The deconfinement phase transition in Yang-Mills theory with general Lie group $G$

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We present numerical results for the deconfinement phase transition in $Sp(2)$ and $Sp(3)$ Yang-Mills theories in $(2+1)$-D and $(3+1)$-D. We then make a conjecture on the order of this phase transition in Yang-Mills theories with general Lie groups $G = SU(N)$, $SO(N)$, $Sp(N)$ and with exceptional groups $G = G(2)$, $F(4)$, $E(6)$, $E(7)$, $E(8)$.

1. Introduction and Overview

Let us consider Yang-Mills theory with gauge symmetry group $G$ on a periodic lattice in $(d+1)$-D with the Wilson action. This theory is invariant under the multiplication by an element of the center subgroup $C(G)$ of $G$ of all time-like links in a given time-slice. This global center symmetry is unbroken at low temperatures and it gets spontaneously broken at the deconfinement phase transition. The Polyakov loop $\Phi$ transforms non-trivially under this symmetry and thus is an order parameter for the deconfinement transition.

Integrating out the spatial degrees of freedom of the Yang-Mills theory, one can write down an effective action for $\Phi$. It describes a scalar model with global symmetry $C(G)$ in $d$-D: the $C(G)$-symmetric confined phase of the gauge theory corresponds to the disordered phase of the scalar model, while the $C(G)$-broken deconfined phase has its counterpart in the ordered phase. Svetitsky and Yaffe [1] argued that the interactions in the effective description are short ranged. Hence, if the deconfinement phase transition in the Yang-Mills theory is second order, approaching criticality, the details of the complicated short-ranged effective scalar action become progressively unimportant. Only the dimensionality $d$ and the symmetry $C(G)$ of the scalar model are relevant.

Thus, one can exploit the universality of the critical behavior to use a simple scalar model to obtain information about the much more complicated Yang-Mills theory. If the phase transition is first order the correlation length does not diverge: there are no universal features and the $G$-symmetric Yang-Mills theory in $(d+1)$-D does not share the critical behaviour with a $C(G)$-symmetric $d$-D scalar model.

Yang-Mills theory on the lattice is naturally formulated in terms of group elements while in the continuum the fundamental field is the gauge potential, living in the algebra. An algebra can generate different groups, however it is natural to expect that lattice Yang-Mills theories whose gauge groups correspond to the same algebra have the same continuum limit [2]. Hence, instead of $SO(N)$ we consider its covering group $Spin(N)$. Keeping this in mind, we look at the center subgroups $C(G)$ of the various simple Lie groups $G$

$C(SU(N)) = \mathbb{Z}(N)$; $C(Sp(N)) = \mathbb{Z}(2)$ \hspace{1cm} (1)

$C(SO(N)) \rightarrow C(Spin(N)) = \begin{cases} \mathbb{Z}(2); & N \text{ odd} \\ \mathbb{Z}(2)^2; & N = 4k \\ \mathbb{Z}(4); & N = 4k+2 \end{cases}$ \hspace{1cm} (2)

$C(G(2)) = C(F(4)) = C(E(8)) = \{1\}$ \hspace{1cm} (3)

$C(E(6)) = \mathbb{Z}(3)$; $C(E(7)) = \mathbb{Z}(2)$ \hspace{1cm} (4)

Many numerical simulations in $(2+1)$-D and $(3+1)$-D have been performed for $SU(N)$ Yang-Mills theory in order to investigate the order of the deconfinement transition and – if it is second order – to check the validity of the Svetitsky-Yaffe

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conjecture. A recent study on this subject can be found in [3]. The currently known results are:

- (3 + 1)-D: for $N = 2$ – consistent with the Svetitsky-Yaffe conjecture – the deconfinement transition is second order, in the universality class of the 3-D Ising model; for $N = 3, 4, 6, 8$ the phase transition is first order with no universal features.
- (2 + 1)-D: fluctuations are stronger and the deconfinement phase transition turns out to be second order for $N = 2, 3, 4$. In agreement with the Svetitsky-Yaffe conjecture, the 2-D universality classes are, respectively, those of the Ising, 3-state Potts, and Ashkin-Teller models.

The $SU(N)$ branch is not a good choice to study the relation between the order of the deconfinement transition and the size of the group. In fact, when increasing the size $N^2 - 1$, also the center $Z(N)$ changes. In order to disentangle these two features we have considered $Sp(N)$ Yang-Mills theory. In this case, the available universality class is fixed and the relevance of the size of the group on the order of the deconfinement transition can be directly addressed.

### 2. $Sp(N)$ Yang-Mills Theory

The matrices $U \in SU(2N)$ with the property

$$U^* = JUJ^+, \quad \text{where} \quad J = i\sigma_2 \otimes 1$$

form the symplectic group $Sp(N)$; $\sigma_2$ is the imaginary Pauli matrix and $1$ is the $N \times N$ unit matrix. From the definition (5), it follows that the Hermitian generators $H$ of $Sp(N)$ satisfy the constraint $H^* = -JHJ^+ = JHJ$. Then we can write the following general forms for $U$ and $H$

$$U = \begin{pmatrix} W & X \\ -X^* & W^* \end{pmatrix}, \quad H = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix}$$

where $A, B, C, D$ are $N \times N$ complex matrices satisfying: $A = A^+, B = B^T, WW^+ + XX^* = 1$ and $WXX^T = XXW^T$. Counting the number of degrees of freedom, it follows that $Sp(N)$ has $N(2N + 1)$ generators and its rank is $N$. For the center elements – since they are multiples of the unit matrix – it holds that $X = 0$ and $W = W^*$; hence $C(\text{Sp}(N)) = Z(2)$. We note that (5) implies that $Sp(N)$ is a pseudo-real group with the special cases $Sp(1) = SU(2)$ and $Sp(2) = Spin(5)$. The $Sp(N)$ Yang-Mills theory on the lattice can be formulated in the usual way in terms of group-valued link variables. We have carried out our simulations using the Wilson plaquette action.

### 3. Numerical Results

In the next two subsections we report on the results of numerical simulations in $Sp(2)$ and $Sp(3)$ Yang-Mills theories in (2 + 1)-D and (3 + 1)-D.

#### 3.1. $Sp(2)$ Yang-Mills Theory

As a first step we have scanned the expectation value of the plaquette from the strong to the weak coupling regime. We find no bulk phase transition that might interfere with the study of the deconfinement transition. In (2 + 1)-D we observe a second order deconfinement transition, signalled by the broadening of the probability distribution of $\Phi$ and, hence, by the increase of the Polyakov loop susceptibility $\chi$ at criticality. A finite size scaling analysis confirms the expectation that the universality class is that of the 2-D Ising model. Fig. 1 shows the collapse of $\chi$ data – collected on lattices of different sizes $L^2 \times 2$ and at various gauge couplings $\beta$ – on a single curve. We also plot rescaled $\chi$ data for $SU(2)$ Yang-Mills theory in (2 + 1)-D: it again deconfines with a second order transition in the 2-D Ising universality class.

The two sets agree excellently. In (3 + 1)-D – contrary to what one might have expected – the probability distribution of $\Phi$ in the critical region shows the coexistence of the symmetric and of the broken phases, indicating that the deconfinement...
transition is first order. Fig. 2 indeed shows that $\chi$ scales with the spatial volume $L^3$ at criticality.

Figure 2. Scaling of $\chi$ in $Sp(2)$ Yang-Mills theory on $L^3 \times 2$ lattices. We estimate $\beta_c = 6.4643(3)$.

3.2. $Sp(3)$ Yang-Mills Theory

The results of $Sp(2)$ Yang-Mills theory show that in (2+1)-D fluctuations are stronger than in (3+1)-D and the deconfinement transition is second order. Expecting that the larger the group the weaker the fluctuations, we have also investigated the deconfinement transition in $Sp(3)$ Yang-Mills theory. Consistent with this picture, we find that (2+1)-D $Sp(3)$ Yang-Mills theory has a first order deconfinement transition. The probability distribution of $\Phi$ in the critical region, indeed displays the coexistence of the broken and of the symmetric phases. In (3+1)-D – similar to the $Sp(2)$ case – $Sp(3)$ Yang-Mills theory deconfines with a first order transition. Fig. 3 shows tunneling events between the coexisting symmetric and the broken phases. $Sp(3)$ Yang-Mills theory also has no bulk phase transition in (2+1)-D and (3+1)-D.

4. Conjecture

Our numerical results show that (2+1)-D $Sp(2)$ Yang-Mills theory has a second order deconfinement transition in the 2-D Ising universality class. However, (3+1)-D $Sp(2)$, (2+1)-D and (3+1)-D $Sp(3)$ Yang-Mills theories deconfine with a non-universal first order transition. Hence, despite the fact that a universality class is available, Yang-Mills theory can have a non-universal first order deconfinement transition. A non-trivial center plays no role in determining the order of this transition. Instead our $Sp(N)$ and the $SU(N)$ results indicate that the order of the deconfinement transition is a dynamical issue related to the size of the gauge group. We conjecture that the difference in the number of the relevant degrees of freedom between the confined phase (color singlet glueballs) and the deconfined phase (gluon plasma) determines the order of the deconfinement transition. Thus, we expect that in (3+1)-D only $SU(2)$ Yang-Mills theory has a second order deconfinement transition; in (2+1)-D, due to stronger fluctuations, only $SU(N)$, $N = 2, 3, 4,$ and $Sp(2)$ Yang-Mills theories should have second order transitions. According to this picture, $E(6)$ and $E(8)$ Yang-Mills theories should also have a first order transition due to the large size of the groups: 78 and 133 generators, respectively. For Yang-Mills theories with trivial center gauge groups $G(2), F(4), E(8)$ there is no compelling reason for a finite temperature phase transition. Then, although a first order transition can not be ruled out, we expect a crossover.

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