Oscillating Flavors in Massless Neutrinos

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By considering Dirac’s equation using quaternions ($\mathbb{H}$) with their greater degree of freedom in imaginaries, it is shown that a model can be created with oscillations among flavors, even if the particles are massless. Furthermore the solutions are spin $\frac{1}{2}$ and have helicities depending on whether their energy is positive or negative.

1 Introduction

To give a short review to set the context, using only complex numbers ($\mathbb{C}$) and following Schiff’s [1] notation and representation, Dirac’s equation for a massless particle is ($\hbar = c = 1$):

$$\left( i\partial_t - i\alpha_c \cdot \nabla \right) |\psi \rangle = 0$$

where the elements of $\alpha_c \in \mathbb{C}$ and

$$\nabla = \partial_x + \partial_y + \partial_z$$

One representation for $\alpha_c$ is given in Appendix A. Equation (1) has four independent solutions corresponding to positive and negative energy having spin up or spin down:

$$|\psi_{\pm} \rangle = u_{\pm} \exp^{i(p \cdot r - Et)}$$

where, for example,

$$u_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{E}{p_x} \\ \frac{p_y + ip_x}{E} \\ 1 \\ 0 \end{pmatrix}$$

Note, in general, that $u^{-1} = (u^*)^T$.

2 Quaternion Solution

Welch [2] has explored Dirac’s equation for massive charged particles and showed that charges $\frac{2}{3}$ and $\frac{1}{3}$ result with a minimum of assumptions using quaternions (see Appendix B). Defining $q$ to be a constant unit imaginary quaternion (see Appendix B) the transition to quaternions could trivially and unproductively follow the complex analysis exactly because any unit imaginary quaternion, $q_i$, is isomorphic to $i \in \mathbb{C}$. Equation (2) becomes

$$|\psi_q \rangle \equiv |\psi_{q_{\pm}} \rangle = u_{q_{\pm}} \exp^{q_i (p \cdot r - Et)}$$

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where, for example, equation 3 becomes

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
-\frac{p_x}{E} \\
-\frac{(p_x + q_x)p_y}{E} \\
0 \\
1
\end{pmatrix}
\]

However, we wish to explore the ramifications possible due to the increased flexibility of quaternions, thus repeating the derivation in \( \mathbb{H} \) that is done in \( \mathbb{C} \) with equation 1 in the form, in which no imaginary unit from \( \in \mathbb{C} \) appears.

\[
|\psi_q > = 0 \quad (4)
\]

with the elements of \( \alpha \in \mathbb{H} \). Requiring that \( |\psi_q > \) also be a solution to Klein-Gordon’s relativistic equation for a massless particle, i.e.,

\[
(\partial^2_t - \nabla^2)|\psi_q > = 0
\]

leads to the necessity that \( \alpha \) be a 2 \( \times \) 2 matrix and obey the commutative relationships:

\[
\alpha_x \alpha_y + \alpha_y \alpha_x = \alpha_y \alpha_z + \alpha_z \alpha_y = \alpha_z \alpha_x + \alpha_x \alpha_z = 0
\]

One such quaternion representation of Dirac’s \( \alpha \) is given in Appendix A. Thus equation 4 is

\[
\left( \begin{array}{cc}
\partial_t & 0 \\
0 & \partial_t
\end{array} \right) - \left( \begin{array}{cc}
0 & i\partial_x \\
-i\partial_x & 0
\end{array} \right) - \left( \begin{array}{cc}
0 & j\partial_y \\
-j\partial_y & 0
\end{array} \right) - \left( \begin{array}{cc}
0 & k\partial_z \\
-k\partial_z & 0
\end{array} \right) \left( \begin{array}{c}
|\psi_{q1} > \\
|\psi_{q2}>
\end{array} \right) = 0
\]

or

\[
\left( \begin{array}{c}
\partial_t \\
i\partial_x + j\partial_y + k\partial_z
\end{array} \right) - \left( \begin{array}{c}
\partial_t \\
-i\partial_x - j\partial_y - k\partial_z
\end{array} \right) \left( \begin{array}{c}
|\psi_{q1} > \\
|\psi_{q2}>
\end{array} \right) = 0 \quad (5)
\]

Let us use as a trial solution

\[
|\psi_q > = u \exp(\theta(\mathbf{p} \cdot \mathbf{r} - Et))
\]

Thus equation 5 can be written:

\[
\left( \begin{array}{c}
\partial_t \\
i\partial_x + j\partial_y + k\partial_z
\end{array} \right) - \left( \begin{array}{c}
\partial_t \\
-i\partial_x - j\partial_y - k\partial_z
\end{array} \right) \left( \begin{array}{c}
u_1 \\
u_2
\end{array} \right) \exp(\theta(\mathbf{p} \cdot \mathbf{r} - Et)) = 0 \quad (6)
\]

\footnote{Given the non-commutative nature of quaternions, an equally valid trial solution, not pursued herein, is

\[
|\psi_q > = \exp(\theta(\mathbf{p} \cdot \mathbf{r} - Et)) u^R
\]
which has the general solutions (see Appendix C).

\[ u^*_2 u_2 = \frac{1}{2} \]
\[ u_1 = \frac{-u_2(\epsilon \cdot p)}{E} \]

Note

\[ u^*_1 u_2 = \frac{(\epsilon \cdot p)}{E} u^*_2 u_2 = \frac{(\epsilon \cdot p)}{2E} \]

If the "spin up" state is the positive energy solution then the "spin down" state is the negative energy solution - i.e., the anti-neutrino, as is seen experimentally. Having only two spin states is indicative of a spin \( \frac{1}{2} \) system. Thus the orthonormal solutions are

\[ u_+ = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \frac{(\epsilon \cdot p)}{E} \\ 1 \end{array} \right) \quad u_- = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -\frac{(\epsilon \cdot p)}{|E|} \end{array} \right) \]

Thus

\[ |\psi_q \rangle = N_+ \frac{1}{\sqrt{2}} \left( \begin{array}{c} \frac{(\epsilon \cdot p)}{E} \\ 1 \end{array} \right) \exp^{i(\mathbf{p} \cdot \mathbf{r} - Et)} + N_- \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -\frac{(\epsilon \cdot p)}{|E|} \end{array} \right) \exp^{i(\mathbf{p} \cdot \mathbf{r} - Et)} \]

\(|\psi_q \rangle\), which has no specific lepton flavor and the most general solution, will be denoted as the "q-neutrino" and is the eigenstate of equation (6). and can be written

\[ |\psi_q \rangle = N_+ |\psi_{q+} \rangle + N_- |\psi_{q-} \rangle \]

where \(|\psi_{q+} \rangle\) and \(|\psi_{q-} \rangle\) are the basis vectors of the Hamiltonian as defined by equation (4). The normalizing factors \(N_+\) and \(N_-\) obey

\[ N_+^2 + N_-^2 = 1 \]

and are chosen to \( \in \mathbb{R} \). It is straightforward to show:

\[ \langle \psi_{q+} | \psi_{q+} \rangle = \langle \psi_{q-} | \psi_{q-} \rangle = 1 \]
\[ \langle \psi_{q-} | \psi_{q+} \rangle = \langle \psi_{q+} | \psi_{q-} \rangle = 0 \]

\section{Lepton Flavors}

Looking at the positive energy state,

\[ |\psi_{q+} \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \frac{-1}{E} (\epsilon \cdot p) \\ 1 \end{array} \right) \exp^{i(\mathbf{p} \cdot \mathbf{r} - Et)} \]
One can expand $|\psi_{q+}\rangle$ using the basis of $\mathbb{H}$ to obtain

$$
|\psi_{q+}\rangle = \frac{1}{\sqrt{2}} \left( \frac{1}{2}(a p_x + b p_y + c p_z) \sin (p \cdot r - E t) \right) 
\cos (p \cdot r - E t)
+ i \frac{1}{\sqrt{2}} \left( - \frac{E}{E} \cos (p \cdot r - E t) - \frac{(c p_y - b p_x)}{E} \sin (p \cdot r - E t) \right) 
\sin (p \cdot r - E t)
+ j \frac{1}{\sqrt{2}} \left( - \frac{E}{E} \cos (p \cdot r - E t) - \frac{(a p_x - c p_z)}{E} \sin (p \cdot r - E t) \right) 
\sin (p \cdot r - E t)
+ k \frac{1}{\sqrt{2}} \left( - \frac{E}{E} \cos (p \cdot r - E t) - \frac{(b p_y - a p_x)}{E} \sin (p \cdot r - E t) \right) 
\sin (p \cdot r - E t)
$$

(7)

or

$$
|\psi_{q+}\rangle = |r_+\rangle + i|e_+\rangle + j|\mu_+\rangle + k|\tau_+\rangle
$$

where an association has been explicitly been made between the lepton flavors and the imaginary bases of quaternions and similarly for $|\psi_{q-}\rangle$.

Note that this equation is invariant under the simultaneous permutations:

$$
i \rightarrow j \rightarrow k \rightarrow i \rightarrow
p_x \rightarrow p_y \rightarrow p_z \rightarrow p_x \rightarrow
a \rightarrow b \rightarrow c \rightarrow a
$$

Thus there is no preferred spatial direction.

4 Inner Product

Before we explore the ramifications of this formulation the issue of the definition of “inner product” of Hilbert vectors $A$ and $B$ ($= A \cdot B$) must be addressed. One mathematical requirement of an “inner product” is that it is commutative, i.e.,

$$A \cdot B = B \cdot A$$

(8)
This clearly is not a problem in \( \mathbb{C} \), a commutative algebra, however in \( \mathbb{H} \), a non-commutative algebra, more care has to be taken since, in general, \( A \cdot B \neq B \cdot A \). The definition used herein is adopted from that used in the special Jordan algebras, i.e.,

\[
A \circ B = \frac{1}{2} (A \cdot B + B \cdot A).
\]

Clearly in \( \mathbb{C} \) equation (9) reduces to equation (8) and thus ordinary quantum mechanics would not be affected by such a generalization. We will extend this change of definition to the bra-key notation, i.e., the \( \langle A | \) and \( | B \rangle \) notation. The bra, \( \langle A | \), is the Hermitian conjugate of the ket \( | A \rangle \). So, by definition,

\[
\langle B | \circ | A \rangle = \frac{1}{2} (\langle B | A \rangle + \langle A | B \rangle)
\]

Using this generalization, it is easy to see from equation (7) that

\[
\langle r | \circ | r \rangle = \frac{1}{2} \left( \frac{(ap_x + bp_y + cp_z)^2}{E^2} \sin^2(p \cdot r - Et) + \cos^2((p \cdot r - Et)) \right) > 0
\]

\[
\langle r | \circ | e \rangle = \langle r | \circ | r \rangle = 0
\]

\[
\langle e | \circ | e \rangle = \left[ \frac{b^2 E}{(p \cdot r - Et)} \cos((p \cdot r - Et)) \right] > 0
\]

\[
\langle e | \circ | \mu \rangle = \langle e | \circ | r \rangle = 0
\]

\[
\langle \mu | \circ | \mu \rangle = \frac{1}{2} \left( \frac{(bp_x - ap_y)E}{(p \cdot r - Et)} \cos((p \cdot r - Et)) \right) > 0
\]

\[
\langle \mu | \circ | \tau \rangle = \langle \mu | \circ | r \rangle = 0
\]

\[
\langle \tau | \circ | \tau \rangle = \frac{1}{2} \left( \frac{(bp_x - ap_y)E}{(p \cdot r - Et)} \cos((p \cdot r - Et)) \right) > 0
\]

5 **Flavor Oscillation**

Oscillation among lepton flavors is well established and the only mechanism theoretically possible in \( \mathbb{C} \) is if the neutrinos have mass. A plane wave, for a free particle has the form \((i \in \mathbb{C})\)

\[
| \psi \rangle = \exp^{i(p \cdot \vec{x} - Et)}
\]

So for a given direction, the only distinguishing parameter of different plane waves is the energy, \( E \). However, in \( \mathbb{H} \), one can have distinct plane waves with the same energy, i.e.,

\[
| \psi_1 \rangle = \exp^{i(p \cdot \vec{x} - Et)}
| \psi_2 \rangle = \exp^{i(p \cdot \vec{x} - Et)}
\]
where \( q_1 \) and \( q_2 \) are differing unit imaginary quaternions and \((i, j, k \in \mathbb{H})\),

\[
q_1 = a_1 i + b_1 j + c_1 k \\
q_2 = a_2 i + b_2 j + c_2 k
\]

and

\[
q_1 * q_1 = q_2 * q_2 = -1
\]

5.1 In \( \mathbb{C} \)

To illustrate simply the consequences of this difference between \( \mathbb{C} \) and \( \mathbb{H} \), we follow the example of Casper [4], a member of the "Super-Kamiokande" Collaboration [3], who derives the requirement of a mass difference within \( \mathbb{C} \).

An outline of the argument is: Let \(|\nu_1\rangle\) and \(|\nu_2\rangle\) be neutrino eigenstates of mass, \(m_1\) and \(m_2\), respectively. The time evolution, for a free particle, by Schrodinger’s equation is:

\[
\begin{pmatrix}
|\nu_1(\vec{x},t)\rangle \\
|\nu_2(\vec{x},t)\rangle
\end{pmatrix}
= \exp^{i\vec{p} \cdot \vec{x}}
\begin{pmatrix}
\exp^{-iE_1t} |\nu_1(0,0)\rangle \\
\exp^{-iE_2t} |\nu_2(0,0)\rangle
\end{pmatrix}
\]

The mass eigenstates can also be expressed in terms of flavor eigenstates (only two flavors are considered for simplicity) as:

\[
|\nu_m\rangle = \alpha |\nu_e\rangle + \beta |\nu_\mu\rangle
\]

where \(\alpha^2 + \beta^2 = \cos^2(\theta) + \sin^2(\theta) = 1\). This can be written:

\[
\begin{pmatrix}
|\nu_1\rangle \\
|\nu_2\rangle
\end{pmatrix}
= \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
|\nu_e\rangle \\
|\nu_\mu\rangle
\end{pmatrix}
\]

Thus

\[
\begin{pmatrix}
|\nu_e(\vec{x},t)\rangle \\
|\nu_\mu(\vec{x},t)\rangle
\end{pmatrix}
= \exp^{i\vec{p} \cdot \vec{x}}
\begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
|\nu_e(0)\rangle \\
|\nu_\mu(0)\rangle
\end{pmatrix}
\]

If we start with all electron neutrinos:

\[
\begin{pmatrix}
|\nu_e(0)\rangle \\
|\nu_\mu(0)\rangle
\end{pmatrix}
= \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

This leads to

\[
<\nu_\mu(\vec{x},t)|\nu_\mu(\vec{x},t)> = \sin^2(2\theta) \sin^2 \left( \frac{E_2 - E_1}{2} t \right)
\]
and since

\[ E_2 - E_1 \approx \frac{m_2^2 - m_1^2}{2p} \]
\[ t \approx |\vec{x}| \equiv L \]
\[ p \approx E \]

\[ P(\nu_e \rightarrow \nu_\mu) = \sin^2 (2\theta) \sin^2 \left( \frac{\Delta m^2 L}{4E} \right) \]

5.2 In $\mathbb{H}$

Let's examine the time evolution of the electron neutrino, $|e>$, as given in equation (7).

\[
|e(\vec{x},t) > = \exp^{\hat{q}(\vec{p} \cdot \vec{r} - Et)} |e(0,0) >
\]
\[ (\hat{q} = \hat{a}i + \hat{b}j + \hat{c}k \text{ is a unit imaginary quaternion.}) \]

Thus

\[
|e(\vec{x},t) > = \frac{1}{\sqrt{2}} \left( \cos (\vec{p} \cdot \vec{r} - Et) + (\hat{a}i + \hat{b}j + \hat{c}k) \sin (\vec{p} \cdot \vec{r} - Et) \right) \begin{pmatrix}
-\frac{p_y}{E} \cos (\vec{p} \cdot \vec{r} - Et) - \frac{(cp_y - bp_z)}{p} \sin (\vec{p} \cdot \vec{r} - Et) \\
\frac{p_x}{E} \cos (\vec{p} \cdot \vec{r} - Et) + \frac{(ap_z - cp_x)}{E} \sin (\vec{p} \cdot \vec{r} - Et)
\end{pmatrix} a \sin (\vec{p} \cdot \vec{r} - Et)
\]

and

\[
< \mu(\vec{x},t) | \circ |e(\vec{x},t) > = \frac{\hat{b}}{2} \sin (\vec{p} \cdot \vec{r} - Et)
\]
\[
\left[ \left( \frac{p_y}{E} \cos (\vec{p} \cdot \vec{r} - Et) + \frac{(ap_z - cp_x)}{E} \sin (\vec{p} \cdot \vec{r} - Et) \right) \left( \frac{p_x}{E} \cos (\vec{p} \cdot \vec{r} - Et) + \frac{(cp_y - bp_z)}{E} \sin (\vec{p} \cdot \vec{r} - Et) \right) + ab \sin^2 (\vec{p} \cdot \vec{r} - Et) \right]
\]

Clearly showing that even though, at $t = 0$, $< \mu | \circ |e > = 0$ the $\mu$ neutrino appears as time evolves.

6 Sterile Neutrinos

The $|r>$ neutrinos from equation (8) have no lepton flavor and are tentatively identified as "sterile neutrinos," however they have the same helicity as the corresponding flavored neutrinos.
7 Conclusions

The primary objective, showing that a formalism does exist wherein neutrinos can oscillate in flavor without having mass, has been demonstrated. The neutrinos in this model are spin $\frac{1}{2}$ and have only one helicity. This model also allows for a flavorless fourth type of neutrino.

The choice for the magnitudes of the three imaginaries, e.g., $a, b, c$ seems to be arbitrary as long as the consistency with $a^2 + b^2 + c^2 = 1$ is maintained. The ubiquity of the combination $(\epsilon \cdot p)$ fuels speculation that to be Lorentz invariant, the $i, j, k$ have to be on equal footing and thus $a = b = c = \frac{1}{\sqrt{3}}$. 

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Appendix A: Dirac’s Matrices

The representation given by Schiff [1] is: In \( \mathbb{C} \)

\[
\begin{align*}
\alpha_x &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \alpha_y &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} & \alpha_z &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}
\end{align*}
\]

and in \( \mathbb{H} \), as used by Rotelli [5]

\[
\begin{align*}
\alpha_x &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} & \alpha_y &= \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} & \alpha_z &= \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}
\end{align*}
\]
Appendix B: Quaternions

Quaternions, \( \mathbb{H} \), are one of only three (\( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{H} \)) finite-dimensional division rings containing the real numbers \( \mathbb{R} \) as a subring - a requirement to preserve probability in quantum mechanics. Quaternion quantum mechanics has been extensively studied and Adler [6] has written the definitive reference. \( \mathbb{H} \) can be loosely viewed as a non-commutative extension of \( \mathbb{C} \). The imaginary quaternion units, \( i, j, k \) are defined by

\[
\begin{aligned}
ii &= jj = kk = -1 \\
ij &= -ji = k, \quad ki = -ik = j, \quad jk = -kj = i
\end{aligned}
\]

A general quaternion \( q \) can be written

\[
q = r + ai + bj + ck
\]

where \( r, a, b, c \in \mathbb{R} \).

Every non-zero quaternion has an inverse.

\[
q^{-1} = \frac{q^*}{(r^2 + a^2 + b^2 + c^2)}
\]

Therefore \( q^*q = (r^2 + a^2 + b^2 + c^2) \in \mathbb{R} \).

Quaternion addition is associative: \( q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3 \) - and defined as

\[
q_1 + q_2 = r_1 + r_2 + (a_1 + a_2)i + (b_1 + b_2)j + (c_1 + c_2)k
\]

and quaternion multiplication (paying heed to the non-commutative nature of the imaginary units) is

\[
q_1q_2 = (r_1r_2 - a_1a_2 - b_1b_2 - c_1c_2) + (r_1a_2 + b_1c_2 + b_1c_2 - c_1b_2)i \\
+ (r_1b_2 - a_1c_2 + b_1r_2 + c_1a_2)j \\
+ (r_1c_2 + a_1b_2 - b_1a_2 + c_1r_2)k
\]

Quaternions are associative under multiplication \( (q_1q_2)q_3 = q_1(q_2q_3) \). A unit imaginary quaternion \( q_i \) is defined as

\[
q_i = ai + bj + ck
\]

( i.e., \( r_i = 0 \) )

where \( q_i^2 = -1 \), which means \( a^2 + b^2 + c^2 = 1 \).

---

2A field is a commutative division ring. A ring is an algebraic structure which generalizes the algebraic properties of the integers and contains two operations usually called addition and multiplication. One example of a field is the familiar complex numbers. A division ring allows for division (except by zero). Every field is a ring but non-commutative rings are not fields.
It should be noted that many algebraic identities in \( \mathbb{C} \) are false in a non-commutative algebra. For example:

\[
\exp^{i\omega} \exp^{j\omega} \neq \exp^{(i+j)\omega}
\]

However Euler’s formula is valid:

\[
\exp^{q\omega} = \cos \omega + q \sin \omega
\]

The notation for division

\[
q_3 = \frac{q_1}{q_2}
\]

is ambiguous (on which side of \( q_1 \) does \( q_2^{-1} \) go?) and should not be used.

Often useful are the identities:

\[
(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}
\]

\[
(q_1 q_2)^* = q_2^* q_1^*
\]

It should also be noted that if \( i \) refers to the \( i \) of \( \mathbb{C} \) rather than of \( \mathbb{H} \) it will be specifically indicated. It is also often convenient to represent \( i, j, k \) as components of a 3-vector \( \epsilon = (i, j, k) \).
Appendix C: Solving for $u$

Assuming that $u_1, u_2 \neq 0$ a function of $x, y, z, t$ and starting with

\[
\left( \begin{array}{ccc}
\frac{\partial_t}{i} & -i\frac{\partial_x}{j} - j\frac{\partial_y}{k} - k\frac{\partial_z}{l} \\
i\frac{\partial_x}{j} + j\frac{\partial_y}{k} + k\frac{\partial_z}{l}
\end{array} \right) \left( \begin{array}{c} u_1 \\
u_2 \end{array} \right) \exp^{\eta(p \cdot r - Et)} = 0
\]

leads to

\[
\begin{align*}
[(u_1\partial_t - u_2(i\partial_x + j\partial_y + k\partial_z)) \exp^{\eta(p \cdot r - Et)} & = 0 \\
[u_1(i\partial_x + j\partial_y + k\partial_z) + u_2\partial_t] \exp^{\eta(p \cdot r - Et)} & = 0
\end{align*} \tag{C.1}
\]

These equations have a solution only if the determinant is zero, therefore, (as in $\mathbb{C}$)

\[E^2 = p_x^2 + p_y^2 + p_z^2\]

and we have both $E_+$ and $E_-$ solutions. From equation \text{(C.1)}

\[
\begin{align*}
-u_1qE - u_2(\epsilon \cdot p)q & = 0 \\
u_1E - u_2(\epsilon \cdot p) & = 0 \\
u_{1+} = u_1 = -\frac{u_{2+}}{E_+}(\epsilon \cdot p) \\
u_{1-} = \frac{u_{2-}}{|E_-|}(\epsilon \cdot p)
\end{align*}
\]

From normalization requirements

\[u_1^*u_1 + u_2^*u_2 = 1\]

Thus

\[\frac{(\epsilon \cdot p)}{E} u_2^*u_2 \frac{(-\epsilon \cdot p)}{E} + u_2^*u_2 = 1\]

and $u_2^*u_2$ must be real and therefore commutes with any quaternion leading to

\[2u_2^*u_2 = 1\]

so $u_2 = \frac{1}{\sqrt{2}}$ to within an arbitrary phase.

---

3Frequent use will be made of $(\frac{\epsilon \cdot p}{E})^{-1} = -(\frac{\epsilon \cdot p}{E})$ and $(\frac{\epsilon \cdot p}{E})(\frac{\epsilon \cdot p}{E}) = -1$
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