ON THE MODULI SPACE OF DEFORMATIONS OF BIHAMILTONIAN HIERARCHIES OF HYDRODYNAMIC TYPE

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Abstract. We investigate the deformation theory of the simplest bihamiltonian structure of hydrodynamic type, that of the dispersionless KdV hierarchy. We prove that all of its deformations are quasi-trivial in the sense of B. Dubrovin and Y. Zhang, that is, trivial after allowing transformations where the first partial derivative $\partial u$ of the field is inverted. We reformulate the question about deformations as a question about the cohomology of a certain double complex, and calculate the appropriate cohomology group.

1. Introduction

In the early 1990's, M. Kontsevich's solution [K] of E. Witten's conjecture established an intriguing connection between the differential equations of the Korteweg-de Vries (KdV) hierarchy and the quantum theory of two-dimensional topological gravity. Their work suggested a deep relationship between integrable hierarchies "of the KdV-type" and a wide class of quantum field theories (QFTs).

Integrability of a differential equation manifests itself in a set of properties, one of which is the existence of a bihamiltonian structure. This means that the system can be written in Hamiltonian form in two distinct ways with respect to two compatible Poisson brackets, permitting the construction of an infinite integrable hierarchy of differential equations containing the original equation. This in turn gives a method of generating infinitely many symmetries and conservation laws. Historically, the KdV equation was the first differential equation found to have such a remarkable set of properties. This equation appeared in the nineteenth century in the study of solitary water waves, but since then has played a prominent role in many distinct areas of mathematics.

In a series of pioneering papers, B. Dubrovin investigated relations between the structure of two-dimensional topological QFTs (TFTs) and the theory of bihamiltonian structures. Dubrovin showed that the tree level, or genus zero, approximation of the TFT is described by a bihamiltonian integrable hierarchy, the genus zero hierarchy, which encodes all of the structure of the model in this approximation.

In this paper, we analyze bihamiltonian deformations of integrable hierarchies and their bihamiltonian structures that are related to generalizations of the Witten conjecture. Following the discoveries of Kontsevich and Witten, Dubrovin and Y. Zhang [D1, D2, D3, DZ1] conjectured the existence of a bihamiltonian deformation of the genus zero hierarchy which encodes the recursion relations among the correlators of the 2d TFT. They further conjectured that this deformed hierarchy should be quasi-trivial, i.e. obtained from the genus zero hierarchy by some generalized coordinate change.

For example, the genus zero hierarchy associated to 2d topological gravity with trivial background is the dispersionless KdV hierarchy. It is known that the KdV hierarchy, which by Kontsevich’s theorem gives a complete description of 2d topological gravity with trivial
background, is a quasi-trivial bihamiltonian deformation of the dispersionless KdV hierarchy. The main theorem, which is presented in Section 3, proves that all bihamiltonian deformations of the dispersionless KdV hierarchy are quasi-trivial.

Given a Poisson bracket on functionals one can associate to it a Poisson bi-vector in the Schouten Lie algebra of functional multi-vectors. This is a super Lie algebra with respect to the Schouten bracket, and it becomes a differential graded (dg) Lie algebra when endowed with the adjoint action of the bi-vector. The moduli space of deformations of the Poisson bracket is then controlled by the cohomology of this dg Lie algebra.

The goal of Section 2 is to develop a convenient setup for carrying out cohomology computations. Of central importance to our study is the notion of the Schouten bracket for functional multi-vectors. The Schouten bracket is due to I. Dorfman and I. Gelfand, but their definition is not well suited to calculations, especially those involving higher multi-vectors. The Schouten graded Lie algebra is isomorphic to the Gerstenhaber algebra of superfunctions on a canonical symplectic supermanifold. This isomorphism identifies the Schouten bracket with an odd Poisson bracket, and enables us to give explicit formulas for the Schouten bracket. We also develop a normal form for functional multi-vectors, which makes the study of obstructions to deformations far more straightforward. We expect this normal form to have applications beyond the problems treated here.

Section 3 begins with a basic introduction to the deformation theory of bihamiltonian structures, or equivalently, of a pair of compatible Poisson brackets. The study of the moduli space of deformations of a bihamiltonian structure reduces to questions about the bihamiltonian cohomology of the pair. This is the cohomology of a double complex, introduced in Section 3.2. Equivalent classes of infinitesimal deformations are parametrized by the cohomology classes of the first bihamiltonian cohomology group, and the obstruction classes to continuing an infinitesimal deformation to higher orders are elements of the second bihamiltonian cohomology group.

In Section 3.3 we determine the space of infinitesimal symmetries of the bihamiltonian structure of the dispersionless KdV hierarchy. In Section 3.5, we prove that all bihamiltonian deformations of this structure are quasi-trivial. We obtain our results by calculating the corresponding cohomology groups.

We follow the summation convention, taking sums over equal upper and lower indices. This work constitutes part of the author’s dissertation [B]. We have striven to give a self-contained treatment. Theorem 3.4 has also been proven independently by Liu and Zhang [LZ2].

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2. The Schouten Bracket and Hamiltonian Operators

2.1. Hamiltonian operators. Let \((u^\alpha)\), \(1 \leq \alpha \leq q\), be coordinates on an open subset \(U\) of a \(q\)-dimensional manifold \(M\). Let \(A_0\) be the algebra of smooth functions \(C^\infty(U)\) in the
variables $u^\alpha$, and let $\mathcal{A}$ be the algebra of differential polynomials in the jet variables $u^\alpha_k$, $k \geq 1$, over the algebra $\mathcal{A}_0$.

The algebra $\mathcal{A}$ is equipped with a natural grading $\mathcal{A} = \bigoplus_k \mathcal{A}_k$, defined by setting

$$\deg u^\alpha_k = k.$$ 

Elements of $\mathcal{A}$ will be referred to as \textit{polynomial densities}. Densities in $\mathcal{A}_k$ are homogeneous of degree $k$.

Let $\mathcal{A}[n]$ be the algebra of differential polynomials in the jet variables $u^\alpha_k$, $k \leq n$, over $\mathcal{A}_0$. The sequence

$$0 \subset \mathcal{A}[0] = \mathcal{A}_0 \subset \mathcal{A}[1] \subset \cdots \subset \mathcal{A}[n] \subset \cdots \subset \mathcal{A}$$

is an increasing filtration of $\mathcal{A}$. If $g \in \mathcal{A}[n]$, we say that $g$ has order $\leq n$.

The derivation representing differentiation with respect to $x$ is given by the \textit{total derivative}

$$\partial = \sum_{j=0}^{\infty} u^\alpha_{j+1} \partial_{u^\alpha_j}.$$ 

A derivation $X$ is \textit{evolutionary vector field} if it satisfies the commutation relation

$$[X, \partial] = 0.$$ 

An evolutionary vector field has the form

$$X = \partial^j f^\alpha \partial_{u^\alpha_j},$$

where the $q$-tuple $(f^1, \ldots, f^q)$ of polynomial densities, $f^\alpha \in \mathcal{A}$, is called the \textit{characteristic} of $X$. It is evident from the commutation relation $[2]$ and the Jacobi identity for the Lie bracket, that the evolutionary vector fields form a Lie algebra, which we denote by $\mathcal{V}^1$.

Since $X$ is uniquely determined by its characteristic, we can identify the space $\mathcal{V}^1$ of evolutionary vector fields with the space $\mathcal{A}^q$.

A \textit{differential operator} is an element $D \in \mathcal{A}^{[\partial]}$; such an operator defines an endomorphism of $\mathcal{A}$. A differential operator $D$ is \textit{homogeneous} of order $k$ if it has the form

$$D = \sum_{j=0}^{k} P_j \partial^j, \quad P_j \in \mathcal{A}_{k-j}.$$ 

The \textit{adjoint} $D^*$ of a differential operator $D$ is given by the formula

$$D^* = \sum_{j=0}^{k} \sum_{i=0}^{k-j} (-1)^{i+j} \binom{j+i}{i} (\partial^i P_{j+i}) \partial^j.$$ 

Define the space of \textit{functionals} by

$$\mathcal{F} = \mathcal{A}/\partial\mathcal{A}.$$ 

The \textit{variational derivative} \n
$$\delta_{u^\alpha} : \mathcal{F} \to \mathcal{A}$$

is given by the formula

$$\delta_{u^\alpha} = \sum_{k=0}^{\infty} (-\partial)^k \partial_{u^\alpha_k}.$$
A \((q \times q)\)-matrix \(D^{\alpha\beta}\) of differential operators defines a bracket
\[
\{F, G\}_D = \sum_{\alpha, \beta} (\delta u^\alpha F) D^{\alpha\beta} (\delta u^\beta G) \mod \partial A
\]
on functionals. This bracket is skew-symmetric if \(D\) is skew-adjoint, that is,
\[
(D^{\alpha\beta})^* = -D^{\beta\alpha}.
\]
If in addition, the bracket satisfies the Jacobi identity, then \(D\) is called a \emph{Hamiltonian operator} or a \emph{Hamiltonian structure}. A pair of Hamiltonian operators \(P\) and \(Q\) is said to form a \emph{bihamiltonian structure}, if for any scalar \(\lambda\) the sum
\[
P + \lambda Q
\]
is again a Hamiltonian operator. The operators \(P\) and \(Q\) are also called \emph{compatible}, and the brackets \(\{\cdot, \cdot\}_P\) and \(\{\cdot, \cdot\}_Q\) are said to form a \emph{Poisson pencil}.

The Schouten graded Lie algebra, introduced by Gelfand and Dorfman \cite{GD}, is a graded vector space \(V = \bigoplus_k V^k\) with super Lie bracket
\[
[[\cdot, \cdot]] : V^k_1 \times V^k_2 \to V^{k_1+k_2-1},
\]
such that \(V^0 \cong \mathcal{C}^0\), \(V^1\) is the space of evolutionary vector fields, and \(V^2\) is isomorphic to the space of skew-adjoint \((q \times q)\) matrices of differential operators. We recall the precise definition of the bracket \([[[\cdot, \cdot]]]\) in the next section. We conclude this section with the following standard characterizations of Hamiltonian and bihamiltonian structures.

**Proposition 2.1.** For a bi-vector \(D \in V^2\), the following conditions are equivalent:

1. \(D\) is Hamiltonian;
2. the associated bracket \(\{\cdot, \cdot\}_D\) is Poisson;
3. the morphism \(d_D = [[D, \cdot]]\) is a differential on \(V\).

**Proposition 2.2.** Two Hamiltonian operators \(P\) and \(Q\) are compatible if and only if
\[
[[P, Q]] = 0.
\]

### 2.2. The Schouten bracket

In this section we give a formula for the Schouten bracket due to Getzler \cite{G}. Let \(T^*[1]M\) be the \(\mathbb{Z}\)-graded manifold underlying the graded vector bundle over \(M\) whose fiber at \(u \in M\) is the cotangent space \(T_u^* M\) concentrated in degree \(-1\). Let \((\theta^\alpha)\) be the coordinates along the fibers of \(T^*[1]U\) dual to the coordinates \((u^\alpha)\) on \(U\).

Let \(\Lambda^\bullet = \mathcal{A}[\theta_{\alpha,k} \mid k \geq 0]\) be the exterior algebra of functions over \(\mathcal{A}\) with generators \(\theta_{\alpha,k}\) in degree 1. Denote by \(\partial_{\theta_{\alpha,k}}\) the graded derivation (partial derivative) such that
\[
\partial_{\theta_{\alpha,k}} \theta_{\beta,\ell} = \delta^\alpha_\beta \delta_{k,\ell}.
\]
Extend the total derivative \(\partial\) on \(\mathcal{A}\) to the algebra \(\Lambda^\bullet\) by the formula
\[
\partial = \sum_{k=0}^\infty (u^\alpha_{k+1} \partial u^\alpha_k + \theta_{\alpha,k+1} \partial_{\theta_{\alpha,k}}).
\]
Denote by \(\delta_{\theta_{\alpha}}\) the variational derivative on \(\Lambda^\bullet\), defined by the formula analogous to \([3]\),
\[
\delta_{\theta_{\alpha}} = \sum_{k=0}^\infty (-\partial)^k \partial_{\theta_{\alpha,k}}.
\]
Let
\[ V^\bullet = \Lambda^\bullet_\infty / \partial \Lambda^\bullet_\infty. \]
Elements of \( V^k \) are denoted \( \int F \, dx, F \in \Lambda^k_\infty \), and are called \textit{functional} \( k \)-\textit{vectors}, or just \( k \)-\textit{vectors}, for short. Notice that, by definition, \( V^0 = \mathcal{F} \), where \( \mathcal{F} \) is the space of functionals introduced in the previous section.

The graded vector space \( V^\bullet \) is a graded Lie algebra, with respect to the \textit{Schouten bracket}
\[ [[\cdot, \cdot]] : V^{k_1} \times V^{k_2} \longrightarrow V^{k_1+k_2-1} \]
defined by the formula
\[
[[\int F \, dx, \int G \, dx]] = \sum_{k,\ell} \int \left( (-1)^{|F|+1} \partial^k \partial_{\theta,\ell} F \cdot \partial^\ell \partial_{u,k} G - \partial^k \partial_{u,\ell} F \cdot \partial^\ell \partial_{\theta,k} G \right) \, dx
\]
\[
= \int \left( (-1)^{|F|+1} \delta_{\theta,\ell} F \cdot \delta_{u,k} G - \delta_{u,\ell} F \cdot \delta_{\theta,k} G \right) \, dx.
\]

For a proof that this bracket satisfies the graded Jacobi identity, see \cite{G}.

In order to have a graded Lie algebra whose bracket has degree zero, it is convenient to replace the graded vector space \( V^\bullet \) by its shifted version
\[ \mathcal{L}^\bullet = V^\bullet + 1. \]

A Hamiltonian operator \( \mathcal{H} \) is then an element of \( \mathcal{L}^1 \) satisfying the Maurer-Cartan equation \( [[\mathcal{H}, \mathcal{H}]] = 0 \). For such an \( \mathcal{H} \), the morphism \( d_{\mathcal{H}} = [[\mathcal{H}, \cdot]] \) on \( \mathcal{L} \) is a derivation of degree 1, and \( (\mathcal{L}, d_{\mathcal{H}}) \) is a dg Lie algebra.

The definition of degree of homogeneity on \( A \) extends to \( \Lambda^\bullet_\infty \), by setting
\[ \deg \theta_{\alpha,k} = k, \quad k \geq 0. \]

Denote by \( \mathcal{L}^k(\ell) \) the space of \((k+1)\)-vectors homogeneous of degree \( k + \ell \).

2.3. \textbf{The graded Lie algebra \( \hat{\mathcal{L}} \).} In this section, take \( q = 1 \), so there is only a single dependent variable \( u \). In the study of deformations of the dispersionless KdV hierarchy, an important role is played by the algebra \( \hat{A} = A[u^{-1}] \). The grading of the algebra \( A \) extends to a grading of \( \hat{A} \) if we assign to \( u^{-1} \) homogeneity degree \(-1\):
\[ \hat{A} = \bigoplus_k \hat{A}_k \]

If \( n > 0 \), let \( \hat{A}[n] = (A[n])[u^{-1}] \).

In parallel to the definition of \( \hat{A} \), let \( \hat{\Lambda}^\bullet_\infty = \Lambda^\bullet_\infty [u^{-1}] \). The derivation \( \partial \) maps \( \hat{\Lambda}^\bullet_\infty \) to itself, and we define the graded Lie algebra \( \hat{\mathcal{L}}^\bullet \) by
\[ \hat{\mathcal{L}}^k = \hat{\Lambda}^{k+1}_\infty / \partial \hat{\Lambda}^{k+1}_\infty. \]

The bracket \( [[\cdot, \cdot]] \) on \( \hat{\mathcal{L}} \), defined by the same formulas as for the Schouten bracket on \( \mathcal{L} \), turns it into a graded Lie algebra. Denote by \( \hat{\mathcal{L}}^k(\ell) \) the space of \((k+1)\)-vectors homogeneous of degree \( k + \ell \). The Schouten bracket is homogeneous of degree 0, in the sense that
\[ [[\hat{\mathcal{L}}^{k_1}(\ell_1), \hat{\mathcal{L}}^{k_2}(\ell_2)]] \subset \hat{\mathcal{L}}^{k_1+k_2}(\ell_1 + \ell_2). \]
2.4. Formal deformations of Hamiltonian operators. Let \( \mathcal{H} \in \mathcal{L}^1 \) be a Hamiltonian operator. Recall that \( \mathcal{H} \) being Hamiltonian is equivalent to the vanishing of the Schouten bracket \([\mathcal{H}, \mathcal{H}]\).

By a formal deformation of \( \mathcal{H} \) we mean a formal power series
\[
\mathcal{H} = \mathcal{H} + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{H}_k, \quad \mathcal{H}_k \in \mathcal{L}^1
\]
such that \([\mathcal{H}, \mathcal{H}] = 0\). The \( n \)-th order deformations of \( \mathcal{H} \) are given by the Maurer-Cartan elements of the dg Lie algebra \( \mathcal{L} \otimes I_n \), where \( I_n \) is the ideal in \( \mathbb{R}[\varepsilon]/(\varepsilon^{n+1}) \) generated by \( \varepsilon \). They are of the form
\[
\mathcal{H}_n = \mathcal{H} + \sum_{k=1}^{n} \varepsilon^k \mathcal{H}_k, \quad \mathcal{H}_k \in \mathcal{L}^1
\]
and the associated brackets on functionals satisfy the Jacobi identity up to order \( n \).

Every first order deformation of \( \mathcal{H} \) determines, and is uniquely determined, by an infinitesimal deformation, i.e. a bi-vector \( \mathcal{H}_1 \in \mathcal{L}^1 \) such that
\[
[[\mathcal{H}, \mathcal{H}_1]] = 0.
\]

The Schouten dg Lie algebra \( (\mathcal{L}, d_{\mathcal{L}}) \) controls the moduli space of deformations of \( \mathcal{H} \) in the following sense. A formal deformation \( \mathcal{H} = \sum_{k=0}^{\infty} \varepsilon^k \mathcal{H}_k \) is called trivial if there is a formal coordinate change
\[
\psi^\alpha \mapsto \psi^\alpha[u^\alpha] = u^\alpha + \varepsilon f^\alpha + \ldots
\]
such that
\[
\mathcal{H} = \mathcal{H} - \varepsilon [[X, \mathcal{H}]] + \ldots,
\]
where \( X = \int f^\alpha \theta^\alpha \, dx \) is the evolutionary vector field with characteristic \( (f^\alpha) \). In particular,
\[
\mathcal{H}_1 = [[\mathcal{H}, X]] = d_{\mathcal{L}} X.
\]

If \( \mathcal{H}_1 \) is an infinitesimal deformation, then the first obstruction cocycle is given by
\[
[[\mathcal{H}_1, \mathcal{H}_1]] \in \mathcal{L}^2.
\]
It is the coefficient of \( \varepsilon^2 \) in \([\mathcal{H}_1, \mathcal{H}_1]]\), and it satisfies
\[
d_{\mathcal{L}} ([[[\mathcal{H}_1, \mathcal{H}_1]]]) = [[[\mathcal{H}, [[\mathcal{H}_1, \mathcal{H}_1]]]]] = 0.
\]

Further, if \( d_{\mathcal{L}} \mathcal{H}_2 = -\frac{1}{2} [[\mathcal{H}_1, \mathcal{H}_1]] \), for some bi-vector \( \mathcal{H}_2 \in \mathcal{L}^1 \), then
\[
\mathcal{H}_2 = \mathcal{H} + \varepsilon \mathcal{H}_1 + \varepsilon^2 \mathcal{H}_2
\]
is a second order deformation. For an \( n \)-th order deformation \( \mathcal{H}_n \), the \( n \)-th obstruction cocycle is the coefficient of \( \varepsilon^{n+1} \) in \([\mathcal{H}_n, \mathcal{H}_n]]\). It is given by
\[
\sum_{i=1}^{n} [[\mathcal{H}_i, \mathcal{H}_{n-i+1}}]]
\]
and by a calculation parallel to the one above, we see that $H_n$ can be extended to an $(n+1)$-st order deformation $H_{n+1} = H_n + \varepsilon^{n+1} H_{n+1}$ if

$$d_{\varepsilon} H_{n+1} = \frac{1}{2} \sum_{i=1}^{n} [[H_i, H_{n-i+1}]].$$

To summarize, we have thus identified the equivalent infinitesimal deformations of the Hamiltonian operator $H$ with the elements of the first cohomology group of the dg Lie algebra $(\mathcal{L}, d_H)$, and the obstruction classes to extending a first order deformation of $H$ to higher order ones with elements of the second cohomology group.

2.5. Hamiltonian operators of hydrodynamic type. A Hamiltonian operator $H$ has hydrodynamic type if it is homogeneous of order 1 (Dubrovin and Novikov [DN]):

$$H_{\alpha\beta} = h_{\alpha\beta}(u) \partial + \Gamma_{\alpha\beta}^\gamma(u) u^\gamma_1.$$  

Skew-symmetry implies that $h_{\alpha\beta}$ is symmetric, and that

$$\Gamma_{\alpha\beta}^\gamma(u) + \Gamma_{\beta\alpha}^\gamma(u) = \partial_\gamma h_{\alpha\beta}(u).$$

Under a change of coordinates, the coefficients $h_{\alpha\beta}$ transform as the components of a symmetric bilinear form on the cotangent bundle, and the coefficients $\Gamma_{\gamma}^{\alpha\beta}(u)$ transform as the associated Christoffel symbols

$$\Gamma_{\gamma}^{\alpha\beta} = -\frac{1}{2} h^{\alpha\delta} h^{\beta\epsilon} (\partial_\delta h_{\epsilon\gamma} - \partial_\epsilon h_{\delta\gamma} + \partial_\gamma h_{\delta\epsilon}).$$

The condition that $H$ is a Hamiltonian operator may be expressed as the flatness condition

$$h^{\alpha\beta} \partial_\beta \Gamma_{\gamma}^{\alpha\beta} - h^{\beta\delta} \partial_\delta \Gamma_{\gamma}^{\alpha\beta} + [\Gamma_{\gamma}^{\alpha\beta}, \Gamma_{\gamma}^{\beta\alpha}] = 0.$$  

We say that the Hamiltonian operator $H$ is nondegenerate if the associated bilinear form $h_{\alpha\beta}$ is nondegenerate, that is, a flat pseudo-metric. In this case, we may choose flat coordinates, i.e. those for which the components $h_{\alpha\beta}$ of the pseudo-metric are constant. In terms of these, the Hamiltonian operator $H$ becomes

$$H^{\alpha\beta} = h^{\alpha\beta} \partial.$$  

If $H$ is a Hamiltonian operator of hydrodynamic type, then the differential $d_{\varepsilon}$ on $\mathcal{L}$ raises the degree by 1, and hence maps $\mathcal{L}^k(\ell)$ to $\mathcal{L}^{k+1}(\ell)$. Thus the dg Lie algebra $(\mathcal{L}, d_{\varepsilon})$ decomposes into the direct sum of subcomplexes

$$\mathcal{L}(\ell) = \bigoplus_k \mathcal{L}^k(\ell)$$

and the cohomology of $(\mathcal{L}, d_{\varepsilon})$ decomposes accordingly into

$$H^k(\mathcal{L}, d_{\varepsilon}) = \bigoplus_{\ell \geq -k} H^k(\mathcal{L}(\ell), d_{\varepsilon}).$$

The next theorem is due to Getzler [G].

**Theorem 2.3.** Let $H$ be a Hamiltonian operator of hydrodynamic type. If $H$ is nondegenerate,

$$H^k(\mathcal{L}(\ell), d_{\varepsilon}) = 0, \quad \text{if } \ell > -k.$$
Now suppose that $\mathcal{H}$ is a nondegenerate Hamiltonian operator of hydrodynamic type, and let $\mathbf{H}$ be a formal deformation of $\mathcal{H}$,

$$\mathbf{H} = \mathcal{H} + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{H}_k,$$

such that the infinitesimal deformation $\mathcal{H}_1$ is homogeneous of degree $p + 1$, that is, $\mathcal{H}_1 \in \mathcal{L}^1(p)$, and for $k > 1$,

$$\mathcal{H}_k \in \bigoplus_{\ell \geq 0} \mathcal{L}^1(\ell).$$

We may show, by induction on $k$, that the deformation $\mathbf{H}$ is equivalent to a deformation such that $\mathcal{H}_k$ is homogeneous of degree $kp + 1$, that is, $\mathcal{H}_k \in \mathcal{L}^1(kp)$. To see this, observe that the right-hand side of the equation

$$d_{\gamma} \mathcal{H}_k = -\frac{1}{2} \sum_{i=1}^{k-1} [[\mathcal{H}_i, \mathcal{H}_{k-i}]],$$

lies in $\mathcal{L}^2(kp)$. By Theorem 2.3, there is an element $\mathcal{J}_k$ of $\mathcal{L}^0$ such that $\mathcal{H}_k + d_{\gamma} \mathcal{J}_k$ is homogeneous of degree $kp + 1$.

A deformation $\mathbf{H}$ with this property is called homogeneous. In this paper, we only consider homogeneous deformations. The second Hamiltonian structure

$$u\partial + \frac{1}{2} u_1 + \frac{3}{2} \varepsilon^2 \partial^3$$

of the small dispersion expansion

(5)

$$\partial_t u = uu_1 + \varepsilon^2 u_3$$

of the KdV equation is an example of such a deformation.

2.6. The normalization operator. The higher variational derivatives on $\Lambda_{\infty}^\bullet$ are defined by

$$\delta_{k,u^\alpha} = \sum_{j=0}^{\infty} (-1)^j \binom{j+k}{k} \partial^j \partial u_{k+j}^\alpha$$

and

$$\delta_{k,\theta^\alpha} = \sum_{j=0}^{\infty} (-1)^j \binom{j+k}{k} \partial^j \partial \theta_{k+j}^\alpha.$$

Note that $\delta_{0,u^\alpha} = \delta_{u^\alpha}$ and $\delta_{0,\theta^\alpha} = \delta_{\theta^\alpha}$ are the variational derivatives introduced earlier.

Let $\mathcal{N} : \Lambda_{\infty}^\bullet \to \Lambda_{\infty}^\bullet$ be the normalization operator

$$\mathcal{N} = \sum_{\alpha} \theta^\alpha \delta_{\theta^\alpha}.$$

Note that $\mathcal{N} \cdot \partial = 0$, since $\delta_{\theta} \cdot \partial = 0$.

Theorem 2.4. If $F \in \Lambda_{\infty}^k$, $\mathcal{N}F - kF \in \partial \Lambda_{\infty}^k$.

Proof. A direct calculation shows that

$$\partial_{\theta^\alpha_i} = \sum_{j=1}^{\infty} \binom{j}{i} \partial^{j-i} \delta_{j,\theta^\alpha}.$$
It follows that if $F \in \Lambda^k_{\infty}$,
\[
k F = \sum_{j=0}^{\infty} \theta_{\alpha,j} \partial_{\theta_{\alpha,j}} F
= \sum_{j=0}^{\infty} \sum_{i=0}^{j} (\partial^i \theta_{\alpha,i})(\partial^{j-i} \delta_{j,\theta_{\alpha}} F)
= \sum_{j=0}^{\infty} \partial^j (\theta_{\alpha} \delta_{j,\theta_{\alpha}} F)
= NF + \partial \sum_{j=0}^{\infty} \partial^j (\theta_{\alpha} \delta_{j+1,\theta_{\alpha}} F).
\]

We will also use the generalization of this theorem with $\Lambda^k_{\infty}$ replaced by $\hat{\Lambda}^k_{\infty}$; the proof is identical.

The normalization operator $N$ may be used to establish the standard identifications of the spaces $L^0$ and $L^1$ mentioned in Section 2.2. For example, if
\[
F = \sum_j f_j^\alpha \theta_{\alpha,j} \in \Lambda^1_{\infty}, \quad f_j^\alpha \in A,
\]
we have \( \int F \, dx = \int NF \, dx \in L^0 \), where the normalization $NF$ is given by the formula
\[
NF = \sum_j \theta_{\alpha} \delta_{\alpha}(f_j^\alpha \theta_{\alpha,j})
= \sum_j \theta_{\alpha} (-\partial)^j f_j^\alpha.
\]
That is, \( \int F \, dx \) is the evolutionary vector field with characteristic \( (\sum_j (-\partial)^j f_j^\alpha) \).

Similarly, if
\[
G = \sum_{j,k} f_{jk}^{\alpha\beta} \theta_{\alpha,j} \theta_{\beta,k} \in \Lambda^2_{\infty},
\]
where \( f_{jk}^{\alpha\beta} = -f_{kj}^{\beta\alpha} \), we have \( \int G \, dx = \frac{1}{2} \int NG \, dx \in L^1 \). Here, the normalization $NG$ is given by the formula
\[
\frac{1}{2} NG = \sum_{j=0}^{\infty} \theta_{\alpha} (-\partial)^j (f_{jk}^{\alpha\beta} \theta_{\beta,k}) = \theta_{\alpha} D^{\alpha\beta} \theta_{\beta},
\]
where $D = (D^{\alpha\beta})$ is a skew-adjoint \((q \times q)\)-matrix of differential operators
\[
D^{\alpha\beta} = \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^i (\partial^{j-i} f_{jk}^{\alpha\beta}) \partial^{k+i}.
\]

The role of the normalization operator $N$ is to implement analogues of these identifications of $L^k$ for arbitrary $k \geq 0$. 9
3. THE BIHAMILTONIAN COHOMOLOGY FOR HAMILTONIAN STRUCTURES OF HYDRODYNAMIC TYPE

3.1. Bihamiltonian structures of hydrodynamic type. The dispersionless KdV equation

\[ \partial_t u = uu_1 \]

is contained in a sequence of commuting flows of one dependent variable \( u \) which make up the dispersionless KdV hierarchy. This hierarchy is associated to the following compatible pair of Hamiltonian operators of hydrodynamic type:

\[ \mathcal{P} = \partial \quad \text{and} \quad \mathcal{Q} = u\partial + \frac{1}{2} u_1. \]

For the choice of Hamiltonian functionals \( H_0 = \frac{1}{3} \int u^2 \, dx \) and \( H_1 = \frac{1}{6} \int u^3 \, dx \), this equation may be written

\[ \partial_t u = \mathcal{P}\delta_u H_1 = \mathcal{Q}\delta_u H_0. \]

The pseudo-differential operator

\[ \mathcal{R} = \mathcal{Q}\mathcal{P}^{-1} = u + \frac{1}{2} u_1 \partial^{-1} \]

is a recursion operator [O] for the dispersionless KdV equation. Letting

\[ \partial_{t_0} u = \mathcal{P}\delta_u H_0 \quad \text{and} \quad \partial_{t_1} u = \partial_t u = \mathcal{P}\delta_u H_1, \]

the equations of the dispersionless KdV hierarchy are given recursively by

\[ \partial_{t_n} u = \mathcal{P}\delta_u H_n = \mathcal{Q}\delta_u H_{n-1} \]

for Hamiltonian functionals \( H_n, n \geq 0 \). The functionals can be recursively determined by the relations \( \{ \cdot, H_n \}_\mathcal{P} = \{ \cdot, H_{n-1} \}_\mathcal{Q} \) starting with the Casimir \( H_{-1} = \frac{1}{3} \int u \, dx \) for \( \{ \cdot, \cdot \}_\mathcal{P} \). They are in involution with respect to both Poisson brackets \( \{ \cdot, \cdot \}_\mathcal{P} \) and \( \{ \cdot, \cdot \}_\mathcal{Q} \).

We can now formulate the problem posed by Dubrovin and Zhang [DZ2]. Suppose we are given a compatible pair \((\mathcal{P}, \mathcal{Q})\) of Hamiltonian operators of hydrodynamic type, such that \( \mathcal{P} \) is nondegenerate. One wishes to classify homogeneous deformations of this pair, modulo the action of the group of Miura transformations, that is, coordinate changes of the form

\[ u^\alpha \mapsto u^\alpha + \sum_{k=1}^{\infty} \varepsilon^k \psi_k^\alpha, \quad \psi_k^\alpha \in \mathcal{A}_k. \]

In this work, we show that every non-trivial deformation of the bihamiltonian structure [7] of the dispersionless KdV hierarchy can be transformed into \((\mathcal{P}, \mathcal{Q})\) by a quasi-Miura transformation, that is, a coordinate change of the form

\[ u \mapsto u + \sum_{k=1}^{\infty} \varepsilon^k \psi_k, \quad \psi_k \in \hat{\mathcal{A}}. \]

We say that every homogeneous deformation of \((\mathcal{P}, \mathcal{Q})\) is quasi-trivial.

Our work was motivated by Lorenzoni [L]: he studied homogeneous deformations of this bihamiltonian structure, and showed by explicit calculation that they were quasi-trivial up to fourth order in \( \varepsilon \).
We conclude this section by showing that it is indeed necessary to consider quasi-Miura transformations. Suppose that \( u \) satisfies the dispersionless KdV equation (6), and that \( v = u + \varepsilon^2 \psi + O(\varepsilon^4) \) satisfies the KdV equation (5) up to terms of order \( \varepsilon^4 \). We calculate that
\[
\partial_t v - v \partial v - \varepsilon^2 \partial^3 v = (\partial_t u + \varepsilon^2 \partial_t \psi) - (uu_1 + \varepsilon^2 u \partial \psi + \varepsilon^2 u_1 \psi) - \varepsilon^2 u_3 + O(\varepsilon^4)
\]
\[
= \varepsilon^2 \big( \partial_t \psi - u \partial \psi - u_1 \psi - u_3 \big) + O(\varepsilon^4) = 0.
\]
This equation for \( \psi \) has no solution in \( \mathcal{A} \), but it does have a solution in \( \hat{\mathcal{A}} \), namely
\[
\psi = -\frac{1}{2} \left( \frac{u_3}{u_1} - \frac{u_2}{u_1^2} \right)
\]
\[
= -\frac{1}{2} \partial^2 \log(u_1) \in \hat{\mathcal{A}}[3].
\]

3.2. Formal deformations of a compatible pair of Hamiltonian operators. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two compatible Hamiltonian operators. We introduce a double complex
\[
C^{**} = \bigoplus C^{m,n}
\]
with \( C^{m,n} = \mathcal{L}^{m+n} \). To see that this is indeed a double complex, observe that the differentials \( d_{\mathcal{P}} \) and \( d_{\mathcal{Q}} \) anticommute: if \( \mathcal{R} \) is an element of \( \mathcal{L} \),
\[
d_{\mathcal{P}}d_{\mathcal{Q}}(\mathcal{R}) = [[\mathcal{P}, [[\mathcal{Q}, \mathcal{R}]]]]
\]
\[
= [[[[\mathcal{P}, \mathcal{Q}]], \mathcal{R}]] - [[\mathcal{Q}, [[\mathcal{P}, \mathcal{R}]]]]
\]
\[
= -d_{\mathcal{Q}}d_{\mathcal{P}}(\mathcal{R}).
\]
The associated total complex $C^\bullet$ with differential $d = d_\mathcal{P} + d_\mathcal{Q}$ is obtained by summing along the anti-diagonals:

$$C^k = \bigoplus_{m+n=k} C^{m,n} = \bigoplus_{m+n=k} \mathcal{L}^{m+n}.$$  

The bihamiltonian cohomology

$$H^\bullet(\mathcal{L}; d_\mathcal{P}, d_\mathcal{Q}) = Z^\bullet(\mathcal{L}; d_\mathcal{P}, d_\mathcal{Q})/B^\bullet(\mathcal{L}; d_\mathcal{P}, d_\mathcal{Q})$$

is the cohomology of this total complex.

A cocycle in $Z^k(\mathcal{L}; d_\mathcal{P}, d_\mathcal{Q})$ is a $(k + 1)$-tuple

$$(c_0, \ldots, c_k) \in \mathcal{L}^k \oplus \cdots \oplus \mathcal{L}^k,$$

such that $d_\mathcal{P}c_0 = 0$, $d_\mathcal{Q}c_i + d_\mathcal{P}c_{i+1} = 0$ for $0 \leq i < k$, and $d_\mathcal{Q}c_k = 0$. This cocycle is a coboundary if there is a $(k-1)$-cochain $(a_0, \ldots, a_{k-1})$ such that $c_0 = d_\mathcal{P}a_0$, $c_i = d_\mathcal{Q}a_{i-1} + d_\mathcal{P}a_i$ for $0 < i < k$, and $c_k = d_\mathcal{Q}a_{k-1}$.

Now suppose that $\mathbf{P} = \mathcal{P} + \sum_{k=1}^\infty \varepsilon^k \mathcal{P}_k$ and $\mathbf{Q} = \mathcal{Q} + \sum_{k=1}^\infty \varepsilon^k \mathcal{Q}_k$ are two compatible formal deformations of $\mathcal{P}$ and $\mathcal{Q}$. Writing the differential $d = d_\mathcal{P} + d_\mathcal{Q}$ in matrix form, we calculate that

$$d(\mathbf{P}_1, \mathbf{Q}_1) = \begin{pmatrix} d_\mathcal{P} & 0 \\ d_\mathcal{Q} & d_\mathcal{P} \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{Q}_1 \end{pmatrix} = \begin{pmatrix} d_\mathcal{P} \mathbf{P}_1 \\ d_\mathcal{Q} \mathbf{P}_1 + d_\mathcal{P} \mathbf{Q}_1 \\ d_\mathcal{Q} \mathbf{Q}_1 \end{pmatrix} \in \mathcal{L}^2 \oplus \mathcal{L}^2 \oplus \mathcal{L}^2.$$

The entries of the latter vector are the coefficients of $\varepsilon$ in the compatibility conditions

$$(\frac{1}{2}[[\mathbf{P}, \mathbf{P}]] = 0, [[\mathbf{P}, \mathbf{Q}], = 0 and \frac{1}{2}[[\mathbf{Q}, \mathbf{Q}]] = 0, respectively.

A first order bihamiltonian deformation consists of first order deformations

$$\mathbf{P}_1 = \mathcal{P} + \varepsilon \mathbf{P}_1 \quad \text{and} \quad \mathbf{Q}_1 = \mathcal{Q} + \varepsilon \mathbf{Q}_1$$

compatible up to first order. The first obstruction cocycle is given by

$$(\frac{1}{2}[[\mathbf{P}_1, \mathbf{P}_1]], [[\mathbf{P}_1, \mathbf{Q}_1]], \frac{1}{2}[[\mathbf{Q}_1, \mathbf{Q}_1]]) \in \mathcal{L}^2 \oplus \mathcal{L}^2 \oplus \mathcal{L}^2,$$

and one easily sees that it vanishes under the differential $d : C^2 \to C^3$. For example, using that $[[\mathbf{P}, \mathbf{Q}_1]] + [[\mathbf{Q}, \mathbf{P}_1]] = 0$, the graded Jacobi identity for the Schouten bracket, and the graded anti-commutativity, we calculate that

$$d_\mathcal{P}[[\mathbf{P}_1, \mathbf{Q}_1]] + \frac{1}{2}d_\mathcal{Q}[[\mathbf{P}_1, \mathbf{P}_1]] = [[\mathbf{P}, [[\mathbf{P}_1, \mathbf{Q}_1]]] + \frac{1}{2}[[\mathbf{Q}, [[\mathbf{P}_1, \mathbf{P}_1]]]]$$

$$= [[[\mathbf{P}, \mathbf{P}_1], \mathbf{Q}_1]] - [[\mathbf{P}_1, [[\mathbf{P}, \mathbf{Q}_1]]]]$$

$$+ \frac{1}{2}[[[[\mathbf{Q}, \mathbf{P}_1], \mathbf{P}_1]] - \frac{1}{2}[[\mathbf{P}_1, [[\mathbf{Q}, \mathbf{P}_1]]]]$$

$$= [[\mathbf{P}_1, [[\mathbf{P}, \mathbf{Q}_1]] + [[\mathbf{Q}, \mathbf{P}_1]]]]$$

$$= 0.$$
Comparing (10) and (11), we see that
$$
\delta f, \ell > 0 \text{ has no nonzero solutions.}
$$

Infinitesimal symmetries.

3.3. dispersionless KdV hierarchy. The Hamiltonian operator $P$ applies to it.

Suppose that $Z \in Z^0(\mathcal{L}(\ell); d_P, d_Q)$, that is, $Z \in L^0$ and $d_P Z = d_Q Z = 0$. Let $h \in \mathcal{A}$ be the characteristic of $Z$. If $\ell = 0$, so that $h \in \mathcal{A}_0$, we see that
$$
d_Q \int h(u) \theta \, dx = - \int \left( u h'(u) + \frac{1}{2} h(u) \right) \theta \, dx,
$$
which cannot be nonzero for smooth $h(u)$. Thus, we may assume that $\ell > 0$.

Theorem 2.3 implies that there exists a functional $\int g \, dx \in L^{-1}(\ell)$ such that
$$
Z = d_P \int g \, dx.
$$

If $\ell = 1$, we see that $g \in \mathcal{A}_0$, and hence
$$
Z = - \int g'(u) \theta_1 \, dx = \int u_1 g''(u) \theta_1 \, dx.
$$

If $\ell = 2$, the density $g = u_1 s(u)$ is a total derivative, and hence $Z = 0$. Thus, assume that $\ell > 2$.

Applying Theorem 2.3 once more to the equation
$$
d_P (d_Q \int g \, dx) = - d_Q d_P \int g \, dx = 0,
$$
we see that there is exists a functional $\int f \, dx \in L^{-1}(\ell)$ such that
$$
d_P \int f \, dx = d_Q \int g \, dx.
$$

We now prove that this is impossible.

The vectors $d_P \int f \, dx = \int \theta_1 \delta f \, dx$ and $d_Q \int g \, dx = \int (u \theta_1 + \frac{1}{2} u_1 \theta) \delta g \, dx$ have characteristics $-\partial \delta f$ and $-\partial \delta (ug) + \frac{1}{2} u_1 \delta g$. Thus, (9) may be written
$$
\frac{1}{2} u_1 \delta g = \partial (\delta (ug) - \delta f).
$$

In particular, $u_1 \delta g$ is a total derivative.

Suppose that $g \in \mathcal{A}_1$, $n > 1$. We have
$$
u_1 \delta g \equiv (-1)^n (1 - n) u_1 u_{n2} \partial_{n2} g + \partial ((-1)^n n u_1 u_{n2-1} \partial_{n2} g) \mod \mathcal{A}[2n - 2].
$$

This cannot be a total derivative unless $\partial_{n2} g = 0$, in which case we may replace $g$ by $g - \partial (u_{n-1} \partial_n g) \in \mathcal{A}[n - 1]$.

Arguing by downward induction, we may assume that $g \in \mathcal{A}[1]$. In other words, $g = u_1^{\ell-1} \gamma(u)$, where $\gamma$ is a smooth function of $u$. We calculate that
$$
\partial \delta f = \partial \delta (ug) - \frac{1}{2} u_1 \delta g
$$
$$
\quad = -(\ell - 2) \partial \left( (\ell - 1) u_2 u_1^{\ell-3} u \gamma(u) + u_1^{\ell-1} u \gamma'(u) + \frac{1}{2} \gamma(u) \right).
$$

This implies that, up to a total derivative, $f$ has the form $u_1^{\ell-1} \eta(u)$, and that
$$
\delta f = -(\ell - 2) ((\ell - 1) u_2 u_1^{\ell-3} \eta(u) + u_1^{\ell-1} \eta'(u))
$$

Comparing (10) and (11), we see that
$$
(\ell - 1) u_2 u_1^{\ell-3} u \gamma(u) + u_1^{\ell-1} (u \gamma'(u) + \frac{1}{2} \gamma(u)) = (\ell - 1) u_2 u_1^{\ell-3} \eta(u) + u_1^{\ell-1} \eta'(u),
$$
which has no nonzero solutions.

In summary, we have proven the following theorem.
Theorem 3.1. The infinitesimal symmetries
\[ H^0(\mathcal{L}; d_P, d_Q) = \{ Z \in \mathcal{L}^0 \mid d_P Z = d_Q Z = 0 \} \]
of the bihamiltonian structure of the dispersionless KdV hierarchy are the vector fields of the form \( d_P \int g(u) \, dx \), or equivalently, vector fields with characteristic of the form \( u_1 h(u) \).

3.4. The bihamiltonian cohomology for structures of hydrodynamic type. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be compatible Hamiltonian operators of hydrodynamic type. The bihamiltonian cohomology decomposes into subspaces
\[ H^k(\mathcal{L}; d_P, d_Q) = \bigoplus_{\ell \geq -k} H^k(\mathcal{L}^{(\ell)}; d_P, d_Q). \]

If \( \mathcal{P} \) is nondegenerate, any cohomology class in \( H^k(\mathcal{L}^{(\ell)}; d_P, d_Q), \ell > -k \), has a representative of the form \((0, \ldots, 0, c)\). The argument, which uses Theorem 3.1, is as follows.

Let \((c_0, \ldots, c_k)\) be a cocycle in \( Z^k(\mathcal{L}^{(\ell)}; d_P, d_Q), \ell > -k \). Since \( d_P c_0 = 0 \) and \( d_P \) is acyclic, we can write the first component \( c_0 \) of the cocycle as a coboundary \( d_P a_0 \) for some \( a_0 \in \mathcal{L}^{k-1}(\ell) \). Subtracting
\[ d(a_0, 0, \ldots, 0) = (d_P a_0, d_Q a_0, 0, \ldots, 0) \]
from \((c_0, \ldots, c_k)\) gives a cocycle in the same cohomology class, with vanishing first component:
\[(c_0, \ldots, c_k) \sim (0, c_1 - d_Q a_0, c_2, \ldots, c_k).\]
In turn, the equation \( d_P(c_1 - d_Q a_0) = 0 \) holds. Iterating the above procedure \( k - 1 \) times, one finally obtains a cocycle \((0, \ldots, 0, c)\) such that
\[(c_0, \ldots, c_k) \sim (0, \ldots, 0, c)\]
and \( d_P c = d_Q c = 0 \). It follows that \( c = d_P Y \) for some \( Y \in \mathcal{L}^{k-1}(\ell) \). Since
\[ d_P d_Q Y = -d_Q d_P Y = 0, \]
we may use the acyclicity of \( d_P \) one last time to see the existence of a cochain \( X \in \mathcal{L}^{k-1}(\ell) \) such that \( d_Q Y = d_P X \). In this way, we obtain the following.

Proposition 3.2. Let \( \mathcal{P} \) and \( \mathcal{Q} \) be a pair of compatible Hamiltonian operators of hydrodynamic type such that \( \mathcal{P} \) is nondegenerate. If \( \ell > -k \), the group \( H^k(\mathcal{L}^{(\ell)}; d_P, d_Q) \) is isomorphic to
\[ \left\{ d_P Y \mid Y \in \mathcal{L}^{k-1}(\ell) \text{ and } d_Q Y = d_P X \text{ for some } X \in \mathcal{L}^{k-1}(\ell) \right\} \bigcup \left\{ d_Q T \mid T \in \mathcal{L}^{k-1}(\ell) \text{ and } d_P T = 0 \right\}. \]

Corollary 3.3. For the two Hamiltonian operators \( \mathcal{P} = \frac{1}{2} \int \theta \theta_1 \, dx \) and \( \mathcal{Q} = \frac{1}{2} \int u \theta_1 \, dx \) of the dispersionless KdV hierarchy, the group \( H^k(\mathcal{L}; d_P, d_Q) \) is isomorphic to
\[ \left\{ d_P Y \mid Y \in \mathcal{L}^{k-1} \text{ and } d_Q Y = d_P X \text{ for some } X \in \mathcal{L}^{k-1} \right\} \bigcup \left\{ d_Q T \mid T \in \mathcal{L}^{k-1} \text{ and } d_P T = 0 \right\}. \]

Proof. In light of Proposition 3.2, we must show that \( H^k(\mathcal{L}^{(-k)}; d_P, d_Q) = 0 \) for \( k \geq 0 \). In fact, the cohomology \( H^k(\mathcal{L}^{(-k)}, d_P) \) is spanned by \( \int 1 \, dx \) and \( \int u \, dx \) for \( k = -1 \), and by \( \int \theta \, dx \) for \( k = 0 \). Thus, the above argument applies as well for \( H^k(\mathcal{L}^{(-k)}, d_P) = 0 \), if \( k > 0 \). Finally, the case \( k = 0 \) was discussed in Section 3.3. \( \square \)
3.5. **Infinitesimal deformations.** A bihamiltonian cohomology class \( c \in H^\bullet(\mathcal{L}; d_P, d_Q) \) is quasi-trivial if its image in \( H^\bullet(\hat{\mathcal{L}}; d_P, d_Q) \) is zero. In other words, \( c = (c_0, \ldots, c_k) \in \mathcal{L}^k \oplus \cdots \oplus \mathcal{L}^k \) is quasi-trivial if there exists \((b_0, \ldots, b_{k-1}) \in \hat{\mathcal{L}}^{k-1} \oplus \cdots \oplus \hat{\mathcal{L}}^{k-1}\) such that
\[
d(b_0, \ldots, b_{k-1}) = (c_0, \ldots, c_k).
\]

We now state our main theorem.

**Theorem 3.4.** All infinitesimal bihamiltonian deformations of the bihamiltonian structure of the dispersionless KdV hierarchy of homogeneous degree greater than 1 are quasi-trivial.

In other words, the image of \( H^1(\mathcal{L}(\ell); d_P, d_Q) \) in \( H^1(\hat{\mathcal{L}}(\ell); d_P, d_Q) \) is zero if \( \ell > 0 \).

Note that \( H^1(\mathcal{L}(0); d_P, d_Q) \) is certainly not trivial: in fact, it may be identified with the space of cocycles \( \{(0, \int s(u)\theta_1 dx) \mid s(u) \in A_0\} \) modulo the one-dimensional space of coboundaries with basis \((0, \int \theta_1 dx) = (-d_P 2\int \theta dx, -d_Q 2\int \theta dx)\).

As part of a more general study of infinitesimal bihamiltonian deformations of “semisimple” bihamiltonian structures in any number of dimensions, Liu and Zhang [LZ1] proved that the kernel of the map
\[
H^1(\mathcal{L}; d_P, d_Q) \rightarrow H^1(\hat{\mathcal{L}}; d_P, d_Q)
\]
has the form
\[
\{(0, \int s(u)\theta_1 \theta_2 dx) \mid s(u) \in A_0\}.
\]
The associated Hamiltonian operator may be calculated by applying the normalization operator \( N \):
\[
\frac{1}{2}N \int s(u)\theta_1 \theta_2 dx = -\theta(s(u)\partial^3 + \frac{3}{2}u_1 s'(u)\partial^2 + \frac{1}{2}(u_2 s'(u) + u_1^2 s''(u))\partial)\theta.
\]
In other words, the equivalence classes of infinitesimal deformations of the bihamiltonian structure \((\partial, u\partial + \frac{1}{2}u_1)\) of the dispersionless KdV hierarchy have the form
\[
(0, s(u)\partial^3 + \frac{3}{2}u_1 s'(u)\partial^2 + \frac{1}{2}(u_2 s'(u) + u_1^2 s''(u))\partial), \quad s(u) \in A_0.
\]
When \( s \) is constant, we recognize the infinitesimal deformation associated to the bihamiltonian structure of the full KdV hierarchy.

3.5.1. **The cocycles.** We start the proof of Theorem 3.4 by studying the equation
\[
(12) \quad \theta \delta_\theta (d_P \int f \theta dx - d_Q \int g \theta dx) = 0, \quad f, g \in A[n].
\]

For \( P = \frac{1}{2}\theta_1 \theta_1 \) and \( Q = \frac{1}{2}u_1 \theta_1 \), the differentials \( d_P \) and \( d_Q \) associated to the Hamiltonian operators \( \mathcal{P} = \int P dx \) and \( \mathcal{Q} = \int Q dx \) are given by the following formulas: if \( F \in \Lambda_\infty \), or
more generally, if \( F \in \hat{\Lambda}_\infty \),
\[
d_P \int F \, dx = -\int \delta \theta P \delta_u F \, dx = -\int \theta_1 \delta_u F \, dx
\]
\[
= -\sum_{k=0}^{\infty} \int \theta_{k+1} \partial_k F \, dx,
\]
\[
d_Q \int F \, dx = -\int (\delta \theta Q \delta_u F + \delta_u Q \delta \theta F) \, dx = -\int ((u \theta_1 + \frac{1}{2} u_1 \theta) \delta_u F + \frac{1}{2} \theta_1 \delta \theta F) \, dx
\]
\[
= -\frac{1}{2} \sum_{k=0}^{\infty} \int ((\partial^k (u \theta_1) + \partial^{k+1} (u \theta)) \partial_k F + \partial^k (\theta \theta_1) \partial_{\theta} F) \, dx.
\]

In particular, if \( h \in \hat{A} \), \( d_P \int h \, dx \) has characteristic \( \partial \delta_u h \), and \( d_Q \int h \, dx \) has characteristic \( (u \partial + \frac{1}{2} u_1) \delta_u h \).

**Proposition 3.5.** Let \( f, g \in \hat{A}[n] \). For \( k \geq 0 \), let

\[
F_k = \partial_k f, \quad \text{and} \quad G_k = \frac{1}{2} \sum_{\ell=0}^{n-k} \left[ \binom{k+\ell}{\ell} + \binom{k+\ell+1}{\ell} \right] u_\ell \partial_{k+\ell} g - \frac{1}{2} \delta_{k,0} g.
\]

Then

\[
\theta \delta \theta (d_P \int f \theta \, dx) = \sum_{k=0}^{n} \theta \theta_{k+1} \left( F_k + \sum_{j=k}^{n} (-1)^j \binom{j+1}{k+1} \partial^{j-k} F_j \right)
\]

and

\[
\theta \delta \theta (d_Q \int g \theta \, dx) = \sum_{k=0}^{n} \theta \theta_{k+1} \left( G_k + \sum_{j=k}^{n} (-1)^j \binom{j+1}{k+1} \partial^{j-k} G_j \right).
\]

**Proof.** We have

\[
\theta \delta \theta \left( d_P \int f \theta \, dx \right) = \theta \delta \theta \left( \sum_{k=0}^{n} \theta \theta_{k+1} F_k \right)
\]

\[
= \theta \sum_{k=0}^{n} \left( \theta_{k+1} F_k + (-1)^k \partial \theta^{k+1} F_k \right)
\]

\[
= \sum_{k=0}^{n} \theta \theta_{k+1} \left( F_k + \sum_{j=k}^{n} (-1)^j \binom{j+1}{k+1} \partial^{j-k} F_j \right).
\]
The formula for \( \theta \delta (d_{\Omega} \int g \, dx) \) is derived similarly:

\[
\theta \delta (d_{\Omega} \int g \, dx) = \frac{1}{2} \theta \delta \left( \theta \sum_{k=0}^{n} \left( \partial^k (u \theta_1) + \partial^{k+1} (u \theta) \right) \partial_k g - \theta \theta_1 g \right)
\]

\[
= \frac{1}{2} \theta \delta \left( \theta \sum_{k=0}^{n} \sum_{\ell=0}^{k} \left( \binom{k}{\ell} + \binom{k+1}{\ell} \right) \partial_{k-\ell+1} u_t \partial_k g - \theta \theta_1 g \right)
\]

\[
= \frac{1}{2} \theta \delta \left( \sum_{k=0}^{n} \theta \theta_{k+1} \sum_{\ell=0}^{n-k} \left( \binom{k}{\ell} + \binom{k+1}{\ell+1} \right) u_t \partial_{k+\ell+1} g - \theta \theta_1 g \right)
\]

\[
= \theta \delta \left( \sum_{k=0}^{n} \theta \theta_{k+1} \left( G_k + \sum_{j=k}^{n} \binom{j+1}{k+1} \partial^{j-k} G_j \right) \right)
\]

3.5.2. The constraints. For \( j \geq 0 \), let us define coefficients

\[
e_j = F_j - G_j
\]

\[
= \partial_j (f - u g) - \frac{1}{2} \sum_{\ell=1}^{n-j} \left( \binom{j+\ell}{\ell} + \binom{j+\ell+1}{\ell+1} \right) u_t \partial_{j+\ell} g + \frac{3}{2} \delta_j, 0 g,
\]

and for \( k \geq 0 \) let

\[(13) \quad S_k = e_k + \sum_{j=0}^{\infty} (-1)^j \binom{j+1}{k+1} \partial^{j-k} e_j.
\]

Assume that \( f, g \in \hat{A}[n] \). Then \( e_j = 0 \) for \( j > n \), and by Proposition 3.5

\[
\theta \delta \left( d_{\Omega} \int f \theta \, dx - d_{\Omega} \int g \theta \, dx \right) = \sum_{k=0}^{n} \theta \theta_{k+1} S_k.
\]

That is, (12) is equivalent to the system

\[ S = \{ S_k = 0 \mid 0 \leq k \leq n \}. \]

Let \( n = 2m > 0 \) be even. Define

\[(14) \quad E_{\ell} = \sum_{j=2^{\ell}}^{2^{m+\ell}} (-1)^j \binom{2m-j}{2^{\ell+1}} \binom{j+1}{2^{\ell+1}} \partial^{j-2\ell} e_j.
\]

**Proposition 3.6.** We have

\[
S_k = \begin{cases} 
\sum_{\ell=0}^{m} \binom{2^{\ell+1}}{k+1} \binom{2m-k-1}{m-\ell} \partial^{2\ell-k} E_{\ell}, & k < n, \\
2E_m, & k = n.
\end{cases}
\]

In particular, the subset \( \{ S_{2i} = 0 \mid 0 \leq i \leq m \} \) of the system \( S \) of equations is equivalent to the system of equations \( E = \{ E_{\ell} = 0 \mid 0 \leq \ell \leq m \}. \)
Proof. The result is clear for \( k = n \), since in this case, \( S_n = 2e_n \) and \( E_m = e_n \). From now on, we assume that \( k < n \).

We wish to prove that 

\[
S_k = \sum_{\ell=0}^{m} \frac{(2\ell+1)}{(n-k-1)_{m-\ell}} \partial^{2\ell-k} E_\ell = \sum_{\ell=0}^{m} \frac{(2\ell+1)}{(n-k-1)_{m-\ell}} \sum_{j=2\ell}^{m+\ell} (-1)^j \binom{n-j}{m-\ell} \binom{j+1}{2\ell+1} \partial^j e_j
\]

= \sum_{j=k}^{n} (-1)^j \left( \sum_{\ell=0}^{\infty} \frac{(n-j)}{(2\ell+1)_{k+1}} \binom{j+1}{2\ell+1} \right) \partial^j e_j.

(In this formula, it is understood that \( \binom{0}{s} = 0 \) if \( s < 0 \).) Comparing with the definition of \( S_k \), we see that we are left to prove that for \( k \leq j \leq n \),

\[
\sum_{\ell=0}^{\infty} \frac{(n-j)}{(n-k-1)_{m-\ell}} \binom{j+1}{2\ell+1} \binom{j+1}{2\ell+1} = \begin{cases} \binom{j+1}{k+1}, & j > k, \\ 1 + (-1)^k, & j = k. \end{cases}
\]

We start with the case where \( j > k \). Making the substitutions \( \alpha = j - k - 1 \), \( \beta = n - j \) and \( \ell = p + j - m \), we have

\[
\sum_{\ell=0}^{\infty} \frac{(n-j)}{(n-k-1)_{m-\ell}} \binom{j+1}{2\ell+1} \binom{j+1}{2\ell+1} = \sum_{p=0}^{\infty} \frac{\beta}{\alpha+\beta} \binom{2p+2j-n+1}{2p+2j-n+1}
\]

\[
= \frac{\beta!(j+1)!}{(\alpha+\beta)!(k+1)!} \sum_{p=0}^{\infty} \frac{(\alpha+p)!}{(\alpha+\beta+2p+1)!(\beta-2p)!} = \frac{\beta!(j+1)!}{(\alpha+\beta)!(k+1)!} \sum_{p=0}^{\infty} \frac{(\alpha+p)!(\alpha+1)_{\beta-2p}}{(\alpha+\beta)!}.\]

By Lemma 3.7 this equals \( \binom{j+1}{k+1} \).

If \( j = k \), then the only possibly nonzero term of the sum

\[
\sum_{\ell=0}^{\infty} \frac{(n-j)}{(n-k-1)_{m-\ell}} \binom{j+1}{2\ell+1} \binom{j+1}{2\ell+1}
\]

is that with \( k = 2\ell \), in which case it equals 2; if \( k \) is odd, the sum vanishes. \( \square \)

Lemma 3.7.

\[
\sum_{p=0}^{\infty} \binom{\alpha+1}{\beta-2p} \binom{\alpha+p}{p} = \binom{\alpha+\beta}{\beta}
\]

Proof. Using the formula

\[
\binom{n+k}{k} = (-1)^k \binom{-n-1}{k},
\]

we see that we must prove that

\[
\sum_{p=0}^{\infty} (-1)^p \binom{\alpha+1}{\beta-2p} \binom{-\alpha-1}{p} = (-1)^\beta \binom{-\alpha-1}{\beta}.
\]

Expanding both sides of the identity

\[
(1 + x)^{\alpha+1} (1 - x^2)^{-\alpha-1} = (1 - x)^{-\alpha-1}
\]

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as power series of \( x \) and extracting the coefficient of \( x^\beta \), the lemma follows.

3.5.3. The induction. The next result is the main part of the proof of Theorem 3.4.

**Theorem 3.8.** Let \( n = 2m > 4 \). Suppose \( f \) and \( g \) are characteristics in \( \hat{A}[n] \) which satisfy the equation

\[
d_{\vec{m}} f \theta dx = d_{\vec{c}} g \theta dx.
\]

Then there are densities \( a, b, c \in \hat{A}[m] \) such that \( \hat{f} = f + \partial \xi a + (u\partial + \frac{1}{2}u_1)\delta a \) and \( \hat{g} = g - \partial \xi b + (u\partial + \frac{1}{2}u_1)\delta c \) lie in \( \hat{A}[n-2] \).

The point of this theorem is that if \( f \) and \( g \) satisfy the equation \( d_{\vec{m}} f \theta dx = d_{\vec{c}} g \theta dx \), then so do \( \hat{f} \) and \( \hat{g} \).

This theorem will be proven in a number of steps.  

Step 1. Since

\[
E_m = c_n
\]

vanishes, we have \( f - ug \in \hat{A}[n-1] \) or, equivalently, \( c_n = \partial_n (f - ug) = 0 \),

Step 2. We now argue by induction on \( 1 \leq t \leq m \) that

\[
\partial_{n-t+1} \partial_n g = 0,
\]

or equivalently, that \( g = u_ng_0 + g_1 \), where \( g_0 \in \hat{A}[n-t] \) and \( g_1 \in \hat{A}[n-1] \). Assume as induction hypothesis that \( \partial_{n-s+1} \partial_n g = 0 \) for \( 1 \leq s < t \). Since \( e_j \in \hat{A}[n] \), we see that

\[
E_m-t = \sum_{j= n-2t}^{n-t} (-1)^j \left( ^{n-j}_t \right) \left( ^{j+1}_{n-2t+1} \right) \partial^{j+2t-n} e_j \in \hat{A}[n+t].
\]

We now argue as follows: the coefficient of \( u_{n+t} \) in \( E_{m-t} \) equals

\[
[u_{n+t}] E_{m-t} = (-1)^{n-t} \left( ^{n-t+1}_{n-2t+1} \right) [u_{n+t}] \partial^t e_{n-t}
\]

\[
= (-1)^{n-t} (n-t + \frac{3}{2}) \left( ^{n-t+1}_{n-2t+1} \right) u_1 \partial_{n-t+1} \partial_n g.
\]

Since \( (n-t + \frac{3}{2}) \left( ^{n-t+1}_{n-2t+1} \right) \neq 0 \), we see that \( \partial_{n-t+1} \partial_n g = 0 \).

Step 3. At this point, we know that \( g = u_ng_0 + g_1 \), where \( g_0 \in \hat{A}[m] \) and \( g_1 \in \hat{A}[n-1] \).

**Lemma 3.9.** If \( h \in \hat{A}[m] \), \( m > 0 \), then

\[
\partial \delta_u (uh) - (u\partial + \frac{1}{2}u_1) \delta_u h \equiv (-1)^m (m + \frac{1}{2}) u_n u_1 \partial^2_m h \mod \hat{A}[n-1].
\]

**Proof.** We have

\[
\delta_u (uh) \equiv (-\partial)^m u \partial_m h \equiv u_\delta h + (-1)^m m u_{n-1} u_1 \partial^2_m h \mod \hat{A}[n-2],
\]

\[
\delta_u h \equiv (-\partial)^m \partial_m h \equiv (-1)^m u_n \partial^2_m h \mod \hat{A}[n-1].
\]

From these equations, the lemma follows easily. 

Let \( h \in \hat{A}[m] \) be a solution of the equation

\[
g_0 = (-1)^m (m + \frac{1}{2}) u_1 \partial^2_m h.
\]

By Lemma 3.9, \( f + \partial \delta_u (u^2 h) - (u\partial + \frac{1}{2}u_1) \delta_u (uh) \) and \( g + \partial \delta_u (uh) - (u\partial + \frac{1}{2}u_1) \delta_u h \) are in \( \hat{A}[n-1] \).
Step 4. In Step 3, we have shown that we may reduce to the case that \( f, g \in \mathring{A}[n - 1] \). We now show that \( \partial^2_{n-1}(f - ug) = 0 \). We have

\[
E_{m-1} = 2e_{n-2} - n\partial e_{n-1} = 0.
\]

Since \( e_{n-2} \in \mathring{A}[n - 1] \) and \( e_{n-1} = \partial_{n-1}(f - ug) \), we see that

\[
[u_n]E_{m-1} = -n[u_n]\partial e_{n-1} = -n\partial^2_{n-1}(f - ug) = 0.
\]

In particular, \( f - (ug + u_{n-1}\partial_{n-1}(f - ug)) \in \mathring{A}[n - 2] \).

Step 5. We now argue by induction on \( 2 \leq t \leq m \) that

\[
\partial_t\partial_{n-1}(f - ug) = 0,
\]

or equivalently, that \( e_{n-1} = \partial_{n-1}(f - ug) \in \mathring{A}[n - t] \). Assume as induction hypothesis that \( \partial_{n-s}e_{n-1} = 0 \) for \( 1 \leq s < t \), so that

\[
\partial_{n-t}e_{n-1} = [u_{2n-t-1}]\partial^{n-1}e_{n-1}.
\]

**Lemma 3.10.** If for \( 2 \leq t \leq m \), and \( 1 \leq s < t \), \( \partial_{n-s}e_{n-1} = 0 \), then

\[
[u_{2n-t-1}]\partial^{n-1}e_{n-1} = 0.
\]

**Proof.** For \( 2 \leq t \leq m \) and \( 1 \leq k \leq t \), let

\[
T_{t,k} = [u_{2n-t-1}]\sum_{j=n-t}^{n-1} (j+1)\binom{n-k}{k-1}(-\partial)^j e_j.
\]

Then

\[
\sum_{k=1}^{n} (-1)^{k+t}\binom{n-k}{n-t}T_{t,k} = -[u_{2n-t-1}]\partial^{n-1}e_{n-1}
\]

since

\[
\sum_{k=1}^{n} (-1)^{j+k+t}\binom{n-k}{n-t}\binom{j+1}{k-1} = \begin{cases} 
0, & n-t < j < n-1, \\
-1, & j = n-1.
\end{cases}
\]

From the definition \([13]\) of the \( S_k \), we see that

\[
T_{t,k} = \begin{cases} 
2[u_{2n-t-1}]\partial^{k-2}S_{k-2}, & 1 < k \leq t, \\
[u_{2n-t-1}]\sum_{j=0}^{n-2}(-\partial)^j S_j, & k = 0.
\end{cases}
\]

Since the functions \( S_k \) vanish, \( T_{t,k} \) also vanish. It follows that \( [u_{2n-t-1}]\partial^{n-1}e_{n-1} = 0 \) for \( 2 \leq t \leq m \).

Step 6. In Step 5, we showed that after the redefinition of Step 3, we have \( f = ug + u_{n-1}e_{n-1} + f_0 \), where \( f_0 \in \mathring{A}[n - 2] \) and \( e_{n-1} \in \mathring{A}[m - 1] \). The next lemma shows that after a further redefinition of \( f \), we may assume \( e_{n-1} = 0 \), that is, \( f - ug \in \mathring{A}[n - 2] \).

**Lemma 3.11.** If \( h \in \mathring{A}[m - 1], m > 0 \), then

\[
\delta_u h \equiv (-1)^{m-1}u_{n-2}\partial^2_{n-1}h \mod \mathring{A}[n - 3].
\]
Proof. We have
\[
\delta_u h \equiv (-\partial)^{m-1} \partial_{m-1} h \mod \hat{A}[n-3],
\]
and the lemma follows. □

Let \( h \in \hat{A}[m-1] \) be a solution of the equation
\[
e_{n-1} = (-1)^{m} \partial_{m-1}^{2} h.
\]
Replacing \( f \) by \( f + \partial \delta_u h \), we see that \( f - ug \in \hat{A}[n-2] \).

Step 7. We now argue by induction on \( 1 \leq t < m \) that
\[
\partial_{n-t} \partial_{n-1} g = 0,
\]
or equivalently, that \( g = u_{n-1}g_0 + g_1 \), where \( g_0 \in \hat{A}[n-t-1] \) and \( g_1 \in \hat{A}[n-2] \). Assume as induction hypothesis that \( \partial_{n-s} \partial_{n-1} g = 0 \) for \( 1 \leq s < t \). Since \( e_j \in \hat{A}[n-1] \), we see that
\[
E_{m-t-1} = \sum_{j=n-2t-2}^{n-t-1} (-1)^{j-1} (n-j) j m \partial^{j+2t-n} e_j \in \hat{A}[n+t].
\]
The coefficient of \( u_{n+t} \) in \( E_{m-t-1} \) equals
\[
[u_{n+t}] E_{m-t-1} = (-1)^{n-t-1} (n-t-1) [u_{n+t}] \partial^{t+1} e_{n-t-1}
\]
\[
= (-1)^{n-t-1} (n-t+1) (n-2t-1)u_1 \partial_{n-t} \partial_{n-1} g.
\]
Since \( (n-t+1) (n-2t-1) \neq 0 \), we see that \( \partial_{n-t} \partial_{n-1} g = 0 \) for \( 1 \leq t < m \).

Step 8. In Step 7, we showed that \( g = u_{n-1}g_0 + g_1 \), where \( g_0 \in \hat{A}[m] \) and \( g_1 \in \hat{A}[n-2] \). We now show that \( g_0 \) is actually in \( \hat{A}[m-1] \), that is, \( \partial_{m} \partial_{n-1} g = 0 \).

Lemma 3.12. If \( e_{n-1} = 0 \), then \( [u_{3m-2}] \partial^{m-1} e_{m-1} = 0 \).

Proof. For \( 1 \leq k \leq m \), let
\[
U_{m,k} = [u_{3m-2}] \sum_{j=m-1}^{n-1} (-1)^{j} (n-j) j m \partial^{j} e_j.
\]
Then
\[
\sum_{k=1}^{n} (-1)^{k+m} (n-m-k) U_{m,k} = [u_{3m-2}] \partial^{m-1} e_{m-1}
\]
since \( e_{n-1} = 0 \), and
\[
\sum_{k=1}^{n} (-1)^{j+k+m} (n-k-j+1) (n-j+1) = \begin{cases} 1, & j = m-1, \\ 0, & m \leq j < n-1. \end{cases}
\]
From the definition of \( S_k \) we see that
\[
U_{m,k} = \begin{cases} 2[u_{3m-2}] \partial^{k-2} S_{k-2}, & 0 \leq k < m, \\ [u_{3m-2}] \sum_{j=0}^{n-2} (-\partial)^{j} S_j, & k = 0. \end{cases}
\]
Since the functions \( S_k \) vanish, the lemma follows. □
To show that $\partial_n \partial_{n-1} g = 0$, we now argue as follows. By definition,

$$e_{m-1} = \partial_{m-1} (f - ug) - \frac{1}{2} \sum_{\ell=1}^{m} \left[(m+\ell-1) + \binom{m+\ell}{\ell}\right] u_{\ell} \partial_{m+\ell-1} g.$$  

Since $f - ug \in \hat{\mathcal{A}}[n - 2]$ and $g - u_{n} g_{0} \in \hat{\mathcal{A}}[n - 2]$, we see that the only contribution to the coefficient of $u_{3m-2}$ in $\partial^{m-1} e_{m-1}$ comes from the term in $e_{m-1}$ with $\ell = 1$, and that

$$[u_{3m-2}] \partial^{m-1} e_{m-1} = -(m + \frac{1}{2}) u_{1} \partial_{m} \partial_{n-1} g.$$  

Since $(m + \frac{1}{2}) \neq 0$, we see that $\partial_{n} \partial_{n-1} g = 0$.

Step 9. We have shown that after redefinitions of $f$ and $g$, we have $g = u_{n-1} g_{0} + g_{1}$, where $g_{0} \in \hat{\mathcal{A}}[m - 1]$ and $g_{1} \in \hat{\mathcal{A}}[n - 2]$, and that $f - ug \in \hat{\mathcal{A}}[n - 2]$. Let $h \in \hat{\mathcal{A}}[m - 1]$ be a solution of the equation

$$g_{0} = (-1)^{m} \partial_{m-1}^{2} h.$$  

By Lemma 3.11, $f - (u \partial + \frac{1}{2} u_{1}) \delta_{\ell} h$ and $g + \partial \delta_{\ell} h$ are in $\hat{\mathcal{A}}[n - 2]$.

This completes the proof of Theorem 3.8.

3.5.4. The proof of Theorem 3.4. Let $c = (c_{0}, c_{1})$ be a cohomology class in $H^{1}(\mathcal{L}; d_{\mathcal{F}}, d_{\mathcal{Q}})$. By Corollary 3.3, there exists $n \geq 0$ and $f$ and $g$ in $\mathcal{A}[n]$ such that $(c_{0}, c_{1})$ is cohomologous to $(0, d_{\mathcal{F}} \int g \theta \, dx)$ and the equation

$$d_{\mathcal{F}} \int f \theta \, dx = d_{\mathcal{Q}} \int g \theta \, dx$$  

holds. We may assume that there exists an integer $\ell$ such that $c$ is homogeneous of degree $\ell + 1$, that is, that $c \in H^{1}(\mathcal{L}(\ell); d_{\mathcal{F}}, d_{\mathcal{Q}})$. Then $f$ and $g$ may be taken to be homogeneous of degree $\ell$; since they are polynomial in the jet variables $\{u_{1}, \ldots, u_{n}\}$, we conclude that $n$ is no larger than $\ell$.

If $\ell > 2$, we may redefine $f$ and $g$ so that they lie in $\hat{\mathcal{A}}[2]$. To see this, we use a downward induction based on Theorem 3.8 to redefine $f$ and $g$ so that they lie in $\hat{\mathcal{A}}[4]$. All the steps up until Step 9 in Subsection 3.5.3 remain valid for $n = 4$, showing that after a further redefinition, we may assume that $f = ug + f_{1}$ and that $g = u_{3} u_{\ell-3} s(u) + g_{1}$, where $f_{1}, g_{1} \in \hat{\mathcal{A}}[2]$. Since $\ell > 2$, the argument in Step 9 may still be used, and in this way, we may redefine $f$ and $g$ so that they lie in $\hat{\mathcal{A}}[2]$.

It is easily checked that Steps 1 and 2 in Subsection 3.5.3 apply, and since $\ell > 2$, Step 3 applies as well. Thus, we are reduced to the case where $f$ and $g$ lie in $\hat{\mathcal{A}}[1]$, and hence

$$f = u_{1} s(u) \quad \text{and} \quad g = u_{1} t(u),$$  

where $s, t \in \mathcal{A}_{0}$. The equation $S_{0} = 0$ may be rewritten as

$$0 = e_{0} - \partial e_{1} = \left(\partial_{0} (f - ug) - \frac{3}{2} u_{1} \partial_{1} g + \frac{3}{2} g\right) - \partial \partial_{1} (f - ug).$$  

Taking the coefficient of $u_{2}$, we see that

$$0 = [u_{2}] (e_{0} - \partial e_{1}) = -[u_{2}] \partial \partial_{1} (f - ug) = -\ell (\ell - 1) u_{1}^{\ell-2} (s(u) - ut(u)).$$  

Thus, $f = ug$, and hence

$$e_{0} - \partial e_{1} = \frac{3}{2} (1 - \ell) g.$$  

Therefore $g$ vanishes, and hence so does the cohomology class $(0, d_{\mathcal{F}} \int g \theta \, dx)$. 



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We now turn to the case $\ell = 2$. By the vanishing of $e_2 = \partial_2(f - ug)$, we see that

$$f = ug + u_1^2 p(u) \quad \text{and} \quad g = u_2 s(u) + u_1^2 t(u),$$

with $s, t, p \in A_0$. In this case, the equation $S_0 = 0$ becomes

$$0 = e_0 - \partial e_1 = (\partial_0 (f - ug) - \frac{3}{2} u_1 \partial_1 g - 2 u_2 \partial_2 g + \frac{3}{2} g) - \partial_1 (f - ug) - \frac{5}{2} u_1 \partial_2 g$$

$$= 2u_2(s(u) - p(u)) + u_1^2\left(\frac{5}{2}s'(u) - p'(u) - \frac{3}{2}t(u)\right).$$

It follows that $s(u) = p(u)$ and that $t(u) = p'(u)$, and hence that $g = \partial(u_1 p(u))$. We calculate that

$$d_\mathcal{P} \int g \theta \, dx = -d_\mathcal{Q} \int u_1 p(u) \theta_1 \, dx = -\int p(u) \theta_1 \theta_2 \, dx.$$

The following lemma shows that the cocycle $(0, \int p(u) \theta_1 \theta_2 \, dx)$ is quasi-trivial.

**Lemma 3.13.**

$$d_\mathcal{Q} d_\mathcal{Q} \int \frac{u_2}{u_1} h(u) \, dx = \frac{3}{2} \int h'(u) \theta_1 \theta_2 \, dx$$

**Proof.** We calculate that

$$d_\mathcal{Q} \int \frac{u_2}{u_1} h(u) \, dx = - \int \left[\frac{3}{2} h(u) \theta_2 + uh'(u)\left(\frac{u_2}{u_1} \theta_1 - \theta_2\right)\right] \, dx.$$ 

Applying $d_\mathcal{P}$, the lemma follows. \qed

The case $\ell = 1$ is uninteresting: since $g = u_1 s'(u)$, $s(u) \in A_0$, we have

$$d_\mathcal{P} \int g \theta \, dx = -d_\mathcal{Q} \int s(u) \theta_1 \, dx = 0.$$
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