Abstract. Calculations of atmospheric refraction are generally based on a simplified model of atmospheric density in the troposphere which assumes that the temperature decreases at a constant lapse rate \( L \) from sea level up to a height \( h_t \approx 11 \text{ km} \), and that afterwards it remains constant. In this model, the ratio \( T_o/L \), where \( T_o \) is the temperature at the observer’s location, determines the length scale in the calculations for altitudes \( h \leq h_t \). But daily balloon measurements across the U.S.A. reveal that in some cases the air temperature actually increases from sea level up to a height \( h_p \) of about one km, and only after reaching a plateau with temperature \( T'_o \) at this height, it decreases at an \textit{approximately} constant lapse rate. Hence, in such cases, the relevant length scale for atmospheric refraction calculations in the altitude range \( h_p \leq h < h_t \) is \( T'_o/L \), and the contribution for \( h \leq h_p \) has to be calculated from actual measurements of air density in this range. Moreover, in three examples considered here, the temperature does not remain constant for \( h_t \leq h \), but continues to decreases to a minimum at \( h_m \approx 16 \text{ km} \), and then increases at higher altitudes at a lower rate. Calculations of atmospheric refraction based on this atmospheric data is compared with the results of simplified models.

Introduction

In current models to calculate atmospheric refraction, it is assumed that in the troposphere the temperature decreases from sea level at a constant rate known as the lapse rate up to a height \( h_t \), and that above this height the temperature remains essentially constant until reaching the stratosphere at \( h > 20 \text{ km} \), where it increases at a slow rate\(^1\). But measurements by balloons released daily into the atmosphere across the U.S.A.\(^2\) reveal that this model is only approximately correct. In particular, in some cases at an altitude below about 1 km, the temperature \textit{increases} instead of decreasing with altitude or decreases at a rate different from the asymptotic lapse rate. Moreover, instead of a tropopause at \( h_t \approx 11 \text{ km} \), the temperature decreases to a minimum at a higher altitude, \( h_t \approx 16 \text{ km} \), and then increases at a slower rate\(^2\). 

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\(^1\) For example, in an influential paper, C.Y. Hohenkerk and A. T Sinclair state: “The temperature decreases at a constant rate within the troposphere, up to the tropopause at about 11 km height. Above the tropopause the temperature remains constant”. They quote a lapse rate \( L \approx 6.5^0 \text{ Celsius/km} \), but their algorithm in current use allows for different rates \(^1\)(see routine from SLALIB at \text{http://star-www.rl.ac.uk/docs/sun67.htx/sun67ss157.html}).

\(^2\) This effect is attributed to the occurrence of an ozone layer.
To illustrate this behavior, three examples of the air temperature dependence on altitude obtained by recent balloon measurements are shown in Figs.1-4. Moreover, in these cases the minimum temperature occurs at about 10°C below the temperature in the conventional model (−57°C), and the estimated lapse rates are different, ranging from $L = 60$ to $80$ degrees Celsius/km.

For sea level observations along the horizon, the air density up to an altitude of 1 km contributes about 1/4 of the total atmospheric refraction (an example is given in Table I), but the effect of a possible temperature increase up to this altitude has been ignored previously\textsuperscript{4}. Instead, in calculations of atmospheric refraction it is assumed that the temperature decreases uniformly at a constant lapse rate $L$ starting at the location of the observations, and consequently, that the temperature $T_o$ at this location determines the relevant length scale $T_o/L$ in the calculation of atmospheric refraction. For these observations, this assumption leads to significant errors in these calculations, but of course it does not affect calculations made for observations at altitudes above about 1 km. But the occurrence of a minimum temperature instead of the assumed constant value in the troposphere, leads to differences in fractions of arc seconds. Our main conclusion is that for an accurate calculation of atmospheric refraction at a given location and time, the parameters of the analytic expression for the atmospheric density should be based on a fit to the atmospheric data obtained at the nearest station.

In the next section the theory of atmospheric refraction for a spherically symmetric atmosphere is extended to the case that the linear decrease in temperature in the troposphere is valid only above a finite height $h_p$. An analytic approximation is derived for the contribution of atmospheric density below this altitude. In section II, we present the parameters of the analytic atmospheric model discussed in section I, obtained by a least square fit to the air pressure and temperature data at three separate stations\textsuperscript{5}(see Table I). For a case when the temperature increases with altitude at low altitudes, the results from a fit to the data at the Oakland station for $h \leq 0.6$ k are compared with the standard calculation for $80^o \leq \phi \leq 90^o$, where $\phi$ is the angle of observation relative to the zenith of the observer (see Table II). For $4 \leq h \leq 16$, a fit to the data for the ratio pressure/temperature proportional to density, is shown in Fig.6. For an illustration of observations above $h \approx 1$ km, the refraction at the Keck observatory\textsuperscript{6} is calculated based on the atmospheric data obtained by the station at Lihue, Kauai, and compared with the refraction table for this observatory (see Table III).

\textsuperscript{3}In a spherically symmetric atmosphere, one would expect that it is not the lapse rate, but the minimum atmospheric temperature, and the altitude where it occurs in the troposphere, that is independent of the observer’s location. This is found to be the case in the three examples discussed here, two of which are separated by a distance equal to 1/5 the Earth’s circumference.

\textsuperscript{4}A detailed discussion of low-altitude refraction has been given by Andrew T. Young\textsuperscript{3}.

\textsuperscript{5}Lihue, Kauai; Oakland, C.A; Buffalo, N.Y.

\textsuperscript{6}This observatory is located on Mauna Kea at 4.207 km above sea level.
Figure 1. Atmospheric temperature, Oakland, California, July 29, 2016.

Figure 2. Atmospheric temperature, Buffalo, N.Y., August 4, 2016.
Figure 3. Atmospheric temperature, Lihue, Kauai, August 20, 2016.

Figure 4. Low altitude atmospheric temperature, Oakland, July 29, 2016.
Theory

The refraction of light traversing a spherically symmetric atmosphere is calculated by the well known integral

\[ R(\phi) = - \int_{r_o}^{\infty} dr \frac{dn}{n dr} \frac{\sin \phi}{\sqrt{(nr/n_o r_o)^2 - \sin^2 \phi}}, \]

where \( n(r) \) is the index of refraction of atmospheric air at a distance \( r \) from the center of the Earth, \( \phi \) is the angle relative to the zenith for observations at an altitude \( h_o \) above sea level, and \( r_o = r_E + h_o \), where \( r_E \) is the mean radius of the Earth [4]. For \( r \geq r_o \), \( n(r) \) is assumed to be determined by the atmospheric air density \( \rho(r) \) according to the relation

\[ n(r) = 1 + c(\lambda) \frac{\rho(r)}{\rho(0)}, \]

where \( c(\lambda) \) is a constant that depends on the wavelength \( \lambda \) of the light beam [5],[6].

Since the density of atmospheric air is very low and decreases with altitude, it is reasonable to assume that it satisfies the ideal gas relation,

\[ \rho = \frac{m}{k} \frac{p}{T}, \]

where \( k \) is Boltzmann’s constant, \( m \) is the mean particle mass of atmospheric air, \( p \) is the pressure, and \( T \) is the temperature. Assuming that parcels of air at \( r \) are in static equilibrium,

\[ \frac{dp}{dr} = -g \rho, \]

where \( g \) is the gravitational acceleration at \( r_o \)[7]. Applying the ideal gas law, Eq.3, we have

\[ p(h) = p(h_o) \exp\left(-\frac{mg}{k} \int_{h_o}^{h} \frac{dh'}{T(h')}\right), \]

where \( h = r - r_E \). Introducing the conventional assumption that for \( h \geq h_o \) the temperature decreases linearly with altitude [7],[8]

\[ T(h) = T_o - L(h - h_o), \]

where \( L = -dT/dh \) is a constant known as the lapse rate, and \( T_o = T(h_o) \) is the temperature at the initial altitude \( h_o \). Substituting this relation in Eq.5 for the pressure, one obtains

\[ p(h) = p(h_o)(1 - \frac{(h - h_o)}{H_o}), \]

where the exponent \( q \), given by

\[ q = \frac{mg}{kL}, \]

\[ \text{The dependence of } g \text{ on altitude can be neglected for the altitudes considered here.} \]
depends only on the lapse rate $L$, while the length parameter

\[ H_o = \frac{T_o}{L}, \]

depends on both the lapse rate $L$, and on the temperature $T_o$ at the onset of the assumed linear temperature decrease with altitude.\(^8\) Hence, according to the ideal gas law, Eq.\(^3\) for $h \geq h_o$ the air density as a function of altitude is given by \(^7,\)\(^8\)

\[ \rho(h) = \frac{p(h) T_o}{T(h)} = \rho(h_o)(1 - \frac{(h - h_o)}{H_o})^{q-1}, \]

and the dependence of the index of refraction on the altitude $h$, Eq.\(^2\) is

\[ n(h) = 1 + c'(\lambda)(1 - \frac{(h - h_o)}{H_o})^{q-1}, \]

where

\[ c'(\lambda) = c(\lambda) \frac{\rho(h_o)}{\rho(0)}. \]

For $h < h_p$, where $h_p$ is approximately one kilometer, the assumption that the temperature decreases with a lapse rate $L$ is not generally valid, and in some cases balloon measurements\(^9\) indicate that the temperature actually increases with altitude from sea level (see Figs 1-4). In this case, the air density can be approximated by a power series in $h$. To second order,

\[ \rho(h) \approx \rho(0) + ah + bh^2, \]

and the integration for the contribution $\delta R(h, \phi)$ to the refraction integral, Eq.\(^1\) in the interval $0 \leq h' \leq h$, can be carried out analytically,

\[ \delta R(h, \phi) = c(\lambda) \sin \phi (A(a,b,h,\phi) + B(b,h,\phi)), \]

where

\[ A(a,b,h,\phi) = (ar_o - br_o^2 \cos^2 \phi)(\sqrt{2(\frac{h}{r_o}) + \cos \phi^2 - \cos \phi}), \]

and

\[ B(b,h,\phi) = \frac{br_o^2}{3} ((\frac{2h}{r_o} + \cos^2 \phi)^{3/2} - \cos^3 \phi)). \]

For illustration, in the case that $\rho(h)$ is given by the the canonical expression, Eq.\(^10\) the coefficients are $a = (q-1)/H_o$, and $b = (q-1)(q-2)/2H_o^2$. For $h << H_o$ this approximation for $\delta R(h, \phi)$ is in good agreement with the exact contribution from the integral, Eq.\(^1\).

\(^8\) Since $T_o = T(0) - h_o L$, Eq.\(^6\) $H_o = T(0)/L - h_o$

\(^9\) These balloons are configured to measure the altitude dependence of air pressure and temperature, while the air density is obtained from the ratio of these quantities, in accordance with the ideal gas law, Eq.\(^4\)
Actually, even at altitudes higher than $h_t$, the temperature decreases linearly only approximately, and therefore Eqs. 6 and 7 for the temperature and pressure are also satisfied only approximately, as will be shown in the next section. In this case, the exponent $q$ and the parameter $H_o$ should be obtained by a least square fit of the data for the ratio pressure/temperature to the theoretical relation, Eq. 10 for the density.

**Numerical calculations**

The parameters for the atmospheric model discussed in the previous section were obtained by a least square fit to the atmospheric data in the range 1 – 16 km at three separate stations on a specific day [10] and the results are shown in Table I. For each column associated with one of these stations, the lapse rate $L$ and the temperature $T_o$ in the first and second column were obtained from a fit to the temperature data, while the exponent $q$, and the scale height $H_o$ in the third and fourth row, associated with the pressure dependence on altitude, Eq. 7 were obtained from a fit to the pressure/temperature ratio vs altitude. The values of $T_o$ and $H_o$ correspond to values at sea level, $h = 0$ km, extrapolated from the fit to the data for $h \geq 1$ km. The fifth row is the minimum temperature $T_m$ of the troposphere at the respective locations. For comparison, the last column shows the conventional values of these parameters.

If the temperature decrease with altitude where truly linear, the theoretical relations for the lapse rate, $L = T_o/H_o$, Eq. 9 and for the exponent in the relation for the pressure, $q = mg/kL$, Eq. 8 should be satisfied exactly, but this is not actually the case. In practice, these two relations are satisfied only approximately, reflecting the fact that the decrease in temperature is only approximately linear. Since the variable that is relevant for the calculation of atmospheric refraction, Eq. 6 is the atmospheric density $\rho$, which according to the ideal gas law, is proportional to the ratio pressure/temperature, Eq. 3, the most appropriate parameters for this calculation should be the values for $q$ and $H_o$ obtained by a least square fit of the theoretical density dependence on altitude, Eq. 10 to the observed altitude dependence of this ratio.

Above the height $h \approx 16$ km, where the minimum temperature $T_m$ occurs, the atmospheric temperature increases with altitude at a slow rate, but the balloon data is not always available at these higher altitudes. In this case the best approximation is to assume that the temperature remains constant, which implies that the density decreases exponentially with a length scale $H_m = kT_m/mg$.

To illustrate the contribution to atmospheric refraction of air density at very low altitudes (less than about one kilometer), in a case that the temperature increased with altitude, consider the data obtained by balloon measurements at Oakland, CA., on July 29, 2016 [2]. Below an altitude of about .6 km, the temperature increased from about 13° to 28° Celsius, where it reached a plateau (see Fig.4) before decreasing approximately linearly to a minimum at about 16 km (see Fig.1 ). The dependence of the ratio pressure/temperature on altitude, is shown in Fig.5. A least square fit of this ratio to a power series to second order in $h$, Eq. 13 gives $a = -.034$ and $b= -.309$, is also shown in this figure. The contribution

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10 Liheu, Kauai on August 20, Oakland, C.A on July 29, and Buffalo, N.Y on August 4.
δR(h, φ) to the refraction, Eq. 14, in the range 80° ≤ φ ≤ 90 is given in Table I, and compared to the contribution calculated with the conventional model. For higher altitudes the pressure/density data is shown in Fig. 6, with a fit based on Eq. 10 with the parameters describe above.

In Table II, a calculation of the atmospheric refraction at the Mauna Kea observatory, for λ = .633 microns, based on a least square fit to the atmospheric pressure and temperature measurements at Lihue, Kauai, is compared with a current refraction table at this observatory [11].

![Figure 5](image.png)

**Figure 5.** Low altitude pressure/temperature data, Oakland, CA. July 29, 2016.

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[11] I would like to thank Drew Phillips for a copy of the refraction table at the Mauna Kea telescope.
Refraction parameters obtained from a least square fit to balloon measurements of atmospheric pressure and temperature at the locations shown. The last column are the parameters in conventional calculations currently in use.

| Refraction Parameters from fit to Troposphere data | Lihue | Oakland | Buffalo | Conventional |
|---------------------------------------------------|-------|---------|---------|---------------|
| $T_o$ (Kelvin)                                    | 319.3 | 310.9   | 307     | 288.15        |
| L(Celsius/km)                                     | 7.91  | 7.19    | 6.95    | 6.5           |
| q(exponent)                                       | 4.31  | 4.59    | 5.16    | 5.25          |
| $H_o$ (km)                                        | 40.35 | 41.71   | 46.3    | 44.33         |
| $T_m$ (Kelvin)                                    | 197   | 203     | 208     | 216           |

The second column is based on the conventional assumption that the lapse rate is $L = 6.5^0$ degrees Celsius/km. The third column is based on a fit to actual atmospheric data obtained at the Oakland station on July 29, 2016.

| Contribution to Atmospheric Refraction (1.6 km altitude), Oakland, CA. July 29.2016 |
|---------------------------------------------------|---|---|---|
| Angle from Zenith in degrees | Conventional calculation. arc minutes | Least square fit to pressure/temperature data. arc minutes |
| 80° | .31 | .71 |
| 81° | .34 | .79 |
| 82° | .38 | .89 |
| 83° | .44 | 1.02 |
| 84° | .51 | 1.18 |
| 85° | .62 | 1.42 |
| 86° | .77 | 1.76 |
| 87° | 1.02 | 2.33 |
| 88° | 1.50 | 3.38 |
| 89° | 2.77 | 5.88 |
| 90° | 8.52 | 11.82 |
Figure 6. Pressure/ Temperature vs. altitude data, Oakland, July 29, 2016. The curve is a fit based on Eq.[10] with the exponent parameter $q = 4.59$, corresponding to a lapse rate $L = 7.19^\circ$ degrees Celsius/km.

Table III

Table of atmospheric refraction at the Mauna Kea observatory, for $\lambda = .633$ microns. The first column are observations angles relative to the observer’s zenith. The second column is a refraction table at the Mauna Kea observatory. The third column was calculated from the conventional parameters. The fourth column was calculated from a least squares fit to the Lihue data for the ratio pressure/temperature dependence on altitude. in Kauai, on August 20, 2016.

| Zenith angle, degrees | Mauna Kea Table, arc seconds | Conventional theory, arc seconds | Lihue data, arc seconds |
|----------------------|-------------------------------|----------------------------------|------------------------|
| 45°                  | 36.70                         | 36.80                            | 36.81                  |
| 50°                  | 43.72                         | 43.83                            | 43.84                  |
| 55°                  | 52.35                         | 52.47                            | 52.50                  |
| 60°                  | 63.41                         | 63.54                            | 63.59                  |
| 65°                  | 78.35                         | 78.48                            | 78.56                  |
| 70°                  | 100.0                         | 100.1                            | 100.2                  |
| 75°                  | 134.7                         | 134.7                            | 135.0                  |
| 80°                  | 200.0                         | 199.6                            | 200.6                  |
| 85°                  | 361.8                         | 361.6                            | 366.5                  |
Summary

Daily balloon measurement of atmospheric air pressure and temperature show that the conventional assumption that the temperature in the troposphere decreases linearly with temperature up to an altitude $h = 11$ km, followed by a constant temperature layer up to $h = 20$ km, the so-called tropopause, is not generally valid. For illustration, plots of the altitude dependence of the temperature data from three separated atmospheric stations, taken at different dates, are shown in Figs.1-4. This data shows that in some cases at low altitudes, the temperature actually increases with altitude up to a height $h \approx 1$ km, and that instead of a tropopause, the temperature continues to decrease to a minimum value at $h \approx 16$ km, and afterwards it increases slowly at higher altitudes. To calculate the contribution of low altitude air density to the refraction integral, the atmospheric data in the range $h$ from 0 to approximately 1km can be fitted by a an expansion in powers of $h$. In the range $1 \leq h \leq 16$, a least square fit to the atmospheric temperature data assuming a linear decrease in temperature with height, Eq.\[8\] determines the lapse rate $L$, while a corresponding fit to the data on the ratio of pressure to temperature by the theoretical relation for the air density $\rho$, Eq.\[10\] determines the exponent $q$. According to the theory, the mean molecular mass of the atmospheric air is $m = qLk/g$, and substituting the values obtained from the Liheu data, Table I, one obtains $m = 28.96$ gram/mole, in remarkably good agreement with the value from measurements of air density at sea level. But for the fit to the data from the Oakland and Buffalo stations (see Table i) this agreement is only approximate, reflecting the fact that the temperature decrease on these locations was only approximately linear.

Finally, it should be pointed out that although calculations of atmospheric refraction based on simplified models are usually given to seven or more significant figures \[9\], this precision is meaningless. Accuracy in calculations of atmospheric refraction to more than seconds of arc is not attainable without fitting the parameters of the analytic model for the dependence of the atmospheric density with height to actual data.

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