Non-paradoxical action of automata groups on infinite words

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Abstract
We show, that groups, defined by wide class of automata, including all polynomial ones, act on the set of infinite words not paradoxical.

1 Introduction and definitions

1.1 Rooted tree of words and it’s isomorphisms

Let $X$ be a finite set, which will be called alphabet with elements called letters. We always suppose $|X| > 1$ (here $|X|$ denotes the cardinality of $X$). Let $X^*$ be the free monoid generated by $X$. The elements of this monoid are finite words $x_1x_2...x_n$, $x_i \in X$, including the empty word $\emptyset$. Denote by $X^w$ the set of all infinite words $x_1x_2...x_n...$, $x_i \in X$. For each $l \in \mathbb{N}$ and $w = w_1...w_l... \in X^w$ set $w[l] := w_1...w_l$. In other words, $w[l]$ is a finite word, formed by first $l$ letters of $w$.

The set $X^*$ is naturally a vertex set of a rooted tree, in which two words are connected by an edge if and only if they are of the form $v$ and $vx$, where $v \in X^*$, $x \in X$. The empty word $\emptyset$ is the root of the tree $X^*$.

A map $f : X \rightarrow X$ is an endomorphism of the tree $X$, if for any two adjacent vertices $v$, $vx \in X^*$ the vertices $f(v)$ and $f(vx)$ are also adjacent, so that there exist $u \in X^*$ and $y \in X$ such that $f(v) = u$ and $f(vx) = uy$. An automorphism is a bijective endomorphism.

1.2 Automata and automorphisms of rooted trees

An automaton $A$ is a quadruple $(X, Q, \pi, \lambda)$, where:

- $X$ is an alphabet;
- $Q$ is a set of states of the automaton;

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• \( \pi : Q \times X \to X \) is a map, called the transition function of the automaton;

• \( \lambda : Q \times X \to X \) is a map, called the output function of the automaton.

An automaton is finite if it has a finite number of states. The maps \( \pi, \lambda \) can be extended on \( Q \times X^* \) by the following recurrent formulas:

\[
\begin{align*}
\pi(q, \emptyset) &= q, \quad \pi(q, xw) = \pi(\pi(q, x), w) \\
\lambda(q, \emptyset) &= \emptyset, \quad \lambda(q, xw) = \lambda(q, x)\lambda(\pi(q, x), w),
\end{align*}
\]

where \( x \in X, q \in Q, \) and \( w \in X^* \) are arbitrary elements. Similarly, the maps \( \pi, \lambda \) are extended on \( Q \times X^w \).

An automaton \( A \) with a fixed state \( q \) is called initial and is denoted by \( A_q \). Every initial automaton defines the automorphism \( \lambda(q, \cdot) \) of the rooted tree \( X^* \), which we also denote by \( A_q(\cdot) = \lambda(q, \cdot) \) (or \( q(\cdot) \) if it is clear, which automaton it belongs to). We denote by \( e \) a trivial state of automaton, i.e., such a state that defines a trivial automorphism of \( X^* \). The action of an initial automaton \( A_q \) can be interpret as the work of a machine, which being in the state \( q \) and reading on the input tape a letter \( x \), goes to the state \( \pi(q, x) \), types on the output tape the letter \( \lambda(q, x) \), then moves both tapes to the next position and proceeds further.

### 1.3 Moore diagrams

An automaton \( A \) can be represented (and defined) by a labelled directed graph, called the Moore diagram, in which the vertices are the states of the automaton and for every pair \((q, x) \in Q \times X\) there is an edge from \( q \) to \( \pi(q, x) \) labelled by \( x|\lambda(q, x) \).

Here is the Moore diagram of automaton, called the adding machine. Consider a word of length \( l \in \mathbb{N} \) as a binary number, with lower digits on the left side. If the automaton gets the word in state \( q \), it adds 1 modulo 2 to it.

### 1.4 Inverse automaton. Composition of automata

An automaton is called invertible, if for each \( q \in Q \) the mapping \( \lambda(q, \cdot) \) is a bijection. Suppose that we have the Moore diagram of invertible automaton \( A \). Let us swap all the left labels with right ones that correspond to them. After renaming the states \( q \to q^{-1} \) we get a Moore diagram of some automaton \( A^{-1} \), which is called the inverse automaton of \( A \). Notice that for each \( q \in Q \) the state \( q^{-1} \) of \( A^{-1} \) defines the automorphism of a rooted tree which is inverse to \( q \).

Further in the article we omit the word "invertible" and consider only invertible automata.
By giving the output of $A = (X, Q, \pi, \lambda)$ to the input of another automaton $B = (X, S, \alpha, \beta)$, we get an application which corresponds to the automaton called the composition of $A$ and $B$ and is denoted by $A \ast B$. This automaton is formally described as the automaton with the set of the states $Q \times S$ and the transition and output functions $\phi, \psi$ defined by

$$
\phi((q, s), x) = (\pi(q, x), \alpha(s, \lambda(q, x)))$$
$$\psi((q, s), x) = \beta(s, \lambda(q, x))$$

Notice that a state $(q, s)$ of $Q \times S$ defines the automorphism of a rooted tree which is a superposition of ones defined by $q$ and $s$.

### 1.5 Paradoxical actions of groups

Let $G$ be a group acting on a set $X$ and suppose $E \subseteq X$. $G$ acts on $E$ paradoxically ($E$ is $G$-paradoxical) if for some positive integers $m, n$ there are pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_n$ of $E$ and $g_1, \ldots, g_n, h_1, \ldots, h_n \in G$ such that $E = \bigcup g_i(A_i) = \bigcup h_i(B_i)$. Loosely speaking, the set $E$ has two disjoint subsets, each of which can be taken apart and rearranged via $G$ to cover all of $E$. Well known example is Banach-Tarski paradox, where subgroup of isometries of $R^3$ act’s on a sphere.
2 Automata, that "almost always" move to the trivial state

In the article we fix some alphabet X and consider words over X only. For arbitrary automata transformation g denote by \( NS(g, l) \) the number of words of length \( l \) which move \( g \) to the non-trivial state.

**Proposition 1.** \( NS(\bullet, \bullet) \) has the following properties:

1. \( NS(g, l) = NS(g^{-1}, l) \)
2. \( NS(gh, l) \leq NS(g, l) + NS(h, l) \),

where \( g, h \) are arbitrary automata transformations, \( l \in \mathbb{N} \)

**Proof.** (1) Let \( w \) be a word with a length \( l \) that doesn’t move \( g \) to the trivial state. Then \( g(w) \) is a word with a length \( l \) that doesn’t move \( g^{-1} \) to the trivial state. Since \( g \) is a bijective mapping on \( X^l \), \( g(w) \) are different for different \( w \). Therefore, \( NS(g^{-1}, l) \leq NS(g, l) \). After replacing \( g \rightarrow g^{-1} \), we get the opposite inequality.

(2) For arbitrary automata transformation \( q \) denote by \( S(q, l) \) the set of words of length \( l \) that move \( q \) to the trivial state; \( NS(q, l) := X^l \setminus S(q, l) \). Then \( |NS(q, l)| = NS(q, l), |S(q, l)| = |X|^l - NS(q, l) \).

As \( g \) is a bijection on \( X^l \), we have \( |g(NS(g, l))| = |NS(g, l)| = NS(g, l) \). If a word \( w \in X^l \) moves \( g \) to the trivial state and so does \( g(w) \) with \( h \), then \( w \) moves \( gh \) to the trivial state. Therefore:

\[
S(gh, l) \supseteq g(S(g, l)) \cap S(h, l) = X^l \setminus (g(NS(g, l)) \cup NS(h, l))
\]

\[
|S(gh, l)| \geq |X|^l - (NS(g, l) + NS(h, l))
\]

\[
NS(g, l) = |NS(gh, l)| \leq NS(g, l) + NS(h, l)
\]

\( \square \)

Denote by \( G_0 \) the set of all automata transformations \( g \) for which the following holds:

\[
NS(g, l) = o(|X|^l), l \rightarrow \infty
\]

To put it simply, "almost" all the words move the transformations to the trivial state.

(1) and (2) imply that \( G_0 \) is a group. Notice that \( G_0 \) includes all the polynomial automata. What do we know about it’s action on \( X^w \)?

**Theorem 1.** \( X^w \) is not \( G_0 \)-paradoxical.

**Proof.** Assume that \( X^w \) is \( G_0 \)-paradoxical. Then there are the elements \( h_1, ..., h_d \) of \( G_0 \) and the partition \( X^w = A_1 \sqcup \ldots \sqcup A_d \) with such property: Let initially every word from \( X^w \) have 1 coin. Then \( h_i \) moves a coin from every \( a_i \in A_i \) to \( h_i(a_i) \), \( 1 \leq i \leq d \). After that every word from \( X^w \) has at least 2 coins.

For arbitrary \( l, s \in \mathbb{N} \), \( s \geq 8 \) consider the set of \( s \cdot |X|^l \) consecutive words \( F := \{w + 1, ..., w + s \cdot |X|^l\} \subset X^w \) and its superset of \( (s + 2) |X|^l \) consecutive...
words

\[ F' := \{ w + 1 - |X|^l, \ldots, w + (s + 1)|X|^l \}, \]

put \( \neg F' := X^w \setminus F' \). Split \( F \) into subsets of \(|X|^l\) consecutive words \( F = F_1 \sqcup \ldots \sqcup F_s \). Notice that in every such subset the beginnings (first \( l \) letters) of words form the set \( X^l \).

Fix an arbitrary \( i \in \{1, \ldots, d\} \). Let \( g_i = h_i^{-1} \). The number of words, which are moved by \( g_i \) from \( F \) to \( \neg F' \) is not greater then \( s \cdot NS(h_i, l) \). Actually, consider arbitrary \( F_j \subset F \), \( 1 \leq j \leq s \). If for some \( v \in F_j \), \( g_i(v) \) is in \( \neg F' \), then \( g_i \) changes a letter with position number greater then \( l \) in word \( v \). It means that if \( g_i \) receives \( v[1..l] \) it doesn’t move to trivial state after reading it.

As \( \{v[1..l] \mid v \in F_j\} = X^l \), number of such words in \( F_j \) is not greater then \( NS(g_i, l) = NS(h_i, l) \). So in \( F \) this number is not greater then \( s \cdot NS(h_i, l) \).

The result can be reworded: the number of words in \( \neg F' \), from which \( h_i \) brings coins to \( F \), is not greater then \( s \cdot NS(h_i, l) \). So, for each \( i \in \{1, \ldots, d\} \), \( h_i \) brings not more then \( s \cdot NS(h_i, l) \) coins to \( F \) from \( \neg F' \). Then all \( h_i \) bring at most \( s \cdot \sum_{i=1}^{d} NS(h_i, l) \) coins. As \( NS(h_i, l) = o(|X|^l) \), \( l \rightarrow \infty \), we can get

\[ s \cdot \sum_{i=1}^{d} NS(h_i, l) \leq \frac{1}{4}s|X|^l \]

by choosing large \( l \). Therefore, if \(|F|\) is quiet large, then the number of coins, which are moved from \( \neg F' \) to \( F \) is not greater then \( \frac{1}{4}|F| \). Since \( s \geq 8 \), \( 2 \leq \frac{1}{4}s \), then at most \( (s + 2)|X|^l \leq \frac{5}{2}s|X|^l = \frac{5}{2}|F| \) coins are moved to \( F \) from \( F' \) (in particular from \( F \)). Together, the number of coins at \( F \) becomes at most \( \frac{5}{2}|F| + \frac{1}{4}|F| = \frac{3}{2}|F| < 2|F| \). It means then some words of \( F \) get less then 2 coins. Contradiction. \( \square \)

**Remark 1.** There are transformations, defined by infinite and not polynomial automata, which satisfy conditions of Theorem 1. For example, all non-trivial states of the following automaton define such transformations:
Each finite word without 0 and 1 moves \( q_i, i \geq 1 \) to non-trivial state. On the other hand, every word that contains 0 moves them to the trivial state. Therefore,

\[ 2^l \leq NS(q_i, l) \leq 3^l, \]

where \( 2^l \) is a number of words without 1 and 0, \( 3^l \) - without 0 only. Since \( \frac{3^l}{2^l} \to 0 \), \( l \to \infty \), all the states \( q_i, i \geq 1 \) are acceptable.
3 Automata, that "almost always" move to cycles

Theorem 1 doesn’t cover some simple automata transformations, which act on \( X^w \) non-paradoxically, such as pictured below:

![Diagram](attachment:image.png)

In theorem 1, "most" of words must move the automaton to the trivial state. In fact, we can generalise the condition.

**Definition 1.** The states \( g_1, ..., g_n \) of the same automaton form an unconditional cycle (UC), if the transition functions \( \pi(g_i, \cdot) \), \( 1 \leq i \leq n \) don’t depend on input data and have the form \( g_1 \rightarrow ... \rightarrow g_n \rightarrow g_1 \).

Apparently, a trivial state is an instance of UC. We will say that a word \( w \in X^w \cup X^* \) moves the automata transformation \( g \) of the automaton \( A \) to a UC in \( s \) steps, if \( A \), receiving \( w \) in \( g \), gets into one of the states of the UC after processing first \( s \) letters. Notice the following:

**Proposition 2.** Assume that \( w \in X^l \) moves \( g \) to an UC of length \( n \) and \( g(w) \) moves \( h \) to an UC of length \( m \). Then \( w \) moves \( gh \) to some UC of length \( \text{LCM}(n, m) \).

**Proof.** We consider \( gh \) as a state \((g, h)\) of \( A * B \). Let \( w \) move \( g \) to the UC \( g_1, ..., g_n \) in \( s \) steps and \( g(w) \) move \( h \) to the UC \( h_1, ..., h_m \) in \( t \) steps. Then after processing first \( \max\{s, t\} \) letters of \( w \) the state \((g, h)\) moves to the UC \((g_1, h_1), ..., (g_n, h_m)\) of length \( \text{LCM}(n, m) \).

For arbitrary automata transformation \( g \) denote by \( NC_g(l) \) a number of words with length \( l \) which don’t move \( g \) to any UC, and by \( NC(g, l) \) - a set of infinite words that start from them. In other words, \( NC(g, l) \subset X^w \) consists of words that don’t move \( g \) to any UC while processing the first \( l \) letters.

**Proposition 3.** \( NC(g, l) \) has the following properties:

1. \( NC(g, l) = NC(g^{-1}, l) \)
2. \( NC(gh, l) \leq NC(g, l) + NC(h, l) \),

where \( g, h \) are arbitrary automata transformations, \( l \in \mathbb{N} \).

**Proof.** (1’) We are going to show that if \( w \in X^l \) moves \( g \) to the UC \( g_1, ..., g_n \) then \( g(w) \in X^l \) moves \( g^{-1} \) to the UC \( g_1^{-1}, ..., g_n^{-1} \). Consider the Moore diagram of some automaton \( A \), that contains \( g \). Starting from \( g \) and moving along left labels that form the word \( w \), we achieve the UC \( g_1, ..., g_n \) in \( k \leq l \) steps. Let us swap all the left labels with right ones that correspond to them. After renaming the states \( h \rightarrow h^{-1} \) we get a Moore diagram of the automaton \( A^{-1} \).
This transformation keeps UC, moreover, starting from $g^{-1}$ and moving along left labels that form $g(w)$, we achieve the UC $g_1^{-1},...,g_n^{-1}$ in $k$ steps. So if $w \in X^l$ moves $g$ to the UC $g_1,...,g_n$ then $g(w) \in X^l$ moves $g^{-1}$ to the UC $g_1^{-1},...,g_n^{-1}$. As $g$ is a bijection, we have $NC(g^{-1},l) \leq NC(g,l)$. Similarly an opposite inequality can be gotten.

(2') Assume that $w \in X^l$ moves $g$ to some UC and $g(w)$ moves $h$ to some (maybe another) UC. Then according to proposition 1 $w$ moves $gh$ to an UC. The rest of the proof is similar to one of (2). We just have to replace $NS \rightarrow NC$, $NS \rightarrow NC$, $S \rightarrow C$.

Now we can see that automata transformations $g$ for which $NC(g,l) = o(|X|^l), l \rightarrow \infty$ form a group. Denote it by $G_1$. Generalising theorem 1, we are going to prove the next statement:

**Theorem 2.** $X^w$ is not $G_1$-paradoxical.

We need two auxiliary lemmas. Let us say that a word $w \in X^w$ is $l$—almost periodic if it has the form $w = uv$ where $u \in X^l$ is arbitrary and $v$ is periodic. Mentioning the period, we mean the shortest repeating word from $X^*$, that starts from $(l+1)th$ position of $w$. For example, word 123010101 as a $3$—almost periodic word has period 01. But as a $4$—almost periodic word it has period 10.

**Lemma 1.** Let $w$ be $l$—almost periodic word with a period of length $t$. Assume that $w$ moves an automata transformation $g$ to UC of length $c$ in at most $l$ steps. Then $g(w)$ is $l$—almost periodic and the length of it’s period divides LCM$(t,c)$.

**Proof.** When the automaton has already processed the first $l$ letters of $w$, the following holds:

- Every $c$ steps the automaton moves to the same state.
- Every $t$ steps the automaton receives the same letter.

Therefore, every LCM$(t,c)$ steps it receives the same letter in the same state, so gives the same data to the output.

Denote by $P^l_n$ the set of all $l$—almost periodic infinite words with the length of period dividing $n!$.

**Lemma 2.** Let $g$ be an automata transformation. Denote by $n(l)$ the maximal length of UC $g$ can move to after processing a word of length $l$. For arbitrary $c \geq n(l)$ there holds:

$g(P^l_c \setminus NC(g,l)) \subseteq P^l_c$

**Proof.** Consider an arbitrary word $w \in P^l_c \setminus NC(g,l)$. Let the automaton $A$, containing $g$, receive $w$ in the state $g$. After processing the first $l$ symbols by $A_g$, the periodic part of $w$ has already started. Besides that, $A_g$ has already moved to the UC (by definition of NC$(g,l)$). Since $w \in P^l_c$, then the length of
it’s period divides $c!$. A length of the UC is not greater then $n(l) \leq c$, so it also divides $c!$. According to lemma 1, $g(w)$ is $l$–almost periodic and length of it’s period divides $\text{LCM}(c!, c!) = c!$. Therefore, $g(w) \in P^l_c$.

Now we are ready to start the proof of theorem 2.

**Proof.** Assume that $X^w$ is $G_1$–paradoxical. Then there are the elements $h_1, \ldots, h_d$ of $G_1$ and the partition $X^w = A_1 \sqcup \ldots \sqcup A_d$ with such property: Let initially every word from $X^w$ has 1 coin. Then $h_i$ moves a coin from every $a_i \in A_i$ to $h_i(a_i)$, $1 \leq i \leq d$. After that every word from $X^w$ has at least 2 coins.

Consider the set $P^l_N$, where $l$ will be defined later and $N = N(l)$ is the maximal length of UC that $h_1, \ldots, h_d$ can move to while processing words of length $l$. Fix an arbitrary $i \in \{1, \ldots, d\}$. Denote $g_i = h_i^{-1}$. According to lemma 2, $g_i(P^l_N \setminus NC_{g_i}(l)) \subseteq P^l_N$. So $g_i$ can move at most $|P^l_N \cap NC_{g_i}(l)|$ words from $P^l_N$ outside it. Estimate the number. Let $\mathbb{T}_N$ be the set of all periods $T \in X^*$ with length dividing $N!$. For each $T \in \mathbb{T}_N$ denote by $P^l_{(T)}$ the set of $l$–almost periodic infinite words with period $T$. Then we have:

\[ P^l_N = \bigcup_{T \in \mathbb{T}_N} P^l_{(T)} \]

\[ P^l_N \cap NC(g_i, l) = \bigcup_{T \in \mathbb{T}_N} (P^l_{(T)} \cap NC(g_i, l)) \]

Notice that for each $v \in X^l$ there is exactly 1 word starting from $v$ in every $P^l_{(T)}$, $T \in \mathbb{T}_N$. Therefore $|P^l_{(T)}| = |X|^l$, in particular

\[ |P^l_{(T)} \cap NC(g_i, l)| = NC(g_i, l) \]

\[ |P^l_N \cap NC(g_i, l)| = \sum_{T \in \mathbb{T}_N} |P^l_{(T)} \cap NC(g_i, l)| = |\mathbb{T}_N| \cdot NC(g_i, l) \]

So, $g_i$ moves $|\mathbb{T}_N| \cdot NC(g_i, l)$ words from $P^l_N$ outside it. It means that $h_i$ brings $|\mathbb{T}_N| \cdot NC(g_i, l)$ coins from $X^w \setminus P^l_N$ to $P^l_N$. Then all $h_1, \ldots, h_d$ bring at most $|\mathbb{T}_N| \sum_{i=1}^d NC(g_i, l)$ coins from $X^w \setminus P^l_N$ to $P^l_N$. As there are $|\mathbb{T}_N| \cdot |X|^l$ words in $P^l_N$ and $NC(g_i, l) = o(|X|^l), l \to \infty$, similarly to theorem 1 we have that $h_1, \ldots, h_d$ don’t bring enough coins to $P^l_N$ for large $l$. \qed
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