BIORTHOGONAL RATIONAL FUNCTIONS OF $R_{II}$ TYPE

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Abstract. In this work, a sequence of orthonormal rational functions that is also biorthogonal to another sequence of rational functions arising from recurrence relations of $R_{II}$ type is constructed. The biorthogonality is proved by a procedure which we call Zhedanov’s method. A particular case is considered that provides a Christoffel type transformation of the generalized eigenvalue problem with a reformulation different from the existing literature.

1. Introduction

Recurrence relations of the form

$$P_{n+1}(z) = \rho_n(z-\nu_n)P_n(z) + \tau_n(z-a_n)(z-b_n)P_{n-1}(z), \quad n \geq 1, \quad (1.1)$$

with initial conditions $P_0(z) = 1$ and $P_1(z) = \rho_0(z-\nu_0)$ are studied extensively [11] to define families of biorthogonal functions having explicit representations in terms of basic hypergeometric functions (see [15] for a recent work). Further, it was shown [11] that if

$$P_n(a_n) \neq 0, \quad P_n(b_n) \neq 0, \quad \tau_n \neq 0, \quad (1.2)$$

then there exists a rational function $\phi_n(z) = \prod_{k=1}^{n} (z-a_k)^{-1} - 1 (z-b_k)^{-1} - 1 P_n(z)$ and a linear functional $M$ defined on the span $\{z^k \phi_n(z) : 0 \leq k \leq n\}$ such that the relation $M(z^k \phi_n(z)) = 0$, for $0 \leq k < n$ holds. Conversely, one can always obtain (1.1) from a sequence of rational functions $\{\phi_n(z)\}_{n=0}^\infty$ having poles at $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ and satisfying a three term recurrence relation. Following [11] (see also [12]), we call (1.1) as recurrence relation of $R_{II}$ type.

Related to such recurrence relations are important concepts of rational functions satisfying both orthogonality and biorthogonality properties. The theory of rational functions orthogonal on the unit circle is developed parallel to that of polynomials orthogonal on the unit circle and is available in the monograph [5]. A sequence of orthonormal rational functions is obtained from the Gram-Schmidt orthonormalization process in the linear space of rational functions which, in fact, can be characterized by the poles of the basis elements as well. In this direction, [3,14], starting from a set of pre-defined poles, the rational functions are characterized by Favard type theorems as well as in terms of three-term recurrence relations similar to that of orthogonal polynomials on the real line [6,10], but with rational coefficients. The effect of poles on the asymptotics of the Christoffel functions associated with the orthogonal rational functions and their interval of orthogonality is also studied [8]. For recent generalizations in the theory, see [3,7,19] and references therein.

Following [14], two sequences of functions $\{R_n(z)\}$ and $\{Q_n(z)\}$ are said to be biorthogonal, if they satisfy

$$M(R_n(z)Q_m(z)) = \kappa_n \delta_{n,m}, \quad \kappa_n \neq 0, \quad n, m \geq 0, \quad (1.3)$$

with respect to a linear functional $M$. We observe that in contrast to the usual orthogonality condition, two different sequences are used for the biorthogonality condition. Further,
The two sequences \( \{ R_\alpha \} \), unlike the case for orthogonal polynomials on the real line [6], the polynomial \( P_n(z) \) satisfying (1.1) is the characteristic polynomial of a matrix pencil \( G_n - zH_n \), where both \( G_n \) and \( H_n \) are tridiagonal matrices [16, 17, 21].

**1.1. Motivation for the problem.** The components of the eigenvectors of the matrix pencil \( G_n - zH_n \) are rational functions with the numerator polynomials \( P_n(z) \) satisfying (1.1). However, these rational functions are not the ones that were used initially to obtain the matrix pencil. In fact, while the three term recurrence relation satisfied by \( \phi_n(z) \) is used to obtain the matrix pencil, the usual process [2] is to partition the poles to form two new sequences of rational functions

\[
p^L_n(z) = \frac{P_n(z)}{\prod_{k=1}^n(z - a_k)}, \quad p^R_n(z) = \frac{P_n(z)}{\prod_{k=1}^n(z - b_k)}
\]

which form the components of the left and right eigenvectors of the matrix pencil \( zG - H \). The two sequences \( \{ p^L_n(z) \}_{n=0}^\infty \) and \( \{ p^R_n(z) \}_{n=0}^\infty \) are then used to define two new sequences of rational functions [2, 9] satisfying the biorthogonality relation (1.3). However we note that two sequences of rational functions that are biorthogonal to each other need not themselves form an orthogonal sequence.

Motivated by the procedure of proving biorthogonality [21], which we call as Zhedanov’s method, the central theme of the manuscript is to study a sequence of rational functions that is both orthogonal as well as biorthogonal. Precisely, we are interested in constructing a sequence of orthogonal rational functions \( \{ \varphi_n(z) \} \) satisfying the following two properties:

(i) The related matrix pencil has the numerator polynomials \( P_n(z) \) as the characteristic polynomials and \( \varphi_n(z) \) as components of the eigenvectors.

(ii) The orthogonal sequence \( \{ \varphi_n(z) \} \) is also biorthogonal to another sequence of rational functions.

We note that such a system exists in the case of polynomials. For instance, the two polynomials \( R_n(z; \alpha, \beta) = zP_1(-n, \alpha + \beta + 1)P_1(2\alpha + 1, 1 - z) \), \( Q_n(z) = R_n(z; \alpha, -\beta) \), \( n \geq 1 \), were proved to be biorthogonal [1] with respect to the weight function \( \omega(\theta) = (2 - 2 \cos \theta)^\alpha (e^{i\beta})^\beta, \theta \in [-\pi, \pi], \) \( \text{Re} \alpha > -1/2 \). The sequence \( \{ R_n(z; \alpha, \beta) \}_{n=0}^\infty \) was later proved to be orthogonal with respect to the weight \( \omega(\theta) = 2^{2\alpha} e^{i\pi \beta} \sin \theta \) if \( \alpha \in \mathbb{R}, \alpha > -1/2 \) and \( i\beta \in \mathbb{R} \) [18]. The present problem serves to find an abstract rational analogue of such cases of orthogonal sequences satisfying biorthogonality properties as well.

The paper is organized as follows. Section 2 introduces the fundamental spaces and the orthogonal rational functions that lead to recurrence relations of \( R_{II} \) type. In Section 3 the reverse procedure, that is, starting with \( R_{II} \) recurrences, and recovering the same orthogonal rational functions via biorthogonality relations is provided. In Section 4 the Christoffel type transform of a particular case of our orthogonal rational functions is discussed.

**2. Fundamental spaces and associated rational functions**

Let \( \{ \alpha_j \}_{j=1}^\infty \) and \( \{ \beta_j \}_{j=0}^\infty \) be two given sequences where \( \beta_0 := 0, \)

\[
\alpha_j, \beta_j \in \mathbb{C} \setminus \{0\}, \quad \alpha_j \neq \infty, \quad j \geq 1.
\]

We define

\[
u_{2j}(z) := \frac{1}{1 - z\beta_j}, \quad \nu_{2j+1}(z) := \frac{1}{z - \alpha_{j+1}}, \quad j \geq 0.
\]
The basis \( \{ u_j \}_{j=0}^n \), \( n \geq 1 \), generates the linear spaces \( L_n = \text{span}\{ u_0, u_1, \cdots, u_n \} \) and \( L = \bigcup_{n=0}^{\infty} L_n \). Equivalently, we also have \( L_n = \text{span}\{ u_0, u_1, \cdots, u_n \} \), where

\[
u_{2j}(z) = \frac{z^{2j}}{\prod_{k=1}^{j}(z - \alpha_k)\prod_{k=1}^{j}(1 - z\beta_k)}, \quad \nu_{2j+1}(z) = \frac{z}{z - \alpha_{j+1}}u_{2j}(z), \quad j \geq 0.
\]

Further, the product spaces \( L_m \cdot L_n \) and \( L \cdot L \) consist of functions of the form \( h_{m,n}(z) = f_m(z)g_n(z) \) and \( h(z) = f(z)g(z) \) respectively, where \( f_m(z) \in L_m \), \( g_n(z) \in L_n \) and \( f(z), g(z) \in L \).

The substar transform \( h_s(z) \) of a function \( h(z) \) is defined as \( h_s(z) = \overline{h(1/z)} \). Let \( \mathfrak{L} \) be a linear functional defined on \( L \cdot L \) such that

\[
\langle f(z), g(z) \rangle := \mathfrak{L}(f(z)g_s(z)),
\]

is Hermitian and positive-definite, and hence defines an inner product on the space \( L \).

Let \( \varphi_j(z) \), \( j \geq 0 \), be the sequence of functions that are orthonormal with respect to \( \mathfrak{L} \) and obtained from the Gram-Schmidt process of the basis \( \{ u_j \}_{j=0}^n \), \( n \geq 1 \). That is \( \varphi_j(z) \), \( j \geq 0 \), satisfy the orthogonality property

\[
\langle \varphi_m(z), \varphi_n(z) \rangle = \mathfrak{L}(\varphi_m(z)\varphi_n(z)) = \delta_{m,n}, \quad m, n = 0, 1, \cdots.
\]

Further, it is clear that \( \varphi_n(z) \) are rational functions of the form \( \varphi_0(z) = 1 \),

\[
\begin{align*}
\varphi_{2j+2}(z) &= \frac{r_{2j+2}(z)}{\prod_{k=1}^{j+1}(z - \alpha_k)\prod_{k=1}^{j+1}(1 - z\beta_k)}, \quad j \geq 0, \\
\varphi_{2j+1}(z) &= \frac{r_{2j+1}(z)}{\prod_{k=1}^{j+1}(z - \alpha_k)\prod_{k=1}^{j+1}(1 - z\beta_k)}, \quad j \geq 0,
\end{align*}
\]

where \( r_n(z) \in \Pi_n \), the linear space of polynomials of degree at most \( n \). Moreover, \( L_{2n} \), \( n \geq 1 \), can now be interpreted as the space of rational functions having poles belonging to the set \( \{ \alpha_1, \cdots, \alpha_n, 1/\beta_1, \cdots, 1/\beta_n \} \) with the order of the pole at \( \alpha_j \) or \( 1/\beta_j \) depending on its multiplicity. The rational function \( \varphi_{2n}(z) \in L_{2n} \) has a simple pole at each of the points \( \alpha_1, \cdots, \alpha_n, 1/\beta_1, \cdots, 1/\beta_n \), and \( \alpha_j \) and \( \beta_j \) are as defined in (2.1). A similar interpretation for \( L_{2n+1} \) follows.

The regularity conditions in the present case can be obtained as follows. The expansion in terms of the basis elements gives

\[
\varphi_{2n}(z) = A_0 + \frac{A_1 z}{z - \alpha_1} + \frac{A_2 z^2}{(z - \alpha_1)(1 - z\beta_1)} + \cdots + \frac{A_{2n} z^{2n}}{\prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n}(1 - z\beta_i)},
\]

so that \( r_{2n}(z) = A_0 \prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n}(1 - z\beta_i) + \cdots + A_{2n} \). Then \( A_{2n} \neq 0 \) if

\[
r_{2n}(\alpha_0) \neq 0 \quad \text{and} \quad r_{2n}(1/\beta_n) \neq 0.
\]

Similarly, for \( \varphi_{2n+1}(z) \), we obtain

\[
r_{2n+1}(\alpha_{n+1}) \neq 0 \quad \text{and} \quad r_{2n+1}(1/\beta_n) \neq 0.
\]

The regularity conditions \( (2.4) \) and \( (2.5) \) are required to guarantee that \( \varphi_{2n}(z) \in L_{2n} \setminus L_{2n-1} \) and \( \varphi_{2n+1}(z) \in L_{2n+1} \setminus L_{2n} \) respectively. Using the definition \( (2.2) \) of the inner product \( \langle \cdot, \cdot \rangle \), the following result is immediate and will be used in deriving the recurrence relations for the orthogonal rational functions \( \varphi_j(z) \).
Lemma 2.1. Let $\gamma_n \in \mathbb{C} \setminus \{0\}$, $n = 1, 2, \ldots$. The following equality

$$\left\langle \frac{1 - z\gamma_n}{z - \gamma_n - 1}, f, g \right\rangle = \left\langle f, \frac{z - \gamma_n}{1 - z\gamma_n - 1} g \right\rangle; \quad \left\langle \frac{z - \gamma_n + 1}{1 - z\gamma_n}, f, g \right\rangle = \left\langle f, \frac{1 - z\gamma_n + 1}{z - \gamma_n} g \right\rangle.$$

holds for the rational functions $f := f(z)$ and $g := g(z)$ in $\mathcal{L}$.

In addition to the conditions (2.4) and (2.5), we also assume $r_{2n}(\beta_{n-1}) \neq 0$, $r_{2n}(1/\alpha_n) \neq 0$, $r_{2n+1}(\beta_n) \neq 0$, $r_{2n+1}(1/\alpha_n) \neq 0$. Here, and in what follows, we consider the sequences $\{\alpha_j\}$ and $\{\beta_j\}$ as defined in (2.1), unless specified otherwise.

Theorem 2.1. The orthonormal rational functions $\{\vec{\phi}(\lambda)\}_{n=0}^\infty$, with $\vec{\phi}_0(\lambda) := 0$ and $\vec{\phi}_0(\lambda) := 1$ satisfy the recurrence relations,

$$\varphi_{2n+1}(z) = \left[ \frac{e_{2n+1}}{z - \alpha_{n+1}} + \frac{d_{2n+1}(z - \beta_n)}{z - \alpha_{n+1}} \right] \varphi_{2n}(z) + c_{2n+1} \frac{1 - z\alpha_n}{z - \alpha_{n+1}} \varphi_{2n-1}(z), \quad (2.6a)$$

$$\varphi_{2n+2}(z) = \left[ \frac{e_{2n+2}}{1 - z\beta_{n+1}} + \frac{d_{2n+2}(1 - z\alpha_{n+1})}{1 - z\beta_{n+1}} \right] \varphi_{2n+1}(z) + c_{2n+2} \frac{z - \beta_n}{1 - z\beta_{n+1}} \varphi_{2n}(z), \quad (2.6b)$$

for $n \geq 0$, where $\beta_0 := 0$, the constants $e_j, d_j \in \mathbb{C}$ and $c_j \in \mathbb{C} \setminus \{0\}$, $j \geq 0$.

Proof. Consider the function

$$\mathcal{W}_{2n}(z) = \frac{1 - z\beta_n}{z - \beta_{n-1}} \varphi_{2n}(z) - \frac{a_{2n}}{z - \beta_{n-1}} \varphi_{2n-1}(z), \quad n \geq 1.$$

We first find the appropriate choice of $a_{2n}$ for which $\mathcal{W}_{2n}(z) \in \mathcal{L}_{2n-1} \setminus \mathcal{L}_{2n-2}$. Using the rational forms (2.3) of $\varphi_{2n}(z)$ and $\varphi_{2n-1}(z)$, we have

$$a_{2n} = \frac{r_{2n}(\beta_{n-1})}{r_{2n-1}(\beta_{n-1})} \neq 0 \implies \mathcal{W}_{2n}(z) \in \mathcal{L}_{2n-1} \setminus \mathcal{L}_{2n-2}.$$

Hence, we can write

$$\mathcal{W}_{2n}(z) = b_{2n}\varphi_{2n-1}(z) + c_{2n}\varphi_{2n-2}(z) + \sum_{j=0}^{2n-3} a_j^{(2n)} \varphi_j(z),$$

where $a_j^{(2n)} = \langle \mathcal{W}_{2n}(z), \varphi_j(z) \rangle$, $j = 0, 1, \ldots, 2n - 3$. However,

$$\frac{z - \beta_n}{1 - z\beta_{n-1}} \varphi_j \in \mathcal{L}_{2n-2} \quad \text{and} \quad \frac{z}{1 - z\beta_{n-1}} \varphi_j \in \mathcal{L}_{2n-2}, \quad j = 0, 1, \ldots, 2n - 3.$$

Using Lemma 2.1 we conclude $a_j^{(2n)} = 0$ for $j = 0, 1, \ldots, 2n - 3$ and hence

$$\varphi_{2n}(z) = \left[ \frac{a_{2n}}{1 - z\beta_n} + b_{2n} \frac{z - \beta_{n-1}}{1 - z\beta_n} \right] \varphi_{2n-1}(z) + c_{2n} \frac{z - \beta_{n-1}}{1 - z\beta_n} \varphi_{2n-2}(z), \quad n \geq 1.$$

However, we note that both $\{1, z - \beta_{n-1}\}$ and $\{1, 1 - z\alpha_n\}$ form a basis for $\Pi_1$ and hence writing $a_{2n} + b_{2n}(z - \beta_{n-1}) = e_{2n} + d_{2n}(1 - z\alpha_n)$, the recurrence relation (2.6b) follows. To prove $c_{2n} \neq 0$, we multiply both sides of (2.6b) by $\frac{1 - z\beta_n}{\prod_{l=1}^{n} (1 - z\alpha_l)}$, so that the definition of the inner product (2.2) gives

$$c_{2n} \left\langle \varphi_{2n-2}(z), \frac{z^{2n-2}}{\prod_{l=1}^{n}(z - \alpha_l) \prod_{i=1}^{n-2}(1 - z\beta_i)} \right\rangle + e_{2n} \left\langle \varphi_{2n-1}(z), u_{2n-1}(z) \right\rangle = 0,$$

which proves $c_{2n} \neq 0, n \geq 1$. 

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To derive the recurrence relation for \( \varphi_{2n+1}(z) \), consider

\[
W_{2n+1}(z) = \frac{z - \alpha_{n+1}}{1 - z\bar{\alpha}_n}\varphi_{2n+1}(z) - \frac{a_{2n+1}}{1 - z\bar{\alpha}_n}\varphi_{2n}(z), \quad n \geq 0,
\]

for \( a_{2n+1} = r_{2n+1}(1/\alpha_n)/r_{2n}(1/\alpha_n) \neq 0 \). As in the case for \( \varphi_{2n}(z) \), we arrive at

\[
\varphi_{2n+1}(z) = \left[ \frac{a_{2n+1} + b_{2n+1}}{z - \alpha_{n+1}} + d_{2n+1} \frac{1 - z\bar{\alpha}_n}{z - \alpha_{n+1}} \right] \varphi_{2n}(z) + e_{2n+1} \left( \frac{1 - z\bar{\alpha}_n}{z - \alpha_{n+1}} \right) \varphi_{2n-1}(z),
\]

for \( n \geq 0 \), which can also be written as (2.6a) since \( \{1, 1 - z\bar{\alpha}_n\} \) and \( \{1, z - \beta_n\} \) both span the linear space \( \Pi_1 \).

To prove \( e_{2n+1} \neq 0 \), we multiply both sides of the recurrence relation (2.6a) by \( \prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n}(z - \beta_i) \). The inner product (2.7) and Lemma 2.1 gives

\[
e_{2n+1} \left( \varphi_{2n-1}(z), \sum_{i=1}^{n}(z - \alpha_i)\prod_{j=1}^{n}(z - \beta_j) \right) = 0,
\]

from which it follows that \( e_{2n+1} \neq 0, n \geq 1. \)

\[\square\]

2.1. \( \varphi_j(z), \quad j \geq 0, \) as components of an eigenvector. The numerator polynomials of orthogonal rational functions satisfy the recurrence relations of \( R_{II} \) type. Indeed, from (2.6a) and (2.6b), it can be shown that

\[
r_{2n+1}(z) = [e_{2n+1} + d_{2n+1}(z - \beta_n)]r_{2n}(z) + c_{2n+1}(1 - z\bar{\alpha}_n)(1 - z\bar{\beta}_n)r_{2n-1}(z), \quad (2.7a)
\]

\[
r_{2n+2}(z) = [e_{2n+2} + d_{2n+2}(1 - z\bar{\alpha}_n)]r_{2n+1}(z) + c_{2n+2}(z - \alpha_{n+1})(z - \beta_n)r_{2n}(z), \quad (2.7b)
\]

for \( n \geq 0 \), where we define \( r_0(z) := 1 \) and \( \beta_0 := 0 \). We use (2.7a) and (2.7b) to obtain a generalized eigenvalue problem such that the zeros of \( r_j(z), \quad j \geq 1, \) are the eigenvalues (that is, \( r_j(z) \) is the characteristic polynomial) while the corresponding rational functions are the components of the corresponding eigenvector.

Consider two infinite matrices \( \mathcal{H} = (h_{i,k})_{i,k \geq 0}^\infty \) and \( \mathcal{G} = (g_{i,k})_{i,k \geq 0}^\infty \), where

\[
\mathcal{H} = \begin{pmatrix}
  d_1 & g_1 & 0 & 0 & \cdots \\
  h_{1,0} & -d_2\bar{\alpha}_1 & g_2 & 0 & \cdots \\
  0 & h_{2,1} & d_3 & g_3 & \cdots \\
  0 & 0 & h_{3,2} & -d_4\bar{\alpha}_2 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
\mathcal{G} = \begin{pmatrix}
  -e_1 + \beta_0 d_1 & \alpha_1 g_1 & 0 & 0 & \cdots \\
  h_{1,0}\beta_0 & -e_2 - d_2 & \alpha_1 g_2 & 0 & \cdots \\
  0 & h_{2,1}/\bar{\alpha}_1 & -e_3 + \beta_1 d_3 & \alpha_2 g_3 & \cdots \\
  0 & 0 & h_{3,2}\beta_1 & -e_4 - d_4 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

with \( g_{2k+2} = -c_{2k+3}/h_{2k+2,2k+1}, \quad g_{2k+1} = -c_{2k+2}/h_{2k+1,2k}, \quad k \geq 0 \). Here, \( \alpha_j, \beta_j, \ e_j, \ d_j \) and \( c_j \) are the constants appearing in the recurrence relations (2.7a) and (2.7b) while \( \{h_{i,j-1}\}_{i=1}^\infty \) is a sequence of arbitrary non-vanishing complex numbers.
Proposition 2.1. [21] Let \( \mathcal{H}_j \) and \( \mathcal{G}_j \) denote the \( j \)th principal minors of \( \mathcal{H} \) and \( \mathcal{G} \) respectively. Then \((-1)^j r_j(\lambda), j \geq 1, \) is the characteristic polynomial of the generalized eigenvalue problem

\[
\mathcal{G}_j \vec{g}_j = \lambda \mathcal{H}_j \vec{g}_j,
\]

where \( \{r_j\} \) satisfies (2.7a) and (2.7b).

The generalized eigenvalue problem (2.8) has \( j - 1 \) free variables \( h_{i, i-1} \) which shows that the matrix pencil associated with the recurrence relations of \( R_{ij} \) type is not unique. We now assign appropriate values to these free variables to obtain an eigenvector \( \vec{g}_j \).

Theorem 2.2. Let the terms of the sequence \( \{h_{i, i-1}\}_{i=1}^{\infty} \) be assigned the values

\[
h_{2i+1, 2i-1} = -c_{2i+1} \alpha_i, \quad h_{2i-1, 2i-2} = c_{2i}, \quad i \geq 1.
\]

Then, \( \vec{g}_j = (\varphi_0, \varphi_1, \cdots, \varphi_j)^T \) is the eigenvector of the generalized eigenvalue problem (2.8) corresponding to the eigenvalue which is a zero of \( r_j(\lambda) \).

Proof. Upon substitution of the values of \( h_{i, i-1} \), the recurrence relations (2.6a) and (2.6b) can be written as \((-e_1 + d_1 \beta_0) \varphi_0 - \alpha_1 \varphi_1 = z[d_1 \varphi_0 - \varphi_1]\) and

\[
\begin{align*}
-c_{2k+3} \varphi_{2k+1} - (e_{2k+3} - d_{2k+3} \beta_{k+1}) \varphi_{2k+2} - \alpha_{k+2} \varphi_{2k+3} \\
= z[-c_{2k+3} \alpha_{k+1} \varphi_{2k+1} + d_{2k+3} \varphi_{2k+2} - \varphi_{2k+3}],
\end{align*}
\]

\[
\begin{align*}
\beta_k c_{2k+2} \varphi_{2k} - (e_{2k+2} + d_{2k+2}) \varphi_{2k+1} + \varphi_{2k+2} \\
= z[c_{2k+2} \varphi_{2k} - d_{2k+2} \alpha_{k+1} \varphi_{2k+1} + \beta_{k+1} \varphi_{2k+2}],
\end{align*}
\]

for \( k \geq 0 \), which can be rearranged to yield the matrix equations

\[
\begin{align*}
\mathcal{G}_{2n} \vec{e}_{2n} &= z\mathcal{H}_{2n} \vec{e}_{2n} - (z - \beta_n) \varphi_{2n+1} \vec{e}_{2n+1}, \\
\mathcal{G}_{2n+1} \vec{e}_{2n+1} &= z\mathcal{H}_{2n+1} \vec{e}_{2n+1} - (z - \alpha_{n+1}) \varphi_{2n+1} \vec{e}_{2n+1},
\end{align*}
\]

where \( \vec{e}_j \) is the \( j \)th column of the unit matrix. Observing the fact that \((z - \beta_n) \varphi_{2n}\) does not vanish for \( z = \beta_n \), \( \vec{g}_j \) becomes an eigenvector for the generalized eigenvalue problem (2.8) with the zeros of \( r_{2n}(z) \) as eigenvalues. Similarly, \( \vec{g}_{2j+1} \) becomes an eigenvector with the zeros of \( r_{2n+1}(z) \) as eigenvalues and the proof is complete. \( \square \)

Theorems 2.1 and 2.2 serve the first step of our construction. That is, we have obtained a sequence of rational functions that is orthogonal with respect to the linear functional \( \mathcal{L} \). These rational functions are also the components of the eigenvector of a matrix pencil whose characteristic polynomials are the numerator polynomials of such rational functions. In the next section, we will discuss the biorthogonality properties of \( \{\varphi_n(z)\} \).

3. A Biorthogonality Relation for the Rational Functions

In the present section, we use the recurrence relations (2.7a) and (2.7b) obtained in Section 2 to define biorthogonality relations involving the orthogonal rational functions \( \{\varphi_j\} \). To start with, we introduce the rational functions \( O_0(z) = 1 \) and

\[
\begin{align*}
O_{2n+1}(z) &= \frac{r_{2n+1}(z)}{\prod_{j=1}^{n+1}(z - \alpha_j) \prod_{j=1}^{n}(1 - z \alpha_j) \prod_{j=0}^{n}(z - \beta_j) \prod_{j=1}^{n+1}(1 - z \beta_j)}, \\
O_{2n+2}(z) &= \frac{r_{2n+2}(z)}{\prod_{j=1}^{n+2}(z - \alpha_j) \prod_{j=1}^{n+1}(1 - z \alpha_j) \prod_{j=0}^{n}(z - \beta_j) \prod_{j=1}^{n+2}(1 - z \beta_j)}.
\end{align*}
\]

(3.1)
for $n \geq 0$. Here $\{r_j\}$ satisfies (2.7a) and (2.7b) so that the sequence $\{O_j(z)\}$ satisfies

\[(z - \alpha_{n+1})(z - \beta_n)O_{2n+1}(z) = [\epsilon_{2n+1} + d_{2n+1}(z - \beta_n)]O_{2n}(z) + c_{2n+1}O_{2n-1}(z),\]

\[(1 - z\bar{\alpha}_n)(1 - z\bar{\beta}_n)O_{2n}(z) = [\epsilon_{2n} + d_{2n}(1 - z\bar{\alpha}_n)]O_{2n-1}(z) + c_{2n}O_{2n-2}(z),\]

for $n \geq 1$. Then, similar to Theorem 3.5 and its following corollary of Ismail and Masson [11], we have

**Theorem 3.1.** Consider the rational functions given by (5.1). Then there exists a linear functional $\mathcal{M}$ on the span of rational functions $\{zO_j(z)\}$ such that the orthogonality relation

$$\mathcal{M}(z^kO_n(z)) = 0, \quad k = 0, 1, \ldots, n - 1,$$

holds. Further, if $\mathcal{M}(1) = m_0$, $\mathcal{M}(z^nO_n(z)) = m_n$, $n \geq 1$, then

\begin{align*}
\alpha_n\bar{\beta}_nm_{2n} + d_{2n}\alpha_nm_{2n-1} - c_{2n}m_{2n-2} &= 0, \quad n \geq 1, \\
m_{2n+1} - d_{2n+1}m_{2n} - c_{2n+1}m_{2n-1} &= 0, \quad n \geq 1.
\end{align*}

(3.2)

We also need the following relations among the leading coefficients $r_j(z)$, $j \geq 1$. If $r_j = \kappa_jz^j + \text{lower order terms}$, then from (2.7a) and (2.7b),

\begin{align*}
\kappa_2n + d_{2n}\alpha_n\kappa_{2n-1} - c_{2n}\kappa_{2n-2} &= 0, \quad n \geq 1, \\
\kappa_{2n+1} - d_{2n+1}\kappa_{2n} - \alpha_n\bar{\beta}_n\kappa_{2n+1} - \bar{\alpha}_n\kappa_{2n-1} &= 0, \quad n \geq 1.
\end{align*}

(3.3)

It is clear that each of the recurrence relations (3.2) and (3.3) involve two arbitrary initial values. We choose $m_0$ and $m_1$ such that $m_1 \neq d_1m_0$. Since $\kappa_0 = 1$ and $\kappa_1 = d_1$, this implies $\kappa_0m_1 - \kappa_1m_0 \neq 0$.

Consider another sequence of rational functions $\{\tilde{\varphi}_j(z)\}^\infty_{j=0}$ where $\tilde{\varphi}_0(z) := 1$,

\begin{align*}
\tilde{\varphi}_{2n+1}(z) &= \frac{r_{2n+1}(z)}{\prod_{j=1}^n(1 - z\alpha_j)}\prod_{j=0}^n(z - \beta_j), \\
\tilde{\varphi}_{2n+2}(z) &= \frac{r_{2n+2}(z)}{\prod_{j=1}^{n+1}(1 - z\alpha_j)}\prod_{j=0}^n(z - \beta_j),
\end{align*}

(3.4)

for $n \geq 0$. Here $\{r_j(z)\}$ satisfy (2.7a) and (2.7b). Let $\tilde{\mathcal{J}}_m(z) = \chi_m^{-1}\tilde{\varphi}_m(z)$, where $\chi_{2m} = \bar{\alpha}_1(\beta_1)^{-1}\cdots\bar{\alpha}_m(\beta_m)^{-1}$ and $\chi_{2m+1} = \bar{\alpha}_1(\beta_1)^{-1}\cdots\bar{\alpha}_m(\beta_m)^{-1}\bar{\alpha}_{m+1}$. Define

$$\tilde{\psi}_{2j}(z) := \frac{c_{2j+2}(\bar{\beta}_j)^{2}}{\alpha_j+1}\tilde{\mathcal{J}}_{2j-1}(z) - \frac{d_{2j+1}}{\alpha_j+1}\tilde{\mathcal{J}}_{2j}(z) + \tilde{\mathcal{J}}_{2n+1}(z), \quad n \geq 1,$$

$$\tilde{\psi}_{2j+1}(z) := \frac{c_{2j+2}(\bar{\beta}_j)^{2}}{\alpha_j+1}\tilde{\mathcal{J}}_{2j}(z) - \frac{d_{2j+2}(\bar{\beta}_j)^{2}}{\alpha_j+1}\tilde{\mathcal{J}}_{2j+1}(z) + \alpha_j+1\tilde{\mathcal{J}}_{2j+2}(z), \quad n \geq 0,$$

with $\tilde{\psi}_0(z) := 1$. The following theorem gives the biorthogonality relations for $\varphi(z)$ constructed in the previous section.

**Theorem 3.2.** The sequences of rational functions $\{\varphi_j(z)\}$ and $\{\tilde{\psi}_j(z)\}$ satisfy the following biorthogonality relations

\begin{align*}
\mathcal{M}(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) &= \frac{c_{2n+1}(m_1\kappa_0 - m_0\kappa_1)}{\chi_{2n+1}}\delta_{2n,m}, \quad (3.5) \\
\mathcal{M}(\varphi_{2n+1}(z) \cdot \tilde{\psi}_m(z)) &= \frac{c_{2n+2}(m_1\kappa_0 - m_0\kappa_1)}{\chi_{2n+2}}\delta_{2n+1,m}, \quad (3.6)
\end{align*}

where $m_j = \mathcal{M}(z^jO_j(z))$ and $\kappa_j$ is the leading coefficient of $r_j(z)$.  

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Proof. For simplicity, we write \( \varphi_j := \varphi_j(z) \) and similar notations follow for others. We divide the proof into the following cases. First, let \( m < 2n \) and \( m \) has even value, say \( m = 2j \). Then

\[
\Re(\varphi_{2n} \cdot \tilde{\psi}_m) = \frac{c_{2j+1} \beta_{j+1}}{\alpha_{j+1}} \Re(\varphi_{2n} \cdot \tilde{J}_{2j-1}) - \frac{d_{2j+1}}{\alpha_{j+1}} \Re(\varphi_{2n} \cdot \tilde{J}_{2j}) + \Re(\varphi_{2n} \cdot \tilde{J}_{2j+1}).
\]

We evaluate the first term. We have \( \Re(\varphi_{2n} \cdot \tilde{J}_{2j-1}) \)

\[
= \frac{1}{\chi_{2j-1}} \Re \left( \prod_{k=1}^{r_{2n}} (z - \alpha_k) \prod_{k=1}^{r_{2j-1}} (1 - z/\beta_k) \right)
\]

\[
= \frac{1}{\chi_{2j-1}} \Re(\mathcal{O}_{2n} \cdot r_{2j-1}(1 - z\alpha_j) \cdots (1 - z\alpha_n)(z - \beta_j) \cdots (z - \beta_{n-1}))
\]

\[
= \frac{(-\alpha_j) \cdots (-\alpha_n) \kappa_{2j-1}}{\chi_{2j-1}} m_{2n}.
\]

A similar evaluation of the remaining two terms yields

\[
\Re(\varphi_{2n} \cdot \tilde{J}_{2j}) = \frac{(-\bar{\alpha}_{j+1}) \cdots (-\bar{\alpha}_n) \kappa_{2j}}{\chi_{2j}} m_{2n},
\]

\[
\Re(\varphi_{2n} \cdot \tilde{J}_{2j+1}) = \frac{(-\bar{\alpha}_{j+1}) \cdots (-\bar{\alpha}_n) \kappa_{2j+1}}{\chi_{2j+1}} m_{2n}.
\]

Using the relations (3.3), we obtain \( \Re(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) = 0 \) for \( m = 2j < 2n \).

In the second case, let \( m > 2n \) and \( m \) has odd value, say \( m = 2j + 1 \). Then

\[
\Re(\varphi_{2n} \cdot \tilde{\psi}_m) = \frac{c_{2j+2} \beta_{j+1}}{\alpha_{j+1}} \Re(\varphi_{2n} \cdot \tilde{J}_{2j}) - \frac{d_{2j+2} \bar{\alpha}_{j+1} \beta_{j+1}}{\alpha_{j+1}} \Re(\varphi_{2n} \cdot \tilde{J}_{2j+1}) + \bar{\alpha}_{j+1} \Re(\varphi_{2n} \cdot \tilde{J}_{2j+2}),
\]

so that, as in the case of \( \tilde{\psi}_{2j}(z) \), we have

\[
\Re(\varphi_{2n} \cdot \tilde{J}_{2j+2}) = \frac{\kappa_{2n} m_{2j+2}}{\chi_{2j+2}}, \quad \Re(\varphi_{2n}(z) \cdot \tilde{J}_{2j}(z)) = \frac{\kappa_{2n} m_{2j}}{\chi_{2j}},
\]

\[
\Re(\varphi_{2n}(z) \cdot \tilde{J}_{2j+1}(z)) = \frac{\kappa_{2n} m_{2j+1}}{\chi_{2j+1}}.
\]

Hence, using (3.2) we have \( \Re(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) = 0 \) for \( m = 2j + 1 > 2n \).

In the third case, we prove the biorthogonality relations (3.5) and (3.6). For \( m = 2n \), we obtain

\[
\Re(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) = \frac{1}{\chi_{2n+1}} (\kappa_{2n} m_{2n+1} - d_{2n+1} \kappa_{2n} m_{2n} - c_{2n+1} \beta_{n} \tilde{\alpha}_{n} \kappa_{2n-1} m_{2n}).
\]

From (3.2), we find that \( m_{2n+1} \kappa_{2n} - d_{2n+1} \kappa_{2n} m_{2n} = c_{2n+1} m_{2n-1} \kappa_{2n} \), so that

\[
\Re(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) = \frac{c_{2n+1}}{\chi_{2n+1}} (\kappa_{2n} m_{2n-1} - \tilde{\alpha}_{n} \beta_{n} \kappa_{2n-1} m_{2n}).
\]

To simplify the numerator in the right hand side above, we note from (3.2) and (3.3) that the following relations

\[
\kappa_{2n} m_{2n-1} - \tilde{\alpha}_{n} \beta_{n} \kappa_{2n-1} m_{2n} = c_{2n} (m_{2n-1} \kappa_{2n-2} - m_{2n-2} \kappa_{2n-1}),
\]

\[
\kappa_{2n-2} m_{2n-1} - \kappa_{2n-1} m_{2n-2} = c_{2n-1} (m_{2n-3} \kappa_{2n-2} - \tilde{\alpha}_{n-1} \beta_{n-1} m_{2n-2} \kappa_{2n-3}),
\]

(3.7)
hold which further imply that
\[
\kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n \kappa_{2n-1}m_{2n} = c_{2n}c_{2n-1} \cdots c_2(m_1 \kappa_0 - m_0 \kappa_1) \neq 0.
\]
The proof of (3.6) follows the exact techniques and line of argument as in the proof of (3.5). Indeed, proceeding as above we obtain, for \( m = 2n + 1 \),
\[
\mathcal{N}(\varphi_{2n+1}(z) \cdot \bar{\psi}_{2n+1}(z)) = \frac{c_{2n+2}(\kappa_{2n}m_{2n+2} - \kappa_{2n+1}m_{2n})}{\chi_{2n+2}}.
\]
Simplifying the numerator in the right hand side above, we note from (3.7) that
\[
m_{2n+1} \kappa_{2n} - \kappa_{2n+1}m_{2n} = c_{2n+1}c_{2n} \cdots c_2(\kappa_0m_1 - m_0 \kappa_1) \neq 0.
\]
The proof of the biorthogonality relations (3.5) and (3.6) for the remaining cases, that is, \( m > 2n, m = 2j \) and \( m < 2n, m = 2j + 1 \), can be obtained with similar arguments, thus completing the proof.

**Remark 3.1.** The technique of using the leading coefficients \( \kappa_n \) and the normalization constants \( m_n \) to prove biorthogonality, as is evident in the present section, is available in the literature, for example, in Zhedanov [21]. However, the difference between the present work and Zhedanov [21] is our second objective of proving biorthogonality for exactly the same rational functions that were used to arrive at the recurrence relations of \( R_{\Pi} \) type for the numerator polynomials \( r_j(z) \) which is also evident from Remark 4.1.

### 4. Spectral Transformation of Christoffel Type

The Christoffel transformation of well-known orthogonal polynomials is abundant in the literature [21, p. 35], [10, Section 2.7] [20]. In the present section, we find a Christoffel type transformation of the orthogonal rational functions given in (2.3) for the special case \( |\beta| = 1 \) and \( \alpha_j = \alpha \in \mathbb{C} \setminus \{0\}, j \geq 1 \). We begin with the recurrence relations (2.7a) and (2.7b) of \( R_{\Pi} \) type for the numerator polynomials \( \{r_n(z)\}_{n=0}^{\infty} \) which are now written, for \( n \geq 0 \), as
\[
\begin{align*}
   r_{2n+1}(z) &= \rho_{2n}(z - \nu_{2n})r_{2n}(z) - \tau_{2n}(z - 1/\bar{\alpha})(z - \beta_n)r_{2n-1}(z), \quad (4.1a) \\
   r_{2n+2}(z) &= \rho_{2n+1}(z - \nu_{2n+1})r_{2n+1}(z) - \tau_{2n+1}(z - \alpha)(z - \beta_n)r_{2n}(z), \quad (4.1b)
\end{align*}
\]
where the new parameters \( \{\rho_n\} \) and \( \{\nu_n\} \) are given by
\[
\begin{align*}
   \rho_{2n} &= d_{2n-1}, \quad \nu_{2n} = (d_{2n+1}\beta_n - c_{2n+1})/d_{2n+1}, \quad \tau_{2n} = -c_{2n+1}\bar{\alpha}\bar{\beta}_n, \\
   \rho_{2n+1} &= -d_{2n+2}\bar{\alpha}, \quad \nu_{2n+1} = (e_{2n+2} + d_{2n+2})/(d_{2n+2}\bar{\alpha}), \quad \tau_{2n+1} = c_{2n+2}.
\end{align*}
\]
The recurrence relations (4.1b) and (4.1a), written in terms of the rational functions \( \varphi_j(z) \), \( j \geq 0 \) (as defined in (2.3)) yield
\[
\begin{align*}
   (z - \alpha)\varphi_{2n+1}(z) &= u_{2n}(z - \nu_{2n})\varphi_{2n}(z) + \lambda_{2n}(z - 1/\bar{\alpha})\varphi_{2n-1}(z), \\
   (z - \beta_{n+1})\varphi_{2n+2}(z) &= u_{2n+1}(z - \nu_{2n+1})\varphi_{2n+1}(z) + \lambda_{2n+1}(z - \alpha)\varphi_{2n}(z), \quad (4.2)
\end{align*}
\]
Moreover, for \( n \geq 0 \), if we define the shift operators \( \Gamma \) and \( \Lambda \) as
\[
\begin{align*}
   \Gamma r_{2n+1} &:= \beta_{n+1}r_{2n+2} - u_{2n+1}\nu_{2n+1}r_{2n+1} - \lambda_{2n+1}\beta_nr_{2n}, \\
   \Gamma r_{2n} &:= \alpha r_{2n+1} - u_{2n}\nu_{2n}r_{2n} - \lambda_{2n}/\bar{\alpha}r_{2n-1}, \\
   \Lambda r_{2n+1} &:= \varphi_{2n+2} - u_{2n+1}\nu_{2n+1}\varphi_{2n+1} - \lambda_{2n+1}\varphi_{2n}, \quad (4.3) \\
   \Lambda r_{2n} &:= \varphi_{2n+1} - u_{2n}\nu_{2n}\varphi_{2n} - \lambda_{2n}\varphi_{2n-1},
\end{align*}
\]
then (4.2) leads to the generalized eigenvalue problem \( \Gamma \tilde{\varphi} = z\Lambda \tilde{\varphi} \) with the eigenvalue \( z \) and the eigenvector \( \tilde{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \cdots)^T \). Let \( \varphi_{2n+1}(z) \) denote the Christoffel
type transform of $ϕ_{2n+1}(ζ)$, $n ≥ 0$, obtained under the action of the operator $D$, where $Dϕ_j(ζ) = ̂ϕ_j(ζ)$. We note that $ϕ_{2n}(ζ)$ is an arbitrary rational function in the present case. Further, we suppose that $D^0ϕ := ̂ϕ$, where $̂ϕ = (̂ϕ_0, ̂ϕ_1, ̂ϕ_2, · · ·)^T$. The following lemma gives information on the action of the operator $D$ on an arbitrary rational function $ϕ_k := ϕ_k(λ)$ which belongs to the space $L_j$.

Lemma 4.1. Let $Dϕ_j := Ω(ζ)(ϕ_{j+1} + ζϕ_j)$, $ϕ_j ∈ L_j$ for $j ≥ 0$, where $Ω(ζ)$ is a function of $z$ but independent of $k$ and hence, is a constant with respect to $D$. Then

$$ζ_{2j+1} = -\frac{θ_{2j+2}}{θ_{2j+1}} \quad \text{and} \quad ζ_{2j} = -\frac{θ_{2j+1}}{θ_{2j}}, \quad j ≥ 0,$$

where $θ_j$ is any function satisfying the recurrence relations (1.2).

Proof. Define another operator $R$ as

$$RΓ = Γ^oD \quad \text{and} \quad RΛ = Λ^oD.$$  

Then, the effect of $R$ on the generalized eigenvalue problem $Γ ̃ϕ = zΛ ̃ϕ$ yields $Γ^o ̃ϕ = zΛ^o ̃ϕ$ which gives the generalized eigenvalue problem for $̂ϕ$. Further, similar to (1.3), we define the shift operator $Γ^o$ by

$$Γ^oϕ_{2n} := ̂α_nϕ_{2n+1} - ̂α_{2n+1}ϕ_{2n} - ̂β_n - ̂λ_{2n+1}ϕ_{2n-1},$$
$$Γ^oϕ_{2n+1} := ̂α_nϕ_{2n+2} - ̂α_{2n+2}ϕ_{2n+1} - ̂λ_{2n}/̂αϕ_{2n},$$

and the shift operator $Λ^o$ by

$$Λ^oϕ_{2n} := ϕ_{2n+1} - ̂u_{2n+1}ϕ_{2n} - ̂λ_{2n+1}ϕ_{2n-1},$$
$$Λ^oϕ_{2n+1} := ϕ_{2n+2} - ̂u_{2n+2}ϕ_{2n+1} - ̂λ_{2n}ϕ_{2n},$$

respectively. We proceed to find the parameters used in (4.5) and (4.6) in terms of the parameters used in the recurrence relations (1.1a) and (1.1b). For this, we use the operator relations defined in (1.3) for $ϕ_{2n}$ and $ϕ_{2n+1}$. Similar to $D$, let the operator $R$ be defined as

$$Rϕ_k := Ω(ζ)(ϕ_{k+1} + ζϕ_k), \quad ϕ_k ∈ L_j, \quad j ≥ 0,$$

where $Ω(ζ)$ is a constant with respect to $R$. Then, we have

$$̂α_n = α_n, \quad ̂π_{2n} = π_{2n+1} + α(ζ_{2n+1} - ζ_{2n+1}), \quad ζ_{2n+1} ̂λ_{2n} = η_{2n+1}λ_{2n},$$
$$̂β_n = β_{n+1}, \quad ̂α ̂α u_{2n} ̂ν_{2n}ζ_{2n} + ̂αλ_{2n} = ̂α ̂α u_{2n} ̂ν_{2n+1}η_{2n+1} + ̂αλ_{2n+1},$$
$$̂π_{2n+1} = π_{2n} + π_{2n+1}ζ_{2n} - β_{n+1}η_{2n}, \quad ζ_{2n} ̂λ_{2n+1} = η_{2n}λ_{2n+1},$$
$$u_{2n} = u_{2n+1} - ζ_{2n+1}ζ_{2n+1}, \quad ζ_{2n+1}u_{2n+1} = u_{2n} - η_{2n}ζ_{2n},$$

where $π_j = u_jν_j$, $π_j = u_jν_j$, and we define $̂λ_{-1} := 0$.

This implies that the operators $Γ^o$ and $Λ^o$ defined in terms of the parameters $̂λ_n$ etc. in (4.5) and (4.6) are well-defined. Now, using (1.4), we note $̂ϕ_{2n+1}$ is an eigenvector with respect to the operators $Γ$ and $Λ$ if, and only if, $̂ϕ_{2n+1}$ is an eigenvector with respect to the operators $Γ^o$ and $Λ^o$. Let $θ_j$ be an eigenvector of the generalized eigenvalue problem $Γθ_j = zΛθ_j$, with the eigenvalue $z$, which is equivalent to $θ_j$ being a solution of the recurrence relation (1.2) with $z$ replaced by $z$. Then, we have ($Γ^o - zΛ^o)θ_{2n+1} = 0$ which gives $ζ_{2n+1} = -θ_{2n+2}/θ_{2n+1}$, $n ≥ 0$. A similar argument for $θ_{2n}$ gives $Dθ_{2n} = 0$, which implies $ζ_{2n} = -θ_{2n+1}/θ_{2n}$, thus completing the proof. □
The expressions for \( \eta_j \) are obtained from the operator relations \( \Lambda^o \mathcal{D} \mathcal{Y}_k = \hat{z} \mathcal{R} \Lambda \mathcal{Y}_k \) for \( \mathcal{Y}_k = \theta_{2n} \) and \( \theta_{2n+1} \) as

\[
\begin{align*}
\eta_{2n} &= -\frac{\theta_{2n+1} - u_{2n} \theta_{2n} - \lambda_{2n} \theta_{2n-1}}{\theta_{2n} - u_{2n-1} \theta_{2n-1} - \lambda_{2n-1} \theta_{2n-2}} \\
\eta_{2n+1} &= -\frac{\theta_{2n+2} - u_{2n+1} \theta_{2n+1} - \lambda_{2n+1} \theta_{2n}}{\theta_{2n+1} - u_{2n} \theta_{2n} - \lambda_{2n} \theta_{2n-1}}.
\end{align*}
\]  

(4.9)

In particular, from (1.18) the following relations

\[
\hat{u}_0 \nu_0 \zeta_0 + \frac{\lambda_0}{\alpha} = u_0 \nu_0 \eta_1 + \frac{\lambda_1}{\alpha} \quad \text{and} \quad \hat{u}_0 = u_1 + \zeta_1 - \eta_1.
\]

(4.10)

hold for \( n = 0 \). We use the relations (4.10) to find the (constant) \( \Omega(z) \) occurring in the definitions of both the operators \( \mathcal{D} \) and \( \mathcal{R} \) leading to the Christoffel type transform of \( \varphi_{2n+1}(z) \). We also remark here that though \( \beta_0 = 0 \), we continue using \( \beta_0 \) in the expressions that follow. The reason is to show explicitly, the role played by \( \beta_0 \) in the calculations.

**Theorem 4.1.** The Christoffel type transform of \( \varphi_{2n+1}(z) \) is given by

\[
\check{\varphi}_{2n+1}(z) = \sigma \frac{z - \alpha_1}{z - \beta_1} \left[ \varphi_{2n+2}(z) - \frac{\varphi_{2n+2}(\hat{z})}{\varphi_{2n+1}(\hat{z})} \varphi_{2n+1}(z) \right]
\]

for a constant \( \sigma \). Further if \( \vec{\varrho} = (\varphi_0 \quad \varphi_1 \quad \cdots)^T \) is the eigenvector for the generalized eigenvalue problem \( \Gamma \vec{\varrho} = z \Lambda \vec{\varrho} \), there exists another generalized eigenvalue problem \( \Gamma^o \vec{\varrho} = z \Lambda^o \vec{\varrho} \), with the same eigenvalue \( z \) for which \( \vec{\varrho} = (\check{\varphi}_0 \quad \check{\varphi}_1 \quad \cdots)^T \) is the eigenvector.

**Proof.** The last part of the theorem is about the existence of generalized eigenvalue problems for the column vectors \( \vec{\varrho} \) and \( \vec{\varrho} \) which follows from the proof of Lemma 4.1. It is also clear that the Christoffel type transform is given by the shift operator \( \mathcal{D} \) and hence we need to find \( \Omega(z) \) which is independent of \( n \). Further, we obtained the functions \( \theta_j, j \geq 0, \) with \( \theta_{-1} = 0, \) that satisfy the recurrence relations (4.2) with \( z \) replaced by \( \hat{z} \). These equations written explicitly are

\[
\begin{align*}
\alpha \theta_{2n+1} - u_{2n} \nu_{2n} \theta_{2n} - (\lambda_{2n}/\alpha) \theta_{2n-1} &= \hat{z}[\theta_{2n+1} - u_{2n} \theta_{2n} - \lambda_{2n} \theta_{2n-1}], \\
\beta_{n+1} \theta_{2n+2} - u_{2n+1} \nu_{2n+1} \theta_{2n+1} - \beta_n \lambda_{2n+1} \theta_{2n} &= \hat{z}[\theta_{2n+2} - u_{2n+1} \theta_{2n+1} - \lambda_{2n+1} \theta_{2n}].
\end{align*}
\]

(4.11a, 4.11b)

Let the Christoffel type transform of \( \varphi_{2n+1}(z) \) be the rational function

\[
\check{\varphi}_{2n+1}(z) = \frac{\hat{r}_{2n+1}(z)}{(z - \alpha)^{n+1} \prod_{j=1}^{n+1} (1 - z \beta_j)} = \frac{\hat{r}_{2n+1}(z)}{(z - \alpha)^{n+1} \prod_{j=1}^{n+1} (1 - z \beta_j)},
\]

where \( \{r_j(\lambda)\} \) satisfies (4.11a) and (4.11b), but with the coefficients \( u \) replaced by \( \hat{u} \) etc. To determine the constant \( \Omega(z) \), we note that the implication

\[
\check{\varphi}_{2n+1} = \Omega(z)(\check{\varphi}_{2n+2} + \zeta_{2n+1} \check{\varphi}_{2n+1}) \implies \Omega(z) = \frac{(z - \beta_1) r_1(z)}{r_2(z) + \zeta_1 (z - \beta_1) r_1(z)}
\]

follows from the values for \( n = 0 \). Further, we obtain \( \frac{\theta_0}{\nu_0} = \frac{u_0(z - \nu_0)}{z - \alpha} \) and \( (z - \beta_1) \frac{\theta_1}{\nu_1} = u_1(\hat{z} - \nu_1) + \lambda_1(\hat{z} - \beta_0) \frac{\theta_0}{\nu_0} \) from (4.11a) and (4.11b) for \( n = 0 \) respectively.

Then, Lemma 4.1 yields

\[
-\zeta_1 = \frac{\theta_0}{\nu_0} = \frac{u_1(\hat{z} - \nu_1)}{\hat{z} - \beta_1} + \frac{\lambda_1(\hat{z} - \beta_0)(\hat{z} - \alpha)}{u_0(\hat{z} - \beta_1)(\hat{z} - \nu_0)},
\]
so that the denominator of \( \Omega(z) \) has the expression
\[
 r_2(\lambda) + \zeta_1(z - \beta_1)r_1(z) = u_0u_1(z - \nu_0)(z - \nu_1) + \lambda_1(z - \beta_0)(z - \alpha) \\
- \frac{z - \beta_1}{\hat{z} - \beta_1} u_0(z - \nu_0) \left[ u_1(\hat{z} - \nu_1) + \frac{\lambda_1(\hat{z} - \beta_0)(\hat{z} - \alpha)}{u_0(\hat{z} - \nu_0)} \right].
\]
Further simplification yields
\[
 \Omega(z) = \frac{z - \beta_1 (\hat{z} - \nu_0)(\hat{z} - \beta_1)\hat{r}_1(z)}{\Upsilon(z)},
\]
where \( \Upsilon(z) = \Upsilon_1 z + \Upsilon_0 \), with
\[
\Upsilon_1 = u_0u_1(\hat{z} - \nu_0)(\nu_1 - \beta_1) + \lambda_1(\beta_1\nu_0 + \beta_0\hat{z} + \alpha\hat{z} - \beta_1\hat{z} - \nu_0\hat{z} - \alpha\beta_0),
\]
\[
\Upsilon_0 = -u_0u_1(\hat{z} - \nu_0)(\nu_1 - \beta_1)\nu_0 + \lambda_1[\nu_0(\beta_1\hat{z} - \alpha\beta_1 - \beta_0\beta_1 + \alpha\beta_0) - \alpha\beta_0(\hat{z} - \beta_1)].
\]
Next, using the relations (4.9) and (4.10), we have \( \hat{r}_1(z) = \hat{u}_0(z - \hat{\nu}_0) \), where \( \hat{u}_0 = u_1 + \zeta_1 - \eta_1 \). Further, \( \hat{u}_0\hat{\nu}_0 = u_1 + \alpha(\zeta_1 - \eta_1) \), which implies \( \hat{u}_0(\alpha - \nu_0) = u_0(\nu_1 - \beta_1) \)
\[
= \frac{u_0(\nu_1 - \beta_1)(\alpha - \hat{z})}{(\hat{z} - \beta_1)} + \frac{\lambda_1(\hat{z} - \alpha)}{u_0(\hat{z} - \nu_0)(\hat{z} - \beta_1)} [\beta_1\hat{z} - \beta_1\nu_0 - \beta_0\hat{z} - \alpha\hat{z} + \nu_0\hat{z} + \alpha\beta_0],
\]
which on further simplification yields
\[
\zeta_0\hat{u}_0(\alpha - \nu_0)(\hat{z} - \beta_1) = u_0u_1(\nu_1 - \beta_1)(\hat{z} - \nu_0) + \lambda_1(\beta_1\nu_0 + \beta_0\hat{z} + \alpha\hat{z} - \beta_1\hat{z} - \nu_0\hat{z} - \alpha\beta_0).
\]
Using the fact that \( -\zeta_0 = u_0(\hat{z} - \nu_0)/(\hat{z} - \alpha) \), we have \( \zeta_0\hat{u}_0(\alpha - \nu_0)(\hat{z} - \beta_1) = \Upsilon_1 \). Further, substituting the value of \( \eta_1 \), we have from the first relation in (4.10)
\[
u_0u_1(\beta_1 - \nu_1)(\hat{z} - \nu_0) + \lambda_1 \left[ \nu_0(\hat{z} - \alpha)(\beta_1 - \beta_0) - \frac{1}{\alpha}(\alpha - \nu_0)(\hat{z} - \beta_1) \right] + \frac{\lambda_0}{\alpha}(\alpha - \nu_0)(\hat{z} - \beta_1) = -\zeta_0(\alpha - \nu_0)(\hat{z} - \beta_1)\hat{u}_0\hat{\nu}_0.
\]
Then, defining \( \hat{\lambda}_0 := \lambda_0 - \beta_0\alpha \) (since \( \beta_0 = 0 \), \( \hat{\lambda}_0 := \lambda_0 \)) yields \( -\zeta_0(\alpha - \nu_0)(\hat{z} - \beta_1)\hat{u}_0\hat{\nu}_0 = \Upsilon_0 \). Hence, we have \( \zeta_0(\alpha - \nu_0)(\hat{z} - \beta_1)\hat{r}_1(z) = \Upsilon(z) \), which means
\[
\Omega(z) = \frac{\hat{z} - \nu_0}{\zeta_0(\alpha - \nu_0)} \frac{z - \beta_1}{z - \hat{z}} = \frac{z - \beta_1}{\zeta_0(\alpha - \nu_0)} \frac{z - \beta_1}{z - \hat{z}},
\]
where \( \sigma = (\hat{z} - \alpha)/(\nu_0(\alpha - \nu_0)) \). Finally, we note that since \( \theta_j \) satisfies (4.11a) and (4.11b), \( \theta_j \) must necessarily be equal to \( \varphi_j(z) \).}

**Remark 4.1.** We would like to emphasize here the use of the relations (4.10) and the second degree polynomial \( r_2(z) \) in deriving the above expressions. This is different from the one given in Zhedanov [21], where the linear polynomial \( r_1(z) \) is used.

We now consider the case \( \varphi_{2n}(z) \). Let \( \hat{\varphi}_{2n}(z) \) denote the Christoffel type transform of \( \varphi_{2n}(z) \), \( n \geq 0 \). In the present case, we use the shift operators \( \Gamma^c \) and \( \Lambda^c \) where, for \( n \geq 0 \), \( \Gamma^c \) is given by
\[
\Gamma^c \varphi_{2n+1} := \hat{\beta}_{n+1} \varphi_{2n} + \hat{u}_{2n+1} \nu_{2n+1} \varphi_{2n} - \hat{\beta}_{n+1} \hat{\lambda}_{2n+1} \varphi_{2n},
\]
\[
\Gamma^c \varphi_{2n+1} := \hat{\alpha} \varphi_{2n+2} - \hat{u}_{2n+2} \nu_{2n+1} \varphi_{2n+1} - \hat{\lambda}_{2n+2} / \hat{\alpha} \varphi_{2n+1},
\]
and \( \Lambda^c \) is same as \( \Lambda^c \), which was defined in the case of \( \varphi_{2n+1}(z) \). The derivation of the expression for \( \hat{\varphi}_{2n}(z) \) follows the same technique as in the case of \( \hat{\varphi}_{2n+1}(z) \). In fact, this technique is used to find the Christoffel type transforms of orthogonal rational functions
with arbitrary poles. However, as remarked earlier, only the polynomial \( r_1(z) \) is used which makes the calculations easier. We state only the result for this case.

**Theorem 4.2.** The Christoffel type transform of \( \varphi_{2n}(z) \) is given by

\[
\hat{\varphi}_{2n}(z) = \sigma \frac{z - \alpha}{z - \bar{z}} \left[ \varphi_{2n+1}(z) - \frac{\varphi_{2n+1}(\bar{z})}{\varphi_{2n}(\bar{z})} \varphi_{2n}(z) \right],
\]

for some constant \( \sigma = (\bar{z} - \alpha)/(\alpha_0(\nu_0 - \alpha)) \). Moreover, if \( \bar{\sigma} = (\varphi_0 \varphi_1 \cdots)^T \) is the eigenvector for the generalized eigenvalue problem \( \Gamma \bar{\sigma} = z \Lambda \bar{\sigma} \), there exists another generalized eigenvalue problem \( \Gamma^e \bar{\sigma} = z \Lambda^e \bar{\sigma}, \) with the same eigenvalue \( z \) for which \( \bar{\sigma} = (\hat{\varphi}_0 \hat{\varphi}_1 \cdots)^T \) is the eigenvector.

**Note 4.1.** The constant \( \sigma \) is same in both the cases of Christoffel type transforms of \( \varphi_{2n}(z) \) and \( \varphi_{2n+1}(z) \).

We conclude this section with information on the moment functionals associated with the Christoffel type transforms. Define the following two linear functionals as

\[
\mathcal{N}_0 := \frac{z - \bar{z}}{z - \beta_1} \mathfrak{M} \quad \text{and} \quad \mathcal{N}_c := \frac{z - \bar{z}}{z - \alpha} \mathfrak{M},
\]

where \( \mathfrak{M} \) is as defined in Theorem 3.1. Further, by multiplication of a functional by a function \( f(z) \mathfrak{M} \) it is understood that \( \mathfrak{M} \) acts on the space of the space of functions \( g(z) \) as \( \mathfrak{M}(f(z)g(z)) \). Then we have

**Theorem 4.3.** The following orthogonality relations hold

\[
\mathcal{N}_0 \left( \frac{z^j}{(1 - z \bar{\alpha})^n \prod_{k=0}^n(z - \beta_k)} \hat{\varphi}_{2n+1}(z) \right) = 0, \quad j = 0, 1, \cdots, 2n,
\]

\[
\mathcal{N}_c \left( \frac{z^j}{(1 - z \bar{\alpha})^n \prod_{k=0}^{n-1}(z - \beta_k)} \hat{\varphi}_{2n}(z) \right) = 0, \quad j = 0, 1, \cdots, 2n - 1,
\]

where \( \mathcal{N}_0 \) and \( \mathcal{N}_c \) are defined in (4.13).

**Proof.** Using Theorem 3.1 it is easy to see that

\[
\mathcal{N}_0 \left( \frac{z^j \hat{\varphi}_{2n+1}(z)}{(1 - z \bar{\alpha})^n \prod_{k=0}^n(z - \beta_k)} \right) = \sigma \mathfrak{M} \left( \frac{z^j (\varphi_{2n+2}(z) + \zeta_{2n+1} \varphi_{2n+1}(z))}{(1 - z \bar{\alpha})^n \prod_{k=0}^n(z - \beta_k)} \right)
\]

\[
= \sigma \mathfrak{M} \left( z^j \left\{ (1 - z \bar{\alpha})\Omega_{2n+2}(z) + \zeta_{2n+1} \Omega_{2n+1}(z) \right\} \right)
\]

\[
= 0, \quad j = 0, 1, 2, \cdots, 2n,
\]

where \( \Omega_j(z) \) are the rational functions defined in (5.1). The proof for the case of \( \hat{\varphi}_{2n}(z) \) is similar and hence omitted.

**References**

1. R. Askey, Discussion of Szegö’s paper “Beiträge zur Theorie der Toeplitzschen Formen”. In: R. Askey, editor. Gabor Szegö. Collected works. Vol. I. Boston, MA: Birkhäuser; 1982: p. 303–305.
2. B. Beckermann, M. Derevyagin and A. Zhedanov, The linear pencil approach to rational interpolation, J. Approx. Theory 162 (2010), no. 6, 1322–1346.
3. A. Bultheel, R. Cruz-Barroso and A. Lasarow, Orthogonal rational functions on the unit circle with prescribed poles not on the unit circle, SIGMA Symmetry Integrability Geom. Methods Appl. 13 (2017), Paper No. 090, 49 pp.
4. A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, Orthogonal rational functions with poles on the unit circle, J. Math. Anal. Appl. 182 (1994), no. 1, 221–243.
5. A. Bultheel, P. González-Vera, E. Hendriksen and O. Njastad, *Orthogonal rational functions*, Cambridge Monographs on Applied and Computational Mathematics, 5, Cambridge University Press, Cambridge, 1999.
6. T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.
7. K. Deckers, M. J. Cantero, L. Moral and L. Velázquez, An extension of the associated rational functions on the unit circle, J. Approx. Theory **163** (2011), no. 4, 524–546.
8. K. Deckers and D. S. Lubinsky, How poles of orthogonal rational functions affect their Christoffel functions, J. Approx. Theory **164** (2012), no. 9, 1184–1199.
9. M. S. Derevyagin and A. S. Zhedanov, An operator approach to multipoint Padé approximations, J. Approx. Theory **157** (2009), no. 1, 70–88.
10. M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, reprint of the 2005 original, Encyclopedia of Mathematics and its Applications, 98, Cambridge Univ. Press, Cambridge, 2009.
11. M. E. H. Ismail and D. R. Masson, Generalized orthogonality and continued fractions, J. Approx. Theory **83** (1995), no. 1, 1–40.
12. M. E. H. Ismail and A. Sri Ranga, $R_{11}$ type recurrences, generalized eigenvalue problem and orthogonal polynomials on the unit circle, [arXiv:1606.08055 [math.CA]]
13. J. D. E. Konhauser, Some properties of biothogonal polynomials, J. Math. Anal. Appl. **11** (1965), 242–260.
14. X. Li, Regularity of orthogonal rational functions with poles on the unit circle, J. Comput. Appl. Math. **105** (1999), no. 1-2, 371–383.
15. H. Rosengren, Rahman's biothogonal rational functions and superconformal indices, Constr Approx (2017), https://doi.org/10.1007/s00365-017-9393-3.
16. V. Spiridonov and A. Zhedanov, Spectral transformation chains and some new biothogonal rational functions, Comm. Math. Phys. **210** (2000), no. 1, 49–83.
17. V. P. Spiridonov and A. S. Zhedanov, Generalized eigenvalue problem and a new family of rational functions bioorthogonal on elliptic grids, in *Special functions 2000: current perspective and future directions (Tempe, AZ)*, 365–388, NATO Sci. Ser. II Math. Phys. Chem., 30, Kluwer Acad. Publ., Dordrecht.
18. A. Sri Ranga, Szegő polynomials from hypergeometric functions, Proc. Amer. Math. Soc. **138** (2010), no. 12, 4259–4270.
19. L. Velázquez, Spectral methods for orthogonal rational functions, J. Funct. Anal. **254** (2008), no. 4, 954–986.
20. A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. **85** (1997), no. 1, 67–86.
21. A. Zhedanov, Bioorthogonal rational functions and the generalized eigenvalue problem, J. Approx. Theory **101** (1999), no. 2, 303–329.

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