Repeated Matching and Learning for Task Allocation

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Expert based online labor platforms (Upwork) need to facilitate the matching of skilled workers and clients. On these platforms the clients do not know the productivities of the workers and the workers do not know the best-suited clients, thus there is adverse selection on both sides. These platforms should facilitate effective learning on both the sides to achieve a desirable matching. The platform faces the following challenges- i) the clients only observe the outputs produced by the workers, which depend on both the workers’ productivities and the efforts exerted, ii) workers are self-interested- the efforts exerted depend on costs privately known to the workers, thus there is moral hazard. We propose an implementation based mechanism design framework for such platforms, which prescribes matching and payment rules. The proposed rules induce a repeated endogenous matching game between the clients and the workers. We arrive at an equilibrium for the game, which we show is unique and mitigates both moral hazard and adverse selection effectively (in wide-range of settings). We propose a new notion of stability- “long-run stability”, which extends the standard notions to environments with incomplete information and learning, and show that the matching in the equilibrium is long-run stable.

Key words: Matching, Mechanism Design

Subject classifications: Games- Non-cooperative, Repeated, Stochastic

Area of review: Games, Information and Networks

1. Introduction

Background. The beginning of online labor markets saw the emergence of crowdsourcing systems for instance Mechanical Turk, which is meant for the completion of micro-tasks at relatively low prices by non-professional labor. However, in recent years expert based online labor markets (for instance Upwork previously oDesk) have grown into a popular medium to match skilled workers (freelancers) to the clients. A client looking to hire a worker on such platforms does not have enough information about the workers and thus faces adverse selection. In some cases, it is possible that the platforms request some background information (for instance resume, feedback from the past clients), but this may not be sufficient because either the workers may have incomplete profiles or may have no significant past experience at all (See Tran-Thanh et al. (2014)). Once the workers are hired they can choose the amount of effort to exert that serves
in their self-interest but if the workers choose low effort levels, then this is not desirable for the clients. Therefore, the workers face the problem of moral hazard. The workers may not have sufficient experience with the tasks in the past and thus they may not know their own productivities on these tasks and hence need to learn them. If the workers and clients are not matched correctly, i.e. the workers not prefer the clients or vice-versa, then it can lead to low overall revenue and instability as workers and clients will desire to search and switch to alternate partners. The existing platforms such as Upwork only provide the background information and facilitate some sample tests, which is not sufficient to guarantee efficient matching, where the efficiency is measured in terms of overall revenue generated subject to the stability of the matching. In this work we seek to answer the following- what should the existing platforms do to improve the matches, i.e. address both moral hazard and adverse selection and thereby improve the overall revenue generated and also achieve stable matchings?

**Setup and challenges.** We approach the above problem from the perspective of implementation theory in mechanism design. The platform designer aims to set the rules for the interaction of the clients with the workers over different time slots in such a way that they enable effective learning and thus lead to better matches. The clients offer tasks that differ in quality, where high-quality tasks generate higher revenues. The productivity of a worker when executing a task depends on both the worker and the task that it performs. The productivity of a worker for a given task measures the speed with which a worker can complete one unit of the task. High productivity workers generate higher revenues. In addition, workers are self-interested: they decide the effort level which they will exert on the task. The effort level is measured as the time spent on the task. The objective is to construct a mechanism that repeatedly matches workers with tasks and prescribes the clients how to pay the workers based on their performance (i.e. output). The mechanism aims to maximize the long-run performance (e.g. the long-run total revenue) while considering the following desiderata:

1. **Incomplete information**- The productivity of the worker for a given task and the cost for exerting the effort is not known to the client and may also not be known to the worker, thus there is a *two-sided adverse selection* [Gale 2001].
2. **Incentives for clients and workers**- The workers decide the effort to exert (*moral hazard*) on a task, which cannot be observed by the clients and seek to optimize their long-term utilities, which is the net of the payment received upon the execution of the tasks and the cost for the effort exerted. We assume that the mechanism can propose to the clients the payment rules for their respective tasks. The proposed payment rules thus need to take into account the incentives of both the clients and the workers.

3. **Stability of matching**- When the set of clients and workers are matched it is desirable that their match persists for a long duration, thus it is desirable that the matches be stable. The environment considered here is quite different from conventional matching setups as it involves both the incomplete information and learning (in the presence of moral hazard). Therefore, an appropriate extension of the definition of stability is needed.

4. **Privacy**- It is desirable for the workers that their output histories and feedbacks are not revealed to public (other workers) and are limited to the concerned clients.

**Mechanism overview and key contributions.**

The main objective of the work is to design a mechanism that maximizes the total revenue generated on the platform while addressing the above challenges. The design of such a mechanism will provide insights that are useful for the improvement of the existing platforms. Next, we describe the main features of the mechanism and our key contributions. The workers and the clients have the option to participate or not in the mechanism. The payment rule proposed by the platform, which clients use to pay the workers, has two properties- the payment per unit output is strictly convex and increasing in the output produced, i.e. the marginal benefit of an extra unit of output increases in the output value, and it is linearly increasing in the quality of the task offered by the client. If the clients choose to participate in the mechanism, then they need to use the proposed payment rule. (We also discuss the extension when the payment rules are set by the clients themselves in Section 3.2.) The matching rule operates in two stages. In the first stage, which we refer to as the evaluation stage, the matching rule is aimed to evaluate the workers over the different tasks (each client has one task and we will use the words clients and tasks interchangeably based on the
context henceforth). In this stage, the workers also evaluate the different tasks as they estimate their own productivities for these tasks. At the end of this stage, the platform requests the workers to submit preference lists for the tasks. The platform ranks the workers based on the output they produced (number of the units of the task completed per unit time) for each client and this serves as the preference list for that client. The aim of the second stage, which we refer to as the operational stage, is to use the preferences from the first stage and combine them to arrive at the final matches. The platform uses these preference lists and executes the Gale-Shapley (G-S) algorithm (Gale and Shapley 1962) with the workers as the suitors to output the final matching, which remains fixed for the rest of the duration of the mechanism.

Before describing the properties of the proposed mechanism and the optimal strategies for the workers, we discuss the insights provided by the structure of the above mechanism for the existing platforms such as Upwork. These platforms do not have a two-stage matching, i.e. evaluation stage followed by the operational stage. The proposed mechanism suggests that these platforms have a short evaluation stage followed by an operational stage (repeating the two stages periodically to adjust for changing populations of workers and clients). In the proposed mechanism it is required that the clients follow the payment and matching rules provided they choose to participate, while on the existing platforms the clients and the workers can decide whether to follow the recommended the payment and the matching rules. In the results we will show that following the recommendations made by the proposed mechanism is useful as it will lead to stable matches and the inefficiency caused by moral hazard and adverse selection is provably mitigated.

The proposed mechanism induces a repeated endogenous matching game. In the proposed mechanism the strategy for the workers comprises of- decision on the amount of effort to exert in the evaluation stage, the preference list to be submitted, the effort to exert once the worker is hired in the operational stage. We arrive at the optimal strategy for the workers (the joint optimal strategy is referred to as the equilibrium strategy). We derive an equilibrium strategy in which every worker exerts maximum effort on all the tasks in the evaluation stage and then submits a ranking for the clients based on its estimate for the long-term utility that it expects to derive when matched to a client in the operational stage. In the operational stage if the worker is assigned to a client that is sufficiently high on its preference list, then the worker continues
to exert maximum effort, else the worker chooses zero effort for all the future slots. For a wide range of settings, we show that the repeated endogenous matching game has a unique equilibrium payoff, which is achieved by the equilibrium strategy discussed above.

In the equilibrium described above, we prove that for a wide-range of settings, moral hazard and adverse selection is effectively mitigated. Specifically, we show that the total long-run revenue on the platform is close to the maximum total long-run revenue generated when the clients and workers are not self-interested (no moral hazard) and there is complete information (no adverse selection). In the existing platforms, there is no guarantee of stability, i.e. there is no guarantee that at any point the clients and workers do not need to search for alternate partners, which makes it likely that the clients and workers repeatedly switch. We propose an extension of the standard notion of stability [Gale and Shapley (1962)] to our setting of matching with incomplete information and learning (in the presence of moral hazard), which we refer to as long-run stability. We prove that the matching achieved by the equilibrium strategy computed for the proposed mechanism is long-run stable (in a wide-range of settings). Thus in a real platform if the matching computed by the proposed mechanism is recommended to the clients and workers and if it is indeed followed, then it will be long-run stable and clients and workers will not need to switch and find alternate partners.

**Prior Work.** There are several ways to categorize works in the area of matching- matching with or without transfers, matching with complete or incomplete information (with or without learning), matching with self-interested or non-self-interested workers, matching in the presence/absence of moral hazard and adverse selection. We do not describe the works in these categories separately, instead we first broadly position our work with respect to the existing works and then describe the works that are closest to us. However, in Table 1 in the Appendix we give a summary of representative works in different categories.

In many real matching setups, the presence of incomplete information is natural, for instance in labor markets and marriage markets the two sides that are matched do not know each other’s types. However, in these markets when the entities on the two-sides are matched to interact (worker producing output for the clients in labor markets, interaction during dating in marriage markets), they use the observations made in
the interaction to learn about each other. The observations made often depend both on the types and the actions (effort in the worker-client setting) taken strategically during the interaction, which makes learning the types separately non-trivial. The interaction of such a learning process (obscured by actions) and its impact on the matching has not been studied in the existing works. More specifically from Table 1, we can see that this is the first work that incorporates different dimensions—matching of self-interested workers and clients (involving transfers) in the presence of incomplete information with learning. Next, we describe the works that are closest to this work.

Our previous works [Xiao et al. (2016a), Xiao et al. (2016b)] have studied matching settings where both the costly effort (moral hazard) and unknown types (adverse selection) play an important role. In [Xiao et al. (2016a)] the workers are assumed to be bounded-rational as they optimize a proxy version of their utility as defined by the conjecture function, while in this work the workers are rational, foresighted and maximize their long-run utilities. In [Xiao et al. (2016a), Xiao et al. (2016b)] the model assumes that the workers know their own qualities perfectly thus there is no learning at the level of workers unlike in the current work. The model proposed in [Xiao et al. (2016a), Xiao et al. (2016b)] only applies to environments where all the tasks are of the same type and thus the productivity of the worker does not vary across the tasks. In comparison, the model in this current work is more general and applies to general matching environments where the tasks can be heterogeneous and is more practical considering the diversity of the tasks that are posted on platforms such as Upwork. In [Xiao et al. (2016a), Xiao et al. (2016b)] the equilibrium matching need not necessarily be efficient—no provable guarantees with regard to mitigation of moral hazard and adverse selection are given. In [Xiao et al. (2016a), Xiao et al. (2016b)] the output produced by the workers is assumed to be a deterministic function of their productivity and effort. However, in real settings the output can be impacted by other unknown factors and we address this issue by extending the definition of output to be a stochastic function of the productivity and the effort.

The literature on matching has two main notions of stability. In [Gale and Shapley (1962)] the first notion of stability was proposed, where it was assumed that all the agents on both the sides know their preferences, i.e. there is complete information. In [Liu et al. (2014) and Bikhchandani (2014)] the notions of stability were
extended to environments with incomplete information. In this work, we propose a notion of stability which applies to environments with incomplete information and learning (in the presence of moral hazard). In the proposed notion we check that in the final matching that is achieved post learning is there any client worker pair that can gain in the long-run by being matched to one another instead of their current partners.

2. Repeated matching mechanism design in a non-stochastic setting

We will first present our main model and problem formulation. There can be several variations of the model where some of the assumptions presented here can be relaxed, we discuss these variations in Section 3. We will use $A$ to represent matrix, $A(i,j)$ for an element of the matrix, $a$ to represent a vector, $a(i)$ for the $i^{th}$ element of the vector, $A$ to represent a set, $a/A$ to represent scalar values.

2.1. Model and Problem Formulation

There is one platform designer, $N$ clients and $N$ workers who desire to be matched. We consider a discrete time infinite horizon model and we write each discrete time slot as $t \in \mathbb{Z}_+$. Each client has one task that it wants to be executed repeatedly in each time slot. In each time slot the clients and workers are matched depending on the matching mechanism explained later. We will assume that in each time slot one worker can be matched to at most one client and vice-versa.

Quality distribution of the tasks. We assume that each client is rational. We define the set of tasks as $S = \{1, \ldots, N\}$. Each task has an associated quality level that represents the revenue generated per unit of the task completed. $g : S \rightarrow [g_{\text{min}}, g_{\text{max}}]$ maps each task to its quality level of the task, where $g_{\text{max}} \geq g_{\text{min}} > 0$. We assume that $g$ is a strictly increasing function without loss of generality. We assume that the qualities of the tasks are known to all (the clients, the workers, and the designer).

Productivity distribution of the workers. We assume that each worker is rational. We define the set of $N$ workers as $\mathcal{N} = \{1, \ldots, N\}$. Each worker $i$’s productivity is a measure of the skill level and it depends on the type of the task that it performs. A higher productivity in performing a task implies that the worker can complete more units of task per unit time, i.e. it has a higher speed. $F : \mathcal{N} \times S \rightarrow [0, f_{\text{max}}]$ is a mapping from every combination of worker and task to a productivity level. We assume that no two workers have the
same productivity for a particular task $x$, i.e. $F(i, x) = F(k, x) \Rightarrow i = k$. We assume that the productivity of the worker in performing a task is not known to anyone (both the client and the worker itself) thus, there is two-sided adverse selection. This is a realistic assumption because in many settings clients do not have much information about the workers and the workers do not know how much output they can produce on a task as they may have never done it before (In Upwork 96% of the workers have no significant past experience, Tran-Thanh et al. (2014)).

**Efforts and outputs of the workers.** Each worker decides (strategically) how much effort $e_i$ to exert, i.e. time invested in working, on the task $x$ it is matched with during a particular time slot. We assume that $e_i \in \mathcal{E} = \{0, \delta, 2\delta, \ldots, e_i^{max}\}$. Therefore, the output- the total number of units of the task $x$ completed is $F(i, x)e_i$ (speed of executing the task times the time spent working on it). The effort exerted by a worker is only known privately to the worker and not observed by anyone else. The revenue that is generated is $[F(i, x)e_i]g(x)$ (number of units of task completed times the revenue per unit of the task). We assume that the revenue generated is observed by all. We define a cost function $C : \mathcal{S} \times \mathcal{N} \to [0, \infty)$. It costs worker $i$ $C(i, x)e_i^2$ to exert effort $e_i$ on task $x$, where $C(i, x) \in [0, e_i^{max}]$, $\forall i \in \mathcal{N}, \forall x \in \mathcal{S}$. The assumption that the cost function is quadratic is made here for simplifying the presentation, while the model and results extend to more general cost functions. We will assume that the worker $i$ knows the cost for exerting the effort $C(i, x)$ for all the tasks $x \in \mathcal{S}$ and no one else knows it. For the convenience of notation and ease of exposition we make the assumption that the worker does not know its productivity but it knows the costs, while in reality the mechanism can be extended to the setting where both the productivities and costs are not known to the worker.

Next, we will define the two main components that are part of the mechanism design problem- matching rule and a payment rule. The matching and the payment rule are known to all. The public knowledge of payment rule and matching rule is perfectly reasonable (typically in practical settings workers know how they will be paid upon working on a task and also how they will be rewarded to do a better or a worse task based on their performance).

**General matching rules.** We first define a general matching rule here later, we will describe the proposed matching rule that is chosen by the designer as a part of the mechanism. If the workers and clients decide to
participate in the mechanism, then they are required to follow the matching rule. We first define the vector of observations made by the designer up to time $t - 1$ (end of time slot $t - 1$) as $h^t_0$, where this vector depends on the mechanism. For instance, the vector of observation may comprise of the outputs produced by the workers and the actions taken by the clients (for e.g., payments made etc.) and the preference list of clients reported by the workers and vice-versa. The observation vector $h^t_0$ for our proposed mechanism is described later. We assign $h^0_0 = \phi$. We denote the set of all the possible histories up to time $t$ as $\mathcal{H}^t_0$ and the set of all the possible histories as $\mathcal{H}_0 = \bigcup_{t=0}^{\infty} \mathcal{H}^t_0$. The matching rule is given as $m : \mathcal{H}_0 \to \Pi(S)$, where $\Pi(S)$ is the set of all possible permutations of $S$. The matching rule maps each history of observations $h^t_0$ to a vector of tasks. $m(h^t_0)[i]$ denotes the $i^{th}$ element of the vector $m(h^t_0)$ and corresponds to the task assigned to worker $i$ following history $h^t_0$.

**General payment rules.** We first describe a general payment rule here, later we will describe the proposed payment rule that is chosen by the designer as a part of the mechanism. If the clients choose to participate in the mechanism, then they need to follow the payment rules. We discuss the extension to the setting where the payment rules are set by the clients themselves in Section 3.2. In each time step the client needs to pay the workers based on the output produced in that time period. The payment rule is given as $p : [0, \infty) \times S \to \mathbb{R}$. $p(w, x)$ is the payment made by the client $x$ to the worker when it produces output $w$ while working on task $x$.

We write the matching rule and the payment rule jointly as $\Omega = (m, p)$. The platform designer’s task is to set $\Omega$ such that it is aligned with a desirable objective (for instance, the total long-run output produced) taking the selfish behavior of the clients and workers into account. Next, we describe the different components of the strategic interaction between clients and the workers.

**Strategy of the workers and clients.** Each worker and client first need to decide whether or not to participate in the mechanism. Next, we define a general strategy of a worker (if it chooses to participate in the mechanism), later we will propose a specific strategy for every worker which will comprise an equilibrium. Each worker only observes its own output and does not observe the outputs of others. Moreover, the workers know the effort they exert while the designer cannot observe the effort of the workers. Hence, we
need to separately define the history of observations for each worker. We write the vector of observations of a worker \( i \) up to time \( t \) as \( \mathbf{h}_i^t \), where this vector depends on the type of the mechanism. We write \( \mathcal{H}_i^t \) to denote the set of all possible observation histories of worker \( i \) up to time \( t \). The set of all the possible observation histories up to \( t = \infty \) is \( \mathcal{H}_i = \bigcup_{t=0}^{\infty} \mathcal{H}_i^t \). We define the strategy of worker \( i \) that maps the history of observations of the worker to effort level as \( \pi_i : \mathcal{H}_i \rightarrow \mathcal{A}_i \), where \( \mathcal{A}_i \) is the set of actions that a worker takes. The set of actions of a worker depend on the mechanism, for instance the mechanism can operate only based on the outputs that the workers produce in which case the action of the workers is just the effort levels or the mechanism can also ask the workers to report their preferences over the tasks in which case the action of the worker is both the effort and the preference list over the tasks.

**The stage game.** In time slot \( t \) worker \( i \) is matched to play a stage game with client \( x = m(h_0^t)[i] \) following an observation history \( h_0^t \), which we describe as follows. The worker decides to exert \( e_i^t \) effort following a private history \( h_i^t (\pi_i(h_i^t)[1] = e_i^t \), where \( \pi_i(h_i^t)[1] \) is the first component of the action vector). We write the output and the revenue generated by worker \( i \) in time slot \( t \) for the client \( x \) as \( W_i(h_0^t, h_i^t, \pi_i|m) = F(i, m(h_0^t)[i])e_i^t \) and \( r_i(h_0^t, h_i^t, \pi_i|m) = F(i, m(h_0^t)[i])g(m(h_0^t)[i])e_i^t \) respectively. The corresponding payment received by the worker \( i \) as \( p(W_i(h_0^t, h_i^t, \pi_i|m), x) \). Therefore, we can compute the utility of the worker \( i \) in the stage game in time slot \( t \) by deducting the cost associated with the effort exerted as follows.

\[
u_i(h_0^t, h_i^t, \pi_i|m, p) = p(W_i(h_0^t, h_i^t, \pi_i|m), x) - C(i, x)(e_i^t)^2
\]

Note that the above utility is quasi-linear (linear in the payments) and is written in such a way that it is clear that worker takes the actions given the knowledge of the matching and the payment rule (announced by the designer). Similarly, we can compute the utility of the client \( x \) (linear in the revenue and the payments made) in time slot \( t \) who is matched to worker \( i \) as

\[
u_x(h_0^t, h_i^t, \pi_i|m, p) = r_i(h_0^t, h_i^t, \pi_i|m) - p(W_i(h_0^t, h_i^t, \pi_i|m), x)
\]

**The repeated endogenous matching game.** In every time slot, a stage game is played between a worker and a client who are matched *endogenously* based on the observation history of the designer, which for instance can include the *past performance* of all the workers. We refer to this repeated game as the “repeated
endogenous matching game”. Formally stated there are $2N$ players in the game where $N$ of them are workers and the remaining $N$ of them are the clients, where a worker and client pair are matched based on $m$ and the utilities for the stage game are given as described above. We define the total long-run revenue generated over the infinite horizon of the repeated game as follows.

$$R(\{\pi_k\}_{k=1}^N|m) = \lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^T \sum_{i=1}^N r_i(h_0^t, h_i^t, \pi_i|m)$$

Similarly, we define the long-run utility for the worker $i$ as

$$U_i(\{\pi_k\}_{k=1}^N|m, p) = \lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^T u_i(h_0^{t+1}, h_i^t, \pi_i|m, p)$$

The long-run utility for the client $x$ is defined as

$$V_x(\{\pi_k\}_{k=1}^N|m, p) = \lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^T v_x(h_0^t, h_i^t, \pi_i|m, p)$$

Note that we are only interested in the settings (set of mechanisms and strategies) for which all of the above limits exist. The above game has incomplete information because the workers’ qualities and utilities are not known and the workers cannot predict the tasks they are matched to in the future. At the risk of repetition, we give a summary of the knowledge structure for our model next.

**Knowledge structure.** The workers and the clients are rational, independent decision makers who do not cooperate in decision making. The total number of time slots in the mechanism, the matching, and the payment rule is public knowledge. The quality of the tasks is known to all however, the productivity of the workers is not known to anyone. The cost of workers to work on the task is only known to the workers privately. The effort exerted by the worker is known to the worker privately. The output produced by the worker is observed by the worker itself, the client with whom the worker is matched and the designer. The structure of the utility (but not the parameters in the utility) of the workers and clients is known to all. The above knowledge structure is common knowledge.

**Objective of the mechanism design problem and associated challenges.** In this section, we will discuss the objective of the mechanism design problem. The designer needs to choose a mechanism $\Omega = (m, p)$- the matching and the payment rule to maximize the total long-run revenue subject to two types of constraints.
The first type of constraints is the individual rationality (IR) constraints, which if satisfied guarantee that the workers and the clients participate in the mechanism. The second type of constraints is the incentive-compatibility (IC) constraints, which guarantee that every worker follows an optimal strategy (given the strategies of others). If the strategy of each worker can satisfy the IC constraint, then the joint strategy of all the workers will be an equilibrium (i.e. no worker will want to deviate). We do not need such a constraint for the client as the client only needs to decide whether or not to participate. Therefore, the design problem is stated as follows.

Design Problem

\[
\max_{m,p} \min_{\{\pi_k\}_{k=1}^N} R(\{\pi_k\}_{k=1}^N | m, p)
\]

s.t. \( V_x(\{\pi_k\}_{k=1}^N | m, p) \geq 0, \forall x \in S \) (IR-clients)

\[
U_i(\{\pi_k\}_{k=1}^N | m, p) \geq 0 \forall i \in N \) (IR-workers)
\]

\[
U_i(\pi_i; \{\pi_k\}_{k=1,k \neq i}^N | m, p) \geq U_i(\pi_i'; \{\pi_k\}_{k=1,k \neq i}^N | m, p) \forall i \in N \forall \pi_i'; (IC-workers)
\]

Note that our formulation above is similar to that of implementation in mechanism design. The designer’s goal is to choose a mechanism such that the total long-run revenue in the worst possible equilibrium of all the possible equilibria that are implementable is maximized. In the above formulation the objective is to maximize the worst case total long-run revenue \( \min_{\{\pi_k\}_{k=1}^N} R(\{\pi_k\}_{k=1}^N | m, p) \), but we can extend the proposed analysis to other objectives for instance total long-run revenue \( R(\{\pi_k\}_{k=1}^N | m, p) \). We describe the technical challenges associated with solving the above problem.

- **Computationally intractable**- The space of possible matching rules, the payment rules, and the strategy of the workers is extremely large thus making the problem computationally intractable. It is even hard to say that there will exist a solution - matching rule, payment rule and the strategy of workers that solve the above optimization problem.

- **Incomplete information**- The designer needs to select \( m \) and \( p \) to maximize the minimum value of long-run revenue achieved by an equilibrium strategy, which depends on both the productivity of the workers \( F(i, x) \) and the costs \( C(i, x) \) that are not known to the designer. In addition, the workers need
to select the optimal equilibrium strategy, which depends on both the quality of the workers $F(i,x)$ and the costs $C(i,x)$ that are not known to the workers. Therefore, the computation of the solution to the above problem (if one exists) may require a separate entity (different from the designer only has limited information) that has access to complete information.

Given the above three challenges, it is unlikely that the above problem can be solved exactly as is. In our approach instead we will solve for a matching and payment rule and strategy for the workers that approximately solve the above design problem. We show that our solution is efficient, i.e. it can achieve high long-run revenue $\min_\pi R(\{\pi_i\}_{i=1}^N | m, p)$ for a wide range of scenarios (Theorem 3) while still satisfying the IR and IC constraints in the design problem. We will also show that the each worker can maximize $U_i(\pi_i, \{\pi_j\}_{j=1,j\neq i}^N | m, p)$ and thus compute its equilibrium strategy in a decentralized manner with access to the limited information. In the next section, we describe our proposed mechanism.

2.2. Proposed Mechanism and its Properties

We first give a brief description of the proposed mechanism. The proposed payment rule is designed to be strictly convex in the output produced by the worker such that it compensates the strictly convex costs for exerting effort and is linearly increasing in the quality of the task. The payment rule is designed in such a way that it ensures that if a worker derives a positive utility from working on a task, then it is the best response for the worker to exert maximum effort thereby, completely eliminating moral hazard.

The proposed matching rule is designed to evaluate each worker on every type of task at least once. Since the worker is evaluated only once we refer to the proposed matching rule as “first impression is the last impression” (FILI). Based on the output of the workers a ranking of the workers over the different tasks is computed and the workers also submit a preference for the tasks to the designer. The designer computes a final matching based on these rankings and preferences, which remains fixed for all the future time slots. The designer will announce both the payment rule and the FILI matching rule to the workers and the clients.

Next, we give a detailed description of the mechanism, denoted as $\Omega^F = (m^F, p^F)$

**Payment rule.** If the worker $i$ works on task $x$ and produces $W(i,x)$ units of output (units of task completed by the worker), then the worker is paid $p^F(W(i,x), x) = \alpha W(i,x)^2 g(x)$ by client $x$, where
\( \alpha \) is a positive constant decided by the designer. We will discuss later as to how can the designer choose the exact value of \( \alpha \) such that an efficient solution to the design problem can be achieved. However, note that the value of \( \alpha \) in the proposed payment rule is required to be less than \( \frac{1}{W_{\text{max}}} \) to ensure that the clients are willing to participate in the mechanism (See Proposition 1.). This payment rule is announced by the designer to all the workers and the clients. Note that the payment rule incentivizes the workers to exert high effort as the marginal benefit from producing more output increases. The payment rule also makes the high quality tasks more desirable. In the extensions section we discuss the scenario where the payment rules are already given and are not decided by the designer.

**Matching rule.** The FILI matching rule \( m^F \) operates in two stages described below.

1. **Evaluation stage** \((0 \leq t \leq N)\). In this stage, the matching is carried out with the aim to evaluate workers’ performances over different tasks. In time slot \( t \), where \( t \leq N \), the worker \( i \) is assigned to task \([(t+i) \mod N]\), where mod is the modulus operator. Observe that in each time slot all the workers are matched to different tasks. Also, each worker is matched to every task exactly once in the first \( N \) time slots, i.e. \( 0 \leq t \leq N - 1 \). At the end of each time slot, the worker, the client and the designer observe the output of the worker on the assigned task. At the end of \( t = N - 1 \) time slot, the designer will know the output of each worker task combination. We write the observation of the designer in the form of a matrix \( W^e \), where \( W^e(i,x) \) is the output of worker \( i \) on task \( x \) in the evaluation stage. At the end of the stage (start of the time slot \( t = N \)), the designer requests the workers to submit their preferences in the form of ranks (strictly ordered) for tasks. The workers form these preferences based on the task qualities and the outputs. These rank submissions are a part of the strategy for the workers, which we describe later. In practical settings not all the tasks on the platform are very different and many of them can be categorized into one type, for instance, translation (each worker has the same productivity for tasks of the same type). In such cases, it is sufficient to evaluate the workers on tasks of different types.

2. **Operational stage** \((t \geq N + 1)\). In this stage, the final matching is computed based on the evaluations in the previous stage and the preferences submitted by the workers. The designer computes the rankings of the workers for the clients based on the outputs \( W^e \) as follows. For every client \( x \) the designer
ranks the workers based on the outputs produced on task $x \{W^x(i, x)\}_{i=1}^N$. The designer computes the matching based on the G-S algorithm [Gale and Shapley 1962] as follows. The designer executes the G-S algorithm with the workers as the proposers and the clients as the acceptors. In each iteration of the algorithm, each worker proposes to its favorite task that has already not rejected it. Each client based on the proposals it gets keeps its favorite worker on hold and rejects the rest. At the end of $N^2 - 2N + 2$ iterations, the matching that is achieved is final. The matching computed above is fixed for the remaining time slots starting from $N + 1$. (Note that the $N^2 - 2N + 2$ iterations are carried out at the start of time slot $N + 1$. Also, the algorithm is executed by the designer and there is no interaction between the workers and the clients here and the algorithm is carried out by the designer).

We describe the above FILI matching in the form of Algorithm at the end. Next, we state a proposition which shows that the workers and clients are always willing to participate in the above mechanism. We define a constant $W_{\text{max}} = f_{\text{max}} \max_{i \in N} \{e_{i, \text{max}}\}$. 

**Proposition 1.** In the proposed mechanism $\Omega^F$ the workers are always willing to participate. If the payment parameter $\alpha \leq \frac{1}{2W_{\text{max}}}$, then all the clients are also willing to participate in the proposed mechanism.

**Proof:** It is easy to see that the workers can always ensure a zero long-run utility (outside option of the worker gives zero utility) by exerting zero effort. Therefore, the participation constraint for the workers is trivially satisfied. If $\alpha \leq \frac{1}{2W_{\text{max}}}$, then the profit per unit output is always greater than or equal to zero which ensures that the clients cannot have a negative profit in any period and so they cannot have a negative long-run profit.

From the above Proposition, we know that the workers and clients are willing to participate in the proposed mechanism. In the rest of the work, we will assume that $\alpha \leq 1/2W_{\text{max}}$. The proposed mechanism induces a repeated endogenous matching game as described in Section 2.1. In the next section, we derive an equilibrium strategy for this repeated endogenous matching game and also show that it has some very useful properties.

**2.2.1. Equilibrium analysis for the repeated endogenous game** The proposed mechanism $\Omega^F$ requires each worker to choose the effort level in every time slot and additionally in the last time slot of the
evaluation stage ($t = N$) each worker is also required to submit a preference/ranking over the clients. Next, we will propose a strategy for each worker $i$, which we refer to as the bang-bang strategy $\pi_i^{bang}$ for reasons that will be explained later. We will show that this strategy leads to maximization of the long-run utility for the worker.

1. **Evaluation stage** ($0 \leq t \leq N$). In each time slot $t$ in this stage, where $t \leq N$, the worker $i$ should exert the maximum effort possible, i.e. $e_{i}^{max}$. In addition, the worker $i$ can estimate its productivity for different tasks when it is matched to them as follows. In time slot $t$ worker $i$ is matched to task $m^F(h^t_0)[i]$, then based on the output generated the worker can estimate the productivity as follows

$$F(i, m^F(h^t_0)[i]) = \frac{W(i, m^F(h^t_0)[i])}{e_{i}^{max}}.$$ 

Hence, at the end of time slot $N - 1$, the worker $i$ will know its qualities over different tasks. At the start of time slot $N$, each worker is required to submit a ranking list for the tasks. Worker $i$ submits a ranking of the tasks in the decreasing order of $\alpha F(i, x)^2 g(x) - C(i, x), \forall x$, which is proportional to the long-run utility the worker expects and submits these rankings to the designer.

2. **Operational stage** ($t \geq N + 1$): The designer executes the G-S algorithm (as described above) and assigns to worker $i$ a task $y$. If $\alpha F(i, y)^2 g(y) - C(i, y) > 0$, then the worker exerts maximum effort in every time slot, otherwise the worker exerts zero effort in every time slot. Note that the ranking list of other workers and clients is not known to worker $i$ and thus it cannot predict the task it will be matched with in the operational stage.

The proposed bang-bang strategy has a desirable simple structure, which is suited for the online labor platforms. Each worker will exert maximum effort in the evaluation stage and then based on whether the worker is assigned a task that is sufficiently high in its preference list it continues to exert maximum effort else, it stops exerting effort. Thus, it means that a client will either be matched to a worker who will either exert maximum effort or exert no effort at all, where the former case is good for the client and in the latter case the client can try to search for some other workers in the future. We will show later, that in many settings it is possible that for a large number of clients the workers exert maximum possible effort. In the next theorem, we show that the proposed bang-bang strategy is a dominant strategy for each worker. Therefore,
if all the workers follow the bang-bang strategy, then the joint strategy will comprise an equilibrium of the repeated endogenous matching game (induced by the proposed mechanism $\Omega^F$), which we refer to as the bang-bang equilibrium (BBE).

**Theorem 1.** **Bang-bang strategy and its properties**

1. The bang-bang strategy is a dominant strategy for each worker.
2. If all the workers follow the bang-bang strategy, then the joint strategy is BBE.
3. The bang-bang strategy can be computed by each worker in a decentralized manner.

The proofs of all the theorems are in the Appendix section. The first two parts of the Theorem establish that the proposed strategy is an equilibrium. Note that we have not established that this is the only equilibrium that can be achieved and that is the reason we next justify why this equilibrium may be selected among the multiple possible equilibria. The amount of information needed to compute the equilibrium strategy can be crucial for equilibrium selection. In the third part of the theorem, we prove that each worker can compute the bang-bang strategy in a decentralized manner with the limited information that they possess. Thus the third part of the above theorem makes a case for the selection of the BBE among other equilibria that are possible.

Next, we will show that in many cases the repeated endogenous matching game has a unique equilibrium payoff (vector of long-run utilities of the workers), which is achieved by the bang-bang equilibrium strategy. Note that the uniqueness in terms of payoff means that there can be multiple equilibrium strategies possible but all of them lead to the same unique equilibrium payoff.

**Uniqueness of the bang-bang equilibrium.** We first state the assumptions that specify the settings under which we can prove that the equilibrium payoff is unique and is achieved by the bang-bang equilibrium strategy.

**Assumption 1.** If a worker $i$ has a higher productivity than another worker $k$ on a task $x$, i.e. $F(i,x) > F(k,x)$, then it has a lower cost $C(i,x) < C(k,x)$ for exerting effort on the same task and this is true for all the tasks $x \in S$. 
Assumption 1 is natural in many settings. It states that if a worker has more experience (and skill) in performing a task, i.e. \( F(i, x) > F(k, x) \), then the worker also has more interest in that task and is willing to spend more time on it, i.e. \( C(i, x) < C(k, x) \).

**Assumption 2.** The productivity of the worker is the same for all tasks, i.e. \( F(i, x) = F(i, y) \), \( \forall x, y \) and is denoted as \( F(i) \). The cost for exerting effort for a worker is the same for all tasks, i.e. \( C(i, x) = C(i, y) \), \( \forall x, y \) and is denoted as \( C(i) \).

The Assumption 2 states that the productivity of a worker across the different tasks is the same. This is natural in the settings where the tasks are homogeneous, i.e. of the same type. For instance, all the tasks can relate to translation (from language A to B). Note that the assumption requires homogeneity in task types but still allows the tasks to have different qualities. For instance, different translation tasks can generate different revenues and thus have different qualities. Moreover, the workers can still have different qualities over the tasks even though the task are of the same type.

**Assumption 3.** We also assume that the maximum effort level \( e_{i}^{max} \) is the same for all the workers.

The Assumption 3 requires that all the workers have the same maximum limit on the amount of effort they can exert. The maximum effort that any worker can exert in a single time slot cannot exceed the total duration for which a task is assigned to a worker, which is equal to the length of a time slot. If the length of the time slot is not large, then it is reasonable to think that the maximum limit on the effort for everyone is the same.

In the next theorem we show that if Assumption 1 and 2 hold, then we will show that the repeated endogenous matching game induced by the proposed mechanism \( \Omega^{F} \) has a unique equilibrium payoff and it is achieved by the bang-bang equilibrium strategy.

**Theorem 2.** **Uniqueness of the equilibrium payoff.** If the Assumption 1, 2, and 3 hold, then the repeated endogenous matching game induced by the proposed mechanism \( \Omega^{F} \) has a unique equilibrium payoff, which is achieved by the bang-bang equilibrium strategy.

In the next section, we establish the conditions under which the proposed mechanism can be shown to be effective in mitigating both moral hazard and adverse selection.
2.2.2. Effective Mitigation of Moral Hazard and Adverse Selection. We first state a performance benchmark that corresponds to an upper bound on the objective, i.e. the long-run revenue achieved, in the design problem. The performance benchmark corresponds to the maximum total long-run revenue achieved when the workers are compliant (no moral hazard)- exert maximum effort and the productivities of the workers are known (no adverse selection). In the next Proposition and Theorem 3 we will require that the Assumption 2 holds. The maximum output that can be produced by worker $i$ (worker $i$’s productivity is $F(i)$ from Assumption 2) on any task in this scenario is $F(i)e_{i}^{\text{max}}$. We write the maximum outputs of workers sorted in the increasing order as follows $\{F(m_{1})e_{m_{1}}^{\text{max}}, ..., F(m_{N})e_{m_{N}}^{\text{max}}\}$, where $m_{i}$ is the index of the worker with the $i^{th}$ highest output. We define an indicator function $I(A)$ which takes the value the condition $A$ is true and zero otherwise.

**Proposition 2.** If Assumption 2 holds, then

- The maximum total long-run revenue generated when the workers are compliant and their productivities are known is $\sum_{x=1}^{N} F(m_{x})g(x)e_{m_{x}}^{\text{max}}$.

- The total long-run revenue generated in the bang-bang equilibrium is

\[
\sum_{x=1}^{N} F(m_{x})g(x)e_{m_{x}}^{\text{max}} I(\alpha F(m_{x})^{2}g(x) - C(m_{x}) > 0)
\]

The difference between the performance benchmark and the performance achieved in the BBE depends on the proportion of the workers for whom $\alpha F(m_{x})^{2}g(x) - C(m_{x}) \leq 0$. Next we show some of the possible scenarios where if $\alpha$ is chosen properly, then the difference between the proposed BBE and the upper bound is small. Before we proceed to the comparison we make a remark about the chosen performance benchmark. The performance benchmark here does not assume that the workers are not self-interested, i.e. they are compliant, and therefore the benchmark takes a very high value. It is possible to improve the benchmark, which takes smaller values. We state the assumption that is needed to make the comparison. In the assumption stated below, we assume that the Assumption 2 and 3 hold.

**Assumption 4.** i) The workers’ productivities are independently drawn from a uniform distribution $U \sim [0, f_{\text{max}}]$. 

ii) The maximum productivity $f^\text{max} > 2$ and the maximum cost $c^\text{max} < \frac{g^\text{min}}{g^\text{max}}$.

The assumption that the productivity distribution of workers is uniform and $f^\text{max} > 2$ ensure that the mean productivity of a worker is not low. The condition $c^\text{max} < \frac{g^\text{min}}{g^\text{max}}$ ensures that the cost for the workers is not too high otherwise incentivizing the workers is impossible while simultaneously ensuring the participation (non-zero profits) of the clients. These conditions ensure that if a worker is given sufficient incentive it will exert high effort. Suppose that designer selects $\alpha = \alpha^* = \frac{c^\text{max}}{2g^\text{min}f^\text{max}}$ (Observe from Assumption 4 ii) that $\alpha^* \leq 1/2W^\text{max}$). In the next theorem, we compare the performance (total long-run revenue) of the proposed mechanism (when $\alpha = \alpha^*$) in the bang-bang equilibrium with the upper bounds stated in the Proposition 1. We will take the expected value of the total long-run revenue as the measure of performance in the comparison in the next theorem, where the expectation is computed respect to the distribution of the productivity of the workers. We define a constant $\Theta = (1 - \frac{2}{f^\text{max}})(\frac{g^\text{min}}{g^\text{max}})$.

**Theorem 3. Efficiency of the proposed mechanism.** If the Assumptions 1-4 hold and if the designer fixes $\alpha = \alpha^*$, then the ratio of the expected total long-run revenue in the bang-bang equilibrium and the expected value of the upper bound (from Proposition 2) is greater than $\Theta$.

From the above comparison, it is clear that if $f^\text{max}$ is high, then $\Theta$ will be high. However, if the variation in the quality of the tasks, i.e. $\frac{g^\text{max}}{g^\text{min}}$ is high, then the guarantees on the performance $\Theta$ are weaker. Observe that the variation in the quality of tasks is high (for a fixed $g^\text{max}$), then the value of $\alpha^*$ is also high. Therefore, the general design insight that we can draw from the above is- if the variation in task qualities is higher, then in order to guarantee the mitigation of moral hazard and adverse selection the clients need to pay a higher proportion of the revenue to the workers. In the above theorem setting $\alpha = \alpha^*$ only requires the designer to know the bounds on the costs, maximum effort and the qualities and it represents one possible choice for which the efficiency guarantee $\Theta$ is high. However, it is possible to find better choices of $\alpha$ possibly with more knowledge about the system provided to the designer. In addition note that the designer can set the values of $\alpha$ taking into consideration the total profit that it generates for the clients instead of the total long-run revenue as in the above theorem.
In this section, we showed that the proposed mechanism is effective in mitigating moral hazard and adverse selection in many possible settings and our aim in the next section is to compare the performance of the proposed mechanism with some alternate mechanisms.

**Numerical Simulations.** We describe the mechanisms that we will compare our performance with below.

1. **Linear payment rules combined with uniform random matching.** In this mechanism linear payment rules are used, i.e. the client pays the workers a fixed amount per unit of the output produced. Such payment rules are the most common in practice (For instance, such rules are common on both Upwork and Amazon Mechanical Turk). The matching on the existing platforms, for instance, Amazon Mechanical Turk and Upwork are close to uniformly random matching, which is justified as follows. In these platforms, the workers and clients arrive asynchronously. The workers decide to contact their favorite clients who in turn need to select the workers with the limited information they possess about the workers, which can be misleading and in many cases not relevant for the particular job (Upwork had 96% workers who...
had no significant past experience for the existing tasks). Since the arrivals of workers and clients occur asynchronously and the selection is based on limited information we argue that the matching is close to a uniform random matching. In the comparison above, we use the optimal linear payment rule, which leads to the maximum total long-run revenue, and requires the clients to pay each worker the entire revenue that is generated by the worker.

2. **Proposed payment rule combined with random matching** - In this mechanism, we will use the proposed payment rule \( p^F \) (see Section 2.1) and use uniformly random matching as the matching rule.

3. **Proposed payment rule with optimal matching** - In this mechanism, we will use the proposed payment rule \( p^F \) (see Section 2.1). We fix the payment rule and find the optimal matching rules, which lead to the maximum total long run revenue generated. The computation of the optimal matching, in this case, requires complete information about the workers and clients.

In Figure 1, we show the comparison of the performance of the proposed mechanism \( \Omega^F \) with the mechanisms described above and the upper bound derived in the Proposition 2. The details of the setup for the numerical simulations can be found in the Appendix. It can be seen that the proposed mechanism leads to large gains over 200 percent than mechanisms that employ linear payment rules combined with random matching. Moreover, the performance is the same as the third mechanism above which employs optimal matching based on complete information. Recall that the upper bound that we choose takes very high values because we assume that the workers are compliant. In the next section, we propose the notion of stability for our setup.

2.2.3. **Long-run Stability of Matching.** In the existing online labor platforms when the workers and clients are matched, there is no guarantee that the two sides have learned about each other sufficiently and are satisfied with their match, i.e. no worker or client will search for a different match in the future. Therefore, it is possible that the workers and clients repeatedly switch in search for better partners and this is inefficient and not desirable for anyone- the clients, the workers, and the designer. In this section, we will establish that our mechanism guarantees that once the clients and workers are matched in the operational stage they will not need to search for alternate partners. To be able to capture the criterion that a worker or
a client does not wish to switch we need to propose a notion of stability that extends the standard notions to such environments, where the matching is carried out repeatedly in the presence of incomplete information and learning (in the presence of moral hazard). The main idea is as follows- if we show that ex-post a final matching is achieved there will be no worker-task pair that is not matched in the final matching but strictly prefers to be matched, then the matching is called “long-run stable”. Next, we provide the formal definition.

We consider a family of matching rules, which satisfy the following condition, \( \lim_{t \to \infty} m(h^t_0) \) exists for all possible sets of histories \( \{h^t_0\}_{t=1}^{\infty} \), i.e. the limit of the matching exists across all possible histories.

Suppose that the worker \( i \) uses a strategy \( \pi_i \) and the joint strategy of all the workers is given as \( \pi = \{\pi_1, \ldots, \pi_N\} \). The joint strategy \( \pi \) will induce a history and the matching rule will take a limiting value depending upon the history. The history for the designer induced by the joint strategy \( \pi \) is denoted as \( h^t_0, \pi \) and the history for the worker \( i \) induced by the joint strategy \( \pi \) is given as \( h^t_0, \pi_i \). We define \( m^*_{\pi} = \lim_{t \to \infty} m(h^t_0, \pi) \). Note that \( m^*_{\pi}[i] \) denotes the task that worker \( i \) gets matched to in the limit. We use the expressions for the long-run utility of worker \( i \) and client \( x = m^*_{\pi}[i] \) as defined in Section 2.1. The expression for the long-run utilities for the worker \( i \) and client \( x \) can be simplified as follows (See the justification in Appendix).

\[
U_i(\pi) = \lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} p \left[ F(i, m^*_{\pi}[i]) \pi_i(h^t_0, \pi) \right] - C(i, m^*_{\pi}[i]) \pi_i(h^t_0, \pi)^2
\]

\[
V_x(\pi) = \lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} F(i, m^*_{\pi}[i]) \pi_i(h^t_0, \pi) g(j) - p \left[ F(i, m^*_{\pi}[i]) \pi_i(h^t_0, \pi) \right]
\]

The above expression for long-run utility shows that the worker’s utility depends on the task assigned in the limit and not on the utility derived in the stages before being matched to this task finally.

We now formalize the condition that no worker-task pair that is not matched by \( m^*_{\pi} \) cannot strictly gain by being matched to one another. Consider worker \( i \) and a task \( y \), where \( y \neq m^*_{\pi}[i] \) and suppose that this worker-task pair was matched instead of \( i \) and \( m^*_{\pi}[i] \). In such a case, the long-run utility achieved by worker \( i \) matched to task \( y \) (in the limit), when the strategy for worker \( i \) is \( \pi'_i \), is given as follows. Note that we are considering the case where the worker \( i \)'s final match, i.e. task \( y \) is known. Therefore, the strategy of others
cannot impact the worker $i$ and client $y$'s long-run utilities. Hence, it is sufficient to consider the strategy $\pi'_i$ to be a function of time only.

$$\hat{U}_{yi}(\pi'_i) = \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} p\left[ F(i, y)\pi'_i(t) \right] - C(i, y)\pi'_i(t)^2$$

$$\hat{V}_{yi}(\pi'_i) = \lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} F(i, y)\pi'_i(t)g(y) - p\left[ F(i, y)\pi'_i(t) \right]$$

Next we define long-run stable matching in terms of the above expressions for long-run utilities.

**Definition 1.** Long-run Stable Matching: A matching rule $m$ is long-run stable with respect to a joint strategy $\pi$ if there exists no worker-task pair $(i, y)$, which is already not matched in the limit of $m$ (where $k \neq m^*_\pi(i)$), and a strategy $\pi'_i$ that leads to a strict increase in the long-run utility for both the worker $i$ and the task $y$, i.e. $\hat{U}_{yi}(\pi'_i) > U_i(\pi)$, $\hat{V}_{yi}(\pi'_i) > V_y(\pi)$.

The above definition captures the main idea that we set out with at the start of this section. If we can show that the proposed mechanism is long-run stable with respect to the bang-bang equilibrium, then it will imply that in the final matching that (in operational stage) is achieved the workers and clients will not need to switch in search of better partners.

We now compare and contrast the difference of the proposed definition of long-run stability with the existing definitions. The standard analysis of stability (Gale and Shapley [1962]) assumes complete information about the preferences. There also exist notions of stability (Liu et al. [2014]) for setups with incomplete information. In our setup as well there is incomplete information because the preferences are not known to the workers and clients themselves. Unlike Liu et al. [2014] the workers and clients interact and learn about each other before being finally matched. In addition, the preferences in our setup depend on the actions, i.e. the effort exerted by workers, which is not the case in the standard setups Gale and Shapley [1962], Liu et al. [2014]. Our definition checks if the matching rule enables efficient learning by checking that in the final matching that is achieved there is no pair of client and worker that would instead prefer to be matched. Moreover, the definition needs to take into consideration the impact of actions taken after matching occurs, which is why we define long-run stability with respect to a joint strategy. Next, we prove that for many settings the proposed mechanism is indeed long-run stable with respect to the bang-bang equilibrium.
**Theorem 4. Long-run Stability of Matching.** If the Assumption 1 and the Assumption 3 hold, then the proposed matching rule \( m_F \) is long-run stable with respect to the bang-bang equilibrium.

We give an intuition into explaining the above Theorem. We start with the stability from the workers’ point of view. Each worker follows the bang-bang strategy and submits the preferences based on rankings in the decreasing order of the utility that it expects to derive from the tasks. Thus we can see that the workers already try to be associated with the best possible tasks. For the clients the ranking is carried out by the designer based on the workers’ outputs. The clients strictly prefer the workers who they expect to generate higher profits from. If \( \alpha \leq \frac{1}{2W_{\max}} \), then a worker with higher output generates a higher profit for the task as long as it is willing to work (exert non-zero effort) on the task (See the Appendix for details). In addition, the Assumption 1 and the Assumption 3 guarantee that a worker who is ranked higher on a task should also have a stronger incentive to work on it in comparison to a worker who has a lower rank. Thus ranking the workers in the order of the outputs correctly reflects the preferences of the tasks as well.

In Theorems 1-3 we discussed the optimality of the strategy from the point of view the workers and approximate optimality from the point of view of the entire system’s performance (total long-run revenue). However, we did not discuss why is the proposed mechanism good for the clients. Theorem 4 above shows that even the clients will have no reason to search for another worker in the final matching achieved. Therefore, the mechanism is desirable from the client point of view as well.

Theorems 4 bears similarity to Theorem 5 in Roth (1989). In Theorem 5 in Roth (1989) it is shown that if the matching rule is worker-optimal and outputs stable outcomes (stability in the sense of Gale and Shapley (1962)), then the truthful revelation of preferences is the dominant strategy for the workers. For our setting to be able to compare with Theorem 5 in Roth we first need to define the true preferences for a worker. In our setting the natural definition for the true preference of a worker \( i \) is based on the long-run utility that the worker expects to derive once it is finally matched to a client. Based on this definition it is clear that the preference list submitted in the bang-bang strategy corresponds to the true preference list. In Theorem 4 we prove that if the proposed matching rule is used, then (from Theorem 1) we know that the bang-bang strategy is the dominant strategy for the worker and it leads to the truthful revelation of preferences and the matching achieved is long-run stable with respect to the bang-bang equilibrium.
Next, we show that how some of the simpler mechanisms that do not promote learning in an efficient manner cannot achieve long-run stability.

**Examples when the long-run stability is not achieved by other matching rules.** First, consider random matching rules that match workers and clients uniformly randomly. These rules certainly do not guarantee any form of stability whatsoever as clients and workers are matched randomly and not based on the preferences. Let us consider some alternate matching rules, which use the performance of the workers to do the matching as follows instead of being uniformly random.

i) In the first time slot, the workers are matched to tasks uniformly randomly. After that workers are assigned tasks based on their performance in the first time slot.

ii) In the first $N$ time slots the workers try all the tasks as explained in the evaluation stage in our mechanism. Each worker’s average output is computed, where the average is taken uniformly across the tasks that the worker performs. The workers are finally matched to the tasks assortatively as follows, the worker with the highest average output gets the task with the highest quality and so on.

We will show that the above two matching rules cannot achieve long-run stable outcomes. Consider the case when there are two workers and two tasks. The productivity of the workers, the cost of the workers and the maximum effort that can be exerted by the workers is given as $F = \begin{bmatrix} 6 & 2 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and, $e_{max} = 1$ respectively. The productivity of the tasks is given as $g = [2 \ 1]$. The payment rules for task 1 and 2 are $p(x, 1)$ and $p(x, 2)$, where it is given that $p(x, 1) > p(x, 2)$. In this setting it is possible to show that both the rules in bullets i) and ii) can lead to worker 1 being matched to task 2 and worker 2 being matched task 1, which is not a long-run stable matching. The proposed matching rule, on the other hand, will lead to the long-run stable matching of worker 1 with task 1 and worker 2 with task 2.

In the section to follow, we will discuss the extensions and variations of the proposed model.

### 3. Extensions and Variations of the Model

#### 3.1. Extension to Stochastic Outputs

The model in Section 2 assumed that the output produced and thus the revenue generated by the workers is a deterministic function of their productivity and effort. In realistic settings the revenue generated is impacted
by other uncertain forces as well. In this section we extend the model to account for such uncertainties. We assume an additive noise based model to quantify the output produced. If a worker $i$ works on task $x$ and exerts effort $e_i$, then the revenue generated is given as $F(i, x)g(x)e_i + Z(i, x)$, where $Z(i, x)$ is a random variable representing noise when worker $i$ works on task $x$. We assume that $Z(i, x)$ has zero mean and has a finite variance $\Sigma(i, x)^2$, where $\Sigma$ is the matrix of variances. The random noises in the revenue generated by different workers are mutually independent random variables. We assume that the mean and the variance of $Z(i, x)$ is known to the designer and the clients. Also, the noise random variables in different time slots are i.i.d.. The definition of a general matching and payment rule continues to be the same as in Section 2.

We define the long-run utility for the workers and clients as the expectation of the long-run utility as defined in Section 2.1, where the expectation is taken with respect to the sequence of noise random variables. The design problem can be stated in the same way as in Section 2.1 by replacing the above definitions of long-term revenue and long-term utilities.

The proposed mechanism $\Omega^F$ can be modified to account for the above uncertainties in a natural way as follows. We give a brief discussion here and a detailed discussion for this extension is given in the Appendix. The payment rule of the modified mechanism has an extra term that adjusts for the extra payments that are made to workers owing to the stochasticity in the revenue. The matching rule in the modified mechanism still comprises of two stages- the evaluation and the operational stage. The main difference is that the evaluation stage is now longer such that each worker can be assigned to every type of task for a sufficient number of time slots and thereby a better estimate of the worker’s output can be computed. The operational stage is exactly the same as in the proposed mechanism in Section 2. The modified mechanism induces a stochastic repeated endogenous matching game. We propose an appropriate extension of the bang-bang strategy (defined in Section 2) as follows. In the evaluation stage, the worker always exerts maximum effort. Each worker also computes an estimate of its own productivity by taking the average of the revenue generated on a task divided by the effort exerted times the task quality. Using these estimates of the productivity the worker submits a preference list for the tasks ordered according to the long-run utility the worker expects to generate. The designer executes the G-S algorithm and the workers and clients are matched. Each worker
decides whether to exert maximum effort or to exert no effort at all based on the utility that it expects to generate from the task assigned. For the proposed modification of the bang-bang strategy, we can extend the results discussed (See the Appendix for details).

3.2. Other extensions

Extension to client selected payment rules. In the model discussed in Section 2, we assumed that the designer prescribes the payment rules to the clients and if the clients choose to participate in the mechanism, then they need to follow the prescribed payment rules. In this extension, we will consider the setting where the payment rules are not prescribed and are instead set/given by the clients. We will consider a natural setting where the payments are linear in the revenue generated by the worker, i.e. each client pays a certain fraction of the revenue generated back to the worker. Suppose that client \( x \) pays a fraction \( \eta_x \) of the revenue generated (We assume that this value \( \eta_x \) is not a decision variable and is given.). We assume that the designer continues to use the same proposed FILI matching rule (defined in Section 2). The above type of payment rules and the FILI matching rule induces a repeated endogenous matching game between the set of workers and clients. We describe the equilibrium strategy for the workers. In the evaluation stage, each worker should always exert maximum effort. At the end of the evaluation stage, each worker needs to submit a preference list, which in this case is computed as follows. For every task each worker computes the maximum long-run utility that it can derive by being assigned to it and then ranks the tasks based on these utilities. In the operational stage, the worker is allocated a task based on the rankings submitted by the workers and the rankings of the workers computed by the designer (based on the outputs as described in Section 2 in the FILI matching rule). Based on the task allocated the worker decides the optimal amount of effort to exert in every time slot in the operational stage. As we can already see that the structure of the equilibrium strategy, in this case, is very similar to the bang-bang strategy in Section 2. The result in Theorem 2 extends directly to this case with the same assumptions and the methodology of the proof is exactly on the same lines. In Theorem 3 we proved that in a wide-range of settings moral hazard and adverse selection are mitigated in BBE. For this extension again we can find appropriate conditions under which a high fraction of the workers will exert maximum or close to maximum effort possible thereby guaranteeing an effective mitigation of
both moral hazard and adverse selection. In Theorem 4 we showed that the proposed matching is long-run stable w.r.t BBE. In this extension, we can show a similar result, which states that the proposed matching is long-run stable w.r.t. to the modified equilibrium strategy described above. The proof of this follows on the same lines as the proof of Theorem 4 given in the Appendix.

**Extension to discounted utilities.** In the model in Section 2 we defined the long-run utilities as the undiscounted average of the stage game utilities. It may seem that the results we discussed depend on the fact that the utilities assume the undiscounted averages, while in some situations it is more plausible to think that the number of time slots in the mechanism cannot be fixed apriori and thus it is more useful to consider discounted utilities. The entire analysis can be extended to discounted utilities. Firstly, we will consider $\epsilon$ approximate notions of the equilibrium. Observe that for the proposed strategies the source of loss is the evaluation stage. We can show that if the discount factors are sufficiently high, then the losses that occur in the evaluation stage are sufficiently low and bounded above by $\epsilon$. In the operational stage, the utility achieved in each stage is at the maximum thus it can be shown that the proposed bang-bang strategy is the $\epsilon$ best response (best response with at most $\epsilon$ loss). The other results in Theorem 2-4 can be extended in a similar manner.

**Extension to worker and client arrivals and exits.** The entire analysis assumes that the total number of workers and clients is fixed. The proposed mechanism can be adapted for worker and client arrivals as follows. First, recall that we assume that one worker works for one client and one client hires one worker. If the number of workers and clients is not the same, then there will be some workers or some clients who will not be matched. For simplicity we assume that in each time slot either a worker/client arrives or leaves. Suppose that a new worker arrives, then this worker can be separately evaluated. Based on the evaluations and the preferences of this worker, the designer checks if there exists a client who prefers this worker more than the existing worker, then the client can hire the incoming worker starting next time slot. Suppose a worker exits or leaves a client, then the client chooses the best possible worker from the set of workers, which can include the workers who are already matched to some client and also the workers who are not matched to any client. There can be similar extensions for the client arrivals/ exits.
4. Conclusion

In this work, we developed a mechanism to learn how to match and incentivize the workers and clients efficiently and thus simultaneously mitigate the problems of adverse selection and moral hazard. The model considered only requires the designer to act on the outputs (stochastic) produced by the selfish workers and also works in settings where the workers may not know their own productivity and must learn it. We show that the proposed mechanism ensures that the designer learns how to mitigate both moral hazard and adverse selection thereby achieving high total long-run revenue. We also show that given the proposed mechanism, the workers find in their self-interest to follow the simple bang-bang strategies that allow them to learn their own qualities and achieve the highest possible utilities among all possible equilibria. We also propose a notion of stability- “long-run stability”, which is tailored to matching environments with incomplete information and learning. In a wide-range of settings our proposed mechanism can achieve long-run stability.

Endnotes

1. Upwork has one billion dollars of annual billing
2. The entire analysis can be extended to setting when the number of clients and workers are not equal.
3. In our setup the clients do not have actions to take besides the decision to participate.
4. If two workers have the same output, then the tie is broken in favor of the worker that has a higher index.
5. Note that these matching rules do not satisfy the condition that for all histories the limiting matching exists, which is needed in the definition of long-run stability.

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ALGORITHM 1: First impression is the last impression (FILL)

\[ h_0^0 = \phi, \ R = 0_{N \times N} \]

for \( t = 0 \) to \( T - 1 \) do

if \( 0 \leq t \leq N - 1 \) then

Evaluation Stage

for \( i = 1 \) to \( N \) do

\[ m^F(h_t^0)[i] = [(t + i) \mod N] \]

end for

Observe the output \( W^e(i, m^F(h_t^0)[i]) \)

end if

if \( t \geq N + 1 \) then

Operational Stage

Observe the preference lists \( \{b_k\}_{k=1}^N \) submitted by the workers for the tasks

Use the G-S algorithm with workers as proposers and tasks as acceptors to compute the matching

\[ m^{GS}(W^e, \{b_k\}_{k=1}^N) \]

for \( i = 1 \) to \( N \) do

\[ m^F(h_t^0)[i] = m^{GS}(W^e, \{b_k\}_{k=1}^N)[i] \]

end for

end if

end for

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| Study                                      | Matching with transfer | Matching with strategic workers | Matching with incomplete info. | MH, AS |
|-------------------------------------------|------------------------|---------------------------------|-------------------------------|--------|
| Gale and Shapley (1962)                   | No                     | No                              | No                            | No, No |
| Shapley and Shubik (1971)                 | Yes                    | No                              | No                            | No, No |
| Roth (1982)                               | No                     | Yes                             | No                            | No, No |
| Roth (1989), Liu et al. (2014)            | No                     | Yes                             | Yes (no learning)            | No, Yes|
| Bikhchandani (2014)                       | No                     | Yes                             | No                            | No, No |
| Rastegari et al. (2013), Lee and Schwarz (2009) | No                     | No                              | Yes (with learning)          | No, Yes|
| Shimer and Smith (2000)                   | Yes                    | Yes                             | No                            | No, No |
| Becker (1974), Grossman et al. (2013)     | Yes                    | Yes                             | Yes (no learning)            | No, No |
| Tran-Thanh et al. (2012)                  | No                     | No                              | Yes (with learning)          | No, Yes|
| Dayama et al. (2015)                      | Yes                    | Yes                             | Yes (no learning)            | No, No |
| Ho et al. (2012)                          | Yes                    | Yes                             | Yes (no learning)            | Yes, No|
| Karger et al. (2014)                      | Yes                    | No                              | Yes (with learning)          | No, Yes|
| Xiao et al. (2016a)                       | Yes                    | Bounded rational (one-step foresight) | Yes (learning out of equilibrium) | Yes, Yes|
| Xiao et al. (2016b)                       | Yes                    | Yes                             | Yes (no learning)            | Yes, Yes|
| This work                                 | Yes                    | Yes                             | Yes (with learning)          | Yes, Yes|

**Table 1** Comparison of works in the area of matching

**APPENDIX**
In all the proofs we will use $I(A)$ as the indicator function. If the condition $A$ holds, then the indicator is one and zero otherwise.

**Appendix A: Proof of Theorem 1.**

There are three parts to the Theorem. In the first part, we need to show that the bang-bang strategy is a best response of a worker to any joint strategy adopted by the rest of the workers, which is equivalent to showing that the bang-bang strategy is the dominant strategy. First, we will simplify the expression for the long-run utility of the worker $i$ when the proposed mechanism $\Omega^e$ is implemented. We write the joint strategy for all the workers as $\pi = (\pi_1, \ldots, \pi_N)$. We write the private history of worker $i$, which is induced by the joint strategy $\pi$ as $h^i_t$. We write the preference list provided by the worker $i$ in the last time slot of the evaluation stage as $b_i = \pi_i(h^i_N, \pi)$. The output produced in time slot $t$ by worker $i$ working on task $j = (t + i) \mod N$ is written as $W^e(i, j) = F(i, j)\pi_i(h^i_t)\pi$.

The G-S algorithm executed by the designer at the beginning of the operational stage takes as input the preference lists $\{b_i\}_{i=1}^N$ and the outputs produced by the workers $W^e$. We represent the output of the G-S algorithm as $m^{GS}(\{b_i\}_{i=1}^N, W^e)$, where $m^{GS}$ is a function that takes the preference lists and performance of workers as input and outputs the matching. Note that here again we do not explicitly write the observation history of the designer $h_0^t$. The joint strategy $\pi$ induces an observation history for the designer, which we write as $h_0^t$. Note that $h_0^t$ and $\{b_i\}_{i=1}^N, W^e$ contain the same relevant information needed for the final matching to be determined by G-S algorithm. For consistency we can state that when $t = N + 1$, $m(h_0^{N+1}, \pi) = m^{GS}(\{b_i\}_{i=1}^N, W^e)$. The expression for the long-run utility for the worker $i$ is given as follows.

$$U_i(\{\pi_k\}_{k=1}^N|m, p) = \lim_{T \to \infty} \inf \frac{1}{T+1} \sum_{t=N+1}^T \left[ \alpha F(i, m^{GS}(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^{GS}(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^{GS}(\{b_k\}_{k=1}^N, W^e)[i]) \right] (e_i^t)^2$$

(1)

In the above expression $e_i^t = \pi_i(h^i_t)$. Note that in the above expression we did not write the utility from the evaluation stage because the number of time slots in evaluation stage are finite $N + 1$ and thus utilities in the evaluation stage do not contribute to the long-run utility.

We denote $\lim_{T \to \infty} \sum_{t=0}^T \frac{1}{T+1} (e_i^t)^2 = \overline{e_i^2}$. Then we can simplify the above utility function as

$$U_i(\{\pi_k\}_{k=1}^N|m, p) = \overline{e_i^2} \left[ \alpha F(i, m^{GS}(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^{GS}(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^{GS}(\{b_k\}_{k=1}^N, W^e)[i]) \right]$$

(2)

Next we want to solve for the optimal strategy $\pi_i$ given the fixed strategy of the rest of the workers $\pi_{-i}$.

Formally stated the optimization problem is given as follows.
\[
\max_{\pi_i} U_i(\{\pi_k\}_{k=1}^N | m, p)
\]

We will first compute an upper bound for the expression \( U_i(\{\pi_k\}_{k=1}^N | m, p) \). Observe that
\[
U_i(\{\pi_k\}_{k=1}^N | m, p) = e^\gamma_i \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
\leq (e_i^{\gamma_i})^2 \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
\leq (e_i^{\gamma_i})^2 \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
\leq (e_i^{\gamma_i})^2 \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
(3)
\]

In the above expression the LHS will achieve the same value as the RHS under the following conditions. If for \( t \geq N + 1 \) and \( \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right) \geq 0 \), then \( e_i = e_i^{\gamma_i} \) and \( e_i = 0 \) zero otherwise. We now compute the optimal value for the maximum for the RHS. Note that the expression in RHS depends only on the actions taken in the evaluation stage. Based on the above inequality we can say that the optimizer of the RHS will be an upper bound of the maximization problem in (2). We first maximize the expression in RHS with respect to the choice of preference lists. Recall that \( b_i = \pi_i(h_i^{N,i})[2] \). We now claim that if worker \( i \) ranks the clients in the order of \( \alpha F(i, j)^2 - C(i, j) \) for all \( j \), it then corresponds to the best choice of the preference list. We denote this preference list as \( b_i^* \) and this result follows from Claim 1.

Thus we can write
\[
U_i(\{\pi_k\}_{k=1}^N | m, p) =
\]
\[
e^\gamma_i \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
\leq (e_i^{\gamma_i})^2 \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
\leq (e_i^{\gamma_i})^2 \left( \alpha F(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i])^2 g(m^GS(\{b_k\}_{k=1}^N, W^e)[i]) - C(i, m^GS(\{b_k\}_{k=1}^N, W^e)[i]) \right)
\]
\[
(4)
\]

Next, we will show that if the preference list is fixed for worker \( i \) to \( b_i^* \), then the optimal choice of effort level (that maximizes the RHS of the above expression) is given on task \( j \) in the evaluation stage, which we denote as \( e_i^{\gamma_i} = e_i^{\gamma_i} \). We do so by arguing that the long-run utility of the worker increases in \( e_i^{\gamma_i} \). If the worker increases \( e_i^{\gamma_i} \) to \( e_i^{\gamma_i} + \delta \), then the ranking of the worker by the task \( j \) can either stay the same or increase. Since other parameters remain the same the ranking of the worker \( i \) on other tasks does not change. Now there are three possibilities.

Suppose when the worker exerts effort \( e_i^{\gamma_i} \), then it is matched to task \( j_1 \). It is possible that rank of task \( j_1 \) in the preference list \( b_i^* \) is greater than task \( j \) or equal or lesser.
If the rank of \( j_1 \) is greater than \( j \), then the worker even after increasing effort on task \( j \) will still be accepted by \( j_1 \) as the ranking of the worker for \( j_1 \) and ranking of \( j_1 \) for all workers is not affected by \( e_{ij}^{eval} \). Thus, in this case, increasing the effort \( e_{ij}^{eval} \) will not change the rank of the task that is assigned.

If the rank of \( j_1 \) is equal to \( j \), then by increasing the effort \( e_{ij}^{eval} \) can only improve worker’s ranking for task \( j \). The ranking of worker \( i \) on tasks ranked higher than task \( j \) is still the same, thus worker \( i \) will be rejected by all those tasks. But since the ranking of worker \( i \) on task \( j \) is the same or higher it means that the worker will be assigned to \( j \).

If the ranking of task \( j_1 \) is lesser than the rank of task \( j \), then note that the ranking of worker on task \( j_1 \) will not change and thus the worker will still be accepted by task \( j_1 \) at least. However, since the worker increases effort on task \( j \) the ranking of the worker can improve on task \( j \). This means that it is possible that the worker is accepted by a strictly higher ranked task. Thus we know that increasing effort \( e_{ij}^{eval} \) can lead to the worker being matched to a task with higher or the same rank as before. A task with higher or the same rank will imply a higher or the same value for the long-run utility of the worker. Hence, the \( e_{ij}^{eval} = e_{i}^{\text{max}} \) is the optimal choice at which the upper bound in the RHS is maximized. This holds for all the tasks that the worker \( i \) is matched to for the first time in the evaluation stage. Observe that the proposed bang-bang strategy achieves the value for the upper bound in the RHS thus it has to be the best response for a worker to every strategy of other workers.

The next part of the theorem follows easily from the fact that since all the workers use their best response strategies the joint strategy has to be an equilibrium.

In the last part of the theorem, we need to show that the bang-bang strategy can be computed by the worker in a decentralized manner with the limited information it possesses. In the proof we optimized the expression for long-run utility and arrived at the optimal policy. The derivation of the optimal policy did not require more information than that is available to the worker \( i \). Basically, the worker \( i \) knows that its utility for any general strategy is in the form of the above expression (1). Based on the expression we can see that the worker \( i \) can derive the upper bound as provided in (3). This means the worker can compute in a decentralized manner that in the operational stage it either needs to execute maximum effort or no effort at all depending on the task assigned to it. The worker \( i \) can also compute that the optimal preference list is \( b_i^* \) because the proof of Claim 1 does not require the worker to know that what the actual preferences of other workers and clients are and it only requires the worker to know that the G-S algorithm is executed with workers as suitors/proposers. In the next part of the computation of the strategy we had shown that the worker should always execute maximum effort in the evaluation stage. This can be determined by the worker independent of whether or not the worker knows the preferences of other clients, preferences of other workers or their costs. Thus the above argument shows that the worker can determine the upper bound on the utility in a decentralized manner. Based on the computation of upper bound the worker can also determine the bang-bang strategy in a decentralized manner that will achieve this upper bound.
The claim below follows from Theorem 5 in [Roth (1982)].

**Claim 1** A worker cannot get a strictly better task in the operational stage by misrepresenting its preferences.

**Proof.** Let us assume that the set of workers is given as $\{i_1, \ldots, i_N\}$. If the true preference list is used by worker $i_1$, then assume that the set of tasks that the workers $\{i_1, \ldots, i_N\}$ are matched to is given as $\{j_1, \ldots, j_N\}$ and we refer to this matching as $m_1$. If a different preference list is used by the worker $i_1$, then assume that the set of tasks that workers $\{i_1, \ldots, i_N\}$ are matched to is $\{j_\alpha(1), \ldots, j_\alpha(N)\}$ and we refer to this matching as $m_2$. Note that both the matching $m_1$ and $m_2$ satisfy the definition of stability as in [Gale and Shapley (1962)] w.r.t their corresponding preference lists. Let us assume that the worker $i_1$ prefers $j_\alpha(1)$ to $j_1$, which means for worker $i_1$ $i_1 : j_\alpha(1) > j_1$. We write the preferences for client $j_\alpha(1)$. We are interested in the workers who are ranked above $i_1$. The preference list is given as $j_\alpha(1) : i_{a_1} > \ldots > i_{a_r} > i_1$. In the matching $m_2$ it has to be true that each of the workers $\{i_{a_j}\}_{j=1}^r$ are matched to a client that they prefer more than $j_\alpha(1)$. If the above statement were not true, then clearly the matching $m_2$ is not stable.

We now discuss the execution of the G-S algorithm when the true preference list is submitted by worker $i_1$. In matching $m_1$ that is achieved $i_1$ is matched to $j_1$. This means that the worker $i_1$ must have been rejected by client $j_\alpha(1)$. Thus, this is only possible if at least one of the workers ranked ahead of $j_\alpha(1)$ was rejected by its match in $m_2$. We first formally state this condition and then justify why it ought to hold. We write the preferences for client $j_\alpha(1)$. We are interested in the workers who are ranked above $i_1$. The preference list is given as $j_\alpha(1) : i_{a_1} > \ldots > i_{a_r} > i_1$. First note that there has to be at least one worker ahead of $i_1$ else, the worker would not be rejected in the first place and the matching $m_1$ won’t be achieved. We know from the argument above that every worker $i_{a_j}$ prefers their corresponding match in $m_2$, i.e. $j_\alpha(a_j)$ to $j_\alpha(i_1)$.

It has to be true that at least one of the workers $i_{a_j}$ is rejected by all the clients that it ranks more than $j_\alpha(1)$ before $i_1$ is rejected by $j_\alpha(1)$. Note that again if this statement is not true, then clearly $i_1$ won’t be rejected by $j_\alpha(1)$. This means $i_{a_j}$ has to be rejected $j_\alpha(a_j)$. Again for $i_{a_j}$ to be rejected by $j_\alpha(a_j)$ at least one worker ahead of $i_{a_j}$ needs to be rejected by all the tasks that it prefers more than $j_\alpha(a_j)$ before $i_{a_j}$ is rejected by $j_\alpha(a_j)$ . Thus it means that this worker say $i_{a_j}$ needs to have been rejected by all the tasks that it prefers more than $j_\alpha(a_j)$ . Note that this worker $i_{a_j}$ has to be different than $i_1$, which we justify next. If $i_{a_j} = i_1$ worker, then it will mean that the worker $i_1$ should have been rejected by all the tasks that it prefers more than $j_\alpha(a_j)$, which includes task $j_\alpha(1)$. This will mean that for worker $i_{a_j}$ to be rejected worker $i_1$ will need to be rejected by $j_\alpha(1)$ first. Remember we previously stated for $i_1$ to be rejected by $j_\alpha(1)$, $i_{a_j}$ has to be rejected first. This leads to a contradiction. Therefore, $i_{a_j}$ has to be different than $i_1$. We again apply the same argument as above to $i_{a_j}$. We know that $i_{a_j}$ has to be rejected by $j_\alpha(i_{a_j})$. Thus it will mean that at least one worker ahead of $i_{a_j}$ in the preference list $j_\alpha(i_{a_j})$ needs to be rejected by all the clients that it
ranks above \( \hat{j}_{\alpha(i, a_j)} \). Based on the reasoning showed above we can show that \( \hat{i}_{a_j} \) has to be different than both \( i_1 \) and \( i_{a_1} \). If we repeatedly apply the above argument, then after \( N \) repetitions it will not be possible to find a distinct such worker. Thus it is not possible that the worker \( i_1 \) will be rejected by \( \hat{j}_{\alpha(1)} \).
Appendix B: Proof of Theorem 2.

Before we provide the proof it is important to remind how we define the uniqueness of the equilibrium for our setup. Each equilibrium strategy has a corresponding equilibrium payoff. If for the repeated game that we analyze all the possible equilibrium strategies lead to the same payoff, then we call the equilibrium to be unique. In this Theorem we will assume that the Assumption 1 and 2 hold. We can combine the Assumption 1 and 2 and interpret them together as follows. Each worker has same productivity across tasks, i.e. worker $i$’s productivity is written as $F(i)$ and the same holds true for the cost of each worker across tasks, i.e. worker $i$’s cost is written as $C(i)$. Moreover, the maximum effort is homogeneous across the workers. Also, if $F(i) > F(k)$, then $C(i) < C(k)$. From assumption 1 and 2 we can see that the preference list for each worker in the bang-bang strategy is the same and corresponds to the ranking of the tasks in the order of their qualities. First, we want to show that there does not exist another equilibrium in which at least one worker $i$ can achieve a higher utility than the utility achieved in bang-bang equilibrium. If the workers play the bang-bang strategy and let us denote the matching that is achieved in the operational stage after the execution of G-S algorithm as $m^{BBE}$, where $m^{BBE}[i]$ is the index of the task assigned to worker $i$. Suppose that there exists another equilibrium in which worker $i$ can strictly gain. Clearly, if the worker $i$ strictly gains in comparison to the utility achieved in BBE, it has to be matched to a task that it strictly prefers to $m^{BBE}[i]$, let that task be denoted as $m^{BBE}[j]$. In this new equilibrium we will show that at least one of the workers that were matched to a task ranked higher or equal to $m^{BBE}[j]$ will be matched to a task that it prefers strictly less than the task that it was matched to in the bang-bang equilibrium. Consider the set of the workers who were matched to tasks ranked higher or equal to $m^{BBE}[j]$ in the matching achieved in BBE and suppose that the number of workers in this set are $N_1$. Now suppose that in this new equilibrium each of these workers in this set is matched to the tasks that they strictly prefer to their matches in BBE. However, note that the total number of tasks that are ranked higher than $m^{BBE}[j]$ are $N_1 - 1$. Therefore, it means that for the $N_1$ workers the total number of possible choices for tasks is at most $N_1 - 1$. Hence, it is not possible to assign each of these workers to a strictly higher task.

Suppose that the worker $k$ is the task with the highest productivity among all the tasks that are assigned to a task lower than their corresponding match $m^{BBE}[k]$, where $m^{BBE}[k]$ has a rank greater than or equal to $m^{BBE}[j]$. Let the task assigned to $k$ be $m^{BBE}[l]$. Also, note that the worker who will be assigned to $m^{BBE}[k]$ will have a productivity lower than $k$, let this worker be $r$. We argue that in this new equilibrium worker $k$ must have either not exerted maximum effort on at least one task ranked ahead of $m^{BBE}[l]$ in the evaluation stage or the preference list that it submits must rank $m^{BBE}[l]$ ahead of at least one task that was ranked higher in the preference list in BBE. Suppose that this is not the case, which means that the worker exerts maximum effort on all the tasks ahead of $m^{BBE}[l]$ and the worker will also rank all the tasks that were ahead of $m^{BBE}[l]$ in the preference list in BBE higher than $m^{BBE}[l]$. We argue that since the worker
maximum effort on all the tasks ahead of \( m^{BBE}[l] \) it will be ranked ahead of \( r \) by \( m^{BBE}[k] \) because it has a higher productivity. We also know that worker \( k \) ranks \( m^{BBE}[k] \) ahead of \( m^{BBE}[l] \). Therefore, the matching achieved is not stable w.r.t the preferences of the workers and the clients. This is a contradiction as the matching achieved must be stable as we use the G-S algorithm. Hence, it must be that the worker must have either not exerted maximum effort on at least one task ranked ahead of \( m^{BBE}[l] \) in the evaluation stage or the preference list that it submits must rank \( m^{BBE}[l] \) ahead of at least one task that was ranked higher in the preference list in BBE. Now the worker \( k \) could instead use the bang-bang strategy in this case. Thus the worker will have to first approach all the tasks ahead of \( m^{BBE}[l] \) where it will be accepted by at least one task ranked higher than or equal to \( m^{BBE}[k] \). Suppose that this is not the case, i.e. no task higher or equal to \( m^{BBE}[k] \) accepts \( k \). Observe that in the matching achieved in the BBE the number of tasks that are ranked higher or equal to \( m^{BBE}[k] \) is the same as the number of workers with productivity higher than or equal to \( F(k) \). Based on this observation and the supposition above it has to be true that at least one of the tasks higher or equal to \( m^{BBE}[k] \) accepts a worker with productivity lower than \( F(k) \), i.e. which is preferred lesser than \( F(k) \) (because \( F(k) \) exerts maximum effort). We also know that this task which is matched to a worker lower than \( F(k) \) is also preferred more by worker \( k \) than its current match. Therefore, the matching that is achieved is not stable. This is a contradiction because the matching achieved by the G-S algorithm has to be stable. Hence, it must be true that using the bang-bang strategy the worker \( k \) is accepted by a task that is at least as high as \( m^{BBE}[k] \). If \( \alpha F(k)^2 g(m^{BBE}[k]) - C(k) > 0 \), then the worker \( k \) will exert maximum effort in the operational stage and thus using the bang-bang strategy in the above equilibrium the worker \( k \) will have a profitable deviation. This contradicts the fact that the above setting corresponds to an equilibrium. Now if \( \alpha F(k)^2 g(m^{BBE}[k]) - C(k) \leq 0 \), then the worker will have no incentive to exert maximum effort. In this case, the deviation is not profitable. But since \( \alpha F(k)^2 g(m^{BBE}[k]) - C(k) \leq 0 \) and we know that worker \( i \) is matched to task \( j \), where \( m^{BBE}[k] \geq m^{BBE}[j] \), it has to be true that \( \alpha F(k)^2 g(m^{BBE}[j]) - C(k) \leq 0 \). From the output of G-S algorithm in the bang-bang equilibrium, we know that \( F(j) < F(k) \). From assumption 1 and assumption 2 we know that since \( F(j) < F(k) \) and \( C(j) > C(k) \). Therefore, \( \alpha F(j)^2 g(m^{BBE}[j]) - C(j) \leq 0 \) which means the utility achieved by the worker \( j \) is 0. Thus this contradicts the fact that we assumed that in another equilibrium worker \( i \) can strictly benefit. Hence, there cannot be another equilibrium in which a worker gains strictly in comparison to BBE.

Up till now we have shown that there cannot exist another equilibrium in which at least one worker gets a strictly higher payoff than BBE. Next, we argue that there cannot be another equilibrium in which at least one worker gets a strictly lower payoff than in BBE. We can develop the proof for this exactly on the same line as the above. Suppose that there is a worker \( k \) with the highest productivity among the workers who are assigned to a task strictly lower than their corresponding match in BBE. Thus we can observe that the worker matched to \( m^{BBE}[k] \) will have a productivity lower than \( k \). For the worker \( k \) to be strictly worse off it has to be true that \( \alpha F(k)^2 g(m^{BBE}[k]) - C(k) > 0 \). Suppose that the worker \( k \) was matched to
\( m^{BBE}[l] \), where \( m^{BBE}[l] < m^{BBE}[k] \). We can argue that it has to be true that the worker \( k \) either did not exert maximum effort on at least one task that is ranked higher than or equal to \( m^{BBE}[l] \) or it submitted a preference list in which the task \( m^{BBE}[l] \) was ranked ahead of at least one of the tasks that was ahead of it in the preference list in the bang-bang strategy. Suppose the above is not true then clearly if the worker ranks every task that has a higher quality than \( m^{BBE}[l] \) ahead of \( m^{BBE}[l] \) in the preference list and also suppose that the worker exerts maximum effort on all the tasks that are higher or equal to \( m^{BBE}[l] \). In this case, it has to be true that the matching that is achieved is not stable, which is a contradiction. The matching is not stable because the worker who is assigned to \( m^{BBE}[k] \) is preferred less by the task to \( k \) and the task \( k \) also prefers \( m^{BBE}[k] \) to \( m^{BBE}[l] \). Hence, for the above to be an equilibrium it has to be true that the worker \( k \) either did not exert maximum effort on at least one task that is ranked higher than or equal to \( m^{BBE}[l] \) or it submitted a preference list in which the task \( m^{BBE}[l] \) was ranked ahead of at least one of the tasks that was ahead of it in the preference list in the bang-bang strategy. Let us consider the case in which the worker \( k \) deviates to the bang-bang strategy. Then on the similar lines as the argument above we can show that at least one worker higher than or equal to \( m^{BBE}[k] \) will have to accept \( k \). If this is not true, then at least one of the tasks that is higher than or equal to \( m^{BBE}[k] \) will have to accept a worker of productivity lower than \( F(k) \) and such a matching will not be stable. Thus deviating to bang-bang strategy will lead to a strictly profitable deviation. Hence, there will be no equilibrium in which at least one worker gets strictly lower payoff than the BBE. We can conclude that in every equilibrium each worker will have the same payoff as in the BBE.
Appendix C: Proof of Proposition 2.

We write the set of outputs as follows \( \{ F(1)e_{max}^1, \ldots, F(N)e_{max}^N \} \) and we write the outputs sorted in the increasing order as follows \( \{ F(m_1)e_{max}^1, \ldots, F(m_N)e_{max}^N \} \). Let us first establish the upper bound on the output. First, we will compute an upper bound on the total revenue that can be generated in one period. Clearly, the revenue generated is monotonic in the effort exerted by any worker. Since we are computing the upper bound here we will assume that each worker exerts maximum effort. Each worker \( i \) should exert maximum effort \( e_{max}^i \) otherwise the effort can always be increased to improve the output. Consider a general matching \( m': N \rightarrow S \), where \( m'[i] \) is the task allocated to worker \( i \).

We can write the total revenue for this matching \( m' \) as follows \( \sum_{i=1}^{N} F(i)e_{max}^i g(m'[i]) \). The inequality given below is a consequence of the rearrangement inequality.

\[
\sum_{i=1}^{N} F(i)e_{max}^i g(m'[i]) \leq \sum_{i=1}^{N} F(m_i)e_{max}^i g(i), \forall M'
\]

Therefore, we can also write the following for every matching rule \( m \) and joint strategy \( \pi \) as defined in Section 2.

\[
\sum_{i=1}^{N} r_i(h_0^i, h_i^i, \pi_i|m) \leq \sum_{i=1}^{N} F(m_i)e_{max}^i g(i)
\]

The above holds true because \( r_i(h_0^i, h_i^i, \pi_i|m) = F(i)g(m(h_0^i))\pi_i(h_i^i) \leq F(i)g(m(h_0^i)|i)e_{max}^i \) and \( m' = m(h_0^i) \). Note that the upper bound is same for each time slot, the same upper bound continues to hold for the long-run average too.

Next we compute the output achieved by the proposed mechanism provided all the workers follow the bang-bang strategy. We know that a worker \( i \) will exert \( e_{max}^i \) in the operational stage if and only if \( \alpha F(i)^2 g(m_{BBE}^i) - c_i \geq 0 \), where \( m_{BBE} \) is the matching achieved in the bang-bang equilibrium. Hence, the total long-run average revenue is given as follows (Note that the revenue from the evaluation stage does not matter for the computation of the long-run average).

\[
\sum_{i=1}^{N} F(m_j)e_{max}^j g(j)I(\alpha F(m_j)^2 g(j) - c_j \geq 0)
\]
Appendix D: Proof of Theorem 3.

We know that $c^\text{max}$ is the upper bound on the cost for any worker. Therefore, if there exists a worker $m_k$ for which $\alpha F(m_k)^2 g(k) - c^\text{max} \geq 0$, then all the workers that work on task $F(m_j) \geq F(m_k)$ and follow the bang-bang strategy will exert maximum effort in the operational stage.

If we substitute $\alpha = \alpha^* = \frac{c^\text{max}}{2g^\text{min} f^\text{max}}$ in $\alpha F(m_k)^2 g(k) - c^\text{max} \geq 0$, then we get the following condition $F(m_k)^2 \geq \frac{2g^\text{min} f^\text{max}}{g(k)}$. Since $g^\text{min} g(k) \leq 1$ it is sufficient that if a worker has a productivity greater than $\sqrt{2f^\text{max}}$, then the worker should exert maximum effort. We can compute the lower bound on the expected revenue achieved in the BBE as follows.

$$E \left[ \sum_{i=1}^{N} F(m_i) g(i) e^\text{max}_{m_i} I(\alpha F(m_i)^2 G(i) - C(m_i) \geq 0) \right] \geq e^\text{max} g^\text{min} E \left[ \sum_{i=1}^{N} F(m_i) I(F(m_i) \geq \sqrt{2f^\text{max}}) \right]$$

$$= e^\text{max} g^\text{min} \sum_{i=1}^{N} E \left[ F(i) I(F(i) \geq \sqrt{2f^\text{max}}) \right]$$

We now compute

$$E[F(i) I(F(i) \geq \sqrt{f^\text{max}})] = \frac{1}{f^\text{max}} \int_{\sqrt{2f^\text{max}}}^{f^\text{max}} x dx = \frac{f^\text{max} - 2}{2}$$

Therefore,

$$E \left[ \sum_{i=1}^{N} F(m_i) G(i) e^\text{max}_{m_i} I(\alpha F(m_i)^2 G(i) - C(m_i) \geq 0) \right] \geq e^\text{max} g^\text{min} N \frac{f^\text{max}}{2} - 1$$

Based on the same method we can get the upper bound long-run output

$$N \frac{f^\text{max}}{2} g^\text{max} e^\text{max}$$

We can take the ratio of the two expressions above and get the final result as follows.

$$\Theta = \frac{g^\text{min}}{g^\text{max}} (1 - \frac{2}{f^\text{max}})$$
Appendix E: Proof of Theorem 4.

Before we give the Proof for Theorem 4 we first need to simplify and arrive at the expressions for the long-run utilities for the workers and clients as given in Section 2.2.3. We are considering the matching functions for which the limit exists across all the histories. Suppose the joint strategy being used is $\pi$. Under this joint strategy the limit of the matching function is given as $m^*_\pi$. The history that is induced by the joint strategy $\pi$ is defined as $h^t_0$ and the $h^t_i$ for worker $i$. Note that the $\lim_{t \to \infty} m(h^t_i, \pi) = m^*_\pi$, where the limit is defined with the standard euclidean norm in the space $\mathbb{R}^N$. Next, we will show that the above limit is attained after a finite number of time slots denoted as $T_{lim}$. Note that the minimum distance between any two distinct matchings is finite and is given as $d_{min}$. From the definition of limit it is clear that after time $t \geq T_{lim}$ the distance between $m(h^t_i, \pi)$ and $m^*_\pi$ is less than $d_{min}$. Therefore, for all $t \geq T_{lim}$

\[
m(h^t_0, \pi) = m^*_\pi
\]

Based on the above simplification we can write the long-run utility of a worker $i$ and client $x = m^*_\pi[i]$ as follows.

\[
U_i(\pi) = \lim_{T \to \infty} \frac{1}{T + 1} \sum_{t=0}^{T} \left[ \frac{1}{T + 1} p(F(i, x)) \pi_i(h^t_i, \pi) - C(i, x) \pi_i(h^t_i, \pi)^2 \right]
\]

Similar justification applies for the clients’ long-run utilities as well.

We write the matching achieved in BBE at the start of the operational stage as $m^{BBE}$ (shorthand for $m^{*}_{\pi, BBE}$ introduced in Section 2.2.3). The long-run utility for worker $k$ in BBE is given as

\[
\left[ \alpha F(k, m^{BBE}[k])^2 g(m^{BBE}[k]) - C(k, m^{BBE}[k]) \right] (e^\text{max}_k)^2 I(\alpha F(k, m^{BBE}[k])^2 g(m^{BBE}[k]) - C(k, m^{BBE}[k]) \geq 0)
\]

The long-run utility for the client $m^{BBE}[m]$ in BBE is given as follows, where $m \neq k$.

\[
(1 - \alpha F(m, m^{BBE}[m]) e^\text{max}_m) I(\alpha F(m, m^{BBE}[m])^2 g(m^{BBE}[m]) - C(m, m^{BBE}[m]) \geq 0) F(m, M^{GS}[m]) e^\text{max}_m
\]

Consider the setup where worker $k$ is matched to client $m^{BBE}[m]$ instead in the operational stage. Our objective here is to show that it is not possible for both worker $k$ and client $m^{BBE}[m]$ to increase their long-run utilities by being matched to one another. If the utility for the worker $k$ strictly increases, then it has to hold true that (\alpha F(k, m^{BBE}[m])^2 g(m^{BBE}[m]) - C(k, m^{BBE}[m])) has to be strictly higher than (\alpha F(k, m^{BBE}[k])^2 g(m^{BBE}[k]) - C(k, m^{BBE}[k])). This has to hold
true because otherwise if \([\alpha F(k, m_{BBE}[m])^2g(m_{BBE}[m]) - C(k, m_{BBE}[m])]\) is strictly lower than 
\([\alpha F(k, m_{BBE}[k])^2g(m_{BBE}[k]) - C(k, m_{BBE}[k])]\), then the maximum utility that worker \(k\) can achieve
by getting matched to \(m_{BBE}[k]\) will always be lesser than the long-run utility that the worker can achieve
by getting matched to task \(m_{BBE}[k]\) in the operational stage of the BBE.

Next, we compute the utility for worker \(k\) when it is matched to \(m_{BBE}[m]\) and the strategy \(\pi_k\)

\[
(\alpha F(k, m_{BBE}[m])^2g(m_{BBE}[m]) - C(k, m_{BBE}[m])) \times \\
I(\alpha F(k, m_{BBE}[m])^2g(m_{BBE}[m]) - C(k, m_{BBE}[m]) \geq 0) \lim_{T \to \infty} \frac{\sum_{t=N,T}^{T} \pi_k(h_k^t)^2}{T+1}
\]

Also, the utility for the client \(m_{BBE}[m]\) in this case is derived as follows.

\[
\lim_{T \to \infty} \frac{1}{T+1} \left[1 - \alpha F(k, m_{BBE}[m])\pi_k(h_k^t)\right] \times \\
I(\alpha F(k, m_{BBE}[m])^2g(m_{BBE}[m]) - C(k, m_{BBE}[m]) \geq 0) F(k, m_{BBE}[m])\pi_k(h_k^t)
\]

Based on the G-S algorithm executed in BBE we know that the rank of worker \(m\) is higher than the rank
of worker \(k\) for task \(m_{BBE}[m]\). From the homogeneity of the maximum effort and Assumption 1, we have
the following.

\[
F(m, m_{BBE}[m])e_{max} \geq F(k, m_{BBE}[m])e_{max} \implies F(m, m_{BBE}[m]) \geq F(k, m_{BBE}[m]) \implies \\
C(k, m_{BBE}[m]) \geq C(m, m_{BBE}[m])
\]

We can show that the following has to hold based on the above.

\[
\alpha F(m, m_{BBE}[m])^2g(m_{BBE}[m]) - C(m, m_{BBE}[m]) \geq \alpha F(k, m_{BBE}[m])^2g(m_{BBE}[m]) - C(k, m_{BBE}[m])
\]

\[
I(\alpha F(m, m_{BBE}[m])^2g(m_{BBE}[m]) - C(m, m_{BBE}[m]) \geq \\
I(\alpha F(k, m_{BBE}[m])^2g(m_{BBE}[m]) - C(k, m_{BBE}[m]) \geq 0)
\]
Observe that the function \((1 - \alpha x)x\) is increasing in \([0, \frac{1}{2\alpha}]\). We assumed that \(\alpha \leq \frac{1}{2W_{max}}\). Therefore, \((1 - \alpha x)x\) is increasing in \(x \in [0, W_{max}]\). We can use the above relations to derive the following condition.

\[
[F(k, m_{BBE}[m])e_k - \alpha F(k, m_{BBE}[m])^2 e_k^2] I(\alpha F(k, m_{BBE}[m])^2 g(m_{BBE}[m]) - C(k, m_{BBE}[m]) \geq 0) \leq \\
[F(k, m_{BBE}[m])e_{max}^k - \alpha F(k, m_{BBE}[m])^2 (e_{max}^k)^2] I(\alpha F(k, m_{BBE}[m])^2 g(m_{BBE}[m]) - C(k, m_{BBE}[m]) \geq 0) \leq \\
[F(m, m_{BBE}[m])e_{max}^m - \alpha F(m, m_{BBE}[m])^2 (e_{max}^m)^2] I(\alpha F(m, m_{BBE}[m])^2 g(m_{BBE}[m]) - C(m, m_{BBE}[m]) \geq 0)
\]

In the above we use the fact

\[W_{max} \geq F(m, m_{BBE}[m])e_{max} \geq F(k, m_{BBE}[m])e_{max} \geq F(k, m_{BBE}[m])\pi_k(h_k^l)\]

This means the client \(m_{BBE}[m]\) cannot have a strict gain at the same time as the worker \(k\). This means that the proposed matching has to be long-run stable.
Appendix F: Details of the simulation setup in Section 2.

In the numerical simulation setup, we assume that the workers’ productivities are homogeneous (as described in Assumption 2.). Each worker productivity is drawn from a uniform distribution, where $U \sim [0, 40]$ (as described in Assumption 4 i)). The cost for exerting effort for a worker $i$ is assumed to be given as $C(i) = 80 - F(i)$, where the cost for exerting the effort decreases with increase in productivity (as described in Assumption 1.). We assume that the maximum effort exerted by the workers is homogeneous, where Assumption 3 holds. The maximum effort each worker can exert is $e^{max} = 25$. Each task’s quality is drawn from a uniform distribution with $U \sim [25, 40]$. The number of workers and the number of tasks are equal. In the comparison, we take the expected total long-run revenue, where the expectation is computed over the joint distribution of the workers’ productivities and task qualities. For the proposed payment rule, we use $\alpha = 1/(f^{max}(g^{max} - g^{min}))$. For the linear payment rule, we assume that the client pays the worker as follows. If the worker $i$ generates output $w$ while working on task $x$, then the payment made to the worker is equal to the revenue generated by the worker and is given as follows $wg(x)$.
Appendix G: Extensions

Repeated matching mechanism design in a stochastic setting. In this section we will extend the model to incorporate the feature that the output produced may not only depend on the productivity and effort as there may be other uncertain forces that impact the output observed. We will assume that the output of the worker will stochastically depend on the worker’s productivity, the effort exerted and the task quality. We assume an additive noise based model to quantify the output produced. Suppose worker $i$ works on task $j$ and exerts effort $e_i$ then the revenue generated is given as

$$F(i,j)e_i g(j) + Z(i,j),$$

where $Z(i,j)$ a random variable representing noise when worker $i$ works on task $j$. We assume that $Z(i,j)$ has zero mean and has a finite variance $\Sigma(i,j)^2$, where $\Sigma$ is the matrix of variances (We do not need further assumptions on the distribution of $Z(i,j)$). The random noises in the output of different workers are mutually independent random variables. We assume that the mean and the variance of $Z(i,j)$ is known to the designer and the clients. Also, the noise random variables in different time slots are i.i.d.

In the model for a non-stochastic setting, we defined the matching and the payment rule in a general manner and they depended on the history of the observations made by the designer. The definition for the matching and the payment rule remain the same in this part but we need to be careful that the history, in this case, will be stochastic. The mechanism has a total of $T + 1$ time slots.

**Matching rule:** The matching rule for this case is given as $m^T : \mathcal{H}_0^{T+1} \rightarrow \Pi(S)$. $m^T(h^*_i)[i]$ denotes the task assigned to worker $i$ following history $h^*_i$. We use superscript $T$ over the matching rule $m^T$ to make the distinction with the mechanism with infinite number of time slots $T = \infty$.

**Payment rule:** The payment rule for this case is given as $p^T : [0, \infty) \times S \rightarrow \mathbb{R}$. $p^T(w,j)$ denotes the amount paid by the client $j$ to a worker for producing output $w$.

**Utility of the workers and the clients:** The worker’s strategy $\pi^T_i : \mathcal{H}_i^{T+1} \rightarrow \{0, \delta, 2\delta, ..., e^{\max}_i\}$ is a mapping from histories of length less than $T + 1$ to effort levels. a worker $i$ exerts effort $e^t_i$ in time slot $t$ following a private history $h^t_i$ and the designer’s observation history $h^*_i$ and generates revenue

$$r_i(h^*_0, h^t_i, \pi^T_i|m) = \left[ F(i, m^T(h^*_0)[i])e^t_i + Z^t_{im^T(h^*_0)[i]} \right] g(m^T(h^*_0)[i])$$

The total expected output when there are $T + 1$ time slots in the mechanism is given as

$$\bar{R}((\pi^T_i)_{i=1}^N|m^T) = E_Z\left[ \frac{1}{T+1} \sum_{t=0}^{T} \sum_{i=1}^{N} r_i(h^*_0, h^t_i, \pi^T_i|m^T) \right]$$

The expectation $E_Z$ in the above expression is taken with respect to the joint distribution of the noise random variables for the $T + 1$ time slots. The utility of the worker $i$ following a private history $h^t_i$ and the designer’s history of $h^*_i$ when it exerts effort $e^t_i$ is
\[
\tilde{U}_i(\{\pi^t_k\}_{k=1}^N | m^T, p^T) = \frac{1}{T+1} \sum_{t=0}^T u_i(\tilde{h}_0^t, \tilde{h}_i^t, \pi^t_i | m^T, p^T)
\]

The expected long-run utility for worker \(i\) when there is a total of \(T + 1\) time slots in the mechanism is

\[
\tilde{U}_i(\{\pi^t_k\}_{k=1}^N | m^T, p^T) = \mathbb{E}_Z[\frac{1}{T+1} \sum_{t=0}^T u_i(\tilde{h}_0^t, \tilde{h}_i^t, \pi^t_i | m^T, p^T)]
\]

The expectation \(\mathbb{E}_Z\) in the above expression is taken with respect to the joint distribution of the noise random variables for the \(T + 1\) time slots. Similarly, we can define the utility for the clients as well. The utility for the client \(j\) who is matched to a worker \(i\) with a private history \(h_i^t\) and when the designer has an observation history of \(h_0^{t+1}\).

\[
v_j(h_0^t, h_i^t, \pi_i^t | m^T, p^T) = r_i(h_0^t, h_i^t, \pi_i | m) - p^T(F(i, m^T(h_0^t))[i]e_i^t, m^T(h_0^t)[i])
\]

The expected long-run utility for client \(j\) when there are a total of \(T + 1\) time slots in the mechanism is

\[
\tilde{V}_j(\{\pi^t_k\}_{k=1}^N | m^T, p^T) = \mathbb{E}_Z[\frac{1}{T+1} \sum_{t=0}^T v_j(h_0^{t+1}, h_i^t, \pi_i^t | m^T, p^T)]
\]

**Knowledge of the designer, the workers, and the clients:** The knowledge structure is the same as in Section 2.

**Objective of the mechanism design problem:** The objective of the designer and the workers can be defined in the same way as in Section 2.

\[
\max_{m^T, p^T} \min_{\{\pi_k^T\}_{k=1}^N} \tilde{R}(\{\pi_k^T\}_{k=1}^N | m^T, p^T) \\
\text{s.t.} \quad \tilde{V}_j(\{\pi_k^T\}_{k=1}^N | m^T, p^T) \geq 0, \ \forall j \in \mathcal{S} \text{ (IR-clients)} \\
\quad \tilde{U}_i(\{\pi_k^T\}_{k=1}^N | m^T, p^T) \geq 0 \ \forall i \in \mathcal{N} \text{ (IR-workers)} \\
\quad \tilde{U}_i(\pi_i^T, \{\pi_k^T\}_{k=1,k \neq i}^N | m^T, p^T) \geq \tilde{U}_i(\pi_i^T, \{\pi_k^T\}_{k=1,k \neq i}^N | m^T, p^T) \ \forall i \in \mathcal{N} \ \forall \pi_i^T \in \Pi_i^T \text{ (IC-workers)}
\]

In the above \(\Pi_i^T\) is the set of all the possible strategies for worker \(i\) when the mechanism has finite number of time slots \(T + 1\). We now describe the proposed mechanism, which we refer to as the modified mechanism because as we will see it is a natural extension of the mechanism proposed in Section 2.

**Modified Mechanism.** In this section, we propose what we call “initial impression is the last impression” (ILLI) mechanism. We used the phrase “first impression” to address the mechanism in the first part when there is no stochasticity and “initial impression” for this case with stochasticity because in the latter case we will need to develop an estimate of the output over multiple initial observations. The ILLI mechanism has two components:
A) Payment rule: We assume that the client follows a similar payment rule as in the first part. Suppose that the worker \(i\) when matched to task \(j\) generates revenue (stochastic) \(R(i,j)\). In this case the client will pay the worker \(\alpha R(i,j)^2/g(j) - \alpha \Sigma(i,j)^2/g(j)\) per unit output.

If the designer uses the same matching rule as proposed in the first part, then the output and the profits can be very bad because the noise in the output can lead to designer identifying the incorrect workers as high productivity ones. We propose a natural extension of the mechanism proposed in the first part. The first stage, i.e. the evaluation stage, will now consist of multiple time slots to evaluate each worker on a particular task. In this stage, the designer will develop an estimate for the normalized outputs of the workers and then use it in the next stage to match them.

B) Matching rule:

- **Evaluation stage \((0 \leq t \leq N\sqrt{T})\):** We will assume that \(T\) is sufficiently large \((T \geq N^2)\) because we will need sufficient number of time slots for evaluation. This stage comprises of the first \(N\sqrt{T} + 1\) time slots. We assume that \(\sqrt{T}\) is an integer for convenience. This stage is divided into phases each of length \(\sqrt{T}\). In phase-\(k\) worker \(i\) is matched to each task \([((k + i) \mod N]\) for \(\sqrt{T}\) time slots. The designer computes the sample mean estimate of the output produced by the worker \(i\) on task \(j\) in phase-\(i\) over the \(\sqrt{T}\) time slots. We write the matrix of the estimates of the outputs as \(\hat{W}^e\), where \(\hat{W}^e(i,j)\) is the estimate of output of worker \(i\) on task \(j\). At the end of the stage in time slot \(t = N\sqrt{T}\), the designer requests the workers to submit their preferences in the form of ranks for tasks. The designer also computes the ranking of workers based on the estimates of normalized outputs \(\hat{W}^e\) as follows. For every task \(j\) the designer orders the workers based on the estimates of normalized outputs for each task \(j\) \(\{\hat{W}^e(i,j)\}_{i=1}^N\).

- **Operational Stage \((N\sqrt{T} + 1 \leq t \leq T)\):** The aim of this stage is to compute the matching for workers based on the observations made in the evaluation stage. The designer computes the matching based on the G-S algorithm as follows. The workers are the proposers and the tasks are the acceptors. In every step, each worker proposes to its favorite task that has already not rejected it. Each task, based on the proposals it gets keeps its favorite worker on hold and rejects the rest. At the end of \(N^2 - 2N + 2\) steps, all the workers the matching that is achieved is final for all the remaining time slots in the mechanism. Each worker then decides the amount of effort to exert on the task assigned in every time slot. The task that a worker is matched to remains the same for the time slots to follow.

**Proposition 3.** In the modified mechanism the workers are always willing to participate. If \(\alpha \leq \frac{1}{2W_{\text{max}}}\), then all the requesters are willing to participate in the mechanism.

The proof of the above proposition works on similar lines as Proposition 1.

The above mechanism induces a stochastic repeated endogenous matching game. In the next section, we propose a strategy, which we will prove will be an equilibrium strategy for the workers.
Stochastic bang-bang strategy

We will propose a strategy, which we refer to as the stochastic bang-bang strategy, that each worker should follow to maximize its long-run utility under the modified mechanism. The stochastic based bang-bang strategy for worker $i$ $\pi_i^{T,\text{bang}}$ is defined as:

- **Evaluation stage**: In each time slot in the evaluation stage the worker $i$ should exert the maximum effort $e_i^{\text{max}}$. Agent $i$ observes the output in time slot $t$. The worker knows the quality of the task assigned to it and the effort that it exerts, which allows the worker to estimate its productivity as follows. The worker is able to observe the revenue generated and it knows the quality of the task and the effort exerted. Hence, the ratio of the output and the task quality times the effort exerted gives an estimate for the productivity in one time period. For $(k - 1)\sqrt{T} \leq t \leq k\sqrt{T}$ we define the match for the worker $i$ is $j = (k + i) \mod N$

$$\tilde{F}(i, j)^t = \frac{r_i(h_i^t, h_0^t, \pi_i^{T,\text{bang}}|m^T)}{g(j)e_i^{\text{max}}}$$

The worker at the end of evaluation stage (after $N\sqrt{T}$ time slots) computes the following estimate for its productivity on every task $j$

$$\hat{F}(i, j) = \frac{1}{\sqrt{T}} \sum_{t=0}^{N\sqrt{T}-1} \tilde{F}(i, m(h_0^t)[i]|m(h_0^t)[i] = j)$$

Based on the above estimates computed by the worker $i$. For each task $j$ worker $i$ computes $\alpha(\hat{F}(i, j))^2G(j) - C(i, j)$ and ranks the tasks based on this computed value. Agent provides the designer with the rankings that it computes in time slot $t = N\sqrt{T}$.

- **Operational stage**: Suppose worker $i$ is allocated a task with index $r$ (based on the G-S algorithm). If $\alpha(\hat{F}(i, r))^2G(r) - C(i, r) > 0$, then the worker exerts maximum effort in every time slot to follow, otherwise the worker exerts zero effort in every time slot.

Next, we will show that the long-run utility achieved by the above strategy is $\epsilon$ close to the optimal.

**Theorem 5.** Stochastic Bang-Bang Equilibrium (S-BBE)

If $T \geq \frac{N^2\Gamma}{\epsilon}$, then

1. The stochastic bang-bang strategy for a worker is the $\epsilon$ best response to other workers’ strategies.
2. If all the workers follow the stochastic bang-bang learning strategy, then it comprises an $\epsilon$ equilibrium referred to as the S-BBE.

**Proof of Theorem 5:** We start by proving the first part. Note that the history of the workers and the designer, in this case, are random variables because the revenue generated is stochastic. We will write the history for worker $i$ as a random variable $H_i^t$ and for the designer as $H_0^t$. We will define the noise random variables and the other random variables to be used in the problem over the following probability space.
\{ \Omega, \mathcal{F}, P \}$ with $\Omega$ as the sample space, $\mathcal{F}$ as the sigma field of events and $P$ as the probability measure. Noise in the output at time $t$ for worker $i$ working on task $j$ is given as the random variable $Z_{ij}^t: \Omega \to \mathbb{R}$, whose mean is zero and variance is $\sigma_{ij}^2$. The random variables $\{Z_{ij}^t, i \in \{1, \ldots, N\}, j \in 1, \ldots, N \}$ are mutually independent. The random variables across time $Z_{ij}^t$ and $Z_{ij}'$ have identical distribution as well.

We define the histories for the worker and the designer as follows. For worker $i$ at time $t$ $H_i^t$ is the random variable that contains the set of observations upto time $t$. For the designer at time $t$ $H_0^t$ is the random variable that contains the set of observations upto time $t$. We initialize $H_i^0 = \phi$ and $H_0^0 = \phi$. The worker observes the stochastic output given as

$$
\tilde{W}_i(H_i^t, H_0^t, \pi_i^T|m^T) = F(i, m^T(H_0^t)[i])\pi_i^T(H_i^t) + Z_{im^T(H_0^t)[i]} \frac{1}{g(m^T(H_0^t)[i])}
$$

The task quality of the task assigned to worker $i$ in time slot $t$ given as $g(m^T(H_0^t)[i])$. The corresponding revenue for the worker is given as

$$
\tilde{R}_i(H_i^t, H_0^t, \pi_i^T|m^T) = \tilde{W}_i(H_i^t, H_0^t, \pi_i^T|m^T)g(m^T(H_0^t)[i])
$$

We denote a realization of random variable normalized output when $\omega$ is the outcome as.

$$
\tilde{W}_i(H_i^t(\omega), H_0^t(\omega), \pi_i^T|m^T) = F(i, M^T(H_0^t(\omega))[i])\pi_i^T(H_i^t(\omega)) + Z_{im^T(H_0^t(\omega))[i]} \frac{1}{g(m^T(H_0^t(\omega))[i])}
$$

The utility of the worker for the particular time slot $t$ is given as $u_i(H_i^t, H_0^t, \pi_i^T|m^T)$. Hence, the history for worker $i$ in time slot $t+1$ is given as $H_i^{t+1} = H_i^t \cup \{\tilde{R}_i(H_i^t, H_0^t, \pi_i^T|m^T)\}$. The designer observes the normalized output for all the workers and its history is given as $H_0^{t+1} = H_0^t \cup \{\{\tilde{R}_i(H_i^t, H_0^t, \pi_i^T|m^T)\}, g(m^T(H_0^t)[i]) \}_{i=1}^N$. Having defined the strategies and the histories, we can define the utility of the worker for the $T$ stage mechanism.

Let us first distinguish how does the matching function work in the two stages.

In the evaluation each task for $\sqrt{T}$ time slots. We will first compute an upper bound for the expected utility in the evaluation stage. We will first show that the highest expected utility a worker $i$ can get in one time slot is given as

$$
\max_j (\alpha F(i, j)^2 g(j) - C(i, j))I(\alpha F(i, j)^2 g(j) - C(i, j)^2 \geq 0)(e_i^{\max})^2
$$

This is derived as follows. We first write the expected utility in a particular time slot $t$.

$$
\alpha E[\tilde{R}_i(H_i^t, H_0^t, \pi_i^T|m^T)^2] - \alpha \frac{\sigma_{im^T(H_0^t)[i]}^2}{g(m^T(H_0^t)[i])} - C(i, m^T(H_0^t)[i])\pi_i(H_i^t)^2
$$
We now simplify the first term in the above expression.

\[
E\left[ \frac{\hat{R}_t(H_t^i, H_{0_t}, \pi_t^i [m^T]^2)}{g(m^T(H_{0_t}^i)[i])} \right] = E\left[ F(i, m^T(H_{0_t}^i)[i])^2 \pi_t^i (H_t^i)^2 g(m^T(H_{0_t}^i)[i]) + 2Z_{im^T(H_{0_t}^i)[i]} F(i, m^T(H_{0_t}^i)[i]) \pi_t^i (H_t^i)^2 \right] + \frac{Z_{im^T(H_{0_t}^i)[i]}^2}{G(m^T(H_{0_t}^i)[i])}
\]

In the above expression there are three random variables. We can show that the expectation of all the three of them will exist. In the first term the random variable only takes finitely many values, which means the expectation has to exist. For the second random variable we know that the expectation of \( E[Z_{ij}] \) exists for all \( j \) and is equal to zero. We use the iterated expectation property to show that expected value of the second term is zero.

\[
E[Z_{im^T(H_{0_t}^i)[i]} F(i, m^T(H_{0_t}^i)[i]) \pi_t^i (H_t^i)] = E[E[Z_{im^T(H_{0_t}^i)[i]} | H_0^i, H_t^i] \pi_t^i (H_t^i) F(i, m^T(H_{0_t}^i)[i])]
\]

The term \( E[Z_{im^T(H_{0_t}^i)[i]} | H_0^i, H_t^i] \) has to equals zero because \( E[Z_{ij}] = 0 \) for all \( j \). Similarly we can simplify the following

\[
E[Z_{im^T(H_{0_t}^i)[i]}^2] = E[E[Z_{im^T(H_{0_t}^i)[i]} | H_0^i] \frac{1}{g(m^T(H_{0_t}^i)[i])}]
\]

Hence, we get the following.

\[
E[F(i, m^T(H_{0_t}^i)[i])^2 \pi_t^i (H_t^i)^2 g(m^T(H_{0_t}^i)[i])] + \frac{\sigma_{im^T(H_{0_t}^i)[i]}^2}{g(m^T(H_{0_t}^i)[i])}
\]

We substitute the above, in the previous expression to get the following.

\[
E[(\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i]) \pi_t^i (H_t^i)^2]
\]

Now we can get an upper bound on the above expression by computing the maximum value that the random variable \( (\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i]) \pi_t^i (H_t^i)^2 \) can take. Observe that if \( (\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i]) \geq 0 \), then the expression is bounded above by \( (\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i])(\epsilon_{\text{max}})^2 \) and if \( (\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i])) < 0 \), then the expression is bounded above by zero. So the upper bound is given as \( (\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i]))) I(\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i]) \geq 0)(\epsilon_{\text{max}})^2 \)

Further \( (\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i])) I(\alpha F(i, m^T(H_{0_t}^i)[i])^2 g(m^T(H_{0_t}^i)[i]) - C(i, m^T(H_{0_t}^i)[i]) \geq 0) \) is bounded above by \( \max_j (\alpha F(i,j))^2 g(j) - C(i,j)) I(\alpha F(i,j)^2 g(j) - C(i,j)) \geq 0 \).

This justifies the upper bound that we give above.
We next write the expected utility for worker $i$ when there is a total of $T + 1$ slots in the mechanism. We will decompose the utility in two parts: first part is the contribution from the evaluation stage and the second part is the contribution from the operational stage.

$$U_i^T(\{\pi_j^T\}_{j=1}^N|m^T) = U_i^{\text{rank},T}(\{\pi_j^T\}_{j=1}^N|m^T) + U_i^{\text{opnl},T}(\{\pi_j^T\}_{j=1}^N|m^T)$$

The first term in the above expression will decay (see the explanation below).

$$U_i^{\text{rank},T}(\{\pi_j^T\}_{j=1}^N|M^T, P^T) \leq \max_j (\alpha F(i,j)^2 G(j) - c_{ij}) I(\alpha F(i,j)^2 G(j) - c_{ij} \geq 0) \frac{N^2 \sqrt{T}}{T + 1}$$

Since the expected utility for a single time slot is finite, the contribution of the utility from the evaluation stage decays to zero as $T$ increases.

We now turn to computing the second term. We write the expression for the utility of the operational stage.

$$\hat{W}^e(i,j) = \frac{1}{\sqrt{T} + 1} \sum_{t=0}^{N \sqrt{T} - 1} \hat{W}_i(H^t_i, H^t_0, \pi^T_i|m^T) I(m^T(H^t_0)[i] = j)$$

The matching function for the operational stage uses the estimates of the normalized outputs and also takes as input the preference lists of the workers for the tasks. So for the worker, the preference list that is submitted is also a random vector as the general strategy that depends on the private history, which is stochastic. We write $\hat{B}_i$ as the random vector to represent the preference list turned in by the worker. We write the matching computed based on the G-S algorithm applied to the set of preferences of the workers and tasks as follows. $m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)$. We can compute the expected utility for a single time slot of the operational stage in a manner exactly similar to that of the evaluation stage and thus get the following expression. For the consistency of notation we state that $m^T(H^t_0)[i] = m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i]$ for all $i$ and $t \geq N \sqrt{T} + 1$.

The expected utility in a single time slot of operational stage computed in a manner exactly similar to that of evaluation stage is given as.

$$E[(\alpha F(i, m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i])^2 g(m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i]) - C(i, m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i])) \pi_i(H^t_i)^2]$$

$$U_i^{\text{opnl},T}(\{\pi_j^T\}_{j=1}^N|m^T) =$$

$$E[(\alpha F(i, m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i])^2 g(m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i]) - C(i, m^{GS}(\hat{W}^e, \{\hat{B}_k\}_{k=1}^N)[i])) \sum_{t=N \sqrt{T} + 1}^{T} \frac{1}{T} \pi_i(H^t_i)^2]$$
We now compute the upper bound for the above expression by maximizing the RHS over the space of strategies for the worker $i$. We will show that the random variable inside the expectation given as

$$
(\alpha F(i, m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i])^2 g(m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i]) - C(i, m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i])) \sum_{t=N\sqrt{T}+1} T \pi_t(H_i^t)^2
$$

is dominated by another random variable, which we define as follows. We will show that any realization of the above random variable is dominated using the ideas exactly as in the Proof of Theorem 1. The random variable defined below dominates the above random variable in (5) based on the same arguments as in the Proof of Theorem 1 applied to each realization of these random variables.

$$
(\alpha F(i, m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i])^2 g(m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i]) - C(i, m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i])) 	imes 
\begin{align*}
&I((\alpha F(i, m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i])^2 g(m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i]) - C(i, m^{GS}(\hat{W}, \{\hat{B}_k\}_{k=1}^N)[i]))) (\varepsilon_i^{max})^2 
&\frac{T - N\sqrt{T}}{T + 1}
\end{align*}
$$

Similar to the proof of the first theorem we claim here that the optimal preference list of the worker will be to order the tasks in the order of $\alpha F(i, j)^2 g(j) - C(i, j)$. We write the preference list based on ranking $\alpha F(i, j)^2 g(j) - C(i, j)$ as $B_i$. We define below another random variable, which dominates the random variable in (6)

$$
(\alpha F(i, m^{GS}(\hat{W}, B_i', \{\hat{B}_k\}_{k=1,k\neq i}^N)[i])^2 g(m^{GS}(\hat{W}, B_i', \{\hat{B}_k\}_{k=1,k\neq i}^N)[i]) - C(i, m^{GS}(\hat{W}, B_i', \{\hat{B}_k\}_{k=1,k\neq i}^N)[i])) 	imes 
\begin{align*}
&I((\alpha F(i, m^{GS}(\hat{W}, B_i', \{\hat{B}_k\}_{k=1,k\neq i}^N)[i])^2 g(m^{GS}(\hat{W}, B_i', \{\hat{B}_k\}_{k=1,k\neq i}^N)[i]) - C(i, m^{GS}(\hat{W}, B_i', \{\hat{B}_k\}_{k=1,k\neq i}^N)[i]))) (\varepsilon_i^{max})^2 
&\frac{T - N\sqrt{T}}{T + 1}
\end{align*}
$$

We can further improve the random variable above in (7) as follows. The $i^{th}$ row of the matrix $\hat{W}$ consists of the random variables that depend on the effort exerted by the worker $i$ in the evaluation stage on the tasks. Based on the same arguments as in the Proof of Theorem 1 we can argue that it is always better to exert maximum effort on every task in the evaluation stage. If the worker $i$ exerts maximum effort in all stages it is evaluated in and submits the list $B_i$, then we write the matching for worker $i$ as $m^*[i]$, which is a random variable in this case unlike the non-stochastic setting.

Based on the above the upper bound can be written as follows.

$$
U_{i}^{opt,T} \leq (\varepsilon_i^{max})^2 \frac{T - N\sqrt{T}}{T + 1} E \left[(\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i])) I((\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) \geq 0)\right]
$$
In the above we can substitute $Y = (\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) I((\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) \geq 0)$. We have derived the upper bound for the long-term average utility in the operational stage above. We have also seen that the long-run utility in the evaluation stage will decay to a small value as $T$ becomes sufficiently large. Our goal next is to compute the long-run utility of the proposed stochastic bang-bang strategy in the operational stage. We write $\hat{B}_i$ to denote the preference list submitted by the worker under the proposed strategy. We know that the worker $i$ exerts the maximum effort in each time slot in evaluation stage. We write the matching for worker $i$ as $m^{prop}(i)$. We also know that the worker develops an estimate for its productivity on each task in the evaluation stage, which we write as $\tilde{F}(i, j)$.

$$U^{\text{opnl}, T}_i = (e_{max}^i)^2 \frac{T - N \sqrt{T}}{T + 1} E[(\alpha \hat{F}(i, m^{prop}(i)) - C(i, m^{prop}(i))) I((\alpha \hat{F}(i, m^{prop}(i)) - C(i, m^{prop}(i)) \geq 0)]$$

$$= (e_{max}^i)^2 \frac{T - N \sqrt{T}}{T + 1} E[X]$$

In the above we use a succinct notation

$$X = (\alpha \hat{F}(i, m^{prop}(i)) - C(i, m^{prop}(i))) I((\alpha \hat{F}(i, m^{prop}(i)) - C(i, m^{prop}(i)) \geq 0)$$

Note that the $m^{prop}(i)$ and $m^*[i]$ will be the same as long as the preference list submitted by worker $i$ is the same. Let us assume that the set $\{\alpha F(i, j)^2 g(j) - C(i, j)\}_{j=1}^N$ consists of only distinct values. The minimum separation between any two elements in the set is $\Delta > 0$. This means that if $\hat{F}(i, j)$ is sufficiently close to $F(i, j)$ within $\Delta_1$ then the ranking given by the worker in our proposed method will be the same as $B'_i$. We compute the condition on $\Delta_1$ next.

$$\alpha(F(i, j) + \Delta_1)^2 g(j) - \alpha F(i, j)^2 g(j) \leq \Delta/2 \implies \alpha \Delta_1^2 g(j) + 2\alpha \Delta_1 F(i, j) g(j) \leq \Delta/2$$

If $\Delta_1 < 1$, then $\alpha \Delta_1^2 g(j) + 2\alpha \Delta_1 F(i, j) g(j) \leq \alpha \Delta_1 g(j) + 2\alpha \Delta_1 F(i, j) g(j) \leq \Delta/2$ Hence, we get the condition $\Delta_1 < \min\{1, \frac{\Delta}{2 g(j) + 2 \alpha F(i, j) g(j)}\}$

There are three possibilities for the outcome

$$\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) > 0 \text{ or }$$

$$\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) < 0 \text{ or }$$

$$\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) = 0$$

If $\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) = 0$, then as long as $m^*[i] = m^{prop}(i)$ there will be no difference in the utility of our proposed strategy and the upper bound.

If $\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) > 0$ or $\alpha F(i, m^*[i])^2 g(m^*[i]) - C(i, m^*[i]) < 0$, then we can define the minimum value

$$\Delta_2 = \min_{k \in \mathbb{N}} |\alpha F(k, m^*[k])^2 g(m^*[k]) - C(k, m^*[k])|$$

s.t. $|\alpha F(k, m^*[k])^2 g(m^*[k]) - C(k, m^*[k])| > 0$
We need an extra condition on $\Delta_1$ to ensure that. Thus we have $\Delta_1 < \min\{1, \frac{\Delta \gamma}{2\alpha g(j) + 2\alpha F(i,j)g(j)}\}$.

We now use Chebyshev’s inequality to compute

$$P(\|\hat{F}(i,j) - F(i,j)\| \leq \Delta_1) \geq 1 - \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}$$

In the above expression we used the fact that $V \gamma [\hat{F}(i,j)] \leq \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}$. This means there can be a difference with a probability at most $1 - \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)} < 1$, which is satisfied if $T \geq \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}$. The previous equation uses Bernoulli’s inequality.

From the above we know that the random variable $X = Y$ with a probability greater than $1 - \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}$ and $X > x^m = \min\{\min_j \epsilon F(i,j)^2 g(j) - C(i,j)\}((\epsilon_i^m)^2, 0$ with remaining probability. We require that the gap between the proposed utility and the upper bound is small. We can write the difference as

$$(\epsilon_i^m)^2 \frac{T - N^2 \sqrt{T}}{T + 1} \times E[Y - X]$$

Now let us define a random variable $D = 0$ when $Y = X$ and $D = y^m - x^m$ when $Y \neq X$ where $y^m$ is the maximum value $Y$ can take and $x^m$ is the smallest value $X$ can take and $y^m > x^m$ (from the definition). Clearly $Y - X \leq D$. Now we compute the upper bound on $E[D]$. $D$ can take the value $y^m - x^m$ with probability at most $\frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}$ and zero otherwise. Therefore,

$$E[D] \leq \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}(y^m - x^m) = \frac{\Gamma}{\sqrt{T} + 1} \leq \frac{\Gamma}{\sqrt{T}}$$

Hence, we can see that if $T$ is sufficiently large, then the above the difference between the proposed strategy’s expected utility and the upper bound is less than $\epsilon/2$. Specifically, if $T \geq 4\Gamma^2/\epsilon^2 = \frac{\sigma_{ij}^2}{\Delta_1(\epsilon_i^m)^2(\sqrt{T} + 1)}(y^m - x^m)^2/\epsilon^2$. For the ranking stage we computed the upper bound on the utility as

$$\max_j (\alpha \epsilon F(i,j)^2 g(j) - C(i,j)) I(\alpha \epsilon F(i,j)^2 g(j) - C(i,j) \geq 0) \frac{\sqrt{T}}{T + 1} \leq y^m \frac{N^2 \sqrt{T}}{T + 1}$$

For the ranking stage compute the lower bound on the utility of the proposed strategy as $x^m \frac{N^2 \sqrt{T}}{T + 1}$. 


Hence, the difference from the ranking stage is given as
\[
(y_{\text{max}} - x_{\text{min}}) \frac{N^2 \sqrt{T}}{T + 1} \leq \frac{(y_{\text{max}} - x_{\text{min}})N^2}{\sqrt{T}}
\].

Specifically, if \( T \geq 4(N^4(y_{\text{max}} - x_{\text{min}}))^2/\epsilon^2 \). Hence,
\[
T_1(\epsilon) = \max \{4(N^4(y_{\text{max}} - x_{\text{min}})/\epsilon^2, A(\frac{\sigma^2 N}{\Delta g(N)^2} (e_{i_{\text{max}}})^2(y_{\text{max}} - x_{\text{min}}))^2/\epsilon^2 \}
\]

This proves the first part. If all the workers follow this strategy, then the second part is immediate from the definition of \( \epsilon \) equilibrium.

**Corollary 1.** The regret for every worker \( i \) decreases as \( \frac{\zeta_i}{\sqrt{T}} \).

**Proof** In the previous proof, we derived the upper bound for the worker’s regret. The upper bound in the previous derivation is actually the highest utility that the worker with the knowledge of its own productivity can possibly achieve. So using the result from the previous theorem we have that the regret has to decrease as \( \frac{\zeta_i}{\sqrt{T}} \), where \( \zeta_i = \max \{\Gamma, (y_{\text{max}} - x_{\text{min}})N^2\} \)

**Theorem 6.** If Assumptions 1-4 hold, then the ratio of the total long-run revenue of the modified mechanism and the upper bound on the total long-run revenue that can be achieved by the proposed mechanism in Section 2 is at least \((1 - \frac{N^2 \eta}{\sqrt{T}})\).

We first write the total mean output for \( T + 1 \) stages as follows.
\[
W(\{\pi_i^T\}_{i=1}^N | m^T) = E[\frac{1}{T+1} \sum_{t=0}^{T} \sum_{i=1}^N W_i(H_i^t, H_0^t, \pi_i^T | m^T)]
\]

We will first compute the long-run revenue generated by the worker \( i \) when all the workers follow the proposed strategy. We write the matching that happens for worker \( i \) when all the workers follow the proposed stochastic bang-bang strategy as \( m^{S-BBE}[i] \). Note that in this case since the worker’s productivity on the task does not depend on the task and the same is true for the cost. This means that the best preference list for all the workers is to rank the tasks in the order of their quality. Again since the workers qualities do not depend on the tasks the tasks rank the workers in the order of their productivity too. Hence, all the workers have the same optimal ranking for the tasks and vice-versa, which means the optimal matching in this case is assortative matching denoted as \( m^{AST} \) (rank the workers in the order of \( F(i)e_{i_{\text{max}}} \) and the tasks in the order of quality \( g(i) \), then match the worker and tasks with the same rank) The challenge though in our setting is that the above requires that the workers know their productivities over the tasks, but the productivity of the workers is not known to the designer and to the workers. This means that if the estimates of the outputs are ranked in the same order as their means (set of values of \( F(i)e_{i_{\text{max}}} \)) then the matching will be optimal, i.e. \( m^{S-BBE} = m^{AST} \). We write \( \hat{F}(i) \) to denote the estimate that the worker has for its own productivity.
The output produced by worker $i$ under the proposed strategy is given as.

$$W_i(\{\pi_k^T\}_{k=1}^N, m^{S-BBE}) = E\left[\frac{1}{T+1} \sum_{t=N\sqrt{T}+1}^T W_i(H_i^t, H_0^t, \pi_i^T|m^{S-BBE})\right]$$

$$= E\left[\frac{1}{T+1} \sum_{t=N\sqrt{T}+1}^T \left(F(i)e_i^{max}\tilde{g}(m^{S-BBE}[i])I(\hat{F}(i)^2g(m^{S-BBE}[i]) - C(i) \geq 0) + Z_{ig}(m^{S-BBE}[i])\right)\right]$$

It is clear that the second term in the above summation equals zero. Note that the first term of the summation does not depend on time $t$ and hence, we can simplify to obtain the following.

$$\frac{T - N\sqrt{T}}{T+1} E\left[F(i)e_i^{max}\tilde{g}(m^{S-BBE}[i])I(\hat{F}(i)^2g(m^{S-BBE}[i]) - C(i) \geq 0)\right]$$

We now aim to obtain a lower bound on the above. We know that if the estimates are sufficiently close to the expected value then the $m^{S-BBE} = m^{AST}$. We use Chebyshev’s inequality in the same manner as we did in the proof of the previous theorem to show that

$$P(|\hat{F}(i) - F(i)| \leq \Delta_1) \geq 1 - \frac{\sigma_{ij}^2}{\Delta_1(g(N)e_i^{max})^2(\sqrt{T} + 1)}$$

Hence, if $F(i)e_i^{max}$ are all distinct and if $\Delta_1$ is sufficiently small, then $|\hat{F}(i) - F(i)| \leq \Delta_1$ for all $i$ will ensure that $m^{S-BBE} = m^{AST}$.

We also need to ensure that $I(\hat{F}(i)^2g(m^{S-BBE}[i]) - C(i) \geq 0) = I(F(i)^2g(m^{AST}) - C(i) \geq 0)$ holds at the same time. We already require $m^{S-BBE} = m^{AST}$ stated in the paragraph above. Suppose $F(i)^2g(m^{AST}[i]) - C(i) > 0$ or $F(i)^2g(m^{AST}[i]) - C(i) < 0$, then there exists $\Delta_1$ for which $|F(i)^2g(m^{AST}) - C(i)| \geq \Delta_1 > 0$. Therefore, $\Delta_1$ is chosen such that difference between $\hat{F}(i)^2g(m^{S-BBE}[i]) - C(i)$ and $F(i)^2g(m^{AST}[i]) - C(i)$ is sufficiently small.

$$\frac{T - N\sqrt{T}}{T+1} E\left[F(i)e_i^{max}\tilde{g}(m^{S-BBE}[i])I(\hat{F}(i)^2g(m^{S-BBE}[i]) - C(i) \geq 0)\right] \geq$$

$$\frac{T - N\sqrt{T}}{T+1} F(i)e_i^{max}\tilde{g}(m^{AST}[i])I(\hat{F}(i)^2g(m^{AST}[i]) - C(i) \geq 0)(1 - \frac{K}{\sqrt{T} + 1}) \approx (1 - \frac{N}{\sqrt{T}})F(i)e_i^{max}\tilde{g}(m^{AST}[i])I(\hat{F}(i)^2g(m^{AST}[i]) - C(i) \geq 0)$$

Therefore, by summing the revenue generated for all the workers we get the result in Theorem we set out to prove.