INTERACTION MEASURES ON THE SPACE OF DISTRIBUTIONS
OVER THE FIELD OF p-ADIC NUMBERS

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Abstract. We construct measures on the space $D'(\mathbb{Q}_p^n)$, $n \leq 4$, of Bruhat-Schwartz distributions over the field of $p$-adic numbers, corresponding to finite volume polynomial interactions in a $p$-adic analog of the Euclidean quantum field theory. In contrast to earlier results in this direction, our choice of the free measure is the Gaussian measure corresponding to an elliptic pseudo-differential operator over $\mathbb{Q}_p^n$. Analogs of the Euclidean $P(\varphi)$-theories with free and half-Dirichlet boundary conditions are considered.

1 Introduction

The basic objects of the Euclidean quantum field theory [2, 9, 13] are probability measures on the space $D'(\mathbb{R}^n)$ or $S'\mathbb{R}^n$ of real distributions. Equivalently, one can speak about a generalized random process $\varphi(f)$, $f \in D(\mathbb{R}^n)$, fixing a probability measure $\mu_0$ on the Borel $\sigma$-algebra $\Sigma$ of $D'(\mathbb{R}^n)$, and considering the probability space $(D'(\mathbb{R}^n), \Sigma, \mu_0)$. The free boson field is described by the measure $\mu_0$ corresponding to the Gaussian process with mean zero and covariance $\langle \varphi(f) \varphi(g) \rangle = (f, (-\Delta + m^2)^{-1}g)$. In order to describe fields with interaction corresponding to a polynomial $P$, it is necessary to define $P(\varphi)$ via some renormalization procedure. Then, for any $g \in D(\mathbb{R}^n)$, the measure

$$d\mu_g(\varphi) = \frac{\exp \{ -\langle P(\varphi), g \rangle \} d\mu_0(\varphi)}{\int \exp \{ -\langle P(\varphi), g \rangle \} d\mu_0(\varphi)}, \quad (1.1)$$

if the expression (1.1) makes sense and indeed defines a measure, is interpreted as an interaction measure in a finite volume. In some cases there exists (in a certain sense) also an infinite volume limit $\lim_{g \to 1} d\mu_g$.

Within the recent tendency to find non-Archimedean analogs of all important objects of mathematical physics, [4, 5, 15] it is natural to look for $p$-adic counterparts of the above constructions. This problem (formulated in Refs. 14, 16) is of a clear mathematical interest as a major problem of the infinite-dimensional non-Archimedean analysis, irrespective of possible physical applications.

As we switch from $\mathbb{R}^n$ to $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$, where $\mathbb{Q}_p$ is the field of $p$-adic numbers, we consider the space $D'(\mathbb{Q}_p^n)$ of Bruhat-Schwartz distributions. It turns out that the first crucial step is the choice of a free covariance. Note that one cannot define differential operators acting on complex-valued functions over $\mathbb{Q}_p$. 
The first results in $p$-adic quantum field theory were obtained in a series of papers by Lerner and Missarov (see e.g. Refs. 7, 8, 10) whose main motivation was to find a continuous analog of hierarchical models. Their choice of a free covariance was
\[
\text{const} \int_{Q^n_p \times Q^n_p} f(x)\|x - y\|^{-\alpha} f(y) \, dx \, dy, \quad \alpha > 0,
\]
which had nice scaling and discretization properties, making it possible to define a Hamiltonian of the discretized interacting field, but did not lead to the construction 11).

In this paper we propose a different approach. We follow the construction of Refs. 13, 2 using (instead of the Laplacian) elliptic pseudo-differential operators over $Q_p$ introduced in Refs 6, 5. Such elliptic operators exist only for $n \leq 4$, and in the "physical" dimension $4$ such an operator is unique up to an isomorphism. Below we assume that $p \neq 2$.

More specifically, let $h(\xi_1, \ldots, \xi_n)$ be a quadratic form with coefficients from $Q_p$, such that the ellipticity (or anisotropy) condition
\[
h(\xi_1, \ldots, \xi_n) \neq 0 \quad \text{if} \quad |\xi_1|_p + \cdots + |\xi_n|_p \neq 0 \tag{1.2}
\]
holds. On the space $L_2(Q^n_p)$ of square integrable complex-valued functions with respect to the additive Haar measure we consider the self-adjoint positive operator $A = F^{-1} M_{h,\alpha} F$, where $M_{h,\alpha}$ is the operator of multiplication by $|h(\xi_1, \ldots, \xi_n)|^\alpha_p$, $\alpha > 0$, $F$ is the Fourier transform (for the main notions and results of non-Archimedean analysis used in this paper see Ref. 5).

If $O$ is a ball in $Q^n_p$ (with respect to the absolute value $\max_{1 \leq j \leq n} |\xi_j|_p$), then $A_O$ will denote the operator on $L_2(O)$ defined as follows. Let $f \in \mathcal{D}(O)$, then $f \in \mathcal{D}(Q^n_p)$, supp $f \subset O$. Extending $f$ onto $Q^n_p$ by zero, we apply the operator $A$ to that extension. Restricting the resulting function to $O$ we obtain a function from $L_2(O)$ which is taken as $A_O f$. This defines an operator on $\mathcal{D}(O)$; its closure $A_O$ is a self-adjoint positive operator on $L_2(O)$.

Let $\mu_0$ be the measure on $\mathcal{D}'(Q^n_p)$ corresponding to the Gaussian process with the mean zero and covariance $\langle \varphi(f) \varphi(g) \rangle = (f, (A + m^2)^{-1} g)$, where $(\cdot, \cdot)$ is the inner product in $L_2(Q^n_p)$. As it is dictated by the condition 12.2, we always assume that $n \leq 4$. We show that for any semibounded polynomial $P$ and any $\alpha \geq \frac{n}{2}$, we can define $P(\varphi)$ to be the Wick renormalization with respect to the measure $\mu_0$, as it is done in the Euclidean $P(\varphi)^2$-model. The resulting non-Gaussian generalized stochastic process will be denoted $:P(\varphi):$. Moreover, with this renormalization the measure 12.1 is well-defined. It may be seen as a $p$-adic counterpart of the Euclidean $P(\varphi)^2$ with free boundary conditions.13

Another option (resembling the $P(\varphi)^2$ with the half-Dirichlet boundary conditions from Ref. 13) is to use the above Wick renormalization with respect to $\mu_0$ while replacing $\mu_0$ in 12.1 by $\mu_0^\theta$, the measure on $\mathcal{D}'(O)$ corresponding, as above, to the operator $A_O$, and taking $g$ to be the indicator function of the ball $O$. We show that 12.1 makes sense in this case too. This approach has some preferences. Though the infinite volume limit is not considered in this paper, we prove that within this "mixed" construction of the interaction measure the corresponding Schwinger functionals (under some assumptions on $P$) depend monotonously on the radius of the ball $O$. This result is based on a version of the lattice (or, rather, graph) approximation, which is of some independent interest.
2 Elliptic operators

In this section we introduce the elliptic pseudo-differential operators that play the role of the Laplacian in our approach. Hereinafter we will assume that \( m \) is some fixed positive constant, \( \alpha \geq \frac{n}{2} \).

2.1 Basic information

The theory of the operator \( A \) defined in the introduction is expounded in detail in Ref. 5. The results needed here can be summarized as follows.

The resolvent \((A + m^2)^{-1}\) is an integral operator of convolution type,

\[
((A + m^2)^{-1} f)(x) = \int_{\mathbb{Q}_p^n} \mathcal{E}(X - Y) f(Y) dY, \quad X \in \mathbb{Q}_p^n.
\]

The investigation of the Green function \( \mathcal{E}(X) = \mathcal{E}(x_1, \ldots, x_n), x_1, \ldots, x_n \in \mathbb{Q}_p \), is based on a procedure of reduction of multi-dimensional pseudo-differential operators over \( \mathbb{Q}_p \) to one-dimensional operators on more general fields.

The vector space \( K = \mathbb{Q}_p^n \ (n \leq 4) \) can be endowed with additional algebraic structures, so that \( K \) is assumed to be a local field (an extension of \( \mathbb{Q}_p \)), if \( n \leq 3 \), or the non-commutative quaternion algebra over \( \mathbb{Q}_p \), if \( n = 4 \). Let \( \| \cdot \| \) be the normalized absolute value, \( \beta \) be a prime element of \( K \), \( q \) be the cardinality of the residue field of \( K \). For \( x \in K, \| x \| = q^N \), we will write \( x = \beta^{-N} u, \| u \| = 1 \).

It is shown in Ref. 5 that for each quadratic form \( h \) satisfying (1.2) we can construct \( K \) in such way that

\[
\mathcal{E}(x_1, \ldots, x_n) = |\det T|_p^{-1} E \left( (T'_{-1} X)_1 e_1 + \cdots + (T'_{-1} X)_n e_n \right) \tag{2.1}
\]

where \( e_1, \ldots, e_n \) is a basis of \( K \) over \( \mathbb{Q}_p \), \( T \in GL(n, \mathbb{Q}_p) \), \( T' \) is the transpose of \( T \), \( E \) is the Green function, that is the integral (convolution) kernel of the resolvent \((\mathfrak{A} + m^2)^{-1}\) of a pseudo-differential operator \( \mathfrak{A} \) over \( K \) with the symbol

\[
a(\xi) = \| \xi \|^{2\alpha/n} \gamma(u\xi), \quad \xi \in K, \tag{2.2}
\]

where \( \gamma \) is a continuous strictly positive function on the group of units \( U \) of \( K \). As usual, the operator \( \mathfrak{A} \) is defined on the space \( D(K) \) of locally constant functions with compact supports as \( \mathfrak{A} = F^{-1}_K M_a F_K \), where \( M_a \) is the operator of multiplication by \( a \), \( F_K \) is the Fourier transform, that is

\[
(F_K f)(\xi) = \int_K \chi(\xi x) f(x) dx, \quad \xi \in K,
\]

where \( \chi \) is a rank zero additive character on \( K \). Below we will often use the notation \( \tilde{f} = F_K f \).

Explicit forms of all the above objects (the choice of the expressions for function \( \gamma \), the basis of coordinate representation \( e_1, \ldots, e_n \), the matrix \( T \) appearing in (2.1) etc.) can be found for various classes of the forms \( h \) in Ref. 5.

The relation (2.1) shows that the Gaussian measure \( \mu_0 \) corresponding to the operator \( A \) and a similar measure constructed from the operator \( \mathfrak{A} \) (we can identify \( K \) and \( \mathbb{Q}_p^n \) via the basis \( \{e_j\} \) are connected by a simple linear transformation. Therefore we will substantiate (1.1) using \( \mathfrak{A} \) instead of \( A \).
order to simplify the notations, below we will ignore the difference between $A$ and $\mathfrak{A}$, and understand $\mu_0$ as the measure corresponding to $\mathfrak{A}$.

Under the above identification of $K$ and $Q^n_p$ a ball $\hat{O}$ in $Q^n_p$ corresponds to a compact set $\Pi$ in $K$ which is not necessary a ball, but the union of a finite number of disjoint balls. Therefore, in the construction of the mixed interaction measure instead of $\mu_0^\Pi$ (defined in the introduction) we will deal with a similar measure $\mu_0^\Pi$.

Later we will often make use of the following technical lemma.

**Lemma 2.1.** Let the function $a(\xi)$ be as in (2.2). If $\alpha \geq \frac{\beta}{n}$, then for any $\kappa \in \mathbb{Z}$, $\kappa \geq 1$,

$$
\int_{\|\xi\|\leq q^\kappa} (a(\xi) + m^2)^{-1} d\xi \leq c_1 \kappa, \tag{2.3}
$$

and for any $\beta > 1$

$$
\int_{\|\xi\|\geq q^\kappa} (a(\xi) + m^2)^{-\beta} d\xi \leq c_2 q^{-\kappa(2\alpha\beta/n - 1)}, \tag{2.4}
$$

where $c_1$ and $c_2$ are positive constants that do not depend on $\kappa$.

**Proof.** First, note that $\gamma(u_\xi)$ in the definition (2.2) is a positive continuous function on a compact set. Thus there exist such positive constants $\gamma_{\min}$ and $\gamma_{\max}$, that

$$
\gamma_{\min} \leq \gamma(u_\xi) \leq \gamma_{\max} \text{ for any } \xi \in K.
$$

The following simple calculations prove (2.3):

$$
\int_{\|\xi\|\leq q^\kappa} (a(\xi) + m^2)^{-1} d\xi \leq m^{-2} \int_{\|\xi\|\leq 1} d\xi + \gamma_{\min}^{-1} \int_{1<\|\xi\|\leq q^\kappa} \|\xi\|^{-2\alpha/n} d\xi
$$

$$
\leq m^{-2} + \gamma_{\min}^{-1} (1 - q^{-1}) \sum_{l=1}^\infty q^{-l(2\alpha/n - 1)} \leq m^{-2} + \gamma_{\min}^{-1} (1 - q^{-1}) \kappa.
$$

Next,

$$
\int_{\|\xi\|\geq q^\kappa} (a(\xi) + m^2)^{-\beta} d\xi \leq \gamma_{\min}^{-1} \int_{\|\xi\|\geq q^\kappa} \|\xi\|^{-2\alpha\beta/n} d\xi
$$

$$
\leq \gamma_{\min}^{-1} (1 - q^{-1}) \sum_{l=\kappa}^{\infty} q^{-l(2\alpha\beta/n - 1)} \leq c_2 q^{-\kappa(2\alpha\beta/n - 1)},
$$

since $\frac{2\alpha\beta}{n} > 0$ and $q > 1$.

Calculations similar to those in proof of (2.3) give the estimate

$$
\int_{\|\xi\|\leq q^\kappa} (a(\xi) + m^2)^{-\beta} d\xi \leq c \kappa, \tag{2.5}
$$

where positive constant $c$ does not depend on $\kappa$. Thus from (2.4) and (2.5) we can conclude that the integral $\int_K (a(\xi) + m^2)^{-\beta} d\xi$ converges for any $\beta > 1$.  

4
2.2 Properties of the operator $A$ and its resolvent

Let us list some properties of the operator $A$ and its Green function $E$. For the proofs see Ref. 5.

The operator $A$ admits a hyper-singular integral representation

$$(A\tilde{z})(x) = \int_K \|y\|^{-2\alpha/n-1} \Omega(u_y)|z(x-y) - z(x)|dy. \quad (2.6)$$

The function $\Omega$ (as well as the function $\gamma$ from (2.23)) is a finite linear combination of continuous (multiplicative) characters of the group $U$. It is important that $\Omega(u) \leq 0$ for all $u \in U$, and that the function $y \mapsto \Omega(u_y)$ is locally constant on $K$.

The Green function $E$ is a non-negative function. If $\alpha > \frac{n}{2}$, then $E$ is continuous on the whole $K$, while for $\alpha = \frac{n}{2}$ the function $E$ is continuous except at the origin where it has a logarithmic singularity

$$E(x) \leq C_1 \log \|x\| + C_2 \quad \text{for} \quad \|x\| \leq 1$$

$(C_1, C_2 \geq 0)$. This property resembles the property of the Green function over $\mathbb{R}^2$. As $\|x\| \to \infty$, $E(x) \leq \text{const}\|x\|^{-2\alpha/n-1}$ (note a misprint in Ref. 5 where the sign is confused in the expression for the order of decay of $E$ in the formula (2.25) of Ref. 5: $\|x\|^\nu$ in that formula should be replaced with $\|x\|^{-\nu-1}$).

2.3 Restriction of $A$ to the union of balls

In order to construct the mixed interaction measure, we need the Green function $E_{\Pi}$ of the operator $A_{\Pi}$. Just as it was explained above for the operator $A_{O}$, $A_{\Pi}$ is defined on a function $f \in \mathcal{D}(\Pi)$ as the function $Af$ restricted to $\Pi$ (note that $f$ equals zero outside $\Pi$). It is clear that $A_{\Pi}$ is symmetric and positive as an operator on $L_2(\Pi)$. Moreover, $A_{\Pi}$ is essentially self-adjoint.

Indeed, let $A_{\Pi}$ be the Friedrichs extension of $A_{\Pi}$. The open compact set $\Pi$ is a union $\bigcup_{i=1}^\nu O_i$ of disjoint balls of the same radius, $O_i = \{x \in K|\|x-x_i\| \leq q^i\}$, where $x_i \in K$, $k \in \mathbb{Z}$, $\|x_i-x_j\| > q^k$ for $i \neq j$. If $y \in K$, and $\|y\|$ is small enough, then the shift operator $(T_yf)(x) = f(x+y)$ is a unitary operator on $L_2(\Pi)$. It follows from (2.6) that $A_{\Pi}$ and $T_y$ commute; then $T_y$ also commute with $A_{\Pi}$. Therefore if $z \in L_2(\Pi)$ is a solution of the equation $(A_{\Pi} + m^2)z = f$ where $f \in \mathcal{D}(\Pi)$ (i.e., $f$ is locally constant), then $z$ is locally constant and belongs to the domain of the operator $A_{\Pi}$. This means that the positive definite operator $A_{\Pi} + m^2$ has a dense range, whence $A_{\Pi}$ is essentially self-adjoint (see Ref. 12, Theorem X.26).

In order to write a hyper-singular integral representation for $A_{\Pi}$, denote

$$R_i = \{y \in K|\|y - (x_j - x_i)\| > q^k \quad \text{for all} \quad j = 1, \ldots, \nu\}.$$ 

Lemma 2.2. Let $x \in O_i$. Then $x - y \in \Pi$ if and only if $y \notin R_i$.

Proof. If $y \notin R_i$, then $\|y - (x_j - x_i)\| \leq q^k$ for some $j$, whence

$$\|(x - y) - x_j\| = \|(x - x_j) - (y - (x_i - x_j))\| \leq q^k$$

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by the ultra-metric property of the absolute value, so that \( x - y \in O_j \).

Conversely, if \( y \in R_i \), then for any \( j \)

\[
\|(x - y) - x_j\| = \|(x - x_j) - (y - (x_i - x_j))\| > q^k. 
\]

Now for \( z \in \mathcal{D}(\Pi) \) we can rewrite (2.6) as follows. If \( x \in O_i, y \in R_i \), we have

\[
(\mathfrak{A}_\Pi z)(x) = \int_{K \setminus R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y)[z(x) - z(x)] \, dy
\]

\[
- z(x) \int_{R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) \, dy, \quad x \in O_i. \tag{2.7}
\]

Proposition 2.1. \((\mathfrak{A}_\Pi + m^2)^{-1}\) is an integral operator on \(L_2(\Pi)\) with a kernel of the form

\[
E_{\Pi}(x, y) = E(x - y) + \Phi(x, y), \quad x, y \in \Pi, \tag{2.8}
\]

where \(\Phi(x, y)\) is locally constant in two variables. The function \(E_{\Pi}\) satisfies the inequality

\[
0 \leq E_{\Pi}(x, y) \leq E(x - y), \quad x, y \in \Pi. \tag{2.9}
\]

Proof. Define a family of functions \(\Psi_{\xi}(x)\) on \(\Pi\) \((\xi \in \Pi)\) setting

\[
\Psi_{\xi}(x) = \int_{R_i} E(x - y - \xi) \|y\|^{-2\alpha/n-1} \Omega(u_y) \, dy, \quad x \in O_i, \quad i = 1, \ldots, \nu. \tag{2.10}
\]

Let \(\Phi(x, \xi)\), for each fixed \(\xi \in \Pi\), be the solution of the equation

\[
(\mathfrak{A}_\Pi + m^2)\Phi(\cdot, \xi) = \Psi_{\xi}. \tag{2.11}
\]

It follows from the local constancy of the function \(y \mapsto \Omega(u_y)\), and the fact that \(\|y\| > q^k\) for any \(y \in R_i\), that \(\Psi_{\xi}(x)\) is a locally constant function on \(\Pi\), uniformly with respect to \(\xi\). Since \(\mathfrak{A}_\Pi\) commutes with small shifts, we find that \(\Phi(x, \xi)\) is locally constant in \(x\), uniformly with respect to \(\xi\).

On the other hand, by (2.10) \(\Psi_{\xi}(x)\) is also locally constant in \(\xi\) (uniformly with respect to \(x \in \Pi\)). Then the uniqueness of a solution of (2.11) implies the local constancy of \(\Phi(x, \xi)\) in \(\xi\), uniformly with respect to \(x\). Hence \(\Phi\) is locally constant in two variables.

Let \(f \in \mathcal{D}(\Pi)\),

\[
v(x) = \int_{\Pi} E_{\Pi}(x, y) f(y) \, dy.
\]

Then \(v = v_1 + v_2\),

\[
v_1(x) = \int_{\Pi} E(x - y) f(y) \, dy, \quad v_2(x) = \int_{\Pi} \Phi(x, y) f(y) \, dy, \quad x \in \Pi;
\]

\[
\bar{v}_1(x) = \begin{cases} v_1(x), & x \in \Pi, \\ 0, & x \in K \setminus \Pi. \end{cases}
\]

\[
x \in K.
\]
Then $\tilde{v}_1 = \tilde{v}_{11} + \tilde{v}_{12}$ where

$$\tilde{v}_{11}(x) = \int_\Pi E(x - y) f(y) \, dy, \quad x \in K;$$

$$\tilde{v}_1(x) = \begin{cases} 0, & x \in \Pi; \\ -\int_\Pi E(x - y) f(y) \, dy, & x \not\in \Pi. \end{cases}$$

If $x \in O_i$, then by Lemma 2.2

$$(\mathfrak{A} \tilde{v}_{12})(x) = \int_{R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) \tilde{v}_{12}(x - y) \, dy$$

$$= -\int_\Pi f(\xi) \, d\xi \int_{R_i} E(x - y - \xi) \|y\|^{-2\alpha/n-1} \Omega(u_y) \, dy$$

$$= -\int_\Pi \Psi_\xi(x) f(\xi) \, d\xi. \quad (2.12)$$

Note that $(\mathfrak{A} + m^2) \tilde{v}_{12} = \mathfrak{A} \tilde{v}_{12}$ on $\Pi$, and $(\mathfrak{A} + m^2) \tilde{v}_{11} = f$ on $\Pi$, so that

$$((\mathfrak{A} + m^2) v_1)(x) = f(x) - \int_\Pi \Psi_\xi(x) f(\xi) \, d\xi, \quad x \in \Pi. $$

Calculating $(\mathfrak{A} + m^2) v_2$ we find that

$$((\mathfrak{A} + m^2) v_2)(x) = -\int_\Pi \Psi_y(x) f(y) \, dy, \quad x \in \Pi;$$

together with 2.2 this yields the required equality $(\mathfrak{A} + m^2) v_2 = f$ on $\Pi$, for any $f \in D(\Pi)$. Since the kernel 2.4 generates a bounded operator on $L^2(\Pi)$, and $D(\Pi)$ is dense in $L^2(\Pi)$, 2.5 actually is the Green function.

Let $f(x) \geq 0$ for all $x \in \Pi$. Suppose that $v$ is not non-negative. Then there exists such $x_0 \in \Pi$ that

$$v(x_0) = \min_{x \in \Pi} v(x) < 0.$$ 

Since $\Omega(u) \leq 0$ on $U$, it follows from 2.4 that $((\mathfrak{A} + m^2) v)(x_0) < 0$. On the other hand, $((\mathfrak{A} + m^2) v)(x_0) = f(x_0) \geq 0$. This contradiction proves the lower bound in 2.9.

In order to prove the upper bound, consider the function

$$w(x) = \int_\Pi [E(x - \xi) - E_\Pi(x, \xi)] f(\xi) \, d\xi,$$

with $f(\xi) \geq 0$ on $\Pi$, $f \in D(\Pi)$. We find again, that if $w(x_0) = \min_{x \in \Pi} w(x) < 0$, then $((\mathfrak{A} + m^2) w)(x_0) < 0$. Denote

$$w_1 = \int_\Pi E(x - \xi) f(\xi) \, d\xi, \quad w_2 = \int_\Pi E_\Pi(x, \xi) f(\xi) \, d\xi.$$

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By (2.7), if \( i \) is such that \( x_0 \in O_i \), then

\[
\begin{align*}
(\mathbb{A}_{\Pi} + m^2) w_1 (x_0) &= \int_{K \setminus R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) [w_1(x_0 - y) - w_1(x_0)] \, dy \\
&\quad - w_1(x_0) \int_{R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) \, dy + m^2 w_1(x_0) \\
&= (\mathbb{A} + m^2) w_1 (x_0) - \int_{R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) w_1(x_0 - y) \, dy \\
&\quad = f(x_0) - \int_{R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) w_1(x_0 - y) \, dy,
\end{align*}
\]

so that

\[
(\mathbb{A}_{\Pi} + m^2) w (x_0) = - \int_{R_i} \|y\|^{-2\alpha/n-1} \Omega(u_y) w_1(x_0 - y) \, dy \geq 0,
\]

since \( w_1(x_0 - y) \geq 0 \) for all \( y \), while \( \Omega(u_y) \leq 0 \). We have come to a contradiction.

Let \( \Pi_1, \Pi_2 \) be open compact subsets of \( K \), such that \( \Pi_1 \subset \Pi_2 \). If \( E_{\Pi_1} \) and \( E_{\Pi_2} \) are the Green functions of the operators \( \mathbb{A}_{\Pi_1} \) and \( \mathbb{A}_{\Pi_2} \) respectively, then

\[
E_{\Pi_1}(x,y) \leq E_{\Pi_2}(x,y) \quad \text{for all} \quad x, y \in \Pi_1. \tag{2.13}
\]

The proof of this inequality is similar to the proof of the upper bound in (2.9).

### 3 The Wick renormalization

In this section we consider (for our situation) the renormalization procedure known as “Wick ordering”. Our purpose here is to define some non-Gaussian process on \( \mathcal{D}'(K) \) which we could identify with power of the Gaussian process \( \varphi(\cdot) \).

#### 3.1 Basic notions

Following Ref. 13, we will interpret our free process \( \varphi \) as the Gaussian process with mean zero indexed by the real Hilbert space \( \mathcal{H} \) obtained by completing \( \mathcal{D}(K) \) with respect to the inner product \((f, g)_\mathcal{H} = ((\mathbb{A} + m^2)^{-1} f, g)\). We fix the probability space as \((\mathcal{D}'(K), \Sigma, \mu_0)\) where \( \Sigma \) is the \( \sigma \)-algebra generated by cylindrical sets, \( \mu_0 \) is the measure defined (via the Minlos theorem) by the characteristic functional

\[
\int_{\mathcal{D}'(K)} e^{i\varphi(f)} \, d\mu_0(\varphi) = e^{-\frac{1}{2}(f, f)_{\mathcal{H}}}, \quad f \in \mathcal{D}(K).
\]

Note that \( \mathcal{D}(K) \) is a nuclear space.\[\]

Below we will write \( L_p(\mathcal{H}) \) instead of \( L_p(\mathcal{D}'(K), \Sigma, \mu_0) \), and \( \Gamma(\mathcal{H}) \) instead of \( L_2(\mathcal{H}) \). Let \( \Gamma(\mathcal{H})_{\leq k} \) be the closure in \( \Gamma(\mathcal{H}) \) of the linear span of all elements
\[ \varphi(f_1) \cdots \varphi(f_l), \quad l \leq k, \] and \( \Gamma(\mathcal{H})_k \) be the orthogonal complement of \( \Gamma(\mathcal{H})_{k-1} \) in \( \Gamma(\mathcal{H})_k \). It is well known (see e.g. Ref. 13) that \( \Gamma(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \Gamma(\mathcal{H})_k \), and if \( \psi \in \bigoplus_{l=0}^{k} \Gamma(\mathcal{H})_l \), then

\[ \|\psi\|_\rho \leq (\rho - 1)^{k/2} \|\psi\|_2 \quad \text{for all} \quad \rho \geq 2, \quad (3.1) \]

where the \( \rho \) subscript denotes the \( L_\rho \)-norm.

As usual, the Wick product \( \varphi(f_1) \cdots \varphi(f_k) \) of the Gaussian random variables \( \varphi(f_1), \ldots, \varphi(f_k) \) is the orthogonal projection of an element \( \varphi(f_1) \cdots \varphi(f_k) \) to \( \Gamma(\mathcal{H})_k \). We will write \( \varphi(f)^k = \varphi(f)^{1} \cdots \varphi(f)^{(k \text{ times})} \). This is a Gaussian random variable with mean zero and the variance \( k! \langle f, f \rangle_{\mathcal{H}} \). More generally, if \( f_1, \ldots, f_k, g_1, \ldots, g_k \in \mathcal{H} \), then

\[ \langle \varphi(g_1) \cdots \varphi(g_k) : \varphi(f_1) \cdots \varphi(f_k) \rangle = \sum_{\pi} (g_{\pi(1)}, f_1)_{\mathcal{H}} \cdots (g_{\pi(k)}, f_k)_{\mathcal{H}} \quad (3.2) \]

where the sum is taken over all the permutations of the indices \( 1, \ldots, k \).

In order to define the Wick power \( \varphi^k \) of the Gaussian generalized stochastic process, we approximate \( \varphi \) by ordinary random Gaussian functions, to which the above operation can be applied. The construction of this approximation is specific for the non-Archimedean case.

Let \( B_\varphi(0) \) be the ball \( \{ x \in K \mid \|x\| \leq q^{-\varphi} \} \) with the center at the origin \( (\varphi = 1, 2, \ldots) \). Denote by \( \Delta_\varphi(x) \) the indicator of \( B_\varphi(0) \), and by \( \delta_\varphi \) delta-like sequence

\[ \delta_\varphi = q_\varphi \Delta_\varphi(x), \quad x \in K. \]

This sequence converges to \( \delta \) in \( \mathcal{D}'(K) \). The Fourier transform \( \hat{\delta}_\varphi \) is the indicator of the ball with radius \( q_\varphi \) centered at the origin. Since \( \delta_\varphi \in \mathcal{D}(K) \), the convolution

\[ \varphi_\varphi(\xi) = (\varphi \ast \delta_\varphi)(\xi), \quad \xi \in K, \]

is an ordinary locally constant function. On the other hand, by the definition of convolution, for fixed \( \xi \), \( \varphi_\varphi(\xi) = \phi(\delta^\varphi(\xi)) \) where \( \delta^\varphi(\xi) = \delta_\varphi(x - \xi) \). Thus we can write the Wick power \( \varphi_\varphi(\xi)^k \), and then associate with it a generalized process by the formula

\[ \varphi_\varphi^k(g) = \int_K \varphi_\varphi(x)^k : g(x) \, dx, \quad g \in \mathcal{D}(K). \]

We can write explicitly [13] that

\[ \varphi_\varphi(x)^k = \sum_{j=0}^{[\frac{k}{2}]} \frac{(-1)^j k!}{2^j j! (k - 2j)!} \varphi_\varphi(x)^{k - 2j} c_\varphi^{2j}, \quad (3.3) \]

and, conversely,

\[ \varphi_\varphi(x)^k = \sum_{j=0}^{[\frac{k}{2}]} \frac{k!}{2^j j! (k - 2j)!} : \varphi_\varphi(x)^{k - 2j} : c_\varphi^{2j}, \]
where
\[ c^2 = \int_{\|\xi\| \leq q'} (a(\xi) + m^2)^{-1} \, d\xi. \]

**Proposition 3.1.** Let \( \alpha \geq \frac{m}{2} \). Then for any \( r \in (1, 2) \) and \( g \in D(K) \) there exist such positive constants \( \tau \) and \( C \) that
\[ \| : \varphi_{x_1}^k : (g) - : \varphi_{x_2}^k : (g) \|_2 \leq C \|g\|_r \, q^{-\tau}, \]  
(3.4)
where \( \kappa = \min\{\kappa_1, \kappa_2\} \).

**Proof.** For a given \( r \in (1, 2) \) we take \( r' = (1 - \frac{1}{r})^{-1}, 2 < r' < \infty \) and \( s = \frac{r'}{2} \). Then by the Hausdorff-Young inequality (see Ref. 3, Theorem 31.22) we have \( \|g\|_{r'} \leq \|g\|_r \). Therefore
\[ \| |g|^2 \|_s \leq \|g\|_r^2, \]  
(3.5)
since \( \| |g|^2 \|_s = \| |g|^2 \|_{r'/2} = \| |g|^2 \|_{r/2} \).

It follows from (3.2) and (3.3) that for \( \kappa_1 \geq \kappa_2 \)
\[ (k!)^{-1} \| : \varphi_{x_1}^k : (g) - : \varphi_{x_2}^k : (g) \|_2 = (g, E_{x_1}^k * g)_{L_2(K)} - (g, E_{x_2}^k * g)_{L_2(K)}, \]  
(3.6)
where
\[ E_{x_i}(x) = \int_K \delta_{x_i}(x - y) E(y) \, dy. \]

The derivation of (3.6) is a straightforward calculation based on the identity \( \delta_{x_1} \ast \delta_{x_2} = \delta_{\min\{x_1, x_2\}} \).

From (3.5) and (3.6) taking in account the Plancherel equality we get
\[ (k!)^{-1} \| \varphi_{x_1}^k : (g) - \varphi_{x_2}^k : (g) \|_2 = \| |g|^2 (\hat{E}_{x_1}^k - \hat{E}_{x_2}^k) \|_{L_2(K)} \leq \| |g|^2 \|_s \| \hat{E}_{x_1}^k - \hat{E}_{x_2}^k \|_{s'} \leq \| g \|^{2} \| \hat{E}_{x_1}^k - \hat{E}_{x_2}^k \|_{s'}, \]
where \( s' > 1 \), namely \( s' - 1 = 1 - 1/s \).

Now it is sufficient to show that the estimate
\[ \| \hat{E}_{x_1}^k - \hat{E}_{x_2}^k \|_{s'} \leq O(q^{-\tau}) \]  
holds for some \( \tau > 0 \). But
\[ \hat{E}_{x_1}^k - \hat{E}_{x_2}^k = (\hat{E}_{x_1} - \hat{E}_{x_2} ) * \hat{E}_{x_2} * \cdots * \hat{E}_{x_2} + \hat{E}_{x_1} * (\hat{E}_{x_1} - \hat{E}_{x_2} ) * \hat{E}_{x_2} * \cdots * \hat{E}_{x_2} + \cdots + \hat{E}_{x_1} * \cdots * \hat{E}_{x_1} (\hat{E}_{x_1} - \hat{E}_{x_2}). \]

From (2.8) and the definition of \( E_{x_i} \) we find that for \( \rho > 1 \) the norm \( \| \hat{E}_{x_i} \|_{\rho} \) is uniformly bounded with respect to \( \kappa = \min\{x_1, x_2\} \). Then by the Young inequality (see Ref. 3, Theorem 31.45) for \( \rho \in (1, 1 + \frac{1}{2}) \) there exists a positive constant, such that
\[ \| \hat{E}_{x_1}^k - \hat{E}_{x_2}^k \|_{s'} \leq \text{const} \| \hat{E}_{x_1} - \hat{E}_{x_2} \|_{\rho}. \]
Therefore the only thing we have to show is the estimate

\[ \| \tilde{E}_{x_1} - \tilde{E}_{x_2} \|_p \leq O(q^{-r\kappa}), \]

but it follows directly from (2.4) and the inequality

\[ \| \tilde{E}_{x_1} - \tilde{E}_{x_2} \|_p \leq \int_{\|\xi\| \geq q^{\kappa}} (a(\xi) + m^2)^{-p} \, d\xi. \]

The last proposition implies that for any function \( g \in D(K) \) the sequence \( \{ :\varphi^k_{\kappa} : (g) \} \) converges, as \( \kappa \to \infty \), to some random variable \( :\varphi^k : (g) \). Moreover, the limit \( :\varphi^k : (g) \) belongs to \( \Gamma(\mathcal{H}_k) \), since \( :\varphi^k_{\kappa} : (g) \) is in \( \Gamma(\mathcal{H}_k) \) for any \( \kappa \).

It follows from (3.6) that

\[ (k!)^{-1} \| :\varphi^k : (g) - :\varphi^k_{\kappa} : (g) \|_2^2 = (g, E^k \ast g)_{L^2(K)} - (g, E^k_{\kappa} \ast g)_{L^2(K)}. \]

Passing to the limit in relation (3.8) yields

\[ \| :\varphi^k : (g) - :\varphi^k_{\kappa} : (g) \|_2 \leq C \|g\|_r q^{-r\kappa}, \quad (3.7) \]

where \( C \) and \( \tau \) are some positive constants, and \( 1 < r < 2 \).

Since \( :\varphi^k : (g) \) belongs to \( \Gamma(\mathcal{H}_k) \), we can apply the inequality (3.1). Taking in account (3.7) we get the following proposition.

**Proposition 3.2.** If \( \alpha \geq \frac{\tau}{2} \), then for any \( r \in (1, 2) \), \( g \in D(K) \), and \( \rho \geq 2 \) there exist such positive constants \( \tau \) and \( C \) that

\[ \| :\varphi^k : (g) - :\varphi^k_{\kappa} : (g) \|_\rho \leq (\rho - 1)^{k/2} C \|g\|_r q^{-r\kappa}. \]

### 3.2 The Wick polynomials

Let \( g \in D(K) \). For any polynomial \( P(X) = a_s X^s + \cdots + a_1 X + a_0 \) we define \( P(\varphi) : (g) = \int_K g(x) : P(\varphi(x)) : \, dx \)

\[ a_s :\varphi^s : (g) + \cdots + a_2 :\varphi^2 : (g) + a_1 \varphi (g) + a_0. \]

Similarly we define the “smoothed” polynomial \( P(\varphi_{\kappa}) : (g) \). In what follows we will assume that \( a_s > 0 \) and \( g(x) > 0 \) for any \( x \in \text{supp} \, g \).

It is quite obvious that if \( s = \text{deg} \, P \) is an even number, then the polynomial \( P \) is bounded from below. So it is natural to expect that under our assumptions the integral

\[ \int_{D'(K)} \exp (- :P(\varphi) : (g)) \, d\mu_0(\varphi) \]

(3.8)

should converge. However, the Wick renormalization procedure usually causes the loss of the semiboundedness. It will take us two additional steps to show convergence of the integral (3.8).

Denote

\[ B = \max_{0 \leq j \leq s} |a_j|, \]

and

\[ D = a_s \|g\|_{L^1(K)} \left( 1 + \max_{0 \leq j \leq s-1} \left( \frac{|a_j|}{a_s} \right)^{s/(s-j)} \right). \]
Proposition 3.3. Let $s = \deg P$ be even; then there exists such a positive constant, that for any $\kappa$

$$:P(\varphi_{\kappa}): (g) \geq -\text{const} \frac{D \kappa s}{2}.$$  

Proof. Using the relation (3.3) we can rewrite $:P(\varphi_{\kappa}(x)):$ as

$$:P(\varphi_{\kappa}(x)) : = \sum_{j=0}^{s} a_j \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!((-1)^l)}{l!(j - 2l)!} \varphi_{\kappa}(x)^{j - 2l} (c_{\kappa}^2)^l$$

$$= a_s \left\{ \varphi_{\kappa}(x)^s + \sum_{j=0}^{s-1} a_j \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!((-1)^l)}{l!(j - 2l)!} \varphi_{\kappa}(x)^{j - 2l} (c_{\kappa}^2)^l 
+ \sum_{k=1}^{s/2} \frac{s!((-1)^k)}{k!(s - 2k)!} \varphi_{\kappa}(x)^{s-2k} (c_{\kappa}^2)^k \right\}.$$  

Let $N_1$ denote the number of terms in the first sum, and $N_2$ the number of terms in the second one; $N = N_1 + N_2$. Then we have

$$:P(\varphi_{\kappa}(x)) : = a_s \sum_{j,l} \left( N^{-1} \varphi_{\kappa}(x)^s + b_{j,l} \frac{a_j}{a_s} \varphi_{\kappa}(x)^{j - 2l} (c_{\kappa}^2)^l \right)$$

$$+ a_s \sum_{k} \left( N^{-1} \varphi_{\kappa}(x)^s + b_k \varphi_{\kappa}(x)^{s-2k} (c_{\kappa}^2)^k \right), \quad (3.9)$$

where $b_{j,l}$ and $d_k$ are some coefficients.

Elementary computations show that for even $s$ and $j < s$ the inequality $X^s - c_j X^j \geq -|c_j|^s/(s-j)$ holds for all $X$. Therefore, each term in the first sum of the right-hand side of (3.9) is bounded from below by the value

$$- C_1 \left| \frac{a_j}{a_s} \right|^{s/(s-j) + 2l} (c_{\kappa}^2)^{s/(s-j) + 2l} \geq - C_1 \left( \left| \frac{a_j}{a_s} \right| + 1 \right)^{s/(s-j)} (c_{\kappa}^2)^{s/2},$$

since $\frac{2l}{s-j+2l} < 1$. By the same reasoning, each term in the second sum is bounded from below by the value

$$- C_2 (c_{\kappa}^2)^{s/2}.$$  

(Here $C_1$ and $C_2$ denote some positive constants.) Thus there exists such a positive constant that

$$:P(\varphi_{\kappa}(x)) : \geq - \text{const} a_s \left( 1 + \max_{0 \leq j \leq s-1} \left( \left| \frac{a_j}{a_s} \right| + 1 \right)^{s/(s-j)} \right) (c_{\kappa}^2)^{s/2}.$$  

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From here and Lemma 2.1 we get

\[
: P(\varphi_\infty)(g) = \int_K : P(\varphi_\infty(x)) : g(x) \, dx \geq \\
- \text{const} \, a_s \| g \|_{L_2(K)} \left( 1 + \max_{0 \leq j \leq s-1} \left( \frac{a_j}{a_0} \right) + 1 \right)^{s/(s-j)} (C_2^2)^{s/2} \\
\geq - \text{const} \, D \, x^{s/2}.
\]

It follows directly from the Proposition 3.1 and the definitions of \( : P(\varphi) : (g) \) and \( : P(\varphi_\infty) : (g) \) that for any \( r \in (1, 2) \) there exist such \( C > 0 \) and \( \tau > 0 \) that

\[
\| : P(\varphi) : (g) - : P(\varphi_\infty) : (g) \|_\rho \leq C (\rho - 1)^{s/2} B \| g \|_\tau \, q^{-\tau \kappa}
\]

(3.10)

for any \( \rho \geq 2 \).

**Proposition 3.4.** Let \( P \) be as in the previous proposition. Then there exist such positive constants \( b \) and \( \gamma \), that

\[
\mu_0 \{ \varphi \mid : P(\varphi) : (g) \leq -b \, x^{s/2} \} \leq e^{-q \gamma} \]

(3.11)

if \( \kappa \) is large enough.

**Proof.** First, we choose a constant \( b \) such that the estimate \( : P(\varphi_\infty) : (g) \geq - \frac{b}{2} \, x^{s/2} \) holds; Proposition 3.3 allows us to make this choice. Then we take \( \kappa_0 \) such that \( b \, \kappa_0^{s/2} > 1 \), so that

\[
: P(\varphi_\infty) : (g) \geq 1 - b \, x^{s/2} \quad \text{for any} \quad \kappa \geq \kappa_0.
\]

If \( : P(\varphi) : (g) \leq -b \, x^{s/2} \), then for all \( \kappa \geq \kappa_0 \) we have

\[
| : P(\varphi) : (g) - : P(\varphi_\infty) : (g) | \geq 1
\]

, and therefore

\[
\mu_0 \{ \varphi \mid : P(\varphi) : (g) \leq -b \, x^{s/2} \} \leq \mu_0 \{ \varphi \mid | : P(\varphi) : (g) - : P(\varphi_\infty) : (g) | \geq 1 \}
\]

\[
\leq \int_{cD(K)} | : P(\varphi) : (g) - : P(\varphi_\infty) : (g) |^\rho \, d\mu_0(\varphi) = \| : P(\varphi) : (g) - : P(\varphi_\infty) : (g) \|_\rho^n
\]

for all \( \rho \geq 2 \). From inequality (3.10) it follows that there exists such a constant \( R \), which does not depend on \( \kappa \), that

\[
\| : P(\varphi) : (g) - : P(\varphi_\infty) : (g) \|_\rho \leq (\rho - 1)^{s/2} R^\rho \, q^{-\tau \kappa},
\]

where \( \tau > 0 \). Then we take \( \rho \) depending on \( \kappa \); namely, we put \( \rho = q^{2\tau \kappa/3} \). We have then

\[
(\rho - 1)^{s/2} \leq \rho^{s/2} \leq q^{\tau \kappa/3}.
\]

Moreover, it is clear that for all sufficiently large \( \kappa \)'s we have \( R^\rho \leq q^{\tau \kappa/3} \), and e \leq q^{\tau \kappa/3}. Thus,

\[
R^\rho (\rho - 1)^{s/2} q^{-\tau \kappa} \leq q^{-\tau \kappa/3} \leq e^\gamma = e^{q^{-2\tau \kappa/3s}} = e^{-q^{-\gamma}}, \quad \text{where} \quad \gamma = \frac{2\tau}{3s}.
\]

\qed
Now everything is prepared for proving the main result of this section which is formulated in the following theorem.

**Theorem 3.1.** Let $P$ be a polynomial bounded from below (that is, $\deg P = 2s$), and $g$ be a nonnegative function from $D'(K)$. Then

$$\exp \left( - \int_K g(x) : P(\varphi(x)) : dx \right) \in \bigcap_{\rho < \infty} L_\rho(D'(K), d\mu_0).$$

(3.12)

**Proof.** Let $f$ be a real-valued function on a probability space $(M, \Sigma, \mu)$ and $m_f(x) = \mu \{ \xi \mid f(\xi) \geq x \}$.

Next let $F$ be a bounded positive function from $C^1(\mathbb{R})$. Then

$$\int F(f(g)) \, d\mu = \int_{-\infty}^{+\infty} F(x) \, dm_f \quad \text{(Stieltjes' integral)}$$

$$= -F(-\infty) + \int_{-\infty}^{+\infty} m_f(x) F'(x) \, dx.$$

By the monotone convergence theorem we have

$$\int e^{f(\xi)} \, d\mu = \int_{-\infty}^{+\infty} e^x \, m_f(x) \, dx,$$

where both sides of the formula can turn into infinity only simultaneously. If the function $f$ is such that $\mu \{ \xi \mid -f(\xi) \geq bx^{s/2} \} \leq e^{-q^{s/2}}$

for all $x \geq x_0$, then

$$\int e^{-f(\xi)} \, d\mu \leq e^{bx^{s/2}} + \int_{bx^{s/2}}^{\infty} e^x \exp \left( -q^{s(x/b)^2/s} \right) \, dx < \infty.$$

From here and the Proposition 3.4 we conclude, that the integral

$$\int_{D'(K)} \exp \left( -:P(\varphi)(g) : \right) \, d\mu_0(\varphi)$$

converges. Thus, we have proved (3.12) for the case of $\rho = 1$. But

$$\|\exp ( -:P(\varphi)(g) : )\|_\rho = \|\exp ( -:P(\varphi)(\rho g) : )\|_1^{1/\rho},$$

which proves (3.12).

Now we can state that the expression

$$d\mu_g(\varphi) = \frac{\exp ( -:P(\varphi)(g) : ) \, d\mu_0}{\int \exp ( -:P(\varphi)(g) : ) \, d\mu_0}$$

(3.13)

defines a probability measure on the space $D'(K)$. 

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4 Concentrated measure and Schwinger functions

In this section we introduce a “concentrated” measure \( d\mu_\Pi \) on the space \( D'(\Pi) \) corresponding to the operator \( A_\Pi \) defined in section 2. We show that the expression similar to that in (3.13) with \( d\mu_0 \) replaced by \( d\mu_\Pi \) makes sense and defines a probability measure on \( D'(\Pi) \). Next we consider the natural counterparts of the “semi-Dirichlet” Schwinger functions corresponding to a bounded region and examine their behaviour as the region increases.

Let \( \Pi \) be a union of balls of the same radius \( q_k \) described in subsection 2.3, and \( A_\Pi \) be the restriction of the operator \( A \) to that union. In this section we accept a little more verbose notations, representing the union in form \( \Pi = \bigcup_{i=1}^{\nu} O_k(x_i) \), where the common radius of balls is mentioned explicitly (subsection 2.3 pointed the ball \( O_k(x_i) \) by saying simply \( O_i \)).

4.1 The measures

The properties of the Green function \( E_\Pi \) of the operator \( A_\Pi \) studied in detail in subsection 2.3 allow us to define on the space \( D'(\Pi) \) a generalized stochastic process \( \varphi \) with mean zero and covariance of form

\[
(f, (A_\Pi + m^2)^{-1} g) = \int_K f(x) E_\Pi(x,y) g(y) \, dy \, dx, \quad f, g \in D(\Pi).
\]

The corresponding Gaussian measure on \( D'(\Pi) \) will be denoted by \( d\mu_\Pi \). At the same time we can consider the restriction of the measure \( d\mu_0 \) to \( D'(\Pi) \); it will be denoted by \( d\mu_0 \) as well. Thus we have two different measures on \( D'(\Pi) \), and therefore two kinds of the Wick renormalization for random variables on \( D'(\Pi) \), such as the generalized process \( \varphi \). Since \( \varphi \) is Gaussian with respect to each of the two measures, one can state (just as in the case of \( \mathbb{R}^2 \); see Ref. 13) that

\[
:\varphi(x)^r : = \sum_{j=0}^\lfloor s/2 \rfloor \frac{r!}{2^j j!(r - 2j)!} :\varphi(x)^{r-2j}:_\Pi \left( \Phi(x,x) \right)^j, \quad (4.1)
\]

for any natural number \( r \), where \( :: \) and \( ::_\Pi \) denote the Wick renormalization with respect to \( d\mu_0 \) and \( d\mu_\Pi \), respectively. From Proposition 2.1 we find that \( \Phi(x,x) \) is a bounded, nonpositive, and locally constant function in \( x \). The formula (4.1) implies that the polynomial \( P(\varphi(x)) : = \sum_{j=0}^s a_j :\varphi(x)^j : \) whose degree \( s \) is even, can be represented as a similar polynomial \( P'(\varphi(x)) :_\Pi = \sum_{j=0}^s a'_j :\varphi(x)^j :_\Pi \) of the same degree. Moreover, if \( a_s > 0 \), then \( a'_s > 0 \); in other words, if the polynomial \( P(\varphi(x)) \) is bounded below, then so is the polynomial \( P'(\varphi(x)) :_\Pi \). Therefore, for proving the convergence of the integral

\[
\int_{D'(\Pi)} e^{-P(\varphi):g} \, d\mu_0 \Pi,
\]

where \( g \) is a nonnegative function from \( D(\Pi) \), \( P \) is the semibounded polynomial, and \( P(\varphi):g = \int_K P(\varphi(x)) \cdot g(x) \, dx \), it is sufficient to show (under the same
conditions) the convergence of
\[
\int_{D''(\Pi)} e^{-:P(\varphi);n(g)} \, d\mu^\Pi_0.
\]
The convergence of the latter integral can be proved in essentially the same way as it was done for \( \int_{D(\Pi)} e^{-:P(\varphi);(g)} \, d\mu_0 \) in the previous section. Together with the formula (4.1) this implies that the expression
\[
d\mu^\Pi_0(\varphi) = \frac{\exp(-:P(\varphi);(g)) \, d\mu^\Pi_0}{\int \exp(-:P(\varphi);(g)) \, d\mu^\Pi_0} \tag{4.2}
\]
defines a probability measure on the space \( D'(\Pi) \).

Concluding this subsection we define a function
\[
S_g^{(\Pi)}(h_1, \ldots, h_r) = Z_{\Pi}^{-1} \int \varphi(h_1) \cdots \varphi(h_r) e^{-:P(\varphi);(g)} \, d\mu^\Pi_0,
\]
where \( \{h_1, \ldots, h_r\} \) is an arbitrary collection of functions from \( D(\Pi) \), and \( Z_{\Pi} = \int e^{-:P(\varphi);(g)} \, d\mu^\Pi_0 \). The function \( S_g^{(\Pi)}(h_1, \ldots, h_r) \) is said to be “the \( r \)-point Schwinger function” of a mixed state corresponding to the region \( \Pi \). In what follows we show that if \( h_i \geq 0 \), then the Schwinger function is nonnegative, and it grows as the region \( \Pi \) increases.

### 4.2 Lattice approximation

Let \( \Pi \) be the union of balls defined at the beginning of the section. For any integer \( l \), such that \( l \leq k \), we can write the following identity based on the geometric features of non-Archimedean fields (see Ref. 15):
\[
\Pi = \bigcup_{i=1}^\eta O_l(x_i),
\]
where \( O_l(x_i) \) denotes a ball \( \{x \in K \mid \|x - x_i\| \leq q_l\} \) of radius \( q_l \), and \( \eta = \eta(l) \geq \nu \) is the number of the balls \( O_l(x_i) \) in the union \( \Pi \). The balls are disjoint, and their centers are more than \( q_l \) apart from each other, that is \( \|x_i - x_j\| \geq q_l \) for \( i \neq j \). This means that we can decompose \( \Pi \) into the union of arbitrarily small parts keeping general structure of the union unchanged, so that all results of subsection 2.3 remain valid. Then let
\[
e_i^{(l)}(x) = q^{-l/2} \Delta_{O_l(x_i)}(x),
\]
where \( \Delta_{O_l(x_i)} \) is the indicator of the ball \( O_l(x_i) \). The collection \( \{e_i^{(l)}\}_{i=1}^\eta \) forms an orthonormal basis in the space \( D_l(\Pi) \), which consists of all functions from \( D_l(\Pi) \) with the following local constancy property: for any \( f \in D_l(\Pi) \) we have \( f(x + x') = f(x) \) if \( \|x'\| \leq q_l \). Note that \( D_l(\Pi) \) is a space of the finite dimension \( \eta(l) \). Applying the formula (2.4) to the function \( z = e_i^{(l)} \) we easily get
\[
\mathfrak{A}_l e_i^{(l)}(x) = \begin{cases} -q^{-l/2} \int_{K \setminus O_l(x_i)} \|y\|^{-2\alpha/n-1} \Omega(u_y) \, dy, & x \in O_l(x_i); \\ q^{-l/2} \int_{\|x_i - x_j\| - y \leq q_l} \|y\|^{-2\alpha/n-1} \Omega(u_y) \, dy, & x \in O_l(x_j), \ i \neq j. \end{cases}
\]
It follows from here that the space \( \mathcal{D}(\Pi) \) is invariant with respect to the operator \( \mathfrak{A}_\Pi \), and so it is with respect to the operators \((\mathfrak{A}_\Pi + m^2)\) and \((\mathfrak{A}_\Pi + m^2)^{-1}\). Moreover, the operator \((\mathfrak{A}_\Pi + m^2)^{-1}\) acts on \( \mathcal{D}(\Pi) \) as a positive operator, since it is a restriction of the positive operator \((\mathfrak{A} + m^2)^{-1}\) to a subspace.

Now consider a matrix \( M^{(\Pi, l)} \) with the elements of form

\[
M^{(\Pi, l)}_{ij} = (e_i^{(l)}, (\mathfrak{A}_\Pi + m^2)^{-1} e_j^{(l)}), \quad i, j = 1, \ldots, \eta.
\]

It is easy to see that this matrix is symmetric and positive definite. Furthermore, since the integral kernel of \((\mathfrak{A}_\Pi + m^2)^{-1}\) is a nonnegative function (see Proposition 2.1), all the elements of \( M^{(\Pi, l)} \) are nonnegative. The elements of its inverse \( N^{(\Pi, l)} \) have the form

\[
N^{(\Pi, l)}_{ij} = (e_i^{(l)}, (\mathfrak{A}_\Pi + m^2)e_j^{(l)}),
\]

and we can express them as

\[
N^{(\Pi, l)}_{ij} = \begin{cases} 
-\int_{K\setminus O_i(x_i)} \|y\|^{-2\alpha/n-1} \Omega(u_y)dy, & i = j; \\
\int \|y\|^{-2\alpha/n-1} \Omega(u_y)dy, & i \neq j.
\end{cases}
\] (4.4)

Since \( \Omega(u) \leq 0 \) for all \( u \in U \), we conclude that

(a) \( N^{(\Pi, l)}_{ii} \geq 0, \quad i = 1, \ldots, \eta; \)

(b) \( N^{(\Pi, l)}_{ij} \leq 0, \quad i \neq j, \quad i, j = 1, \ldots, \eta. \)

There is one more important fact, which follows directly from (4.4). Suppose that we added a number of new balls of the radius \( q^l \) to the union \( \Pi \); denote that extended union by \( \Pi' \). Again, for any \( l \leq k \) we can represent \( \Pi' \) as a union of balls of radius \( q^l \); moreover, once we have such a representation for \( \Pi \), we can choose the representation for \( \Pi' \) to be exactly the same plus some additional balls of radius \( q^l \). The corresponding matrix \( N^{(\Pi', l)} \) will then consist of all elements of \( N^{(\Pi, l)} \) plus additional ones. More precisely, if \( \Pi \subset \Pi' \), and \( i, j \in \Pi \), then

\[
N^{(\Pi, l)}_{ij} = N^{(\Pi', l)}_{ij},
\] (4.5)

because the matrix elements have exactly the same form, according to (4.4). (Here and below \( i \in \Pi \) means that \( O_i \subset \Pi \).)

Next we define a Gaussian stochastic process \( \varphi_{\delta} \) on \( \mathcal{D}(\Pi) \) with the covariance

\[
\langle \varphi_{\delta}(e_i^{(l)}) \varphi_{\delta}(e_j^{(l)}) \rangle = M^{(\Pi, l)}_{ij},
\]

where \( \delta = q^l \). The Gaussian random variables \( \varphi_{\delta}(e_i) \) form a jointly-Gaussian collection with the characteristic function

\[
c(t_1, \ldots, t_\eta) = \exp \left( -\frac{1}{2} \sum_{i,j=1}^\eta M^{(\Pi, l)}_{ij} t_i t_j \right),
\]

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where \( t_i \in \mathbb{R} \), \( l = \overline{1, \eta} \). The corresponding Gaussian measure on the space \( D'^{(l)}(\Pi) \cong D_l(\Pi) \), which is essentially a measure on \( \mathbb{R}^\eta \), will be denoted by \( d\mu^\Pi_\delta \).

It is a well known fact (see e.g. Ref. 13), that if \( F \) is a function on \( \mathbb{R}^\eta \), then

\[
\int_{D'^{(l)}(\Pi)} F(\varphi_\delta(e_1^{(l)}), \ldots, \varphi_\delta(e_\eta^{(l)})) \, d\mu^\Pi_\delta(\varphi_\delta)
\]

\[
= (2\pi)^{-\eta/2} \left( \det M(\Pi, l) \right)^{-1/2} \int_{\mathbb{R}^\eta} F(t_1, \ldots, t_\eta) e^{-\frac{1}{2} \sum_{i,j=1}^\eta X^{(l)(i)} t_i t_j} \, d^n t. \tag{4.6}
\]

For any function \( g \in D(\Pi) \) we put

\[
\varphi_\delta(g) = \sum_{i=1}^\eta g(x_i) \varphi_\delta(e_i^{(l)}),
\]

where \( x_i \) is the center of the ball \( O_i \). The random variable \( \varphi_\delta(g) \) can be considered as defined on the probability space \( (D'^{(l)}(\Pi), d\mu_0) \). This allows us to define its Wick powers by the formula

\[
:\varphi_\delta^r(g) = \sum_{i=1}^\eta g(x_i) :\varphi_\delta^r(e_i^{(l)}) :,
\]

where the renormalization in the right-hand side is taken with respect to the free measure \( d\mu_0 \). Since \( \varphi_\delta(e_i) \) is an ordinary random variable, there are no difficulties with such a definition.

For an arbitrary polynomial \( P(X) = a_s X^s + a_{s-1} X^{s-1} + \cdots + a_1 X + a_0 \) and any function \( g \in D(\Pi) \) we define

\[
:\!P(\varphi_\delta) : (g) = \sum_{j=0}^s \sum_{i=1}^\eta a_j :\varphi_\delta^j(e_i^{(l)}) : g(x_i).
\]

Finally, we define the function

\[
S_{\delta, g}^{(l)(\Pi)}(h_1, \ldots, h_r) = Z_{\Pi^{-1}} \int \varphi_\delta(h_1) \cdots \varphi_\delta(h_r) e^{-P(\varphi_\delta)(g)} \, d\mu^\Pi_\delta, \tag{4.7}
\]

where \( Z_{\Pi} = \int e^{-P(\varphi_\delta)(g)} \, d\mu^\Pi_\delta \), the polynomial \( P \) is bounded from below (i.e., \( \deg P = s \) is even), \( g \) is a nonnegative function from \( D(\Pi) \), and \( h_i \in D(\Pi), i = \overline{1, r} \). The function \( S_{\delta, g}^{(l)(\Pi)}(h_1, \ldots, h_r) \) can be regarded as the lattice approximation for the \( r \)-point Schwinger function \( S_g^{(l)(\Pi)}(h_1, \ldots, h_r) \) defined above. Note that for \( g \in D_l(\Pi) \)

\[
\varphi_\delta(g) = \varphi(g) \quad \text{and} \quad :\varphi_\delta^k: (g) = :\varphi^k: (g) \quad \text{for any} \ k \in \mathbb{Z}.
\]

Moreover, if \( h_i \in D_l(\Pi) \) then the integral in the right-hand side of \( \overline{17, 18} \) coincides with the integral

\[
\int \varphi(h_1) \cdots \varphi(h_r) e^{-P(\varphi)(g)} \, d\mu_0
\]
by the definition of the cylindrical measure \(d\mu_0\). Since for any fixed function from \(D(\Pi)\) there exists such an integer \(k_0\), that for all \(k \geq k_0\) the function belongs to \(D_I(\Pi)\), for all sufficiently small \(\delta\)'s the Schwinger function and its lattice approximations are the same thing. Thus, all facts proven to be true for those approximations with arbitrary \(\delta\), automatically hold for the Schwinger function itself.

### 4.3 Griffiths inequalities

The Griffiths inequalities are standard correlation inequalities, which are usually used for proving the properties of Schwinger functions. To formulate them we will need some more notions \[13\].

**Definition 4.1.** A polynomial \(Q\) is said to be even, if it of form
\[
Q(X) = a_sX^s + a_{s-2}X^{s-2} + \cdots + a_2X^2 + a_0,
\]
where \(s\) is an even number.

**Definition 4.2.** A probability measure \(\mu\) on \(\mathbb{R}^r\) is said to correspond to an even Ising ferromagnet, if it is of form
\[
\mu = Z^{-1} \exp \left( -\sum_{i,j} b_{i,j} x_i x_j \right) d\nu_1 \ldots d\nu_r,
\]
where \(Z\) is a normalization constant, \(b_{i,j} \leq 0\) for \(i \neq j\), \(\nu_i = \exp(\lambda_i x_i) d\nu'_i\), \(\lambda_i \geq 0\), and \(d\nu'_i\) is a measure which is invariant under reflection, that is \(d\nu'_i(\xi) = d\nu'_i(-\xi)\).

The fact stated in the following lemma follows directly from the definitions above and the definition of the measure \(d\mu_{\Pi,\delta}\). (Recall here that the space \(D_I(\Pi)\) is essentially \(\mathbb{R}^n\), so the measure \(d\mu_{\Pi,\delta}\) is a Gaussian measure on \(\mathbb{R}^n\).)

**Lemma 4.1.** If the polynomial \(P\) is of form \(P(X) = Q(X) - \lambda X\), where \(Q(X)\) is an even polynomial, \(g\) is a nonnegative function from \(D(\Pi)\), and \(\lambda\) is a non-negative constant, then the measure
\[
d\nu_{\delta,g} = Z^{-1}_{\Pi} \exp \left( \frac{-1}{2} \sum_{i,j} N_{i,j}(\Pi, l) t_i t_j \right) \prod_{j \in \Pi} e^{-\sum_{i,j} P(t_i) (g(x_j))} dt_j
\]
corresponds to an even Ising ferromagnet.

**Proof.** Indeed, we have
\[
d\nu_{\delta,g} = Z^{-1}_{\Pi} \exp \left( \frac{-1}{2} \sum_{i,j} N_{i,j}^{(\Pi, l)} t_i t_j \right) \prod_{j \in \Pi} d\nu_j,
\]
where
\[
d\nu_j = \exp \{ \lambda g(x_j) t_j \} \exp \left\{ -\sum_{i,j} P(t_i) (g(x_j)) \right\} dt_j.
\]
Note that the numbers \(-N_{i,j}^{(\Pi, l)}\) are nonnegative for \(i \neq j\), since the non-diagonal elements of the matrix \(N^{(\Pi, l)}\) are non-positive which was proven above. All the rest is provided by nonnegativeness of \(\lambda\) and \(g\), and by evenness of \(Q\). \(\square\)
Next we state a version of Griffiths’ theorem (see Ref. 13, Theorem VIII.3) for even Ising ferromagnets.

**Theorem 4.1 (Griffiths).** Let the measure $d\mu$ correspond to an even Ising ferromagnet, and let $\xi^i = \xi^i_1 \ldots \xi^i_r$. Then

(a) (Griffiths’ first inequality) $\int \xi^i d\mu(\xi) \geq 0$;

(b) (Griffiths’ second inequality) $\int \xi^{i_1} \cdot \xi^{i_2} d\mu \geq (\int \xi^{i_1} d\mu) (\int \xi^{i_2} d\mu)$.

It follows from the last theorem that if \{ $h_1, \ldots, h_r$ \} is a collection of nonnegative functions from $D_\delta(\Pi)$, then the Schwinger function $S_{\delta, g}(h_1, \ldots, h_r)$ is nonnegative. Indeed, since $h_i \geq 0$, it is sufficient to show that $S_{\delta, g}(e_{i_1}, \ldots, e_{i_r}) \geq 0$, but that follows from the Griffiths’ first inequality and lemma 4.1.

### 4.4 Monotonous increase

Here we show the monotonous increase of the Schwinger functions. As noted above, it is sufficient to consider the lattice approximations for that functions, where the “lattice” consists of balls $O_l$ of arbitrarily small radius $\delta = q^l$.

**Theorem 4.2.** Let the polynomial $P$ be of form $P(X) = Q(X) - \lambda X$, where $Q(X)$ is an even polynomial, and $\lambda$ is a nonnegative constant. Let then $\Pi, \Pi'$ be the unions of $\eta$ and $\eta'$ balls of the radius $q^l$, respectively, such that $\Pi \subset \Pi'$. Finally, let $g$ be a nonnegative function from $D(\Pi)$, and \{ $h_1, \ldots, h_r$ \} be a collection of nonnegative functions from $D(\Pi)$. Then

$$S_{\delta, g}^{(\Pi)}(h_1, \ldots, h_r) \leq S_{\delta, g}^{(\Pi')} (h_1, \ldots, h_r).$$

**Proof.** Since $h_i \geq 0$, it is sufficient to establish the inequality

$$S_{\delta, g}^{(\Pi)} (e_{i_1}^{(l)}, \ldots, e_{i_r}^{(l)}) \leq S_{\delta, g}^{(\Pi')} (e_{i_1}^{(l)}, \ldots, e_{i_r}^{(l)}),$$

$e_{i_k}^{(l)} \in D(\Pi)$. By the formula (4.6) we can write

$$S_{\delta, g}^{(\Pi)} (e_{i_1}^{(l)}, \ldots, e_{i_r}^{(l)}) = Z^{\Pi^{-1}}_{\Pi} \int_{R^q} t_{k_1} \cdots t_{k_r} \exp \left( -\frac{1}{2} \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi, l)} t_it_{i'} \right) \prod_{j \in \Pi} e^{-P(t_j):(g)} dt_j,$$

$$= Z^{-1} \int_{R^{q'}} t_{k_1} \cdots t_{k_r} \exp \left( -\frac{1}{2} \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi, l)} t_it_{i'} - \frac{1}{2} \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_it_{i'} \right) \times \prod_{j \in \Pi'} e^{-P(t_j):(g)} dt_j,$$
where

\[ Z = \int_{\mathbb{R}^l} \exp \left( -\frac{1}{2} \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi, l)} t_i t_{i'} - \frac{1}{2} \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} \right) \times \prod_{j \in \Pi'} e^{-P(t_j):g} \, dq^l t, \]

so that the integral with respect to the variables \( \{t_j\}_{j \in \Pi' \setminus \Pi} \) reduces because of the normalization multiplier \( Z^{-1} \).

Using the equality (4.5) we can rewrite \( S_{\delta, g}^{(\Pi)} (\epsilon_{k_1}, \ldots, \epsilon_{k_r}) \) as

\[ Z^{-1} \int_{\mathbb{R}^l} t_{k_1} \cdots t_{k_r} \exp \left( -\frac{1}{2} \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} - \frac{1}{2} \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} \right) \times \prod_{j \in \Pi'} e^{-P(t_j):g} \, dq^l t. \]

It is important that the renormalization of the Wick polynomial \( P \) is taken with respect to the free measure \( \mu_0 \) and does not depend on the region \( \Pi \) or \( \Pi' \).

The function \( S_{\delta, g}^{(\Pi')} (\epsilon_{k_1}, \ldots, \epsilon_{k_r}) \) has the same form as \( S_{\delta, g}^{(\Pi)} (\epsilon_{k_1}, \ldots, \epsilon_{k_r}) \), except that the sum \( \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} + \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} \) is added by the term \( \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} \), and the normalization multiplier is changed appropriately. We put

\[ F(\tau) = Z^{-1} \int_{\mathbb{R}^l} t_{k_1} \cdots t_{k_r} \exp \left( -\frac{1}{2} \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} - \frac{1}{2} \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} \right) \times \prod_{j \in \Pi'} e^{-P(t_j):g} \exp \left( -\tau/2 \sum_{i, i' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi, l)} t_i t_{i'} \right) \, dq^l t, \]

where \( Z_\tau \) is the corresponding normalization multiplier, and \( 0 \leq \tau \leq 1 \). Then, we have \( F(0) = S_{\delta, g}^{(\Pi)} (\epsilon_{k_1}, \ldots, \epsilon_{k_r}) \), and \( F(1) = S_{\delta, g}^{(\Pi')} (\epsilon_{k_1}, \ldots, \epsilon_{k_r}) \). Let us show that \( F(1) \geq F(0) \). We have

\[ \frac{dF}{d\tau} = \frac{1}{2} \sum_{i \in \Pi, i' \in \Pi' \setminus \Pi} \left( -N_{i, i'}^{(\Pi', l)} \langle t_{k_1} \cdots t_{k_r} t_i \rangle \tau - \langle t_{k_1} \cdots t_{k_r} \rangle \tau \langle t_i \rangle \right), \]

(4.8)
where $\langle \cdot \rangle_\tau$ denotes the mean with respect to the measure
\[
d\mu_\tau = Z_\tau^{-1} \exp \left( -\frac{1}{2} \sum_{i, i' \in \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} + \frac{1}{2} (1 - \tau) \sum_{i \in \Pi, \nu' \in \Pi' \setminus \Pi} N_{i, i'}^{(\Pi', l)} t_i t_{i'} \right) \times \prod_{j \in \Pi'} e^{-P(t_j) - (g) d\eta'_j} t.
\]

It is easy to see that the measure $d\mu_\tau$ corresponds to an even Ising ferromagnet. So the derivative (4.8) is nonnegative, since $-N_{i, i'}^{(\Pi', l)} \geq 0$ for $i \in \Pi, i' \in \Pi' \setminus \Pi$ by the property of non-diagonal elements of the matrix $N^{(\Pi', l)}$ proven near the beginning of subsection 4.2, and the expression enclosed in braces in the right-hand part of (4.8) is nonnegative by the Griffiths’ second inequality.

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References

[1] F. Bruhat, *Distributions sur un groupe localement compact et applications à l’étude des représentations des groupes p-adiques*, Bull. Soc. Math. France, 89 (1961) 43-75.

[2] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View* (Springer, New York, 1981).

[3] E. Hewitt and K.A.Ross, *Abstract Harmonical Analysis. Vol.2* (Springer, Berlin, 1970).

[4] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models* (Kluwer, Dordrecht, 1997).

[5] A.N. Kochubei, *Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields* (Marcel Dekker, New York, 2001).

[6] A.N. Kochubei, *On p-adic Green functions*, Theor. Math. Phys. 96 (1993) 854-865.

[7] E.Y. Lerner and M.D. Missarov, *Scalar models of p-adic quantum field theory and the hierarchical models*, Theor. Math. Phys. 78 (1989) 177-184.

[8] E.Y. Lerner and M.D. Missarov, *p-Adic Feynman and string amplitudes*, Comm. Math. Phys. 121 (1989) 35-48.

[9] J. Magnen, *Constructive methods and results*, in XI-th International Congress of Mathematical Physics, ed. D. Iagolnitzer (International Press, Cambridge, MA, 1995) 121-141.
[10] M.D. Missarov, $p$-Adic $\varphi^4$-theory as a functional equation problem, Lett. Math. Phys. 39 (1997) 253-260.

[11] R.S. Phillips, The extensions of dual subspaces invariant under an algebra, in Proc. Intern. Sympos. Linear Spaces, Jerusalem, 1960 (Pergamon Press, Oxford, 1961) 366-398.

[12] M. Reed and B. Simon, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness (Academic Press, New York, 1975).

[13] B. Simon, The $P(\varphi)_2$ Euclidean Quantum Field Theory (Princeton University Press, 1974).

[14] V.S. Varadarajan, Non-Archimedean models for space-time, Mod. Phys. Lett. A 16 (2001) 387–395.

[15] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, $p$-Adic Analysis and Mathematical Physics (World Scientific, Singapore, 1994).

[16] V.S. Vladimirov and I.V. Volovich, $p$-Adic quantum mechanics, Comm. Math. Phys. 123 (1989) 659–676.