Deformations of Galois representations arising from degenerate extensions

Adam Logan
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1. Introduction

This paper is inspired by that of Boston and Mazur [B-M], and work on this problem was begun when
the author was a graduate student of Barry Mazur (supported by an NSF Graduate Fellowship). In the paper
[B-M], the authors study the deformation theory of a certain type of $S_3$-extensions of $\mathbb{Q}$, which they term
neat, and more specifically that of generic $S_3$-extensions, which satisfy an additional condition. Restricting
their numerical study to one particular family of neat extensions, they note that all such extensions seem to
satisfy their genericity condition.

Their principal result on generic $S_3$-extensions can be summarized as follows:

**Theorem 1.1** ([B-M], prop. 13) Let $L/\mathbb{Q}$ be a neat $S_3$-extension for the prime $p$ (we will define this in
Section 2, below). The universal deformation ring of the natural representation of its Galois group into
$GL_2(F_p)$ is isomorphic to $\mathbb{Z}_p[[T_1, T_2, T_3]]$. If $L/\mathbb{Q}$ is generic, then:

(a) The inertially reducible locus is composed of the union of two smooth hypersurfaces in the universal
deformation space.

(b) The globally dihedral locus is equal to the inertially dihedral locus and is a smooth hypersurface.

(c) The ordinary locus consists in a smooth analytic curve in the deformation space.

(d) The inertially ample locus is equal to the complement of the union of three hypersurfaces, any two of
which meet transversely.

They also show that generic $S_3$-extensions actually exist:

**Proposition 1.2** ([B-M], prop. 9) Let $a$ be an integer such that $27 + 4a^3$ is positive, prime, and less than
$10^4$. Then the splitting field of the polynomial $x^3 + ax + 1$ is a generic $S_3$-extension for the prime $27 + 4a^3$.
(For the degree 6 extension $x^3 + 100x + 27$ is neat, we may suppose that $e_1$ is not a $p$th power in $L_{p_1}$, and we may
also arrange things so that $e_2$ is not a $p$th power in $L_{p_2}$ or $L_{p_3}$. If $e_2$ is not a $p$th power in $L_{p_1}$ either,
then $L$ is generic.

2. Basics

We start with some fundamental definitions borrowed from [B-M], with very slight modifications.

**Definition.** (Cf. [B-M], Definition 2.) Let $L/\mathbb{Q}$ be a totally complex $S_3$-extension in which $p$ splits as
$(p_1p_2p_3)^2$, and let $S$ be the set of finite ramified primes of $L$. We say that $L$ is admissible for $p$, or $L$ is neat,
if:

1. Any global unit of $L$ which is locally a $p$th power at all elements of $S$ is globally a $p$th power.
2. The class number of $L$ is prime to $p$.
3. The completion of $L$ at any element of $S$ does not contain $p$th roots of $1$. (In particular, it follows that
the cardinality of the residue field is not congruent to $1$ mod $p$.) Let $L$ be a neat $S_3$-extension of $\mathbb{Q}$, and
let $p_1, p_2, p_3$ be the primes of $L$ lying above $p$. Let $e_1, e_2$ be a basis for global units mod $p$th powers.
Since we are assuming that $L$ is neat, we may suppose that $e_1$ is not a $p$th power in $L_{p_1}$, and we may
also arrange things so that $e_2$ is not a $p$th power in $L_{p_2}$ or $L_{p_3}$. If $e_2$ is not a $p$th power in $L_{p_1}$ either,
then $L$ is generic.

**Definition.** The degeneracy index of $L$ at $p$ will be the largest integer $i$ such that $e_2$ is a $p^i$th power in $L_{p_1}$. 
(For the degree 4 extension $x^3 + 25x + 27$ is neat, we may suppose that $e_1$ is not a $p$th power in $L_{p_1}$, and we may
also arrange things so that $e_2$ is not a $p$th power in $L_{p_2}$ or $L_{p_3}$. If $e_2$ is not a $p$th power in $L_{p_1}$ either,
then $L$ is generic.)

The authors of [B-M] pay particular attention to the Galois closures of cubic fields of the form $\mathbb{Q}(x)$,
where $x^3 + ax + 1 = 0$, for $a$ an integer such that $27 + 4a^3$ is positive and prime. They show that the first
seven such fields are generic $S_3$-extensions of $\mathbb{Q}$, using a simple numerical criterion. As noted above, I have extended this verification to all $a < 500000$, and find it hard to believe that there are any counterexamples. However, if one does not restrict to these particular cubic fields, it becomes easy to find degenerate $S_3$-extensions. (The tables of number fields available by anonymous FTP from megrez.math.u-bordeaux.fr greatly facilitate such a search.)

The rest of the paper will be devoted to modifying the proofs and results of Boston and Mazur so that they apply in the degenerate case. That is, we will determine the natural subspaces of the universal extensions. (The tables of number fields available by anonymous FTP from megrez.math.u-bordeaux.fr greatly facilitate such a search.)

3. Definitions and Notations

We now recall some more definitions from [B-M].

**Definition.** Let $L$ be an $S_3$-extension of $\mathbb{Q}$, and let $p$ be a rational prime greater than 3 which decomposes in $L$ as $p_1p_2p_3$. (We assume that such a prime exists.) Let $S$ be the set of ramified primes of $L$. Let $P$ be the Galois group over $L$ of the maximal pro-$p$ extension of $L$ unramified away from $p$, or outside $S$ (in the situations we will be considering, these are the same), $G$ its Galois group over $\mathbb{Q}$, $L_p$ the completion of $L$ at $p_1$, $P_p$ the Galois group over $L_p$ of its maximal pro-$p$ extension, and $G_p$ the Galois group of the maximal pro-$p$ extension of $L_p$ over $\mathbb{Q}_p$. We also fix an embedding of $\mathbb{Q}$ into $\mathbb{Q}_p$, and thus of $\text{Gal} L_p$ into $\text{Gal} L$, such that the inertia subgroup $P_p^0$ maps to the inertia subgroup for $p_1$.

**Proposition 3.1** $P$ is a free pro-$p$ group on 4 generators, and $P_p$ is a free pro-$p$ group on 3 generators.

**Proof.** [B-M, props. 3, 4].

We take $\sigma$ (resp. $\tau$) to be an element of order 2 (resp. 3) in $S_3$. Following one of the notations in [B-M], we will let $P$ be generated by $u, \tau(u), \tau^2(u), v$, where $u$ conjugated by $\tau$ is, obviously, $\tau(u), \tau(v) = v$, $\sigma(u) = u$, and $\sigma(v) = v^{-1}$. On the other hand, $P_p$ will be generated by $\xi, \eta, \phi$, with $\xi$ and $\eta$ generating the inertia and the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acting as $+1$ on $\xi, \phi$ and $-1$ on $\eta$.

**Definition.** Let $E$ be the group of global units of $L$. It is the direct product of a free abelian group of rank 2 with a cyclic group of order 2. For any place $v$ of $L$, let $E_v$ be the group of units in the ring of integers of $L_v$.

**Definition.** For any topological group $T$, let $\hat{T}$ be its $p$-Frattini quotient. More generally, let $\hat{T}$ be the maximal quotient of $T$ which is an abelian pro-$p$ group with exponent dividing $p^3$ (that is, $\hat{T} = T/(T,T)^3$).

**Definition.** Let $K$ be a cubic extension of $\mathbb{Q}$. Let $L$ be its Galois closure, and let $S$ be the set of finite ramified primes of $L$. Global class field theory gives us a map from the idèle class group of $L$ to the abelianization of its absolute Galois group. This induces a map $\oplus_{v \in S} E_v \rightarrow \hat{P}$ which is trivial on the image in $\oplus_{v \in S} E_v$ of the global units. Under the conditions that the class number of $L$ be prime to $p$ and that no completion of $L$ at a prime in $S$ contain the $p$th roots of 1, this map is surjective.

In this situation we say, as above, that $L$ is neat for $p$, or for $S$, if the map $E \rightarrow \oplus_{v \in S} E_v$ is injective. In this case, we consider a map $E \rightarrow \hat{E}_1$. If it too is injective, we are in the generic situation treated by Boston and Mazur. Otherwise, the extension is termed degenerate, as remarked above, and the degeneracy index is the largest $i$ for which the map $iE \rightarrow i\hat{E}_1$ has cyclic image.

At this point we give our promised example of a degenerate $S_3$-extension.

**Example 3.2** Let $K$ be the field $\mathbb{Q}(x)$, where $x^3 + 7x - 12 = 0$, and $L$ its Galois closure. We claim that $L$ is a degenerate $S_3$-extension of $\mathbb{Q}$, of degeneracy index 1, with $p = 5$. Here the set of ramified primes is $\{5, 263\}$, both of which split as $(p_1p_2p_3)^2$, so it is easy to check that the completions there do not contain fifth roots of 1. The class number of $L$ is 2.

Using $\mathfrak{gp}$, it is easy to check that the units of the cubic subfields of $L$ generate the full unit group of $K$. A fundamental unit of $K$ is $14x - 19$. $K$ has a unique embedding into $\mathbb{Q}_5$, in which the image of $x$ is congruent to 62 mod 125, so that the image of the fundamental unit is congruent to $-1$ mod 25, but not mod 125, and is therefore a fifth power but not a 25th power. I assert that the image of $x$ under the embedding of $K$ into $\mathbb{Q}_5(\sqrt[5]{10})$ is not a fifth power. This will essentially complete the verification that $L$ is neat.

In fact, it is easy to show that a unit of $\mathbb{Q}_5(\sqrt[5]{10})$ which is congruent to 1 modulo $m$, the maximal ideal, is a fifth power iff it is congruent to 1 modulo $m^5$. However, a root of $x^3 + 7x - 12$ which does not belong to
$\mathbb{Q}_5$ is congruent modulo $m^2$ to $4 \pm \sqrt{10}$, and thus the unit is congruent modulo $m^2$ to $2 \mp \sqrt{10}$ and cannot be a fifth power (multiply by 3$^3$).

So, if we take a unit $u = \pm u_1^2 u_2^5$ of $L$, where $u_1, u_2$ are fundamental units in different cubic subfields of $L$, then in one completion $u$ is a fifth power iff $5|a$, and in another iff $5|b$. Thus $u$ is a fifth power locally iff it is a fifth power globally, which completes the proof that $L$ is neat.

**Example 3.3** Let $p$ be a prime, $n$ a positive integer, and $r$ and $s$ integers with $p^n|(r+s)$. Suppose that the polynomial $x^2 + rx^2 + sx - 1$ is irreducible and does not have all real roots, and let $K$ be its root field and $L$ its splitting field as above. Suppose further that $p$ decomposes in $K$ as $p\mathfrak{p}_2$. (This cannot be arranged for all $p$, since it requires the polynomial $x^3 - 8x^3 + 18x^2 - 27$ to have a root mod $p$, but it can for some, for example with $p = 5, r = -29, s = 4, n = 2$.) Clearly $K$ has a unit congruent to 1 modulo $p^n$ in the embedding of $K$ into $\mathbb{Q}_p$. This must be the $p^{n-1}$-th power of a local unit, and it seems that it is always possible to arrange for $L$ to be neat by varying $r, s$ within their congruence classes mod $p^n$. Constructions like this lead me to believe that for all primes $p > 3$, there are $S_3$-extensions with all degeneracy indices at $p$, but I cannot prove it. Among other problems, it can never be possible to estimate the class number of $K$ as being less than $p$, as did Mazur [M, section 1.13], because for a fixed $p$ the discriminant must grow as $n$ increases.

### 4. Representation Theory

Let $G$ be a finite group and $F$ a field of characteristic prime to $\text{card}G$. Then, of course, the group algebra $F[G]$ is semisimple. It is isomorphic to a direct sum of matrix algebras over $F$ iff all irreducible representations of $G$ (say there are $c$) can be defined over $F$.

Suppose we are in this case, and let $R$ be a local Artinian ring with residue field $F$. Since $R[G] \cong F[G] \otimes_F R$, it is clear that $R[G]$ is a direct sum of $c$ matrix algebras. Now, representations of $G$ with coefficients in $R$ correspond naturally to $R[G]$-modules free over $R$. These, then, correspond to $R^n$-modules free over $R$, that is, to $c$-tuples of free $R$-modules. In turn, these correspond canonically to $c$-tuples of $F$-modules, whence to $F[G]$-modules or to representations. In summary, a representation of $G$ to $M_n(R)$ is uniquely determined up to conjugacy by its reduction to $M_n(F)$. The usual theorems on reducibility of representations then follow for representations to $R$. For example, if we have a representation $\rho$ to $R$ which is an extension of representations, it must in fact be their sum, for $\rho$ and the sum have the same reduction to $M_n(F)$.

We will apply these ideas with $F = F_p$ and $G = S_3$. In essence, they allow us to immediately take over all results about module decompositions given in [B-M] without change here. Since $P$, the Galois group of the maximal pro-$p$ extension of $L$ unramified away from $p$ over $L$, is a free pro-$p$ group, the groups $\bar{P}$ are all free modules over $\mathbb{Z}/p^i\mathbb{Z}$ of the same rank.

**Proposition 4.1** $A$ is a semidirect product of $P$ by $S_3$ such that for any $j$, $A$ acts on $\bar{j}\bar{P}$ by $V + \chi$, where $\chi$ is the nontrivial 1-dimensional representation of $S_3$ with coefficients in $F_p$ and $V$ is the natural 3-dimensional representation of $S_3$. For any $j$, $A_j$ is a semidirect product of $P_j$ by $\mathbb{Z}/2\mathbb{Z}$, with $\mathbb{Z}/2\mathbb{Z}$ acting on $\bar{j}\bar{P}_j$ by $1 + 1 + \chi$. In the global case, the inertia subgroup maps to the space spanned by a basis vector of $V$ and $\chi$ (not a submodule, since different choices of prime above $p$ give different inertia subgroups); in the local case, to $1 + \chi$.

**Proof.** [B-M, props. 7 and 8], together with the above to remove the restriction $j = 1$ made there. $\square$

Boston and Mazur study the exact sequence of $p$-Frattini quotients

$$0 \to \bar{E} \to \bar{E}_1 \oplus \bar{E}_2 \oplus \bar{E}_3 \to \bar{P} \to 0.$$ 

Likewise we will study the exact sequence of $p^i$-quotients. That is, we define a map $\Pi_{j}^i$ to be that given by class field theory from $\bar{j}\bar{E}_k$ to $\bar{j}\bar{P}$. Its image will be denoted $\bar{j}\bar{P}_k$, and by class field theory $\bar{j}\bar{P}_k$ is the image of the inertia subgroup $j\bar{P}_k^0$ in $\bar{j}\bar{P}$.

**Proposition 4.2** Let $L$ be an $S_3$-extension of $Q$, degenerate for $p$ with degeneracy index $i$. Then the intersection of any two, or all three, of the $j\bar{P}_k$ is isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$, where $l = \min(i, j)$. (In the case $j = 1$, this reduces to the results in the first part of [B-M, section 2.3].)

**Proof.** We consider the cases $i < j$, $j \leq i$ separately. In the case $j \leq i$, the image of $P_1$ is isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$, and it is stable under the action of the involution of the Galois group which fixes $p_1$. The rest of the proof in [B-M, section 2.3] can now be taken over word for word. $\square$
We now consider the case $i < j$. Everything is compatible with the inclusion maps $j \bar{E}_k \to j_{i+1}E_k$ and $j \bar{P} \to j_{i+1}P$, so the image must contain a $\mathbb{Z}/p^r\mathbb{Z}$-subgroup, and no more elements of order dividing $p^i$. If there is an element of higher order in the intersection $j \bar{P}_k \cap j \bar{P}_e$, say $y$, coming from $y_k$ and $y_{e_k}$, then the element $y_k \oplus y_{k_e} \oplus 0$ would be in the image of $j \bar{E}$, by exactness. This would immediately imply that the degeneracy index of $L$ is greater than $i$. ⌣

5. Linking Local and Global Presentations

We have already described (in Proposition 4.1) the presentations of the local and global Galois groups $G_p, G$. Now we must show how they behave under the map $G_p \to G$ (in particular, what happens when we restrict this to a map $\Pi_p \to \Pi$). This is where the difference between the generic and degenerate situations becomes important.

**Proposition 5.1** (Cf. [B-M, lemma 2.4.4].) Suppose that $i$, the degeneracy index of $L$, is at least $j$, and let $\xi, \eta$ be generators of the inertia subgroup of $j \bar{P}_p$ such that the nontrivial element of $\text{Gal}(L_p/\mathbb{Q})$ acts as $+1$ on $\xi$ and $-1$ on $\eta$. Let $r, s$ be the images of $\xi, \eta$ in $j \bar{\Pi}$, and let $R, S$ be the $S_3$-stable subspaces that they generate. Then $R \cong 1 + \chi + \epsilon$ and $S \cong \chi$.

**Proof.** Recall that $j \bar{\Pi} \cong 1 + \chi + \epsilon$. Because $L$ has no unramified extensions of degree $p$, the $S_3$-stable subspace of $j \bar{\Pi}$ generated by the image of a local inertia group—that is, $R + S$—must be the whole thing. Also, $R, S$ must be quotients of the inductions of 1 and $1 + \epsilon$ from $A_p = \mathbb{Z}/2\mathbb{Z}$ to $A = S_3$, respectively. On the other hand, if $i < j$, neither $R$ nor $S$ can be one-dimensional over $\mathbb{Z}/p^r\mathbb{Z}$, by Proposition 4.2. Thus, the statement on $R$ follows if we prove the statement about $S$.

Corresponding to $\xi, \eta$, let $a, b$ be generators of $j \bar{E}_1$ such that $a^2 = a, b^2 = 1/b$. Let $v$ be the global unit of $L$ which is a $p^r$th power in $L_{p_1}$. Then the image of $v$ in $j \bar{E}_{p_2}$ must be a multiple of $b$. Indeed, on the one hand, the product of the three global conjugates of $v$ can be taken to be 1, and on the other hand, the two conjugates that are not in $Q_p$ are local conjugates in $L_p$, so when reduced to $j \bar{E}_{L_p}$ they have the same coefficient of $a$, which must therefore be 0. On the other hand, the coefficient of $b$ must be a unit, for otherwise $v$ would be a $p^r$th power everywhere locally, a possibility excluded by our hypotheses.

In particular, the image of $j \bar{E}_L$ in $\oplus_k j \bar{E}_k$ is spanned by $(b, -b, 0)$ and $(0, b, -b)$. It follows that the intersection of the images of the $j \bar{E}_k$ in $j \bar{P}$ is the image of $(b, 0, 0)$, which is obviously the $\chi$ subspace as claimed. ⌣

A curious consequence of this proposition is as follows:

**Corollary 5.2** Let $L/\mathbb{Q}$ be an $S_3$-extension of degeneracy index at least $j$ for $p$, and let $F$ be the subextension of $Q(\zeta_{p^{i+1}})$ which is of degree $p^i$ over $\mathbb{Q}$. Then the class group of the compositum $L \vee F$ contains a subgroup isomorphic to $(\mathbb{Z}/p^r\mathbb{Z})^2$.

**Proof.** It is sufficient, of course, to construct an unramified extension with this Galois group. The point is simply that the inertia groups for $p_i$ in the extension cut out by $j \bar{P}$ are isomorphic to $(\mathbb{Z}/p^r\mathbb{Z})^2$ and have an intersection isomorphic to $(\mathbb{Z}/p^r\mathbb{Z})^2$, which cuts out the extension $M$, say. On the other hand, $L \vee F \subset M$ and is totally ramified over $L$ at each $p_i$. Thus, $M/(L \vee F)$ is unramified at these primes, and (by definition of $P$) at all others as well. ⌣

We must now specify the relation between local and global presentations more precisely.

**Proposition 5.3** (Cf. [B-M, prop. 10].) Let $L$ be an admissible $S_3$-extension of $\mathbb{Q}$ of degeneracy index $i$ for the prime $p$. Then we may take the local and global systems of generators such that the image of $\xi$ is $u$ and, in the induced map on quotients $j \bar{\Pi}_p \to j \bar{\Pi}$, the image of $\eta$ is $v$, if $i \geq j$.

**Proof.** By [B-M, prop. 7 and addendum] we may take the image of $\xi$ to be $u$. The statement about $\eta$ follows from the last lemma, similarly to the proof of [B-M, prop. 10]. ⌣

We have now accumulated all necessary information about the Galois groups and can proceed to study the universal deformation.

6. The Universal Deformation

Let $L$ be an admissible $S_3$-extension; for the moment, the index of degeneracy does not matter. There is a Galois representation $\bar{\rho} : G \to GL_2(\mathbb{F}_p)$, unique up to conjugacy, which factors through $\text{Gal} L/\mathbb{Q}$ and maps it injectively into $GL_2(\mathbb{F}_p)$. For concreteness, we fix elements $\sigma, \tau$ in $S_3$ of order 2 and 3 respectively.
and map them to
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{pmatrix}.
\]

We will be studying deformations of $\bar{\rho}$ to complete local noetherian rings with residue field $F_p$. The universal deformation has been completely described.

**Proposition 6.1** The universal deformation ring is the power series ring $\mathbb{Z}_p[[T_1, T_2, T_3]]$, and the universal deformation may be given as follows:

\[
\sigma \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tau \to \begin{pmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{pmatrix}, u \to \begin{pmatrix} 1 + T_1 & 0 \\ 0 & 1 + T_1 \end{pmatrix}, v \to \begin{pmatrix} (1 - 3T_3^2)^{1/2} & T_3 \\ -3T_3 & (1 - 3T_3^2)^{1/2} \end{pmatrix}.
\]

**Proof.** This is [B-M], prop. 11, and a detailed proof is given there. □

To understand the universal deformation more fully, we must understand the image of $\eta$. Since $\eta$ conjugated by $\sigma$ is $\eta^{-1}$, the image of $\eta$ must have determinant 1 and equal diagonal entries, so it is, say,

\[
\begin{pmatrix} (1 + fg)^{1/2} & f \\ g & (1 + fg)^{1/2} \end{pmatrix}.
\]

**Proposition 6.2** Modulo $m$, the power series $f$ is congruent to $T_3$, and $g$ to $-3T_3$.

**Proof.** As [B-M], prop. 12, except that here the image of $1\bar{\eta}$ under the natural map $\bar{\Pi}_p \to \Pi$ is $\bar{v}$. □

We can now determine some of the natural subspaces. We will be considering representations of the Galois group into $GL_2(\mathbb{Z}_p)$ which are deformations of the representation into $GL_2(F_p)$. Thus they come from the universal deformation, and are described by a continuous homomorphism $\mathbb{Z}_p[[T_1, T_2, T_3]] \to \mathbb{Z}_p$. Such a homomorphism $\alpha$ is described by giving $\alpha(T_1), \alpha(T_2), \alpha(T_3)$; the space of such is therefore naturally identified with $p\mathbb{Z}_p \times p\mathbb{Z}_p \times p\mathbb{Z}_p$, which is a 3-dimensional $p$-adic manifold. The only visible difference between our situation and the generic one is that here $f$ and $g$ are not transversal. Presumably the order of contact of their zero loci is equal to the degeneracy locus of the extension, but I do not see how to prove this.

**Proposition 6.3** (cf. [B-M], prop. 13.) The inertially reducible locus is the union of the hypersurfaces $f = 0$ and $g = 0$. The ordinary locus is the smooth curve defined by $T_1 = g = 0$.

**Proof.** Identical to the proofs given in [B-M]. □

It is still true that a representation is inertially dihedral iff $T_1 = T_2$ or $f = g = 0$. If, as is presumably the case, the loci $f = 0$ and $g = 0$ are distinct, the argument in [B-M] goes through to show that $f = g = 0$ implies $T_1 = T_2$. (This sounds like something that should be easy to prove, but I have not managed to.) It would then follow, just as in [B-M], that the inertially ample locus is the complement of the union of the inertially reducible and inertially dihedral loci.

**References**

[B-M] N. Boston, B. Mazur, *Explicit universal deformations of Galois representations*. In *Algebraic Number Theory*, Adv. Stud. Pure Math. 17, 1–21.

[M] B. Mazur, *Deforming Galois representations*. In *Galois groups over $\mathbb{Q}$*, MSRI Publications 16, 385–437.