Aharonov–Bohm interferometry with a tunnel-coupled wire

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Abstract

Recent experiments (Yamamoto et al 2012 Nature Nanotechnology 7 247) used the transport of electrons through an Aharonov–Bohm (AB) interferometer and two coupled channels (at both ends of the interferometer) to demonstrate a manipulable flying qubit. Results included in-phase and anti-phase (AB) oscillations of the two outgoing currents as a function of the magnetic flux, for strong and weak inter-channel coupling, respectively. Here we present new experimental results for a three terminal interferometer, with a tunnel coupling between the two outgoing wires. We show that in some limits, this system is an even simpler realization of the ‘two-slit’ experiment. We also present a simple tight-binding theoretical model which imitates the experimental setup. For weak inter-channel coupling, the AB oscillations in the current which is reflected from the device are very small, and therefore the oscillations in the two outgoing currents must cancel each other, yielding the anti-phase behavior, independent of the length of the coupling regime. Technically, the tight binding equations within the two coupled wires have four solutions for each electronic energy. In the
‘anti-phase’ region all of these solutions are wave-like, oscillating with the distance along the wires. As the coupling between the wires increases, two of these solutions become evanescent, and their amplitudes decay as the electron moves in the wires. In this regime, the amplitudes of the two remaining ‘running’ waves are proportional to each other, with a ratio which is practically flux-independent. As a result, the two outgoing currents are proportional to each other, yielding the ‘in phase’ behavior. For larger coupling all the solutions are evanescent, and the outgoing currents become very small.

Keywords: electron transport, flying qubit, AB effect

1. Introduction

In a recent paper, [1] some of us demonstrated a scalable flying qubit architecture in a four-terminal setup. Electrons were transported via an Aharonov–Bohm (AB) ring into two channel wires that have a tunable tunnel coupling between them. The superposition of the two electron states between the two outgoing channels can be considered as a flying qubit, [2] which can be manipulated by the various gate voltages on the system. That paper also exhibited an interesting variation of the relative phases of the AB oscillations in the currents in the two outgoing channels, as function of the coupling between them. In the present paper we present similar experimental results for a new (simpler) three-terminal setup, and then present a simple theoretical model which reproduces them.

The early experiments on two-terminal ‘closed’ AB interferometers [3] exhibited a phase rigidity: the minima and maxima of the AB oscillations in the outgoing current stayed at the same values of the magnetic flux through the interferometer ring, irrespective of the details of a quantum dot which was placed on one arm of the interferometer. At most, the phase of the oscillation jumped by π, interchanging the minima and maxima. This rigidity was due to the Onsager relation, by which unitarity and time reversal symmetry imply that the conductance through the interferometer must be an even function of the magnetic field [4]. One way to overcome this rigidity, and to measure the phase of the transmission amplitude through the quantum dot, was to open the interferometer, allowing leaks of electrons out of the interferometer ring and thus breaking the unitarity condition needed for the Onsager relation [5]. Indeed, experiments with open interferometers [6] yielded a continuous shift in the oscillation phases. In a ‘two-slit’ geometry (as in Young’s classical diffraction experiment), the electronic wave passes only once through each branch of the interferometer, and then this phase shift should be equal to the desired transmission phase. However, this ‘two-slit’ limit is achieved only when the electronic leaks are very large, and therefore the remaining visibility (i.e. the amplitude of the AB oscillations) is very small [7].

In the present paper we discuss an alternative way to cross between the ‘two-slit’ and the ‘two-terminal’ limits, i.e. between the case in which the oscillation phase reflects the scattering phase through one branch of the interferometer and the case of full phase rigidity. This is achieved by having two outgoing wires, namely by our novel three-terminal setup, consisting of an AB ring and a coupled-wire [8]. A priori, the current in each outgoing wire need not obey the Onsager phase rigidity, because electrons ‘leak’ through the other outgoing wire. However, we
show that the strength of the tunnel-coupling between the outgoing wires can cause the crossover between the above two limits.

Section 2 describes our experimental setup, as shown in figure 1. This is similar to that of [1], but now we use only three (and not four) terminals, one for the incoming current (on the left) and two for the outgoing coupled channels (on the right). Section 2 also presents results for the two outgoing currents, see figure 2. These results are also similar to those found in [1]. For strong inter-wire tunnel-coupling, the AB oscillations in both wires have the same phases, and they exhibit phase rigidity as in the ‘two-terminal’ case. For weak tunnel coupling, the two oscillations have opposite phases (which we call ‘anti-phase’). In this limit the phases of the oscillations vary smoothly with the gate voltage on the upper branch of the interferometer, implying a relation between the measured phase and the phase of the electronic wave function on that branch.

In the rest of this paper we develop a simple and minimal theoretical tight-binding model, which imitates the experimental setup. The model, which is constructed from one-dimensional wires, is shown in figure 3. The AB loop is modeled by a triangle (ABC), and the tunnel-coupling between the outgoing wires (BD and CF) is modeled by $N+1$ transverse wires, each having a tunneling energy $V$. The electrons ‘enter’ through the left hand side lead. A fraction $R$ of them are reflected from the point $A$, and fractions $T^\uparrow$ and $T^\downarrow$ are transmitted to two leads on the right hand side (from points $D$ and $F$). Since the following sections, which describe the solution of this model, are somewhat technical, we first give a qualitative summary of our results. The tight-binding equations are derived in section 3. For the special case $N = 0$ (no coupling between the outgoing wires) we end up with three coupled linear equations, and these are solved analytically in appendix A. This case already exhibits the ‘anti-phase’ behavior. To obtain the ‘in-phase’ behavior, we need the solution for $N \gg 1$. Section 4 shows that within the tunnel-coupled wires there exist four basic solutions. For small $V$ and an appropriate choice of the other parameters, these solutions are all ‘running’ waves, which oscillate along the coupled wires with fixed amplitudes. For an intermediate range of $V$, two of these solutions may become
evanescent, i.e. decaying exponentially along the wires. For larger $V$ all four solutions become evanescent.

For the full network with $N > 1$ we need to solve five coupled linear equations, and this is done in section 5. The results are expressed in terms of the reflection coefficient $R$ and the two transmission coefficients $T^\uparrow$ and $T^\downarrow$. As we show, the AB oscillations in $R$ always exhibit phase rigidity, with maxima and minima at integer multiples of the unit flux. At small $V$, these oscillations in $R$ are small, and current conservation requires that the oscillations in the outgoing currents (which are not phase locked) cancel each other, yielding the ‘anti-phase’ behavior. The
maxima and minima of these oscillations vary continuously with the gate voltage $V_M$, and can therefore be used to measure the phase of the waves on each branch of the interferometer. In this regime the three terminal interferometer can be used as an alternative to the open two-terminal interferometer, for phase measurements. In contrast, the ‘in-phase’ behavior appears only for a long region of tunnel-coupling between the wires, $N \gg 1$, and for the intermediate values of $V$, for which one finds only two ‘running’ wave solutions on the coupled wires. For this case, the ratio of the wave amplitudes on the two wires, and therefore also the ratio of the two outgoing currents, are practically independent of the magnetic flux. Therefore, both currents have the same flux dependence (up to a constant multiplicative factor). Since the sum of these currents is equal to the current on the left hand side, the AB oscillations on both outgoing currents are phase locked (as those in $R$). This explains the ‘in-phase’ behavior. Section 6 presents our conclusions.

2. Experiments

We employed an AB ring connected to a tunnel-coupled wire shown in figure 1. This three-terminal geometry is the simplest for realizing the two-slit experiment even compared with the four-terminal geometry employed in the previous work [1]. Our device is fabricated from a modulation doped AlGaAs/GaAs heterostructure (depth of 2DEG: 125 nm, carrier density: $1.9 \times 10^{11}$ cm$^{-2}$, mobility: $2 \times 10^6$ cm$^2$ Vs$^{-1}$) using a standard Schottky gate technique. By varying the tunneling gate voltage $V_T$ we can modulate in-situ the tunnel coupling energy $V$. By applying a voltage to the gate $V_M$ the phase acquired in one of the two paths can be varied. To observe the quantum interference, a low energy excitation current (excitation energy across the overall sample: $50 \mu$eV) is injected into the quantum wire on the left, and the output currents $I_{\uparrow}$ and $I_{\downarrow}$ are measured simultaneously by sweeping the magnetic field at each gate configuration. All experiments were performed using the dilution refrigerator with a base temperature of 70 mK.

When we apply a relatively small negative voltage on $V_T$, we can deplete the center region of the ring to form an AB ring while keeping strong coupling between the parallel quantum wires. This is because the gate electrode deposited to define the tunnel coupling is narrow. In such a strong coupling case, the two output currents $I_{\uparrow}$ and $I_{\downarrow}$ oscillate in-phase as shown in figures 2(a) and (d). Namely, the two output contacts work equally and the interferometer effectively works as a standard AB ring in a two-terminal setup. The total current oscillates with a period of $S/\hbar$, where $S$ is the area enclosed by the AB ring. This standard AB interference is subjected to phase rigidity, as would result from the Onsager law. Below we explain why this law applies in this limit. The phase of the AB oscillation can thus only take the values 0 or $\pi$ at zero magnetic field and as a consequence leads to phase jumps when the AB phase is modulated by changing a voltage $V_M$ applied to a side gate of the AB ring as shown in figure 2(g). The gate voltage irregularly shifts the phase of the AB oscillation, implying that the observed AB oscillation is not an ideal two-path interference, but a complicated multi-path interference.

In contrast, our device can also be tuned into the weak coupling regime by applying a large negative voltage on $V_T$. When increasing the negative $V_T$ to reduce the coupling, crossover from the in-phase to anti-phase oscillation occurs. For the intermediate coupling, the in-phase and anti-phase oscillations are mixed. When the negative $V_T$ is further increased and properly tuned so that the coupled wire works as a beam splitter to yield high visibility, and the potential
change at the transition region between the AB ring and the coupled wire is small enough, the observed two output currents only oscillate with opposite phases with almost the same amplitude (see figures 2(c) and (f)). In other words, the total outgoing current \( I_{\text{tot}} = I_{\uparrow} + I_{\downarrow} \) has very small AB oscillations. This result strongly suggests that backscattering does not contribute to the main oscillation. Furthermore, when the phase difference between \( I_{\uparrow} \) and \( I_{\downarrow} \) is exactly \( \pi \), the phase of the oscillation evolves smoothly and linearly with \( V_{M} \) without any jump (see figure 2(i)). These results are in contrast to what is observed for the standard two-terminal AB interferometer as well as the results for the strong coupling and the intermediate coupling regimes (figures 2(g) and (h)). The observed interference for the weak coupling regime does not suffer from the multi-path contribution that modulates the total current, but captures the phase difference between the two paths that linearly shifts with \( V_{M} \), suggesting the realization of a true two-path interference.

Note the correspondence between the anti-phase (in-phase) and the smooth phase shift with \( V_{M} \) (\( \pi \) phase jumps) is observed more clearly in the three-terminal geometry than in the four-terminal geometry used in the previous work [1] due to the absence of the fourth terminal. For the four-terminal setup, we sometimes see by chance the smooth phase shift even when the observed oscillations are in-phase due to the current leaking into the fourth terminal. Such relative simplicity in the three-terminal geometry also helps theoretical analysis. In what follows, we show that the anti-phase oscillation and the smooth phase shift in the three-terminal setup are reproduced in a simple tight-binding model and prove that the measured phase shift is the bare phase shift of the upper path.

In the experiment, the visibility of both the in-phase and the anti-phase oscillations is suppressed by the existence of many transmitting channels with different tunnel couplings. However, as we show below, a model with a single channel in each wire captures the above mentioned observed features.

### 3. Theoretical model

We now describe a minimal theoretical model, which captures all the physical ingredients of the experimental setup. Although the model does not contain exactly all the structural details of the experimental system, it does give a qualitative and physical understanding of the main experimental results. In the model, each wire in the experiment is modeled by a one-dimensional line, as shown in figure 3. Since the transverse confinement of each wire is strong and the transverse spreading of the wave function is much smaller than other length scales of the sample, this simple representation of replacing each quantum wire with a single channel is usually sufficient. Furthermore, the currents through the system are usually dominated by a small number of (transverse) channels within each wire; the electronic wave functions which correspond to the other channels are usually evanescent, decaying because of scattering from the disorder in the semiconductor material [9]. Different channels would be penetrated by different Aharonov–Bohm fluxes, and the interference between them would yield beats in the Aharonov–Bohm oscillations. On flux ranges which are small compared to these beats the oscillations due to each loop can be described by a single period, and they are calculated from by one-dimensional wires surrounding that loop.

The wires include one incoming lead on the left hand side, two outgoing leads on the right hand side, the triangle ABC which represents the Aharonov–Bohm interferometer, and the two
wires \( BD \) and \( CF \), which are coupled by \( N + 1 \) transverse tunneling bonds. For further simplicity, each wire is described by a periodic sequence of sites, which are coupled via a tight binding model. Although the detailed solutions for continuum wires and discrete tight-binding sites may differ from each other, [10] these differences are usually not important in the middle of the energy bands (where the density of states is practically constant), and the qualitative results are the same. The rest of this section presents the detailed tight-binding equations for the electronic wave functions on the various wires. At the end, we derive five coupled linear equations in five unknown amplitudes. Solving for the five unknown amplitudes then yields the reflection and transmission coefficients, \( R, T^\uparrow \) and \( T^\downarrow \).

As usual, the tight binding model contains single-level sites \( n \), with site energies \( \epsilon_n \) and localized electronic states \( |n\rangle \), and nearest-neighbor hopping matrix elements \( J_{nm} \). The general Hamiltonian is written as

\[
\mathcal{H} = \sum_n \epsilon_n |n\rangle \langle n| - \sum_{\langle nm \rangle} (J_{nm}|n\rangle \langle m| + \text{h.c.}),
\]

where \( \langle nm \rangle \) denotes a bond between the neighboring sites \( n \) and \( m \). The Schrödinger equation for the electron’s wave function \( |\psi\rangle \equiv \sum_m |m|\psi|m\rangle \equiv \sum_m \psi(m)|m\rangle \) is therefore

\[
(\epsilon - \epsilon_n)\psi(n) = -\sum_m J_{nm}\psi(m),
\]

where \( \epsilon \) is the energy of the electron.

**The leads.** The system is connected to three leads, one on the left hand side and two on the right hand side (see figure 3). These leads are described by one-dimensional chains, with zero site energies and with only constant nearest-neighbor hopping matrix elements \( J_{n,n+1} = J_{n,n-1} \equiv J \) (all other \( J_{nm} \)'s on the leads are zero). The left lead has \( n \leq 0 \), and the two right-hand side outgoing leads have \( n \geq n_1 \). Within each lead, the Schrödinger equation is therefore

\[
e\psi(n) = -J [\psi(n - 1) + \psi(n + 1)],
\]

with the plane wave solutions \( \psi = e^{\pm ikn} \) and \( \epsilon = -2J \cos(k) \), where the dimensionless wave number \( k \) contains the lattice constant. In the following we look for a scattering solution, in which a wave \( e^{ikn} \) which comes from the left is scattered by the system. Thus we set

\[
\psi_{in}(n) = e^{ikn} + re^{-ikn}, \quad n \leq 0,
\]

\[
\psi_{out}^\uparrow(n) = t_1 e^{ik(n-n_1)}, \quad n \geq n_1,
\]

\[
\psi_{out}^\downarrow(n) = t_1 e^{ik(n-n_1)}, \quad n \geq n_1.
\]

Here, \( r \) denotes the amplitude of the reflected wave, while \( t^\uparrow \) and \( t^\downarrow \) denote the amplitudes of the two transmitted waves, in the upper and lower outgoing leads. These amplitudes determine the two outgoing currents and the reflected current, via the Landauer formula [11]. In the linear response limit (zero bias between the left and right leads) and at zero temperature, the ratios of the two outgoing currents and of the reflected current to the incoming one are given by the two transmission and one reflection coefficients,

\[
T^\uparrow \equiv |t^\uparrow|^2, \quad T^\downarrow \equiv |t^\downarrow|^2, \quad R = |r|^2,
\]
and current conservation implies the relation
\[ T^\dagger + T^\dagger + R = 1. \] (6)

**The interferometer.** The interferometer is modeled by a triangle \( ABC \) of single-level sites. On the upper and lower arms of the triangle (\( AB \) and \( AC \)) we place one-dimensional chains of sites, of lengths \( n_u \) and \( n_d \) (in the figure, \( n_u = 3, n_d = 2 \)), with uniform site energies \( \epsilon_u \) and \( \epsilon_d \) and with uniform nearest-neighbor hopping energies \( j_u \) and \( j_d \). The vertical arm \( BC \) closes the interferometer loop, and it has a tunneling matrix element \( V e^{i\phi} \) (or \( V e^{-i\phi} \)) between the sites \( B \) and \( C \) (or \( C \) and \( B \)). Here, \( \phi \) is the magnetic flux which penetrates the triangle, in units of the unit quantum flux times \( 2\pi \). We have used gauge invariance to place the Aharonov–Bohm phase, resulting from this flux, on the single bond \( BC \).

Within the upper (\( \ell = u \)) and lower (\( \ell = d \)) arms, the Schrödinger equation has the one-dimensional form
\[ (\epsilon - \epsilon_\ell)\psi_\ell(n) = -j_\ell \left[ \psi_\ell(n - 1) + \psi_\ell(n + 1) \right], \] (7)
with the solutions
\[ \psi_\ell(n) = \left[ \sin (k_\ell n)\psi_\ell(n_\ell) + \sin [k_\ell(n_\ell - n)]\psi (A) \right]/\sin (k_\ell n_\ell), \] (8)
where the wave numbers \( k_\ell = \arccos[(\epsilon_\ell - \epsilon)/(2j_\ell)] \) again contain the appropriate lattice constant.

At the corners of the triangle, we have \( \psi (A) = \psi_u(0) = \psi_d(0) = 1 + r, \ psi (B) = \psi_u(n_u) \) and \( \psi (C) = \psi_d(n_d) \). The tight binding equation at the site \( A \) is
\[ \epsilon (1 + r) = -j_u \psi_u(1) - j_d \psi_d(1) - J \left( e^{-ik} + re^{ik} \right). \] (9)
Substituting equation (8) into equation (9), the latter equation relates the three functions \( \psi (A), \ psi (B) \) and \( \psi (C) \).

**The tunnel-coupled wires.** The sequence of \( N \) rectangular loops within the rectangle \( BDFC \) represents the two coupled wires, \( BD \) and \( CF \), and the tunneling between them. Each such loop is penetrated by a flux \( \phi \) (in the same units) through it, and each vertical bond represents the tunneling matrix element \( V e^{i\phi} \) (or \( V e^{-i\phi} \)), with \( \phi_n = \phi + n\phi \) (again, we use gauge invariance to place the phases on the vertical bonds). In most of the following calculations we imitate the experiment and assume that the area of \( BDFC \) is equal to one third of the area of the interferometer loop \( ABC \), and therefore we use \( \phi = \Phi/(3N) \). To imitate the continuous tunnel wires one would like \( N \) to be very large, and therefore \( \phi \) is very small. In practice we perform calculations at several large values of \( N \), and ensure that the results do not vary much with \( N \). The tight binding for the sites on the upper and lower branches of the ladder, which we denote by \( \psi^\dagger(n) \) and by \( \psi^\dagger(n) \), respectively (\( n = 0, 1, 2, ..., N - 1 \)), are
\[ (\epsilon - \epsilon_\uparrow)\psi^\dagger(n) = -j_\uparrow \left[ \psi^\dagger(n + 1) + \psi^\dagger(n - 1) \right] - V e^{-i\phi^\dagger}\psi^\dagger(n), \] \[ (\epsilon - \epsilon_\downarrow)\psi^\dagger(n) = -j_\downarrow \left[ \psi^\dagger(n + 1) + \psi^\dagger(n - 1) \right] - V e^{i\phi^\dagger}\psi^\dagger(n), \] (10)
where \( \epsilon_\uparrow \) and \( \epsilon_\downarrow \) are the (constant) site energies, which model gate voltages applied to each wire separately, while \( j_\uparrow \) and \( j_\downarrow \) represent the corresponding nearest-neighbor hopping energies.

At the two ends of the tunnel coupled wires, the wave functions obey the boundary conditions,
\[ \psi_1(0) = \psi(B) = \psi_u(n_u), \quad \psi^1(0) = \psi(C) = \psi_d(n_d), \]
\[ \psi_1(N) = \psi(D) = t^1, \quad \psi^1(N) = \psi(F) = t^1. \] (11)

Therefore, the equations for the sites \( D \) and \( F \) are
\[ (\epsilon - \epsilon_1)\psi_1(N) = -j_1\psi_1(N - 1) - Jt^1e^{i\theta} - Ve^{-i\theta}\psi_1(N), \]
\[ (\epsilon - \epsilon_1)\psi^1(N) = -j_1\psi^1(N - 1) - Jt^1e^{i\theta} - Ve^{-i\theta}\psi^1(N). \] (12)

Similarly, the equations for the wave functions at the corners \( B \) and \( C \) of the interferometer are
\[ (\epsilon - \epsilon_1)\psi_1(0) = -j_u\psi_u(n_u - 1) - j_1\psi_1(1) - Ve^{-i\theta}\psi_1(0), \]
\[ (\epsilon - \epsilon_1)\psi^1(0) = -j_d\psi_d(n_d - 1) - j_1\psi^1(1) - Ve^{-i\theta}\psi^1(0). \] (13)

Equations (10) can be used to express all the functions \( \psi_1(n) \) and \( \psi^1(n) \) in terms of the four boundary functions \( \psi(B) \), \( \psi_1(N) \) = \( \psi(D) \), \( \psi_1(0) \) = \( \psi(C) \) and \( \psi^1(0) \) = \( \psi(F) \).

In the next section we show that these four unknown functions can be replaced by four amplitudes of four basic solutions, \( \{A_{\ell}\}, \ell = 1, 2, 3, 4 \). Equation (9) then relates the \( A_{\ell} \)'s to \( \psi(A) = 1 + r \). Together with equations (12) and (13), we end up with five linear equations in the five unknowns \( r \) and \( \{A_{\ell}\} \). Their solution yields the reflection coefficient \( R = |r|^2 \) and the two transmission coefficients \( T^i = |t^i|^2 \) and \( T^\dagger = |t^\dagger|^2 \).

Without the tunnel-coupled wires, i.e., when \( N = 0 \), one does not need the wave functions \( \psi_1 \) on these wires, and therefore one needs to solve only three linear equations in the three unknowns \( r, t^1 \) and \( t^\dagger \). As shown in appendix A, one can then obtain analytic solutions for these amplitudes, which allow a detailed analysis of the results. For \( n_u = n_d = 1 \), the results (figure A1) are ‘anti-phase’ for all \( V \), even when the reflection is large. In this case, the AB oscillations also exhibit phase rigidity, and the location of the extrema do not depend on the gate voltage. When \( n_u > 1 \) and when \( V \) is small (figure A2), one still obtains the ‘anti-phase’ behavior, and one also obtains a smooth shift of the extrema with the gate voltage. Thus, the ‘anti-phase’ behavior is generic for the three-terminal interferometer, and does not require the tunnel-coupled wires. However, at larger \( V \) the results without these wires become complicated, and one never finds the ‘in-phase’ behavior. In order to obtain both types of behavior, we now describe results with the tunnel-coupled wires.

4. The solutions within the tunnel-coupled wires

We first discuss the solution of the tight-binding equations within the ladder, equation (10). As we show below, these equations have four basic solutions, and the general solution is a linear combination of these solutions. Depending on the parameters, these solutions can be either wave-like (‘running’) or evanescent. For a large tunnel coupling between the wires, \( V \gg J \), all the solutions become evanescent, i.e. they decay as they propagate into the coupled wires. This results in very small transmission coefficients and with a reflection coefficient \( R \) close to one (see equation (6)). This \( R \sim 1 \) is realistic when there is only a single transmitting channel in each wire. But it cannot be found experimentally because there are many transmitting channels in real experiments which have all different \( V \) as well as finite inter-channel scatterings. As \( V \) decreases, one sometimes finds two evanescent solutions and two running solutions. As we explain below, this is the region where one finds the ‘in-phase’ behavior: for the remaining
‘running’ wave, the amplitudes $t^\uparrow$ and $t^\downarrow$ have a fixed (almost $\phi$-independent) ratio, and therefore the two outgoing currents are proportional to each other, with the same phase. At very low $V$ one has a very small reflection, and this yields the ‘anti-phase’ behavior: from equation (6), $T^\downarrow = 1 - R - T^\uparrow \approx 1 - T^\uparrow$, and therefore the oscillating parts in the two currents are opposite to each other.

A wave-like solution for equations (10) can be found by setting $\psi^\uparrow(n) = e^{i(Kn - \phi/2)} u^\uparrow$, $\psi^\downarrow(n) = e^{i(Kn + \phi/2)} u^\downarrow$ (where again the dimensionless wave number $K$ contains the lattice constant along the ladder). When $K$ is a real number, these functions oscillate with fixed amplitudes, and therefore they represent ‘running’ solutions. When $K$ is complex, these functions decay (or grow) exponentially, and then they are called evanescent. The amplitudes $u^\uparrow, \downarrow$ must then obey the linear equations

\[
\begin{align*}
(e - e_\uparrow + 2j_\uparrow \cos (K - \phi/2))u^\uparrow + Vu^\uparrow &= 0, \\
Vu^\uparrow + (e - e_\downarrow + 2j_\downarrow \cos (K + \phi/2))u^\downarrow &= 0.
\end{align*}
\]

Therefore, the wave-numbers $K$ are the solutions of the determinant equation,

\[
\begin{pmatrix}
(e - e_\uparrow + 2j_\uparrow \cos (K - \phi/2)) & (e - e_\downarrow + 2j_\downarrow \cos (K + \phi/2)) \\
(e - e_\downarrow + 2j_\downarrow \cos (K + \phi/2)) & (e - e_\uparrow + 2j_\uparrow \cos (K - \phi/2))
\end{pmatrix} - V^2 = 0,
\]

i.e.,

\[
4j_\uparrow j_\downarrow \cos^2 K + 2[(e - e_\uparrow)j_\downarrow + (e - e_\downarrow)j_\uparrow] \cos (\phi/2) \cos K \\
+ (e - e_\uparrow)(e - e_\downarrow) - V^2 - 4j_\uparrow j_\downarrow \sin^2 (\phi/2)
= 2[(e - e_\uparrow)j_\downarrow - (e - e_\downarrow)j_\uparrow] \sin (\phi/2) \sin K.
\]

As stated, we need results for large $N$ and therefore for small $\phi$. At $\phi = 0$ we have a quadratic equation, with two solutions, $\cos K = c_\pm \equiv [(e_\uparrow - e)j_\downarrow + (e_\downarrow - e)j_\uparrow \pm \sqrt{(e_\uparrow - e)j_\downarrow - (e_\downarrow - e)j_\uparrow}^2 + 4V^2/(j_\uparrow j_\downarrow)]/4$. For small $V$, the two solutions for $\cos K$ remain in the range $-1 \gg c_\pm \gg 1$, and therefore the two solutions correspond to two pairs of waves, running in opposite directions with wave numbers $K_{1,2} = \pm \arccos [c_+]$ and $K_{3,4} = \pm \arccos [c_-]$. However, as $V$ increases one of $|c_\pm|$ crosses the value 1 when

\[
V^2 = \left[e - e_\uparrow \pm 2j_\uparrow \cos (\phi/2)\right]\left[e - e_\downarrow \pm 2j_\downarrow \cos (\phi/2)\right].
\]

Above these values of $|V|$, $K_{1,2}$ and/or $K_{3,4}$ become complex, and the corresponding waves become evanescent. Figure 4 shows an example of regions in the $e_\uparrow - V$ plane, for the special parameters $j_\uparrow = j_\downarrow = J$, and $e = e_\downarrow = 0$. The numbers on the diagram (4, 2 or 0) indicate the number of ‘running’ solutions, with real $K$s, in each region.

The two lines in figure 4 cross each other when $j_\uparrow (e - e_\uparrow) + j_\downarrow (e - e_\downarrow) = 0$. For the parameters used in the figure, this happens when $e_\uparrow = 0$. At this crossing point, one ‘jumps’ directly from four running solutions to no such solutions. In particular, this always happens for the symmetric case, when $e_\uparrow = e_\downarrow$ and $j_\uparrow = j_\downarrow$. In order to find only two running solutions one must therefore use parameters which deviate from this point. Below we achieve this by using non-zero values of $e_\uparrow$, which implies some asymmetry between the two wires.
For a large but finite \( N \) we need to solve equation (16) for a finite small \( \phi \). In practice we do that by searching a solution for \( K \) close to each of the four solutions found at \( \phi = 0 \). For each value of \( \phi \) this yields four waves, with wave numbers \( K_1, K_2, K_3, K_4 \) (an alternative, which gives the same results, is to square equation (16), solve a quartic equation for the four solutions \( \cos K_\ell \) and then choose the right signs of \( \sin K_\ell \) which satisfy equation (16)). As \( V \) increases at fixed \( j \), at first all the four \( K \)'s are real, then two of them become complex and then the other two also become complex. As we shall see below, each of these regimes ends up with a different qualitative behavior of the two outgoing currents.

For each of the four \( K \)'s, the corresponding amplitudes of the wave functions obey the relation

\[
 u_\ell^\dagger = -u_\ell^\dagger \left[ e - j_1 + 2j_1 \cos (K_\ell - \phi/2) \right] / V. \tag{18}
\]

The general solution for the \( N + 1 \) sites \( (n = 0, 1, 2, \ldots, N) \) on the \( N \) squares of the ladder can now be written as a linear combination of the four solutions,

\[
 \psi^\uparrow(n) = \sum_{\ell=1}^{4} A_\ell e^{(K_n - \phi)/2}, \\
 \psi^\downarrow(n) = -\sum_{\ell=1}^{4} \left[ e - j_1 + 2j_1 \cos (K_\ell - \phi/2) \right] A_\ell e^{(K_n + \phi)/2} / V, \tag{19}
\]

with the four yet unknown amplitudes \( \{A_\ell\} \). As explained at the end of the previous section, equations (9), (12) and (13) finally yield the amplitudes \( r, t^\dagger \) and \( t^\dagger \).

Figure 4 is central for the interpretation of the results presented below. The ‘anti-phase’ behavior arises at small \( V \), in the region denoted by ‘4’ in the figure. The ‘in-phase’ arises in the intermediate region, denoted by ‘2’ in the figure. In that regime the outgoing currents are dominated by the two ‘running’ waves. For large \( N \), \( \phi \) is small, and therefore the ratio of the two outgoing currents, derived from equation (18), is practically flux-independent. This yields the ‘in phase’ behavior. The next section elaborates on the details of these behaviors.
5. Results

To imitate a flat density of states ($\frac{d\epsilon}{dk}$) in the external leads (within the present tight-binding model), one usually chooses electron energies near the center of the band, $\epsilon = 0$ or $k = \pi/2$. For $N = 0$ we present analytic results in appendix A. Here we concentrate on large $N$. As stated, we set $\phi = \Phi/(3N)$. Below we plot results as a function of the flux through the interferometer $ABC$, namely $\Phi$. For reasonable values of $\Phi$ and sufficiently large $N$ this implies relatively small values of $\phi$. For the results presented below we used $N = 1001$ and $N = 401$ (the numerical solutions of the five linear equations become difficult for large $N$ and large $V$, when the exponentially small (or large) factors $e^{iKN}$ (which arise in the evanescent cases) vary by many orders of magnitude). These results change only slightly (quantitatively but not qualitatively) when we used other (large) values of $N$.

In order to observe the dependence of the results on the gate voltage on the upper arm of the interferometer, we need to have $n_u > 1$ (see appendix A). Below we present typical results with $n_u = 5$. The results do not change qualitatively for a wide range of this and other parameters. For small $V$ we always find ‘anti-phase’ behavior, even when the two wires are symmetric. As explained below, to see the ‘in-phase’ behavior we need to have only two ‘running’ waves, and this happens only with some asymmetry between the wires (see figure 4). Indeed, with some such anisotropy and for appropriately chosen values of large $V$ (explained below) we find ‘in-phase’ behavior. This behavior requires the coupled wires, and did not appear in the simpler three-terminal interferometer presented in appendix A.

Figure 5. Typical results for the tunneling-coupled wires, for weak coupling: $N = 1001$, $\phi = \Phi/(3N)$, $\epsilon_t = .1J$, $\epsilon_1 = 0$, $j_u = j_j = j_l = J$, $j_d = .2J$, $n_u = 5$, $n_d = 1$, $\epsilon = 0$ and $V = 0.001J$. (a) Transmissions $T^1$ (red) and $T\underline{1}$ (blue) and reflection $R$ (green), for $\epsilon_u = .1J$. (b) An enlarged version of (a), showing $T^1 - \langle T\underline{1}\rangle$, $T\underline{1} - \langle T^1\rangle$ and $R - \langle R\rangle$. (c) Maxima (red) and minima (blue) of $T^1$ versus $\epsilon_u$ (all other parameters are the same as in (a)).
We start with weak coupling, \(V = 0.001J\). Figure 5(a) shows results for a small coupling \(V\), for the parameters as indicated. Due to the anisotropy between the two branches of the interferometer, \(j_{ud} = \frac{0.2J}{\pi}\), \(T_{\uparrow}\) is much larger than \(T_{\downarrow}\). However, when we shift each of the transmissions by their average value, as shown in figure 5(b), it is obvious that the AB oscillations of the two transmissions exhibit the same kind of ‘anti-phase’ behavior as also seen without the coupled wires in appendix A (and as seen experimentally). The main reason for the ‘anti-phase’ behavior can be attributed to the weak oscillations in \(R\) (which are practically zero in figure 5(b)). The relation \(T_{\uparrow} + T_{\downarrow} = 1 - R\) then requires that the oscillating terms in the two transmissions must have opposite signs, so they cancel each other in the sum.

Figure 5(c) shows the variation of the minima and maxima of \(T_{\uparrow}\) with \(\epsilon_{\uparrow}\). Similar to figure A2(b), these maxima and minima move smoothly with the gate voltage, without any jumps. These maxima and minima reflect the energy levels of the upper arm AB of the interferometer. Therefore, we conclude that shifts of the oscillations in the two transmissions reflect the basic ‘two-slit’ interference around the AB loop.

Figure 6 exhibits typical results for an intermediate value of the tunnel coupling, \(V = 1.7J\). The phases and their shifts depend on the parameters, and one cannot identify either the ‘in-phase’ or the ‘anti-phase’ behavior. We now turn to large \(V\), within the intermediate range in figure 4. Figure 7 is similar to figure 5, except that \(V = 1.95J\) and \(N = 401\). As can be seen from figures 7(a) and (b), the two outgoing transmissions are practically identical to each other, and therefore they are fully in phase with each other. Furthermore, they both are symmetric under \(\Phi \leftrightarrow -\Phi\). As required by the relation \(T_{\uparrow} + T_{\downarrow} = 1 - R\), the reflection coefficient \(R\) exhibits opposite oscillations, with a double amplitude (see figure 7(b)). The maxima and minima of \(T_{\uparrow}\), shown in figure 7(c), exhibit full ‘phase locking’; they remain at integer multiples of \(\pi\). (For
some values of $\epsilon_t$ one observes additional red points between these integer values; these indicate the splitting of the maxima into pairs of maxima, due to higher integer Fourier components. One always finds only integer Fourier components, but with amplitudes which vary with the parameters.) This phase locking is similar to that observed in the two-terminal interferometer [3]. Apart from being locked, maxima occasionally turn into minima and vice versa. This must arise from crossing of resonances on the branch AB of the interferometer. All of these features are very similar to those seen experimentally.

For large $V$, electrons are strongly reflected from the points B and C of the structure. This explains the large value of $R$, and the small values of the transmitted currents. These features also appeared for the three terminal interferometer, without the tunnel coupling between the outgoing wires. To understand the other features observed in figure 7, we note that the point $\epsilon_t = 0.1J$, $V = 1.95J$ is just above the lower line in figure 4 (it is difficult to solve the linear equations for larger $V$, when the imaginary part of the evanescent wave numbers are large and the corresponding wave functions go beyond the accuracy limits of the computer. This is why we stay close to this line, and also why we only present numbers for $N = 401$). Within the region between the two lines in figure 4, two of the four solutions within the ‘ladder’ decay exponentially, and we are left with only two running waves, which result from one of the solutions $c_+$ or $c_-$. For $\epsilon_t > 0$, the first evanescent waves (as we cross the lower line in figure 4) are associated with $c_+$ becoming larger than 1. For the parameters in that figure, this happens at $V^2 = 2j_r (2j_r - \epsilon_t)$, namely at $V \approx 1.949J$. At that point one has $c_- = \epsilon_t/(2j_r) - 1 \approx -0.95$, and the two remaining running waves have $K \approx \pm \arccos(c_-) \approx \pm 2.824$. At the same neighborhood, equation (18) yields
where we neglected the small \(\phi\). Since this ratio applies to both the running solutions, and since the evanescent solutions can be neglected, we conclude that the same ratio applies to their linear combinations, \(\psi\uparrow(n)\) and \(\psi\downarrow(n)\). In particular,

\[
T^\uparrow / T^\downarrow = \left| t^\uparrow / t^\downarrow \right|^2 \approx \left| e - e^1 + 2j \epsilon \right|^2 / V^2,
\]

practically independent of \(\Phi\) (at least for fluxes which are not too large). Since this ratio is independent of \(\Phi\), the two transmissions are proportional to each other, and therefore they are exactly in phase. Deviations will occur only at large fluxes, when \(\phi\) becomes significant.

For the parameters used in figure 7, this equation yields \(T^\uparrow / T^\downarrow \approx 1.05\). This explains the practical overlap of the two transmissions. For other parameters the ratio need not be so close to unity, but the in-phase behavior will persist whenever we have two evanescent waves. The Onsager relation requires that the full current through the system should be an even function of the flux. This implies that both \(R\) and \(T^\uparrow + T^\downarrow\) should be such even functions. Indeed, all our calculations show that \(R\) is even in \(\Phi\). Once we demonstrated that the two transmissions are proportional to each other, each of them must be proportional to their sum, and therefore each of them separately must be an even function of \(\Phi\). Combined with the periodicity in \(\Phi\), this explains the phase rigidity of the results.

6. Conclusion

We have demonstrated a novel two-slit experiment, using an AB ring connected to a tunnel-coupled wire in a three-terminal setup. Our simple tight binding model captures most of the observed behavior of the currents in the three terminal interferometer. For small tunnel-coupling \(V\), one always has the anti-phase behavior. In that limit, the dependence of the the phase of the Aharonov–Bohm oscillations in the outgoing currents on the gate voltage acting on one arm of the interferometer follows the phase of the electronic wave function on that arm. This three terminal interferometer thus behaves like the two-slit or open interferometer. For large \(V\) one needs to tune the inter-wire coupling to a regime where one has only two running wave solutions. In that regime, the ratio between the two outgoing currents is practically flux independent, and therefore they are in phase, with phase rigidity. Although our model does not include all the details of the interferometer, such as finite widths of the quantum wires, influence of the multiple transport channels and accompanying electron–electron interaction, the simple and analytically solvable model provides the guiding principles for realizing a ‘two-slit’ experiment and for a reliable phase measurement in the three-terminal setup.

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Appendix A. Three terminal interferometer without tunnel-coupled wires

In this appendix we present analytic results for the case \( N = 0 \). Without the coupled wires, the outgoing leads connect directly to sites \( B \) and \( C \), and we have \( \psi (B) = t^1 \) and \( \psi (C) = t^1 \). Therefore, equations (12) and (13) are replaced by

\[
\begin{align*}
\epsilon t^1 &= -\epsilon_0 e^{ik} t^1 - V e^{-i\Phi} t^1 - j_u \psi_d(n_u - 1), \\
\epsilon t^1 &= -\epsilon_0 e^{ik} t^1 - V e^{i\Phi} t^1 - j_d \psi_d(n_d - 1), \\
\epsilon(1 + r) &= -j_u \psi_d(1) - j_d \psi_d(1) - J \left( e^{-ik} + re^{ik} \right). 
\end{align*}
\] (A.1)

From equation (8), one has \( \psi_f(1) = (1 + r) y_f + t^1 x_f \) and \( \psi_f(n_f - 1) = (1 + r) x_f + t^1 y_f \), where \( x_f = j_f \sin (k_f) / \sin (k_f n_f) \), \( y_f = j_f \sin [k_f(n_f - 1)] / \sin (k_f n_f) \). Substitution into equation (A.1) yields three equations in \( r \), \( t^1 \) and \( t^1 \), with the analytic solutions

\[
\begin{align*}
t^1 &= \left[ x_u \left( Je^{-ik} - y_d \right) + V x_d e^{-i\Phi} \right](1 + r)/d, \\
t^1 &= \left[ x_d \left( Je^{-ik} - y_u \right) + V x_u e^{i\Phi} \right](1 + r)/d, \\
1 + r &= -2iJ \sin (k)d/D, 
\end{align*}
\] (A.2)

where

\[
\begin{align*}
d &= \left( Je^{-ik} - y_u \right) \left( Je^{-ik} - y_d \right) - V^2, \\
D &= \left( Je^{-ik} - y_u - y_d \right) d - x_u^2 \left( Je^{-ik} - y_d \right) \\
&\quad - x_d^2 \left( Je^{-ik} - y_u \right) - 2x_u x_d V \cos \Phi. 
\end{align*}
\] (A.3)

Interestingly, \( r \) depends on \( \Phi \) only via the term with \( \cos \Phi \) in the denominator \( D \). Therefore, \( R = |r|^2 = 1 - T^1 - T^1 \) is an even function of the flux, as might be expected from the Onsager relation. Therefore, \( R \) also exhibits phase rigidity. The same phase rigidity is maintained when one includes the tunnel-coupled wires.

Setting \( \epsilon = 0 \) (or \( k = \pi/2 \)), these results yield

\[
\begin{align*}
T^1 &= 4J^2 \left[ x_u^2 \left( J^2 + y_d^2 \right) + x_d^2 V^2 + 2x_u x_d V \left( J \sin \Phi - y_d \cos \Phi \right) \right] |D|^2, \\
T^1 &= 4J^2 \left[ x_d^2 \left( J^2 + y_u^2 \right) + x_u^2 V^2 - 2x_u x_d V \left( J \sin \Phi + y_u \cos \Phi \right) \right] |D|^2. 
\end{align*}
\] (A.4)

The denominator has the general form \( D = Q - 2x_u x_d V \cos \Phi + iP \), where

\[
\begin{align*}
P &= J \left( J^2 - y_u^2 - y_d^2 - 3y_u y_d + x_u^2 + x_d^2 + V^2 \right) \equiv P_0 + JV^2, \\
Q &= 2J^2 - y_u y_d + V^2 \left( y_u + y_d \right) + x_u^2 y_d + x_d^2 y_u \equiv Q_0 + \left( y_u + y_d \right)V^2, 
\end{align*}
\] (A.5)

so that \( |D|^2 = P^2 + (Q - 2x_u x_d V \cos \Phi)^2 \).

For \( n_u = n_d = 1 \) one has \( y_u = y_d = 0 \), and therefore also \( Q = 0 \), and \( |D|^2 = P^2 + 4x_u^2 x_d^2 V^2 \cos^2 \Phi \), while both numerators in equation (A.4) contain the term \( \pm 2x_u x_d JV \sin \Phi \). It turns out that in this special case the numerators determine the locations of the maxima and minima of the two transmissions, and therefore the two outgoing currents...
always have opposite phases, with maxima or minima at \( \Phi = (1/2 + m)\pi \) (with integer \( m \)) for all the values of the various parameters. Examples are shown in figure A1.

In the special case \( n_u = n_d = 1 \), the model contains no dependence on the gate voltages on the branches of the AB interferometer; for example, the site energy \( \epsilon_u \) is included in the model only for sites between \( A \) and \( B \) in figure 3, which requires \( n_u > 1 \). To investigate the dependence of the results on the gate voltage \( V_M \) (figure 1), which is represented by the site energy \( \epsilon_u \), we thus studied the model with \( n_u > 1 \). For a small tunnel coupling, \( V = 0.1J \), typical results are shown in figure A2(a). This figure was drawn for \( n_u = 5 \), a ‘gate voltage’ on the upper arm of the interferometer \( \epsilon_u = .5J \) and \( n_d = 1 \) (so that \( x_d = j_d \) and \( y_d = 0 \)). As seen in the figure, the oscillations in \( R \) have a very small amplitude. Since \( T^\dagger + T^\dagger = 1 - R \), this means that the oscillations in the two outgoing transmissions must be in ‘anti-phase’, as indeed seen in the same figure.

Figure A2(b) shows the shifts \( \beta \) in the maxima and minima of \( T^\dagger \) with the ‘gate voltage’ \( \epsilon_u \) on the upper arm of the interferometer. When the shifts are small, they are linear in this gate voltage. This means that they are also linear in the wave vector \( k_u \), and therefore also in the shift in the optical path of the electron wave function on the upper branch. A measurement of this phase shift yields this bare phase shift, as in the two-slit interferometer! We have thus demonstrated that our system can be used for measuring phase shifts.

The above anti-phase behavior, and the smooth variation of the phases of both outgoing currents with the gate voltage, appear only for small \( V \). For large \( V \), both of the outgoing

**Figure A1.** Typical results for the simple three-terminal interferometer, with \( k = \pi/2 \) (i.e. \( \epsilon = 0 \)), \( j_u = J \), \( j_d = 0.3J \), \( n_u = n_d = 1 \), red: \( T^\dagger \), blue: \( T^\dagger \), green: \( R \). (a) \( V = 0.1J \). (b) \( V = 3J \).

**Figure A2.** (a) Same as figure A1(a), but with \( n_u = 5 \) and \( \epsilon_u = 0.5J \). (b) The locations of the maxima (red) and minima (blue) in \( T^\dagger \) (denoted by \( \beta \)) as functions of the ‘gate voltage’ \( \epsilon_u \).
currents are small, proportional to $1/V^2$, and the reflection $R$ is close to 1. The visibility (i.e. the amplitude of the AB oscillations) is even smaller, of order $1/V^3$. In this limit, the details of the AB oscillations depend on the parameters of the device, but they never reach the ‘in-phase’ behavior.

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