The McKay-Thompson series of Mathieu Moonshine modulo two

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Abstract

In this note, we describe the parity of the coefficients of the McKay-Thompson series of Mathieu moonshine. As an application, we prove a conjecture of Cheng, Duncan and Harvey stated in connection with umbral moonshine for the case of Mathieu moonshine.

1 Introduction

In 2010, Eguchi, Ooguri, and Tachikawa discovered a phenomenon connecting the Mathieu group $M_{24}$ and the elliptic genus of a K3 surface. To describe their observation, we let $q = e^{2\pi i \tau}$ and consider the function

$$\Sigma(\tau) = -8(\mu(1/2; \tau) + \mu(\tau/2; \tau) + \mu((\tau + 1)/2; \tau))$$

$$= q^{-\frac{1}{8}}(-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + \cdots)$$

where

$$\mu(z; \tau) = \frac{ie^{\pi i z}}{\theta_1(z; \tau)} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1 - q^n e^{2\pi inz}}$$

$$\theta_1(z; \tau) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}.$$
The function $\Sigma(\tau)$ is a Mock modular form and counts the decomposition of the elliptic genus of a K3 surface into representations of the $\mathcal{N}=4$ Virasoro algebra.

The Mathieu moonshine phenomenon is that the first five coefficients appearing in the Fourier expansion of $\Sigma(\tau)$ divided by 2, 

$$\{45, 231, 770, 2277, 5796\},$$

are equal to dimensions of irreducible representations of $M_{24}$ and further coefficients can be written as simple sums of dimensions of the irreducible representations of $M_{24}$, for example $13915 = 3520 + 10395$. The reason for this mysterious phenomenon is still unknown.

This observation suggested the existence of a virtual graded $M_{24}$-module $K = \bigoplus_{n=-1}^{\infty} K_n q^{n/8}$ such that for $n \geq 0$ the $K_n$ are honest $M_{24}$-representations. In analogy to the monstrous moonshine case [5], one can consider for an element $g$ in the conjugacy class $\ell X$ of $M_{24}$ the so-called McKay-Thompson series

$$\Sigma_{\ell X}(\tau) = \sum_{n=-1}^{\infty} \text{Tr}(g|K_n) q^{n/8}.$$ 

In [12, 7] (cf. also [1, 11, 2]), candidates for the 26 McKay-Thompson series for the Mathieu moonshine have been proposed. We note that the McKay-Thompson series for the $M_{24}$ conjugacy classes in the pairs $(7A, 7B)$, $(14A, 14B)$, $(21A, 21B)$, $(15A, 15B)$ and $(23A, 23B)$ are equal to each other and we denote these cases shortly by $\ell AB$. We list in the appendix the 21 different McKay-Thompson series. Using explicit formulas, it can be shown that all the McKay-Thompson series have integer coefficients.

Elementary character theory [19] implies that the $\Sigma_{\ell X}(\tau)$ together uniquely determine the $M_{24}$-module $K$ if it exists. Terry Gannon [13] showed that this is indeed the case.

**Theorem 1.1** (Mathieu moonshine module). The McKay-Thompson series as in [12, 7] determine a virtual graded $M_{24}$-module $K = \bigoplus_{n=-1}^{\infty} K_n q^{n/8}$. For $n \geq 0$, the $K_n$ are honest (and not only virtual) $M_{24}$-representations. Furthermore, $K_n$ can be decomposed as a direct sum of $M_{24}$-representations of the form $\lambda \oplus \bar{\lambda}$ where $\lambda$ is irreducible.

This implies that if an irreducible representation $\lambda$ is real, i.e. $\lambda \cong \bar{\lambda}$, then the multiplicity of an irreducible $\lambda$ in $K_n$ is even.
Unlike for the monstrous moonshine case where a vertex operator algebra structure was constructed in [10], there are yet no known underlying algebraic structures on the Mathieu moonshine module $K$.

To illustrate our main result, we consider the McKay-Thompson series for $7AB$:

$$
\Sigma_{7AB}(\tau) = \frac{1}{8} \left( \sum(\tau) \eta(\tau)^3 - 14 \phi_2^{(7)}(\tau) \right) / \eta(\tau)^3 
$$

$$
= -2q^{-1/8} - q^{7/8} + 4q^{31/8} - 2q^{47/8} + 2q^{55/8} - 3q^{63/8} + 6q^{87/8} - 6q^{103/8} 
- 4q^{119/8} + 8q^{143/8} - 6q^{159/8} + 4q^{167/8} - 7q^{175/8} + 12q^{199/8} + \cdots .
$$

One observes that the coefficient of $q^{n/8}$ in $\Sigma_{7AB}(\tau)$ is odd if $n = 7m^2$, where $m$ is odd. In general we show:

**Theorem 1.2.** For a conjugacy class $\ell X$ of $M_{24}$, the coefficient of $q^{n/8}$ in $\Sigma_{\ell X}(\tau)$ is odd if and only if $\ell X \in \{7AB, 14AB, 15AB, 23AB\}$ and $n = \ell m^2$, where $m$ is odd, or $\ell X = 21AB$ and $n = \ell m^2$, where $m$ is odd and not divisible by 3.

For congruences of the Fourier coefficients of $\Sigma_{\ell X}(\tau)$ for other primes, we refer to the references [15, 17, 16]. One reason for considering the parity of the Fourier coefficients is that it explains the appearance of certain irreducible representations of $M_{24}$. The following conjecture was made in [3], which we state for the case of the Mathieu moonshine only.

**Conjecture 1.1 ([3], Conj. 5.11).** Let $n = \ell m^2 \equiv 7 \pmod{8}$. Then the $M_{24}$-representation $K_n$ determined by the coefficients of $q^{n/8}$ of the McKay-Thompson series contains the following conjugate pairs of irreducible representations:

- For $\ell = 7$, one of the pairs $(\chi_3, \chi_4)$, $(\chi_{12}, \chi_{13})$ or $(\chi_{15}, \chi_{16})$;
- for $\ell = 15$, the pair $(\chi_5, \chi_6)$;
- for $\ell = 23$, the pair $(\chi_{10}, \chi_{11})$.

Here, $\chi_i$ where $1 \leq i \leq 26$ denotes the $i$-th irreducible representation as listed in the ATLAS [4].

The paper is organized as follows. In Section 2 we study the cases $7AB, 14AB, 15AB, 21AB, 23AB$. In Section 3 we study the remaining cases. In the final section, we use Theorem 1.2 to prove Conjecture 1.1.
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2 The non-even cases

In this section, we prove Theorem 1.2 for the cases $7AB$, $14AB$, $15AB$, $21AB$ and $23AB$.

The strategy is to obtain the parity properties of $\Sigma_{\ell X}(\tau)$ from the parity properties of a modular form for a group $\Gamma_0(N)$ which in turn can be proven by an application of Sturm’s theorem [20]. The theorem allows to obtain a divisibility property of the Fourier coefficients of a modular form of weight $k$ for $\Gamma_0(N)$ by verifying this property only for the first $n$ coefficients where $n$ is an explicit given bound depending only on $N$ and $k$,

$$n \geq \frac{k}{12}[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)].$$

We let

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$$

be the Dedekind $\eta$-function and consider for $N \geq 2$ the Eisenstein series

$$\phi_2^{(N)}(\tau) = \frac{24}{N-1} q \frac{d}{dq} \log \left( \frac{\eta(N\tau)}{\eta(\tau)} \right)$$

$$= 1 + \frac{24}{N-1} \sum_{k=1}^{\infty} \sigma_1(k)(q^k - Nq^{Nk})$$

of weight 2 for $\Gamma_0(N)$.

Proof of Theorem 1.2 for the case $7AB$. Define the function $f_m(\tau)$ as follows:

$$f_m(\tau) = \frac{1}{4} \left( \vartheta_3 \left( \frac{m\tau}{8} \right) - \vartheta_4 \left( \frac{m\tau}{8} \right) \right)$$
where \( \vartheta_3(\tau) = 1 + \sum_{m=1}^{\infty} 2q^{m^2} \) and \( \vartheta_4(\tau) = 1 + \sum_{m=1}^{\infty} 2(-q)^{m^2} \). Then

\[
f_7(\tau) = \frac{1}{4} \left( \vartheta_3\left(\frac{7\tau}{8}\right) - \vartheta_4\left(\frac{7\tau}{8}\right) \right) = q^{7/8} + q^{63/8} + q^{175/8} + q^{243/8} + \ldots
\]

We call \( f_7(\tau) \) the "parity function."

We have to show that all coefficients of \( \Sigma_{7AB}(\tau) + f_7(\tau) \) are even since the coefficient of \( q^{n/8} \) in \( f_7(\tau) \) is odd if and only if \( n = 7m^2 \), where \( m \) is odd. This follows if all coefficients of \( (\Sigma_{7AB}(\tau) + f_7(\tau))\eta(\tau)^3 \) are even since \( \eta(\tau)^{-3} \) has integral coefficients.

Let

\[
E_2(\tau) = 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)q^m.
\]

By [6] p. 43, Example 4, we have

\[
\Sigma(\tau) = -2 \left( E_2(\tau) + 24 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{\frac{n(n+1)}{2}}}{1 - q^n} \right) / \eta(\tau)^3.
\]

Thus

\[
(\Sigma_{7AB}(\tau) + f_7(\tau))\eta(\tau)^3 = \frac{1}{8} \left( \Sigma(\tau)\eta(\tau)^3 - 14 \phi_2^{(7)}(\tau) \right) + \eta(\tau)^3 f_7(\tau)
\]

\[
= -\frac{1}{4} \left( E_2(\tau) + 24 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{\frac{n(n+1)}{2}}}{1 - q^n} + 7\phi_2^{(7)}(\tau) \right) + \eta(\tau)^3 f_7(\tau).
\]

First, we observe that the constant term is even. Since all coefficients, except for the constant term, of the function

\ [-\frac{1}{4} \left( E_2(\tau) + 24 \sum_{n=1}^{\infty} \frac{(-1)^n n q^{\frac{n(n+1)}{2}}}{1 - q^n} \right) \]

are even, it is enough to show that the coefficients of the following function, again except for the constant term,

\ [-\frac{7}{4} \phi_2^{(7)}(\tau) + \eta(\tau)^3 f_7(\tau) = -\frac{7}{4} - 6q - 24q^2 - 28q^3 - 44q^4 - 42q^5 + \ldots \]

are even.
Define the “correction function”

\[
\frac{7}{4} \vartheta_3(\tau)^4 = \frac{7}{4} + 14q + 42q^2 + 56q^3 + 42q^4 + 84q^5 + \cdots.
\]

The coefficients of \(\frac{7}{4} \vartheta_3(\tau)^4\) without the constant term are even, hence it is enough to show that the coefficients of the function

\[
-\frac{7}{4} \varphi_2(\tau) + \eta(\tau)^3 f_7(\tau) + \frac{7}{4} \vartheta_3(\tau)^4 = 8q + 18q^2 + 28q^3 - 2q^4 + 42q^5 + \cdots
\]

are even. We are going to prove this using Sturm’s theorem. The theta functions \(\vartheta_3(\tau)\) and \(\vartheta_4(\tau)\) can be expressed as a quotient of \(\eta\)-functions, namely

\[
\vartheta_3(\tau) = \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2} \quad \text{and} \quad \vartheta_4(\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}.
\]

It follows using [18, Theorem 1.64] that \(\eta(8\tau)^3 f_7(8\tau)\) is a modular form of weight 2 for \(\Gamma_0(448)\) and hence

\[
\sum_{m=1}^{\infty} a_{7AB}(m) q^m := -\frac{7}{4} \varphi_2(8\tau) + \eta(8\tau)^3 f_7(8\tau) + \frac{7}{4} \vartheta_3(8\tau)^4
\]

\[
= 8q^8 + 18q^{16} + 28q^{24} + \cdots
\]

is also a modular form of weight 2 for \(\Gamma_0(448)\). Using Sturm’s theorem [20] (see also [18, Theorem 2.58]) and the fact that \([\text{SL}_2(\mathbb{Z}) : \Gamma_0(448)] = 768\), the computer verification that \(a_{7AB}(m) \equiv 0 \pmod{2}\) for \(m \leq 129\) shows \(a_{7AB}(m) \equiv 0 \pmod{2}\) for all \(m\).

This completes the proof of the case \(7AB\).

Since the other cases can be handled in complete analogy, we collect the relevant information in Table 1. For the definition of \(\Sigma_{23AB}(\tau)\) in Appendix A, we use functions \(f_{23,1}(\tau)\) and \(f_{23,2}(\tau)\) which are modular forms for \(\Gamma_0(23)\), explicitly given in [8, Appendix A.1].

3 The even cases

In this section, we prove that all coefficients of the McKay-Thompson series for the remaining conjugacy classes of \(M_{24}\) are divisible by two.
Table 1: Data for the proofs in Section 2

| ℓX | parity function | correction function | Γ | [SL₂(ℤ) : Γ] | Sturm bound |
|-----|------------------|---------------------|---|---------------|-------------|
| 7AB | \( f_7(\tau) \) | \( \frac{1}{7} \theta_3(\tau)^4 \) | \( \Gamma_0(448) \) | 768 | 129 |
| 14AB | \( f_7(\tau) \) | \( \frac{23}{12} \phi_2^{(2)}(\tau) \) | \( \Gamma_0(448) \) | 768 | 129 |
| 15AB | \( f_{15}(\tau) \) | \( \frac{23}{12} \phi_2^{(2)}(\tau) \) | \( \Gamma_0(960) \) | 2304 | 385 |
| 21AB | \( f_7(\tau) - f_{63}(\tau) \) | \( 2\theta_3(\tau)^4 \) | \( \Gamma_0(4032) \) | 9216 | 1537 |
| 23AB | \( f_{23}(\tau) \) | \( \frac{23}{12} \phi_2^{(2)}(\tau) \) | \( \Gamma_0(1472) \) | 2304 | 385 |

We remark that the case 1A is clear since \( \Sigma_{1A}(\tau) = \Sigma(\tau) \) and equation (2). It is trivial to see that for the cases 2B, 3B, 4A, 4C, 6B, 10A, 12A, 12B, the coefficients of \( \Sigma_{\ell X}(\tau) \) are even because \( \Sigma_{\ell X}(\tau) \) is \(-2\) times an \( \eta \)-product (see the appendix).

3.1 The cases 2A, 3A, 4B, 5A, 6A, 8A

We give the detailed proof for the case of 2A.

Proof. The McKay-Thompson series for 2A is

\[
\Sigma_{2A}(\tau) = \frac{1}{3} \left( \Sigma(\tau)\eta(\tau)^3 - 4\phi_2^{(2)}(\tau) \right) / \eta(\tau)^3
= -2q^{-1/8} - 6q^{7/8} + 14q^{15/8} - 28q^{23/8} + 42q^{31/8} + \cdots
\]

The coefficients of \( \Sigma(\tau)\eta(\tau)^3 \), except for the constant term, are divisible by 24 (cf. (2)). Moreover, also the coefficients of \( \phi_2^{(2)}(\tau) \), again except for the constant term, are divisible by 24. The constant term of the function \( \Sigma(\tau)\eta(\tau)^3 - 4\phi_2^{(2)}(\tau) \) is 6. Therefore, the coefficients of \( \Sigma_{2A}(\tau) \) are divisible by two.

The proof for the other cases is analogous. The relevant information can be read off from Table 2 together with the appendix.

3.2 The case 11A

Finally, it remains to show that the Fourier coefficients of the McKay-Thompson series for the case 11A are divisible by two. Because of the extra term
Table 2: Data for the proofs in Section 3.1

| ℓX | N | divisor of φ₂⁵⁸⁰[N](τ) − 1 |
|-----|---|--------------------------|
| 2A  | 2 | 24                       |
| 3A  | 3 | 12                       |
| 5A  | 5 | 6                        |
| 4B  | 2, 4 | 24, 8                  |
| 8A  | 4, 8 | 8, 24/7                 |
| 6A  | 2, 3, 6 | 24, 12, 24/5         |

\(\frac{264}{5}(\eta(\tau)\eta(11\tau))^2\) in \(\Sigma_{11A}(\tau)\), it is not obvious that the coefficients except for the constant term in the numerator for \(\Sigma_{11A}(\tau)\) as given in the appendix are divisible by 24. Therefore, we proceed similar to the cases in Section 3.1.

**Proof.** The McKay-Thompson series for 11A is

\[
\Sigma_{11A}(\tau) = \frac{1}{12} \left( \frac{\Sigma(\tau)\eta(\tau)^3 - 22\phi_{2}^{(11)}(\tau) + \frac{264}{5}(\eta(\tau)\eta(11\tau))^2}{\eta(\tau)^3} \right).
\]

The coefficients of \(\Sigma(\tau)\eta(\tau)^3\), except for the constant term are divisible by 24. We need to prove that the coefficients of the function:

\[
22\phi_{2}^{(11)}(\tau) - \frac{264}{5}(\eta(\tau)\eta(11\tau))^2 = 22 + 264q^2 + 264q^3 + 264q^4 + \cdots
\]

are also (except for the constant term) divisible by 24. The coefficients of \(22\phi_{2}^{(2)}(\tau)\) (except for the constant term) are divisible by 24, hence it is enough to show that the coefficients of the function

\[
\sum_{m=1}^{\infty} a_{11A}(m)q^m := 22\phi_{2}^{(11)}(\tau) - \frac{264}{5}(\eta(\tau)\eta(11\tau))^2 - 22\phi_{2}^{(2)}(\tau)
\]

\[
= -528q - 264q^2 - 1848q^3 - 264q^4 - 2904q^5 - 1584q^6 - 3696q^7 + \cdots
\]

are divisible by 24. Note that this function is a modular form for \(\Gamma_0(22)\) [14, p. 130, Proposition 19]. Using Sturm’s theorem [18 Theorem 2.58] again and the fact that \([\text{SL}_2(\mathbb{Z}) : \Gamma_0(22)] = 36\), the verification that \(a_{11A}(m) \equiv 0 \pmod{24}\) for \(m \leq 7\) shows \(a_{11A}(m) \equiv 0 \pmod{24}\) for all \(m\). Therefore, the coefficients of \(\Sigma_{11A}(\tau)\) are divisible by two. \(\square\)

Sections 2 and 3 together prove Theorem 1.2.
4 Multiplicities of Irreducibles

Recall that $\chi_i$ denotes the $i$-th irreducible $M_{24}$-representation as in [4]. We will use Theorem 1.2 to prove the following result about the multiplicities of the $M_{24}$-representation $\lambda \oplus \bar{\lambda}$ with irreducible $\lambda$ inside the Mathieu moonshine module constituents $K_n$:

**Theorem 4.1.** Let $n = \ell m^2 \equiv 7 \pmod{8}$ and let $K_n$ be the degree $n$-part of the Mathieu moonshine module $K$. Then the following numbers are odd (and therefore positive):

- For $\ell = 7$, the multiplicity of $\chi_3 \oplus \chi_4$ in $K_n$ plus the multiplicity of $\chi_{12} \oplus \chi_{13}$ in $K_n$;
- for $\ell = 7$ and $m$ divisible by 3, the multiplicity of $\chi_{15} \oplus \chi_{16}$ in $K_n$;
- for $\ell = 15$, the multiplicity of $\chi_5 \oplus \chi_6$ in $K_n$;
- for $\ell = 23$, the multiplicity of $\chi_{10} \oplus \chi_{11}$ in $K_n$.

**Proof.** Let

\[ K_n = \bigoplus_{i=1}^{26} m_{\chi_i} \chi_i \]

be the decomposition of the $M_{24}$-representation $K_n$ into its irreducible constituents, i.e. $m_{\chi_i}$ is the multiplicity in which $\chi_i$ occurs in $K_n$. Taking on both sides of (3) the trace for an element $g \in M_{24}$ we have

\[ \text{Tr}(g|K_n) = \sum_{i=1}^{26} m_{\chi_i} \text{Tr}(g|\chi_i) \]

and $\text{Tr}(g|K_n)$ is the coefficient of $q^{n/8}$ in $\Sigma_{\ell X}(\tau)$ with $\ell X$ the conjugacy class of $g$. We note that if $K_n$ is non-zero then $n$ is odd, i.e. we only need to consider the cases $n = \ell m^2$ with $m$ odd.

First, we consider the cases $\ell = 15$ and $\ell = 23$ and let $g$ be an element of type $15A$ or $23A$, respectively. If $n$ is of the form $\ell m^2$ with $m$ odd, the left-hand side of (4) is odd by Theorem 1.2. For the right-hand side an inspection of the character table of $M_{24}$ [4] shows that if $\lambda \neq \bar{\lambda}$ one has that $\text{Tr}(g|\lambda) = \text{Tr}(g|\bar{\lambda})$ is integral unless $(\lambda, \bar{\lambda}) = (\chi_5, \chi_6)$ or $(\chi_{10}, \chi_{11})$, respectively, in which case $\text{Tr}(g|\lambda + \bar{\lambda}) = -1$. Using Theorem 1.1 it follows that $m_\lambda = m_{\bar{\lambda}}$ has then to be odd.

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For the cases $\ell = 7$, we let $g$ be an element of type $7A$. Here, only the characters $(\chi_3, \chi_4)$ or $(\chi_{12}, \chi_{13})$ can provide an odd contribution to the right-hand side of (4) and so in total an odd number of those pairs has to appear. For $m$ divisible by 3, we take in addition an element $g$ of type $21A$. Here $(\chi_3, \chi_4)$, $(\chi_{12}, \chi_{13})$ and $(\chi_{15}, \chi_{16})$ provide an odd contribution. Since the total number of pairs of type $(\chi_3, \chi_4)$ and $(\chi_{12}, \chi_{13})$ is odd and by Theorem 1.2 the left-hand side of (4) is even, it follows that the number of pairs $(\chi_{15}, \chi_{16})$ is odd, too. 

It is clear that Theorem 4.1 implies Conjecture 1.1.

The argument in the proof can also be applied to the elements $g$ of type $\ell X$ as studied in Section 3. One directly recovers Theorem 1.2 for those cases. However, Theorem 1.1 as stated is a refinement which uses parts of our original arguments in its proof.

We remark that it is likely that the methods in proving Theorem 1.2 and Theorem 1.1 could also be applied to the other cases of umbral moonshine.

A McKay-Thompson series

We list all McKay-Thompson series using the ATLAS [4] names for the conjugacy classes of $M_{24}$. We refer to Section 2 for the functions used.
1A: $\Sigma_{1A}(\tau) = \Sigma(\tau)$

2A: $\Sigma_{2A}(\tau) = \frac{1}{4} \left( \Sigma(\tau) \eta(\tau)^3 - 4 \phi_2^{(2)}(\tau) \right) / \eta(\tau)^3$

2B: $\Sigma_{2B}(\tau) = -2 \eta(\tau)^5 / \eta(2\tau)^4$

3A: $\Sigma_{3A}(\tau) = \frac{1}{4} \left( \Sigma(\tau) \eta(\tau)^3 - 6 \phi_2^{(3)}(\tau) \right) / \eta(\tau)^3$

3B: $\Sigma_{3B}(\tau) = -2 \eta(\tau)^3 / \eta(3\tau)^2$

4A: $\Sigma_{4A}(\tau) = -2 \eta(2\tau)^8 / (\eta(\tau)^3 \eta(4\tau)^4)$

4B: $\Sigma_{4B}(\tau) = \frac{1}{12} \left( \Sigma(\tau) \eta(\tau)^3 + 2 \phi_2^{(2)}(\tau) - 12 \phi_2^{(4)}(\tau) \right) / \eta(\tau)^3$

4C: $\Sigma_{4C}(\tau) = -2 \eta(\tau) \eta(2\tau)^2 / \eta(4\tau)^2$

5A: $\Sigma_{5A}(\tau) = \frac{1}{6} \left( \Sigma(\tau) \eta(\tau)^3 - 10 \phi_2^{(5)}(\tau) \right) / \eta(\tau)^3$

6A: $\Sigma_{6A}(\tau) = \frac{1}{12} \left( \Sigma(\tau) \eta(\tau)^3 + 6 \phi_2^{(2)}(\tau) + 6 \phi_2^{(3)}(\tau) - 30 \phi_2^{(6)}(\tau) \right) / \eta(\tau)^3$

6B: $\Sigma_{6B}(\tau) = -2 \eta(2\tau)^2 \eta(3\tau)^2 / (\eta(\tau) \eta(6\tau)^2)$

7AB: $\Sigma_{7AB}(\tau) = \frac{1}{8} \left( \Sigma(\tau) \eta(\tau)^3 - 14 \phi_2^{(7)}(\tau) \right) / \eta(\tau)^3$

8A: $\Sigma_{8A}(\tau) = \frac{1}{12} \left( \Sigma(\tau) \eta(\tau)^3 + 6 \phi_2^{(4)}(\tau) - 28 \phi_2^{(8)}(\tau) \right) / \eta(\tau)^3$

10A: $\Sigma_{10A}(\tau) = -2 \eta(2\tau) \eta(5\tau) / \eta(10\tau)$

11A: $\Sigma_{11A}(\tau) = \frac{1}{12} \left( \Sigma(\tau) \eta(\tau)^3 - 22 \phi_2^{(11)}(\tau) + \frac{264}{5} (\eta(\tau) \eta(11\tau))^2 \right) / \eta(\tau)^3$

12A: $\Sigma_{12A}(\tau) = -2 \eta(4\tau)^2 \eta(6\tau)^3 / (\eta(2\tau) \eta(3\tau) \eta(12\tau)^2)$

12B: $\Sigma_{12B}(\tau) = -2 \eta(\tau) \eta(4\tau) \eta(6\tau) / (\eta(2\tau) \eta(12\tau))$

14AB: $\Sigma_{14AB}(\tau) = \frac{1}{24} \left( \Sigma(\tau) \eta(\tau)^3 + \frac{3}{2} \phi_2^{(2)}(\tau) + 14 \phi_2^{(7)}(\tau) - \frac{182}{3} \phi_2^{(14)}(\tau) + 112 \eta(\tau) \eta(2\tau) \eta(7\tau) \eta(14\tau) \right) / \eta(\tau)^3$

15AB: $\Sigma_{15AB}(\tau) = \frac{1}{24} \left( \Sigma(\tau) \eta(\tau)^3 + \frac{3}{2} \phi_2^{(3)}(\tau) + 5 \phi_2^{(5)}(\tau) - \frac{105}{2} \phi_2^{(15)}(\tau) + 90 \eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau) \right) / \eta(\tau)^3$

21AB: $\Sigma_{21AB}(\tau) = -\frac{1}{3} \left( 7 \eta(7\tau)^3 / (\eta(3\tau) \eta(21\tau)) - \eta(\tau)^3 / \eta(3\tau)^2 \right)$

23AB: $\Sigma_{23AB}(\tau) = \frac{1}{24} \left( \Sigma(\tau) \eta(\tau)^3 - 46 \phi_2^{(23)}(\tau) + \frac{276}{11} f_{23,1}(\tau) + \frac{1922}{11} f_{23,2}(\tau) \right) / \eta(\tau)^3$
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