Research Article

Modified 2D Proca Theory: Revisited under BRST and (Anti-)chiral Superfield Formalisms

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Within the framework of Becchi-Rouet-Stora-Tyutin (BRST) approach, we discuss mainly the fermionic (i.e., off-shell nilpotent) (anti-)BRST, (anti-)co-BRST, and some discrete dual symmetries of the appropriate Lagrangian densities for a two (1 + 1)-dimensional (2D) modified Proca (i.e., a massive Abelian 1-form) theory without any interaction with matter fields. One of the novel observations of our present investigation is the existence of some kinds of restrictions in the case of our present Stückelberg-modified version of the 2D Proca theory which is not like the standard Curci-Ferrari (CF) condition of a non-Abelian 1-form gauge theory. Some kinds of similarities and a few differences between them have been pointed out in our present investigation. To establish the sanctity of the above off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries, we derive them by using our newly proposed (anti-)chiral superfield formalism where a few specific and appropriate sets of invariant quantities play a decisive role. We express the (anti-)BRST and (anti-)co-BRST conserved charges in terms of the superfields that are obtained after the applications of (anti-)BRST and (anti-)co-BRST invariant restrictions and prove their off-shell nilpotency and absolute anticommutativity properties, too. Finally, we make some comments on (i) the novelty of our restrictions/obstructions and (ii) the physics behind the negative kinetic term associated with the pseudoscalar field of our present theory.

1. Introduction

One of the simplest gauge theories is the well-known Maxwell $U(1)$ gauge theory which can be generalized to the Proca theory by incorporating a mass term in the Lagrangian density for the bosonic field (thereby rendering the latter field to acquire three degrees of freedom in the physical four (3 + 1)-dimensional (4D) flat Minkowskian spacetime). The beautiful gauge symmetry of the Maxwell theory (generated by the first-class constraints) is not respected by the Proca theory because the latter is endowed with the second-class constraints in the terminology of Dirac’s prescription for the classification scheme of constraints (see, e.g., [1–3] for details). By exploiting the theoretical potential and power of the celebrated Stückelberg formalism (see, e.g., [4]), the beautiful gauge symmetry can be restored by invoking a new pure real scalar field in the theory. This happens because the second-class constraints of the Proca theory get converted into the first-class constraints which generate the gauge symmetry transformations (see, e.g., [5, 6]) for the Stückelberg-modified version of the Proca theory in any arbitrary dimension of spacetime. As a consequence, the modified version of the Proca theory is an example of the massive gauge theory.

The purpose of our present investigation is to concentrate on the two (1 + 1)-dimensional (2D) Stückelberg-modified version of the Proca theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism and show the existence of fermionic (anti-)BRST and (anti-)co-BRST symmetry transformations as well as other kinds of discrete and continuous symmetries which provide the physical realizations of the de Rham cohomological operators of differential geometry [7–11]. In other words, we prove that the massive 2D Abelian 1-form gauge theory (i.e., the
Stückelberg-modified version of the 2D Proca theory is a field-theoretic example of Hodge theory. In this context, it is pertinent to point out that we have already shown, in our earlier work [12], that the above modified 2D Proca theory is a tractable field-theoretic model for the Hodge theory. However, the fermionic (anti-)BRST and (anti-)co-BRST symmetries of the theory have been shown to be nilpotent and absolutely anticommuting in nature (only on the on-shell). The question of the existence of the off-shell nilpotent fermionic symmetries has not been discussed, in detail, in our previous works. We accomplish this goal cogently in our present endeavor.

At this juncture, it would be worthwhile to state a few words about the cohomological operators of differential geometry. On a compact manifold without a boundary, the set of three operators \((d, \delta, \Delta)\) constitute the de Rham cohomological operators of differential geometry [7–11] where \((\delta)d\) are the (co-)exterior derivatives (with \(d^2 = 0\)) and \(\Delta = (d + \delta)^2 = (d, \delta)\) is the Laplacian operator with an underlying algebra: \(d^2 = \delta^2 = 0, \Delta = \{d, \delta\}, [\Delta, d] = [\Delta, \delta] = 0\) which is popularly known as the Hodge algebra. The models which provide the physical realizations of the above operators, in the language of symmetries and/or conserved quantities, are a set of examples of Hodge theory.

Against the backdrop of the discussions on the models for the Hodge theory, we would like to state that we have established that any arbitrary Abelian \(p\)-form \((p = 1, 2, 3 \cdots )\) gauge theory is a model for the Hodge theory in \(D = 2p\) dimensions of spacetime (see, e.g., [13–15] for details). However, these models are for the massless fields because these are field-theoretic examples of gauge theories. In addition, we have shown that the \(N = 2\) supersymmetric quantum mechanical models [16–20] are also examples for the Hodge theory. These latter models are, however, massive but they are not gauge theories because these are not endowed with the first-class constraints in the terminology of Dirac’s classification scheme for constraints (see, e.g., [1–3]). Thus, the Stückelberg-modified 2D Proca theory is very special because, for this field-theoretic model, mass and gauge invariance coexist together at the classical level and, at the quantum level, many discrete and continuous internal symmetries exist for this theory within the framework of BRST formalism. We discuss these symmetries extensively in our present endeavor.

In our present investigation, we have demonstrated the existence of two equivalent Lagrangian densities for the 2D Proca theory (within the framework of BRST formalism) which respect the off-shell nilpotent and absolutely anticommuting (anti-)BRST and (anti-)co-BRST symmetry transformations (separately and independently). We have also shown, for the first time, the existence of some restrictions in the case of our present 2D massive Abelian 1-form gauge theory which are distinctly different from the usual CF condition that exists for the non-Abelian 1-form gauge theory [21]. We have obtained the correct expressions for the conserved (anti-)BRST and (anti-)co-BRST charges which are found to be off-shell nilpotent and absolutely anticommuting (separately and independently). To verify the sanctity of the (anti-)BRST and (anti-)co-BRST symmetries (and corresponding conserved charges), we have applied our newly proposed (anti-)chiral superfield approach to BRST formalism [22–25] and proven their nilpotency and absolute anticommutativity properties. We have captured the existence of the new type of restrictions/obstructions within the framework of (anti-)chiral superfield approach to BRST formalism while proving the invariance of the Lagrangian densities under the (anti-)BRST and (anti-) co-BRST symmetry transformations (cf. Section 6). We have also shown that the discrete and continuous symmetries of the equivalent Lagrangian densities are such that both of them represent the field-theoretic examples of Hodge theory (independently and separately).

We would like to state a few words about the geometrical superfield approach [26–33] to BRST formalism (SFABF) which leads to the derivation of the (anti-)BRST symmetries and the CF-type condition [21] in the context of (non-)Abelian 1-form gauge theories. Within the framework of SFABF, a given \(D\)-dimensional gauge theory is generalized onto a \((D, 2)\)-dimensional supermanifold which is parameterized by the superspace coordinates \(Z^\mu = (x^\mu, \theta, \bar{\theta})\) where \(x^\mu\) (with \(\mu = 0, 1, \cdots D – 1\)) are the bosonic coordinates associated with the \(D\)-dimensional Minkowski space and a pair of Grassmannian variables \((\theta, \bar{\theta})\) satisfy: \(\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0\). We invoke the theoretical strength of celebrated horizontality condition to obtain the (anti-)BRST symmetries and the CF condition [21]. In the process, we also provide the geometrical basis for the abstract mathematical properties (i.e., nilpotency and absolute anticommutativity) that are associated with the (anti-)BRST symmetries and corresponding conserved charges. In our recent works [22–25], we have simplified the above SFABF by considering only the (anti-)chiral superfields on the \((D, 1)\)-dimensional super-submanifolds of the general \((D, 2)\)-dimensional supermanifold and obtained the (anti-)BRST symmetries by demanding the Grassmannian independence of the (anti-)BRST invariant quantities at the quantum level. The novel observation, in this context, has been the result that the conserved (anti-)BRST charges turn out to be absolutely anticommuting even within the framework of the (anti-)chiral superfield approach to BRST formalism [22–25] where only one Grassmannian variable is taken into account. This observation should be contrasted with the applications of the (anti-)chiral supervariable approach to the \(N = 2\) SUSY quantum mechanical models where the absolute anticommutativity is not respected. We would like to mention that the absolute anticommutativity property of the (anti-)BRST conserved charges is obvious when we take the full expansions of the superfields that are defined on the \((D, 2)\)-dimensional supermanifold.

Our present investigation is essential and interesting on the following counts. First and foremost, we wish to discuss the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries (in detail) for our present modified version of 2D Proca theory in contrast to our earlier work [12] where we have discussed only the on-shell nilpotent version of the above fermionic symmetries. Second, there are some very
interesting discrete symmetries in the theory which have not been discussed in [12]. These discrete symmetries are essential for the proof of equivalence of the coupled Lagrangian densities of our 2D massive Abelian 1-form gauge theory. Third, for the first time, we find a set of nontrivial restrictions/obstructions in our Stückelberg-modified version of the Proca (i.e., massive Abelian 1-form) theory which are not like the usual CF condition [21] of the non-Abelian 1-form gauge theory. We dwell briefly on the key differences and some kinds of similarities of these different restrictions. Fourth, we apply the (anti-)chiral superfield approach to derive the nilpotent (anti-)BRST and (anti-)co-BRST symmetries to prove the sanctity of these nilpotent transformations. Fifth, the proof of absolute anticommutativity of the nilpotent (anti-)BRST and (anti-)co-BRST symmetries is a novel observation within the framework of (anti-)chiral superfield approach. Sixth, the existence of a pseudoscalar field with a negative kinetic term and its physical relevances are pointed out at the fag ends of Sections 7 and 8. Finally, there are some novel observations in our present investigation that we point out at the fag end of our present paper. At the moment, we do not know the reasons behind the existence of these novel features in the context of our Stückelberg-modified version of the 2D Proca gauge theory (cf. Section 7 below).

Our present paper is organized as follows. First of all, to set the notions, we recapitulate the bare essentials of our earlier work [12] and discuss the on-shell nilpotent symmetries of the theory in the Lagrangian formulation. We also show the existence of the equivalent two Lagrangian densities for our modified version of 2D Proca (i.e., a massive Abelian 1-form) gauge theory in Section 2. Our Section 3 is devoted to the discussion of the off-shell nilpotent version of (anti-)BRST, (anti-)co-BRST symmetries, and the existence of restrictions/obstructions on the theory. In Section 4, we derive the conserved currents and corresponding charges. We also prove the off-shell nilpotency and absolute anticommutativity properties associated with them. Our Section 5 deals with the derivations of all the conserved and nilpotent charges and their proof of the off-shell nilpotency and absolute anticommutativity within the framework of our newly proposed (anti-)chiral superfield approach [22–25]. Section 6 contains the proof of the invariance(s) of the Lagrangian densities within the framework of (anti-)chiral superfield approach. In this section, we prove the sanctity of the underlying CF-type restrictions of our theory, too. We devote time, in Section 7, on the discussion of our new restrictions for the coupled Lagrangian densities and discuss their some kinds of similarities and distinct differences with the standard CF condition that exists in the case of non-Abelian 1-form gauge theory [21]. We also briefly comment on the negative kinetic term (associated with the pseudoscalar field of our present modified version of 2D Proca theory). Finally, we summarize the key results of our present investigation and point out a few future directions for further investigations in Section 8.

In Appendices A, B, and C, we discuss a few explicit computations. The essence of these Appendices are essential for the full appreciation of the key results of our present paper. Appendix D is devoted to a concise discussion of bosonic and ghost symmetries of the two equivalent Lagrangian densities of our theory to prove that both of them represent models for the Hodge theory provided we consider all the discrete and continuous symmetries together.

1.1. Convention and Notations. We choose the background 2D Minkowskian flat spacetime metric $g_{\mu\nu}$ with the signatures $(+1,-1)$ so that $P \cdot Q = P_\mu Q^\mu = (P_0 Q_0 - P_1 Q_1)$ for the non null 2D vectors $P_\mu$ and $Q_\mu$, where the Greek indices $\mu, \nu, \lambda, \cdots = 0, 1$ and Latin indices $i, j, k, \cdots = 1$ (because there is only one space direction in our theory). We also take the Levi-Civita tensor $\varepsilon_{\mu\nu}$ such that $\varepsilon_{01} = 1$ and $\varepsilon_{\mu\nu}\varepsilon^{\nu\lambda} = 2$, etc. We denote, in the whole body of our text, the (anti-)BRST and (anti-)co-BRST symmetries of all varieties (and in all contexts) by the symbols $s_{(a)b}$ and $s_{(a)id}$, respectively. We also adopt the convention of the left-derivative w.r.t. all the fermionic fields and use the notations $\Omega = \partial_\nu \Gamma^\nu$ and $\Psi = \partial \Psi / \partial t$, etc., for a generic field $\Psi$. We focus only on the internal symmetries of our 2D theory and spacetime symmetries of the 2D Minkowskian spacetime manifold do not play any crucial role in our whole discussion.

2. Preliminaries: Lagrangian Formulation and Various Kinds of Symmetries

We begin with the celebrated Proca (i.e., a massive Abelian 1-form) theory in any arbitrary $D$-dimension of spacetime. This theory, with the rest mass $m$ for the vector boson, is described by the following Lagrangian density (see, e.g., [4]).

$$\mathcal{L}_{(p)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu,$$

where the antisymmetric field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is derived from the 2-form $F^{(2)} = (1/2!) \left( dx^\mu \wedge dx^\nu \right) = dA^{(1)}$ where the nilpotent $(d^2 = 0)$ exterior derivative $d = dx^\mu \partial_\mu$ (with $\mu = 0, 1 \cdots D - 1$) acts on a 1-form $(A^{(1)} = dx^\mu A_\mu)$ to produce the 2-form $F^{(2)}$ w.r.t. to the vector potential $A_\mu$. This theory is endowed with the second-class constraints, and therefore, it does not respect any kind of gauge symmetry. However, one can exploit the theoretical strength of the Stückelberg formalism [4] and replace $A_\mu$ by

$$A_\mu \longrightarrow A_\mu + \frac{1}{m} \partial_\mu \varphi,$$

where $\varphi$ is a pure scalar field. The resulting Stückelberg’s modified Lagrangian density

$$\mathcal{L}_{(s)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \mu_\mu \varphi \partial^\mu \varphi + m A_\mu \partial^\mu \varphi,$$

is endowed with the first-class constraints (with $\Pi^0 = 0$,

$$\Pi^i = E_i, \partial E_i = \nabla \cdot \overrightarrow{E}$$

$$\Pi^0 \equiv 0, \overrightarrow{\nabla} \cdot \overrightarrow{E} + m \Pi_\varphi = 0,$$

is a key result of our present investigation that we point out at the fag end of Sections 7 and 8. Finally, there are some novel observations in our present investigation that we point out at the fag end of our present paper.
where $\Pi^\mu = -F^0\mu$ and $\Pi_\phi = \dot{\phi} + mA_0$ are the momenta w.r.t. $A_\mu$ and $\phi$ and $\bar{E}$ is the electric field (present as a component in $F_{\mu\nu}$). The generator of the infinitesimal gauge transformations ($\delta_\phi$) can be written, in terms of the above first-class constraints, as [5, 6]

$$
G = \int d^{(D-1)}x \left[ \sum_{i=0}^{0} \bar{\Pi} \left( \sum_i \left( \nabla \cdot \bar{E} \mp m \bar{\Pi} \phi \right) \right) \right].
$$

where $\Sigma(x)$ is the gauge transformation parameter (with $\bar{\Sigma} = \partial \Sigma / \partial t$). The above generator leads to the following gauge transformation for a generic field $\Psi$, namely,

$$
\delta_\phi \Psi = i[\Psi, G], \quad \Psi = A_\mu, \phi,
$$

where we have to use the following equal-time canonical commutators (with $\hbar = c = 1$)

$$
\begin{align*}
[A_0(\bar{x}, t), \Pi_0(y, t)] &= i\delta^{(D-1)}(\bar{x} - y), \\
[A_i(\bar{x}, t), E_i(y, t)] &= i\delta^{(D-1)}(\bar{x} - y), \\
[\phi(\bar{x}, t), \Pi_\phi(y, t)] &= i\delta^{(D-1)}(\bar{x} - y),
\end{align*}
$$

and the rest of the equal-time commutators are taken to be zero. Ultimately, we obtain the following infinitesimal gauge transformations ($\delta_\phi$), namely,

$$
\delta_\phi A_\mu = \partial_\mu \Sigma, \quad \delta_\phi \phi = \pm m \Sigma,
$$

which are valid in any arbitrary $D$-dimension of spacetime.

For the definition of the propagator for the massive vector field $A_\mu$ and for the purpose of quantization of the St"uckelberg-modified Lagrangian density $\mathcal{L}_{(s)}$, we have to incorporate the gauge-fixing term. The ensuing Lagrangian density $\mathcal{L}_{(s)}^{(g)}$ is as follows:

$$
\mathcal{L}_{(s)}^{(g)} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \mp m A_\mu \partial^\mu \phi - \frac{1}{2} (\partial \cdot A \mp m \phi)^2,
$$

which does not respect the gauge symmetry transformations (8) unless we put a restriction from outside equal to $(\square + m^2) \Sigma = 0$. In the special case of two $(1 + 1)$-dimensional (2D) theory, the Lagrangian density (9) takes the following form:

$$
\mathcal{L}_{(s)}^{(2D)} = -\frac{1}{2} E^2 + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \mp m A_\mu \partial^\mu \phi - \frac{1}{2} (\partial \cdot A \mp m \phi)^2,
$$

because, in 2D spacetime, we have only $F_{01} = -F_{10} = E = -e^{i \phi} \partial_\phi A_0$ as the existing (nonzero) component of $F_{\mu\nu}$ (because there is no magnetic field in this theory). The above gauge-fixed Lagrangian density has the following generalized form (see, e.g., [12]):

$$
\mathcal{L}_{(2D)}^{(g)} = \frac{1}{2} \left( E \mp m \phi \right)^2 \pm m E \phi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} A_\mu A^\mu
$$

$$
+ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \mp m A_\mu \partial^\mu \phi - \frac{1}{2} (\partial \cdot A \pm m \phi)^2.
$$

In the above, we have generalized $(E^2/2)$ in the same manner as the St"uckelberg formalism generalizes $(m^2/2)A_\mu A^\mu$ term in the Lagrangian density (3). To be precise, we have incorporated a pseudoscalar field $(\phi)$ in our theory because the electric field $E$ is a pseudoscalar in 2D spacetime. It will be worthwhile to point out that all the basic fields of our 2D theory (i.e., $A_\mu, \phi$) have mass dimension zero in the natural units (where $\hbar = c = 1$).

2.1. Discrete Symmetries and (Dual-)Gauge Symmetries. We shall now concentrate on the most generalized version of the 2D Lagrangian density (11) for our further discussions. In this connection, it can be checked that under the following discrete symmetry transformations

$$
A_\mu \longrightarrow \pm \varepsilon_{\mu \nu} A^\nu, \quad \phi \longrightarrow \mp i \phi, \quad \phi \longrightarrow \mp i \phi,
$$

the 2D Lagrangian density $\mathcal{L}_{(2D)}^{(g)}$ remains invariant (because $E \longrightarrow \mp i (\partial \cdot A)$, $(\partial \cdot A) \longrightarrow \mp i E$ due to $A_\mu \longrightarrow \pm \varepsilon_{\mu \nu} A^\nu$) modulo some total spacetime derivaties. Furthermore, it is very interesting to point out that under the following (dual-)gauge transformations ($\delta_\phi$)

$$
\begin{align*}
\delta_\phi A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu \Omega, \\
\delta_\phi \phi &= \mp m \Omega, \\
\delta_\phi \phi &= 0, \\
\delta_\phi (\partial \cdot A \mp m \phi) &= 0, \\
\delta_\phi E &= \mp \square \Omega,
\end{align*}
$$

$$
\delta_\phi \left( E \mp m \phi \right) = \mp (\square + m^2) \Omega,
$$

$$
\begin{align*}
\delta_\phi A_\mu &= \partial_\mu \Sigma, \\
\delta_\phi \phi &= \pm m \Sigma, \\
\delta_\phi \phi &= 0, \\
\delta_\phi E &= 0, \\
\delta_\phi (\partial \cdot A \pm m \phi) &= \mp (\square + m^2) \Sigma,
\end{align*}
$$

the 2D Lagrangian density transforms as
\[ \delta_{dg} \mathcal{L}^{(2D)} = \partial_{\mu} \left[ m \epsilon^\nu (m A_{\nu} \Omega \pm \phi \partial_{\nu} \Omega) \pm m \hat{\phi} \partial_{\mu} \Omega \right] \]
\[ + \left( E + m \hat{\phi} \right) \left( (\mp m^2) \Omega, \right) \]
\[ \delta_{g} \mathcal{L}^{(2D)} = - (\partial \cdot A + m \phi) \left( \Box + m^2 \right) \sum, \]

where \( \Sigma(x) \) and \( \Omega(x) \) are the infinitesimal gauge and dual-gauge transformation parameters. In other words, \( \Sigma(x) \) and \( \Omega(x) \) are the pure scalar and pseudoscalar, respectively.

At this stage, a few comments are in order. First of all, there are two equivalent gauge-fixed Lagrangian densities that are hidden in (11), namely,

\[ \mathcal{L}^{(1)} = \frac{1}{2} \left( E - m \hat{\phi} \right)^2 + m E \hat{\phi} - \frac{1}{2} \partial_{\mu} \partial_{\nu} \phi + \frac{m^2}{2} A_{\mu} A^\mu \]
\[ + \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi - m A_{\mu} \partial_{\mu} \phi - \frac{1}{2} (\partial \cdot A + m \phi)^2, \]

\[ \mathcal{L}^{(2)} = \frac{1}{2} \left( E + m \hat{\phi} \right)^2 - m E \hat{\phi} - \frac{1}{2} \partial_{\mu} \partial_{\nu} \phi + \frac{m^2}{2} A_{\mu} A^\mu \]
\[ + \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + m A_{\mu} \partial_{\mu} \phi - \frac{1}{2} (\partial \cdot A - m \phi)^2, \]

which are connected to each other by a discrete symmetry transformations: \( \phi \rightarrow \phi, \phi \rightarrow \phi, A_{\mu} \rightarrow A_{\mu}. \) Second, it is obvious that the (dual-)gauge transformation parametrizes \( \Omega \) and \( \Sigma \) is constrained by the same type of restrictions (i.e., \((\mp m^2) \Omega = 0, (\Box + m^2) \Sigma = 0\)) from outside if we wish to have perfect (dual-)gauge symmetries in the theory. Third, we note that only one pair of ghost and antighost fields would be good enough to take care of these restrictions for the perfect “quantum” (dual-)gauge (i.e., BRST-type) symmetries within the framework of BRST formalism. Fourth, one of the decisive features of the (dual-)gauge symmetries is the observation that the gauge-fixing and kinetic terms of our 2D theory remain invariant under these symmetries, respectively.

2.2. On-Shell Nilpotent Symmetries and Discrete Symmetries.

In our earlier work [12], we have taken up one of the above Lagrangian densities (i.e., (11)) for the generalizations of the (dual-)gauge symmetries at the “quantum” level within the framework of BRST formalism. For instance, the following (anti-)BRST symmetries (which are the generalizations of the gauge symmetries (13), namely,

\[ s_{ab} A_{\mu} = \partial_{\mu} \hat{C}, s_{ab} \phi = i (\partial \cdot A + m \phi), s_{ab} \phi = m \hat{C}, \]
\[ s_{ab} \hat{C} = 0, s_{ab} E = s_{ab} \hat{\phi} = 0, s_{ab} (\partial \cdot A + m \phi) = (\pm m^2) \hat{C}, \]
\[ s_{b} A_{\mu} = \partial_{\mu} \hat{C}, s_{b} \phi = - i (\partial \cdot A + m \phi), s_{b} \phi = m \hat{C}, \]
\[ s_{b} \hat{C} = 0, s_{b} E = s_{b} \hat{\phi} = 0, s_{b} (\partial \cdot A + m \phi) = (\pm m^2) \hat{C}, \]

leave the following Lagrangian density invariant (modulo a total spacetime derivative

\[ \mathcal{L}_{(B_1)} = \frac{1}{2} \left( E - m \hat{\phi} \right)^2 + m E \hat{\phi} - \frac{1}{2} \partial_{\mu} \partial_{\nu} \phi \]
\[ - \frac{m^2}{2} A_{\mu} A^\mu + \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi - m A_{\mu} \partial_{\mu} \phi \]
\[ - \frac{1}{2} (\partial \cdot A + m \phi)^2 - i \partial_{\mu} \hat{C} \partial_{\nu} \hat{C} + m^2 \hat{C}, \]

which is a generalization of the gauge-fixed 2D Lagrangian density \( \mathcal{L}^{(1)} \) to the “quantum” level (within the framework of BRST formalism where the last two terms, in the Lagrangian density (18), are the Faddeev-Popov ghost terms). It should be noted that the fermionic (i.e., \( \hat{C}^2 = \hat{C} = 0, \hat{C} \hat{C} + \hat{C} = 0 \)) (anti-)ghost fields \( \hat{C} \) are introduced in the theory to maintain the unitarity at any arbitrary order of perturbative computations.

A few comments are in order at this juncture. First, we note that the total kinetic term (associated with the gauge field) remains invariant (i.e., \( s_{(a)b} E = s_{(a)b} \hat{\phi} = 0 \)) under the nilpotent (anti-)BRST symmetry transformations \( s_{(a)b}. \) Second, the (anti-)BRST symmetries are on-shell nilpotent \( s_{(a)b}^2 = 0 \) as we have to use the relevant EOMs: \((\mp m^2) \hat{C} = 0, (\Box + m^2) \hat{C} = 0 \) for the proof of the nilpotency property. Third, the Lagrangian density \( \mathcal{L}_{(B_1)} \) as the general version of (16), can also be obtained from \( \mathcal{L}_{(B_1)} \) by the replacements: \( \phi \rightarrow \phi, \phi \rightarrow \phi, A_{\mu} \rightarrow A_{\mu}, C \rightarrow C, \hat{C} \rightarrow \hat{C}. \) Fourth, the (anti-)BRST symmetries for the Lagrangina density \( \mathcal{L}_{(B_1)} \) can also be obtained from (17) by the above replacements (i.e., \( \phi \rightarrow \phi, \phi \rightarrow \phi, A_{\mu} \rightarrow A_{\mu}, C \rightarrow C, \hat{C} \rightarrow \hat{C}. \)) Finally, we conclude that both the Lagrangian densities \( \mathcal{L}_{(B_1)} \) and \( \mathcal{L}_{(B_2)} \) are equivalent and they describe the same 2D Stückelberg-modified massive Abelian 1-form gauge theory.

In addition to the on-shell nilpotent (anti-)BRST symmetries (17), there is another set of on-shell nilpotent \( s_{(a)b}^2 = 0 \) (anti-)co-BRST (or (anti-)dual BRST) symmetries \( s_{(a)d} \) in our theory because under these (i.e., \( s_{(a)d} \)) transformations

\[ s_{ad} A_{\mu} = - \epsilon_{\mu \nu} \partial_{\nu} C, s_{ad} \hat{C} = i \left( E - m \hat{\phi} \right), s_{ad} \phi = 0, \]
\[ s_{ad} C = 0, s_{ad} E = \Box C, s_{ad} \partial (\partial \cdot A + m \phi) = 0, s_{ad} \hat{\phi} = - m \hat{C}, \]
\[ s_{d} A_{\mu} = - \epsilon_{\mu \nu} \partial_{\nu} \hat{C}, s_{d} \phi = - i (E - m \hat{\phi}), s_{d} \phi = 0, \]
\[ s_{d} \hat{C} = 0, s_{d} E = \Box \hat{C}, s_{d} \partial (\partial \cdot A + m \phi) = 0, s_{d} \hat{\phi} = - m \hat{C}, \]

the Lagrangian density \( \mathcal{L}_{(B_1)} \) (cf. Equation (18)) remains invariant, modulo some total spacetime derivatives, as listed below:

\[ s_{ad} \mathcal{L}_{(B_1)} = \partial_{\mu} \left[ m \epsilon^\nu (m A_{\nu} \phi + \phi \partial_{\nu} \phi + E \partial_{\nu} C) \right], \]
\[ s_{d} \mathcal{L}_{(B_1)} = \partial_{\mu} \left[ m \epsilon^\nu (m A_{\nu} \phi + \phi \partial_{\nu} \phi + E \partial_{\nu} C) \right]. \]
As a consequence, the action integral \( S = \int d^2x \mathcal{L}_{(B_1)} \) remains perfectly invariant under the on-shell nilpotent (anti-)co-BRST symmetry transformations.

We comment on some of the salient features of the (anti-)co-BRST symmetries at this specific point of our discussion. First, we note that the total gauge-fixing term of the Lagrangian densities remains invariant under the (anti-)co-BRST symmetry transformations (i.e., \( s_{(a)d} (\partial \cdot A) = 0, s_{(a)d} \phi = 0 \)). Second, the mathematical origin of the gauge-fixing term (corresponding to the gauge field) is hidden in the coexterior derivative of differential geometry because we note that \( \delta A^{(1)} = -s d A^{(1)} = (\partial \cdot A) \) where \( \delta = -s d * \) is the coexterior derivative and * is the Hodge duality operation on the 2D Minkowskian spacetime manifold. The other part of the gauge-fixing term (i.e., \( m \phi \)) has been added/subtracted on the dimensional ground (in the natural units). Third, the (anti-)co-BRST symmetries are absolutely anticommuting and nilpotent of order two provided we take the advantage of EOMs. Finally, we note that the following interesting relationships (21) can be obtained from \( \mathcal{L}_{(B_1)} \) and Equation (19) by the replacements: \( \phi \rightarrow \tilde{\phi}, \phi \rightarrow i \tilde{\phi}, C \rightarrow \mp i C, \tilde{C} \rightarrow \mp i C \),

\[
A_{\mu} \longrightarrow \pm i \epsilon_{\mu \nu} A^{\nu}, \phi \rightarrow \pm i \tilde{\phi}, \phi \rightarrow \pm i \tilde{\phi}, C \longrightarrow \mp i C, \tilde{C} \longrightarrow \mp i C,
\]

(21)

leave the Lagrangian densities \( \mathcal{L}_{(B_1)} \) and \( \mathcal{L}_{(B_1)} \) invariant (modulo some total spacetime derivatives). The existence of these discrete symmetries is very important for us as these symmetries provide the physical realizations of the Hodge duality * operation of the differential geometry because we note that the following interesting relationships

\[
s_{(a)d} = \pm * s_{(a)b} * s_{(a)b} = \mp * s_{(a)d} * s_{(a)d},
\]

(22)

are true provided we take the above mathematical connections in their operator form. In the above relationships, * is nothing but the discrete symmetry transformations (21). Thus, we note that it is the interplay between the discrete and continuous symmetries of our 2D BRST invariant theory that provides the physical realizations of the celebrated relationship of differential geometry where the (co-)exterior derivatives are connected to each other by the relationships: \( \delta \pm \ast d * \) [7–11]. There is another very important relationship that is governed and dictated by the discrete symmetry transformations in (21). For instance, it can be checked that the direct application of the discrete symmetry transformations (21) on (17) and (19) leads to the following mappings:

\[
s_{(a)} s_{(a)} d \rightarrow s_{(a)} d, s_{(a)} a \rightarrow s_{(a)} d.
\]

(23)

In other words, the (anti-)co-BRST and (anti-)BRST symmetries (that have been listed in (19) and (17)) are also connected with each other by the direct application of the discrete symmetry transformations (21). Let us take an example to illustrate this point clearly. We note that \( s_{(a)} A_{\mu} = \partial_{\mu} C \).

Now we apply directly the discrete symmetry transformations (21) on it. Taking into account the mapping listed in (23), we have to take \( s_{(d)} \rightarrow s_{(d)} \) and, after that, we obtain the following (from \( s_{(d)} A_{\mu} = \partial_{\mu} C \)), namely,

\[
s_{(d)} (A_{\mu} = \partial_{\mu} (\mp i C)) \rightarrow s_{(d)} (\mp i \epsilon_{\mu \nu} A^{\nu} = \partial_{\mu} (\mp i C)),
\]

(24)

where * is nothing but the discrete symmetry transformations (21). From the above relationship, it is obvious that we have obtained the dual-BRST symmetry transformation \( s_{(d)} \) (from the given BRST symmetry transformation \( s_{(d)} \)) on the gauge field of our theory which amounts to \( s_{(d)} A_{\mu} = -\epsilon_{\mu \nu} \partial_{\nu} C \). Thus, the discrete symmetry transformations (21) provide a direct relationship between \( s_{(a)d} \) and \( s_{(a)b} \). It can be checked that the mappings, given in Equation (23), are correct and very useful.

We end this section with the remarks that there are various kinds of discrete symmetries in the theory which connect equivalent Lagrangian densities \( \mathcal{L}_{(B_1)} \) and \( \mathcal{L}_{(B_1)} \) as well as the on-shell nilpotent and absolutely anticommuting (anti-)BRST and (anti-)co-BRST symmetry transformations. In the next section, we shall discuss about the coupled (but equivalent) Lagrangian densities, off-shell nilpotent fermionic symmetries, and the corresponding CF-type restrictions.

### 3. Off-Shell Nilpotent Symmetries, Discrete Symmetries, and Some Kinds of Restrictions

We have seen that the Lagrangian densities \( \mathcal{L}_{(B_1)} \) and \( \mathcal{L}_{(B_1)} \) respect the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations. These Lagrangian densities can be generalized in the following fashion (i.e., \( \mathcal{L}_{(B_1)} \rightarrow \mathcal{L}_{(B_1)} \) and \( \mathcal{L}_{(B_1)} \rightarrow \mathcal{L}_{(B_1)} \)):

\[
\mathcal{L}_{(B_1)} = \mathcal{B} (E - m \phi) - \frac{1}{2} \mathcal{B}^2 + m E \phi - \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi \\
+ \frac{1}{2} A_{\mu} A^\mu + \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi - m A_{\mu} \partial^\mu \phi + B (\partial \cdot A + m \phi) \\
+ \frac{1}{2} B^2 - i \partial_{\mu} \tilde{C} \partial^\mu C + i m^2 C C,
\]

\[
\mathcal{L}_{(B_1)} = \mathcal{B} (E + m \phi) - \frac{1}{2} \mathcal{B}^2 - m E \phi - \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi \\
+ \frac{1}{2} A_{\mu} A^\mu + \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi + m A_{\mu} \partial^\mu \phi + \tilde{B} (\partial \cdot A - m \phi) \\
+ \frac{1}{2} B^2 - i \partial_{\mu} \tilde{C} \partial^\mu C + i m^2 C C.
\]

(25)
In the above, we have linearized the kinetic term as well as the gauge-fixing term by invoking the Nakanishi-Lautrup type auxiliary fields $(\mathcal{B}, \mathcal{R}, B, \bar{B})$. It is elementary to check that the following (anti-)BRST symmetries $(s_{(a)b})$, namely,

\[
s^{(1)}_{ab} A_{\mu} = \partial_{\mu} \bar{C}, \quad s^{(1)}_{ab} E = s^{(1)}_{ab} \bar{\phi} = s^{(1)}_{ab} \bar{C} = 0, \quad s^{(1)}_{ab} B = s^{(1)}_{ab} \mathcal{B} = 0,
\]

leave the action integral $S = \int d^4x \mathcal{L}_{(b)}$ invariant because the Lagrangian density $\mathcal{L}_{(b)}$ transforms to a total spacetime derivative (i.e., $s^{(1)}_{ab} \mathcal{L}_{(b)} = \partial_{\mu} [B \partial^{\mu} C]$), $s^{(1)}_{ab} \mathcal{L}_{(b)} = 0$ under the (anti-)BRST symmetry transformations. We note that the above (anti-)BRST symmetry transformations $s_{(a)b}$ are off-shell nilpotent $[(s_{(a)b})^2 = 0]$ and absolutely anticommuting $s^{(1)}_{ab} + s^{(2)}_{ab} = 0$ in nature. They leave the total kinetic terms $B(E - m\phi) - (1/2)B^2 + mE\phi - (1/2)\partial_{\mu} \phi \partial^{\mu} \phi$ for the 1-form gauge field and a pseudoscalar field invariant. We recall here that the kinetic term of the gauge field has its origin in the exterior derivative $d$.

There is another set of (anti-)BRST symmetry transformations $(s_{(a)b})$ that leave the action integral $S = \int d^4x \mathcal{L}_{(b)}$ invariant because the Lagrangian density $\mathcal{L}_{(b)}$ respects the following off-shell nilpotent $[(s_{(a)b})^2 = 0]$ and absolutely anticommuting $(s^{(2)}_{ab} s^{(2)}_{ab} + s^{(2)}_{ab} s^{(2)}_{ab} = 0)$ (anti-)BRST symmetry transformations $(s_{(a)b})$, namely,

\[
s^{(2)}_{ab} A_{\mu} = \partial_{\mu} \bar{C}, \quad s^{(2)}_{ab} E = s^{(2)}_{ab} \bar{\phi} = s^{(2)}_{ab} \bar{C} = 0, \quad s^{(2)}_{ab} B = s^{(2)}_{ab} \mathcal{B} = 0,
\]

because the Lagrangian density $\mathcal{L}_{(b)}$ transforms to a total spacetime derivative (under the (anti-)BRST symmetry transformations $(s_{(a)b})$). It can be, once again, checked that the total kinetic terms $\mathcal{B}(E - m\phi) - (1/2)\mathcal{B} - mE \phi - (1/2)\partial_{\mu} \phi \partial^{\mu} \phi$ for the Abelian 1-form gauge field and pseudoscalar field remain invariant under the (anti-)BRST transformations $s^{(2)}_{ab}$.

There is an interesting discrete symmetry in the theory which relates $\mathcal{L}_{(b)}$ with $\mathcal{L}_{(b)}$ and $s_{(a)b}$ with $s_{(a)b}$. These symmetry transformations are as follows:

\[
B \leftrightarrow \bar{B}, \quad \mathcal{B} \leftrightarrow \bar{\mathcal{B}}, \quad \phi \leftrightarrow -\phi, \quad \bar{\phi} \leftrightarrow -\bar{\phi},
\]

\[
A_{\mu} \leftrightarrow A_{\mu}, \quad C \leftrightarrow \bar{C}, \quad \bar{C} \leftrightarrow \bar{C}.
\]

In other words, only the auxiliary fields and analogues of Stückelberg’s fields transform but the original basic fields $(A_{\mu}, C, \bar{C})$ do not transform at all under the discrete transformations (28). Thus, we note that the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ are equivalent due to the existence of the discrete symmetry transformations in (28). It would be very interesting to apply the (anti-)BRST symmetry transformations $s_{(a)b}$ on $\mathcal{L}_{(b)}$ and $s_{(a)b}$ on $\mathcal{L}_{(b)}$.

In this context, we note that the following are true, namely,

\[
s^{(2)}_{ab} \mathcal{L}_{(b)} = \partial_{\mu} [B \partial^{\mu} C + m^2 A^2 C + 2m\phi \partial^{\mu} \phi] - [B + B + 2(\partial \cdot A)] \mathcal{C},
\]

\[
s^{(2)}_{ab} \mathcal{L}_{(b)} = \partial_{\mu} [B \partial^{\mu} C + 2m^2 A^2 C + 2m \phi \partial^{\mu} \phi] - [B + B + 2(\partial \cdot A)] \mathcal{C},
\]

\[
s^{(2)}_{ab} \mathcal{L}_{(b)} = \partial_{\mu} [B \partial^{\mu} C + 2m^2 A^2 C - 2m \phi \partial^{\mu} \phi] - [B + B + 2(\partial \cdot A)] \mathcal{C},
\]

where we have used the following nilpotent transformations:

\[
s^{(1)}_{ab} B = -2\mathcal{C}, \quad s^{(1)}_{ab} \mathcal{B} = -2\mathcal{C}, \quad s^{(1)}_{ab} \mathcal{B} = 0, \quad s^{(1)}_{ab} \mathcal{B} = 0,
\]

\[
s^{(2)}_{ab} B = -2\mathcal{C}, \quad s^{(2)}_{ab} \mathcal{B} = -2\mathcal{C}, \quad s^{(2)}_{ab} \mathcal{B} = 0, \quad s^{(2)}_{ab} \mathcal{B} = 0,
\]

in addition to the (anti-)BRST symmetry transformations (26) and (27). We note that if we impose the following restriction

\[
B + B + 2(\partial \cdot A) = 0,
\]

we find that the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ both respect both types of (anti-)BRST symmetry transformations $s_{(a)b}$ as well as $s_{(a)b}$ in a beautiful fashion. It should be pointed out that, at this stage, we can not use the EOM $(\mathcal{C} + m^2 \mathcal{C}) = 0$.

We provide here the origin of the restriction (31) as well as the (anti-)BRST symmetry transformations (30) (in addition to the (anti-)BRST symmetry transformations listed in
(26) and (27)). First of all, we note that Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(s)}$ lead to the following EL-EOMs, namely,

$$\mathcal{B} = E - m\phi,$$
$$\mathcal{B} = E + m\phi,$$
$$B = -[(\partial \cdot A) - m\phi],$$
$$\bar{B} = -[(\partial \cdot A) + m\phi],$$

which result in the following combinations of restrictions:

$$B + \bar{B} + 2(\partial \cdot A) = 0,$$
$$B - \bar{B} + 2m\phi = 0,$$
$$\mathcal{B} + \mathcal{B} - 2E = 0,$$
$$\mathcal{B} - \mathcal{B} + 2m\phi = 0.$$  

(32)

If these restrictions are to be imposed from outside, these have to be (anti-)BRST invariant. This requirement leads to the derivation of the (anti-)BRST symmetry transformations listed in (30). We would like to comment that, on the constrained hypersurface in the 2D Minkowskian spacetime manifold where the restriction $B + \bar{B} + 2(\partial \cdot A) = 0$ is valid, we obtain the following (anti-)BRST symmetry transformations:

$$s^{(1)}_{a} \mathcal{L}_{(b)} = \partial_{\mu} [\mathcal{B}\partial^{\mu}C + 2m^2A^{\mu}C + 2m\phi\partial^{\mu}C],$$
$$s^{(1)}_{ab} \mathcal{L}_{(b)} = \partial_{\mu} [\mathcal{B}\partial^{\mu}C + 2m^2A^{\mu}C + 2m\phi\partial^{\mu}C],$$
$$s^{(2)}_{a} \mathcal{L}_{(b)} = \partial_{\mu} [\mathcal{B}\partial^{\mu}C + 2m^2A^{\mu}C - 2m\phi\partial^{\mu}C],$$
$$s^{(2)}_{ab} \mathcal{L}_{(b)} = \partial_{\mu} [\mathcal{B}\partial^{\mu}C + 2m^2A^{\mu}C - 2m\phi\partial^{\mu}C],$$
$$s^{(1)}_{b} \mathcal{L}_{(b)} = \partial_{\mu} [\mathcal{B}\partial^{\mu}C],$$
$$s^{(2)}_{b} \mathcal{L}_{(b)} = \partial_{\mu} [\mathcal{B}\partial^{\mu}C].$$  

(34)

Hence, we note that both the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(s)}$ respect both the nilpotent (anti-)BRST symmetries $s^{(1)}_{a,b}$ as well as $s^{(2)}_{a,b}$. Provided we use the restriction $B + \bar{B} + 2(\partial \cdot A) = 0$. In other words, the action integrals $S = \int d^2x \mathcal{L}_{(b)}$ and $S_2 = \int d^2x \mathcal{L}_{(s)}$ are invariant under $s^{(1)}_{a,b}$ as well as $s^{(2)}_{a,b}$ on a hypersurface in the 2D Minkowskian space which is defined by the field equation $B + \bar{B} + 2(\partial \cdot A) = 0$. We shall discuss more about this restriction in our Section 7 (see below).

In addition to the above (anti-)BRST symmetry transformations $s^{(1,2)}_{a,b}$, the Lagrangian density $\mathcal{L}_{(b)}$ respects the following off-shell nilpotent $[(s^{(1)}_{a,b})^2 = 0]$ and absolutely anticommuting $s^{(1)}_{a} s^{(1)}_{b} + s^{(2)}_{a} s^{(2)}_{b} = 0$ (anti-)co-BRST symmetry transformations $s^{(1)}_{a,b}$:

$$s^{(1)}_{a} A_{\mu} = -\epsilon_{\mu\nu}\partial^{\nu}C, s^{(1)}_{a} (\partial \cdot A) = s^{(1)}_{a} \phi = s^{(1)}_{a} C = 0, s^{(1)}_{a} B = s^{(1)}_{a} \bar{B} = 0,$$
$$s^{(1)}_{a} \bar{B} = -mC, s^{(1)}_{a} \bar{B} = +i\mathcal{B}, s^{(1)}_{a} (E - m\phi) = (C + m^2)C, s^{(1)}_{a} E = \Box C,$$
$$s^{(1)}_{b} A_{\mu} = -\epsilon_{\mu\nu}\partial^{\nu}C, s^{(1)}_{b} (\partial \cdot A) = s^{(1)}_{b} \phi = s^{(1)}_{b} C = 0, s^{(1)}_{b} B = s^{(1)}_{b} \bar{B} = 0,$$
$$s^{(1)}_{b} \bar{B} = -mC, s^{(1)}_{b} \bar{B} = -i\mathcal{B}, s^{(1)}_{b} (E - m\phi) = (C + m^2)C, s^{(1)}_{b} E = \Box C.$$

(35)

A few noteworthy points, at this stage, are as follows. First, we note that the total gauge-fixing term remains invariant $[s^{(1)}_{a} (\partial \cdot A + m\phi) = 0]$ which owes its origin to the exterior derivative $\delta = -d *$ (because $\delta A^{(1)} = (\partial \cdot A)$ and the extra term $m\phi$ has been added to it on the dimensional ground). Second, we note that the Lagrangian density $\mathcal{L}_{(b)}$ transforms, under the (anti-)co-BRST symmetry transformations, as follows:

$$s^{(1)}_{a} \mathcal{L}_{(b)} = \partial_{\mu} \left[ \mathcal{B}\partial^{\mu}C + m\epsilon^{\mu\nu} (mA_{\nu}C + \phi\partial_{\nu}C) + m\phi\partial^{\mu}C \right],$$
$$s^{(1)}_{a} \mathcal{L}_{(b)} = \partial_{\mu} \left[ \mathcal{B}\partial^{\mu}C + m\epsilon^{\mu\nu} (mA_{\nu}C + \phi\partial_{\nu}C) + m\phi\partial^{\mu}C \right].$$  

(36)

As a consequence, we observe that the action integral $S = \int d^2x \mathcal{L}_{(b)}$ respects the (anti-)co-BRST symmetry transformations $s^{(1)}_{a,b}$

It can be checked that the following (anti-)co-BRST symmetry transformation $s^{(2)}_{a,b}$ which are off-shell nilpotent $[(s^{(2)}_{a,b})^2 = 0]$ and absolutely anticommuting $s^{(2)}_{a} s^{(2)}_{b} = 0$ in nature, namely,

$$s^{(2)}_{a} A_{\mu} = -\epsilon_{\mu\nu}\partial^{\nu}C, s^{(2)}_{a} (\partial \cdot A) = s^{(2)}_{a} \phi = s^{(2)}_{a} C = 0, s^{(2)}_{a} B = s^{(2)}_{a} \bar{B} = 0,$$
$$s^{(2)}_{a} \bar{B} = +mC, s^{(2)}_{a} \bar{B} = +i\mathcal{B}, s^{(2)}_{a} (E + m\phi) = (C + m^2)C, s^{(2)}_{a} E = \Box C,$$
$$s^{(2)}_{b} A_{\mu} = -\epsilon_{\mu\nu}\partial^{\nu}C, s^{(2)}_{b} (\partial \cdot A) = s^{(2)}_{b} \phi = s^{(2)}_{b} C = 0, s^{(2)}_{b} B = s^{(2)}_{b} \bar{B} = 0,$$
$$s^{(2)}_{b} \bar{B} = +mC, s^{(2)}_{b} \bar{B} = -i\mathcal{B}, s^{(2)}_{b} (E + m\phi) = (C + m^2)C, s^{(2)}_{b} E = \Box C.$$  

(37)

leave the action integral $S = \int d^2x \mathcal{L}_{(b)}$ invariant because the Lagrangian density $\mathcal{L}_{(b)}$ transforms, under the above fermionic symmetry transformations $s^{(2)}_{a,b}$, as follows:

$$s^{(2)}_{a} \mathcal{L}_{(b)} = \partial_{\mu} \left[ \mathcal{B}\partial^{\mu}C + m\epsilon^{\mu\nu} (mA_{\nu}C - \phi\partial_{\nu}C) - m\phi\partial^{\mu}C \right],$$
$$s^{(2)}_{a} \mathcal{L}_{(b)} = \partial_{\mu} \left[ \mathcal{B}\partial^{\mu}C + m\epsilon^{\mu\nu} (mA_{\nu}C - \phi\partial_{\nu}C) - m\phi\partial^{\mu}C \right].$$  

(38)
because all the well-defined physical fields vanish at $x = \pm \infty$ due to the Gauss divergence theorem. We note that, once again, the gauge-fixing term for the Abelian 1-form gauge field, owing its origin to the coexterior derivative $\delta = -d \ast$ (with $\delta^2 = 0$), remains invariant $[s_{(a)id}^{(2)}(\partial \cdot A - m \phi) = 0]$ under the (anti-)co-BRST symmetry transformation $s_{(a)id}^{(2)}$.

As we have done for the (anti-)BRST symmetry transformations $s_{(a)id}^{(1,2)}$, it would be very interesting to find out the applications of $s_{(a)id}^{(1)}$ on the Lagrangian density $\mathcal{L}_{(b)}$, and $s_{(a)id}^{(2)}$ on the Lagrangian density $\mathcal{L}_{(b)}$. With the following inputs, namely,

\begin{align}
 s_{(a)id}^{(1)}(B) &= 0, s_{(a)id}^{(1)}(\tilde{B}) = 2 \Box C, s_{(a)id}^{(1)}(\tilde{B}) = 2 \Box C, \\
 s_{(a)id}^{(2)}(B) &= 0, s_{(a)id}^{(2)}(\tilde{B}) = 2 \Box C, s_{(a)id}^{(2)}(\tilde{B}) = 2 \Box C,
\end{align}

we obtain the following results:

\begin{align}
 s_{(a)id}^{(1)}(\mathcal{L}_{(b)}) &= \partial^\mu \left[ \mathcal{B} \partial^\nu C + m \epsilon_{\mu \nu} (m A_\nu C - \phi \partial_\nu C) + m \phi \partial^\nu C \right] \\
 &\quad - \left[ \mathcal{B} + \tilde{B} - 2 E \right] (\Box + m^2) C, \\
 s_{(a)id}^{(2)}(\mathcal{L}_{(b)}) &= \partial^\mu \left[ \tilde{\mathcal{B}} \partial^\nu C + m \epsilon_{\mu \nu} (m A_\nu C - \phi \partial_\nu C) + m \phi \partial^\nu C \right] \\
 &\quad - \left[ \mathcal{B} + \tilde{B} - 2 E \right] (\Box + m^2) C, \\
 s_{(a)id}^{(2)}(\mathcal{L}_{(b)}) &= \partial^\mu \left[ \tilde{\mathcal{B}} \partial^\nu C + m \epsilon_{\mu \nu} (m A_\nu C + \phi \partial_\nu C) - m \phi \partial^\nu C \right] \\
 &\quad - \left[ \mathcal{B} + \tilde{B} - 2 E \right] (\Box + m^2) C.
\end{align}

Thus, if we impose the restriction $(B + \tilde{B} - 2 E = 0)$ from Equation (33), we shall be able to note that the both the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ respect both the (anti-)co-BRST symmetry transformations $s_{(a)id}^{(1,2)}$. In other words, on the 2D hypersurface (defined by the restrictions (33)), the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ respect both the sets of (anti-)BRST and (anti-)co-BRST symmetries. These symmetries are off-shell nilpotent $[s_{(a)id}^{(1,2)}]_2 = 0$ and absolutely anticommuting in a couple of pairs (separately and independently). For a instance, we have the validity of the following:

\begin{align}
 s_{(a)id}^{(1)}(s_{(a)id}^{(1)}) &= 0, s_{(a)id}^{(2)}(s_{(a)id}^{(2)}) = 0, \\
 s_{(a)id}^{(1)}(s_{(a)id}^{(2)}) &= 0, s_{(a)id}^{(2)}(s_{(a)id}^{(1)}) = 0.
\end{align}

We discuss, in Appendix A, all the other combinations of the anticommutators which are not found to be true. Thus, we note that $\mathcal{L}_{(b)}$ supports well-defined (anti-)BRST symmetries $s_{(a)id}^{(1)}$ and (anti-)co-BRST symmetries $s_{(a)id}^{(2)}$. On the other hand, the well-defined (i.e., off-shell nilpotent and absolutely anticommuting) symmetry transformations $s_{(a)id}^{(2)}$ and $s_{(a)id}^{(1)}$ are respected by the Lagrangian density $\mathcal{L}_{(b)}$ in a perfect manner. However, it is observed that $s_{(a)id}^{(1)}$ and $s_{(a)id}^{(2)}$ are the symmetry transformations for $\mathcal{L}_{(b)}$ (as well as $s_{(a)id}^{(2)}$ and $s_{(a)id}^{(1)}$ are respected by $\mathcal{L}_{(b)}$) provided we invoke the (anti-)BRST and (anti-)co-BRST invariant restrictions $(B + \tilde{B} + 2(\partial \cdot A) = 0, \mathcal{B} + \tilde{\mathcal{B}} - 2 E = 0)$, namely,

\begin{align}
 s_{(a)id}^{(1)}(B + \tilde{B} + 2(\partial \cdot A)) &= 0, \\
 s_{(a)id}^{(1)}(\mathcal{B} + \tilde{\mathcal{B}} - 2 E) &= 0, \\
 s_{(a)id}^{(2)}(\mathcal{B} + \tilde{\mathcal{B}} - 2 E) &= 0,
\end{align}

which are the physical restrictions/conditions because of their invariance properties under the basic fermionic symmetries: $s_{(a)id}^{(1,2)}$. Furthermore, we lay emphasis on the fact that the restrictions in (33) also remain invariant under the discrete symmetry transformations (28). Hence, these restrictions are physical for our theory.

There are some discrete symmetries in our theory which provide the physical realizations of the Hodge duality operation of differential geometry. These are nothing but the generalization of discrete symmetries (21) that we have discussed in our previous section. We note the following discrete transformations, in this context, namely,

\begin{align}
 A_\mu &\longrightarrow \pm i \epsilon_{\mu \nu} A^\nu, \\
 \phi &\longrightarrow \pm i \phi, \\
 \dot{\phi} &\longrightarrow \pm i \dot{\phi}, \\
 C &\longrightarrow \mp i C, \\
 \mathcal{C} &\longrightarrow \mp i \mathcal{C}, \\
 B &\longrightarrow \pm i B, \\
 \mathcal{B} &\longrightarrow \pm i \mathcal{B}, \\
 \mathcal{B} &\longrightarrow \pm i \mathcal{B},
\end{align}

leave the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ invariant (separately and independently). It is evident that the transformations $(\partial \cdot A) \longrightarrow \mp i \mathcal{E}, \mathcal{E} \longrightarrow \mp i (\partial \cdot A)$ are true due to the discrete symmetry transformation $A_\mu \longrightarrow \pm i \epsilon_{\mu \nu} A^\nu$ on the basic gauge field $A_\mu$. As argued in the
previous section, the discrete symmetry transformations (43) also lead to

\[
\begin{align*}
{s^{(1)}_{(a)b}} & \longleftrightarrow s^{(1)}_{(a)d}, \\
{s^{(2)}_{(a)b}} & \longleftrightarrow s^{(2)}_{(a)d},
\end{align*}
\]

(44)
as can be explicitly checked by taking into account Equations (26), (27), (35), and (37). Furthermore, we also note that we have the validity of the following:

\[
\begin{align*}
{s^{(1)}_{(a)d}} &= \pm s^{(1)}_{(a)b}, \\
{s^{(2)}_{(a)b}} &= \pm s^{(2)}_{(a)d},
\end{align*}
\]

(45)
where * is the discrete symmetry transformations in (43).

4. Conserved Currents and Charges: Nilpotency and Absolute Anticommutativity Properties

In this section, first of all, we derive the conserved currents by exploiting the basic ideas behind the celebrated Noether theorem and deduce the simple forms of conserved charges corresponding to them. In this context, we first concentrate on the Lagrangian density \( \mathcal{L}_{(b)} \) and using the continuous, nilpotent \( [s^{(1)}_{(a)b}]^2 = 0, (s^{(1)}_{(a)d})^2 = 0 \) and absolutely anticommuting \( (s^{(1)}_{(a)b} + s^{(1)}_{(a)d}) = 0, s^{(1)}_{(a)b} s^{(1)}_{(a)d} = 0 \) (anti-)BRST and (anti-)co-BRST symmetry transformations [cf. Equations (26) and (35)], we derive the following Noether currents:

\[
\begin{align*}
J^{\mu}_{(ab)} &= -\varepsilon^{\mu\nu}\left( B + m\phi \right) \partial_{\nu}(B + m\phi) + B\partial^\mu \phi + mC\partial^\mu \phi - m^2 A^\mu C, \\
J^{\mu}_{(b)} &= -\varepsilon^{\mu\nu}\left( B + m\phi \right) \partial_{\nu}(B + m\phi) + B\partial^\mu \phi + mC\partial^\mu \phi - m^2 A^\mu C, \\
J^{\mu}_{(ad)} &= -\varepsilon^{\mu\nu}\left( B + m\phi \right) \partial_{\nu}(B + m\phi) + B\partial^\mu \phi + mC\partial^\mu \phi - m^2 A^\mu C, \\
J^{\mu}_{(d)} &= -\varepsilon^{\mu\nu}\left( B + m\phi \right) \partial_{\nu}(B + m\phi) + B\partial^\mu \phi + mC\partial^\mu \phi - m^2 A^\mu C.
\end{align*}
\]

(46)
The conservation law \( \delta_{\mu} J^{\mu}_{(r)} = 0 \), \( r = b, ab, d, ad \) of these Noether currents can be proven by the following Euler-Lagrange (EL) equations of motion (EOMs):

\[
\begin{align*}
\left( \square + m^2 \right) C &= 0, \quad \left( \square + m^2 \right) \phi &= 0, \\
\square \phi &= m(\partial \cdot A) + mB, \quad \phi &= mB - mE, \\
\varepsilon^{\mu\nu}(\partial_{\nu}B + m\partial_{\nu}\phi) &= \partial^\mu B + m^2 A^\mu - m\partial^\mu \phi = 0,
\end{align*}
\]

(47)
that are derived from the variation of the action integral w.r.t. Lagrangian density \( \mathcal{L}_{(b)} \).

The conserved currents of (46) lead to the following explicit expressions for the conserved charges \( (Q^{(1)}_{(r)} = \int dx f^{(0)}_{(r)}, r = b, ab, d, ad) \), namely,

\[
\begin{align*}
Q^{(1)}_{b} &= \int dx f^{(0)}_{(b)} = \int dx \left[ B\partial_{\gamma}C + m\phi(\partial_{\gamma}C) + B\phi + mC\phi - m^2 CA_{\gamma} \right], \\
Q^{(1)}_{ab} &= \int dx f^{(0)}_{(ab)} = \int dx \left[ B\partial_{\gamma}C + m\phi(\partial_{\gamma}C) + B\phi + mC\phi - m^2 CA_{\gamma} \right], \\
Q^{(1)}_{d} &= \int dx f^{(0)}_{(d)} = \int dx \left[ B\partial_{\gamma}C + m\phi(\partial_{\gamma}C) + B\phi + mC\phi - m^2 CA_{\gamma} \right], \\
Q^{(1)}_{ad} &= \int dx f^{(0)}_{(ad)} = \int dx \left[ B\partial_{\gamma}C + m\phi(\partial_{\gamma}C) + B\phi + mC\phi - m^2 CA_{\gamma} \right].
\end{align*}
\]

(48)
which reduce to the following simple forms by using the strength of EL-EOMs (47):

\[
\begin{align*}
Q^{(1)}_{b} &= \int dx \left[ B\partial_{\gamma}C - B\partial_{\gamma}C \right], \\
Q^{(1)}_{ab} &= \int dx \left[ B\partial_{\gamma}C - B\partial_{\gamma}C \right], \\
Q^{(1)}_{d} &= \int dx \left[ B\partial_{\gamma}C - B\partial_{\gamma}C \right], \\
Q^{(1)}_{ad} &= \int dx \left[ B\partial_{\gamma}C - B\partial_{\gamma}C \right].
\end{align*}
\]

(49)

We would like to point out that, in the derivation of (49), we have used the Gauss divergence theorem to drop all the total space derivatives of terms and we have used the expressions for \( B \) and \( B \) that are deduced from the last entry of EL-EOM in (47). The above conserved charges (48) and (49) are the generators of the continuous symmetry transformations (26) and (35) which can be verified using the generic definition (6) where we have to take into account \( G = (Q^{(1)}_{(ab)}, Q^{(1)}_{(ab)}) \) and the generic field \( \Psi = A_{\mu}, C, \phi, \phi \).

The absolute anticommutativity and off-shell nilpotency of the above conserved charges \( Q^{(1)}_{(r)}(r = b, ab, d, ad) \) can be proven. In this context, first of all, we prove the off-shell nilpotency property by using the following standard formula, namely:

\[
\begin{align*}
\bar{s}_{b} Q^{(1)}_{b} &= -i \left\{ Q^{(1)}_{b}, Q^{(1)}_{b} \right\} = 0, \\
\bar{s}_{d} Q^{(1)}_{d} &= -i \left\{ Q^{(1)}_{d}, Q^{(1)}_{d} \right\} = 0, \\
\bar{s}_{ab} Q^{(1)}_{ab} &= -i \left\{ Q^{(1)}_{ab}, Q^{(1)}_{ab} \right\} = 0, \\
\bar{s}_{ad} Q^{(1)}_{ad} &= -i \left\{ Q^{(1)}_{ad}, Q^{(1)}_{ad} \right\} = 0,
\end{align*}
\]

(50)
where we have exploited the basic definition of the generator for the continuous symmetry transformations (26) and (35). In the above proof, it is straightforward to use the continuous symmetry transformations (26) and (35) and apply them...
directly on the concise forms of the conserved charges (49). In other words, we have to use the l.h.s. of the equations given in (50). In exactly similar manner, to prove the absolute anticommutativity of the conserved charges \( Q_{(r)b}^{(1)} \), we take into account the following expressions:

\[
\begin{align*}
S_{b}^{(1)} Q_{ab}^{(1)} &= -i \left\{ Q_{ab}^{(1)}, Q_{b}^{(1)} \right\} = 0 \equiv s_{ab}^{(1)} Q_{b}^{(1)}, \\
S_{d}^{(1)} Q_{ad}^{(1)} &= -i \left\{ Q_{ad}^{(1)}, Q_{d}^{(1)} \right\} = 0 \equiv s_{ad}^{(1)} Q_{d}^{(1)}.
\end{align*}
\]

(51)

It is obvious that one can compute the expressions \( s_{b}^{(1)} Q_{ab}^{(1)}, s_{d}^{(1)} Q_{ad}^{(1)}, \) and \( s_{ad}^{(1)} Q_{d}^{(1)} \) from the direct applications of the transformations (26) and (35) to verify that the following anticommutativity properties of the conserved charges

\[
\begin{align*}
Q_{b}^{(1)} Q_{ab}^{(1)} + Q_{ab}^{(1)} Q_{b}^{(1)} &= 0, \\
Q_{d}^{(1)} Q_{ad}^{(1)} + Q_{ad}^{(1)} Q_{d}^{(1)} &= 0,
\end{align*}
\]

(52)

are satisfied. Thus, we have already demonstrated that, for the Lagrangian density \( \mathcal{L}_{(b)} \), the (anti-)BRST and (anti-)co-BRST charges obey the off-shell nilpotency and absolute anticommutativity properties in a perfect manner.

Now we focus on the (anti-)BRST and (anti-)co-BRST symmetries (cf. Equations (27) and (37)) that are associated with the Lagrangian density \( \mathcal{L}_{(b)} \). It can be checked that the Noether theorem leads to the following expressions for the currents \( \mathcal{J}_{(r)}^{(1)} \), \( r = b, ab, d, ad \), namely,

\[
\begin{align*}
\mathcal{J}_{(ab)}^{\mu} &= -\epsilon^{\nu\mu} \left( \partial_{\nu} - m \phi \right) \partial_{\beta} C + \partial_{\beta} \partial_{\mu} C - m C \partial_{\mu} \phi - m^{2} A^{\mu} C, \\
\mathcal{J}_{(b)}^{\mu} &= -\epsilon^{\nu\mu} \left( \partial_{\nu} - m \phi \right) \partial_{\beta} C + \partial_{\beta} \partial_{\mu} C - m C \partial_{\mu} \phi - m^{2} A C, \\
\mathcal{J}_{(ad)}^{\mu} &= -\epsilon^{\nu\mu} \left( m^{2} A_{\nu} C - m \phi \partial_{\nu} C + B \partial_{\nu} C \right) + \partial_{\nu} \partial_{\mu} C - m C \partial_{\mu} \phi, \\
\mathcal{J}_{(d)}^{\mu} &= -\epsilon^{\nu\mu} \left( m \phi \partial_{\nu} C + m^{2} A_{\nu} C + B \partial_{\nu} C \right) + \partial_{\nu} \partial_{\mu} C - m C \partial_{\mu} \phi,
\end{align*}
\]

(53)

where we have used the continuous symmetry transformations (27) and (37). The conservation law (i.e., \( \partial_{\mu} \mathcal{J}_{(r)}^{\mu} = 0, r = b, ab, d, ad \)) can be proven by using the following EL-EOMs:

\[
\begin{align*}
&\left( \square + m^{2} \right) C = 0, \quad \left( \square + m^{2} \right) \bar{B} = 0, \\
&\square \phi = -m \partial \cdot A - m \bar{B}, \quad \square \bar{\phi} = -m \bar{B} + m E, \\
&\epsilon^{\mu\nu} \left( m \partial_{\mu} \phi - m \bar{B} \right) - \partial_{\nu} \bar{B} + \partial_{\mu} A^{\mu} + m \partial_{\nu} \phi = 0,
\end{align*}
\]

(54)

which are derived from the Lagrangian density \( \mathcal{L}_{(b)} \). The above conserved currents \( \mathcal{J}_{(r)}^{(1)} \) (with \( r = b, ab, d, ad \)) lead to the following expressions for charges:

\[
\begin{align*}
Q_{b}^{(2)} &= \int dx \mathcal{J}_{(b)}^{0} = \int dx \left[ \bar{B} \partial_{1} C - m \bar{\phi} \partial_{1} C + \bar{B} \partial_{1} C + m C \partial_{1} \phi - m^{2} C A_{0} \right], \\
Q_{ab}^{(2)} &= \int dx \mathcal{J}_{(ab)}^{0} = \int dx \left[ \bar{B} \partial_{1} C - m \bar{\phi} \partial_{1} C + \bar{B} \partial_{1} C + m C \partial_{1} \phi - m^{2} C A_{0} \right], \\
Q_{d}^{(2)} &= \int dx \mathcal{J}_{(d)}^{0} = \int dx \left[ \bar{B} \partial_{1} C + B \partial_{1} C - m C \partial_{1} \phi + m^{2} A C - m \phi \partial_{1} C \right], \\
Q_{ad}^{(2)} &= \int dx \mathcal{J}_{(ad)}^{0} = \int dx \left[ \bar{B} \partial_{1} C + B \partial_{1} C - m C \partial_{1} \phi + m^{2} A C - m \phi \partial_{1} C \right],
\end{align*}
\]

(55)

which are the generators for the continuous symmetry transformations (27) and (37). This statement can be verified by replacing \( G \) by the charges \( Q_{(r)b}^{(2)}, Q_{(r)d}^{(2)} \) and the generic field \( \Psi \) by the fields \( A_{\mu}, C, \bar{B}, \partial_{1}, \phi, \phi \) in the basic definition (6).

The explicit expressions for the conserved charges (55) can be expressed in a concise form by using the following EOMs that are derived form (54), namely,

\[
\begin{align*}
\bar{B} &= \partial_{1} \bar{B} - m \partial_{1} \phi + m^{2} A_{0} + m \phi, \\
\bar{B} &= \partial_{1} \bar{B} - m \partial_{1} \phi + m^{2} A_{1} + m \phi.
\end{align*}
\]

(56)

At this stage, first of all, we use Gauss’s divergence theorem and drop all the total space derivative terms. After this, we use Equation (56). The substitutions of the above equations, in the explicit forms of the conserved charges (55), lead to the following:

\[
\begin{align*}
Q_{b}^{(2)} &= \int dx \left[ \bar{B} \partial_{1} C - \bar{B} C \right], \\
Q_{ab}^{(2)} &= \int dx \left[ \bar{B} \partial_{1} C - \bar{B} C \right], \\
Q_{d}^{(2)} &= \int dx \left[ \bar{B} \partial_{1} C - \bar{B} C \right], \\
Q_{ad}^{(2)} &= \int dx \left[ \bar{B} \partial_{1} C - \bar{B} C \right].
\end{align*}
\]

(57)

It is now straightforward to prove the off-shell nilpotency and absolute anticommutativity properties of the above charges by exploiting the basic ideas behind the relationship between the continuous symmetry transformations and their generators. For instance, it can be explicitly checked that the following are true, namely,

\[
\begin{align*}
S_{b}^{(2)} Q_{b}^{(2)} &= -i \left\{ Q_{b}^{(2)}, Q_{b}^{(2)} \right\} = 0, \\
S_{ab}^{(2)} Q_{ab}^{(2)} &= -i \left\{ Q_{ab}^{(2)}, Q_{ab}^{(2)} \right\} = 0, \\
S_{d}^{(2)} Q_{d}^{(2)} &= -i \left\{ Q_{d}^{(2)}, Q_{d}^{(2)} \right\} = 0, \\
S_{ad}^{(2)} Q_{ad}^{(2)} &= -i \left\{ Q_{ad}^{(2)}, Q_{ad}^{(2)} \right\} = 0.
\end{align*}
\]

(58)
In the above, we note that it is elementary exercise to compute the l.h.s. of the expressions directly by taking into account the continuous symmetry transformations ((27), (37)) and expressions for the conserved charges (57). At the level of the conserved charges, the relations in (58) imply the following relationships:

\[
\begin{aligned}
\left[ Q_{(a,b)}^{(2)} \right]^2 &= 0, \\
\left[ Q_{(a,d)}^{(2)} \right]^2 &= 0, \\
Q_{(a,b)}^{(2)} Q_{(a,d)}^{(2)} + Q_{(a,b)}^{(2)} Q_{(a,d)}^{(2)} &= 0, \\
Q_{(a,b)}^{(2)} Q_{(a,d)}^{(2)} + Q_{(a,b)}^{(2)} Q_{(a,d)}^{(2)} &= 0,
\end{aligned}
\]

which prove the off-shell nilpotency of the conserved charges along with the absolute anticommutativity between the pairs \((Q_{(a,b)}^{(2)}, Q_{(a,b)}^{(2)})\) and \((Q_{(a,b)}^{(2)}, Q_{(a,d)}^{(2)})\).

We end this section with the remarks that the pairs \((s_1^{(1)}, s_2^{(1)}), (s_1^{(1)}, s_2^{(2)}), (s_1^{(2)}, s_2^{(2)})\) and \((s_1^{(2)}, s_2^{(2)})\) anticommutate among themselves (separately and independently). However, it has been found that \(even s_1^{(1)} \) and \(s_2^{(2)}\) do not absolutely anticommutate with each other. We discuss all these, in detail, in Appendix A where we compute all the possible anticommutators among all this fermionic transformation operators \(s_{(a,b)}, s_{(a,d)}, s_{(a,d)}\), and \(s_{(a,d)}\). As a result of these observations, we find that the pairs of the conserved charges \((Q_{(a,b)}^{(1)}, Q_{(a,d)}^{(1)}), (Q_{(a,b)}^{(1)}, Q_{(a,d)}^{(2)}), (Q_{(a,b)}^{(2)}, Q_{(a,d)}^{(2)}), \) and \((Q_{(a,b)}^{(2)}, Q_{(a,d)}^{(2)})\) absolutely anticommutate but other possible pairs of the conserved charges do not absolutely anticommutate even if we impose the restrictions (33). These computations have been incorporated in Appendix B.

5. (Anti-)chiral Superfield Approach: Nilpotent Symmetries and Conserved Charges

To verify the sanctity of all the off-shell nilpotent and absolutely anticommuting (anti-)BRST and (anti-)co-BRST symmetry transformations, we exploit the potential and power of our newly proposed (anti-)chiral superfield approach to BRST formalism [22–25].

5.1. Off-Shell Nilpotent (Anti-)BRST Symmetries and Conserved Charges: (Anti-)chiral Superfield Formalism. First of all, we concentrate on the derivation of the off-shell nilpotent symmetries \(s_1^{(1)}\) for the Lagrangian density \(\mathcal{L}_{(b)}\). Towards this goal in mind, we generalize the 2D basic and auxiliary fields \(A_{\mu}, C, \phi, B, \mathcal{B}\) (onto a (2, 1)-dimensional antichiral supermanifold) as

\[
A_\mu(x) \longrightarrow B_\mu(x, \bar{\theta}) = A_\mu(x) + \bar{\theta} R_\mu(x), \quad C(x) \longrightarrow F(x, \bar{\theta}) = C(x) + i \bar{\theta} B_1(x),
\]

where the (2, 1)-dimensional antichiral super-submanifold is characterized by the superspace coordinates \(Z^M = (x^\mu, \bar{\theta})\). The bosonic coordinates \(x^\mu\) (with \(\mu = 0, 1\) describe the 2D Minkowskian spacetime manifold and \(\bar{\theta}\) is a fermionic \((\bar{\theta} = 0)\) Grassmannian variable. To be precise, the \((2, 1)\)-dimensional anti-chiral supermanifold is a super-submanifold of the general \((2, 2)\)-dimensional supermanifold (parameterized by \(Z^M = (x^\mu, \bar{\theta}, \tilde{\bar{\theta}})\) on which our 2D theory is generalized.

In the above expansions (60), the fields \((R_\mu, B_2, f_1, B_3, f_2, f_3, f_4)\) are called as the secondary fields which are to be determined in terms of the basic and auxiliary fields of our 2D theory (described by the Lagrangian densities \(\mathcal{L}_{(b)}\) and \(\mathcal{L}_{(b)}\)) by invoking one of the key ideas of the (anti-)chiral superfield formalism where we demand that all the BRST invariant quantities \((i.e., physical quantities)\) at the quantum level must be independent of the Grassmannian variable \(\bar{\theta}\) (which happens to be merely a mathematical artifact). The fermionic nature of \(\bar{\theta}\) ensures that \((R_\mu, f_2, f_3, f_4)\) are fermionic and \((B_2, B_3)\) are the bosonic secondary fields in the expansion (60) for all the basic and auxiliary antichiral superfields (defined on the \((2, 1)\)-dimensional antichiral super-submanifold of the general \((2, 2)\)-dimensional supermanifold as the generalizations of the 2D ordinary fields).

Towards our goal of determining the secondary fields in terms of the basic and auxiliary fields of the Lagrangian density \(\mathcal{L}_{(b)}\), we note that the following very useful and interesting quantities (which are obtained from the symmetry transformations (26)), namely,

\[
\begin{aligned}
s_1^{(1)} C &= s_1^{(1)} B = s_1^{(1)} \mathcal{B} = 0, s_1^{(1)} (m A_\mu - \partial_\mu \phi) = 0, s_1^{(1)} (C \phi) = 0, \\
s_2^{(1)} (A^\mu \partial_\mu B + i \bar{\theta} \partial_\mu \bar{\theta} C) &= 0, s_1^{(1)} (m C + i B) = 0, \\
s_2^{(1)} (B \bar{\theta} - \bar{\theta} B) &= 0, s_1^{(1)} (A^\mu \partial_\mu C) = 0, s_1^{(1)} (\phi) = 0,
\end{aligned}
\]

are BRST invariant. As a consequence, these useful and interesting quantities are physical at the quantum level (and, hence, at the classical level, they ought to be gauge invariant). Such quantities, according to the basic tenets of (anti-)chiral superfield approach to BRST formalism [22–25], must be independent of the Grassmannian \(\bar{\theta}\) variable. For instance,
we note that the following equalities are true, namely,

\[
\begin{align*}
    s_b^{(1)} C &= 0 \implies F^{(b)}(x, \overline{\theta}) = C(x) + \overline{\theta}(0) \equiv C(x) + \overline{\theta}(s_b^{(1)} C(x)), \\
    s_b^{(1)} B &= 0 \implies \tilde{B}^{(b)}(x, \overline{\theta}) = B(x) + \overline{\theta}(0) \equiv B(x) + \overline{\theta}(s_b^{(1)} B(x)), \\
    s_b^{(1)} \phi &= 0 \implies \tilde{\phi}^{(b)}(x, \overline{\theta}) = \phi(x) + \overline{\theta}(0) \equiv \phi(x) + \overline{\theta}(s_b^{(1)} \phi(x)), \\
    s_b^{(1)} \overline{\theta} &= 0 \implies \tilde{\overline{\theta}}^{(b)}(x, \overline{\theta}) = \overline{\theta}(x) + \overline{\theta}(0) \equiv \overline{\theta}(s_b^{(1)} \overline{\theta}(x)),
\end{align*}
\]

(62)

where the superscripts \((b)\) denote the \textit{antichiral} superfields that have been obtained after the applications of the BRST invariant restrictions on the antichiral superfields. In other words, we have taken into account \(F(x, \overline{\theta}) = C(x), B(x, \overline{\theta}) = B(x), \phi(x, \overline{\theta}) = \phi(x), \tilde{\overline{\theta}}(x) = \overline{\theta}(x)\) which lead to the precise determination of the secondary fields as \(B_1(x) = 0, f_1(x) = 0, f_2(x) = 0, \) and \(f_3(x) = 0.\) As a consequence, we have already determined \(s_b^{(1)} C = 0, s_b^{(1)} B = 0, s_b^{(1)} \phi = 0, s_b^{(1)} \overline{\theta} = 0\) which are nothing but the coefficients of \(\overline{\theta}\) in the expansions of the \textit{antichiral} superfields which have been obtained after the applications of the BRST invariant restrictions (61). In other words, we note that \(\partial_\overline{\theta} F^{(b)}(x, \overline{\theta}) = s_b^{(1)} C, \partial_\overline{\theta} B^{(b)}(x, \overline{\theta}) = s_b^{(1)} B, \partial_\overline{\theta} \phi^{(b)}(x, \overline{\theta}) = s_b^{(1)} \phi, \overline{\partial}_\theta B^{(b)}(x, \overline{\theta}) = s_b^{(1)} B \) which physically imply that the translations of the antichiral superfields (with superscripts \((b)\)) along \(\overline{\theta}\)-direction of the \((2, 1)\)-dimensional antichiral super-submanifold generates the BRST symmetry transformations for the corresponding \textit{ordinary} 2D fields (defined on the \((1 + 1)\)-dimensional (2D) \textit{ordinary} flat Minkowskian spacetime manifold).

We discuss a bit more about the determination of \textit{secondary} fields in terms of the basic and auxiliary fields of our 2D theory described by the Lagrangian density \(L_{(b)}\) (cf. Equation (25)). It is elementary to check that the following equalities

\[
\begin{align*}
    s_b^{(1)} (\phi C) &= 0 \implies \Phi(x, \overline{\theta}) F^{(b)}(x, \overline{\theta}) = \phi(x) C(x), \\
    s_b^{(1)} (A^\mu \partial_\mu C) &= 0 \implies B^\mu(x, \overline{\theta}) \partial_\mu F^{(b)}(x, \overline{\theta}) = A^\mu(x) \partial_\mu C(x),
\end{align*}
\]

(63)

lead to the nontrivial solutions \(R_\mu = \kappa_1 \partial_\mu C \text{ and } f_1 = \kappa_2 C(x)\) where \(\kappa_1\) and \(\kappa_2\) are some numerical constants. With these as inputs, we now observe the following:

\[
\begin{align*}
    B^{(m)}_\mu(x, \overline{\theta}) &= A^\mu(x) + \overline{\theta}(\kappa_1 \partial_\mu C(x)), \Phi^{(m)}(x, \overline{\theta}) = \phi(x) + \overline{\theta}(\kappa_2 C(x)),
\end{align*}
\]

(64)

where the superscript \((m)\) on the \textit{antichiral} superfields denotes the \textit{modified} version of the antichiral superfields \(B^\mu(x, \overline{\theta})\) and \(\Phi(x, \overline{\theta})\). At this stage, we utilize

\[
\begin{align*}
    s_b^{(1)} (m A_\mu - \partial_\mu \phi) &= 0 \implies m B^{(m)}_\mu(x, \overline{\theta}) - \partial_\mu \Phi^{(m)}(x, \overline{\theta}) \\
    &= m A_\mu(x) - \partial_\mu \phi(x),
\end{align*}
\]

(65)

which leads to a relationship between \(\kappa_1\) and \(\kappa_2\) as: \(m \kappa_1 = \kappa_2.\) Finally, the other BRST invariant quantities and their generalizations onto the \((2, 1)\)-dimensional \textit{antichiral} supermanifolds imply the following restrictions on the superfields:

\[
\begin{align*}
    s_b^{(1)} [m C C - i B \phi] &= 0 \implies m F(x, \overline{\theta}) F^{(b)}(x, \overline{\theta}) \\
    &= m C(x) C(x) - i B(x) \phi(x), \\
    s_b^{(1)} [A^\mu \partial_\mu B + i \partial_\mu \overline{C} \partial^\mu C] &= 0 \implies B^{(b)}_\mu(x, \overline{\theta}) \partial_\mu F^{(b)}(x, \overline{\theta}) \\
    &+ i \partial_\mu F(x, \overline{\theta}) \partial_\mu C(x) \\
    &= A^\mu(x) \partial_\mu B(x) + i \partial_\mu \overline{C} \partial^\mu C(x), \\
    B^{(b)}(x, \overline{\theta}) \tilde{F}(x, \overline{\theta}) - \tilde{B}^{(b)}(x, \overline{\theta}) \hat{F}(x) &= B(x) \hat{C}(x) - B(x) \hat{C}(x),
\end{align*}
\]

(66)

which lead to the derivation of constants and all the secondary fields in terms of the basic and auxiliary fields of the Lagrangian density \(L_{(b)}\) (cf. Equation (25)) as:

\[
\begin{align*}
    \kappa_1 &= 1, \\
    \kappa_2 &= m, \\
    R_\mu(x) &= \partial_\mu C(x), \\
    B_2(x) &= B(x), \\
    f_1(x) &= m C(x).
\end{align*}
\]

(67)

As a side remark, we would like to mention that, in Appendix C, we determine the value of constant \(\kappa_1 = +1\) in an explicit fashion. As a consequence of the above, we have the following super expansions:

\[
\begin{align*}
    B^{(b)}_\mu(x, \overline{\theta}) &= A_\mu(x) + \overline{\theta}(\partial_\mu C) \equiv A_\mu(x) + \overline{\theta}(s_b^{(1)} A_\mu(x)), \\
    \tilde{F}^{(b)}(x, \overline{\theta}) &= \hat{C}(x) + \overline{\theta}(i B) \equiv \hat{C}(x) + \overline{\theta}(s_b^{(1)} \hat{C}(x)), \\
    \Phi^{(b)}(x, \overline{\theta}) &= \phi(x) + \overline{\theta}(m C) \equiv \phi(x) + \overline{\theta}(s_b^{(1)} \phi(x)),
\end{align*}
\]

(68)

in addition to Equation (62). Thus, we have derived all the BRST symmetry transformations for \(L_{(b)}\) and proven their sanctity within the framework of (anti-)chiral superfield approach.

For the derivation of the off-shell nilpotent anti-BRST symmetry transformations \(s_b^{(1)}\), we generalize the basic and
auxiliary fields \( A_\mu, C, \tilde{C}, \phi, \tilde{\phi}, B, \mathcal{B} \) of the theory (onto a (2, 1)-dimensional \textit{chiral} super-submanifold) as follows:

\[
A_\mu(x) \longrightarrow B_\mu(x, \theta) = A_\mu(x) + \theta \tilde{R}_\mu(x), \quad C(x) \longrightarrow F(x, \theta) = C(x) + i \theta \tilde{B}_1(x),
\]

\[
\tilde{C}(x) \longrightarrow \tilde{F}(x, \theta) = \tilde{C}(x) + i \theta \tilde{B}_2(x), \quad \phi(x) \longrightarrow \Phi(x, \theta) = \phi(x) + \theta \tilde{f}_1(x),
\]

\[
\tilde{\phi}(x) \longrightarrow \tilde{\Phi}(x, \theta) = \tilde{\phi}(x) + \theta \tilde{f}_2(x), \quad B(x) \longrightarrow \tilde{B}(x, \theta) = B(x) + \theta \tilde{f}_3(x),
\]

\[
\mathcal{B}(x) \longrightarrow \tilde{\mathcal{B}}(x, \theta) = \mathcal{B}(x) + \theta \tilde{f}_4(x),
\]

(69)

where the superspace coordinates \( Z^M = (x^\mu, \theta) \) characterize the (2, 1)-dimensional \textit{chiral} supermanifold. Here \( x^\mu \) (with \( \mu = 0, 1 \)) are the 2D bosonic coordinates and \( \theta \) is a \textit{fermionic} (i.e., \( \theta^2 = 0 \)) Grassmannian variable. The secondary fields \( (\tilde{R}_\mu, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4) \) are fermionic in nature, whereas \( (B_1, B_2) \) are bosonic (due to the fermionic nature of \( \theta \)). To determine the secondary fields in terms of the \textit{basic} and \textit{auxiliary} fields of the Lagrangian density \( L_{(b)} \), we obtain the following useful and interesting anti-BRST invariant quantities:

\[
s_{ab}^{(1)} \tilde{C} = s_{ab}^{(1)} B = 0, \quad s_{ab}^{(1)} (m A_\mu - \partial_\mu \phi) = 0, \quad s_{ab}^{(1)} (\tilde{C} \phi) = 0,
\]

\[
s_{ab}^{(1)} \left[ A^{i\mu} \partial_\mu B + i \partial_\mu \tilde{C} \partial_\mu C \right] = 0, \quad s_{ab}^{(1)} \left[ m \tilde{C} C - i B \phi \right] = 0,
\]

\[
s_{ab}^{(1)} \left( B \tilde{C} - \tilde{B} C \right) = 0, \quad s_{ab}^{(1)} \left( A^{i} \partial_\mu \tilde{C} \right) = 0, \quad s_{ab}^{(1)} (\tilde{\phi}) = 0.
\]

(70)

Following the \textit{basic tenets of (anti)-chiral superfield approach to BRST formalism}, we note that the following restrictions have to be imposed on the \textit{chiral} superfields:

\[
\tilde{F}(x, \theta) = \tilde{C}(x), \quad \tilde{B}(x, \theta) = B(x), \quad \tilde{\mathcal{B}}(x, \theta) = \mathcal{B}(x),
\]

\[
m B_\mu(x, \theta) - \partial_\mu \Phi(x, \theta) = m A_\mu(x) - \partial_\mu \phi(x, \theta) = \tilde{\phi}(x),
\]

\[
B^{\mu}(x, \theta) \partial_\mu \tilde{F}(x, \theta) = A^{i\mu}(x) \partial_\mu \tilde{C}(x), \quad \tilde{F}(x, \theta) \Phi(x, \theta) = \tilde{C}(x) \phi(x),
\]

\[
B^{\mu}(x, \theta) \partial_\mu \tilde{B}(x, \theta) + i \partial_\mu \tilde{F}(x, \theta) = A^{i\mu}(x) \partial_\mu \tilde{B}(x, \theta) + i \partial_\mu \tilde{F}(x, \theta) = A^{i\mu}(x) \partial_\mu \tilde{B}(x, \theta) + i \partial_\mu \tilde{F}(x, \theta).
\]

(71)

The above restrictions are \textit{physical} because of the fact that any anti-BRST invariant quantity (at the \textit{quantum} level) is a gauge invariant quantity (at the \textit{classical} level). Hence, such quantities should be independent of the \textit{mathematical} quantity \( \theta \) (as this Grassmannian variable is \textit{not} a physical quantity \textit{but} it is a purely mathematical artifact).

The equalities in (71) lead to the determination of secondary fields, in terms of the auxiliary and basic fields of the Lagrangian density \( L_{(b)} \), as follows:

\[
\tilde{R}_\mu = \partial_\mu \tilde{C}, \quad \tilde{B}_1 = -B, \quad \tilde{f}_1 = m C, \quad \tilde{f}_2 = \tilde{f}_3 = \tilde{f}_4 = B_2 = 0.
\]

(72)

The above deduction has been performed on exactly similar lines of arguments (see, e.g., Appendix C) as we have done for the determination of the BRST symmetry \( (s_b^{(1)}) \). The substitutions of \textit{all} the secondary fields into the expansion (69) lead to the following:

\[
B^{(ab)}(x, \theta) = A_\mu(x) + \theta (\partial_\mu \tilde{C}) \equiv A_\mu(x) + \theta \left( s_{ab}^{(1)} A_\mu(x) \right),
\]

\[
\tilde{F}^{(ab)}(x, \theta) = C(x) + \theta (-i B) \equiv C(x) + \theta \left( s_{ab}^{(1)} C(x) \right),
\]

\[
\tilde{F}^{(ab)}(x, \theta) = C(x) + \theta (0) \equiv C(x) + \theta \left( s_{ab}^{(1)} C(x) \right),
\]

\[
\Phi^{(ab)}(x, \theta) = \phi(x) + \theta (m C) \equiv \phi(x) + \theta \left( s_{ab}^{(1)} \phi(x) \right),
\]

\[
\tilde{\phi}^{(ab)}(x, \theta) = \tilde{\phi}(x) + \theta (0) \equiv \tilde{\phi}(x) + \theta \left( s_{ab}^{(1)} \tilde{\phi}(x) \right),
\]

\[
B^{(ab)}(x, \theta) = B(x) + \theta (0) \equiv B(x) + \theta \left( s_{ab}^{(1)} B(x) \right),
\]

\[
\tilde{\mathcal{B}}^{(ab)}(x, \theta) = \tilde{\mathcal{B}}(x) + \theta (0) \equiv \tilde{\mathcal{B}}(x) + \theta \left( s_{ab}^{(1)} \tilde{\mathcal{B}}(x) \right).
\]

(73)

where the anti-BRST symmetry transformations \( (s_{ab}^{(1)}) \) have been listed in Equation (26) and they appear on the r.h.s. of the super expansions of all the \textit{chiral} superfields of our theory as the coefficient of \( \theta \). Hence, we conclude that we have derived \textit{all} the anti-BRST symmetry transformations (26) and we have obtained a relationship and a mapping

\[
\frac{\partial \Omega^{(ab)}(x, \theta)}{\partial \theta} \equiv s_{ab}^{(1)} \omega(x), \quad s_{ab}^{(1)} \longleftrightarrow \frac{\partial}{\partial \theta},
\]

(74)

which illustrate that the anti-BRST symmetry transformations for the \textit{ordinary} generic field \( \omega(x) \) are nothing but the translation of the generic \textit{chiral} superfields \( (\Omega^{(ab)}(x, \theta)) \), derived after the application of the anti-BRST invariant restrictions (71), along the \( \theta \)-direction of the (2, 1)-dimensional \textit{chiral} super-submanifold. Hence, we have established
the mapping $\partial_\theta \longrightarrow s_{ab}^{(1)}$ which implies that the nilpotency $(s_{ab}^{(1)})^2 = 0$ of the anti-BRST symmetry $(s_{ab}^{(1)})$ is due to the nilpotency $(\partial_\theta^2 = 0)$ of the translational generator $(\partial_\theta)$.

At this stage, we wish to capture the off-shell nilpotency and absolute anticommutativity of the conserved (anti-)BRST charges $Q_{(a,b)}^{(1)}$, that have been expressed in a concise form in Equation (49). Taking the helps from the expansions (62), (68), and (73), it can be checked that we have the following expressions:

$$Q_{ab}^{(1)} = \frac{\partial}{\partial \bar{\theta}} \int d^{D-1}x \left[ i \bar{\Phi}^{(b)}(x, \bar{\theta}) F^{(b)}(x, \bar{\theta}) - i \bar{\Phi}^{(b)}(x, \bar{\theta}) F^{(b)}(x, \bar{\theta}) \right],$$

$$Q_{ab}^{(1)} = \frac{\partial}{\partial \bar{\theta}} \int d^{D-1}x \left[ i \bar{\Phi}^{(b)}(x, \bar{\theta}) F^{(b)}(x, \bar{\theta}) - i \bar{\Phi}^{(b)}(x, \bar{\theta}) F^{(b)}(x, \bar{\theta}) \right].$$

Taking the help of the basic principles behind the definition of a generator for the corresponding continuous symmetry transformation (cf. Equation (6)), we obtain the following:

$$s_{ab}^{(1)} = -i \left\{ Q_{ab}^{(1)}, Q_{ab}^{(1)} \right\} = 0 \implies (s_{ab}^{(1)})^2 = 0,$$

$$s_{b}^{(1)} Q_{ab}^{(1)} = -i \left\{ Q_{b}^{(1)}, Q_{ab}^{(1)} \right\} = 0 \implies (s_{b}^{(1)})^2 = 0,$$

$$s_{b}^{(1)} Q_{ab}^{(1)} = -i \left\{ Q_{b}^{(1)}, Q_{ab}^{(1)} \right\} = 0 \implies (s_{b}^{(1)} Q_{ab}^{(1)})^2 = 0,$$

$$s_{b}^{(1)} Q_{ab}^{(1)} = -i \left\{ Q_{b}^{(1)}, Q_{ab}^{(1)} \right\} = 0 \implies Q_{b}^{(1)} Q_{ab}^{(1)} + Q_{b}^{(1)} (s_{b}^{(1)}) = 0.$$

Thus, we note that we have captured the off-shell nilpotency and absolute anticommutativity of the conserved charges within the framework of our newly proposed (see, e.g., [22–25]) (anti-)chiral superfield approach to BRST formalism (cf. Equations (77) and (78)).

Against the backdrop of the above discussions, we concentrate now on the derivation of (anti-)BRST symmetries $s_{(a,b)}^{(2)}$ for the Lagrangian density $L_{(b)}$ within the framework of (anti-)chiral superfield approach to BRST formalism [22–25]. For this purpose, first of all, we take into account the (anti-)chiral superfield expansions given in (69) and (60) with the following replacements: $B(x) \longrightarrow B(x)$, $\mathcal{B}(x) \longrightarrow \mathcal{B}(x)$, $B(x, \bar{\theta}) \longrightarrow \dot{B}(x, \bar{\theta})$, $B(x, \bar{\theta}) \longrightarrow \dot{B}(x, \bar{\theta})$, $\mathcal{B}(x, \bar{\theta}) \longrightarrow \dot{\mathcal{B}}(x, \bar{\theta})$. We note that the secondary fields in the expansions (69) and (60) remain the same. For the derivation of the (anti-)BRST symmetry transformations $s_{(a,b)}^{(2)}$, we check that the following are the (anti-)BRST invariant quantities:

$$s_{ab}^{(2)} C = s_{ab}^{(2)} B = s_{ab}^{(2)} \mathcal{B} = 0, s_{ab}^{(2)} \left( m A_{\mu} + \partial_{\mu} \phi \right) = 0, s_{ab}^{(2)} (\dot{C} \phi) = 0,$$

$$s_{ab}^{(2)} \left[ A^\mu \partial_\mu B + i \partial_{\mu} C \partial^\mu C \right] = 0, s_{ab}^{(2)} \left[ m C C + i B \phi \right] = 0,$$

$$s_{ab}^{(2)} \left( \dot{B} \dot{C} - \dot{C} \dot{B} \right) = 0, s_{ab}^{(2)} \left( A^\mu \partial_\mu \dot{C} \right) = 0, s_{ab}^{(2)} \left( \dot{\phi} \right) = 0.$$
\[ s^{(2)}_{\theta} C = s^{(2)}_{\theta} \vec{B} = s^{(2)}_{\theta} \vec{\omega} = 0, s^{(2)}_{\theta} (m A_{\mu} + \partial_\mu \phi) = 0, s^{(2)}_{\theta} (C \phi) = 0, \]
\[ s^{(2)}_{\theta} [A^\mu \partial_\mu \vec{B} + i \partial_\mu C \vec{B} \partial^\mu C] = 0, s^{(2)}_{\theta} [m C C + i \vec{B} \phi] = 0, \]
\[ s^{(2)}_{\theta} \left( \vec{B} \hat{C} - \hat{B} \vec{C} \right) = 0, s^{(2)}_{\theta} (A^\mu \partial_\mu C) = 0, s^{(2)}_{\theta} (\phi) = 0. \]

\[ (80) \]

According to the basic tenets of the (anti-)chiral superfield approach, first of all, the anti-BRST invariant quantities (79) have to be generalized onto the chiral (2, 1)-dimensional super-submanifold and BRST invariant quantities (80) have to be generalized onto (2, 1)-dimensional antichiral super-submanifold. After that, we demand the following restrictions on the chiral superfields for the derivation of exact \( s^{(2)}_{\theta} \), namely,

\[ \tilde{F}(x, \theta) = \tilde{C}(x), \tilde{B}(x, \theta) = \tilde{B}(x), \tilde{\omega}(x, \theta) = \tilde{\omega}(x), \]
\[ m B_\mu(x, \theta) + \partial_\mu \Phi(x, \theta) = m A_\mu(x) \]
\[ + \partial_\mu \phi(x, \theta), \tilde{F}(x, \theta) = \tilde{C}(x) \phi(x), \]
\[ B^\mu(x, \theta) \partial_\mu \tilde{F}(x, \theta) = A^\mu(x) \partial_\mu \tilde{C}(x) + \partial^{\mu} C(x) \partial^{\mu} \tilde{C}(x), \]
\[ m \tilde{F}(x, \theta) F(x, \theta) + \frac{i}{2} \tilde{B}(x, \theta) \Phi(x, \theta) = m \tilde{C}(x) C(x) + i \tilde{B}(x) \phi(x), \]
\[ \tilde{B}(x, \theta) \tilde{F}(x, \theta) - \tilde{F}(x, \theta) \tilde{B}(x, \theta) F(x, \theta) = \tilde{B}(x) \tilde{C}(x) - \tilde{B}(x) C(x). \]

\[ (81) \]

The arguments for the derivation of the secondary fields, in terms of the basic and auxiliary fields of the Lagrangian density \( L(b_i) \), go along the similar lines as we have done for the derivations of \( s^{(1)}_{ab} \). We, ultimately, obtain the following (see also, e.g., Appendix C):

\[ \tilde{R}_\mu = \partial_\mu \tilde{C}, \]
\[ \tilde{B}_i = -B_i, \]
\[ \tilde{f}_1 = -m \tilde{C}, \]
\[ \tilde{f}_2 = \tilde{f}_3 = \tilde{f}_4 = \tilde{B}_2 = 0. \]

\[ (82) \]

The substitutions of these secondary fields into the appropriate super expansions of the chiral superfields lead to the following:

\[ B^{(AB)}_\mu(x, \theta) = A_\mu(x) + \theta \left( \partial_\mu \tilde{C} \right) = A_\mu(x) + \theta \left( s^{(2)}_{ab} A_\mu(x) \right), \]
\[ F^{(AB)}_\mu(x, \theta) = C(x) + \theta \left( -i \tilde{B} \right) \equiv C(x) + \theta \left( s^{(2)}_{ab} C(x) \right), \]
\[ \tilde{F}^{(AB)}_\mu(x, \theta) = \tilde{C}(x) + \theta (0) \equiv \tilde{C}(x) + \theta \left( s^{(2)}_{ab} \tilde{C}(x) \right), \]
\[ \phi^{(AB)}_\mu(x, \theta) = \phi(x) + \theta (0) \equiv \phi(x) + \theta \left( s^{(2)}_{ab} \phi(x) \right), \]
\[ \tilde{\phi}^{(AB)}_\mu(x, \theta) = \tilde{\phi}(x) + \theta (0) \equiv \tilde{\phi}(x) + \theta \left( s^{(2)}_{ab} \tilde{\phi}(x) \right), \]
\[ \tilde{s}^{(AB)}_\mu(x, \theta) = \tilde{B}(x) + \theta (0) \equiv \tilde{B}(x) + \theta \left( s^{(2)}_{ab} \tilde{B}(x) \right), \]
\[ \tilde{\omega}^{(AB)}_\mu(x, \theta) = \tilde{\omega}(x) + \theta (0) \equiv \tilde{\omega}(x) + \theta \left( s^{(2)}_{ab} \tilde{\omega}(x) \right), \]

\[ (83) \]

where on the r.h.s., we have found the coefficients of \( \theta \) as the anti-BRST symmetry transformations \( s^{(2)}_{ab} \) that have been listed in Equation (27). In other words, we have already derived the anti-BRST symmetry transformations \( s^{(2)}_{ab} \) for the Lagrangian density \( L(b_i) \). We also note that superscript \((AB)\) on the chiral superfields (cf. l.h.s. of (83)) denotes the superfields that have been obtained after the applications of the restrictions (81).

For the derivation of the BRST symmetry transformations \( s^{(2)}_{\theta} \), we generalize the BRST invariant quantities (80) onto (2, 1)-dimensional antichiral super-submanifold and invoke the following restrictions on the antichiral superfields:

\[ F(x, \tilde{\theta}) = C(x), \tilde{B}(x, \tilde{\theta}) = \tilde{B}(x), \tilde{\omega}(x, \tilde{\theta}) = \tilde{\omega}(x), \]
\[ m B_\mu(x, \tilde{\theta}) + \partial_\mu \Phi(x, \tilde{\theta}) = m A_\mu(x) \]
\[ + \partial_\mu \phi(x, \tilde{\theta}), F(x, \tilde{\theta}) = C(x) \phi(x), \]
\[ B^\mu(x, \tilde{\theta}) \partial_\mu \tilde{F}(x, \tilde{\theta}) = A^\mu(x) \partial_\mu C(x), F(x, \tilde{\theta}) \Phi(x, \tilde{\theta}) \]
\[ = C(x) \phi(x), \]
\[ B^\mu(x, \tilde{\theta}) \partial_\mu \tilde{B}(x, \tilde{\theta}) + i \partial_\mu \tilde{F}(x, \tilde{\theta}) = A^\mu(x) \partial_\mu \tilde{B}(x) + i \partial_\mu \tilde{C}(x) \partial^\mu \tilde{C}(x), \]
\[ m \tilde{F}(x, \tilde{\theta}) F(x, \tilde{\theta}) + i \tilde{B}(x, \tilde{\theta}) \Phi(x, \tilde{\theta}) \]
\[ = m \tilde{C}(x) C(x) + i \tilde{B}(x) \phi(x), \]
\[ \tilde{F}(x, \tilde{\theta}) \tilde{B}(x, \tilde{\theta}) - \tilde{B}(x, \tilde{\theta}) \tilde{F}(x, \tilde{\theta}) = \tilde{B}(x) \tilde{C}(x) - \tilde{B}(x) C(x). \]

\[ (84) \]

The above restrictions lead to the determination of the secondary fields of the appropriate antichiral superfields.
(cf. Equation (60)), in the terms of the basic and auxiliary fields of the Lagrangian density $L_{(b)}$, as follows:

$$
R_{\mu} = \partial_{\mu} C, \\
B_{2} = B, \\
f_{1} = -m C, \\
f_{2} = f_{3} = f_{4} = B_{1} = 0.
$$

(85)

In the derivation of (85), the arguments and discussions have been taken on the similar lines as that in the context of the derivation of $\tilde{s}_{b}^{(1)}$ (cf. Appendix C, too). The substitutions of (85), into the appropriate super expansions of the antichiral superfields, lead to

$$
\begin{align*}
B_{\mu}^{(B)}(x, \tilde{\theta}) &= A_{\mu}(x) + \tilde{\theta} (\partial_{\mu} C) \equiv A_{\mu}(x) + \tilde{\theta} \left( s_{b}^{(2)} A_{\mu}(x) \right), \\
F^{(B)}(x, \tilde{\theta}) &= C(x) + \tilde{\theta}(0) \equiv C(x) + \tilde{\theta} \left( s_{b}^{(2)} C(x) \right), \\
\tilde{F}^{(B)}(x, \tilde{\theta}) &= \tilde{C}(x) + \tilde{\theta}(0) \equiv \tilde{C}(x) + \tilde{\theta} \left( s_{b}^{(2)} \tilde{C}(x) \right), \\
\tilde{\Phi}^{(B)}(x, \tilde{\theta}) &= \phi(x) + \tilde{\theta}(-m C) \equiv \phi(x) + \tilde{\theta} \left( s_{b}^{(2)} \phi(x) \right), \\
\tilde{\phi}^{(B)}(x, \tilde{\theta}) &= \tilde{\phi}(x) + \tilde{\theta}(0) \equiv \tilde{\phi}(x) + \tilde{\theta} \left( s_{b}^{(2)} \tilde{\phi}(x) \right), \\
\tilde{B}_{2}^{(B)}(x, \tilde{\theta}) &= \tilde{B}(x) + \tilde{\theta}(0) \equiv \tilde{B}(x) + \tilde{\theta} \left( s_{b}^{(2)} \tilde{B}(x) \right), \\
\tilde{\Phi}^{(B)}(x, \tilde{\theta}) &= \tilde{\Phi}(x) + \tilde{\theta}(0) \equiv \tilde{\Phi}(x) + \tilde{\theta} \left( s_{b}^{(2)} \tilde{\Phi}(x) \right),
\end{align*}
$$

(86)

where on the r.h.s. of (86), we have obtained the BRST symmetry transformation $s_{b}^{(2)}$ as the coefficients of $\tilde{\theta}$ (which have been quoted in Equation (27)). The superfield $(B)$ on the antichiral superfields denotes the superfields that have been obtained after the applications of the BRST invariant restrictions (84) and which lead to the determination of the BRST symmetry transformation $s_{b}^{(2)}$ as the coefficients of $\tilde{\theta}$ in their super expansions.

Against the backdrop of the super expansions (83) and (86), we capture the off-shell nilpotency and absolute anticommutativity of the conserved charges $Q_{(a|b)}$ which are associated with the Lagrangian density $\mathcal{L}_{(b)}$. For this purpose, we take into account the concise forms of the nilpotent and conserved (anti-)BRST charges $Q_{(a|b)}^{(2)}$ that are listed in Equation (57). It can be checked that we have the following expressions for $Q_{(a|b)}^{(2)}$ (cf. Equation (57)) in terms of the superfields (derived in Equations (83) and (86)), Grassmannian differentials ($d\tilde{\theta}, d\theta$), and corresponding partial derivatives ($\partial_{\theta}, \partial_{\tilde{\theta}}$), namely,

$$
\begin{align*}
Q_{b}^{(2)} &= \frac{\partial}{\partial \theta} \left[ d^{3-1} \left[ i \tilde{F}^{(B)}(x, \theta) F^{(B)}(x, \theta) \right] \\
&- i \tilde{F}^{(B)}(x, \theta) \tilde{F}^{(B)}(x, \theta) \right], \\
&= \frac{\partial}{\partial \theta} \left[ d^{3-1} \left[ i \tilde{F}^{(B)}(x, \theta) F^{(B)}(x, \theta) \right] \\
&- i \tilde{F}^{(B)}(x, \theta) \tilde{F}^{(B)}(x, \theta) \right].
\end{align*}
$$

(87)

where the superfields with superscripts $(B)$ and $(AB)$ have already been explained earlier. A close look at (87) implies that we have $already$ the following (due to $\partial_{\theta} = \partial_{\tilde{\theta}} = 0$):

$$
\begin{align*}
\partial_{\theta} Q_{b}^{(2)} &= 0, \\
\partial_{\tilde{\theta}} Q_{b}^{(2)} &= 0, \\
\partial_{\theta} Q_{ab}^{(2)} &= 0, \\
\partial_{\tilde{\theta}} Q_{ab}^{(2)} &= 0.
\end{align*}
$$

(88)

These relations are crucial for capturing the off-shell nilpotency and absolute anticommutativity of the charges $Q_{(a|b)}^{(2)}$ in view of the observations that $\partial_{\theta} \leftrightarrow s_{ab}^{(2)}, \partial_{\tilde{\theta}} \leftrightarrow s_{b}^{(2)}$. To be more precise, it can be checked that the relationships of (88) can be expressed, in the ordinary 2D spacetime in terms of the (anti-)BRST symmetry transformations $(s_{(a|b)}^{(2)})$, as follows:

$$
\begin{align*}
\begin{array}{c}
s_{b}^{(2)} Q_{b}^{(2)} = -i \left\{ Q_{b}^{(2)}, Q_{b}^{(2)} \right\} = 0 \implies \left[ Q_{b}^{(2)} \right]^{2} = 0, \\
s_{ab}^{(2)} Q_{ab}^{(2)} = -i \left\{ Q_{ab}^{(2)}, Q_{ab}^{(2)} \right\} = 0 \implies \left[ Q_{ab}^{(2)} \right]^{2} = 0, \\
s_{ab}^{(2)} Q_{b}^{(2)} = -i \left\{ Q_{b}^{(2)}, Q_{ab}^{(2)} \right\} = 0 \implies Q_{b}^{(2)} Q_{ab}^{(2)} + Q_{ab}^{(2)} Q_{b}^{(2)} = 0, \\
s_{b}^{(2)} Q_{ab}^{(2)} = -i \left\{ Q_{b}^{(2)}, Q_{ab}^{(2)} \right\} = 0 \implies Q_{ab}^{(2)} Q_{b}^{(2)} + Q_{b}^{(2)} Q_{ab}^{(2)} = 0.
\end{array}
\end{align*}
$$

(89)

The above relationships, in a very explicit fashion, demonstrate the nilpotency and absolute anticommutativity of the (anti-)BRST charges $Q_{(a|b)}^{(2)}$ where we have exploited the key ideas behind the intimate connection between the continuous symmetry transformations and their generators (cf. Equation (6)). An interesting result is the observation that...
the nilpotency of the BRST charge $Q_b^{(2)}$ is connected with the nilpotency ($\partial (2/\Theta) = 0$) of the translational generator $\partial_\Theta$ but its absolute anticommutativity, with the anti-BRST charge, is deeply related with the nilpotency ($\partial^2_\Theta = 0$) of the translational generator $\partial_\Theta$. Geometrically, the translation of BRST charge $Q_b^{(2)}$ along $\Theta$-direction of the (2, 1)-dimensional antichiral super-submanifold is related with its nilpotency. However, the translation of the same charge along $\theta$-direction of the chiral super-submanifold leads to the observation of absolute anticommutativity of the BRST charge with anti-BRST charge (i.e., $Q_b^{(2)} + Q_b^{(2)} = 0$). Similar kinds of statements can be made for the anti-BRST charge $Q_a^{(2)}$ as well.

We end this subsection with the remarks that the nilpotency ($\partial^2_\Theta = 0$, $\partial (2/\Theta) = 0$) properties of the translational generators ($\partial_\Theta$, $\partial_\theta$) are deeply connected with the off-shell nilpotency ($|\partial_{\Theta s}^{(1,2)}|^2 = 0$, $|\partial_{\Theta s}^{(1,2)}|^2 = 0$) of the (anti)-BRST symmetry transformations $s_{\Theta s}^{(1)}$ and corresponding conserved charges $Q_{ab}^{(1)}$, for the Lagrangian densities $L_{(b_1,b_2)}$, which have been considered for our present discussions on the modified 2D Proca theory.

5.2. Off-Shell Nilpotent (Anti-)co-BRST Symmetries and Conserved Charges: (Anti-)chiral Superfield Approach. We exploit the (anti-)chiral super expansions of (69) and (60) to derive, first of all, the nilpotent (anti-)co-BRST symmetry transformations $s_{\Theta s}^{(1)}$ for the Lagrangian density $L_{(b_1)}$. Towards this goal in mind, we note that the following (anti-)co-BRST invariant quantities

$$s_{\Theta s}^{(1)} = s_{\Theta s}^{(1)} (\partial + A + m \phi) = 0, \quad s_{\Theta s}^{(1)} B = 0,$$

are to be generalized onto a (2, 1)-dimensional (anti-)chiral super-submanifold and we have to demand specific restrictions on the (anti-)chiral superfields to obtain the secondary fields of (69) and (60) in terms of the basic and auxiliary fields of the Lagrangian density $\mathcal{L}_{(b_1)}$.

We concentrate on the derivation of the anti-co-BRST symmetry transformation $s_{\Theta s}^{(1)}$ by imposing the following restrictions on the antichiral superfields

$$F(x, \bar{\theta}) = C(x), \quad \bar{\mathcal{B}}(x, \bar{\theta}) = \mathcal{B}(x, \bar{\theta}), \quad \bar{F}(x, \bar{\theta}) = \bar{B}(x, \bar{\theta})$$

$$B(x, \bar{\theta}) = \partial_\mu B^\mu (x, \bar{\theta}) = \partial_\mu A^\mu (x),$$

$$\Phi(x, \bar{\theta}) = \phi(x, \bar{\theta}) - \epsilon^\mu \partial_\mu \Phi(x, \bar{\theta}) = m A^\mu (x) - \epsilon^\mu \partial_\mu \phi(x, \bar{\theta}),$$

$$\bar{\mathcal{B}}(x, \bar{\theta}) \dot{F}(x, \bar{\theta}) - \dot{\bar{\mathcal{B}}}(x, \bar{\theta}) \bar{F}(x, \bar{\theta}) = \mathcal{B}(x) \bar{C}(x) - \bar{\mathcal{B}}(x) C(x),$$

$$\Phi(x, \bar{\theta}) F(x, \bar{\theta}) = \phi(x, \bar{\theta}) (x, \bar{\theta}) + e^\mu B_\mu (x, \bar{\theta}) \partial_\mu \Phi(x, \bar{\theta})$$

$$= m A_\mu (x) - \epsilon^\mu \partial_\mu \phi(x, \bar{\theta}),$$

$$\bar{\mathcal{B}}(x, \bar{\theta}) \dot{F}(x, \bar{\theta}) - \dot{\bar{\mathcal{B}}}(x, \bar{\theta}) \bar{F}(x, \bar{\theta}) = \mathcal{B}(x) \bar{C}(x) - \bar{\mathcal{B}}(x) C(x),$$

(91)

which lead to the determination of some of the trivial expressions for the secondary fields in the super expansions (60) as follows:

$$\partial_\mu R^\mu = 0,$$

$$B_\mu = 0,$$

$$f_1 = f_3 = f_4 = 0.$$

The substitution of $B_1 = f_1 = f_3 = f_4 = 0$, in the expansions (60), leads to the following:

$$F^{(x, \bar{\theta})} = C(x) + \bar{\mathcal{B}}(x, \bar{\theta}) = \mathcal{B}(x, \bar{\theta}),$$

$$\bar{F}^{(x, \bar{\theta})} = \bar{F}(x, \bar{\theta}) + \bar{B}(x, \bar{\theta}) = \bar{B}(x, \bar{\theta}),$$

$$\Phi^{(x, \bar{\theta})} = \phi(x, \bar{\theta}) + \bar{\mathcal{B}}(x, \bar{\theta}),$$

$$\Phi^{(x, \bar{\theta})} = \phi(x, \bar{\theta}) + \bar{\mathcal{B}}(x, \bar{\theta}).$$

(93)

which shows that we have already derived the transformations $s_{\Theta s}^{(1)} C(x) = s_{\Theta s}^{(1)} \mathcal{B}(x) = s_{\Theta s}^{(1)} B(x) = s_{\Theta s}^{(1)} \phi(x) = 0$ as the coefficients of $\bar{\mathcal{B}}$ in the expansions for the superfields with the superscript $(ad)$. The latter symbol denotes that the antichiral superfields, on the l.h.s. of (93), have been obtained after the applications of the restrictions (91). The arguments and discussions for the determination of the secondary fields in terms of the basic and auxiliary fields of the Lagrangian density $L_{(b_1)}$ go along similar lines as we have done in the previous Subsection 5.1 (see also, Appendix
C for details). Ultimately, we obtain the following expressions for the secondary fields:

\[ R_\mu(x) = -\varepsilon_{\mu\nu} \partial^\nu C(x), \]
\[ f_2(x) = -m C, \]
\[ B_2(x) = \bar{\mathcal{B}}(x). \]  

(94)

The substitutions of the above values into the appropriate expansions for the (2, 1)-dimensional antichiral superfields lead to the following:

\[ P^{(ad)}_\mu(x, \bar{\theta}) = A_\mu + \bar{\theta}(-\varepsilon_{\mu\nu} \partial^\nu C) \equiv A_\mu(x) + \bar{\theta}\left(\frac{s_{ad}^{(1)}}{s_{ad}^{(1)}} A_\mu(x)\right), \]
\[ \bar{F}^{(ad)}(x, \bar{\theta}) = C(x) + \bar{\theta}(i \mathcal{D}) \equiv C(x) + \bar{\theta}\left(\frac{s_{ad}^{(1)}}{s_{ad}^{(1)}} C(x)\right), \]
\[ \bar{\phi}^{(ad)}(x, \bar{\theta}) = \bar{\phi}(x) + \bar{\theta}(-m C) \equiv \bar{\phi}(x) + \bar{\theta}\left(\frac{s_{ad}^{(1)}}{s_{ad}^{(1)}} \bar{\phi}(x)\right). \]  

(95)

From Equations (93) and (95), it is crystal clear that we have computed all the anti-co-BRST symmetry transformations \( s_{ad}^{(1)} \) for all the fields of the Lagrangian density \( L_{(b)} \).

To determine all the secondary fields of (69) in terms of the basic and auxiliary fields of the Lagrangian density \( \mathcal{L}_{(b)} \), we have to invoke the co-BRST (i.e., dual-BRST) invariant quantities of Equation (90) and generalize them onto the (2, 1)-dimensional chiral super-submanifold with the following restrictions:

\[ \bar{F}(x, \theta) = \bar{C}(x), \bar{\mathcal{B}}(x, \theta) = \bar{\mathcal{B}}(x), \bar{B}(x, \theta) = B(x), \partial_\mu B^\mu(x, \theta) = \partial_\mu A^\mu(x), \]
\[ \Phi(x, \theta) = \phi(x), m B_\mu(x, \theta) = -\varepsilon_{\mu\nu} \partial^\nu \phi(x), \]
\[ -m A_\mu(x) = -\varepsilon_{\mu\nu} \partial^\nu \phi(x), \]
\[ \bar{\mathcal{B}}(x, \theta) \bar{F}(x, \theta) = \bar{\mathcal{B}}(x) C(x), \]
\[ \Phi(x, \theta) \bar{F}(x, \theta) = \phi(x) C(x), e^{\nu_0} B_\mu(x, \theta) \partial_\mu \bar{\mathcal{B}}(x, \theta) \]
\[ -i \partial_\mu \bar{F}(x, \theta) \partial^\mu F(x, \theta) = e^{\nu_0} A_\mu(x) \partial_\mu \bar{\mathcal{B}}(x), \]
\[ -i \partial_\mu \bar{C}(x) \partial^\mu C(x). \]  

(96)

We demand that the chiral superfields (and their useful combinations) on the l.h.s. of the above equations must be independent of the Grassmannian variable \( \theta \) because the co-BRST invariant quantities (for a model of a Hodge theory) are a set of physical quantities at the quantum level. The above restrictions lead to the following relationships between the secondary fields of the expansions (69) and the basic and auxiliary fields of \( L_{(b)} \), namely,

\[ \bar{R}_\mu = -\varepsilon_{\mu \nu} \partial^\nu C, \]
\[ \bar{B}_1 = -\mathcal{D}, \]
\[ \bar{f}_2 = -m C, \]
\[ \bar{B}_2 = 0. \]

(97)

The substitutions of the above secondary fields into the expansions (69) lead to

\[ B^{(d)}_\mu(x, \theta) = A_\mu + \theta(-\varepsilon_{\mu\nu} \partial^\nu C) \equiv A_\mu(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} A_\mu(x)\right), \]
\[ \bar{F}^{(d)}(x, \theta) = \bar{C}(x) + \theta(0) \equiv \bar{C}(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} C(x)\right), \]
\[ F^{(d)}(x, \theta) = C(x) + \theta(-i \mathcal{D}) \equiv C(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} C(x)\right), \]
\[ \Phi^{(d)}(x, \theta) = \phi(x) + \theta(0) \equiv \phi(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} \phi(x)\right), \]
\[ \bar{\Phi}^{(d)}(x, \theta) = \bar{\phi}(x) + \theta(-m C) \equiv \bar{\phi}(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} \bar{\phi}(x)\right), \]
\[ B^{(d)}(x, \theta) = B(x) + \theta(0) \equiv B(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} B(x)\right), \]
\[ \bar{B}^{(d)}(x, \theta) = \bar{\mathcal{B}}(x) + \theta(0) \equiv \bar{\mathcal{B}}(x) + \theta\left(\frac{s_{d}^{(1)}}{s_{d}^{(1)}} \bar{\mathcal{B}}(x)\right), \]  

(98)

where the coefficients of \( \theta \) on the r.h.s. of the above expansions are nothing but the co-BRST symmetry transformations (35) and the superscript \( (d) \) on the chiral superfields, on the l.h.s., denotes the superfields that have been obtained after the applications of the restrictions (96) and which lead to the determination of the co-BRST symmetry transformations \( s_{d}^{(1)} \) for the Lagrangian density \( L_{(b)} \) of our 2D modified Proca theory.

Taking the helps of expansions in (93), (95), and (98), we can now express the co-BRST and anti-co-BRST charges in the following explicit forms:

\[ Q_{(d)}^{(1)} = \int d\theta \left( i \bar{F}^{(d)}(x, \theta) \Phi^{(d)}(x, \theta) - i \bar{F}^{(d)}(x, \theta) \Phi^{(d)}(x, \theta) \right) \]
\[ = \int d\theta \left( i \bar{F}^{(d)}(x, \theta) \Phi^{(d)}(x, \theta) - i \bar{F}^{(d)}(x, \theta) \Phi^{(d)}(x, \theta) \right), \]

(99)
A close and clear observation of the above expressions for the (anti-)co-BRST charges \( Q^{(1)}_{(a)d} \) immediately implies the following (due to \( \partial_\theta = \partial(2/\theta) = 0 \)), namely,

\[
\begin{align*}
\partial_\theta Q^{(1)}_d &= 0, \\
\partial_\theta Q^{(1)}_{ad} &= 0, \\
\partial_\theta Q^{(1)}_{a} &= 0, \\
\partial_\theta Q^{(1)}_{ad} &= 0,
\end{align*}
\] (100)

which encompass, in their folds, the off-shell nilpotency and absolute anticommutativity of the (anti-)co-BRST charges \( Q^{(1)}_{(a)d} \). The above statement becomes transparent when we express (100) in the 2D ordinary spacetime (with the identifications: \( \partial_\theta \longleftrightarrow s^{(1)}_{ad}, \partial_\theta \longleftrightarrow s^{(1)}_{d} \)), in the language of the continuous symmetry transformations \( s^{(1)}_{(a)d} \) (and corresponding conserved charges \( Q^{(1)}_{(a)d} \)) for the Lagrangian density \( L_{(b_1)} \), namely,

\[
\begin{align*}
s^{(1)}_{(a)d} Q^{(1)}_d &= -i \left\{ Q^{(1)}_d, Q^{(1)}_d \right\} = 0 \Rightarrow \left[ Q^{(1)}_d \right]^2 = 0, \\
s^{(1)}_{ad} Q^{(1)}_d &= -i \left\{ Q^{(1)}_d, Q^{(1)}_d \right\} = 0 \Rightarrow \left[ Q^{(1)}_d \right]^2 = 0, \\
s^{(1)}_{(a)d} Q^{(1)}_{ad} &= -i \left\{ Q^{(1)}_{ad}, Q^{(1)}_{ad} \right\} = 0 \Rightarrow Q^{(1)}_{ad} Q^{(1)}_{ad} + Q^{(1)}_{ad} Q^{(1)}_{ad} = 0, \\
s^{(1)}_{d} Q^{(1)}_d &= -i \left\{ Q^{(1)}_d, Q^{(1)}_d \right\} = 0 \Rightarrow Q^{(1)}_d Q^{(1)}_d + Q^{(1)}_d Q^{(1)}_d = 0,
\end{align*}
\] (101)

which demonstrate the off-shell nilpotency and absolute anticommutativity of the conserved (anti-)co-BRST charges \( Q^{(1)}_{(a)d} \) for the Lagrangian density \( L_{(b_1)} \).

As we have discussed various aspects of (anti-)co-BRST symmetries \( s^{(1)}_{(a)d} \) and corresponding charges \( Q^{(1)}_{(a)d} \) for the Lagrangian density \( L_{(b_1)} \), we can do the same for the Lagrangian density \( L_{(b_2)} \). Towards this objective in mind, first of all, we note that the following (anti-)co-BRST invariant quantities w.r.t. \( s^{(2)}_{(a)d} \), namely,

\[
\begin{align*}
s^{(2)}_{ad} C &= 0, s^{(2)}_{ad} (\partial \cdot A - m \phi) = 0, s^{(2)}_{(a)d} B = s^{(2)}_{ad} \bar{B} = s^{(2)}_{(a)d} \bar{B} = 0, \\
s^{(2)}_{ad} \left[ m A_\mu + \epsilon_{\mu
u} \partial^\nu \phi \right] &= 0, s^{(2)}_{ad} \left[ e^{\nu
u} A_\mu \partial^\nu \bar{\phi} - i \partial_\mu \bar{C} \partial^\nu C \right] = 0, \\
s^{(2)}_{ad} \left[ e^{\nu
u} A_\mu \partial^\nu \bar{C} \right] = 0, s^{(2)}_{(a)d} \left[ \bar{C} \partial^\nu C \right] = 0, \\
s^{(2)}_{d} \left[ m A_\mu + \epsilon_{\mu
u} \partial^\nu \phi \right] &= 0, s^{(2)}_{d} \left[ e^{\nu
u} A_\mu \partial^\nu \bar{\phi} - i \partial_\mu \bar{C} \partial^\nu C \right] = 0, \\
s^{(2)}_{d} \left[ e^{\nu
u} A_\mu \partial^\nu \bar{C} \right] = 0, s^{(2)}_{(a)d} \left[ \bar{C} \partial^\nu C \right] = 0,
\end{align*}
\] (102)

are to be generalized onto (2, 1)-dimensional (anti-)chiral super-submanifolds (of the general (2, 2)-dimensional supermanifold) and we have to invoke specific restrictions on them so that we could derive the (anti-)co-BRST symmetry transformations \( s^{(2)}_{(a)d} \) for the Lagrangian density \( L_{(b_2)} \) within the framework of (anti-)chiral superfield formalism.

First and foremost, we concentrate on the derivation of anti-co-BRST symmetry transformations \( s^{(2)}_{(a)d} \). In this context, the following restrictions on the antichiral superfields (emerging from a close look at (102)), namely,

\[
\begin{align*}
F(x, \bar{\phi}) &= C(x), \bar{\phi} \left( x, \bar{\phi} \right) = \bar{B}(x), \bar{B} \left( x, \bar{\phi} \right) \\
&= \bar{B}(x), \partial_\mu B^\mu \left( x, \bar{\phi} \right) = \partial_\mu A^\mu(x), \\
\Phi(x, \bar{\phi}) &= \phi(x), m B_\mu \left( x, \bar{\phi} \right) + \epsilon_{\mu\nu} \partial^\nu \phi(x, \bar{\phi}) \\
&= m A_\mu(x) + \epsilon_{\mu\nu} \partial^\nu \phi(x, \bar{\phi}), \\
\bar{\phi} \left( x, \bar{\phi} \right) &\not= \bar{B}(x), \bar{B} \left( x, \bar{\phi} \right) \\
&= \bar{B}(x), \bar{B} \left( x, \bar{\phi} \right) = \partial_\mu \bar{B}(x), \\
&= m \epsilon_{\mu\nu} \partial^\nu \phi(x, \bar{\phi}) \\
&\not= m A_\mu(x) + \epsilon_{\mu\nu} \partial^\nu \phi(x, \bar{\phi}), \\
&= m A_\mu(x) + \epsilon_{\mu\nu} \partial^\nu \phi(x, \bar{\phi}),
\end{align*}
\] (103)

lead to the derivation of secondary fields of super expansions (60) in terms of the auxiliary and basic fields of \( L_{(b_1)} \) as follows:

\[
\begin{align*}
R_\mu &= -\epsilon_{\mu\nu} \partial^\nu C, \\
B_2 &= i \bar{B}, \\
f_2 &= m C, \\
B_1 &= f_1 = f_3 = f_4 = 0.
\end{align*}
\] (104)

Substitutions of these secondary fields into the super expansions (60) lead to the following:

\[
\begin{align*}
B^{(AD)}_\mu \left( x, \bar{\phi} \right) &= A_\mu + \bar{B} \left( -\epsilon_{\mu\nu} \partial^\nu C \right) \equiv A_\mu(x) + \bar{B} \left( s^{(2)}_{ad} A_\mu(x) \right), \\

\bar{F}^{(AD)} \left( x, \bar{\phi} \right) &= C(x) + \bar{B} \left( i \bar{B} \right) \equiv C(x) + \bar{B} \left( s^{(2)}_{ad} C(x) \right), \\

\bar{F}^{(AD)} \left( x, \bar{\phi} \right) &= C(x) + \bar{B} \left( 0 \right) \equiv C(x) + B \left( s^{(2)}_{ad} C(x) \right), \\

\Phi^{(AD)} \left( x, \bar{\phi} \right) &= \phi(x) + \bar{B} \left( m C \right) \equiv \phi(x) + \bar{B} \left( s^{(2)}_{ad} \phi(x) \right), \\

\Phi^{(AD)} \left( x, \bar{\phi} \right) &= \phi(x) + \bar{B} \left( 0 \right) \equiv \phi(x) + \bar{B} \left( s^{(2)}_{ad} \phi(x) \right), \\

\bar{B}^{(AD)} \left( x, \bar{\phi} \right) &= B(x) + \bar{B} \left( 0 \right) \equiv B(x) + \bar{B} \left( s^{(2)}_{ad} B(x) \right), \\

\bar{B}^{(AD)} \left( x, \bar{\phi} \right) &= \bar{B}(x) + \bar{B}(0) \equiv \bar{B}(x) + \bar{B} \left( s^{(2)}_{ad} \bar{B}(x) \right).
\end{align*}
\] (105)
where the superscript \((AD)\) stands for the super expansions of the chiral superfields that have been obtained after the application of restrictions from Equation (102). It should be noted that we have derived all the anti-co-BRST symmetry transformations \(s_{(a)d}^{(2)}\) as the coefficients of \(\bar{B}\) in the final super expansions (105).

Taking into account the co-BRST invariant quantities from Equation (102) and generalizing them onto the \((2, 1)\)-dimensional chiral super-submanifold, we demand the following restrictions on these specific combinations of chiral superfields:

\[
\begin{align*}
F(x, \theta) &= C(x), \quad \bar{B}(x, \theta) = \bar{B}(x), \quad B(x, \theta) = B(x), \\
\Phi(x, \theta) &= \phi(x), \quad m B_u(x, \theta) + \epsilon_{\mu \nu} \partial^\nu \phi(x, \theta) \\
\bar{B}(x) F(x, \theta) &= \bar{B}(x) \hat{F}(x, \theta) = \bar{B}(x) \hat{F}(x, \theta) = \bar{B}(x) \tilde{C}(x), \\
\Phi(x, \theta) \bar{B}(x) F(x, \theta) &= \phi(x) \tilde{C}(x), \quad e^{\nu \rho} A^\rho(x, \theta) \partial^\nu \phi(x, \theta) \\
&= i \partial^\rho \Phi(x, \theta) \partial^\rho \phi(x, \theta) \\
&= e^{\nu \rho} A^\rho(x, \theta) \partial^\nu \phi(x, \theta) - i \partial^\rho \tilde{C}(x, \theta) \partial^\rho \phi(x, \theta). 
\end{align*}
\]

(106)

due to the basic tenets of (anti-)chiral superfield approach to BRST formalism where we demand that all the co-BRST invariant quantities must be independent of the Grassmannian variables \(\theta\). As a consequence of the restrictions in (106), we obtain the following expressions for the secondary fields (cf. Equation (69)) in terms of the basic and auxiliary fields of the Lagrangian density \(\mathcal{L}_{(b_2)}\), namely,

\[
\begin{align*}
\bar{R}_u &= - \epsilon_{\mu \nu} \partial^\nu \bar{C}, \\
\bar{B}_1 &= - \bar{\partial} \bar{B}, \\
f_2 &= m \bar{C}, \\
\bar{f}_1 &= \bar{f}_3 = \bar{f}_4 = \bar{B}_2 = 0. 
\end{align*}
\]

(107)

The substitutions of these secondary fields into the super expansions (69) lead to the following super expansions for the chiral superfields:

\[
\begin{align*}
B^{(2)}(x, \theta) &= A_\nu(x) + \theta (- \epsilon_{\mu \nu} \partial^\nu \bar{C} \equiv A_\nu(x) + \theta \left( s_{(a)d}^{(2)} A_\nu(x) \right), \\
\bar{F}^{(D)}(x, \theta) &= \bar{C}(x) + \theta (0) \equiv \bar{C}(x) + \theta \left( s_{(a)d}^{(2)} C(x) \right), \\
F^{(D)}(x, \theta) &= C(x) + \theta (0) \equiv C(x) + \theta \left( s_{(a)d}^{(2)} C(x) \right), \\
\Phi^{(D)}(x, \theta) &= \phi(x) + \theta (0) \equiv \phi(x) + \theta \left( s_{(a)d}^{(2)} \phi(x) \right), \\
\bar{\Phi}^{(D)}(x, \theta) &= \bar{\phi}(x) + \theta (m \bar{C}) \equiv \bar{\phi}(x) + \theta \left( s_{(a)d}^{(2)} \bar{\phi}(x) \right), \\
B^{(D)}(x, \theta) &= \bar{B}(x) + \theta (0) \equiv \bar{B}(x) + \theta \left( s_{(a)d}^{(2)} \bar{B}(x) \right), \\
\bar{\Phi}^{(D)}(x, \theta) &= \bar{\phi}(x) + \theta (m \bar{C}) \equiv \bar{\phi}(x) + \theta \left( s_{(a)d}^{(2)} \bar{\phi}(x) \right), \\
\Phi^{(D)}(x, \theta) &= \phi(x) + \theta (0) \equiv \phi(x) + \theta \left( s_{(a)d}^{(2)} \phi(x) \right). 
\end{align*}
\]

(108)

where the chiral superfields (with superscript \((D)\)) denote the superfields that have been obtained after the application of the restrictions quoted in (106). A close look at (108) reveals that we have already derived the co-BRST symmetry transformations \(s_{(a)d}^{(2)}\) (for \(L_{(b_2)}\)) as the coefficients of \(\theta\) in the above chiral super expansions.

At this stage, we can use the super expansions (105) and (108) to express the (anti-)co-BRST charges \(Q_{(a)d}^{(2)}\) connected with the nilpotent transformations \(s_{(a)d}^{(2)}\) for the Lagrangian density \(L_{(b_2)}\) as follows:

\[
\begin{align*}
Q_{(a)d}^{(2)} &= \int \! dx \left[ \partial \left( i \bar{F}^{(D)}(x, \theta) F^{(D)}(x, \theta) + i \Phi^{(D)}(x, \theta) \partial^\nu \bar{C}(x, \theta) \right) \right] \\
&= \int \! dx \left[ d \theta \left( i \bar{F}^{(D)}(x, \theta) F^{(D)}(x, \theta) + i \Phi^{(D)}(x, \theta) \partial^\nu \bar{C}(x, \theta) \right) \right],
\end{align*}
\]

(109)

It is now very clear that we have the following very interesting and informative relationships (due to the nilpotency \(\partial_{\theta}^2 = \partial(2/\theta)\) properties of \((\partial_{\theta}, \partial_{\bar{\theta}})\)), namely,

\[
\begin{align*}
\partial_{\theta} Q_{(a)d}^{(2)} &= 0, \\
\partial_{\bar{\theta}} Q_{(a)d}^{(2)} &= 0, \\
\partial_{\theta} Q_{(a)d}^{(2)} &= 0, \\
\partial_{\bar{\theta}} Q_{(a)d}^{(2)} &= 0.
\end{align*}
\]

(110)

The above equation actually captures the off-shell nilpotency and absolute anticommutativity of the conserved charges \(Q_{(a)d}^{(2)}\). This statement becomes very transparent when we express (110) in the ordinary space (with the mappings: \(s_d \mapsto \partial_{\theta}, s_{ad} \mapsto \partial_{\bar{\theta}}\)) and exploit the idea behind the continuous symmetry transformations and their relationships with their generators (cf. Equation (6)), namely,
Thus, we have captured the off-shell nilpotency and absolute anticommutativity of the conserved charges $Q^{(2)}_{\text{ch}}$ within the framework of (anti-)chiral supermanifold approach to BRST formalism by using the super expansions (62), (68), and (71) which are primarily connected with the nilpotency $(\partial_{\theta}^2 - \partial (2/\theta) = 0)$ of the translational generators $(\partial_{\theta}, \partial_{\theta})$ along the chiral and antichiral super-submanifolds.

6. Invariance of the Lagrangian Densities: Chiral and Antichiral Superfield Approach

In this section, first of all, we capture the (anti-)BRST invariance(s) of the Lagrangian densities $\mathcal{L}(b_1)$ and $\mathcal{L}(b_2)$ within the framework of our (anti-)chiral superfield approach to BRST formalism by using the super expansions (62), (68), (73), (83), and (86) which have been obtained after the (anti-)BRST invariant restrictions on the (anti-)chiral superfields (defined on the $(2,1)$-dimensional (anti-)chiral super-submanifolds of the general $(2,2)$-dimensional supermanifolds). We note, in this connection, that the ordinary Lagrangian density $\mathcal{L}(b_1)$ can be generalized onto the $(2,1)$-dimensional (anti-)chiral super-submanifolds (of the general $(2,2)$-dimensional supermanifold) as follows:

$$\mathcal{L}(b_1) \rightarrow \tilde{\mathcal{L}}^{(ac)}(b_1) = \tilde{\mathcal{B}}^{(ab)}(x, \bar{\theta}) \left( \tilde{E}^{(ab)}(x, \bar{\theta}) - m \Phi^{(ab)}(x, \bar{\theta}) \right)$$

where the superscripts $(c)$ and $(ac)$ on the super Lagrangian densities denote the generalizations of the ordinary Lagrangian densities to their chiral and antichiral counterparts. Furthermore, we note that superscripts $(b)$ and $(ab)$ on the (anti-)chiral superfields denote that these superfields have been obtained after the applications of the (anti-)BRST invariant restrictions (cf. Equations (62), (68), and (73)). In addition, we have to use the following:

$$\tilde{B}^{(ab)}(x, \bar{\theta}) = \tilde{B}^{(ab)}(x, \bar{\theta}) = B(x), \tilde{E}^{(ab)}(x, \bar{\theta}) = \tilde{E}^{(ab)}(x, \bar{\theta}) = E(x), \tilde{F}^{(ab)}(x, \bar{\theta}) = \tilde{F}^{(ab)}(x, \bar{\theta}) = F(x), \tilde{B}^{(ac)}(x, \bar{\theta}) = \tilde{B}^{(ac)}(x, \bar{\theta}) = B(x), \tilde{E}^{(ac)}(x, \bar{\theta}) = \tilde{E}^{(ac)}(x, \bar{\theta}) = E(x), \tilde{F}^{(ac)}(x, \bar{\theta}) = \tilde{F}^{(ac)}(x, \bar{\theta}) = F(x),$$

due to the fact that we have $s_\mu(E, B, B, C, \phi) = 0$ and $s_{ab}(E, B, B, \tilde{C}, \phi) = 0$. In other words, we have some (anti-)chiral superfields with superscripts $(b)$ and $(ab)$ which are actually ordinary fields. In view of the mappings $s_\mu^{(1)} \rightarrow \partial_\mu$ and $s_{ab}^{(1)} \rightarrow \partial_\mu$, we observe the following (as far as the super Lagrangian densities are concerned), namely:

$$\frac{\partial}{\partial \theta} \left[ \tilde{\mathcal{L}}^{(c)}(b_1) \right] = \partial_\mu (\bar{B} \partial_\mu \tilde{C}) \equiv s_\mu^{(1)} \mathcal{L}(b_1),$$

$$\frac{\partial}{\partial \theta} \left[ \tilde{\mathcal{L}}^{(ac)}(b_1) \right] = \partial_\mu (\bar{B} \partial_\mu \tilde{C}) \equiv s_\mu^{(1)} \mathcal{L}(b_1),$$

$$\frac{\partial}{\partial \theta} \left[ \tilde{\mathcal{L}}^{(b_1)} \right] = \tilde{\mathcal{L}}^{(b_1)}$$

(114)
which show the (anti-)BRST invariance of the ordinary action integral $S = \int d^2 x L_{(b)}$ that can be also written as super action integrals $\tilde{S} = \int d\theta \int d^2 x L^{(ac)}_{(b)} = \int d\theta \int d^2 x L^{(c)}_{(b)}$.

As we have captured the (anti-)BRST invariance of the Lagrangian density $L_{(b)}$, we can also express the (anti-)BRST invariance of the Lagrangian density $L_{(b)}$. For this purpose, we have to generalize the ordinary 2D Lagrangian density to its counterparts super Lagrangian densities on the (anti-)chiral super-submanifolds as follows:

$$
\mathcal{L}_{(b)} \rightarrow \mathcal{L}^{(AC)}_{(b)} = \mathcal{L}^{(B)}_{(b)}(x, \vec{\partial}) \left( E^{(B)}_{(b)}(x, \vec{\partial}) + m \Phi^{(B)}_{(b)}(x, \vec{\partial}) \right)
$$

where the superscripts $(B)$ and $(AB)$, on the superfields, denote the (anti-)chiral superfields that have been obtained after the applications of the (anti-)BRST invariant restrictions for the Lagrangian density $L_{(b)}$, and, on the l.h.s., the superscripts $(C)$ and $(AC)$, on the super Lagrangian densities stand for the chiral and antichiral versions of the ordinary Lagrangian density $L_{(b)}$ which contain chiral and antichiral superfields. It should be noted that some of the (anti-)chiral superfields, with superscripts $(B)$ and $(AB)$, are actually ordinary 2D fields. For instance, we observe that the following are true, namely,

$$
\mathcal{L}_{(b)} \rightarrow \mathcal{L}^{(C)}_{(b)} = \mathcal{L}^{(B)}_{(b)}(x, \theta) \left( E^{(B)}_{(b)}(x, \theta) + m \Phi^{(B)}_{(b)}(x, \theta) \right)
$$

In view of the mappings $s^2_{(ab)} \mapsto \partial_{\bar{\theta}}$ and $s^2_{(ab)} \mapsto \partial_{\theta}$, it is elementary to check that the following are true in the context of super Lagrangian densities, namely,

$$
\frac{\partial}{\partial \theta} \left[ \mathcal{L}_{(b)}^{(C)} \right] = \frac{\partial}{\partial \mu} \left( B \delta^{\mu} C \right) \equiv s^2_{(ab)} \mathcal{L}_{(b)}^{(2)},
$$

Hence, we have captured the (anti-)BRST invariance of the Lagrangian density $L_{(b)}$ because the corresponding action integral(s)

$$
S = \int \theta d^2 x \mathcal{L}_{(b)}^{(C)} \mathcal{L}^{(C)}_{(b)} \equiv \int \theta d^2 x \mathcal{L}_{(b)}^{(AC)},
$$

vanish due to Gauss’s divergence theorem where all the physical fields vanish off at $x = \pm \infty$.

We now explain the existence of the restriction (i.e., $B + B + 2(\partial \cdot A) = 0$) within the framework of the (anti-)chiral superfield approach. Towards this goal in mind, first of all, we derive the nontrivial (anti-)BRST symmetry transformations (30) for the auxiliary fields $B(x), B(x)$ (i.e., $s^1_{b}(B) = -2\partial C \equiv s^2_{(ab)} B$ and $s^1_{(ab)} B = -2\partial C \equiv s^2_{(ab)} B$). In this context, it can be seen that $s^1_{(b)}(B + B + 2(\partial \cdot A)) = 0$ and $s^1_{(ab)}(B + B + 2(\partial \cdot A)) = 0$ imply the following restrictions on the (anti-)chiral superfields:
\[ \tilde{B}^{(b)}(x, \bar{\theta}) + \tilde{B}(x, \bar{\theta}) = B(x) + B(x) + 2(\partial \cdot A)(x), \]
\[ \tilde{B}^{(ab)}(x, \theta) + \tilde{B}(x, \theta) + 2\partialA\tilde{B}^{(b)}(x, \bar{\theta}) = B(x) + B(x) + 2(\partial \cdot A)(x), \]
\[ \tilde{B}(x, \bar{\theta}) + \tilde{B}^{(b)}(x, \theta) + 2\partialA\tilde{B}^{(B)}(x, \bar{\theta}) = B(x) + B(x) + 2(\partial \cdot A)(x), \]
\[ \tilde{B}(x, \theta) + \tilde{B}^{(ab)}(x, \theta) + 2\partialA\tilde{B}^{(B)}(x, \theta) = B(x) + B(x) + 2(\partial \cdot A)(x). \]

(119)

In the above, the auxiliary fields \( B(x) \) and \( \tilde{B}(x) \) have been generalized to the (anti-)chiral super-submanifolds with the following super expansions:

\[ B(x) \Longrightarrow \tilde{B}(x, \theta) = B(x) + \theta \tilde{f}_3(x), \]
\[ B(x) \Longrightarrow \tilde{B}(x, \bar{\theta}) = B(x) + \bar{\theta} \tilde{f}_5(x), \]
\[ B(x) \Longrightarrow \tilde{B}(x, \theta) = B(x) + \theta \tilde{f}_5(x), \]
\[ B(x) \Longrightarrow \tilde{B}(x, \bar{\theta}) = B(x) + \bar{\theta} \tilde{f}_3(x). \]

(120)

We also note that the super expansions for the superfields \( B(x, \bar{\theta}) \) and \( \tilde{B}(x, \theta) \) have been also given in (60) and (69), respectively. In Equation (119), it is very clear that we have the trivial equalities: \( \tilde{B}^{(b)}(x, \bar{\theta}) = \tilde{B}^{(ab)}(x, \bar{\theta}) \) \( B(x) = B(x) \) due to \( s^{(1)}_{ab}B(x) = 0 \) as well as \( \tilde{B}^{(b)}(x, \theta) = \tilde{B}^{(AB)}(x, \theta) = B(x) \) due to our observations: \( s^{(2)}_{ab}B(x) = 0 \). Plugging in these values and (120) into (119) yields the following expressions for the secondary fields:

\[ f_3(x) \equiv \tilde{f}_3(x) = -2\square C, \]
\[ \tilde{f}_5(x) \equiv \tilde{f}_5(x) = -2\square C. \]

(121)

It is straightforward to note that the secondary fields \( (f_3(x), \tilde{f}_5(x)) \) in (120) are fermionic in nature because of the fermionic nature of \( (\theta, \bar{\theta}) \). Thus, we have obtained \( s^{(1)}_{ab}B(x) = -2\square C \equiv s_{ab}^{(2)}B(x) \) and \( s^{(1)}_{ab}B(x) = -2\square C = s_{ab}^{(2)}B(x) \). In other words, we have the following (anti-)chiral super expansions for the (anti-)chiral superfields:

\[ \tilde{B}^{(AB)}(x, \theta) = B(x) + \theta (-2\square C) \equiv B(x) + \theta \tilde{s}_{ab}^{(2)}B(x), \]
\[ \tilde{B}^{(B)}(x, \bar{\theta}) = B(x) + \bar{\theta} (-2\square C) \equiv B(x) + \bar{\theta} \tilde{s}_{ab}^{(2)}B(x). \]

(122)

We note that the coefficients of \( \theta \) and \( \bar{\theta} \), in the above super expansions, are nothing but the (anti-)BRST symmetry transformations \( s_{ab}^{(2)} \). We further point out that the following emerges from the restrictions (119), namely,

\[ \tilde{s}^{(ab)}(x, \theta) = B(x) + \theta (-2\square C) \equiv B(x) + \theta \tilde{s}_{ab}^{(1)}B(x), \]
\[ \tilde{s}^{(b)}(x, \bar{\theta}) = B(x) + \bar{\theta} (-2\square C) \equiv B(x) + \bar{\theta} \tilde{s}_{b}^{(1)}B(x), \]

(123)

which show that we have already derived the (anti-)BRST symmetry transformations \( s^{(1)}_{ab}B(x) = -2\square C \) and \( s_{b}^{(1)}B(x) = -2\square C \) as the coefficients of \( \theta \) and \( \bar{\theta} \).

Taking into account the inputs from (122) and (123), we can generalize the ordinary Lagrangian densities \( \mathcal{L} \) in the following forms:

\[ \mathcal{L}^{(\text{ch})} \longrightarrow \tilde{\mathcal{L}}^{(\text{ch})} = \tilde{\mathcal{L}}^{(AB)}(x, \bar{\theta}) \tilde{E}(x, \bar{\theta}) \]
\[ - m \partialA \tilde{\Phi}^{(B)}(x, \bar{\theta}) - \frac{1}{2} \tilde{\Phi}^{(B)}(x, \bar{\theta}) \tilde{\Phi}^{(B)}(x, \bar{\theta}) \]
\[ + m \tilde{B}^{(b)}(x, \bar{\theta}) \tilde{B}^{(b)}(x, \bar{\theta}) \]
\[ + \tilde{\Phi}^{(B)}(x, \bar{\theta}) \tilde{\Phi}^{(B)}(x, \bar{\theta}) \]
\[ + m \tilde{\Phi}^{(B)}(x, \bar{\theta}) \tilde{\Phi}^{(B)}(x, \bar{\theta}) \]
\[ + \tilde{f}^{(b)}(x, \bar{\theta}) \tilde{f}^{(b)}(x, \bar{\theta}) \]
\[ + \tilde{f}^{(B)}(x, \bar{\theta}) \tilde{f}^{(B)}(x, \bar{\theta}) \],

(124)
where the (anti-)chiral superfields with superscripts \((B)\) and \((AB)\) have already been explained earlier. It should be noted that we also have the following:

\[
\begin{align*}
\tilde{B}^{(B)}(x, \bar{\theta}) &= \tilde{B}^{(AB)}(x, \theta) = \tilde{B}(x), \\
\tilde{\Phi}^{(B)}(x, \bar{\theta}) &= \tilde{\Phi}^{(AB)}(x, \theta) = \tilde{\Phi}(x), \\
\tilde{L}^{(B)}(x, \bar{\theta}) &= \tilde{L}^{(AB)}(x, \theta) = \tilde{L}(x), \\
\tilde{F}^{(B)}(x, \bar{\theta}) &= \tilde{F}^{(AB)}(x, \theta) = \tilde{F}(x), \\
\tilde{E}^{(B)}(x, \bar{\theta}) &= \tilde{E}^{(AB)}(x, \theta) = \tilde{E}(x),
\end{align*}
\]

which show that there are (anti-)chiral superfields in the super Lagrangian density \((124)\) that are, in reality, the ordinary fields (defined on the 2D Minkowskian spacetime manifold). In exactly similar fashion, we can generalize the ordinary Lagrangian density \(L^{(b)}\) onto the \((2, 1)\)-dimensional (anti-)chiral super-submanifolds (of the general \((2, 2)\)-dimensional supermanifold) as follows (with the trivial inputs: \(\tilde{B}^{(b)}(x, \bar{\theta}) = \tilde{B}^{(ab)}(x, \theta) = B(x), \tilde{\Phi}^{(b)}(x, \bar{\theta}) = \tilde{\Phi}^{(ab)}(x, \theta) = \Phi(x), \tilde{L}^{(b)}(x, \bar{\theta}) = \tilde{L}^{(ab)}(x, \theta) = L(x), \tilde{F}^{(b)}(x, \bar{\theta}) = \tilde{F}^{(ab)}(x, \theta) = F(x), \tilde{E}^{(b)}(x, \bar{\theta}) = \tilde{E}^{(ab)}(x, \theta) = E(x)\), etc.):

\[
\begin{align*}
\mathcal{L}^{(b)}(x, \theta) &\longrightarrow \tilde{\mathcal{L}}^{(ac)}(x, \bar{\theta}) \\
&= \tilde{B}^{(b)}(x, \bar{\theta}) \tilde{F}^{(b)}(x, \bar{\theta}) \\
&+ m \tilde{\Phi}^{(b)}(x, \bar{\theta}) \\
&- \frac{1}{2} \tilde{\Phi}^{(b)}(x, \bar{\theta}) \tilde{\Phi}^{(b)}(x, \bar{\theta}) \\
&- m \tilde{\Phi}^{(b)}(x, \bar{\theta}) \tilde{\Phi}^{(b)}(x, \bar{\theta}) \\
&- \frac{1}{2} \partial_{\mu} \tilde{\Phi}^{(b)}(x, \bar{\theta}) \partial^{\mu} \tilde{\Phi}^{(b)}(x, \bar{\theta}) \\
&+ \frac{m^2}{2} B_{\mu}^{(b)}(x, \bar{\theta}) B^{(b)}(x, \bar{\theta}) \\
&+ \frac{1}{2} \partial_{\mu} \tilde{\Phi}^{(b)}(x, \bar{\theta}) \partial^{\mu} \tilde{\Phi}^{(b)}(x, \bar{\theta}) \\
&+ m B_{\mu}^{(b)}(x, \bar{\theta}) \partial^{\mu} \Phi^{(b)}(x, \bar{\theta}) \\
&+ \tilde{B}^{(b)}(x, \bar{\theta}) \tilde{\Phi}^{(b)}(x, \bar{\theta}) \\
&- m \Phi^{(b)}(x, \bar{\theta}) \\
&+ \frac{1}{2} \tilde{B}^{(b)}(x, \bar{\theta}) B^{(b)}(x, \bar{\theta}) \\
&- i \partial_{(b)} \tilde{\Phi}^{(b)}(x, \bar{\theta}) B^{(b)}(x, \bar{\theta}) \\
&+ i m^2 \tilde{F}^{(b)}(x, \bar{\theta}) F^{(b)}(x, \bar{\theta}),
\end{align*}
\]

where all the superscripts \((b)\) and \((ab)\) (on the r.h.s.) have been explained earlier and the superscripts \((C)\) and \((AC)\) on the Lagrangian densities (on the l.h.s.) denote the generalizations of the ordinary Lagrangian densities to the corresponding super Lagrangian densities so that we can study the variation of Lagrangian density \(\mathcal{L}^{(b)}\) w.r.t. \(s_{(ab)1}\) and Lagrangian density \(\tilde{\mathcal{L}}^{(b)}\) with respect to \(s_{(b)1}\). Keeping in our mind the mappings: \(s_{(b)1}^{(1)} \leftrightarrow s_{(ab)1}^{(2)}\), we note the following:

\[
\frac{\partial}{\partial \theta} \tilde{\mathcal{L}}^{(C)}(x, \bar{\theta}) = \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}^{(AC)}(x, \bar{\theta}) = \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}^{(ac)}(x, \bar{\theta}) = \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}^{(b)}(x, \bar{\theta}),
\]

which show that we have captured the existence of restrictions within the framework of (anti-)chiral superfield approach to BRST formalism because we can have the symmetry
invariance on the r.h.s. of (127) iff $B + 2 (\partial \cdot A) = 0$ (pro-
vided we do not impose the mass-shell conditions: $(\Box + m^2) C = (\Box + m^2) \bar{C} = 0$ from outside).

We now concentrate on encapsulating the (anti-)co-
BRST invariance of the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}^{(a)}$ within the framework (anti-)chiral superfield approach to BRST formalism. Towards this goal in mind, we generalize the *ordinary* Lagrangian densities onto (2, 1)-dimensional (anti-)chiral super-submanifolds as follows (with the trivial inputs: $F^{(d)}(x, \theta) = C(x), \Phi^{(d)}(x, \theta) = (\tilde{\partial} \cdot A)(x), \tilde{\mathcal{B}}^{(d)}(x, \theta) = \mathcal{B}(x), \Phi^{(d)}(x, \theta) = \Phi(x)$). Similarly, we observe that $s_{ad}^{(1)}(C, (\tilde{\partial} \cdot A), \mathcal{B}, \Phi) = 0$ implies that we have $F^{(ad)}(x, \bar{\theta}) = C(x), \Phi^{(ad)}(x, \theta) = \mathcal{B}(x), \Phi^{(ad)}(x, \theta) = \Phi(x)$, etc.:

$$\mathcal{L}_{(b)}^{(a)} \longrightarrow \tilde{\mathcal{L}}_{(b)}^{(a)} = \tilde{\mathcal{B}}^{(ad)}(x, \bar{\theta}) [\tilde{E}^{(ad)}(x, \bar{\theta}) - m \tilde{\Phi}^{(ad)}(x, \bar{\theta})]$$

$$- \frac{1}{2} \tilde{\mathcal{B}}^{(ad)}(x, \bar{\theta}) \tilde{\mathcal{B}}^{(ad)}(x, \bar{\theta}) + m \mathcal{E}^{(ad)}(x, \bar{\theta})$$

$$+ \frac{1}{2} (\tilde{\partial})^{(ad)}(x, \bar{\theta}) \tilde{\Phi}^{(ad)}(x, \bar{\theta}) - \frac{1}{2} \mathcal{B}^{(ad)}(x, \bar{\theta}) \mathcal{B}^{(ad)}(x, \bar{\theta}) + m \mathcal{B}^{(ad)}(x, \bar{\theta})$$

$$+ \frac{1}{2} \mathcal{B}^{(ad)}(x, \bar{\theta}) \mathcal{B}^{(ad)}(x, \bar{\theta}) - \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + m \mathcal{D}^{(ad)}(x, \bar{\theta})$$

$$+ \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + m \mathcal{D}^{(ad)}(x, \bar{\theta})$$

where the superscript $(ac, ad)$ on the super Lagrangian density denotes that the superfields, contained in it, are antichiral which have been obtained after the anti-co-BRST invariant restrictions (cf. Equations (93) and (95)). In exactly similar fashion, we note that the super Lagrangian density, with superscript $(c, d)$, incorporates chiral superfields that have been obtained after the applications of the co-BRST invariant restrictions (cf. Equation (98)). It is straightforward now to check that

$$\frac{\partial}{\partial \bar{\theta}} \left[ \tilde{\mathcal{L}}_{(b)}^{(ac, ad)} \right] = \partial_{\mu} \left[ \partial \Phi \tilde{C} + m \partial \Phi (m \mathcal{A} + \mathcal{B} \partial \mathcal{A}) + m \partial \Phi \tilde{C} \right]$$

$$\equiv s_{ad}^{(1)} \mathcal{L}_{(b)}^{(ac, ad)}$$

The above equation captures the (anti-)co-BRST invariance of the action integral $S = \int d^2x \mathcal{L}_{(b)}^{(ad)}$ as it matches precisely with our earlier observation in Equation (36).

We would like to capture now the (anti-)co-BRST invariance of the Lagrangian density $\mathcal{L}_{(b)}^{(ad)}$ within the framework of (anti-)chiral superfield approach to BRST formalism. In this connection, first of all, we generalize the *ordinary* Lagrangian density $\mathcal{L}_{(b)}^{(ad)}$ onto the suitably chosen (2, 1)-dimensional (anti-)chiral super-submanifolds (of the general (2, 2)-dimensional supermanifold) as follows:

$$\mathcal{L}_{(b)}^{(ad)} \longrightarrow \tilde{\mathcal{L}}_{(b)}^{(ad)} = \tilde{\mathcal{B}}^{(ad)}(x, \bar{\theta}) [\tilde{E}^{(ad)}(x, \bar{\theta}) - m \tilde{\Phi}^{(ad)}(x, \bar{\theta})]$$

$$+ \frac{1}{2} \mathcal{B}^{(ad)}(x, \bar{\theta}) \mathcal{B}^{(ad)}(x, \bar{\theta}) - \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + m \mathcal{D}^{(ad)}(x, \bar{\theta})$$

$$+ \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + m \mathcal{D}^{(ad)}(x, \bar{\theta})$$

$$+ \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + \frac{1}{2} \mathcal{D}^{(ad)}(x, \bar{\theta}) \mathcal{D}^{(ad)}(x, \bar{\theta}) + m \mathcal{D}^{(ad)}(x, \bar{\theta})$$

(128)
where the super Lagrangian density, with superscript \((AC, AD)\), contains the antichiral superfields that have been obtained after the applications of anti-co-BRST invariant restrictions (cf. Equation (102)). In exactly similar fashion, we note that the super Lagrangian density \(\mathcal{L}^{(C,D)}\) incorporates the chiral superfields that have been obtained after the applications of the co-BRST invariant restrictions (cf. Equation (102)). At this stage, taking the helps of the mappings: \(s^{(2)} \mapsto \partial \), \(s^{(2)} \mapsto \partial \), we observe the following:

\[
\begin{align*}
&\partial \mathcal{L}^{(AC, AD)} = \partial \left[ \mathcal{L}^{(C, D)} + m \phi^\nu (m A_\nu - \phi \partial_\nu C) \right] = \mathcal{L}^{(2)}_{(b)} , \\
&\partial \mathcal{L}^{(C, D)} = \partial \left[ \mathcal{L}^{(D)} + m \phi^\nu (m A_\nu - \phi \partial_\nu C) \right] = \mathcal{L}^{(2)}_{(b)} .
\end{align*}
\]

Thus, we note that we have captured the (anti-)co-BRST invariance of the action integral \(S = \int d^2 x \mathcal{L}^{(b)}\) because we observe that Equation (131) matches with Equation (38). It goes without saying that there are some superfields, with superscripts \((C, D)\) and \((AC, AD)\), which are actually ordinary fields on the 2D Minkowskian spacetime manifold as is evident from the observations \(s^{(2)}(\mathcal{B}, \phi) = 0\) as well as \(s^{(2)} C = 0, s^{(2)} ad C = 0\), etc.

We concentrate, at this stage, on capturing the restriction: \(\mathcal{B} + \mathcal{B} = 2 E = 0\) within the framework of the (anti-)chiral superfield approach to BRST formalism. In this context, we generalize the Lagrangian density \(\mathcal{L}^{(b)}\) to the \((2, 1)\)-dimensional (anti-)chiral super-submanifolds as follows:

\[
\mathcal{L}^{(2, 1)}(b) \rightarrow \mathcal{L}^{(AC, AD)} = \mathcal{B} (x, \phi) \left( E^{(D)}(x, \phi) + m \phi^{(D)}(x, \phi) \right)
\]

where all the superscripts and their implicants have been clarified earlier. It is now straightforward to check that the following are true, namely,

\[
\begin{align*}
&\mathcal{L}^{(2, 1)}(b) \rightarrow \mathcal{L}^{(AC, AD)} = \mathcal{B} (x, \phi) \left( E^{(D)}(x, \phi) + m \phi^{(D)}(x, \phi) \right) \\
&- m \mathcal{B}^{(A)}(x, \phi) \left( E^{(D)}(x, \phi) + m \phi^{(D)}(x, \phi) \right)
\end{align*}
\]
\[ \frac{\partial}{\partial \theta} \left[ \mathcal{L}^{(ACAD)}_{(b_1)} \right] = \partial_{\mu} \left[ \mathcal{B} \partial^\mu C + m e^{\mu \nu} (m A_{\nu} C + \phi \partial_{\nu} C) - m \phi \partial^\mu C \right] \\
- \left[ \mathcal{B} \mathcal{B} - 2 E \right] (\Box + m^2) C \equiv s_{ad}^{(2)} \mathcal{L}_{(b_1)}^{(2)}, \]

\[
\frac{\partial}{\partial \theta} \left[ \mathcal{L}^{(CD)}_{(b_1)} \right] = \partial_{\mu} \left[ \mathcal{B} \partial^\mu C + m e^{\mu \nu} (m A_{\nu} C + \phi \partial_{\nu} C) - m \phi \partial^\mu C \right] \\
- \left[ \mathcal{B} \mathcal{B} - 2 E \right] (\Box + m^2) C \equiv s_{ad}^{(2)} \mathcal{L}_{(b_1)}^{(2)}, \]

(133)

where we have taken into account the mappings: \( s_{ad}^{(2)} \rightarrow \partial_{\theta}, \)

\[ s_{ad}^{(2)} \rightarrow \partial_{\theta}. \]

Thus, we note that we have captured the restriction: \( \mathcal{B} + \mathcal{B} - 2 E = 0 \) which has appeared in Equation (40) in connection with the applications of the (anti-)BRST symmetry transformations \( s_{ad}^{(2)} \) on Lagrangian density \( \mathcal{L}_{(b_1)} \). It goes without saying that there are some superfields, with superscripts \( (C, D) \) that are primarily ordinary fields because of the observations: \( s_{ad}^{(2)}[\mathcal{B}, (\partial \cdot A), \phi] = 0 \)
as well as \( s_{ad}^{(2)} C = 0, s_{ad}^{(2)} C = 0, \) etc.

As we have captured the restriction in connection with the applications of \( s_{ad}^{(2)} \) on the Lagrangian density \( \mathcal{L}_{(b_1)} \), in exactly similar fashion, we now present the existence of the above restriction in the context of the applications of \( s_{ad}^{(1)} \) on the Lagrangian density \( \mathcal{L}_{(b_1)} \). Towards this goal in mind, we generalize the Lagrangian density \( \mathcal{L}_{(b_2)} \) onto \( (2, 1) \)-dimensional (anti-)chiral super-submanifolds as follows:

\[
\mathcal{L}_{(b_2)} \rightarrow \mathcal{L}_{(b_2)}^{(acad)} = \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \left( E^{(ad)} \right) \left( x, \bar{\theta} \right) \\
+ m \phi^{(ad)} \left( x, \bar{\theta} \right) \\
- \frac{1}{2} \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \\
- m \mathcal{E}^{(ad)} \left( x, \bar{\theta} \right) \phi^{(ad)} \left( x, \bar{\theta} \right) \\
- \frac{1}{2} \partial_{\mu} \phi^{(ad)} \left( x, \bar{\theta} \right) \partial^\mu \phi^{(ad)} \left( x, \bar{\theta} \right) \\
+ \frac{m^2}{2} B_{\mu}^{(ad)} \left( x, \bar{\theta} \right) B^{(ad)} \left( x, \bar{\theta} \right) \\
+ \frac{1}{2} \partial_{\mu} E^{(ad)} \left( x, \bar{\theta} \right) \partial^\mu \phi^{(ad)} \left( x, \bar{\theta} \right) \\
+ \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \left[ \partial_{\mu} B^{(ad)} \left( x, \bar{\theta} \right) \right] \\
- m \phi^{(ad)} \left( x, \bar{\theta} \right) \\
+ \frac{1}{2} \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \\
- i \partial_{\mu} F^{(ad)} \left( x, \bar{\theta} \right) \phi^{(ad)} \left( x, \bar{\theta} \right) \\
+ i m^2 \mathcal{F}^{(ad)} \left( x, \bar{\theta} \right) \mathcal{F}^{(ad)} \left( x, \bar{\theta} \right), \]

(134)

where all the symbols and superscripts have been clarified earlier. It is now elementary exercise to note that the following are true, namely,

\[
\frac{\partial}{\partial \theta} \left[ \mathcal{L}_{(b_2)}^{(acad)} \right] = \partial_{\mu} \left[ \mathcal{B} \partial^\mu C + m e^{\mu \nu} (m A_{\nu} C + \phi \partial_{\nu} C) - m \phi \partial^\mu C \right] \\
+ \frac{m^2}{2} \mathcal{B}_{\mu}^{(ad)} \left( x, \bar{\theta} \right) B^{(ad)} \left( x, \bar{\theta} \right) \\
+ \frac{1}{2} \partial_{\mu} E^{(ad)} \left( x, \bar{\theta} \right) \partial^\mu \phi^{(ad)} \left( x, \bar{\theta} \right) \\
+ \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \left[ \partial_{\mu} B^{(ad)} \left( x, \bar{\theta} \right) \right] \\
- m \phi^{(ad)} \left( x, \bar{\theta} \right) \\
+ \frac{1}{2} \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \mathcal{B}^{(ad)} \left( x, \bar{\theta} \right) \\
- \frac{1}{2} \partial_{\mu} F^{(ad)} \left( x, \bar{\theta} \right) \partial^\mu \mathcal{F}^{(ad)} \left( x, \bar{\theta} \right) \\
+ i m^2 \mathcal{F}^{(ad)} \left( x, \bar{\theta} \right) \mathcal{F}^{(ad)} \left( x, \bar{\theta} \right), \]

\[
\mathcal{L}_{(b_2)} \rightarrow \mathcal{L}_{(b_2)}^{(acad)} = \mathcal{B}^{(ad)} \left( x, \theta \right) \left( E^{(ad)} \left( x, \theta \right) + m \phi^{(ad)} \left( x, \theta \right) \right) \\
- \frac{1}{2} \mathcal{B}^{(ad)} \left( x, \theta \right) \mathcal{B}^{(ad)} \left( x, \theta \right) \\
- m \mathcal{E}^{(ad)} \left( x, \theta \right) \phi^{(ad)} \left( x, \theta \right) \\
- \frac{1}{2} \partial_{\mu} \phi^{(ad)} \left( x, \theta \right) \partial^\mu \phi^{(ad)} \left( x, \theta \right) \\
+ \mathcal{B}^{(ad)} \left( x, \theta \right) \left[ \partial_{\mu} B^{(ad)} \left( x, \theta \right) \right] \\
+ m B_{\mu}^{(ad)} \left( x, \theta \right) \partial^\mu \phi^{(ad)} \left( x, \theta \right) \\
+ \mathcal{B}^{(ad)} \left( x, \theta \right) \left[ \partial_{\mu} B^{(ad)} \left( x, \theta \right) \right] \\
- i \partial_{\mu} F^{(ad)} \left( x, \theta \right) \partial^\mu \mathcal{F}^{(ad)} \left( x, \theta \right) \\
+ i m^2 \mathcal{F}^{(ad)} \left( x, \theta \right) \mathcal{F}^{(ad)} \left( x, \theta \right), \]

(135)

The above equation shows that we have proven the sanctity of the restriction within the framework of (anti-)chiral superfield approach to BRST formalism (provided we do not take into account the mass-shell conditions: \( \Box + m^2 C = 0, \Box + m^2 C = 0 \) for the (anti-)ghost fields. It should be pointed that some of the superfields, with superscripts \( (ac, ad) \) and \( (c, d) \) are actually ordinary fields. For instance, we have \( \mathcal{F}^{(ad)} \left( x, \theta \right) = C \left( x \right) , \mathcal{F}^{(ad)} \left( x, \theta \right) = C \left( x \right) , \phi^{(ad)} \left( x, \theta \right) = \phi^{(ad)} \left( x, \theta \right) = \phi \left( x \right) , \) etc.

7. CF-Type Restrictions and Pseudoscalar Field with Negative Kinetic Term: A Few Comments

In this section, we dwell a bit on the existence of the (anti-)BRST and (anti-)co-BRST invariant restrictions (e.g., \( B + B + 2 \left( \partial \cdot A \right) = 0, B + B - 2 E = 0 \) that have appeared
(cf. Equation (33)) in our BRST analysis of the 2D modified version of the Proca theory and discuss their drastic differences and some similarities vis-à-vis the usual CF condition [21] that exists in the BRST analysis of the non-Abelian 1-form gauge theory. In the case of the latter theory (defined in any arbitrary dimension of spacetime), the coupled [34] Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_\beta \), in the Curci-Ferrari gauge [35, 36], are as follows [34, 37]:

\[
\mathcal{L}_B = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} + B \cdot (\partial_{\mu} A^\mu) + \frac{1}{2} \left( B \cdot B + \bar{B} \cdot \bar{B} \right) - i \partial_{\mu} \bar{C} \cdot D^\mu C, \\
\mathcal{L}_\beta = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} - \bar{B} \cdot (\partial_{\mu} A^\mu) + \frac{1}{2} \left( B \cdot B + \bar{B} \cdot \bar{B} \right) - i D_{\mu} \bar{C} \cdot \partial^\mu C, 
\]

where \( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i (A_{\mu} \times A_{\nu}) \) is the field strength tensor which is derived from the 2-form \( F^{(2)} = d A^{(1)} + i (A^{(1)} \wedge A^{(1)}) \) with the 1-form \( A^{(1)} = d x^\mu A_{\mu} \). The \( \text{SU}(N) \) non-Abelian symmetry transformations are expressed in terms of the generator \( T^a \) which obey the \( \text{SU}(N) \) Lie algebra \( [T^a, T^b] = f^{abc} T^c \). where \( a, b, c = 1, 2 \cdots \), \( N^2 - 1 \) are the group indices in the \( \text{SU}(N) \) Lie algebraic space where the cross and dot products between two non-null vectors \( (P^a, Q^a) \) are defined as \( (P \times Q)^a = f^{abc} P^b Q^c \) and \( P \cdot Q = P^a Q^a \). The covariant derivatives \( D_{\mu} C = \partial_{\mu} C + i (A_{\mu} \times C) \) and \( D_{\mu} \bar{C} = \partial_{\mu} \bar{C} + i (A_{\mu} \times \bar{C}) \) are in the adjoint representation of the \( \text{SU}(N) \) Lie algebra. For the \( \text{SU}(N) \) algebra, the structure constants \( f^{abc} \) can be chosen to be totally antisymmetric in all the indices (To be precise, for a specific representation of \( T^a \), the structure constants \( f^{abc} \) become totally antisymmetric for any arbitrary Lie algebra (see, e.g., [38] for details.) (see, e.g., [38] for details). In the above coupled Lagrangian densities \( B \cdot T = B^a T^a \) and \( B = \bar{B} \cdot T = \bar{B}^a T^a \) are the Nakanishi-Lautrup type auxiliary fields which obey the Curci-Ferrari (CF) condition [21] as follows:

\[
B + \bar{B} + (C \times \bar{C}) = 0, \tag{137}
\]

where \( C^a \) and \( \bar{C}^a \) are the fermionic [i.e., \( (C^a)^2 = (\bar{C}^a)^2 = 0 \), \( C^a C^b + C^b C^a = 0 \), \( \bar{C}^a \bar{C}^b + \bar{C}^b \bar{C}^a = 0 \), \( C^a \bar{C}^b + \bar{C}^b C^a = 0 \), \( \text{etc.} \)] ghost and antighost fields which are needed in the theory to maintain the unitary at any arbitrary order of perturbative computations.

The CF condition (137) emerges out when we equate \( \mathcal{L}_B \) and \( \mathcal{L}_\beta \) and demand their equivalence (modulo a total spacetime derivative term). To be precise, we have the following:

\[
\mathcal{L}_B \equiv \mathcal{L}_\beta \implies B + \bar{B} + (C \times \bar{C}) = 0. \tag{138}
\]

In other words, the very existence of the coupled Lagrangian densities \( L_B \) and \( L_\beta \) depends crucially on the CF condition. Furthermore, the absolute anticommutativity of the nilpotent (anti-)BRST symmetry transformations (i.e., \( s_b s_{ab} + s_{ab} s_b = 0 \)) is valid only when the CF condition is satisfied in the non-Abelian 1-form gauge theory. This also gets reflected at the level of the conserved and nilpotent charges \( Q_{(ab)} \) because the absolute anticommutativity of these charges (i.e., \( (Q_{a} Q_{ab} + Q_{ab} Q_{b} = 0) \)), once again, crucially depends on the existence of CF condition \( (B + \bar{B} + (C \times \bar{C}) = 0) \). In addition, we note that the applications of the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations \( s_{(ab)} \) on the Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_\beta \) of the non-Abelian 1-form gauge theory lead to the following (see, e.g., [39]):

\[
s_b \mathcal{L}_B = \partial_\mu \left[ B \cdot \partial^\mu C \right], \quad s_{ab} \mathcal{L}_B = -\partial_\mu \left[ B \cdot \partial^\mu \bar{C} \right], \quad s_b \mathcal{L}_\beta = \partial_\mu \left[ (B + (C \times \bar{C})) \cdot \partial^\mu C \right] - \left\{ B + \bar{B} + (C \times \bar{C}) \right\} \cdot D_\mu \partial^\mu C, \tag{139}
\]

The off-shell nilpotent (anti-)BRST symmetry transformations for the coupled Lagrangian densities (136) are \( s_{ab} A_\mu = D_\mu C, s_{ab} \bar{C}_\mu = -(i/2) (C \times \bar{C}), s_{ab} C_\mu = i B, s_{ab} \bar{B}_\mu = 0, s_{ab} F_{\mu \nu} = i (F_{\mu \nu} \times C), s_{ab} \partial_\mu A^\mu = \partial_\mu D^\mu C, s_{ab} \partial_\mu \bar{C} = i (B \times C), s_{ab} \partial_\mu \partial^\mu C = 0, s_{ab} \partial_\mu \partial^\mu \bar{C} = 0, s_{ab} \partial_\mu \partial^\mu \bar{C} = 0 \), \( s_{ab} \partial_\mu }
Thus, we find that all the four restrictions (that have been pointed out in (33)) appear very naturally in the equality: $\mathcal{L}_{(b)} = \mathcal{L}_{(b)}$ (modulo some total spacetime derivatives). Therefore, we conclude that one of the solutions of (141) is nothing but the restrictions derived in (33). This observation is exactly similar to our observation in the context of non-Abelian 1-form gauge theory (cf. Equation (138)). We now focus on the symmetry properties of the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ which have been illustrated in Equation (34). We observe that both the Lagrangian densities respect both the (anti-)co-BRST and (anti-)BRST symmetries of the (anti-)co-BRST symmetry transformations $s^{(12)}_{(a)b}$ on the constrained hypersurface where $B + B + 2(\bar{\theta} \cdot A) = 0$ is satisfied. Thus, there is, once again, similarity between our 2D modified Proca (i.e., massive Abelian 1-form gauge) theory and the non-Abelian 1-form gauge theory (cf. Equation (140)). We point out that the restriction $B + B + 2E = 0$ also appears in (141) which is, once again, similar to the observation in 2D non-Abelian theory in the context of the existence of the off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetries [39].

We would like to point out here that both the factorized terms in Equation (141) are zero separately and independently because both of them owe their origins to mathematically independent cohomological operators of differential geometry. For instance, as pointed out earlier, the restriction $(\mathcal{B} + \mathcal{B} - 2E = 0)$ owes its origin to the exterior derivative $d = dx^\mu \delta^\mu_p$ (with $d^2 = 0$) because the 2-form $F^{(2)} = dA^{(1)} \equiv (1/2)(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ defines the field strength tensor $F_{\mu\nu}$ which possesses only one nonzero component in 2D (that is nothing but the electric field $E$). In exactly similar fashion, we note that the restriction: $B + B + 2(\bar{\theta} \cdot A) = 0$ owes its origin to the coexterior derivative $\delta = -d \ast$ because we observe that $\delta A^{(1)} = -d \ast A^{(1)} = (\bar{\theta} \cdot A)$ which defines the gauge-fixing term of the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ (where we generalize it to $[\bar{\theta} \cdot A \pm m \phi]$) on the dimensional ground. Since the cohomological operators ($d$) and ($\delta$) are linearly independent of each other, we argue that both the terms of Equation (141) would vanish off on their own. At present level of our understanding, we do not know as to why the restrictions $\mathcal{B} + \mathcal{B} - 2E = 0$ and $B + B + 2(\bar{\theta} \cdot A) = 0$ are picked out, from Equation (33), in the discussions of the (anti-)co-BRST and (anti-)BRST symmetry transformations of the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(b)}$ but the other constraints $B - B + 2m \phi = 0$ and $\mathcal{B} + \mathcal{B} + 2m \phi = 0$ are not utilized by the (anti-)co-BRST and (anti-)BRST symmetries of our 2D Proca theory.

We would like to mention a few things connected with the 2D non-Abelian 1-form gauge theory which we have discussed in our earlier work [39] where we have shown the existence of the (anti-)co-BRST symmetries (in addition to (anti-)BRST symmetries). In fact, we have derived the off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetry transformation for the 2D non-Abelian 1-form gauge theory under which the Lagrangian densities and, specifically, the gauge-fixing term remain invariant. To be precise, we have considered the following coupled Lagrangian densities for the 2D non-Abelian gauge theory [34, 37] in the Curci-Ferrari gauge [35, 36] for our discussions, namely,

$$
\mathcal{L}_B = \mathcal{B} \cdot E - \frac{1}{2} \mathcal{B} \cdot \mathcal{B} + B \cdot (\bar{\partial}_\mu A^\mu) \\
\left. \right| + \frac{1}{2} \left( B \cdot B + B \cdot \bar{B} \right) - i \bar{\partial}_\mu \bar{C} \cdot D^\mu C,
$$

where $\mathcal{B} = \mathcal{B} \cdot T$ is the Nakanoishi-Lautrup type auxiliary field which has been invoked to linearize the kinetic term ($(1/2) E \cdot E = -(1/4) F_{\mu\nu} F^{\mu\nu}$) for the 2D non-Abelian theory. It is clear that, in 2D spacetime, we have only one existing component of $F_{\mu\nu}$ (i.e., $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i(A_\mu \times A_\nu) \equiv E$). We have found out that the following are true, namely,

$$
\begin{align*}
\mathcal{E}_B & = \mathcal{E}_B \cdot A = \partial_\mu \left[ \mathcal{B} \cdot \partial^\mu C \right], \\
\partial_\mu \mathcal{L}_B & = \partial_\mu \left[ \mathcal{B} \cdot \partial^\mu C \right], \\
\mathcal{E}_B & = \partial_\mu \left[ \mathcal{B} \cdot \partial^\mu C - e^{\mu \nu} (\partial_\nu A \times C) \right] C \left| + i(\bar{\partial}_\nu A^\nu) \cdot (\mathcal{B} \times C), \right.
\end{align*}
$$

which demonstrate that for the both the (anti-)co-BRST symmetries to be respected by both Lagrangian densities, we need to invoke the following restrictions:

$$
\begin{align*}
\mathcal{B} \times C & = 0, \\
\mathcal{B} \times \bar{C} & = 0.
\end{align*}
$$

The off-shell nilpotent and absolutely anticommuting (anti-)co-BRST symmetries for the 2D coupled Lagrangian densities (142) are e.g. $s_{\mathcal{A}d} A_\mu = -e_\rho \partial^\rho C$, $s_{\mathcal{A}d} \mathcal{C} = 0$, $s_{\mathcal{A}d} C = i B$, $s_{\mathcal{A}d} B = 0$, $s_{\mathcal{A}d} E = D_\mu \partial^\mu C$, $s_{\mathcal{A}d} B = 0$, $s_{\mathcal{A}d} (\bar{\partial}_\mu A^\mu) = 0$, and $s_{\mathcal{A}d} A_\mu = -e_\rho \partial^\rho C$, $s_{\mathcal{A}d} \mathcal{C} = -i B$, $s_{\mathcal{A}d} C = 0$, $s_{\mathcal{A}d} B = 0$, $s_{\mathcal{A}d} E = D_\mu \partial^\mu C$, $s_{\mathcal{A}d} B = 0$, $s_{\mathcal{A}d} (\bar{\partial}_\mu A^\mu) = 0$ (see, e.g., [39] for details). It should be pointed out, at this stage, that the restriction (cf. Equation (40)) that emerges out in our discussions on the 2D modified Proca theory (i.e., $\mathcal{B} + \mathcal{B} - 2E = 0$) is analogous to the restrictions $\mathcal{B} = \mathcal{C}$ and $\mathcal{B} \times \bar{C} = 0$ that are essential for the BRST analysis of the 2D non-Abelian theory. Hence, there is some kind of similarity.

Now we pin-point a few distinct differences between the restrictions of our 2D modified Proca theory and standard non-Abelian 1-form gauge theory. In the context of the latter, we know that the (anti-)BRST symmetry transformations $s_{(a)b}$ absolutely anticommute (i.e., $s_b s_{ab} + s_{ab} s_b = 0$) only
when we impose the CF condition $B + \tilde{B} + (C \times \tilde{C}) = 0$. This observation is not true in the context of our 2D modified Proca theory because we observe that only the pairs $(s^{(1)}_{ab}, s^{(1)}_{b})$ and $(s^{(2)}_{ab}, s^{(2)}_{b})$ absolutely anticommute without any recourse to the restrictions (33). Except the above pairs, we point out that the rest of the fermionic symmetry transformations $s^{(1)}_{a\alpha\beta}$ and $s^{(2)}_{a\alpha\beta}$ do not absolutely anticommute (even if the restrictions (33) are imposed from outside). These kinds of results are also true in the case of (anti-)co-BRST symmetry transformations $s^{(1,2)}_{a\alpha\beta}$. We have collected all such possible anticommutators in Appendix A. The above observations, at the symmetry level, are also reflected at the level of conserved charges because we find that the pairs $(Q^{(1)}_{\bar{a}}, Q^{(1)}_{a\alpha\beta})$, $(Q^{(2)}_{\bar{a}}, Q^{(2)}_{a\alpha\beta})$, $(Q^{1}_{\bar{a}}, Q^{1}_{a\alpha\beta})$, and $(Q^{2}_{\bar{a}}, Q^{2}_{a\alpha\beta})$ absolutely anticommute but other possible combinations do not absolutely anticommute even if we impose the restrictions (33) from outside. We have collected all these results in Appendix B. Furthermore, we note that, in the CF condition (137), there is no gauge field. However, we find that in the restriction $B + \tilde{B} + 2(\partial \cdot A) = 0$ (connected with the (anti-)BRST symmetries), the gauge field appears in the form of Lorentz gauge (i.e., $(\partial \cdot A)$) and electric field appears in $\mathcal{A} + \tilde{\mathcal{A}} - 2E = 0$ which is the restriction in the context of (anti-)co-BRST symmetry transformations for our 2D modified version of Proca theory.

At this juncture, we comment on the appearance of a pseudoscalar field $\phi$ in our theory which is endowed with the negative kinetic term but it possesses a properly well-defined mass (as it satisfies the Klein-Gordon equation $(\Box + m^2)\phi = 0$). In fact, we observe that this pseudoscalar field is essential for our discussion because we have shown the existence of a set of appropriate discrete symmetries (cf. Equation (43)) which provide the physical realizations of the Hodge duality * operation of differential geometry (cf. Equation (45)). Thus, the appearance of such kind of term is very natural in our whole discussion. We would like to point out that such kinds of fields have become very popular in the realm of cosmology where these kinds of fields have been christened as the "ghost" fields (which are distinctly different from the fermionic Faddeev-Popov ghost terms) (see, e.g., [40–48]). Such kinds of fields have also been proposed as the candidates for the dark matter and dark energy in modern literature (see, e.g., [49, 50]). In the context of the dark energy, these fields have no mass (which is the massless limit of the massive field theory with only the negative kinetic term(s) for the field(s) but without any explicit mass term).

We end this section with the remark that we have generalized our present discussion to the 4D massive theory of Abelian 2-form gauge theory [51] where we have discussed the physical implications of the existence of such kinds of fields (which are endowed with negative kinetic terms but properly defined mass) in the context of bouncing, cyclic, and self-accelerated models of Universe [52–57].

8. Conclusions

In our present investigation, we have considered the Stückelberg-modified version of the 2D Proca theory and shown that there are two Lagrangian densities for this theory which respect the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations besides respecting the ghost-scale and bosonic continuous symmetries. There exists a couple of discrete symmetries, too, in our theory which make both the above Lagrangian densities represent a couple of field-theoretic examples of Hodge theory because all the above symmetries, taken together, provide the physical realizations of the de Rham cohomological operators [7–11] of differential geometry at the algebraic level.

We have applied the (anti-)chiral superfield approach to derive the fermionic (anti-) BRST and (anti-)co-BRST symmetry transformations where we have defined the superfields on the $(2, 1)$-dimensional (anti-)chiral supermanifolds of the general $(2, 2)$-dimensional supermanifold on which our 2D theory has been generalized. One of the key results of our present endeavor has been the proof of the off-shell nilpotency and absolute anticommutativity of the (anti-)BRST and (anti-)co-BRST charges. For instance, we have shown that the off-shell nilpotency $(Q^2_{\bar{a}} = 0)$ of the BRST charge is deeply connected with nilpotency $((\partial_{\theta})^2 = 0)$ of the translational generator $(\partial_{\theta})$ along the $\theta$-direction of the antichiral super-submanifold. However, the absolute anticommutativity of the BRST charge with the anti-BRST charge has been found to be encoded in the nilpotency $(Q^2_{\bar{a}} = 0)$ of the translational generator $(\partial_{\theta})$ along the $\theta$-direction of the chiral super-submanifold of the general $(2, 2)$-dimensional supermanifold. Similar kinds of statements could be made in connection with the other fermionic charges (e.g., anti-BRST and (anti-)co-BRST charges) of our present theory. These observations are completely novel results within the framework of the superfield approach to BRST formalism (cf. Section 5 for details).

The novel observations, in our discussions on 2D modified Proca theory, are (i) the introduction of a pseudoscalar field (on symmetry ground) which is endowed with the negative kinetic term and (ii) the existence of the restrictions which are found to have some kinds of similarities and a few distinct differences with the standard CF condition [21] that exists in the realm of BRST approach to non-Abelian 1-form gauge theory in any arbitrary dimension of spacetime. Thus, we note that our restrictions (cf. (33)) exist only in 2D modified model of Proca theory but the standard CF condition [21] exists for the non-Abelian 1-form theory in any arbitrary $D$-dimension of spacetime. Our restrictions do not play any role in the proof of absolute anticommutativity property (cf. Section 7). The existence of a pseudoscalar field with negative kinetic term is important because it is a precursor to the existence of such kinds of fields in 4D theory where it is expected to play important role in the cosmological models of Universe and it might provide a clue to the ideas behind the dark matter/dark energy (see, Section 7 for details).
In our present endeavor, we have concentrated only on the 2D massive Abelian (i.e., Stückelberg-modified) 1-form gauge theory. However, we expect that this analysis could be generalized to the physical four (3 + 1)-dimensions of spacetime. In this context, we would like to mention our recent work [51] on the 4D massive Abelian 2-form gauge theory where we have shown the existence of a pseudoscalar and an axial-vector fields with negative kinetic terms. Both the above models (i.e., the 2D modified Proca and 4D Abelian 2-form theories) are the massive field-theoretic examples of Hodge theory. We plan to apply our ideas to the massive 6D Abelian 3-form gauge theory and find out the consequences therein. The massless version of the latter theory has already been proven to be a tractable field-theoretic example of Hodge theory in our earlier work (see, e.g., [14]).

We speculate that the massive models of Hodge theory would solve the problem of dark matter/dark energy from the point of view of symmetries as these field-theoretic models would invoke some new kinds of fields endowed with a few exotic physical properties. These theories might turn out to be useful in the context of cosmology, too, where one requires the existence of “ghost” fields (i.e., scalar fields with negative kinetic terms) [40–48]. We are, at present, very much involved with the massive version of gauge theories and we plan to prove these theories to be the models for the Hodge theory. In the process, we shall be discussing about the field/particles with exotic properties [58] which might turn out to be useful in the context of various kinds of cosmological models. There is yet another direction that could be explored in the future even in the case of an interacting 2D Proca theory where there exists a coupling between the massive Abelian 1-form gauge field and the Dirac fields (see, e.g., [4] for details).

Appendix

A. On the Absolute Anticommutativity of Nilpotent Symmetries

We have already seen that the pairs \((s_b^{(1)}, s_{ab}^{(1)})\), \((s_b^{(2)}, s_{ab}^{(2)})\), \((s_d^{(1)}, s_{ad}^{(1)})\), and \((s_d^{(2)}, s_{ad}^{(2)})\) absolutely anticommute (separately and independently) without any recourse to the restrictions that have been listed in Equation (33). However, we show here that the other combinations of the above fermionic symmetries do not absolutely anticommute. In this context, we note that the following combinations of the (anti-)BRST symmetries \(s^{(1,2)}_{a[b}\)

\[
\{s_b^{(1)}, s_b^{(2)}\}, \{s_b^{(1)}, s_{ab}^{(2)}\}, \{s_d^{(1)}, s_{ad}^{(2)}\}, \{s_d^{(2)}, s_{ad}^{(2)}\}, \{s_{ab}^{(1)}, s_{ab}^{(2)}\}.
\]  

(A.1)

are the nontrivial anticommutators which we have to apply on all the fields of our theory (that has been described by the Lagrangian densities \(\mathcal{L}_{(b)}\) and \(\mathcal{L}_{(d)}\)). It is straightforward, in this connection, that the following is true, namely,

\[
\{s_b^{(1)}, s_b^{(2)}\} \Psi = 0,
\]

(A.2)

\[
\Psi = A_\mu, C, \phi, \bar{\phi}, B, \bar{B}, \mathcal{R}, \mathcal{\bar{R}},
\]

except the nonzero anticommutator:

\[
\{s_b^{(1)}, s_{ab}^{(2)}\} C = -4\Box C.
\]

(A.3)

Thus, we conclude that the BRST symmetry transformations \(s_b^{(1)}\) and \(s_{ab}^{(2)}\) do not absolutely anticommute with each other. Hence, their corresponding charges \(Q_b^{(1)}\) and \(Q_{ab}^{(2)}\) would also not absolutely anticommute with each other (cf. Appendix B below).

We focus now on the computation of the anticommutator \(\{s_{ab}^{(1)}, s_{ab}^{(2)}\}\) for our theory. In this connection, we observe the following:

\[
\{s_b^{(1)}, s_{ab}^{(2)}\} \Psi = 0,
\]

(A.4)

\[
\Psi = \bar{\phi}, \mathcal{R}, \mathcal{\bar{R}}.
\]

However, the other fields (e.g., \(A_\mu, C, \bar{C}, \phi, B, \bar{B}, B, B\)) obviously do not satisfy (A.4). We list here, the nontrivial anticommutators (acting on these fields), as follows:

\[
\{s_b^{(1)}, s_{ab}^{(2)}\} A_\mu = i\partial_\mu (B - \bar{B})
\]

\[
= -2im\partial_\mu \phi, \{s_b^{(1)}, s_{ab}^{(2)}\} C = 2\Box C,
\]

\[
\{s_b^{(1)}, s_{ab}^{(2)}\} \bar{C} = -2\Box \bar{C}, \{s_{ab}^{(1)}, s_{ab}^{(2)}\} \phi
\]

\[
= -im (B + \bar{B}) = 2im (\partial \cdot A),
\]

\[
\{s_b^{(1)}, s_{ab}^{(2)}\} B = -2\Box B, \{s_{ab}^{(1)}, s_{ab}^{(2)}\} B = 2\Box B,
\]

(A.5)

where we have also exploited the restrictions (33) to demonstrate that even if we impose them from outside, the above anticommutators are not zero. We proceed ahead and compute the anticommutator \(\{s_{ab}^{(1)}, s_{ab}^{(2)}\}\). In this context, we observe the following:

\[
\{s_{ab}^{(1)}, s_{ab}^{(2)}\} \Psi = 0,
\]

(A.6)

\[
\Psi = A_\mu, C, \phi, \bar{\phi}, B, \bar{B}, \mathcal{R}, \mathcal{\bar{R}}.
\]

However, we note that the following is true, namely,

\[
\{s_{ab}^{(1)}, s_{ab}^{(2)}\} C = 4\Box C,
\]

(A.7)

which demonstrates that the absolute anticommutativity between \(s_{ab}^{(1)}\) and \(s_{ab}^{(2)}\) is not satisfied for our 2D theory because one of the fields (i.e., \(C\)) does not respect it.
We concentrate now on the last nontrivial anticommutator amongst the (anti-)BRST symmetry transformations \( s_{(1,2)}^{(1,2)} \). In this connection, we note the following:

\[
\begin{align*}
\{ s_{b}^{(1,2)} , s_{ab}^{(1)} \} \Psi &= 0, \\
\Psi &= \bar{\phi} B, C, B, \bar{B}, \tag{A.8}
\end{align*}
\]

which is just like our observation in (A.4). The nonvanishing and nontrivial anticommutators in this regards are as follows:

\[
\begin{align*}
\{ s_{b}^{(1,2)} , s_{ab}^{(1)} \} A_{\mu} &= -i \partial_{\mu} (B - B) = 2 i m \partial_{\mu} \phi, \\
\{ s_{b}^{(2)} , s_{ab}^{(1)} \} C &= 2 \Box C, \\
\{ s_{b}^{(2)} , s_{ab}^{(1)} \} = -2 \Box C, \\
\{ s_{b}^{(1,2)} , s_{ab}^{(1)} \} B &= -2 i \Box B, \\
\{ s_{b}^{(1)} , s_{ab}^{(1)} \} B &= 2 \Box B, \tag{A.9}
\end{align*}
\]

where we have used the restrictions (33) to demonstrate that the anticommutator \( \{ s_{b}^{(1,2)} , s_{ab}^{(1)} \} \) is not zero (in spite of their imposition of (33) from outside).

At this stage, we now take up the computation of the possible anticommutators amongst \( s_{(1,2)}^{(1,2)} \) with our background knowledge that the pairs \( (s_{b}^{(1,2)} , s_{ab}^{(1)} ) \) and \( (s_{b}^{(2)} , s_{ab}^{(1)} ) \) absolutely anticommutate without any use of the restrictions (33). The nontrivial anticommutators from the four fermionic operators \( (s_{(1,2)}^{(1,2)} ) \) are as follows:

\[
\begin{align*}
\{ s_{d}^{(1)} , s_{d}^{(2)} \} , \{ s_{d}^{(1)} , s_{ad}^{(1)} \} , \{ s_{ad}^{(1)} , s_{d}^{(2)} \} , \{ s_{ad}^{(1)} , s_{ad}^{(2)} \} . \tag{A.10}
\end{align*}
\]

It turns out that the following general observation is correct:

\[
\begin{align*}
\{ s_{d}^{(1)} , s_{d}^{(2)} \} \Psi &= 0, \\
\Psi &= A_{\mu} C, \phi, \bar{\phi}, B, \bar{B}, B, \bar{B}, \tag{A.11}
\end{align*}
\]

The above anticommutator proves the fact that the fermionic operators \( s_{d}^{(1)} \) and \( s_{d}^{(2)} \) are not absolutely anticommuting in nature. Next, we focus on the evaluation of \( \{ s_{ad}^{(1)} , s_{ad}^{(2)} \} \) where we find that \( \{ s_{ad}^{(1)} , s_{ad}^{(2)} \} \Psi = 0, \Psi = A_{\mu} C, \phi, \bar{\phi}, B, \bar{B}, B, \bar{B}, \) for the generic field \( \Psi \) of our theory. However, we observe that the following is true, namely,

\[
\begin{align*}
\{ s_{ad}^{(1)} , s_{ad}^{(2)} \} C &= -4 i \Box C. \tag{A.12}
\end{align*}
\]

Hence, \( s_{ad}^{(1)} \) and \( s_{ad}^{(2)} \) do not absolutely anticommutate with each other (just like \( s_{d}^{(1)} \) and \( s_{d}^{(2)} \)).

We take up now the anticommutator \( \{ s_{d}^{(1)} , s_{ad}^{(2)} \} \). In this context, we note that the following nontrivial anticommutators are true, namely,

\[
\begin{align*}
\{ s_{d}^{(2)} , s_{ad}^{(1)} \} A_{\mu} &= -i \epsilon_{\mu \nu} \partial^{\nu} (B - B) \\
&= -i m \epsilon_{\mu \nu} \partial^{\nu} , \tag{A.13}
\end{align*}
\]

Thus, we find that even the impositions of the restrictions (33) do not help in making \( s_{d}^{(2)} \) and \( s_{ad}^{(1)} \) absolutely anticommuting in nature. However, we observe that

\[
\begin{align*}
\{ s_{d}^{(2)} , s_{ad}^{(1)} \} \Psi &= 0, \\
\Psi &= \phi, B, B. \tag{A.14}
\end{align*}
\]

In other words, we get the result that \( s_{d}^{(2)} \) and \( s_{ad}^{(1)} \) absolutely anticommutate only for the fields \( \phi, B, B \). The last nontrivial anticommutator \( \{ s_{d}^{(1)} , s_{ad}^{(2)} \} \) is found to be absolutely anticommuting only for the fields \( \phi, B, B \). However, we find that the following nontrivial and nonzero anticommutators exist, namely,

\[
\begin{align*}
\{ s_{ad}^{(1)} , s_{d}^{(2)} \} A_{\mu} &= i \epsilon_{\mu \nu} \partial^{\nu} (B - B) \\
&= -i m \epsilon_{\mu \nu} \partial^{\nu} , \tag{A.15}
\end{align*}
\]

We end this Appendix with the remarks that all the fermionic transformations \( s_{(1,2)}^{(1,2)} \) and \( s_{ad}^{(1,2)} \) do not absolutely anticommutate amongst themselves.

**B. On the Absolute Anticommutativity of Nilpotent Charges**

We have already witnessed and verified that the pairs \( (Q_{(1,2)}, Q_{(1)}^{(1,2)}), (Q_{(1,2)}, Q_{(1)}^{(1,2)}), (Q_{(2)}, Q_{(2)}^{(2)}), \) and \( (Q_{(2)}^{(2)}, Q_{(2)}^{(2)}) \) absolutely anticommutate with each other (separately and independently). We have also noted that all these charges are off-shell nilpotent \( ([Q_{(1,2)}^{(1,2)}, 0 - [Q_{(1,2)}^{(1,2)}, 0] \because of the following are true, namely,
\[ s_b^{(1)} Q_b^{(1)} = 0, s_{ab}^{(1)} Q_{ab}^{(1)} = 0, s_d^{(1)} Q_d^{(1)} = 0, s_{ad}^{(1)} Q_{ad}^{(1)} = 0, \]
\[ s_b^{(2)} Q_b^{(2)} = 0, s_{ab}^{(2)} Q_{ab}^{(2)} = 0, s_d^{(2)} Q_d^{(2)} = 0, s_{ad}^{(2)} Q_{ad}^{(2)} = 0, \]
where we have used the relationship between the continuous symmetry transformations and their generators (cf. Equation (6)). For example, we note that \( s_b^{(1)} Q_b^{(1)} = -i \{ Q_b^{(1)}, Q_b^{(1)} \} = 0 \Rightarrow (Q_b^{(1)})^2 = 0 \). In fact, we have applied the fermionic transformations (26), (27), (35), and (37) directly on the charges (49) and (57) for the purpose of computations of (B.1). We use the expressions for the charges (cf. Equations (49) and (57)) and nilpotent symmetry transformations (cf. Equations (26), (27), (35) and (37)) to compute all the possible nontrivial anticommutators amongst the conserved and off-shell nilpotent charges \( (Q^{(1,2)}_{i(ab)}) \). These basic nontrivial anticommutators for the (anti-)BRST charges are as follows:

\[ \{ Q_b^{(1)}, Q_b^{(2)} \}, \{ Q_b^{(1)}, Q_{ab}^{(2)} \}, \{ Q_b^{(2)}, Q_{ab}^{(1)} \}, \{ Q_{ab}^{(1)}, Q_{ab}^{(2)} \}. \]

The above brackets can be computed from the following direct applications of the nilpotent (anti-)BRST symmetries (26) and (27) on the charges (49) and (57), namely,

\[ s_b^{(1)} Q_b^{(2)} = -i \{ Q_b^{(2)}, Q_b^{(1)} \}, s_{ab}^{(1)} Q_{ab}^{(2)} = -i \{ Q_{ab}^{(2)}, Q_b^{(1)} \}, \]
\[ s_b^{(2)} Q_b^{(1)} = -i \{ Q_b^{(2)}, Q_b^{(1)} \}, s_{ab}^{(2)} Q_{ab}^{(1)} = -i \{ Q_{ab}^{(2)}, Q_{ab}^{(1)} \}. \]

The explicit computations of the l.h.s of the above equation are as follows:

\[ s_b^{(1)} Q_b^{(2)} = 4 m^2 \int dx \hat{C} \hat{C}, s_{ab}^{(1)} Q_{ab}^{(2)} = \int dx \left[ 2 m^2 \left( \hat{C} \hat{C} - \hat{C} \hat{C} \right) + i \left( \hat{B} \hat{B} - \hat{B} \hat{B} \right) \right], \]
\[ s_b^{(2)} Q_b^{(1)} = \int dx \left[ 2 m^2 \left( \hat{C} \hat{C} - \hat{C} \hat{C} \right) + i \left( \hat{B} \hat{B} - \hat{B} \hat{B} \right) \right], s_{ab}^{(2)} Q_{ab}^{(1)} = 4 m^2 \int dx \hat{C} \hat{C}. \]

Thus, it is crystal clear that the nontrivial anticommutators, listed in Equation (B.2), are nonvanishing. Hence, we conclude that, except the anticommutators \( \{ Q_b^{(1)}, Q_{ab}^{(1)} \} \) and \( \{ Q_b^{(2)}, Q_{ab}^{(2)} \} \), rest of the nontrivial anticommutators are nonzero (i.e., nonvanishing).

We perform similar exercise with the nilpotent charges \( Q_{i(ab)}^{(1,2)} \) and observe that the following nontrivial anticommutators amongst these (anti-)co-BRST charges, namely,

\[ \{ Q_d^{(1)}, Q_d^{(2)} \}, \{ Q_{ad}^{(1)}, Q_{ad}^{(2)} \}, \{ Q_d^{(2)}, Q_ad^{(1)} \}, \{ Q_{ad}^{(1)}, Q_{ad}^{(2)} \}, \]

are to be evaluated using the basic principle of the continuous symmetry transformations and their generators (cf. Equation (6)). In this connection, we find that

\[ s_d^{(1)} Q_d^{(2)} = -i \{ Q_d^{(2)}, Q_d^{(1)} \} = 4 m^2 \int dx \hat{C} \hat{C}, \]
\[ s_d^{(2)} Q_d^{(1)} = -i \{ Q_d^{(2)}, Q_d^{(1)} \} = \int dx \left[ 2 m^2 \left( \hat{C} \hat{C} - \hat{C} \hat{C} \right) + i \left( \hat{B} \hat{B} - \hat{B} \hat{B} \right) \right] \]
\[ + i \left( \hat{B} \hat{B} - \hat{B} \hat{B} \right), \]
\[ s_{ad}^{(1)} Q_d^{(2)} = -i \{ Q_d^{(2)}, Q_{ad}^{(1)} \} = \int dx \left[ 2 m^2 \left( \hat{C} \hat{C} - \hat{C} \hat{C} \right) + i \left( \hat{B} \hat{B} - \hat{B} \hat{B} \right) \right] \]
\[ + i \left( \hat{B} \hat{B} - \hat{B} \hat{B} \right), \]
\[ s_{ad}^{(2)} Q_d^{(1)} = -i \{ Q_d^{(2)}, Q_{ad}^{(1)} \} = 4 m^2 \int dx \hat{C} \hat{C}. \]

The above equation encapsulates the results that the nontrivial anticommutators amongst \( Q^{(1,2)}_{i(ab)} \) are nonzero establishing the fact that the absolute anticommutativity amongst the (anti)-co-BRST charges is not true even if we impose the restrictions (33) from outside. Only the exceptions to these observations are as follows:

\[ s_d^{(1)} Q_d^{(1)} = -i \{ Q_d^{(1)}, Q_d^{(1)} \} \equiv s_d^{(1)} Q_d^{(1)}, \]
\[ s_d^{(2)} Q_d^{(2)} = -i \{ Q_d^{(2)}, Q_d^{(2)} \} \equiv s_d^{(2)} Q_d^{(2)}, \]

where there is no need of any kind of restrictions from (33) because the pairs \( (Q_d^{(1)}, Q_d^{(1)}) \) and \( (Q_d^{(2)}, Q_d^{(2)}) \) absolutely anticommute (separately and independently).

**C. On the Derivation of \( \kappa_1 = +1 \)**

We provide here an explicit computation of our result \( \kappa_1 = +1 \) in the case of determination of the secondary field \( R_\mu(x) \) for the expansion (cf. Equation (64)).

\[ B_\mu \left( x, \overline{\theta} \right) = A_\mu(x) + \overline{\theta} R_\mu(x) = A_\mu(x) + \overline{\theta} \left( \kappa_1 \partial_\mu C(x) \right), \]

where we have taken \( R_\mu(x) = \kappa_1 \partial_\mu C \) (because of the restriction: \( B^\mu(x, \overline{\theta}) \partial_\mu C = A^\mu(x) \partial_\mu C(x) \) which leads to \( R_\mu(x) = \kappa_1 \partial_\mu C \). In fact, the constant \( \kappa_1 \) is just a numerical constant which has to be determined precisely. We further note that a close look at Equations (63) and
(65) shows that \( m \kappa_1 = \kappa_2 \) where \( \kappa_2 \) is a numerical constant in
\[
\Phi^{(m)}(x, \bar{\theta}) = \phi(x) + \bar{\theta} (\kappa_2 C(x)). \tag{C.2}
\]

Taking into account the top restriction in (66), we observe that we have \( m B_2(x) = \kappa_1 B(x) \) which reduces to \( B_2(x) = \kappa_1 B(x) \) due to our earlier derived relationship: \( m \kappa_1 = \kappa_2 \). At this stage, the last restriction in (66) yields the following:
\[
B_2(x) \partial B(x) = B_2(x) B(x) \rightarrow \frac{1}{B} \frac{dB}{dt} = \frac{1}{B} \frac{dB}{dt}. \tag{C.3}
\]

Integrating the above equation w.r.t. the “time” variable \( t \), we obtain the following:
\[
\ln B_2(x) = \ln (B(x)) + C, \tag{C.4}
\]
where \( C \) is a numerical constant. Substituting \( B_2 = \kappa_1 B \), we obtain \( \ln (\kappa_1) = C \) which shows that \( C \) can be made equal to zero by the choice \( \kappa_1 = 1 \). The latter choice immediately leads to \( \kappa_2 = m \). These lead to the explicit expressions for (C.1) and (C.2) as the ones which lead to the derivation of \( \phi_1^{(1)} \), namely,
\[
\begin{align*}
B_\mu^{(b)}(x, \bar{\theta}) &= A_\mu(x) + \bar{\theta} (\partial_\mu C)(x) \\
&= A_\mu(x) + \bar{\theta} (\phi^{(1)}_b A_\mu(x)), \\
\Phi^{(b)}(x, \bar{\theta}) &= \phi(x) + \bar{\theta} (m C(x)) \\
&= \phi(x) + \bar{\theta} (\phi^{(1)}_b \phi(x)),
\end{align*}
\tag{C.5}
\]
which match with what we have already quoted in Equation (68).

### D. Bosonic and Ghost-Scale Symmetry Transformations

To establish that the Lagrangian densities \( L_{(b)} \) and \( L_{(g)} \) represent the field-theoretic models for the Hodge theory, we define the bosonic symmetry transformations [59, 60]:
\[
\begin{align*}
\phi^{(1)}_w A_\mu &= \partial_\mu B + \varepsilon_{\mu \nu} \partial^\nu B, \\
\phi^{(1)}_w \phi &= m B, \\
\phi^{(2)}_w A_\mu &= \partial_\mu B + \varepsilon_{\mu \nu} \partial^\nu B, \\
\phi^{(2)}_w \phi &= -m B, \\
\phi^{(1)}_w (\partial \cdot A) &= \Box B, \\
\phi^{(2)}_w (\partial \cdot A) &= -\Box B,
\end{align*}
\tag{D.2}
\]

We note that the key feature of the above symmetry transformations is the observation that the (anti-)ghost fields do not transform at all under \( \phi^{(1)}_w \). It is straightforward to check that the following are true, namely,
\[
\begin{align*}
\phi^{(1)}_w \mathcal{L}_{(b)} &= \partial_\mu [B \partial^\nu B - B \partial^\nu \phi + \mu (\partial \partial^\nu B) - m \phi \partial^\nu B], \\
\phi^{(2)}_w \mathcal{L}_{(b)} &= \partial_\mu [B \partial^\nu B - B \partial^\nu \phi + \mu (\partial \partial^\nu B) - m \phi \partial^\nu B] - m A_\mu B + m \partial^\mu B,
\end{align*}
\tag{D.3}
\]
which demonstrate that the action integrals \( \phi^{(1)}_w \mathcal{L}_{(b)} \) and \( \phi^{(2)}_w \mathcal{L}_{(b)} \) remain invariant under the bosonic symmetry transformations \( \phi^{(1,2)}_w \).

In addition to the above continuous bosonic symmetry transformations \( \phi^{(1,2)}_w \), the Lagrangian densities \( \mathcal{L}_{(b)} \) and \( \mathcal{L}_{(g)} \) respect the following infinitesimal and continuous scale symmetry transformations \( \phi^{(1)}_g \) [59, 60]:
\[
\begin{align*}
\phi^{(1)}_g C &= + C, \\
\phi^{(1)}_g \bar{C} &= - \bar{C}, \\
\phi^{(1)}_g \Psi &= 0,
\end{align*}
\tag{D.4}
\]
where the global scale parameter has been taken equal to one for the sake of brevity. In other words, we observe that the ghost and antighost fields (with ghost numbers +1 and -1, respectively) transform under the ghost-scale symmetry transformations but all the other fields (with ghost number zero) do not transform at all. It can be checked that all the six continuous symmetries of the Lagrangian densities \( \mathcal{L}_{(b)} \) obey the algebra [59, 60]:
\[
\begin{align*}
\{ \phi^{(1)}_b, \phi^{(1)}_d \}^2 &= \{ \phi^{(1)}_d, \phi^{(1)}_a \}^2 = \{ \phi^{(1)}_a, \phi^{(1)}_b \}^2 = 0, \\
\{ \phi^{(1)}_b, \phi^{(1)}_a \} &= 0, \\
\{ \phi^{(1)}_a, \phi^{(1)}_b \} &= 0, \\
\{ \phi^{(1)}_d, \phi^{(1)}_a \} &= 0, \\
\{ \phi^{(1)}_d, \phi^{(1)}_b \} &= 0, \\
\{ \phi^{(1)}_d, \phi^{(1)}_d \} &= 0,
\end{align*}
\tag{D.5}
\]
It is clear that the above symmetry transformations have been defined with the help of basic fermionic symmetry transformations \( \phi^{(1,2)}_{ab} \) and \( \phi^{(1,2)}_{ad} \) for the Lagrangian densities \( \mathcal{L}_{(b)} \) and \( \mathcal{L}_{(g)} \). The explicit forms of the bosonic symmetry transformations for all the fields (modulo a factor of \( (-i) \)) for both the Lagrangian densities are as follows:
We perform the above exercise for $\mathcal{L}_{(b)}$ and obtain the following:

$$
\begin{align*}
\left(\mathcal{L}_b^{(2)}\right)^2 &= \left(\mathcal{L}_d^{(2)}\right)^2 = \left(\mathcal{L}_{\Delta}^{(2)}\right)^2 = 0, \\
\{\mathcal{L}_b^{(2)}, \mathcal{L}_d^{(2)}\} &= \{\mathcal{L}_b^{(2)}, \mathcal{L}_{\Delta}^{(2)}\} = 0, \\
\{\mathcal{L}_d^{(2)}, \mathcal{L}_{\Delta}^{(2)}\} &= \{\mathcal{L}_b^{(2)}, \mathcal{L}_{\Delta}^{(2)}\} = 0,
\end{align*}
$$

We conclude that the Lagrangian densities $\mathcal{L}_{(b)}$ and $\mathcal{L}_{(a)}$ represent a couple of field-theoretic models for the Hodge theory (separately and independently) because the interplay between the discrete and continuous symmetries of these Lagrangian densities provides the physical realizations of the de Rham cohomological operators of differential geometry at the algebraic level as is evident from the mappings (D.8).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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