), it holds

$$\|Q(\theta^*_G - \theta^*)\| \leq \|QD_G^{-2}G^2\theta^*\| + 2\sqrt{\delta_{3,G}(r_b)}. \quad (2.13)$$

One can apply \(Q = D\) for the prediction problem and \(Q = \sqrt{n}I_p\) for the estimation problem. Putting together the results of Theorem 2.1 through 2.3 allows to bound the error \(Q(\tilde{\theta}_G - \theta^*)\) with a high probability and establish a bound for the risk \(\mathbb{E}\|Q(\tilde{\theta}_G - \theta^*)\|^q\) with any \(q > 0\).

2.3 Posterior concentration

Now we turn to the properties of the shifted posterior \(\theta_G - \tilde{\theta}_G\ \mid Y\). Our first result shows that it concentrates on the elliptic set \(\bar{B}(r_0) = \{u: \|\bar{D}u\| \leq r_0\}\) with \(\bar{D}^2 = \mathbb{F}(\tilde{\theta}_G)\) and a proper value \(r_0 \geq C\sqrt{p_G} + C\sqrt{x}\) for \(p_G = p_G(\tilde{\theta}_G);\) see \((D\mid G)\). For this we bound from above the random quantity

$$\rho(r_0) \overset{\text{def}}{=} \frac{\int \mathbb{I}(\|\bar{D}u\| > r_0) \exp\{L_G(\tilde{\theta}_G + u)\} du}{\int \mathbb{I}(\|\bar{D}u\| \leq r_0) \exp\{L_G(\tilde{\theta}_G + u)\} du}. \quad (2.14)$$

Obviously \(P(\theta_G - \tilde{\theta}_G \notin \bar{B}(r_0) \mid Y) \leq \rho(r_0)\). Therefore, small values of \(\rho(r_0)\) indicate a concentration of \(\theta_G - \tilde{\theta}_G\ \mid Y\) on the set \(\bar{B}(r_0)\).

Let \(\Theta^0\) be an open subset of \(\Theta\) that contains the concentration set \(A_G(r_0) = \{\theta: \|D_G(\theta - \theta^*_G)\| \leq r_G\}\) of \(\tilde{\theta}_G\); see Theorem 2.1. Our results rely on maximum of the quantities \(\delta_m(\theta, u)\) from \((L_\theta)\) over the all \(\theta \in \Theta^0\) and \(u \in U^0\) with \(\|\sqrt{F(\theta)}u\|\) bounded:

$$\delta_m(r_0) = \delta_m(\Theta^0, r) \overset{\text{def}}{=} \sup_{\theta \in \Theta^0} \sup_{u: \|\sqrt{F(\theta)}u\| \leq r} \|\delta_m(\theta, u)\|, \quad m = 3, 4. \quad (2.15)$$
Theorem 2.4. Let conditions of Theorem 2.1 be satisfied. Let, for some fixed values \( r_0 \) and \( x > 0 \), it hold

\[
\hat{\delta}(r_0) \overset{\text{def}}{=} 4\delta_3^2(r_0) + 4\delta_4(r_0) \leq 1/2, \tag{2.16}
\]

\[
C_0 \overset{\text{def}}{=} 1 - 3r_0^{-2}\delta_3(r_0) \geq 1/2.
\]

\[
C_0r_0 \geq 2\sqrt{\mathbb{P}_G(\theta)} + \sqrt{x}, \quad \theta \in \Theta^o. \tag{2.17}
\]

Then, on the random set \( \Omega(x) \) from Theorem 2.1, the quantity \( \rho(r_0) \) from (2.14) fulfills

\[
\rho(r_0) \leq \frac{1}{1 - \hat{\delta}(r_0)} \frac{\exp\{-x/2\}}{1 - \exp\{-x/2\}}, \tag{2.18}
\]

where \( \bar{\mathbb{P}}_G = \mathbb{P}_G(\bar{\theta}_G) \).

Remark 2.2. The bound (2.16) is meaningful if \( \delta_3(r_0) \) is small. If \( \delta_3(r_0) \asymp r_0^3/\sqrt{n} \), then by (2.17) we need \( p_G^3(\theta) \ll n \) for a sensible concentration result.

Remark 2.3. It has been already mentioned that \( p_G(\theta) \asymp p_{Y|G} \) for \( \theta \in \Theta^o \) and hence, \( r_0 \asymp r_G \). Therefore, the concentration results for the penalized MLE \( \bar{\theta}_G \) and for the posterior \( \mathbb{P}_G|Y \) look similar, but there is one essential difference. The properly shifted and scaled posterior \( D_G(\bar{\theta}_G - \hat{\theta}_G) \) well concentrates in a rather small elliptic vicinity \( B_G(r_G) = \{u : \|D_Gu\| \leq r_G\} \). In other words, \( D_G(\bar{\theta}_G - \hat{\theta}_G) \) belongs to the ball in \( \mathbb{R}^p \) of radius \( r_G \) with a high probability. This holds true even if \( p = \infty \). In the contrary, the shifted and scaled posterior \( D_G(\bar{\theta}_G - \hat{\theta}_G) | Y \) does not concentrate on a ball in \( \mathbb{R}^\infty \) for any radius \( r \). Our result of Theorem 2.4 claims concentration on a larger set \( \bar{B}(r_0) \), also with an elliptic shape, but larger axes corresponding to \( \bar{D} \) instead of \( D_G \).

2.4 Gaussian approximation for the posterior

The concentration result can be restated in the form that the centered posterior \( \theta_G - \bar{\theta}_G \) concentrates on the random set

\[
\bar{B}(r_0) \overset{\text{def}}{=} \{u \in \mathbb{R}^p : \|\bar{D}u\| \leq r_0\}.
\]

Now we aim to show that, after restricting to this set, the posterior can be well approximated by a Gaussian distribution \( \mathcal{N}(\bar{\theta}_G, \bar{D}_G^{-2}) \). This allows to apply the results for the Gaussian case to more general non-Gaussian models. In what follows we use that \( \bar{\theta}_G \) is random on the original probability space, however, it can be considered as fixed under the posterior measure. By \( \mathbb{P}' \) we denote a standard normal distribution of a random
vector $\gamma \in \mathbb{R}^p$ given $\bar{D}_G = D_G(\bar{\theta}_G)$. In our first result we distinguish between the class $\mathcal{B}_s(\mathbb{R}^p)$ of centrally symmetric Borel sets and the class $\mathcal{B}(\mathbb{R}^p)$ of all Borel sets in $\mathbb{R}^p$.

**Theorem 2.5.** Let the conditions of Theorem 2.4 hold and $\rho(x_0)$ satisfy (2.18). It holds on the set $\Omega(x)$ from Theorem 2.1 for any centrally symmetric Borel set $A \in \mathcal{B}_s(\mathbb{R}^p)$

$$\mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) \geq \frac{1 - \diamond(x_0)}{\{1 + \diamond(x_0) + \rho(x_0)\}} \mathbb{P}'(\bar{D}_G^{-1}\gamma \in A) - \rho(x_0),$$

$$\mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) \leq \frac{1 + \diamond(x_0)}{\{1 - \diamond(x_0)\}} \mathbb{P}'(\bar{D}_G^{-1}\gamma \in A) + \rho(x_0).$$

For any measurable set $A \in \mathcal{B}(\mathbb{R}^p)$, similar bounds hold with $\delta_3(x_0)$ in place of $\diamond(x_0)$.

The first result of the theorem for can be represented in the form

$$\left| \mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) - \mathbb{P}'(\bar{D}_G^{-1}\gamma \in A) \right| \lesssim \mathbb{P}'(\bar{D}_G^{-1}\gamma \in A) \{\diamond(x_0) + e^{-x}\} + \rho(x_0).$$

Here and below “$a \lesssim b$” means $a \leq Cb$ with an absolute constant $C$ that possibly depends on the constants from our conditions. The second statement applies to any $A \in \mathcal{B}(\mathbb{R}^p)$ and hence, it bounds the distance in total variation between the posterior and its Gaussian approximation $\bar{D}_G^{-1}\gamma$.

**Corollary 2.6.** Suppose that $x_0$ satisfies the conditions (2.16) and (2.17) with $x = 2\log n$. It holds on $\Omega(x)$

$$\sup_{A \in \mathcal{B}_s(\mathbb{R}^p)} \left| \mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) - \mathbb{P}'(\bar{D}_G^{-1}\gamma \in A) \right| \lesssim \diamond(x_0) + 1/n,$$

$$\sup_{A \in \mathcal{B}(\mathbb{R}^p)} \left| \mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) - \mathbb{P}'(\bar{D}_G^{-1}\gamma \in A) \right| \lesssim \delta_3(x_0) + 1/n. $$

Comparison of two bounds of Corollary 2.6 reveals that the use of symmetric credible sets improves the accuracy of Gaussian approximation from $\delta(x_0)$ to $\diamond(x_0) \propto \delta_3^2(x_0) + \delta_4(x_0)$. In typical regular cases, $\delta_3(x_0) \propto \sqrt{x_0}/n$ while $\diamond(x_0) \propto x_0^3/n$. The choice $x = 2\log n$ and $x_0 = C(\sqrt{\rho_G} + \sqrt{\log n})$ yields $\rho(x_0) \leq 1/n$, and the only leading term in the error of approximation is $\diamond(x_0) \propto p_G^3/n$, and this is the guaranteed approximation error in the BvM approximation under symmetricity. The bound in TV-distance ensures an error $\delta_3(x_0) \propto \sqrt{p_G^3/n}$.

It is natural to expect that a small departure from symmetricity still yields a good approximation. We present one result of this flavor. Let $a$ be a possibly random vector in $\mathbb{R}^p$. The next result assumes that $a$ is sufficiently small so that $\|\bar{D}_G a\| \leq 1$. 
Theorem 2.7. Let the conditions of Theorem 2.4 hold. Suppose that \( x_0 \) satisfies the conditions (2.16) and (2.17) with \( x = 2 \log n \). Let a random vector \( a \) satisfy \( \| D_G a \| \leq 1 \) a.s. It holds on the set \( \Omega(x) \) from Theorem 2.1 for any set \( A \in \mathcal{B}_s(\mathbb{R}^p) \)

\[
\left| P(\vartheta_G - \tilde{\vartheta}_G - a \in A \mid Y) - P'(\tilde{D}_G^{-1} \gamma - a \in A) \right| 
\lesssim \left\{ \hat{\varphi}(x_0) + \delta_3(x_0)\| \tilde{D}_G a \| \right\} P'(\tilde{D}_G^{-1} \gamma \in A) + 1/n. 
\tag{2.19}
\]

We accomplish the result of this theorem with the bounds on the first two moments of the posterior. The results claim that posterior mean and variance are close to that of normal \( N(\tilde{\vartheta}_G, \tilde{D}_G^{-2}) \). Let \( \overline{\vartheta}_G = \mathbb{E}(\vartheta_G \mid Y) \) and \( a = \overline{\vartheta}_G - \tilde{\vartheta}_G \). Note that symmetricity arguments do not apply to the posterior mean, therefore one can expect an accuracy of order \( \| a \| \simeq \delta_3(x_0) \simeq \sqrt{p_2} \). In the contrary, the posterior covariance can be estimated with a higher accuracy of order \( \hat{\varphi}(x_0) \), again by symmetricity arguments. The next results describes the moments of \( Q(\vartheta_G - \tilde{\vartheta}_G) \) for an arbitrary linear operator \( Q \) with \( Q^T Q \leq D_G^2 \).

Theorem 2.8. Let the conditions of Theorem 2.4 hold. Let \((L_0)\) be fulfilled with \( x_0 \) satisfying (2.17) with \( \sqrt{p}(\vartheta_G + 1) \) in place of \( \sqrt{p}(\vartheta_G) \), and \( \hat{\varphi}(x_0) \) be given by (2.16) and (2.17) with \( x = 2 \log n \). Then \( \overline{\vartheta}_G = \mathbb{E}(\vartheta_G \mid Y) \) fulfills on the set \( \Omega(x) \) for any linear operator \( Q \)

\[
\|Q(\overline{\vartheta}_G - \tilde{\vartheta}_G)\| \lesssim \delta_3(x_0) \sqrt{p_{Q|G}} + n^{-1} p_{Q|G}, \tag{2.20}
\]

where \( p_{Q|G} = \text{tr}(Q \tilde{D}_G^{-2} Q^T) \). The posterior variance \( \text{Var}(\vartheta_G \mid Y) \) fulfills on \( \Omega(x) \)

\[
\left| I - \tilde{D}_G \text{Var}(\vartheta_G \mid Y) \tilde{D}_G \right| = \sup_{z \in \mathcal{B}_p} \mathbb{E} \left( \left| \langle z, \tilde{D}_G(\vartheta_G - \tilde{\vartheta}_G) \rangle \right|^2 \mid Y \right) - 1 \lesssim \hat{\varphi}(x_0),
\]

\[
\mathbb{E} \left( \left| Q(\vartheta_G - \tilde{\vartheta}_G) \right|^2 \mid Y \right) - p_{Q|G} \lesssim \hat{\varphi}(x_0) p_{Q|G}.
\]

Remark 2.4. The result (2.20) of Theorem 2.8 does not apply to \( Q = \tilde{D}_G \) for \( p = \infty \) because \( \tilde{D}_G(\vartheta_G - \tilde{\vartheta}_G) \) is nearly standard normal given \( Y \) and \( p_{Q|G} = \infty \). However, it well applies to \( Q = \tilde{D} \) yielding

\[
\| \tilde{D}(\vartheta_G - \tilde{\vartheta}_G) \| \lesssim \delta_3(x_0) \sqrt{p_G}, \quad \mathbb{E} \left( \left| \tilde{D}(\vartheta_G - \tilde{\vartheta}_G) \right|^2 \mid Y \right) \approx p_G.
\]

Another typical choice of \( Q \) is \( Q = \tilde{D}_G \Pi_m \), where \( \Pi_m \) is the projector on the first \( m \) eigenvectors of \( \tilde{D}_G \). Then \( p_{Q|G} = m \).
The results on posterior concentration (Theorem 2.4), and on Gaussian approximation (Corollary 2.6) suggest elliptic credible sets of the form \( \{ \theta : \| Q(\theta - \tilde{\theta}_G)\| \leq r \} \). A natural question here is whether one can use the posterior mean \( \bar{\theta}_G \) as a proxy for \( \tilde{\theta}_G \) leading to the sets \( \{ \theta : \| Q(\theta - \bar{\theta}_G)\| \leq r \} \) centered at \( \bar{\theta}_G \). To check that the shift by the vector \( a = \bar{\theta}_G - \tilde{\theta}_G \) does not significantly affect the credible probability, one can try to apply Theorem 2.8 and Gaussian comparison bound from Theorem A.1. However, the set \( E_Q(r) \) shifted by a vector \( a \), is not anymore centrally symmetric around \( \tilde{\theta}_G \), and the nice bounds for accuracy of Gaussian approximation from Theorem 2.5 and Corollary 2.6 do not apply. Instead, we use the bound (2.19) of Theorem 2.7 expecting that \( a \) is small and \( \| \tilde{D}_G a \| \ll 1 \) due to (2.20) of Theorem 2.8. This and the Gaussian comparison bound of Theorem A.1 yield the following result.

**Theorem 2.9.** Let the conditions of Theorem 2.8 be satisfied, let \( Q = Q \Pi \) where \( \Pi \) is a projector in \( \mathbb{R}^p \). Then it holds on the set \( \Omega(x) \) shown in Theorem 2.1

\[
\sup_{r>0} \left| P \left( \| Q(\theta_G - \bar{\theta}_G)\| \leq r \mid Y \right) - P' \left( \| Q\tilde{D}_G^{-1} \gamma\| \leq r \right) \right| \lesssim \diamondsuit(r_0) \sqrt{\bar{p} \Pi} + n^{-1} \bar{p} \Pi
\]

with \( \bar{p} \Pi \overset{\text{def}}{=} \text{tr}(\Pi \tilde{D}_G \Pi \tilde{D}_G^{-1} \Pi) \).

**Remark 2.5.** If \( \tilde{D}_G \Pi = \Pi \tilde{D}_G \), then \( \bar{p} \Pi = \text{dim}(\Pi) \). In particular, if \( Q = \tilde{D}_G \Pi_m \) for the eigenprojector \( \Pi_m \) as in Remark 2.4, then \( \bar{p} \Pi = \bar{p} \Pi_m = m \). One can see that the use of the posterior mean instead of the penalized MLE \( \tilde{\theta}_G \) is justified under a bit stronger condition “\( \diamondsuit(r_0) \sqrt{\bar{p} \Pi} + n^{-1} \bar{p} \Pi \) is small” compared to “\( \diamondsuit(r_0) \) is small”.

### 2.5 Prior comparison and prior impact

The classical BvM result claims that the prior impact asymptotically washes out, as the sample size increases. The posterior becomes close to the normal distribution with the same distribution as for the MLE \( \bar{\theta} \), namely, to \( N(\bar{\theta}, D^{-2}) \). It is well understood that a general BvM result is impossible in a infinite dimensional nonparametric set-up whatever sample size is. In this section we want to quantify the accuracy of the BvM approximation using the obtained bounds on the Gaussian approximation of the posterior and the results on Gaussian comparison. As in previous sections we show that restricting to the class of elliptic sets helps to improve the bounds. We slightly change the statement of the problem and consider it as a problem of prior impact. Let \( G^{-2} \) and \( G_1^{-2} \) be prior covariance matrices for two different priors. We want to compare their posteriors. A special case of interest is when \( G_1 \) is fixed and \( G^2 \) approaches zero. The limiting case corresponds to the Bvm approximation. By Theorem 2.5 the posterior \( \theta_G \) is nearly
Gaussian $\mathcal{N}(\overline{\theta}_G, \overline{D}_G^{-2})$. Similarly the posterior $\theta_G$ is close to $\mathcal{N}(\overline{\theta}_G, \overline{D}_G^{-2})$. We are interested if elliptic credible sets calibrated for for the simple $G_1$-prior can be used for the more complex $G$-prior. The Gaussian approximation reduces this question to Gaussian comparison; see Götze et al. (2019) or Theorem A.1. Motivated by the above discussion we assume $G^2 \leq G_1^2$.

**Theorem 2.10.** Let the conditions of Theorem 2.8 be satisfied for two priors $\mathcal{N}(0, G^{-2})$ and $\mathcal{N}(0, G_1^{-2})$ with $G^2 \leq G_1^2$. Then it holds on a set $\Omega(x)$ with $\mathbb{P}(\Omega(x)) \geq 1 - 2/n$

$$\sup_r \left| \mathbb{P} \left( \|Q(\theta_G - \overline{\theta}_G)\| \leq r \mid Y \right) - \mathbb{P} \left( \|Q(\theta_{G_1} - \overline{\theta}_{G_1})\| \leq r \mid Y \right) \right| \leq \delta_3(r_0) + n^{-1} + \frac{1}{\|Q\overline{D}_G^{-2}Q^\top\|_{Fr}} \left\{ \text{tr}((Q\overline{D}_G^{-2} - \overline{D}_G^{-2})Q^\top) + \|Q(\overline{\theta}_G - \overline{\theta}_{G_1})\|^2 \right\}. \quad (2.21)$$

**Remark 2.6.** The last term in the bound (2.21) comes from the Gaussian comparison result of Theorem A.1. It includes the “variance” part that relates two covariance operators $Q\overline{D}_G^{-2}Q^\top$ and $Q\overline{D}_{G_1}^{-2}Q^\top$, and the “squared bias” term $\|Q(\overline{\theta}_G - \overline{\theta}_{G_1})\|^2$. Applicability of the prior $G_1$ in place of $G$ is justified under “small bias” condition $\|Q(\overline{\theta}_G - \overline{\theta}_{G_1})\|^2 \ll \|Q\overline{D}_G^{-2}Q^\top\|_{Fr}$, and under the “variance” condition $\overline{p}_G - \overline{p}_{G_1} \ll \|Q\overline{D}_G^{-2}Q^\top\|_{Fr}$.

### 2.6 Some extensions

Here we list some possible straightforward extensions of the results presented above. We do not present exact statements, just explain how the obtained results can be used for studying the standard asymptotic questions like the rate of contraction and Bayesian confidence sets.

#### 2.6.1 Contraction rate

In statistical literature, one usually aims at bounding the distance between the support of the posterior and the true value $\theta^*$. The difference $\theta_G - \theta^*$ can be decomposed as

$$\theta_G - \theta^* = (\theta_G - \overline{\theta}_G) + (\overline{\theta}_G - \theta^*) = (\theta_G - \overline{\theta}_G) + (\overline{\theta}_G - \overline{\theta}_G^*) + (\overline{\theta}_G^* - \theta^*).$$

For any $Q \leq D_G$, the results of Section 2.2 allow to bound with a high probability

$$\|Q(\overline{\theta}_G - \overline{\theta}_G^*)\|^2 \lesssim \text{tr}(QD_G^{-2}V)^2.$$

Moreover, usually $V^2 \approx D^2 \leq D_G^2$ and $\text{tr}(QD_G^{-2}V)^2 \asymp \text{tr}(Q^2D_G^{-2})$. Similarly, the results of Section 2.3 yield with high probability for a proper constant $c$

$$\mathbb{P}(\|Q(\theta_G - \overline{\theta}_G)\| \geq r_Q \mid Y) \leq n^{-1}, \quad r_Q^2 \geq c \{ \text{tr}(Q^2\overline{D}_G^{-2}) + \log n \}.$$
As \( \text{tr}(Q^2 \tilde{D}_G^{-2}) \approx \text{tr}(Q^2 D_G^{-2}) \), we conclude: the posterior enjoys essentially the same concentration properties as the penalized MLE \( \tilde{\theta}_G \).

For the bias \( \theta^*_G - \theta^* \), by Theorem 2.3, for any \( Q \leq D \)

\[
\|Q(\theta^*_G - \theta^*)\|^2 \approx \|QD_G^{-1}G^2\theta^*\|^2 \leq \|G\theta^*\|^2.
\]

The bias-variance trade-off corresponds to the relation

\[
\|QD_G^{-1}G^2\theta^*\|^2 \propto \text{tr}(Q\tilde{\theta}_G) \propto \text{tr}(QD_G^{-2}V)^2.
\]

**Corollary 2.11.** Assume that

\[
\|QD_G^{-2}G^2\theta^*\|^2 \lesssim \text{tr}(Q^2 D_G^{-2}), \quad \text{tr}(QD_G^{-2}V)^2 \lesssim \text{tr}(Q^2 D_G^{-2}).
\]

Then it holds on \( \Omega(x) \) for some fixed \( c, c_1, c_2 \)

\[
P\left( \|Q(\theta_G - \theta^*)\|^2 \geq c \text{tr}(Q^2 D_G^{-2}) + c_1 \log n \mid Y \right) \leq c_2 n^{-1}.
\]

A prior ensuring the bias-variance trade-off leads to the optimal contraction rate which corresponds to the optimal penalty choice in penalized maximum likelihood estimation.

**Example 2.1.** Consider \( V^2 = D^2 = n\sigma^{-2}I_p \) and \( Q = D \), and let \( \theta^* \) belong to a Sobolev ball \( B_s(1) = \{ \theta : \sum_j \theta_j^2 j^{2s} \leq 1 \} \) for \( s > 1/2 \). Define \( G^2 = p_s \text{diag}\{1, \ldots, j^{2s}, \ldots\} \), where \( p_s \) is fixed by the relation \( p_s^{2s+1} \approx \sigma^{-2}n \) yielding \( p_s = (\sigma^{-2}n)^{1/(2s+1)} \). By definition \( p_s j^{2s} + \sigma^{-2}n \leq 2\sigma^{-2}n \) for \( j \leq p_s \) and one easily estimate under \( s > 1/2 \)

\[
p_G = \text{tr}(D^2 D_G^{-2}) = \sum_j \frac{\sigma^{-2}n}{p_s j^{2s} + \sigma^{-2}n} \lesssim p_s, \quad \|G\theta^*\|^2 \leq p_s,
\]

implying the bias-variance trade-off \( p_G \propto \|G\theta^*\|^2 \propto (\sigma^{-2}n)^{1/(2s+1)} \) and the contraction rate \( (\sigma^{-2}n)^{-s/(2s+1)} \) for the posterior \( \theta \mid Y \). Namely, it holds on \( \Omega(x) \)

\[
P\left( \sigma^{-1} \sqrt{n} \|\theta - \theta^*\| \geq c (\sqrt{p_G} + \sqrt{\log n}) \mid Y \right) \leq n^{-1}.
\]

**2.6.2 Coverage probability**

One of the main questions of nonparametric Bayes approach is whether one can use Bayesian credible sets as frequentist confidence sets. Corollary 2.6 suggests to consider credible sets of the form

\[
\mathcal{A}_{Q|G}(r) \overset{\text{def}}{=} \{ \theta : \|Q(\tilde{\theta}_G - \theta)\| \leq r \},
\]
where $Q \leq D$ and $r = r_\alpha$ is fixed to ensure
\[
P'(\|Q\tilde{D}_G^{-1}\gamma\| > r_\alpha) \leq \alpha.
\]
Our results allow to reduce this question to reliability of pMLE-based confidence sets. Indeed, note first that by definition, it holds for the true parameter $\theta^*$:
\[
P(\theta^* \in \mathcal{A}_Q|G(r)) = P(\|Q(\tilde{\theta}_G - \theta^*)\| \leq r).
\]
Suppose that \textbf{(EV)} holds with $V = D$ and moreover, the standardized score $D^{-1}\nabla \zeta$ is asymptotically standard normal. The Fisher expansion (2.8) $\tilde{\theta}_G - \theta^*_G \approx D_G^{-2}\nabla \zeta$ of Theorem 2.2 combined with the CLT $D^{-1}\nabla \zeta \overset{w}{\rightarrow} \gamma$ for a standard normal $\gamma$ reduces the latter question to Gaussian probability
\[
P(\|Q(\tilde{\theta}_G - \theta^*)\| \leq r) \approx P(\|Q(D_G^{-2}D\gamma + \theta^*_G - \theta^*)\| \leq r).
\]
By Gaussian comparison Theorem A.1, the impact of the bias $\theta^*_G - \theta^*$ is negligible under the undersmoothing condition $\|Q(\theta^*_G - \theta^*)\|^2 \ll \text{tr}(QD_G^{-2}D)^2$. Combining with the bound (2.13) of Theorem 2.3 yields the sufficient condition
\[
\frac{\|QD_G^{-2}G^2\theta^*\|^2}{\text{tr}(QD_G^{-2}D)^2} = o(1). \tag{2.22}
\]
Under this condition, we can derive in view of $D_G^{-2}D \leq D^{-1}_G$
\[
1 - \alpha \approx P'(\|Q\tilde{D}_G^{-1}\gamma\| \leq r_\alpha) \approx P(\|QD_G^{-1}\gamma\| \leq r_\alpha) \leq P(\|QD_G^{-2}D\gamma\| \leq r_\alpha)
\]
that is, the credible set $\mathcal{A}_Q|G(r_\alpha)$ is an asymptotically valid confidence set. We conclude that “small bias condition” (2.22) and correct noise specification ensure frequentist validity of the credible sets. The key observation is that the variance $D_G^{-2}\text{Var}(\nabla \zeta)D_G^{-2}$ of the pMLE is smaller than the variance $D_G^{-2}$ of the posterior.

2.6.3 A family of priors and uniform Gaussian approximation

Consider a more general situation when a family of Gaussian priors $\mathcal{N}(0, G^{-1}_\kappa)$, $\kappa \in \mathcal{M}$, is given. For each of them, under appropriate conditions, one can state the Gaussian approximation result as in Corollary 2.6 or Theorem 2.5. The choice of a prior by empirical or full Bayes approaches requires to state this approximation uniformly in $\kappa \in \mathcal{M}$. Surprisingly, in the contrary to the classical frequentist model selection, such
a uniform approximation can be stated in a straightforward way. In fact, all the results about posterior distribution are stated conditionally on the data after restricting to the random set \( \Omega(\mathbf{x}) \) on which a deviation bound \( \|D_{G}^{-1}\nabla \zeta\| \leq z(B_{V}(\mathbf{x})) \) holds. All we need is a uniformly in \( \varkappa \) version of this bound. Note that the conditions \( (\mathcal{L}) \), \( (E) \) \( (EV) \) do not involve any prior. The further conditions \( (V|G) \), \( (D|G) \), \( (\mathcal{L}_{0}) \) will be assumed uniformly in \( G \in \{G_{\varkappa}\} \). Conditions \( (V|G) \), \( (D|G) \) only require that the penalization by any \( G_{\varkappa}^{2} \) is strong enough to ensure a uniform upper bound on the values \( p_{V|G}(\theta) \) and \( p_{G}(\theta) \). Condition \( (\mathcal{L}_{0}) \) depends on \( G \) only through the sets \( \Theta^{0} \) and \( U^{0} \) which typically can be taken the same for all \( G_{\varkappa} \). In the case when the family \( \{G_{\varkappa}\} \) contains the smallest covariance \( G_{\min}^{2} \leq G_{\varkappa}^{2} \), it suffices to check conditions \( (V|G) \), \( (D|G) \), and \( (\mathcal{L}_{0}) \) for \( G_{\min}^{2} \) only. The next result describes the uniform properties of the estimators \( \hat{\theta}_{\varkappa} \) and the posteriors \( \vartheta_{\varkappa} \). Everywhere we write the subindex \( \varkappa \) in place of \( G_{\varkappa} \). In particular, \( D_{\varkappa}^{2} = \mathbb{F}_{\varkappa}(\theta_{r}) = \mathbb{F}(\theta_{r}) + G_{\varkappa}^{2} \) with \( \mathbb{F}(\theta) = -\nabla^{2} \mathbb{E}L(\theta) \).

**Theorem 2.12.** Let the conditions \( (\mathcal{L}) \), \( (E) \) \( (EV) \) be fulfilled. Let also \( \{\mathcal{N}(0, G_{\varkappa}^{-2}) \}, \varkappa \in \mathbb{M} \) be a family of Gaussian priors, and let the conditions \( (V|G) \), \( (D|G) \), \( (\mathcal{L}_{0}) \) be satisfied uniformly in \( \varkappa \in \mathbb{M} \). Let also there exist a random set \( \Omega(\mathbf{x}) \) with \( \mathbb{P}(\Omega(\mathbf{x})) \geq 1 - e^{-\mathbf{x}} \) and an upper function \( z(B_{V}(\mathbf{x})) \) of \( \varkappa \) such that it holds on \( \Omega(\mathbf{x}) \)

\[
\|D_{\varkappa}^{-1}\nabla \zeta\| \leq z(B_{V}(\mathbf{x})), \quad \forall \varkappa \in \mathbb{M}.
\]

Then all the statements of Theorem 2.1 through 2.10 are fulfilled on \( \Omega(\mathbf{x}) \) uniformly in \( \varkappa \in \mathbb{M} \). In particular, each of \( \hat{\theta}_{\varkappa} \) concentrates on the elliptic set

\[
\mathcal{A}_{\varkappa}(r_{\varkappa}) = \{\theta: \|D_{\varkappa}(\theta - \theta_{r})\| \leq r_{\varkappa}\},
\]

with the center at \( \theta_{r}^{*} \), while the posterior \( \vartheta_{\varkappa} | Y \) concentrates on a larger vicinity

\[
\tilde{B}_{\varkappa}(r_{\varkappa}) = \{\theta: \|\mathbb{F}^{1/2}(\tilde{\theta}_{\varkappa})(\theta - \tilde{\theta}_{\varkappa})\| \leq r_{\varkappa}\}
\]
centered at \( \tilde{\theta}_{\varkappa} \). Each posterior \( \vartheta_{\varkappa} | Y \) can be approximated by the Gaussian \( \mathcal{N}(\tilde{\theta}_{\varkappa}, \tilde{D}_{\varkappa}^{-2}) \), the error of approximation is given by Corollary 2.6 or Theorem 2.5.

This result follows from the fact that after restricting to the set \( \Omega(\mathbf{x}) \) we only operate with deterministic function \( \mathbb{E}L(\theta) \) and use its local smoothness properties from \( (\mathcal{L}_{0}) \).

**Remark 2.7.** The probabilistic bound \( \mathbb{P}(\|D_{\varkappa}^{-1}\nabla \zeta\| > z(B_{V}(\mathbf{x}))) \leq e^{-\mathbf{x}} \) follows from \( (EV) \) for each \( \varkappa \in \mathbb{M} \). If \( \mathbb{M} \) is a finite set and \( |\mathbb{M}| \) is its cardinality then a uniform version of this bound can be easily obtained by an increase of \( \mathbf{x} \) to \( \mathbf{x} + \log |\mathbb{M}| \).
2.6.4 Non-Gaussian priors

The assumption of a Gaussian prior was essential in the stated results. However, the applied technique allows to extend the approach to a broad class of non-Gaussian priors under some mild assumptions. For simplicity we restrict ourselves to the case of Gaussian likelihood $L(\theta)$ which is a quadratic function of $\theta$, so that $-\nabla^2 L(\theta) = F$ for a symmetric operator $F$ in $\mathbb{R}^p$. Now, let the prior has a log-concave density $\Pi(\theta)$ which we also assume to be a sufficiently smooth. More precisely, we suppose that

$$G^2(\theta) \overset{\text{def}}{=} -\nabla^2 \log \Pi(\theta) \geq 0.$$

Define

$$F_G(\theta) \overset{\text{def}}{=} F + G^2(\theta)$$

and redefine $\delta_m(\theta, u)$ in (2.3) using $\log \Pi(\theta)$ in place of $\mathbb{E}L(\theta)$. It is rather straightforward to see that with this exchange, all the previous results continue to apply without any change.

3 Examples

In this section we illustrate the general results of the Section 2 by applying to nonparametric density estimation and generalized regression. Log-density model is a popular example in statistical literature related to BvM Theorem and nonparametric Bayes study. We mention Castillo and Nickl (2014), Castillo and Rousseau (2015) among many others. Generalized regression model includes the logit model for binary response or classification problems, Poisson and Cox regression, several reliability models and so on. The related BvM results can be found e.g. in Castillo and Nickl (2014), Ghosal and van der Vaart (2017) and references therein. The results on Gaussian approximation of the posterior are typically asymptotic and do not provide any accuracy guarantees for this approximation. Our results are stated for finite samples and deliver the quantitative and tight bounds on the accuracy of this approximation in terms of effective dimension of the problem.

3.1 Nonparametric log-density estimation

Suppose we are given a random sample $X_1, \ldots, X_n$ in $\mathbb{R}^d$. The i.i.d. model assumption means that all these random variables are independent identically distributed from some measure $P$ with a density $f(x)$ with respect to a $\sigma$-finite measure $\mu_0$ in $\mathbb{R}^d$. This
density function is the target of estimation. By definition, the function \( f \) is non-negative, measurable, and integrates to one:

\[
\int f(x) \mu_0(dx) = 1.
\]

Here and below, the integral \( \int \) without limits means the integral over the whole space \( \mathbb{R}^d \). If \( f(\cdot) \) has a smaller support \( X \), one can restrict integration to this set. To recover \( f \) from observed data \( X_1, \ldots, X_n \), this function is usually assumed to possess some smoothness properties.

Below we parametrize the model by a linear decomposition of the log-density function. Let \( \{ \psi_j(x), j = 1, \ldots, p \} \) be a collection of functions in \( \mathbb{R}^d \) satisfying

\[
\int \exp \{ t|\psi_j(x)| \} \mu_0(dx) < \infty \quad (3.1)
\]

for some \( t > 0 \). Here the dimension \( p \) is either infinity or very large. Denote by \( \Theta_1 \) the subset in \( \mathbb{R}^p \) of all \( \theta \) satisfying

\[
\int \exp \left\{ \sum_{j=1}^p \theta_j \psi_j(x) \right\} \mu_0(dx) < \infty.
\]

The Hölder inequality and condition (3.1) imply that \( \Theta_1 \) is a convex set in \( \mathbb{R}^p \). Below we assume that the log of the unknown density function \( f(x) \) can be expanded using the basis \( \psi_j \). For each \( \theta \in \Theta_1 \), define

\[
\ell(x, \theta) \overset{\text{def}}{=} \sum_{j=1}^p \theta_j \psi_j(x) - \phi(\theta),
\]

where \( \phi(\theta) \) is a constant given by

\[
\int \exp \left\{ \sum_{j=1}^p \theta_j \psi_j(x) \right\} \mu_0(dx) = e^{\phi(\theta)}.
\]

Equivalently

\[
\phi(\theta) = \log \int e^{\langle \Psi(x), \theta \rangle} \mu_0(dx), \quad (3.2)
\]

where \( \Psi(x) \) is a vector with components \( \psi_j(x) \). By the Jensen inequality, the function \( \phi(\theta) \) is convex. Linear log-density modeling assumes that

\[
\log f(x) = \ell(x, \theta^*) = \sum_{j=1}^p \theta_j^* \psi_j(x) - \phi(\theta^*) \quad (3.3)
\]
for some $\theta^* \in \Theta_1$. A nice feature of such representation is that the function $\log f(x)$ in the contrary to the density itself does not need to be non-negative. One more important benefit of using the log-density is that the stochastic part of the corresponding log-likelihood expression has a linear structure w.r.t. the parameter $\theta$. Indeed, the log-likelihood $L(\theta)$ reads as
\[
L(\theta) = \sum_{i=1}^n \ell(X_i, \theta) = \sum_{i=1}^n \langle \Psi(X_i), \theta \rangle - n\phi(\theta) = \langle S, \theta \rangle - n\phi(\theta)
\]
with a random vector $S$ given by
\[
S \overset{\text{def}}{=} \sum_{i=1}^n \Psi(X_i), \quad \Psi(X_i) \overset{\text{def}}{=} (\psi_1(X_i), \psi_2(X_i), \ldots)^\top.
\]
Now we discuss how the general conditions of Section 2.1 can be verified for the log-density model. First note that the generalized linear structure of the model yields automatically conditions $\mathcal{L}$ and $\mathcal{E}$. Indeed, convexity of $\phi(\cdot)$ implies that $\mathbb{E}L(\theta) = \langle \mathbb{E}S, \theta \rangle - n\phi(\theta)$ is concave. Further, for the stochastic component $\zeta(\theta) = L(\theta) - \mathbb{E}L(\theta)$, it holds
\[
\zeta(\theta) = L(\theta) - \mathbb{E}L(\theta) = \sum_{i=1}^n \langle \Psi(X_i) - \mathbb{E}\Psi(X_i), \theta \rangle,
\]
and $\mathcal{E}$ follows.

Now we proceed with the other conditions. To simplify our presentation, we assume that $X_1, \ldots, X_n$ are indeed i.i.d. This can be easily relaxed at cost of more complicated notations. Then it holds
\[
\mathbb{E}S = \sum_{i=1}^n \mathbb{E}\Psi(X_i) = n \mathbb{E}\Psi(X_1) = n \overline{\Psi}
\]
with $\overline{\Psi} = \mathbb{E}\Psi(X_1)$.

We further assume that the underlying density $f(x)$ can indeed be represented in the form (3.3) for some parameter vector $\theta^*$, and $\phi(\theta^*) = 0$. It also holds
\[
\theta^* = \arg\max_{\theta \in \Theta_1} \mathbb{E}L(\theta) = \arg\max_{\theta \in \Theta_1} \{\langle \mathbb{E}S, \theta \rangle - n\phi(\theta)\} = \arg\max_{\theta \in \Theta_1} \{\langle \overline{\Psi}, \theta \rangle - \phi(\theta)\}.
\]
An important assumption on the model is that the “true” point $\theta^*$ is an interior point of $\Theta_1$. It appears that $\mathcal{EV}$ requires the function $\phi(\theta)$ to be two times continuously
differentiable in a small vicinity of \( \theta^* \). Namely, define for a small value \( g > 0 \)

\[
\mathbb{F}_1 \overset{\text{def}}{=} \nabla^2 \phi(\theta^*),
\]

\[
\mathcal{U}_g = \{ u : \| \mathbb{F}_1^{1/2} u \| \leq g \} = \{ u : \langle \nabla^2 \phi(\theta^*) u, u \rangle \leq g^2 \},
\]

\[
\Theta_g = \theta^* + \mathcal{U}_g = \{ \theta = \theta^* + u : \| \mathbb{F}_1^{1/2} u \| \leq g \}.
\]

Below, see Lemma C.1, we show that \((EV)\) is fulfilled with \( V^2 = n \nabla^2 \phi(\theta^*) \) and \( g = \sqrt{n} \) under the following condition:

\[
(\phi g) \quad \text{With } \nu_0^2 \geq 1, \text{ it holds}
\]

\[
\sup_{u \in \mathcal{U}_g} \sup_{t \in [0,1]} \frac{\langle \nabla^2 \phi(\theta^* + tu) u, u \rangle}{\langle \nabla^2 \phi(\theta^*) u, u \rangle} \leq \nu_0^2. \tag{3.4}
\]

For a given operator \( G^2 \), the corresponding penalized log-likelihood reads \( L_G(\theta) = L(\theta) - \| G\theta \|^2 / 2 \), and the penalized MLE \( \tilde{\theta}_G \) and the target \( \theta_G^* \) are

\[
\tilde{\theta}_G = \arg \max_{\theta} L_G(\theta) = \arg \max_{\theta} \left\{ \langle \theta, S \rangle - n \phi(\theta) - \frac{1}{2} \| G\theta \|^2 \right\},
\]

\[
\theta_G^* = \arg \max_{\theta} \mathbb{E} L_G(\theta) = \arg \max_{\theta} \left\{ \mathbb{E} \langle \theta, S \rangle - n \phi(\theta) - \frac{1}{2} \| G\theta \|^2 \right\}.
\]

Below we suppose that \( \theta_G^* \) is an internal point of \( \Theta_g \). This allows to relate \( \nabla^2 \phi(\theta) \) and \( \mathbb{F}_1 = \nabla^2 \phi(\theta^*) \) for \( \theta \) close to \( \theta_G^* \). Further, define

\[
\mathbb{F}(\theta) = -\nabla^2 \mathbb{E} L(\theta) = n \nabla^2 \phi(\theta),
\]

\[
\mathbb{F}_G(\theta) = -\nabla^2 \mathbb{E} L_G(\theta) = n \nabla^2 \phi(\theta) + G^2,
\]

\[
B_G(\theta) = \mathbb{F}_G^{-1/2}(\theta) \mathbb{F}(\theta) \mathbb{F}_G^{-1/2}(\theta),
\]

\[
p_G(\theta) = \text{tr} B_G(\theta).
\]

We write \( D^2_G = \mathbb{F}_G(\theta_G^*) \) and \( p_G = p_G(\theta_G^*) \). We assume that the penalization by \( \| G\theta \|^2 / 2 \) does a good job and the effective dimension \( p_G \) is significantly smaller than the real parameter dimension \( p \) yielding \((D|G)\). We also fix \( V^2 = \mathbb{F}(\theta^*) \) and \((V|G)\) is fulfilled under \((\phi g)\) with \( p_{V|G} \leq \nu_0^2 p_G \), and \( \lambda_{V|G} \leq \nu_0^2 \).

We also need to quantify the values \( \delta_m(\theta, u) \) from (2.3) in a small vicinity of \( \theta_G^* \). Fix \( \theta \in \Theta \) and define a measure \( P_\theta \) by the relation:

\[
\frac{dP_\theta}{d\mu_0}(x) = \exp \{ \langle \Psi(x), \theta \rangle - \phi(\theta) \}.
\]
The identity (3.2) ensures that $P_\theta$ is a probabilistic measure. All what we need for bounding the values $\delta_m(\theta, u)$ is that the scalar product $\langle \Psi(X_1), u \rangle$ has bounded fourth moment w.r.t. $E_\theta$.

**Suppose that for some constant $\kappa \geq 1$ and $m = 3$ or $m = 4$**

$$
\sup_{\theta \in \Theta} \sup_{u \in U} E_\theta \left| \langle \Psi(X_1) - E_\theta \Psi(X_1), u \rangle \right|^m \\
\leq \kappa^m \left\{ E_\theta \left| \langle \Psi(X_1) - E_\theta \Psi(X_1), u \rangle \right|^2 \right\}^{m/2}.
$$

(3.5)

We are now well prepared for specifying the concentration result of Theorem 2.1 and other results of Section 2.

**Theorem 3.1.** Suppose that $(\phi_\varrho)$ and $(\Psi u)$ hold for some $\varrho > 0$ such that

$$
\sqrt{n} \varrho \geq 4 \sqrt{\varrho G + \log n}
$$

and $\theta_{G}^* \in \Theta_{\varrho/2}$. Define

$$
z_G \overset{\text{def}}{=} \sqrt{\varrho G} + \sqrt{2 \log n}.
$$

Then for $n$ large enough, namely, for $n \geq C z_G$ with a constant $C$ depending on $\kappa$ and $\nu_0$ only, it holds on a set $\Omega_n$ with $P(\Omega_n) \geq 1 - 3/n$

$$
\| D_G(\tilde{\theta}_G - \theta_{G}^*) \| \leq 2z_G,
$$

$$
\| D_G(\tilde{\theta}_G - \theta_{G}^*) - D_G^{-1} \nabla \zeta \| \lesssim z_G^2 n^{-1/2},
$$

$$
\left| L_G(\tilde{\theta}_G) - L_G(\theta_{G}^*) - \frac{1}{2} \| D_G^{-1} \nabla \zeta \|^2 \right| \lesssim z_G^3 n^{-1/2}.
$$

(3.7)

All these expansions are meaningful under the condition $z_G^2 = o(n)$, or, equivalently, $\varrho G \ll n$.

Now we continue with the properties of the posterior $\theta_{G} \mid X$ for a Gaussian prior $\mathcal{N}(0, G^2)$. For applying the general results of Theorem 2.4 through 2.10 we only have to check the conditions, this has been already done above.

**Theorem 3.2.** Suppose that the conditions of Theorem 3.1 hold. Define $r_0 = c_0 z_G$ and let $n \geq C r_0^2$ with constants $C, C_0$ specified below and depending on $\kappa$ and $\nu_0$ only. Then Theorem 2.4 through 2.10 continue to apply to the posterior $\theta_{G} \mid X$. In particular,
on a set $\Omega_n$ with $P(\Omega_n) \geq 1 - 3/n$, it holds with $\tilde{D}^2 = F(\theta_G)$ and $\tilde{D}^2_G = F_G(\theta_G)$

$P\left( \|\tilde{D}(\theta_G - \tilde{\theta}_G)\| > r_0 \mid X \right) \lesssim n^{-1}$,

$$\sup_{A \in B_s(\mathbb{R}^p)} \left| P(\theta_G - \tilde{\theta}_G \in A \mid X) - P(\tilde{D}^{-1}_G \gamma \in A) \right| \lesssim \frac{2G}{n} ,$$

$$\sup_{A \in B_s(\mathbb{R}^p)} \left| P(\theta_G - \tilde{\theta}_G \in A \mid X) - P(\tilde{D}^{-1}_G \gamma \in A) \right| \lesssim \frac{2G}{n^{1/2}} .$$

### 3.2 Generalized regression

Now we discuss how the general results apply to generalized regression. Suppose we are given independent data $Y_1, \ldots, Y_n$ which follow the model

$$Y_i \sim P_{v_i} \in \mathcal{P}, \quad i = 1, \ldots, n, \quad (3.8)$$

where $\mathcal{P} = (P_v, v \in \mathcal{T})$ be an exponential family with a canonical parameter. The latter means that $\mathcal{P}$ is dominated by a $\sigma$-finite measure $\mu$ and

$$\log \frac{dP_v}{d\mu}(y) = vy - \phi(v) + \ell(y)$$

for a convex function $\phi(v)$ of a univariate parameter $v$. A typical example is given by the logistic regression with binary observations $Y_i$. Then $\phi(v) = \log(1 + e^v)$.

The model (3.8) yields $\mathbb{E}Y_i = \phi'(v_i)$ and $\text{Var}(Y_i) = \phi''(v_i)$. We, however, do not assume that the model is correct. The value $v_i$ is just defined by the canonical link $\mathbb{E}Y_i = f_i = \phi'(v_i)$.

Generalized regression assumes that the $v_i$’s in (3.8) are values of the regression function $f(X_i)$ at deterministic design points $X_1, \ldots, X_n$. A linear basis expansion $f(x) = \sum_j \theta_j \psi_j(x) = \langle \psi(x), \theta \rangle$ leads to a generalized linear model

$$Y_i \sim P_{(\Psi_i, \theta)}$$

with $\theta = (\theta_1, \ldots, \theta_p)^\top \in \mathbb{R}^p$ and $\Psi_i = (\psi_1(X_i), \ldots, \psi_p(X_i))^\top \in \mathcal{X} \subset \mathbb{R}^p$. Note that our approach allows to consider $p = \infty$. This model yields the log-likelihood

$$L(\theta) = \sum_{i=1}^n Y_i \langle \Psi_i, \theta \rangle - \phi(\langle \Psi_i, \theta \rangle) .$$

Below we study the properties of the posterior on the parameter $\theta$ for a Gaussian prior $\mathcal{N}(0, G^{-2})$. First we check the conditions. Define $\varepsilon_i = Y_i - \mathbb{E}Y_i$. Then the stochastic
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Component of the log-likelihood is linear in $\theta$ and (E) is fulfilled with

$$
\zeta(\theta) = L(\theta) - EL(\theta) = \sum_{i=1}^{n} \varepsilon_i \langle \Psi_i, \theta \rangle,
$$

$$
\nabla \zeta = \sum_{i=1}^{n} \varepsilon_i \Psi_i,
$$

and

$$
V^2 \overset{\text{def}}{=} \text{Cov}(\nabla \zeta) = \sum_{i=1}^{n} \sigma_i^2 \Psi_i \otimes \Psi_i
$$

with $\sigma_i^2 = E \varepsilon_i^2$. Further,

$$
-\nabla^2 L(\theta) = \sum_{i=1}^{n} \phi''(\langle \Psi_i, \theta \rangle) \Psi_i \otimes \Psi_i = F(\theta) > 0
$$

because $\phi$ is strictly convex. This yields (L). Let us fix some Gaussian prior $\mathcal{N}(0, G^{-2})$ and define $\theta^*_G$ by as the unique maximizer of the value $EL_G(\theta) = EL(\theta) - \|G^2 \theta\|^2/2$. This leads to the equation $\nabla EL_G(\theta) = 0$ or

$$
\sum_{i=1}^{n} \{f_i - \phi'(\langle \Psi_i, \theta \rangle)\} \Psi_i = \sum_{i=1}^{n} \{\phi'(v_i) - \phi'(\langle \Psi_i, \theta \rangle)\} \Psi_i = G^2 \theta.
$$

We assume that the matrix $F_G(\theta) = F(\theta) + G^2$ is well posed for all $\theta \in \Theta^\circ$. Here the set $\Theta^\circ$ is compact and large enough to contain $\theta^*_G$ with its vicinity. Now we fix $V^2$ e.g. by (3.9) and assume $(V|G), (D|G)$ to be fulfilled. (EV) can be derived under exponential moments condition on the errors $\varepsilon_i = Y_i - EY_i$ yielding by Theorem A.5 the probabilistic bound

$$
\|D_G^{-1} \nabla \zeta\| \leq z \overset{\text{def}}{=} \sqrt{F_G} + \sqrt{2 \log n}
$$

on a set $\Omega_n$ with $P(\Omega_n) \geq 1 - 3/n$; cf. Panov and Spokoiny (2015). Finally, (L0) follows from smoothness of the function $\phi(v)$ and the remainders $\delta_3$ and $\delta_4$ satisfy for $\theta \in \Theta^\circ$ and $\|u\| = o(n^{1/2})$

$$
\delta_3(\theta, u) = \sum_{i=1}^{n} (\langle \Psi_i, u \rangle)^3 \phi^{(3)}(\langle \Psi_i, \theta \rangle),
$$

$$
\delta_4(\theta, u) = \sum_{i=1}^{n} (\langle \Psi_i, u \rangle)^4 \phi^{(4)}(\langle \Psi_i, \theta \rangle).
$$
Under standard conditions on the design \((Ψ_i)\) and on smoothness of \(φ(·)\), one can derive

\[
δ_3(θ, u) ≤ C∥u∥^3 n^{-1/2},
\]
\[
δ_4(θ, u) ≤ C∥u∥^4 n^{-1}.
\]

This immediately implies all the expansions in (3.7) for the penalized MLE \(\tilde{θ}_G\) and also the properties of the posterior \(θ_G | Y\) listed in Theorem 3.2 under the same condition \(p^3_G ≤ n\).

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A Tools

A.1 Gaussian comparison

Let $\mathbb{H}$ be a Hilbert space and $\Sigma_\xi$ be a covariance operator of an arbitrary Gaussian random element in $\mathbb{H}$. By $\{\lambda_j\}_{j \geq 1}$ we denote the set of its eigenvalues arranged in the non-increasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \ldots$, and let $\lambda_\xi \overset{\text{def}}{=} \text{diag}(\lambda_j)_{j=1}^{\infty}$. Note that $\sum_{j=1}^{\infty} \lambda_j < \infty$. Introduce the following quantities

$$A^2_{\xi} \overset{\text{def}}{=} \sum_{j=k}^{\infty} \lambda_j^2, \quad k = 1, 2,$$

**Theorem A.1** (Götze et al. (2019)). Let $\xi$ and $\eta$ be Gaussian elements in $\mathbb{H}$ with zero mean and covariance operators $\Sigma_\xi$ and $\Sigma_\eta$ respectively. Then for any $a \in \mathbb{H}$

$$\sup_{x > 0} |P(\|\xi - a\| \leq x) - P(\|\eta\| \leq x)| \lesssim \left( \frac{1}{(A_1 A_2)^{1/2}} + \frac{1}{(A_1 A_2)^{1/2}} \right) \left( \|\lambda_\xi - \lambda_\eta\|_1 + \|a\|^2 \right).$$

Moreover, assume that

$$3\|\Sigma_\xi\|^2 \leq \|\Sigma_\xi\|^2_{\text{Fr}} \quad \text{and} \quad 3\|\Sigma_\eta\|^2 \leq \|\Sigma_\eta\|^2_{\text{Fr}}.$$

Then for any $a \in \mathbb{H}$

$$\sup_{x > 0} |P(\|\xi - a\| \leq x) - P(\|\eta\| \leq x)| \lesssim \left( \frac{1}{\|\Sigma_\xi\|^2_{\text{Fr}}} + \frac{1}{\|\Sigma_\eta\|^2_{\text{Fr}}} \right) \left( \|\lambda_\xi - \lambda_\eta\|_1 + \|a\|^2 \right).$$

A.2 Deviation bounds for Gaussian quadratic forms

The next result explains the concentration effect of $\langle B\gamma, \gamma \rangle$ for a standard Gaussian vector $\gamma$ and a symmetric trace operator $B$ in $\mathbb{R}^p$, $p \leq \infty$. We use a version from Laurent and Massart (2000).

**Theorem A.2.** Let $\gamma$ be a standard normal Gaussian element in $\mathbb{R}^p$ and $B$ be symmetric non-negative trace operator in $\mathbb{R}^p$. Then with $p = \text{tr}(B)$, $v^2 = \text{tr}(B^2)$, and $\lambda = \|B\|_{op}$, it holds for each $x \geq 0$

$$P\left( \langle B\gamma, \gamma \rangle > z^2(B, x) \right) \leq e^{-x},$$

$$z(B, x) \overset{\text{def}}{=} \sqrt{p + 2vx^{1/2} + 2\lambda x}.$$
In particular, it implies
\[ \mathbb{P}(\|B^{1/2}\gamma\| > p^{1/2} + (2\lambda x)^{1/2}) \leq e^{-x}. \]

Also
\[ \mathbb{P}(\langle B\gamma, \gamma \rangle < p - 2\sqrt{x}/2) \leq e^{-x}. \]

If \( B \) is symmetric but non necessarily positive then
\[ \mathbb{P}(\|B\gamma\| > 2\sqrt{x} + 2\lambda x) \leq 2e^{-x}. \]

As a special case, we present a bound for the chi-squared distribution corresponding to \( B = I_p \), \( p < \infty \). Then \( \text{tr}(B) = p \), \( \text{tr}(B^2) = p \) and \( \lambda(B) = 1 \).

**Corollary A.3.** Let \( \gamma \) be a standard normal vector in \( \mathbb{R}^p \). Then for any \( x > 0 \)
\[ \mathbb{P}(\|\gamma\|^2 \geq p + 2\sqrt{px} + 2x) \leq e^{-x}, \]
\[ \mathbb{P}(\|\gamma\| \geq \sqrt{p} + \sqrt{2x}) \leq e^{-x}, \]
\[ \mathbb{P}(\|\gamma\|^2 \leq p - 2\sqrt{px}) \leq e^{-x}. \]

### A.3 Deviation bounds for non-Gaussian quadratic forms

This section collects some probability bounds for non-Gaussian quadratic forms. The presented results can be viewed as a slight improvement of the bounds from Spokoiny (2012). The proofs are very similar to ones from Spokoiny (2012) and are omitted by the space reasons.

Let a random vector \( \xi \in \mathbb{R}^p \) has some exponential moments. More exactly, suppose for some fixed \( g > 0 \) that
\[ \log \mathbb{E} \exp(\langle \gamma, \xi \rangle) \leq \|\gamma\|^2/2, \quad \gamma \in \mathbb{R}^p, \|\gamma\| \leq g. \quad (A.1) \]

First we present a bound for the norm \( \|\xi\| \) assuming \( p \lesssim g^2 \). For ease of presentation, assume below that \( g \) is sufficiently large, namely, \( 0.3g \geq \sqrt{p} \). In typical examples of an i.i.d. sample, \( g \asymp \sqrt{n} \). Define
\[ x_c \overset{\text{def}}{=} g^2/4, \]
\[ z_c^2 \overset{\text{def}}{=} p + \sqrt{pg^2 + g^2/2} = g^2(1/2 + \sqrt{p/g^2 + p/g^2}), \]
\[ g_c \overset{\text{def}}{=} \frac{g(1/2 + \sqrt{p/g^2 + p/g^2})^{1/2}}{1 + \sqrt{p/g^2}}. \]
Note that with $\alpha = \sqrt{p/g^2} \leq 0.3$, one has
\[ z_c^2 = g^2 (1/2 + \alpha + \alpha^2), \quad g_c = g \frac{(1/2 + \alpha + \alpha^2)^{1/2}}{1 + \alpha}, \]
so that $z_c^2/g^2 \in [1/2, 1]$ and $g_c^2/g^2 \in [1/2, 1]$.

**Theorem A.4.** Let (A.1) hold and $0.3 g \geq \sqrt{p}$. Then for each $x > 0$
\[
\mathbb{P}(\|\xi\|^2 \geq z(p, x)) \leq 2e^{-x} + 8.4e^{-x} I(x < x_c),
\]
(A.2)

where $z(p, x)$ is defined by
\[
z(p, x) \overset{\text{def}}{=} \begin{cases} 
(p + 2\sqrt{p}x + 2x)^{1/2}, & x \leq x_c, \\
z_c + 2g_c^{-1}(x - x_c), & x > x_c. 
\end{cases}
\]

Depending on the value $x$, we have two types of tail behavior of the quadratic form $\|\xi\|^2$. For $x \leq x_c = g^2/4$, we have the same deviation bounds as in the Gaussian case with the extra-factor two in the deviation probability. Remind that one can use a simplified expression $(p + 2\sqrt{p}x + 2x)^{1/2} \leq \sqrt{p} + \sqrt{2x}$. For $x > x_c$, we switch to the special regime driven by the exponential moment condition (A.1). Usually $g^2$ is a large number (of order $n$ in the i.i.d. setup) and the second term in (A.2) can be simply ignored.

Next we present a bound for a quadratic form $\langle B\xi, \xi \rangle$, where $\xi$ satisfies (A.1) and $B$ is a given symmetric non-negative operator in $\mathbb{R}^p$. Here we relax $p < \infty$ to $\text{tr} \, B < \infty$. Define
\[
p = \text{tr}(B), \quad v^2 = \text{tr}(B^2), \quad \lambda = \lambda_{\text{max}}(B).
\]

For ease of presentation, suppose that $0.3 g \geq \sqrt{p}$ so that $\alpha = \sqrt{p/g^2} \leq 0.3$. The other case only changes the constants in the inequalities. Define also
\[
x_c \overset{\text{def}}{=} \frac{g^2}{4}, \quad z_c^2 \overset{\text{def}}{=} p + v\lambda + \lambda g^2/2, \quad g_c \overset{\text{def}}{=} \sqrt{p/\lambda + v/\lambda + g^2/2} \quad 1 + v/(\lambda g).
\]

**Theorem A.5.** Let (A.1) hold and $0.3 g \geq \sqrt{p/\lambda}$. Then for each $x > 0$
\[
\mathbb{P}(\langle B\xi, \xi \rangle \geq z^2(B, x)) \leq 2e^{-x} + 8.4e^{-x} I(x < x_c),
\]
where \( z(B, x) \) is defined by

\[
    z(B, x) \overset{\text{def}}{=} \begin{cases} 
    \sqrt{p + 2vx^{1/2} + 2\lambda x}, & x \leq x_c, \\
    z_c + 2\lambda(x - x_c)/g_c, & x > x_c. 
    \end{cases} \tag{A.3}
\]

Similarly to the case \( B = I_p \), the upper quantile \( z(B, x) = \sqrt{p + 2vx^{1/2} + 2\lambda x} \) can be upper bounded by \( \sqrt{p + 2\lambda x} \):

\[
    z(B, x) \leq \begin{cases} 
    \sqrt{p + 2\lambda x}, & x \leq x_c, \\
    z_c + 2\lambda(x - x_c)/g_c, & x > x_c. 
    \end{cases}
\]

A.4 Taylor expansions

Here we collect some useful bounds for various Taylor-type expansions for a smooth function. Let \( f \) be a four time differentiable function on \( \mathbb{R}^p \). Here \( p \leq \infty \). By \( f^{(m)}(x, u) \) we denote the \( m \)th directional derivative at \( x \):

\[
    f^{(m)}(x, u) \overset{\text{def}}{=} \frac{d^m}{dt^m} f(x + tu) \bigg|_{t=0}.
\]

In particular, \( f'(x, u) = \langle \nabla f(x), u \rangle \) and \( f''(x, u) = \langle \nabla^2 f(x)u, u \rangle \). Below we assume that some open set \( X \subseteq \mathbb{R}^p \) is fixed, and, in addition, for each \( x \in X \), and a centrally symmetric convex set \( \mathcal{U}(x) \) are fixed and

\[
    \frac{1}{m!} f^{(m)}(x, u) = \delta_m(x, u) \leq \delta_m, \quad x \in X, \; u \in \mathcal{U}, \; m = 3, 4 \tag{A.4}
\]

for some constants \( \delta_m \) depending on \( X \) and \( \mathcal{U} \). All bounds will be given in terms of \( \delta_3 \) and \( \delta_4 \). The construction can be extended by making \( \mathcal{U} \) dependent on \( x \in X \) at cost of more complicated notation.

**Lemma A.6.** Suppose (A.4) with \( \delta_m \leq 1 \) for \( m = 3, 4 \). Then for any point \( x \in X \)

\[
    \left| \frac{1}{2} \left( e^{f(x+u) - f(x)} - f'(x)u + e^{f(x-u) - f(x)} + f'(x)u \right) \right| - e^{f''(x, u)/2} \leq e^{f''(x, u)/2} (4\delta_3^2 + 4\delta_4). \tag{A.5}
\]

Furthermore,

\[
    \left| e^{(x+u) - f(x) - f'(x)u} - e^{f''(x, u)/2} \right| \leq \delta_3 e^{f''(x, u)/2}. \tag{A.6}
\]
Proof. Taylor expansions of the forth order imply
\[
\begin{align*}
    f(x + u) - f(x) - f'(x, u) - \frac{1}{2} f''(x, u) - \frac{1}{6} f^{(3)}(x, u) &= \rho_1, \quad |\rho_1| \leq \delta_4, \\
    f(x - u) - f(x) + f'(x, u) - \frac{1}{2} f''(x, u) + \frac{1}{6} f^{(3)}(x, u) &= \rho_2, \quad |\rho_2| \leq \delta_4.
\end{align*}
\]
Further, define \( \varkappa = f^{(3)}(x, u)/6 \), so that \( |\varkappa| \leq \delta_3 \leq 1 \). Then
\[
e^{f(x+u)-f(x)-f'(x,u)} + e^{f(x-u)-f(x)+f'(x,u)} - 2e^{f''(x,u)/2}
\]
\[
e^{f''(x,u)/2} (e^{\varkappa+\rho_1} + e^{-\varkappa+\rho_2} - 2).
\]
The function
\[
g(s) \overset{\text{def}}{=} \frac{1}{2} \exp(s \varkappa + \rho_1) + \frac{1}{2} \exp(-s \varkappa + \rho_2) - 1
\]
fulfills
\[
|g(0)| = \left| \frac{1}{2} e^{\rho_1} + \frac{1}{2} e^{\rho_2} - 1 \right| \leq |\rho_1| + |\rho_2|,
\]
\[
|g'(0)| = \frac{1}{2} |\varkappa (e^{\rho_1} - e^{\rho_2})| \leq |\rho_1| + |\rho_2|
\]
and for any \( s \in [0,1] \) by simple algebra due to \( |\varkappa| \leq 1 \) and \( |\rho_m| \leq 1 \) for \( m = 1,2 \)
\[
|g''(s)| \leq \frac{1}{2} |\varkappa^2 \left\{ \exp(s \varkappa + \rho_1) + \exp(-s \varkappa + \rho_2) \right\}|
\]
\[
\leq \frac{|\varkappa|^2 e}{2} (e^{|\varkappa|} + e^{-|\varkappa|}) < 8|\varkappa|^2,
\]
and thus
\[
|g(1)| \leq \sup_{s \in [0,1]} |g(0) + g'(0) + \frac{1}{2} g''(s)| \leq 4|\varkappa|^2 + 2|\rho_1| + 2|\rho_2|,
\]
and (A.5) follows. The bound (A.6) can be obtained in a similar way using the Taylor expansion of the third order. \( \square \)

Now we study the modulus of continuity for the gradient \( \nabla f(x) \) and the Hessian \( \nabla^2 f(x) \).

Lemma A.7. Suppose (A.4) with \( \delta_3 \leq 1 \). Let \( x \in X \) and \( u \in U \) be such that \( x+u \in X \). Then, for any \( w \in U \)
\[
\left| \langle w, \nabla f(x + u) - \nabla f(x) - \nabla^2 f(x) u \rangle \right| \leq C \delta_3,
\]
\[
\left| \langle w, \{ \nabla^2 f(x + u) - \nabla^2 f(x) \} u \rangle \right| \leq C \delta_3.
\]
Proof. Let us fix any \( x^0 \in X \) and \( w^0 \in U \) and define the function
\[
g(t) \overset{\text{def}}{=} f(x^0 + tw^0) + f(x^0 - tw^0) - 2f(x^0) - t^2 f''(x^0, w^0).
\]
The Taylor expansion of the third order yields
\[
|g(1)| = |f(x^0 + w^0) + f(x^0 - w^0) - 2f(x^0) - f''(x^0, w^0)| \leq 2\delta_3(x^0, w^0).
\]
We apply this bound for \( x^0 = x \) and \( x^0 = x + u \) and take the difference between them. This implies
\[
|f''(x, w^0) - f''(x + u, w^0)| \leq |f(x + w^0) + f(x - w^0) - 2f(x) - f(x + u + w^0) - f(x + u - w^0) + 2f(x + u)| + 2\delta_3(x, w^0) + 2\delta_3(x + u, w^0). \tag{A.8}
\]
For given \( x, u, w \), and \( \overline{x} = x + u/2 \), define
\[
g(t) \overset{\text{def}}{=} f(\overline{x} + tw) - f(\overline{x} - tw) + f(\overline{x} + t(u + w)) - f(\overline{x} - t(u - w)) - 2f(\overline{x} + tu) + 2f(\overline{x} - tu).
\]
It is straightforward to see that \( g(0) = g'(0) = g''(0) = 0 \). Moreover, in view of \( u \in U \) and \( (u \pm w)/2 \in U \), it holds \( \delta_3(\overline{x}, u/2) = \delta_3(\overline{x}, u)/8 \) and for any \( |t| \leq 1/2 \)
\[
\frac{1}{6}|g^{(3)}(t)| \leq \frac{5\delta_3}{2}.
\]
By Taylor expansion of the third order we derive
\[
|g(1/2)| \leq \sup_{t \in [0,1]} \frac{1}{6}|g^{(3)}(t)| \leq \frac{5\delta_3}{2}.
\]
Note that \( g(1/2) \) is exactly the expression in the right hand-side of (A.8) with \( w^0 = w/2 \). The use of \( \delta_3(x^0, w^0) = \delta_3(x^0, w)/8 \) together with (A.8) yields (A.7) with \( C = 3 \).

Now we specify the result to the case of an elliptic set \( U \) of the form
\[
U = \{ u : \|Qu\| \leq r \} \tag{A.9}
\]
for a positive invertible operator \( Q \) and \( r > 0 \).

Lemma A.8. Let \( U \) be given by (A.9) with \( Q > 0 \), and let \( x \in X \) and \( u \in U \) be such that \( x + u \in X \). Then
\[
\|Q^{-1}\{\nabla f(x + u) - \nabla f(x) - \nabla^2 f(x)u\}\| \leq Cr^{-1}\delta_3, \tag{A.10}
\]
\[
\|Q^{-1}\{\nabla^2 f(x) - \nabla^2 f(x + u)\}Q^{-1}\| \leq Cr^{-2}\delta_3.
\]
Proof. For any \( w \in U \), it holds by Lemma A.7
\[
\left| \langle w, \{\nabla^2 f(x + u) - \nabla^2 f(x)\} \rangle w \right| = \left| \langle Qw, Q^{-1}\{\nabla^2 f(x + u) - \nabla^2 f(x)\}Q^{-1}(Qw) \rangle \right| \leq C\delta_3.
\]
As this bound holds for all \( w \in U \) with \( \|Qw\| \leq r \), the result follows. \( \square \)

The result of Lemma A.6 can be extended to the integral of \( e^{f(x+u)} \) over \( u \in U \).

**Lemma A.9.** Let \( U \) be a subset in \( \mathbb{R}^p \). Suppose (A.4) with \( \delta_m \leq 1 \) for \( m = 3, 4 \). Then for any point \( x \in X \) and any centrally symmetric set \( A \subset U \)
\[
\int_A e^{f(x+u)-f(x)-f'(x,u)} \, du - \int_A e^{f''(x,u)/2} \, du \leq \hat{\delta} \int_A e^{f''(x,u)/2} \, du \tag{A.11}
\]
with \( \hat{\delta} = 4\delta_3^2 + 4\delta_4 \) and for any vector \( z \)
\[
\left| \int_A \langle z, u \rangle^2 e^{f(x+u)-f(x)-f'(x,u)} \, du - \int_A \langle z, u \rangle^2 e^{f''(x,u)/2} \, du \right| \leq \hat{\delta} \int_A \langle z, u \rangle^2 e^{f''(x,u)/2} \, du \tag{A.12}
\]
If \( A \) is not centrally symmetric then
\[
\int_A e^{f(x+u)-f(x)-f'(x,u)} \, du - \int_A e^{f''(x,u)/2} \, du \leq \delta_3 \int_A e^{f''(x,u)/2} \, du. \tag{A.13}
\]

Proof. By symmetricity of \( U \), it holds
\[
\int_A e^{f(x+u)-f(x)-f'(x,u)} \, du = \frac{1}{2} \int_A \left(e^{f(x+u)-f(x)-f'(x,u)} + e^{f(x-u)-f(x)+f'(x,u)}\right) \, du,
\]
and the first result is proved by (A.5). The same symmetricity arguments apply to (A.12). The final bound for any \( A \) follows from (A.6). \( \square \)

The bound (A.14) can be specified to the case of a massive set \( U \). We assume that \( f \) is concave and \( H^2 \triangleq -\nabla^2 f(x) \geq 0 \).

**Lemma A.10.** Let \( f(\cdot) \) be strictly concave with \( H^2 = -\nabla^2 f(x) > 0 \). Suppose (A.4) with \( \delta_3 \leq 1 \). For a linear operator \( Q \), it holds
\[
\left\| \int_U Qu e^{f(x+u)-f(x)-f'(x,u)} \, du \right\| \leq \delta_3 \int_{U} \|Qu\| e^{f''(x,u)/2} \, du, \tag{A.14}
\]
Let also \( U \) be massive in the sense that
\[
P(H^{-1} \gamma \in U) \geq 1/2 \tag{A.15}
\]
with $\gamma$ standard normal in $\mathbb{R}^p$. Then for any linear operator $Q$, it holds
\[
\left\| \int_{U} Q u e^{f(x+u)-f(x)-f^\prime(x,u)}/2 \, du \right\| \leq 2\delta_3 E \| Q H^{-1} \|.
\]
Proof. The bound \((A.14)\) follows in a way similar to the proof of Lemma \(A.9\) using \((A.6)\) instead of \((A.5)\). We apply \((A.14)\) yielding in view of $f''(x,u) = -\|Hu\|^2$
\[
\left\| \int_{U} Q u e^{f(x+u)-f(x)-f^\prime(x,u)}/2 \, du \right\| \leq \delta_3 \int_{U} \| Q u \| e^{f''(x,u)/2} \, du \\
\leq 2\delta_3 \frac{\int_{U} \| Q u \| e^{f''(x,u)/2} \, du}{\int e^{f''(x,u)/2} \, du} \leq 2\delta_3 E \| Q H^{-1} \|
\]
as required. \hfill \Box

All the bounds presented above assume that $U$ is a symmetric subset of $\mathbb{R}^p$, in particular, an ellipsoid centred at zero. Now we check what happens under a small departure from symmetricity.

**Lemma A.11.** Let $f$ be concave with $H^2 = -\nabla^2 f(x)$. Let also $U$ be centrally symmetric massive set; see \((A.15)\) and let $a \in U$ be fixed and $U + a \subset E(x_0) = \{ u : \|Hu\| \leq x_0 \}$. Suppose \((A.4)\) with $\delta_m = \delta_m(x_0) \leq 1$ for $m = 3, 4$. Then
\[
\left\| \int_{U} e^{f(x+u+a)-f(x)-f^\prime(x,u+a)} \, du - \int_{U} e^{f''(x,u+a)/2} \, du \right\| \\
\lesssim \left\{ \hat{\diamond}(x_0) + (\|Ha\| + \|Ha\|^2)\delta_3(x_0) \right\} \int_{U} e^{f''(x,u)/2} \, du.
\]
Proof. Define
\[
h(t) \overset{\text{def}}{=} q_0 \int_{U} e^{f(x+u+a)-f(x)-f^\prime(x,u+a)} \, du \\
v(t) \overset{\text{def}}{=} q_0 \int_{U} e^{f''(x,u+a)/2} \, du
\]
with
\[
q_0 \overset{\text{def}}{=} \left( \int_{U} e^{f''(x,u)/2} \, du \right)^{-1}.
\]
Then we have to bound the difference $h(t) - v(t)$ for $t \leq 1$. For this, we bound the first two derivatives of $h(t)$. By \((A.11)\) Lemma \(A.9\), it holds $|h(0) - v(0)| \leq \hat{\diamond}(x_0)$. Further,
in view of $f''(x, u + ta) = -\|H(u + ta)\|^2$

$$h'(0) = q_0 \int_{\mathcal{U}} \langle a, \nabla f(x + u) - \nabla f(x) \rangle e^{f(x + u) - f(x) - f'(x, u)} du,$$  

(A.16)

$$v'(0) = q_0 \int_{\mathcal{U}} \langle Ha, Hu \rangle e^{f''(x, u)/2} du = 0$$

because $\mathcal{U}$ is centrally symmetric. By (A.10) of Lemma A.8 and (A.11) of Lemma A.9

$$\left| \int_{\mathcal{U}} \langle a, \nabla f(x + u) - \nabla f(x) - \nabla^2 f(x) u \rangle e^{f(x + u) - f(x) - f'(x, u)} du \right|$$

$$\leq C \|Ha\| x_0^{-1} \delta_3(x_0) \int_{\mathcal{U}} e^{f(x + u) - f(x) - f'(x, u)} du$$

$$\leq C \|Ha\| x_0^{-1} \delta_3(x_0) (1 + \diamond(x_0)) \int_{\mathcal{U}} e^{f''(x, u)/2} du$$

(A.17)

for $\diamond(x_0) = 4\delta_3^2(x_0) + 4\delta_4(x_0)$. Further we use $f''(x, u) = -\|Hu\|^2$, $-\langle a, \nabla^2 f(x) u \rangle = \langle Ha, Hu \rangle$, and it follows by Lemma A.10 that

$$\left| q_0 \int_{\mathcal{U}} \langle a, \nabla^2 f(x) u \rangle e^{f(x + u) - f(x) - f'(x, u)} du \right| \leq 2\delta_3(x_0) IE \|\langle Ha, \gamma \rangle \| \leq 2\delta_3(x_0) \|Ha\|.$$ 

Putting this together with (A.16), (A.17) yields

$$|h'(0)| \lesssim \delta_3(x_0) \|Ha\|.$$ 

For the second derivative,

$$h''(t) = q_0 \int_{\mathcal{U}} \left\{ \langle a, \nabla f(x + u + ta) - \nabla f(x) \rangle^2 + \langle a, \nabla^2 f(x + u + ta) a \rangle \right\}$$

$$\times e^{f(x + u + ta) - f(x) - f'(x, u + ta)} du.$$ 

Similarly, by the use of $f''(x, u) = -\|Hu\|^2$ we derive

$$v''(t) = q_0 \int_{\mathcal{U}} \left\{ \langle a, H^2(u + ta) \rangle^2 - \langle a, H^2 a \rangle \right\} e^{-\|H(u + ta)\|^2/2} du.$$ 

Now by (A.14) of Lemma A.9

$$\left| \langle a, \nabla^2 f(x + u + ta) a \rangle - \langle a, \nabla^2 f(x) a \rangle \right|$$

$$= \left| \langle Ha, H^{-1}\{\nabla^2 f(x + u + ta) - \nabla^2 f(x)\}H^{-1}Ha \rangle \right| \lesssim \|Ha\|^2 x_0^{-2} \delta_3(x_0).$$ 

Similarly

$$\left| \langle a, \nabla f(x + u + ta) - \nabla f(x) + H^2(u + ta) \rangle \right| \lesssim \|Ha\| x_0^{-1} \delta_3(x_0).$$
and by \( \|H(u + ta)\| \leq r_0 \) it holds \( \left| \langle a, H^2(u + ta) \rangle \right| \leq \|Ha\| r_0 \) and

\[
\langle a, \nabla f(x + u + ta) - \nabla f(x) \rangle^2 - \langle a, H^2(u + ta) \rangle^2 \\
\lesssim \|Ha\|^2 r_0^{-2} \delta_3^2(x_0) + 2\|Ha\| r_0^{-1} \delta_3(x_0) \|Ha\| r_0 \\
\lesssim \|Ha\|^2 \delta_3(x_0).
\]

We conclude that

\[
h''(t) = q_0 \int_{\mathcal{U}} \left\{ \langle a, H^2(u + ta) \rangle^2 - \langle a, H^2 a \rangle + \tau(a, u, t) \right\} e^f(x + u + ta) - f(x) - f'(x, u + ta) du
\]

where \( |\tau(a, u, t)| \lesssim \|Ha\|^2 \delta_3(x_0) \). Lemma A.6 helps to bound

\[
|h''(t) - v''(t)| \lesssim \|Ha\|^2 \delta_3(x_0)
\]

uniformly in \( |t| \leq 1 \). This yields with some \( \rho \in [0, 1] \)

\[
|h(1) - v(1)| \leq |h(0) - v(0)| + |h'(0) - v'(0)| + |h''(\rho) - v''(\rho)|/2 \\
\lesssim \diamond(x_0) + (\|Ha\| + \|Ha\|^2) \delta_3(x_0)
\]

which completes the proof. \( \square \)

### A.5 Concavity and tail bounds

Let \( f(x) \) be a function on \( \mathbb{R}^p \). Previous results describe the local behavior of \( f(x + u) \) for \( u \in \mathcal{U} \) under local smoothness conditions. Now we derive some upper bounds on \( f(x + u) \) for \( u \) large using that \( f \) is concave. More precisely, we fix \( x \) and \( u \) and bound the values \( f(x + tu) - f(x) - tf'(x, u) \) for \( u \in \mathcal{U} \) and large \( t \).

**Lemma A.12.** Suppose (A.4) with \( \delta_m \leq 1 \) for \( m = 3, 4 \). Let \( x + \mathcal{U} \subset X \). Let the function \( f(x + tu) \) be concave in \( t \). Then it holds for any \( u \in \mathcal{U} \) and for \( t > 1 \)

\[
f(x + tu) - f(x) - \langle \nabla f(x), u \rangle t \leq \left( t - \frac{1}{2} \right) \left\{ \langle \nabla^2 f(x)u, u \rangle - 3\delta_3 \right\} \\
\text{ (A.18)}
\]

**Proof.** The Taylor expansion of the third order for \( g(t) = f(x + tu) \) at \( t = 0 \) yields

\[
\left| g(1) - g(0) - g'(0) - \frac{1}{2} g''(0) \right| \leq \delta_3.
\]

Similarly one obtains

\[
g'(1) - g'(0) = g'(1) - g'(0) - g''(0) + g''(0) \leq g''(0) + 3\delta_3.
\]
Concavity of $g(\cdot)$ implies
\[ g(t) - g(1) \leq (t - 1)g'(1). \]

We summarize that
\[
\begin{align*}
g(t) - g(0) - tg'(0) &= g(t) - g(1) - (t - 1)g'(1) + (t - 1)\{g'(1) - g'(0)\} + g(1) - g(0) - g'(0) \\
&\leq (t - 1)\{g''(0) + 3\delta_3\} + \frac{1}{2}g''(0) + \delta_3 \\
&\leq (t - 1/2)\{g''(0) + 3\delta_3\}.
\end{align*}
\]

This implies the assertion in view of $g''(0) = \langle \nabla^2 f(x)u, u \rangle$. □

Now we specify the result of Lemma A.12 for the elliptic set $\mathcal{U}(x_0)$ defined by the condition $-\langle \nabla^2 f(x)u, u \rangle \leq r_0^2$. We write $\delta_3(x_0)$ in place of $\delta_3(x, \mathcal{U}(x_0))$. We aim at bounding from above the value $f(x + u) - f(x)$ for $u$ with $-\langle \nabla^2 f(x)u, u \rangle = r^2 > r_0^2$.

**Lemma A.13.** Consider $x \in \mathcal{X}$ and $\mathcal{U} = \mathcal{U}(x_0) = \{u: -\langle \nabla^2 f(x)u, u \rangle \leq r_0^2\}$. Let $f(x + u)$ be concave in $u$. Then for any $u$ with $-\langle \nabla^2 f(x)u, u \rangle = r^2 > r_0^2$, it holds
\[
f(x + u) - f(x) - \langle \nabla f(x), u \rangle \leq -(rr_0 - r_0^2/2)\{1 - 3r_0^{-2}\delta_3(r_0)\}. \tag{A.19}
\]

*Proof.* Define $t = r/r_0$ and $u^o = ur_0/r$, so that $-\langle \nabla^2 f(x)u, u \rangle = r^2$ and $u^o \in \mathcal{U}(x_0)$. Then it holds by (A.18)
\[
\begin{align*}
f(x + u) - f(x) - \langle \nabla f(x), u \rangle &= f(x + tu^o) - f(x) - \langle \nabla f(x), u \rangle \\
&\leq -(r/r_0 - 1/2)\{r_0^2 - 3\delta_3(r_0)\} \\
&= -(rr_0 - r_0^2/2)\{1 - 3r_0^{-2}\delta_3(r_0)\}
\end{align*}
\]
and the result follows. □

The result is meaningful if $3r_0^{-2}\delta_3(r_0) < 1$. Then with $c_0 = 1 - 3r_0^{-2}\delta_3(r_0)$, we obtain for any $u$ with $-\langle \nabla^2 f(x)u, u \rangle = r^2 > r_0^2$
\[
f(x + u) - f(x) - \langle \nabla f(x), u \rangle \leq -c_0(rr_0 - r_0^2/2).
\]
A.6 Gaussian integrals

Let $\mathcal{T}$ be a linear operator in $\mathbb{R}^p$, $p \leq \infty$, with $\|\mathcal{T}\|_{\text{op}} \leq 1$. By $\mathcal{T}^\top$ we denote the adjoint operator for $\mathcal{T}$. Given positive $r_0$ and $C_0$, consider the following ratio

$$
\frac{\int_{\|\mathcal{T}u\| > r_0} \exp\left(-C_0\|\mathcal{T}u\| + \frac{1}{2}C_0r_0^2 + \frac{1}{2}\|\mathcal{T}u\|^2 - \frac{1}{2}\|u\|^2\right)du}{\int_{\|\mathcal{T}u\| \leq r_0} \exp\left(-\frac{1}{2}\|u\|^2\right)du}.
$$

Obviously, one can rewrite this value as ratio of two expectations

$$
\mathbb{E}\left\{\exp\left(-C_0r_0\|\mathcal{T}\gamma\| + \frac{1}{2}C_0r_0^2 + \frac{1}{2}\|\mathcal{T}\gamma\|^2\right) \mathbb{I}(\|\mathcal{T}\gamma\| > r_0)\right\} / \mathbb{P}(\|\mathcal{T}\gamma\| \leq r_0),
$$

where $\gamma \sim \mathcal{N}(0, I_p)$. Note that without the linear term $-C_0\|\mathcal{T}\gamma\|$ in the exponent, the expectation in the numerator can be infinite. We aim at describing $r_0$ and $C_0$-values which ensure that the probability in denominator is close to one while the expectation in the numerator is small.

**Lemma A.14.** Let $\mathcal{T}$ be a linear operator in $\mathbb{R}^p$ with $\|\mathcal{T}\|_{\text{op}} \leq 1$. Define $p_r = \text{tr}(\mathcal{T}^\top\mathcal{T})$. For any $C_0, r_0$ with $1/2 < C_0 \leq 1$ and $C_0r_0 = 2\sqrt{p_r} + \sqrt{x}$ for $x > 0$

$$
\mathbb{E}\left\{\exp\left(-C_0r_0\|\mathcal{T}\gamma\| + \frac{1}{2}C_0r_0^2 + \frac{1}{2}\|\mathcal{T}\gamma\|^2\right) \mathbb{I}(\|\mathcal{T}\gamma\| > r_0)\right\} \leq C e^{-(p_r+x)/2}
$$

and

$$
\mathbb{P}(\|\mathcal{T}\gamma\| \leq r_0) \geq 1 - \exp\left\{-\frac{1}{2}(r_0 - \sqrt{p_r})^2\right\} \geq 1 - e^{-(p_r+x)/2}.
$$

**Remark A.1.** The result applies even if the full dimension $p$ is infinite and $\gamma$ is a Gaussian element in a Hilbert space, provided that $p_r = \text{tr}(\mathcal{T}^\top\mathcal{T})$ is finite, that is, $\mathcal{T}^\top\mathcal{T}$ is a trace operator.

**Proof.** Define

$$
\Phi(r) \overset{\text{def}}{=} \mathbb{P}(\|\mathcal{T}\gamma\| \geq r),
$$

$$
f(r) \overset{\text{def}}{=} \exp\left(-C_0r_0r + \frac{C_0r_0^2}{2} + \frac{r^2}{2}\right).
$$

Then

$$
\mathbb{E}\left\{\exp\left(-C_0r_0\|\mathcal{T}\gamma\| + \frac{1}{2}C_0r_0^2 + \frac{1}{2}\|\mathcal{T}\gamma\|^2\right) \mathbb{I}(\|\mathcal{T}\gamma\| > r_0)\right\} = -\int_{r_0}^{\infty} f(r) d\Phi(r) = f(r_0)\Phi(r_0) + \int_{r_0}^{\infty} f'(r)\Phi(r) dr.
$$
Now we use that \( \Phi(\sqrt{p_r} + \sqrt{2x}) \leq e^{-x} \) for any \( x > 0 \). This can be rewritten as

\[
\Phi(r) \leq \exp\left\{ -\frac{1}{2}(r - \sqrt{p_r})^2 \right\}
\]

for \( r > \sqrt{p_r} \). In particular, in view of \( r_0 \geq 2\sqrt{p_r} + \sqrt{x} \)

\[
f(r_0)\Phi(r_0) \leq \Phi(r_0) \leq \exp\left\{ -\frac{1}{2}(r_0 - \sqrt{p_r})^2 \right\} \leq \exp\left\{ -\frac{1}{2}(p_r + x) \right\}.
\]

Now we use that \( f'(r) = (r - C_0r_0)f(r) \) and

\[
\int_{r_0}^{\infty} f'(r)\Phi(r) \, dr = \int_{r_0}^{\infty} (r - C_0r_0)f(r)\Phi(r) \, dr
\]

\[
\leq \int_{r_0}^{\infty} (r - C_0r_0)\exp\left\{ -\frac{1}{2}(r - \sqrt{p_r})^2 - C_0r_0r + \frac{C_0r_0^2}{2} + \frac{r^2}{2} \right\} \, dr
\]

\[
= \int_{r_0}^{\infty} (r - C_0r_0)\exp\left\{ -(C_0r_0 - \sqrt{p_r})r + \frac{C_0r_0^2}{2} - \frac{p_r}{2} \right\} \, dr
\]

\[
= \int_{0}^{\infty} (x + r_0 - C_0r_0)\exp\left\{ -(C_0r_0 - \sqrt{p_r})(x + r_0) + \frac{C_0r_0^2}{2} - \frac{p_r}{2} \right\} \, dx.
\]

The use of \( \int_{0}^{\infty} e^{-x} \, dx = \int_{0}^{\infty} xe^{-x} \, dx = 1 \) yields

\[
\int_{r_0}^{\infty} f'(r)\Phi(r) \, dr \leq \left( \frac{x_0 - C_0r_0}{C_0r_0 - \sqrt{p_r}} + \frac{1}{(C_0r_0 - \sqrt{p_r})^2} \right) \exp\left\{ r_0\sqrt{p_r} - \frac{C_0r_0^2}{2} - \frac{p_r}{2} \right\}.
\]

It remains to check that for \( C_0 \in (1/2, 1) \) and \( C_0r_0 = 2\sqrt{p_r} + \sqrt{x} \)

\[
-x_0\sqrt{p_r} + \frac{C_0r_0^2}{2} + \frac{p_r}{2} \geq \frac{x + p_r}{2}.
\]

The result follows. \( \square \)

Now we consider Gaussian integrals with an additional quadratic multiplier.

**Lemma A.15.** Let \( T \) be a linear operator in \( \mathbb{R}^p \) with \( \|T\|_{op} \leq 1 \). Let \( z \in \mathbb{R}^\infty \) be a unit norm vector: \( \|z\| = 1 \). Define \( p_r = \text{tr}(T^\top T) \). For any positive \( C_0, r_0 \) with \( 1/2 < C_0 \leq 1 \) and \( C_0r_0 > 2\sqrt{p_r} + 1 + \sqrt{x} \)

\[
\mathbb{E}\left\{ |\langle z, \gamma \rangle|^2 \exp\left( -C_0r_0\|T\gamma\| + \frac{C_0r_0^2}{2} + \frac{1}{2}\|T\gamma\|^2 \right) \mathbb{I}(\|T\gamma\| > r_0) \right\}
\]

\[
\leq C e^{-(p_r + x)/2}.
\]  \hspace{1cm} (A.20)
Proof. Define $T_z$ by $T_z^\top T_z = T^\top T + z \otimes z$. Obviously $\|T_z \gamma\| \geq \|T \gamma\|$, $|\langle z, \gamma \rangle| \leq \|T_z \gamma\|$. Further, $r^2/2 - C_0 r_0$ grows in $r \geq r_0$ in view of $C_0 \leq 1$. Therefore,

$$\langle z, \gamma \rangle^2 \exp\left(-C_0 r_0 \|T \gamma\| + \frac{C_0 r_0^2}{2} + \frac{1}{2}\|T \gamma\|^2\right) \mathbb{I}(\|T \gamma\| > r_0) \leq \|T_z \gamma\|^2 \exp\left(-C_0 r_0 \|T \gamma\| + \frac{C_0 r_0^2}{2} + \frac{1}{2}\|T_z \gamma\|^2\right) \mathbb{I}(\|T_z \gamma\| > r_0)$$

Now we can follow the line of the proof of Lemma A.14. Consider

$$\Phi_\lambda(x) = \mathbb{P}(\|T_z \gamma\| \geq x) \leq \exp\left\{-\frac{1}{2}(x - \sqrt{p_\lambda})\right\},$$

$$f(x) \overset{\text{def}}{=} x^2 \exp\left(-C_0 r_0 x + \frac{C_0 r_0^2}{2} + \frac{x^2}{2}\right)$$

with $p_\lambda = \text{tr} T_z^\top T_z = p + 1$. Then

$$\mathbb{E}\left\{\langle z, \gamma \rangle^2 \exp\left(-C_0 r_0 \|T \gamma\| + \frac{C_0 r_0^2}{2} + \frac{1}{2}\|T \gamma\|^2\right) \mathbb{I}(\|T \gamma\| > r_0)\right\} \leq - \int_{r_0}^{\infty} f(x) d\Phi_\lambda(x) = f(r_0) \Phi_\lambda(r_0) + \int_{r_0}^{\infty} f'(x) \Phi_\lambda(x) dx.$$

Now we can continue as in the proof of Lemma A.14. \qed

The bound (A.20) can be easily extended to the case of a more general functional $Q \gamma$ in place of $\langle \lambda, \gamma \rangle$.

Lemma A.16. Let $T$ be a linear operator in $\mathbb{R}^p$ with $\|T\|_{op} \leq 1$ and $p_r = \text{tr}(T^\top T) < \infty$. Let $A$ be a bounded linear operator with $\text{tr}(A^\top A) < \infty$. For any positive $C_0, r_0$ with $1/2 < C_0 \leq 1$ and $C_0 r_0 > 2 \sqrt{p_r + 1} + \sqrt{x}$

$$\mathbb{E}\left\{\|A \gamma\|^2 \exp\left(-C_0 r_0 \|T \gamma\| + \frac{C_0 r_0^2}{2} + \frac{1}{2}\|T \gamma\|^2\right) \mathbb{I}(\|T \gamma\| > r_0)\right\} \lesssim \text{tr}(A^\top A) e^{-(p_r + x)/2}.$$

Proof. We use the Karhunen-Loeve decomposition of $A^\top A$:

$$\|A \gamma\|^2 = \sum_j \mu_j \langle z_j, \gamma \rangle^2$$

with orthogonal unit vectors $z_j$ and $\sum_j \mu_j = \text{tr}(A^\top A)$, and apply the result of Lemma A.15 to each term of this decomposition. \qed

## B Proofs of the main results

This section collects the proofs of the main theorems.
B.1 Proof of Theorem 2.1

The idea of the proof is to show that for each \( u \) with \( \| D_G u \| = r_G \), the derivative of the function \( L_G(\theta^*_G + t u) \) in \( t \) is negative for \( |t| \geq 1 \). This yields that the point of maximum of \( L_G(\theta) \) cannot be outside of \( A_G(r_G) \). Let us fix any \( u \) with \( \| D_G u \| \leq r \).

We use the decomposition

\[
L_G(\theta^*_G + t u) - L_G(\theta^*_G) = \langle \nabla \zeta, u \rangle + E L_G(\theta^*_G + t u) - E L_G(\theta^*_G).
\]

With \( f(t) = E L_G(\theta^*_G + t u) \), it holds

\[
\frac{d}{dt} L_G(\theta^*_G + t u) = \langle \nabla \zeta, u \rangle + f'(t).
\] (B.1)

The bound (2.5) implies on \( \Omega(x) \)

\[
|\langle \nabla \zeta, u \rangle| = |\langle D_G^{-1} \nabla \zeta, D_G u \rangle| \leq r z(B_{G|x}, x).
\] (B.2)

By definition of \( \theta^*_G \), it also holds \( f''(0) = 0 \). Condition (L0) implies

\[
|f'(t) - t f''(0)| = |f'(t) - f'(0) - t f''(0)| \leq 3 t^2 \delta_{3,G}(r_G).
\]

For \( t = 1 \), we obtain

\[
f'(1) \leq f''(0) + 3 \delta_{3,G}(r_G) = -\langle D_G^2 u, u \rangle + 3 \delta_{3,G}(r_G) = -r_G^2 + 3 \delta_{3,G}(r_G).
\]

If \( 3 \delta_{3,G}(r_G) \leq \rho r_G^2 \) for \( \rho < 1 \), then \( f'(1) < 0 \). Concavity of \( f(t) \) and \( f'(0) = 0 \) imply that \( f'(t) \) decreases in \( t \) for \( t > 1 \). Further, on \( \Omega(x) \) by (B.2)

\[
\left. \frac{d}{dt} L_G(\theta^*_G + t u) \right|_{t=1} \leq \langle \nabla \zeta, u \rangle - r_G^2 + 3 \delta_{3,G}(r_G)
\]

\[
\leq r_G z(B_{G|x}, x) - r_G^2 + 3 \delta_{3,G}(r_G) \leq r_G z(B_{G|x}, x) - (1 - \rho) r_G^2 < 0
\]

for \( r_G > (1 - \rho)^{-1} z(B_{G|x}) \). As \( \frac{d}{dt} L_G(\theta^*_G + t u) \) decreases with \( t \geq 1 \) together with \( f'(t) \) due to (B.1), the same applies to all such \( t \). This implies the assertion.

B.2 Proof of Theorem 2.2

To show (2.10), we use that \( \tilde{\theta}_G \in A_G(r_G) \) and \( \nabla L_G(\tilde{\theta}_G) = 0 \). Therefore,

\[
L_G(\tilde{\theta}_G + u) - L_G(\tilde{\theta}_G) = L_G(\tilde{\theta}_G + u) - L_G(\tilde{\theta}_G) - \langle \nabla L_G(\tilde{\theta}_G), u \rangle.
\]

Let us fix any \( \theta \in A_G(r_G) \) and \( u \) with \( \| D_G u \| \leq r \), and consider

\[
f(t) = f(t, u) \overset{\text{def}}{=} L_G(\theta + t u) - L_G(\theta) - \langle \nabla L_G(\theta), u \rangle t.
\]
As the stochastic term of \( L(\theta) \) and thus, of \( L_G(\theta) \) is linear in \( \theta \), it cancels in this expression, and it suffices to consider the deterministic part \( \mathbb{E}L_G(\theta) \). Obviously \( f(0) = 0 \), \( f'(0) = 0 \). Moreover, \( f''(0) = \langle \nabla^2 \mathbb{E}L_G(\theta) \rangle u, u \rangle = -\langle D^2_G(\theta) u, u \rangle < 0 \). Taylor expansion of the third order implies

\[
|f(1) - \frac{1}{2}f''(0)| \leq |\delta_3(\theta', u)|, \quad \theta' \in [\theta, \theta + u].
\]

In particular, for any \( \theta \in A_G(\Gamma_G) \)

\[
\left| \mathbb{E}L_G(\theta^*_G) - \mathbb{E}L_G(\theta) - \frac{1}{2} \| D_G(\theta - \theta^*_G) \|_2^2 \right| \leq \delta_{3,G}(\Gamma_G). \tag{B.3}
\]

We now use that by Theorem 2.1, \( u = \theta^*_G - \tilde{\theta}_G \) fulfills \( \| D_G u \| \leq r_G \) on \( \Omega(x) \). Therefore, for \( \theta \in A_G(\Gamma_G) \)

\[
\left| L_G(\theta) - L_G(\tilde{\theta}_G) - \frac{1}{2} \| D_G(\theta - \tilde{\theta}_G) \|_2^2 \right|
= \left| L_G(\theta) - L_G(\tilde{\theta}_G) - \langle \nabla L_G(\tilde{\theta}_G), \theta - \tilde{\theta}_G \rangle - \frac{1}{2} \| D_G(\theta - \tilde{\theta}_G) \|_2^2 \right| \leq \delta_{3,G}(\Gamma_G).
\]

The result (2.10) follows. Further, as \( \tilde{\theta}_G \in A_G(\Gamma_G) \), it holds

\[
L_G(\tilde{\theta}_G) - L_G(\theta^*_G) - \frac{1}{2} \| D^{-1}_G \nabla \zeta \|_2^2 = \max_{\theta \in A_G(\Gamma_G)} \left\{ L_G(\theta) - L_G(\theta^*_G) - \frac{1}{2} \| D^{-1}_G \nabla \zeta \|_2^2 \right\}
= \max_{\theta \in A_G(\Gamma_G)} \left\{ \langle \theta - \theta^*_G, \nabla \zeta \rangle + \mathbb{E}L_G(\theta) - \mathbb{E}L_G(\theta^*_G) - \frac{1}{2} \| D^{-1}_G \nabla \zeta \|_2^2 \right\}
\leq \max_{\theta \in A_G(\Gamma_G)} \left\{ \langle D_G(\theta - \theta^*_G), D^{-1}_G \nabla \zeta \rangle - \frac{1}{2} \| D_G(\theta - \theta^*_G) \|_2^2 - \frac{1}{2} \| D^{-1}_G \nabla \zeta \|_2^2 \right\} + \delta_{3,G}(\Gamma_G)
\leq \max_{\theta \in A_G(\Gamma_G)} \left\{ -\frac{1}{2} \| D_G(\theta - \theta^*_G) - D^{-1}_G \nabla \zeta \|_2^2 \right\} + \delta_{3,G}(\Gamma_G)
\]

and similarly \( L_G(\tilde{\theta}_G) - L_G(\theta^*_G) - \frac{1}{2} \| D^{-1}_G \nabla \zeta \|_2^2 \geq -\delta_{3,G}(\Gamma_G) \). This two-sided bound yields as (2.8) as (2.9).

The last statement (2.11) of the theorem follows directly from Lemma A.8 with \( Q = D_G \) and \( f(\theta) = \mathbb{E}L_G(\theta) \).

### B.3 Proof of Theorem 2.3

The definition of \( \theta^* \) and \( \theta^*_G \) implies that

\[
\mathbb{E}L_G(\theta^*_G) \geq \mathbb{E}L_G(\theta^*), \quad \mathbb{E}L(\theta^*_G) \leq \mathbb{E}L(\theta^*).
\]
As $\mathbb{E} L_G(\theta) = \mathbb{E} L(\theta) - \|G\theta\|^2/2$, it follows that

$$
\mathbb{E} L_G(\theta^*_G) - \mathbb{E} L_G(\theta^*) \leq \frac{1}{2} \|G\theta^*\|^2 - \frac{1}{2} \|G\theta^*_G\|^2 \leq \frac{1}{2} \|G\theta^*\|^2.
$$

(B.4)

The bound (B.3) with $\theta = \theta^*$ implies the first statement of (2.12).

Further we show that $\|G\theta^*\| \leq r_b/2$ implies $\|D_G(\theta^*_G - \theta^*)\| \leq r_b$. Indeed, suppose the opposite inequality. Define $u = r_b D_G(\theta^* - \theta^*_G)/\|D_G(\theta^*_G - \theta^*)\|$, so that $\|u\| = r_b$.

The function $f(t) = \mathbb{E} L_G(\theta^*_G) - \mathbb{E} L_G(\theta^*_G + tu)$ is convex in $t$ and $\theta^*_G + tu \in \Theta^o$ for $|t| \leq 1$. Using the approximation (B.3) for $\theta = \theta^*_G + u$ implies

$$
\mathbb{E} L_G(\theta^*_G) - \mathbb{E} L_G(\theta^*_G + tu) \geq \frac{\|u\|^2}{2} \geq \frac{r_b^2}{4}
$$

and concavity of $\mathbb{E} L_G(\theta)$ together with $\nabla \mathbb{E} L_G(\theta^*_G) = 0$ implies

$$
\mathbb{E} L_G(\theta^*_G) - \mathbb{E} L_G(\theta^*_G + tu) \geq \frac{r_b^2}{4}
$$

for $t \geq 1$. This contradicts to the bounds (B.4) and $\|G\theta^*\|^2 \leq r_b^2/2$.

Now for any $\theta$ with $\|D_G(\theta^*_G - \theta)\| \leq r_b$

$$
\left| \mathbb{E} L_G(\theta^*_G) - \mathbb{E} L_G(\theta) - \frac{1}{2} \|D_G(\theta - \theta^*_G)\|^2 \right| \leq \delta_{3,G}(r_b).
$$

(B.5)

Further we use that $\theta^* = \arg\max \mathbb{E} L(\theta)$ and $\mathbb{E} L_G(\theta) = \mathbb{E} L(\theta) - \|G\theta\|^2/2$. By (B.5) in view of $\|D_G(\theta^*_G - \theta^*)\| \leq r_b$ and $D_G^2 = D^2 + G^2$

$$
\mathbb{E} L(\theta^*) - \mathbb{E} L_G(\theta^*_G) = \max_{\theta \in A_G(r_b)} \{ \mathbb{E} L_G(\theta) + \frac{1}{2} \|G\theta\|^2 - \mathbb{E} L_G(\theta^*_G) \}
$$

$$
\leq \max_{\theta \in A_G(r_b)} \left\{ -\frac{1}{2} \|D_G(\theta - \theta^*_G)\|^2 + \frac{1}{2} \|G\theta\|^2 \right\} + \delta_{3,G}(r_b)
$$

$$
= \max_{\theta \in A_G(r_b)} \left\{ -\frac{1}{2} \|D\theta - D^{-1}D_G^2\theta^*_G\|^2 + \frac{1}{2} \|D^{-1}D_G^2\theta^*_G\|^2 \right\} + \delta_{3,G}(r_b)
$$

A similar inequality holds from below with another sign for $\delta_{3,G}$-term yielding for the maximizer $\theta^*$ the bound

$$
\|D\theta^* - D^{-1}D_G^2\theta^*_G\|^2 \leq 4\delta_{3,G}(r_b).
$$

Equivalently, using again $D_G^2 = D^2 + G^2$

$$
\|D^{-1}D_G^2(\theta^* - \theta^*_G) - D^{-1}G^2\theta^*\|^2 \leq 4\delta_{3,G}(r_b).
$$

As $D^2 \leq D_G^2$, this also implies

$$
\|D_G(\theta^* - \theta^*_G) - D_G^{-1}G^2\theta^*\|^2 \leq 4\delta_{3,G}(r_b).
$$
B.4 Proof of the Theorem 2.4

Let \( \tilde{\theta}_G = \text{argmax}_\theta L_G(\theta) \) be the penalized MLE of the parameter \( \theta \). We aim at bounding from above the quantity

\[
\rho(x_0) = \frac{\int_{\|Du\| > r_0} \exp\{L_G(\tilde{\theta}_G + u)\} du}{\int_{\|Du\| \leq r_0} \exp\{L_G(\theta + u)\} du}
\]

with \( \bar{D}^2 = \mathbb{F}(\tilde{\theta}_G) \) for \( \mathbb{F}(\theta) = -\nabla^2 \mathbb{E}L(\theta) \) and \( D(\theta) = \sqrt{\mathbb{F}(\theta)} \).

**Step 1** The use of \( \nabla L_G(\tilde{\theta}_G) = 0 \) allows to represent

\[
\rho(x_0) = \frac{\int_{\|Du\| > r_0} \exp\{L_G(\tilde{\theta}_G + u) - L_G(\tilde{\theta}_G)\} du}{\int_{\|Du\| \leq r_0} \exp\{L_G(\theta + u) - L_G(\theta)\} du}
\]

Now we study this expression for any possible value \( \theta \) from the concentration set of \( \tilde{\theta}_G \).

Consider \( f(\theta) = \mathbb{E}L(\theta) \). As the stochastic term of \( L(\theta) \) and thus, of \( L_G(\theta) \) is linear in \( \theta \), it holds

\[
L_G(\theta + u) - L_G(\theta) - \langle \nabla L_G(\theta), u \rangle = f(\theta + u) - f(\theta) - \langle \nabla f(\theta), u \rangle.
\]

Therefore, it suffices to bound the ratio

\[
\rho(x_0, \theta) \overset{\text{def}}{=} \frac{\int \mathbb{I}(\|D(\theta)u\| > r_0) \exp\{f(\theta + u) - f(u) - \langle \nabla f(\theta), u \rangle\} du}{\int \mathbb{I}(\|D(\theta)u\| \leq r_0) \exp\{f(\theta + u) - f(u) - \langle \nabla f(\theta), u \rangle\} du}
\]

uniformly in \( \theta \) from the set \( \{\theta: \|D_G(\theta - \tilde{\theta}_G)\| \leq r_G\} \); see Theorem 2.1.

**Step 2** First we present some bounds for the denominator of \( \rho(\theta) \). Lemma A.9 yields

\[
\int_{\|D(\theta)u\| \leq r_0} \exp\{f(\theta + u) - f(u) - \langle \nabla f(\theta), u \rangle\} du
\]

\[
\geq (1 - \hat{\chi}(r_0)) \int_{\|D(\theta)u\| \leq r_0} \exp\left(-\frac{\|D_G(\theta)u\|^2}{2}\right) du,
\]

\[
\int_{\|D(\theta)u\| \leq r_0} \exp\{f(\theta + u) - f(u) - \langle \nabla f(\theta), u \rangle\} du
\]

\[
\leq (1 + \hat{\chi}(r_0)) \int_{\|D(\theta)u\| \leq r_0} \exp\left(-\frac{\|D_G(\theta)u\|^2}{2}\right) du,
\]
where $D_G^2(\theta) = F_G(\theta) = -\nabla^2 f(\theta)$ and $\Diamond(x_0)$ is given by (2.16). Moreover, after a proper normalization, the integral $\int_{\|D(\theta)u\| \leq x_0} \exp(-\|D_G(\theta)u\|^2/2) \, du$ can be viewed as the probability of the Gaussian event. Namely

$$\frac{\det D_G(\theta)}{(2\pi)^{p/2}} \int_{\|D(\theta)u\| \leq x_0} \exp\left(-\frac{\|D_G(\theta)u\|^2}{2}\right) \, du = \mathbb{P}\left(\|D(\theta)D_G^{-1}(\theta)\gamma\| \leq x_0\right)$$

for a standard normal $\gamma \in \mathbb{R}^p$. The choice $x_0 \geq \sqrt{p\gamma(\theta)} + \sqrt{2x}$ yields by Corollary A.3

$$\mathbb{P}\left(\|D(\theta)D_G^{-1}(\theta)\gamma\| \leq x_0\right) \geq 1 - e^{-x}.$$ If the error term $\Diamond(x_0)$ is small, we obtain a sharp bound for the integral in the denominator of $\rho(x_0, \theta)$ from (B.6).

**Step 3** Now we bound the integral on the exterior of $\mathcal{U}^o = \{u: \|D(\theta)u\| \leq x_0\}$. Linearity of stochastic term in $L_G(\theta) = L(\theta) - \|G\theta\|^2/2$ and quadraticity of the penalty term imply

$$L_G(\theta + u) - L_G(\theta) - \langle \nabla L_G(\theta), u \rangle = \mathbb{E}L(\theta + u) - \mathbb{E}L(\theta) - \langle \nabla \mathbb{E}L(\theta), u \rangle - \frac{1}{2}\|Gu\|^2.$$ Now we apply Lemma A.13 with $f(\theta + u) = \mathbb{E}L(\theta + u)$. This function is concave and it holds $-\langle \nabla^2 f(\theta)u, u \rangle = \|D(\theta)u\|^2$. The bound (A.19) yields for any $u$ with $\|D(\theta)u\| = r > x_0$

$$L_G(\theta + u) - L_G(\theta) - \langle \nabla L_G(\theta), u \rangle = f(\theta + u) - f(\theta) - \langle \nabla f(\theta), u \rangle - \|Gu\|^2/2 \leq -c_0(\|D(\theta)u\| x_0 - r_0^2/2) - \|Gu\|^2/2$$

$$= -c_0(\|D(\theta)u\| x_0 - r_0^2/2) - \|D_G(\theta)u\|^2/2 + \|D(\theta)u\|^2/2.$$ with $c_0 = 1 - 3r_0^{-2}\delta_3(x_0) \geq 1/2$ and $D_G^2(\theta) = D^2(\theta) + G^2$.

Now we can use the result about Gaussian integrals from Section A.5. With $\mathcal{T} = D(\theta)D_G^{-1}(\theta)$, it holds by Lemma A.14

$$\frac{\det D_G(\theta)}{(2\pi)^{p/2}} \int \mathbb{I}(\|D(\theta)u\| > x_0) \exp\{L_G(\theta + u) - L_G(\theta) - \langle \nabla L_G(\theta), u \rangle\} \, du \leq \mathbb{E}\left\{\exp\left(-c_0x_0\mathcal{T}\gamma + \frac{c_0r_0^2}{2} + \frac{1}{2}\|\mathcal{T}\gamma\|^2\right) \mathbb{I}(\|\mathcal{T}\gamma\| > x_0)\right\} \leq c e^{-(p_G(\theta) + x)/2}. $$

Putting together of Step 1 through Step 3 yields the statement about $\rho(x_0)$. 
B.5 Proof of Theorem 2.5 and Corollary 2.6

We proceed similarly to the proof of Theorem 2.4. Fix any centrally symmetric set \( A \). First we restrict the posterior probability to the set \( \bar{B}(x_0) = \{ u : \| \bar{D}u \| \leq r_0 \} \). Then we apply the quadratic approximation of the log-likelihood function \( L(\theta) \). Denote \( A(x_0) = A \cap \bar{B}(x_0) \). Obviously, \( A(x_0) \) is centrally symmetric as well. Further,

\[
\mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) = \frac{\int_A \exp\{L_G(\bar{\theta}_G + u)\} du}{\int_{\mathbb{R}^p} \exp\{L_G(\theta_G + u)\} du}
\]

\[
\leq \frac{\int_{A(x_0)} \exp\{L_G(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) - \langle \nabla L_G(\bar{\theta}_G), u \rangle \} du}{\int_{\| \bar{D}u \| \leq r_0} \exp\{L_G(\theta_G + u) - L_G(\theta_G) - \langle \nabla L_G(\theta_G), u \rangle \} du} + \rho(x_0).
\]

Now we apply the bounds from the proof of Theorem 2.4 yielding

\[
\mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) \leq \frac{\{1 + \hat{\diamond}(x_0)\} \int_{A(x_0)} \exp\{-\| \bar{D}_G u \|^2/2 \} du}{\{1 - \hat{\diamond}(x_0)\} \int_{\| \bar{D}u \| \leq r_0} \exp\{-\| D_G u \|^2/2 \} du} + \rho(x_0)
\]

This implies the upper estimate for the posterior probability. Now we prove the lower bound. It holds in a similar way that

\[
\mathbb{P}(\theta_G - \bar{\theta}_G \in A \mid Y) = \frac{\int_A \exp\{L_G(\bar{\theta}_G + u)\} du}{\int_{\mathbb{R}^p} \exp\{L_G(\theta_G + u)\} du}
\]

\[
\geq \frac{\int_{A(x_0)} \exp\{L_G(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) - \langle \nabla L_G(\bar{\theta}_G), u \rangle \} du}{\{1 - \hat{\diamond}(x_0)\} \mathbb{P}(\| DD_G^{-1} \gamma \| \leq r_0) + c e^{-r_0^2 / 2}}
\]

\[
\geq \frac{\{1 - \hat{\diamond}(x_0)\} \{\mathbb{P}(\| DD_G^{-1} \gamma \| \leq r_0) - \rho(x_0)\}}{\{1 + \hat{\diamond}(x_0)\} \mathbb{P}(\| DD_G^{-1} \gamma \| \leq r_0) + c e^{-r_0^2 / 2}}.
\]

For the case of an arbitrary possibly non-symmetric \( A \), the proof is similar with the use of (A.13) instead of (A.12).

B.6 Proof of Theorem 2.7

This result can be proved in the same line as Theorem 2.5 using Lemma A.11.
B.7 Proof of Theorem 2.8

It holds

$$\overline{\theta}_G - \bar{\theta}_G = \int (\theta - \bar{\theta}_G) \exp L_G(\theta) d\theta \overline{\exp L_G(\theta)}.$$ 

The use of $\nabla L_G(\bar{\theta}_G) = 0$ helps to represent with $\bar{B}(r_0) = \{u: \|\bar{D}u\| \leq r_0\}$

$$Q(\overline{\theta}_G - \bar{\theta}_G) = \left( \frac{\int_{\|\overline{\theta}_G\| \leq r_0} + \|\overline{\theta}_G\| > r_0} {\int \exp L_G(\bar{\theta}_G) - \langle \nabla L_G(\bar{\theta}_G), u \rangle \exp \{ f(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) \} d\theta} \right).$$

Now, with $f_{\theta}(u) = E L_G(\theta + u)$, define $f(u)$ by using $\theta = \bar{\theta}_G$, that is, $f(u) = f_{\bar{\theta}_G}(u)$.

Linearity of the stochastic part of $L_G(\theta)$ implies

$$L_G(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) - \langle \nabla L_G(\bar{\theta}_G), u \rangle = f(u) - f(0) - f'(0, u),$$

and it holds

$$\|Q(\overline{\theta}_G - \bar{\theta}_G)\| \leq \rho_0(r_0) + \rho_1(r_0)$$

with

$$\rho_0(r_0) \overset{\text{def}}{=} \left\{ \frac{\int_{\|\overline{\theta}_G\| \leq r_0} Qu \exp \{ f(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) \} d\theta} {\int \exp \{ f(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) \} d\theta} \right\},$$

$$\rho_1(r_0) \overset{\text{def}}{=} \left\{ \frac{\int_{\|\overline{\theta}_G\| > r_0} Qu \exp \{ f(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) \} d\theta} {\int \exp \{ f(\bar{\theta}_G + u) - L_G(\bar{\theta}_G) \} d\theta} \right\}.$$ 

As $-\nabla^2 f(0) = \bar{D}_G^2$, Lemma A.10 and Theorem 2.4 imply

$$\rho_0(r_0) \lesssim \frac{\delta_3 \|Q\bar{D}_G^{-1}\|}{\|\bar{D}_G^{-1}\|} \lesssim \delta_3 \sqrt{\bar{p}_Q|G}$$

with $\bar{p}_Q|G = \text{tr}(Q \bar{D}_G^{-2} Q^\top)$. For bounding the term $\rho_1(r_0)$, we apply the bound from Lemma A.13 and then Lemma A.14 and A.15 with $T = \bar{D}_G^{-1} \bar{D}_G$ and $p_r = \text{tr}(T^\top T) = \bar{p}_G$.

The use of $2\|Q\bar{u}\| \leq 1 + \|Q\bar{u}\|^2$ yields on $\Omega(x)$

$$\rho_1(r_0) \lesssim \bar{p}_Q|G \exp\{-(\bar{p}_G + x)/2\}.$$
The second moment of the expression \( \langle u, \tilde{D}_G (\vartheta_G - \tilde{\vartheta}_G) \rangle \) given \( Y \) and a unit vector \( z \) is evaluated similarly. One gets

\[
\mathbb{E} \left[ \langle z, \tilde{D}_G (\vartheta_G - \tilde{\vartheta}_G) \rangle^2 \mid Y \right] - 1 = \frac{\int \langle z, \tilde{D}_G (\vartheta_G - \tilde{\vartheta}_G) \rangle^2 \exp L_G(\vartheta) \, d\vartheta}{\int \exp L_G(\vartheta) \, d\vartheta} - 1
\]

\[
= \left( \int_{\|\tilde{D}u\| \leq r_0} + \int_{\|\tilde{D}u\| > r_0} \right) \frac{[\langle z, \tilde{D}_G u \rangle^2 - 1] \exp \{ f(u) - f(0) - f'(0, u) \} \, du}{\int \exp \{ f(u) - f(0) - f'(0, u) \} \, du}
\]

\[
= \rho_2 + \rho_3.
\]

Similarly to (B.7), one can get \( |\rho_3| \leq \tilde{p}_Q \exp \{ - (\tilde{p}_G + x)/2 \} \). For the value \( |\rho_2| \), we use symmetry of \( U \circ \tilde{B} = \tilde{B}(r_0) = \{ u : \| \tilde{D}u \| \leq r_0 \} \) and Lemma A.9 yielding \( |\rho_2| \lesssim \Diamond \).

### B.8 Proof of Theorem 2.9

Taking into account the Gaussian approximation result from Corollary 2.6, we only have to compare the posterior probability of \( \| Q(\vartheta_G - \overline{\vartheta}_G) \| \leq r \) with \( \mathbb{P}'(\| Q \tilde{D}_G^{-1} \gamma \| \leq r) \).

Let \( a \) be defined as

\[
a = \tilde{\vartheta}_G - \overline{\vartheta}_G.
\]

As \( Q = Q\Pi \) for a projector \( \Pi \), it also holds with \( a_0 = \Pi a \)

\[
\| Q(\vartheta_G - \overline{\vartheta}_G) \| = \| Q(\vartheta_G - \tilde{\vartheta}_G - a_0) \|
\]

and

\[
\mathbb{P}'(\vartheta_G - \overline{\vartheta}_G \in \mathcal{E}_Q(r) \mid Y) = \mathbb{P}'(\vartheta_G - \tilde{\vartheta}_G - a_0 \in \mathcal{E}_Q(r) \mid Y)
\]

Now Theorem 2.7 implies

\[
\left| \mathbb{P}'(\vartheta_G - \tilde{\vartheta}_G - a_0 \in \mathcal{E}_Q(r) \mid Y) - \mathbb{P}'(\tilde{D}_G^{-1} \gamma - a_0 \in \mathcal{E}_Q(r)) \right| \lesssim c \left\{ \Diamond + \delta_3 \| \tilde{D}_G a_0 \| + n^{-1} \right\}.
\]

Theorem 2.8 yields that the norm of \( \tilde{D}_G a_0 \) can be bounded on \( \Omega(x) \) as

\[
\| \tilde{D}_G a_0 \| = \| \tilde{D}_G \Pi a \| \lesssim \delta_3 \sqrt{\bar{p}_{\Pi}} + n^{-1} \bar{p}_{\Pi}
\]

with \( \bar{p}_{\Pi} = \text{tr}(\Pi \tilde{D}_G^{-2} \Pi \tilde{D}_G^{-2} \Pi) \). It remains to compare two Gaussian probabilities of \( \| Q \tilde{D}_G^{-1} \gamma \| \leq r \) and of \( \| Q(\tilde{D}_G^{-1} \gamma - a_0) \| \leq r \). For this one can apply the Pinsker
inequality. However, the Gaussian comparison result of Theorem A.1 provides a more precise bound in view of the elliptic shape of the considered credible sets:
\[
|P((Q(D_G^{-1} \gamma) - a_0) \leq r) - P((Q(D_G^{-1} \gamma) \leq r)| \leq \frac{\|Qa_0\|^2}{\|QD_G^{-2}Q^\top\|_F}.
\]
Now the assertion follows by one more application of Theorem 2.8 in view of $\bar{P}_{Q|G} \leq \|QD_G^{-2}Q^\top\|_F^2 \bar{p}_\Pi$.

### B.9 Proof of Theorem 2.10

We assume that all the conditions are fulfilled for the smaller prior covariance $G^2$, and all error terms correspond to that prior. Corollary 2.6 implies on a set of probability at least $1 - 1/n$ for any measurable set $A$
\[
|P(\theta_G - \bar{\theta}_G \in A \mid Y) - P'(D_G^{-1} \gamma \in A)\| \leq \delta_3 + n^{-1}.
\]
Similarly, again on a set of probability at least $1 - 1/n$
\[
|P'(\theta_{G_1} - \bar{\theta}_{G_1} \in A \mid Y) - P'(D_{G_1}^{-1} \gamma \in A)\| \leq \delta_3 + n^{-1}.
\]
Define $a = \bar{\theta}_G - \bar{\theta}_{G_1}$. The bound of Theorem A.1 yields
\[
|P'(\|Q(D_G^{-1} \gamma - a)\| \leq r) - P'(\|Q(D_{G_1}^{-1} \gamma) \leq r)| \leq \frac{1}{\|QD_G^{-2}Q^\top\|_F}\{\text{tr}(Q(D_G^{-2} - D_{G_1}^{-2})Q^\top) + \|Qa\|^2\}.
\]
Putting all bounds together completes the proof.

### C Proof of Theorems 3.1 and 3.2

We just check the conditions of the general results from Section 2.

**Lemma C.1.** Let, for a fixed $\varrho > 0$, the function $\phi(\theta)$ be well defined on the set $\Theta_{\varrho}$ and satisfy $(\phi g)$. Then $(EV)$ holds with $V^2 = n \nabla^2 \phi(\theta^*)$, and $g = \sqrt{n} g$.

**Proof.** For each $\theta \in \Theta_1$, due to the i.i.d. structure of the data, it holds from (3.2) in view of the i.i.d. assumption
\[
\log E \exp\{\langle S, \theta \rangle\} = n \log E \exp\{\langle \Psi(X_1), \theta \rangle\}.
\]
Let \( u \in \mathbb{R}^p \) be such that \( \|Vu\| = 1 \). Then for \( |\lambda| \leq g = \sqrt{\rho} \), we have \( \theta^* + \lambda u \in \Theta_\phi(\theta^*) \), and, in view of \( \phi(\theta^*) = 0 \) and \( \nabla\phi(\theta^*) = \nabla \)
\[
\log \mathbb{E} \exp \left\{ \lambda \langle \nabla \zeta, u \rangle \right\} = n \left\{ \phi(\theta^* + \lambda u) - \lambda \langle \Psi, u \rangle \right\}
\]
\[
= n \left\{ \phi(\theta^* + \lambda u) - \phi(\theta^*) - \lambda \langle \nabla \phi(\theta^*), u \rangle \right\}
\]
\[
= \frac{n}{2} \langle \nabla^2 \phi(\theta^* + \rho \lambda u) \lambda u, \lambda u \rangle
\]
for \( \rho \in [0, 1] \). It remains to note that \( \|Vu\|^2 = n \langle \nabla^2 \phi(\theta^*) u, u \rangle = 1 \) yielding \( (EV) \) by \( (C.1) \) and \( (3.4) \). \( \square \)

**Remark C.1.** For small \( \rho \), the vicinity \( \Theta_\rho \) is small and \( \nu_0 \) is close to one.

Condition \( (3.6) \) implies that \( 0.3g \geq \sqrt{\rho G} \) and \( x_c = \frac{g^2}{4} \geq 4 \log n \). Condition \( (EV) \) and Theorem A.5 ensure for \( z_G = \sqrt{\rho G} + \sqrt{2 \log n} \) the probability bound \( (2.5) \)
\[
\|D_G^{-1} \nabla \zeta\| \leq z_G \quad \text{on a random set } \Omega_n \text{ with } \mathbb{P}(\Omega_n) \geq 1 - 3/n.
\]
Now we turn to \( (L_0) \). It holds \( \mathbb{E} L(\theta) = n \left\{ \langle \Psi, \theta \rangle - \phi(\theta) \right\}, \)
and
\[
\delta_m(\theta, u) = -n \frac{d^m}{m! dt^m} \phi(\theta + tu) \bigg|_{t=0}, \quad m = 3, 4.
\]
The function \( \phi(\theta) \) is analytic in the domain \( \Theta_1 \), so \( (L_0) \) is trivial. However, to apply our results we need a quantitative bound on the values \( \delta_m(\theta, u), \quad m = 3, 4 \).

The definition \( (3.2) \) of \( \phi(\theta) \) implies
\[
\nabla \phi(\theta) = E_\theta [\Psi(X_1)],
\]
\[
\nabla^2 \phi(\theta) = E_\theta [\Psi(X_1) \otimes \Psi(X_1)] - E_\theta \Psi(X_1) \otimes E_\theta \Psi(X_1) = Var_\theta [\Psi(X_1)].
\]

**Lemma C.2.** Under \( (\Psi u) \), it holds for any \( \theta \in \Theta_\phi(\theta^*_G) \) and \( u \in \mathcal{U}_\phi(\theta) \)
\[
\delta_3(\theta, u) \lesssim n \langle \nabla^2 \phi(\theta) u, u \rangle^{3/2}, \quad \delta_4(\theta, u) \lesssim n \langle \nabla^2 \phi(\theta) u, u \rangle^2.
\]

**Proof.** For any fixed \( \theta \) and \( u \), consider
\[
h(t) = \langle E_\theta \Psi(X_1), \theta + tu \rangle - \phi(\theta + tu)
\]
\[
= -\log \int \exp \{ \langle \Psi(x) - E_\theta \Psi(X_1), \theta + tu \rangle \} \mu_0(dx).
\]
Obviously, \( \delta_m(\theta, u) = nh^{(m)}(0), \quad m \geq 2 \). Define
\[
q_m(t) \overset{\text{def}}{=} \int (\Psi(x) - E_\theta \Psi(X_1), u)^m e^{\langle \Psi(x) - E_\theta \Psi(X_1), \theta + tu \rangle} \mu_0(dx), \quad m \geq 0.
\]
It holds \( q'_m(t) = q_{m+1}(t) \) and, also
\[
\frac{q_m(0)}{q_0(0)} = E_\theta (\Psi(X_1), u)^m, \quad m \geq 1,
\]
\[
q_2(0) = \langle \nabla^2 \phi(\theta) u, u \rangle.
\]
By (3.5) we deduce from \( \langle \nabla^2 \phi(\theta) u, u \rangle \leq \rho^2 \) that
\[
\left| \frac{q_m(0)}{q_0(0)} \right| \leq \kappa^m \left( \frac{q_2(0)}{q_0(0)} \right)^m \leq \kappa^m \rho^m.
\]

Straightforward calculus yields
\[
\begin{align*}
    h''(t) &= -\frac{q_2(t)}{q_0(t)} + \frac{q_1^2(t)}{q_0^2(t)}, \\
    h^{(3)}(t) &= -\frac{q_3(t)}{q_0(t)} + \frac{3q_2(t) q_1(t)}{q_0^2(t)} - \frac{2q_1^3(t)}{q_0^3(t)}, \\
    h^{(4)}(t) &= -\frac{q_4(t)}{q_0(t)} + \frac{4q_3(t) q_1(t)}{q_0^2(t)} + \frac{3q_2^2(t)}{q_0^3(t)} - \frac{12q_2(t) q_1^2(t)}{q_0^4(t)} + \frac{6q_1^4(t)}{q_0^5(t)},
\end{align*}
\]
so that
\[
|h^{(3)}(0)| \leq C_3 \kappa^3 \rho^3, \quad |h^{(4)}(0)| \leq C_4 \kappa^4 \rho^4
\]
for some absolute constants \( C_3, C_4 \). This implies (C.2). \( \square \)

The result of this lemma can be written in the following form: if \( u \) fulfills \( \| \sqrt{F(\theta)} u \| \leq r \) for \( F(\theta) = n \nabla^2 \phi(\theta) \), then
\[
\delta_3(\theta, u) \lesssim \kappa^3 r^3 n^{-1/2}, \quad \delta_4(\theta, u) \lesssim \kappa^4 r^4 n^{-1}.
\]
This also implies for \( \delta_{3,G}(r) \) from (2.6)
\[
\delta_{3,G}(r) \leq \kappa^3 r^3 n^{-1/2}, \quad \delta_{4,G}(r) \leq \kappa^4 r^4 n^{-1}
\]
and similarly for \( \delta_3(r) \) from (2.15). Now we are prepared for proving the main results.

Let \( z_G = \sqrt{p_G} + \sqrt{2 \log n} \) and \( r_G = 2z_G \). First note that \( (\phi \rho, G) \) implies for any \( \theta \) with \( \| D_G(\theta - \theta^*_G) \| \leq r_G \) that \( \| \sqrt{F(\theta)}(\theta - \theta^*_G) \| \leq \nu_0 r_G \). Further, Lemma C.2 and (C.3) help to check (2.7) for \( n \gtrsim \kappa^3 r_G \) and the claims of Theorem 3.1 follow from Theorems 2.1 and 2.2. The proof of Theorem 3.2 is similar.