Variations on the Dirac string

Brad Cownden
Department of Physics & Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada

Andrew R. Frey
Department of Physics, University of Winnipeg, 515 Portage Ave, Winnipeg, Manitoba R3B 2E9, Canada and Department of Physics & Astronomy, University of Manitoba Winnipeg, Manitoba R3T 2N2, Canada

(Dated: July 31, 2018)

Dirac’s original solution of the nontrivial Bianchi identity for magnetic monopoles [1], which redefines the fieldstrength along the Dirac string, diagonalizes the gauge and monopole degrees of freedom. We provide a variant of the Dirac string, which we motivate through a formal expansion of the Bianchi identity. We show how to use our variant prescription to study monopole electrodynamics without reference to a dual potential and provide some applications.

I. INTRODUCTION

Magnetic monopoles, while experimentally elusive [2], hold a special place in particle physics; they complete electric-magnetic duality, provide a reason for charge quantization, and arise in the spontaneous breaking of many grand unification models. At the same time, there is a long tradition of reformulating the description of the monopole’s interaction with the electromagnetic field since Dirac’s original work [1,3]:

• Monopoles arise as solitons of broken gauge theories; the microscopic description as a semiclassical field configuration describes the monopole’s interactions but requires tracking the full degrees of freedom of that gauge theory. While of course necessary for processes like monopole creation and annihilation (see eg [4]), it may be computationally excessive when an effective description is valid.

• In the absence of electric charges, the fieldstrength can be defined in terms of a dual potential \( \star F \equiv d \tilde{A} \), which couples to monopoles in the same way the vector potential couples to charges. Of course, if the fields of both charges and monopoles are of interest, the fieldstrengths from both the potential and dual potential must be superposed in a “democratic” formalism (which makes it ill-suited to the quantum mechanics of charges and monopoles). This type of approach has been advocated at least since [5].

• As pointed out originally in [6] and [7], the vector potential is not a globally defined function but a section of a fiber bundle. When the fieldstrength has a nontrivial Bianchi identity, the potential is defined in at least two coordinate patches; the potentials in the two patches are related by a gauge transformation in the overlap of the two patches. While mathematically rigorous, treating the potential in this manner mixes the gauge and monopole degrees of freedom because the overlap region moves with the monopole. This formalism also obscures the coupling between the potential and monopole (for example, [8] switch to a dual potential formalism to derive the monopole’s classical equation of motion, while [9] following [10] combined Dirac’s formalism below with the rigorous gauge patching procedure).

• The potential in a single coordinate patch extends to all of spacetime except for a half-line singularity extending from the monopole (Dirac’s famous string singularity). In [1], Dirac noted that the electromagnetic fieldstrength can be written in terms of a globally defined potential and an extra term supported on the string singularity. This approach separates the electromagnetic and monopole degrees of freedom and elucidates the monopole-fieldstrength coupling. However, it suffers some conceptual difficulties, including a singular fieldstrength along the (arbitrarily chosen) string and a constraint that electric charges cannot intersect the Dirac string worldsheet. Pragmatically, the semi-infinite string can be awkward.

In this paper, we present a new hybridization of Dirac’s string formalism with the rigorous gauge patching prescription by allowing the Dirac string to end on an unphysical reference monopole, and we review the derivation of Dirac’s string prescription from the dual potential formalism with an emphasis on the origin of the string. We also advocate for a particular configuration for the string and give a simple physical picture for its origin. The fixed prescription for the string means that the string worldsheet embedding coordinated depend on the monopole position along the entire worldsheet, unlike Dirac’s original formalism; our approach presents computational advantages in several circumstances. The extension of our results to higher-dimensional monopole-like branes appears in the companion paper [10] by one of us. Throughout this paper, we work in 4-dimensional Minkowski spacetime for simplicity. We list our conventions, particularly signs, in an appendix.
II. DIRAC'S STRING VARIABLES

The Maxwell equations and Bianchi identities with magnetic sources are

\[ d \star F = - \star j \, , \quad dF = - \star j \, , \]

where \( j^\mu \) is the electric current and \( j^\mu \) the magnetic current. For a single monopole, the current is

\[ j^\mu = g \int_M d\tau \partial_\tau X^\mu(\tau) \delta^4(x - X) = gu^\mu(t) \delta^3(\vec{x} - \vec{X}) \, , \]

where \( g \) is the monopole charge, \( M \) is the monopole’s worldline, \( \tau \) the worldline time, and \( X^\mu \) the monopole position. The second equality follows in a fixed reference frame with a worldline static gauge \( \tau = t \); \( u^\mu \) is the monopole’s 4-velocity.

Dirac’s key observation stems from the fact that any conserved current (in Minkowski spacetime) can be written as the divergence of a two-form, or \( \star j = d \star H \) for some \( H \). In fact, the field strength defined as

\[ \star H = \mathrm{d} \delta G(x) \equiv \int_N \mathrm{d}Y^\mu \wedge \mathrm{d}Y^\nu \delta^4(x - Y) \, , \]

where \( N \) is the worldsheet of a string with boundary \( \partial N = M - M_* \) (\( M_* \) is a so-far arbitrary worldline). \( Y^\mu(\tau, \sigma) \) is the string position at worldsheet coordinates \( \tau, \sigma \) with \( \sigma \) increasing from \( M_* \) to \( M \), and \( \partial = d\sigma \partial_\tau + d\tau \partial_\sigma \) is the worldsheet exterior derivative. This has divergence

\[ (\star d \star G) = \int_N \partial \delta^4(x - Y) = (1/g)j^\mu - (1/g)j^\mu \]

in terms of the currents of monopoles along \( M \) and \( M_* \). As a result, the field strength defined as \( F = dA - g \star G \) automatically solves the Bianchi identity for the dynamical monopole along \( M \) as long as \( A \) includes the potential for a monopole with reference worldline \( M_* \) (with appropriate gauge patching).

It is worth pausing now to discuss the configuration of the string. In Dirac’s formalism, the string is a dynamical object with no kinetic term (in modern parlance, a tensionless string), so the worldsheet \( N \) is completely arbitrary except for the specification of its boundary.

Dirac furthermore locates the worldline \( M_* \) at spatial infinity, so the string becomes semi-infinite. However, we are free to specify the configuration of the string; a simple choice in a given reference frame is to choose the reference monopole to be stationary \( (M_* \) at constant position \( \vec{X}_* \)) and the string to be the straight line from \( \vec{X}_* \) to the monopole position \( \vec{X}(t) \) at each time. We can build an arbitrary worldsheet in this way by then extending a Dirac string from \( M_* \) to another reference worldline \( M_{**} \), and so on, and then letting the reference worldlines approach each other. On the other hand, with a single line segment, the worldsheet embedding coordinates \( Y^\mu \) depend on the monopole position at all worldsheet at all worldsheet points \( \tau, \sigma \) unlike in Dirac’s formalism. We motivate this prescription for the Dirac string configuration in section [IV].

It is also instructive to consider the Dirac quantization of charge for different formalisms. With distinct gauge patches, quantization enforces the requirement that a single-value wavefunction in one coordinate patch remains single-valued after the gauge transformation in passing to another patch. In Dirac’s original formalism, there is only a single gauge, but the wavefunction picks up a phase under motion of the string equal to \( g \) times the electric flux through the surface swept out by the worldsheet — the surface is noncontractible since the charge cannot intersect the worldsheet, as we see below. Since the surface swept out by the string is closed (including the point at infinity), charge quantization is given by the condition that the wavefunction remain single-valued if we sweep the string around the charge. When the reference monopole is at a finite position, both mechanisms for charge quantization are possible. In particular, if the string worldsheet is arbitrary but \( M_* \) is fixed, we can sweep the string in a closed surface around a charge, as per Dirac. With a set prescription for the worldsheet like discussed above, however, the worldsheet only moves if we move the reference position \( \vec{X}_* \), so the string sweeps out open surfaces, removing the quantization condition. On the other hand, the potential must be defined in patches around the reference point, so single-valuedness of the wavefunction still leads to charge quantization.

The Dirac string formalism also provides a direct means of finding the monopole equation of motion, which is otherwise carried out indirectly through the dual potential formalism. Including the string coupling \( G \) in the field-strength, the variation of the Maxwell action with respect to the monopole and worldsheet positions is

\[ \delta S_{Max} = \frac{g}{4} \epsilon_{\mu \nu \lambda \rho} \int d^4x F^{\mu \nu}(x) \delta G^{\lambda \rho}(x) \]

\[ = \frac{g}{4} \epsilon_{\mu \nu \lambda \rho} \int d^4x \int_N \left[ \partial \delta^4(x - Y) \right] \left\{ \partial Y^\lambda \wedge \partial Y^\rho \delta Y^\alpha \partial Y^\nu \delta^4(x - Y) + \left[ \partial \delta^4(x - Y) \right] \left\{ \partial Y^\lambda \wedge \partial Y^\rho \right\} \delta^4(x - Y) \right\} \, . \]

We can now restrict to a worldsheet integral, first converting the derivative on the delta function to one with respect to \( x^\alpha \) and integrating by parts. Then, with all
partial derivatives now with respect to Y,
\[
\delta S_{\text{Max}} = \frac{\mathcal{g}}{2} \int_{\mathcal{M}} \left\{ \tilde{d} \left[ F^{\mu\nu} (Y) \delta Y^\lambda \tilde{d} Y^\rho \right] \\
+ \frac{1}{2} \partial_\mu F^{\mu \nu} (Y) \left( \tilde{d} Y^\alpha \wedge \tilde{d} Y^\lambda \delta Y^\nu \right) \\
+ \tilde{d} Y^\lambda \wedge \tilde{d} Y^\rho \delta Y^\alpha + \tilde{d} Y^\rho \wedge \tilde{d} Y^\alpha \delta Y^\lambda \right\}.
\]

The latter lines of (6) reorganize to
\[
\frac{\mathcal{g}}{2} \int_{\mathcal{M}} \left( d \star F \right)_{\nu \lambda \rho} \delta Y^\nu \wedge \tilde{d} Y^\lambda \delta Y^\rho,
\]
which would yield an interaction between the (unphysical) string — and therefore the monopole — with the electric current when the gauge fields are on shell. To avoid this unphysical result, Dirac imposed the additional condition that charges not intersect the string. The first term, on the other hand, gives an integral over \( \mathcal{M} \) (assuming \( \mathcal{M} \) is fixed)\(^3\) so it combines with the variation of the monopole’s kinetic term to give the magnetic Lorentz force equation
\[
\partial_\tau p_\mu = -g (\star F)_{\mu \nu} \partial_\tau X^\nu.
\]

Meanwhile, the electric charge couples to the redefined potential \( A \), so its equation of motion turns out as usual except for contact terms with the Dirac string, which are forbidden due to Dirac’s condition.

It is important to note that our derivation of the monopole equation of motion treated the potential \( A \) as an independent degree of freedom in contrast to the case without a Dirac string and with gauge patches. In that case, there is no coupling between the monopole and gauge field, only a hidden dependence of \( A \) on \( X^\mu \) which should be treated as an explicit dependence. The Dirac string removes this explicit dependence from the vector potential, but \( A \) does still have a “hidden” explicit dependence on the arbitrary reference position. The equation of motion for the reference position is
\[
F^{\mu \nu} /\Delta X^\lambda = 0,
\]
which enforces the condition that the fieldstrength is independent of the reference position. This condition determines the explicit dependence of the potential on \( X^\mu \).

### III. THE STRING FROM THE DUAL POTENTIAL

The dual potential formalism is useful in many applications since it translates the electrodynamics of monopoles into the more familiar electrodynamics of charges (and can even allow for the interaction of charges and monopoles since electromagnetism is linear). Here we review a derivation of the Dirac string from the dual potential, simplified from versions presented in \([11, 12]\).

In particular, we start with the dual fieldstrength and potential and a magnetic current in the presence of electric charges (which lead to a Dirac string for the dual fieldstrength) in order to emphasize that the Poincaré duality itself leads to the Dirac string for the fieldstrength in the presence of monopoles.

In form notation, the action for the dual electromagnetism with a single monopole is
\[
S = \int \left( -\frac{1}{2} \tilde{F} \wedge \star \tilde{F} + \tilde{A} \wedge \star \tilde{j} \right),
\]
where \( \tilde{A} , \tilde{F} \) are the dual potential and fieldstrength and \( \tilde{j} \) is the monopole current \( \tilde{f} \). To dualize back to the “usual” potential in the absence of monopoles, we treat \( \tilde{F} \) as the independent variable and add a Lagrange multiplier term \( A d \tilde{F} \) to enforce the Bianchi identity for \( \tilde{F} \) since \( \tilde{A} \) does not appear. Solving the equation of motion for \( \tilde{F} \) and substituting gives the usual Maxwell action.

With the monopole current, we must first find a way to eliminate the dual potential from (9). We proceed by recalling that any conserved current can be written as the divergence of a 2-form (in other words, any closed form in Minkowski spacetime is co-exact). In fact, the Maxwell equation \( d \star \tilde{F} = \star \tilde{j} \) shows that \( \tilde{j} \) can be written in terms of the monopole’s field strength (which also satisfies \( d \tilde{F} = 0 \)). We have also seen above that the Dirac string coupling \( G \) is another such two form up to the subtraction of a current along the reference worldline \( \mathcal{M} \), so we can write
\[
S = \int \left[ \left( -\frac{1}{2} \tilde{F} \wedge \star \tilde{F} + \tilde{A} \wedge \star (gG + \tilde{F}_s) \right) \right.
\]
\[
= \int \left[ \left( -\frac{1}{2} \tilde{F} \wedge \star \tilde{F} + \tilde{F} \wedge \star (gG + \tilde{F}_s) - \tilde{A} \wedge \star d \tilde{F} \right) \right]
\]
where \( \tilde{F}_s \) is the dual fieldstrength of the reference monopole and we have added the Lagrange multiplier term. This last term can be rewritten as \( \tilde{F} d \tilde{A}' \) up to a total derivative, so the equation of motion is \( \star \tilde{F} = \star (gG + \tilde{F}_s) - d \tilde{A}' \). The action is therefore classically equivalent to
\[
S = -\int \left( \frac{1}{2} \left( d \tilde{A}' - \tilde{F}_s - g * G \right) \wedge \star \left( d \tilde{A}' - \tilde{F}_s - g * G \right) \right).
\]

We recognize \( - \tilde{F}_s = F_s \) as the reference monopole’s field strength, so we define \( d \tilde{A}' + F_s = d A \) in terms of a potential \( A \) with the appropriate gauge patches for the reference monopole. We are left with \( \tilde{F} = d A - g * G \) and the usual Maxwell action \( -\int F * F / 2 \) including the Dirac string.

### IV. AN INTERPRETATION OF THE STRING

We can gain new insight into the Dirac string through the Bianchi identity. For clarity, we pick a reference
frame and work in static gauge for the monopole (ie, worldline time $t = Y^0$). In non-relativistic notation, the Bianchi identity is

$$\vec{\nabla} \cdot \vec{B} = g \delta^3(\vec{x} - \vec{X}) \ , \ \vec{\nabla} \times \vec{E} + \partial_t \vec{B} = -g \partial_t \vec{X} \delta^3(\vec{x} - \vec{X}) . \quad (12)$$

Our goal is to diagonalize the degrees of freedom, removing the explicit dependence of $A_i$ on the monopole position. We carry out a formal expansion of the monopole current around a static reference monopole at fixed position $\vec{X}_s$; therefore, we end up with a static gauge patching procedure around $\vec{X}_s$ and will see that the remaining terms can be organized as a contribution to the field-strength.

We begin by writing

$$\delta^3(\vec{x} - \vec{X}) \equiv \delta^3(\vec{x} - \vec{X}_s) \ + \ \sum_{n=1}^{\infty} \frac{1}{n!} \delta X^{i_1} \cdots \delta X^{i_n} \nabla^{x}_{i_1} \cdots \nabla^{x}_{i_n} \delta^3(\vec{x} - \vec{X}_s) ,$$

where $\nabla^x$ is the gradient with respect to $\vec{X}_s$ and $\delta \vec{X} = \vec{X} - \vec{X}_s$. Converting one of the derivatives to one with respect to $\vec{x}$, the terms in the sum can be written as a divergence, so defining

$$\vec{B} = \vec{\nabla} \times \vec{A} - g \delta \vec{X} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\delta \vec{X})^{n,i_1 \cdots i_n} (\nabla^x)_{i_1 \cdots i_n} \delta^3(\vec{x} - \vec{X}_s) \quad (14)$$

where $\vec{A}$ includes the potential of a fixed monopole at $\vec{X}_s$ solves the scalar equation of (12). Subtracting this contribution from the vector part of the Bianchi identity similarly extracts a curl from the expansion of the delta function. We can therefore define

$$\vec{E} = -\vec{\nabla} \Phi - \partial_t \vec{A} + g (\partial_t \vec{X} \times \delta \vec{X}) \ + \ \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} (\delta \vec{X})^{n,i_1 \cdots i_n} (\nabla^x)_{i_1 \cdots i_n} \delta^3(\vec{x} - \vec{X}_s) \quad (15)$$

to solve the Bianchi identity.

To confirm that this approach separates the gauge and monopole degrees of freedom, we can vary the Maxwell action with respect to $\vec{X}$ and, with some manipulation, find the magnetic Lorentz force equation by treating the potentials as independent variables. As with the Dirac string, the variation of the action also includes terms proportional to the sourceless Maxwell equations and derivatives of $\delta^3(\vec{x} - \vec{X}_s)$.

As an alternate approach, we can consider the Dirac string with $M_4$, a static worldline at $\vec{X}_s$. Then, as we suggested in the section above, take $\mathcal{N}$ at any fixed time to be the line segment from $\vec{X}_s$ to $\vec{X}$. Then we can choose a static gauge $^t = Y^0(\tau, \sigma) = \tau$ and $\vec{Y}(\tau, \sigma) = \vec{X}_s + \sigma \delta \vec{X}(t)$ with $0 \leq \sigma \leq 1$. Then the Dirac string coupling becomes

$$G^{0i} = \int_0^1 d\sigma \delta X^i \delta^3(\vec{x} - \vec{X}_s - \sigma \delta \vec{X}) , \quad (16)$$

and similarly

$$G^{ij} = (\partial_i X^j - \partial_j X^i) \sum_{n=0}^{\infty} \frac{1}{n!} (\delta \vec{X})^{n,i_1 \cdots i_n} \times (\nabla^x)_{i_1 \cdots i_n} \delta^3(\vec{x} - \vec{X}_s) \int_0^1 d\sigma \sigma^{n+1} \quad (17)$$

after expanding the delta function. Carrying out the integral and using the usual relation between the field-strength $F_{\mu\nu}$ and fields $\vec{E}, \vec{B}$, we find (14,15).

So the particular solution of the Bianchi identity that we found by expanding the delta function around $\vec{X}_s$ is a particular realization of the Dirac string. As a result, we see that the expansion of the delta function separates the explicit dependence of the field-strength on the monopole position from the potential. This also tells us that the Dirac string is a way of treating a monopole's motion as a fluctuation (even a large one) around a fixed position. The key difference with Dirac's arbitrary string is that this interpretation suggests treating the embedding coordinates of the string as dependent on the monopole position along the entire worldsheet. In fact, we have done this explicitly in deriving (14,15).

For future reference, it is useful to give the exact expressions for the fields with the linear string configuration:

$$\vec{E} = -\vec{\nabla} \Phi - \partial_t \vec{A} + g \left( \partial_t \vec{X} \times \delta \vec{X} \right) \int_0^1 d\sigma \delta^3 \left( \vec{x} - \vec{X}_s - \sigma \delta \vec{X} \right)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} - g \delta \vec{X} \int_0^1 d\sigma \delta^3 \left( \vec{x} - \vec{X}_s - \sigma \delta \vec{X} \right) . \quad (18)$$

The displacement $\vec{D}$ and field $\vec{H}$ are determined as usual from $\vec{E}, \vec{B}$ in linear media, including the Dirac string contribution.

**V. THE STRING AS A SOURCE**

The Dirac string in the fieldstrength automatically solves the Bianchi identity, so the Bianchi identity no longer determines the fieldstrength associated with the dynamic monopole. Instead, as Dirac realized, the string

---

\(^4\) This is the approach taken in [13] for D3-branes.
The effective charge is

$$\rho_{\text{eff}} = j_{\text{eff}}^0$$

$$= -g \left( \partial_t \vec{X} \times \delta \vec{X} \right) \cdot \vec{\nabla} \left[ \int_0^1 d\sigma \sigma \delta^3(\vec{x} - \vec{X}_* - \sigma \delta \vec{X}) \right],$$

where the gradient is with respect to \( \vec{x} \). Evaluating the effective current in the static gauge is slightly subtler. We have

$$j_{\text{eff},i} = -g \epsilon_{i0jk} \int dt \int_0^1 d\sigma \left( \sigma \partial_\tau X^j \delta X^k \right)$$

$$\times \partial_t \left[ \delta(t - \tau) \delta^3(\vec{x} - \vec{X}_* - \sigma \delta \vec{X}) \right]$$

$$+ g \epsilon_{ijk} \int_0^1 d\sigma \delta X^k \partial_j \delta^3(\vec{x} - \vec{X}_* - \sigma \delta \vec{X}).$$ \hspace{1cm} (20)

We must take care to convert the \( t \) derivative to a \( \tau \) derivative and integrate by parts where the delta function in time is differentiated. We are left with

$$\vec{j}_{\text{eff}} = g \partial_t \left[ \left( \partial_t \vec{X} \times \delta \vec{X} \right) \int_0^1 d\sigma \delta^3(\vec{x} - \vec{X}_* - \sigma \delta \vec{X}) \right]$$

$$- g \delta \vec{X} \times \vec{\nabla} \left[ \int_0^1 d\sigma \delta^3(\vec{x} - \vec{X}_* - \sigma \delta \vec{X}) \right].$$ \hspace{1cm} (21)

This is worth two comments. First, the current for arbitrary linear motion of the monopole, with \( \vec{X}_* \) chosen to lie on the line, is

$$j_{\text{eff}} = -g \delta \vec{X} \times \vec{\nabla} \left[ \int_0^1 d\sigma \delta^3(\vec{x} - \vec{X}_* - \sigma \delta \vec{X}) \right].$$ \hspace{1cm} (22)

Specifying a string configuration that depends explicitly on the monopole position along the worldsheet gives a well-defined effective current from the string. In principle, we can now solve for the electromagnetic fields for arbitrary monopole motion as a superposition of the magnetic field from the fixed reference monopole and the effective current. The linear string configuration seems particularly well suited to this type of calculation.

**VI. APPLICATIONS**

As we have noted previously, there are technical difficulties in the theory of electrodynamics with monopoles. Here we present several possible applications in which the linear Dirac string configuration yields simplifications.

We indicated above that one such application is a direct determination of radiation from moving monopoles, including energy loss in dielectric materials (such as Cherenkov radiation),\(^6\) which may be useful for monopole search experiments. While an infinite, arbitrarily moving Dirac string is unwieldy, the linear Dirac string configuration gives a well-defined current, which is simply a growing or shrinking solenoid in many contexts. In the presence of materials, as noted, it is necessary to write the fields nonrelativistically as \( E, B \) and include the permittivity and permeability in defining \( \tilde{D}, \tilde{H} \).

We have emphasized that the Dirac string formulation separates the gauge and monopole degrees of freedom; as a result, it provides a basis for Hamiltonian and therefore quantum treatments. In this form, the introduction of extra unphysical degrees of freedom for the Dirac string leads to constraints \([10]\). On the other hand, when the Dirac string takes the linear configuration, the entire string depends on the monopole position. In the variation of the action, these appear through terms proportional to the sourceless Maxwell equations (see equation \([7]\)) and are trivial on-shell. On the other hand, these terms can contribute off-shell, for example in the path integral. Understanding how these contribute to the quantum mechanics of monopoles is an interesting question. Meanwhile, the analogous terms for D3-branes (see the discussion below) also play an important role in the 4-dimensional effective action of type IIB string theory \([10, 13]\).

The Dirac string formalism straightforwardly extends to curved spacetimes and higher dimensions, and the linear configuration for the string becomes a worldsheet with geodesics as constant-time slices. A second endpoint to the string on \( M \) therefore allows us to use the Dirac string for monopoles on compact slices. While the magnetic Gauss law constraint (net magnetic charge on a compact manifold vanishes) means that any Dirac string can end on oppositely charged physical monopoles, an

---

6 See \([13, 15]\) for the energy loss rate in different approximations.
arbitrary reference endpoint allows for a cleaner separation of the dynamics of the different monopoles. Furthermore, it allows us to avoid having multiple Dirac strings end on one monopole in the case that the monopoles on a compact manifold carry different numbers of magnetic quanta (for example, there are two monopoles of charge $+1$ and one of charge $-2$). Again, in the case of higher-dimensional monopole-like branes, monopole charge can dissolve into the flux of other fields, so the Dirac string from a monopole may not even have another monopole on which to end, necessitating the reference endpoint.

Finally, higher-dimensional branes of string theory are magnetic sources for various rank form fields. As in electromagnetism, a mathematically rigorous treatment considers the potentials as sections, while a Dirac-like formalism allows the separation of the gauge and brane degrees of freedom (see [13]; one of us will detail this formalism in the forthcoming [10]). The formalism therefore provides an alternative to the democratic (dual potential) formalism for the derivation of brane equations of motion. It is particularly helpful for a careful accounting of degrees of freedom, as needed in dimensional reduction. Determining the lower-dimensional effective action is also an off-shell calculation, and extra terms from the generalized Dirac string worldvolume (which vanish on-shell) are critical to account for all the kinetic terms required by supergravity [10] [13].

**ACKNOWLEDGMENTS**

AF would like to thank K. Dasgupta, N. Afshordi, and L. Boyle for interesting discussions. BC and AF are supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant program. Part of this work was supported by the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research and Innovation.

**Appendix: Conventions**

Here we briefly lay out our conventions, including signs. To start, we take the mostly plus metric convention with $\epsilon_{0123} = +1$. The Hodge star for a differential $p$-form is given by $(\ast F)_{\mu_1 \cdots \mu_p} = (1/p!)\epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p} F_{\nu_1 \cdots \nu_p}$, so $\ast \ast F = (-1)^{p(4-p)+1}$.

With standard conventions (see [14] [17]), the Maxwell equations with magnetic currents included are

$$\nabla \cdot \mathbf{E} = \rho, \nabla \times \mathbf{B} = \partial_t \mathbf{E} = \mathbf{j},$$
$$\nabla \cdot \mathbf{B} = \partial_t \mathbf{E} = \nabla \times \mathbf{E} + \partial_t \mathbf{B} = -\mathbf{j},$$

(A.1)

where $\rho, \mathbf{j}$ are the electric charge and current and $\partial_t, \nabla$ are the magnetic. In relativistic notation, we take $A^\mu = (\Phi, \vec{A})$, $j^\mu = (\rho, \vec{j})$ (and likewise for the magnetic current), so $F_{\mu\nu} = -\epsilon_{\nu,ij} B_k$. The Maxwell equations become

$$\partial_\mu F^{\mu\nu} = -j^\nu, \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\mu\lambda} + \partial_\lambda F_{\mu\nu} = -\epsilon_{\mu\nu\lambda\rho} j^\rho,$$
(A.2)

or $d F = -j$ and $dF = -*j$ in terms of forms. The dual field strength $\tilde{F} \equiv \ast F$ therefore satisfies the dual Maxwell equations $d\tilde{F} = -*j$ and $d\tilde{F} = +*j$. As a result, the dual electric current is usually defined as $-\tilde{j}$; for simplicity of comparison, we do not introduce this sign.

Finally, we define the Dirac string coupling $G$ as a form integral over the string worldsheet coordinates $\tau, \sigma$.

We choose the orientation by taking integration measure $d^2 \sigma = d\tau \wedge d\sigma = -d\sigma \wedge d\tau$.

---

[1] Paul A. M. Dirac, “The theory of magnetic poles,” Phys. Rev. 74, 817–830 (1948).

[2] B. Acharya et al. (MoEDAL), “Search for magnetic monopoles with the MoEDAL forward trapping detector in 2.11 fb$^{-1}$ of 13 TeV proton-proton collisions at the LHC,” Phys. Lett. B782, 510–516 (2018) arXiv:1712.09849 [hep-ex].

[3] Paul A. M. Dirac, “Quantized singularities in the electromagnetic field,” Proc. Roy. Soc. Lond. A133, 60–72 (1931) [278(1931)].

[4] Tanmay Vachaspati, “Creation of magnetic monopoles in classical scattering,” Phys. Rev. Lett. 117, 181601 (2016) arXiv:1607.07460 [hep-th].

[5] P. Rohrlich, “Classical theory of magnetic monopoles,” Phys. Rev. 150, 1104–1111 (1966).

[6] Tai Tsun Wu and Chen Ning Yang, “Dirac monopole without strings: monopole harmonics,” Nucl. Phys. B107, 365 (1976).

[7] Tai Tsun Wu and Chen Ning Yang, “Dirac’s monopole without strings: classical Lagrangian theory,” Phys. Rev. D14, 437–445 (1976) [509 (1976)].

[8] K. Hirata, “Classical Lagrangian theory of Dirac’s monopole: avoiding Dirac’s veto,” Phys. Lett. 81B, 169–172 (1979).

[9] Richard A. Brandt and Joel R. Primack, “Avoiding Dirac’s veto in monopole theory,” Phys. Rev. D15, 1798–1802 (1977).

[10] Andrew R. Frey, “Dirac branes for Dirichlet branes,” (2018), in preparation.

[11] Stanley Deser, A. Gomberoff, M. Henneaux, and C. Teitelboim, “P-brane dyons and electric magnetic duality,” Nucl. Phys. B520, 179–204 (1998) arXiv:hep-th/9712189 [hep-th].

[12] K. Lechner and P. A. Marchetti, “Duality invariant quantum field theories of charges and monopoles,” Nucl. Phys. B569, 529–576 (2000) arXiv:hep-th/9906079 [hep-th].
[13] Brad Cownden, Andrew R. Frey, M. C. David Marsh, and Bret Underwood, “Dimensional reduction for D3-brane moduli,” JHEP 12, 139 (2016), arXiv:1609.05904 [hep-th].

[14] John David Jackson, Classical electrodynamics (Wiley, 1975 (2nd ed), 1998 (3rd ed)).

[15] J. Gea-Banacloche, Kevin E. Cahill, and D. Rossbach, “Energy loss by slow magnetic monopoles,” Lett. Nuovo Cim. 37, 145 (1983).

[16] A. P. Balachandran, R. Ramachandran, J. Schechter, Kameshwar C. Wali, and Heinz Rupertsberger, “Hamiltonian formulation of monopole theories with strings,” Phys. Rev. D13, 354 (1976).

[17] David J. Griffiths, Introduction to electrodynamics (Pearson, 2012 (4th ed)).