The Berezin–Simon quantization for Kähler manifolds and their path integral representations

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Abstract
The Berezin–Simon (BS) quantization is a rigorous version of the “operator formalism” of quantization procedure. The goal of the paper is to present a rigorous real-time (not imaginary-time) path-integral formalism corresponding to the BS operator formalism of quantization; Here we consider the classical systems whose phase space \( M \) is a (possibly non-compact) Kähler manifold which satisfies some conditions, with a Hamiltonian \( H : M \to \mathbb{R} \). For technical reasons, we consider only the cases where \( H \) is smooth and bounded. We use Güneysu’s extended version of the Feynman–Kac theorem to formulate the path-integral formula.

1 Introduction
In this paper, we set the “classical phase space” \( M \) to be a (possibly non-compact) Kählerian manifold which is a submanifold of a (possibly infinite-dimensional) complex projective space \( \mathbb{P}H \), where \( H \) is a complex Hilbert space. The phase space \( M \) admits a quantization procedure, which we called the Glauber-Sudarshan-type quantization in [15]; However instead we will call it the Berezin–Simon (BS) quantization in this paper, since we follow the formulation of the quantization procedure given in Simon [12], which is based on Berezin’s works; See [12] and references therein. A BS quantization is an “operator formalism” of quantization procedure. The goal of the paper is to present a path-integral representation of the BS quantization; Roughly, we present a rigorous path-integral formalism corresponding to the BS operator formalism of quantization.

The previous paper [15] had a similar goal, but it was very restricted in that we confined ourselves to the cases where the phase space \( M \) is a compact homogeneous space, which is a submanifold of a projective space \( \mathbb{P}H \) where \( H \) is a finite-dimensional Hilbert space. Thus this paper will be viewed as a considerable extension of [15].

Our main mathematical tool for the path-integral formulation is the Feynman–Kac formula on a vector bundle over a Riemannian manifold given by [5], together with the Bochner-Kodaira-Nakano identity for Kählerian manifolds. A Feynman-Kac formula itself is seen as a mathematical justification of the imaginary-time path integral method in quantum physics, but we devise a method to use it for real-time path integrals.

Roughly speaking, this paper is situated in the following context of past rigorous studies on path integrals.

Feynman’s original idea [4] is to represent the time evolution of a quantum system, as well as the expectation values of observables in it, by an integral on the space of paths on the configuration space of the system. As is well known, if we consider the “imaginary time” evolution instead of real time evolution, so-called the Wick rotation, a large part of the idea can be made rigorous by the Feynman–Kac theorem and its generalizations, and this “imaginary time + Feynman–Kac” approach is the most successful one. However, note that in the imaginary-time approaches, it is difficult to deal with time-dependent Hamiltonians, as well as non-unitary time evolutions occurring in open systems. This implies that it is hard to apply the imaginary-time methods to e.g. the theories of quantum
information/probability, where time-dependent Hamiltonians and non-unitary time evolutions (e.g. decoherences) frequently occur.

On the other hand, the notion on configuration-space path integrals are believed to be derived from more general notion of \textit{phase-space} path integrals. In some sense, the latter ones may be more fundamental if we consider a path integral as a procedure of quantization of a classical system: The main stream of the rigorous studies of quantization (e.g. the theories of geometric/deformation quantization) are formulated on phase spaces. Unlike imaginary-time configuration-space path integrals, little is known about the rigorous justification of general phase-space path integrals (in real or imaginary time).

However in 1985, Daubechies and Klauder [3] gave an important rigorous result on \textit{coherent-state path integrals}, which can be viewed as a sort of phase-space path integral formula, representing real-time evolution for some class of Hamiltonians, in terms of Brownian motions and stochastic integrals. Yamashita [14] studied phase-space path integrals in a similar idea but for other class of Hamiltonians, and with an emphasis on geometric meaning of them. In these results, mainly the phase spaces are assumed to be flat, i.e., $M \cong \mathbb{R}^{2n} \cong \mathbb{C}^n$. Yamashita [15] is seen as an attempt to apply such methods on some sort of (non-flat) compact phase spaces, which arise in irreducible unitary representation of semisimple Lie groups, e.g., SU($n$), SO($n$), Sp($n$), etc. Then we are given a question whether these works [14, 15] can be unified and extended for more general phase spaces, or not. This paper is intended to be an affirmative answer to this question.

However note that we consider only the \textit{bounded} Hamiltonians in this paper, hence not all of the results of [3] is contained in our result. Since this boundedness assumption is quite unsatisfactory for applications to realistic physical systems, we are required to loosen this assumption, but the treatment of general unbounded Hamiltonians appears to be extremely difficult. A hopeful approach will be to examine some moderate assumptions such as that the classical Hamiltonian $H(x)$ is bounded from below and increases as $H(x) \sim |x|^2$; Another hopeful approach will be to consider a “solvable” (or “algebraically tractable”) set of Hamiltonians which are generators of (representations of) a Lie group, e.g. symplectic groups, Poincare groups, etc. The latter approach will be related to the construction of unitary representations of a Lie group in terms of orbit method/geometric quantization [6, 13].

2 Projective representations of BS quantizations

2.1 BS quantization

Let $\mathcal{H}$ be a complex Hilbert space, and $\mathbb{P}\mathcal{H}$ denote the set of orthogonal projections onto one-dimensional subspaces of $\mathcal{H}$, that is,

$$\mathbb{P}\mathcal{H} := \{ |v\rangle\langle v| : v \in \mathcal{H}, \|v\| = 1 \}.$$ 

Let $\mathcal{H}^\times := \mathcal{H} \setminus \{ 0 \}$, and define $\text{pr} : \mathcal{H}^\times \rightarrow \mathbb{P}\mathcal{H}$ to be the map from $v \in \mathcal{H}^\times$ to the orthogonal projection onto $\mathbb{C}v$, i.e.,

$$\text{pr}(v) := \frac{|v\rangle\langle v|}{\|v\|^2}, \quad v \in \mathcal{H}^\times.$$ 

Let $M$ be a subset of $\mathbb{P}\mathcal{H}$ with the measure $\mu$. The measure space $(M, \mu)$ is called a \textbf{family of coherent states} on $\mathcal{H}$ if

$$\int_M p \, d\mu(p) = I.$$ 

Let $\text{Coh}(\mathcal{H})$ be the set of families of coherent states on $\mathcal{H}$. For a function $f : \mathbb{P}\mathcal{H} \rightarrow \mathbb{C}$, let

$$Q(f) := \int_M f(p) p \, d\mu(p),$$

if the integral exists. In this paper, we call the operation $f \mapsto Q(f)$ the \textbf{BS quantization} on $(M, \mu)$.
\[ \text{pr}^{-1}(M) := \{ v \in H^\times \mid \text{pr}(v) \in M \} = \bigcup_{p \in M} \text{ran}(p) \setminus \{0\} \]

\[ S(H) := \{ v \in H \mid ||v|| = 1 \}, \quad S(M) := S(H) \cap \text{pr}^{-1}(M) \]

The projection \( \text{pr}_{|S(M)} : S(M) \to M \) defines a \( U(1) \)-bundle over \( M \). (Precisely, the term “bundle” should be used only when \( M \) is a (smooth) manifold.) Let \( \mu_\mathbb{S} \) be the \( U(1) \)-invariant measure on \( S(M) \) determined by

\[ \mu(E) = \mu_\mathbb{S}(\text{pr}^{-1}_\mathbb{S}(E)), \quad \forall E \subset M, \text{ measurable} \]

where \( \text{pr}_\mathbb{S} := \text{pr}|_{S(M)} \). For measurable functions \( f_1, f_2 : S(M) \to \mathbb{C} \) (or \( f_1, f_2 : \text{pr}^{-1}(M) \to \mathbb{C} \)), let

\[ \langle f_1 | f_2 \rangle_\mathbb{S} := \int_{S(M)} \overline{f_1(v)} f_2(v) \text{d}\mu_\mathbb{S}(v), \]

if the integral exists.

Let \( \mathcal{H}^* \) denote the dual of \( \mathcal{H} \). For any \( u \in \mathcal{H} \), define \( u^* \in \mathcal{H}^* \) by

\[ u^*(v) := \langle u | v \rangle, \quad v \in \mathcal{H}. \]

We denote \( u^*|_{S(M)} \) simply by \( u^* \). Then we find that

\[ \langle u_1^* | u_2^* \rangle_\mathbb{S} = \langle u_2 | u_1 \rangle, \quad \forall u_1, u_2 \in \mathcal{H}. \]

The inner product \( \langle \cdot | \cdot \rangle_\mathbb{S} \) defines the Hilbert space \( L^2(S(M)) = L^2(S(M), \mu_\mathbb{S}) \).

For \( \ell \in \mathbb{Z} \), let

\[ \Gamma_\ell(\mathcal{H}^\times) := \{ f : \mathcal{H}^\times \to \mathbb{C} : f(\lambda v) = \lambda^\ell f(v), \ \forall \lambda \in \mathbb{C}^\times, v \in \mathcal{H}^\times \}, \]

\[ \Gamma_{\ell,M} \equiv \Gamma_\ell(\text{pr}^{-1}(M)) := \{ f : \text{pr}^{-1}(M) \to \mathbb{C} : f(\lambda v) = \lambda^\ell f(v), \ \forall \lambda \in \mathbb{C}^\times, v \in \text{pr}^{-1}(M) \}, \]

\[ \Gamma^L_{\ell,M} := \{ f \in \Gamma_{\ell,M} : f|_{S(M)} \in L^p(S(M), \mu_\mathbb{S}) \}, \quad 1 \leq p \leq \infty. \]

For each \( \ell \in \mathbb{Z} \), \( \Gamma^L_{\ell,M} \) is a Hilbert space with the inner product \( \langle \cdot | \cdot \rangle_\mathbb{S} \), and naturally viewed as a closed subspace of \( L^2(S(M), \mu_\mathbb{S}) \). In this paper, we deal with the cases where \( \ell = 1, 0, -1 \), and our main concern is the case \( \ell = 1 \).

For each \( u \in \mathcal{H} \), we see \( u^*|_{\mathcal{H}^\times} \in \Gamma^L_{1,M} \). Hence \( \mathcal{H}^* \) is viewed as a closed subspace of \( \Gamma^L_{1,M} \).

For \( F : M \to \mathbb{C} \), define \( \tilde{F} : \text{pr}^{-1}(M) \to \mathbb{C} \) by

\[ \tilde{F}(v) := F(\text{pr}(v)). \]

Then we see \( \tilde{F} \in \Gamma_{0,M} \); It follows that \( \mathbb{C}^M \) (the space of functions \( M \to \mathbb{C} \)) can be identified with \( \Gamma_{0,M} \); We can also identify \( L^p(M, \mathbb{C}) \) with \( \Gamma^L_{0,M} \).

Let \( F \in L^\infty(M, \mathbb{C}) \), then \( F \) acts on \( \Gamma^L_{1,M} \) as a pointwise multiplication operator \( M_F \):

\[ (M_F f)(v) := \tilde{F}(v)f(v), \quad v \in \text{pr}^{-1}(M), \ f \in \Gamma^L_{1,M}. \]

Let \( E_{\mathcal{H}^*} \) be the orthogonal projection from \( \Gamma^L_{1,M} \) onto \( \mathcal{H}^* \). We see

\[ E_{\mathcal{H}^*} = \int_{S(M)} \text{pr}(v^*) \text{d}\mu_\mathbb{S}(v). \tag{2.1} \]

where \( \text{pr}(v^*) \) is the orthogonal projection from \( \Gamma^L_{1,M} \) onto \( \mathbb{C}v^* \). For \( f \in \Gamma^L_{1,M}, v \in \text{pr}^{-1}(M) \), we have

\[ (E_{\mathcal{H}^*} f)(v) = \langle v^* | E_{\mathcal{H}^*} f \rangle_\mathbb{S} = \langle v^* | f \rangle_\mathbb{S} = \int_{S(M)} \langle x | v \rangle f(x) \text{d}\mu_\mathbb{S}(x). \]
That is, the integral kernel $E_{H^*}(u, v)$ of $E_{H^*}$ is given by

$$E_{H^*}(v, u) = \langle u | v \rangle , \quad u, v \in S(M). \quad (2.2)$$

For $F \in L^\infty(M, \mathbb{C})$, define the operator $\hat{Q}(F)$ on $\Gamma^L_{1,M}$ by

$$\hat{Q}(F) := \int_{S(M)} \tilde{F}(v) \mathfrak{pr}(v^*) d\mu_S(v). \quad (2.3)$$

We see

$$\hat{Q}(F)v^* = (\hat{Q})(F^*)v^*, \quad \langle v^* | \hat{Q}(F)u^* \rangle_S = \langle u | \hat{Q}(F)v \rangle, \quad u, v \in H.$$

We call the operation $\hat{Q}$ the BS quantization on $\Gamma^L_{1,M}$.

The following theorem says that all the information of the BS quantization $\hat{Q}$ (or $Q$) is essentially contained in the projection operator $E_{H^*}$.

**Theorem 2.1.** For any $F \in L^\infty(M, \mathbb{C})$, we have

$$\hat{Q}(F) = E_{H^*} M_F E_{H^*}.$$

**Proof.** Let $u \in H$ and $v \in H^\times$. Then we have

$$(E_{H^*} M_F E_{H^*} u^*)(v) = (E_{H^*} M_F u^*)(v) = \langle v^* | M_F u^* \rangle_S$$

$$= \int_{S(M)} d\mu_S(x) \overline{v(x)\hat{F}(x)} u^*(x)$$

$$= \int_{S(M)} d\mu_S(x) (v|x) \hat{F}(x) \langle u | x \rangle$$

$$= \int_{S(M)} d\mu_S(x) \langle v^* | x^* \rangle_S \langle x^* | u^* \rangle_S \hat{F}(x)$$

$$= \langle v^* | \hat{Q}(F)u^* \rangle_S = \langle \hat{Q}(F)u^* | v \rangle.$$

On the other hand, if $f \in \Gamma^L_{1,M}$ is orthogonal to $H^*$,

$$\hat{Q}(F) f = 0 = E_{H^*} M_F E_{H^*} f.$$

$\square$

### 2.2 Some lemmas

Generally, for a bounded operator $A$ on $H$, define the operator $\tilde{A}$ on $\Gamma^L_{1,M}$ by

$$\tilde{A}v^* := (A^* v)^*, \quad v \in H,$$

$$\tilde{A}f := 0 \quad \text{if } f \in H^{*\perp} \subset \Gamma^L_{1,M}.$$

**Lemma 2.2.** Let $A$ be a bounded operator $A$ on $H$. Define $K_A : S(M) \times S(M) \to \mathbb{C}$ by

$$K_A(v, u) = \langle u | Av \rangle = \langle v^* | \tilde{A}u^* \rangle_S, \quad u, v \in S(M).$$

Then $K_A$ is the integral kernel of $\tilde{A}$, i.e.,

$$\langle \tilde{A}f | v \rangle = \int_{S(M)} d\mu_S(u) K_A(v, u) f(u), \quad \forall f \in \Gamma^L_{1,M}; \ a.e. \ v \in S(M).$$

Especially, the kernel of $p = |v\rangle \langle v| \in M$ is

$$K_p(v', v'') = \langle v'' | p v' \rangle = \langle v'' | v \rangle \langle v | v' \rangle. \quad (2.4)$$
Proof. Since $\tilde{A}f = \tilde{A}E_{H^*}f$, we can choose $w \in H$ such that $w^* = E_{H^*}f$. 

\[
\int_{S(M)} d\mu_{\tilde{w}}(u) \langle u|Av \rangle w^*(u) \\
= \int_{S(M)} d\mu_{\tilde{w}}(u) \langle w|u \rangle \langle u|Av \rangle \\
= \langle w|Av \rangle = (A^*w^*)(v) = (\tilde{A}w^*)(v) = (\tilde{A}f)(v)
\]

\[
\square
\]

Fix $H \in L^\infty(M)$. For $t \in \mathbb{R}$, define the bounded operator $Q_t(F)$ and $\tilde{Q}_t(F)$ on $H$ and $\Gamma^{1,2}_{t,M}$ respectively, by 

\[
Q_t(F) := e^{itQ(H)}F e^{-itQ(H)}, \quad \tilde{Q}_t(F) := e^{it\tilde{Q}(H)}F e^{-it\tilde{Q}(H)}.
\]

Note that 

\[
\tilde{Q}_t(F) = \int_{S(M)} \tilde{F}(v) \text{pr}_t(v^*) d\mu_{\tilde{w}}(v) = \int_{S(M)} \tilde{F}(v) \text{pr}(v^*) d\mu_{\tilde{w}}(v), \quad \text{by (2.5)}
\]

where 

\[
\text{pr}_t(v^*) := e^{it\tilde{Q}(H)} \text{pr}(v^*) e^{-it\tilde{Q}(H)}, \quad v_t := e^{itQ(H)}v, \quad p_t := e^{it\tilde{Q}(H)}p e^{-it\tilde{Q}(H)}.
\]

Lemma 2.3. For $F \in L^\infty(M, \mathbb{C})$, 

\[
(\tilde{Q}_t(F)s)(v) = \int_{S(M)} K_t(v, u)s(u)d\mu_{\tilde{w}}(u), \quad s \in \Gamma^{1,2}_{1,M},
\]

where 

\[
K_t(v, u) = \langle v^*|\tilde{Q}_t(F)u^* \rangle = \langle u|Q_t(F)v \rangle \\
= \int_{S(M)} d\mu_{\tilde{w}}(x) \tilde{F}(x) \langle u|x_1\rangle \langle x_1|v \rangle = \int_{M} d\mu(p) F(p) \langle u|p_tv \rangle
\]

Proof. By (2.5), 

\[
(\tilde{Q}_t(F)s)(v) = \int_{S(M)} \tilde{F}(u) (\text{pr}_t(u^*)s)(v) d\mu_{\tilde{w}}(u) \\
= \int_{S(M)} d\mu_{\tilde{w}}(u) \tilde{F}(u) u^*_t(v) \langle u^*_t|s \rangle \\
= \int_{S(M)} d\mu_{\tilde{w}}(u) \tilde{F}(u) u^*_t(v) \int_{S(M)} d\mu_{\tilde{w}}(x)u^*_t(x)s(x) \\
= \int_{S(M)} d\mu_{\tilde{w}}(x) \left( \int_{S(M)} d\mu_{\tilde{w}}(u) \tilde{F}(u) \langle x|u^*_t \rangle \langle u^*_t|v \rangle \right) s(x) \\
= \int_{S(M)} d\mu_{\tilde{w}}(x) \left( \int_{S(M)} d\mu_{\tilde{w}}(u) \tilde{F}(u) \langle v^*|u^*_t \rangle \langle u^*_t|x^* \rangle \right) s(x) \\
= \int_{S(M)} d\mu_{\tilde{w}}(x) \langle v^*|\tilde{Q}_t(F)x^* \rangle s(x).
\]

\[
\square
\]

Lemma 2.4. For $t_1, ..., t_N \in \mathbb{R}$, and $H, F_1, ..., F_N \in L^\infty(M, \mathbb{C})$, 

\[
\text{Tr}\tilde{Q}_{t_1}(F_1) \cdots \tilde{Q}_{t_N}(F_N) = \int_{M} d\mu(p_1) \cdots \int_{M} d\mu(p_N) \text{Tr}(p_1 \cdot \cdots \cdot p_N, t_1 \cdots \cdot t_N) \prod_{j=1}^{N} F_j(p_j), \quad (2.6)
\]

where 

\[
p_{j,t_j} := e^{it_j\tilde{Q}(H)}p_j e^{-it_j\tilde{Q}(H)}.
\]
Proof. Let \( v_{N+1} := v_1 \). Then by Lemma 2.3,

\[
\text{Tr} \tilde{Q}_{t_1}(F_1) \cdots \tilde{Q}_{t_N}(F_N) = \int_{\mathbb{R}(M)} d\mu(v_1) \cdots d\mu(v_N) \prod_{j=1}^N \int_{\mathbb{R}(M)} d\mu(p_j) F_j(p_j) \langle v_j \rangle_{p_j,t_j,v_{j+1}} \]

\[
= \int_{\mathbb{R}(M)} d\mu(p_1) \cdots d\mu(p_N) \text{Tr}(p_{1,t_1} \cdots p_{N,t_N}) \prod_{j=1}^N F_j(p_j).
\]

\[\square\]

3 Kähler manifold

In the following sections, we assume the assumptions:

**Assumption 3.1.**

1. \( M \) is a (finite-dimensional) Kähler submanifold of \( \mathbb{P}H \).
2. \( M \) is complete as a Riemannian manifold (or equivalently, complete as a metric space).
3. For any \( f \in \Gamma^1_{1,M} \), \( f \in \mathcal{H}^\ast \) if and only if \( f \) is holomorphic.
4. Let \( \text{vol} \) be the volume form on \( M \) (as a Riemannian manifold). Then there exists a constant \( C > 0 \) s.t. the measure \( \mu := C \cdot |\text{vol}| \) satisfies \( (M, \mu) \in \text{Coh}(\mathcal{H}) \), i.e., \( \int_M d\mu(p) = 1 \).

In the following we will explain the precise meaning of these assumptions.

Note that \( \mathbb{P}H \) has the natural topology induced by that of \( \mathcal{H} \), and so \( M \subset \mathbb{P}H \) has the topology of a subspace of \( \mathbb{P}H \). Assume that there exist open sets \( U_i \subset \mathbb{C}^n \) (\( i \in \mathcal{I} \), some index set), and holomorphic maps \( \psi_i : U_i \to \mathcal{H}^\times := \mathcal{H} \setminus \{0\} \) (\( i \in \mathcal{I} \)) such that

1. \( \{\text{pr} \circ \psi_i(U_i)\}_{i \in \mathcal{I}} \) is an open cover of \( \mathcal{M} \).
2. \( \text{pr} \circ \psi_i \) is injective for all \( i \in \mathcal{I} \).

Then we find that \( \{\text{pr} \circ \psi_i\}_{i \in \mathcal{I}} \) gives an atlas of \( M \) as a complex manifold. We assume without loss of generality that \( 0 \in U_i \) and \( \langle \psi_i(0) | \psi_i(z) \rangle = 1 \) for all \( z \in U_i \), \( i \in \mathcal{I} \). We write \( U_i' := \text{pr} \circ \psi_i(U_i) \subset \mathcal{M} \), and locally identify \( U_i' \) with \( U_i \); we use the coordinates \( z = (z_1, ..., z_n) \) on \( U_i \) also as coordinates on \( U_i' \).

Let \( d \) denote the exterior derivative on \( \mathcal{M} \), which is decomposed to the holomorphic and antiholomorphic parts: \( d = d' + d'' \). For example, on \( U_i \), for \( f \in \mathcal{C}^\infty(U_i, \mathbb{C}) \), \( d'f \) and \( d''f \) are explicitly written as

\[
d'f = \sum_k \frac{\partial}{\partial z_k} f \, dz_k, \quad d''f = \sum_k \frac{\partial}{\partial \overline{z}_k} f \, d\overline{z}_k.
\]

Define \( h_i : U_i \to \mathbb{R} \) by

\[
h_i(z) := ||\psi_i(z)||^2, \quad z \in U_i.
\]

Define the 2-form \( \omega \) on \( U_i \) by

\[
\omega := -i \, d''d' \log h_i.
\]

It turns out that \( \omega \) becomes a globally defined closed 2-form on \( \mathcal{M} \), and hence \( (\mathcal{M}, \omega) \) is a symplectic manifold. In fact, \( \omega \) is naturally defined on whole projective space \( \mathbb{P}H \) as follows: Define \( h : \mathcal{H}^\times \to \mathbb{R} \) by \( h(v) := ||v||^2 \). Then the 2-form \( \omega := -i \, d''d' \log h \) on \( \mathcal{H}^\times \) is defined even when \( \dim \mathcal{H} = \infty \). Since \( \omega \) is invariant under the action of \( \mathcal{C}^\times := \mathbb{C} \setminus \{0\} \) on \( \mathcal{H}^\times \), \( \omega \) can be viewed as a 2-form on \( \mathbb{P}H \cong \mathcal{H}^\times / \mathcal{C}^\times \).

For tangent vector fields \( X, Y \) on \( \mathcal{M} \), let

\[
g(X, Y) := \omega(X, JY),
\]

where \( J \) is the complex structure on \( \mathcal{M} \). Then \( g \) becomes a Riemannian metric on \( \mathcal{M} \), and moreover we find that \( (\mathcal{M}, g, \omega) \) is a Kähler manifold. We call \( h_i \) a **Kähler potential** on \( U_i' \).
Therefore, we find that if $M$ is a (finite-dimensional) complex submanifold of $\mathbb{P}H$, $M$ satisfies Assumption 3.1 (1), even when $\dim H = \infty$.

Assumption 3.1 (2) says that every geodesic line of $M$ can be extended for arbitrarily large values of its canonical parameter. This is equivalent to say that $M$ is a complete metric space with respect to the distance function $d$ induced by the Riemannian metric $g$.

Assumption 3.1 (3) says that for any $f \in \Gamma_{1,M}^2$ if $f \circ \psi_i : U_i \to \mathbb{C}$ is holomorphic for all $i \in I$, then $f \in \mathcal{H}^*$. (The converse always holds.)

### 3.1 Line bundle

Let $f \in \Gamma_{1,M}^\ell$ ($\ell \in \mathbb{Z}$) and $v \in \text{pr}^{-1}(M)$. We define the value of $f$ at $p = \text{pr}(v) \in M$, denoted $f(p) = f(\text{pr}(v))$, to be

$$f(p) := f|_{\text{ran}(\text{pr})\setminus\{0\}}, \quad \text{equivalently,} \quad f(\text{pr}(v)) := f|_{\mathbb{C} \times v},$$

that is, $f(\text{pr}(v))$ is the function $\mathbb{C} \times v \to \mathbb{C}$ defined by

$$f(\text{pr}(v))(ov) := f(ov) = \alpha^tf(v), \quad \alpha \in \mathbb{C}^\times.$$

Let

$$\mathcal{O}_{1,M} := \{f(p)|f \in \Gamma_{1,M}, \ p \in M\};$$

then the natural projection $\mathcal{O}_{1,M} \to M$, $f(p) \mapsto p$ defines a complex line bundle over $M$, where each $f \in \Gamma_{1,M}$ is a section of the line bundle. The space $\mathbb{A}(\mathcal{O}_{1,M})$ of $\mathcal{O}_{1,M}$-valued $r$-forms are usually defined.

For $f \in \Gamma_{1,M}$, let $\text{supp}(f)$ denote the support of $f$ as a map $M \to \mathcal{O}_{1,M}$, and then for any $\alpha \in \mathbb{A}(\mathcal{O}_{1,M})$, the support $\text{supp}_\mathbb{A}(\alpha) \subset M$ of $\alpha$ is naturally defined.

Here, recall Lemma 2.2; The integral kernel $K_A$ of an operator $\hat{A}$ on $\Gamma_{1,M}^2$ was a $\mathbb{C}$-valued function on $\mathbb{S}(M) \times \mathbb{S}(M)$ there. However, if $\Gamma_{1,M}^2$ is viewed as a space of sections $s : M \to \mathcal{O}_{1,M}$, the an integral kernel $K$ of an operator on $\Gamma_{1,M}^2$ should be a map such that $K(p_1, p_2) \in \text{Hom}(\mathcal{O}_{1,M}, p_1, \mathcal{O}_{1,M}, p_2)$ for all $p_1, p_2 \in M$, where $\mathcal{O}_{1,M} \to M$ is the fiber of the line bundle $\mathcal{O}_{1,M}$ at $p \in M$. Equivalently, $K$ is a section of the external tensor product bundle $\mathcal{O}_{1,M} \otimes \mathcal{O}_{1,M}^* \to M \times M$. Note that $\mathcal{O}_{1,M}$ is naturally identified with the dual space of $\text{ran}(p)$. Hence we can define $K_A(p_1, p_2) \in \text{Hom}(\mathcal{O}_{1,M}, p_1, \mathcal{O}_{1,M}, p_2)$ as follows. For any $v_1^* \in \mathcal{O}_{1,M}(p_1) = \text{ran}(p_1)^*$ with $v_2 \in \text{ran}(p_2)$, define $K_A(p_1, p_2) v_1^* \in \mathcal{O}_{1,M}(p_2)$ by

$$(K_A(p_1, p_2) v_1^*)(v_2) := \langle v_2 | A v_1 \rangle = K_A(v_1, v_2), \quad v_1 \in \text{ran}(p_1). \quad (3.1)$$

Sometimes we simply write $A(p_1, p_2)$ (resp. $A(v_1, v_2)$) for $K_A(p_1, p_2)$ (resp. $K_A(v_1, v_2)$). Especially we have

$$\langle E_{\mathcal{H}^*}(p_1, p_2) v_1^* | v_2 \rangle = \langle v_2 | v_1 \rangle = E_{\mathcal{H}^*}(v_1, v_2), \quad v_1 \in \text{ran}(p_1). \quad (3.2)$$

Let $f_1, f_2 \in \Gamma_{1,M}^\ell$ ($\ell \in \mathbb{Z}$). Define $\langle f_1 | f_2 \rangle_{\text{pr}} : M \to \mathbb{C}$ by

$$\langle f_1 | f_2 \rangle_{\text{pr}}(\text{pr}(v)) := f_1 \left( \frac{v}{\|v\|} \right) f_2 \left( \frac{v}{\|v\|} \right) = \|v\|^{-2} f_1(v) f_2(v), \quad v \in \text{pr}^{-1}(M)$$

so that if $f_1, f_2 \in \Gamma_{1,M}^\ell$ then

$$\langle f_1 | f_2 \rangle_{\mathbb{C}} = \int_M \langle f_1 | f_2 \rangle_{\text{pr}} \text{vol}.$$

$\langle | \rangle_{\text{pr}}$ is called the (pointwise) **Hermitian metric** of the line bundle $\mathcal{O}_{1,M}$.

Let $e_i := \psi_i(0)$, then $e_i^*$ is a local holomorphic section of $\mathcal{O}_{1,M}$ on $U_i$; Precisely, if we identify the fiber of $\mathcal{O}_{1,M}$ over $p \in M$ with $\text{ran}(p)^*$, the value $e_i^*(p)$ of the section $e_i^* \in \Gamma_{1,M}$ at $p \in U_i$ is the function $\text{ran}(p) \to \mathbb{C}$, $v \mapsto \langle e_i | v \rangle$. Define $h_i : U_i \to \mathbb{R}$ by

$$h_i(z) := \langle e_i^* | e_i^* \rangle_{\text{pr}}(z), \quad z \in U_i.$$

Without loss of generality, assume $\langle \psi_i(0) | \psi_i(z) \rangle = 1$, and we find

$$h_i(z) = ||\psi_i(z)||^{-2} e_i^*(\psi_i(z)) e_i^*(\psi_i(z)) = ||\psi_i(z)||^{-2} \langle e_i | \psi_i(z) \rangle \langle e_i | \psi_i(z) \rangle = ||\psi_i(z)||^{-2} = \frac{1}{h_i(z)}.$$
Let $\Gamma_{\mathcal{O}_M}^\infty (\ell \in \mathbb{Z})$ denote the subspace of $\Gamma_{\mathcal{O}_M}$ which consists of the smooth functions. Any section $s \in \Gamma_{\mathcal{O}_M}^\infty$ is uniquely expressed locally on $U_i \subset M$ by

$$s(v) = f(v)e_i^* (v) = f(v) \langle e_i | v \rangle, \quad f \in \Gamma_{\mathcal{O}_M}^\infty, \quad v \in \mathbb{C}^x \psi_i(U_i),$$

which is simply written as $s = fe_i^*$. The antiholomorphic exterior derivative $d''$ of $s = fe_i^*$ is defined on $U_i$ by

$$d''(fe_i^*) := (d'' f) \otimes e_i^*.$$

We find that $d'' s \in \mathcal{A}^1(\mathcal{O}_M)$ is globally well-defined on $M$. This is naturally extended to any $\mathcal{O}_M$-valued $r$-forms $d'' : \mathcal{A}^*(\mathcal{O}_M) \rightarrow \mathcal{A}^{r+1}(\mathcal{O}_M)$; More precisely, let $\mathcal{A}^{\rho,q}(\mathcal{O}_M)$ denote the space of $\mathcal{O}_M$-valued differential forms of type $(\rho, q)$, then $d'' : \mathcal{A}^{\rho,q}(\mathcal{O}_M) \rightarrow \mathcal{A}^{\rho,q+1}(\mathcal{O}_M)$.

The local Chern connection form $\theta$ of the line bundle $\mathcal{O}_M$ on $U_i$ is defined by

$$\theta := d' \log h_i = -d' \log h_i = \sum_k \left( h_i^{-1} \frac{\partial}{\partial z_k} h_i \right) dz_k.$$ 

The Chern connection $\nabla : \mathcal{A}^0(\mathcal{O}_M) \rightarrow \mathcal{A}^1(\mathcal{O}_M)$ is defined as follows: for any section $s \in \Gamma_{\mathcal{O}_M}^\infty$ which $s = fe_i^*$ on $U_i \subset M$, $\nabla s$ is defined locally on $U_i$ as

$$\nabla s := (df) \otimes e_i^* + f \nabla e_i^*, \quad \nabla e_i^* := \theta \otimes e_i^*.$$

The Chern connection $\nabla$ is Hermitian, i.e., for any vector field $X$,

$$X \langle s_1 | s_2 \rangle = \langle \nabla_X s_1 | s_2 \rangle + (s_1 | \nabla_X s_2 \rangle), \quad s_1, s_2 \in \Gamma_{\mathcal{O}_M}^\infty.$$

Hence, for any piecewise smooth curve $c$ on $M$, the parallel transport $/ / (c)$ along $c$ is unitary w.r.t. the Hermitian metric $\langle \cdot | \cdot \rangle_{pt}$. Especially, the holonomy group at any point $p \in M$ w.r.t. $\nabla$ is $U(1)$. Let $\psi(z) := \psi(z)/\|\psi(z)\|$. The normalized frame $e_i^* \in \Gamma_{\mathcal{O}_M}$ on $U_i$ is determined by

$$e_i^* (\psi(z)) = 1 = h^{-1/2}(z)e_i^*(\psi(z)), \quad z \in \mathbb{C},$$

which is written $e_i^* := h_i^{-1/2}e_i^*$ in short. (Here, recall $e_i^*(\psi(z)) = \langle \psi_i(0) | \psi_i(z) \rangle = 1$ and $h_i(z)^{-1} = \|\psi_i(z)\| = h_i(z).$) Precisely, the value $e_i^*(p)$ of the section $e_i^*$ at $p \in U_i$ is the function

$$\text{ran}(p) \rightarrow \mathbb{C}, \quad \zeta \psi(z) \mapsto h_i^{-1/2}(z)e_i^* | \zeta \psi(z) = \zeta, \quad \zeta \in \mathbb{C}^d, \quad \psi(z) \in \text{ran}(p), \quad \zeta \in \mathbb{C}.$$

For $s = fe_i^*, \quad f \in C^\infty(U_i, \mathbb{C})$, we have

$$\nabla s = (df) \otimes e_i^* + f \theta_{\text{ant}} \otimes e_i^*,$$

where

$$\theta_{\text{ant}} := \frac{1}{2} \sum_k \left[ \left( \frac{\partial}{\partial x_k} \log h_i \right) \, dy_k - \left( \frac{\partial}{\partial y_k} \log h_i \right) \, dx_k \right], \quad z_k = x_k + iy_k. \quad (3.3)$$

which is a $\mathbb{R}$-valued 1-form; Since $i\mathbb{R} = u(1)$ (the Lie algebra of $U(1)$), this says that the Chern connection $\nabla$ determines a connection on the associated $U(1)$ principal bundle, explicitly written by $i\theta_{\text{ant}}$.

For $s = fe_i^*$, let

$$\nabla' s := (d' f) \otimes e_i^* + f (\theta \otimes e_i^*),$$

then we have

$$\nabla = \nabla' + d''.$$

We find that $\nabla : \mathcal{A}^0(\mathcal{O}_M) \rightarrow \mathcal{A}^1(\mathcal{O}_M)$ and $\nabla' : \mathcal{A}^{(0,0)}(\mathcal{O}_M) \rightarrow \mathcal{A}^{(1,0)}(\mathcal{O}_M)$ are globally well-defined on $M$. 
For general differential forms $\alpha \in A^r(\mathcal{O}_1, \mathcal{M})$, $\nabla \alpha \in A^{r+1}(\mathcal{O}_1, \mathcal{M})$ is defined by

$$
\nabla \alpha := \sum_j (d\alpha_j \otimes s_j + (-1)^r \alpha_j \wedge \nabla s_j)
$$

where $\alpha = \sum_j \alpha_j \otimes s_j$ ($\alpha_j \in A^r(\mathcal{M}, \mathbb{C})$, $s_j \in \Gamma_{1,M}^\infty$).

It is shown that there exists $\Theta \in A^2(\mathcal{M}, \mathbb{C})$ such that

$$
\nabla^2 \alpha = \Theta \wedge \alpha, \quad \forall \alpha \in A^k(\mathcal{O}_1, \mathcal{M}), \; k = 0, ..., n. \tag{3.4}
$$

$\Theta$ is called the curvature form of the Chern connection $\nabla$. Generally, the curvature form $\Theta$ is locally expressed by the Hermitian metric $h$ by

$$
\Theta = d''d' \log h_i,
$$

and so we have

$$
\omega = i\Theta \tag{3.5}
$$

in our case.

### 3.2 The Bochner-Kodaira-Nakano identity

Let $\Gamma_{1,M,c}^\infty$ denote the space of compactly supported smooth sections of $\mathcal{O}_1, \mathcal{M}$, and $A^*_r(\mathcal{O}_1, \mathcal{M})$ the space of compactly supported $\mathcal{O}_1, \mathcal{M}$-valued $r$-forms on $\mathcal{M}$. The inner product $\langle \cdot | \cdot \rangle_\theta$ is naturally extended to $A^*_r(\mathcal{O}_1, \mathcal{M})$, and the formal adjoint operators are usually defined on these spaces: $\nabla^*, \nabla''$ and $d''$ are the formal adjoints of $\nabla$, $\nabla''$, and $d''$, respectively. We use the notation $\Delta_X := X^*X + XX^*$ for any operator $X$ on $A^*_r(\mathcal{O}_1, \mathcal{M})$, if the formal adjoint $X^*$ of $X$ exists. For example,

$$
\Delta_{d''} := d'' d'' + d'' d'' d'', \quad \Delta_{\nabla^*} := \nabla^* \nabla + \nabla \nabla^*.
$$

The Lefschetz map $L : A^{p,q}(\mathcal{O}_1, \mathcal{M}) \to A^{p+1,q+1}(\mathcal{O}_1, \mathcal{M})$ is defined by

$$
L(\alpha) := \omega \wedge \alpha.
$$

Since $\omega = i\Theta$ holds in our case, we have $L(\alpha) = i\Theta \wedge \alpha = i\nabla^2 \alpha$, i.e.,

$$
L = i\nabla^2. \tag{3.6}
$$

Let $P_r$ denote the natural projection onto $A^r(\mathcal{O}_1, \mathcal{M})$: If $\alpha = \sum_{k=0}^{\dim \mathcal{M}} \alpha_k$, $\alpha_k \in A^k(\mathcal{O}_1, \mathcal{M})$, then $P_r(\alpha) := \alpha_r$.

**Lemma 3.2.** (See e.g. [1, Ch.5]; [7, Eq.(3.2.37)]) $[L^*, L] = \sum_r^n (n-r)P_r$, where $n = \dim \mathcal{M}$.

**Theorem 3.3.** (Bochner–Kodaira–Nakano identity) (See e.g. [1, Ch.5])

$$
2\Delta_{d''} = \Delta_{\nabla^*} + [i\nabla^2, L^*] \tag{3.7}
$$

By (3.6): $L = i\nabla^2$, Lemma 3.2 and the Bochner–Kodaira–Nakano identity (3.7), we have

**Corollary 3.4.** On the line bundle $\mathcal{O}_1, \mathcal{M}$ over $\mathcal{M}$, we have

$$
2\Delta_{d''} = \Delta_{\nabla^*} + [L, L^*] = \Delta_{\nabla^*} - \sum_{r=0}^n (n-r)P_r, \quad n = \dim \mathcal{M}.
$$

Especially,

$$
2\Delta_{d''} s = (\Delta_{\nabla^*} - n) s = (\nabla^* \nabla - n) s, \quad \forall s \in A^0(\mathcal{O}_1, \mathcal{M}) = \Gamma_{1,M}^\infty.
$$

Generally, the operators $\Delta_{\nabla^*}$ and $\Delta_{d''}$ are shown to be essentially self-adjoint, and we write the self-adjoint extensions these operators again by $\Delta_{\nabla^*}$ and $\Delta_{d''}$, respectively.
Corollary 3.5. \[ E_{H^*} = \text{s-lim}_{\nu \to \infty} \exp \left[ -\nu (\nabla^* \nabla - n) \right], \]

where \( \text{s-lim} \) denotes the limit in the strong topology on \( B(\Gamma^2_{1, M}) \), the space of bounded operators on \( \Gamma^2_{1, M} \).

We say that \( \Delta_{\nu'} \) has a spectral gap if \( \inf (\text{spec}(\Delta_{\nu'}) \setminus \{ 0 \}) > 0 \), where \( \text{spec}(\Delta_{\nu'}) \) is the spectrum of \( \Delta_{\nu'} \). In other words, \( \Delta_{\nu'} \geq \alpha (I - E_{H^*}) \) for some \( \alpha > 0 \).

Corollary 3.6. If \( \Delta_{\nu'} \) has a spectral gap, then \[ E_{H^*} = \lim_{\nu \to \infty} \exp \left[ -\nu (\nabla^* \nabla - n) \right], \]

where \( \lim \) denotes the limit in operator norm on \( B(\Gamma^2_{1, M}) \).

4 Asymptotic representation

Let \( \mathcal{L} \) be an arbitrary complex Hilbert space, and \( \mathcal{K} \) be a closed subspace of \( \mathcal{L} \). Let \( H \) be a bounded self-adjoint operator on \( \mathcal{L} \), and \( A \) a possibly unbounded positive semidefinite operator on \( \mathcal{L} \) such that \( \ker A = \mathcal{K} \subset \mathcal{L} \). Assume that \( A \) has a spectral gap, i.e., there exists \( \alpha > 0 \) such that the spectrum of \( A \) satisfies \( \text{spec}(A) \cap (0, \alpha) = \emptyset \), equivalently \( A^2 \geq \alpha A \), or \( A \geq \alpha E_K^0 \) where \( E_K^0 := 1 - E_K \).

This section we show the following theorem:

Theorem 4.1. For all \( t \geq 0 \),

\[ \lim_{\nu \to \infty} e^{-t(\nu A + iH)} v = e^{-itE_K HE_K} v, \quad \forall v \in \mathcal{K} = \ker A, \]
\[ \lim_{\nu \to \infty} e^{-t(\nu A + iH)} v = 0, \quad \forall v \in \mathcal{K}^\perp. \]

That is,

\[ \text{s-lim}_{\nu \to \infty} e^{-t(\nu A + iH)} = e^{-itE_K HE_K} E_K = E_K e^{-itE_K HE_K} E_K. \]

Note that we can give a precise meaning to the operator \( e^{-t(\nu A + iH)} \) as follows. We find that \( T_\nu := \nu A + iH \) is a closed operator satisfying \( \Re (\nu T_\nu v) \geq 0 \) for all \( v \in \text{dom}(T_\nu) = \text{dom}(A) \). Hence \( T_\nu \) generates the strongly continuous contraction semigroup \( \{ e^{\nu T_\nu t} \}_{t \geq 0} \) by the Hille–Yosida Theorem [9].

Recall the notations and assumptions in the previous section. Set

\[ \mathcal{L} := \Gamma^2_{1, M}, \quad \mathcal{K} := \mathcal{H}^* \subset \Gamma^2_{1, M}, \quad A := 2\Delta_{\nu'} |_{\Gamma^2_{1, M}} = \nabla^* \nabla - n, \quad n := \text{dim}_\mathbb{R} M, \]

in Theorem 4.1, then we have the following:

Corollary 4.2. Assume that \( \Delta_{\nu'} \) has a spectral gap. Then for any “classical Hamiltonian” \( H \in L^\infty(\mathcal{M}, \mathbb{R}) \), the “quantum time evolution” \( e^{-itQ(H)} \) is expressed as

\[ e^{-itQ(H)} E_{H^*} = E_{H^*} e^{-itQ(H)} E_{H^*} = \text{s-lim}_{\nu \to \infty} \exp \left[ -t (\nu (\nabla^* \nabla - n) + iH) \right], \quad t \geq 0, \]

where \( H \) in the rhs is viewed as a multiplication operator \( M_H \) on \( \Gamma^2_{1, M} \).

A similar statement and its proof are found in [14], but that proof was somewhat erroneous, and confusing in notations. Thus we will present another proof here. To complete the proof, we need some lemmas.

Lemma 4.3. Let \( v \in \mathcal{L} \) and \( v_1 := e^{-t(A+iH)} v, \ t \geq 0 \). If \( v_t \in \text{dom} A \) and \( \frac{d}{dt} \langle v_t | Av_t \rangle \geq 0 \), then we have

\[ \| Av_t \| \leq \| H \| \| v_t \|, \quad \text{and} \quad \langle v_t | Av_t \rangle \leq \alpha^{-1} \| H \|^2 \| v_t \|^2. \]
Proof. We see \( \frac{d}{dt} \langle v_t | A v_t \rangle = -2 \left( \langle A v_t | A v_t \rangle + R (iH v_t | Av_t) \right) \) and \( |(iH v_t | Av_t)| \leq H \|v_t\| \|A v_t\|. \) Hence \( \frac{d}{dt} \langle v_t | A v_t \rangle \geq 0 \) implies \( \|A v_t\|^2 \leq H \|v_t\| \|A v_t\| \), i.e. \( \|A v_t\| \leq H \|v_t\|. \) Since \( A^2 \geq \alpha A \), we have \( \| (\alpha A)^{1/2} v_t \| \leq H \|v_t\|. \) This is equivalent to \( \langle v_t | A v_t \rangle \leq \alpha^{-1} H^2 \|v_t\|^2 \). \qed

**Lemma 4.4.** Let \( v \in \mathcal{K} = \ker A \), and \( v_t := e^{-t(A+iH)} v, \ t \geq 0. \) Then for all \( t \geq 0 \), we have \( v_t \in \text{dom} A \) and

\[
\|A v_t\| \leq H \|v\|, \quad \langle v_t | A v_t \rangle \leq \frac{H^2 \|v\|^2}{\alpha}, \tag{4.1}
\]

\[
\left\| E_\alpha^\perp v_t \right\| \leq \frac{H \|v\|}{\alpha}. \tag{4.2}
\]

**Proof.** The first and second inequalities are the direct consequences of Lemma 4.3. Recall \( \alpha^{-1} A \geq E_\alpha^\perp \), then the third follows from

\[
\| E_\alpha^\perp v_t \|^2 = \langle v_t | E_\alpha^\perp v_t \rangle \leq \alpha^{-1} \langle v_t | A v_t \rangle \leq \alpha^{-2} H^2 \|v\|^2 .
\]

**Proof of Theorem 4.1:** Substitute \( \nu A \) for \( A \), and so \( \nu \alpha \) for \( \alpha \) in the above lemmas, and let \( v \in \mathcal{K} \) and

\[ v_t := v_t^{(\nu)} := e^{-t(\nu A+iH)} v \]

We find

\[
\left( \frac{d}{dt} + iE_\nu H E_\nu \right) E_\nu v_t^{(\nu)} = -iE_\nu H E_\nu^\perp v_t^{(\nu)},
\]

and hence

\[
\left\| \left( \frac{d}{dt} + iE_\nu H E_\nu \right) E_\nu v_t^{(\nu)} \right\| = \left\| E_\nu H E_\nu^\perp v_t^{(\nu)} \right\| \leq H \left\| E_\nu^\perp v_t^{(\nu)} \right\|
\]

\[
\leq (4.2) \frac{H \|v\|}{\nu \alpha} = \frac{H^2 \|v\|}{\nu \alpha} \tag{4.3}
\]

Thus we have

\[
\lim_{n \to \infty} \left\| \left( \frac{d}{dt} + iE_\nu H E_\nu \right) E_\nu v_t^{(\nu)} \right\| = 0 .
\]

This implies \( u_t := \lim_{n \to \infty} E_\nu v_t^{(\nu)} \) exists, and satisfies

\[
\left( \frac{d}{dt} + iE_\nu H E_\nu \right) u_t = 0 , \quad \text{i.e.} \quad u_t = e^{-tE_\nu H E_\nu} v.
\]

On the other hand, again by (4.2)

\[
\lim_{n \to \infty} \left\| E_\nu^\perp v_t^{(\nu)} \right\| \leq \lim_{n \to \infty} \frac{H \|v\|}{\nu \alpha} = 0 .
\]

Hence we have

\[
\lim_{\nu \to \infty} v_t^{(\nu)} = \lim_{\nu \to \infty} E_\nu v_t^{(\nu)} = u_t .
\]

\[ \Box \]
5 Path integral representation

In this section, first we state a Feynman–Kac formula on a Riemannian manifold. Our main source is Güneysu [5].

Let \((M, g)\) be a complete Riemannian manifold. The Laplace-Beltrami operator on \(M\) is denoted with

\[
\Delta = -\text{d}^* \text{d} : C^\infty(M) \rightarrow C^\infty(M).
\]

The following fact is known (see e.g. [5], Theorem 2.24):

**Theorem 5.1.** There exists a unique minimal positive fundamental solution

\[
p : (0, \infty) \times M \times M \rightarrow (0, \infty), \quad (t, x, y) \mapsto p_t(x, y)
\]

of the heat equation

\[
\frac{\partial}{\partial t} h(t, x) = \frac{1}{2} \Delta_x h(t, x).
\]

The map \(p\) is called the minimal heat kernel of \(M = (M, g)\).

**Definition 5.2.** Let \(X\) be a continuous semi-martingale with values in \(M\) in the time interval \(T = [t_0, t_1]\) or \([t_0, \infty)\). Fix a smooth principal bundle \(\pi : P \rightarrow M\) with structure group \(G\) and the associated Lie algebra \(\mathfrak{g}\), and a connection 1-form \(\alpha_0 \in \mathfrak{A}^1(P, \mathfrak{g})\). A continuous semi-martingale \(U\) on \(P\) defined in the time interval \(T\) is called a horizontal lift of \(X\) to \(P\) (with respect to the connection \(\alpha_0\)), if \(\pi(U) = X\) and

\[
\int \alpha_0(\text{d}U) = 0,
\]

where \(\text{d}\) denotes the Stratonovich integral.

**Proposition 5.3.** ([5], Theorem 2.15) There is a unique horizontal lift \(U\) of \(X\) to \(P\) with \(U_0 = u_0\) \(P\)-a.s.

Let \(E \rightarrow M\) be a Hermitian vector bundle with a fixed Hermitian connection \(\nabla\). Let \(\pi : P(E) \rightarrow M\) be the \(U(d)\)-principal bundle corresponding to \((E, (\cdot, \cdot)_g)\), that is,

\[
P(E) = \bigcup_{x \in M} \{ u | u : C^d \overset{\sim}{\rightarrow} E_x \text{ is an isometry} \}.
\]

The Hermitian connection \(\nabla\) on the vector bundle \(E \rightarrow M\) is induced by a unique connection 1-form \(\alpha_0 \in \mathfrak{A}^1(P(E), \mathfrak{g})\) on \(P\), \(\mathfrak{g} := u(d)\).

Let \(E \boxtimes E^* \rightarrow M \times M\) be the external tensor product bundle corresponding to \(E\), that is,

\[
E \boxtimes E^*|_{(x,y)} = E_x \otimes E_y^* = \text{Hom}(E_y, E_x).
\]

**Proposition 5.4.** ([5], Proposition and definition 2.17) Let \(X\) be a continuous semi-martingale with values in \(M\) in the time interval \(T = [t_0, t_1]\) or \([t_0, \infty)\). Let \(U\) be a horizontal lift of \(X\) to \(P(E)\) w.r.t. the connection 1-form \(\alpha_0\). Then the continuous adapted process given by

\[
//^X := UU_0^{-1} : T \times \Omega \rightarrow E \boxtimes E^*
\]

does not depend on the particular choice of the lift \(U\), and \(//^X\) is called the stochastic parallel transport in \(E\) along \(X\).

Let \(V \in L^2_{\text{loc}}(M, \mathbb{R})\) and assume that \(V\) is bounded from below, and is in the local Kato class. Define a self-adjoint operator

\[
H(V) := \frac{1}{2} \nabla^* \nabla + V
\]

on \(\Gamma_{L^2}(M, E)\), the Hilbert space of \(L^2\) sections of \(E\); Precisely, \(\nabla^* \nabla/2 + V\) is shown to be essentially self-adjoint on the domain of smooth sections with compact support, and \(H(V)\) denotes the self-adjoint extension of it.
Proposition 5.5. If $M$ is geodesically complete with Ricci curvature bounded from below and a positive injectivity radius, then there exists the Brownian bridge measures $\mathbb{P}_{t}^{x,y}$ in a way that the expectation values $\mathbb{E}_{t}^{x,y}[\bullet]$ are a rigorous version of the conditional expectation values $\mathbb{E}^{x,y}[B_{t}(x) = y]$. 

Theorem 5.6. ([5], Theorem 1.2) Let $M$ satisfy the conditions of Prop. 5.5, and $B$ a Brownian motion on $M$ in the time interval $[0, \infty)$. For any $t > 0$, define the section $K_{t}$ of the bundle $E \boxtimes E^{*} \to M \times M$, i.e., $K_{t}(x, y) \in \text{Hom}(E_{x}, E_{y})$ for all $x, y \in M$, by

$$K_{t}(x, y) := p_{t}(x, y)\mathbb{E}_{t}^{x,y}[\mathcal{F}_{t}^{B} / \mathcal{F}_{t}^{B,-1}], \quad \mathcal{F}_{t}^{B} := e^{-\int_{0}^{t}V(B, s)ds}, \quad x, y \in M. \quad (5.1)$$

Then $K_{t}$ defines an bounded integral kernel for the operator $e^{-tH(V)}$. We write $e^{-tH(V)}(x, y) := K_{t}(x, y)$.

Here, $V$ is assumed to be real-valued. We want to extend this formula to the cases where $V$ is complex-valued. However, in that case $H(V)$ is not self-adjoint, and the analysis appears to become far more difficult. The extension is easy only when $V$ is bounded, and its proof is similar to the case where $V$ is real-valued. We will consider this case in the following.

For each $x, y \in M$ and $0 \leq t_{0} < t_{1}$, let $C_{x,y}([t_{0}, t_{1}], M)$ denote the space of continuous functions $x : [t_{0}, t_{1}] \to M$ such that $x(t_{0}) = x$ and $x(t_{1}) = y$. Then we can choose $C_{x,y}([0, t], M)$ (or more generally $C_{x,y}([t_{0}, t_{0} + t], M)$ as a natural sample space $\Omega$ of the probability measure $\mathbb{P}_{t}^{x,y}$. It follows from Theorem 5.6 that there exists a finite measure $\mu_{0, \nu|x,y} := p_{t}(x, y)\mathbb{E}_{t}^{x,y} \in C_{x,y}([0, t], M)$ such that

$$e^{-tH(V)}(x, y) = \int_{C_{x,y}([0, t], M)} e^{-\int_{0}^{t}V(x, s)ds} / \int_{t}^{s} d\mu_{0, \nu|x,y}(x). \quad (5.2)$$

In the following we use this form of path integral formula instead of the probabilistic form (5.1), mainly because the appearance of (5.2) is more similar to physicists’ naive path integral formulas than that of (5.1).

Recall the definitions and notations in previous sections, together with Assumption 3.1; Set $M := M$, and $E$ to be the Hermitian line bundle $\mathcal{O}_{1,M}$ over $M$. Let $\nabla$ be the Chern connection on $\mathcal{O}_{1,M}$.

For each $\nu > 0$, consider the measure $\mu_{0, \nu|x,y} \in C_{x,y}([0, \nu], M)$. Let $x_{t}^{(\nu)} := x_{\nu t}$, and define the measure $\mu_{0, t;x,y}^{(\nu)}$ on $C_{x,y}([0, t], M)$ by the time rescaling $t \mapsto \nu t$;

$$d\mu_{0, t;x,y}^{(\nu)}(x^{(\nu)}) := d\mu_{0, \nu t;x,y}(x).$$

Theorem 5.7. Let $M$ satisfy Assumption 3.1 and the conditions of Proposition 5.5 as a Riemannian manifold. Furthermore assume that the operator $\Delta_{\nu}$ on the line bundle $\mathcal{O}_{1,M}$ has a spectral gap. Let $H \in L^{\infty}(\mathcal{M}, \mathbb{R})$ (a “classical Hamiltonian”), and $U_{\nu} := e^{-itQ_{\nu}H}$, the corresponding “quantum time evolution”. (Note that $U_{\nu} = e^{-itQ_{\nu}H}E_{H^{\ast}} = E_{H^{\ast}}e^{-itQ_{\nu}H}E_{H^{\ast}}$.) Then for any $t > 0$, the integral kernel of $U_{\nu}$ on $M \times M$ is expressed as

$$\widetilde{U}_{\nu}(p_{1}, p_{2}) = \lim_{\nu \to \infty} e^{\nu t} \int_{C_{p_{1}, p_{2}}([0, \nu], M)} e^{-\int_{0}^{\nu}H(x, s)ds} / \int_{t}^{\nu} d\mu_{0, t;p_{1}, p_{2}}^{(\nu)}(x), \quad p_{1}, p_{2} \in M,$

where $n = \dim_{\mathbb{R}} M$.

Proof. Noticing Cor. 4.2 and

$$\exp[-t(\nu(\nabla^{\ast}\nabla - n) + iH)] = e^{\nu t} \exp[-\nu t (\nabla^{\ast}\nabla + \nu^{-1}H)],$$

let $V := i\nu^{-1}H$. We see

$$/\mathcal{P}_{t}(x^{(\nu)}) = /\mathcal{P}_{t}(x), \quad \int_{0}^{t} H(x^{(\nu)}_{s})ds = \int_{0}^{\nu t} \nu^{-1}H(x_{s})ds.$$
Therefore, by (5.2),
\[
\exp \left[ -t \left( \mathbf{v} \left( \nabla^s \nabla - n \right) + iH \right) \right] (p_1, p_2) \\
= e^{\mu t} \exp \left[ -\mu t \left( \nabla^s \nabla + iW^s \cdot H \right) \right] (p_1, p_2) \\
= e^{\mu t} \int_{C_{p_1, p_2}(0, [0, \mu t], M)} e^{-\int_0^t \mu r^{-1} H(x^s) ds} / / \mu t^{-1} (x) d\mu_{0, \mu t; p_1, p_2}(x) \\
= e^{\mu t} \int_{C_{p_1, p_2}(0, [0, t], M)} e^{-i \int_0^t H(x^s) ds} / / t^{-1} (x^s) d\mu_{0, t; p_1, p_2}(x^s),
\]
and hence
\[
e^{-iQ_t(H)}(p_1, p_2) = \lim_{\nu \to \infty} \exp \left[ -t \left( \mathbf{v} \left( \nabla^s \nabla - n \right) + iH \right) \right] (p_1, p_2) \\
= \lim_{\nu \to \infty} e^{\mu t} \int_{C_{p_1, p_2}(0, [0, t], M)} e^{-i \int_0^t H(x^s) ds} / / t^{-1} (x^s) d\mu_{0, t; p_1, p_2}(x^s).
\]

Recall Lemma 2.2 and Eq. (3.1). Then we can define the integral kernel of \( \tilde{U}_t \) on \( S(M) \times S(M) \) (not on \( M \times M \)) by
\[
\tilde{U}_t(v_1, v_2) := \langle v_2 | e^{-iQ_t(H)} | v_1 \rangle = \langle \tilde{U}_t(p_1, p_2) v_2 \rangle(v_1), \quad v_k \in \text{ran}(p_k), \ k = 1, 2.
\]
Similarly, the parallel transport \( / /_t^{x_1, x_2} \) \( \in \text{Hom}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_2) \) can also viewed as a function on \( \text{ran}(p_1) \times \text{ran}(p_2) \):
\[
/ /_t^{x_1, x_2}(v_1, v_2) := (/ /_t^{x_1, x_2})^* (v_2), \quad (v_1, v_2) \in \text{ran}(p_1) \times \text{ran}(p_2).
\]
If \( t_1 > t_2 \), let \([t_1, t_2]\) refer to the closed interval \([t_2, t_1]\), and let \( \mu^{(\nu)}_{t_1, t_2; p_1, p_2} \) denote the same measure as \( \mu^{(\nu)}_{t_1, t_2; p_1, p_2} \).

**Corollary 5.8.** For any \( t_1, t_2 \in \mathbb{R} \), \( v_1, v_2 \in S(M) \) with \( p_k := \text{pr}(v_k) \in M \) \( (k = 1, 2) \),
\[
\langle v_2 | e^{-i(t_2 - t_1)Q_t(H)} | v_1 \rangle = \tilde{U}_{t_2 - t_1}(v_1, v_2) \\
= \lim_{\nu \to \infty} e^{\mu t_2 - t_1} \int_{C_{p_1, p_2}(0, [t_2, t_1], M)} e^{-i \int_0^{t_2} H(x^s) ds} / /_{t_2 - t_1} (v_1, v_2) d\mu_{t_2 - t_1; p_1, p_2}(x).
\]

For \( k = 1, ..., N \), let \( t_1 < \cdots < t_N \), \( p_k \in M \), \( v_k \in \text{ran}(p_k) \), \( \|v_k\| = 1 \), and \( x_k : [t_k, t_{k+1}] \to M \) \( (k = 1, ..., N - 1) \) be a path (of a Brownian motion) on \( M \) with \( x_k(t_k) = p_k, x_k(t_{k+1}) = p_{k+1} \). Let \( x : [t_1, t_N] \to M \) be the concatenation of the paths \( x_1, ..., x_{N-1} \). When \( N = 3 \), we have
\[
/ /_{t_N}^{x_1, x_3}(v_3, v_3) = (/ /_{t_3}^{x_2, x_3} / /_{t_2}^{x_1, x_2}) (v_3, v_3) = \left( / /_{t_2}^{x_2, x_3} / /_{t_3}^{x_1, x_2} \right) (v_3) = (v_3 | / /_{t_2}^{x_2, x_3} / /_{t_3}^{x_1, x_2} v_3) = \prod_{k=1}^{N-1} / /_{t_k}^{x_k, x_{k+1}}(v_k, v_{k+1}),
\]
and in general
\[
/ /_{t_N}^{x_1, x_N}(v_1, v_N) = (/ /_{t_N}^{x_{N-1}, x_N} / /_{t_{N-1}}^{x_1, x_{N-1}})(v_1, v_N) = \prod_{k=1}^{N-1} / /_{t_k}^{x_k, x_{k+1}}(v_k, v_{k+1})\]

Let \( \tilde{p} = (p_1, ..., p_N) \in M^N \), \( \tilde{t} = (t_1, ..., t_N) \), \( 0 \leq t_1 < \cdots < t_N \).
\[
C_{\tilde{p}, \tilde{t}} := \{ x \in C([t_1, t_N], M) | x(t_j) = p_j, \ j = 1, ..., N \}
\]
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Consider the measures \( \mu_{i,j}^{(\nu)} \) on \( C_{p_j,p_{j+1}}([t_j,t_{j+1}],\mathcal{M}) \), and define the measure \( \mu_{i,p}^{(\nu)} \) on \( C_{t,B} \) by
\[
d\mu_{i,p}^{(\nu)}(x) = \prod_{j=1}^{N-1} d\mu_{i,j}^{(\nu)}(p_{j-1},p_j \mid [t_j,t_{j+1}])
\]

**Theorem 5.9.** For any \( 0 \leq t_1 < \cdots < t_N, v_1, \ldots, v_N \in \mathbb{S}(\mathcal{M}) \) with \( p_k := p(v_k) \in \mathcal{M} \) \((k = 1, \ldots, N)\), we have
\[
\prod_{k=1}^{N-1} (\bar{U}_{t_{k+1}-t_k}(v_k,v_{k+1})) = \lim_{\nu \to \infty} e^{\nu t_1} \int_{C_{t,B}} e^{-i \int_{t_1}^{t_N} H(x_s)ds} / t_N^{-1}(v_1,v_N) d\mu_{i,p}^{(\nu)}(x).
\]

**Remark 5.10.** We see
\[
\prod_{k=1}^{N-1} (\bar{U}_{t_{k+1}-t_k}(v_k,v_{k+1}))
\]
\[
= \langle v_N | e^{-i(t_N-t_{N-1})Q(H)} v_{N-1} \rangle \cdots \langle v_2 | e^{-i(t_2-t_1)Q(H)} v_1 \rangle
\]
\[
= \langle v_N | e^{-i(t_N-t_{N-1})Q(H)} p_{N-1} \cdots e^{-i(t_2-t_1)Q(H)} p_2 e^{-i(t_2-t_1)Q(H)} v_1 \rangle
\]
\[
= \langle v_N | e^{-i(t_N-Q(H))} p_{N-1} \cdots p_2, v_1 \rangle e^{-i(t_N-t_1)Q(H)} v_1 \rangle
\]
where \( p_{t,t} := e^{it_j Q(H)} p_j e^{-it_j Q(H)} \). Thus, this is invariant under the transformation \( v_k \mapsto e^{i\theta_k} v_k, \theta_k \in \mathbb{R} \), for \( k = 2, \ldots, N-1 \). Furthermore, if \( v_1 = v_N \), this equals
\[
\text{Tr}(p_{N-1} \cdots p_2, v_1 e^{-i(t_N-t_1)Q(H)})
\]
and hence is invariant under the above transformation for all \( v_k, k = 1, \ldots, N \) with \( \theta_1 = \theta_N \).

**Proof.** By Corollary 5.8,
\[
\prod_{k=1}^{N-1} (\bar{U}_{t_{k+1}-t_k}(v_k,v_{k+1})
\]
\[
= \prod_{k=1}^{N-1} \lim_{\nu \to \infty} e^{\nu(t_{k+1}-t_k)n} \int_{C_{p_j,p_{j+1}}([t_k,t_{k+1}],\mathcal{M})} e^{-i \int_{t_k}^{t_{k+1}} H(x_s)ds} / t_{k+1}^{-1}(v_k,v_{k+1}) d\mu_{i,j}^{(\nu)}(p_k,p_{k+1})(x_k)
\]
\[
= \lim_{\nu \to \infty} e^{\nu t_1} \prod_{k=1}^{N-1} \int_{C_{p_j,p_{j+1}}([t_k,t_{k+1}],\mathcal{M})} d\mu_{i,j}^{(\nu)}(p_k,p_{k+1})(x_k)
\]
\[
= \lim_{\nu \to \infty} e^{\nu t_1} \prod_{k=1}^{N-1} \int_{C_{t,B}} e^{-i \int_{t_k}^{t_{k+1}} H(x_s)ds} / t_{k+1}^{-1}(v_k,v_{k+1}) d\mu_{i,p}^{(\nu)}(x).
\]

\[\square\]

**6 Representation of quantum probability**

Recall (2.6), then we want a path integral formula for \( \text{Tr}(p_{1,t_1} \cdots p_{N,t_N}) \) to represent the trace of product of some operators, which often has a physical meaning as a quantum probability or an expectation value of some physical quantity. Roughly speaking, if we set \( t_1 = t_N \) in Theorem 5.9, then we could get it. However, Theorem 5.9 is meaningful only when \( 0 \leq t_1 < \cdots < t_N \). If we want to extend Theorem 5.9 for any \( t_1, \ldots, t_N \in \mathbb{R} \), we need to introduce a curve parameter \( t \) other than the time parameter \( t \).
Assume that $H : M \to \mathbb{R}$ is bounded and smooth. Consider the manifold $M \times \mathbb{R}$ with the natural Riemannian metric. Then $\mathcal{O}_{1,M} \times \mathbb{R}$ is a line bundle of $M \times \mathbb{R}$. Define the connection $\nabla^{(H)}$ on $\mathcal{O}_{1,M} \times \mathbb{R}$ by
\[
\nabla_{X}^{(H)} s := \nabla_{X} s + iH \partial_{t}s, \quad X : \text{ a vector field on } M, \quad s : \text{ a section of } \mathcal{O}_{1,M} \times \mathbb{R}, \quad \partial_{t} := \frac{\partial}{\partial t},
\]
where $\nabla$ is the Chern connection on $M$. For a path $X$ on $M \times \mathbb{R}$, let $/\!/(\cdot,H)\!(X)$ or $/\!/(\cdot,H)\!x$ denote the parallel transport along $X$ w.r.t. $\nabla^{(H)}$.

Let $t_{0} \leq t_{1}$. For $x \in C([t_{0}, t_{1}], M)$, define $\tilde{x}, \tilde{x}^{-1} \in C([0, 1], M \times \mathbb{R})$ by
\[
\tilde{x}(t) := (x(t), t), \quad t := t(t_{1} - t_{0}) + t_{0}, \quad t \in [0, 1]
\]
\[
\tilde{x}^{-1}(t) := x(1 - t), \quad t \in [0, 1].
\]
Notice that $\tilde{x}^{-1} = (\tilde{x})^{-1}$ differs from $\tilde{x}^{-1}$. If $x$ is piecewise smooth, we find
\[
/\!/(H), \tilde{x}^{-1} = e^{-i \int_{t_{0}}^{t} \! H(x(s)) ds} /\!/(\cdot, \tilde{x}^{-1}). \tag{6.1}
\]

When $x$ is a Brownian motion, this equation can be understood as a stochastic one.

For $p_{0}, p_{1} \in M$ and $t_{0}, t_{1} \in \mathbb{R}$, let
\[
\tilde{C}_{t_{0}, t_{1}; p_{0}, p_{1}} := \begin{cases} 
\{ \tilde{x} | x \in C_{p_{0}, p_{1}}([t_{0}, t_{1}], M) \} & \text{if } t_{0} \leq t_{1} \\
\{ \tilde{x}^{-1} \} \in C_{p_{1}, p_{0}}([t_{1}, t_{0}], M) \} & \text{if } t_{0} > t_{1}
\end{cases}
\]
so that if $X_{1} \in \tilde{C}_{t_{0}, t_{1}; p_{0}, p_{1}}$ and $X_{2} \in \tilde{C}_{t_{1}, t_{2}; p_{1}, p_{2}}$, then the concatenation $X_{2} \bullet X_{1} \in C([0, 1], M \times \mathbb{R})$ is defined:
\[
(X_{2} \bullet X_{1})(t) := \begin{cases} 
X_{1}(2t) & (0 \leq t \leq \frac{1}{2}) \\
X_{2}(2t - 1) & \left( \frac{1}{2} < t \leq 1 \right)
\end{cases}
\]

If $t_{0} \leq t_{1}$ (resp. $t_{0} > t_{1}$), the measure $\mu^{(\nu)}_{t_{0}, t_{1}; p_{1}, p_{2}}$ on $C_{p_{0}, p_{1}}([t_{0}, t_{1}], M)$ (resp. $C_{p_{1}, p_{0}}([t_{1}, t_{0}], M)$) induces the measure $\tilde{\mu}^{(\nu)}_{t_{0}, t_{1}; p_{0}, p_{1}}$ on $\tilde{C}_{t_{0}, t_{1}; p_{0}, p_{1}}$. Let $\tilde{p} = (p_{1}, ..., p_{N}) \in M^{N}, \tilde{t} = (t_{1}, ..., t_{N}) \in \mathbb{R}^{N}$. Define the space of loops $\text{Loop}_{\tilde{p}, \tilde{t}} \subset C([0, 1], M \times \mathbb{R})$ by
\[
\text{Loop}_{\tilde{p}, \tilde{t}} := \left\{ X_{N} \bullet \cdots \bullet X_{1} | X_{k} \in \tilde{C}_{p_{k+1}, p_{k+1}; t_{k}, t_{k+1}}, \ k = 1, ..., N, \ p_{N+1} := p_{1}, \ t_{N+1} := t_{1} \right\}
\]
where $X_{N} \bullet \cdots \bullet X_{1} := X_{N} \bullet (X_{N-1} \bullet (\cdots \bullet X_{1} \cdots))$, although the order of the concatenations is not important in the following; That is, we could alternatively define $X_{N} \bullet \cdots \bullet X_{1}$ by, say, $(\cdots (X_{N} \bullet X_{N-1}) \cdots \bullet X_{1})$.

Define the measure $\mu_{\tilde{t}, \tilde{p}}^{(\nu)}$ on $\text{Loop}_{\tilde{p}, \tilde{t}}$ by
\[
d\mu_{\tilde{t}, \tilde{p}}^{(\nu)}(X) := \prod_{k=1}^{N} d\mu_{t_{k+1}; t_{k+1}, p_{k+1}}^{(\nu)}(X_{k}), \quad X = X_{N} \bullet \cdots \bullet X_{1} \in \text{Loop}_{\tilde{p}, \tilde{t}}.
\]

where $p_{N+1} := p_{1}, \ t_{N+1} := t_{1}$. For $X \in \text{Loop}_{\tilde{p}, \tilde{t}}$, of course, $/\!/(H)(X) \in U(1)$ refers to the (stochastic) holonomy along the loop $X$.

Lemma 6.1. For any $t_{1}, t_{2} \in \mathbb{R}, \upsilon_{1}, \upsilon_{2} \in S(M)$ with $p_{k} := \text{pr}(v_{k}) \in M, \ k = 1, 2,$
\[
\langle \upsilon_{2} | e^{-i(t_{2} - t_{1})Q^{(H)}} | \upsilon_{1} \rangle = \lim_{\nu \to \infty} e^{\nu \nu^{\upsilon_{2}}(t_{2} - t_{1})} \int_{C_{p_{1}, p_{2}; t_{1}, t_{2}, M}} /\!/(t_{2} - t_{1})X_{1}^{-1}(v_{1}, v_{2})d\mu_{t_{1}, t_{2}; p_{1}, p_{2}}^{(\nu)}(X).
\]

Proof. Directly follows from Corollary 5.8 and (6.1).

Theorem 6.2. Let $\tilde{p} = (p_{1}, ..., p_{N}) \in M^{N}, \tilde{t} = (t_{1}, ..., t_{N}) \in \mathbb{R}^{N}$. Then
\[
\text{Tr}(p_{1}, t_{1} \cdots p_{N}, t_{N}) = \lim_{\nu \to \infty} e^{\nu \nu^{T'}} \int_{\text{Loop}_{\tilde{p}, \tilde{t}}} /\!/(H)(X)d\mu_{\tilde{p}, \tilde{t}}^{(\nu)}(X) \tag{6.2}
\]
where $T := \sum_{k=1}^{N} |t_{k+1} - t_{k}| (t_{N+1} := t_{1}), \ p_{1}, ..., p_{N} \in M (k = 1, ..., N).
Proof. Let \(v_1, \ldots, v_N \in \mathcal{S}(\mathcal{M})\) with \(p_k = \rho(v_k) \in \mathcal{M}\). Then we see
\[
\overline{\text{Tr}(p_{1, t_1} \cdots p_{N, t_N})} = \text{Tr}(p_{N, t_N} \cdots p_{1, t_1}) = \prod_{j=1}^{N} (v_j + 1 | e^{-i(t_{j+1} - t_j)Q(H)}v_j) - 1
\]
where \(v_{N+1} := v_1, t_{N+1} := t_1\). Hence (6.2) follows from Lemma 6.1.

Let \(\mathcal{B}(\mathcal{M})\) denote the family of Borel sets of \(\mathcal{M}\). Let \(E_0(S) := \mathcal{Q}(\chi_S) = \int_S \rho \, d\mu(p), \quad S \in \mathcal{B}(\mathcal{M})\).
\[
E_t(S) := e^{-it\mathcal{Q}(H)}E_0(S)e^{it\mathcal{Q}(H)} = \int_S \rho \, d\mu(p), \quad t \in \mathbb{R}.
\]
For each \(t \in \mathbb{R}\), \(E_t(\bullet)\) is called a positive operator valued measure (POVM) on \(\mathcal{M}\). Let \(\rho\) be a density operator on \(\mathcal{H}\), i.e., \(\rho \geq 0\), \(\text{Tr}\rho = 1\), and let \(0 \leq t_1 \leq \cdots \leq t_N\). Then the value
\[
P_\rho(\vec{t}, \vec{S}) := \text{Tr}E_{t_N}(S_N) \cdots E_{t_1}(S_1) \rho E_{t_1}(S_1) \cdots E_{t_N}(S_N), \quad \vec{S} := (S_1, \ldots, S_N)
\]
is interpreted as the joint probability that under the condition that the state at time 0 is \(\rho\), the position in the phase space \(\mathcal{M}\) is measured to be in \(S_1\) at time \(t_1\), and then the position in the phase space \(\mathcal{M}\) is measured to be in \(S_2\) at time \(t_2\), etc. Of course, these measurements are somewhat “fuzzy” in that even if \(S, S' \in \mathcal{B}(\mathcal{M})\) satisfy \(S \cap S' = \emptyset\), the probability \(P_\rho((t, t), (S, S'))\) can be non-zero; Any error-free quantum measurement on \(\mathcal{M}\) is impossible by the uncertainty principle.

**Corollary 6.3.** Let \(\rho\) be a density operator which have the representation \(\rho = \mathcal{Q}(f_\rho)\) for some Borel function \(f_\rho : \mathcal{M} \to \mathbb{R}\). Let
\[
(F_1, \ldots, F_{2N+1}) := (\chi_{S_N}, \ldots, \chi_{S_1}, f_\rho, \chi_{S_1}, \ldots, \chi_{S_N}),
\]
\((\tau_1, \ldots, \tau_{2N+1}) := (t_N, \ldots, t_1, t_0, t_1, \ldots, t_N), \quad t_0 := 0.
\]
Then we have the path-integral representation of the quantum probability
\[
P_\rho(\vec{t}, \vec{S}) = \lim_{\nu \to \infty} e^{2nT\nu} \int_{\mathcal{M}^{2N+1}} \rho(\vec{p}) \int_{\text{Loop}_{\vec{p}, \nu}} \rho(\vec{p}, \vec{u}) - 1 \prod_{j=1}^{2N+1} F_j(p_j),
\]
where \(\vec{p} = (p_1, \ldots, p_{2N+1}), \vec{u} = (\tau_1, \ldots, \tau_{2N+1})\), and
\[
\int_{\mathcal{M}^{2N+1}} \rho(\vec{p}) \int_{\text{Loop}_{\vec{p}, \nu}} \rho(\vec{p}, \vec{u}) - 1 \prod_{j=1}^{2N+1} F_j(p_j).
\]
Proof. By Theorem 6.2 and Eq.(2.6), we find
\[
P_\rho(\vec{t}, \vec{S}) = \text{Tr}\mathcal{Q}_{t_N}(\chi_{S_N}) \cdots \mathcal{Q}_{t_1}(\chi_{S_1}) \mathcal{Q}(f_\rho) \mathcal{Q}_{t_1}(\chi_{S_1}) \cdots \mathcal{Q}_{t_N}(\chi_{S_N})
\]
\[
= \int_{\mathcal{M}} \rho(\vec{p}) \int_{\mathcal{M}^{2N+1}} \rho(\vec{p}, \vec{u}) - 1 \prod_{j=1}^{2N+1} F_j(p_j)
\]
\[
= \int_{\mathcal{M}} \rho(\vec{p}) \int_{\mathcal{M}^{2N+1}} \rho(\vec{p}, \vec{u}) \left[ \prod_{j=1}^{2N+1} F_j(p_j) \right] \lim_{\nu \to \infty} e^{nT\nu} \int_{\text{Loop}_{\vec{p}, \nu}} \rho(\vec{p}, \vec{u}) - 1 \prod_{j=1}^{2N+1} F_j(p_j),
\]
with
\[
T := \sum_{k=1}^{N} (\tau_{k+1} - \tau_k) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) = 2t_N.
\]
Remark 6.4. Note that there exist many density operators $\rho$ which do not have the representation $\rho = Q(f)$. However, the set of density operators of the form $Q(f)$ is dense in the space of density operators, which has a metric induced by the trace norm, and hence any density operator can be approximated by the operator of the form $Q(f)$.

Remark 6.5. We also emphasize that the formulas for quantum joint probabilities as Corollary 6.3 will not be formulated in the imaginary-time path integral; there will be no direct relation between the quantum joint probability (with real-time evolution) and the imaginary-time path integral.

7 Example: Glauber coherent states

Consider the case where $M \subset \mathbb{P}H$ is homeomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Let $\psi : \mathbb{C}^n \to H^\times$ be a holomorphism such that $\text{pr} \circ \psi : \mathbb{C}^n \to M$ is a single global coordinate chart of $M$. Without loss of generality, assume $\langle \psi(0) | \psi(z) \rangle = 1$ for all $z \in \mathbb{C}^n$.

The line bundle $\mathcal{O}_{1,M}$ over $M$, which is topologically trivial, i.e. $\cong M \times \mathbb{C}$, has the Hermitian metric $\langle \cdot | \cdot \rangle_{\text{pt}}$ given by $\langle f_1 | f_2 \rangle_{\text{pt}}(\text{pr}(v)) := \|v\|^2 \overline{f_1(v)}f_2(v)$, where $v \in \text{pr}^{-1}(M)$, $f_1, f_2 \in \Gamma_{1,M}$. A global Kähler potential $h : M \to \mathbb{R}$ is given by

$$h(z) = \frac{1}{h(z)} = \|\psi(z)\|^2.$$

The corresponding global symplectic form is $\omega := -i d'd' \log h$, and the Riemannian metric is $g(X, Y) := \omega(X, jY)$. Let $e := \psi(0)$ then $e^*$ is a global holomorphic section of $\mathcal{O}_{1,M}$. Precisely, if we identify the fiber of $\mathcal{O}_{1,M}$ over $p \in M$ with $\text{ran}(p)^*$, the value $e^*(p)$ of the section $e^* \in \Gamma_{1,M}$ at $p \in M$ is the function $\text{ran}(p) \to \mathbb{C}$, $v \mapsto \langle e | v \rangle$. The normalization of $e^*$ is $\hat{e}^* := h^{-1/2} e^*$; precisely, the value $\hat{e}^*(p)$ of the section $\hat{e}^* \in \Gamma_{1,M}$ at $p \in M$ is the function

$$\text{ran}(p) \to \mathbb{C}, \quad \zeta(z) \mapsto \zeta, \quad z \in \mathbb{C}^n, \psi(z) \in \text{ran}(p), \zeta \in \mathbb{C}.$$

Here the Chern connection form w.r.t. the holomorphic frame $e^*$ is a $\mathbb{C}$-valued 1-form on $\mathbb{C}^n$ defined by $\theta := d' \log h = -d' \log h_*$. The Chern connection is globally defined by

$$\nabla(f e^*) := (df) \otimes e^* + f \theta \otimes e^*, \quad f \in \Gamma_0^\infty(M) .$$

The Chern connection form $\theta_{\text{nor}}$ on $\mathbb{C}^n$ w.r.t. the normalized frame $e^*$ is defined by (3.3). Let $C : [0, 1] \to \mathbb{C}^n$ be a piecewise smooth curve, so that $x := \text{pr} \circ \psi \circ C$ is a piecewise smooth curve on $M$. If $C$ is a loop, i.e. $C(0) = C(1)$, the parallel transport (holonomy) $/X$ becomes a scalar:

$$/X \equiv /X = e^{\int_C \theta_{\text{nor}}} \in U(1).$$

Assume the conditions of Corollary 6.3. Let $X \in \text{Loop}_{\mathbb{C}^n}$, with $X(t) = (x(t), \gamma(t))$ ($t \in [0, 1]$) where $x : [0, 1] \to M$, $\gamma : [0, 1] \to \mathbb{R}$. Consider the manifold $\hat{M} \times \mathbb{R}$ as in Sec. 6, and let $H : M \to \mathbb{R}$ be a bounded smooth Hamiltonian. The connection on the line bundle $\mathcal{O}_{1,M} \times \mathbb{R}$ over $M \times \mathbb{R}$ defined in Sec. 6 is explicitly given by the global $\mathbb{R}$-valued 1-form

$$\Theta_H := \theta_{\text{nor}} + H dt ,$$

on $\mathbb{C}^n \times \mathbb{R}$, w.r.t. the normalized frame of $\mathcal{O}_{1,M} \times \mathbb{R}$, and hence the holonomy along $X$ becomes

$$/X(H)(X) = \exp \left( i \int_X \Theta_H \right) , \quad \hat{X}(t) := (\psi^{-1}(x(t)), \gamma(t)) \in \mathbb{C}^n \times \mathbb{R} .$$

Let $\text{Loop}_{\mathbb{C}^n} := \{ X | X \in \text{Loop}_{\mathbb{C}^n} \}$. The measure $\mu^{(\nu)}$ on $\text{Loop}_{\mathbb{C}^n}$ induces the measure $\hat{\mu}^{(\nu)}$ on $\hat{\text{Loop}}_{\mathbb{C}^n}$ by the bijection $X \mapsto \hat{X}$. Now the quantum joint probability formula in Corollary 6.3 is rewritten in a somewhat more familiar and intuitive form:
Theorem 7.1. Under the conditions of Corollary 6.3,

\[ P_\nu(\vec{r}, \vec{S}) = \lim_{\nu \to \infty} e^{2n N \nu} \int_{\mathbb{R}^{2N + 1}} \frac{d\nu^{2N + 1}(\vec{p})}{\cosh \nu} \int_{\text{Log}(\mu, \nu)} d\mu^{2N + 1}(\vec{X}) e^{i \int_{x} \Theta_H} \prod_{j=1}^{2N + 1} F_j(p_j). \]

Example 7.2. Next let us consider the most basic but important case. Let \( n = 1 \). Let \( a^*, a \) be usual creation/annihilation operators on a Hilbert space \( \mathcal{H} \), which satisfy \( [a, a^*] = 1 \), and assume that \( \{a, a^*\} \) is irreducible on \( \mathcal{H} \). Let \( v_0 \in \mathbb{S}(\mathcal{H}) \) be a “vacuum state,” i.e. \( av_0 = 0 \). Set

\[ \psi(z) = e^{za^*}v_0, \quad z \in \mathbb{C} \]

then the corresponding Kähler potential, the symplectic form, and the Riemannian metric are calculated as

\[ h(z) = \|\psi(z)\|^2 = e^{2|z|^2}, \]
\[ \omega = -i d'd' \log h = i dz \wedge d\bar{z} = 2dx \wedge dy, \quad g = 2dx \otimes dy. \]

Thus \( \mathbb{M} \cong \mathbb{C} \) as Kähler manifolds; The volume measure on \( \mathbb{M} \) is the usual Lebesgue measure \( dxdy \) on \( \mathbb{C} \cong \mathbb{R}^2 \) times 2. Note that the normalized vector \( |z\rangle := \psi(z) := e^{-|z|^2/2}\psi(z) \) is nothing other than the coherent state in the usual sense (i.e. the Glauber coherent state). We can also check \( \langle \psi(0) | \psi(z) \rangle = 1 \). The overcompleteness relation

\[ \frac{1}{\pi} \int_{\mathbb{C}} |z\rangle\langle z| dxdy = I, \quad z = x + iy \]

implies that we should take the measure \( \mu \) on \( \mathbb{M} \) as \( d\mu(z) := \frac{1}{\pi} dxdy \). Now the BS quantization of \( f : \mathbb{M} \to \mathbb{R} \) is given by

\[ Q(f) = \int_{\mathbb{C}} \hat{f}(z)|z\rangle\langle z| d\mu(z), \quad \text{where} \quad \hat{f}(z) := f(\text{pr}(z)) = f(|z\rangle\langle z|). \]

The r.h.s. is often called the Glauber–Sudarshan representation of the operator of l.h.s.

\[ e^* = \psi(0)^* = v_0^* \]

is a holomorphic section of \( \mathcal{O}_{1,\mathbb{M}} \). By (3.3) we find

\[ \theta_{\text{nor}} = ydx - xdy, \quad z = x + iy. \]

Let \( C : [0, 1] \to \mathbb{C} \) be a piecewise smooth curve, with \( x := \text{pr} \circ \psi \circ C \). Then we find that the operation of the parallel transport \( /\!\!/_{\mathcal{T}}^x \) on \( \mathcal{O}_{1,\mathbb{M}} \) is explicitly written as

\[ /\!\!/_{\mathcal{T}}^x(C(0)) = e^{i \int_{C(0)}^{C(1)} \theta_{\text{nor}}(C(t))}, \]

where we used the bra notation \( |z\rangle := \psi(z)^*, \quad z \in \mathbb{C} \). Let \( \Theta_H := \theta_{\text{nor}} + Hdt \), then the quantum joint probability w.r.t. the time evolution generated by the quantum Hamiltonian \( Q(H) \) is given by Theorem 7.1.

Note that the paths occurring in the (stochastic) line integral in this formula are closed, and hence even if we substitute an arbitrary \( \theta'_{\text{nor}} := \theta_{\text{nor}} + \alpha \) with \( d\alpha = 0 \) for \( \theta_{\text{nor}} \), we get the same probability. To fix the 1-form \( \theta_{\text{nor}} \) is analogous to a gauge fixing in physics; This suggests that Theorem 7.1 realizes a somewhat “gauge-invariant” formulation of quantization. However, of course, different gauge fixings of a classical system can lead to different quantizations, where the difference is experimentally observable, in general. Thus we can only say that Theorem 7.1 may reduce the “gauge-dependence” of the notion of quantizations.

8 Toward the geometric path integral quantizations

The results in the previous sections are given in the situation where the BS quantization \( Q(f) \) is considered only when the function \( f \) is bounded, and so \( Q(f) \) becomes a bounded operator. The various difficulties in dealing with unbounded operators put obstacles in the generalizations of those
results, and so it may be very difficult to formulate a general theory. (One can see such difficulty also from the case of deformation quantization; The theory of deformation quantization was presented by [2] in 1978. The theory achieved a great success in the algebraic level, when Kontsevich [8] proved its generality, that there exists a deformation quantization for every Poisson manifold. However, the theories to realize the deformation quantizations in terms of the operators in the Hilbert spaces (e.g. the strict deformation quantization of Rieffel [10, 11]) seem to remain incomplete even on the problems concerning only bounded operators.)

Nevertheless, we conjecture that the results of the previous sections can be generalized for most situations which are “physically relevant.” To clarify this conjecture, we consider the notion of “geometric path integral quantization” in this section.

Let $M$ be a complete Kähler manifold, and assume the condition of Proposition 5.5 as a Riemannian manifold. (Forget the assumption in Sec. 2 that $M$ is a submanifold of some projective space $PH$.) Physically, we interpret $M$ as a classical-mechanical phase space, whose symplectic form is the Kähler form $\omega$. Recall the notion of prequantization bundle in geometric quantization [13].

**Definition 8.1.** A symplectic manifold $(M, \omega)$ is prequantizable when there exists a Hermitian line bundle, called a prequantization bundle, $\pi : L \rightarrow M$ with connection $\nabla$, whose curvature form $\Theta$ is proportional to the symplectic 2-form, $\Theta = -i\omega/h$. (Cf. Eq.(3.5))

Note: In this paper, we set $h = 1$, and recall that $\Theta$ is defined to be a $u(1) = i\mathbb{R}$-valued 2-form here.

For quantization, we must assume that $M$ is prequantizable. Moreover we assume that the prequantization bundle $L \rightarrow M$ is holomorphic.

**Conjecture 8.2.** Assume that $H : M \rightarrow \mathbb{R}$ satisfy some adequate conditions as a classical-mechanical Hamiltonian. Then Cor. 6.3 can be generalized for such $H$.

**Remark 8.3.** In the typical cases, $H(x)$ is smooth, bounded from below, and increases in a polynomial order as $\|x\| \rightarrow \infty$. Hence we could take the above “adequate conditions” as such conditions. However also note the important cases such as the Hamiltonian of a hydrogen atom, which is not bounded from below.

Note that since a Brownian motion on $M$ is well-defined, our path integral is also well-defined for each fixed $\nu$. The above assertion says that the probability $P_\nu(\tilde{T}, \tilde{S})$ calculated via our path integral (with the limit $\nu \rightarrow \infty$) coincides with the value calculated via BS quantization. However, since $M$ is not assumed to be a subset of $PH$ here, BS quantization is not defined yet, and so the above assertion is still too vague. We will explain further this point in the following.

Consider the Hilbert space $K := \Gamma_{\text{hol}}^L \subset \Gamma^{L^2}$, where $\Gamma^{L^2}$ is the space of $L^2$ sections of $B$, and $\Gamma_{\text{hol}}^L$ is the closed subspace consisting of holomorphic sections. Here we consider $K$ as the quantum state space, following the method of holomorphic quantization.

Let $K_A$ be the integral kernel of an operator $A$ on $\Gamma^{L^2}$, i.e.,

$$(As)(x_1) = \int_M K_A(x_1, x_2)s(x_2)\, dx_2, \quad s \in \Gamma^{L^2}, \ x_1 \in M.$$ 

where $dx_2$ denotes the integral w.r.t. the volume form $\text{vol}$ on $M$. $K_A$ is a map such that $K_A(x_1, x_2) \in \text{Hom}(L_{x_2}, L_{x_1})$ for all $x_1, x_2 \in M$, where $L_x$ is the fiber of the line bundle $L$ at $x \in M$; Equivalently, $K_A$ is a section of the external tensor product bundle $L \boxtimes L^* \rightarrow M \times M$.

Let $E_K$ denote the orthogonal projection from $\Gamma^{L^2}$ onto $K$. For each $x \in M$, define $\nu_x \in \mathbb{S}(K) \subset \Gamma^{L^2}$ by

$$\nu_x(x') := C_x K_{E_K}(x', x), \quad x, x' \in M, \ C_x > 0.$$ 

Define $P : M \rightarrow PK$ by

$$P(x) := |\nu_x\rangle \langle \nu_x|, \quad x \in M.$$ 

Assume that this map $P$ is an embedding of the Kähler manifold $M$ into $PK$, viewed as a possibly infinite-dimensional Kähler manifold. This assumption says that $M$ may be identified with its range $P(M) \subset PK$, in other words, that $M$ can be seen as a submanifold of the projective space $PK$. In this situation the BS quantization can be defined for $M$, and so the meaning of Conjecture 8.2 becomes clearer.
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