Radu groups acting on trees are CCR

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Abstract

We classify the irreducible unitary representations of closed simple groups of automorphisms of trees acting 2-transitively on the boundary and whose local action at every vertex contains the alternating group. As an application, we confirm Claudio Nebbia’s CCR conjecture on trees for \((d_0, d_1)\)-semi-regular trees such that \(d_0, d_1 \in \Theta\), where \(\Theta\) is an asymptotically dense set of positive integers.

1 Introduction

In this document topological groups are second-countable, locally compact groups are Hausdorff and the word “representation” stands for strongly continuous unitary representation on a separable complex Hilbert space. A locally compact group \(G\) is called \textbf{CCR} if the operator \(\pi(f)\) is compact for all irreducible representation \(\pi\) of \(G\) and all \(f \in L^1(G)\). For totally disconnected locally compact groups this property is equivalent to ask that every irreducible representation of \(G\) is admissible see [Neb99]. We recall that an irreducible representation \(\pi\) of totally disconnected locally compact group \(G\) is \textbf{admissible} if for every compact open subgroup \(K \leq G\) the space \(H^K_{\pi}\) of \(K\)-invariant vectors is finite dimensional. A very important property of CCR groups is that they are \textbf{type I} groups [BdlH20, Definition 6.E.7. and Proposition 6.E.11]. Loosely speaking, type I groups are the locally compact groups all of whose representations can be written as a unique direct integral of irreducible representations, thus reducing the study of arbitrary representations to considerations on irreducible representations. Concerning groups

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of automorphisms of trees, Nebbia’s work highlighted surprising relations between the action on the boundary and the regularity of representation theory. To be more precise, he showed in [Neb99] that any closed unimodular CCR subgroup \( G \leq \text{Aut}(T) \) of the group of automorphisms of a regular tree \( T \) necessarily acts transitively on the boundary \( \partial T \). Further progress going in that direction were recently achieved by Houdayer and Raum [HR19] and with higher level of generality by Caprace, Kalantar and Monod [CKM22]. Among other things, they showed that a closed non-amenable type I subgroup acting minimally on a locally finite tree \( T \) acts 2-transitively on the boundary \( \partial T \) [CKM22, Corollary D]. Going in the other direction, Nebbia conjectured in [Neb99] that any closed subgroup of automorphisms of a regular tree acting transitively on the boundary is CCR. His conjecture naturally extends to the case of semi-regular trees.

**Conjecture** (CCR conjecture on trees [Neb99]). Let \( T \) be a \((d_0, d_1)\)-semi-regular tree with \( d_0, d_1 \geq 3 \) and let \( G \leq \text{Aut}(T) \) be a closed subgroup acting transitively the boundary of \( T \). Then \( G \) is CCR.

We recall from [BM00, Lemma 3.1.1] that, for a locally finite tree \( T \), closed subgroups \( G \leq \text{Aut}(T) \) are non-compact and act transitively on the boundary \( \partial T \) if and only if they act 2-transitively on \( \partial T \). Furthermore, the existence of such a group implies that the tree is semi-regular. In particular, since compact groups are automatically CCR we deduce that the hypothesis of semi-regularity is non-restrictive in the conjecture.

One of the first evidence supporting the conjecture was provided by Bernstein and Harish-Chandra’s works. Among other things, they proved that rank one semi-simple algebraic groups over local-fields are uniformly admissible [Ber74], [HC70]. We recall that a totally disconnected locally compact group \( G \) is **uniformly admissible** if for every compact open subgroup \( K \), there exists a positive integer \( k_K \) such that \( \dim(H^K_\pi) < k_K \) for all irreducible representation \( \pi \) of \( G \). In particular, uniformly admissible groups are CCR. Concerning non-linear groups, the conjecture was supported by the complete classification of the irreducible representations of the full group of automorphisms of a semi-regular tree and more generally of closed subgroups acting transitively on the boundary and satisfying the Tits independence property [Ol’77], [Ol’80], [Cho94], [FTN91], [Ama03] (those classifications lead to the conclusion that they are uniformly admissible [Cio15]).

Our paper concerns closed subgroups acting 2-transitively on the boundary \( \partial T \) and whose local action at every vertex \( v \) contains the alternating group of corresponding degree. We recall that for each vertex \( v \in V(T) \), the stabilizer \( \text{Fix}_G(v) \) of \( v \) acts on the set \( E(v) \) of edges containing \( v \). The image of \( \text{Fix}_G(v) \) in \( \text{Sym}(E(v)) \) for this natural projection map is called the **local**
action of $G$ at $v$ and we denote this group by $G(v)$. When the degree of each vertex is bigger than 6, those groups of automorphisms of trees have been extensively studied and classified by Radu in [Rad17]. For that reason we call them Radu groups. It is not hard to realize that those groups are type I. Indeed, each Radu group $G$ contains a cocompact subgroup $H$ that is conjugate in $\text{Aut}(T)$ to the semi-regular version of the universal group of Burger-Mozes $\text{Alt}_{(i)}(T)^+$ see [Rad17, page 4]. Since $H$ is both open and cocompact in $G$, [Kal73, Theorem 1] ensures that $G$ is type I if and only if $H$ is type I. On the other hand, when the degree of each vertex is bigger than 4, $\text{Alt}_{(i)}(T)^+$ acts transitively on the boundary and satisfies the Tits independence property. It follows from [Ama03] and [Cio15] that $H$ is a type I group which proves that every Radu group is Type I. The purpose of these notes is to go further. Inspired by Ol’shanskii’s work and the recent progress achieved in the abstraction of his framework [Sem21], we give a classification of the irreducible representations of simple Radu groups and deduce a description of the irreducible representations of any Radu groups. Among other things, this provides the following contribution to Nebbia’s CCR conjecture on trees.

**Theorem A.** Let $T$ be a $(d_0,d_1)$-semi-regular tree with $d_0, d_1 \geq 6$. Then, Radu groups are uniformly admissible and hence CCR.

To put this result into the perspective of Radu’s paper, we recall that the local action $G(v) \leq \text{Sym}(E(v))$ at every vertex $v \in T$ of a closed subgroup $G \leq \text{Aut}(T)$ that is 2-transitive on the boundary is a 2-transitive subgroup of $\text{Sym}(E(v))$ [BM00, Lemma 3.1.1]. On the other hand, [Rad17, Proposition B.1 and Corollary B.2] ensure that

$$\Theta = \{d \geq 6 | \text{each finite 2-transitive subgroup of } \text{Sym}(d) \text{ contains } \text{Alt}(d)\}$$

is asymptotically dense in $\mathbb{N}$ and its ten smallest elements are 34, 35, 39, 45, 46, 51, 52, 55, 56 and 58. All together, this implies the following.

**Theorem B.** Nebbia’s CCR conjecture on trees is confirmed for any $(d_0,d_1)$-semi-regular tree with $d_0, d_1 \in \Theta$ where $\Theta$ is the asymptotically dense subset of $\mathbb{N}$ defined above.

We now explain how we obtained a classification of the irreducible representations of simple Radu groups. We first recall that the irreducible representations of a closed automorphism group $G \leq \text{Aut}(T)$ of a locally finite tree $T$ splits in three categories. An irreducible representation $\pi$ of $G$ is called:

- **spherical** if there exists a vertex $v \in V(T)$ such that $\pi$ admits a non-zero $\text{Fix}_G(v)$-invariant vector where $\text{Fix}_G(v) = \{g \in G | gv = v\}$.
• **special** if it is not spherical and there exists an edge $e \in E(T)$ such that $\pi$ admits a non-zero $\text{Fix}_G(e)$-invariant vector where $\text{Fix}_G(e) = \{ g \in G | gv = v \ \forall v \in e \}$ is the fixator of the edge $e$.

• **cuspidal** if it is neither spherical nor special.

The spherical and special representations are classified since the end of the 70’s at the level of generality of the conjecture that is for any closed non-compact subgroup $G \leq \text{Aut}(T)$ acting transitively on the boundary of the tree see [Mat77], [Ol’77] and [Ol’80]. Furthermore, we recall that Matsumoto’s work emphasises a strong connection between those kinds of representations and the irreducible representations of Hecke algebras. To be more precise, we recall that a group acting 2-transitively on the boundary is either type-preserving or admits an index 2 closed type-preserving subgroup acting 2-transitively on the boundary. Since [CC15, Corollary 3.6.] ensures that every such group $G$ comes from a B-N pair, each spherical or special representation of $G$ defines an irreducible representation of the associated Hecke algebra $C_c(B \backslash G/B)$ of continuous compactly supported $B$-bi-invariant functions $f : G \to \mathbb{C}$ where $B = \text{Fix}_G(e)$ is the pointwise fixator of an edge $e \in E(T)$. Matsumoto’s works enlightened the fact that this correspondence is actually bijective see [Mat77, Chapter 5, Section 6].

The cuspidal representations on the other hand are not classified at the level of generality of the conjecture. Nevertheless, a complete classification of those representations was achieved for certain families of groups. Among non-linear groups for instance, Ol’shanskii’s work lead to a classification of the cuspidal representations for any closed group of automorphisms of a semi-regular tree satisfying the Tits independence property see [Ol’77], [Ama03]. The main idea leading to this classification was to exploit the independence of the action on the tree to obtain a particular factorization on a well chosen basis of neighbourhood of the identity made by compact open subgroups. When it comes to Radu groups, [Rad17] highlighted the fact that those groups are defined by local conditions. Among other things, when $T$ is a $(d_0, d_1)$-semi regular tree with $d_0, d_1 \geq 4$, Radu introduced a family of groups $G_{(i)}(Y_0, Y_1)$ indexed by two finite subsets $Y_0, Y_1 \subseteq \mathbb{N}$ see Definition 3.3 below. Furthermore, he showed that those groups are abstractly simple and that they exhaust the list of simple Radu groups when $d_0, d_1 \geq 6$. Our paper takes advantage of the recent abstraction of Ol’shanskii’s framework developed in [Sem21] and the description of those groups provided by Radu to obtain a classification their cuspidal representations see Section 4. The author would like to underline that an application of Ol’shanskii’s machinery on Radu groups already exists in [Sem21, Section 4] since they satisfy a generalisation of the Tits independence property (the property IP$_k$ defined in
However, unless the property IP$_k$ coincides with the Tits independence property ($k = 1$), this approach never leads to a classification of all cuspidal representation of the group. However, the approach considered in the present paper relies on the independence provided by local conditions rather than the property IP$_k$. In particular, by contrast with the approach developed in [Sem21], Section 4 below leads to a description of every cuspidal representation of the groups $G_{(i)}(Y_0, Y_1)$.

**Theorem C.** In a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$, the cuspidal representations of $G_{(i)}(Y_0, Y_1)$ are in bijective correspondence given by induction with a family of irreducible representations of compact open subgroups. This correspondence is explicitly described by [Theorem 4.11, Section 4].

Among other things, this proves the cuspidal representations are induced from compact open subgroups and therefore square-integrable. Since [HC70, Corollary of Theorem 2] ensures, for every compact open subgroup $K \leq G$, the existence a positive integer $k_K$ such that $\dim(H^K\pi) \leq k_K$ for all square-integrable representations $\pi$ of $G$ and as a consequence of the classification of the spherical representations (see Section 2) this leads to the conclusion that the groups $G_{(i)}(Y_0, Y_1)$ are uniformly admissible (see Section 5). On the other hand, when $d_0, d_1 \geq 6$ Radu’s classification ensure that every Radu group $G$ belongs to a finite chain $H_n \geq \ldots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all $t$ and $H_0$ is conjugate in the group of type-preserving automorphisms $\text{Aut}(T)^+$ to one of those $G_{(i)}^+(Y_0, Y_1)$. Since Mackey’s machinery allows one to describe the irreducible representation of a locally compact $G$ in terms of the irreducible representations of any of its closed subgroup $H$ of index 2 (see Appendix A), this leads to a description of the cuspidal representations of any other Radu group. Furthermore, this also shows that every other Radu group is uniformly admissible see Lemma A.8.

**Structure of the paper**

In Section 2 we recall the classification of spherical and special representations of any closed non-compact subgroup $G \leq \text{Aut}(T)$ acting transitively one the boundary see [Mat77], [Ol’77] and [Ol’80]. The purpose of Section 3 is to recall Radu’s classification of Radu groups [Rad17] and the definition of the $G_{(i)}(Y_0, Y_1)$. In Section 4, we recall the notion of Ol’shanskii’s factorization developed in [Sem21] and obtain a classification of the cuspidal representations of the $G_{(i)}(Y_0, Y_1)$. The complete classification of the irreducible representations of $G_{(i)}(Y_0, Y_1)$ resulting from Sections 2 and 4 is then used in Section 5 to prove uniform admissibility. Finally, the purpose of
Appendix A is to recall the procedure allowing one to describe the irreducible representation of a locally compact $G$ in terms of the irreducible representations of any of its closed subgroup $H$ of index 2. In particular, this appendix provides a way to obtain the irreducible representations of any Radu groups from the irreducible representations of the abstractly simple Radu groups $G_{(i)}(Y_0, Y_1)$ and shows that other Radu groups are also uniformly admissible.

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## 2 Spherical and special representations

Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 3$. We recall that a tree $T$ is called $(d_0, d_1)$-semi-regular if there exists a bipartition $V(T) = V_0 \sqcup V_1$ of $T$ such that each vertex of $V_i$ has degree $d_i$ and every edge of $T$ contains exactly one vertex in each $V_i$. As explained in the introduction, the irreducible representations of any closed subgroup $G \leq \text{Aut}(T)$ of the group of automorphisms of such a tree splits in three categories. Those representations are either spherical, special or cuspidal. The purpose of the present section is to recall the classification of spherical and special representations of any closed non-compact subgroup $G \leq \text{Aut}(T)$ acting transitively on the boundary (this covers every Radu group). This classification is a classical result known since the end of the 70’s and we claim no originality. Furthermore, we refer to [Mat77], [Ol’77] and [FTN91] for details.

The details of this classification are gathered in Theorems 2.2, 2.3, 2.4 below. We now recall preliminaries required for the statement of those results. Given a locally compact group $G$ and a compact subgroup $K \leq G$, we say that $(G, K)$ is a Gelfand pair if the convolution algebra $C_c(K \backslash G / K)$ of compactly supported, continuous $K$-bi-invariant functions on $G$ is commutative. Now, let $(G, K)$ be a Gelfand pair and let $\mu$ be the left-Haar measure of $G$ renormalised in such a way that $\mu(K) = 1$. A function $\varphi : G \to \mathbb{C}$ is called $K$-spherical if it is a $K$-bi-invariant continuous function with $\varphi(1_G) = 1$ and

$$\int_K \varphi(\kappa g' \kappa) \, d\mu(\kappa) = \varphi(g)\varphi(g') \quad \forall g, g' \in G.$$  

The interests of those notions lay in the following result.

**Theorem 2.1** ([Lan85, Chapter IV. §3, Theorem 3 and 9]). Let $(G, K)$ be a Gelfand pair. For every irreducible representation $\pi$ of $G$ we have that
\( \dim(\mathcal{H}_K^\pi) \leq 1 \). Furthermore, there is a bijective correspondence \( \pi \to \varphi_\pi \) with inverse map given by the GNS construction between the equivalence classes of irreducible representations of \( G \) with non-zero \( K \)-invariant vectors and the \( K \)-spherical functions of positive type on \( G \) (the function \( \varphi_\pi \) is the function \( \varphi_\pi(g) = \langle \pi(g)\xi, \xi \rangle \) corresponding to any unit vector \( \xi \in \mathcal{H}_K^\pi \)).

We are finally ready to recall the details of the classification of spherical and special representations for any non-compact closed subgroups \( G \leq \text{Aut}(T) \) acting transitively on the boundary of \( T \). We recall that those groups act transitively on the edges of \( T \) and have therefore either one or two orbits of vertices. We treat those cases separately.

**Theorem 2.2** ([Neb99, Chapter II]). Let \( T \) be a \( d \)-regular tree, let \( v \in V(T) \) and let \( G \leq \text{Aut}(T) \) be a closed non-compact subgroup acting transitively on the vertices of \( T \) and the boundary \( \partial T \). Then, \( (G, \text{Fix}_G(v)) \) is a Gelfand pair and every spherical representation of \( G \) admits a non-zero \( \text{Fix}_G(v) \)-invariant vector. Furthermore, the equivalence classes of spherical representations of \( G \) are in bijective correspondence with the interval \( [-1; 1] \) via the map \( \phi_v : \pi \mapsto \varphi_\pi(\tau_v) \) where \( \tau_v \) is any element of \( G \) such that \( d(\tau_v v, v) = 1 \) and \( \varphi_\pi \) is the unique \( \text{Fix}_G(v) \)-spherical function of positive type attached to \( \pi \). Under this correspondence, the trivial representation corresponds to \( 1 \).

The following theorem is obtained from [Mat77] but is formulated differently for coherence of our expository.

**Theorem 2.3** ([Mat77, Chapter 5, Section 6]). Let \( T \) be a \( (d_0, d_1) \)-semi-regular tree with \( d_0, d_1 \geq 3 \), let \( v \in V(T) \), let \( v' \) be any vertex at distance one from \( v \) and let \( G \leq \text{Aut}(T) \) be a closed non-compact subgroup of type-preserving automorphisms acting transitively on the boundary \( \partial T \). Then, there is exactly one spherical representation \( \pi_v \) of \( G \) with a non-zero \( \text{Fix}_G(v) \)-invariant vector but no non-zero \( \text{Fix}_G(v') \)-invariant vector. Furthermore, \( (G, \text{Fix}_G(v)) \) is a Gelfand pair and apart from the two exceptional representations \( \pi_v \) and \( \pi_{v'} \), every spherical representation of \( G \) admits, for all \( w \in V(T) \), a non-zero \( \text{Fix}_G(w) \)-invariant vector. In addition, if \( v' \) has degree \( d' \), the equivalence classes of spherical representations admitting a non-zero \( \text{Fix}_G(v) \)-invariant vector are in bijective correspondence with the interval \([-\frac{1}{d-1}; 1]\) via the map \( \phi_v : \pi \mapsto \varphi_\pi(\tau_v) \) where \( \tau_v \) is an element of \( G \) such that \( d(\tau_v v, v) = 2 \). Under this correspondence, the exceptional spherical representation \( \pi_v \) corresponds to \(-\frac{1}{d-1}\) and the trivial representation corresponds to 1. Finally, if \( \pi \) is a non-exceptional spherical representation of \( G \) we have that

\[
\phi_{v'}(\pi) = \frac{d(d'-1)}{d'(d-1)} \phi_v(\pi) + \frac{d-d'}{d'(d-1)}.
\]

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To describe the special representations, let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 3$, let $e \in E(T)$ and let $G \leq \text{Aut}(T)$ be a closed subgroup acting transitively on the edges of $T$. We define $\mathcal{L}(e)$ as the subspace of Fix$_G(e)$-right invariant square-integrable functions $\varphi : G \to \mathbb{C}$ satisfying

$$\int_{\text{Fix}_G(v)} \varphi(gk) \, d\mu(k) = 0 \quad \forall g \in G, \forall v \in e.$$ 

Notice that $\mathcal{L}(e)$ is a closed left invariant subspace of $L^2(G)$ and let $\sigma : G \to \mathcal{U}(\mathcal{L}(e))$ be the unitary representation of $G$ defined by $\sigma(t)\varphi(g) = \varphi(t^{-1}g)$ $\forall g, t \in G, \forall \varphi \in \mathcal{L}(e)$. If $G$ is transitive on the vertices of $T$, we choose an inversion $h \in G$ of the edge $e$ and consider the map $\nu : \mathcal{L}(e) \to \mathcal{L}(e)$ defined by $\nu(\varphi)(g) = \varphi(gh)$ $\forall \varphi \in \mathcal{L}(e), \forall g \in G$. This map is well defined since for all $\varphi \in \mathcal{L}(e)$, for all $g \in G$ and every $v \in e$ we have

$$\int_{\text{Fix}_G(v)} (\nu \varphi)(gk) \, d\mu(k) = \int_{\text{Fix}_G(v)} \varphi(ghk) \, d\mu(k) = \int_{\text{Fix}_G(h^{-1}v)} \varphi(ghk) \, d\mu(k) = 0.$$ 

On the other hand, since every element of $\mathcal{L}(e)$ is Fix$_G(e)$-right invariant, notice that $\nu$ is an involution and that it does not depend on our choice of inversion of the edge $e$. For every $\epsilon \in \{-1, 1\}$, we let $\mathcal{L}(e)_\epsilon$ be the eigenspace of $\nu$ and we $\sigma^\epsilon : G \to \mathcal{U}(\mathcal{L}(e)_\epsilon)$ be the unitary representation of $G$ defined by $\sigma^\epsilon(t)\varphi(g) = \varphi(t^{-1}g)$ $\forall g, t \in G, \forall \varphi \in \mathcal{L}(e)_\epsilon$. We are now ready to state the classification of special representations.

**Theorem 2.4** ([FTN91, Chapter III, Section 2], [Mat77, Section 5.6]). Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 3$, let $e \in E(T)$ and let $G \leq \text{Aut}(T)$ be a closed non-compact subgroup acting transitively on the boundary $\partial T$. Every special representation of $G$ is square integrable and admits a Fix$_G(f)$-invariant vector for every $f \in E(T)$. Furthermore:

1. If $G$ acts transitively on $V(T)$, $(\sigma^{-1}, \mathcal{L}(e)_{-1})$ and $(\sigma^{+1}, \mathcal{L}(e)_{+1})$ are representatives of the two equivalence classes of special representations.

2. If $G$ has two orbits on $V(T)$, $(\sigma, \mathcal{L}(e))$ is a representative of the unique equivalence class of special representations.
Let $T$ be a $(d_0, d_1)$ semi-regular tree with $d_0, d_1 \geq 4$ and bipartition $V(T) = V_0 \sqcup V_1$ and let $\text{Aut}(T)^+\text{ denotes the group of type-preserving automorphisms of } T \text{ that is the set of automorphisms of } T \text{ which leave } V_0 \text{ and } V_1 \text{ invariants.}$ The purpose of this section is to recall the classification of Radu groups [Rad17]. To this end, we set $H_T = \{G \leq \text{Aut}(T) \mid G \text{ is closed and 2-transitive on } \partial T\}$ and $H_T^+ = \{G \leq \text{Aut}(T)^+ \mid G \text{ is closed and 2-transitive on } \partial T\}$. If $d_0 \neq d_1$, notice that every automorphisms of $T$ is type-preserving so that $H_T^+ = H_T$. We recall that for each vertex $v \in V(T)$, the stabilizer $\text{Fix}_G(v)$ of $v$ acts on the set $E(v)$ of edges containing $v$ and that the image of $\text{Fix}_G(v)$ in $\text{Sym}(E(v))$ for this projection map (which we denote by $G(v)$) is called the local action of $G$ at $v$. Furthermore, we recall in the light of [BM00, Lemma 3.1.1], that every group $G \in H_T^+$ is transitive on $V_0$ and $V_1$. Hence, all the groups $G(v)$ with $v \in V_0$ (respectively $v \in V_1$) are permutation isomorphic to the same group $F_0 \leq \text{Sym}(d_0)$ (respectively $F_1 \leq \text{Sym}(d_1)$). We recall that the groups $G \in H_T$ such that $G(v) \cong F_t \geq \text{Alt}(d_t)$ for every vertex $v \in V_t(T)$ and for $t \in \{0, 1\}$ are called Radu groups. Those groups have been extensively studied and classified by N. Radu in [Rad17] when $d_0, d_1 \geq 6$. The purpose of this section is to recall his classification.

We start by recalling a few definitions needed to describe Radu groups. For every vertex $v \in V(T)$ and every positive integer $r \in \mathbb{N}$ let $S(v, r) = \{w \in V(T) \mid d(v, w) = r\}$ be the set of vertices of $T$ at distance $r$ from $v$.

**Definition 3.1.** A legal coloring $i : V(T) \to \mathbb{N}$ of $T$ is the concatenation of a pair of maps $i_0 : V_0 \to \{1, \ldots, d_1\}$ and $i_1 : V_1 \to \{1, \ldots, d_0\}$ such that $i_0|_{S(v, 1)} : S(v, 1) \to \{1, \ldots, d_1\}$ and $i_1|_{S(w, 1)} : S(w, 1) \to \{1, \ldots, d_0\}$ are bijections for all $v \in V_1$ and $w \in V_0$.

Given a legal coloring $i$ of $T$ and an automorphism $g \in \text{Aut}(T)$, the local action of $g$ at a vertex $v \in V(T)$ is defined as the following permutation:

$$\sigma_{(v)}(g, v) = i|_{S(gv, 1)} \circ g \circ (i|_{S(v, 1)})^{-1} \in \begin{cases} \text{Sym}(d_0) & \text{if } v \in V_0 \\ \text{Sym}(d_1) & \text{if } v \in V_1. \end{cases}$$
Remark 3.2. If $d_0 = d_1$, the tree $T$ is a regular tree and this notion of legal coloring and local action of an element differ from the notion of legal coloring and local action used to define the universals Burger Mozes groups in [BM00]. Indeed, with our definition, the closed subgroup $G \leq \text{Aut}(T)$ of all automorphisms of trees $g \in G$ such that $\sigma(i)(g, v) = \text{id} \ \forall v \in V$ is not transitive on the set of vertices of $T$ (not even transitive on $V_0$).

Now, let $T$ be a $(d_0,d_1)$-semi-regular tree with $d_0,d_1 \geq 4$ and let $i$ be a legal coloring of $T$. For every vertex $v \in V(T)$ and every finite set $Y \subseteq \mathbb{N}$ let

$$S_Y(v) = \bigcup_{r \in Y} S(v, r)$$

and for every set of vertices $B \subseteq V(T)$ let

$$\text{Sgn}_i(g, B) = \prod_{w \in B} \text{sgn}(\sigma(i)(g, w))$$

where $\text{sgn}(\sigma(i)(g, w))$ is the sign of the local action $\sigma(i)(g, w)$ of the automorphism $g$ at $w$ for the legal coloring $i$. The following groups will have a central importance in the rest of this paper.

Definition 3.3. For all (possibly empty) finite sets $Y_0,Y_1$ of $\mathbb{N}$ and every legal coloring $i$ of $T$, we set

$$G_i^+(Y_0, Y_1) = \left\{ g \in \text{Aut}(T)^+ \left| \begin{array}{ll} \text{Sgn}_i(g, S_{Y_0}(v)) = 1 & \text{for each } v \in V_{t_0}, \\ \text{Sgn}_i(g, S_{Y_1}(v)) = 1 & \text{for each } v \in V_{t_1} \end{array} \right. \right\},$$

where $t_0 = \max(Y_0) \mod 2$, $t_1 = (1 + \max(Y_1)) \mod 2$ and $\max(\emptyset) = 0$.

Remark 3.4. Notice, that the choices of $t_0$ and $t_1$ are made in such a way that the vertices of $S_{Y_0}(v)$ with $v \in V_{t_0}$ at maximal distance from $v$ and the vertices of $S_{Y_1}(w)$ with $w \in V_{t_1}$ at maximal distance from $w$ have opposite types.

Notice that $G_i^+(\emptyset, \emptyset) = \text{Aut}(T)^+$ is the full group of type-preserving automorphisms and that $G_i^+(\{0\}, \{0\})$ is a subgroup of each $G_i^+(Y_0, Y_1)$. Furthermore, if $T$ is a $d$-regular tree notice that $G_i^+(\{0\}, \{0\})$ is conjugate to $U(\text{Alt}(d))^+$ where $G^+ = G \cap \text{Aut}(T)^+$ and $U(\text{Alt}(d))$ is the universal Burger-Mozes group of the alternating group see [BM00].

As we recall below, when $d_0,d_1 \geq 6$, every abstractly simple Radu group is of the form $G_i^+(Y_0, Y_1)$ for some finite $Y_0,Y_1 \subseteq \mathbb{N}$ and some legal coloring $i$ of $T$. Furthermore, when $d_0,d_1 \geq 6$, every Radu group $G$ belongs to a finite
chain $H_n \geq \ldots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all $t$ and $H_0$ is conjugate in $	ext{Aut}(T)^+$ to one of those $G^+_{(i)}(Y_0, Y_1)$. In particular, using the Appendix A the irreducible representations of every Radu groups can be obtained from the irreducible representations of the $G^+_{(i)}(Y_0, Y_1)$ and vice versa. We now recall more precisely the statements proved in [Rad17] that will be used in this paper.

**Theorem** ([Rad17, Theorem A]). Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$ and let $i$ be a legal coloring of $T$. Then, for every finite subsets $Y_0, Y_1 \subseteq \mathbb{N}$ the group $G^+_{(i)}(Y_0, Y_1)$ belongs to $\mathcal{H}^+_T$ and is abstractly simple.

The following results ensure that every Radu group contains a conjugate of such a $G^+_{(i)}(Y_0, Y_1)$ with finite index. To formulate this precisely, we introduce some notations. For every locally compact group $G$ we let $G^{(\infty)}$ be the intersection of all normal cocompact closed subgroups of $G$. We recall that for any group $H \in \mathcal{H}^+_T$, Burger and Mozes proved that $H^{(\infty)}$ belongs to $\mathcal{H}^+_T$ and is topologically simple [BM00, Proposition 3.1.2] (in our cases, it is even abstractly simple). Finally, we let $G^+_T(i)$ be the set of groups $G^+_{(i)}(Y_0, Y_1)$ with non-empty finite $Y_0, Y_1 \subseteq \mathbb{N}$ such that $y = \text{max}(Y_i) \mod 2$ for each $y \in Y_t$ with $y \geq \text{max}(Y_{t-1})$ ($t \in \{0, 1\}$).

**Theorem** ([Rad17, Theorem B]). Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 6$, let $i$ be a legal coloring of $T$ and let $G \in \mathcal{H}^+_T$ be such that $G(v) \cong F_0 \cong \text{Alt}(d_0)$ for each $v \in V_0$ and $G(w) \cong F_1 \cong \text{Alt}(d_1)$ for each $w \in V_1$. Then, we have $[G : G^{(\infty)}] \in \{1, 2, 4\}$ and $G^{(\infty)}$ is conjugate in $\text{Aut}(T)^+$ to an element of $G^+_T(i)$.

When $T$ is a $d$-regular tree, a similar result holds for all $G \in \mathcal{H}_T - \mathcal{H}^+_T$.

**Theorem** ([Rad17, Corollary C]). Let $T$ be a $d$-regular tree with $d \geq 6$ and let $i$ be a legal coloring of $T$ and let $G \in \mathcal{H}_T - \mathcal{H}^+_T$ be such that $G(v) \cong F \geq \text{Alt}(d)$ for each $v \in V(T)$. Then, we have $[G : G^{(\infty)}] \in \{2, 4, 8\}$ and $G^{(\infty)}$ is conjugate to $G^+_{(i)}(Y, Y)$ for some finite subset $Y$ of $\mathbb{N}$.

The following theorem follows from Radu’s description of Radu groups.

**Theorem 3.5.** Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 6$ and let $i$ be a legal coloring of $T$. Every Radu group $G$ admits a finite chain $H_n \geq \ldots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all $t$ and $H_0$ is conjugate in $\text{Aut}(T)^+$ to $G^+_{(i)}(Y_0, Y_1)$.  

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4 Cuspidal representations of the simple Radu groups

Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$ and let $V(T) = V_0 \cup V_1$ be the associated bipartition. Let $i$ be a legal coloring of $T$ and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. Recall from Section 3 that $G^+_{(i)}(Y_0, Y_1)$ (Definition 3.3) is a closed abstractly simple subgroups of $\text{Aut}(T)^+$ acting 2-transitively on the boundary $\partial T$ and whose local action at every vertex contains the alternating group. Furthermore, when $d_0, d_1 \geq 6$, Radu’s classification ensures that every simple Radu group is of this form. Our current purpose is to describe the irreducible representations of $G^+_{(i)}(Y_0, Y_1)$ and show that this group is uniformly admissible, hence CCR.

We recall from the introduction that the irreducible representations of $G^+_{(i)}(Y_0, Y_1)$ splits in three categories. Those are either spherical, special or cuspidal. A classification of the spherical and special representations of any subgroup $G \leq \text{Aut}(T)$ acting 2-transitively on the boundary is already given in Section 2. In particular, this classification applies to the spherical and special representations of $G^+_{(i)}(Y_0, Y_1)$. Our current purpose is to give a description of the cuspidal representations of those groups. As announced in the introduction, our idea is to take advantage of the recent abstraction of Ol’shanskii’s framework developed in [Sem21] and the description of those groups provided by Radu. The main concept developed in [Sem21], is the concept of Ol’shanskii’s factorization see Definition 4.2 below. We recall that such a factorization leads to a description of the irreducible representations admitting particular invariant vectors as induced representations from compact open subgroups. The author would like to recall that an Ol’shanskii’s factorization for Radu groups is already provided by [Sem21, Section 4] since they satisfy a generalisation of the Tits independence property (the property $\text{IP}_k$ defined in [BEW15]). However, this approach never leads to a description of every cuspidal representations of $G^+_{(i)}(Y_0, Y_1)$ unless the property property $\text{IP}_k$ coincides with the Tits independence property (that is $k = 1$). By contrast, the approach developed in the present section relies on the independence provided by local conditions given by Definition 3.3 rather than the property $\text{IP}_k$ and a description of every cuspidal representation of $G^+_{(i)}(Y_0, Y_1)$ is obtained in Section 4.4 below.

4.1 Preliminaries

The purpose of this section is to recall the axiomatic framework developed in [Sem21] (we refer to this paper for details). This machinery will then
be used in the following sections to obtain a description of the cuspidal representations of the Radu groups $G^+_{i(1)}(Y_0, Y_1)$.

Let $G$ be a totally disconnected locally compact group, let $B$ denote the set of compact open subgroups of $G$, $P(B)$ denote the power set of $B$ and let $C : B \rightarrow P(B)$ be the map sending a compact open subgroup to its conjugacy class in $G$.

Let $S$ be a basis of neighbourhoods of the identity consisting of compact open subgroups of $G$ and let $F_S = \{ C(U) | U \in S \}$. We equip $F_S$ with the partial order given by the reverse inclusion of representatives ($C(U) \leq C(V)$ if there exists $\tilde{U} \in C(U)$ and $\tilde{V} \in C(V)$ such that $\tilde{V} \subseteq \tilde{U}$). For a poset $(P, \leq)$ and an element $x \in P$, we recall that the height of $x$ in $(P, \leq)$ is $L_x - 1$ where $L_x$ is the maximal length of a strictly increasing chain in $P_{\leq x} = \{ y \in P | y \leq x \}$ if such a maximal length exists and we say that the height is infinite otherwise.

**Definition 4.1.** A basis of neighbourhoods of the identity $S$ consisting of compact open subgroups of $G$ is called a **generic filtration** of $G$ if the height of every element in $F_S$ is finite.

Every generic filtration $S$ of $G$ splits as a disjoint union $S = \bigsqcup_{l \in \mathbb{N}} S[l]$ where $S[l]$ denotes the set of elements $U \in S$ such that $C(U)$ has height $l$ in $F_S$. The element of $S[l]$ are called the elements at **depth** $l$. Since $S$ is a basis of neighbourhood of the identity consisting of compact open subgroups of $G$, notice that for every irreducible representation $\pi$ of $G$, there exists a group $U \in S$ such that $\pi$ admits non-zero $U$-invariant vector. In particular, for every irreducible representation $\pi$ of $G$ there exists a smallest non-negative integer $l_\pi \in \mathbb{N}$ such that $\pi$ admits non-zero $U$-invariant vectors for some $U \in S[l_\pi]$. This $l_\pi$ is called the **depth** of $\pi$ with respect to $S$.

The key notion developed in [Sem21] is the notion of **factorization** at depth $l$ for a generic filtration $S$ that we now recall.

**Definition 4.2.** Let $G$ be a non-discrete unimodular totally disconnected locally compact group, let $S$ be a generic filtration of $G$ and let $l$ be a strictly positive integer. We say that $S$ **factorizes at depth** $l$ if the following conditions hold:

1. For all $U \in S[l]$ and every $V$ in the conjugacy class of an element of $S$ such that $V \nsubseteq U$, there exists $W$ in the conjugacy class of an element of $S[l - 1]$ such that:

$$U \subseteq W \subseteq VU = \{ vu | u \in U, v \in V \}.$$
2. For all $U \in S[l]$ and every $V$ in the conjugacy class of an element of $S$, the set
$$N_G(U,V) = \{ g \in G | g^{-1}Vg \subseteq U \}$$
is compact.

Furthermore, the generic filtration $S$ of $G$ is said to **factorize** at depth $l$ if in addition for all $U \in S[l]$ and every $W$ in the conjugacy class of an element of $S[l - 1]$ such that $U \subseteq W$ we have
$$W \subseteq N_G(U,U) = \{ g \in G | g^{-1}Ug \subseteq U \}.$$Since $G$ is unimodular, notice that the set $N_G(U,U)$ coincides with the normalizer $N_G(U)$ of $U$ in $G$.

The relevance of this notion is given by [Sem21, Theorem A] which lead to a description of the irreducible representations at height $l$ in terms of a family of irreducible representations of finite groups called $S$-standard representations (Definition 4.10 below) if the generic filtration factorizes at height $l$.

### 4.2 Generic filtration for $G^+_{(i)}(Y_0, Y_1)$

Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$ and let $V(T) = V_0 \cup V_1$ be the associated bipartition. Let $i$ be a legal coloring of $T$ and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. The purpose of this section is to explicit a generic filtration for $G^+_{(i)}(Y_0, Y_1)$. To this end, let $\mathfrak{T}_0$ be the family of subtrees of $T$ defined by
$$\mathfrak{T}_0 = \{ B_T(v, r) | v \in V(T), r \geq 1 \} \cup \{ B_T(e, r) | e \in E(T), r \geq 0 \}$$
and consider the basis of neighbourhoods of the identity given by the fixators of those trees
$$S_0 = \{ \text{Fix}_G(T) | T \in \mathfrak{T}_0 \}.$$In [Sem21], one introduced the following definition:

**Definition 4.3.** A group $G \leq \text{Aut}(T)$ is said to satisfy the hypothesis $H_0$ if for all $T, T' \in \mathfrak{T}_0$ we have that
$$\text{Fix}_G(T') \leq \text{Fix}_G(T)$$if and only if $T \subseteq T'$.

**Lemma 4.4** ([Sem21, Lemma 4.12]). Let $G \leq \text{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis $H_0$. Then, $S_0$ is a generic filtration of $G$ and the sets $S_0[l]$ can be described as follows:
• If \( l \) is even \( S_0[l] = \{ \text{Fix}_G(B_T(e, \frac{1}{2})) | e \in E(T) \} \).

• If \( l \) is odd \( S_0[l] = \{ \text{Fix}_G(B_T(v, (\frac{l-1}{2})) | v \in V(T) \} \).

We come back to our case \( G = G^+(i)(Y_0, Y_1) \).

**Lemma 4.5.** The group \( G^+(i)(Y_0, Y_1) \) satisfies the hypothesis \( H_0 \).

**Proof.** For every set \( X \), let \( \Delta_X \) denote the diagonal \( \Delta_X = \{(x, x) | x \in X \} \subseteq X \times X \). We choose two functions

\[
\psi_0 : \{1, ..., d_0\} \times \{1, ..., d_0\} - \Delta\{1, ..., d_0\} \longrightarrow \text{Sym}(d_0) : (k, l) \mapsto \psi_0(k, l)
\]

\[
\psi_1 : \{1, ..., d_1\} \times \{1, ..., d_1\} - \Delta\{1, ..., d_1\} \longrightarrow \text{Sym}(d_1) : (k, l) \mapsto \psi_1(k, l)
\]

such that \( \psi_i(k, l) \) is a non-trivial element of \( \text{Alt}(d_i) \) which fixes \( k \) but not \( l \). Notice that the existence of such functions is guaranteed from the fact that \( d_0, d_1 \geq 4 \). For shortening of the formulation we denote by \( G \) be the group \( G^+(i)(Y_0, Y_1) \). Let \( T, T' \) be two subtrees of \( \Sigma_0 \). If \( T \subseteq T' \), we clearly have that \( \text{Fix}_G(T') \subseteq \text{Fix}_G(T) \). Now, let us suppose that \( T \not\subseteq T' \). In order to prove that \( G \) satisfies the hypothesis \( H_0 \), we need to show that \( \text{Fix}_G(T') \not\subseteq \text{Fix}_G(T) \). Since \( T \not\subseteq T' \), there exists a vertex \( v \in T \) that does not belong to \( T' \). Let \( \gamma \) be the smallest geodesic from \( v \) to \( T' \), let \( v' \) be the vertex of \( \gamma \) that is adjacent to \( v \) and let \( t \in \{0, 1\} \) be such that \( v' \in V_t \). Let \( w \) be the neighbour of \( v' \) which is the closest to \( T' \). Notice that this vertex exists and is unique since \( T \) and \( T' \) are complete. The definition of \( v' \) ensures that \( T' \subseteq T(v', v') = \{ x \in V(T) \mid d_T(x, v') < d_T(x, v) \} \). Now, notice the existence of an automorphism \( g \in \text{Fix}_{\text{Aut}(T)^+}(T(v', v)) \) such that \( \sigma_0(g, v') = \psi_i(i(w), i(v)) \) and \( \sigma_+(g, x) \) is even for every \( x \in V(T) \). In particular, \( g \in G^+(i)(Y_0, Y_1) \cap \text{Fix}_{\text{Aut}(T)^+}(T') \). However, \( g \) does not fix \( v \) by construction. This implies that \( g \not\in \text{Fix}_G(T) \). \( \square \)

In particular, Lemma 4.4 ensures that \( S_0 \) is a generic filtration of \( G^+(i)(Y_0, Y_1) \).

### 4.3 Factorization

Let \( T \) be a \((d_0, d_1)\)-semi-regular tree with \( d_0, d_1 \geq 4 \) and let \( V(T) = V_0 \cup V_1 \) be the associated bipartition. Let \( i \) be a legal coloring of \( T \) and let \( Y_0, Y_1 \subseteq \mathbb{N} \) be two finite subsets. We have shown in Section 4.2, that \( S_0 \) is a generic filtration of \( G^+(i)(Y_0, Y_1) \). The purpose of the present section is to prove that this generic filtration factorizes \(^+ \) at all depth \( l \geq 1 \).

We start with some notations that will be used in the proof. For every two distinct vertices \( v, w \in V(T) \), let \([v, w] \) be the unique geodesic between...
Suppose that \(d(v, w) = n\), let \(v = v_0, v_1, ..., v_n = w\) be the sequence of vertices corresponding to \([v, w]\) in \(T\), let

\[p_{[v, w]} : [v, w] - \{v\} \rightarrow [v, w] : v_i \mapsto v_{i-1}\]

and let

\[T(v, w) = \{x \in V(T) | d_T(x, p_{[v, w]}(w)) < d_T(x, w)\}\]

\[= \{x \in V(T) | d_T(x, v_{n-1}) < d_T(x, w)\}.\]

Figure 1: The set \(T(v, w)\)

The following intermediate result is the key ingredient required to prove the factorization of the generic filtration \(S_0^+ \circ (Y_0, Y_1)\) at all depth \(l \geq 1\).

**Proposition 4.6.** For all \(l, l' \in \mathbb{N}\) such that \(l \geq 1\) and \(l' \geq l\), for all \(U\) in the conjugacy class of an element of \(S_0[l]\) and every \(V\) in the conjugacy class of an element of \(S_0[l']\) such that \(V \not\subseteq U\), there exists a subgroup \(W \in S_0[l-1]\) such that \(U \subseteq W \subseteq VU\).

**Proof.** To shorten the proof and for clarity of the argument, parts of the reasoning are proved in Lemmas 4.7 and 4.8 below. Since the proof is quite long and technical, we start by giving an idea of its structure. We begin the proof by identifying the group \(W\) from \(U\) and \(V\). We then prove that each element of \(W\) decomposes as a product of an element of \(V\) and an element of
$U$. The proof of this decomposition is where the technicalities come from. It is achieved by a compactness argument taking advantage from the fact that $G^+_{(o)}(Y_0, Y_1)$ is defined by local actions conditions.

Let $G = G^+_{(o)}(Y_0, Y_1)$. As announced at the beginning of the proof, we start by identifying $W$. Notice that $\Sigma_0$ is stable under the action of $G$. Furthermore, for every $g \in \text{Aut}(T)^+$ and for every subtree $T$ of $T$, we have that $g\text{Fix}_G(T)g^{-1} = \text{Fix}_G(gT)$. In particular, there exist $T, T' \in \Sigma_0$ such that $U = \text{Fix}_G(T)$ and $V = \text{Fix}_G(T')$. Since $V \not\subseteq U$, notice that $T \not\subseteq T'$. If $l$ is even, Lemma 4.4 ensures that $T = B_T(e, \frac{l}{2})$ for some edge $e \in E(T)$. Furthermore, since $T \not\subseteq T'$ and since $l' \geq l$, there exists a unique vertex $v \in e$ such that $T' \subseteq T(v, w) \cup B_T(v, \frac{1}{2})$ where $w$ denotes the other vertex of $e$. In that case, we let $T_W = B_T(v, \frac{1}{2})$. If on the other hand $l$ is odd, Lemma 4.4 ensures that $T = B_T(w, \frac{l+1}{2})$ for some vertex $w \in V(T)$. Furthermore, since $T \not\subseteq T'$ and since $l' \geq l$, there exists a unique vertex $v \in B_T(w, 1) - \{w\}$ such that $T' \subseteq T(v, w) \cup B_T(\{v, w\}, \frac{l+1}{2})$. In that case, we let $T_W = B_T(\{v, w\}, \frac{l+1}{2})$. In both cases, we set $W = \text{Fix}_G(T_W)$.

By construction, notice that $W \in S_0[l - 1]$ and that $U \subseteq W$ (since $T_W \subseteq T$). Our purpose is therefore to show that $W \subseteq UV$. To this end, let $\alpha \in W$ and let us show the existence of an element $\alpha_0 \in U$ such that $\alpha|_{\bar{T}} = \alpha_0|_{\bar{T}}$. We start by explaining why the existence of $\alpha_0$ settles the proof. Indeed, if $\alpha_0$ exists, notice that the automorphism $\alpha_1 = \alpha_0^{-1} \circ \alpha$ is an element of $G$ for which $\alpha_1|_{\bar{T}} = \text{id}|_{\bar{T}}$. In particular, we have that $\alpha_1 \in \text{Fix}_G(T')$, $\alpha_0 \in \text{Fix}_G(T)$ and by construction $\alpha = \alpha_0 \circ \alpha_1$ which proves that $W \subseteq UV$. Applying the inverse map on both sides of the inclusion we obtain that $W \subseteq U'$ which settles the proof.

Now, let us prove the existence of $\alpha_0$. As announced, at the beginning of the proof, we are going to use a compactness argument taking advantage from the fact that $G^+_{(o)}(Y_0, Y_1)$ is defined by local actions conditions. To be more precise, we are going to define a descending chain of non-empty compact sets $\Omega_n \subseteq \text{Aut}(T)^+$ and an increasing chain of finite subtrees $R_n$ of $T$ such that $T = \bigcup_{n \in \mathbb{N}} R_n$ and such that for all $h \in \Omega_n$ we have:

- $h \in \text{Fix}_G(T)$ and $h|_{\bar{T}} = \alpha|_{\bar{T}}$,
- $\text{Sgn}_{(i)}(h, S_{t_0}(v)) = 1$ for all $v$ in $V_{t_0} \cap R_n$,
- $\text{Sgn}_{(i)}(h, S_{t_1}(v)) = 1$ for all $v$ in $V_{t_1} \cap R_n$.

We recall that in the above $t_0 = \max(Y_0) \mod 2$, $t_1 = (1 + \max(Y_1)) \mod 2$ and $\max(\emptyset) = 0$. Let us first show that this settles the existence of $\alpha_0$. Since the $\Omega_n$ form a descending chain of non-empty compact sets in a Hausdorff space we obtain $\bigcap_{n \in \mathbb{N}} \Omega_n \neq \emptyset$. Let $\alpha_0 \in \bigcap_{n \in \mathbb{N}} \Omega_n$. Since $\alpha_0 \in \Omega_0$, notice
that \( \alpha_0|_{T} = \text{id}|_{T} \). To see that \( \alpha_0 \) is as desired, we are left to show that \( \alpha_0 \in G_{i}^\| (Y_0, Y_1) \). However, for every \( v \in V_t(T) \), there exists a positive integer \( n \in \mathbb{N} \) such that \( v \in R_n \) and since \( \alpha_0 \in \Omega_n \) we have that \( \text{Sgn}_{i}(\alpha_0, S_{Y_0}(v)) = 1 \). This proves that \( \alpha_0 \) is as desired.

We are left to define the descending chain of non-empty compact sets \( \Omega_n \subseteq \text{Aut}(T)^+ \). Suppose that \( \max(Y_0) \leq \max(Y_1) \) (the proof for \( \max(Y_1) \leq \max(Y_0) \) is similar). Let \( \gamma \) be the smallest geodesic of \( T \) containing both the centre of \( T \), the centre of \( T' \) and oriented from \( T \) to \( T' \) (note that the centre is either a vertex or an edge depending on the values of \( l \) and \( l' \)). Since \( T \not\subseteq T' \) and since \( l' \geq l \) notice that \( \gamma \) contains at least two vertices.

Figure 2: The tree \( T_W \) and the geodesic \( \gamma \)

The increasing chain of finite subtrees \( R_n \) of \( T \) such that \( T = \bigcup_{n \in \mathbb{N}} R_n \) that we are going to use is \( R_n = B_T(\gamma, n) \). We let

\[
\Omega_{-1} = \left\{ h \in \text{Aut}(T)^+ \mid h|_{T} = \text{id}|_{T} \quad \text{and} \quad h|_{T'} = \alpha|_{T'} \right\} .
\]

Since \( \alpha \in \text{Fix}_{G}(T_W) \) and since \( T_W \) contains every vertices of \( T \cap T' \) notice that \( \Omega_{-1} \) is not empty. Now, since \( \max(Y_0) \leq \max(Y_1) \), notice that there exists a unique \( r \in \mathbb{N} \) such that \( \max(Y_0) + 2r \leq \max(Y_1) \leq \max(Y_0) + 2r + 1 \) (where one of this inequality is an equality). We let

\[
\Omega_0 = \left\{ h \in \Omega_{-1} \mid \begin{array}{l}
\text{Sgn}_{i}(g, S_{Y_0}(v)) = 1 \quad \text{for each} \quad v \in B_T(\gamma, 2r) \cap V_0, \\
\text{Sgn}_{i}(g, S_{Y_1}(v)) = 1 \quad \text{for each} \quad v \in B_T(\gamma, 0) \cap V_1,
\end{array} \right\}.
\]

Lemma 4.7 below ensures that this set is not empty. From there, we define the sets \( \Omega_n \) by induction on \( n \). For every \( n \geq 1 \), let \( h_n \) an element of \( \Omega_{n-1} \)
and let
\[
\Omega_n = \left\{ h \in \Omega_{n-1} \mid \begin{array}{l}
h \big|_{B_T(\gamma, n+\max(Y_1))} = h_n \big|_{B_T(\gamma, n+\max(Y_1))}, \\
\text{Sgn}_{(i)}(h, S_{Y_0}(w)) = 1 \quad \forall w \in B_T(\gamma, n+2r) \cap V_{t_0}, \\
\text{Sgn}_{(i)}(h, S_{Y_1}(w)) = 1 \quad \forall w \in B_T(\gamma, n) \cap V_{t_1}
\end{array} \right\}.
\]

For this induction to make sense, it is important for \(\Omega_n\) to be not empty for all \(n \geq 1\). This is proved by Lemma 4.8 below which ensures that \(\Omega_n\) is a non-empty compact set. The result follows.

Our current purpose is to prove Lemmas 4.7 and 4.8. To this end, we introduce some formalism that will be used in both proofs. For all \(v \in V(T)\), we are going to need an automorphism \(h(v) \in \text{Aut}(T)^+\) that will be used to create an element of \(\omega_{n+1}\) from an element of \(\Omega_n\). We start by choosing four functions:

\[
\begin{align*}
\phi_0 : \{1, \ldots, d_0\} &\longrightarrow \text{Sym}(d_0) : k \mapsto \phi_0(k) \\
\phi_1 : \{1, \ldots, d_1\} &\longrightarrow \text{Sym}(d_1) : k \mapsto \phi_1(k) \\
\tilde{\phi}_0 : \{1, \ldots, d_0\} \times \{1, \ldots, d_0\} &\longrightarrow \text{Sym}(d_0) : (k, l) \mapsto \tilde{\phi}_0(k,l) \\
\tilde{\phi}_1 : \{1, \ldots, d_1\} \times \{1, \ldots, d_1\} &\longrightarrow \text{Sym}(d_1) : (k, l) \mapsto \tilde{\phi}_1(k,l)
\end{align*}
\]

such that \(\phi_t(k)\) is an odd permutation of \(\text{Sym}(d_t)\) which fixes \(k\) and \(\tilde{\phi}_t(k,l)\) is an odd permutation of \(\text{Sym}(d_t)\) which fixes \(k\) and \(l\).

If \(v \in V(T) - \gamma\) we choose \(w \in \gamma\) and let \(h(v) \in \text{Aut}(T)^+\) be such that:

1. \(h(v) \in \text{Fix}_{\text{Aut}(T)^+}(T(p[w,v](v), v))\).
2. \(\sigma_{(i)}(h(v), v) = \tilde{\phi}_t(i(p[w,v](v)))\) where \(t \in \{0, 1\}\) is such that \(v \in V_t\).

Notice that for all \(v \in V(T) - \gamma\) and every \(w, w' \in \gamma\) \(p[w,v](v) = p[w',v](v)\) (recall the definition of \(p[w,v]\) from page 16) so that our choice of \(w \in \gamma\) does not change the two properties that \(h(v)\) must satisfy.
If $v \in \gamma$, we have two cases. Remember that $\gamma$ has at least two vertices. If $v$ is an end of $\gamma$ let $w$ be the unique vertex of $\gamma$ that is adjacent to $v$ and choose an automorphism $h(v) \in \text{Aut}(T)^+$ such that:

1. $h(v) \in \text{Fix}_{\text{Aut}(T)}(T(w, v))$.
2. $\sigma_{(i)}(h(v), v) = \phi_t(i(w))$ where $t \in \{0, 1\}$ is such that $v \in V_t$.

On the other hand, if $v$ is not an end of $\gamma$, let $w_1, w_2$ be the two neighbours of $v$ which belong to $\gamma$ and choose an automorphism $h(v) \in \text{Aut}(T)^+$ such that:

1. $h(v) \in \text{Fix}_{\text{Aut}(T)}(T(w_1, v) \cup T(w_2, v))$.
2. $\sigma_{(i)}(h(v), v) = \tilde{\phi}_t(i(w_1), i(w_2))$ where $t \in \{0, 1\}$ is such that $v \in V_t$.

We are now ready to prove Lemmas 4.7 and 4.8.

**Lemma 4.7.** The set

$$
\Omega_0 = \left\{ h \in \Omega_{-1} \mid \begin{array}{l}
\text{Sgn}_{(i)}(g, S_{Y_0}(v)) = 1 \text{ for each } v \in B_T(\gamma, 2r) \cap V_{t_0}, \\
\text{Sgn}_{(i)}(g, S_{Y_1}(v)) = 1 \text{ for each } v \in B_T(\gamma, 0) \cap V_{t_1}
\end{array} \right\}.
$$

is not empty.

**Proof.** We recall that $\max(Y_0) \leq \max(Y_1)$, that $r \in \mathbb{N}$ is the unique integer such that $\max(Y_0) + 2r \leq \max(Y_1) \leq \max(Y_0) + 2r + 1$, that $t_0 = \max(Y_0)$
mod 2 and that $t_1 = \max(Y_1) + 1 \mod 2$. Remember from the proof of Proposition 4.6 that $\Omega_{-1}$ is not empty and let $h_0 \in \Omega_{-1}$. We are going to modify the element $h_0$ with the automorphisms $h(v)$ in order to obtain an element of $\Omega_0$. A concrete example of the procedure is given on a 4-regular tree with $Y_0 = \{0\}$ and $Y_1 = \{1, 2\}$ by figures 4, 5 and 6. In those figures:

- The hollow vertices are those concerned by the current and previous steps.
- The vertices circled in purple are the vertices for which we desire to change the sign $\text{Sgn}_{(i)}(h_0, S_{Y_0}(v))$ or $\text{Sgn}_{(i)}(h_0, S_{Y_1}(v))$ (depending on the step) without affecting the sign of other hollow vertices.
- The vertices circled in yellow are the vertices for which a change of the local action is applied in order to fulfil the desired change of sign (note that for our choice $Y_0 = \{0\}$ those vertices are also the vertices circled in purple).

Let $\{w_{0,0}, ..., w_{0,m_0}\}$ be the set of vertices $w \in B_T(\gamma, 0) \cap V_0$ such that $\text{Sgn}_{(i)}(h_0, S_{Y_0}(w)) = -1$. For all $j = 0, 1, ..., m_0$ we choose a vertex

$$v_{0,j} \in \bigcap_{w \in \gamma - \{w_{0,j}\}} T(w_{0,j}, w)$$

such that $d(v_{0,j}, w_{0,j}) = \max(Y_0)$. In particular, notice that $v_{0,j} \in S_{Y_0}(w_{0,j})$ but that $v_{0,j} \not\in S_{Y_0}(w)$ for every $w \in B_T(\gamma, 0) \cap V_0$. Furthermore, since $\text{Sgn}_{(i)}(h_0, S_{Y_0}(w_{0,j})) = -1$, $h_0|_T = \text{id}|_T$, $h_0|_{T'} = \alpha|_{T'}$, and due to the form of $T$ and $T'$, the vertices $v_{0,j}$ must be such that the automorphisms $h(v_{0,0}), ..., h(v_{0,m_0})$ fix $T \cup T'$ pointwise. In particular, the automorphism

$$h_{0,0} = h_0 \circ h(v_{0,0}) \circ ... \circ h(v_{0,m_0})$$

satisfies $h_{0,0}|_T = h_0|_T = \text{id}|_T$, $h_{0,0}|_{T'} = h_0|_{T'} = \alpha|_{T'}$, and

$$\text{Sgn}_{(i)}(h_{0,0}, S_{Y_0}(w)) = 1 \forall w \in \gamma \cap V_0.$$
If \( r \neq 0 \), we iterate this procedure. For every \( 1 \leq \nu \leq 2r \), let \( \{ w_{\nu,0}, \ldots, w_{\nu,m_\nu} \} \) be the set of vertices \( w \in B_T(\gamma, \nu) \cap V_0 \) such that

\[
\text{Sgn}_{(i)}(h_{\nu-1,0}, S_{Y_0}(w)) = -1.
\]

For all \( j = 0, 1, \ldots, m_\nu \) we choose a vertex

\[
v_{\nu,j} \in \bigcap_{w \in B_T(\gamma, \nu) \cap V_0 \setminus \{ w_{\nu,j} \}} T(w_{\nu,j}, w)
\]

such that \( d(v_{\nu,j}, w_{\nu,j}) = \max(Y_0) \). Hence, notice that \( v_{\nu,j} \in S_{Y_0}(w_{\nu,j}) \) but that \( v_{\nu,j} \notin S_{Y_0}(w) \) for every \( w \in B_T(\gamma, \nu) \cap V_0 \setminus \{ w_{\nu,j} \} \). Furthermore, notice that the automorphisms \( h_{(v_{\nu,0})}, \ldots, h_{(v_{\nu,m_\nu})} \) fix \( T \cup T' \) pointwise. In particular, the automorphism

\[
h_{\nu,0} = h_{\nu-1,0} \circ h_{(v_{\nu,0})} \circ \ldots \circ h_{(v_{\nu,m_\nu})}
\]

satisfies that \( h_{\nu,0}|_T = h_{\nu-1,0}|_T = \text{id}|_T \), \( h_{\nu,0}|_{T'} = h_{\nu-1,0}|_{T'} = \alpha|_{T'} \), and

\[
\text{Sgn}_{(i)}(h_{\nu,0}, S_{Y_0}(w)) = 1 \quad \forall w \in B_T(\gamma, \nu) \cap V_0.
\]
Figure 5: Step II of the proof of Lemma 4.7

Consider the element $h_{2r,0}$ that we have just constructed. This element behaves as desired for the condition given by $Y_0$ on the vertices of $B_T(\gamma, 2r) \cap V_{t_0}$. However, nothing ensures that the condition given by $Y_1$ on the vertices of $B_T(\gamma, 0) \cap V_{t_1}$ is yet satisfied. We now take care of this task.

Let $\{w_0, \ldots, w_m\}$ be the set of vertices $w \in B_T(\gamma, 0) \cap V_{t_1}$ such that $\text{Sgn}_{(i)}(h_{2r,0}, S_{Y_1}(w)) = -1$. For all $j = 0, 1, \ldots, m$ let $v_j \in \bigcap_{w \in \gamma - \{w_j\}} T(w_j, w)$ such that $d(v_j, w_j) = \max(Y_1)$. In particular, notice that $v_j \in S_{Y_1}(w_j)$ but that $v_j \not\in S_{Y_1}(w)$ for every $w \in B_T(\gamma, 0) \cap V_{t_1}$. Furthermore, since $\max(Y_0) + 2r \leq \max(Y_1)$ notice from Remark 3.4 that $v_j \not\in S_{Y_0}(v)$ for every $v \in B_T(\gamma, 2r) \cap V_{t_0}$. On the other hand, just as before, the automorphisms $h_{(w_0)}, \ldots, h_{(v_m)}$ fix $T \cup T'$ pointwise. In particular, $h_1 = h_{2r,0} \circ h_{(w_0)} \circ \ldots \circ h_{(v_m)}$ satisfies:

- $h_1|_T = h_{2r,0}|_T = \text{id}|_T$ and $h_1|_{T'} = h_{2r,0}|_{T'} = \alpha|_{T'}$.
- $\text{Sgn}_{(i)}(h_1, S_{Y_0}(w)) = 1 \ \forall w \in B_T(\gamma, 2r) \cap V_{t_0}$.
- $\text{Sgn}_{(i)}(h_1, S_{Y_1}(w)) = 1 \ \forall w \in B_T(\gamma, 0) \cap V_{t_1}$.
This proves that $h_1 \in \Omega_0$ and therefore that $\Omega_0$ is not empty.

**Lemma 4.8.** For all $n \geq 1$, the set

$$
\Omega_n = \left\{ h \in \Omega_{n-1} \left| \begin{array}{l}
h|_{B_T(\gamma, n+\max(Y_1))} = h_n|_{B_T(\gamma, n+\max(Y_1))}, \\
\sgn_{(i)}(h, S_{Y_0}(w)) = 1 \quad \forall w \in B_T(\gamma, n+2r) \cap V_t, \\
\sgn_{(i)}(h, S_{Y_1}(w)) = 1 \quad \forall w \in B_T(\gamma, n) \cap V_{t_1}
\end{array} \right. \right\}.
$$

is a non-empty compact subsets of $\text{Aut}(T)^+$.  

**Proof.** We show that $\Omega_n$ is not empty by induction. Lemma 4.7 ensures that $\Omega_0$ is not empty. Suppose that $\Omega_{n-1}$ is not empty and let $h_n \in \Omega_{n-1}$ be the automorphism appearing in the definition of $\Omega_n$. Just as in the proof of Lemma 4.7, we are going to modify $h_n$ with the automorphisms $h(v)$ in order to obtain an element of $\Omega_n$. A concrete example of the procedure is given by figures 7 and 8 on a 4-regular tree with $Y_0 = \{0\}$ and even $\max(Y_1)$ (with the same conventions as before and where the vertices concerned by the current step are circled in pink). Let $\{\tilde{w}_0, \ldots, \tilde{w}_k\}$ be the set of vertices $w \in B_T(\gamma, n + 2r) \cap V_t$ such that $\sgn_{(i)}(h_n, S_{Y_0}(\tilde{w}_j)) = -1$. For each $j = 0, 1, \ldots, k$ we choose a vertex $\tilde{v}_j \in \bigcap_{w \in B_T(\gamma, n+2r) \cap V_t \setminus \{\tilde{w}_j\}} T(\tilde{w}_j, w)$ such that $d(\tilde{v}_j, \tilde{w}_j) = \max(Y_0)$. Hence, notice that $\tilde{v}_j \not\in S_{Y_0}(w)$ for all $w \in B_T(\gamma, n + 2r) \cap V_t \setminus \{\tilde{w}_j\}$. Furthermore, since $\max(Y_1) - 1 \leq 2r + \max(Y_0)$ notice from Remark 3.4 that $\tilde{v}_j \not\in S_{Y_1}(w)$ for all $w \in B_T(\gamma, n - 1) \cap V_t$. Furthermore, since $\sgn_{(i)}(h_n, S_{Y_0}(\tilde{w}_j)) = -1$, $h_n|_T = \text{id}|_T$, $h_n|_{T'} = \alpha|_{T'}$ and due to the form of $T$ and $T'$, notice that the automorphisms $h_{(\tilde{v}_0)}, \ldots, h_{(\tilde{v}_k)}$ fix $T \cup T'$ pointwise. In particular, $\hat{h}_n = h_n \circ h_{(\tilde{v}_0)} \circ \ldots \circ h_{(\tilde{v}_k)}$ satisfies:
\[ \hat{h}_n \big|_{B_T(\gamma, n - 1 + \max(Y_1))} = h_n \big|_{B_T(\gamma, n - 1 + \max(Y_1))}. \]

- \( \text{Sgn}_{(i)}(h_n, S_{Y_0}(w)) = 1 \ \forall w \in B_T(\gamma, n + 2r) \cap V_{t_0}. \)
- \( \text{Sgn}_{(i)}(h_n, S_{Y_1}(w)) = 1 \ \forall w \in B_T(\gamma, n - 1) \cap V_{t_1}. \)

Figure 7: Step I of the proof of Lemma 4.8

Now, let \( \{w_0, ..., w_m\} \) be the set of vertices \( w \in B_T(\gamma, n) \cap V_{t_1} \) such that \( \text{Sgn}_{(i)}(h_n, S_{Y_1}(w)) = -1. \) For each \( j = 0, 1, ..., m, \) we choose a vertex \( v_j \in \bigcap_{w \in B_T(\gamma, n) - \{w_j\}} T(w_j, w) \) such that \( d(v_j, w_j) = \max(Y_1). \) Since \( \max(Y_0) + 2r \leq \max(Y_1) \) notice from Remark 3.4 that \( v_j \notin S_{Y_0}(w) \) for every \( w \in B_T(\gamma, n + 2r) \cap V_{t_0} \) and \( v_j \notin S_{Y_1}(w) \) for every \( w \in B_T(\gamma, n) \cap V_{t_1} - \{w_j\}. \) Just as before, notice that the automorphisms \( h_{(v_0)}, ..., h_{(v_m)} \) fix \( T \cup T' \) pointwise. In particular, \( h_{n+1} = \hat{h}_n \circ h_{(v_0)} \circ ... \circ h_{(v_m)} \) satisfies:

- \( h_{n+1} \big|_{B_T(\gamma, n - 1 + \max(Y_1))} = h_n \big|_{B_T(\gamma, n - 1 + \max(Y_1))}. \)
- \( \text{Sgn}_{(i)}(h_{n+1}, S_{Y_0}(w)) = 1 \ \forall w \in B(\gamma, n + 2r) \cap V_{t_0}. \)
- \( \text{Sgn}_{(i)}(h_{n+1}, S_{Y_1}(w)) = 1 \ \forall w \in B(\gamma, n) \cap V_{t_1}. \)
This proves that $\Omega_n$ is not empty. We now show that $\Omega_n$ is compact for every integer $n \geq 1$. To this end, notice that $\Omega_n$ is a closed subset of $\text{Aut}(T)^+$ and that

$$\Omega_n \subseteq h_n \text{Fix}_{\text{Aut}(T)^+}(B_T(\gamma, n + \max(Y_1))).$$

Since the right hand side is a compact subset of $\text{Aut}(T)^+$ the results follows.

We are finally able to prove the result announced at the beginning of the Section 4.3.

**Theorem 4.9.** The generic filtration $\mathcal{S}_0$ of $G_{(i)}^+(Y_0, Y_1)$ factorizes$^+$ at all depth $l \geq 1$.

**Proof.** For shortening of the formulation, we let $G = G_{(i)}^+(Y_0, Y_1)$. To prove that $\mathcal{S}_0$ factorizes$^+$ a depth $l \geq 1$, we shall successively verify the three conditions of the Definition 4.2.

First, we need to prove that for every $U$ in the conjugacy class of an element of $\mathcal{S}_0[l]$ and every subgroup $V$ in the conjugacy class of an element
of $S_0$ with $V \not\subset U$, there exists a $W$ in the conjugacy class of an element of $S_0[l-1]$ such that

$$U \subseteq W \subseteq VU.$$  

Let $U$ and $V$ be as above. If $V$ is conjugate to an element of $S_0[l']$ for some $l' \geq l$, the result follows directly from Proposition 4.6. Therefore, let us suppose that $l' < l$. By the definition of $S_0$ and since $\mathcal{S}_0$ is stable under the action of $G$, there exist two subtrees $T, T' \in \mathcal{S}_0$ such that $U = \text{Fix}_G(T)$ and $V = \text{Fix}_G(T')$. There are two cases. Either, $T' \subseteq T$ and there exists a subtree $P \in \mathcal{S}_0$ such that $T' \subseteq P \subseteq T$ and $\text{Fix}_G(R) \in S_0[l-1]$. In that case

$$\text{Fix}_G(T) \subseteq \text{Fix}_G(P) \subseteq \text{Fix}_G(T') \subseteq \text{Fix}_G(T') \text{Fix}_G(T)$$

and the result follows. Or else, $T' \not\subseteq T$ and since $l' < l$, there exists a subtree $P \in \mathcal{S}_0$ such that $T' \subseteq P \not= T$ and $\text{Fix}_G(P) \in S_0[l]$. In particular, Proposition 4.6 ensures the existence of a subgroup $W \in S_0[l-1]$ such that $U \subseteq W \subseteq \text{Fix}_G(P)U \subseteq \text{Fix}_G(T')U$. This proves the first condition.

Next, we need to prove that $N_G(U, V) = \{g \in G | g^{-1}V g \subseteq U \}$ is compact for every $V$ in the conjugacy class of an element of $S_0$. Just as before, notice that there exists a $T' \in \mathcal{S}_0$ such that $V = \text{Fix}_G(T')$. Since $G$ satisfies the hypothesis $H_0$, notice that

$$N_G(U, V) = \{g \in G | g^{-1}V g \subseteq U \} = \{g \in G | g^{-1}\text{Fix}_G(T')g \subseteq \text{Fix}_G(T)\} = \{g \in G | \text{Fix}_G(g^{-1}T') \subseteq \text{Fix}_G(T)\} = \{g \in G | gT \subseteq T'\}.$$  

In particular, since both $T$ and $T'$ are finite subtrees of $T$, $N_G(U, V)$ is a compact subset of $G$ which proves the second condition.

Finally, we need to prove for every $W$ in the conjugacy class of an element of $S_0[l-1]$ with $U \subseteq W$ that

$$W \subseteq N_G(U, U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$  

For the same reasons as before, there exists $R \in \mathcal{S}_0$ such that $W = \text{Fix}_G(R)$. Furthermore, since $U \subseteq W$ and since $G$ satisfies the hypothesis $H_0$, notice that $R \subseteq T$. Moreover, since $\text{Fix}_G(R)$ has depth $l-1$, notice that $R$ contains every interior vertex of $T$. Since $G$ is unimodular and satisfies the hypothesis $H_0$, this implies that

$$\text{Fix}_G(R) \subseteq \{h \in G | hT \subseteq T\} = \{h \in G | \text{Fix}_G(T) \subseteq \text{Fix}_G(hT)\} = \{h \in G | h^{-1}\text{Fix}_G(T)h \subseteq \text{Fix}_G(T)\} = N_G(U, U)$$

which proves the third condition.  

\[\square\]
4.4 Description of cuspidal representations

The purpose of this section is to give a description of the cuspidal representations of $G^+_d(Y_0, Y_1)$. This is done by Theorem 4.11 below but requires some preliminaries. We refer to [Sem21] for proofs and details about the formalism.

Let $G$ be a non-discrete unimodular totally disconnected locally compact group $G$ and let $S$ be a generic filtration of $G$ factorizing at depth $l$. Then, for every irreducible representation $\pi$ of $G$ at depth $l$, [Sem21, Theorem A] ensures the existence of a unique conjugacy class $C_\pi \in F_S = \{C(U) | U \in S\}$ at height $l$ such that $\pi$ admits non-zero $U$-invariant vectors for any $U \in C_\pi$. The conjugacy class $C_\pi$ is called the seed of $\pi$. We define the group of automorphisms $\text{Aut}_G(C)$ of the seed $C$ as the quotient $N_G(U)/U$ corresponding to any $U \in C$. This group $\text{Aut}_G(C)$ is finite and does not depend up to isomorphism on our choice of representative $U \in C$. Now, let $p_U : N_G(U) \to N_G(U)/U$ denote the quotient map, let

$$\tilde{\mathcal{H}}_S(U) = \{W \mid \exists g \in G \text{ s.t. } gWg^{-1} \in S[l-1] \text{ and } U \subseteq W\},$$

and set

$$\mathcal{H}_S(C) = \{p_U(W) | W \in \tilde{\mathcal{H}}_S(U)\}.$$  

Notice that $\mathcal{H}_S(C)$ does not depend on our choice of representative $U \in C$.

**Definition 4.10.** An irreducible representation $\omega$ of $\text{Aut}_G(C)$ is called $S$-standard if it has no non-zero $H$-invariant vector for any $H \in \mathcal{H}_S(C)$.

The importance of this notion is given by [Sem21, Theorem A] which ensures that the irreducible representations of $G$ at depth $l$ with seed $C$ are obtained from the $S$-standard representations of $\text{Aut}_G(C)$ if $S$ factorizes at depth $l$. To be more precise, we recall that every irreducible representation $\omega$ of $\text{Aut}_G(C) \cong N_G(U)/U$ can be lifted to an irreducible representation $\omega \circ p_U$ of $N_G(U)$ acting trivially on $U$ and with representation space $\mathcal{H}_\omega$. The lifted representation can then be induced to $G$. The resulting representation

$$T(U, \omega) = \text{Ind}^G_{N_G(U)}(\omega \circ p_U)$$

is an irreducible representation of $G$ with seed $C(U)$. Conversely, if $\pi$ is an irreducible representation of $G$ with seed $C$, notice that $\mathcal{H}_\pi(U)$ is a non-zero $N_G(U)$-invariant subspace of $\mathcal{H}_\pi$ for every $U \in C$. In particular, the restriction $(\pi|_{N_G(U)}, \mathcal{H}_\pi(U))$ is a representation of $N_G(U)$ whose restriction to $U$ is trivial. This representation passes to the quotient group $N_G(U)/U$ and defines an $S$-standard representation $\omega_\pi$ of $\text{Aut}_G(C)$.
We now come back to the case we are interested in this paper. Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$, let $V(T) = V_0 \sqcup V_1$ be the associated bipartition and let

$$\mathcal{F}_0 = \{B_T(v, r) | v \in V(T), r \geq 1 \} \cup \{B_T(e, r) | e \in E(T), r \geq 0 \}.$$ 

Let $i$ be a legal coloring of $T$, let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets and consider the group $G^+_i(Y_0, Y_1)$. For shortening of the formulation, let $G = G^+_i(Y_0, Y_1)$. We have shown in Sections 4.2 and 4.3 that

$$S_0 = \{\text{Fix}_G(T) | T \in \mathcal{F}_0 \}.$$ 

is a generic filtration of $G^+_i(Y_0, Y_1)$ that factorizes at all depth $l \geq 1$. In particular, [Sem21, Theorem A] provides a bijective correspondence between irreducible representations of $G^+_i(Y_0, Y_1)$ at depth $l$ with seed $C \in \mathcal{F}_{S_0}$ and the $S_0$-standard representations of $\text{Aut}_G(C)$. We start by identifying those seeds and show that the cuspidal representations of $G$ are precisely the irreducible representations of $G$ at depth $l \geq 1$ with respect to $S_0$. In light of Lemma 4.4 we consider the partition $\mathcal{F}_0 = \bigsqcup_{l \in \mathbb{N}} \mathcal{F}_0[l]$ where:

- $\mathcal{F}_0[l] = \{B_T(e, \frac{l}{2}) | e \in E(T) \}$ if $l$ is even.
- $\mathcal{F}_0[l] = \{B_T(v, \frac{l+1}{2}) | v \in V(T) \}$ if $l$ is odd.

Notice that for all $l \in \mathbb{N}$, $\mathcal{F}_0[l]$ is stable under the action of $G$. Furthermore, notice if $l$ is even or if $l$ is odd and $G$ is transitive on the vertices that the set $\mathcal{F}_0[l]$ consists of a single $G$-orbit. On the other hand, if $l$ is odd and $G$ has two orbits of vertices notice that the set $\mathcal{F}_0[l]$ consists of two $G$-orbits namely $\{B_T(v, \frac{l+1}{2}) | v \in V_0 \}$ and $\{B_T(v, \frac{l+1}{2}) | v \in V_1 \}$. In particular, in light of Lemma 4.5, there are either one or two elements of $\mathcal{F}_{S_0} = \{C(U) | U \in S_0 \}$ at height $l$ and each such element is of the form

$$C = \{\text{Fix}_G(T) | T \in \mathcal{O} \}$$

where $\mathcal{O}$ is a $G$-orbit of $\mathcal{F}_0[l]$. We deduce easily that the irreducible representations of $G^+_i(Y_0, Y_1)$ at depth $l \geq 1$ with respect to $S_0$ are the cuspidal representations of $G$. Indeed, $\pi$ is an irreducible representation at depth $l \geq 1$ with respect to $S_0$ if and only if $\pi$ does not admit a non-zero $V$-invariant vector for any $V$ in a conjugacy class $C$ at depth 0 that is for any $V \in \{\text{Fix}_G(e) | e \in E(T) \}$. Now, let $\pi$ be a cuspidal representations of $G$, let $C_\pi \in \mathcal{F}_{S_0}$ be the seed of $\pi$, let $U \in C_\pi$ and let $T \in \mathcal{F}_0$ be such that $U = \text{Fix}_G(T)$. Since $S_0$ factorizes at all depth $l \geq 1$, [Sem21, Theorem A] ensures that $\pi$ is induced from an irreducible representation of $N_G(U)$ that
passes to the quotient $\text{Aut}_G(C_\pi) \cong N_G(U)/U$. Furthermore, since $G$ satisfies the hypothesis $H_0$, notice that

$$N_G(U) = \{ g \in G | gUg^{-1} = U \} = \{ g \in G | g\text{Fix}_G(T)g^{-1} = \text{Fix}_G(T) \} = \{ g \in G | \text{Fix}_G(gT) = \text{Fix}_G(T) \} = \{ g \in G | gT = T \} = \text{Stab}_G(T)$$

is exactly the stabilizer of $T$ in $G^+(Y_0,Y_1)$. In particular, $\text{Aut}_G(C_\pi)$ can be identified with the automorphism group of $T$ obtained by restricting the action of $\text{Stab}_G(T)$ to $T$. Moreover, since $G$ satisfies the hypothesis $H_0$ notice that

$$\tilde{H}_S(U) = \{ \text{Fix}_G(R) | R \in T_0, R \subseteq T \text{ and } R \text{ is maximal for this property} \}$$

is the set of fixators (in $\text{Aut}_G(C_\pi)$) of subtrees $R \in T_0$ satisfying $R \subseteq T$ and which are maximal for this property. In particular, the $S_0$-standard representations of $\text{Aut}_G(C_\pi)$ are the irreducible representations of the group of automorphisms of $T$ obtained by restricting the action of $\text{Stab}_G(T)$ and which do not admit any non-zero invariant vector for the fixator of any subtree $R$ of $T$ which belongs to $\mathfrak{T}_0$ and is maximal for this property. The above discussion together with [Sem21, Theorem A] leads to following description of the cuspidal representations of $G^+(Y_0,Y_1)$.

**Theorem 4.11.** Let $T$ be a $(d_0,d_1)$-semi-regular tree with $d_0,d_1 \geq 4$, let $i$ be a legal coloring of $T$, let $Y_0,Y_1 \subseteq \mathbb{N}$ be two finite subsets, let $G = G^+_{(i)}(Y_0,Y_1)$, consider the generic filtration $S_0$ of $G$ (defined in Section 4.2) and let us use the above notations. Then, the cuspidal representations of $G$ are exactly the irreducible representations at depth $l \geq 1$ with respect to $S_0$. Furthermore, if $\pi$ is a cuspidal representation at depth $l$ we have that:

1. $\pi$ has no non-zero $\text{Fix}_G(R)$-invariant vector for any $R \in \bigsqcup_{r<l} \mathfrak{T}_0[r]$.

2. There exists a unique conjugacy class $C_\pi \in \mathcal{F}_{S_0}$ at height $l$ such that $\pi$ admits a non-zero $U$-invariant vector for any (hence for all) $U \in C_\pi$. Equivalently, there exists a unique $G$-orbit $\mathcal{O}$ of $\mathfrak{T}_0[l]$ such that $\pi$ admits a non-zero $\text{Fix}_G(T)$-invariant vector for any (hence for all) $T \in \mathcal{O}$. Furthermore, $\mathcal{O}$ is the only orbit of $\mathfrak{T}_0$ under the action of $G$ such that $C_\pi = \{ \text{Fix}_G(T) | T \in \mathcal{O} \}$. 

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• If $O$ is the unique $G$-orbit of $\Sigma_0[l]$ corresponding to $\pi$ and if $T \in O$, $\pi$ admits a non-zero diagonal matrix coefficient supported in $\text{Stab}_G(T)$. In particular, $\pi$ is square-integrable its equivalence class is isolated in the unitary dual $\hat{G}$ for the Fell topology.

Furthermore for every $C \in F_{S_0}$ at height $l \geq 1$ with corresponding $G$-orbit $O$ in $\Sigma_0[l]$ that is $C = \{\text{Fix}_G(T)|T \in O\}$, there exists a bijective correspondence between the equivalence classes of irreducible representations of $G$ with seed $C$ and the equivalence classes of $S_0$-standard representations of $\text{Aut}_G(C)$. More precisely, for every $T \in O$ the following holds:

1. If $\pi$ is a cuspidal representation of $G$ with seed $C$, $(\omega_{\pi}, \mathcal{H}^{\text{Fix}_G(T)}_{\pi})$ is an $S_0$-standard representation of $\text{Aut}_G(C)$ such that

$$\pi \cong T(\text{Fix}_G(T), \omega_{\pi}) = \text{Ind}_{\text{Stab}_G(T)}^G((\omega_{\pi} \circ \text{Fix}_G(T))).$$

2. If $\omega$ is an $S_0$-standard representation of $\text{Aut}_G(C)$, the representation $T(\text{Fix}_G(T), \omega)$ is a cuspidal of $G$ with seed in $C$.

Furthermore, if $\omega_1$ and $\omega_2$ are $S_0$-standard representations of $\text{Aut}_G(C)$, we have that $T(\text{Fix}_G(T), \omega_1) \cong T(\text{Fix}_G(T), \omega_2)$ if and only if $\omega_1 \cong \omega_2$. In particular, the above two constructions are inverse of one an other.

### 4.5 Existence of cuspidal representations

Just as in the above sections, let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$, let $V(T) = V_0 \sqcup V_1$ be the associated bipartition and let

$$\Sigma_0 = \{B_T(v, r)|v \in V(T), r \geq 1\} \sqcup \{B_T(e, r)|e \in E(T), r \geq 0\}.$$ 

Let $i$ be a legal coloring of $T$, let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets and consider the group $G^+_{(i)}(Y_0, Y_1)$. For shortening of the formulation and when it leads to no confusion we let $G = G^+_{(i)}(Y_0, Y_1)$. We have shown in Sections 4.2 and 4.3 that

$$S_0 = \{\text{Fix}_G(T)|T \in \Sigma_0\}.$$ 

is a generic filtration of $G^+_{(i)}(Y_0, Y_1)$ that factorizes at all depth $l \geq 1$. In particular, [Sem21, Theorem A] provided a bijective correspondence between the irreducible representations of $G^+_{(i)}(Y_0, Y_1)$ at depth $l$ with seed $C \in F_{S_0}$ and the $S_0$-standard representations of $\text{Aut}_G(C)$. This lead to a description of the cuspidal representations of $G^+_{(i)}(Y_0, Y_1)$ see Theorem 4.11. On the other hand, none of those results yet ensures the existence of a cuspidal representation of $G^+_{(i)}(Y_0, Y_1)$. The purpose of this section is to prove the
existence of a cuspidal representation with seed $C$ for each conjugacy class $C \in \mathcal{F}_{S_0}$ at height $l \geq 1$. From the description of cuspidal representations of $G^+_0(Y_0, Y_1)$ provided by Theorem 4.11, it is equivalent to prove the following theorem.

**Theorem 4.12.** Let $G = G^+_0(Y_0, Y_1)$ and let $C \in \mathcal{F}_{S_0}$ be a conjugacy class at height $l \geq 1$. Then, there exists a $S_0$-standard representations of $\text{Aut}_G(C)$.

The proof of this theorem is gathered in the few results below. We start by recalling a result from [Sem21].

**Proposition 4.13** ([Sem21, Proposition 2.29]). Let $T$ be a locally finite tree, let $G \leq \text{Aut}(T)$ be a closed subgroup, let $\mathcal{T}$ be a finite subtree of $T$ and let $\{T_1, T_2, ..., T_s\}$ be a set of distinct finite subtrees of $T$ contained in $\mathcal{T}$ such that $T_i \cup T_j = \mathcal{T}$ for every $i \neq j$. Suppose that $\text{Stab}_G(\mathcal{T})$ acts by permutation on the set $\{T_1, T_2, ..., T_s\}$ and that $\text{Fix}_G(\mathcal{T}) \subseteq \text{Fix}_G(T_i) \subseteq \text{Stab}_G(\mathcal{T})$. Then, there exists an irreducible representation of $\text{Stab}_G(\mathcal{T})/\text{Fix}_G(\mathcal{T})$ without non-zero $\text{Fix}_G(\mathcal{T})/\text{Fix}_G(\mathcal{T})$-invariant vector for every $i = 1, ..., s$.

The following proposition ensures the existence of $S_0$-standard representation of $\text{Aut}_G(C)$ for all $C \in \mathcal{F}_{S_0}$ with even height $l \geq 1$.

**Proposition 4.14.** Let $G = G^+_0(Y_0, Y_1)$ and let $C \in \mathcal{F}_{S_0}$ be a conjugacy class at even height $l \geq 1$. Then, there exists a $S_0$-standard representation of $\text{Aut}_G(C)$.

*Proof.* Since $l$ is even, Lemma 4.4 ensures the existence of an edge $e \in E(T)$ and an integer $r \geq 1$ such that $B_T(e, r) \in \mathcal{F}_0$ and $C = C(\text{Fix}_G(B_T(e, r)))$. For shortening of the formulation, we let $\mathcal{T} = B_T(e, r)$. As observed in Section 4.4 we have that $\text{Aut}_G(C) \cong \text{Stab}_G(\mathcal{T})/\text{Fix}_G(\mathcal{T})$ and

$$\mathcal{H}_{S_0}(\text{Fix}_G(\mathcal{T})) = \left\{ \text{Fix}_G(\mathcal{R})/\text{Fix}_G(\mathcal{T}) | \mathcal{R} \subseteq \mathcal{T} \right\}$$

and $\mathcal{R}$ is maximal for this property

$$= \left\{ \text{Fix}_G(B_T(v, r))/\text{Fix}_G(B_T(e, r)) | v \in e \right\}.$$

Let $v_0, v_1$ be the two vertices of $e$, set $T_i = B_T(v_i, r)$ and notice that $T_0 \cup T_1 = \mathcal{T}$. Moreover, notice that $\text{Stab}_G(\mathcal{T}) = \{ g \in G | ge = e \}$ acts by permutations on the set $\{T_0, T_1\}$. Furthermore, since $G$ satisfies the hypothesis $H_0$ notice that $\text{Fix}_G(\mathcal{T}) \subseteq \text{Fix}_G(T_i) \subseteq \text{Stab}_G(\mathcal{T})$. In particular, the hypotheses of Proposition 4.13 are satisfied and the result follows. \hfill $\square$

A similar reasoning leads to a proof of the existence of $S_0$-standard representation of $\text{Aut}_G(C)$ for all $C \in \mathcal{F}_{S_0}$ with odd height $l > 1$. 

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Lemma 4.15. Let $G = G_{(0)}^+(Y_0, Y_1)$ and let $C \in \mathcal{F}_{S_0}$ be a conjugacy class at odd height $l > 1$. Then, there exists a $S_0$-standard representation of $\text{Aut}_G(C)$.

Proof. Since $l$ is even, Lemma 4.4 ensures the existence of a vertex $v \in V(T)$ and an integer $r \geq 1$ such that $B_T(v, r + 1) \in \mathcal{F}_0$ and $C = C(B_T(v, r + 1))$. For shortening of the formulation, we let $T = B_T(v, r + 1)$ of $T$. As observed in Section 4.4 we have that $\text{Aut}_G(C) \cong \text{Stab}_G(T)/\text{Fix}_G(T)$ and

$$\mathcal{S}_{S_0}(\text{Fix}_G(T)) = \{ \text{Fix}_G(\mathcal{R})/\text{Fix}_G(T) | \mathcal{R} \in \mathcal{F}_0, \mathcal{R} \not\subseteq T \}$$

and $\mathcal{R}$ is maximal for this property.

Let $\{w_1, ..., w_d\}$ be the leaves of $B_T(v, 1)$. For every $i = 1, ..., d$ let $T_i = (B_T(v, 1) \setminus \{w_i\})^{(r-1)}$ where $\mathcal{R}^{(i)} = \{ v \in V(T) | \exists w \in \mathcal{R} \text{ s.t. } d_T(v, w) \leq t \}$. Notice that $T_i \cup T_j = T$. On the other hand, in our case

$$\text{Stab}_G(T) = \{ g \in G | gv = v \} = \text{Fix}_G(v)$$

and $\text{Fix}_G(v)$ acts by permutation on the set $\{T_1, ..., T_d\}$. Finally, notice that $v \in T_i$ and therefore that $\text{Fix}_G(T_i) \subseteq \text{Stab}_G(T)$ for all $i = 1, ..., d$. Furthermore, since $G$ contains $G_{(0)}^+(\{0\}, \{0\})$, notice that $\text{Fix}_G(T) \not\subseteq \text{Fix}_G(T) \not\subseteq \text{Stab}_G(T)$. In particular, [Sem21, Proposition 2.27] ensures the existence of an irreducible representation of $\text{Aut}_G(C)$ without non-zero $\text{Fix}_G(T_i)/\text{Fix}_G(T)$-invariant vectors. The result follows from the fact that for every edge $e \in E(B_T(v, 1))$ there exists some $i \in \{1, ..., d\}$ such that $B_T(e, r) \subseteq T_i$ and hence $\text{Fix}_G(B_T(e, r))/\text{Fix}_G(T) \subseteq \text{Fix}_G(T_i)/\text{Fix}_G(T)$. 

The next lemma treats the remaining case $l = 1$ where Proposition 4.13 does not apply.

Lemma 4.16. Let $G = G_{(0)}^+(Y_0, Y_1)$ and let $C \in \mathcal{F}_{S_0}$ be a conjugacy class at height 1. Then, there exists a $S_0$-standard representation of $\text{Aut}_G(C)$.

Proof. Lemma 4.4 ensures the existence of a vertex $v \in V(T)$ such that $C = C(\text{Fix}_G(B_T(v, 1)))$. For shortening of the formulation, we let $T = B_T(v, r)$. We recall as observed in Section 4.4 that $\text{Aut}_G(C) \cong \text{Stab}_G(T)/\text{Fix}_G(T)$ where $\text{Stab}_G(T) = \{ g \in G | gT \subseteq T \} = \{ g \in G | gv = v \} = \text{Fix}_G(v)$. In particular, $\text{Aut}_G(C)$ can be realised as a the group of automorphisms of $B_T(v, 1)$ obtained by restricting the action of $\text{Fix}_G(v)$. Furthermore, we have that

$$\mathcal{S}_{S_0}(\text{Fix}_G(T)) = \{ \text{Fix}_G(\mathcal{R})/\text{Fix}_G(T) | \mathcal{R} \in \mathcal{F}_0, \mathcal{R} \not\subseteq T \}$$

and $\mathcal{R}$ is maximal for this property.

$$= \{ \text{Fix}_G(f)/\text{Fix}_G(B_T(v, 1)) | f \in E(B_T(v, 1)) \}.$$
Let $d$ be the degree of $v$ in $T$, let $X = E(B_T(v, 1))$ and let $e \in X$. Since $G$ is a closed subgroup of $\operatorname{Aut}(T)$ acting 2-transitively on the boundary $\partial T$ [BM00, Lemma 3.1.1] ensures that $G(v)$ is 2-transitive. In particular, $\operatorname{Aut}_G(C)$ is 2-transitive. Let $X = E(B_T(v, 1))$ and let $e \in X$. Since $G$ is a closed subgroup of $\operatorname{Aut}(T)$ acting 2-transitively on the boundary $\partial T$ [BM00, Lemma 3.1.1] ensures that $G(v)$ is 2-transitive. In particular, $\operatorname{Aut}_G(C)$ is 2-transitive and $X = E(B_T(v, 1))$ ensures the existence of an irreducible representation $\sigma$ of $\operatorname{Aut}_G(C)$ without non-zero $\operatorname{Fix}_{\operatorname{Aut}_G(C)}(e)$-invariant vector. Since $\operatorname{Fix}_G(v)$ is transitive on $E(B_T(v, 1))$, this representation does not admit a non-zero $\operatorname{Fix}_{\operatorname{Aut}_G(C)}(f)$-invariant vector for any $f \in E(B_T(v, 1))$. The lemma follows from the fact that $\operatorname{Fix}_{\operatorname{Aut}_G(C)}(f) = \operatorname{Fix}_G(f)/\operatorname{Fix}_G(B_T(v, 1))$. 

5 Simple Radu groups are CCR

Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 4$. Let $i$ be a legal coloring of $T$ and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. The purpose of this section is to exploit the classification of the irreducible representations of $G^+_i(Y_0, Y_1)$ obtained in Section 2 and Section 4.4 to prove that $G^+_i(Y_0, Y_1)$ is uniformly admissible and hence CCR.

We recall that a totally disconnected locally compact group $G$ is uniformly admissible if for every compact open subgroup $K$, there exists a positive integer $k_K$ such that $\dim(H^K_\pi) < k_K$ for all irreducible representation $\pi$ of $G$. In particular, uniformly admissible groups are CCR. The following classical result ensures that the spherical representations of $G$ are uniformly admissible.

**Theorem 5.1.** Let $T$ be a $(d_0, d_1)$-semi-regular tree with $d_0, d_1 \geq 3$, let $G \leq \operatorname{Aut}(T)$ be a closed non-compact subgroup acting transitively on the boundary of $T$ and let $v \in V(T)$. Then, for every integer $n \geq 1$ there exists a constant $k_n \in \mathbb{N}$ such that $\dim(H^K_\pi) < k_n$ for every spherical representation $\pi$ of $G$ admitting a non-zero $\operatorname{Fix}_G(v)$-invariant vector and where $K_n = \operatorname{Fix}_G(B_T(v, n))$.

**Proof.** Let $K = \operatorname{Fix}_G(v)$ and let $\mu$ be the Haar measure of $G$ renormalised in such a way that $\mu(K) = 1$. Theorems 2.2 and 2.3 ensure that $(G, K)$ is a Gelfand pair and we observe that $\dim(H^K_\pi) = 1$. Now, let $\xi$ be a unit vector of $H^K_\pi$ and let $\eta$ be a unit vector of $H^K_\pi$. Notice that

$$
\varphi_{\xi, \eta} : G \rightarrow \mathbb{C} : g \mapsto \langle \pi(g)\xi, \eta \rangle
$$

is $K$-right invariant and $K_n$-left invariant continuous function. On the other
hand, since $\dim(H^K) = 1$, notice for all $g, h \in G$ that

$$\int_K \varphi_{\xi,\eta}(gkh) \, d\mu(k) = \langle \int_K \pi(gkh)\xi, \eta \rangle$$

$$= \langle \pi(h)\xi, \int_K \pi(k^{-1})\pi(g^{-1})\eta \rangle$$

$$= \langle \pi(h)\xi, \alpha(\eta, g)\xi \rangle = \overline{\alpha(\eta, g)} \varphi_{\xi,\xi}(h)$$

for some $\alpha(\eta, g) \in \mathbb{C}$. However, $\varphi_{\xi,\xi}(1_G) = 1$ and hence $\alpha(\eta, g) = \varphi_{\xi,\eta}(g)$. This implies for all $g, h \in G$ that

$$\int_K \varphi_{\xi,\eta}(gkh) \, d\mu(k) = \varphi_{\xi,\eta}(g)\varphi_{\xi,\xi}(h). \quad (5.1)$$

Since $\varphi_{\xi,\eta}$ is $K$-right invariant and $K_n$-left invariant notice that it can be realised as a function $\phi : Gv \to \mathbb{C}$ on the orbit $Gv$ of $v$ in $V(T)$ that is constant on the $K_n$-orbits of $v$. On the other hand, since $K_n$ is an open subgroup of the compact group $K$ and since $K$, the index of $K_n$ in $K$ is finite. Since $K$ is transitive on the boundary of the tree, this implies that $K_n$ has finitely many orbit on $\partial T$. In particular, there exists an integer $N_n \geq \max\{2, n\}$ such that $\partial T(w, v)$ is contained in a single $K_n$-orbit for all $w \in \partial B_T(v, N_n)$ where $\partial T(w, v)$ is the set of ends of $T(w, v) = \{u \in V(T) \mid d_T(u, w) < d_T(u, v)\}$ which are not vertices. Now, let $t \in G$ be such that $d_T(v, tv) = 2$. Notice that equality $(5.1)$ ensures that the sum of values of $\phi$ on the vertices at distance 2 from $gv$ is equal to $\phi(gv)\varphi_{\xi,\xi}(t)$. In particular, this implies that $\phi$ is determined entirely by the values it takes in $B_T(v, N_n)$. Hence, the space $L_n$ of function $\varphi : G \to \mathbb{C}$ which are $K$-right invariant, $K_n$-left invariant and satisfy the equality $(5.1)$ has finite dimension bounded by the cardinality $k_n$ of $B_T(v, N_n)$. Furthermore, since $\pi$ is irreducible notice that $\xi$ is cyclic and therefore that the linear map $\Psi_n : H^K \to L_n : \eta \to \varphi_{\xi,\eta}$ is injective. This proves as desired that $\dim(H^K) \leq \dim(L_n) \leq k_n < +\infty$. \hfill $\square$

This allows one to prove what we announced at the beginning of the section.

**Theorem 5.2.** $G^+(1) = \pi \cdot \pi' \cdot \pi'' \cdot \pi''' \cdot \pi''''$ is uniformly admissible, hence CCR.

**Proof.** For shortening of the formulation, let $G = G^+(1) = \pi \cdot \pi' \cdot \pi'' \cdot \pi''' \cdot \pi''''$. Let $K \leq G$ be a compact open subgroup, let $v \in V(T)$ and let $K_n = \text{Fix}_G(B_T(v, n))$ for every $n \in \mathbb{N}$. Since $(K_n)_{n \in \mathbb{N}}$ is a basis of neighbourhood of the identity there exists some $N \in \mathbb{N}$ such that $K_N \subseteq K$. In particular, Theorem 5.1 ensures the existence of a positive integer $k^1_K$ such that $\dim(H^K) \leq k^1_K$ for
every spherical representation $\pi$ of $G$. On the other hand, [HC70, Corollary of Theorem 2] ensures the existence of a positive integer $k^2_K$ such that $\dim(H^K) \leq k^2_K$ for every irreducible square-integrable representation $\pi$ of $G$ and Theorems 2.4 and 4.11 ensures that the special and cuspidal representations are square-integrable. The result therefore follows from the fact that each irreducible $\pi$ of $G$ is either spherical, special or cuspidal. $\square$
Appendix

A Irreducible representations of a group and subgroups of index 2

The purpose of this appendix is to explicit the relations between the irreducible unitary representations of a locally compact group \(G\) and the irreducible representations of its closed subgroups \(H \leq G\) of index 2. Among other things, Theorem A.2 below explicit the correspondence between those representations. Furthermore, when \(G\) is a totally disconnected locally compact group, we show that \(G\) is uniformly admissible if and only if \(H\) is uniformly admissible see Lemma A.8.

The relevance of this appendix is given by Theorem 3.5 which ensures that every Radu group \(G\) lies in a finite chain \(H_n \geq \ldots \geq H_0\) with \(n \in \{0, 1, 2, 3\}\) such that \(H_n = G\), \([H_t : H_{t-1}] = 2\) for all \(t\) and where \(H_0\) is conjugate in \(\text{Aut}(T)^+\) to \(G^+_{(i)}(Y_0, Y_1)\) if \(T\) is a \((d_0, d_1)\)-semi-regular tree with \(d_0, d_1 \geq 6\). As a direct consequence, we therefore obtain a description of the irreducible representations of those groups and observe that they are uniformly admissible. To be more precise, the spherical and special representations of any Radu group \(G\) are classified by Section 2 and a description of the cuspidal representations of those groups can be obtained from the description of cuspidal representations of \(G^+_{(i)}(Y_0, Y_1)\) given in Sections 4 by applying Theorem A.2 \(n\) times (where \(n\) is as above).

A.1 Preliminaries

Let \(G\) be a locally compact group and let \(H \leq G\) be a closed normal subgroup of finite index (in particular \(H\) is open in \(G\)). The purpose of this section is to recall three operations that can be applied either to the representations of \(G\) or to those of \(H\).

We start with the conjugation of representations. For every \(g \in G\), we denote by 
\[ c(g) : G \to G : h \mapsto ghg^{-1} \]
the conjugation map and for every representation \(\pi\) of \(H\) we define the morphism \(\pi^g = \pi \circ c(g)\). Since \(H\) is normal in \(G\), notice that \(\pi^g\) is a well defined representation of \(H\) on the Hilbert space \(\mathcal{H}_\pi\). This representation is called the conjugate representation of \(\pi\) by \(g\). Furthermore, notice that the conjugate representation \(\pi^g\) depends up to equivalence only on the coset \(gH\).
and that $\pi^g$ is irreducible if and only if $\pi$ is irreducible. In particular, the action by conjugation of $G$ on $\hat{H}$ pass to the quotient $G/H$.

Now, let $G'$ be an other topological group, let $\phi : G \to G'$ be a continuous group homomorphism, let $\pi$ be a representations of $G$ and let $\chi$ be a unitary character of $G'$. We define the twisted representation $\pi^\chi$ as the representation of $G$ on $\mathcal{H}_\pi$ given by

$$\pi^\chi(g) = \chi(\phi(g))\pi(g) \quad \forall g \in G.$$ 

Notice that this representation is still continuous and unitary since $\chi$, $\pi$ and $\phi$ are continuous group homomorphism and since $\chi$ and $\pi$ are unitary. Furthermore, notice that $\pi^\chi \cong \pi$ if $\chi$ is the trivial representation of $G'$.

**Lemma A.1.** $\pi^\chi$ is irreducible if and only if $\pi$ is irreducible.

**Proof.** Since $\chi(g)$ is a unitary complex number for every $g \in G$, notice that, for every $\xi \in \mathcal{H}_\pi$, the subspace of $\mathcal{H}_\pi$ spanned by $\{\pi(g)\xi \mid g \in G\}$ is the same than the subspace spanned by $\{\chi(g)\pi(g)\xi \mid g \in G\} = \{\pi^\chi(g)\xi \mid g \in G\}$. The result therefore follows from the fact that a representation is irreducible if and only if every non-zero vector is cyclic.

Finally, we recall the notion of **induction.** Since most of the complexity vanishes when $H$ is an open subgroup of $G$ (because the quotient space $G/H$ is discrete) and since it is the only set up encountered in this notes, we will work under this hypothesis. We refer to [KT13, Chapters 2.1 and 2.2] for details. Let $G$ be a locally compact group, let $H \leq G$ be an open subgroup and let $\sigma$ be a representation of $H$. The induced representation $\text{Ind}^G_H(\sigma)$ is a representation of $G$ with representation space given by

$$\text{Ind}^G_H(\mathcal{H}_\sigma) = \left\{ \phi : G \to \mathcal{H}_\sigma \left| \phi(gh) = \sigma(h^{-1})\phi(g), \sum_{gh \in G/H} \langle \phi(g), \phi(g) \rangle < +\infty \right. \right\}.$$ 

For $\psi, \phi \in \text{Ind}^G_H(\mathcal{H}_\sigma)$, we let

$$\langle \psi, \phi \rangle_{\text{Ind}^G_H(\mathcal{H}_\sigma)} = \sum_{gh \in G/H} \langle \psi(g), \phi(g) \rangle.$$ 

Equipped with this inner product, $\text{Ind}^G_H(\mathcal{H}_\sigma)$ is a separable complex Hilbert space. The induced representation $\text{Ind}^G_H(\sigma)$ is the representation of $G$ on $\text{Ind}^G_H(\mathcal{H}_\sigma)$ defined by

$$[\text{Ind}^G_H(\sigma)(h)] \phi(g) = \phi(h^{-1}g) \quad \forall \phi \in \text{Ind}^G_H(\mathcal{H}_\sigma) \text{ and } \forall g, h \in G.$$
A.2 THE EXPLICIT CORRESPONDENCE

We come back to the context we are interested in. Let \( G \) be a locally compact group and let \( H \leq G \) be a closed subgroup of index 2 in \( G \). In particular, \( H \) is an normal open subgroup of \( G \) and the quotient \( G' = G/H \) is isomorphic to the cyclic group of order two. We let \( \tau \) denote the only irreducible non-trivial representation of \( G/H \). Let \( t \in G - H \), let \( \phi : G \to G/H \) be the canonical projection on the quotient and, for every representation \( \pi \) of \( G \), let \( \pi^{\tau} \) be the twisted representation of \( G \) as defined in the Section A.1. The purpose of this section is to prove the following theorem.

**Theorem A.2.** For every irreducible representation \( \pi \) of \( G \) we have that:

- \( \pi \not\cong \pi^{\tau} \) if and only if \( \text{Res}^G_H(\pi) \) is an irreducible representation of \( H \) and in that case \( \text{Res}^G_H(\pi) \cong \text{Res}^G_H(\pi)^t \).

- \( \pi \cong \pi^{\tau} \) if and only if \( \text{Res}^G_H(\pi) \cong \sigma \oplus \sigma^t \) for some irreducible representation \( \sigma \) of \( H \) and in that case \( \sigma \not\cong \sigma^t \).

For every irreducible representation \( \sigma \) of \( H \), we have that:

- \( \sigma \not\cong \sigma^t \) if and only if \( \text{Ind}^G_H(\sigma) \) is an irreducible representation of \( G \) and in that case \( \text{Ind}^G_H(\sigma) \cong \text{Ind}^G_H(\sigma)^\tau \).

- \( \sigma \cong \sigma^t \) if and only if \( \text{Ind}^G_H(\sigma) \cong \pi \oplus \pi^{\tau} \) for some irreducible representation \( \pi \) of \( G \) and in that case \( \pi \not\cong \pi^{\tau} \).

Furthermore:

1. Every irreducible representation \( \pi \) of \( G \) satisfies \( \pi \leq \text{Ind}^G_H(\sigma) \) for some irreducible representation \( \sigma \) of \( H \).

2. Every irreducible representation \( \sigma \) of \( H \) satisfies \( \sigma \leq \text{Res}^G_H(\pi) \) for some irreducible representation \( \pi \) of \( G \).

The proof of this theorem is gathered in the following few results.

**Lemma A.3.** Let \( \pi \) be an irreducible representation of \( G \). Then exactly one of the following happens:

- \( \text{Res}^G_H(\pi) \) is an irreducible representation of \( H \) and \( \text{Res}^G_H(\pi) \cong \text{Res}^G_H(\pi)^t \).

- \( \text{Res}^G_H(\pi) \cong \sigma \oplus \sigma^t \) for some irreducible representation \( \sigma \) of \( H \).
Lemma A.4. Let $\pi$ be an irreducible representation of $G$ such that $\pi \cong \pi^\tau$. Then, $\text{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$ for some irreducible representations $\sigma$ of $H$.

Proof. Lemma A.3 ensures that $\text{Res}_H^G(\pi)$ is either irreducible or split as desired. Suppose for a contradiction that $\sigma = \text{Res}_H^G(\pi)$ is irreducible and let
\( \mathcal{U} : \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi \) be the unitary operator intertwining \( \pi \) and \( \pi^r \). Notice that \( \pi \) and \( \sigma \) have the same representation space \( \mathcal{H}_\pi \). Furthermore, for every \( h \in H \), we have that \( \pi(h) = \pi^r(h) = \sigma(h) \). In particular, \( \mathcal{U} \) is a unitary operator that intertwines \( \sigma \) with itself. Since \( \sigma \) is irreducible this implies that \( \mathcal{U} \) is a scalar multiple of the identity. However, this is impossible since for every \( h \in H \) and every \( \xi \in \mathcal{H}_\pi \) we have

\[ \mathcal{U} \pi(h) \xi = \pi^r(h) \mathcal{U} \xi = -\pi(h) \mathcal{U} \xi. \]

We obtain as desired that \( \text{Res}_G^H(\pi) \cong \sigma \oplus \sigma^t \) for some irreducible representation \( \sigma \) of \( H \) when \( \pi \cong \pi^r \).

We now recall the weak version of Frobenius reciprocity that will be used for the rest of the proof of Theorem A.2.

**Theorem A.5** ([Mac76, Corollary 1 of Theorem 3.8]). Let \( G \) be a locally compact group and let \( H \leq G \) be a closed subgroup of \( G \). Then, for every representation \( \pi \) of \( G \) and every representation \( \sigma \) of \( H \) we have

\[ I(\text{Ind}_G^H(\sigma), \pi) \leq I(\sigma, \text{Res}_H^G(\pi)), \]

where \( I(\pi_1, \pi_2) \) is the dimension of the space of intertwining operators between the two representations \( \pi_1 \) and \( \pi_2 \). Furthermore, if the index of \( H \) in \( G \) is finite, this relation becomes an equality.

**Proposition A.6.** Let \( \sigma \) be an irreducible representation of \( H \). Then, we have

\[ \text{Res}_H^G(\text{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t. \]

Furthermore, the followings hold:

- \( \sigma \not\cong \sigma^t \) if and only if \( \text{Ind}_H^G(\sigma) \) is an irreducible representation of \( G \) and in that case \( \text{Ind}_H^G(\sigma) \cong \text{Ind}_H^G(\sigma)^r \).
- \( \sigma \cong \sigma^t \) if and only if \( \text{Ind}_H^G(\sigma) \cong \pi \oplus \pi^r \) for some irreducible representation \( \pi \) of \( G \).

**Proof.** We start by showing that \( \text{Res}_H^G(\text{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t \). We set

\[ \mathcal{L} = \{ \varphi \in \text{Ind}_H^G(\mathcal{H}_\sigma) | \text{supp}(\varphi) \subseteq H \} \]

and \( \mathcal{L}^t = \{ \varphi \in \text{Ind}_H^G(\mathcal{H}_\sigma) | \text{supp}(\varphi) \subseteq H^t \} \).

By definition of \( \text{Ind}_H^G(\mathcal{H}_\sigma) \) and since \( G = H \cup H^t \), it is clear that

\[ \text{Ind}_H^G(\mathcal{H}_\sigma) = \mathcal{L} \oplus \mathcal{L}^t. \]
Now, notice that $\mathcal{U} : \mathcal{L} \to \mathcal{H}_\sigma : \varphi \mapsto \varphi(1_G)$ is a unitary operator and that for every $h \in H$ and every $\varphi \in \mathcal{L}$ we have
\[
\sigma(h)\mathcal{U}\varphi = \sigma(h)\varphi(1_G) = \varphi(h^{-1}) = [\text{Ind}^G_H(\sigma)(h)]\varphi(1_G) = \mathcal{U}[\text{Ind}^G_H(\sigma)(h)]\varphi.
\]
In particular, this proves that $(\text{Res}^G_H(\text{Ind}^G_H(\sigma)), \mathcal{L}) \cong (\sigma, \mathcal{H}_\sigma)$. Similarly, notice that $\mathcal{U}' : \mathcal{L}' \to \mathcal{H}_\sigma : \varphi \mapsto \varphi(t^{-1})$ is a unitary operator and that for every $h \in H$ and every $\varphi \in \mathcal{L}'$ we have
\[
\sigma'(h)\mathcal{U}'\varphi = \sigma(tht^{-1})\varphi(t^{-1}) = \varphi(t^{-1}th^{-1}t^{-1}) = \varphi(h^{-1}t^{-1}) = [\text{Ind}^G_H(\sigma)(h)]\varphi(t^{-1}) = \mathcal{U}'[\text{Ind}^G_H(\sigma)(h)]\varphi.
\]
This proves that $(\text{Res}^G_H(\text{Ind}^G_H(\sigma)), \mathcal{L}') \cong (\sigma', \mathcal{H}_{\sigma'})$ and we obtain as desired that $\text{Res}^G_H(\text{Ind}^G_H(\sigma)) \cong \sigma \oplus \sigma'$.

Since Theorem A.5 ensures that
\[
I(\text{Ind}^G_H(\sigma), \text{Ind}^G_H(\sigma)) = I(\sigma, \text{Res}^G_H(\text{Ind}^G_H(\sigma))
\]
this implies that $\text{Ind}^G_H(\sigma)$ is irreducible (that is $I(\text{Ind}^G_H(\sigma), \text{Ind}^G_H(\sigma)) = 1$) if and only if $\sigma \not\sim \sigma'$. Furthermore, in that case, Theorem A.5 ensures that
\[
I(\text{Ind}^G_H(\sigma), \text{Ind}^G_H(\sigma)^\tau) = I(\sigma, \text{Res}^G_H(\text{Ind}^G_H(\sigma)^\tau)) = I(\sigma, \text{Res}^G_H(\text{Ind}^G_H(\sigma))) = I(I(\text{Ind}^G_H(\sigma), \text{Ind}^G_H(\sigma)) = 1
\]
which proves that $\text{Ind}^G_H(\sigma) \cong \text{Ind}^G_H(\sigma)^\tau$ and settles the first case.

On the other hand, Theorem A.5 ensures that $\sigma \cong \sigma'$ if and only if $\text{Ind}^G_H(\sigma)$ is not irreducible. In that case, notice that $\text{Ind}^G_H(\mathcal{H}_\sigma)$ must split as a sum of two non-zero closed $G$-invariant subspaces $M$ and $M'$. On the other hand, since $\text{Res}^G_H(\text{Ind}^G_H(\sigma))$ splits as a sum of two irreducible representations of $H$, and since every $G$-invariant subspace is $H$-invariant, $M$ and $M'$ do not admit any proper invariant subspaces. This proves that $\text{Ind}^G_H(\sigma) \cong \pi \oplus \pi'$ for some irreducible representations $\pi$ and $\pi'$ of $G$. On the other hand, since $\text{Res}^G_H(\pi) = \text{Res}^G_H(\pi^\tau)$, Theorem A.5 ensures that
\[
I(\text{Ind}^G_H(\sigma), \pi) = I(\sigma, \text{Res}^G_H(\pi)) = I(\sigma, \text{Res}^G_H(\pi^\tau)) = I(\text{Ind}^G_H(\sigma), \pi^\tau)
\]
for every irreducible representation $\pi$ of $G$. In particular, if $\pi \not\cong \pi^\tau$, we obtain that $\text{Ind}^G_H(\sigma) \cong \pi \oplus \pi^\tau$. On the other hand, if $\pi \cong \pi^\tau$, notice from Lemma A.4 that $\text{Res}^G_H(\pi) \cong \sigma \oplus \sigma'$. Hence, since $\sigma \cong \sigma'$, Theorem A.5 implies that
\[
I(\text{Ind}^G_H(\sigma), \pi) = I(\sigma, \text{Res}^G_H(\pi)) = I(\sigma, \sigma' \oplus \sigma') > 1
\]
which proves that $\text{Ind}^G_H(\sigma) \cong \pi \oplus \pi \cong \pi \oplus \pi^\tau$. \qed
The first part of Theorem A.2 follows from Lemma A.3, Lemma A.4, Proposition A.6 and from the impossibility to have simultaneously that \( \pi \cong \pi^r \) and that \( \sigma \cong \sigma^t \). Indeed, if \( \pi \cong \pi^r \), Lemma A.4 ensures that \( \text{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t \). However, if \( \sigma \cong \sigma^t \), Proposition A.6 ensures that \( \text{Ind}_H^G(\sigma) \cong \pi \oplus \pi^r \). In particular, if those conditions were satisfied simultaneously one would obtain that

\[
\text{Res}_H^G \left( \text{Ind}_H^G(\sigma) \right) \cong \sigma \oplus \sigma^t \oplus \sigma \oplus \sigma^t \cong 4\sigma
\]

which impossible since Proposition A.6 ensures that

\[
\text{Res}_H^G(\text{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t \cong 2\sigma.
\]

The following result completes the proof of Theorem A.2.

**Lemma A.7.** Every irreducible representation \( \pi \) of \( G \) satisfies \( \pi \leq \text{Ind}_H^G(\sigma) \) for some irreducible representation \( \sigma \) of \( H \) and every irreducible representation \( \sigma \) of \( H \) satisfies \( \sigma \leq \text{Res}_H^G(\pi) \) for some irreducible representation \( \pi \) of \( G \).

**Proof.** Let \( \pi \) be an irreducible representation of \( G \) and let us show that \( \pi \leq \text{Ind}_H^G(\sigma) \) for some irreducible representation \( \sigma \) of \( H \). Notice from Theorem A.5 that

\[
I(\text{Ind}_H^G \left( \text{Res}_H^G(\pi) \right), \pi) = I(\text{Res}_H^G(\pi), \text{Res}_H^G(\pi)) \geq 1
\]

which proves that \( \pi \leq \text{Ind}_H^G \left( \text{Res}_H^G(\pi) \right) \). If \( \text{Res}_H^G(\pi) \) is irreducible, the result follows trivially. On the other hand, if \( \text{Res}_H^G(\pi) \) is not irreducible, Lemma A.3 ensures that \( \text{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t \) for some irreducible representation \( \sigma \) of \( H \). In particular, since \( \pi \leq \text{Ind}_H^G \left( \text{Res}_H^G(\pi) \right) \cong \text{Ind}_H^G(\sigma) \oplus \text{Ind}_H^G(\sigma^t) \) we obtain either that \( \pi \leq \text{Ind}_H^G(\sigma) \) or that \( \pi \leq \text{Ind}_H^G(\sigma^t) \).

Now, let \( \sigma \) be an irreducible representation of \( H \) and let us show that \( \sigma \leq \text{Res}_H^G(\pi) \) for some irreducible representation \( \pi \) of \( G \). Notice from Theorem A.5 that

\[
I(\sigma, \text{Res}_H^G \left( \text{Ind}_H^G(\sigma) \right)) = I(\text{Ind}_H^G(\sigma), \text{Ind}_H^G(\sigma)) \geq 1
\]

which proves that \( \sigma \leq \text{Res}_H^G \left( \text{Ind}_H^G(\sigma) \right) \). If \( \text{Ind}_H^G(\sigma) \) is irreducible, the result follows trivially. On the other hand, if \( \text{Ind}_H^G(\sigma) \) is not irreducible, Proposition A.6 ensures that \( \text{Ind}_H^G(\sigma) \cong \pi \oplus \pi^r \) for some irreducible representation \( \pi \) of \( G \). In particular, since \( \pi \leq \text{Ind}_H^G \left( \text{Res}_H^G(\pi) \right) \cong \text{Ind}_H^G(\sigma) \oplus \text{Ind}_H^G(\sigma^t) \) we obtain either that \( \pi \leq \text{Ind}_H^G(\sigma) \) or that \( \pi \leq \text{Ind}_H^G(\sigma^t) \).

Finally, as a consequence of Theorem 5.2 and of the correspondence provided by Theorem A.2, the following lemma ensures that every Radu group on a \((d_0, d_1)\)-semi-regular tree with \(d_0, d_1 \geq 6\) is uniformly admissible.
Lemma A.8. Let $G$ be a totally disconnected locally compact group and let $H$ be a closed subgroup of index 2. Then, $G$ is uniformly admissible if and only if $H$ is uniformly admissible.

Proof. Suppose that $G$ is uniformly admissible and let $K$ be a compact open subgroup of $H$. Since $H$ has index 2 in $G$, it is a clopen subgroup of $G$ which implies that $K$ is a compact open subgroup of $G$. Since $G$ is uniformly admissible, there exists a constant $k_K \in \mathbb{N}$ such that $\dim(\mathcal{H}_\pi^K) \leq k_K$ for every irreducible representation $\pi$ of $G$. Let $\sigma$ be an irreducible representation of $H$. Theorem A.2 ensures that $\text{Ind}_H^G(\sigma)$ is either irreducible or splits as a sum of two irreducible representations of $G$. On the other hand, notice from Theorem A.5 that

$$I(\sigma, \text{Res}_H^G(\text{Ind}_H^G(\sigma))) = I(\text{Ind}_H^G(\sigma), \text{Ind}_H^G(\sigma)) \geq 1$$

which implies that $\sigma \leq \text{Res}_H^G(\text{Ind}_H^G(\sigma))$. All together, this proves that

$$\dim(\mathcal{H}_\sigma^K) \leq \dim(\mathcal{H}_{\text{Ind}_H^G(\sigma)}^K) \leq 2k_K$$

and $H$ is uniformly admissible.

Suppose now that $H$ is uniformly admissible and let $K$ be a compact open subgroup of $G$. Since $H$ has index 2 in $G$, it is a clopen subgroup of $G$ and $K \cap H$ is a compact open subgroup of $H$. Since $H$ is uniformly admissible, this implies the existence of a constant $k_{K \cap H}$ such that $\dim(\mathcal{H}_\sigma) \leq k_{K \cap H}$ for every irreducible representation $\sigma$ of $H$. Furthermore, Theorem A.2 ensures that $\text{Res}_H^G(\pi)$ is either an irreducible representation of $G$ or splits as a direct sum of 2 irreducible representations of $H$. This implies that

$$\dim(\mathcal{H}_\pi^K) \leq \dim(\mathcal{H}_{\text{Res}_H^G(\pi)}^{K \cap H}) = \dim(\mathcal{H}_{\text{Res}_H^G(\pi)}^{K \cap H}) \leq 2k_{K \cap H}.$$ 

Hence, $G$ is uniformly admissible. \qed
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