Second-order integrable Lagrangians and WDVV equations

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Abstract
We investigate integrability of Euler-Lagrange equations associated with 2D second-order Lagrangians of the form
\[ \int f(u_{xx}, u_{xy}, u_{yy}) \, dx \, dy. \]
By deriving integrability conditions for the Lagrangian density \( f \), examples of integrable Lagrangians expressible via elementary functions, Jacobi theta functions and dilogarithms are constructed. A link of second-order integrable Lagrangians to WDVV equations is established. Generalisations to 3D second-order integrable Lagrangians are also discussed.

MSC: 35Q51, 37K05, 37K10, 37K20, 53D45.

Keywords: Second-order Lagrangians, systems of hydrodynamic type, integrability (diagonalisability) conditions, Jacobi theta functions, Chazy equation, WDVV equations.
1 Introduction and summary of the main results

We investigate second-order Lagrangians

\[ \int f(u_{xx}, u_{xy}, u_{yy}) \, dxdy, \quad (1) \]

such that the corresponding Euler-Lagrange equations are integrable (in the sense to be explained below). Examples of integrable Lagrangians (1) have appeared in the mathematical physics literature, thus, the Lagrangian density

\[ f = u_{xy}(u_{xx}^2 - u_{yy}^2) + \alpha(u_{xx}^2 - u_{yy}^2) + u_{xy}(\beta u_{xx} + \gamma u_{yy}) \quad (2) \]

governs integrable geodesic flows on a 2-torus which possess a fourth-order integral polynomial in the momenta [3]. Similarly, the density

\[ f = u_{yy}^2 + u_{xx}^2 u_{yy} + u_{xx} u_{xy}^2 + \frac{1}{4} u_{xx}^4 \quad (3) \]

governs integrable Newtonian equations possessing a fifth-order polynomial integral. In Section 2 we investigate the integrability aspects of 2D Lagrangians (1). Our main results can be summarised as follows.
The Euler-Lagrange equation coming from Lagrangian (1) can be represented as a four-component Hamiltonian system of hydrodynamic type (Section 2.1). The requirement of its hydrodynamic integrability (which is equivalent to the vanishing of the corresponding Haantjes tensor) leads to an involutive system of third-order PDEs for the Lagrangian density \( f \) (Section 2.2). Analysis of the integrability conditions reveals that integrable Lagrangians (1) locally depend on six arbitrary functions of one variable. Furthermore, the integrability conditions are themselves integrable – a standard phenomenon in the theory of integrable systems.

The class of integrable Lagrangians (1) is invariant under the symplectic group \( \text{Sp}(4, \mathbb{R}) \); under this action the Lagrangian density \( f \) transforms as a genus two Siegel modular form of weight \(-1\) (Section 2.3). In particular, the integrability conditions can be represented via \( \text{Sp} \)-invariant operations known as generalised Rankin-Cohen (Eholzer-Ibukiyama) brackets (Section 2.4).

Potentials \( U(x, t) \) of classical Newtonian equations \( \ddot{x} = -U_x \) that possess a fifth-order polynomial integral are governed by a Lagrangian (1) with density (3) (Section 2.5).

Integrable Lagrangians (1) are related to WDVV prepotentials of the form
\[
F(t_1, t_2, t_3, t_4) = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + W(t_2, t_3, t_4);
\]
here \( W \) is a partial Legendre transform of the Lagrangian density \( f \) (Section 2.6). This correspondence works both ways: using known solutions of WDVV equations one can construct new integrable Lagrangians (1). Conversely, integrable Lagrangian densities \( f \) give rise to WDVV prepotentials. Examples of this kind are given in Sections 2.7.4 and 2.7.5.

In Section 3 we investigate 3D second-order Lagrangians of the form
\[
\int f(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt}) \, dx dy dt.
\]

Our results can be summarised as follows:

- Integrable Lagrangians in 3D are governed by a third-order PDE system for the Lagrangian density \( f \) which comes from the requirement that all travelling wave reductions of 3D Lagrangians to 2D are integrable in the sense of Section 2 (Section 3.1).
- The class of integrable Lagrangians (1) is invariant under the symplectic group \( \text{Sp}(6, \mathbb{R}) \); the Lagrangian density \( f \) transforms as a genus three Siegel modular form of weight \(-1\) (Section 3.2).
- Examples of integrable Lagrangians (1) are constructed in Section 3.3. These include the densities
\[
f = u_{yy} - u_{xx}u_{xt} + u_{xx}^2 u_{yy} + u_{xx}u_{xy}^2 + \frac{1}{4} u_{xx}^4,
\]
\[
f = (u_{xy} - u_{tt} - u_{xx}u_{xt} + \frac{1}{3} u_{xx}^3)^{3/2},
\]
\[
f = u_{xt}^2 (u_{xt}u_{yt} - u_{xx}u_{xt}^2)^{3/2},
\]
coming from the theory of dispersionless KP hierarchy (Section 3.3.1).

Classification of integrable densities of the form
\[
f = f(u_{xy}, u_{xt}, u_{yt})
\]
is given in Section 3.3.2. Here the generic case is quite non-trivial, involving spherical trigonometry and Schl"afly-type formulae, and is expressed in terms of the Lobachevky function \( L(s) = -\int_0^s \ln \cos \xi \, d\xi \).
In Section 4 we discuss examples of integrable dispersive deformations of integrable Lagrangian densities \([4]\). The general problem of constructing such deformations is largely open.

Finally, we recall that paper \([22]\) gives a characterisation of 3D first-order integrable Lagrangians of the form

\[
\int f(u_x, u_y, u_t) \, dx dy dt.
\]

It was pointed out in \([23]\) that the generic integrable Lagrangian density of this type is an automorphic function of its arguments. Note that 2D first-order Lagrangian densities \(f(u_x, u_y)\) lead to linearisable Euler-Lagrange equations and, therefore, are automatically integrable. On the contrary, for second-order Lagrangian densities \(f(u_{xx}, u_{xy}, u_{yy})\), the 2D case \([1]\) is already nontrivial.

2 Integrable Lagrangians in 2D

In this section we consider second-order integrable Lagrangians of type \([1]\),

\[
\int f(u_{xx}, u_{xy}, u_{yy}) \, dx dy.
\]

2.1 Hydrodynamic form of Euler-Lagrange equations

The Euler-Lagrange equation corresponding to Lagrangian \([1]\) is a fourth-order PDE for \(u(x, y)\):

\[
\left( \frac{\partial f}{\partial u_{xx}} \right)_{xx} + \left( \frac{\partial f}{\partial u_{xy}} \right)_{xy} + \left( \frac{\partial f}{\partial u_{yy}} \right)_{yy} = 0.
\]

(5)

Setting \(a = u_{xx}, b = u_{xy}, c = u_{yy}\) we can rewrite \([5]\) in the form

\[
b_x = a_y, \quad c_x = b_y, \quad (f_a)_{xx} + (f_b)_{xy} + (f_c)_{yy} = 0.
\]

(6)

Introducing the auxiliary variable \(p\) via the relations

\[
p_y = -(f_a)_x, \quad p_x = (f_b)_x + (f_c)_y,
\]

we can rewrite \([6]\) as a first-order four-component conservative system

\[
a_y = b_x, \quad b_y = c_x, \quad (f_c)_y = (p - f_b)_x, \quad p_y = -(f_a)_x
\]

(7)

or, in matrix form,

\[
\mathbf{R} \mathbf{w}_y = \mathbf{S} \mathbf{w}_x
\]

where \(\mathbf{w} = (a, b, c, p)^T\) and

\[
\mathbf{R} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
f_{ac} & f_{bc} & f_{cc} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-f_{ab} & -f_{bb} & -f_{bc} & 1 \\
-f_{aa} & -f_{ab} & -f_{ac} & 0
\end{pmatrix}.
\]

Assuming \(f_{cc} \neq 0\) we obtain a four-component system of hydrodynamic type,

\[
\mathbf{w}_y = \mathbf{V}(w) \mathbf{w}_x, \quad \mathbf{V}(w) = \mathbf{R}^{-1} \mathbf{S}.
\]

(8)

Remark 1. System \([7]\) can be put into a Hamiltonian form. For that purpose we introduce the new dependent variables \((\mathbf{A}, \mathbf{B}, \mathbf{C})\) which are related to \((a, b, c)\) via partial Legendre transform,

\[
A = a, \quad B = b, \quad C = f_c, \quad h = cf - f, \quad h_A = -f_a, \quad h_B = -f_b, \quad h_C = c.
\]
In the new variables, system (7) takes the form $(P = p)$

$$A_y = B_x, \quad B_y = (hC)_x, \quad C_y = (P + hB)_x, \quad P_y = (hA)_x,$$

which is manifestly Hamiltonian:

$$\begin{pmatrix} A \\ B \\ C \\ P \end{pmatrix}_y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{d}{dx} \begin{pmatrix} H_A \\ H_B \\ H_C \\ H_P \end{pmatrix},$$

with the Hamiltonian density $H = h(A, B, C) + BP$.

### 2.2 Integrability conditions

Since hydrodynamic type system (7) is conservative, its integrability by the generalised hodograph method [39] is equivalent to the diagonalisability of the corresponding matrix $V(w)$ from (8). This is equivalent to the vanishing of the corresponding Haantjes tensor [24]. Recall that the Nijenhuis tensor of the matrix $V(w) = (v_i^j(w))$ is defined as

$$N_{jk}^i = v_j^s \partial_{w^s} v_k^i - v_k^s \partial_{w^s} v_j^i - v^i_s (\partial_{w^j} v_k^s - \partial_{w^k} v_j^s),$$

where we adopt the notation $w = (a, b, c, p)^T = (w^1, w^2, w^3, w^4)^T$. The Haantjes tensor is defined as

$$H_{jk}^i = N_{ip}^j v_p^k v'^{r}_j + N_{jk}^p v'^{r}_{p} - N_{jp}^p v'^{r}_v - N_{jk}^p v'^{r}_v - N_{jk}^p v'^{r}_v.$$

It is easy to see that both tensors are skew-symmetric in the low indices. The requirement of vanishing of the Haantjes tensor leads to a system of PDEs (integrability conditions) for the Lagrangian density $f(a, b, c)$ which can be represented in symmetric conservative form:

$$\begin{align*}
(f_{ab} f_{cc} - f_{ac} f_{bc})_a &= (f_{bc} f_{aa} - f_{ab} f_{ac})_c, \\
(f_{aa} f_{cc} - f_{ac}^2)_a &= (f_{bb} f_{cc} - f_{ab}^2)_c, \\
(f_{aa} f_{cc} - f_{ac}^2)_c &= (f_{cc} f_{bb} - f_{bc}^2)_a, \\
(f_{bb} f_{cc} - f_{bc}^2)_b &= 2(f_{ab} f_{cc} - f_{ac} f_{bc})_c, \\
(f_{bb} f_{aa} - f_{ab}^2)_b &= 2(f_{bc} f_{aa} - f_{ac} f_{ab})_a.
\end{align*}$$

Integrability conditions (10) are invariant under the discrete symmetries $a \leftrightarrow c$ and $b \rightarrow -b$. Indeed, under the interchange of $a$ and $c$ equation (10) stays the same, while $10_2$, $10_3$ and $10_4$, $10_5$ get interchanged. Strictly speaking, the vanishing of the Haantjes tensor gives only the first four of relations (10), however, one can show that the fifth follows from the first four. We prefer to keep all of them for symmetry reasons.

Our next goal is to show that the system of integrability conditions (10) is in involution, and its general solution depends on six arbitrary functions of one variable.

**Theorem 1** The general 2D integrable Lagrangian density $f$ depends on six arbitrary functions of one variable.

**Proof:**

Let us introduce the new dependent variables

$$s = (s_1, s_2, s_3, s_4, s_5, s_6)^T = (f_{aa}, f_{ab}, f_{ac}, f_{bb}, f_{bc}, f_{cc})^T,$$
which satisfy the obvious consistency conditions such as
\[
(s_1)_b = (s_2)_a, \quad (s_1)_c = (s_3)_a, \quad (s_2)_b = (s_4)_a, \quad (s_2)_c = (s_3)_b = (s_6)_a, \\
(s_3)_c = (s_6)_a, \quad (s_4)_c = (s_5)_b, \quad (s_5)_c = (s_6)_b.
\]
Taking these consistency conditions along with the four integrability conditions (10) (also rewritten in terms of \(s\) -variables) we obtain a system of twelve first-order quasilinear equations for \(s_1(a, b, c)\) which can be represented in the form of two six-component systems of hydrodynamic type,
\[
s_a = P(s)s_c, \quad s_b = Q(s)s_c,
\]
where \(P, Q\) are the following \(6 \times 6\) matrices:
\[
P = \frac{1}{s_6} \begin{pmatrix}
2s_3 + s_4 & -2s_2 & -s_1 & s_1 & 0 & 0 \\
2s_5 & 0 & -2s_2 & 0 & s_1 & 0 \\
s_6 & 0 & 0 & 0 & 0 & 0 \\
6 & 2s_5 & -2s_3 - s_4 & 0 & 0 & s_1 \\
0 & s_6 & 0 & 0 & 0 & 0 \\
0 & 0 & s_6 & 0 & 0 & 0
\end{pmatrix},
\]
\[
Q = \frac{1}{s_6} \begin{pmatrix}
2s_5 & 0 & -2s_2 & 0 & s_1 & 0 \\
s_6 & 2s_5 & -2s_3 - s_4 & 0 & 0 & s_1 \\
0 & s_6 & 0 & 0 & 0 & 0 \\
0 & 2s_6 & -2s_5 & 2s_5 & -2s_3 - s_4 & 2s_2 \\
0 & 0 & s_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_6 & 0
\end{pmatrix}.
\]
Equations (11) possess six conserved densities \(s_2, s_4, s_5, s_1s_5 - s_2s_3, s_2s_6 - s_3s_5, s_1s_6 + s_2s_5 - s_3s_4 - s_3^2\) which satisfy the equations
\[
(s_2)_a = (s_1)_b, \quad (s_4)_a = (s_2)_b, \quad (s_5)_a = (s_3)_b, \\
(s_2)_c = (s_3)_b, \quad (s_4)_c = (s_5)_b, \quad (s_5)_c = (s_6)_b,
\]
\[
2(s_1s_5 - s_2s_3)_a = (s_1s_4 - s_2^2)_b, \quad 2(s_2s_6 - s_3s_5)_a = (s_1s_6 - s_3^2)_b, \\
2(s_1s_5 - s_2s_3)_c = (s_1s_6 - s_3^2)_b, \quad 2(s_2s_6 - s_3s_5)_c = (s_4s_6 - s_5^2)_b,
\]
\[
(s_1s_6 + s_2s_5 - s_3s_4 - s_3^2)_a = (s_1s_5 - s_2s_3)_b, \\
(s_1s_6 + s_2s_5 - s_3s_4 - s_3^2)_c = (s_2s_6 - s_3s_5)_b.
\]
Direct calculation shows that systems (11) commute, that is, \(s_{ab} = s_{ba}\). Thus, equations (11) are in involution, and their general common solution depends on six arbitrary functions of one variable, namely, the Cauchy data \(s_i(0, 0, c)\). This finishes the proof.

Remark 2. Relations (10) and (12) imply that there exists a potential \(\rho\) such that
\[
\rho_{aa} = f_{aa}f_{bb} - f_{ab}^2, \quad \rho_{ac} = f_{aa}f_{cc} - f_{ac}^2, \quad \rho_{cc} = f_{cc}f_{bb} - f_{bc}^2, \\
\rho_{ab} = 2(f_{bc} f_{aa} - f_{ac} f_{ab}), \quad \rho_{bc} = 2(f_{ab} f_{cc} - f_{ac} f_{bc}), \\
\rho_{bb} = 2(f_{ab} f_{bc} - f_{ac} f_{bb} + f_{aa} f_{cc} - f_{ac}^2).
\]

Remark 3. System (10) possesses a Lax pair
\[
\psi_a = \lambda K \psi, \quad \psi_b = \lambda L \psi, \quad \psi_c = \lambda M \psi,
\]
where \( \lambda \) is a spectral parameter and the \( 4 \times 4 \) matrices \( K, L, M \) are defined as

\[
K = \begin{pmatrix}
0 & f_{ac} & 0 & 1 \\
-f_{aa} & -f_{ab} & 0 & 0 \\
-\frac{1}{2} \rho_{ab} & -\rho_{ac} & -f_{ab} & f_{ac} \\
-\rho_{aa} & -\frac{1}{2} \rho_{ab} & -f_{aa} & 0
\end{pmatrix}, \quad
L = \begin{pmatrix}
0 & f_{bc} & 1 & 0 \\
-f_{ab} & -f_{bb} & 0 & 1 \\
-\frac{1}{2} \rho_{bb} & -\rho_{bc} & -f_{bb} & f_{bc} \\
-\rho_{ab} & -\frac{1}{2} \rho_{bb} & -f_{ab} & 0
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
0 & f_{cc} & 0 & 0 \\
-f_{ac} & -f_{bc} & 1 & 0 \\
-\frac{1}{2} \rho_{bc} & -\rho_{cc} & -f_{bc} & f_{cc} \\
-\rho_{ac} & -\frac{1}{2} \rho_{bc} & -f_{ac} & 0
\end{pmatrix}.
\]

Remark 4. We have verified that both systems (11) are linearly degenerate and non-diagonalisable (their Haantjes tensor does not vanish). This suggests that integrable Lagrangian densities (1) are related to the associativity (WDVV) equations where analogous commuting six-component systems were obtained in [21], see also [34, 35] for related results. Such a link indeed exists, and is discussed in Section 2.6.

2.3 Equivalence group in 2D

Let \( U \) be the \( 2 \times 2 \) Hessian matrix of the function \( u(x,y) \). Integrable Lagrangians of type (1) are invariant under \( \text{Sp}(4, \mathbb{R}) \)-symmetry

\[
U \rightarrow (AU + B)(CU + D)^{-1}, \quad f \rightarrow \frac{f}{\det(CU + D)},
\]

where the matrix

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

belongs to the symplectic group \( \text{Sp}(4, \mathbb{R}) \) (here \( A, B, C, D \) are \( 2 \times 2 \) matrices). Symmetry (14) suggests a relation to Siegel modular forms (the density \( f \) transforms as a genus two Siegel modular form of weight \(-1\)). This symmetry corresponds to linear symplectic transformations of the four-dimensional jet space with coordinates \( u_{xx}, u_{yy}, x, y \). Furthermore, integrable Lagrangians (1) are invariant under rescalings of \( f \), as well as under the addition of a ‘null-Lagrangian’, namely, transformations of the form

\[
f \rightarrow \lambda_0 f + \lambda_1 (u_{xx} u_{yy} - u_{xy}^2) + \lambda_2 u_{xx} + \lambda_3 u_{xy} + \lambda_4 u_{yy} + \lambda_5,
\]

which do not effect the Euler-Lagrange equations. Transformations (14) and (15) generate a group of dimension \( 10 + 6 = 16 \) which preserves the class of integrable Lagrangians (1). These equivalence transformations will be utilised to simplify the classification results in Section 2.7. For instance, modulo equivalence transformations the Lagrangian density (2) is equivalent to \( f = u_{xy}(u_{xx}^2 - u_{yy}^2) \).

2.4 Integrability conditions via generalised Rankin-Cohen brackets

Integrability conditions (10) possess a compact formulation via higher genus Rankin-Cohen (Eholzer-Ibukiyama) brackets for Siegel modular forms [19]. This does not come as something unexpected, indeed, the integrability conditions possess \( \text{Sp}(4) \)-invariance (14) and, therefore, should be expressible via \( \text{Sp}(4) \)-invariant operations. Here we mainly follow [31, 25], which specialised the general results of [19] to the genus two case. Let us introduce two matrix differential operators

\[
R = \begin{pmatrix}
\frac{1}{2} \partial_a & \frac{1}{2} \partial_b \\
\frac{1}{2} \partial_b & \frac{1}{2} \partial_c
\end{pmatrix}, \quad
S = \begin{pmatrix}
\frac{1}{2} \partial_a & \frac{1}{2} \partial_b \\
\frac{1}{2} \partial_b & \frac{1}{2} \partial_c
\end{pmatrix},
\]

and define the operators \( P_0, P_1, P_2 \) via the expansion

\[
\det(R + \lambda S) = P_0 + \lambda P_1 + \lambda^2 P_2.
\]
Explicitly, we have

\[ P_0 = \partial_a \partial_e - \frac{1}{4} \partial_\xi^2, \quad P_1 = \partial_a \partial_e + \partial_e \partial_a - \frac{1}{2} \partial_b \partial_b, \quad P_2 = \partial_b \partial_e - \frac{1}{4} \partial_\xi^2. \]

Let us also define two operators \( Y_1, Y_2 \) depending on the auxiliary parameters \( \xi = (\xi_1, \xi_2) \) by the formulae

\[ Y_1 = \xi R\xi^T = \xi_1^2 \partial_a + \xi_1 \xi_2 \partial_b + \xi_2^2 \partial_e, \quad Y_2 = \xi S\xi^T = \xi_1^2 \partial_a + \xi_1 \xi_2 \partial_b + \xi_2^2 \partial_e. \]

Finally, we introduce the \( \xi \)-dependent operator

\[ v = (\partial_a \partial_b - \partial_b \partial_a) \xi_1^2 + 2(\partial_a \partial_e - \partial_e \partial_a) \xi_1 \xi_2 + (\partial_b \partial_e - \partial_e \partial_b) \xi_2^2. \]

Then integrability conditions \([10]\) can be represented in the Hirota-type bilinear form

\[ (P_1 Y_1 v - 2P_0 Y_2 v)[f(a, b, c) \cdot f(\tilde{a}, \tilde{b}, \tilde{c})]\bigg|_{\tilde{a}=a, \tilde{b}=b, \tilde{c}=c} = 0. \quad (16) \]

Here the left-hand side is a homogeneous quartic in \( \xi_1, \xi_2 \), with five nontrivial components. Equating them to zero we obtain all of the five integrability conditions \([10]\).

**Remark 5.** It follows from \([31]\), Proposition 2.3, that if \( f \) transforms as in \([14]\), that is, as a weight \(-1\) Siegel modular form, then the left-hand side of \([16]\) transforms as a vector-valued Siegel modular form with values in the representation \( \text{Sym}_4 \otimes \text{det} \text{GL}(2, \mathbb{C}) \).

**Remark 6.** The principal symbol of the Euler-Lagrange equation \([5]\) is given by a compact formula in terms of the operator \( Y_1 \):

\[ Y_1^2[f] = f_{aa} \xi_1^4 + 2f_{ab} \xi_1^3 \xi_2 + (2f_{ac} + f_{bb}) \xi_1^2 \xi_2^2 + 2f_{bc} \xi_1 \xi_2^3 + f_{cc} \xi_2^4. \]

This expression transforms as a vector-valued Siegel modular form with values in the representation \( \text{Sym}_4 \otimes \text{det}^{-1} \).

### 2.5 Integrable Lagrangians and classical Newtonian equations

Here we sketch the derivation of the Lagrangian density \([3]\). Consider a classical Newtonian equation

\[ \ddot{x} = -U_x \]

where \( U(x, t) \) is the potential function, \( x = x(t) \), and dot denotes differentiation by \( t \). This equation can be written in the canonical Hamiltonian form

\[ \dot{x} = p, \quad \dot{p} = -U_x. \]

To be Liouville integrable, this Hamiltonian system should be equipped with an extra first integral \( F(t, x, p) \) such that

\[ \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial p} \dot{p} = F_t + pF_x - U_x F_p = 0. \quad (17) \]

First integrals \( F \) polynomial in the momentum \( p \) were thoroughly investigated in \([13]\), and later in \([27, 33]\). In particular, the following cases have been studied:

\[ F = \frac{p^3}{3} + Up + V, \quad F = \frac{p^4}{4} + Up^2 + Vp + W, \quad F = \frac{p^5}{5} + Up^3 + Vp^2 + Wp + Q. \]

In the last (fifth-order) case, equation \([17]\) implies the following quasilinear system for the coefficients:

\[ U_t + V_x = 0, \quad V_t + W_x = 3UU_x, \quad W_t + Q_x = 2VU_x, \quad Q_t = WU_x. \]
Let us introduce a potential $u$ such that $U = u_{xx}$, $V = -u_{xt}$, $W = \frac{3}{2}u_{xx}^2 + u_{tt}$. Then the first two equations will be satisfied identically, while the last two imply

$$Q_x = -2u_{xt}u_{xxx} - 3u_{xx}u_{xxt} - u_{ttt}, \quad Q_t = \frac{3}{2}u_{xx}^2u_{xxx} + u_{tt}u_{xxx}.$$ 

The compatibility condition of these equations for $Q$ leads to a fourth-order PDE for $u$,

$$u_{ttt} + \frac{3}{2}u_{xx}^2u_{xxx} + 3u_{xx}u_{xxt}^2 + u_{tt}u_{xxx} + 2u_{xt}u_{xxt} + 3u_{xx}u_{xxt} + 3u_{xxx}u_{xxx} + 3u_{xxt}^2 = 0,$$

which is nothing but the Euler-Lagrange equation for the second-order Lagrangian

$$S = \int \left[ u_{tt}^2 + u_{xx}^2u_{tt} + u_{xx}u_{xt}^2 + \frac{1}{4}u_{xx}^4 \right] dx dt,$$

whose density is identical to (9) up to relabelling $t \leftrightarrow y$.

## 2.6 Integrable Lagrangians and WDVV equations

Let $F(t_1, \ldots, t_n)$ be a function of $n$ independent variables such that the symmetric matrix

$$\eta_{ij} = \partial_i \partial_j F$$

is constant and non-degenerate (thus, $t_1$ is a marked variable), and the coefficients

$$c_{jk}^i = \eta^{is} \partial_s \partial_j \partial_k F$$

satisfy the associativity condition $c_{ij}^k c_{kj}^l = c_{jl}^k c_{ik}^l$, here $i, j, k \in \{1, \ldots, n\}$. These requirements impose a nonlinear system of third-order PDEs for the prepotential $F$, the so-called associativity (WDVV) equations which were discovered in the beginning of 1990s by Witten, Dijkgraaf, Verlinde and Verlinde in the context of two-dimensional topological field theory. Geometry and integrability of WDVV equations has been thoroughly studied by Dubrovin, culminating in the remarkable theory of Frobenius manifolds [15]. An important ingredient of this theory is an integrable hydrodynamic hierarchy whose ‘primary’ part is defined by $n - 1$ commuting Hamiltonian flows

$$\partial_{T_n} t_i = c_{nk}^i \partial_X t_k = \partial_X (\eta^{is} \partial_s \partial_n F) \tag{18}$$

where $T_n$ are the higher ‘times’; here $T_1 = X$. The flows (18) are manifestly Hamiltonian with the Hamiltonian operator $\eta^{is} \partial_X$ and the Hamiltonian density $\partial_n F$. Note that WDVV equations are equivalent to the requirement of commutativity of these flows.

We will need a particular case of the general construction when $n = 4$ and the matrix $\eta$ is anti-diagonal, which corresponds to prepotentials $F(t_1, t_2, t_3, t_4) = \frac{1}{2}t_1^2t_2 + t_1t_2t_3 + W(t_2, t_3, t_4)$. \tag{19}

The corresponding primary flows (18) take the form

$$\begin{align*}
\partial_{T_4} t_1 &= \partial_X (\partial_4 \partial_2 F), \quad \partial_{T_2} t_2 = \partial_X (\partial_3 \partial_2 F), \quad \partial_{T_2} t_3 = \partial_X (\partial_2 \partial_2 F), \quad \partial_{T_2} t_4 = \partial_X (\partial_1 \partial_2 F), \\
\partial_{T_3} t_1 &= \partial_X (\partial_3 \partial_3 F), \quad \partial_{T_3} t_2 = \partial_X (\partial_3 \partial_3 F), \quad \partial_{T_3} t_3 = \partial_X (\partial_2 \partial_3 F), \quad \partial_{T_3} t_4 = \partial_X (\partial_1 \partial_3 F), \\
\partial_{T_4} t_1 &= \partial_X (\partial_4 \partial_4 F), \quad \partial_{T_4} t_2 = \partial_X (\partial_3 \partial_4 F), \quad \partial_{T_4} t_3 = \partial_X (\partial_2 \partial_4 F), \quad \partial_{T_4} t_4 = \partial_X (\partial_1 \partial_4 F), \tag{20}
\end{align*}$$

which are Hamiltonian systems with the Hamiltonian densities

$$\begin{align*}
\partial_2 F &= t_1t_3 + \partial_2 W, \quad \partial_3 F = t_1t_2 + \partial_3 W, \quad \partial_4 F = \frac{1}{2}t_1^2 + \partial_4 W,
\end{align*}$$
respectively. In compact form, equations (20) can be represented as
\[ \partial_{\xi_i} \xi_i = \partial_X (\partial_{\xi_{i-1}} \partial_{\xi_i} F), \quad i = 1, 2, 3, 4, \quad \alpha = 2, 3, 4. \]
Setting \((t_1, t_2, t_3, t_4) = (P, B, C, A)\) we obtain
\[ F = \frac{1}{2} P^2 A + PBC + W(B, C, A). \]
In this case WDVV equations reduce to the following system of four PDEs for \(W\):
\[ W_{AAA} = W_{ABC}^2 + W_{ABB} W_{ACC} - W_{AAB} W_{BCC} - W_{AAC} W_{BCC}, \]
\[ W_{AAB} = W_{BBB} W_{ACC} - W_{ABB} W_{BCC}, \]
\[ W_{AAC} = W_{ABB} W_{CCC} - W_{ACC} W_{BCC}, \]
\[ 2W_{ABC} = W_{BBB} W_{CCC} - W_{BBC} W_{BCC}. \]
(21)
The corresponding primary flows (20) take the form
\[ A_{T_2} = C_X, \quad B_{T_2} = (P + W_{BC}) X, \quad C_{T_2} = (W_{BB}) X, \quad P_{T_2} = (W_{AB}) X, \]
\[ A_{T_3} = B_X, \quad B_{T_3} = (W_{CC}) X, \quad C_{T_3} = (P + W_{BC}) X, \quad P_{T_3} = (W_{AC}) X, \]
\[ A_{T_4} = P_X, \quad B_{T_4} = (W_{AC}) X, \quad C_{T_4} = (W_{BA}) X, \quad P_{T_4} = (W_{AA}) X. \]
(22)
Note that system (22) coincides with (9) under the identification \(h = W_C\), thus establishing a link between WDVV equations and integrable Lagrangians. This link can be summarised as follows:
- Take prepotential of type (19), set \((t_2, t_3, t_4) = (B, C, A)\) and define \(h(A, B, C) = W_C\).
- Reconstruct Lagrangian density \(f(a, b, c)\) by applying partial Legendre transform to \(h(A, B, C)\):
\[ a = A, \quad b = B, \quad c = h_C, \quad f = Ch_C - h, \quad f_a = -h_A, \quad f_b = -h_B, \quad f_c = C. \]
Examples of calculations of this kind will be given in Section 2.7.4.

Remark 7. Conversely, given a Lagrangian density \(f(a, b, c)\), the corresponding prepotential \(W(A, B, C)\) can be reconstructed by the formulae
\[ W_{AA} = -\rho_a, \quad W_{AB} = -\frac{1}{2} \rho_b, \quad W_{AC} = -f_a, \]
\[ W_{BB} = -\rho_c, \quad W_{BC} = -f_b, \quad W_{CC} = c. \]
\[ A = a, \quad B = b, \quad C = f_c, \]
where \(\rho\) is defined by formulae (13), see Section 2.7.5.

2.7 Examples of integrable Lagrangians in 2D

In this section we present explicit examples of integrable Lagrangian densities \(f\) obtained by assuming a suitable ansatz for \(f\) and computing the corresponding integrability conditions (10). This gives a whole range of integrable densities expressible via polynomials, elementary functions, Jacobi theta functions and dilogarithms.
2.7.1 Integrable Lagrangian densities of the form \( f = g(u_{xx}, u_{yy}) \)

In this case the integrability conditions lead to the only constraint \( g_{aa}g_{cc} - g_{ac}^2 = k \) where \( k = \text{const} \). Its solutions can be represented parametrically, thus, for \( k = 0 \) (parabolic case) and \( k = -1 \) (hyperbolic case) we obtain the general solution in parametric form:

\[
a = p'(w)v + q'(w), \quad c = v, \quad f = w[p'(w)v + q'(w)] - [p(w)v + q(w)],
\]

and

\[
a = p'(w + v) + q'(w - v), \quad c = v, \quad f = w[p'(w + v) + q'(w - v)] - [p(w + v) + q(w - v)],
\]

respectively; here \( p \) and \( q \) are arbitrary functions and prime denotes differentiation.

2.7.2 Integrable Lagrangian densities of the form \( f = e^{u_{xx}}g(u_{xy}, u_{yy}) \)

We will show that the generic integrable density of this form corresponds to

\[
g(b, c) = [\Delta(ic/\pi)]^{-1/8} \theta_1(b, ic/\pi)
\]

where \( \Delta \) is the modular discriminant and \( \theta_1 \) is the Jacobi theta function. The details are as follows. Substituting \( f = e^u g(b, c) \) into the integrability conditions \(^{[10]}\) one obtains

\[
\begin{align*}
gg_{bccc} &= 3gg_{c}g_{b} - 2gg_{bc}g_{c}, \quad (23) \\
gg_{bbbb} &= gg_{bb} + 4gg_{bc} - 4gg_{c}, \quad (24) \\
gg_{c} &= gg_{c}g_{bc} + 2gg_{c} - 2(g_{bc})^2, \quad (25) \\
gg_{bb} &= 2gg_{bc} - gg_{bb} + 2gg_{c} - 2(g_{c})^2. \quad (26)
\end{align*}
\]

This over-determined system for \( g \) is in involution, and can be solved as follows. First of all, equation \(^{(24)}\) implies

\[
\left(\frac{gg_{b}}{g}\right)_b = \left(\frac{gg_{c}}{g}\right)_b,
\]

so that one can set

\[
g_{c} = \frac{1}{4}(gg_{bb} - hg)
\]

where \( h \) is a function of \( c \) only. Using \(^{(27)}\), both \(^{(23)}\) and \(^{(26)}\) reduce to

\[
gg_{bb}g_{b} - 4gg_{bb}g_{b} + 3g_{b}^2 = 4h(gg_{bb} - g_{b}^2) - 4h'g_{b}^2; \quad (28)
\]

here prime denotes differentiation by \( c \). Modulo \(^{(27)}\) and \(^{(28)}\), equation \(^{(25)}\) implies

\[
gg_{bb}g_{b}^2 + gg_{bb}(4g_{b}^3 - 6gg_{bb}g_{b}) - 3g_{b}^2g_{bb} + 4gg_{bb}^3 = 4h(gg_{bb} - g_{bb})^2 + 8h'g_{b}^2(g_{b}^2 - gg_{bb}) + \frac{8}{3}h''g_{b}^4. \quad (29)
\]

Note that \(^{(29)}\) can be obtained from \(^{(28)}\) by differentiating it with respect to \( c \), and using \(^{(27)}\), \(^{(28)}\). Similarly, differentiating \(^{(29)}\) with respect to \( c \) we obtain the Chazy equation \(^{[8]}\) for \( h \):

\[
h'''' = 2hh'' - 3h'^2. \quad (30)
\]

Equations \(^{(28)}\) and \(^{(29)}\) can be simplified by the substitution \( v = -(\ln g)_{bb} \), which implies

\[
v_{bb} = 6v^2 + 4hv + 4h'
\]

and

\[
v_{b}^2 = 4v^3 + 4hv^2 + 8h'v + \frac{8}{3}h''
\]

\[

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\]
Note that the change of independent variables $x$ allows for the general solution of system (34) to be constructed in [7]:

$$g_c = \frac{1}{4}(g_{bb} - h g), \quad v = -(\ln g)_{bb}, \quad v_b^2 = 4v^3 + 4hv^2 + 8h'v + \frac{8}{3}h''; \quad (33)$$

here $h$ solves the Chazy equation [30]. We recall that modulo the natural $SL(2, \mathbb{R})$-symmetry [9], the Chazy equation possesses three non-equivalent solutions: $h = 0$, $h = 1$ and $h = \frac{1}{2} \Delta$ where $\Delta$ is the modular discriminant. These three solutions (which correspond to rational, trigonometric and elliptic cases of the Weierstrass $\wp$-function equation in (33)) are considered separately below. Note that both the rational and trigonometric cases lead to degenerate Lagrangians, so only the elliptic case is of interest.

**Rational case** $h = 0$. In this case equations (33) simplify to

$$g_c = \frac{1}{4}g_{bb}, \quad v = -(\ln g)_{bb}, \quad v_b^2 = 4v^3,$$

which are straightforward to solve. Modulo unessential constants the generic solution of these equations is $g = e^{2\mu b + \mu^2 c}(b + \mu c)$ where $\mu = \text{const}$. The corresponding Lagrangian density $f$ takes the form

$$f = e^{u_{xx} + 2\mu u_{xy} + \mu^2 u_{yy}}(u_{xy} + \mu u_{yy}).$$

Note that the change of independent variables $x = \tilde{x}$, $y = \tilde{y} + \mu \tilde{x}$ brings this Lagrangian to the degenerate form $\tilde{f} = e^{u_{xx} \tilde{x} \tilde{y}}$ (the order of the corresponding Euler-Lagrange equation can be reduced by two by setting $v = \tilde{u}_{\tilde{x}}$).

**Trigonometric case** $h = 1$. In this case equations (33) simplify to

$$g_c = \frac{1}{4}(g_{bb} - g), \quad v = -(\ln g)_{bb}, \quad v_b^2 = 4v^3 + 4v^2,$$

which are also straightforward to solve. Modulo unessential constants the generic solution of these equations is $g = e^{2\mu_b + \mu^2 c} \sinh(b + \mu c)$ where $\mu = \text{const}$. The corresponding Lagrangian density $f$ takes the form

$$f = e^{u_{xx} + 2\mu u_{xy} + \mu^2 u_{yy}} \sinh(u_{xy} + \mu u_{yy}).$$

Note that the same change of variables as in the rational case brings this Lagrangian to the degenerate form $\tilde{f} = e^{u_{xx} \sinh \tilde{x} \tilde{y}}$.

**Elliptic case** $h = \frac{1}{2} \Delta$, see e.g. [38]. Here the modular discriminant $\Delta$ is given by the formula

$$\Delta(c) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i c},$$

recall that $h$ has the $q$-expansion

$$h(c) = \pi i E_2 = \pi i \left( 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \right)$$

where $E_2$ is the Eisenstein series (here $\sigma_1(n)$ is the divisor function). Setting $g(b, c) = [\Delta(c)]^{-1/8} w(b, c)$ we see that the first equation (33) becomes the heat equation for $w$:

$$w_c = \frac{1}{4} w_{bb}, \quad v = -(\ln w)_{bb}, \quad v_b^2 = 4v^3 + 4hv^2 + 8h'v + \frac{8}{3}h''; \quad (34)$$

The general solution of system (34) was constructed in [7]:

$$w(b, c) = \Delta^{1/8} \sigma(b g_2, g_3) e^{b^2 h/6}.$$
where $\sigma$ is the Weierstrass sigma function with the invariants $g_2 = \frac{1}{3}h^2 - 8h$, $g_3 = -\frac{8}{27}h^3 + \frac{8}{3}hh' - \frac{8}{3}h''$. Note that $\Delta = \pi^6(g_2^3 - 27g_3^2)$. Thus,

$$g(b, c) = \sigma(b, g_2, g_3)e^{2\pi h/6}.$$  

**Remark 8.** An alternative (real-valued) representation of the general solution of system (34) in terms of the Jacobi theta function $\theta_1$ is as follows:

$$w(b, c) = \theta_1(b, ic/\pi) = 2 \sum_{n=0}^{\infty} (-1)^n e^{-(n+1/2)^2c} \sin((2n + 1)b);$$

here for $h$ in the last equation [34] one has to use

$$\frac{i}{\pi} \frac{h(ic/\pi)}{\Delta} = -1 + 24 \sum_{n=1}^{\infty} \sigma_1(n)e^{-2nc} = -1 + 24(e^{-2c} + 3e^{-4c} + 4e^{-6c} + 7e^{-8c} + \ldots),$$

which is another (real-valued) solution of the Chazy equation (note that the Chazy equation is invariant under the scaling symmetry $h(c)\rightarrow \lambda h(\lambda c)$). Thus,

$$g(b, c) = [\Delta(ic/\pi)]^{-1/6}\theta_1(b, ic/\pi).$$

Note that the function $\Delta^{-1/6}\theta_1(z, \tau)$ appears in the theory of weak Jacobi forms (it is a holomorphic weak Jacobi form of weight $-1$ and index $1/2$).

2.7.3 *Integrable Lagrangian densities polynomial in $e^{u_{xx}}$ and $e^{u_{yy}}$*

Here we describe integrable Lagrangian densities $f$ that are linear/quadratic in $e^{u_{xx}}$ and $e^{u_{yy}}$, the coefficients being functions of $u_{xy}$ only.

**Linear case:**

$$f = p_0 + p_1 e^a + p_2 e^c.$$

Substituting this ansatz into the integrability conditions (and assuming $p_1, p_2$ to be nonzero) we obtain a system of ODEs for the coefficients $p_i(b)$ which, modulo equivalence transformations, can be simplified to

$$p_1 = p_2 = p, \quad p'' = p, \quad p_0' = \alpha/p;$$

here $\alpha = \text{const}$ (which can be set equal to 1 if nonzero) and prime denotes differentiation by $b$. Modulo equivalence transformations, these equations possess two essentially different solutions:

$$f = \alpha e^{-b} + (e^a + e^c)e^b \quad \text{and} \quad f = \alpha q(b) + (e^a + e^c) \sinh b,$$

where the function $q(b)$ satisfies $q'' = \frac{1}{\sinh b}$. This implies $q' = \ln \frac{1-e^b}{1+e^b}$, and another integration gives

$$q(b) = Li_2(-e^b) - Li_2(e^b)$$

where $Li_2$ is the dilogarithm function: $(Li_2(x))' = -\frac{\ln(1-x)}{x}$.

**Quadratic case:**

$$f = p_0 + p_1 e^a + p_2 e^c + p_3 e^{2a} + p_4 e^{a+c} + p_5 e^{2c}.$$

Substituting this ansatz into the integrability conditions we obtain a large system of ODEs for the coefficients $p_i(b)$ which, modulo equivalence transformations, lead to the following integrable densities (here we only present those examples that are not reducible to the linear case by a change of variables):

$$f = e^{k+b+a+c}, \quad f = e^{\frac{4}{3}b+a+c} + e^{\frac{4}{3}b+2c}, \quad f = \alpha e^{-\frac{1}{\sqrt{3}}b} + \alpha e^\frac{1}{\sqrt{3}b+a} + e^\frac{1}{\sqrt{3}b+c} + e^\frac{3}{\sqrt{3}b+a+c},$$

$$f = pe^{2a} + 2p^2 e^{a+c} + pe^{2c}, \quad p = \cosh \left( \frac{2}{\sqrt{3}} b \right).$$
2.7.4 Integrable Lagrangian densities from WDVV prepotentials

In this section we discuss polynomial prepotentials $F$ of type (19) associated with finite Coxeter groups $W$ as given in [16], p. 107. Applying the procedure outlined at the end of Section 2.6 we compute the corresponding integrable Lagrangian densities $f$ which, in general, will be algebraic functions of $a, b, c$ (presented below up to appropriate scaling factors).

Group $W(A_4)$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{1}{2} t_2^3 + \frac{1}{3} t_3^4 + 6 t_2 t_3 t_4 + 9 t_3^2 t_4^2 + 24 t_2^3 t_4^2 + \frac{216}{5} t_4^3;$$

$$f = (c - 48 a^3 - 12 ab)^{3/2}.$$ 

Swapping $t_2$ and $t_3$ (which is an obvious symmetry of WDVV equations) and following the same procedure gives a polynomial density $f$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{1}{2} t_2^3 + \frac{1}{3} t_3^4 + 6 t_3 t_2 t_4 + 9 t_3^2 t_4^2 + 24 t_2^3 t_4^2 + \frac{216}{5} t_4^3;$$

$$f = 54 a^4 - 6 a^2 c + \frac{1}{6} c^2 - 6 b^2 a.$$

Group $W(B_4)$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + t_3^3 + \frac{3}{3} t_2 t_3^2 t_4 + \frac{t_3 t_4}{4} + 3 t_2 t_3 t_4^2 + 6 t_2^2 t_4^2 + \frac{t_3^4}{5} t_4 + \frac{18}{7} t_4^9;$$

$$f = 2 a C^3 + (3 a^3 + b) C^2 - 3 a b^2,$$

where $C$ is defined by the quadratic equation $3 a C^2 + (6 a^3 + 2 b) C + \frac{6}{9} a^2 (6 a^3 + 5 b) = c$. Swapping $t_2$ and $t_3$ gives a polynomial density $f$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{t_2^3}{3} t_3 + 3 t_2 t_3 t_4 + \frac{t_3 t_4}{4} + 3 t_2 t_3 t_4^2 + 6 t_2^2 t_4^2 + \frac{t_3^4}{5} t_4 + \frac{18}{7} t_4^9;$$

$$f = 12 a^6 + 12 a^4 b - 2 a^3 c - b a c + \frac{1}{12} c^2 - \frac{1}{3} b^3.$$

Group $W(D_4)$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{t_2^3}{4} t_4 + 6 t_2 t_3 t_4 + \frac{54}{35} t_4^7;$$

$$f = \frac{c^2}{12 a} - 6 b a^3.$$

Group $W(F_4)$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{t_3^4}{18} + \frac{3 t_3 t_4}{4} + \frac{t_2^3}{6} t_4 + \frac{t_2^2 t_4}{28} + \frac{t_3 t_4}{24} + \frac{t_3}{7} \cdot 13;$$

$$f = \frac{1}{\sqrt{a}} \left( a^7 + 14 b a^3 - 14 c \right)^{3/2}.$$ 

Swapping $t_2$ and $t_3$ gives a rational density $f$:

$$F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{t_3^4}{18} + \frac{3 t_3 t_4}{4} + \frac{t_2^3}{2} t_3 + \frac{t_2^2 t_4}{60} + \frac{t_2^2 t_4}{28} + \frac{t_3 t_4}{24} + \frac{t_3}{7} \cdot 13;$$

$$f = \frac{a^3}{600} - \frac{1}{10} c a^4 - \frac{1}{2} b^2 a^3 + \frac{3 c^2}{2 a}.$$ 

Group $W(H_4)$:
\[
F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{2 t_2^3 t_4}{3} + t_3^5 t_4 + \frac{t_2 t_3^3 t_4^3}{240} + \frac{t_2^2 t_3 t_4^5}{18} + \frac{t_3^4 t_4^7}{15} + \frac{t_3^4 t_4^7}{3^5 \cdot 3^7} \cdot 5
\]
\[+ \frac{t_2 t_3^2 t_4^9}{2 \cdot 3^4 \cdot 5} + \frac{8 t_2^2 t_4^{11}}{3^4 \cdot 5^2 \cdot 11} + \frac{t_3^3 t_4^{13}}{2^2 \cdot 3^6 \cdot 5^2} + \frac{2 t_3^2 t_4^{19}}{3^8 \cdot 5^3 \cdot 19} + \frac{32 t_4^{31}}{3^13 \cdot 5^5 \cdot 29 \cdot 31}; \]
\[f = \frac{a}{16} C^4 + \frac{a^7}{135} C^3 + \frac{a^3 b}{6} C^2 + \frac{a_1}{2^2 \cdot 3^5 \cdot 5^2} C^2 - a^3 b^2, \]
where \( C \) is defined by the cubic equation \( \frac{a}{12} C^3 + \frac{a_7}{90} C^2 + \frac{a^3 b}{3} C + \frac{a_3}{2 \cdot 3^3 \cdot 5} C^2 + \frac{a^3 b}{3 \cdot 5^3} C^2 + \frac{4 a^9}{3 \cdot 5^3} = c. \)
Swapping \( t_2 \) and \( t_3 \) gives a rational density \( f \):
\[
F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{2 t_3^3 t_4}{3} + \frac{t_2^3 t_4}{240} + \frac{t_2 t_3^2 t_4}{18} + \frac{t_2 t_3^2 t_4^3}{15} + \frac{t_4^7}{3^3 \cdot 3^7} \cdot 5
\]
\[+ \frac{t_2^2 t_3 t_4^9}{2 \cdot 3^4 \cdot 5} + \frac{8 t_2^2 t_4^{11}}{3^4 \cdot 5^2 \cdot 11} + \frac{t_3^3 t_4^{13}}{2^2 \cdot 3^6 \cdot 5^2} + \frac{2 t_3^2 t_4^{19}}{3^8 \cdot 5^3 \cdot 19} + \frac{32 t_4^{31}}{3^13 \cdot 5^5 \cdot 29 \cdot 31}; \]
\[f = \frac{32 a^2}{3^5 \cdot 5^5 \cdot 11^2} + \frac{4 c a^{10}}{5^2 \cdot 34 \cdot 11} + \frac{2 b^2 a^6}{5^2 \cdot 3^5 \cdot 5^2} - \frac{1}{30 b c a^4} - \frac{1}{18} b^3 a^3 + \frac{c^2}{8 a}. \]

Non-polynomial prepotentials [19] associated with extended affine Weyl groups can be found in [13]:
\[
F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 - \frac{1}{12} t_2^2 t_4^3 + \frac{1}{720} t_2^5 = \frac{1}{36288} t_3^8 + 2 t_2 t_3 e^4 + \frac{1}{6} t_3^4 e^4 + \frac{1}{2} e^{24 t_4} + \frac{1}{6} t_3^2 ;
\]
\[f = \frac{c^2}{756} + \frac{1}{48} b C^4 + \frac{4}{3} e^a C^3 - 2 b c a + \frac{b^3}{2 C^2}, \]
where \(- \frac{c^2}{64} + \frac{1}{36} b C^3 + 2 C^2 e^a - \frac{1}{6} b^2 + \frac{b^3}{3 c c} = c. \)
Swapping \( t_2 \) and \( t_3 \) gives:
\[
F = 12 g_1(a) - \frac{1}{6} b c^3 + \frac{1}{2} e^{2 a} b - 2 e^a b.
\]

Modular prepotentials [2] [32] give rise to modular Lagrangian densities (as an example we took prepotential 4.2.2. from [32]):
\[
F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 - \frac{1}{4} t_2^2 t_3^2 (t_4) + \frac{t_2^3 t_4 (t_4)}{2} + t_1^2 t_4 (t_4) + t_3^2 t_4 (t_4); \]
\[f = \frac{1}{12 g_1(a)} [c + \frac{1}{2} b^2 (a)]^2 - g_3(a) b^4. \]
Swapping \( t_2 \) and \( t_3 \) gives:
\[
F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 - \frac{1}{4} t_2^2 t_3^2 (t_4) + \frac{t_3^3 t_4 (t_4)}{2} + t_1^2 t_4 (t_4) + t_2^3 t_4 (t_4); \]
\[f = 24 C^5 g_4(a) + 8 b C^2 g_3(a), \]
where \( C \) is defined by the algebraic equation
\[
30 C^4 g_4 + 12 b C^2 g_3 - \frac{1}{2} b^2 C = c.
\]
Here \( g_3 = K g_1^3, \ g_4 = \frac{K_2}{30} (g_1 - \frac{1}{2} g_3) \gamma \) where the functions of \( \gamma (a) \) and \( g_1(a) \) satisfy the ODEs
\[
\gamma' = \frac{1}{2} \gamma^2 - 72 K g_1^4, \quad \gamma'' = 2 \gamma g_1 - g_1 \gamma', \]
\( K = \text{const} \). The above ODE system falls within Bureau’s class and its solutions are given in terms of the Schwarzian triangle functions [32].
2.7.5 WDVV prepotentials from integrable Lagrangian densities

In view of the correspondence between integrable Lagrangians and WDVV prepotentials

\[ F(t_1, t_2, t_3, t_4) = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + W(t_2, t_3, t_4) \]

described in Section 2.6, integrable Lagrangian densities \( f(a, b, c) \) constructed in this paper give rise to prepotentials some of which are apparently new. Here we list some examples (omitting details of calculations; we will only present the corresponding function \( W \)).

**Example 1.** The polynomial Lagrangian density from Section 2,

\[ f = b(a^2 - c^2), \]

gives rise to the prepotential

\[ W = \frac{1}{15} t_4^2 - t_1 t_2 t_3 + \frac{1}{3} t_4 t_1^3 - \frac{t_3^2}{12 t_2}. \]

**Example 2.** Lagrangian densities from Section 2.7.3 (linear case): the density

\[ f = \alpha e^{-b} + (e^a + e^c) e^b \]

gives rise to the prepotential

\[ W = -\alpha e^{t_4} - e^{-t_2} t_3 - e^{t_4} e^{t_2} t_3 - \frac{1}{2} t_2 t_3^2 + \frac{t_3^2}{2} \ln t_3; \]

the density

\[ f = \alpha q(b) + (e^a + e^c) \sinh b \]

gives rise to the prepotential

\[ W = \frac{1}{8} e^{2t_4} - e^{t_4} t_3 \sinh t_2 - \alpha e^{t_4} - \alpha q(t_2) t_3 + \frac{1}{2} t_3^2 \ln \frac{t_3}{\sinh t_2} - \frac{3}{4} t_3^2; \]

Here

\[ q(t_2) = \text{Li}_2(-e^{t_2}) - \text{Li}_2(e^{t_2}) \]

where \( \text{Li}_2 \) is the dilogarithm function.

**Example 3.** Lagrangian densities from Section 2.7.3 (quadratic case): the density

\[ f = e^{kb+a+c} \]

gives rise to the prepotential

\[ W = -\frac{1}{2} t_4 t_3^2 - \frac{k}{2} t_2 t_3^2 + \frac{t_3^2}{2} \ln t_3; \]

the density

\[ f = e^{\frac{1}{\sqrt{2}} b+a+c} + e^{\frac{2}{\sqrt{2}} b+2c} \]

gives rise to the prepotential

\[ W = \frac{t_3^2}{2} \ln t_3 - \frac{1}{2} t_4 t_3^2 - \frac{2}{\sqrt{3}} t_2 t_3^2 - \gamma e^{\frac{2}{\sqrt{3}} t_2+2t_4} t_3; \]

the density

\[ f = \alpha e^{-\frac{1}{\sqrt{2}} b} + \alpha e^{\frac{1}{\sqrt{2}} b+a} + e^{\frac{1}{\sqrt{2}} b+c} + e^{\frac{3}{\sqrt{2}} b+a+c} \]
Example 4. The Lagrangian density $f = e^g(b, a)$ from Section 2.7.2 gives rise to the prepotential (recall that system (10) is invariant under the interchange $a \leftrightarrow c$; for our convenience we choose $f = e^g(b, a)$ instead of $f = e^g(b, c)$):

$$W = \frac{t_3^2}{2} \ln \frac{t_3}{g(t_2, t_4)},$$

Here

$$g(t_2, t_4) = \left[\Delta(it_4/\pi)\right]^{-1/2} \theta_1(t_2, it_4/\pi)$$

where $\Delta$ is the modular discriminant and $\theta_1$ is the Jacobi theta function. Note the formula $\Delta^{1/8}(it_4/\pi) = \sqrt{2\pi} \theta_1(0, it_4/\pi)$ where prime denotes derivative by $t_2$. The corresponding solution of WDVV equations is related to Whitham averaged one-phase solutions of NLS/Toda equations [17], see also [11].

Example 5. The Lagrangian density $f = \sqrt{b(ab + \beta b)(\alpha b + \beta c)}$ from Section 3.4 gives rise to the prepotential

$$W = -\frac{\beta t_2}{4} (a t_4 + \beta t_2) \ln t_3 - \frac{\alpha}{2 \beta} t_2 t_3^2 + \frac{\beta^2 t_2^2}{8} \ln t_2 + \frac{1}{8} (a t_4 + \beta t_2)^2 \ln(a t_4 + \beta t_2).$$

3 Integrable Lagrangians in 3D

In this section we consider second-order integrable Lagrangians of the form (4).

$$\int f(u_{11}, u_{12}, u_{22}, u_{13}, u_{23}, u_{33}) \, dx_1 dx_2 dx_3,$$

here $u_{ij} = u_{x_i x_j}$.

3.1 Integrability conditions

Let us require that all travelling wave reductions of a 3D Lagrangian density to two dimensions are integrable in the sense of Sections 2.2 and 2.3. This gives the necessary conditions for integrability which, in our particular case, prove to be also sufficient. The computational details are as follows.

Consider a travelling wave reduction of a 3D Lagrangian density $f(u_{11}, u_{12}, u_{22}, u_{13}, u_{23}, u_{33})$ obtained by setting $u(x_1, x_2, x_3) = v(x, y) + Q$ where $x = x_1 + x_3, \, y = x_2 + x_3, \, s_1 = \text{const}$, and $Q$ is an arbitrary homogeneous quadratic form in $x_1, x_2, x_3$. We have

$$u_{11} = s_1^2 v_{xx} + \zeta_1, \quad u_{12} = s_1 s_2 v_{xy} + \zeta_2, \quad u_{22} = s_2^2 v_{yy} + \zeta_3, \quad u_{13} = s_1 s_3 (v_{xx} + v_{xy}) + \zeta_4, \quad u_{23} = s_2 s_3 (v_{xy} + v_{yy}) + \zeta_5, \quad u_{33} = s_3^2 (v_{xx} + 2v_{xy} + v_{yy}) + \zeta_6,$$

where $\zeta_i$ are the coefficients of the quadratic form $Q$. Setting $v_{xx} = a, \, v_{xy} = b, \, v_{yy} = c$ we obtain the reduced 2D Lagrangian density $f$ in the form

$$f(a, b, c) = f(u_{11}, u_{12}, u_{22}, u_{13}, u_{23}, u_{33}) = f(s_1^2 a + \zeta_1, s_1 s_2 b + \zeta_2, s_2^2 c + \zeta_3, s_1 s_3 (a + b) + \zeta_4, s_2 s_3 (b + c) + \zeta_5, s_3^2 (a + 2b + c) + \zeta_6).$$

We have the following differentiation rules:

$$\partial_a = s_1^2 \partial_{u_{11}} + s_1 s_3 \partial_{u_{13}} + s_3^2 \partial_{u_{33}},$$

$$\partial_b = s_1 s_2 \partial_{u_{12}} + s_1 s_3 \partial_{u_{13}} + s_2 s_3 \partial_{u_{23}} + 2 s_3^2 \partial_{u_{33}},$$

$$\partial_c = s_2^2 \partial_{u_{22}} + s_2 s_3 \partial_{u_{23}} + s_3^2 \partial_{u_{33}}.$$
Substituting partial derivatives of the reduced density $f(a, b, c)$ into the 2D integrability conditions (10) we obtain homogeneous polynomials of degree ten in $s_1, s_2, s_3$ whose coefficients are expressed in terms of partial derivatives of the original 3D density $f(u_{ij})$. Equating to zero the coefficients of these polynomials we obtain 3D integrability conditions for $f$ (note that due to the presence of arbitrary constants $\zeta_i$ the arguments of $f$ can be viewed as independent of $s_1, s_2, s_3$). The integrability conditions can be represented in compact Hirota-type form analogous to (16):

$$\left( P_1 Y_1 v - 2 P_0 Y_2 v \right) [ f(u_{ij}) \cdot f(\bar{u}_{ij}) ] \bigg|_{\bar{u}_{ij} = u_{ij}} = 0.$$  \hspace{1cm} (36)

Here the operators on the left-hand side of (36) are identical to that from Section 2.4, with the only difference that we substitute expressions (35) (and their tilded versions) for $\partial_a, \partial_b, \partial_c$ and $\partial_{\bar{a}}, \partial_{\bar{b}}, \partial_{\bar{c}}$. Thus, $\partial_a = s_1 s_3 \partial_{u_{11}} + s_1 s_3 \partial_{u_{13}} + s_2 s_3 \partial_{u_{33}}$, $\partial_{\bar{a}} = s_1 s_3 \partial_{\bar{u}_{11}} + s_1 s_3 \partial_{\bar{u}_{13}} + s_2 s_3 \partial_{\bar{u}_{33}}$, etc. The left-hand side of (36) is an $Sp(6)$-invariant operation which transforms a function $f$ defined on the space of $3 \times 3$ symmetric matrices $u_{ij}$ into a homogeneous form of degree four in $\xi_1, \xi_2$ and degree ten in $s_1, s_2, s_3$.

### 3.2 Equivalence group in 3D

Let $U$ be the $3 \times 3$ Hessian matrix of the function $u(x_1, x_2, x_3)$. Integrable Lagrangians of type (4) are invariant under $Sp(6)$-symmetry

$$U \rightarrow (AU + B)(CU + D)^{-1}, \hspace{1cm} f \rightarrow \frac{f}{\det(CU + D)},$$  \hspace{1cm} (37)

where the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to the symplectic group $Sp(6, \mathbb{R})$ (here $A, B, C, D$ are $3 \times 3$ matrices). Note that symmetry (37) suggests a relation to genus three Siegel modular forms. Furthermore, integrable Lagrangians (1) are invariant under rescalings of $f$, as well as under the addition of a ‘null-Lagrangian’, namely, transformations of the form

$$f \rightarrow \lambda_0 f + \sum \lambda_{\sigma} U_{\sigma},$$  \hspace{1cm} (38)

where $U_{\sigma}$ denote all possible minors of the Hessian matrix $U$. Transformations (37) and (38) generate a group of dimension $21 + 15 = 36$ which preserves the class of integrable Lagrangians (4).

### 3.3 Examples of integrable Lagrangians in 3D

In this section we give some explicit examples of 3D integrable Lagrangian densities $f$.

#### 3.3.1 Integrable Lagrangians associated with the dKP hierarchy

Here we construct three explicit integrable Lagrangian densities arising in the context of the dKP hierarchy:

$$f = u_{yy}^2 - u_{xx} u_{xt} + u_{xx}^2 u_{yy} + u_{xx} u_{xy}^2 + \frac{1}{4} u_{xx}^4,$$

$$f = (u_{xy} - u_{tt} - u_{xx} u_{xt} + \frac{1}{3} u_{xx}^3)^{3/2},$$

$$f = u_{xt}^{-2}(u_{xt} u_{yt} - u_{xx} u_{xt}^2)^{3/2}.$$  \hspace{1cm} (41)

These examples come from the following dKP flows.
Case 1. The fifth-order flow of the dKP hierarchy comes from the dispersionless Lax representation

\[ p_y = \left( \frac{p^2}{2} + w \right)_x, \quad p_t = \left( \frac{p^5}{5} + wp^3 + vp^2 + bp + c \right)_x, \]

which gives rise to the equations

\[ w_y = v_x, \quad b_x = v_y + 3wv_x, \quad c_x = b_y + 2vw_x, \quad w_t = bw_x + cy. \]

Setting \( w = u_{xx}, v = u_{xy} \) and \( b = u_{yy} + \frac{3}{2}u_{xx}^2 \) we obtain two equations for \( c, \)

\[ c_x = u_{yy}u_{xx} + 3u_{xx}u_{xy} + 2u_{xy}u_{xxx}, \quad c_y = u_{xx} - u_{yy}u_{xxx} - \frac{3}{2}u_{xx}^2u_{xxx}, \]

whose compatibility condition results in the following fourth-order PDE for \( u: \)

\[ u_{yyyy} - u_{xx}u_{xxyy} + 2u_{xy}u_{xxyy} + 3u_{xxy}^2 + 3u_{xxxy}u_{xyy} + u_{yy}u_{xxxx} + \frac{3}{2}u_{xx}^2u_{xxx} + 3u_{xx}u_{xxx}^2 = 0. \]

This is the Euler-Lagrange equation corresponding to the polynomial Lagrangian density (39). Note that the two-dimensional density (3) is just the stationary \((t\text{-independent})\) reduction of (39).

Case 2. Another flow of the dKP hierarchy is associated with the Lax representation

\[ p_t = \left( \frac{p^3}{3} + wp + v \right)_x, \quad p_y = \left( \frac{p^5}{5} + wp^3 + vp^2 + bp + c \right)_x, \]

which gives rise to the equations

\[ b_t = -bw_x + wb_x - 2vv_x + vy, \quad c_t = wc_x - bw_x + vy, \]

\[ w_t = -2wv_x + bx, \quad v_t = -2wv_x - 2vv_x + cx. \]

Setting \( w = u_{xx}, b = u_{xt} + u_{xx}^2 \) and \( v^2 = u_{xy} - u_{tt} - u_{xx}u_{xt} + \frac{1}{3}u_{xx}^3 \) we obtain two equations for \( c, \)

\[ c_t = v_y + u_{xx}v_t + 2vu_{xx}u_{xxx} + (u_{xx}^2 - u_{xt})v_x, \quad c_x = v_t + 2vu_{xxx} + 2u_{xx}v_x, \]

whose compatibility condition yields

\[ v_{tt} + (vu_{xx})_{xt} = v_{xy} + (v(u_{xx}^2 - u_{xt}))_{xx}. \]

This PDE is the Euler-Lagrange equation corresponding to the density (40).

Case 3. This example comes from the dispersionless Lax pair

\[ p_t = \left( \frac{r}{p - q} \right)_x, \quad p_y = \left( \frac{p^3}{3} + wp + v \right)_x, \]

which gives rise to the equations

\[ q_y = q^2q_x + qw_x + wq_x + v_x, \quad r_y = q^2r_x + 2qrq_x + rw_x + wr_x, \quad w_t = -r_x, \quad v_t = -qr_x - rq_x. \]

Setting \( w = u_{xx}, r = -u_{xt}, q^2 = \frac{u_{xx}}{u_{xt}} - u_{xx}, \) the second and the third equations will be satisfied identically, while the first and the fourth imply

\[ v_x = q_y - \left( \frac{1}{3}q^3 + qu_{xx} \right)_x, \quad v_t = (u_{xt}q)_x, \]

whose consistency condition gives

\[ q_{yt} - \frac{1}{3}q \left( \frac{u_{yt}}{u_{xt}} + 2u_{xx} \right)_{xt} = (u_{xt}q)_{xx}. \]

This is the Euler-Lagrange equation corresponding to the density (41).
3.3.2 Integrable Lagrangian densities of the form $f = f(u_{xy}, u_{xt}, u_{yt})$

We will show that the general integrable density of this form is expressible in terms of the Lobachevsky function $L(s) = -\int_0^s \cos \xi \, d\xi$, a special function which features in Lobachevsky’s formulae for hyperbolic volumes [28].

Using the notation $u_{xy} = v_3$, $u_{xt} = v_2$, $u_{yt} = v_1$ and $f_{ij} = f_{i,v_j}$, one can show that the integrability conditions (which we do not present here explicitly) can be rewritten as simple relations for the $2 \times 2$ minors of the Hessian matrix $F = Hess f$. Namely, they are equivalent to the conditions that the minors

$$f_{11}f_{22} - f_{12}^2 = a_3, \quad f_{11}f_{33} - f_{13}^2 = a_2, \quad f_{22}f_{33} - f_{23}^2 = a_1,$$

and

$$f_{12}f_{13} - f_{11}f_{23} = p_1, \quad f_{12}f_{23} - f_{22}f_{13} = p_2, \quad f_{13}f_{23} - f_{33}f_{12} = p_3,$$

are such that $a_i = \text{const}$ and $p_i$ is a function of the argument $v_i$ only. This gives the inverse of the Hessian matrix $F$ in the form

$$\left( \begin{array}{ccc}
    f_{11} & f_{12} & f_{13} \\
    f_{12} & f_{22} & f_{23} \\
    f_{13} & f_{23} & f_{33}
\end{array} \right)^{-1} = \frac{1}{\det F} \left( \begin{array}{ccc}
    a_1 & p_3 & p_2 \\
    p_3 & a_2 & p_1 \\
    p_2 & p_1 & a_3
\end{array} \right).$$

(42)

Taking the determinant of both sides we obtain

$$\det F = \sqrt{a_1a_2a_3 - a_1p_1^2 - a_2p_2^2 - a_3p_3^2 + 2p_1p_2p_3}.$$  

(43)

Inverting the matrix identity (42) gives

$$\left( \begin{array}{ccc}
    f_{11} & f_{12} & f_{13} \\
    f_{12} & f_{22} & f_{23} \\
    f_{13} & f_{23} & f_{33}
\end{array} \right) = \frac{1}{\det F} \left( \begin{array}{ccc}
    a_2a_3 - p_1^2 & p_1p_2 - a_3p_3 & p_1p_3 - a_2p_2 \\
    p_1p_2 - a_3p_3 & a_1a_3 - p_2^2 & p_2p_3 - a_1p_1 \\
    p_1p_3 - a_2p_2 & p_2p_3 - a_1p_1 & a_1a_2 - p_3^2
\end{array} \right).$$

(44)

where we use (43) for $\det F$. The consistency conditions of equations (44) lead to simple ODEs for the functions $p_i(v_i)$:

$$p_1' = c(p_1^2 - a_2a_3), \quad p_2' = c(p_2^2 - a_1a_3), \quad p_3' = c(p_3^2 - a_1a_2),$$

where $c$ is yet another arbitrary constant. The further analysis depends on how many constants among $a_i$ are equal to zero.

**All constants are zero.** In this case without any loss of generality one can set $p_i = 1/v_i$ which leads to the integrable Lagrangian density

$$f = \sqrt{u_{xy}u_{xt}u_{yt}}.$$  

**Two constants are zero.** Then one can also set $p_i = 1/v_i$. Modulo (complex) rescalings this leads to the Lagrangian density

$$f = \sqrt{u_{xy}u_{xt}(2u_{yt} - u_{xy}u_{xt})} - 2u_{yt} \arctan \sqrt{\frac{2u_{yt}}{u_{xy}u_{xt}}} - 1.$$  

**One constant is zero.** This leads to the Lagrangian density

$$f = (u_{xt} - u_{xy}) \arctan \sqrt{2u_{xt}u_{xy} \coth u_{xt} - u_{xy}^2 - (u_{xt} + u_{xy})} \arctan \frac{\sqrt{2u_{xt}u_{xy}\coth u_{xt} - u_{xy}^2}}{u_{xt} + u_{xy}}.$$  

**All constants are nonzero.** This case is more interesting. Setting $a_i = 1, \ c = -1$ we obtain $p_i' = 1 - p_i^2$ so that $p_i = \tanh v_i$. Equations (44) can be integrated once to yield

$$f_1 = \arcsin \frac{p_1 - p_2p_3}{\sqrt{1 - p_2^2}\sqrt{1 - p_3^2}}, \quad f_2 = \arcsin \frac{p_2 - p_1p_3}{\sqrt{1 - p_1^2}\sqrt{1 - p_3^2}}, \quad f_3 = \arcsin \frac{p_3 - p_1p_2}{\sqrt{1 - p_1^2}\sqrt{1 - p_2^2}}.$$
Choosing 

\[ f_{p_1} = \frac{1}{1-p_1^2} \arcsin \frac{p_1-p_2 p_3}{\sqrt{1-p_2^2} \sqrt{1-p_3^2}}; \]

\[ f_{p_2} = \frac{1}{1-p_2^2} \arcsin \frac{p_2-p_1 p_3}{\sqrt{1-p_1^2} \sqrt{1-p_3^2}}; \]

\[ f_{p_3} = \frac{1}{1-p_3^2} \arcsin \frac{p_3-p_1 p_2}{\sqrt{1-p_1^2} \sqrt{1-p_2^2}}, \]

or, in differentials,

\[ df = \frac{dp_1}{1-p_1^2} \arcsin \frac{p_1-p_2 p_3}{\sqrt{1-p_2^2} \sqrt{1-p_3^2}} + \frac{dp_2}{1-p_2^2} \arcsin \frac{p_2-p_1 p_3}{\sqrt{1-p_1^2} \sqrt{1-p_3^2}} + \frac{dp_3}{1-p_3^2} \arcsin \frac{p_3-p_1 p_2}{\sqrt{1-p_1^2} \sqrt{1-p_2^2}}, \]

(45)

In the original variables \( v_1, v_2, v_3 \) relation (45) takes the form

\[ df = \arcsin(\cosh v_2 \cosh v_3 \tanh v_1 - \sinh v_2 \sinh v_3) \, dv_1 \]

\[ + \arcsin(\cosh v_1 \cosh v_3 \tanh v_2 - \sinh v_1 \sinh v_3) \, dv_2 \]

\[ + \arcsin(\cosh v_1 \cosh v_2 \tanh v_3 - \sinh v_1 \sinh v_2) \, dv_3. \]

**Remark 9.** Relation (45) has an unexpected link to spherical trigonometry. On the unit sphere \( S^2 \), consider a spherical triangle \( \triangle ABC \) with interior angles \( A, B, C \) and side lengths \( a, b, c \) (so that side \( a \) lies opposite the angle \( A \), etc.). The spherical laws of cosines are

\[
\begin{align*}
\cos a &= \cos b \cos c + \sin b \sin c \cos A, \\
\cos b &= \cos a \cos c + \sin a \sin c \cos B, \\
\cos c &= \cos a \cos b + \sin a \sin b \cos C,
\end{align*}
\]

(46)

and

\[
\begin{align*}
\cos A &= -\cos B \cos C + \sin B \sin C \cos a, \\
\cos B &= -\cos A \cos C + \sin A \sin C \cos b, \\
\cos C &= -\cos A \cos B + \sin A \sin B \cos c,
\end{align*}
\]

(47)

respectively. Note that the map \((A, B, C) \to (a, b, c)\) sending angles of a spherical triangle to its side lengths is integrable in the sense of multidimensional consistency [36], and is closely related to the discrete Darboux system [5] [20]. Setting

\[ p_1 = \cos a, \quad p_2 = \cos b, \quad p_3 = \cos c \]

(48)

and using (46) we can rewrite (45) in the following Schlafly-type form:

\[ df = (A - \pi/2) \frac{da}{\sin a} + (B - \pi/2) \frac{db}{\sin b} + (C - \pi/2) \frac{dc}{\sin c}, \]

(49)

Recall that the classical Schlafly formula expresses the differential of the volume of a spherical polyhedron in terms of its side lengths and dihedral angles. Expression (49), which can be viewed as a two-dimensional Schlafly formula, has appeared in [12] [29] as a special case of a one-parameter family of closed Schlafly-type forms associated with spherical triangles (case \( h = 0 \) of Theorem 3.2(b) in [29]). Note that the function \( f \) of a spherical triangle defined by (49) is essentially the volume of the ideal hyperbolic octahedron which is the convex hull of the six intersection points of the three circles on the sphere at infinity bounding the spherical triangle \( \triangle ABC \) [29], Appendix C, see also [30]. This function \( f \) is related to the ‘capacity’ function of a spherical triangle. Expressions similar to (49) have appeared before in the context of variational principles for circle packings and triangulated surfaces [10] [4] [6] [12].
Integration of \( \text{(45)} \) is quite non-trivial, the answer is given in terms of the Lobachevsky function. The computations below are based on formula \( \text{(49)} \) and the spherical cosine laws \( \text{(46), (47)} \). Using \( \frac{da}{\sin a} = \frac{1}{2} d \ln \frac{1 - \cos a}{1 + \cos a} \) we can rewrite \( df \) in the form

\[
df = -\frac{1}{2} d \left( \ln \frac{1 - \cos a}{1 + \cos a} + \ln \frac{1 - \cos b}{1 + \cos b} + \ln \frac{1 - \cos c}{1 + \cos c} \right)
\]

\[
+ \frac{3}{2} d \ln \frac{1 - \cos a}{1 + \cos a} + \frac{B}{2} d \ln \frac{1 - \cos b}{1 + \cos b} + \frac{C}{2} d \ln \frac{1 - \cos c}{1 + \cos c}.
\]

Equivalently,

\[
df = -\frac{1}{2} d \left( \ln \frac{1 - \cos a}{1 + \cos a} + \ln \frac{1 - \cos b}{1 + \cos b} + \ln \frac{1 - \cos c}{1 + \cos c} \right)
\]

\[
+ d \left( \frac{A}{2} \ln \frac{1 - \cos a}{1 + \cos a} + \frac{B}{2} \ln \frac{1 - \cos b}{1 + \cos b} + \frac{C}{2} \ln \frac{1 - \cos c}{1 + \cos c} \right)
\]

\[
- \frac{1}{2} \left( \ln \frac{1 - \cos a}{1 + \cos a} dA + \ln \frac{1 - \cos b}{1 + \cos b} dB + \ln \frac{1 - \cos c}{1 + \cos c} dC \right).
\]

Let us rewrite the last term of this expression as a total differential. Note that using \( \text{(47)} \) we have

\[
\frac{1 - \cos a}{1 + \cos a} = \frac{\sin B \sin C - \cos A - \cos B \cos C}{\sin B \sin C + \cos A + \cos B \cos C} = -\cos A - \cos B + \cos C = \frac{\cos 2\pi - A - B - C}{\cos \frac{A + B - C}{2} \cos \frac{A - B - C}{2}}.
\]

With similar formulae for \( \frac{1 - \cos b}{1 + \cos b} \) and \( \frac{1 - \cos c}{1 + \cos c} \) we obtain:

\[
-\frac{1}{2} \left( \ln \frac{1 - \cos a}{1 + \cos a} dA + \ln \frac{1 - \cos b}{1 + \cos b} dB + \ln \frac{1 - \cos c}{1 + \cos c} dC \right)
\]

\[
= -\frac{1}{2} \ln \frac{\cos 2\pi - A - B - C}{\cos \frac{A + B - C}{2} \cos \frac{A - B - C}{2}} dA
\]

\[
- \frac{1}{2} \ln \frac{\cos 2\pi - A - B - C}{\cos \frac{A + B - C}{2} \cos \frac{A - B - C}{2}} dB
\]

\[
- \ln \frac{\cos 2\pi - A - B - C}{\cos \frac{A + B - C}{2} \cos \frac{A - B - C}{2}} dC
\]

\[
= \ln \cos \frac{2\pi - A - B - C}{2} d \left( \frac{2\pi - A - B - C}{2} \right) + \ln \cos \frac{A + B - C}{2} d \left( \frac{A + B - C}{2} \right)
\]

\[
+ \ln \cos \frac{A + C - B}{2} d \left( \frac{A + C - B}{2} \right) + \ln \cos \frac{B + C - A}{2} d \left( \frac{B + C - A}{2} \right).
\]

On integration, we obtain the final formula for \( f \):

\[
f = -\frac{1}{2} \left( \ln \frac{1 - \cos a}{1 + \cos a} + \ln \frac{1 - \cos b}{1 + \cos b} + \ln \frac{1 - \cos c}{1 + \cos c} \right)
\]

\[
+ \frac{3}{2} d \ln \frac{1 - \cos a}{1 + \cos a} + \frac{B}{2} d \ln \frac{1 - \cos b}{1 + \cos b} + \frac{C}{2} d \ln \frac{1 - \cos c}{1 + \cos c}
\]

\[
- L \left( \frac{2\pi - A - B - C}{2} \right) - L \left( \frac{A + B - C}{2} \right) - L \left( \frac{A + C - B}{2} \right) - L \left( \frac{B + C - A}{2} \right),
\]

where \( L(s) = -\int_0^s \ln \cos \xi \ d\xi \) is the Lobachevsky function. In the original variables \( v_1, v_2, v_3 \) defined as \( p_1 = \cos a = \tanh v_1, p_2 = \cos b = \tanh v_2, p_3 = \cos c = \tanh v_3 \) this gives:

\[
f = \frac{\pi}{2} \left( v_1 + v_2 + v_3 \right) - (Av_1 + Bv_2 + Cv_3)
\]

\[
- L \left( \frac{2\pi - A - B - C}{2} \right) - L \left( \frac{A + B - C}{2} \right) - L \left( \frac{A + C - B}{2} \right) - L \left( \frac{B + C - A}{2} \right);
\]

hence \( A, B, C \) are defined, as functions of \( v_1, v_2, v_3 \), via the spherical cosine laws:

\[
\cos A = \frac{\cos a - \cos \theta \cos \varphi \cos \psi}{\sin \psi \sin \varphi} = \cosh v_2 \cosh v_3 \tan v_1 - \sinh v_2 \sinh v_3,
\]

\[
\cos B = \frac{\cos b - \cos a \cos \theta \cos \psi}{\sin \theta \sin \varphi} = \cosh v_1 \cosh v_3 \tan v_2 - \sinh v_1 \sinh v_3,
\]

\[
\cos C = \frac{\cos c - \cos \theta \cos \varphi \cos \psi}{\sin \theta \sin \varphi} = \cosh v_1 \cosh v_2 \tan v_3 - \sinh v_1 \sinh v_2.
\]
Note that the linear term $\frac{2}{5}(v_1 + v_2 + v_3)$ can be ignored as it does not affect the Euler-Lagrange equations corresponding to the density $f$.

**Euler-Lagrange equation.** In terms of the side lengths $a, b, c$ and angles $A, B, C$ of a spherical triangle, the Euler-Lagrange equation corresponding to the density $f$ takes the form

$$A_{yt} + B_{xt} + C_{xy} = 0,$$

(50)

where we keep in mind the spherical cosine laws (46), (47). Indeed, the first relation is equivalent to $(f_1)_{yt} + (f_2)_{xt} + (f_3)_{xy} = 0$, while the second set of relations comes from the consistency conditions of relations (48) rewritten in the form $u_{xy} = \arctanh(\cos c)$, $u_{xt} = \arctanh(\cos b)$, $u_{yt} = \arctanh(\cos a)$.

**Remark 10.** Similar analysis of the Lagrangian densities $f = f(u_{xx}, u_{yy}, u_{tt})$ gives no interesting examples: one can show that in this case the integrability conditions imply that all $2 \times 2$ minors of the Hessian matrix of $f$ must necessarily be constant, thus leading to quadratic densities $f$ with linear Euler-Lagrange equations.

### 3.4 2D densities as travelling wave reductions of 3D densities

Given a 3D integrable Lagrangian density $f(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt})$ one can apply a travelling wave ansatz, $u(x, y, t) = u(\xi, \eta)$ where $\xi = a_1 x + a_2 y + a_3 t$, $\eta = b_1 x + b_2 y + b_3 t$, to obtain an integrable 2D Lagrangian density of the form $f(u_{\xi}, u_{\eta})$. In fact, modulo linear transformations of $\xi$ and $\eta$ it is sufficient to assume $\xi = x + \alpha t$, $\eta = y + \beta t$. Applying this construction to the density $f = \sqrt{u_{xy}u_{xt}u_{yt}}$ found in Section 3.3.2 one obtains 2D integrable densities of the form

$$f = \sqrt{u_{\xi\eta}(\alpha u_{\xi} + \beta u_{\eta})(\alpha u_{\xi} + \beta u_{\eta})}.$$

### 4 Dispersive deformations of integrable Lagrangian densities

Some integrable Lagrangian densities possess integrable dispersive deformations (both in 2D and 3D). Here we give three examples (a complete classification of integrable dispersive deformations is a non-trivial open problem).

**Example 1.** The Lagrangian density (39),

$$f = u_{yy}^2 - u_{xx}u_{xt} + u_{xx}^2u_{yy} + u_{xx}u_{xy}^2 + \frac{1}{4}u_{xx}^4,$$

(Section 3.3.1 case 1) possesses integrable dispersive deformation

$$f_\epsilon = u_{yy}^2 - u_{xx}u_{xt} + u_{xx}^2u_{yy} + u_{xx}u_{xy}^2 + \frac{1}{4}u_{xx}^4 - \epsilon^2u_{xx}u_{xx}^2 - \epsilon^2\frac{1}{2}u_{xy}^2 + \frac{3}{80}u_{xxxx}^2,$$

here $\epsilon$ is a deformation parameter. The corresponding (dispersive) Euler-Lagrange equation has the Lax pair

$$\epsilon\psi_y = \frac{\epsilon^2}{2}\psi_{xx} + a\psi, \quad \epsilon\psi_t = \frac{\epsilon^3}{5}\psi_{xxxx} + \epsilon^2b\psi_{xx} + \epsilon c\psi_x + w\psi,$$

where $a = u_{xx}$, $b = u_{xy} + \frac{\epsilon^2}{4}u_{xxxx}$, $c = u_{yy} + \frac{3}{2}u_{xx} + cu_{xx} + \frac{5\epsilon^2}{4}u_{xxxx}$, and the variable $w$ is defined by the equations

$$w_x = u_{yyyy} + 3u_{xx}u_{xy} + 2u_{xy}u_{xxx} + \frac{3\epsilon}{2}u_{xxx}^2 + \epsilon u_{xy} + \epsilon u_{xx}^2 + \frac{3\epsilon^2}{4}u_{xxx} + \frac{3\epsilon^3}{8}u_{xxxx},$$

$$w_y = u_{xx} + \frac{\epsilon}{2}u_{xy} + \frac{\epsilon^2}{4}u_{xxx} + \frac{3\epsilon^3}{8}u_{xxxx} + \frac{\epsilon^4}{80}u_{xxxxx}. $$

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The stationary reduction of Example 1 provides dispersive deformation of the two-dimensional density (3).

**Example 2.** The Lagrangian density (40),

\[ f = \left( u_{xy} - u_{tt} - u_{xx}u_{xt} + \frac{1}{3}u_{xx}^3 \right)^{3/2}, \]

(Section 3.3.1 case 2) possesses integrable dispersive deformation

\[ f_\epsilon = \left( u_{xy} - u_{tt} - u_{xx}u_{xt} + \frac{1}{3}u_{xx}^3 + \frac{\epsilon^2}{12}(4u_{xx}u_{xxxx} + 3u_{xxx}^2 - 4u_{xxxx}) + \frac{\epsilon^4}{45}u_{xxxxxx} \right)^{3/2}. \]

The corresponding dispersive Euler-Lagrange equation has the Lax pair

\[
\begin{align*}
\epsilon \psi_t &= \frac{\epsilon^3}{3} \psi_{xxx} + \epsilon w \psi_x + v \psi, \\
\epsilon \psi_y &= \frac{\epsilon^5}{5} \psi_{xxxxx} + \epsilon^3 w \psi_{xxx} + \epsilon^2 (v + \epsilon w) \psi_{xx} + \epsilon b \psi_x + c \psi,
\end{align*}
\]

where

\[ w = u_{xx}, \quad v = f_t^{1/3} + \frac{\epsilon}{2} u_{xx}, \quad b = u_{xx} + u_{xx}^2 + \frac{2\epsilon^2}{3} u_{xxxx} + \epsilon v_x, \]

and the function \( c \) is determined by the equations

\[
\begin{align*}
c_x &= v_t + 2(u_{xx} v)_x + \frac{2}{3}\epsilon^2 v_{xxx}, \\
c_t &= v_y + (u_{xx} v)_t + \left( v u_{xx}^2 - v u_{xx} - \epsilon v v_x - \frac{\epsilon^2}{3} u_{xx} v_{xx} - \frac{2\epsilon^2}{3} v_x u_{xxx} + \frac{\epsilon^2}{3} c_x - \frac{\epsilon^4}{5} v_{xxxx} \right)_x.
\end{align*}
\]

**Example 3.** The Lagrangian density (41),

\[ f = u_{x}^{-2} \left( u_{xt} u_{yt} - u_{xx} u_{xt}^2 \right)^{3/2}, \]

(Section 3.3.1 case 3) possesses integrable dispersive deformation

\[ f_\epsilon = u_{x}^{-2} \left( u_{xt} u_{yt} - u_{xx} u_{xt}^2 + \frac{\epsilon^2}{4} u_{xx}^2 - \frac{\epsilon^2}{3} u_{xxx} u_{xx} \right)^{3/2}. \]

The corresponding dispersive Euler-Lagrange equation comes from the Lax pair

\[
\begin{align*}
\epsilon \psi_y &= \frac{\epsilon^3}{3} \psi_{xxx} + \epsilon w \psi_x + v \psi, \\
\epsilon^2 \psi_{xt} &= \epsilon q \psi_t + r \psi,
\end{align*}
\]

where

\[ w = u_{xx}, \quad r = -u_{xx}, \quad q = \left( \frac{f_\epsilon}{u_{xx}} \right)^{1/3} + \frac{\epsilon}{2} \frac{u_{xx}}{u_{xx}}, \]

and the variable \( v \) is defined by the equations

\[
\begin{align*}
v_t &= (u_{xx} q)_x, \quad v_x = q_y - \left( u_{xx} q + \frac{1}{3} q^3 + \epsilon qq_x + \frac{\epsilon^2}{3} q_{xx} \right)_x.
\end{align*}
\]

5 **Concluding remarks**

Here we list some problems for further study.
• **Multi-dimensional Lagrangians.** It would be of interest to describe multi-dimensional versions of second-order integrable Lagrangians. Thus, anti-self-dual four-manifolds with a parallel real spinor are described by the integrable 4D Dunajski system \[18\]

\[
a_{zt} + a_{yz} + u_{xx}a_{yy} + u_{yy}a_{xx} - 2u_{xy}a_{xy} = 0,
\]

\[
u_{zt} + u_{yz} + u_{yy}u_{xx} - u_{xy}^2 = a,
\]

which can be written as a single fourth-order PDE for the function \(u\). This PDE comes from the second-order Lagrangian

\[
\int (u_{zt} + u_{yz} + u_{yy}u_{xx} - u_{xy}^2)^2 \, dx dy dz dt.
\]

Similarly, anti-self-dual scalar flat four-manifolds (Flaherty-Park spaces, see \[37\] and references therein) are governed by the equations

\[
u_{xz}(\ln F)_{yt} - u_{xy}(\ln F)_{zt} - u_{zt}(\ln F)_{xy} + u_{yt}(\ln F)_{zz} = 0,
\]

\[
u_{xz}u_{yt} - u_{xy}u_{zt} = F,
\]

which are equivalent to a single fourth-order PDE for \(u\). The corresponding Lagrangian is

\[
S = \int [F \ln F - F] \, dx dy dz dt,
\]

where one has to substitute \(F = u_{xz}u_{yt} - u_{xy}u_{zt}\).

• **Multi-component Lagrangians.** Our approach can be generalised in a straightforward way to describe 2-field integrable Lagrangians of the form

\[
\int f(u_x, u_y, v_x, v_y) \, dx dy,
\]

as well as their 3D analogues,

\[
\int f(u_x, u_y, u_t, v_x, v_y, v_t) \, dx dy dt.
\]

• **Higher-order quasilinear PDEs.** Similarly, one can classify third-order integrable PDEs of the form

\[
a_1u_{xxx} + a_2u_{xxy} + a_3u_{xyy} + a_4u_{yyy} = 0
\]

where the coefficients \(a_i\) are functions of the second-order derivatives \(u_{xx}, u_{xy}, u_{yy}\) only. This problem also has a natural 3D analogue.

**Acknowledgments**

We thank A. Basalaev, A. Bobenko, L. Bogdanov, Yu. Brezhnev, D. Guzzetti, A. Mednykh, V. Shramchenko, I. Strachan and A. Verbovetsky for useful discussions. EVF was supported by the EPSRC grant EP/N031369/1. MVP was supported by the Russian Foundation for Fundamental Research (grant 18-51-18007) and an LMS scheme 4 grant. LX was supported by the National Natural Science Foundation of China (grant numbers: 11501312 and 11775121).
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