Probing of quantum dissipative chaos by purity

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In this paper, the purity of quantum states is applied to probe chaotic dissipative dynamics. To achieve this goal, a comparative analysis of regular and chaotic regimes of nonlinear dissipative oscillator (NDO) is performed on the base of excitation number and the purity of oscillatory states. While the chaotic regime is identified in our semiclassical approach by means of strange attractors in Poincaré section and with the Lyapunov exponent, the state in the quantum regime is treated via the Wigner function. Specifically, interesting quantum purity effects that accompany the chaotic dynamics are elucidated in this paper for NDO system driven by either: (i) a time-modulated field, or (ii) a sequence of pulses with Gaussian time-dependent envelopes.

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I. INTRODUCTION

Nonlinear dissipative systems demonstrating chaotic behaviour in their dynamics are still the subject of much attention [1], [2], [3]. The early studies of dissipative chaotic systems date back to the works [4]. The investigations of quantum chaotic systems are distinctly connected with the quantum-classical correspondence problem in general, and with environment induced decoherence and dissipation in particular. It has been recognized [5] that decoherence has rather unique properties for systems whose classical analogs display chaos. Several methods have been proposed to determine whether a quantum dissipative system exhibits chaotic behaviour. At this point, we note that quite generally, chaos in classical conservative and dissipative systems with noise, has completely different properties. For example, strange attractors on Poincaré section can appear only in dissipative systems. The most successful approach that probes quantum dissipative chaos seems to be quantum tomographic methods based on the measurement of the Wigner function, which is a quantum quasi-distribution in phase-space. In this way, the connection between quantum and classical treatment of chaos can be realized by means of a comparison between strange attractors in the classical Poincaré section and the contour plots of the Wigner functions [6], [7], [8]. However, such manifestation of chaos seems to be hardly realized by experiments because Wigner function can only be obtained through data post-processing. On the other hand, alternative methods that probe quantum dissipative chaos involve considerations of entropic characteristics, analysis of statistics of excitation number [9], [10], and a method based on the fidelity decay [11].

II. PURITY AND MODELS OF NONLINEAR OSCILLATORS

The purity of the quantum states which is defined via the density matrix of the system as $\text{Tr}(\rho^2)$, is connected to the linear entropy $S_L$ in the following manner:

$$S_L = 1 - \text{Tr}(\rho^2).$$

(1)

Note that $S_L$ can be obtained from the von Neumann entropy

$$S = \text{Tr}(\rho \ln(\rho))$$

(2)

as a lower-order approximation.

For an ensemble of mixed states the density matrix

$$\rho = \sum P_{\psi_j} |\psi_j\rangle\langle\psi_j|,$$

(3)

where $P_{\psi_j}$ is the probability of occurrence of state $\psi_j$. In
this case, the purity takes the form

\[ P = Tr(\rho^2) = \sum_{l=0}^{\infty} P_{\psi_j}^2, \]  

(4)

In particular, for thermal light with a photon population \( P_{\psi_j} = \frac{\pi}{(n+1)^{\frac{3}{2}}} \), the purity can be calculated as

\[ P = \frac{1}{2\pi + 1}, \]  

(5)

where \( \pi \) is the mean number of thermal photons. From Eq. 5 it is evident that purity decreases as excitation number increases.

In this paper, we employ quantum purity as a tool to analyse quantum dynamics of NDO which allows us to determine whether or not the system has reached the chaotic regimes. It is easy to realize that in general the purity of an ensemble of oscillatory states strongly depend on the operational regimes of NDO. More specifically, an increase in the excitation number would raise the number of mixing oscillatory states, leading to a decrease in purity which is apparent from Eq. 5. In addition, diffusion and decoherence of oscillatory modes are also relevant to the level of purity. Thus, in the following, we analyse the purity using the density matrix of NDO \( \rho(t) \) for both the regular and chaotic regimes.

We consider a model of anharmonic oscillator driven by external field with time-modulated amplitude that is based on the following Hamiltonian in the rotating wave approximation:

\[ H = \hbar \Delta a^+ a + \hbar (a^+ a)^2 + \hbar (f(t)a^+ + f(t)^* a), \]  

(6)

where \( a^+ \) and \( a \) are the oscillatory creation and annihilation operators respectively, \( \chi \) is the nonlinearity strength, and \( \Delta = \omega_0 - \omega \) is the detuning between the mean frequency of the driving field and the frequency of the oscillator. In the case of a constant amplitude, i.e. \( f(t) = \Omega \), this Hamiltonian describes a nonlinear oscillator driven by a monochromatic force. In this paper, we shall consider two cases of driving force: (i) \( f(t) = f_0 + f_1 \exp(\delta t) \), where \( \delta \) is the frequency of modulation; and (ii)

\[ f(t) = \Omega \sum_{n=0}^{\infty} e^{-t-t_0-n\tau^2/2}. \]  

(7)

For the latter case, the time dependent interaction term is proportional to the amplitude of the driving field \( \Omega \), and consists of Gaussian pulses with duration \( T \) separated by time intervals \( \tau \).

The evolution of the system of interest is governed by the following master equation for the reduced density matrix of the oscillatory mode in the interaction picture:

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_{i=1,2} \left( L_i \rho L_i^+ - \frac{1}{2} L_i^+ L_i \rho - \frac{1}{2} \rho L_i^+ L_i \right), \]  

(8)

where \( L_1 = \sqrt{(N+1)\gamma} a \) and \( L_2 = \sqrt{N\gamma} a^+ \) are the Lindblad operators, \( \gamma \) is a dissipation rate, and \( N \) denotes the mean number of quanta of a heath bath. To study the pure quantum effects, we focus below on cases of very low reservoir temperatures which, however, still ought to be larger than the characteristic temperature \( T \gg T_{cr} = \hbar \gamma/k_B \).

In the semiclassical approach, the corresponding equation of motion for the dimensionless amplitude of oscillatory mode has the following form:

\[ \frac{d\alpha}{dt} = -i[\Delta + 2|\alpha|^2 \chi |\alpha| + i f(t) \Omega - \gamma \alpha]. \]  

(9)

This equation modifies the standard Duffing equation in the case of NDO with time-dependent coefficient. Note, that quantum effects in a NDO with a time-modulated driving force have been studied in the context of quantum stochastic resonance [12], quantum dissipative chaos [10, 11], quantum interference assisted by a bistability [12], and generation of superposition of Fock states [13].

These models seem experimentally feasible and can be realized in several experimental schemes. We mention nano-electromechanical systems and nano-optomechanical systems based on various nonlinear oscillators [13, 10], and superconducting devices based on the nonlinearity of the Josephson junction (JJ) [17, 18], that exhibits a wide variety of quantum oscillatory phenomena.

### III. PURITY AS AN INDICATOR OF CHAOS

In this section, we investigate the purity of quantum oscillatory states for various regimes of NDO. To achieve this goal, we perform detailed comparative analysis of purity for both the case of regular and chaotic dynamics. In general, the purity of an ensemble of oscillatory states strongly depends on the level of the excitation number. Indeed, a larger number of mixing oscillatory states would lead to a decrease of purity, which can also be seen from Eq. 5. Therefore, in our comparative analysis, we consider regimes of regular and chaotic dynamics with the same levels of oscillatory excitation number. We shall consider two schemes of nonlinear dissipative oscillator (NDO) in this analysis: a NDO driven by a continuously modulated pump field; and a NDO under the action of a periodic sequence of identical pulses with Gaussian envelope.

The time evolution of NDO driven by an external time-modulated coherent force field cannot be solved analytically, and hence suitable numerical methods have to be employed. We shall determine the excitation number and the Wigner functions of oscillatory mode numerically on the base of master equation by means of the quantum state diffusion method [20]. For the semiclassical approach, we shall calculate the Lyapunov exponent and the Poincaré section according to the framework of Eq. 9.
of points \((X,Y)\) have chosen dynamics. Note that to calculate the Poincaré section, we confirm that the regime is indeed chaotic, we plot the trajectories and the corresponding purity for the same parameters corresponding to the chaotic regime. In order to show that the lower purity is a result of chaotic dynamics, we present the purity dynamics for the parameters: \(\Delta/\gamma = -15, \chi/\gamma = 2, f_0/\gamma = 10.2, f_1/\gamma = 10.2, \delta/\gamma = 5\).

A. NDO under time-modulated force

Let us begin our study by considering a NDO under the action of a periodically modulated driving force, \(f(t) = f_0 + f_1 \exp(\delta t)\). As shown in our earlier analysis, for extended time scale exceeding the damping rate, the asymptotic dynamics of the system is regular in the limit of small and large values of the modulation frequency in comparison with the decay rate, and also when one of the perturbation forces \(f_0\) and \(f_1\) is much larger than the other. Furthermore, the dynamics of the system is chaotic if the parameters \(f_0\) and \(f_1\) are approximately equal to each other. In Figs. 1 and 2, the mean excitation number of the averaged quantum trajectories and the corresponding purity for the same parameters versus time interval are plotted for parameters corresponding to the chaotic regime. In order to confirm that the regime is indeed chaotic, we plot the classical Poincaré section in Fig. 1(c) which displays a strange attractor with fractal structure typical of chaotic dynamics. Note that to calculate the Poincaré section, we have chosen \(x_0\) and \(y_0\) as an arbitrary initial phase space point of the system at the time \(t_0\). We then define a constant phase map in the \((X,Y)\) plane by the sequence of points \((X_n,Y_n) = (X(t_n), Y(t_n))\) at \(t_n = t_0 + \frac{n}{\delta}\) \(n = 0, 1, 2, \ldots\). Demonstration of chaos in the quantum treatment is presented in Fig. 1(d) via the Wigner function. The Wigner function exhibits a helical structure with contour plots concentrating approximately around the attractor, reflecting the intrinsic presence of chaotic dynamics.

Next, we present results of the mean excitation number and the purity of two chaotic regimes in greater detail (see Figs. 1 and 2). From the figures, we observe a time-dependent modulation with a period of \(2\pi/\delta\) which corresponds to the period of the driving force at the over-transient time interval. Comparing the results of the two figures, we conclude that the magnitude of the purity for the regime of NDO shown in Fig. 2(b) is lower than that in Fig. 1(b). This outcome probably reflects the fact that for the second case, the NDO is in a deeper chaotic regime characterized by a smaller purity even when the excitation number depicted in Fig. 2(a) does not essentially deviate from that of Fig. 1(a).

In order to show that the lower purity is a result of chaotic dynamics, we present the purity dynamics for the case of regular dynamics in Fig. 3(b). For the sake of comparison, we select the parameters for both the regular and chaotic regimes so that they have approximately the same oscillatory excitation number as shown in Figs. 1(a) and 2(a). Note that the relatively small amplitude of the modulation in the mean excitation number and the purity is due to the small ratio \(f_1/f_0\) chosen in this regime. The regularity of the dynamics is confirmed via calculation of the Wigner function (see Fig. 3(c)) which is almost Gaussian in this case. By comparing the results of Fig. 3 with those of Figs. 1 and 2, we observe that the level of purity of the oscillatory mode for regular dynamics \(P \approx 0.7\), essentially exceeds that of \(P \approx 0.09\) and \(P \approx 0.03\)
for the case of chaotic dynamics.

B. NDO driven by Gaussian pulses

First, we briefly discuss the regimes of NDO driven by a sequence of Gaussian pulses. Depending on the parameters, we observe that this system is able to exhibit both regular and chaotic dynamics. The phenomenon of chaos is found to occur when the strength of the pulse trains $\Omega$ is sufficiently large, with the nonlinearity strength $\chi$ and pulse duration $T$ assuming appropriate values. In fact, this system was found to possess chaotic, deep chaotic and regular regime in an analogous fashion to the system discussed in the last section. However, in this section, we shall explore new regimes within this system, that will give rise to a low-level of excitation number as in the previous section. The other novelty in this section is the investigation on the connection between the purity of the quantum states and the Lyapunov exponent of the semiclassical dynamics.

As before, we shall adjust the parameters while keeping the oscillatory excitation number to be approximately constant. The results are shown in Figs. 4, 5 and 6. As illustrated in Fig. 4, the semiclassical dynamics display a strange chaotic attractor in the Poincaré section, with a Lyapunov exponent of 0.187. Note that the excitation number and purity take an average value of about 2.5 and 0.305 respectively in this case. Then, by setting the system into the deep chaotic regime (with a Lyapunov exponent of 0.305), we observe a drop in the purity to a mean value of 0.22 with an average excitation number of about 3. This reduction in the purity as the excitation number is about the same and the corresponding semiclassical dynamics become more chaotic is consistent with the results obtained for time-periodic modulated driving discussed in the last section. And when we put the system in the regular domain (Lyapunov exponent $=-0.1693$) while maintaining the mean excitation number at 2, we notice a sharp increase in the purity to an average value of 0.922. Thus, we have again observed a larger purity for regular dynamics against chaotic dynamics as we fix oscillatory excitation number within a close range, which indicates a direct relationship between the purity of the state and the underlying oscillatory dynamics.

Let us now examine this relationship in greater detail. To do this, we first obtain a plot of the maximum oscillatory excitation number against the parameters $\Omega/\gamma$ and $\chi/\gamma$, which is shown in Fig. 7. This allows us to determine the set of $\Omega/\gamma$ and $\chi/\gamma$ that gives approximately the same maximum excitation number. We take the set of $\Omega/\gamma$ and $\chi/\gamma$ that gives excitation number in the range $3.6958$ to $5.5217$. With this, we plot the maximum purity and Lyapunov exponent against $\Omega/\gamma$, which is illustrated in Fig. 8. We observe the subtle relationship between purity and the corresponding semiclassical dynamical behaviour in this figure. When the semiclassical dynamics is regular, the purity is high; and when

FIG. 4. (a) The excitation number, (b) purity dynamics, and (c) Poincaré section, for the parameters: $\Omega/\gamma = 15, \chi/\gamma = 0.7, \Delta/\gamma = -15, \gamma T = 10.2, \gamma \tau = 2\pi/5$.

FIG. 5. (a) The excitation number, and (b) purity dynamics. The parameters are: $\Omega/\gamma = 20.4, \chi/\gamma = 0.7, \Delta/\gamma = -15, \gamma T = 0.1, \gamma \tau = 2\pi/5$.

FIG. 6. (a) The excitation number, (b) purity dynamics, and (c) the Wigner function indicating regular dynamics, for the parameters: $\Omega/\gamma = 12, \chi/\gamma = 0.1, \Delta/\gamma = -15, \gamma T = 0.1, \gamma \tau = 2\pi/5$. 
FIG. 7. A plot of the maximum oscillatory excitation number against the parameter $\Omega/\gamma$ and $\chi/\gamma$. The values of the rest of the parameters are: $\Delta/\gamma = -15$, $\gamma T = 0.1$, $\gamma \tau = 2\pi/5$.

FIG. 8. The maximum purity $P$ of the quantum states and the associated Lyapunov exponents of the corresponding semiclassical dynamical behaviour, versus the strength of the Gaussian pulse train $\Omega/\gamma$. Note that for each $\Omega/\gamma$, we have selected a specific $\chi/\gamma$ to ensure that the maximum excitation number lies within the range 3.6958 to 5.5217. The values of the rest of the parameters are: $\Delta/\gamma = -15$, $\gamma T = 0.1$, $\gamma \tau = 2\pi/5$.

The semiclassical dynamics is chaotic, the purity reduces to a low value. In other words, there is a close link between an enhanced mixedness in the quantum domain with the presence of dynamical chaos in the semiclassical domain. In Fig. 8 we observe the occurrence of a sharp transition, which happens at the same strength of the Gaussian driving field for both the purity and the Lyapunov exponent. These results illustrate the good quantum-to-classical correspondence in this system.

IV. SUMMARY

We have applied the purity of an ensemble of oscillatory states to analyse and identify the chaotic behaviour of two different NDO systems. In particular, we have calculated varying levels of purity of quantum states for both the regular and chaotic regimes of NDO driven by a continuously modulated pump field, as well as for NDO under the action of a periodic sequence of identical pulses with Gaussian envelopes. It is important to note that these models seem feasible experimentally, and they can be achieved in several practical schemes involving nanomechanical oscillators as well as superconducting devices based on nonlinearity of the Josephson junction. In fact, we believe that the NDO scheme operating with trains of Gaussian pulses is closer to practical realization via laser- or microwave pulses (see, for example [17] and [21]).

Finally, we have demonstrated that for both systems, the level of purity of oscillatory modes for regular dynamics essentially exceeds the analogous case of chaotic dynamics. Moreover, we have shown the occurrence of a drop in purity when the NDO moves into the deep chaotic regime which is characterized by a larger positive Lyapunov exponent.
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