On Quantization of the Electrical Charge–Mass

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Abstract

Suggested a non-linear, non-gauge invariant model of Maxwell equations, based on the Kaluza-Klein theory. The spectrum of elementary charges and masses is obtained.

PAC numbers: 11.10.-z; 11.10.Lm

1 Introduction

Quantization of the electrical charge and mass still remains a challenge for modern physics. Several attempts were made to solve this problem, e.g. [1, 2]. As an alternative to these consider the following approach. Consider fields \((g_{ab}, \tilde{A}_a)\) in four-dimensional spacetime, where \(g_{ab}\) is the metrics and \(\tilde{A}_a\) is dimensionless electromagnetic potential, related to the one in standard units by the rule \(\tilde{A}_a = \ell A_a\), where \(\ell\) is dimensional constant. Take the following actions for a particle and fields (electromagnetic and gravitational):

\[
S_p = -m_1 c_s \int \sqrt{G_{ab} dx^a dx^b} - m_2 c_s \int \tilde{A}_a dx^a; \quad (1)
\]

\[
S_f = -\frac{1}{16\pi c_s \ell^2} \int d\Omega \sqrt{-g} \tilde{F}^{ab} \tilde{F}_{ab}; \quad (2)
\]

\[
S_g = -\frac{c_s^3}{16\pi k} \int d\Omega \sqrt{-g} R(g). \quad (3)
\]
Here $\tilde{F}_{ab} = \partial_a A_b - \partial_b A_a$, $\tilde{F}^{ab} = G^{ac} G^{bd} \tilde{F}_{cd}$, tensor $G^{ab}$ is inverse to $G_{ab} = g_{ab} - A_a A_b$. The mass of the particle is denoted by $m_1$; ‘mass’ $m_2 = \ell^{-1} c^{-2} e$ is related to the electrical charge, $e$. The speed of light is denoted by $c_*$, and $k$ is the Newtonian constant of gravitational interactions. This theory may be regarded as a model of the Kaluza-Klein theory. If $\tilde{A}_a$ were a gauge potential and transformations $\tilde{A}_a = \tilde{A}_a' + \partial_a f$ were allowed, one might assume due to (1) that $G_{ab}$ is gauge-invariant. Then, from definition of $G_{ab}$ one would obtain that metrics $g_{ab}$ transforms by the rule $g_{ab} = g_{ab}' + 2 \tilde{A}_{(a} \partial_{b)f + \partial_a f \partial_b f}$. Thus, as it follows from (1)–(3), this theory is not gauge-invariant. Define densities, $\mu_1$ and $\mu_2$, accordingly,

$$m_i = \int \mu_i \sqrt{-g} dV, \quad (i = 1, 2.)$$  \hfill (4)

Here $dV$ is element of three-dimensional volume. From the least action principle it follows that ‘generalized’ Maxwell equations are:

$$\tilde{F}^{ab} + G^{ad} F_{dc} \tilde{F}^{bc} A_b = -4\pi \ell^2 J^a;$$  \hfill (5)

$$\tilde{F}_{[abc]} = 0.$$

Here semicolon denotes the covariant derivative associated with $g_{ab}$, i.e. equations $g_{ab;c} = 0$ take place. The electrical current,

$$J^a = \left( \mu_2 c_*^2 - \mu_1 c_*^2 \tilde{A}_b U^b \right) \frac{U^a}{U^0}.$$  \hfill (7)

Here

$$U^a = \frac{dx^a}{\sqrt{G_{cd} dx^c dx^d}}.$$  \hfill (8)
The stress-energy, entering Einstein equations for $g_{ab}$ consists of two parts; $T_{ab}$, pertaining to the particle, and $t_{ab}$, pertaining to electromagnetic field,

$$T_{ab} = \mu_1 c^2 U_a U_b; \quad (9)$$

$$t_{ab} = \frac{1}{4\pi\ell^2} \left( -G^{ad} \tilde{F}_{dc} \tilde{F}^{bc} + \frac{1}{4} g^{ab} \tilde{F}_{cd} \tilde{F}^{cd} \right). \quad (10)$$

Below the attention is focused on electromagnetic interactions in flat space-time ($g_{ab} = \eta_{ab} = \text{diag}(1, -1, -1, -1)$).

## 2 Non-Relativistic Limit

In non-relativistic limit one neglects terms of the order $\frac{v}{c^2}$, where $v$ denotes speeds of particles. Take $x^a = (c^*_t, x^\alpha)$, $\alpha = 1, 2, 3$. The potential is described by $\tilde{A}_a(x^b) = (\sin \Psi(x^b), 0, 0, 0)$, generated by the only non-vanishing component of the current,

$$J^0 = \mu_2 c^2 - \mu_1 c^2 \frac{\tilde{A}_0}{\sqrt{G_{00}}}. \quad (11)$$

Then (5) is equivalent to

$$\Delta \Psi = -4\pi\ell^2 c^2 (\mu_2 \cos \Psi - \mu_1 \sin \Psi); \quad (12)$$

here $\Delta$ denotes the Laplacian.

One may write down a class of exterior solutions, corresponding to the charge distribution $J^0$,

$$\tilde{A}_0(x^a) = \sin[\ell \phi(x^a) + \psi]. \quad (13)$$
Here $\phi(x^a)$ is the Coulomb potential generated by current component $j^0(x^a) = \cos(\ell \phi(x^a) + \psi) J^0(x^a)$, and $\psi$ is the constant of integration. The exterior solution for a pointlike charge, $e$, in spherical coordinates, $x^a = (c_s t, r, \theta, \varphi)$,

$$\tilde{A}_0 = \sin \left( \frac{\ell e}{r} + \psi \right).$$  \hspace{1cm} (14)

Since field equations are non-linear, the superposition principle doesn’t work, and asymptotic on spatial infinity should be the same for any matter distribution.

To find respective ‘gravitational’ potential $G_{ab}$ and electromagnetic potential, one has to rescale the speed of light. Really, on spatial infinity $ds_1^2 \equiv G_{ab} dx^a dx^b = \cos^2 \psi c^2_s dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$, hence, one may introduce observable speed of light, $|c|$, where $c = c_s \cos \psi$; then metrics $H_{ab}$ is defined accordingly, $ds_1^2 = H_{ab} dx^a dx^b$, where $x^a = (ct, x^\alpha)$. In the case considered above,

$$ds_1^2 = \frac{\cos^2 \left( \frac{\ell e}{r} + \psi \right)}{\cos^2 \psi} c^2 dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$ \hspace{1cm} (15)

For ‘interval’ $ds_2^2 \equiv -\ell A_a dx^a$ one obtains,

$$ds_2 = - (\cos \psi)^{-1} \tilde{A}_0 c dt - \tilde{A}_a dx^\alpha,$$ \hspace{1cm} (16)

where $A_a$ is electromagnetic potential in the standard units. In considered above spherically-symmetric case,

$$A_0(r) = \frac{\sin \left( \frac{\ell e}{r} + \psi \right)}{\ell \cos \psi}.$$ \hspace{1cm} (17)
For $\ell e \ll r$ one obtains from (15) and (17),

\[
H_{00} = 1 - \frac{2\ell e \tan \psi}{r} + O(r^{-2}); \quad (18)
\]

\[
A_0 = \ell^{-1} \tan \psi + \frac{e}{r} + O(r^{-2}). \quad (19)
\]

3 Quantizing the Electrical Charge

Consider flat spacetime with motionless electrical charge, described in spherical coordinate system. Take $\tilde{A}_a = \begin{bmatrix} \sin \frac{\phi(c)}{r}, 0, 0 \end{bmatrix}$. Equation (5) is equivalent to the following,

\[
\Phi'' = 4\pi \ell^2 c^2 r \left( -\mu_2 \cos \frac{\Phi}{r} + \mu_1 \sin \frac{\Phi}{r} \right); \quad (20)
\]

here prime denotes differentiation over $r$.

One may take the following model of the current (7) for the particle. (a) For the density take

\[
-\mu_1 + i\mu_2 = \frac{p^2}{4\pi \ell^2 c_s^2} \exp(i\psi) \frac{\arctan \sqrt{\tilde{A}_a \tilde{A}^a} + \psi}{\sin \left( \arctan \sqrt{\tilde{A}_a \tilde{A}^a} + \psi \right)}. \quad (21)
\]

Here $\tilde{A}^a = G^{ab} \tilde{A}_b$, and $p, \psi$ are real normalizing constants, found from (4). (b) For four velocity $U^a$ take

\[
U^a = \frac{{\tilde{A}}^a}{\sqrt{\tilde{A}_b \tilde{A}^b}}. \quad (22)
\]

(c) Define boundary, $S$, of the particle by equation,

\[
\left( \arctan \sqrt{\tilde{A}_a \tilde{A}^a} + \psi \right)_S = 0. \quad (23)
\]
It is assumed that in the exterior of the boundary $\mu_1 + i\mu_2 = 0$.

Solving (20) with (21), one obtains, $\tilde{A}_- = \sin \left( C \sin \frac{pr}{pr} - \psi \right)$ and $\tilde{A}_+ = \sin \left( \frac{a}{r} + b \right)$, respectively interior and exterior solutions. Here $C$, $a$, and $b$ are constants of integration.

Define boundary of the particle, as a surface of the sphere $r = R$. Then, from (23) it follows, (a) $\sin(pR) = 0$. Demanding continuity of the potential together with its first derivative, one obtains two relations: (b) $b = -\frac{a}{R} - \psi$; (c) $a = -CR \cos(pR)$. From (a) it follows that two cases take place: (i) $a = -C \frac{2 \pi n}{p}$ with $R = \frac{2 \pi n}{p}$ and (ii) $a = C \frac{\pi (2n+1)}{p}$ with $R = \frac{\pi (2n+1)}{p}$. In both cases $n = 0, 1, 2, \ldots$. For the exterior potential one obtains two branches, corresponding to cases (i) and (ii) above. Assuming that both branches have the same asymptotic on spatial infinity, one obtains, $C = \pi N$, $N = 0, \pm 1, \pm 2, \ldots$. On the other hand, demanding that mass-charge be finite, one has to put $N = \pm 1$. Thus, in the standard units, introducing $e$, charge of proton, one obtains,

$$A_{0(1)} = \ell^{-1} \cos^{-1} \psi \sin \left( N \frac{(2n)\ell e}{3r} + \psi \right),$$

$$A_{0(2)} = \ell^{-1} \cos^{-1} \psi \sin \left( -N \frac{(2n+1)\ell e}{3r} + \psi \right).$$

Both branches originate from the following interior ($r \leq R$) solution:

$$A_{0(-)} = \ell^{-1} \cos^{-1} \psi \sin \left( \pi N \frac{\sin(pr)}{pr} - \psi \right).$$

Here $p = 3\pi^2 \ell^{-1} e^{-1}$. The respective mass-charge density is

$$\mu_1 - i\mu_2 = -\frac{p^2 \exp(i\psi)}{4\ell^2 c^2} \frac{v(pr)}{\sin(\pi v(pr))}. \quad (27)$$
and it doesn’t depend on $N$. Thus, one may take $N = 1$. In (27) $v(x) = \frac{\sin x}{x}$.

Expanding solutions (24) and (25), one obtains,

$$A_{0(1)} = A_* + \frac{e}{3} \left( \frac{2n}{r} \right) - \frac{s}{2} \left( \frac{2n}{r} \right)^2 - \frac{w}{6} \left( \frac{2n}{r} \right)^3 + \cdots$$  \hspace{1cm} (28)

$$A_{0(2)} = A_* - \frac{e}{3} \left( \frac{2n + 1}{r} \right) - \frac{s}{2} \left( \frac{2n + 1}{r} \right)^2 + \frac{w}{6} \left( \frac{2n + 1}{r} \right)^3 + \cdots$$  \hspace{1cm} (29)

Here $A_* = \ell^{-1} \tan \psi$, and the following definitions are used for charges,

$$e = 3\pi^2 \ell^{-1} p^{-1};$$  \hspace{1cm} (30)

$$s = \pi^4 \ell^{-1} p^{-2} \tan \psi;$$  \hspace{1cm} (31)

$$w = \pi^6 \ell^{-1} p^{-3};$$  \hspace{1cm} (32)

or, expressing $p, \psi,$ and $\ell$ through $e, s,$ and $w$, one obtains,

$$p^2 = \frac{\pi^4 e}{3w};$$  \hspace{1cm} (33)

$$\tan \psi = \frac{\sqrt{3}s}{\sqrt{ew}};$$  \hspace{1cm} (34)

$$\ell = \frac{3\sqrt{3}}{e} \sqrt{\frac{w}{e}}.$$  \hspace{1cm} (35)

The spectrum of charges consists of two parts; the first one is $q_{2n} = \frac{2n}{3}e$, ($n = 1, 2, ...$), and the second one is $q_{2n'+1} = -\frac{2n'+1}{3}e$ ($n' = 0, 1, 2, ...$), these correspond to quarks and leptons.

### 4 Quantizing the Mass

Insofar, parameter $\psi$ isn’t fixed yet. To fix it, one may use (4) together with quantization rule for charge. From (4) and (27) it follows

$$m_{1(k)} - im_{2(k)} = -\frac{\pi \exp(i\psi)}{pl^2c_e^2} \alpha_k;$$  \hspace{1cm} (36)
\[ \alpha_k = \int_0^{\pi k} \frac{x^2 v(x) dx}{\sin (\pi v(x))} \, . \] (37)

Here \( k = 2n \) for electrical charge \( \frac{2n}{3} e \), and \( k = 2n + 1 \) for electrical charge \( -\frac{2n+1}{3} e \). On the other hand, \( m_{2(2n)} = -\frac{\pi^2 (2n)}{p^2 c^2} \), and \( m_{2(2n+1)} = \frac{\pi^2 (2n+1)}{p^2 c^2} \). It follows, then, \( \sin \psi_{2n} = -\frac{\pi (2n)}{\alpha_{2n}} \), and \( \sin \psi_{2n+1} = \frac{\pi (2n+1)}{\alpha_{2n+1}} \). Hence, one obtains spectrum of masses,

\[
\begin{align*}
    m_{2n} &= \frac{e}{3\pi cc_s^2} \sqrt{\alpha_{2n}^2 - 4\pi^2 n^2} ; \ n = 1, 2, ... \quad (38) \\
    m_{2n'+1} &= \frac{e}{3\pi cc_s^2} \sqrt{2n' + 1)^2} ; \ n' = 0, 1, 2, ... \quad (39)
\end{align*}
\]

here masses \( m_{2n} \) correspond to charges \( q_{2n} \), and masses \( m_{2n'+1} \) correspond to charges \( q_{2n'+1} \). For the first three alphas one obtains, \( \alpha_1 \approx 6.27; \alpha_2 \approx 30.2; \alpha_3 \approx 93.5 \). The respective masses for two quarks are \( m_{-\frac{1}{3}} = .059 m_e \) and \( m_{\frac{4}{3}} = .32 m_e \), where \( m_e \) is mass of an electron.

5 Conclusion

In this work we obtained spectrum of electrical charges, corresponding to leptons, \( (e, \mu, \tau) \), and quarks \( (u, d, s, c, t, b) \). It appears, that spectrum of masses \( [38] \) and \( [39] \) doesn’t describe quarks \( (s, c, t, b) \). On the other hand, correct spectrum of electrical charges allows to believe that the analogous result may follow from the Kaluza-Klein theory without assumption of topological closeness of the extra-dimension.

Definition of density \( [21] \) is made accordingly in order to obtain the simplest differential equation for potential \( [20] \). A more consistent and realistic
approach would be considering constant densities, $\mu_1$ and $\mu_2$. The price to be paid for that is to deal with non-linear equation for potential. We thought of doing that as unworthy, since the theory anyway is just a model of Kaluza-Klein theory, so that the effort of solving non-linear equation for potential should be made in frames of Kaluza-Klein.

The masses are calculated for isolated particles. In realistic situations of many interacting particles constant $\psi$ should be found for the complete closed system which would shift the masses (but not the electrical charges).

6 Acknowledgments

I wish to thank Boris S Tsirelson for helping me with numeric computations.

References

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