The Laplacian coflow on almost-abelian Lie groups

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July 26, 2018

Abstract

We find explicit solutions of the Laplacian coflow of $G_2$--structures on seven-dimensional almost-abelian Lie groups. Moreover, we construct new examples of solitons for the Laplacian coflow which are not eigenforms of the Laplacian and we exhibit a solution, which is not a soliton, having a bounded interval of existence.

1 Introduction

A $G_2$-structure on a seven-dimensional manifold $M$ is given by a 3-form $\varphi$ on $M$ with pointwise stabilizer isomorphic to the exceptional group $G_2 \subset SO(7)$. The 3-form $\varphi$ induces a Riemannian metric $g_{\varphi}$, an orientation and so a Hodge star operator $\star_{\varphi}$ on $M$. It is well-known \cite{7} that $\varphi$ is parallel with respect to the Levi-Civita connection of $g_{\varphi}$ if and only if $\varphi$ is closed and coclosed and that when this happens the holonomy of $g_{\varphi}$ is contained in $G_2$.

The different classes of $G_2$-structures can be described in terms of the exterior derivatives $d\varphi$ and $d\star_{\varphi}\varphi$ \cite{4,7}. If $d\varphi=0$, then the $G_2$-structure is called closed (or calibrated in the sense of Harvey and Lawson \cite{13}) and if $\varphi$ is closed and coclosed and that when this happens the holonomy of $g_{\varphi}$ is contained in $G_2$.

The flows of $G_2$-structures were first considered by Bryant in \cite{4}. In particular, he considered the Laplacian flow of closed $G_2$-structures. Recently, Lotay and Wei investigated the properties of the Laplacian flow in the series of papers \cite{19,20,21}. The Laplacian coflow has been originally proposed by Karigiannis, McKay and Tsui in \cite{15} and, for an initial coclosed $G_2$-form $\varphi_0$ with $\star_{\varphi_0}\varphi_0 = \phi_0$, it is given by

$$\frac{\partial}{\partial t} \phi(t) = -\Delta_t \phi(t), \quad d\phi(t) = 0, \quad \phi(0) = \phi_0,$$  \hspace{1cm} (1)
where $\phi(t)$ is the Hodge dual 4-form of a $G_2$-structure $\varphi(t)$ with respect to the Remannian metric $g_{\varphi(t)}$. This flow preserves the condition of the $G_2$-structure being coclosed and it was studied in [15] for warped products of an interval, or a circle, with a compact 6-manifold $N$ which is taken to be either a nearly Kähler manifold or a Calabi-Yau manifold. No general result is known about the short time existence of the coflow (1). In [2] the Laplacian coflow on the seven-dimensional Heiseberg group has been studied, showing that the solution is always ancient, that is it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$. Other examples of flows of $G_2$-structures are the modified Laplacian coflow [11, 12] and Weiss and Witt’s heat flow [24]. The first one is a flow of coclosed $G_2$-structures obtained by adding a fixing term to the Laplacian coflow in order to ensure weak parabolicity in the exact directions. The second one is the gradient flow associated to the functional which measures the full torsion tensor of a $G_2$-structure; generally it does not preserve any special class of $G_2$-structures but it can be modified to fix the underlying metric (see [3]).

As for the Ricci flow (and other geometric flows), for the Laplacian coflow it is interesting to consider self-similar solutions which are evolving by diffeomorphisms and scalings. If $x_t$ is a 1-parameter family of diffeomorphisms generated by a vector field $X$ on $M$ with $x_0 = \text{Id}_M$ and $c_t$ is a positive real function on $M$ with $c_0 = 1$, then a coclosed $G_2$-structure $\phi(t) = c_t(x_t)^*\phi_0$ is a solution of the coflow (1) if and only if $\phi_0$ satisfies

$$-\Delta_0 \phi_0 = L_X \phi_0 + c_0' \phi_0 = d(X \cdot \phi_0) + c_0' \phi_0,$$

where by $L_X$ and $X \cdot$ we denote respectively the Lie derivative and the contraction with the vector field $X$. A coclosed $G_2$-structure satisfying the previous equation is called soliton.

As in the case of the Ricci flow, the soliton is said to be expanding, steady, or shrinking if $c_0'$ is positive, zero, or negative, respectively. By Proposition 4.3 in [15], if $M$ is compact, then there are no expanding or steady soliton solutions of (1), other than the trivial case of a torsion-free $G_2$-structure in the steady case. Examples of solitons for the Laplacian flow have been constructed in [5, 16, 17, 18, 22, 8].

In this paper, we study the coflow (1) on almost abelian Lie groups, i.e., on solvable Lie groups with a codimension-one abelian normal subgroup. Coclosed and closed left-invariant $G_2$-structures on almost-abelian Lie groups have been studied by Freibert in [9, 10]. General obstructions to the existence of a coclosed $G_2$-structure on a Lie algebra of dimension seven with non-trivial center have been given in [1].

By [16] the Laplacian coflow on homogeneous spaces can be completely described as a flow of Lie brackets on the ordinary euclidean space, the so-called bracket flow. In particular, Lauret showed in [16] that any left-invariant closed Laplacian flow solution $\varphi(t)$ on an almost abelian Lie group is immortal, i.e., defined in the interval $[0, +\infty)$. Moreover, he proved that the scalar curvature of $g_{\varphi(t)}$ is strictly increasing and converges to zero as $t$ goes to $+\infty$.

In Section 3 we find an explicit description of the left-invariant solutions to the Laplacian coflow on almost-abelian Lie groups under suitable assumptions on the initial data, showing that the solution is ancient.

In Section 4 we show sufficient conditions for a left-invariant coclosed $G_2$-structure on an
almost-abelian Lie group to be a soliton for the Laplacian coflow. In particular we construct new examples of solitons which are not eigenforms of the Laplacian.

2 Preliminaries

A $k$-form on an $n$-dimensional real vector space is stable if it lies in an open orbit of the linear group $GL(n, \mathbb{R})$. In this section we review the theory of stable forms in dimensions six and seven. We refer to [6, 14], and the references therein, for more details. Throughout the sections we denote by $\vartheta$ and by $\ast$ the actions of the endomorphism group and the general linear group respectively.

2.1 Linear $G_2$–structures

A 3-form $\varphi$ on a seven-dimensional real vector space $V$ is stable if the $\Lambda^7(V^*)$-valued bilinear form $b_\varphi$, defined by

$$b_\varphi(x, y) = \frac{1}{6} (x \varphi) \wedge (y \varphi) \wedge \varphi, \quad x, y \in V,$$

is nondegenerate. In this case $\varphi$ defines an orientation $vol_\varphi$ by $\sqrt[9]{\det b_\varphi}$ and a bilinear form $g_\varphi$ by $b_\varphi = g_\varphi vol_\varphi$. A stable 3-form $\varphi$ is said to be positive, and we will write $\varphi \in \Lambda^3_+ V^*$.

It is a well-known fact that the action of $GL(V)$ on $\Lambda^3_+ V^*$ is transitive and the stabiliser of every $\varphi \in \Lambda^3_+ V^*$ is a subgroup of $SO(g_\varphi)$ isomorphic to $G_2$. Therefore, if we assume that $||\varphi||_{g_\varphi} = 7$ we get a one-to-one correspondence between normalized positive 3-forms on $V$ and presentations of $G_2$ inside $GL(V)$.

We denote by $\ast_\varphi$ the Hodge operator induced by $\varphi$ and we will always write $\phi$ to indicate the Hodge dual form $\ast_\varphi \varphi$ of $\varphi$. Precisely $\phi$ belongs to the $GL(V)$–orbit, denoted by $\Lambda^4_+ V^*$, of positive 4-forms. It is another basic fact that the stabilisers of $\varphi$ and $\phi$ under $GL^+(V)$ are equal and therefore the choice of $\phi$ and of an orientation $vol$ is sufficient to define $\varphi$.

We will refer to a presentation of $G_2$ inside $GL(V)$ as a linear $G_2$–structure on $V$, and we will call $\varphi$ and $\phi$ the fundamental forms associated to the linear $G_2$–structure.

On $V$ there exists always a $g_\varphi$-orthonormal and positive oriented co-frame $(e^1, \ldots, e^7)$, called an adapted frame, such that

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

$$\phi = e^{1234} + e^{3456} + e^{1256} - e^{2467} + e^{1367} + e^{1457} + e^{2357}.$$

2.2 Linear $SU(3)$–structures

Let $U$ be a real vector space of dimension six. A 2-form $\omega$ on $U$ is stable if it is nondegenerate, i.e., if $\omega^3 \neq 0$.

Given a 3-form $\psi$ on $U$, the equivariant identification of $\Lambda^5 U^*$ with $U \otimes \Lambda^6 U^*$ allows us to define the operator

$$K_\psi : U \to U \otimes \Lambda^6 U^*, \quad x \mapsto (x \varphi) \wedge \psi.$$
We can consider the trace of its second iterate
\[ \lambda(\psi) = \frac{1}{6} \text{tr}(K_\psi^2) \in (\Lambda^6 U^*) \otimes (\Lambda^6 U^*), \]
where
\[ K_\psi^2 : U \to U \otimes (\Lambda^6 U^*) \otimes (\Lambda^6 U^*). \]
Then \( \psi \) is stable if and only if \( \lambda(\psi) \neq 0 \). If \( \lambda(\psi) < 0 \), the 3-form \( \psi \) is called negative. In this case we will write \( \psi \in \Lambda^3 U \). Here the basic fact is that the action of \( \text{GL}^+(U) \) is transitive on \( \Lambda^3 U \) with stabiliser of \( \psi \) isomorphic to \( \text{SL}(3, \mathbb{C}) \), where the associated complex structure \( J_\psi \) and complex volume form \( \Psi \) on \( U \) are given respectively by
\[ J_\psi = \frac{1}{\sqrt{-\lambda}} K_\psi, \quad \Psi = -J_\psi^* \psi + i \psi. \]
It is important to note that the 3-form \( J_\psi^* \psi \) is still negative and that it defines the same complex structure of \( \psi \).

If \( \psi \) is a negative 3-form and \( \omega \) a stable 2-form, then \( \omega \) is of type \((1,1)\) with respect to \( J_\psi \), meaning that \( J_\psi^* \omega = \omega \), if and only if \( \psi \wedge \omega = 0 \). In this case we can define a symmetric bilinear form \( h \) on \( U \) by
\[ h(x, y) = \omega(x, J_\psi y), \quad x, y \in U. \]
When \( h \) is positive definite, the couple \((\omega, \psi)\) is said to be a positive couple and it defines a linear \( \text{SU}(3) - \)structure, meaning that its stabiliser in \( \text{GL}(U) \) is isomorphic to \( \text{SU}(3) \). In this case \( h \) is hermitian with respect to \( J_\psi \) and \( \Psi = -J_\psi^* \psi + i \psi \) is a complex volume form. A positive couple is said to be normalized if
\[ 2 \omega^3 = 3 \psi \wedge J_\psi^* \psi. \]
If a positive couple \((\omega, \psi)\) is normalized, then there exists an \( h \)-orthonormal and positive oriented co-frame of \( U \), called an adapted frame, \( (f^1, J^* f^1, f^2, J^* f^2, f^3, J^* f^3) \) such that
\[
\begin{align*}
\omega &= f^1 \wedge J^* f^1 + f^2 \wedge J^* f^2 + f^3 \wedge J^* f^3, \\
\psi &= -f^2 \wedge f^4 \wedge f^6 + f^1 \wedge J^* f^3 \wedge J^* f^6 + J^* f^1 \wedge f^4 \wedge J^* f^5 + J^* f^2 \wedge J^* f^3 \wedge f^5.
\end{align*}
\]
Therefore, if we denote by \( *_h \) the Hodge operator on \( U \) associated to \( h \), it follows that
\[ *_h \omega = \frac{1}{2} \omega^2, \quad *_h \psi = J_\psi^* \psi. \]

### 2.3 From \( G_2 \) to \( \text{SU}(3) \)

Given a linear \( G_2 - \)structure \( \varphi \) on \( V \), with fundamental forms \( \varphi \) and \( \phi \), the six-dimensional sphere
\[ S^6 = \{ x \in V \mid g_\varphi(x, x) = 1 \} \subset V \]
is $G_2$–homogeneous and, for any non-zero vector $v \in S^6$, there is an induced linear SU(3)–structure on the $g_\varphi$-orthogonal complement $U = (\text{span} < v >) ^\perp$. This structure is constructed as follows. Let

$$\omega = v - \varphi, \quad \psi = -v - \phi.$$  

Then $(\omega, \psi)$ is a positive couple on $U$ defining the linear SU(3)–structure. It is then clear that the restriction of an adapted co-frame of $(V, \varphi)$, with $v = e_7$, to $U$ gives an adapted frame of $(U, \omega, \psi)$ and it follows that

$$\varphi = \omega \wedge e^7 - J_\varphi^* \psi, \quad \phi = \frac{1}{2} \omega^2 + \psi \wedge e^7.$$  

3 Explicit solutions to the Laplacian coflow on almost-abelian Lie groups

We recall that a Lie group $G$ is said to be almost-abelian if its Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal $\mathfrak{h}$. Such a Lie algebra will be called almost-abelian, and it can be written as a semidirect product $\mathfrak{g} = \mathbb{R}x \ltimes_A \mathfrak{h}$. We point out that an almost-abelian Lie algebra is nilpotent if and only if the operator $\text{ad}_x|_{\mathfrak{h}}$ is nilpotent.

Freibert showed in [9] that if $\mathfrak{g}$ is a 7-dimensional almost-abelian Lie algebra, then, the following are equivalent:

1. $\mathfrak{g}$ admits a coclosed $G_2$-structure $\varphi$.
2. For any $x \in \mathfrak{g} \setminus \mathfrak{h}$, $\text{ad}(x)|_{\mathfrak{h}} \in \mathfrak{gl}(\mathfrak{h})$ belongs to $\mathfrak{sp}(\mathfrak{h}, \omega)$, where $\omega$ is a non-degenerate 2-form $\omega$ on $\mathfrak{h}$.
3. For any $x \in \mathfrak{g} \setminus \mathfrak{h}$, the complex Jordan normal form of $\text{ad}(x)|_{\mathfrak{h}}$ has the property that for all $m \in \mathbb{N}$ and all $\lambda \neq 0$ the number of Jordan blocks of size $m$ with $\lambda$ on the diagonal is the same as the number of Jordan blocks of size $m$ with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m - 1$ with 0 on the diagonal is even.

In this section we obtain an explicit description of the solutions to the Laplacian coflow on almost-abelian Lie groups under suitable assumptions on the initial data.

Let $G$ be a seven-dimensional, simply-connected, almost-abelian Lie group equipped with an invariant coclosed $G_2$–structure $\varphi_0$ with 4-form $\phi_0$ and let $\mathfrak{h}$ be a codimension one abelian ideal of the Lie algebra $\mathfrak{g}$ of $G$. By Proposition 4.5 in [23], if we choose a vector $e_7$ in the orthogonal complement of $\mathfrak{h}$ with respect to $g_{\varphi_0}$ such that $g_{\varphi_0}(e_7, e_7) = 1$, the forms

$$\omega_0 = e_7 - \varphi_0, \quad \psi_0 = -e_7 - \phi_0,$$  

(2)

define an SU(3)-structure $(\omega_0, \psi_0)$ on $\mathfrak{h}$. Let $\eta = e_7 - g_{\varphi_0}$. Then we can identify $\mathfrak{g}^*$ with $\mathfrak{h}^* \oplus \mathbb{R}\eta$ and we have $d\eta = 0$, since $\eta$ vanishes on the commutator $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$. Moreover

$$d\alpha = \eta \wedge \vartheta(A)\alpha,$$  

for every $\alpha \in \Lambda^p h^*$, where $A = ad_{e_{\alpha}}|_h$. In particular, if $\phi$ is any 4-form on $g$, we can consider the decomposition
\[
\phi = \phi^{(4)} + \phi^{(3)} \wedge \eta, \quad \phi^{(i)} \in \Lambda^i h^*, \quad i = 3, 4.
\] (3)
So $(\phi_0)^{(4)} = 1/2 \omega_0^2$ and $(\phi_0)^{(3)} = \psi_0$. Finally let us observe that $d\phi_0 = 0$ if and only if $\nabla(A)(\omega_0^3) = 2\nabla(A)(\omega_0)\wedge\omega_0 = 0$, which means that $A \in sp(h, \omega_0)$, since $\omega_0$ is a nondegenerate 2-form.

**Lemma 3.1.** Let $U$ be a real vector space of dimension 6 endowed with a linear $\text{SU}(3)$-structure $(\omega, \psi)$ and $A \in sp(\omega)$ be normal with respect to the inner product $h$ defined by $(\omega, \psi)$. Denote by $J$ the complex structure induced by $\psi$ and by $S$ and $L$ the symmetric and skew-symmetric part of $A$, respectively. Then there exist $\theta \in [0, 2\pi]$ and a basis $(e_1, e_2, e_3, Je_1, Je_2, Je_3)$ of $U$ such that
\[
\omega = e^1 \wedge J^* e^1 + e^2 \wedge J^* e^2 + e^3 \wedge J^* e^3, \\
\Psi = (e^1 + iJ^* e^1) \wedge (e^2 + iJ^* e^2) \wedge (e^3 + iJ^* e^3)
\] (4)
and
\[
S(e_i) = s_i(\cos(\theta)e_i + \sin(\theta)Je_i), \quad S(Je_i) = -s_i(\cos(\theta)e_i + \sin(\theta)Je_i), \quad i = 1, 2, 3,
\] (5)
where the real numbers $\{\pm s_i, i = 1, 2, 3\}$ are the eigenvalues of $S$ (counted with their multiplicities), and $JV_{s_i} = V_{-s_i}$, where $V_{s_i}$ denotes the eigenspace of $S$ associated to the eigenvalue $s_i$. Moreover,

1. if $s_j = 0$, then $Le_j = l_Je_j$ and $LJ e_j = -l_J e_j$, for $l_j \in \mathbb{R}$;
2. if $s_j \neq 0$ with multiplicity $m_j$, then $L|_{V_{s_j} + V_{-s_j}}$ is given by the block matrix
\[
L = \begin{bmatrix} L' & 0 \\ 0 & L' \end{bmatrix}, \quad L' \in \mathfrak{so}(m_j),
\]
with respect to the basis $(e_{i_1}, \ldots, e_{i_m}, Je_{i_1}, \ldots, Je_{i_m})$ of $V_{s_j} \oplus V_{-s_j}$.

**Proof.** Clearly $S$ and $L$ belong to $sp(\omega)$ since $A$ does. Therefore we have
\[
h(x, SJy) = h(Sx, Jy) = -\omega(Sx, y) = \omega(x, Sy) = -h(x, JSy), \quad x, y \in V.
\]
Thus $SJ = -JS$ and, similarly, $LJ = JL$.

The spectrum of $S$ must be real and centrally symmetric, since $S$ is symmetric and anti-commutes with $J$. Let $\{\pm s_i, i = 1, 2, 3\}$ be the spectrum of $S$. Denote by $V_{s_i}$ the eigenspace of $S$ associated with the eigenvalue $s_i$, and by $m(s_i)$ its multiplicity. It is then clear that, since $[S, L] = 0$ and $SJ = -JS$, $L$ preserves each eigenspace $V_{s_i}$ and $JV_{s_i} = V_{-s_i}$.

Now we show that on each $J$–invariant subspace $W_{s_i} = V_{s_i} \oplus V_{-s_i}$, both $S$ and $L$ are given as in the statement with respect to some orthonormal basis. First let us consider the case when $s_i = 0$ is an eigenvalue of $S$. Clearly its multiplicity $m_0 = m(0)$ is even and
the restriction $L|_{V_0}$ of $L$ to the eigenspace $V_0$ belongs to $\mathfrak{sp}(m_0,\mathbb{R}) \cap \mathfrak{so}(m_0) = \mathfrak{u}(m_0/2)$. Therefore we can diagonalize $L$ over $\mathbb{C}$ as a complex matrix finding the desired expression; indeed its eigenvalues are all imaginary numbers.

Now let $s_i \neq 0$ and $m(s_i) = m_i$. Then $W_{s_i}$ has real dimension $2m_i$ and there exists an orthonormal basis $(e_{r_1}, \ldots, e_{r_m})$ of $V_{s_i}$ such that, $L|_{V_i}$ has the following expression with respect to the orthonormal basis $(e_{r_1}, \ldots, e_{r_m}, e_{r_{m+1}}, \ldots, e_{r_{m+i}})$

$$L = \begin{bmatrix} L_1 & L_3 \\ -L_3^\dagger & L_2 \end{bmatrix}, \quad L_1, L_2 \in \mathfrak{so}(m_i), \quad L_3 \in \mathfrak{gl}(m_i, \mathbb{R}),$$

where by $\dagger$ we denotes the transpose. So, by $LJ = JL$ and $LS = SL$ we get $L_3 = 0$ and $L_1 = L_2$.

Putting all together the basis of $W_i$ we get an orthonormal basis $(e_1, e_2, e_3, J e_1, J e_2, J e_3)$ of $U$ but, generally, the basis is not an adapted frame with respect to the linear SU(3)-structure $(\omega, \Psi)$. Indeed $\Psi = (e^1 + e^* e^1) \wedge (e^2 + i J^* e^2) \wedge (e^3 + i J^* e^3)$ does not necessarily coincide with $\Psi$. However there exists a complex number $z$ of modulus 1 such that $z^{-1} \Psi_0 = \Psi$. If we take a cubic root $w$ of $z$ and we consider the linear map $Q$ defined by $Q = \text{Re}(w) \text{id} + \text{Im}(w)J$ we get that $Q^* \Psi_0 = \Psi$. The transformation $Q$ commutes with $J$ and preserves each vector subspace $W_{s_i}$. Moreover

$$Q^* S = Q S Q^{-1} = (\text{Re}(w) \text{id} + \text{Im}(w)J) S (\text{Re}(w) \text{id} - \text{Im}(w)J)$$
$$= S (\text{Re}(w) \text{id} - \text{Im}(w)J) (\text{Re}(w) \text{id} - \text{Im}(w)J)$$
$$= S \{ (\text{Re}(w)^2 - \text{Im}(w)^2) \text{id} - 2 (\text{Re}(w) \text{Im}(w)) J \}$$
$$= \cos(\theta) S + \sin(\theta) JS,$$

$$Q^* L = Q L Q^{-1} = (\text{Re}(w) \text{id} + \text{Im}(w)J) L (\text{Re}(w) \text{id} - \text{Im}(w)J)$$
$$= L (\text{Re}(w)^2 + \text{Im}(w)^2) \text{id}$$
$$= L.$$

Therefore the new basis $(Q e_1, Q e_2, Q e_3, J Q e_1, J Q e_2, J Q e_3)$ satisfies all the requested properties. \hfill \Box

**Lemma 3.2.** Let $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ be an almost-abelian Lie algebra endowed with a coclosed $G_2$-structure $\varphi_0$. Let $(\omega_0, \psi_0)$ be the induced SU(3)-structure on $\mathfrak{h}$ defined by (2), with $\eta(e_7) \neq 0$, $\eta|_\mathfrak{h} = 0$ and $\|\eta\|_{g_{\varphi_0}} = 1$. The solution $\phi_t$ of the Laplacian coflow on $\mathfrak{g}$

$$\begin{cases}
\dot{\phi}_t = -\Delta t \phi_t, \\
D\phi_t = 0, \\
\phi_0 = *_0 \varphi_0,
\end{cases} \tag{6}$$

is given by

$$\phi_t = \frac{1}{2} \omega_0^2 + p_t \wedge \eta,$$
where \( p_t \) is a time-dependent negative 3-form on \( \mathfrak{h} \) solving
\[
\begin{align*}
\dot{p}_t &= -\varepsilon(p_t)^2 \vartheta(A) \vartheta(B_t)p_t, \\
p_0 &= \psi_0,
\end{align*}
\]
where \( \varepsilon(p_t) \) is a function such that \( (\omega_0, \varepsilon(p_t)p_t) \) defines an SU(3)-structure on \( \mathfrak{h} \) and \( B_t \) is the adjoint of \( A = \text{ad}_{\varepsilon_t} |_{\mathfrak{h}} \) with respect to the scalar product \( h_t \) induced by the SU(3)-structure \( (\omega_0, \varepsilon(p_t)p_t) \).

**Proof.** By Cauchy theorem the system of ODEs (6) admits a unique solution. Let \( \phi_t \) be the solution of (6) and \( \varepsilon_t \) be the norm \( ||\eta||_t \) with respect to the scalar product \( g_t \) induced by \( \phi_t \). Then we can write
\[
\phi_t = \frac{1}{2} \omega_t^2 + \psi_t \wedge \frac{1}{\varepsilon_t} \eta,
\]
where the couple \( (\omega_t, \psi_t) \) defines an SU(3)-structure on \( \mathfrak{h} = \text{Ker}(\eta) \). To see this observe that if we define \( x_t \) by \( g_t(x_t, y) = \eta(y), \) for any \( y \in \mathfrak{g}, \) then \( \mathfrak{h} = \text{Ker}(\eta) = \{ y \in \mathfrak{g} \mid g_t(x_t, y) = 0 \} \). Therefore, for every \( t \), the 4-form \( \phi_t \) defines an SU(3)-structure \( (\omega_t, \psi_t) \) on \( \mathfrak{h} \).

With respect to the decomposition (3) we can write \( \phi_t \) as \( \phi_t = \phi_t^{(4)} + \phi_t^{(3)} \wedge \eta \) with
\[
\phi_t^{(4)} = \frac{1}{2} \omega_t^2, \quad \phi_t^{(3)} = \frac{1}{\varepsilon_t} \psi_t.
\]

Since the cohomology class of \( \phi_t \) is fixed by the flow, i.e., \( \phi_t = \phi_0^t + dA_t \) it turns out that
\[
\dot{\phi}_t = \phi_t^{(4)} + \phi_t^{(3)} \wedge \eta = da_t \in d\Delta^3 \mathfrak{g}^* \subseteq \Lambda^3 \mathfrak{h}^* \wedge \mathbb{R} \eta.
\]

Therefore \( \dot{\phi}_t^{(4)} = 0 \), i.e., \( \omega_t \equiv \omega_0 \) and consequently
\[
\phi_t = \frac{1}{2} \omega_0^2 + \psi_t \wedge \frac{1}{\varepsilon_t} \eta.
\]

Now define \( \eta_t = \frac{1}{\varepsilon_t} \eta \) and denote by \( \ast_{g_t} \) and \( \ast_{h_t} \) the star Hodge operators on \( \mathfrak{g} \) and \( \mathfrak{h} \) with respect to \( g_t \) and \( h_t \) respectively. Note that
\[
\ast_{g_t} \phi_t = \omega_0 \wedge \eta_t - \ast_{h_t} \psi_t,
\]

since
\[
\ast_{g_t} \beta = \ast_{h_t} \beta \wedge \eta_t, \quad \ast_{g_t} (\beta \wedge \eta_t) = (-1)^k \ast_{h_t} \beta,
\]

for every \( k \)-form \( \beta \) on \( \mathfrak{h} \).

Then
\[
\Delta_t \phi_t = d \ast_{g_t} d \ast_{g_t} \phi_t = d \ast_t d (\omega \wedge \eta_t - \ast_t \psi_t)
\]
\[
= - d \ast_t (\eta \wedge \vartheta(A) \ast_t \psi_t) = - d \ast_t (\varepsilon_t \eta_t \wedge \vartheta(A) \ast_t \psi_t)
\]
\[
= - \varepsilon_t d \ast_t (\vartheta(A) \ast_t \psi_t)
\]
\[
= \varepsilon_t (\vartheta(A) \ast_t \vartheta(A) \ast_t \psi_t) \wedge \eta
\]
\[
= \varepsilon_t (\vartheta(A) \ast_t \vartheta(A) \ast_t \psi_t) \wedge \eta_t.
\]
On the other hand we have
\[ \dot{\phi}_t = \dot{\psi}_t \wedge \frac{1}{\varepsilon_t} \eta - \psi_t \wedge \frac{\dot{\varepsilon}_t}{\varepsilon_t^2} \eta = \dot{\psi}_t \wedge \eta_t - \frac{\dot{\varepsilon}_t}{\varepsilon_t} \psi_t \wedge \eta_t. \]

Imposing \( \dot{\phi}_t = -\Delta_t \phi_t \) we get
\[ \frac{d}{dt} \psi_t - \frac{d}{dt} (\varepsilon_t) \varepsilon_t^{-1} \psi_t = -\varepsilon_t^2 (\vartheta(A) *_t \vartheta(A) *_t \psi_t). \tag{8} \]

Consider the 3-form \( p_t = \varepsilon_t^{-1} \psi_t \). It is clear that \( p_t \) is a negative 3-form, compatible with \( \omega_0 \) and defining the same complex structure \( J_t \) induced by \( \psi_t \). Moreover it satisfies the condition
\[ -6 p_t \wedge J_t^* p_t = 4 \varepsilon_t^{-2} \omega_0^3. \]

Then, by (8) we obtain
\[ \varepsilon_t \dot{p}_t + \dot{\varepsilon}_t p_t - \varepsilon_t \dot{p}_t = -\varepsilon_t^3 (\vartheta(A) *_t \vartheta(A) *_t p_t). \]
and thus the following equation in terms of the 3-form \( p_t \)
\[ \dot{p}_t = -\varepsilon(p_t)^2 (\vartheta(A) *_t \vartheta(A) *_t p_t), \quad p_0 = \psi_0, \tag{9} \]
where the function \( \varepsilon(p_t) = \varepsilon_t = ||\eta||_t \) is defined in terms of the 3-form \( p_t \) by
\[ 6 p_t \wedge J_t^* p_t = 4 \varepsilon(p_t)^{-2} \omega_0^3. \]

It is easy to see that \( *_t \vartheta(A)*_t \) is the \( h_t \)–adjoint operator of \( \vartheta(A) \) on \( \Lambda^3 h^* \). Indeed, if \( \alpha, \beta \in \Lambda^3 h^* \), then
\[ \langle ((*_t \vartheta(A)*_t) \alpha, \beta \rangle_\omega^3_0/6 = -\beta \wedge \vartheta(A)(*_t \alpha) = \vartheta(A)(\beta) \wedge *_t (\alpha) = (\alpha, \vartheta(A)(\beta))_t \omega^3_0/6, \]
where in the second equality we have used that \( A \) is traceless and consequently that \( \vartheta(A) \) acts trivially on 6-forms.

Now let \( B_t \) be the \( h_t \)–adjoint of \( A \) on \( h \). We claim that \( *_t \vartheta(A)*_t \alpha = \vartheta(B_t) \alpha \) for any 3-form \( \alpha \) on \( h \). To see this let \( (e_1, \ldots, e_6) \) be an \( h_t \)–ortonormal basis of \( h \), so\(^1\)
\[ (B_t)^{i}_j = \sum_{a,b} (A)^a_{b} (h_t)_{a j} (h_t)^{b i} = (A)^{j}_{i}, \quad i, j = 1, \ldots, 6. \]

On the other hand, for any choice of ordered triples \( (i, j, k) \) and \( (a, b, c) \), we get
\[ h_t \left( \vartheta(A) e^{ijk}, e^{abc} \right) = -h_t \sum_{l,m,n} \left( A^i_l e^{ljk} + A^j_m e^{imk} + A^k_n e^{ijn}, e^{abc} \right) \]
\[ = - \sum_{l,m,n} h_t \left( A^i_l e^{i'j'k'} + A^j_m e^{i'j'k'} + A^k_n e^{i'j'k'}, e^{abc} \right) \]
\[ = -(A^i_a + A^j_b + A^k_c) \]

\(^1\)Note that we are not using the Einstein notation.
Proof. Consider the following system as we claimed and (7) holds.

Theorem 3.3. Let $h$ since $G$ is coclosed and (2) holds. Moreover assume that $A$ satisfies $\theta = 0$. Then the solution $p_t$ of (7) is ancient and it is given by

$$ p_t = -b_1(t)e^{246} + b_2(t)e^{136} + b_3(t)e^{145} + b_4(t)e^{235}, \quad t \in \left(-\infty, \frac{1}{8\left(s_1^2 + s_2^2 + s_3^2\right)}\right), $$

where $b_i(t) = e^{-\sigma_i \epsilon(t)}$ for suitable constants $\sigma_i$ and

$$ \epsilon(t) = \int_0^t \frac{1}{1 - 8\left(s_1^2 + s_2^2 + s_3^2\right)} u \, du. $$

Proof. Consider the following system

$$ \begin{cases} 
\dot{x} = -f(t)^2 \vartheta(A)\varrho(A)x, \\
x_0 = \psi_0, 
\end{cases} $$

(10)

where $f(t)$ is a positive function which will be defined later. Moreover, let $(f_1, f_2, f_3, f_4, f_5, f_6) = (e_1, J_0e_1, e_2, J_0e_2, e_3, J_0e_3)$ be an adapted frame of $\mathfrak{h}$ such that $\omega_0$ and $\psi_0$ are given by (1) and $A$ has the normal form (5). It is clear that

$$ \vartheta(A)\varrho(A)\psi_0 = - (s_1 + s_2 + s_3)^2 f^{246} + (s_1 + s_2 - s_3)^2 f^{136} + (s_1 - s_2 + s_3)^2 f^{145} + (-s_1 + s_2 + s_3)^2 f^{235}. $$
So
\[ \vartheta(A)\vartheta(A)\psi_0 = -\sigma_1 f^{246} + \sigma_2 f^{136} + \sigma_3 f^{145} + \sigma_4 f^{235}, \]
for suitable constants \( \sigma_1, \sigma_2, \sigma_3 \) and \( \sigma_4 \). The solution of (10) is then given by
\[ \chi_t = -b_1(t) f^{246} + b_2(t) f^{136} + b_3(t) f^{145} + b_4(t) f^{235}, \quad (11) \]
where \( b_i(t) = e^{-\sigma_i \epsilon(t)} \) for a function \( \epsilon(t) \) satisfying \( \dot{\epsilon}(t) = f(t)^2 \). In order to determine the function \( f(t) \), note that, for every \( t \) where it is defined, the 3-form \( \chi_t \) is negative, compatible with \( \omega_0 \) and it defines a complex structure \( J_t \), given by
\[ J_t = \frac{2}{\sqrt{-\nu_t}} \begin{bmatrix} 0 & -b_4 b_1 & 0 & 0 & 0 & 0 \\ b_2 b_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_3 b_1 & 0 & 0 \\ 0 & 0 & b_2 b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_2 b_1 \\ 0 & 0 & 0 & 0 & b_3 b_4 & 0 \end{bmatrix}, \quad \nu_t = -4b_1^2 b_2^2 b_3^2 b_4^2, \quad (12) \]
with respect to the adapted frame \( (f_1, f_2, f_3, f_4, f_5, f_6) \). Moreover,
\[ 6 \chi_t \wedge J_t^* \chi_t = 4b_1^2 b_2^2 b_3^2 b_4^2 \omega_0^3. \]
The previous condition is satisfied if we choose \( f(t) \) such that
\[ f(t)^{-2} = b_1^2 b_2^2 b_3^2 b_4^2 = e^{-2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) \int_0^t f(u)^2 du} = e^{-8(s_1^2 + s_2^2 + s_3^2) \int_0^t f(u)^2 du}. \]
The above identity is satisfied if and only if the function \( F_t = \int_0^t f(u)^2 du \) solves the following Cauchy problem
\[ \begin{cases} \dot{F}_t = e^{8\delta F_t}, \\ F_0 = 0, \end{cases} \]
where \( \delta = s_1^2 + s_2^2 + s_3^2 \). Integrating we get
\[ t = \frac{1 - e^{-8\delta F_t}}{8\delta}. \]
Therefore
\[ F_t = \frac{\ln(1 - 8\delta t)}{-8\delta} \]
and consequently \( f(t) = \frac{1}{\sqrt{1 - 8\delta t}} \). Finally we observe that the metric \( h_t \) defined by \( (\omega_0, J_t) \) is positive definite. Moreover, the endomorphism \( H_t \), defined by \( g_0(x, H_t y) = h_t(x, y) \) for
any \( x, y \in \mathfrak{h} \), has the following matrix representation

\[
H_t = \frac{2}{\sqrt{-\nu_t}} \begin{bmatrix}
  b_2b_3 & 0 & 0 & 0 & 0 & 0 \\
  0 & b_1b_4 & 0 & 0 & 0 & 0 \\
  0 & 0 & b_2b_4 & 0 & 0 & 0 \\
  0 & 0 & 0 & b_1b_3 & 0 & 0 \\
  0 & 0 & 0 & 0 & b_3b_4 & 0 \\
  0 & 0 & 0 & 0 & 0 & b_1b_2
\end{bmatrix}, \tag{13}
\]

with respect to the adapted frame \((f_1, f_2, f_3, f_4, f_5, f_6)\). Now we claim that \( \chi_t \), given by

\[
\epsilon(t) = \int_0^t f(u)^2 \, du = \int_0^t \frac{1}{1 - 8 (s^2 + s^2 + s^2)} \, du,
\]

is the solution of (7). To see this, first observe that the choice of \( f(t) \) ensures that \( \epsilon(\chi_t)^2 = f(t)^2 \). The only thing we have to prove is that the adjoint \( C_t \) of \( A \) with respect to \( h_t \) is constant. It is clear that \( C_t = H_t^{-1}B_0H_t \) and then the claim is equivalent to show that \([H_t, B_0] = 0\). With respect to the adapted frame \((f_1, f_2, f_3, f_4, f_5, f_6)\) the endomorphism \( H_t \) is diagonal as well as \( B_0 = A \), and then the claim follows. Thus the solution \( p_t \) of (7) is given by \( \chi_t \), and in particular \( B_t \equiv B_0 \). \( \square \)

**Remark 3.4.** The previous proof can be adapted to the case \( \theta = \pi \). Indeed, if \( \theta = \pi \) then \( \vartheta(A) \vartheta(A) \psi_0 \) is again a linear combination of elements of the form \( e^a \wedge e^b \wedge J^*_t e^c \) with coefficients given by a suitable choice of \( \pm (s_a + s_b - s_c) \). On the other hand, when \( \theta \) is different from 0 and \( \pi \), it turns out that \( \vartheta(A) \vartheta(A) \psi_0 \) is a linear combination of elements \( e^a \wedge e^b \wedge J^*_t e^c \) and \( e^a \wedge J^*_t e^b \wedge J^*_t e^c \). Therefore the derivative of \( J^*_t \) at \( t = 0 \) is much more complicated than in the other cases (see Remark 3.6).

**Theorem 3.5.** Let \((\mathfrak{g} = \mathbb{R} e_7 \ltimes_{+} \mathfrak{h}, \varphi_0)\) be an almost-abelian Lie algebra endowed with a coclosed G2-structure \( \varphi_0 \). Let \((\omega_0, \psi)\) be the induced SU(3)-structure on \( \mathfrak{h} \) defined by (2), with \( \eta(e_7) \neq 0 \), \( \eta|_{\mathfrak{h}} = 0 \) and \( \|\eta\|_{g_{\varphi_0}} = 1 \). Suppose that \( A = \text{ad}_{e_7}|_{\mathfrak{h}} \) is skew-symmetric with respect to the inner product \( h_0 = g_0|_{\mathfrak{h}} \) and define \( l = l_1 + l_2 + l_3 \), where \( l_1, l_2 \) and \( l_3 \) are as in Lemma 3.7. Then the solution \( p_t \) of (7) is given by

\[
p_t = b(t)\psi_0,
\]

where \( b(t) = e^{-t^2 \int_0^t \varepsilon_u^2 \, du} \) and \( \varepsilon_t \) is a positive function given by

\[
\varepsilon_t = \frac{1}{\sqrt{1 - 2l^2 t}}.
\]

In particular, \( p_t \) is an ancient solution, defined for every \( t \) in \(( -\infty, \frac{1}{2l^2} ) \).

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Proof. Let $f_t$ be a positive function which will be fixed later and let us consider the following system

$$
\begin{aligned}
\dot{\chi}_t &= -f_t^2 \vartheta(A) \vartheta(-A) \chi_t, \\
\chi_0 &= \psi_0.
\end{aligned}
$$

(14)

Moreover, let $(f_1, \ldots, f_6) = (e_1 J_0 e_1, e_2 J_0 e_2, e_3 J_0 e_3)$ be an adapted frame such that $\omega_0$ and $\psi_0$ are given by (4) and $A$ has the normal form (5). It is clear that

$$
\vartheta(A) \vartheta(A) \psi_0 = -l^2 \psi_0.
$$

Therefore the solution of (14) is given by

$$
\chi_t = b(t) \psi_0,
$$

where $b(t) = e^{-l^2 f_t^2} f_t^2 du$.

The 3-form $\chi_t$ is negative, compatible with $\omega_0$ and it defines a constant complex structure $J_t \equiv J_0$. Moreover, it satisfies

$$
6 \chi_t \wedge I_t \chi_t = 4 b^2(t) \omega^3.
$$

Now we choose $f_t$ so that

$$
f_t^{-2} = b(t)^2 = e^{-2l^2 f_t^2} f_t^2 du.
$$

To do this we solve the system

$$
\begin{aligned}
\dot{F}_t &= e^{2l^2 F_t} du, \\
F_0 &= 0,
\end{aligned}
$$

and then we put $f_t = \sqrt{F_t}$. Integrating by $t$ we get

$$
t = \frac{1 - e^{2l^2 F_t}}{2l^2}.
$$

Thus

$$
F_t = \frac{\ln(1 - 2l^2 t)}{-2l^2}
$$

and consequently $\varepsilon_t = \frac{1}{\sqrt{1 - 2l^2 t}}$.

Now it is easy to show that $\chi_t$ is a solution of (7). Indeed, the choice of $f_t$ ensures that $\varepsilon(\chi_t)^2 = f_t^2$ and moreover that the metric $h_t$ induced by $\omega_0$ and $\chi_t$ is constant. Therefore the adjoint of $A$ is constantly equal to $-A$ and, as a consequence, the solution $p_t$ of (7) is given by $\chi_t$.

Remark 3.6. It is not hard to prove that if $A$ is normal with respect to $h_0$, then the solution $p_t$ of (7) is given by

$$
p_t = -b_1(t) f^2 \gamma_6 + b_2(t) f^{13 \gamma_6} + b_3(t) f^{145} + b_4(t) f^{235} + c_1(t) f^{135} - c_2(t) f^{245} - c_3(t) f^{236} - c_4(t) f^{145},
$$
where \((f_1, \ldots, f_6) = (e_1, J_0 e_1, e_2, J_0 e_2, e_3, J_0 e_3)\) is an adapted frame of \(\mathfrak{h}\) and \(A\) is given by \([5]\). Unfortunately in this case we cannot find an explicit solution of \([7]\).

However, note that if we write \(p_t = (x_t)^* \psi_0\), for \([x_t] \in \text{GL}(\mathfrak{h})/\text{SL}(3, \mathbb{C})\), then \(x_t\) belongs to \(\text{GL}^+(2, \mathbb{R})^3\) acting on \(< e_1, J_0 e_1 > \oplus < e_2, J_0 e_2 > \oplus < e_3, J_0 e_3 >\). Therefore \([x_t] = ([x_t^{(1)}], [x_t^{(2)}], [x_t^{(3)}]) \in ((\text{GL}^+(2, \mathbb{R})/\text{SO}(2))^3\).

### 4 Solitons for the Laplacian coflow on almost-abelian Lie groups

In this section we find sufficient conditions for a left-invariant coclosed \(G_2\)-structure on an almost-abelian Lie group \(G\) to be a soliton for the Laplacian coflow.

Let \(\mathfrak{g}\) be a Lie algebra. We recall the following

**Definition 4.1.** Let \(\mathfrak{g}\) be a seven-dimensional Lie algebra endowed with a coclosed \(G_2\)-structure \(\varphi_0\). A solution \(\phi_t\) to the Laplacian coflow \([6]\) on \(\mathfrak{g}\) is **self-similar** if

\[\phi_t = c_t (x_t)^* \phi_0,\]

for a real-valued function \(c_t\) and a \(\text{GL}(\mathfrak{g})\)-valued function \(x_t\).

It is well-known that a solution \(\phi_t\) of \([6]\) is self-similar if and only if the Cauchy datum \(\phi_0\) at \(t = 0\) is a **soliton**, namely if it satisfies

\[-\Delta_0 \phi_0 = -4c \phi_0 + \vartheta(D) \phi_0,\]

for some real number \(c\) and some derivation \(D\) of the Lie algebra \(\mathfrak{g}\) (see \([16]\)). A soliton is said to be **expanding** if \(c < 0\), **shrinking** if \(c > 0\) and **steady** if \(c = 0\).

Let \(K_t\) be the stabiliser of \(\phi_t\) and fix a \(K_t\)-invariant decomposition of \(\text{End}(\mathfrak{g})\). Since \(\phi_t\) is stable at any time \(t\), there exists a time-dependent endomorphism \(X_t\) of \(\mathfrak{g}\), transversal to the Lie algebra of \(K_t\) (in the sense that, for every \(t\), \(X_t\) takes values in an \(ad\)-invariant complement of the Lie algebra of \(K_t\)), such that

\[-\Delta_t \phi_t = \vartheta(X_t) \phi_t.\]

Therefore \(\phi_0\) is a soliton on \(\mathfrak{g}\) if and only if

\[X_0 = c \text{Id} + D.\]

Suppose now that \((\mathfrak{g} = \mathbb{R} e_7 \ltimes_A \mathfrak{h}, \varphi_0)\) is an almost-abelian Lie algebra endowed with a coclosed \(G_2\)-structure \(\varphi_0\). In Lemma \([3.2]\) we have seen that, with no further assumptions on \(A = ad_{e_7}|_{\mathfrak{h}}\), the Laplacian coflow reads as

\[
\begin{cases}
\frac{d}{dt} \phi_t = -\varepsilon_t \vartheta(A) \vartheta(B_t) \psi_t \wedge \eta, \\
\phi_0 = *_0 \varphi_0.
\end{cases}
\]
We can show that the term $-\varepsilon_t \vartheta(A)\vartheta(B_t)\psi_t \wedge \eta$ can be re-written as

$$-\varepsilon_t (\vartheta(A)\vartheta(B_t)\psi_t) \wedge \eta = \vartheta(X_t)\phi_t,$$

for a time-dependent endomorphism $X_t$ of $\mathfrak{g}$ in the following way.

Let $(\omega_t, \psi_t)$ be the SU(3)-structure on $\mathfrak{h}$ induced by $\phi_t$. By Lemma 3.1 there exist $\theta(t) \in [0, 2\pi]$ and an adapted frame of $\mathfrak{h}$ such that $\eta = \varepsilon_t e^\theta$ and the symmetric part $S(t)$ of $A$ has the normal form [5]. More precisely, let

$$a(t) = \cos(\theta(t)), \quad b(t) = \sin(\theta(t)).$$

With respect to the adapted frame at time $t$, $S(t)$ has the form [5], so it is given by

$$S(t) = \begin{bmatrix} S_1(t) & 0 & 0 & 0 \\ 0 & S_2(t) & 0 & 0 \\ 0 & 0 & S_3(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$S_i(t) = \begin{bmatrix} a(t)s_i(t) & b(t)s_i(t) \\ b(t)s_i(t) & -a(t)s_i(t) \end{bmatrix}.$$

Define $l(t)$ to be the imaginary part of the complex trace of the skew-symmetric part $L(t)$ of $A$ at time $t$ and let

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t) & 0 & 0 & 0 \\ 0 & \Sigma_2(t) & 0 & 0 \\ 0 & 0 & \Sigma_3(t) & 0 \\ 0 & 0 & 0 & -s^2(t) \end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l(t)^2 \end{bmatrix}$$

(16)

where

$$\Sigma_1(t) = \begin{bmatrix} -2s_2(t)s_3(t) + 4a(t)^2s_2(t)s_3(t) & -4a(t)b(t)s_2(t)s_3(t) \\ -4a(t)b(t)s_2(t)s_3(t) & 2s_2(t)s_3(t) - 4a(t)^2s_2(t)s_3(t) \end{bmatrix},$$

$$\Sigma_2(t) = \begin{bmatrix} -2s_1(t)s_3(t) + 4a(t)^2s_1(t)s_3(t) & -4a(t)b(t)s_1(t)s_3(t) \\ -4a(t)b(t)s_1(t)s_3(t) & 2s_1(t)s_3(t) - 4a(t)^2s_1(t)s_3(t) \end{bmatrix},$$

$$\Sigma_3(t) = \begin{bmatrix} -2s_1(t)s_2(t) + 4a(t)^2s_1(t)s_2(t) & -4a(t)b(t)s_1(t)s_2(t) \\ -4a(t)b(t)s_1(t)s_2(t) & 2s_1(t)s_2(t) - 4a(t)^2s_1(t)s_2(t) \end{bmatrix},$$

and $s(t)^2 = s_1(t)^2 + s_2(t)^2 + s_3(t)^2$. We claim that

$$X_t = -\varepsilon_t (\Sigma(t) + \Lambda(t) - [S(t), L(t)]).$$

(17)

To prove this first observe that

$$\vartheta(A)\vartheta(B_t)\psi_t = \vartheta(S(t) + L(t))\vartheta(S(t) - L(t))\psi_t = (\vartheta(S(t))\vartheta(S(t)) - \vartheta(L(t))\vartheta(L(t)) - \vartheta([S(t), L(t)]) \psi_t.$$
Then a direct computation shows that
\[
\vartheta(\Sigma(t))\phi_t = \vartheta(\Sigma(t))(\psi_t \land \eta) = (\vartheta(S(t))\vartheta(S(t))\psi_t) \land \eta.
\]
On the other hand if we change the adapted frame so that \(L_t\) has the form \([5]\) we see that, as already seen in the proof of Theorem 3.5, \(l(t) = l_1(t) + l_2(t) + l_3(t)\) and
\[
-\vartheta(L(t))\vartheta(L(t))\psi_t = l_2(t)\psi_t,
\]
which is an expression independent on the choice of the adapted frame. Thus
\[
\vartheta(\Lambda(t))\phi_t = \vartheta(\Lambda(t))(\psi_t \land \eta) = l(t)^2\psi_t \land \eta = -(\vartheta(L(t))\vartheta(L(t))\psi_t) \land \eta,
\]
proving the claim.

**Theorem 4.2.** Let \((\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)\) be an almost-abelian Lie algebra endowed with a coclosed \(G_2\)-structure \(\varphi_0\). The 4-form \(\phi_0 = *_0\varphi_0\) is a soliton for the Laplacian coflow if and only if \(A\) satisfies
\[
[-\Sigma(0) + 1/2[A, A^\dagger], A] = \delta A, \tag{18}
\]
where \(A^\dagger\) denotes the transpose of \(A\) with respect to the underlying metric on \(\mathfrak{h}\), \(\delta = l_0^2 + s_0^2 - c\) for a constant \(c \in \mathbb{R}\) and \(\Sigma(0)\) the endomorphism \([16]\). If \(\phi_0\) is a soliton, then the solution \(\phi_t\) to the Laplacian coflow is given by
\[
\phi_t = c(t)e^{f(t)D} \phi_0, \quad c(t) = (1 - 2ct)^2, \quad f(t) = -\frac{1}{2c}\ln(1 - 2ct), \quad t < \frac{1}{2c}, \tag{19}
\]
where the derivation \(D\) of \(\mathfrak{g}\) is given by \(X_0 - c\text{Id}\), with \(X_0\) as in \([17]\).

**Proof.** In the light of the previous results we can write down the soliton equation for the Laplacian coflow as follows. Suppose that \(\phi_0\) is a soliton, that is, \(X_0 = c\text{Id} + D\) for some \(c \in \mathbb{R}\) and a derivation \(D\) of \(\mathfrak{h}\). Then, by \([16\text{ Theorem 4.10}]\), \(19\) holds. Therefore, \(A\) corresponds to a soliton if and only if there exists \(c \in \mathbb{R}\) such that \(D = X_0 - c\text{Id}\) is a derivation of \(\mathfrak{g}\).

This condition can be read as a system of algebraic equations for \(c\) and the elements of the matrix associated to \(A\). Note that \(De_7 = \delta e_7\), with \(\delta = l_0^2 + s_0^2 - c\). Hence, denoting by \(\mu_A\) the Lie bracket structure defined by \(A\),
\[
[D, A]v = DAv - ADv = D\mu_A(e_7, v) - \mu_A(e_7, Dv) = \mu_A(De_7, v) = \delta Av, \quad v \in \mathfrak{h}.
\]
This reads as
\[
[D, A] = \delta A. \tag{20}
\]
Finally, writing \(D = X_0 - c\text{Id}\) for \(X_0\) as in \([17]\) and recalling that \([A, A^\dagger] = -2[S(0), L(0)]\), we derive \([18]\) from \([20]\).
We call \[ (18) \] the soliton equation of the almost-abelian Laplacian coflow.

Notice that we can split the soliton equation into two coupled equations, for the symmetric and skew symmetric parts of \( A \), in the following way. Since the commutator of two symmetric matrices is skew-symmetric and the commutator of a symmetric matrix and a skew-symmetric one is symmetric, we find

\[
[-\Sigma(0) + [S(0), L(0)], L(0)] = \delta S(0), \quad [-\Sigma(0) + [S(0), L(0)], S(0)] = \delta L(0).
\]

We have just proved the following result.

**Corollary 4.3.** If a soliton \( \phi_0 \) of the Laplacian coflow on the almost-abelian Lie algebra \( g = \mathbb{R} e_7 \ltimes_A \mathfrak{h} \) is an eigenform of the Laplacian then it is harmonic, namely \( A \in \mathfrak{su}(6) \).

**Proof.** Clearly, if \( \phi_0 \) is an eigenform of the Laplacian, then \( D = 0 \), hence \( X_0 = c \text{Id} \). Taking the trace of \( X_0|_h = c \text{Id}|_h \) we find \( c = 0 \). \( \Box \)

**Corollary 4.4.** Let \( (g = \mathbb{R} e_7 \ltimes_A \mathfrak{h}, \varphi_0) \) be an almost-abelian Lie algebra endowed with a coclosed \( G_2 \)-structure \( \varphi_0 \). Assume that \( A \) is normal with respect to the underlying metric. Then \( \varphi_0 = *_0 \varphi_0 \) is soliton on \( g \) if and only if \( 4b(1 - 4a^2)s_1s_2s_3 = 0 \) and \( [\Sigma(0), L(0)] = 0 \). In such a case \( \delta = 0 \) and hence \( c \geq 0 \).

**Proof.** By hypotheses, equations \[ (21) \] reduce to

\[
- [\Sigma(0), S(0)] = \delta L(0), \quad - [\Sigma(0), L(0)] = \delta S(0).
\]

Using the normal form \[ (5) \], a direct computation shows that

\[
[\Sigma(0), S(0)] = 4b(1 - 4a^2)s_1s_2s_3J_0.
\]

If \( \delta \) was different from zero, then \( S(0) \) would be invertible (each \( s_i \) should be non-zero) and therefore \( L(0) \) would be a non-zero multiple of \( J_0 \), contradicting Lemma \[ 3.1 \] Thus \( \delta = 0 \), that is \( 4b(1 - 4a^2)s_1s_2s_3 = 0 \). Clearly, if \( L(0) \neq 0 \), the equation \( [\Sigma(0), L(0)] = 0 \) is not generically satisfied. \( \Box \)

**Corollary 4.5.** Let \( (g = \mathbb{R} e_7 \ltimes_A \mathfrak{h}, \varphi_0) \) be an almost-abelian Lie algebra endowed with a coclosed \( G_2 \)-structure \( \varphi_0 \) and suppose that \( A = \text{ad}_{e_7}|_h \) is skew-symmetric with respect to the underlying metric. Then the solution to the Laplacian coflow obtained in Theorem \[ 3.5 \] is a soliton.

**Remark 4.6.** Differently from the Laplacian flow studied in \[ 16 \] there exist almost-abelian Lie algebras \( g = \mathbb{R} e_7 \ltimes_A \mathfrak{h} \) with \( A \) symmetric and admitting coclosed \( G_2 \)-structures that are no solitons for the Laplacian coflow. Indeed, in the light of Corollary \[ 4.4 \] it is enough to choose a symmetric matrix \( A \) and a suitable \( G_2 \)-structure for which the constant \( 4b(1 - 4a^2)s_1s_2s_3 \) is non-zero. For instance we can consider the \( G_2 \)-structure

\[
\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}
\]
on the Lie algebra $\mathfrak{g} = \mathbb{R}e_7 \ltimes A \mathfrak{h}$, where $\mathfrak{h} = \mathbb{R} < e_1, \ldots, e_6>$ and

$$A = \text{ad}_{e_7}|_h = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Moreover we are able to prove that the interval of existence of the corresponding solution is bounded. To this aim, and in analogy with the proof of Theorem 3.3, observe that the solution to (10) has the following expression:

$$\chi_t = -b_1(t)e^{246} + b_2(t)e^{136} + b_3(t)e^{145} + b_4(t)e^{235}.$$ 

Indeed

$$\vartheta(A)\vartheta(A)\chi_t = -(3b_1(t) - 2b_2(t) - 2b_3(t) - 2b_4(t))e^{246} + (-2b_1(t) + 3b_2(t) + 2b_3(t) + 2b_4(t))e^{136} + (-2b_1(t) + 2b_2(t) + 3b_3(t) + 2b_4(t))e^{145} + (-2b_1(t) + 2b_2(t) + 3b_3(t) + 3b_4(t))e^{235}.$$ 

and therefore the vector-valued function $(b_1(t), b_2(t), b_3(t), b_4(t))$ satisfies a linear ODE whose matrix is

$$-f_t^2 \begin{pmatrix} 3 & -2 & -2 & -2 \\ -2 & 3 & 2 & 2 \\ -2 & 2 & 3 & 2 \\ -2 & 2 & 2 & 3 \end{pmatrix}.$$ 

Taking into account that this matrix is symmetric, with eigenvalues $-9f_t^2, -f_t^2, -f_t^2, -f_t^2$ and eigenvectors $(-1, 1, 1, 1), (1, 1, 0, 0), (1, 0, 1, 0)$ and $(1, 0, 0, 1)$, it follows that

$$2b_1(t) = -e^{-9f_t^2}f_0^2\int_0^t f_u^2du + 3e^{-f_t^2}f_0^2du, \quad 2b_2(t) = 2b_3(t) = 2b_4(t) = e^{-9f_t^2}f_0^2\int_0^t f_u^2du + e^{-f_t^2}f_0^2du.$$ 

The function $F_t = \int_0^t f_u^2du$ can be fixed, as we did in Theorem 3.3 by imposing

$$1 = F_t b_1^2(t)b_2^2(t)b_3^2(t)b_4^2(t) = F_t (e^{-9F_t} + 3e^{-F_t})^2 (e^{-9F_t} + e^{-F_t})^6 \frac{1}{32}.$$ 

This guarantees that $\chi_t$ actually solves (7) (note also that $A$ is symmetric for any time).

Notice that the previous equation, after integration, ensures that, since $F_t \geq 0$ if and only if $t \geq 0$, the solution extinguishes in finite time. With an analogous argument we see that $F_t$ cannot exist for any negative time. To be more precise let $I$ be the maximal interval of existence of $F$, then

$$32t = \int_0^{F_t} (-e^{-9x} + 3e^{-x})^2(e^{-9x} + e^{-x})^6 dx, \quad t \in I.$$
We immediately see that \( \sup_I < +\infty \). On the other hand, if \( \inf_I = -\infty \) then \( F_t \) should be unbounded near \(-\infty\): indeed when \( M < F_t < 0 \) it turns out that

\[
32t = \int_0^{F_t} (-e^{-9x} + 3e^{-x})^2(e^{-9x} + e^{-x})^6 dx > \int_0^M (-e^{-9x} + 3e^{-x})^2(e^{-9x} + e^{-x})^6 dx.
\]

Therefore it would exist a sufficiently large negative time \( t \) such that \( 0 = -e^{-9F_t} + 3e^{-F_t} = 2b_1(t) \). Clearly this cannot happen because \( \chi_t \) must be a stable and negative form. By these considerations we also deduce that the only negative singular time \( \tau \) for the monotone function \( F_t \) satisfies \( b_1(\tau) = 0 \), that is \( F_\tau = -1/8\ln(3) \).

We will now construct an explicit example of soliton on a nilpotent almost-abelian Lie group.

**Example 4.7.** Let \( \mathfrak{g} \) be the nilpotent almost-abelian Lie algebra with structure equations

\[
de^1 = e^{27},
\]
\[
de^j = 0, \quad j = 2, 4, 6, 7,
\]
\[
de^3 = e^{47},
\]
\[
de^5 = e^{67}.
\]

Then in this case we have \( \mathfrak{h} = \mathbb{R} < e_1, \ldots, e_6 > \) and

\[
A = \text{ad}_{e_7}|_{\mathfrak{h}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Consider the \( G_2 \)-structure \( \varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \). The 4-form

\[
\phi_0 = \ast_{\varphi_0} \varphi_0 = e^{1234} + e^{3456} + e^{1256} - e^{2467} + e^{1367} + e^{1457} - e^{2357}
\]

is closed and thus \( \varphi \) defines a coclosed \( G_2 \)-structure. The basis \( (e_1, \ldots, e_7) \) is orthonormal with respect to \( g_{\varphi_0} \) and one can check that \( A \) is not normal. We will apply Theorem 4.2 to show that \( \phi_0 \) is a soliton for the Laplacian coflow. First observe that \( S(0) \) and \( L(0) \), on \( \mathfrak{h} \), restrict to

\[
\begin{pmatrix}
0 & 1/2 & 0 & 0 & 0 & 0 \\
-1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & -1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 0 & -1/2 & 0
\end{pmatrix}.
\]
So $S(0)$ is in normal form and therefore the matrix $\Sigma(0)$, restricted to $\mathfrak{h}$, turns out to be
\[
\begin{pmatrix}
-1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 \\
\end{pmatrix}.
\]
A direct computation then shows that $[S(0),L(0)] = \Sigma(0)$ on $\mathfrak{h}$, which leads to $[-\Sigma(0) + [S(0),L(0)],A] = 0$, so $A$ solves the soliton equation for $\delta = 0$. In particular we have $s_1(0) = s_2(0) = s_3(0) = 1/2$ and $l(0) = l_1(0) + l_2(0) + l_3(0) = 3/2$. Thus
\[
s^2(0) = 3/4, \quad l^2(0) = 9/4
\]
and $c = 3$. Then the associated derivation $D$ is given by $D = X - 3 \text{Id}$, with
\[
X = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{pmatrix},
\]
and the existence interval is $(-\infty, 1/6)$. Note that $\phi_0$ is not an eigenform of the Laplacian since
\[
\vartheta(X)\phi_0 = -3(-e^{2467} + e^{1367} + e^{1457} - e^{2357}).
\]

**Acknowledgements**

The authors are supported by the project PRIN “varietà reali e complesse: geometria, topologia e analisi armonica” and by GNSAGA. of I.N.d.A.M. The authors also thank the anonymous referees for all the useful improvements they suggested.

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