ANALYSIS OF GALERKIN METHODS FOR THE FULLY NONLINEAR MONGE-AMPÈRE EQUATION*

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Abstract. This paper develops and analyzes finite element Galerkin and spectral Galerkin methods for approximating viscosity solutions of the fully nonlinear Monge-Ampère equation $\det(D^2 u^\varepsilon) = f (> 0)$ based on the vanishing moment method which was developed by the authors in [17, 15]. In this approach, the Monge-Ampère equation is approximated by the fourth order quasilinear equation $-\varepsilon \Delta^2 u^\varepsilon + \det D^2 u^\varepsilon = f$ accompanied by appropriate boundary conditions. This new approach allows one to construct convergent Galerkin numerical methods for the fully nonlinear Monge-Ampère equation (and other fully nonlinear second order partial differential equations), a task which has been impracticable before. In this paper, we first develop some finite element and spectral Galerkin methods for approximating the solution $u^\varepsilon$ of the regularized fourth order problem. We then derive optimal order error estimates for the proposed numerical methods. In particular, we track explicitly the dependence of the error bounds on the parameter $\varepsilon$, for the error $u^\varepsilon - u^\varepsilon_h$. Due to the strong nonlinearity of the underlying equation, the standard perturbation argument for error analysis of finite element approximations of nonlinear problems does not work here. To overcome the difficulty, we employ a fixed point technique which strongly makes use of the stability property of the linearized problem and its finite element approximations. Finally, using the Aygris finite element method as an example, we present a detailed numerical study of the rates of convergence in terms of powers of $\varepsilon$ for the error $u^0 - u^\varepsilon_h$, and numerically examine what is the “best” mesh size $h$ in relation to $\varepsilon$ in order to achieve these rates.

1. Introduction

Fully nonlinear partial differential equations (PDEs) are those equations which are nonlinear in the highest order derivative(s) of the unknown function(s). In the case of the second order equations, the general form of the fully nonlinear PDEs is given by

\begin{equation}
F(D^2 u^0, Du^0, u^0, x) = 0,
\end{equation}

where $D^2 u^0(x)$ and $Du^0(x)$ denote respectively the Hessian and the gradient of $u^0$ at $x \in \Omega \subset \mathbb{R}^n$. $F$ is assumed to be a nonlinear function in at least one of entries of $D^2 u^0$. Fully nonlinear PDEs arise from many scientific and engineering fields including differential geometry, optimal control, mass transportation, geostrophic fluid, meteorology (cf. [7, 8, 21, 20, 27] and the references therein).
In this paper we focus our attention on a prototypical fully nonlinear second order PDE, the well-known Monge-Ampère equation. Our goal is to develop and analyze finite element and spectral Galerkin methods for approximating viscosity solutions of the Dirichlet problem for the Monge-Ampère equation (cf. [22]):

\begin{align}
\det(D^2u^0) &= f \quad \text{in } \Omega \subset \mathbb{R}^n, \\
u^0 &= g \quad \text{on } \partial\Omega,
\end{align}

where \( \Omega \subset \mathbb{R}^n \) is a convex domain with smooth or piecewise smooth boundary \( \partial\Omega \). \( \det(D^2u^0(x)) \) denotes the determinant of \( D^2u^0(x) \). Clearly, the Monge-Ampère equation is a special case of (1.1) with \( F(D^2u^0, Du^0, u^0, x) = \det(D^2u^0) - f \). We refer the reader to [4, 8, 21, 22] and the references therein for the derivation, applications, and properties of the Monge-Ampère equation.

For fully nonlinear second order PDEs, in particular for the Monge-Ampère equation, it is well-known that their Dirichlet problems do not have classical solutions in general even \( f, g \) and \( \partial\Omega \) are smooth, if \( \Omega \) is not strictly convex (see [21]). So it is imperative to develop some weak solution theories for these problems. However, because of the fully nonlinearity of these equations, unlike in the case of linear and quasilinear PDEs, the usual weak solution theory based on integration by parts does not work for fully nonlinear PDEs, hence, other “nonstandard” weak solution theories must be sought. In the case of the Monge-Ampère equation, the first such theory was due to A. D. Aleksandrov, who introduced a notion of generalized solutions and proved the Dirichlet problem with \( f > 0 \) has a unique generalized solution in the class of convex functions (cf. [1], also see [9]). But major advances on analysis of problem (1.2)-(1.3) has only been achieved many years later after the introduction and establishment of the viscosity solution theory (cf. [7, 12, 22]). We recall that the notion of viscosity solutions was first introduced by Crandall and Lions [11] in 1983 for the first order fully nonlinear Hamilton-Jacobi equations. It was quickly extended to second order fully nonlinear PDEs, with dramatic consequences in the wake of a breakthrough of Jensen’s maximum principle [25] and the Ishii’s discovery [24] that the classical Perron’s method could be used to infer existence of viscosity solutions. It should be noted that there also exist nonconvex solutions to problem (1.2)-(1.3), so if the convexity requirement is dropped, solutions to problem (1.2)-(1.3) are not unique. We shall refer this uniqueness property as conditional uniqueness. Nonuniqueness or conditional uniqueness is often expected for the Dirichlet problems of fully nonlinear second order PDEs. It should also be noted that unlike in the case of fully nonlinear first order PDEs, the terminology “viscosity solution” loses its original meaning in the case of fully nonlinear second order PDEs.

For the Dirichlet Monge-Ampère problem (1.2)-(1.3) with \( f > 0 \), we recall that [22] a convex function \( u^0 \in C^0(\overline{\Omega}) \) satisfying \( u^0 = g \) on \( \partial\Omega \) is called a viscosity subsolution (resp. viscosity supersolution) of (1.2) if for any \( \varphi \in C^2 \) there holds \( \det(D^2\varphi(x_0)) \leq f(x_0) \) (resp. \( \det(D^2\varphi(x_0)) \geq f(x_0) \)) provided that \( u^0 - \varphi \) has a local maximum (resp. a local minimum) at \( x_0 \in \Omega \). \( u^0 \in C^0(\overline{\Omega}) \) is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution. Form this definition we can see that the notion of viscosity solutions is not variational and not defined using the more familiar integration by parts approach which in fact is not possible because of the fully nonlinearity of the PDE. On the other hand, the
strategy of shifting derivatives onto the test functions at extremal points in the definition of viscosity solutions can be viewed as a “differentiation by parts” approach. It can be shown that a viscosity solution satisfies the PDE in the classical sense at every point where the viscosity solution has second order continuous derivatives. However, the theory does not tell what equations or relations the viscosity solution satisfies at other points. Numerically, this non-variational nature of the viscosity solution theory poses a daunting challenge for computing viscosity solutions because it makes it impossible for one to directly approximate viscosity solutions using any Galerkin type numerical methods including finite element, spectral and discontinuous Galerkin methods, which all are based on variational formulations of PDEs. In addition, it is extremely difficult (if all possible) to mimic “differentiation by parts” approach at the discrete level, so there seems have no hope to develop a discrete viscosity solution theory.

To overcome the above difficulties, recently in [15, 17] we introduced a new approach, called the vanishing moment method, for establishing the existence of viscosity solutions for fully nonlinear second order PDEs, in particular, for problem (1.2)-(1.3). In addition, this new approach gives arise a new notion of weak solutions, called moment solutions, for fully nonlinear second order PDEs. Furthermore, the vanishing moment method is constructive, so practical and convergent numerical methods can be developed based on the approach for computing the viscosity solutions of fully nonlinear second order PDEs such as problem (1.2)-(1.3). The main idea of the vanishing moment method is to approximate a fully nonlinear second order PDE by a quasilinear higher order PDE. The notion of moment solutions and the vanishing moment method are natural generalizations of the original definition of viscosity solutions and the vanishing viscosity method introduced for the Hamilton-Jacobi equations in [11]. We now briefly recall the definitions of moment solutions and the vanishing moment method, and refer the reader to [15, 17] for a detailed exposition.

Firstly, the vanishing moment method approximates the fully nonlinear equation (1.1) by the following quasilinear fourth order PDE:

\[
-\varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, Du^\varepsilon, u^\varepsilon, x) = 0 \quad (\varepsilon > 0),
\]

which holds in domain \( \Omega \). Suppose the Dirichlet boundary condition \( u^0 = g \) is given on \( \partial \Omega \), then it is natural to impose the same boundary condition on \( u^\varepsilon \), hence,

\[
uar\]u^\varepsilon = g \quad \text{on} \quad \partial \Omega.
\]

Secondly, in addition to boundary condition (1.5), one more boundary condition must be imposed to ensure uniqueness of solutions. In [15] we proposed to use one of the following boundary conditions:

\[
\Delta u^\varepsilon = \varepsilon, \quad \text{or} \quad D^2 u^\varepsilon \nu \cdot \nu = \varepsilon \quad \text{on} \quad \partial \Omega,
\]

where \( \nu \) stands for the unit outward normal to \( \partial \Omega \). Although both boundary conditions work well numerically, the first boundary condition is more convenient for standard Galerkin type methods such as finite methods, spectral and discontinuous Galerkin methods, while the second boundary condition fits mixed finite element methods (cf. [15, 18]) better. Hence, in this paper we shall use the first boundary condition. We also refer the reader to [17] for the heuristic argument why these boundary conditions were chosen in the first place.
To sum up, the vanishing moment method consists of approximating second order boundary value problem (1.3)–(1.1) by fourth order boundary value problem (1.4)–(1.5), (1.6). In the case of the Monge-Ampère equation, this then results in approximating boundary value problem (1.2)–(1.3) by the following problem:

\[
-\varepsilon \Delta^2 u^\varepsilon + \det(D^2 u^\varepsilon) = f \quad \text{in } \Omega, \\
u^\varepsilon = g \quad \text{on } \partial \Omega, \\
\Delta u^\varepsilon = \varepsilon \quad \text{on } \partial \Omega.
\]

(1.7) \hspace{1cm} (1.8) \hspace{1cm} (1.9)

It was proved in [15] that if \( f \geq 0 \) in \( \Omega \) then problem (1.7)–(1.9) has a unique solution \( u^\varepsilon \) which is a convex function over \( \Omega \). Moreover, \( u^\varepsilon \) uniformly converges as \( \varepsilon \to 0 \) to the unique viscosity solution of (1.2)–(1.3). As a byproduct, this also shows that (1.2)–(1.3) has a unique moment solution which coincides with the unique viscosity solution. Furthermore, it was proved that there holds the following a priori bounds which will be used frequently later in this paper:

\[
\| u^\varepsilon \|_{H^j} = O(\varepsilon^{-\frac{j-1}{2}}), \quad \| u^\varepsilon \|_{W^{2,\infty}} = O(\frac{1}{\varepsilon}), \quad \| \text{cof}(D^2 u^\varepsilon) \|_{L^\infty} = O(\frac{1}{\varepsilon})
\]

for \( j = 2, 3 \). Where \( \text{cof}(D^2 u^\varepsilon) \) denotes the cofactor matrix of the Hessian \( D^2 u^\varepsilon \). With the help of the vanishing moment method, the difficult task of computing the unique convex viscosity solution of the fully nonlinear second order Monge-Ampère problem (1.2)–(1.3), which has multiple solutions (i.e. there are non-convex solutions), is now reduced to a feasible task of computing the unique regular solution of the quasilinear fourth order problem (1.7)–(1.9). In particular, this allows one to use and/or adapt the wealthy amount of existing numerical methods, in particular, finite element methods to solve problem (1.2)–(1.3) via problem (1.7)–(1.9).

The specific goal of this paper is to develop and analyze Galerkin methods for approximating the solution of (1.7)–(1.9) in 2-D and 3-D. When deriving error estimates of the proposed numerical methods, we are particularly interested in obtaining error bounds that show explicit dependence on \( \varepsilon \). Aygris confirming finite element method in 2-D and Legendre spectral Galerkin method in both 2-D and 3-D are specifically considered in the paper although our analysis applies to any conforming Galerkin method for problem (1.7)–(1.9). We note that finite element approximations of fourth order PDEs, in particular, the biharmonic equation, were carried out extensively in 1970’s in the two-dimensional case (see [10] and the references therein),and have attracted renewed interests lately for generalizing the well-know 2-D finite elements to the 3-D case (cf. [34, 35, 33]) and for developing discontinuous Galerkin methods in all dimensions (cf. [16, 28]). Clearly, all these methods can be readily adapted to discretize problem (1.7)–(1.9) although their convergence analysis do not come easy due to the strong nonlinearity of the PDE (1.7). Also, the standard perturbation technique for deriving error estimates for numerical approximations of mildly nonlinear problems does not work for problem (1.7)–(1.9). We refer the reader to [18, 29] for further discussions in this direction.

We also like to mention that a few results on numerical approximations of the Monge-Ampère equation as well as related equations have recently been reported in the literature. Oliker and Prussner [31] constructed a finite difference scheme for computing Aleksandrov measure induced by \( D^2 u \) in 2-D and obtained the solution \( u \) of problem (1.7)–(1.9) as a by-product. Baginski and Whitaker [2] proposed a finite difference scheme for Gauss curvature equation (cf. [17] and the references therein) in 2-D by mimicking the unique continuation method (used to prove existence of
the PDE) at the discrete level. In a series of papers (cf. [13] and the references therein) Dean and Glowinski proposed an augmented Lagrange multiplier method and a least squares method for problem (1.7)–(1.9) and the Pucci’s equation (cf. [7, 21]) in 2-D by treating the Monge-Ampère equation and Pucci’s equation as a constraint and using a variational criterion to select a particular solution. Very recently, Oberman [30] constructed some wide stencil finite difference scheme which fulfill the convergence criterion established by Barles and Souganidis in [3] for finite difference approximations of fully nonlinear second order PDEs. Consequently, the convergence of the proposed wide stencil finite difference scheme immediately follows from the general convergence framework of [3]. Numerical experiments results were reported in [31, 30, 2, 13], however, convergence analysis was not addressed except in [30].

The remainder of this paper is organized as follows. In Section 2, we first derive the weak formulation for problem (1.7)-(1.9) and then present our confirming finite element and spectral Galerkin methods based on this weak formulation. In Section 3, we study the linearization of problem (1.7)–(1.9) and its Galerkin approximations. The results of this section, which are of independent interests in themselves, will play an important role in our error analysis for the numerical method introduced in Section 2. In Section 4, we establish optimal order error estimates in the energy norm for the proposed confirming finite element and spectral Galerkin methods. Our main ideas are to use a fixed point technique and to make strong use of the stability property of the linearized problem which is analyzed in Section 3. In addition, we derive the optimal order error estimate in the $L^2$-norm for $u^\varepsilon - u_0^\varepsilon$ using a duality argument. Finally, in Section 5, we first run some numerical tests to validate our theoretical error estimate results. We then present a detailed computational study for determining the “best” choice of mesh size $h$ in terms of $\varepsilon$ in order to achieve the optimal rates of convergence and for estimating the rates of convergence for both $u^0 - u_0^\varepsilon$ and $u^0 - u^\varepsilon$ in terms of powers of $\varepsilon$.

2. Formulation of Galerkin Methods

Standard space notations are adopted in this paper, we refer to [6, 21, 10] for their exact definitions. In addition, $\Omega$ denotes a bounded domain in $\mathbb{R}^n$. $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ denote the $L^2$-inner products on $\Omega$ and on $\partial\Omega$, respectively. $C$ is used to denote a generic $\varepsilon$-independent positive constant. We also introduce the following special space notation:

$V := H^2(\Omega), \quad V_0 := H^2(\Omega) \cap H^1_0(\Omega), \quad V_g := \{ v \in V; v|_{\partial\Omega} = g \}$.

Testing (1.7) with $v \in V_0$ yields

$$\epsilon \int_{\Omega} \Delta u^\varepsilon \Delta v dx + \int_{\Omega} \det(D^2u^\varepsilon)v dx = \int_{\Omega} fv dx - \int_{\partial\Omega} \varepsilon^2 \frac{\partial v}{\partial n} ds.$$  (2.1)

Based on (2.1), we define the variational formulation of (1.7)–(1.9) as follows: Find $u^\varepsilon \in V_g$ such that

$$-\varepsilon(\Delta u^\varepsilon, \Delta v) + \langle \det(D^2u^\varepsilon), v \rangle = (f, v) - \left\langle \varepsilon^2, \frac{\partial v}{\partial n} \right\rangle_{\partial\Omega} \quad \forall v \in V_0.$$  (2.2)

**Remark 2.1.** We note that $\det(D^2u^\varepsilon) = \frac{1}{n} \Phi^\varepsilon D^2u^\varepsilon = \frac{1}{n} \sum_{i=1}^{n} \Phi^\varepsilon_{ij} u_{x_i x_j}$ $j = 1, 2, \ldots n$, where $\Phi^\varepsilon$ is the cofactor matrix of $D^2u^\varepsilon$. Thus, using the divergence free property
of the cofactor matrix $\Phi^\varepsilon$ (cf. Lemma 3.1) we can define the following alternative variational formulation for (2.2).

$$-\varepsilon(\Delta u^\varepsilon, \Delta \psi) - \frac{1}{n}(\Phi^\varepsilon D u^\varepsilon, D \psi) = \langle f, \psi \rangle - \left\langle \varepsilon^2 \frac{\partial \psi}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall \psi \in V_0.$$  

However, we shall not use the above weak formulation in this paper although it is interesting to compare Galerkin methods based on the two different but equivalent weak formulations.

In this paper, we shall consider two types of Galerkin approximations for (2.2). The first type is the confirming finite element Galerkin method in 2-D. Aygris finite element will be used as a specific example of this class of methods although our analysis is applicable to other confirming finite element such as Bell element, Bogner–Fox–Schmit element, and Hsieh–Clough–Tocher element (cf. [10]). The second type of the methods is the spectral Galerkin method in 2-D and 3-D. Legendre spectral Galerkin method will be used as a specific example although our analysis also applies to other spectral methods (with necessary assumptions on the domain in the case of Fourier spectral method).

To formulate the finite element method in 2-D, let $T_h$ be a quasuniform triangular or rectangular partition of $\Omega \subset \mathbb{R}^2$ with mesh size $h \in (0,1)$ and $V^h \subset V$ denote a confirming finite element space consisting of piecewise polynomial functions of degree $r \geq 5$ such that for any $v \in V \cap H^k(\Omega)$

$$\inf_{v_h \in V^h} \| v - v_h \|_{H^\ell} \leq h^{\ell-j} \| v \|_{H^j}, \quad j = 0, 1, 2; \quad \ell = \min\{r+1, s\}. \tag{2.4}$$

We recall that $r = 5$ in the case of Aygris element (cf. [10]).

Let

$$V^h_g = \{ v_h \in V^h; \ v_h|_{\partial \Omega} = g \}, \quad V^h_0 = \{ v_h \in V^h; \ v_h|_{\partial \Omega} = 0 \}.$$

Based on the weak formulation (2.2), we define our finite element Galerkin method as follows. Find $u^h_N \in V^h_g$ such that

$$-\varepsilon(\Delta u^h_N, \Delta v_h) + (\det(D^2 u^h_N), v_h) = \langle f, v_h \rangle - \left\langle \varepsilon^2 \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall v_h \in V^h_0. \tag{2.5}$$

To formulate the spectral Galerkin method, we assume that $\Omega$ is a rectangular domain and let $L_j$ denote the $j$th order Legendre polynomial of single variable and define the following finite dimensional spaces: For $N \geq 2$, let

$$V^N := \text{span}\{L_0(x_1), L_2(x_1), \cdots, L_N(x_1)\} \quad \text{when } n = 1,$$

$$V^N := \text{span}\{L_i(x_1)L_j(x_2); \ i, j = 1, 2, \cdots, N\} \quad \text{when } n = 2,$$

$$V^N := \text{span}\{L_i(x_1)L_j(x_2)L_k(x_3); \ i, j, k = 1, 2, \cdots, N\} \quad \text{when } n = 3.$$ 

It is well-known that $V^N$ has the following approximation property (cf. [5]):

$$\inf_{v_N \in V^N} \| v - v_N \|_{H^j} \leq C N^{\ell-j-s} \| v \|_{H^s}, \quad 0 \leq j \leq \min\{s, N+1\} \tag{2.6}$$

for any $v \in V \cap H^s(\Omega)$.

Again, based on the weak formulation (2.2), our spectral Galerkin method for (2.2) is defined as seeking $u^N_N \in V^N \cap V_0$ such that for any $v_N \in V^N \cap V_0$

$$-\varepsilon(\Delta u^N_N, \Delta v_N) + (\det(D^2 u^N_N), v_N) = \langle f, v_N \rangle - \left\langle \varepsilon^2 \frac{\partial v_N}{\partial \nu} \right\rangle_{\partial \Omega}. \tag{2.7}$$
Clearly, Galerkin methods (2.5) and (2.7) have the exact same form as the variational problem (2.2). The only difference is that the infinite dimensional space $V$ in (2.2) is replaced by the finite dimensional subspace $V^h$ and $V^N$, respectively. Letting $h := \frac{1}{N}$, $u^h := u^N$ and $V^h := V^N$ in the definition of the spectral method, (2.7) can be rewritten as the exactly same form as (2.5). For this reason, we shall abuse the notation by using $u^h$ to denote the solution of (2.5) and the solution of (2.7) with understanding that $h = \frac{1}{N}$. Since the convergence analyses for (2.5) and for (2.7) are essentially same, we shall only present the detailed analysis for (2.5) and make comments about that for (2.7) in case there is a meaningful difference.

Let $u_\varepsilon$ be the solution to (2.2) and $u_\varepsilon^h$ the solution of (2.5) or (2.7). As mentioned in Section 1, the main task of this paper is to derive optimal error estimates for $u_\varepsilon - u_\varepsilon^h$. To this end, we first need to prove existence and uniqueness of $u_\varepsilon^h$. It turns out both tasks are not easy due to the strong nonlinearity in (2.5) and (2.7). Unlike the continuous PDE case where $u_\varepsilon$ is proved to be convex for all $\varepsilon$ (cf. [17]), it is not clear whether $u_\varepsilon^h$ preserves the convexity even for small $h$. Without a guarantee of convexity for $u_\varepsilon^h$, we could not establish any stability result for $u_\varepsilon^h$. This is the main obstacle for proving existence and uniqueness for (2.5) and (2.7). In addition, again due to the strong nonlinearity, the standard perturbation technique for deriving error estimate for numerical approximations of mildly nonlinear problems does not work here. To overcome the difficulty, our idea is to adopt a combined fixed point and linearization technique which was used by the authors in [19], where a nonlinear singular second order problem known as the inverse mean curvature flow was studied. We note that by using this technique we are able to simultaneously prove existence and uniqueness for $u_\varepsilon^h$ and also derive the desired error estimates in the next two sections, we shall give the detailed account of this technique and apply it to problems (2.5) and (2.7).

3. Linearization and its Finite Element Approximation

To analyze (2.5) and (2.7), we shall study the linearization of (1.7) to establish the required technical tools. First, we recall the following divergence-free row property for the cofactor matrices, which will be used frequently in later sections. We refer the reader to [14] p. 440 for a short proof of the lemma.

**Lemma 3.1.** Given a vector-valued function $v = (v_1, v_2, \cdots, v_n) : \Omega \rightarrow \mathbb{R}^n$. Assume $v \in [C^2(\Omega)]^n$. Then the cofactor matrix $\text{cof}(Dv)$ of the gradient matrix $Dv$ of $v$ satisfies the following row divergence-free property:

\begin{equation}
\text{div}(\text{cof}(Dv))_i = \sum_{j=1}^{n} \partial_{x_j}(\text{cof}(Dv))_{ij} = 0 \quad \text{for } i = 1, 2, \cdots, n,
\end{equation}

where $(\text{cof}(Dv))_i$ and $(\text{cof}(Dv))_{ij}$ denote respectively the $i$th row and the $(i, j)$-entry of $\text{cof}(Dv)$.

3.1. Linearization. It is easy to check that for a given smooth function $w$ there holds

\begin{equation}
det(D^2u_\varepsilon + tw) = det(D^2u_\varepsilon) + t\text{tr}(\Phi_\varepsilon D^2w) + \cdots + t^n \det(D^2w).
\end{equation}
The linearization of \( M^\varepsilon(u^\varepsilon) := -\varepsilon \Delta^2 u^\varepsilon + \text{det}(D^2 u^\varepsilon) \) at the solution \( u^\varepsilon \) is given by

\[
L_{u^\varepsilon}(w) := \lim_{t \to 0} \frac{M^\varepsilon(u^\varepsilon + tw) - M^\varepsilon(u^\varepsilon)}{t} = -\varepsilon \Delta^2 w + \Phi^\varepsilon : D^2 w = -\varepsilon \Delta^2 w + \text{div}(\Phi^\varepsilon D w),
\]

where \( \Phi^\varepsilon \) denotes the cofactor matrix of \( D^2 u^\varepsilon \) and we have used Lemma 3.1 to get the last equality. Also, using Lemma 3.1 it is easy to check that \( L_{u^\varepsilon} \) is self-adjoint, i.e., the adjoint operator \( L_{u^\varepsilon}^* \) of \( L_{u^\varepsilon} \) coincides with \( L_{u^\varepsilon} \).

We now consider the following linear problem.

\[
(3.3) \quad L_{u^\varepsilon}(v) = \varphi \quad \text{in } \Omega, \\
(3.4) \quad v = 0 \quad \text{on } \partial \Omega, \\
(3.5) \quad \Delta v = q \quad \text{on } \partial \Omega.
\]

Multiplying the PDE by a test function \( w \in V_0 \) and integrating over \( \Omega \) we get the following weak formulation for (3.3)–(3.5): Find \( v \in V_0 \) such that

\[
(3.6) \quad B[v, w] = \langle \varphi, w \rangle - \varepsilon \left\langle q, \frac{\partial w}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall w \in V_0,
\]

where

\[
B[v, w] := \varepsilon \int_{\Omega} \Delta v \Delta w \, dx + \int_{\Omega} \Phi^\varepsilon Dv \cdot Dw \, dx.
\]

The next theorem ensures the well-posedness of the above variational problem.

**Theorem 3.2.** Suppose \( \partial \Omega \in C^2 \). Then for every \( \varphi \in V_0^* \) and \( q \in H^{-\frac{1}{2}}(\partial \Omega) \) there exists a unique solution \( v \in V_0 \) to problem (3.6). Furthermore, there exists a positive constant \( C_1(\varepsilon) \) such that

\[
(3.7) \quad \|v\|_{H^2} \leq C_1(\varepsilon) \left( \|\varphi\|_{(H^1(\Omega))^*} + \|q\|_{H^{-\frac{1}{2}}(\partial \Omega)} \right).
\]

**Proof.** It is easy to check that \( B[\cdot, \cdot] \) is a bounded and coercive bilinear form on \( V_0 \times V_0 \) with coercive constant \( C_2(\varepsilon) = O(\varepsilon) \). Also, the right-hand side of (3.6) defines a bounded linear functional on \( V_0 \) (see [29] for details). So the assertions of the theorem follows immediately from an application of Lax-Milgram Theorem [21]. \( \square \)

For more regular data, the above theorem can be improved to the following one (see [23] for a similar proof).

**Theorem 3.3.** Suppose \( \varphi \in (H^s(\Omega) \cap H^1_0(\Omega))^* \), \( q \in H^{s-\frac{1}{2}}(\partial \Omega) \), \( \partial \Omega \in C^s \), \( (s \geq 2) \) and \( v \) is the unique solution to (3.6). Then \( v \in H^s(\Omega) \cap H^1_0(\Omega) \), and there exists a \( C_s(\varepsilon) > 0 \) such that

\[
(3.8) \quad \|v\|_{H^s} \leq C_s(\varepsilon) \left( \|\varphi\|_{(H^1(\Omega))^*} + \|q\|_{H^{s-\frac{1}{2}}(\partial \Omega)} \right).
\]

**Remark 3.1.** From the definition of \( C_2(\varepsilon) \), we see that in Theorem 3.2, \( C_1(\varepsilon) = O\left(\frac{1}{\varepsilon}\right) \). Currently, the explicit dependence of \( C_s(\varepsilon) \) on \( \varepsilon \) in Theorem 3.3 remains unknown for general \( s \geq 3 \). Finally, we note that Theorem 3.2 can easily be extended to the case of nonhomogeneous boundary data which we summarize in the following theorem.
Theorem 3.4. For every \( \varphi \in V^* \), \( g \in H^{\frac{3}{2}}(\partial \Omega) \), \( q \in H^{-\frac{1}{2}}(\partial \Omega) \), there exists a unique weak solution \( w \in V \) to
\[
L_w^*(w) = \varphi \quad \text{in } \Omega, \\
w = q \quad \text{on } \partial \Omega, \\
\Delta w = q \quad \text{on } \partial \Omega.
\]
Furthermore, there exists \( C > 0 \) such that
\[
\|w\|_{H^2} \leq C \left( \|\varphi\|_{(H^3_0 \cap H^3)(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\partial \Omega)} + \|q\|_{H^{-\frac{1}{2}}(\partial \Omega)} \right). 
\]

We refer the reader to [23] for a similar proof.

3.2. Finite element approximation of linearized problem. Let \( V_0^h \subset V_0 \) be one of the finite-dimensional subspace of \( V_0 \) as defined in the previous subsection (e.g. Aygris finite element and Legendre spectral element), and \( v \in V_0 \) denote the solution of (3.6). Based on the variational equation (3.6), our Galerkin method for (3.3) is defined as seeking \( v_h \in V_0^h \) such that
\[
B[v_h, w_h] = \langle \varphi, w_h \rangle - \varepsilon \left( q, \frac{\partial w_h}{\partial \nu} \right)_{\partial \Omega} \quad \forall w_h \in V_0^h. 
\]

Our objective in this subsection is to first prove existence and uniqueness for problem (3.10) and then derive optimal order error estimates in various norms.

Theorem 3.5. Suppose \( v \in V_0 \cap H^s(\Omega) \) (\( s \geq 3 \)). Then there exists a unique \( v_h \in V_0^h \) satisfying (3.10). Furthermore, we have the following estimates:
\[
\|v_h\|_{H^2(\Omega)} \leq C_3(\varepsilon) \left( \|\varphi\|_{(H^3_0 \cap H^3)(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\partial \Omega)} \right),
\]
\[
\|v - v_h\|_{H^2(\Omega)} \leq C_4(\varepsilon) h^{\ell-2} \|v\|_{H^\ell(\Omega)},
\]
\[
\|v - v_h\|_{H^\ell(\Omega)} \leq C_5(\varepsilon) h^{\ell-1} \|v\|_{H^\ell(\Omega)},
\]
\[
\|v - v_h\|_{L^2(\Omega)} \leq C_6(\varepsilon) h^\ell \|v\|_{H^\ell(\Omega)},
\]
where \( \ell = \min\{r + 1, s\} \) in the case of the finite element Galerkin method, and \( \ell = \min\{N + 1, s\} \) in the case of the spectral Galerkin method.

Proof. Estimate (3.11) follows immediately from setting \( w_h = v_h \) in (3.10) and using the coercivity of the bilinear form \( B[\cdot, \cdot] \).

To derive the error estimate in the \( H^2 \)-norm, we use the error equation,
\[
B[v - v_h, w_h] = 0 \quad \forall w_h \in V_0^h.
\]
Using the coercivity property of \( B[\cdot, \cdot] \), we have
\[
C_2(\varepsilon) \|v - v_h\|_{H^2} \leq B[v - v_h, v - v_h] = B[v - v_h, v] - B[v - v_h, v_h] = B[v - v_h, v] = B[v - v_h, v - I_h v],
\]
where \( C_2(\varepsilon) = O(\varepsilon) \), \( I_h v \) denotes the Galerkin interpolant of \( v \) onto \( V_0^h \). On noting that
\[
B(v - v_h, v - I_h v) \leq \frac{C}{\sqrt{\varepsilon}} \|v - v_h\|_{H^2} \|v - I_h v\|_{H^2},
\]
we have
\[
\|v - v_h\|_{H^2} \leq \frac{C}{C_2(\varepsilon) \sqrt{\varepsilon}} \|v - I_h v\|_{H^2} \leq C_4(\varepsilon) h^{\ell-2} \|v\|_{H^\ell},
\]
where \( C_2(\varepsilon) \geq C_3(\varepsilon) \geq C_4(\varepsilon) \geq C_5(\varepsilon) \geq C_6(\varepsilon) \geq 0 \).
where \( C_4(\varepsilon) = O\left(\varepsilon^{-\frac{3}{2}}\right) \). Thus, (3.12) holds.

Next, we derive the \( H^1 \)-norm error estimate using a duality argument. Define \( e_h := v - v_h \) and consider the following problem:

\[
\begin{align*}
L_{u^*}(\psi) &= \Delta e_h & \text{in } \Omega, \\
\psi &= 0 & \text{on } \partial\Omega, \\
\Delta \psi &= 0 & \text{on } \partial\Omega.
\end{align*}
\]

Using (3.8), we have

\[
\|\psi\|_{H^3} \leq C_s(\varepsilon)\|\Delta e_h\|_{H^{-1}}.
\]

Since \( \|\Delta e_h\|_{H^{-1}} = \sup\{\langle \Delta e_h, u \rangle | \ u \in H_0^1(\Omega), \|u\|_{H_0^1} \leq 1\} \), we have

\[
\langle \Delta e_h, u \rangle = \langle \nabla e_h, \nabla u \rangle \leq \|\nabla e_h\|_{L^2}\|\nabla u\|_{L^2} \leq \|\nabla e_h\|_{L^2}\|u\|_{H_0^1} = \|\nabla e_h\|_{L^2}.
\]

It follows that

\[
\|\Delta e_h\|_{H^{-1}} \leq \|\nabla e_h\|_{L^2}.
\]

Thus,

\[
\|\nabla e_h\|_{L^2}^2 = \langle \Delta e_h, e_h \rangle = B[e_h, \psi] = B[e_h, \psi - I_h \psi] \\
\leq \frac{C}{\sqrt{\varepsilon}}\|\psi - I_h \psi\|_{H^2}\|e_h\|_{H^2} \leq \frac{hC}{\sqrt{\varepsilon}}\|\psi\|_{H^3}\|e_h\|_{H^2} \\
\leq \frac{hC_s(\varepsilon)}{\sqrt{\varepsilon}}\|\Delta e_h\|_{H^{-1}}\|e_h\|_{H^2} \leq \frac{hC_s(\varepsilon)}{\sqrt{\varepsilon}}\|\nabla e_h\|_{L^2}\|e_h\|_{H^2}.
\]

Hence,

\[
\|\nabla e_h\|_{L^2} \leq \frac{hC_s(\varepsilon)}{\sqrt{\varepsilon}}\|e_h\|_{H^2}.
\]

Combining the above with (3.12), we get (3.13) with \( C_5 = O\left(C_s(\varepsilon)^{-\frac{3}{2}}\right) \).

To derive the error in the \( L^2 \)-norm, we consider the following problem:

\[
\begin{align*}
L_{u^*}(\psi) &= e_h & \text{in } \Omega, \\
\psi &= 0 & \text{on } \partial\Omega, \\
\Delta \psi &= 0 & \text{on } \partial\Omega.
\end{align*}
\]

On noting (3.8) implies that

\[
\|\psi\|_{H^2} \leq C_s(\varepsilon)\|e_h\|_{L^2}.
\]

Thus,

\[
\|e_h\|_{L^2}^2 = (e_h, v - v_h) = B[v - v_h, \psi] = B[v - v_h, \psi - I_h \psi] \\
\leq \frac{C}{\sqrt{\varepsilon}}\|v - v_h\|_{H^2}\|\psi - I_h \psi\|_{H^2} \\
\leq \frac{h^2C}{\sqrt{\varepsilon}}\|v - v_h\|_{H^2}\|\psi\|_{H^4} \leq \frac{h^2C_s(\varepsilon)}{\sqrt{\varepsilon}}\|v - v_h\|_{H^2}\|e_h\|_{L^2}.
\]

Dividing by \( \|e_h\|_{L^2} \), we get (3.14) with \( C_6 = O\left(C_s(\varepsilon)^{-\frac{3}{2}}\right) \). The proof is complete. \( \square \)

**Remark** (a) In the case of Aygris finite element method, \( r = 5 \) in (3.11)–(3.14).

(b) In the case of Legendre spectral method, \( h = \frac{1}{N} \) in (3.11)–(3.14), where \( N \) stands for the highest degree of polynomials in \( V^N \).
4. Error Analysis for Galerkin Methods \((2.5)\) and \((2.7)\)

The goal of this section is to derive optimal order error estimates for the Galerkin methods \((2.5)\) and \((2.7)\). Our main idea is to use a combined fixed point and linearization technique which was introduced by the authors in [19]. Once again, we only present the detailed analysis for \((2.5)\) since the analysis for \((2.7)\) is essentially same.

First, we define a linear operator \(T : V^h \to V^h\). For any \(w_h \in V^h\), let \(T(w_h) \in V^h\) denote the solution of following problem:

\[
B[w_h - T(w_h), \psi_h] = \varepsilon(\Delta w_h, \Delta \psi_h) - (\det(D^2 w_h), \psi_h) + (f, \psi_h) - \left(\varepsilon, \frac{\partial \psi_h}{\partial \nu}\right)_{\partial \Omega} \quad \forall \psi_h \in V^h_0.
\]

By Theorem 3.5, we see \(T(w_h)\) is uniquely defined. Notice that the right-hand side of \((4.1)\) is the residual of \(w_h\) to equation \((2.5)\). It is easy to see that any fixed point \(w_h\) of the mapping \(T\) (i.e. \(T(w_h) = w_h\)) is a solution to problem \((2.5)\) and vice-versa. In the following we shall show that the mapping \(T\) indeed has a unique fixed point in a small neighborhood of \(I_h u^\varepsilon\). To this end, we set

\[
B_h(\rho) := \{v_h \in V^h; ||v_h - I_h u^\varepsilon||_{H^2} \leq \rho\}.
\]

In the rest of the section, we assume \(u^\varepsilon \in H^s(\Omega)\) and set \(\ell = \min\{r + 1, s\}\).

**Lemma 4.1.** There exists a constant \(C_\gamma(\varepsilon) = O(\varepsilon^{-n}) > 0\) \((n = 2, 3)\) such that

\[
||I_h u^\varepsilon - T(I_h u^\varepsilon)||_{H^2} \leq C_\gamma(\varepsilon) h^{\ell-2} ||u^\varepsilon||_{H^r}.
\]

**Proof.** To simplify notation, let \(\omega_h := I_h u^\varepsilon - T(I_h u^\varepsilon)\) and let \(\eta_h := I_h u^\varepsilon - u^\varepsilon\). We then have

\[
B[\omega_h, \omega_h] = \varepsilon(\Delta(I_h u^\varepsilon), \Delta \omega_h) - (\det(D^2(I_h u^\varepsilon)) - f, \omega_h) - \left(\varepsilon, \frac{\partial \omega_h}{\partial \nu}\right)_{\partial \Omega}
\]

\[
= \varepsilon(\Delta \eta_h, \Delta \omega_h) + (\det(D^2 u^\varepsilon) - (\det(D^2(I_h u^\varepsilon)), \omega_h)
\]

\[
\leq \varepsilon||\Delta \eta_h||_{L^2}||\Delta \omega_h||_{L^2} + ||\det(D^2 u^\varepsilon) - \det(D^2(I_h u^\varepsilon))||_{L^2}||\omega||_{L^2}.
\]

Using the Mean Value Theorem we have

\[
\det(D^2(I_h u^\varepsilon)) - \det(D^2 u^\varepsilon) = \tilde{\Phi} : D^2 \eta_h,
\]

where \(\tilde{\Phi} = \text{cof}(\tau D^2(I_h u^\varepsilon) + (1-\tau)D^2 u^\varepsilon)\) for some \(\tau \in [0, 1]\).

Next, when \(n = 2\), we bound \(||\tilde{\Phi}||_{L^\infty}\) as follows:

\[
||\tilde{\Phi}||_{L^\infty} = ||\text{cof}(\tau D^2(I_h u^\varepsilon) + (1-\tau)D^2 u^\varepsilon)||_{L^\infty}
\]

\[
= ||\tau D^2(I_h u^\varepsilon) + (1-\tau)D^2 u^\varepsilon||_{L^\infty}
\]

\[
\leq ||D^2(I_h u^\varepsilon)||_{L^\infty} + ||D^2 u^\varepsilon||_{L^\infty}
\]

\[
\leq C ||D^2 u^\varepsilon||_{L^\infty} \leq \frac{C}{\varepsilon}.
\]

Similarly, when \(n = 3\), we can show that \(||\tilde{\Phi}||_{L^\infty} = O(\varepsilon^{-2})\). Hence,

\[
B[\omega_h, \omega_h] \leq \varepsilon||\Delta \eta_h||_{L^2}||\Delta \omega_h||_{L^2} + \frac{C}{\varepsilon^{n-1}} ||D^2 \eta_h||_{L^2}||\omega_h||_{L^2}
\]

\[
\leq \frac{C}{\varepsilon^{n-1}} ||\eta_h||_{H^2}||\omega_h||_{H^2}.
\]
Using the coercivity of the bilinear form $B[\cdot, \cdot]$ we get

$$\|\omega_h\|_{H^2} \leq \frac{C}{\varepsilon^{n-1}C_2(\varepsilon)} \|\eta_h\| \leq \frac{C}{\varepsilon^{n-1}C_2(\varepsilon)} h^{\ell-2} \|u^\varepsilon\|_{H^2}.$$ 

Thus, (4.3) holds with $C_7(\varepsilon) = \frac{C}{\varepsilon^{n-1}C_2(\varepsilon)} = O(\varepsilon^{-n})$. \hfill $\square$

**Lemma 4.2.** There exists an $h_0 > 0$ and $0 < \rho_0 < 1$ such that for $h \leq h_0$, the mapping $T$ is a contracting mapping in the ball $B_h(\rho_0)$ with a contraction factor $\frac{1}{2}$. That is, for any $v_h, w_h \in B_h(\rho_0)$, there holds

$$\|T(v_h) - T(w_h)\|_{H^2} \leq \frac{1}{2} \|v_h - w_h\|_{H^2}.$$ 

**Proof.** For any $\psi_h \in V_0^h$, using the definition of $T(v_h)$ and $T(w_h)$ we get

$$B[T(v_h) - T(w_h), \psi_h] = (\Phi^\varepsilon(Dv_h - Dw_h), D\psi_h) + (\det(D^2v_h) - \det(D^2w_h), \psi_h)$$

Let $v_h^\mu, w_h^\mu$ denote the standard mollifications of $v_h$ and $w_h$, respectively. Adding and subtracting these terms and using the Mean Value Theorem yield

$$B[T(v_h) - T(w_h), \psi_h] = (\Phi^\varepsilon(Dv_h - Dw_h), D\psi_h) + (\det(D^2v_h) - \det(D^2w_h), \psi_h)$$

where $\Psi_h = \text{cof}(D^2v_h^\mu + \tau(D^2w_h^\mu - D^2v_h^\mu))$, $\tau \in [0, 1]$.

Using Lemma 3.1 we have

$$B[T(v_h) - T(w_h), \psi_h] \leq C \left\{ \|\Phi^\varepsilon - \Psi_h\|_{L^2} \|v_h - w_h\|_{H^2} \|\psi_h\|_{H^2} + \|\Psi_h\|_{L^2} \|\psi_h\|_{H^2} \|v_h - v_h^\mu\|_{H^2} 
+ \|w_h - w_h^\mu\|_{H^2} + \|\det(D^2v_h) - \det(D^2v_h^\mu)\|_{L^2} 
+ \|\det(D^2w_h) - \det(D^2w_h^\mu)\|_{L^2} \|\psi_h\|_{L^2} \right\},$$

where we have used Sobolev’s inequality.

Next, we derive an upper bound for $\|\Phi^\varepsilon - \Psi_h\|_{L^2}$ when $n=2$ as follows:

$$\|\Phi^\varepsilon - \Psi_h\|_{L^2} = \|\Phi^\varepsilon - \text{cof}(D^2v_h^\mu + \tau(D^2w_h^\mu - D^2v_h^\mu))\|_{L^2}$$

$$\leq 2\|D^2v_h^\mu - (D^2v_h^\mu + \tau(D^2w_h^\mu - D^2v_h^\mu))\|_{L^2}$$

$$\leq C h^{\ell-2} \|u^\varepsilon\|_{H^2} + 2\rho_0 + 2\|v_h - v_h^\mu\|_{H^2} + \|w_h - w_h^\mu\|_{H^2}.$$
Similarly, when $n=3$, we can show
\[ \| \Phi^\varepsilon - \Psi_h \|_{L^2} \leq \frac{C}{\varepsilon} \left( h^{\ell-2} \| u^\varepsilon \|_{H^\ell} + 2\rho_0 + 2\| v_h - u_h^\varepsilon \|_{H^2} + \| w_h - u_h^\varepsilon \|_{H^2} \right). \]

Using this result above, we have
\[ B[T(v_h) - T(w_h), \psi_h] \leq \frac{C}{\varepsilon^{n-2}} \left\{ \left( h^{\ell-2} \| u^\varepsilon \|_{H^\ell} + \rho_0 + \| v_h - u_h^\varepsilon \|_{H^2} + \| w_h - u_h^\varepsilon \|_{H^2} \right) \| v_h - w_h \|_{H^2} \\
+ \| \Psi_h \|_{L^2} \left( \| v_h - u_h^\varepsilon \|_{H^2} + \| w_h - u_h^\varepsilon \|_{H^2} \right) \\
+ \| \det(D^2v_h) - \det(D^2u_h^\varepsilon) \|_{L^2} + \| \det(D^2w_h) - \det(D^2u_h^\varepsilon) \|_{L^2} \right\} \| \psi_h \|_{H^2}. \]

Setting $\mu \to 0$ yields
\[ B[T(v_h) - T(w_h), \psi_h] \leq \frac{C}{\varepsilon^{n-2}} \left( h^{\ell-2} \| u^\varepsilon \|_{H^\ell} + \rho_0 \right) \| v_h - w_h \|_{H^2} \| \psi_h \|_{H^2}. \]

Using the coercivity of the bilinear form $B[\cdot, \cdot]$ we get
\[ \| T(v_h) - T(w_h) \|_{H^2} \leq \frac{C}{C_2(\varepsilon)^{n-2}} \left( h^{\ell-2} \| u^\varepsilon \|_{H^\ell} + \rho_0 \right) \| v_h - w_h \|_{H^2}. \]

Choosing $\rho_0 = \frac{C_2(\varepsilon)^{n-2}}{C} h^{\ell-2} \| u^\varepsilon \|_{H^\ell}$ and $h_0 = \left( \frac{C_2(\varepsilon)^{n-2}}{C\| u^\varepsilon \|_{H^\ell}} \right)^{\frac{1}{\ell+2}}$, then for $h \leq h_0$ we have
\[ \| T(v_h) - T(w_h) \|_{H^2} \leq \frac{1}{2} \| v_h - w_h \|_{H^2}. \]

The proof is complete. \hfill \Box

**Theorem 4.3.** Let $\rho_1 = 2C_2(\varepsilon)h^{\ell-2} \| u^\varepsilon \|_{H^\ell}$. Then there exists an $h_1 > 0$ such that for $h \leq \min\{h_0, h_1\}$, there exists a unique solution $u_h^\varepsilon$ of (2.5) in the ball $B_h(\rho_1)$. Moreover, there exists a constant $C_8(\varepsilon) = O(\varepsilon^{-n})$ such that
\[ \| u^\varepsilon - u_h^\varepsilon \|_{H^2} \leq C_8(\varepsilon)h^{\ell-2} \| u^\varepsilon \|_{H^\ell}, \quad \ell = \min\{r + 1, s\}. \]

**Proof.** Let $v_h \in B_h(\rho_1)$ and $h_1 = \left( \frac{C_2(\varepsilon)^{n-2}}{\| u^\varepsilon \|_{H^\ell}} \right)^{\frac{1}{\ell+2}}$. Then $h \leq \min\{h_0, h_1\}$ implies that $\rho_1 \leq \rho_0$. Thus, using the triangle inequality and Lemmas 1.1 and 4.2 we have
\[ \| I_h u^\varepsilon - T(v_h) \|_{H^2} \leq \| I_h u^\varepsilon - T(I_h u^\varepsilon) \|_{H^2} + \| T(I_h u^\varepsilon) - T(v_h) \|_{H^2} \leq C_7(\varepsilon)h^{\ell-2} \| u^\varepsilon \|_{H^\ell} + \frac{1}{2} \| I_h u^\varepsilon - v_h \|_{H^2} \leq \frac{\rho_1}{2} + \frac{\rho_1}{2} = \rho_1. \]

Hence, $T(v_h) \in B_h(\rho_1)$. In addition, from Lemma 4.2 we know that $T$ is a contracting mapping. Thus, the Brouwer fixed theorem [21] infers that $T$ has a unique fixed point $u_h^\varepsilon \in B_h(\rho_1)$, which is the unique solution to (2.5).

To get the error estimate, we use the triangle inequality to get
\[ \| u^\varepsilon - u_h^\varepsilon \|_{H^2} \leq \| u^\varepsilon - I_h u^\varepsilon \|_{H^2} + \| I_h u^\varepsilon - u_h^\varepsilon \| \leq Ch^{\ell-2} \| u^\varepsilon \|_{H^\ell} + \rho_1 = C_8(\varepsilon)h^{\ell-2} \| u^\varepsilon \|_{H^\ell}, \]

where $C_8(\varepsilon) := CC_7(\varepsilon) = O(\varepsilon^{-n})$. The proof is complete. \hfill \Box
Theorem 4.4. In addition to the hypothesis of Theorem 4.3, assume that the linearized equation is $H^4$-regular with the regularity constant $C_s(\varepsilon)$. Then there holds

\begin{equation}
\|u^\varepsilon - u^\varepsilon_h\|_{L^2} \leq C_0(\varepsilon) \left[ \frac{h^5}{\sqrt{\varepsilon}} \|u^\varepsilon\|_{H^4} + \varepsilon^{2-n} C_8(\varepsilon) h^{2n-4} \|u^\varepsilon\|_{H^4}^2 \right],
\end{equation}

where $C_0(\varepsilon) = C_s(\varepsilon) C_C(\varepsilon)$.

Proof. Let $e^\varepsilon_h := u^\varepsilon - u^\varepsilon_h$ and $u^\varepsilon_h$ denote a standard mollification of $u^\varepsilon_h$. It is easy to verify that $e^\varepsilon_h$ satisfies the following error equation:

$$
\varepsilon (\Delta e^\varepsilon_h, \Delta \psi_h) + (\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h), \psi_h) = 0 \quad \forall \psi_h \in V^h_0.
$$

Using the Mean Value Theorem and Lemma 3.1 we have

$$
0 = \varepsilon (\Delta e^\varepsilon_h, \Delta \psi_h) + (\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h), \psi_h) + (\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h), \psi_h)
$$

where $\Phi = \text{cof}(D^2 u^\varepsilon_h + \tau (D^2 u^\varepsilon_h - D^2 u^\varepsilon_h)), \tau \in [0, 1]$.

Next, the $H^4$-regular assumption implies that (cf. Theorem 3.2) there exists a unique solution $\psi$ to the following problem:

\begin{align*}
L^\varepsilon(\psi) &= e^\varepsilon_h \quad \text{in } \Omega, \\
\psi &= 0 \quad \text{on } \partial \Omega, \\
\Delta \psi &= 0 \quad \text{on } \partial \Omega.
\end{align*}

Moreover, there holds

\begin{equation}
\|\psi\|_{H^4} \leq C_s(\varepsilon) \|e^\varepsilon_h\|_{L^2}.
\end{equation}

Thus,

\begin{align*}
\|e^\varepsilon_h\|_{L^2}^2 &= \langle e^\varepsilon_h, e^\varepsilon_h \rangle = \varepsilon (\Delta e^\varepsilon_h, \Delta \psi) + (\Phi^\varepsilon D\psi, D e^\varepsilon_h) \\
&= \varepsilon (\Delta e^\varepsilon_h, \Delta (\psi - I_h \psi)) + (\Phi^\varepsilon D e^\varepsilon_h, D (\psi - I_h \psi)) + \varepsilon (\Delta e^\varepsilon_h, \Delta (I_h \psi)) \\
&\quad + (\Phi^\varepsilon D e^\varepsilon_h, D (I_h \psi)) - \varepsilon (\Delta e^\varepsilon_h, \Delta (I_h \psi)) - (\Phi^\varepsilon D (u^\varepsilon - u^\varepsilon_h), D (I_h \psi)) \\
&\quad - (\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h), I_h \psi) \\
&= \varepsilon (\Delta e^\varepsilon_h, \Delta (\psi - I_h \psi)) + (\Phi^\varepsilon D e^\varepsilon_h, D (\psi - I_h \psi)) \\
&\quad + (\Phi^\varepsilon D e^\varepsilon_h - \Phi^\varepsilon D (u^\varepsilon - u^\varepsilon_h), D (I_h \psi)) - (\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h), I_h \psi) \\
&\quad = \varepsilon (\Delta e^\varepsilon_h, \Delta (\psi - I_h \psi)) + (\Phi^\varepsilon D e^\varepsilon_h, D (\psi - I_h \psi)) + (\Phi^\varepsilon D (u^\varepsilon - u^\varepsilon_h), D (I_h \psi)) \\
&\quad + (\Phi^\varepsilon D (u^\varepsilon - u^\varepsilon_h), D (I_h \psi)) + (\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h), I_h \psi) \\
&\leq \|\Delta e^\varepsilon_h\|_{L^2} \|\psi - I_h \psi\|_{L^2} + C \|\Phi^\varepsilon\|_{L^2} \|e^\varepsilon_h\|_{H^2} \|\psi - I_h \psi\|_{H^2} \\
&\quad + C \|\Phi^\varepsilon - \Phi\|_{L^2} \|e^\varepsilon_h\|_{H^2} \|I_h \psi\|_{H^2} + C \|\Phi\|_{L^2} \|u^\varepsilon_h - u^\varepsilon\|_{H^2} \|I_h \psi\|_{H^2} \\
&\quad + \|\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h)\|_{L^2} \|I_h \psi\|_{L^2},
\end{align*}

where we have used Sobolev’s inequality. Continuing, we have

\begin{equation}
\|e^\varepsilon_h\|_{L^2}^2 \leq C \left\{ \varepsilon h^2 \|e^\varepsilon_h\|_{H^2} + h^2 \|\Phi^\varepsilon\|_{L^2} \|e^\varepsilon_h\|_{H^2} + \|\Phi^\varepsilon - \Phi\|_{L^2} \|e^\varepsilon_h\|_{H^2} \\
&\quad + \|\Phi\|_{L^2} \|u^\varepsilon_h - u^\varepsilon\|_{L^2} + \|\det(D^2 u^\varepsilon_h) - \det(D^2 u^\varepsilon_h)\|_{L^2} \right\} \|\psi\|_{H^4},
\end{equation}
We now bound $\|\Phi^\varepsilon - \tilde{\Phi}\|_{L^2}$ for $n=2$ as follows:

\begin{equation}
\|\Phi^\varepsilon - \tilde{\Phi}\|_{L^2} = \|\text{cof}(D^2u^\varepsilon) - \text{cof}(D^2u^\varepsilon - \tau(D^2u^\varepsilon - D^2u^\varepsilon))\|_{L^2}
\end{equation}

\begin{equation*}
= \|D^2u^\varepsilon - D^2u^\varepsilon - \tau(D^2u^\varepsilon - D^2u^\varepsilon)\|_{L^2}
\end{equation*}

\begin{equation*}
\leq \|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2} + \|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2}
\end{equation*}

\begin{equation*}
+ \|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2} + \|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2}
\end{equation*}

\begin{equation*}
= 2\|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2} + 2\|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2}.
\end{equation*}

Similarly, when $n=3$, we have

\begin{equation}
\|\Phi^\varepsilon - \tilde{\Phi}\|_{L^2} \leq \frac{C}{\varepsilon}(\|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2} + \|D^2u^\varepsilon - D^3u^\varepsilon\|_{L^2}).
\end{equation}

Substituting (4.8)-(4.9) into (4.7) we obtain

\begin{equation*}
\|\epsilon_h^\varepsilon\|^2_{L^2} \leq C \left( \varepsilon h^2 \|e_h^\varepsilon\|_{H^2} + h^2 \|\Phi^\varepsilon\|_{L^2} \|e_h^\varepsilon\|_{H^2} + \varepsilon^{2-n} \|e_h^\varepsilon\|_{H^2} \|e_h^\varepsilon\|_{H^2} \right) \|\psi\|_{H^4}
\end{equation*}

\begin{equation*}
+ \|\Phi^\varepsilon\|_{L^2} \|\epsilon_h^\varepsilon\|_{H^2} + \|\Phi_h^\varepsilon\|_{L^2} \|\epsilon_h^\varepsilon\|_{H^2} + \|\epsilon_h^\varepsilon\|_{L^2} \|\epsilon_h^\varepsilon\|_{L^2} \|\epsilon_h^\varepsilon\|_{L^2}.
\end{equation*}

Setting $\mu = 0$ and using (4.6) yield

\begin{equation}
\|\epsilon_h^\varepsilon\|^2_{L^2} \leq C \left( \varepsilon h^2 \|e_h^\varepsilon\|_{H^2} + h^2 \|\Phi^\varepsilon\|_{L^2} \|e_h^\varepsilon\|_{H^2} + \varepsilon^{2-n} \|e_h^\varepsilon\|_{H^2} \|e_h^\varepsilon\|_{H^2} \right) \|\psi\|_{H^4}
\end{equation}

\begin{equation}
\leq C_s(\varepsilon) \left( \varepsilon h^2 \|e_h^\varepsilon\|_{H^2} + h^2 \|\Phi^\varepsilon\|_{L^2} \|e_h^\varepsilon\|_{H^2} + \varepsilon^{2-n} \|e_h^\varepsilon\|_{H^2} \|e_h^\varepsilon\|_{H^2} \right) \|e_h^\varepsilon\|_{L^2}.
\end{equation}

Hence,

\begin{equation}
\|e_h^\varepsilon\|_{L^2} \leq C_s(\varepsilon) \left( \varepsilon h^2 \|e_h^\varepsilon\|_{H^2} + h^2 \|\Phi^\varepsilon\|_{L^2} \|e_h^\varepsilon\|_{H^2} + \varepsilon^{2-n} \|e_h^\varepsilon\|_{H^2} \right)
\end{equation}

\begin{equation}
\leq C_s(\varepsilon) \left( \frac{h^2}{\varepsilon} \|e_h^\varepsilon\|_{H^2} + \varepsilon^{2-n} \|e_h^\varepsilon\|_{H^2} \right)
\end{equation}

\begin{equation}
\leq C_s(\varepsilon) \left( h^2 \frac{C_s(\varepsilon)}{\varepsilon} \|u^\varepsilon\|_{H^4} + \varepsilon^{2-n} C_s(\varepsilon) h^{2\ell-4} \|u^\varepsilon\|_{H^4} \right)
\end{equation}

\begin{equation}
= C_s(\varepsilon) \left[ h^2 \frac{C_s(\varepsilon)}{\varepsilon} \|u^\varepsilon\|_{H^4} + \varepsilon^{2-n} C_s(\varepsilon) h^{2\ell-4} \|u^\varepsilon\|_{H^4} \right],
\end{equation}

where $C_s := C_s(\varepsilon) C_s(\varepsilon)$.

Remark: It can be shown that the corresponding error estimates for spectral Galerkin method (2.7) are

\begin{equation}
\|u^\varepsilon - u_N^\varepsilon\|_{H^2} \leq C_s(\varepsilon) N^{2-l} \|u^\varepsilon\|_{H^4},
\end{equation}

\begin{equation}
\|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq C_s(\varepsilon) \left[ N^{-l} \varepsilon^{-\frac{3}{2}} \|u^\varepsilon\|_{H^4} + \varepsilon^{2-n} C_s(\varepsilon) N^{4-2\ell} \|u^\varepsilon\|_{H^4} \right],
\end{equation}

provided that $u^\varepsilon \in H^s(\Omega)$. Where $\ell = \min\{N + 1, s\}$. We refer the reader to [24] for a detailed proof.

5. Numerical Experiments and Rates of Convergence

In this section, we provide several 2-D numerical experiments to gauge the efficiency of the finite element method developed in the previous sections. We numerically find the “best” choice of the mesh size $h$ in terms of $\varepsilon$, and rates of convergence for both $u^0 - u^\varepsilon$ and $u^\varepsilon - u_h^\varepsilon$. All tests given below are done on the domain $\Omega = [0, 1]^2$. We refer the reader to [17, 29] for more extensive 2-D and 3-D numerical simulations.
ε  | |u_h^ε - u^ε| |L_2^2 | |u_h^ε - u^ε| |H_1 | |u_h^ε - u^ε| |H_2^2
---|---|---|---|---
0.75 | 0.109045862 | 0.528560309 | 3.39800721
0.5  | 0.11340196  | 0.548262711 | 3.524120741
0.1  | 0.08043631  | 0.401646611 | 2.657326288
0.075| 0.06932532  | 0.352221521 | 2.900677852
0.05 | 0.053925875 | 0.283684684 | 2.567362628
0.025| 0.03202484  | 0.18559903  | 2.270039867
0.0125| 0.019792385 | 0.117524466 | 1.928506935
0.005| 0.007871272 | 0.062721607 | 1.544066061
0.0025| 0.004115832 | 0.038522721 | 1.301171395
0.00125| 0.002124611 | 0.023464656 | 1.095145652
0.0005| 0.00087474  | 0.012073603 | 0.871227869

Table 1. Test 1a: Change of \( |u^0 - u_h^ε| \) w.r.t. \( ε \) (\( h = 0.009 \))

| ε  | \( |u_h^ε - u_h^0| |L_2^2 | \( |u_h^ε - u_h^0| |H_1^2 | \( |u_h^ε - u_h^0| |H_2^2 | \( \sqrt{ε} |u_h^ε - u_h^0| |H_2^2 | \( \frac{ε}{2^4} \)
---|---|---|---|---|---|
0.75 | 0.145394483 | 0.610328873 | 3.651396376
0.5  | 0.226680392 | 0.775360561 | 4.190909460
0.1  | 0.804363102 | 1.270118105 | 5.462612693
0.075| 0.924337601 | 1.286131149 | 5.542863492
0.05 | 1.078517501 | 1.268676476 | 5.619560909
0.025| 1.280993735 | 1.173813343 | 5.708480320
0.0125| 1.437826805 | 1.051170776 | 5.767580991
0.005| 1.574254363 | 0.887014744 | 5.806196004
0.0025| 1.646326257 | 0.774544111 | 5.819015381
0.00125| 1.699688442 | 0.663680706 | 5.824308630
0.0005| 1.749480266 | 0.53994796  | 5.826251909

Table 2. Test 1a: Change of \( |u^0 - u_h^ε| \) w.r.t. \( ε \) (\( h = 0.009 \))

**Test 1:** For this test, we calculate \( |u^0 - u_h^ε| \) for fixed \( h = 0.009 \), while varying \( ε \) in order to approximate \( |u^ε - u^0| \). We use Argyris elements and set to solve problem (2.5) with the following test functions:

(a). \( u^0 = e^{(x^2+y^2)/2}, \quad f = (1+x^2+y^2)e^{(x^2+y^2)/2}, \quad g = e^{(x^2+y^2)/2}. \)

(b). \( u^0 = x^4 + y^2, \quad f = 24x^2, \quad g = x^4 + y^2. \)

After having computed the error, we divide by various powers of \( ε \) to estimate the rate at which each norm converges. Tables 2 and 3 clearly show that \( |u^0 - u_h^ε| |H_2 = O(ε^{1/4}). Since we have fixed \( h \) very small, then \( |u^0 - u^ε| |H_2 \approx |u^0 - u_h^ε| |H_2 = O(ε^{1/4}). Based on this heuristic argument, we predict that \( |u^0 - u^ε| |H_2 = O(ε^{1/4}). Similarly, from Tables 2 and 3 we see that \( |u^0 - u^ε| |L_2 \approx O(ε) \) and \( |u^0 - u^ε| |H_1 \approx O(ε^{1/2}).

**Test 2.** The purpose of this test is to calculate the rate of convergence of \( |u^ε - u_h^ε| \) for fixed \( ε \) in various norms. As in Test 1, we use Argyris elements and solve problem (2.5) with boundary condition \( \Delta u^ε = ε \) on \( ∂Ω \) being replaced by \( \Delta u^ε = φ^ε \) on \( ∂Ω \). We use the following test functions:
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\[
\epsilon \left\| u_0 - u_\epsilon \right\|_{L^2} \quad \left\| u_0 - u_\epsilon \right\|_{H^1} \quad \left\| u_0 - u_\epsilon \right\|_{H^2}
\]

| \( \epsilon \) | 0.75  | 0.179911089 | 0.896016741 | 5.98759668 |
|----------|------|-------------|-------------|------------|
| 0.5      | 0.177287901 | 0.883816723 | 5.982088348 |
| 0.1      | 0.102586549 | 0.549713562 | 4.822739159 |
| 0.075    | 0.085457592 | 0.47264786  | 4.537189438 |
| 0.05     | 0.063960926 | 0.374513017 | 4.150185418 |
| 0.025    | 0.036755952 | 0.24464464  | 3.552739159 |
| 0.0125   | 0.020291198 | 0.157714933 | 3.032842066 |
| 0.005    | 0.008967657 | 0.087384209 | 2.451390014 |
| 0.0025   | 0.004761813 | 0.055425626 | 2.080704688 |
| 0.00125  | 0.002502224 | 0.034885527 | 1.762185899 |
| 0.0005   | 0.001054596 | 0.018689724 | 1.410593138 |
| 0.00025  | 0.000544002 | 0.011565172 | 1.189359491 |
| 0.000125 | 0.000279021 | 0.007112    | 0.999834091 |
| 0.00005  | 0.000114659 | 0.003700268 | 0.787092117 |

Table 3. Test 1b: Change of \( \left\| u^0 - u_\epsilon \right\| \) w.r.t. \( \epsilon \) (\( h = 0.009 \))

| \( \epsilon \) | 0.75  | 0.239881452 | 1.034631013 | 6.434091356 |
|----------|------|-------------|-------------|------------|
| 0.5      | 0.354575803 | 1.249905596 | 7.113942026 |
| 0.1      | 1.025865488 | 1.738346916 | 8.576177747 |
| 0.075    | 1.139434558 | 1.725865966 | 8.670049892 |
| 0.05     | 1.279218511 | 1.738346916 | 8.670049892 |
| 0.025    | 1.470238076 | 1.547268561 | 8.776573597 |
| 0.0125   | 1.623295808 | 1.410654242 | 9.070313735 |
| 0.005    | 1.793531488 | 1.235799336 | 9.218704868 |
| 0.0025   | 1.904725072 | 1.108512517 | 9.39594248 |
| 0.00125  | 2.001778889 | 0.986711711 | 9.37183749 |
| 0.0005   | 2.109122626 | 0.835829887 | 9.43204851 |
| 0.00025  | 2.176009824 | 0.731445692 | 9.45827896 |
| 0.000125 | 2.23216725  | 0.636116593 | 9.45612506 |
| 0.00005  | 2.293174219 | 0.523296856 | 9.360155452 |

Table 4. Test 1a: Change of \( \left\| u^0 - u_\epsilon \right\| \) w.r.t. \( \epsilon \) (\( h = 0.009 \))

(a). \( u^\epsilon = 20x^6 + y^6, \quad f^\epsilon = 18000x^4y^4 - \epsilon(7200x^2 + 360y^2), \)
\[ g^\epsilon = 20x^6 + y^6, \quad \phi^\epsilon = 600x^4 + 30y^4. \]

(b). \( u^\epsilon = x\sin(x) + y\sin(y), \quad f^\epsilon = (2\cos(x) - x\sin(x))(2\cos(y) - y\sin(y)) \)
\[ - \epsilon(x\sin(x) - 4\cos(x) + y\sin(y) - 4\cos(y)), \]
\[ g^\epsilon = x\sin(x) + y\sin(y), \quad \phi^\epsilon = 2\cos(x) - x\sin(x) + 2\cos(y) - y\sin(y). \]

After recording the error, we divided each norm by the power of \( h \) expected to be the convergence rate by the analysis in the previous section. As seen by Tables 6 and 8 the error converges quicker than anticipated in all the norms.

**Test 3.** In this section, we fix a relation between \( \epsilon \) and \( h \) to determine a "best" choice for \( h \) in terms of \( \epsilon \) such that the global error \( u^0 - u_\epsilon^h \) is the same convergence
rate as that of $u^0 - u^\varepsilon$. We solve problem (2.5) with the following test functions and parameters.

(a). $u^0 = x^4 + y^2$, \hspace{1em} f = 24x^2, \hspace{1em} g = x^4 + y^2.$

(b). $u^0 = 20x^6 + y^6$, \hspace{1em} f = 18000x^4y^4, \hspace{1em} g = 20x^6 + y^6.$

To see which relation gives the sought-after convergence rate, we compare the data with a function, $y = \beta x^\alpha$, where $\alpha = 1$ in the $L^2$-case, $\alpha = \frac{1}{2}$ in the $H^1$-case.

| $h$           | $\|u^0 - u_h\|_{L^2}$ | $\|u^0 - u_h\|_{H^1}$ | $\|u^0 - u_h\|_{H^2}$ |
|---------------|------------------------|------------------------|------------------------|
| 0.083333333  | 0.00268815             | 0.0000986132           | 0.000312985            |
| 0.05          | 0.91661E-05            | 0.00968512             | 0.00312985             |
| 0.030656967   | 5.42931E-06            | 0.00968512             | 0.00312985             |
| 0.023836565   | 1.29176E-06            | 0.00968512             | 0.00312985             |
| 0.015988237   | 1.04141E-05            | 0.00968512             | 0.00312985             |
| 0.012833175   | 1.18929E-05            | 0.00968512             | 0.00312985             |

| $h$           | $\|u^0 - u_h\|_{L^2}$ | $\|u^0 - u_h\|_{H^1}$ | $\|u^0 - u_h\|_{H^2}$ |
|---------------|------------------------|------------------------|------------------------|
| 0.083333333  | 0.062935746            | 0.038607272            | 0.021431653            |
| 0.05          | 0.033106867            | 0.019099981            | 0.012157901            |
| 0.030656967   | 0.232558269            | 0.019099981            | 0.012157901            |
| 0.023836565   | 0.232558269            | 0.019099981            | 0.012157901            |
| 0.015988237   | 0.232558269            | 0.019099981            | 0.012157901            |
| 0.012833175   | 0.232558269            | 0.019099981            | 0.012157901            |
and $\alpha = \frac{1}{4}$ in the $H^2$-case. The constant, $\beta$, is determined using a least squares fitting algorithm based on the data.

Figures 3-6 and 1-2 show that when $h = \varepsilon^{\frac{1}{2}}$, $\|u^0 - u_h^e\|_{H^2} \approx O(\varepsilon^{\frac{1}{4}})$ and $\|u^0 - u_h^e\|_{L^2} \approx O(\varepsilon)$. We can conclude from the data that the relation, $h = \varepsilon^{\frac{1}{2}}$ is the "best choice" for $h$ in terms of $\varepsilon$. It can also be seen from Figures 3-4 that when $h = \varepsilon$, $\|u^0 - u_h^e\|_{H^1} \approx O(\varepsilon^{\frac{1}{2}})$.

![Graphs](image)

**Figure 1.** Test 3a. $L^2$ Error

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Figure 3. Test 3a. $H^1$ Error

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Test 3b. $H^1$ Error

Figure 4. Test 3b. $H^1$ Error

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Figure 5. Test 3a. $H^2$ Error
Figure 6. Test 3b. $H^2$ Error