GLOBAL BEHAVIOR OF DELAY DIFFERENTIAL EQUATIONS
MODEL OF HIV INFECTION WITH APOPTOSIS

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Abstract. In this paper, a class of delay differential equations model of HIV infection dynamics with nonlinear transmissions and apoptosis induced by infected cells is proposed, and then the global properties of the model are considered. It shows that the infection-free equilibrium of the model is globally asymptotically stable if the basic reproduction number $R_0 < 1$, and globally attractive if $R_0 = 1$. The positive equilibrium of the model is locally asymptotically stable if $R_0 > 1$. Furthermore, it also shows that the model is permanent, and some explicit expressions for the eventual lower bounds of positive solutions of the model are given.

1. Introduction. The human immunodeficiency virus (HIV) is a slowly replicating retrovirus that causes the acquired immunodeficiency syndrome (AIDS) [52]. HIV infects vital cells such as helper T cells, specifically CD4$^+$ T cells, dendritic cells and macrophages in the human immune system [3]. The infection of HIV which can self-replicate leads to low levels of CD4$^+$ T cells through a number of mechanisms, including apoptosis of uninfected bystander cells (see, e.g., [17, 18, 45]). The cell-mediated immunity is easily inactivated, and the body becomes gradually very susceptible to opportunistic infections if CD4$^+$ T cells fall below a critical level.

Mathematical models have made considerable contributions for deep understanding of the dynamics of HIV infection. In 1981, Anderson and May [1] originally proposed a classic model describing dynamics of microparasites with a free-living infective stage in a population of invertebrate hosts. Afterwards, Nowak and Bangham [40] developed this classic model to the model known as the basic model (ordinary differential equations model) of dynamics of HIV infection, which is well-accepted and is investigated by theorists and experimentalists (see, e.g., [14, 16, 39, 42, 43, 44] and the references therein).

The basic model was further developed to consider the time course of HIV viral infection and replication kinetics by delay differential equations (DDEs) models [2, 6, 22, 32, 41, 42, 51]. For the infection process, the principle of mass action cannot account for the contributions of the interaction between uninfected cells and viral particles to the growth rate of the cells. To consider the contributions, the
bilinear infection rate is replaced with the saturation infection rate [53], Holling type II functional response [25], Beddington-DeAngelis functional response [23], Crowley-Martin functional response [34], or more general nonlinear infection rate [24, 31, 37, 46].

With the development in the field of science and technology, we had got a clear recognition concerning the death of CD4$^+$ T cells. Embretson et al. [11] had pointed out that only a small subsection (< 0.1%) of CD4$^+$ T cells were productively infected though the high viral burden and turnover throughout the course of HIV infection. Then it was demonstrated by Gougeon et al. [19] that the extent of apoptosis was significantly higher in CD4$^+$ T, CD8$^+$ T and B cells in comparison to uninfected individuals, and was correlated with disease progression. Following the discussion of Carbonari et al. [4] (see also Selliah and Finkel [45]), the number of apoptotic CD4$^+$ T cells in lymphoid tissue is larger than that of productively infected CD4$^+$ T cells, which implies the occurrence of bystander cell death (see Fig. 1). Thus, HIV gene expression products can produce toxic substances, which directly or indirectly induce apoptosis in uninfected CD4$^+$ T cells. Furthermore, experiments [21, 38] show the result that apoptosis occurs in the absence of viral replication when infected and uninfected cells are cultured together. These data suggest that viral proteins associate with uninfected CD4$^+$ T cells, and induce an apoptotic signal which induces death of uninfected CD4$^+$ T cells (see, e.g., [5, 12, 33, 45]).

\[
\begin{align*}
    x'(t) &= s - dx(t) - cx(t)y(t) - \beta x(t)v(t), \\
    y'(t) &= e^{-\mu \tau} \beta x(t - \tau)v(t - \tau) - py(t), \\
    v'(t) &= ky(t) - uv(t),
\end{align*}
\]

where $x(t)$, $y(t)$, and $v(t)$ denote the population of uninfected cells, infected cells, and viruses at time $t$, respectively. All parameters of (1) are assumed to be positive constants, and the biological meanings are as follows. The parameter $s$ represents the rate at which new uninfected cells are generated. The parameters $d$ and $c$ indicate the specific death rate of uninfected cells, and the apoptosis rate of uninfected cells induced by infected cells, respectively. The parameter $\beta$ is the rate constant characterizing infection of the cells. The term $e^{-\mu \tau}$ denotes the survival probability of infected cells in the latent period $\tau(\geq 0)$, from the entry of viral particles into the cell to the release of new viral particles. Here, $1/\mu$ is the average lifetime of the cell during the latent period. The parameter $p$ represents the death rate of infected cells either due to the action of the virus or the immune system. The infected cells
produce new viruses at the rate \( k \) during their lifespan which on average has the length \( 1/p \). The parameter \( u \) indicates the rate at which viruses are cleared from the system. Here, the mass product type function \( cx(t)y(t) \) is applied for the apoptosis effect of infected cells on uninfected cells, and [7] has analyzed the stability of equilibria and the uniform persistence for (1).

Notice that the apoptosis effect considered in (1) is different from the cell-to-cell infection of HIV studied in some literature. For instance, Lai and Zou [35] recently investigated a dynamical model that incorporated both the cell-to-cell infection and the virus-to-cell infection mode. They considered a well-mixed situation and no multiple infection for both modes of transmission, analyzed the stability of equilibria and the persistence for the model. Afterwards, Lai and Zou [36] also studied the virus dynamics model which combined cell-free virus transmission and cell-to-cell transfer of HIV, and the effects of cell-to-cell transfer of HIV with logistic uninfected cell growth. They analyzed the stability of equilibria and Hopf bifurcation for the model. In cell-to-cell infection mode, a large number of viral particles can be transferred from infected cells to uninfected cells through the formation of virological synapses. With this mechanism, the infected cells can cause uninfected cells to become infected ones. In (1), the cell-to-cell transfer was not considered for mathematical tractability, since it is difficult to show the global dynamical behavior of (1).

In this paper, based on (1) and [24, 31, 37, 46], we consider the following DDEs model with more complex and general nonlinear infection processes.

\[
\begin{aligned}
x'(t) &= s - g(y(t))x(t) - f(x(t), v(t)), \\
y'(t) &= e^{-\mu\tau}f(x(t-\tau), v(t-\tau)) - py(t), \\
v'(t) &= ky(t) - uv(t).
\end{aligned}
\]  

(2)

Here, the function \( g(y(t)) \) which satisfies some prescribed conditions, denotes the combined rate of specific death of uninfected cells and apoptosis induced by infected cells; the function \( f(x(t), v(t)) \) under some prescribed conditions, indicates the rate of uninfected cells becoming infected by the HIV viral particles. All other parameters in (2) have completely the same biological meanings as that of (1). The global analysis of (2) is more complicated than that of (1), and we propose a specific analyze method for proving the permanence of (2).

In the following, we suppose that the function \( f(x, v) \) has the similar properties as assumed in [24], that is,

\( \text{(H1)} \) Both partial derivatives \( f_x(x, v) \) and \( f_v(x, v) \) are continuous for \( x, v \geq 0 \) as well as \( f_x(x, v) > 0 \) for all \( x \geq 0, v > 0 \), and \( f_v(x, v) > 0 \) for all \( x > 0, v \geq 0 \). Moreover, \( f(0, v) = f(x, 0) = 0 \) for all \( x, v > 0 \).

For the function \( g(y) \), we suppose that

\( \text{(H2)} \) \( g(y) \geq g(0) > 0 \) and \( g'(y) > 0 \) for \( y \geq 0 \).

It is easy to see that \( g(y) = d + cy \) and \( f(x(t), v(t)) = \beta x(t)v(t) \) for (1). We consider the following the initial conditions of (2):

\[
x(\theta) = \phi_1(\theta), \quad y(\theta) = \phi_2(\theta), \quad v(\theta) = \phi_3(\theta), \quad \theta \in [-\tau, 0],
\]

(3)

where \( \phi = (\phi_1, \phi_2, \phi_3)^T \in C \) such that \( \phi_i(\theta) \geq 0 \) \((i = 1, 2, 3)\). Here, \( C \) denotes the Banach space \( C([-\tau, 0], \mathbb{R}^3) \) of continuous functions mapping from the interval \([-\tau, 0]\) to \( \mathbb{R}^3 \) equipped with the sup-norm.

The organization of this paper is as follows. In Section 2, we prove the global existence, nonnegativity and boundedness of the solutions of (2), and the existence
of positive equilibria. In Section 3, by using the method of Lyapunov functionals, we prove the global properties of infection-free equilibrium, and we consider the local asymptotic stability of positive equilibrium and the permanence of (2). Furthermore, we obtain an explicit expression on the eventual lower bound of each positive solution of (2) based on some techniques shown in [50].

2. Global existence, nonnegativity and boundedness of solutions and existence of positive equilibria. First, we have the following properties about the well-posedness of (2).

Theorem 2.1. Suppose that the functions \( f(x, v) \) and \( g(y) \) satisfy the conditions (H1) and (H2). Then the solution \((x(t), y(t), v(t))^{T}\) of (2) with the initial conditions (3) exists, is unique and nonnegative on \([0, +\infty)\), which satisfies

\[
\limsup_{t \to +\infty} x(t) \leq \frac{s}{g(0)}, \quad \limsup_{t \to +\infty} y(t) \leq \frac{se^{-\mu \tau}}{l}, \quad \limsup_{t \to +\infty} v(t) \leq \frac{se^{-\mu \tau}}{lu},
\]

where \( l = \min\{g(0), p\} \).

Proof. From local existence and uniqueness theorem of solutions of DDEs (see, e.g., [28, 29]), the solution \((x(t), y(t), v(t))^{T}\) of (2) with the initial conditions (3) is existent and unique on \([0, T_0]\) for some constant \( T_0 > 0 \). First, let us show that the solution \((x(t), y(t), v(t))^{T}\) is nonnegative on \([0, T_0]\). For any given \( b \in (0, T_0) \) and any given \( \varepsilon > 0 \), \((x(t, \varepsilon), y(t, \varepsilon), v(t, \varepsilon))^{T}\) is the solution of the following model

\[
\begin{aligned}
&x'(t) = s - g(y(t))x(t) - f(x(t), v(t)), \\
y'(t) &= e^{-\mu \tau} f(x(t - \tau), v(t - \tau)) - p y(t) + \varepsilon, \\
v'(t) &= ky(t) - uw(t) + \varepsilon,
\end{aligned}
\]

with the initial conditions (3) for \( t \in [0, b] \). The solution \((x(t, \varepsilon), y(t, \varepsilon), v(t, \varepsilon))^{T}\) is a continuous function with the parameter \( \varepsilon \), thus the solution \((x(t, \varepsilon), y(t, \varepsilon), v(t, \varepsilon))^{T}\) is uniformly existent on \([0, b]\) for sufficiently small \( \varepsilon > 0 \). Assume that \( x(t, \varepsilon) \) loses its positivity on \((0, b]\). According to the nonnegativity of \((x(0, \varepsilon), y(0, \varepsilon), v(0, \varepsilon))^{T}\) and the first equation of (5), it has that there exists a \( t \in (0, b] \) such that

\[
\bar{t} = \sup\{t \in (0, b] \mid x(s, \varepsilon) > 0, s \in (0, t]\}. 
\]

Then \( x(\bar{t}, \varepsilon) = 0 \) and \( x'(\bar{t}, \varepsilon) \leq 0 \). Hence, it follows easily from the first equation of (5) that \( x'(\bar{t}, \varepsilon) = s > 0 \). Clearly, this is a contradiction. Therefore, \( x(t, \varepsilon) > 0 \) for all \( t \in (0, b] \). Further, with the similar argument, we can verify from the second and the third equations of (5) that \( y(t, \varepsilon) > 0, v(t, \varepsilon) > 0 \) for \( t \in (0, b] \).

Now, let \( \varepsilon \to 0^+ \); then it follows for any \( t \in (0, b] \) that \( x(t, 0) \geq 0, y(t, 0) \geq 0, v(t, 0) \geq 0 \). Consider any arbitrary \( b \in (0, T_0) \), then it follows for any \( t \in (0, T_0) \) that

\( x(t) \geq 0, y(t) \geq 0, v(t) \geq 0 \).

Next, we consider the boundedness of solutions of (2). Define \( H(t) \) as follows,

\[
H(t) = e^{-\mu \tau} x(t - \tau) + y(t).
\]

Taking derivative of \( H(t) \) along with the solution of (2) for \( t \in [0, T_0] \), we obtain that

\[
H'(t) \leq e^{-\mu \tau} s - e^{-\mu \tau} g(0) x(t - \tau) - py(t) \\
\leq e^{-\mu \tau} s - l H(t),
\]

where \( l = \min\{g(0), p\} \). By the nonnegativity of \( x(t) \) and \( y(t) \), and applying the well-known comparison principle to the first equation of (2) and (6), we know that \( x(t) \)
and \( y(t) \) are all bounded on \([0, T_0]\) and let 
\[ M = \sup_{t \in [0, T_0]} H(t) < +\infty. \]
Consequently, from the third equation of (2), it follows that
\[ v'(t) \leq kM - uv(t). \]
Hence, we also have that \( v(t) \) is bounded on \([0, T_0]\). Therefore, with the aid of the basic theory of DDEs (see, for example, [28, 29]), the existence and uniqueness of the solution \((x(t), y(t), v(t))^T\) of (2) on \([0, T_0]\) can be extended to \([0, +\infty)\). That is, by using similar argument as above, we obtain that the solution \((x(t), y(t), v(t))^T\) of (2) with the initial conditions (3) is also nonnegative on \([0, +\infty)\).

Now, let us show that (4) holds. First, from the first equation of (2), it is easy to obtain that \( \limsup_{t \to +\infty} x(t) \leq s/g(0) \). In addition, we have from (6) that, \( \limsup_{t \to +\infty} H(t) \leq se^{-\mu\tau}/l \). From the nonnegativity of \( x(t) \) and \( y(t) \), it follows that \( \limsup_{t \to +\infty} y(t) \leq se^{-\mu\tau}/l \). Again from the third equation of (2), we have \( \limsup_{t \to +\infty} v(t) \leq ske^{-\mu\tau}/lu \). Then the proof of Theorem 2.1 is complete. \( \Box \)

**Remark 1.** It is easy to see that the solution \((x(t), y(t), v(t))^T\) of (2), with the initial conditions (3) which satisfies \( \phi_i(0) > 0 \) \((i = 1, 2, 3)\) is positive.

By using some similar techniques in [9, 10], the basic reproductive number for (2) is denoted as
\[ R_0 = \frac{k}{pu} e^{-\mu\tau} f_v(x_0, 0), \]
where \( x_0 = s/g(0) \). Here, the first term \( k/pu \) indicates the average number of virus particles emerging from each virus-producing cell, and the term \( f_v(x_0, 0) \) represents the maximal average number of cells infected by each virion. The model (1) with the linear term \( d + cy \) and the bilinear incidence rate \( \beta xv \) was considered by Cheng, Ma and Guo [7], where the basic reproductive number is simplified to \( \frac{2\beta k}{d_{pu}} e^{-\mu\tau} \).

The model (2) always has an infection-free equilibrium \( E_0 = (x_0, 0, 0)^T \). Generally, if \( R_0 > 1 \), then it has a positive equilibrium \( E_1 = (x^*, y^*, v^*)^T \), which satisfies
\[
\begin{aligned}
s &= g(y^*) x^* + f(x^*, v^*), \\
p y^* &= e^{-\mu\tau} f(x^*, v^*), \\
u v^* &= ky^*.
\end{aligned}
\]

Next, we give the following theorem which provides the existence conditions of positive equilibria. First, we assume the following.

(H3) \( f_v(x, v) \) is decreasing with respect to \( v \) for \( x, v > 0 \).

(H4) \( g'(y) \) is decreasing for \( y \geq 0 \).

**Theorem 2.2.** Assume that the functions \( f(x, v) \) and \( g(y) \) satisfy the conditions (H1) and (H2). If \( R_0 > 1 \), then there exists at least one positive equilibrium \( E_1 \). In addition, if the conditions (H3) and (H4) are satisfied, then the positive equilibrium \( E_1 \) is unique.

**Proof.** Let the right-hand sides of the three equations in (2) equal zero, and it follows that
\[ s - g(y)x = f(x, v) = e^{\mu\tau} py = e^{\mu\tau} pu \frac{e^{\mu\tau}}{k} v. \]
Further,
\[
\begin{aligned}
x &= \frac{sk - e^{\mu\tau} pu v}{kg (uv/k)}, \\
y &= \frac{u}{k} v, \\
v &\in [0, v_0], \\
v_0 &= \frac{ske^{-\mu\tau}}{pu}.
\end{aligned}
\]
From (9), we have the following equation for $v \in (0, v_0]$.

$$H(v) \equiv \frac{f \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right) v}{v} - \frac{e^{\mu_\tau} pv}{k} = 0.$$  

It is clear from (H1)-(H2) and $R_0 > 1$ that

$$\lim_{v \to 0^+} H(v) = e^{\mu_\tau} pu(R_0 - 1)/k > 0,$$

and $H(v_0) = -e^{\mu_\tau} pu/k < 0$. Therefore, it follows from the continuity of the function $H(v)$ on $(0, v_0]$ that there exists at least one $v^* \in (0, v_0)$ such that $H(v^*) = 0$. Consequently, by (9), it follows that $x^* > 0$ and $y^* > 0$, and there exists at least one positive equilibrium $E_1$.

Derivation yields for $H(v)$,

$$H'(v) = \frac{F(v)}{v^2},$$

where

$$F(v) = -su'g(uv/k) - e^{\mu_\tau} pu(g(uv/k) - g'(uv/k)uv/k)f_x \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right) v$$

$$+ f_v \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right) v - f \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right).$$

According to (H3), it follows that

$$f \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})}, v \right) - f_v \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})}, v \right)$$

$$= \left( f_v \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right) v \right) v - f_v \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right) \left( f_v \left( \frac{sk - e^{\mu_\tau} pvuv}{kg(\frac{uv}{k})} \right) v \right) \geq 0 \ (\xi_1 \in (0, v), v \in (0, v_0),$$

and it follows from (H4) that

$$g' \left( \frac{uv}{k} \right) \frac{uv}{k} = g(0) + g' \left( \frac{uv}{k} \right) \frac{uv}{k} \geq g(0) + \frac{uv}{k} g' \left( \frac{uv}{k} \right) \geq 0 \ (\xi_2 \in (0, uv/k), v \in (0, v_0)).$$

Hence it follows easily that $F(v) < 0$ for $v \in (0, v_0]$. In consequence, $H'(v) < 0$ for $v \in (0, v_0]$. On the other hand, it can be seen that $\lim_{v \to 0^+} H(v) > 0$, and $H(v_0) < 0$. Therefore, $H(v_0) = 0$ has a unique root in $(0, v_0)$, and the positive equilibrium $E_1$ is unique. The proof is complete. 

**Remark 2.** From the proof of Theorem 2.2, it is easy to see that if $R_0 \leq 1$, then (2) has no positive equilibria under the conditions (H1)-(H4).

3. **Stability and permanence.** In this section, by using some techniques of constructing Lyapunov functionals (see, e.g., [13, 20, 30, 31]), we shall give the global properties of infection-free equilibrium $E_0$ of (2), and then we obtain the local property of positive equilibrium $E_1$ and permanence of (2). First, we assume the following.

(H5) $x - x_0 - \int_{x_0}^x \lim_{v \to 0^+} f(x_0, v) \, ds \to +\infty$ as $x \to 0^+$ or $x \to +\infty$;

(H6) $\frac{x}{f(x, v)}$ is increasing with respect to $x$ for $x, v > 0$;

(H7) $f_v(x, 0)$ is strictly increasing with respect to $x > 0$. 


Theorem 3.1. Suppose that the conditions (H1)-(H7) are satisfied.

(i) If $R_0 < 1$, then the infection-free equilibrium $E_0$ is globally asymptotically stable.

(ii) If $R_0 = 1$, then the infection-free equilibrium $E_0$ is globally attractive.

Proof. To start the proof, let we define

$$G = \{ \phi = (\phi_1, \phi_2, \phi_3)^T \in C \mid \|\phi_1\| \leq x_0, \phi_i \geq 0, i = 1, 2, 3 \}. $$

From Theorem 2.1, it follows that $G$ attracts all the solutions of (2). For any $\phi = (\phi_1, \phi_2, \phi_3)^T \in G$, let $(x(t), y(t), v(t))^T$ be the solution of (2) with the initial function $\phi$. We claim that $x(t) \leq x_0$ for $t \geq 0$. Actually, if not, there exists a $t_0 > 0$ such that $x(t_0) > x_0$ and there exists an $\bar{t} < t_0$ such that

$$\bar{t} = \inf \{ t \in [0, t_0] \mid x(\eta) < x_0, \eta \in [t, t_0] \}. $$

Then $x(\bar{t}) = x_0$ and $x(t) > x_0$ for $t \in (\bar{t}, t_0)$. Hence it follows from Lagrange's mean value theorem that there exists a $t_1 \in (\bar{t}, t_0)$ such that

$$x'(t_1) = \frac{x(t_0) - x(\bar{t})}{t_0 - \bar{t}} = \frac{x(t_0) - x_0}{t_0 - \bar{t}} > 0. \quad (10) $$

Clearly, $x(t_1) > x_0$ and thus we have that

$$x'(t_1) = s - g(0)x(t_1) - (g(y(t_1)) - g(0))x(t_1) - f(x(t_1), v(t_1))$$

$$< -(g(y(t_1)) - g(0))x(t_1) - f(x(t_1), v(t_1))) \leq 0. $$

Obviously, this is a contradiction to (10). Therefore, $G$ is a positively invariant with respect to (2).

If $R_0 < 1$, let us define a functional $L_1$ on $G$ as follows,

$$L_1 = \phi_1(0) - x_0 - \int_{x_0}^{\phi_1(0)} \lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} dq + a_1 \phi_2(0)$$

$$\quad + a_1 e^{-\mu t} \int_{-\tau}^{0} f(\phi_1(\theta), \phi_3(\theta)) d\theta + a_2 \phi_3(0), $$

where $a_1, a_2$ ($> 0$) are determined later. It can be seen that $L_1$ is continuous on the subset $G$ in $C$ and $L_1 \equiv 0$ if and only if $\phi \equiv E_0$. Clearly, it follows from (H5) that $L_1$ is positive definite with respect to $E_0$, and it is also provided with the property of infinite. From (H1)-(H7), the time derivative of $L_1$ along the solution of (2) is given as follows.

$$L_1' = \left(1 - \lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} \right) \left[g(0)(x_0 - x) - (g(y) - g(0))x - f(x, v) \right]$$

$$+ e^{-\mu t} a_1 f(x, v) - a_1 py + a_2 ky - a_2 uv$$

$$= g(0) (x_0 - x) \left(1 - \lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} \right)$$

$$\lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} \left[(g(y) - g(0))x + f(x, v) \right]$$

$$\quad - \left[(g(y) - g(0))x + f(x, v) \right] e^{-\mu t} a_1 f(x, v) - a_1 py + a_2 ky - a_2 uv$$

$$\leq g(0) (x_0 - x) \left(1 - \lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} \right) - \left( a_1 p - a_2 k \right) \lim_{v \to 0^+} \frac{g'(0)f(x_0, v)x}{f(x, v)} \right) y$$

$$\quad - \left( a_2 uv - e^{-\mu t} a_1 f(x, v) - f(x, v) \lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} \right) - (g(y) - g(0))x - f(x, v)$$

$$\leq g(0) (x_0 - x) (f_v(x_0, 0) - f_v(x, 0)) - (a_1 p - a_2 k - g'(0)x_0) y$$

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\[- (a_2u - a_1 e^{-\mu t} f_v(x_0, 0) - f_v(x_0, 0)) v - (g(y) - g(0))x - f(x, v).\]

In the last inequality above, the following inequalities have been used.
\[\frac{g'(0)f(x_0, v)x}{f(x, v)} \leq \frac{g'(0)f(x_0, x_0)}{f(x_0, v)} = g'(0)x_0 (x, v > 0),\]
\[f(x, v) = f_v(x, \xi) v \leq f_v(x, 0) v (\xi \in [0, v], x > 0, v \geq 0),\]
\[f(x, v) \lim_{v \to 0^+} \frac{f(x_0, v)}{f(x, v)} \leq f_v(x_0, 0) \frac{f_v(x_0, 0)}{f_v(x_0, 0)} = f_v(x_0, 0) v (x > 0, v \geq 0).\]

Solving the following simultaneous equations
\[
\begin{align*}
    a_1 p - a_2 k - g'(0)x_0 &= 0, \\
    a_2 u - a_1 e^{-\mu t} f_v(x_0, 0) - f_v(x_0, 0) &= 0,
\end{align*}
\]
yields
\[
\begin{align*}
    a_1 &= \frac{kf_v(x_0, 0) + x_0 u g'(0)}{pu - ke^{-\mu t} f_v(x_0, 0)}, \\
    a_2 &= \frac{pf_v(x_0, 0) + x_0 e^{-\mu t} g'(0)f_v(x_0, 0)}{pu - ke^{-\mu t} f_v(x_0, 0)}.
\end{align*}
\]

Hence, it follows that
\[L' \leq L \equiv \frac{g(0)}{f_v(x_0, 0)} (x_0 - x) (f_v(x, 0) - f_v(x_0, 0)) - (g(y) - g(0))x - f(x, v) \leq 0.\]

Hence, this shows that \(L_1\) is a Lyapunov functional on \(G\). From (H5) and \(L'_1 \leq 0\), the solution of (2) with any initial function \(\phi \in G\) is also bounded on \([0, +\infty)\). In addition, it is clear that \(L\) is negative definite with respect to \(E_0\). Therefore, it follows from Corollary 5.2 of Kuang [29, p. 30] that \(E_0\) is globally asymptotically stable.

If \(R_0 = 1\), i.e. \(ke^{-\mu t} f_v(x_0, 0) = pu\), let us define the following functional on \(G\),
\[L_2 = k\phi_2(0) + p\phi_3(0) + ke^{-\mu t} \int_{-\tau}^{0} f(\phi_1(\theta), \phi_3(\theta))d\theta.\]

The time derivative of \(L_2\) along the solution of (2) is given by
\[
L'_2 = ke^{-\mu t} f(x, v) - pvu \\
\leq (ke^{-\mu t} f_v(x, 0) - pu) v \\
= \frac{pu}{f_v(x_0, 0)} (f_v(x, 0) - f_v(x_0, 0)) v \\
\leq 0.
\]

Hence, \(L_2\) is also a Lyapunov functional on \(G\). Define
\[E = \{ \phi = (\phi_1, \phi_2, \phi_3)^T \in G \mid G = G' = L'_2 = 0 \}, \]
and we have that
\[E \subset \{ \phi \in G \mid \phi_1(0) = x_0 \text{ or } \phi_3(0) = 0 \}.\]

Let \(M\) be the largest set in \(E\) which is invariant with respect to (2). Clearly, \(M\) is not empty since \((x_0, 0, 0)^T \in M\). For any \(\phi \in M\), let \((x(t), y(t), v(t))^T\) be the solution of (2) with the initial function \(\phi\). From the invariance of \(M\), it follows that \((x(t), y(t), v(t))^T \in M \subset E\) for any \(t \in \mathbb{R}\). Thus, for each \(t \in \mathbb{R}\), it follows that
\[x(t) = x_0 \text{ or } v(t) = 0.\]
If $x(t) = x_0$ for some $t$, it follows that $x'(t) = 0$ by $x(t) \leq x_0$ and the differentiability of $x(t)$. Thus, the first equation of (2) implies that

$$ s - g(y(t))x_0 - f(x_0, v(t)) = -(g(y(t)) - g(0))x_0 - f(x_0, v(t)) = 0. $$

It follows that $y(t) = 0$ and $v(t) = 0$. Therefore, it follows that $v(t) \equiv 0$ for any $t \in \mathbb{R}$. It is obviously seen that $y(t) \equiv 0$ from the third equation of (2). The first equation of (2) and the invariance of $M$ imply that $x(t) \equiv x_0$ for any $t \in \mathbb{R}$. Consequently, $M = \{E_0\}$. The classical Lyapunov-LaSalle invariance principle (see Theorem 5.3 of Kuang [29, p. 30]) shows that $E_0$ is globally attractive. The proof of Theorem 3.1 is complete.

Next, we consider the local asymptotic stability of positive equilibrium $E_1$ of (2), first, we assume the following.

(H8) $f_s(x, v)$ is decreasing with respect to $x$ for $x, v > 0$.

**Theorem 3.2.** Suppose that the conditions (H1)-(H4) and (H8) are satisfied. If $R_0 > 1$, then the infection-free equilibrium $E_0$ is unstable, and the positive equilibrium $E_1$ is locally asymptotically stable.

According to the detailed analysis for the roots of the transcendental characteristic equation of the corresponding linearized system of (2) at $E_1$, it is not difficult to prove Theorem 3.2. Hence, we omit its proof here.

**Definition 3.3.** (29) (2) is said to be permanent if there are positive constants $\nu_i$ and $M_i$ ($i = 1, 2, 3$) such that

$$\begin{align*}
\nu_1 &\leq \liminf_{t \to - \infty} x(t) \leq \limsup_{t \to + \infty} x(t) \leq M_1, \\
\nu_2 &\leq \liminf_{t \to - \infty} y(t) \leq \limsup_{t \to + \infty} y(t) \leq M_2, \\
\nu_3 &\leq \liminf_{t \to - \infty} v(t) \leq \limsup_{t \to + \infty} v(t) \leq M_3
\end{align*}
$$

hold for any positive solution of the model (2) with the initial conditions (3). Here $\nu_i$ and $M_i$ ($i = 1, 2, 3$) are independent of (3). In particular, (2) is said to be uniformly persistent if the left-hand side inequalities in (11) hold, that is,

$$\begin{align*}
\nu_1 &\leq \liminf_{t \to - \infty} x(t), \\
\nu_2 &\leq \liminf_{t \to - \infty} y(t), \\
\nu_3 &\leq \liminf_{t \to - \infty} v(t).
\end{align*}
$$

It is well-known that two traditional methods in persistence theory are as follows: one using Morse decompositions and the other using acyclic coverings (see [8, 15, 26, 27, 47, 48, 49, 54] for details). In the following, we shall consider the permanence of (2) with the initial conditions (3) by using some analysis techniques which are different from the traditional approaches. First, we assume the following.

(H9) $\frac{s}{2|\overline{e}^p|}$ is increasing with respect to $v$ for $x, v > 0$.

**Theorem 3.4.** Suppose that conditions (H1)-(H2) (which ensure the existence of positive equilibria), (H6) and (H9) are satisfied. If $R_0 > 1$, then (2) is permanent, and each positive solution $(x(t), y(t), v(t))\mathbb{R}$ of (2) with initial conditions (3) satisfies

$$\begin{align*}
\liminf_{t \to - \infty} x(t) &\geq \frac{s}{g(se^{-\mu \tau}/l) + f_z(0, ske^{-\mu \tau}/lu)} \equiv \nu_1, \\
\liminf_{t \to - \infty} y(t) &\geq \frac{y^*}{2} e^{-pA(h)} \equiv \nu_2, \\
\liminf_{t \to - \infty} v(t) &\geq \frac{k\gamma^*}{2u} e^{-pA(h)} \equiv \nu_3.
\end{align*}
$$
where \( A(h) = T_0 + T_1 + T_2 + a + \tau \) and

\[
T_0 = -\frac{1}{u} \ln \frac{e^{\mu T} y^* l}{2 s}, \quad T_1 = \frac{x^*}{s - bx^*}, \quad b = g \left( \frac{y^*}{2} \right) + \frac{f(x^*, \nu^*)}{x^*},
\]

\[
T_2 = -\frac{1}{b} \ln \left( 1 - \frac{bx^*}{s h} \right), \quad a = \frac{q}{u(1 - q)}, \quad q = \frac{f(x^*, \nu^*)}{f(x^*/h, \nu^*)}, \quad h \in (bx^*/s, 1).
\]

**Proof.** Note from Theorem 2.1 that, we only need to show that (12) holds. Let us consider a positive solution \((x(t), y(t), v(t))^T\) of (2) with initial conditions (3).

Again from Theorem 2.1, for any given \( \varepsilon > 0 \), there exists some \( t_0 > 0 \) such that \( y(t) \leq se^{-\mu \tau}/l + \varepsilon, \ v(t) \leq ske^{-\mu \tau}/lu + \varepsilon \) for all \( t \geq t_0 \). Hence, we have from the first equation of (2) that,

\[
x'(t) = s - \left( g(y(t)) + \frac{f(x(t), v(t))}{x(t)} \right) x(t)
\]

\[
\geq s - \left[ g \left( \frac{se^{-\mu \tau}}{l} + \varepsilon \right) + f_x \left( 0, \frac{ske^{-\mu \tau}}{lu} + \varepsilon \right) \right] x(t).
\]

Thus, it follows that

\[
\liminf_{t \to +\infty} x(t) \geq \frac{s}{g \left( \frac{se^{-\mu \tau}}{l} + \varepsilon \right) + f_x \left( 0, \frac{ske^{-\mu \tau}}{lu} + \varepsilon \right)}.
\]

Consider any arbitrary \( \varepsilon > 0 \), then we have that

\[
\liminf_{t \to +\infty} x(t) \geq \nu_1.
\]

Next, let us show that \( \liminf_{t \to +\infty} y(t) \geq \nu_2 \).

First, from (8), it follows that

\[
\frac{e^{\mu \tau} p u}{k} = f(x^*, \nu^*), \quad \frac{k y^*}{u} = v^*.
\]

According to this solution of (2), we define that

\[
V(t) = y(t) + \frac{p}{k} v(t) + e^{\mu \tau} \frac{f(x^*, \nu^*)}{v^*} \int_{t - \tau}^{t} v(\theta) d\theta.
\]

(13)

Then the derivative of \( V(t) \) is given by

\[
V'(t) = e^{\mu \tau} \left( f(x(t - \tau), v(t - \tau)) - \frac{f(x^*, \nu^*)}{v^*} v(t - \tau) \right).
\]

(14)

There exists a \( T > 0 \) such that for all \( t \geq T \), we have that

\[
v(t) \leq \frac{ske^{-\mu \tau}}{lu} + \frac{k y^*}{2u}.
\]

Note that \( x^* = s/(g(y^*) + f(x^*, \nu^*)/x^*) \) and \( b = g(y^*/2) + f(x^*, \nu^*)/x^* \), it follows that \( x^* < s/b \). In addition, there exists a \( T_0 > 0 \) such that \( se^{-\mu \tau}/lu = y^*/2 \) and for any given \( T_2 = -\ln (1 - bx^*/s) / b \), it always has that \( x^* < s(1 - e^{-bT_2}) / b = x^0 < s/b \), i.e., \( T_2 = -\ln (1 - bx^*/s) / b \) for each \( h \equiv x^0 / x^0 \in (bx^*/s, 1) \).

Let us first claim that for any \( t_0 \geq T > 0 \), it is impossible to satisfy \( y(t) \leq y^*/2 \) for all \( t \geq t_0 \). In fact, otherwise, there exists a \( T_0 \geq T \) such that \( y(t) \leq y^*/2 \) for all \( t \geq t_0 \). Then from the third equation of (2), we have for \( t \geq t_0 \) that,

\[
v'(t) = ky(t) - uv(t) \leq \frac{k y^*}{2} - uv(t),
\]
Thus, it follows that
\[ v(t) \leq \frac{ky^*}{2u} + \left( v(t_0) - \frac{ky^*}{2u} \right) e^{-u(t-t_0)} \]
\[ \leq \frac{ky^*}{2u} + \frac{sk}{\ell u} e^{-u(t-t_0)}. \]

Thus, it follows that \( v(t) \leq ky^*/u = v^* \) for \( t \geq t_0 + T_0 \). Therefore, it follows from (14) for \( t \geq t_0 + T_0 + \tau \) that
\[
V'(t) = e^{-\mu \tau} \left( \frac{f(x(t-\tau), v(t-\tau))}{v(t-\tau)} - \frac{f(x^*, v^*)}{v^*} \right) v(t-\tau) \\
\geq e^{-\mu \tau} \left( \frac{f(x(t-\tau), v^*)}{v^*} - \frac{f(x^*, v^*)}{v^*} \right) v(t-\tau).
\]

Now, we divide the two cases to show that \( x(t) \geq x^* \) for \( t \geq t_0 + T_0 + T_1 \), where
\[ T_1 = \frac{x^*}{s - g(y^*/2)x^* - f(x^*, v^*)}. \]

(a) If \( x(t_0 + T_0) \geq x^* \), then \( x(t) \geq x^* \) for \( t \geq t_0 + T_0 \).

In fact, if not, there exists a \( t_2 > t_0 + T_0 \) such that \( x(t_2) < x^* \). Denote
\[ \bar{t}_1 = \inf \{ t \in [t_0 + T_0, t_2] \mid x(s) < x^*, s \in [t, t_2] \}, \]
\[ \bar{t}_2 = \sup \{ t \geq t_2 \mid x(s) < x^*, s \in [t_2, t] \}. \]

Then, it is obvious to see that \( \bar{t}_1 < \bar{t}_2 \) and \( x(\bar{t}_1) = x^* \). If \( \bar{t}_2 < +\infty \), then \( x(\bar{t}_2) = x^* \) and \( x(t) < x^* \) for \( t \in (\bar{t}_1, \bar{t}_2) \). If \( \bar{t}_2 = +\infty \), then \( x(t) < x^* \) for \( t > \bar{t}_1 \). It follows from the first equation of (2) for \( t \in [\bar{t}_1, \bar{t}_2] \) that,
\[
x'(t) \geq s - g \left( \frac{y^*}{2} \right) x^* - f(x^*, v^*) > 0. \tag{15}
\]

From (15), it follows that \( x(t) \) is strictly increasing on \( [\bar{t}_1, \bar{t}_2] \). Hence, if \( \bar{t}_2 < +\infty \), then \( x^* = x(\bar{t}_1) < x(\bar{t}_2) = x^* \). This is a contradiction. If \( \bar{t}_2 = +\infty \), from (15), this is a contradiction to the boundedness of \( x(t) \). Therefore, the conclusion (a) holds.

(b) If \( x(t_0 + T_0) < x^* \), and let
\[
\bar{t}_2 = \sup \{ t \geq t_0 + T_0 \mid x(s) < x^*, s \in [t_0 + T_0, t] \}. \tag{16}
\]

Then \( \bar{t}_2 < t_0 + T_0 + T_1 \) and \( x(t) \geq x^* \) for \( t \geq \bar{t}_2 \).

In fact, if \( \bar{t}_2 = +\infty \), then \( x(t) < x^* \) for \( t \geq t_0 + T_0 \). From (15), this is a contradiction to the boundedness of \( x(t) \). Thus, \( \bar{t}_2 < +\infty \) and then \( x(\bar{t}_2) = x^* \).

It is similar to the proof of conclusion (a), we have that \( x(t) \geq x^* \) for \( t \geq \bar{t}_2 \). Consequently, it follows from (15) that,
\[
\bar{t}_2 < t_0 + T_0 + T_1.
\]

Hence, the conclusion (b) holds.

Therefore, it follows from the conclusions (a) and (b) that \( x(t) \geq x^* \) for \( t \geq t_0 + T_0 + T_1 \).

From the first equation of (2), it follows for \( t \geq t_0 + T_0 + T_1 \) that
\[
x'(t) \geq s - \left( g \left( \frac{y^*}{2} \right) + \frac{f(x(t), v^*)}{x(t)} \right) x(t) \\
\geq s - bx(t).
\]
Then, for \( t \geq t_0 + T_0 + T_1 \),
\[
x(t) \geq e^{-b(t_0 - T_0 - T_1)} \left[ x(t_0 + T_0 + T_1) + s \int_{t_0 + T_0 + T_1}^{t} e^{b(\theta - t_0 - T_0 - T_1)} \, d\theta \right] \geq \frac{s}{b} \left[ 1 - e^{-b(t_0 - T_0 - T_1)} \right].
\]
Hence, for \( t \geq t_0 + T_0 + T_1 + T_2 \), it follows that
\[
x(t) > \frac{s}{b} \left( 1 - e^{-bT_2} \right) = x^0 > x^*.
\]
Thus, we have from (17) for \( t \geq t_0 + T_0 + T_1 + T_2 + \tau \) that,
\[
V'(t) \geq e^{-\mu \tau} \left( \frac{f(x^0, v^*}){v^*} - \frac{f(x^*, v^*)}{v^*} \right) v(t - \tau).
\]
Define
\[
t_m = t_0 + T_0 + T_1 + T_2,
ym = \min_{\theta \in [-\tau, 0]} y(t_m + \tau + \theta),
v_m = \min_{\theta \in [-\tau, 0]} v(t_m + \tau + \theta),
m = \min \left\{ y_m, \frac{uv_m}{k} \right\}.
\]
Now, we shall show that \( y(t) \geq m \) for \( t \geq t_m \). If not, there exists a \( T_3 \geq 0 \) such that \( y(t) \geq m \) for \( t_m \leq t \leq t_m + \tau + T_3 \), \( y(t_m + \tau + T_3) = m \) and \( y'(t_m + \tau + T_3) \leq 0 \). For \( t_m \leq t \leq t_m + \tau + T_3 \), we have that
\[
v'(t) = ky(t) - uv(t) \geq km - uv(t).
\]
Integration yields for (18),
\[
v(t) \geq \left( v(t_m) - \frac{km}{u} \right) e^{ut_m - ut} + \frac{km}{u} \geq (v(t_m) - v_m) e^{ut_m - ut} + \frac{km}{u} \geq \frac{km}{u}.
\]
Then, for \( \frac{e^{-\mu \tau}}{k} = \frac{f(x^*, v^*)}{v^*} \) and \( t = t_m + \tau + T_3 \), we have that
\[
y'(t) = v(t - \tau) \frac{e^{-\mu \tau} f(x(t - \tau), v(t - \tau))}{v(t - \tau)} - py(t) \geq pm \left( \frac{e^{-\mu \tau} f(x(t - \tau), v^*)}{v^*} - 1 \right) = pm \left( \frac{f(x(t - \tau), v^*)}{f(x^*, v^*)} - 1 \right) > pm \left( \frac{f(x^0, v^*)}{f(x^*, v^*)} - 1 \right)
\]
> 0.

Obviously, this is a contradiction to \( y'(t_m + \tau + T_3) \leq 0 \). Hence, \( y(t) \geq m \) for \( t \geq t_m \). On the other hand, we have from (19) that, for \( t \geq t_m \),

\[
v(t) \geq \frac{km}{u}.
\]

Hence, it follows for \( t \geq t_m + \tau \) that

\[
V'(t) > e^{-\nu \tau} \frac{km}{u} \left( \frac{f(x^0, v^*)}{v^*} - \frac{f(x^*, v^*)}{v^*} \right),
\]

which means that \( V(t) \to +\infty \ (t \to +\infty) \). Nevertheless, from the boundedness of \((x(t), y(t), v(t))^T\), for \( t \geq t_m + \tau \), it follows from (13) that

\[
V(t) \leq \frac{1}{2} \left( \frac{1}{u} + \frac{p}{u} + pr \right) y^*,
\]

which is a contradiction to \( V(t) \to +\infty \ (t \to +\infty) \). Therefore, the claim is proved.

From the above claim, we are remaining to consider the following two cases.

(i) \( y(t) \geq y^*/2 \) for all large \( t \);

(ii) \( y(t) \) oscillates about \( y^*/2 \) for all large \( t \).

Next, we show that \( y(t) \geq \nu_2 \) for all large \( t \). Obviously, we only need to consider the case (ii). Let \( t_1, t_2 \geq T \) be sufficiently large such that

\[
y(t_1) = y(t_2) = \frac{y^*}{2}, \quad y(t) < \frac{y^*}{2} \ (t_1 < t < t_2).
\]

If \( t_2 - t_1 \leq A(h) \), we have from the second equation of (2) that

\[
y'(t) > -py(t),
\]

which means that for \( t \in (t_1, t_2) \),

\[
y(t) > y(t_1) e^{-p(t-t_1)}.
\]

(20)

It is clear that for \( t \in (t_1, t_2) \),

\[
y(t) > \frac{y^*}{2} e^{-pA(h)} = \nu_2.
\]

If \( t_2 - t_1 > A(h) \), it is easy to see that \( y(t) \geq \nu_2 \) for \( t \in [t_1, t_1 + A(h)] \). Next, proceeding similarly as the proof for the aforementioned claim, we can prove that \( y(t) \geq \nu_2 \) for \( t \in [t_1 + A(h), t_2] \). In fact, if not, there exists a \( t_4 \geq 0 \) such that \( y(t) \geq \nu_2 \) for \( t_1 \leq t \leq t_1 + A(h) + T_3 \), \( y(t_1 + A(h) + T_4) = \nu_2 \) and \( y'(t_1 + A(h) + T_4) \leq 0 \). First, for \( t_1 \leq t \leq t_1 + A(h) + T_4 \), we have that

\[
v'(t) = Ky(t) - uv(t) \geq qk\nu_2 - uv(t),
\]

where \( q = \frac{f(x^0, v^*)}{f(x^*, v^*)} < 1 \). Hence, the following two subcases to be discussed.

(ii) If \( v(t_1) \geq \frac{qk\nu_2}{u} \), then for \( t_1 \leq t \leq t_1 + A(h) + T_4 \),

\[
v(t) \geq \left( v(t_1) - \frac{qk\nu_2}{u} \right) e^{u(t-t_1)} + \frac{qk\nu_2}{u} \geq \frac{qk\nu_2}{u}.
\]

Further, for \( t_1 \leq t \leq t_1 + A(h) + T_4 \), we have from the third equation of (2) that,

\[
v'(t) = Ky(t) - uv(t) \leq \frac{Ky^*}{2} - uv(t),
\]
thus,
\[ v(t) \leq \frac{ky^*}{2u} + \left( v(t_1) - \frac{ky^*}{2u} \right) e^{-u(t-t_1)} \]

\[ \leq \frac{ky^*}{2u} + \frac{ske^{-\mu T}}{lu} e^{-u(t-t_1)}. \]

Then, it follows that \( v(t) \leq \frac{ky^*}{u} = v^* \) for \( t \geq t_1 + T_0 \). In addition,
\[ \frac{e^{\mu T} pu}{k} = f(x^*, v^*). \]

Hence, for \( t = t_1 + A(h) + T_4 \), we have that
\[ y'(t) = v(t - \tau) e^{-\mu T} f(x(t - \tau), v(t - \tau)) - pu \]
\[ \geq pu_2 \left( \frac{e^{-\mu T} qk f(x(t - \tau), v^*)}{v^*} - 1 \right) \]
\[ = pu_2 \left( \frac{q f(x^0, v^*)}{f(x^*, v^*)} - 1 \right) \]
\[ > pu_2 \left( q f(x^0, v^*) - 1 \right) \]
\[ = 0. \]

This is a contradiction to \( y'(t_1 + A(h) + T_4) \leq 0 \).

(ii) If \( v(t_1) < \frac{q \nu_2}{u} \), and let
\[ \bar{t} = \sup \left\{ t \in [t_1, t_1 + A(h) - \tau + T_4] \mid v(s) < \frac{q \nu_2}{u}, s \in [t_1, t] \right\}. \]

Then for \( \bar{t} \in (t_1, t_1 + A(h) - \tau + T_4] \), \( v(\bar{t}) \leq \frac{q \nu_2}{u} \). On the other hand, we have from third equation of (2) for \( t \in [t_1, \bar{t}] \) that,
\[ v'(t) = ky(t) - uv(t) \]
\[ \geq \nu_2 (1 - q), \]
consequently,
\[ k\nu_2 (1 - q) (\bar{t} - t_1) \leq v(\bar{t}) - v(t_1) < \frac{q \nu_2}{u}. \]

Thus, \( \bar{t} - t_1 < \frac{q}{u(1-q)} = a \), which implies that \( \bar{t} \in (t_1, t_1 + A(h) - \tau + T_4) \) and \( v(\bar{t}) = \frac{q \nu_2}{u} \). By
\[ v'(t) \geq qk \nu_2 - uv(t), \]
we have that for \( t \in [\bar{t}, t_1 + A(h) + T_4], \)
\[ v(t) \geq \left( v(\bar{t}) - \frac{q \nu_2}{u} \right) e^{u(\bar{t} - t)} + \frac{q \nu_2}{u} = \frac{q \nu_2}{u}. \]

Hence, by using similar approach in the proof of (ii)1, (21) holds, i.e., \( y'(t_1 + A(h) + T_4) > 0 \). This is a contradiction to \( y'(t_1 + A(h) + T_4) \leq 0 \).

In view of the proofs of (ii)1 and (ii)_2, therefore, \( y(t) \geq \nu_2 \) for \( t \in [t_1, t_2] \). On account that this kind of interval \([t_1, t_2]\) is chosen in an arbitrary way, we conclude that \( y(t) \geq \nu_2 \) for all large \( t \) in the case (ii). Thus, \( \liminf_{t \to +\infty} y(t) \geq \nu_2 \). Further, we have from the third equation of (2) that \( \liminf_{t \to +\infty} v(t) \geq \nu_3 \). Now, the proof of Theorem 3.4 is complete.
\[ \square \]
Remark 3. In fact, under the condition (H1), the condition (H9) can be replaced by (H3) since (H9) implies that (H9) holds, similarly, (H6) can be replaced by (H8) since (H8) implies that (H6) holds.

Remark 4. Note that if (H6) is replaced by (H8) in the conditions of Theorem 3.4, it is not difficult to verify that
\[ A(h_0) \] is the minimum value of \[ A(h) \] on \((bx^*/s, 1)\), i.e.
\[ \liminf_{t \to +\infty} y(t) \geq \frac{y^*}{2} e^{-pA(h_0)} = \frac{y^*}{2} e^{-pA(h)} \equiv \nu_2, \]
where \(h_0\) is a unique root in \((bx^*/s, 1)\) of the equation
\[ f(x^*, v^*) f_x(x^*/h, v^*) = \frac{uh}{sh - bx^*}. \]

Remark 5. From the proof of Theorem 3.4, it is not difficult to find that
\[ \liminf_{t \to +\infty} y(t) \geq \frac{y^*}{2} e^{-p(A(h_1) - a)} = \frac{y^*}{2} e^{-p(T_0+T_1+T_2+\tau)}, \]
where \(h_1\) is a unique root in \((bx^*/s, 1)\) of the equation
\[ \frac{f(x^*, v^*)}{u(f(x^*/h, v^*) - f(x^*, v^*))} = -\frac{1}{b} \ln (1 - bx^*/sh). \]

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