Strong approximations in a charged-polymer model

Yueyun Hu  
Université Paris 13

Davar Khoshnevisan  
University of Utah

November 19, 2009

Abstract

We study the large-time behavior of the charged-polymer Hamiltonian $H_n$ of Kantor and Kardar [Bernoulli case] and Derrida, Griffiths, and Higgs [Gaussian case], using strong approximations to Brownian motion. Our results imply, among other things, that in one dimension the process $\{H_{\lfloor nt\rfloor}\}_{0 \leq t \leq 1}$ behaves like a Brownian motion, time-changed by the intersection local-time process of an independent Brownian motion. Chung-type LILs are also discussed.

Keywords: Charged polymers, strong approximation, local time.

AMS 2000 subject classification: Primary: 60K35; Secondary: 60K37.

Running Title: Polymer measures, strong approximations.

1 Introduction

Consider a sequence $\{q_i\}_{i=1}^\infty$ of independent, identically-distributed mean-zero random variables, and let $S := \{S_i\}_{i=0}^\infty$ denote an independent simple random walk on $\mathbb{Z}^d$ starting from 0. For $n \geq 1$, define

$$H_n := \sum_{1 \leq i < j \leq n} q_i q_j 1_{\{S_i = S_j\}};$$  \hspace{1cm} (1.1)

this is the Hamiltonian of a so-called “charged polymer model.” See Kantor and Kadar [12] in the case that the $q_i$’s are Bernoulli, and Derrida, Griffiths,
and Higgs [8] for the case of Gaussian random variables. Roughly speaking, \( q_1, q_2, \ldots \) are random charges that are placed on a polymer path modeled by the trajectories of \( S \); and one can construct a Gibbs-type polymer measure from the Hamiltonian \( H_n \).

We follow Chen [4] (LIL and moderate deviations), Chen and Khooshnevisan [5] (comparison between \( H_n \) and the random walk in random scenery model), and Asselah [1] (large deviations in high dimensional case), and continue the analysis of the Hamiltonian \( H_n \). We assume here and in the sequel that

\[
E(q_1^2) = 1 \quad \text{and} \quad E(|q_1|^{p(d)}) < \infty \quad \text{where} \quad p(d) := \begin{cases} 
6 & \text{if} \ d = 1, \\
4 & \text{if} \ d \geq 2.
\end{cases} \tag{1.2}
\]

**Theorem 1.1.** On a possibly-enlarged probability space, we can define a version of \( \{H_n\}_{n=1}^{\infty} \) and a one-dimensional Brownian motion \( \{\gamma(t)\}_{t \geq 0} \) such that the following holds almost surely:

\[
H_n = \begin{cases} 
\frac{1}{\sqrt{2}} \gamma \left( \int_{-\infty}^{\infty} (\ell_x^n)^2 \, dx \right) + o(n^{\frac{3}{4} - \epsilon}) & \text{if} \ d = 1 \text{ and } 0 < \epsilon < \frac{1}{24}, \\
\frac{1}{\sqrt{2\pi}} \gamma(n \log n) + O(n^{\frac{1}{2} + \epsilon} \log \log n) & \text{if} \ d = 2, \\
\sqrt{\kappa} \gamma(n) + o(n^{\frac{1}{2} - \epsilon}) & \text{if} \ d \geq 3 \text{ and } 0 < \epsilon < \frac{1}{8},
\end{cases} \tag{1.3}
\]

where \( \{\ell_t^x\}_{t \geq 0, x \in \mathbb{R}} \) denotes the local times of a linear Brownian motion \( B \) independent of \( \gamma \), and \( \kappa := \sum_{k=1}^{\infty} P\{S_k = 0\} \).

It was shown in [5] that when \( d = 1 \) the distribution of \( H_n \) converges, after normalization, to the “random walk in random scenery.” The preceding shows that the stochastic process \( \{H_{[nt]}\}_{0 \leq t \leq 1} \) does not converge weakly to the random walk in random scenery; rather, we have the following consequence of Brownian scaling for all \( T > 0 \): As \( n \to \infty \),

\[
\left\{ \frac{H_{[nt]}}{n^{3/4}} \right\}_{0 \leq t \leq T} \overset{D([0,T])}{\to} \left\{ \frac{1}{\sqrt{2}} \gamma \left( \int_{-\infty}^{\infty} (\ell_x^T)^2 \, dx \right) \right\}_{0 \leq t \leq T}. \tag{1.4}
\]

With a little bit more effort, we can also obtain strong limit theorems.
Let us state the following counterpart to the LILs of Chen [4], as it appears to have novel content.

**Theorem 1.2.** Almost surely: (i) If \( d = 1 \), then

\[
\liminf_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{3/4} \max_{0 \leq k \leq n} |H_k| = (a^*)^{3/4} \frac{\pi}{4},
\]

where \( a^* = 2.189 \pm 0.0001 \) is a numerical constant (0.6); (ii) If \( d = 2 \), then

\[
\liminf_{n \to \infty} \sqrt{\frac{\log \log n}{n \log n}} \max_{0 \leq k \leq n} |H_k| = \frac{\sqrt{\pi}}{4};
\]

(iii) If \( d \geq 3 \), then

\[
\liminf_{n \to \infty} \sqrt{\frac{\log \log n}{n}} \max_{0 \leq k \leq n} |H_k| = \pi \sqrt{\frac{\kappa}{8}},
\]

where \( \kappa \) was defined in Theorem 1.1.

Theorems 1.1 and 1.2 are proved respectively in Sections 2 and 3.

### 2 Proof of Theorem 1.1

Let \( W \) be a one-dimensional Brownian motion starting from 0. By the Skorohod embedding theorem, there exists a sequence of stopping times \( \{T_n\}_{n=1}^{\infty} \) such that \( \{T_n - T_{n-1}\}_{n=1}^{\infty} \) (with \( T_0 = 0 \)) are i.i.d., and:

\[
E(T_1) = E(q_1^2) = 1, \quad \text{Var}(T_1) \leq \text{const} \cdot E(q_1^4) < \infty, \quad \text{and}
\]

\[
\{W(T_n) - W(T_{n-1})\}_{n=1}^{\infty} \overset{\text{law}}{=} \{q_n\}_{n=1}^{\infty}.
\]

Throughout this paper, we take the following special construction of the charges \( \{q_i\}_{i=1}^{\infty} \):

\[
q_n := W(T_n) - W(T_{n-1}) \quad \text{for } n \geq 1.
\]

3
Next, we describe how we choose a special construction of the random walk $S$, depending on $d$.

If $d = 1$, then on a possibly-enlarged probability space let $B$ be another one-dimensional Brownian motion, independent of $W$. By using a theorem of Révész [14], we may construct a one-dimensional simple symmetric random walk $\{S_i\}_{i=1}^\infty$ from $B$ such that almost surely,

$$\sup_{x \in \mathbb{Z}} |L_n^x - \ell_n^x| = n^{\frac{1}{4} + o(1)} \quad \text{as} \quad n \to \infty,$$

where $L_n^x := \sum_{i=1}^n 1\{S_i = x\}$, \quad (2.3)

and $\ell_n^x$ denotes the local times of $B$ at $x$ up to time $n$.

If $d \geq 2$, then we just choose an independent simple symmetric random walk $\{S_n\}_{n=1}^\infty$, after enlarging the probability space, if we need to.

Now we define the Hamiltonians $\{H_n\}_{n=1}^\infty$ via the preceding constructions of $\{q_i\}_{i=1}^\infty$ and $\{S_n\}_{n=1}^\infty$. That is,

$$H_n = \sum_{1 \leq i < j \leq n} (W(T_i) - W(T_{i-1}))(W(T_j) - W(T_{j-1}))(1_{S_i = S_j}) \quad (2.4)$$

where, for all integers $n \geq 1$ and reals $s \geq 0$,

$$G_n(s) := \sum_{1 \leq i < j \leq n} 1_{S_i = S_j} (W(T_i) - W(T_{i-1}))(1_{T_{j-1} \leq s < T_j}). \quad (2.5)$$

By the Dambis, Dubins–Schwarz representation theorem [15, Theorem 1.6, p. 170], after possibly enlarging the underlying probability space, we can find a one-dimensional Brownian motion $\gamma$ such that $\int_0^t G_n \, dW$ is equal to $\gamma(\int_0^t |G_n(s)|^2 \, ds)$ for $t \geq 0$. We stress the fact that if $d = 1$, then $\gamma$ is independent of $B$. This is so, because the bracket between the two continuous martingales vanishes: $\langle \int_0^t G_n \, dW, B \rangle_t = 0$ for $t \geq 0$. Consequently, the following holds for all $n \geq 1$: Almost surely,

$$H_n = \gamma(\Xi_n), \quad \text{where} \quad \Xi_n := \int_0^{T_n} |G_n(s)|^2 \, ds. \quad (2.6)$$
Proposition 2.1. The following holds almost surely:

\[ \Xi_n = \begin{cases} 
\frac{1}{2} \int_{-\infty}^{\infty} (\ell_n^x)^2 \, dx + O(n^{\frac{3}{2}-\epsilon}) & \text{if } d = 1 \text{ and } 0 < \epsilon < \frac{1}{12}, \\
\frac{1}{2\pi} n \log n + O(n \log \log n) & \text{if } d = 2, \\
\kappa n + O(n^{1-\epsilon}) & \text{if } d \geq 3 \text{ and } 0 < \epsilon < \frac{1}{4}. 
\end{cases} \]  

(2.7)

We prove this proposition later. First, we show that in case \( d = 1 \), the preceding proposition estimates \( \Xi_n \) correctly to leading term.

Lemma 2.2. If \( d = 1 \), then a.s., \( \int_{-\infty}^{\infty} (\ell_n^x)^2 \, dx = n^{\frac{3}{2}+o(1)} \) as \( n \to \infty \).

Proof. This is well known; we include a proof for the sake of completeness.

Because \( \int_{-\infty}^{\infty} \ell_n^x \, dx = n \), we have \( \int_{-\infty}^{\infty} (\ell_n^x)^2 \, dx \leq n \sup_{-\infty < x < \infty} \ell_n^x \), and this is \( n^{\frac{3}{2}+o(1)} \) \cite{13}. For the converse bound we apply the Cauchy–Schwarz inequality to find that \( n^2 = (\int_{-\infty}^{\infty} \ell_n^x \, dx)^2 \leq \int_{-\infty}^{\infty} (\ell_n^x)^2 \, dx \cdot \text{Osc}_{[0,n]} B \), where \( \text{Osc}_{[0,n]} B := \sup_{[0,n]} B - \inf_{[0,n]} B = n^{\frac{1}{2}+o(1)} \) by Khintchine’s LIL. This completes the proof. \( \square \)

Let us complete the proof of Theorem 1.1, first assuming Proposition 2.1. That proposition will then be proved subsequently.

Proof of Theorem 1.1. We shall consider only the case \( d = 1 \); the cases \( d = 2 \) and \( d \geq 3 \) are proved similarly. We apply the Csörgő–Révész modulus of continuity of Brownian motion \cite{7, Theorem 1.2.1} to \( H_n = \gamma(\Xi_n) \)—see (2.6)—with the changes of variables, \( t = n^{\frac{3}{2}+o(1)} \) and \( a(t) = n^{\frac{3}{2}-\epsilon} \); then apply Lemma 2.2 to see that \( |\gamma(\Xi_n) - \gamma(\frac{t}{2} \int_{-\infty}^{\infty} (\ell_n^x)^2 \, dx)| = O(n^{\frac{3}{2}-\epsilon}) \) as \( n \to \infty \) a.s. \( \square \)

Lemma 2.3. The following holds almost surely:

\[ \sum_{1 \leq i < k \leq n} 1\{S_i = S_k\} = \begin{cases} 
\frac{1}{2} \int_{-\infty}^{\infty} (\ell_n^x)^2 \, dx + n^{\frac{3}{2}+o(1)} & \text{if } d = 1, \\
\frac{1}{2\pi} n \log n + O(n \log \log n) & \text{if } d = 2, \\
\kappa n + n^{\frac{1}{2}+o(1)} & \text{if } d \geq 3. 
\end{cases} \]  

(2.8)
Proof. In the case that $d = 2$, this result follows from Bass, Chen and Rosen [2]; and in the case $d \geq 3$, from Chen [4, Theorem 5.2]. Therefore, we need to only check the case $d = 1$.

We begin by writing $\sum_{1 \leq i < k \leq n} \mathbf{1}_{\{S_i = S_k\}} = \frac{1}{2} \sum_{x \in \mathbb{Z}} (L_n^x)^2 - \frac{1}{2}$. According to Bass and Griffin [3, Lemma 5.3], $\sup_{x \in \mathbb{Z}} \sup_{y \in [x, x+1]} |L_n^x - \ell_n^y| = n^{\frac{1}{4} + o(1)}$ as $n \to \infty$ a.s. This and (2.3) together imply that $|L_n^x - \ell_n^y| = n^{\frac{1}{4} + o(1)}$ uniformly over all $y \in [x, x+1]$ and $x \in \mathbb{Z}$ [a.s.], whence

$$\left| \sum_{x \in \mathbb{Z}} (L_n^x)^2 - \int_{-\infty}^{\infty} (\ell_n^y)^2 \, dy \right| \leq n^{\frac{1}{4} + o(1)} \cdot \sum_{x \in \mathbb{Z}} \int_{x}^{x+1} (L_n^x + \ell_n^y) \, dy. \quad (2.9)$$

Since the latter sum is equal to $2n$, the lemma follows. \square

Lemma 2.4. The following holds a.s.: As $n \to \infty$,

$$\sum_{1 \leq i < k \leq n} \mathbf{1}_{\{S_i = S_k\}} \left( q_i^2 - 1 \right) = \begin{cases} n^{\frac{1}{6} + o(1)} & \text{if } d = 1, \\ n^{\frac{1}{4} + o(1)} & \text{if } d \geq 2. \end{cases} \quad (2.10)$$

Proof. We can let $M_n$ denote the double sum in the lemma, and check directly that $M_n = \sum_{1 \leq i \leq n-1} (L_n^{S_i} - L_i^{S_i})(q_i^2 - 1)$. Let $\mathcal{S}$ denote the $\sigma$-algebra generated by the entire process $S$. Then, conditionally on $\mathcal{S}$, each $M_n$ is a sum of independent random variables. By Burkholder’s inequality [10, Theorem 2.10, p. 34], for all even integers $p \geq 2$,

$$\mathbb{E}(|M_n|^p) \leq \text{const} \cdot \mathbb{E} \left( \left| \sum_{1 \leq i \leq n-1} \left( L_n^{S_i} - L_i^{S_i} \right)^2 (q_i^2 - 1)^2 \right|^{p/2} \right). \quad (2.11)$$

According to the generalized Hölder inequality,

$$\mathbb{E} \left( \prod_{k=1}^{p/2} (q_{i_k}^2 - 1)^2 \right) \leq \prod_{k=1}^{p/2} \left\{ \mathbb{E} \left( |q_{i_k}^2 - 1|^p \right) \right\}^{2/p} = \mathbb{E} \left( |q_1^2 - 1|^p \right). \quad (2.12)$$

Another application of the generalized Hölder inequality, together with an
appeal to the Markov property, yields

\[
E \left( \prod_{k=1}^{\frac{p}{2}} \left( L_n^{S_{ik}} - L_{ik}^{S_{ik}} \right)^2 \right) \leq \prod_{k=1}^{\frac{p}{2}} \left\{ E \left( |L_n^{S_{ik}} - L_{ik}^{S_{ik}}|^p \right) \right\}^{2/p} = \prod_{k=1}^{\frac{p}{2}} \left\{ E \left( |L_n^{S_{ik}} - L_{ik}^{S_{ik}}|^p \right) \right\}^{2/p}.
\] (2.13)

Therefore, we can apply the local-limit theorem to find that

\[
E \left( |M_n|^p \right) \leq \text{const} \cdot \begin{cases} \frac{n^p}{2} & \text{if } d = 1, \\ \frac{n^{\frac{p}{2}+o(1)}}{2} & \text{if } d \geq 2. \end{cases}
\] (2.14)

The lemma follows from this and the Borel–Cantelli lemma. □

Lemma 2.5. The following holds almost surely: As \( n \to \infty \),

\[
\sum_{1 \leq i < k \leq n} \mathbf{1}_{\{S_i = S_k\}} q_i^2 (T_k - T_{k-1} - 1) = \begin{cases} n^{1+o(1)} & \text{if } d = 1, \\ n^{\frac{1}{2}+o(1)} & \text{if } d \geq 2. \end{cases}
\] (2.15)

Proof. Let \( N_n \) denote the double sum in the lemma, and note that

\[
N_n = \sum_{2 \leq k \leq n} \beta_{k-1} (T_k - T_{k-1} - 1), \ 	ext{where} \\
\beta_{k-1} := \sum_{1 \leq i \leq k-1} q_i^2 \mathbf{1}_{\{S_i = S_k\}}.
\] (2.16)

Recall that \( \mathcal{S} \) denotes the \( \sigma \)-algebra generated by the entire process \( S \) and observe that, conditionally on \( \mathcal{S} \), \( \{N_n\}_{n=1}^\infty \) is a mean-zero martingale with

\[
E(N_n^2 | \mathcal{S}) = \text{Var}(T_1) \cdot \sum_{2 \leq k \leq n} E \left( \beta_{k-1}^2 | \mathcal{S} \right) = \text{Var}(T_1) \cdot \sum_{2 \leq k \leq n} \left( \text{Var}(q_1^2) + \left| E(q_1^2) \right|^2 \cdot L_{k-1}^{S_k} \right) L_{k-1}^{S_k}.
\] (2.17)
This and Doob’s inequality together show that
\[
E \left( \max_{1 \leq k \leq n} N_k^2 \right) \leq \text{const} \cdot n \left( \max_{a \in \mathbb{Z}} E(L_n^a) + \max_{a \in \mathbb{Z}} E(|L_n^a|^2) \right).
\] (2.18)

By the local-limit theorem, the preceding is at most a constant multiple of \(n(\sum_{1 \leq i \leq n} i^{-d/2})^2\). The Borel–Cantelli lemma finishes the proof. \(\Box\)

**Proof of Proposition 2.1.** Recall the definition of each \(q_i\). With that in mind, we can decompose \(\Xi_n\) as follows:

\[
\Xi_n = \sum_{1 \leq k \leq n} \int_{T_{k-1}}^{T_k} ds \left( \sum_{1 \leq i < k} 1\{S_i = S_k\} q_i \right)^2 = \Xi_n^{(1)} + \Xi_n^{(2)},
\] (2.19)

where,

\[
\Xi_n^{(1)} := \sum_{1 \leq i < k \leq n} 1\{S_i = S_k\} q_i^2 (T_k - T_{k-1}),
\] (2.20)

and

\[
\Xi_n^{(2)} := 2 \sum_{1 \leq i < j < k \leq n} 1\{S_i = S_j = S_k\} q_i q_j (T_k - T_{k-1}).
\] (2.21)

Since

\[
\Xi_n^{(1)} = \sum_{1 \leq i < k \leq n} 1\{S_i = S_k\} + \sum_{1 \leq i < k \leq n} 1\{S_i = S_k\} \left( q_i^2 - 1 \right)
+ \sum_{1 \leq i < k \leq n} 1\{S_i = S_k\} q_i^2 \left( T_k - T_{k-1} - 1 \right),
\] (2.22)

Lemmas 2.2, 2.3, 2.4, and 2.5 together imply that \(\Xi_n^{(1)}\) has the large-\(n\) asymptotics that is claimed for \(\Xi_n\). In light of Lemma 2.2, it suffices to show that almost surely the following holds as \(n \to \infty\):

\[
\Xi_n^{(2)} = \begin{cases} 
n^{12 + o(1)} & \text{if } d = 1, \\
n^{4 + o(1)} & \text{if } d \geq 2.
\end{cases}
\] (2.23)
We can write \( \Xi_n^{(2)} = 2 \sum_{k=3}^{n} \tau_{k-1}(S_k)(T_k - T_{k-1}) \), where

\[
\tau_{k-1}(z) := \sum_{1 \leq i < j \leq k-1} 1 \{ S_i = S_j = z \} q_i q_j \quad \text{for } z \in Z \text{ and } k > 1. \tag{2.24}
\]

In particular, we can write

\[
\Xi_n^{(2)} := 2 (a_n + b_n), \quad \text{where} \tag{2.25}
\]

\[
a_n := \sum_{k=3}^{n} \tau_{k-1}(S_k)(T_k - T_{k-1} - 1) \quad \text{and} \quad b_n := \sum_{k=3}^{n} \tau_{k-1}(S_k). \tag{2.26}
\]

Recall that \( \mathcal{S} \) denotes the \( \sigma \)-algebra generated by the process \( S \). It follows that, conditional on \( \mathcal{S} \), the process \( \{ a_n \}_{n=1}^{\infty} \) is a mean-zero martingale, and

\[
E \left( a_n^2 \mid \mathcal{S} \right) = \text{Var}(T_1) \cdot \sum_{k=3}^{n} E \left( |\tau_{k-1}(S_k)|^2 \mid \mathcal{S} \right). \tag{2.27}
\]

The latter conditional expectation is also computed by a martingale computation. Namely, we write \( \tau_{k-1}(z) = \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} 1 \{ S_i = S_j = z \} q_i q_j \) for all \( z \in Z \) in order to deduce that

\[
E \left( |\tau_{k-1}(z)|^2 \mid \mathcal{S} \right) = \sum_{j=2}^{k-1} 1 \{ S_j = z \} L_{j-1}^z \leq (L_{k-1}^z)^2. \tag{2.28}
\]

It follows from Doob’s maximal inequality that

\[
E \left( \max_{1 \leq k \leq n} a_k^2 \right) \leq 4 \text{Var}(T_1) \cdot E \left( \sum_{k=3}^{n} (L_{k-1}^S)^2 \right). \tag{2.29}
\]

By time reversal, we can replace \( L_{k-1}^S \) by \( L_{k-1}^\circ \). Therefore, the local-limit theorem implies that \( E(\max_{1 \leq k \leq n} a_k^2) \leq \text{const} \cdot n(\sum_{i=1}^{n} i^{-d/2})^2 \), and hence almost surely as \( n \to \infty \), (2.23) is satisfied with \( \Xi_n^{(2)} \) replaced by \( a_n \) [the Borel–Cantelli lemma]. It suffices to prove that (2.23) holds if \( \Xi_n \) is replaced by \( b_n \).
We can write 

\[ b_n := b_{n,n} \]

where

\[ b_{n,k} = \sum_{j=2}^{n-1} \theta_{j-1,k} q_j \quad \text{for} \quad \theta_{j-1,k} := \sum_{i=1}^{j-1} 1_{\{S_i = S_j\}} q_i \left( L_{k_i}^{S_i} - L_{j_i}^{S_i} \right). \]  

(2.30)

For each fixed integer \( k \geq 1 \), \( \{b_{n,k}\}_{n \geq 3} \) is a mean-zero martingale, conditional on \( S \). Therefore, Burkholder’s inequality yields

\[
E \left( |b_{n,k}|^p \mid S \right) \leq \text{const} \cdot E \left( \left\| \sum_{j=2}^{n-1} \theta_{j-1,k}^2 q_j^2 \right\|_{S}^{p/2} \right),
\]

where the implied constant is nonrandom and depends only on \( p \). Since

\[
|\sum_{j=1}^{n-1} x_j|^{p/2} \leq n^{p/2} |\sum_{j=1}^{n-1} x_j|^{p/2}
\]

for all real \( x_1, \ldots, x_{n-1} \), we can apply the preceding with \( k := n \) to obtain

\[
E \left( |b_n|^p \mid S \right) \leq \text{const} \cdot E(|q_1|^p) \cdot n^{p/2} \sum_{j=2}^{n-1} E \left( |\theta_{j-1,n}|^p \mid S \right). \]

(2.32)

Yet another application of Burkholder’s inequality yields

\[
E \left( |\theta_{j-1,n}|^p \mid S \right) \leq \text{const} \cdot E(|q_1|^p) \cdot \left( \sum_{i=1}^{j-1} 1_{\{S_i = S_j\}} q_i^2 \left( L_{n_i}^{S_i} - L_{j_i}^{S_i} \right)^2 \right)^{p/2},
\]

(2.33)

since

\[
E(q_1^2 \cdots q_{p/2}^2) \leq E(|q_1|^p)
\]

for all \( 1 \leq i_1, \ldots, i_{p/2} < j \). We take expectations and apply the Markov property and time reversal to find that

\[
E \left( |\theta_{j-1,n}|^p \right) \leq \text{const} \cdot E(|q_1|^p) \cdot E \left( (L_{j-1}^0)^{p/2} \right) E \left( (L_{n-j}^0)^{p/2} \right).
\]

(2.34)

It follows readily that

\[
E \left( |b_n|^p \right) \leq \text{const} \cdot E(|q_1|^p) \cdot n^{p/2} E \left( (L_n^0)^{p/2} \right) E \left( (L_n^0)^{p/2} \right) E \left( (L_n^0)^p \right).
\]

(2.35)

This, the local-limit theorem, and the Borel–Cantelli lemma together imply
that (2.23) holds with $b_n$ in place of $\Xi^{(2)}_n$. The proposition follows. □

3 Proof of Theorem 1.2

In view of Theorem 1.1 and the LIL for the Brownian motion, it suffices to consider only the case $d = 1$, and to establish the following:

$$\liminf_{n \to \infty} \left( \frac{L_2 n}{n} \right)^{3/4} \max_{0 \leq k \leq n} |\gamma(\alpha(k))| = (a^*)^{3/4} \frac{\pi}{\sqrt{8}}, \quad (3.1)$$

where

$$L_2 x := \log \log (x \lor 1) \quad \text{and} \quad \alpha(t) := \int_{-\infty}^\infty (\ell_x^t)^2 \, dx. \quad (3.2)$$

It is known that [6, Theorem 3], $\sup_{x \in \mathbb{R}} \sup_{1 \leq k \leq n} (\ell_x^k - \ell_x^{k-1}) = o((\log n)^{-1/2})$ almost surely [P]. This implies readily that $\max_{0 \leq k \leq n} (\alpha(k+1) - \alpha(k-1)) = O(n \sqrt{\log n})$ a.s. Therefore, it follows from [7, Theorem 1.2.1] that (3.1) is equivalent to the following:

$$\liminf_{t \to \infty} \left( \frac{L_2 t}{t} \right)^{3/4} \sup_{0 \leq s \leq t} |\gamma(\alpha(s))| = (a^*)^{3/4} \frac{\pi}{\sqrt{8}}. \quad (3.3)$$

Brownian scaling implies that $\alpha(t)$ and $t^{3/2} \alpha(1)$ have the same distribution. On one hand, Proposition 1 of [11] tells us that the limit $C := \lim_{\lambda \to \infty} \exp\{a^* \lambda^{2/3}\} \mathbb{E} \exp(-\lambda \alpha(1))$ exists and is positive and finite. On the other hand, we can write $\gamma^*(t) := \sup_{0 \leq s \leq t} |\gamma(s)|$ and appeal to Lemma 1.6.1 of [7] to find that for all $t, y > 0$,

$$\frac{2}{\pi} \exp\left(-\frac{\pi^2 t}{8 y^2}\right) \leq P\{\gamma^*(t) < y\} \leq \frac{4}{\pi} \exp\left(-\frac{\pi^2 t}{8 y^2}\right). \quad (3.4)$$

Therefore uniformly for all $t > 0$ and $x \in (0, 1]$,

$$P\left\{\sup_{0 \leq s \leq t} |\gamma(\alpha(s))| < xt^{3/4}\right\} = P\{\gamma^*(\alpha(1)) < x\} \leq \frac{4}{\pi} \mathbb{E} e^{-\pi^2 \alpha(1)/(8x^2)}$$

$$\leq \text{const} \cdot \exp\left(-\frac{\alpha^*}{x^{4/3}} \left(\frac{\pi^2}{8}\right)^{2/3}\right). \quad (3.5)$$
This and an application of the Borel–Cantelli lemma together yield one half of the (3.3); namely, (3.3) where “=” is replaced by “≥.” In order to derive the other half we choose \( t_n := n \) and \( c > (a^*)^{3/4} \pi / \sqrt{8} \), and define

\[
A_n := \left\{ \omega : \sup_{0 \leq s \leq t_n} |\gamma(\alpha(s))| < c \left( \frac{t_n}{L_2 t_n} \right)^{3/4} \right\}.
\]
(3.6)

Every \( A_n \) is measurable with respect to \( \mathcal{F}_{t_n} := \sigma\{\gamma(\alpha(t_n))\} \vee \sigma\{B_v : v \leq t_n\} \). In light of the 0-1 law of Paul Lévy, and since \( c > (a^*)^{3/4} \pi / \sqrt{8} \) is otherwise arbitrary, it suffices to prove that

\[
\sum_{n=1}^{\infty} P( A_n | \mathcal{F}_{t_n-1} ) = \infty \quad \text{a.s.}
\]
(3.7)

The argument that led to (3.5) can be used to show that for all \( v \geq 0 \),

\[
P\left\{ \gamma^*(v + \alpha(t)) < x t^{3/4} \right\} \geq \frac{2}{\pi} \exp\left( -\frac{\pi^2 v}{8 x^2 t^{3/2}} \right) E e^{-\pi^2 \alpha(1)/(8 x^2)}
\geq \text{const} \cdot \exp\left( -\frac{\pi^2 v}{8 x^2 t^{3/2}} - \frac{a^*}{x^{4/3}} \left( \frac{\pi^2}{8} \right)^{2/3} \right).
\]
(3.8)

In order to prove (3.7), let us choose and fix a large integer \( n \) temporarily. We might note that

\[
\sup_{0 \leq s \leq t_n} |\gamma(\alpha(s))| = \gamma^*(\alpha(t_n)) \leq \gamma^*(\alpha(t_{n-1})) + \widetilde{\gamma}^*(\alpha(t_n) - \alpha(t_{n-1})),
\]
(3.9)

where \( \widetilde{\gamma}(s) = \widetilde{\gamma}_n(s) := \gamma(s + \alpha(t_{n-1})) - \gamma(\alpha(t_{n-1})) \) and \( \widetilde{\gamma}^*(s) := \sup_{0 \leq v \leq s} |\widetilde{\gamma}(v)| \) for \( s \geq 0 \). Of course, \( \widetilde{\gamma} \) is a Brownian motion independent of \( \mathcal{F}_{t_{n-1}} \). Moreover, we can write \( \ell^x_{t_n} = \ell^x_{t_{n-1}} + \widetilde{\ell}^x_B_{t_n-1} \), where \( \widetilde{\ell} \) denotes the local time process of the Brownian motion \( \widetilde{B}(s) := B(s + t_{n-1}) - B(t_{n-1}) \), \( s \geq 0 \). Clearly, \((\widetilde{\gamma}, \widetilde{B})\) is a two-dimensional Brownian motion, independent of \( \mathcal{F}_{t_{n-1}} \).
Observe that
\[
\alpha_n - \alpha_{n-1} = \int dx \left[ (\ell^x_{t_n})^2 - (\ell^x_{t_{n-1}})^2 \right] = \int dx \ell^x_{t_n-t_{n-1}} (\ell^x_{t_{n-1}} + \ell^x_{t_{n-1}-t_n}) \\
\leq (t_n - t_{n-1})\ell^*_n + \bar{\alpha}_{t_n-t_{n-1}},
\]
where \(\ell^*_n := \sup_{x \in \mathbb{R}} \ell^x_{t_n-1} \). Therefore, we obtain
\[
\gamma^*(\alpha_{n-1}) \leq \gamma^*(\alpha_{n-1}) + \tilde{\gamma}^*(t_n - t_{n-1})\ell^*_n + \bar{\alpha}_{t_n-t_{n-1}}.
\]
(3.11)
Let \(\varepsilon > 0\) be such that \(2\varepsilon < c - (a^*)^{3/4} \pi / \sqrt{8}\), and define
\[
D_n := \left\{ \gamma^*(\alpha_{n-1}) < \varepsilon \left( \frac{t_n}{L_2 t_n} \right)^{3/4}, \ell^*_n \leq \sqrt{3t_{n-1} L_2 t_{n-1}} \right\}.
\]
(3.12)
Clearly, \(D_n\) is \(\mathcal{F}_{t_{n-1}}\)-measurable. Let \(v_n := t_n \sqrt{3t_{n-1} L_2 t_{n-1}}\). Since
\[
A_n \supset D_n \cap \left\{ \tilde{\gamma}^*(v_n + \bar{\alpha}_{t_n}) \leq (c - \varepsilon) \left( \frac{t_n}{L_2 t_n} \right)^{3/4} \right\},
\]
(3.13)
we can deduce from (3.8) that
\[
P(A_n | \mathcal{F}_{t_{n-1}})
\geq \text{const} \cdot 1_{D_n} \exp \left( -\frac{\pi^2 v_n}{8} \left( \frac{L_2 t_n}{t_n} \right)^{3/2} \right) - a^* \left( \frac{\pi^2}{8(c - \varepsilon)^2} \right)^{2/3} L_2 t_n
\geq \text{const} \cdot 1_{D_n} (n \ln n)^{-a^*(\pi^2/8)^{2/3}(c - \varepsilon)^{-4/3}},
\]
(3.14)
where we have used the fact that \(v_n/t_n^{3/2} \sim 1/n\). Because \(a^*(\pi^2/8)^{2/3}(c - \varepsilon)^{-4/3} < 1\), (3.7) implies that almost surely, \(1_{D_n} = 1\) for all \(n\) large. Indeed, the LIL tells us that almost surely for all large \(n\), \(\ell^*_n \leq \sqrt{3t_{n-1} L_2 t_{n-1}}\), and \(\gamma^*(\alpha_{n-1}) \leq \sqrt{3\alpha_{t_{n-1}} L_2 \alpha_{t_{n-1}}} \). Since \(\alpha(t) = \int (\ell^x_t)^2 dx \leq t \ell^*_t\), we find that \(\alpha_{t_{n}} \leq (3\alpha_{t_{n-1}} L_2 \alpha_{t_{n-1}})^{2/3} \). Since \(t_{n-1}/t_n \sim 1/n\), it follows that almost surely, \(\ell^*_n \leq \varepsilon (t_n/L_2 t_n)^{3/4}\) for all large \(n\) and prove that \(D_n\) realizes eventually for all large \(n\). The proof of Theorem 1.2 is complete. \(\square\)
Acknowledgements. We thank the University of Paris for generously hosting the second author (D.K.), during which time much of this research was carried out.

References

[1] Asselah, Amine, Annealed large deviation estimates for the energy of a polymer, *Preprint* (2009) [arXiv:0812.0443v2].

[2] Bass, R.F., Chen, X. and Rosen, J., Moderate deviations and laws of the iterated logarithm for the renormalized self-intersection local times of planar random walks. *Electron. J. Probab.* **11** (2006), no. 37, 993–1030 (electronic).

[3] Bass, Richard F. and Griffin, Philip S., The most visited site of Brownian motion and simple random walk. *Z. Wahrschein. Verw. Gebiete* **70** (1985), no. 3, 417–436.

[4] Chen, X., Limit laws for the energy of a charged polymer. *Ann. Inst. Henri Poincaré: Probab. Stat.* **44** (2008), no. 4, 638–672.

[5] Chen, X. and Khoshnevisan, D., From charged polymers to random walk in random scenery. *IMS Lecture Notes. Optimality: The 3rd E.L. Lehmann Symposium* Vol. 57 (2009) 237–251.

[6] Csáki, E., Csörgő, M., Földes, A., and Révész, P., How big are the increments of the local time of a Wiener process? *Ann. Probab.* **11** (1983), no. 3, 593–608.

[7] Csörgő, M. and Révész, P., *Strong Approximations in Probability and Statistics*, Academic Press, Inc., New York-London, 1981.

[8] Derrida, B., Griffiths, B. and Higgs, P.G., A Model of Directed Walks with Random Self-Interactions. *Europhys. Lett.*, **18**, no. 4 (1992) pp. 361–366.

[9] Erdős, P.; and Taylor, S.J., Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* **11** (1960) 137–162.

[10] Hall, P. and Heyde, C. C., *Martingale Limit Theory and Its Applications*, Academic Press, New York, 1980.

[11] van der Hofstad, R., den Hollander, F. and König, W., Central limit theorem for Edwards model. *Ann. Probab.* **25** (1997), no. 2 573–597.

[12] Kantor, Y. and Kardar, M., Polymers with Random Self-Interactions. *Europhys. Lett.* **14**, No. 5 (1991) 421–426.
[13] Kesten, Hary, An iterated logarithm law for local time. *Duke Math. J.* **32** (1965) 447–456.

[14] Révész, P., Local time and invariance. In: *Analytical Methods in Probability Theory* (Oberwolfach, 1980), pp. 128–145, Lecture Notes in Math., 861, Springer, Berlin-New York, 1981.

[15] Revuz, Daniel and Yor, Marc, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, 1991.

**Yueyun Hu.** Département de Mathématiques, Université Paris XIII, 99 avenue J-B Clément, F-93430 Villetaneuse, France,  Email: yueyun@math.univ-paris13.fr

**Davar Khoshnevisan.** Department of Mathematics, University of Utah, 155 South 1440 East, JWB 233, Salt Lake City, Utah 84112-0090, USA,  Email: davar@math.utah.edu