THE SET OF NUMERICAL SEMIGROUPS OF A GIVEN MULTPLICITY AND FROBENIUS NUMBER

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Abstract. We study the structure of the family of numerical semigroups with fixed multiplicity and Frobenius number. We give an algorithmic method to compute all the semigroups in this family. As an application we compute the set of all numerical semigroups with given multiplicity and genus.

Introduction

Let $\mathbb{N}$ be the set of non-negative integers. A numerical semigroup is a submonoid $S$ of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ has finitely many elements; the cardinality of $\mathbb{N} \setminus S$ is the genus of $S$, denoted here $g(S)$. If $S$ is a numerical semigroup, its multiplicity, $m(S)$, is the smallest positive integer belonging to $S$. The largest integer not belonging to $S$ is the Frobenius number of $S$, denoted $F(S)$.

There are different methods for the computation of the set of numerical semigroups with a fixed Frobenius number (see [15, 2]). These methods admit no a priori filtration through multiplicities which makes it very expensive, in computing resources, to compute families of numerical semigroups with small multiplicity and relatively large Frobenius number. The quotient of the Frobenius number and the multiplicity is closely related to the depth of the semigroups (see Remark 5, and to have families with depth greater than two has interest in the context of the Bras’ or Wilf’s conjectures (see [5, 8, 9, 7]).

The main aim of this paper is to give an algorithmic method that, given two integers $m$ and $F$, computes the set of all numerical semigroups with Frobenius number $F$ and multiplicity $m$. We denote this set $L(m, F)$.

Following the same strategy as in [2], we define an equivalence relation $\sim$ on $L(m, F)$ such that each equivalence class contains one and

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only one irreducible numerical semigroup. Recall that a numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. We denote by $\mathcal{I}(m, F)$ the set of all irreducible numerical semigroups with multiplicity $m$ and Frobenius number $F$. Thus, our equivalence relation establish a bijection $\mathcal{L}(m, F)/\sim \cong \mathcal{I}(m, F)$ which allow us to compute $\mathcal{L}(m, F)$ by determining the equivalence class modulo $\sim$ of each semigroup in $\mathcal{I}(m, F)$. Proceeding in this way, we divide our main objective in two subtask, namely 1) compute $\mathcal{I}(m, F)$ and 2) determine the class modulo $\sim$ of an element of $\mathcal{I}(m, F)$. To this end, we have formulated Algorithms 22 and 35 respectively. And we have also include “non-polished” implementations in GAP [11] of our algorithms that takes advantage of the GAP package functionalities NumericalSgps [6]. Soon, more polished implementations of our algorithms will be included in the development version of NumericalSgps available at https://gap-packages.github.io/numericalsgps

This paper is organized as follows. In Section 1 we give the necessary and sufficient conditions on the integers $F$ and $m$ for the existence of numerical semigroups with Frobenius number $F$ and multiplicity $m$ (Proposition 3). In Section 2 we define the equivalence relation $\sim$ on $\mathcal{L}(m, F)$ mentioned above. The main result here is Theorem 7 which states that in each equivalence class modulo $\sim$ there is one and only one irreducible numerical semigroup. In Section 3 we study in depth the structure of $\mathcal{I}(m, F)$. We show that $\mathcal{I}(m, F)$ has an structure of rooted tree that allows us to formulate an algorithm for the computation of all the semigroups in $\mathcal{I}(m, F)$. The root of the tree is the semigroup $C(m, F)$ that we completely determine in Proposition 16, we emphasize here the important role that plays the ratio of a numerical semigroup (see Definition ref ratio). Of course, the main result in this section is Algorithm 22. However, we would like to draw attention to the interpretation of the Kunz coordinates of $\mathcal{I}(m, F)$ as solutions of a particular integer program (see (2)) which converts our algorithm of a solver to these problems. Once we have developed an algorithm for the computation of $\mathcal{I}(m, F)$, in Section 4, we perform the computation of the classes module $\sim$ which leads to Algorithm 35. Finally, in Section 5 we show that it is possible to adapt our algorithms for the computation of the set of numerical semigroups with a (suitable) given multiplicity and genus.

We close this paper with a remark that points out towards the computation of the set of numerical semigroups with a (suitable) given depth and genus which in our opinion will be valuable for the researchers dealing with Wilf’s and Bras’ conjectures.

1. Preliminaries

In this section we describe the conditions that $m$ and $F$ must satisfy for the existence of numerical semigroups with multiplicity $m$ and Frobenius
number $F$. But first, we need to introduce some notation and recall a couple of well-known results.

Let $A$ be a nonempty subset of $\mathbb{N}$. We write $\langle A \rangle$ for the submonoid of $(\mathbb{N}, +)$ generated by $A$, that is,

$$\langle A \rangle := \left\{ \sum_{i=1}^{n} \lambda_i a_i \mid n \in \mathbb{N} \setminus \{0\}, \ a_1, \ldots, a_n \in A, \text{and} \ \lambda_1, \ldots, \lambda_n \in \mathbb{N} \right\}.$$ 

If $M$ is a submonoid of $(\mathbb{N}, +)$ and $A$ is a subset of $M$ such that $M = \langle A \rangle$ then we say that $A$ is a system of generators of $M$. Moreover, if $M \neq \langle A' \rangle$ for all $A' \subsetneq A$, then we say that $A$ is a minimal system of generators of $M$.

**Lemma 1.** [17, Corollary 2.8] Let $M$ be a submonoid of $(\mathbb{N}, +)$. Then $M$ has a unique minimal system of generators, which in addition is finite.

Given a submonoid $M$ of $(\mathbb{N}, +)$, we denote by $\text{msg}(M)$ the minimal system of generators of $M$.

The following result follows from [17, Lemma 2.3].

**Lemma 2.** Let $S$ be a numerical semigroup and $x \in S$. Then $x \in \text{msg}(S)$ if and only if $S \setminus \{x\}$ is a numerical semigroup.

Given $\{x_1 \leq x_2 \leq \cdots \leq x_n\} \subseteq \mathbb{N}$ we denote by $\{x_1, x_2, \ldots, x_n, \rightarrow\}$ the set $\{x_1, x_2, \ldots, x_n\} \cup \{x \in \mathbb{N} \mid x > x_n\}$.

Let $a$ and $b$ be two integers, we say that $a$ divides $b$ if there exists an integer $c$ such that $b = ca$, in this case, we write $a|b$. Otherwise, we will write $a \nmid b$.

Let $m$ and $F$ be two integers and let $\mathcal{L}(m, F)$ be set of numerical semigroups with multiplicity $m$ and Frobenius number $F$. By definition, if $m \leq 0$ or $F \leq -1$ then $\mathcal{L}(m, F) = \emptyset$. Moreover, if $m = 1$, then $\mathcal{L}(m, F) \neq \emptyset$ if and only if $F = -1$. In this case, $\mathcal{L}(m, F) = \{\mathbb{N}\}$.

**Proposition 3.** Let $m$ and $F$ be two integers. If $(m, F) \neq (1, -1)$, then $\mathcal{L}(m, F) \neq \emptyset$ if and only if $F \geq m - 1 \geq 1$ and $m \nmid F$.

**Proof.** If $S \in \mathcal{L}(m, F)$ then $m - 1 \notin S$ and thus $F \geq m - 1 \geq 1$. Furthermore $F \notin S$ and $\langle m \rangle \subseteq S$. Hence $F \notin \langle m \rangle$ which implies that $m \nmid F$. Conversely, if $F \geq m - 1$ and $m \nmid F$, then $S = \langle m \rangle \cup \{F + 1, \rightarrow\}$ is a numerical semigroup of multiplicity $m$ and Frobenius number $F$, that is, $S \in \mathcal{L}(m, F)$. \qed

Notice that $\langle m \rangle \cup \{F + 1, \rightarrow\}$ is the (unique) minimal (with respect to the inclusion) element in $\mathcal{L}(m, F)$, that is to say, $\langle m \rangle \cup \{F + 1, \rightarrow\} \subseteq S$ for every $S \in \mathcal{L}(m, F)$.

**Proposition 4.** Let $m$ be an integer greater than or equal to two.

(a) $\mathcal{L}(m, m - 1) = \{\{0, m \rightarrow\}\}

(b) If $m < F < 2m$, then $\mathcal{L}(m, F) = \{\{0, m\} \cup A \cup \{F + 1, \rightarrow\} \mid A \subseteq \{m + 1, \ldots, F - 1\}\}.$
(c) If \( L(2, F) \neq \emptyset \), then \( F \) is odd and furthermore \( L(2, F) = \{ \langle 2, F + 2 \rangle \} \).

Proof. (a) If \( S \) is a numerical semigroup with multiplicity \( m \) and Frobenius number \( F = m - 1 \), then \( S \subseteq \{ 0, m, \rightarrow \} \). Now, since \( \langle m \rangle \cup \{ F + 1, \rightarrow \} = \{ 0, m, \rightarrow \} \), our claim follows.

(b) If \( m < F < 2m \), by [4, Proposition 6(b)], one has that the numerical semigroup \( \{ 0, m \} \cup \{ F \} \) is the (unique) maximal (with respect to the inclusion) element of \( L(m, F) \). Thus, \( \langle m \rangle \cup \{ F + 1, \rightarrow \} \subseteq S \subseteq \{ 0, m \} \), for every \( S \in L(m, F) \). Therefore, \( \langle m \rangle \cup \{ F + 1, \rightarrow \} \subseteq S \subseteq \{ 0, m \} \), for every \( S \in L(m, F) \). Moreover, given \( S \in L(2, F) \) and an odd integer \( x \in S \), it follows that \( x + N \subseteq S \) thus \( x \geq F + 2 \) and we are done.

(c) If \( L(2, F) \neq \emptyset \), then \( F \) cannot be a multiple of 2, that is, \( F \) is odd.

Remark 5. Let \( S \) be a numerical semigroup with multiplicity \( m \) and Frobenius number \( F \), and write \( F + 1 = qm - r \) for some integers \( q \) and \( r \) with \( 0 \leq r < m \). The integer \( q \) is called the depth of \( S \) (see [9]). Observe that \( S \) has depth 2 if and only \( m < F < 2m \). Therefore, Proposition 4(b) can be understood as a characterization of the numerical semigroups with depth 2.

Depth equal to two has a particular relevance, since Bras’ conjecture holds in the restricted class of numerical semigroups having this depth [9]. On the other hand, in [8] it is shown that Wilf’s conjecture is true for depth lesser than or equal to three. These facts make valuable to obtain families of numerical semigroups with high depth.

2. A partition of \( L(m, F) \)

The goal of this section is to give a partition \( L(m, F) \) which will lead to the structure of an algorithmic procedure to compute \( L(m, F) \). As a consequence of Proposition 4 we concentrate on the case \( m \geq 3 \) and \( F > 2m \), that is to say, we will study the families with depth and multiplicity greater than or equal to three.

Given a numerical semigroup \( S \), we will write

\[ \theta(S) = \{ s \in S \mid m(S) < s < \frac{F(S)}{2} \}, \]

and let \( \sim \) be relation on \( L(m, F) \) by defined by

\[ S \sim S' \text{ if and only if } \theta(S) = \theta(S'); \]

clearly \( \sim \) is an equivalence relation on \( L(m, F) \). Let \( [S] \) denote the class of \( S \in L(m, F) \) modulo \( \sim \), that is, \( [S] := \{ S' \in L(m, F) \mid S \sim S' \} \).

Lemma 6. The class of \( S \in L(m, F) \) modulo \( \sim \) is closed under union and intersection of its elements. That is to say, \( [S] \) has a lattice structure given by the inclusion.
Proof. Assume that \( \{S_1, S_2\} \subseteq [S] \) and let us prove that \( \{S_1 \cap S_2, S_1 \cup S_2\} \subseteq [S] \). Clearly, \( S_1 \cap S_2 \in \mathcal{L}(m, F) \). Moreover, \( \theta(S_1) = \theta(S_2) = \theta(S) \) and thus \( \theta(S_1 \cap S_2) = \theta(S) \). Hence \( S_1 \cap S_2 \in [S] \). Now, we claim that \( S_1 \cup S_2 \) is a semigroup. Indeed, if \( x \in S_1 \setminus S_2 \) and \( y \in S_2 \setminus S_1 \), then \( x \) and \( y \) are greater than \( F/2 \), because \( \theta(S_1) = \theta(S_2) \). Therefore \( x + y > F \) and, consequently, \( x + y \in S_1 \cup S_2 \). Hence we have \( S_1 \cup S_2 \in \mathcal{L}(m, F) \). Finally, since \( \theta(S_1 \cup S_2) = \theta(S_1) = \theta(S_2) = \theta(S) \) we conclude that \( S_1 \cup S_2 \in [S] \). □

Observe that
\[
Z([S]) = \bigcap_{S' \in [S]} S' \quad \text{and} \quad U([S]) = \bigcup_{S' \in [S]} S'
\]
are the minimum and the maximum (with respect to the inclusion) of \( [S] \), respectively. In particular, both belong to \( \mathcal{L}(m, F) \) by Lemma 6.

One of the keys to this work is the following result which shows that \( U([S]) \) is the unique irreducible numerical semigroup belonging to \( [S] \).

**Theorem 7.** Let \( m \) and \( F \) be positive integers such that \( m \geq 3 \), \( F > 2m \) and \( m \mid F \). If \( S \in \mathcal{L}(m, F) \), then \( [S] \cap \mathcal{J}(m, F) = \{U([S])\} \). Moreover \( U([S]) = S \cup \{x \in \mathbb{N} \setminus S \mid F - x \notin S \text{ and } x > \frac{F}{2}\} \).

For the sake of completeness, we include the following result which will be used both in the proof of Theorem 7 and in the following sections.

**Lemma 8.** Let \( S \) be a numerical semigroup.

(a) \( S \) is irreducible if and only if \( S \) is maximal among all numerical semigroups with Frobenius number \( F(S) \).

(b) If \( h = \max\{x \in \mathbb{N} \setminus S \mid F(S) - x \notin S \text{ and } x \neq \frac{F(S)}{2}\} \), then \( S \cup \{h\} \) is a numerical semigroup with \( F(S \cup \{h\}) = F(S) \).

(c) \( S \) is irreducible if and only if \( \{x \in \mathbb{N} \setminus S \mid F(S) - x \notin S \text{ and } x \neq \frac{F(S)}{2}\} \) is the empty set.

(d) \( S \cup \{x \in \mathbb{N} \setminus S \mid F(S) - x \notin S \text{ and } x > \frac{F(S)}{2}\} \) is an irreducible numerical semigroup.

**Proof.** It is an immediate consequence of Lemmas 4 and 5. □

**Proof of Theorem 7.** First, let us see that \( U([S]) \in \mathcal{J}(m, F) \). If \( U([S]) \) is not irreducible then, by using Lemma 8, we have that there exists \( h \in \mathbb{N} \setminus S \) such that \( \frac{F}{2} < h < F \) and \( U([S]) \cup \{h\} \in \mathcal{L}(m, F) \). Besides, as \( \theta(U([S]) \cup \{h\}) = \theta(U([S])) = \theta(S) \), we have that \( U([S]) \cup \{h\} \in [S] \), in contradiction with the maximality of \( U([S]) \).

Now, we prove that if \( S' \in [S] \cap \mathcal{J}(m, F) \) then \( S' = U([S]) \). In fact, as \( S' \in [S] \) then \( S' \subseteq U([S]) \). Since both \( S' \) and \( U([S]) \) belong to \( \mathcal{J}(m, F) \), from Lemma 8(a), we conclude that \( S' = U([S]) \).

Finally, by applying Lemma 8(d), we get that \( S \cup \{x \in \mathbb{N} \setminus S \mid F - x \notin S \text{ and } x > \frac{F}{2}\} \in \mathcal{J}(m, F) \cap [S] \). Therefore \( S \cup \{x \in \mathbb{N} \setminus S \mid F - x \notin S \text{ and } x > \frac{F}{2}\} = U([S]) \). □
The following results are immediate consequences of Theorem 7.

Corollary 9. The map \( \varphi : \mathcal{L}(m, F) \rightarrow \mathcal{I}(m, F) \) such that \( S \mapsto U([S]) \) is surjective and \( \varphi(S) = \varphi(S') \) if and only if \( \theta(S) = \theta(S') \). In particular, \( \mathcal{L}(m, F)/\sim = \mathcal{I}(m, F) \).

Corollary 10. Let \( m \) and \( F \) be positive integers such that \( m \geq 3 \), \( F > 2m \) and \( m \nmid F \). Then \( \mathcal{L}(m, F) = \bigsqcup_{S \in \mathcal{I}(m, F)} [S] \), where \( \bigsqcup \) means disjoint union.

Remark 11. In view of Corollary 10, in order to determine explicitly the elements in the set \( \mathcal{L}(m, F) \) we will need
1) an algorithm to compute the set \( \mathcal{I}(m, F) \)
2) an algorithm to compute the set \([S]\), for each \( S \in \mathcal{I}(m, F) \).

These algorithms will be developed in Sections 3 and 4, respectively.

3. An algorithm for the computation of \( \mathcal{I}(m, F) \)

Our main goal in this section is to describe all the elements in \( \mathcal{I}(m, F) \).
To do that, we first describe the conditions that \( m \) and \( F \) must verify such that there exists at least one irreducible numerical semigroup with multiplicity \( m \) and Frobenius number \( F \).

Lemma 12. [4, Lemma 8]. Let \( S \) be a numerical semigroup. Then \( S \) is irreducible if and only if \( g(S) = \left\lceil \frac{F(S)+1}{2} \right\rceil \), being \( \lceil - \rceil \) the ceiling operator.

Proposition 13. Let \( m \) and \( F \) be positive integers such that \( F \geq 3 \). Then \( \mathcal{I}(m, F) \neq \emptyset \) if and only if \( m \leq \frac{F+2}{2} \) and \( m \nmid F \).

Proof. Let \( S \in \mathcal{I}(m, F) \). If \( m\mid F \) then we have \( F \in \langle m \rangle \subseteq S \), which is impossible. As \( \{1, \ldots, m-1\} \cup \{F\} \in \mathbb{N} \setminus S \) it follows that \( m \leq g(S) \), from Lemma 12 we get that \( m \leq \frac{F+2}{2} \). Conversely, it is clear that \( S = \langle m \rangle \cup \{F+1, \rightarrow\} \) belongs to \( \mathcal{L}(m, F) \). By using Lemma 8(d) we obtain that \( \overline{S} = S \cup \{x \in \mathbb{N} \setminus S \mid F - x \not\in S \text{ and } x > \frac{F}{2}\} \) is in \( \mathcal{I}(m, F) \). \( \square \)

It is well known (see for instance [14]) that the class of irreducible numerical semigroups is the disjoint union of the two sub-classes of particular interest, which are called: symmetric and pseudo-symmetric numerical semigroups (see [12, 10, 1]). There are several characterizations for these class of numerical semigroups. The next result is one of many and it will be used extensively in what follows.

Lemma 14. [17, Proposition 4.4]. Let \( S \) be a numerical semigroup.
(a) \( S \) is symmetric if and only if \( F(S) \) is odd and \( x \in \mathbb{N} \setminus S \) implies \( F(S) - x \in S \).
(b) \( S \) is pseudo-symmetric if and only if \( F(S) \) is even and \( x \in \mathbb{N} \setminus S \) implies that either \( F(S) - x \in S \) or \( x = \frac{F(S)}{2} \).
Note that a numerical semigroup is symmetric (respectively pseudo-symmetric) if it is irreducible with Frobenius number odd (respectively even).

**Definition 15.** Given a numerical semigroup $S$, the smallest element in $S \setminus \langle m(S) \rangle$ is called the ratio of $S$ and will be denoted by $r(S)$.

Notice that if $S$ is a numerical semigroup and $\text{msg}(S) = \{n_1 < n_2 < \cdots < n_p\}$, then we have $n_1 = m(S)$ and $r(S) = n_2$.

**Proposition 16.** Let $m$ and $F$ be positive integers. If $F \geq 3$, $m \leq \frac{F+2}{2}$ and $m \nmid F$, then there exists a unique irreducible numerical semigroup $C(m, F)$ with Frobenius number $F$, multiplicity $m$ and ratio greater than $\frac{F}{2}$. Moreover,

(a) If $F$ is odd and $1 \leq r \leq m$ the smallest integer such that $\frac{F+1}{2} + x \pmod{m}$ is congruent to $r$ modulo $m$, then $C(m, F)$ is a numerical semigroup with minimal system of generators $\{m\} \cup \left\{\frac{F+1}{2} + x \mid x \in \{0, \ldots, m-1\} \setminus (m-r, r-1)\right\}$.

(b) If $F$ is even and $1 \leq r \leq m$ the smallest integer such that $\frac{F+1}{2}$ is congruent to $r$ modulo $m$, then $C(m, F)$ is a numerical semigroup with minimal system of generators $\{3, F/2 + 3, F + 3\}$, if $m = 3$, or $\{m\} \cup \left\{\frac{F+2}{2} + x \mid x \in \{0, \ldots, m-1\} \setminus (m-r, r-2)\right\}$, if $m \neq 3$.

**Proof.** Let $S$ be an irreducible numerical semigroup such that $F(S) = F$, $m(S) = m$ and $r(S) > \frac{F}{2}$.

If $F$ is odd. It is clear that $\left\{\frac{F+1}{2}, \ldots, \frac{F+1}{2} - m\right\} \setminus \left\{\frac{F+1}{2} - r\right\} \subseteq \mathbb{N} \setminus S$ and by applying (a) in Lemma [14] we deduce that $\left\{\frac{F+1}{2}, \ldots, \frac{F+1}{2} + m - 1\right\} \setminus \left\{\frac{F+1}{2} + r - 1\right\} \subseteq S$. Consequently, $S$ contains $C(m, F)$. In order to prove the equality, it suffices to show that both have the same genus and, by Lemma [12] it is enough to prove that $C(m, F)$ is irreducible. Since $F$ is odd, this is the same as $C(m, F)$ to be a symmetric numerical semigroup. If $x > \frac{F}{2}$ and $x \notin C(m, F)$ and considering the set of system of generators of $C(m, F)$ we deduce that $x = \frac{F+1}{2} + r - 1 + km$ for some $k \in \mathbb{N}$ and thus $F - x = \frac{F+1}{2} - r - km$. Since $\frac{F+1}{2}$ is $r$ modulo $m$, we obtain $F - x \in \langle m \rangle \subseteq C(m, F)$. From Lemma [14] we can guarantee that $C(m, F)$ is a symmetric numerical semigroup.

Suppose now that $F$ is even. Clearly $C(3, F)$ is the numerical semigroup with minimal system of generators $\{3, F/2 + 3, F + 3\}$. So, we assume $m > 3$.

Then $\left\{\frac{F}{2}, \ldots, \frac{F}{2} - m\right\} \setminus \left\{\frac{F}{2} - (r-1)\right\} \subseteq \mathbb{N} \setminus S$. By using (b) in Lemma [14] we get that $\left\{\frac{F+2}{2}, \ldots, \frac{F+2}{2} + m - 1\right\} \setminus \left\{\frac{F+2}{2} + r - 2\right\} \subseteq S$ and thus $S$ contains $C(m, F)$. To conclude the proof, we need to prove that $C(m, F)$ is irreducible, as in the previous case. To see this, it suffices to check that $C(m, F)$ is a pseudo-symmetric numerical semigroup. If $x > \frac{F}{2}$ and $x \notin C(m, F)$ we get that $x = \frac{F}{2} + r - 1 + km$ for some $k \in \mathbb{N}$ and so $F - x = \frac{F+2}{2} - r - km$ is a multiple of $m$. Consequently, $F - x \in \langle m \rangle \subseteq C(m, F)$. By applying Lemma [14](b), we have that $C(m, F)$ is a pseudo-symmetric numerical semigroup. \qed
The following result is a sufficient condition for a numerical semigroup to be irreducible.

**Proposition 17.** [3 Proposition 2.5]. Let $S$ be an irreducible numerical semigroup with Frobenius number $F$. If $x \in \text{msg}(S)$ verifies $x < F$, $2x - F \notin S$, $3x \neq 2F$ and $4x \neq 3F$, then $(S \setminus \{x\}) \cup \{F - x\}$ is an irreducible numerical semigroup with Frobenius number $F$.

**Corollary 18.** If $S$ is an irreducible numerical semigroup such that $r(S) < F(S)/2$, then

$$
\overline{S} = (S \setminus \{r(S)\}) \cup \{F - r(S)\}
$$

is an irreducible numerical semigroup with $F(\overline{S}) = F(S)$ and $r(\overline{S}) > r(S)$.

**Proof.** By hypothesis, $r(S) < 2r(S) < F(S)$, $3r(S) \neq 2F$ and $4r(S) \neq 3F$. In particular, $2r(S) - F(S) < 0$. So $2r(S) - F \notin S$ and, by Proposition 17 we can conclude that $\overline{S}$ is an irreducible numerical semigroup with $F(\overline{S}) = F(S)$. Furthermore, we have $r(\overline{S}) > r(S)$ due to $F(S) - r(S) > r(S)$. \(\square\)

Consider the following binary relation on $\mathcal{S}(m, F)$: $T \preceq S$ if and only if $T = S$ or $r(T) < F/2$ and $S = (T \setminus \{r(T)\}) \cup \{F - r(T)\}$.

**Proposition 19.** For each $S \in \mathcal{S}(m, F)$ there exists a subset $\{S_0, \ldots, S_n\}$ of $\mathcal{S}(m, F)$, such that $S = S_0 < \ldots < S_{n-1} < S_n = C(m, F)$.

**Proof.** If $r(S) > F/2$, then $S = S_0 = C(m, F)$ by Proposition 16. Otherwise, by Corollary 18 the exists $S_1 \in \mathcal{S}(m, F)$ such that $S = S_0 < S_1$ and $r(S_1) > r(S)$. By repeating this argument with $S_1$, we will obtain either $S_1 = C(m, F)$ or $S_2 \succ S_1$ with $r(S_2) > r(S_1)$. Since this process cannot continue indefinitely, our claim follows. \(\square\)

**Example 20.** Let $S = \langle 6, 8, 9 \rangle = \{0, 6, 8, 9, 12, 14, 15, 16, 17, 18, 20, -\} \}$ is an irreducible numerical semigroup with $m(S) = 6$ and $F(S) = 19$.

$$
S_1 = (S_0 \setminus \{8\}) \cup \{11\} = \langle 6, 9, 11, 14, 16 \rangle,
S_2 = (S_1 \setminus \{9\}) \cup \{10\} = \langle 6, 10, 11, 14, 15 \rangle = C(6, 19).
$$

Clearly, $S < S_1 < S_2 = C(6, 19)$.

**Theorem 21.** Let $m$ and $F$ be positive integers. If $F \geq 3$, $m \leq \frac{F+2}{2}$ and $m \nmid F$, then set of elements $T$ with $T < S$ in $\mathcal{S}(m, F)$, is equal to

$$
\{ \langle S \setminus \{x\} \rangle \cup \{F - x\} \mid x \in \alpha(S) \}
$$

where

$$
\alpha(S) := \left\{ x \in \text{msg}(S) \left| \begin{array}{l}
\frac{F}{2} < x < F , \ 2x - F \notin S \\
3x \neq 2F , \ 4x \neq 3F \\
m(S) < F - x < r(S)
\end{array} \right. \right\}.
$$

**Proof.** Let $S \in \mathcal{S}(m, F)$ and $x \in \text{msg}(S)$ such that $\frac{F}{2} < x < F$, $2x - F \notin S$, $3x \neq 2F$, $4x \neq 3F$ and $m(S) < F - x < r(S)$. By Proposition 17 we know that $T = (S \setminus \{x\}) \cup \{F - x\}$ is an irreducible numerical semigroup with Frobenius number $F$. Furthermore, since $m(S) < F - x < r(S)$ we obtain
\( m(T) = m \) and \( r(T) = F - x < F/2 \). Hence, we have \( S = (T \setminus \{r(T)\}) \cup \{F - r(T)\} \) and so \( T \prec S \).

Now, let \( S \) and \( T \in \mathcal{I}(m, F) \) such that \( T \prec S \). Since \( r(T) < \frac{F}{3} \) and \( S = (T \setminus \{r(T)\}) \cup \{F - r(T)\} \), then \( T = (S \setminus \{F - r(T)\}) \cup \{F - (F - r(T))\} \).

To conclude the proof it is enough to see that \( F - r(T) \in \text{msg}(S) \) verifies the conditions given in the definition of \( \alpha(S) \). Clearly \( F - r(T) \in \text{msg}(S) \) because \( F - r(T) \notin T \). Since \( r(T) < \frac{F}{3} \) we obtain \( \frac{F}{2} < F - r(T) < F \).

Moreover, as \( 2r(T) \in T \setminus \{r(T)\} \) then \( 2r(T) \in S \) and thus \( F - 2r(T) \notin S \). Consequently, \( 2(F - r(T)) \notin S \). If \( 3(F - r(T)) = 2F \) we would obtain that \( F = 3r(T) \in S \), which is impossible. Furthermore, if \( 4(F - r(T)) = 3F \) we would get that \( F = 4r(T) \in S \), which is impossible. Finally, since \( S = (T \setminus \{r(T)\}) \cup \{F - r(T)\} \) and \( F - r(T) > r(T) \) then \( m(S) < r(T) < r(S) \). □

Now, we are ready to formulate our algorithm that will allow us to compute the set \( \mathcal{I}(m, F) \).

**Algorithm 22.**

**INPUT:** \( m \) and \( F \) be positive integers with \( F \geq 3 \), \( m \leq \frac{F+2}{3} \) and \( m \nmid F \).

**OUTPUT:** The set \( \mathcal{I}(m, F) \).

1. Set \( A := \{C(m, F)\} \) and \( \mathcal{I}(m, F) := A \).
2. While \( A \neq \emptyset \) do.
   1. For \( S \in A \)
      1.1. Compute \( \alpha(S) \).
      1.2. Set \( E(S) := \{(S \setminus \{x\}) \cup \{F - x\} \mid x \in \alpha(S)\} \).
   2. Set \( A := \{E(S) \mid S \in A\} \) and \( \mathcal{I}(m, F) := \mathcal{I}(m, F) \cup A \).
3. Return \( \mathcal{I}(m, F) \).

This algorithm computes the set \( \mathcal{I}(m, F) \) starting from \( C(m, F) \) whose existence is proved in Proposition 10. The correctness of the algorithm relies on Theorem 21.

We end this section by exploring the tree structure of \( \mathcal{I}(m, F) \) which is somehow the structure of our algorithm and by analyzing a GAP [11] implementation of our algorithm.

**The tree of \( \mathcal{I}(m, F) \).** A graph \( G \) is a pair \( (V, E) \), where \( V \) is a nonempty set whose elements are called vertices, and \( E \) is a subset of \( \{(v, w) \in V \times V \mid v \neq w\} \). The elements of \( E \) are called edges of \( G \). A path of length \( n \) connecting the vertices \( v \) and \( w \) of \( G \) is a sequence of distinct edges of the form \( (v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n) \) with \( v_0 = v \) and \( v_n = w \). In this case, \( w \) is said to be a child of \( v \).

A graph \( G \) is a tree if there exists a vertex \( r \) (known as the root of \( G \)) such that for every other vertex \( v \) of \( G \), there exist a unique path connecting \( v \) and \( r \).

Let \( m \) and \( F \) positive integers such that \( F \geq 3, m \leq \frac{F+2}{3} \) and \( m \nmid F \). Let \( G(\mathcal{I}(m, F)) \) be the graph with vertex set equal to \( \mathcal{I}(m, F) \) and such that \( (S, T) \in \mathcal{I}(m, F) \times \mathcal{I}(m, F) \) is an edge if and only if \( S \prec T \).
**Corollary 23.** The graph $G(ℐ(m,F))$ is a tree with root equal $C(m,F).

*Proof.* Let $S ∈ ℐ(m,F)$. By Proposition [19] there exists a path from $S$ to $C(m,F)$. If there exists another path from $S$ to $C(m,F)$, then there are $S'$ and $S''$ such that $S ≺ S'$ and $S ≺ S''$ but this is not possible by the definition of $≺$. □

**Corollary 24.** If $(S,T)$ is an edge of $G(ℐ(m,F))$, then $S ⊄ T$ and $T ⊄ S$.

*Proof.* Since $T ≺ S$, by definition, $S = (T \setminus \{r(T)\}) ∪ \{F − r(T)\}$, then $r(T) ∈ T \setminus S$ and $F − r(T) ∈ S \setminus T$. □

This last corollary says that no edge of $G(ℐ(m,F))$ is an edge of the tree introduced in [16].

**GAP Computations.** The following GAP code is an implementation of Algorithm [22]. This implementation requires the GAP package NumericalSgps [6].

```gap
sons:=function(s,F)
   local small, m, r, msg, candidates;
   small:=SmallElementsOfNumericalSemigroup(s);
   m:=small[2];
   r:=First(small,i->RemInt(i,m) <> 0);
   msg:=function(x)
      return Filtered(small, y-> (y<x) and (x-y) in small)=[0];
   end;
   candidates:=Filtered(small, x-> (x>F/2) and (x<F) and not(2*x-F in small) and not(3*x=2*F) and not(4*x=3*F) and (F-x>m) and (F-x<r) and (msg(x)));
   return List(candidates, x-> NumericalSemigroupBySmallElements(Set(Concatenation(Difference(small,[x]),[F-x]))));
end;

IrreducibleNumericalSemigroupsWithMultiplicityAndFrobeniusNumber := function(m,F)
   local p,r,msgCmf, Cmf, A, Irrmf, s, lsons;
   p := RemInt(F,2);
   r := RemInt((F+2-p)/2,m);if r = 0 then r:=m; fi;
   msgCmf := (F+2-p)/2 + Difference([0 .. (m-1)], [m-r, r-(2-p)]);
   Cmf := NumericalSemigroupByGenerators(Concatenation([m,F+m],
      msgCmf));
   A:=[Cmf];
   Irrmf:=A;
   while A<>[] do
      s:=A[1];
      A:=A[2..Length(A)];
      lsons:=sons(s,F);
      Append(Irrmf,lsons);
      Append(A,lsons);
   end;
```

Observe that, since we do not need to compute the whole set of irreducible numerical semigroups with Frobenius $F$ numbers and then restrict our search to those of multiplicity $m$, our algorithm supposes a real improvement in computation time means. For instance, we can compute $\mathcal{I}(20, 70)$ in 0.149 seconds, whereas the computation of set of irreducible numerical semigroups with Frobenius number equal to 70 spent 1.175 seconds. Both computations were performed running GAP 4.8.8 in an Intel(R) Core(TM) i5-2450M CPU 2.50GHz, by the latest version on the package NumericalSgps.

**On the Kunz coordinates of $\mathcal{I}(m, F)$.** The last part of this section is devoted to study of the structure of $\mathcal{I}(m, F)$ as solution set of certain integer programs. In the following, $S$ will denote a numerical semigroup of multiplicity $m$ and Frobenius number $F$.

**Definition 25.** The Apéry set of $S$ with respect to $m$ is the set

$$\text{Ap}(S, m) := \{s \in S \mid s - m \not\in S\}.$$  

It is known that $\{\text{Ap}(S, m)\backslash\{0\} \cup \{m\}\}$ is a (non-necessarily minimal) system of generators of $S$ (see [17, Lema 1.6]). Moreover one has that $\text{Ap}(S, m) = \{0 = w(0), w(1), \ldots, w(m - 1)\}$, where $w(i), i \in \{0, \ldots, m - 1\}$, is the least element in $S$ whose remainder under division by $m$ is $i$, that is, $w(i) = q_i m + i$ for some $q_i \in \mathbb{N}\backslash\{0\}$, $i = 0, \ldots, m - 1$. Therefore, $(q_1, \ldots, q_{m - 1}) \in (\mathbb{N}\backslash\{0\})^{m-1}$ characterizes $\text{Ap}(S, m)$ and vice versa. The vector $(q_1, \ldots, q_{m - 1})$ is called the Kunz coordinate vector of $S$.

An easy computation shows that

$$w(i) + w(j) = (q_i + q_j)m + i + j \geq \begin{cases} q_{i+j}m + i + j & \text{if } i + j \leq m - 1 \\ q_{i+j-m}m + i + j - m & \text{if } i + j > m \end{cases}$$

for every $i, j \in \{0, \ldots, m-1\}$. Thus, we can see $(q_1, \ldots, q_{m - 1}) \in (\mathbb{N}\backslash\{0\})^{m-1}$ as a solution of the following system of inequalities:

$$
\begin{align*}
x_i &\geq 1 & 1 \leq i \leq m - 1, \\
x_i + x_j - x_{i+j-m}\delta &\geq -\delta & 1 \leq i \leq j \leq m - 1,
\end{align*}
$$

where $\delta = \lfloor \frac{i+j}{m} \rfloor$ being $\lfloor - \rfloor$ the floor operator.

Moreover, since $F = \max\{w(i) - m \mid i = 1, \ldots, m - 1\}$ (see [17, Proposition 2.12]), a set of extra constrains must be considered.

Thus, putting all this together we obtain the convex lattice polytope defined by:

$$
\begin{align*}
x_i &= \frac{F-i}{m} + 1, & \text{if } i \equiv F \mod m \\
1 \leq x_i &< \frac{F-i}{m} + 1, & i = 1, \ldots, m - 1, \ i \not\equiv F \mod m, \\
x_i + x_j - x_{i+j-m}\delta &\geq -\delta & 1 \leq i \leq j \leq m - 1, \ i + j \not\equiv m \\
x_i \in \mathbb{N}, & & i = 1, \ldots, m - 1,
\end{align*}
$$

(1)
where \( \delta = \left\lfloor \frac{i+j}{m} \right\rfloor \).

The following result is a direct consequence of Proposition 3.

**Corollary 26.** If \( F \geq m - 1 \geq 1 \) and \( m \nmid F \), then there exists, at least, one integer point satisfying (1).

The following results will useful in the sequel.

**Proposition 27.** Let \( m \) and \( F \) be positive integers. With the same notation as in Section 3. If \( T \) and \( S \in \mathcal{I}(m, F) \) and \( T < S \), then

\[
\text{Ap}(S, m) = \left( \text{Ap}(T, m) \setminus \{r(T), m + F - r(T)\} \right) \cup \{F - r(T), m + r(T)\}.
\]

**Proof.** By definition, \( r(T) < F/2 \) and \( S = (T \setminus \{r(T)\}) \cup \{F - r(T)\} \). Since \( r(T) \) is a minimal generator, it belongs to \( \text{Ap}(T, m) \), and \( r(T) \notin S \). But \( m + r(T) \in S \) and \( m + r(T) \equiv r(T) \mod m \), and thus \( m + r(T) \in \text{Ap}(S, m) \). On the other hand, \( r(T) - m \notin T \), then, by Lemma 14, \( F - r(T) + m \in T \) and \( F - r(T) + m \in \text{Ap}(T, m) \). Now, since \( r(T) \notin S \), then \( F - r(T) \in S \) and, by Lemma 14 again, \( F - r(T) \equiv F - r(T) + m \mod m \), we conclude that \( F - r(T) \in \text{Ap}(S, m) \).

Now, we exhibit the equations of the convex sets in \( \mathbb{R}^{m-1} \) whose integral points are the Kunz coordinates of the irreducible numerical semigroups.

The next result is well known and is easy to prove.

**Lemma 28.** Let \( S \) be a numerical semigroup, \( m \in S \setminus \{0\} \) and \( \text{Ap}(S, m) = \{0, q_1 m + 1, \ldots, q_{m-1} m + m - 1\} \), then \( g(S) = q_1 + \cdots + q_{m-1} \).

From Lemmas 12 and 28 we obtain the following result.

**Proposition 29.** If \( F \geq 3, m \leq \frac{F+2}{2} \) and \( m \nmid F \), then the sum of the Kunz coordinates vector of \( S \in \mathcal{I}(m, F) \) is equal to \( \left\lfloor \frac{F+1}{2} \right\rfloor \).

Therefore, the Kunz coordinate vector of the \( S \in \mathcal{I}(m, F) \) lies in the hyperplane \( x_1 + \cdots + x_{m-1} = \left\lfloor \frac{F+1}{2} \right\rfloor \), for some positive integer \( \left\lfloor \frac{F+1}{2} \right\rfloor \) that only depends on \( m \) and \( F \).

The next result is a consequence of [17] Propositions 4.10 and 4.15.

**Proposition 30.** Let \( q \) and \( r \) be the quotient and the remainder of the division of \( \left\lfloor \frac{F+1}{2} \right\rfloor \) by \( m \) and let \( q_1, \ldots, q_{m-1} \) be the Kunz coordinates vector of \( S \in \mathcal{I}(m, F) \). Then \( q_i + q_j + \delta_i = 2q + 1 + \delta_2 \) for every \( 1 \leq i \leq j \leq m-1 \) such that \( i + j \equiv F \mod m \).

In conclusion, the Kunz coordinates of \( S \in \mathcal{I}(m, F) \) must verify the following:

\[
\begin{align*}
x_1 + \ldots + x_{m-1} &= \left\lfloor \frac{F+1}{2} \right\rfloor \\
x_i &= \frac{F-i}{m} + 1, \quad &i \equiv F \mod m \\
1 \leq x_i < \frac{F-i}{m} + 1, \quad &i = 1, \ldots, m-1, \quad i \neq F \mod m, \\
x_i + x_j - x_{i+j-m-1} &\geq -\delta_1 \quad &1 \leq i \leq j \leq m-1, \quad i + j \neq m \\
x_i + x_j + \delta_i &= 2q + 1 + \delta_2 \quad &1 \leq i \leq j \leq m-1, \quad i + j \equiv F \mod m \\
x_i &\in \mathbb{N}, \quad &i = 1, \ldots, m-1,
\end{align*}
\]
where $\delta_1 = \lfloor \frac{i+j}{m} \rfloor$ and $\delta_2 = \lfloor \frac{2r-1}{m} \rfloor$.

Note that as a consequence of [17, Propositions 4.10 and 4.15] we have that the integer solutions of the previous system are the Kunz coordinates of an irreducible numerical semigroup with Frobenius number $F$ and multiplicity $m$. Therefore, our Algorithm 22 can be seen as a solver of (2).

4. An algorithm for the computation of the classes

In this section we address the problem of finding an algorithm to compute the set $[S]$, for each $S \in \mathcal{I}(m, F)$, that is, the second algorithm mentioned in Remark 11.

We say that a numerical semigroup $S$ is homogeneous if it has no minimal generator in the interval $\left[\frac{F(S)}{2}, F(S)\right]$. We denote by $H(m, F)$ the set of homogeneous numerical semigroups with multiplicity $m$ and Frobenius number $F$.

**Proposition 31.** Let $m$ and $F$ be positive integers such that $m \geq 3$, $F > 2m$, $m \nmid F$. If $S \in \mathcal{L}(m, F)$, then $[S] \cap H(m, F) = \{Z([S])\}$. Moreover

$$Z([S]) = \langle \theta(S) \cup \{m\} \rangle \cup \{F + 1, \rightarrow\}.$$ 

**Proof.** If $Z([S])$ does not belong to $H(m, F)$, there exists a minimal generator $x$ of $Z([S])$ such that $\frac{F}{2} < x < F$. By using Lemma 2 we deduce that $Z([S]) \setminus \{x\} \in [S]$ in contradiction with the minimality of $Z([S])$.

Next we see that if $S' \in [S] \cap H(m, F)$ then $S' = Z([S])$. Since $S' \in [S]$ it follows that $Z([S]) \subseteq S'$. For the other inclusion, we consider $x \in S'$ and we distinguish three cases.

- If $x < \frac{F}{2}$ then we have $x \in Z([S])$, since $S' \sim (Z([S]))$
- If $\frac{F}{2} < x < F$ then $x \in Z([S])$, because $S'$ has no minimal generators in the interval $\left[\frac{F}{2}, F\right]$.
- If $x > F$ then $x \in Z([S])$ because $F(Z([S])) = F$.

Finally, observe that $\langle \theta(S) \cup \{m\} \rangle \cup \{F + 1, \rightarrow\} \in H(m, F)$ and thus $\langle \theta(S) \cup \{m\} \rangle \cup \{F + 1, \rightarrow\} = Z([S])$. $\square$

As a consequence of Theorem 7 and Proposition 31, we obtain the following result.

**Corollary 32.** Let $m$ and $F$ be positive integers such that $m \geq 3$, $F > 2m$ and $m \nmid F$. If $S \in \mathcal{I}(m, F)$, then a numerical $S'$ belongs to $[S]$ if and only if $Z([S]) \subseteq S' \subseteq S$.

For a given numerical semigroup $S \in \mathcal{I}(m, F)$, we write

$$D(S) = S \setminus Z([S]).$$

Given two subsets $A$ and $B$ of $\mathbb{N}$, we write $A + B$ for the set $\{a + b \mid a \in A, b \in B\}$. 

Lemma 33. Let \( m \) and \( F \) be positive integers such that \( m \geq 3 \), \( F > 2m \), \( m \nmid F \). If \( S \in \mathcal{S}(m, F) \) and \( B \subseteq D(S) \), then

\[
\overline{S} = Z([S]) \cup (B + Z([S])) \cap D(S) \in [S].
\]

Moreover, all the elements in \([S]\) are in that form.

**Proof.** Since \( Z([S]) \subseteq \overline{S} \subseteq S \), in order to prove that \( \overline{S} \in [S] \) it suffices to see that \( \overline{S} \) is a numerical semigroup by Corollary 32. In fact, it is enough to prove that the sum of two elements of \( B \) belongs to \( \overline{S} \). Since \( B \subseteq D(S) \), all the elements in \( B \) are greater than \( \frac{F}{2} \). Hence the sum of two elements in \( B \) belongs to \( Z([S]) \subseteq \overline{S} \) and we are done.

Let \( S' \in [S] \), by Corollary 32 there exists \( B \subseteq D(S) \) such that \( S' = Z([S]) \cup B \). As \( S' \) is a numerical semigroup, we conclude that \( S' = Z([S]) \cup (B + Z([S])) \cap D(S) \).

\[
\bigcup_{b \in B} (\{b\} + Z([S])) \cap D(S).
\]

Given \( S \in \mathcal{S}(m, F) \) and \( B \subseteq D(S) \), we write

\[
T(B) = \bigcup_{b \in B} (\{b\} + Z([S])) \cap D(S).
\]

The next result is a reformulation of Lemma 33 with this new notation.

**Proposition 34.** Let \( m \) and \( F \) be positive integers such that \( m \geq 3 \), \( F > 2m \), \( m \nmid F \). If \( S \in \mathcal{S}(m, F) \) and \( A = \{T(B) \mid B \subseteq D(S)\} \), then \([S] = \{Z([S]) \cup X \mid X \in A\}\).

Observe that the lattice structure of \([S]\) is same as the lattice structure of \( A = \{T(B) \mid B \subseteq D(S)\}\).

Now, we are ready to give an algorithmic procedure to compute the class \([S]\) from \( S \in \mathcal{S}(m, F) \).

**Algorithm 35.**

**Input:** \( S \in \mathcal{S}(m, F) \).

**Output:** \([S]\).

1. Set \( Z([S]) := \theta(S) \cup \{m\} \cup \{F + 1, \rightarrow\} \) and \( D(S) := S \setminus Z([S]) \).
2. Compute the set \( A = \{T(B) \mid B \subseteq D(S)\} \).
3. Return the set \( \{Z([S]) \cup X \mid X \in A\} \).

Let us see in an example how our algorithm works.

**Example 36.** By Lemma 12 we have that \( S = \langle 5, 7, 9, 11 \rangle \in \mathcal{S}(5, 13) \). Let us compute \([S]\) by using Algorithm 35. In this case, \( \theta(S) = \varnothing \), therefore \( Z([S]) = \langle 5 \rangle \cup \{14, \rightarrow\} \) and \( D(S) = \{7, 9, 11, 12\} \). Since, \( T(7) = \{7, 12\}, T(9) = \{9\}, T(11) = \{11\} \) and \( T(12) = \{12\} \), we obtain that

\[
A = \{\varnothing, \{9\}, \{11\}, \{12\}, \{7, 12\}, \{9, 11\}, \{9, 12\}, \{11, 12\}, \{7, 9, 12\}, \{7, 11, 12\}, \{9, 11, 12\}, \{7, 9, 11, 12\}\}.
\]

and thus \([S] = \{5 \cup X \cup \{14, \rightarrow\} \mid X \in A\} \).
GAP Computations. Notice that the above algorithm completes the computation of $\mathcal{L}(m, F)$ (see Remark \[\text{11}\]). Thus, we can use Algorithms \[\text{22}\] and \[\text{35}\] to compute the whole set of numerical semigroups with fixed multiplicity and Frobenius number. To this end, we have written the following GAP code which requires the GAP package NumericalSgps \[\text{[6]}\].

```gap
NumericalSemigroupsWithMultiplicityAndFrobeniusNumber := function(m,F)
local L,IrrmF,Lmf,S,small,T2,genZ,smallZ,SZ,pow,B,b,TB,TBD,bS;
IrrmF:= IrreducibleNumericalSemigroupsWithMultiplicityAndFrobeniusNumber (m,F);
Lmf:=IrrmF;
for S in IrrmF do
    small:=SmallElementsOfNumericalSemigroup(S);
    T2:=Intersection(small,[m .. Int(F/2)]);
    genZ:=Union(T2,[F+1 .. (F+m)]);
    SZ:=NumericalSemigroupByGenerators(genZ);
    smallZ:=SmallElements(SZ);
    D:=Difference(small,smallZ);
    pow:=Combinations(D);
    for B in pow do
        TB:=[];
        for b in B do
            TB:=Concatenation(TB,b+smallZ);
        od;
        TBD:=Intersection(TB,D);
        bS:=NumericalSemigroupByGenerators(Union(genZ,TBD));
        Lmf:=Concatenation(Lmf,[bS]);
    od;
    return Set(Lmf);
od;
end;
```

Again, this code is faster than the one currently implemented in GAP. For example, for $F = 25$ and $m = 11$, we have obtained the 896 numerical semigroups with multiplicity $m$ and Frobenius number $F$ is 0.092 seconds whereas the GAP command

```gap
Filtered(NumericalSemigroupsWithFrobeniusNumber(25),
i->Multiplicity(i)=11);
```

included in the package NumericalSgps took 2.788 seconds. Notice that the command above compute first the whole set of numerical semigroups with Frobenius number 25 and then it filters the set by the given multiplicity.

We finish this section observing that we can combine that algorithm with the algorithm \[\text{35}\] to calculate simultaneously the elements in the \([S]\) classes, because we can compute the set $Z([S])$ by using the construction give in Theorem \[\text{21}\]. More precisely:
Corollary 37. Let \( m \) and \( F \) be positive integers such that \( F \geq 3 \), \( m \leq \frac{F+2}{2} \) and \( m \mid F \). If \( T \) and \( S \in \mathcal{J}(m, F) \) with \( S < T \), then

\[
Z([S]) = \langle \theta(T) \cup \{m, F-x\} \rangle \cup \{F+1, \rightarrow\},
\]

for some \( x \in \text{msg}(T) \) such that \( \frac{F}{2} < x < F \), \( 2x - F \not\in S \), \( 3x \neq 2F \), \( 4x \neq 3F \) and \( m(T) < F - x < r(T) \).

Proof. By Theorem 21, \( T = (S \setminus \{x\}) \cup \{F-x\} \) for some \( x \in \text{msg}(S) \) such that \( \frac{F}{2} < x < F \), \( 2x - F \not\in S \), \( 3x \neq 2F \), \( 4x \neq 3F \) and \( m(S) < F - x < r(S) \). Then \( \langle \theta(S) \cup \{m\} \rangle = \langle \theta(T) \cup \{m, F-x\} \rangle \) and, by Proposition 31, we are done.

This opens a door to potentially faster implementations of our algorithm for the computation of \( \mathcal{L}(m, F) \).

On the (binomial ideal) structure of \([S] \). We end this section by noticing that the structure of \([S] \) can be described in terms of certain binomial ideals in a polynomial ring over a field, where \([S] \) is the class of \( S \in \mathcal{L}(m, F) \) for the equivalence relation defined by \( \theta(S) \) in Section 2. Recall that by Lemma 10, \([S] \) is a semigroup of sets with respect to the union. Moreover, by Proposition 34, \([S] \) is generated by \( T(d), d \in D(S) \). Therefore, if \( D(S) = \{d_1, \ldots, d_n\} \), we have the following semigroup homomorphism

\[
\varphi : \mathbb{N}^n \to [S]; e_i \mapsto Z([S]) \cup T(d_i), \ i = 1, \ldots, n,
\]

by convention \( \varphi(0) = \emptyset \). Associated to this homomorphism, we have the following binomial ideal

\[
I_{[S]} = \langle X^u - X^v \mid \varphi(u) = \varphi(v) \rangle \subseteq \mathbb{k}[X_1, \ldots, X_n],
\]

where \( X^u = X_1^{u_1} \cdots X_n^{u_n}, \ u = (u_1, \ldots, u_n) \in \mathbb{N}^n \).

Example 38. The binomial ideal associated to \([S] \) in Example 36 is the ideal generated by \( \{X_2^2 - X_1, \ldots, X_4^2 - X_4, X_3X_4 - X_4\} \).

Observe that the dimension of \( \mathbb{k}[X_1, \ldots, X_n]/I_{[S]} \) as \( \mathbb{k} \)-vector space is the same as the cardinality of \([S] \); equivalently, as the cardinality of \( A = \{T(B) \mid B \subseteq D(S)\} \) by Proposition 34. In fact, \( I_{[S]} \) is the ideal of a finite set of points \( \mathcal{Z} \) with coordinates in \( \{0, 1\} \) and there is natural one-to-one correspondence with \( \mathcal{Z} \to A; P = (x_1, \ldots, x_n) \mapsto \{d_i \mid x_i = 1\} \). Thus, we can compute the structure of \([S] \) through the primary decomposition of \( I_{[S]} \) and vice versa.

5. The set of numerical semigroups of a given multiplicity and genus

Following the same idea than the expressed in Remark 11 we can formulate an analogous type algorithm to compute the set of all numerical semigroups with multiplicity \( m \) and genus \( g \) that we denote by \( \hat{\mathcal{L}}(m, g) \).

The next result characterizes the integers \( m \) and \( g \) such that there exists a numerical semigroup with multiplicity \( m \) and genus \( g \).
Lemma 39. [13] Proposition 2.1. If \((m, g) \in \mathbb{N} \setminus \{0\} \times \mathbb{N}\), then \(\mathcal{L}(m, g) \neq \emptyset\) if and only if \((m, g) = (1, 0)\) or \(2 \leq m \leq g + 1\).

Since \(\mathcal{L}(g + 1, g) = \{0, g + 1 \rightarrow\}\), from now on we will assume \(2 \leq m \leq g\).

The following result determines a necessary and sufficient condition for an integer \(F\) to be the Frobenius number of a numerical semigroup of multiplicity \(m\) and genus \(g\).

Proposition 40. [13] Theorem 2.4. Let \(m\), \(g\) and \(F\) be positive integers. If \(2 \leq m \leq g\) and \(m \nmid F\), then there exists \(S \in \mathcal{L}(m, g)\) with \(F(S) = F\) if and only if \(\lceil \frac{mg}{m-1} \rceil - 1 \leq F \leq 2g - 1\).

We denote by \(\mathcal{L}(m, g, F)\) the set of all numerical semigroups with multiplicity \(m\), genus \(g\) and Frobenius number \(F\). As a consequence of Proposition 40 we have the following result.

Corollary 41. Let \(m\), \(g\) and \(F\) be positive integers. If \(2 \leq m \leq g\) and \(B_{m,g} := \{F \in \{\lceil \frac{mg}{m-1} \rceil - 1, \ldots, 2g - 1\} \mid m \nmid F\}\), then

\[
\mathcal{L}(m, g) = \bigcup_{F \in B_{m,g}} \mathcal{L}(m, g, F).
\]

Therefore, to compute all elements in \(\mathcal{L}(m, g)\), it is enough to compute the elements in \(\mathcal{L}(m, g, F)\) for each \(F \in B_{m,g}\).

The next algorithm is a reformulation of Algorithm 35 for the computation of the set of elements in the class of \(S \in \mathcal{I}(m, F)\) with respect to \(\sim\) with genus \(g\).

Algorithm 42.

**INPUT:** \(S \in \mathcal{I}(m, F)\) and \(g\).

**OUTPUT:** \(\{T \in [S] \mid g(T) = g\}\)

1. Set \(Z([S]) := \langle 0 \rangle \cup \{m\}\) and \(D(S) := S \setminus Z([S])\).
2. Set \(A = \{T(B) \mid B \subseteq D(S)\text{ and }\#T(B) := g(Z([S]) - g)\}\).
3. Return the set \(\{Z([S]) \cup X \mid X \in A\}\).

We illustrate the above algorithm with the following example.

Example 43. From Example 36 we know that \(S = \langle 5, 7, 9, 11\rangle \in \mathcal{I}(5, 13)\). By using Algorithm 42 we compute the set \(\{T \in [S] \mid g(T) = 10\}\). Since \(Z([S]) = \langle 5\rangle \cup \{14, \rightarrow\}\), \(D(S) = \{7, 9, 11, 12\}\) and \(g(Z([S])) = 11\), we have that

\[
A = \{T(B) \mid B \subseteq D(S)\text{ and }\#T(B) = 11 - 10 = 1\} = \{\{9\}, \{11\}, \{12\}\}.
\]

Hence, \(\{T \in [S] \mid g(T) = 10\} = \{\langle 5\rangle \cup X \cup \{14, \rightarrow\} \mid X \in A\} = \{\{5, 9, 16, 17\}, \{5, 11, 14, 17, 18\}, \{5, 12, 14, 16, 18\}\}.

Observe that by just controlling the cardinality of \(T(B)\) suitably, we can modify our GAP code in Section 4 to produce the corresponding function which computes \(\mathcal{L}(m, g, F)\).
Remark 44. For $q > 1$ and $g \geq 2$ we set
\[ B_{m,g} := \left\{ F \in \left\{ \left\lfloor \frac{mg}{m-1} \right\rfloor - 1, \ldots, 2g-1 \right\} \cap \{ (q-1)m+1, \ldots,qm \} \mid m \nmid F \right\} \]
for each $m = 2, \ldots, g$. Clearly,
\[ \bigcup_{m=2}^{g} \left( \bigcup_{F \in B_{m,g}} \mathcal{L}(m, g, F) \right) \]
is the set of numerical semigroups with genus $g$ and depth $q$ (see Remark 5). If $q = 2$, this formula is rather explicit by Proposition 4. For $q > 2$, we can take advantage of our algorithms to compute this set.

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THE SET $L(m, F)$

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