A SCHEMATIC MODEL OF GAUGED S-DUALITY SPONTANEOUS BREAKDOWN

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Abstract. We present a schematic solution of the $SL(2,\mathbb{R})$ BRST equation of gauged $S$-duality using modular forms of the Ramanujan $\Delta$ function type. The solutions with spontaneously broken gauged $S$-duality with the gauge group $\bar{\Gamma}_0(12)$ satisfy the desired conditions for the dimensionality of space-time and the gauge symmetries of low energy theory in the zero slope limit $\alpha' \to 0$.

Dynamical gauged $S$-duality, proposed by the author, has been used to describe the dynamics of $D$-brane fields.\cite{1,2} The gauged $S$-duality is defined only by its algebraic structure and, in this formalism, we cannot always evaluate the path-integrals. Nevertheless, we can obtain some non-perturbative properties of the theory by solving the $SL(2,\mathbb{R})$ BRST equation of gauged $S$-duality.

The search for the solutions of the BRST equation brings us back to Ramanujan’s $\Delta$ function.\cite{3} This function is one of the most celebrated modular forms and is defined by

$$\Delta[q] = q \prod_{\ell=1}^{\infty} \left(1 - q^{\ell}\right)^{24}, \quad q \in \exp(2\pi i h)$$

for the Poincaré upper half plane $h$.

It is worth pointing out that the modularity of this $\Delta$ function is related to the modulus of the world sheet as seen in, for example, the GSO projection in superstring theory and the Mandelstam-type open/closed string duality in the partition function of bosonic string theory. The number 24 differs from the critical dimension of bosonic string theory by the dimension of the tachyonic states.

We would like to obtain the partition function of $M$-field theory. The partition function of dynamical gauged $S$-duality obeys the equation

$$\left(\sum_{I=1}^{3} \Theta^I \left(Q\mathcal{A}^I + \frac{i}{2} \mathcal{C}^I\right) - \alpha' \Lambda\right) \psi[g_s, \vec{t}, \vec{t}] = 0$$

where

$$\Theta^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Theta^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Theta^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The gauge potential $\mathcal{A}$ and ghost $\mathcal{C}$ for coefficients of $SL(2,\mathbb{R})$ group factors $\varepsilon^I$ are defined by

$$\mathcal{A} = \sum_{I=1}^{3} \varepsilon^I \exp\left(\sum_{n=1}^{\infty} (s_n g_n^I + \bar{s}_n \bar{g}_n^I)\right), \quad \mathcal{C} = \sum_{I=1}^{3} \varepsilon^I \exp\left(\sum_{n=1}^{\infty} (s_n \theta_n^I + \bar{s}_n \bar{\theta}_n^I)\right)$$

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with fermionic operators $q^I_n$ and Grassmann numbers $\theta^I_n$.

We define the quadratic term as the external product of ghosts
\begin{equation}
\mathcal{R} = \mathcal{C} \times \mathcal{C}
\end{equation}
and the ‘cosmological constant’ $\Lambda$ is the eigenvalue of the Laplacian in the time variables $s_n$.

\begin{equation}
\Delta \left[ \bar{s}_n s_n \right] \mathcal{A} \bigg|_{\bar{s}=s=0} = -\Lambda \mathcal{A} \bigg|_{\bar{s}=s=0}, \quad \Delta \left[ \bar{s}_n s_n \right] = \sum_{n=1}^{\infty} \left[ \frac{\partial^2}{\partial s_n^2} + \frac{\partial^2}{\partial \bar{s}_n^2} \right].
\end{equation}

The operator $Q$ is the sum of BRST charges of the $U(n)$ Chan-Paton factors $q_n$ and $\bar{q}_n$.

\begin{equation}
Q^I = i \sum_{n=1}^{\infty} (q^I_n + \bar{q}^I_n).
\end{equation}

This operator forces the $D$-brane field to have $SU(n)$ Yang-Mills physical states as its components.

In the paper Ref.1, we did not distinguish the time parameter $s_n$ in the BRST equation and the time variable $t_n$ in the physical state. In this paper, we distinguish them and we regard the former time $s_n$ as a constant parameter when we solve BRST equation.

Now we explain the role of each term in the BRST charge $Q$. When we take the gauge $s_n = s_0$ for all $n$, if $SL(2, \mathbb{R})$ is not spontaneously broken, then $Q$ is nilpotent and the first term of the BRST equation becomes the first term of the quantity
\begin{equation}
\sum_{I=1}^{3} \Theta^I \varepsilon^I Q^I \psi.
\end{equation}

If the $SL(2, \mathbb{R})$ symmetry is spontaneously broken, the polarized terms are added.

The second term of BRST equation is
\begin{equation}
\sum_{I=1}^{3} \Theta^I \varepsilon^I \left( \sum_{\ell=0}^{\infty} \left( \frac{2 \text{Re}(\epsilon^I_n)}{\ell!} \right)^\ell \right) \psi
\end{equation}
where we have defined new Grassmann numbers $\epsilon^I_n = \sum_{i \neq i} \theta^I_{n'}. \quad$ We denote the $i$-th term in BRST charge by $Q^{(i)}$.

It is worth noting that the first term of $Q$ can be regarded as an annihilation-like operator acting on all unitary Yang-Mills physical states
\begin{equation}
Q^{(1)} : \mathcal{H}_{c_n} \rightarrow \mathcal{H}_{c_n}
\end{equation}
where $\mathcal{H}_c$ is the Hilbert space of physical states $\psi_n (\epsilon = \prod_n \epsilon_n)$ that are the terms in the expansion of the wave function $\psi$ corresponding to $n$ ghosts. The $n$-th component of the expansion of the second term in terms of $\epsilon_n$ can be regarded as a creation-like operator for ghosts
\begin{equation}
Q^{(2)}_{\epsilon_n} : \mathcal{H}_1 \rightarrow \mathcal{H}_{c_n} \quad \text{and} \quad \mathcal{H}_{c_n}^2
\end{equation}
since the contribution of $\text{Re}(\epsilon)$ factors from Eq. 9 is up to quadratic order.

\textcolor{red}{1}\text{We need to be careful of the index } I = 1, 2, 3 \text{ of } SL(2, \mathbb{R}) \text{ in each charge } q^I, \text{ because these charges have meaning only when they are summed over this index.}
The terms $Q^{(i)}$ for $i = \{1, 2\}$ are physically interpreted as the kinetic term and interaction term of $D$-particle fields. These terms impose linear relations for each set of three $\tau$ numbers in the expansion $\psi = \sum_{n=0}^{\infty} \tau_n \psi^n$:

$$\sum_{k=0}^{2} c_k \tau(\epsilon_n)^k = 0$$

for $(\epsilon_n)^0 = 1$ and certain coefficients $c_k$. This equation is just the linear relation between three coefficients in Fourier expansions of modular forms that are related to each other by the Hecke operators of the generators of Fuchsian group. [5]

Since the BRST equation $Q\psi = 0$ is $SL(2, \mathbb{R})$ covariant, the space of solutions must have $SL(2, \mathbb{R})$ rotational symmetry.

So, we schematically deduce the properties of solutions of Eq. (2) in terms of modular forms.

Our schematic model of internal gauge symmetries and space-time requires:

1. The solution in modular form such that the weight, sum of cusps and index of the congruence group are $2n$, $24m$ and $24n$ for integers $n$ and $m$. The integer 24 comes from the critical dimension of bosonic string theory minus two. Of course the coefficients of the Fourier expansion of this solution satisfy linear relations of the type shown in Eq. (12).

2. The cusps of the modular form correspond to the classical saddle-point solution with spontaneous $S$-duality breakdown. The cusp $i\infty$ corresponds to gravity and is always included. The number of gauge symmetries and their ranks determine the internal space according to the Kaluza-Klein reduction. The standard model demands three kinds of gauge interactions besides gravity. These gauge interactions would be unified in the higher rank terms of the physical state $\psi$. In the modular form, gauge symmetries are encoded as ideals (a notion defined in algebraic number theory) in an infinite product expression. An ideal $I_\ell = \ell \mathbb{Z}$ consists of the $\ell$-th saddle point ($\ell = 1, 2, \cdots, \infty$) of a factor $\psi^{Nn/24} \prod_\ell (1 - \psi^{\ell} \psi^{\ell/2} / 2 - \cdots)$, depends on the coupling constant of $D$-particle fields by the factor of ‘chemical potential’ $\alpha'\Lambda$, corresponding to the dimension of the derivative by time variable.

3. The genus of the gauged $S$-duality moduli space is desired to be a low one. This requires the axion-dilaton degenerations of $T$-dual type IIB string theory from the original $M$-theory.

The schematic form of the solution when the gauged $S$-duality is not broken, i.e., $\Lambda = 0$, is

$$\psi[\chi] = \chi \prod_{\ell=1}^{\infty} \left(1 - \chi^{\ell}\right)^{24}.$$
for the $U(\ell)$ BRST charge $q_{\ell}$. The wave function $\chi^\ell$, which needs to have no mass dimension, can be expanded in terms of the number of ghosts

$$\chi^\ell = \sum_{\ell} \chi^\ell \prod_{i=1}^{\ell} c^i,$$

where the coefficients $\chi^\ell$ are independent of the Grassmann numbers $c$.

The degeneracy of vacua in $\psi[\chi]$, that leads to the flat ten dimensionality of space-time, breaks due to the existence of $\alpha' \Lambda \neq 0$ in the BRST charge Eq. (2). We call this gauged S-duality spontaneous breakdown (GSSB).

The schematic form of the solution of GSSB that satisfies these requirements for genus zero is

$$\psi^{(0)}[\chi] = \chi \prod_{\ell=1}^{\infty} \left(1 - \chi^\ell\right)^2 \left(1 - \left[\bigotimes_1^3 u_{\alpha'} \Lambda \chi\right]^{\ell}\right)^2 \left(1 - \left[\bigotimes_1^4 u_{\alpha'} \Lambda \chi\right]^{\ell}\right),$$

where the factor $u(\vec{s}, \vec{\bar{s}})$ with length dimension is a function of given time parameters in BRST equation.

The solution $\psi^{(0)}$ is a modular form of weight 2. It has the symmetry group of the level 12 congruence subgroup of $SL(2, \mathbb{Z})$, known as $\Gamma_0(12)$. The cusps of this group are

$$1^2 \cdot 3^2 \cdot 4^1 \cdot 12^1,$$

where we denote by $n^k$ the cusp with $k$ orbits and width $n$. The saddle points of $\psi^{(0)}$ on $\mathfrak{h}$ with a cusp whose width is $n$ are

$$\chi^{(0)}_n \simeq \frac{\omega_n}{u_{\alpha'} \Lambda}, \quad (\omega_n)^n = 1.$$

Here, the equality $a \simeq b$ indicates that $a$ and $b$ are in the same conjugacy class of $SL(2, \mathbb{Z})$.

In the following we make three important remarks about these schematic solutions.

On the dimensionality of space-time.

The solution $\psi^{(0)}$ has the factor

$$\chi^{\frac{1}{2}} \prod_{\ell=1}^{\infty} \left(1 - \chi^\ell\right)^2.$$

There is no factor of $\sqrt{\alpha'}$, which leads to the existence of the four-dimensional element of flat space-time. We remark that GSSB breaks the flat directions of the moduli space of vacua non-perturbatively since the flatness of each direction of space-time is perturbatively stable so that a $\Delta[\mathbf{q}]$ type solution is possible.

We also draw attention to the fact that our formulation is completely background independent, and even the dimensionality is not assumed.

We note that the factor $\chi^{\frac{1}{2}}$ in Eq. (16) arises from the consistency with modularity. The product of Dedekind $\eta$ functions $\prod_{i=1}^{k} \eta(r_i \chi)$ is a modular form with weight $k/2$ when $\sum_i r_i = 24$. 
On the gauge symmetries.

However tempting it might be to regard the width of a cusp as the rank of a corresponding gauge symmetry, this will not work. The rank of the gauge symmetry does not appear in each solution explicitly. The gauge symmetries are based on ideals and if one ideal is included by two other ideals, i.e., $I_3 \subset I_1 \cap I_2$ the first interaction unifies the other two interactions. In the present paper, we can only conclude the number of gauge symmetries and their relations with each other since we may need to solve the BRST equation exactly including some self-consistency conditions to specify the concrete form of the variable $\chi$.

On the parameters in the BRST equation.

The equation of the $D$-particle field in our gauges has an infinite number of time parameters
\begin{equation}
(20)
  s_1, \ s_2, \ \cdots
\end{equation}
and the parameter as the eigenvalue of Laplacian
\begin{equation}
(21)
  -\Lambda.
\end{equation}
As seen before, due to the modular symmetry, in the schematic solution Eq.(16) these time parameters have finite degrees of freedom governed by a finite number of ideals
\begin{equation}
(22)
  I_1, \ I_2, \ \cdots, \ I_n
\end{equation}
which correspond to the cusps of the physical state solution.

When we take the Dirac quantization condition on breaking $SL(2, \mathbb{R})$ to $SL(2, \mathbb{Z})$ around the saddle points of the physical state or gauge potential on the upper plane, the Goldstone mode $\tilde{\psi}$ with the constraint $\Delta \tilde{\psi} = 0$ has the set of discrete length spectra composed of closed geodesics $C^{(\ell_1)}_m$, conjugate classes of geodesics on $\mathfrak{h}$ promoted by time variable $s_m$, on the moduli space compact Riemann surface $\Sigma$. The isometry parameters $s_m^{(\ell_1)}$ of the geodesics break the $U(N_m)$ gauge symmetry since the nilpotency of BRST charge $q^{(\ell_1)}_I$ stands at $\vec{s} = 0$.

The parameters are
\begin{equation}
(23)
  \{s_m^{(\ell_1)}, \Lambda^{(\ell_2)}\}_{m=1}^n
\end{equation}
where
\begin{equation}
(24)
  s_m^{(\ell_1)} = \oint_{C^{(\ell_1)}_m} ds
\end{equation}
with line element $ds$ on $\Sigma$ and
\begin{equation}
(25)
  0 = \Lambda^{(0)} < \Lambda^{(1)} < \cdots.
\end{equation}

The non-zero eigenvalue of Laplacian $\Lambda > 0$ breaks $SL(2, \mathbb{R})$ symmetry spontaneously.

Using ’t Hooft’s argument on naturalness, we assume that nature adopts the first eigenvalues of these discrete spectra. So, we can conclude that $s_m^{(1)}$ and $\Lambda^{(1)}$ are the natural parameters in the BRST equation.

\footnote{This way of describing the $U(N)$ gauge symmetries by Goldstone modes $\tilde{\psi}$ resembles that of dual resonance models.}

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