Cartesian-closedness and subcategories of \((L, M)\)-fuzzy \(Q\)-convergence spaces

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Abstract
In this paper, we first construct the function space of \((L, M)\)-fuzzy \(Q\)-convergence spaces to show the Cartesian-closedness of the category \((L, M)\)-QC of \((L, M)\)-fuzzy \(Q\)-convergence spaces. Secondly, we introduce several subcategories of \((L, M)\)-QC, including the category \((L, M)\)-KQC of \((L, M)\)-fuzzy Kent \(Q\)-convergence spaces, the category \((L, M)\)-LQC of \((L, M)\)-fuzzy \(Q\)-limit spaces and the category \((L, M)\)-PQC of \((L, M)\)-fuzzy pretopological \(Q\)-convergence spaces, and investigate their relationships.

Keywords \((L, M)\)-fuzzy \(Q\)-convergence · Cartesian-closedness · Function space · Bireflective (bicoreflective) subcategory

1 Introduction
In general topology, function spaces of topological spaces cannot be constructed in a satisfactory way. This means the category of topological spaces with continuous mappings as morphisms is not Cartesian-closed. In order to overcome this deficiency, the concept of filter convergence spaces (convergence spaces in short) was proposed and discussed (Choquet 1948; Fischer 1959; Kent 1964; Kowalsky 1954). In Preuss (2002), Preuss gave a systematical collection of convergence structures, including function spaces and subcategories of convergence spaces as well as their connections with topological spaces.

With the development of fuzzy set theory, many mathematical structures have been generalized to the fuzzy case (Arqub and Al-Smadi 2020; Arqub et al. 2016, 2017; Li and Wang 2020; Xiu 2020; Zhang and Pang 2020). In the theory of fuzzy topology (Chang 1968; Kubiak 1985; Šostak 1985), many types of fuzzy convergence structures have been proposed, such as stratified \(L\)-generalized convergence structure (Jäger 2001, 2016b; Li and Jin 2012, 2014), \(L\)-fuzzifying convergence structure (Pang 2018; Xu 2001; Yao 2009), \(L\)-convergence tower structure (Flores et al. 2006; Jäger 2016a; Pang 2019), \(L\)-ordered convergence structure (Fang 2010a, b), (Enriched) \(L\)-fuzzy \((Q\)-)convergence structure (Pang 2014a, b; Pang and Zhao 2016, 2017), \(\top\)-convergence structure (Fang and Yue 2017, 2021; Jin et al. 2019; Yu and Fang 2017; Yue and Fang 2020) and so forth. Fuzzy convergence structures are usually discussed from two aspects. On the one hand, the categorical relationship between fuzzy convergence structures and fuzzy topologies is discussed. For example, Yu and Fang (Yu and Fang 2017) showed that the category of strong \(L\)-topological spaces can be embedded in the category of \(\top\)-convergence spaces as a reflective subcategory and the category of topological \(\top\)-convergence spaces is isomorphic to that of strong \(L\)-topological spaces. On the other hand, the categorical properties of fuzzy convergence spaces are investigated. Zhang et al. (2019) showed the monoidal closedness of the category of \(L\)-generalized convergence spaces. Pang and Zhao (2017) established the categorical properties among subcategories of enriched \((L, M)\)-fuzzy convergence spaces. Recently, Pang (Pang 2018, 2019) discussed the Cartesian-closedness, extensionality and productivity of quotient mappings of subcategories of \(L\)-fuzzifying convergence spaces and stratified \(L\)-generalized convergence tower spaces.

In the theory of fuzzy convergence spaces, many researchers usually show the Cartesian-closedness of fuzzy convergence spaces by constructing the corresponding function space, i.e., the power object in the category of fuzzy convergence structures.
spaces. Actually, there are different approaches to show the Cartesian-closedness of a category (Preuss 2002). For example, a topological category $A$ is Cartesian-closed if and only if the functor $A \times - : A \to A \times B$ preserves final epi-sinks for each object $A$ in $A$. In this approach, Pang and Li showed the Cartesian-closedness of the categories of $(L, M)$-fuzzy convergence spaces (Pang 2014b) and $L$-fuzzy $Q$-convergence spaces (Li 2016), respectively. Later, (Pang and Zhao 2016) introduced the concept of stratified $(L, M)$-fuzzy $Q$-convergence spaces and proved that the resulting category is Cartesian-closed. From a theoretical aspect, Cartesian-closedness of a category ensures the existence of its corresponding function space. However, the researchers failed to construct the corresponding function space although they showed the Cartesian-closedness of the categories of their corresponding fuzzy convergence spaces. By this motivation, we will focus on the function space of $Q$-fuzzy Kent convergence spaces. Suppose that $f : X \to Y$ is a mapping. Define $f^\leftarrow : L^X \to L^Y$ and $f^\rightarrow : L^Y \to L^X$ by $f^\leftarrow (y) = \bigvee f(x) = y$ for $A \in L^X$ and $y \in Y$, and $f^\rightarrow (B) = B \circ f$ for $B \in L^Y$, respectively.

**Definition 2.2** (Pang 2014a) For each $x_L \in J(L^X)$, we define $\hat{q}(x_L) : L^X \to M$ as follows:

$$\forall A \in L^X, \hat{q}(x_L)(A) = \begin{cases} \top_M, & x_L \hat{q}A, \\ \bot_M, & \text{otherwise.} \end{cases}$$

Then, $\hat{q}(x_L)$ is an $(L, M)$-fuzzy filter on $X$.

**Definition 2.3** (Pang and Zhao 2016) A mapping $q : F_{LM}(X) \to L^X$ is called an $(L, M)$-fuzzy $Q$-convergence structure on $X$ provided that each $b \in L$. An element $a$ in $L$ is called prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$.

For a nonempty set $X$, $L^X$ denotes the set of all $L$-subsets on $X$. $L^X$ is also a complete lattice when it inherits the structure of the lattice $L$ in a natural way, by defining $\lor$, $\land$ and $\leq$ pointwisely. The smallest element and the largest element in $L^X$ are denoted by $\bot_L$ and $\top_L$, respectively. For each $x \in X$ and $a \in L$, the $L$-subset $x_a$, defined by $x_a(y) = a$ if $y = x$, and $x_a(y) = \bot_L$ if $y \neq x$, is called a fuzzy point. The set of nonzero coprime elements in $L^X$ is denoted by $J(L^X)$. It is easy to see that $J(L^X) = \{x_L \mid x \in X, \lambda \in J(L)\}$. Let $x_L$ be a fuzzy point in $J(L^X)$, then $x_L \hat{q}A$, denoted by $x_L \hat{q}A$, if $\lambda \not\in \lambda' \leftarrow (x)$ for each $\lambda \in J(L^X)$.

**Definition 2.1** (Yao 2012) A mapping $F : L^X \to M$ is called an $(L, M)$-fuzzy filter on $X$ if it satisfies

(1) $F(\bot_L) = \bot_M, F(\top_L) = \top_M$;

(2) $F(A \cup B) = F(A) \cap F(B)$.

The family of all $(L, M)$-fuzzy filters on $X$ is denoted by $F_{LM}(X)$.

Throughout this paper, both $L$ and $M$ denote completely distributive lattices and $\perp$ is an order-reversing involution on $L$. The smallest element and the largest element in $L$ ($M$) are denoted by $\bot_L$ ($\bot_M$) and $\top_L$ ($\top_M$), respectively. For $a, b \in L$, we say that $a$ is wedge below $b$, in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$. An element $a$ in $L$ is called coprime if $a \leq b \lor c$ implies $a \leq b$ or $a \leq c$. The set of nonzero coprime elements in $L$ is denoted by $J(L)$. A complete lattice $L$ is completely distributive if and only if $b = \bigvee\{a \in J(L) \mid a \prec b\}$ for $b \in L$. An element $a$ in $L$ is called prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$.

**2 Preliminaries**

Throughout this paper, both $L$ and $M$ denote completely distributive lattices and $\perp$ is an order-reversing involution on $L$. The smallest element and the largest element in $L$ ($M$) are denoted by $\bot_L$ ($\bot_M$) and $\top_L$ ($\top_M$), respectively. For $a, b \in L$, we say that $a$ is wedge below $b$, in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$. An element $a$ in $L$ is called coprime if $a \leq b \lor c$ implies $a \leq b$ or $a \leq c$. The set of nonzero coprime elements in $L$ is denoted by $J(L)$. A complete lattice $L$ is completely distributive if and only if $b = \bigvee\{a \in J(L) \mid a \prec b\}$ for each $b \in L$. An element $a$ in $L$ is called prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$.

For a nonempty set $X$, $L^X$ denotes the set of all $L$-subsets on $X$. $L^X$ is also a complete lattice when it inherits the structure of the lattice $L$ in a natural way, by defining $\lor$, $\land$ and $\leq$ pointwisely. The smallest element and the largest element in $L^X$ are denoted by $\bot_L$ and $\top_L$, respectively. For each $x \in X$ and $a \in L$, the $L$-subset $x_a$, defined by $x_a(y) = a$ if $y = x$, and $x_a(y) = \bot_L$ if $y \neq x$, is called a fuzzy point. The set of nonzero coprime elements in $L^X$ is denoted by $J(L^X)$. It is easy to see that $J(L^X) = \{x_L \mid x \in X, \lambda \in J(L)\}$. Let $x_L$ be a fuzzy point in $J(L^X)$, then $x_L \hat{q}A$, denoted by $x_L \hat{q}A$, if $\lambda \not\in \lambda' \leftarrow (x)$ for each $\lambda \in J(L^X)$.
Let $(L, M)$-fuzzy $Q$-convergence structure $q$ on $X$, the pair $(X, q)$ is called an $(L, M)$-fuzzy $Q$-convergence space.

A continuous mapping between $(L, M)$-fuzzy $Q$-convergence spaces $(X, q_X)$ and $(Y, q_Y)$ is a mapping $f : X \rightarrow Y$ such that $\lambda \in \mathcal{J}(L_X)$ and $\lambda \leq \lambda'$ imply $\tau(q_X(x)) \leq \tau(q_Y(f(x)))$ for each $x \in X$, or equivalently, $q_X(f(F)) \leq q_Y(f(F))(f(x))$ for each $F \in \mathcal{F}_{LM}(X)$ and $x \in X$.

It is easy to check that $(L, M)$-fuzzy $Q$-convergence spaces and their continuous mappings form a category, denoted by $(L, M)$-$QC$.

**Example 2.4** (Pang and Zhao 2016) Let $X$ be a nonempty set.

1. Define $q^c_X : \mathcal{F}_{LM}(X) \rightarrow L_X$ as follows:
   \[
   \forall F \in \mathcal{F}_{LM}(X), q^c_X(F) = \begin{cases} \bigvee_{q_X(x) \leq F} q_X(x) & \text{if } \exists \lambda \in J(L_X) \text{ s.t. } q_X(x) \leq \lambda F; \\ \mathbb{T}_L & \text{otherwise.} \end{cases}
   \]

   It is easy to verify that $q^c_X$ is an $(L, M)$-fuzzy $Q$-convergence structure on $X$.

2. Define $q^e_X : \mathcal{F}_{LM}(X) \rightarrow L_X$ as follows:
   \[
   \forall F \in \mathcal{F}_{LM}(X), q^e_X(F) = \mathbb{T}_L.
   \]

   It is easy to check that $q^e_X$ is an $(L, M)$-fuzzy $Q$-convergence structure on $X$.

In order to provide an example from the aspect of fuzzy topology, we first recall the following definition.

**Definition 2.5** (Hohle and Sostak 1999) A stratified $(L, M)$-fuzzy topology on $X$ is a mapping $\tau : L^X \rightarrow M$ which satisfies:

(LFT1) $\tau(\mathbb{1}_L) = \tau(\mathbb{T}_L) = \mathbb{T}_M$;

(LFT2) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$;

(LFT3) $\tau(\bigvee_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau(A_j)$;

(SLFT) $\forall a \in L, \tau(a) = \mathbb{T}_M$.

For a stratified $(L, M)$-fuzzy topology $\tau$ on $X$, the pair $(X, \tau)$ is called a stratified $(L, M)$-fuzzy topological space.

**Example 2.6** (Pang and Zhao 2016) Let $(X, \tau)$ be a stratified $(L, M)$-fuzzy topological space and define $q^e_\tau : \mathcal{F}_{LM}(X) \rightarrow L_X$ as follows:

\[
\forall F \in \mathcal{F}_{LM}(X), q^e_\tau(F) = \bigvee_{\mathcal{Q}_F^\tau \subseteq F} q^\tau_{\mathcal{Q}_F^\tau},
\]

where $\mathcal{Q}_F^\tau : L_X \rightarrow M$ is defined by $\mathcal{Q}_F^\tau(A) = \bigvee_{q_X(x) \leq A} \tau(B)$ for each $A \in L_X$. Then $q^e_\tau$ is an $(L, M)$-fuzzy $Q$-convergence structure on $X$.

Notice that $(L, M)$-fuzzy $Q$-convergence structures in Definition 2.3 are exactly stratified $(L, M)$-fuzzy $Q$-convergence structures in Pang and Zhao (2016). In this paper, we will focus on this kind of fuzzy convergence structures and explore the concrete form of its function spaces as well as its subcategories.

**Definition 2.7** (Pang and Zhao 2016) Let $\{(X_j, q_j)\}_{j \in J}$ be a family of $(L, M)$-fuzzy $Q$-convergence spaces and $\{p_k : \prod_{j \in J} X_j \rightarrow X_k\}_{k \in K}$ be the source formed by the family of the projection mappings $\{p_k : \prod_{j \in J} X_j \rightarrow X_k\}_{k \in K}$. Then, the $(L, M)$-fuzzy $Q$-convergence structure on $\prod_{j \in J} X_j$ defined by

\[
\forall F \in \mathcal{F}_{LM}(X), q^e(F) = \bigvee_{j \in J} p_j^e(q_j(p_j^e(F)))
\]

is called the product $(L, M)$-fuzzy $Q$-convergence structure, which is denoted by $\prod_{j \in J} q_j$. The pair $\left(\prod_{j \in J} X_j, \prod_{j \in J} q_j\right)$ is called the product space. For the product of two $(L, M)$-fuzzy $Q$-convergence spaces $(X, q_X)$ and $(Y, q_Y)$, we usually write $(X \times Y, q_X \times q_Y)$ or $(X \times Y, q_X \times q_Y)$.

**Theorem 2.8** (Pang and Zhao 2016) $(L, M)$-$QC$ is a topological category.

For other notions related to category theory, we refer to Adamek et al. (1990); Preuss (2002).

### 3 Function space of $(L, M)$-fuzzy $Q$-convergence spaces

In this section, we will construct the function space of $(L, M)$-fuzzy $Q$-convergence spaces. By means of the constructed function space, we will show the Cartesian-closedness of $(L, M)$-$QC$.

In order to guarantee the existence of the product of $(L, M)$-fuzzy filters, we assume that $\bot_L$ is prime in this section.

Let $(X, q_X)$ and $(Y, q_Y)$ be $(L, M)$-fuzzy $Q$-convergence spaces, $[X, Y]$ be the set of all continuous mappings from $(X, q_X)$ to $(Y, q_Y)$ and let $ev : [X, Y] \times X \rightarrow Y$ defined by $ev(f, x) = f(x)$ be the evaluation mapping. For each $H \in \mathcal{F}_{LM}([X, Y])$ and $f \in [X, Y]$, we denote two subsets of $L$ as follows:

\[
\mathcal{R}_H(f) = \{v \in J(L) \mid \forall \mu \leq v, \forall a \in L, \mu \leq a' \implies H(a) = \mathbb{T}_M\}
\]
Lemma 3.1 \( \lambda \) 11462 B. Pang, L. Zhang

Then, we define \( q_{[X,Y]} : \mathcal{F}_{LM}([X,Y]) \rightarrow L^{[X,Y]} \) as follows:

\[
q_{[X,Y]}(\mathcal{H})(f) = \bigvee R_\mathcal{H}(f) \wedge \bigvee S_\mathcal{H}(f).
\]

In order to show \( q_{[X,Y]} \) is an \((L,M)\)-fuzzy \( Q \)-convergence structure on \([X,Y] \), the following lemma is necessary.

Lemma 3.1 (Pang 2014b) Let \( f_\lambda \in J(L^{[X,Y]}) \) and \( \mathcal{F} \in \mathcal{F}_{LM}(X) \). Then \( ev_\Rightarrow(\hat{q}(f_\lambda) \times \mathcal{F}) \geq f_\Rightarrow(\mathcal{F}) \).

Now let us show that \( q_{[X,Y]} \) defined above is an \((L,M)\)-fuzzy \( Q \)-convergence structure on \([X,Y] \).

Theorem 3.2 \( q_{[X,Y]} \) is an \((L,M)\)-fuzzy \( Q \)-convergence structure on \([X,Y] \).

Proof It suffices to verify that \( q_{[X,Y]} \) satisfies (LMQC1)–(LMQC3). Indeed,

(LMQC1) Take each \( f_\lambda \in J(L^{[X,Y]}) \). In order to show \( f_\lambda \leq q_{[X,Y]}(\hat{q}(f_\lambda)) \), i.e.,

\[
\lambda \leq q_{[X,Y]}(\hat{q}(f_\lambda))(f) = \bigvee R_{\hat{q}(f_\lambda)}(f) \wedge \bigvee S_{\hat{q}(f_\lambda)}(f),
\]

we only need to show that (1) \( \lambda \leq \bigvee R_{\hat{q}(f_\lambda)}(f) \) and (2) \( \lambda \leq \bigvee S_{\hat{q}(f_\lambda)}(f) \).

For (1), take each \( \mu \in J(L) \) with \( \mu \leq \lambda \) and \( a \in L \) with \( \mu \not\leq a' \). Then it follows that \( \lambda \not\leq a' \), which means \( f_\lambda \hat{q}(a) \), i.e., \( \hat{q}(f_\lambda)(\hat{a}) = \top_M \). This implies \( \lambda \in R_{\hat{q}(f_\lambda)}(f) \). Thus, we obtain \( \lambda \leq \bigvee R_{\hat{q}(f_\lambda)}(f) \).

For (2), take each \( \mu \in J(L) \) with \( \mu \leq \lambda \) and \( \mathcal{F}, x \in \mathcal{F}_{LM}(X) \times X \) with \( x_\lambda \leq q_X(\mathcal{F}) \). Then it follows from Lemma 3.1 that

\[
q_Y(ev_\Rightarrow(\hat{q}(f_\lambda) \times \mathcal{F}))(f(x)) \geq q_Y(f_\Rightarrow(\mathcal{F}))(f(x)) \geq q_X(\mathcal{F})(x) \geq \mu.
\]

That is, \( f(x)_\mu \leq q_Y(ev_\Rightarrow(\hat{q}(f_\lambda) \times \mathcal{F})) \). This shows \( \lambda \in S_{\hat{q}(f_\lambda)}(f) \). Thus, we have \( \lambda \leq \bigvee S_{\hat{q}(f_\lambda)}(f) \).

By (1) and (2), we have

\[
\lambda \leq \bigvee R_{\hat{q}(f_\lambda)}(f) \wedge \bigvee S_{\hat{q}(f_\lambda)}(f) = q_{[X,Y]}(\hat{q}(f_\lambda))(f),
\]

which means \( f_\lambda \leq q_{[X,Y]}(\hat{q}(f_\lambda)) \).

(LMQC2) Straightforward.

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where the third equality holds since \( p_{\gamma} \circ \hat{x} = \text{id}_{\gamma} \). Now for each \( A \in L^X \) with \( y_{\mu} \leq q_{\gamma}(\mathcal{G}) \) and (LMQC3) that \( \mathcal{G}(A) = \top \). Then,
\[
(p_{\gamma} \circ \hat{x})^{-\tau}(\mathcal{G})(A) = \mathcal{G}((p_{\gamma} \circ \hat{x})^{-\tau}(A)) = \mathcal{G}(A) = \top,
\]
where the second equality holds since
\[
(p_{\gamma} \circ \hat{x})^{-\tau}(A)(y) = A(p_{\gamma} \circ \hat{x}(y)) = A(p_{\gamma}(x, y)) = A(x).
\]
This shows \( (p_{\gamma} \circ \hat{x})^{-\tau}(\mathcal{G}) \geq \hat{x}(y_{\mu}) \). Then,
\[
q_{X \times Y}(\hat{x}(y)) = q_{X}(p_{\gamma}(x, y)) \land q_{\gamma}(\mathcal{G})(y)
\]
which means \( \hat{x}(y_{\mu}) \leq q_{X \times Y}(\hat{x}^{-\tau}(\mathcal{G})) \). This proves that \( \hat{x} : (Y, q_{\gamma}) \rightarrow (X \times Y, q_{X \times Y}) \) is continuous. Considering the continuity of \( f : (X \times Y, q_{X \times Y}) \rightarrow (Z, q_{\gamma}) \), we obtain \( f_{\hat{x}} = f \circ \hat{x} \) (as the composition of two continuous mappings \( \hat{x} \) and \( f \)) is continuous, as desired. \( \square \)

**Lemma 3.5** (Pang 2014b) Let \( \mathcal{F} \in \mathcal{F}_{LM}(X) \), \( \mathcal{G} \in \mathcal{F}_{LM}(Y) \) and \( f : X \times Y \rightarrow Z \) be a mapping. Then, \( ev^{\tau}(\varphi(f))^{\tau}(\mathcal{F}) \times \mathcal{G}) = f^{\tau}(\mathcal{F} \times \mathcal{G}) \).

**Theorem 3.6** If \( f : (X \times Y, q_{X \times Y}) \rightarrow (Z, q_{\gamma}) \) is continuous, then \( \varphi(f) : (X, q_{\gamma}) \rightarrow ([Y, Z], q_{[Y,Z]}) \) is continuous.

**Proof** By Lemma 3.4, we know the mapping \( \varphi(f) \) is well defined. Take each \( x_{\lambda} \in J(L^X) \) and \( \mathcal{F} \in \mathcal{F}_{LM}(X) \)

\[
\varphi(f)(x_{\lambda})(\lambda) \leq q_{X}(\mathcal{F}),
\]

In order to show \( \varphi(f)(x_{\lambda})(\lambda) \leq q_{X}(\mathcal{F}) \), it suffices to show \( \lambda \in \mathcal{R}_{\varphi(f)}^{\tau}(\mathcal{F})(\varphi(f)(x)) \).

\[
\begin{align*}
\mathcal{F}(\lambda) \in \mathcal{F}_{LM} & \land \mathcal{F}(\lambda) \leq \mathcal{G}(\mathcal{F}) \\forall \mathcal{G} \in \mathcal{F}_{LM}(X) \\
\mathcal{F}(\lambda) \in \mathcal{F}_{LM} & \land \mathcal{F}(\lambda) \leq \mathcal{G}(\mathcal{F}) \\forall \mathcal{G} \in \mathcal{F}_{LM}(X).
\end{align*}
\]

i.e., \( \varphi(f)(x_{\lambda})(\lambda) \leq q_{Z}(ev^{\tau}(\varphi(f))^{\tau}(\mathcal{F} \times \mathcal{G})) \).

By Theorems 3.2, 3.3 and 3.6, we have

**Theorem 3.7** The category \((L, M)\)-QC is Cartesian-closed.

Actually, Pang and Zhao (2016) showed the Cartesian-closedness of the category \((L, M)\)-fuzzy Q-convergence spaces (which is called stratified \((L, M)\)-fuzzy Q-convergence space in Pang and Zhao (2016)). However, they failed to construct the corresponding function spaces. In this section, we provide the concrete form of the corresponding function spaces, which gives an answer to the question proposed in Pang and Zhao (2016).

\section{\((L, M)\)-fuzzy Kent Q-convergence spaces}

In this section, we will generalize the notion of Kent convergence spaces to the \((L, M)\)-fuzzy case and study its relationship with \((L, M)\)-fuzzy Q-convergence spaces.

**Definition 4.1** An \((L, M)\)-fuzzy Q-convergence structure \( q \) on \( X \) is called an \((L, M)\)-fuzzy Kent Q-convergence structure if it satisfies

\[
\text{(LMKQC)} \forall \mathcal{F} \in \mathcal{F}_{LM}(X), x_{\lambda} \in J(L^X), x_{\lambda} \leq q(\mathcal{F}) \text{ implies } x_{\lambda} \leq q(\mathcal{F} \cap \hat{\mathcal{G}}(x_{\lambda})).
\]

For an \((L, M)\)-fuzzy Kent Q-convergence structure \( q \) on \( X \), the pair \((X, q)\) is called an \((L, M)\)-fuzzy Kent Q-convergence space.

The full subcategory of \((L, M)\)-QC, consisting of \((L, M)\)-fuzzy Kent Q-convergence spaces, is denoted \((L, M)\)-KQC.

Next let us establish the relationship between \((L, M)\)-fuzzy Kent Q-convergence spaces and \((L, M)\)-fuzzy Q-convergence spaces.

**Lemma 4.2** Let \((X, q)\) be an \((L, M)\)-fuzzy Q-convergence space and define \( q' : \mathcal{F}_{LM}(X) \rightarrow L^X \) by for each \( \mathcal{F} \in \mathcal{F}_{LM}(X) \),
\[
q'(\mathcal{F}) = \bigvee \{x_{\lambda} \in J(L^X) : \exists \mathcal{G} \in \mathcal{F}_{LM}(X) \text{s.t. } x_{\lambda} \leq q(\mathcal{G}) \text{ and } \mathcal{G} \cap \hat{\mathcal{G}}(x_{\lambda}) \leq \mathcal{F} \}.
\]

Then \( q' \) is an \((L, M)\)-fuzzy Kent Q-convergence structure on \( X \).

**Proof** It is enough to show that \( q' \) satisfies (LMQC1)–(LMQC3) and (LMKQC). Indeed, (LMQC1) and (LMQC2) are straightforward.
(LMQC3) Take each \( x_\lambda \in J(L^X) \), \( F \in F_{LM}(X) \) and \( a \in L \) such that \( x_\lambda \leq q'(F) \) and \( \lambda \notin a' \). This implies
\[
q'(F)(x) = \bigvee \{ \lambda \in J(L) \mid \exists G \in F_{LM}(X), \text{s.t. } x_\lambda \leq q(G) \land G \land \hat{q}(x_\lambda) \leq F \} \leq a'.
\]
Then there exists \( \lambda_a \in J(L) \) such that \( \lambda_a \notin a' \) and there exists \( G \in F_{LM}(X) \) such that \( x_\lambda \leq q(G) \) and \( G \land \hat{q}(x_\lambda) \leq F \). Since \( q \) satisfies (LMQC3), it follows from \( x_\lambda \leq q(G) \) and \( \lambda_a \notin a' \) that \( G(q) = \top_M \), and further \( F(q) \geq G(q) \land \hat{q}(x_\lambda) = \top_M \).

(LMKQC) Take each \( x_\lambda \in J(L^X) \) and \( F \in F_{LM}(X) \) such that \( x_\lambda \leq q'(F) \), i.e., \( \lambda \leq q'(F)(x) \). Then for each \( \mu < \lambda \), there exists \( \lambda_1 \in J(L) \) such that \( \mu \leq \lambda_1 \) and there exists \( G \in F_{LM}(X) \) such that \( x_\lambda \leq q(G) \) and \( G \land \hat{q}(x_\lambda) \leq F \). Thus it follows that \( x_\lambda \leq q(G) \) and \( G \land \hat{q}(x_\lambda) \leq F \land \hat{q}(x_\lambda) \leq F \land \hat{q}(x_\lambda) \). This implies
\[
\mu \leq \bigvee \{ v \in J(L) \mid \exists G \in F_{LM}(X), \text{s.t. } x_v \leq q(G) \land G \land \hat{q}(x_v) \leq F \land \hat{q}(x_\lambda) \}
\]
\[
= q'(F \land \hat{q}(x_\lambda))(x).
\]
By the arbitrariness of \( \mu \), we obtain \( \lambda \leq q'(F \land \hat{q}(x_\lambda))(x) \). That is to say, \( x_\lambda \leq q'(F \land \hat{q}(x_\lambda))(x) \), as desired. \( \square \)

**Theorem 4.3** \((L, M)\)-KQC is a bireflective subcategory of \((L, M)\)-QC.

**Proof** Let \((X, q)\) be an \((L, M)\)-fuzzy \(Q\)-convergence space. By Lemma 4.2, we know \((X, q')\) is an \((L, M)\)-fuzzy \(Q\)-convergence space. Next we claim that \(id_X : (X, q) \longrightarrow (X, q')\) is the \((L, M)\)-KQC-bireflector. To this end, we need to show:

1. \(id_X : (X, q) \longrightarrow (X, q')\) is continuous.
2. For each \((L, M)\)-fuzzy \(Q\)-convergence space \((Y, q_Y)\) and each mapping \( f : X \longrightarrow Y \), the continuity of \( f : (X, q) \longrightarrow (Y, q_Y)\) implies the continuity of \( f : (X, q') \longrightarrow (Y, q_Y)\).

For (1), it is easy to verify that \( q(F) \leq q'(F) \) for each \( F \in F_{LM}(X) \).

For (2), take each \( x_\lambda \in J(L^X) \) and \( F \in F_{LM}(X) \) such that \( x_\lambda \leq q'(F) \). For each \( \mu \in J(L) \) with \( \mu < \lambda \), it follows that \( \mu < q'(F)(x) \). Then there exists \( \lambda_1 \in J(L) \) such that \( \mu \leq \lambda_1 \) and there exists \( G \in F_{LM}(X) \) such that \( x_\lambda \leq q(G) \) and \( G \land \hat{q}(x_\lambda) \leq F \). Since \( f : (X, q) \longrightarrow (Y, q_Y)\) is continuous, it follows that \( f(x)_{\lambda_1} \leq q_Y(f^\omega(G)) \). By (LMKQC), we have
\[
f(x)_{\lambda_1} \leq q_Y(f^\omega(f(x)_{\lambda_1})) \land f^\omega(G) = q_Y(f^\omega(\hat{q}(x_{\lambda_1}) \land G)) \leq q_Y(f^\omega(F)).
\]
This implies \( f(x)_{\mu} \leq q_Y(f^\omega(F)) \). By the arbitrariness of \( \mu \), we obtain \( f(x)_{\lambda_1} \leq q_Y(f^\omega(F)) \). This proves the continuity of \( f : (X, q') \longrightarrow (Y, q_Y) \). \( \square \)

**Lemma 4.4** Let \((X, q)\) be an \((L, M)\)-fuzzy \(Q\)-convergence space and define \( q' : F_{LM}(X) \longrightarrow L^X \) by
\[
\forall F \in F_{LM}(X), \quad q'(F) = \bigvee \{ x_\lambda \in J(L^X) \mid \forall \mu < \lambda, x_\mu \leq q(F \land \hat{q}(x_\mu)) \}.
\]
Then, \( q' \) is an \((L, M)\)-fuzzy \(Q\)肯特\(Q\)-convergence structure on \(X\).

**Proof** (LMQC1) and (LMQC2) are easy to be verified and omitted.

(LMQC3) Take each \( x_\lambda \in J(L^X) \), \( F \in F_{LM}(X) \) and \( a \in L \) such that \( x_\lambda \leq q'(F) \) and \( \lambda \notin a' \). It follows that
\[
q'(F)(x) = \bigvee \{ \lambda \in J(L) \mid \forall \mu < \lambda, x_\mu \leq q(F \land \hat{q}(x_\mu)) \} \leq a'.
\]
Then, there exists \( \lambda_a \in J(L) \) such that \( \lambda_a \notin a' \) and for each \( \mu < \lambda_a, x_\mu \leq q(F \land \hat{q}(x_\mu)) \). Since \( \lambda_a \notin a' \), there exists \( \mu_a < \lambda_a \) such that \( \mu_a \notin a' \). This implies \( x_{\mu_a} \leq q(F \land \hat{q}(x_{\mu_a})) \) and \( \mu_a \notin a' \). Since \( q \) satisfies (LMQC3), we have \( (F \land \hat{q}(x_{\mu_a}))(a) = \top_M \). This implies \( F(a) = \top_M \).

(LMKQC) Take each \( x_\lambda \in J(L^X) \), \( F \in F_{LM}(X) \) such that \( x_\lambda \leq q'(F) \), i.e., \( \lambda \leq q'(F)(x) \). For each \( v \in J(L) \) with \( v < \lambda \), it follows that
\[
v < q'(F)(x) = \bigvee \{ \lambda_1 \in J(L) \mid \forall \mu < \lambda_1, x_\mu \leq q(F \land \hat{q}(x_\mu)) \} \leq q'(F)(x).
\]
Then, there exists \( \lambda_1 \in J(L) \) such that \( v \leq \lambda_1 \) and for each \( \mu < \lambda_1, x_\mu \leq q(F \land \hat{q}(x_\mu)) \). Thus, for each \( \mu \in J(L) \) with \( \mu < v \), it follows that
\[
q(F \land \hat{q}(x_\lambda) \land \hat{q}(x_\mu)) = q(F \land \hat{q}(x_\mu) \land \hat{q}(x_\lambda)) \geq \mu.
\]
This implies
\[
q'(F \land \hat{q}(x_\lambda))(x) = \bigvee \{ \gamma \in J(L) \mid \forall \mu < \gamma, x_\mu \leq q(F \land \hat{q}(x_\mu)) \} \geq v.
\]
By the arbitrariness of \( v \), we obtain \( \lambda \leq q'(F \land \hat{q}(x_\lambda))(x) \), that is, \( x_\lambda \leq q'(F \land \hat{q}(x_\lambda))(x) \), as desired. \( \square \)

**Theorem 4.5** \((L, M)\)-KQC is a bireflective subcategory of \((L, M)\)-QC.

**Proof** Let \((X, q)\) be an \((L, M)\)-fuzzy \(Q\)-convergence space. By Lemma 4.4, we obtain \( q' \) is an \((L, M)\)-fuzzy \(Q\)-convergence structure on \(X\). Next we claim that \( id_X : (X, q') \longrightarrow (X, q) \) is the \((L, M)\)-KQC-bireflector.

For this it suffices to show:
1. \( id_X : (X, q') \longrightarrow (X, q) \) is continuous.
(2) For each \((L, M)\)-fuzzy Kent \(Q\)-convergence space \((Y, q_Y)\) and each mapping \(f : Y \to X\), the continuity of \(f : (Y, q_Y) \to (X, q)\) implies the continuity of \(f : (Y, q_Y) \to (X, q^\vee)\).

For (1), it is easy to show \(q^\vee(F) \leq q(F)\) for each \(F \in F_{LM}(X)\).

For (2), take each \(G \in F_{LM}(Y)\) and \(y_\lambda \in J(L_Y)\) such that \(y_\lambda \leq q_Y(G)\). Then, for each \(\mu < \lambda\), it follows that \(y_\mu \leq q_Y(G)\). Since \((Y, q_Y)\) satisfies (LMQC), we have \(y_\mu \leq q_Y(G \wedge \hat{q}(y_\mu))\). By the continuity of \(f : (Y, q_Y) \to (X, q)\), we obtain \(f(y)_\mu \leq q(f(\hat{q}(G) \wedge \hat{q}(f(y)_\mu)))\). From the definition of \(q^\vee\), we get

\[
q^\vee(f(\hat{q}(G)))(f(y)) = \bigvee\{v \in J(L) \mid \forall \mu < v, \quad f(y)_\mu \leq q(f(\hat{q}(G) \wedge \hat{q}(f(y)_\mu))) \geq \lambda\}.
\]

This shows \(f(y)_\lambda \leq q^\vee(f(\hat{q}(G)))\), as desired. \(\square\)

**Lemma 5.4** (Preuss 2002) Suppose that \(A\) is a topological category. If \(B\) is a bicoreflective (full and isomorphic closed) subcategory of \(A\) which is closed under formation of finite products in \(A\), then \(B\) is Cartesian-closed whenever \(A\) is Cartesian-closed.

**Theorem 5.7** Suppose that \(\bot_L\) is prime in \(L\). Then, \((L, M)\)-KQC is a Cartesian-closed.

**Proof** By Theorem 4.3, we know \((L, M)\)-KQC is closed under formation of finite product in \((L, M)\)-QC. Further, it is easy to see that \((L, M)\)-KQC is a full and isomorphic closed subcategory of \((L, M)\)-QC. Then, it follows from Theorems 2.8, 3.7 and 4.5, and Lemma 4.6 that \((L, M)\)-KQC is Cartesian-closed. \(\square\)

### 5 \((L, M)\)-fuzzy \(Q\)-limit spaces

In this section, we will propose the concept of \((L, M)\)-fuzzy \(Q\)-limit spaces, which is a generalization of limit spaces in general topology. Then, we will study its relationship with \((L, M)\)-fuzzy Kent \(Q\)-convergence spaces from a categorical aspect.

**Definition 5.1** An \((L, M)\)-fuzzy \(Q\)-convergence structure \(q\) on \(X\) is called an \((L, M)\)-fuzzy \(Q\)-limit structure if it satisfies

\[
\text{LMQC} \quad \forall F, G \in F_{LM}(X), q(F) \wedge q(G) \leq q(F \wedge G).
\]

For an \((L, M)\)-fuzzy \(Q\)-limit structure \(q\) on \(X\), the pair \((X, q)\) is called an \((L, M)\)-fuzzy \(Q\)-limit space.

The full subcategory of \((L, M)\)-QC, consisting of \((L, M)\)-fuzzy \(Q\)-limit spaces, is denoted by \((L, M)\)-LQC.

Obviously, \((\text{LMLQC})\) implies \((\text{LMKQC})\). That is to say, an \((L, M)\)-fuzzy \(Q\)-limit space is an \((L, M)\)-fuzzy Kent \(Q\)-convergence space. Thus, \((L, M)\)-LQC is a full subcategory of \((L, M)\)-KQC.

In order to show the further relationship between \((L, M)\)-fuzzy Kent \(Q\)-convergence spaces and \((L, M)\)-fuzzy \(Q\)-limit spaces, we first give the following lemma.

**Lemma 5.2** Let \((X, q)\) be an \((L, M)\)-fuzzy Kent \(Q\)-convergence space and define \(q^J : F_{LM}(X) \to L^X\) by for each \(F \in F_{LM}(X)\),

\[
q^J(F) = \bigvee\{x_i \in J(L^X) \mid \exists F_1, \ldots, F_n \in F_{LM}(X) \text{ s.t. } x_i \leq q(F_i) \}.
\]

Then, \(q^J\) is an \((L, M)\)-fuzzy \(Q\)-limit structure on \(X\).

**Proof** (LMQC1) and (LMQC2) are obvious. It suffices to show (LMQC3) and (LMLQC).

LMQC3) Take each \(F \in F_{LM}(X)\), \(x_\lambda \in J(L^X)\) and \(a \in L\) such that \(x_\lambda \leq q^J(F)\) and \(a \neq a'\). Then, \(q^J(F)(x) \neq a'\). By the definition of \(q^J(F)\), there exists \(\lambda \in J(L)\) such that \(\lambda \neq a'\) and there exist \(F_1, F_2, \ldots, F_n \in F_{LM}(X)\) such that \(x_\lambda \leq q(F_1)\) and \(F \geq \bigwedge_{i=1}^n F_i\). Since \(x_\lambda \leq q(F_1)\) and \(\lambda \neq a'\), it follows that \(F_1(q) = \bigvee_{j=1}^n F_j(q)\) for each \(i = 1, \ldots, n\). This implies \(F_1(q) \geq \bigwedge_{i=1}^n F_i(q)\).

LMLQC) Take \(F, G \in F_{LM}(X)\) and \(x_\lambda \in J(L^X)\) such that \(x_\lambda \leq q^J(F) \wedge q^J(G)\). For each \(\mu \in J(L)\) with \(\mu < \lambda\), it follows that \(\mu \leq q^J(F)(x)\) and \(\mu \leq q^J(G)(x)\). Then, there exist \(\lambda_1, \lambda_2 \in J(L)\) and \(F_1, F_2, \ldots, F_n, G_1, G_2, \ldots, G_m \in F_{LM}(X)\) such that \(\mu \leq \lambda_1, \mu \leq \lambda_2, x_\lambda_1 \leq q(F_1), x_\lambda_2 \leq q(G_1), F \geq \bigwedge_{i=1}^n F_i \) and \(G \geq \bigwedge_{j=1}^m G_j\). Let \(\{H_k \mid k = 1, 2, \ldots, m + n\} = \{F_i \mid i = 1, 2, \ldots, n\} \cup \{G_j \mid j = 1, 2, \ldots, m\}\). Then \(x_\mu \leq q(H_k)\) and \(F \wedge G \geq \bigwedge_{k=1}^{m+n} H_k\).

This implies

\[
q^J(F \wedge G)(x) = \bigvee\{v \in J(L) \mid \exists H_1, \ldots, H_p \in F_{LM}(X) \text{ s.t. } x_v \}
\]

\[
q^J(H_k) \wedge q^J(G) \geq q^J(H_k) \geq \mu.
\]

By the arbitrariness of \(\mu\), we obtain \(\lambda \leq q^J(F \wedge G)(x)\), that is, \(x_\lambda \leq q^J(F \wedge G)\), as desired. \(\square\)

**Theorem 5.3** \((L, M)\)-LQC is a bireflective subcategory of \((L, M)\)-KQC.

**Proof** Let \((X, q)\) be an \((L, M)\)-fuzzy Kent \(Q\)-convergence space. By Lemma 5.2, we know \(q^J\) is an \((L, M)\)-fuzzy \(Q\)-limit structure on \(X\). Next we claim that \(id_X : (X, q) \to (X, q^J)\) is the \((L, M)\)-LQC-bireflector. For this, it suffices to verify
Proof

It suffices to verify that \( q(Y, q) \rightarrow (X, q^I) \) is continuous.

(2) For each \((L, M)\)-fuzzy \(Q\)-limit space \((Y, q_Y)\) and each mapping \( f : X \rightarrow Y \), the continuity of \( f : (X, q) \rightarrow (Y, q_Y) \) implies the continuity of \( f : (X, q^I) \rightarrow (Y, q_Y) \).

For (1), it follows immediately from \( q(F) \leq q^I(F) \) for each \( F \in \mathcal{F}_{LM}(X) \).

For (2), take each \( F \in \mathcal{F}_{LM}(X) \) and \( x_\lambda \in J(L) \) such that \( x_\lambda \leq q(F) \). Then, for each \( \mu \prec \lambda \), there exists \( \lambda_{\mu} \in J(L) \) such that \( \mu \leq \lambda_{\mu} \) and there exist \( F_1, \ldots, F_n \in \mathcal{F}_{LM}(X) \) such that \( x_{\lambda_{\mu}} \leq q(F_i) \) and \( F \geq \bigwedge_{i=1}^n F_i \). Since \( f : (X, q) \rightarrow (Y, q_Y) \) is continuous, it follows that \( f(x)_{\lambda_{\mu}} \leq q_Y(f^\rightarrow(F_i)) \) for each \( i = 1, \ldots, n \). Then, we have

\[
\begin{align*}
 f(x)_{\mu} &\leq f(x)_{\lambda_{\mu}} \leq \bigwedge_{i=1}^n q_Y(f^\rightarrow(F_i)) \\
 &\leq q_Y(\bigwedge_{i=1}^n f^\rightarrow(F_i)) = q_Y(f^\rightarrow(\bigwedge_{i=1}^n F_i)) \\
 &\leq q_Y(\bigwedge_{i=1}^n F_i).
\end{align*}
\]

By the arbitrariness of \( \mu \), we obtain \( f(x)_{\lambda} \leq q_Y(f^\rightarrow(F)) \). This proves \( f : (X, q^I) \rightarrow (Y, q_Y) \) is continuous. \( \Box \)

By Theorems 4.3 and 5.3, we have

**Corollary 5.4** \((L, M)\)-LQC is a bireflective subcategory of \((L, M)\)-QC.

Next we discuss the Cartesian-closedness of \((L, M)\)-LQC. To this end, the following two lemmas are necessary.

**Lemma 5.5** (Pang 2014b) Suppose that \( \perp_L \) is prime in \( L \). Let \( F, K \in \mathcal{F}_{LM}(X) \) and \( G \in \mathcal{F}_{LM}(Y) \). Then,

\[
(\mathcal{F} \land K) \times G = (F \times G) \land (K \times G).
\]

**Lemma 5.6** Suppose that \( \perp_L \) is prime in \( L \). \((L, M)\)-LQC is closed under the formation of power objects in \((L, M)\)-QC.

**Proof** For \((L, M)\)-fuzzy \(Q\)-limit spaces \((X, q_X)\) and \((Y, q_Y)\). Let \( q_{X,Y} \) be the corresponding \((L, M)\)-fuzzy \(Q\)-convergence structure on \([X, Y] \) in \((L, M)\)-QC. That is,

\[
q_{X,Y}(\mathcal{H})(f) = \bigvee R_{\mathcal{H}}(f) \land \bigvee S_{\mathcal{H}}(f).
\]

It suffices to verify that \( q_{X,Y} \) satisfies (LMLQC). Take each \( f_{X,Y} \in J(L^{[X,Y]}) \), \( \mathcal{H}, \mathcal{K} \in \mathcal{F}_{LM}([X, Y]) \) such that \( f_{X,Y} \leq q_{X,Y}(\mathcal{H}) \land q_{X,Y}(\mathcal{K}) \), that is,

\[
\lambda \leq q_{X,Y}(\mathcal{H}) \land q_{X,Y}(\mathcal{K})(f).
\]

For each \( \mu \in J(L) \) with \( \mu \prec \lambda \), there exist \( v_1, v_2, \gamma_1, \gamma_2 \in J(L) \) such that \( \mu \leq v_1 \land v_2 \land \gamma_1 \land \gamma_2 \) and

(1) for each \( v \in J(L) \) with \( v \leq v_i \) (\( i = 1, 2 \)) and for each \( a \in L \) with \( v \nleq a' \), \( \mathcal{H}(a) = \top_M \) and \( \mathcal{K}(a) = \top_M \), which implies \( (\mathcal{H} \land \mathcal{K})(a) = \top_M \).

\[
(2) \text{for each } v \in J(L) \text{ with } v \leq \gamma_i \ (i = 1, 2) \text{ and for each } (F, x) \in \mathcal{F}_{LM}(X) \times X, x_v \leq q_X(F) \text{ implies }
\]

\[
f(x)_v \leq q_Y(\mathcal{H}^\rightarrow(F) \land q_Y(\mathcal{K}^\rightarrow(F))) = q_Y(\mathcal{H}^\rightarrow(F) \land (\mathcal{K} \times F))) = q_Y(\mathcal{K}^\rightarrow(F)). \tag{by Lemma 5.5}
\]

Then for each \( \gamma \in J(L) \) with \( \gamma \leq \mu \), it follows that \( \gamma \leq v_1 \land v_2 \land \gamma_1 \land \gamma_2 \). Further, for each \( a \in L \) with \( \gamma \nleq a' \) and for each \( (F, x) \in \mathcal{F}_{LM}(X) \times X \), we have \( (\mathcal{H} \land \mathcal{K})(a) = \top_M \) and \( x_v \leq q_X(F) \) implies \( f(x)_v \leq q_Y(\mathcal{H}^\rightarrow(F)). \text{This shows } \mu \in R_{\mathcal{H} \land \mathcal{K}}(f) \land S_{\mathcal{H} \land \mathcal{K}}(f). \text{This means}
\]

\[
\mu \leq \bigvee R_{\mathcal{H} \land \mathcal{K}}(f) \land \bigvee S_{\mathcal{H} \land \mathcal{K}}(f) = q_{X,Y}(\mathcal{H} \land \mathcal{K})(f).
\]

By the arbitrariness of \( \mu \), we have \( \lambda \leq q_{X,Y}(\mathcal{H} \land \mathcal{K})(f) \), i.e., \( f_{X,Y} \leq q_{X,Y}(\mathcal{H} \land \mathcal{K}), \text{as desired}. \Box \)

**Lemma 5.7** (Preuss 2002) Suppose that \( A \) is a topological category. If \( B \) is a bireflective (full and isomorphic closed) subcategory of \( A \) which is closed under formation of power objects in \( A \), then \( B \) is Cartesian-closed whenever \( A \) is Cartesian-closed.

**Theorem 5.8** Suppose that \( \perp_L \) is prime in \( L \). Then, \((L, M)\)-LQC is Cartesian-closed.

**Proof** It follows immediately from Theorems 2.8, 5.3 and 5.6, and Lemma 5.7. \( \Box \)

**Remark 5.9** It is required that \( \perp_L \) should be prime in several conclusions. This requirement seems to be strong. However, the real unit interval \( I = [0, 1] \) at least fulfills this requirement. Moreover, \( I \) fulfills the assumption of being completely distributive lattice with an order reversing involution.

### 6 (L, M)-fuzzy pretopological and topological Q-convergence spaces

In this section, we will introduce the concept of \((L, M)\)-fuzzy pretopological \(Q\)-convergence spaces and discuss its relationship with \((L, M)\)-fuzzy \(Q\)-limit spaces and \((L, M)\)-fuzzy topological \(Q\)-convergence spaces (Pang and Zhao 2016). For this, we first recall the following notation.

For an \((L, M)\)-fuzzy \(Q\)-convergence space \((X, q)\), define \( \mathcal{F}_{\alpha}^{X} : L^X \rightarrow M \) by

\[
\mathcal{F}_{\alpha}^{X} = \bigwedge_{x \leq q(F)} \mathcal{F}.
\]

Then \( \mathcal{F}_{\alpha}^{X} \) is an \((L, M)\)-fuzzy filter on \( X \) satisfying \( \mathcal{F}_{\alpha}^{X} \leq q(x) \).
Definition 6.1 An \((L, M)\)-fuzzy \(Q\)-convergence structure \(q\) on \(X\) is called pretopological if it satisfies
\[
(LMPQC) \quad x_\lambda \leq q(F^{q}_{x_\lambda}).
\]

For an \((L, M)\)-fuzzy pretopological \(Q\)-convergence structure \(q\) on \(X\), the pair \((X, q)\) is called an \((L, M)\)-fuzzy pretopological \(Q\)-convergence space.

The full subcategory of \((L, M)\)-QC, consisting of \((L, M)\)-fuzzy pretopological \(Q\)-convergence spaces, is denoted by \((L, M)\)-PQC.

Lemma 6.2 If \((X, q)\) is an \((L, M)\)-fuzzy pretopological \(Q\)-convergence space, then \((X, q)\) is an \((L, M)\)-fuzzy \(Q\)-limit space.

Proof It suffices to show that \((LMPQC)\) implies \((LMLQC)\). Take each \(F, G \in F_{LM}(X)\), \(x_\lambda \in J(L^X)\) such that \(x_\lambda \leq q(F)\) and \(x_\lambda \leq q(G)\). By the definition of \(F^{q}_{x_\lambda}\), it follows that \(F^{q}_{x_\lambda} \leq F\) and \(F^{q}_{x_\lambda} \leq G\). This implies \(F^{q}_{x_\lambda} \leq F \land G\). Thus, \(x_\lambda \leq q(F^{q}_{x_\lambda}) \leq q(F \land G)\). By the arbitrariness of \(x_\lambda\), we obtain \(q(F) \land q(G) \leq q(F \land G)\), as desired. \(\square\)

Lemma 6.3 Let \((X, q)\) be an \((L, M)\)-fuzzy \(Q\)-limit space and define \(q^p: F_{LM}(X) \to L^X\) by
\[
\forall F \in F_{LM}(X), \quad q^p(F) = \bigwedge \{x_\lambda \in J(L^X) \mid F^{q}_{x_\lambda} \leq F\}.
\]

Then \(q^p\) is an \((L, M)\)-fuzzy pretopological \(Q\)-convergence structure on \(X\).

Proof (LMQC1) and (LMQC2) are straightforward.

(LMQC3) Take each \(F \in F_{LM}(X)\), \(x_\lambda \in J(L^X)\) and \(a \in L\) such that \(x_\lambda \leq q^p(F)\) and \(a \not\leq a'\). It follows that
\[
q^p(F)(x) = \bigwedge \{\lambda \in J(L) \mid F^{q}_{x_\lambda} \leq F\} \not\leq a'.
\]

Then, there exists \(\lambda_a \in J(L)\) such that \(F^{q}_{x_{\lambda_a}} \leq F\) and \(\lambda_a \not\leq a'\). This implies
\[
F(a) = F^{q}_{x_{\lambda_a}}(a) = \bigwedge_{x_{\lambda_a} \leq q(F)} F(a) = \top_M.
\]

(LMPQC) For each \(x_\lambda \in J(L^X)\) and \(F \in F_{LM}(X)\) with \(x_\lambda \leq q^p(F)\), take each \(\mu \in J(L)\) such that \(\mu < \lambda\). It follows that
\[
\mu < \lambda \leq q^p(F)(x) = \bigwedge \{v \in J(L) \mid F^{q}_{x_\mu} \leq F\}.
\]

Then, there exists \(v \in J(L)\) such that \(\mu \leq F^{q}_{x_\mu} \leq F\). This implies \(F^{q}_{x_\mu} \leq F^{q}_{x_\lambda} \leq F\). So we have \(F^{q}_{x_\mu} \leq q^p(F)\).

Then, it follows that
\[
\mu \leq \bigwedge \{y \in J(L) \mid F^{q}_{x_\mu} \leq F^{q}_{x_\lambda} \} = q^p(F^{q}_{x_\lambda})(x).
\]

By the arbitrariness of \(\mu\), we get \(x_\lambda \leq q^p(F^{q}_{x_\lambda})(x)\), i.e., \(x_\lambda \leq q^p(F^{q}_{x_\lambda})\), as desired. \(\square\)

Theorem 6.4 \((L, M)\)-PQC is a bireflective subcategory of \((L, M)\)-LQC.

Proof Let \((X, q)\) be an \((L, M)\)-fuzzy \(Q\)-limit convergence space. By Lemma 6.3, we know \(q^p\) is an \((L, M)\)-fuzzy pretopological \(Q\)-convergence structure on \(X\). Next we claim that \(id_X: (X, q) \to (X, q^p)\) is the \((L, M)\)-PQC-bireflector. For this, it suffices to verify

1. \(id_X: (X, q) \to (X, q^p)\) is continuous.

For (1), take each \(x_\lambda \in J(L^X)\) and \(F \in F_{LM}(X)\) such that \(x_\lambda \leq q(F)\). Then it follows that \(F^{q}_{x_\lambda} \leq F\), which means \(x_\lambda \leq q^p(F)\). This shows \(q(F) \leq q^p(F)\).

For (2), take each \(F \in F_{LM}(X)\) and \(x_\lambda \in J(L^X)\) such that \(x_\lambda \leq q^p(F)\). Then, for each \(\mu \in J(L)\) with \(\mu < \lambda\), it follows that
\[
\mu < q^p(F)(x) = \bigwedge \{v \in J(L) \mid F^{q}_{x_\mu} \leq F\}.
\]

This means there exists \(v \in J(L)\) such that \(F^{q}_{x_\mu} \leq F\) and \(\mu \leq v\). Then it follows that
\[
F^{q}_{f(x)} = \bigwedge_{f(x) \leq F(q)} \{H \leq \bigwedge \{f \in F^{q}_{x_\lambda} \mid F \leq q^{p}(F)\} \leq q^{p}(F)\}
\]

which implies \(f(x) \leq q(F^{q}_{x_\lambda}) \leq q^{p}(F^{q}_{x_\lambda}) \leq q^{p}(F)\). By the arbitrariness of \(\mu\), we obtain \(f(x) \leq q^{p}(F)\). This proves \(f: (X, q^p) \to (Y, q^p)\) is continuous. \(\square\)

Next let us recall the definition of \((L, M)\)-fuzzy topological \(Q\)-convergence structures in Pang (2014b).

Definition 6.5 (Pang 2014b) An \((L, M)\)-fuzzy pretopological \(Q\)-convergence structure \(q\) on \(X\) is called topological if it satisfies
\[
(LMTQC) \quad \forall x_\lambda q^{p}_{\lambda}(A) = \bigwedge_{x_\lambda q^{p}_{\lambda}(B) \subseteq A} \bigwedge_{y_\mu q^{p}_{\mu}(B) \subseteq B} F^{q}_{x_\lambda}(A) = \bigwedge_{x_\lambda q^{p}_{\lambda}(B) \subseteq A} \bigwedge_{y_\mu q^{p}_{\mu}(B) \subseteq B} F^{q}_{x_\lambda}(A).
\]

For an \((L, M)\)-fuzzy topological \(Q\)-convergence structure \(q\) on \(X\), the pair \((X, q)\) is called an \((L, M)\)-fuzzy topological \(Q\)-convergence space.
The full subcategory of $(L, M)$-PQC, consisting of $(L, M)$-fuzzy topological $Q$-convergence spaces, is denoted by $(L, M)$-TQC.

Actually, combining Theorem 3.7 in Pang (2013) and Theorem 5.3 in Pang and Zhao (2016), the authors had shown the relationship between $(L, M)$-fuzzy pretopological $Q$-convergence structures and $(L, M)$-fuzzy topological $Q$-convergence structures without the stratification condition (LMQC3). However, most of the proofs can be adopted. So we only present the final result and omit the proofs.

**Theorem 6.6** $(L, M)$-TQC is a bireflective subcategory of $(L, M)$-PQC.

The following graph collects the main results of the previous sections:

$$
\begin{array}{c}
(L, M)-QC \\
\downarrow \quad \downarrow \text{(Cartesian-closed)} \\
\downarrow \quad \downarrow \\
(L, M)-QQC \\
\downarrow \text{(Cartesian-closed)} \\
(L, M)-TQC \\
\end{array}
$$



7 Conclusions

In this paper, we mainly constructed the function space of $(L, M)$-fuzzy $Q$-convergence spaces, which ensured the Cartesian-closedness of the category $(L, M)$-QC of $(L, M)$-fuzzy $Q$-convergence spaces. This gave an answer to the problem proposed by Pang and Zhao in Pang and Zhao (2016). That is, what is the concrete form of the function space of the category of $(L, M)$-fuzzy $Q$-convergence spaces. Furthermore, we made some investigations on subcategories of $(L, M)$-QC, including the category $(L, M)$-KQC of $(L, M)$-fuzzy Kent $Q$-convergence spaces, the category $(L, M)$-LQC of $(L, M)$-fuzzy $Q$-limit spaces and the category $(L, M)$-PQC of $(L, M)$-fuzzy pretopological $Q$-convergence spaces. Concretely, we investigated the relationships among these categories as well as the Cartesian-closedness of $(L, M)$-KQC and $(L, M)$-LQC. In the future, we will consider further categorical properties of $(L, M)$-fuzzy $Q$-convergence spaces, such as extensionality and productivity of quotient mappings.

**Declaration**

**Conflict of interest** The authors declare that they have no conflict of interest regarding the publication of this paper.

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