Quantum criticality of the Lipkin-Meshkov-Glick Model in terms of fidelity susceptibility

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We study the critical properties of the Lipkin-Meshkov-Glick Model in terms of the fidelity susceptibility. By using the Holstein-Primakoff transformation, we obtain explicitly the critical exponent of the fidelity susceptibility around the second-order quantum phase transition point. Our results provide a rare analytical case for the fidelity susceptibility in describing the universality class in quantum critical behavior. The different critical exponents in two phases are non-trivial results, indicating the fidelity susceptibility is not always extensive.

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I. INTRODUCTION

The Lipkin-Meshkov-Glick (LMG) model [1] was introduced in nuclear physics. It describes a cluster of mutually interacting spins in a transverse magnetic field. In condensed matter physics, this model is associated with a system of infinite coordination number. In earlier time, scaling behaviors of critical observables have been studied by mean field analysis [2], while recently the finite-size scaling of this model was studied by the $1/N$ expansion in the Holstein-Primakoff single boson representation [3] and by the continuous unitary transformations (CUT) [4, 5, 6]. Meanwhile, a rich structure of four different regions is revealed in the parameter space through a careful scrutiny on the spectrum [7]. Besides, the quantum criticality has been investigated by studying its entanglement properties [8, 9, 10, 11, 12]. Both the first- and second-order quantum phase transitions (QPTs) [13] have been revealed, in the antiferromagnetic and the ferromagnetic cases respectively [8, 9].

Regarding the QPT itself, the ground state of a system would undergo a significant structural change at certain critical point. This primary observation suggests a new description of QPTs in terms of fidelity [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], a concept introduced in quantum information theory [12]. Mathematically it is the overlap between different regions is revealed in the parameter space through a careful scrutiny on the spectrum [7]. Besides, the quantum criticality has been investigated by studying its entanglement properties [8, 9, 10, 11, 12]. Both the first- and second-order quantum phase transitions (QPTs) [13] have been revealed, in the antiferromagnetic and the ferromagnetic cases respectively [8, 9].

In this paper, we explicitly compute the ground-state fidelity susceptibility and its critical exponent of the LMG model. Numerical analysis is also performed to check with our analytic calculations. We show that, the $1/N$ expansion in the Holstein-Primakoff transformation is sufficient to determine the critical exponent of the fidelity susceptibility $\chi_f$. In addition, we revealed two distinct critical exponents in two phases which is not a general feature. Therefore, our findings not only suggest another route on understanding the quantum criticality of the LMG model, but also show the fidelity susceptibility is not always extensive in describing the universality class of a quantum many-body system.

This paper consists of five sections. In Sec. II we review the Hamiltonian, symmetry, and conserved quantities of the LMG model. In Sec. III we diagonalize the model Hamiltonian and compute the fidelity susceptibility in the anisotropic model. In Sec. IV we perform finite size scaling analysis and discuss the scaling relation between different exponents. Finally, we give a brief summary in Sec. V.

II. THE MODEL HAMILTONIAN

The Hamiltonian of the LMG model reads

$$H = -\frac{\lambda}{N} \sum_{i<j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - h \sum_i \sigma_i^z, \quad (1)$$

$$= -\frac{2\lambda}{N} (S_z^2 + \gamma S_y^2) - 2hS_z + \frac{\lambda}{2} (1 + \gamma), \quad (2)$$

$$= -\frac{\lambda}{N} (1 + \gamma) (S^2 - S_z^2 - N/2) - 2hS_z$$

$$- \frac{\lambda}{2N} (1 - \gamma) (S_x^2 + S_y^2), \quad (3)$$

where $\sigma_\kappa (\kappa = x, y, z)$ are the Pauli matrices, $S_\kappa = \sum_i \sigma_i^\kappa/2$, and $S_\pm = S_x \pm iS_y$. The prefactor $1/N$ is necessary to ensure finite energy per spin in the thermodynamic limit. It is understood that the total spin and the parity are the conserved quantities, i.e.,

$$[H, S^2] = \left[H, \prod_i \sigma_i^z \right] = 0. \quad (4)$$

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In addition, in the isotropic case $\gamma = 1$, one has $[H, S_z] = 0$ and simultaneous eigenstates can be found. In the main context, the following parameter space is considered: $\lambda = 1, |\gamma| < 1, h \geq 0$. We take $h \geq 0$ as the spectrum is invariant under the transformation $h \leftrightarrow -h$. In addition, as a common practice we only consider the maximum spin sector $S = N/2$ which contains the lowest energy state.

III. CRITICAL BEHAVIOR OF THE FIDELITY SUSCEPTIBILITY

We briefly review the concept of the fidelity susceptibility here. Suppose there is a Hamiltonian of a general form as

$$H = H_0(\gamma) + f(h)H_f,$$

for $H_f$ is defined as the driving term of the system, which simply does not commute with $H_0$. The function $f(h)$ coupled to $H_f$ is often considered as the linear external field $f(h) = h$. Then the fidelity susceptibility is defined as \[18, 19\]

$$\chi_f = \left[ \frac{df(h)}{dh} \right] \sum_{n \neq 0} \frac{|\langle n| H_f | 0 \rangle|^2}{E_n - E_0},$$

where $E_n$ and $|n\rangle$ stand for the $n$th eigenenergies and eigenstates of the (whole) Hamiltonian respectively.

The fidelity susceptibility is well-defined for a non-degenerate ground state of the continuous variable $h$, but it is not suitable to deal with states with good quantum numbers. The LMG model undergoes ground state level crossing when $\gamma = 1$, the ground states are assigned the magnetization as the quantum numbers.

We put our focus on the fidelity susceptibility for an arbitrary isotropy $|\gamma| < 1$. One resolution is to use the Bethe-Ansatz solution \[27, 28\], which is rather complicated. So we adopt the $1/N$ expansion method which was used extensively by Dusuel and Vidal \[4, 5\], that corresponds to the large $N$ limit.

The $1/N$ expansion method is done under the Holstein-Primakoff boson representation \[3\] framework. In low energy spectrum the spin operators in the $S = N/2$ subspace are mapped into boson operators:

$$S_z = S - a^\dagger a, \quad S_+ = (2S - a^\dagger a)^{1/2}a = N^{1/2}(1 - a^\dagger a/N)^{1/2}a = S^z, \quad S_- = 2S_+ = a^\dagger a^{1/2},$$

where $a(a^\dagger)$ is the standard bosonic annihilation (creation) operator satisfying $[a, a^\dagger] = 1$. The above transformation is valid when $h \geq 1$, but when $0 < h < 1$ it can also be used through semi-classical treatment \[4, 5\]. This representation is also known as the spin-wave theory. It is well adapted to the computation of the low-energy physics when $(a^\dagger a)/N \ll 1$. After inserting these expressions of the spin operators in Eq. \[4\], one can approximate the square roots as one and express the result in normal ordered form with respect to the boson vacuum state. Keeping terms of order $(1/N)^{-1}, (1/N)^{-1/2}$ and $(1/N)^0$ for $h \geq 1$ (in which the approximation is justified), the Hamiltonian becomes

$$H = -hN + (2h - 1 + \gamma)a^\dagger a - \frac{1 - \gamma}{2}(a^2 + a^\dagger),$$

The above Hamiltonian can be diagonalized by a standard Bogoliubov transformation

$$a^\dagger = \cosh(\Theta/2)b^\dagger + \sinh(\Theta/2)b,$$

$$a = \sinh(\Theta/2)b^\dagger + \cosh(\Theta/2)b,$$

where $b(b^\dagger)$ is the quasi-bosonic annihilation (creation) operator, and

$$\tanh[\Theta(h \geq 1)] = \frac{1 - \gamma}{2h - 1 + \gamma},$$

then the Hamiltonian is diagonalized as

$$H = -h(N + 1) + 2\sqrt{(h - 1)(h - \gamma)} \left( b^\dagger b + \frac{1}{2} \right).$$

Thus the low-energy spectrum of the model is mapped to the spectrum of a simple harmonic oscillator. The eigenstates are just $|n\rangle$, where $b^\dagger b|n\rangle = n|n\rangle$. We consider the driving Hamiltonian $H_f$ responsible for the QPT,

$$H_f = -\sum_i \sigma^i_\xi = -2S_z.$$

By transforming them into combinations of $b$ and $b^\dagger$ operators, the fidelity susceptibility is calculated as

$$\chi_f = \frac{(1 - \gamma)^2}{32(h - 1)^2(h - \gamma)^2}.$$

The derivation above is only valid for $h \geq 1$, for $0 < h < 1$ the calculation is actually similar to the above case of $h \geq 1$, provided that one first rotates the $z$ axis to bring it along the classical spin direction. We do not show it explicitly here, but interested readers are recommended to refer to Ref. \[4, 5\]. We simply quote the main result, after all the procedures the Hamiltonian becomes:

$$H = -\frac{(1 + h^2)}{2} N - \frac{1 - \gamma}{2} + 2\sqrt{(1 - h^2)(1 - \gamma)} \left( b^\dagger b + \frac{1}{2} \right).$$

The driving Hamiltonians also takes a different form:

$$-\sum_i \sigma^i_\xi = -2S_z$$

$$= -2\left( -\sqrt{1 - h^2} S_z + \hbar \tilde{S}_z \right),$$

for the HP transformation is done on the $\tilde{S}$ operators. The fidelity susceptibilities are then obtained accordingly:

$$\chi_f = \frac{N}{4 \sqrt{(1 - h^2)(1 - \gamma)}} + \frac{h^2(h^2 - \gamma)^2}{32(1 - \gamma)^2(1 - h^2)^2}. $$
Thus we obtained $\chi_f$ of the anisotropic LMG model in large $N$ limit. We first see the effect of isotropy to the fidelity susceptibility. It dominates when $h < 1$, but fades out for large $h$. Especially in the isotropic limit, when $\gamma \to 1$, $\chi_f$ diverges when $h < 1$, but tends to zero when $h > 1$. This is the effect of the level-crossing points in the thermodynamic limit. They together form a region of criticality, and the system undergoes continuous level crossing. The fidelity susceptibility responds drastically while moving along $h$. But when $h > 1$, there are no further critical points, $\chi_f$ naturally measures zero when moving along $h$ because we have $[H_0, H_I] = 0$.

An interesting observation is $\chi_f$ behaves extensively when $h < 1$ even in the large $N$ limit. When discarding the extensive part of Eq. (17), we arrive a zero point at $h = \sqrt{7}$, which does not fit with numerical analysis [Fig. 1]. This discrepancy may be eliminated by adopting other transformations of the driving Hamiltonian. Particularly, the flow of operators in the LMG model haven been studied by the continuous unitary transformation (CUT) method [4, 5]. However, such discrepancy would not hinder us from getting the correct critical exponent of the fidelity susceptibility.

Let us emphasize the intensive property of the fidelity susceptibility, which measures the average response to some driving Hamiltonians. Its divergence should correspond to a critical point of a second-order QPT rather than to the increasing system size. In order to predict the critical exponent correctly, we should average the fidelity susceptibility whenever necessary. To the leading order, Eq. (17) becomes

$$\chi_f = \frac{1}{N} \frac{1}{4(1-h^2)(1-\gamma)}.$$  \hspace{1cm} (18)

Then it comes to a key result of our paper: $\chi_f$ bears different critical exponents across the critical point. It diverges as $(1-h)^{\frac{1}{2}}$ when $h < 1$, $(h-1)^{\frac{1}{2}}$ when $h > 1$. It is unlike the Ising model in a transverse field [15] nor the one-dimensional asymmetric Hubbard model [22], where the critical exponent is a single number over the phases.

### IV. Finite Size Scaling Analysis

To illustrate the scaling behavior of the fidelity susceptibility, we perform the exact diagonalization (ED) to solve the spectrum of $H$ and then calculate the corresponding fidelity susceptibility numerically.

Let us recall the fidelity susceptibility scaling analysis performed in the asymmetric Hubbard model [22]. According to the scaling ansatz [29] and the obvious power-law divergence observed in Fig. 1, the rescaled fidelity susceptibility around its maximum point at $h_{\text{max}}$ is a simple function of a scaling variable, i.e.

$$\frac{\chi_f - \chi_f}{\chi_f} = f[N^\nu(h-h_{\text{max}})],$$  \hspace{1cm} (19)

where $f(x)$ is the scaling function and $\nu$ is the correlation length critical exponent. This function is universal and does not depend on the system size, as shown in Fig. 2 for cases of $\gamma = 0.5, 0$ and $\gamma = -0.5$. Remarkably, the critical exponent $\nu$ for three cases are very close. This observation strongly implies that $\nu$ is a universal constant and does not depend on the parameters $\gamma$ and $h$.

In recent studies of the fidelity susceptibility in critical phenomena, it was pointed out that the intensive fidelity susceptibility scales generally like [21, 22]

$$\chi_f \propto N^{-\frac{1}{2}}(h-h_{\text{max}})^{\nu},$$  \hspace{1cm} (20)

around the critical point. In the last section, we have already obtained

$$\alpha = \left\{\begin{array}{ll} 2, & h > 1 \\ \frac{1}{2}, & 0 \leq h < 1 \end{array}\right.,$$  \hspace{1cm} (21)

which is also a universal constant. Then if the maximum point of the intensive fidelity susceptibility scales like

$$\chi_f \propto N^{\mu},$$  \hspace{1cm} (22)

the scaling ansatz also implies another important relation, i.e.

$$\alpha = \frac{\mu}{\nu}.$$  \hspace{1cm} (23)

We try to confirm this equality in numerically. In Fig. 2, Eq. (19) is best fitted with $\nu = 0.665$. The case to determine $\mu$ is more subtle. It is because Eq. (19) remains the same form even for averaged $\chi_f$, but the maximum of $\chi_f$ does not. To resolve this problem, we first determine $\mu$ from the “bare” $\chi_f$. By using least square fit method, we evaluated “bare” $\mu$ for at different $\gamma$. The numerical details are shown in table 1. However, the exponent $\mu$ does not converge perfectly. We compare the $\mu$ obtained in a range of $[2^{12}, 2^{16}]$, and those from the range $[2^8, 2^{16}]$. The results converge better for larger scaling regions. According to the trend of $\mu$ in larger system sizes, we roughly estimate $\mu = 1.33$ with three effective digits [Fig. 3].

When $h > 1$, $\chi_f$ is observed to be intensive [Fig. 1]. With the estimated $\mu$ and $\nu$, the equality (23) is satisfied with $\alpha = 2$.
and driving parameter is a function of

\[ \gamma \]

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\[ \max \] of the fidelity susceptibility.

FIG. 2: (color online) The finite size scaling analysis is performed for the case of power-law divergence at \( \gamma = 0.5 \) (LEFT), \( \gamma = 0 \) (MIDDLE) and \( \gamma = -0.5 \) (RIGHT) for system sizes \( N = 2^n \) \((n = 12, 13, 14, 15, 16) \). The fidelity susceptibility is considered as a function of system size and driving parameter is a function of \( N(h - h_{\max}) \) only, with the correlation length critical exponent \( \nu = 0.665 \).

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