Universal Minimality in TV Regularization with Rectilinear Anisotropy

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Abstract

In this article, we study theoretical aspects of the $\ell^1$-anisotropic Rudin-Osher-Fatemi (ROF) model on hyperrectangular domains of arbitrary dimension. Our main result states that the solution of the ROF model minimizes a large class of convex functionals, including all $L^p$-norms, over a certain neighbourhood of the datum. This neighbourhood arises naturally in the dual formulation and is given by the TV subdifferential at zero, scaled by the regularization parameter and translated by the datum. In order to prove the main result, we exploit the recently established fact that, for piecewise constant datum, the ROF model reduces to a finite-dimensional problem. We then show a finite-dimensional version of the main result, before translating it back to the original setting and finishing the proof by means of an approximation argument.

1 Introduction

This article is concerned with the $\ell^1$-anisotropic Rudin-Osher-Fatemi (ROF) model \cite{ROF} on hyperrectangular domains

$$\Omega = \prod_{i=1}^{d} (a_i, b_i), \quad a_i < b_i, \quad (1.1)$$

of arbitrary dimension $d \geq 1$. For $f \in L^2(\Omega)$ and $\alpha > 0$ it consists in finding the unique minimizer $u_\alpha \in L^2(\Omega) \cap BV(\Omega)$ of

$$\frac{1}{2} \|f - u\|_{L^2(\Omega)}^2 + \alpha J(u),$$

where $J : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is the total variation (TV) seminorm with $\ell^1$-anisotropy

$$J(u) = \sup \left\{ \int_{\Omega} u \text{ div } H \, dx : H \in C_0^\infty(\Omega, \mathbb{R}^d), |H_i| \leq 1 \text{ for } 1 \leq i \leq d \right\}. \quad (1.2)$$

This particular anisotropy makes the model a natural choice for image processing applications where the datum $f$ has an underlying rectilinear geometry, see for example \cite{BO, T-S, DF, S}. Recently, see \cite{K-S} and \cite{K-S}, it was indeed proved that $u_\alpha$ inherits the piecewise constant structure of such data.

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The ROF model can equivalently be formulated as the constrained minimization problem

$$\min_{u \in f - \alpha \partial J(0)} \| u \|_{L^2(\Omega)}.$$ 

Our main result, Theorem 4.6, states that the solution $u_\alpha$ of the $\ell_1$-anisotropic ROF model actually minimizes a considerably larger class of functionals over the set $f - \alpha \partial J(0)$, namely

$$\int_\Omega \varphi(u_\alpha) \, dx \leq \int_\Omega \varphi(u) \, dx$$

(1.3)

for all $u \in f - \alpha \partial J(0)$ and every convex function $\varphi : \mathbb{R} \to \mathbb{R}$. An immediate consequence, for example, is the estimate

$$\| u_\alpha \|_{L^p(\Omega)} \leq \| f \|_{L^p(\Omega)},$$

for $1 \leq p \leq \infty$. While Theorem 4.6 does not imply finiteness of these norms for $p > 2$, we do have that $u_\alpha$ is in $L^p(\Omega)$ whenever $f$ is.

Results of the type (1.3) already exist in the literature. The one that resembles Theorem 4.6 most closely is probably [16, Thm. 4.46]. It states that, for $\Omega \subset \mathbb{R}^d$ being a bounded, open and connected domain with Lipschitz boundary, the minimizer of the isotropic ROF model has a property analogous to (1.3). The proof in [16] relies on a characterization of the subdifferential of the isotropic TV by means of the coarea formula. The case $d = 1$ is also considered in [13, Lem. 1] where an alternative proof based on a variational inequality is given. Our proof on the other hand makes use of piecewise constant approximations and is briefly sketched at the end of this introduction.

Another result related to Theorem 4.6 is [9, Thm. 5.3, Rem. 5.4]. Consider the set

$$B_1 = \{ H \in C^\infty_c(\Omega, \mathbb{R}^d) : |H_i| \leq 1 \text{ for } 1 \leq i \leq d \}$$

from (1.2). In [9] the authors show that $\text{cl}_{L^1}(\text{div} \, B_1)$, that is, the $L^1$-closure of $\text{div} \, B_1$, is invariant $\varphi$-minimal. This means that for every $g \in L^1(\Omega)$ there is an element $u_\alpha^* \in \text{cl}_{L^1}(\text{div} \, B_1)$ such that

$$\int_\Omega \varphi(u_\alpha^* - g) \, dx \leq \int_\Omega \varphi(u - g) \, dx$$

(1.4)

holds for every $u \in \text{cl}_{L^1}(\text{div} \, B_1)$ and all convex $\varphi : \mathbb{R} \to \mathbb{R}$. Theorem 4.6 is related to this result in the following way. First, adopting the terminology of [9], the subdifferential $\partial J(0)$ is an invariant $\varphi$-minimal subset of $L^2(\Omega)$. Next, since $J$ is the support function of $\text{div} \, B_1$, we have $\partial J(0) = \text{cl}_{L^2}(\text{div} \, B_1) \subset \text{cl}_{L^1}(\text{div} \, B_1)$. Finally and most notably however, Theorem 4.6 characterizes the element $u_\alpha^*$ of best approximation in terms of the ROF minimizer for datum $f$.

In the discrete setting, one early instance of a result in the spirit of (1.3) is shown in [1, Prop. 1.1, Cor.]. Let $g^*$ denote the isotonic regression of the vector $g$ with respect to a fixed vector $w$ of positive weights. Then $g^*$ minimizes all functionals of the form

$$u \mapsto \sum_i \varphi(u_i) w_i,$$

where $\varphi$ is a convex function, over the class of vectors $u$ such that $g - u$ is in the dual cone $C^*$ to the convex cone $C$ of isotonic vectors. A particular case of this result, considering $\varphi(t) = \sqrt{1 + t^2}$, turns out to be closely related to the characterization
Recently, see [8, Thm. 4.3], a characterization of invariant \( \varphi \)-minimal sets in \( \mathbb{R}^n \), where a discrete analogue of (1.4) holds, has been established.

We conclude the overview of related work with an example which can be seen as a discrete analogue of Theorem 4.6. Let \((V, E)\) be a graph and define the total variation of a vertex function \( u : V \to \mathbb{R} \) as

\[
J(u) = \sum_{(v, \bar{v}) \in E} |u(v) - u(\bar{v})|.
\]

Then, as is shown in [6, Thm. 3.2], the minimizer \( u_\alpha \) of

\[
\frac{1}{2} \sum_{v \in V} |u(v) - f(v)|^2 + \alpha J(u)
\]

satisfies

\[
\sum_{v \in V} \varphi(u_\alpha(v)) \leq \sum_{v \in V} \varphi(u(v))
\]

for every \( u \in f - \alpha \partial J(0) \) and all convex \( \varphi : \mathbb{R} \to \mathbb{R} \).

The organization of this article is as follows. In section 2, as a first step towards Theorem 4.6, we establish a generalization of the above mentioned [6, Thm. 3.2], which we believe is of independent interest. More precisely, let \( u_\alpha \) be the minimizer of the following weighted version of (1.5)

\[
\frac{1}{2} \sum_{v \in V} w(v)|f(v) - u(v)|^2 + \alpha J_W(u),
\]

where \( J_W(u) = \sum_{(v, \bar{v}) \in E} W(v, \bar{v})|u(v) - u(\bar{v})| \) with \( w > 0 \) and \( W > 0 \) being vertex and edge weights, respectively. Then, Theorem 2.4 shows that

\[
\sum_{v \in V} w(v)\varphi(u_\alpha(v)) \leq \sum_{v \in V} w(v)\varphi(u(v))
\]

holds for every \( u \in f - \alpha \partial J_W(0) \) and convex \( \varphi : \mathbb{R} \to \mathbb{R} \). The proof of Theorem 2.4 requires a new result, Proposition 2.1 on invariant \( \varphi \)-minimal subsets of \( \mathbb{R}^n \). The proof of Proposition 2.1 is postponed to the appendix.

Next, section 3 collects background results from [7, 10] regarding the preservation of piecewise constancy of the \( \ell^1 \)-anisotropic ROF model. In section 4, the main result is proved. The connection between the discrete and continuous ROF models is made explicit in Proposition 4.1. This, in turn, enables us to translate Theorem 2.4 to the continuous setting yielding a weaker version of the main result, Proposition 4.4, which only holds for piecewise constant data. An approximation argument finishes the proof of Theorem 4.6. Finally, in section 5, some consequences of Theorem 4.6 are discussed.

2 The weighted graph setting

2.1 Invariant \( \varphi \)-minimal sets

We recall the notion of invariant \( \varphi \)-minimal subsets of \( \mathbb{R}^n \) as introduced in [8].

**Definition 2.1.** A set \( M \subset \mathbb{R}^n \) is called invariant \( \varphi \)-minimal if for any \( a \in \mathbb{R}^n \) there exists an element \( x_a \in M \) such that

\[
\sum_{i=1}^n \varphi(a_i - x_{a,i}) \leq \sum_{i=1}^n \varphi(a_i - x_i)
\]

holds for all \( x \in M \) and all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \).
For our purposes the following result, which shows that every invariant \( \varphi \)-minimal set in addition fulfills a weighted version of inequality (2.1), will be useful.

**Proposition 2.1.** Let \( M \subset \mathbb{R}^n \) be a bounded, closed and convex set that is invariant \( \varphi \)-minimal. Then for any \( a \in \mathbb{R}^n \) and any \( w \in \mathbb{R}_{>0}^n \) there exists an element \( x_{a,w} \in M \) such that

\[
\sum_{i=1}^n w_i \varphi \left( \frac{a_i - x_{a,w,i}}{w_i} \right) \leq \sum_{i=1}^n w_i |\varphi'(a_i)| \left( \frac{a_i - x_i}{w_i} \right)
\]

holds for all \( x \in M \) and all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \).

**Proof.** See the appendix. \( \square \)

### 2.2 The ROF model on weighted graphs

Let \((V,E)\) be an oriented graph. That is, \( V = \{v_1, \ldots, v_n\}, E \subset V \times V \), and if \((v_i,v_j) \in E\), then \((v_j,v_i) \notin E\) must hold. In addition, we endow both \( V \) and \( E \) with positive weight functions \( w \in \mathbb{R}_{>0}^V \) and \( w \in \mathbb{R}_{>0}^E \), respectively.

**Definition 2.2.** The weighted divergence operator \( \text{div}_{W,w} : \mathbb{R}^E \to \mathbb{R}^V \) maps edge functions to vertex functions. For \( H \in \mathbb{R}^E \) it is given by

\[
\text{div}_{W,w} H(v) = \frac{1}{w(v)} \left( \sum_{e=(v,v) \in E} W(e) H(e) - \sum_{e=(w,v) \in E} W(e) H(e) \right).
\]

This definition is inspired by a discrete version of the divergence theorem, where \( v \) represents a subset of Euclidean space with volume \( w(v) \) and each edge \( e \) incident to \( v \) forms a part of its boundary with area \( W(e) \). For more details see the proof of Lemma 1.2.

For \( \alpha > 0 \) we set \( B_{\alpha} = \{ H \in \mathbb{R}^E : -\alpha \leq H \leq \alpha \} \).

**Definition 2.3.** The weighted total variation \( J_W : \mathbb{R}^V \to \mathbb{R} \) is defined as

\[
J_W(u) = \sup_{h \in \text{div}_{W,w} B_1} \sum_{v \in V} w(v) u(v) h(v),
\]

where \( \text{div}_{W,w} B_1 \) denotes the image of \( B_1 \) under the weighted divergence operator.

The following lemma shows that \( J_W(u) \) is independent of both vertex weights and edge orientation.

**Lemma 2.2.**

\[
J_W(u) = \sup_{h \in \text{div}_{W,1} B_1} \sum_{v \in V} u(v) h(v) \quad \text{(2.2)}
\]

\[
= \sum_{(v_i,v_j) \in E} W(v_i,v_j) |u(v_i) - u(v_j)| \quad \text{(2.3)}
\]

**Proof.** Representation (2.2) follows immediately from Definitions 2.2 and 2.3. We can rewrite

\[
J_W(u) = \sup_{H \in B_1} \sum_{v \in V} u(v) (\text{div}_{W,1} H)(v)
\]

\[
= \sup_{H \in B_1} \sum_{(v_i,v_j) \in E} W(v_i,v_j) (u(v_j) - u(v_i)) H(v_i,v_j). \quad \text{(2.4)}
\]
The supremum in (2.4) is attained for $H \in B_1$ if and only if
\[ H(v_i, v_j) \in \begin{cases} 
\{1\}, & u(v_i) < u(v_j), \\
[-1, 1], & u(v_i) = u(v_j), \\
\{-1\}, & u(v_i) > u(v_j),
\end{cases} \]
giving (2.3).

Below we define the subdifferential in a general Hilbert space setting, which will allow us to reuse it in later sections.

**Definition 2.4.** Let $F : \mathcal{H} \to \mathbb{R} \cup \{\pm \infty\}$ be an extended real-valued function defined on a Hilbert space $\mathcal{H}$. Suppose $u \in H$ is such that $F(u) \in \mathbb{R}$. Then the subdifferential of $F$ at $u$ is given by
\[ \partial F(u) = \{ u^* \in H : F(v) - F(u) \geq \langle u^*, v - u \rangle \text{ for all } v \in H \} . \]
If $F(u) \notin \mathbb{R}$, then $\partial F(u) = \emptyset$.

**Remark 2.1.** According to (2.2) $J_W$ is the support function of the closed and convex set $\text{div}_{W,1} B_1$. Therefore $\text{div}_{W,1} B_1 = \partial J_W(0)$.

We now turn to the ROF model on a weighted graph
\[
\min_{u \in \mathbb{R}^V} \frac{1}{2} \| f - u \|_{2,w}^2 + \alpha J_W(u),
\]
where $f \in \mathbb{R}^V$ and parameter $\alpha > 0$ are given, and $\| \cdot \|_{p,w}$ is the $\ell^p$-norm on $\mathbb{R}^V$ with weight $w$, that is,
\[ \| g \|_{p,w}^p = \sum_{v \in V} w(v) |g(v)|^p, \quad 1 \leq p < \infty. \]
We denote by $u_\alpha$ the ROF minimizer, that is, the unique solution to (2.5). The following two results are generalizations of [6, Prop. 3.1, Thm. 3.2] to weighted graphs. Lemma 2.3 characterizes $u_\alpha$ as the element of minimal weighted $\ell^2$-norm in a certain neighbourhood of the datum $f$.

**Lemma 2.3.** Problem (2.5) is equivalent to
\[
\min_{u \in \mathbb{R}^V} \frac{1}{2} \| f - u \|_{2,w}^2 + \alpha J_W(u),
\]
\[
\min_{u^* \in \mathbb{R}^V} \frac{1}{2} \| f - u^*/w \|_{2,w}^2 + (\alpha J_W)^*(u^*),
\]
where $(\alpha J_W)^*$ denotes the convex conjugate of $\alpha J_W$. See, for instance, [5, Chap. III, Sect. 4]. As the conjugate of a support function is a characteristic function, we have
\[ (\alpha J_W)^*(u^*) = \begin{cases} 
0, & u^* \in \alpha \partial J_W(0), \\
+\infty, & u^* \notin \alpha \partial J_W(0),
\end{cases} \]
recall Remark 2.1. It follows that the unique solution $u_\alpha^*$ to the dual problem (2.6) is given by
\[ u_\alpha^* = \arg \min_{u^* \in \alpha \partial J_W(0)} \| f - u^*/w \|_{2,w}. \]
Since the primal and dual solutions are related by \( u_\alpha = f - u^*_\alpha / w \), see [3] Chap. III, Rem. 4.2, we obtain
\[
\| u_\alpha \|_{2,w} = \| f - u^*_\alpha / w \|_{2,w} = \min_{u^* \in \partial J_W(0)} \| f - u^* / w \|_{2,w} = \min_{u \in f - \frac{\partial J_W}{w}(0)} \| u \|_{2,w}.
\]

It turns out that \( u_\alpha \) minimizes not only \( \| \cdot \|_{2,w} \) over the set \( f - \frac{\partial J_W}{w}(0) \).

**Theorem 2.4.** The ROF minimizer \( u_\alpha \) satisfies
\[
\sum_{v \in V} \varphi(u_\alpha(v)) w(v) = \min_{u \in f - \frac{\partial J_W}{w}(0)} \sum_{v \in V} \varphi(u(v)) w(v)
\]
for every convex function \( \varphi : \mathbb{R} \to \mathbb{R} \).

**Proof.** From [3] Thm. 2.4, Rem. 2.5 it follows that the bounded, closed and convex set \( \alpha \partial J_W(0) = \text{div} W_\alpha B_\alpha \) is invariant \( \varphi \)-minimal. Proposition 2.1 then gives that there exists an unique element \( x_\alpha \in \alpha \partial J_W(0) \) satisfying
\[
\sum_{v \in V} w(v) \varphi \left( \frac{w(v)f(v) - x_\alpha(v)}{w(v)} \right) = \min_{x \in \alpha \partial J_W(0)} \sum_{v \in V} w(v) \varphi \left( \frac{w(v)f(v) - x(v)}{w(v)} \right) \tag{2.7}
\]
for every convex function \( \varphi : \mathbb{R} \to \mathbb{R} \). From Lemma 2.3 we know that
\[
\sum_{v \in V} (u_\alpha(v))^2 w(v) = \min_{u \in f - \frac{\partial J_W}{w}(0)} \sum_{v \in V} (u(v))^2 w(v). \tag{2.8}
\]
Comparing (2.7), for \( \varphi(t) = t^2 \), with (2.8) then gives \( w f - x_\alpha = w u_\alpha \) and the result follows. \( \square \)

**Remark 2.2.** Theorem 2.4 implies in particular that \( u_\alpha \) minimizes \( \| \cdot \|_{p,w} \) for all \( p \in [1, \infty) \) over the set \( f - \frac{\partial J_W}{w}(0) \). Observing that \( \| u \|_{p,w} \to \| u \|_\infty \) as \( p \to \infty \), it follows that \( u_\alpha \) also minimizes \( \| \cdot \|_\infty \) over this set.

### 3 The \( \ell^1 \)-anisotropic ROF model

In this section we return to the continuous setting by stating several results from [7, 10] related to the \( \ell^1 \)-anisotropic ROF model. Regarding the next lemma, which is analogous to Lemma 2.3, recall the definitions of \( J \), given in (1.2), and of the subdifferential (Definition 2.4).

**Lemma 3.1.** For every \( f \in L^2(\Omega) \) the minimization problem
\[
\min_{u \in L^2(\Omega)} \frac{1}{2} \| u - f \|_{L^2}^2 + \alpha J(u) \tag{3.1}
\]
is equivalent to
\[
\min_{u \in f - \alpha \partial J(0)} \| u \|_{L^2}.
\]

**Proof.** This equivalence can be shown in essentially the same way as Lemma 2.3. For details see [7, Lem. 1]. \( \square \)
3.1 PCR functions

Let $G$ be a grid of $\mathbb{R}^d$, that is, a finite collection of affine hyperplanes, each orthogonal to one of the coordinate axes. In addition, we assume that $G$ covers the entire boundary of $\Omega$ (recall (1.1)), that is, $\partial \Omega \subset \bigcup G$. The grid $G$ defines a natural partition $Q(G)$ of $\Omega$ into smaller open hyperrectangles: a subset $R$ of $\Omega$ belongs to $Q(G)$, if it does not intersect with $\bigcup G$ while its boundary is covered by $\bigcup G$.

We denote by $\text{PCR}_G(\Omega)$, or simply $\text{PCR}_G$, the set of all functions $g : \Omega \to \mathbb{R}$ which are almost everywhere equal to finite linear combinations of indicator functions of elements of $Q(G)$. That is, a $g \in \text{PCR}_G$ satisfies

$$g(x) = \sum_{i=1}^N c_i 1_{R_i}(x)$$

for almost every $x \in \Omega$ and some $R_i \in Q(G)$ and $c_i \in \mathbb{R}$. The indicator function $1_A$ of a set $A$ is defined by

$$1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

3.2 Preservation of piecewise constancy

The averaging operator $A_G : L^1(\Omega) \to \text{PCR}_G(\Omega)$ associated to a grid $G$ is defined by

$$A_G g = \sum_{i=1}^N \left( \frac{1}{|R_i|} \int_{R_i} g(s) \, ds \right) 1_{R_i},$$

where $\{R_1, \ldots, R_N\} = Q(G)$ and $|R_i|$ denotes the $d$-dimensional volume of $R_i$. It is an immediate consequence of the next lemma that $A_G$ is a contraction on $L^p(\Omega)$ for $1 \leq p < \infty$.

Lemma 3.2. For every $u \in L^1(\Omega)$ and convex $\varphi : \mathbb{R} \to \mathbb{R}$

$$\int_\Omega \varphi (\langle A_G u(x) \rangle) \, dx \leq \int_\Omega \varphi (u(x)) \, dx.$$  

Proof. This is an application of Jensen’s inequality. See [7, Lem. 2].

The following result, stating that $A_G$ maps $\partial J(0)$ into itself, crucially relies on the fact that the rectilinearity of the grid is compatible with the anisotropy of $J$.

Theorem 3.3. $A_G (\partial J(0)) \subset \partial J(0)$.

Proof. See [7, Thm. 1]. Note that the additional assumption in [7, Thm. 1], that $G$ should cover $\partial \Omega$, was already imposed in Section 3.1.

The final result of this section asserts that the $\ell^1$-anisotropic ROF model respects the piecewise constant structure of $\text{PCR}_G$ data. A proof for $d = 1$, where the situation is comparatively simple, can be found in [10, Lem. 4.34]. The case $d = 2$ was resolved in [10] while the general result ($d \geq 1$) was established in [7].

Theorem 3.4. Let $f \in \text{PCR}_G$. Then the minimizer $u_\alpha$ of the $\ell^1$-anisotropic ROF functional with datum $f$ also lies in $\text{PCR}_G$.

Proof. The proof combines Lemmas 3.1 and 3.2 with Theorem 3.3. For details we refer to [7, Thm. 2] from which Theorem 3.4 follows.

Theorem 3.4 implies that, for $\text{PCR}_G$ datum, the $\ell^1$-anisotropic ROF model becomes a finite-dimensional problem. Thus, $u_\alpha$ can be found by minimizing a discrete energy of the form (2.5). This fact is exploited below in order to prove our main result Theorem 4.6.
4 Universal minimality of the ROF minimizer

This section consists of two subsections. In the first one we relate the weighted graph and PCR settings to each other. Its culmination is Proposition 4.4, which is a translation of Theorem 2.4 to the continuous setting and requires new notions as well as some intermediate lemmas. In the second subsection the main result, Theorem 4.6, is established. The basis of the proof is a combination of Proposition 4.4 with an approximation procedure detailed in Lemma 4.5.

4.1 Relations between weighted graph and PCR settings

Let $G$ be a grid with associated partition $Q(G)$ of $\Omega$. Consider the set of all common sides of pairs of elements of $Q(G)$

$$S(G) = \{S_{ij} = R_i \cap R_j : R_i, R_j \in Q(G), \mathcal{H}^{d-1}(S_{ij}) > 0\}, \quad (4.1)$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure.

**Definition 4.1.** Define the oriented graph $(V_G, E_G)$ by identifying $V_G$ with $Q(G)$ and $E_G$ with $S(G)$. That is, there is a vertex $v_i \in V_G$ for every $R_i \in Q(G)$, as well as one arbitrarily oriented edge $(v_i, v_j)$ whenever $R_i \cap R_j \in S(G)$. The weight functions $w \in \mathbb{R}^{V_G}$ and $W \in \mathbb{R}^{E_G}$ associated to the graph $(V_G, E_G)$ are defined by

$$w(v_i) = |R_i|, \quad W(v_i, v_j) = \mathcal{H}^{d-1}(S_{ij}). \quad (4.2)$$

With this construction at hand we can identify the two spaces $PCR_G$ and $\mathbb{R}^{V_G}$ by means of the following isomorphism

$$\iota : PCR_G \to \mathbb{R}^{V_G}, \quad \iota(f)(v_i) = f|_{R_i}. \quad (4.3)$$

The next lemma shows that the two ROF models introduced in Sections 2.2 and 3 respectively, are equivalent if the weighted graph is constructed from the grid $G$ associated to the datum $f \in PCR_G$.

**Lemma 4.1.** Let $f \in PCR_G$. Then $u_\alpha$ minimizes

$$\frac{1}{2} \|f - u\|^2_{L^2} + \alpha J(u) \quad (4.3)$$

over $L^2(\Omega)$ if and only if $\iota(u_\alpha)$ minimizes

$$\frac{1}{2} \|\iota(f) - u\|^2_{\mathbb{R}^{V_G}} + \alpha J_W(u) \quad (4.4)$$

over $\mathbb{R}^{V_G}$.

**Proof.** By Theorem 3.4 minimization of (4.3) can be restricted to $PCR_G$. For $u \in PCR_G$ functional (4.3) simplifies to

$$\frac{1}{2} \sum_{i=1}^N |R_i| \left(\left|f|_{R_i} - u|_{R_i}\right|^2 + \alpha \sum_{i,j=1}^N \mathcal{H}^{d-1}(S_{ij}) \left|u|_{R_i} - u|_{R_j}\right|^2\right).$$

Next, identify $u \in PCR_G$ with $\iota(u) \in \mathbb{R}^{V_G}$. Recalling Lemma 2.2 and the definition of weights in (4.2), it follows that the functionals (4.3) and (4.4) coincide. \qed

Next, we introduce a space of piecewise affine vector fields, which allows us to characterize the set $A_G(\partial J(0))$ in Lemma 4.3 and, in further consequence, will enable us to prove the intermediate result Proposition 4.4.
Definition 4.2. Denote by $\text{PAR}_0^0(\Omega)$, or simply $\text{PAR}_0^0$, the set of all vector fields $F = (F_1, \ldots, F_d): \Omega \to \mathbb{R}^d$ which satisfy the following conditions.

- Each component $F_i$ is continuous in $x_i$ and satisfies the boundary condition $F_i(x_i = a_i) = F_i(x_i = b_i) = 0$, recall (1.1).
- For every $R_k \in \mathcal{Q}(G)$ and $1 \leq i \leq d$ there are $c_k^i, d_k^i \in \mathbb{R}$ such that $F_i|_{R_k} = c_k^i x_i + d_k^i$.

We can identify the space $\text{PAR}_0^0$ with the space of edge functions on the graph $(V_G, E_G)$ constructed in Definition 4.1 using the following isomorphism

$$\kappa: \text{PAR}_0^0 \to \mathbb{R}^{E_G}, \quad \kappa(F)(v_i, v_j) = \nu_{ij} \cdot F|_{S_{ij}}.$$

Here $\nu_{ij} \in \mathbb{R}^n$ denotes the unit normal of $S_{ij}$, recall (4.1), pointing in the reversed direction of the corresponding edge $(v_i, v_j)$. Note that the continuity condition in the definition of $\text{PAR}_0^0$ ensures that $\nu_{ij} \cdot F$ is always well-defined and constant on $S_{ij}$, as $\nu_{ij} = \pm e_k$ for some $k = 1, \ldots, d$ where $\{e_k\}_{k=1}^d$ is the standard basis of $\mathbb{R}^d$.

With the notions introduced above we have the following relation between the weak divergence and weighted graph divergence. See also the commutative diagram in Figure 1.

Lemma 4.2. For every $F \in \text{PAR}_0^0$ we have

$$\iota(\text{div} \, F) = \text{div}_{W,w} \kappa(F).$$

Proof. Let $F \in \text{PAR}_0^0$ and $v_i \in V_G$. Using the fact that $\text{div} \, F$ is piecewise constant we can write

$$\iota(\text{div} \, F)(v_i) = \text{div} \, F|_{R_i} = \frac{1}{|R_i|} \int_{R_i} \text{div} \, F(x) \, dx.$$

Now, with $n$ denoting the outward unit normal of $R_i$, the divergence theorem yields

$$\frac{1}{|R_i|} \int_{\partial R_i} n(x) \cdot F(x) \, dS.$$

Next we decompose the integral over $\partial R_i$ into a sum of integrals over the sides of $R_i$. Observe that if $\partial R_i \cap \partial \Omega$ is nonempty then $n \cdot F = 0$ there, by definition of $\text{PAR}_0^0$. Therefore we obtain

$$\frac{1}{|R_i|} \sum_{j: S_{ij} \in \mathcal{S}(G)} \int_{S_{ij}} n(x) \cdot F(x) \, dS.$$

Since all the integrands are constant this equals

$$\frac{1}{|R_i|} \sum_{j: S_{ij} \in \mathcal{S}(G)} \mathcal{H}^{d-1}(S_{ij})(n \cdot F)|_{S_{ij}}.$$

Taking into account the fact that $\nu_{ij} = \pm n|_{S_{ij}}$ we can rewrite

$$\frac{1}{|R_i|} \sum_{j: S_{ij} \in \mathcal{S}(G)} \mathcal{H}^{d-1}(S_{ij})(\nu_{ij} \cdot n|_{S_{ij}})(\nu_{ij} \cdot F|_{S_{ij}}).$$
Finally, we replace all quantities by their graph analogues and observe that the inner product $\nu_{ij} \cdot n|_{S_{ij}} = -1$, if and only if the corresponding edge points away from $v_i$.

$$\frac{1}{w(v_i)} \left( \sum_{e=(v_i,v_j) \in E_G} W(e) \kappa(F)(e) - \sum_{e=(v_i,v_j) \in E_G} W(e) \kappa(F)(e) \int_{a_i}^{b_i} g_i \, dx_i = 0 \right)$$

$$div_{\kappa} \kappa(F)(v_i)$$

The next lemma characterizes $A_G(\partial J(0))$ as divergences of vector fields in $PAR_G^0$ which are bounded by 1 in each component. Since $A_G(\partial J(0)) \subset \partial J(0)$ by Theorem 3.3 the next lemma also provides simple examples of subgradients of $J$.

**Lemma 4.3.**

$$\text{div} \left\{ U \in PAR_G^0 : \max_{1 \leq i \leq d} \| U_i \|_{L^\infty} \leq 1 \right\} = A_G(\partial J(0)).$$

**Proof.** We prove the equality in two steps:

$$\text{div} \left\{ U \in PAR_G^0 : \max_{1 \leq i \leq d} \| U_i \|_{L^\infty} \leq 1 \right\} = \Gamma_G = A_G(\partial J(0)), \quad (4.5)$$

where the auxiliary set $\Gamma_G$ is given by

$$\Gamma_G = \left\{ \sum_{i=1}^d g_i : g_i \in \Gamma_i^G, 1 \leq i \leq d \right\},$$

$$\Gamma_i^G = \left\{ g \in PCR_G : \sup_{s \in (a_i, b_i)} \left| \int_{a_i}^{s} g \, dx_i \right| \leq 1, \int_{a_i}^{b_i} g \, dx_i = 0 \right\}.$$
for $1 \leq i \leq d$. Define the vector field $U = (U_1, \ldots, U_d) : \Omega \to \mathbb{R}^d$ by

$$U_i(x) = \int_{a_i}^{x_i} g_i(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_d) \, ds.$$ 

The function $U_i$ clearly is piecewise affine and continuous in $x_i$. That $U_i$ is bounded by 1 and vanishes at the endpoints follows directly from the properties of $g_i$. Thus we have shown the first equality in (4.5).

Regarding the second equality we recall the proof of [7, Thm. 1], where it is shown that $A_G(\partial J(0)) \subset \Gamma_G \subset \partial J(0)$. Observing that the first two sets only consist of PCR$^G$ functions we even have $A_G(\partial J(0)) \subset \Gamma_G \subset \partial J(0) \cap \text{PCR}_G$.

It remains to show that $\partial J(0) \cap \text{PCR}_G \subset A_G(\partial J(0))$. But this is immediate since for every $u \in \partial J(0) \cap \text{PCR}_G$ we have $A_G u = u$.

The following result is a specialization of Theorem 4.6 for data $f \in \text{PCR}_G$ and will be used in its proof.

**Proposition 4.4.** Given data $f \in \text{PCR}_G$, the ROF minimizer $u_{\alpha}$ satisfies

$$\int_{\Omega} \varphi(u_{\alpha}(x)) \, dx = \min \left\{ \int_{\Omega} \varphi(u(x)) \, dx : u \in f - \alpha A_G(\partial J(0)) \right\}$$

for every convex function $\varphi : \mathbb{R} \to \mathbb{R}$.

**Proof.** By Lemma 4.1, $\iota(u_{\alpha})$ minimizes the corresponding ROF functional on the graph $\Gamma_G$. Next, pick an arbitrary convex function $\varphi : \mathbb{R} \to \mathbb{R}$ and apply Theorem 2.4 to obtain

$$\sum_{v \in V} \varphi(\iota(u_{\alpha})(v)) \, w(v) = \min \left\{ \sum_{v \in V} \varphi(\iota(v)) \, w(v) : u \in \iota(f) - \text{div} W_B \right\}.$$ 

By means of the inverse isomorphism $\iota^{-1}$ we can write equivalently

$$\int_{\Omega} \varphi(u_{\alpha}(x)) \, dx = \min \left\{ \int_{\Omega} \varphi(u(x)) \, dx : u \in \iota^{-1}(f) - \text{div} W_B \right\}.$$ 

Taking into account Lemmas 4.2 and 4.3 it follows that

$$\iota^{-1}(f) - \text{div} W_B = f - \iota^{-1}(\text{div} W_B) = f - \text{div} \kappa^{-1}(B) = f - \text{div} \left\{ U \in \text{PAR}_G^0 : \max_{1 \leq i \leq d} \| U_i \|_{L^\infty} \leq \alpha \right\} = f - \alpha A_G(\partial J(0)).$$

$\square$
4.2 The main result

From now on, we consider a general datum $f \in L^2(\Omega)$. The next result, applied in the proof of Theorem 4.6, concerns an approximating sequence of the ROF minimizer $u_\alpha$.

**Lemma 4.5.** Let $\{G_m\}$ be a sequence of grids such that the longest diagonal of the hyperrectangles in the partition $Q(G_m)$ of $\Omega$ goes to 0 as $m \to \infty$. For a given $f \in L^2(\Omega)$, let

$$u_{\alpha,m} = \arg\min_{u \in L^2(\Omega)} \left( \frac{1}{2} \|A_{G_m}f - u\|^2_{L^2} + \alpha J(u) \right),$$

(i.e. $u_{\alpha,m}$ is the ROF minimizer for the datum $A_{G_m}f$). The sequence $\{u_{\alpha,m}\}$ has a subsequence $\{u_{\alpha,m_k}\}$ that converges weakly to $u_\alpha$ in $L^2$.

**Proof.** The assumption on the sequence of grids $\{G_m\}$ ensures that $A_{G_m}h \to h$ in $L^2$ for $h \in L^2(\Omega)$. Next, recalling Lemma 3.1, the minimization problem (4.6) is equivalent to

$$u_{\alpha,m} = \arg\min_{u \in \text{arg min } \alpha \partial J(0)} \|u\|_{L^2}. \quad (4.7)$$

As $A_{G_m}(\partial J(0)) \subseteq \partial J(0)$, recall Theorem 3.3 $A_{G_m}u_\alpha \in A_{G_m}f - \alpha \partial J(0)$ holds. Combining this observation with (4.7) and the fact that $A_{G_m}$ is a contraction on $L^2$ gives

$$\|u_{\alpha,m}\|_{L^2} \leq \|A_{G_m}u_\alpha\|_{L^2} \leq \|u_\alpha\|_{L^2}.$$ 

So, the sequence $\{u_{\alpha,m}\}$ is bounded in $L^2$ and therefore contains a weakly convergent subsequence $\{u_{\alpha,m_k}\}$ in $L^2$. Let $u_*$ denote the weak limit of $\{u_{\alpha,m_k}\}$. From

- $u_{\alpha,m_k} \in A_{G_m}f - \alpha \partial J(0),$
- $A_{G_m}f$ converges in $L^2$ to $f,$
- $\partial J(0)$ is a weakly closed set in $L^2,$

follows that $u_* \in f - \alpha \partial J(0)$. The weak lower semicontinuity of $\|\cdot\|_{L^2}$ gives that $\|u_*\|_{L^2} \leq \|u_\alpha\|_{L^2}$ and as $u_\alpha$ is the unique $L^2$-minimizer in $f - \alpha \partial J(0)$ we conclude that $u_* = u_\alpha.$ \hfill \Box

We are now ready to prove the main result.

**Theorem 4.6.** Given datum $f \in L^2(\Omega)$, the ROF minimizer $u_\alpha$ satisfies

$$\int_{\Omega} \varphi(u_\alpha(x)) \, dx \leq \int_{\Omega} \varphi(u(x)) \, dx$$

for every $u \in f - \alpha \partial J(0)$ and every convex function $\varphi : \mathbb{R} \to \mathbb{R}$.

**Proof.** Let $\{G_m\}$ be a sequence of grids according to Lemma 4.5. From this lemma it follows that the sequence of ROF minimizers $\{u_{\alpha,m}\}$ for data $\{A_{G_m}f\}$ contains a subsequence $\{u_{\alpha,m_k}\}$ that converges weakly to $u_\alpha$ in $L^2$.

Consider an arbitrary element $u \in f - \alpha \partial J(0)$. It is clear that $A_{G_m}u \in A_{G_m}f - \alpha A_{G_m}(\partial J(0))$. Next, as $A_{G_m}f \in PCRG_m$, by Theorem 3.4 $u_{\alpha,m} \in PCRG_m$ and it is then an immediate consequence that also $u_{\alpha,m} \in A_{G_m}f - \alpha A_{G_m}(\partial J(0))$. Applying first Proposition 4.4 and then Lemma 3.2 we obtain the inequalities

$$\int_{\Omega} \varphi(u_{\alpha,m_k}(x)) \, dx \leq \int_{\Omega} \varphi((A_{G_m}u)(x)) \, dx \leq \int_{\Omega} \varphi(u(x)) \, dx,$$
which are valid for every convex $\varphi : \mathbb{R} \to \mathbb{R}$.

The remaining step is to verify weak lower semicontinuity of $h \mapsto \int_{\Omega} \varphi (h(x)) \, dx$ in $L^2(\Omega)$. This, however, is a direct consequence of the convexity of $\varphi$. See, for instance, [11 Thm. 3.20]. Therefore

$$\int_{\Omega} \varphi (u_\alpha(x)) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} \varphi (u_{\alpha, m_k}(x)) \, dx \leq \int_{\Omega} \varphi (u(x)) \, dx.$$  

5 Discussion

We conclude this article by discussing several consequences of Theorem 4.6.

First, the ROF solution minimizes all $L^p$-norms, $1 \leq p \leq \infty$, over the set $f - \alpha \partial J(0)$. By choosing $\varphi(\cdot) = |\cdot|^p$, $1 \leq p < \infty$, Theorem 4.6 implies that

$$\|u_\alpha\|_{L^p} \leq \|u\|_{L^p}$$

for all $u \in f - \alpha \partial J(0)$. A limiting argument shows that the inequality also holds for $p = \infty$. Note, however, that the norms might be infinite for $p > 2$.

It follows that the ROF model preserves integrability of the datum. Inequality (5.1) in particular gives

$$\|u_\alpha\|_{L^p} \leq \|f\|_{L^p}$$

for all $p \in [1, \infty]$. Therefore, $f \in L^p(\Omega)$ implies $u_\alpha \in L^p(\Omega)$.

Theorem 4.6 is an $\ell^1$-anisotropic analogue to [16, Thm. 4.46], which concerns the isotropic ROF model. While it might be possible to adapt the arguments of [16 Thm. 4.46] to the anisotropic setting, we think that our proof is of independent interest revealing precise connections between the discrete and continuous settings. On the other hand, since the essential Theorem 3.4 fails to hold for the isotropic ROF model (see [12 Prop. 6]), a proof based on the spaces $PCR_G$ cannot be directly applied to that setting.

In [9] invariant $\varphi$-minimal sets were introduced as subsets of $L^1(\Omega)$. We can extend this definition to $L^p(\Omega)$, $1 \leq p \leq \infty$, in the following way. A set $M \subset L^p(\Omega)$ is invariant $\varphi$-minimal, if for every $g \in L^p(\Omega)$ there is an element $u^*_g \in M$ of best approximation in the sense that

$$\int_{\Omega} \varphi (u^*_g - g) \, dx \leq \int_{\Omega} \varphi (u - g) \, dx$$

holds for all $u \in M$ and all convex functions $\varphi$. This property is invariant under translation and scaling, meaning that, for every $\beta \in \mathbb{R}$ and $h \in L^p(\Omega)$, the set $h + \beta M$ is again invariant $\varphi$-minimal. Theorem 4.6 implies that $\alpha \partial J(0)$ is an invariant $\varphi$-minimal set in $L^2(\Omega)$ for every $\alpha > 0$. The scaled subdifferential of the isotropic total variation $\alpha \partial J_2(0)$ is another example of an invariant $\varphi$-minimal set in $L^2(\Omega)$ by [16 Thm. 4.46]. Both results provide a characterization of the element $u^*_f, \alpha \in \alpha \partial J(0)$ of best approximation to $f \in L^2(\Omega)$ in terms of the respective ROF minimizer

$$u^*_f, \alpha = f - u_\alpha.$$  

Note that $u^*_f, \alpha$ is nothing but the solution of the Fenchel dual of the ROF problem.

It is possible to extend the results of this article to the case of $\Omega$ being a finite union of hyperrectangles, which is the setting of [7]. In order to do so the boundary condition in Definition 4.2 of $PAR^0_G$ has to be modified. First, note that $\partial \Omega$ is a finite union of subsets of affine hyperplanes. The modified boundary condition for
a vector field $F \in \mathcal{P}_G^0$ should require that $F_i$ vanishes on all those subsets of $\partial \Omega$ that are perpendicular to the $x_i$-axis. Accordingly, the set $\Gamma_G$ (recall the proof of Lemma 4.3) must be defined as in [7]. All other parts of this article can remain unchanged.

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A Proof of Proposition 2.1

In order to prove Proposition 2.1, we first recall the notion of special cone property which was introduced in [8]. By $\{e_i\}_{i=1}^n$ we denote the standard basis of $\mathbb{R}^n$.

Definition A.1. Let $M \subset \mathbb{R}^n$ be closed and convex. For $x \in M$, consider all vectors $s = e_i - e_j$ such that $x + \varepsilon s \in M$ for some $\varepsilon > 0$. Denote by $S_x$ the set of all such vectors at $x$ and let $K_x = \{y \in \mathbb{R}^n : y = \sum_{s \in S_x} \lambda_s s, \lambda_s \geq 0\}$ be the convex cone generated by these vectors. The set $M$ is said to have the special cone property if $M \subset x + K_x$ for each $x \in M$.

Remark A.1. In [8], vectors of the type $e_i$ and $e_i + e_j$ are also considered in the definition of the special cone property. These vectors are required for the characterization of invariant $K$-minimal sets, a notion related to invariant $\varphi$-minimal sets.

The following characterization of invariant $\varphi$-minimal sets was established in [8, Thm. 3.2, Thm. 4.2].

Theorem A.1. A bounded, closed and convex set $M \subset \mathbb{R}^n$ is invariant $\varphi$-minimal if and only if it has the special cone property.

The next lemma will turn out to be useful for proving Proposition 2.1.

Lemma A.2. Given $s = e_k - e_l$ and $b \in \mathbb{R}^n$, consider the closed line segment $L_{c,d} = \{x \in \mathbb{R}^n : x = b + ts, t \in [c, d] \subset \mathbb{R}\}$.

For every $a \in \mathbb{R}^n$ and $w \in \mathbb{R}^n_{>0}$ there exists an element $x_{a,w} \in L_{c,d}$ such that

$$\sum_{i=1}^n w_i \varphi \left( \frac{a_i - x_{a,w,i}}{w_i} \right) \leq \sum_{i=1}^n w_i \varphi \left( \frac{a_i - x_i}{w_i} \right)$$

holds for all $x \in L_{c,d}$ and every convex function $\varphi : \mathbb{R} \to \mathbb{R}$.

Proof. Let $a \in \mathbb{R}^n$ and $w \in \mathbb{R}^n_{>0}$ be given. Consider an arbitrary convex function $\varphi : \mathbb{R} \to \mathbb{R}$ and let

$$\psi_{a,w}(x) = \sum_{i=1}^n w_i \varphi \left( \frac{a_i - x_i}{w_i} \right).$$

It is straightforward to show, using Jensen’s inequality, that $\psi_{a,w}$ is minimized on the line $L = \{x \in \mathbb{R}^n : x = b + ts, t \in \mathbb{R}\}$ by $x_{a,w} = b + t_{a,w} s$ where

$$t_{a,w} = \frac{(a_k - b_k)w_l - (a_l - b_l)w_k}{w_k + w_l}.$$
Note that \( \varphi \) does not influence \( t_{a,w} \) and \( x_{a,w} \). On \( L \), \( \psi_{a,w} \) may be viewed as a function of the parameter \( t \in \mathbb{R} \) and as such it is decreasing for \( (-\infty, t_{a,w}] \) and increasing for \( [t_{a,w}, \infty) \). Therefore, when restricted to the closed segment \( L_{c,d} \) of \( L \), \( \psi_{a,w} \) has a minimum at \( x_{a,w} = b + cs \) if \( t_{a,w} < c \), at \( x_{a,w} = b + t_{a,w}s \) if \( c \leq t_{a,w} \leq d \), and at \( x_{a,w} = b + ds \) if \( t_{a,w} > d \). In all cases, \( x_{a,w} \) does not depend on \( \varphi \). \( \square \)

We are now ready to prove Proposition 2.1. The structure of the proof is similar to the one of [8] Thm. 3.1.

**Proof of Proposition 2.1** Let \( a \in \mathbb{R}^n \) and \( w \in \mathbb{R}_{+}^n \) be given and fix a convex function \( \varphi : \mathbb{R} \to \mathbb{R} \). As in the proof of Lemma A.2 let

\[
\psi_{a,w}(x) = \sum_{i=1}^{n} w_i \varphi\left(\frac{a_i - x_i}{w_i}\right). \tag{A.1}
\]

Since \( \psi_{a,w} \) is continuous on \( M \) there exist at least one element in \( M \) where \( \psi_{a,w} \) attains a minimum. Let \( M_{a,w} \) denote the set of all minimizers of \( \psi_{a,w} \) in \( M \), i.e.

\[
M_{a,w} = \left\{ y \in M : \psi_{a,w}(y) = \min_{x \in M} \psi_{a,w}(x) \right\}.
\]

Note that \( M_{a,w} \) is a closed and convex subset of \( M \). Let \( x_{a,w} \) denote the element in \( M_{a,w} \) which satisfies

\[
\sum_{i=1}^{n} w_i \left(\frac{a_i - x_{a,w,i}}{w_i}\right)^2 = \min_{x \in M_{a,w}} \sum_{i=1}^{n} w_i \left(\frac{a_i - x_i}{w_i}\right)^2. \tag{A.2}
\]

Uniqueness of \( x_{a,w} \) follows from the strict convexity of \( x \to \sum_{i=1}^{n} w_i \left(\frac{a_i - x_i}{w_i}\right)^2 \). Equivalently, (A.2) can be formulated as

\[
||a - x_{a,w}||_{2,1/w} = \min_{x \in M_{a,w}} ||a - x||_{2,1/w}
\]

where

\[
||x||_{2,1/w} = \sqrt{\langle x, x \rangle_{1/w}}
\]

is the norm associated to the weighted inner product on \( \mathbb{R}^n \) given by

\[
\langle x, y \rangle_{1/w} = \sum_{i=1}^{n} x_i y_i \frac{1}{w_i}.
\]

The remaining part of the proof is devoted to show that \( x_{a,w} \) is the element of best approximation in \( M \) of \( a \) with respect to \( || \cdot ||_{2,1/w} \). As this element is unique, it then follows that \( x_{a,w} \) is independent of the specific \( \varphi \) and minimizes (A.1) in \( M \) for all convex functions.

Now, \( M \) is a set in the class characterized by Theorem A.1 and has therefore the special cone property. Take a vector \( s \in S_{a,w} \), and recall that \( x_{a,w} + \varepsilon s \in M \) for some \( \varepsilon > 0 \). The convexity of \( M \) gives that the entire line segment \( L_{0,c} = \{ x : x = x_{a,w} + ts, t \in [0, c] \} \) is in \( M \). By construction, \( x_{a,w} \) minimizes (A.1) on \( L_{0,c} \). Further, if there are several minimizers of (A.1) on \( L_{0,c} \), \( x_{a,w} \) has the smallest \( || \cdot ||_{2,1/w} \) norm among them. Taking into account these two properties of \( x_{a,w} \), Lemma A.2 gives that \( x_{a,w} \) minimizes \( || \cdot ||_{2,1/w} \) on \( L_{0,c} \). We can then derive

\[
||a - x_{a,w}||_{2,1/w}^2 \leq ||a - (x_{a,w} + ts)||_{2,1/w}^2
\]

for all \( a \in a_{x,ws}, s \in S_{a,w}, t \in [0, c] \). Therefore, when restricted to the closed segment \( L_{c,d} \) of \( L \), \( \psi_{a,w} \) has a minimum at \( x_{a,w} = b + cs \) if \( t_{a,w} < c \), at \( x_{a,w} = b + t_{a,w}s \) if \( c \leq t_{a,w} \leq d \), and at \( x_{a,w} = b + ds \) if \( t_{a,w} > d \). In all cases, \( x_{a,w} \) does not depend on \( \varphi \). \( \square \)
for \( t \in [0, \varepsilon] \). Therefore, for \( t \) in this interval,
\[
-2t(a - x_{a,w}, s)_{1/w} + t^2(s, s)_{1/w} \geq 0
\]
which in turn gives that
\[
\langle a - x_{a,w}, s \rangle_{1/w} \leq 0. \tag{A.3}
\]
Consider now a general element \( x \in x_{a,w} + K_{x_{a,w}}, \) i.e. \( x = x_{a,w} + \sum_{s \in S_{x_{a,w}}} \lambda_s s \) where \( \lambda_s \geq 0 \) (recall Definition A.1). We have
\[
\|a - x\|_{2,1/w}^2 = \|a - (x_{a,w} + \sum_{s \in S_{x_{a,w}}} \lambda_s s)\|_{2,1/w}^2
\]
\[
= \|a - x_{a,w}\|_{2,1/w}^2 - 2 \sum_{s \in S_{x_{a,w}}} \lambda_s \langle a - x_{a,w}, s \rangle_{1/w} + \sum_{s, r \in S_{x_{a,w}}} \lambda_s \lambda_r \langle s, r \rangle_{1/w}.
\]
The inequality \( \text{(A.3)} \) implies that
\[
\sum_{s \in S_{x_{a,w}}} \lambda_s \langle a - x_{a,w}, s \rangle_{1/w} \leq 0. \tag{A.4}
\]
Further,
\[
\sum_{s, r \in S_{x_{a,w}}} \lambda_s \lambda_r \langle s, r \rangle_{1/w} \geq 0 \tag{A.5}
\]
as \( \langle \cdot, \cdot \rangle_{1/w} \in \mathbb{R}^{|S_{x_{a,w}}| \times |S_{x_{a,w}}|} \) is a Gram matrix and therefore positive semidefinite. From \( \text{(A.4)} \) and \( \text{(A.5)} \), we get the estimate
\[
\|a - x_{a,w}\|_{2,1/w} \leq \|a - x\|_{2,1/w}. \tag{A.6}
\]
In view of \( \text{(A.6)} \), which holds for any \( x \in x_{a,w} + K_{x_{a,w}}, \) and \( M \subset x_{a,w} + K_{x_{a,w}}, \) the element \( x_{a,w} \) turns out to be the unique best approximation in \( M \) of \( a \) with respect to \( \|\cdot\|_{2,1/w} \). As \( \varphi \) was arbitrarily chosen, \( x_{a,w} \) minimizes \( \text{(A.1)} \) in \( M \) given any choice of convex function.

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