On Truncations of the Exact Renormalization Group

Tim R. Morris

CERN TH-Division
CH-1211 Geneva 23
Switzerland

Abstract

We investigate the Exact Renormalization Group (ERG) description of ($Z_2$ invariant) one-component scalar field theory, in the approximation in which all momentum dependence is discarded in the effective vertices. In this context we show how one can perform a systematic search for non-perturbative continuum limits without making any assumption about the form of the lagrangian. The approximation is seen to be a good one, both qualitatively and quantitatively. We then consider the further approximation of truncating the lagrangian to polynomial in the field dependence. Concentrating on the non-perturbative three dimensional Wilson fixed point, we show that the sequence of truncations $n = 2, 3, \ldots$, obtained by expanding about the field $\varphi = 0$ and discarding all powers $\varphi^{2n+2}$ and higher, yields solutions that at first converge to the answer obtained without truncation, but then cease to further converge beyond a certain point. Within the sequence of truncations, no completely reliable method exists to reject the many spurious solutions that are also generated. These properties are explained in terms of the analytic behaviour of the untruncated solutions – which we describe in some detail.

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* On leave from Southampton University, U.K.
In ref.[3] we claimed that a sequence of truncations of the field dependence of
the ERG[1][2] do not work in general (in the sense of providing dependable and
in principle arbitrarily accurate results), because these do not converge beyond a
certain point, and because no completely reliable method exists to reject the many
spurious solutions that are also generated. In this letter we verify this claim for the
relatively simple $O(p^0)$ case described in the abstract. We compute the truncations
to high order ($n = 25$), and show how this behaviour may be understood, accurately
and at a deeper level, in terms of the analytic behaviour of the untruncated solutions
– which is described here in much more detail than was possible in ref.[3]. (We
must emphasise here the distinction between a momentum/derivative expansion in
which higher space-time derivative terms are discarded but no approximation is
made in the field dependence – these approximations do appear to converge[4][3] –
and truncations of the field dependence of the lagrangian which in general do not
converge and can give even qualitatively wrong results).

We point out that the analytic properties of the untruncated solutions allow for the
possibility of searching, within the momentum/derivative expansion [3]–[5], system-
atically through the infinite dimensional space of non-perturbative lagrangians for
new continuum limits. It need hardly be stated that, firstly very little is known
about the possible existence of continuum theories in such a space, and secondly, if
new fixed points were found, they could have profound implications. We have done
such a search for $O(N)$ scalar field theory in $D = 4$ and 3 dimensions, for the cases
$N = 1, 2, 3, 4$, in the $O(p^0)$ approximation. However, we found there to be only the
known fixed points at this level: Gaussian for $D = 3$ and 4, and the Wilson fixed
points in 3 dimensions.

Finally we construct two better methods of approximation by expansion. The
simplest way to solve the $O(p^0)$ equations, is directly numerically[3] however
(see also later, and ref.[10]). The truncations, if carefully interpreted, can give
moderately accurate results – thus the simpler low order truncations may be of some
use in situations where more reliable and accurate calculations are prohibitively
difficult to perform. This situation seems very reminiscent of approximations (also
involving truncations of the operator basis) to the “real space renormalization
group” investigated in the late 1970’s[12]. For recent work on approximations to the
ERG see for example refs.[3]–[11]; From Alford’s work[8], and the numerical solution, it is clear that expansion of $\varphi$ around the semi-classical minimum of the effective potential, results at low orders of truncation, in convergence to three decimal places or more for the $O(p^0)$ $\nu$ (or the other exponents related by scaling). It would be interesting to better understand the reliability of this method and its limiting accuracy (also for $\omega$). Its behaviour no doubt is otherwise similar to the truncations we discuss, and for the same underlying causes. Effectively this expansion was part of the calculation in ref.[9], where the authors claim to obtain quite accurate values for exponents of 3D $N$-vector models. At higher orders in the momentum expansion we expect that truncations become more limited in accuracy and reliability, if only because it ought now to involve the much more complicated calculation of the expansion of several functions[3] in polynomials, each with their own limitations in accuracy. Indeed, even in ref.[5], where we will consider truncations in the field dependence of higher momentum terms ($O(p^n)$ with $n > 0$) only, making no expansion in the potential, the results support this hypothesis.

Here we will concentrate on the case of sharp cutoff and $O(p^0)$, i.e. the remarkable equation[13] first studied without further approximation, by Hasenfratz and Hasenfratz[10]. We start with a simple alternative derivation. The equivalent smooth cutoff equation[3] is qualitatively very similar so that much the same analysis, and all our general conclusions, apply equally well to the smooth case. We work in $D$ euclidean dimensions with a single real scalar field $\varphi$. The partition function is defined as

$$\exp W_{\Lambda}[J] = \int D\varphi \exp\{-\frac{1}{2} \varphi . \Delta_{\Lambda}^{-1} . \varphi - S_{\Lambda_0}[\varphi] + J.\varphi\}.$$  (1)

$$\Delta_{\Lambda} \equiv \Delta(q, \Lambda) = \theta_{\varepsilon}(q, \Lambda)/q^2$$ is a free massless propagator times a smooth (everywhere positive) infrared cutoff function $\theta_{\varepsilon}(q, \Lambda)$. The sharp cutoff limit is given by the Heaviside function:

$$\theta_{\varepsilon}(q, \Lambda) \rightarrow \theta(q - \Lambda) \quad \text{as} \quad \varepsilon \rightarrow 0.$$  (2)

From (1) we derive the flow equation for $W_{\Lambda}$:

$$\frac{\partial}{\partial \Lambda} W_{\Lambda}[J] = -\frac{1}{2} \left\{ \frac{\delta W_{\Lambda}}{\delta J} \cdot \frac{\delta \Delta_{\Lambda}^{-1}}{\delta \Lambda} \cdot \frac{\delta W_{\Lambda}}{\delta J} + \text{tr} \left( \frac{\partial \Delta_{\Lambda}^{-1}}{\partial \Lambda} \cdot \frac{\delta^2 W_{\Lambda}}{\delta J \delta J} \right) \right\}.$$  (3)

\[1\] For more information on this, see ref[4].
which on rewriting in terms of the interaction part of the Legendre effective action via $\Gamma_{\Lambda}[\varphi] + \frac{1}{2} \varphi \Delta^{-1}_{\Lambda} \varphi = -W_{\Lambda}[J] + J \varphi$, $\varphi = \delta W_{\Lambda}/\delta J$, gives

$$\frac{\partial}{\partial \Lambda} \Gamma_{\Lambda}[\varphi] = -\frac{1}{2} \text{tr} \left[ \frac{1}{\Delta_{\Lambda}} \frac{\partial \Delta_{\Lambda}}{\partial \Lambda} \left( 1 + \Delta_{\Lambda} \frac{\delta^2 \Gamma_{\Lambda}}{\delta \varphi \delta \varphi} \right)^{-1} \right].$$

(3)

$\Gamma_{\Lambda}$ generates the one particle irreducible parts of the Wilson effective action (Wegner-Houghton effective action[2] in the sharp cutoff limit) [4]. If we now make the approximation of discarding all momentum dependence from $\Gamma_{\Lambda}$ in (3), we obtain

$$\frac{\partial}{\partial \Lambda} V(\varphi, \Lambda) = -\frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \left\{ \frac{\partial \theta_{\varepsilon}(q, \Lambda)}{\partial \Lambda} \theta_{\varepsilon}(q, \Lambda) \left[ 1 + \theta_{\varepsilon}(q, \Lambda) V''(\varphi, \Lambda)/q^2 \right] \right\},$$

(4)

where we have introduced the potential through $\Gamma_{\Lambda} = \int d^D x V(\varphi(x), \Lambda)$. Primes denote differentiation with respect to $\varphi$. The integrand (in curly brackets) in (4) may be written

$$\frac{\partial}{\partial \Lambda} \left\{ \ln \theta_{\varepsilon}(q, \Lambda) - \ln \left[ 1 + \theta_{\varepsilon}(q, \Lambda) V''(\varphi, \tilde{\Lambda})/q^2 \right] \right\} \bigg|_{\tilde{\Lambda}=\Lambda}.$$

The first term above yields a field independent vacuum energy which we drop from $V$. Taking the sharp cutoff limit (2) we thus obtain for the integrand

$$\delta(q - \Lambda) \ln (1 + V''(\varphi, \Lambda)/q^2).$$

The integral in (4) is now trivial. Factoring out the scale $\Lambda$ by writing $\varphi \mapsto \Lambda^{D/2-1} \varphi/\zeta$, $V(\zeta^{-1} \varphi \Lambda^{D/2-1}, \Lambda) \mapsto \zeta^{-2} \Lambda^{D/2} V(\varphi, t)$, and $t = \ln(\Lambda_0/\Lambda)$, where the factor $\zeta = (4\pi)^{D/4} \sqrt{\Gamma(D/2)}$ is chosen for convenience, we obtain the advertised equation[13][10]:

$$\frac{\partial}{\partial t} V(\varphi, t) + (D/2 - 1) \varphi V'(\varphi, t) - D V(\varphi, t) = \ln \left[ 1 + V''(\varphi, t) \right].$$

(5)

The anomalous dimension $\eta = 0$ in this case since a non-zero $\eta$ results from non-trivial wavefunction renormalization. The reader may be puzzled as to why we obtain the same equation as that for the Wilson effective potential[10] given that ours is the Legendre effective potential. In fact these are one and the same, since at zero momentum all vertices of the Wilson effective action $S_{\Lambda}$ coincide with $\Gamma_{\Lambda}$ [4].
Now we set $D = 3$. From (5), a fixed point effective potential $V(\varphi, t) \equiv V(\varphi)$ must satisfy
\begin{equation}
\frac{1}{2} \varphi V'(\varphi) - 3V(\varphi) = \ln [1 + V''(\varphi)] , \\
\text{and} \quad V'(0) = 0
\end{equation}
(by $\varphi \leftrightarrow -\varphi$ invariance). At first sight these equations appear to have many solutions, which may be parametrized by $V(0) = -\ln(1 + \sigma)/3$, $\sigma \Lambda^2$ being the semi-classical effective mass-squared. Actually this is not the case[10], because all but two solutions end at a singularity of the form $V(\varphi) \sim 2(1 - \varphi/\varphi_c) \ln(\varphi_c - \varphi)$, $\varphi_c$ some positive constant, or more precisely, as a series in decreasingly singular terms,
\begin{equation}
V = \ln(x) \left( x - \frac{3}{8} x^2 - \frac{25}{432} x^3 - \frac{5 \varphi_c^2}{384} x^4 + \frac{3169}{27648} x^4 + \cdots \right) \\
+ \ln(x)^2 \left( \frac{25}{288} x^3 - \frac{25}{1152} x^4 - \cdots \right) + O(\ln(x)^4 x^5) ,
\end{equation}
where $x = 1 - \varphi^2/\varphi_c^2$. We now justify this statement, by dividing the behaviour of $V(\varphi)$ into three classes.

First of all, if $V$ ends at a singularity, it, or some derivative of it diverges there. By considering how the various terms can balance in (6), one sees that a singularity must be of the form above. Secondly, if $V$ does not end at a singularity but instead it and its first two derivatives tend to a limit ($\pm \infty$ included) as $\varphi \to \infty$, then again, considering the balance of terms in (6) one sees that either $V(\varphi) \to 0$ or $V(\varphi)$ satisfies
\begin{equation}
V(\varphi) = A \varphi^6 - \frac{4}{3} \ln \varphi - \frac{2}{9} - \frac{1}{3} \ln(30A) - \frac{1}{150A \varphi^4} + O(1/\varphi^6) ,
\end{equation}
for some positive constant $A$, as $\varphi \to \infty$. Linearizing in (6) about the first possibility, i.e. setting $V(\varphi) \mapsto V(\varphi) + \delta V(\varphi)$, one finds that $\delta V$ must behave for large $\varphi$ as a linear combination of $\varphi^6$ and $\exp(\varphi^2/4)$. Since both of these corrections are excluded if we require also $V(\varphi) + \delta V(\varphi) \to 0$ as $\varphi \to \infty$, we conclude that $V(\varphi)$ is identically zero in this case, which indeed is the trivial Gaussian solution to (6). (For a study of perturbations about the Gaussian in eqn.(5), see ref.[10]). Linearizing about the second possibility, i.e. taking $V(\varphi)$ to be as in (8), one finds that $\delta V$ must behave for large $\varphi$ as a linear combination of $\varphi^6$ and $\exp(5A \varphi^6/2)$. Now the
first correction merely perturbs the coefficient $A$, while the second is excluded if $V + \delta V$ also satisfies (8). It follows that the space of solutions obeying (8) divide into isolated one-parameter subsets, each parametrized by $A$. For both possibilities the exponentially growing corrections were the result of linearizing the singular behaviour (7), since they involve balancing the same terms in (6). Thirdly, $V$ or one of its first two derivatives may be defined for all finite real $\varphi$ but not tend to a limit as $\varphi \to \infty$. Studying (6) one sees that this requires solutions to become infinitely oscillatory as $\varphi \to \infty$. We do not supply a proof that this does not happen, but we feel confident we can rule it out because we saw no hint of it in our extensive numerical and analytic investigations (see below).

Therefore, apart from the trivial solution $V(\varphi) \equiv 0$, any global fixed point solution of (6) must satisfy two boundary conditions: (8) and $V'(0) = 0$. We thus expect at most a countable number of such solutions; we find only one. It may be characterized by $\sigma = \sigma_* = -0.46153372 \cdots$ (or $A = A_* = .003033 \cdots$) and is displayed in fig.1. (We describe the search below). This is the same solution as in ref.[10] of course and is a fair approximation to the Wilson fixed point. To obtain the critical exponents and

Fig.1. The solution $V(\varphi)$ which approximates the Wilson fixed point.
operator spectrum one linearizes about this solution in (5), i.e. one writes \( V(\varphi, t) = V(\varphi) + \delta V(\varphi, t) \), with \( \delta V(\varphi, t) \propto v(\varphi) \exp(\lambda t) \). If one excludes the exponentially growing solution (which is \( \exp(5A\varphi^6/2) \) again) one obtains a discrete spectrum, and deduces from the eigenvalues the critical exponents, in reasonable agreement with the best estimates, as already described in ref.[3] (cf. also [10] and later). It is surely the case that the exponentially growing solution is again a linearization of singular behaviour, this time in the flowing solution \( V(\varphi, t) \).

Note that it is wrong to conclude from fig.1, e.g. by naïve semi-classical considerations, that the Wilson fixed point describes a spontaneously broken theory: indeed this is inconsistent with the fact that the field theory is scale invariant at this point. The remaining quantum corrections for momenta \( q < \Lambda \), which generically give positive contributions to the mass-squared, in this case exactly cancel the negative mass-squared \( \sigma_* \Lambda^2 \), and as \( \Lambda \to 0 \) one recovers the complete Legendre effective potential as \( V(\varphi) = A_* \varphi^6 \) (from (8) in unscaled units). Note also that it is not necessary to consider separately the question of the physical stability of solutions to (6). This is because there are no solutions that are unbounded from below (contrary to the statement in ref.[10]), while solutions that end on the singularity (7) are clearly physically unacceptable: the potential does not diverge at \( \varphi_c \), it simply ceases to exist – or is complex thereafter. (In the Gaussian case the stability is seen to hold automatically once perturbations about the fixed point are considered, as a consequence of the \( \ln \) in (5)).

The general structure of the solutions is as follows. For \( \sigma \) close but less than \( \sigma_* \), the solutions look very similar to fig.1 but end at some \( \varphi = \varphi_c \) in a singularity (7), as is most easily seen by plotting \( V'' \sim 2/\varphi_c/(\varphi_c - \varphi) \). As \( \sigma \) approaches \( \sigma_* \) from below, \( \varphi_c \) moves out along the real axis, but very slowly, so that for \( \varphi_c > 3 \) we require \( \sigma_* - \sigma < \Delta \) where \( \Delta \approx .005 \), while \( \sigma \) must approximate \( \sigma_* \) to high precision for say \( \varphi_c \gtrsim 4 \). For \( \sigma \) at the same proximity but above \( \sigma_* \) the singularity splits into a complex conjugate pair close to the real axis, with real positions \( \text{Re}(\varphi_c) \) in approximately the same place. These move closer to, and out along, the real axis as \( \sigma \to \sigma_*^+ \). The distance from the real axis is also a sensitive function of \( \sigma - \sigma_* \). (Perhaps the position of these singularities is given by \( \varphi_c \sim (\sigma_* - \sigma)^{-\tau} \), when \( \sigma \approx \sigma_* \), \( \sigma \lesssim \sigma_* \), for some small positive constant exponent \( \tau \)). On the real
axis, at $\varphi \approx \text{Re}(\varphi_c)$, $V''$ turns over steeply and drops rapidly to a value just greater than $-1$ and approximately constant over a large range. In this range $V$ slowly “rolls over” as

$$V(\varphi) = -\frac{1}{3} \ln \delta + c \varphi - \frac{1}{2} \varphi^2 + \delta \int_0^\varphi d\psi (\varphi - \psi) \exp\left\{\psi^2 - \frac{5}{2}c\psi\right\} + \cdots$$  \hspace{1cm} (9)

where $\varphi = c + O(\delta)$ is the point inside the range where the potential reaches a maximum, $\delta$ is small ($\delta \to 0$ as $\sigma \to \sigma^+$), and $|\varphi - c| \ll \sqrt{\ln(1/\delta)}$. Eventually, at some position $\varphi = \varphi_c'$ outside this range, $V$ encounters another singularity of the form (7).

For $\sigma = -1 + \delta$, $\delta \to 0^+$, the solution $V(\varphi)$ is governed by (9) with $c = 0$, ending in a singularity $\varphi_c' \sim \sqrt{\ln(1/\delta)}$. As $\sigma \to 0^-$, the singularities move out to infinity in such a way that $V(\varphi)$ tends pointwise to zero. For $\sigma > 0$, $V(\varphi)$ grows monotonically (with real increasing $\varphi$) ending in a singularity (7), which moves out slowly as $\sigma$ is increased.

Returning to the true solution, $V(\varphi)$ at $\sigma = \sigma_*$, we note that $V$ has a four-fold symmetry in the complex plane: complex conjugation $\times (\varphi \leftrightarrow -\varphi)$. Factoring out this symmetry, if one carefully integrates out along rays $\varphi = re^{i\vartheta}$, with $0 \leq \vartheta \leq \pi/2$, one can determine the position of the closest singularity. It appears at $r = r_* = 3.12$, $\vartheta = \vartheta_* = .257\pi$. (There are others with $r > r_*$).

We see that it is possible to make a systematic search for new continuum limits without making any assumption about the form of the bare potential. Indeed if such a potential can be tuned to a critical point where a continuum limit is recovered, then the corresponding Wilson effective potential must satisfy (6) at that point. On the other hand, if we find an acceptable solution of (6), then because such a solution is scale invariant, this equally well serves as the critical bare potential. Thus a search over the infinite dimensional space of bare potentials reduces to a one dimensional search for effective potentials obeying (6) that do not end in a singularity. In fig.2 we plot the inverse of the position of the real singularity against $\sigma$. The Gaussian and Wilson fixed points are clearly seen as sharp downward spikes, at $\sigma = 0$ and $\sigma = \sigma_*$ respectively. We have performed the equivalent search in $O(N)$ scalar field theory in $D = 3, 4$ dimensions for $N$ from 1 to 4, as mentioned in the introduction – finding only the expected fixed points. (The relevant equation for
general $N$ is given for example in ref.[10]). Of course the restriction to considering only general potentials is a result of our approximation; at higher orders in the momentum expansion a larger space of lagrangians can be searched.

**Fig.2.** The inverse of the position of the real singularity $\varphi_c$ as a function of $\sigma$, sampled with a stepsize $\delta\sigma = 1/120$. The downward spikes reach further towards zero with a finer mesh. The various features are explained in the description of the analytic structure given earlier.

(Note that equation (6) is stiff[14], the higher order equations[3] more severely so. To obtain an accurate representation of the solution at $\sigma = \sigma_*$ one can binary chop between the slow rollover behaviour (9) for $\sigma > \sigma_*$ and the singular behaviour (7) for $\sigma < \sigma_*$, but a more efficient method is to require (8) at some large value of $\varphi$. One can then either shoot to the origin\(^2\) – determining $A$ to zero $V'(0)$, or use relaxation.)

We can now give an intuitive explanation, based on the simplified context of (6), for why the truncations in fact converge at first but then cease to further converge beyond a certain maximum $n$. The point is that, as well as the true solutions,

\(^2\) For $N \neq 1$ one would need to shoot to an intermediate fitting point.
there are many ‘bad’ solutions with singular field dependence on the real axis. Very bad solutions have (real or complex) singular field dependence very close to the origin \( \varphi = 0 \), causing the coefficients of \( \varphi^m \) (\( m \)-point Green functions or vertices in general) to diverge rapidly with \( m \) according to the appropriate radius of convergence. Naturally, the polynomial field dependence of the truncations, for which the \( 2n + 2 \) vertex vanishes,\(^3\) tend therefore to better approximate the Taylor series of a true solution. Increasing \( n \) will tend at first to further improve the approximation, by in effect ensuring that the singularities are forced further from the origin. However a non-trivial true solution also has singularities for complex \( \varphi \) at (and in general beyond) some radius \(|\varphi| = r^*\). Therefore the truncations cannot be expected to converge to better results than would be obtained from ‘moderately bad’ solutions with singular field dependence only at or beyond the radius \( r^* \).

Now let us make this argument much more precise. If we wish to ensure that the potential \( V(\varphi) \) is an even function, then the Taylor expansion must be done about the origin. It is helpful to write

\[
V(\varphi) = -\frac{1}{3} \ln(1 + \sigma) + \frac{1}{2} \sigma \varphi^2 + 4 \sum_{k=2}^{\infty} \frac{\alpha_{2k}(\sigma)}{2k(2k-1)} \varphi^{2k} .
\] (10)

Plugging this into eqn.(6) we deduce \( \alpha_4 = -\frac{1}{4} \sigma(1 + \sigma) \), \( \alpha_6 = \frac{1}{48} \sigma(1 + \sigma)(1 + 7\sigma) \), \( \alpha_8 = -\frac{1}{48} \sigma^2(1 + \sigma)(1 + 3\sigma) \), \( \vdots \), and for \( k \geq 4 \) that \( \alpha_{2k} \) is given as \( \sigma^2 \) times \((1 + \sigma)\) times a polynomial in \( \sigma \) – as follows from \( D = 3 \) being an upper critical dimension\[7\], the slow rollover behaviour as \( \sigma \to -1 \), and the recurrence relation

\[
\frac{\alpha_{2k+2}}{1 + \sigma} = \frac{k - 3}{2k(2k-1)} \alpha_{2k} + \sum_{m=2}^{k} \frac{(-4)^{m-1}}{m} \sum_{k_1, \ldots, k_m \geq 1, k_1 + \cdots + k_m = k} \frac{\alpha_{2k_1+2} \cdots \alpha_{2k_m+2}}{(1 + \sigma)^m} ,
\]

respectively. The \( n^{\text{th}} \) truncation is defined by setting \( \alpha_{2n+2}(\sigma) = 0 \). We concentrate on large \( n \), where the solutions \( \sigma \) that result from this, can be understood from the asymptotic expressions for \( \alpha_{2k} \). To leading order in \( 1/k \), if the closest singularities (7) are on the real axis at \( \varphi_c = \pm r \) one has \( \alpha_{2k} \sim 1/r^{2k} \), while if they are complex then there are four in the form \( \varphi_c = \pm r \exp(\pm i\vartheta) \) and one has \( \alpha_{2k} \sim 2 \cos(2k\vartheta)/r^{2k} \).

(In fact the \( \alpha_{2k} \) asymptote to these expressions even for quite small \( k \); for example

\[^3\] For a discussion of truncations in general see for example ref[4].
one finds for $\sigma = \sigma_*$, that the asymptotic expressions give the right sign for $k > 2$ and are within a factor 2 for $k \geq 7$). Recalling the analytic behaviour of $V(\varphi)$ as a function of $\sigma \approx \sigma_*$, we see that for $\sigma < \sigma_* - \Delta$ the coefficients $\alpha_{2n+2}(\sigma)$ are all positive, so there are no solutions $\sigma$ in this region for large $n$. On the other hand for $\sigma = \sigma_*$ the coefficients are controlled by the singularities $\varphi_c = \pm r_* \exp(\pm i \vartheta_*)$ and thus the signs of the $\alpha_{2n+2}(\sigma_*)$ are very closely four-fold periodic in $n$ in the pattern $++--$, as a consequence of $\vartheta_* \approx \pi/4$. For the negative pair, it follows by continuity in $\sigma$ that $\alpha_{2n+2}(\sigma)$ must vanish for some $\sigma$ in the range $\sigma_* - \Delta < \sigma < \sigma_*$. For the positive pair, one has to look at $\sigma > \sigma_* + \Delta$; here the closest singularities are complex with an angle $\vartheta$ which is a rapidly increasing function of $\sigma$. This implies that there is always a zero of $\alpha_{2n+2}(\sigma)$ close to but greater than $\sigma_* + \Delta$. Together, these solutions $\sigma$ are the real ones that best approximate the Wilson fixed point. They are shown in fig.3 up to $n = 25$. (For the higher $n$ one must work to an accuracy of at least 20 significant figures to avoid round off errors). One clearly sees the four-fold periodicity with amplitude $\approx \Delta$, and that, as expected, the upper solutions are slightly worse approximations. Indeed the average of the last four solutions gives $\sigma = -0.4607$ which is greater than $\sigma_*$ by $\approx \Delta/6$.

The four-fold periodicity is transferred to the critical exponents, which may be computed by linearizing, in (5), as $\alpha_{2k}(t) = \alpha_{2k}(\sigma) + \varepsilon \beta_{2k} e^{\lambda t}$ (where we have used separation of variables). The neatest method of computing the eigenvalues $\lambda$ is by the linear recurrence relations between the $\beta_{2k}$, imposing $\beta_{2n+2} = 0$. All truncations, but $n = 23$, have one positive eigenvalue as required, which yields $[1] \nu = 1/\lambda$. The results are shown in fig.4. One clearly sees again the limiting periodic behaviour about the exact solution ($\nu = .6895$), this time with amplitude $\approx .008$. The exponent for the first correction to scaling is given by $\omega = -\lambda$ where $\lambda$ is the least negative eigenvalue. This depends much more sensitively on the approximations and is even complex for $n = 19, 22, 23$. It bounces about the exact answer ($\omega = .5952$) with an amplitude $\approx .15$. These truncations have been considered before, in refs.[6][7] to $n = 11, 8$ respectively, but without the deeper understanding it was possible at these orders to interpret the numerical results as indicating convergence. (The much worse behaviour for $O(N)$ scalar field theory with $N = 2, 3$ [7] is surely due to the exact solution having complex singularities much closer to the origin).
Fig.3. The solutions $\sigma$ that best approximate the exact answer ($\sigma = \sigma_*$, shown as a continuous horizontal line) for the truncations $n$ up to $n = 25$. The solutions not displayed ($n = 2, 3$) lie outside the range on the $\sigma$ axis. The dotted lines are $\sigma = \sigma_* \pm .005$. Recall that $\Delta \approx .005$.

Of course these approximate solutions are not the only solutions for $\sigma$. For $n$ too small the asymptotic pattern has not set in. Even for $n$ large, there are many ‘spurious’ solutions $\alpha_{2n+2} = 0$ in the range $\sigma_* < \sigma < 0$, which must be there by continuity arguments; away from the boundaries of this range, one finds that they slowly drift leftwards with increasing $n$ (asymptoting towards $\sigma = \sigma_*$), reflecting the fact that the closest singularities have an angle $\vartheta$ which is a slowly increasing function of $\sigma$ in this range. Looking only at the truncations, how can one tell that these solutions are spurious? There is surely no completely reliable method. There is no good reason to require the truncated potential to be bounded below and indeed the first cases to violate that criterion are truncations $n = 6, 7$ in figs.3,4; nor is it compelling to assume the solutions must be real: the approximations are bad (compared to their neighbours) for truncations 22 and 23, unless one chooses certain solutions with small imaginary parts ($\sigma = -.4572 + .0059i$ and $-.4566 + .0027i$ respectively – only the real parts are shown in figs.3,4). For $n = 23$ even
Fig. 4. The exponent $\nu$ for truncations up to $n = 25$. The exact answer is shown as a horizontal line.

the requirement that the approximation should have only one relevant direction, breaks down: there are no such solutions. The case we chose has $\lambda = 1.472 + .0253i$, which we use to compute $\nu = .6791 - .0117i$, and a less relevant direction with $\lambda = .509 + 1.695i$. The spurious solutions generally, but not always, have about the same number of positive eigenvalues $\lambda$ as negative eigenvalues: which is many for large $n$. The best one can do[6] to eliminate spurious solutions, is to look numerically for convergence/stability with increasing $n$. That this is not good enough is nicely demonstrated in ref.[6] where the slow drift of a sequence of spurious solutions $\sigma$, with two relevant eigenvalues, is mistaken for convergence to a tricritical point.

Finally we mention two better expansion methods. The first is the analytic equivalent of shooting to an intermediate fitting point, namely we require the Taylor expansion (10) and its derivative to agree with the asymptotic expansion (8) and its derivative, at some given intermediate point $\varphi_f$. In this way one obtains just one solution as expected, with $\sigma \approx \sigma_*$ and $A \approx A_*$. By comparing a pair of truncated Taylor expansions with a pair of truncated asymptotic expansions over a range of $\varphi_f$ (to, at the same time, estimate the error and determine the ‘best’ $\varphi_f$) one can
extract moderately accurate bounding values for $\sigma_*$, for example within a range of width .04 by using Taylor series to $\varphi^8$ and $\varphi^{10}$ and asymptotic series to $\varphi^{-10}$ and $\varphi^{-14}$. This method will be reliable providing the asymptotic series is accurate for $\varphi > a$ (when $A \approx A_*$) for some $a$ such that $a < r_*$, which is certainly case here. Presumably it works in general if there are no singularities in the region $a < \text{Re}(\varphi) < \infty$, except that it cannot give unlimited accuracy since the expansion (8) converges only in the Poincaré sense.

\[\text{Fig.5. Results for the exponent } \nu \text{ against } n, \text{ in an expansion method that utilises the fact that } \vartheta_* \approx \pi/4. \text{ The exact answer is shown as a horizontal line. Note the much finer vertical scale compared to fig.4.}\]

In the second method we assume that $\sigma_*$ has been determined to the accuracy required of the eigenvalues. Linearising about the fixed point position of the nearest singularity as $\vartheta = \vartheta_* + \varepsilon \psi e^{\lambda t}$ and $r = r_* + \varepsilon s e^{\lambda t}$, one obtains the asymptotic behaviour of $\beta_{2k}$:

$$
\beta_{2k} \sim -4k \left\{ s \cos(2k\vartheta_*)/r_*^{2k+1} + \psi \sin(2k\vartheta_*)/r_*^{2k} \right\}.
$$

Thus to leading order in $1/n$,

$$
(n - 1)\beta_{2n+2} \alpha_{2n-2}(\sigma_*) - (n + 1)\beta_{2n-2} \alpha_{2n+2}(\sigma_*) \sim -8(n^2 - 1)\psi \sin(4\vartheta_*)/r_*^{4n}.
$$
Since $\vartheta_* \approx \pi/4$ we can expect that it is a good approximation to choose $\lambda$ such that the left hand side vanishes. Doing so, we find a spectacular improvement in convergence, cf. fig.5, which nicely provides further confirmation for our theory. By looking at running averages of four points, and assuming no systematic shift from neglecting the right hand side of the above, we extract $\nu = .689457(8)$. Similarly we find $\omega = .5955(5)$.

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