Symmetric Reduction of Regular Controlled Lagrangian System with Momentum Map

Hong Wang
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P.R.China
E-mail: hongwang@nankai.edu.cn

March 12, 2021

Abstract. In this paper, following the ideas in Marsden et al. [18], we set up the regular reduction theory of a regular controlled Lagrangian (RCL) system with symmetry and momentum map, by using Legendre transformation and Euler-Lagrange vector field, and this reduction is an extension of symmetric reduction theory of a regular Lagrangian system under regular controlled Lagrangian equivalence conditions. Considering the completeness of reduction, in order to describe uniformly the RCL systems defined on a tangent bundle and on its regular reduced spaces, we first define a kind of RCL systems on a symplectic fiber bundle. Then we give a good expression of the dynamical vector field of the RCL system, such that we can describe the RCL-equivalence for the RCL systems. Moreover, we introduce regular point and regular orbit reducible RCL systems with symmetries and momentum maps, by using the reduced Lagrange symplectic forms and the reduced Euler-Lagrange vector fields, and prove the regular point and regular orbit reduction theorems for the RCL systems and regular Lagrangian systems, which explain the relationships between RpCL-equivalence, RoCL-equivalence for the reducible RCL systems with symmetries and RCL-equivalence for the associated reduced RCL systems, as well as the relationship of equivalences of the regular reducible Lagrangian systems, $R_p$-reduced Lagrangian systems and $R_o$-reduced Lagrangian systems.

Keywords: regular controlled Lagrangian system, Legendre transformation, RCL-equivalence, momentum map, regular point reduction, regular orbit reduction.

AMS Classification: 70H33, 53D20, 70Q05.

Contents

1 Introduction 2

2 Legendre Transformation, Regular Lagrangian System and Its Reduction 4

3 Regular Controlled Lagrangian System and RCL-Equivalence 10

4 Regular Point Reduction of the RCL System 13

5 Regular Orbit Reduction of the RCL System 18
1 Introduction

It is well known that the theory of controlled mechanical systems has formed an important subject in recent thirty years. Its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system in a way that helps both for analysis and design. Thus, it is natural to study the controlled mechanical systems by combining with the analysis of dynamic systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. Following the theoretical development of geometric mechanics, a lot of important problems about this subject are being explored and studied.

In 2005, we hope to study the mechanical system with control from geometrical viewpoint. I and my students read the following two papers of Professor Marsden and his students on a seminar of Nankai University, see Chang et al. [8, 9]. We found that there are some serious wrong of rigor for the definitions of controlled Lagrangian (CL) system, controlled Hamiltonian (CH) system and the reduced CH systems, the reduced CL system, as well as CH-equivalence, CL-equivalence and the reduced CH-equivalence, the reduced CL-equivalence in this two papers. In Marsden et al. [18], we have corrected and renewed carefully some of these wrong definitions for the regular controlled Hamiltonian (RCH) system and the reduced RCH systems. In this paper, we shall consider the cases of the regular controlled Lagrangian (RCL) system and the reduced RCL systems.

There are the following three aspects of these wrong for CL system and the reduced CL systems in [8, 9]:

(1) The authors define CL system by using a wrong expression. In fact, the CL system is defined in [8, 9], by using the following expression

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = F(q, \dot{q}) + u(q, \dot{q}),$$

where $\mathcal{EL}$ is the Euler-Lagrange operator and the bundle maps $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$, and $F : TQ \to T^*Q$, and the control $u : TQ \to W(\subset T^*Q)$. This expression (1.1) can not be an equation, because the left side of (1.1) is defined on the second order tangent bundle $T^{(2)}Q$, and the right side of (1.1) is defined on the tangent bundle $TQ$, and $T^{(2)}Q$ and $TQ$ are different spaces. Thus, it is impossible to define the CL system by using a wrong expression. The similar wrong appears in the definition 2.4 of [9] for the reduced CL system. In addition, it is worthy of noting that the use of the above wrong expression (1.1) has led to the wrong of the method of controlled Lagrangians to judge the stabilization of mechanical systems, see Bloch et al. [4–7], and this is also a serious problem should to be corrected carefully in our next paper.

(2) The authors didn’t consider the phase spaces of CL system and the reduced CL system, that is, all of CL systems and the reduced CL systems given in [8, 9], have not the spaces on which these systems are defined, see Definition 2.1 in [8] and Definition 2.1, 2.3 in [9]. Thus, it is impossible to give the actions of a Lie group on the phase spaces of CL systems and their momentum maps, also impossible to determine precisely the reduced phase spaces of CL systems.

(3) The authors didn’t consider the momentum map of the CL system with symmetry, and didn’t consider yet the change of geometrical structures of the phase spaces of the CL systems, and hence cannot determine precisely the geometrical structures of phase spaces of the reduced CL systems. In fact, it is not that all of CL systems in [9] have same phase space $TQ$, same action of Lie group $G$, and same reduced phase space $TQ/G$. Different structures of geometry determine the different CL systems, the the different reduced CL systems and their phase spaces. Moreover, it is
also impossible to give precisely the relationship of the equivalences for the reduced CL systems, if don’t consider the different Lie group actions and momentum maps.

To sum up the above statement, we think that there are a lot of serious wrong of rigor for the definitions of CL system and its reduced CL systems, as well as CL-equivalence and the reduced CL-equivalence in Chang et al. [8,9], and we want to correct their work. It is important to find these wrong, but the more important is to correct well these wrong. It is worthy of noting that we can not define directly a CL system on the second order tangent bundle $T^{(2)}Q$ or on the tangent bundle $TQ$. Because for $\mu \in \mathfrak{g}^*$, a regular value of the momentum map $J\gamma, G\mu$ is the isotropy subgroup of the coadjoint $G$-action at the point $\mu$, the reduced second order tangent bundle $(T^{(2)}Q)/G\mu$ or the reduced tangent bundle $(TQ)/G\mu$ may not be a second order tangent bundle or not be a tangent bundle of a configuration manifold. If we define directly a CL system with symmetry on the second order tangent bundle $T^{(2)}Q$ or on the tangent bundle $TQ$, then the reduced CL system may not have definition. Thus, in order to set up the regular reduction theory for the CL system with symmetry, in this paper, following the ideas in Marsden et al. [18], we have to correct and renew carefully these wrong definitions in Chang et al. [8,9].

In this paper, we first consider that a regular controlled Lagrangian (RCL) system is a regular Lagrangian system with external force and control. In general, an RCL system under the action of external force and control is not a regular Lagrangian system, however, it is a dynamical system closely related to a regular Lagrangian system, and it can be explored and studied by extending the methods for external force and control in the study of regular Lagrangian systems. In consequence, we can set up the regular reduction theory for an RCL system with symmetry and momentum map, by analyzing carefully the geometrical and topological structures of the phase space and the reduced phase spaces of the corresponding regular Lagrangian system.

A brief of outline of this paper is as follows. In the second section, we review some relevant definitions and basic facts about the regular Lagrangian system and its regular point and regular orbit reductions; we also analyse the geometrical structures of the phase space and the reduced phase spaces of the regular Lagrangian system, which will be used in subsequent sections. An RCL system is defined by using a (Lagrangian) symplectic form on a symplectic fiber bundle and on the tangent bundle of a configuration manifold, respectively, and a good expression of the dynamical vector field for the RCL system is given, and RCL-equivalence is introduced in the third section. From the fourth section we begin to discuss the RCL systems with symmetries and momentum maps by combining with regular reduction theory of a regular Lagrangian system. The regular point and regular orbit reducible RCL systems are considered respectively in the fourth section and the fifth section, and prove the regular point and regular orbit reduction theorems for the RCL systems to explain the relationships between the RpCL-equivalence, RoCL-equivalence for the reducible RCL systems with symmetries and the RCL-equivalence for the associated reduced RCL systems. We also study the relationship of equivalences of the regular reducible Lagrangian systems, $R^{t\gamma}$-reduced Lagrangian systems and $R^{\gamma}$-reduced Lagrangian systems. These research work develop the theory of symmetric reduction for the RCL systems with symmetries and momentum maps, and make us have much deeper understanding and recognition for the structures of the regular controlled mechanical systems.
2 Legendre Transformation, Regular Lagrangian System and Its Reduction

In this section, we review some relevant definitions and basic facts about Legendre transformation, the regular Lagrangian system and its regular point and regular orbit reductions; we also analyse the geometrical structures of the phase space and the reduced phase spaces for a regular Lagrangian system, which will be used in subsequent sections. We shall follow the notations and conventions introduced in Abraham and Marsden [1], Abraham et al. [2], Marsden [13], Marsden et al. [14], Marsden and Ratiu [17], and Ortega and Ratiu [22]. For convenience, we assume that all manifolds in this paper are real, smooth and finite dimensional. In particular, we always assume that $Q$ is a smooth manifold with coordinates $q^i$, and $TQ$ its tangent bundle with coordinates $(q^i, \dot{q}^i)$, and $T^*Q$ its cotangent bundle with coordinates $(q^i, p_i)$, which is the canonical cotangent bundle coordinates of $T^*Q$ and $\theta_0 = p_i dq^i$ and $\omega_0 = -d\theta_0 = dq^i \wedge dp_i$ are canonical one-form and canonical symplectic form on $T^*Q$, respectively, where summation on repeated indices is understood.

**Definition 2.1** Assume that $Q$ is an $n$-dimensional smooth manifold and the function $L : TQ \to \mathbb{R}$. Then the map $FL : TQ \to T^*Q$ defined by

$$FL(v)w := \left. \frac{d}{dt} \right|_{t=0} L_q(v + tw), \ \forall \ v, w \in T_q Q,$$

(2.1)

is fiber-preserving smooth map, which is called the fiber derivative of $L$, where $L_q$ denotes the restrictions of $L$ to the fiber over $q \in Q$. If $FL : TQ \to T^*Q$ is a local diffeomorphism, then $L : TQ \to \mathbb{R}$ is called a regular Lagrangian; and if $FL : TQ \to T^*Q$ is a diffeomorphism, then $L$ is called hyperregular.

In the finite dimensional case, the local expression of the map $FL : TQ \to T^*Q$ is given by

$$FL(q^i, \dot{q}^i) = (q^i, \partial L / \partial \dot{q}^i) = (q^i, p_i).$$

(2.2)

The change of data from $(q^i, \dot{q}^i)$ on $TQ$ to $(q^i, p_i)$ on $T^*Q$, which is given by the map $FL : TQ \to T^*Q$, is called a Legendre transformation. From Marsden and Ratiu [17], we know that the Lagrangian $L$ is regular, if the matrix $(\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)$ is invertible. In the following by using the Legendre transformation, we can give a definition of a regular Lagrangian system as follows.

**Definition 2.2** (Regular Lagrangian System) Assume that $Q$ is a smooth manifold, and $\theta_0$ and $\omega_0$ are the canonical one form and the canonical symplectic form on the cotangent bundle $T^*Q$, and the function $L : TQ \to \mathbb{R}$ is hyperregular. Denote $\theta^L := (FL)^* \theta_0$ and $\omega^L := (FL)^* \omega_0$, where the bundle map $(FL)^* : T^*T^*Q \to T^*TQ$. Then $\theta^L$ and $\omega^L$ are called the Lagrangian one-form and the Lagrangian symplectic form on the tangent bundle $TQ$, respectively. Define an action $A : TQ \to \mathbb{R}$ given by $A(v) := FL(v)v$, $\forall v \in T_q Q$ and an energy $E_L : TQ \to \mathbb{R}$ given by $E_L := A - L$. If there exists a vector field $\xi_L$ on $TQ$, such that the Euler-Lagrange equation $\xi_L \omega^L = dE_L$ holds, then $\xi_L$ is called an Euler-Lagrange vector field of $L$, and the triple $(TQ, \omega^L, L)$ is called a regular Lagrangian system.

In the finite dimensional case, the local expression of $\theta^L$ and $\omega^L$ are given by

$$\theta^L = \frac{\partial L}{\partial q^i} dq^i, \quad \omega^L = \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge d\dot{q}^j,$$

where summation on repeated indices is understood. Moreover, we know that the energy $E_L$ is conserved along the flow of the Euler-Lagrange vector field $\xi_L$, if $\xi_L$ satisfies a second order equation,
that is, $T\tau_Q \circ \xi_L = id_{TQ}$, where the map $T\tau_Q : TTQ \to TQ$, is the tangent map of the projection $	au_Q : TQ \to Q$. Moreover, in a local coordinates of $TQ$, an integral curve $(q(t), \dot{q}(t))$ of $\xi_L$ satisfies the following Euler-Lagrange equations:

$$\frac{dq^i}{dt} = \dot{q}^i, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n.$$  

If $L$ is regular, then $\xi_L$ satisfies always the second order equation.

Furthermore, by using the Legendre transformation, the following proposition gives a description of the equivalence between the regular Lagrangian system $(TQ, \omega, L)$ and the Hamiltonian system $(T^*Q, \omega_0, H)$ under the hyperregular case of $L$, see Marsden and Ratiu [17].

**Proposition 2.3** Assume that $L : TQ \to \mathbb{R}$ is a hyperregular Lagrangian on $TQ$, and define a function $H := E_L \cdot (FL)^{-1} : T^*Q \to \mathbb{R}$. Then $H$ is a hyperregular Hamiltonian on $T^*Q$, and the Hamiltonian vector field $X_H \in TT^*Q$ and the Euler-Lagrange vector field $\xi_L \in TTQ$ are $FL$-related, i.e. $T(FL) \cdot \xi_L = X_H \cdot FL$, where $T(FL) : TTQ \to TT^*Q$ is the tangent map of $FL : TQ \to T^*Q$, and the integral curves of $\xi_L$ are mapped by $FL$ onto integral curves of $X_H$.

It is well-known that Hamiltonian reduction theory is one of the most active subjects in the study of modern analytical mechanics and applied mathematics, in which a lot of deep and beautiful results have been obtained, see the studies given by Abraham and Marsden [1], Abraham et al. [2], Arnold [3], Libermann and Marle [11], Marsden [13], Marsden et al. [14, 15], Marsden and Perlmutter [16], Marsden and Ratiu [17], Marsden and Weinstein [19], Meyer [20], Nijmeijer and Van der Schaft [21] and Ortega and Ratiu [22], in which the Marsden-Weinstein reduction for the Hamiltonian system with symmetry and momentum map is most important and foundational. Now, for a regular Lagrangian system with symmetry and momentum map, we can also give its regular point reduction as follows.

Let $Q$ be a smooth manifold and $TQ$ its tangent bundle with the induced Lagrangian symplectic form $\omega^L$. Assume that $\Phi : G \times Q \to Q$ be a smooth left action of a Lie group $G$ on $Q$, which is free and proper, then the tangent lifted left action $\Phi^T : G \times TQ \to TQ$ is also free, proper. Moreover, assume that the action is symplectic with respect to $\omega^L$, and admits an $Ad^*$-equivariant momentum map $J_L : TQ \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$. For a regular value of $J_L$, $\mu \in \mathfrak{g}^*$, denote $G_\mu = \{g \in G | Ad^*_g \mu = \mu\}$ the isotropy subgroup of the co-adjoint $G$-action at the point $\mu \in \mathfrak{g}^*$. Since $G_\mu(\subset G)$ acts freely and properly on $Q$ and on $TQ$, then $Q_\mu = Q/G_\mu$ is a smooth manifold and the canonical projection $\rho_\mu : Q \to Q_\mu$ is a surjective submersion. It follows that $G_\mu$ acts also freely and properly on $J_L^{-1}(\mu)$, so that the space $(TQ)_\mu = J_L^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form $\omega^L_\mu$ uniquely characterized by the relation

$$\tau^*_\mu \cdot \omega^L_\mu = j^*_\mu \cdot \omega^L.$$  

The map $j_\mu : J_L^{-1}(\mu) \to TQ$ is the inclusion and $\tau_\mu : J_L^{-1}(\mu) \to (TQ)_\mu$ is the projection. The pair $((TQ)_\mu, \omega^L_\mu)$ is called the regular point reduced space of $(TQ, \omega^L)$ at $\mu$.

Let $L : TQ \to \mathbb{R}$ be a $G$-invariant hyperregular Lagrangian, the flow $F_t$ of the Euler-Lagrange vector field $\xi_L$ leaves the connected components of $J_L^{-1}(\mu)$ invariant and commutes with the $G$-action, so it induces a flow $f^L_t$ on $(TQ)_\mu$, defined by $f^L_t \cdot \tau_\mu = \tau_\mu \cdot F_t \cdot j_\mu$, and the vector field $\xi_{l_\mu}$ generated by the flow $f^L_t$ on $(TQ)_\mu$, $\omega^L_\mu$ is the reduced Euler-Lagrange vector field with the associated regular point reduced Lagrangian function $l_\mu : (TQ)_\mu \to \mathbb{R}$ defined by $l_\mu \cdot \tau_\mu = L \cdot j_\mu$, and the reduced Euler-Lagrange equation $i_{\xi_{l_\mu}} \omega^L_\mu = dE_{l_\mu}$ holds, where the reduced energy $E_{l_\mu} : (TQ)_\mu \to \mathbb{R}$ is given by $E_{l_\mu} := A_{l_\mu} - l_\mu$, and the reduced action $A_\mu : (TQ)_\mu \to \mathbb{R}$ is given by
\[ A_\mu \cdot \tau_\mu = A \cdot j_\mu, \] and the Euler-Lagrange vector fields \( \xi_L \) and \( \xi_{\mu} \) are \( \tau_\mu \)-related. Thus, we can introduce a kind of regular point reducible Lagrangian systems as follows.

**Definition 2.4** (Regular Point Reducible Lagrangian System) A 4-tuple \((TQ, G, \omega^L, L)\), where the hyperregular Lagrangian \( L : TQ \to \mathbb{R} \) is \( G \)-invariant, is called a regular point reducible Lagrangian system, if there exists a point \( \mu \in g^* \), which is a regular value of the momentum map \( J_L \), such that the point regular reduced system, that is, the 3-tuple \(((TQ)_\mu, \omega^L_\mu, l_\mu)\), where \((TQ)_\mu = J_L^{-1}(\mu)/G_\mu, \tau_\mu = \omega^L_\mu, \) is a regular Lagrangian system, which is simply written as \( R_p \)-reduced Lagrangian system. Where \(((TQ)_\mu, \omega^L_\mu)\) is the \( R_p \)-reduced space, the function \( l_\mu : (TQ)_\mu \to \mathbb{R} \) is called the \( R_p \)-reduced Lagrangian.

We know that the orbit reduction of a Hamiltonian system is an alternative approach to symplectic reduction given by Kazhdan, Kostant and Sternberg [10] and Marle [12], which is different from the Marsden-Weinstein reduction. Now, for a regular Lagrangian system with symmetry and momentum map, we can also give its regular orbit reduction as follows, which is different from the above regular point reduction.

Assume that \( \Phi : G \times Q \to Q \) is a smooth left action of a Lie group \( G \) on \( Q \), which is free and proper, then the tangent lifted left action \( \Phi^T : G \times TQ \to TQ \) is also free and proper. Moreover, assume that the action is symplectic with respect to \( \omega_L \), and admits an \( Ad^* \)-equivariant momentum map \( J_L : TQ \to g^* \). For a regular value of the momentum map \( J_L, \mu \in g^* \), \( \mathcal{O}_\mu = G \cdot \mu \subset g^* \) is the \( G \)-orbit of the co-adjoint \( G \)-action through the point \( \mu \). Since \( G \) acts freely and properly and symplectically on \( TQ \) with respect to \( \omega_L \), then the quotient space \( (TQ)_{\mathcal{O}_\mu} = J_L^{-1}(\mathcal{O}_\mu)/G \) is a regular quotient symplectic manifold with the reduced symplectic form \( \omega^L_{\mathcal{O}_\mu} \) uniquely characterized by the relation

\[ j^*_{\mathcal{O}_\mu} \cdot \omega^L = \tau^*_{\mathcal{O}_\mu} \cdot \omega^L_{\mathcal{O}_\mu} + (J_L)_{\mathcal{O}_\mu} \cdot \omega^L_{\mathcal{O}_\mu}, \tag{2.4} \]

where \((J_L)_{\mathcal{O}_\mu}\) is the restriction of the momentum map \( J_L \) to \( J_L^{-1}(\mathcal{O}_\mu) \), that is, \((J_L)_{\mathcal{O}_\mu} = J_L \cdot j_{\mathcal{O}_\mu}\) and \( \omega^L_{\mathcal{O}_\mu} \) and \( \omega^L_{\mathcal{O}_\mu} \) are the +symplectic structures on the orbit \( \mathcal{O}_\mu \) given by

\[ \omega^L_{\mathcal{O}_\mu}(\nu)(\xi, \eta) = \omega^L_{\mathcal{O}_\mu}(\nu)(\xi_{\mathcal{O}_\mu}^*(\nu), \eta_{\mathcal{O}_\mu}^*(\nu)) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in \mathcal{O}_\mu, \xi, \eta \in g, \xi_{\mathcal{O}_\mu}^*, \eta_{\mathcal{O}_\mu}^* \in g^*. \tag{2.5} \]

The maps \( j_{\mathcal{O}_\mu} : J_L^{-1}(\mathcal{O}_\mu) \to TQ \) and \( \tau_{\mathcal{O}_\mu} : J_L^{-1}(\mathcal{O}_\mu) \to (TQ)_{\mathcal{O}_\mu} \) are natural injection and the projection, respectively. The pair \(((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu})\) is called the regular orbit reduced space of \((TQ, \omega^L)\) at the point \( \mu \).

Let \( L : TQ \to \mathbb{R} \) be a \( G \)-invariant hyperregular Lagrangian, the flow \( F_t \) of the Euler-Lagrange vector field \( \xi_L \) leaves the connected components of \( J_L^{-1}(\mathcal{O}_\mu) \) invariant and commutes with the \( G \)-action, so it induces a flow \( f_t^{\mathcal{O}_\mu} \) on \((TQ)_{\mathcal{O}_\mu}\), defined by \( f_t^{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = \tau_{\mathcal{O}_\mu} \cdot F_t \cdot j_{\mathcal{O}_\mu} \), and the vector field \( \xi_{\mathcal{O}_\mu} \) generated by the flow \( f_t^{\mathcal{O}_\mu} \) on \((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}\) is the reduced Euler-Lagrange vector field with the associated regular orbit reduced Lagrangian function \( i_{\xi_{\mathcal{O}_\mu}} \omega^L_{\mathcal{O}_\mu} = dE_{\mathcal{O}_\mu} \) holds, where the reduced energy \( E_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R} \) given by \( E_{\mathcal{O}_\mu} = A_{\mathcal{O}_\mu} - l_{\mathcal{O}_\mu} \), and the reduced action \( A_{\mathcal{O}_\mu} : (TQ)_{\mathcal{O}_\mu} \to \mathbb{R} \) given by \( A_{\mathcal{O}_\mu} \cdot \tau_{\mathcal{O}_\mu} = A \cdot j_{\mathcal{O}_\mu} \), and the Euler-Lagrange vector fields \( \xi_L \) and \( \xi_{\mathcal{O}_\mu} \) are \( \tau_{\mathcal{O}_\mu} \)-related. Thus, we can introduce a kind of the regular orbit reducible Lagrangian systems as follows.

**Definition 2.5** (Regular Orbit Reducible Lagrangian System) A 4-tuple \((TQ, G, \omega^L, L)\), where the hyperregular Lagrangian \( L : TQ \to \mathbb{R} \) is \( G \)-invariant, is called a regular orbit reducible Lagrangian system, if there exists an orbit \( \mathcal{O}_\mu, \mu \in g^* \), where \( \mu \) is a regular value of the momentum map \( J_L \), such that the regular orbit reduced system, that is, the 3-tuple \(((TQ)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu})\), where
(TQ)\omega = J_L^{-1}(O_\mu)/G, \tau_{(TQ)\mu} = j_0^* - (J_L)^* \cdot \omega_{(TQ)\mu}, \ell_{(TQ)\mu} : (TQ)\omega \rightarrow \mathbb{R}, is a regular Lagrangian system, which is simply written as $R_o$-reduced Lagrangian system. Where $((TQ)\omega, \omega_{(TQ)\mu})$ is the $R_o$-reduced Lagrangian system, which is simply written as $R_o$-reduced Lagrangian.

In the following we shall give a precise analysis for the geometrical structures of the regular point reduced space $((TQ)\mu, \omega_\mu^T)$ and the regular orbit reduced space $((TQ)\mu, \omega_\mu^T)$. Assume that the Lagrangian $L : TQ \rightarrow \mathbb{R}$ is hyperregular, then the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is a diffeomorphism. If the cotangent lift $G$-action $\Phi^{T} : G \times T^*Q \rightarrow T^*Q$ is free, proper and symplectic with respect to the canonical symplectic form $\omega_0$ on $T^*Q$, and has an $Ad^*$-equivariant momentum map $J : T^*Q \rightarrow g^*$ given by $<J(\alpha_q), \xi> = \alpha_q(\xi_Q(q))$, where $\alpha_q \in T_q^*Q$ and $\xi \in g^$, $\xi_Q(q)$ is the value of the infinitesimal generator $\xi_Q$ of the $G$-action at $q \in Q$, $\langle, \rangle : g^* \times g^* \rightarrow \mathbb{R}$ is the duality pairing between the dual $g^*$ and $g$. Then we have that the following theorem holds.

**Theorem 2.6** The momentum map $J_L : TQ \rightarrow g^*$ given by $J_L = J \cdot FL$, is $Ad^*$-equivariant, if the Lagrangian $L : TQ \rightarrow \mathbb{R}$ is hyperregular, and the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is $(\Phi^T, \Phi^{T*})$-equivariant. Moreover, if $\mu \in g^*$ is a regular value of the momentum map $J$, then $\mu$ is also a regular value of the momentum map $J_L$.

**Proof:** We first prove that the momentum map $J_L : TQ \rightarrow g^*$ is $Ad^*$-equivariant. Since the Lagrangian $L : TQ \rightarrow \mathbb{R}$ is hyperregular, then the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is a diffeomorphism. Because the momentum map $J : T^*Q \rightarrow g^*$ is $Ad^*$-equivariant, we have that $Ad^* : J = J \cdot \Phi^{T*}$. Note that the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is $(\Phi^T, \Phi^{T*})$-equivariant, then we have that $\Phi^{T*} \cdot \mathcal{F}L = \mathcal{F}L \cdot \Phi^T$. From the following commutative Diagram-1,

![Diagram-1](image)

we can obtain that

$$Ad^* \cdot J_L = Ad^* \cdot J \cdot \mathcal{F}L = J \cdot \Phi^{T*} \cdot \mathcal{F}L = J \cdot \mathcal{F}L \cdot \Phi^T = J_L \cdot \Phi^T.$$  

Thus, the momentum map $J_L : TQ \rightarrow g^*$ is $Ad^*$-equivariant.

Next, if $\mu \in g^*$ is a regular value of the momentum map $J$, then there exists an $\alpha \in T^*Q$, such that $J(\alpha) = \mu$. Since the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is a diffeomorphism, we have that $v = \mathcal{F}L^{-1}(\alpha) \in TQ$, such that

$$J_L(v) = J \cdot \mathcal{F}L(\mathcal{F}L^{-1}(\alpha)) = J(\alpha) = \mu.$$  

Thus, $\mu \in g^*$ is also a regular value of the momentum map $J_L$.

For a given $\mu \in g^*$, a regular value of the momentum map $J : T^*Q \rightarrow g^*$, denote by $G_\mu$ the isotropy subgroup of the co-adjoint $G$-action at the point $\mu$, then the Marsden-Weinstein reduced space $(T^*Q)_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with the symplectic form $\omega_\mu$ uniquely characterized by the relation

$$\pi_\mu \cdot \omega_\mu = i_\mu \cdot \omega_0. \tag{2.6}$$  

The map $i_\mu : J^{-1}(\mu) \rightarrow T^*Q$ is the inclusion and $\pi_\mu : J^{-1}(\mu) \rightarrow (T^*Q)_\mu$ is the projection. From Marsden et al. [14], we know that the classification of symplectic reduced spaces of a cotangent
bundle is given as follows. (1) If \( \mu = 0 \), the symplectic reduced space of cotangent bundle \( T^*Q \) at \( \mu = 0 \) is given by \( ((T^*Q)_{\mu}, \omega_\mu) = (T^*(Q/G), \hat{\omega}_0) \), where \( \hat{\omega}_0 \) is the canonical symplectic form of cotangent bundle \( T^*(Q/G) \). Thus, the symplectic reduced space \( ((T^*Q)_{\mu}, \omega_\mu) \) at \( \mu = 0 \) is a symplectic vector bundle. (2) If \( \mu \neq 0 \), and \( G \) is Abelian, then \( G_\mu = G \), in this case the regular point symplectic reduced space \( ((T^*Q)_{\mu}, \omega_\mu) \) is symplectically diffeomorphic to vector bundle \( (T^*(Q/G), \hat{\omega}_0 - B_\mu) \), where \( B_\mu \) is a magnetic term. (3) If \( \mu \neq 0 \), and \( G \) is not Abelian and \( G_\mu \neq G \), in this case the regular point symplectic reduced space \( ((T^*Q)_{\mu}, \omega_\mu) \) is symplectically diffeomorphic to a symplectic fiber bundle over \( T^*(Q/G_\mu) \) with fiber to be the co-adjoint orbit \( O_\mu \), see the cotangent bundle reduction theorem—bundle version, also see Marsden and Perlmutter [16]. Comparing the regular point reduced spaces \( ((TQ)_\mu, \omega^L_\mu) \) and \( ((T^*Q)_\mu, \omega_\mu) \) at the point \( \mu \), we have the following theorems.

**Theorem 2.7** Assume that the Lagrangian \( L : TQ \rightarrow \mathbb{R} \) is hyperregular, and the Legendre transformation \( FL : TQ \rightarrow T^*Q \) is \( (\Phi^T, \Phi^T*) \)-equivariant, then the regular point reduced space \( ((TQ)_\mu, \omega^L_\mu) \) of \( (TQ, \omega^L) \) at \( \mu \) is symplectically diffeomorphic to the regular point reduced space \( ((T^*Q)_\mu, \omega_\mu) \) of \( (T^*Q, \hat{\omega}_0) \) at \( \mu \), and hence is also symplectically diffeomorphic to a symplectic fiber bundle.

**Proof:** Since the Lagrangian \( L : TQ \rightarrow \mathbb{R} \) is hyperregular, then the Legendre transformation \( FL : TQ \rightarrow T^*Q \) is a diffeomorphism. Because \( FL \) is \( (\Phi^T, \Phi^T*) \)-equivariant, that is, \( \Phi^T \cdot FL = FL \cdot \Phi^T \), then we can define a map \( (FL)_\mu : (TQ)_\mu \rightarrow (T^*Q)_\mu \) given by \( (FL)_\mu \cdot \tau_\mu = \pi_\mu \cdot FL \), and \( i_\mu \cdot FL = FL \cdot j_\mu \), see the following commutative Diagram-2, which is well-defined and a diffeomorphism.

![Diagram-2](image)

We shall prove that \( (FL)_\mu \) is symplectic, that is, \( (FL)^*_\mu \cdot \omega_\mu = \omega^L_\mu \). In fact, from (2.6) and (2.3), we have that

\[
\tau^*_\mu \cdot (FL)^*_\mu \cdot \omega_\mu = ((FL)_\mu \cdot \tau_\mu)^* \cdot \omega_\mu = (\pi_\mu \cdot FL)^* \cdot \omega_\mu = (FL)^* \cdot \pi^*_\mu \cdot \omega_\mu
\]

\[
= (FL)^* \cdot j^*_\mu \cdot \omega_0 = (i_\mu \cdot FL)^* \cdot \omega_0 = (FL \cdot j_\mu)^* \cdot \omega_0
\]

\[
= j^*_\mu \cdot (FL)^* \cdot \omega_0 = j^*_\mu \cdot \omega^L = \tau^*_\mu \cdot \omega^L_\mu.
\]

Notice that \( \tau_\mu \) is surjective, and hence \( (FL)^*_\mu \cdot \omega_\mu = \omega^L_\mu \). Thus, the regular point reduced space \( ((TQ)_\mu, \omega^L_\mu) \) of \( (TQ, \omega^L) \) at \( \mu \) is symplectically diffeomorphic to the regular point reduced space \( ((T^*Q)_\mu, \omega_\mu) \) of \( (T^*Q, \hat{\omega}_0) \) at \( \mu \). From Marsden et al. [14], we know that the space \( ((T^*Q)_\mu, \omega_\mu) \) is symplectically diffeomorphic to a symplectic fiber bundle, and hence \( ((TQ)_\mu, \omega^L_\mu) \) is also symplectically diffeomorphic to a symplectic fiber bundle. 

For a given \( \mu \in g^* \), a regular value of the momentum map \( J : T^*Q \rightarrow g^* \), the regular orbit reduced space \( (T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G \) is a regular quotient symplectic manifold with the symplectic form \( \omega_{O_\mu} \) uniquely characterized by the relation

\[
i^*_\mu \cdot \omega_0 = \pi^*_\mu \cdot \omega_{O_\mu} + J^*_\mu \cdot \omega^+_{O_\mu},
\]

(2.7)

where \( J_{O_\mu} \) is the restriction of the momentum map \( J \) to \( J^{-1}(O_\mu) \), that is, \( J_{O_\mu} = J \cdot i_{O_\mu} \), and \( \omega^+_{O_\mu} \) is the \( + \)-symplectic structure on the orbit \( O_\mu \) given by

\[
\omega^+_{O_\mu}(\nu)(\xi_{O_\mu}(\nu), \eta_{O_\mu}(\nu)) = <\nu, [\xi, \eta] >, \ \forall \nu \in O_\mu, \ \xi, \eta \in g, \ \xi_{O_\mu}, \eta_{O_\mu} \in g^*.
\]

(2.8)
Theorem 2.8 Assume that the Lagrangian \( L: TQ \to \mathbb{R} \) is hyperregular, and the Legendre transformation \( FL: TQ \to T^*Q \) is \((\Phi^T, \Phi^{T*})\)-equivariant, then the regular orbit reduced space \(((T^*Q)_{\mu}, \omega_{\mu}^L)\) of \((TQ, \omega_L)\) at the orbit \( O_{\mu} \) is symplectically diffeomorphic to the regular orbit reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\) of \((TQ, \omega_0)\) at \( O_{\mu} \). From Ortega and Ratiu [22] and the regular reduction diagram, we know that the regular orbit reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\) is symplectically diffeomorphic to the regular point reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\), and hence is also symplectically diffeomorphic to a symplectic fiber bundle. Comparing the regular orbit reduced spaces \(((TQ)_{\mu}, \omega_{\mu}^L)\) and \(((T^*Q)_{\mu}, \omega_{\mu})\) at the orbit \( O_{\mu} \), we have that the following theorem holds.

**Proof:** Since the Lagrangian \( L: TQ \to \mathbb{R} \) is hyperregular, then the Legendre transformation \( FL: TQ \to T^*Q \) is a diffeomorphism. Because \( FL \) is \((\Phi^T, \Phi^{T*})\)-equivariant, that is, \( \Phi^{T*} \circ FL = FL \circ \Phi^T \), then we can define a map \((FL)_{\mu}: (TQ)_{\mu} \to (T^*Q)_{\mu}\) given by \((FL)_{\mu} \circ \tau_{\mu} = \pi_{\mu} \circ FL\), and \( i_{\mu} \circ FL = FL \circ j_{\mu} \), see the following commutative Diagram-3, which is well-defined and a diffeomorphism.

\[
\begin{array}{ccc}
J_L^{-1}(O_{\mu}) & \xrightarrow{FL} & J_L^{-1}(O_{\mu}) \\
\downarrow \tau_{O_{\mu}} & & \downarrow \pi_{O_{\mu}} \\
(TQ)_{O_{\mu}} & \xrightarrow{(FL)_{O_{\mu}}} & (T^*Q)_{O_{\mu}}
\end{array}
\]

Diagram-3

We shall prove that \((FL)_{O_{\mu}}\) is symplectic, that is, \((FL)_{O_{\mu}}^* \cdot \omega_{O_{\mu}} = \omega_{O_{\mu}}^L\). In fact, from (2.7), (2.5) and (2.4), we have that

\[
\tau_{O_{\mu}} \cdot (FL)_{O_{\mu}}^* \cdot \omega_{O_{\mu}} = ((FL)_{O_{\mu}} \cdot \tau_{O_{\mu}})^* \cdot \omega_{O_{\mu}} = (\pi_{O_{\mu}} \cdot FL)^* \cdot \omega_{O_{\mu}}
\]

\[
= (FL)^* \cdot \tau_{O_{\mu}} \cdot \omega_{O_{\mu}} = (FL)^* \cdot (i_{O_{\mu}} \cdot \omega_0 - J_{O_{\mu}}^* \cdot \omega_{O_{\mu}}^L)
\]

\[
= (FL)^* \cdot i_{O_{\mu}} \cdot \omega_0 - (FL)^* \cdot (J_{O_{\mu}} \cdot \omega_{O_{\mu}}^L)
\]

\[
= (i_{O_{\mu}} \cdot FL)^* \cdot \omega_0 - (J_{O_{\mu}} \cdot FL)^* \cdot \omega_{O_{\mu}}^L
\]

\[
= (FL \cdot j_{O_{\mu}})^* \cdot \omega_0 - (J \cdot FL \cdot j_{O_{\mu}})^* \cdot \omega_{O_{\mu}}^L
\]

\[
= j_{O_{\mu}}^* \cdot (FL)^* \cdot \omega_0 - (J \cdot FL \cdot j_{O_{\mu}})^* \cdot \omega_{O_{\mu}}^L
\]

\[
= j_{O_{\mu}}^* \cdot \omega_0 - (J \cdot FL \cdot j_{O_{\mu}})^* \cdot \omega_{O_{\mu}}^L
\]

\[
= \tau_{O_{\mu}}^* \cdot \omega_{O_{\mu}}^L.
\]

Notice that \( \tau_{O_{\mu}} \) is surjective, and hence \((FL)_{O_{\mu}}^* \cdot \omega_{O_{\mu}} = \omega_{O_{\mu}}^L\). Thus, the regular orbit reduced space \(((TQ)_{O_{\mu}}, \omega_{O_{\mu}}^L)\) of \((TQ, \omega_L)\) at the orbit \( O_{\mu} \) is symplectically diffeomorphic to the regular orbit reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\) of \((T^*Q, \omega_0)\) at the orbit \( O_{\mu} \). From Ortega and Ratiu [22] and the regular reduction diagram, we know that the regular orbit reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\) at the orbit \( O_{\mu} \) is symplectically diffeomorphic to the regular point reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\) of \((T^*Q, \omega_0)\) at \( \mu \), and hence \(((TQ)_{O_{\mu}}, \omega_{O_{\mu}}^L)\) is symplectically diffeomorphic to the regular point reduced space \(((T^*Q)_{\mu}, \omega_{\mu})\) at \( \mu \), and is also symplectically diffeomorphic to a symplectic fiber bundle.
Thus, from the above discussion, we know that the regular point or regular orbit reduced space for a regular Lagrangian system defined on a tangent bundle may not be a tangent bundle. Considering the completeness of the symmetric reduction, if we may define an RCL system on a symplectic fiber bundle, then it is possible to describe uniformly the RCL systems on $TQ$ and their regular reduced RCL systems on the associated reduced spaces.

3 Regular Controlled Lagrangian System and RCL-Equivalence

In order to give a proper definition of CL system, following the ideas in Marsden et al. [18], we first define a CL system on $TQ$ by using the Lagrangian symplectic form, and such system is called a regular controlled Lagrangian (RCL) system, and then regard a regular Lagrangian system on $TQ$ as a special case of an RCL system without external force and control. Thus, the set of the regular Lagrangian systems on $TQ$ is a subset of the set of RCL systems on $TQ$. On the other hand, since the regular reduced system of a regular Lagrangian system with symmetry defined on the tangent bundle $TQ$ may not be a regular Lagrangian system on a tangent bundle. So, we can not define directly an RCL system on the tangent bundle $TQ$. However, from Theorem 2.7 and Theorem 2.8, we know that the regular point reduced space $((TQ)_\mu, \omega^L_\mu)$ of $(TQ, \omega^L)$ at $\mu$ is symplectically diffeomorphic to a symplectic fiber bundle over $T(Q/G_\mu)$ with fiber to be the co-adjoint orbit $O^\mu_\mu$, and the regular orbit reduced space $((TQ)_{O_\mu}, \omega^L_{O_\mu})$ of $(TQ, \omega^L)$ at the orbit $O_\mu$ is also symplectically diffeomorphic to a symplectic fiber bundle. In consequence, if we may define an RCL system on a symplectic fiber bundle, then it is possible to describe uniformly the RCL system on $TQ$ and its regular reduced RCL systems on the associated reduced spaces, and we can study regular reduction of the RCL systems with symmetries and momentum maps, as an extension of the regular reduction theory of the regular Lagrangian systems under regular controlled Lagrangian equivalence conditions, and set up the regular reduction theory of the RCL system on a tangent bundle, by using momentum map and the associated reduced Lagrangian symplectic form and from the viewpoint of completeness of regular reduction.

In this section, we first define an RCL system on a symplectic fiber bundle, then we obtain the RCL system on a tangent bundle as a special case, by using the Legendre transformation and the Lagrangian symplectic form on the tangent bundle of a configuration manifold, and give a good expression of the dynamical vector field of the RCL system, such that we can discuss RCL-equivalence. In consequence, we can study the RCL systems with symmetries by combining with the symmetric reduction of the regular Lagrangian systems with symmetries. For convenience, we assume that all controls appearing in this paper are the admissible controls.

Let $(E, M, N, \pi, G)$ be a fiber bundle and $(E, \omega_E)$ be a symplectic fiber bundle. If a function $L : E \to \mathbb{R}$ is hyperregular Lagrangian, and there is an action function $A : E \to \mathbb{R}$ and an Euler-Lagrange vector field $\xi_L$ satisfy the equation $i_{\xi_L} \omega_E = dE_L$, where $E_L : E \to \mathbb{R}$ is an energy function given by $E_L := A - L$. Then $(E, \omega_E, L)$ is a regular Lagrangian system. Moreover, if considering the external force and control, we can define a kind of regular controlled Lagrangian (RCL) system on the symplectic fiber bundle $E$ as follows.

**Definition 3.1 (RCL System)** An RCL system on $E$ is a 5-tuple $(E, \omega_E, L, F^L, C^L)$, where $(E, \omega_E, L)$ is a regular Lagrangian system, and the function $L : E \to \mathbb{R}$ is called the (hyperregular) Lagrangian, a fiber-preserving map $F^L : E \to E$ is called the (external) force map, and a fiber submanifold $C^L$ of $E$ is called the control subset.
Sometimes, $C^L$ is also denoted the set of fiber-preserving maps from $E$ to $C^L$. When a feedback control law $u^L : E \rightarrow C^L$ is chosen, the 5-tuple $(E, \omega_E, L, F^L, u^L)$ is a closed-loop dynamical system. In particular, when $Q$ is a smooth manifold, and $TQ$ its tangent bundle, and $T^*Q$ its cotangent bundle with a canonical symplectic form $\omega_0$, assume that $L : TQ \rightarrow \mathbb{R}$ is a hyperregular Lagrangian on $TQ$, and the Legendre transformation $F^L : TQ \rightarrow T^*Q$ is a diffeomorphism, then $(TQ, \omega^L)$ is a symplectic vector bundle, where $\omega^L = F^L(\omega_0)$. If we take that $E = TQ$, from above definition we can obtain an RCL system on the tangent bundle $TQ$, that is, 5-tuple $(TQ, \omega^L, L, F^L, C^L)$.

In order to describe the dynamics of the RCL system $(E, \omega_E, L, F^L, C^L)$ with a control law $u^L : E \rightarrow C^L$, we need to give a good expression of the dynamical vector field of the RCL system. We shall use the notations of vertical lift maps of a vector along a fiber introduced in Marsden et al. [18]. In fact, for a smooth manifold $E$, its tangent bundle $TE$ is a vector bundle, and for the fiber bundle $\pi : E \rightarrow M$, we consider the tangent mapping $T\pi : TE \rightarrow TM$ and its kernel $\ker(T\pi) = \{ \rho \in TE | T\pi(\rho) = 0 \}$, which is a vector subbundle of $TE$. Denote by $VE := \ker(T\pi)$, which is called a vertical bundle of $E$. Assume that there is a metric on $E$, and we take a Levi-Civita connection $A$ on $TE$, and denote by $HE := \ker(A)$, which is called a horizontal bundle of $E$, such that $TE = HE \oplus VE$. For any $x \in M$, $a_x, b_x \in E_x$, any tangent vector $\rho(b_x) \in T_{b_x}E$ can be split into horizontal and vertical parts, that is, $\rho(b_x) = \rho^h(b_x) \oplus \rho^v(b_x)$, where $\rho^h(b_x) \in H_{b_x}E$ and $\rho^v(b_x) \in V_{b_x}E$. Let $\gamma$ be a geodesic in $E_x$ connecting $a_x$ and $b_x$, and denote by $\rho^\gamma_x(a_x)$ a tangent vector at $a_x$, which is a parallel displacement of the vertical vector $\rho^v(b_x)$ along the geodesic $\gamma$ from $b_x$ to $a_x$. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then $T\pi(\rho^\gamma_x(a_x)) = 0$, and hence $\rho^\gamma_x(a_x) \in V_{a_x}E$. Now, for $a_x, b_x \in E_x$ and tangent vector $\rho(b_x) \in T_{b_x}E$, we can define the vertical lift map of a vector along a fiber given by

$$\text{vlift} : TE_x \times E_x \rightarrow TE_x; \quad \text{vlift}(\rho(b_x), a_x) = \rho^\gamma_x(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of $\gamma$. If $F^L : E \rightarrow E$ is a fiber-preserving map, for any $x \in M$, we have that $F^L_x : E_x \rightarrow E_x$ and $TF^L_x : TE_x \rightarrow TE_x$, then for any $a_x \in E_x$ and $\rho \in TE_x$, the vertical lift of $\rho$ under the action of $F^L$ along a fiber is defined by

$$(\text{vlift}(F^L_x)^\gamma(\rho)(a_x) = \text{vlift}((TF^L_x \rho)(F^L_x(a_x)), a_x) = (TF^L_x \rho)^\gamma_x(a_x),$$

where $\gamma$ is a geodesic in $E_x$ connecting $F^L_x(a_x)$ and $a_x$.

In particular, when $\pi : E \rightarrow M$ is a vector bundle, for any $x \in M$, the fiber $E_x = \pi^{-1}(x)$ is a vector space. In this case, we can choose the geodesic $\gamma$ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $\rho^\gamma_x(a_x) = \rho^v(b_x)$. Moreover, when $E = TQ$, by using the local trivialization of $TTQ$, we have that $TTQ \cong TQ \times TQ$ (locally). Because of $T\pi : TQ \rightarrow Q$, and $T\tau_Q : TTQ \rightarrow TQ$, then in this case, for any $v_x, w_x \in T_xQ, x \in Q$, we know that $(0, w_x) \in V_{w_x}T_xQ$, and hence we can get that

$$\text{vlift}((0, w_x)(w_x), v_x) = (0, w_x)(v_x) = \frac{d}{ds}{\bigg|}_{s=0} (v_x + sw_x),$$

which coincides with the definition of vertical lift map along fiber in Marsden and Ratiu [17].

For a given RCL System $(TQ, \omega^L, L, F^L, C^L)$, the dynamical vector field of the associated regular Lagrangian system $(TQ, \omega^L, L)$ is the Euler-Lagrange vector field $\xi_L$, such that $\xi_L \omega^L = dE_L$. If considering the external force $F^L : TQ \rightarrow TQ$, by using the above notation of vertical lift map of a vector along a fiber, the change of $\xi_L$ under the action of $F^L$ is that

$$\text{vlift}(F^L)\xi_L(v_x) = \text{vlift}((TF^L \xi_L)(F^L(v_x)), v_x) = (TF^L \xi_L)^v_x(v_x),$$

11
where \( v_x \in T_xQ, x \in Q \) and the geodesic \( \gamma \) is a straight line in \( T_xQ \) connecting \( F^L(v_x) \) and \( v_x \). In the same way, when a feedback control law \( u^L : TQ \to CL \) is chosen, the change of \( \xi_L \) under the action of \( u^L \) is that

\[
vlift(u^L) \xi_L(v_x) = vlift((Tu^L \xi_L)(u^L(v_x)), v_x) = (Tu^L \xi_L)^\gamma(v_x).
\]

In consequence, we can give an expression of the dynamical vector field of the RCL system as follows.

**Theorem 3.2** The dynamical vector field of an RCL system \((TQ, \omega^L, L, F^L, CL)\) with a control law \( u^L \) is the synthetic of the Euler-Lagrange vector field \( \xi_L \) and its changes under the actions of the external force \( F^L \) and control \( u^L \), that is,

\[
\xi(TQ, \omega^L, L, F^L, u^L)(v_x) = \xi_L(v_x) + vlift(F^L) \xi_L(v_x) + vlift(u^L) \xi_L(v_x),
\]

for any \( v_x \in T_xQ, x \in Q \). For convenience, it is simply written as

\[
\xi(TQ, \omega^L, L, F^L, u^L) = \xi_L + vlift(F^L) + vlift(u^L). \tag{3.1}
\]

Where \( vlift(F^L) = vlift(F^L) \xi_L \) and \( vlift(u^L) = vlift(u^L) \xi_L \) are the changes of \( \xi_L \) under the actions of \( F^L \) and \( u^L \). We also denote that \( vlift(CL) = \{ vlift(u^L) \xi_L \mid u^L \in CL \} \). It is worthy of noting that, in order to deduce and calculate easily, we always use the simple expression of dynamical vector field \( \xi(TQ, \omega^L, L, F^L, u^L) \). Moreover, we also use the simple expressions for \( R_p \)-reduced vector field \( \xi((TQ)_i, \omega^L_i, L, F^L_i, u^L_i) \) and \( R_o \)-reduced vector field \( \xi((TQ)_i, \omega^L_i, L, F^L_i, u^L_i) \) in Section 4 and Section 5.

From the expression (3.1) of the dynamical vector field of the RCL system, we know that under the actions of the external force \( F^L \) and control \( u^L \), in general, the dynamical vector field may not be an Euler-Lagrange vector field, and hence the RCL system may not be yet a regular Lagrangian system. However, it is a dynamical system closed relative to a regular Lagrangian system, and it can be explored and studied by extending the methods for external force and control in the study of the regular Lagrangian system. In particular, it is worthy of noting that the energy \( E_L \) is conserved along the flow of the Euler-Lagrange vector field \( \xi_L \), if \( \xi_L \) satisfies the second order equation \( T\tau_Q \circ \xi_L = id_{TQ} \). Note that \( T\tau_Q \cdot vlift(F^L) = T\tau_Q \cdot vlift(u^L) = 0 \), then from the expression (3.1) we have that \( T\tau_Q \circ \xi(TQ, \omega^L, L, F^L, u^L) = id_{TQ} \), that is, the dynamical vector field of the RCL system satisfies always the second order equation.

On the other hand, for two given regular Lagrangian systems \((TQ_i, \omega^L_i, L_i), i = 1, 2\), we say them to be equivalent, if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that their Euler-Lagrange vector fields \( \xi_{L_i}, i = 1, 2 \) satisfy the condition \( \xi_{L_2} \cdot T\varphi = T(T\varphi) \cdot \xi_{L_1} \), where the map \( T\varphi : TQ_1 \to TQ_2 \) is the tangent map of \( \varphi \), and the map \( T(T\varphi) : TTQ_1 \to TTQ_2 \) is the tangent map of \( T\varphi \). It is easy to see that the condition \( \xi_{L_2} \cdot T\varphi = T(T\varphi) \cdot \xi_{L_1} \) is equivalent the fact that the map \( T\varphi : TQ_1 \to TQ_2 \) is symplectic with respect to their Lagrangian symplectic forms \( \omega^L_i \) on \( TQ_i, i = 1, 2 \).

For two given RCL systems \((TQ_i, \omega^L_i, L_i, F^L_i, CL_i), i = 1, 2\), we also want to define their equivalence, that is, to look for a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that \( \xi(TQ_2, \omega^L_2, L_2, F^L_2, CL_2) \cdot T\varphi = T(T\varphi) \cdot \xi(TQ_1, \omega^L_1, L_1, F^L_1, CL_1) \). But, it is worthy of noting that, when an RCL system is given, the force map \( F^L : TQ \to TQ \) is determined, but the feedback control law \( u^L : TQ \to CL \) could be chosen. In order to emphasize explicitly the impact of external force and control in the study of the RCL systems, by using the above expression (3.1) of the dynamical vector field of the RCL system, we can describe the feedback control law how to modify the structure of the RCL system, and the regular controlled Lagrangian matching conditions and RCL-equivalence are induced as follows.
**Definition 3.3 (RCL-equivalence)** Suppose that we have two RCL systems \((TQ_1, \omega^L_1, L_1, F^L_1, C^L_1), i = 1, 2\), we say them to be RCL-equivalent, or simply, \((TQ_1, \omega^L_1, L_1, F^L_1, C^L_1) \sim_{RCL} (TQ_2, \omega^L_2, L_2, F^L_2, C^L_2)\), if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\), such that the following regular controlled Lagrangian matching conditions hold:

**RCL-1:** The control subsets \(C^L_i, i = 1, 2\) satisfy the condition \(C^L_i = T\varphi(C^L_i)\), where the map \(T\varphi : TQ_1 \to TQ_2\) is tangent map of \(\varphi\).

**RCL-2:** For each control law \(u^L_i : TQ_1 \to C^L_i\), there exists the control law \(u^L_2 : TQ_2 \to C^L_2\), such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is, \(\xi(TQ_2, \omega^L_2, L_2, F^L_2, u^L_2) \cdot T\varphi = T(T\varphi) \cdot [\xi_{TQ_1, \omega^L_1, L_1, F^L_1, u^L_1}]\), where the map \(T(T\varphi) : TTTQ_1 \to TTTQ_2\) is the tangent map of \(T\varphi\).

From the expression (3.1) of the dynamical vector field of the RCL system and the condition \(\xi(TQ_2, \omega^L_2, L_2, F^L_2, u^L_2) \cdot T\varphi = T(T\varphi) \cdot [\xi_{TQ_1, \omega^L_1, L_1, F^L_1, u^L_1}]\), we have that

\[
(\xi_{L_2} + \vlift(F^L_2)\xi_{L_2} + \vlift(u^L_2)\xi_{L_2}) \cdot T\varphi = T(T\varphi) \cdot [\xi_{L_1} + \vlift(F^L_1)\xi_{L_1} + \vlift(u^L_1)\xi_{L_1}].
\]

By using the notation of vertical lift map of a vector along a fiber, for \(v_x \in T_xQ_1, x \in Q_1\), we have that

\[
T(T\varphi) \cdot \vlift(F^L_2)\xi_{L_1}(v_x) = T(T\varphi) \cdot \vlift((T F^L_1) \cdot \xi_{L_1})(F^L_1(v_x)), v_x)
\]

\[
= \vlift((T(T\varphi) \cdot T F^L_1) \cdot T(T\varphi^{-1}) \cdot \xi_{L_1})(T(T\varphi \cdot F^L_1 \cdot T\varphi^{-1}) \cdot (T\varphi \cdot v_x)), T\varphi \cdot v_x)
\]

\[
= \vlift((T((T\varphi \cdot F^L_1 \cdot T\varphi^{-1}) \cdot \xi_{L_1})(T(T\varphi \cdot F^L_1 \cdot T\varphi^{-1}) \cdot (T\varphi \cdot v_x)), T\varphi \cdot v_x)
\]

\[
= \vlift(T\varphi \cdot F^L_1 \cdot T\varphi^{-1}) \cdot \xi_{L_1}(T\varphi \cdot v_x),
\]

where the map \(T\varphi^{-1} : TQ_2 \to TQ_1\). In the same way, we have that \(T(T\varphi) \cdot \vlift(u^L_1)\xi_{L_1} = \vlift(T\varphi \cdot u^L_1 \cdot T\varphi^{-1}) \cdot \xi_{L_1} \cdot T\varphi\). Thus, the explicit relation between the two control laws \(u^L_i : TQ_i \to C^L_i, i = 1, 2\) in RCL-2 is given by

\[
(\vlift(u^L_2) - \vlift(T\varphi \cdot u^L_1 \cdot T\varphi^{-1})) \cdot T\varphi = -\xi_{L_2} \cdot T\varphi + T(T\varphi) \cdot (\xi_{L_1}) + (\vlift(u^L_2) - \vlift(F^L_2) \cdot \vlift(T\varphi \cdot F^L_2 \cdot T\varphi^{-1})) \cdot T\varphi.
\]

From the above relation we know that, when two RCL systems \((TQ_i, \omega^L_i, L_i, F^L_i, C^L_i), i = 1, 2\), are RCL-equivalent with respect to \(T\varphi\), the corresponding regular Lagrangian systems \((TQ_i, \omega^L_i, L_i), i = 1, 2\), may not be equivalent with respect to \(T\varphi\). If two corresponding regular Lagrangian systems are also equivalent with respect to \(T\varphi\), then the control laws \(u^L_i : TQ_i \to C^L_i, i = 1, 2\) and the external forces \(F^L_i : TQ_i \to TQ_i, i = 1, 2\) in RCL-2 must satisfy the following condition

\[
\vlift(u^L_2) - \vlift(T\varphi \cdot u^L_1 \cdot T\varphi^{-1}) = -\vlift(F^L_2) + \vlift(T\varphi \cdot F^L_2 \cdot T\varphi^{-1}).
\]

In the following we shall introduce the regular point and regular orbit reducible RCL systems with symmetries, and show a variety of relationships of their regular reducible RCL-equivalences.

## 4 Regular Point Reduction of the RCL System

We know that, when the external force and control of an RCL system \((TQ, \omega^L, L, F^L, C^L)\) are both zeros, that is, \(F^L = 0\) and \(C^L = \emptyset\), in this case the RCL system is just a regular Lagrangian system \((TQ, \omega^L, L)\). Thus, we can regard a regular Lagrangian system on \(TQ\) as a spacial case of the RCL system without external force and control. In consequence, the set of regular Lagrangian systems with symmetries on \(TQ\) is a subset of the set of RCL systems with symmetries on \(TQ\). If we first
admit the regular point reduction of a regular Lagrangian system with symmetry, then we may study the regular point reduction of an RCL system with symmetry, as an extension of the regular point reduction of a regular Lagrangian system under the regular controlled Lagrangian equivalence conditions. In order to do these, in this section we consider the RCL system with symmetry and momentum map, and first give the regular point reducible RCL system and theRpCL-equivalence, then prove the regular point reduction theorems for the RCL system and regular Lagrangian system.

We know that, if an RCL system with symmetry and momentum map is regular point reducible, then the associated regular Lagrangian system must be regular point reducible. Thus, from Definition 2.4 and Theorem 2.7, if the Legendre transformation is equivariant, then we can introduce a kind of regular point reducible RCL systems as follows.

Definition 4.1 (Regular Point Reducible RCL System) A 6-tuple (\(TQ, G, \omega^L, L, F^L, C^L\)), where the hyperregular Lagrangian \(L : TQ \to \mathbb{R}\), the fiber-preserving map \(F^L : TQ \to TQ\) and the fiber submanifold \(C^L\) of \(TQ\) are all \(G\)-invariant, is called a regular point reducible RCL system, if the Legendre transformation \(FL : TQ \to T^*Q\) is \((\Phi^T, \Phi^{T^*})\)-equivariant, and there exists a point \(\mu \in \mathfrak{g}^*\), which is a regular value of the momentum map \(J_L\), such that the \(G\)-invariant external force map \(F^L(J_L^{-1}(\mu)) \subset J_L^{-1}(\mu)\), and \(f^L_J \cdot \tau_\mu = J_L^{-1}(\mu)\), such that we can define the \(G\)-invariant control subset \(C^L\cap J_L^{-1}(\mu)\) can be reduced and \(R\)-reduced control subset is \(C^L = \tau_\mu(C^L \cap J_L^{-1}(\mu))\).

It is worthy of noting that for the regular point reducible RCL system \((TQ, G, \omega^L, L, F^L, C^L)\), the \(G\)-invariant external force map \(F^L : TQ \to TQ\) has to satisfy the conditions \(F^L(J_L^{-1}(\mu)) \subset J_L^{-1}(\mu)\), and \(f^L_J \cdot \tau_\mu = J_L^{-1}(\mu)\), such that we can define the \(G\)-invariant control subset \(C^L\cap J_L^{-1}(\mu)\) can be reduced and \(R\)-reduced control subset is \(C^L = \tau_\mu(C^L \cap J_L^{-1}(\mu))\).

Assume that the dynamical vector field \(\xi_{(TQ,G,\omega^L,L,F^L,u^L)}\) of a given regular point reducible RCL system \((TQ, G, \omega^L, L, F^L, C^L)\) with a control law \(u^L \in C^L\) can be expressed by

\[
\xi_{(TQ,G,\omega^L,L,F^L,u^L)} = \xi_L + \text{vlift}(F^L) + \text{vlift}(u^L).
\]

Then, for the regular point reducible RCL system we can also introduce the regular point reducible controlled Lagrangian equivalence (RpCL-equivalence) as follows.

Definition 4.2 (RpCL-equivalence) Suppose that we have two regular point reducible RCL systems \((TQ_i, G_i, \omega^L_i, L_i, F^L_i, C^L_i)\), \(i = 1, 2\), we say them to be RpCL-equivalent, or simply, 
\((TQ_1, G_1, \omega^L_1, L_1, F^L_1, C^L_1) \sim_{RLC} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, C^L_2)\), if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\) such that the following regular point reducible controlled Lagrangian matching conditions hold:

RpCL-1: For \(\mu_i \in \mathfrak{g}_i^*\), the regular reducible points of the RCL systems \((TQ_i, G_i, \omega^L_i, L_i, F^L_i, C^L_i)\), \(i = 1, 2\), the map \(\varphi \cdot j_{\mu_1}^{-1} : (J_L)_{\mu_1}^{-1}(\mu_1) \to (J_L)_{\mu_2}^{-1}(\mu_2)\) is \((G_{1\mu_1}, G_{2\mu_2})\)-equivariant and \(C^L_i \cap (J_L)_{\mu_2}^{-1}(\mu_2) = (T\varphi)_{\mu_2}(C^L_i \cap (J_L)_{\mu_1}^{-1}(\mu_1))\), where \(\mu = (\mu_1, \mu_2)\), and denote by \(j_{\mu_2}^{-1}(S)\) the pre-image of a subset \(S \subset TQ_2\) for the map \(j_{\mu_2} : (J_L)_{\mu_2}^{-1}(\mu_2) \to TQ_2\).

RpCL-2: For each control law \(u^L_1 : TQ_1 \to C^L_1\), there exists the control law \(u^L_2 : TQ_2 \to C^L_2\), such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is, 
\[
\xi_{(TQ_2,G_2,\omega^L_2,L_2,F^L_2,u^L_2)} \cdot T\varphi = T(T\varphi) \cdot \xi_{(TQ_1,G_1,\omega^L_1,L_1,F^L_1,u^L_1)}.\]
It is worthy of noting that for the regular point reducible RCL system, the induced equivalent map $T\varphi$ also keeps the equivariance of $G$-action at the regular point. If a feedback control law $u^L : TQ \to CL$ is chosen, and $u^L \in CL \cap J^{-1}_L(\mu)$, and $CL \cap J^{-1}_L(\mu) \neq \emptyset$, then the $R\mu$-reduced control law $u^L_\mu : (TQ)_\mu \to CL \cap J^{-1}_L(\mu)$, and $u^L_\mu \cdot T\mu = \tau_\mu \cdot u^L : j_\mu$. The $R\mu$-reduced RCL system $((TQ)_\mu, \omega^L_\mu, l_\mu, f^L_\mu, u^L_\mu)$ is a closed-loop regular dynamical system with the $R\mu$-reduced control law $u^L_\mu$. Assume that its dynamical vector field $\xi((TQ)_\mu, \omega^L_\mu, l_\mu, f^L_\mu, u^L_\mu)$ can be expressed by

$$\xi((TQ)_\mu, \omega^L_\mu, l_\mu, f^L_\mu, u^L_\mu) = \xi l_\mu + \vlift(f^L_\mu) + \vlift(u^L_\mu), \quad (4.2)$$

where $\xi l_\mu$ is the $R\mu$-reduced Euler-Lagrange vector field, $\vlift(f^L_\mu) = \vlift(f^L_\mu) \xi l_\mu$, $\vlift(u^L_\mu)$ are the changes of $\xi l_\mu$ under the actions of the $R\mu$-reduced external force $f^L_\mu$ and the $R\mu$-reduced control law $u^L_\mu$, and the dynamical vector fields of the RCL system and the $R\mu$-reduced RCL system satisfy the condition

$$\xi((TQ)_\mu, \omega^L_\mu, l_\mu, f^L_\mu, u^L_\mu) \cdot \tau_\mu = T\tau_\mu \cdot \xi((TQ,G,\omega^L_\mu, l^L, f^L, u^L) \cdot j_\mu, \quad (4.3)$$

see Marsden et al. [18], Wang [24]. Then we can obtain the following regular point reduction theorem for the RCL system, which explains the relationship between the RpCL-equivalence for the regular point reducible RCL system with symmetry and the RCL-equivalence for the associated $R\mu$-reduced RCL system.

**Theorem 4.3** Two regular point reducible RCL systems $(TQ_i, G_i, \omega^L_{i1}, l_{i1}, F^L_{i1}, C^L_{i1})$, $i = 1, 2$, are $R\mu$-equivalent if and only if the associated $R\mu$-reduced RCL systems $((TQ_i)_\mu, \omega^L_{i\mu}, l_{i\mu}, f^L_{i\mu}, C^L_{i\mu})$, $i = 1, 2$, are RCL-equivalent.

**Proof:** If $(TQ_1, G_1, \omega^L_1, l_1, F^L_1, C^L_1)$ $\sim$ $(TQ_2, G_2, \omega^L_2, l_2, F^L_2, C^L_2)$, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that for $\mu_i \in G^*_i$, $i = 1, 2$, $(T\varphi)_\mu = \frac{\mu}{G}_i \cdot T\varphi \cdot j_\mu : (J^L_1)_i^{-1}(\mu_1) \to (J^L_2)_i^{-1}(\mu_2)$ is $(G_{1\mu}, G_{2\mu})$-equivariant, and $(C^L_1 \cap (J^L_1)_i^{-1}(\mu_1))$ and RpCL-2 holds. From the following commutative Diagram-4:

$$\begin{array}{ccc}
TQ_1 & \xleftarrow{j_{\mu_1}} & (J^L_1)_i^{-1}(\mu_1) \\
\downarrow T\varphi & \downarrow (T\varphi)_\mu & \downarrow (T\varphi)_\mu \cdot G \\
TQ_2 & \xleftarrow{j_{\mu_2}} & (J^L_2)_i^{-1}(\mu_2)
\end{array}$$

Diagram-4

we can define a map $(T\varphi)_\mu (G) : (TQ_1)_\mu \to (TQ_2)_\mu$ such that $(T\varphi)_\mu (G) \cdot \tau_{\mu_1} = \tau_{\mu_2} \cdot (T\varphi)_\mu$. Because $(T\varphi)_\mu : (J^L_1)_i^{-1}(\mu_1) \to (J^L_2)_i^{-1}(\mu_2)$ is $(G_{1\mu}, G_{2\mu})$-equivariant, $(T\varphi)_\mu (G)$ is well-defined. We shall show that $(C^L_2 \cap (J^L_2)_i^{-1}(\mu_2))$ is regular point reducible RCL systems, then $C^L_{2\mu} \cap (J^L_2)_i^{-1}(\mu_1) \neq \emptyset$ and $C^L_{i\mu} = \tau_{\mu_i} (C^L_i \cap (J^L_i)_i^{-1}(\mu_i))$, $i = 1, 2$, $\mu_i \in G^*_i$. From $C^L_i \cap (J^L_i)_i^{-1}(\mu_i)$, we have that

$$C^L_{2\mu} = \tau_{\mu_2} (C^L_2 \cap (J^L_2)_i^{-1}(\mu_2)) = \tau_{\mu_2} \cdot (T\varphi)_\mu (C^L_i \cap (J^L_i)_i^{-1}(\mu_i)) = (T\varphi)_\mu (G) \cdot \tau_{\mu_i} (C^L_i \cap (J^L_i)_i^{-1}(\mu_i)) = (T\varphi)_\mu (G) \cdot (C^L_{i\mu}).$$

Thus, the condition RCL-1 holds. On the other hands, for the $R\mu$-reduced control law $u^L_{1\mu} : (TQ_1)_\mu \to C^L_{1\mu}$, we have the control law $u^L_{1} : TQ \to C^L_1$, such that $u^L_{1\mu} \cdot \tau_{\mu_1} = \tau_{\mu_1} \cdot u^L_{1} \cdot j_1$. From the condition RpCL-2 we have that there exists the control law $u^L_{2} : TQ \to C^L_2$, such that $\xi((TQ_2,G_2,\omega^L_2, l_2, F^L_2, u^L_{2})) \cdot T\varphi = (T\varphi) \cdot \xi((TQ_1,G_1,\omega^L_1, l_1, F^L_1, u^L_{1}))$. But, for the control law $u^L_{2} : TQ \to C^L_2,$
we have the $R_p$-reduced control law $u_{2\mu_2}^L : (TQ_2)_\mu \to C_{2\mu_2}^L$, such that $u_{2\mu_2}^L \cdot \tau_{\mu_2} = \tau_{\mu_2} \cdot u_{2\mu_2}^L \cdot j_{\mu_2}$.

Note that for $i = 1, 2$, from (4.3), we have that

$$\xi((TQ_i)_{\mu_i} \cdot \omega_{i\mu_i} \cdot f_{i\mu_i} \cdot u_{i\mu_i}^L) \cdot \tau_{\mu_i} = T\tau_{\mu_i} \cdot \xi((TQ_i, G, \omega_i, L, F_i^L, u_i^L) \cdot j_{\mu_i},$$

(4.4)

and from the commutative Diagram-4, $(T\varphi)_{\mu/G} \cdot \tau_{\mu_1} = \tau_{\mu_2} \cdot (T\varphi)_{\mu}$ and $j_{\mu_2} \cdot (T\varphi)_{\mu} = (T\varphi) \cdot j_{\mu_1}$, then we have that

$$\xi((TQ_2)_{\mu_2} \cdot \omega_{2\mu_2} \cdot f_{2\mu_2} \cdot u_{2\mu_2}^L) \cdot (T\varphi)_{\mu/G} \cdot \tau_{\mu_2} = \xi((TQ_2)_{\mu_2} \cdot \omega_{2\mu_2} \cdot f_{2\mu_2} \cdot u_{2\mu_2}^L) \cdot \tau_{\mu_2} \cdot (T\varphi)_{\mu}$$

(4.5)

that is, the condition RCL-2 holds. So, the $R_p$-reduced RCL systems $((TQ_i)_{\mu_i}, \omega_{i\mu_i}, l_{i\mu_i}, f_{i\mu_i}, C_{i\mu_i}^L)$, $i = 1, 2$, are RCL-equivalent.

Conversely, assume that the $R_p$-reduced RCL systems $((TQ_i)_{\mu_i}, \omega_{i\mu_i}, l_{i\mu_i}, f_{i\mu_i}, C_{i\mu_i}^L)$, $i = 1, 2$, are RCL-equivalent, then there exists a diffeomorphism $(T\varphi)_{\mu/G} : (TQ_1)_{\mu_1} \to (TQ_2)_{\mu_2}$, such that $C_{2\mu_2}^L = (T\varphi)_{\mu/G}(C_{1\mu_1}^L)$. $\mu_i \in g_i^*$, $i = 1, 2$ and for the $R_p$-reduced control law $u_{2\mu_2}^L : (TQ_1)_{\mu_1} \to C_{2\mu_2}^L$, there exist the $R_p$-reduced control law $u_{2\mu_2}^L : (TQ_2)_{\mu_2} \to C_{2\mu_2}^L$, such that (4.5) holds. Then from commutative Diagram-4, we can define a map $(T\varphi)_{\mu} : (J_L)_{\mu}^{-1}((\mu_1)) \to (J_L)_{\mu}^{-1}((\mu_2))$ such that $(T\varphi)_{\mu} \cdot (T\varphi)_{\mu/G} \cdot \tau_{\mu_1}$, and the map $T\varphi : TQ_1 \to TQ_2$ such that $T\varphi \cdot j_{\mu_1} = j_{\mu_2} \cdot (T\varphi)_{\mu}$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$, whose tangent lift is just $T\varphi : TQ_1 \to TQ_2$. Moreover, for above definition of $(T\varphi)_{\mu}$, we know that $(T\varphi)_{\mu}$ is $(G_{1\mu_1}, G_{2\mu_2})$-equivariant. In fact, for any $z_1 \in (J_L)_{\mu}^{-1}(\mu_1)$, $g_i \in G_{i\mu_i}, i = 1, 2$ such that $z_2 = (T\varphi)_{\mu}(z_1)$, $[z_2] = (T\varphi)_{\mu/G}(z_1)$, then we have that

$$(T\varphi)_{\mu}(\Phi_{1g_1}(z_1)) = \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2} \cdot (T\varphi)_{\mu}(\Phi_{1g_1}(z_1)) = \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2} \cdot (T\varphi)_{\mu} \cdot (g_1 z_1)$$

(4.6)

Here we denote by $\tau_{\mu_1}^{-1}(S)$ the pre-image of a subset $S \subset (TQ_1)_{\mu_1}$ for the map $\tau_{\mu_1} : (J_L)_{\mu_1}^{-1}(\mu_1) \to (TQ_1)_{\mu_1}$, and for any $z_1 \in (J_L)_{\mu_1}^{-1}(\mu_1)$, $\tau_{\mu_1}^{-1}(\mu_1) = z_1$. So, we obtain that $(T\varphi)_{\mu} \cdot \Phi_{1g_1} = \Phi_{2g_2} \cdot (T\varphi)_{\mu}$. Moreover, we have that

$$C_{22}^L \cap (J_L)_{\mu_2}^{-1}(\mu_2) = \tau_{\mu_2}^{-1} \cdot \tau_{\mu_2} \cdot (T\varphi)_{\mu/G}(C_{1\mu_1}^L) = \tau_{\mu_2}^{-1} \cdot (T\varphi)_{\mu/G} \cdot (C_{1\mu_1}^L) \cap (J_L)_{\mu_1}^{-1}(\mu_1)$$

(4.7)

Thus, the condition RpCl-1 holds. In the following we shall prove that the condition RpCL-2 holds. For the above $R_p$-reduced control laws $u_{2\mu_2}^L : (TQ_i)_{\mu_i} \to C_{i\mu_i}^L$, $i = 1, 2$, there exist the control laws $u_{i}^L : TQ_i \to C_i^L$, such that $u_{i\mu_i} \cdot \tau_{\mu_i} = \mu_i \cdot u_{i\mu_i} \cdot j_{\mu_i}$, $i = 1, 2$. we shall prove that

$$\xi((TQ_2, G, \omega_2, l_2, F_2, u_2^L) \cdot T\varphi = T(T\varphi) \cdot \xi((TQ_1, G, \omega_1, L_1, F_1^L, u_1^L)).$$
In fact, from (4.4) we have that
\[
\begin{align*}
T((T \varphi)_{\mu/G}) \cdot \xi((TQ_1)_{\mu_1} \omega_{\mu_1, l_{\mu_1}, f_{\mu_1, u_{\mu_1}}}) \cdot \tau_{\mu_1} &= T((T \varphi)_{\mu/G}) \cdot T \tau_{\mu_1} \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot \tilde{j}_{\mu_1} \\
&= T((T \varphi)_{\mu/G}) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot \tilde{j}_{\mu_1} = T(\tau_{\mu_2} \cdot (T \varphi)_\mu) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot \tilde{j}_{\mu_1} \\
&= T \tau_{\mu_2} \cdot T(T \varphi) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot \tilde{j}_{\mu_1}.
\end{align*}
\]

On the other hand,
\[
\begin{align*}
\xi((TQ_2)_{\mu_2} \omega_{\mu_2, l_{\mu_2}, f_{\mu_2, u_{\mu_2}}}) \cdot (T \varphi)_{\mu/G} \cdot \tau_{\mu_1} &= \xi((TQ_2)_{\mu_2} \omega_{\mu_2, l_{\mu_2}, f_{\mu_2, u_{\mu_2}}}) \cdot \tau_{\mu_2} \cdot (T \varphi)_\mu \\
&= T \tau_{\mu_2} \cdot \xi((TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)) \cdot \tilde{j}_{\mu_2} \cdot (T \varphi)_\mu = T \tau_{\mu_2} \cdot \xi((TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)) \cdot T \varphi \cdot \tilde{j}_{\mu_1}.
\end{align*}
\]

From (4.5) we have that
\[
T \tau_{\mu_2} \cdot \xi((TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)) \cdot T \varphi \cdot \tilde{j}_{\mu_1} = T \tau_{\mu_2} \cdot T(T \varphi) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot \tilde{j}_{\mu_1}.
\]

Note that the map \( j_{\mu_1} : (J_L)_1^{-1}(\mu_1) \to TQ_1 \) is injective, and \( T \tau_{\mu_2} : T(J_L)_2^{-1}(\mu_2) \to T(TQ_2)_{\mu_2} \) is surjective, hence, we have that
\[
\xi((TQ_2,G_2,\omega_2^L,L_2,F_2^L,u_2^L)) \cdot T \varphi = T(T \varphi) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)).
\]

It follows that the theorem holds. \( \blacksquare \)

It is worthy of noting that, when the external force and control of a regular point reducible RCL system \((TQ,G,\omega^L,L,F^L,C^L)\) are both zeros, that is, \(F^L = 0\) and \(C^L = \emptyset\), in this case the RCL system is just a regular point reducible Lagrangian system \((TQ,G,\omega^L,L)\). Then the following theorem explains the relationship between the equivalence for the regular point reducible Lagrangian systems with symmetries and the equivalence for the associated \(R_p\)-reduced Lagrangian systems.

**Theorem 4.4** Two regular point reducible Lagrangian systems \((TQ_i,G_i,\omega_i^L,L_i), \; i = 1, 2, \) are equivalent if and only if the associated \(R_p\)-reduced Lagrangian systems \((TQ_i)_{\mu_i},\omega_i^L_{\mu_i}, l_{\mu_i}), \; i = 1, 2, \) are equivalent.

**Proof:** If two regular point reducible Lagrangian systems \((TQ_i,G_i,\omega_i^L,L_i), \; i = 1, 2, \) are equivalent, then there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \) such that \( T \varphi : TQ_1 \to TQ_2 \) is symplectic with respect to their Lagrangian symplectic forms \( \omega_i^L, \; i = 1, 2, \) that is, \( \omega_i^L = (T \varphi)^* \cdot \omega_i^L \), and for \( \mu_i \in g_i^*, \; i = 1, 2, \) \( (T \varphi)_\mu = j_{\mu_2}^{-1} \cdot T \varphi \cdot j_{\mu_1} : (J_L)_1^{-1}(\mu_1) \to (J_L)_2^{-1}(\mu_2) \) is \((G_{1\mu_1},G_{2\mu_2})\)-equivariant.

From the above commutative Diagram-4, we can define a map \( (T \varphi)_{\mu_i/G} : (TQ_1)_{\mu_i} \to (TQ_2)_{\mu_i} \), such that \( (T \varphi)_{\mu_i/G} \cdot \tau_{\mu_1} = \tau_{\mu_2} \cdot (T \varphi)_\mu \). Since \( (T \varphi)_\mu : (J_L)_1^{-1}(\mu_1) \to (J_L)_2^{-1}(\mu_2) \) is \((G_{1\mu_1},G_{2\mu_2})\)-equivariant, then \( (T \varphi)_{\mu}/G \) is well-defined. In order to prove that the associated \(R_p\)-reduced Lagrangian systems \(((TQ_i)_{\mu_i},\omega_i^L_{\mu_i}, l_{\mu_i}), \; i = 1, 2, \) are equivalent, in the following we shall show that \( (T \varphi)_{\mu_i/G} \) is symplectic with respect to their \(R_p\)-reduced Lagrangian symplectic forms \( \omega_i^L_{\mu_i}, \; i = 1, 2, \) that is, \( (T \varphi)_{\mu_i/G} \omega_i^L_{\mu_2} = \omega_i^L_{\mu_1} \). In fact, since \( T \varphi : TQ_1 \to TQ_2 \) is symplectic with respect to their Lagrangian symplectic forms, the map \( (T \varphi)^* : \Omega^2(TQ_2) \to \Omega^2(TQ_1) \) satisfies \( (T \varphi)^* \omega_i^L = \omega_i^L \). From (2.3) we know that, \( j_{\mu_i}^* \omega_i^L = \tau_{\mu_i}^* \omega_i^L_{\mu_i}, \; i = 1, 2, \) from the following commutative Diagram-5,

\[
\begin{array}{ccc}
\Omega^2(TQ_2) & \xrightarrow{(T \varphi)^*} & \Omega^2((J_L)_2^{-1}(\mu_2)) \\
\downarrow & & \downarrow \\
\Omega^2(TQ_1) & \xrightarrow{(T \varphi)_{\mu_i}^*} & \Omega^2((J_L)_1^{-1}(\mu_1)) \\
\end{array}
\]

17
we have that
\[
\tau^*_{i_1} \cdot (T\varphi)^*_{\mu/G} \omega^L_{2\mu_2} = ((T\varphi)^*_{\mu/G} \cdot \tau^*_{i_1}) \cdot \omega^L_{2\mu_2} = (\tau_{\mu_2} \cdot (T\varphi)^*_{\mu}) \cdot \omega^L_{2\mu_2}
\]
\[
= (j_{\mu_1}^{-1} \cdot T\varphi \cdot j_{\mu_1})^* \cdot \tau^*_{\mu_2} \cdot \omega^L_{2\mu_2}
\]
\[
= j_{\mu_1}^* \cdot (T\varphi)^* \cdot (j_{\mu_1}^{-1})^* \cdot j_{\mu_2} \cdot \omega^L_{2\mu_2}
\]
\[
= j_{\mu_1}^* \cdot (T\varphi)^* \cdot \omega^L_{2\mu_2} = j_{\mu_1}^* \cdot \omega^L_{2\mu_1} = \tau_{\mu_1}^* \cdot \omega^L_{2\mu_1}.
\]
Notice that \(\tau_{\mu_1}\) is surjective, thus, \((T\varphi)^*_{\mu/G} \omega^L_{2\mu_2} = \omega^L_{1\mu_1}.

Conversely, assume that the \(R_p^p\)-reduced Lagrangian systems \(((TQ_i)_{\mu_1}, \omega^L_{1\mu_1}, l_{i\mu_1}), i = 1, 2\), are equivalent, then there exists a diffeomorphism \((T\varphi)_{\mu/G} : (TQ_1)_{\mu_1} \to (TQ_2)_{\mu_2}\), which is symplectic with respect to their \(R_p^p\)-reduced Lagrangian symplectic forms \(\omega^L_{i\mu_1}, i = 1, 2\). From the above commutative Diagram-4, we can define a map \((T\varphi)_{\mu} : (J_L)_{l_1}^{-1}(\mu_1) \to (J_L)_{l_2}^{-1}(\mu_2)\), such that \(\tau_{\mu_2} \cdot (T\varphi)_{\mu} = (T\varphi)_{\mu/G} \cdot \tau_{\mu_1}\), and the map \(T\varphi : TQ_1 \to TQ_2\), such that \(T\varphi \cdot j_{\mu_1} = j_{\mu_2} \cdot (T\varphi)_{\mu},\) as well as a diffeomorphism \(\varphi : Q_1 \to Q_2\), whose tangent map is just \(T\varphi : TQ_1 \to TQ_2\). From definition of \((T\varphi)_{\mu}\), we know that \((T\varphi)_{\mu}\) is \((G_{1\mu_1} \cdot G_{2\mu_2})\)-equivariant. In the following we shall show that \(T\varphi\) is symplectic with respect to the Lagrangian symplectic forms \(\omega^L_{i\mu}, i = 1, 2\), that is, \(\omega^L_{i} = (T\varphi)^* \cdot \omega^L_{2\mu_2}\). Because \((T\varphi)^*_{\mu/G} : (TQ_1)_{\mu_1} \to (TQ_2)_{\mu_2}\) is symplectic with respect to their \(R_p^p\)-reduced Lagrangian symplectic forms, the map \((T\varphi)^*_{\mu/G} : \Omega^2((TQ_2)_{\mu_2}) \to \Omega^2((TQ_1)_{\mu_1})\), satisfies \((T\varphi)^*_{\mu/G} \cdot \omega^L_{2\mu_2} = \omega^L_{1\mu_1}\). From (2.3) we know that, \(j^*_{\mu_1} \cdot \omega^L_{i\mu} = \tau^*_{i_1} \cdot \omega^L_{i\mu_1}, i = 1, 2\), from the commutative Diagram-5, we have that
\[
\tau^*_{i_1} \cdot (T\varphi)^* \cdot \omega^L_{2\mu_2} = ((T\varphi)^*_{\mu/G} \cdot \omega^L_{2\mu_2} = ((T\varphi)^*_{\mu/G} \cdot \tau^*_{i_1} \cdot \omega^L_{2\mu_2}
\]
\[
= (\tau_{\mu_2} \cdot (T\varphi)^* \cdot \omega^L_{2\mu_2} = (j_{\mu_1}^{-1} \cdot T\varphi \cdot j_{\mu_1}) \cdot \tau_{\mu_2} \cdot \omega^L_{2\mu_2}
\]
\[
= j_{\mu_1} \cdot (T\varphi)^* \cdot \omega^L_{2\mu_2} = j_{\mu_1} \cdot (T\varphi)^* \cdot \omega^L_{2\mu_1}
\]
Notice that \(j_{\mu_1}\) is injective, and hence, \(\omega^L_{i\mu} = (T\varphi)^* \cdot \omega^L_{1\mu_1}\). Thus, the regular point reducible Lagrangian systems \((TQ_i, G_i, \omega^L_{i\mu}, L_i), i = 1, 2\), are equivalent.

Thus, the regular point reduction Theorem 4.3 for the RCL systems can be regarded as an extension of the regular point reduction Theorem 4.4 for the regular Lagrangian systems under regular controlled Lagrangian equivalence conditions.

**Remark 4.5** If \((TQ, \omega^L)\) is a connected symplectic manifold, and \(J_L : TQ \to g^*\) is a non-equivariant momentum map with a non-equivariance group one-cocycle \(\sigma : G \to g^*\), which is defined by \(\sigma(g) := J_L(g \cdot z) - Ad_{g^{-1}}^* J_L(z),\) where \(g \in G\) and \(z \in TQ\). Then we know that \(\sigma\) produces a new affine action \(\Theta : G \times g^* \to g^*\) defined by \(\Theta(g, \mu) := Ad_{g^{-1}}^* \mu + \sigma(g),\) where \(\mu \in g^*\), with respect to which the given momentum map \(J_L\) is equivariant. Assume that \(G\) acts freely and properly on \(TQ\), and \(G_{\mu}\) denotes the isotropy subgroup of \(\mu \in g^*\) relative to this affine action \(\Theta\) and \(\mu\) is a regular value of \(J_L\). Then the quotient space \((TQ)_{\mu} := J_L^{-1}(\mu)/G_{\mu}\) is also a symplectic manifold with the symplectic form \(\omega^L_{\mu}\) uniquely characterized by (2.3). In this case, we can also define the regular point reducible RCL system \((TQ, G, \omega^L, L, F^L, C^L)\) and RpCL-equivalence, and prove the regular point reduction theorem for the RCL system by using the above similar way.

### 5 Regular Orbit Reduction of the RCL System

Since the set of regular Lagrangian systems with symmetries on \(TQ\) is a subset of the set of RCL systems with symmetries on \(TQ\). If we first admit the regular orbit reduction of a regular La-
grangian system with symmetry, then we may study the regular orbit reduction of an RCL system with symmetry, as an extension of the regular orbit reduction of a regular Lagrangian system under the regular controlled Lagrangian equivalence conditions. In order to do these, in this section we consider the RCL system with symmetry and momentum map, and first give the regular orbit reducible RCL system and the RoCL-equivalence, then prove the regular orbit reduction theorems for the RCL system and regular Lagrangian system.

Note that, if an RCL system with symmetry and momentum map is regular orbit reducible, then the associated regular Lagrangian system must be regular orbit reducible. Thus, from Definition 2.5 and Theorem 2.8, if the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is $(\Phi^T, \Phi^{T*})$-equivariant, then we can introduce a kind of regular orbit reducible RCL systems as follows.

**Definition 5.1 (Regular Orbit Reducible RCL System)** A 6-tuple $(TQ, G, \omega^L, L, F^L, C^L)$, where the hyperregular Lagrangian $L : TQ \rightarrow \mathbb{R}$, the fiber-preserving map $F^L : TQ \rightarrow TQ$ and the fiber submanifold $C^L$ of $TQ$ are all $G$-invariant, is called a regular orbit reducible RCL system, if the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$ is $(\Phi^T, \Phi^{T*})$-equivariant, and there exists an orbit $O_\mu$, $\mu \in g^*_*$, where $\mu$ is a regular map, the momentum map $J_\mu$, such that the regular orbit reduced system, that is, the 5-tuple $\{(TQ)\mu_1, \omega^L_\mu_1, l\mu_1, J_{\mu_1}^T, C_{\mu_1}^T\}$, where $\{(TQ)\mu_1 = J^{-1}_L(O_\mu)/G, \omega^L_\mu_1 = J^*_L(O_\mu)\}$, $l\mu_1 = l\cdot j\mu_1$, $J_{\mu_1}^T = J^T/\mu_1$, and $C_{\mu_1}^T = C^T/\mu_1$ is an RCL system, which is simply written as RoCL-reduced RCL system. Where $\{(TQ)\mu_1, \omega^L_\mu_1\}$ is the Ro-reduced space, the function $l\mu_1 : (TQ)\mu_1 \rightarrow \mathbb{R}$ is called the Ro-reduced Lagrangian, the fiber-preserving map $f_{\mu_1}^L : (TQ)\mu_1 \rightarrow (TQ)\mu_2$ is called the Ro-reduced (external) force map, $C_{\mu_1}^L$ is a fiber submanifold of $\{(TQ)\mu_1\}$, and is called the Ro-reduced control subset.

It is worthy of noting that for the regular orbit reducible RCL system $(TQ, G, \omega^L, L, F^L, C^L)$, the $G$-invariant external force map $F^L : TQ \rightarrow TQ$ has to satisfy the conditions $F^L(J^{-1}_L(O_\mu)) \subseteq J^{-1}_L(O_\mu)$, and $f_{\mu_1}^L \cdot \tau_{\mu_1} = \tau_{\mu_2} \cdot F^L \cdot j\mu_1$, such that we can define the Ro-reduced external force map $f_{\mu_1}^L : (TQ)\mu_1 \rightarrow (TQ)\mu_2$. The condition $C^L \cap J^{-1}_L(O_\mu) \neq \emptyset$ in above definition makes that the G-invariant control subset $C^L \cap J^{-1}_L(O_\mu)$ can be reduced and the Ro-reduced control subset is $C_{\mu_1}^L = \tau_{\mu_1}(C^L \cap J^{-1}_L(O_\mu))$.

Assume that the dynamical vector field $\xi_{(TQ,G,\omega^L,L,F^L,u^L)}$ of a given regular orbit reducible RCL system $(TQ, G, \omega^L, L, F^L, C^L)$ with a control law $u^L \in C^L$ can be expressed by

$$\xi_{(TQ,G,\omega^L,L,F^L,u^L)} = \xi_L + \text{vlift}(F^L) + \text{vlift}(u^L). \quad (5.1)$$

Then, for the regular orbit reducible RCL system we can also introduce the regular orbit reducible controlled Lagrangian equivalence (RoCL-equivalence) as follows.

**Definition 5.2 (RoCL-equivalence)** Suppose that we have two regular orbit reducible RCL systems $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, C^L_i)$, $i = 1, 2$, we say them to be RoCL-equivalent, or simply, $(TQ_1, G_1, \omega^L_1, L_1, F^L_1, C^L_1) \sim_{\text{RoCL}} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, C^L_2)$, if there exists a diffeomorphism $\varphi : Q_1 \rightarrow Q_2$ such that the following regular orbit reducible controlled Lagrangian matching conditions hold:

**RoCL-1:** For $O_\mu_i$, $\mu_i \in g^*_*$, the regular reducible orbits of RCL systems $(TQ_i, G_i, \omega^L_i, L_i, F^L_i, C^L_i)$, $i = 1, 2$, the map $(T\varphi)\mu_1 = j_{\mu_2}^{-1} \cdot T\varphi \cdot j_{\mu_1} : (J_L)_i^{-1}(O_{\mu_1}) \rightarrow (J_L)_2^{-1}(O_{\mu_2})$ is $(G_1, G_2)$-equivariant, $\mu = (\mu_1, \mu_2)$, and denote by $j_{\mu_2}^{-1}(S)$ the pre-image of a subset $S \subset TQ_2$ for the map $j_{\mu_2} : (J_L)_2^{-1}(O_{\mu_2}) \rightarrow TQ_2$. Then, for the regular orbit reducible RCL system we can also introduce the regular orbit reducible controlled Lagrangian equivalence (RoCL-equivalence) as follows.
RoCL-2: For each control law \( u_i^1 : TQ_1 \rightarrow C_i^L \), there exists the control law \( u_i^2 : TQ_2 \rightarrow C_i^L \), such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is,

\[
\xi((TQ_2, G_2, \omega^L_2, L_2, F^L_2, u^2_i)) \cdot T\varphi = T(T\varphi) \cdot \xi((TQ_1, G_1, \omega^L_1, L_1, F^L_1, u^1_i)).
\]

It is worthy of noting that for the regular orbit reducible RCL system, the induced equivalent map \( T\varphi \) not only keeps the equivariance of \( G \)-action on their regular orbits, but also keeps the restriction of the \((+)^{\text{symplectic}}\) structure on the regular orbit to \( J_{T^{-1}}^{-1}(O_\mu) \). If a feedback control law \( u^L : TQ \rightarrow C_i^L \) is chosen, and \( u^L \in C^L \cap J_{T^{-1}}^{-1}(O_{\mu_i}) \), and \( C^L \cap J_{T^{-1}}^{-1}(O_{\mu_i}) \neq \emptyset \), then the \( R_{\theta} \)-reduced control law \( u^L_{O_{\mu_i}} : (TQ)_{O_\mu_i} \rightarrow C_i^L_{O_{\mu_i}} = \tau_{O_\mu_i}(C^L \cap J_{T^{-1}}^{-1}(O_{\mu_i})) \), and \( u^L_{O_\mu_i} \cdot \tau_{O_\mu_i} = \tau_{O_\mu_i} \cdot u^L \cdot j_{O_{\mu_i}} \). The \( R_{\theta} \)-reduced RCL system \( ((TQ)_{O_\mu_i}, \omega^L_{O_{\mu_i}}, l_{O_{\mu_i}}, F^L_{O_{\mu_i}}, u^L_{O_{\mu_i}}) \) is a closed-loop regular dynamical system with the \( R_{\theta} \)-reduced control law \( u^L_{O_{\mu_i}} \). Assume that its dynamical vector field \( \xi_{((TQ)_{O_\mu_i}, \omega^L_{O_{\mu_i}}, l_{O_{\mu_i}}, F^L_{O_{\mu_i}}, u^L_{O_{\mu_i}})} \) can be expressed by

\[
\xi_{((TQ)_{O_\mu_i}, \omega^L_{O_{\mu_i}}, l_{O_{\mu_i}}, F^L_{O_{\mu_i}}, u^L_{O_{\mu_i}})} = \xi_{l_{O_{\mu_i}}} + \textup{vlift}(f^L_{O_{\mu_i}}) + \textup{vlift}(u^L_{O_{\mu_i}}),
\]

where \( \xi_{l_{O_{\mu_i}}} \) is the \( R_{\theta} \)-reduced Euler-Lagrange vector field, \( \textup{vlift}(f^L_{O_{\mu_i}}) = \textup{vlift}(f^L_{O_{\mu_i}}) \xi_{l_{O_{\mu_i}}} \), \( \textup{vlift}(u^L_{O_{\mu_i}}) = \textup{vlift}(u^L_{O_{\mu_i}}) \xi_{l_{O_{\mu_i}}} \) are the changes of \( \xi_{l_{O_{\mu_i}}} \) under the actions of the \( R_{\theta} \)-reduced external force \( f^L_{O_{\mu_i}} \) and the \( R_{\theta} \)-reduced control law \( u^L_{O_{\mu_i}} \), and the dynamical vector fields of the RCL system and the \( R_{\theta} \)-reduced RCL system satisfy the condition

\[
\xi_{((TQ)_{O_\mu_i}, \omega^L_{O_{\mu_i}}, l_{O_{\mu_i}}, F^L_{O_{\mu_i}}, u^L_{O_{\mu_i}})} \cdot \tau_{O_\mu_i} = T \tau_{O_\mu_i} \cdot \xi_{((TQ, G, \omega^L, L, F^L, u^L))} \cdot j_{O_{\mu_i}},
\]

see Marsden et al. [18], Wang [24]. Then we can obtain the following regular orbit reduction theorem for the RCL system, which explains the relationship between the RoCL-equivariance for the regular orbit reducible RCL system with symmetry and the RCL-equivariance for the associated \( R_{\theta} \)-reduced RCL system.

**Theorem 5.3** If two regular orbit reducible RCL systems \( (TQ_i, G_i, \omega^L_i, L_i, F^L_i, C_i^L), i = 1, 2, \) are RoCL-equivalent if and only if the associated \( R_{\theta} \)-reduced RCL systems \( ((TQ_i)_{O_{\mu_i}}, \omega^L_{O_{\mu_i}}, l_{O_{\mu_i}}, F^L_{O_{\mu_i}}, C^L_{O_{\mu_i}}), i = 1, 2, \) are RCL-equivalent.

**Proof:** If \( (TQ_1, G_1, \omega^L_1, L_1, F^L_1, C^L_1) \overset{\text{RoCL}}{\sim} (TQ_2, G_2, \omega^L_2, L_2, F^L_2, C^L_2) \), then there exists a diffeomorphism \( \varphi : Q_1 \rightarrow Q_2 \), such that for \( O_{\mu_i}, \mu_i \in g_i^* \), the regular reducible orbits, the map \( (T\varphi)_{O_{\mu_i}} = j_{O_{\mu_i}}^{-1} \cdot T\varphi \cdot j_{O_{\mu_i}} : (JL)_{1}^{-1}(O_{\mu_1}) \rightarrow (JL)_2^{-1}(O_{\mu_2}) \) is \( (G_1, G_2) \)-equivariant, and \( C^L_1 \cap (JL)_1^{-1}(O_{\mu_1}) = (T\varphi)_{O_{\mu_i}}(C^L_1 \cap (JL)_1^{-1}(O_{\mu_1})) \), and RoCL-2 holds. From the following commutative Diagram-6:

\[
\begin{array}{cccc}
TQ_1 & \leftarrow & (JL)_1^{-1}(O_{\mu_1}) & \rightarrow & (TQ_1)_{O_{\mu_1}} \\
T\varphi & \downarrow & (T\varphi)_{O_{\mu_i}} & \downarrow & (T\varphi)_{O_{\mu_i}/G} \\
TQ_2 & \leftarrow & (JL)_2^{-1}(O_{\mu_2}) & \rightarrow & (TQ_2)_{O_{\mu_2}}
\end{array}
\]

we can define a map \( (T\varphi)_{O_{\mu_i}/G} : (TQ_1)_{O_{\mu_1}} \rightarrow (TQ_2)_{O_{\mu_2}} \) such that \( (T\varphi)_{O_{\mu_i}/G} \cdot \tau_{O_{\mu_1}} = \tau_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_i}} \). Because \( (T\varphi)_{O_{\mu_i}} : (JL)_1^{-1}(O_{\mu_1}) \rightarrow (JL)_2^{-1}(O_{\mu_2}) \) is \( (G_1, G_2) \)-equivariant, \( (T\varphi)_{O_{\mu_i}/G} \) is well-defined. We shall show that \( C^L_{2O_{\mu_2}} = (T\varphi)_{O_{\mu_i}/G}(C^L_{1O_{\mu_1}}) \). In fact, because \( (TQ_i, G_i, \omega^L_i, L_i, F^L_i, C^L_i), i = 1, 2, \) are regular orbit reducible RCL systems, then \( C_i^L \cap (JL)_i^{-1}(O_{\mu_i}) \neq \emptyset \) and \( C^L_{1O_{\mu_1}} = \tau_{O_{\mu_1}}(C_i^L \cap (JL)_i^{-1}(O_{\mu_i})) \), \( i = 1, 2 \). From \( C^L_2 \cap (JL)_2^{-1}(O_{\mu_2}) = (T\varphi)_{O_{\mu_i}}(C^L_i \cap (JL)_1^{-1}(O_{\mu_1})) \), and we have that

\[
C^L_{2O_{\mu_2}} = \tau_{O_{\mu_2}}(C^L_2 \cap (JL)_2^{-1}(O_{\mu_2})) = \tau_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_i}}(C^L_i \cap (JL)_1^{-1}(O_{\mu_1})) = (T\varphi)_{O_{\mu_i}/G}(C^L_{1O_{\mu_1}}).
\]
Thus, the condition RCL-1 holds. On the other hand, for the $R_o$-reduced control law $u_{2O_{\mu_1}}^L : (TQ_2)_{O_{\mu_1}} \to C_{1O_{\mu_1}}^L$, we have the control law $u_{2O_{\mu_1}}^L : TQ_1 \to C_{1O_{\mu_1}}^L$, such that $u_{1O_{\mu_1}}^L \cdot \tau_{O_{\mu_1}} = \tau_{O_{\mu_1}} \cdot u_{2O_{\mu_1}}^L \cdot j_{O_{\mu_1}}$.

From the condition RoCL-2 we know that there exists the control law $u_{2}^L : TQ_2 \to C_{1}^L$, such that $\xi(TQ_2,G_2,\omega_2,L_2,F_2^L,u_2^L) \cdot T\varphi = T(\varphi) \cdot \xi(TQ_1,G_1,\omega_1,L_1,F_1^L,u_1^L)$. But, for the control law $u_{2}^L : TQ_2 \to C_{1}^L$, we have the $R_o$-reduced control law $u_{2O_{\mu_2}}^L : (TQ_2)_{O_{\mu_2}} \to C_{2O_{\mu_2}}$, such that $u_{2O_{\mu_2}}^L \cdot \tau_{O_{\mu_2}} = \tau_{O_{\mu_2}} \cdot u_{2}^L \cdot j_{O_{\mu_2}}$.

Note that for $i = 1, 2$, from (5.3), we have that

$$\xi((TQ_1)_{O_{\mu_i}} \omega_{iO_{\mu_i}}^L \lambda_{2O_{\mu_i}} \lambda_{2O_{\mu_i}}, j_{2O_{\mu_i}}^L \mu_{iO_{\mu_i}}^L u_{2O_{\mu_i}}^L) \cdot \tau_{O_{\mu_i}} = T\tau_{O_{\mu_i}} \cdot \xi((TQ_1,G_i,\omega_i^L,L_i,F_i^L,u_i^L)) \cdot j_{O_{\mu_i}},$$

(5.4)

and from the commutative Diagram-6, $(T\varphi)_{O_{\mu_i}/G} \cdot \tau_{O_{\mu_i}} = \tau_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_i}}$ and $j_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_i}} = (T\varphi) \cdot j_{O_{\mu_i}}$, then we have that

$$\xi((TQ_2)_{O_{\mu_2}} \omega_{2O_{\mu_2}}^L \lambda_{2O_{\mu_2}} \lambda_{2O_{\mu_2}}, j_{2O_{\mu_2}}^L \mu_{iO_{\mu_2}}^L u_{2O_{\mu_2}}^L) \cdot (T\varphi)_{O_{\mu_i}} = T\tau_{O_{\mu_2}} \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot j_{O_{\mu_i}},$$

$$= \xi((TQ_2)_{O_{\mu_2}} \omega_{2O_{\mu_2}}^L \lambda_{2O_{\mu_2}} \lambda_{2O_{\mu_2}}, j_{2O_{\mu_2}}^L \mu_{iO_{\mu_2}}^L u_{2O_{\mu_2}}^L) \cdot (T\varphi)_{O_{\mu_i}} = T\tau_{O_{\mu_2}} \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot j_{O_{\mu_i}},$$

$$= T\tau_{O_{\mu_2}} \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot j_{O_{\mu_i}} = T((T\varphi)_{O_{\mu_i}/G} \cdot \tau_{O_{\mu_i}}) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot j_{O_{\mu_i}},$$

$$= T((T\varphi)_{O_{\mu_i}/G}) \cdot \xi((TQ_1,G_1,\omega_1^L,L_1,F_1^L,u_1^L)) \cdot j_{O_{\mu_i}} = T((T\varphi)_{O_{\mu_i}/G}) \cdot \tau_{O_{\mu_1}} = \tau_{O_{\mu_1}},$$

Since $\tau_{O_{\mu_1}} : (J_L)^{-1}(O_{\mu_1}) \to (TQ_1)_{O_{\mu_1}}$ is surjective, thus,

$$\xi((TQ_2)_{O_{\mu_2}} \omega_{2O_{\mu_2}}^L \lambda_{2O_{\mu_2}} \lambda_{2O_{\mu_2}}, j_{2O_{\mu_2}}^L \mu_{iO_{\mu_2}}^L u_{2O_{\mu_2}}^L) \cdot (T\varphi)_{O_{\mu_i}/G} = T((T\varphi)_{O_{\mu_i}/G}) \cdot \xi((TQ_1)_{O_{\mu_1}} \omega_{1O_{\mu_1}}^L \lambda_{1O_{\mu_1}} \lambda_{1O_{\mu_1}}, j_{1O_{\mu_1}} \mu_{iO_{\mu_1}}^L u_{1O_{\mu_1}}^L),$$

(5.5)

that is, the condition RCL-2 holds. So, the $R_o$-reduced RCL systems $((TQ_1)_{O_{\mu_i}}, \omega_{iO_{\mu_i}}^L, \lambda_{iO_{\mu_i}}, j_{iO_{\mu_i}} \mu_{iO_{\mu_i}}^L, C_{iO_{\mu_i}}^L), i = 1, 2$, are RCL-equivalent.

Conversely, assume that the $R_o$-reduced RCL systems $((TQ_1)_{O_{\mu_i}}, \omega_{iO_{\mu_i}}^L, \lambda_{iO_{\mu_i}}, j_{iO_{\mu_i}} \mu_{iO_{\mu_i}}^L, C_{iO_{\mu_i}}^L), i = 1, 2$, are RCL-equivalent, then there exists a diffeomorphism $(T\varphi)_{O_{\mu_i}/G} : (TQ_1)_{O_{\mu_1}} \to (TQ_2)_{O_{\mu_2}}$, such that $C_{2O_{\mu_2}}^L = (T\varphi)_{O_{\mu_i}/G}(C_{1O_{\mu_1}}^L), \forall O_{\mu_1}, \mu_i \in \mathbf{g}^*_1, i = 1, 2$ and for the $R_o$-reduced control law $u_{2O_{\mu_1}}^L : (TQ_1)_{O_{\mu_1}} \to C_{1O_{\mu_1}}^L$, there exists the $R_o$-reduced control law $u_{2O_{\mu_2}}^L : (TQ_2)_{O_{\mu_2}} \to C_{2O_{\mu_2}}$, such that (5.5) holds. Then from commutative Diagram-6, we can define a map $(T\varphi)_{O_{\mu_i}} : (J_L)_{1}^{-1}(O_{\mu_i}) \to (J_L)_{2}^{-1}(O_{\mu_i}), \forall \mu_i \in \mathbf{g}^*_1, i = 1, 2$ such that $T\varphi \cdot j_{O_{\mu_1}} = j_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_i}}$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$, whose tangent lift is just $T\varphi : TQ_1 \to TQ_2$. Moreover, for above definition of $(T\varphi)_{O_{\mu_i}}$, we know that $(T\varphi)_{O_{\mu_i}}$ is $(G_1,G_2)$-equivariant. In fact, for any $z_i \in (J_L)_{1}^{-1}(O_{\mu_i}), g_i \in G_i, i = 1, 2$ such that $z_2 = (g_1 \varphi)(z_1)$, $[z_2] = (T\varphi)_{O_{\mu_i}/G}(g_1 \varphi)(z_1)$, then we have that

$$(T\varphi)_{O_{\mu_i}}(\Phi_{1g_1}(z_1)) = \tau_{O_{\mu_1}}^{-1} \cdot \tau_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_i}}(\Phi_{1g_1}(z_1)) = \tau_{O_{\mu_1}}^{-1} \cdot \tau_{O_{\mu_2}} \cdot (T\varphi)_{O_{\mu_1}}(g_1 z_1) = \tau_{O_{\mu_1}}^{-1} \cdot (T\varphi)_{O_{\mu_1}/G} \cdot (T\varphi)_{O_{\mu_1}}(z_1) = \tau_{O_{\mu_2}}^{-1} \cdot (T\varphi)_{O_{\mu_1}/G} \cdot (T\varphi)_{O_{\mu_1}}(z_1) = \tau_{O_{\mu_2}}^{-1} \cdot \tau_{O_{\mu_2}}(g_2 z_2) = \Phi_{2g_2}(z_2) = \Phi_{2g_2} \cdot (T\varphi)_{O_{\mu_i}}(z_1).$$
Here we denote by \( \tau_{\mathcal{O}_{\mu_1}}^{-1}(S) \) the pre-image of a subset \( S \subset (TQ_1)\mathcal{O}_{\mu_1} \) for the map \( \tau_{\mathcal{O}_{\mu_1}} : (J_L)_{1}^{-1}(\mathcal{O}_{\mu_1}) \rightarrow (TQ_1)\mathcal{O}_{\mu_1} \), and for any \( z_1 \in (J_L)_{1}^{-1}(\mathcal{O}_{\mu_1}) \), \( \tau_{\mathcal{O}_{\mu_1}}^{-1} \cdot \tau_{\mathcal{O}_{\mu_1}}(z_1) = z_1 \). So, we obtain that \( (T\varphi)\mathcal{O}_{\mu} \cdot \Phi_{1g_1} = \Phi_{2g_2} \cdot (T\varphi)\mathcal{O}_{\mu} \). Moreover, we have that

\[
\mathcal{C}_L^T \cap (J_L)_2^{-1}(\mathcal{O}_{\mu_2}) = \tau_{\mathcal{O}_{\mu_2}}^{-1} \cdot \tau_{\mathcal{O}_{\mu_2}} (\mathcal{C}_L^T \cap (J_L)_2^{-1}(\mathcal{O}_{\mu_2}))
\]

Thus, the condition RoCL-1 holds. In the following we shall prove that the condition RoCL-2 holds. For the above \( R_{\omega} \)-reduced control laws \( u^L_{i\mathcal{O}_{\mu_1}} : (TQ_i)\mathcal{O}_{\mu_1} \rightarrow \mathcal{C}_L^T, i = 1, 2 \), there exist the control laws \( u^L_t : TQ_i \rightarrow \mathcal{C}_L^T \), such that \( u^L_{i\mathcal{O}_{\mu_1}} \cdot \tau_{\mathcal{O}_{\mu_1}} = \tau_{\mathcal{O}_{\mu_1}} \cdot u^L_t \cdot j_{\mathcal{O}_{\mu_1}}, i = 1, 2 \). We shall prove that

\[
\xi((TQ_2, G, \omega^L_2, L_2, F^L_2, u^L_2)) \cdot T\varphi = T(T\varphi) \cdot \xi((TQ_1, G, \omega^L_1, L_1, F^L_1, u^L_1))
\]

In fact, from (5.4) we have that

\[
T((T\varphi)\mathcal{O}_{\mu} / G) = \xi((TQ_1)\mathcal{O}_{\mu_1} \cdot \omega^L_1 \cdot \mathcal{O}_{\mu_1} \cdot f^L_1 \cdot \mathcal{O}_{\mu_1} \cdot u^L_1) \cdot \tau_{\mathcal{O}_{\mu_1}}
\]

On the other hand,

\[
\xi((TQ_2)\mathcal{O}_{\mu_2} \cdot \omega^L_2 \cdot \mathcal{O}_{\mu_2} \cdot f^L_2 \cdot \mathcal{O}_{\mu_2} \cdot u^L_2) \cdot \tau_{\mathcal{O}_{\mu_2}} \cdot (T\varphi)\mathcal{O}_{\mu} / G
\]

From (5.5) we have that

\[
T\tau_{\mathcal{O}_{\mu_2}} \cdot \xi((TQ_2, G, \omega^L_2, L_2, F^L_2, u^L_2)) \cdot T\varphi \cdot j_{\mathcal{O}_{\mu_1}} = T\tau_{\mathcal{O}_{\mu_2}} \cdot T(T\varphi) \cdot \xi((TQ_1, G, \omega^L_1, L_1, F^L_1, u^L_1)) \cdot j_{\mathcal{O}_{\mu_1}}
\]

Note that the map \( j_{\mathcal{O}_{\mu_1}} : (J_L)_1^{-1}(\mathcal{O}_{\mu_1}) \rightarrow TQ_1 \) is injective, and \( T\tau_{\mathcal{O}_{\mu_2}} : (J_L)_2^{-1}(\mathcal{O}_{\mu_2}) \rightarrow T(TQ_2)\mathcal{O}_{\mu_2} \) is surjective, hence, we have that

\[
\xi((TQ_2, G, \omega^L_2, L_2, F^L_2, u^L_2)) \cdot T\varphi = T(T\varphi) \cdot \xi((TQ_1, G, \omega^L_1, L_1, F^L_1, u^L_1))
\]

It follows that the theorem holds. ■

It is worthy of noting that, when the external force and control of a regular orbit reducible RCL system \( (TQ, G, \omega^L, L, F^L, C^L) \) are both zeros, that is, \( F^L = 0 \) and \( C^L = \emptyset \), in this case the RCL system is just a regular orbit reducible Lagrangian system \( (TQ, G, \omega^L, L) \). Then the following theorem explains the relationship between the equivalence for the regular orbit reducible Lagrangian systems with symmetries and the equivalence for the associated \( R_{\omega} \)-reduced Lagrangian systems.
Theorem 5.4 If two regular orbit reducible Lagrangian systems \((TQ_i, G_i, \omega^L_i, L_i), i = 1, 2\), are equivalent, then their associated \(R_o\)-reduced Lagrangian systems \(((TQ)_L)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu})\), \(i = 1, 2\), must be equivalent. Conversely, if the \(R_o\)-reduced Lagrangian systems \(((TQ)_L)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu})\), \(i = 1, 2\), are equivalent, and the induced map \((T\varphi)_\mu : (J_L)_1^{-1}(\mathcal{O}_\mu) \rightarrow (J_L)_2^{-1}(\mathcal{O}_\mu)\), such that \((J_L)_1^{\ast}_{\mathcal{O}_\mu} : \omega^L_{\mathcal{O}_\mu} \rightarrow (J_L)_2^{\ast}_{\mathcal{O}_\mu} \omega^L_{20\mathcal{O}_\mu^2}\), then the regular orbit reducible Lagrangian systems \((TQ_i, G_i, \omega^L_i, L_i), i = 1, 2\), are equivalent.

Proof: If two regular orbit reducible Lagrangian systems \((TQ_i, G_i, \omega^L_i, L_i), i = 1, 2\), are equivalent, then there exists a diffeomorphism \(\varphi : Q_1 \rightarrow Q_2\), such that \(T\varphi : TQ_1 \rightarrow TQ_2\) is symplectic with respect to their Lagrangian symplectic forms \(\omega^L_i, i = 1, 2\), and for \(\mathcal{O}_\mu, \mu \in \mathfrak{g}^*_\mathcal{O}\), \(i = 1, 2\), \((T\varphi)_{\mathcal{O}_\mu} = j_{\mathcal{O}_\mu}^{-1} \cdot T\varphi \cdot j_{\mathcal{O}_\mu} : (J_L)_1^{-1}(\mathcal{O}_\mu) \rightarrow (J_L)_2^{-1}(\mathcal{O}_\mu)\) is \((G_1, G_2)\)-equivariant. From the above commutative Diagram-6, we can define a map \((T\varphi)_{\mathcal{O}_\mu / G} : (TQ)_1_{\mathcal{O}_\mu} \rightarrow (TQ)_2_{\mathcal{O}_\mu}\), such that \((T\varphi)_{\mathcal{O}_\mu / G} = \tau_{\mathcal{O}_\mu}(T\varphi)_{\mathcal{O}_\mu}\). Since \((T\varphi)_{\mathcal{O}_\mu} : (J_L)_1^{-1}(\mathcal{O}_\mu) \rightarrow (J_L)_2^{-1}(\mathcal{O}_\mu)\) is \((G_1, G_2)\)-equivariant, then \((T\varphi)_{\mathcal{O}_\mu / G}\) is well-defined. In order to prove that the associated \(R_o\)-reduced Lagrangian systems \(((TQ)_L)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu})\), \(i = 1, 2\), are equivalent, in following we shall prove that \((T\varphi)_{\mathcal{O}_\mu / G}\) is symplectic with respect to their \(R_o\)-reduced Lagrangian symplectic forms \(\omega^L_{\mathcal{O}_\mu}, i = 1, 2\), that is, \((T\varphi)_{\mathcal{O}_\mu / G} : \omega^L_{\mathcal{O}_\mu} = \omega^L_{\mathcal{O}_\mu}\). In fact, since \(T\varphi : TQ_1 \rightarrow TQ_2\) is symplectic with respect to their Lagrangian symplectic forms, and the map \((T\varphi)^{\ast} : \Omega^2(TQ_2) \rightarrow \Omega^2(TQ_1)\) satisfies \((T\varphi)^{\ast} \cdot \omega^L_2 = \omega^L_1\). From (2.4) we have that \(j_{\mathcal{O}_\mu}^{\ast} \cdot \omega^L_2 = \tau_{\mathcal{O}_\mu} \cdot \omega^L_{\mathcal{O}_\mu} + (J_L)_1^{\ast}_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2}, i = 1, 2\), and \((J_L)_1^{\ast}_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2} = \tau_{\mathcal{O}_\mu}(T\varphi)_{\mathcal{O}_\mu}\). From the following commutative Diagram-7,

\[
\begin{array}{ccc}
\Omega^2(TQ_2) & \xrightarrow{(T\varphi)^{\ast}} & \Omega^2((J_L)_2^{-1}(\mathcal{O}_\mu_2)) \\
\downarrow \tau_{\mathcal{O}_\mu} & & \downarrow \tau_{\mathcal{O}_\mu} \\
\Omega^2(TQ_1) & \xrightarrow{(T\varphi)^{\ast}_{\mathcal{O}_\mu}} & \Omega^2((J_L)_1^{-1}(\mathcal{O}_\mu_1))
\end{array}
\]

we have that

\[
\tau_{\mathcal{O}_\mu}^{\ast} \cdot (T\varphi)^{\ast}_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2} = ((T\varphi)_{\mathcal{O}_\mu / G} \cdot \tau_{\mathcal{O}_\mu})^{\ast} \cdot \omega^L_{20\mathcal{O}_\mu^2}
\]

\[
= (\tau_{\mathcal{O}_\mu} \cdot (T\varphi)_{\mathcal{O}_\mu})^{\ast} \cdot \omega^L_{20\mathcal{O}_\mu^2}
\]

\[
= ((T\varphi)_{\mathcal{O}_\mu})^{\ast} \cdot \tau_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2}
\]

\[
= (j_{\mathcal{O}_\mu}^{-1} \cdot T\varphi \cdot j_{\mathcal{O}_\mu})^{\ast} \cdot j_{\mathcal{O}_\mu}^{\ast} \cdot \omega^L_{20\mathcal{O}_\mu^2} - (T\varphi)_{\mathcal{O}_\mu} \cdot (J_L)_2^{\ast}_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2}
\]

\[
= j_{\mathcal{O}_\mu}^{\ast} \cdot (T\varphi)^{\ast} \cdot \omega^L_2 - (J_L)_1^{\ast}_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2}
\]

\[
= j_{\mathcal{O}_\mu}^{\ast} \cdot \omega^L_1 - (J_L)_1^{\ast}_{\mathcal{O}_\mu} \cdot \omega^L_{20\mathcal{O}_\mu^2}
\]

\[
= \tau_{\mathcal{O}_\mu}^{\ast} \cdot \omega^L_{1\mathcal{O}_\mu}.
\]

Because \(\tau_{\mathcal{O}_\mu}\) is surjective, thus, \(((T\varphi)_{\mathcal{O}_\mu / G})^{\ast} \cdot \omega^L_{20\mathcal{O}_\mu^2} = \omega^L_{1\mathcal{O}_\mu}\).

Conversely, assume that the \(R_o\)-reduced Lagrangian systems \(((TQ)_L)_{\mathcal{O}_\mu}, \omega^L_{\mathcal{O}_\mu}, l_{\mathcal{O}_\mu})\), \(i = 1, 2\), are equivalent, then there exists a diffeomorphism \((T\varphi)_{\mathcal{O}_\mu / G} : (TQ)_1_{\mathcal{O}_\mu} \rightarrow (TQ)_2_{\mathcal{O}_\mu}\), which is symplectic with respect to the \(R_o\)-reduced Lagrangian symplectic forms \(\omega^L_{\mathcal{O}_\mu}, i = 1, 2\), that is, \((T\varphi)_{\mathcal{O}_\mu / G} \cdot \omega^L_{20\mathcal{O}_\mu^2} = \omega^L_{1\mathcal{O}_\mu}\). Thus, from the above commutative Diagram-6, we can define a map \((T\varphi)_{\mathcal{O}_\mu} : (J_L)_1^{-1}(\mathcal{O}_\mu_1) \rightarrow (J_L)_2^{-1}(\mathcal{O}_\mu_2)\), such that \(\tau_{\mathcal{O}_\mu} \cdot (T\varphi)_{\mathcal{O}_\mu} = (T\varphi)_{\mathcal{O}_\mu / G} \cdot \tau_{\mathcal{O}_\mu}\), and map
$T\varphi : TQ_1 \to TQ_2$, such that $j_{O_2} \cdot (T\varphi)_{O_1} = T\varphi \cdot j_{O_1}$, as well as a diffeomorphism $\varphi : Q_1 \to Q_2$, whose tangent map is just $T\varphi : TQ_1 \to TQ_2$. From definition of $(T\varphi)_{O_1}$ we know that $(T\varphi)_{O_1}$ is $(G_1, G_2)$-equivariant.

Now we shall show that $T\varphi$ is symplectic with respect to the Lagrangian symplectic forms $\omega_i^L$, $i = 1, 2$, that is, $\omega_i^L = (T\varphi)^* \omega_i^L$. In fact, since $(T\varphi)_{O_1}/G : (TQ_1)_{O_1} \to (TQ_2)_{O_2}$ is symplectic with respect to their $R_0$-reduced Lagrangian symplectic forms, the map $((T\varphi)_{O_1}/G)^* : \Omega^2((TQ_2)_{O_2}) \to \Omega^2((TQ_1)_{O_1})$ satisfies $((T\varphi)_{O_1}/G)^* \cdot \omega_i^L_{O_2} = \omega_i^L_{O_1}$. From (2.4) we have that $j_{O_1}^* \cdot \omega_i^L = \tau_{O_1}^* \cdot \omega_i^L_{O_1} + (J_L)^*_{O_1} \cdot \omega_i^L_{O_1}$.

\[
j_{O_1}^* \cdot \omega_i^L = \tau_{O_1}^* \cdot \omega_i^L_{O_1} + (J_L)^*_{O_1} \cdot \omega_i^L_{O_1} = ((T\varphi)_{O_1}/G)^* \cdot \omega_i^L_{O_2} + (J_L)^*_{O_1} \cdot \omega_i^L_{O_1} = ((T\varphi)_{O_1}/G)^* \cdot \omega_i^L_{O_2} + (J_L)^*_{O_1} \cdot \omega_i^L_{O_1} = (j_{O_1}^{-1} \cdot T \varphi \cdot j_{O_1})^* \cdot \tau_{O_2} \cdot \omega_i^L_{O_2} + (J_L)^*_{O_1} \cdot \omega_i^L_{O_1} = j_{O_1}^* \cdot (T\varphi)^* \cdot \omega_i^L_{O_2} + (J_L)^*_{O_1} \cdot \omega_i^L_{O_1} = j_{O_1}^* \cdot (T\varphi)^* \cdot \omega_i^L_{O_2} - ((T\varphi)_{O_1})^* \cdot (J_L)^*_{O_1} \cdot \omega_i^L_{O_1} \]

Notice that $j_{O_1}^*$ is injective, and by our hypothesis,

\[
(j_L)^*_{O_1} \cdot \omega_i^L_{O_1} = ((T\varphi)_{O_1})^* \cdot (J_L)^*_{O_1} \cdot \omega_i^L_{O_1},
\]

then $\omega_i^L = (T\varphi)^* \omega_i^L$. Thus, the regular orbit reducible Lagrangian systems $(TQ_i, G_i, \omega_i^L, L_i)$, $i = 1, 2$, are equivalent.

Thus, the regular orbit reduction Theorem 5.3 for the RCL systems can be regarded as an extension of the regular orbit reduction Theorem 5.4 for the regular Lagrangian systems under regular controlled Lagrangian equivalence conditions.

**Remark 5.5** If $(TQ, \omega^L)$ is a connected symplectic manifold, and $J_L : TQ \to g^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \to g^*$, which is defined by $\sigma(g) := J_L(g \cdot z) - \text{Ad}_{g^{-1}} \cdot J_L(z)$, where $g \in G$ and $z \in TQ$. Then we know that $\sigma$ produces a new affine action $\Theta : G \times g^* \to g^*$ defined by $\Theta(g, \mu) := \text{Ad}^*_{g^{-1}} \mu + \sigma(g)$, where $\mu \in g^*$, with respect to which the given momentum map $J_L$ is equivariant. Assume that $G$ acts freely and properly on $TQ$, and $O_\mu = G : \mu \subset g^*$ denotes the $G$-orbit of the point $\mu \in g^*$ with respect to the above affine action $\Theta$, and $\mu$ is a regular value of $J_L$. Then the quotient space $(TQ)_{O_\mu} = J_L^{-1}(O_\mu)/G$ is also a symplectic manifold with the symplectic form $\omega^L_{O_\mu}$ uniquely characterized by (2.4). In this case, we can also define the regular orbit reducible RCL system $(TQ, G, \omega^L, L, F^L, C^L)$ and RoCL-equivalence, and prove the regular orbit reduction theorem for the RCL system by using the above similar way.

It is worthy of noting that the research idea and work in Marsden et al. [18] are very important. The authors not only correct and renew carefully some wrong definitions for CH system and its reduced CH systems, as well as CH-equivalence and the reduced CH-equivalence in Chang et al. [8, 9], but also set up the regular reduction theory of regular controlled Hamiltonian systems on a symplectic fiber bundle, from the viewpoint of completeness of Marsden-Weinstein reduction. In this paper, following the ideas in Marsden et al. [18], we correct and renew carefully some wrong definitions for CL system and its reduced CL systems, as well as CL-equivalence and the
reduced CL-equivalence in Chang et al. [8, 9], and set up the regular reduction theory of regular controlled Lagrangian systems on a symplectic fiber bundle, by analyzing carefully the geometrical and topological structures of the phase space and the reduced phase space of the regular Lagrangian system. Note that some developments around the work in Marsden et al. [18] are given in Wang and Zhang [28], Ratiu and Wang [23], Wang [24], and Wang [25], and some applications are given in Wang [26, 27]. Thus, it is natural idea to develop a variety of reduction theory and applications for regular controlled Lagrangian systems, in particular, in celestial mechanics, hydrodynamics and plasma physics. In addition, it is also an important topic for us to explore and reveal the deeply internal relationships between the geometrical structures of phase spaces and the dynamical vector fields of the controlled mechanical systems. In particular, it is an important task for us to correct and develop well the research work of Professor Jerrold E. Marsden, such that we never feel sorry for his great cause.

References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, second ed., Addison-Wesley, Reading, MA, 1978.

[2] R. Abraham, J.E. Marsden, T.S. Ratiu, Manifolds, Tensor Analysis and Applications, in: Applied Mathematical Science, vol. 75, Springer-Verlag, New York, 1988.

[3] V.I. Arnold, Mathematical Methods of Classical Mechanics, second ed., in: Graduate Texts in Mathematics, vol. 60, Springer-Verlag, 1989.

[4] A.M. Bloch, D.E. Chang, N.E. Leonard and J.E. Marsden, Controlled Lagrangians and the stabilization of mechanical systems II: potential shaping, IEEE Trans, Automatic Control, 46 (2001), 1556–1571.

[5] A.M. Bloch and N.E. Leonard, Symmetries, conservation laws, and control, In “Geometry, Mechanics and Dynamics, Volume in Honour of the 60th Birthday of J.E. Marsden” (eds. P.Newton, P.Holmes and A. Weinstein), Springer, New York, 2002.

[6] A.M. Bloch, N.E. Leonard and J.E. Marsden, Controlled Lagrangian and the stabilization of mechanical systems I: the first matching theorem, IEEE Trans, Automatic Control, 45(2000), 2253–2270.

[7] A.M. Bloch, N.E. Leonard and J.E. Marsden, Controlled Lagrangians and the stabilization of Euler-Poincaré mechanical systems, Int. J. Nonlinear and Robust Control, 11(2001), 191-214.

[8] D.E. Chang, A.M. Bloch, N.E. Leonard, J.E. Marsden and C.A. Woolsey, The equivalence of controlled Lagrangian and controlled Hamiltonian systems, ESAIM Control, Optimisation and Calculus of Variations, 8(2002), 393–422.

[9] D.E. Chang and J.E. Marsden, Reduction of controlled Lagrangian and Hamiltonian systems with symmetry, SIAM J. Control Optimization, 43(1)(2004), 277–300.

[10] D. Kazhdan, B. Kostant and S. Sternberg, Hamiltonian group actions dynamical systems of Calogero type, Comm. Pure Appl. Math. 31(1978), 481-508.

[11] P. Libermann, C.M. Marle, Symplectic Geometry and Analytical Mechanics, Kluwer Academic Publishers, 1987.
[12] C.M. Marle, Symplectic manifolds, dynamical groups and Hamiltonian mechanics, In: Differential Geometry and Relativity, (M. Cahen and M. Flato, eds.), D. Reidel, Boston, 1976, 249-269.

[13] J.E. Marsden, Lectures on Mechanics, in: London Mathematical Society Lecture Notes Series, vol. 174, Cambridge University Press, 1992.

[14] J.E. Marsden, G. Misiolek, J.P. Ortega, M. Perlmutter, T.S. Ratiu, Hamiltonian Reduction by Stages, in: Lecture Notes in Mathematics, vol. 1913, Springer, 2007.

[15] J.E. Marsden, R. Montgomery, T.S. Ratiu, Reduction, Symmetry and Phases in Mechanics, in: Memoirs of the American Mathematical Society, vol. 88, American Mathematical Society, Providence, Rhode Island, 1990.

[16] J.E. Marsden and M. Perlmutter, The orbit bundle picture of cotangent bundle reduction, C. R. Math. Acad. Sci. Soc. R. Can., 22, 33-54 (2000).

[17] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, second ed., in: Texts in Applied Mathematics, vol. 17, Springer-Verlag, New York, 1999.

[18] J.E. Marsden, H. Wang and Z.X. Zhang, Regular reduction of controlled Hamiltonian systems with symplectic structure and symmetry, Diff. Geom. Appl., 33(3), 13-45 (2014), (arXiv: 1202.3564, a revised version).

[19] J.E. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1974) 121-130.

[20] K.R. Meyer, Symmetries and integrals in mechanics, In Peixoto M. (eds), Dynamical Systems, Academic Press, 259–273 (1973).

[21] H. Nijmeijer and A.J. Van der Schaft, Nonlinear Dynamical Control Systems, Springer-Verlag, 1990.

[22] J.P. Ortega, T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, in: Progress in Mathematics, vol. 222, Birkhäuser, 2004.

[23] T.S. Ratiu and H. Wang, Poisson reduction by controllability distribution for a controlled Hamiltonian system, (arXiv: 1312.7047).

[24] H. Wang, The geometrical structure of phase space of the controlled Hamiltonian system with symmetry, (arXiv: 1802.01988, a revised version).

[25] H. Wang, Hamilton-Jacobi theorems for regular reducible Hamiltonian systems on a cotangent bundle, Jour. Geom. Phys., 119 82-102, (2017).

[26] H. Wang, Dynamical equations of the controlled rigid spacecraft with a rotor, (arXiv: 2005.02221).

[27] H. Wang, Symmetric reduction and Hamilton-Jacobi equation for the controlled underwater vehicle-rotor system, (arXiv: 1310.3014, a revised version).

[28] H.Wang and Z.X.Zhang, Optimal reduction of controlled Hamiltonian system with Poisson structure and symmetry, Jour. Geom. Phys., 62 (5)(2012), 953-975.