Factorizations and Physical Representations

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Abstract

A Hilbert space in M dimensions is shown explicitly to accommodate representations that reflect the prime numbers decomposition of M. Representations that exhibit the factorization of M into two relatively prime numbers: the kq representation (J. Zak, Phys. Today, 23 (2), 51 (1970)), and related representations termed $q_1q_2$ representations (together with their conjugates) are analysed, as well as a representation that exhibits the complete factorization of M. In this latter representation each quantum number varies in a subspace that is associated with one of the prime numbers that make up M.

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I Introduction

Information and computation may be understood in terms of classical physics [1]. However, the extension of these ideas to the quantum domain [2] enriches our understanding of both information theory and quantum mechanics. Thus quantum computers, where entanglement and superposition of states are essential elements, allow computations believed to be intractable on any classical computer. A most often quoted example is Shor’s [3] quantum algorithm for factorizing numbers, while there is no known efficient classical algorithm for factoring. In this paper we study the relation of factorizability to quantum physics. Thus we wish to find and characterize physical representations which reflect the prime factorization of M, the dimensionality of the space of the problem. Our study is based on Schwinger’s [4] general theory of quantum mechanics in finite dimensional space in terms of unitary operators.

Schwinger [4] showed that M-dimensional vector spaces allow the construction of two unitary operators, U and V (in his notation), that form a complete operator basis, i.e. they suffice to
construct all possible operators of the physical system. This means that if an operator commutes with both \( U \) and \( V \) it is, necessarily, a multiple of the unit operator. These operators have a period \( M \), i.e.,

\[
U^M = V^M = 1,
\]

where \( M \) is the smallest integer for which this equality holds. The eigenvalues of both \( U \) and \( V \) are distinct: they are the \( M \) roots of unity, i.e. with \( |x\rangle \) the eigenfunctions of \( U \),

\[
U|x\rangle = e^{i(2\pi)n/M}|x\rangle, \quad |x + M\rangle = |x\rangle, \quad x = 1, \ldots, M.
\]

The operator \( V \) is defined over these eigenvectors as

\[
V|x\rangle = |x - 1\rangle.
\]

Schwinger then showed that the absolute value of the overlap between \( \text{any} \) eigenfunction of \( U \), \( |x\rangle \) and any one of \( V \), \( |p\rangle \), is a constant:

\[
|\langle p|x\rangle| = \frac{1}{\sqrt{M}}.
\]

Vector bases with this attribute are referred [5, 6] to as conjugate vector bases. It was further noted by Schwinger [4] that alternative conjugate vector bases may be constructed. For example, we may let \( U \to U' = U^n \) for \( n < M \) such that it has no common factor with \( M \). \( U' \) has, clearly, the same period and eigenvalues as \( U \). The corresponding \( V' \) that satisfies the relevant equation, Eq. (2), was shown to be some power of \( V \).

Our aim in this paper is to expand Schwinger’s analysis and stress its relation to factorization of \( M \), the dimensionality of the space. We choose to consider a specific example of the \( M \)-dimensional space, namely \( M \) points on a line, i.e., we consider discretized and truncated spatial coordinate \( x \) and its conjugate momentum \( p \) as our \( M \)-dimensional space. This may be realized by imposing boundary conditions on the spatial coordinate, \( \psi(x) \), and on their Fourier transforms, \( F(p) \) (we take \( \hbar = 1 \) [7]):

\[
\psi(x + Mc) = \psi(x), \quad F(p + \frac{2\pi}{Mc}) = F(p).
\]

Here \( M \) is an integer - it is the dimensionality of the Hilbert space, and we term \( c \) the “quantization length”. As a consequence of the above boundary conditions we have that the value of the spatial coordinate, \( x \), and the value of the momentum, \( p \), are discrete and finite:

\[
x = sc, s = 1, \ldots, M; \quad p = \frac{2\pi}{Mc}t, t = 1, \ldots, M.
\]

In this case we may replace the operators \( x \) and \( p \) by the unitary operators

\[
\tau(M) = e^{i\frac{2\pi}{Mc}x}; \quad T(c) = e^{ipc}.
\]

These operators satisfy the basic commutator relation

\[
\tau(M)T(c) = T(c)\tau(M)e^{-i\frac{2\pi}{M}}.
\]
They exhibit the dimensionality (i.e. periodicity) automatically (cf. Eq. (1)):

$$\tau(M)^M = [T(c)]^M = 1,$$

and we may associate Schwinger’s operator $U$ with $\tau(M)$ and his $V$ with $T(c)$ (henceforth $c=1$).

For our analysis it is convenient to represent the number $M$ in terms of prime numbers, $P_j$,

$$M = \prod_{j=1}^{N} P_j^{n_j}, \quad P_j \neq P_i, \; j \neq i,$$

where the $n_j$ are integers, and more concisely we denote $P_j^{n_j}$ by $m_j$, i.e.

$$M = \prod_{j=1}^{N} m_j.$$

We find thus that the greatest common divisor (gcd) among the $m_j$s is 1:

$$\text{gcd}(m_j, m_i) = 1, \; \forall j \neq i,$$

i.e. distinct $m_j$s are relatively prime. Our aim is to construct representations that reflect explicitly this factorization of $M$. In our study of the $kq$ representation [8, 9, 10] the above was used to show that the number of $kq$ representations, $\chi(M)$, having conjugate representations that can be accommodated in the $M$ dimensional space, is simply related to the number of primes, $N$, that appear in $M$ (cf. Eq. (8)):

$$\chi(M) = 2^{N-1}.$$  \hfill (10)

It should be noted that the familiar finite dimensional Fourier representation is included in this counting. This is reviewed in section II. In section III we consider a novel representation, closely related to the $kq$ representation, that we call $q_1q_2$ representation [10]. Here the relation between the number of representations follows much the same reasoning as for the $kq$ representation. In section IV we develop a representation that exhibits explicitly the number of prime numbers that comprise $M$ (cf. Eq. (8)). It is in this section that the central point of this paper is presented, i.e. we exhibit the inter-relation between the dimensionality of the space under investigation and representations that reflect its prime number constituents. For the analysis in this section we note that what was required above was less restrictive than having all the involved numbers relatively prime, i.e. that among every pair of them Eq. (9) holds. What is required is that the numbers are relatively prime numbers $[\text{mod } M]$. This is defined as follows [9, 10, 11]: two numbers $M_1, M_2$ such that their product $M_1M_2 = M$ are said to be relatively prime $[\text{mod } M]$ if the equation,

$$tM_1 + sM_2 = 0 \; [\text{mod } M]$$

has, for the integers $[s, t]$, only the trivial solution, viz $s = M_1, \; t = M_2$. (Note: from their definition $s = 1, \ldots, M_1$ and $t = 1, \ldots, M_2$.) This does not preclude a nontrivial common
divisor for \( M_1 \) and \( M_2 \). This more relaxed requirement allows representations, presented in this section, wherein every prime number that makes up the dimensionality, \( M \), can be associated with a subspace which may be labeled by an appropriate quantum number. In section V we note the relation between the number of conjugate \( kq \) representations (which is the same as \( q_1q_2 \) (or \( k_1k_2 \)) representations) and the number of solutions to the equation \( x^2 = 1 \mod M \), which is used in number theory to factorize a given integer \( M \) into two relatively prime factors. The last section, section VI, is devoted to some conclusions and discussion.

II The \( kq \) representation and factorization

Schwinger [4] noted that \( U \) and \( V \), with their powers and products, generate \( M^2 \) operators which allow expressing all operators in terms of them. We shall study space dimensionalities, \( M_s \), which are not prime numbers, i.e. \( N > 1 \) in Eq. (8). We now briefly review our previous results [9, 10] to introduce a somewhat different notation that is convenient for our later generalization: Consider bi-partitioning the product that represents \( M \) (Eq. (8)) into two factors,

\[
M = M_1M_2. \tag{12}
\]

Here \( M_1 \) incorporates one part of the \( N \) factors of Eq. (8) and \( M_2 \) contains the other part. Our way of bi-partitioning implies that the two numbers, \( M_1 \) and \( M_2 \), are relatively prime, viz. \( gcd(M_1, M_2) = 1 \). We now introduce:

\[
L_1 = \frac{M}{M_1}, \quad L_2 = \frac{M}{M_2}.
\]

In the case at hand we simply have \( L_1 = M_2, \quad L_2 = M_1 \), however in section IV this definition will prove very useful. \( L_1 \) and \( L_2 \) are also relatively prime \( mod \) \( M \), cf. Eq. (11), i.e. the equation

\[
sL_1 + tL_2 = 0 \mod M \quad \tag{13}
\]

has only the trivial solution for the integers \([s, t]\), viz \( s = M_1, \quad t = M_2 \). This implies that the equation (we take \( c = 1 \)),

\[
x = sL_1 + tL_2 \mod M; \quad x = 1, ..., M; \quad s = 1, ..., M_1; \quad t = 1, ..., M_2, \tag{14}
\]

has a unique solution \( x \) for every pair \([s, t]\), with \( x \) running over its whole range of \( M \) values. We note that, in general, the pair \([s, t]\) that corresponds to \( x = 1 \) is not \([s = 1, t = 1]\). We will now show how to modify Eq. (14) to attain this simpler relation among the solutions: Let us consider the replacements \( s \rightarrow s'N_1 \mod M_1 \), \( t \rightarrow t'N_2 \mod M_2 \) with \( N_1 \) relative prime to \( L_2 \) and \( N_2 \) relative prime to \( L_1 \). Such replacements retain a unique correspondence \( s \leftrightarrow s' \mod M_1 \) and \( t \leftrightarrow t' \mod M_2 \) [12]. In these new variables Eq. (14) is

\[
x = s'N_1L_1 + t'N_2L_2 \mod M; \quad x = 1, ..., M; \quad s' = 1, ..., M_1; \quad t' = 1, ..., M_2. \tag{15}
\]
We may now choose the $N_i$ to assure that the solution $x = 1$ corresponds to the pair $[s' = 1, t' = 1]$ by solving

$$1 = N_1 L_1 + N_2 L_2 \ [mod M],$$

i.e. \[11\]

$$N_2 = L_2^{-1} \ [mod M_2], \text{ and } N_1 = L_1^{-1} \ [mod M_1]. \tag{16}$$

Now Eq. (14) can be rewritten with the solution $x = 1$ corresponding to $s = t = 1$ as

$$x = s N_1 L_1 + t N_2 L_2 \ [mod M]. \tag{17}$$

An alternative presentation of the above which will be useful in later sections is as follows: Recalling that $L_1$ and $L_2$ are relatively prime $[mod M]$ Eq. (14) may be regarded as the solution of a set of two congruences,

$$x = s \ [mod M_1]$$
$$x = t \ [mod M_2]. \tag{18}$$

The solution of these is \[11\]

$$x = s N_1 L_1 + t N_2 L_2 \ [mod M]. \tag{19}$$

To define a $kq$ representation, we use the two commuting operators $[8, 9]$

$$\tau(M_2) = e^{i \frac{2\pi}{M_2} x}; \ T(N_1 L_1) = e^{i p N_1 L_1}. \tag{20}$$

Since $N_1 L_1 = 1 \ [mod M_1]$, the equation $[e^{i p N_1 L_1}]_{M_1} = 1$ is a minimal equation (i.e., $M_1$ is the smallest number for which it is satisfied). Therefore the eigenvalues of $T(N_1 L_1)$ are $e^{i \frac{2\pi}{M_1} k}$, $k = 1, \ldots M_1$. (In [9] we used $e^{i p M_2}$ instead of the present $T(N_1 L_1)$; these two operators have the same eigenvalues and eigenstates, but enumerated differently. The advantage of $T(N_1 L_1)$ is that it shifts the eigenvalues of $\tau(M_1)$ by unity whereas $e^{i p M_2}$ shifts them by $M_2$.) The common eigenvectors of these operators are given by

$$\tau(M_2)|k_1, q_2\rangle = e^{i \frac{2\pi}{M_2} q_2}|k_1, q_2\rangle$$
$$T(N_1 L_1)|k_1, q_2\rangle = e^{i \frac{2\pi}{M_1} k_1}|k_1, q_2\rangle. \tag{21}$$

They define an $M$-dimensional $kq$ representation that is associated with the particular factorization of $M = M_1 M_2$. The indices are always associated with the range of the variable, thus, e.g. $q_2 = 1, \ldots M_2$. In the following we shall omit, unless clarity requires otherwise, the numerical indices of $q$ and $k$, i.e. $q_2 \rightarrow q$, $k_1 \rightarrow k$, with similar omission for such indices which will be introduced later. It should be noted that in this notation operators of different indices commute as is illustrated in Eq. (20). To construct the conjugate vector basis $[9, 6]$ we consider the conjugate pair of (commuting) operators:

$$\tau(M_1) = e^{i \frac{2\pi}{M_1} x}, \ T(N_2 L_2) = e^{i p N_2 L_2}, \tag{22}$$
and their eigenfunctions

$$\tau(M_1)|K_2, Q_1\rangle = e^{i\frac{2\pi}{M}Q_1}|K_2, Q_1\rangle; \quad Q_1 = 1, \ldots, M_1,$$

$$T(N_2L_2)|K_2, Q_1\rangle = e^{i\frac{2\pi}{N}K_2}|K_2, Q_1\rangle; \quad K_2 = 1, \ldots, M_2. \quad (23)$$

The basic commutation relations for our operators are:

$$T(N_1L_1)\tau(M_1) = \tau(M_1)T(N_1L_1)e^{i\frac{2\pi}{M}Q_1},$$

$$T(N_2L_2)\tau(M_2) = \tau(M_2)T(N_2L_2)e^{i\frac{2\pi}{N}K_2}, \quad (24)$$

with all other operators commuting. Hence we have

$$T(N_1L_1)\tau(M_1)|k, q\rangle = e^{i\frac{2\pi}{M}Q_1}e^{i\frac{2\pi}{N}K_2}\tau(M_1)|k, q\rangle, \quad (25)$$

indicating that $\tau(M_1)|k, q\rangle$ is, up to a phase factor, $|k + 1, q\rangle$. In a similar fashion one can show that $T(N_1L_1)|k, q\rangle$ is, again up to a phase factor, $|k, q - 1\rangle$. Now, since $k$ is defined $mod\; M_1$ and $q$ is defined $mod\; M_2$, successive application of either (and both) $\tau(M_1), T(N_2L_2)$ on any one vector $|k, q\rangle$ will generate, uniquely, all the vectors in the set $|k, q\rangle$. Thus all the states of one set may be generated by the operators of the other set [13].

Returning to our factorization of $M$ in terms of relative primes, Eq. (8), we find that only bi-partitionings of the $N$ primes are allowed: $P^n$ may not be split by breaking it up into two powers, say $P^{n_1}$ and $P^{n_2}, \quad n = n_1 + n_2$ with one factor in $M_1$ and the other in $M_2$, i.e., the bi-partitionings are among the groups of $m_i$s (Eq. (8)). Thus the number of $kq$ representations that form a complete operator basis for an $M$ dimensional physical system equals the number of possible bi-partitionings of $M$ into products of distinct primes that make $M$ (Eq. (8)), i.e., $2^{N-1}$ [9].

To conclude this section we give a new derivation for the overlap $\langle kq|KQ\rangle$: Recalling our discussion above, we supplement Eq. (21) with

$$\tau(M_1)|k, q\rangle = |k + 1, q\rangle \quad \text{and} \quad T(N_2L_2)|k, q\rangle = |k, q - 1\rangle, \quad (26)$$

and Eq. (23) with

$$\tau(M_2)|K, Q\rangle = |K + 1, Q\rangle, \quad \text{and} \quad T(N_1L_1)|K, Q\rangle = |K, Q - 1\rangle. \quad (27)$$

These are valid up to phase factors, that are, conveniently, chosen to be null [4]. We now evaluate $\langle kq|A|KQ\rangle$, where $A$ stands for each of the four operators that generate the complete operator basis for the case under study,

$$\tau(M_1), T(N_1L_1), \tau(M_2) \text{ and } T(N_2L_2).$$

This leads to the four relations

$$e^{i\frac{2\pi}{M}Q}\langle kq|KQ\rangle = \langle kq|K + 1, Q\rangle, \quad (28)$$

$$e^{i\frac{2\pi}{N}K}\langle kq|KQ\rangle = \langle k - 1, q|K, Q\rangle, \quad (28)$$

$$e^{i\frac{2\pi}{M}Q}\langle kq|KQ\rangle = \langle k, q|K, Q - 1\rangle, \quad (28)$$

$$e^{i\frac{2\pi}{N}K}\langle k, q|KQ\rangle = \langle k, q + 1|KQ\rangle. \quad (28)$$
These are solved by
\[ \langle kq|KQ \rangle = \frac{e^{i(KqM_1-kQM_2)\frac{2\pi}{M}}}{\sqrt{M}}, \] (29)
which implies the conjugacy of the two vector bases [5, 6].

III The \(q_1q_2\) representation

The choice of the two unitary commuting operators \(\tau(M_2)\) and \(T(N_1L_1)\) (Eq. (21)) as the ones (corresponding to Schwinger’s U) that define our vector space basis, i.e. the choice of a kq representation to study the system, is optional. An alternative choice is the two unitary and commuting operators \(\tau(M_2)\) and \(\tau(M_1)\) [10]. We now discuss such a choice - it leads to the representation that we choose to call the \(q_1q_2\) representation, since its labels may be considered as designating the spatial coordinates. This representation is closely related to the kq representation. It exists only when \(M_1\) and \(M_2\) are relatively prime, in which case the kq representation has a conjugate KQ representation. The common eigenfuctions of \(\tau(M_1)\) and \(\tau(M_2)\) are \(|q_1,q_2\rangle\).

Thus, with
\[ \tau(M_1) = e^{i\frac{2\pi}{M_1}q_1} = \tau(M)^{M_2} \]
\[ \tau(M_2) = e^{i\frac{2\pi}{M_2}q_2} = \tau(M)^{M_1}, \] (30)
the eigenvector equations are
\[ \tau(M_1)|q_1,q_2\rangle = e^{i\frac{2\pi}{M_1}q_1}|q_1,q_2\rangle; \quad q_1 = 1, \ldots, M_1, \]
\[ \tau(M_2)|q_1,q_2\rangle = e^{i\frac{2\pi}{M_2}q_2}|q_1,q_2\rangle; \quad q_2 = 1, \ldots, M_2. \] (31)

These provide an alternative vector basis for the M dimensional space. The complete operator basis includes, in addition, the unitary operators,

\(T(N_1L_1)\) and \(T(N_2L_2)\).

The eigenvector equations for these operators are
\[ T(N_1L_1)|k_2,k_1\rangle = e^{i\frac{2\pi}{M_1}k_1}|k_2,k_1\rangle, \]
\[ T(N_2L_2)|k_2,k_1\rangle = e^{i\frac{2\pi}{M_2}k_2}|k_2,k_1\rangle. \] (32)

These, too, span the space and form the conjugate vector basis to \(|q_1,q_2\rangle\). A convenient way to demonstrate this is by showing that the absolute value of the overlap of any member of one basis with the other is independent of either vector [5, 6]. We may get the expression for the overlap \(\langle q_1,q_2|k_1,k_2\rangle\) in much the same way that we got Eq. (29). The result is
\[ \langle q_1,q_2|k_1,k_2\rangle = \frac{e^{i(q_1k_1M_2+q_2k_2M_1)\frac{2\pi}{M}}}{\sqrt{M}}, \] (33)
assuring that the two vector bases are conjugate. We now obtain the overlap \( \langle x|q_1, q_2 \rangle \) where \( |x\rangle \) is the eigenvector of \( \tau(M) \) with eigenvalue \( e^{i \frac{2\pi}{M} x} \). The method is similar to the one we used above for the overlap of the vectors belonging to conjugate vector bases. Thus, since \( \tau(M_1) = [\tau(M)]^{M_2} \), we have

\[
\langle x|\tau(M_1)|q_1, q_2 \rangle = \langle x|q_1, q_2 \rangle e^{i \frac{2\pi}{M_1} q_1} = \langle x|[\tau(M)]^{M_2}|q_1, q_2 \rangle = e^{i \frac{2\pi}{M_2} x} \langle x|q_1, q_2 \rangle .
\]

Using a similar equation with \( \tau(M_2) \) replacing \( \tau(M_1) \), we obtain

\[
x = q_1 [\text{mode } M_1], \\
x = q_2 [\text{mode } M_2].
\]

Noting that \( \gcd(M_1, M_2) = 1 \) and using the Chinese Remainder Theorem [11, 14], we have that the unique solution is

\[
x = q_1 N_1 L_1 + q_2 N_2 L_2 [\text{mode } M].
\]

Here \( N_i = L_i^{-1} \mod M_i \), \( i = 1, 2; \) (cf. Eq.(16)). Thus we obtain

\[
\langle x|q_1, q_2 \rangle = \Delta(x - q_1 N_1 L_1 - q_2 N_2 L_2),
\]

with \( \Delta(y) = 1 \) when \( y = 0 \mod M \), and is zero otherwise. The relation for the conjugate vector basis \( |k_1, k_2\rangle \) can be handled similarly and we get

\[
\langle k|k_1, k_2 \rangle = \Delta(k - k_1 L_1 - k_2 L_2).
\]

We now comment briefly on some localization attributes of wave functions when described in this representation. We consider a state \( |\psi\rangle \) that is smeared over one spatial label but is localized in the other:

\[
\langle q_1, q_2|\psi \rangle = \frac{\delta_{q_1, M_1}}{\sqrt{M_2}}.
\]

In the \( k_1 k_2 \) space we have

\[
\langle k_1 k_2|\psi \rangle = \frac{1}{\sqrt{M_1 M_2}} \sum_{q_2} e^{2\pi i q_2 \frac{k_2}{M_2}} = \frac{\delta_{k_2, M_2}}{\sqrt{M_1}}.
\]

Thus states spread over \( q_2 \) and localized in \( q_1 \) are, in the conjugate basis, spread in \( k_1 \) and localized in \( k_2 \), with the localization exhibiting the factorization of \( M \).

**IV Complete factorization**

We now proceed and obtain a representation in which each prime number in the expression for \( M \) (cf. Eq. (8)) has characteristics of a degree of freedom [4]. We define

\[
L_j \equiv \prod_{k \neq j} P_k^{n_k} = \frac{M}{m_j}; \quad m_j = P_j^{n_j}.
\]
Now consider
\[
\tau(m_j) = \tau(M)^{L_j} = U_j = e^{i\frac{2\pi}{m_j}x},
\]
\[
T(N_jL_j) = T(c)^{N_jL_j} = V_j = e^{iN_jL_j}\tau.
\]
(42)

We have clearly
\[
U_j^{m_j} = V_j^{m_j} = 1,
\]
which defines the dimensionality of of the relevant coordinates (see below), and
\[
U_iU_j = U_jU_i, \quad V_iV_j = V_jV_i, \quad \text{and} \quad V_iU_j = U_jV_i, \quad \forall \; i \neq j.
\]
(44)

However (cf. [4])
\[
V_iU_i = U_iV_i e^{i\frac{2\pi}{m_i}}.
\]
(45)

We define the N-indexed wave function \(|q_1,...,q_N\rangle\) as the eigenfunction of the N (commuting) operators \(\tau(m_j), \; j = 1,...,N\).
\[
\tau(m_j)|q_1,...,q_j,..,q_N\rangle \equiv U_j|q_1,...,q_j,...,q_N\rangle = e^{i\frac{2\pi}{m_j}q_j}|q_1,...,q_j,...,q_N\rangle; \quad q_j = 1,...,m_j.
\]
(46)

Since the \(m_j\)s are relatively prime and the equation \(\tau(m_j)^{m_j} = 1\) is a minimal equation, its \(m_j\) eigenfunctions are distinct and different for each index \(j\). We now relate this wavefunction to the eigenfunction of \(\tau(M)\) by the same procedure that we used above: We establish the correspondence between the M eigenvectors of \(\tau(M)\) and those of \(\tau(m_j)\). We have N equations of the form
\[
\langle x|e^{i\frac{2\pi}{m_j}x}|q_1,...,q_j,...,q_N\rangle = e^{i\frac{2\pi}{m_j}q_j}\langle x|q_1,...,q_j,...,q_N\rangle = \langle x|[e^{i\frac{2\pi}{m_j}x}]^{L_j}|q_1,...,q_j,...,q_N\rangle = e^{i\frac{2\pi}{m_j}x}\langle x|q_1,...,q_j,...,q_N\rangle.
\]
(47)

Thus we must have
\[
x = q_1 \mod m_1
\]
\[
x = q_2 \mod m_2
\]
\[
\ldots
\]
\[
x = q_N \mod m_N.
\]
(48)

Since \(\gcd(m_i,m_j) = 1\), for all \(i \neq j\), we have by the Chinese Remainder Theorem [11, 14] that
\[
\langle x|q_1,...,q_j,...,q_N\rangle = \Delta(x - \Sigma_{j=1}^{N} q_jN_jL_j).
\]
(49)

This associates each of the M values of \(x\) with a unique set of the \(q_j\)s.

The M eigenvectors of the commuting operators \(T(N_jL_j), \; j = 1,...,N\) satisfy
\[
T(N_jL_j)|k_1,...,k_j,...,k_N\rangle = e^{i\frac{2\pi}{m_j}k_j}|k_1,...,k_j,...,k_N\rangle, \; k_j = 1,...,m_j.
\]
(50)
By a procedure analogous to the one used above to derive Eq. (49), we get here

\[ \langle k|k_1,...,k_N \rangle = \Delta(k - \sum_{j=1}^{N} k_j L_j). \]  

(51)

The overlap is evaluated to be

\[ \langle k_1...k_N|q_1...q_N \rangle = e^{i(\sum_{j=1}^{N} k_j q_j L_j) \frac{2\pi}{M}}. \]  

(52)

In the above, the conjugate vector bases representations \(|q_1,...q_j,...q_N\rangle\) and \(|k_1,...,k_j,...,k_N\rangle\) exhibit the prime numbers constituents of M. Each index j may be viewed as defining a subspace that is associated with the prime \(P_j\). We refer to this representation as the completely factorized representation.

V Characterization of factorization

In this section we characterize the possible bi-factorizations of M into two relative primes by the roots of an equation implied by the Chinese Remainder Theorem. In principle one might expect that such a process could be reversed, i.e. by noting the characteristics of the factorizable physical system, given in some space dimensionality M, one may deduce the factors involved. However we address ourselves to the former issue. Thus we will show, in parallel with the number theory analysis, that the eigenvalues of unitary operators which form a complete operator basis, [4], for a given space dimensionality, M, reflect the factors that make up the number M.

Our analysis above and, in particular, the completely factorized representation as such, allows viewing the N distinct prime constituents of M, Eq.(8), as N degrees of freedom (cf. [4, 6, 15]). Now the relation between \(|x\rangle\), the eigenfunction of \(\tau(M)\) which deals with the space as a whole (Eq. (4)), to the eigenfunction \(|q_1,...,q_N\rangle\) of \(\tau(m_r)\), that reflects the subspaces, each associated with a particular prime \(P_r\) (and dimensionality \(m_r\)) is given by Eq.(49)

\[ \langle x|q_1,...,q_N \rangle = \Delta(x - q_1 N_1 L_1 - q_2 N_2 L_2 - ... - q_N N_N L_N). \]

As was noted in the previous section this equation brings into our analysis the results of the Chinese Remainder Theorem [11, 14]. This theorem implies the following

\[ x = 1 \ [mod M] \iff q_r = 1 \ [mod m_r], \text{ for all } r \]

\[ x^2 = 1 \ [mod M] \iff q_r^2 = 1 \ [mod m_r], \text{ for all } r. \]  

(53)

The equation \(x^2 = 1 \ [mod M]\) has several solutions. We will henceforth designate the solutions by \(a_s\). We have immediately that, if \(a_s\) is a solution, viz \(a_s^2 = 1 \ [mod M]\), so is \(-a_s\), i.e the solutions appear in pairs.
We now argue that the number of pairs of solutions is $2^{N-1}$. Thus we may associate each solution with a conjugate pair of the $kq$-representation (or equivalently with the $q_1 q_2$ and $k_1 k_2$ representations) that can be accommodated in $M$ dimensions. The trivial solution, $a_s = 1$, is always (i.e. even if $M$ is (power of) prime) present. It corresponds to the trivial factorization, $M = 1 \cdot M$ that we associate with the Fourier representation [9, 10]. We now show that the number of solutions to $x^2 = 1 \ [mod \ M]$ equals $2^{N-1}$. The proof is direct: Eq.(53) implies that
\[ x^2 = 1 \ [mod \ M] \Rightarrow q_r = \pm 1 \ [mod \ m_r] \text{ for } r = 1, ..., N. \]
This gives $2^N$ possibilities. But only half of these are distinct since the two solutions $a_s = \pm 1$ give equivalent factorization but in a reverse order (if $a_s$ satisfies $(a_s + 1)(a_s - 1) = 0 \ [mod \ M]$, then $-a_s$ satisfies $(a_s - 1)(a_s + 1) = 0 \ [mod \ M]$), and as the order of the factors is immaterial the two lead to one bi-factorization. Note that similar reasoning introduces a factor $1/2$ in counting the number of $kq$-representations; there this was interpreted as having each distinct bi-factorization leading to a distinct conjugate pair of vector bases - the $kq$ and $KQ$ [9]. Thus $2^{N-1}$ gives the number of $kq$ conjugate pairs and the number of solutions of $x^2 = 1 \ [mod \ M]$, both expressing the bi-factorization of $M$ into coprime numbers.
To clarify the above we now consider, in some detail, a simple example: Let $M = 105 = 3 \cdot 5 \cdot 7$. Thus we have
\[
\begin{align*}
m_1 &= 3, \ N_1 = 2, \ L_1 = 35; \\
m_2 &= 5, \ N_2 = 1, \ L_2 = 21; \\
m_3 &= 7, \ N_3 = 1, \ L_3 = 15.
\end{align*}
\]
(54)
There are $2^2 = 4$ pairs of (distinct) solutions
\[
\begin{align*}
q_1 &= q_2 = q_3 = 1, \Rightarrow a_1 = 1 \ [mod \ 105], \\
q_1 &= q_2 = 1, \ q_3 = -1, \Rightarrow a_2 = 76 \ [mod \ 105], \\
q_1 &= 1, \ q_2 = q_3 = -1, \Rightarrow a_3 = 34 \ [mod \ 105], \\
q_1 &= q_3 = 1, \ q_2 = -1, \Rightarrow a_4 = 64 \ [mod \ 105].
\end{align*}
\]
(55)
The four other solutions may be obtained by reversing the signs of the $a_s$ which is obtained by changing the signs of all three $q_r$ in each set. One can readily check that $a_s^2 = 1 \ [mod \ 105]$ in all cases. Now we have it that, for each $s$ ($s = 2, 3, 4$)
\[ (a_s + 1)(a_s - 1) = 0 \ [mod \ 105]. \]
Inserting the values of the $a_s$ from $s = 2$ to $s = 4$ (skipping the trivial case of $s = 1$) we get the following expressions for $(a_s + 1)(a_s - 1)$
\[
\begin{align*}
s &= 2 : & 5 \cdot 11(15)(7), \\
s &= 3 : & 11(15)(7), \\
s &= 4 : & 3 \cdot 13(21)(5),
\end{align*}
\]
(56)
all evidently zero \([mod\ 105]\). We see that every distinct root leads to a distinct bi-factorization. Since the bi-factors must be distinct in every case, so must be the \(a_s\).

To summarize, we have shown that among the eigenstates of the completely factorized representation, those distinguished by \(q_j = \pm 1\ (j = 1, \ldots N)\) correspond uniquely to the relatively prime bi-factorization of \(M\).

VI Conclusions and discussion

Shor’s discovery [3] of an algorithm for factorization with quantum computers forms a central step in the development of quantum information theory. The number theoretic basis of the factorization method in Shor’s algorithm has been studied extensively [11]. In this paper we give what may be viewed as a study of the physics of factorization, i.e. the inter-relation between the dimensionality of the space under investigation and the representations that reflect its prime number constituents. To this end we elaborate on Schwinger’s [4] analysis of unitary operator bases for finite dimensional Hilbert spaces and show, in what we consider to be a physical language, that a natural representation is available which exhibits the prime number constituents of \(M\). In such a representation each of the \(N\) prime numbers present in the prime factorization of \(M\) defines a subspace. We give the operator basis acting in such subspaces. We further show that different, when possible, bi-factorizations of \(M\) may be viewed as different conjugate pairs of vector bases that may be associated with the \(kq\) representations [7, 8], or \(q_1q_2\) and \(k_1k_2\) representations. It was shown that the factorization of the dimensionality of the space as a number is equivalent to the breakup of the space into subspaces each representing a distinct degree of freedom that reflects a prime number that is among the prime constituents of \(M\).

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[12] An illustrative example is: \( L_1 = 5, \ L_2 = 3; \ N_1 = 2, \ N_2 = 2 \) which leads to the correspondences \( s = 1, 2, 3 \leftrightarrow s' = 2, 1, 3; \ t = 1, 2, 3, 4, 5 \leftrightarrow t' = 3, 1, 4, 2, 5. \)

[13] It is via this attribute that the necessity of having \( M_1 \) and \( M_2 \) as relative primes emerges. If \( M_1 \) and \( M_2 \) have a common factor the \( |k, q⟩ \) set defined by \( \tau(M_2) \) and \( T(N_1 L_1) \) is complete. Thus, to have a kq representation what is required is that \( M \) is not a prime number. To have conjugate kq representations it is necessary that \( M \) be factorized into a product of two relative primes. To illustrate this consider an example with \( M = 12 \) bi-factorized by \( M_1 = 2, \ M_2 = 6 \). One can readily check that applying \( \tau(M_1) \) and \( T(L_2) \) will shift both \( k \) and \( q \) by multiples of 2 only. In this case we may consider the operator

\[
F(U) = \lambda_1 \sum_{k,q = \text{even}} |k,q⟩⟨k,q| + \lambda_2 \sum_{k,q = \text{odd}} |k,q⟩⟨k,q|, \quad \lambda_1 \neq \lambda_2.
\]

This operator commutes with both our U-like operators, \( \tau(M_2) \) and \( T(N_1 L_1) \), and our V-like ones, \( \tau(M_1) \) and \( T(N_2 L_2) \), while it is not a multiple of the unit operator. Hence in cases where the bi-factorization involves numbers which are not relatively prime, one is not led to a complete operator basis. Thus the bi-factorization must be without having the same prime (cf. Eq.(8)) occuring in both terms.
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