SU(2) finite temperature phase transition
and Michael sum rules

Yingcai Peng and Richard W. Haymaker

Department of Physics and Astronomy,
Louisiana State University, Baton Rouge, Louisiana 70803-4001

We studied the finite temperature phase transition of SU(2) gauge theory on
four-dimensional Euclidean lattices by Monte Carlo simulations, and measured the
flux distributions of a $q\bar{q}$ pair in both confined and unconfined phases. We reviewed
and generalized Michael sum rules to include finite temperature effects. To compare
our flux data with predictions of Michael sum rules, we studied the behavior of string
tension with temperature. Our data agree well with the generalized sum rules.

PACS number(s): 11.15.Ha
I. INTRODUCTION

The finite temperature phase transition of QCD [1] has been studied extensively in lattice gauge theory (LGT). It has been shown that at low temperatures QCD is in the confined phase, quarks are confined by a linear interquark potential, \( V(r) \sim \kappa r \), with \( \kappa \) the string tension. However, at sufficiently high temperatures the system is in the unconfined phase. Quarks are unconfined because the interquark potential is a screened Coulomb potential [2–4], \( V(r) \sim e^{-mr}/r \).

It is believed that quark confinement is due to the string formation in the confined phase, that is, the color field between quarks forms a string-like flux tube. In LGT one can probe the spacial distribution of the energy density between a \( q\bar{q} \) pair by using a local operator (e.g., plaquette). Some strong numerical evidence for string formation has been found [5–8]. Here we measured the flux data at various temperatures. The color flux energy and action of the \( q\bar{q} \) pair are related to the interquark potential energy \( V(r) \) by the Michael sum rules [9]. In this paper our aim is to check these sum rules with our finite temperature flux data.

The Michael sum rule for energy is the LGT version of the relation in classical electrostatics in which the work to assemble a charge distribution is equal to the integral over all space of the electrostatic energy density. The Michael sum rule for energy and a similar relation for action were tested on large lattices by Haymaker and Wosiek [10]. Except for an unresolved discrepancy arising in the determination of self energies, both sum rules were satisfied by the flux data. In this paper we rederive the sum rules for the case of finite temperature and check them with our flux data.

In the following we shall only consider pure SU(2) gauge theory. To estimate the prediction of the Michael sum rules at finite temperatures we need to know the lattice scaling relation \( a(\beta) \) and the behavior of the string tension with temperature, \( \kappa(T) \). For this purpose we adopted the lattice asymptotic scaling relation \( a(\beta) \) [11] to the non-perturbative region by allowing the scale constant to vary slowly with \( \beta \). Also the string tension was measured at finite temperature, and it was found that the string tension data agree very
well with the fitting function,

$$\kappa(T) = \kappa_0 (1 - \frac{\theta}{T_c})^\alpha \quad \text{(for } T < T_c).$$

(1)

This paper is organized as follows. In Sec. II we review some basic properties of the finite
temperature phase transition for SU(2) gauge theory. The scaling relation $a(\beta)$ is studied, and the flux measurement techniques in LGT is presented. In Sec. III we describe our
Monte Carlo calculations, and present some numerical results for global quantities, then we
analyze the behavior of string tension $\kappa$ with temperature. In Sec. IV we present a complete
derivation for Michael sum rules and generalize them to include finite size effects, then we
give a detailed comparison of these sum rules with the flux data. Finally Sec. V presents
the summary and conclusions.

II. THEORETICAL BACKGROUND

In this section we present the fundamental ideas of LGT in studying the finite temper-
ature phase transition of QCD, and emphasize aspects that are relevant for our numerical
investigations.

A. SU(2) Finite Temperature Phase Transition

In our study of SU(2) LGT we used the standard Wilson action [11]

$$S(U) = \beta \sum_P (1 - \frac{1}{2} \text{Tr} U_P),$$

(2)

where $\beta=4/g^2$ with $g$ to be the lattice coupling constant, and $U_P$ is the product of link
variables around a plaquette,

$$U_P(n) = U_{\mu}(n)U_{\nu}(n + \mu)U_{\mu}^{-1}(n + \nu)U_{\nu}^{-1}(n),$$

(3)

where the indices $\mu, \nu$ represent the orientation of the plaquette, $n$ the position of the
plaquette on the lattice.
On a four-dimensional Euclidean lattice of the size, \( N_t \times N_s^2 \times N_z \), the temperature \( T \) is related to the temporal extent of the lattice \([12]\). If one chooses \( N_t \) as the time direction, then one can define the temperature as,

\[
T = \frac{1}{N_t a},
\]

where \( a \) is the lattice spacing, which is a function of \( \beta \).

To study the finite temperature phase transition a convenient order parameter is provided by the Polyakov loop closed in the time direction \([13]\),

\[
P(r) = \frac{1}{2} \text{Tr} \prod_{\tau=1}^{N_t} U_t(r, \tau),
\]

where \( U_t(r, \tau) \) is the link variable oriented in \( N_t \) direction at the site \((r, \tau)\). It is well-known for pure SU\((N)\) gauge theory that the expectation value of the Polyakov loop \( \langle P \rangle \) plays the role of an order parameter \([13]\) in the finite temperature phase transition, that is, in the infinite volume limit one has

- in the confined phase, \( T < T_c \), \( \langle P \rangle = 0 \);
- in the unconfined phase, \( T > T_c \), \( \langle P \rangle \neq 0 \);

because of the spontaneous breaking of the global \( Z_N \) symmetry \([14]\). Here \( T_c \) is the transition temperature.

The expectation value \( \langle P \rangle \) is associated with the free energy \( F_q \) of an isolated quark \( q \) in the infinite volume limit \([14]\),

\[
|\langle P \rangle| \sim e^{-L_t F_q},
\]

where \( L_t = N_t a \), is the inverse temperature. Further, the correlation function \( \langle P(0)P^\dagger(r) \rangle \) is related to the potential energy \( V(r) \) of a \( q\bar{q} \) pair in the infinite volume limit,

\[
\langle P(0)P^\dagger(r) \rangle \sim \text{const.} \exp[-L_t V(r)].
\]

In the confined phase, \( T < T_c \), Eq. (7) becomes, for large \( r \) \([2]\),

\[
\lim_{r \to \infty} \langle P(0)P^\dagger(r) \rangle \sim \exp[-L_t kr],
\]

\[
= \langle P \rangle^2 = 0,
\]
because of $\langle P \rangle = 0$ in the confined phase, where $\kappa$ is the string tension. Eq. (8) implies that the $q\bar{q}$ potential in the confined phase is linear, e.g., $V(r) \sim \kappa r$. On the other hand, in the unconfined phase, $T > T_c$, and in the infinite volume limit it is found that for SU(2) gauge theory [2],

$$\lim_{r \to \infty} \langle P(0)P^\dagger(r) \rangle = \text{const.}[1 + L_t \frac{3}{\beta 4\pi r} \exp(-mr)]$$

$$= \langle P \rangle^2 \neq 0,$$

(9)

for $\langle P \rangle \neq 0$ in the unconfined phase. Eq. (9) implies that the $q\bar{q}$ potential in the unconfined phase is a screened Coulomb potential, e.g., $V(r) \sim \frac{e^{-mr}}{r}$ with $m^{-1}$ the Debye screening length. To the lowest order of perturbation theory one has [2]

$$m^2 = \frac{2}{3} g^2 T^2.$$  

(10)

Ref. [3] gives a detailed discussion about the $q\bar{q}$ potential in the unconfined phase. We shall not discuss it any more in the following.

The transition temperature $T_c$ can be determined by Monte Carlo calculations of LGT. For SU(2) one recent result is [15,16],

$$\beta_c = 2.2985 \pm 0.0006. \quad \text{(for } N_t = 4)$$

(11)

This transition point $\beta_c$ was calculated on large lattices (e.g., $4 \times 26^3$), on which finite size effects are small, the results can be considered to be the transition point in the infinite volume limit.

To obtain the transition temperature $T_c$ in physical units, one must know the scaling relation between the lattice spacing $a$ and $\beta$. If we assume the scaling relation $a(\beta)$ is given by the asymptotic relation for SU(2) [11],

$$a(\beta) = \Lambda_L^{-1}(\frac{6}{11} \pi^2 \beta^2 \frac{24}{20} \exp[-\frac{3}{11} \pi^2 \beta]} = \Lambda_L^{-1} f(\beta),$$

(12)

where $\Lambda_L$ is the lattice scaling constant, then one can calculate the transition temperature $T_c$ in physical units from Eqs. (4), (11) and (12),

5
\[ T_c \Lambda_L^{-1} = \frac{1}{N_t f(\beta_c)}, \]
\[ = 42.11 \pm 0.06. \]  

This is consistent with the results of Refs. [17,22]. From Eqs. (12) and (13) one can also see that \( \beta < \beta_c \) implies \( T < T_c \), and \( \beta > \beta_c \) corresponds to \( T > T_c \) for fixed \( N_t \). Now we proceed to study the lattice scaling constant \( \Lambda_L \).

B. The Scaling Relation \( a(\beta) \)

We follow Refs. [19,20] in choosing a scale such that in the limit of zero temperature and infinite volumes the string tension is

\[ \sqrt{\kappa_0} = 0.44 \text{ GeV}, \]  

which was determined from the Regge trajectory of experimental data. Here for definiteness we use the data from the real world to discuss SU(2) gauge theory.

From Eq. (14) and the string tension data of Monte Carlo calculations in Refs. [8,19] one can extract a relation between the lattice spacing \( a \) and lattice coupling constant \( \beta \). The result is listed in Table I, which is consistent with the similar result of Ref. [10]. In Table I the values of \( \Lambda_L^{-1} \) were calculated from the asymptotic scaling relation in Eq. (12) by substituting the corresponding values of \( a \) and \( \beta \) in this table. One can see that the values of \( \Lambda_L^{-1} \) changes slowly with \( \beta \) in the region, \( 2.22 \leq \beta \leq 2.5 \). This implies that the perturbative asymptotic scaling relation in Eq. (12) is not exactly valid in this region. To study the non-perturbative physics and obtain the relation \( a(\beta) \) for other \( \beta \) values in this region, we use the function in Eq. (12) to fit the data of Table I, with \( \Lambda_L^{-1} \) considered as a function of \( \beta \). For simplicity, we chose a quadratic function to fit the \( \Lambda_L^{-1} \) values. The result is

\[ a(\beta) = \Lambda_L^{-1}(\beta) f(\beta), \]

with \( \Lambda_L^{-1}(\beta) = d_1 + d_2 \beta + d_3 \beta^2 \) (fm),  

(15)
where the coefficients $d_1 = 59.37 \pm 0.86$, $d_2 = -5.96 \pm 0.39$ and $d_3 = -3.28 \pm 0.17$, and the function $f(\beta)$ is given by Eq. (12).

Using the scaling relation of Eq. (15) we can transform the Monte Carlo data from lattice units to physical units. Now the transition temperature $T_c$ in Eq. (13) can be expressed explicitly in physical units, i.e.,

$$T_c = 1.487 \pm 0.140 \ (1/\text{fm}).$$

(16)

Also, by using the scaling relation one can estimate the transition point $\beta_c$ on lattices of any temporal size $N_t$, if we assume the transition temperature $T_c$ in Eq. (16) is independent of the lattice size $N_t$ because it is a physical observable. For example, the estimated values of $\beta_c$ for $N_t = 6$ and 8 are

$$\beta_c \simeq 2.42 \quad \text{(for } N_t = 6),$$

$$\beta_c \simeq 2.50 \quad \text{(for } N_t = 8).$$

(17)

In Sec. IV we shall use the scaling relation of Eq. (15) to discuss the predictions of Michael sum rules. In the following we proceed to describe the methods used in measuring the $q\bar{q}$ flux distributions.

**C. $q\bar{q}$ Flux Distributions**

To measure the flux distributions of a $q\bar{q}$ pair, we calculated the quantity [6,8],

$$f_{\mu\nu}(r,x) = \frac{\beta}{a^4} \left[ \frac{\langle P(0)P^\dagger(r)\Box_{\mu\nu} \rangle}{\langle P(0)P^\dagger(r) \rangle} - \langle \Box_{\mu\nu} \rangle \right],$$

(18)

where the Polyakov loop $P(r)$ is defined in Eq. (5), and $\Box_{\mu\nu} = \frac{1}{4} \text{Tr}(U_P)$ is the plaquette variable with $U_P$ defined in Eq. (3). The plaquette has 6 different orientations, $(\mu, \nu) =$ (2, 3), (1, 3), (1, 2), (1, 4), (2, 4), (3, 4).

To reduce the fluctuations of the quantity $P(0)P^\dagger(r)\Box$, in practical calculations we measure the quantity [6]
\[ f'_{\mu\nu}(r, x) = \frac{\beta}{a^4} \left[ \frac{\langle P(0)P^\dagger(r)\square_{\mu\nu}(x) \rangle - \langle P(0)P^\dagger(r)\square_{\mu\nu}(x_R) \rangle}{\langle P(0)P^\dagger(r) \rangle} \right], \] (19)

as the flux distribution instead of Eq. (18), where the reference point \( x_R \) was chosen to be far from the \( q\bar{q} \) sources. This replacement does not change the measured average due to the cluster decomposition theorem. The six components of \( f'_{\mu\nu} \) in Eq. (19) correspond to the components of the chromo-electric and chromo-magnetic fields \((\mathcal{E}, \mathcal{B})\) in Minkowski space, i.e.,

\[ f'_{\mu\nu} \rightarrow \frac{1}{2} \left( -\mathcal{B}_1^2, -\mathcal{B}_2^2, -\mathcal{B}_3^2, \mathcal{E}_1^2, \mathcal{E}_2^2, \mathcal{E}_3^2 \right). \] (20)

The total electric and magnetic energy densities are defined as

\[ \rho_{\text{el}} = \frac{1}{2} \langle \mathcal{E}^2 \rangle = \frac{1}{2} \left[ \langle \mathcal{E}_1^2 \rangle + \langle \mathcal{E}_2^2 \rangle + \langle \mathcal{E}_3^2 \rangle \right], \]

\[ \rho_{\text{ma}} = \frac{1}{2} \langle \mathcal{B}^2 \rangle = \frac{1}{2} \left[ \langle \mathcal{B}_1^2 \rangle + \langle \mathcal{B}_2^2 \rangle + \langle \mathcal{B}_3^2 \rangle \right]. \] (21)

To improve the statistical accuracy we used the multihit techniques [6], that is, we make the replacement for Polyakov loops in Eq. (19),

\[ P(r) \rightarrow \bar{P}(r) = \frac{1}{2} \text{Tr} \prod_{\tau=1}^{N_t} \bar{U}_t(r, \tau), \] (22)

where \( \bar{U}_t(r, \tau) \) is given by

\[ \bar{U}_t = \frac{\int U_t \exp[\beta \text{Tr}(U_tX_t^\dagger + \text{h.c.})]dU_t}{\int \exp[\beta \text{Tr}(U_tX_t^\dagger + \text{h.c.})]dU_t}, \]

\[ = X_t \frac{I_2(\beta \lambda)}{\lambda I_1(\beta \lambda)}, \] (23)

with \( X_t^\dagger \) to be the neighborhood of \( U_t \), i.e., \( U_tX_t^\dagger = \sum \square_P \), the sum extends over all plaquettes containing \( U_t \). And \( \lambda = \sqrt{\text{det}(X_t)} \), \( I_1 \) and \( I_2 \) are the modified Bessel functions. This technique is limited to the situations where the neighboring links \( X_t \) of Polyakov loops \( P \) do not overlap with the plaquette \( \square_{\mu\nu} \) in Eq. (19).
III. NUMERICAL RESULTS FOR GLOBAL QUANTITIES

In this section we present the numerical results and analyses for global quantities in our Monte Carlo calculations. We follow Ref. [17] to choose our lattices of the geometry $N_t \times N_s^2 \times N_z$, with $N_t \ll N_s \ll N_z$. If $N_t$ is chosen as the time direction, we can simulate SU(2) gauge theory at finite temperatures ($T = 1/N_t a$) in large volumes ($V = N_s^2 N_z a^3$). Typically, we choose $N_t = 4$ and 6, $N_s = 7, 9, 11$, and $N_z = 65$ in most cases and $N_z = 37$ in a few cases with $N_t = 6$. The lattice coupling constant $\beta$ is in the range $2.25 \leq \beta \leq 2.40$ for $N_t = 4$, and $2.30 \leq \beta \leq 2.50$ for $N_t = 6$. We updated the lattice configurations by using the standard Metropolis algorithm alternated with overrelaxation method. We typically thermalized for 4000 sweeps, and made one measurement every 10 sweeps. The total number of measurements for each data is about 2000. The actual number may vary by a small amount in each case. The calculations were done on LSU’s IBM 3090 mainframe.

A. Measurements of The Order Parameter $\langle |P| \rangle$

and $\langle P(0)P^\dagger(z) \rangle$

As we discussed in Sec. II A the expectation $\langle P \rangle$ plays the role of an order parameter in the infinite volume limit. However, on a finite lattice this quantity is always zero if the computation time is taken to be infinite, because the system would flip between two ordered states, and the values of the Polyakov loop $P$ would flip sign after some iterations. Therefore, we take the expectation value of the modulus of the Polyakov loop $\langle |P| \rangle$ as the “order parameter” on finite lattices [13,14].

The measured data of $\langle |P| \rangle$ are plotted in Fig. 1, which shows a rapid increase of $\langle |P| \rangle$ at $\beta \approx 2.30$ for $N_t = 4$, and another at $\beta \approx 2.42$ for $N_t = 6$. This implies that a phase transition occurs at $\beta_c \approx 2.30$ for $N_t = 4$, or $\beta_c \approx 2.42$ for $N_t = 6$, in the infinite volume limit, which are consistent with Eqs. (11) and (17). For cases of $N_t = 4$ the data measured on the lattice $4 \times 9^2 \times 65$ agree very well with those from $4 \times 11^2 \times 65$, both approach the
infinite volume limit. However, for cases of $N_t = 6$ the data from the lattice $6 \times 7^2 \times 65$ have some discrepancies with the data from $6 \times 11^2 \times 37$. This may be due to finite-volume effects, as discussed in Ref. [15], because in these cases $N_s$ (or $N_z$) are not large enough to approach the infinite volume limit.

We also measured the correlations of two Polyakov loops, $\langle P(0)P^\dagger(z) \rangle$, with the separation $z$ along $N_z$ direction. Some typical results are shown in Figs. 2 and 3, both were drawn as a logarithmic plot. Fig. 2 shows $\langle P(0)P^\dagger(r) \rangle$ versus $r$ in the confined phase, from this figure one can see that the correlation $\langle P(0)P^\dagger(r) \rangle \to 0$ as $r \to \infty$, as predicted by Eq. (8). Also, the linear behavior of the quantity $\ln \langle P(0)P^\dagger(r) \rangle$ with $r$ indicates that the $q\bar{q}$ potential $V(r)$ is linear in the confined phase. And the slope of the straight lines corresponds to the string tension $\kappa$. This figure shows that the string tension decreases with $\beta$ (or temperature). Fig. 3 is similar to Fig. 2, but is in the unconfined phase. It shows that the correlation $\langle P(0)P^\dagger(r) \rangle \not\to 0$ as $r \to \infty$, as predicted by Eq. (9).

**B. Measurements of String Tension $\kappa$**

The string tension $\kappa$ can be extracted from the correlation $\langle P(0)P^\dagger(r) \rangle$ according to Eq. (8). In our measurements we choose the correlation along the longest extent $N_z$, i.e., $r = r e_z$, and follow Ref. [17] to extract the string tension data from the relation,

$$\langle P(0)P^\dagger(z) \rangle_c = \text{const.} \left(e^{-Lt\kappa z} + e^{-Lt\kappa(L_z - z)} \right), \tag{24}$$

where on the L.H.S. of this equation the connected correlation $\langle P(0)P^\dagger(z) \rangle_c$ is defined as, $\langle P(0)P^\dagger(z) \rangle - \langle P \rangle^2$. So the dependence on the coordinates $t, x, y$ can be suppressed, and one can extract the effective string tension. On the R.H.S. the new term $e^{-Lt\kappa(L_z - z)}$ accounts for the finite size effects along the correlation direction $N_z$, with $L_z = N_z a$.

The raw string tension data are given in Table II. Some data were measured in the transition region, e.g., $\beta = 2.30$ for $(N_t = 4)$. In this case we assume Eq. (24) be still valid. Since our lattices have the geometry, $N_z \gg N_s \gg N_t$, we find that finite size effects are
small as \( N_s/N_t \geq 2 \) in the limit of \( N_z \to \infty \). The detailed analysis about the finite size effects is presented elsewhere [5]. This conclusion also agree with the results of Refs. [17,18]. Therefore, we expect that the data measured on lattices \( 4 \times 9^2 \times 65 \) and \( 4 \times 11^2 \times 65 \) can be viewed as the string tension in the large volume limit. However, the data from the lattice \( 6 \times 7^2 \times 65 \) might have some large finite size effects because \( N_s = 7 \) is not large enough for \( N_t = 6 \). For data measured on the lattice \( 6 \times 11^2 \times 37 \) the transverse size \( N_s = 11 \) is large, but some large finite size effects may be caused by the relatively small longitudinal size \( N_z \).

C. Behaviour of \( \kappa \) With Temperature

In Sec. IV when we discuss Michael sum rules and analyze the finite temperature effects on the flux data, we need to know the relation between string tension \( \kappa \) and temperature. In this part we proceed to analyze the string tension data.

Using the scaling relation in Eq. (15) we can transform the string tension data in Table II from lattice units to physical units. The results are shown in Table III. From this table one can see that the string tension \( \kappa \) decreases with temperature \( T \) (or \( \beta \)) in the confined phase. If we assume that \( \kappa \) is a continuous function of \( T \) (or \( \beta \)), one has

\[
\frac{\partial \kappa}{\partial T} < 0 \quad \text{(for } T < T_c), \\
\text{or} \quad \frac{\partial \kappa}{\partial \beta} < 0 \quad \text{(for } \beta < \beta_c). \tag{25}
\]

This behavior is also confirmed in other recent papers [21,22].

Since the string tension \( \kappa \) decreases with temperature \( T \) in the confined phase and is expected to vanish at the transition point \( T_c \), for simplicity, we then use the simple function with power behavior in Eq. (1) to fit the string tension near \( T_c \), that is,

\[
\kappa(T) = \kappa_0(1 - \frac{T}{T_c})^\alpha \quad \text{(for } T < T_c). \tag{26}
\]

This is a simplified form of the fitting function used in Ref. [22]. Here the constant \( \kappa_0 \) represents the string tension \( \kappa \) in the limit of zero temperature and infinite volumes, which
can be given by Eq. (14), i.e., \( \kappa_0 \approx 0.981 \) GeV/fm. The transition temperature \( T_c \) is given by Eq. (16) in physical units.

As we discussed above, the string tension data measured on lattices \( 6 \times 7^2 \times 65 \) and \( 6 \times 11^2 \times 37 \) may have large finite size effects. By viewing the data in Table III we find that the string tension data from the lattice \( 6 \times 7^2 \times 65 \) are consistent with the data from lattices \( 4 \times 9^2 \times 65 \) and \( 4 \times 11^2 \times 65 \), on which finite size effects can be neglected, as we discussed before. However, data from the lattice \( 6 \times 11^2 \times 37 \) have large discrepancies with other data. This suggests that finite size effects on string tension are sensitive to the longitudinal size \( N_z \), but not as much to the transverse size \( N_s \).

We then use the string tension data from lattices of small finite size effects (i.e., \( 4 \times 11^2 \times 65 \) and \( 6 \times 7^2 \times 65 \)), to fit the function in Eq. (26). The only fitting parameter is the exponential index \( \alpha \), the best fit result is,

\[
\alpha = 0.35 \pm 0.04. \tag{27}
\]

Fig. 4 shows this fitting function and the measured string tension data of small finite size effects. In this figure one can see that a few data point very close to the transition point \( T_c \) have large discrepancies from the fitting function, because fluctuations are large in the transition region. We then excluded these data at, \( T \approx 1.48 \) (1/fm), from our fitting process. Also all other data above \( T_c \) were excluded because they would correspond to complex values for the fitting function. Below the transition region our data agree very well with the conjectured behavior in Eq. (26).

To get the relation between \( \kappa \) and \( \beta \), we also fit the string tension data with the following function similar to Eq. (26),

\[
\kappa(\beta) = \kappa_0(1 - \frac{\beta}{\beta_c})^\delta \quad (\text{for } \beta < \beta_c). \tag{28}
\]

The transition point \( \beta_c = 2.2985 \pm 0.0006 \) for \( N_t = 4 \) can be chosen from Eq. (11), and \( \beta_c = 2.42 \pm 0.01 \) for \( N_t = 6 \) from Eq. (17). We find that the best fit results for the exponential index \( \delta \) are
\[ \delta = 0.22 \pm 0.03 \quad \text{for } N_t = 4, \]
\[ \delta = 0.14 \pm 0.04 \quad \text{for } N_t = 6. \] (29)

In Fig. 5 we plot the result for the case of \( N_t = 4 \). Also, in this figure the data above the transition point, \( \beta_c = 2.2985 \), were excluded from our fitting process. Again our data agree well with the fitting function. In Sec. IV B we shall use the relation \( \kappa(\beta) \) to check the Michael sum rules.

**IV. MICHAEL SUM RULES AND FLUX DISTRIBUTIONS**

To study the detailed properties of the finite temperature phase transition, we measured the \( q\bar{q} \) flux distributions at various temperatures using the technique discussed in Sec. subsec: mfd. The flux energies (action) are related to the potential energy \( V(r) \) of the \( q\bar{q} \) pair by Michael sum rules, which were first derived by C. Michael [9]. In the derivation of these sum rules some scaling relations on asymmetric lattices are used, which were studied extensively by A. Hasenfratz and P. Hasenfratz [24], and F. Karsch [25]. The original work of C. Michael is somewhat terse and drawing on other references. In order to generalize it we feel it is helpful to present a thorough derivation, including the other references. In this section we shall give a complete derivation of Michael sum rules, and present our supplements and generalizations. One result is delegated to the Appendix. Then we compare our flux data with the predictions of these sum rules.

**A. Review and Generalizations of Michael Sum Rules**

For a static \( q\bar{q} \) system at the spatial separation \( r \) the Michael sum rules states that, in the limit of zero temperature, infinite volume and infinitesimal lattice spacing \( (a \to 0) \), the electric and magnetic flux energies of the system in Minkowski space are [9],

\[ E_{el}(r) = \sum_s \frac{1}{2} a^3 \langle \mathcal{E}^2 \rangle = \frac{1}{2} V(r) \left[ \frac{\partial \ln a}{\partial \ln \beta} - 1 \right] - \frac{1}{2a} \left[ \beta \frac{\partial f}{\partial \beta} - f \right], \] (30)
$$E_{ma}(r) = \sum_s \frac{1}{2} a^3 \langle B^2 \rangle = \frac{1}{2} V(r) \left[ \frac{\partial \ln a}{\partial \ln \beta} + 1 \right] + \frac{1}{2a} \left[ \beta \frac{\partial f}{\partial \beta} + f \right].$$

(31)

where the sums extends over the whole space around the $q\bar{q}$ pair, $V(r)$ is the potential energy and $f/a$ is the self energy of the $q\bar{q}$ sources. $E_{el}(r)$ and $E_{ma}(r)$ denote the electric and magnetic parts of the flux energy, $\langle \mathcal{E}^2 \rangle$ and $\langle \mathcal{B}^2 \rangle$ are given by Eq. (21).

1. The Action Sum Rule

To derive Eqs. (30) and (31), it is convenient to work in the transfer matrix formalism of LGT. We choose the temporal gauge [23], that is, all temporal links are trivial, $U_t(n) = e^{igaA_\mu(n)} = 1$. One remark about the temporal gauge of a finite lattice is that along a time axis one can not choose all temporal links trivial, $U_t = 1$, because of the restriction that a Polyakov loop in the time direction is gauge invariant, which has values other than the trivial one, $P(\vec{r}) = 1$. Therefore, on a finite lattice one can choose all temporal links to be trivial, except one link on each time axis, which can be chosen far from the operators under considerations.

Let’s consider a Wilson loop $W$ of the temporal size $na$ and the spatial size $r$, as shown in Fig. 6. The time directed pathes represent the $q\bar{q}$ sources, the space directed pathes $P_r(0)$ and $P_r(n)$ create and annihilate the static $q\bar{q}$ pair from and to the vacuum. In the temporal gauge the expectation of Wilson loop becomes,

$$\langle W \rangle = \langle P_r(0)P_r(n) \rangle,$$

(32)

where the expectation is evaluated in the partition function form,

$$\langle W \rangle = \frac{\int d[U] e^{-\beta S'} W}{\int d[U] e^{-\beta S'}}.$$

(33)

with $\beta S' = S$, the action given by Eq. (2), i.e., $S' = \sum (1 - \Box)$, where $\Box$ is defined in Eq. (18), the sum is over all plaquettes on the lattice.

Using the transfer matrix approach [23,7] and assuming a discrete spectrum for the lowest eigenstates of the transfer matrix, one can evaluate Eq. (32) as
\[
\langle W \rangle = \frac{1}{Z} \text{Tr}(P_r(0)T_{qq}^n P_r(n)T^{N_t-n}) \\
= \frac{1}{\sum_{\alpha} \lambda_{\alpha}^N} \sum_{\mu, \nu} \langle \mu | P_r | \nu, r \rangle \lambda_{\nu}^n(r) \langle \nu, r | P_r | \mu \rangle \lambda_{\mu}^{N_t-n} \\
= \frac{1}{\sum_{\alpha} \lambda_{\alpha}^N} \sum_{\mu, \nu} d_{\mu \nu} d_{\nu \mu} \lambda_{\nu}^n(r) \lambda_{\mu}^{N_t-n},
\] (34)

where \( T_{qq} \) is the transfer matrix projected into the \( q\bar{q} \) sector of the Hilbert space [7], \( T \) is the transfer matrix for the vacuum. \(|\nu, r\rangle\) and \(|\mu\rangle\) are the eigenstates of \( T_{qq} \) and \( T \) respectively, and \( \lambda_{\nu}(r) \), \( \lambda_{\mu} \) are the corresponding eigenvalues, with \( \lambda_{\nu}(r) = e^{-aE_{\nu}(r)} \) and \( \lambda_{\mu} = e^{-aE_{\mu}} \). The coefficient \( d_{\mu \nu} = \langle \mu | P_r | \nu, r \rangle \). In the limit of \( N_t \rightarrow \infty \), the partition function \( Z = \lambda_0^{N_t} \), and in Eq. (34) the dominant contributions correspond to \( \mu = 0 \), that is,

\[
\langle W \rangle = \sum_{\nu} d_{0 \nu}^2 \left( \frac{\lambda_{\nu}(r)}{\lambda_0} \right)^n \quad \text{(as } N_t \rightarrow \infty \text{).}
\] (35)

If the temporal size \( n \) of the Wilson loop \( W \) is very large \( (n \rightarrow \infty) \), the dominant term in Eq. (35) is given by \( \nu = 0 \),

\[
\langle W \rangle = d_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n = d_{00}^2 e^{-na(E_0(r)-E_0)}, \quad \text{(as } N_t, n \rightarrow \infty \text{)}
\] (36)

where the energy of the vacuum \( E_0 \) is usually chosen to be zero, in the following we will take this choice which implies \( \lambda_0 = 1 \).

Now we consider the \( \beta \)-derivative of Eq. (36), that is,

\[
\frac{\partial \langle W \rangle}{\partial \beta} = -\langle WS' \rangle + \langle W \rangle \langle S' \rangle = \frac{\partial}{\partial \beta} \left[ d_{00}^2 e^{-naE_0(r)} \right],
\] (37)

where we have taken \( E_0 = 0 \), and \( S' \) is defined after Eq. (33).

Consider a plaquette \( \square(m) \) outside the Wilson loop \( W \) in the time direction (i.e., \( 0 < n < m \)), e.g., the plaquette \( P_1 \) in Fig. 6, where we draw a plaquette with a time extension. In the limit of infinite large \( n \) and infinitesimal lattice spacing, \( a \rightarrow 0 \), one can neglect the time extension of \( P_1 \). The contribution of \( P_1 \) to the L.H.S. of Eq. (37) is \( \langle W \square(m) \rangle - \langle W \rangle \langle \square(m) \rangle \). As \( \square(m) \) is far from \( W \) \( (m-n \gg 1) \), one has \( \langle W \square(m) \rangle \approx \langle W \rangle \langle \square(m) \rangle \), then the contribution vanishes, \( \langle W \square(m) \rangle - \langle W \rangle \langle \square(m) \rangle \rightarrow 0 \). However, when \( \square(m) \) is close to \( W \) \( (m \approx n) \), one expects that this contribution does not vanish. Now let us show this explicitly,
\[ \langle W \Box(m) \rangle - \langle W \rangle \langle \Box(m) \rangle \quad (\text{for } 0 < n < m) \]

\[ = \frac{1}{Z} \text{Tr}(P_r(0) T_{qq}^n P_r(n) T^{m-n} \Box(m) T^{N_t-m}) - \langle W \rangle \langle \Box(m) \rangle \]

\[ \xrightarrow{N_t \to \infty} \sum_{\mu, \nu} d_{\mu 0} \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^n d_{\nu \mu} \left( \frac{\lambda_\nu}{\lambda_0} \right)^{m-n} \langle \nu | \Box(m) | 0 \rangle 
- \sum_{\mu} d_{\mu 0}^2 \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^n \langle 0 | \Box(m) | 0 \rangle, \quad (38) \]

where in the last step we have used Eq. (35)

From Eq. (38) one can see that, as \( m - n \gg 1 \), the dominant term of \( \langle W \Box(m) \rangle \) is given by \( \nu = 0 \), which would be cancelled by the product \( \langle W \rangle \langle \Box(m) \rangle \). So one has \( \langle W \Box(m) \rangle - \langle W \rangle \langle \Box(m) \rangle \approx 0 \), as \( m - n \gg 1 \).

However, for \( m \approx n \) the term of \( \nu = 0 \) of the quantity, \( \langle W \Box(m) \rangle \), in Eq. (38) is cancelled by the product \( \langle W \rangle \langle \Box(m) \rangle \). The major contribution comes from the term of \( \nu = 1 \). In the limit of \( n \to \infty \) Eq. (38) becomes,

\[ \langle W \Box(m) \rangle - \langle W \rangle \langle \Box(m) \rangle \quad (\text{for } 0 < n < m) \]

\[ \approx \sum_{\mu} d_{\mu 0} \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^n d_{\mu 1} \left( \frac{\lambda_1}{\lambda_0} \right)^{m-n} \langle 1 | \Box(m) | 0 \rangle 
- \sum_{\mu} e^{-naE_0(r)} d_{\mu 0} d_{\mu 1} \lambda_{1}^{m-n} \langle 1 | \Box(m) | 0 \rangle. \quad (39) \]

where in the last step we have used the fact that \( \lambda_0 = 1 \), and the dominant term is \( \mu = 0 \) as \( n \to \infty \). Eq. (39) implies that when the plaquette \( \Box(m) \) is close to the Wilson loop in the time direction, it gives a contribution of the order \( e^{-naE_0(r)} \), because the coefficients \( d_{00}, d_{01} \) and \( \lambda_{1}^{m-n} \) are of order of unity in this case.

Finally for the plaquette \( \Box(m) \) inside the Wilson loop \( W \) in the time direction (i.e., \( 0 < m < n \)), e.g., the plaquette \( P_2 \) in Fig. 6, its contribution to the L.H.S. of Eq. (37) is,

\[ \langle W \Box(m) \rangle - \langle W \rangle \langle \Box(m) \rangle \quad (0 < m < n) \]

\[ = \frac{1}{Z} \text{Tr}(P_r(0) T_{qq}^m \Box(m) T^{n-m} P_r(n) T^{N_t-n}) - \langle W \rangle \langle \Box(m) \rangle \]

\[ \xrightarrow{N_t \to \infty} \sum_{\mu, \nu} d_{\mu \nu} d_{\nu 0} \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^m \left( \frac{\lambda_\nu(r)}{\lambda_0} \right)^{n-m} \langle \mu, r | \Box(m) | \nu, r \rangle 
- \langle W \rangle \langle \Box(m) \rangle. \quad (40) \]
In the limit of \( n \to \infty \), and for cases that both \( m \) and \( n - m \) are large, the dominant term of Eq. (40) is given by \( \mu = \nu = 0 \), that is,

\[
\langle W \square(m) \rangle - \langle W \rangle \langle \square(m) \rangle \quad (0 < m < n)
\]

\[
\approx d_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n [\langle 0, r | \square(m) | 0, r \rangle - \langle 0 | \square(m) | 0 \rangle]
\]

\[
= d_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n (\square)_r - 0 ,
\]

(41)

where we denote \((\square)_r - 0 = [\langle 0, r | \square(m) | 0, r \rangle - \langle 0 | \square(m) | 0 \rangle]\), and Eq. (36) is used.

From the above discussion we can conclude that for plaquettes outside the Wilson loop \( W \) in the time direction, their contributions to the L.H.S. of Eq. (37) can be neglected as they are far enough from \( W \). For the Wilson loop of large temporal size, i.e., \( n \to \infty \), the major contribution to the L.H.S. of Eq. (37) comes from plaquettes inside \( W \). The contribution of one plaquette inside \( W \) is given by Eq. (41). Summing over contributions from all such plaquettes gives the dominant term of the L.H.S. in Eq. (37), that is,

\[
\frac{\partial \langle W \rangle}{\partial \beta} = -\langle WS' \rangle + \langle W \rangle \langle S' \rangle
\]

\[
\approx nd_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n \left( \sum_s \square \right)_r - 0 ,
\]

(42)

where the sum \( \sum_s \) is over all plaquettes in the spatial volume at one fixed time value \( m \) ( \( 0 < m < n \) ). The factor of \( n \) comes from summing equal contributions from plaquettes on each time slice. The dominant term of the R.H.S. in Eq. (37) is, as \( n \to \infty \),

\[
\frac{\partial}{\partial \beta} [d_{00}^2 e^{-naE_0(r)}] = -nd_{00}^2 e^{-naE_0(r)} \frac{\partial [aE_0(r)]}{\partial \beta} .
\]

(43)

Collecting Eqs. (37), (42) and (43) yields,

\[
-\frac{\partial aE_0(r)}{\partial \beta} = \left( \sum_s \square \right)_r - 0 ,
\]

(44)

where the R.H.S. can be measured by calculating the quantity, \( \langle W \square \rangle / \langle W \rangle \) in LGT. In the continuum limit \((a \to 0)\) the R.H.S. of Eq. (44) becomes,

\[
\left( \sum_s \square \right)_r - 0 = \frac{1}{\beta} \sum_s \frac{1}{2} a^4 [\langle E^2 \rangle]_r - 0 - \langle B^2 \rangle]_r - 0 = \frac{a}{\beta} A ,
\]

(45)
where $\mathcal{E}$ and $\mathcal{B}$ are the color electric and color magnetic fields in Minkowski space, $A$ is the integration of the action density over the spatial volume.

On the L.H.S. of Eq. (44) if we write the $q\bar{q}$ colour field energy $E_0(r)$ in the form [9],

$$E_0(r) = V(r) + f(\beta)/a,$$

where $V(r)$ is the potential energy, $f/a$ the self-energy. Then one can obtain the following relation from Eq. (44) in the continuum limit,

$$A = \frac{\beta \partial a E_0(r)}{a} = \frac{\beta \partial a V(r)}{a} + \frac{\beta \partial f}{a \partial \beta}.$$  

This is just the Michael’s action sum rule, which can be obtained by subtracting Eq. (31) from Eq. (30) because of $A = E_{el} - E_{ma}$, if we assume that $V(r)$ scales and hence is independent of $\beta$.

2. Energy Sum Rules and Generalizations

To derive the energy sum rules in Eqs. (30) and (31) we need to study the color electric and color magnetic fields separately. Let us consider an asymmetric lattice with the time-spacing $a_t$, the spatial-spacing $a_s (= a)$ and the asymmetry $\xi = a_s/a_t$ [9]. The action for SU($N$) LGT becomes,

$$S_A = -\beta_t \sum \Box_t - \beta_s \sum \Box_s + \text{const.},$$

where $\Box_t$ is a plaquette with a time extent and $\Box_s$ is a space-like plaquette. In the continuum limit ($a_s, a_t \rightarrow 0$) the action must become the classical action,

$$S_A \rightarrow \frac{1}{4} \int d^4x (F_{\mu\nu}^c)^2,$$

In this limit the time-like plaquette becomes

$$\beta_t \Box_t = \beta_t \Box_{j4} = \beta_t \frac{1}{N} tr(e^{ia_s g_t F_{j4}})$$

$$\approx \beta_t (1 - \frac{1}{4N} a_t^2 a_s^2 g_t^2 (F_{j4}^c)^2),$$
where $N$ is for SU($N$). $j = 1, 2, 3$ and $c$ is the colour index. To get the correct continuum action, Eq. (49), we require that in Eq. (50),

$$\frac{\beta_t}{2N} a_t^2 a_s^2 g_t^2 = a_s a_t,$$

from which we obtain the relation between $\beta_t$ and $g_t$.

$$\beta_t = \frac{2N a_s}{g_t^2 a_t} = \frac{2N}{g_t^2} \xi. \quad (51)$$

Similarly one can obtain the relation between $\beta_s$ and $g_s$,

$$\beta_s = \frac{2N a_t}{g_s^2 a_s} = \frac{2N}{g_s^2} \xi^{-1}. \quad (52)$$

In the weak coupling limit, $\beta_s$, $\beta_t$ can be expanded in terms of the coupling $\beta$ of the corresponding symmetric lattice [9,24],

$$\beta_s \xi = \frac{2N}{g_s^2} = \beta(a) + 2N c_s(\xi) + O(\beta^{-1}), \quad (53a)$$

$$\beta_t \xi^{-1} = \frac{2N}{g_t^2} = \beta(a) + 2N c_t(\xi) + O(\beta^{-1}); \quad (53b)$$

where the coefficients $c_s(\xi)$ and $c_t(\xi)$ satisfy the conditions [9,25],

$$c_s(\xi)|_{\xi=1} = c_t(\xi)|_{\xi=1} = 0,$$

$$\frac{\partial}{\partial \xi} [c_s(\xi) + c_t(\xi)]|_{\xi=1} = (c'_s + c'_t)|_{\xi=1} = b_0 = 11N/48\pi^2. \quad (54)$$

where $c'_s$ ($c'_t$) denotes the $\xi$-derivative of $c_s$ ($c_t$).

With these definitions we can now study the color electric and magnetic fields of a $q\bar{q}$ sources. Again we consider a Wilson loop $W$ of the size $na_t$ in the time direction and $r$ in the space direction on the asymmetric lattice. We can obtain the following result in this case, if we repeat the steps that lead to Eq. (36),

$$\langle W \rangle = a_{00}^2 e^{-na_t E_0(r)}, \quad (as \ N_t, n \rightarrow \infty ) \quad (55)$$

where $E_0(r)$ is the ground-state energy of the $q\bar{q}$ pair, and $\langle W \rangle$ is defined by the partition function formalism in Eq. (33), with the action $\beta S'$ replaced by $S_A$ in Eq. (48).
By taking the derivatives of Eq. (55) with respect to $\beta_s$ and $\beta_t$ respectively, one can get the following relations similar to Eq. (44),

\[ -\frac{\partial a_t E_0(r)}{\partial \beta_t} = (\sum_s \Box_t)_{r-0}, \]
\[ -\frac{\partial a_t E_0(r)}{\partial \beta_s} = (\sum_s \Box_s)_{r-0}. \] (56a)

where the sum $\sum_s$ is over the whole spatial volume as before. The R.H.S. of Eqs. (56a) and (56b) become the total color electric and color magnetic energies of the $q\bar{q}$ pair respectively in the continuum limit ($a_s, a_t \to 0$). On the symmetric lattice ($\xi = 1$) one has,

\[ \lim_{a \to 0} (\sum_s \Box_t)_{r-0} = \frac{1}{\beta} \sum_s \frac{1}{2} a^4 (E^2)_{r-0}, \] (57a)
\[ \lim_{a \to 0} (\sum_s \Box_s)_{r-0} = -\frac{1}{\beta} \sum_s \frac{1}{2} a^4 (B^2)_{r-0}; \] (57b)

where $E$ and $B$ are the color electric and magnetic fields in Minkowski space.

To evaluate the L.H.S. of Eqs. (56a) and (56b) we need to resort following relations which relate the quantities on the asymmetric lattice with the quantities on the equivalent symmetric lattice,

\[ \left( \frac{\partial F(\beta_s, \beta_t)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial F}{\partial \beta} + \frac{1}{\beta} \frac{\partial F}{\partial \xi} \right)_{\xi=1}, \] (58a)
\[ \left( \frac{\partial F(\beta_s, \beta_t)}{\partial \beta_s} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial F}{\partial \beta} - \frac{1}{\beta} \frac{\partial F}{\partial \xi} \right)_{\xi=1}; \] (58b)

where $F(\beta_s, \beta_t)$ is a function of $\beta_s$, $\beta_t$. The proof is given in the Appendix.

Since we can write the energy $E_0(r)$ of Eqs. (58a) and (58b) in the form, $E_0(r) = V(r) + f(\beta)/a$, from Eq. (46), then the L.H.S. of Eq. (56a) becomes,

\[ \left( \frac{\partial a_t E_0(r)}{\partial \beta_t} \right)_{\xi=1} = \left( \frac{\partial a_t V(r)}{\partial \beta_t} \right)_{\xi=1} + \left( \frac{\partial (f/\xi)}{\partial \beta_t} \right)_{\xi=1}. \] (59)

Applying Eq. (58a) to above equation, and using the fact that the potential $V(r)$ is independent of $\beta$ in the limit of infinite lattice sizes and infinitesimal lattice spacing ($a \to 0$), one can obtain,
\[
\left( \frac{\partial a_t V(r)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial a_t V(r)}{\partial \beta} + \frac{1}{\beta} \frac{\partial a_t V(r)}{\partial \xi} \right)_{\xi=1} \\
= \frac{1}{2} V(r) \left( \frac{\partial a}{\partial \beta} - \frac{a}{\beta} \right),
\]
where we have used \( \left( \frac{\partial a}{\partial \beta} \right)_{\xi=1} = \frac{\partial a}{\partial \beta} \) and \( \left( \frac{\partial a}{\partial \xi} \right)_{\xi=1} = -a \). Also the second term on the L.H.S. of Eq. (59), \( \left( \frac{\partial (f/\xi)}{\partial \beta_t} \right)_{\xi=1} \), can be evaluated as,
\[
\left( \frac{\partial (f/\xi)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial f(\beta)}{\partial \beta} - \frac{f(\beta)}{\beta} \right).
\]
where the self-energy \( f(\beta)/a \) depends on \( \beta \) and \( a \). Substituting Eqs. (60) and (61) into Eq. (59) yields
\[
\left( \frac{\partial a_t E_0(r)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} V(r) \left( \frac{\partial a}{\partial \beta} - \frac{a}{\beta} \right) + \frac{1}{2} \left( \frac{\partial f(\beta)}{\partial \beta} - 1 \right).
\]
Similarly, the L.H.S. of Eq. (56b) can be evaluated by applying Eq. (58b), the result is,
\[
\left( \frac{\partial a_t E_0(r)}{\partial \beta_s} \right)_{\xi=1} = \frac{1}{2} V(r) \left( \frac{\partial a}{\partial \beta} + \frac{a}{\beta} \right) + \frac{1}{2} \left( \frac{\partial f(\beta)}{\partial \beta} + \frac{f(\beta)}{\beta} \right).
\]
From Eqs. (56a), (56b), (57a), (57b) and (62), (63) one can get the energy sum rules in Eqs. (30) and (31).

Now we proceed to consider the generalization of these sum rules to the finite temperature case. In the limit of finite temperature \( T \) and infinite volume the potential \( V(r) \) would be a function of \( r \) and \( T \) (or \( \beta \)) because of \( T = 1/N_t a(\beta) \). For example, in the confined phase \( V(r) \sim \kappa r \), with the string tension \( \kappa = \kappa(\beta) \) for fixed lattice size \( N_t \), as discussed in Sec. III C. In this case one can write, \( V(r) = V(r, \beta) \). Then Eq. (60) should be rewritten as,
\[
\left( \frac{\partial a_t V(r, \beta)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial a V(r, \beta)}{\partial \beta} - \frac{a}{\beta} V(r, \beta) \right).
\]
Similarly one has
\[
\left( \frac{\partial a_t V(r, \beta)}{\partial \beta_s} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial a V(r, \beta)}{\partial \beta} + \frac{a}{\beta} V(r, \beta) \right).
\]
The self-energy part \( f(\beta)/a \) does not change in this case. Therefore, at finite temperature Michael sum rules in Eqs. (30) and (31) should be modified as following,
These modified sum rules can be obtained from the original ones in Eqs. (30) and (31) by the replacement,

\[ V(r) \frac{\partial a}{\partial \beta} \rightarrow \frac{\partial [aV(r, \beta)]}{\partial \beta}. \]  

We expect that the modified sum rules would account for the finite temperature effects on the \( qq \bar{q} \) system. Although Eqs. (66) and (67) are derived by considering the Wilson loop representation of the \( qq \bar{q} \) pair, we expect that they can be applied in describing the flux distributions \( f'_{\mu \nu} \) in Eq. (19), which involves the Polyakov loops, because both the Wilson loop, \( W \), and Polyakov loops, \( P(0)P^\dagger(r) \), represent a \( q \bar{q} \) pair.

**B. Analysis of Flux Data**

We calculated the flux distributions \( f'_{\mu \nu} \) of Eq. (19) on various lattices, which cover the regions of both confined and unconfined phases. The detailed analysis of the \( qq \bar{q} \) flux distributions is presented elsewhere [5]. In this part we shall compare our flux data with the predictions of modified Michael sum rules in Eqs. (66) and (67).

As we mentioned in Sec. II C, we used the multihit technique [6] in the flux measurements. The flux data is only good when the test charge (plaquette) is away from the \( qq \bar{q} \) sources (Polyakov loops). Thus we only consider the flux data on the middle transverse slice between the \( qq \bar{q} \) pair at large separations (i.e., \( r \geq 3a \)).

In the confined phase string formation is expected to occur. When the separation, \( r \), of a \( qq \bar{q} \) pair is large enough, the flux energy stored in the flux tube per unit length between the \( qq \bar{q} \) should equal to the string tension \( \kappa \). In this case one can obtain the color electric and magnetic energy of the flux tube per unit length from Eqs. (66) and (67),
The corresponding flux action is

$$\sigma_A = \sigma_{el} - \sigma_{ma} = -\frac{\beta}{a} \frac{\partial [a\kappa]}{\partial \beta},$$

(71)

which is the integration of the action density on the transverse plane. And the corresponding flux energy is

$$\sigma_E = \sigma_{el} + \sigma_{ma} = \frac{dV(r)}{dr} = \kappa(\beta),$$

(72)

which agrees with our expectation for string formation.

Eq. (71) can also be obtained from the action sum rule of Eq. (47). The R.H.S. of Eq. (71) is, $\frac{\beta}{a} \frac{\partial \ln a}{\partial \ln \beta} = \kappa \frac{\partial \ln a}{\partial \ln \beta} + \beta \frac{\partial \kappa}{\partial \beta}$. The first term can be estimated from the scaling relation of Eq. (15), e.g., $\frac{\partial \ln a}{\partial \ln \beta} \sim -8$ for $\beta \sim 2.4$, which is consistent with the flux measurement results of Ref. [10]. The second term $\beta \frac{\partial \kappa}{\partial \beta}$ can be estimated from the fitting function in Eq. (28), and it is negative because $\kappa$ decreases with $\beta$ for fixed lattice size $N_t$, as shown in Eq. (25). So Eq. (71) implies that $\sigma_A \gg \sigma_E = \kappa$ for cases we consider. Also, Eqs. (69) and (70) show that the electric part of the flux energy $\sigma_{el}$ is positive, but the magnetic part $\sigma_{ma}$ negative, their magnitudes are of the same order with $\sigma_{el}$ slightly larger. These predictions can be summarized as below,

$$\sigma_{el} > 0 \quad \text{and} \quad \sigma_{ma} < 0,$$

$$\sigma_E = \sigma_{el} + \sigma_{ma} = \kappa > 0,$$

$$\sigma_A \gg \sigma_E;$$

(73)

where these flux energies (action) are defined in the confined phase.

From our flux data we can calculate the values of $\sigma_{el}$, $\sigma_{ma}$, $\sigma_E$ and $\sigma_A$, which are the energies (action) stored in the center slice per unit length between the $q\bar{q}$ pair. We find
that our data agree with the predictions of Eq. (73) qualitatively. However, it is difficult to measure $\sigma_E$ accurately because of the strong cancellation between two terms, $\sigma_{el}$ and $\sigma_{ma}$, which have opposite signs. This problem was also discussed by Refs. [9,10].

In the following we proceed to study the prediction of the action sum rule in Eq. (71). As we mentioned above, the R.H.S. of Eq. (71) contains two terms, the first term, $\kappa \frac{\partial \ln a}{\partial \ln \beta}$, was predicted by the original Michael sum rules in Eqs. (30) and (31), the second one, $\beta \frac{\partial \kappa}{\partial \beta}$, is a new term, which is only predicted by the modified sum rules of Eqs. (66) and (67). We expect that the second term describes finite temperature effects.

From the scaling relation $a(\beta)$ of Eq. (15) one can estimate the quantity,

$$- \frac{\partial \ln a}{\partial \ln \beta} = - \frac{51}{121} + \frac{3\pi^2}{11} \beta - \frac{d_2 + 2d_3\beta}{\Lambda L^{-1}(\beta)} \beta,$$

with $\Lambda L^{-1}(\beta)$ defined in Eq. (15). And by using the fitting function $\kappa(\beta)$ in Eq. (28) one can calculate the derivative,

$$\frac{\partial \kappa}{\partial \beta} = -\kappa_0 \frac{\beta_c}{\beta_c} \delta (1 - \frac{\beta}{\beta_c})^{\delta-1},$$

with the constants $\kappa_0$ and $\delta$ given by Eqs. (14) and (29).

By similar considerations that lead to Eq. (71), one can also obtain the prediction about $\sigma_A$ from the original Michael sum rules in Eqs. (30) and (31). The result is

$$(\sigma_A)_0 = -\kappa \frac{\partial \ln a}{\partial \ln \beta},$$

which is just the first term of the R.H.S. in Eq. (71).

In Table IV we list the predictions of Eq. (71) obtained by substituting Eqs. (74) and (75), and the measured $\sigma_A$ data in the confined phase. Since in the confined phase the value of $\sigma_A$ should not change with the $q\bar{q}$ separation $r$ in the limit of $r \to \infty$, we then choose the $\sigma_A$ data at a moderate value of $r$ as the asymptotic value of $\sigma_A$ ($r \to \infty$), because error bars are large at very large $r$. In Table IV most $\sigma_A$ data were calculated from the flux measurements of $q\bar{q}$ pair at $r = 4a$, in few cases we choose the data at $r = 3a$ or $5a$, depending on the quality of data.
In Fig. 7 we plot the predictions of Eqs. (71) and (76) respectively, and compare them with the measured $\sigma_A$ data on lattices of $N_t = 4$. From this figure one can see that the measured data are consistent with the prediction of Eq. (71), but have large difference from the prediction of Eq. (76). Especially, in the transition region ($\beta \approx \beta_c$) Eq. (76) predicts that $\sigma_A$ approaches zero as $\beta \to \beta_c$. However, our $\sigma_A$ data have large values in this region, which agree with the prediction of Eq. (71). As the temperature decreases ($T \to 0$ or $\beta \to 0$), both predictions coincide. However, it is difficult to obtain data of clear signal in the small $\beta$ region (i.e., $\beta < 2.25$).

We also notice that in Fig. 7 the $\sigma_A$ data in the confined region ($\beta < \beta_c$) have some discrepancies from the prediction of Eq. (71). This may be due to finite-size effects and finite lattice spacing effects of lattices, and to the large fluctuations in the confined phase because confinement corresponds to disorder.

To compare the behaviors of $\sigma_A$ in both phases, in Fig. 7 we also plot the $\sigma_A$ data in the unconfined region (i.e., $\beta > \beta_c$). These data were measured at large $q\bar{q}$ separations, i.e., $r = 6a$, which still have clear signals, as shown in Table V. Since in the unconfined phase there is no string formation, one expects that $\sigma_A$ vanishes at large $r$. From this figure one can see that near the transition point, i.e., $\beta \sim 2.30$, the $\sigma_A$ data in the unconfined region decrease rapidly with $\beta$, and becomes very small beyond the transition region. This agrees with the expectation. The fact that our $\sigma_A$ data in the unconfined region do not vanish may be due to the following factors, in the transition region finite-size effects are large, beyond this region the contribution from the self-energy of the $q\bar{q}$ pair still exists, because the $q\bar{q}$ separation, $r = 6a$, is not large enough to approach the asymptotic region.

V. SUMMARY

We studied the SU(2) finite temperature phase transition by Monte Carlo simulations. To transform the measurements from lattice units to physical units the lattice asymptotic scaling relation $a(\beta)$ is extrapolated into the non-perturbative region. The behavior of string
tension with temperature is also studied. We find that the string tension data agree very well with the fitting function, \( \kappa(T) = \kappa_0(1 - \frac{T}{T_c})^\alpha \) for \( T < T_c \).

We also measured the flux distribution of the \( q\bar{q} \) pair at various temperatures. To check Michael sum rules with the flux data, a complete derivation of the sum rules are presented, and a generalization of the sum rules is suggested to account for the finite temperature effects. We found that our flux data are consistent with the prediction of the generalized sum rules. Our data shows explicitly that the \( q\bar{q} \) flux distribution has different behaviors in the two phases. In the confined phase the asymptotic value of the center slice action, \( \sigma_A (r \to \infty) \), has large values. However, across the transition region, \( \sigma_A \) becomes very small in the unconfined phase. This agrees with our expectation that string formation occurs in the confined phase, but disappears in the unconfined phase.

**ACKNOWLEDGMENTS**

We wish to thank C. Michael, B.A. Berg, J. Wosiek, D.A. Browne and V. Singh for many fruitful discussions on this problem. This research was supported by the U.S. Department of Energy under Grant No. DE-FG05-91ER40617.
REFERENCES

[1] A.M. Polyakov, Phys. Lett. 72 B, 477 (1978); L. Susskind, Phys. Rev. D 20, 2610 (1979).

[2] J. Kuti, et al, Phys. Lett. 98 B, 199 (1981).

[3] J. Engels, F. Karsch and H. Satz, Nucl. Phys. B315, 419 (1989).

[4] H.J. Rothe, Lattice Gauge Theories-An Introduction, World Scientific Publishing Co. 1992; and references therein.

[5] Y. Peng and R.W. Haymaker, LSU report, LSUHE No. 141-1993, to be appear in Phys. Rev. D 47, Num. 11 (1993).

[6] J. Wosiek and R.W. Haymaker, Phys. Rev. D 36, 3297 (1987).

[7] R.W. Haymaker, Y. Peng, V. Singh and J. Wosiek, Nucl. Phys. B (Proc. Suppl.) 17, 558 (1990).

[8] R. Sommer, Nucl. Phys. B306, 181 (1988).

[9] C. Michael, Nucl. Phys. B280, 13 (1987).

[10] R.W. Haymaker and J. Wosiek, Phys. Rev. D 43, 2676 (1991).

[11] M. Creutz, Quarks, Gluons and Lattices, Cambridge University Press, 1983.

[12] C.W. Bernard, Phys. Rev. D 9, 3312 (1974); L. Dolan and R. Jackiw, ibid. 9, 3320 (1974).

[13] G. Curci and R. Tripiccione, Phys. Lett. 151 B, 145 (1985); B.A. Berg, et al, Phys. Lett. 209 B, 319 (1988); B.A. Berg, et al, Phys. Rev. Lett. 62, 2433 (1989).

[14] L.D. McLerran and B. Svetitsky, Phys. Rev. D 24, 450 (1981); and Phys. Lett. 98 B, 195 (1981).

[15] J. Engels, J. Fingberg and M. Weber, Nucl. Phys. B332, 737 (1990).
[16] J. Engels and V.K. Mitrjushkin, Phys. Lett. 282 B, 415 (1992).

[17] B.A. Berg, A.H. Billoire, Phys. Rev. D 40, 550 (1989).

[18] B.A. Berg, A. Billoire and R. Salvador, Phys. Rev. D 37, 3774 (1988).

[19] S. Perantonis, A. Huntley and C. Michael, Nucl. Phys. B326, 544 (1989).

[20] C. Michael, Phys. Lett. 283 B, 103 (1992).

[21] L.D. Debbio, et al, CERN-TH. 6282/91, IFUP-TH 37/91.

[22] J. Engels, et al, Nucl. Phys. B280, 577 (1987).

[23] J.B. Kogut, Rev. Mod. Phys. 51, 659 (1979); and Rev. Mod. Phys. 55, 775 (1983).

[24] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B193, 210 (1981).

[25] F. Karsch, Nucl. Phys. B205, 285 (1982).
To prove Eqs. (58a) and (58b), we notice that Eqs. (53a) and (53b) implies that $\beta_s$, $\beta_t$ and $\beta$, $\xi$ are two equivalent sets of variables, that is, $F(\beta_s, \beta_t) = F(\beta_s(\beta, \xi), \beta_t(\beta, \xi))$. Then we take the partial derivatives of $F$ with respect to $\beta$ and $\xi$ respectively, one has

$$
\left( \frac{\partial F}{\partial \beta} \right)_\xi = \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_s(\beta, \xi)} \frac{\partial \beta_s}{\partial \beta} + \left( \frac{\partial F}{\partial \beta} \right)_{\beta_t(\beta, \xi)} \frac{\partial \beta_t}{\partial \beta}, \quad (A1)
$$

$$
\left( \frac{\partial F}{\partial \xi} \right)_\beta = \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_s(\beta, \xi)} \frac{\partial \beta_s}{\partial \xi} + \left( \frac{\partial F}{\partial \beta} \right)_{\beta_t(\beta, \xi)} \frac{\partial \beta_t}{\partial \xi}. \quad (A2)
$$

From Eqs. (53a) and (53b) one can get that, when $\beta$ is large (i.e., $\beta \to \infty$),

$$
\left( \frac{\partial \beta_t}{\partial \beta} \right)_{\xi=1} = \left( \frac{\partial \beta_s}{\partial \beta} \right)_{\xi=1} = 1;
$$

$$
\left( \frac{\partial \beta_s}{\partial \xi} \right)_{\xi=1} = -\beta + 2Nc'_s|\xi=1,
$$

$$
\left( \frac{\partial \beta_t}{\partial \xi} \right)_{\xi=1} = \beta + 2Nc'_t|\xi=1. \quad (A3)
$$

where $N$ denotes for $SU(N)$, and $c'$ the $\xi$-derivative of $c$. Substituting Eq. (A3) into Eqs. (A1) and (A2) yields

$$
\left( \frac{\partial F}{\partial \beta} \right)_{\xi=1} = \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_s(\beta, \xi)} \frac{\partial \beta_s}{\partial \beta} + \left( \frac{\partial F}{\partial \beta_t} \right)_{\beta_t(\beta, \xi)} \frac{\partial \beta_t}{\partial \beta}, \quad (A4)
$$

$$
\left( \frac{\partial F}{\partial \xi} \right)_{\xi=1} = \left[ \frac{\partial F}{\partial \beta_t}(\beta + 2Nc'_t) + \frac{\partial F}{\partial \beta_s}(-\beta + 2Nc'_s) \right]_{\xi=1}. \quad (A5)
$$

F. Karsch has studied the coefficients $c_s$ and $c_t$ [25]. His results show that the derivatives $c'_s|\xi=1$, $c'_t|\xi=1$ vanish for sufficient large lattice coupling constant, $\beta$. For example, as $\beta \geq 2.2$, the values of $c'_s|\xi=1$ and $c'_t|\xi=1$ of $SU(2)$ LGT are about 0.1, which is much less than the values of $\beta$. For large $\beta$ cases Eqs. (A4) and (A5) become

$$
\left( \frac{\partial F}{\partial \beta} \right)_{\xi=1} = \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_s(\beta, \xi)} \frac{\partial \beta_s}{\partial \beta} + \left( \frac{\partial F}{\partial \beta_t} \right)_{\beta_t(\beta, \xi)} \frac{\partial \beta_t}{\partial \beta}, \quad (A6)
$$

$$
\left( \frac{\partial F}{\partial \xi} \right)_{\xi=1} = \beta \left( \frac{\partial F}{\partial \beta_t} \right)_{\beta_t(\beta, \xi)} - \beta \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_s(\beta, \xi)}. \quad (A7)
$$

After carrying out some simple algebra, we can obtain Eqs. (58a) and (58b).
TABLES

TABLE I. The correspondence of the lattice spacing $a$ and the coupling constant $\beta$ for SU(2) LGT, extracted from the string tension data of Refs. [8,19].

| $\beta$ | $a(\beta)$ (fm) | $\Lambda^{-1}_{L}$ (fm) |
|---------|-----------------|-------------------------|
| 2.22    | 0.1981 (61)     | 27.41 (84)              |
| 2.30    | 0.1616 (47)     | 27.32 (79)              |
| 2.40    | 0.1210 (5)      | 26.30 (11)              |
| 2.50    | 0.0843 (8)      | 23.57 (22)              |

TABLE II. The raw string tension data $\kappa$ measured in lattice units on lattices of the size, $4 \times 9^2 \times 65$, $4 \times 11^2 \times 65$, $6 \times 7^2 \times 65$ and $6 \times 11^2 \times 37$.

| $N_t = 4$ | $4 \times 9^2 \times 65$ | $4 \times 11^2 \times 65$ | $N_t = 6$ | $6 \times 7^2 \times 65$ | $6 \times 11^2 \times 37$ |
|-----------|--------------------------|--------------------------|-----------|--------------------------|--------------------------|
| $\beta$   | $\sqrt{\kappa a}$      | $\sqrt{\kappa a}$      | $\beta$   | $\sqrt{\kappa a}$      | $\sqrt{\kappa a}$      |
| 2.25       | 0.301 (38)               | 0.313 (32)               | 2.30       | 0.310 (48)               | 0.377 (27)               |
| 2.28       | 0.242 (33)               | 0.243 (29)               | 2.36       | 0.248 (21)               | 0.284 (13)               |
| 2.29       | 0.224 (17)               | 0.181 (33)               | 2.40       | 0.172 (30)               | 0.217 (17)               |
| 2.30       | 0.216 (19)               | 0.203 (17)               | 2.42       | 0.200 (17)               | 0.177 (41)               |
TABLE III. The string tension data $\kappa$ measured in physical units on lattices of the size, $4 \times 9^2 \times 65$, $4 \times 11^2 \times 65$, $6 \times 7^2 \times 65$ and $6 \times 11^2 \times 37$, which were calculated from the data in Table II by using the scaling relation $a(\beta)$ in Eq. (15).

| $N_t$ | 4 $\times 9^2 \times 65$ | 4 $\times 11^2 \times 65$ |
|-------|-----------------|-----------------|
| $\beta$ | $T \Lambda_L^{-1}$ | $T$ (1/fm) | $\kappa$ (GeV/fm) | $\kappa$ (GeV/fm) |
| 2.25 | 37.29 | 1.271 | 0.46 (12) | 0.50 (10) |
| 2.28 | 40.20 | 1.401 | 0.36 (10) | 0.37 (9) |
| 2.29 | 41.22 | 1.447 | 0.33 (5) | 0.22 (8) |
| 2.30 | 42.27 | 1.494 | 0.33 (6) | 0.29 (5) |

$N_t = 6$

| $\beta$ | $T \Lambda_L^{-1}$ | $T$ (1/fm) | $\kappa$ (GeV/fm) | $\kappa$ (GeV/fm) |
|-------|-----------------|-----------------|
| 2.30 | 28.18 | 0.996 | 0.68 (21) | 1.00 (14) |
| 2.36 | 32.76 | 1.213 | 0.64 (11) | 0.84 (8) |
| 2.40 | 36.23 | 1.386 | 0.40 (14) | 0.64 (10) |
| 2.42 | 38.10 | 1.482 | 0.62 (11) | 0.49 (23) |

TABLE IV. The predictions from Eq. (71) and the data of center slice action $\sigma_A$, which were measured on lattices $4 \times 9^2 \times 65$, $4 \times 11^2 \times 65$, $6 \times 7^2 \times 65$ and $6 \times 11^2 \times 37$. The quantities $\kappa(\beta)$, $\beta \frac{\partial \kappa}{\partial \beta}$, $\kappa \frac{\partial a}{\partial \ln \beta}$ and $\sigma_A$ have the physical unit GeV/fm. The values of $\kappa(\beta)$ were estimated from Eq. (28). Near the transition point (i.e., $\beta_c \sim 2.30$ for $N_t = 4$) no stable prediction is obtained.

| $N_t = 4$ |
|-------|-----------------|-----------------|
| $\beta$ | $\kappa(\beta)$ | $-\kappa \frac{\partial \sigma_A}{\partial \ln \beta}$ | $-\beta \frac{\partial \kappa}{\partial \beta}$ | $-\kappa \frac{\partial a}{\partial \ln \beta}$ | $-\beta \frac{\partial \kappa}{\partial \beta}$ | $(\sigma_A)_{4 \times 9^2 \times 65}$ | $(\sigma_A)_{4 \times 11^2 \times 65}$ |
| 2.25 | 0.42 | 3.03 (47) | 4 (1) | 7 (2) | 12.18 (23) | 8.70 (22) |
| 2.28 | 0.34 | 2.50 (48) | 9 (3) | 12 (3) | 10.95 (14) | 9.97 (13) |
| 2.29 | 0.29 | 2.12 (47) | 17 (6) | 19 (7) | 10.44 (12) | 9.39 (10) |
TABLE V. The $\sigma_A$ data in the unconfined phase ($\beta < \beta_c$), which were measured on lattices, $4 \times 9^2 \times 65$ and $4 \times 11^2 \times 65$. The data are in the physical unit GeV/fm.

| $\beta$     | 2.30 | 2.32 | 2.34 | 2.36 | 2.40 |
|-------------|------|------|------|------|------|
| $\langle \sigma_A \rangle_{4 \times 9^2 \times 65}$ | 11.77 (11) | 10.40 (9) | 1.78 (4) | 1.28 (4) |
| $\langle \sigma_A \rangle_{4 \times 11^2 \times 65}$ | 11.03 | 3.81 (4) | 1.77 (3) | 1.32 (4) |
FIGURES

FIG. 1. Monte Carlo data for $\langle |P| \rangle$ vs. $\beta$ with the standard Wilson action, calculated from lattices $4 \times 9^2 \times 65$ (circles), $4 \times 11^2 \times 65$ (squares), $6 \times 7^2 \times 65$ (triangles) and $6 \times 11^2 \times 37$ (diamonds).

FIG. 2. Monte Carlo data of $\langle P(0)P(r) \rangle$ vs. $r/a$ in the confined phase, calculated on the lattice, $4 \times 11^2 \times 65$ with $\beta = 2.25$ (squares), $\beta = 2.28$ (triangles) and $\beta = 2.29$ (diamonds). These data are all in the confined region, with $\beta < \beta_c$ ($\beta_c \sim 2.30$ for $N_t = 4$). The plot is in a frame with logarithmic $y$ axis.

FIG. 3. Monte Carlo data of $\langle P(0)P(r) \rangle$ vs. $r/a$ in the unconfined phase, calculated on the lattice $4 \times 9^2 \times 65$ with $\beta = 2.36$ (squares), and $\beta = 2.40$ (triangles).

FIG. 4. The plot of $\kappa$ vs. $T$ (1/fm) near the transition point $T_c \sim 1.487$ (1/fm). Data were calculated on lattices, $4 \times 9^2 \times 65$ (circles), $4 \times 11^2 \times 65$ (squares) and $6 \times 7^2 \times 65$ (triangles). The solid line is the fitting function in Eq. (26) with $\alpha = 0.35 \pm 0.04$. The string tension $\kappa$ is in the physical unit GeV/fm.

FIG. 5. $\kappa$ vs. $\beta$ for $N_t = 4$, the transition point is chosen as $\beta_c = 2.2985$. Data were calculated on lattices, $4 \times 9^2 \times 65$ (squares), $4 \times 11^2 \times 65$ (triangles). The solid line is the fitting function in Eq. (28) with $\delta = 0.22 \pm 0.03$. The string tension $\kappa$ is in the physical unit GeV/fm.

FIG. 6. The Wilson loop $W$ of the temporal size $na$ and the spatial size $r$. The plaquette $P_1$ is outside the Wilson loop $W$, and the plaquette $P_2$ is inside the Wilson loop $W$.

FIG. 7. The plot of the predictions of $\sigma_A$ vs. $\beta$ from Eq. (71) (solid lines), and Eq. (76) (dashed lines) in the confined region ($\beta < \beta_c$). The two solid lines represent the upper and lower limits predicted by Eq. (71). The data were measured on lattices, $4 \times 9^2 \times 65$ (squares) and $4 \times 11^2 \times 65$ (triangles). For comparison the data in the unconfined region ($\beta > \beta_c$) are also shown, which were measured on the same lattices, $4 \times 9^2 \times 65$ (circles) and $4 \times 11^2 \times 65$ (diamonds). Here $\sigma_A$ has the physical unit GeV/fm, and the transition point $\beta_c$ is indicated by the up arrow.
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9305013v1
This figure "fig3-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9305013v1
This figure "fig4-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9305013v1
This figure "fig5-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9305013v1