EXISTENCE AND BLOW UP OF SOLUTIONS TO THE 2D BURGERS EQUATION WITH SUPERCritical DISSIPATION

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Abstract. This paper is concerned with the Cauchy problem for a fractal Burgers equation in two dimensions. When $\alpha \in (0, 1)$, the same problem has been studied in one dimensions, we can refer to [1, 17, 24]. In this paper, we study well-posedness of solutions to the Burgers equation with supercritical dissipation. We prove the local existence with large initial data and global existence with small initial data in critical Besov space by energy method. Furthermore, we show that solutions can blow up in finite time if initial data is not small by contradiction method.

1. Introduction. We consider the 2D Burgers equation with fractional dissipation

$$\begin{cases}
\partial_t u + k(-\Delta)^{\frac{\alpha}{2}} u = \sum_{i=1}^2 u \partial_i u, & t > 0, x \in \mathbb{R}^2, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^2,
\end{cases}$$

where $k \geq 0, \alpha \in (0, 1)$, $\partial_i = \frac{\partial}{\partial x_i}$, $u = u(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is real valued unknown function and $u_0(x)$ is the initial function. The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined by the Fourier transform:

$$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}} u)(\xi) = |\xi|^\alpha \hat{u}(\xi).$$

The fractional Laplacian is a nonlocal operator, and for its integral formula, the reader may refer to [15]. There are many different mathematical problems and physical problems relevant to this operator. For example, the fractional Laplacian operator yields the anomalous diffusion, which is related to the dynamics of electrons in a semiconductor [19]. In the stochastic process [3], it is related to random trajectories, generalizing the concept of Brownian motion, which may contain jump discontinuities. By using the fractional Laplacian, we can construct some simple
models to simulate properties of solutions for some important equations and systems. Recently, many people get interested in the Navier-Stokes equation [22] and quasi-geostrophic equation [12, 34] with fractional dissipation, and it also appears naturally in the Keller-Segel models [5, 25]. As for the basic properties of the fractional Laplacian operator, it has been studied in a number of papers, such as [4, 7, 21, 27, 35].

The study of fractal Burgers equation has a long history and has been investigated extensively from different aspects. As is known to all, the Burgers equation can also be viewed as the simplest in the family of partial differential equations modeling the nonlinearity in the Euler and Navier-Stokes equations. Furthermore, if some dissipative terms are added to the Burgers equation, then it is possible to simulate the properties of solutions for some important equations and systems. In addition, we want to compare the effect of dissipative term and the one of nonlinear term in the study of well-posedness of classical solutions for the fractal Burgers equation. As we know, when \( k = 0 \), the equation (1) is Burgers equation, and the local existence and blow up in finite time have been proved for some smooth initial data. If \( k > 0 \) and \( \alpha = 0 \), the equation (1) is the Burgers equation with damping [33], the solution has similar properties as the Burgers equation. If \( k > 0 \) and \( \alpha = 2 \), the equation (1) is the classical viscous Burgers equation, and the global existence result is well known [29].

When \( k > 0 \), the cases \( \alpha > 1, \alpha = 1, \) and \( \alpha < 1 \) are called subcritical, critical and supercritical, respectively. We introduce some previous results of well-posedness for the fractal Burgers equation. First, in the subcritical case, the order of derivation to the fractional dissipative term is larger than the one of nonlinear term. For one dimension, Dong, Du and Li [17] obtained the global solutions in Gevrey function class, Kiselev, Nazarov and Shterenberg [24] studied the global solutions in Sobolev spaces, and Droniou, Gallouët and Vovelle [18] got the global solutions in Lebesgue space by splitting time technique. For multi-dimensions, Wang and Wang [32] studied the global large solutions with periodic initial data, Li and Rong [26] got the time decay of solutions with large initial data. For the proof of these facts, authors used the energy method and maximum principle, the details can refer to the proof of global solutions for the quasi-geostrophic equation case [14, 15]. Secondly, in the critical case, the dissipation balances nonlinearity. The relevant results of the global existence to solutions have been proved, some good methods and ideas come from the dissipative quasi-geostrophic equation [13, 23]. For one dimension, Dong et al [17] and Kiselev et al [24] got the global solutions in Sobolev spaces by the modulus of continuity method, Miao and Wu [28] also got the global solutions in Besov spaces by the modulus of continuity method and Littlewood-Paley decomposition. For higher dimensional, Chan and Czubak [9] got the global solutions by parabolic De Giorgi’s method, Constantin and Vicol [13] obtained the global existence of solutions in Schwartz class by nonlinear maximum principle. Finally, for the supercritical case, the dissipation less than nonlinearity. In one dimension, Alibaud, Droniou and Vovelle [1] got the global solutions if the initial data has some suitable boundedness, and proved that the entropy solutions can blow up in finite time by time splitting and characteristic method. Kiselev et al [24] obtained that the classical solutions can blow up in finite time by time splitting and iterative method. Also, Dong et al [17] got the local existence and blow up of solutions by weight function and functional method without small assumption. However, for multi-dimensional case, there is no result.
Another motivation is from the quasi-geostrophic equation

\[\begin{align*}
\partial_t u + k(-\Delta)^{\frac{\alpha}{2}} u - v \cdot \nabla u &= 0, \quad t > 0, x \in \mathbb{R}^2, \\
v &= \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi), \quad t > 0, x \in \mathbb{R}^2, \\
u &= (-\Delta)^{\frac{\alpha}{2}} \psi, \quad t > 0, x \in \mathbb{R}^2.
\end{align*}\]  

We know that the non-dissipative \((k = 0)\) quasi-geostrophic equation share many similar properties with 3D Euler equation. And for the dissipative quasi-geostrophic equation, many people have studied it and results can be referred to [8, 11, 12, 14, 15]. While \(k > 0\) and \(\alpha < 1\), whether the solutions of equation (2) can develop singularities in finite time remains an open problem. Hopefully, similar to the supercritical fractal Burgers equation, we can find some good ideas to explore the quasi-geostrophic equation.

For the fractal Burgers equation, when \(k > 0\) and \(\alpha > 1\), there was a good priori estimates for the solution of (1), so the global existences has been resolved successfully by energy method. While the case \(k > 0\) and \(\alpha \leq 1\), it is difficult to apply the energy method. For \(k > 0\) and \(\alpha = 1\), the global existences of solution has been proved by some new methods, for example, modulus of continuity, parabolic De-Giorgi’s estimates and nonlinear maximum principle. But for \(k > 0\) and \(\alpha < 1\), due to the lack of regularity and good priori estimates, it is difficult to get the global existence. For one dimension, the bound of dissipative term [1, 24] and pointwise estimates [17] can be obtained by time splitting technique. The nonlinear term effect is stronger than dissipative term for some initial data, thus many people have shown that the solution can blow up in finite time. While for higher dimension case, the solution of supercritical fractal Burgers equation may blow up in finite time for some large initial data, but there are the technical difficulties for applying previous methods. So we have no well-posedness results for multi-dimensional case.

In this paper, our main focus is the well-posedness of solution for 2D fractal Burgers equation in supercritical case, and without loss of generality, we consider the case \(k = 1\). The local existence and global existence with small data are proved by standard iterative and energy methods. For the proof of blow up in finite time with large data, it is our main work in this paper. We use the contradiction method. For some classes of initial data, new conservation properties of the solution are established. The characteristic method and conservation properties are used in the proof.

The followings are our main results and the usual inhomogeneous, homogeneous Besov space and Chemin-Lerner space are denoted by \(B^s_{p,r}(\mathbb{R}^2)\), \(\dot{B}^s_{p,r}(\mathbb{R}^2)\) and \(\dot{L}_2^2 \dot{B}_{p,r}^s(\mathbb{R}^2)\), the details can refer to Section 2.

The first theorem establishes the local wellposedness of (1) in critical Besov space.

**Theorem 1.1.** Assume that \(0 < \alpha < 1\), the initial data \(u_0 \in B^s_{p,r}(\mathbb{R}^2)\), \(s = \frac{2}{p} + 1 - \alpha, 2 \leq p \leq 4, 1 \leq r \leq 2\), then there exist \(T > 0\) and a unique solution \(u\) of (1), such that

\[u \in \dot{L}^2(0, T; \dot{B}^{s+\frac{2}{p}}_{p,r}(\mathbb{R}^2)) \cap C([0, T); B^s_{p,r}(\mathbb{R}^2)).\]

When there are small assumptions on initial data, the equation (1) admits a global solution. Our main result is:

**Theorem 1.2.** Assume that \(0 < \alpha < 1\) and initial data \(u_0\) has some assumptions in Theorem 1.1. Furthermore, if there exists a positive constant \(\epsilon\), such that \(\|u_0\|_{B^s_{p,r}} \leq \epsilon\), then...
\( \epsilon \), then the unique solution \( u \) of (1) is global existence and we have
\[
u \in L^2(0, \infty; \dot{B}^{s+\frac{2}{p}, r}_p(\mathbb{R}^2)) \cap L^\infty([0, \infty); B^s_{p,r}(\mathbb{R}^2)).
\]

**Remark 1.** The homogeneous Besov space \( \dot{B}^s_{p,r} \) is important as it is the scaling invariant function space. In fact, if \( u(t, x) \) is solution of (1), then \( u(\lambda t, \lambda x) = \lambda^{\alpha-1} u(\lambda^\alpha t, \lambda x) \) are also solution of (1). The \( \dot{B}^s_{p,r} \) norm of \( u(t, x) \) is invariant under this scaling. Moreover, for the global well-posedness, the smallness assumption is imposed only on the homogenous norm of the initial data.

The last result is about the blow-up of solutions corresponding to a class of initial data. Here \( \mathcal{A}(M, \Omega) \) denote a class of functions definition precisely below (Definition 5.2).

**Theorem 1.3.** Assume that \( 0 < \alpha < 1 \). Let \( u_0 \in L^\infty(\mathbb{R}^2), \partial_1 u_0 + \partial_2 u_0 \in B^1_{1,1}(\mathbb{R}^2) \) with \( s \geq 4 \). Furthermore, if there exist \( M \) and \( \Omega \), such that \( u_0 \in \mathcal{A}(M, \Omega) \), then the corresponding solution of equation (1) blow up in finite time. More precisely, there exists \( T_0 < \infty \), such that
\[
\lim_{t \to T_0} \|\partial_1 u + \partial_2 u\|_{L^\infty(\mathbb{R}^2)} = +\infty.
\]

**Remark 2.** For the set \( \mathcal{A}(M, \Omega) \), \( M \) is positive constant, \( \Omega \subset \mathbb{R}^2 \) is bound domain. And in the proof of the Theorem 1.3, the \( M \) is fixed, we need \( \Omega \) to be small.

**Remark 3.** We believe that the Theorem 1.3 remain valid in the case of the spatial domain \( \mathbb{R}^d (d \geq 1) \), but for simplicity we do not pursue this here.

This paper is organized as follows. In Section 2, we introduce the Besov space, its basic properties and some important estimates. In Section 3, we prove the local existence and uniqueness of solution in Besov space. In Section 4, we prove that the solution is global existence with small initial data. In Section 5, we prove that the solution can blow up in finite time if initial data has no smallness assumption, and some important propositions and lemmas are also proved in this section.

Throughout the paper, \( C \) will stand for universal constants that may change from line to line.

2. Preliminaries. In this section, we recall some facts on the Littlewood-Paley decomposition, the inhomogeneous Besov space, homogeneous Besov space, and we introduce some classical properties for these spaces. For more details, the reader may refer to [2, 10, 11, 30].

Let \((\phi, \chi)\) be a couple of smooth functions valued in \([0, 1]\), such that \( \phi \) is supported in the shell \( \{\xi \in \mathbb{R}^d | \frac{3}{4} \leq |\xi| \leq \frac{4}{3}\} \), \( \chi \) is supported in the ball \( \{\xi \in \mathbb{R}^d | |\xi| \leq \frac{4}{3}\} \) and
\[
\forall \xi \in \mathbb{R}^d, \chi(\xi) + \sum_{q \in \mathbb{N}} \phi(2^q \xi) = 1.
\]

Let \( \mathcal{S}'(\mathbb{R}^d) \) be the temperate distributions, which is the dual of Schwartz class \( \mathcal{S}(\mathbb{R}^d) \). For any \( u \in \mathcal{S}(\mathbb{R}^d) \), the Fourier transform and Fourier inverse transform of \( u \) are defined by
\[
\mathcal{F}[u](\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x) dx,
\]
and
\[
\mathcal{F}^{-1}[u](x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} u(\xi) d\xi.
\]

For \( u \in \mathcal{S}'(\mathbb{R}^d) \), we can define inhomogeneous dyadic blocks as follows. Let
\[
\Delta_q u = 0 \quad \text{if} \quad q \leq -2, \\
\Delta_{-1} u = \chi(D) u = \hat{h} * u \quad \text{with} \quad \hat{h} = \mathcal{F}^{-1} \chi, \\
\Delta_q u = \phi(2^{-q}D) u = 2^d \int h(2^q y) u(x-y) dy \quad \text{with} \quad h = \mathcal{F}^{-1} \phi, \quad \text{if} \quad q \geq 0.
\]

For all temperate distributions \(u\), one can prove
\[
u = \sum_{q \in \mathbb{Z}} \Delta_q u \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d).
\]

**Remark 4.**

1. The low frequency cut-off operator \(S_q\) is defined by
\[
S_q u = \sum_{p \leq q-1} \Delta_p u = \chi(2^{-q}D) u = \int \hat{h}(2^q y) u(x-y) dy \quad \forall \ q \in \mathbb{N}.
\]

2. For any \(u, v \in \mathcal{S}'(\mathbb{R}^d)\), the following properties hold:
\[
\Delta_q \Delta_p u \equiv 0 \quad \text{if} \quad |p-q| \geq 2, \\
\Delta_q (S_{q-1} u \Delta_p v) \equiv 0 \quad \text{if} \quad |p-q| \geq 5.
\]

3. By Young’s inequality, there exists a positive constant \(C\) independent of \(p\) and \(q\) such that
\[
\|\Delta_q u\|_{L^p} \leq C \|u\|_{L^p}, \quad \|S_q u\|_{L^p} \leq C \|u\|_{L^p}, \quad u \in \mathcal{S}'(\mathbb{R}^d).
\]

**Definition 2.1.**

Let \(1 \leq p, r \leq \infty\) and \(s \in \mathbb{R}\), the inhomogeneous Besov space \(B^s_{p,r}\) is defined by
\[
B^s_{p,r}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \|u\|_{B^s_{p,r}} < \infty\},
\]
where
\[
\|u\|_{B^s_{p,r}} = \begin{cases} 
\left( \sum_{q \geq 0} (2^q \|\Delta_q u\|_{L^p})^r \right)^{\frac{1}{r}} + \|S_0 u\|_{L^p}, & \text{if} \quad 1 \leq r < \infty, \\
\sup_{q \geq 0} 2^q \|\Delta_q u\|_{L^p} + \|S_0 u\|_{L^p}, & \text{if} \quad r = \infty.
\end{cases}
\]

The homogeneous dyadic blocks are defined by
\[
\hat{\Delta}_q u = \phi(2^{-q}D) u \quad \forall q \in \mathbb{Z},
\]
and the low frequency cut-off operator \(\hat{S}_q\) is defined by
\[
\hat{S}_q u = \chi(2^{-q}D) u \quad \forall q \in \mathbb{Z}.
\]

**Definition 2.2.**

Let \(1 \leq p, r \leq \infty\) and \(s \in \mathbb{R}\), the homogeneous Besov space \(\hat{B}^s_{p,r}\) is defined by
\[
\hat{B}^s_{p,r}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d) \mid \|u\|_{\hat{B}^s_{p,r}} < \infty\},
\]
where
\[
\|u\|_{\hat{B}^s_{p,r}} = \begin{cases} 
\left( \sum_{q \in \mathbb{Z}} (2^q \|\hat{\Delta}_q u\|_{L^p})^r \right)^{\frac{1}{r}}, & \text{if} \quad 1 \leq r < \infty, \\
\sup_{q \in \mathbb{Z}} 2^q \|\hat{\Delta}_q u\|_{L^p}, & \text{if} \quad r = \infty,
\end{cases}
\]
and \(\mathcal{P}(\mathbb{R}^d)\) denote the set of all polynomials on \(\mathbb{R}^d\).
For inhomogeneous Besov space and homogeneous Besov space, if $s > 0$, then $s \leq 1$. There exists a positive constant $C$ such that for any $s \in \mathbb{R}$ and $u \in S'(\mathbb{R}^d)$, the following inequality is known as Bernstein's inequality [10, 11].

\[ \|\Delta_j u\|_{L^p} \leq \|(-\Delta)^{\frac{s}{2}} u\|_{L^p} \leq C 2^j s \|\Delta_j u\|_{L^p} \]

Let $1 \leq p \leq q \leq \infty$. There exists a positive constant $C$ such that for any $s \in \mathbb{R}$ and $u \in S'(\mathbb{R}^d)$, the following inequality holds.

\[ \|\Delta_j u\|_{L^q} \leq C 2^{\left(\frac{j}{p} - \frac{j}{q}\right)} \|\Delta_j u\|_{L^p}. \]
The following lemma has been proved in Chemin and Danchin [2].

**Lemma 2.5.** Let $1 \leq p, q, r, r_1, r_2 \leq \infty$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, $\beta < 1$, $\gamma > -1$ and $T \in (0, \infty]$. Suppose that

$$
\beta - \gamma + d \min\{1, \frac{2}{p}\} > 0 \quad \text{and} \quad \beta + \frac{d}{p} > 0.
$$

Then there exists a positive $C_{q,j}$ such that

$$
\|u, \Delta_j v\|_{L_T^q L^p} \leq C_{q,j} 2^{-j(\frac{d}{2} + \beta - \gamma - 1)} \|\nabla u\|_{L_T^q B_{p,q}^{\frac{d}{2} + \beta - 1}} \|v\|_{L_T^q B_{p,q}^{\frac{d}{2} - \gamma - 1}},
$$

where $\{C_{q,j}\}_{j \in \mathbb{Z}}$ is a positive series such that $\|C_{q,j}\|_{s} = (\sum_{j \in \mathbb{Z}} C_{q,j}^s)^{\frac{1}{s}} \leq 1$.

The following positive lemma has been proved in Cordoba [15] and Ju [20].

**Lemma 2.6.** Let $0 \leq s \leq 2$, $x \in \mathbb{R}^d$, and $u, (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^d)$ with $p \geq 2$. we get

$$
\frac{2}{p} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx \leq \int_{\mathbb{R}^d} |u|^{p-2} u (-\Delta)^{\frac{s}{2}} u dx.
$$

By Lemma 2.4 and 2.6, the following Bernstein type inequality has been proved by Chen, Miao and Zhang [11].

**Lemma 2.7.** Let $2 \leq p < \infty$ and $s \in [0, 1]$. There exists a positive constant $C$ such that for any $j \in \mathbb{Z}$ and $u \in \mathcal{S}'(\mathbb{R}^2)$

$$
C^{-1} 2^{|j|} \|\Delta_j u\|_{L^p} \leq \|(-\Delta)^{\frac{s}{2}} (|\Delta_j u|^\frac{s}{2})\|_{L^2} \leq C 2^{\frac{|j|}{p}} \|\Delta_j u\|_{L^p}.
$$

The following lemma is well-known for the estimate of the fractional integral [36].

**Lemma 2.8.** Let $1 < p, q < \infty$ and $0 < s < \frac{d}{q}$. If

$$
\frac{1}{q} = \frac{1}{p} - \frac{s}{d},
$$

then there exists a positive constant $C$ such that

$$
\|u\|_{L^q} \leq C \|u\|_{H^s_x}, \quad \text{for} \quad u \in H^s_x(\mathbb{R}^d).
$$

3. **Local existence and uniqueness.** In this section, we prove Theorem 1.1. The iterative method is used in the proof (see [11, 28, 30]).

**Proof of Theorem 1.1.** To simplify the presentation, we divide the proof into four steps.

**Step 1.** Approximation solution.

Let $u^1$ be a solution of

$$
\begin{cases}
\partial_t u^1 + (-\Delta)^{\frac{s}{2}} u^1 = 0, & t > 0, x \in \mathbb{R}^d, \\
u^1(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
$$

thus, $u^1 = e^{-t(-\Delta)^{\frac{s}{2}}} u_0$. For $k = 2, 3, \cdots$, let $u^k$ be a solution of the following linearized equation

$$
\begin{cases}
\partial_t u^k + (-\Delta)^{\frac{s}{2}} u^k = u^{k-1}(\partial_1 u^k + \partial_2 u^k), & t > 0, x \in \mathbb{R}^2, \\
u^k(0, x) = u_0(x), & x \in \mathbb{R}^2.
\end{cases}
$$

(6)
By the condition of Theorem 1.1, it is clear that there exist $T > 0$ such that $u^1 \in L^2_T(B^{s+\frac{2}{p}}_{p,r}(\mathbb{R}^2))$. And we claim that
\[ u^k \in \tilde{L}^2(0,T;B^{s+\frac{2}{p}}_{p,r}(\mathbb{R}^2)) \cap L^\infty([0,T);B^s_{p,r}(\mathbb{R}^2)), \quad \forall k \in \mathbb{N}, \tag{7} \]
it is proved in Step 2. it is proved in Step 2.

**Step 2.** Uniform bounds.

In this part, we show that if $T > 0$ is sufficiently small, we have
\[ \|u^k\|_{L^2_TB^{s+\frac{2}{p}}_{p,r}} \leq 2\|u_0\|_{B^{s+\frac{2}{p}}_{p,r}}, \tag{8} \]
and there exists a nonnegative constant $C(T)$ such that
\[ \|u^k\|_{L^2_TB^{s+\frac{2}{p}}_{p,r}} \leq C(T). \tag{9} \]

Applying the operator $\tilde{\Delta}_j$ to the both sides of the equation (6), one has
\[ \partial_t\tilde{\Delta}_j u^k + (-\Delta)^{\frac{2}{p}}\tilde{\Delta}_j u^k = u^{k-1}\tilde{\Delta}_j(\partial_1 u^k + \partial_2 u^k) + [\tilde{\Delta}_j, u^{k-1}](\partial_1 u^k + \partial_2 u^k). \tag{10} \]
Multiplying the both sides of the equation (10) by $|\tilde{\Delta}_j u^k|^{p-2}\tilde{\Delta}_j u^k$ and integrating over $\mathbb{R}^2$, we obtain that
\[ \frac{1}{p} \frac{d}{dt}\|\tilde{\Delta}_j u^k\|_{L^p}^p + \int_{\mathbb{R}^2} (-\Delta)^{\frac{2}{p}}\tilde{\Delta}_j u^k|\tilde{\Delta}_j u^k|^{p-2}\tilde{\Delta}_j u^k \, dx 
= \int_{\mathbb{R}^2} [u^{k-1}\tilde{\Delta}_j(\partial_1 u^k + \partial_2 u^k)]|\tilde{\Delta}_j u^k|^{p-2}\tilde{\Delta}_j u^k \, dx 
+ \int_{\mathbb{R}^2} [\tilde{\Delta}_j, u^{k-1}](\partial_1 u^k + \partial_2 u^k)|\tilde{\Delta}_j u^k|^{p-2}\tilde{\Delta}_j u^k \, dx. \tag{11} \]

By Lemma 2.4 and 2.6, the second term of the left hand side in (11) can be estimated as
\[ \frac{\lambda}{p} 2^{\alpha j}\|\tilde{\Delta}_j u^k\|_{L^p}^p \leq \int_{\mathbb{R}^2} (-\Delta)^{\frac{2}{p}}\tilde{\Delta}_j u^k|\tilde{\Delta}_j u^k|^{p-2}\tilde{\Delta}_j u^k \, dx \]
for some constant $\lambda > 0$. Applying Hölder’s inequality in (11), one has
\[
\frac{d}{dt}\|\tilde{\Delta}_j u^k\|_{L^p}^p + \lambda 2^{\alpha j}\|\tilde{\Delta}_j u^k\|_{L^p}^p \leq C(\|u^{k-1}\tilde{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p} + ||[\tilde{\Delta}_j, u^{k-1}](\partial_1 u^k + \partial_2 u^k)\|_{L^p})\|\tilde{\Delta}_j u^k\|_{L^p}^{p-1}. \tag{12} 
\]

Thus, we obtain that
\[ \frac{d}{dt}\|\tilde{\Delta}_j u^k\|_{L^p} + \lambda 2^{\alpha j}\|\tilde{\Delta}_j u^k\|_{L^p} \leq C(\|u^{k-1}\tilde{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p} + ||[\tilde{\Delta}_j, u^{k-1}](\partial_1 u^k + \partial_2 u^k)\|_{L^p}), \tag{12} \]
which yields that
\[ \|\tilde{\Delta}_j u^k\|_{L^p} \leq e^{-\lambda 2^{\alpha j}t}\|\tilde{\Delta}_j u^k_0\|_{L^p} + Ce^{-\lambda 2^{\alpha j}t} * R_j(t), \tag{13} \]
where
\[ R_j(t) = \|u^{k-1}\tilde{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p} + ||[\tilde{\Delta}_j, u^{k-1}](\partial_1 u^k + \partial_2 u^k)\|_{L^p} \]
and
\[ e^{-\lambda 2^{\alpha j}t} * R_j(t) = \int_0^t e^{-\lambda 2^{\alpha j}(t-\tau)} R_j(\tau) \, d\tau. \]
We estimate $\|u^k\|_{L^2 T \dot{B}^\alpha_{p,r}}$. Multiplying the both sides of (13) by $2^j\mathbf{s}$ and taking $L^r$ norm with respect to $j \in \mathbb{Z}$, we have
\begin{equation}
\|u^k(t)\|_{\dot{B}^\alpha_{p,r}} \leq \|u_0\|_{\dot{B}^\alpha_{p,r}} + C \left\| 2^j \int_0^t R_j(\tau) d\tau \right\|_{L^r}.
\end{equation}
By Hölder inequality, Lemma 2.4 and 2.8, the first term of the right hand side of (12) can be estimated as
\begin{align*}
\|u^{k-1}_j (\partial_1 u^k + \partial_2 u^k)\|_{L^p} & \leq \|u^{k-1}\|_{L^{p_1}} \|\Delta_j (\partial_1 u^k + \partial_2 u^k)\|_{L^{p_2}} \\
& \leq C \|(-\Delta)^{\frac{\sigma+\frac{\sigma}{2}}{p}} u^{k-1}\|_{L^p} \|(-\Delta)^{\frac{\sigma}{p}} \Delta_j (\partial_1 u^k + \partial_2 u^k)\|_{L^p} \\
& \leq 2^{-sj} \|(-\Delta)^{\frac{\sigma+\frac{\sigma}{2}}{p}} u^{k-1}\|_{L^p} \|(-\Delta)^{\frac{\sigma}{p}} \Delta_j (\partial_1 u^k + \partial_2 u^k)\|_{L^p},
\end{align*}
where $\frac{1}{p_1} = \frac{1}{p} - \frac{s+\frac{\sigma}{2}}{2}$ and $\frac{1}{p_2} = \frac{s+\frac{\sigma}{2}}{2} - \frac{1}{p} - \frac{\sigma}{4}, \sigma = \alpha - 2$. Hence if $r \leq 2$, we have
\begin{align*}
\left\| 2^j \int_0^t \|u^{k-1}_j (\partial_1 u^k + \partial_2 u^k)\|_{L^p} d\tau \right\|_{L^r} & \leq C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \left\| \left( \int_0^T \|(-\Delta)^{\frac{\sigma+\frac{\sigma}{2}}{p}} \Delta_j (\partial_1 u^k + \partial_2 u^k)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \right\|_{L^r} \\
& \leq C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|\partial_1 u^k + \partial_2 u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \\
& \leq C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|\partial_1 u^k + \partial_2 u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}-1}_{p,r}} \\
& \leq C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{2}}_{p,r}},
\end{align*}
where we use some properties in Proposition 1 and Remark 7, we can read it in Section 2. According to the Lemma 2.5 with $r = 1, r_1 = r_2 = 2, \beta = 1 - \frac{\sigma}{2}$ and $\gamma = \frac{\sigma}{2} - 1$ to the second term of the right hand side of (12), we obtain
\begin{equation}
\int_0^t \|\Delta_j (u^{k-1})(\partial_1 u^k + \partial_2 u^k)\|_{L^p} d\tau \leq C 2^{-sj} C_{r,j} \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{2}}_{p,r}},
\end{equation}
which implies that
\begin{equation}
\left\| 2^j \int_0^t \|\Delta_j (u^{k-1})(\partial_1 u^k + \partial_2 u^k)\|_{L^p} d\tau \right\|_{L^r} \leq C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{2}}_{p,r}}.
\end{equation}
From (16) and (17), we obtain
\begin{equation}
\left\| 2^j \int_0^t R_j(\tau) d\tau \right\|_{L^r} \leq C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{2}}_{p,r}},
\end{equation}
thus, it implies by (14) that
\begin{equation}
\|u^k(t)\|_{L^2_T \dot{B}^\alpha_{p,r}} \leq \|u_0\|_{\dot{B}^\alpha_{p,r}} + C \|u^{k-1}\|_{L^2_T \dot{B}^{s+\frac{\sigma}{4}}_{p,r}} \|u^k\|_{L^2_T \dot{B}^{s+\frac{\sigma}{2}}_{p,r}}.
\end{equation}
By Hölder inequality, we have that

\[ 2^{(s+\frac{2}{p})j} \| \Delta_j u_k \|_{L^2_t B^s_{p,r}}^2 \leq 2^{(s+\frac{2}{p})j} \left( \int_0^T e^{-\lambda^{2^{s+1}t}} dt \right)^{\frac{1}{p}} \| \Delta_j u_0 \|_{L^p_t} + 2^{(s+\frac{2}{p})j} \left( \int_0^T (e^{-\lambda^{2^{s+1}t}} + R_j(t))^2 dt \right)^{\frac{1}{2}} \]

\[ \leq 2^{\frac{2}{p}j} \left( \int_0^T e^{-\lambda^{2^{s+1}t}} dt \right)^{\frac{1}{2}} \left( 2^{sj} \| \Delta_j u_0 \|_{L^p_t} + 2^{sj} \int_0^T R_j(t) dt \right). \]

We know that

\[ 2^{\frac{2}{p}j} \left( \int_0^T e^{-\lambda^{2^{s+1}t}} dt \right)^{\frac{1}{2}} \leq \lambda^{-\frac{1}{2}} (1 - e^{-\lambda^{2^{s+1}T}})^{\frac{1}{2}} \leq \lambda^{-\frac{1}{2}}, \]

then it follows by taking the \( l' \) norm with respect to \( j \) that

\[ \| u^k \|_{L^2_t B^{s+\frac{2}{p}}_{p,r}} \leq C \| u^k \|_{L^2_t B^{s+\frac{2}{p}}_{p,r}} + C \| u^k \|_{L^2_t B^{s+\frac{2}{p}}_{p,r}} \| u^k \|_{L^2_t B^{s+\frac{2}{p}}_{p,r}}, \]

where

\[ C^*(T) = \left( \sum_{j \in \mathbb{Z}} 2^{sjr} (1 - e^{-\lambda^{2^{s+1}T}})^{\frac{1}{2}} \| \Delta_j u_0 \|_{L^p_t} \right) \frac{1}{2}. \]

From \( \lim_{T \to 0} C^*(T) = 0 \) and (20), we can obtain that there exists sufficient small \( T_0 > 0 \), for \( T \in (0, T_0) \), such that

\[ \| u^k \|_{L^2_t B^{s+\frac{2}{p}}_{p,r}} \leq 2C^*(T), \]

and by (19) and (21), one has

\[ \| u^k \|_{L^p_t B^s_{p,r}} \leq 2 \| u_0 \|_{B^s_{p,r}} \]

for sufficiently small \( T > 0 \).

Next we prove (8). For the equation (6), multiplying the both side by \( |u^k|^{p-2} u^k \) and integrating over \( \mathbb{R}^2 \), we obtain

\[ \frac{1}{p} \frac{d}{dt} \| u^k \|_{L^p}^p + \int_{\mathbb{R}^2} (-\Delta)^{\frac{r}{2}} u^k |u^k|^{p-2} u^k dx = \int_{\mathbb{R}^2} [u^{k-1}(\partial_1 u^k + \partial_2 u^k)] |u^k|^{p-2} u^k dx. \]

By Hölder inequality, we have that

\[ \int_{\mathbb{R}^2} [u^{k-1}(\partial_1 u^k + \partial_2 u^k)] |u^k|^{p-2} u^k dx \]

\[ \leq \| u^{k-1}(\partial_1 u^k + \partial_2 u^k) \|_{L^p} \| u^k \|_{L^p}^{p-1} \]

\[ \leq C \| u^{k-1} \|_{B^{s+\frac{2}{p}}_{p,r}} \| u^k \|_{B^{s+\frac{2}{p}}_{p,r}} \| u^k \|_{L^p}^{p-1} \]

\[ \leq C \| u^{k-1} \|_{L^p} + \| u^{k-1} \|_{B^{s+\frac{2}{p}}_{p,r}} \| u^k \|_{B^{s+\frac{2}{p}}_{p,r}} \| u^k \|_{L^p}^{p-1}. \]

By Lemma 2.6, hence we infer that

\[ \frac{d}{dt} \| u^k \|_{L^p} \leq C \| u^{k-1} \|_{L^p} + \| u^{k-1} \|_{B^{s+\frac{2}{p}}_{p,r}} \| u^k \|_{B^{s+\frac{2}{p}}_{p,r}}. \]
Combing the integral in $[0, T]$ of (23) and embedding properties, one has
\[ \|u^k\|_{L^p} \leq \|u_0\|_{L^p} + C(T^{\frac{p}{2}} \|u^{k+1}\|_{L^\infty} + \|u^{k-1}\|_{L^\infty} + \|\dot{\Delta}^\frac{p}{2}\|) \|u^k\|_{L^2} + \|\dot{\Delta}^\frac{p}{2}\| \].
\[ (24) \]
For $T$ is small enough, taking the $L^\infty_T$ norm for above inequality, we obtain
\[ \|u^k\|_{L^\infty_T L^p} \leq 2\|u_0\|_{L^p}. \]
We denote $C(T) = 2C* (T)$, if $T$ is small enough, according to the Remark 6, we can get (8). So (7), (8) and (9) are proved.

**Step 3.** Strong convergence.

We only have to prove that $\{u^k\}$ is Cauchy sequence in $L^\infty_T B_{p,r}^s \cap L^2_T B_{p,r}^{s+\frac{p}{2}}$. We put $\tilde{u}^k = u^{k+1} - u^k$, according to the equation (6), $\tilde{u}^k$ satisfies that
\[
\begin{align*}
\partial_t \tilde{u}^k + (-\Delta)^{\frac{p}{2}} \tilde{u}^k &= u^k \partial_t \tilde{u}^k + \partial_2 \tilde{u}^k + \tilde{u}^{k-1}(\partial_1 u^k + \partial_2 u^k), \quad t > 0, x \in \mathbb{R}^2, \\
\tilde{u}^k(0, x) &= 0, \quad x \in \mathbb{R}^2.
\end{align*}
\]
Applying the operator $\dot{\Delta}$ to the sides of the equation (25), one has
\[
\begin{align*}
\partial_t \dot{\Delta}_j \tilde{u}^k + (-\Delta)^{\frac{p}{2}} \dot{\Delta}_j \tilde{u}^k &= u^k \partial_t \dot{\Delta}_j \tilde{u}^k + \partial_2 \dot{\Delta}_j \tilde{u}^k + [\dot{\Delta}_j, u^k](\partial_1 \tilde{u}^k + \partial_2 \tilde{u}^k) \\
&+ \tilde{u}^{k-1} \dot{\Delta}_j(\partial_1 u^k + \partial_2 u^k) + [\dot{\Delta}_j, \tilde{u}^{k-1}](\partial_1 u^k + \partial_2 u^k).
\end{align*}
\]
Multiplying the both sides of the equation (26) by $|\dot{\Delta}_j \tilde{u}^k|^{p-2} \dot{\Delta}_j \tilde{u}^k$ and integrating over $\mathbb{R}^2$. By the similar argument as in the derivation of (11), one has
\[
\frac{d}{dt} \|\dot{\Delta}_j \tilde{u}^k\|_{L^p} + \lambda 2^{\alpha_j} \|\dot{\Delta}_j \tilde{u}^k\|_{L^p} \leq \int_{\mathbb{R}^2} u^k \dot{\Delta}_j(\partial_1 \tilde{u}^k + \partial_2 \tilde{u}^k)|\dot{\Delta}_j \tilde{u}^k|^{p-2} \dot{\Delta}_j \tilde{u}^k \, dx \\
+ \int_{\mathbb{R}^2} [\dot{\Delta}_j, u^k](\partial_1 \tilde{u}^k + \partial_2 \tilde{u}^k)|\dot{\Delta}_j \tilde{u}^k|^{p-2} \dot{\Delta}_j \tilde{u}^k \, dx \\
+ \int_{\mathbb{R}^2} \dot{\Delta}_j(\tilde{u}^{k-1} |\partial_1 u^k + \partial_2 u^k|) |\dot{\Delta}_j \tilde{u}^k|^{p-2} \dot{\Delta}_j \tilde{u}^k \, dx.
\]
For (27), in the same way as in the derivation of (12), we obtain that
\[
\frac{d}{dt} \|\dot{\Delta}_j \tilde{u}^k\|_{L^p} + \lambda 2^{\alpha_j} \|\dot{\Delta}_j \tilde{u}^k\|_{L^p} \leq C(\|u^k \dot{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p} + \|\dot{\Delta}_j, u^k\|)(\partial_1 \tilde{u}^k + \partial_2 \tilde{u}^k)\|_{L^p} \\
+ \|\dot{\Delta}_j, \tilde{u}^{k-1}|\partial_1 u^k + \partial_2 u^k\|_{L^p} + \|\tilde{u}^{k-1} \dot{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p}.
\]
\[ (28) \]
From (28) and the argument in Step 2. If we set
\[ R_j(t) = C(\|u^k \dot{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p} + \|\dot{\Delta}_j, u^k\|)(\partial_1 \tilde{u}^k + \partial_2 \tilde{u}^k)\|_{L^p} \\
+ \|\dot{\Delta}_j, \tilde{u}^{k-1}|\partial_1 u^k + \partial_2 u^k\|_{L^p} + \|\tilde{u}^{k-1} \dot{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p}), \]
then
\[ \|\dot{\Delta}_j \tilde{u}^k\|_{L^p} \leq e^{-\lambda 2^{\alpha_j} t} R_j(t). \]
We estimate $\|\tilde{u}^k\|_{L^\infty_T B_{p,r}^{s+\frac{p}{2}}}$. In the same way as in the estimate of (16), the first term and the forth term of $R_j(t)$ can be estimated as
\[
\left\| 2^{j_s} \int_0^T \|u^k \dot{\Delta}_j(\partial_1 u^k + \partial_2 u^k)\|_{L^p} \, dt \right\|_{L^r} \leq CC(T)\|\tilde{u}^k\|_{L^\infty_T B_{p,r}^{s+\frac{p}{2}}}
\]
and
\[ \left\| 2^{js} \int_0^T \hat{u}^{k-1} \hat{\Delta}^i \hat{u}^k \right\|_{L^p} \leq CC(T) \left\| \hat{u}^{k-1} \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}}. \]

By Lemma 2.5 and (17), we deduce that
\[ \left\| 2^{js} \int_0^T \left[ \hat{\Delta}^i, u^k \right] (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^p} \leq CC(T) \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \]
and
\[ \left\| 2^{js} \int_0^T \left[ \hat{\Delta}^i, \hat{u}^{k-1} \right] (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^p} \leq CC(T) \left\| \hat{u}^{k-1} \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}}. \]

Thus we have
\[ \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \leq CC(T) \left( \left\| \hat{u}^{k-1} \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} + \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \right). \tag{30} \]

As the same way in (24), we obtain
\[ \left\| \hat{u}^k \right\|_{L^2_t L^p} \leq CC(T) \left( \left\| \hat{u}^{k-1} \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} + \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \right) , \]
so we conclude that
\[ \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \leq CC(T) \left( \left\| \hat{u}^{k-1} \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} + \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \right) . \]

Multiplying the both sides of (29) by \( 2^{s+\frac{3}{2}} \) and taking the \( L^p \) norm, applying Young's inequality, one has
\[ \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \leq \left\| 2^{(s+\frac{3}{2})j} \left( \int_0^T \left( e^{-\lambda 2^{\alpha_j} t} \hat{R}_j(t) \right)^2 dt \right)^{\frac{1}{2}} \right\|_{L^p} \]
\[ \leq \left\| 2^{(s+\frac{3}{2})j} \left( \int_0^T \hat{R}_j^2(t) dt \right)^{\frac{1}{2}} \right\|_{L^p} . \]

Next we estimate \( \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \). For the first term of the \( \hat{R}_j(t) \), in the same way of (15), we also obtain
\[ \left\| \hat{u}^k \hat{\Delta}_j (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^p} \leq 2^{-(s-\frac{3}{2})j} \left\| (-\Delta)^{\frac{s+\frac{3}{2}}{2}} \hat{u}^k \right\|_{L^p} \left\| (-\Delta)^{\frac{s+\frac{3}{2}}{2}} \hat{\Delta}_j (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^p} , \]
where \( \sigma = \alpha - 2 \), so we infer that
\[ 2^{(s+\frac{3}{2})j} \left( \int_0^T \left\| \hat{u}^k \hat{\Delta}_j (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^p}^2 dt \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_0^T \left( \left\| (-\Delta)^{\frac{s+\frac{3}{2}}{2}} \hat{u}^k \right\|_{L^p} \left\| (-\Delta)^{\frac{s+\frac{3}{2}}{2}} \hat{\Delta}_j (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^p}^2 \right)^{\frac{1}{2}} \right) \]
\[ \leq \left\| (-\Delta)^{\frac{s+\frac{3}{2}}{2}} \hat{u}^k \right\|_{L^2_t L^p} \left\| (-\Delta)^{\frac{s+\frac{3}{2}}{2}} \hat{\Delta}_j (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^2_t L^p} \]
\[ \leq \left\| \hat{u}^k \right\|_{L^2_t B^{s+\frac{3}{2}}_{p,r}} \cdot 2^{(s+\frac{3}{2})j} \left\| \hat{\Delta}_j (\partial_1 \hat{u}^k + \partial_2 \hat{u}^k) \right\|_{L^2_t L^p} . \]
According to Lemma 2.5, we infer that this implies that

\[ u \in L^\infty \cap L^2B_{p,r}^{s+\frac{\alpha}{2}}, \]

thus we have

\[ u \in L^\infty B_{p,r}^s, \]

the approximation equation, we can get a solution to (1) in 

\[ L^\infty B_{p,r}^s. \]

By the same argument, we obtain

\[ 2^{(s-\frac{\alpha}{2})j} \left( \int_0^T \| u_j \|_2 \right)^{\frac{1}{j}} \leq CC(T) \| u \|_L^\infty B_{p,r}^s. \]

According to Lemma 2.5, we infer that

\[ 2^{(s-\frac{\alpha}{2})j} \left( \int_0^T \| \Delta_j u \|_2 \right)^{\frac{1}{j}} \leq CC(T) \| \hat{u} \|_L^\infty B_{p,r}^s. \]

Thus, we have

\[ \| \hat{u} \|_L^\infty B_{p,r}^{s+\frac{\alpha}{2}} \leq CC(T) \left( \| \hat{u} \|_L^\infty B_{p,r}^s + \| \hat{u} \|_L^\infty B_{p,r}^s \right) \]

We denote the space 

\[ X = L^\infty B_{p,r}^s \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \]

and the norm

\[ \| f \|_X = \| f \|_L^\infty B_{p,r}^s + \| f \|_L^\infty B_{p,r}^s, \forall f \in X. \]

According to the \( T \to 0, (30) \) and (31), then there exists a positive constant \( C < 1 \) for sufficient small \( T > 0 \), such that

\[ \| u^{k+1} - u^k \|_X \leq C \| u^k - u^{k-1} \|_X. \]

This implies that \( \{ u_k \}_{k \in \mathbb{N}} \) is a Cauchy sequence in \( L^\infty B_{p,r}^s \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \), then \( u^k \) converges strongly to \( u \) in \( L^\infty B_{p,r}^s \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \). Thus by passing to the limit into the approximation equation, we can get a solution to (1) in \( L^\infty B_{p,r}^s \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \).

**Step 4. Uniqueness.**

Let \( u_1 \) and \( u_2 \) be two solutions of equation (1) with the same initial data and belong to the space \( L^\infty B_{p,r}^s \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \cap \tilde{L}^\infty \). Let \( u_{1,2} := u_1 - u_2 \), then we have

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u_{1,2} + (-\Delta) \tilde{u}_{1,2} = u_1 (\partial_1 u_{1,2} + \partial_2 u_{1,2}) + u_{1,2} (\partial_1 u_2 + \partial_2 u_2), & \quad t > 0, x \in \mathbb{R}^2, \\
u_{1,2}(0, x) = 0, & \quad x \in \mathbb{R}^2.
\end{array} \right.
\end{align*}
\]
According to the same way of (25) and (32), there exists a positive constant $C < 1$, such that
\[ \|u_1 - u_2\|_{\mathcal{X}} \leq C\|u_1 - u_2\|_{\mathcal{X}}. \] (34)
Thus $u_1 = u_2$, the uniqueness is proved.
Moreover, it is easy to show that $u \in C([0, T); B^s_{p,r})$, The reader can refer to [11, 16]. We complete the proof of Theorem 1.1.

4. Global existence for small data. In this section, we prove Theorem 1.2. Some important formulas of Section 3 are used in the proof.

Proof of Theorem 1.2. Let $k$ tend to infinity in (20), we can obtain
\[ \|u\|_{L^\infty_T B^s_{p,r}} \leq C(\|u_0\|_{B^s_{p,r}} + \|u\|_{L_T^\infty B^s_{p,r}}^2), \] (35)
where
\[ C^*(T) = \left( \sum_{j \in \mathbb{Z}} 2^{sj}\rho(1 - e^{-\lambda^2s^j+1T}) \|\hat{\Delta}_j u_0\|_{L^p_L} \right)^{\frac{1}{p}} \leq \|u_0\|_{B^s_{p,r}}. \]

Then there exists a positive constant $C$ such that
\[ \|u\|_{L^\infty_T B^{s+\frac{2}{p}}_{p,r}} \leq C(\|u_0\|_{B^{s+\frac{2}{p}}_{p,r}} + \|u\|_{L_T^\infty B^{s+\frac{2}{p}}_{p,r}}^2). \] (36)
By maximum principle and (36), we have
\[ \|u\|_{L^\infty_T B^{s+\frac{2}{p}}_{p,r}} \leq \|u_0\|_{B^{s+\frac{2}{p}}_{p,r}} + C\|u\|_{L_T^\infty B^{s+\frac{2}{p}}_{p,r}}^2. \] (37)
According to (35) and the condition of Theorem 1.2, we obtain
\[ \|u\|_{L^\infty_T B^{s+\frac{2}{p}}_{p,r}} \leq C(\|u_0\|_{B^{s+\frac{2}{p}}_{p,r}} + \|u\|_{L_T^\infty B^{s+\frac{2}{p}}_{p,r}}^2), \] (38)
thus, if $\epsilon < \frac{1}{4C^2}$, then for any $T \in (0, \infty)$, we deduce that
\[ \|u\|_{L^\infty_T B^{s+\frac{2}{p}}_{p,r}} \leq \frac{1 - \sqrt{1 - 4C^2\epsilon}}{2C} < \infty, \] (39)
and from (37) and (39), we have
\[ \|u\|_{L^\infty_T B^{s+\frac{2}{p}}_{p,r}} \leq \|u_0\|_{B^{s+\frac{2}{p}}_{p,r}} + \frac{(1 - \sqrt{1 - 4C^2\epsilon})^2}{4C} < \infty. \] (40)
Then the solution of equation (1) does not blow up in finite time, namely,
\[ u \in \tilde{L}^2(0, \infty; \tilde{B}^{s+\frac{2}{p}}_{p,r}(\mathbb{R}^2)) \cap L^\infty([0, \infty); \tilde{B}^s_{p,r}(\mathbb{R}^2)). \]

We complete the proof of Theorem 1.2.
5. Blow up in finite time. In this section, we prove Theorem 1.3. Before the proof is presented, we establish some propositions and lemmas. For the fractional Laplacian operator, there is an important lemma as follows.

**Lemma 5.1.** Let $\varphi \in W^{1,\infty}(\mathbb{R}^2)$, then for $0 < \alpha < 1$, there exists a constant $C = C(\alpha, \varphi) > 0$ such that

$$\|(-\Delta)^{\frac{\alpha}{2}} \varphi\|_{L^\infty} \leq C.$$

*Proof.* For $\varphi \in W^{1,\infty}(\mathbb{R}^2)$, $(-\Delta)^{\frac{\alpha}{2}} \varphi$ can be expressed as (see e.g. [15])

$$(-\Delta)^{\frac{\alpha}{2}} \varphi = C_\alpha \int_{\mathbb{R}^2} \frac{\varphi(x) - \varphi(y)}{|x - y|^{2+\alpha}} dy,$$

where

$$C_\alpha = \frac{\Gamma(1 + \frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}} \Gamma(1 - \frac{\alpha}{2})} \quad \text{and} \quad \Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt.$$

From which it follows that

$$(-\Delta)^{\frac{\alpha}{2}} \varphi = C_\alpha \int_{|x - y| \leq 1} \frac{\varphi(x) - \varphi(y)}{|x - y|^{2+\alpha}} dy + C_\alpha \int_{|x - y| > 1} \frac{\varphi(x) - \varphi(y)}{|x - y|^{2+\alpha}} dy = I_1 + I_2.$$

Furthermore, since $\alpha \in (0, 1)$, the first term $I_1$ can be estimated as

$$|I_1| \leq C_\alpha \int_{|x - y| \leq 1} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{2+\alpha}} dy \leq C_\alpha \|\nabla \varphi\|_{L^\infty} \int_{|x - y| \leq 1} \frac{1}{|x - y|^{1+\alpha}} dy < \infty,$$

and the second term $I_2$ can be estimated as

$$|I_2| \leq C_\alpha \int_{|x - y| > 1} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{2+\alpha}} dy \leq 2C_\alpha \|\varphi\|_{L^\infty} \int_{|x - y| > 1} \frac{1}{|x - y|^{2+\alpha}} dy < \infty.$$

Combining (42) and (43), the lemma is proved. \qed

**Remark 8.** The proof method of Lemma 5.1 is similar to Bournaveas, Calvez [6] and Sugiyama, Yamamoto, Kato [30], we recall the proof for convenience. And the $W^{1,\infty}(\mathbb{R}^2)$ is inhomogeneous Sobolev space when $p = \infty, s = 1$.

The next result establish the conservation properties of the solutions corresponding to a class of initial data. The following proposition plays an important role in the proof of Theorem 1.3.

**Proposition 2 (Conservation properties of solutions).** Assume that $0 < \alpha < 1$. Let initial data $u_0 \in L^{\infty}(\mathbb{R}^2)$, $\partial_1 u_0 + \partial_2 u_0 \in B^{s}_{2,1}(\mathbb{R}^2)$ with $s \geq 4$, and $\partial_1 u_0 + \partial_2 u_0 \geq 0$, then there exists the local solution $u$ of (1) remain $\partial_1 u + \partial_2 u \geq 0$ for $0 \leq t < T$, $T$ is maximal lifespan of $u$, and we have

$$\int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) dx = \int_{\mathbb{R}^2} (\partial_1 u_0 + \partial_2 u_0) dx, \quad 0 \leq t < T.$$

*Proof.* For the initial data $u_0(x)$, according to the proof of method in Theorem 1.1, it is easy to get the local existence of solution. Let $u$ be the corresponding maximal lifespan solution. Applying the $\partial_1, \partial_2$ to the both sides of equation (1) respectively, and summing up, we obtain that

$$\begin{cases}
\partial_t (\partial_1 u + \partial_2 u) + (-\Delta)^{\frac{\alpha}{2}} (\partial_1 u + \partial_2 u) = \nabla \cdot ((\partial_1 u + \partial_2 u)v), & t > 0, x \in \mathbb{R}^2, \\
v = (u, u), & t > 0, x \in \mathbb{R}^2, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^2.
\end{cases}$$
We divide the proof into two parts.

**Part 1.** Nonnegativity of $\partial_1 u + \partial_2 u$.

For $\partial_1 u_0 + \partial_2 u_0 \in B^s_{1,1}(\mathbb{R}^2)$ with $s \geq 4$. Some embedding relationship are as follows

$$
B^s_{1,1} \hookrightarrow B^{s-2(1-\frac{1}{p})}_{p,1} \hookrightarrow B^0_{p,1}, \quad B^s_{1,1} \hookrightarrow B^{s-2(1-\frac{1}{p})}_{p,1}, \quad p \geq 2,
$$

so we implies that

$$
\partial_1 u_0 + \partial_2 u_0 \in L^1(\mathbb{R}^2), \quad \partial_1 u_0 + \partial_2 u_0 \in B^{s-2(1-\frac{1}{p})}_{p,1}(\mathbb{R}^2).
$$

For (44), the proof similar to Theorem 1.1, we can conclude that there exists $T > 0$, such that

$$
\partial_1 u + \partial_2 u \in C((0, T]; B^{s-2(1-\frac{1}{p})}_{p,1}(\mathbb{R}^2)). \tag{45}
$$

For $0 < \alpha < 1$, by (44) and (45), one has

$$
\partial_1(\partial_1 u + \partial_2 u) \in C((0, T]; B^{s-2(1-\frac{1}{p})-1}_{p,1}(\mathbb{R}^2)),
$$

so we have

$$
\partial_1 u + \partial_2 u \in C^1((0, T]; B^{s-2(1-\frac{1}{p})-1}_{p,1}(\mathbb{R}^2)). \tag{46}
$$

We use the contradiction method. Let us denote $\Omega_T = (0, T] \times \mathbb{R}^2$, we can imply $\partial_1 u + \partial_2 u$ is bounded by (46). We assume that $\partial_1 u + \partial_2 u \geq 0$ is not true, then we can imply that there exists a constant $\delta > 0$ such that

$$
\inf_{(t, x) \in \Omega_T} (\partial_1 u + \partial_2 u)(t, x) = -\delta < 0. \tag{47}
$$

And we know that there exists $(t_0, x_0) \in \Omega_T$ satisfies

$$
(\partial_1 u + \partial_2 u)(t_0, x_0) = -\delta, \quad \partial_1(\partial_1 u + \partial_2 u)(t_0, x_0) = 0, \quad i = 1, 2. \tag{48}
$$

By (44), we can deduce that

$$
\begin{align*}
\partial_i(\partial_1 u + \partial_2 u)(t_0, x_0) &= \nabla \cdot ((\partial_1 u + \partial_2 u)v)(t_0, x_0) - (-\Delta)^{\frac{\alpha}{2}}(\partial_1 u + \partial_2 u)(t_0, x_0) \\
&= (\partial_1 u + \partial_2 u)^2(t_0, x_0) - (-\Delta)^{\frac{\alpha}{2}}(\partial_1 u + \partial_2 u)(t_0, x_0).
\end{align*} \tag{49}
$$

Since $\partial_1 u + \partial_2 u$ attains minimum at $(t_0, x_0)$, we obtain

$$
\begin{align*}
-(-\Delta)^{\frac{\alpha}{2}}(\partial_1 u + \partial_2 u)(t_0, x_0) &= C_\alpha \int_{\mathbb{R}^2} \frac{(\partial_1 u + \partial_2 u)(t_0, y) - (\partial_1 u + \partial_2 u)(t_0, x_0)}{|y - x_0|^\alpha + 2} dy \\
&\geq 0. \tag{50}
\end{align*}
$$

Combing with (48), (49) and (50), we can imply that

$$
\partial_i(\partial_1 u + \partial_2 u)(t_0, x_0) \geq (\partial_1 u + \partial_2 u)^2(t_0, x_0) = \delta^2. \tag{51}
$$

But this is obviously a contradiction to the fact that $\partial_1 u + \partial_2 u$ attains minimum at $(t_0, x_0) \in \Omega_T$. So we prove that

$$
(\partial_1 u + \partial_2 u)(t, x) \geq 0, \quad (t, x) \in \Omega_T.
$$

**Part 2.** $L^1$ conservation of $\partial_1 u + \partial_2 u$.

Integrating over $\mathbb{R}^2$ in (44), one has

$$
\frac{d}{dt} \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) dx + \int_{\mathbb{R}^2} (-\Delta)^{\frac{\alpha}{2}}(\partial_1 u + \partial_2 u) dx = \int_{\mathbb{R}^2} \nabla \cdot ((\partial_1 u + \partial_2 u)v) dx. \tag{52}
$$

According to (41), for any integrable function \( \varphi \), we have
\[
\int_{\mathbb{R}^2} (-\Delta)^{\alpha/2} \varphi(x) dx = C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \varphi(x) - \varphi(y) |x-y|^{2+\alpha} dx dy
\]
\[
= C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\varphi(x)}{|x-y|^{2+\alpha}} dx dy - C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\varphi(y)}{|x-y|^{2+\alpha}} dx dy
\]
\[
= C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\varphi(x)}{|x-y|^{2+\alpha}} dx dy - C_\alpha \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\varphi(x)}{|x-y|^{2+\alpha}} dx dy
\]
\[
= 0.
\]

Thus, for the second term of the left hand side in (52), we get
\[
\int_{\mathbb{R}^2} (-\Delta)^{\alpha/2} (\partial_1 u + \partial_2 u) dx = 0.
\]

And for the term of the right hand side in (52), due to the conservation condition, we obtain
\[
\int_{\mathbb{R}^2} \nabla \cdot ((\partial_1 u + \partial_2 u) v) dx = 0.
\]

From (52), (54) and (55), we imply that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) dx = 0.
\]

For \( 0 \leq t \leq T \), taking the integral in \([0,t]\) for two sides of (56), we have
\[
\int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) dx = \int_{\mathbb{R}^2} (\partial_1 u_0 + \partial_2 u_0) dx.
\]

This completes the proof of Proposition 1. \( \square \)

**Definition 5.2.** Let \( M \) be positive constant and \( \Omega \subset \mathbb{R}^2 \) is bound domain, the set \( A(M, \Omega) \) consists of functions \( f \) satisfying the following conditions:

1. \( f \) is antisymmetric about the \( x_1 + x_2 = 0 \). Namely, for any \( x^1, x^2 \in \mathbb{R}^2 \) is symmetric about the \( x_1 + x_2 = 0 \), then
   \[
   f(x^1) = -f(x^2).
   \]

2. For any \( x \in \mathbb{R}^2 \), \( \partial_1 f + \partial_2 f \geq 0 \), and
   \[
   M = \int_{\mathbb{R}^2} (\partial_1 f + \partial_2 f) dx = \int_{\Omega} (\partial_1 f + \partial_2 f) dx.
   \]

**Remark 9.** (1) For the equation (1), if the initial data \( u_0(x) \) satisfy the first condition of Definition 5.2, by the time splitting method (see e.g. [1]), the corresponding solutions \( u \) satisfies
   \[
   u(t, x^1) = -u(t, x^2), \quad t \geq 0.
   \]

(2) If the initial data \( u_0(x) \) satisfy the second condition of Definition 5.2, that is, the mass of the \( \partial_1 u_0 + \partial_2 u_0 \) is concentrated in the \( \Omega \).

If the solutions of equation (1) are global for a class of initial data, we can obtain the follow important lemma.

**Lemma 5.3.** Assume that \( 0 < \alpha < 1 \) and initial data \( u_0 \) satisfies the condition of Theorem 1.1. Let \( u \) be the corresponding solution and assume it is global. Then for fixed constant \( C^* \in (\frac{1}{2}, 1) \), there exists finite time \( T \) and bound domain \( \Omega(t) \), for any \( 0 \leq t \leq T \), we have
\[
\mu(\Omega(t)) \leq \mu(\Omega),
\]

where \( \mu(\cdot) \) denotes the measure.
where \( \mu \) is volume, and
\[
\int_{\Omega(t)} (\partial_1 u + \partial_2 u) \, dx \geq C^* \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) \, dx.
\]

Proof. Without loss of generality, we only consider the parts of \( u_0 \geq 0 \), then we can easy to get
\[
u(t,x) \geq 0. \tag{58}
\]
For the equation of (1), we use the characteristic method. We denote the characteristic curve which passes through \( x = (x_1, x_2) \) at \( t = 0 \) by \( X(t) = (X_1(t), X_2(t)) \). This curve satisfies
\[
\begin{cases}
\frac{dX_i(t)}{dt} = -u(t, X(t)), \\
X_i(0) = x_i, \quad i = 1, 2.
\end{cases} \tag{59}
\]
For any \( t \geq 0 \), according to the definition of (59), we can assume two characteristic curve
\[
X^1(t) = (X^1_1(t), X^1_2(t)), \quad X^1(0) = x^1 = (x^1_1, x^1_2) \in \Omega,
\]
\[
X^2(t) = (X^2_1(t), X^2_2(t)), \quad X^2(0) = x^2 = (x^2_1, x^2_2) \in \Omega,
\]
and
\[
x^1_1 - x^2_1 = x^1_2 - x^2_2.
\]
For \( A = (a_1, a_2), B = (b_1, b_2) \), we denote
\[
A \geq B \iff a_1 \geq b_1, a_2 \geq b_2.
\]
Due to \( u \) is global, if \( x^1 \geq x^2 \), we get
\[
X^1(t) \geq X^2(t).
\]
By \( \partial_1 u + \partial_2 u \geq 0 \), we deduce that
\[
u(t, X^1(t)) \geq u(t, X^2(t)),
\]
then for any \( t \), by (58) and (59), we obtain that
\[
\frac{dX^1_i(t)}{dt} \leq \frac{dX^2_i(t)}{dt}, \quad i = 1, 2,
\]
then we have
\[
X^1(t) - X^2(t) \leq x^1 - x^2, \tag{60}
\]
we denote \( \Omega(t) = \{ X(t) \in \mathbb{R}^2 | X(0) = x \in \Omega \} \), then \( \Omega(t) \) is bound domain, and we have
\[
\mu(\Omega(t)) \leq \mu(\Omega). \tag{61}
\]
By the definition of the characteristics of (59), one has
\[
\begin{cases}
\frac{du(t, X(t))}{dt} = -(-\Delta)^{\frac{\alpha}{2}} u(t, X(t)), \\
u(t, X(t))|_{t=0} = u_0(x).
\end{cases} \tag{62}
\]
For \( 0 < \alpha < 1 \) and the solution is global, by Lemma 5.1 and the properties of \( u(t, X(t)) \), we can assume that there exists a constant \( C_0 = C_0(u_0, \alpha) > 0 \), such that
\[
\|(-\Delta)^{\frac{\alpha}{2}} u(t, X(t))\|_{L^\infty} \leq C_0. \tag{63}
\]
For the initial data \( u_0 \), there exists a transformation of coordinates, for any \( x \in \mathbb{R}^2 \), we have
\[
\eta = \eta(x), \quad \zeta = \zeta(x),
\]
For any $T$ such that taking the integral in $[0, T)$, there exist $J$ and $ζ$ where $x$ and there exist $X$ and $R$.

By the same argument, we imply that (62) and (67), we obtain

$$
\int_Ω (\partial_1 u + \partial_2 u) dx = \sqrt{2} \int_Ω \partial n dx
$$

and

$$
\int_Ω (\partial_1 u + \partial_2 u) dx = \sqrt{2} \int_Ω \partial n dx
$$

By (66), (68) and (69), we obtain

$$
\| (-Δ)^{\frac{σ}{2}} (u(t, X^2(t)) - u(t, X^1(t))) \|_{L^∞} \leq 2C_0.
$$

By (66), (68) and (69), we obtain

$$
\int_Ω (\partial_1 u + \partial_2 u) dx = \sqrt{2} \int_Ω \partial n dx
$$

taking the integral in $[0, T]$ for two sides of (70), one has

$$
\int_Ω (\partial_1 u + \partial_2 u) dx \geq M - 2√2(b - a)C_0 t.
$$
Then for $C^* \in (\frac{1}{2}, 1)$, let 
\[ T = \frac{(1 - C^*)M}{2\sqrt{2(b-a)}C_0}, \]
for any $0 \leq t \leq T$, we obtain 
\[ \int_{\Omega(t)} (\partial_1 u + \partial_2 u) dx \geq M - 2\sqrt{2(b-a)}C_0 T = C^* M. \quad (72) \]
Then for any $0 \leq t \leq T$, we have 
\[ \int_{\Omega(t)} (\partial_1 u + \partial_2 u) dx \geq C^* \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) dx. \quad (73) \]
The lemma is proved. \hfill \Box

Now we prove main theorem. In the supercritical case, Theorem 1.3 gives the existence of blowing-up solutions for a class of initial data.

**Proof of Theorem 1.3.** We argue by contradiction. For the initial data $u_0$, let $u$ be the corresponding solution of (1), and it is global. We define a non-negative function $\omega(x) \in C^\infty_c(\mathbb{R}^2)$, it is satisfies 
\[ 0 \leq \omega(x) \leq 1, \quad \omega(x) = 1, \quad x \in \Omega. \]
By nonlinear maximum principle (see e.g. [19]), we can assume that there exists a constant $C(\alpha) > 0$, such that 
\[ (-\Delta)^{\frac{\alpha}{2}} \omega(x) \geq \frac{C(\alpha)}{||\omega||^2_{L^2}}, \quad x \in \Omega. \quad (74) \]
For $0 \leq t \leq T$, we can define weight function $\tilde{\omega}(t, x)$. We denote 
\[ \tilde{\omega}(t, X(t)) = \tilde{\omega}(0, X(0)), \quad (75) \]
where $X(t)$ is satisfies (59) and if it passes through $x_0$ at $t = 0$, we define 
\[ \tilde{\omega}(0, x_0) = \omega(x_0). \]
For any $x_0 \in \mathbb{R}^2$ and fixed $t$, let $x = X(t)$, we obtain 
\[ \tilde{\omega}(t, x) = \tilde{\omega}(t, X(t)). \]
We can assume that $\tilde{\omega}(t, x) \in C^\infty_c(\mathbb{R}^2)$. By Lemma 5.3 and (74), we imply that 
\[ (-\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \geq \frac{C(\alpha)}{||\tilde{\omega}(t)||^2_{L^2}}, \quad x \in \Omega(t), \quad (76) \]
and by the definition of $\tilde{\omega}(t, x)$, we have 
\[ \frac{d}{dt}\tilde{\omega}(t, x) = 0. \]
So we can infer that 
\[ \partial_t((\partial_1 u + \partial_2 u)(t, x)) \tilde{\omega}(t, x) \]
\[ = \partial_t(\partial_1 u + \partial_2 u)(t, x) \tilde{\omega}(t, x) + (\partial_1 u + \partial_2 u)(t, x) \partial_t \tilde{\omega}(t, x) \quad (77) \]
\[ = \partial_t(\partial_1 u + \partial_2 u)(t, x) \tilde{\omega}(t, x) + (\partial_1 u + \partial_2 u)(t, x)(\partial_1 \tilde{\omega} + \partial_2 \tilde{\omega})u. \]
For the equation of (44), we first compute the truncated integral by \( \tilde{\omega}(t, x) \), we have
\[
\int_{\mathbb{R}^2} \partial_t (\partial_1 u + \partial_2 u) \tilde{\omega}(t, x) \, dx
= \frac{d}{dt} \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) \tilde{\omega}(t, x) \, dx - \int_{\mathbb{R}^2} u (\partial_1 u + \partial_2 u) (\partial_1 \tilde{\omega} + \partial_2 \tilde{\omega})(t, x) \, dx
\]
\[
= \int_{\mathbb{R}^2} \nabla \cdot ((\partial_1 u + \partial_2 u) v) \tilde{\omega}(t, x) \, dx - \int_{\mathbb{R}^2} (-\Delta)^{\frac{\alpha}{2}} (\partial_1 u + \partial_2 u) \tilde{\omega}(t, x) \, dx
\]
\[
= - \int_{\mathbb{R}^2} ((\partial_1 u + \partial_2 u) v) \cdot \nabla \tilde{\omega}(t, x) \, dx - \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) (-\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx
\]
\[
= - \int_{\mathbb{R}^2} u (\partial_1 u + \partial_2 u) (\partial_1 \tilde{\omega} + \partial_2 \tilde{\omega})(t, x) \, dx - \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) (-\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx.
\]
Then we can deduce that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) \tilde{\omega}(t, x) \, dx = - \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx
\]
\[
= - \int_{\Omega(t)} (\partial_1 u + \partial_2 u) (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx - \int_{\mathbb{R}^2 \setminus \Omega(t)} (\partial_1 u + \partial_2 u) (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx.
\] (78)

For \( 0 < t \leq T \), if we denote
\[
C_1(t) = \min_{x \in \Omega(t)} \{ (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \} \geq \frac{C(\alpha)}{\| \tilde{\omega}(t) \|_{L^2}^\alpha},
\] (79)
then first term of the right hand side in (78), by Proposition 2, Lemma 5.3, we obtain
\[
\int_{\Omega(t)} (\partial_1 u + \partial_2 u) (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx \geq C_1(t) C^* \int_{\mathbb{R}^2} (\partial_1 u_0 + \partial_2 u_0) \, dx,
\] (80)
and for the second term of the right hand side in (78), if we denote
\[
C_2(t) = \min_{x \in \mathbb{R}^2} \{ (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \} < 0,
\] (81)
then we have
\[
- \int_{\mathbb{R}^2 \setminus \Omega(t)} (\partial_1 u + \partial_2 u) (\Delta)^{\frac{\alpha}{2}} \tilde{\omega}(t, x) \, dx \leq - C_2(t) (1 - C^*) \int_{\mathbb{R}^2} (\partial_1 u_0 + \partial_2 u_0) \, dx.
\] (82)
From (78), (80) and (82), we deduce that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) \tilde{\omega}(t, x) \, dx
\]
\[
\leq - (C_1(t) C^* + C_2(t) (1 - C^*)) \int_{\mathbb{R}^2} (\partial_1 u_0 + \partial_2 u_0) \, dx.
\] (83)
If \( \Omega \) is small, according to (61) and the definition of \( \tilde{\omega}(t, x) \), we have
\[
\int_0^T (C_1(t) C^* + C_2(t) (1 - C^*)) \, dt > 1,
\] (84)
and by Proposition 2 and the definition of \( \tilde{\omega}(t, x) \), we deduce that
\[
\int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u) \tilde{\omega}(t, x) \, dx < \int_{\mathbb{R}^2} (\partial_1 u_0 + \partial_2 u_0) \, dx.
\] (85)
From (83), (84) and (85), we deduce that there exists $0 < t_0 < T$, such that
\[
\int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u)(t_0, x)\tilde{\omega}(t_0, x)dx < 0.
\] (86)

However, by the Proposition 2 and the definition of $\tilde{\omega}(t, x)$, we have
\[
\int_{\mathbb{R}^2} (\partial_1 u + \partial_2 u)\tilde{\omega}(t, x)dx \geq 0,
\]
but this is a contradiction with (86). Because of the $\partial_1 u + \partial_2 u$ is $L^1(\mathbb{R}^2)$ conservation and nonzero, thus there exists $0 < T_0 < \infty$, we have
\[
\lim_{t \to T_0} \|\partial_1 u + \partial_2 u\|_{L^\infty(\mathbb{R}^2)} = +\infty.
\]
We complete the proof of Theorem 1.3. \hfill \Box

**Remark 10.** Without loss of generality, in (72), we can assume
\[
M = 1, \quad b - a = 1, \quad 2\sqrt{2}(b - a)C_0T = \frac{1}{4}M,
\]
so we have
\[
T = \frac{1}{8\sqrt{2}C_0}.
\] (87)

According to (61), (76) and (83), if $\Omega$ is small, then there exists a constant $C_3(t) > \frac{1}{2}$, such that
\[
\frac{3}{4} C_1(t) + \frac{1}{4} C_2(t) \geq C_3(t)C_1(t) \geq C(\alpha) \frac{C_3(t)}{\|\tilde{\omega}(t)\|^\alpha_{L^2}}.
\] (88)

By (63), we can assume that there exist $x_0 > 0$ and $C > 0$, such that $C_0 = \frac{C}{x_0}$ and
\[
\|(-\Delta)^\frac{\alpha}{2} u(t, x)\|_{L^\infty} \leq \frac{C}{x_0},
\]
then for
\[
u_0 t - \frac{1}{2} \frac{C}{x_0^2} t^2 = x_0.
\] (89)

Due to $0 < \alpha < 1$, if $x_0$ is small, let $t_1 > 0$ be the solution of (89), we deduce that
\[
t_1 \ll T,
\]
and we can obtain that
\[
\|\tilde{\omega}(t_1)\|_{L^2} \ll \|\omega\|_{L^2}.
\]

According to (87) and (88), if $\Omega$ is small, we have
\[
\int_0^T \left(\frac{3}{4} C_1(t) + \frac{1}{4} C_2(t)\right) dt \\
\geq C(\alpha) \int_0^T \frac{C_3(t)}{\|\tilde{\omega}(t)\|^\alpha_{L^2}} dt \\
= C(\alpha) \int_0^{t_1} \frac{C_3(t)}{\|\tilde{\omega}(t)\|^\alpha_{L^2}} dt + C(\alpha) \int_{t_1}^T \frac{C_3(t)}{\|\tilde{\omega}(t)\|^\alpha_{L^2}} dt > 1.
\]

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