FINITE GROUP ACTIONS ON HOMOLOGY SPHERES AND MANIFOLDS WITH NONZERO EULER CHARACTERISTIC

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Abstract. Let X be a smooth manifold belonging to one of these three collections: (1) open acyclic manifolds, (2) compact manifolds (possibly with boundary) with nonzero Euler characteristic, and (3) homology spheres. We prove the existence of a constant C such that any finite group acting effectively and smoothly on X has an abelian subgroup of index at most C. The proof uses a result on finite groups by Alexandre Turull and the author which is based on the classification of finite simple groups.

1. Introduction

Inspired by a classic result of Jordan [6] on finite subgroups of $\text{GL}(n, \mathbb{C})$ (see Theorem 5.3 below), Étienne Ghys conjectured at the end of the 90’s that for any compact and connected smooth manifold X there is a constant C such that any finite group G acting smoothly and effectively on X has an abelian subgroup A of index at most C (see Question 13.1 in [3], and footnote 1 in [10]). The particular case in which X is a sphere was also independently asked by Walter Feit [1] and later by Bruno Zimmermann [14, §5] (this case of the conjecture is more directly related to Jordan’s theorem, since smooth group actions on spheres can be seen as nonlinear analogues of actions on spheres induced by subgroups of the orthogonal group).

Partial cases of Ghys’ conjecture have been proved in [9] (for closed manifolds admitting a nonzero top dimensional cohomology class expressible a product of one dimensional classes), [10] (for compact manifolds whose cohomology is torsion-free and supported in even degrees — in particular, even dimensional spheres), and in [11] (for compact 4-manifolds with nonzero Euler characteristic). The results in [10, 11] go beyond proving Ghys conjecture, as they imply not only the existence of an abelian subgroup of bounded index, but also the existence of points fixed by the entire abelian subgroup. An important difference between the proofs in [9, 11] and [10] is that whereas the former only use basic facts about finite groups, the latter relies on a result by Alexandre Turull and the author [12] whose proof uses the classification of finite simple groups (CFSG).

Ghys’ conjecture has also been proved by Zimmermann [15] for compact 3-manifolds using the Thurston’s geometrization conjecture (now Perelman’s theorem).

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1I thank I. Hambleton for this information.
If one considers open connected manifolds instead of compact ones the statement of Ghys’ conjecture becomes false in general: an example was provided by Popov in [13]. However, the statement is known to be true for some low dimensional Euclidean spaces. The cases $\mathbb{R}$ and $\mathbb{R}^2$ are easy exercises. The case $\mathbb{R}^3$ follows from the work of Meeks and Yau on minimal surfaces, see Theorem 4 in [8]. Guazzi, Mecchia and Zimmermann proved in [4] the conjecture for $\mathbb{R}^4$ and reproved the case $\mathbb{R}^3$ with much more elementary methods than those of [8]. Finally, Zimmermann has proved in [15] the cases of $\mathbb{R}^5$ and $\mathbb{R}^6$ using [12] (and, hence, the CFSG). It is thus an interesting question to ask what hypothesis one has to impose on an open manifold for it to satisfy Ghys’ conjecture (a natural guess seems to be that its integral homology must be finitely generated as an abelian group, but we can not offer any strong evidence supporting this guess).

In this paper we prove the following, using [12].

**Theorem 1.1.** Let $X$ be a smooth manifold belonging to one of these three collections: (1) open acyclic manifolds, (2) compact manifolds (possibly with boundary) with nonzero Euler characteristic, and (3) homology spheres. There exists a constant $C$ such that any finite group acting effectively and smoothly on $X$ has an abelian subgroup of index at most $C$.

This result is proved in the paper in three separate theorems: Theorem 4.1 (open acyclic manifolds), Theorem 4.2 (compact manifolds with nonzero Euler characteristic) and Theorem 5.1 (homology spheres).

We next make some remarks on the result for compact manifolds with nonzero Euler characteristic. While in the present paper we manage to prove Ghys conjecture for a wider class of compact manifolds than [10], the strategy we follow here does not allow us to extend the fixed point theorem in [10]. However, it does not seem unreasonable to expect that the latter should be true for all compact manifolds with nonzero Euler characteristic: in particular, we expect that for any such manifold $X$ there should exist a constant $C$ such that if a finite group acts smoothly on $X$ then there is a point $x \in X$ whose isotropy group satisfies $[G : G_x] \leq C$ (this was proved in dimension 4 in [11]); it seems also reasonable (but maybe a little bit bolder) to conjecture that the statement is true as well for actions of finite groups on open acyclic manifolds.

As for the method of proofs, both here and in [10] we use [12], and hence rely on the CFSG. However, the actual geometric result proved in the present paper is much weaker than that of [10]: while here we optimize the use of [12] and only consider actions of groups of the form $Q \ltimes P$, with $Q$ a $q$-group and $P$ a $p$-group ($p,q$ are primes), the geometric arguments in [10] prove the existence of abelian subgroups of arbitrary finite groups acting on the given manifold (with the inconvenient that the upper bound on the index depends on the number of different prime divisors of the cardinal of the group — and this is why we finally need to use [12]).

In Section 2 we recall the main result in [12] and prove a slight strengthening of it. In Section 3 we consider the situation of an action of a finite group on a manifold which fixes a submanifold on which the induced action is abelian, and prove a lemma which plays a
crucial role in our proofs. In Section 4 we treat the cases of open acyclic manifolds and compact manifolds with nonzero Euler characteristic, and finally in Section 5 we treat the case of homology spheres.

As usual in the theory of finite transformation groups (see e.g. [1]), in this paper \(\mathbb{Z}_p\) denotes \(\mathbb{Z}/p\mathbb{Z}\) (i.e., not the \(p\)-adic integers).

2. Testing Jordan’s property on \(\{p,q\}\)-groups

Suppose that \(\mathcal{C}\) is a set of finite groups. We denote by \(\mathcal{T}(\mathcal{C})\) the set of all \(T \in \mathcal{C}\) such that there exist primes \(p\) and \(q\), a Sylow \(p\)-subgroup \(P\) of \(T\) (which might be trivial), and a normal Sylow \(q\)-subgroup \(Q\) of \(T\), such that \(T = PQ\). (In particular, if \(T \in \mathcal{T}(\mathcal{C})\) then \(|T| = p^\alpha q^\beta\) for some primes \(p\) and \(q\) and some nonnegative integers \(\alpha, \beta\).)

Let \(C\) and \(d\) be positive integers. We say that a set of groups \(\mathcal{C}\) satisfies (the Jordan property) \(\mathcal{J}(C,d)\) if each \(G \in \mathcal{C}\) has an abelian subgroup \(A\) such that \([G:A] \leq C\) and \(A\) can be generated by at most \(d\) elements. For convenience, we will say that \(\mathcal{C}\) satisfies the Jordan property, without specifying any constants, whenever there exist some \(C\) and \(d\) such that \(\mathcal{C}\) satisfies \(\mathcal{J}(C,d)\).

The following is the main result in [12]:

**Theorem 2.1.** Let \(d\) and \(M\) be positive integers. Let \(\mathcal{C}\) be a set of finite groups which is closed under taking subgroups and such that \(\mathcal{T}(\mathcal{C})\) satisfies \(\mathcal{J}(M,d)\). Then there exists a positive integer \(C_0\) such that \(\mathcal{C}\) satisfies \(\mathcal{J}(C_0,d)\).

We next prove a refinement of this theorem. Given a set of finite groups \(\mathcal{C}\), we define \(\mathcal{A}(\mathcal{C})\) exactly like \(\mathcal{T}(\mathcal{C})\) but imposing additionally that the Sylow subgroups are abelian (in particular, \(\mathcal{A}(\mathcal{C}) \subset \mathcal{T}(\mathcal{C})\)). In concrete terms, a group \(G \in \mathcal{C}\) belongs to \(\mathcal{A}(\mathcal{C})\) if and only if there exist primes \(p\) and \(q\), an abelian Sylow \(p\)-subgroup \(P \subseteq G\) and a normal abelian Sylow \(q\)-subgroup \(Q \subseteq G\), such that \(G = PQ\). Denote also by \(\mathcal{P}(\mathcal{C})\) the set of groups \(G \in \mathcal{C}\) with the property that there exists a prime \(p\) such that \(G\) is a \(p\)-group.

If \(G\) is a (possibly infinite) group we denote by \(\mathcal{C}(G)\) the set of finite subgroups of \(G\) and we let \(\mathcal{A}(G) := \mathcal{A}(\mathcal{C}(G))\).

**Corollary 2.2.** Let \(d\) and \(M\) be positive integers. Let \(\mathcal{C}\) be a set of finite groups which is closed under taking subgroups and such that \(\mathcal{T}(\mathcal{C}) \cup \mathcal{A}(\mathcal{C})\) satisfies \(\mathcal{J}(M,d)\). Then there exists a positive integer \(C_0\) such that \(\mathcal{C}\) satisfies \(\mathcal{J}(C_0,d)\).

**Proof.** Let \(d\) and \(M\) be positive integers, suppose that \(\mathcal{C}\) is a set of finite groups which closed under taking subgroups, and assume that \(\mathcal{T}(\mathcal{C}) \cup \mathcal{A}(\mathcal{C})\) satisfies \(\mathcal{J}(M,d)\). Let \(C := M^2(M!)^d\). We claim that \(\mathcal{T}(\mathcal{C})\) satisfies \(\mathcal{J}(C,d)\). This immediately implies our result, in view of Theorem 2.1. To justify the claim it suffices to prove the following fact.

Let \(G\) be a finite group, let \(p, q\) be distinct prime numbers, let \(P \subseteq G\) be a \(p\)-Sylow subgroup, let \(Q \subseteq G\) be a normal \(q\)-Sylow subgroup, and assume that \(G = PQ\); if there exist abelian subgroups \(P_0 \subseteq P\) and \(Q_0 \subseteq Q\) such
that \([P : P_0] \leq M, [Q : Q_0] \leq M\), and \(Q_0\) can be generated by at most \(d\) elements, then there exists some \(G' \in \mathcal{A}(G)\) such that \([G : G'] \leq C\).

To prove this, define \(Q' := \bigcap_{\phi \in \text{Aut}(Q)} \phi(Q_0)\), where \(\text{Aut}(Q)\) denotes the group of automorphisms of \(Q\). Clearly \(Q'\) is an abelian characteristic subgroup of \(Q\), so it is normal in \(G\) (because \(Q\) is normal in \(G\)). Define \(G' := P_0Q'\). Then \(G' \in \mathcal{A}(G)\), so we only need to prove that \([G : G'] \leq C\).

Suppose that \(\{g_1, \ldots, g_d\}\) is a generating set of \(Q_0\), with \(\delta \leq d\), and such that if \(\Gamma_j \subseteq Q_0\) is the cyclic group generated by \(g_j\) then \(Q_0 \simeq \prod \Gamma_j\) (such generating set exists because \(Q_0\) is abelian and can be generated by at most \(d\) elements). If \(\Gamma \subseteq Q_0\) is a subgroup of index at most \(M\), then \(g^{Mj}_j\) belongs to \(\Gamma\) for each \(j\). Consequently, the subgroup \(Q'' \subseteq Q_0\) generated by \(\{g^{M!}_1, \ldots, g^{M!}_d\}\) is contained in any subgroup \(\Gamma \subseteq Q_0\) of index at most \(M\). In particular \(Q'' \subseteq Q'\), because for any \(\phi \in \text{Aut}(Q)\) we have \([Q_0 : Q_0 \cap \phi(Q_0)] \leq M\).

On the other hand, \([Q_0 : Q''] \leq (M!)^d \leq (M!)^d\), so a fortiori \([Q_0 : Q'] \leq (M!)^d\). Since \([G : G'] = [P : P_0][Q : Q'] = [P : P_0][Q : Q_0][Q_0 : Q']\), the result follows. \(\square\)

3. Actions of finite groups on real vector bundles

For any manifold \(Y\) we define \(\mathcal{A}(Y) := \mathcal{A}(\text{Diff}(Y))\). If \(E \to Y\) is a real vector bundle, we denote by \(\text{Diff}(E \to Y)\) the group of smooth bundle automorphisms lifting arbitrary diffeomorphisms of the base \(Y\) (equivalently, diffeomorphisms of \(E\) which map fibers to fibers and whose restriction to each fiber is a linear map). Let also \(\mathcal{A}(E \to Y) := \mathcal{A}(\text{Diff}(E \to Y))\). The natural morphism of groups \(\text{Diff}(E \to Y) \to \text{Diff}(Y)\) induces a map \(\pi : \mathcal{A}(E \to Y) \to \mathcal{A}(Y)\). Using this notation, we have the following.

**Lemma 3.1.** Assume that \(Y\) is connected and let \(r = \text{rk} E\) be the rank of \(E\). Any \(G \in \mathcal{A}(E \to Y)\) whose projection \(\pi(G) \in \mathcal{A}(Y)\) is abelian has an abelian subgroup \(A \subseteq G\) satisfying \([G : A] \leq r!\).

**Proof.** Take a group \(G \in \mathcal{A}(E \to Y)\) such that \(\pi(G)\) is abelian. There exist two distinct primes \(p\) and \(q\), a \(p\)-Sylow subgroup \(P \subseteq G\), and a normal \(q\)-Sylow subgroup \(Q \subseteq G\), such that \(G = PQ\). Furthermore, both \(P\) and \(Q\) are abelian. Let \(Q_0 \subseteq Q\) be the subgroup consisting of those elements whose action on \(E\) lifts the identity on \(Y\). Since \(Q\) is normal in \(G\), there is an action of \(P\) on \(Q\) given by conjugation. This action preserves \(Q_0\), and the induced action on \(Q/Q_0\) is trivial because \(\pi(G)\) is abelian. The complexified vector bundle \(E \otimes \mathbb{C}\) splits as a direct sum of subbundles indexed by the characters of \(Q_0\),

\[
E \otimes \mathbb{C} = \bigoplus_{\rho \in \text{Mor}(Q_0, \mathbb{C}^*)} E_\rho,
\]

where \(v \in E_\rho\) if and only if \(\eta \cdot v = \rho(\eta)v\) for any \(\eta \in Q_0\). The action of \(P\) on \(E \otimes \mathbb{C}\) permutes the summands \(\{E_\rho\}\). In concrete terms, if we define for any \(\rho \in \text{Mor}(Q_0, \mathbb{C}^*)\) and \(\gamma \in P\) the character \(\rho_\gamma \in \text{Mor}(Q_0, \mathbb{C}^*)\) by \(\rho_\gamma(\eta) = \rho(\gamma^{-1} \eta \gamma)\) for any \(\eta \in Q_0\), then we have \(\gamma \cdot E_\rho = E_{\rho_\gamma}\). Since there are at most \(r\) nonzero summands in (1) (this follows from \(\text{rk} E = r\) and the fact that \(Y\) is connected), the subgroup \(P_0 \subseteq P\) consisting of those
elements which preserve each nonzero subbundle $E_\rho$ satisfies $[P : P_0] \leq r!$. Furthermore, each element of $P_0$ commutes with all the elements in $Q_0$, because $P_0$ acts linearly on $E \otimes \mathbb{C}$ preserving the summands in (1) and the action of $Q_0$ on each summand is by homothecies. Hence, the action of $P_0$ on $Q$ given by conjugation gives a morphism

$$P_0 \to B := \{ \phi \in \text{Aut}(Q) \mid \phi(q) = q \text{ for each } q \in Q_0, \quad \phi(q)q^{-1} \in Q_0 \text{ for each } q \in Q \}.$$ 

A simple computation (using the fact that $Q$ is an abelian $q$-group) shows that $B$ is $q$-group. Since $P_0$ is a $p$-group and $p \neq q$, it follows that any morphism $P_0 \to B$ is trivial. In other words, $P_0$ commutes with $Q_0$. Setting $A := P_0Q$, the result follows. □

Lemma 3.2. Suppose that $X$ is a smooth connected manifold and that $G \in \mathcal{A}(X)$. Assume that there is a $G$-invariant connected submanifold $Y \subseteq X$. Let $G_Y \subseteq \text{Diff}(Y)$ be the group consisting of all diffeomorphisms of $Y$ which are induced by restricting to $Y$ the action of the elements of $G$. Let $r := \dim X - \dim Y$. If $G_Y$ is abelian, then there is an abelian subgroup $A \subseteq G$ satisfying $[G : A] \leq r!$.

Proof. There is an inclusion of vector bundles $TY \hookrightarrow TX|_Y$. Denote the quotient bundle by $E := TX|_Y/TY \to Y$ (this is the normal bundle of the inclusion $Y \hookrightarrow X$). Let $\text{Diff}(X,Y)$ be the group of diffeomorphisms of $X$ which preserve $Y$. There is a natural restriction map $\rho : \text{Diff}(X,Y) \to \text{Diff}(E \to Y)$ given by restricting the diffeomorphisms in $\text{Diff}(X,Y)$ to the first jet of the inclusion $Y \hookrightarrow X$, which gives a bundle automorphism $TX|_Y \to TX|_Y$ preserving $TY$, and then projecting to an automorphism of $E$. Furthermore, if $\Gamma \subseteq \text{Diff}(X,Y)$ is a finite group then $\rho|_\Gamma : \Gamma \to \rho(\Gamma)$ is injective. This follows from the general fact that if a smooth action of a finite group $\Gamma$ on a manifold $M$ fixes a point $x$, then the linearization at $x$ of the action defines an injective morphism $\Gamma \hookrightarrow \text{GL}(T_xM)$ — see e.g. [10, Lemma 2.1]. Applying this to the group $G \in \mathcal{A}(X)$ in the statement of the lemma we obtain a group $G_E := \rho(G) \in \mathcal{A}(E \to Y)$ which is isomorphic to $G$. Furthermore, the image $\pi(G_E) \in \mathcal{A}(Y)$ coincides with $G_Y$, which by hypothesis is abelian. We are thus in the setting of Lemma 3.1, so we deduce that $G_E$ (and hence $G$) has an abelian subgroup of index at most $r!$. □

4. Actions of finite groups on open acyclic manifolds and on compact manifolds with $\chi \neq 0$

Theorem 4.1. For any $n$ there is a constant $C$ such that any finite group acting smoothly and effectively on an open acyclic smooth $n$-dimensional manifold $X$ has an abelian subgroup of index at most $C$.

The proof that follows does not actually use the hypothesis that the manifold is open. We include this condition in the statement to stress the fact that the theorem is true for open manifolds. The case of compact acyclic manifolds (necessarily with boundary) is included in the statement of Theorem 4.2.
Proof. Fix \( n \) and let \( X \) be an open acyclic smooth \( n \)-dimensional manifold. Let \( \mathcal{C} \) be the set of all finite subgroups of \( \text{Diff}(X) \). By Corollary 2.2 it suffices to prove that \( \mathcal{P}(\mathcal{C}) \cup \mathcal{A}(\mathcal{C}) \) satisfies the Jordan property. We prove it first for \( \mathcal{P}(\mathcal{C}) \) and then for \( \mathcal{A}(\mathcal{C}) \).

Let \( G \in \mathcal{P}(\mathcal{C}) \) be a finite \( p \)-group acting on \( X \), where \( p \) is a prime. By Smith theory the fixed point set \( X^G \) is nonempty (see [1] Corollary III.4.6 for the case \( G = \mathbb{Z}_p \) and use induction for the general case). Let \( x \in X^G \). Linearizing the action of \( G \) at \( x \) we get an injective morphism \( G \hookrightarrow \text{GL}(T_xX) \approx \text{GL}(n, \mathbb{R}) \) (see [10] Lemma 2.1). It follows from Jordan’s theorem (see Theorem 5.3 below) that there is an abelian subgroup \( A \subseteq G \) such that \( [G : A] \leq C_n \), where \( C_n \) depends only on \( n \). Furthermore, since \( A \) can be identified with a subgroup of \( \text{GL}(n, \mathbb{R}) \), it can be generated by at most \( n \) elements. We have thus proved that \( \mathcal{P}(\mathcal{C}) \) satisfies \( \mathfrak{J}(C_n, n) \). The same argument also proves that any elementary \( p \)-group acting on \( X \) has rank at most \( n \).

Now let \( G \in \mathcal{A}(\mathcal{C}) \). By definition, there are two distinct primes \( p \) and \( q \), an abelian \( p \)-Sylow subgroup \( P \subseteq G \), and a normal abelian \( q \)-Sylow subgroup \( Q \subseteq G \), such that \( G = PQ \). Let \( Y := X^Q \). By Smith theory, \( Y \) is a \( \mathbb{Z}_p \)-acyclic manifold (see again [1] Corollary III.4.6 plus induction); in particular, \( Y \) is nonempty and connected. Since \( Q \) is normal in \( G \), the action of \( G \) on \( X \) preserves \( Y \). Finally, since the elements of \( Q \) act trivially on \( Y \), the action of \( G \) on \( Y \) given by restriction defines an abelian subgroup of \( \text{Diff}(Y) \). This means that we are in the setting of Lemma 3.2 and we can deduce that \( G \) has an abelian subgroup \( A \subseteq G \) of index at most \( n! \). Since, as explained in the previous paragraph, any elementary \( p \)-group acting on \( X \) has rank at most \( n \), \( A \) can be generated by at most \( n \) elements. We have thus proved that \( \mathcal{A}(\mathcal{C}) \) satisfies \( \mathfrak{J}(n!, n) \), and the proof of the theorem is now complete. \( \square \)

**Theorem 4.2.** Let \( X \) be a compact connected smooth manifold, possibly with boundary, and satisfying \( \chi(X) \neq 0 \). There exists a constant \( C \) such that any finite group acting smoothly and effectively on \( X \) has an abelian subgroup of index at most \( C \).

Proof. Let \( \mathcal{C} \) be the set of all finite subgroups of \( \text{Diff}(X) \). We will again deduce the theorem from Corollary 2.2 so we only need to prove that \( \mathcal{P}(\mathcal{C}) \cup \mathcal{A}(\mathcal{C}) \) satisfies the Jordan property.

To prove that \( \mathcal{P}(\mathcal{C}) \) satisfies the Jordan property, let \( s := q^r \) be the biggest prime power dividing \( \chi(X) \). Let \( p \) be any prime, and consider a \( p \)-group \( G \in \mathcal{P}(G) \). By [10] Lemma 2.5, there is a point \( x \in X \) whose stabiliser \( G_x \) satisfies \( [G : G_x] \leq p^r \), where \( p^r \) divides \( \chi(X) \). In particular, \( [G : G_x] \leq s \). By [10] Lemma 2.1 there is an inclusion \( G_x \hookrightarrow \text{GL}(T_xX) \approx \text{GL}(n, \mathbb{R}) \), where \( n = \dim X \). By the same argument as in the proof of Theorem 4.11 there is an abelian subgroup \( A \subseteq G_x \) of index \( [G_x : A] \leq C_n \) (with \( C_n \) depending only on \( n \)) and which can be generated by \( n \) elements. Hence, \( \mathcal{P}(\mathcal{C}) \) satisfies \( \mathfrak{J}(sC_n, n) \).

We now prove that \( \mathcal{A}(\mathcal{C}) \) satisfies the Jordan property. Let \( G \in \mathcal{A}(\mathcal{C}) \). By [10] Lemma 2.3 there exists a constant \( C_X \), depending on \( X \) but independent of \( G \), such that \( G \) has a subgroup \( G_0 \subseteq G \) of index at most \( C_X \) and whose action on \( H^*(X; \mathbb{Z}) \) is trivial. Clearly \( G_0 \in \mathcal{A}(\mathcal{C}) \), so there exist two distinct primes \( p \) and \( q \), an abelian \( p \)-Sylow subgroup
$P \subseteq G_0$, and a normal abelian $q$-Sylow subgroup $Q \subseteq G_0$, such that $G_0 = PQ$. By the arguments used before (involving [10, Lemma 2.5]) there is a point $x \in X$ whose stabiliser $Q_x$ satisfies $[Q : Q_x] \leq s$ for some positive integer $s$ depending only on $\chi(X)$. Since $Q_x$ is abelian and we have an inclusion $Q_x \hookrightarrow \text{GL}(T_x X)$, we know that $Q_x$ can be generated by at most $n$ elements. Define

$$Q' := \bigcap_{\phi \in \text{Aut}(Q)} \phi(Q_x).$$

By the arguments at the end of the proof of Corollary 2.2, we have $[Q_x : Q'] \leq (s!)^n$. Since $Q' \subseteq Q_x$, we have $x \in X^{Q'}$, so $X^{Q'}$ is nonempty. Also, $Q'$ does not contain any elementary $q$-group of rank greater than $n$ (because it is a subgroup of $G_x$), so it can be generated by at most $n$ elements. Since the action of $Q$ on $H^*(X; \mathbb{Z})$ is trivial, Corollary 3.2] gives the bound

$$\sum_j b_j(X^{Q'}; \mathbb{F}_q) \leq \sum_j b_j(X; \mathbb{F}_q) \leq K := \sum_j \max\{b_j(X; \mathbb{Z}_p) \mid p \text{ prime}\},$$

where $K$ is finite because $X$ is compact. In particular, $X^{Q'}$ has at most $K$ connected components. On the other hand, $Q'$ is a characteristic subgroup of $Q$, and since $Q$ is normal in $G_0$ it follows that $Q'$ is also normal in $G_0$. Hence the action of $P$ on $X$ preserves $X^{Q'}$. Since the latter has at most $K$ connected components, there exists a subgroup $P_0 \subseteq P$ of index $[P : P_0] \leq K$ and a connected component $Y \subseteq X^{Q'}$ such that $P_0$ preserves $Y$. There is also a subgroup $P' \subseteq P_0$ which fixes some point in $X$ and such that $[P_0 : P'] \leq s$ ([10, Lemma 2.5]) which implies as before that $P'$ can be generated by at most $n$ elements. Let $G' := P'Q'$. We can bound

$$[G_0 : G'] = [P : P_0][P_0 : P'][Q : Q_x][Q_x : Q'] \leq sKs(s!)^n.$$ 

On the other hand, $G'$ preserves $Y$, and its induced action on $Y$ is abelian (because $Q'$ acts trivially on $Y$ and $P'$ is abelian). By Lemma 3.2 there exists an abelian subgroup $G'' \subseteq G'$ satisfying $[G' : G''] \leq n!$. It follows that

$$[G : G''] = [G : G_0][G_0 : G'][G' : G''] \leq M := C_XsKs(s!)^nn!.$$ 

Since both $P'$ and $Q'$ can be generated by at most $n$ elements, $G'$ does not contain any elementary $p$-group or $q$-group of rank greater than $n$, which implies that $G''$ can be generated by $n$ elements. We have thus proved that $A(\mathcal{E})$ satisfies $\mathcal{J}(M, n)$, so the proof of the theorem is complete. \(\square\)

5. Actions on homology spheres

Recall some standard terminology: given a ring $R$ and an integer $n \geq 0$, an $R$-homology $n$-sphere is a topological $n$-manifold $M$ satisfying $H_*(M; R) \simeq H_*(S^n; R)$. A homology $n$-sphere is a $\mathbb{Z}$-homology $n$-sphere. By the universal coefficient theorem any homology $n$-sphere is a $\mathbb{Z}_p$-homology $n$-sphere for any prime $p$. Standard properties of topological manifolds imply that for any prime $p$ and any integer $n$ any $\mathbb{Z}_p$-homology $n$-sphere is compact and orientable.
Theorem 5.1. For any $n$ there is a constant $C$ such that any finite group acting smoothly and effectively on a smooth homology $n$-sphere has an abelian subgroup of index at most $C$.

Before proving the theorem we collect a few facts which will be used in our arguments. In what follows $p$ denotes an arbitrary prime.

Let $G$ be a finite $p$-group acting on a $\mathbb{Z}_p$-homology $n$-sphere $S$. Then the fixed point set $S^G$ is a $\mathbb{Z}_p$-homology $(n(G))$-sphere for some integer $-1 \leq n(G) \leq n$, where a $(-1)$-sphere is by convention the empty set (see [1, IV.4.3] for the case $G = \mathbb{Z}_p$; the general case follows by induction, see [1, IV.4.5]). Furthermore, if $S$ is smooth and the action of $G$ is smooth and nontrivial, then $S^G$ is a smooth proper submanifold of $S$ and $n(G) < n$.

If $G \cong (\mathbb{Z}_p)^r$ is an elementary abelian $p$-group acting on a $\mathbb{Z}_p$-homology $n$-sphere $S$, we know by what we have just said that for any subgroup $H \subseteq G$ the fixed point locus $S^H$ is a $\mathbb{Z}_p$-homology $(n(H))$-sphere for some integer $n(H) \leq n$. The Borel formula ([1, Theorem XIII.2.3]) is the following:

\[(2)\quad n - n(G) = \sum_{H \subseteq G \text{ subgroup}} (n(H) - n(G)).\]

Suppose that $G$ is a finite $p$-group acting smoothly on a smooth $\mathbb{Z}_p$-homology $n$-sphere $S$. For any subgroup $H \subseteq G$ let $n(H)$ be the integer such that $S^H$ is a $\mathbb{Z}_p$-homology $n(H)$-sphere (as before, if $S^H = \emptyset$ then $n(H) = -1$). Dotzel and Hamrick prove in [2] that there exists a real representation $\rho : G \to \text{GL}(V)$ such that $\dim_{\mathbb{R}} V^H = n(H) + 1$ for each subgroup $H \subseteq G$. This implies that $\dim_{\mathbb{R}} V = n + 1$ (take $H = \{1\}$) and that, if the action is effective, $\rho$ is injective (because for any nontrivial $H \subseteq G$ we have $n(H) < n$). Consequently, if $G$ acts effectively on $S$ then we can identify $G$ with a subgroup of $\text{GL}(n + 1, \mathbb{R})$ (which, when $S = S^n$, does not mean that the original action of $G$ on $S^n$ is necessarily linear!). In particular, if $G \cong (\mathbb{Z}_p)^r$ acts effectively on $S$, then $r \leq n + 1$ ($r$ can also be bounded using a general theorem of Mann and Su [7]).

Let $p$ be now an odd prime, let $d$ be a positive integer, and consider a morphism $\psi : \mathbb{Z}_d \to \text{Aut}(\mathbb{Z}_p)$. Suppose that the image under $\psi$ of a generator $g \in \mathbb{Z}_d$ is multiplication by some $y \in \mathbb{Z}_p^\times$ (here we use additive notation on $\mathbb{Z}_p$). The following is a very slight modification of a result of Guazzi and Zimmermann [5, Lemma 2]:

Lemma 5.2. If $\mathbb{Z}_p \times_{\psi} \mathbb{Z}_d$ acts effectively on a smooth $\mathbb{Z}_p$-homology $n$-sphere $S$ and the restriction of the action to $\mathbb{Z}_p$ is free then $y^{n+1} = 1 \in \mathbb{Z}_p^\times$.

The original result of Guazzi and Zimmermann does not require the restriction of the action to $\mathbb{Z}_p$ to be free, but it requires the action of $\mathbb{Z}_p \times_{\psi} \mathbb{Z}_d$ to be orientation preserving. The proof we give of Lemma 5.2 is essentially the same as [5, Lemma 2]; we give it to justify that the result is valid without assuming that $\mathbb{Z}_p \times_{\psi} \mathbb{Z}_d$ acts orientation-preservingly.
Proof. Since $S$ is smooth and compact, $p$ is odd, and the action of $\mathbb{Z}_p$ is free, $n$ must be odd, say $n = 2\nu + 1$. Let $\pi : S/\mathbb{Z}_p \to B\mathbb{Z}_p$ be the Borel construction and let $\zeta : S/\mathbb{Z}_p \to B\mathbb{Z}_p$ be the composition of the natural homotopy equivalence $S/\mathbb{Z}_p \simeq S/\mathbb{Z}_p$ (which exists because $\mathbb{Z}_p$ acts freely on $S$) with $\pi$. A simple argument using the Serre spectral sequence for $\zeta$ proves that the map $H^k(\zeta) : H^k(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^k(S/\mathbb{Z}_p; \mathbb{Z}_p)$ is an isomorphism for $0 \leq k \leq n$ (note that, since $p$ is odd, the action of $\mathbb{Z}_p$ on $S$ is orientation preserving, so the second page of the Serre spectral sequence for $\zeta$ has entries $H^u(B\mathbb{Z}_p; \mathbb{Z}_p) \otimes H^v(S/\mathbb{Z}_p)$). We have $H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \simeq \Lambda(\alpha) \otimes \mathbb{Z}_p[\beta]$ where $\deg \alpha = 1$, $\deg \beta = 2$, and $\beta = b(\alpha)$, where $b$ denotes the Bockstein. The action of $\mathbb{Z}_d$ on $\mathbb{Z}_p$ induces an action $\phi^* : \mathbb{Z}_d \to \text{Aut}(H^*(B\mathbb{Z}_p; \mathbb{Z}_p))$ satisfying $\phi^*(g)(\alpha) = y\alpha$, and by naturality and linearity of the Bockstein we also have $\phi^*(g)(\beta) = y\beta$. This implies that the action of $g$ on $H^n(B\mathbb{Z}_p; \mathbb{Z}_p) = \mathbb{Z}_p(\alpha/\beta)$ is multiplication by $y^{1+\nu}$. Since the map $\zeta$ is $\mathbb{Z}_d$-equivariant, the action of $g$ on $H^n(S/\mathbb{Z}_p; \mathbb{Z}_p)$ is also multiplication by $y^{1+\nu}$. This is the reduction mod $p$ of the action of $g$ on $H^n(S/\mathbb{Z}_p; \mathbb{Z}) \simeq \mathbb{Z}$ (the isomorphism follows from the fact that $S$ is compact and orientable, and that the action of $\mathbb{Z}_p$ is orientation preserving). Since the action of $g$ is by a diffeomorphism, it follows that $y^{1+\nu}$ is the reduction mod $p$ of $1$, so $y^{2(1+\nu)} = y^{n+1} = 1 \in \mathbb{Z}_p$. 

Finally, we mention a classic theorem of Jordan, on which Ghys’ conjecture is inspired:

**Theorem 5.3** (Jordan). For any $n \in \mathbb{N}$ there exists a constant $C_n$ such that if $G$ is a finite subgroup of $\text{GL}(n, \mathbb{R})$ then $G$ has an abelian subgroup $A \subseteq G$ of index at most $C_n$.

5.1. **Proof of Theorem [5.1]** Fix some $n \geq 1$, let $S$ be a smooth homology $n$-sphere, and let $\mathcal{C}$ be the set of finite subgroups of $\text{Diff}(S)$. We are going to use Corollary [2.2] so we need to prove that $\mathcal{P}(\mathcal{C}) \cup \mathcal{A}(\mathcal{C})$ satisfies the Jordan property. As in the proofs of the other two theorems of this paper, we treat separately $\mathcal{P}(\mathcal{C})$ and $\mathcal{A}(\mathcal{C})$.

If $G \in \mathcal{P}(\mathcal{C})$ then by the theorem of Dotzel and Hamrick [2] we may identify $G$ with a subgroup of $\text{GL}(n+1, \mathbb{R})$. By Theorem [5.3] there is an abelian subgroup $A \subseteq G$ of index bounded above by a constant $C_{n+1}$ depending on $n$ but not on $G$. Furthermore, $A$ can be generated by at most $n+1$ elements. Consequently, $\mathcal{P}(\mathcal{C})$ satisfies $\mathcal{J}(C_{n+1}, n+1)$.

The fact that $\mathcal{A}(\mathcal{C})$ satisfies the Jordan property is a consequence of the following lemma, combined with the existence of a uniform upper bound, for any prime $p$, on the rank of elementary $p$-groups acting effectively on $S^n$ (such bound follows, as we said, either from the theorem of Dotzel and Hamrick [2] or from that of Mann and Su [7]).

**Lemma 5.4.** Given two integers $m \geq 0, r \geq 1$ there exists an integer $K_{m,r} \geq 1$ such that for any two distinct primes $p$ and $q$, any abelian $p$-group $P$, any abelian $q$-group $Q$, any morphism $\phi : P \to \text{Aut}(Q)$, any smooth $\mathbb{Z}_p$-homology $m$-sphere $S$, and any smooth and effective action of $G := Q \rtimes_\phi P$ on $S$, the following holds. If the $p$-rank of $P$ is at most $r$, then there is an abelian subgroup $A \subseteq G$ satisfying $[G : A] \leq K_{m,r}$.

**Proof.** Fix some integer $r \geq 1$. We prove the lemma, for this fixed value of $r$, using induction on $m$. The case $m = 0$ being obvious, we may suppose that $m > 0$ and
assume that Lemma 5.4 is true for smaller values of $m$. Let $p, q, P, Q, \phi, G, S$ be as in the statement of the lemma (so in particular the $p$-rank of $P$ is at most $r$), and take a smooth effective action of $G$ on $S$.

Suppose first that $S^Q \neq \emptyset$. Then $S^Q$ is a smooth $\mathbb{Z}_q$-homology sphere of smaller dimension than $S$. Furthermore, since $Q$ is normal in $G$, the action of $G$ on $S$ preserves $S^Q$. Let $G_0 \subseteq \text{Diff}(S^Q)$ be the diffeomorphisms of $S^Q$ induced by restricting the action of the elements of $G$ on $S$ to $S^Q$. Then $G_0$ is a quotient of $G$, and this implies that $G_0 \simeq Q_0 \rtimes P_0$, where $P_0$ (resp. $Q_0$) is a quotient of $P$ (resp. $Q$). Hence we may apply the inductive hypothesis to the action of $G_0$ on $S^Q$ and deduce that there exists an abelian subgroup $A_0 \subseteq G_0$ satisfying $[G_0 : A_0] \leq K_{m-1,r}$. Let $\pi : G \to G_0$ be the quotient map (i.e., the restriction of the action to $A_0$). Then $[G : G'] \leq K_{m-1,r}$ and the action of $G'$ satisfies the hypothesis of Lemma 3.2. Hence, there is an abelian subgroup $A \subseteq G'$ satisfying $[G' : A] \leq m!$. By the previous estimates $A$ is an abelian subgroup of $G$ of index $[G : A] \leq m!K_{m-1,r}$.

Now assume that $S^Q = \emptyset$. Let $Q' := \{ \eta \in Q \mid \eta^q = 1 \} \subseteq Q$. Then $Q' \simeq (\mathbb{Z}_q)^l$. Since $Q'$ is a characteristic subgroup of $Q$ and $Q$ is normal in $G$, $Q'$ is normal in $G$. We distinguish two cases.

Suppose first that $l \geq 2$. The Borel formula (2) applied to the action of $Q'$ on $S$ gives
\[
m + 1 = \sum_{H \subseteq Q' \text{ subgroup} \atop [Q', H] = p} (n(H) + 1).
\]
All summands on the RHS are nonnegative integers. So at least one summand is strictly positive, and there are at most $m + 1$ strictly positive summands. Hence, the set
\[\mathcal{H} := \{ H \text{ subgroup of } Q' \mid S^H \neq \emptyset, [Q' : H] = q \}\]
is nonempty and has at most $m + 1$ elements. The action of $G$ on $Q'$ by conjugation permutes the elements of $\mathcal{H}$, so there is a subgroup $G' \subseteq G$ fixing some element $H \in \mathcal{H}$ and such that $[G : G'] \leq m + 1$. The fact that $G'$ fixes $H$ as an element of $\mathcal{H}$ means that $H$ is a normal subgroup of $G'$, so the action of $G'$ on $S$ preserves $S^H \neq \emptyset$. We now proceed along similar lines to the previous case. Let $G'' \subseteq \text{Diff}(S^H)$ be the diffeomorphisms of $S^H$ induced by restricting the action of the elements of $G'$ on $S$ to $S^H$. Then $G''$ is a quotient of $G'$ and $S^H$ is a smooth $\mathbb{Z}_q$-homology sphere of dimension strictly smaller than $m$; we may thus apply the inductive hypothesis and deduce the existence of an abelian subgroup $A' \subseteq G''$ of index at most $K_{m-1,r}$. Letting $G_a \subseteq G'$ be the preimage of $A'$ under the projection map $G' \to G''$ we apply Lemma 3.2 to the action of $G_a$ near the submanifold $S^H \subseteq S$ and conclude that $G_a$ has an abelian subgroup $A$ of index at most $m!$. Then $[G : A] \leq (m + 1)m!K_{m-1,r}$.

Finally, suppose that $l = 1$. In this case $Q'$ acts freely on $S$ and $Q \simeq (\mathbb{Z}_q)^r$. If $q = 2$ then $|\text{Aut}(\mathbb{Z}_q^r)| = (q - 1)q^{s-1}$ is equal to $2^{s-1}$ and since $p \neq 2$ the morphism $\phi : P \to \text{Aut}(Q)$ is trivial, which means that $G$ is abelian and there is nothing to prove. Assume then that $q$ is odd. Taking an isomorphism $P \simeq \mathbb{Z}_{p^1} \times \cdots \times \mathbb{Z}_{p^c}$ (with $c \leq r$, by our assumption on the $p$-rank of $P$) we may apply Lemma 5.2 to the restriction of $\phi$ to
each summand, $\phi|_{\mathbb{Z}_{q^e_i}} : \mathbb{Z}_{q^e_i} \to \text{Aut}(\mathbb{Z}_q)$ and conclude that there is a subgroup $\Gamma_i \subset \mathbb{Z}_{q^e_i}$ of index at most $m + 1$ such that $\phi(\Gamma_i)$ contains only the trivial automorphism of $\mathbb{Z}_q$, i.e., $\Gamma_i$ commutes with $Q' \simeq \mathbb{Z}_q$. Let $P_0 := \Gamma_1 \times \cdots \times \Gamma_c$. Then $[P : P_0] \leq (m + 1)^c$.

Finally, since, if we identify $\mathbb{Z}_q$ with the $q$-torsion of $\mathbb{Z}_q$, the group
\[
\{\alpha \in \text{Aut}(\mathbb{Z}_q) \mid \alpha(t) = t \text{ for every } t \in \mathbb{Z}_q\}
\]
is a $q$-group, it follows that $P_0$ not only commutes with $Q'$, but also with all the elements of $Q$. Consequently $G_0 := P_0Q$ is an abelian group, and we have
\[
[G : G_0] \leq (m + 1)^c \leq (m + 1)^r.
\]
This completes the proof of the induction step, and with it that of Lemma 5.4. □

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