Chords of 2-Factors in Planar Cubic Bridgeless Graphs

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Abstract
We show that every edge in a 2-edge-connected planar cubic graph is either contained in a 2-edge-cut or is a chord of some cycle contained in a 2-factor of the graph. As a consequence, we show that every edge in a cyclically 4-edge-connected planar cubic graph with at least six vertices is contained in a perfect matching whose removal disconnects the graph. We obtain a complete characterization of 2-edge-connected planar cubic graphs that have an edge such that every 2-factor containing the edge is a Hamiltonian cycle, and also of those that have an edge such that the complement of every perfect matching containing the edge is a Hamiltonian cycle. Another immediate consequence of the main result is that for any two edges contained in a facial cycle of a 2-edge-connected planar cubic graph, there exists a 2-factor in the graph such that both edges are contained in the same cycle of the 2-factor. We conjecture that this property holds for any two edges in a 2-edge-connected planar cubic graph, and prove it in the case the graph is also bipartite. The main result is proved in the dual form by showing that every plane triangulation admits a vertex 3-coloring such that no face is monochromatic and there is exactly one specified edge between a specified pair of color classes.

Keywords Cubic · Planar · 2-Factor · Perfect matching · Chord

Mathematics Subject Classification 05C70 · 05C38

1 Introduction
A classical result in graph theory is Petersen’s theorem [10] that every 2-edge-connected cubic graph has a perfect matching, and hence a 2-factor obtained by taking the complement of the matching. This can be strengthened to show that
there exists a perfect matching including any specified edge, and a 2-factor including any two specified edges in the graph [11]. More can be said about the structure of the 2-factor if the cubic graph has additional properties, in particular, for 2-edge-connected planar cubic graphs. It is well-known that the four color theorem is equivalent to the statement that every 2-edge-connected planar cubic graph is 3-edge-colorable. This is equivalent to the statement that every such graph has a 2-factor in which all cycles have even lengths. It was shown in [3] that every 2-edge-connected planar cubic graph with at least six vertices has a disconnected 2-factor, that is, a 2-factor with more than one component. Recently, a stronger statement was proved in [8], that if every 2-factor in a 2-edge-connected planar cubic graph $G$ contains only cycles of length $k$, for some $k \geq 3$, then $k = 4$ and $G$ must be $K_4$, the complete graph with four vertices.

Häggkvist [6] showed that every edge in a bipartite cubic graph is either contained in a 2-edge-cut or is a chord of some cycle contained in a 2-factor of the graph. Barnette [2] conjectured that every 3-edge-connected planar cubic bipartite graph has a Hamiltonian cycle, which is a connected 2-factor, and this question is still open. We show that Häggkvist’s result also holds for all 2-edge-connected planar cubic graphs.

**Theorem 1** Every edge in a 2-edge-connected planar cubic graph is either contained in a 2-edge-cut or is a chord of some cycle contained in a 2-factor of the graph.

An edge in a cubic graph is a chord of some cycle if and only if it is not contained in an edge-cut of size at most two. Thus another way of stating Theorem 1 is that if an edge in a 2-edge-connected planar cubic graph is a chord of some cycle then it is a chord of a cycle contained in a 2-factor. As a consequence, we get a simpler and completely different proof of the result in [3] and in fact prove much stronger statements. We show that every edge in a cyclically 4-edge-connected planar cubic graph with at least six vertices is contained in a perfect matching whose complement is a disconnected 2-factor. We call such a perfect matching a separating perfect matching. We obtain a constructive characterization of 2-edge-connected planar cubic graphs that have an edge that is not contained in any separating perfect matching, and also of those 2-edge-connected planar cubic graphs that have an edge such that every 2-factor containing the edge is a Hamiltonian cycle. It may be noted that a characterization of cubic bipartite graphs in which every 2-factor is a Hamiltonian cycle has been conjectured [5], but the problem is still open.

Another consequence of our result is that for any two edges contained in a facial cycle of a 2-edge-connected planar cubic graph, there exists a 2-factor in the graph such that both edges are contained in the same cycle of the 2-factor. We conjecture that this property holds for any two edges in the graph. It is also possible that it holds for any two edges in a connected cubic bipartite graph. We prove it for any two edges in a connected planar cubic bipartite graph.
Theorem 2 Let $G$ be a connected planar cubic bipartite graph and let $f_1, f_2$ be any two edges in $G$. There exists a 2-factor in $G$ such that $f_1$ and $f_2$ are contained in the same cycle of the 2-factor.

The Petersen graph, which is the smallest 2-edge-connected cubic graph that is not 3-edge-colorable, also shows that in general, a 2-edge-connected cubic graph may not have any 2-factor in which some cycle has a chord. In this case, no edge is contained in a 2-edge-cut and no edge is a chord of a cycle contained in a 2-factor. It is an interesting question to determine which 2-edge-connected cubic graphs have a 2-factor in which some cycle has a chord. It is possible that every 3-edge-colorable cubic graph has such a 2-factor, although it is not true that every edge that is not contained in a 2-edge-cut is a chord of some cycle contained in a 2-factor in such a graph. It has been conjectured [7] that every 3-edge-colorable cubic graph has an edge $e$ such that the graph obtained by deleting $e$ and contracting one other edge incident with each endpoint of $e$ is also 3-edge-colorable. Perhaps a stronger statement that combines both properties may hold. If every 3-edge-colorable cubic graph has a 3-edge-coloring such that some 2-edge-colored cycle has a chord, it would imply both the results. This statement can be verified easily for 2-edge-connected planar cubic graphs, even without assuming that all of them are 3-edge-colorable. It also holds for cubic bipartite graphs by Häggkvist’s result, since any such graph must have an edge not contained in a 2-edge-cut. However, unlike in the bipartite case, it is not clear whether every edge in a 2-edge-connected planar cubic graph that is not contained in a 2-edge-cut is a chord of a 2-edge-colored cycle in some 3-edge-coloring of the graph.

One reason for studying 2-factors in 2-edge-connected planar cubic graphs is that results on these have a dual version in terms of vertex colorings of the dual graphs. The planar dual of a 2-edge-connected planar cubic graph is a plane triangulation, which is a loopless plane graph embedded in the plane so that each face has three edges on its boundary. The most famous example of this is of course Tait’s reformulation of the four color theorem. Tait in fact falsely conjectured that all 3-edge-connected planar cubic graphs are Hamiltonian, which would have proved the four color theorem. Penaud [9] first noted that the existence of a 2-factor in a 2-edge-connected planar cubic graph implies that any plane triangulation has a non-monochromatic 2-coloring, that is a coloring of the vertices with two colors such that no three vertices on the boundary of a face have the same color. This is also called a polychromatic 2-coloring of a plane graph. A polychromatic $k$-coloring is a vertex coloring with $k$ colors such that every color occurs on the boundary of each face. The problem is to find the largest $k$ for which a plane graph has a polychromatic $k$-coloring. This problem was studied in [1], where also it is shown that every plane triangulation has a polychromatic 2-coloring. Although for 2-colorings, polychromatic and non-monochromatic colorings are the same, this is not so if the number of colors is more than two. Since we need non-monochromatic 3-colorings, we will use the same term for 2-colorings also.

As shown in [4], the existence of a disconnected 2-factor in a 2-edge-connected planar cubic graph implies the existence of a strict 3-coloring of the vertices of
the dual plane triangulation such that no face is monochromatic and also no face is rainbow, that is, some two vertices on the boundary of the face have the same color. In general, it is shown in [4] that the existence of a 2-factor with at least \( k \) components is equivalent to the existence of a strict \((k + 1)\)-coloring of the dual such that no face is monochromatic or rainbow. Although this problem is NP-complete for arbitrary \( k \), it is still open for any fixed \( k \geq 3 \).

The dual version of finding a 2-factor with a chord is to find a non-monochromatic 2-coloring of the vertices of the dual triangulation such that the subgraph formed by the monochromatic edges has a bridge. This is equivalent to finding a non-monochromatic 3-coloring such that for some two color classes there is exactly one edge (the bridge) between the color classes. To find a 2-factor in which a specified edge is a chord of some cycle, we fix the edge between the color classes to be the dual of the edge required to be a chord. We will use this dual formulation and prove the existence of such a 3-coloring by induction. The dual result may also be of independent interest. We prove the following theorem and show that it implies Theorem 1.

**Theorem 3** Let \( G \) be a loopless plane triangulation with vertices \( v_1, v_2, v_3 \) on the boundary of the external face. There exists a vertex coloring \( f : V(G) \rightarrow \{a, b, c\} \) such that no face is monochromatic, and for any edge \( uv \), \( f(u) = a \) and \( f(v) = b \) if and only if \( u = v_1 \) and \( v = v_2 \).

In Sect. 2 we introduce some terminology and prove some basic results. Section 3 gives the proof of Theorem 3. Section 4 gives a characterization of 2-edge-connected planar cubic graphs that have an edge not contained in any separating perfect matching, and also of those 2-edge-connected planar cubic graphs that have an edge such that every 2-factor containing the edge is a Hamiltonian cycle. In Sect. 5 we prove Theorem 2.

**2 Terminology and Basic Results**

The notation and terminology used is mostly standard. We will only clarify the terms that are specific to this work. We consider undirected graphs that may have multiple edges but no self-loops. A graph is \( k \)-edge-connected if it cannot be disconnected by removing less than \( k \) edges. An edge whose removal increases the number of connected components in a graph is called a bridge. A graph is cubic if the degree of every vertex is three. The cubic graph with two vertices and three edges joining them is denoted by \( K_3^2 \). A cubic graph is said to be cyclically \( k \)-edge-connected if removing any set of less than \( k \) edges results in a graph with at most one component containing a cycle. A plane graph is a graph that has been embedded in the plane. Any such embedding divides the plane into connected regions called faces. The unbounded region will be called the external face and all other faces are said to be internal. We will identify a face by the sequence of vertices and edges on its boundary in circular order. A plane triangulation is a plane graph such that
every face is bounded by three edges. The dual of any plane cubic graph is a plane triangulation and vice-versa. Since we consider only bridgeless cubic graphs, their duals will be triangulations without any self-loops. This implies the plane triangulations are necessarily 2-connected and no vertex or edge is repeated on the boundary of any face. A plane near-triangulation is a 2-connected plane graph such that every internal face is bounded by three edges. The boundary of a near-triangulation is the sequence of vertices that are on the boundary of the external face, in circular order. The vertices and edges that are on the boundary of the external face are called boundary vertices and edges, respectively, and the other vertices and edges are said to be internal. A chord of a near-triangulation is an internal edge both of whose endpoints are boundary vertices.

A 3-coloring of a graph $G$ is a function $f : V(G) \to \{a, b, c\}$ that assigns one of three colors to each vertex in $G$. A face of a near-triangulation is said to be monochromatic in a 3-coloring $f$ if all vertices on the boundary of the face have the same color. The coloring $f$ is said to be non-monochromatic if there is no monochromatic face in $f$. A 2-coloring with colors $\{a, c\}$ is a function $f : V(G) \to \{a, c\}$ that assigns one of two colors to each vertex, and it is non-monochromatic if both colors occur on the boundary of every face.

A non-monochromatic 3-coloring $f$ of a near-triangulation $G$ with boundary $v_1 v_2 \ldots v_l$ is said to be special if for any edge $uv$ in $G$, $f(u) = a$ and $f(v) = b$ if and only if $u = v_1$ and $v = v_2$. An adjacent pair of vertices $(u, v)$ is called an $ab$-pair in a coloring $f$ if $f(u) = a$ and $f(v) = b$. Thus a 3-coloring of $G$ is special if it is non-monochromatic and the only $ab$-pair is $(v_1, v_2)$. For a string $s = s_1 s_2 \ldots s_l$ of length $l$ over the alphabet $\{a, b, c\}$, we say that $G$ has a special 3-coloring $f$ with colors $s$ assigned to the boundary if $f$ is a special 3-coloring of $G$ such that $f(v_i) = s_i$ for $1 \leq i \leq l$. We say that $s$ is feasible for $G$ if there exists a special 3-coloring of $G$ with colors $s$ assigned to the boundary. Note that we can choose the vertex $v_1$ in the boundary arbitrarily and label the other vertices in circular order, either clockwise or anti-clockwise. In the case of a triangulation, we can choose the external face also arbitrarily, so that any pair of adjacent vertices in a triangulation may be labeled $v_1, v_2$.

We now state some basic results using the above terminology.

**Lemma 1** (Penaud) Let $G$ be a plane triangulation without self-loops having the boundary $v_1 v_2 v_3$. Then there exists a non-monochromatic 2-coloring of $G$ with colors $\{a, c\}$ such that the boundary is assigned colors $aac$.

**Proof** This follows from the stronger form of Petersen’s theorem. Consider a perfect matching in the dual of $G$ that contains the dual edge of $v_1 v_2$. Deleting the edges in this matching gives a 2-factor which is a collection of disjoint cycles in the plane. The regions into which the plane is divided by these cycles can be 2-colored so that adjacent regions get distinct colors. This can be done by assigning a color to a region depending on the parity of the number of cycles in the 2-factor whose interior contains the region. A vertex in $G$ is assigned the color of the region containing the face of the dual cubic graph corresponding to the vertex. Then no face in the triangulation will be monochromatic since the vertex in the dual corresponding to the face
will be contained in some cycle in the 2-factor, and all three faces incident with the vertex cannot be on the same side of the cycle. Since the dual edge of \(v_1v_2\) is in the matching, \(v_1\) and \(v_2\) will get the same color. We can assume this color is \(a\), without loss of generality, and since \(v_1v_2v_3\) is a face in \(G\), \(v_3\) must have a different color. \(\square\)

A small modification of this gives the coloring equivalent of a 2-factor with a chord.

**Lemma 2** Let \(G\) be a 2-edge-connected plane cubic graph and \(uv\) an edge in \(G\) that is not contained in a 2-edge-cut. Let \(G'\) be the dual of \(G\) with the endpoints of the dual edge of \(uv\) labeled as \(v_1\) and \(v_2\). Then \(G\) has a 2-factor in which some cycle has \(uv\) as a chord if and only if the dual \(G'\) has a special 3-coloring.

**Proof** Suppose \(G\) has such a 2-factor. We first assign colors \(a, c\) as in the proof of Lemma 1 so that no face in \(G'\) is monochromatic. Let \(C\) be the cycle in the 2-factor such that \(uv\) is a chord of \(C\). Without loss of generality, we can assume that the chord \(uv\) is in the interior of \(C\) and the region \(R\) in the interior of \(C\) that has \(C\) as part of its boundary is colored \(a\). In particular, this implies \(v_1, v_2\) are colored \(a\). The chord \(uv\) splits \(R\) into two regions, \(R_1\) which contains the face corresponding to \(v_1\) and \(R_2\) which contains the face corresponding to \(v_2\). We now change the color of all vertices in \(G'\) corresponding to faces contained in the region \(R_2\) to \(b\). This gives a special 3-coloring of the dual triangulation \(G'\). Conversely, if there exists a special 3-coloring of \(G'\), the edges in \(G\) corresponding to the edges in \(G'\) whose endpoints have the same color, and the edge \(uv\), form a perfect matching in \(G\). Since there is no 2-edge-cut in \(G\) containing the edge \(uv\), there is a single edge between \(v_1\) and \(v_2\) in \(G'\). This edge must be a bridge in the subgraph of \(G'\) containing the edges having endpoints of the same color and the edge \(v_1v_2\), since deleting it separates \(v_1\) from \(v_2\). Therefore the dual edge \(uv\) must be a chord of some cycle in the 2-factor, since it must become a self-loop after contracting all the edges in the 2-factor. \(\square\)

We will be working with plane near-triangulations and we define a simple operation on them that gives a smaller near-triangulation. This will be used in inductive arguments. Let \(uv\) be an internal edge in a plane near-triangulation \(G\). Suppose \(w_1, w_2\) are distinct vertices such that \(uvw_1\) and \(uvw_2\) are the two faces in \(G\) whose boundary contains the edge \(uv\) and \(w_2\) is an internal vertex. Let \(G'\) be the graph obtained from \(G\) by deleting the edges \(uv, uw_2, vw_2\) and identifying the vertices \(w_1\) and \(w_2\). Then \(G'\) is also a near-triangulation with the same boundary as \(G\), although it may contain a loop if \(w_1\) was adjacent to \(w_2\) in \(G\). We say \(G'\) is obtained from \(G\) by cutting the edge \(uv\). Figure 1 shows an application of this operation.

**Lemma 3** Let \(uv\) be an internal edge in a near-triangulation \(G\) with boundary \(v_1v_2 \ldots v_l\). Let \(w_1, w_2\) be distinct vertices such that \(uvw_1\) and \(uvw_2\) are faces in \(G\). Suppose \(w_2\) is an internal vertex in \(G\) and is not adjacent to \(v_1\). Let \(G'\) be the near-triangulation obtained by cutting the edge \(uv\). If \(G'\) has a special 3-coloring \(f'\) such
that \( f'(w_1) = b \) and \( f'(u) = c \), then \( G \) has a special 3-coloring \( f \) with \( f(x) = f'(x) \) for all \( x \in V(G') \).

**Proof** Define the coloring \( f \) of \( G \) by \( f(x) = f'(x) \) for all \( x \in V(G') \) and \( f(w_2) = b \). We show that \( f \) is a special 3-coloring of \( G \). The only faces in \( G \) that are not faces in \( G' \) are the face \( uvw_1 \), the face \( uvw_2 \), and the other faces that have \( w_2 \) on their boundary. For any such face \( xyw_2 \) other than \( uvw_2, xyw_1 \) is a face in \( G' \). Since \( xyw_1 \) is not monochromatic in \( f' \) and \( f(w_2) = f'(w_1) = b \), the face \( xyw_2 \) is not monochromatic in \( f \). Since \( f(u) = c \) and \( f(w_1) = f(w_2) = b \), the faces \( uw_1 \) and \( uw_2 \) are not monochromatic in \( f \). The only edges in \( G \) that are not edges in \( G' \) are \( uv \) and the edges incident with \( w_2 \). If \( xw_2 \) is an edge in \( G \), we have \( x \neq v_1 \), since \( v_1 \) is not adjacent to \( w_2 \). Since \( xw_1 \) is an edge in \( G' \) and \( f'(w_1) = b \), we have \( f'(x) = f(x) \neq a \) hence \( (x, w_2) \) is not an \( ab \)-pair in \( f \). Since \( f(u) = c \), neither \( (u, v) \) nor \( (v, u) \) can form an \( ab \)-pair in \( f \). Therefore the only \( ab \)-pair in \( f \) is \( (v_1, v_2) \) and \( f \) is a special 3-coloring of \( G \). \( \square \)

In Sects. 4 and 5, we will be working with 2-edge-connected planar cubic graphs. We define the 2-cut-reduction operation on these that will be used for inductive arguments. Suppose a 2-edge-connected planar cubic graph \( G \) has a 2-edge-cut \( \{u_1u_2, v_1v_2\} \). Let \( C_1 \) and \( C_2 \) be the components of \( G - \{u_1u_2, v_1v_2\} \) such that \( u_1, v_1 \in V(C_1) \) and \( u_2, v_2 \in V(C_2) \). Let \( G_1, G_2 \) be the 2-edge-connected planar cubic graphs obtained by adding edges \( e_1 = u_1v_1 \) and \( e_2 = u_2v_2 \) to \( C_1 \) and \( C_2 \), respectively. We call \( G_1 \) and \( G_2 \) the 2-cut reductions of \( G \) with respect to the cut \( \{u_1u_2, v_1v_2\} \). The following lemma, which is straightforward to verify, gives a relation between 2-factors in \( G \) and those in \( G_1, G_2 \).

![Fig. 1 Cutting an edge](image-url)
Lemma 4 Let $G_1$ and $G_2$ be the 2-cut reductions of a 2-edge-connected planar cubic graph $G$ with respect to a 2-edge-cut $\{u_1u_2, v_1v_2\}$. Then $F$ is a 2-factor in $G$ containing the edges $u_1u_2, v_1v_2$ iff $F_1 = (F \cap E(G_1)) \cup \{e_1\}$ and $F_2 = (F \cap E(G_2)) \cup \{e_2\}$ are 2-factors in $G_1$ and $G_2$, respectively. Further, $F$ is a Hamiltonian cycle iff both $F_1$ and $F_2$ are. Similarly, $F$ is a 2-factor in $G$ not containing the edges $u_1u_2, v_1v_2$ iff $F_1 = F \cap E(G_1)$ and $F_2 = F \cap E(G_2)$ are 2-factors in $G_1, G_2$, not containing the edges $e_1, e_2$, respectively. \hfill \square

3 Main Result

In this section we prove Theorem 3. Lemma 2 then implies Theorem 1. In order to prove Theorem 3, we need to prove another statement involving near-triangulations with a four sided boundary, simultaneously by induction.

Lemma 5 Let $G$ be a near-triangulation with boundary $v_1v_2v_3v_4$. Then at least two of the assignments of colors in $S = \{abbc, abca, abcc\}$ to the boundary are feasible for $G$.

We can classify four sided near-triangulations into three types depending on which two of the three assignments in $S$ are feasible for them. Note that for some near-triangulations all three assignments may be feasible, in which case they can be assumed to be any type. The three types correspond to the simplest four sided near-triangulations, two with no internal vertex and a chord assumed to be any type. The three types correspond to the simplest four sided near-triangulations all three assignments may be feasible, in which case they can be feasible for them. Note that for some which two of the three assignments in $S$ are feasible for it. We show that every four sided near-triangulation is of one of these types. We say a near-triangulation is of type $T_1$ if the assignments $abbc, abca$ are feasible for it. It is of type $T_2$ if the assignments $abca, abcc$ are feasible for it and of type $T_3$ if $abbc, abcc$ are feasible assignments.

Proof of Theorem 3 and Lemma 5 Suppose the theorem and/or the lemma is false and consider a counterexample $G$ to either with the minimum number of vertices.

We first show that $G$ must be simple and cannot contain a separating triangle, that is, a triangle that has vertices of $G$ in its interior as well as exterior.

Suppose $G$ has multiple edges, and let $e_1, e_2$ be two edges in $G$ with the same endpoints $p, q$. Let $C$ be the 2-cycle containing the two edges. Since every face is three sided, there must be vertices in the interior as well as exterior of $C$ and at least one of the edges, say $e_1$, is an internal edge. Let $G'$ be the graph obtained from $G$ by deleting all the vertices and edges in the interior of $C$ and the edge $e_1$. Then $G'$ is also a triangulation or a near-triangulation with the same boundary but fewer vertices than $G$. We show that any assignment of colors to the boundary that is feasible for $G'$ is also feasible for $G$. Suppose $f'$ is a special 3-coloring of $G'$. Let $G''$ be the triangulation obtained from $G$ by deleting the vertices and edges in the exterior of the cycle $C$ and the edge $e_1$, and let the boundary of $G''$ be the boundary of a face that contains $e_2$. Relabel the vertex $p$ as $v_1$ and $q$ as $v_2$ in $G''$. Suppose
\[ \{f'(p)\} \cup \{f'(q)\} \neq \{a, b\}. \] Without loss of generality, assume \( b \notin \{f'(p)\} \cup \{f'(q)\}. \) Lemma 1 implies, after swapping colors and relabeling vertices if necessary, that there exists a non-monochromatic 2-coloring \( f'' \) of \( G'' \) with colors \( \{a, c\} \) such that \( f''(p) = f'(p) \) and \( f''(q) = f'(q) \). If \( \{f'(p)\} \cup \{f'(q)\} = \{a, b\} \), then we may assume, without loss of generality, \( p = v_1, f'(p) = a, q = v_2, \) and \( f'(q) = b \). The minimality of \( G \) implies \( G'' \) has a special 3-coloring \( f'' \) such that \( f''(p) = a \) and \( f''(q) = b \). In both cases, setting \( f(v) = f'(v) \) for all \( v \in V(G') \) and \( f(v) = f''(v) \) for all \( v \in V(G'') \) gives a special 3-coloring of \( G \) with the same assignment of colors to the boundary as in \( G' \). This contradicts the assumption that \( G \) is a counterexample.

Suppose \( G \) is simple but contains a separating triangle \( C \) with vertices \( p, q, r \). Again, let \( G' \) be the near-triangulation obtained from \( G \) by deleting the vertices and edges in the interior of \( C \) and \( G'' \) the triangulation obtained from \( G \) by deleting the vertices and edges in the exterior of \( C \). Suppose \( f' \) is any special 3-coloring of \( G' \) with some assignment of colors to the boundary. Since \( C \) is an internal face in \( G', \{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\} \geq 2 \). If \( \{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\} = 2 \), then \( \{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\} \neq \{a, b\} \), since we may assume \( r \notin \{v_1, v_2\} \) and either \( f'(r) \neq f'(p) \) or \( f'(r) \neq f'(q) \). Again, we may assume without loss of generality that the color \( b \) is not assigned to any of the vertices \( p, q, r \) in \( f' \). Lemma 1 then implies, again after swapping colors and relabeling vertices if necessary, that there exists a non-monochromatic 2-coloring \( f'' \) of \( G'' \) with colors \( \{a, c\} \) such that \( f''(v) = f'(v) \) for all \( v \in \{p, q, r\} \). If \( \{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\} = 3 \), we may assume without loss of generality that \( p = v_1, q = v_2, f'(p) = a, f'(q) = b \) and \( f'(r) = c \). Considering \( v_1v_2r \) to be the boundary of \( G'' \), by the minimality of \( G \), there is a special 3-coloring \( f'' \) of \( G'' \) such that \( f''(v) = f'(v) \) for all \( v \in \{p, q, r\} \). In either case, setting \( f(v) = f'(v) \) for all \( v \in V(G') \) and \( f(v) = f''(v) \) for all \( v \in V(G'') \) gives a special 3-coloring of \( G \) with the same assignment of colors to the boundary as in \( G' \). Again, this contradicts the assumption that \( G \) is a counterexample.

We now assume \( G \) is simple and has no separating triangles and consider cases depending on whether \( G \) is a triangulation or a near-triangulation with a four sided boundary.

**Case 1:** Suppose \( G \) is a triangulation. If \( G \) contains a vertex of degree 2, since there are no multiple edges, \( G \) must be a triangle, in which case, the assignment \( abc \) of colors to the boundary is a special 3-coloring of \( G \). If \( G \) contains a vertex of degree 3, since there are no separating triangles, \( G \) must be \( K_4 \). However, in this case, assigning color \( c \) to the internal vertex and \( abc \) to the boundary gives a special 3-coloring of \( G \). So we may assume every vertex in \( G \) has degree at least 4.

We next show that \( v_5 \) must have degree exactly 4. Suppose not and let \( v_2, v_4, v_5, v_6 \) be the neighbors of \( v_3 \) in circular order, such that \( v_2v_3v_4 \) and \( v_3v_4v_5 \) and \( v_3v_5v_6 \) are internal faces in \( G \). In any special 3-coloring of \( G \), \( v_3 \) must be assigned color \( c \), since it is adjacent to \( v_1 \) and \( v_2 \). The vertex \( v_5 \) is not adjacent to either \( v_1 \) or \( v_2 \), otherwise there is a separating triangle in \( G \), separating \( v_4 \) and \( v_6 \). Let \( G' \) be the triangulation obtained from \( G \) by cutting the edge \( v_3v_4 \). Then \( G' \) has the same boundary as \( G \) and no self-loops, hence by the minimality of \( G \), it has a special 3-coloring \( f' \) with the colors \( abc \) assigned to the boundary. We can now apply Lemma 3 taking \( u = v_3 \), \( w_1 = v_2, w_2 = v_5 \) and \( v = v_4 \). This gives a special 3-coloring of \( G \), a contradiction.
Note that this argument fails if the degree of \( v_3 \) is exactly 4, since \( v_3 \) is now adjacent to \( v_1 \).

We now assume the degree of \( v_3 \) is exactly 4 and as before, let \( v_2, v_4, v_5, v_1 \) be its four neighbors in circular order. Let \( G' \) be the near-triangulation obtained from \( G \) by deleting the vertex \( v_3 \), having the boundary \( v_1v_2v_4v_5 \) in circular order. The minimality of \( G \) implies \( G' \) is of one of the types \( T_1, T_2, \) or \( T_3 \). In all three cases, there exists a special 3-coloring \( f' \) of \( G' \) with the boundary assigned colors \( abbc \) or \( abca \). In either case, setting \( f(v_3) = c \) and \( f(v) = f'(v) \) for all \( v \in V(G') \) gives a special 3-coloring of \( G \), with colors \( abc \) assigned to the boundary. This contradicts the assumption that \( G \) is a counterexample.

Case 2: Suppose \( G \) is a four sided near-triangulation. Suppose \( G \) has a chord, in which case there are no internal vertices in \( G \). If the chord is \( v_1v_3 \), the assignments \( abca \) and \( abcc \) are special 3-colorings of \( G \), but \( abbc \) is not since the vertex \( v_1 \) with color \( a \) is adjacent to the vertex \( v_3 \) of color \( b \). In this case, \( G \) is of type \( T_2 \). If the chord is \( v_2v_4 \) then the assignments \( abbc \) and \( abcc \) are special 3-colorings and \( G \) is of type \( T_3 \).

We may assume \( G \) has no chords. Let \( v_5 \) be the internal vertex in \( G \) such that \( v_1v_2v_5 \) is the internal face in \( G \) containing the edge \( v_1v_2 \). If \( v_5 \) is adjacent to both \( v_3 \) and \( v_4 \), then there are no other vertices in \( G \). Assigning color \( c \) to \( v_5 \) and either colors \( abbc \) or \( abc \) to the boundary, gives a special 3-coloring of \( G \). Thus \( G \) is of type \( T_1 \).

Suppose \( v_5 \) is adjacent to \( v_3 \) but not to \( v_4 \). Let \( G' \) be the triangulation obtained from \( G \) by deleting the vertex \( v_2 \) and adding the edge \( v_1v_3 \) in the interior of the cycle \( v_1, v_5, v_3, v_4 \). Then \( G' \) is a triangulation with boundary \( v_1v_3v_5 \) and by the minimality of \( G \), \( G' \) has a special 3-coloring \( f' \) with colors \( abc \) assigned to the boundary. Since \( v_4 \) is adjacent to both \( v_1 \) and \( v_3 \), we must have \( f'(v_4) = c \). Setting \( f(v_2) = b \) and \( f(v) = f'(v) \) for all \( v \in V(G') \) gives a special 3-coloring of \( G \) with colors \( abbc \) assigned to the boundary. Similarly, Lemma 1 implies that \( G' \) has a non-monochromatic 2-coloring \( f'' \) with colors \( acc \) assigned to the boundary. In this case, \( v_4 \) may be colored either \( a \) or \( c \). Setting \( f(v_2) = b \) and \( f(v) = f''(v) \) for all \( v \in V(G') \), gives a special 3-coloring of \( G \) with either \( abca \) or \( abcc \) assigned to the boundary. Then \( G \) is of type \( T_1 \) or \( T_3 \), depending on the color of \( v_4 \) in \( f'' \). A symmetrical argument holds if \( v_5 \) is adjacent to \( v_4 \) but not to \( v_3 \).

We may now assume \( v_5 \) is not adjacent to either \( v_3 \) or \( v_4 \). The degree of \( v_5 \) is at least 4, since \( G \) has no separating triangles. Let \( v_6, v_7 \) be vertices adjacent to \( v_5 \) such that \( v_2v_5v_6 \) and \( v_3v_6v_7 \) are faces in \( G \). If the degree of \( v_5 \) is at least 5, then \( v_7 \) is not adjacent to either \( v_1 \) or \( v_2 \), and we can again apply Lemma 3 by cutting the edge \( v_5v_6 \), taking \( u = v_5, w_1 = v_2, w_2 = v_7 \) and \( v = v_6 \). Therefore \( G \) is of the same type as the near-triangulation \( G' \) obtained by cutting edge \( v_5v_6 \), a contradiction. So we may assume the degree of \( v_5 \) is exactly 4. Lemma 3 cannot be used now as \( v_7 \) is adjacent to \( v_1 \).

Suppose there exists a cycle \( C \) of length four \( v_1, v_2, p, q \) in \( G \) such that \( \{ p, q \} \not\subset \{ v_5, v_6, v_7 \} \) and \( \{ p, q \} \not\subset \{ v_3, v_4 \} \). Let \( G' \) be the near-triangulation obtained from \( G \) by deleting the vertices in the exterior of the cycle \( C \), with the boundary \( v_1v_2pq \). Since at least one of \( v_3, v_4 \) must be in the exterior of \( C \), \( G' \) has fewer vertices than \( G \). The minimality of \( G \) implies \( G' \) is of one of the three types. We replace the vertices in the interior of \( C \) by an equivalent smaller subgraph of the same type.
Note that at least two of the vertices $v_5, v_6, v_7$ must be contained in the interior of $C$. Let $H$ be the graph obtained from $G$ by deleting the vertices in the interior of $C$. If $G'$ is of type $T_1$, add a new vertex $r$ to $H$ in the interior of the cycle $C$ and join it to all four vertices $v_1, v_2, p, q$ and call the resulting graph $G''$. If $G'$ is of type $T_2$, add the edge $v_1 p$ in the interior of $C$ to construct $G''$ from $H$. If $G'$ is of type $T_3$, add the edge $v_2 q$ in the interior of $C$ to construct $G''$ from $H$. In all cases, $G''$ is a near-triangulation with the same boundary but fewer vertices than $G$, and by the minimality of $G$, is of one of the three types. We claim that $G$ is of the same type as $G''$. Let $f''$ be any special 3-coloring of $G''$. If $G''$ was constructed by adding the vertex $r$ in the interior of $C$, since $r$ is adjacent to $v_1$ and $v_2$, we must have $f''(r) = c$. This implies that either $f''(p) = b$ and $f''(q) = c$ or $f''(p) = c$ and $f''(q) = a$. Since in this case $G'$ was of type $T_1$, there exists a special 3-coloring $f'$ of $G'$ such that $f'(p) = f''(p)$ and $f'(q) = f''(q)$. Similarly, in the other cases, it can be argued that the possible values of $f''(p)$ and $f''(q)$ are such that there exists a special 3-coloring $f'$ of $G'$ such that $f'(p) = f''(p)$ and $f'(q) = f''(q)$. In all cases, we have $f'(v_1) = f''(v_1) = a$ and $f'(v_2) = f''(v_2) = b$. Defining $f(v) = f'(v)$ for all $v \in V(G')$ and $f(v) = f''(v)$ for all $v \in V(G'')$ gives a special 3-coloring of $G$ with the same assignment of colors to the boundary as in the coloring of $G''$. This implies $G$ is of the same type as $G''$, a contradiction.

We may now assume there is no such cycle $C$ in $G$. This implies that $v_7$ is not adjacent to $v_3$ and $v_6$ is not adjacent to $v_4$. Suppose $v_4$ is adjacent to $v_4$ and $v_6$ is adjacent to $v_3$. We claim that $G$ is of type $T_1$ in this case. Let $G'$ be the triangulation obtained from $G$ by deleting the vertices $v_1, v_2, v_5$ and adding the edge $v_3 v_7$ in the exterior of the cycle $v_3, v_4, v_7, v_6$. Let the boundary of $G'$ be $v_7 v_3 v_4$. The minimality of $G$ implies that $G'$ has a special 3-coloring $f'$ with colors $abc$ assigned to the boundary. Since $v_6$ is adjacent to both $v_3$ and $v_7$, we have $f'(v_6) = c$. Setting $f(v_1) = a$, $f(v_2) = b$, $f(v_3) = c$ and $f(v) = f'(v)$ for all $v \in V(G')$ gives a special 3-coloring of $G$ with colors $abc$ assigned to the boundary. Similarly, by deleting the vertices $v_1, v_2, v_5$ and adding the edge $v_4 v_6$ to construct a triangulation $G''$ with boundary $v_4, v_6, v_3$, we can find a special 3-coloring of $G$ with colors $abca$ assigned to the boundary. Thus $G$ is of type $T_1$, a contradiction.

Suppose, without loss of generality, that $v_6$ is not adjacent to $v_3$. The argument in the other case is symmetrical, after relabeling the vertices and swapping the colors $a$ and $b$. Let $u \neq v_5$ be the internal vertex in $G$ such that $v_5 v_6 u$ is the other face in $G$ containing the edge $v_2 v_6$. We show that we can find special 3-colorings of $G$ with the additional condition that $u$ is colored $c$. Let $v_2 = w_1, w_2, \ldots, w_d = v_6$ be the vertices adjacent to $u$ in circular order, starting with $v_2$ and ending with $v_6$, where $d \geq 4$ is the degree of $u$. Note that if $u$ is adjacent to $v_3$, we must have $w_2 = v_3$, otherwise $G$ has a separating triangle. The vertex $v_1$ cannot be adjacent to any vertex $w_i$ for $i > 1$, otherwise $v_1, v_2, u, w_i$ is a 4-cycle in $G$. This implies $v_4, v_7$ are not adjacent to $u$. Also, there is no edge $w_i w_j$, for $1 < |i - j| < d - 1$, else $G$ has a separating triangle.

Suppose $d$ is even. Construct a near-triangulation $G'$ from $G$, as shown in Fig. 2, by cutting the edges $u w_{2i}$ for $1 \leq i \leq d/2$ in this order. The vertices $w_3, w_5, \ldots, w_{d-1}$ will be identified with the vertex $w_1 = v_2$ by this. The vertex $u$ will have degree 2 in $G'$ with $v_2 = w_1$ and $v_6 = w_d$ as its two neighbors. There will be two edges between $v_6$ and $v_2$, corresponding to the edges $v_2 v_6 = w_1 w_d$ and $w_{d-1} w_d$ in $G$, and $u$ is the
only vertex contained in the interior of the 2-cycle formed by the two edges. There will be no self-loops in \( G' \). The minimality of \( G \) implies \( G' \) is of one of the three types. We show that \( G \) is of the same type. Consider any special 3-coloring \( f' \) of \( G' \) with some assignment of colors to the boundary. By definition, \( f'(v_2) = f'(w_1) = b \). Since the only faces containing the vertex \( u \) have vertices \( u, v_2, v_6 \) on their boundary, we can recolor vertex \( u \) as \( c \) if necessary, without creating any monochromatic face or any \( ab \)-pair. We may therefore assume \( f'(u) = c \). Lemma 3 now gives a special 3-coloring of \( G' \), with the same assignment of colors to the boundary. Hence \( G \) is of the same type as \( G' \).

If \( d \) is odd, this argument cannot be used directly, since the resulting near-triangulation will have a self-loop corresponding to the edge \( v_2v_6 = w_1w_d \). To avoid this, we first construct a near-triangulation \( G'' \) by replacing the edge \( v_2v_6 \) by the edge \( uv_5 \). Now \( u \) has even degree \( d + 1 \) and we construct \( G' \) from \( G'' \) by cutting edges \( uw_{2i} \) for \( 1 \leq i < (d + 1)/2 \). In \( G' \), the vertex \( v_6 \) will get identified with the vertex \( v_2 \), and an application of Lemma 3 will give a special 3-coloring of \( G'' \) in which \( v_2, v_6 \) are colored \( b \), while \( v_5, u \) are colored \( c \). This is also a special 3-coloring of \( G \). Hence \( G \) will be of the same type as \( G' \).

\[ \square \]

4 Disconnected 2-Factors

In this section, we consider disconnected 2-factors in 2-edge-connected planar cubic graphs. This is equivalent to finding separating perfect matchings in the graph. While the existence of these in all 2-edge-connected planar cubic graphs with at least six vertices was shown in [3], here we consider their existence with the additional restriction that they should include or exclude a specified edge. We say an edge \( e \) in a 2-edge-connected planar cubic graph is a forcing edge if every 2-factor containing \( e \) is a Hamiltonian cycle. We say it is an anti-forcing edge if every 2-factor not containing \( e \) is a Hamiltonian cycle. If \( e \) is a forcing edge in a graph, we say an edge \( e' \neq e \) is forced by \( e \) if every 2-factor containing \( e \), which
must be a Hamiltonian cycle, also contains $e'$. Similarly, $e'$ is anti-forced by $e$ if every 2-factor not containing $e$ must contain $e'$.

We give constructive characterizations of 2-edge-connected planar cubic graphs that contain a forcing or an anti-forcing edge. We first show using Theorem 1 that cyclically 4-edge-connected planar cubic graphs except $K_2^3$ and $K_4$ do not contain any such edges.

**Theorem 4** Let $G$ be a 2-edge-connected planar cubic graph and let $uv$ be any edge in $G$ such that there is no edge parallel to it and $G - \{u, v\}$ is 2-edge-connected. Then there exists a separating perfect matching in $G$ including the edge $uv$.

**Proof** Consider any plane embedding of $G$. Since there is no edge parallel to $uv$, both $u$, $v$ must have two neighbors each not in $\{u, v\}$. Let $u_1, u_2$ be the other neighbors of $u$ and $v_1, v_2$ the other neighbors of $v$ such that $u_1uvv_1$ are consecutive vertices on the boundary of a face of $G$, as are $u_2uvv_2$. The vertices $u_1, u_2, v_1, v_2$ must all be distinct otherwise there is a bridge in $G - \{u, v\}$. Let $G'$ be the 2-edge-connected planar cubic graph obtained from $G - \{u, v\}$ by adding two new vertices $u', v'$ adjacent to each other, joining $u'$ to $u_1, v_1$ and $v'$ to $u_2, v_2$. Since $G - \{u, v\}$ is 2-edge-connected, $u'v'$ is not contained in a 2-edge-cut in $G'$. Theorem 1 implies there is a 2-factor in $G'$ such that $u'v'$ is a chord of some cycle $C$ in the 2-factor. The cycle $C$ must contain the edges $u_1u', u'v_1, u_2v'$ and $v'v_2$. The planarity of $G$ implies $C - \{u', v'\}$ is the disjoint union of a path from $u_1$ to $u_2$ and a path from $v_1$ to $v_2$. Adding the edges $u_1u, u_2u, v_1v$ and $v_2v$ to these two paths gives two disjoint cycles in $G$ which include all vertices in $C$ except $u', v'$. These two cycles, along with the cycles other than $C$ in the 2-factor of $G'$ (if any), give a disconnected 2-factor in $G$ not containing the edge $uv$. The complement of this is a separating perfect matching in $G$ containing $uv$.  

An immediate consequence of Theorem 4 is that every edge in a cyclically 4-edge-connected planar cubic graph, other than $K_2^3$ and $K_4$, is contained in a separating perfect matching. Any edge $uv$ in such a graph satisfies the condition $G - \{u, v\}$ is 2-edge-connected. This also implies that every edge in such a graph is contained in a disconnected 2-factor. The result in [3] that every 2-edge-connected planar cubic graph except $K_2^3$ and $K_4$ has a disconnected 2-factor also follows easily from this. If the graph is not cyclically 4-edge-connected, it must have a cyclic cut of size at most 3. If it has a 2-edge-cut then any perfect matching containing an edge in the cut must also contain the other edge and is a separating perfect matching. Petersen’s theorem implies there exists such a matching. On the other hand, if the graph $G$ is 3-edge-connected, any edge $uv$ contained in a cyclic cut of size 3 must satisfy $G - \{u, v\}$ is 2-edge-connected, otherwise there is a 2-edge-cut in $G$.

We next give a recursive characterization of 2-edge-connected planar cubic graphs that have a forcing edge. Let $H_n$ for $n \geq 1$ denote the graph obtained from a cycle $v_0, v_1, \ldots, v_{2n-1}$ of length $2n$ by adding the edges $v_0v_n$ and $v_iv_{2n-i}$ for $1 \leq i < n$. Note that $H_1$ is $K_2^3$, $H_2$ is $K_4$ and $H_3$ is the prism $K_3 \times K_2$. 

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**Theorem 5** Let \( G \) be a 2-edge-connected planar cubic graph and \( e = uv \) an edge in \( G \). Then \( e \) is a forcing edge in \( G \) if and only if one of the following conditions holds.

1. \( G \) is \( H_n \) for some \( n \geq 1 \) and \( e \) is the edge \( v_0v_n \). In this case, the edges forced by \( e \) are the edges \( v_iv_{2n-i} \) for \( 1 \leq i < n \) (Fig. 3a).

2. There exists a 2-edge-cut \( \{u_1u_2, v_1v_2\} \) in \( G \), containing the edge \( e \) with \( u_1 = u, u_2 = v \), such that the edges \( e_1, e_2 \) are forcing edges in the 2-cut reductions \( G_1, G_2 \) of \( G \) with respect to this 2-edge-cut. In this case, the edges forced by \( e \) in \( G \) are the edges forced by \( e_1 \) in \( G_1 \), the edges forced by \( e_2 \) in \( G_2 \), and the edge \( v_1v_2 \) (Fig. 3b).

3. There exists a 2-edge-cut \( \{u_1u_2, v_1v_2\} \) in \( G \) not containing the edge \( e \), such that \( e, e_2 \) are forcing edges in the 2-cut-reductions \( G_1, G_2 \) of \( G \) with respect to this cut, and \( e_1 \) is an edge forced by \( e \) in \( G_1 \). In this case, the edges forced by \( e \) in \( G \) are the edges forced by \( e \) in \( G_1 \), except \( e_1 \), the edges forced by \( e_2 \) in \( G_2 \), and the edges \( u_1u_2, v_1v_2 \) (Fig. 3c).

![Fig. 3 Forcing and forced edges](image-url)
Proof It is easy to verify using Lemma 4 that if $G$ and $e$ satisfy one of the three conditions, then $e$ is a forcing edge in $G$ and the edges forced by $e$ are exactly those specified. We only need to prove the converse. Let $G$ be any 2-edge-connected planar cubic graph and $e = uv$ a forcing edge in $G$.

Suppose $G$ is 3-edge-connected. We prove that $G$ must be $H_n$ for some $n \geq 1$ and $e$ must be the edge $v_0v_n$. The proof is by induction on the number of vertices. If $G$ has only two vertices, it must be $K_3$ and the result holds. Suppose $G$ has $2n$ vertices for some $n > 1$. Let $u_1, u_2$ be the neighbors of $u$ other than $v$. Theorem 4 implies $G - \{u, u_1\}$ as well as $G - \{u, u_2\}$ contains a bridge, otherwise there is a separating perfect matching containing $uu_1$ or $uu_2$, whose complement is a disconnected 2-factor in $G$ containing the edge $uv$, a contradiction. Let $pq$ be a bridge in $G - \{u, u_1\}$, let $C_1, C_2$ be the components of $(G - \{u, u_1\}) - pq$ and assume without loss of generality that $p$ is in $C_1$ and $q$ in $C_2$. Since $G$ is 3-edge-connected, $u_2$ and $v$ belong to different components, and without loss of generality, $u_2$ is in $C_1$. Note that $u_1$ also has one neighbor in $C_1$ and one in $C_2$. If the component $C_1$ is non-trivial, it must contain at least three vertices and must be 2-connected, otherwise $G$ has a 2-edge-cut. The same holds for $C_2$. This implies that if $C_1$ is not trivial, we can find a 2-edge-connected subgraph in $G - \{u, u_2\}$ containing both $v$ and $u_1$. This is obtained by adding a path to $C_2$ that goes through $u_1$ and $pq$, and uses a path in $C_1 - \{u_2\}$ between $p$ and the vertex in $C_1$ that is adjacent to $u_1$. This contradicts the fact that there must be a bridge that separates $v$ and $u_1$ in $G - \{u, u_2\}$. Therefore $C_1$ must be trivial and contain only the vertex $p = u_2$. This implies $u, u_1, u_2$ form a triangle in $G$. Let $G'$ be obtained from $G$ by contracting the triangle $u, u_1, u_2$ to the vertex $u$. If $uv$ is contained in a disconnected 2-factor in $G'$, then it is also contained in a disconnected 2-factor in $G$, obtained by adding appropriate edges from the contracted triangle, a contradiction. Otherwise $G'$ must be $H_{n-1}$ and $uv$ must be the edge $v_0v_{n-1}$. This implies that $G$ must be $H_n$ and $uv$ the edge $v_0v_n$. Figure 3a shows the graph $H_4$. The forcing edge is shown by a thick line and the forced edges by a dotted line.

Suppose $G$ has a 2-edge-cut $\{u_1u_2, v_1v_2\}$ containing the edge $e$, with $u_1 = u, u_2 = v$. Let $G_1, G_2$ be the 2-cut reductions of $G$ with respect to this cut. It follows from Lemma 4 that if $e$ is a forcing edge in $G$, then $e_1, e_2$ must be forcing edges in $G_1$ and $G_2$, respectively. This implies $G$ satisfies condition 2. Figure 3b shows this step in the recursive construction.

The only other possibility is that $G$ has a 2-edge-cut $\{u_1u_2, v_1v_2\}$ not containing the edge $e$. Let $G_1, G_2$ be the 2-cut reductions of $G$ with respect to this cut. Without loss of generality assume $e$ is an edge in $G_1$. If $e$ is a forcing edge in $G$, every 2-factor containing $e$ must contain the edges $u_1u_2, v_1v_2$. This implies $e$ must be a forcing edge in $G_1$, $e_2$ a forcing edge in $G_2$, and any 2-factor in $G_1$ containing $e$ must contain the edge $e_1$. Thus $e_1$ must be forced by $e$ in $G_1$, and $G$ satisfies condition 3. Figure 3c shows this step in the recursive construction.

We next give a recursive characterization of 2-edge-connected planar cubic graphs with an anti-forcing edge $e$. 

$\square$
Theorem 6 Let $G$ be a 2-edge-connected planar cubic graph and $e$ an edge in $G$. Then $e$ is an anti-forcing edge in $G$ iff $G$ and $e$ satisfy one of the following conditions.

1. $G$ is $K_3^2$ and $e$ is any edge in $G$. The other two edges are anti-forced by $e$.
2. There exist disjoint 3-edge-connected planar cubic graphs $G_1, G_2$ with anti-forcing edges $u_1v_1, u_2v_2$, respectively, such that $G$ is obtained from $G_1 \cup G_2$ by deleting edges $u_1v_1, u_2v_2, w_1w_2$ for some neighbors $w_1, w_2$ of $u_1, u_2$, respectively, and adding the edges $u_1u_2, v_1v_2, u_1w_2, u_2w_1$. The edge $e$ is the edge $u_1u_2$ and the edges anti-forced by $e$ in $G$ are the edges $u_1w_2$, $u_2w_1$, the edges in $G_1$ other than $u_1w_1$ that are anti-forced by $u_1v_1$, and the edges in $G_2$ other than $u_2w_2$ that are anti-forced by $u_2v_2$ (Fig. 4a).
3. There exists a 2-edge-cut $\{u_1u_2, v_1v_2\}$ in $G$ with 2-cut reductions $G_1, G_2$ such that $e \neq e_1$ is an anti-forcing edge in $G_1$, $e_2$ is a forcing edge in $G_2$ and $e_1$ is anti-forced by $e$ in $G_1$. The edges anti-forced by $e$ in $G$ are the edges $u_1u_2, v_1v_2$, the edges other than $e_1$ that are anti-forced by $e$ in $G_1$, and the edges in $G_2$ that are forced by $e_2$ (Fig. 4b).

Proof We first show that if $G$ and $e$ satisfy these properties then $e$ is an anti-forcing edge in $G$. This is clear if $G$ is $K_3^2$ and it follows from Lemma 4 if $G$ satisfies condition 3. Suppose $G$ satisfies condition 2. Any 2-factor $F$ in $G$ that does not contain $u_1u_2$ must contain the edges $u_1w_2, u_2w_1$ and it does not contain the edge $v_1v_2$. Therefore $(F \cup \{u_1w_1, u_2w_2\}) \setminus \{u_1w_2, u_2w_1\}$ is a 2-factor in $G_1 \cup G_2$ not including the edges $u_1v_1, u_2v_2$. This implies it is the union of Hamiltonian cycles in $G_1$ and $G_2$, and therefore $F$ is a Hamiltonian cycle in $G$. It is easy to check that the edges anti-forced by $u_1u_2$ in $G$ are those specified.

We prove the converse. If $G$ has only two vertices it must be $K_3^2$ and satisfies condition 1. Suppose $G$ is 3-edge-connected with at least four vertices and let
e = u_1u_2 be an anti-forcing edge in G. Since there cannot be any edge parallel to e, Theorem 4 implies G − {u_1, u_2} has a bridge v_1v_2. Let C_1, C_2 be the components of (G − {u_1, u_2}) − v_1v_2 such that v_i ∈ V(C_i) for i ∈ {1, 2}. Since G is 3-edge-connected each of u_1, u_2 has one neighbor in C_1, C_2. Let x_1, w_2 be the neighbors of u_1, and w_1, x_2 the neighbors of u_2 in C_1, C_2, respectively. Let G_1 be the graph obtained from C_1 by adding the vertex u_1 and the edges u_1v_1, u_1w_1, u_1x_1 and G_2 the graph obtained from C_2 by adding the vertex u_2 and the edges u_2v_2, u_2w_2, u_2x_2. Suppose there is a disconnected 2-factor F_1 in G_1 not containing u_1v_1 and hence containing u_1w_1. There exists a 2-factor F_2 in G_2 not containing u_2v_2 and hence containing u_2w_2. Then (F_1 ∪ F_2 ∪ {u_1w_2, u_2w_1}) \{u_1w_1, u_2w_2\} is a disconnected 2-factor in G not containing the edge u_1u_2, a contradiction. Therefore the edge u_1v_1 must be an anti-forcing edge in G_1, and by the same argument, u_2v_2 is an anti-forcing edge in G_2. Therefore G satisfies condition 2. Figure 4a shows this step in the construction, with anti-forcing edges shown by thick lines and anti-forced edges by dotted lines.

Suppose G has a 2-edge-cut {u_1u_2, v_1v_2}. An anti-forcing edge cannot be contained in a 2-edge-cut, since any 2-factor not including it cannot include the other edge in the cut, and hence must be disconnected. Let G_1, G_2 be the 2-cut reductions of G with respect to the 2-edge-cut, and assume without loss of generality, e ≠ e_1 is an edge in G_1. Lemma 4 then implies that e is an anti-forcing edge in G_1, e_2 is a forcing edge in G_2, and e_1 must be anti-forced by e in G_1. Therefore G satisfies condition 3. Figure 4b shows this step in the construction.

5 Extension

A possible extension of Theorem 1 is that for any two edges in a 2-edge-connected planar cubic graph, there exists a 2-factor in the graph such that both edges are contained in the same cycle of the 2-factor. If the two edges are on the boundary of a face, we can subdivide the two edges and add an edge joining the two vertices of degree 2 in the interior of the face. The added edge is not contained in a 2-edge-cut in the resulting 2-edge-connected planar cubic graph, and Theorem 1 implies it is a chord of some cycle in a 2-factor of the graph. This gives a 2-factor in the original graph such that both edges are contained in the same cycle. It is also possible that the same property holds for all connected cubic bipartite graphs. We prove it for connected planar cubic bipartite graphs.

Proof of Theorem 2 Suppose there exists a counterexample and let G be one with minimum number of vertices. Since G is cubic and bipartite, it cannot have a bridge. Suppose G has a 2-edge-cut {u_1u_2, v_1v_2}. Let G_1, G_2 be the 2-cut reductions of G with respect to this cut. Then G_1, G_2 are also connected planar cubic bipartite graphs with fewer vertices than G and satisfy the theorem. We label the edge e_1 as f_1 if f_1 is in the 2-edge-cut and label e_2 as f_2 if f_2 is in the 2-edge-cut. If f_1, f_2 are now contained in the same graph, say G_1, there exists a 2-factor F_1 in G_1 such that f_1, f_2 are in the same cycle of F_1. If F_1 does not contain the edge e_1, let F_2 be any 2-factor
in $G_2$ not containing $e_2$, which exists by the stronger form of Petersen’s theorem. Then by Lemma 4, $F_1 \cup F_2$ is a 2-factor in $G$ such that $f_1$ and $f_2$ are contained in the same cycle of the 2-factor. If $F_1$ contains $e_1$ then let $F_2$ be a 2-factor in $G_2$ containing $e_2$. Then by Lemma 4, $F = (F_1 \cup F_2 \cup \{u_1u_2,v_1v_2\}) \setminus \{e_1,e_2\}$ is a 2-factor in $G$ such that $f_1,f_2$ are contained in the same cycle of $F$. Suppose $f_1,f_2$ are in different graphs, say $f_1$ is in $G_1$ and $f_2$ in $G_2$. There is a 2-factor $F_1$ in $G_1$ such that $f_1$ and $e_1$ are in the same cycle of $F_1$. Similarly, there is a 2-factor $F_2$ in $G_2$ such that $f_2$ and $e_2$ are in the same cycle of $F_2$. Then Lemma 4 implies there is a 2-factor $F = (F_1 \cup F_2 \cup \{u_1u_2,v_1v_2\}) \setminus \{e_1,e_2\}$ in $G$ such that $f_1,f_2$ are contained in the same cycle of $F$.

We may now assume $G$ is 3-edge-connected. If $G$ is $K^2_3$ the result is obviously true. If $G$ has at least four vertices then there are no parallel edges and since $G$ is bipartite, every face in a plane embedding of $G$ has even size at least 4. Euler’s formula implies $G$ has at least six faces of size 4, and hence at least two such that they do not contain any of the edges $f_1,f_2$ on their boundary. Let $v_1,v_2,v_3$ be the boundary of such a face and let $u_i$ be the neighbor of $v_i$ not on the boundary of the face, for $1 \leq i \leq 4$. The $u_i$ must be distinct vertices since $G$ is planar cubic and bipartite. If $f_1$ or $f_2$ is one of the edges $uv_i$, we may assume without loss of generality, $f_1$ is the edge $u_1v_1$. If $f_2$ is also incident with some vertex $v_j$, we may assume, by symmetry, it is one of $u_2v_2$ or $u_3v_3$. Let $G'$ be the graph obtained from $G$ by deleting the vertices $v_1,v_2,v_3,v_4$ and adding the edges $u_4u_4$ and $u_2u_3$. Then $G'$ is a planar cubic bipartite graph and must be connected, otherwise $G$ has a 2-edge-cut. Label the added edge $u_1u_4$ as $f_1$ if the edge $u_1v_1$ was the original edge $f_1$. Similarly, label $u_2u_3$ as $f_2$ if either $u_2v_2$ or $u_3v_3$ was the original edge $f_2$. Then $G'$ has a 2-factor $F'$ such that $f_1$ and $f_2$ are in the same cycle of $F'$. If $F'$ does not contain any of the added edges $u_1u_4,u_2u_3$, then adding the edges in the 4-cycle $v_1,v_2,v_3,v_4$ to $F'$ gives a 2-factor in $G$ such that $f_1,f_2$ are in the same cycle. If $F'$ contains $u_1u_4$ but not $u_2u_3$, then $(F'\setminus \{u_1u_4\}) \cup \{u_1v_1,v_1v_2,v_2v_3,v_3v_4,u_4v_4\}$ is a 2-factor in $G$ such that $f_1,f_2$ are contained in the same cycle. The same argument holds if $F'$ contains $u_2u_3$ but not $u_1u_4$. If $F'$ contains both, then $(F'\setminus \{u_1u_4,u_2u_3\}) \cup \{u_1v_1,u_2v_2,u_3v_3,u_4v_4,v_1v_4,v_2v_3\}$ is the required 2-factor in $G$. \hfill $\square$

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