Non-adiabatic optomechanical Hamiltonian of a moving dielectric membrane in a cavity

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(Dated: January 30, 2013)

Abstract

We formulate a non-relativistic Hamiltonian in order to describe the interaction between a moving dielectric membrane and radiation pressure. Such a Hamiltonian is derived without making use of the single-mode adiabatic approximation and linear approximation, and hence it enables us to incorporate multi-mode effects and general (non-relativistic) motion of the membrane in cavity optomechanics. By performing second quantization, we show how a set of generalized Fock states can be constructed to represent quantum states of the membrane and cavity field. In addition, we discuss examples showing how photon scattering among different cavity modes would modify the interaction strengths and the mechanical frequency of the membrane.

PACS numbers: 42.50.Wk, 42.50.Pq, 07.10.Cm
I. INTRODUCTION

The interplay between optical and mechanical degrees of freedom via radiation pressure is at the heart of cavity optomechanics [1–4]. With the recent advances in cooling techniques in optomechanical setups [5–10], it is becoming possible to access quantum ground states, and study the interplay at the quantum level experimentally. This may lead to novel applications in quantum information based on optomechanical coupling [11–15]. Specifically, the system formed by a movable dielectric membrane inside an optical cavity provides a basic configuration to explore quantum phenomena in macroscopic objects [16–21], such as non-classical mechanical states [22, 23] and cavity QED effects [24, 25]. Technically, the system has the advantage that it enables a strong and tunable optomechanical coupling, with the possibility of achieving nonlinearity in the membrane’s displacement [21].

The theoretical analysis of a moving membrane system is mainly based on the Hamiltonian [19–25]

\[
H \approx \hbar \omega (\hat{x}_m) a^\dagger a + \hbar \Omega_m b^\dagger b \\
\approx \hbar \left( \omega_0 + \hat{x}_m \frac{\partial \omega(x)}{\partial x} \bigg|_{x_0} \right) a^\dagger a + \hbar \Omega_m b^\dagger b, \tag{1}
\]

where \(a\) and \(b\) are the annihilation operators for optical and mechanical modes, respectively, \(\hat{x}_m \propto (b + b^\dagger)\) is the displacement operator of dielectric motion, and \(\Omega_m\) is the natural frequency of the membrane. The radiation pressure coupling between the two degrees of freedom is contained in the membrane-position-dependent field frequency \(\omega (\hat{x}_m)\), which can be linearized around an equilibrium position \(x_0\). If the first derivative of field frequency vanishes, then the coupling is dominantly quadratic in membrane displacements [20, 21].

We note that the essential assumption of the model (1) is that all photons would stay in the same cavity mode throughout the evolution because of the slow motion of the dielectric membrane. This is an adiabatic single-mode approximation in which photon scattering among different cavity modes are neglected. However, we point out that the mode coupling induced by membrane’s motion can become significant in non-adiabatic regimes. This happens especially when the oscillation frequency of the dielectric is close to the frequency spacing between two cavity modes, in which case transitions between the two modes can be resonantly enhanced. Indeed, by exploiting such a kind of resonance, Dobrindt and Kippenberg have recently indicated an optomechanical displacement transducer with a high
sensitivity [26]. Hence a natural question of the moving-membrane system is how a Hamiltonian model can be rigorously formulated, without employing the adiabatic single-mode approximation. Such a Hamiltonian would provide us with a basis of studying the quantum mechanics of field-membrane systems. A better understanding of the field-membrane interaction also opens up possibilities of new schemes to manipulate quantum states of both the light field and mechanical motion.

A key to address the field-membrane interaction is to treat both the field and the moving membrane consistently as dynamical variables. A similar problem for a moving perfect-mirror system has been treated in Ref. [27]. The case of moving-dielectric system has yet been formulated, although the field Hamiltonian for a dielectric or partially transparent mirror moving in a prescribed trajectory has been discussed in the context of dynamical Casimir effect [28–30]. A major conceptual difficulty of the problem is that the normal modes associated with cavity field depend on the position of the dielectric, which enters as a dynamical variable. These field modes change with time as the dielectric membrane moves, which in turn affect the radiation pressure on the membrane. A consistent approach to the coupled field-membrane dynamics is hence essential to tackle the problem. Recently, Biancofiore et al. [31] have constructed a Hamiltonian based on a linearized form of the radiation pressure coupling (i.e., to first order in $x_m$). While their Hamiltonian may also address non-adiabatic photon scattering among cavity modes, it remains unclear how the constructed Hamiltonian can be extended beyond the linear coupling.

The main purpose of this paper is to provide a Hamiltonian formulation that describes the optomechanical coupling without using the single-mode adiabatic approximation and linear approximation. Based on the interaction between macroscopic dielectric and electromagnetic field, we first derive the Lagrangian and Hamiltonian of the classical counterpart of the system. The canonical quantization of the Hamiltonian and the membrane-position dependent Fock states are then introduced. Our Hamiltonian indicates how photons defined by such Fock states can be coupled to various cavity modes through the motion of the mirror. In the regime where adiabatic single-mode approximation and linear approximation are applicable, our Hamiltonian can be reduced to the usual form (1). Near the end of this paper we indicate some physical consequences arising from the involvement of the multiple cavity modes.
II. THE CLASSICAL LAGRANGIAN AND EQUATIONS OF MOTION

We begin by considering a one-dimensional optical cavity of length $l$ formed by two perfectly reflecting end mirrors. A movable membrane of rigid uniform dielectric is placed inside the cavity. The cavity field is specified by its vector potential $A = A(x,t)\hat{e}_z$ ($0 < x < l$) under transverse gauge, with the boundary conditions $A(x = 0, t) = A(x = l, t) = 0$. We assume non-birefringent dielectric so that the two polarizations of the field are decoupled, and hence it suffices to consider a linearly polarized field. The dielectric is specified by its mass $m$, its center-of-mass coordinate $q = q(t)\hat{e}_x$ ($d/2 < q < l - d/2$), and the dielectric constant

$$
\epsilon(x, q) = \begin{cases} 
1 + \chi, & q - d/2 < x < q + d/2 \\
1, & \text{otherwise}
\end{cases}
$$

(2)

where $d$ and $\chi$ are the width and susceptibility of the dielectric, respectively. We have used the convention $\epsilon_0 = \mu_0 = 1$ (i.e., $c = 1$), and assumed non-magnetic dielectric $\mu = \mu_0$. We have also assumed a non-dispersive dielectric so that $\epsilon$ does not depend on the field frequency.

The motion of the dielectric affects the electromagnetic fields in the cavity, which in turn modifies the radiation pressure on the dielectric. To study the complete dynamics of the system, $q(t)$ must be included as a dynamical degree of freedom. The system is specified by the Lagrangian

$$
L = \frac{1}{2}mq^2 - V(q) + \int_0^l dx \mathcal{L}_F,
$$

(3)
where \( V(q) \) is the external mechanical potential on the dielectric, and \( \mathcal{L}_F \) is the Lagrangian (linear) density of the field after eliminating the electronic degrees of freedom of the dielectric. To find \( \mathcal{L}_F \), we go to an inertial frame \( S' \) in which the dielectric membrane is instantaneously at rest. Assuming the acceleration of the membrane does not change the macroscopic properties of the dielectric, the field Lagrangian density in \( S' \) is given by the familiar form: 

\[
\mathcal{L}_F' = \frac{1}{2} (\epsilon \mathbf{E}'^2 - B'^2) \]

where \( \mathbf{E}' = E'e_z \) and \( \mathbf{B}' = B'e_y \) are the electric and magnetic fields in \( S' \), respectively. Now as the motion of the dielectric in the laboratory frame \( S \) is perpendicular to both fields \( \dot{q} = \dot{q}(t)e_x \), we can relate the fields between \( S \) and \( S' \) by the Lorentz transformation: 

\[
\mathbf{E}' = \gamma (\mathbf{E} + \dot{q} \times \mathbf{B}) \quad \text{and} \quad \mathbf{B}' = \gamma (\mathbf{B} - \dot{q} \times \mathbf{E}),
\]

with 

\[
\gamma = \left(1 - \dot{q}^2\right)^{-1/2}.
\]

In terms of the vector potential, \( \mathbf{E} = -(\partial_t A)e_z \) and \( \mathbf{B} = -(\partial_x A)e_y \), the Lagrangian density \( \mathcal{L}_F \) in the space between the mirrors reads

\[
\mathcal{L}_F = \frac{1}{2} \left[ (\epsilon - \dot{q}^2) \left( \frac{\partial A}{\partial t} \right)^2 - (1 - \epsilon \dot{q}^2) \left( \frac{\partial A}{\partial x} \right)^2 + 2 (\epsilon - 1) \dot{q} \left( \frac{\partial A}{\partial t} \right) \left( \frac{\partial A}{\partial x} \right) \right]. \tag{4}
\]

This agrees with the earlier work by Barton et al. \[28\] and by Salamone \[29\] (in the case \( \mu = 1 \)). We see that the Lagrangian density (4) appears to be more complicated compared with that in the primed frame. This can be understood because in the laboratory frame, a dielectric with polarization \( \mathbf{P} \) moving at velocity \( \dot{q} \) processes a magnetization \( \mathbf{M} = -\dot{q} \times \mathbf{P} \), which must not be neglected in the regime of non-adiabatic dielectric motion. After some calculations, it can be shown that the Lagrangian density (4) contains the interaction terms corresponding to \( \mathbf{P} \cdot \mathbf{E} + \mathbf{M} \cdot \mathbf{B} \).

In this paper we confine our study for non-relativistic motion \( \dot{q} \ll 1 \), so that Eq. (4) is approximated by (up to first order in \( \dot{q} \)),

\[
\mathcal{L}_F = \frac{1}{2} \left[ \epsilon (\partial_t A)^2 - (\partial_x A)^2 \right] + \dot{q} (\epsilon - 1) (\partial_t A) (\partial_x A). \tag{5}
\]

Together with Eq. (3), we obtain

\[
L = \frac{1}{2} m \dot{q}^2 - V(q) + \int_0^l dx \left\{ \frac{1}{2} \left[ \epsilon (\partial_t A)^2 - (\partial_x A)^2 \right] + \dot{q} (\epsilon - 1) (\partial_t A) (\partial_x A) \right\}, \tag{6}
\]

which is the non-relativistic Lagrangian of our membrane-field model.

To justify this Lagrangian, we need to examine whether it consistently generates the equations of motion for both the fields and the membrane within the accuracy limited by the approximation made in (6). First the Euler-Lagrange equation of \( A(x, t) \) derived from
\( \epsilon \partial_t^2 A - \partial_x^2 A + 2 \dot{q}(\epsilon - 1) (\partial_x \partial_t A) + \ddot{q}(\epsilon - 1) \partial_x A = 0, \quad (7) \)

where we have discarded terms involving \( \dot{q}^2 \) for consistency, and made use of the relation \((\partial_t + \dot{q} \partial_x)\epsilon = 0\).

From Eq. (7), the effects of membrane’s motion on the field appear in the \( \epsilon \) terms, the third term with a velocity dependence, and the last term that is proportional to the membrane’s acceleration. If \( \ddot{q} = 0 \), then Eq. (7) is simply the wave equation obtained by transforming the wave equation \( \epsilon \partial_t^2 A' - \partial_x^2 A' = 0 \) in \( S' \) frame to \( S \) frame up to first order in \( \dot{q} [28, 32] \). The acceleration dependent term therefore acts like a source term in the wave equation. However, for an oscillating membrane with a mechanical frequency \( \Omega \) and field frequency \( \omega \), the ratio of the acceleration dependent term to the velocity dependent term in Eq. (7) is of the order \( \Omega/\omega \), which is much smaller than one.

Next, the Euler-Lagrange equation of motion of the membrane based on (6) is given by

\[
m \ddot{q} = -\frac{\partial V(q)}{\partial q} + 1 \frac{1}{2} \left[ 1 + \chi \left( \frac{\partial A}{\partial x} \right)^2 \right]^{q-d/2}.
\]

The second term on the right-hand side of Eq. (8) corresponds to a radiation pressure force from the field. Such a force term is consistent with that obtained from the Lorentz force density \( f' = (\partial_t P') \times B' \) appearing in \( S' [33] \). This can be shown by using \( P' = \chi E' \) and the wave equation in dielectric rest frame, then a straightforward transform on the force to the laboratory frame in the non-relativistic limit would yield the same radiation force expression in Eq. (8), apart from a term that is about \( \dot{q}/c \) times smaller. Note that our Lagrangian (6) gives a wave equation that is accurate up to \( O(\dot{q}) \), but the accuracy is lower by one order of \( \dot{q} \) for the membrane’s equation of motion. This is because of the partial derivative \( \partial/\partial \dot{q} \) in the membrane’s Euler-Lagrange equation.

III. THE HAMILTONIAN AND QUANTIZATION

The Hamiltonian associated with \( L \) is defined by

\[
H (\Pi, A, p, q) \equiv \pi \dot{q} + \int_0^l dx \left[ \Pi (\partial_t A) - L (A, \partial_t A, q, \dot{q}) \right],
\]

(9)
where $p$ and $\Pi(x,t)$ are canonical momenta conjugate to $q$ and $A(x,t)$, respectively,

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} + \int_0^l dx \, (\epsilon - 1) \left( \partial_t A \right) \left( \partial_x A \right)$$

(10)

$$\Pi = \frac{\partial L}{\partial \dot{t} A} = \epsilon \left( \partial_t A \right) + \dot{q} \left( \epsilon - 1 \right) \left( \partial_x A \right) .$$

(11)

We see that the dielectric canonical momentum $p$ is not equal to its kinetic momentum $m\dot{q}$ for non-zero fields. The explicit expression of the Hamiltonian (9) now reads

$$H = \frac{1}{2m'} (p + \Lambda)^2 + V(q) + \frac{1}{2} \int_0^l dx \left[ \frac{\Pi^2}{\epsilon} + \left( \partial_x A \right)^2 \right] ,$$

(12)

with $\Lambda$ given by

$$\Lambda = - \int_0^l dx \left( \frac{\epsilon - 1}{\epsilon} \right) \Pi \left( \partial_x A \right) ,$$

(13)

and $m'$ is identified as a ‘renormalized mass’ defined by

$$m' = m \left[ 1 - \frac{1}{m} \int_0^l dx \frac{(\epsilon - 1)^2}{\epsilon} \left( \frac{\partial A}{\partial x} \right)^2 \right] ^{1/2} .$$

(14)

The form of Hamiltonian (12) is similar to the minimal coupling Hamiltonian in electrodynamics with $\Lambda$ somehow playing the role of the vector potential in the kinetic energy term.

At this point we would like to comment on the renormalized mass $m'$ defined in Eq. (14) [34]. First, it might look peculiar that the renormalized mass $m'$ depends only on the magnetic field energy inside the membrane, but we point out that this is an artifact due to the truncation of the Lagrangian (4) up to $\dot{q}$. If we retain $\dot{q}^2$ terms in (4), it can be shown that $m'$ appears to depend on the electric field energy as well. Second, in quantum theory, the vacuum field energy would make the integral in Eq. (14) divergent if all the field frequencies are counted. In practice, however, a physical dielectric membrane must become transparent (i.e. $\epsilon \rightarrow 1$) at high field frequencies, so there is only a finite range of field frequencies contributing. It is useful to estimate the order of magnitude if the field frequencies are counted up to $\omega_c = 10^{17}$ Hz in the ultra-violet range. We then find that for a $l = 1$ cm cavity, the vacuum contribution is of the order $10^{-28}$ kg, many orders of magnitude lighter than a pico-gram membrane in typical optomechanical setup. If the cavity is filled with photon excitations in a single mode, then a similar consideration shows that the photon number has to be as high as $10^{15}$ for the mass correction to be comparable to the mass of
FIG. 2: A sketch of a few mode functions. Evidently the mode functions depends on the position of the dielectric.

the dielectric. Hence the mass correction can be safely neglected as long as we restrict our dielectric model to optical field frequencies and a sufficiently massive membrane. From now on, we will take $m' \approx m$, and the Hamiltonian reads

$$H = \frac{1}{2m} (p + \Lambda)^2 + V(q) + \frac{1}{2} \int_0^l dx \left[ \frac{\Pi^2}{\epsilon} + (\partial_x A)^2 \right]. \quad (15)$$

To quantize the system, we promote the dynamical variables $q, p, A(x), \Pi(x)$ into operators by postulating the commutation relations $[\hat{q}, \hat{A}(x)] = [\hat{q}, \hat{\Pi}(x)] = [\hat{p}, \hat{A}(x)] = [\hat{p}, \hat{\Pi}(x)] = 0, \quad [\hat{q}, \hat{p}] = i\hbar, \quad [\hat{A}(x), \hat{\Pi}(x')] = i\hbar \delta(x-x').$ The quantum Hamiltonian takes the same expression as (15), but with $\Lambda$ defined in (13) symmetrized as

$$\hat{\Lambda}(\hat{q}) = - \int_0^l dx \left( \frac{\epsilon - 1}{2\epsilon} \right) \left[ \hat{\Pi} \left( \partial_x \hat{A} \right) + \left( \partial_x \hat{A} \right) \hat{\Pi} \right]. \quad (16)$$

A. Instantaneous normal-mode projection

The field operators can be projected onto any set of complete orthonormal modes. For instance, we may use the set of mode functions $\{\varphi_k(x, q_0)\}$ defined by

$$\frac{\partial^2 \varphi_k(x, q_0)}{\partial x^2} + \epsilon(x, q_0) \omega_k^2(q_0) \varphi_k(x, q_0) = 0 \quad (17)$$
with a vanishing boundary condition at \( x = 0 \) and \( x = l \). Here \( q_0 \) is a reference position (c-number), say, an equilibrium position of the dielectric. The orthonormality relation between these mode functions is written as: \( \int_0^l dx \epsilon(x, q_0) \varphi_k(x, q_0) \varphi_j(x, q_0) = \delta_{kj} \). Note that we have explicitly labelled the mode functions and frequencies as \( \varphi_k(x, q_0) \) and \( \omega_k(q_0) \) to emphasize their dependence on \( q_0 \), i.e., \( \varphi_k(x, q_0) \) would be the normal-mode of the field if the membrane had been fixed at \( x = q_0 \).

By substituting \( \hat{A}(x) = \sum_k \hat{Q}_k \varphi_k(x, q_0) \) and \( \hat{\Pi}(x) = \sum_k \hat{P}_k \epsilon(x, q_0) \varphi_k(x, q_0) \), the Hamiltonian reads

\[
H = \frac{1}{2m} \left[ \hat{p} + \sum_{k,j} \frac{\xi_{kj}}{2} \left( \hat{P}_k \hat{Q}_j + \hat{Q}_j \hat{P}_k \right) \right]^2 + V(q) + \frac{1}{2} \left( \sum_{k,j} \eta_{kj} \hat{P}_k \hat{P}_j + \sum_k \omega_k^2 \hat{Q}_k^2 \right),
\]

where

\[
\begin{align*}
\hat{Q}_k &= \int_0^l dx \epsilon(x, q_0) \varphi_k(x, q_0) \hat{A}(x) \\
\hat{P}_k &= \int_0^l dx \varphi_k(x, q_0) \hat{\Pi}(x) \\
\eta_{kj}(\hat{q}) &= \int_0^l dx \epsilon^{-1}(x, \hat{q}) \varphi_k(x, q_0) \varphi_j(x, q_0) \\
\xi_{kj}(\hat{q}) &= -\int_0^l dx \left[ \frac{\epsilon(x, \hat{q}) - 1}{\epsilon(x, \hat{q})} \right] \epsilon(x, q_0) \varphi_k(x, q_0) \frac{\partial \varphi_j(x, q_0)}{\partial x}.
\end{align*}
\]

In this way the standard commutation relations: \([\hat{q}, \hat{Q}_k] = [\hat{q}, \hat{P}_k] = [\hat{p}, \hat{Q}_k] = [\hat{p}, \hat{P}_k] = 0\), \([\hat{Q}_j, \hat{P}_k] = i\hbar \delta_{kj}\) are preserved. Since \( \varphi_k(x, q_0) \) is not an instantaneous normal-mode of the cavity, the field part of the Hamiltonian (18) is not diagonalized.

To reveal the physical picture it is always desirable to cast the field part of the Hamiltonian in a diagonal basis. This can be achieved by performing a unitary transformation \( H' = T_1^* HT_1 \) (where \( T_1 \) is defined in Appendix A):

\[
H' = \frac{1}{2m} \left[ \hat{p} + \sum_{k,j} g_{kj} \hat{P}_k \hat{Q}_j \right]^2 + V(\hat{q}) + \frac{1}{2} \sum_k \left[ \hat{P}_k^2 + \omega_k^2(\hat{q}) \hat{Q}_k^2 \right],
\]

with

\[
g_{kj}(\hat{q}) = -\int_0^l dx \left\{ \epsilon(x, \hat{q}) \frac{\partial \varphi_j(x, \hat{q})}{\partial \hat{q}} + [\epsilon(x, \hat{q}) - 1] \frac{\partial \varphi_j(x, \hat{q})}{\partial x} \right\} \varphi_k(x, \hat{q})
\]

which satisfies \( g_{kj} = -g_{jk} \), so that \( g_{kk} = 0 \). The field part of \( H' \) is now diagonalized with
\( \dot{q} \)-dependent frequencies \( \omega_k(\dot{q}) \). In particular, the transformed field operators are given by

\[
T_1^\dagger \hat{A}(x) T_1 = \sum_k \left( T_1^\dagger \hat{Q}_k T_1 \right) \varphi_k(x, q_0) = \sum_k \hat{Q}_k \varphi_k(x, \dot{q}) \quad (25)
\]

\[
T_1^\dagger \hat{\Pi}(x) T_1 = \sum_k \left( T_1^\dagger \hat{P}_k T_1 \right) \epsilon(x, q_0) \varphi_k(x, q_0) = \sum_k \hat{P}_k \epsilon(x, \dot{q}) \varphi_k(x, \dot{q}) \quad (26)
\]

according to Appendix A. \( \varphi_k(x, \dot{q}) \) is defined in Eq. (17) but with \( q_0 \) replaced by \( \dot{q} \). Noting that \( \dot{q} \) is the position operator of the membrane, \( \varphi_k(x, \dot{q}) \) corresponds to an instantaneous normal-mode of the cavity. Therefore \( \hat{Q}_k \) and \( \hat{P}_k \) become the expansion of transformed \( \hat{A}(x) \) and \( \hat{\Pi}(x) \) using the instantaneous normal-modes respectively:

\[
\hat{Q}_k = \int_0^l dx \left[ T_1^\dagger \hat{A}(x) T_1 \right] \epsilon(x, \dot{q}) \varphi_k(x, \dot{q}) \quad (27)
\]

\[
\hat{P}_k = \int_0^l dx \left[ T_1^\dagger \hat{\Pi}(x) T_1 \right] \varphi_k(x, \dot{q}). \quad (28)
\]

It should be noted that \( \hat{Q}_k \) and \( \hat{P}_k \) are the same operators defined in (19) and (20) and hence they are independent of \( \dot{q} \). Therefore, the \( \dot{q} \)-dependence of \( T_1(\dot{q}) \), \( \epsilon(x, \dot{q}) \) and \( \varphi_k(x, \dot{q}) \) must have ‘cancelled out’ each other in (27) and (28).

**B. Generalized Fock spaces**

To represent the quantum state of the system, we introduce the \( \dot{q} \)-dependent annihilation and creation operators for each cavity field mode:

\[
a_k(\dot{q}) = \sqrt{\frac{1}{2\hbar \omega_k(\dot{q})}} \left[ \omega_k(\dot{q}) \hat{Q}_k + i \hat{P}_k \right], \quad (29)
\]

\[
a_k^\dagger(\dot{q}) = \sqrt{\frac{1}{2\hbar \omega_k(\dot{q})}} \left[ \omega_k(\dot{q}) \hat{Q}_k - i \hat{P}_k \right], \quad (30)
\]

which satisfy the commutation relation \([a_k(\dot{q}), a_j^\dagger(\dot{q})] = \delta_{kj}\). Since \( a_k(\dot{q}) \) depends on \( \dot{q} \), for each position of the dielectric we have a set of Fock states associated with that position. These states can be labelled as \(|\{n_k\}, q\rangle\), where \( \{n_k\} = \{n_1, n_2, n_3, \ldots\} \) denotes the occupation number of each photon mode. \(|\{n_k\}, q\rangle\) is a simultaneous eigenstate of the photon-number operator \( a_k^\dagger(\dot{q}) a_k(\dot{q}) \) and the position operator \( \dot{q} \) i.e. \( a_k^\dagger(\dot{q}) a_k(\dot{q}) |\{n_k\}, q\rangle = n_k |\{n_k\}, q\rangle \) and \( \dot{q} |\{n_k\}, q\rangle = q |\{n_k\}, q\rangle \). Such a set of eigenstates are orthonormal and complete, so that any quantum state of the whole system \(|\Psi\rangle\) can be expanded in the basis of these eigenstates.
i.e.
\[ |\Psi\rangle = \sum_{\{n_k\}} \int_{d/2}^{l-d/2} C(\{n_k\}, q)|\{n_k\}, q\rangle dq, \]  
(31)
where \( C(\{n_k\}, q) \) is the probability amplitude.

With the help of the \( \hat{q} \)-dependent annihilation and creation operators, the Hamiltonian (23) becomes
\[ H' = \left( \hat{p} + \Gamma \right)^2 \frac{2m}{2m^2} + V(\hat{q}) + \sum_k \hbar \omega_k(\hat{q}) \left( a_k^\dagger a_k + \frac{1}{2} \right), \]  
(32)
where we have used a shorthand \( a_k = a_k(\hat{q}) \) for convenience, and
\[ \Gamma(\hat{q}) = -\frac{i\hbar}{2} \sum_{k,j} g_{k,j}(\hat{q}) \sqrt{\frac{\omega_k}{\omega_j}} \left( a_k a_j - a_k^\dagger a_j^\dagger + a_j^\dagger a_k - a_j a_k^\dagger \right). \]  
(33)

The vacuum field energy appearing in (32) leads to the Casimir force on the dielectric (e.g. see [35] for calculations of a setup similar to our case). We may replace the vacuum energy by the Casimir potential energy. However, since the Casimir energy is feebly small compared with \( V(\hat{q}) \) in typical optomechanical experiments, its effect should be negligible.

C. A unitary transformation

The canonical momentum operator \( \hat{p} \) in the Hamiltonian (32) differs from the kinetic momentum \( m\dot{q} \) due to \( \Gamma(\hat{q}) \). We may apply a unitary transformation to the Hamiltonian to make the two momenta coincide. This procedure is analogous to the case of atom-field interaction, where one transforms the minimal-coupling Hamiltonian into the electric-dipole form under electric-dipole approximation. The transformation is defined with the unitary operator
\[ T_2(\hat{q}) = \exp \left[ -\frac{i}{\hbar} \int_{q_0}^{\hat{q}} dq' \Gamma(q') \right]. \]  
(34)

The transformed Hamiltonian \( \tilde{H} = T_2^\dagger H' T_2 \) reads (Appendix [33] )
\[ \tilde{H} = \frac{\hat{p}^2}{2m} + u(\hat{q}) + \sum_k \hbar \omega_k a_k^\dagger a_k + \sum_{k,j} \hbar \left[ \xi_{kj}^{(+)} \left( a_k^\dagger a_j + a_j^\dagger a_k \right) + \xi_{kj}^{(-)} \left( a_k^\dagger a_j^\dagger + a_k a_j \right) \right], \]  
(35)
where

\[ u(\hat{q}) = V(\hat{q}) + \frac{1}{2} \sum_k \hbar \left( \omega_k(\hat{q}) + 2\xi_{kk}^{(+)}(\hat{q}) \right) \]  

(36)

\[ \xi_{kj}^{(\pm)}(\hat{q}) = \frac{1}{4} \sqrt{\omega_k \omega_j} \left[ 2\lambda_{kj} \frac{\omega_k}{\omega_j} + \sum_l \lambda_{lk} \lambda_{lj} \frac{\omega_l^2}{\omega_k \omega_j} \pm (\lambda_{kj} - \lambda_{jk}) \right] \]  

(37)

\[ \lambda_{kj}(\hat{q}) = f_{kj} + \frac{1}{2!} \sum_h f_{kh} f_{kj} + \frac{1}{3!} \sum_{h,l} f_{kh} f_{hl} f_{lj} + \ldots \]  

(38)

\[ f_{kj}(\hat{q}) = \int_{q_0}^{\hat{q}} dq' g_{kj}(q'). \]  

(39)

The Hamiltonian (35) is the main result of this paper, which determines the coupling strengths \( \xi_{kj}^{(\pm)} \) once the mode functions are known. We stress that it is applicable to general motion of the membrane, since no assumption of the motion (except \( \dot{q} \ll 1 \)) have been made.

It should be noted that the Hamiltonian (35) contains a photon-number non-conserving part proportional to \( (a_k^\dagger a_j^\dagger + a_k a_j) \), which is responsible for dynamical Casimir effect [36].

We remark that the transformation has modified the mode function associated with \( a_k \), as the transformed cavity field operators (related to that in (15) by the combined unitary transform defined via \( T = T_1 T_2 \)) read

\[ T^\dagger \hat{A}(x) T = \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \left( a_k + a_k^\dagger \right) \tilde{\varphi}_k(x, \hat{q}) \]  

(40)

\[ T^\dagger \hat{\Pi}(x) T = -i \sum_k \sqrt{\frac{\hbar \omega_k}{2}} \left( a_k - a_k^\dagger \right) \epsilon(x, \hat{q}) \tilde{\varphi}_k(x, \hat{q}). \]  

(41)

In other words, the mode functions have been changed to \( \tilde{\varphi}_k(x, \hat{q}) \equiv \varphi_k(x, \hat{q}) + \sum_j \lambda_{jk}(\hat{q}) \varphi_j(x, \hat{q}) \) instead of \( \varphi_k(x, \hat{q}) \). We show in Appendix [3] that \( \{ \tilde{\varphi}_k(x, \hat{q}) \} \) indeed forms an orthonormal complete set of mode functions.

**D. Linear approximation**

The Hamiltonian \( \hat{H} \) in (35) exhibits nonlinear feature in the field-membrane coupling. In most practical situations, the potential \( V(\hat{q}) \) bounds the dielectric membrane about an equilibrium position \( q_0 \) such that \( \hat{x}_m = \hat{q} - q_0 \) is small compared with the field wavelengths
concerned. Therefore we can perform the expansion

\[ \omega_k(\hat{q}) \approx \omega_{k0} + \hat{x}_m \left( \frac{\partial \omega_k}{\partial q} \right)_{q=q_0} \quad (42) \]

\[ a_k(\hat{q}) \approx a_{k0} + \frac{\hat{x}_m}{2\omega_{k0}} \left( \frac{\partial \omega_k}{\partial q} \right)_{q=q_0} a_{k0}^{\dagger} \quad (43) \]

\[ \lambda_{kj} \approx \hat{x}_m g_{kj}^{(0)} \quad (44) \]

where \( \omega_{k0} = \omega_k(q = q_0) \) and \( g_{kj}^{(0)} = g_{kj}(q = q_0) \) are the frequency and coupling constant associated with the equilibrium position respectively, and

\[ a_{k0} = \sqrt{\frac{1}{2\hbar \omega_{k0}}} \left( \omega_{k0} \hat{Q}_k + i \hat{P}_k \right) \quad (45) \]

is the annihilation operator linearized in \( \omega_k(\hat{q}) \). The linearized \( a_{k0} \) commutes with both \( \hat{q} \) and \( \hat{p} \). The linearized \( \tilde{H} \) reads

\[ \tilde{H} = \frac{\hat{p}^2}{2m} + u(\hat{x}_m) + \sum_k \hbar \omega_{k0} a_{k0}^{\dagger} a_{k0} + \hat{x}_m F_0, \quad (46) \]

where \( F_0 \) is the normal-ordered radiation pressure force

\[ F_0 = \frac{\hbar}{2} \sum_{k,j} \left[ \left( \frac{\partial \omega_k}{\partial q} \right)_{q=q_0} \right] \delta_{kj} + \omega_{k0} \sqrt{\frac{\omega_{k0}}{\omega_{j0}}} g_{kj}^{(0)} \left( a_{k0}^{\dagger} a_{j0}^{\dagger} + a_{k0} a_{j0} + a_{k0}^{\dagger} a_{j0} + a_{j0}^{\dagger} a_{k0} \right). \quad (47) \]

This agrees with the work in Ref. [31], and the corrections to the coupling term are of the order \( \hat{x}_m^2 \).

**IV. SOME PHYSICAL CONSEQUENCES OF THE HAMILTONIAN**

Current experimental and theoretical study on optomechanical systems are mainly in the regime of single-mode and adiabatic approximations. While our formulation does not require these approximations, we show in this section how our Hamiltonian model can be reduced to simpler forms when these approximations are applicable. In addition, we indicate some possible modifications due to the interaction with multiple cavity modes.

**A. Frequency shift in single-mode limit**

Single-mode approximation is applicable when the cavity field is dominantly contributed by a single mode \( k \), and photon excitations in other modes are negligible. As \( g_{kk} = 0 \), the
single-mode consideration immediately gives $\Gamma \approx 0$ from (33), and so the Hamiltonian (32) becomes

$$H' \approx \frac{\hat{p}^2}{2m} + u(\hat{q}) + \hbar \omega_k(\hat{q}) a_k^\dagger(\hat{q}) a_k(\hat{q}).$$

(48)

By linearizing in $\hat{x}_m = \hat{q} - q_0$ and employing rotating-wave approximation (RWA), we have

$$H' \approx \frac{\hat{p}^2}{2m} + u(\hat{q}) + \hbar \omega_{k0} a_{k0}^\dagger a_{k0} + \hat{x}_m \left( \hbar \left. \frac{\partial \omega_k}{\partial \hat{q}} \right|_{\hat{q}=q_0} \right) a_{k0}^\dagger a_{k0},$$

(49)

where $\omega_{k0}$ and $a_{k0}$ are the linearized frequency and annihilation operator defined in the previous section. Equivalently, (49) can also be obtained from (46) and (47) under single-mode approximation together and neglecting the counter-rotating terms.

The above Hamiltonians (48) and (49) are what one would expect when the dielectric motion is adiabatically slow. In obtaining (48) we have simply ‘dropped out’ the $j \neq k$ terms in (32), hence scattering processes between photon modes are neglected. However, it should be noted that while only the field mode $k$ dominates, a photon in that mode can make (virtual) transitions to other modes then back to the $k$ mode. Such a process induces a shift in $\omega_k(q)$, which is the leading order correction to (48). To determine the correction, we work on our full multi-mode Hamiltonian (35) and examine the Heisenberg equation of motion for $a_j(\hat{q})$,

$$\dot{a}_j = -i \omega_j a_j + \frac{1}{4} \left( \frac{\hat{q}}{\omega_j} \frac{\partial \omega_i}{\partial \hat{q}} a_j^\dagger + a_j^\dagger \frac{1}{\omega_j} \frac{\partial \omega_j}{\partial \hat{q}} \hat{q} \right) - i \sum_l \left[ \left( \xi_{jl}^{(+)} + \xi_{lj}^{(+)} \right) a_l + \left( \xi_{jl}^{(-)} + \xi_{lj}^{(-)} \right) a_l^\dagger \right].$$

(50)

We have used the equation of motion $\dot{\hat{q}} = \hat{p}/m$ and the relation

$$\frac{\partial a_j}{\partial \hat{q}} = \frac{1}{2 \omega_j} \frac{\partial \omega_j}{\partial \hat{q}} a_j^\dagger$$

(51)

which follows from (29) and (30). The terms in (50) that contain $a_j^\dagger$ and $a_l^\dagger$ are fast-rotating, and can be neglected in the spirit of rotating-wave approximation (RWA). Provided further that the oscillation frequency of the membrane $\Omega$ is low compared with the frequency difference $|\omega_k - \omega_j|$, we may adiabatically eliminate $a_j (j \neq k)$ in the equation of motion of $a_k$:

$$\dot{a}_k \approx -i \left[ \omega_k + 2 \xi_{kk}^{(+)} + \sum_{k,j \neq k} \frac{\left( \xi_{kj}^{(+)} + \xi_{jk}^{(+)} \right)^2}{\left( \omega_k + 2 \xi_{kk}^{(+)} \right) - \left( \omega_j + 2 \xi_{jj}^{(+)} \right)} \right] a_k,$$

(52)
Hence as a leading order non-adiabatic correction, the frequency $\omega_k$ in (48) should be modified into $\omega_k + \Delta_k$, where

$$\Delta_k = 2\xi_{kk}^{(+)} + \sum_{k,j}^{k,j \neq k} \frac{\left(\xi_{kj}^{(+)} + \xi_{jk}^{(+)}\right)^2}{\left(\omega_k + 2\xi_{kk}^{(+)}\right) - \left(\omega_j + 2\xi_{jj}^{(+)}\right)}.$$  (53)

We emphasize that this frequency shift is caused by the interaction with other field modes, as is evident in the expression of (53). Hence the shift is essentially a multi-mode effect, even though it is calculated under the single-mode limit.

If the dielectric membrane is bounded by a harmonic potential $u(\hat{x}_m) \approx \frac{m\Omega^2\hat{x}_m^2}{2}$, with the membrane equilibrium position $q_0$ at an extremum of $\omega_k(\hat{q})$ (for example, at $q_0 = l/2$), the linearized single-mode Hamiltonian under RWA reads

$$\tilde{H} \approx \frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega^2\hat{x}_m^2 + \hbar\omega_{k0}a_{k0}^{\dagger}a_{k0}$$

$$+ \hbar\hat{x}_m^2 \left[ \frac{\omega_{k0}''}{2} + \sum_j g^{(0)2}_{kj} \frac{\left(\omega_{k0} - \omega_{j0}\right)^2 \left(\omega_{k0} + \omega_{j0}\right)}{4\omega_{k0}\omega_{j0}} \right] a_{k0}^{\dagger}a_{k0}$$  (54)

where $\omega_{k0}'' = \frac{\partial^2 \omega_k}{\partial q^2}_{q=q_0}$. The last term of (54) comes from the Taylor expansion of $\Delta_k$, which has a leading order of $\hat{x}_m^2$. It is intriguing to note that the last term can be viewed in two ways: it can be regarded as a shift in field frequency proportional to $\hat{x}_m^2$ (hence proportional to phonon number under RWA [20]), as well as a shift in the mechanical frequency $\Omega$ that is proportional to the photon number.

Eq. (54) shows that besides $\omega_{k0}''$, the frequency shift is also contributed by the presence of other field modes. To compare the effect between these two contributions, we have performed numerical calculations based on parameters on the experimental setup of [20] where $l = 0.06$ m, $n \equiv \sqrt{1 + \chi} = 2.2$, $d = 50$ nm. With $q_0 = l/2$, a mode with $\omega_{k0} \approx 1.77 \times 10^{15}$ Hz (corresponding to $\lambda \approx 1064$ nm) is present with $\omega_{k0}'' \approx -3.68 \times 10^5$ Hz nm$^{-2}$. From our calculation, modes with non-zero $g^{(0)2}_{kj}$ contribute about 0.22 Hz nm$^{-2}$ to the sum in (54).

While the contribution from each mode is feebly small, we found that their contributions are approximately the same over a wide spectral interval. Since these contributing modes have a frequency spacing of about $3 \times 10^{10}$ Hz, so if we include modes of $10^{15} - 10^{16}$ Hz in our model, the correction would sum to $\approx 0.7 \times 10^5$ Hz nm$^{-2}$, which is only a few times smaller than the contribution from $\omega_{k0}''$. However, we must point out that our result depends on the number of modes included in the model. For this reason, the use of our calculation
is limited to an order-of-magnitude estimate. A more detailed analysis should include the dispersion effect of the dielectric, so as to address the spectral interval unambiguously.

B. Resonant mode transitions

Resonant mode transitions occur when the mechanical frequency of the membrane $\Omega$ is comparable with the frequency spacing of neighboring field modes. As an illustrative example, let us consider the case where $q = q_0 = l/2$ and a harmonic potential $u(\hat{x}_m) \approx m\Omega^2\hat{x}_m^2/2$. If the index of refraction of the membrane is sufficiently high, the eigen-frequencies of cavity modes distribute as doublets. Now suppose the mechanical frequency $\Omega$ is close to the frequency difference of two cavity modes in a doublet, say, $k_1$ and $k_2$, then the two modes can be resonantly coupled. Neglecting other non-resonant cavity modes, we approximate the Hamiltonian (46) as

$$\tilde{H} = \hbar \Omega \left( b^\dagger b + \frac{1}{2} \right) + \hbar \left( \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 \right) + \hbar \eta \left( b + b^\dagger \right) \left( a_1^\dagger a_2 + a_2^\dagger a_1 \right), \quad (55)$$

where $\omega_1$ ($\omega_2$) and $a_1$ ($a_2$) are the linearized frequency and annihilation operator of the $k_1$ ($k_2$) mode, respectively, $b$ is the annihilation operator of the dielectric motion, $\eta$ is the coupling frequency defined by

$$\eta = g_{12}^{(0)} \sqrt{\frac{\hbar}{8m\Omega}} \frac{(\omega_1^2 - \omega_2^2)}{\sqrt{\omega_1\omega_2}}. \quad (56)$$

Note that due to symmetry of the system,

$$\left. \frac{\partial \omega_{k_1}(q)}{\partial q} \right|_{q=q_0} = \left. \frac{\partial \omega_{k_2}(q)}{\partial q} \right|_{q=q_0} = 0, \quad (57)$$

hence the usual radiation pressure term (49) is absent, and the field-membrane coupling to first order in $\hat{x}_m$ describes the scattering between the two field modes (as in [26]). In particular, the case of $\Omega = \omega_2 - \omega_1$ (assuming $\omega_2 > \omega_1$) corresponds to a resonance at which the mode coupling can be resonantly enhanced. If a rotating-wave approximation is made, the interaction terms in Eq. (55) would take the same form as that appears in parametric down conversion in nonlinear optics.

As a remark we note that the field frequency spacing can be reduced by increasing the cavity length or tuning the refractive index and thickness of the membrane. Recent study [21] also shows that avoided crossings of transverse field modes (due to broken symmetry
of the cavity along its lateral dimensions) can provide a frequency spacing of the order \( \sim 1 \) MHz, which is comparable to the mechanical frequency achievable in current optomechanical experiments \[37\]. Therefore it would be possible for the membrane frequency to match the field frequency spacing.

V. CONCLUSION

To conclude, we have presented a non-relativistic Lagrangian and a Hamiltonian for a one-dimensional coupled membrane-field system in the \( \dot{q} \ll 1 \) regime. The classical equations of motion of both the field and membrane are obtained within the accuracy limited by the approximation made in the Lagrangian. Our Hamiltonian \[35\] should capture optomechanical processes not described by the single-mode adiabatic approximation and linear approximation. For example, we have indicated that the presence of multiple cavity-modes can modify the single-mode Hamiltonian [see Eq. \[51\]], and possibly give rise to parametric down conversion type interaction if the membrane frequency matches the frequency spacing of cavity modes. With the explicit form of interaction strengths between the membrane and various cavity modes in \[35\], one can further study quantum dynamics resulted from optomechanical coupling in non-adiabatic regimes.

There are interesting subtle features that emerge in developing the Hamiltonian model. First, the velocity-dependent coupling in the Lagrangian \[5\], which is necessary to recover the leading radiation pressure force term in \[8\], causes the membrane to have a canonical momentum different from its kinetic momentum. The unitary transformation \( T_2 \) can eliminate the difference, but it turns out that \( T_2 \) also changes the field operators accordingly. In this paper we have shown that the transformation \( T_2 \) on the fields can be interpreted as a modification of mode functions. Another subtle feature is the use of instantaneous mode functions in diagonalizing the field part of the Hamiltonian \[23\]. The Fock states associated with instantaneous modes are defined by the \( \dot{q} \)-dependent photon creation/annihilation operators. It may not be convenient to perform calculations directly based on such creation and annihilation operators, but these operators can always be expanded around the equilibrium position of the membrane in order to obtain the relevant interaction terms. Using our approach, such an expansion can be carried out to the first order of \( \dot{x}_m \) (as in Sec. IIID), or to higher orders as desired. Finally, we should point out that since our model is based on
the non-dispersive approximation of the dielectric, a further investigation should incorporate the absorptive and dispersive properties. This requires an extension of the current theory of quantized field in dispersive media \[38\] to a moving media.

**Acknowledgments**—This work is partially supported by a grant from the Research Grants Council of Hong Kong, Special Administrative Region of China (Project No. CUHK401810).

**Appendix A: The \(T_1(\hat{q})\) transformation**

The unitary operator \(T_1(\hat{q})\) is defined as

\[
T_1(\hat{q}) = \exp \left[-\frac{i}{\hbar} \sum_{k,j} \int_{q_0}^{\hat{q}} dq' \zeta_{kj}(q') \left(\hat{P}_k \hat{Q}_j + \hat{Q}_j \hat{P}_k\right) / 2\right],
\]  

(A1)

with

\[
\zeta_{kj}(q') = \int_0^l dx \epsilon(x, q') \frac{\partial \varphi_j(x, q')}{\partial q'} \varphi_k(x, q').
\]  

(A2)

Here \(\varphi_k(x, q')\) are mode functions defined in Eq. (17) but with \(q_0\) replaced by \(q'\). The integrand \(\zeta_{kj}(q')\) in \(A1\) is a c-number except at the upper limit \(\hat{q}\). In other words, the integral in the exponential of \(A1\) can be viewed as an anti-derivative of \(\zeta_{kj}(q')\), evaluated at the two end-points. It follows that

\[
p' = T_1^* \hat{p} T_1 = \hat{p} - i\hbar T_1^* \frac{\partial T_1}{\partial \hat{q}} = \hat{p} - \sum_{k,j} \frac{\zeta_{kj}}{2} \left(\hat{P}_k \hat{Q}_j + \hat{Q}_j \hat{P}_k\right).
\]  

(A3)

To consider the transformation on \(\hat{Q}_k\) and \(\hat{P}_k\), we decompose \(T_1\) into a (continuous) product of infinitesimal transform

\[
T_1 = \prod_{q' = q_0}^{\hat{q}} \left[I - \frac{i}{\hbar} dq' \sum_{k,j} \zeta_{kj}(q') \left(\hat{P}_k \hat{Q}_j + \hat{Q}_j \hat{P}_k\right) / 2\right] \equiv \prod_{q' = q_0}^{\hat{q}} \left(I - \frac{i}{\hbar} dq' K(q')\right)
\]  

(A4)

where the product is ‘\(q'\)-ordered’, i.e. the leftmost term is associated with \(q' = q_0\), the rightmost with \(q' = \hat{q}\). One can check that \(A1\) and \(A4\) are equivalent by expanding the product of \(A4\) into sums of integrals. Each infinitesimal transform acts on \(\hat{Q}_k\) and \(\hat{P}_k\) as

\[
\left(I + \frac{i}{\hbar} dq' K(q')\right) \hat{Q}_k \left(I - \frac{i}{\hbar} dq' K(q')\right) = \hat{Q}_k + dq' \sum_j \zeta_{kj}(q') \hat{Q}_j
\]  

(A5)

\[
\left(I + \frac{i}{\hbar} dq' K(q')\right) \hat{P}_k \left(I - \frac{i}{\hbar} dq' K(q')\right) = \hat{P}_k - dq' \sum_j \zeta_{jk}(q') \hat{P}_j
\]  

(A6)
with correction only in second order infinitesimals. Hence the infinitesimal transform modifies \( \hat{A}(x) \) and \( \hat{\Pi}(x) \) by

\[
\left( I + \frac{i}{\hbar} dq' K(q') \right) \left( \sum_k \hat{Q}_k \varphi_k(x, q') \right) \left( I - \frac{i}{\hbar} dq' K(q') \right) = \sum_k \hat{Q}_k \varphi_k(x, q') + dq' \sum_{k,j} \hat{Q}_j \zeta_{kj}(q') \varphi_k(x, q')
\]

\[
= \sum_k \hat{Q}_k \left[ \varphi_k(x, q') + dq' \frac{\partial \varphi_k(x, q')}{\partial q'} \right]
\]

\[
= \sum_k \hat{Q}_k \varphi_k(x, q' + dq'), \tag{A7}
\]

and

\[
\left( I + \frac{i}{\hbar} dq' K(q') \right) \left( \sum_k \hat{P}_k \epsilon(x, q') \varphi_k(x, q') \right) \left( I - \frac{i}{\hbar} dq' K(q') \right) = \sum_k \hat{P}_k \epsilon(x, q') \varphi_k(x, q') - dq' \sum_{k,j} \hat{P}_j \epsilon(x, q') \zeta_{jk}(q') \varphi_k(x, q')
\]

\[
= \sum_k \hat{P}_k \left[ \epsilon(x, q') \varphi_k(x, q') + dq' \frac{\partial}{\partial q'} \left( \epsilon(x, q') \varphi_j(x, q') \right) \right]
\]

\[
= \sum_k \hat{P}_k \epsilon(x, q' + dq') \varphi_k(x, q' + dq'). \tag{A8}
\]

We have used the completeness relation

\[
\sum_k \epsilon(x, q') \varphi_k(x, q') \varphi_k(x', q') = \delta(x, x'), \tag{A9}
\]

and integration by parts

\[
\int_0^l dx' \epsilon(x', q') \frac{\partial \varphi_j(x', q')}{\partial q'} \varphi_k(x', q') = - \int_0^l dx' \frac{\partial}{\partial q'} \left[ \epsilon(x', q') \varphi_k(x', q') \right] \varphi_j(x', q'). \tag{A10}
\]

It follows that combining all the infinitesimal transform readily gives (25) and (26), which, together with (A3), lead to (23). Note that we need not symmetrize \( \hat{P}_k \hat{Q}_j \) in (23) because \( g_{kj} = -g_{jk} \), and hence \( g_{kk} = 0 \). This assertion can be proven by performing integration by parts on (24), and noting that the dielectric constant \( \epsilon \) is fixed in its rest frame, so that \( \partial_v \epsilon = \gamma (\partial_t + \dot{q} \partial_x) \epsilon = \gamma \dot{q} (\partial_q + \partial_x) \epsilon = 0 \), leading to \( \partial_q \epsilon = -\partial_x \epsilon \).
Appendix B: The $T_2(\hat{q})$ transformation

With the unitary transform $T_2(\hat{q})$ defined in (34), the canonical momentum operator $\hat{p}$ transforms as

$$T_2^\dagger \hat{p} T_2 = \hat{p} - \sum_{k,j} g_{kj}(\hat{q}) \hat{P}_k \hat{Q}_j = \hat{p} - \Gamma$$

so that $p$ becomes the kinetic momentum $m \dot{q}$ in the new basis. The transform field variables $\tilde{Q}_k = T_2^\dagger \hat{Q}_k T_2$ and $\tilde{P}_k = T_2^\dagger \hat{P}_k T_2$ reads

$$\tilde{Q}_k = \hat{Q}_k + \sum_j \lambda_{kj} \hat{Q}_j$$  \hspace{1cm} (B2)

$$\tilde{P}_k = \hat{P}_k + \sum_j \lambda_{kj} \hat{P}_j$$  \hspace{1cm} (B3)

with $\lambda_{kj}$ defined in (38). Hence the fields in the new frame (i.e. $T^\dagger A(x)T$ and $T^\dagger \Pi(x)T$) can be expanded in terms of $\tilde{Q}_k$ and $\tilde{P}_k$ by (40) and (41) respectively.

Next, we show that $\tilde{\varphi}(x, \hat{q})$ is orthonormal and complete. Using the fact $[\hat{Q}_j, \hat{P}_k] = i\hbar \delta_{kj}$ for unitary transformation, together with (B2) and (B3), we have the identity

$$\lambda_{kj} + \lambda_{jk} + \sum_l \lambda_{kl} \lambda_{jl} = 0.$$  \hspace{1cm} (B4)

Furthermore, by examining the form of $\lambda_{kj}$ with $f_{kj} = -f_{jk}$, it can be shown that

$$\sum_l \lambda_{kl} \lambda_{lj} = \sum_l \lambda_{lk} \lambda_{lj}.$$  \hspace{1cm} (B5)

These two properties of $\lambda_{kj}$ readily lead to

$$\int_0^l dx \epsilon(x, \hat{q}) \tilde{\varphi}_k(x, \hat{q}) \tilde{\varphi}_j(x, \hat{q}) = \int_0^l dx \epsilon(x, \hat{q}) \varphi_k(x, \hat{q}) \varphi_j(x, \hat{q}) = \delta_{kj},$$  \hspace{1cm} (B6)

$$\sum_k \epsilon(x, \hat{q}) \tilde{\varphi}_k(x, \hat{q}) \tilde{\varphi}_k(x', \hat{q}) = \sum_k \epsilon(x, \hat{q}) \varphi_k(x, \hat{q}) \varphi_k(x', \hat{q}) = \delta(x, x'),$$  \hspace{1cm} (B7)

hence $\{\tilde{\varphi}_k(x, \hat{q})\}$ indeed forms an orthonormal complete set of mode functions.

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