Sample Complexity of Automata Cascades

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Abstract

Every automaton can be decomposed into a cascade of basic automata. This is the Prime Decomposition Theorem by Krohn and Rhodes. We show that cascades allow for describing the sample complexity of automata in terms of their components. In particular, we show that the sample complexity is linear in the number of components and the maximum complexity of a single component, modulo logarithmic factors. This opens to the possibility of learning automata representing large dynamical systems consisting of many parts interacting with each other. It is in sharp contrast with the established understanding of the sample complexity of automata, described in terms of the overall number of states and input letters, which implies that it is only possible to learn automata where the number of states is linear in the amount of data available. Instead our results show that one can learn automata with a number of states that is exponential in the amount of data available.

1 Introduction

Automata are fundamental in computer science. They are one of the simplest models of computation, with the expressive power of regular languages, placed at the bottom of the Chomsky hierarchy. They are also a mathematical model of finite-state dynamical systems. In learning applications, automata allow for capturing targets that exhibit a time-dependent behaviour: namely, functions over sequences such as time-series, traces of a system, histories of interactions between an agent and its environment. Automata are typically viewed as state diagrams, where states and transitions are the building blocks of an automaton, cf. (Hopcroft and Ullman 1979). Accordingly, classes of automata are described in terms of the number of states and input letters. Learning from such classes requires an amount of data that is linear in the number of states and letters (Ishigami and Tani 1997). Practically it means that, in order to learn large dynamical systems made of many components, the amount of data required is exponential in the number of components, as the number of states of a system will typically be exponential in the number of its stateful components.

We propose automata cascades as a structured, modular, way to describe automata as complex systems made of many components connected in an acyclic way. Our cascades are strongly based on the theory of Krohn and Rhodes, which says that every automaton can be decomposed into a cascade of basic components called prime automata (Krohn and Rhodes 1965). Conversely, the theory can be seen as prescribing which components to use in order to build certain classes of automata, and hence obtain a certain expressivity. For example, we can cascade so-called flip-flop automata in order to build all counter-free automata, and hence obtain the expressivity of well-known logics such as monadic first-order logic on finite linearly-ordered domains (McNaughton and Papert 1971) and the linear temporal logic on finite traces LTLf (De Giacomo and Vardi 2013).

We focus on cascades as a means to learn automata. Our cascades are designed for a fine control of their sample complexity. We show that—ignoring logarithmic factors—the sample complexity of automata cascades is at most linear in the product of the number of components and the maximum complexity of a single component. Notably, the complexity of a single component does not grow with the number of components in the cascade. We carry out the analysis both in the setting where classes of automata cascades are finite, and in the more general setting where they can be infinite. For both cases, we obtain bounds of the same shape, with one notable difference that for infinite classes we incur a logarithmic dependency on the maximum length of a string.

Figure 1: Diagram of a fully-connected cascade of three components, based on a figure in (Maler 1990). Boxes denote components. Arrows depict flow of information: component A reads the external input; component B reads the external input together with the output of A; component C reads the external input together with the outputs of A and B. The overall output is the output of C.
Overall, our results show that the sample complexity of automata can be decoupled from the the number of states and input letters. Rather, it can be described in terms of the components of a cascade capturing the automaton. Notably, the number of states of such an automaton can be exponential in the number of components of the cascade.

We see the opportunity for cascades to unlock a greater potential of automata in learning applications. On one hand, automata come with many favourable, well-understood, theoretical properties, and they admit elegant algorithmic solutions. On the other hand, the existing automata learning algorithms have a complexity that depends directly on the number of states. This hurts applications where the number of states grows very fast, such as non-Markov reinforcement learning (Toro Icarte et al. 2018; De Giacomo et al. 2019; Gaon and Brafman 2020; Brafman and De Giacomo 2019; Abadi and Brafman 2020; Xu et al. 2020; Neider et al. 2021; Jothimurugan et al. 2021; Ronca, Paludo Licks, and De Giacomo 2022).

Given our favourable sample complexity results, automata cascades have a great potential to extend the applicability of automata learning in large complex settings.

Before concluding the section, we introduce our running example, that is representative of a class of tasks commonly considered in reinforcement learning. It is based on an example from (Andreas, Klein, and Levine 2017).

**Example 1. Consider a Minecraft-like domain, where an agent has to build a bridge by first collecting some raw materials.**

**2 Preliminaries**

**Functions.** For $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, their composition $f \circ g : X \rightarrow Z$ is defined as $(f \circ g)(x) = g(f(x))$.

For $f : X \rightarrow Y$ and $h : X \rightarrow Z$, their cross product $f \times h : X \rightarrow Y \times Z$ is defined as $(f \times h)(x) = (f(x), h(x))$.

A class of functions is **uniform** if all functions in the class have the same domain and codomain. Given a tuple $t = \langle x_1, \ldots, x_n \rangle$, and a subset $J \subseteq [1, n]$, the projection $\pi_J(t)$ is $\langle x_{j_1}, \ldots, x_{j_m} \rangle$ where $j_1, \ldots, j_m$ is the sorted sequence of elements of $J$. Furthermore, $\pi^a_m$ denotes the class of all projections $\pi_J$ for $J$ a subset of $[1, a]$ of cardinality $m$. We write $I$ for the identity function, and $\log$ for $\log_2$.

**String Functions and Languages.** An **alphabet** $\Sigma$ is a set of elements called **letters**. A **string over an alphabet** $\Sigma$ is an expression $\sigma_1 \ldots \sigma_r$ where each letter $\sigma_i$ is from $\Sigma$. The **empty string** is denoted by $\varepsilon$. The set of all strings over $\Sigma$ is denoted by $\Sigma^*$. A factored alphabet is of the form $X^a$ for some set $X$ called the **domain** and some integer $a \geq 1$ called the **arity**. A **string function** is of the form $f : \Sigma^* \rightarrow \Gamma$ for $\Sigma$ and $\Gamma$ alphabets. Languages are a special case of string functions; namely, when $f : \Sigma \rightarrow \{0, 1\}$ is an indicator function, it can be equivalently described by the set $\{x \in \Sigma^* \mid f(x) = 1\}$, that is called a **language**.

**3 Learning Theory**

We introduce the problem of learning, following the classical perspective of statistical learning theory (Vapnik 1998).

**Learning Problem.** Consider an input domain $X$ and an output domain $Y$. For example, in the setting where we classify strings over an alphabet $\Sigma$, the input domain $X$ is $\Sigma^*$ and the output domain $Y$ is $\{0, 1\}$. The **problem of learning** is that of choosing, from an **admissible class** $\mathcal{F}$ of functions from $X$ to $Y$, a function $f$ that best approximates an unknown **target function** $f_0 : X \rightarrow Y$, not necessarily included in $\mathcal{F}$. The quality of the approximation of $f$ is given by the overall discrepancy of $f$ with the target $f_0$. On a single domain element $x$, the discrepancy between $f(x)$ and $f_0(x)$ is measured as $L(f(x), f_0(x))$, for a given **loss function** $L : Y \times Y \rightarrow \{0, 1\}$. The overall discrepancy is the expectation $\mathbb{E}[L(f(x), f_0(x))]$ with respect to an underlying probability distribution $P$, and it is called the **risk** of the function, written $R(f)$. Then, the goal is to choose a function $f \in \mathcal{F}$ that minimises the risk $R(f)$, when the underlying probability distribution $P$ is unknown, but we are given a **sample** $Z_t$ of i.i.d. elements $x_t \in X$ drawn according to $P(x_t)$ together with their labels $f_0(x_t)$; specifically, $Z_t = z_1, \ldots, z_t$ with $z_t = (x_t, f_0(x_t))$.

**Sample Complexity.** We would like to establish the minimum sample size $\ell$ sufficient to identify a function $f \in \mathcal{F}$ such that

$$R(f) - \min_{f \in \mathcal{F}} R(f) \leq \epsilon$$

with probability at least $1 - \eta$. We call such $\ell$ the sample complexity of $\mathcal{F}$, and we write it as $S(\mathcal{F}, \epsilon, \eta)$. When $\epsilon$ and $\eta$ are considered fixed, we write it as $S(\mathcal{F})$.

**Sample Complexity Bounds for Finite Classes.** When the set of admissible functions $\mathcal{F}$ is finite, its sample complexity can be bounded in terms of its cardinality, cf. (Shalev-Shwartz and Ben-David 2014). In particular,

$$S(\mathcal{F}, \epsilon, \eta) \in O \left( (\log |\mathcal{F}| - \log \eta) / \epsilon^2 \right).$$

Then, for fixed $\epsilon$ and $\eta$, the sample complexity $S(\mathcal{F})$ is $O(\log |\mathcal{F}|)$, and hence finite classes can be compared in terms of their cardinality.

**4 Automata**

This section introduces basic notions of automata theory, with some inspiration from (Ginzburg 1968; Maler 1990).

An automaton is a mathematical description of a stateful machine that returns an output letter on every input string. At its core lies the mechanism that updates the internal state upon reading an input letter. This mechanism is captured by the notion of semiautomaton. An $n$-state semiautomaton is a tuple $D = (\Sigma, Q, \delta, q_{init})$ where: $\Sigma$ is an alphabet called the **input alphabet**; $Q$ is a set of $n$ states called states; $\delta : Q \times \Sigma \rightarrow Q$ is a function called **transition function**; $q_{init} \in Q$ is called **initial state**. The transition function is recursively extended to non-empty strings as $\delta(q, \sigma_1 \sigma_2 \ldots \sigma_m) = \delta(\delta(q, \sigma_1), \sigma_2 \ldots \sigma_m)$, and to the empty string as $\delta(q, \varepsilon) = q$. The result of executing a semiautomaton $D$ on an input string is $D(\sigma_1 \ldots \sigma_m) = \text{state}$.
Figure 2: State diagram of the automaton for Example 2.

4.1 Existing Sample Complexity Results

Classes of automata are typically defined in terms of the cardinality $k$ of the input alphabet (assumed to be finite) and the number $n$ of states. The existing result on the sample complexity of automata is for such a family of classes.

**Theorem 1** (Ishigami and Tani, 1997). Let $A(k,n)$ be the class of $n$-state acceptors over the input alphabet $[1,k]$. Then, the sample complexity of $A(k,n)$ is $\Theta(k \cdot n \cdot \log n)$.

Consequently, learning an acceptor from the class of all acceptors with $k$ input letters and $n$ states requires an amount of data that is at least $k \cdot n$. Such a dependency is also observed in the existing automata learning algorithms, e.g., (Angluin 1987; Ron, Singer, and Tishby 1996, 1998; Clark and Thollard 2004; Palmer and Goldberg 2007; Balle, Castro, and Gavalda 2013, 2014). More recently, there has been an effort in overcoming the direct dependency on the cardinality $k$ of the input alphabet, through symbolic automata (Mens and Maler 2015; Maler and Mens 2017; Argyros and D’Antoni 2018), but their sample complexity has not been studied.

5 Automata Cascades

We present the formalism of automata cascades, strongly based on the cascades from (Krohn and Rhodes 1965)—see also (Ginzburg 1968; Maler 1990; Dömösi and Nehaniv 2005). The novelty of our formalism is that every cascade component is equipped with mechanisms for processing of inputs and outputs. This allows for (i) controlling the complexity of components as the size of cascade increases; and (ii) handling large (and even infinite) input alphabets.

**Definition 1.** An automata cascade is a sequence of automata $A_1 \times \cdots \times A_d$ where each $A_i$ is called a component of the cascade and it is of the form

$$(\Sigma_1 \times \cdots \times \Sigma_d, Q_i, \delta_i, q_i^{\text{init}}, \Sigma_{i+1}, \theta_i).$$

The function implemented by the cascade is the one implemented by the automaton $\langle \Sigma_1, Q, \delta, q^{\text{init}}, \Sigma_{d+1}, \theta \rangle$ having set of states $Q = Q_1 \times \cdots \times Q_d$, initial state $q^{\text{init}} = \langle q_1^{\text{init}}, \ldots, q_d^{\text{init}} \rangle$, transition and output functions defined as

$$(\delta(q_1, \ldots, q_d), \sigma) = (\delta_1(q_1, \sigma_1), \ldots, \delta_d(q_d, \sigma_d)),$$

$$\theta(q_1, \ldots, q_d, \sigma) = \theta_d(q_d, \sigma_d),$$

where each component reads the recursively-defined input

$$\sigma_1 = \sigma, \text{ and } \sigma_{i+1} = (\sigma_i, \theta_i(q_i, \sigma_i)).$$

A cascade is simple if $\theta_i(q, \sigma) = q_i$ for every $i \in [1, d - 1]$.

The components of a cascade are arranged in a sequence. Every component reads the input and output of the preceding component, and hence, recursively, it reads the external input.
input together with the output of all the preceding components. A cascade architecture is depicted in Figure 1. The external input is the input to the first component, and the overall output is the one of the last component.

As components of a cascade, we consider automata on factored alphabets that first apply a projection operation on their input, then apply a map to a smaller internal alphabet, and finally transition based on the result of the previous operations.

**Definition 2.** An \( n \)-state automaton is a tuple \( A = (X^n, J, \Pi, \phi, Q, \delta, q_{\text{init}}, \Gamma, \theta) \) where \( X^n \) is the factored input alphabet; \( J \subseteq [1, a] \) is the dependency set and its cardinality \( m \) is the degree of dependency; \( \Pi \) is the finite internal alphabet; \( \phi : X^m \to \Pi \) is an input function that operates on the projected input tuples; \( Q \) is a set of \( n \) states; \( \delta : Q \times \Pi \to Q \) is the transition function on the internal letters; \( q_{\text{init}} \in Q \) is the initial state; \( \Gamma \) is the output alphabet; and \( \theta : Q \times X^m \to \Gamma \) is an output function that operates on the projected input tuples. The automaton induced by \( A \) is the automaton \( A' = (\Sigma, Q, \delta, \Pi, q_{\text{init}}, \Gamma, \theta_J) \) where the input alphabet is \( \Sigma = X^n \), the transition function is \( \delta_J(q, \sigma) = \delta(q, \phi(\pi_J(\sigma))) \), and the output function is \( \theta_J(q, \sigma) = \theta(q, \pi_J(\sigma)) \). The core semiautomaton of \( A \) is the core semiautomaton of \( A' \). The string function implemented by \( A \) is the one implemented by the induced automaton.

The above definition adds two key aspects to the standard definition of automaton. First, the projection operation \( \pi_J \), that allows for capturing the dependencies between the components in a cascade. Although every component receives the output of all the preceding components, it may use only some of them, and hence the others can be projected away. The dependency set \( J \) corresponds to the indices of the input tuple that are relevant to the component. Second, the input function \( \phi \), that maps the result of the projection operation to an internal letter. The rationale is that many inputs trigger the same transitions, and hence they can be mapped to the same internal letter. This particularly allows for decoupling the size of the core semiautomaton from the cardinality of the input alphabet—in line with the mechanisms of symbolic automata (Maler and Mens 2017; Argyros and D’Antoni 2018).

### 6 Expressivity of Automata Cascades

Cascades can be built out of any set of components. However, the theory by Krohn and Rhodes identifies a set of prime automata that is a sufficient set of components to build cascades, as it allows for capturing all automata. They are, in a sense, the building blocks of automata. Moreover, using only some prime automata, we obtain specialised expressivity results.

#### 6.1 Prime Components

Prime automata are partitioned into two classes. The first class of prime automata are flip-flops, a kind of automaton that allows for storing one bit of information.

**Definition 3.** A flip-flop is a two-state automaton \( \langle X^n, J, \Pi, \phi, Q, \delta, q_{\text{init}}, \Gamma, \theta \rangle \) where \( \Pi = \{\text{set}, \text{reset}, \text{read}\} \).

**Definition 4.** An \( n \)-counter is an \( n \)-state automaton \( \langle X^n, J, \Pi, \phi, Q, \delta, q_{\text{init}}, \Gamma, \theta \rangle \) where \( \Pi = \{\text{inc}, \text{read}\} \), \( Q = [0, n - 1] \), and the transition function satisfies the following two identities:

\[
\delta(i, \text{read}) = i, \quad \delta(i, \text{inc}) = i + 1 \pmod{n}.
\]

An \( n \)-counter is prime if \( n \) is a prime number.

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**Figure 3:** Some of the simplest prime automata: two flip-flops (left), and a 5-counter (right).

\[ Q = \{0, 1\} \] and the transition function satisfies the following three identities:

\[
\delta(q, \text{read}) = q, \quad \delta(q, \text{set}) = 1, \quad \delta(q, \text{reset}) = 0.
\]

Figure 3 depicts two flip-flops. The top flip-flop implements a rewritable bit of memory, since it can be set, reset, and read—e.g., it captures whether ‘an event has just happened’. The bottom flip-flop implements a write-once bit of memory, since it cannot be reset—e.g., it captures whether ‘an event has ever happened’.

We capture the task of our running example with a cascade where each task is captured exactly by a flip-flop.

**Example 3.** The sequence task of our running example is captured by the cascade

\[
A_{\text{wood}} \times A_{\text{iron}} \times A_{\text{fire}} \times A_{\text{steel}} \times A_{\text{factory}}
\]

where each component is a flip-flop that outputs its current state. The diagram for the cascade is shown in Figure 4, where getWood corresponds to \( A_{\text{wood}} \), and similarly for the other components. All components read the input, and only \( A_{\text{factory}} \) also reads the output of the other components. Thus, the dependency set of \( A_{\text{wood}}, A_{\text{iron}}, A_{\text{fire}}, \) and \( A_{\text{steel}} \) is the singleton \( \{1\} \), and the dependency set of \( A_{\text{factory}} \) is \( \{1, 2, 3, 4, 5\} \)—note that the indices correspond to positions of the components in the cascade. Then, \( A_{\text{wood}} \) has input function \( \phi_{\text{wood}}(x) \) that returns set if \( x = \text{wood} \), and returns read otherwise. Similarly, \( A_{\text{iron}}, A_{\text{fire}}, \) and \( A_{\text{steel}} \). Instead, the component \( A_{\text{factory}} \) has input function \( \phi_{\text{factory}}(x, \text{wood}, \text{iron}, \text{fire}, \text{steel}) \) that returns set if

\[
(x = \text{factory}) \land (\text{[wood} \land \text{iron} \land \text{fire}) \lor \text{steel})
\]

and returns read otherwise.

The second class of prime automata is a class of automata that have a correspondence with simple groups from group theory. Their general definition is beyond the scope of this paper. For that, see (Ginzburg 1968). Here we present the class of prime counters, as a subclass that seems particularly relevant from a practical point of view.

**Definition 4.** An \( n \)-counter is an \( n \)-state automaton \( \langle X^n, J, \Pi, \phi, Q, \delta, q_{\text{init}}, \Gamma, \theta \rangle \) where \( \Pi = \{\text{inc}, \text{read}\} \), \( Q = [0, n - 1] \), and the transition function satisfies the following two identities:

\[
\delta(i, \text{read}) = i, \quad \delta(i, \text{inc}) = i + 1 \pmod{n}.
\]

An \( n \)-counter is prime if \( n \) is a prime number.
An $n$-counter implements a counter modulo $n$. The modulo aspect can be seen both as a bug and a feature. We may be interested just in counting; then, the modulo stands for the overflow due to finite memory. On the other hand, we might be interested in counting modulo $n$; for instance, to capture periodic events such as ‘something happens every 24 hours’.

A 5-counter, that is a prime counter, is depicted in Figure 3.

**Example 4.** Resuming our running example, say that now, in order to use the factory, we need (i) 13 pieces of wood, 5 pieces of iron, and fire, or alternatively (ii) 7 pieces of steel. From the cascade in Example 3, it suffices to change the cascade components $A_{\text{wood}}, A_{\text{iron}}, A_{\text{steel}}$ into counters (e.g., 16-counters) and change the input function of $A_{\text{factory}}$ so that $\phi_{\text{factory}}(x, \text{wood, iron, fire, steel})$ returns set if

\[
(x = \text{factory}) \land 
\left[ \left( (\text{wood} \geq 13) \land (\text{iron} \geq 5) \land \text{fire} \right) \lor (\text{steel} \geq 7) \right],
\]

and it returns read otherwise. The rest is left unchanged. The cascade, despite its simplicity, corresponds to an automaton that has over 700 states. Furthermore, suppose that we need to learn to detect wood, iron, fire, steel from video frames represented as vectors of $\mathbb{R}^n$. It suffices to replace the input function of the corresponding components with a function over $\mathbb{R}^n$ such as a neural network.

6.2 Expressivity Results

A key aspect of automata cascades is their expressivity. As an immediate consequence of the Krohn-Rhodes theorem (Krohn and Rhodes 1965)—see also (Ginzburg 1968; Maler 1990; Dömösi and Nehaniv 2005)—we have that simple cascades of prime automata capture all automata.

**Theorem 2.** Every automaton is captured by a simple cascade of prime automata. Furthermore, every counter-free automaton is captured by a simple cascade of flip-flops. The converse of both claims holds as well.

The second part of the theorem is worth noting, since counter-free acceptors have the expressivity of star-free regular languages (Schützenberger 1965)—i.e., the languages that can be specified by regular expressions without using the Kleene star—which is the expressivity of well-known logics such as monadic first-order logic on finite linearly-ordered domains (McNaughton and Papert 1971), and the linear temporal logic on finite traces LTL$_f$ (De Giacomo and Vardi 2013).

**Corollary 1.** The expressivity of simple cascades of prime automata is the regular languages. The expressivity of simple cascades of flip-flops is the star-free regular languages.

7 Sample Complexity of Automata Cascades

We study the sample complexity of class of automata (and cascades thereof) built from given classes of input functions, semiautomata, and output functions. This allows for a fine-grained specification of automata and cascades.

**Definition 5.** The class $\mathcal{A}(\Phi, \Delta, \Theta)$ over a given input alphabet $X^\Theta$ consists of each automaton with input function from a given class $\Phi$, core semiautomaton from a given class $\Delta$, and output function from a given class $\Theta$.

\[|\mathcal{A}| \leq |\pi_a| \cdot |\Phi| \cdot |\Delta| \cdot |\Theta|,\]

and its sample complexity is asymptotically bounded as $S(\mathcal{A}) \in O \left( \log |\pi_a| + \log |\Phi| + \log |\Delta| + \log |\Theta| \right)$.

**Theorem 4.** The cardinality of a class of automata cascades $\mathcal{C} = \mathcal{A}_1 \ltimes \cdots \ltimes \mathcal{A}_d$ where the automata classes are $\mathcal{A}_i = \mathcal{A}(\Phi_i, \Delta_i, \Theta_i; a_i, m_i)$ is bounded as

\[|\mathcal{C}| \leq \prod_{i=1}^d |\pi_{a_i}| \cdot |\Phi_i| \cdot |\Delta_i| \cdot |\Theta_i|,\]

and its sample complexity is asymptotically bounded as $S(\mathcal{C}) \in O \left( d \cdot (\log |\pi_{a_i}| + \log |\Phi_i| + \log |\Delta_i| + \log |\Theta_i|) \right)$, where dropping the indices denotes the maximum.

Consequently, the complexity of a cascade is bounded by the product of the number of components and a second factor that bounds the complexity of a single component.

7.2 Aspects and Implications of The Results

The term $\log |\pi_a|$, accounting for the projection functions, ranges from 0 to $\min(m, a_1 + d - m) \cdot \log(a_1 + d)$ where $a_1$ is the input arity of the first component, and hence of the
external input. In particular, it is minimum when we allow each component to depend on all or none of its preceding components, and it is maximum when each component has to choose half of its preceding components as its dependencies.

The term $\log |\Phi|$ plays an important role, since the set of input functions has to be sufficiently rich so as to map the external input and the outputs of the preceding components to the internal input. First, its cardinality can be controlled by the degree of dependency $m$; for instance, taking all Boolean functions yields $\log |\Phi| = 2^m$. Notably, it does not depend on $d$, and hence, the overall sample complexity grows linearly with the number of cascade components as long as the degree of dependency $m$ is bounded. Second, the class of input functions can be tailored towards the application at hand. For instance, in our running example, input functions can be chosen from a class $\Phi$ such that $\log |\Phi|$ is linear in the number of tasks—see Example 5 below.

The term $\log |\Delta|$ is the contribution of the number of semiautomata—note that the number of letters and states of each semiautomaton plays no role. Interestingly, very small classes of semiautomata suffice to build very expressive cascades, by the results in Section 5. In general, it is sufficient to build $\Delta$ out of prime components. But we can also include other semiautomata implementing some interesting functionality.

The term $\log |\Theta|$ admits similar considerations as $\log |\Phi|$. It is worth noting that one can focus on simple cascades, by the results in Section 5, where output functions of all but last component are fixed. Then, the contribution to the sample complexity is given by the class of output functions of the last component.

Overall, the sample complexity grows linearly with the number $d$ of components, modulo a logarithmic factor. Specifically, the term $\log |\pi_m^n|$ has only a logarithmic dependency on $d$, and the other terms $\log |\Phi|$, $\log |\Delta|$, and $\log |\Theta|$ are independent of $d$.

**Corollary 2.** Let us recall the quantities from Theorem 4, and fix the quantities $a$, $m$, $\Phi$, $\Delta$, and $\Theta$. Then, the sample complexity of $C$ is bounded as $S(C) \in O(d \cdot \log d)$.

**Example 5.** The cascade described in Example 3 has one component per task, and all components have the same output function and semiautomaton. The input function $\Phi_{\text{factory}}$ is 2-term monotone DNF over 9 propositional variables. Every other component has an input function that is 1-term monotone DNF over 5 propositional variables. Using these observations, we can design a class of cascades for similar sequence tasks where the goal task depends on two groups of arbitrary size, having $d$ basic tasks overall. Such a class will consist of cascades of $d$ components. For every $i \in [1, d-1]$, the class of input functions $\Phi_i$ is the class of 1-term monotone DNF over $d$ variables; and $\Phi_d$ is the class of 2-term monotone DNF over $2d - 1$ variables. For all $i \in [1, d]$, $\Delta_i$ is a singleton consisting of a flip-flop semiautomaton; and $\Theta_i$ is a singleton consisting of the function that returns the state. The cardinality of $\Phi_i$ is $e^2 \cdot 2^{4d-4}$, and hence its logarithm is less than $4d$—see Corollary 5 of (Schmitt 2004). By Theorem 4, considering that we have $d$ cascade components, we obtain that our class of cascades has sample complexity $O(d^2)$. At the same time, the minimum automaton for the considered family of sequence tasks has $\Omega(2^d)$ states. Going back to the bound of Theorem 1, if we had to learn from the class of all automata with $2^d$ states, we would incur a sample complexity exponential in the number of tasks.

**7.3 Learning Theory for Infinite Classes**

Infinite classes of cascades naturally arise when the input alphabet is infinite. In this case, one may consider to pick input and output functions from an infinite class. For instance, the class of all threshold functions over the domain of integers, or as in Example 4 the neural networks over the vectors of real numbers.

When considering infinite classes of functions, the sample complexity bounds based on cardinality become trivial. However, even when the class is infinite, the number of functions from the class that can be distinguished on a given sample is finite, and the way it grows as a function of the sample size allows for establishing sample complexity bounds. In turn, such growth can be bounded in terms of the dimension of the class of functions, a single-number characterisation of its complexity. Next we present these notions formally.

**Growth and Dimension.** Let $F$ be a class of functions from a set $X$ to a finite set $Y$. Let $X_\ell = x_1, \ldots, x_\ell$ be a sequence of $\ell$ elements from $X$. The set of patterns of a class $F$ on $X_\ell$ is

$$F(X_\ell) = \{ (f(x_1), \ldots, f(x_\ell)) \mid f \in F \},$$

and the number of distinct patterns of class $F$ on $X_\ell$ is

$$N(F, X_\ell) = |F(X_\ell)|.$$ The growth of $F$ is

$$G(F, \ell) = \sup_{X_\ell} N(F, X_\ell).$$

The growth of $F$ can be bounded using its dimension, written as $\dim(F)$. When $|Y| = 2$, we define the dimension of $F$ to be its VC dimension (Vapnik and Chervonenkis 1971)—see also (Vapnik 1998). It is the largest integer $h$ such that $G(F, h) = 2^h$ and $G(F, h + 1) < 2^{h+1}$ if such an $h$ exists, and infinity otherwise. When $|Y| > 2$, we define the dimension of $F$ to be its graph dimension (Natarajan 1989; Haussler and Long 1995). It is defined by first binarising the class of functions. For a given function $f : X \to Y$, its binarisation $f_{\text{bin}} : X \times Y \to \{0, 1\}$ is defined as $f_{\text{bin}}(x, y) = 1$ if $f(x) = y$. The binarisation of $F$ is $F_{\text{bin}} = \{ f_{\text{bin}} \mid f \in F \}$. Then, the graph dimension of $F$ is defined as the VC dimension of its binarisation $F_{\text{bin}}$.

The growth of $F$ can be bounded in terms of its dimension (Haussler and Long 1995), as follows:

$$G(F, \ell) \leq (e \cdot \ell \cdot |Y|)^{\dim(F)}.$$

**Sample Complexity.** The sample complexity can be bounded in terms of the dimension, cf. (Shalev-Shwartz and Ben-David 2014). In particular,

$$S(F, \epsilon, \eta) \in O \left( \min \{ (\dim(F) \cdot \log |Y| - \log \eta) / \epsilon^2 \right).$$

For fixed $\epsilon, \eta$, and $Y$, the sample complexity is $O(\dim(F))$, and hence arbitrary classes over the same outputs can be compared in terms of their dimension.
7.4 Results for Infinite Classes of Cascades

We generalise our sample complexity bounds to infinite classes of automata and cascades. The bound have the same shape of the bounds derived in Section 7.1 for finite classes, with the dimension replacing the logarithm of the cardinality. One notable difference is the logarithmic dependency on the maximum length $M$ of a string. It occurs due to the (stateless) input and output functions. Their growth is on single letters, regardless of the way they are grouped into strings. Thus, the growth on a sample of $\ell$ strings is the growth on a sample of $\ell \cdot M$ letters.

The bounds are derived using a functional description of automata and cascades. It is a shift of perspective, from stateful machines that process one letter at a time, to blackboxes that process an entire input string at once. We first introduce function constructors to help us to build such descriptions.

Definition 6 (Function constructors). For $f : \Sigma \to \Gamma$, $f^* : \Sigma^* \to \Gamma$ is defined as $f^*(\sigma_1 \ldots \sigma_n) = f(\sigma_1)$. For $g : \Sigma^* \to \Gamma$, $g : \Sigma^* \to \Gamma^*$ is defined as $g(\sigma_1 \ldots \sigma_n) = g(\sigma_1) \ldots g(\sigma_1 \ldots \sigma_n)$; furthermore, $g^* : \Sigma^* \to \Gamma^*$ is defined as $g^*(\sigma_1 \ldots \sigma_n) = g(\sigma_1 \ldots \sigma_{n-1})$.

Lemma 1. The function $A$ implemented by an automaton $(X^\alpha, J, \Pi, \phi, Q, \delta, q_{\text{init}}, \Gamma, \theta)$ can be expressed as

$$A = \overline{\delta} \circ ((\phi^* \circ \overline{D}) \times I^*) \circ \theta,$$

where $D$ is the function implemented by the semiautomaton $(\Pi, Q, \delta, q_{\text{init}})$.

Note that $I^*$ propagates the projected input to the output function. Also note that the output function reads the state before the last update. The functional description allows us to bound the growth, by making use of the fact that the growth of composition and cross product of two classes of functions is upper bounded by the product of their respective growths. From there, we derive the dimension, and the sample complexity.

Theorem 5. Let $A$ be a class $A(\Phi, Q, \delta, a, m, \Pi, \Gamma)$, let $M$ be the maximum length of a string, and let $w = \log |\pi_m^w| + \log |\Delta| + \dim(\Phi) + \dim(\Theta) \geq 2$.

(i) The growth of $A$ is bounded as:

$$G(A, \ell) \leq |\pi_m^w| \cdot |\Delta| \cdot G(\Phi, \ell \cdot M) \cdot G(\Theta, \ell).$$

(ii) The dimension of $A$ is bounded as:

$$\dim(A) \leq 2 \cdot d \cdot w \cdot \log(w \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|).$$

(iii) The sample complexity of $A$ is bounded as:

$$S(A) \in O(d \cdot \log(w \cdot M \cdot |\Pi| \cdot |\Gamma|)).$$

Next we show the functional description of an automata cascade. It captures the high-level idea that each component of a cascade can be fully executed before executing any of the subsequent automata, since it does not depend on them.

Lemma 2. The function implemented by an automata cascade $A_1 \times \cdots \times A_d$ with $d \geq 2$ can be expressed as

$$(I^* \times A_1) \circ \cdots \circ (I^* \times A_{d-1}) \circ A_d.$$

The cross product with $I^*$ in the functional description above captures the fact that the input of each component is propagated to the next one.

Then, a bound on the growth is derived from the functional description, and hence the dimension and sample complexity bounds.

Theorem 6. Let $C$ be a class $A_1 \times \cdots \times A_d$ where automata classes are $A_i = A(\Phi_i, D_i, \Theta_i; a_i, m_i, \Pi_i, \Gamma_i)$, let $M$ be the maximum length of a string, and let $w = \log(|\pi_m^w| + \log |\Delta| + \dim(\Phi) + \dim(\Theta)) \geq 2$ where dropping the indices denotes the maximum.

(i) The growth of $C$ is bounded as:

$$G(C, \ell) \leq \prod_{i=1}^d |\pi_m^w| \cdot |\Delta| \cdot G(\Phi_i, \ell \cdot M) \cdot G(\Theta_i, \ell \cdot M).$$

(ii) The dimension of $C$ is bounded as:

$$\dim(C) \leq 2 \cdot d \cdot w \cdot \log(d \cdot w \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|).$$

(iii) The sample complexity of $C$ is bounded as:

$$S(C) \in O(d \cdot w \cdot \log(d \cdot w \cdot M \cdot |\Pi| \cdot |\Gamma|)).$$

By a similar reasoning as in the results for finite classes, the sample complexity has a linear dependency on the number of cascade components, modulo a logarithmic factor.

Corollary 3. Let us recall the quantities from Theorem 6, and fix the quantities $a_i, m_i, \Phi, \Delta, \Theta, M, \Pi, \Gamma$. Then, the sample complexity of $C$ is bounded as $S(C) \in O(d \cdot \log d)$.

8 Related Work

The main related work is the sample complexity bounds from (Ishigami and Tani 1997), stated in Theorem 1. Our bounds are qualitatively different, as they allow for describing the sample complexity of a richer variety of classes of automata, and cascades thereof. With reference to Theorem 3, their specific case is obtained when: the input alphabet is non-factor and hence $|\pi_m^w| = 1$; $\Phi = I$ and hence $|\Pi| = |\Sigma| = k$; $\Delta$ is the class of all semiautomata on $k$ letters and $n$ states, and hence $|\Delta| = n^{k \cdot n}$; and $\Theta$ is the class of all indicator functions on $n$ states, and hence $|\Theta| = 2^n$. In this case, it is easy to verify that our bound matches theirs.

Learning automata expressed as a cross product is considered in (Moerman 2018). They correspond to cascades where all components read the input, but no component reads the output of the others. The authors provide a so-called active learning algorithm, that asks membership and equivalence queries. Although it is a different setting from ours, it is interesting that they observe an exponential gain in some specific cases, compared to ignoring the product structure of the automata.

The idea of decoupling the input alphabet from the core functioning of an automaton is found in symbolic automata. The existing results on learning symbolic automata are in the active learning setting (Berg, Jonsson, and Raffelt 2006; Mens and Maler 2015; Maler and Mens 2017; Argyros and D’Antoni 2018).
9 Conclusion

Given the favourable sample complexity of automata cascades, the next step is to devise learning algorithms, able to learn automata as complex systems consisting of many components implementing specific functionalities.

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A Proofs

A.1 Proof of Theorem 2 (Expressivity of Automata Cascades)

We first introduce the cascade product between semiautomata. Given two semiautomata

\[ D_1 = (\Sigma, Q_1, \delta_1, q_{1\text{init}}) \]
\[ D_2 = (\Sigma \times Q_2, \delta_2, q_{2\text{init}}) \]

their cascade product \( D_1 \times D_2 \) yields the semiautomaton \( D = (\Sigma, Q, \delta, q_{\text{init}}) \) where

\[ Q = Q_1 \times Q_2 \]
\[ q_{\text{init}} = (q_{1\text{init}}, q_{2\text{init}}) \]
\[ \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, (q_1, a))) \]

The cascade product on semiautomata is left-associative. Hence, given semiautomata \( D_i = (\Sigma \times Q_i \times \cdots \times Q_{i-1}, Q_i, \delta_i, q_{i\text{init}}) \) for \( i \in [1, d] \), we can build a semiautomaton as \( D_1 \times \cdots \times D_d \).

The proof of Theorem 2 is based on the following theorem, which is an immediate consequence of the Krohn-Rhodes Prime Decomposition Theorem (Krohn and Rhodes 1965), together with Theorem 5.6 of (Ginzburg 1968).

**Theorem 7** (Krohn and Rhodes + Ginzburg). Every automaton \( A \) is captured by an automaton \( A' \) whose core semiautomaton is expressed by a cascade product \( A_1 \times \cdots \times A_d \) where each \( A_i \) is a prime semiautomaton. Furthermore, every counter-free automaton \( A \) is captured by an automaton \( A' \) whose core semiautomaton is expressed by a cascade product \( A_1 \times \cdots \times A_d \) where each \( A_i \) is a flip-flop semiautomaton. The converse of both claims holds as well.

**Theorem 2**. Every automaton is captured by a simple cascade of prime automata. Furthermore, every counter-free automaton is captured by a simple cascade of flip-flops. The converse of both claims holds as well.

**Proof.** Consider an automaton \( A = (\Sigma, Q, \delta, q_{\text{init}}, \Gamma, \theta) \). By Theorem 7, automaton \( A \) is captured by an automaton \( A' = (\Sigma, Q', \delta', q_{\text{init}}, \Gamma', \theta') \) such that its semiautomaton \( (\Sigma, Q', \delta', q_{\text{init}}) \) is expressed by a cascade product \( D_1 \times \cdots \times D_d \) where each \( D_i \) is a prime semiautomaton of the form \( (\Sigma \times Q_i, \delta_i, q_{i\text{init}}) \). For each \( i \in [1, d-1] \), let \( A_i \) be the automaton with core semiautomaton \( D_i \) and output function \( \theta_i(q_i, (\sigma, q_{1}, \ldots, q_{i-1})) = q_i \). Also, let \( A_d \) be the automaton with core semiautomaton \( D_d \) and output function \( \theta_d(q_d, (\sigma, q_1, \ldots, q_{d-1})) = \theta((q_1, \ldots, q_d), \sigma) \). Now, consider the cascade \( C = (A_1, \ldots, A_d) \) made of the automata just introduced. Since the cascade is simple and its components are prime, it suffices to show that \( A \) is captured by \( C \). Namely, that \( A \) and \( C \) implement the same function. Let \( A_C = (\Sigma, Q_C, \delta_C, q_{\text{init}}, \Gamma, \theta_C) \) be the automaton corresponding to the cascade \( C \). Consider an input string \( \sigma_1 \ldots \sigma_m \). Let \( q = (q_1, \ldots, q_d) \) be the state of \( A_C \) after reading \( \sigma_1 \ldots \sigma_{m-1} \); namely, \( q = \delta_C(q_{\text{init}}, \sigma_1 \ldots \sigma_{m-1}) \). The function implemented by \( C \) is

\[ C(\sigma_1 \ldots \sigma_m) = \theta_d(q_d, (\sigma_m, q_1, \ldots, q_{d-1})) = \theta((q_1, \ldots, q_d), \sigma_m) = A'(\sigma_1 \ldots \sigma_m) = A(\sigma_1 \ldots \sigma_m) \]

For the second claim, it suffices to note that \( A_1, \ldots, A_d \) can be taken to be flip-flops when \( A \) is counter-free, by Theorem 7.

Next we show the converse of the two claims. The converse of the first claim holds immediately by the definition of cascade, which says that the function it implements is the one implemented by the corresponding automaton. For the converse of the second claim, consider a simple cascade \( C = (A_1, \ldots, A_d) \). For each \( i \in [1, d] \), the core semiautomaton \( D_i \) of \( A_i \) is a flip-flop. Also, for each \( i \in [1, d-1] \) the output function of \( A_i \) is \( \theta_i(q_i, (\sigma, q_1, \ldots, q_{i-1})) = q_i \). Thus, the core semiautomaton of the automaton corresponding to \( C \) is expressed by the cascade product \( D_1 \times \cdots \times D_d \). Thus, by Theorem 7, cascade \( C \) is captured by a counter-free automaton.

A.2 Proof of Theorem 3 (Cardinality and Sample Complexity of Finite Classes of Automata)

**Theorem 3.** The cardinality of a class of automata \( \mathcal{A} = \mathcal{A}(\Phi, \Delta, \Theta; a, m) \) is bounded as

\[ |\mathcal{A}| \leq |\pi_m^a| \cdot |\Phi| \cdot |\Delta| \cdot |\Theta| \]

and its sample complexity is asymptotically bounded as

\[ S(\mathcal{A}) \in O\left( \log |\pi_m^a| + \log |\Phi| + \log |\Delta| + \log |\Theta| \right) \]

**Proof.** We bound the number of automata in the class \( \mathcal{A}(m, \Phi, \Delta, \Theta) \). Consider that each automaton in the class is of the form

\[ A = (X^a, J, \Pi, \phi, Q, \delta, q_{\text{init}}, \Gamma, \theta) \]

To establish the cardinality of the class, it suffices to count the number of distinct components of the tuple above. The input alphabet \( X^a \) is fixed. The number of possible sets of indices of \( J \subseteq [1, a] \) is the cardinality of the set \( |\pi_m^a| = \binom{a}{m} \). The number of input functions \( \phi \) is \( |\Phi| \). Each input function determines the internal alphabet \( \Pi \). Each semiautomaton \( (\Pi, \phi, Q, \delta, q_{\text{init}}) \) is from \( \Delta \), and hence we have at most \( |\Delta| \). The number of output functions \( \theta \) is \( |\Theta| \), and the output alphabet \( \Gamma \) is determined by the output function. The bound follows by taking the product of the sets of components whose cardinality has been discussed above.

For the case where \( \Delta \) is the set of all semiautomaton with \( k \) letters and \( n \) states, we have \( |\Delta| \leq n^{k \cdot n} \)—assuming w.l.o.g. that \( Q = [1, n] \) and \( q_{\text{init}} = 1 \).

The sample complexity is immediate by taking the logarithm of the cardinality bound established above.
A.3 Proof of Theorem 4 (Cardinality and Sample Complexity of Finite Classes of Automata Cascades)

Note that the theorem is restated by defining some of the quantities more clearly.

**Theorem 4.** The cardinality of a class of automata cascades \( C = A_1 \times \cdots \times A_d \) where the automata are from \( A_i = A(\Phi_i, \Delta_i, \Theta_i; a_i, m_i) \) is bounded as

\[
|C| \leq \prod_{i=1}^{d} |\pi_{m_i}^a| \cdot |\Phi_i| \cdot |\Delta_i| \cdot |\Theta_i|,
\]

and its sample complexity is asymptotically bounded as

\[
S(C) \in O \left( d \cdot (\log |\pi_{m_i}^a| + \log |\Phi| + \log |\Delta| + \log |\Theta|) \right),
\]

where dropping the indices denotes the maximum as follows:

\[
\pi_{m_i}^a = \arg \max \pi_{m_i} |\pi_{m_i}^a| \quad \text{the class of projection functions with maximum cardinality},
\]
\[
\Phi = \arg \max \Phi_i |\Phi_i| \quad \text{the class of input functions with maximum cardinality},
\]
\[
\Delta = \arg \max \Delta_i |\Delta_i| \quad \text{the class of semiautomata with maximum cardinality},
\]
\[
\Theta = \arg \max \Theta_i |\Theta_i| \quad \text{the class of output functions with maximum cardinality}.
\]

**Proof.** We bound the number of cascades in the class \( C = A_1 \times \cdots \times A_d \). In order to do so, we bound the cardinality of each class of automata \( A_i = A(\Phi_i, \Delta_i, \Theta_i; a_i, m_i) \). Thus, by the cardinality bound in Theorem 3, we have that

\[
|A_i| \leq |\pi_{m_i}^a| \cdot |\Phi_i| \cdot |\Delta_i| \cdot |\Theta_i|.
\]

Thus, the bound on \( |C| \) is given by \( \prod_{i=1}^{d} |A_i| \). Then, for the sample complexity, we loosen the bound taking the maximum of each indexed quantity. Namely,

\[
|C| \leq \prod_{i=1}^{d} |A_i| \leq \prod_{i=1}^{d} |\pi_{m_i}^a| \cdot |\Phi| \cdot |\Delta| \cdot |\Theta|,
\]

where \( \pi_{m_i}^a, \Phi, \Delta, \) and \( \Theta \) are defined in the statement of the theorem. Then,

\[
|C| \leq |\pi_{m_i}^a|^d \cdot |\Phi|^d \cdot |\Delta|^d \cdot |\Theta|^d.
\]

The sample complexity bound follows by taking the logarithm of the cardinality bound established above. \( \square \)

A.4 Further Details on the Growth and Sample Complexity

This section restates two claims from Section 7.3, providing details on the way they are derived. The following Proposition 1 restates the upper bound on the growth in terms of the dimension.

**Proposition 1.** Let \( F \) be a class of functions from \( X \) to \( Y \). Then,

\[
G(F, \ell) \leq \sum_{i=0}^{\dim(F)} \left( \ell \right)^i \cdot (|Y| - 1)^i \leq \sum_{i=0}^{\dim(F)} \left( \ell \right)^i \cdot |Y|^i \leq (e \cdot \ell \cdot |Y|)^{\dim(F)}.
\]

**Proof.** The first inequality is Corollary 3 of (Haussler and Long 1995). We loosen it by substituting \((|Y| - 1)^i \) with \(|Y|^i \) and then applying the well-known bound on the partial sum of binomials \( \sum_{i=0}^{d} \binom{m}{i} \leq (e \cdot m)^d \) to derive the last inequality. \( \square \)

The following Proposition 2 restates the asymptotic bound on the sample complexity in terms of the dimension.

**Proposition 2.** The following asymptotic bound on the sample complexity holds true:

\[
S(F, \epsilon, \eta) \in O \left( (\dim(F) \cdot \log |Y| - \log \eta) / \epsilon^2 \right).
\]

**Proof.** By Point 2 of Theorem 19.3 of (Shalev-Shwartz and Ben-David 2014), the proposition holds with the Natarajan dimension of \( F \) in place of \( \dim(F) \). Then the proposition follows immediately since \( \dim(F) \) is an upper bound on the Natarajan dimension, by Equations (24) and (26) of (Haussler and Long 1995). \( \square \)
A.5 Basic Propositions

In this section we state and prove some basic propositions that we use in the next sections of the appendix, where we prove the results for Section 7.4.

The next Proposition 3 describes a relationship between the growth of $F$ and the growth of its binarisation $F_{\text{bin}}$.

**Proposition 3.** Let $F$ be a class of functions. Then, $G(F_{\text{bin}}, \ell) \leq G(F, \ell)$.

**Proof.** Let $Z_\ell = \langle x_1, y_1 \rangle, \ldots, \langle x_\ell, y_\ell \rangle$ and let $X_\ell = x_1, \ldots, x_\ell$. The proposition is immediate by observing that every tuple in

$$F(X_\ell) = \{\{f(x_1), \ldots, f(x_\ell)\} : f \in F\}$$

corresponds to at most one tuple in

$$F_{\text{bin}}(Z_\ell) = \{\{1[y_1 = f(x_1)], \ldots, 1[y_\ell = f(x_\ell)]\} : f \in F\}.$$

\[\square\]

The next Propositions 4 and 5 provide bounds on the growth for composition and cross product of functions. They are well-known. For instance, they are left as an exercise in the book (Shalev-Shwartz and Ben-David 2014). However, we were not able to find a peer-reviewed reference for the proofs. We report proofs based on the lecture notes (Kakade and Tewari 2008).

**Proposition 4.** Let $F_1 : X \rightarrow W$, and let $F_2 : W \rightarrow Y$. Then, $G(F_1 \circ F_2, \ell) \leq G(F_1, \ell) \cdot G(F_2, \ell)$.

**Proof.** Let $X_\ell = x_1, \ldots, x_\ell$ be a sequence of $\ell$ elements from $X$. Let $F = F_1 \circ F_2$. We have

$$F(X_\ell) = \{\{f_2(f_1(x_1)), \ldots, f_2(f_1(x_\ell))\} : f_1 \in F_1, f_2 \in F_2\} = \bigcup_{U_\ell \in F_1(X_\ell)} \{\{f_2(u_1), \ldots, f_2(u_\ell)\} : f_2 \in F_2\},$$

and therefore,

$$N(F, X_\ell) = |F(X_\ell)| \leq \sum_{U_\ell \in F_1(X_\ell)} |\{\{f_2(u_1), \ldots, f_2(u_\ell)\} : f_2 \in F_2\}|$$

$$\leq \sum_{U_\ell \in F_1(X_\ell)} N(F_2, U_\ell)$$

$$\leq \sum_{U_\ell \in F_1(X_\ell)} G(F_2, \ell)$$

$$= N(F_1, X_\ell) \cdot G(F_2, \ell)$$

$$\leq G(F_1, \ell) \cdot G(F_2, \ell).$$

Since $X_\ell$ is arbitrary, this concludes the proof. \[\square\]

**Proposition 5.** Let $F_1 : X \rightarrow Y$ and let $F_2 : X \rightarrow Z$. Then, $G(F_1 \times F_2, \ell) \leq G(F_1, \ell) \cdot G(F_2, \ell)$.

**Proof.**

$$G(F_1 \times F_2, \ell) = \sup_{X_\ell} N(F_1 \times F_2, X_\ell) = \sup_{X_\ell} \left[ N(F_1, X_\ell) \cdot N(F_2, X_\ell) \right] \leq G(F_1, \ell) \cdot G(F_2, \ell).$$

\[\square\]

The next Propositions 6–8 describe properties of the growth with respect to string functions.

**Proposition 6.** Let $F$ be a class of functions $\Sigma \rightarrow \Gamma$. Then, $G(F^*, \ell) \leq G(F, \ell)$.

**Proof.** Let $X_\ell = x_1, \ldots, x_\ell$ be a sequence of strings over $\Sigma$, and let $\sigma_i$ be the last letter of $x_i$. Since $f^*(x_i) = f(\sigma_i)$ for every function $f \in F$, we have that $F^*(X_\ell) = F^*(\sigma_1, \ldots, \sigma_\ell) = F(\sigma_1, \ldots, \sigma_\ell)$. Therefore,

$$N(F^*, X_\ell) = |F^*(X_\ell)| = |F(\sigma_1, \ldots, \sigma_\ell)| = N(F, \sigma_1, \ldots, \sigma_\ell) \leq G(F, \ell).$$

Since $X_\ell$ is arbitrary, this concludes the proof. \[\square\]

**Proposition 7.** Let $F_1 : \Sigma^* \rightarrow Y$ and let $F_2 : \Sigma^* \rightarrow Z$. Then, $G(F_1 \times F_2, \ell) = G(F_1 \times F_2, \ell)$.
Proof. Consider a function \( f_1 \in \mathcal{F}_1 \), a function \( f_2 \in \mathcal{F}_2 \), and a sequence \( X_\ell = x_1, \ldots, x_\ell \) of strings over \( \Sigma \). We have that
\[
(f_1 \times f_2)(X_\ell) = (f_1 \times f_2)(x_1), \ldots, (f_1 \times f_2)(x_\ell),
\]
and
\[
(f\bar{1} \times f\bar{2})(X_\ell) = (f\bar{1} \times f\bar{2})(x_1), \ldots, (f\bar{1} \times f\bar{2})(x_\ell).
\]
Now, consider \( x_i = \sigma_1 \ldots \sigma_s \). We have
\[
(f\bar{1} \times f\bar{2})(x_i) = (f_1 \times f_2)(\sigma_1) \ldots (f_1 \times f_2)(\sigma_1 \ldots \sigma_s) = (f_1(\sigma_1), f_2(\sigma_1) \ldots (f_1(\sigma_1 \ldots \sigma_s), f_2(\sigma_1 \ldots \sigma_s))
\]
and
\[
(f\bar{1} \times f\bar{2})(x_i) = (f\bar{1}(x_i), f\bar{2}(x_i)) = (f_1(\sigma_1) \ldots f_1(\sigma_1 \ldots \sigma_s), f_2(\sigma_1) \ldots f_2(\sigma_1 \ldots \sigma_s)).
\]
Note that the output of \( f\bar{1} \times f\bar{2}(x_i) \) and \( f\bar{1} \times f\bar{2}(x_i) \) contain the same occurrences of letters, arranged differently. Hence, the same holds for \( (f\bar{1} \times f\bar{2})(X_\ell) \) and \( (f\bar{1} \times f\bar{2})(X_\ell) \). Therefore,
\[
|\{(f\bar{1} \times f\bar{2})(X_\ell) \mid f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2\}| = |\{(f\bar{1} \times f\bar{2})(X_\ell) \mid f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2\}|
\]
and hence
\[
N(f\bar{1} \times f\bar{2}, X_\ell) = N(f\bar{1} \times f\bar{2}, X_\ell).
\]
Since \( X_\ell \) is arbitrary, this concludes the proof. \( \square \)

**Proposition 8.** Let \( \mathcal{F} \) be a class of functions \( \Sigma^* \rightarrow \Gamma \). Then, \( G(\mathcal{F}, \ell) \leq G(\mathcal{F}, \ell \cdot M) \) where \( M \) is the maximum length of a string.

**Proof.** Consider a string \( x = \sigma_1 \ldots \sigma_s \) with \( \sigma_i \in \Sigma \). Consider the sequence of non-empty prefixes of \( x \),
\[
x^p = \sigma_1, \sigma_1 \sigma_2, \ldots, \sigma_1 \sigma_2 \ldots \sigma_s.
\]
For every \( f \in \mathcal{F} \), the following two identities hold:
\[
f(x) = f(\sigma_1) \ldots f(\sigma_1 \ldots \sigma_s), \tag{1}
\]
\[
f(x^p) = f(\sigma_1), \ldots, f(\sigma_1 \ldots \sigma_s). \tag{2}
\]
Now, let \( X_\ell = x_1, \ldots, x_\ell \) be a sequence of \( \ell \) strings over \( \Sigma \), and let \( X_\ell^p = x_1^p, \ldots, x_\ell^p \) where \( x_i^p \) is the sequence of non-empty prefixes of \( x_i \) defined as above. Then, the identities (1) and (2) imply \( N(f\bar{1}, X_\ell) = N(F, X_\ell^p) \). Thus, \( N(F, X_\ell^p) \leq G(F, \ell \cdot M) \), since \( X_\ell^p \) is a sequence of length at most \( \ell \cdot M \), and hence \( N(F, X_\ell^p) \leq G(F, \ell \cdot M) \). Since \( X_\ell \) is arbitrary, we conclude that \( G(\mathcal{F}, \ell) \leq G(\mathcal{F}, \ell \cdot M) \). \( \square \)

The next Propositions 9–11 provide some algebraic identities for functions expressed in terms of composition, cross product, and constructors introduced in Definition 6.

**Proposition 9.** Let \( f_1 : \Sigma^* \rightarrow \Gamma \), and let \( f_2 : \Gamma^* \rightarrow Y \). Then, \( f_1 \circ f_2 = f_1 \circ f_2 \).

**Proof.**
\[
(f_1 \circ f_2)(\sigma_1 \ldots \sigma_n)
= (f_1 \circ f_2)(\sigma_1) \ldots (f_1 \circ f_2)(\sigma_1 \ldots \sigma_n)
= f_2(f_1(\sigma_1)) \ldots f_2(f_1(\sigma_1 \ldots \sigma_n))
= f_2(f_1(\sigma_1)) \ldots f_2(f_1(\sigma_1 \ldots \sigma_n))
= f_2(f_1(\sigma_1) \ldots f_1(\sigma_1 \ldots \sigma_n))
= f_2(f_1(\sigma_1) \ldots f_1(\sigma_1 \ldots \sigma_n))
= f_2(f_1(\sigma_1) \ldots f_1(\sigma_1 \ldots \sigma_n))
= (f_1 \circ f_2)(\sigma_1 \ldots \sigma_n).
\]

**Proposition 10.** Let \( f_1 : \Sigma^* \rightarrow \Gamma \), and let \( f_2 : \Gamma \rightarrow Y \). Then, \( f_1 \circ f_2 = f_1 \circ f_2 \).
The function implemented by $\phi$ is the one implemented by the induced automaton $f(\theta)$ = $\phi$, where $D$ is the function implemented by the semiautomaton $\langle Q, Q, \delta, qinit \rangle$.

Proof. The function implemented by $A$ is the one implemented by the induced automaton $A' = (\Sigma, Q, \delta, qinit, \Gamma, \theta)$, where $\delta(q, \sigma) = \delta(q, \phi(\pi_j(\sigma)))$, and the output function is $\theta(q, \sigma) = \theta(q, \pi_j(\sigma))$. In turn, the function implemented by $A'$ is $A'(\sigma_1 \ldots \sigma_m) = \theta_j(\delta_j(\phi(qinit), \sigma_1 \ldots \sigma_{m-1}), \sigma_m)$. Thus, the function implemented by $A$ is $A(\sigma_1 \ldots \sigma_m) = \theta_j(\delta_j(\phi(qinit), \sigma_1 \ldots \sigma_{m-1}), \sigma_m).

A.6 Proof of Lemma 1 (Functional Description of an Automaton)

Lemma 1. The function $A$ implemented by an automaton $\langle X^n, J, \Pi, \phi, Q, \delta, qinit, \Gamma, \theta \rangle$ can be expressed as

$A = \pi_j \circ ((\sigma^m \times D^o) \times I^*) \circ \theta$, where $D$ is the function implemented by the semiautomaton $\langle Q, Q, \delta, qinit \rangle$.

Proof. The function implemented by $A$ is the one implemented by the induced automaton $A' = (\Sigma, Q, \delta, qinit, \Gamma, \theta_j)$, where $\delta_j(q, \sigma) = \delta(q, \phi(\pi_j(\sigma)))$, and the output function is $\theta_j(q, \sigma) = \theta(q, \pi_j(\sigma))$. In turn, the function implemented by $A'$ is $A'(\sigma_1 \ldots \sigma_m) = \theta_j(\delta_j(qinit, \sigma_1 \ldots \sigma_{m-1}), \sigma_m)$. Thus, the function implemented by $A$ is $A(\sigma_1 \ldots \sigma_m) = \theta_j(\delta_j(qinit, \sigma_1 \ldots \sigma_{m-1}), \sigma_m).

Observe that

$\phi(\pi_j(\sigma_1)) \ldots \phi(\pi_j(\sigma_m)) = \phi(\pi_j(\sigma_1)) \pi_j(\sigma_m) = \phi(\pi_j(\sigma_1)) \pi_j(\sigma_m)

Thus,

$\delta_j(\phi(qinit), \sigma_1 \ldots \sigma_m) = \delta(qinit, \phi(\pi_j(\sigma_1)) \pi_j(\sigma_m)) = \delta(qinit, \phi(\pi_j(\sigma_1)) \pi_j(\sigma_m)) = \delta(qinit, \phi(\pi_j(\sigma_1)) \pi_j(\sigma_m))$.

Now, going back to (3), we have

$A(\sigma_1 \ldots \sigma_m) = \theta_j(\delta_j(qinit, \sigma_1 \ldots \sigma_{m-1}), \sigma_m)$

The last equality is by Proposition 11. This concludes the proof.
A.7 Proof of Theorem 5 (Growth and Sample Complexity of Infinite Classes of Automata)

We prove Theorem 5 in two lemmas. First, we prove the bound on the growth in Lemma 3, then we prove the bound on the dimension in Lemma 4. Then, the asymptotic bound on the sample complexity follows immediately by Proposition 2.

Lemma 3. Let $A$ be a class of automata $A(\Phi, \Delta, \Theta; a, m, \Pi, \Gamma)$, and let $M$ be the maximum length of a string. The growth of $A$ is bounded as:

$$G(A, \ell) \leq |\pi_m^n| \cdot |\Delta| \cdot G(\Phi, \ell \cdot M) \cdot G(\Theta, \ell).$$

Proof. By Lemma 1 the class of functions implemented by $A$ can be expressed as

$$A = (\pi_m^n)^* \circ ((\Phi^* \circ \Delta^o) \times I^*) \circ \Theta.$$

Then,

$$G(A, \ell) = G((\pi_m^n)^* \circ ((\Phi^* \circ \Delta^o) \times I^*) \circ \Theta, \ell)$$

by Lemma 1,

$$\leq G((\pi_m^n)^*, \ell) \cdot G((\Phi^* \circ \Delta^o) \times I^*, \ell) \cdot G(\Theta, \ell)$$

by Proposition 4,

$$\leq |\pi_m^n| \cdot G((\Phi^* \circ \Delta^o), \ell) \cdot G(I^*, \ell) \cdot G(\Theta, \ell)$$

by Proposition 4,

$$\leq |\pi_m^n| \cdot G(\Phi^*, \ell \cdot M) \cdot G(\Theta, \ell)$$

by Proposition 8,

$$\leq |\pi_m^n| \cdot |\Delta| \cdot G(\Phi, \ell \cdot M) \cdot G(\Theta, \ell)$$

by Proposition 6.

This proves the lemma. \qed

Lemma 4. Let $A$ be a class of automata $A(\Phi, \Delta, \Theta; a, m, \Pi, \Gamma)$, let $M$ be the maximum length of a string, and let $w = \log |\pi_m^n| + \log |\Delta| + \dim(\Phi) + \dim(\Theta) \geq 2$. The dimension of $A$ is bounded as:

$$\dim(A) \leq 2 \cdot w \cdot \log(w \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|).$$

Proof. By Lemma 3 the upper bound on the growth $G(A, \ell)$ is

$$G(A, \ell) \leq |\pi_m^n| \cdot |\Delta| \cdot G(\Phi, \ell \cdot M) \cdot G(\Theta, \ell).$$

Applying Proposition 1 to $G(\Phi, \ell \cdot M)$ and $G(\Theta, \ell)$ we obtain,

$$G(A, \ell) \leq |\pi_m^n| \cdot |\Delta| \cdot (e \cdot \ell \cdot M \cdot |\Pi| \cdot |\Gamma|)^h \cdot (e \cdot \ell \cdot |\Gamma|)^g.$$

We loosen the bound on $G(A, \ell)$ by collecting the factors with an exponent,

$$G(A, \ell) \leq |\pi_m^n| \cdot |\Delta| \cdot (\ell \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|)^{h+g}.$$

Next we show the claimed upper bound on the dimension of $A$. By the definition of dimension, in order to show that a number $\ell$ is an upper bound on the dimension of $A$, it suffices to show that $G(A_{\text{bin}}, \ell) < 2^\ell$—note that $G(A_{\text{bin}}, \ell) = G(A, \ell)$ when the output alphabet has cardinality two. Then, since $G(A_{\text{bin}}, \ell) \leq G(A, \ell)$ by Proposition 3, a number $\ell$ satisfies $G(A_{\text{bin}}, \ell) < 2^\ell$ if it satisfies $G(A, \ell) < 2^\ell$, and in turn if it satisfies

$$|\pi_m^n| \cdot |\Delta| \cdot (\ell \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|)^{h+g} < 2^\ell.$$

We show that the former inequality is satisfied for $\ell = 2 \cdot w \cdot \log(w \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|) = 2 \cdot w \cdot \log(w \cdot e \cdot C)$, where we introduce $C = M \cdot |\Pi| \cdot |\Gamma|$ for brevity. We keep proceeding by sufficient conditions. Specifically, at each step, either we replace the r.h.s. with a smaller expression, we simplify a common term between the two sides, or we just rewrite an expression in an equivalent...
This concludes the proof.

From the definition of cascade, at each step, the input to component

Lemma 5.

We prove Theorem 6 in two lemmas. First, we prove the bound on the growth of cascades in Lemma 5, then we prove the

A.9 Proof of Theorem 6 (Growth and Sample Complexity of Infinite Classes of Automata Cascades)

Proof. Let \( A_1 \times \cdots \times A_d \) be the input to component \( A_d \), and let \( \Phi_i, \Delta_i, \Theta_i; a_i, m_i, \Pi_i, \Gamma_i \), and let \( M \) be the maximum length of a string. The growth of \( C \) is bounded as:

\[
G(C, \ell) \leq \prod_{i=1}^{d} |\pi_m^a| \cdot |\Delta| \cdot G(\Phi_i, \ell \cdot M) \cdot G(\Theta_i, \ell \cdot M).
\]

Proof. By Lemma 2 the class of functions implemented by \( C \) can be expressed as,

\[
C = (I^* \times A_1) \circ \cdots \circ (I^* \times A_{d-1}) \circ A_d.
\]
Then,
\[
G(C, \ell) = G\left( (I^* \times A_1) \circ \cdots \circ (I^* \times A_{d-1}) \circ A_d, \ell \right)
\]
by Lemma 2,
\[
\leq G(A_d, \ell) \cdot \prod_{i=1}^{d-1} G(I^* \times A_i, \ell)
\]
by Proposition 4,
\[
\leq G(A_d, \ell) \cdot \prod_{i=1}^{d-1} G(I^*, \ell) \cdot G(A_i, \ell)
\]
by Proposition 5 and 7,
\[
= G(A_d, \ell) \cdot \prod_{i=1}^{d-1} G(A_i, \ell)
\]
since \(I^*\) is singleton.

Next we derive an upper bound on \(G(\overline{A}, \ell)\) for an arbitrary class \(A = A(\Phi, \Delta, \Theta; a, m)\).
\[
G(\overline{A}, \ell) = G\left( (\pi_m^a)^* \circ \left( (\Theta^* \circ \Delta)^* \times I^* \right) \circ \Theta, \ell \right)
\]
by Lemma 1,
\[
\leq G\left( (\pi_m^a)^* \circ \left( (\Theta^* \circ \Delta)^* \times I^* \right), \ell \right) \cdot G(\Theta^*, \ell)
\]
by Proposition 10 and 4,
\[
\leq G\left( (\pi_m^a)^* \circ \left( (\Theta^* \circ \Delta)^* \times I^* \right), \ell \right) \cdot G(\Theta^*, \ell)
\]
by Proposition 9,
\[
\leq G\left( (\pi_m^a)^* \circ \left( (\Theta^* \circ \Delta)^* \times I^* \right), \ell \right) \cdot G(\Theta^*, \ell)
\]
by Proposition 4,
\[
\leq \pi_m^a \cdot G\left( (\Theta^* \circ \Delta)^* \times I^*, \ell \right) \cdot G(\Theta^*, \ell)
\]
since \(\pi_m^a\) is finite,
\[
\leq \pi_m^a \cdot G\left( (\Theta^* \circ \Delta)^* \times I^*, \ell \right) \cdot G(\Theta^*, \ell)
\]
by Proposition 7,
\[
\leq \pi_m^a \cdot G\left( (\Theta^* \circ \Delta)^*, \ell \right) \cdot G(I^*, \ell) \cdot G(\Theta^*, \ell)
\]
by Proposition 5,
\[
\leq \pi_m^a \cdot G\left( (\Theta^* \circ \Delta)^*, \ell \right) \cdot G(\Theta^*, \ell)
\]
since \(I^*\) is a single function,
\[
\leq \pi_m^a \cdot G\left( \Theta^*, \ell \right) \cdot G(\Theta^*, \ell)
\]
by Proposition 9,
\[
\leq \pi_m^a \cdot \Delta \cdot G\left( \Theta^*, \ell \right) \cdot G(\Theta^*, \ell)
\]
by Proposition 4,
\[
\leq \pi_m^a \cdot |\Delta| \cdot G(\Theta^*, \ell) \cdot G(\Theta^*, \ell)
\]
since \(\Delta\) is a finite class,
\[
\leq \pi_m^a \cdot |\Delta| \cdot G(\Theta^*, \ell) \cdot G(\Theta^*, \ell)
\]
by Proposition 8,
\[
\leq \pi_m^a \cdot |\Delta| \cdot G(\Theta, \ell) \cdot G(\Theta, \ell)
\]
by Proposition 6.

Using Lemma 3 we derive an upper bound on the \(G(A_d, \ell)\).
\[
G(A_d, \ell) \leq \prod_{i=1}^{d} |\pi_m^a| \cdot |\Delta_d| \cdot G(\Phi_d, \ell \cdot M) \cdot G(\Theta_d, \ell) \leq \prod_{i=1}^{d} |\pi_m^a| \cdot |\Delta_d| \cdot G(\Phi_d, \ell \cdot M) \cdot G(\Theta_d, \ell \cdot M).
\]

Then,
\[
G(C, \ell) \leq \prod_{i=1}^{d} |\pi_m^a| \cdot |\Delta_i| \cdot G(\Phi_i, \ell \cdot M) \cdot G(\Theta_i, \ell \cdot M).
\]

The lemma is proved.

Lemma 6. Let \(C\) be a class \(C = A_1 \times \ldots \times A_d\) of cascades where \(A_i = A(\Phi_i, \Delta_i, \Theta_i; a_i, m_i, \Pi_i, \Gamma_i)\). Let \(M\) be the maximum length of a string. Let \(h = \max_i \dim(\Phi_i)\) be maximum dimension of a class of input functions, let \(g = \max_i \dim(\Theta_i)\) be the maximum dimension of a class of output functions, let \(\Delta = \arg \max_{\Delta_i} |\Delta_i|\) be the class of semiautomata with maximum cardinality, and let \(\pi_m^a = \arg \max_{\pi_m^a} |\pi_m^a|\). Let \(w = \log |\pi_m^a| + |\Delta| + h + g \geq 2\). The dimension of \(C\) is bounded as:
\[
\dim(C) \leq 2 \cdot d \cdot w \cdot \log(d \cdot w \cdot e \cdot M \cdot |\Pi| \cdot |\Gamma|),
\]

Proof. By Lemma 5 the upper bound on \(G(C, \ell)\) is
\[
G(C, \ell) \leq \prod_{i=1}^{d} |\pi_m^a| \cdot |\Delta_i| \cdot G(\Phi_i, \ell \cdot M) \cdot G(\Theta_i, \ell \cdot M).
\]
Applying Proposition 1 to terms $G(\Phi_1, \ell \cdot M), G(\Theta_1, \ell \cdot M)$ we obtain

$$G(C, \ell) \leq \prod_{i=1}^{\ell} \left( |P| \cdot (e \cdot \ell \cdot M \cdot |P|) \right)^{h_i} \cdot \left( e \cdot \ell \cdot |P| \right)^{g_i}.$$  

We loosen the bound by collecting the factors with the exponent and by taking the maximum of each indexed quantity,

$$G(C, \ell) \leq |P|^d \cdot |D| \cdot (e \cdot \ell \cdot M \cdot |P|)^{d(h+g)}$$

Next we show the claimed upper bound on the dimension of $C$. By the definition of dimension, in order to show that a number $\ell$ is an upper bound on the dimension, it suffices to show that $G(C, \ell) < 2^\ell$. Then, since $G(C, \ell) \leq G(C, \ell)$ by Proposition 3, a number $\ell$ satisfies $G(C, \ell) < 2^\ell$ if it satisfies $G(C, \ell) < 2^\ell$, and in turn if it satisfies

$$\left( |P|^d \cdot |D| \cdot (e \cdot \ell \cdot M \cdot |P|)^{d(h+g)} < 2^\ell \right).$$

We show that the former inequality is satisfied for $\ell = 2 \cdot d \cdot w \cdot \log(d \cdot w \cdot e \cdot M \cdot |P| \cdot |P|) = 2 \cdot d \cdot w \cdot \log(d \cdot w \cdot e \cdot C)$, where we introduce $C = M \cdot |P| \cdot |P|$ for brevity. We keep proceeding by sufficient conditions. Specifically, at each step, either we replace the r.h.s. with a smaller expression, we simplify a common term between the two sides, or we just rewrite an expression in an equivalent form.

$$\left( |P|^d \cdot |D| \cdot (e \cdot \ell \cdot M \cdot |P|)^{d(h+g)} < 2^\ell \right).$$

The term on the r.h.s. is strictly greater than 1 whenever $d \cdot w \geq 2$, considering that $C \geq 1$. The lemma is proved. \hfill \Box

### B Formal Description of The Running Example

The sequence task of the running example (Example 1) is formalised by the following Temporal Datalog program—for Temporal Datalog see (Ronca et al. 2022). Let wood, iron, fire, steel, and factory be unary unary predicates. They are used to model the input. For example, wood(17) is in the input trace if the agent collects wood at time 17, and factory(5) is in the input trace if the agent attempts to use the factory at time 5. Additionally, let getWood, getIron, getFire, getSteel, and useFactory be unary predicates. They are defined by the following rules.

\[
\begin{align*}
\text{wood}(t) &\rightarrow \text{getWood}(t) \quad (4) \\
\text{getWood}(t) &\rightarrow \text{getWood}(t+1) \quad (5) \\
\text{iron}(t) &\rightarrow \text{getIron}(t) \quad (6) \\
\text{getIron}(t) &\rightarrow \text{getIron}(t+1) \quad (7) \\
\text{fire}(t) &\rightarrow \text{getFire}(t) \quad (8) \\
\text{getFire}(t) &\rightarrow \text{getFire}(t+1) \quad (9) \\
\text{steel}(t) &\rightarrow \text{getSteel}(t) \quad (10) \\
\text{getSteel}(t) &\rightarrow \text{getSteel}(t+1) \quad (11) \\
\text{getSteel}(t-1) \land \text{factory}(t) &\rightarrow \text{useFactory}(t) \quad (12) \\
\text{getWood}(t-1) \land \text{getIron}(t-1) \land \text{getFire}(t-1) \land \text{factory}(t) &\rightarrow \text{useFactory}(t) \quad (13) \\
\text{useFactory}(t) &\rightarrow \text{useFactory}(t+1) \quad (14)
\end{align*}
\]

Then, at any given time point $\tau$, the task has been completed if useFactory($\tau$) is entailed by the input facts and the rules above.
References for The Appendix

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