Slowly Synchronizing Automata
with Idempotent Letters of Low Rank

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Abstract

We use a semigroup-theoretic construction by Peter Higgins in order to produce, for each even \(n\), an \(n\)-state and 3-letter synchronizing automaton with the following two features:
1) all its input letters act as idempotent selfmaps of rank \(\frac{n^2}{2}\);
2) its reset threshold is asymptotically equal to \(\frac{n^2}{2}\).

1 Background and overview

A complete deterministic finite automaton (DFA) is a triple \(\langle Q, \Sigma, \delta \rangle\), where \(Q\) and \(\Sigma\) are finite sets called the state set and the input alphabet respectively, and \(\delta : Q \times \Sigma \rightarrow Q\) is a totally defined map called the transition function. Let \(\Sigma^*\) stand for the collection of all finite words over the alphabet \(\Sigma\), including the empty word. The transition function extends to a function \(Q \times \Sigma^* \rightarrow Q\), still denoted \(\delta\), in the following natural way: for every \(q \in Q\) and \(w \in \Sigma^*\), we set \(\delta(q, w) := q\) if \(w\) is empty and \(\delta(q, w) := \delta(\delta(q, v), a)\) if \(w = va\) for some \(v \in \Sigma^*\) and some \(a \in \Sigma\). Thus, every word \(w \in \Sigma^*\) induces the selfmap \(q \mapsto \delta(q, w)\) of the set \(Q\); we say that \(w\) is idempotent if so is the selfmap induced by \(w\), that is, if \(\delta(q, w) = \delta(q, w^2)\) for each \(q \in Q\).

When we deal with a fixed DFA, we simplify our notation by suppressing the sign of the transition function; this means that we introduce the DFA

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*Supported by the Ministry of Science and Higher Education of the Russian Federation, project no. 1.580.2016, and the Competitiveness Enhancement Program of Ural Federal University.
as a pair \( (Q, \Sigma) \) rather than a triple \( (Q, \Sigma, \delta) \) and write \( q.w \) for \( \delta(q, w) \) and \( Q.w \) for \( \{ \delta(q, w) \mid q \in Q \} \).

A DFA \( \mathcal{A} = (Q, \Sigma) \) is called synchronizing if there exists a word \( w \in \Sigma^* \) whose action resets \( \mathcal{A} \), that is, \( w \) leaves the automaton in one fixed state, regardless of the state at which \( w \) is applied. This means that \( q.w = q'.w \) for all \( q, q' \in Q \). Any word \( w \) with this property is said to be a reset word for the automaton, and the minimum length of reset words for \( \mathcal{A} \), denoted \( \text{rt}(\mathcal{A}) \), is called the reset threshold of \( \mathcal{A} \).

Synchronizing automata serve as transparent and useful models of error-resistant systems in many applied areas (system and protocol testing, information coding, robotics). At the same time, synchronizing automata surprisingly arise in some parts of pure mathematics (symbolic dynamics, theory of substitution systems, and others). Basics of the theory of synchronizing automata as well as its diverse connections and applications are discussed, for instance, in the survey [24] and in the chapter [13] of the forthcoming “Handbook of Automata Theory”. Here we focus on only one aspect of the theory, namely, on the question of how the reset threshold of a synchronizing automaton depends on the number of states.

A DFA with two input letters is called binary. In 1964, Černý [3] constructed for each \( n > 1 \), a binary synchronizing automaton \( \mathcal{C}_n \) with \( n \) states and reset threshold \( (n - 1)^2 \). Recall the definition of \( \mathcal{C}_n \). If we denote the states of \( \mathcal{C}_n \) by \( 1, 2, \ldots, n \) and the input letters by \( \sigma_1 \) and \( \sigma_2 \), the actions of the letters are as follows:

\[
i.\sigma_1 := \begin{cases} i & \text{if } i < n, \\ 1 & \text{if } i = n\end{cases} \quad i.\sigma_2 := \begin{cases} i + 1 & \text{if } i < n, \\ 1 & \text{if } i = n.\end{cases}
\]

The automaton \( \mathcal{C}_n \) is shown in Fig. 1.

The automata in the Černý series are well-known in the connection with the famous Černý conjecture about the maximum reset threshold for synchronizing automata with \( n \) states, see [24]. The automata \( \mathcal{C}_n \) provide the lower bound \( (n - 1)^2 \) for this maximum, and the conjecture claims that these automata represent the worst possible case since it has been conjectured that every synchronizing automaton with \( n \) states can be reset by a word of length \( (n - 1)^2 \). The conjecture, first stated in the 1960s, resists researchers’ efforts for more than 50 years. The best upper bound achieved so far is cubic in \( n \); it is due to Shitov [21] who has slightly improved the bound established by Szykula [22]. In turn, Szykula’s bound is only slightly better than the upper bound \( \frac{n^2 - n}{6} \) established by Pin [16] and Frankl [6] approx. 35 years ago.
Why is the Černý conjecture so surprisingly hard? Here we mention only one of the difficulties encountered by the theory of synchronizing automata, namely, the shortage of examples of slowly synchronizing automata, i.e., automata with reset threshold close to the square of the number of states. It has already been observed in the literature that with a very restricted number of examples in hand, it was hard to verify various guesses and assumptions that had arisen when researchers were searching for approaches to the Černý conjecture. That is why the history of investigations in the area abounds in “false trails”, i.e., auxiliary hypotheses that looked promising at first but were disproved after some time.

For brevity, a DFA with \( n \) states is referred to as an \( n \)-automaton. The series found in [3] still remains the only known infinite series of \( n \)-automata with reset threshold \((n-1)^2\). Besides that, we know only a few isolated examples of such automata, the largest (with respect to the state number) being the 6-automaton discovered by Kari [12]; see [24] for a complete list of known synchronizing \( n \)-automata with reset threshold \((n-1)^2\). Moreover, even infinite series of synchronizing \( n \)-automata whose reset thresholds are asymptotically equal to \((n-1)^2\) turns out to be extremely rare, especially those consisting of non-binary automata. We call a non-binary synchronizing automaton \( \mathcal{A} = (Q, \Sigma, \delta) \) proper if for every letter \( a \in \Sigma \), the DFA \( \langle Q, \Sigma^a, \delta^a \rangle \) is not synchronizing, where \( \Sigma^a := \Sigma \setminus \{a\} \) and \( \delta^a \) is the restriction of \( \delta \) to the set \( Q \times \Sigma^a \); in other terms, \( \mathcal{A} \) is proper whenever \(|\Sigma| > 2\) and each reset word of \( \mathcal{A} \) involves all letters of \( \Sigma \). The largest known proper DFA with reset threshold that matches the Černý’s bound is the 3-letter 5-automaton found by Roman [18]. We mention also two infinite series of proper 3-letter
synchronizing \( n \)-automata with reset thresholds \( n^2 - 3n + 3 \) and \( n^2 - 3n + 2 \) constructed by Kisielewicz and Szykula [14] and two infinite series of proper synchronizing \( n \)-automata with \( k \) letters (\( k \) is any integer greater than 2) and reset thresholds \( n^2 - (k + 2)n + 2k + 1 \) and \( n^2 - (k + 2)n + 2k + 2 \) invented by Dzyga, Ferens, Gusev, and Szykula [5, Section 6].

It appears that the border value of reset threshold, at which one begins to observe proper non-binary synchronizing \( n \)-automata more frequently, is situated somewhere near the value \( \frac{n^2}{2} \). In the literature, one can find several constructions of infinite series of non-binary synchronizing \( n \)-automata such that their reset thresholds are asymptotically equal to \( \frac{n^2}{2} \) and, besides that, the automata in the series possess some specific property such as having a sink state [19], admitting a directed Eulerian walk [23], or being capable to perform an arbitrary selfmap of the state set [8]. For further examples of proper non-binary synchronizing automata with relatively large reset threshold, we refer to [2,17].

In the present note we provide a tool for constructing new series of synchronizing automata with similar parameters but rather peculiar extra properties. Namely, we describe a simple transformation that, given an arbitrary synchronizing \( n \)-automaton \( \mathcal{A} \) with \( k \) input letters, produces a synchronizing automaton \( \mathbb{H}(\mathcal{A}) \) with \( 2n \) states and \( k + 1 \) input letters such that every letter of \( \mathbb{H}(\mathcal{A}) \) acts as an idempotent selfmap on the state set and \( \text{rt}(\mathbb{H}(\mathcal{A})) = 2 \text{rt}(\mathcal{A}) \). If applied to a synchronizing \( n \)-automaton whose reset threshold is close to \( (n - 1)^2 \), the transformation results in a synchronizing automaton with \( m := 2n \) states and reset threshold close to \( \frac{m^2}{2} \). In particular, if one applies it to the automata \( \mathcal{C}_n \) in the Černý series, one gets a new series of proper synchronizing automata \( \mathcal{B}_m := \mathbb{H}(\mathcal{C}_m) \) with \( m := 2n \) states and 3 idempotent input letters such that the reset threshold of \( \mathcal{B}_m \) is equal to \( 2(n - 1)^2 = \frac{m^2}{2} - 2m + 2 \).

An additional feature of the transformation \( \mathcal{A} \mapsto \mathbb{H}(\mathcal{A}) \) is that it leads to automata in which all letters have relatively low rank. Recall that the rank of a letter \( a \in \Sigma \) with respect to a DFA \( \mathcal{A} = (Q, \Sigma) \) is defined as the cardinality of the set \( Q.a \). In an overwhelming majority of known examples of slowly synchronizing \( n \)-automata, their input letters all have ranks close to \( n \). Gusev [9] investigated the following question: if all input letters of a synchronizing automata \( \mathcal{A} = (Q, \Sigma) \) have rank at most \( r < |Q| \), how does the reset threshold of \( \mathcal{A} \) depend on \( r \)? He provided a lower bound (about which he expressed the hope that it might be close to optimal) via rather a tricky construction that yields for each \( r \), a binary synchronizing automaton whose letters have rank \( r \) and whose reset threshold is at least \( r^2 - r - 1 \).
In our automata of the form $H(A)$, the rank $r$ of each input letter is equal to one half of number of states, whence, as we see from the aforementioned example of the series $B_m := H(C_n)$, the reset threshold of such automata as a function of $r$ may attain the value $2r^2 - 4r + 2$. This improves the lower bound from [9]; we recall, however, that automata in Gusev’s construction are binary while the automata $B_m$ are not.

Our transformation $\mathcal{A} \rightarrow H(\mathcal{A})$ is a straightforward adaptation of a construction suggested by Higgins [11] in the realm of semigroup theory. Higgins used this construction to give a new simple proof of the following result independently obtained in [7] and [15]: an arbitrary (finite) semigroup may be embedded into another (finite) semigroup in which every element is the product of two idempotents. Our contribution consists in observing that, when restated in automata-theoretic terms, Higgins’s construction leads to a transformation that preserves both the property of being synchronizing and the order of magnitude of reset threshold.

We describe the transformation and prove its main properties in Section 2. In Section 3 we establish a tight upper bound for the reset threshold of synchronizing automata with two idempotent input letters. Here the reset threshold is always less than the number of states, in a strong contrast to the non-binary case.

A preliminary version of this paper appeared in July 2018 as preprint [26] and was submitted to a journal in October 2018. Later the author learned that the transformation presented in Section 2 was independently found by Don and Zantema; their preprint [4] appeared in December 2018.

## 2 The transformation

Take a DFA $\mathcal{A} = \langle Q, \Sigma \rangle$ with $Q := \{1, 2, \ldots, n\}$ and $\Sigma := \{\sigma_1, \ldots, \sigma_k\}$. Let $Q' := \{1', 2', \ldots, n'\}$ be a copy of $Q$ such that $Q$ and $Q'$ is disjoint. Consider the automaton $H(\mathcal{A})$ with the state set $R := Q \cup Q'$ and the input alphabet $\Theta := \{a_1, \ldots, a_k, b\}$ in which the action of the letters is defined as follows:

\begin{align*}
i.a_j &:= i \quad \text{and} \quad i'.a_j := i.\sigma_j \quad \text{for all} \quad i = 1, \ldots, n, \quad j = 1, \ldots, k; \\
i.b &:= i'.b := i' \quad \text{for all} \quad i = 1, \ldots, n.
\end{align*}

For an illustration, Fig. [2] shows the automaton that one gets if the above construction is applied to the Černý automaton $C_n$ from Fig. [1].

We start with registering two properties of the automaton $H(\mathcal{A})$ that immediately follow from the definitions [1] and [2].
Lemma 1. For every DFA \( \mathcal{A} \), the DFA \( \mathbb{H}(\mathcal{A}) = \langle R, \Theta \rangle \) is such that each letter in \( \Theta \) acts on \( R \) as an idempotent selfmap and has rank \( \frac{|R|}{2} \).

Now we establish the relation between reset thresholds of \( \mathcal{A} \) and \( \mathbb{H}(\mathcal{A}) \).

Theorem 2. The automaton \( \mathbb{H}(\mathcal{A}) \) is synchronizing if and only if so is \( \mathcal{A} \), and if this is the case, then \( \text{rt}(\mathbb{H}(\mathcal{A})) = 2 \text{rt}(\mathcal{A}) \).

Proof. First suppose that \( \mathcal{A} = \langle Q, \Sigma \rangle \) with \( Q = \{1, 2, \ldots, n\} \) and \( \Sigma = \{\sigma_1, \ldots, \sigma_k\} \) is a synchronizing automaton. Let \( w \in \Sigma^* \) be a reset word of \( \mathcal{A} \) of minimum length. Consider the morphism \( \chi: \Sigma^* \to \Theta^* \) defined by the rule \( \chi(\sigma_j) := ba_j \) for \( j = 1, \ldots, k \) and let \( W := \chi(w) \). Clearly, \( |W| = 2|w| \).

From (1) and (2) we readily conclude that
\[
i.\sigma_j = i.\chi(\sigma_j) = i'.\chi(\sigma_j)
\]
for all \( i = 1, \ldots, n \) and all \( j = 1, \ldots, k \). From (3), we get \( Q.w = R.W \), whence \( W \) is a reset word for the automaton \( \mathbb{H}(\mathcal{A}) \). We see that \( \mathbb{H}(\mathcal{A}) \) is synchronizing and
\[
\text{rt}(\mathbb{H}(\mathcal{A})) \leq 2 \text{rt}(\mathcal{A})
\]
since \( \text{rt}(\mathbb{H}(\mathcal{A})) \leq |W| = 2|w| \) and \( |w| = \text{rt } \mathcal{A} \) by the choice of the word \( w \).
Conversely, suppose that the automaton $\mathbb{H}(\mathcal{A})$ is synchronizing, and let $U \in \Theta^*$ be its reset word of minimum length so that $\text{rt}(\mathbb{H}(\mathcal{A})) = |U|$. Since the letter $b$ acts as an idempotent on $R$, the word $b^2$ does not occur in $U$ as a factor. (Otherwise we could reduce $b^2$ to $b$ and obtain a shorter reset word.) Further, since by (1) the image of each of the letters $a_1, \ldots, a_k$ is contained in $Q$ and the letters $a_1, \ldots, a_k$ act as the identity map on $Q$, we conclude that $r.a_s a_t = r.a_s$ for all $r \in R$ and all $s, t \in \{1, \ldots, k\}$. Therefore, none of the words $a_s a_t$ can occur in $U$ as a factor. Thus, in $U$, every occurrence of the letter $b$, except its possible occurrence as the last letter of $U$, is followed by an occurrence of one of the letters $a_1, \ldots, a_k$, and for each $j = 1, \ldots, k$, every occurrence of the letter $a_j$, except its possible occurrence as the last letter of $U$, is followed by an occurrence of the letter $b$. In other terms, the occurrences of $b$ in the word $U$ alternate with the occurrences of $a_1, \ldots, a_k$.

Suppose that $b$ is the last letter of $U$. Then $U = U'b$ for some $U' \in \Theta^*$ and $U'$ does not reset $\mathbb{H}(\mathcal{A})$ and $|U'| < |U|$ and $U$ was chosen to be a reset word of minimum length. Therefore, $|R.U'| \geq 2$. The last letter of $U'$ is one of the letters $a_1, \ldots, a_k$, whence $R.U' \subseteq Q$. However, since the letter $b$ bijectively maps $Q$ onto $Q'$, it cannot merge the states in any subset of $Q$. Hence, $|R.U| = |R.U'b| = |R.U'| \geq 2$, and this contradicts our choice of the word $U$. Thus, $U$ cannot end with $b$.

Now we aim to show that $U$ starts with $b$. Again, we argue by contradiction. Suppose that $U = a_j U''$ for some $j \in \{1, \ldots, k\}$ and some $U'' \in \Theta^*$. Observe that $U''$ starts with $b$, does not end with $b$, and the occurrences of $b$ in $U''$ alternate with the occurrences of $a_1, \ldots, a_k$. Thus, $U''$ can be decomposed as a product of factors of the form $ba_j$, $j = 1, \ldots, k$, that is, $U''$ belongs to the image of the morphism $\chi : \Sigma^* \rightarrow \Theta^*$ introduced in the first paragraph of the proof. Consider the word $v \in \Sigma^*$ defined by $v := \chi^{-1}(U'')$. It is clear that $|v| = \frac{|U''|}{2}$, and from (3), it readily follows that $Q.v = Q.U''$. By (1), the letter $a_j$ fixes all states in $Q$, whence $Q.U'' = Q.a_j U'' = Q.U \subseteq R.U$. Since $U$ is a reset word for $\mathbb{H}(\mathcal{A})$, the set $R.U$ is a singleton and so is the set $Q.v = Q.U''$. We conclude that $v$ is a reset word for $\mathcal{A}$, whence $\text{rt}(\mathcal{A}) \leq |v|$. Using the inequality (3), we arrive at the conclusion that

$$\text{rt}(\mathbb{H}(\mathcal{A})) \leq 2 \text{rt}(\mathcal{A}) \leq 2|v| = |U''| < |U|,$$

which contradicts the choice of the word $U$.

Summing up the facts established so far, the word $U$ starts with the letter $b$, ends with one of the letters $a_1, \ldots, a_k$, and the occurrences of $b$ in $U$ alternate with the occurrences of $a_1, \ldots, a_k$. This ensures that $U$ is a product of factors of the form $ba_j$, $j = 1, \ldots, k$, and we can apply to $U$ the inverse of
the morphism $\chi$ as we did in the preceding paragraph for the word $U''$. Let $u := \chi^{-1}(U)$. Then $|u| = \frac{|U|}{2}$, and from (3) we conclude that $Q.u = R.U$. Hence, $u$ is a reset word for $\mathcal{A}$, and $\text{rt}(\mathcal{A}) \leq |u| = \frac{|U|}{2} = \frac{\text{rt}(H(\mathcal{A}))}{2}$. Combining this inequality with (4), we obtain the equality $\text{rt}(H(\mathcal{A})) = 2 \text{rt}(\mathcal{A})$. 

By Lemma[1] and Theorem[2], the transformation $\mathcal{A} \mapsto H(\mathcal{A})$ applied to the automata in Černý’s series produces a series of automata that witnesses the following fact:

**Corollary 3.** For each even $n$, there exists a 3-letter proper synchronizing $n$-automaton with reset threshold $\frac{n^2}{2} - 2n + 2$ whose letters are idempotents of rank $\frac{n}{2}$.

### 3 The binary case

A DFA with one input letter is called *unary*. It is known and easy to verify that for every unary synchronizing automaton, its reset threshold is strictly less than the number of states. Therefore the application of the transformation $\mathcal{A} \mapsto H(\mathcal{A})$ to a unary synchronizing automaton cannot produce a binary synchronizing automaton with idempotent input letters and reset threshold greater than or equal to the number of states. This does not exclude the possibility that such binary synchronizing automata can be constructed in some other way. In this section, we address this issue. We prove that the reset threshold of every synchronizing $n$-automaton with two idempotent input letters does not exceed $n - 1$ and this bound is tight. The upper bound $n - 1$ has been established for several types of synchronizing $n$-automata, see, e.g., [19], but it appears that automata with two idempotent input letters do not fall into any previously analyzed class.

The following arguments involve a few standard concepts of automata theory which we recall here for the sake of completeness.

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be a DFA. If $S \subseteq Q$ is such that $\delta(s, a) \in S$ for all $s \in S$ and $a \in \Sigma$, one can consider the DFA $\mathcal{S} := (S, \Sigma, \tau)$, where the function $\tau$ is defined as the restriction of $\delta$ to the set $S \times \Sigma$, that is, $\tau(s, a) := \delta(s, a)$ for all $s \in S$ and $a \in \Sigma$. Any such DFA is said to be a subautomaton of $\mathcal{A}$. Clearly, if $\mathcal{A}$ is synchronizing, then so is each of its subautomata.

An equivalence $\pi$ on the state set $Q$ of $\mathcal{A}$ is called a congruence if $(p, q) \in \pi$ implies $(\delta(p, a), \delta(q, a)) \in \pi$ for all $p, q \in Q$ and all $a \in \Sigma$. If $\pi$ is a congruence and $q$ is a state, $[q]_\pi$ stands for the $\pi$-class containing $q$. The
quotient $\mathcal{A}/\pi$ is the DFA $\langle Q/\pi, \Sigma, \delta_\pi \rangle$, where $Q/\pi := \{[q]_\pi \mid q \in Q\}$ and the function $\delta_\pi$ is defined by the rule $\delta_\pi([q]_\pi, a) := [\delta(q, a)]_\pi$. Again, if $\mathcal{A}$ is synchronizing, then so is each of its quotients.

A state $s$ of a DFA is called a sink if $s.a = s$ for each input letter $a$. Clearly, if a synchronizing automaton has a sink, then the sink must be unique, and every reset word must bring the automaton to the sink.

Let $\mathcal{A} = \langle Q, \Sigma \rangle$ be a DFA and $p, q \in Q$. We say that $q$ is reachable from $p$ if there is a word $w \in \Sigma^*$ such that $p.w = q$. A DFA is called strongly connected if every state in it is reachable from every other state. A strongly connected automaton with more than one state cannot have any sink.

It is actually a part of synchronizing automata folklore that estimating reset threshold for a “sufficiently robust” class of synchronizing automata reduces to two special cases: the case of automata with a unique sink and the case of strongly connected automata. Here it is convenient to employ this folklore fact in the following form, which is a special case of [25, Proposition 2.1].

**Lemma 4.** Let $\mathcal{C}$ be any class of automata closed under taking subautomata and quotients, and let $\mathcal{C}_n$ stand for the class of all $n$-automata in $\mathcal{C}$. If each synchronizing automaton in $\mathcal{C}_n$ which either is strongly connected or possesses a unique sink has a reset word of length $n - 1$, then the same holds for all synchronizing automata in $\mathcal{C}_n$.

**Proposition 5.** The reset threshold of every synchronizing $n$-automaton with two idempotent input letters does not exceed $n - 1$.

**Proof.** The class of automata with two idempotent input letters is closed under taking subautomata and quotients. Hence Lemma 4 applies, and it suffices to verify the bound of Proposition 5 for synchronizing automata in this class which either are strongly connected or have a unique sink.

First, consider the strongly connected case. In this case we will prove a much stronger fact: the reset threshold of any strongly connected synchronizing $n$-automaton $\mathcal{A} = \langle Q, \Sigma \rangle$ with $|\Sigma| = 2$ and the letters in $\Sigma$ being idempotents does not exceed $\max\{1, n - 1\}$. We may suppose that $n > 1$; otherwise, there is nothing to prove. Take a state $q_0 \in Q$. There must be a letter $a \in \Sigma$ such that $p_0 := q_0.a \neq q_0$ as $q_0$ would be a sink otherwise. Since $\mathcal{A}$ is strongly connected, there exists a word $w \in \Sigma^*$ such that $p_0.w = q_0$. Let $w$ be chosen to be a word of minimum length with the latter property. Since $p_0.a = q_0.a^2 = q_0, a = p_0$, we see that $w \neq aw'$ because otherwise the suffix $w'$ would be a shorter word with the property $p_0.w' = q_0$. Thus, denoting by $b$ the other letter in $\Sigma$, we conclude that $w$ starts with $b$ and,
moreover, the occurrences of $b$ and $a$ alternate in $w$. The last letter of $w$ cannot be $a$ because $q_0$ would be a fixed point of $a$ otherwise, and this is not the case. Summarizing, we see that $aw$ starts with $a$, ends with $b$, and $a$ and $b$ alternate in $aw$, that is, $aw = (ab)^k$ for some $k \geq 1$.

If $k > 1$, let $q_i := q_0(ab)^i$ and $p_i := q_0(ab)^i a$ for $i = 1, \ldots, k - 1$. Then for each $i = 0, \ldots, k - 1$, we have

$$q_i.a = p_i, \quad q_i.b = q_i, \quad p_i.a = p_i, \quad p_i.b = q_{i+1}(\text{mod } k).$$

From this, it is easy to deduce that $q_0.v \neq q_1.v$ for each word $v \in \Sigma^*$. Indeed, write $v$ as a product of powers of $a$ and $b$ and let $\ell$ be the number of ‘switches’ from a power of $a$ to a power of $b$; that is, $\ell$ is the number of occurrences of the word $ab$ as a factor in $v$. Then a direct calculation based on (5) gives the following values of $q_0.v$ and $q_1.v$: if $v$ ends with $a$, then $q_0.v = q_{\ell(\text{mod } k)}$ and $q_1.v = q_{\ell+1(\text{mod } k)}$; if $v$ ends with $b$, then $q_0.v = q_{\ell(\text{mod } k)}$ and $q_1.v = q_{\ell+1(\text{mod } k)}$. Hence, the automaton $\mathcal{F}$ is not synchronizing, a contradiction.

Thus, $k = 1$, and we have $q_0.a = p_0$, $q_0.b = q_0$, $p_0.a = p_0$, $p_0.b = q_0$, that is, the set $\{q_0, p_0\}$ is closed under the action of letters in $\Sigma$. Since the automaton $\mathcal{F}$ is strongly connected, this is only possible provided that $n = 2$ and $Q = \{q_0, p_0\}$. The automaton $\mathcal{F}$ is then nothing but the classical flip-flop, see Fig. 3. Of course, every word of length 1 is a reset word for the flip-flop.

![Figure 3: Flip-flop](image)

Now consider the case of automata with a unique sink. Let $\mathcal{F} = \langle Q, \Sigma \rangle$ be a synchronizing $n$-automaton with two idempotent input letters and a unique sink, which we denote $z$. If $p$ and $q$ are two different states in $Q$, we say that $p$ is a predecessor of $q$ in $\mathcal{F}$ if $q$ is the image of $p$ under the action of a letter in $\Sigma$. First, we show that there exists a state without predecessors in $\mathcal{F}$. Arguing by a contradiction, assume that every state has a predecessor in $\mathcal{F}$. Since the number of states is finite, this assumption implies that for some $k > 1$, there is a sequence $S$ of states $q_0, q_1, \ldots, q_{k-1}$ such that $q_i$ is a predecessor of $q_{i+1(\text{mod } k)}$ for each $i = 0, 1, \ldots, k - 1$. Clearly, $z \notin S$, as the sink is not a predecessor of any state. Now take any $q_i \in S$ and let
Let $a \in \Sigma$ be such that $q_i - 1 \equiv \mod{k}$. Then $q_i \cdot a = q_i$ since the letter $a$ is idempotent, and if $b$ is the other letter in $\Sigma$, then $q_i \cdot b = q_{i+1} \equiv \mod{k}$. We see that both $q_i \cdot a$ and $q_i \cdot b$ belong to $S$, that is, the set $S$ is closed under the action of letters in $\Sigma$. Hence, $z$ is not reachable from any state in $S$, while any reset word must bring all states in $Q$, including those in $S$, to $z$. This is a contradiction.

We show that $\mathcal{S}$ has a reset word of length $n - 1$ by induction on $n$. The induction basis $n = 1$ is obvious; now assume that $n > 1$. As shown in the preceding paragraph, there is a state $q \in Q$ that has no predecessors in $\mathcal{S}$. Clearly, $q \neq z$ because $\mathcal{S}$ is synchronizing and the sink $z$ must be reachable from every state in $Q$, whence $z$ must have a predecessor. Then the subautomaton $\mathcal{S}'$ obtained by the restriction of the transition function of $\mathcal{S}$ to the set $Q' := Q \setminus \{q\}$ is synchronizing and has $n - 1$ states, two idempotent input letters, and a unique sink. The induction assumption applies to $\mathcal{S}'$, whence there exists a word $w$ of length $n - 2$ such that $Q' \cdot w = \{z\}$. Since $q \neq z$, there exists a letter $a \in \Sigma$ such that $q \cdot a \neq q$, that is, $q \cdot a \in Q'$. Therefore, $Q \cdot a \cdot w = \{z\}$, whence $aw$ is a reset word of length $n - 1$ for $\mathcal{S}$. \hfill $\Box$

Finally, we exhibit a series of $n$-automata $\mathcal{S}_n$, $n = 3, 4, \ldots$, showing that the bound of Proposition 5 is tight. The state set of $\mathcal{S}_n$ is $\{1, 2, \ldots, n\}$; the action of the input letters $a$ and $b$ is defined as follows:

$$
i \cdot a = \begin{cases} i & \text{if } i \text{ is odd or } i = n, \\ i + 1 & \text{if } i \text{ is even and } i < n\end{cases}, \quad i \cdot b = \begin{cases} i & \text{if } i \text{ is even or } i = n, \\ i + 1 & \text{if } i \text{ is odd and } i < n.\end{cases}$$

See Fig. 4 for an illustration. By the construction, the letters $a$ and $b$ act as idempotent selfmaps. Clearly, the state $n$ is a unique sink of $\mathcal{S}_n$. It is easy to see that $\mathcal{S}_n$ is synchronizing, and the reset threshold of $\mathcal{S}_n$ is at least $n - 1$ since $n - 1$ is the length of the shortest word that brings the state $1$ to the sink.

![Figure 4: The automaton $\mathcal{S}_n$](image)

Quite interestingly, Gusev [10] has constructed for each odd $n$, a binary synchronizing $n$-automaton with reset threshold $\frac{n^2 - 3n + 4}{2}$, in which one letter
Figure 5: Gusev’s automaton for $n = 7$

is idempotent and the other acts almost as an idempotent in the sense that it does not fix exactly one state in its image. Fig. 5 shows the 7-state automaton from Gusev’s series; observe that it differs from $I_7$ by a single transition! Thus, even a minimum possible step out of idempotency causes a drastic leap in terms of the value of reset threshold.

4 Conclusion

We have described a transformation $A \mapsto H(A)$ that converts an arbitrary DFA with $n$ states and $k$ input letters into a DFA with $2n$ states and $k + 1$ idempotent input letters of rank $n$. The transformation preserves the property of being synchronizing and doubles the reset threshold. Observe in passing that the transformation preserves several other properties relevant in the context of synchronization: say, $A$ is strongly connected or proper if and only if so is $H(A)$.

Informally, our main result (Theorem 2) shows that synchronization problems for the class of DFAs with idempotent input letters are as difficult as in the general case. This informal statement can be precisely expressed in the language of computational complexity—one can use Theorem 2 to transfer numerous hardness results collected in the theory of synchronizing automata (see [13, Section 2] for an overview) to automata whose letters
act as idempotent selfmaps. Theorem 2 also implies that a quadratic in \( n \) upper bound on the reset threshold of synchronizing \( n \)-automata with idempotent input letters would lead to a quadratic upper bound in the general case—here we recall that the best upper bound known so far is cubic. In this connection, we mention a result by Rystsov [20] who established the upper bound \( 2(n - 1)^2 \) on the reset threshold of synchronizing \( n \)-automata whose input letters are either permutations or idempotents of rank \( n - 1 \).

Acknowledgements. The author is very much indebted to the anonymous referees of the previous version for their valuable remarks (in particular, for spotting inaccuracies in the original formulations of Lemma 1 and Corollary 3) and suggesting an improvement in Theorem 2 and for providing several relevant references.

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