Covers for self-dual supercuspidal representations of the Siegel Levi subgroup of classical $p$-adic groups

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Abstract

We study components of the Bernstein category for a $p$-adic classical group (with $p$ odd) with inertial support a self-dual positive level supercuspidal representation of a Siegel Levi subgroup. More precisely, we use the method of covers to construct a Bushnell-Kutzko type for such a component. A detailed knowledge of the Hecke algebra of the type should have number-theoretic implications.

Introduction

Let $F_0$ be a locally compact nonarchimedean local field, and $G$ the $F_0$-points of a connected reductive quasi-split algebraic group $G$ defined over $F_0$. The smooth representation theory of $G$ plays a role in automorphic forms, and is therefore of deep interest to number theorists. A theorem of Bernstein shows that, in the terminology of [6], it is enough to understand the subcategories $\mathcal{R}^p(G)$ of smooth representations with supercuspidal support in a given inertial class $s$. The class $s = [L, \pi]_G$ is determined by a Levi subgroup $L \subset G$ and an irreducible supercuspidal representation $\pi$ of $L$. The theory of types and covers [6] gives one a method for trying to understand the subcategories $\mathcal{R}^p(G)$. In particular, if one knows supercuspidal types for all Levi subgroups of $G$, and how to construct $G$-covers of all such types, then one has a complete set of types for $G$. Here, we study the case where $F$ has odd residual characteristic, $G$ is a classical group, and $s = [L, \pi]_G$ with $L$ a Siegel Levi subgroup and $\pi$ an irreducible supercuspidal representation fixed by the non-trivial Weyl group element. While this work gives a complete set of $G$-covers (and hence types) in this case, the theory cannot be considered complete without knowledge of the associated Hecke algebras. As shown in [15], a precise understanding of such Hecke algebras gives information on reducibility of induced representations and therefore determines poles of certain Langlands $L$-functions. These considerations will be the subject of future work.

Let $G$ be any of the groups $Sp_{2M}$, $SO_{2M}$, $SO_{2M+1}$, $U_n$ or $SO^*_n$, i.e., a symplectic, quasi-split special orthogonal group, or quasi-split unitary group. (Here $SO^*_n$ stands for a quasi-split but non-split special orthogonal group.) Let $B = TN$ be a Borel subgroup with unipotent radical $N$ and

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maximal torus $T$. There is a choice of standard maximal parabolic subgroup $P = LU$ containing $B$ whose Levi component $L$ is isomorphic to either $\text{GL}_M \times G_0$, or $\text{Res}_{F/F_0}(\text{GL}_M) \times G_0$; here $G_0$ is an anisotropic group and $F/F_0$ is the quadratic extension defining the unitary group. We study the theory of types for the inertial class $s = [L, \pi]_G$, where $\pi$ is fixed by the non-trivial element $w_0$ of the Weyl group $N_G(L)/Z_G(L)$. The work of Bushnell and Kutzko gives one a type $(J_L, \lambda_L)$ for the inertial class $s_L = [L, \pi_L]_L$. The purpose of this paper is to construct a cover $(J, \lambda)$ of $(J_L, \lambda_L)$ for each group $G$. (This was done by Blondel when $G = \text{Sp}_{2M}$; her methods are somewhat different.) Therefore, by [6], the pair $(J, \lambda)$ is a type for $s$. One then knows that the category $\mathcal{H}_s(G)$ is isomorphic to the category $\mathcal{H}(G, \lambda)\text{-Mod}$ of unital (left) modules over the $\lambda$-spherical Hecke algebra.

This isomorphism of categories has implications both for the location of poles of certain $L$-functions, and for the classification of local galois representations. One example of this may be seen in recent work of Kutzko and Morris, where $G$ is one of the groups $\text{Sp}_{2M}$, $\text{SO}_{2M}$, or $\text{SO}_{2M+1}$, and $\pi$ is a level zero self-dual cuspidal representation. Specific information about the Hecke algebra $\mathcal{H}(G, \lambda)$ is employed to give a purely local proof of Shahidi’s theorem on the reducibility of parabolic induction, which can also be described in terms of the poles of $L$-functions. On the other hand, recent work of Henniart, building on work of Harris and Taylor and Henniart, relates this to the classification of local galois representations by their images (e.g. symplectic, orthogonal). In the case of the level zero representations in [15], the classification of galois representations obtained via the Hecke algebra isomorphism was known already. However, in the general situation of this paper, the corresponding classification of galois representations is not fully understood and has been sought for some time. We hope to address this problem in a sequel.

We now give a summary of the contents of the paper. In [11] we review the theory of types for self-dual representations, via results of Blondel, which is crucial to the construction (see [2] Proposition 2.2). In [12] we fix a self-dual representation $\pi$ of $L$, and use these results to define a well-adapted inner product on an $F$-vector space $V$, which defines the group $G$. In [14] we show that there is a choice of simple stratum (see [5]), defining the type in $\pi$, which is particularly well suited. In particular, it gives rise to a skew semisimple stratum (see [23]) in $\text{End}_F(V)$, which underpins the construction of $G$-cover.

We construct the $G$-cover $(J, \lambda)$ in [22], following the recipe of [21] §7. The first stages, when we are working with pro-$p$ groups, are performed first in $\text{Aut}_F(V)$ (see [21]), and then transferred to $G$ using the Glauberman correspondence (see [22]). To verify that $(J, \lambda)$ is indeed a $G$-cover, we construct an invertible element $f$ in the Hecke algebra $\mathcal{H}(G, \lambda)$, which is supported on a strongly $(P, J)$-positive element in the centre of $L$ (see [5]). To do this, we use an argument which harks back to the work of Borel [3]: we find two invertible elements of the Hecke algebra $\mathcal{H}(G, \lambda)$, which each have support in a compact subgroup of $G$, and whose convolution is supported on a strongly positive element of $L$ (see Lemmas 2.3, 2.10); a suitable power of this is the required element $f$.

Finally, in [23] we prove a result on the Hecke algebra $\mathcal{H}(G, \lambda)$, following the techniques and philosophy of [5] §5 (see also [18]). We show that the subalgebra of elements with support in a fixed maximal compact subgroup is isomorphic to an algebra of the form $\mathcal{H}(G', \rho')$, where $G'$ is a (possibly disconnected) classical group over a finite field, and $\rho'$ is an irreducible cuspidal representation of the Siegel Levi subgroup of $G'$. In many cases, the calculations in [15] will give the parameters of these Hecke algebras; in general, the calculations will be similar. This will simplify the computation of Hecke algebra $\mathcal{H}(G, \lambda)$; these specific calculations, and their implications to the classification of galois representations, are left to future work.
1 Preliminaries

Let $F$ be a locally compact nonarchimedean local field. Let $\mu$ be an automorphism of $F$ with $\mu^2 = 1$; we allow the possibility that $\mu$ is trivial. We set $F_0 = F^{\mu}$ to be the fixed points of $\mu$. Let $\mathfrak{o}_F$ be the ring of integers in $F$, and $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$. Denote by $k_F$ the residual field $\mathfrak{o}_F/\mathfrak{p}_F$, and let $q_F = |k_F|$. We adopt similar notation for $F_0$ and for any extension of $F_0$.

Let $p$ denote the characteristic of $k_F$. We assume that $p$ is not 2 throughout.

For $r$ a real number, we write $\lfloor r \rfloor$ greatest integer less than or equal to $r$, and $\lceil r \rceil$ for the least integer greater than or equal to $r$.

1.1 Self-dual representations of $GL_M(F)$

We begin by looking at the self-dual representations of $GL_M(F)$, following Blondel [2, §2.2]. Let $W$ be an $M$-dimensional $F$-vector space with basis $\mathcal{B} = \{w_1, \ldots, w_M\}$, equipped with the non-degenerate hermitian form $(,)_{W}$ given by $(w_i, w_j)_W = \delta_{i+j,M+1}$. Denote by $\overline{\varepsilon}_W$ the adjoint involution on $A_W = \text{End}_F(W)$ and by $\varepsilon_W$ the involution $g \mapsto \overline{\varepsilon}_W(g^{-1})$ of $G_W = \text{Aut}_F(W) \simeq GL_M(F)$. Then, writing $a \in A_W$ with respect to the basis $\mathcal{B}$, we have

$$\overline{\varepsilon}_W(a) = \dagger a^\mu,$$

where $a^\mu \in A_W$ is obtained by applying $\mu$ to the coefficients of $a$, and $\dagger$ denotes transpose with respect to the off-diagonal. For $\rho$ a representation of a subgroup $J$ of $G_W$, we write $\rho^{\varepsilon_W}$ for the representation of $\varepsilon_W(J)$ given by $\rho^{\varepsilon_W}(\varepsilon_W(j)) = \rho(j)$, for $j \in J$.

Let $\pi$ be an irreducible supercuspidal representation of $G_W$ such that $\pi \simeq \pi^{\varepsilon_W}$. Note that, if $F = F_0$, then, by Gel’fand and Kazhdan [10], this is equivalent to $\pi$ being self-contragredient.

Let $(J_W, \lambda_W)$ be a maximal simple type in $G_W$ corresponding to the inertial equivalence class $[G_W, \pi]_{G_W}$. We will need to use the construction of $\lambda_W$ quite explicitly so we recall it briefly here (see [5] for more details).

It begins with a simple stratum $[\mathfrak{A}_W, n_W, 0, \beta]$, where $\mathfrak{A}_W$ is a principal hereditary $\mathfrak{o}_F$-order in $A_W$, with Jacobson radical $\mathfrak{p}_W$, and $n_W \in \mathbb{N}$ is such that $\beta \in \mathfrak{p}_{W}^{-n_W} \setminus \mathfrak{p}_{W}^{1-n_W}$. Further, $E = F[\beta]$ is a field extension of $F$ and $E^\times$ normalizes $\mathfrak{A}_W$. We set $B_W$ to be the $A_W$-centralizer of $E$ and put $\mathfrak{B}_W = \mathfrak{A}_W \cap B_W$, a maximal hereditary $\mathfrak{o}_E$-order in $B_W$. We also fix a uniformizer $\varpi_E$ of $E$.

From the stratum are defined certain subgroups $H^k_W = J^k(\beta, \mathfrak{A}_W)$ and $J^k_W = J^k(\beta, \mathfrak{A}_W)$, for $k \geq 0$, along with some sets $\mathcal{C}(\mathfrak{A}_W, k, \beta)$ of characters of $H^k_W$ called simple characters (see [5, §3]). The construction of the type continues with a simple character $\theta_W \in \mathcal{C}(\mathfrak{A}_W, 0, \beta)$. There is then a unique irreducible representation $\eta_W$ of $J^1_W$ which contains $\theta_W$.

Now we take $\kappa_W$ to be a $\beta$-extension of $\eta_W$, that is, one of a certain family of representations of $J_W = J^0_W$ which restrict to $\eta_W$. We also recall the construction of $\kappa_W$ here, from [5, §5.1–2]. Let $\mathfrak{B}^m_W$ be a minimal hereditary $\mathfrak{o}_E$-order in $B_W$ contained in $\mathfrak{B}_W$ and let $\mathfrak{A}^m_W \subset \mathfrak{A}_W$ be the unique hereditary $\mathfrak{o}_F$-order in $A_W$ which is normalized by $E^\times$ and such that $\mathfrak{A}^m_W \cap B_W = \mathfrak{B}^m_W$. Then $[\mathfrak{A}^m_W, n^m_W, 0, \beta]$ is also a simple stratum, for some integer $n^m_W$, and we can define simple characters associated to this stratum also. Moreover, there is a canonical bijection $\tau_{\mathfrak{A}_W, \mathfrak{A}^m_W, \beta} : \mathcal{C}(\mathfrak{A}_W, 0, \beta) \rightarrow \mathcal{C}(\mathfrak{A}^m_W, 0, \beta)$ (see [5, §3.6]). Let $\theta_W^m$ be the transfer of $\theta_W$ under this bijection. There is a unique irreducible representation $\eta^m_W$ of $J^1(\beta, \mathfrak{A}^m_W)$ which contains $\theta_W^m$. 

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Now we form the group $\tilde{J}_W^1 = U^1(\mathfrak{A}_W^r)J_W^1$. There is a unique representation $\tilde{\eta}_W$ of $\tilde{J}_W^1$ such that $\tilde{\eta}_W|_{J_W^1} = \eta_W$ and $\tilde{\eta}_W, \eta_W^r$ induce equivalent irreducible representations of $U^1(\mathfrak{A}_W^r)$. Then $\kappa_W$ is any representation of $J_W$ such that $\kappa_W|_{\tilde{J}_W^1} = \tilde{\eta}_W$. Finally, $J_W/J_W^1 \cong U(\mathfrak{B}_W^r)/U(\mathfrak{B}_W)$ is isomorphic to $GL_r(k_E)$, where $r = M/[E : F]$. Then there is a cuspidal representation $\rho_W$ of $J_W/J_W^1$ such that

$$\lambda_W = \kappa_W \otimes \rho_W.$$ 

By conjugating $(J_W, \lambda_W)$ if necessary, we may and do assume $\mathfrak{A}_W$ is standard; that is, with respect to our chosen basis $\mathcal{B}$, it consists of matrices with entries in $\mathfrak{o}_F$ which are upper block triangular modulo $\mathfrak{p}_F$. Note that this means that $\bar{\varepsilon}_W(\mathfrak{A}_W) = \mathfrak{A}_W$.

**Proposition 1.1 (cf. [2, 2.2 Proposition]).**

(i) There exists $\sigma \in U(\mathfrak{A}_W)$ such that $J_W$ is stable under $\sigma : g \mapsto \sigma \varepsilon_W(g)\sigma^{-1}$ and $\lambda_W$ is equivalent to $\lambda_W \circ \sigma$.

(ii) Such an element $\sigma$ is unique up to left multiplication by $J_W$. It satisfies:

(a) $\sigma \varepsilon_W(\sigma) \in J_W$ and $\varepsilon_E^{-1}\sigma \varepsilon_W(\varepsilon_E^{-1}\sigma) \in J_W$.

(b) The map $\sigma$ stabilizes $H_W^1$ and $J_W^1$ and we have $\theta_W = \theta_W \circ \sigma$.

(c) The lattices $\mathfrak{A}_W, \mathfrak{A}_W^1$ are stable under $X \mapsto \sigma \varepsilon_W(X)\sigma^{-1}$.

The proof is identical to that of [2, 2.2 Proposition].

### 1.2 Simple characters

We record here, the following useful lemma, from [2, 4.3 Lemma]. Note that this uses very strongly the condition that $p \neq 2$.

**Lemma 1.2 ([2, 4.3 Lemma 1]).** Suppose $V'\pi$ is any $F$-vector space and $[\mathfrak{A}'', n'', 0, \beta'']$ is a simple stratum in $A'' = \text{End}_F(V''')$, where $\mathfrak{A}''$ is a hereditary $\mathfrak{o}_F$-order in $V'''$.

(i) $[\mathfrak{A}'', n'', 0, \frac{1}{2}\beta'']$ is a simple stratum in $A''$, with $H^k(\frac{1}{2}\beta'', \mathfrak{A}'') = H^k(\beta'', \mathfrak{A}'')$, for each $k \geq 0$, and similarly for $J_k$.

(ii) For each $m \geq 0$, the map $\theta \mapsto \theta^2$ is a bijection from $\mathcal{C}(\mathfrak{A}'', m, \frac{1}{2}\beta'')$ onto $\mathcal{C}(\mathfrak{A}'', m, \beta'')$ which is compatible with the canonical bijections $\tau$ of [5, §3.6]. We denote the inverse bijection by $\theta \mapsto \theta^{1/2}$.

We will write $\theta_L = \theta^{1/2}_W \in \mathcal{C}(\mathfrak{A}_W, 0, \frac{1}{2}\beta)$ and $\theta_L^m = (\theta_L^m)^{1/2} \in \mathcal{C}(\mathfrak{A}_W^m, 0, \frac{1}{2}\beta)$. We also let $\eta_L$ be the unique irreducible representation of $J^1(\beta, \mathfrak{A}_W)$ which contains $\theta_L$, and likewise $\eta_L^m$.

Note also that we have $(\theta_L \circ \sigma)^2 = \theta_L^2 \circ \sigma = \theta_L$. In particular, since the squaring map is a bijection, we see that $\theta_L \circ \sigma = \theta_L$.

### 1.3 Classical groups

Let $V_0$ be an $F$-vector space equipped with a nondegenerate totally isotropic $\nu$-hermitian form $\langle \cdot, \cdot \rangle_0$, with $\nu = \pm 1$. Thus $\langle v_0, w_0 \rangle_0 = \nu \langle w_0, v_0 \rangle_0^0$, for all $v_0, w_0 \in V_0$. We write $G_0 = \text{Aut}_F(V_0)$ and denote by $\mathcal{G}_0^+$ the (anisotropic) group corresponding to the form $\langle \cdot, \cdot \rangle_0$:

$$\mathcal{G}_0^+ := \{ g_0 \in G_0 : \langle g_0v_0, g_0w_0 \rangle_0 = \langle v_0, w_0 \rangle_0 \text{ for all } v_0, w_0 \in V_0 \}.$$


We put $G_0 := \{ g_0 \in \bar{G}_0^+ : \det_{G_0/F}(g_0) = 1 \}$. We allow the possibility that $V_0 = \{0\}$.

Set $V = W \oplus V_0 \oplus W$ and define a form $\langle , \rangle$ on $V$ by

$$\langle (v_1, v_0, v_2), (w_1, w_0, w_2) \rangle = \langle v_1, \sigma^{-1}w_2 \rangle_W + \langle v_0, w_0 \rangle_0 + \nu \langle \sigma^{-1}v_2, w_1 \rangle_W,$$

for $v_1, v_2, w_1, w_2 \in W$ and $v_0, w_0 \in V_0$, where $\sigma$ is the element given by Proposition 1.1. Note that $\langle , \rangle$ is now a nondegenerate $\nu$-hermitian form in which the two copies of $W$ are (dual) maximal isotropic spaces, and $V_0$ is the maximal anisotropic space. Let $N = \dim_F V$; then $N = 2M + D$, where $D = \dim_F V_0$.

We put $A = \text{End}_F(V) \simeq \mathbb{M}(N, F)$. We set $G = A^\times = \text{Aut}_F(V) \simeq GL_N(F)$ and $\bar{G}^+ = \{ g \in G : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V \}$,

a unitary, symplectic or orthogonal group over $F$. We also put $G := \{ g \in \bar{G}^+ : \det_{G/F}(g) = 1 \}$.

More generally, for $H$ a subgroup of $G$, we will write $\mathcal{P}^+$ for the intersection $\mathcal{P}^+ = H \cap \bar{G}^+$, and $\mathcal{P}$ for the intersection $\mathcal{P} = H \cap \bar{G}$.

We denote by $\bar{\varepsilon}$ the adjoint involution on $A$ determined by the form; that is, for $a \in A$, $\bar{\varepsilon}(a)$ is the unique element of $A$ such that

$$\langle av, w \rangle = \langle v, \bar{\varepsilon}(a)w \rangle, \quad \text{for all } v, w, \in V,$$

We have an involution $\varepsilon$ on $G$ given by $\varepsilon(g) = \bar{\varepsilon}(g^{-1})$, for $g \in G$, so that $\bar{G}^+ = G^{\varepsilon}$.

Note that $\bar{\varepsilon}$ induces an involution on $A_W$ via the two embeddings of $W$ into the first (respectively last) factor of $V = W \oplus V_0 \oplus W$, and we also denote this involution by $\bar{\varepsilon}$. For any $v, w \in W$ and $a \in A_W$, we have $\langle a(v, 0, 0), (0, 0, w) \rangle = \langle (v, 0, 0), \bar{\varepsilon}(a)(0, 0, w) \rangle$. So, by definition of the form $\langle , \rangle$,

$$\langle av, \sigma^{-1}w \rangle_W = \langle v, \sigma^{-1}\bar{\varepsilon}(a)w \rangle_W, \text{ that is } \langle v, \bar{\varepsilon}(a)w \sigma^{-1}w \rangle_W = \langle v, \sigma^{-1}\bar{\varepsilon}(a)w \rangle_W.$$

Thus, for any $a \in A_W$, we have $\bar{\varepsilon}(a) = \sigma\bar{\varepsilon}_W(a)\sigma^{-1}$.

Given a representation $\rho$ of a subgroup $J$ of $G_W$, we denote by $\rho^\varepsilon$ the representation of $\varepsilon(J)$ given by $\rho^\varepsilon(\varepsilon(j)) = \rho(j)$, for $j \in J$.

For $L$ an $\mathfrak{o}_F$-lattice in $V$, we define its dual lattice to be

$$L^\# = \{ v \in V : \langle v, L \rangle \subseteq \mathfrak{p}_F \}.$$

We note that, since $\bar{G}_0^+$ stabilizes lattices $L_0 \supset L_0^\# \supset \mathfrak{p}_F L_0$ in $V_0$ (see [16, §1.8]), it is contained in a maximal $\varepsilon$-stable compact open subgroup of $G_0$ (namely, the $G_0$-stabilizer of these lattices).

Let $P$ be the parabolic subgroup of $G$ stabilizing the self-dual flag

$$\{0\} \subseteq W \oplus \{0\} \oplus \{0\} \subseteq (W \oplus \{0\} \oplus \{0\})^\perp = W \oplus V_0 \oplus \{0\} \subsetneq V,$$

with unipotent radical $U$. Let $L$ denote the Levi component of $P$ which stabilizes each copy of $W$ along with $V_0$. So $L \cong GL_M(F) \times GL_D(F) \times GL_M(F)$. Let $P_-$ denote the opposite parabolic subgroup, $P_- = LU_-$. We also put $A_L = A_W \oplus A_0 \oplus A_W = \text{Lie}(L)$. 5
We set $\mathcal{P} = P \cap \mathcal{G}$, the Siegel parabolic subgroup of $\mathcal{G}$, with unipotent radical $\mathcal{U} = U \cap \mathcal{G}$, and $\mathcal{L} = L \cap \mathcal{G}$, a Levi component of $\mathcal{P}$. Then $\mathcal{L} \cong GL_M(F) \times \mathcal{G}_0$; in block matrix form, we identify $G_W \times \mathcal{G}_0$ with $\mathcal{L}$ via the isomorphism

$$i(g, g_0) = \begin{pmatrix} g & 0 & 0 \\ 0 & g_0 & 0 \\ 0 & 0 & \varepsilon(g) \end{pmatrix},$$

for $g \in G_W$, $g_0 \in \mathcal{G}_0$.

If $\rho$ is a representation of a subgroup $J_W$ of $G_W$, and $\mathcal{J}_0$ is a subgroup of $\mathcal{G}_0$, then we denote by $i(\rho)$ the representation of $i(J_W \times \mathcal{J}_0)$ given by $i(\rho)(i(j, j_0)) = \rho(j)$, for $j \in J_W$, $j_0 \in \mathcal{J}_0$. If the group $\mathcal{J}_0$ is not specified then we take it to be the whole of $\mathcal{G}_0$.

More generally, if $\mathcal{J}$ is a subgroup of $\mathcal{G}$ such that $\mathcal{J} \cap \mathcal{L} = i(J_W \times \mathcal{J}_0)$ and $\mathcal{J}$ has an Iwahori decomposition with respect to $(\mathcal{L}, \mathcal{P})$, then we denote by $i(\rho)$ the representation of $\mathcal{J}$ given by

$$i(\rho)(u- l u) = i(\rho)(l),$$

for $u- \in \mathcal{J} \cap \mathcal{U}$, $l \in \mathcal{J} \cap \mathcal{L}$, $u \in \mathcal{J} \cap \mathcal{U}$, whenever this defines a representation.

It will also be useful, later, to put $V' = W \oplus \{0\} \oplus W \subset V$, equipped with the restriction of the form $\langle , \rangle$. We put $A' = \text{End}_F(V')$, $G' = \text{Aut}_F(V')$, $\varepsilon$ the involution of $G'$ associated to the form, and $\mathcal{P}' = (G')^\varepsilon$. We also let $P'$ be the maximal parabolic subgroup of $G$ which stabilizes the flag

$$\{0\} \subseteq V' \subseteq V,$$

with unipotent radical $U'$. Let $L'$ denote the Levi component of $P'$ which stabilizes the decomposition $V = V' \oplus V_0$, so that $L' \simeq G' \times G_0$, and let $P'^-$ denote the opposite parabolic subgroup, with unipotent radical $U'^-$. We note that, while $L'$ is stable under the involution $\varepsilon$, $\mathcal{P}'$ is not a Levi subgroup of $\mathcal{G}$.

We consider the inertial class $s = [(\mathcal{L}, i(\pi))]_\mathcal{L}$ and a type $(J_L, \lambda_L)$ for it, where

$$J_L := i(J_W) = i(J_W \times \mathcal{G}_0) \quad \text{and} \quad \lambda_L := i(\lambda_W).$$

We are going to construct a $\mathcal{G}$-cover of this type, which will give a $\mathcal{G}$-type for the inertial class $s = [(\mathcal{L}, i(\pi))]_\mathcal{G}$.

### 1.4 Semisimple characters

We continue with the notation of the previous sections. Let $\mathcal{L}_W = \{L_k^W : k \in \mathbb{Z}\}$ be the $\mathfrak{a}_F$-lattice chain in $W$ corresponding to $\mathfrak{a}_W$, so that

$$\mathfrak{a}_W = \{a \in A_W : aL_k^W \subseteq L_k^W \text{ for all } k \in \mathbb{Z}\},$$

normalized so that $L_0^W = \mathfrak{o}_F w_1 \oplus \cdots \oplus \mathfrak{o}_F w_M$. Let $e_W$ denote the $\mathfrak{o}_F$-period of $\mathfrak{a}_W$. We define $\mathcal{L}'$ to be the $\mathfrak{o}_E$-lattice chain of $\mathfrak{o}_F$-period $2e_W$ in $V'$ given by

$$\cdots \supset L_k^W \oplus L_k^W \supset L_k^W \oplus L_k^W \supset L_k^W \oplus L_{k+1}^W \supset \cdots$$

It is straightforward to check, since $\sigma \in U(\mathfrak{a}_W)$, we have

$$(L_k^W \oplus L_k^W)^\# = L_k^{e_W-k} \oplus L_k^{e_W-k} \quad \text{and} \quad (L_k^W \oplus L_{k+1}^W)^\# = L_{e_W-k-1}^{e_W-k} \oplus L_{e_W-k}^{e_W-k}$$
so that \( L' \) is a self-dual lattice chain. We write \( L' = \{ L'_k : k \in \mathbb{Z} \} \), where we number the lattices so that \( L'_0 = L'_W + L'_W = (L'_0)' \).

Now let \( \Lambda' \) be the \( \sigma_F \)-lattice sequence of period \( 4e_W \) in \( V' = W \oplus W \) given by

\[
\Lambda'(k) = L'_{\lfloor \frac{1}{2}k \rfloor}
\]

so that every lattice of \( L' \) occurs twice in the sequence and \( \Lambda'(k)' = \Lambda'(1-k) \), for \( k \in \mathbb{Z} \).

We consider the element \( \frac{1}{2} \beta \oplus \frac{1}{2} \beta \) to be the self-dual \( \Lambda'(1) \) and \( \Lambda'(0) \), which we will also call \( \frac{1}{2} \beta \). Then \([L', 2nW_0, \frac{1}{2} \beta] \) and \([\Lambda', 4nW_0, \frac{1}{2} \beta] \) are both simple strata in \( A' \) so we can define the orders \( 5, 3 \) and the groups \( H^k, J^k \) for them, and also simple characters (see [7]). These are in fact the same, up to a scaling of the index (loc. cit.); so, for example,

\[
H^k(\frac{1}{2} \beta, \Lambda') = H^k(\frac{1}{2} \beta, L').
\]

In particular, \( H^1(\frac{1}{2} \beta, \Lambda') = H^1(\frac{1}{2} \beta, L') \) and we shall denote this group \( H^1 \). Similarly, we put \( J^1 = J^1(\frac{1}{2} \beta, \Lambda') = J^1(\frac{1}{2} \beta, L') \) and \( J^k = J^k(\frac{1}{2} \beta, \Lambda') = J^k(\frac{1}{2} \beta, L') \). Likewise, the simple characters of \( H^1 \) are the same: \( C(\Lambda', 0, \frac{1}{2} \beta) = C(\Lambda', 0, \frac{1}{2} \beta) \).

Moreover, the groups \( H^1, J^1 \) and \( J \) are described in [2] 2.2 Lemma. So, for example, with respect to the basis \( B \cup B \) of \( V' \) (which, we note, is not a Witt basis),

\[
H^1 = \left( \begin{array}{ccc}
H^1_W & 3_W \\
\varphi(E & H^1_W)
\end{array} \right).
\]

Finally, we have the bijection \( \tau_{3_W, \varphi, \frac{1}{2} \beta} : C(\mathcal{W}, 0, \frac{1}{2} \beta) \rightarrow C(\mathcal{W}, 0, \frac{1}{2} \beta) \) from [5] §3.6]. Let \( \theta' \) be the image of \( \theta_L \) under this map. Note that, by [5] §7.1–2, \( \theta' \) is trivial on \( H^1 \cap U \) and \( H^1 \cap U_\perp \), while

\[
\theta'_{\mid H^1 \cap L} = \theta_L \otimes \theta_L.
\]

It is straightforward to check that Proposition [4.1] together with our definition of the form \( \langle , \rangle \), implies that \( \theta' \) is fixed by the involution \( \varepsilon \) (cf. [2] 2.3 Corollary).

Now let \( \mathcal{L}_0 \) be the unique self-dual \( \sigma_F \)-lattice chain in \( V_0 \)

\[
\cdots \supseteq L_0 \supseteq \Lambda_0 \supseteq L_0^\# \supseteq p_F L_0 \supseteq p_F L_0^\# \supseteq \cdots
\]

Note that we may have \( L_0 = L_0^\# \) or \( L_0^\# = p_F L_0 \) so this lattice chain has \( \sigma_F \)-period \( e_0 = 1 \) or \( 2 \). Let \( \Lambda_0 \) be the self-dual \( \sigma_F \)-lattice sequence of period \( 4e_W \) given by

\[
\Lambda_0(k) = \begin{cases}
p_F L_0 & \text{if } \left[ \frac{k}{4e_W} \right] = 2j, \\
p_F L_0^\# & \text{if } \left[ \frac{k}{4e_W} \right] = 2j + 1,
\end{cases}
\]

so that every lattice of \( \mathcal{L}_0 \) appears with equal multiplicity \( 4e_W/e_0 \). Note also that \( \Lambda_0(k)' = \Lambda_0(1-k) \), for \( k \in \mathbb{Z} \). Moreover, the filtration of \( A_0 \) determined by \( \Lambda_0 \) is the same as that determined by \( \mathcal{L}_0 \) up to a scaling of the index. In particular, \( a_0(\Lambda_0) = \mathfrak{A}(\mathcal{L}_0) \) and \( a_1(\Lambda_0) = \mathfrak{P}(\mathcal{L}_0) \).

Finally, we define \( \Lambda \) to be the \( \sigma_F \)-lattice sequence in \( V \) given by

\[
\Lambda(k) = \Lambda'(k) \oplus \Lambda_0(k), \quad \text{for } k \in \mathbb{Z}.
\]
Then, by construction, $\Lambda$ is self-dual and of $\sigma_{F}$-period $e = 4e_{W}$. We consider the element $\frac{1}{2} \beta \oplus 0 \in \mathcal{A}' \oplus A_{0}$ (in fact, in $\mathcal{A}_{L}$); by abuse of notation, we will still call this element $\frac{1}{2} \beta$. Then $[\Lambda, n, 0, \frac{1}{2} \beta]$ is a semisimple stratum in $A$, where $n = 4n_{W}$. (See [23 §3.1] for the definition of semisimple stratum, which is more general than the definition in [21 §3.3]; in particular, null strata are thought of as simple strata so, alternatively, the definition in [21] could be used with “simple” replaced by “simple or null” everywhere. The results of [21] all remain valid in this situation – the proofs are the same and they are also proved in [23].)

We put $J = J(\frac{1}{2} \beta, \Lambda)$ (see [21] or [23]) and similarly for $J^{1}$, $H^{1}$, $\mathfrak{J}$, $\mathfrak{J}$ etc. In matrix form we have

$$\mathfrak{J} = \begin{pmatrix} 0 & \mathfrak{a}_{[n/2]}(\Lambda) & \varpi_{E}^{-1} \mathfrak{H}_{1}^{\mathfrak{J}} \\ \mathfrak{a}_{[n/2]}(\Lambda) & \mathfrak{M}(\mathfrak{L}_{0}) & \mathfrak{a}_{[n/2]}(\Lambda) \\ \mathfrak{H}_{1}^{\mathfrak{J}} & \mathfrak{a}_{[n/2]}(\Lambda) & 0 \end{pmatrix}$$

and there are similar decompositions for $\mathfrak{J}^{k}$ and $\mathfrak{J}^{k}$, $k \geq 0$. Since $J'$ is stable under the involution $\varepsilon$ and $\Lambda$ is self-dual, we see that $J$ is stable under the involution $\varepsilon$, and likewise for $J^{1}$ and $H^{1}$.

Let $\theta$ be the unique semisimple character (see [21 §3.3] or [23 §3.2]) of $H^{1}$ such that

$$\theta|_{H^{1}} = \theta'.$$

Now $\theta$ is trivial on $H^{1} \cap U'$ and $H^{1} \cap U'$ by definition so, since $\theta'$ is trivial on $H^{1} \cap U$ and $H^{1} \cap U_{-}$, we see that $\theta$ is in fact trivial on $H^{1} \cap U$ and $H^{1} \cap U_{-}$. Since $\theta|_{H^{1}}$ is fixed by $\varepsilon$ and $\theta|_{U'(\mathfrak{L}_{0})}$ is trivial (by definition), we see that $\theta$ is fixed by $\varepsilon$. Moreover, since $\theta_{L}^{2} = \theta_{W}$, we have

$$\theta|_{\mathfrak{T}} = i(\theta_{W}).$$

**Proposition 1.3** (cf. [2, 2.3 Theorem]). There exists a semisimple stratum $[\Lambda, n, 0, \alpha]$ in $A$ with $\alpha \in A_{L}$ and $\alpha = -\varepsilon(\alpha)$ such that $\theta \in \mathcal{C}(\Lambda, 0, \alpha)$.

**Proof** Let $\varphi : \mathcal{A}' \to \mathcal{A}'$ be the involution given by conjugation by

$$h = \begin{pmatrix} I_{M} & 0 \\ 0 & -I_{M} \end{pmatrix}.$$ 

Then $(\theta')^{\varphi} = \theta'$, since it is trivial on the unipotent parts, and $(\theta')^{\varepsilon} = \theta'$. Thus, $\theta'$ is invariant under the subgroup $\Omega$ of $\text{Aut}(\mathcal{G}')$ generated by $\varepsilon$ and $\varphi$. As $\varepsilon \varphi = \varphi \varepsilon$, we see $\Omega \simeq \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, and thus by [21 Theorem 6.3] (see the Remarks at the bottom of page 139 there also), there is a choice of $\alpha$ which is fixed by $\Omega$ such that $\theta' \in \mathcal{C}(\mathcal{A}', 0, \alpha)$. Then $[\Lambda, n, 0, \alpha]$ is a semisimple stratum with $\alpha \in \mathcal{A}' \cap A_{L}$ and $\alpha = -\varepsilon(\alpha)$, such that $\theta \in \mathcal{C}(\Lambda, 0, \alpha)$. □

In particular, replacing $\frac{1}{2} \beta$ by the element $\alpha$ found in Proposition 1.3, we may (and do) assume that $[\Lambda, n, 0, \frac{1}{2} \beta]$ is a skew semisimple stratum.

As in the construction of the type $(J_{W}, \lambda_{W})$, we will also need the transfer of $\theta$ to a minimal order, which we give now. Let $\mathfrak{B}'$ denote the centralizer in $\mathcal{A}'$ of $E$ and put $\mathfrak{B}' = \mathfrak{A}' \cap B'$, an $\sigma_{E}$-order of period 2 with radical $\Omega'$. Recall that we had, in §1.1, a minimal $\sigma_{E}$-order $\mathfrak{B}_{W}^{m} \subset \mathfrak{B}_{W}$. Let $\mathfrak{B}_{W}^{m} \subset \mathfrak{B}'$ be the $\varepsilon$-stable minimal $\sigma_{E}$-order in $B'$ given by

$$\mathfrak{B}_{W}^{m} = (\mathfrak{B}_{W}^{m} \oplus \varepsilon(\mathfrak{B}_{W}^{m})) + \Omega',$$
In our situation, we recall the following, from [21, Theorem 3.14, Corollary 4.2 and Proposition 4.3]:

\[ \Lambda^m(k)^\# = \Lambda^m(1 - k), \quad \text{for all } k \in \mathbb{Z}. \]

Note that \([\mathcal{L}^m, 2n_W^m, 0, \frac{1}{2}\beta]\) and \([\Lambda^m, 4n_W^m, 0, \frac{1}{2}\beta]\) are simple strata in \(V'\) whose associated groups and characters are the same up to a scaling of index.

Now let \(\Lambda^m\) be the \(\sigma_E\)-lattice sequence in \(V\) defined by

\[ \Lambda^m(k) = \Lambda^m(k) \oplus \Lambda_0 \left( \frac{k}{r} \right). \]

It is a self-dual lattice sequence of period \(re\) such that \(a_0(\Lambda^m) \subset a_0(A)\). Then \([\Lambda^m, n^m, 0, \frac{1}{2}\beta]\) is a semisimple stratum in \(A\), where \(n^m = 4n_W^m = nr\).

We put \(\theta^m = \tau_{\Lambda^m, \frac{1}{2}\beta}(\theta')\), a simple character of \(H^1(\frac{1}{2}\beta, \Lambda^m)\); then also \(\theta^m = \tau_{\Lambda^m, \frac{1}{2}\beta}(\theta_L^m)\).

Finally, let \(\theta^m\) be the unique semisimple character of \(H^1_m = H^1(\frac{1}{2}\beta, \Lambda^m)\) such that

\[ \theta^m|_{H^1(\frac{1}{2}\beta, \Lambda^m)} = \theta^m. \]

In the language of [23 §3.5], we have \(\theta^m = \tau_{\Lambda, \Lambda^m, \frac{1}{2}\beta}(\theta). \)

## 2 Covers and Hecke Algebras

We continue with the notation above, so we have \([\Lambda, n, 0, \frac{1}{2}\beta]\) a skew semisimple stratum in \(A\), with \(\beta \in A_L\), and \(\theta \in C(\Lambda, 0, \frac{1}{2}\beta)\) such that

\[ \theta|_{\mathbb{F}^L} = i(\theta_W). \]

We also have \(E = F[\beta]\) and \(B\) the \(A\)-centralizer of \(E\). The galois involution \(\mu\) extends to \(E\) (as \(\overline{\epsilon}\)), and we write \(E_0\) for the fixed subfield, which has index 2. We fix a uniformizer \(\varpi_E\) of \(E\) such that \(\varpi_E^m = \pm \varpi_E\).

### 2.1 G-Covers

We begin by doing some work in \(G\), before using Glauberman’s correspondence to transfer this information to \(\overline{G}\).

Recall that, given \(\rho\) a representation of a subgroup \(H\) of \(G\) and \(g \in G\), the intertwining space \(I_g(\rho|H)\) is

\[ I_g(\rho|H) = \text{Hom}_{H \rtimes gH}(\rho, g\rho), \]

where \(gH = gHg^{-1}\) and \(g\rho\) is the representation of \(gH\) given by \(g\rho(ghg^{-1}) = \rho(h)\), for \(h \in H\). The \(G\)-intertwining \(I_G(\rho|H)\) of \(\rho\) is then defined to be

\[ I_G(\rho|H) = \{g \in G : I_g(\rho|H) \neq 0\}. \]

In our situation, we recall the following, from [21 Theorem 3.14, Corollary 4.2 and Proposition 4.3]:
Lemma 2.1. (i) \( I_G(\theta) = J^1 B^x J^1 \).

(ii) There exists a unique irreducible representation \( \eta \) of \( J^1 \) which contains \( \theta \). Moreover, \( \dim \eta = (J^1 : H^1)^\frac{1}{2} \) and, for \( g \in G \),

\[
\dim I_g(\eta|J^1) = \begin{cases} 
1 & \text{if } g \in J^1 B^x J^1; \\
0 & \text{otherwise}.
\end{cases}
\]

From [5] §7.1 (together with [21] §3.3), \( H^1 \) has an Iwahori decomposition with respect to \((L,P)\) and

\[
H^1 \cap L = H^1(\frac{1}{2}\beta, A_W) \times U^1(\Lambda_0) \times H^1(\frac{1}{2}\beta, A_W).
\]

There are also similar decompositions for \( J^1 \) and for \( J \). Moreover, \( \theta \) is trivial on \( H^1 \cap U \) and \( H^1 \cap U_- \), and the restriction of \( \theta \) to \( H^1 \cap L \) takes the form

\[
\theta|_{H^1 \cap L} = \theta_L \otimes 1 \otimes \theta_L.
\]

We recall ([21] Proposition 4.1] – see also [5] §3.4]) that the pairing

\[
(j,j') \mapsto \theta[j,j'], \quad \text{for } j,j' \in J^1
\]

induces a nondegenerate alternating bilinear form \( k_\theta \) on \( J^1/H^1 \); likewise, we have a nondegenerate alternating bilinear form \( k_{\theta_L} \) on \( J^1(\frac{1}{2}\beta, A_W)/H^1(\frac{1}{2}\beta, A_W) \). Then, exactly as in [5] Proposition 7.2.3], we get:

Lemma 2.2 (cf. [5] Proposition 7.2.3]). (i) The subspaces \( J^1 \cap U_-/H^1 \cap U_- \) and \( J^1 \cap U/H^1 \cap L \) are both totally isotropic for the form \( k_\theta \) and orthogonal to the subspace \( J^1 \cap L/H^1 \cap L \).

(ii) The restriction of \( k_\theta \) to the group

\[
J^1 \cap L/H^1 \cap L = J^1(\frac{1}{2}\beta, A_W)/H^1(\frac{1}{2}\beta, A_W) \times J^1(\frac{1}{2}\beta, A_W)/H^1(\frac{1}{2}\beta, A_W)
\]

is the orthogonal sum of the pairings \( k_{\theta_L}, k_{\theta_L} \).

(iii) We have an orthogonal sum decomposition

\[
\frac{J^1}{H^1} = \frac{J^1 \cap L}{H^1 \cap L} \perp \left( \frac{J^1 \cap U_-}{H^1 \cap U_-} \times \frac{J^1 \cap U}{H^1 \cap U} \right).
\]

We define the groups

\[
H^1_P = H^1(J^1 \cap U), \quad J^1_P = H^1(J^1 \cap P), \quad J_P = H^1(J \cap P).
\]

Since \( J^1 \) normalizes \( \theta \) and \( \theta|_{H^1 \cap U} \) is trivial, we can define the character \( \theta_P \) of \( H^1_P \) by

\[
\theta_P(hu) = \theta(h), \quad \text{for } h \in H^1, u \in J^1 \cap U.
\]

As in [5] §7.2], we immediately get:
Corollary 2.3 (cf. [5, Propositions 7.2.4, 7.2.9]). There is a unique irreducible representation \( \eta_P \) of \( J_P \) such that \( \eta_P|_{H_P} \) contains \( \theta_P \). Moreover, \( \eta \simeq \text{Ind}_{J_P}^{H_P} \eta_P \), and for each \( b \in B^x \), there is a unique \((J_P, J_P)\)-double coset in \( J^1 b J^1 \) which intertwines \( \eta_P \).

We note also that we certainly have \( \eta_P|_{J_P \cap L} \simeq \eta_L \otimes 1 \otimes \eta_L \).

Proposition 2.4. We have \( I_G(\theta_P) = J_P B^x J_P \) and hence \( I_G(\eta_P) = J_P B^x J_P \).

Proof. We have \( I_G(\theta_P) = I_G(\eta_P) \) so, by Corollary 2.3, we need only check that all of \( B^x \) intertwines \( \theta_P \). Since \( B^x \subset L' \), and \( B^x \) is trivial on \( H^1 \cap U' \) and \( H^1 \cap U'' \), we need only check that \( B^x \) intertwines \( \theta_P|_{H^1 \cap L'} = \theta_P' \otimes 1 \), where \( \theta_P' \) is the character of \( H^1 P = H^1 (J^1 \cap U) \) obtained by trivial extension, as above. Since \( G_0 \) clearly intertwines the trivial representation \( 1 \), we need to check that \( B'^x \), the \( G' \)-centralizer of \( \beta \), intertwines \( \theta_P' \).

To ease notation, we omit the superscripts \(^{'}\); indeed we have just reduced to the case where \( V_0 = 0 \). Then we are in the situation of [5, §7.1–2]. We note that \( J_P = (U(\mathfrak{B}) \cap L) J_P \). Since \( U(\mathfrak{B}) \cap L \) normalizes \( \theta_L \otimes \theta_L \), while \( \theta_P \) is trivial on \( H^1 \cap U \) and \( H^1 \cap U'' \), we deduce that \( J_P \) normalizes \( \theta_P \).

Recall that we have our \((\tilde{z}\text{-stable})\) minimal \( \sigma_E \)-order \( \mathfrak{B}^m \subset \mathfrak{B} \) from §7.3 (where we are still omitting the superscripts \(^{'}\)). Let \( v_1, \ldots, v_r \) be an \( E \)-basis for \( W \) and \( v_{r+i} = \sigma v_i \), for \( i = 1, 2, \ldots, r \), so that, with respect to the basis \( v_1, \ldots, v_{2r} \) of \( W \oplus W \), we have \( \mathfrak{B}^m \) in standard form. We put \( V^{(i)} = \tilde{E} v_i \), for \( i = 1, \ldots, 2r \). Let \( P_0 \subset P \) denote the parabolic subgroup of \( G \) which is the stabilizer of the flag

\[
0 \subset V^{(1)} \subset V^{(1)} \oplus V^{(2)} \subset \cdots \subset \bigoplus_{i=1}^{2r} V^{(i)} = V
\]

Let \( U_0 \supset U \) be the unipotent radical of \( P_0 \) and let \( L_0 \subset L \) be the Levi component of \( P_0 \) which stabilizes the decomposition

\[
V = \bigoplus_{i=1}^{2r} V^{(i)}.
\]

Also, let \( P_0^- = L_0 U_0^- \) denote the opposite parabolic subgroup.

For \( i = 1, \ldots, 2r \), let \( \Lambda^{(i)} \) denote the lattice \emph{chain} (i.e. ignore repetitions) given by intersection of \( \Lambda \) with \( V^{(i)} \); note that \( [\Lambda^{(i)}, n, 0, \frac{1}{r} \beta] \) is a simple stratum in \( A^{(i)} = \text{End}_E(V^{(i)}) \). Moreover, since each \( V^{(i)} \) is a one-dimensional \( E \)-vector space, the lattice chain \( \Lambda^{(i)} \) is uniquely determined by the property of being normalized by \( E^x \). Hence, we can (and do) identify \( \Lambda^{(i)} \) with \( \Lambda^{(1)} \), for each \( i \).

From [6, Examples 10.9, 10.10], \( H^1 \) has an Iwahori decomposition with respect to \((L_0, P_0)\) and

\[
H^1_P \cap L_0 = H^1 \cap L_0 = \prod_{i=1}^{2r} H^1(\beta, \Lambda^{(i)}).
\]

There is also a similar decomposition for \( J^1 \) (though not for \( J \)) and hence also for \( H^1_J \). Moreover (loc. cit.), \( \theta \) is trivial on \( H^1 \cap U_0 \) and \( H^1 \cap U_0^- \), and the restriction of \( \theta \) to \( H^1 \cap L_0 \) takes the form

\[
\theta|_{H^1 \cap L_0} = \bigotimes_{i=1}^{2r} \theta^{(1)}
\]

where \( \theta^{(1)} \) is the simple character transfer of \( \theta_L \) to \( H^1(\beta, \Lambda^{(1)}) \).
Since $U(\mathfrak{B}^m) \subset (U(\mathfrak{B}) \cap P)U^1(\mathfrak{B})$, we need only show that some set of double coset representatives for $U(\mathfrak{B}^m) \setminus B^\times / U(\mathfrak{B}^m)$ intertwines $\theta_P$. By the Bruhat decomposition, we may take these representatives to be of the form

$$b = yz$$

where, with respect to the basis $v_1, \ldots, v_{2r}$, $y$ is a permutation matrix and $z = \text{diag}(z_1, \ldots, z_{2r})$ is a diagonal matrix with entries $z_i \in \mathbb{C}$. Then $b$ normalizes $H_P^1 \cap L_0$ and indeed, since $\mathbb{C}$ normalizes $\theta(1)$, normalizes $\theta_P|_{H_P^1 \cap L_0}$.

Since $b$ normalizes $L_0$, $H_P^1$ also has an Iwahori decomposition with respect to $(L_0, P_0^b)$, whence, by uniqueness of Iwahori decompositions, $H_P^1 \cap bH_P^1$ has an Iwahori decomposition with respect to $(L_0, P_0)$. Writing $g \in H_P^1 \cap bH_P^1$ in this decomposition as $g = u_-lu$, we have

$$\theta_P(g) = \theta_P(l) = b\theta_P(l).$$

To conclude the Proposition, we need only show that $\theta_P(b^{-1}ub) = 1$, and likewise for $u_-$. Since both are similar, we treat only the first of these. Since $b^{-1}ub \in H_P^1$, we can write $b^{-1}ub = u'lu'$ in the Iwahori decomposition of $H_P^1$. However, by elementary row and column operations, we can write $b^{-1}ub = x_{-}x$, with $x_{-} \in U_{-}$ and $x \in U_0$. Now uniqueness of Iwahori decompositions implies that $u'_{-} = x_{-}, l' = 1$ and $u' = x$. Hence $\theta_P(b^{-1}ub) = \theta_P(u'_-\theta_P(u) = 1$ as required, since $\theta_P$ is trivial on $H_P^1 \cap U_0$ and $H_P^1 \cap U_0^{-}$.

We will also need similar results for our character $\theta^m$ of $H^1_m = H^1_m(\frac{1}{2}B, A^m)$. Let $\eta^m$ be the unique irreducible representation of $J^1_m = J^1_m(\frac{1}{2}B, A^m)$ which contains $\theta^m$. As above, $H^1_m$ and $J^1_m$ have Iwahori decompositions with respect to $(L, P)$ and we can define the character $\theta^m_P$ of $H^1_m, P = H^1_m(J^1_m \cap U)$ by trivial extension of $\theta^m$. The same proofs (indeed, they are somewhat simpler) as those of Lemma 2.2, Corollary 2.3 and Proposition 2.4 show that

$$I_G(\theta^m_P) = J^1_m, P B^\times J^1_m, P,$$

where $J^1_m, P = H^1_m(J^1_m \cap P)$, that there is a unique irreducible representation $\eta^m_P$ of $J^1_m, P$ which contains $\theta^m_P$, and that $\eta^m = \text{Ind}_{J^1_m, P}^{H^1_m, P} \eta^m_P$.

Let $\theta^m_L$ denote the transfer to $H^1_m(\frac{1}{2}B, A^m_W)$ of $\theta_L$ (this is just the restriction of $\theta^m$ to one of the copies of $G_W \subset L$). Then, if $\eta^m_L$ denotes the unique irreducible representation of $J^1_m(\frac{1}{2}B, A^m_W)$ which contains $\theta^m_L$, we have

$$\eta^m_P|_{H^1_m \cap L} \cong \eta^m_L \otimes 1 \otimes \eta^m_L.$$

### 2.2 $\mathcal{G}$-Covers

Now we will transfer the information obtained in the last section to $\mathcal{G}$, using Glauberman’s correspondence (see [9], or [21] for the situation here). Let $\Omega$ denote a 2-group of automorphisms of $G$. Recall that if $H$ is a pro-$p$ subgroup of $G$ and $H^\Omega$ is the group of $\Omega$-fixed points, then there is a bijection, denoted $\rho \leftrightarrow g_{\Omega}(\rho)$ between (equivalence classes of) irreducible representations of $H$ with $\rho^\omega \simeq \rho$, for all $\omega \in \Omega$, and (equivalence classes of) irreducible representations of $H^\Omega$. Further, this correspondence commutes with irreducible restriction and irreducible induction. Recall also that the representation $g_{\Omega}(\rho)$ is characterized as the unique component of $\rho|_{H^\Omega}$ appearing with odd multiplicity.
We will usually apply this correspondence with \( \Omega \) the group of automorphisms of \( G \) consisting of \( \varepsilon \) and the identity, in which case we will just write \( g \) for the correspondence. Note also that, for \( \varepsilon \)-stable pro-\( p \) subgroups \( H \) of \( G \), we have \( H^\varepsilon = \overline{\Pi}^+ = \overline{\Pi} \).

We write \( \vartheta = g(\theta) = \theta|\overline{\Pi} \), a skew semisimple character (see \[21\ §3.4\]). We also set
\[
\overline{G}_E^+ = B \cap \overline{G}^+ \quad \text{and} \quad \overline{G}_E = B \cap \overline{G}
\]
so that \( \overline{G}_E^+ \) (respectively \( \overline{G}_E \)) is the direct product of a unitary group, for the quadratic extension \( E/E_0 \), and the anisotropic group \( \overline{G}_0^+ \) (respectively \( \overline{G}_0 \)).

We recall first the following, from \[21\ Theorem 3.16\] (see also the Remarks following \textit{op. cit.} Corollary 4.2), and \[23\ Proposition 3.3.1\):

**Lemma 2.5.**

(i) \( I_{\overline{G}_E}(\vartheta) = \mathcal{J}^1 \mathcal{G}_E^+ \mathcal{J}^1 \) and \( I_{\overline{G}}(\vartheta) = \mathcal{J}^1 \mathcal{G}_E \mathcal{J}^1 \).

(ii) There exists a unique irreducible representation \( \overline{\eta} \) of \( \mathcal{J}^1 \) which contains \( \vartheta \). Moreover, \( \overline{\eta} = g(\eta) \),
\[
\dim I(g(\eta), I_{\mathcal{J}^1}) = \begin{cases} 1 & \text{if } g \in \mathcal{J}^1 \mathcal{G}_E \mathcal{J}^1; \\ 0 & \text{otherwise.} \end{cases}
\]

We also put \( \overline{J}_P = g(\theta_P) = \theta_P|\overline{\Pi} \). Note that we have
\[
\overline{J}_P = i(\overline{\theta}_W).
\]

We remark here that, since \( \mathcal{J}_P \cap \overline{\mathcal{U}} = \overline{\mathcal{J}}_P \cap \overline{\mathcal{U}} \), and likewise for \( \overline{\mathcal{U}}_- \), this implies (see \[1\ Lemma 1(ii)\]) that
\[
(\mathcal{J}_P \cap \overline{\mathcal{U}})(\mathcal{J}_P \cap \overline{\mathcal{U}}_-) \subset (\mathcal{J}_P \cap \overline{\mathcal{U}}_-)(\overline{\theta}_W)(\mathcal{J}_P \cap \overline{\mathcal{U}}).
\]

In particular (by \textit{loc. cit.}), if \( \rho \) is a representation of a subgroup \( K \) of \( J_W \times G_0 \) which restricts to a multiple of \( \theta_W \), then \( i(\rho) \) is a well-defined representation of \( (\mathcal{J}_P \cap \overline{\mathcal{U}}_-)(\overline{\theta}_W)(\mathcal{J}_P \cap \overline{\mathcal{U}}) \).

From Proposition \[2.4\] together with \[21\ Corollary 2.5\] and \[22\ Theorem 2.3\], we get:

**Proposition 2.6.** \( I_{\overline{G}_E}(\vartheta_P) = \mathcal{J}_P \mathcal{G}_E^+ \mathcal{J}_P \) and \( I_{\overline{G}}(\vartheta_P) = \mathcal{J}_P \mathcal{G}_E \mathcal{J}_P \).

Let \( \vartheta^m = g(\theta^m) = \theta^m|\overline{\Pi} \). Let \( \overline{\eta} \) be the unique irreducible representation of \( \mathcal{J}_m^1 \) which contains \( \vartheta \), as in Lemma \[2.2\] and let \( \overline{\eta}^m \) be the unique irreducible representation of \( \mathcal{J}_m^1 \) which contains \( \vartheta^m \). Let \( \vartheta_P = g(\theta_P) \) be the trivial extension of \( \theta \) to \( \overline{\theta}_P \), and, similarly, \( \overline{\eta}^m_P \) the trivial extension of \( \overline{\eta}^m \) to \( \overline{\mathcal{J}}_m^1 \).

We also put \( \overline{\eta}_P = g(\eta_P) \) and \( \overline{\eta}^m_P = g(\eta^m_P) \). By taking \( \varepsilon \)-fixed points in \( \mathcal{J}_m^1 \), we can imitate Lemma \[2.2\] and Corollary \[2.3\] to show that there is a unique irreducible representation of \( \mathcal{J}_m^1 \) which contains \( \vartheta_P \) and, since \( \eta_P|\overline{\mathcal{J}}_m^1 \) is a multiple of \( \vartheta_P \), we see that this must be \( \overline{\eta}_P \). Likewise, \( \overline{\eta}^m_P \) is the unique irreducible representation of \( \overline{\mathcal{J}}_m^1 \) containing \( \overline{\eta}^m_P \). We also have
\[
\overline{\eta}_P = i(\overline{\eta}_W) \quad \text{and} \quad \overline{\eta}^m_P = i(\overline{\eta}^m_W),
\]

since the representations on the right restrict to multiples of \( i(\overline{\theta}_W) = \overline{\theta}_P \) and \( i(\overline{\theta}^m_W) = \overline{\theta}^m_P \) respectively.
Since $U(\Lambda) \cap B$ normalizes $J^1_P$ and $U(\Lambda^m)$ is contained in $U(\Lambda)$, we can form the group $\bar{J}^1_P = (U^1(\Lambda^m) \cap G_E)\bar{J}^1_P$. We also recall that we have the representation $\bar{\eta}_W$ of $\bar{J}^1_W = U^1(\mathfrak{B}_W)J^1_W$ (see §1.2) and observe that $\bar{J}^1_P \cap \mathfrak{T} = i(\bar{J}^1_W)$.

**Proposition 2.7 (cf. [5, Propositions 5.1.15, 5.1.19]).** There is a unique representation $\bar{\eta}_P$ of $\bar{J}^1_P$ such that

(i) $\bar{\eta}_P|\bar{J}^1_P = \eta_P$;

(ii) $\bar{\eta}_P$ and $\eta^m_P$ induce equivalent irreducible representations of $\overline{U}^1(\Lambda^m)$.

Moreover, $\bar{\eta}_P = i(\bar{\eta}_W)$ and

$$\dim I_g(\bar{\eta}_P) = \begin{cases} 1 & \text{if } g \in \bar{J}^1_P \overline{G}^+ \bar{J}^1_P; \\ 0 & \text{otherwise}. \end{cases}$$

**Proof** Let $\Omega_1$ be the group of automorphisms of $\overline{G}^+$ generated by conjugation by

$$h_1 = \begin{pmatrix} -I_M & I_D \\ I_D & -I_M \end{pmatrix}.$$

Then $(\overline{G}^+)^{\Omega_1} = (\overline{G}^+)^{x} \times \overline{G}^+_0$. We will denote the Glauberman correspondence $g_{\Omega_1}$ on representations of pro-$p$ subgroups of $\overline{G}$ by $g_1$. We note that, since all our representations are trivial on $U'$ and $U^-$, they are fixed by $\Omega_1$ and, moreover, $g_1$ is just restriction for these representations (since their restrictions are irreducible).

Let $\Omega_2$ be the group of automorphisms of $(\overline{G}^+)^{\Omega_1}$ generated by conjugation by

$$h_2 = \begin{pmatrix} I_M & I_D \\ I_D & -I_M \end{pmatrix}.$$

Then $((\overline{G}^+)^{\Omega_1})^{\Omega_2} = \mathfrak{T}$. We will denote the Glauberman correspondence $g_{\Omega_2}$ on representations of pro-$p$ subgroups of $(\overline{G}^+)^{\Omega_1}$ by $g_2$. As above, since all our representations are trivial on $U$ and $U^-$, their transfers to $((\overline{G}^+)^{\Omega_1}$ are fixed by $\Omega_2$ and, moreover, $g_2$ is again just restriction for these representations (since their restrictions are still irreducible).

We write $g_{\Omega}$ for the composition $g_2 \circ g_1$. Then we have

$$g_{\Omega}(\bar{\eta}_P) = i(\eta_W) \quad \text{and} \quad g_{\Omega}(\eta^m_P) = i(\eta^m_W).$$

We define $\bar{\eta}_P$ to be $\bar{i}(\bar{\eta}_W)$. Then we clearly have $g_{\Omega}(\bar{\eta}_P) = i(\bar{\eta}_W)$ and $\bar{\eta}_P|\bar{J}^1_P = \bar{\eta}_P$. Also, since the intertwining of $\bar{\eta}_P$ is contained in that of $\eta_P$,

$$\mathcal{I}_{\overline{G}^+}(\bar{\eta}_P) \cap \overline{U}^1(\Lambda^m) \subset \bar{J}^1_P \overline{G}^+ \bar{J}^1_P \cap \overline{U}^1(\Lambda^m) = (\overline{U}^1(\Lambda^m) \cap \overline{G}_E)\bar{J}^1_P = \bar{J}^1_P,$$
so the induced representation \( \text{Ind}_{J_P}^G \eta_P \) is irreducible. Likewise, \( \text{Ind}_{J_{m,P}}^U \eta_P \) is irreducible. Then, since Glauberman’s correspondence commutes with irreducible induction, we have

\[
g_{\Omega} \left( \text{Ind}_{J_P}^U \eta_P \right) = \text{Ind}_{j_P}^{U_1} (U \eta_W) = \text{Ind}_{j_P}^{U_1} (\eta_W) = g_{\Omega} \left( \text{Ind}_{j_{m,P}}^U \eta_P \right).
\]

Condition (ii) now follows as \( g_{\Omega} \) is injective.

Now we show that these two conditions determine \( \eta_P \) uniquely. For this, we need only show that \( \eta_P \) occurs in \( \text{Ind}_{j_{m,P}}^U \eta_P \) with multiplicity one. We use the Mackey formula to compute the restriction

\[
\text{Res}_{J_P}^{U_1} \text{Ind}_{J_P}^U \eta_P.
\]

If \( x \in \mathcal{U}_1(\Lambda^m) \) intertwines \( \eta_P \) with \( \eta_P \), then it intertwines \( \eta_P \) with itself so lies in \( J_1 \), as above. Thus the multiplicity of \( \eta_P \) in \( \text{Ind}_{j_{m,P}}^U \eta_P \) is equal to its multiplicity in \( \eta_P \), which we know to be one.

Finally, we must show that all of \( \mathcal{G}^+_E \) intertwines \( \eta_P \). So suppose \( b \in \mathcal{G}^+_E \). Since \( b \) intertwines \( \eta_P \), it certainly intertwines

\[
\text{Ind}_{j_P}^{U_1} \eta_P \cong \text{Ind}_{j_{m,P}}^U \eta_P.
\]

We deduce that there exist \( u, v \in \mathcal{U}_1(\Lambda^m) \) such that \( uv \) intertwines \( \eta_P \). In particular, \( uv \) intertwines \( \eta_P \) so there exist \( j_1, j_2 \in J_1 \) such that \( j_1uvj_2 \in \mathcal{G}^+_E \). Note that this element also still intertwines \( \eta_P \).

Now

\[
\mathcal{U}_1(\Lambda^m)b\mathcal{U}_1(\Lambda^m) \cap B^x = \mathcal{U}_1(\Lambda^m)b\mathcal{U}_1(\Lambda^m) \cap L' \cap B^x = (\mathcal{U}_1(\Lambda^m) \cap L')b(\mathcal{U}_1(\Lambda^m) \cap L') \cap B^x.
\]

Now we apply [22] Lemma 2.1] applied with \( \Gamma = \Omega_1 \), and we see

\[
\mathcal{U}_1(\Lambda^m)b\mathcal{U}_1(\Lambda^m) \cap B^x = (\mathcal{U}_1(\Lambda^m) \cap B^x)b(\mathcal{U}_1(\Lambda^m) \cap B^x),
\]

by [5] Theorem 1.6.1.

Then, applying [22] Lemma 2.1] again, with \( \Gamma = \{1, \varepsilon\} \), we get

\[
\mathcal{U}_1(\Lambda^m)b\mathcal{U}_1(\Lambda^m) \cap \mathcal{G}^+_E = (\mathcal{U}_1(\Lambda^m) \cap \mathcal{G}_E)b(\mathcal{U}_1(\Lambda^m) \cap \mathcal{G}_E).
\]

So there exist \( u', v' \in \mathcal{U}_1(\Lambda^m) \cap \mathcal{G}_E \) such that \( u'v' = j_1uvj_2 \) and, since \( u', v' \in J_1 \), we see that \( b \) intertwines \( \eta_P \).

Now we define \( \pi_P = \tilde{i}(\lambda_W) \), a representation of \( J_P \), so that \( \pi_P |_{j_P} \cong \tilde{\eta}_P \). We also put \( \pi_P = \tilde{i}(\rho_W) \), a representation of \( J_P \) trivial on \( J_P \), and

\[
\pi_P = \tilde{i}(\lambda_W) = \pi_P \otimes \pi_P.
\]
Writing matrices with respect using the basis $B$ of $W$ in each of the two copies of $W$, we set

$$w_1 = \begin{pmatrix} \nu \in W(\sigma) & I_D & \sigma \end{pmatrix},$$

where $I_D$ is the $D \times D$ identity matrix.

**Proposition 2.8.** $\iota_{\sigma}^{*}(\bar{x}_P) = J_P N_{\mathfrak{g}_E}((\overline{U}(\Lambda) \cap \overline{G}_E) J_P$.

**Proof.** Notice first (cf. [2, 4.1]) that the normalizer $N_{\mathfrak{g}_E}(\overline{U}(\Lambda) \cap \overline{G}_E)$ is $(\overline{U}(\Lambda) \cap \overline{G}_E) \overline{W}(\overline{U}(\Lambda) \cap \overline{G}_E)$, where

$$\overline{W} = \{i(\varpi_E^a, 1_{V_0}) : a \in \mathbb{Z}\} \cup \{i(\varpi_E^a, 1_{V_0}) w_1 : a \in \mathbb{Z}\},$$

where $\varpi_E$ is our fixed uniformizer of $E \subset A_W$ and $1_{V_0}$ is the identity map on $V_0$.

Since the elements of $\overline{W}$ normalize $J_P \cap \overline{L}$, while $\varpi_E$ normalizes $\lambda_W$ and $\lambda_W = \lambda_W \circ \sigma$ (see Proposition 5.3.2), we see that the elements of $\overline{W}$ normalize $i(\lambda_W)$. On the other hand, they either preserve $\overline{U}$ and $\overline{U}_-$ or interchange them so we see that every element of $\overline{W}$ intertwines $\bar{x}_P = i(\lambda_W)$. Hence we have $\iota_{\sigma}^{*}(\bar{x}_P) \supset J_P N_{\mathfrak{g}_E}(\overline{U}(\Lambda) \cap \overline{G}_E) J_P$.

The proof of the opposite containment, which is a variant of the proof of [5, Proposition 5.3.2], is inspired by [2, page 551]; in place of [2, 4.2 Lemma], we use [24, Proposition 1.1], which is a slight generalization of [17, Proposition 4.13]. It is almost identical to the proof of [24, Proposition 6.14], except that the definition of $\bar{\pi}_P$ there is a priori slightly different.

Suppose $g \in \mathcal{G}$ intertwines $\bar{x}_P = \pi_P \circ \pi_P$, so that $g \in \iota_{\sigma}^{*}(\bar{\pi}_P J_P) = J_P G_E J_P$, as $\pi_P$ is trivial on $J_P$. Thus, we may assume $g$ lies in $G_E$. Moreover $J_P \cap G_E = U(\Lambda) \cap G_E$ is a parahoric subgroup of $G_E$ containing the Iwahori subgroup $U(\Lambda^m) \cap G_E$. Therefore, we may further assume $g$ is a distinguished double coset representative for $\overline{U}(\Lambda) \cap G_E \setminus G_E / U(\Lambda) \cap G_E$ (see [17, §3] or [21, §1] for this notion).

Since $\dim I_g(\bar{\pi}_P J_P) = 1$, we can imitate the proof of [5, Proposition 5.3.2] to get that any non-zero intertwining operator in $I_g(\bar{x}_P) J_P)$ has the form $S \circ T$, with $S \in I_g(\bar{\pi}_P J_P)$ and $T$ an endomorphism of the space of $\overline{\pi}_P$. Now the operator $S$ also intertwines the restriction $\kappa|_{\bar{J}_P^1} = \overline{\pi}_P$ so, again as in [5, Proposition 5.3.2], it follows that $T$ belongs to $I_g(\overline{\pi}_P |_{\bar{J}_P^1})$. In particular, $g$ intertwines $\overline{\pi}_P |_{\bar{J}_P^1} \overline{\mathfrak{g}_E}$.

But $\bar{J}_P^1 \cap G_E = U^1(\Lambda^m) \cap G_E$ is the radical of the Iwahori subgroup $U(\Lambda^m) \cap G_E$ of $G_E$ contained in $U(\Lambda) \cap G_E$. By [24, Proposition 1.1] and the Remarks that follow it, we conclude that we can assume that $g$ normalizes $U(\Lambda) \cap G_E$, as required. \hfill \blacksquare

Recall that we write $J_T = i(J_W)$ and $\lambda_T = i(\lambda_W)$. In order to prove that $(\overline{J}_P, \overline{\lambda}_P)$ is a cover of $(J_T, \lambda_T)$, the only thing remaining is to find a strongly $(\overline{\mathfrak{T}}, \overline{J}_P)$-positive element in the centre of $\mathfrak{T}$ which supports an invertible element of the spherical Hecke algebra $H(\mathfrak{g}, \overline{x}_P)$. To achieve this, we look at the $\overline{x}_P$-spherical Hecke algebras of two parahoric subgroups whose intersections with $G_E$ are non-conjugate maximal compact open subgroups of $G_E$.

Let $\mathcal{L}^{(1)}$ be the self-dual $\mathfrak{g}_E$-lattice chain of $\mathfrak{g}_E$-period $e_W$ in $V = W \oplus W$ given by

$$\cdots \supset L_k^W \oplus L_k^W \supset L_{k+1}^W \oplus L_{k+1}^W \supset \cdots,$$
so that $\mathcal{L}'^{(1)}$ consists of every second lattice of $\mathcal{L}'$. Let $\Lambda^{(1)}$ be the self-dual $\sigma_E$-lattice sequence in $V'$ in which every lattice of $\mathcal{L}'^{(1)}$ occurs twice and with the indexing chosen such that
\[
\Lambda^{(1)}(k)^\# = \Lambda^{(1)}(1 - k), \quad \text{for all } k \in \mathbb{Z}.
\]

Let $\Lambda^{(1)}$ be the $\sigma_E$-lattice sequence in $V$ defined by
\[
\Lambda^{(1)}(k) = \Lambda^{(1)}(k) \oplus \Lambda_0(2k).
\]

It is a self-dual lattice sequence of period $2e_W = e/2$ such that $\Lambda_0(\Lambda^{(1)}) \supset \Lambda_0(\Lambda)$. Put $\mathcal{K}_1 = \mathcal{U}(\Lambda^{(1)})$.

We define $\mathcal{K}_2 = \mathcal{U}(\Lambda^{(2)})$ by the same process, starting from the self-dual $\sigma_E$-lattice chain $\mathcal{L}'^{(2)}$ in $V'$ given by
\[
\cdots \supset L_k^W \oplus L_{k+1}^W \supset L_{k+1}^W \oplus L_{k+2}^W \supset \cdots
\]

Then $\mathcal{U}(\Lambda) \subset \mathcal{K}_1 \cap \mathcal{K}_2$ so, in particular, $J_P \subset \mathcal{K}_1 \cap \mathcal{K}_2$.

Note that the element $w_1$ lies in $\mathcal{W} \cap \mathcal{K}_1$. We also set $w_2 = i(w_E^{-1}, 1_{V_0})w_1 \in \mathcal{W} \cap \mathcal{K}_2$.

Lemma 2.9. (i) $\mathcal{H}(\mathcal{K}_1, \lambda_P) = \langle f_1, f_{w_1} \rangle$ where $f_1$ is supported on $J_P$ and $f_{w_1}$ is supported on $J_{Pw_1}J_P$.

(ii) $\mathcal{H}(\mathcal{K}_2, \lambda_P) = \langle f_1, f_{w_2} \rangle$, with $f_1$ as in (i) and $f_{w_2}$ supported on $J_{Pw_2}J_P$.

Proof Both parts follow from the following consideration. For $i = 1, 2$ the $\mathcal{K}_i$-intertwining of $\lambda_P$ is given by
\[
I_{\mathcal{K}_i}(\lambda_P) = (J_P N_{\mathcal{G}_E}(\mathcal{U}(\Lambda) \cap \mathcal{G}_E) J_P) \cap \mathcal{K}_i = J_P N_{\mathcal{G}_E}(\mathcal{U}(\Lambda) \cap \mathcal{G}_E) J_P \cap \mathcal{K}_i.
\]

But $(N_{\mathcal{G}_E}(\mathcal{U}(\Lambda) \cap \mathcal{G}_E) \cap \mathcal{K}_i = \{1, w_i\})$. Moreover, the restriction of $\lambda_P$ to $J_P \cap \mathcal{L}$ is irreducible and $w_i$ normalizes $J_P \cap \mathcal{L}$ so the intertwining space $I_{w_i}(\lambda_P)$ is 1-dimensional.

Lemma 2.10. Consider $f_{w_1}, f_{w_2}$ as elements of $\mathcal{H}(\mathcal{G}, \lambda_P)$. Then the convolution $f_{w_1} * f_{w_2}$ is supported on $J_{Pw_1}w_2J_P$.

Proof We know $f_{w_1} * f_{w_2}$ is supported on
\[
J_{Pw_1}w_2J_P = J_P \left( w_1(J_P \cap \mathcal{U})^{-1} \right) \left( w_2^{-1}(J_P \cap \mathcal{U}) \right) J_P.
\]

Since $J_P$ contains $w_1(J_P \cap \mathcal{U})^{-1}$, $w_1(J_P \cap \mathcal{U})^{-1}$ and $w_2^{-1}(J_P \cap \mathcal{U})$, the lemma follows.

We will prove that $f_\zeta := f_{w_1} * f_{w_2}$ is invertible. To accomplish this we prove that $f_{w_1}$ and $f_{w_2}$ are each invertible. In each case we know
\[
f_{w_i} * f_{w_i} = c_{i}f_1 + d_if_{w_i}
\]
by Lemma 2.9. Thus, we only need to show $c_{i} \neq 0$ for each $i$.

Lemma 2.12. In equation (2.11) $c_{i} \neq 0$ for $i = 1, 2$.  

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Proof We treat only the case \( i = 1 \), since the other case is identical. We need to check that \( f_{w_1} \ast f_{w_1}(1) \neq 0 \). Since
\[
f_{w_1}(x) = \begin{cases} 0 & \text{if } x \notin \mathcal{J}_P w_1 \mathcal{J}_P \\ \chi^\mathcal{J}_P(j_1) f_{w_1}(w_1) \chi^\mathcal{J}_P(j_2) & \text{if } x = j_1 j_2, \text{with } j_1, j_2 \in \mathcal{J}_P,
\end{cases}
\]
we can write
\[
f_{w_1} \ast f_{w_1}(1) = \int_{\mathcal{K}_1} f_{w_1}(y) f_{w_1}(y^{-1}) dy = \frac{1}{|\mathcal{J}_P \cap w_1 \mathcal{J}_P|} \int_{\mathcal{J}_P \times \mathcal{J}_P} f_{w_1}(j_1 j_2) f_{w_1}(j_2^{-1} w_1^{-1} j_1^{-1}) d j_1 d j_2 = \frac{|\mathcal{J}_P|}{|\mathcal{J}_P \cap w_1 \mathcal{J}_P|} \int_{\mathcal{J}_P} \chi^\mathcal{J}_P(j_1) f_{w_1}(w_1^{-1}) \chi^\mathcal{J}_P(j_1^{-1}) d j_1.
\]
Now \( w_1 \) intertwines \( \widetilde{\lambda}_P' = \iota(\lambda_W^\alpha) \) and normalizes \( \mathcal{J}_P \cap \mathcal{L} = i(\mathcal{L}) \) so \( f_{w_1}(w_1) \) is an equivalence \( \iota^1_i(\lambda_W^\alpha) \simeq i(\lambda_W^\alpha) \). But \( w_1^{-1} = h_1^{-1} w_1 \), where \( h_1 = i(\nu \sigma \varepsilon \sigma(\pi), 1_{V_0}) \) and, by Proposition 1.11 \( \nu \sigma \varepsilon \sigma(\pi) \in \mathcal{L}_W \); hence \( w_1^{-1} \in \mathcal{J}_P w_1 \) and \( f_{w_1}(w_1^{-1}) \) is an equivalence \( i(\lambda_W^\alpha) \simeq \iota^1_i(\lambda_W^\alpha) \). Thus \( f_{w_1}(w_1) f_{w_1}(w_1^{-1}) \) is an equivalence of \( i(\lambda_W^\alpha) \) and hence a scalar \( c \neq 0 \). Thus
\[
f_{w_1} \ast f_{w_1}(1) = c |\mathcal{J}_P|^{-2} |\mathcal{J}_P \cap w_1 \mathcal{J}_P| \neq 0.
\]
Therefore, \( f_{w_1} \) is invertible, as required. \( \blacksquare \)

**Lemma 2.13.** For each \( k \in \mathbb{N} \), the \( k \)-fold convolution \( \chi^k \) is supported on \( \mathcal{J}_P \mathcal{J}_P \).

**Proof** This is by simple induction on \( k \), since
\[
\mathcal{J}_P \chi^k \mathcal{J}_P \chi^k \mathcal{J}_P = \mathcal{J}_P \left( \chi(\mathcal{J}_P \cap \mathcal{U}) \chi^{-1} \right) \left( \chi(\mathcal{J}_P \cap \mathcal{L}) \chi^{-1} \right) \chi^{-1} \left( \chi^{-1} \mathcal{J}_P \chi \mathcal{U} \right) \chi^k \mathcal{J}_P,
\]
while \( \mathcal{J}_P \) contains \( \chi(\mathcal{J}_P \cap \mathcal{U}) \chi^{-1} \), \( \chi(\mathcal{J}_P \cap \mathcal{L}) \chi^{-1} \) and \( \chi^{-1} \left( \mathcal{J}_P \chi \mathcal{U} \right) \chi^k \) (cf. Lemma 2.10). \( \blacksquare \)

In particular, \( \chi^k(\mathcal{J}_P \mathcal{U}) \) is an invertible element of \( \mathcal{H}(\mathcal{G}, \lambda_P) \) which is supported on the double coset \( \mathcal{J}_P \mathcal{P} \), where
\[
\lambda_F = i(\mathcal{U} \lambda_1 \lambda_2)
\]
is a strongly \( (\mathcal{P}, \mathcal{J}_P) \)-positive element of the centre of \( \mathcal{L} \). We conclude:

**Theorem 2.14.** Let \( \pi_W \) be an irreducible supercuspidal representation of \( G_W \simeq GL_M(F) \), with \( \pi_W \simeq \pi_W \). Using the notation above, the pair \( (\mathcal{J}_P, \lambda_P) \) is a \( \mathcal{G} \)-cover of \( (\mathcal{J}_P, \lambda_\mathcal{J}) \). In particular, it is an \( \mathcal{S} \)-type, with \( \mathcal{S} = [\mathcal{L}, i(\pi_W)]_\mathcal{G} \).

**Proof** By construction, \( \mathcal{J}_P \) is decomposed with respect to \( \mathcal{P} \) so (i) and (ii) of \( [\mathcal{K}] \) Definition 8.1] are satisfied for \( \mathcal{P} \), and likewise for \( \mathcal{P}_- \). By Lemma 2.12 the strongly \( (\mathcal{P}, \mathcal{J}_P) \)-positive element \( \lambda_F \) supports an invertible element of \( \mathcal{H}(\mathcal{G}, \lambda_P) \). Thus \( \mathcal{J}_P, \lambda_P \) also satisfies (iii) of \( [\mathcal{K}] \) Definition 8.1] for \( \mathcal{P} \), and is therefore a \( \mathcal{G} \)-cover of \( (\mathcal{J}_P, \lambda_\mathcal{J}) \). Then \( [\mathcal{K}] \) Theorem 8.3 implies \( \mathcal{J}_P, \lambda_P \) is an \( \mathcal{S} \)-type. \( \blacksquare \)
2.3 Hecke Algebras

In this section we derive results analogous to those of Chapter 5 of [5]. In particular, we show that the Hecke algebra of our type, $\mathcal{H}(G, \lambda)$, can be computed by using analogous computations in a case where the representation $\pi$ of $T$ is of level zero. For many of these situations, [15] will give us the parameters of the Hecke algebra.

We fix $i = 1$ or $2$ and, in the notation of the previous section, we put $w = w_i$. Note that, for $g \in G_\mathcal{W}$ and $g_0 \in G_0$, we have $w_i i(g, g_0) = i(\bar{\sigma}(g), g_0)$ (and a similar result for $w_2$) so, by Proposition [14] we have $w_i i(\theta_W) = i(\theta_W)$, and similarly for $\eta_W$. In particular, $w$ intertwines the representation $\pi_P = i(\eta_W)$ of $J_P$.

Now $\kappa_W \circ \bar{\sigma}$ is also a $\beta$-extension of $\eta_W$ so $\kappa_W \circ \bar{\sigma} \simeq \kappa_W \otimes \chi_W$; for some character $\chi_W$ of $U(G_\mathcal{W})/U^1(G_\mathcal{W}) \simeq J_W/J_{W,1}^1$ which factors through the determinant $\det_{B_{W,E}}$. Then

$$w_i(\kappa_W) \simeq i(\kappa_W \circ \bar{\sigma}) = i(\kappa_W) \otimes i(\chi_W)$$

and $w$ intertwines $\pi_P$ with $\pi_P \otimes \chi_P$, where $\chi_P = i(\chi_W)$. Note also that, since $J/W \simeq J_P/J_{P} \simeq J/W/J_{W,1}^1$, we can extend $\chi_P$ to the character $\chi = i(\chi_W)$ of $J$. Set $\pi = \text{Ind}_{J_P}^{J} \pi_P$; since $\mathcal{I}_{G}(\pi_P) = \mathcal{J}_P G_E J_P$, we know that $\pi$ is irreducible and further, by [5] Proposition 4.1.3], $w$ intertwines $\pi$ with $\pi \otimes \chi$.

In the notation of the previous section, we put $\Lambda^M = \Lambda^{(i)}$. Then $[\Lambda^M, n_M, 0, \frac{1}{2} \beta]$ is a skew semisimple stratum and we set $\mathcal{J}_M = \mathcal{J}(\frac{1}{2} \beta, \Lambda^M)$, and similarly $\mathcal{J}_M$ and $\mathcal{P}_M$. Let $\mathcal{G}_M$ denote the transfer of $\mathcal{G}$ to $\mathcal{J}_M$. By [21] Corollary 4.2], there is a unique irreducible representation $\mathcal{H}_M$ of $\mathcal{J}_M$ which extends $\mathcal{B}_M$, and by [21] Theorem 4.1], we may choose a $\beta$-extension, $\mathcal{X}_M$ of $\mathcal{X}_M$ to $\mathcal{J}_M$ – we recall here what we mean by $\beta$-extension:

Recall the lattice sequence $\Lambda^m$ from [14] such that $U(\Lambda^m) \cap G_E$ is an Iwahori subgroup of $U(\Lambda) \cap G_E$; then we have $\mathcal{G}_M^m$ the transfer of $\mathcal{G}_M$ and $\mathcal{P}_m$ the unique irreducible representation of $\mathcal{J}^1(\frac{1}{2} \beta, \Lambda^m)$ containing $\mathcal{G}_M$. We abbreviate $U(\Lambda) \cap G_E = U(\Lambda_{\mathcal{G}_E})$ (and similarly for other lattice sequences) and define $\mathcal{J}_M = U(\Lambda_{\mathcal{G}_E}) J^1_{\mathcal{P}}$ and $\mathcal{J}_{W,1}^1 = U^1_m(\Lambda_{\mathcal{G}_E}) J^1_{\mathcal{P}}$. Thus, $J_M \supset J_{M,1} \supset J_{W,1}^1$ and $J_{W,1}^1$ is a pro-$p$ Sylow subgroup of $J_M$. By [21] Proposition 3.7], there is a unique irreducible representation, $\mathcal{H}_M$ of $\mathcal{J}_M$ which extends $\mathcal{P}_M$ and such that $\mathcal{H}_M$ and $\mathcal{X}_M$ induce equivalent irreducible representations of $U^1(\Lambda_M)$. Then a $\beta$-extension $\mathcal{X}_M$ is an extension to $\mathcal{J}_M$ of $\mathcal{H}_M$.

Similarly (as in [5] Proposition 5.2.5) – see [21] Lemma 4.2], there is a unique irreducible representation, $\mathcal{H}_M$ of $J_M$ which extends $\mathcal{P}_M$ and such that

$$\text{Ind}_{J_M}^{U(\Lambda)} \mathcal{H}_M \simeq \text{Ind}_{\mathcal{J}}^{U(\Lambda) \cap G_E} \mathcal{H}_M \simeq \text{Ind}_{J_M}^{U(\Lambda) \cap G_E} \mathcal{H}_M \simeq \text{Ind}_{\mathcal{J}_M}^{U(\Lambda)} \mathcal{X}_M. \tag{2.15}$$

Moreover, as in [5] Proposition 5.2.6], we have $\mathcal{H}_M|_{J_{M,1}} = \mathcal{H}_M$ and, as in [19] Proposition 2.9], the $G_E$- intertwining of $\mathcal{H}_M$ and $\mathcal{X}_M$ are the same; indeed, the same proof shows that $w$ intertwines $\mathcal{H}_M$ with $\mathcal{X}_M$, where $\chi_M = i(\chi_W)$ on $\mathcal{J}_M$.

Lemma 2.16. There is a choice of $\kappa_W$ for which $\kappa_W \circ \bar{\sigma} \simeq \kappa_W$.

Proof Fix some choice of $\kappa_W$, which fixes the character $\chi_W$. Recall that we have $\mathcal{X}_M|_{J_{M,1}} = \mathcal{H}_M$ and, by construction of $\beta$-extensions in [21] Theorem 4.1]. Thus, if we let $\hat{\kappa}_M = \mathcal{X}_M|_{J_{M,1}}$, we must
have \( \hat{\kappa}_M \simeq \hat{\mu}_M \otimes \psi_M \) for a character \( \psi_M \) of \( \mathcal{U}(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}) \), that is, a character of the Siegel Levi subgroup of \( \mathcal{U}(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}) \) which is trivial on the maximal unipotent radical \( \mathcal{U}^1(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}) \).

Then \( \psi_M \) factorizes through the determinant on the Levi subgroup and we can write \( \psi_M = i(\psi_W) \), for some character \( \psi_W \) of \( J_M/J_M \).

Now \( w \in J_M \) so \( w \) certainly intertwines \( \hat{\kappa}_M \) with itself. Hence \( w \) intertwines \( \hat{\mu}_M \otimes \psi_M \) with \( \hat{\mu}_M \otimes \psi_M \). Chasing back through the constructions above, we see that \( w \) intertwines \( \pi_P \otimes \psi_M \) with \( \pi_P \otimes \psi_M \). Since \( w \) normalizes \( L \), this implies that conjugation by \( w \) gives an equivalence \( \kappa_W \otimes \psi_W \simeq \kappa_W \otimes \chi_w(\psi_W \circ \sigma) \). By \( \{5, \text{Theorem 5.2.2}\} \), we deduce that \( \psi_W = \chi_w(\psi_W \circ \sigma) \) and, in particular, \( \kappa_W \otimes \psi_W \) is a \( \beta \)-extension with the required property.

We now choose \( \kappa_W \) as in Lemma \( \ref{lem:psi} \) and make the same constructions as before: \( \pi_P, \pi \) and \( \hat{\mu}_M \). Comparing \( \hat{\mu}_M \) with \( \hat{\kappa}_M \), we again get a character \( \psi_M \) but now with the property that \( \psi_M \otimes \psi_M = \psi_M \). If \( E/E_0 \) is unramified then this implies that \( \psi_M = \psi_0 \), for some character \( \psi_0 \). Then \( \psi_0 \) extends to a character of \( J_M \) and, replacing \( \kappa_M \) by \( \kappa_M \otimes \psi_0^{-1} \), we may assume \( \psi_M = 1 \). In the ramified case, the condition on \( \psi_M \) is that \( \psi_M = 1 \) but it is (at least in principle) possible that \( \psi_M = 1 \).

We write \( \psi_M = i(\psi_W) \); then we may assume \( \psi_W = 1 \).

**Proposition 2.17.** With the notation as above, we have a support-preserving algebra isomorphism 

\[
\mathcal{H}(J_M, \pi_P) \simeq \mathcal{H}(\mathcal{U}(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}), i(\rho_W \otimes \psi_W)).
\]

**Proof** From the isomorphisms in \( \ref{iso:1} \) and \( \{5, \text{Corollary 4.1.5}\} \), we get a support-preserving isomorphism 

\[
\mathcal{H}(J_M, \pi_P) \simeq \mathcal{H}(J_M, \pi \otimes i(\rho_W)).
\]

Similarly, as in \( \{5, \text{Proposition 5.5.13}\} \), since \( \psi_W = 1 \) we have a support-preserving isomorphism 

\[
\mathcal{H}(J_M, \pi \otimes i(\rho_W)) \simeq \mathcal{H}(J_M, (\hat{\mu}_M \otimes \psi_M) \otimes (i(\rho_W \otimes \psi_W))).
\]

Finally, we have support-preserving isomorphisms 

\[
\mathcal{H}(J_M, (\hat{\mu}_M \otimes \psi_M)) \otimes (i(\rho_W \otimes \psi_W)) \simeq \mathcal{H}(J_M, i(\rho_W \otimes \psi_W)) 
\]

\[
\simeq \mathcal{H}(\mathcal{U}(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}), i(\rho_W \otimes \psi_W)),
\]

where the first isomorphism follows from the fact that \( \hat{\mu}_M \otimes \psi_M = \hat{\kappa}_M \) extends to a representation \( \kappa_M \) of \( J_M \) (cf. \( \{5, \text{Lemma 5.6.3}\} \)), and the second by reduction modulo \( J_M \), since \( J_M/\mathcal{J}_M \equiv \mathcal{U}(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}) \). Putting these isomorphisms together gives the isomorphism of the Proposition.

**Remarks 2.18.** (i) Note that, writing \( B_M \) for the self-dual \( \sigma_E \)-order \( a_0(\Lambda_{E}) \cap B' \), we have 

\[
\mathcal{U}(\Lambda_{E})/\mathcal{U}^1(\Lambda_{E}) \simeq \mathcal{U}(B_M)/\mathcal{U}^1(\mathcal{B}_M) \times \mathcal{G}_0/\mathcal{G}_0^1 \quad \text{and} \quad \mathcal{U}(\mathcal{A}_{E})/\mathcal{U}^1(\mathcal{A}_{E}) \simeq \mathcal{U}(\mathcal{B}')/\mathcal{U}^1(\mathcal{B}') \times \mathcal{G}_0/\mathcal{G}_0^1,
\]

where \( \mathcal{G}_0^1 \) is the pro-\( p \) radical of the anisotropic group \( \mathcal{G}_0 \). Then we have an isomorphism 

\[
\mathcal{H}(\mathcal{U}(\mathcal{A}_{E})/\mathcal{U}^1(\mathcal{A}_{E}), i(\rho_W \otimes \psi_W)) \simeq \mathcal{H}(\mathcal{U}(\mathcal{B}_M)/\mathcal{U}^1(\mathcal{B}_M), i(\rho_W \otimes \psi_W)).
\]
The quotient $\overline{U}(\mathfrak{B}_M')/\overline{U}^1(\mathfrak{B}_M')$ is a unitary (if $E/E_0$ is unramified), symplectic or orthogonal group over $k_{E_0}$ and the Hecke algebra on the right is described in [12]. Alternatively, reduction modulo $\overline{U}^1(\mathfrak{B}_M')$ gives a support-preserving isomorphism

$$\mathcal{H}((\overline{U}(\mathfrak{B}_M')/\overline{U}^1(\mathfrak{B}_M'), i(\rho_W \otimes \psi_W)) \simeq \mathcal{H}(\overline{U}(\mathfrak{B}_M'), i(\rho_W \otimes \psi_W)),$$

and the latter is described in [16].

(ii) Since (in the case where $E/E_0$ is ramified) we have $\chi_W^2 = 1$, we may replace our choice of $\kappa_W$ by $\kappa_W \otimes \chi_W$ (which has the same property of being fixed by $\overline{\sigma}$); this replaces $\rho_W$ by $\rho_W \otimes \chi_W$, another self-dual cuspidal representation of $\overline{U}(\mathfrak{B}_M)$, and we lose the character $\chi_W$ from the RHS of the isomorphism in Proposition 2.17. However, we cannot do this independently for the two choices $\Lambda^{(1)}, \Lambda^{(2)}$ for $\Lambda^M$. In particular, if we choose (as we always can) to dispose with the character $\chi_W$ in one case, then it may still be non-trivial in the other.

(iii) Since $(J_P, \lambda_P)$ is a cover of $(J_L, \lambda_L)$, by [6, Corollary 7.12] we have a canonical embedding of Hecke algebras $t_P : \mathcal{H}(J_L, \lambda_L) \hookrightarrow \mathcal{H}(\overline{G}, \overline{\lambda}_P)$ and we identify $\mathcal{H}(\overline{J}_M, \overline{\lambda}_P)$ (which is just the algebra of Laurent polynomials in a single variable) with its image $\mathcal{B}_P$. We also put $\mathcal{K} = \mathcal{H}(\overline{J}_M, \overline{\lambda}_P)$. Then [8, Theorem 1.5] implies that the map

$$\mathcal{B}_P \otimes_{\mathcal{K}} \mathcal{K} \rightarrow \mathcal{H}(\overline{G}, \overline{\lambda}_P)$$

$$f \otimes \phi \mapsto \overline{f} \ast \phi$$

is an isomorphism of $(\mathcal{B}_P, \mathcal{K})$-bimodules.

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