A Study of Two-Temperature Non-Equilibrium Ising Models: Critical Behavior and Universality

P. Tamayo  
*Theoretical Division  
Los Alamos National Laboratory MS-B285  
Los Alamos, NM 87545  

and  
Thinking Machines Corp.  
245 First St.  
Cambridge, MA 02142

F. J. Alexander*  
*Center for Nonlinear Studies  
Los Alamos National Laboratory MS-B258  
Los Alamos, NM 87545

R. Gupta  
*Theoretical Division  
Los Alamos National Laboratory MS-B285  
Los Alamos, NM 87545

ABSTRACT

We study a class of 2D non-equilibrium Ising models based on competing dynamics induced by contact with heat-baths at two different temperatures. We make a comparative study of the non-equilibrium versions of Metropolis, heat bath/Glauber and Swendsen-Wang dynamics and focus on their critical behavior in order to understand their universality classes. We present strong evidence that some of these dynamics have the same critical exponents and belong to the same universality class as the equilibrium 2D Ising model. We show that the bond version of the Swendsen-Wang update algorithm can be mapped into an equilibrium model at an effective temperature.

*Present address: Institute for Scientific Computing Research L-416, Lawrence Livermore National Laboratory, Livermore, CA 94550
1.- Introduction

Despite attempts at constructing a rigorous theory for non-equilibrium statistical mechanics, there is still no formalism to parallel the one which exists for equilibrium systems. As a result, there are few analytical methods with which to deal with non-equilibrium systems. In general, non-equilibrium systems display rich and complex behavior such as phase separation, pattern formation, turbulence [1, 2], and it is therefore useful to first study simple model systems.

There exist model non-equilibrium systems described by stationary distributions that are comparatively easy to study. An open system maintained in a non-equilibrium steady state by an external temperature or density gradient is one example [3]. Another class of non-equilibrium steady states is obtained when the system is closed, and the dynamics is a local competition of two dynamics at different temperatures [14]. It is this type of system that we address in this paper. Since these non-equilibrium systems display behavior qualitatively similar to equilibrium systems, such as phase transitions, it is important to ask if their critical properties and universality classes are the same.

Flows of renormalized probability distributions and fixed points are normally independent of the details of a Hamiltonian. Perhaps these flows and fixed points are even independent of the existence of a Hamiltonian and a Boltzmannian distribution. If one were able to establish equivalences (or near equivalences) of universality classes between equilibrium and non-equilibrium models, then we will be able to answer many questions about the behavior of these systems without a complete formulation equivalent to the one for equilibrium systems.

Simple non-equilibrium spin-flip stochastic systems have received considerable attention in the literature over the last 10 years. Reviews and general discussions about non-equilibrium phase transitions and stationary states can be found in refs. [4, 5, 6, 7]. Studies of driven diffusive systems can be found in Wang et al [8], Marro et al [9], Garrido et al [10] and Grinstein et al [11].

Grinstein et al [12] studied the statistical mechanics of probabilistic cellular automata using time-dependent Ginzburg-Landau theory. They suggested that any non-equilibrium spin-flip dynamics with up-down symmetry belongs to the same universality class as the equilibrium Ising model. Their argument is based on the observation that under the renormalization group in \(d = 4 - \epsilon\), the dynamical fixed point of the Ising model is stable with respect to all additional analytic terms which preserve the lattice geometry and the spin up-down symmetry.

Kanter and Fisher [13] analyzed the existence of ordered phases in stochastic Ising systems with short-range interactions. They found that the existence of universality for those systems might depend on the details of the interaction and casted doubts on the general applicability of the argument of Grinstein et al [12].

A two-temperature Glauber Ising model was introduced by Garrido, Labarta and Marro [14] (we will refer to this model as GLM) to investigate stationary non-equilibrium states. They obtained a mean field solution and performed some Monte Carlo simulations on 2D lattices. They found critical behavior qualitatively similar to the equilibrium case.

The non-equilibrium behavior of competing dynamics such as spin-flip vs spin-exchange has been studied by Garrido et al [15] using hydrodynamic macroscopic equations and Monte Carlo data, and by Wang and Lebowitz [16] using a Monte Carlo renormalization group method. They found evidence of equilibrium Ising behavior and Ising-like exponents.

Tomé et al [17] studied the GLM model for the case when one of the temperatures is negative. They used a dynamical pair approximation to analyze antiferromagnetic steady states and obtained
the corresponding phase diagram. Their mean field renormalization group calculations show evidence in favor of equivalence with the equilibrium Ising universality class.

Böte et al. [18] studied a model similar to GLM in which each sublattice is in equilibrium at a different temperature. They performed Monte Carlo simulations and found strong evidence that the model belongs to the equilibrium Ising universality class.

Marques [19, 20] used a mean field renormalization group calculation to obtain a phase diagram and calculated the exponent \( \nu \) for the GLM model. Her results compared well with the equilibrium Ising values. Later she extended this technique to two different three state systems which retain the up-down symmetry [21] to test Grinstein et al. conjecture [12], and found good agreement with equilibrium Ising exponents.

Recently de Oliveira [22] analyzed the isotropic majority-vote non-equilibrium model by Monte Carlo and finite size scaling. He found very good agreement for the critical exponents between this model and the equilibrium Ising model, and also for Binder’s cumulant. In a separate paper de Oliveira et al. [23] studied a family of non-equilibrium spin models with up-down symmetry parameterized by a Glauber-like transition rate. They also found good evidence that the critical exponents for this family (except for the limit case of the voter model) are the same as the equilibrium Ising model.

Considerable numerical and analytic evidence has been accumulating in favor of universality and equilibrium Ising exponents for some of these models but the results are not definite yet. Most analytical methods used, such as mean-field RG, are of an approximate nature. Monte Carlo simulations have focused mainly on qualitative behavior and the calculations of critical exponents have not been carried out with high resolution. To better understand and test this equivalence we have undertaken a detailed, high resolution Monte Carlo study of two-temperature Ising models and explore the question of universality using different local and non-local update dynamics.

The paper is organized as follows. In Section 2 we will describe the different two-temperature non-equilibrium models that are the subject of this study. Section 3 presents detailed results for the Metropolis non-equilibrium dynamics including critical exponents and cumulant behavior. Section 4 focuses on the Swendsen-Wang [24] non-equilibrium dynamics. Section 5 contains a comparative study of all the dynamics. Extension to many temperature model is discussed in Section 6 and the conclusions are presented in Section 7.

### 2. Two-temperature non-equilibrium dynamics

We begin with the two dimensional Ising model on the square lattice with Hamiltonian

\[
H = -J\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j
\]

where \( \beta \) is the inverse temperature and \( J \) the coupling. A dynamics for the model can be described in terms of a time dependent probability distribution \( P(\sigma,t) \) which evolves according to a master equation,

\[
\frac{dP(\sigma,t)}{dt} = \sum_{\sigma'} \{ W(\sigma' \leftarrow \sigma) P(\sigma,t) - W(\sigma \leftarrow \sigma') P(\sigma',t) \}
\]
where \( W(\sigma' \leftarrow \sigma) \) is the transition rate from configuration \( \sigma \) to \( \sigma' \). We are interested in stationary probability distributions,

\[
\frac{dP(\sigma, t)}{dt} = 0, \quad P(\sigma, t) = P(\sigma)
\]

In the case of equilibrium systems \( P(\sigma) \) is not only stationary but has the form of a Boltzmann distribution parameterized by the inverse temperature \( \beta \),

\[
P(\sigma) = Z^{-1} \exp(-\beta H) \quad (4)
\]

where \( Z = \sum_{\sigma} \exp(-\beta H(\sigma)) \). In our case we are interested in stationary non-equilibrium distributions produced by the local competition of equilibrium dynamics at different temperatures. The usual condition to obtain an equilibrium Monte Carlo dynamics is to make \( W(\sigma' \leftarrow \sigma) \) obey detailed balance (note that imposing detailed balance on the \( W \) is a sufficient but not necessary condition),

\[
W(\sigma' \leftarrow \sigma) P(\sigma) = W(\sigma \leftarrow \sigma') P(\sigma') \quad (5)
\]

In the two-temperature model one considers a composite rate \( W(\sigma' \leftarrow \sigma) \),

\[
W(\sigma' \leftarrow \sigma) = pW_1(\sigma' \leftarrow \sigma) + (1 - p)W_2(\sigma' \leftarrow \sigma) \quad (6)
\]

with competing \( W_1 \) and \( W_2 \). At each time step the transition probability will be chosen at random to be \( W_1 \) with probability \( p \) or \( W_2 \) with probability \( (1 - p) \). \( W_1 \) and \( W_2 \) individually correspond to equilibrium transition rates obeying detailed balance with respect to temperatures \( \beta_1 \) and \( \beta_2 \), i.e.

\[
\frac{W_i(\sigma' \leftarrow \sigma)}{W_i(\sigma \leftarrow \sigma')} = \frac{e^{\beta_i H(\sigma)}}{e^{\beta_i H(\sigma')}} \quad (7)
\]

At each time step the dynamics obeys detailed balance locally with respect to \( \beta_1 \) or \( \beta_2 \) and the spins act as if in instantaneous contact with one of two heat baths. The overall effect is to produce a non-equilibrium dynamics which reduces to the equilibrium model when \( \beta_1 = \beta_2 \). From the master equation one can prove that for a combined dynamics of this sort there is indeed a stationary regime given by the condition \([4]\),

\[
p\langle \sigma W_1 \rangle + (1 - p)\langle \sigma W_2 \rangle = 0 \quad (8)
\]

The induced global probability distribution \( P(\sigma) \) does not, in general, correspond to a local known Hamiltonian and it depends on the details of the dynamics, i.e. the particular choice of \( W_1 \) and \( W_2 \). This is in contrast to equilibrium simulations where the Boltzmann distribution is independent of the particular choice of \( W \). The combination of \( W_1 \) and \( W_2 \) produces a stationary state analogous to a system being driven by an external potential. The ensemble of stationary configurations exhibits physical properties qualitatively similar to equilibrium (i.e. ordering, cluster formation, phase transitions, etc.). This is therefore one of the simplest ways to generate non-equilibrium models from equilibrium ones. In order to investigate the properties of stationary distribution on the dynamics we study three different update algorithms: Metropolis, Glauber (or equivalently heat-bath) and Swendsen-Wang. Furthermore each of these dynamics has a “bond” and “spin” version as we explain below.

We start by defining the GLM dynamics \([4]\). The transition rate for spin \( i \) takes the standard Glauber form,

\[
W_i = \frac{1}{2} \alpha [1 - \tanh(J\beta_i \sigma_i \sum_{|j-i|=1} \sigma_j)] \quad (9)
\]
where $\beta_1$ is chosen to be equal to $\beta_1$ with probability $p$ and $\beta_2$ with probability $(1-p)$. In one dimension this dynamics is always equivalent to an equilibrium model at an effective temperature $\beta_{\text{eff}}$ given by

$$
\tanh(2J\beta_{\text{eff}}) = p\tanh(2J\beta_1) + (1-p)\tanh(2J\beta_2)
$$

(10)

Similarly “bond” dynamics is defined by

$$
W_i = \frac{1}{2}\alpha[1 - \tanh(J\sigma_i \sum_{|j-i|=1} \beta_j\sigma_j)]
$$

(11)

where $\beta_j$ is $\beta_1$ with probability $p$ or $\beta_2$ with probability $(1-p)$ and in this way the temperature is selected independently for each bond. These transition rates with $\alpha = 1$ also define the heat-bath algorithm, i.e. the two algorithms are equivalent and we only need to discuss one. Henceforth we shall refer to this as the Glauber dynamics.

Metropolis non-equilibrium dynamics are defined in a similar way. The relevant acceptance factor for spin $i$ is given by,

$$
A_i = e^{-2J\sigma_i \beta \sum_j \sigma_j}
$$

(12)

where, as in the Glauber case, $\beta$ is either $\beta_1$ or $\beta_2$. In the bond version we have,

$$
A_i = e^{-2J\sigma_i \sum_j \beta_j\sigma_j}
$$

(13)

where each bond $j$ is chosen independently with temperature $\beta_1$ or $\beta_2$.

Finally we define non-equilibrium Swendsen-Wang spin and bond versions. In the bond version the percolation probability for bond $ij$ is chosen to be,

$$
\pi_{ij} = 1 - e^{-2J\beta_{ij}}
$$

(14)

where $\beta_{ij}$ is $\beta_1$ with probability $p$ or $\beta_2$ with probability $(1-p)$. The subsequent percolation, cluster finding and flipping steps are done in the same way as in the original Swendsen-Wang dynamics. In section 4 we show that this dynamics can be mapped to an equilibrium system at an intermediate “effective” temperature $\beta_{\text{eff}}$. The spin version is defined similarly except that the four bonds contributing to the update of each red (or black) site are chosen to be at the same temperature $\beta_1$ or $\beta_2$.

All these algorithms can be implemented very efficiently on a parallel computer such as the CM-5. In this paper we present detailed results for the Metropolis-spin and Swendsen-Wang-bond cases and make some comparisons with the other dynamics.

### 3. Metropolis non-equilibrium dynamics

We performed a careful investigation of the spin version of the Metropolis dynamics. Our update algorithm is parallel, so we simultaneously update all the red/black sites on the (checkerboard) lattice. For each sublattice site $i$ we choose temperature $\beta_1$ or $\beta_2$ independently using a uniformly distributed random number. Then we compute the change of energy with the spin flipped,

$$
\Delta E_i = 2\sigma_i \sum_j \sigma_j.
$$

(15)
If $\Delta E_i$ is less or equal to zero then the flip is always accepted, otherwise it is accepted with probability $\text{exp}(\beta_i \Delta E_i)$.

To address the question of the existence of an equivalent equilibrium system we consider the local transition rate for the combined dynamics,

$$e^{-\beta_{\text{eff}} \Delta E} = pe^{-\beta_1 \Delta E} + (1 - p)e^{-\beta_2 \Delta E}.$$  \hspace{1cm} (16)

This equation has to be satisfied for all values of $\Delta E$, however, only the cases for $\Delta E > 0$ are temperature dependent$^1$. In one-dimension there is only one relevant case ($\Delta E = 4$) and the equation is satisfied with a $\beta_{\text{eff}}$ equal to

$$\beta_{\text{eff}} = -\frac{1}{4} \log[pe^{-4\beta_1} + (1 - p)e^{-4\beta_2}],$$ \hspace{1cm} (17)

as was reported by Garrido et al$^{[12]}$. In two-dimensions one has to satisfy two equations (for $\Delta E = 4$ and $8$),

$$pe^{-4\beta_1} + (1 - p)e^{-4\beta_2} = e^{-4\beta_{\text{eff}}^4},$$ \hspace{1cm} (18)

$$pe^{-8\beta_1} + (1 - p)e^{-8\beta_2} = e^{-8\beta_{\text{eff}}^8}.$$ \hspace{1cm} (19)

For fixed $p$, $\beta_1$ and $\beta_2$ each equation has the solution

$$\beta_{\text{eff}}^4 = -\frac{1}{4} \log[pe^{-4\beta_1} + (1 - p)e^{-4\beta_1}],$$ \hspace{1cm} (20)

$$\beta_{\text{eff}}^8 = -\frac{1}{8} \log[pe^{-8\beta_1} + (1 - p)e^{-8\beta_2}],$$ \hspace{1cm} (21)

which are shown in Fig. 1 as a function of $\beta_2$ for fixed $\beta_1 = 0.35$ and $p = 0.5$. One can see that only for $\beta_1 = \beta_2$ is $\beta_{\text{eff}}^4 = \beta_{\text{eff}}^8$. This shows that for the Metropolis spin algorithm the two-temperature system cannot be described by a nearest neighbor effective Hamiltonian. The same is true for a triangular lattice with three (six) nearest-neighbors as there are two (three) independent equations.

Monte Carlo simulations show that the configurations generated over time are qualitatively similar to equilibrium configurations and the system exhibits an Ising-like phase transition. We have explored the critical behavior of this dynamics on $16^2$, $32^2$, $64^2$ and $128^2$ lattices. To locate the critical point we set $p = 0.5$ and $\beta_1 = 0.35$, and searched for the transition as a function of $\beta_2$. Our best estimate for the critical point is $\beta_2 = 0.6372(5)$. The scaling region around this critical point appears to be rather narrow, and the calculation of critical exponent is therefore quite difficult. We choose this particular set of values for $\beta_1$ and $\beta_2$ as they are far from the Ising critical point ($\beta_1 = \beta_2 = 0.440678$), and hope that any non-equilibrium effects will be manifest. We computed the critical exponents using finite size scaling and Binder’s cumulant analysis. We accumulated measurements over $5 \times 10^6$ for $16^2$, $10^7$ steps for $32^2$ and $64^2$ lattices and $9 \times 10^6$ for $128^2$.

The data for magnetization and susceptibility for the four different lattice sizes are given in Table 1 and plotted in Figs. 2 and 3. These figures show a change in the curvature between $\beta_2 = 0.6370$ and $\beta_2 = 0.6372$. On the basis of these data we estimate that the critical coupling is $\beta_2^* = 0.6371(1)$. The slopes give an estimate for the exponents $\beta/\nu$ and $\gamma/\nu$, assuming that the corrections to the leading finite size scaling forms $m \sim L^{-\beta/\nu}$ and $\chi \sim L^{\gamma/\nu}$ are negligible. The results are

$$\beta/\nu = 0.122(2)$$ \hspace{1cm} (22)

$$\gamma/\nu = 1.73(2),$$ \hspace{1cm} (23)

$^1$We thank R. Swendsen for bringing this to our attention.
where the errors are determined as follows. We first compute the statistical error in the magnetization and susceptibility for each data point. The error in the exponents is then obtained from the mean square fit to a straight line on a log-log plot.

The data for Binder’s cumulant $\mathbb{E}^2$,

$$U = 1 - \frac{\langle M^4 \rangle}{3 \langle M^2 \rangle^2}$$  \hspace{1cm} (24)

are also given in Table 1 and plotted in Figure 4 as a function of $\beta^2$. The value of this cumulant at the critical point is conjectured to be a universal number independent of lattice size $. Figures 5a and b show the cumulant values for the two pairs of lattice sizes, $(32^2, 64^2)$ and $(64^2, 128^2)$, at different temperatures around the critical point. The solid straight is the best fit to the data and the dashed line in the figures corresponds to $U_{L^2} = U_L$. The point of crossing of these two lines gives an estimate of the critical point. The crossing takes place at

$$U^*(32 \text{ vs } 64) = 0.610(2)$$  \hspace{1cm} (25)

$$U^*(64 \text{ vs } 128) = 0.605(10)$$  \hspace{1cm} (26)

and the corresponding estimates for $\beta^2$ are $0.6370(2)$ and $0.6360(12)$ respectively. These values are consistent with the estimate from finite size scaling given above. Our longest runs were done at $\beta_2 = 0.6372$, at which temperature our estimates

$$U_{16} = 0.611(1)$$  \hspace{1cm} (27)

$$U_{32} = 0.611(2)$$  \hspace{1cm} (28)

$$U_{64} = 0.612(6)$$  \hspace{1cm} (29)

$$U_{128} = 0.615(7)$$  \hspace{1cm} (30)

compare very well with the value $U^* = 0.611(1)$ computed by Bruce for the 2D equilibrium Ising model, and $U^*_{64} = 0.611(5)$ independently by us. Thus, we shall henceforth call $\beta_2 = 0.6372$ the critical point.

From the scaling of the cumulant one can obtain an estimate of $\nu$,

$$\left. \frac{dU_L}{d\beta} \right|_{\beta_0} = L^{1/\nu} G(L^{1/\nu} \epsilon)$$  \hspace{1cm} (31)

where $\epsilon = (\beta_c - \beta)/\beta$. To do this we first compute the slope $dU_L/d\beta$ for $L = 32$, 64 and 128 using the data points near the critical point and then fit these values versus $L$ using the above expression to obtain an estimate for $\nu$. We find,

$$\nu = 0.99(5).$$  \hspace{1cm} (32)

Another estimate can be computed from $dU_L/dU_L$, in the critical region because this quantity should scale as

$$\frac{dU_L}{dU_L} \sim \left( \frac{L}{L'} \right)^{1/\nu}.$$  \hspace{1cm} (33)

Using the more precise data shown in Fig. 5a for $L = 32$ and 64 we obtain $\nu = 0.95(8)$. The agreement between these results and the values for the equilibrium Ising model is quite good ($\nu = 1$, $\beta/\nu = 0.125$, $\gamma/\nu = 1.75$, $U^* = 0.611(1)$) and provides strong evidence for their equivalence.
To further confirm this equivalence we measured the probability distribution for the magnetization and energy at $\beta_1 = 0.35$ and $\beta_2 = 0.6372$ and compare them with those for the equilibrium model in Figs. 7a and b. The agreement is somewhat better for the magnetization than for the energy and qualitatively the distribution functions are equilibrium-like. The data are slightly more disordered than those for the critical equilibrium model, suggesting that $\beta_2^* \approx 0.6372$.

The bond version of the Metropolis dynamics has not been studied as extensively, and we postpone its discussion until Section 5 where we compare the different dynamics.

4. Swendsen-Wang non-equilibrium dynamics

We start with the bond contribution to a global probability distribution,

$$ P_{ij} = pe^{-\beta_1 (1-\sigma_i \sigma_j)} + (1-p)e^{-\beta_2 (1-\sigma_i \sigma_j)} \quad (34) $$

If we assume that $P_{ij}$ is always equivalent to an equilibrium distribution with coupling $\beta_{\text{eff}}$, i.e.

$$ P_{ij} = pe^{-\beta_{\text{eff}} (1-\sigma_i \sigma_j)} = e^{-\beta_{\text{eff}} (1-\sigma_i \sigma_j)} \quad (35) $$

then there exists a solution satisfying this equation for the two possible values of the bond energy,

$$ pe^{-2\beta_1} + (1-p)e^{-2\beta_2} = e^{-2\beta_{\text{eff}}}, \quad [\sigma_i = -\sigma_j] \quad (36) $$

$$ p + (1-p) = 1, \quad [\sigma_i = \sigma_j] \quad (37) $$

The solution is,

$$ p = \frac{e^{-2\beta_{\text{eff}}} - e^{-2\beta_2}}{e^{-2\beta_1} - e^{-2\beta_2}} \quad (38) $$

or equivalently, for a set of values $\beta_1$, $\beta_2$ and $p$, there always exists a $\beta_{\text{eff}}$ given by,

$$ \beta_{\text{eff}} = -\frac{1}{2} \log[p(e^{-2\beta_1} - e^{-2\beta_2}) + e^{-2\beta_2}] \quad (39) $$

that corresponds to an Ising equilibrium system. This implies that this dynamics is nothing more than equilibrium dynamics in disguise. If we set $\beta_{\text{eff}} = \beta_c$ we find lines of critical points in the $\beta_1 - \beta_2$ plane given by

$$ \beta_1 = -\frac{1}{2} \log[\frac{1}{p}(e^{-2\beta_c} - (1-p)e^{-2\beta_2})] \quad (40) $$

A set of these critical lines is shown in Fig. 6 for different values of $p$. Our simulations confirm this equivalence. Notice that for some extreme values of $\beta_1$ ($\beta_2$) there is no positive value of $\beta_2$ ($\beta_1$) that yields critical behavior.

The reason for the equivalence is that for the Swendsen-Wang bond dynamics there are, independent of the number of spatial dimensions, only two equations of constraint as each bond is independently in contact with the heat bath. One is the trivial condition $p + (1-p) = 1$ and the second is the desired result given in Eq. [35]. In a different context, a similar analysis of different ways to satisfy local equations for block percolation in equilibrium systems has been made in ref [28].

Lastly, we have measured autocorrelation times at criticality with $\beta_1 = 0.1$, $\beta_2 = 2.318$, and $p = 0.5$ ($\beta_{\text{eff}} = 0.440687$) to see if the stochastic choice of temperatures accelerates the decorrelation.
process. We find that the autocorrelation times are comparable to the values for the standard equilibrium Swendsen-Wang values. This is as expected since the two models are locally equivalent.

5. Other dynamics and comparisons

We have investigated the spin and bond version of Glauber dynamics for only three combinations of $\beta_1$ and $\beta_2$ using $32^2$ lattices. The ensemble size in these runs is $\sim 50,000$ update sweeps, significantly smaller than for Metropolis or Swendsen-Wang dynamics. In Figs. 8a, 8b, and 8c we compare the equation of state ($\langle E \rangle$ vs $\langle m^2 \rangle$) for the various dynamics for the three different combinations of $\beta_1$ and $\beta_2$. Figure 8a shows the results for $\beta_1 = 0.4$, $\beta_2 = 0.6$, Figure 8b corresponds to $\beta_1 = 0.1$, $\beta_2 = 2.318$ and Figure 8c to $\beta_1 = 0.2$, $\beta_2 = 0.738464$. These three sets of temperatures were chosen such that the corresponding $\beta_{eff}$ for Swendsen-Wang bond dynamics corresponds to a cold ($\beta_{eff} = 0.49$), critical ($\beta_{eff} = 0.4406868$) and hot ($\beta_{eff} = 0.40$) system respectively. All the simulations were done with $p = 0.5$. The dashed line corresponds to the result for the equilibrium Ising model obtained on $L = 32^2$ lattices. The errors in the data are roughly equal to the size of the symbols.

From Figs. 8a-c it is clear that the equation of state depends very sensitively on the dynamics and there is a large spread in the results for all three choices of temperatures. In all cases, except for the Swendsen-Wang spin version, the results lie very close to the line characterizing the equilibrium Ising model. The deviations from the equilibrium results in all three versions of the 2-temperature spin dynamics are in the direction of a more ordered system. The situation is reversed for the bond dynamics: compared to the equilibrium values the two-temperature results are less ordered. Qualitatively, the data with the various dynamics show the following ordering: Metropolis spin, Glauber spin, Metropolis bond, Glauber bond, and Swendsen-Wang bond. This pattern is also shown in Figs. 9a and b where we give the magnetization and energy probability distributions for $\beta_1 = 0.35$ and $\beta_2 = 0.6372$ (a critical point for the Metropolis spin dynamics). All cases appear to have the same functional form but their position and amplitudes are rescaled. The magnetization probability distributions are Gaussian near the peak but have long tails toward $m = 0$. Further work is needed to explore the possibility that all of these probability distributions are just rescaled forms of the equilibrium distribution and correspond to different $\beta_{eff}$'s.

6. Three-Temperature Model

To study the general case of the competition of many temperatures we extended the analysis to the three-temperature Metropolis spin dynamics. The three temperatures are chosen with probabilities $p_1$, $p_2$ and $p_3$:

$$W(\sigma' \leftarrow \sigma) = p_1 W_1(\sigma' \leftarrow \sigma) + p_2 W_2(\sigma' \leftarrow \sigma) + p_3 W_3(\sigma' \leftarrow \sigma).$$  

We fixed $\beta_1 = 0.35$, $\beta_3 = 0.6372$, and $p_1 = p_2 = p_3 = 1/3$ and varied $\beta_2$ about the equilibrium critical value $\beta_2 = 0.4406868$. We expected the system to display critical behavior for $\beta_2 = 0.4406868$.

The system effectively displays critical behavior in that region as can be seen in Fig. 10 where results for Binder’s cumulant are shown for $\beta_2$ in the range $[0.424, 0.456]$. Furthermore, the probability distribution functions for $\beta_2 = 0.4406868$ match the ones for the two-temperature model.
discussed in Section 4 and the equilibrium model at criticality as can be seen in Figs. 11a and b. We do find an apparent narrowing of the critical region compared with the two-temperature models and the statistical quality of the data are not accurate enough to measure the exponents.

Based on the study of the 3 temperature model we make the following conjecture for the Metropolis spin dynamics. A model in contact with an arbitrary number of heat-baths will display Ising critical behavior provided each temperature or a pair of them is tuned to the critical value. As the number of pairs of temperatures increase the critical region becomes narrower, making the measurement of critical exponents and the study of critical behavior more difficult.

7. Conclusions

We present high statistics results showing that for the Metropolis spin dynamics the stationary states produced by the two-temperature model are very similar to equilibrium states. On basis of the agreement of the critical exponents and Binder’s cumulant we conclude that the two-temperature Metropolis spin dynamics is in the same universality class as the Ising model. We also show that the bond version of the two-temperature Swendsen-Wang dynamics can be mapped into an equilibrium Ising model at an intermediate effective temperature. Thus, for these cases our results agree with the conjecture of Grinstein et al. that any non-equilibrium spin-flip dynamics which preserves up-down symmetry belongs to the same universality class as the equilibrium Ising model. Assuming that the system evolves into an equilibrium distribution after some thermalization steps, one can measure the critical exponents and flow of renormalized couplings using the Monte Carlo renormalization group method. We hope to investigate this possibility in the future.

Our results for the Metropolis bond, Glauber spin and bond, and Swendsen-Wang spin dynamics are qualitative. Further work is required to confirm that they too belong to the same universality class as the equilibrium Ising model.

We have extended the two-temperature critical Metropolis spin dynamics to the three temperature case. We find that the system shows critical behavior when the third temperature is tuned to $\beta = 0.4406868$. Based on this we conjecture that the Ising critical behavior is preserved as long as one adds pairs of temperatures that are, by themselves, critical. The critical region appears to become narrower as the number of pairs of temperature values are increased and the statistical quality of the data deteriorates. This makes the calculation of critical exponents and the study of the models more difficult.

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References

[1] J.D. Gunton, M. San Miguel and P. S. Sahni in Phase Transitions and Critical Phenomena, edited by C. Domb and J. L. Lebowitz Academic, NY (1983).

[2] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys., 65, 851 (1993).

[3] H. Spohn, J. Phys. A, 16, 4275-4291 (1983).

[4] J. Marro, in Fluctuations and Stochastic Phenomena in Condensed Matter, Springer-Verlag Lecture notes in Physics, vol. 268 (1986).

[5] A. I. Lopez-Lacomba, P. L. Garrido and J. Marro, J. Phys. A 23 3809 (1990).

[6] J. Marro, J. L. Vallés and J. M. Gonzalez-Miranda, Phys. Rev. B 35 3372 (1987).

[7] P. L. Garrido and J. Marro, Phys. Rev. Lett. 62 1929 (1989).

[8] J. S. Wang, K. Binder and J. L. Lebowitz, J. Stat. Phys. 56 783 (1989).

[9] J. Marro, P. L. Garrido and J. L. Vallés, Phase Transitions, Gordon and Breach Science pubs. 1990.

[10] P. L. Garrido, J. Marro, and R. Dickman, Annals of Phys. 1990.

[11] G. Grinstein, J. Jayaprakash and J. E. S. Socolar, Phys. Rev. E 48 R643 (1993).

[12] G. Grinstein, C. Jayaprakash, and Y. He, Phys. Rev. Lett. 55 2527 (1985).

[13] I. Kanter and D. S. Fisher, Phys. Rev. A 40 5327 (1989).

[14] P. L. Garrido, A. Labarta, and J. Marro, J. Stat. Phys. 49 551 (1987).

[15] P. L. Garrido, J. Marro, and J. M. Gonzalez-Miranda, Phys. Rev. A 40 5802 (1990).

[16] J. S. Wang and J. L. Lebowitz, J. Stat. Phys. 51 893 (1988).

[17] T. Tomé, M. J. de Oliveira, and M. A. Santos, J. Phys. A 24 3677 (1991).

[18] H. W. J. Blöte, J. R. Herringa, A. Hoogland, and R. K. P. Zia, J. Phys. A 23 3799 (1990).

[19] M. C. Marques, J. Phys. A 22 4493 (1989).

[20] M. C. Marques, Phys. Lett. A 145 379 (1990); Physica A 163 915 (1990).

[21] M. C. Marques, J. Phys. A 26 1559 (1993).

[22] M. J. de Oliveira, J. Stat. Phys. 66 273 (1992).

[23] M. J. de Oliveira, J. F. F. Mendes, and M. A. Santos, J. Phys. A 26 2317 (1993).

[24] R. Swendsen and J. S. Wang, Phys. Rev. Lett. 58 86 (1987).

[25] K. Binder, Z. Phys. B 43 119 (1981).
[26] A. D. Bruce, *J. Phys. A* **18** L873 (1985).

[27] K. Binder and A. P. Young, *Rev. Mod. Phys.* **58** 801 (1986).

[28] R. Brower and P. Tamayo, *Physica A* **193** 314 (1992).

[29] R. Swendsen. Private communications.
Table I: Data for Susceptibility, Magnetization and Binder’s cumulant for the 2 temperature Metropolis spin dynamics ($\beta_1 = 0.35$, $p = 0.5$)

| $\beta_2$ | $L$ | $\chi$ | $|m|$ | $U$ |
|-----------|-----|-------|-------|-----|
| 0.6372    | 16  | 9.11(5)| 0.7043(3) | 0.611(1) |
| 0.6300    | 32  | 33.5(3) | 0.629(2) | 0.603(3) |
| 0.6350    | 32  | 32.0(4) | 0.639(2) | 0.608(4) |
| 0.6370    | 32  | 30.2(8) | 0.647(3) | 0.611(11) |
| 0.6372    | 32  | 30.6(3) | 0.6464(7) | 0.611(2) |
| 0.6375    | 32  | 30.6(3) | 0.6463(8) | 0.611(3) |
| 0.6400    | 32  | 30.0(6) | 0.651(2) | 0.613(7) |
| 0.6475    | 32  | 27.2(4) | 0.666(1) | 0.619(3) |
| 0.6500    | 32  | 26.3(6) | 0.671(2) | 0.621(5) |
| 0.6550    | 32  | 25.1(4) | 0.679(1) | 0.624(4) |
| 0.6600    | 32  | 23.3(7) | 0.689(2) | 0.628(5) |
| 0.6300    | 64  | 114(4)  | 0.566(4) | 0.600(13) |
| 0.6350    | 64  | 112(5)  | 0.580(4) | 0.605(12) |
| 0.6370    | 64  | 103(2)  | 0.592(2) | 0.611(4) |
| 0.6372    | 64  | 101(2)  | 0.594(2) | 0.612(6) |
| 0.6375    | 64  | 101(2)  | 0.594(2) | 0.612(5) |
| 0.6400    | 64  | 96(5)   | 0.605(2) | 0.616(3) |
| 0.6475    | 64  | 76(5)   | 0.635(4) | 0.629(14) |
| 0.6500    | 64  | 68(3)   | 0.644(3) | 0.633(11) |
| 0.6550    | 64  | 64(5)   | 0.655(3) | 0.636(11) |
| 0.6600    | 64  | 56(3)   | 0.669(2) | 0.640(4) |
| 0.6350    | 128 | 409(51) | 0.52(1)  | 0.597(40) |
| 0.6370    | 128 | 357(22) | 0.538(6) | 0.607(22) |
| 0.6372    | 128 | 315(12) | 0.550(2) | 0.615(7) |
| 0.6375    | 128 | 328(18) | 0.550(3) | 0.615(11) |
| 0.6400    | 128 | 257(20) | 0.573(5) | 0.627(21) |
| 0.6475    | 128 | 197(19) | 0.611(3) | 0.639(9) |
| 0.6500    | 128 | 148(10) | 0.626(3) | 0.645(10) |
| 0.6550    | 128 | 133(15) | 0.644(3) | 0.649(9) |
| 0.6600    | 128 | 91(10)  | 0.664(2) | 0.654(7) |
Fig. 1.- Plot of $\beta_{\text{eff}}^4$ and $\beta_{\text{eff}}^8$ (see eq. 20) versus $\beta_2$ (at $\beta_1 = 0.35$ and $p = 0.5$) for the non-equilibrium Metropolis spin dynamics.
Fig. 2.- Scaling behavior of Magnetization as a function of $\beta_2$ (with $\beta_1 = 0.35$ and $p = 0.5$ held fixed) on different lattice sizes for the non-equilibrium Metropolis spin dynamics.
Fig. 3.- Scaling behavior of susceptibility as a function of $\beta_2$ (with $\beta_1 = 0.35$ and $p = 0.5$ held fixed) on different lattice sizes for the non-equilibrium Metropolis spin dynamics.
Fig. 4.- Cumulant values as a function of $\beta_2$ (at $\beta_1 = 0.35$ and $p = 0.5$) and on different lattice sizes for the non-equilibrium Metropolis spin dynamics.
Fig. 5a.- Cumulant values for $32^2$ versus $64^2$ lattices at temperatures around the critical point for the non-equilibrium Metropolis spin dynamics.
Fig. 5b. - Cumulant values for $64^2$ versus $128^2$ lattices at temperatures around the critical point for the non-equilibrium Metropolis spin dynamics.
Fig. 6.- Critical lines (see eq. 40) for the Swendsen-Wang bond dynamics.
Fig. 7a.- Magnetization probability distributions $P(|m|)$ for the Metropolis spin dynamics at $\beta_1 = 0.35$, $\beta_2 = 0.6372$, $p = 0.5$ and $L = 32$ (critical point). The probability distributions for the 2D and 3D equilibrium Ising model are also shown for comparison.
Fig. 7b.- Energy probability distributions $P(-E)$ for the Metropolis spin dynamics at $\beta_1 = 0.35$, $\beta_2 = 0.6372$, $p = 0.5$ and $L = 32$ (critical point). The probability distributions for the 2D and 3D equilibrium Ising model at criticality are also shown for comparison.
Fig. 8a.- Equation of state data for different dynamics with $\beta_1 = 0.4$, $\beta_2 = 0.6$, $p = 0.5$ and $L = 32$ ($\beta_{\text{eff}}^{\text{SW bond}} = 0.49$)
Fig. 8b.- Equation of state data for different dynamics with $\beta_1 = 0.1$, $\beta_2 = 2.318$, $p = 0.5$ and $L = 32$ ($\beta_{SW}^{SW_{bond}} = 0.4406868$)
Fig. 8c.- Equation of state data for different dynamics with $\beta_1 = 0.2$, $\beta_2 = 0.738464$, $p = 0.5$ and $L = 32$ ($\beta_{SW_{bond}}^{SW} = 0.4$)
Fig. 9a.- Magnetization probability distributions $P(|m|)$ for the spin and bond versions of Metropolis, Glauber and Swendsen-Wang dynamics at $\beta_1 = 0.35$, $\beta_2 = 0.6372$, $p = 0.5$ and $L = 32$ (critical point).
Fig. 9b.- Energy probability distributions $P(-E)$ for the spin and bond versions of Metropolis, Glauber and Swendsen-Wang dynamics at $\beta_1 = 0.35$, $\beta_2 = 0.6372$, $p = 0.5$ and $L = 32$ (critical point).
Fig. 10.- Cumulant values as a function of $\beta_2$ (at $\beta_1 = 0.35$ and $\beta_3 = 0.6372$) and $L = 32$ and 64 lattice sizes for the three-temperature Metropolis spin dynamics.
Fig. 11a.- Magnetization probability distributions $P(|m|)$ for the two temperature and three temperature Metropolis spin dynamics at the critical point ($L = 32$).
Fig. 11b.- Energy probability distributions $P(-E)$ for the two temperature and three temperature Metropolis spin dynamics at the critical point ($L = 32$).