SIMPLE LOOPS ON 2-BRIDGE SPHERES IN 2-BRIDGE LINK COMPLEMENTS

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Abstract. The purpose of this note is to announce complete answers to the following questions. (1) For an essential simple loop on a 2-bridge sphere in a 2-bridge link complement, when is it null-homotopic in the link complement? (2) For two distinct essential simple loops on a 2-bridge sphere in a 2-bridge link complement, when are they homotopic in the link complement? We also announce applications of these results to character varieties and McShane’s identity.

1. Introduction

Let $K$ be a knot or a link in $S^3$ and $S$ a punctured sphere in the complement $S^3 - K$ obtained from a bridge sphere of $K$. Then the following natural question arises.

Question 1.1. (1) Which essential simple loops on $S$ are null-homotopic in $S^3 - K$?

(2) For two distinct essential simple loops on $S$, when are they homotopic in $S^3 - K$?

A refined version of the first question for 2-bridge spheres of 2-bridge links was proposed in the second author’s joint work with Ohtsuki and Riley [20, Question 9.1(2)], in relation with epimorphisms between 2-bridge links. It may be regarded as a special variation of a question raised by Minsky [9, Question 5.4] on essential simple loops on Heegaard surfaces of 3-manifolds.

The purpose of this note is to announce a complete answer to Question 1.1 for 2-bridge spheres of 2-bridge links established by the series of papers [11, 12, 13, 14] and to explain its application to the study of character varieties and McShane’s identity [15].
The key tool for solving the question is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups. We note that it has been proved by Weinbaum [32] and Appel and Schupp [5] that the word and conjugacy problems for prime alternating link groups are solvable, by using small cancellation theory (see also [10] and references in it). Moreover, it was shown by Sela [24] and Préaux [21] that the word and conjugacy problems for any link group are solvable. A characteristic feature of our work is that it gives complete answers to special (but also natural) word and conjugacy problems for the link groups of 2-bridge links, which form a special (but also important) family of prime alternating links. (See [1, 4] for the role of 2-bridge links in Kleinian group theory.)

This note is organized as follows. In Sections 2, 3 and 4, we describe the main results, applications to character varieties and McShane’s identity. The remaining sections are devoted to explanation of the idea of the proof of the main results. In Section 5, we describe the two-generator and one-relator presentation of the 2-bridge link group to which small cancellation theory is applied, and give a natural decomposition of the relator, which plays a key role in the proof. In Section 6, we introduce a certain finite sequence associated with the relator and state its key properties. In Section 7, we recall small cancellation theory and present a characterization of the “pieces” of the symmetrized subset arising from the relator. In Sections 8 and 9, we describe outlines of the proofs of the main results.

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2. Main results

For a rational number $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope $r$, which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope $\infty$ and $r$ (see Figure 1). The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the Conway sphere $(S^2, P) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where $H$ is the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the $\pi$-rotations around the points in the lattice $\mathbb{Z}^2$. Let $S$ be the 4-punctured sphere $S^2 - P$ in the link complement $S^3 - K(r)$. Any essential simple loop in $S$, up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbb{Q}}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto $S$. The (unoriented) essential simple loop in $S$ so obtained is denoted by $\alpha_s$. We
also denote by \(\alpha_s\) the conjugacy class of an element of \(\pi_1(S)\) represented by (a suitably oriented) \(\alpha_s\). Then the link group \(G(K(r)) := \pi_1(S^3 - K(r))\) is identified with \(\pi_1(S)/\langle\langle\alpha_\infty, \alpha_r\rangle\rangle\), where \(\langle\langle\cdot\rangle\rangle\) denotes the normal closure.

\[\begin{figure}
\includegraphics[width=0.5\textwidth]{fig1.png}
\end{figure}\]

**Figure 1.** \((S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))\) with \(r = 1/3\). Here \((B^3, t(r))\) and \((B^3, t(\infty))\), respectively, are the inside and the outside of the bridge sphere \(S^2\).

Let \(\mathcal{D}\) be the Farey tessellation, whose ideal vertex set is identified with \(\hat{\mathbb{Q}}\). For each \(r \in \hat{\mathbb{Q}}\), let \(\Gamma_r\) be the group of automorphisms of \(\mathcal{D}\) generated by reflections in the edges of \(\mathcal{D}\) with an endpoint \(r\), and let \(\hat{\Gamma}_r\) be the group generated by \(\Gamma_r\) and \(\Gamma_\infty\). Then the region, \(R\), bounded by a pair of Farey edges with an endpoint \(\infty\) and a pair of Farey edges with an endpoint \(r\) forms a fundamental domain of the action of \(\hat{\Gamma}_r\) on \(\mathbb{H}^2\) (see Figure 2). Let \(I_1\) and \(I_2\) be the closed intervals in \(\hat{\mathbb{R}}\) obtained as the intersection with \(\hat{\mathbb{R}}\) of the closure of \(R\). Suppose that \(r\) is a rational number with \(0 < r < 1\). (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write

\[
\begin{align*}
\frac{1}{m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_k}}} &=: [m_1, m_2, \ldots, m_k],
\end{align*}
\]
where \( k \geq 1, (m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k \), and \( m_k \geq 2 \). Then the above intervals are given by \( I_1 = [0, r_1] \) and \( I_2 = [r_2, 1] \), where

\[
\begin{align*}
r_1 &= \begin{cases} 
[m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is odd}, \\
[m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even},
\end{cases} \\
r_2 &= \begin{cases} 
[m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd}, \\
[m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is even}.
\end{cases}
\end{align*}
\]

![Figure 2. A fundamental domain of \( \hat{\Gamma}_r \) in the Farey tessellation (the shaded domain) for \( r = 5/17 = [3, 2, 2] \).](image)

We recall the following fact (\cite[Proposition 4.6 and Corollary 4.7]{20} and \cite[Lemma 7.1]{11}) which describes the role of \( \hat{\Gamma}_r \) in the study of 2-bridge link groups.

**Proposition 2.1.** (1) If two elements \( s \) and \( s' \) of \( \hat{\mathbb{Q}} \) belong to the same orbit \( \hat{\Gamma}_r \)-orbit, then the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(r) \).

(2) For any \( s \in \hat{\mathbb{Q}} \), there is a unique rational number \( s_0 \in I_1 \cup I_2 \cup \{\infty, r\} \) such that \( s \) is contained in the \( \hat{\Gamma}_r \)-orbit of \( s_0 \). In particular, \( \alpha_s \) is homotopic to \( \alpha_{s_0} \) in \( S^3 - K(r) \). Thus if \( s_0 \in \{\infty, r\} \), then \( \alpha_s \) is null-homotopic in \( S^3 - K(r) \).

Thus the following question naturally arises (see \cite[Question 9.1(2)]{20}).

**Question 2.2.** (1) Does the converse to Proposition 2.1(2) hold? Namely, is it true that \( \alpha_s \) is null-homotopic in \( S^3 - K(r) \) if and only if \( s \) belongs to the \( \hat{\Gamma}_r \)-orbit of \( \infty \) or \( r \) ?

(2) For two distinct rational numbers \( s, s' \in I_1 \cup I_2 \), when are the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) homotopic in \( S^3 - K(r) \) ?
The following theorem proved in [11] gives a complete answer to Question 2.2(1).

**Theorem 2.3.** The loop $\alpha_s$ is null-homotopic in $S^3 - K(r)$ if and only if $s$ belongs to the $\tilde{\Gamma}_r$-orbit of $\infty$ or $r$. In other words, if $s \in I_1 \cup I_2$, then $\alpha_s$ is not null-homotopic in $S^3 - K(r)$.

This has the following application to the study of epimorphisms between 2-bridge link groups (see [11], Section 2) for precise meaning.

**Corollary 2.4.** There is an upper-meridian-pair-preserving epimorphism from $G(K(s))$ to $G(K(r))$ if and only if $s$ or $s + 1$ belongs to the $\tilde{\Gamma}_r$-orbit of $r$ or $\infty$.

The following theorem proved in [12, 13, 14] gives a complete answer to Question 2.2(2).

**Theorem 2.5.** Suppose that $r$ is a rational number such that $0 < r \leq 1/2$. For distinct $s, s' \in I_1 \cup I_2$, the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if one of the following holds.

1. $r = 1/p$, where $p \geq 2$ is an integer, and $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$, where $(p_i, q_i)$ is a pair of relatively prime positive integers.
2. $r = 3/8$, namely $K(r)$ is the Whitehead link, and the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$.

The proof of Theorem 2.5 reveals the structure of the normalizer of an element of $G(K(r))$ represented by $\alpha_s$. This enables us to show the following.

**Theorem 2.6.** Let $r$ be a rational number such that $0 < r \leq 1/2$. Suppose $K(r)$ is hyperbolic, i.e., $r = q/p$ and $q \not\equiv \pm 1 \pmod{p}$, and let $s$ be a rational number contained in $I_1 \cup I_2$.

1. The loop $\alpha_s$ is peripheral if and only if one of the following holds.
   - (i) $r = 2/5$ and $s = 1/5$ or $s = 3/5$.
   - (ii) $r = n/(2n + 1)$ for some integer $n \geq 3$, and $s = (n + 1)/(2n + 1)$.
   - (iii) $r = 2/(2n + 1)$ for some integers $n \geq 3$, and $s = 1/(2n + 1)$.

2. The conjugacy class $\alpha_s$ is primitive in $G(K(r))$ with the following exceptions.
   - (i) $r = 2/5$ and $s = 2/7$ or $3/4$. In this case $\alpha_s$ is the third power of some primitive element in $G(K(r))$.
   - (ii) $r = 3/7$ and $s = 2/7$. In this case $\alpha_s$ is the second power of some primitive element in $G(K(r))$. 


(iii) \( r = 2/7 \) and \( s = 3/7 \). In this case \( \alpha_s \) is the second power of some primitive element in \( G(K(r)) \).

At the end of this section, we describe a relation of Theorem 2.3 with the question raised by Minsky in [9, Question 5.4]. Let \( M = H_+ \cup_S H_- \) be a Heegaard splitting of a 3-manifold \( M \). Let \( \Gamma_\pm := MCG(H_\pm) \) be the mapping class group of \( H_\pm \), and let \( \Gamma_+^0 \) be the kernel of the map \( MCG(H_\pm) \to \text{Out}(\pi_1(H_\pm)) \). Identify \( \Gamma_+^0 \) with a subgroup of \( MCG(S) \), and consider the subgroup \( \langle \Gamma_\pm^0, \Gamma_-^0 \rangle \) of \( MCG(S) \). Now let \( \Delta_\pm \) be the set of (isotopy classes of) simple loops in \( S \) which bound a disk in \( H_\pm \). Let \( Z \) be the set of essential simple loops in \( S \) which are null-homotopic in \( M \). Note that \( Z \) contains \( \Delta_\pm \) and invariant under \( \langle \Gamma_\pm^0, \Gamma_-^0 \rangle \). In particular, the orbit \( \langle \Gamma_\pm^0, \Gamma_-^0 \rangle(\Delta_+ \cup \Delta_-) \) is a subset of \( Z \). Then Minsky posed the following question.

**Question 2.7.** When is \( Z \) equal to the orbit \( \langle \Gamma_\pm^0, \Gamma_-^0 \rangle(\Delta_+ \cup \Delta_-) \)?

The above question makes sense not only for Heegaard splittings but also bridge decompositions of knots and links. In particular, for 2-bridge links, the groups \( \Gamma_\infty \) and \( \Gamma_r \) in our setting correspond to the groups \( \Gamma_+^0 \) and \( \Gamma_0^- \), and hence the group \( \hat{\Gamma}_r \) corresponds to the group \( \langle \Gamma_\pm^0, \Gamma_-^0 \rangle \). To make this precise, recall the bridge decomposition \( (S^3, K(r)) = (B_3^3, t(\infty)) \cup (B_3^3, t(r)) \), and let \( \hat{\Gamma}_+ \) (resp. \( \hat{\Gamma}_- \)) be the mapping class group of the pair \( (B_3^3, t(\infty)) \) (resp. \( (B_3^3, t(r)) \)), and let \( \hat{\Gamma}_+^0 \), \( \hat{\Gamma}_-^0 \) be the kernel of the natural map \( \hat{\Gamma}_+ \to \text{Out}(\pi_1(B_3^3 - t(\infty))) \) (resp. \( \hat{\Gamma}_- \to \text{Out}(\pi_1(B_3^3 - t(r))) \)). Identify \( \hat{\Gamma}_\pm^0 \) with a subgroup of the mapping class group \( MCG(S) \) of the 4-times punctured sphere \( S \). Recall that the Farey tessellation \( D \) is identified with the curve complex of \( S \) and there is a natural epimorphism from \( MCG(S) \) to the automorphism group \( \text{Aut}(D) \) of \( D \), whose kernel is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^2 \). Then the groups \( \Gamma_\infty \) and \( \Gamma_r \), respectively, are identified with the images of \( \Gamma_\infty^0 \) and \( \Gamma_-^0 \) by this epimorphism. Moreover, the sets \( \{\alpha_\infty\} \) and \( \{\alpha_r\} \), respectively, correspond to the sets \( \Delta_+ \) and \( \Delta_- \). Theorem 2.3 says that the set \( Z \) of simple loops in \( S \) which are null-homotopic in \( S^3 - K(r) \) is equal to the orbit \( \langle \Gamma_\infty, \Gamma_r \rangle(\Delta_+ \cup \Delta_-) \). Thus Theorem 2.3 may be regarded as an answer to the special variation of Question 2.7.

3. Application to character varieties

In this section and the next section, we assume \( r = q/p \), where \( p \) and \( q \) are relatively prime positive integers such that \( q \neq \pm 1 \pmod{p} \). This is equivalent to the condition that \( K(r) \) is hyperbolic, namely the link complement \( S^3 - K(r) \) admits a complete hyperbolic structure of finite volume. Let \( \rho_r \) be the
PSL(2, \mathbb{C})-representation of \( \pi_1(S) \) obtained as the composition

\[
\pi_1(S) \to \pi_1(S) / \langle \langle \alpha_\infty, \alpha_r \rangle \rangle \cong \pi_1(S^3 - K(r)) \to \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}),
\]

where the last homomorphism is the holonomy representation associated with the complete hyperbolic structure.

Now, let \( T \) be the once-punctured torus obtained as the quotient \( (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2 \), and let \( O \) be the orbifold \( (\mathbb{R}^2 - \mathbb{Z}^2)/\hat{H} \) where \( \hat{H} \) is the group generated by \( \pi \)-rotations around the points in \( (\mathbb{Z}/2\mathbb{Z})^2 \). Note that \( O \) is the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle \( \pi \). The surfaces \( T \) and \( S \), respectively, are \( \mathbb{Z}/2\mathbb{Z} \)-covering and \( (\mathbb{Z}/2\mathbb{Z})^2 \)-covering of \( O \), and hence their fundamental groups are identified with subgroups of the orbifold fundamental group \( \pi_1(O) \) of indices 2 and 4, respectively. The PSL(2, \mathbb{C})-representation \( \rho_r \) of \( \pi_1(S) \) extends, in a unique way, to that of \( \pi_1(O) \) (see [4, Proposition 2.2]), and so we obtain, in a unique way, a PSL(2, \mathbb{C})-representation of \( \pi_1(T) \) by restriction. We continue to denote it by \( \rho_r \). Note that \( \rho_r : \pi_1(T) \to \text{PSL}(2, \mathbb{C}) \) is type-preserving, i.e., it satisfies the following conditions.

1. \( \rho_r \) is irreducible, i.e., its image does not have a common fixed point on \( \partial \mathbb{H}^3 \).
2. \( \rho_r \) maps a peripheral element of \( \pi_1(T) \) to a parabolic transformation.

By extending the concept of a geometrically infinite end of a Kleinian group, Bowditch [7] introduced the notion of the end invariants of a type-preserving PSL(2, \mathbb{C})-representation of \( \pi_1(T) \). Tan, Wong and Zhang [30] (cf. [26]) extended this notion (with slight modification) to an arbitrary PSL(2, \mathbb{C})-representation of \( \pi_1(T) \). (To be precise, [30] treats SL(2, \mathbb{C})-representations. However, the arguments work for PSL(2, \mathbb{C})-representations.)

To recall the notion of end invariants, let \( \mathcal{C} \) be the set of free homotopy classes of essential simple loops on \( T \). Then \( \mathcal{C} \) is identified with \( \hat{\mathbb{Q}} \), the vertex set of the Farey tessellation \( \mathcal{D} \) by the following rule. For each \( s \in \hat{\mathbb{Q}} \), let \( \beta_s \) be the essential simple loop on \( T \) obtained as the image of a line of slope \( s \) in \( \mathbb{R}^2 - \mathbb{Z}^2 \). Then the correspondence \( s \mapsto \beta_s \) gives the desired identification \( \hat{\mathbb{Q}} \cong \mathcal{C} \). The projective lamination space \( \mathcal{PL} \) is then identified with \( \hat{\mathbb{R}} := \mathbb{R} \cup \{ \infty \} \) and contains \( \mathcal{C} \) as the dense subset of rational points.

**Definition 3.1.** Let \( \rho \) be a PSL(2, \mathbb{C})-representation of \( \pi_1(T) \).

1. An element \( X \in \mathcal{PL} \) is an end invariant of \( \rho \) if there exists a sequence of distinct elements \( X_n \in \mathcal{C} \) such that \( X_n \to X \) and such that \( \{ |\text{tr} \rho(X_n)| \} \) is bounded from above.
2. \( \mathcal{E}(\rho) \) denotes the set of end invariants of \( \rho \).
In the above definition, it should be noted that $|\text{tr}\rho(X_n)|$ is well-defined though $\text{tr}\rho(X_n)$ is defined only up to sign. Note also that the condition that $\{|\text{tr}\rho(X_n)|\}_n$ is bounded from above is equivalent to the condition that the hyperbolic translation lengths of the isometries $\rho(X_n)$ of $\mathbb{H}^3$ are bounded from above.

Tan, Wong and Zhang [26, 30] showed that $E(\rho)$ is a closed subset of $\mathcal{P}\mathcal{L}$ and proved various interesting properties of $E(\rho)$, including a characterization of those representations $\rho$ with $E(\rho) = \emptyset$ or $\mathcal{P}\mathcal{L}$, generalizing a result of Bowditch [7]. They also proposed an interesting conjecture [30, Conjecture 1.8] concerning possible homeomorphism types of $E(\rho)$. The following is a modified version of the conjecture of which Tan [25] informed the authors.

Conjecture 3.2. Suppose $E(\rho)$ has at least two accumulation points. Then either $E(\rho) = \mathcal{P}\mathcal{L}$ or a Cantor set of $\mathcal{P}\mathcal{L}$.

They constructed a family of representations $\rho$ which have Cantor sets as $E(\rho)$, and proved the following supporting evidence to the conjecture.

Theorem 3.3. Let $\rho : \pi_1(T) \to \text{SL}(2, \mathbb{C})$ be discrete in the sense that the set $\{\text{tr}(\rho(X)) \mid X \in \mathbb{C}\}$ is discrete in $\mathbb{C}$. Then if $E(\rho)$ has at least three elements, then $E(\rho)$ is either a Cantor set of $\mathcal{P}\mathcal{L}$ or all of $\mathcal{P}\mathcal{L}$.

The above theorem implies that the end invariants $E(\rho_r)$ of the representation $\rho_r$ induced by the holonomy representation of a hyperbolic 2-bridge link $K(r)$ is a Cantor set. But it does not give us the exact description of $E(\rho_r)$. By using the main results stated in Section 2, we can explicitly determine the end invariants $E(\rho_r)$. To state the theorem, recall that the limit set $\Lambda(\hat{\Gamma}_r)$ of the group $\hat{\Gamma}_r$ is the set of accumulation points in the closure of $\mathbb{H}^2$ of the $\hat{\Gamma}_r$-orbit of a point in $\mathbb{H}^2$.

Theorem 3.4. For a hyperbolic 2-bridge link $K(r)$, the set $E(\rho_r)$ is equal to the limit set $\Lambda(\hat{\Gamma}_r)$ of the group $\hat{\Gamma}_r$.

We would like to propose the following conjecture.

Conjecture 3.5. Let $\rho : \pi_1(T) \to \text{PSL}(2, \mathbb{C})$ be a type-preserving representation such that $E(\rho) = \Lambda(\hat{\Gamma}_r)$. Then $\rho$ is conjugate to the representation $\rho_r$.

4. Application to McShane’s identity

In his Ph.D. thesis [17], McShane proved the following surprising theorem.
Theorem 4.1. Let $\rho : \pi_1(T) \to \text{PSL}(2, \mathbb{R})$ be a type-preserving fuchsian representation. Then

$$2 \sum_{s \in \mathbb{Q}} \frac{1}{1 + e^{l_\rho(\beta_s)}} = \frac{1}{2}$$

In the above identity, $l_\rho(\beta_s)$ denotes the translation length of the orientation-preserving isometry $\rho(\beta_s)$ of the hyperbolic plane. This identity has been generalized to cusped hyperbolic surfaces by McShane himself [18], to hyperbolic surfaces with cusps and geodesic boundary by Mirzakhani [19], and to hyperbolic surfaces with cusps, geodesic boundary and conical singularities by Tan, Wong and Zhang [27]. A wonderful application to the Weil-Petersson volume of the moduli spaces of bordered hyperbolic surface was found by Mirzakhani [19]. Bowditch [7] (cf. [6]) showed that the identity in Theorem 4.1 is also valid for all quasifuchsian representations of $\pi_1(T)$, where $l_\rho(\beta_s)$ is regarded as the complex translation length of the orientation-preserving isometry $\rho(\beta_s)$ of the hyperbolic 3-space. Moreover, he gave a nice variation of the identity for hyperbolic once-punctured torus bundles, which describes the cusp shape in terms of the complex translation lengths of essential simple loops on the fiber torus [8]. Other 3-dimensional variations have been obtained by [2, 3, 26, 27, 28, 29, 30, 31].

As an application of the main results stated in Section 2, we can obtain yet another 3-dimensional variation of McShane’s identity, which describes the cusp shape of a hyperbolic 2-bridge link in terms of the complex translation lengths of essential simple loops on the bridge sphere. This proves a conjecture proposed by the first author in [23].

To describe the result, note that each cusp of the hyperbolic manifold $S^3 - K(r)$ carries a Euclidean structure, well-defined up to similarity, and hence it is identified with the quotient of $C$ (with the natural Euclidean metric) by the lattice $Z \oplus Z \lambda$, generated by the translations $[\zeta \mapsto \zeta + 1]$ and $[\zeta \mapsto \zeta + \lambda]$ corresponding to the meridian and (suitably chosen) longitude respectively. This $\lambda$ does not depend on the choice of the cusp, because when $K(r)$ is a two-component link there is an isometry of $S^3 - K(r)$ interchanging the two cusps. We call $\lambda$ the modulus of the cusp and denote it by $\lambda(K(r))$.

Theorem 4.2. For a hyperbolic 2-bridge link $K(r)$ with $r = q/p$, the following identity holds:

$$2 \sum_{s \in \text{int} I_1} \frac{1}{1 + e^{l_{\text{er}}(\beta_s)}} + 2 \sum_{s \in \text{int} I_2} \frac{1}{1 + e^{l_{\text{er}}(\beta_s)}} + \sum_{s \in \partial I_1 \cup \partial I_2} \frac{1}{1 + e^{l_{\text{er}}(\beta_s)}} = -1.$$
Further the modulus $\lambda(K(r))$ of the cusp torus of the cusped hyperbolic manifold $S^3 - K(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$
\lambda(K(r)) = \begin{cases} 
8 \sum_{s \in \text{int} I_1} \frac{1}{1 + e^{p \alpha_s}} + 4 \sum_{s \in \partial I_1} \frac{1}{1 + e^{p \alpha_s}} & \text{if } p \text{ is odd,} \\
4 \sum_{s \in \text{int} I_1} \frac{1}{1 + e^{p \alpha_s}} + 2 \sum_{s \in \partial I_1} \frac{1}{1 + e^{p \alpha_s}} & \text{if } p \text{ is even.}
\end{cases}
$$

The main results stated in Section 2 are used to establish the absolute convergence of the infinite series.

5. Presentations of 2-bridge link groups

In the remainder of this note, $p$ and $q$ denote relatively prime positive integers such that $1 \leq q \leq p$ and $r = q/p$. Theorems 2.3 and 2.5 are proved by applying the small cancellation theory to a two-generator and one-relator presentation of the link group $G(K(r))$. To recall the presentation, let $a$ and $b$, respectively, be the elements of $\pi_1(B^3 - t(\infty), x_0)$ represented by the oriented loops $\mu_1$ and $\mu_2$ based on $x_0$ as illustrated in Figure 3. Then $\pi_1(B^3 - t(\infty), x_0)$ is identified with the free group $F(a, b)$. Note that $\mu_i$ intersects the disk, $\delta_i$, in $B^3$ bounded by a component of $t(\infty)$ and the essential arc, $\gamma_i$, on $\partial(B^3, t(\infty)) = (S^2, P)$ of slope $1/0$, in Figure 3.

![Figure 3. \(\pi_1(B^3 - t(\infty), x_0) = F(a, b)\), where $a$ and $b$ are represented by $\mu_1$ and $\mu_2$, respectively.](image)

To obtain an element, $u_r$, of $F(a, b)$ represented by the simple loop $\alpha_r$ (with a suitable choice of an orientation and a path joining $\alpha_r$ to the base point $x_0$), note that the inverse image of $\gamma_1$ (resp. $\gamma_2$) in $\mathbb{R}^2 - \mathbb{Z}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges in Figure 4. Let
Figure 4. The line of slope $2/5$ gives $\hat{u}_{2/5} = bab^{-1}a^{-1}$ and $u_{2/5} = a\hat{u}_{2/5}b\hat{u}_{2/5}^{-1} = abab^{-1}aba^{-1}b^{-1}$. Since the inverse image of $\gamma_1$ (resp. $\gamma_2$) in $\mathbb{R}^2 - \mathbb{Z}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges, a positive intersection with a single arrowed (resp. double arrowed) edge corresponds to $a$ (resp. $b$).

$L(r)$ be the line in $\mathbb{R}^2$ of slope $r$ passing through the origin, and let $L^+(r)$ be the line in $\mathbb{R}^2 - \mathbb{Z}^2$ obtained by slightly modifying $L(r)$ near each of the lattice points in $L(r)$ so that $L^+(r)$ takes an upper circuitous route around it, as illustrated in Figure 4. Pick a base point $z_0$ from the intersection of $L^+(r)$ with the second quadrant, and consider the sub-line-segment of $L^+(r)$ bounded by $z_0$ and $z_4 := z_0 + (2p, 2q)$. Then the image of the sub-line-segment in $S$ is homotopic to the loop $\alpha_s$. Let $u_r$ be the word in $\{a, b\}$ obtained by reading the intersection of the line-segment with the vertical lattice lines (= the inverse images of $\gamma_1$ and $\gamma_2$) as in Figure 4. Then $u_r \in F(a, b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop $\alpha_r$, and we obtain the following two-generator one-relator presentation.

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle$$

$$\cong F(a, b)/\langle\langle u_r \rangle\rangle \cong \langle a, b \mid u_r \rangle.$$

To describe the explicit formula for $u_r$, set $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$ where $\lfloor x \rfloor$ is the greatest integer not exceeding $x$. Then we have the following (cf. [22, Proposition 1]). Let

$$\epsilon_i = (-1)^{\lfloor iq/p \rfloor},$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding $x$. 

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(1) If $p$ is odd, then
\[ u_{q/p} = a\hat{u}_{q/p}b^{(-1)^q}u_{q/p}, \]
where $\hat{u}_{q/p} = b^{q}a^{q_2} \ldots b^{q_{p-2}}a^{q_{p-1}}$.

(2) If $p$ is even, then
\[ u_{q/p} = a\hat{u}_{q/p}a^{q_1-1}u_{q/p}, \]
where $\hat{u}_{q/p} = b^{q_1}a^{q_2} \ldots a^{q_{p-2}}b^{q_{p-1}}$.

In the above formula, $\hat{u}_{q/p}$ is obtained from the open interval of $L(r)$ bounded by $(0,0)$ and $(p,q)$.

**Figure 5.** The decomposition of the relator $u_r = v_1v_2v_3v_4$

We now describe a natural decomposition of the word $u_r$, which plays a key role in the proof of the main results. Let $r_i = q_i/p_i$ ($i = 1, 2$) be the rational number introduced in Section 2. Then $(p,q) = (p_1 + p_2, q_1 + q_2)$ and the parallelogram in $\mathbb{R}^2$ spanned by $(0,0)$, $(p_1, q_1)$, $(p_2, q_2)$ and $(p,q)$ does not contain lattice points in its interior. Consider the infinite broken line, $L_b(r)$, obtained by joining the lattice points
\[ \ldots, (0,0), (p_2, q_2), (p,q), (p+p_2, q+q_2), (2p, 2q), \ldots \]
which is invariant by the translation $(x,y) \mapsto (x+p, y+q)$. Let $L_b^+(r)$ be the topological line obtained by slightly modifying $L_b(r)$ near each of the lattice points in $L_b(r)$ so that $L_b^+(r)$ takes an upper or lower circuitous route around it according as the lattice point is of the form $d(p,q)$ or $d(p,q) + (p_2,q_2)$ for some $d \in \mathbb{Z}$, as illustrated in Figure 5. We may assume the base points $z_0$ and $z_4$ in $L^+(r)$ also lie in $L_b^+(r)$. Then the sub-arcs of $L^+(r)$ and $L_b^+(r)$
bounded by \( z_0 \) and \( z_4 \) are homotopic in \( \mathbb{R}^2 - \mathbb{Z}^2 \) by a homotopy fixing the end points. Moreover, the word \( u_r \) is also obtained by reading the intersection of the sub-path of \( L^+_b(r) \) with the vertical lattice lines. Pick a point \( z_1 \in L^+_b(r) \) whose \( x \)-coordinate is \( p_2 + (\text{small positive number}) \), and set \( z_2 := z_0 + (p, q) \) and \( z_3 := z_1 + (p, q) \). Let \( L^+_{b,i}(r) \) be the sub-path of \( L^+_b(r) \) bounded by \( z_{i-1} \) and \( z_i \) \( (i = 1, 2, 3, 4) \), and consider the subword, \( v_i \), of \( u_r \) corresponding to \( L^+_{b,i}(r) \). Then we have the decomposition

\[
 u_r = v_1 v_2 v_3 v_4,
\]

where the lengths of the subwords \( v_i \) are given by \( |v_1| = |v_3| = p_2 + 1 \) and \( |v_2| = |v_4| = p_1 - 1 \). This decomposition plays a key role in the following section.

6. Sequences associated with the simple loop \( \alpha_r \)

We begin with the following observation.

(1) The word \( u_r \) is reduced, i.e., it does not contain \( xx^{-1} \) or \( x^{-1}x \) for any \( x \in \{a, b\} \). It is also cyclically reduced, i.e., all its cyclic permutations are reduced.

(2) The word \( u_r \) is alternating, i.e., \( a^{\pm 1} \) and \( b^{\pm 1} \) appear in \( u_r \) alternately, to be precise, neither \( a^{\pm 2} \) nor \( b^{\pm 2} \) appears in \( u_r \). It is also cyclically alternating, i.e., all its cyclic permutations are alternating.

This observation implies that the word \( u_r \) is determined by the \( S \)-sequence defined below and the initial letter (with exponent).

**Definition 6.1.** (1) Let \( w \) be a nonempty reduced word in \( \{a, b\} \). Decompose \( w \) into

\[
 w \equiv w_1 w_2 \cdots w_t,
\]

where, for each \( i = 1, \ldots, t-1 \), all letters in \( w_i \) have positive (resp. negative) exponents, and all letters in \( w_{i+1} \) have negative (resp. positive) exponents. (Here the symbol \( \equiv \) means that the two words are not only equal as elements of the free group but also visibly equal, i.e., equal without cancellation.) Then the sequence of positive integers \( S(w) := (|w_1|, |w_2|, \ldots, |w_t|) \) is called the \( S \)-sequence of \( v \).

(2) Let \((w)\) be a nonempty reduced cyclic word in \( \{a, b\} \) represented by a word \( w \). Decompose \((w)\) into

\[
 (w) \equiv (w_1 w_2 \cdots w_t),
\]

where all letters in \( w_i \) have positive (resp. negative) exponents, and all letters in \( w_{i+1} \) have negative (resp. positive) exponents (taking subindices modulo \( t \).
Then the cyclic sequence of positive integers $CS(w) := ([|w_1|, |w_2|, \ldots, |w_t|])$ is called the cyclic $S$-sequence of $(w)$. Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

In the above definition, by a cyclic word, we mean the set of all cyclic permutations of a cyclically reduced word. By $(v)$, we denote the cyclic word associated with a cyclically reduced word $v$.

Definition 6.2. For a rational number $r$ with $0 < r \leq 1$, let $u_r$ be the word in $\{a, b\}$ defined in Section 5. Then the symbol $S(r)$ (resp. $CS(r)$) denotes the $S$-sequence of $u_r$ (resp. cyclic $S$-sequence $CS(u_r)$ of $(u_r)$), which is called the $S$-sequence of slope $r$ (resp. the cyclic $S$-sequence of slope $r$).

We can easily observe the following.

$$S(r) = S(u_r) = (S(v_1), S(v_2), S(v_3), S(v_4)),$$

$$CS(r) = CS(u_r) = ([S(v_1), S(v_2), S(v_3), S(v_4)]),$$

where $u_r = v_1v_2v_3v_4$ is the natural decomposition of $u_r$ obtained at the end of the last section. It is also not difficult to observe $S(v_1) = S(v_3)$ and $S(v_2) = S(v_4)$. By setting $S_1 := S(v_1) = S(v_3)$ and $S_2 := S(v_2) = S(v_4)$, we have the following key propositions.

Proposition 6.3. The decomposition $S(r) = (S_1, S_2, S_1, S_2)$ satisfies the following.

1. Each $S_i$ is symmetric, i.e., the sequence obtained from $S_i$ by reversing the order is equal to $S_i$. (Here, $S_1$ is empty if $k = 1$.)
2. Each $S_i$ occurs only twice in the cyclic sequence $CS(r)$.
3. Set $m := \lfloor q/p \rfloor$. Then $S(r)$ consists of only $m$ and $m + 1$, and $S_1$ begins and ends with $m + 1$, whereas $S_2$ begins and ends with $m$.

Proposition 6.4. Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 6.3. For a rational number $s$ with $0 < s \leq 1$, suppose that the cyclic $S$-sequence $CS(s)$ contains both $S_1$ and $S_2$ as a subsequence. Then $s \notin I_1 \cup I_2$. 

7. Small cancellation conditions for 2-bridge link groups

A subset $R$ of the free group $F(a, b)$ is called symmetrized, if all elements of $R$ are cyclically reduced and, for each $w \in R$, all cyclic permutations of $w$ and $w^{-1}$ also belong to $R$.

Definition 7.1. Suppose that $R$ is a symmetrized subset of $F(a, b)$. A nonempty word $v$ is called a piece if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv vc_1 \equiv \ldots \equiv c_k w_2$.
and \( w_2 \equiv vc_2 \). Small cancellation conditions \( C(p) \) and \( T(q) \), where \( p \) and \( q \) are integers such that \( p \geq 2 \) and \( q \geq 3 \), are defined as follows (see [16]).

1. Condition \( C(p) \): If \( w \in R \) is a product of \( n \) pieces, then \( n \geq p \).
2. Condition \( T(q) \): For \( w_1, \ldots, w_n \in R \) with no successive elements \( w_i, w_{i+1} \) an inverse pair \((i \mod n)\), if \( n < q \), then at least one of the products \( w_1w_2, \ldots, w_{n-1}w_n, w_nw_1 \) is freely reduced without cancellation.

The following proposition enables us to apply small cancellation theory to the group presentation \( \langle a, b \mid w_r \rangle \) of \( G(K(r)) \).

**Proposition 7.2.** Let \( r = \frac{a}{b} \) be a rational number such that \( 0 < r < 1 \), and let \( R \) be the symmetrized subset of \( F(a, b) \) generated by the single relator \( u_r \) of the group presentation \( G(K(r)) = \langle a, b \mid w_r \rangle \). Then \( R \) satisfies \( C(4) \) and \( T(4) \).

This proposition follows from the following characterization of pieces, which in turn is proved by using Proposition [6.3].

**Proposition 7.3.** (1) A subword \( w \) of the cyclic word \( (u_r^{\pm 1}) \) is a piece if and only if \( S(w) \) does not contain \( S_1 \) as a subsequence and does not contain \( S_2 \) in its interior, i.e., \( S(w) \) does not contain a subsequence \((\ell_1, S_2, \ell_2)\) for some \( \ell_1, \ell_2 \in \mathbb{Z}_+ \).
2. For a subword \( w \) of the cyclic word \( (u_r^{\pm 1}) \), \( w \) is not a product of two pieces if and only if \( S(w) \) either contains \((S_1, S_2)\) as a proper initial subsequence or contains \((S_2, S_1)\) as a proper terminal subsequence.

8. **Outline of the proof of Theorem 2.3**

Let \( R \) be the symmetrized subset of \( F(a, b) \) generated by the single relator \( u_r \) of the group presentation \( G(K(r)) = \langle a, b \mid w_r \rangle \). Suppose on the contrary that \( \alpha_s \) is null-homotopic in \( S^3 - K(r) \), i.e., \( u_s = 1 \) in \( G(K(r)) \), for some \( s \in I_1 \cup I_2 \). Then there is a *van Kampen diagram* \( M \) over \( G(K(r)) = \langle a, b \mid R \rangle \) such that the boundary label is \( u_s \). Here \( M \) is a simply connected 2-dimensional complex embedded in \( \mathbb{R}^2 \), together with a function \( \phi \) assigning to each oriented edge \( e \) of \( M \), as a *label*, a reduced word \( \phi(e) \) in \( \{a, b\} \) such that the following hold.

1. If \( e \) is an oriented edge of \( M \) and \( e^{-1} \) is the oppositely oriented edge, then \( \phi(e^{-1}) = \phi(e)^{-1} \).
2. For any boundary cycle \( \delta \) of any face of \( M \), \( \phi(\delta) \) is a cyclically reduced word representing an element of \( R \). (If \( \alpha = e_1, \ldots, e_n \) is a path in \( M \), we define \( \phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_n) \).)

We may assume \( M \) is reduced, namely it satisfies the following condition: Let \( D_1 \) and \( D_2 \) be faces (not necessarily distinct) of \( M \) with an edge \( e \subseteq \partial D_1 \cap \partial D_2 \),
and let \( e\delta_1 \) and \( \delta_2 e^{-1} \) be boundary cycles of \( D_1 \) and \( D_2 \), respectively. Set \( \phi(\delta_1) = f_1 \) and \( \phi(\delta_2) = f_2 \). Then we have \( f_2 \neq f_1^{-1} \). Moreover, we may assume the following conditions:

1. \( d_M(v) \geq 3 \) for every vertex \( v \in M - \partial M \).
2. For every edge \( e \) of \( \partial M \), the label \( \phi(e) \) is a piece.
3. For a path \( e_1, \ldots, e_n \) in \( \partial M \) of length \( n \geq 2 \) such that the vertex \( e_i \cap e_{i+1} \) has degree 2 for \( i = 1, 2, \ldots, n - 1 \), \( \phi(e_1)\phi(e_2) \cdots \phi(e_n) \) cannot be expressed as a product of less than \( n \) pieces.

Since \( R \) satisfies the conditions \( C(4) \) and \( T(4) \) by Proposition 7.2, \( M \) is a \([4,4]\)-map, i.e.,

1. \( d_M(v) \geq 4 \) for every vertex \( v \in M - \partial M \);
2. \( d_M(D) \geq 4 \) for every face \( D \in M \).

Here, \( d_M(v) \) is the degree of \( v \), denotes the number of oriented edges in \( M \) having \( v \) as initial vertex, and \( d_M(D) \) is the degree of \( D \), denotes the number of oriented edges in a boundary cycle of \( D \).

Now, for simplicity, assume that \( M \) is homeomorphic to a disk. (In general, we need to consider an extremal disk of \( M \).) Then by the Curvature Formula of Lyndon and Schupp (see [16, Corollary V.3.4]), we have

\[
\sum_{v \in \partial M} (3 - d_M(v)) \geq 4.
\]

By using this formula, we see that there are three edges \( e_1, e_2 \) and \( e_3 \) in \( \partial M \) such that \( e_1 \cap e_2 = \{v_1\} \) and \( e_2 \cap e_3 = \{v_2\} \), where \( d_M(v_i) = 2 \) for each \( i = 1, 2 \). Since \( \phi(e_1)\phi(e_2)\phi(e_3) \) is not expressed as a product of two pieces, we see by Proposition 7.3 that the boundary label of \( M \) contains a subword, \( w \), with \( S(w) = (S_1, S_2, \ell) \) or \( (\ell, S_2, S_1) \). This in turn implies that the \( S \)-sequence of the boundary label contains both \( S_1 \) and \( S_2 \) as subsequences. Hence, by Proposition 6.4, we have \( s \notin I_1 \cup I_2 \), a contradiction.

9. Outline of the Proof of Theorem 2.5

Suppose, for two distinct \( s, s' \in I_1 \cup I_2 \), the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(r) \). Then there is a reduced annular \( R \)-diagram \( M \) such that \( u_s \) is an outer boundary label and \( u_{s'}^{\pm 1} \) is an inner boundary label of \( M \). Again we can see that \( M \) is a \([4,4]\)-map and hence we have the following curvature formula:

\[
\sum_{v \in \partial M} (3 - d_M(v)) \geq 0.
\]
By using this formula, we obtain the following very strong structure theorem for $M$, which plays key roles throughout the series of papers \cite{12, 13, 14}.

**Theorem 9.1.** Figure 6(a) illustrates the only possible type of the outer boundary layer of $M$, while Figure 6(b) illustrates the only possible type of whole $M$. (The number of faces per layer and the number of layers are variable.)

In the above theorem, the outer boundary layer of the annular map $M$ is the submap of $M$ consisting of all faces $D$ such that the intersection of $\partial D$ with the outer boundary of $M$ contains an edge, together with the edges and vertices contained in $\partial D$.

![Figure 6](image)

**Figure 6.**

The first paper \cite{12} of the series devoted to proof of Theorem 2.5 treats the case when the 2-bridge link is a $(2, p)$-torus link, the second paper \cite{13} treats the case of 2-bridge links of slope $n/(2n+1)$ and $(n+1)/(3n+2)$, where $n \geq 2$ is an arbitrary integer, and the third paper \cite{14} treats the general case. The two families treated in the second paper play special roles in the project in the sense that the treatment of these links form a base step of an inductive proof of the theorem for general 2-bridge links. We note that the figure-eight knot is both a 2-bridge link of slope $n/(2n+1)$ with $n = 2$ and a 2-bridge link of slope $(n+1)/(3n+2)$ with $n = 1$. Surprisingly, the treatment of the figure-eight knot, the simplest hyperbolic knot, is the most complicated. This reminds us of the phenomenon in the theory of exceptional Dehn filling that the figure-eight knot attains the maximal number of exceptional Dehn fillings.
References

[1] C. Adams, Hyperbolic 3-manifolds with two generators, Comm. Anal. Geom. 4 (1996), 181–206.
[2] H. Akiyoshi, H. Miyachi and M. Sakuma, A refinement of McShane’s identity for quasi-fuchsian punctured torus groups, In the Tradition of Ahlfors and Bers, III, Contemporary Math. 355 (2004), 21–40.
[3] H. Akiyoshi, H. Miyachi and M. Sakuma, Variations of McShane’s identity for punctured surface groups, Proceedings of the Workshop “Spaces of Kleinian groups and hyperbolic 3-manifolds”, London Math. Soc., Lecture Note Series 329 (2006), 151–185.
[4] H. Akiyoahi, M. Sakuma, M. Wada, and Y. Yamashita, Punctured torus groups and 2-bridge knot groups (I), Lecture Notes in Mathematics 1909, Springer, Berlin, 2007.
[5] K. I. Appel and P. E. Schupp, The conjugacy problem for the group of any tame alternating knot is solvable, Proc. Amer. Math. Soc. 33 (1972), 329–336.
[6] B. H. Bowditch, A proof of McShane’s identity via Markoff triples, Bull. London Math. Soc. 28 (1996), 73–78.
[7] B. H. Bowditch, Markoff triples and quasi-fuchsian groups, Proc. London Math. Soc. 77 (1998), 697–736.
[8] B. H. Bowditch, A variation of McShane’s identity for once-punctured torus bundles, Topology 36 (1997), 325–334.
[9] C. Gordon, Problems, Workshop on Heegaard Splittings, 401–411, Geom. Topol. Monogr. 12, Geom. Topol. Publ., Coventry, 2007.
[10] K. Johnsgard, The conjugacy problem for the groups of alternating prime tame links is polynomial-time, Trans. Amer. Math. Soc. 349 (1977), 857–901.
[11] D. Lee and M. Sakuma, Epimorphisms between 2-bridge link groups: Homotopically trivial simple loops on 2-bridge spheres, Proc. London Math. Soc., to appear, arXiv:1004.2571
[12] D. Lee and M. Sakuma, Homotopically equivalent simple loops on 2-bridge spheres in 2-bridge link complements (I), arXiv:1010.2232
[13] D. Lee and M. Sakuma, Homotopically equivalent simple loops on 2-bridge spheres in 2-bridge link complements (II), arXiv:1103.0850
[14] D. Lee and M. Sakuma, Homotopically equivalent simple loops on 2-bridge spheres in 2-bridge link complements (III), preliminary notes.
[15] D. Lee and M. Sakuma, A variation of McShane’s identity for 2-bridge links, in preparation.
[16] R. C. Lyndon and P. E. Schupp, Combinatorial group theory, Springer-Verlag, Berlin, 1977.
[17] G. McShane, A remarkable identity for lengths of curves, Ph.D. Thesis, University of Warwick, 1991.
[18] G. McShane, Simple geodesics and a series constant over Teichmüller space, Invent. Math. 132 (1998), 607–632.
[19] M. Mirzakhani, Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces, Invent. Math. 167 (2007), 179–222.
[20] T. Ohtsuki, R. Riley, and M. Sakuma, Epimorphisms between 2-bridge link groups, Geom. Topol. Monogr. 14 (2008), 417–450.
[21] J. P. Préaux, *Conjugacy problems in groups of oriented geometrizable 3-manifolds*, Topology 45 (2006), 171–208.

[22] R. Riley, *Parabolic representations of knot groups. I*, Proc. London Math. Soc. 24 (1972), 217–242.

[23] M. Sakuma, *Variations of McShane’s identity for the Riley slice and 2-bridge links*, In “Hyperbolic Spaces and Related Topics”, R.I.M.S. Kokyuroku 1104 (1999), 103–108.

[24] Z. Sela, *The conjugacy problem for knot groups*, Topology 32 (1993), 363–369.

[25] S. P. Tan, *Private communication*, May, 2011.

[26] S. P. Tan, Y. L. Wong, and Y. Zhang, *SL(2, C) character variety of a one-holed torus*, Electron. Res. Announc. Amer. Math. Soc. 11 (2005), 103–110.

[27] S. P. Tan, Y. L. Wong, and Y. Zhang, *Generalizations of McShane’s identity to hyperbolic cone-surfaces*, J. Differential Geom. 72 (2006), 73–112.

[28] S. P. Tan, Y. L. Wong, and Y. Zhang, *Necessary and sufficient conditions for McShane’s identity and variations*, Geom. Dedicata 119 (2006), 119–217.

[29] S. P. Tan, Y. L. Wong, and Y. Zhang, *Generalized Markoff maps and McShane’s identity*, Adv. Math. 217 (2008), 761–813.

[30] S. P. Tan, Y. L. Wong, and Y. Zhang, *End invariants for SL(2, C) characters of the one-holed torus*, Amer. J. Math. 130 (2008), 385–412.

[31] S. P. Tan, Y. L. Wong, and Y. Zhang, *McShane’s identity for classical Schottky groups*, Pacific J. Math. 237 (2008), 183–200.

[32] C. M. Weinbaum, *The word and conjugacy problems for the knot group of any tame, prime, alternating knot*, Proc. Amer. Math. Soc. 30 (1971), 22–26.

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