Homomorphically Encrypted Linear Contextual Bandit

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Abstract

Contextual bandit is a general framework for online learning in sequential decision-making problems that has found application in a large range of domains, including recommendation system, online advertising, clinical trials and many more. A critical aspect of bandit methods is that they require to observe the contexts –i.e., individual or group-level data– and the rewards in order to solve the sequential problem. The large deployment in industrial applications has increased interest in methods that preserve the privacy of the users. In this paper, we introduce a privacy-preserving bandit framework based on asymmetric encryption. The bandit algorithm only observes encrypted information (contexts and rewards) and has no ability to decrypt it. Leveraging homomorphic encryption, we show that despite the complexity of the setting, it is possible to learn over encrypted data. We introduce an algorithm that achieves a \(\tilde{O}(\sqrt{T})\) regret bound in any linear contextual bandit problem, while keeping data encrypted.

1 Introduction

Contextual bandits have become a key part of several applications such as marketing, healthcare and finance where it is a desirable to provide a personalized service [Bouneffouf et al. 2017; Sawant et al. 2018; Bastani & Bayati 2020]. Existing algorithms depend on users’ features, the “contexts”, to tailor their recommendations. In many cases, they may disclose sensitive information, as personal (e.g., age, gender, etc.) or geolocalized features, commonly used in recommendation systems. Privacy awareness has increased over years and users are less willing to disclose information and are more and more concerned about how personal data are used. To preserve privacy, many applications outside recommendation have adopted end-to-end encryption to guarantee that data is readable only by the users.

In this paper, we introduce and study the setting of encrypted linear contextual bandit. At each round, the bandit algorithm observes encrypted features, pulls an arm and observe an encrypted reward, that is used to improved the quality of recommendations. Using asymmetric encryption techniques (e.g., homomorphic encryption) allow to obtain end-to-end encryption since the algorithm has no ability to decrypt users’ data (it does not know the secure key). The performance of the learning agent is evaluated in terms of regret w.r.t. pulling the optimal arm at each round. In this setting, we address the following question: is it possible to achieve sub-linear regret when contexts and rewards are encrypted?

Related work. To prevent information leakage, the bandit literature has mainly focused on Differential Privacy (DP) (e.g., Tossou & Dimitrakakis 2016; Shariff & Sheffet 2018). While standard \((\epsilon, \delta)\)-DP enforces statistical diversity of the output of an algorithm, it does not provide guarantees on the security of user data that can be accessed directly by the algorithm. A stronger privacy notion, called local DP, requires data being privatized before being accessed by the algorithm. While it may be conceptually similar to encryption, i) it does not provide the same security guarantee as encryption (having access to a large set of samples may allow some partial denoising [Cheu et al. 2019]); and ii) it has a large impact on the regret of the algorithm. For example, Zheng et al. [2021] have recently analyzed \(\epsilon\)-LDP in contextual linear bandit and derived an algorithm with \(O(T^{3/4}/\epsilon)\) regret bound to be compared with \(\tilde{O}(\sqrt{T})\) regret of non-private algorithms.

In federated learning (a.k.a., collaborative multi-agent), Wang et al. [2020]; Zhu et al. [2020] have shown how the DP and local DP guarantee can provide a higher level of privacy for less impact on the regret using
collaboration between users. Secure Multi-Party Computation (MPC) (e.g., [Damgard et al., 2011]) is another collaborative approach to privacy-preserving machine learning that allows to divide computations between parties, and guarantees it is not possible for any of them to learn anything about the others. Recently, [Hannun et al., 2019] have empirically investigated using MPC for bandits. Their setting is different than the one considered here as they assume that each party provides a subset of the features observed at each round.

Finally, Homomorphic Encryption (HE) (e.g., [Halevi, 2017]) aims at providing a set of tools to perform computation on encrypted data, enabling to outsource computations to potentially untrusted third parties (in our setting the bandit algorithm) since data cannot be decrypted. To the best of our knowledge, HE has been only used by [Ciucanu et al., 2019, 2020] to encrypt rewards in bandit problems. Their setting is different than the one considered here and inherently simpler: i) they do not consider contextual problem; ii) they allow the use of a trusted party that has the ability to decrypt data. In particular, ii) makes the design of the algorithm much easier but requires users to trust the third party which, in turn, can lead again to privacy/security concerns.

**Contributions.** Our main contributions can be summarized as follows: 1) We introduce and formalize the problem of secure contextual bandit with asymmetric key encryption. 2) Leveraging ideas of OFUL (Abbasi-Yadkori et al., 2011) and HE, we derive a bandit algorithm for learning over encrypted data. We prove a $O(\sqrt{T})$ regret bound, showing that i) is possible to learn with encrypted information; ii) preserving users’ data security has negligible impact on the learning process. This is a large improvement w.r.t. $\varepsilon$-LDP which has milder security guarantees and where the best known bound is $O(T^{3/4}/\varepsilon)$. 3) We discuss practical limitations of HE and ways of improving the efficiency of the proposed algorithm. We also report preliminary numerical simulations confirming the theoretical results.

## 2 Preliminaries

In this section, we briefly recall the basics of linear contextual bandits and introduce the concepts of HE.

### 2.1 Linear Contextual Bandits

Consider a standard contextual linear bandit setting with $K \in \mathbb{N}$ arms and $T \in \mathbb{N}$ steps [Auer, 2002; Abbasi-Yadkori et al., 2011]. At each time $t \in [T] := \{1, \ldots, T\}$, a learner first observes a set of features $(x_{t,a})_{a \in [K]} \subseteq \mathbb{R}^d$, then selects an action $a_t \in [K]$ and finally observes a bounded reward: $r_t = \langle x_{t,a_t}, \theta^* \rangle + \eta_t$ where $\theta^* \in \mathbb{R}^d$ is an unknown feature vector and $\eta_t$ is a conditionally independent zero-mean, bounded noise.

We rely on the following standard assumption on the features and the unknown parameter $\theta^*$.

**Assumption 1.** There exists $S > 0$ such that $\|\theta^*\|_2 \leq S$ and there exists $L \geq 1$ such that, for all time $t \in [T]$ and arm $a \in [K]$, $\|x_{t,a}\|_2 \leq L$ and $r_t \in [-1, 1]$.

Unlike most of non-parametric contextual bandits settings (e.g., [Perchet et al., 2013]), we do not assume any prior on the distribution of the features $(x_{t,a})_a$. The performance of an algorithm $\mathfrak{A}$ over $T$ steps is measured by the pseudo-regret, the difference between the maximal (expected) cumulative reward collected by always selecting the same arm and the learner cumulative reward, defined formally as: $R_T = \sum_{t=1}^T \langle \theta^*, x_{t,a_t} \rangle - \langle \theta^*, x_{t,a_t} \rangle$ where $a_t^* = \arg\max_{a \in [K]} \langle \theta^*, x_{t,a} \rangle$.

At each time $t \in [T]$, the state of the art OFUL algorithm computes a ridge regression, $\theta_t = V_t^{-1} \sum_{i=1}^{t-1} x_{i,a_i} r_i$ where $V_t = \lambda I + \sum_{i=1}^{t-1} x_{i,a_i} x_{i,a_i}^\top$ is the $\lambda$-regularized design matrix. Then it sets its confidence set around $\theta^*$ as $C_t = \{\theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_t\|_{V_t} \leq \beta_t\}$, where $\|x\|_V = \sqrt{x^\top V x}$ is the weighted norm w.r.t. any positive matrix $V \in \mathbb{R}^{d \times d}$, and, for some confidence $\delta > 0$,

$$\beta_t = \sigma \sqrt{d \log(1 + L^2 t/\lambda) + \log(1/\delta)} + S \sqrt{\lambda}.$$  

Finally, OFUL plays the arm $a_t$ that maximizes $\max_{a \in [K]} \max_{\theta \in C_t} \langle \theta, x_{t,a} \rangle$. With high probability, the regret of OFUL is bounded as:

$$R_T \leq 4 \beta_T \sqrt{T L \log(\lambda + TL/d)}$$

(1)

with $\beta_T = S \sqrt{\lambda} \sigma \sqrt{2 \log(1/\delta)} + d \log(1 + TL/(\lambda \delta))$. From now on, we shall further assume that $\lambda < L^2$.

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1 As usual, boundedness can be relaxed to sub-Gaussianity.
2.2 Homomorphic Encryption (HE)

Homomorphic Encryption \cite{Halevi} is a probabilistic encryption method that enables third-party to perform some computations (addition and multiplication) on encrypted data. Formally, given two original messages $m_1$ and $m_2 \in \mathbb{R}$, the addition (resp. multiplication) of their encrypted versions (called ciphertexts) is equal to the encryption of their sum $m_1 + m_2$ (resp. $m_1 \times m_2$).\footnote{Most schemes also support Single Instruction Multiple Data (SIMD), i.e., the same operation on multiple data points in parallel.} We consider a generic homomorphic scheme that generates a public key $pk$ distributed widely, used to encrypt messages, and a private key $sk$ used to decrypt encrypted messages. This private key is, contrary to the public key, assumed to be kept private. This asymmetry allows agent A to encrypt messages with the public key of another agent B, outsource computations on those messages to a distant server, and send back the result to agent B. This process happens without anyone except agents A and B being able to access the content of the messages.

More precisely, we shall consider Leveled Fully Homomorphic encryption (LFHE) schemes for real numbers. This type of schemes supports both operations (addition and multiplication) but only for a fixed and finite number of operations, referred to as the depth. This limitation is a consequence of the fact that HE is a probabilistic approach. While noise injection allows to achieve high security, the fact that the noise level grows with operations limit the maximum number of operations that can be performed (e.g., \cite{Albrecht}). Exceeding the depth results in indecipherable ciphertexts (the encrypted message). For most LFHE schemes, the depth refers to the maximum number of multiplication, as the noise growth is more severe for multiplication than addition. The security of a LFHE schemes is defined by $\kappa \in \mathbb{N}$, usually $\kappa \in \{128, 192, 256\}$. A $\kappa$-bit level of security means that an attacker has to perform roughly $2^\kappa$ operations to break the encryption scheme, i.e., to decrypt ciphertext without the secret key. Formally, an LFHE scheme is defined by:

- $\text{KeyGen}(N, D, \kappa)$: takes as input the maximum depth $D$ (e.g., max. number of multiplications), a security parameter $\kappa$ and the degree $N$ of polynomials used as ciphertexts (App.~A). It outputs a secret key $sk$ and a public key $pk$.
- $\text{Enc}_{pk}(m)$: encrypts the message $m \in \mathbb{R}^d$ with the public key $pk$. The output is a ciphertext $ct$.
- $\text{Dec}_{sk}(ct)$: decrypts the ciphertext $ct$ of $m \in \mathbb{R}^d$ using the secret key $sk$. The output is the message $m$.
- $\text{Add}(ct_1, ct_2)$: for ciphertexts $ct_1$ and $ct_2$ of messages $m_1$ and $m_2$, it outputs ciphertext $ct$ of $m_1 + m_2$:
  \[
  \text{Dec}_{sk}\left(\text{Add}(\text{Enc}_{pk}(m_1), \text{Enc}_{pk}(m_2))\right) = m_1 + m_2.
  \]
- $\text{Mult}(ct_1, ct_2)$: similar to $\text{Add}$ but for ciphertexts $ct_1$ and $ct_2$ of messages $m_1$ and $m_2$ and output ciphertext $ct$ of $m_1 \cdot m_2$.

Minimizing the parameter $D$ is essential as it is the main bottleneck for performance, especially when generating keys. This computational cost comes from the fact that the higher $D$ is the bigger the dimension of the ciphertext space, namely $N \geq \mathcal{O}(\kappa D)$, hence greatly impacting the complexity of operations. The details of those functions is presented more thoroughly in App.~A. The CKKS algorithm \cite{Cheon} is an example of LFHE scheme that supports operations on real numbers. Other LFHE schemes (e.g., \cite{Fan, Brakerski, Brakerski14}) support only computations on integers. Admittedly, they could be used by taking into account the quantization error necessary to encode real numbers as large integers.

Other HE schemes. Most HE schemes \cite{Rivest, ElGamal, Paillier} are Partially Homomorphic because only support either additions or multiplications, but not both. Other schemes that support any number of operations are called Fully Homomorphic encryption (FHE) schemes. Most LFHE schemes can be turned into FHE schemes thanks to the bootstrapping technique introduced by \cite{Gentry, Boneh}. However, the computational cost of this technique is extremely high (and sometimes even prohibitive). It is thus important to optimize the design of the algorithm to minimize its multiplicative depth and (possibly) avoid bootstrapping \cite{Ducas, Zhao, Acar}.
In this paper, we consider the contextual bandit problem, where contexts and rewards are encrypted before being observed by the learner; this setting is called encrypted contextual bandit (Alg. 1). As before, we denote by \(\tilde{x}_{t,a}\) and \(\tilde{r}_t\) the values observed by the algorithm that are now encrypted versions of the original values \(x_{t,a}\) and \(r_t\). Formally, at time \(t \in [T]\), the learner \(A\) observes encrypted features \(\tilde{x}_{t,a} = \text{Enc}_{pk}(x_{t,a})\) for all actions \(a \in A\), and the encrypted reward \(\tilde{r}_t = \text{Enc}_{pk}(r_t)\) associated to the selected action \(a_t\). The learner may know the public key but not the secure key. As a consequence, the learner is not able to decrypt messages, thus it never observes the true contexts and rewards. This also means that all the internal statistics built by the bandit algorithm will be potentially encrypted. While several schemas can be used in this encrypted framework, in this paper we focus on homomorphic encryption. HE has been used in several fields (e.g. Kim et al., 2016; Archer et al., 2017; Kim et al., 2020) due to its ability of providing security, while allowing computation on encrypted data.

First, in this framework, it is important that the encryption scheme is probabilistic. In general, this is an essential property for asymmetric encryption since anyone can perform encryption via the public key. If the encryption was deterministic, that would allow to check ciphertexts and break security (Matthews, 1970). Practically, encrypting twice the same message with an HE scheme, gives two different results since HE schemas add noise (usually a discrete Gaussian noise) to the ciphertext after the encryption operation. Second, the learning algorithm may decide to save the encrypted data to try to break the encryption. HE relies on the hardness of the Learning With Error problem (Albrecht et al., 2015) to guarantee security. To break an HE scheme, an attacker has to perform at least \(2^\kappa\) operations to be able to differentiate noise from messages in a given ciphertext, see (Albrecht et al., 2018) for a survey on the actual number of operations needed to break HE schemes with most of the known attacks. Although collecting multiple ciphertexts may speedup some attacks, the security of any HE scheme is still guaranteed as long as long the number of ciphertexts observed by an attacker is polynomial in \(N\) (Regev, 2009).

3 First Steps Toward an Encrypted OFUL

From now on, we consider a contextual linear bandit problem with secure interaction protocol obtained through homomorphic encryption. Our objective is to design the first learning algorithm with sublinear regret for this setting. Since OFUL is a simple and efficient algorithm for learning in the non-secure protocol, we will leverage similar principles for the secure scenario. However, there are many, both theoretical and practical, challenges in adapting OFUL to the secure setting. The first one is the mere computations of optimistic estimates of the rewards that are compatible with the limited computational capabilities of a secure algorithm. Indeed, 1) computing ridge regressions (see Sec. 2.1) is extremely difficult with HE as finding the inverse of a matrix is not directly feasible for a leveled scheme (Esperança et al., 2017). 2) Similarly, computing the weighted norm \(\|x_{t,a}\|_{V_t^{-1}}\) requires to compute the square root of an encrypted number, a non-homomorphic operation. Finally, 3) computing the maximum element (or maximum index) of a list of encrypted values is non-trivial for the algorithm alone, as the algorithm cannot observe the values to compare. We thus need to design approximate algorithms leveraging only addition and multiplications for approximating these operations in the encrypted space.

This section aims at providing the basic components for building a bandit algorithm for the encrypted

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3 A trusted third party can generate a public and secret keys. The public key is distributed to encrypt contexts and the secret key is unknown to the learner (see Sec. 5).
linear bandit problem. As a first step, we address the issue mentioned above by modifying OFUL to leverage only homomorphic friendly operations. While not explicitly mentioned, all the values should be considered not encrypted. We provide a very incremental presentation in order to explain the difficulties of making OFUL homomorphic compatible. At the end (Sec. 3.4), we mention how to use these approximations in the encrypted scenario.

### 3.1 Homomorphic Friendly Ridge Regression

First step is to solve the ridge regression by leveraging an approximate inversion scheme. Given a matrix $V \in \mathbb{R}^d$ with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_d > 0$ and $c \in \mathbb{R}$ such that for all $i \in [d]$, $\lambda_i \in \text{Conv}(\{ z \in \mathbb{R} \mid |z-c| \leq c \}, 2c) \setminus \{0, 2c\}$, Guo & Higham (2006) showed that computing $X_{k+1} = 2X_k - X_k^2V$ with $X_0 = \frac{1}{2}I_d$ converges quadratically to $V^{-1}$. However, this algorithm is not numerically stable, contrary to the following one Guo & Higham (2006), initialized at $X_0 = \frac{1}{2}I_d$ and $M_0 = \frac{1}{2}V$:

$$X_{k+1} = X_k(2I_d - M_k), \quad M_{k+1} = (2I_d - M_k)M_k.$$  

Proposition 1 gives the speed of convergence for any symmetric positive definite matrix; its proof is delayed to App. B.1.

**Proposition 1.** Given a symmetric positive definite matrix $V \in \mathbb{R}^{d \times d}$, $c \geq \text{Tr}(V)$ and a precision level $\varepsilon > 0$, the iterate in (2) satisfies $\| X_k - V^{-1} \| \leq \varepsilon$ for any $k \geq k_1(\varepsilon)$ with $k_1(\varepsilon) = \frac{1}{\lambda^2} \ln \left( \frac{\ln(\lambda) + \ln(\varepsilon)}{\ln(1 - \frac{1}{2})} \right)$, where $\lambda \leq \lambda_d$ is a lower bound to the minimal eigenvalue of $V$ and $\| \cdot \|$ is the matrix spectral-norm.

We aim at computing the inverse of the regularized design matrix $V_t$. Therefore, $\lambda_d \geq \lambda > 0$ and by setting $c = \lambda_d + L^2t$ it holds that $c \geq \text{Tr}(V_t) \geq \max_i \{ \lambda_i \}$, for any step $t \in [T]$ Abbasi-Yadkori et al. (2011). Iterations (2) are compatible with HE as defined solely in terms of additions and (matrix) multiplications.

### 3.2 Computing Optimistic Ellipsoid Width.

At each iteration $t$, the iterative procedure in (2) computes an $\varepsilon_t$-approximation $A_t := X_{k_1(\varepsilon_t)}$ of $V_t^{-1}$ that can be used to estimate $\tilde{\theta}_t = A_t \sum_{l=1}^{t-1} x_{l,a_t}r_{l}$. Prop. 1 bounds the difference between the standard $\theta_t$ and the homomorphic friendly estimate $\tilde{\theta}_t$ (see App. B.3 for the proof).

**Corollary 1.** With the precision $\varepsilon_t = \left( \frac{Lt^{3/2}L^2t + 1}{\lambda} \right)^{-1}$ in Prop. 1, it holds, at any time $t$, that $\| \theta_t - \tilde{\theta}_t \|_{V_t} \leq t^{-1/2}$.

This result, along with Thm. 2 in Abbasi-Yadkori et al. (2011), implies that, at all time steps $t$, $\theta^* \in \tilde{C}_t := \{ \theta \in \mathbb{R}^d \mid \| \theta - \tilde{\theta}_t \|_{V_t} \leq \tilde{\beta}_t \}$ with probability at least $1 - \delta$, where $\tilde{\beta}_t = t^{-1/2} + S\sqrt{\lambda} + \sigma \sqrt{d \ln(1 + L^2t/\lambda)} + \ln(n2^{t/2}/(6\delta))$ is the inflated confidence interval (see Prop. 5 in App. B.4). The optimistic reward can then be computed for any $x \in \mathbb{R}^d$ as:

$$\max_{\theta \in \tilde{C}_t} \langle \theta, x \rangle = \langle \tilde{\theta}_t, x \rangle + \tilde{\beta}_t \| x \|_{V_t^{-1}} \geq \langle \theta^*, x \rangle$$  

(3)

The first problem with computing the maximum is the evaluation of the norm $\| x \|_{V_t^{-1}}$. We need to control the deviation between performing this operation with $V_t^{-1}$ and $A_t$. Leveraging Prop. 1 the definition of $\varepsilon_t$ in Cor. 1 and the fact that $\| x \|_2 \leq L$, it holds that

$$\| x \|_{V_t^{-1}}^2 - \| x \|_{A_t}^2 \leq L^2\| V_t^{-1} - A_t \| \leq L^2 \lambda / 2 (\lambda + L^2t)^{-1/2}$$

The last step to compute an optimistic estimate of the reward is the evaluation of the square root of $\| x \|_{A_t}^2 + L(t^{3/2}\sqrt{\lambda + L^2t})^{-1}$.  

\[\text{Conv}(E)\] is the convex hull of set $E$.  

5
Approximate Square Root. Given \( z \in [0, 1] \), the following Newton iterates can be used to compute \( \sqrt{z} \) with HE:

\[
q_{k+1} = q_k \left( 1 - \frac{v_k}{2} \right), \quad v_{k+1} = v_k^2 \left( \frac{v_k - 3}{4} \right)
\]

where \( q_0 = z \) and \( v_0 = z - 1 \). Cheon et al. (2019a) showed that this algorithm converges exponentially fast to \( \sqrt{z} \). It remains to control the error of this approximation scheme.

**Proposition 2.** For any \( z \in \mathbb{R}_+ \), \( c_1, c_2 > 0 \) with \( c_2 \geq z \geq c_1 \) and a precision \( \varepsilon > 0 \), let \( q_k \) be the result of \( k \) iterations of Eq. (4), with \( q_0 = \frac{z}{c_2} \) and \( v_0 = \frac{z}{c_2} - 1 \). Then, \( |q_k \sqrt{c_2} - \sqrt{z}| \leq \varepsilon \) for any \( k \geq k_0(\varepsilon) := \frac{1}{\ln(\varepsilon) - \ln(\sqrt{c_2})} \).

Therefore, for any \( t \geq 1 \) and \( x_{t,a} \in \mathbb{R}^d \), the specific choices \( c_1 = L(t^{3/2} \sqrt{\lambda} + L^2 t)^{-1} \), \( c_2 = c_1 + L^2 \lambda^{-1/2} \left( 1 + \lambda^{-1/2} \right) \), \( z = c_1 + \|\hat{x}_{t,a}\|_2^2 \) and \( \varepsilon = \frac{1}{t} \) gives:

\[
\max_{\theta \in c_i} (x_{t,a}, \theta) \leq (\tilde{\theta}_t, x_{t,a}) + \tilde{\beta}_t \left[ \sqrt{c_2} q_k(1/t) + \frac{1}{t} \right] := \rho_a(t).
\]

The estimate \((\rho_a(t))_{a \leq K}\) involves only quantities that can be computed thanks to additions and multiplications, thus it is HE compatible.

### 3.3 Homomorphic Friendly Approximate Argmax

The last challenge due to HE is that algorithms can not directly compute \( \arg \max_{a \leq K} \rho_a(t) \). The maximum index can be computed in several ways – e.g., pair-wise comparisons, comparison with the maximum – but all involving at least a comparison. However, as simple as it may seem, comparing two encrypted numbers \( a, b \in [0, 1] \) (\( \text{Comp}(a, b) = \mathbb{I}_{a > b} \)) is not directly possible. The problem resides in the fact that if data are encrypted, the algorithm cannot observe the values, it can for instance build an approximation of the sign function (or \( \text{Comp}(a, b) \)) but not evaluate it since it is encrypted. It is necessary to ask a third party, having access to the secure key \( sk \), to decrypt the approximate value and to send back a single bit representing the condition (e.g., positive or negative).

Recently, Cheon et al. (2019a) introduced an homomorphic compatible algorithm, called **NewComp**, that builds a polynomial approximation of \( \text{Comp}(a, b) \) for any \( a, b \in [0, 1] \). This algorithm allows to compute an HE friendly approximation of \( \max_{a \in K} \rho_a(t) \) for any \( a, b \in [0, 1] \). We leverage these ideas and derive **acomp**, a homomorphic compatible algorithm that computes an approximation of the maximum index (see Alg. 8 in App. B.3). Precisely, **acomp** does not directly compute \( \arg \max_{a \leq K} \rho_a(t) \) but an approximate vector \( b_t \equiv (\mathbb{I}_{a = \arg \max \rho_a(t)})_{a \leq K} \). The maximum index is the value \( a \) such that \( (b_t)_a \) is greater than a threshold accounting for the approximation error. If \( b_t \) is encrypted, the algorithm relies on a third party to evaluate this condition.

The **acomp** algorithm is composed by two phases. First, **acomp** computes an approximation \( M \) of \( \max_{i \leq K} \rho_i(t) \) by comparing each pair \((\rho_i(t), \rho_j(t))\) with \( i < j \leq K \). Second, each value \( \rho_a(t) \) is compared with this approximated maximum value \( M \) to obtain \((b_t)_a\), an approximate computation of \( \mathbb{I}_{(\rho_a(t)) > M} \). The following corollary shows that if an entry of the vector \( b_t \) is big enough, then the difference between max\( \rho_a(t) \) and any arm with \( 4(b_t)_a \geq t^{-1} \) is bounded by \( O(1/t) \) (proof in App. B.3).

**Corollary 2.** At any time \( t \in [T] \), any arm \( a \in [K] \) satisfying \((b_t)_a \geq \frac{1}{4t}\) is such that

\[
\rho_a(t) \geq \max_{k \leq K} \rho_k(t) - \frac{1}{t} \left( 1 + \tilde{\beta}^*(t) \right)
\]

where \( \tilde{\beta}^*(t) = \tilde{\beta}(t) \left[ \frac{t}{L} + \sqrt{\frac{L}{\lambda^2 \sqrt{\lambda}} + t L^2 \sqrt{\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}}} \right] \)

This shows that the action selected may not be the optimistic one selected by OFUL. As shown in the next section, this has little impact on the final regret of the algorithm as the approximation error decreases fast enough.
3.4 Discussion

These approximations can be used in the non-secure setting to build an approximate version of OFUL. More interesting is to use them in the encrypted scenario. Indeed, since all operations are homomorphic compatible (involve only additions and multiplications), we can combine these iterates to build a bandit algorithm for the encrypted protocol – name this algorithm SecOFUL. In the encrypted protocol, the algorithm has access to encrypted contexts $\tilde{x}_{t,a}$, rewards $\tilde{r}_t$, and, as a consequence, all the statistics built by the algorithm are encrypted (e.g., $V_t \mapsto \tilde{V}_t$ and $\tilde{\theta}_t \mapsto \tilde{\tilde{\theta}}_t$). Thanks to the properties of HE, we have for example that $\tilde{V}_t$ is equal to the encryption of the original design matrix $V_t$ (i.e., $\tilde{V}_t = \text{Enc}_{pk}(V_t)$). All the terms computed by SecOFUL are encrypted under the public key $pk$, thus not decryptable by the algorithm, which guarantee that no information about users’ data is exposed to the algorithm. Finally, in secure settings, since $b_t$ is encrypted, the algorithm does not know the action to play. $b_t$ is sent to the user who decrypts it and selects the action to play (the user is the only one having access to $pk$). Note $b_t$ does not expose any information about the algorithm and does not introduce any security concerns (e.g., the user may gain information about other users).

The following statement shows that SecOFUL essentially suffers the same regret of the unsecure OFUL (App. C.2).

**Corollary 3.** Under Asm. 4, for any $\delta > 0$ and $T \geq d$, there exist a universal constants $C_1 > 0$ such, with probability at least $1 - \delta$ by:

$$R_T \leq C_1 \beta^* \cdot \left( \sqrt{T \ln \left( \frac{L^2 T}{\lambda d} \right)} + L\sqrt{L} \ln(T) \right)$$

with $\beta^* = 1 + \sqrt{S} + \sigma \sqrt{d \left( \ln \left( 1 + \frac{L^2 T}{\lambda d} \right) + \ln \left( \frac{2\pi T^2}{65} \right) \right)}$.

Despite its good regret bound, this algorithm is impractical because of the extremely high number of multiplications performed. Nonetheless, this theoretical result is reported here to highlight that the impact of the several approximations to the regret bound is only logarithmic. In the following, we continue the path toward a secure algorithm and consider a low switching condition for updating the estimate $\tilde{\theta}_t$.

4 Tractable Secure LinUCB

Inverting the design matrix at each step incurs a large multiplicative depth over iterations. The simplest way of reducing the multiplicative depth is to reduce the number of times the ridge regression is solved. This means that the policy will not be updated at each time step but rather (hopefully) only when necessary. Reducing the number of policy changes is exactly the aim of low switching algorithms (see e.g., Abbasi-Yadkori et al., 2011; Bai et al., 2019; Calandriello et al., 2020; Dong et al., 2020). Low switching algorithms are mostly divided into two main categories. One where there exists a fixed schedule known ahead of time (e.g., Perchet et al., 2016) where the policy is updated. The other when the policy is changed if some data dependent condition has been fulfilled (e.g., Abbasi-Yadkori et al., 2011). Han et al. (2020) showed that a regret of order $\sqrt{T}$ is not attainable using a fixed known-ahead-of-time schedule. Hence, we focus on a dynamic, data-dependent, conditions.

4.1 Low Switching Cost for Linear Contextual Bandit

Abbasi-Yadkori et al. (2011) introduced a low switching variant of OFUL (RSOFUL) that recomputes the ridge regression only when the following condition $\det(V_{t+1}) \geq (1 + C)\det(V)$ is met, with $V$ the design matrix after the last update. The regret of RSOFUL scales as $\tilde{O} \left( d \sqrt{(1+C)T} \right)$. In the secure setting, computing the determinant of a matrix is costly (see e.g. Kaltofen & Villard, 2005) and requires multiple matrix multiplications. As a consequence, the complexity of checking the above condition with HE would unfortunately overweight the benefits introduced by the low switching regime, rendering this technique non practical. Instead of this determinant-based condition, we consider a trace-based condition, inspired by the update rule for GP-BUCB Desautels et al. (2014); Calandriello et al. (2020).
Trace-based Condition. The “batch j” is defined as the set of time steps between j-th and (j + 1)-th updates of \( \hat{\theta} \). We note \( t_j \) the first time step of batch \( j \). For any time \( t \), the design matrix, now noted \( \nabla_j = \lambda I + \sum_{i=1}^{t-1} x_{t,a_i}x_{t,a_i}^\top \), is only updated at the beginning of each batch \( j \) as is the approximate inverse \( A_j \) and approximate \( \hat{\theta}_j \). We introduce the following trace-based condition. At step \( t \), the current batch \( j \) is ended if and only if this is met:

\[
C \leq \text{Tr} \left( \sum_{t=t_j+1}^{t-1} \nabla_j^{-1} x_{t,a_j}x_{t,a_j}^\top \right) = \sum_{t=t_j+1}^{t-1} \|x_{t,a_j}\|^2_{\nabla_j^{-1}}
\]  

(6)

During batch \( j \), the regret incurred by an optimistic based algorithm is bounded by \( \sum_{t=t_j}^{t-1} \beta(t) \approx \sum_{t=t_j}^{t-1} \|x_{t,a_j}\|^2_{\nabla_j^{-1}} \) up to a constant. Therefore, Condition (6) chooses to update a batch as soon as the regret incurred during the current batch exceeds a certain threshold.

This condition rely on the inverse of \( \nabla_j \). To obtain an HE compatible condition, we resort to the iterative computation of the inverse through iterates (2). The following result shows how to account for this approximation in the test.

Proposition 3. Let \( \varepsilon_j = (L_{t_j}^{3/2} \sqrt{\lambda + L_{t_j}^2})^{-1} \) and \( A_j = X_{k_t(e_j)} \) as in Eq. (2) be the approximation of \( \nabla_j^{-1} \). Then:

\[
\left| \text{Tr} \left( \sum_{t=t_j+1}^{t-1} (A_j - \nabla_j^{-1}) x_{t,a_j}x_{t,a_j}^\top \right) \right| \leq L^2 \varepsilon_j (t - 1 - t_j)
\]

Since all the operations are now homomorphic compatible we can evaluate the RHS in (6) on encrypted values \( \nabla_j \) and \( x_{t,a_j} \). Finally, since the switching condition involves data-dependent encrypted quantities, we leverage the same idea as before. We compute an (encrypted) homomorphic approximation of the sign that is sent to the user or trusted server for the evaluation. They send back to the algorithm a single bit denoting whether the condition is verified or not. This is a very cheap operation. See App. C.1.1 for details.

Controlling the Approximation Error. In non-encrypted setting, Condition (6) can be used to dynamically control the growth of the regret due to the freeze of the policy because the latter is bounded by \( O\left( \sum_{j=0}^{M_T} \sum_{t=t_j+1}^{t_{j+1}} \|x_{t,a_j}\|_{\nabla_j^{-1}} \right) \). But in the secure setting, the regret can not be solely bounded as previously. The condition for updating the batch has to take into account the approximation error introduced by all the approximate operations. Let \( M_T \) be the total number of batches, then the contribution of the approximations to the regret scales as \( \sum_{j=0}^{M_T-1} O((t_{j+1} - t_j)^2 \varepsilon_j) \). We thus introduce an additional condition aiming at explicitly controlling the length of each batch. Let \( \eta > 0 \), then a new batch is started if Condition (6) is met or if:

\[
t \geq (1 + \eta)t_j
\]

(7)

This ensures that the additional regret term grows proportionally to the total number of batches \( M_T \).

Since all these operations are homomorphic compatible, we can carry out them in the secure protocol. Note that \( t_j \) and \( t \) are not encrypted values because do not depend on users’ data. Thus the comparison is “simple”. Our final tractable SecOFUL algorithm (named SecOFULLS) is detailed in Alg. 2. This algorithm is similar to SecOFUL in the action selection process but only requires to perform a ridge regression at each new batch.

4.2 Regret Analysis

The regret analysis of SecOFULLS is decomposed in two parts. First, we show that, thanks to Conditions (6) and (7), the number of batches of SecOFULLS is bounded by \( O(\ln(T)) \). Second, we control the regret per batch.
Algorithm 2: Tractable Secure LinUCB

**Input:** horizon: $T$, regularization factor: $\lambda$, failure probability: $\delta$, feature bound: $L$, $\theta^*$ norm bound: $S$, dimension: $d$, batch growth: $\eta$, trade condition: $C$

Set $\theta_1 = 0$, $V_1 = \text{Encpk}(\lambda I)$, $A_1 = \text{Encpk}(\lambda^{-1} I)$, $\delta_0 = 0$, $j = 0$ and $t_0 = 1$

for $t = 1, \ldots, T$ do

Set $\tilde{\beta}(t) = \sigma \sqrt{d \ln \left( \left(1 + \frac{L^2}{\lambda S} \right) \left( \frac{\sigma^2}{\delta t} \right) \right) + t_j^{-1/2} + S \sqrt{\lambda} + \epsilon_j}$

Receive encrypted context $\hat{x}_t \in \mathbb{R}^{K \times d}$

for $a = 1, \ldots, K$ do

Compute approximate square root $sq_a(t) = \sqrt{\text{Enc}q_{\theta_0}(1/t)}$ as in Prop. 3 with $z = (\langle x_{t,a}, A_c x_{t,a} \rangle + \epsilon_j)$, $c_1 = \epsilon_j$ and $c_2 = c_1 + L^2 \left( \lambda^{-1} + \lambda^{-1/2} \right)$

Set $\rho_a(t) = \langle x_{t,a}, \tilde{\theta}_j \rangle + \tilde{\beta}(t)(sq_a(t) + t^{-1})$

Set $\tilde{\rho}_a(t) = \rho_a(t) - \rho_{\text{min}}$ with $\rho_{\text{max}} = r_{\text{max}} + 2 \tilde{\beta}(t) \left[ \frac{L}{\lambda^2} \lambda \epsilon_k \epsilon + \frac{2L}{\lambda^2} \sqrt{\lambda + L^2} + L^2 \left( 1 + \frac{1}{\sqrt{\lambda}} \right) \right]$

end for

Compute comparison vector $b_t \in \mathbb{R}^d$ using $\text{comp}$ (see Alg. 1 in App. B.5) with precision $\epsilon_t = (4.1t)^{-1}$

Select action $a$ to pull such that $(b_t)_a > (4t)^{-1}$

Receive encrypted $\hat{r}_t$

Update $\hat{V}_{t+1} = \hat{V}_t + \hat{x}_{t,a} \hat{x}_{t,a}^\top$ and $\hat{y}_{t+1} = \hat{y}_t + \hat{r}_t \hat{x}_{t,a}$

Evaluate trade condition by computing $\delta_t$ with $\epsilon = 0.45$ and $\epsilon_t = L^2(\lambda^{-1} + \lambda^{-1/2})(t - 1 - t_j)$ (see App. C.1.1).

if $\delta_t \geq 0.45$ or $t \geq (1 + \eta)T_j$ then

$t_{j+1} = t$, $j = j + 1$

Compute $A_{j+1} = X_{k_1(\epsilon_{j+1}/L^2)}$ in Prop. 1 ($V = \hat{V}_{j+1}$, $c = \lambda d + L^2 t_{j+1}$)

Compute $\theta_{j+1} = A_{j+1} \hat{g}_{j+1}$

end if

end for

Proposition 4. For any $T > 1$, if $C - \frac{L}{\sqrt{\lambda + L^2}} > \frac{1}{4}$, the number of episodes $M_T$ of SECOFULLS (see Alg. 2) is bounded by:

$$M_T \leq 1 + \frac{d \ln \left( 1 + \frac{\lambda^2}{C} \right)}{2 \ln \left( \frac{1}{4} + C - \frac{L^2}{\lambda + L^2} \right)} + \frac{\ln(T)}{\ln(1 + \eta)}$$

This leads to SECOFULLS, a low-switching, faster and more computationally tractable variant of SECOFUL. The multiplicative depth of this algorithm is in general smaller by roughly a factor $(1 + \eta)$ as SECOFUL. However, the total number of multiplications computed to compute $\hat{\theta}$ is $T/M_T$-times smaller thanks to the low-switching condition. This leads to a vast improvement in computational complexity. Note that at each round $t$, SECOFULLS has still to compute the upper-confidence bound on the reward and the maximum action. However, this operations will not impact the total multiplicative depth of the algorithm.

Leveraging this result, when any of the conditions [6]–[7] is satisfied, the regret can be controlled in the same way as the non-batched case, up to a multiplicative constant. The final regret bound is as follows (proof in App. C.2).

Theorem 1. Under Assm. 4 for any $\delta > 0$ and $T \geq d$, there exists universal constants $C_1, C_2 > 0$ such that the regret of SECOFULLS (Alg. 2) is bounded with probability at least $1 - \delta$ by:

$$R_T \leq C_1 \beta^* \left( \frac{5}{4} + C \right) dT \ln \left( \frac{TL}{\lambda d} \right) + L^{3/2} \ln(T)$$

$$+ C_2 \cdot \beta^* M_T \max \left\{ \sqrt{L} + \frac{\eta}{\sqrt{\lambda}}, \eta^2 + \frac{L}{\sqrt{\lambda + L^2}} \right\}$$

with $\beta^* = 1 + \sqrt{\lambda} S + \sigma \sqrt{d \ln \left( 1 + \frac{L^2 T}{\lambda} \right) + \ln \left( \frac{2^2 T^2}{60} \right)}$ and $M_T$ as in Prop. 4.

It is worth noticing that the main order of the regret is the same of SECOFUL, while being more tractable. As in Prop. 3, the first term of the regret highlights the impact of the approximation of the square root
and maximum. These approximations have a very similar impact as in the non-batched algorithm since are performed at each round to select the action for a specific context. The second term shows the impact of the approximation of the inverse. It clearly depends on the number of batches since the inverse is updated only once per batch. By Prop. 4, we can notice that even this term has a logarithmic impact on the regret. Finally, the last term is the regret incurred due to low-switch of the optimistic algorithm. We can notice that the parameter $C$ regulates a trade-off between regret and computational complexity. This term is also the regret incurred by running OFUL with trace condition instead of the determinant-based condition. This further stress that the cost of encryption on the regret is only logarithmic.

5 Discussion and Extensions

In this section, we present a numerical validation of the proposed algorithm in a secure linear bandit problem and we discuss limitations and possible extensions.

**Numerical simulation.** Despite the focus of the paper is mainly theoretical, we illustrate the performance of the proposed algorithm on a toy example. This test does not want to be a claim on the practical aspect of the approach but we simply aim to empirically validate the theoretical findings. We consider a simple linear contextual bandit problem with 4 contexts in dimension 2 and 2 arms. As baselines, we consider OFUL, RSOFUL and RSOFUL-Tr (a version of RSOFUL where we replaced the determinant-based condition with the trace-condition in (6)). We run these baselines on non-encrypted data and compare the performance with SecOFULLS working with encrypted data. In the latter case, at each step, contexts and rewards are encrypted using the CKKS (Cheon et al., 2017) scheme with parameter $\kappa = 128$, $D = 100$ and $N = 2^{16}$. For the implementation, we use the PALISADE library PAL (2020). The noise of the reward is $\sigma = 0.5$. Finally, we use $C = 1$ and $\eta = 0.1$ in Condition (6) and Condition (7). The regularization parameter is set to 1 and $L = 5.5$. Fig. 5 shows the regret of the algorithms averaged over 25 repetitions.

We start noticing that while the non-encrypted low-switching algorithms (i.e., RSOFUL and RSOFUL-Tr) recompute the ridge regression only 11 times on average, their performance is only slightly affected by this and it is comparable to the one of OFUL. The reduced number of updates is a significant improvement in light of the current limitation in the multiplicative depth of homomorphic schemes. This was the enabling factor to implement SecOFULLS. Note that the update condition in SecOFULLS (combining (6) and (7)) increases the number of updates to about 20 on average. As expected, the successive approximations and low-switching combined worsen the regret of SecOFULLS. However, this small loss in performance comes with a very high security on users’ data.

**Computational Complexity.** Even though we minimized the number of total multiplications and additions the total runtime of SecOFULLS is still significative, several orders of magnitude higher compared to the unencrypted setting. We believe that a significant speed up can be obtained by optimizing the way encryption is done and how matrix multiplication is handled. For example, implementation optimization have been carried out in Blatt et al. (2020) to increase the speed of computation of a logistic regression on genomics data and obtaining a practical homomorphic algorithm. However, we stress out that SecOFULLS is agnostic to the homomorphic scheme used, hence any improvement in the HE literature can be leveraged.
by our algorithm. As mentioned in the introduction, bootstrapping procedures can be used for converting a leveled schema into a Fully HE scheme. This mechanism, together with the low-switching nature of our algorithm, can be the enabling tool for scaling this approach to large problems.

There are many other approaches possible for increasing the computational efficiency of our algorithm, for example user-side computation or using a trusted execution environment [Sabt et al. (2015)]. We decided to design an algorithm where the major computation (except for comparisons) is done server-side because this is what happen in standard “unencrypted setting”. The objective was to make as secure as possible this protocol so that the server can leverage the information coming from all users. However, if we assume that users have greater computation capabilities, the algorithm can delegate some computations to the user (see e.g., Blatt et al. 2020). For example, instead of approximating an inverse, the algorithm can generate a random (invertible) matrix \(N_t\), homomorphically compute \(V_tN_t\) and sends to, the now noisy matrix, \(V_tN_t\), to the user. The latter decrypts, inverts, reencrypts the inverse and sends it to the algorithm (see [Bost et al. 2015, Sec. 8] for details about delegating matrix inversion). A similar scenario, can be imagined for computing a square root or a matrix multiplication. This protocol requires users to perform computationally heavy operations (inverting a matrix) locally. To ensure security with this delegation, it is needed to perform a verification step (see e.g., Bost et al. 2015) further increasing communications between the user and the bandit algorithm. We believe that an interesting direction for future work is to integrate this protocol in a distributed setting (i.e., federated learning). Using a server-side trusted execution environment to compute matrix inversion would also speed up computations as operations are executed in the clear in private regions of the memory (also called enclave).

**Multi-users Setting.** Finally, assuming that all contexts and rewards are encrypted under a common public key may be unrealistic. For contextual bandits, contexts usually represent different users described by their features \(x_t\). Hence, distributing a single key to every user, though possible in most applications, is difficult. If each user encrypts its context/reward with its own public key \(pk_i\) (and secret key \(sk_i\)), SecOFULLS can be used with KeySwitching [Fan & Vercauteren, Brakerski (2012), Brakerski et al. (2014)].

This operation takes as input a ciphertext \(c_1\) decipherable by a secret key \(sk_1\) and transform it into a ciphertext \(c_2\) decipherable by a secret key \(sk_2\). A user then send an encrypted context/reward to the bandit algorithm which then perform a key switching (see App. A.3 for more details) such that all ciphertexts received are decipherable by the same key and compatible for homomorphic operations. KeySwitching can be done such that the bandit algorithm can not access the data; keeping the same level of security as if a key was distributed to all users.

### 6 Conclusion

We studied the problem of linear contextual bandit with encrypted information and designed SecOFULLS, the first algorithm for this problem with regret guarantees. We leave as open question the design of an algorithm tailored to the characteristics of the HE and extensions to other algorithms (e.g., Thompson sampling) or settings (e.g., reinforcement learning).
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The LWE problem consists in distinguishing between noisy pairs \((x, Ax + \epsilon_t)\) and \((r, \epsilon_t)\) with \(z = (\langle x_t, a \rangle, A_t x_t, a) + \epsilon_t\), \(c_1 = \epsilon_t + 2\epsilon_t\lambda = 2^{t - 1} + 2\epsilon_t + L^2\left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}\right)\).

A.1 Encoding and Decoding of Messages.

In CKKS, the space of message is defined as \(C^{N/2}\) but rather on an integer polynomial ring \(R = \mathbb{Z}[X]/(X^N + 1)\) (the plaintext space) \(\text{Seidenberg} (1978)\). Encoding a message \(m\) \(\in\mathbb{C}^{N/2}\) into the plaintext space \(R\) is not as straightforward as using a classical embedding of a vector into a polynomial because we need the coefficients of the resulting polynomial to be integers. To solve this issue the CKKS scheme use a more sophisticated construction that the canonical embedding, based on the subring \(\mathbb{H} = \{z \in \mathbb{C}| z_j = \bar{z}_{N-j}, j \leq N/2\}\) which is isomorphic to \(\mathbb{C}^{N/2}\). Finally, using a canonical embedding \(\sigma: R \rightarrow \sigma(R) \subset \mathbb{H}\) and the \textit{coordinate-wise random rounding} technique developed in \text{Lyubashevsky et al.} (2013b), the CKKS scheme is able to construct an isomorphism between \(\mathbb{C}^{N/2}\) and \(R\).

A.2 Encryption and Decryption of Ciphertexts.

Most public key scheme relies on the hardness of the Learning with Error (LWE) problem introduced in \text{Regev} (2009). The LWE problem consists in distinguishing between noisy pairs \((a_i, \langle a_i, s \rangle + \epsilon_i)\) \(\in\mathbb{Z}/q\mathbb{Z}\) and uniformly sampled pairs in \((\mathbb{Z}/q\mathbb{Z})^n = \mathbb{Z}/q\mathbb{Z}\) where \(\langle e_i \rangle_{i\leq n}\) are random noises and \(q \in \mathbb{N}\). However, building a cryptographic public key system based on LWE is computationally inefficient. That’s why CKKS relies on the Ring Learning with Error (RLWE) introduced in \text{Lyubashevsky et al.} (2013a) which is based on the same idea as LWE but with working with polynomials \(\mathbb{Z}[X]/(X^N + 1)\) instead on integer in \(\mathbb{Z}/q\mathbb{Z}\). RLWE (and LWE) problem are assumed to be difficult to solve and are thus used as bases for cryptographic system. The security of those schemes can be evaluated thanks to \text{Albrecht et al.} (2015) which gives practical bounds on the number of operations needed for known attacks to solve the LWE (RLWE) problem.
The CKKS scheme samples a random $s$ on $\mathcal{R}$ and defines the secret key as $sk = (1, s)$. It then samples a vector $a$ uniformly on $\mathcal{R}/qL$ (with $qL = 2^L$ where $L$ is the depth of the scheme) and an error term $e$ sampled on $\mathcal{R}$ (usually each coefficient is drawn from a discrete Gaussian distribution). The public key is then defined as $pk = (a, -as + e)$. Finally, to encrypt a message $m \in \mathbb{C}^{N/2}$ identified by a plaintext $m \in \mathcal{R}$ the scheme samples an encrypting noise $\nu \sim \mathcal{O}(0.5)$\footnote{A random variable $X \sim \mathcal{O}(0.5)$ such that $X \in \{0, 1, -1\}^N$, $(X_i)_{i \leq N}$ are i.i.d such that for all $i \leq N P(X_i = 0) = 1/2, P(X_i = 1) = 1/4$ and $P(X_i = -1) = 1/4$.} The scheme then samples $c_0, c_1 \in \mathbb{Z}^N$ two independent random variable from any distribution on $\mathcal{R}$, usually a discrete Gaussian distribution. The ciphertext associated to the message $m$ is then $[(\nu \cdot pk + (m + c_0, c_1))|_{qL}]$ with $\lfloor \cdot \rfloor_{qL}$ the modulo operator and $qL = 2^L$. Finally, to decrypt a ciphertext $c = (c_0, c_1) \in \mathcal{R}^2$ (with $l$ the level of the ciphertext, that is to say the depth of the ciphertext), the scheme computes the plaintext $m' = (c_0 + c_1 s)p$\footnote{for any $n \in \mathbb{N}$, $[,]_n$ is the remainder of the division by $n$} and returns the message $m'$ associated to the plaintext $m'$.

### A.3 Key Switching

Homomorphic Encryption schemes need all ciphertexts to be encrypted under the same public key in order to perform additions and multiplications. As we mentioned in Sec.\footnote{A random variable $X \sim \mathcal{O}(0.5)$ such that $X \in \{0, 1, -1\}^N$, $(X_i)_{i \leq N}$ are i.i.d such that for all $i \leq N P(X_i = 0) = 1/2, P(X_i = 1) = 1/4$ and $P(X_i = -1) = 1/4$.} one way to circumvent this issue is to use a KeySwitching operation. The KeySwitching operation takes as input a ciphertext $c_1$ encrypted thanks to a public key $pk_1$ associated to a secret key $sk_1$ and transform it into a cyphertext encrypting the same message as $c_1$ but under a different secret key $sk_2$.

The exact KeySwitching procedure for each scheme is different. In this work we use the CKKS scheme which is inspired by the BGV scheme\cite{Brakerski14}. In this scheme KeySwitching relies on two operations BitDecomp and PowerOf2, described below.

1. BitDecomp$(c, q)$ takes as input a ciphertext $c \in \mathbb{R}^N$ with $m$ the size of the ring dimension used in CKKS and an integer $q$. This algorithm decomposes $c$ in its bit representation $(u_0, \ldots, u_{\lfloor \log_2(q) \rfloor}) \in \mathbb{R}^{N \times \lfloor \log_2(q) \rfloor}$ such that $c = \sum_{j=0}^{\lfloor \log_2(q) \rfloor} 2^j u_j$

2. PowerOf2$(c, q)$ takes as input a ciphertext $c \in \mathbb{R}^N$ and an integer $q$. This algorithm outputs $(c, 2c, \ldots, 2^{\lfloor \log_2(q) \rfloor} c) \in \mathbb{R}^{m \times \lfloor \log_2(q) \rfloor}$

The KeySwitching operation can then be decomposed as:

- the first party responsible for $sk_1$ generates a new (bigger, in the sense that the parameter $N$ is bigger than $sk_1$) public key $pk_1$ still associated to $sk_1$
- the owner of secret key $sk_2$ computes PowerOf2$(sk_2)$ and add it to $pk_1$. This object is called the KeySwitchingKey.
- the new cyphertext is computed by multiplying BitDecomp$(c_1)$ with the KeySwitchingKey. This gives a new cyphertext decryptable with the secret key $sk_2$ and encrypted under a new public key $pk_2$

---

**Algorithm 4 KeySwitching Procedure**

**Input:** Cyphertext: $c$, User: $u$, User public key/secret key: $pk_u$, $sk_u$, Bandit Algorithm: $\mathfrak{A}$, Trusted Third Party: $\mathfrak{B}$, integer $q$

Alg. $\mathfrak{A}$ receives cyphertext $c$ encrypted with key $pk_u$

- $\mathfrak{B}$ sends public key $pk$ to $u$
- $u$ computes $\text{Enc}_{pk_u}(sk_u) = \text{Enc}_{pk_u}($PowerOf2$(sk_u, q) + pk)$
- $u$ sends $\text{Enc}_{pk_u}(sk_u)$ to $\mathfrak{A}$
- $\mathfrak{A}$ computes the new cyphertext $c' = \text{Enc}_{pk_u}(\text{BitDecomp}(c, q)^\top)\text{Enc}_{pk_u}(sk_u) = \text{Enc}_{pk_u}(\text{Enc}_{pk}(c))$
- $u$ decrypts $c'$ and sends the result to $\mathfrak{A}$

Alg. $\mathfrak{A}$ allows us to perform the KeySwitching in a private manner for the CKKS scheme. Indeed, the key switch operation requires to decompose a secret key thanks to the PowerOf2 procedure. If not done in a secure fashion this could lead to a leak of the frist private key. It is thus necessary to ensure that this key
is not distributed in the clear. However, our private procedure requires communication between the bandit algorithm $A$ and the user $u$. In particular, the user still needs to receive the public key from the trusted third party. However, the user does not need to be known ahead of time as previously.

B Toward an Encrypted OFUL (Proofs of Sec. 3)

In this section, we provide the proof of the results of Sec. 3. That is to say, the speed of convergence of iterating Eq. (2) or Eq. (4), how to build a confidence intervals around $\theta^*$ and how the approximate argmax is computed in Alg. 3 and Alg. 2.

B.1 Computing an Approximate Inverse

First, we show Prop. 1. The proof of convergence the Newton method for matrix inversion is rather standard but the proof of convergence for the stable method (Eq. (2)) is often not stated. We derive it here for completeness. First, we recall Prop. 1.

**Proposition.** Given a symmetric positive definite matrix $V \in \mathbb{R}^{d \times d}$, $c \geq \text{Tr}(V)$ and a precision level $\varepsilon > 0$, the iterate in (2) satisfies

$$\|X_k - V^{-1}\| \leq \varepsilon$$

for any $k \geq k_1(\varepsilon)$ with

$$k_1(\varepsilon) = \frac{1}{\ln(2)} \ln \left( \frac{\ln(\lambda) + \ln(\varepsilon)}{\ln \left( 1 - \frac{\lambda}{c} \right)} \right)$$

where $\lambda \leq \lambda_d$ is a lower bound to the minimal eigenvalue of $V$ and $\| \cdot \|$ is the matrix spectral-norm.

**Proof.** of Prop. 1. After $k$ iterations of Eq. (2), we have that $V X_k = M_k$. Indeed we proceed by induction:

- For $k = 0$, $M_0 = \frac{1}{c} V = VX_0$
- For $k + 1$ given the property at time $k$, $V X_{k+1} = VX_k(2I_d - M_k) = M_k(2I_d - M_k) = M_{k+1}$

Let’s note $E_k = X_k - V^{-1}$ and $\tilde{E}_k = M_k - I_d$ then:

$$E_{k+1} = (X_{k+1} V - I_d) V^{-1} = (M_{k+1} - I_d) V^{-1}$$

$$= -(M_k^2 - 2M_k + I_d) V^{-1}$$

$$= -(M_k - I_d)^2 V^{-1} = -\tilde{E}_k^2 V^{-1}$$

where the second equality is possible because $V$ and $(X_k)_{k \in \mathbb{N}}$ commute as for all $k \in \mathbb{N}$, $X_k$ is a polynomial function of $V$.

Therefore, we have for any $k \in \mathbb{N}$:

$$\|E_{k+1}\| = \|\tilde{E}_k^2 V^{-1}\| \leq \|V^{-1}\| \times \|\tilde{E}_k\|^2$$

But at the same time:

$$\|\tilde{E}_{k+1}\| = \|M_{k+1} - I_d\| = \|M_k(2I_d - M_k) - I_d\| = \| (M_k - I_d)^2 \| \leq \|\tilde{E}_k\|^2$$

thus iterating Eq. (10), we have that for all $k \in \mathbb{N}$, $\|\tilde{E}_k\| \leq \|\tilde{E}_0\|^{2^k}$. And then $\|\tilde{E}_k\| \leq \|\tilde{E}_0\|^{2^k} \|V^{-1}\|$, therefore using that any $V$ symmetric definite positive $\|V^{-1}\| = \|V\|^{-1}$ then for all $k \in \mathbb{N}$:

$$\|E_k\| \leq \left( \frac{V}{c} - I_d \right)^{2^k} \|V\|^{-1}$$
The next step to build an optimistic algorithm is to compute a confidence ellipsoid around the estimate \( \tilde{\theta}_k \). Therefore, for \( q \)

\[
\text{Prop. 2.} \quad \text{Because } 0 \leq c_1 < x < c_2, \text{ we have that } \frac{x}{c_2} \in (0, 1), \text{ hence thanks to Lemma 2 of Cheon et al. (2019b), we have that after } k \text{ iterations:}
\]

\[
\left| q_k - \frac{x}{\sqrt{c_2}} \right| \leq \left( 1 - \frac{x}{4c_2} \right)^{2k+1}
\]

where \( q_k \) is the \( k \)-th iterate from iterating Eq. (4) with \( q_0 = \frac{x}{c_2} \) and \( v_0 = \frac{x}{c_2} - 1 \). Then because \( x \geq c_1 \), we have that \( 1 - \frac{x}{4c_2} \leq 1 - \frac{c_1}{4c_2} \). That is to say:

\[
\left| q_k - \frac{x}{\sqrt{c_2}} \right| \leq \left( 1 - \frac{c_1}{4c_2} \right)^{2k+1}
\]

Therefore, for \( k \geq \frac{1}{\ln(2)} \ln \left( \frac{\ln(\lambda) - \ln(\sqrt{\lambda})}{2\ln(1 + \sqrt{\lambda})} \right) \), we have:

\[
\sqrt{c_2} \left| q_k - \frac{x}{c_2} \right| \leq \varepsilon
\]

\[\square\]

**B.2 Computing an Approximate Square Root**

The proof of Prop. 2 is very similar to the proof of Prop. 1 thanks the analysis of the convergence speed in Cheon et al. (2019b). Let’s recall Prop. 2:

**Proposition.** For any \( z \in \mathbb{R}_+ \), \( c_1, c_2 > 0 \) with \( c_2 \geq z \geq c_1 \) and a precision \( \varepsilon > 0 \), let \( q_k \) be the result of \( k \) iterations of Eq. (4), with \( q_0 = \frac{x}{c_2} \) and \( v_0 = \frac{x}{c_2} - 1 \). Then, \( |q_k \sqrt{c_2} - \sqrt{z}| \leq \varepsilon \) for any \( k \geq k_0(\varepsilon) := \frac{1}{\ln(2)} \ln \left( \frac{\ln(\lambda) - \ln(\sqrt{\lambda})}{2\ln(1 + \sqrt{\lambda})} \right) \).

**Proof.** of Prop. 2. Because \( 0 \leq c_1 < x < c_2 \), we have that \( \frac{x}{c_2} \in (0, 1) \), hence thanks to Lemma 2 of Cheon et al. (2019b), we have that after \( k \) iterations:

\[
\left| q_k - \frac{x}{\sqrt{c_2}} \right| \leq \left( 1 - \frac{x}{4c_2} \right)^{2k+1}
\]

where \( q_k \) is the \( k \)-th iterate from iterating Eq. (4) with \( q_0 = \frac{x}{c_2} \) and \( v_0 = q_0 - 1 \). Then because \( x \geq c_1 \), we have that \( 1 - \frac{x}{4c_2} \leq 1 - \frac{c_1}{4c_2} \). That is to say:

\[
\left| q_k - \frac{x}{\sqrt{c_2}} \right| \leq \left( 1 - \frac{c_1}{4c_2} \right)^{2k+1}
\]

Therefore, for \( k \geq \frac{1}{\ln(2)} \ln \left( \frac{\ln(\lambda) - \ln(\sqrt{\lambda})}{2\ln(1 + \sqrt{\lambda})} \right) \), we have:

\[
\sqrt{c_2} \left| q_k - \frac{x}{c_2} \right| \leq \varepsilon
\]

\[\square\]

**B.3 Computing an Optimistic Ellipsoid Width.**

The next step to build an optimistic algorithm is to compute a confidence ellipsoid around the estimate \( \tilde{\theta}_t \) such that the true parameter \( \theta^* \) belongs to this confidence ellipsoid with high probability. First, we need an estimate of the distance between \( \theta^* \) and \( \tilde{\theta}_t \), that is the object of Cor. 1. The proof of Cor. 1 is based on the fact that the approximated inverse is closed enough to the true inverse. Let’s recall Cor. 1 first.

**Corollary.** Define the precision \( \varepsilon_t = \left( Lt^3/2 \sqrt{L^2t + \lambda} \right)^{-1} \) in Prop. 1. Then, for any time \( t \), we have that

\[
\|\tilde{\theta}_t - \tilde{\theta}_t\|_{V_t} \leq t^{-1/2}
\]
Proof. of Cor. \[\] Let’s note $A_t$, the result of iterating Eq. \[\], $k_1(ε_t)$ times with $V = V_t$ and $c = λd + L^2t$. Thanks to the definition of $θ_t$ and $θ_t$, we have:

$$
\|θ_t - ̂θ_t\|_{V_t} = \left\| V_t^{1/2} (V^{-1}_t - A_t) \sum_{l=1}^{t-1} \tilde{r}_l \tilde{x}_{l,a} \right\|_2 \\
= \left\| (V^{-1}_t - A_t) V_t^{1/2} \sum_{l=1}^{t-1} \tilde{r}_l \tilde{x}_{l,a} \right\|_2 \\
\leq \|A_t - V_t^{-1}\| \left\| V_t^{1/2} \sum_{l=1}^{t-1} \tilde{r}_l \tilde{x}_{l,a} \right\|_2
$$

(17)

(18)

(19)

But $\text{Tr}(V_t) ≤ λd + L^2t$ and $\lambda_{\text{min}}(V_t) ≥ λ$. Therefore thanks to Prop. \[\] $A_t$ is such that:

$$
\|A_t - V_t^{-1}\| ≤ ε_t
$$

(20)

We also have that:

$$
\left\| V_t^{1/2} \sum_{l=1}^{t-1} \tilde{r}_l \tilde{x}_{l,a} \right\|_2 ≤ \|\sqrt{V_t}\| \left\| \sum_{l=1}^{t-1} \tilde{r}_l \tilde{x}_{l,a} \right\|_2 \\
\leq Lt \sqrt{\|V_t\|} \leq Lt \sqrt{λ + L^2t}
$$

(21)

(22)

because $\tilde{r}_l ∈ [-1, 1]$ for all $l ≤ t$ and $\lambda_{\text{max}}(V_t) ≤ λ + L^2t$. Finally, we have that:

$$
\|θ_t - ̂θ_t\|_{V_t} ≤ ε_t Lt \sqrt{λ + L^2t} ≤ t^{-1/2}
$$

(23)

\[\]

\[\]

\[\]

B.4 Approximate Confidence Ellipsoid

Finally thanks to Cor. \[\] we can now show that with high probability $θ^*$ belongs to the inflated confidence intervals $\bar{C}_t$ for all time $t$. That is the object of Prop. \[\]

Proposition 5. For any $δ > 0$, we have that with probability at least $1 - δ$:

$$
θ^* ∈ \bigcap_{t=1}^{+∞} C_t(δ) := \left\{ θ \mid \|θ - ̂θ_t\|_{V_t} ≤ \bar{β}(t) \right\}
$$

(24)

with $\bar{β}(t) = t^{-1/2} + √λS + σ \sqrt{d(\ln(1 + L^2t/(λd)) + \ln(π^2t^2/(6δ)))}

Proof. of Prop. \[\] Using Cor. \[\] and Thm. 2 in Abbasi-Yadkori et al. (2011), we have that for any time $t$ that with probability at least $1 - δ$:

$$
\|θ^* - ̂θ_t\|_{V_t} ≤ \|θ_t - ̂θ_t\|_{V_t} + \|θ^* - ̂θ_t\|_{V_t} \\
≤ t^{-1/2} + √λS + σ \sqrt{d(\ln(1 + L^2t/(λd)) + \ln(1/δ))}
$$

(25)

(26)

where $̂θ_t$ computed as in Alg. \[\] and $θ_t$ is the ridge regression estimate computed at every time step in OFUL. Taking a union bound with high-probability event means that with probability at least $1 - \frac{6δ}{π^2}$, we have:

$$
\|θ^* - ̂θ_t\|_{V_t} ≤ \|θ_t - ̂θ_t\|_{V_t} + \|θ^* - ̂θ_t\|_{V_t} \\
≤ t^{-1/2} + √λS + σ \sqrt{d(\ln(1 + L^2t/(λd)) + \ln(π^2t^2/(6δ)))}
$$

(27)

(28)

\[\]
B.5 Homomorphic Friendly Approximate Argmax

As mentionned in Sec. 3.3, an homomorphic algorithm can not directly compute the argmax of a given list of values. In this work, we introduce the algorithm Alg. 8 to compute the comparaison vector \( b_t \) with (\( \rho_a(t) \)) the UCBs defined in Sec. 3.2. This algorithm is divided in two parts. First, it computes an approximate maximum, \( M \) thanks to Alg. 7 and then compares each values (\( \rho_a(t) \)) to this approximate maximum \( M \) thanks to the algorithm NewComp of Cheon et al. (2019a) (recalled as Alg. 5).

Algorithm 5 NewComp

**Input:** Entry numbers: \( a, b \in [0, 1] \), \( n \) and depth \( d \)

Set \( x = a - b \)

for \( k = 1, \ldots, d \) do

Compute \( x = f_n(x) = \sum_{i=0}^{n} \frac{1}{i!} x (1-x^2)^i \)

end for

Return: \( (x + 1)/2 \)

Algorithm 6 NewMax

**Input:** Entry numbers: \( a, b \in [0, 1] \), \( n \) and depth \( d \)

Set \( x = a - b, y = \frac{a+b}{2} \)

for \( k = 1, \ldots, d \) do

Compute \( x = f_n(x) = \sum_{i=0}^{n} \frac{1}{i!} x (1-x^2)^i \)

end for

Return: \( y + \frac{a+b}{2} \cdot x \)

Algorithm 7 amax

**Input:** Entry numbers: \( (a_i)_{i \leq K} \), \( n \) and depth \( d \)

Set \( m = a_1 \)

for \( i = 2, \ldots, K \) do

Compute \( m = \max\{m, a_i \} \) thanks to NewMax in Cheon et al. (2019a) with parameter \( a = m, b = a_i, n \) and \( d \)

end for

Rescaling the UCB index: In order to use the HE-friendly algorithms of Cheon et al. (2019a), we need to rescale the UCB-index to be in \([0, 1]\). Determining the range of the indexes is the object of the following proposition.

**Proposition 6.** For every time \( t \geq 1 \), assuming \( r_l \in [-1, 1] \) for any \( l \leq t \) and \( L \geq 1 \) then for any \( \delta > 0 \) we have that with probability at least \( 1 - \delta \):

\[
-1 \leq \rho_a(t) \leq 2 \hat{\beta}(t) \left[ 2t^{-1} + L \sqrt{\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}} + \sqrt{\frac{L}{t^{3/2} \sqrt{\lambda} + L^2 t}} \right]
\]  

(29)

where \( \rho_a(t) = \langle \theta_t, x_{t,a} \rangle + \hat{\beta}(t) \left[ q_{k_0(t+1)} + t^{-1} \right] \) the UCB index of arm \( a \) at time \( t \).

**Proof.** For \( \delta > 0 \), let’s note \( E = \bigcap_{t=1}^{+\infty} \{ \theta^* \in \hat{C}_t(\delta) \} \) then thanks to Prop. 5 \( P(E) \geq 1 - \delta \). Under the event \( E \), we have for any arm \( a \):

\[
-1 \leq \langle x_{t,a}, \theta^* \rangle \leq \rho_a(t) \leq \langle x_{t,a}, \theta^* \rangle + 2 \hat{\beta}(t) \left[ q_{k_0(t+1)} + t^{-1} \right]
\]

(30)
Algorithm 8 \text{acomp}

\textbf{Input:} Entry numbers: \((a_i)_{i \leq K}\), precision \(\varepsilon\)

Set \(d = 1 + \left\lfloor 3.2 + \frac{\ln(1/\varepsilon)}{\ln(3/2)} + \frac{\ln(1/(\varepsilon^2)) - 2}{\ln(2)} \right\rfloor\) and \(d' = \frac{1}{\ln(3/2)} \ln\left(\frac{\alpha \ln(\frac{1}{\varepsilon})}{\ln(2)} - 2\right)\) with \(\alpha = \frac{3}{2} + \frac{5.2 \ln(3/2)}{\ln(4)} + \frac{\ln(3/2)}{2 \ln(2)}\)

Compute \(M = \text{amax}\left((a_i)_{i \leq K}, n, d\right)\)

\begin{algorithmic}
\For {i = 2, \ldots, K}
\State Set \(b_i = \text{NewComp}(a_i, M, n, d')\)
\EndFor
\end{algorithmic}

On the other hand thanks to Prop. 2 we have that \(q_{k_0(t-1)} \leq \sqrt{\|x\|^2_{A_t} + \frac{L}{t^{3/2} \sqrt{\lambda} + L^2 t} + t^{-1}}\) and also \(\|x\|^2_{A_t} \leq L^2 \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}\right)\).

Indeed because \(A_t\) is a polynomial function of \(V_t\), we have that \(A_t\) is symmetric and \(A_t V_t = V_t A_t\), hence \(A_t\) and \(V_t^{-1}\) are diagonalizable in the same basis therefore \(\|A_t - V_t^{-1}\| = \max_{i \leq d} |\lambda_i(A_t) - \lambda_i(V_t^{-1})|\) with \(\lambda_i(M)\) the \(i\)-th biggest eigenvalue of \(M\). Hence:

\[\lambda_1(A_t) \leq \frac{1}{\lambda} + \frac{1}{L t^{3/2} \sqrt{\lambda} + L^2 t} \tag{31}\]

and:

\[\lambda_d(A_t) \geq \frac{1}{\lambda + L^2 t} - \frac{1}{L t^{3/2} \sqrt{\lambda} + L^2 t} > 0 \tag{32}\]

for \(t \geq 2\). Therefore, we have that for any arm \(a\):

\[\rho_a(t) \leq \langle \theta^*, x_{t,a} \rangle + 2 \bar{\beta}(t) \left(2t^{-1} + L \left[\frac{1}{\lambda} + \frac{1}{L t^{3/2} \sqrt{\lambda} + L^2 t}\right]\right) \tag{33}\]

\[\Box\]

\textbf{Computing the Comparison Vector:} The algorithm Alg. 8 operates on values in \([0, 1]\) therefore using Prop. 3 we can compute rescaled UCB index, noted \(\tilde{\rho}_a(t) \in [0, 1]\). We are then almost ready to prove Cor. 2 we just need two lemmas which relates the precision of Alg. 8 and Alg. 7 to the precision of \text{NewComp} and \text{NewMax} of Cheon et al. (2019a).

The first lemma (Lem. 1) gives a lower bound on the depth needed for Alg. 7 to achieve a given precision.

\textbf{Lemma 1.} For any sequences \((a_i)_{i \leq K} \in [0,1]^K\), for any precision \(0 < \varepsilon < K/4, n \in \mathbb{N}^*\) and

\[d(\varepsilon, n) \geq \frac{\ln\left(\frac{\ln(\frac{\varepsilon}{n})}{\ln(2)} - 2\right)}{\ln(c_n)} \tag{34}\]

with \(c_n = \frac{2n+1}{4^n}\left(\begin{array}{c}2n \end{array}\right)\). Noting \(M\) the result of Alg. 7 with parameter \((a_i)_{i \leq K}, n\) and \(d(\varepsilon, n)\), we have that:

\[\max_i a_i \leq \varepsilon \tag{35}\]

\textbf{Proof.} of Lemma 1. Thanks to Corollary 4 in Cheon et al. (2019a), we have that for any \(n\) and depth 
\[d \geq \frac{\ln\left(\frac{\ln(\frac{\varepsilon}{n}) - 2}{\ln(c_n)}\right)}{\ln(c_n)} \tag{with \(c_n = \frac{2n+1}{4^n}\left(\begin{array}{c}2n \end{array}\right)\)}\)

and number \(a, b\):

\[|\text{NewMax}(a, b, n, d) - \max\{a, b\}| \leq \varepsilon \tag{36}\]

Let’s note \(m_k\) the iterate \(m\) of Alg. 7 at step \(k \in [K]\) in the for loop. We show that by induction \(|m_k - \max_{i \in \{k\}} a_i| \leq k \varepsilon\).
• By definition $m_1 = a_1$ and $|m_1 - \max_{i \leq 1} a_i| = 0$

• Using that $|\max\{a,c\} - \max\{b,c\}| \leq |a - b|$ for any $a,b,c \in \mathbb{R}$, we have:

$$
\left|m_{k+1} - \max_{i \leq k+1} a_i\right| = \left|\text{NewMax}(m_k, a_{k+1}, n, d) - \max\{m_k, a_{k+1}\} + \max\{m_k, a_{k+1}\} - \max_{i \leq k} a_{i+1}\right|
\leq \left|\text{NewMax}(m_k, a_{k+1}, n, d) - \max\{m_k, a_{k+1}\}\right| + \left|\max\{m_k, a_{k+1}\} - \max_{i \leq k} a_{i+1}\right|
\leq \varepsilon + |m_k - \max a_i| \leq (k+1)\varepsilon
$$

Finally, because $M = m_K$, we just need to choose $d \geq \frac{\ln(\frac{\ln(K/\varepsilon)}{\ln(c_\varepsilon)})}{\ln(2)}$ to get the result. \(\square\)

The next lemma (Lem. 2) has the same purpose of Lem. 1 but this time for Alg. \(\text{8}^\text{a}\). The proof is based on properties of the polynomial function used by the algorithm \text{NewComp} in order to predict the result of the comparaison when the margin condition of \text{NewComp} (that is to say the result of the comparaison of $a,b \in [0,1]$ is valid if and only if $|a - b| \geq \varepsilon$ for some $\varepsilon > 0$) is not satisfied.

**Lemma 2.** For $\varepsilon \in (0,1/4)$ and sequence $(a_i)_{i \leq K} \in [0,1]^K$, let’s denote $(b_i)_{i \leq K}$ te result of Alg. \(\text{8}^\text{a}\) ruuned with parameter $(a_i)_{i \leq K}$, $n = 1$, $d' = d_2(\varepsilon)$ and $d = d_3(\varepsilon)$ with:

$$
d_2(\varepsilon) = \left[3.2 + \frac{\ln(1/\varepsilon)}{\ln(c_\varepsilon)} + \frac{\ln(1/\varepsilon)}{\ln(n+1)} \right] + 1
$$

$$
d_3(\varepsilon) \geq \frac{1}{\ln(c_\varepsilon)} \ln \left[\frac{\alpha}{\ln(2)} - 2\varepsilon\right]
$$

where $\alpha = \frac{3}{2} + \frac{5.2}{\ln(4)} + \frac{5.2}{\ln(n+1)}$. Then selecting any $i \leq K$ such that $b_i \geq \varepsilon$ (and there is at least one such index $i$), we have that $a_i \geq \max_{k \leq K} a_k - 2\varepsilon$.

**Proof of Lemma 2**. Thanks to Corollary 1 in [Cheon et al. (2019b)], we have that for each $i \leq K$, $|b_i - \text{Comp}(a_i, M)| \leq \varepsilon$ as soon as $|a_i - M| \geq \varepsilon$ and $d' = \left[3.2 + \frac{\ln(1/\varepsilon)}{\ln(c_\varepsilon)} + \frac{\ln(\ln(\frac{1}{\varepsilon})/\ln(2))}{\ln(n+1)} \right] + 1$. For $i \in [K]$, we have that:

• If $\max_{k \leq K} a_k \geq a_i \geq M + \varepsilon$ then $\text{Comp}(a_i, M) = 1$, $|b_i - 1| \leq \varepsilon$ and $a_i \geq \max_{k \leq K} a_k - |\max_{k \leq K} a_k - M| - \varepsilon$

• If $a_i \leq M - \varepsilon$ then $\text{Comp}(a_i, M) = 0$, thus $|b_i| \leq \varepsilon$ and $a_i \leq \max_{k \leq K} a_k + |\max_{k \leq K} a_k - M| - \varepsilon$

Therefore for any $a_i$ such that $|a_i - M| \geq \varepsilon$ then the resulting $b_i$ is either bounded by $1 - \varepsilon$ or $\varepsilon$.

The second option is if $|a_i - M| \leq \varepsilon$ then the \text{NewComp} algorithm provides no guarantee to the result of the algorithm. However the algorithm applies a function $f_n^d$ multiple times to its input. For every $x \in [-1,1]$:

$$|f_n(x)| \leq c_n |x| \text{ and } f_n([-1,1]) \subset [-1,1]$$

with $c_n = \frac{2^n + 1}{\frac{1}{2} n^2}$. Hence:

$$\forall x \in [-1,1] \quad |f_n^{(d)}(x)| \leq c_n |f_n^{(d-1)}(x)| \leq c_n^{d'} |x|$$

But if $|a_i - M| \leq \varepsilon$, $f_n^{(d)}(a_i - M) \leq c_n^{d'} |a_i - M| \leq c_n^{d'} \varepsilon$ thus $|b_i - \frac{1}{2^n} | \leq \frac{c_n^{d'} \varepsilon}{2^n}$.

Finally for each $i$, we only three option for $b_i$:

\footnote{For all $x \in [-1,1]$, $f_n(x) = \sum_{i=0}^{n} \frac{1}{2^n} (2^i) x (1 - x^2)^i$.}
• If $|a_i - M| \leq \varepsilon$ then $|b_i - \frac{1}{t}| \leq \frac{d't}{2}$ and $a_i \geq \max_k a_k - (\varepsilon + \max_k a_k - M) \geq \max_k a_k - 2\varepsilon$

• If $|a_i - M| \geq \varepsilon$ and $a_i \leq M - \varepsilon$ then $|b_i| \leq \varepsilon$ and $a_i \leq \max_k a_k + |M - \max_k a_k| - \varepsilon \leq \max_k a_k$

• If $|a_i - M| \geq \varepsilon$ and $a_i \geq M + \varepsilon$ then $|b_i - 1| \leq \varepsilon$ and $a_i \geq \max_k a_k - (|M - \max_k a_k| + \varepsilon) \geq \max_k a_k - 2\varepsilon$

To finish the proof, we just need to ensure that there exists at least one $i$ such that $b_i \geq \varepsilon$ but noting $i^* = \arg \max_k a_k$, if the amax algorithm is used with depth $d$ such that:

$$d \geq \frac{1}{\ln(C_d)} \ln \left( \frac{\alpha \ln \left( \frac{1}{\lambda} \right) - 2}{\ln(2)} \right)$$

where $\alpha = \frac{3}{2} + \frac{5.2 \ln(c_a)}{\ln(4)} + \frac{\ln(c_a)}{\lambda \ln(n+1)}$, we have $|a_i| - M \leq \varepsilon$ and $i^* \geq \frac{1}{2} - \frac{\varepsilon^{\varepsilon}}{2} > \varepsilon$. Hence there always exists an index $i$ such that $b_i \geq \varepsilon$.

Finally, thanks to Lem. 2 we are finally ready to show Cor. 2. The proof of this corollary simply amounts to choose the right precision for NewComp algorithm at every step of Alg. 8. First let’s recall Cor. 2.

**Corollary.** For any time $t$, selecting any arm $a$ such that $(b_i)_a \geq \frac{1}{4t}$ then:

$$\rho_a(t) \geq \max_{k \leq K} \rho_k(t) - \frac{1}{t} \left( 1 + \tilde{\beta}^*(t) \right)$$

where $\tilde{\beta}^*(t) = \tilde{\beta}(t) \left[ 2t^{-1} + \sqrt{\frac{L}{t^{1/2} \sqrt{\lambda + L}}} + L \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda}} \right]$.\n
**Proof.** of Cor. 2 Using Lem. 2 with $\varepsilon = \frac{1}{4t}$ yields the following result:

$$\rho_i(t) \geq \max_{k \leq K} \tilde{\rho}_k(t) - \frac{1}{2t}$$

But for any $i \leq K$, $\tilde{\rho}_i(t) = (\rho_i(t) + 1)/ \left( 2 + 2\tilde{\beta}(t) \left[ 2t^{-1} + \sqrt{\frac{L}{t^{1/2} \sqrt{\lambda + L}}} + L \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda}} \right] \right)$. Hence the result.

## C Low Switching Condition and Regret (Proofs for Sec. 4)

In this appendix, we present the analysis of the regret of SecOFULLS. The proof is decomposed in two steps. The first one is the analysis of the number of batches for any time $T$. That is the object of the Sec. C.1. The second part of the proof amounts to bounding the regret as a function of the number of batches (Sec. C.2).

### C.1 Number of batches of SecOFULLS (Proof of Prop. 4)

We first prove Prop. 4 which states that the total number of batches for SecOFULLS is logarithmic in $T$ contrary to SecOFUL where the parameter are updated a linear number of times. The proof of this proposition is itself divided in multiple steps. First, we show how using NewComp to compare the parameter $C$ and $\text{Tr} \left( A_j \sum_{l=t+1}^{t-1} x_{l,a_l} x_{l,a_l}^T \right)$ (for any batch $j$) relate to the comparison of $C$ and $\text{Tr} \left( V_j^{-1} \sum_{l=t+1}^{t-1} x_{l,a_l} x_{l,a_l}^T \right)$. Then, we show how Condition 6 relates to the det-based condition used in RSOFUL which allows us to finish the proof of Prop. 4 following the same reasoning as in Abbasi-Yadkori et al. (2011).

#### C.1.1 Homomorphically Friendly Comparison for Condition 6

We first prove the following proposition, bounding the error made by our algorithm when comparing $\text{Tr} \left( A_j \sum_{l=t+1}^{t-1} x_{l,a_l} x_{l,a_l}^T \right)$ with instead of $\text{Tr} \left( V_j^{-1} \sum_{l=t+1}^{t-1} x_{l,a_l} x_{l,a_l}^T \right)$.
Proposition 7. For an batch $j$, time $t \geq t_j + 1$, $\varepsilon < 1/2$ and $\varepsilon' > 0$, let’s note $\delta_t$ the result of NewComp applied with parameters $a = \frac{\text{Tr}(A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top)}{L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)}$, $b = \frac{C}{L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)}$, $n = 1$ and $d_5(\varepsilon)$ such that:

$$d_5(\varepsilon) \geq 3.2 + \frac{\ln(1/\varepsilon)}{\ln(e_n)} + \frac{\ln \left( \frac{1}{\varepsilon} \right) / \ln(2) - 2}{\ln(n + 1)}$$

then:

- **if $\delta_t > \varepsilon$**:

$$C - \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j) \leq \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right)$$

(48)

- **else if $\delta_t \leq \varepsilon$**:

$$\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) \leq C + \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)$$

(49)

**Proof.** of Prop. 7. Let’s differentiate the two cases when $\delta_t$ is bigger than $\varepsilon$ or not.

If $\delta_t > \varepsilon$: we proceed by disjunction of cases.

- If $\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) - C > \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)$:

  $$\left| \delta_t - \text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right), C \right) \right| \leq \varepsilon$$

thanks to Cor. 1 in [Cheon et al. (2019)] for the precision of NewComp. We also used the fact that for any $x, y \in \mathbb{R}$ and $z \in \mathbb{R}_+$, $\text{Comp}(x/z, y/z) = \text{Comp}(x, y)$. Using the equation above:

$$\text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right), C \right) \geq \delta_t - \varepsilon > 0$$

because we assumed here that $\delta_t > \varepsilon$. This readily implies that $\text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right), C \right) = 1$ because $\text{Comp}(a, b) \in \{0, 1\}$ for any $a, b \in [0, 1]$. But, because we are in the case that:

$$\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) - C > \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)$$

we have that either $\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) > C + \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)$ or $\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) < C - \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)$. Hence, because $\text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right), C \right) = 1$, we have that $\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) > C$ that is to say:

$$\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^\top \right) \geq C + \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j) \geq C - \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)$$

(50)

with the convention that $0/0 = 0$ and $C/0 = 1$.
If $\delta_t \leq \varepsilon$: Again, we distinguish the two different cases possible.

- If $\left| \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) - C \right| > \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j)$:

  Using Cor. 1 from Cheon et al. (2019a), we have once again that:

  $$ \left| \delta_t - \text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right), C \right) \right| \leq \varepsilon $$

  Therefore $\text{Comp} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T, C \right) \leq \delta_t + \varepsilon \leq 2\varepsilon < 1$ (because $\varepsilon < 1/2$). But $\text{Comp}(a, b) \in \{0, 1\}$ for any $a, b \in [0, 1]$ which means that $\text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right), C \right) = 0$. But we assumed that

  $$ \left| \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) - C \right| > \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) $$

  in other words:

  $$ \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) > C + \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) \geq C $$

  or

  $$ \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) < C - \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) $$

  But $\text{Comp} \left( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right), C \right) = 0$, it is thus only possible that

  $$ \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) \leq C - \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) $$

- If $\left| \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) - C \right| \leq \varepsilon'$:

  In this case, by definition we have

  $$ C - \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) \leq \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) \leq C + \varepsilon' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) $$

The previous proposition ensures that when a batch is ended because $\delta_t > 0.45$ then we have, for a small enough $\varepsilon'$, that, $\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a_j} x_{l,a_j}^T \right) \geq C'$ for some constant $C'$. However, thanks to Prop. 1 we have that for any batch $j$, that for all $l \in \{t_j + 1, \ldots, t - 1\}$:

$$ \|x\|_{\bar{V}_j^{-1}}^2 - \|x\|_{A_j}^2 \leq L^2 \|\bar{V}_j^{-1} - A_j\| \leq \frac{L}{t_j^{3/2} \sqrt{\lambda} + L^2 t_j} $$

(53)
Summing over all time steps $l \in [t_j + 1, t - 1]$, we have that:

$$
\left| \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) - \text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \right| \leq \frac{L(t-1-t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
$$

(54)

Therefore when $\delta_t > 0.45$, we that:

$$
\text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \geq C - \varepsilon_t' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j) - \frac{L(t-1-t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
$$

(55)

but $\varepsilon_t' = \frac{1}{4tL^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j)}$ so:

$$
\text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \geq C - \frac{1}{4t} - \frac{L(t-1-t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
$$

(56)

But if $\delta_t \leq 0.45$:

$$
\text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \leq C + \varepsilon_t' L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t-1-t_j) + \frac{L(t-1-t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
$$

(57)

or using the definition of $\varepsilon_t'$:

$$
\text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \leq C + \frac{1}{4t} + \frac{L(t-1-t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
$$

(58)

C.1.2 Impact on the Growth of the determinant:

In this section, we study the impact on the determinant of the determinant of the design matrix when Condition 6 is satisfied for some constant $C'$. Our result is based on the classic following lemma.

**Lemma 3.** For any positive definite symmetric matrix $A, B$ and symmetric semi-positive definite matrix $C$ such that $A = B + C$ we have that:

$$
\frac{\det(A)}{\det(B)} \geq 1 + \text{Tr} \left( B^{-1/2} C B^{-1/2} \right)
$$

(59)

Proof. of Lemma 3 Using that $A = B + C$:

$$
\frac{\det(A)}{\det(B)} = \det \left( I_d + B^{-1/2} C B^{-1/2} \right) \geq 1 + \text{Tr} \left( B^{-1/2} C B^{-1/2} \right) = 1 + \text{Tr} \left( B^{-1} C \right)
$$

(60)

The last inequality is a consequence of the following inequality:

$$
\forall n \in \mathbb{N}^*, \forall \alpha \in \mathbb{R}^n_+, \quad 1 + \sum_{i=1}^{n} a_i \leq \prod_{i=1}^{n} (1 + a_i)
$$

(61)

Indeed $I_d + B^{-1/2} C B^{-1/2}$ is symmetric definite positive hence its eigenvalues are positive. \hfill \Box

Therefore, using the lemma above applied to the design matrix, $V_t = \tilde{V}_j + \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T$ for $t \geq t_j + 1$, we have that:

$$
\det(V_t) \geq \left( 1 + \text{Tr} \left( \tilde{V}_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \right) \det(\tilde{V}_j)
$$

(62)
C.1.3 Putting Everything Together:

We are finally ready to prove an upper-bound on the number of batches. First, let’s recall Prop. 4.

**Proposition.** If \( C - \frac{L\eta}{\sqrt{\lambda + L^2}} > \frac{1}{4} \), the number of episodes in Alg. 2 \( M_T \) for \( T \) steps, is bounded by:

\[
M_T \leq 1 + \frac{d \ln \left( 1 + \frac{L^2 \eta}{\lambda + L^2} \right)}{2 \ln \left( \frac{3}{4} + C - \frac{L\eta}{\sqrt{\lambda + L^2}} \right)} + \frac{\ln(T)}{\ln(1 + \eta)}
\]

(63)

**Proof.** of Lem. 3. Let’s define for \( i \geq 1 \), the macro-episode:

\[
n_i = \min \{ n > n_{i-1} | \delta_t > \varepsilon_t \}
\]

(64)

with \( n_0 = 0 \). That is to say macro-episodes are episodes such that the norm of the context has grown too big. It means that for all episode between two macro-episodes the batches are ended because the current batch is too long. Therefore for macro-episode \( i \), thanks to Eq. (54) and Prop. 7

\[
C - \varepsilon_{n_i} L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (n_i - 1 - t_j) - \frac{L(n_i - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \leq \text{Tr} \left( \sum_{l=t_j + 1}^{n_i - 1} x_t^T x_l \right)
\]

(65)

where \( \varepsilon_{n_i} = \left( 4n_i L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (n_i - 1 - t_j) \right)^{-1} \) as defined in Alg. 2. But the batch \( j \) for which \( n_i = t_j + 1 \) is such that \( t_j + 1 - t_j \leq \eta + 1 \) (by the Condition 7). Hence:

\[
\frac{L(n_i - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \leq \frac{L\eta}{\sqrt{\lambda + L^2}}
\]

(66)

Therefore by Lem. 3 we have that:

\[
\det \tilde{V}_{j+1} \geq \left( 1 + C - \left( \frac{3}{4} + \frac{L\eta}{\sqrt{\lambda + L^2}} \right) \right) \det \tilde{V}_j
\]

(67)

because for all \( t \), \( \varepsilon_{n_t} L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) (t - 1 - t_j) \leq \frac{1}{4} \) and because for all episode \( j \) between two macro-episodes \( i \) and \( i + 1 \) the determinant of the design matrix is an increasing function of the episode (because for two matrices \( M, N \) symmetric semi-definite positive \( \det(M + N) \geq \det(M) \)).

Thus \( \det \tilde{V}_{j+1} \geq \left( \frac{3}{4} + C - \frac{L\eta}{\sqrt{\lambda + L^2}} \right) \det \tilde{V}_{j-1} \) where \( j_i \) is the episode such that \( n_i = t_{j_i + 1} \). Therefore the number of macro-episodes \( M_1 \) is such that:

\[
\left( \frac{3}{4} + C - \frac{L\eta}{\sqrt{\lambda + L^2}} \right)^{M_1 - 1} \leq \frac{\det(\tilde{V}_{M_T})}{\det(\tilde{V}_0)}
\]

(68)

where \( \tilde{V}_{M_T} \) is the design matrix after \( T \) steps (or \( M_T \) batches) and \( \tilde{V}_0 = \lambda I_d \). This upper bound gives that:

\[
M_1 \leq 1 + \frac{\ln \left( \frac{\det(\tilde{V}_{M_T})}{\det(\tilde{V}_0)} \right)}{\ln \left( \frac{3}{4} + C - \frac{L\eta}{\sqrt{\lambda + L^2}} \right)}
\]

(69)

if \( \frac{3}{4} + C - \frac{L\eta}{\sqrt{\lambda + L^2}} > 1 \). Moreover, thanks to Lemma 10 in Abbasi-Yadkori et al. (2011), the log-determinant of the design matrix is bounded by: \( \ln \left( \frac{\det(\tilde{V}_{M_T})}{\det(\tilde{V}_0)} \right) \leq d \ln \left( 1 + \frac{L^2 t}{\lambda \sigma} \right) \). In addition, there is at most \( 1 + \frac{\ln(n_{i+1}/n_i)}{\ln(1 + \eta)} \) batches between macro-episode \( i \) and \( i + 1 \). Therefore:

\[
M_T \leq \sum_{i=0}^{M_1 - 1} 1 + \frac{1}{\ln(1 + \eta)} \ln(n_{i+1}/n_i) = M_1 + \frac{\ln(T)}{\ln(1 + \eta)} \leq 1 + \frac{d \ln \left( 1 + \frac{L^2 T}{\lambda \sigma} \right)}{2 \ln \left( \frac{3}{4} + C - \frac{L\eta}{\sqrt{\lambda + L^2}} \right)} + \frac{\ln(T)}{\ln(1 + \eta)}
\]

(70)
C.2 Regret Upper Bound (Proof of Thm. 1)

Now that we have shown an upper-bound on the number of bathes for the SecOFULLS algorithm, we are ready to prove the regret bound of Thm. 1. The proof of this theorem follows the same logic as the regret analysis of OFUL. That is to say, we first show a high-probability upper bound on the regret thanks to optimism and then proceed to bound each term of the bonus used in SecOFULLS.

We first show the following lemma giving a first upper bound on the regret relating the error due to the approximation of the argmax and optimism.

Lemma 4. For any \( \delta > 0 \), the regret of Alg. 2 is bounded with probability at least \( 1 - \delta \) by:

\[
R_T(\text{SecOFULLS}) := \sum_{t=1}^{T} (\theta^*, x_{t,a_t}^* - x_{t,a_t}) \leq \sum_{j=0}^{M_T-1} \sum_{t=j+1}^{t_{j+1}} \left[ 2t^{-1} + \sqrt{\frac{L}{t_j^{3/2} L^2 + \lambda} + L \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right)} \right] + \sum_{j=0}^{M_T-1} \sum_{t=j+1}^{t_{j+1}} 2\tilde{\beta}(j) [\delta_{a_t}(t) + t^{-1}] 
\]

where for every time step \( t \), \( a_t^* = \arg \max_{a \in \mathcal{K}} \langle x_{t,a}, \theta^* \rangle \) and \( M_T \) is the number of batches.

Proof. Let's define \( E \) the event that all confidence ellipsoids, \( \hat{C}_j \), contains \( \theta^* \) with probability at least \( 1 - \delta \). That is to say \( E = \left\{ \theta^* \in \bigcap_{j=1}^{\infty} \hat{C}_j(\delta) \right\} \). Thanks to Prop. 5, \( \mathbb{P}(E) \geq 1 - \delta \). Because \( E \) is included in the event described by Prop. 5, for any \( t \geq 1 \) inside batch \( j \):

\[
\max_a \rho_{a_t}(t) - \rho_{a_t}(t) \leq \frac{1}{t} \left( 1 + \tilde{\beta}(j) \left[ 2t^{-1} + \sqrt{\frac{L}{t_j^{3/2} L^2 + \lambda} + L \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right)} \right] \right) + 2\tilde{\beta}(j) [\delta_{a_t}(t) + t^{-1}] 
\]

In addition, we have that under event \( E \):

\[
\rho_{a_t}(t) - \langle \theta^*, x_{t,a_t} \rangle = \langle \tilde{\theta}_j - \theta^*, x_{t,a_t} \rangle + \tilde{\beta}(j) [\delta_{a_t}(t) + t^{-1}] 
\]

But still conditioned on the event \( E \), \( \langle \tilde{\theta}_j - \theta^*, x_{t,a_t} \rangle \leq \| \tilde{\theta}_j - \theta^* \| \| x_{t,a_t} \| \| \tilde{\gamma}_j \| \leq \tilde{\beta}(j) \| x_{t,a_t} \| \| \tilde{\gamma}_j \| \leq \tilde{\beta}(j) [\delta_{a_t}(t) + t^{-1}] \).

Putting the last two equations together, for every step \( t \leq T \):

\[
\langle \theta^*, x_{t,a_t}^* - x_{t,a_t} \rangle \leq \frac{1}{t} \left( 1 + \tilde{\beta}(j) \left[ 2t^{-1} + \sqrt{\frac{L}{t_j^{3/2} L^2 + \lambda} + L \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right)} \right] \right) + 2\tilde{\beta}(j) [\delta_{a_t}(t) + t^{-1}] 
\]

Bounding 1. We now proceed to bound each term in Eq. (71). The following lemma is used to 1.

Lemma 5. For all \( t \geq 1 \):

\[
\sum_{j=0}^{M_T-1} \sum_{t=j+1}^{t_{j+1}} \frac{1}{t} \left( 1 + \tilde{\beta}(j) \left[ 2t^{-1} + \sqrt{\frac{L}{t_j^{3/2} L^2 + \lambda} + L \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right)} \right] \right) \leq O(\ln(T)^{3/2}) 
\]
Hence:

**Lemma 6.** For all the following lemma.

Proof. of Lem. 5. Because of Lem. 6. For any time

Bounding each component of the sum of (1) in Eq. (71) individually, we get:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} \frac{1}{t} \leq (1 + \ln(T))
\]

Hence:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} \frac{1}{t} \left( 1 + \tilde{\beta}(j) \left[ \frac{2}{t} + \sqrt{\frac{L}{t_j^{3/2} L_j^2 + \lambda}} + L \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda}} \right] \right) \leq (1 + \ln(T)) \left[ 1 + \left( 1 + \sqrt{\lambda} + \frac{1}{\lambda} + \frac{L}{\sqrt{\lambda}} \right) \right]
\]

Lem. 5 shows that the error from our procedure to select the argmax induces only an additional logarithmic cost in \( T \) compared with the regret of directly selecting the argmax of the UCBs \((\bar{\rho}_a(t))_{a \leq K} \).

**Bounding (2).** We are now left with bounding the second term in Eq. (71). This term is usually the one that appears in regret analysis for linear contextual bandits. First, (2) can be further broke down thanks to the following lemma.

**Lemma 6.** For all \( t \geq 1 \),

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} 2\tilde{\beta}(j) \left[ s_{a_j}(t) + t^{-1} \right] \leq \sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} \frac{4\tilde{\beta}(j)}{t} + \sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} 2\tilde{\beta}(j) ||x_{t,a_j}|| \tilde{V}_j^{-1}
\]

\[
\leq \sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} \frac{2L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
\]

Proof. of Lem. 6. For any time \( t \geq 1 thanks to Prop. 2, we have:

\[
s_{a_j}(t) \leq \frac{1}{t} + \sqrt{\frac{||x_{t,a_j}||_j^2 L_j^{3/2} + L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}}
\]

\[
\leq \frac{1}{t} + \sqrt{\frac{L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} + \frac{L}{2} ||x_{t,a_j}||_j^2 \tilde{V}_j^{-1} + ||x_{t,a_j}||_j^2 \tilde{V}_j^{-1}
\]

\[
\leq \frac{1}{t} + \sqrt{\frac{2L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} + ||x_{t,a_j}||_j^2 \tilde{V}_j^{-1}
\]

\[
\leq \frac{1}{t} + \sqrt{\frac{2L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} + ||x_{t,a_j}||_j^2 \tilde{V}_j^{-1}
\]
We proceed to bound each term \( \circ \), \( \circ \). Bounding \( \circ \) is similar to the analysis of OFUL. ON the other hand bounding \( \circ \) is why we introduced Condition \( \circ \).

The following lemma bounds \( \circ \) which is simply a numerical error due to the approximation of the square root.

**Lemma 7.** For any \( T \geq 1 \),

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_j+1} \frac{4\tilde{\beta}(j)}{t} \leq 4 \left( 1 + \sqrt{AS} + \sigma \sqrt{d \left( \ln \left( 1 + \frac{L^2T^2}{\lambda d} \right) + \ln \left( \frac{\pi^2T^2}{6\delta} \right) \right)} \right) (1 + \ln(T))
\]  

(86)

*Proof.* Using the upper bound on the \( \tilde{\beta}(j) \) shown in the proof of Lem. 5 we get the result. \( \Box \)

We are finally left with the two terms \( \circ \) and \( \circ \). The first term, \( \circ \), will be compared to the bonus used in OFUL so that we can use Lemma 11 in Abbasi-Yadkori et al. (2011) to bound it. But first, we need to show how the norm for two different matrices \( A \) and \( B \) relates to each other.

**Lemma 8.** For any context \( x \in \mathbb{R}^d \) and symmetric semi-definite matrix \( A, B \) and \( C \) such that \( A = B + C \) then:

\[
\|x\|_{B^{-1}}^2 \leq \lambda_{\max} \left( I_d + B^{-1/2}CB^{-1/2} \right) \|x\|_{A^{-1}}^2 \leq \left( 1 + \text{Tr} \left( B^{-1/2}CB^{-1/2} \right) \right) \|x\|_{A^{-1}}^2
\]  

(87)

where \( \lambda_{\max}(\cdot) \) returns the maximum eignevalue of a matrix.

*Proof.* of Lemma 8 We have by definition of \( A \) and \( B \):

\[
\langle x, A^{-1}x \rangle = \langle x, (B + C)^{-1}x \rangle = \langle x, B^{-1/2}(I_d + B^{-1/2}CB^{-1/2})^{-1}B^{-1/2}x \rangle
\]  

(88)

\[
= \langle B^{-1/2}x, (I_d + B^{-1/2}CB^{-1/2})^{-1}(B^{-1/2}x) \rangle
\]  

(89)

\[
\geq \lambda_{\min} \left( I_d + B^{-1/2}CB^{-1/2} \right) \|B^{-1/2}x\|^2
\]  

(90)

\[
\geq \lambda_{\max} \left( I_d + B^{-1/2}CB^{-1/2} \right) \|x\|_{B^{-1}}^2
\]  

(91)

Hence:

\[
\|x\|_{B^{-1}}^2 \leq \lambda_{\max} \left( I_d + B^{-1/2}CB^{-1/2} \right) \|x\|_{A^{-1}}^2
\]  

(92)

The result follows from Weyl’s inequality \cite{HornJohnson91}, that is to say for all symmetric matrix \( M, N \)

\[
\lambda_{\max}(M + N) \leq \lambda_{\max}(M) + \lambda_{\max}(N)
\]  

And the fact that all eigenvalues of \( B^{-1/2}CB^{-1/2} \) are positive hence \( \lambda_{\max}(B^{-1/2}CB^{-1/2}) \leq \text{Tr}(B^{-1/2}CB^{-1/2}) = \text{Tr}(CB^{-1/2}) \).

\( \Box \)

We are now able to bound \( \circ \) using Lemma 11 in Abbasi-Yadkori et al. (2011).

**Lemma 9.** If \( \lambda \geq L^2 \) we have:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_j+1} 2\beta \|x_{t,a}\|_{\tilde{V}^{-1}_j} \leq \beta^* \sqrt{2d\ln \left( 1 + \frac{TL^2}{\lambda d} \right)} \left[ \sqrt{T \left( 1.25 + C + L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) \right)} + \sqrt{M_T \left( \frac{L^2}{\lambda + L^2 \beta^*} \right)} \right]
\]  

(93)

with \( \beta^* = 1 + \sqrt{AS} + \sigma \sqrt{d \left( \ln \left( 1 + \frac{L^2T^2}{\lambda d} \right) + \ln \left( \frac{\pi^2T^2}{6\delta} \right) \right)} \)

*Proof.* of Lem. 9 For any time \( t \) in batch \( j \), we have thanks to Lem. 8 that:

\[
\|x_{t,a}\|_{\tilde{V}^{-1}_j} \leq \sqrt{1 + \text{Tr} \left( \tilde{V}^{-1}_j \sum_{t=t_j+1}^{t_j+1} x_{t,a}x_{t,a}^T \right)} \|x_{t,a}\|_{\tilde{V}^{-1}_j}
\]  

(94)
with \( V_t = \lambda I_d + \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \). The rest of the proof relies on bounding \( \text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \). To do so, we have the following inequality, see Eq. (54):

\[
\text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) - \frac{L(t - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \leq \text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \leq \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) + \frac{L(t - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}
\]

Therefore during batch \( j \), \( \delta_t \leq 0.45 \) because the batch is not over so no condition is satisfied. Thanks to Prop. 7 with \( \varepsilon' = \frac{\varepsilon}{4L(t-1-t_j)} \):

\[
\forall t \in \{t_j + 1, \ldots t_j+1-1\} \quad \text{Tr} \left( A_j \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \leq C + \frac{t - 1 - t_j}{4t}
\]

But for \( t = t_j+1 \) we have either that \( \delta_{t+1} > 0.45 \) or \( t_{j+1} \geq (1 + \eta)t_j \):

- If \( \delta_{t+1} \leq 0.45 \), then \( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t_{j+1}-1} x_{l,a} x_{l,a}^T \right) \leq C + \frac{t_{j+1}-1-t_j}{4t} \)

- If \( \delta_{t+1} > 0.45 \), then \( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t_{j+1}-1} x_{l,a} x_{l,a}^T \right) \geq C - \frac{t_{j+1}-1-t_j}{4t} \) but \( \delta_{t_j+1} \leq 0.45 \) thus \( \text{Tr} \left( A_j \sum_{l=t_j+1}^{t_{j+1}-1} x_{l,a} x_{l,a}^T \right) \leq \frac{t_{j+1}-1-t_j}{4t} \). Therefore, using that \( \|x_{t,a}\|_{A_j}^2 \leq \lambda_{\max}(A_j)\|x_{t,a}\|_2^2 \leq L^2 \left( \frac{1}{\lambda} + \frac{1}{L\sqrt{\lambda + L^2}} \right) \). Hence, we have that:

\[
\text{Tr} \left( A_j \sum_{l=t_j+1}^{t_{j+1}-1} x_{l,a} x_{l,a}^T \right) \leq C + \frac{t - 1 - t_j}{4t} + L^2 \left( \frac{1}{\lambda} + \frac{1}{L\sqrt{\lambda + L^2}} \right) \quad (95)
\]

To sum up, for all \( t_j + 1 \leq t \leq t_{j+1} \):

\[
\text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right) \leq C + \frac{t - 1 - t_j}{4t} + L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) + \frac{L(t - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \quad (96)
\]

Overall, we have that:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}-1} \|x_{t,a}\|_{V_t^{-1}} \leq \sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}-1} \sqrt{1 + \text{Tr} \left( V_j^{-1} \sum_{l=t_j+1}^{t-1} x_{l,a} x_{l,a}^T \right)} \|x_{t,a}\|_{V_t^{-1}} \quad (97)
\]

\[
\leq \sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}-1} \|x_{t,a}\|_{V_t^{-1}} \left( 1 + C + \frac{t - 1 - t_j}{4t} + L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) + \frac{L(t - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \right) \quad (98)
\]

\[
\leq \frac{5}{4} + C + L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right) \sum_{t=1}^{T} \|x_{t,a}\|_{V_t^{-1}} + \sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}-1} \|x_{t,a}\|_{V_t^{-1}} \sqrt{\frac{L(t - 1 - t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} \quad (99)
\]

where the last inequality is due to Cauchy-Schwarz inequality. The first term in inequality Eq. (99) is bounded by using Lemma 29 in Ruan et al. (2020),

\[
\sum_{t=1}^{T} \|x_{t,a}\|_{V_t^{-1}}^2 \leq 2 \ln \left( \frac{\det(V_T)}{\det(V_0)} \right) \leq 2d \ln \left( 1 + \frac{TL^2}{\lambda d} \right) \quad (100)
\]
In addition, the last term in Eq. (99) is bounded by:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} \frac{L(t-1-t_j)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \leq \sum_{j=0}^{M_T-1} \frac{L(t_{j+1} - t_j)^2}{2t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \quad (101)
\]

But because of the second condition the length of batch \( j \), \( t_{j+1} - t_j \leq \eta t_j + 1 \). Therefore:

\[
\sum_{j=0}^{M_T-1} \frac{L(\eta t_j + 1)^2}{2t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \leq \sum_{j=0}^{M_T-1} \frac{L(\eta^2 t_j^2 + 1)}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}} \leq \sum_{j=0}^{M_T-1} \frac{\eta^2 + L}{(\lambda + L^2)^{3/2}} \leq \eta^2 M_T + \frac{LM_T}{(\lambda + L^2)^{3/2}} \quad (104)
\]

Putting everything together we get:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} \|x_{t,a_t}\|_V^{-1} \leq \sqrt{\frac{5}{4} + C + L^2 \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}\right)} \sqrt{2Td \ln \left(1 + \frac{TL^2}{\lambda d}\right)} \quad (105)
\]

\[
+ \sqrt{2d \ln \left(1 + \frac{TL^2}{\lambda d}\right)} \left(\eta^2 M_T + \frac{LM_T}{(\lambda + L^2)^{3/2}}\right) \quad (106)
\]

Hence the result using the upper bound on \( \tilde{\beta}(j) \) proved in Lem. 5.

Finally, the last term to bound is (C). Doing so is pretty similar to the end of the proof of Lem. 9.

**Lemma 10.** For all \( T \geq 1 \),

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} 2\tilde{\beta}(j) \sqrt{\frac{2L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} \leq 2\sqrt{2L} \max_j \tilde{\beta}(j) \left[ M_T + \sum_{j=0}^{M_T-1} \eta \sqrt{\frac{\sqrt{T_j}}{\lambda + L^2 T_j}}\right] \quad (107)
\]

with \( \beta^* = 1 + \sqrt{\lambda S} + \sigma \sqrt{d \ln \left(1 + \frac{L^2 T}{\lambda d}\right) + \ln \left(\frac{\pi^2 T^2}{65}\right)} \) and \( M_T \) the number of episodes.

**Proof.** of Lem. 10 We have:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} 2\tilde{\beta}(j) \sqrt{\frac{2L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} \leq 2\sqrt{2L} \max_j \tilde{\beta}(j) \sum_{j=0}^{M_T-1} \sqrt{\frac{1}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} (t_{j+1} - t_j) \quad (108)
\]

But Condition C ensures that for any batch \( j \), \( t_{j+1} - t_j \leq \eta t_j + 1 \), thus equation above can be bounded by:

\[
\sum_{j=0}^{M_T-1} \sum_{t=t_j+1}^{t_{j+1}} 2\tilde{\beta}(j) \sqrt{\frac{2L}{t_j^{3/2} \sqrt{\lambda + L^2 t_j}}} \leq 2\sqrt{2L} \max_j \tilde{\beta}(j) \left[ M_T + \sum_{j=0}^{M_T-1} \eta \sqrt{\frac{\sqrt{T_j}}{\lambda + L^2 T_j}}\right] \quad (109)
\]

\[
\leq 2\sqrt{2L} \max_j \tilde{\beta}(j) \left[ M_T + \sum_{j=0}^{M_T-1} \frac{\eta}{\sqrt{L}}\right] \quad (110)
\]

\[
\leq 2\sqrt{2L} M_T \max_j \tilde{\beta}(j) \left[ 1 + \frac{\eta}{\sqrt{L}}\right] \quad (111)
\]

Hence the result.
Finally, we can finish the proof of Thm. 1. We recall the Thm. 1

**Theorem.** Under Asm. 7, for any \( \delta > 0 \) and \( T \geq d \), there exists universal constants \( C_1, C_2 > 0 \) such that the regret of SecOFULLS (Alg. 3) is bounded with probability at least \( 1 - \delta \) by:

\[
R_T \leq C_1 \beta^* \left( \sqrt{\frac{5}{4} + C} dT \ln \left( \frac{TL}{\lambda d} \right) + \frac{L^{3/2}}{\sqrt{\lambda}} \ln(T) \right) + C_2 \beta^* M_T \max \left\{ \sqrt{L + \eta, \eta^2 + \frac{L}{\sqrt{\lambda + L^2}}} \right\}
\]

**Proof.** of Thm. 1. For any \( \delta > 0 \), let’s define the event \( E \) as in the proof of Lem. 4. Then conditioned on this event, we have using Lem. 4:

\[
R_T(\text{SecOFULLS}) \leq \sum_{j=0}^{M_T - 1} \sum_{t=t_j+1}^{t_j+1} \frac{4}{T} \left( 1 + 2 \bar{\beta}(j) \left[ 2t^{-1} + \frac{L}{t_j^{3/2} \sqrt{t_j L^2 + \lambda}} + L \sqrt{\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}} \right] \right)
\]

But using Lem. 5 to bound the first term of the RHS equation above also Lem. 6, 7, 8, and 10 to the bound the second term, we get:

\[
R_T(\text{SecOFULLS}) \leq \left( 1 + \ln(T) \right) \left[ 1 + \beta^* \left( 6 + L \sqrt{\frac{1}{\lambda}} + \frac{1}{\sqrt{\lambda}} + \frac{L}{\sqrt{\lambda}} \right) \right]
\]

\[
+ \beta^* \sqrt{2d \ln \left( 1 + \frac{TL^2}{\lambda d} \right)} \left[ \sqrt{\frac{5}{4} + C + L^2 \left( \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} \right)} \sqrt{T} + \left( \eta^2 + \frac{L}{(\lambda + L^2)^{3/2}} \right) M_T \right]
\]

\[
+ 2\sqrt{2L} M_T \beta^* \left[ 1 + \frac{\eta}{\sqrt{L}} \right]
\]

with \( \beta^* = 1 + \sqrt{X S + \sigma} \sqrt{d \ln \left( 1 + \frac{L^2 T}{\lambda d} + \ln \left( \frac{\pi^2 T^2}{60} \right) \right)} \) and \( M_T = 1 + \frac{d \ln \left( \frac{1 + \frac{L^2 T}{\lambda d}}{X + C - \frac{\ln \left( \frac{\pi^2 T^2}{60} \right)}{\sqrt{\lambda + L^2}} \right)} + \frac{\ln(T)}{\ln(1 + \eta^2)} \)

To conclude, we provide the proof of Cor. 3. This result is based on the fact that Alg. 3 is an instance of Alg. 2 with \( C = 0 \) and \( \eta = 0 \). However, taking \( C = \eta = 0 \) means that the number of batches in Lem. 9 and 10 is equal to \( T \). We show here how this modify the proof. First let’s recall Cor. 3

**Corollary.** Under Asm. 7, for any \( \delta > 0 \) and \( T \geq d \), there exist a universal constants \( C_1 > 0 \) such, with probability at least \( 1 - \delta \) by:

\[
R_T \leq C_1 \beta^* \cdot \left( \sqrt{T \ln \left( \frac{L^2 T}{\lambda d} \right)} + \frac{L \sqrt{T}}{\sqrt{\lambda \ln(T)}} \right)
\]

with \( \beta^* = 1 + \sqrt{X S + \sigma} \sqrt{d \ln \left( 1 + \frac{L^2 T}{\lambda d} + \ln \left( \frac{\pi^2 T^2}{60} \right) \right)} \).

**Proof.** of Cor. 3. Alg. 2 and 3 essentially uses the same function except when decide to update the parameters of the parameters. That is to say for the Alg. 3 we have \( M_T = T \), and \( t_{j+1} = 1 + t_j \) for all \( j \leq M_T \). Hence, we can still use Lem. 4 to ensure that with probability at least \( 1 - \delta \) that:

\[
R_T(\text{SecOFULL}) \leq \sum_{t=1}^{T} \frac{4}{T} \left( 1 + 2 \bar{\beta}(t) \left[ 2t^{-1} + \frac{L}{t_j^{3/2} \sqrt{t_j L^2 + \lambda}} + L \sqrt{\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}} \right] \right)
\]

\[
+ \sum_{t=1}^{T} 2 \bar{\beta}(t) \left[ s_k(t) + t^{-1} \right]
\]
Again the first term in the equation above can be bounded using Lem. 5:

\[
\sum_{t=1}^{T} \frac{4}{t} \left( 1 + 2\tilde{\beta}(t) \left[ \frac{2}{t} + \sqrt{\frac{L}{t^{3/2}\lambda^{2} + \lambda}} + L \left( \frac{1}{\lambda} + \frac{1}{\lambda^{1/2}} + \frac{L}{\sqrt{\lambda}} \right) \right] \right) \leq (1 + \ln(T)) \left[ 1 + \beta^* \left( 2 + L \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda^{1/2}} + \frac{L}{\sqrt{\lambda}}} \right) \right]
\]

with \( \beta^* = 1 + \sqrt{\lambda}d + \sigma \sqrt{d \left( \ln \left( 1 + \frac{L^2T}{\lambda d} \right) + \ln \left( \frac{\pi^2T^2}{6d} \right) \right)} \). In addition, the second term of Eq. (114) follows the same reasoning as in the proof of Thm. 1. First, we use Lem. 6 to decompose the second term.

\[
\sum_{t=1}^{T} \frac{2\tilde{\beta}(t)}{t} [\text{sq}_{t+1}(t) + t^{-1}] \leq \sum_{t=1}^{T} \frac{4\tilde{\beta}(t)}{t} + \sum_{t=1}^{T} 2\tilde{\beta}(t) \|x_{t,a}\|_{V^{-1}} := \text{(A)} + \sum_{t=1}^{T} 2\tilde{\beta}(t) \sqrt{\frac{2L}{t^{3/2}\lambda + L^2t}} := \text{(B)}
\]

Bounding (A) is possible thanks to Lem. 7. Whereas bounding (B) and (C) is a bit different than in the proof of Thm. 1. Bounding (B) is exactly Lem. 29 in Ruan et al. (2020). And finally, (C) can be bounded using standard sum-integral comparison.

\[
\text{(C)} \leq 2\sqrt{2L}\beta^* \sum_{t=1}^{T} \frac{1}{t} \leq 2\sqrt{2L}\beta^*(1 + \ln(T))
\]

Putting everything together we have:

\[
R_T(\text{SecOFUL}) \leq \beta^* \left( (1 + \ln(T))(4 + 2\sqrt{2L}) + 2\sqrt{2T\ln \left( 1 + \frac{L^2T}{\lambda d} \right)} \right) \leq (1 + \beta^* \left( 2 + L \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda^{1/2}} + \frac{L}{\sqrt{\lambda}}} \right) \right)
\]

with \( \beta^* = 1 + \sqrt{\lambda}d + \sigma \sqrt{d \left( \ln \left( 1 + \frac{L^2T}{\lambda d} \right) + \ln \left( \frac{\pi^2T^2}{6d} \right) \right)} \). \(\square\)

### D Implementation Details:

In this section, we further detail how SecOFULLS is implemented. In particular, we present the matrix multiplication and matrix-vector operations. For the experiments, we used the PALISADE library (development version v1.10.4) PAL (2020). This library automatically chooses most of the parameters used for the CKKS scheme. In particular the ring dimension of the ciphertext space is chosen automatically. In the end, the user only needs to choose four parameters: the maximum multiplicative depth (here chosen at 200), the number of bits used for the scaling factor (here 50), the batch size that is to say the number of plaintext slots used in the ciphertext (here 8) and the security level (here chosen at 128 bits for Fig. 5).

#### D.1 Matrix/Vector Encoding

Usually, when dealing with matrices and vectors in homomorphic encryption there are multiple ways to encrypt those. For example, with a vector \( y \in \mathbb{R}^d \) one can create \( d \) ciphertexts encrypting each value \( y_i \) for all \( i \leq d \). This approach is nonetheless expensive in terms of memory. An other approach is to encrypt directly the whole vector in a single ciphertext. A ciphertext is a polynomial \( (X \mapsto \sum_{i=0}^{N} a_i X^i) \) where each coefficient is used to encrypt a value of \( y (a_i = y_i \) for \( i \leq d \)). This second method is oftentimes preferred as it reduces memory usage.
It is possible to take advantage of this encoding method in order to facilitate computations, e.g., matrix multiplication, matrix-vector operation or scalar product. In this work, we need to compute the product of square matrices of size $d \times d$, and thus choose to encrypt each matrix/vector as a unique ciphertext (assuming $d \leq N$).

We have two different encoding for matrices and vectors. For a matrix $A = (a_{i,j})_{i \in \{0,\ldots,p-1\}, j \in \{0,\ldots,q-1\}}$ with $p, q \in \mathbb{N}$, we first transform $A$ into a vector of size $pq$, $\tilde{a} = (a_{0,0}, a_{0,1}, \ldots, a_{0,q-1}, \ldots, a_{1,0}, \ldots, a_{1,q-1}, \ldots, a_{p-1,q-1})$. This vector is then encrypted into a single ciphertext.

But for a vector $y \in \mathbb{R}^d$, we create a bigger vector of dimension $pq$ (here $p$ is a parameter of the encoding method for vectors), $\tilde{y} = (y_j)_{i \in \{0,\ldots,p-1\}, j \in \{0,\ldots,q-1\}} = (y_0, \ldots, y_{q-1}, y_{p+0}, \ldots, y_{pq-1})$. We choose those two encodings because the homomorphic multiplication operation of PALISADE only perform a coordinate-wise multiplication between two ciphertexts. Therefore, using this encoding, a matrix-vector product for a matrix $A \in \mathbb{R}^{p \times d}$, a vector $y \in \mathbb{R}^{q}$, a public key $pk$ can be computed as:

$$c_A \times c_y = \text{Enc}_{pk}(\tilde{a} \cdot \tilde{y}) = \text{Enc}_{pk}((a_{0,0}y_0, a_{0,1}y_1, \ldots, a_{0,q-1}y_{q-1}, a_{1,0}y_0, a_{1,1}y_1, \ldots, a_{1,q-1}y_{q-1}, \ldots, a_{p-1,q-1}y_{pq-1}))$$

with $\tilde{a}$ the encoding of $A$, $c_A = \text{Enc}_{pk}(\tilde{a})$, $\tilde{y}$ the encoding of $y$ of dimension $pq$ and $c_y = \text{Enc}_{pk}(\tilde{y})$, $\cdot$ the elementwise multiplication operation. Then using EvalSumCol (an implementation of the SumColVec method from Han et al. in the PALISADE library) to compute partial sums of the coefficients of $c_A \times c_y$, we get:

$$\text{EvalSumCol}(c_A \times c_y, p, q) = \text{Enc}_{pk}\left(\left(\sum_{j=0}^{q-1} a_{0,j}y_j, \ldots, \sum_{j=0}^{q-1} a_{1,j}y_j, \ldots, \sum_{j=0}^{q-1} a_{p-1,j}y_j\right)\right)$$

Finally, the matrix-vector product $Ay$ is computed by $\text{EvalSumCol}(c_A \times c_y, p, q)$ taking the coefficient at $(j + j \cdot p)_{j \in [p]}$.

### D.2 Matrix Multiplication

Using the encoding of App. D.1, we have a way to compute a matrix-vector product therefore computing the product between two square matrices $M, N \in \mathbb{R}^{p \times p}$ can be done using a series of matrix-vector products. However, this approach requires $p$ ciphertexts to represent a matrix. We then prefer to use the method introduced in Sec. 3 of Jiang et al. (2018). This method relies on the following identity for any matrices $M, N \in \mathbb{R}^{p \times p}$ and $i, j \in \{0, \ldots, p-1\}$:

$$\begin{equation}
(MN)_{i,j} = \sum_{k=0}^{p-1} M_{i,k}N_{k,j} = \sum_{k=0}^{p-1} M_{i,[i+k+j]p}N_{[i+k+j]p,j} = \sum_{k=0}^{p-1} \sigma(M)_{i,[i+k+j]p} \tau(N)_{[i+k+j]p,j} = \sum_{k=0}^{p-1} (\sigma^k \circ \sigma(M))_{i,j} (\psi^k \circ \tau(N))_{i,j}
\end{equation}
$$

where we define $\sigma, \tau, \psi$ and $\phi$ as:

- $\sigma(M)_{i,j} = M_{i,[i+j]p}$
- $\tau(M)_{i,j} = M_{[i+j]p,j}$
- $\psi(M)_{i,j} = M_{[i+1]p}$
- $\phi(M)_{i,j} = M_{[i+1]p,j}$
Table 1: Ratio of running time for SecOFULLS as a function of $\kappa$ for the bandit problem of Sec. 5. We use the running time with $\kappa = 128$ bits and $T = 130$ steps as a reference to compute the ratio between this time and the total time for $\kappa \in \{192, 256\}$.

| $\kappa$ (Bits) | Ratio Execution Time |
|-----------------|----------------------|
| 128             | 1                    |
| 192             | 1.016                |
| 256             | 1.026                |

and $[.]_p$ is the modulo operator. Therefore, using Eq. (119) we have that computing the product between $M$ and $N$ can simply be done by computing a component-wise multiplication between $(\phi^k \circ \sigma(M))_{i,j}$ and $(\psi^k \circ \tau(N))_{i,j}$ for all $k \in \{0, \ldots, p - 1\}$. Those quantities can in turn be easily computed thanks to a multiplication between a plaintext and a ciphertext (this does not impact the depth of the ciphertext).

D.3 Influence of the Security Level

Finally, we investigate the influence of the security level $\kappa$ on the running time and regret of SecOFULLS. As mentionned in Sec. 5 the security parameter $\kappa$ ensures that an attacker has to perform at least $2^\kappa$ operations in order to decrypt a ciphertext encrypted using an homomorphic encryption scheme. But, the security parameter also has an impact on the computational efficiency of our algorithm. Indeed the dimension $N$ of the ciphertext space, i.e., the degree of the polynomials in $\mathbb{Z}[X]/(X^N + 1)$, increases with the multiplicative depth $D$ and $\kappa$. However, this means that our algorithm has to compute operations with polynomials of higher dimensions hence more computationally demanding.

The library PALISADE allows us to choose $\kappa \in \{128, 192, 256\}$. We executed SecOFULLS with the same parameter and the same environment of Sec. 5 except for the parameter $\kappa$ which now varies in $\{128, 192, 256\}$. First, we investigate the regret of for each parameter $\kappa$, this parameter should have no impact on the regret SecOFULLS, as showed in Fig. 2.

Second, we investigate the running time for each $\kappa \in \{128, 192, 256\}$. Table 1 shows the ratio between the total computation time of 130 steps using the environment described in Sec. 5 with SecOFULLS for different security parameters and $\kappa = 128$ bits. In order to investigate only the effect of the parameter $\kappa$, the results in Table 1 are expressed as a ratio. For reference, the total time for $T = 130$ steps and $\kappa = 128$ bits was 20 hours and 39 minutes. As we observe in Table 1 the impact on the security parameter is around 1% and 2% of the total computation time for 128 bits. This increase in computation time represents between 20 and 40 minutes of computation which in some applications can be prohibitive.