RELATIVE ADJONT TRANSCENDENTAL CLASSES
AND ALBANESE MAPS OF
COMPACT KÄHLER MANIFOLDS WITH NEF RICCI CURVATURE

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1. Introduction

Let \( p : X \to Y \) be a holomorphic surjective map, where \( X \) and \( Y \) are compact Kähler manifolds. We denote by \( W \subseteq Y \) an analytic set containing the singular values of \( p \), and let \( X_0 := p^{-1}(Y \setminus W) \). Let \( \{ \beta \} \in H^{1,1}(X, \mathbb{R}) \) be a real cohomology class of \((1,1)\)-type, which contains a non-singular, semi-positive definite representative \( \beta \).

Our primary goal in this note is to investigate the positivity properties of the class

\[
c_1(K_{X/Y}) + \{ \beta \},
\]

which are inherited from similar fiberwise properties.

In this perspective, the main statement we obtain here is as follows.

**Theorem 1.1.** Let \( p : X \to Y \) be a surjective map. We consider a semi-positive class \( \{ \beta \} \in H^{1,1}(X, \mathbb{R}) \), such that the adjoint class \( c_1(K_{X_y}) + \{ \beta \}|_{X_y} \) is Kähler for any \( y \in Y \setminus W \). Then the relative adjoint class

\[
c_1(K_{X/Y}) + \{ \beta \}
\]

contains a closed positive current \( \Theta \), which equals a (non-singular) semi-positive definite form on \( X_0 \).

As a consequence of the proof of the previous result, the current \( \Theta \) will be greater than a Kähler metric when restricted to any relatively compact open subset of \( X_0 \), provided that \( \beta \) is a Kähler metric. Also, if \( \beta \geq p^*(\gamma) \) for some \((1,1)\)-form \( \gamma \) on \( Y \), then we have

\[
\Theta \geq p^*(\gamma).
\]

We remark that if the class \( \beta \) is the first Chern class of a holomorphic \( \mathbb{Q} \)-line bundle \( L \), that is to say, if

\[
\{ \beta \} \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q}),
\]

then there are many results concerning the positivity of the twisted relative canonical bundle, cf. [2], [3], [4], [6], [14], [17], [18], [21], [22], [23], [24], [25], [28], [31], [35], [39], [40], [41] to quote only a few.
The references [34], [36] are particularly important for us; indeed, a large part of the arguments presented by G. Schumacher in [34], [35] will be used in our proof, as they rely on the complex Monge-Ampère equation as substitute for the theory of linear bundles used in the other works quoted above (see section 3.2 of this paper).

Before stating a few consequences of our main result, we recall the following metric version of the usual notion of nef line bundle in algebraic geometry, as it was introduced in [10].

**Definition 1.2.** Let \((X, \omega)\) be a compact complex manifold endowed with a hermitian metric, and let \(\{\rho\}\) be a real \((1,1)\) class on \(X\). We say that \(\{\rho\}\) is nef (in metric sense) if for every \(\varepsilon > 0\) there exists a function \(f_\varepsilon \in \mathcal{C}^\infty(X)\) such that

\[
\rho + \sqrt{-1} \partial \bar{\partial} f_\varepsilon \geq -\varepsilon \omega.
\]

Thus the class \(\{\rho\}\) is nef if it admits non-singular representatives with arbitrary small negative part. It was established in [11] that if \(X\) is projective and if \(\{\rho\}\) is the first Chern class of a line bundle \(L\), then \(L\) is nef in the algebro-geometric sense if and only if \(L\) is nef in metric sense.

Let \(\mathcal{X} \to \mathbb{D}\) be a non-singular Kähler family over the unit disk. Then we have the following (direct) consequence of Theorem 1.1.

**Corollary 1.3.** We assume that the bundle \(K_{\mathcal{X}_t}\) is nef, for any \(t \in \mathbb{D}\). Then \(K_{\mathcal{X}/\mathbb{D}}\) is nef.

We remark that in the context of the previous corollary, much more is expected to be true. For example, if the Kähler version of the invariance of plurigenera turns out to be true, then it would be enough to assume in Corollary 1.3 that \(K_{\mathcal{X}_0}\) is pseudo-effective in order to derive the conclusion that \(K_{\mathcal{X}/\mathbb{D}}\) is pseudo-effective.

The second application of Theorem 1.1 concerns the Albanese morphism associated to a compact Kähler manifold \(X\). We denote by \(q := h^0(X, \mathcal{O}_X)\) the irregularity of \(X\), and let

\[
\text{Alb}(X) := H^0(X, T_\mathcal{X}^*)^*/H_1(X, \mathbb{Z})
\]

be the Albanese torus of associated to \(X\). We recall that the Albanese map \(\alpha_X : X \to \text{Alb}(X)\) is defined as follows

\[
\alpha_X(p)(\gamma) := \int_{p_0}^p \gamma
\]

modulo the group \(H_1(X, \mathbb{Z})\), i.e. modulo the integral of \(\gamma\) along loops at \(p_0\).

We assume that \(-K_X\) is nef, in the sense of the definition above. It was conjectured by J.-P. Demailly, Th. Peternell and M. Schneider in [12] that \(\alpha_X\) is surjective; some particular cases of this problem are
established in [12], [32], [7]. If $X$ is assumed to be projective, then the surjectivity of the Albanese map was established by Q. Zhang in [44], by using in an essential manner the char $p$ methods. More recently, in the article [45], the same author provides an alternative proof of this result, based on the semi-positivity of direct images.

We settle here the conjecture in full generality.

**Theorem 1.4.** Let $X$ be a compact Kähler manifold such that $-K_X$ is nef. Then its Albanese morphism $\alpha_X : X \to \text{Alb}(X)$ is surjective.

Besides Theorem 1.1, our proof is using some ideas from [12] and [5]; we refer to the first paragraph for a detailed discussion about the connections with these articles.

Our paper is organized as follows. In the first paragraph we review the proof of Theorem 1.4 under the additional assumption that $X$ is projective. As we have already mentioned, in this case the result is known, but the proof we will present is different from the arguments in [44]: actually, it can be seen as a simplified variation of some of the arguments invoked in [45] (see also [9]). It is based upon a version of Theorem 1.1 under the hypothesis that the class $\{\beta\}$ corresponds to a line bundle (this result is completely covered by the literature on the subject, cf. [4], [37]).

This serves us as a motivation for the second paragraph, where we prove Theorem 1.1. In a word, we show that the so-called fiberwise twisted Kähler-Einstein metric endows the bundle $K_{X/Y}|_{X_0}$ with a metric whose curvature is bounded from below by $-\beta$. Thus, the twisted version of the psh variation of the Kähler-Einstein metric established in [34] holds true. Finally, we show that the local weights of the metric constructed in this way are bounded near the analytic set $X \setminus X_0$. This is by no means automatic, given the tools which are involved in the proof (the approximation theorem in [13], together with a precise version of the Ohsawa-Takegoshi extension theorem, [4]). The difficulty steams from the fact that in order to establish the estimates for the said weights we cannot rely on the geometry of the manifold $X_y$, as $y$ is approaching a singular value of the map $p$.

Finally, a complete proof of the Corollary 1.3 and Theorem 1.4 is provided, together with a few questions/comments.

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2. Surjectivity of the Albanese map: review of the projective case

As we have already mentioned in the introduction, our proof for the surjectivity of the Albanese map corresponding to the compact Kähler manifolds with nef anti-canonical class relies heavily on Theorem 1.1. However, if the manifold $X$ is projective, then the following stronger version of Theorem 1.1 was obtained in [4], [37].

**Theorem 2.1.** Let $(F, h_F)$ be a line bundle on $X$, endowed with a metric with semi-positive definite curvature form. We assume moreover that for some generic $y \in Y$ the bundle $k(mK_y + F)$ admits a section which is $L^{1/m}$-integrable with respect to $h_F^{1/m}$, where $m$ is a positive integer. Then the bundle $mK_{Y/X} + F$ is pseudo-effective.

As a consequence, we infer the following statement.

**Corollary 2.2.** Let $p : X \to Y$ be a surjective map between non-singular projective manifolds. We consider $L \to X$ a nef line bundle, such that $H^0(X_y, K_{X_y} + L|_{X_y}) \neq 0$. Then the bundle $K_{Y/X} + L$ is pseudo-effective.

**Proof.** Let $A \to X$ be a very ample line bundle. Then for each positive integer $m$ we define the bundle

$$L_m := mL + A;$$

it is ample, hence it can be endowed with a metric $h_m$ with positive definite curvature. We consider the bundle $mK_{X/Y} + L_m$; by hypothesis, there exists a section $u \in H^0(X_y, K_{X_y} + L|_{X_y})$, so the bundle

$$mK_{Y/X} + L_m|_{X_y}$$

admits a non-trivial section, e.g. $u^{\otimes m} \otimes s_A$ where $s_A$ is a non-zero section of $A$. By Theorem 2.1 the bundle $mK_{Y/X} + L_m$ is pseudo-effective; as $m \to \infty$, we infer that $K_{Y/X} + L$ is pseudo-effective. 

We will explain next the relevance of the previous result in the proof of Theorem 1.4 under the assumption that $X$ is projective; we first recall a few notions.

Let $X$ be a non-singular manifold such that $-K_X$ is nef, and let

$$\alpha_X : X \to \text{Alb}(X)$$

be its Albanese morphism. We assume that $\alpha_X$ is not surjective, and let $Y \subset \text{Alb}(X)$ be the image of $\alpha_X$.

We denote by $\pi_Y : \hat{Y} \to Y$ the desingularization of $Y$, and let $p : \hat{X} \to \hat{Y}$ be the map obtained by resolving the indeterminacy of the rational map $X \dasharrow \hat{Y}$. 
We apply Corollary 2.2 with the following data
\[ X := \hat{X}, \quad Y := \hat{Y} \]
and \( L := \pi_X^*(-K_X) \); here we denote by \( \pi_X : \hat{X} \to X \) the modification of \( X \), so that we have
\[ \pi_Y \circ p = \alpha_X \circ \pi_X. \]
The hypothesis required by Corollary 2.2 are quickly seen to be verified: indeed, the nefness of the bundle \( L \) is due to the fact that \( -K_X \) is nef, and if we denote by \( E \) the effective divisor such that
\[ (2) \quad K_{\hat{X}} = \pi^*(K_X) + E \]
then we see that \( K_{\hat{X}} + L \) is simply equal to \( E|_{\hat{X}} \). This bundle is clearly effective.

Hence we infer that
\[ (3) \quad K_{\hat{X}/\hat{Y}} + \pi_X^*(-K_X) \]
is pseudo-effective. But this bundle equals \( E - p^*(K_\hat{Y}) \); let \( \Lambda \) be a closed positive current in the class corresponding to \( E - p^*(K_\hat{Y}) \). Since the Kodaira dimension of \( K_\hat{Y} \) is at least 1 (we refer to [20] for a justification of this property), we obtain two \( \mathbb{Q} \)-effective divisors say \( W_1 \neq W_2 \) linearly equivalent with \( K_\hat{Y} \).

As a conclusion, we obtain two different closed positive currents belonging to the class of the exceptional divisor \( E \), namely \( \Lambda + p^*(W_j) \) for \( j = 1, 2 \). This gives a contradiction, since any closed positive current linearly equivalent to \( E \) must be \( \pi_X \)-contractible, so its support is contained in the support of \( E \). The existence of two closed positive currents having the support contained in \( E \) whose cohomology classes coincide shows that one of the irreducible components of the support of \( E \) must be equal to a linear combination of the other components in \( H^{1,1}(X, \mathbb{R}) \). This is of course absurd. \( \square \)

**Remark 2.3.** The proof above shows that Theorem 1.4 still holds if \( X \) is projective, and if we replace the hypothesis \(-K_X \text{ nef}\) with the hypothesis \(-K_X \text{ pseudo-effective}, and the multiplier ideal sheaf associated to some of its positively curved metrics is equal to the structural sheaf\). The arguments are absolutely similar.

**Remark 2.4.** Let \( X \) be a Fano manifold, and let \( p : X \to Y \) be a submersion onto a non-singular manifold \( Y \). Then it follows that \( Y \) is Fano as well (see [26], [16]). This result can be obtained via the following elegant argument, very recently found and explained to us by S. Boucksom. By the results e.g. in [2], the direct image of the bundle
\[ K_{X/Y} + L \]
is positive provided that $L$ is an ample line bundle. We take $L = -K_X$ and we are done. A similar idea, $\varepsilon$-close to our arguments in this section can be found in the article \cite{12} by J.-P. Demailly, Th. Peternell and M. Schneider (cf. the proof of their Theorem 2.4).

3. Twisted Kähler-Einstein metrics and their variation

In this paragraph we are going to prove Theorem 1.1. We start (cf. section 3.1) with a few notations/remarks concerning the metrics induced on $K_{X/Y}$ by a (1,1)-form on $X$ which is positive definite along the fibers of $p : X \to Y$. The next section 3.2 is the main part of our paper: we show that the fiberwise twisted Kähler-Einstein metric (which exists and it is unique, thanks to the fact that the class $c_1(K_{X_Y}) + \{\beta\}|_{X_0}$ is Kähler, for each $y \in Y_0$) endow the bundle $K_{X/Y}|_{X_0}$ with a metric whose curvature is strictly greater than $-\beta$. As we have already mentioned, at this point we will adapt to our setting the computations in \cite{34}, \cite{36}. In the last subsection of this paragraph, we show that this metric extends across the singularities of the map $p$.

3.1. Metrics for the relative canonical bundle of a fibration.

Let $p : X \to Y$ be a surjective map; here $X$ and $Y$ are assumed to be compact Kähler manifolds. We are using the notations/conventions in the introduction, so that the restriction $p : X_0 \to Y \setminus W$ becomes a surjective, smooth, proper map between two complex manifolds.

We consider a smooth (1,1)-form $\rho$ on $X$, whose restriction to the fibers of $p$ is positive definite. Then $\rho$ induces a metric on the bundle $K_{X/Y}$ as follows.

Let $x \in X$ be a point, and let $U$ be a coordinate set of $X$ centered at $x$. We denote by $z^1, \ldots, z^{n+d}$ a coordinate system on $U$, and we equally introduce $t^1, \ldots, t^d$ coordinates near the point $y = p(x)$. This data induces a trivialization of the relative canonical bundle, with respect to which the weight of the metric we want to introduce is given by the function $\Psi_U$, defined by the equality

\begin{equation}
\rho^n \wedge \prod_{j=1}^{d} \sqrt{-1} p^* (dt^j \wedge dt^j) = e^{\Psi_U} \prod_{i=1}^{n+d} \sqrt{-1} dz^i \wedge dz^i.
\end{equation}

Here the dimension of $X$ is assumed to be $n + d$, and the dimension of $Y$ equals $d$. The functions $(\Psi_U)$ glue together as weights of a globally defined metric denoted by $h^{\rho}_{X/Y}$ on the relative canonical bundle; the corresponding curvature form is simply $\sqrt{-1} \partial \bar{\partial} \Psi_U$.

We remark that we can very well define the function $\Psi_U$ even if the point $x$ projects into a singular value of $p$; however, the resulting metric
$h^\rho_{X/Y}$ will be identically $\infty$ along the zero set of the Jacobian of the map $p$ (in other words, the weight $\Psi_U$ acquires a log pole). Thus, the metric $h^\rho_{X/Y}$ will be singular in general, even if to start with we are using a non-singular metric $\rho$.

A simple example is provided by the map $p : \hat{X} \to X$, the blow-up of a manifold $X$ along a subset $W$. As one can see right away from the formula (4), the metric $h^\rho_{X/Y}$ is equal to the singular metric associated to the exceptional divisor (so it is independent of $\rho$).

In the next section, we will evaluate the positivity of the curvature of the metric $h^\rho_{X/Y}$; for this purpose, we have to find a lower bound for the quantity

$$\sum_{\alpha,\beta} \frac{\partial^2 \Psi_U}{\partial z^\alpha \partial z^\beta}(x)v^\alpha v^\beta,$$

where $v := \sum_{\alpha} v^\alpha \frac{\partial}{\partial z^\alpha}$

is a tangent vector at $X$ in $x$. In order to obtain a lower bound for the quantity in (2), it is enough to consider a well-chosen restriction of our initial map $p$, namely

$$\tilde{p} : X_D \to \mathbb{D}$$

where $\mathbb{D} \subset Y$ is a disk containing the point $p(x)$, such that the vector $v$ belongs to the tangent space at $x$ to the complex manifold $X_D := p^{-1}(\mathbb{D})$.

Such a choice is clearly possible, and we formulate next our conclusion as follows.

**Remark 3.1.** Let $\gamma$ be a real $(1,1)$-form on $X$. We assume that for each generic enough disk $\mathbb{D} \subset Y$ so that the analytic subset $X_D := p^{-1}(\mathbb{D})$ of $X$ is non-singular we have

$$\Theta_{h^\rho_{X_D/\mathbb{D}}}(K_{X_D/\mathbb{D}}) \geq \gamma|_{X_D}.$$

Then we have

$$\Theta_{h^\rho_{X/Y}}(K_{X/Y}) \geq \gamma$$

on $X$.

This is absolutely clear, given the formula (4). Therefore, we will restrict our attention to families over 1-dimensional bases, as long as we are only interested in the curvature properties of the metric $h^\rho_{X/Y}$.

After these general considerations, we show here that in the context of Theorem 1.1 a very special $(1,1)$-form $\rho$ can be obtained as follows.

Let $\omega$ be a Kähler form on $X$; as explained at the beginning of the current section, we can construct a metric $h^\omega_{X/Y}$ on the bundle $K_{X/Y}$ induced by it. We recall that we denote by $\beta$ a semi-positive definite form on $X$ given by hypothesis of Theorem 1.1.
Moreover, we know that for each $y \in Y \setminus W$, the cohomology class
\begin{equation}
\{\Theta_{h_{X/Y}}(K_{X/Y}) + \beta\}|_{x_y}
\end{equation}
is Kähler, again by hypothesis of 1.1. Therefore, by the main result of S.-T. Yau in [43] we have.

**Theorem 3.2** ([43]). There exists a unique function $\varphi_y \in C^\infty(X_y)$ such that
\begin{equation}
\Theta_{h_{X/Y}}(K_{X/Y}) + \beta|x_y + \sqrt{-1}\partial\bar{\partial}\varphi_y > 0,
\end{equation}
and such that $\varphi_y$ is the solution of the next Monge-Ampère equation
\begin{equation}
(\Theta_{h_{X/Y}}(K_{X/Y}) + \beta|x_y + \sqrt{-1}\partial\bar{\partial}\varphi_y)^n = e^{\varphi_y}\omega^n.
\end{equation}

We stress on the fact that the differential form
\begin{equation}
\Theta_{h_{X/Y}}(K_{X/Y}) + \beta|x_y
\end{equation}
is not necessarily positive definite, but still the equation (8) admits a solution, since the cohomology class corresponding to it (6) is Kähler. Hence the function $\varphi_y$ can be seen to be equal to the sum of two functions: a potential whose Hessian added to (9) makes it positive definite, and the solution of the Monge-Ampère given by the main theorem in [43]. The potential we have to add is by no means unique, but it is the case for the resulting function $\varphi_y$ (as one can see thanks to the usual arguments in [43]).

Also, an important fact is that the function
\begin{equation}
\varphi(x) := \varphi_y(x) \in C^\infty(X_0),
\end{equation}
where $y = p(x)$ is smooth. That is to say, the function obtained by piecing together the solutions $\varphi_y$ of the (8) is smooth on $X_0$; this is a standard consequence of the usual estimates for the Monge-Ampère operator.

We consider next the (1,1)-form
\begin{equation}
\rho := \Theta_{h_{X/Y}}(K_{X/Y}) + \beta + \sqrt{-1}\partial\bar{\partial}\varphi
\end{equation}
on the manifold $X_0$. A first remark is that $\rho$ is definite positive when restricted to $X_y$, for any $y \in Y_0$; this is contained in Yau’s result [43]. Thus, we can define a metric $h_{X/Y}^\rho$ on the bundle $K_{X/Y}|_{X_0}$; given the equation (5), its curvature is rapidly computed as follows.

\begin{equation}
\Theta_{h_{X/Y}^\rho}(K_{X/Y}) = \rho - \beta
\end{equation}
(we are using the relations (5) and (4) in order to derive this).

There are two main steps in the proof of Theorem 1.1, namely.

(i) Show that the form $\rho$ is definite positive on $X_0$; this will imply the positivity of the form $\Theta_{h_{X/Y}^\rho}(K_{X/Y}) + \beta$ on $X_0$. 

(ii) Show that the metric \( h_{X/Y}^\rho \) extends across the set \( X \setminus X_0 \). As soon as this second step is performed, we can infer the positivity of \( \Theta_{h_{X/Y}^\rho} (K_{X/Y}) + \beta \) as current on \( X \).

These points will be addressed in the next two paragraphs.

3.2. The computation. As we have already mentioned, in order to analyze the positivity properties of the form \( \rho \) in (11) it is enough to restrict ourselves to a family over a 1-dimensional base. Therefore we will assume that the map

\[ p : X \to Y \]

is a proper fibration over a 1-dimensional manifold \( Y \); we equally assume that \( X \) is non-singular, and so it is the generic fiber of \( p \).

Let \( x \) be a point of \( X \) such that the fiber \( X_y := p^{-1}(y) \) is non-singular; here we denote by \( y := p(x) \). Let \( t \) be the coordinate on \( Y \) centered at \( y \), and let \((z^1, ..., z^n)\) be local coordinates on the manifold \( X_y \) so that the functions \((t, z^1, ..., z^n)\) are local coordinates for \( X \) at \( x \) (we use here the notation \( t \) for the inverse image of the coordinate on \( Y \) via the map \( p \), to avoid some notational complications).

With the notations in section 3.1, we write the form \( \rho \) in coordinates as follows

\[
\rho = \sqrt{-1} g_{\bar{\alpha}} dt \wedge d\bar{\bar{\alpha}} + \sqrt{-1} \sum_{\alpha} g_{\bar{\alpha} \bar{\beta}} d\bar{z}^\alpha \wedge d\bar{\bar{\beta}} + \sqrt{-1} \sum_{\alpha} g_{\bar{\alpha} \bar{\gamma}} dt \wedge dz^\gamma + \sqrt{-1} \sum_{\alpha, \gamma} g_{\bar{\alpha} \bar{\beta}} dz^\alpha \wedge dz^\gamma.
\]

We already know that \( \rho \) is positive definite when restricted to \( X_y \), hence it has at least \( n \) positive eigenvalues. In local writing as above, this implies that the matrix \((g_{\bar{\alpha} \bar{\beta}})\) is invertible; we denote the coefficients of its inverse by \((g^{\bar{\gamma} \bar{\alpha}})\) (with the convention that the 1st index is the line index of the associated matrix). In order to show that the \( n + 1 \)th eigenvalue (in the “base direction”) is equally positive, we consider the form \( \rho^{n+1} \) on \( X \). As it is well-known, we have

\[
\rho^{n+1} = c(\rho) \rho^n \wedge \sqrt{-1} dt \wedge d\bar{t}
\]

where the function \( c(\rho) \) defined globally on \( X_y \) by the preceding equality can be expressed locally near \( x \) in coordinates as follows

\[
c(\rho) = g_{\bar{\alpha} \bar{\bar{\alpha}}} - \sum_{\alpha, \gamma} g_{\bar{\alpha} \bar{\beta}} g_{\bar{\gamma} \bar{\beta}}.
\]

Our next goal will be to show that we have \( c(\rho)|_{X_y} > 0 \), as this is equivalent to the fact that \( \rho \) is positive definite. The method (cf. [34]) is to show that this function is the solution of a certain elliptic equation on \( X_y \). The computations to follow are straightforward.
Let
\begin{equation}
\Box_y := - \sum_{i,j} g_i^j \frac{\partial^2}{\partial z^i \partial z^j}
\end{equation}
be the Laplace operator (with positive eigenvalues) associated to the metric $\rho|_{X_y}$. We will evaluate the expression
\begin{equation}
\Box_y c(\rho)
\end{equation}
by using the local coordinates fixed above; we can (and will) assume that $(z^1, ..., z^n)$ are geodesic for the metric $\rho|_{X_y}$ at the point $x_0$.

We first evaluate the expression
\begin{equation}
- \sum_{i,j} g_i^j \frac{\partial^2 g_t}{\partial z^i \partial z^j},
\end{equation}
a first observation, which will be used many times in what follows is that
\begin{equation}
\frac{\partial^2 g_{\bar{\imath}}}{\partial z^i \partial \bar{z}^j} = \frac{\partial^2 g_{\bar{\imath}}}{\partial \bar{t} \partial \bar{t}}
\end{equation}
since $\rho$ is locally $\partial \bar{\partial}$ exact, given the expression (11). For any indexes $(i, j)$ we have
\begin{equation}
g_i^j \frac{\partial^2 g_t}{\partial t \partial \bar{t}} = \frac{\partial}{\partial t} \left( g_i^k \frac{\partial g_{\bar{\imath}}}{\partial \bar{t}} \right) - \frac{\partial g_i^k}{\partial t} \frac{\partial g_{\bar{\imath}}}{\partial \bar{t}}
\end{equation}
The term $\frac{\partial g_{\bar{\imath}}}{\partial \bar{t}}$ can be written in terms of the $t$-derivative of $g_{\bar{\imath}}$ since we have
\begin{equation}
\sum_s \frac{\partial g_{\bar{\imath}}}{\partial t} g_{k\bar{s}} = - \sum_s g_{\bar{\imath}k} \frac{\partial g_{\bar{\imath}}}{\partial t}
\end{equation}
which implies that
\begin{equation}
\sum_{s,k} \frac{\partial g_{\bar{\imath}}}{\partial t} g_{k\bar{s}} g_\bar{k} = - \sum_{s,k} g_{\bar{\imath}k} g_\bar{k} \frac{\partial g_{\bar{\imath}}}{\partial t}
\end{equation}
and thus we get
\begin{equation}
\frac{\partial g_{\bar{\imath}}}{\partial \bar{t}} = - \sum_{s,k} g_{\bar{\imath}k} g_\bar{k} \frac{\partial g_{\bar{\imath}}}{\partial t}.
\end{equation}
We notice that we have
\begin{equation}
\sum_{i,j} g_i^j \frac{\partial g_{\bar{\imath}}}{\partial t} = \frac{\partial}{\partial t} \log(g)
\end{equation}
where $g := \det(g_{\alpha \bar{\beta}})$; in conclusion, we obtain the following identity
\begin{equation}
- \sum_{i,j} g_i^j \frac{\partial^2 g_t}{\partial z^i \partial z^j} = - \frac{\partial^2 \log(g)}{\partial t \partial \bar{t}} - \sum_{s,k,i,j} g_{\bar{s}k} g_\bar{k} \frac{\partial g_{\bar{\imath}}}{\partial t} \frac{\partial g_{\bar{\imath}}}{\partial \bar{t}}.
\end{equation}
In local coordinates, the equation (8) implies that we have
\begin{equation}
\frac{\partial^2 \log(g)}{\partial t \partial \bar{t}} = g_{\pi} - \beta_{\pi}
\end{equation}
so we get
\begin{equation}
- \sum_{i,j} g^{\gamma}_{\bar{\gamma}} \frac{\partial^2 g_{\pi \bar{\pi}}}{\partial z^i \partial z^j} = -g_{\pi \bar{\pi}} + \beta_{\pi \bar{\pi}} - \sum_{s,k,i,j} g^{\gamma}_{\bar{\gamma}} g^{\mu}_{\bar{\mu}} \frac{\partial g_{\pi \bar{\pi}}}{\partial t} \frac{\partial g_{\gamma \bar{\gamma}}}{\partial \bar{t}}
\end{equation}

We will detail next the computation for the factor
\begin{equation}
\sum_{i,j,\alpha,\gamma} g^{\gamma}_{\bar{\gamma}} \frac{\partial^2}{\partial z^i \partial z^j} (g^{\alpha \gamma} g_{\pi \bar{\pi}} g_{\alpha \bar{\gamma}});
\end{equation}
given that the coordinates \((z^\alpha)\) are geodesic, the only terms we have
to evaluate are the following
\begin{equation}
I_1 := \sum_{i,j,\alpha,\gamma} g^{\gamma}_{\bar{\gamma}} g_{\pi \bar{\pi}} g^{\alpha \gamma} \frac{\partial^2 g^{\alpha \gamma}}{\partial z^i \partial z^j},
\end{equation}
as well as
\begin{equation}
I_2 := \sum_{i,j,\alpha,\gamma} g^{\gamma}_{\bar{\gamma}} g_{\pi \bar{\pi}} g_{\alpha \bar{\gamma}} \frac{\partial^2 g^{\alpha \gamma}}{\partial z^i \partial z^j}, \quad I_3 := \sum_{i,j,\alpha,\gamma} g^{\gamma}_{\bar{\gamma}} \frac{\partial g^{\alpha \gamma}}{\partial z^i} \frac{\partial g_{\pi \bar{\pi}}}{\partial z^j},
\end{equation}
\begin{equation}
I_4 := \sum_{i,j,\alpha,\gamma} g^{\gamma}_{\bar{\gamma}} \frac{\partial g^{\alpha \gamma}}{\partial z^i} \frac{\partial g_{\bar{\gamma} \pi}}{\partial z^j}.
\end{equation}
In order to simplify the term \(I_1\), we observe that at \(x\) we have
\begin{equation}
\frac{\partial^2 g^{\alpha \gamma}}{\partial z^i \partial z^j} = R^{\alpha \gamma}_{ij}
\end{equation}
hence we get
\begin{equation}
I_1 = \sum_{i,j,\alpha,\gamma} g^{\gamma}_{\bar{\gamma}} g_{\pi \bar{\pi}} g^{\alpha \gamma} R^{\alpha \gamma}_{ij}.
\end{equation}
We observe that \(\sum_{i,j} g^{\alpha \gamma} R^{\alpha \gamma}_{ij} = \sum_{p,q} \text{Ricci}_{pq} g^{\alpha \gamma} g^{\pi \bar{\pi}}\) where we denote by
(Ricci_{pq}) the coefficients of the Ricci curvature of the metric \(\rho\). By
using the equation (8) we infer that
\begin{equation}
I_1 = - \sum_{p,q,\alpha,\gamma} g_{\alpha \gamma} g_{\pi \bar{\pi}} g^{\alpha \gamma} (g_{\pi \bar{\pi}} - \beta_{\pi \bar{\pi}}) =
- \sum_{p,q,\gamma} g_{\gamma} g_{\pi \bar{\pi}} g^{\alpha \gamma} + \sum_{p,q,\alpha,\gamma} g_{\alpha \gamma} g^{\alpha \gamma} g_{\pi \bar{\pi}} \beta_{\pi \bar{\pi}}
\end{equation}
The other terms are evaluated in a similar way; we have
\[ I_2 = \sum_{i,j,\alpha,\gamma} g_{ji} g_{\bar{\alpha} \bar{\gamma}} \frac{\partial^2 g_{\bar{\gamma}}}{\partial z^i \partial z^j} = \sum_{i,j,\alpha,\gamma} g_{ji} g_{\bar{\alpha} \bar{\gamma}} \frac{\partial^2 g_{\bar{\gamma}}}{\partial z^i \partial \bar{t}} \]
\[ = \sum_{i,j,\alpha,\gamma} g_{\bar{\alpha} \bar{\gamma}} \frac{\partial^2 \log(g)}{\partial \bar{z} \partial \bar{t}} = \sum_{i,j,\alpha,\gamma} g_{\bar{\alpha} \bar{\gamma}} (g_{\bar{\gamma}} - \beta_{\bar{\gamma}}) = \sum_{i,j,\alpha,\gamma} g_{\bar{\alpha} \bar{\gamma}} g_{\bar{\gamma}} - \sum_{i,j,\alpha,\gamma} g_{\bar{\alpha} \bar{\gamma}} \beta_{\bar{\gamma}}. \]

as well as
\[ I_3 = \sum_{i,j,\alpha,\gamma} g_{ji} g_{\bar{\alpha} \bar{\gamma}} \frac{\partial g_{\bar{\gamma}}}{\partial z^j} \frac{\partial g_{\bar{\alpha}}}{\partial z^i} = \sum_{i,j,\alpha,\gamma} g_{ji} g_{\bar{\alpha} \bar{\gamma}} \frac{\partial g_{\bar{\gamma}}}{\partial \bar{t}} \frac{\partial g_{\bar{\alpha}}}{\partial \bar{t}}. \]

We want to have an intrinsic interpretation of the factor \( I_4 \), so we consider next the vector field
\[ (33) \quad v := \frac{\partial}{\partial t} - \sum_{i,j} g_{ji} \frac{\partial}{\partial z^j}; \]

as it is well-known \cite{36, 34}, \( v \) is the horizontal lift of \( \frac{\partial}{\partial t} \) with respect to the metric \( \rho \). Then we see that we have
\[ (34) \quad I_4 = |\partial v|^2, \]

and by combining all the equalities above, we infer that we have the compact formula
\[ (35) \quad \Box_X c(\rho) = -c(\rho) + |\partial v|^2 + \beta(v, v) \]

The Ricci curvature of the metric \( \rho|_{X_0} \) is bounded from below by -1, by the equation \( (8) \); hence precisely as in \cite{34}, we infer that we have
\[ (36) \quad \inf_{X_0} c(\rho) \geq C \int_{X_0} (|\partial v|^2 + |v|^2) dV_{\rho} \]

where \( C \) only depends on the diameter of the fiber \((X_0, \rho)\). Hence, the form \( \rho \) is positively defined in the base directions as well.

In conclusion, the fiberwise twisted Kähler-Einstein metric \( \rho \) defines a Kähler metric on \( X_0 \), the pre-image of the set \( Y \setminus W \). Given the equation \( (12) \), this means that the curvature of the metric \( h_{X/Y|X_0}^R \) is bounded by \(-\beta\). In the next section, we will show that this metric extends across the singularities of \( p \), and therefore the proof of theorem 1.1 will be complete. \( \square \)
3.3. Extension across the singularities. We will use the same notations as in the previous section. Given the point \( x_0 \in X \), we will derive an upper bound of the metric induced by \( \rho \) on the relative canonical bundle.

Let \( \Omega \) be an open coordinate set in \( X \) centered at \( x_0 \). We consider \( \Omega_y := \Omega \cap X_y \); we denote by \( \psi_\beta \) a local potential of the Kähler metric \( \beta \) on \( \Omega \). We recall that we denote by \( \Psi_\Omega \) the local weight of the metric on \( K_{X/Y} \) induced by the metric \( \omega \) (so implicitly we assume that we have fixed a coordinate system \((z^{\alpha})_{\alpha=1,\ldots,n+d} \) on \( \Omega \) and \((t^{\gamma})_{\gamma=1,\ldots,d} \) near \( y := p(x_0) \)). The function to be bounded from above is

\[
\tau_y := \Psi_\Omega + \psi_\beta + \varphi|_{\Omega_y}
\]

At first glance this may look very simple, since we have \( \varphi|_{\Omega_y} = \varphi_y \), the solution of \((8)\), and thus the function \((37)\) is psh on \( \Omega_y \). But the bound one can obtain from the meanvalue inequality are not good enough for our purposes, since they depend on the geometry of the manifold \( X_y \): this is precisely what we want to avoid, as \( y \) approaches the singular values of \( p \).

The idea is first to approximate the function \((37)\) with log of absolute values of holomorphic functions; then we show that the holomorphic functions involved in this process admit an extension to \( \Omega \), where the use of Cauchy inequalities is “legitimate”, since the manifold \( X \) is non-singular and compact.

We recall next the following approximation result, cf. \cite{13}.

**Theorem 3.3.** \cite{13} Let \( H^{(m)}_y \) be the Hilbert space defined as follows

\[
H^{(m)}_y := \{ f \in \mathcal{O}(\Omega_y) \text{ such that } \|f\|^2_y = \int_{\Omega_y} |f|^2 e^{-m\tau_y - \|z\|^2 (dd^c \tau_y)^n} < \infty \}.
\]

Then we have

\[
\tau_y(x) = \lim_{m \to \infty} \sup_{f \in H^{(m)}_y, \|f\|^2_y \leq 1} \frac{1}{m} \log |f(x)|^2
\]

for every \( x \in \Omega_y \).

In the statement above, the fact that the manifold \( \Omega_y \) is Stein is of course crucial, since without this hypothesis we cannot approximate \( \tau_y \) by using global holomorphic functions. The importance of the volume element \((dd^c \tau_y)^n\) will become clear in a moment.

Let \( f \in H^{(m)}_y \) be a holomorphic function, such that \( \|f\|^2_y \leq 1 \). By Hölder inequality we have

\[
\int_{\Omega_y} |f|^{2/m} e^{-\tau_y (dd^c \tau_y)^n} \leq \left( \int_{\Omega_y} (dd^c \tau_y)^n \right)^{\frac{m-1}{m}} \leq C
\]
where \( C \) can be taken to be the maximum between 1 and the volume of the fiber \( X_y \) with respect to the Kähler class \( c_1(K_{X_y}) + \{\beta\} \); hence, it is a constant independent of \( m \) and \( y \).

We use now again the equation (8): in local coordinates, it can be written as

\[
(dd^c \tau_y)^n = e^{\tau_y - \varphi_\beta} \left| \frac{dz}{dt} \right|^2
\]

where the notations are (hopefully...) self-explanatory. Then the estimate above implies

\[
\int_{\Omega_y} |f|^{2/m} e^{-\varphi_\beta} \left| \frac{dz}{dt} \right|^2 \leq C
\]

We now invoke the \( L^{2/m} \) version of the Ohsawa-Takegoshi theorem obtained in [4]; it implies the existence of a holomorphic function \( F \), such that:

(a) The restriction of \( F \) to \( \Omega_y \) is equal to \( f \).

(b) There exists a numerical constant \( C_0 > 0 \) independent of \( m \) such that

\[
\int_{\Omega} |F|^{2/m} e^{-\varphi_\beta} \left| \frac{dz}{dt} \right|^2 \leq C_0 \int_{\Omega_y} |f|^{2/m} e^{-\varphi_\beta} \left| \frac{dz}{dt} \right|^2
\]

In particular, this implies that the value \( |F(x_0)|^{2/m} \) is bounded from above by a constant which is independent of \( y \) and on \( m \). Hence the weight function \( \tau_y \) have the same property (by the restriction statement (a) above), and this implies that the metric \( h^{\rho_{X/Y}}|_{X_0} \) extends as a singular metric for the relative canonical bundle of the fibration \( p \); moreover, its curvature current is greater than \(-\beta\). Theorem 1.1 is therefore completely proved. \( \square \)

**Remark 3.4.** As we have already mentioned, in the “linear” context, the positivity properties of the relative adjoint bundles of type \( K_{X/Y} + L \) is established in a very explicit way, by showing that the fiberwise Bergman kernel has a psh variation. The only assumptions which are needed to obtain a non-trivial positively curved metric is the positivity of \( L \), together with the existence of an \( L^2 \) section of \( K_{X_y} + L|_{X_y} \).

In order to compare this theory with our previous considerations, let \( p : X \rightarrow Y \) be a map such that \( K_{X_y} \) is ample, for some generic \( y \in Y \). In the article [37], H. Tsuji shows that his method of iterating the Bergman kernels can be used to construct inductively a metric on the bundle \( mK_{X/Y} + A \), for any \( m \geq 1 \) (here we denote by \( A \) some positive enough line bundle). Moreover, he shows that the metric obtained on \( K_{X/Y} \) by a limiting process is precisely the fiberwise Kähler-Einstein metric considered in G. Schumacher paper [34]. As we have seen, the method in [34] has the advantage that it offers a lower bound for the
curvature of the metric constructed on the relative adjoint class, but on the other hand, it should be further extended e.g. to encompass the case where the adjoint class has base points when restricted to fibers.

4. Further corollaries, consequences and comments

4.1. Proof of Corollary 1.2. Let \( p : \mathcal{X} \to D \) be a Kähler family. We denote by \( \beta \) a Kähler metric on \( \mathcal{X} \). For each \( \varepsilon > 0 \) and for each \( t \in D \), the class

\[
c_1(K_{\mathcal{X}_t}) + \varepsilon \{ \beta \}
\]

is Kähler, since by hypothesis the canonical bundle \( K_{\mathcal{X}_t} \) is nef. The family \( p \) is assumed to be non-singular, hence by the results we have obtained in the section 3.2, the class

\[
c_1(K_{\mathcal{X}/D}) + \varepsilon \{ \beta \}
\]

is Kähler. This means that \( K_{\mathcal{X}/D} \) is nef, so Corollary 1.2 is proved.

4.2. Proof of Theorem 1.4. Let \( X \) be a compact Kähler manifold, such that \( -K_X \) is nef in the sense of the definition 1.3 in the introduction. We denote by \( \alpha_X : X \to \text{Alb}(X) \) the Albanese morphism of \( X \).

As we have seen in the section 1, a successful approach towards the subjectivity of the map \( \alpha_X \) in the projective case is using in an essential manner the positivity properties of the relative canonical bundle associated to the desingularization of \( \alpha_X \). In the general case we will follow basically the same line of arguments, except that the Kähler version of the results 2.1, 2.1 is less general.

Indeed, as a consequence of Theorem 1.1 we infer the next statement.

**Corollary 4.1.** Let \( p : X \to Y \) be a surjective map between two compact Kähler manifolds, which are assumed to be non-singular. Let \( L \to X \) be a nef line bundle, such that the adjoint system \( K_{X_y} + L|_{X_y} \) is equally nef for any \( y \in Y \) generic. Then the bundle \( K_{X/Y} + L \) is pseudo-effective.

We recall that in the projective case, instead of the nefness of the adjoint bundle \( K_{X_y} + L|_{X_y} \) we have assumed that this bundle has a non-trivial section. Also, even if in the statement of the preceding corollary there is no transcendental class involved, its proof is using the full force of Theorem 1.1: the class \( \{ \beta \} := c_1(L) + \varepsilon \omega \) is Kähler, for every positive real \( \varepsilon \). This being said, the Corollary 4.1 is a direct consequence of Theorem 1.1.

In order to be able to use Corollary 4.1, we first consider a desingularization \( \hat{Y} \) of the \( \alpha_X(X) \). We denote by \( \overline{X} \) the fibered product of \( X \)
with $\hat{Y}$ over the base $Y$. This variety may be singular, but its singular loci projects into an analytic set of $X$ whose co-dimension is at least 2. This is seen e.g. by considering the rational map $X \dashrightarrow \hat{Y}$ obtained by composing the inverse of $\pi_Y$ with the Albanese map $\alpha_X$: this map is defined outside a set of co-dimension at least 2, and the set $\overline{X}$ is non-singular at each point of the pre-image of this set. Next we invoke the desingularisation result of H. Hironaka and infer the existence of a map $\hat{X} \rightarrow \overline{X}$ which is an isomorphism outside the singular set of $\overline{X}$. We note that this procedure does not use the fact that $X$ is projective. Let $\pi_X : \hat{X} \rightarrow X$ be the map obtained by composing the desingularization of $\overline{X}$ with the natural map $\overline{X} \rightarrow X$; the manifolds/maps constructed above have the next important properties

(i) The generic fiber of the map $p : \hat{X} \rightarrow \hat{Y}$ is disjoint from the support of the exceptional divisor associated to the map $\pi_X$ defined by the relation

$$K_{\hat{X}} = \pi_X^*(K_X) + E.$$

(ii) The divisor $E$ is $\pi_X$-contractible.

The rest of the proof of Theorem 1.4 is identical to the arguments invoked in the projective case; the hypothesis of Corollary 4.1 are verified, by the properties (i) and (ii) above.

**Remark 4.2.** In a forthcoming paper, we will investigate the singular version of Theorem 1.1; the precise statement can be easily guessed from the projective case ([25], [40]), as follows. Let $\{\beta\}$ be a $(1,1)$-class on $X$, admitting a representative $\Theta = \beta + \sqrt{-1} \partial \overline{\partial} f$ which is a closed positive current, such that

$$\int_X e^{-f} dV < \infty$$

(i.e. $(X, \Theta)$ is the analogue of a klt pair in algebraic geometry). If the class

$$c_1(K_{X_y}) + \{\beta\}|_{X_y}$$

contains a Kähler current for each $y \in Y$ generic, then the class $c_1(K_{X/Y}) + \{\beta\}$ should contain a Kähler current. In order to adapt the argument used in this note for the proof of such a result, it seems to us that there are serious technical difficulties to overcome. Nevertheless, if $\Theta$ is allowed to be singular only along a SNC divisor, and if the class ([44]) is Kähler, then a slight generalization of the results in [8], [19] are enough to conclude. However, it is highly desirable to prove the result in singular context, as the article by J. Kollár [25] shows it: one should allow base points for the positive representatives of the
class (11). We refer to the work of O. Fujino [15] for new results and an overview of related topics from the algebraic geometry point of view.

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