1+1+2 gravitational perturbations on LRS class II space-times: GEM scalar harmonic amplitudes

R. B. Burston
Max Planck Institute for Solar System Research, 37191 Katlenburg-Lindau, Germany
E-mail: burston@mps.mpg.de

Abstract. This is the third in a series of papers which considers first-order gauge-invariant and covariant gravitational perturbations to locally rotationally symmetric (LRS) class II space-times. In this paper we complete our analysis of the first-order gravito-electromagnetic (GEM) system by showing how to derive three decoupled equations governing the GEM scalar fields. One of these is for the gravito-magnetic scalar, whereas another two arise from the 2-gradient of the gravito-electric scalar.

PACS numbers: 04.25.Nx, 04.20.-q, 04.40.-b, 03.50.De, 04.20.Cv
1. Introduction

Paper I in this series [1] uses similar methods that we used to decouple electromagnetic (EM) perturbations [2, 3] to show how to decouple the gravito-electromagnetic (GEM) 2-tensor harmonic amplitudes. This was followed by Paper II which showed that further decoupling is achieved when the complex GEM 2-vector is combined with the shear 2-tensors describing the 2/3-sheets [4]. Summarily, from Paper’s I and II we have found eight specific combinations that decouple which are

Decoupled polar perturbations:  \[ \left\{ (E_T + H_T), (E_T - H_T) \right\}, \]

\[ \left\{ \left( E_V + \frac{3}{2} E r \zeta_T \right) - \left( H_V + \frac{3}{2} E r \Sigma_T \right) \right\}, \left\{ \left( E_V + \frac{3}{2} E r \zeta_T \right) + \left( H_V + \frac{3}{2} E r \Sigma_T \right) \right\} \]  \hspace{1cm} (1)

Decoupled axial perturbations:  \[ \left\{ (H_T + \bar{E}_T), (H_T - \bar{E}_T) \right\}, \]

\[ \left\{ \left( H_V + \frac{3}{2} E r \bar{\Sigma}_T \right) + \left( \bar{E}_V - \frac{3}{2} E r \bar{\zeta}_T \right) \right\}, \left\{ \left( H_V + \frac{3}{2} E r \bar{\Sigma}_T \right) - \left( \bar{E}_V - \frac{3}{2} E r \bar{\zeta}_T \right) \right\} \]  \hspace{1cm} (2)

This paper concentrates on decoupling the quantities governing the 1+1+2 GEM scalars.

2. Complex 1+1+2 GEM system

The 1+1+2 complex GEM system was first expressed in Paper I and subsequently new dependent variables were chosen in Paper II and the new GEM system becomes

\[ (L_u - \frac{5}{2} \Sigma + \frac{5}{3} \theta) \Xi_{\mu} + i \epsilon_\mu^\alpha \left( L_n + \frac{5}{2} \phi \right) \Xi_\alpha + i \epsilon_\mu^\alpha (\delta^2 - K - 3 E) \Psi_\alpha \]

\[ -3 E \left[ \left( \Sigma - \frac{2}{3} \theta \right) \Gamma_\mu + i \phi \epsilon_\mu^\alpha \Gamma_\alpha \right] = T_\mu, \]  \hspace{1cm} (3)

\[ (L_n + 2 \phi) \Psi_\mu - \frac{3}{2} \Sigma \epsilon_\mu^\alpha \Psi_\alpha - \frac{1}{2} \Xi_\mu + (\delta^2 + K + 3 E) \Gamma_\mu = \tilde{G}_\mu, \]  \hspace{1cm} (4)

\[ (L_u - 2 \Sigma + \frac{4}{3} \theta) \Psi_\mu - i \epsilon_\mu^\alpha \left( A - \frac{1}{2} \phi \right) \Psi_\alpha + i \frac{1}{2} \epsilon_\mu^\alpha \Xi_\alpha + i(\delta^2 + K + 3 E) \epsilon_\mu^\alpha \Gamma_\alpha = \tilde{F}_\mu, \]  \hspace{1cm} (5)

\[ (L_u + \frac{3}{2} \Sigma + \theta) \Gamma_\mu - i \epsilon_\mu^\alpha \left( L_n + 2 A + \frac{1}{2} \phi \right) \Gamma_\alpha + i \frac{1}{2} \epsilon_\mu^\alpha \Psi_\alpha = \delta^\alpha F_\mu^\alpha. \]  \hspace{1cm} (6)

where the new variables were defined

\[ \Xi_\mu := (\delta^2 + K) C_\mu - 3 E \left[ \phi \delta^\alpha \zeta_{\mu \alpha} + \left( \Sigma - \frac{2}{3} \theta \right) \delta^\alpha \Sigma_{\mu \alpha} + \delta^\alpha \Pi_{\mu \alpha} \right], \]  \hspace{1cm} (7)

\[ \Psi_\mu := (\delta^2 + K) \Phi_\mu + i 3 E \epsilon_\mu^\alpha \delta^\beta \Lambda_{\alpha \beta}; \]  \hspace{1cm} (8)

\[ \Gamma_\mu := \delta^\alpha \Phi_{\mu \alpha}. \]  \hspace{1cm} (9)
3. Decoupling $\Xi_\mu$ and its harmonic amplitudes

In order to decouple $\Xi_\mu$, we construct higher-order derivatives by first taking the Lie derivative of (3) with respect to $u^\mu$ and after much manipulation, we find

$$\left[ \left( \mathcal{L}_u - 4 \mathcal{S} + \frac{11}{3} \theta \right) \mathcal{L}_u - (\mathcal{L}_n + A + 5 \phi) \mathcal{L}_n - V \right] \Xi_\mu = \mathcal{M}_\mu, \quad (10)$$

where the potential and energy-momentum source are

$$V := \delta^2 + 19 K + 12 \mathcal{E}, \quad (11)$$

$$\mathcal{M}_\mu := \left( \mathcal{L}_u - \frac{3}{2} \mathcal{S} + 2 \theta \right) T_\mu - i \epsilon_\mu^\alpha \left( \mathcal{L}_n + A + \frac{5}{2} \phi \right) T_\alpha + 3 \mathcal{E} \left( \Sigma - \frac{2}{3} \theta \right) \delta^\alpha \mathcal{F}_{\mu\alpha} + i 3 \mathcal{E} \phi \epsilon_\mu^\alpha \delta^\beta \mathcal{F}_{\alpha\beta} + (\delta^2 - K - 3 A)(\delta^2 + K)(\mathcal{G}_\mu - i \epsilon_\mu^\alpha \mathcal{F}_{\alpha}). \quad (12)$$

This clearly demonstrates the decoupling of $\Xi_\mu$ from the remaining first-order quantities.

Furthermore, the differential operator in (10) is real and thus it is possible to consider the real and imaginary parts separately.

3.1. Harmonic expansion

The dependent variable $\Xi_\mu$ and the energy-momentum source $\mathcal{M}_\mu$ are expanded into 2-vector harmonics according to

$$\Xi_\mu = \Xi_\nu Q_\nu + \bar{\Xi}_\nu \bar{Q}_\nu \quad \text{and} \quad \mathcal{M}_\mu = \mathcal{M}_\nu Q_\nu + \bar{\mathcal{M}}_\nu \bar{Q}_\nu. \quad (13)$$

Therefore, (10) naturally decouples into

$$\left[ \left( \mathcal{L}_u - 5 \mathcal{S} + \frac{13}{3} \theta \right) \mathcal{L}_u - (\mathcal{L}_n + 6 \phi + A) \mathcal{L}_n - \bar{V} \right] \Xi_\nu = \mathcal{M}_\nu, \quad (14)$$

$$\left[ \left( \mathcal{L}_u - 5 \mathcal{S} + \frac{13}{3} \theta \right) \mathcal{L}_u - (\mathcal{L}_n + 6 \phi + A) \mathcal{L}_n - \bar{V} \right] \bar{\Xi}_\nu = \bar{\mathcal{M}}_\nu. \quad (15)$$

and the new potential has been defined

$$\bar{V} := -\frac{k^2}{r^2} + 30 K + 21 \mathcal{E}. \quad (16)$$

Now since the differential operators in (14)-(15) are purely real, the real and imaginary parts may be considered separately. It is of interest to see how the harmonic amplitudes of $\Xi_\mu$ are related back to the harmonic amplitudes of the GEM scalars and other 1+1+2 quantities. By using the definition (7) it can be shown that

$$\Re[\Xi_\nu] = r p \left\{ X_\nu - \frac{3}{2} \mathcal{E} r \left[ \phi \zeta_r + \left( \Sigma - \frac{2}{3} \theta \right) \Sigma_r + \Pi_r \right] \right\}, \quad (17)$$

$$\Re[\bar{\Xi}_\nu] = r p \left\{ \bar{X}_\nu + \frac{3}{2} \mathcal{E} r \left[ \phi \bar{\zeta}_r + \left( \Sigma - \frac{2}{3} \theta \right) \Sigma_r + \bar{\Pi}_r \right] \right\}, \quad (18)$$

$$\Im[\Xi_\nu] = p \mathcal{H}_s, \quad (19)$$

$$\Im[\bar{\Xi}_\nu] = 0, \quad (20)$$

where

$$p := \frac{1}{r} \left( 2 K - \frac{k^2}{r^2} \right). \quad (21)$$
Therefore, there are only 3 equations here. One of them governs the gravito-magnetic scalar $H_s$ whereas the remaining two govern the 2-gradient of the gravito-electric scalar combined with the 2-divergence of the 2-tensors for the shear of the 2/3-sheets and the anisotropic stress 2-tensor.

The factors $p$ and $r$ arise from a harmonic expansion of the 2-Laplacian in terms of harmonics and they can be differentiated and factorized if desired. For example, the decoupled equation governing $H_s$ becomes

$$p \left[ (L_u - 2 \Sigma + \frac{7}{3} \theta) L_u - (L_n + A + 3 \phi) L_n - V_H \right] H_S = \Im[\mathcal{M}_V]$$

where

$$V_H := \frac{k^2}{r^2} + \frac{3}{2} \phi^2 - 6 \mathcal{E} - \frac{3}{2} \left( \Sigma - \frac{2}{3} \theta \right) \left( \Sigma + \frac{1}{3} \theta \right).$$

Thus the three quantities which each decouple are

Decoupled polar perturbations: \( \left\{ X_V - \frac{3}{2} \mathcal{E} r \left[ \phi \zeta_T + \left( \Sigma - \frac{2}{3} \theta \right) \Sigma_T + \Pi_T \right] \right\} \), \( (24) \)

Decoupled axial perturbations: \( \left\{ \bar{X}_V + \frac{3}{2} \mathcal{E} r \left[ \phi \bar{\zeta}_T + \left( \Sigma - \frac{2}{3} \theta \right) \bar{\Sigma}_T + \bar{\Pi}_T \right], H_S \right\} \). \( (25) \)

3.2. Reduction to the covariant Schwarzschild space-times

We now consider the covariant Schwarzschild space-time such that these results may be related to previous work. The only non-vanishing LRS class II scalars for this case are \((A, \phi, \mathcal{E})\) for which we now have

$$\Re[\Xi_{\mu}] := (\delta^2 + K) X_{\mu} - 3 \mathcal{E} \phi \delta^0 \zeta_{\mu \alpha},$$

$$\Im[\Xi_{\mu}] := (\delta^2 + K) \delta_{\mu} H_S.$$ \( (27) \)

Clarkson and Barret \[5\] showed how to derive a Regge-Wheeler (RW) 2-tensor $W_{\mu \nu}$ in this case and one can show that this tensor is related to \( (26) \) according to

$$- \frac{6 \mathcal{E}}{r^2} \delta^0 W_{\mu \alpha} = \Re[\Xi_{\mu}].$$

The imaginary part is related to work from the Newman-Penrose formalism in \[7\]. The decoupled equation we have derived for $H_S$ reduces to

$$p \left[ (L_u - 2 \Sigma + \frac{7}{3} \theta) L_u - (L_n + A + 3 \phi) L_n + \frac{\ell(\ell + 1)}{r^2} - \frac{3}{2} \phi^2 + 6 \mathcal{E} \right] H_S = 0.$$ \( (29) \)

A scaling is introduced to facilitate comparison with the work of Price \[7\], $H'_S := r^3 H_S,$

$$\frac{p}{r^3} \left( L_u L_u - (L_n + A) L_n + \frac{\ell(\ell + 1)}{r^2} + 3 \mathcal{E} \right) H'_S = 0.$$ \( (30) \)

We now expand everything in terms of coordinates, transform to the “tortoise” coordinate, $r_\star$, and omit the common factor to reveal,

$$\left( \partial^2_t - \partial^2_{r_\star} + V_{RW} \right) H'_S = 0,$$ \( (31) \)

where the standard RW potential is

$$V_{RW} := \left( 1 - \frac{2 M}{r} \right) \left( \frac{\ell(\ell + 1)}{r^2} - \frac{6 M}{r^3} \right),$$ \( (32) \)
and $M$ is the mass. This corresponds to Price’s result that $r^3 \Im[\Psi_0]$ is a RW quantity \cite{7} and $\Psi_0$ is a Newman-Penrose scalar.

4. Summary

We have provided a comprehensive analysis of the first-order 1+1+2 complex GEM system spanning three papers. This paper delivered the final three decoupled quantities and for completeness, we write the total of 11 decoupled quantities that we have found,

Decoupled polar perturbations: \[ \left\{ \begin{array}{l} (\mathcal{E}_T + \mathcal{H}_T), \\
(\mathcal{E}_T - \mathcal{H}_T), \\
\left[ \left( \mathcal{E}_V + \frac{3}{2} \mathcal{E} r \zeta_T \right) - \left( \mathcal{H}_V + \frac{3}{2} \mathcal{E} r \Sigma_T \right) \right], \\
\left[ \left( \mathcal{E}_V + \frac{3}{2} \mathcal{E} r \zeta_T \right) + \left( \mathcal{H}_V + \frac{3}{2} \mathcal{E} r \Sigma_T \right) \right], \\
X_V - \frac{3}{2} \mathcal{E} r \left[ \phi_T + \left( \Sigma - \frac{2}{3} \theta \right) \Sigma_T + \Pi_T \right] \end{array} \right\} \] \hspace{1cm} (33)

Decoupled axial perturbations: \[ \left\{ \begin{array}{l} (\mathcal{H}_T + \bar{\mathcal{E}}_T), \\
(\mathcal{H}_T - \bar{\mathcal{E}}_T), \\
\left[ \left( \mathcal{H}_V + \frac{3}{2} \mathcal{E} r \Sigma_T \right) + \left( \bar{\mathcal{E}}_V - \frac{3}{2} \mathcal{E} r \bar{\zeta}_T \right) \right], \\
\left[ \left( \mathcal{H}_V + \frac{3}{2} \mathcal{E} r \Sigma_T \right) - \left( \bar{\mathcal{E}}_V - \frac{3}{2} \mathcal{E} r \bar{\zeta}_T \right) \right], \\
\bar{X}_V + \frac{3}{2} \mathcal{E} r \left[ \phi_T + \left( \Sigma - \frac{2}{3} \theta \right) \Sigma_T + \bar{\Pi}_T \right], \\
\mathcal{H}_S \end{array} \right\} \] \hspace{1cm} (34)

References

[1] Burston R B 2007 1+1+2 gravitational perturbations on LRS class II space-times: GEM vector harmonic amplitudes (gr-qc)
[2] Burston R B 2007 1+1+2 electromagnetic perturbations on LRS class II space-times: Decoupling vector and scalar harmonic amplitudes submitted to Class. Quantum Grav.
[3] Burston R B and Lun A W C 2007 1+1+2 electromagnetic perturbations on LRS space-times: Regge-Wheeler and Bardeen-Press equations submitted to Class. Quantum Grav.
[4] Burston R B 2007 1+1+2 gravitational perturbations on LRS class II space-times: Decoupling GEM tensor harmonic amplitudes submitted to Class. Quantum Grav.
[5] Clarkson C and Barrett R 2003 Class. Quantum Grav. 20 3855-84
[6] Regge T and Wheeler J 1957 Phys. Rev. 108 1063
[7] Price R H (1972) Phys. Rev. D 5 2439-54