The Speed–Robustness Trade-Off for First-Order Methods with Additive Gradient Noise

Bryan Van Scoy\textsuperscript{1} \quad Laurent Lessard\textsuperscript{2}

Abstract

We study the trade-off between convergence rate and sensitivity to stochastic additive gradient noise for first-order optimization methods. Ordinary Gradient Descent (GD) can be made fast-and-sensitive or slow-and-robust by increasing or decreasing the stepsize, respectively. However, it is not clear how such a trade-off can be navigated when working with accelerated methods such as Polyak’s Heavy Ball (HB) or Nesterov’s Fast Gradient (FG) methods. We consider three classes of functions: (1) smooth strongly convex quadratics, (2) smooth strongly convex functions, and (3) functions that satisfy the Polyak–Lojasiewicz property and have one-sided Lipschitz gradients. For each function class, we present a tractable way to compute the convergence rate and sensitivity to additive gradient noise for a broad family of first-order methods, and we present algorithm designs that trade off these competing performance metrics. Each design consists of a simple analytic update rule with two states of memory, similar to HB and FG. Moreover, each design has a scalar tuning parameter that explicitly trades off convergence rate and sensitivity to additive gradient noise. We numerically validate the performance of our designs by comparing their convergence rate and sensitivity to those of many other algorithms, and through simulations on Nesterov’s “bad function”.

1 Introduction

We consider the problem of designing robust first-order methods for unconstrained minimization. Given a differentiable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, consider solving the optimization problem

$$x^* \in \arg \min_{x \in \mathbb{R}^d} f(x),$$

where the algorithm only has access to gradient measurements corrupted by additive stochastic noise. Specifically, the algorithm can sample the oracle $g(x) := \nabla f(x) + w$, where $w$ is zero-mean and independent across queries. This form of additive noise arises in various applications, such as the following:

- To protect sensitive data, optimization algorithms may intentionally perturb the gradient by Gaussian noise in order to obtain differential privacy [7].

\textsuperscript{1}B. Van Scoy is with the Department of Electrical and Computer Engineering, Miami University, Oxford, OH, USA. Email: bvanscoy@miamioh.edu

\textsuperscript{2}L. Lessard is with the Department of Mechanical and Industrial Engineering, Northeastern University, Boston, MA, USA. Email: l.lessard@northeastern.edu
For some engineering systems, the gradient can only be obtained through noisy measurements [9].

In risk minimization in the context of learning algorithms, the objective is to minimize the expectation of the loss function over the population distribution [30,34,39].

A number of iterative algorithms have been proposed to solve this problem, and most have tunable parameters. For example, the well-known Gradient Descent (GD) method uses the iteration

\[
\text{Gradient Descent (GD): } x^{t+1} = x^t - \alpha \, g(x^t),
\]

where \( t \) is the iteration index and the stepsize \( \alpha \) is a tunable parameter. Fig. 1 illustrates how the error \( \|x^t - x^*\| \) evolves under GD applied to strongly convex quadratic functions for different fixed choices of \( \alpha \). We observe that convergence of the error is characterized by an initial transient phase followed by a stationary phase. In the transient phase, the magnitude of the gradient is much larger than that of the noise, and the error converges at a linear rate. When the gradient becomes small enough that the noise significantly corrupts the gradient, the average error of the iterates converges to a constant value.

![Figure 1: Trade-off between convergence rate and steady-state error (sensitivity to noise). Three different tunings of Gradient Descent (GD) with additive gradient noise are applied to random strongly convex quadratic functions on \( \mathbb{R}^{10} \). Half of the Hessian eigenvalues are at \( m = 1 \), the other half at \( L = 10 \). The initialization is \( x^0 = 1000e_1 \). Gradient noise is normally distributed \( \mathcal{N}(0,I) \) and i.i.d. across iterations. The plot shows mean and \( \pm 1 \) standard deviation of the error \( \|x^t - x^*\| \) for 1000 sample trajectories. Faster convergence comes at the cost of a larger steady-state error, and the trade-off is mediated by the stepsize \( \alpha \). On the right panel, iterations are plotted on a log scale to show a larger range of \( \alpha \) values.](image)

To characterize the transient phase, we use the convergence rate \( r > 0 \), defined in (9), which is the worst-case contraction rate of the error when there is no noise. Likewise, we characterize the stationary phase using the sensitivity \( \gamma > 0 \), defined in (10), which is the asymptotic root mean squared error of the iterates. This two-phase behavior is typical of stochastic methods.\(^1\) The fundamental trade-off observed in Fig. 1 is that \( r \) can only be made small (faster initial convergence) at the expense of a larger \( \gamma \) (larger steady-state error).

\(^1\)In the literature, stepsize is also known as learning rate. The transient phase is also known as the search or burn-in phase. The stationary phase is also known as the convergence or steady-state phase.
Gradient Descent is easy to interpret and tune: the choice of stepsize directly mediates the trade-off between convergence rate and sensitivity. Unfortunately, GD is generally slow to converge because it does not exploit the structure present in smooth strongly convex functions, for example. For such functions, alternative methods can provide faster convergence rates. Two such methods are Polyak’s Heavy Ball [51] and Nesterov’s Fast Gradient [47], which use the iterations

\[
\text{Heavy Ball (HB): } x^{t+1} = x^t - \alpha g(x^t) + \beta (x^t - x^{t-1}), \\
\text{Fast Gradient (FG): } x^{t+1} = x^t - \alpha g(x^t + \beta (x^t - x^{t-1})) + \beta (x^t - x^{t-1}).
\]

In the noise-free setting (exact gradient oracle) and under suitable regularity assumptions about the function \(f\) such as smoothness and strong convexity, both HB and FG can be tuned in a way that achieves a faster worst-case performance than GD. When using a noisy gradient oracle, these methods exhibit a stationary phase similar to that of GD in Fig. 1. A trade-off between convergence rate and sensitivity to noise must also exist for HB and FG, but there are now two parameters to tune, so it is unclear how they should be modified to mediate this trade-off.

The goal of this work is to study the trade-off between rate and sensitivity for first-order algorithms, and to design algorithms that trade off these competing performance metrics. We consider three well-studied classes of functions \(f : \mathbb{R}^d \to \mathbb{R}\) characterized by scalar parameters \(m\) and \(L\) that satisfy \(0 < m \leq L < \infty\). The function classes are defined as follows.

- **Smooth strongly convex quadratics** \((Q_{m,L})\). Functions such that, for some \(f^* \in \mathbb{R}\), \(y^* \in \mathbb{R}^d\), and \(Q = Q^T \in \mathbb{R}^{d \times d}\) with eigenvalues in the interval \([m, L]\), the function has the form \(f(y) = \frac{1}{2}(y - y^*)^T Q(y - y^*) + f^*\).

- **Smooth strongly convex functions** \((F_{m,L})\). Differentiable functions for which \(f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} ||y - x||^2\) and \(||\nabla f(x) - \nabla f(y)|| \leq L||x - y||\) for all \(x, y \in \mathbb{R}^d\).

- **PL functions with Lipschitz gradients** \((PL_{m,L})\). Differentiable functions that attain a minimal value (call it \(f^*\)), for which \(f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} ||y - x||^2\) for all \(x, y \in \mathbb{R}^d\) and \(\frac{1}{2}||\nabla f(x)||^2 \geq m(f(x) - f^*)\) for all \(x \in \mathbb{R}^d\). We further discuss this function class in Section 6.

These classes of functions are nested in the sense that

\[Q_{m,L} \subseteq F_{m,L} \subseteq PL_{m,L}\]

with equality occurring when \(m = L\). Moreover, all functions in these classes have a set of optimal points \(Y^* = \arg \min_{y \in \mathbb{R}^d} f(y)\) such that \(\nabla f(y^*) = 0\) for all \(y^* \in Y^*\).

**Remark 1.** Each of the function classes is defined in terms of the parameters \(m\) and \(L\) that describe lower and upper bounds on the function. In our definitions, these parameters may not be tight bounds, so the function classes include degenerate functions; for instance, the function

\[f(y) = \frac{1}{2}||y||^2 \text{ with } m = L = 0\]

has eigenvalues in the interval \([m, L]\), that \(m\)-strong convexity implies the PL condition with the same value \(m\) is shown in [37, Appendix B], and that the Lipschitz condition \(||\nabla f(x) - \nabla f(y)|| \leq L||x - y||\) implies the (one-sided) Lipschitz condition \(f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} ||y - x||^2\) follows from the discussion in Remark 7. The fact that the function classes are equal when \(m = L\) follows from the fact that the PL condition implies quadratic growth with the same constant [37, Appendix A] which, along with the Lipschitz condition, implies that the function is quadratic.
$x \mapsto \frac{1}{2} \| x \|^2$ is in both $Q_{1,1}$ and $Q_{1,10}$. In Section 3, we show how our analysis can be modified to exclude such degenerate functions for the class $Q_{m,L}$.

For each function class, we design algorithms in the three-parameter family:

$$x^{t+1} = x^t - \alpha g(x^t + \eta (x^t - x^{t-1})) + \beta (x^t - x^{t-1}),$$  \hspace{1cm} (5)$$

where $\alpha$, $\beta$, and $\eta$ are scalar parameters. This algorithm parameterization was first introduced in [41] and is further discussed in Section 2.2. Note that GD, HB, and FG are special cases of the three-parameter family (5) that use $\beta = \eta = 0$, $\eta = 0$, and $\eta = \beta$, respectively.

### 1.1 Main contributions

Our three main contributions are as follows.

1. For each function class $Q_{m,L}$, $F_{m,L}$, and $PL_{m,L}$, we present a method for efficiently bounding the worst-case convergence rate and sensitivity to additive gradient noise for a wide class of algorithms. The computational effort required to find the rate and sensitivity bounds for a given algorithm is independent of problem dimension $d$ and takes fractions of a second on a laptop.

2. We present two new algorithms, RHB and RAM, for the function classes $Q_{m,L}$ and $F_{m,L}$, respectively. Each algorithm has the form (5), where $(\alpha, \beta, \eta)$ are explicit algebraic functions of the function class parameters $m$ and $L$ along with a single tuning parameter that directly trades off convergence rate and sensitivity. In particular, the tuning parameter $r$ is an upper bound on the worst-case convergence rate over the function class, with a slower rate yielding an algorithm that is less sensitive to gradient noise.

3. We demonstrate through several empirical studies that our algorithm designs effectively trade-off convergence rate and sensitivity to gradient noise. We use a brute-force approach to numerically compare our algorithms against a large sampling of other algorithms. We also show that our algorithm designs compare favorably to (i) popular algorithms such as nonlinear conjugate gradient and quasi-Newton methods, and (ii) existing algorithm designs that use either fewer or more parameters.

Our results can be concisely summarized in Fig. 2, which illustrates the trade-off between convergence rate and noise sensitivity. Our proposed tunable algorithms RHB and RAM are designed for the function classes $Q_{m,L}$ and $F_{m,L}$, respectively, and GD is shown for $PL_{m,L}$. Each design is a curve in the rate–sensitivity space, parameterized by the tuning parameter. Fig. 2 also compares popular algorithms HB, FG, GD, and Triple Momentum (TM) with their recommended tunings (defined in Table 1), which show up as individual points.

**Paper organization.** The rest of the paper is organized as follows. In the following subsections, we provide additional background on the problem and describe our analysis and design methodology in more detail. In Section 2, we give a detailed description of the problem setting and performance measures we will be using. Sections 3 to 6 treat the function classes $Q_{m,L}$, $F_{m,L}$, and

---

3For the function class $PL_{m,L}$, our methodology was unable to construct an algorithm that outperforms Gradient Descent, so we instead provide the rate of convergence and sensitivity to noise for this method.
Figure 2: Illustration of the trade-off between steady-state error (sensitivity to noise) and convergence rate for several algorithms. Our proposed tunable algorithms RHB and RAM are designed for the function classes $Q_{m,L}$ and $F_{m,L}$, respectively, and GD is shown for the function class $PL_{m,L}$. The left panel shows $L = 5$ and the right panel shows $L = 50$. Both panels use $m = 1$, $d = 1$, and $\sigma^2 = 1$ (noise variance). Also shown are other algorithms with their recommended tunings, shown as individual colored dots (see Table 1).

Function class $Q_{m,L}$
- RHB
- HB

Function class $F_{m,L}$
- RAM
- TM
- FG

Function class $PL_{m,L}$
- GD
- GD, $1/L$

In each of these sections, we develop computationally tractable approaches to computing the convergence rate and noise sensitivity, and we present our algorithm designs. In Section 7, we present empirical studies supporting the effectiveness of our designs and comparing them to existing algorithms. We provide concluding remarks in Section 8.

1.2 Literature review

Complexity bounds. The noise-free setting is well-studied, and algorithms have been discovered that achieve optimal worst-case iteration complexity for a variety of different function classes. We discuss these results in Sections 3.2, 5.3 and 6.3, where we present our algorithm designs for $Q_{m,L}$, $F_{m,L}$, and $PL_{m,L}$, respectively.

In the additive gradient noise setting, there are fundamental lower bounds on the asymptotic convergence rate of iterative methods. No matter what iterative scheme is used, $E\|x^t - x^*\|^2$ cannot decay asymptotically to zero faster than $1/t$. Roughly, this is because the rate at which error can decay is limited by the concentration properties of the gradient noise [30,31]. This optimal asymptotic rate is achieved by Gradient Descent with a diminishing stepsize that decreases like $1/t$ [10, Theorem 4.7] (GDDS).

From an asymptotic standpoint, there is no benefit to using anything more complicated than GDDS. However, GDDS can perform poorly in practice. This is because algorithms are typically run for a predetermined number of iterations (or time budget), or until a predetermined error level is reached.

Stochastic approximation. In the additive gradient noise setting, a variety of techniques have been proposed to improve transient performance. For the case of least squares regression, for instance, a dramatic speedup can be achieved via careful manipulation of the stepsize or by using acceleration [28,35]. Kulunchakov and Mairal [38] also show that any algorithm that converges
linearly for smooth and strongly convex objectives in the noiseless setting can be converted into an
algorithm that converges optimally (up to log factors) when additive noise is included.

Ghadimi and Lan [30] propose the AC-SA algorithm, which is a modification of Nesterov’s Fast Gradient method that has near-optimal iteration complexity in the smooth and strongly convex setting. Besides characterizing the convergence of the average iterates, they also estimate the performance of the algorithm on a single problem instance. Building on this work, Cohen et al. [15] propose the algorithm AGD+, a “lazy” (dual averaging) counterpart of AC-SA for the smooth and convex setting, along with its extension $\mu$AGD+ for the strongly convex setting. In the strongly convex setting, $\mu$AGD+ improves on the convergence bound of the AC-SA algorithm, but does not achieve the lower complexity bound. In particular, applying [15, Corollary B.5] with $\gamma_i = c \sqrt{m/L}$ yields the bound

$$\frac{m}{2} \mathbb{E} \|y^t - y^\star\|^2 \leq \mathbb{E} [f(y^t) - f^\star] \leq \left( 1 - c \sqrt{\frac{m}{L}} \right)^t \frac{L - m}{2} \|x^0 - x^\star\|^2 + \frac{a^2 c^2}{\sqrt{mL}}$$

where we used the definition of strong convexity to bound the distance to optimality in terms of the optimality gap. The parameter $c \in (0, \sqrt{L/m}]$ can be chosen to trade off rate and sensitivity for this algorithm; however, we show that the resulting trade-off is strictly suboptimal (see Fig. 4).

None of the above algorithms achieve the optimal iteration complexity in the smooth strongly convex setting. One way to achieve the optimal complexity, however, is to exploit the rapid convergence of the transient phase by using a piecewise constant stepsize, effectively dividing the convergence into epochs or stages. This can be done on a predetermined schedule, or by using a statistical test to detect the phase transition which then triggers the parameter change [14, 15]. In Ghadimi and Lan’s companion paper [31], for instance, they propose a multi-stage version of AC-SA that achieves the optimal expected rate of convergence (up to constants) in this setting. Aybat et al. build on Nesterov’s Accelerated Stochastic Gradient method to construct the Multi-Stage Accelerated Stochastic Gradient method (M-ASG) [5]. This algorithm simultaneously achieves the optimal iteration complexity (again, up to constants) in both the deterministic and stochastic cases without requiring knowledge of the noise parameters.

The aforementioned multi-stage algorithms tune the single-stage algorithm stepsizes and the restart schedule to achieve desirable transient and asymptotic convergence, and they achieve the optimal complexity up to constants. In this work, we focus on analyzing and designing single-stage algorithms with fixed stepsizes that can be tuned via a single scalar parameter to explicitly trade-off convergence rate and sensitivity to noise. In other words, our algorithms can easily be adjusted to be made faster (and more sensitive to noise), or more robust to noise (and slower). When tuned to be as fast as possible, our algorithm designs recover algorithms that are known to converge with the optimal linear rate in the noise-free setting for the respective function classes (see Section 7).

Our algorithm designs may be used to converge at a linear rate to a predetermined noise level, or the trade-off parameter may be adjusted over time to construct multi-stage variants.

Other noise models. While we restrict our attention to additive stochastic gradient noise in the present work, other inexact oracles have been studied in the literature.

---

4While we do not propose a restart mechanism or time-varying parameter schedule in this work, we illustrate such an approach in Section 7.4 through a hand-tuned schedule.
When the gradient noise is due to sampling a finite-sum objective function, viable strategies include incremental gradient or variance reduction methods such as SVRG [36], SAGA [19], and many others [2, 26, 48, 61, 65].

A prominent alternative model is to assume deterministic bounded noise, which may be additive or multiplicative. For instance, Devolder et al. [22] propose an inexact deterministic first-order oracle and show that, under this oracle, acceleration necessarily results in an accumulation of gradient errors in the unconstrained smooth (not strongly convex) case. For this setting, the same authors propose the Intermediate Gradient Method, a family of first-order methods that can be tuned to trade off convergence rate and sensitivity to gradient errors [21]. The same authors also extended this oracle to analyze inexact first-order methods in the strongly convex case [20]; there is still a trade-off between rate and sensitivity, but errors no longer accumulate and can converge linearly to a within a constant ball about the optimal solution. Multiplicative deterministic noise has been studied in [41], for which the Robust Momentum (RM) [16] was designed to trade off convergence rate and sensitivity. Stochastic Gradient Descent has also been analyzed under a hybrid additive and multiplicative deterministic oracle [33].

1.3 Methodology

We now provide an overview of our analysis and design framework and point to related techniques. Our method for bounding the convergence rate $r$ and sensitivity $\gamma$ for the classes $F_{m,L}$ and $PL_{m,L}$ relies on solving a small linear matrix inequality (LMI). This idea builds upon several related works.

- The performance estimation problem (PEP) framework [24, 57] uses LMIs to directly search for a problem instance that achieves the worst-case performance. The PEP framework has been successfully applied to numerous algorithms, such as the proximal gradient method [58], operator splitting methods [52], and gradient descent using an exact line search with noisy search directions [17]. Of particular interest, [54] uses PEP to analyze iterative algorithms under various noise models when the objective is smooth and convex; this approach searches for a time-varying potential (or Lyapunov) function whose existence certifies a performance bound. To obtain closed-form expressions for the bound, a sequence of increasingly large LMIs is solved, and the numerical solution is used as a guide to find expressions for the potential function parameters as a function of the iteration. Alternatively, small PEPs can be formulated and solved to search for potential functions with fixed parameters as in [18]. Typical potential functions include the function value suboptimality $f(x^t) - f^*$, squared iterate suboptimality $\|x^t - x^*\|^2$, and squared gradient norm $\|\nabla f(x^t)\|^2$. In contrast to the standard PEP approach, we search over a parameterized class of Lyapunov functions that may depend on a finite number of sequential iterates.

- Viewing algorithms as discrete-time dynamical systems, Integral Quadratic Constraints (IQC) from control theory [27, 41, 45] may be used to search for worst-case performance guarantees. This approach also leads to LMIs and characterizes the asymptotic performance of an algorithm, although the ensuing performance bounds may not be tight in general. Moreover, while IQCs may be used to analyze the function classes $Q_{m,L}$ and $F_{m,L}$, no known IQCs exist for $PL_{m,L}$, as the PL condition depends on function values and IQCs do not involve function values.

---

5 Also known as worst-case or adversarial noise.
Finally, LMIs may be used to directly search for a Lyapunov function, which is a generalized notion of “energy” stored in the system. If a fraction of the energy dissipates at each iteration, this is akin to proving convergence at a specified rate [32, 40, 56]. Similar to IQCs, Lyapunov functions provide asymptotic (albeit more interpretable) performance guarantees.

In the present work, we adopt a Lyapunov approach most similar to [56], but we generalize it to include both convergence rate and sensitivity to noise. We also explain in Section 5.4 how the Lyapunov approach is related to IQCs.

Given an LMI that establishes the performance (convergence rate or sensitivity to noise) of a given first-order method, the next step is to design algorithms that exploit this trade-off. Several approaches have been proposed.

- One can parameterize a family of candidate algorithms and search over algorithm parameters to effectively trade off convergence rate and sensitivity to gradient noise. For example, for every fixed rate, one could seek algorithm parameters that minimize the bound on the sensitivity. Such a problem is typically non-convex, so one must resort to exhaustive search [41] or nonlinear numerical solvers that find local optima.

- Using convex relaxations or other heuristics such as coordinate descent, the algorithm design problem can be solved approximately. While this approach may lead to conservative designs, it has the benefit of being automated, flexible, and amenable to efficient convex solvers [45].

- In certain settings, the controller parameters can be eliminated from the LMI entirely, yielding bounds that hold for all algorithms. This approach has been used to show that the Triple Momentum (TM) method achieves the optimal worst-case rate over the class $F_{m,L}$ [42, 53].

As in the above approaches, we parameterize a family of algorithms, (5), and search over the algorithm parameters. However, our approach to algorithm design is algebraic rather than numeric. We find analytic solutions to the non-convex semidefinite programs that arise when the algorithm parameters are treated as decision variables. This approach enables us to find explicit analytic expressions for our algorithm parameters. Nevertheless, we still make use of numerical approaches in order to validate our choice of parameterization and the efficacy of our designs, and to compare the performance of our designs to that of existing algorithms; see Section 7. Our design procedure is outlined as follows.

1. **Choose the function class.** For each algorithm design, we choose one of the function classes $Q_{m,L}$, $F_{m,L}$, or $PL_{m,L}$. We typically choose $L/m = 10$ or 100, since larger ratios of $L/m$ may cause numerical issues for the LMI solvers while smaller ratios are typically irrelevant in practice.

2. **Numerically find an effective algorithm.** As described previously, directly searching over algorithm parameters is often a non-convex problem. However, we can use exhaustive search or a nonlinear numerical solver to find an algorithm that achieves a particular trade-off between rate and sensitivity. Ideally, this algorithm would be Pareto-optimal with respect to the rate–sensitivity trade off. In practice, however, we optimize the trade-off between the bounds on rate and sensitivity without any guarantees on optimality.

3. **Find the active constraints.** Using the numerical values of the algorithm parameters, we determine the active constraints. For analyses based on LMIs, the associated matrices will
drop rank at the optimal solution, giving rise to a set of equations: if a matrix has rank $\nu$, then all of its $(\nu + 1) \times (\nu + 1)$ minors are zero.

4. **Solve the set of equations.** We then solve the set of active constraints to obtain algebraic expressions for the algorithm parameters. For analyses based on LMIs, the set of active constraints is a system of polynomial equations, such as the minors of a rank-deficient matrix being zero. One way to solve such a system is to compute a Gröbner basis [13], which provides a means of characterizing all solutions to a set of polynomial equations, similar to Gaussian elimination for linear systems. While many software implementations exist to compute such a basis, these algorithms are typically inefficient for the large system of equations produced by the LMIs. Instead of describing all solutions, we use Mathematica [62] to search for the single solution that describes the numerical algorithm and its corresponding solution to the LMI obtained previously. Some heuristics that we use in solving the set of equations are:

- first eliminate variables that appear linearly, since the resulting system after back-substitution remains polynomial, and
- observe whether any expressions factor, in which case we can use the numerical solution to determine which factor is zero.

Our design process results in simple algebraic expressions for the algorithm parameters as well as any other variables used in the analysis (such as the solution to an LMI$^6$. While we provide no formal guarantees of Pareto-optimality of our designed algorithms, we show numerically in Section 7 that our designs provide an effective trade-off between convergence rate and sensitivity to noise.

2 Problem setting and assumptions

To solve optimization problem (1), we consider iterative first-order methods described by linear time-invariant dynamics of the form

$$
\begin{align*}
\xi^{t+1} &= A\xi^t + B(u^t + w^t), \\
y^t &= C\xi^t, \\
u^t &= \nabla f(y^t),
\end{align*}
$$

(7a)
(7b)
(7c)

where $\xi^t \in \mathbb{R}^{n \times d}$ is the state of the algorithm, $y^t \in \mathbb{R}^{1 \times d}$ is the point at which the gradient is evaluated, $u^t \in \mathbb{R}^{1 \times d}$ is the (exact) value of the gradient, $w^t \in \mathbb{R}^{1 \times d}$ is the gradient noise, and $t$ is the iteration. The state is the memory of the algorithm because its size reflects the number of past iterates that must be stored at each timestep. Solutions of the dynamical system are called trajectories. For example, given an algorithm $(A, B, C)$, particular function $f$, initial condition $\xi^0$, and noise distribution $w \sim \mathbb{P}$, a trajectory is any sequence $(\xi^t, u^t, y^t, w^t)_{t \geq 0}$ that satisfies (7). This general framework encompasses a wide variety of fixed-parameter first-order iterative methods; we discuss this in more detail in Section 2.2.

**Remark 2** (important notational convention). Throughout the paper, we represent iterates as row vectors, so that the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, and $C \in \mathbb{R}^{1 \times n}$ act on the columns of the

$^6$For the function class $F_{m,L}$, the analytic solution corresponding to the numerical algorithm is quite complex (involving the roots of polynomials of large degree), so we instead find an algorithm with simple stepsizes that trades off rate and sensitivity almost as well as the algorithm obtained numerically.
iterates while the gradient oracle $\nabla f : \mathbb{R}^{1 \times d} \to \mathbb{R}^{1 \times d}$ acts on the rows. Because of this convention, $\| \cdot \|$ denotes the Frobenius norm, for example, $\| \xi \|^2 := \sum_{i=1}^{n} \sum_{j=1}^{d} (\xi_{ij})^2$. This framework decouples the state dimension of the algorithm ($n$) from the dimension of the objective function's domain ($d$).

In our model (7), the noise corrupts the gradient $u^t$, although it is straightforward to modify our analysis to other scenarios, such as when the noise corrupts the entire state $\xi^t$.

We make the following assumptions regarding the algorithm (7).

**Assumption 1** (algorithm form). The matrix $A$ has an eigenvalue at 1, and the associated eigenvector $v \in \mathbb{R}^n$ satisfies $Cv \neq 0$.

The assumption on the algorithm form implies the existence of a deterministic fixed point for any optimizer of (1). Each function in the three function classes has a set of optimizers $Y^* := \arg \min_{y \in \mathbb{R}^d} f(y)$ that satisfies $\nabla f(y^*) = 0$ for all $y^* \in Y^*$. For each such point, Assumption 1 implies the existence of a corresponding fixed point $\xi^*$ such that $\xi^* = A \xi^*$ and $y^* = C \xi^*$. If we initialize our algorithm at the fixed point $\xi^0 = \xi^*$ and there is no gradient noise, subsequent iterates will remain at $\xi^*$.

Since we are doing worst-case analysis, we may assume without loss of generality that an optimal point is at the origin. We formalize this fact in the following lemma.

**Lemma 2.1** (fixed point shifting). Let $F \in \{Q_{m,L}, F_{m,L}, PL_{m,L}\}$ be one of the function classes defined in Section 1. Consider a function $f \in F$ with an optimal point $y^*$, optimal function value $f^* := f(y^*)$, and optimal gradient $u^* = 0$. Then the function $\tilde{f}(y) = f(y + y^*) - f^*$ has the property that $\tilde{f} \in F$ has an optimal point, optimal function value, and optimal gradient all zero.

Let $(A, B, C)$ be an algorithm of the form (7) satisfying Assumption 1. Define the shifted coordinates $\tilde{\xi}^t := \xi^t - \xi^*$ and similarly for $\tilde{y}^t$ and $\tilde{u}^t$. Then these shifted coordinates also satisfy the dynamics (7) applied to the function $\tilde{f}$, that is,

$$
\tilde{\xi}^{t+1} = A \tilde{\xi}^t + B(\tilde{u}^t + w^t),
$$

$$
\tilde{y}^t = C \tilde{\xi}^t,
$$

$$
\tilde{u}^t = \nabla \tilde{f}(\tilde{y}^t).
$$

Lemma 2.1 implies that we can make the following assumption without loss of generality.

**Assumption 2.** The optimal function value $f^*$ is zero, and the set of optimal points $Y^*$ contains zero.

In the remainder of this section, we provide definitions of the convergence rate and noise sensitivity introduced in Section 1, and we parameterize the set of algorithms considered for design.

### 2.1 Performance evaluation

As alluded to in Section 1, we consider the trade-off between *convergence rate* and *sensitivity* to additive stochastic gradient noise. Let $A = (A, B, C)$ denote an algorithm as defined in (7) and let $F \in \{Q_{m,L}, F_{m,L}, PL_{m,L}\}$ denote one of the families of functions defined in Section 1.
**Convergence rate.** The convergence rate describes the first phase of convergence observed in Fig. 1: exponential decrease of the error. In this regime, gradients are relatively large compared to the noise, so we assume $w_t = 0$ for all $t \geq 0$. For any algorithm $A$ with initial point $\xi^0$ and fixed point $\xi^*$ and any function $f \in \mathcal{F}$, consider the trajectory $(\xi^0, \xi^1, \ldots)$ produced by $A$. We define the linear convergence rate as

$$
\text{RATE}(A, \mathcal{F}) := \inf \left\{ r > 0 \mid \sup_{f \in \mathcal{F}} \sup_{\xi^0 \in \mathbb{R}^{n \times d}} \sup_{t \geq 0} \frac{\|\xi^t - \xi^*\|}{r^t \|\xi^0 - \xi^*\|} < \infty \right\}.
$$

In other words, if the convergence rate $r = \text{RATE}(A, \mathcal{F})$ is finite, then for all $\varepsilon > 0$, there exists a constant $c$ such that the trajectory satisfies the bound $\|\xi^t - \xi^*\| \leq c(r + \varepsilon)^t \|\xi^0 - \xi^*\|$ for all functions $f \in \mathcal{F}$, initial points $\xi^0 \in \mathbb{R}^{n \times d}$, and iterations $t \geq 0$. This definition corresponds to the conventional notion of linear convergence rate used in the worst-case analysis of deterministic algorithms. If $r < 1$, the algorithm is said to be *globally linearly convergent*, and for all $\varepsilon > 0$, we have $\|y^t - y^*\| = O((r + \varepsilon)^t \|\xi^0 - \xi^*\|)$. Smaller $r$ corresponds to a faster (worst-case) convergence rate.

**Sensitivity.** The sensitivity $\gamma$ characterizes the steady-state phase of convergence observed in Fig. 1. The steady-state error depends on the noise characteristics. We make the following assumptions on the noise sequence.

**Assumption 3 (Noise sequence).** We assume that the noise sequence $w^0, w^1, \ldots$ has joint distribution $\mathbb{P} \in \mathcal{P}_\sigma$, with parameter $\sigma$ to be defined shortly. We assume the set of admissible joint distributions $\mathcal{P}_\sigma$ satisfies:

1. Independence across time. For all $\mathbb{P} \in \mathcal{P}_\sigma$, if $w \sim \mathbb{P}$, then $w^t$ and $w^\tau$ are independent for all $t \neq \tau$. Then we may characterize the joint distribution $\mathbb{P} \in \mathcal{P}_\sigma$ by its associated marginal distributions $(\mathbb{P}^0, \mathbb{P}^1, \ldots)$. We do not assume the $\mathbb{P}^t$ are necessarily identical.

2. Zero-mean and bounded covariance. For all $(\mathbb{P}^0, \mathbb{P}^1, \ldots) \in \mathcal{P}_\sigma$, we have $E_{w^t \sim \mathbb{P}^t} (w^t) = 0$ and $E_{w^t \sim \mathbb{P}^t} ((w^t)^T w^t) \leq \sigma^2 I_d$. We assume $\sigma$ is known for the purpose of our analysis.

Our assumptions on the noise imply that $\mathcal{P}_\sigma$ is completely characterized by the variance bound $\sigma^2$. For any fixed algorithm $A$, function class $\mathcal{F}$, initial point $\xi^0$, and family of noise distributions $\mathcal{P}_\sigma$, consider the stochastic iterate sequence $y^0, y^1, \ldots$ produced by $A$. We define the noise sensitivity as

$$
\text{SENS}(A, \mathcal{F}, \sigma^2) := \sup_{f \in \mathcal{F}} \sup_{\xi^0 \in \mathbb{R}^{n \times d}} \sup_{\mathbb{P} \in \mathcal{P}_\sigma} \limsup_{T \to \infty} \sqrt{E_{w^\sim \mathbb{P}} \frac{1}{T} \sum_{t=0}^{T-1} \|y^t - y^*\|^2}.
$$

A smaller sensitivity is desirable because it means the algorithm achieves small error in spite of gradient perturbations. This definition is similar to that used in recent works exploring first-order algorithms with additive gradient noise using techniques from robust control [6, 45, 46].

**Remark 3.** Some authors [45, 46] compute the sensitivity with respect to the squared-norm of the state $\|\xi^t - \xi^*\|^2$ as opposed to that of the output. This quantity, however, depends on the state-space realization; performing a similarity transformation $(A, B, C) \mapsto (T A T^{-1}, T B, C T^{-1})$ for some invertible matrix $T$ does not change the input $u^t$ or output $y^t$ of the system, but it does map the state as $\xi^t \mapsto T \xi^t$ and therefore scales the sensitivity when measured with respect to the
Instead, we compute the sensitivity with respect to the squared-norm of the output $\|y^t - y^*\|^2$, which is invariant under similarity transformations. It is straightforward to modify our analysis to compute the sensitivity with respect to other quantities, such as the squared-norm of the gradient $\|u^t\|^2$ or the function values $f(y^t) - f^*$, as we shall see in Section 6.

2.2 Algorithm parameterization

While our analysis applies to the general algorithm model (7), for the purpose of design we will further restrict the class of algorithms to those with state dimension $n = 2$. At first, it may appear that algorithms of the form (7) have $n^2 + 2n$ degrees of freedom since we are free to choose $A, B, C$ however we like, so describing the case $n = 2$ should require 8 parameters. However, many of these parameters are redundant, and if we further assume the fixed point condition of Assumption 1, the case $n = 2$ can be completely described by the three-parameter family of algorithms (5).

Lemma 2.2 (three-parameter family). Any algorithm of the form (7) with state dimension $n = 2$ that satisfies Assumption 1 is equivalent to an algorithm in the three-parameter family (5) for some $(\alpha, \beta, \eta)$. By equivalent, we mean that both algorithms produce the same sequence of iterates $(y^0, y^1, \ldots)$ given appropriate initialization.

Proof. To see that the two algorithm forms are equivalent, we first observe that the three-parameter family (5) is precisely the general algorithm form (7) with

\[
\hat{\xi}^t = \begin{bmatrix} x^t \\ x^{t-1} \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 1 + \beta & -\beta \\ 1 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} -\alpha \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 + \eta & -\eta \end{bmatrix},
\]

(11)

where the “hat” symbol denotes this particular choice of state and matrices. We will show that any algorithm of the form (7) with state $\xi^t$ and matrices $(A, B, C)$ is equivalent to one with state $\hat{\xi}^t$ and matrices $(\hat{A}, \hat{B}, \hat{C})$ for some choice of $(\alpha, \beta, \eta)$. To do so, consider the transfer function [3, §2.7], which is the linear map from the $z$-transform of $(u^t + w^t)$ to the $z$-transform of $y^t$. For the algorithm (7), the transfer function is given by the rational function: $G(z) = C(zI - A)^{-1}B$. When $n = 2$, the transfer function $G(z)$ is a rational function with numerator degree at most one and denominator degree at most two. The roots of the denominator polynomial of $G(z)$ correspond to the eigenvalues of $A$, and by Assumption 1, one of those eigenvalues is fixed at one. This leaves three degrees of freedom: the root of the numerator polynomial, the remaining root of the denominator polynomial, and the constant scaling factor. One such parameterization of all such transfer functions is

\[
G(z) = -\alpha \frac{(1 + \eta)z - \eta}{(z - 1)(z - \beta)}
\]

where the three constants $(\alpha, \beta, \eta)$ parameterize the three degrees of freedom. This is precisely the transfer function of $(\hat{A}, \hat{B}, \hat{C})$. Putting this all together, we have that any algorithm of the form (7) with state dimension $n = 2$ that satisfies Assumption 1 has the same transfer function as the three-parameter algorithm (5) for some $(\alpha, \beta, \eta)$. Because the algorithms have the same transfer function, there exists a state-space similarity transformation that relates the states of the two algorithms [66, §3.3]. A similarity transformation is described by an invertible matrix $T \in \mathbb{R}^{n \times n}$ that relates the states of the two algorithms by $\hat{\xi}^t = T \xi^t$. Under this transformation, the state-space matrices of the two algorithms are related by $(\hat{A}, \hat{B}, \hat{C}) = (TAT^{-1}, TB, CT^{-1})$. If we initialize these algorithms with initial state $\xi^0$ and $\hat{\xi}^0 = T \xi^0$ respectively, they will have the same
trajectories \((y^0, y^1, \ldots)\) and are therefore equivalent. See [3, §5.3] for details on constructing this similarity transformation.

We will refer to specific algorithms from the three-parameter family (5) by their triplet of parameters \((\alpha, \beta, \eta)\). All algorithms in this class satisfy Assumption 1. By choosing specific values for the parameters, the algorithm (5) recovers several known algorithms. We describe these algorithms in Table 1, as they will serve as useful benchmarks for our designs. Later in Section 7.3, we provide numerical evidence that further justifies our choice of parameterization.

**Table 1:** Comparison of different algorithms with their recommended/standard tunings. For RM, the tuning parameter satisfies \(1 - \sqrt{\frac{m}{L}} \leq r \leq 1 - \frac{m}{L} \). When \(r = 1 - \sqrt{\frac{m}{L}}\), RM becomes TM; when \(r = 1 - \frac{m}{L}\), RM becomes GD with \(\alpha = \frac{1}{L}\).

| Algorithm name          | \(\alpha\)                   | \(\beta\)                   | \(\eta\)                   |
|-------------------------|------------------------------|------------------------------|------------------------------|
| Gradient Descent (GD)   | \(\frac{1}{L}\) or \(\frac{2}{L+m}\) | 0                            | 0                            |
| Heavy Ball (HB), [51]   | \(\frac{4}{(\sqrt{L}+\sqrt{m})^2}\) | \(\frac{\sqrt{L-\sqrt{m}}}{\sqrt{L+\sqrt{m}}^2}\) | 0                            |
| Fast Gradient (FG), [47]| \(\frac{1}{L}\)             | \(\frac{\sqrt{L-\sqrt{m}}}{\sqrt{L+\sqrt{m}}^2}\) | \(\frac{\sqrt{L-\sqrt{m}}}{\sqrt{L+\sqrt{m}}^2}\) |
| Triple Momentum (TM), [60]| \(\frac{L+\sqrt{m}}{mL}\) | \(\frac{L}{L+\sqrt{m}L}\) | \(\frac{L}{L+\sqrt{m}L}\) |
| Robust Momentum (RM), [16]| \(\frac{(1-r)^2(1+r)}{m}\) | \(\frac{L^3}{L-m}\)          | \(\frac{(L-m)(1-r)^2(1+r)}{m}\) |

Different triples \((\alpha, \beta, \eta)\) generally correspond to different algorithms, with one important exception: Gradient Descent has a degenerate family of parameterizations.

**Proposition 2.3.** Gradient Descent with parameterization \((\alpha, 0, 0)\) can also be parameterized by \((\alpha(1-\beta), \beta, \frac{\beta}{1-\beta})\) for any choice of \(\beta \neq 1\).

Proposition 2.3 follows from the fact that when we substitute \((\alpha, \beta, \eta) \mapsto (\alpha(1-\beta), \beta, \frac{\beta}{1-\beta})\) into (5), the update equation can be rearranged to obtain

\[
\left(\frac{x^{t+1} - \beta x^t}{1-\beta}\right) = \left(\frac{x^t - \beta x^{t-1}}{1-\beta}\right) - \alpha \nabla f \left(\frac{x^t - \beta x^{t-1}}{1-\beta}\right) - \alpha w^t. \tag{12}
\]

In other words, it is simply Gradient Descent applied to the quantity \(y^t := \frac{1}{1-\beta}(x^t - \beta x^{t-1})\). The equivalence can also be seen by observing that the transfer function \(G(z)\) simplifies to \(G(z) = \frac{z}{z-\beta}\) due to a pole-zero cancellation of the factor \((z - \beta)\), so we recover the transfer function of GD.

In the next three sections, we focus on the function classes \(Q_{m,L}\), \(F_{m,L}\), and \(PL_{m,L}\), respectively. For each class, we provide a tractable approach for computing (bounds on) the convergence rate and sensitivity, and we design algorithms of the form (5) that effectively trade off these performance metrics.

### 3 Smooth strongly convex quadratic functions

We begin with the class \(Q_{m,L}\) of strongly convex quadratics. We will first derive exact formulas for the worst-case convergence rate and the sensitivity to additive gradient noise.
3.1 Performance bounds for $Q_{m,L}$

The quadratic case has been treated extensively in recent works on algorithm analysis. In particular, the convergence rate has been completely characterized for this function class [41], and closed-form expressions for the sensitivity of the GD, HB, and FG methods have been obtained [6, 46]. We now present versions of these results adapted to our algorithm class of interest. When $f$ is a positive definite quadratic and we shift the fixed-point to zero using Lemma 2.1, we can write $\nabla f(y) = yQ$ for some $Q \in \mathbb{R}^{d \times d}$ satisfying $Q = Q^T > 0$ (recall that the iterates are row vectors, so $y \in \mathbb{R}^{1 \times d}$). The algorithm’s dynamics (8) become

$$\xi^{t+1} = A\xi^t + BC\xi^t Q + B\tilde{w}^t \quad \text{and} \quad y^t = C\xi^t. \quad (13)$$

If we diagonalize $Q$, we can split the dynamics into $d$ decoupled systems, each of the form:

$$\hat{\xi}^{t+1} = (A + qBC)\hat{\xi}^t + B\hat{\tilde{w}}^t \quad \text{and} \quad \hat{y}^t = C\hat{\xi}^t, \quad (14)$$

where $q$ is an eigenvalue of $Q$, and we now have $\hat{\xi}^t \in \mathbb{R}^{n \times 1}$, $\hat{y}^t \in \mathbb{R}$, and $\hat{\tilde{w}}^t \in \mathbb{R}$. The convergence rate and sensitivity of the original system can be obtained by analyzing the simpler system (14). In particular, the convergence rate is the spectral radius of the system matrix $A + qBC$. With regards to the sensitivity, we observe that since the covariance of $\tilde{w}^t$ is bounded by $\sigma^2 I_d$, the covariance of $\hat{\tilde{w}}^t$ is bounded by $\sigma^2$. Due to the way the sensitivity is defined in (10), we can compute the sensitivity separately for each decoupled system (14) and sum them together to obtain the sensitivity for the original system. We summarize the results in the following proposition.

**Proposition 3.1** ($Q_{m,L}$ analysis). Consider an algorithm $A = (A, B, C)$ defined in (7) satisfying Assumption 1 applied to a strongly convex quadratic $f \in Q_{m,L}$ defined in Section 1, and assume the noise sequence satisfies Assumption 3. Let $q_1, \ldots, q_d$ denote the eigenvalues of the Hessian $\nabla^2 f$. The algorithm has convergence rate

$$\text{RATE}(A, \{f\}) = \sup_{i \in \{1, \ldots, d\}} \rho(A + q_i BC).$$

If $\text{RATE}(A, \{f\}) < 1$, the algorithm has sensitivity

$$\text{SENS}(A, \{f\}, \sigma^2) = \sigma \sqrt{\sum_{i=1}^{d} B^T P_{q_i} B},$$

where $P_{q_i}$ is the unique solution to the linear equation

$$(A + q_i BC)^TP_{q_i} (A + q_i BC) - P_{q_i} + C^TC = 0. \quad (15)$$

Here, $\rho(\cdot)$ denotes the spectral radius (largest eigenvalue magnitude). The linear matrix equation is a Lyapunov equation, and the fact that it has a unique solution is provided in [25, Theorem 11.5]. Results similar to Proposition 3.1 have appeared in the context of algorithm analysis for quadratic functions in [6, 46], and make use of the fact that the sensitivity is equivalent to the $\mathcal{H}_2$ norm of the system, which can be computed using a Lyapunov approach.
Proposition 3.1 describes the performance on a single quadratic objective function. Over the entire function class \( Q_{m,L} \), the convergence rate is

\[
\text{Rate}(A, Q_{m,L}) = \sup_{q \in [m,L]} \rho(A + qBC).
\]

If \( \text{Rate}(A, Q_{m,L}) < 1 \), the worst-case sensitivity over \( Q_{m,L} \) is attained by a (degenerate) quadratic function for which all eigenvalues of the Hessian are at the value of \( q \) that maximizes \( B^T P_q B \), which yields

\[
\text{Sens}(A, Q_{m,L}, \sigma^2) = \sigma \sqrt{d} \cdot \sup_{q \in [m,L]} \sqrt{B^T P_q B}
\]

where \( P_q \) satisfies the Lyapunov equation (15). These expressions for the rate and sensitivity may be difficult to evaluate if the matrices \((A, B, C)\) are large. Fortunately, the sizes of these matrices only depend on the state dimension \( n \), which is typically small \((n \leq 2\) for all methods in Table 1). The dimension \( d \) of the function domain does not appear in the expression for the convergence rate, and only appears as a proportionality constant in the expression for the sensitivity.

3.2 Algorithm design for \( Q_{m,L} \)

For strongly convex quadratic functions, first-order methods can achieve exact convergence in \( d \) iterations when there is no gradient noise, where \( d \) is the dimension of the domain of \( f \). One such example is the Conjugate Gradient (CG) method [49, Thm. 5.4]. However, when the number of iterations \( t \) satisfies \( t < d \), exact convergence is not possible in general. Nesterov’s lower bound [47, Thm. 2.1.13] demonstrates that for any \( t \geq 0 \), one can construct a function \( f \in Q_{m,L} \) with domain dimension \( d > t \) such that:

\[
\|y^t - y^*\| \geq \left( \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}} \right)^t \|y^0 - y^*\|.
\]

This lower bound holds for any first-order method such that \( y^t \) is a linear combination of \( y^0 \) and (the exact) past gradients \( \nabla f(y^0), \ldots, \nabla f(y^{t-1}) \). This class includes not only CG but also methods with unbounded memory.

In the regime \( t < d \), the CG method matches Nesterov’s lower bound [49, Thm. 5.5] and is therefore optimal in terms of worst-case rate. However, it is not clear how CG should be adjusted to be robust in the presence of additive gradient noise, since it has no tunable parameters.

The Heavy Ball method (3), when tuned as in Table 1, also matches Nesterov’s lower bound when applied to the function class \( Q_{m,L} \) [51, §3.2.1], but has a simpler implementation than CG: its updates are linear and its parameters are constant.

We adopted the three-parameter class \((\alpha, \beta, \eta)\) described in Section 2.2 as our search space for optimized algorithms because the Heavy Ball method is a special case of this family and Heavy Ball can achieve optimal performance on \( Q_{m,L} \) when there is no noise. Substituting the three-parameter algorithm (11) into Proposition 3.1, we obtain the following result.

\textsuperscript{7}Neither \( \rho(A + qBC) \) nor \( P_q \) are convex functions of \( q \) in general.
Corollary 3.2 (Q_{m,L} analysis, reduced). Consider the three-parameter algorithm \( A = (\alpha, \beta, \eta) \) defined in (5) applied to a strongly convex quadratic \( f \in Q_{m,L} \) defined in Section 1, and assume the noise sequence satisfies Assumption 3. The algorithm has convergence rate

\[
\text{Rate}(A, Q_{m,L}) = \max_{q \in \{m,L\}} \left\{ \frac{\sqrt{\beta - \alpha \eta q}}{2} \left( |\beta + 1 - \alpha q - \alpha \eta q| + \Delta \right) \right\} \quad \text{if } \Delta < 0 \tag{17}
\]

where \( \Delta := (\beta + 1 - \alpha q - \alpha \eta q)^2 - 4(\beta - \alpha \eta q) \).

If \( \text{Rate}(A, Q_{m,L}) < 1 \), the algorithm has sensitivity

\[
\text{Sens}(A, Q_{m,L}, \sigma^2) = \sigma \sqrt{d} \cdot \max_{q \in \{m,L\}} \sqrt{h(q)}, \tag{18}
\]

where \( h(q) := \frac{\alpha (1+\beta+(1+2\eta)\alpha q)}{q(1-\beta+\alpha \eta q)(2+2\beta-(1+2\eta)\alpha q)}. \)

Proof. We first prove the expression for the rate in (17). Substituting the algorithm form (11) into Proposition 3.1, we obtain

\[
\text{Rate}(A, Q_{m,L}) = \sup_{q \in \{m,L\}} \rho \left( \begin{bmatrix} \beta + 1 - \alpha (\eta + 1)q & -\beta + \alpha \eta q \\ 1 & 0 \end{bmatrix} \right). \tag{19}
\]

We will prove that the function inside the supremum, \( \phi(q) \), is quasiconvex [12, §3.4]. The characteristic polynomial associated with the matrix in (19) is \( \chi(z) = z^2 + (\alpha(\eta + 1)q - \beta - 1)z + (\beta - \alpha \eta q) \).

Given a polynomial with real coefficients, a necessary and sufficient condition for its roots to lie inside the unit circle is given by the Jury test [25, §4.5]. For the quadratic polynomial \( z^2 + a_1z + a_0 \), the Jury test amounts to the inequalities \( 1 + a_1 + a_0 > 0 \), \( 1 - a_1 + a_0 > 0 \), and \(-1 < a_0 < 1\).

Applying the Jury test to \( \chi(rz) \), we find that \( \phi(q) < r \) if and only if

\[
\begin{align*}
(1-r)(\beta - r) + \alpha(\eta r - \eta + r)q & > 0, \quad (20a) \\
(1+r)(\beta + r) - \alpha(\eta r + \eta + r)q & > 0, \quad (20b) \\
r^2 + \beta - \alpha \eta q & > 0, \quad (20c) \\
r^2 - \beta + \alpha \eta q & > 0. \quad (20d)
\end{align*}
\]

The inequalities (20) are linear in \( q \), so the sublevel sets \( \{ q \mid \phi(q) < r \} \) are open intervals, which are convex. Therefore, \( \phi \) is quasiconvex and attains its supremum over \( q \in [m, L] \) at one of the endpoints, \( q = m \) or \( q = L \). The explicit formula for \( \phi(q) \) can be found by applying the quadratic formula to find the roots of \( \chi(z) \). In (17), \( \Delta \) is the discriminant of \( \chi(z) \) and the two cases correspond to whether the roots are real or complex.

We next prove the expression for the sensitivity in (18). Substituting the algorithm form (11) into Proposition 3.1, we can explicitly solve the linear equation for \( P_q \) in (18) to obtain \( h(q) = B^TP_qB \), where \( h(q) \) is defined in (18). When \( r = 1 \), the Jury conditions (20) reduce to:

\[
\begin{align*}
\alpha q & > 0, \tag{21a} \\
2\beta + 2 - \alpha(2\eta + 1)q & > 0, \tag{21b} \\
1 + \beta - \alpha \eta q & > 0, \tag{21c} \\
1 - \beta + \alpha \eta q & > 0. \tag{21d}
\end{align*}
\]
We will prove that when the rate is strictly less than one, \( h(q) \) is a positive and convex function of \( q \). Evaluating \( h(q) \) and \( h''(q) \) and performing algebraic manipulations, we obtain:

\[
\begin{align*}
  h(q) &= \frac{a^2(2\eta + 1)^2}{2(1 - \beta + a\eta)(2\beta + a\eta^2 + 2(2\eta + 1)q)} + \frac{a^2}{2a\eta - \beta + 1}, \tag{22a} \\
  h''(q) &= \frac{\alpha^4(2\eta + 1)^2}{4(1 - \beta + a\eta)^3(2\beta + 2 - \alpha(2\eta + 1)q)^3 + \alpha\eta(3(2\eta + 1) + (1 - \beta)^2 + (1 - \beta)^2)}.
\end{align*}
\]

In the form (22), it is clear that whenever the rate is strictly less than one (i.e., when (21) holds) we have \( h(q) > 0 \) and \( h''(q) > 0 \). Therefore, the quantity under the square root in (18) is always positive and \( h(q) \) is convex, so it attains its supremum over \( q \in [m, L] \) at one of the endpoints, \( q = m \) or \( q = L \).

In Corollary 3.2, the suprema from Proposition 3.1 are replaced by a simple maximum; we only need to check the endpoints of the interval \([m, L]\) because both the rate and sensitivity are quasiconvex functions of \( q \) when \( n \leq 2 \).

For the function class \( Q_{m,L} \), the parameters \( m \) and \( L \) are lower and upper bounds on the eigenvalues of the Hessian, but these bounds need not be tight. If we instead consider the function class \( \hat{Q}_{m,L} \) for which \( m \) and \( L \) are tight lower and upper bounds on the eigenvalues of the Hessian, the sensitivity is

\[
\text{SENS}(A, \hat{Q}_{m,L}, \sigma^2) = \sigma \max\{\sqrt{(d - 1)h(m) + h(L)}, \sqrt{h(m) + (d - 1)h(L)}\},
\]

which is smaller than that for \( Q_{m,L} \), especially when the dimension \( d \) is small.

Our proposed algorithm for the class \( Q_{m,L} \) is a special tuning of Heavy Ball, which we named Robust Heavy Ball (RHB). The RHB algorithm was found by careful analysis of the analytic expressions for the rate and sensitivity in Corollary 3.2. The algorithm is described in the following theorem.

**Theorem 3.3** (Robust Heavy Ball, RHB). Consider the function class \( Q_{m,L} \) and let \( r \) be a parameter chosen with \( \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}} \leq r < 1 \). Then under Assumption 3, the algorithm \( A \) of the form (5) with tuning \( \alpha = \frac{1}{m}(1 - r)^2, \beta = r^2, \) and \( \eta = 0 \) achieves the following performance metrics\(^8\)

\[
\text{RATE}(A, Q_{m,L}) = r \quad \text{and} \quad \text{SENS}(A, Q_{m,L}, \sigma^2) = \frac{\sigma \sqrt{d}}{m} \sqrt{\frac{1 - r^4}{(1 + r)^4}}.
\]

**Proof.** We first show that the parameter \( r \) is in fact the converge rate of the algorithm. Applying Corollary 3.2, we substitute the algorithm parameters \( \alpha = \frac{1}{m}(1 - r)^2, \beta = r^2, \) and \( \eta = 0 \) into (17) to obtain

\[
\Delta = -(1 - r)^4 \left( \frac{q}{m} - 1 \right) \left( \left( \frac{1 + r}{1 - r} \right)^2 - \frac{q}{m} \right).
\]

\(^8\)Over the function class \( Q_{m,L} \) of quadratics for which \( m \) and \( L \) are tight bounds on the eigenvalues of the Hessian, the sensitivity of the Robust Heavy Ball method is \( \sigma \sqrt{\frac{d+1}{m^2} + \frac{1 - r^2}{(1 + r)^2}} \).
Rearranging the inequalities \( m \leq q \leq L \) and \( \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}} \leq r < 1 \) yields \( 1 \leq \frac{q}{m} \leq \frac{L}{m} \leq \left( \frac{1 + r}{1 - r} \right)^2 < \infty \). Thus, we conclude that \( \Delta \leq 0 \) and we have \( \text{rate}(\mathcal{A}, Q_{m,L}) = \sqrt{\beta} = r \), as required.

We next use Corollary 3.2 to obtain the sensitivity of the algorithm. Substituting the algorithm parameters into (18), we observe that

\[
h(m) - h(L) = \frac{(1 - r) (r^2 + 1) \left( \frac{L}{m} - 1 \right) \left( \left( \frac{1 + r}{1 - r} \right)^2 - \frac{L}{m} \right)}{Lm(r + 1)^3 \left( 1 + \left( \frac{1 + r}{1 - r} \right)^2 - \frac{L}{m} \right)} \geq 0,
\]

so the quantity \( h(q) \) is maximized when \( q = m \). If follows that the sensitivity is

\[
\text{SENS}(\mathcal{A}, Q_{m,L}, \sigma^2) = \sigma \sqrt{d} \sqrt{h(m)} = \frac{\sigma \sqrt{d} \sqrt{1 - r^4}}{m \sqrt{(1 + r)^4}}.
\]

Although we set out to design an algorithm in the three-parameter class \((\alpha, \beta, \eta)\), our designed algorithm RHB uses \( \eta = 0 \) (Heavy Ball). Our numerical experiments in Section 7.1 suggest that this parameter choice yields the most effective trade-off between rate and sensitivity. In other words, it is unnecessary to use a nonzero \( \eta \) when optimizing over the class \( Q_{m,L} \).

If we choose the smallest possible \( r = \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}} \) (fastest possible convergence rate), then we recover Polyak’s tuning of HB, whose convergence rate matches Nesterov’s lower bound. It is straightforward to check that the sensitivity is a monotonically decreasing function of \( r \), so as the convergence rate slows down \((r \ increases)\), the algorithm becomes less sensitivity to noise. Thus, \( r \) is a single tunable parameter that enables us to trade off convergence rate with sensitivity to noise.

## 4 Lyapunov analysis

In contrast to the \( Q_{m,L} \) case, the function classes \( F_{m,L} \) and \( PL_{m,L} \) are not readily parameterizable. We therefore adopt a Lyapunov approach to certify performance bounds on the rate of convergence and sensitivity to noise for a given algorithm. We now describe the main ideas of our approach, which we then apply to the function classes \( F_{m,L} \) and \( PL_{m,L} \) in the following sections.

### 4.1 Lifted dynamics

To obtain performance bounds, we use a lifting approach that searches for certificates of performance that depend on a finite history of past algorithm iterates and function values. The main idea is to lift the state to a higher dimension so that we can search over a broader class of certificates to reduce the conservativeness of the bound. We denote the lifting dimension by \( \ell \geq 0 \), which dictates the dimension of the lifted state, and we use bold to indicate quantities related to the lifted dynamics.

Given a trajectory of the system, we define the following augmented vectors, each consisting of
\( \ell + 1 \) consecutive iterates of the system:

\[
y^t := \begin{bmatrix} y^t \\ \vdots \\ y^{t-\ell} \end{bmatrix}, \quad u^t := \begin{bmatrix} u^t \\ \vdots \\ u^{t-\ell} \end{bmatrix}, \quad \text{and} \quad f^t := \begin{bmatrix} f^t \\ \vdots \\ f^{t-\ell} \end{bmatrix},
\]

where \( y^t, u^t \in \mathbb{R}^{(\ell+1) \times d} \) and \( f^t \in \mathbb{R}^{\ell + 1} \) with \( f^t := f(y^t) \) (recall our convention that algorithm inputs and outputs \( u^t, y^t \) are row vectors). Also, define the truncation matrices \( Z, Z_+ \in \mathbb{R}^{\ell \times (\ell + 1)} \) as

\[
Z_+ := [I_\ell \quad 0_{\ell \times 1}] \quad \text{and} \quad Z := [0_{\ell \times 1} \quad I_\ell].
\]

Multiplying an augmented vector on the left by \( Z \) removes the most recent iterate at time \( t \), while multiplication by \( Z_+ \) removes the last iterate at time \( t - \ell \). Using these augmented vectors, we then define the augmented state as

\[
x^t := \begin{bmatrix} \xi^t \\ Zy^t \\ Zu^t \end{bmatrix} \in \mathbb{R}^{(n+2\ell) \times d} \tag{25}
\]

which consists of the current state \( \xi^t \) as well as the \( \ell \) previous inputs \( y^{t-1}, \ldots, y^{t-\ell} \) and outputs \( u^{t-1}, \ldots, u^{t-\ell} \) of the original system. Since the dynamics of this augmented state must be consistent with those of the original system, the associated augmented dynamics for the state \( x^t \) with inputs \( (u^t, w^t) \), which is the same as in (7), and augmented outputs \( (y^t, u^t) \) are

\[
x^{t+1} = \begin{bmatrix} A \\ Z_+ e_1 C \quad Z_+ Z^T \\ 0 \quad 0 \quad Z_+ Z^T \end{bmatrix} x^t + \begin{bmatrix} B \\ 0 \quad 0 \quad Z_+ e_1 \end{bmatrix} u^t + \begin{bmatrix} B \\ 0 \end{bmatrix} w^t,
\]

\[
\begin{bmatrix} y^t \\ u^t \end{bmatrix} = \begin{bmatrix} e_1 C \quad Z^T \\ 0 \quad 0 \quad Z^T \\ C \quad D \end{bmatrix} x^t + \begin{bmatrix} 0 \\ e_1 \end{bmatrix} u^t,
\]

where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{\ell+1} \). We can recover the iterates of the original system by projecting the augmented state and the input as

\[
\xi^t = \begin{bmatrix} I_n \quad 0_{n \times (2\ell+1)} \end{bmatrix} x^t, \quad y^t = \begin{bmatrix} C \\ 0_{1 \times (2\ell+1)} \end{bmatrix} x^t, \quad \text{and} \quad u^t = \begin{bmatrix} 0_{1 \times (n+2\ell)} \quad 1 \end{bmatrix} u^t. \tag{27}
\]

**State reduction for the noise-free case.** When there is no noise \( (w^t = 0) \), the augmented state (25) has linearly dependent rows. This leads to the definition of a reduced state \( x^t \in \mathbb{R}^{(n+\ell) \times d} \)

\[
x^t = \begin{bmatrix} \xi^t \\ Zy^t \\ Zu^t \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A^t B & AB & \cdots & A^{t-1} B \\ CA^{t-1} & 0 & CB & \cdots & CA^{t-2} B \\ \vdots & 0 & \ddots & \vdots & \vdots \\ CA & 0 & \cdots & CB \\ C \end{bmatrix} \\ 0_{\ell \times 1} \quad I_\ell \end{bmatrix} \begin{bmatrix} \xi^{t-\ell} \\ Zu^t \end{bmatrix} = : \Psi x^t. \tag{28}
\]
The associated augmented (noise-free) dynamics for this reduced state are

\[
x_{r+1} = \begin{bmatrix} A & B e^{T} + Z T \\ 0 & Z T \end{bmatrix} x_{r} + \begin{bmatrix} 0 \\ Z T \end{bmatrix} e_{1} u, \tag{29a}
\]

\[
\begin{bmatrix} y' \\ u' \end{bmatrix} = \begin{bmatrix} e_{1} C \Psi_{11} + Z T \Psi_{21} \\ 0 \\ e_{1} C \Psi_{12} + Z T \Psi_{22} \end{bmatrix} x_{r} + \begin{bmatrix} 0 \\ e_{1} \end{bmatrix} u, \tag{29b}
\]

where \( e_{\ell+1} = (0, 0, \ldots, 1) \in \mathbb{R}^{\ell+1} \) and \( \Psi_{ij} \) denotes the \((i, j)\) block of the \(3 \times 2\) block-matrix \( \Psi \) defined in (28). We can recover the iterates of the original system as before, using

\[
\xi' = \begin{bmatrix} \Psi_{11} & \Psi_{12} & 0_{n \times 1} \\ \Psi_{12} & \Psi_{12} & 0 \\ X_{r} \end{bmatrix} x', \quad y' = \begin{bmatrix} C \Psi_{11} & C \Psi_{12} \\ 0 \\ Y_{r} \end{bmatrix} x', \quad \text{and} \quad u' = \begin{bmatrix} 0_{1 \times (n+\ell)} & 1 \end{bmatrix} u. \tag{30}
\]

Our analysis is based on searching for a function that certifies either a particular convergence rate (assuming no noise) or a level of sensitivity (assuming noise). The function depends on the lifted state as well as the augmented vector of function values, that is, we search for certificates of the form

\[
V(x, f) := \text{tr}(x^{T} P x) + p^{T} Z f, \tag{31}
\]

where \( P \) and \( p \) are parameters that we search over, and the matrix \( Z \) is defined in (24). This is quadratic in the lifted state and linear in the \(\ell\) previous function values. Here, the lifted state is either \( x' \) defined in (25) or the reduced state \( x_{r} \) defined in (28), depending on whether there is noise or not, respectively.

### 4.2 Bounds on the rate of convergence

To bound the rate of convergence of an algorithm, we search for a function of the form (31) that decreases along trajectories and is lower bounded by the squared norm of the iterates.

**Lemma 4.1** (Lyapunov analysis for rate of convergence). Consider an algorithm \( A = (A, B, C) \) defined in (7) satisfying Assumption 1 applied to a function \( f \in \mathcal{F} \) with no gradient noise (\( u' = 0 \) for all \( t \)). If there exists a function \( V \) of the form (31) and a constant \( r > 0 \) such that, for all functions \( f \in \mathcal{F} \) and all iterations \( t \geq \ell \), the iterates \( x_{r} \) of the lifted system (29) and function values \( f' \) in (23) with lifting dimension \( \ell \) satisfy the conditions

(i) Lower bound condition: \( V(x_{r}, f') \geq \|x_{r} - \xi\|^{2} \)

(ii) Decrease condition: \( V(x_{r+1}^{*}, f^{(t+1)}) \leq r^{2} V(x_{r}^{*}, f') \),

then \( \text{Rate}(A, \mathcal{F}) \leq r \).

**Proof.** By applying the lower bound condition followed by the decrease condition at each iteration \( t \geq \ell \), we obtain the chain of inequalities

\[
\|x_{r} - \xi\|^{2} \leq V(x_{r}, f') \leq r^{2} V(x_{r-1}, f^{(t-1)}) \leq \ldots \leq r^{2(t-\ell)} V(x_{r}, f').
\]
From Eq. (9), we obtain rate\( (\mathcal{A}, \mathcal{F}) \leq r \), as required.

Lemma 4.1 provides a way to bound the rate of convergence by searching for a function that satisfies the lower bound and decrease conditions. In the following sections, we show how the search for such a function can be cast as a semidefinite program.

### 4.3 Bounds on the sensitivity to noise

To bound the sensitivity of an algorithm to noise, we search for a function of the form (31) that satisfies the following lower bound and decrease conditions.

**Lemma 4.2** (Lyapunov analysis for sensitivity to noise). Consider an algorithm \( \mathcal{A} = (A, B, C) \) defined in (7) satisfying Assumption 1 applied to a function \( f \in \mathcal{F} \) with additive gradient noise satisfying Assumption 3 with variance \( \sigma^2 \). If there exists a function \( V \) of the form (31) and a constant \( \gamma > 0 \) such that, for all functions \( f \in \mathcal{F} \) and all iterations \( t \geq \ell \), there exists an optimizer \( y^* \in \text{arg min}_y f(y) \) such that the iterates \( x^t \) of the lifted system (26) and function values \( f^t \) in (23) with lifting dimension \( \ell \) satisfy the conditions

(i) Lower bound condition: \( \mathbb{E} V(x^t, f^t) \geq 0 \) and

(ii) Decrease condition: \( \mathbb{E} V(x^{t+1}, f^{t+1}) - \mathbb{E} V(x^t, f^t) + \mathbb{E} \| y^t - y^* \|^2 \leq \gamma^2 \),

then \( \text{sens}(\mathcal{A}, \mathcal{F}, \sigma^2) \leq \gamma \).

**Proof.** Averaging the decrease condition over \( t = \ell, \ldots, \ell + T - 1 \), the sum telescopes and we obtain

\[
\frac{1}{T} \mathbb{E} V(x^{\ell+T}, f^{\ell+T}) - \frac{1}{T} \mathbb{E} V(x^\ell, f^\ell) + \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \| y^{t+\ell} - y^* \|^2 \leq \gamma^2.
\]

Applying the lower bound condition to the first term and taking the limit superior as \( T \to \infty \), the second term vanishes and we obtain

\[
\limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \| y^t - y^* \|^2 = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \| y^{t+\ell} - y^* \|^2 \leq \gamma^2,
\]

which implies from (10) that \( \text{sens}(\mathcal{A}, \mathcal{F}, \sigma^2) \leq \gamma \). 

### 5 Smooth strongly convex functions

We now consider the class \( \mathcal{F}_{m,L} \) of strongly convex functions whose gradient is Lipschitz continuous. We first provide a means of searching for a function \( V \) of the form (31) that satisfies Lemma 4.1 to certify the convergence rate for a given algorithm by solving a linear matrix inequality, and we then use this LMI to design an algorithm that trades-off rate and sensitivity for this function class.

#### 5.1 Performance bounds for \( \mathcal{F}_{m,L} \)

A useful characterization of the function class \( \mathcal{F}_{m,L} \) is given by the *interpolation conditions*. These are necessary and sufficient conditions on a sequence of points to be interpolable by a function in
the class. In our analysis, we will make use of these conditions in our search for a suitable function $V$. We state the result from [57, Thm. 4], rephrased to match our notation.

**Proposition 5.1** (Interpolation conditions for $F_{m,L}$). Let $y^1, \ldots, y^k \in \mathbb{R}^{1 \times d}$ and $u^1, \ldots, u^k \in \mathbb{R}^{1 \times d}$ and $f^1, \ldots, f^k \in \mathbb{R}$. The following two statements are equivalent.

1) There exists a function $f \in F_{m,L}$ such that $f(y^i) = f^i$ and $\nabla f(y^i) = u^i$ for $i = 1, \ldots, k$.

2) For all $i, j \in \{1, \ldots, k\}$,

$$
\text{tr}((u^i)^T(y^i - y^j)) - (f^i - f^j) + \frac{1}{2(L - m)} \text{tr}\left[\begin{bmatrix} y^i - y^j \\ u^i - u^j \end{bmatrix}^T \begin{bmatrix} -mL & m \\ m & -1 \end{bmatrix} \begin{bmatrix} y^i - y^j \\ u^i - u^j \end{bmatrix}\right] \geq 0. \tag{32}
$$

We now develop a version of Proposition 5.1 that holds for the augmented vectors defined in (23).

**Lemma 5.2.** Consider a function $f \in F_{m,L}$, and let $y^* \in \mathbb{R}^{1 \times d}$ denote the optimizer, $u^* = 0 \in \mathbb{R}^{1 \times d}$ the optimal gradient, and $f^* \in \mathbb{R}$ the optimal value. Let $y^1, \ldots, y^{l-f} \in \mathbb{R}^{1 \times d}$ be a sequence of iterates, and define $u^{l-i} := \nabla f(y^{l-i})$ and $f^{l-i} := f(y^{l-i})$ for $i = 0, \ldots, l$. Using these values, define the augmented vectors $y^{i*}, u^{i*}, f^{i*}$ as in (23). Finally, define the index set $I := \{1, \ldots, l + 1, *\}$ and let $e_i$ denote the $i^{th}$ unit vector in $\mathbb{R}^{l+1}$ with $e_* := 0 \in \mathbb{R}^{l+1}$. Then the inequality

$$
\text{tr}\left[\begin{bmatrix} y^{i*} \\ u^{i*} \end{bmatrix}^T \Pi(\Lambda) \begin{bmatrix} y^{i*} \\ u^{i*} \end{bmatrix} + \pi(\Lambda)^T f^{i*}\right] \geq 0 \tag{33}
$$

holds for all $\Lambda \in \mathbb{R}^{(l+2) \times (l+2)}$ such that $\Lambda \geq 0$ (elementwise), where

$$
\Pi(\Lambda) := \sum_{i,j \in I} \Lambda_{ij} \begin{bmatrix} -mL(e_i - e_j)(e_i - e_j)^T & (e_i - e_j)(me_i - Le_j)^T \\ (me_i - Le_j)(e_i - e_j)^T & -(e_i - e_j)(e_i - e_j)^T \end{bmatrix}, \tag{34a}
$$

$$
\pi(\Lambda) := 2(L - m) \sum_{i,j \in I} \Lambda_{ij} (e_i - e_j). \tag{34b}
$$

**Proof.** For each $i \in I$, define the vectors $\tilde{y}_i := e_i^T y^{i*}$ and $\tilde{u}_i := e_i^T u^{i*}$ and $\tilde{f}_i := e_i^T f^{i*}$. By definition, the points $(\tilde{y}_i, \tilde{u}_i, \tilde{f}_i)$ are interpolated by the function $f \in F_{m,L}$, so by Proposition 5.1 the interpolation conditions (32) are satisfied. The proof is then completed by noting that the inequality (33) can be expanded as

$$
\sum_{i,j \in I} \Lambda_{ij} \left(-mL\|\tilde{y}_i - \tilde{y}_j\|^2 + 2(\tilde{y}_i - \tilde{y}_j)(m\tilde{u}_i - L\tilde{u}_j)^T - \|	ilde{u}_i - \tilde{u}_j\|^2 + 2(L - m)(\tilde{f}_i - \tilde{f}_j)\right) \geq 0,
$$

which is a nonegative weighted combination of the interpolation conditions, where the interpolation condition between index $i$ and $j$ is scaled by the nonnegative quantity $2(L - m)\Lambda_{ij} \geq 0$.

Since $V$ is quadratic in the augmented state and linear in the augmented function values, we can efficiently search for such Lyapunov functions using the following linear matrix inequalities that make use of the characterization of smooth strongly convex functions in Lemma 5.2.
Theorem 5.3 (F_{m,L} analysis). Consider an algorithm \( A = (A, B, C) \) defined in (7) satisfying Assumption 1 applied to a function \( f \in F_{m,L} \) defined in Section 1 with additive gradient noise satisfying Assumption 3. Define the truncation matrices in (24), the augmented state space and projection matrices in (26)–(30), and the multiplier functions in (34). Then the algorithm satisfies the following convergence rate and sensitivity bounds.

1) If there exist \( r > 0 \), \( P = P^T \in \mathbb{R}^{(n+\ell) \times (n+\ell)} \), \( p \in \mathbb{R}^\ell \), and \( \Lambda_1, \Lambda_2 \geq 0 \) such that

\[
\begin{bmatrix}
A_r & B_r \\
I & 0
\end{bmatrix}^T \begin{bmatrix}
P & 0 \\
0 & -r^2P
\end{bmatrix} \begin{bmatrix}
A_r & B_r \\
I & 0
\end{bmatrix} + \begin{bmatrix}
C_r & D_r
\end{bmatrix}^T \Pi(\Lambda_1) \begin{bmatrix}
C_r & D_r
\end{bmatrix} \leq 0 \tag{35a}
\]

\[
(Z_+ - r^2Z)^TP + \pi(\Lambda_1) \leq 0 \tag{35b}
\]

\[
X_r^TX_r - \begin{bmatrix}
I & 0
\end{bmatrix}P \begin{bmatrix}
I & 0
\end{bmatrix} + \begin{bmatrix}
C_r & D_r
\end{bmatrix}^T \Pi(\Lambda_2) \begin{bmatrix}
C_r & D_r
\end{bmatrix} \leq 0 \tag{35c}
\]

\[
-Z^TP + \pi(\Lambda_2) \leq 0 \tag{35d}
\]

then \( \text{rate}(A, F_{m,L}) \leq r \).

2) If there exist \( P = P^T \in \mathbb{R}^{(n+2\ell) \times (n+2\ell)} \), \( p \in \mathbb{R}^\ell \), and \( \Lambda_1, \Lambda_2 \geq 0 \) such that

\[
\begin{bmatrix}
A & B \\
I & 0
\end{bmatrix}^T \begin{bmatrix}
P & 0 \\
0 & -P
\end{bmatrix} \begin{bmatrix}
A & B \\
I & 0
\end{bmatrix} + \begin{bmatrix}
C & D
\end{bmatrix}^T \Pi(\Lambda_1) \begin{bmatrix}
C & D
\end{bmatrix} + Y^TY \leq 0 \tag{36a}
\]

\[
(Z_+ - Z)^TP + \pi(\Lambda_1) \leq 0 \tag{36b}
\]

\[
-X_r^TX_r - \begin{bmatrix}
I & 0
\end{bmatrix}P \begin{bmatrix}
I & 0
\end{bmatrix} + \begin{bmatrix}
C & D
\end{bmatrix}^T \Pi(\Lambda_2) \begin{bmatrix}
C & D
\end{bmatrix} \leq 0 \tag{36c}
\]

\[
-Z^TP + \pi(\Lambda_2) \leq 0 \tag{36d}
\]

then \( \text{sens}(A, F_{m,L}, \sigma^2) \leq \sqrt{\sigma^2d \cdot (H^TPH)} \).

Proof. Consider a trajectory \((\xi^t, u^t, y^t, w^t)\) of the dynamics (7) with \( w^t = 0 \). From Lemma 2.1, we can assume without loss of generality that the fixed point is zero. Form the augmented vectors and state as described in Section 5.1. Multiply the LMIs (35a) and (35c) on the right and left by \( \text{col}(x_r^t, u^t) \in \mathbb{R}^{n+\ell+1} \) and its transpose, respectively, and take the trace. Also, take the inner product of (35b) and (35d) with \( f^t \), which is valid because \( f^t \) is elementwise nonnegative. The resulting inequalities are:

\[
\text{tr}(x_r^{t+1})^TPx_r^{t+1} - r^2\text{tr}(x_r^t)^TPx_r^t + \text{tr}\left(y_r^t\right)^T\Pi(\Lambda_1)\left(y_r^t\right) \leq 0, \tag{37a}
\]

\[
p^T(Z_+ - r^2Z)f^t + \pi(\Lambda_1)^Tf^t \leq 0, \tag{37b}
\]

\[
\|\xi^t\|^2 - \text{tr}(x_r^t)^TPx_r^t + \text{tr}\left(y_r^t\right)^T\Pi(\Lambda_1)\left(y_r^t\right) \leq 0, \tag{37c}
\]

\[
-p^TZf^t + \pi(\Lambda_2)^Tf^t \leq 0. \tag{37d}
\]

Summing (37a)+(37b) and (37c)+(37d) and applying (33), we recover the lower bound and decrease properties in Lemma 4.1 and the rate bound result follows.

For the second part of the proof, we do not restrict \( w^t = 0 \), and perform similar operations to the inequalities (36) as in the first part, except that we multiply the linear matrix inequalities by
col(\(x^t, u^t\)) instead to obtain the inequalities

\[
\begin{align*}
\text{tr} (x^{t+1} - H w^t)^{T} P (x^{t+1} - H w^t) - \text{tr} (x^t)^{T} P x^t + \text{tr} \begin{bmatrix} y^t \end{bmatrix}^{T} \Pi(A_1) \begin{bmatrix} y^t \end{bmatrix} + \|y^t\|^2 & \leq 0, \quad (38a) \\
p^T (Z_+ - Z) f^t + \pi(A_1)^{T} f^t & \leq 0, \quad (38b) \\
- \text{tr} (x^t)^{T} P x^t + \text{tr} \begin{bmatrix} y^t \end{bmatrix}^{T} \Pi(A_2) \begin{bmatrix} y^t \end{bmatrix} & \leq 0, \quad (38c) \\
- p^T Z f^t + \pi(A_2)^{T} f^t & \leq 0. \quad (38d)
\end{align*}
\]

Summing (38a)+(38b) and (38c)+(38d) and applying (33), we obtain the inequalities

\[
\begin{align*}
V(x^{t+1}, f^{t+1}) - V(x^t, f^t) + \|y^t\|^2 - 2 \text{tr} (A x^t + B u^t)^{T} PH w^t - \text{tr}(w^t)^{T} H^{T} PH w^t & \leq 0, \quad (39a) \\
- V(x^t, f^t) & \leq 0. \quad (39b)
\end{align*}
\]

Taking the expectation of both inequalities, the term \(-2(A x^t + B u^t)^{T} PH w^t\) in the first inequality vanishes because \(w^t\) is zero-mean and is independent of \(x^t\) and \(u^t\), which only depend on \(w^{t-1}, w^{t-2}, \ldots\). Also, since \(w^t\) has covariance bound \(\sigma^2 I\), we have \(\mathbb{E}((w^t)^{T} H^{T} PH w^t) \leq \sigma^2 d(H^{T} PH)\). Thus, the previous inequalities imply the decrease and lower bound conditions in Lemma 4.2 with \(\gamma^2 = \sigma^2 d(H^{T} PH)\), which implies the bound on the sensitivity.

**Remark 4.** When the lifting dimension \(\ell\) is zero, the lifted system is identical to the original system. In other words, \(\xi^t = x^t = \bar{x}^t\). The system matrices also satisfy \(A = A = A_r\), and similarly for \(B\). As \(\ell\) is increased, the LMIs (35) and (36) have the potential to yield less conservative bounds on the rate and sensitivity, respectively.

**Remark 5.** In Theorem 5.3, the LMI (35) has fewer inequalities and variables than the LMI (36) because it makes use of the reduced state \(\bar{x}^t\) instead of the full state \(x^t\). The reduced state can only be used when computing the rate because this case has no noise.

Both bounds for the class \(F_{m,L}\) in Theorem 5.3 can be evaluated and optimized efficiently. The sizes of the LMIs depend only on \(n\) and \(\ell\), which are typically small.

### 5.2 Choosing the lifting dimension

Solving the LMIs in Theorem 5.3 first requires choosing a lifting dimension \(\ell\). Increasing \(\ell\) yields potentially tighter bounds on the rate and sensitivity, but at the expense of making the LMIs larger and more time-consuming to solve.

To decide which lifting dimension to use, we performed pilot tests to observe how the rate and sensitivity bounds improved as \(\ell\) was increased. In Table 2, we show some representative results, evaluating the performance of Nesterov’s Fast Gradient method with or without balancing\(^9\). By definition, the computed bounds should decrease (or stay the same) as we increase \(\ell\), so any observed increases must be due to numerical solver error. We also found empirically that the sensitivity LMI (36) becomes poorly conditioned when either the lifting dimension \(\ell\) or condition ratio \(L/m\) is large.

---

\(^9\)As a heuristic to improve the conditioning of the LMI, we applied a balancing technique described in Appendix A.
Table 2: Numerical values for Nesterov’s Fast Gradient method with standard tuning (see Table 1), with additional parameters $d = 1$, $\sigma = 1$, $m = 1$, and $L = 100$. Results obtained by computationally solving the LMIs from Theorem 5.3 using various values of the lifting dimension $\ell$. Numerical results are more reliable when balancing is used. Last column indicates the wall clock time for computing sensitivity with balancing (one LMI solve).

| $\ell$ | Rate       | Balanced rate | Sensitivity | Balanced sens. | Time (sec.) |
|--------|------------|---------------|-------------|----------------|-------------|
| 1      | 0.9279379028 | 0.9279330969  | 0.2007657395 | 0.2007653112  | 0.0053      |
| 2      | 0.9279357392 | 0.9279330772  | error       | 0.1859082519  | 0.0094      |
| 3      | 0.9279401124 | 0.9279330707  | error       | 0.1837282849  | 0.0169      |
| 4      | 0.9279420203 | 0.9279330510  | error       | 0.1835113705  | 0.0323      |
| 5      | 0.9279418892 | 0.9279330248  | error       | 0.1834909008  | 0.0796      |
| 6      | 0.9279397845 | 0.9279329461  | error       | 0.1834856744  | 0.1042      |
| 7      | 0.9279395616 | 0.9279328346  | 0.1834918867 | 0.1834813977  | 0.2000      |
| 8      | 0.9279494816 | 0.9279329133  | 0.1834920874 | 0.1834817113  | 0.3434      |
| 9      | 0.9279330379 | 0.9279327232  | 0.1834930651 | 0.1834868332  | 0.5947      |
| 10     | 0.9279350966 | 0.9279327756  | error       | 0.1834840969  | 1.0601      |

In Table 2, we observe that the rate reaches its minimal value at $\ell = 1$, beyond which there is no improvement. Meanwhile, the sensitivity continues to improve as $\ell$ increases.

5.3 Algorithm design for $F_{m,L}$

For the case with no noise, the Triple Momentum (TM) method [60] attains the fastest-known worst-case convergence rate of $1 - \frac{\sqrt{m}}{L}$ over the function class $F_{m,L}$. Recent work by Drori and Taylor [23,55] has confirmed that this rate is in fact optimal.

We adopted the three-parameter class $(\alpha, \beta, \eta)$ described in Section 2.2 as our search space for optimized algorithms, because it includes TM as a special case, as well Nesterov’s Fast Gradient (FG) method, which is a popular choice for this function class. Our proposed algorithm, which we call the Robust Accelerated Method (RAM), uses a parameter $r$ to trade off convergence rate and sensitivity to noise.

To design RAM, we applied the procedure outlined in Section 1.3 to the rate LMI (35) with lifting dimension $\ell = 1$. While using a larger lifting dimension could potentially lead to an algorithm design that better trades off rate and sensitivity, a larger lifting dimension leads to larger LMIs for which it becomes intractable to find closed-form algorithm parameters due to the complexity of the system of polynomial equations that describes the optimal solution. Our main result concerning RAM is Theorem 5.4, which provides the exact convergence rate of the algorithm. While we do not construct a bound on the sensitivity, we show in Section 7 that it effectively trades off rate and sensitivity. To obtain a numerical bound on the sensitivity for a given parameter $r$, one can use the analysis result in Theorem 5.3.

**Theorem 5.4** (Robust Accelerated Method, RAM). Consider the function class $F_{m,L}$, and let $r$
be a parameter chosen with $1 - \sqrt{\frac{m}{L}} \leq r < 1$. Then, the algorithm $A$ of the form (5) with tuning

$$\alpha = \frac{(1 + r)(1 - r)^2}{m}, \quad \beta = r \frac{L(1 - r + 2r^2) - m(1 + r)}{(L - m)(3 - r)},$$

and

$$\eta = r \frac{L(1 - r^2) - m(1 + 2r - r^2)}{(L - m)(3 - r)(1 - r^2)}$$

achieves the performance metric $\text{rate}(A, F_{m, L}) = r$.

**Proof.** To prove that RAM converges with rate $r$ when there is no noise, we provide a feasible solution to the LMI (35) with lifting dimension $\ell = 1$. In fact, our solution has the same structure as the weighted off-by-one IQC formulation in (40), where the positive definite matrix $Q > 0$ is given by

$$Q = \frac{m}{(3 - r)(1 - r)^2(1 + r)^3(m - L(1 - r)^2)} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix},$$

with

$$q_{11} = r \left(2m^2(1 - r) + 2Lm(4 + r - 2r^2 + r^3) - L^2(1 - r^2)^2\right),$$

$$q_{12} = r \left(-2m^2(1 - r) - 2Lm(1 + r)^2 + L^2(4 - r)(1 - r^2)^2\right),$$

$$q_{13} = (3 - r)(1 - r^2)(-m(1 + r^2) + L(1 + r - 2r^2 - r^3 + r^4)),$$

$$q_{22} = r \left(2m^2(1 - r) - 2Lm(2 - 3r - 4r^2 + r^3) + L^2(2 - 4r + r^2)(1 - r^2)^2\right),$$

$$q_{23} = r \left(3 - r)(1 - r^2)(m(-1 + 2r + r^2) - L(-1 + r - r^3 + r^4)\right),$$

$$q_{33} = r (3 - r)^2(1 - r^2)^2.$$

Using Mathematica [62], it is straightforward to verify that, for all function class parameters $0 < m \leq L$ with $1 - \sqrt{\frac{m}{L}} \leq r < 1$, this is a feasible solution to a modified version of the LMI (35) in which the term $X_r^T X_r$ is replaced by $X_r^T T^T Q T X_r$, where

$$T = \begin{bmatrix} A & B \\ -LC & 1 \end{bmatrix}.$$

Since $T^T Q T$ is strictly positive definite, we have $X_r^T T^T Q T X_r \succeq c X_r^T X_r$, where $c > 0$ is the minimum eigenvalue of $T^T Q T$. Therefore, scaling the entire solution by $1/c$ provides a feasible solution to the original LMI (35). Theorem 5.3 then implies that RAM has convergence rate at least $r$. To show that the convergence rate is exactly $r$, note that the spectral radius of $A + mBC$ is precisely $r$, so the bound is achieved by the function $f(y) = \frac{m}{2} \|y\|^2$. $\blacksquare$

**Remark 6.** When the convergence factor $r$ is set to its minimum value of $1 - \sqrt{\frac{m}{L}}$, the Robust Accelerated Method in Theorem 5.4 reduces to the Triple Momentum Method (see Table 1).

### 5.4 Comparison with IQCs from robust control

We now compare our analysis in Theorem 5.3 for computing the convergence rate and sensitivity for the function class $F_{m, L}$ with the use of integral quadratic constraints (IQCs) [44] from robust control.
In the robust control framework, the gradient of the objective function is interpreted as a nonlinear or uncertain component of the algorithm, and the goal is to certify that properties of the algorithm (convergence rate or sensitivity) are robust to all uncertainties within the function class. One such formulation for certifying an upper bound on the convergence rate consists of the LMI in [41, Eq. 3.8] with the weighted off-by-one IQC in [41, Lemma 10], which we summarize as follows.

Proposition 5.5 (Weighted off-by-one IQC). The convergence rate of an algorithm \( A = (A, B, C) \) over the function class \( F_{m,L} \) is upper bounded by the constant \( r > 0 \) if there exists a positive definite matrix \( Q \succ 0 \) that satisfies the LMI

\[
\begin{bmatrix}
\hat{A}^TQ \hat{A} - r^2Q & \hat{A}^TQ \hat{B} \\
\hat{B}^TQ \hat{A} & \hat{B}^TQ \hat{B}
\end{bmatrix} + \begin{bmatrix}
\hat{C} & \hat{D}
\end{bmatrix}^T \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
\hat{C} & \hat{D}
\end{bmatrix} \leq 0,
\]

where the matrices \((\hat{A}, \hat{B}, \hat{C}, \hat{D})\) are given by

\[
\hat{A} = \begin{bmatrix}
A & 0 \\
-LC & 0
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
B \\
1
\end{bmatrix}, \quad \hat{C} = \begin{bmatrix}
LC & r^2 \\
-mC & 0
\end{bmatrix}, \quad \hat{D} = \begin{bmatrix}
-1 \\
1
\end{bmatrix}.
\]

We now show that feasibility of this LMI implies feasibility of our analysis LMI in (35) with lifting dimension \( \ell = 1 \). If \( Q \succ 0 \) satisfies the LMI for the weighted off-by-one IQC, then the LMI in (35) has the feasible solution

\[
p = 2(L-m), \quad \Lambda_1 = \begin{bmatrix}
0 & 0 & 0 \\
r^2 & 0 & 0 \\
1-r^2 & 0 & 0
\end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad \text{and}
\]

\[
P = \left[ \begin{array}{cc}
A & B \\
-LC & 1
\end{array} \right]^T Q \left[ \begin{array}{cc}
A & B \\
-LC & 1
\end{array} \right] - (L-m) m \begin{bmatrix}
C^T C \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
-mC & 1 \\
0 & -mC & 1
\end{bmatrix}^T [-mC & 1]. \quad (40a)
\]

In this case, a Lyapunov function for the system is

\[
V(\xi^t, u^t, f^t) = Q \begin{bmatrix}
\xi^{t+1} \\
\zeta^{t+1}
\end{bmatrix} + 2(L-m) \left( f^t - \frac{m}{2} \|y^t\|^2 \right) - \|u^t - my^t\|^2 \quad (41)
\]

where \( \xi^{t+1} = A\xi^t + Bu^t \) and \( \zeta^{t+1} := u^t - Ly^t \). Therefore, using the weighted off-by-one IQC can be interpreted as searching over this restricted class of Lyapunov functions. Even though our analysis for computing the convergence rate in Theorem 5.3 is more general, the weighted off-by-one IQC formulation appears to be general enough to prove tight results. For example, while RAM was designed using the more general analysis, its Lyapunov function has the special form (41).

In the recent work [45], Zames–Falb multipliers [64] (of which the weighted off-by-one IQC is a special case) are used to formulate LMIs for computing both the convergence rate and the sensitivity. Just as with the weighted off-by-one IQC, using general Zames–Falb multipliers can also be interpreted as searching over a restricted class of Lyapunov functions, although a detailed comparison is beyond the scope of this work.

While the weighted off-by-one IQC formulation appears to achieve tight bounds on the convergence rate, computing tight bounds on the sensitivity requires the more general LMI (36); see Table 2.
6 PL and Lipschitz functions

The Polyak–Lojasiewicz (PL) condition, first introduced in [50], is a weaker condition than strong convexity, and is typically combined with some sort of smoothness condition. In this paper, we consider the class $\text{PL}_{m,L}$, which are functions that achieve some minimum value $f^\star$ and satisfy the following pair of conditions for all $x, y \in \mathbb{R}^d$.

\begin{align*}
\text{PL condition: } & \quad \frac{1}{2} \|\nabla f(x)\|^2 \geq m(f(x) - f^\star), \quad (42) \\
\text{Lipschitz: } & \quad f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2. \quad (43)
\end{align*}

The PL condition implies that whenever $\nabla f(y) = 0$, we must have $f(y) = f^\star$. Therefore, all local minima are global minima. However, there may be multiple globally optimal points; a trivial example is the function $f(y) = 0$. For this reason, we do not refer to the optimal $y^\star$ but rather an optimal $y^\star$.

Remark 7. The classical definition of Lipschitz gradients is that the inequality (i) $\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|$ holds for all $x, y$. Applying Cauchy–Schwarz, we deduce that (ii) $(x - y)^T (\nabla f(x) - \nabla f(y)) \leq L \|x - y\|^2$, which implies that $\frac{1}{2} \|x\|^2 - f(x)$ is a convex function, and therefore (43) holds. If we further assume that $f$ is itself convex, then the converse holds and the aforementioned “Lipschitz conditions” (i), (ii), (43) are all equivalent. In our study of the PL condition, we do not assume $f$ is convex, so the various Lipschitz conditions are not interchangeable. In this paper, we use the (43), which is the weakest of the Lipschitz conditions and is the one conventionally used in the literature [37].

Many prevalent optimization models that are convex but not strongly convex nevertheless satisfy the PL condition, which can be used to prove useful convergence properties. Examples include: least squares, logistic regression, Lasso regression, and support vector machines [37].

Some prevalent non-convex function also satisfy the PL condition or local variants. Examples include invex functions [37] and certain over-parameterized neural networks; see [43] and references therein.

6.1 Performance evaluation using function values

Since functions $f \in \text{PL}_{m,L}$ need not have unique optimal values, the definitions of RATE and SENS introduced in Section 2.1 cannot be used. Instead, we will redefine these quantities to be in terms of function values $f^t - f^\star$,

\begin{align*}
\text{RATE}(A, \mathcal{F}) & := \inf \left\{ r > 0 \left| \sup_{f \in \mathcal{F}} \sup_{\xi^0 \in \mathbb{R}^n \times d} \sup_{t \geq 0} \frac{(f^t - f^\star)}{r^2 t (f^0 - f^\star)} < \infty \right. \right\}, \quad (44a) \\
\text{SENS}(A, \mathcal{F}, \sigma^2) & := \sup_{f \in \mathcal{F}} \sup_{\xi^0 \in \mathbb{R}^n \times d} \sup_{P \in \mathcal{P}} \limsup_{T \to \infty} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (f^t - f^\star) \right]. \quad (44b)
\end{align*}

We use a square root in (44b) so that this our redefined sensitivity is compatible with the original definition in (10).
6.2 Performance bounds for PL<sub>m,L</sub>

Our analysis for the PL<sub>m,L</sub> class will parallel that of F<sub>m,L</sub> in Section 5. In particular, we will use the same lifted dynamics Eqs. (23) to (30) and the same form of the LMIs Eqs. (35) and (36) as in Theorem 5.3. The only difference is that the Π(Λ) and π(Λ) quantities from Lemma 5.2 must be replaced by their appropriate PL<sub>m,L</sub> counterparts. This leads to the following result.

**Lemma 6.1.** Consider a function \( f \in PL_{m,L} \), and let \( y^* \in \mathbb{R}^{1 \times d} \) denote an optimizer, \( u^* = 0 \in \mathbb{R}^{1 \times d} \) the optimal gradient, and \( f^* \in \mathbb{R} \) the optimal value. Let \( y^t, \ldots, y^{t-\ell} \in \mathbb{R}^{1 \times d} \) be a sequence of iterates, and define \( u^{t-i} := \nabla f(y^{t-i}) \) and \( f^{t-i} := f(y^{t-i}) \) for \( i = 0, \ldots, \ell \). Using these values, define the augmented vectors \( \tilde{y}^t, \tilde{u}^t, \tilde{f}^t \) as in (23). Finally, define the index set \( I := \{1, \ldots, \ell + 1, *\} \) and let \( e_i \) denote the \( i^{th} \) unit vector in \( \mathbb{R}^{\ell+1} \) with \( e_* := 0 \in \mathbb{R}^{\ell+1} \). Then the inequality

\[
\text{tr} \left[ \begin{bmatrix} y^t^T \\ u^t^T \end{bmatrix} \right] \Pi(\lambda, \Lambda) \left[ \begin{bmatrix} y^t^T \\ u^t^T \end{bmatrix} \right] + \pi(\lambda, \Lambda)^T f^t \geq 0
\]  

(45)

holds for all \( \lambda \in \mathbb{R}^{\ell+1}, \Lambda \in \mathbb{R}^{(\ell+2) \times (\ell+2)} \) such that \( \lambda, \Lambda \geq 0 \) (elementwise), where

\[
\Pi(\lambda, \Lambda) := \left[ \begin{array}{cc} 0 & 0 \\ 0 & \text{diag}(\lambda) \end{array} \right] + \sum_{i,j \in I} \Lambda_{ij} \left[ \begin{array}{ccc} L (e_i - e_j)(e_i - e_j)^T & -(e_i - e_j)e_i^T \\ -e_i(e_i - e_j)^T & 0 \end{array} \right],
\]

(46a)

\[
\pi(\lambda, \Lambda) := -2m\lambda + 2 \sum_{i,j \in I} \Lambda_{ij} (e_i - e_j).
\]

(46b)

**Proof.** For each \( i \in I \), define the vectors \( \tilde{y}_i := e_i^T y^t \) and \( \tilde{u}_i := e_i^T u^t \) and \( \tilde{f}_i := e_i^T f^t \). Expanding (45), we obtain

\[
\sum_{i \in I} \lambda_i \left( \|\tilde{u}_i\|^2 - 2m\tilde{f}_i \right) + \sum_{i,j \in I} \Lambda_{ij} \left( L\|\tilde{y}_i - \tilde{y}_j\|^2 - 2 (\tilde{y}_i - \tilde{y}_j)\tilde{u}_i^T + 2(\tilde{f}_i - \tilde{f}_j) \right) \geq 0,
\]

which is a nonnegative weighted combination of the PL condition (42) and the Lipschitz condition (43).

Analysis for the PL<sub>m,L</sub> case is carried out analogously to the F<sub>m,L</sub> case. We now state the analogous version of Theorem 5.3 for the class PL<sub>m,L</sub>.

**Theorem 6.2 (PL<sub>m,L</sub> analysis).** Consider an algorithm \( A = (A, B, C) \) defined in (7) satisfying Assumption 1 applied to a function \( f \in PL_{m,L} \) defined in Section 1 with additive gradient noise satisfying Assumption 3. Define the truncation matrices in (24), the augmented state space and projection matrices in (26)–(30), and the multiplier functions in (46). Then the algorithm satisfies the following convergence rate and sensitivity bounds.

1) If there exist \( r > 0 \), \( P = P^T \in \mathbb{R}^{(n+\ell) \times (n+\ell)} \), \( p \in \mathbb{R}^\ell \), and \( \lambda_1, \lambda_1, \lambda_2, \lambda_2 \geq 0 \) such that

\[
\begin{bmatrix} A_r & B_r \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -r^2P \end{bmatrix} \begin{bmatrix} A_r & B_r \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_r & D_r \end{bmatrix}^T \Pi(\lambda_1, \Lambda_1) \begin{bmatrix} C_r & D_r \end{bmatrix} \leq 0
\]

(47a)

\[
(Z + r^2Z)^T p + \pi(\lambda_1, \Lambda_1) \leq 0
\]

(47b)

\[
- \begin{bmatrix} I & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} C_r & D_r \end{bmatrix}^T \Pi(\lambda_2, \Lambda_2) \begin{bmatrix} C_r & D_r \end{bmatrix} \leq 0
\]

(47c)

\[
e_1 - Z^T p + \pi(\lambda_2, \Lambda_2) \leq 0
\]

(47d)
then \( \text{rate}(\mathcal{A}, PL_{m,L}) \leq r \).

2) If there exist \( P = P^T \in \mathbb{R}^{(n+2\ell) \times (n+2\ell)} \), \( p \in \mathbb{R}^\ell \), and \( \lambda_1, \Lambda_1, \lambda_2, \Lambda_2 \geq 0 \) such that

\[
\begin{bmatrix}
  A & B \\
  I & 0
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
0 & -P
\end{bmatrix}
\begin{bmatrix}
  A & B \\
  I & 0
\end{bmatrix}
\begin{bmatrix}
  C & D
\end{bmatrix}
\Pi(\lambda_1, \Lambda_1)
\begin{bmatrix}
  C & D
\end{bmatrix} \leq 0
\]

\( e_1 + (Z_+ - Z)^T p + \pi(\lambda_1, \Lambda_1) \leq 0 \)

\[
- [I \ 0]^T P [I \ 0] + \begin{bmatrix}
  C & D
\end{bmatrix}^T \Pi(\lambda_2, \Lambda_2) \begin{bmatrix}
  C & D
\end{bmatrix} \leq 0
\]

\[-Z^T p + \pi(\lambda_2, \Lambda_2) \leq 0 \]

then \( \text{sens}(\mathcal{A}, PL_{m,L}, \sigma^2) \leq \sqrt{\sigma^2 d \cdot (H^T PH)} \).

**Proof.** The proof is analogous to that of Theorem 5.3. Note that Lemmas 4.1 and 4.2 must also be slightly modified to use function values. Specifically, the lower bound condition in Lemma 4.1 should be changed to \( V(x_t^r, f_t^*) \geq (f_t^r - f^*) \) and the decrease condition in Lemma 4.2 to \( E V(x_{t+1}, f_{t+1}) - E V(x_t, f_t) + E(f_t^r - f^*) \leq \gamma^2 \).

As with Theorem 5.3 and as discussed in Section 5.2, solving the LMIs in Theorem 6.2 also requires selecting a lifting dimension \( \ell \). Larger \( \ell \) can potentially lead to less conservative bounds, but requires solving larger LMIs. We performed a similar pilot test to that in Section 5.2, and this time, it was revealed that \( \ell = 1 \) is sufficient to achieve the tightest bounds certifiable for both convergence rate and sensitivity to noise.

### 6.3 Algorithm design for \( PL_{m,L} \)

In 1963, Polyak [50] showed that Gradient Descent with stepsize \( \alpha \in (0, 2/L) \) applied to functions in \( PL_{m,L} \) satisfies the upper bound \( f^t - f^* \leq (1 - m\alpha(2 - L\alpha))^t(f^0 - f^*) \). Karimi et al. [37] show how the PL condition relates to many other conditions in the literature, and they use it to analyze coordinate descent, sign-based gradient descent, and stochastic gradient methods. Abbaszadehpeivasti et al. [1] use the PEP approach to obtain the convergence rate of GD over all stepsizes in the interval \((0, 2/L)\) for functions in \( PL_{m,L} \) that also satisfy a curvature condition.

A lower bound on the worst-case number of iterations for an algorithm to achieve an \( \varepsilon \)-optimal point of a PL function with Lipschitz gradients is \( \Omega(L/m \log \frac{1}{\varepsilon}) \) gradient evaluations [63]. This bound is achieved by Gradient Descent, showing that it is tight. For this reason, we will not look to design accelerated algorithms for \( PL_{m,L} \), but rather analyze the convergence rate and sensitivity to noise of Gradient Descent as a function of stepsize. The rate result is due to [50], while the sensitivity result is new.

**Corollary 6.3** (Gradient descent for PL). Consider the function class \( PL_{m,L} \). Then under Assumption 3, the algorithm \( \mathcal{A} = (\alpha, 0, 0) \) with \( 0 \leq \alpha \leq \frac{2}{L} \) of the form (5) achieves the performance metrics

\[
\text{rate}(\mathcal{A}, PL_{m,L}) \leq \sqrt{1 - m\alpha(2 - L\alpha)},
\]

\[
\text{sens}(\mathcal{A}, PL_{m,L}, \sigma^2) \leq \sqrt{\frac{\sigma^2 d}{2m} \cdot \frac{L\alpha}{2 - L\alpha}}.
\]
Proof. In Eq. (35), substitute \( p = 1 \), and
\[
P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda_1 = \Lambda_2 = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}, \quad \lambda_1 = \begin{bmatrix} 0 \\ (\alpha(2 - L\alpha)/2) \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 \end{bmatrix}.
\]
Then (35) reduces to \( \alpha(2 - L\alpha) \geq 0 \) and \( \rho^2 \geq 1 - m\alpha(2 - L\alpha) \). The first inequality holds because \( 0 \leq \alpha \leq \frac{2}{T} \), and if \( \rho^2 \) is chosen to satisfy the second inequality, Theorem 5.3 implies that
\[
\text{rate}(A, PL_{m,L}) \leq \sqrt{1 - m\alpha(2 - L\alpha)},
\]
as required. The corresponding Lyapunov function in this case is simply \( f^t - f^* \).

For the second result, in Eq. (36), substitute \( p = \frac{1}{m\alpha(2 - L\alpha)} \) and
\[
P = \frac{p}{2} \begin{bmatrix} L & -L \\ -L & L \end{bmatrix}, \quad \Lambda_1 = \Lambda_2 = \begin{bmatrix} 0 \\ p/2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = \begin{bmatrix} 0 \\ \frac{1}{2m} \end{bmatrix}.
\]
Then (36) reduces to \( \alpha(2 - L\alpha) \geq 0 \), which holds because \( 0 \leq \alpha \leq \frac{2}{T} \). Finally,
\[
H^TPH = \alpha^2 P_{11} = \frac{L\alpha}{2m(2 - L\alpha)},
\]
so the result follows from Theorem 5.3.

Remark 8. The analytic solutions of the LMIs in the proof of Theorem 6.2 can be used to construct conventional proofs using standard inequalities. For example, the proof associated with the convergence rate is as follows. Define the quantity
\[
\Omega := (\hat{f}^{t+1} - \rho^2 \hat{f}^t) + \frac{\alpha(2 - L\alpha)}{2} \left( \|\nabla f(y^t)\|^2 - 2m\hat{f}^t \right) + \left( \frac{L}{2} \|\tilde{y}^t - \tilde{y}^{t+1}\|^2 - (\tilde{y}^t - \tilde{y}^{t+1})^T \nabla f(y^t) + (\hat{f}^t - \hat{f}^{t+1}) \right).
\]
Since \( 0 \leq \alpha \leq \frac{2}{T} \) and \( f \in PL_{m,L} \), the second and third bracketed quantities are nonnegative, from which we conclude that \( \Omega \) satisfies
\[
\Omega \geq \hat{f}^{t+1} - \rho^2 \hat{f}^t
\]
Now substitute the algorithm update \( \tilde{y}^{t+1} = y^t - \alpha \nabla f(y^t) \) into (49) and after simplification, many terms cancel and we are left with \( \Omega = (1 - \rho^2 - 2m\alpha(2 - L\alpha)) \hat{f}^t \). Substitute this finding into (50), and obtain \( (1 - \rho^2 - 2m\alpha(2 - L\alpha)) \hat{f}^t \geq \hat{f}^{t+1} - \rho^2 \hat{f}^t \). Whenever \( \rho^2 \geq 1 - 2m\alpha(2 - L\alpha) \), the left-hand side is nonpositive, and thus the right-hand side is nonpositive and therefore \( \hat{f}^{t+1} \leq \rho^2 \hat{f}^t \), as required. Although the proof above is more complicated than the one presented in [37, Thm. 1], it was automatically generated from a solution to the LMI (35).

Remark 9. Corollary 6.3 bounds the convergence rate and sensitivity defined in terms of function values using (44). Indeed, the LMIs (35)–(36) are infeasible. Had they been feasible, this would imply convergence to every optimal point, i.e., the optimal point would be unique, which is not true in general for the class \( PL_{m,L} \).

In Section 5.4, we compared our results for \( F_{m,L} \) with those achieved using the IQC approach from robust control. For the \( PL_{m,L} \) case, there are no such IQCs results since IQCs do not explicitly use function values.
7 Numerical validation

In this section, we use numerical experiments to verify that our algorithm designs: effectively trade off convergence rate and noise sensitivity, have correct asymptotic convergence, use an adequate number of parameters (they are neither under- nor over-parameterized), and outperform popular iterative schemes when applied to a worst-case test function.

7.1 Empirical validation

The main results of the previous sections provided means to efficiently compute upper bounds on the convergence rate and sensitivity for any algorithm of the form (7) for each function class \( Q_{m,L}, F_{m,L}, \) and \( PL_{m,L} \). We also provided algorithms in the three-parameter family \( A = (\alpha, \beta, \eta) \) in (5) with a single tunable parameter that mediates the trade-off between rate and sensitivity. We summarize these results in Table 3.

| Section | Function class | Analysis result | Algorithm design |
|---------|----------------|-----------------|------------------|
| Section 3 | \( Q_{m,L} \) | Corollary 3.2 | RHB, Theorem 3.3 |
| Section 5 | \( F_{m,L} \) | Theorem 5.3 | RAM, Theorem 5.4 |
| Section 6 | \( PL_{m,L} \) | Theorem 6.2 | GD, Corollary 6.3 |

To empirically validate the performance of our designs, we perform a brute-force search over algorithms \( (\alpha, \beta, \eta) \) and make a scatter plot of the convergence rate and sensitivity. To facilitate sampling, the following result provides bounds on admissible tuples \( (\alpha, \beta, \eta) \).

**Lemma 7.1 (algorithm parameter restriction).** Consider the three-parameter algorithm \( A = (\alpha, \beta, \eta) \) defined in Section 2.2. Let \( F \in \{ Q_{m,L}, F_{m,L}, PL_{m,L} \} \). If \( \text{Rate}(A, F) < 1 \), then:

\[
0 < \alpha < \frac{4}{L}, \quad -\frac{2}{L-m} < \alpha \eta < \frac{2}{L-m}, \quad \text{and} \quad \begin{cases} 
-1 + L(\alpha \eta) < \beta < 1 + m(\alpha \eta) & \text{if } \alpha \eta \geq 0 \\
-1 + m(\alpha \eta) < \beta < 1 + L(\alpha \eta) & \text{if } \alpha \eta < 0.
\end{cases}
\]

**Proof.** The Jury criterion for stability \((r < 1)\) is given in (21). Combining (21b) + 2 · (21d) together with (21a), we obtain: \(0 < \alpha \eta < 4\). This must hold for all \( q \in [m, L]\), so we conclude that \(0 < \alpha < \frac{4}{L}\). Combining (21c) and (21d), we obtain \(-1 + \alpha \eta q < \beta < 1 + \alpha \eta q\). This must hold for all \( q \in [m, L]\). We consider two cases. When \( \alpha \eta \geq 0 \), the \( \beta \) range reduces to \(-1 + L(\alpha \eta) < \beta < 1 + m(\alpha \eta)\) and consequently, \( \alpha \eta < \frac{2}{L-m} \). When \( \alpha \eta < 0 \), we instead obtain \(-1 + m(\alpha \eta) < \beta < 1 + L(\alpha \eta)\) and \(-\frac{2}{L-m} < \alpha \eta\), thus completing the proof. Although these bounds are derived for the function class \( Q_{m,L}\), the nestedness properties \( Q_{m,L} \subseteq F_{m,L} \) and \( Q_{m,L} \subseteq PL_{m,L} \) implies that these necessary conditions on \( (\alpha, \beta, \eta) \) also hold for \( F_{m,L} \) and \( PL_{m,L} \).

From Lemma 7.1, we see that \( \alpha, \alpha \eta, \) and \( \beta \) each have finite ranges. So a convenient way to grid the space of possible \( (\alpha, \beta, \eta) \) values is to first grid over \( \alpha \), then \( \alpha \eta \), then \( \beta \), in a nested fashion, extracting the associated \( (\alpha, \beta, \eta) \) values at each step. Due to the multiplicative nature of the parameter \( \alpha \), we opted to sample \( \alpha \) logarithmically in the range \([10^{-5}, \frac{4}{L}]\), but to sample \( \alpha \eta \) and \( \beta \) linearly in their associated intervals.
Strongly convex quadratics \((Q_{m,L})\). We show our brute-force search for the class \(Q_{m,L}\) in Fig. 3 for \(Q_{1,10}\) and \(Q_{1,100}\). For this figure, we used the sampling approach based on Lemma 7.1 with \(500 \times 201 \times 200\) samples for \(\alpha\), \(\alpha \eta\), and \(\beta\), respectively.

![Figure 3: Plot of \(\gamma^2\) vs. \(r^2\) for algorithms applied to the function class \(Q_{1,10}\) (left panel) and \(Q_{1,100}\) (right panel), found using Corollary 3.2. Each point in the point cloud corresponds to an algorithm \((\alpha, \beta, \eta)\). The Pareto-optimal front coincides with the Robust Heavy Ball method (Theorem 3.3), tuned using \(r \in \left[\frac{\sqrt{L+m}}{L+\sqrt{m}}, 1\right]\) to mediate the trade-off. The normalized squared sensitivity is \(\gamma^2/\sigma^2d\), where \(\gamma\) is the sensitivity.

In Fig. 3, each algorithm \((\alpha, \beta, \eta)\) corresponds to a single gray dot. The curve labeled RHB shows each possible tuning as we vary the parameter \(r\). We observe that RHB perfectly traces out the boundary of the point cloud, which represents the Pareto-optimal algorithms. In other words, for a fixed convergence rate \(r\), RHB with parameter \(r\) achieves this rate and is also as robust as possible to additive gradient noise (smallest \(\gamma\)).

Fig. 3 reveals that Gradient Descent (GD) with \(0 < \alpha < \frac{2}{L+m}\) is outperformed by by RHB on the function class \(Q_{m,L}\). We also plot the performance of GD for \(\alpha > \frac{2}{L+m}\), which is even worse as this leads to slower convergence and increased sensitivity. Fig. 3 also reveals that the Fast Gradient (FG) method is strictly suboptimal compared to RHB based on the analysis in Theorem 5.3, although the optimality gap appears to shrink as \(L/m\) gets larger.

Smooth strongly convex functions \((F_{m,L})\). We show our brute-force search for the class \(F_{m,L}\) in Fig. 4 for \(F_{1,10}\) and \(F_{1,100}\). For this figure, we used the same sampling approach as in Fig. 3, with \(200 \times 51 \times 50\) samples. When applying Theorem 5.3, we used a lifting dimension \(\ell = 1\) to compute the rate and \(\ell = 6\) to compute the sensitivity. For more details on these choices, see Section 5.2.

The Robust Momentum (RM) method from [16] interpolates between TM and GD with \(\alpha = \frac{1}{L}\) and does trade off convergence rate for sensitivity, but based on our analysis in Theorem 5.3 it is strictly outperformed by our proposed Robust Accelerated Method (RAM). The gap in performance between RM and RAM appears to shrink as \(L/m\) gets larger.

---

\(^{10}\)We opted to plot \(\gamma^2\) vs. \(r^2\) rather than \(\gamma\) vs. \(r\) because the former leads to a convex feasible region that looks more like conventional Pareto trade-off plots.
Squared convergence rate

Normalized squared sensitivity

| Algorithm | Sensitivity |
|-----------|-------------|
| RM        | 0.74, 0.44  |
| GD, 2/(L + m) | 0.22057   |
| GD, 1/L   | 0.1676     |

While Fig. 4 indicates that RAM effectively trades off rate and robustness, it is strictly suboptimal\(^\text{11}\). Suboptimality becomes most apparent when \(L/m\) is small and \(r\) is close to 1. For example, consider RAM with the parameter choice \(r = 0.9, m = 1, L = 2, d = 1, \) and \(\sigma = 1,\) which corresponds to \((\alpha, \beta, \eta) = (0.019, 0.66, -3.631579).\) Solving the LMIs in Theorem 5.3 yields the rate 0.9000 and sensitivity 0.22057. However, if we change \(\eta\) and use the tuning \((\alpha, \beta, \eta) = (0.019, 0.66, 0.00)\) instead, we obtain the same rate with the strictly smaller sensitivity 0.1676. Larger optimality gaps can be found by making \(L/m\) even closer to 1, although such cases are not practical.

**Remark 10.** The point cloud in the left panel of Fig. 4 (\(L/m = 10\)) is denser than that of the right panel (\(L/m = 100\)), even though the same number of sample points is used in both experiments. The reason for this difference is that the point cloud on the right is spread over a relatively larger range of \(\gamma\) values (we truncated the vertical axis). In other words, desirable algorithm tunings are harder to find by random sampling when \(L/m\) is larger.

**PL functions with Lipschitz gradients (\(PL_{m,L}\)).** We show our brute-force search for the class \(PL_{m,L}\) in Fig. 5 for \(PL_{1,10}\). For this figure, we used the same sampling approach as in Figs. 3 and 4, with 200 \(\times\) 51 \(\times\) 50 samples. When applying Theorem 6.2, we used a lifting dimension \(\ell = 1\) to compute the rate and \(\ell = 6\) to compute the sensitivity.

For this function class, our brute-force numerical search did not find a three-parameter algorithm that outperforms Gradient Descent, whose performance we characterized in Corollary 6.3. Interestingly, stepsizes in the interval \([\frac{1}{L}, \frac{2}{L+2m}]\) produce convergent algorithms with rate strictly less than one, but are strictly suboptimal in that there exists a smaller stepsize that yields both faster convergence rate and smaller sensitivity.

\(^{11}\)Suboptimality is difficult to verify for the function class \(F_{m,L}\) since the results depend on the lifting dimension \(\ell.\) However, we performed extensive numerical computations to ensure that \(\ell\) is sufficiently large, see Section 7.
Convergence rate $\rho^2$

Figure 5: Plot of $\gamma^2$ vs. $r^2$ for algorithms applied to the function class $PL_{1,10}$, found using Theorem 6.2. Each point in the point cloud corresponds to an algorithm $(\alpha, \beta, \eta)$. We used a lifting dimension $\ell = 1$ for computing $r$ and $\ell = 6$ for computing $\gamma$. The normalized squared sensitivity is $\gamma^2/(\sigma^2d)$, where $\gamma$ is the sensitivity.

7.2 Asymptotic convergence rate

Figs. 3 to 5 show that as the rate $r \rightarrow 1$, the sensitivity $\gamma \rightarrow 0$. In the limit of arbitrarily slow convergence, we obtain the desirable behavior of complete noise attenuation. However, these trade-off plots give limited insight on the asymptotic convergence rate as $r \rightarrow 1$. As mentioned in Section 1.2, the fastest possible decay of the mean squared error of any algorithm is $1/t$. Therefore, for any fixed initial condition and objective function, there must exist some constant $c_0 > 0$ such that

$$E\|y^t - y^*\|^2 > \frac{c_0}{t+1} \quad \text{for all } t \geq 0.$$  \hspace{1cm} (51)

For any fixed tuning, the expected norm of the iterates is bounded by $E\|y^t - y^*\|^2 = \max\{c_1^2 \gamma^2 r^{2t}, \gamma^2\}$ (as in Fig. 1), where $c_1 > 1$ is determined by the initial condition. The phase transition occurs when $c_1 r^t = 1$, which is when $t = \frac{\log c_1}{\log r}$. Substituting $t$ into (51) and rearranging the inequality assuming $0 < r < 1$, we obtain

$$\gamma^2 \left(\frac{-1}{\log r}\right) > \frac{c_0}{\log c_1 - \log r} \xrightarrow{r \rightarrow 1} \frac{c_0}{\log c_1} > 0.$$  

The quantity $-1/\log r$ may be interpreted as the iteration complexity; it is proportional to the number of iterations required to guarantee that the error is less than some prescribed amount. For an algorithm with optimal scaling as $r \rightarrow 1$, a log-log plot of $\gamma^2$ vs. $-1/\log r$ is therefore a line of slope $-1$ as $r \rightarrow 1$. As shown in Fig. 6, our algorithm designs appear to have optimal or
near-optimal scaling as \( r \to 1 \). An asymptotic slope steeper than \(-1\) is not possible, for then a decay faster than \(1/t\) could be achieved with a suitably chosen piecewise constant \( r \) schedule.

For cases where we have an explicit formula for the sensitivity \( \gamma \) in terms of the rate \( r \) such as for RHB (Theorem 3.3), we can evaluate the limit as \( r \to 1 \) from the left of \( \gamma^2 \left( \frac{-1}{\log r} \right) \) and as expected, it is finite and equal to \( \frac{d^{a^2}}{4m^2} \).

### 7.3 Justifying the three-parameter algorithm family

A natural question to ask is whether something as general as our three-parameter family (5) is needed to achieve the performance of our designs. Several recent works have restricted their attention to optimizing algorithms with two parameters \((\alpha, \beta)\) in either Nesterov’s FG or Polyak’s HB form [6, 29, 46]. From our results in Sections 3.2 and 7.1, the HB form is sufficient for the class \( Q_{m,L} \). However, neither the HB or the FG forms are sufficient for \( F_{m,L} \). While some algorithms in these restricted classes achieve acceleration, they are incapable of obtaining the same trade-off between convergence rate and sensitivity as a properly-tuned three-parameter method, as illustrated in Fig. 7 (left panel). Indeed, even when there is no noise, no algorithm in the FG or HB families achieves the optimal convergence rate for the function class \( F_{m,L} \), which is attained by Van Scoy et al.’s Triple Momentum (TM) method [60].

Alternatively, we could ask whether three parameters are enough, and whether adding more could lead to further improvements. As explained in Section 2.2, any algorithm with \( n = 2 \) states can be represented by three parameters. In general, we would need \( 2n - 1 \) parameters to represent an algorithm with \( n \) states. In principle, our methodology of Section 1.3 can still be applied, but the associated semidefinite programs become substantially more difficult to solve and we were unable to find better designs.

An alternative approach was presented in the recent work [45], which uses a convex synthesis procedure and bilinear matrix inequalities to numerically construct algorithms that trade off convergence rate and sensitivity. As shown in Fig. 7 (right panel), these synthesized algorithms do not achieve...
the same performance metrics as our method RAM, despite using up to $n = 6$ states.

**Figure 7:** Left: Regions of the rate vs. sensitivity trade-off space for $F_{1,100}$ covered by the three-parameter family $(\alpha, \beta, \eta)$, the Nesterov (Fast Gradient) family $(\alpha, \beta, \beta)$, and the Polyak (Heavy Ball) family $(\alpha, \beta, 0)$. The FG and HB families are not expressive enough to capture the whole trade-off space. The normalized squared sensitivity is $\gamma^2/(\sigma^2 d)$, where $\gamma$ is the sensitivity. Right: Comparison of RAM with the numerically synthesized algorithms (using a state dimension up to $n = 6$) from [45] for $F_{1,50}$. RAM outperforms in spite of using only two states of memory. We plot the log of the normalized sensitivity vs. the rate to match [45, Fig. 6].

### 7.4 Simulation of a worst-case test function

We simulated various algorithms on Nesterov’s lower-bound function, which is a quadratic with a tridiagonal Hessian [47, §2.1.4]. We used $d = 100$ with $m = 1$ and $L = 10$ and initialized each algorithm at zero. The results are reported in Fig. 8. We tested both a low noise ($\sigma = 10^{-5}$, left column) and a higher noise ($\sigma = 10^{-2}$, right column) regime. We recorded the mean and standard deviation of the error across 100 trials for each algorithm (the trials differ only in the noise realization).

Fig. 8 shows that our Robust Heavy Ball (RHB) method from Theorem 3.3 trades off convergence rate (the slope of the initial decrease) with sensitivity to noise (the value of the steady-state error), and compares favorably to a variety of other methods. The other methods we tested (first row of Fig. 8) are generally suboptimal compared to RHB, in the sense that there is some choice of tuning parameter $r$ such that RHB is both faster and has smaller steady-state error.

In addition to gradient descent (GD) and Nesterov’s method (FG), we tested Nonlinear Conjugate Gradient (NLCG) with Polak-Ribière (PR) update scheme.\(^{12}\) NLCG performs similarly to RHB with the most aggressive tuning, which is equivalent to the Heavy Ball method. We also tested the popular quasi-Newton methods [49] Broyden–Fletcher–Goldfarb–Shanno (BFGS) and Symmetric Rank-One (SR1), which performed strictly worse than RHB. Both NLCG and BFGS involve line searches. That is, given a current point $y \in \mathbb{R}^d$ and search direction $s \in \mathbb{R}^d$, we must find $\alpha \in \mathbb{R}$ such that we minimize $f(y + \alpha s)$. In practice, inexact line searches are performed at each timestep, with a stopping criterion such as the Wolfe conditions. To show these algorithms in the most charitable

\(^{12}\)We also tested other popular NLCG update schemes: Fletcher–Reeves, Hestenes–Stiefel, and Dai–Yuan; all produced similar trajectories to PR.
Figure 8: Simulations of various algorithms with low noise ($\sigma = 10^{-5}$, left column) and higher noise ($\sigma = 10^{-2}$, right column). Each algorithm was simulated on Nesterov’s lower-bound quadratic function, with $m = 1$, $L = 10$, and dimension $d = 100$. Shaded regions indicate ±1 standard deviations about the mean across 100 trials (different noise realizations). Different tunings of our proposed Robust Heavy Ball (RHB) from Theorem 3.3 effectively trade off convergence rate and steady-state error (sensitivity to noise). Bottom row: the red curve shows RHB with (hand-tuned) piecewise constant $r$, where the red dots indicate switch points.

possible light, we used exact line searches but substituted the noisy gradient oracle. Specifically, with $f(y) = \frac{1}{2}(y - y^*)^T Q (y - y^*)$, the optimal stepsize is $\alpha^* = - (s^T \nabla f(y)) / (s^T Q s)$. We used this formula, but replaced $\nabla f(y)$ by the noisy gradient $\nabla f(y) + w$. In other words, we assumed exact knowledge of $Q$ but not of $y^*$.

In the second row of Fig. 8, we duplicate the settings of the first row, except plot iterations on a log scale as in Fig. 1. Here, we also show a hand-tuned version of RHB with piecewise constant parameter $r^{13}$. Every time $r$ is changed, we re-initialize the algorithm by setting $x_{t-1} = x_t$. When the error is large compared to the noise level, the algorithm matches Nesterov’s lower bound (16), which is the lower bound associated with this particular function that holds for any algorithm when $t < d$. When the error is small compared to the noise level, the algorithm matches the asymptotic

---

13We use a hand-tuned schedule to illustrate our results; more systematic scheduling methods have been proposed, such as the RESTART+SLOWDOWN method in [15].
lower bound (slope of $-1/2$) described in Sections 1.2 and 7.2.\textsuperscript{14}

8 Concluding remarks

For each of the function classes $Q_{m,L}$, $F_{m,L}$, and $PL_{m,L}$, we have provided (i) efficient methods for computing the convergence rate and noise sensitivity for a broad class of first-order methods, and (ii) first-order algorithms designs, each with a single tunable parameter that directly trades off convergence rate versus sensitivity.

An interesting future direction is to explore adaptive versions of these algorithms, for example where the parameter $r$ is increased over time. We showed in Fig. 8 that a hand-tuned piecewise constant version of RHB can match both Nesterov’s lower bound and the gradient lower bound in the asymptotic regime, so more sophisticated adaptive schemes such as those described in Section 1.2 might also work.

It may also be possible to adjust parameters continually (rather than in a piecewise fashion), but proving the convergence of adaptive algorithms is generally more challenging. For example, the well-known ADMM algorithm is often tuned adaptively to improve transient performance, even when convergence guarantees only hold for fixed parameters [11, §3.4.1]. Nevertheless, LMI-based approaches have been successfully used to prove convergence of algorithms with time-varying parameters [27,32].

Another interesting open question is whether our analysis is tight. For the function classes $F_{m,L}$ and $PL_{m,L}$, our bounds depend on the lifting dimension $\ell$, and it is an open question how large $\ell$ needs to be in order to obtain tight bounds on the convergence rate and noise sensitivity.

Disclosure Statement

No potential conflict of interest was reported by the authors.

Acknowledgments

We would like to thank Jason Lee for his helpful comments and insights regarding the statistical aspects of this work. L. Lessard is partially supported by the National Science Foundation under Grants No. 2136945 and 2139482.

\textsuperscript{14}The lower bound is $1/t$ for the squared error, hence $1/\sqrt{t}$ for the error, which appears as a line of slope $-1/2$ on the log-log scale of Fig. 8, bottom row.
References

[1] H. Abbaszadehpeivasti, E. de Klerk, and M. Zamani. Conditions for linear convergence of the gradient method for non-convex optimization. *Optimization Letters*, 17(5):1105 – 1125, 2023.

[2] Z. Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, page 1200–1205, New York, NY, USA, 2017. Association for Computing Machinery.

[3] P. J. Antsaklis and A. N. Michel. *Linear systems*. Springer Science & Business Media, 2006.

[4] M. ApS. *The MOSEK optimization suite 9.2.49*, 2021.

[5] N. Aybat, A. Fallah, M. Gürbüzbalaban, and A. Ozdaglar. A universally optimal multistage accelerated stochastic gradient method. *Advances in Neural Information Processing Systems*, 32, 2019.

[6] N. S. Aybat, A. Fallah, M. Gurbuzbalaban, and A. Ozdaglar. Robust accelerated gradient methods for smooth strongly convex functions. *SIAM Journal on Optimization*, 30(1):717–751, 2020.

[7] R. Bassily, A. Smith, and A. Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, 2014.

[8] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah. Julia: A fresh approach to numerical computing. *SIAM Review*, 59(1):65–98, 2017.

[9] B. Birand, H. Wang, K. Bergman, and G. Zussman. Measurements-based power control - a cross-layered framework. In *Optical Fiber Communication Conference/National Fiber Optic Engineers Conference 2013*, page JTh2A.66. Optica Publishing Group, 2013.

[10] L. Bottou, F. E. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *SIAM Review*, 60(2):223–311, 2018.

[11] S. Boyd, N. Parikh, and E. Chu. *Distributed optimization and statistical learning via the alternating direction method of multipliers*. Now Publishers Inc, 2011.

[12] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

[13] B. Buchberger. Gröbner bases: A short introduction for systems theorists. In R. Moreno-Díaz, B. Buchberger, and J. Luis Freire, editors, *Computer Aided Systems Theory — EUROCAST 2001*, pages 1–19, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

[14] J. Chee and P. Toulis. Convergence diagnostics for stochastic gradient descent with constant learning rate. In A. Storkey and F. Perez-Cruz, editors, *Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics*, volume 84 of *Proceedings of Machine Learning Research*, pages 1476–1485. PMLR, 09–11 Apr 2018.

[15] M. Cohen, J. Diakonikolas, and L. Orecchia. On acceleration with noise-corrupted gradients. In J. Dy and A. Krause, editors, *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 1019–1028. PMLR, 10–15 Jul 2018.
[16] S. Cyrus, B. Hu, B. Van Scoy, and L. Lessard. A robust accelerated optimization algorithm for strongly convex functions. In American Control Conference, pages 1376–1381, June 2018.

[17] E. de Klerk, F. Glineur, and A. B. Taylor. On the worst-case complexity of the gradient method with exact line search for smooth strongly convex functions. Optimization Letters, 11:1185–1199, 2017.

[18] E. De Klerk, F. Glineur, and A. B. Taylor. Worst-case convergence analysis of inexact gradient and newton methods through semidefinite programming performance estimation. SIAM Journal on Optimization, 30(3):2053–2082, 2020.

[19] A. Defazio, F. Bach, and S. Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In Z. Ghahramani, M. Welling, C. Cortes, N. Lawrence, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems, volume 27. Curran Associates, Inc., 2014.

[20] O. Devolder, F. Glineur, and Y. Nesterov. First-order methods with inexact oracle: the strongly convex case. Core discussion paper; 2013/16, Université catholique de Louvain, May 2013.

[21] O. Devolder, F. Glineur, and Y. Nesterov. Intermediate gradient methods for smooth convex problems with inexact oracle. Core discussion paper; 2013/17, Université catholique de Louvain, May 2013.

[22] O. Devolder, F. Glineur, and Y. Nesterov. First-order methods of smooth convex optimization with inexact oracle. Mathematical Programming, 146(1-2):37–75, 2014.

[23] Y. Drori and A. Taylor. On the oracle complexity of smooth strongly convex minimization. Journal of Complexity, 68(C), Feb 2022.

[24] Y. Drori and M. Teboulle. Performance of first-order methods for smooth convex minimization: a novel approach. Mathematical Programming, 145(1):451–482, 2014.

[25] M. S. Fadali and A. Visioli. Digital control engineering: analysis and design. Academic Press, 2013.

[26] C. Fang, C. J. Li, Z. Lin, and T. Zhang. SPIDER: Near-optimal non-convex optimization via stochastic path integrated differential estimator. In Proceedings of the 32nd International Conference on Neural Information Processing Systems, page 687–697, Red Hook, NY, USA, 2018. Curran Associates Inc.

[27] M. Fazlyab, A. Ribeiro, M. Morari, and V. M. Preciado. Analysis of optimization algorithms via integral quadratic constraints: Nonstrongly convex problems. SIAM Journal on Optimization, 28(3):2654–2689, 2018.

[28] R. Ge, S. M. Kakade, R. Kidambi, and P. Netrapalli. The step decay schedule: A near optimal, geometrically decaying learning rate procedure for least squares. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché-Buc, E. Fox, and R. Garnett, editors, Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019.
[29] E. Ghadimi, H. R. Feyzmahdavian, and M. Johansson. Global convergence of the heavy-ball method for convex optimization. In 2015 European Control Conference (ECC), pages 310–315, 2015.

[30] S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization I: A generic algorithmic framework. *SIAM Journal on Optimization*, 22(4):1469–1492, 2012.

[31] S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization, II: Shrinking procedures and optimal algorithms. *SIAM Journal on Optimization*, 23(4):2061–2089, 2013.

[32] B. Hu and L. Lessard. Dissipativity theory for Nesterov’s accelerated method. In *International Conference on Machine Learning*, pages 1549–1557, Aug. 2017.

[33] B. Hu, P. Seiler, and L. Lessard. Analysis of biased stochastic gradient descent using sequential semidefinite programs. *Mathematical Programming*, 187(0):383–408, Mar. 2020.

[34] P. Jain, S. M. Kakade, R. Kidambi, P. Netrapalli, and A. Sidford. Accelerating stochastic gradient descent for least squares regression. In S. Bubeck, V. Perchet, and P. Rigollet, editors, *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 545–604. PMLR, 06–09 Jul 2018.

[35] P. Jain, S. M. Kakade, R. Kidambi, P. Netrapalli, and A. Sidford. Accelerating stochastic gradient descent for least squares regression. In S. Bubeck, V. Perchet, and P. Rigollet, editors, *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 545–604. PMLR, 06–09 Jul 2018.

[36] R. Johnson and T. Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc., 2013.

[37] H. Karimi, J. Nutini, and M. Schmidt. Linear convergence of gradient and proximal-gradient methods under the polyak-lojasiewicz condition. In *Joint European conference on machine learning and knowledge discovery in databases*, pages 795–811. Springer, 2016.

[38] A. Kulunchakov and J. Mairal. A generic acceleration framework for stochastic composite optimization. In *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019.

[39] G. Lan. An optimal method for stochastic composite optimization. *Mathematical Programming*, 133(1):365–397, 2012.

[40] L. Lessard. The analysis of optimization algorithms: A dissipativity approach. *IEEE Control Systems Magazine*, 42(3):58–72, 2022.

[41] L. Lessard, B. Recht, and A. Packard. Analysis and design of optimization algorithms via integral quadratic constraints. *SIAM Journal on Optimization*, 26(1):57–95, 2016.

[42] L. Lessard and P. Seiler. Direct synthesis of iterative algorithms with bounds on achievable worst-case convergence rate. In *American Control Conference*, July 2020.
[43] C. Liu, L. Zhu, and M. Belkin. Loss landscapes and optimization in over-parameterized non-linear systems and neural networks. *Applied and Computational Harmonic Analysis*, 59:85–116, 2022.

[44] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, 1997.

[45] S. Michalowsky, C. Scherer, and C. Ebenbauer. Robust and structure exploiting optimisation algorithms: an integral quadratic constraint approach. *International Journal of Control*, 94(11):2956–2979, 2021.

[46] H. Mohammadi, M. Razaviyayn, and M. R. Jovanović. Robustness of accelerated first-order algorithms for strongly convex optimization problems. *IEEE Transactions on Automatic Control*, 66(6):2480–2495, 2021.

[47] Y. Nesterov. *Lectures on convex optimization, second edition*, volume 137. Springer, 2018.

[48] L. M. Nguyen, J. Liu, K. Scheinberg, and M. Takáč. SARAH: A novel method for machine learning problems using stochastic recursive gradient. In *Proceedings of the 34th International Conference on Machine Learning - Volume 70*, ICML’17, page 2613–2621. JMLR.org, 2017.

[49] J. Nocedal and S. Wright. *Numerical optimization*. Springer Science & Business Media, 2006.

[50] B. T. Polyak. Gradient methods for the minimisation of functionals. *USSR Computational Mathematics and Mathematical Physics*, 3(4):864–878, 1963.

[51] B. T. Polyak. *Introduction to optimization*. Translations series in mathematics and engineering. Optimization Software, Inc., 1987.

[52] E. Ryu, A. Taylor, C. Bergeling, and P. Giselsson. Operator splitting performance estimation: Tight contraction factors and optimal parameter selection. *SIAM Journal on Optimization, Society for Industrial and Applied Mathematics*, 30:2251–2271, 2020.

[53] C. Scherer and C. Ebenbauer. Convex synthesis of accelerated gradient algorithms. *SIAM Journal on Control and Optimization*, 59(6):4615–4645, 2021.

[54] A. Taylor and F. Bach. Stochastic first-order methods: non-asymptotic and computer-aided analyses via potential functions. In A. Beygelzimer and D. Hsu, editors, *Proceedings of the Thirty-Second Conference on Learning Theory*, volume 99 of *Proceedings of Machine Learning Research*, pages 2934–2992. PMLR, 25–28 Jun 2019.

[55] A. Taylor and Y. Drori. An optimal gradient method for smooth strongly convex minimization. *Mathematical Programming*, 2022.

[56] A. Taylor, B. Van Scoy, and L. Lessard. Lyapunov functions for first-order methods: Tight automated convergence guarantees. In *International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 4897–4906, Stockholmsmässan, Stockholm Sweden, Jul 2018. PMLR.

[57] A. B. Taylor, J. M. Hendrickx, and F. Glineur. Smooth strongly convex interpolation and exact worst-case performance of first-order methods. *Mathematical Programming*, 161(1-2):307–345, 2017.
A. B. Taylor, J. M. Hendrickx, and F. Glineur. Exact worst-case convergence rates of the proximal gradient method for composite convex minimization. *Journal of Optimization Theory and Applications*, 178(2):455–476, 2018.

M. Udell, K. Mohan, D. Zeng, J. Hong, S. Diamond, and S. Boyd. Convex optimization in Julia. *SC14 Workshop on High Performance Technical Computing in Dynamic Languages*, 2014.

B. Van Scoy, R. A. Freeman, and K. M. Lynch. The fastest known globally convergent first-order method for minimizing strongly convex functions. *IEEE Control Systems Letters*, 2(1):49–54, 2017.

C. Wang, X. Chen, A. J. Smola, and E. P. Xing. Variance reduction for stochastic gradient optimization. In *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc., 2013.

Wolfram Research, Inc. Mathematica, Version 12.3.1. Champaign, IL, 2021.

P. Yue, C. Fang, and Z. Lin. On the lower bound of minimizing polyak-lojasiewicz functions. *arXiv preprint arXiv:2212.13551*, 2022.

G. Zames and P. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal on Control*, 6(1):89–108, 1968.

D. Zhou, P. Xu, and Q. Gu. Stochastic nested variance reduction for nonconvex optimization. *Journal of Machine Learning Research*, 21(103):1–63, 2020.

K. Zhou, J. C. Doyle, and K. Glover. *Robust and optimal control*. Prentice-Hall, Inc., 1996.
A Numerical considerations for LMIs

This section explains the details of how we numerically solved the linear matrix inequalities in and Theorem 5.3 and Theorem 6.2. We used the Julia programming language version 1.6 [8] along with the Convex.jl package version 0.14 [59] to model the optimization problems that were then solved using Mosek version 9.3 [4]. All simulations and numerical experiments were conducted on a laptop computer with conventional hardware.

We found empirically that the sensitivity LMIs (36) becomes poorly conditioned when either the lifting dimension \( \ell \) or condition ratio \( L/m \) is large. One way to improve the conditioning of the problem is via a similarity transform. For example, given any invertible matrix \( T \), the LMI (36) for the function class \( F_{m,L} \) is feasible if and only if it is feasible under the transformation

\[
(A, B, H, C, Y) \mapsto \left( TA^{-1}, TB, TH, CT^{-1}, Y \begin{bmatrix} T^{-1} & 0 \\ 0 & 1 \end{bmatrix} \right),
\]

which corresponds to transforming the lifted state as \( x^t \mapsto Tx^t \) and the solution transforms as \( P \mapsto T^{-T}PT^{-1} \) with \( p, \Lambda_1, \Lambda_2, \) and \( \gamma \) unchanged. A similar transformation is also possible for computing the rate via (35). Transforming the LMIs via an appropriately chosen \( T \) can lead to improved conditioning and solver performance.

As a heuristic, we use a transformation inspired by the balanced realization [66, §3.9]. Given a scalar parameter \( \theta > \rho(A) \), let \( W_o \) and \( W_c \) be the unique and positive definite solutions to the matrix equations

\[
A^TW_oA - \theta^2 W_o + C^TC = 0,
\]

\[
AW_cA^T - \theta^2 W_c + [B \ H][B \ H]^T = 0.
\]

Denote the matrix square roots of the solutions as \( W = \sqrt{W_c} \) and \( Z = \sqrt{W_o} \), and compute the singular value decomposition \( WZ^T = U\Sigma V^T \), where \( U \) and \( V \) are orthogonal and \( \Sigma \) is diagonal. The state transformation matrix for the balanced realization is then \( T = \sqrt{\Sigma^{-1}V^TZ} \). The matrices \( W_o \) and \( W_c \) are the (scaled) observability and controllability Gramians of the system. The balanced realization is constructed such that, in the transformed coordinates, the controllability and observability Gramians are both equal to the diagonal matrix \( \Sigma \) of singular values. In our computations, we set \( \theta = 1.1 \cdot \rho(A) \).

Our observations in Section 5.2 regarding the lifting dimension were robust across all algorithms we tested. So for all subsequent numerical simulations, including those of Fig. 4, we used \( \ell = 1 \) for computing the rate and \( \ell = 6 \) for the sensitivity, both with balancing. In computing the rate, we used a bisection search with tolerance \( 10^{-6} \). Using these parameters, finding the rate and sensitivity each took about 100 ms. Consequently each plot in Fig. 4 took roughly 30 hours to compute.