Microscopic approach to a class of 1D quantum critical models

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Abstract
Starting from the finite volume form factors of local operators, we show how and under which hypothesis the $c = 1$ free boson conformal field theory in two-dimensions emerges as an effective theory governing the large-distance regime of multi-point correlation functions for a large class of one-dimensional massless quantum Hamiltonians. In our approach, in the large-distance critical regime, the local operators of the initial model are represented by well suited vertex operators associated to the free boson model. This provides an effective field theoretic description of the large distance behaviour of correlation functions in 1D quantum critical models. We develop this description starting from the first principles and directly at the microscopic level, namely in terms of the properties of the finite volume matrix elements of local operators.

Keywords: integrable models, correlation functions, critical phenomena

Introduction
It is widely believed that the spectrum of a quantum Hamiltonian is intimately related to certain overall properties of the large-distance asymptotic behaviour of its correlation functions. If, in the infinite volume $L$ limit, a given model’s spectrum exhibits a gap between a \textit{finitely} degenerated ground state and the tower of excited states above it, then the correlation functions are expected to decay exponentially fast in the distance of separation between the various operators involved in the correlator. This behaviour changes for gapless models, viz those whose ground state, in the $L \to +\infty$ limit, becomes directly connected to the continuum of excited states above it. For those models, the correlation functions are expected to decay algebraically in terms of the distance of separation. The powers of the distance which arise in this algebraic decay are called critical exponents. Models having the same values of...
their critical exponents are said to belong to the same universality class. In fact, it is believed that the features determining a given universality class are rather sparse in the sense that solely certain overall symmetries of the model should fix it [17]. Unfortunately, the lack of any exactly solvable and truly interacting many-body quantum critical model in dimensions higher than one did not allow, so far, for a direct and explicit check of these properties. However, extensive numerical data and experimental results do speak in favour of such an interplay [22].

The situation improves drastically in the case of one spatial dimension4, where a plethora of exactly solvable models arises: the Luttinger model (see [20] and references therein), $1 + 1$ dimensional conformal field theories [3, 12] or models that are solvable by one of the versions of the Bethe Ansatz [6] such as the celebrated XXZ spin-1/2 chain or the 1D Bose gas at arbitrary repulsive coupling. The abundance of exact solutions turns these one-dimensional models into a laboratory allowing one to test the universality principle in many concrete situations.

It is worth reminding that, in all these models, two scales coexist: a microscopic scale $\delta$ which is related to the lattice spacing or, more generally proportional to the inverse of the Fermi momentum, and a macroscopic scale $L$ corresponding to the volume or number of sites of the model. Scale invariance can only be realized at distances $\Delta x$ between local operators that range between these two scales but are far from them. This imposes that the ratio of the two scales $L/\delta$ is very large in such a way that there exist a whole range of distances verifying $\delta \ll \Delta x \ll L$. For $\Delta x$ close to either the microscopic or the macroscopic scales the scaling properties of the correlation functions get modified in drastic ways.

The very structure of a conformal field theory imposes a simple form for the transformation of its operators under scaling. In its turn, this imposes that the correlation function exhibit an algebraic in the distance pre-factor. In particular, two-point functions are purely algebraic. The exponents driving the power-law behaviour of a correlator are then constructed in terms of the scaling dimensions of the operators that are being averaged. The data issuing from conformal field theories can thus be used so as to provide one with quite explicit predictions for the large-distance decay of correlators in massless quantum models in one spatial dimension. In fact, there, one can be slightly more precise in respect to the two pillars on which these predictions build

- in the critical regime $\delta \ll \Delta x \ll L$, correlation functions should exhibit conformal invariance as argued by Polyakov [45]. This suggests that the leading contribution to the long-distance asymptotics should be reproduced by correlation functions of appropriate operators in a two-dimensional conformal field theory. Still, the identification of which conformal field theory is to be used and which are the ‘appropriate’ operators is left open at this stage.
- The $1/L$ corrections to the model’s ground and low-lying excited state’s energies contain the information on the central charge of the conformal field theory describing the asymptotics and the scaling dimensions of the tower of operators describing the correlation functions as argued by Cardy [7, 9].
- The choice of ‘appropriate’ operators, on the conformal field theory side, is done by advocating that these should inherit the symmetry that is satisfied by the correlation function one starts with.

To rephrase the above, Polyakov’s argument justifies why a conformal field theory should emerge as an effective large-distance theory in the domain $\delta \ll \Delta x \ll L$ of the model

4 In the language of classical statistical mechanics, this corresponds to the case of a two-dimensional model [50].
while Cardy’s observation permits one to extract, from the knowledge of the structure of a model’s excitations, the quantities which would characterize the effective conformal field theory describing the model’s large-distance regime.

A similar line of though is followed by exploiting the Luttinger liquid model as a tool for providing the critical exponents. Is is argued that a model belongs to the Luttinger liquid universality class if it has the same form of the low-lying excitations above its ground state. The parameters describing these excitations fix the Luttinger liquid model that is pertinent for the model of interest. The critical exponents of the model one starts with are then deduced from the ones computed explicitly for the Luttinger liquid model [18, 19, 37].

Independently of which of the above methods one chooses to employ, one needs to access to the $1/L$ corrections to the energy spectrum of a model’s Hamiltonian so as to be able to carry out effective predictions. The extraction of such corrections from the spectra of various quantum integrable models was initiated in the series of works [10, 11, 30–32] and led to the identification of the central charge and scaling dimensions, for numerous quantum integrable models. It is worth mentioning the work [15] where the $1/L$ corrections to the free energy of certain non-integrable perturbations of the two-dimensional Ising model were obtained, on rigorous grounds through constructive field theory methods. By using Cardy’s form of the $1/L$ corrections, this work demonstrated that, for sufficiently small perturbations, the model still has the same central charge $1/2$ as the Ising model.

Although effective in the sense that producing explicit answers, the above techniques are more of a list of prescriptions than a well argued from the first principles line of though that joins, argument after argument, the structure of a given microscopic model with the resur- gence of an effective field theoretic description in the large-distance regime. Several attempts have been made in the literature to bring some elements of rigour or, at least, some $ab\ initio$ justification of the principle. In a series of works [44, 46], by using the tools developed in [42], Polyakov argued the arisal of a universal behaviour on the basis of perturbative field theoretic calculations. Although constituting an important step forward, these reasonings did not allow for any control on the magnitude of the contributions that he dropped from his calculations on the basis of some hand-waving arguments, without mentioning the problems inherent to lacks of convergence in perturbative handling in quantum field theory. A rigorous approach allowing one to prove the power-law decay of certain two-point functions in interacting one-dimensional fermion models that are a sufficiently small perturbation of a free fermion model was first proposed by Pinson and Spencer [43], further developed by Mas-tropietro [38] and then generalized in [14] to the case of multi-point energy—energy correlation functions in certain non-integrable perturbations of the two-dimensional Ising model. An adaptation of this approach [4, 5] also allowed to establish the universality of certain properties of Luttinger liquid type for a class of sufficiently small perturbations of a free fermion model. The method relies on the possibility to provide a rigorous construction of the path integral for such models in finite volume and then study its infinite volume and scaling limits through rigorous renormalization group methods. Finally, we should also mention the approach for proving the universality of certain types of bi-dimensional Ising lattices—which are related to one-dimensional massless quantum Hamiltonians by means of the correspondence proposed in [50]—that was developed by Smirnov in [49]. Finally, we would also like to mention the derivation of the large-distance asymptotic behaviour of spin–spin correlation functions in the massless regime of the XXZ chain on the basis of Bethe Ansatz methods by the authors in collaboration with Kitanine et al [23]. The Bethe Ansatz approach was sub- sequently brought to a satisfactory level of rigour in [35] by one of the authors. The proof of the long-distance asymptotic behaviour solely relies on convergence hypothesis for certain auxiliary series of multiple integrals.
The present paper introduces a convenient microscopic description for the spectrum, space of states and matrix elements of local operators in a class of one-dimensional massless quantum models. Our setting allows us to construct, in the large-distance regime $\delta \ll \Delta x \ll L$, a one-to-one map (up to higher order corrections in the distance) between the relevant to the large-distance regime sub-space of the model’s Hilbert space and the free boson Hilbert space. The local operators of the original model are then represented in terms of a collection of vertex operators acting in the free boson Hilbert space. We should mention that our work takes its roots in the series of papers [25, 27, 28] where the authors, in collaboration with Kitanine, Slavnov and Terras, developed a form factor series based approach to the analysis of the long-distance and large-time asymptotic behaviour of correlation functions in quantum integrable models. The setting we introduce in the present paper, as shown in the works [24, 26, 36], is clearly verified for a large-class of quantum integrable models, the XXZ spin-1/2 chain being the most prominent example. However, we do trust that the overall hypothesis that we lay down for the structure of the model’s observable is quite universal and at least encompasses several instances of one-dimensional quantum liquids [16]. This is supported, e.g. by perturbative calculations around a free model [4, 5, 47]. Furthermore all objects that we handle are standard within the phenomenological approach to interacting Fermi systems [16, 39].

Going slightly deeper into the details of the physics’ jargon, the main assumptions on which our setting builds are that

- in the large-volume limit, the relevant part to the critical regime of the model’s spectrum is purely constructed in terms of particle–hole excitations\(^5\);
- the form factors—expectation values of local operators—taken between two states realized in terms of excitations that are all located in an immediate vicinity of the Fermi zone admit a structure descending from a local repulsion principle between the interacting momenta of the particle and hole building up the excited state.

The main result of the paper can be phrased as follows. Let $\mathcal{O}_1(x_1), \ldots, \mathcal{O}_r(x_r)$ be local operators located at at positions $x_1, \ldots, x_r$ and associated with some one-dimensional quantum model in finite (but large) volume $L$ and having the Hilbert space $\mathcal{H}_{\text{phys}}$. The operators $\mathcal{O}_i$ is assumed to induce solely transitions between spin sectors differing by some fixed integer $\sigma$. Then, in the large $\delta \ll |x_i - x_j| \ll L$ limit, the $r$-point ground-to-ground state expectation value of these operators satisfies

$$\langle \mathcal{O}_{i_1}(x_{i_1}) \cdots \mathcal{O}_{i_r}(x_{i_r}) \rangle_{\mathcal{H}_{\text{phys}}} \cong \langle \mathcal{E}_{i_1}(\omega_{i_1}) \cdots \mathcal{E}_{i_r}(\omega_{i_r}) \rangle_{\mathcal{H}_{\text{eff}}},$$

where the $\cong$ sign is to be understood as an equality up to the first leading correction arising in each oscillating harmonics in the spacing difference. The operator $\mathcal{E}_i$, $i = 1, \ldots, r$, appearing in the rhs of (0.1) act in the Hilbert space $\mathcal{H}_{\text{eff}}$ associated to the free boson model while the expectation value on the lhs of (0.1) is taken in the Hilbert space $\mathcal{H}_{\text{phys}}$ of the initial model; they are built up from the free boson vertex operators $\mathcal{V}(\nu, \kappa; \omega)$:

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\(^5\) In the following, for simplicity, we shall assume a purely particle–hole excitation spectrum. This allows one to lighten the discussions. We stress that this assumption is not a limitation in that, for instance, the bound states are expected to solely produce exponentially small corrections in the distance.
Formula acting in the free boson model \( \mathcal{C}(\omega_i) \) can be found in the core of the paper. We do stress that \( \omega_i \) is grasped by the effective operator \( \mathcal{O} \) acting in the \( \mathcal{H} \) and the values \( \nu_i \); \( \kappa \) taken \( \mathcal{C}(\omega_i) \) is the vertex operator; see (1.27), which connects states belonging to sectors differing by the charge \( \kappa \); \( \nu_i, \kappa, |\omega_i| \) are the values of the so-called shift function relatively to the ground state, see [2.29], and evaluated at the left \( (-q) \) or right \( (q) \) end of the Fermi zone. We do stress that it is a function of the charge \( \kappa \) and of the operator’s spin \( o_x \); \( \rho(\nu_i(q) - \alpha_i + \kappa) \) and \( \rho(\nu_i(-q) + \kappa) \) is the tower of scaling dimensions associated with the operator \( \mathcal{O} \); \( \mathcal{F}_\kappa(\mathcal{O}_i), \kappa \in \mathbb{Z} \), is the tower of typical form factors of the physical operator \( \mathcal{O}_i \) taken between the ground state and the lowest-lying excited state belonging to the \( \kappa \)-Umklapp sector in \( \mathcal{H}_{\text{phys}} \).

We refer the reader to section 3.3.1 where explicit examples for the above-listed quantities are given in the case of the fundamental local operators for the XXZ chain. In section 3.3.2 we show how the structure of the Luttinger-liquid critical exponents can be recovered within our formalism.

What equation (0.1) states is that, in the weak sense—i.e. for expectation values—and for large distances of separation all the essence of the physical operator \( \mathcal{O}_i(\omega_i) \) acting in the Hilbert space \( \mathcal{H}_{\text{phys}} \) is grasped by the effective operator \( \mathcal{C}(\omega_i) \) acting in the free boson model’s Hilbert space \( \mathcal{H}_{\text{eff}} \). Formula (0.2) provides an effective map, in the critical regime, between local operators of the physical models and vertex operators in the corresponding conformal field theory. This map is completely determined by the typical form factors \( \mathcal{F}_\kappa(\mathcal{O}_i), \kappa \in \mathbb{Z} \), of the physical operators \( \mathcal{O}_i(\omega_i) \) and the values \( \nu_i(\pm q) \) taken by the so-called shift function on the left/right endpoint of the Fermi zone. A more detailed description of the correspondence and the constituents of the operator \( \mathcal{C}_i \) can be found in the core of the paper. We do stress that the expectation value appearing in the rhs of (0.1) is trivial to estimates by means of the free fermion exchange relations, see (D.5) and (D.6) and that the correspondence (0.1) and (0.2) only provides one with the leading order to each oscillatory harmonics. This leading order only involves primary operators. However, should one be interested in sub-leading corrections then one would need to supplement the expansion (0.2) with additional terms that would involve descendants. Obtaining these terms is, in principle, possible within the method developed in the present paper, although would demand more computations.

On top of providing the microscopic origin of the appearance of a \( c = 1 \) conformal field theory as an effective theory in the large-distance regime, our setting supports the interpretation of the universality hypothesis that has been advocated by one of the authors in [35]. Namely, within our approach, the leading power-law contribution associated with a given oscillating harmonic in the large-distance asymptotic expansion of a correlator stems from a
saddle-point like contribution that is extracted from the form factor series expansion of the correlator. In the large-distance regime studied in the present paper, the saddle-point is located on the two ends of the Fermi zone. In the time-dependent case, one should as well incorporate all the possible critical points of the dressed momentum/energy combination $xp − te$, see [27]. It is the singular structure, viz local behaviour, of an operator’s form factor in the vicinity of the saddle-point that fixes the value of the critical exponents. Thus, two models belonging to the same universality class—in the sense that their appropriate correlators share the same critical exponents—have to share the same singularity structure of their form factors. Note that the regular part of the form factors can differ from one model to another in that they solely impact the value of the amplitudes, viz the non-universal part of the large-distance asymptotic expansion. Recall that the standard formulation of the universality hypothesis states that two models sharing the same overall symmetries share the same critical exponents. In this light, it would appear that the set of symmetries of a given model completely fixes the singularity structure of its form factors. However, it is not clear for us at the moment how one could derive the singular structure of a model’s form factors solely building on its symmetries. Nonetheless, it seems more satisfactory to us to think of universality in terms of classes of models sharing the same singular structure of their form factors. Indeed, this criterion is precise and can be checked, in certain cases, by explicit calculations, say in a perturbative regime. The matter is that there is, a priori, no criterion allowing one to say that one has identified all the symmetries of a model that are pertinent for fixing its universality class. It could well be that two models have apparently the same symmetry structure but that one has missed some important yet quite oblivious symmetry of one of the models that would imply that, in fact, the models belong to two distinct universality classes. It is worth, in this respect, to remind that two apparently very similar models, the 2D Ising and the the eight-vertex model in its lattice model formulation belong to fundamentally distinct universality classes.

The paper is organized as follows. The present section is the introduction. In section 1 we recall the free fermion based description of the space of states of the free boson model. We also present new formulae for the expectation values of exponents of specific current operators. In section 2 we present the general framework—properties of the model’s spectrum and form factors—that allow us to derive the correspondence with the free boson model. Finally, in section 3, we establish the main result of the paper, namely the set of formulae (0.1) and (0.2) that appeared in the introduction. The paper contains four appendices. In appendix A, we briefly review the special functions which arise in the course of our study. In appendix B we compute some two-dimensional integrals which are of interest to the problem. In appendix C, we prove proposition 1.1. Finally, in appendix D, we show how one can recover, starting from the formalism developed in the present paper, the value of the multipoint restricted sums that were first introduced in [28].

1. A free fermion description of the free boson model

In the present section, we recall the free fermion based construction of the space of states for the free boson model. Our presentation basically follows the notations and conventions that can be found in the excellent review paper [1]. The various results found in this review originate from a long series of developments which started with the cornerstone work of Kyoto’s school (Jimbo, Miwa and Sato) in the late ’70s on holonomic quantum fields [21].

See also [40] for some later developments of the method.
1.1. Overall definitions and generalities

1.1.1. The space of states. We consider a set of fermionic operators $\{\psi_n^a\}_{n \in \mathbb{Z}}$ and their * associates $\{\psi_n^a\}_{n \in \mathbb{Z}}$ which satisfy the anti-commutation relations

$$\{\psi_n, \psi_m\} = \{\psi_n^a, \psi_m^a\} = 0 \quad \text{and} \quad \{\psi_n, \psi_m^a\} = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol. We also assume the existence of a vacuum vector $|0\rangle$ which satisfies the properties

$$\psi_n |0\rangle = 0 \quad \text{for} \quad n < 0 \quad \text{and} \quad \psi_n^a |0\rangle = 0 \quad \text{for} \quad n \geq 0.$$  

(1.2)

The dual vacuum $\langle 0 |$ fulfills the analogous properties

$$\langle 0 | \psi_n^a = 0 \quad \text{for} \quad n < 0 \quad \text{and} \quad \langle 0 | \psi_n = 0 \quad \text{for} \quad n \geq 0.$$  

(1.3)

Starting from the vacuum (resp. the dual vacuum) one constructs vectors (resp. dual vectors) through a repetitive action of the fermion operators. Such vectors are parametrized, in a natural way, by the sets

$$\mathcal{J}_{n; m} = \left\{ \{p_n\}_1^{p_n}; \{h_n\}_1^{h_n} \right\}$$  

(1.4)

consisting of two collections of ordered integers $1 \leq p_1 < \cdots < p_{p_n}$ and $1 \leq h_1 < \cdots < h_{h_n}$. It is convenient, for a deeper physical insight, to think of the integers $\{p_n\}_1^{p_n}$ as a labelling of the particle-like excitations and to think of the integers $\{h_n\}_1^{h_n}$ as a label for the hole-like excitations. To be more precise, to each set $\mathcal{J}_{n; m}$ defined as above, one associates the vector

$$|\mathcal{J}_{n; m}\rangle = \psi_{-h_1}^a \cdots \psi_{-h_{h_n}}^a \cdot \psi_{p_{p_n}-1} \cdots \psi_{p_{p_n}-1}^a |0\rangle,$$

(1.5)

and the dual vector

$$\langle \mathcal{J}_{n; m} | = \langle 0 | \psi_{p_{p_n}-1}^a \cdots \psi_{p_{p_n}-1}^a \cdot \psi_{-h_1} \cdots \psi_{-h_{h_n}}^a.$$  

(1.6)

The Hilbert space $\mathfrak{h}$ of the model is then defined as the span of the vectors introduced above

$$\mathfrak{h} = \text{span} \left\{ |\mathcal{J}_{n; m}\rangle \text{ with } n_p, n_h \in \mathbb{N} \text{ and } 1 \leq p_1 < \cdots < p_{p_n}, 1 \leq h_1 < \cdots < h_{h_n}, p_n, h_n \in \mathbb{N}^* \right\}.$$  

(1.7)

Note that, when the number of particle and hole-like integers coincide, i.e. $n_p = n_h = n$, one can identify the set $\mathcal{J}_{n; m}$ with a Young diagram. The one-to-one map is obtained by interpreting the integers $\{\{p_n\}_1^{p_n}; \{h_n\}_1^{h_n}\}$ as the Frobenius coordinates of the Young diagram.

Such an identification is also possible when $n_p \neq n_h$. The bijection is, however, more complicated. We refer the interested reader to the proof of lemma 1.1 where one can find all of the necessary details for constructing such a bijection. Note that such a bijection was constructed for the first time, although indirectly, in appendix A.3 of [25].

The Hilbert space $\mathfrak{h}$ also enjoys of another basis $|\mathcal{Y}; \ell\rangle$ which makes the connection with a Young diagram $\mathcal{Y}$ explicit. This new basis singles out the collection of the so-call vacuum and dual vacuum states having a prescribed charge $\ell$:

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6 We adopt the slightly unusual convention where the origin of the Frobenius coordinates is the diagonal so that these start from $1$. 
\[ |\ell\rangle = \begin{cases} \psi_{\ell-1} \cdots \psi_0 |0\rangle & \ell > 0 \\ \psi_{\ell} \cdots \psi_{\ell-1} |1\rangle & \ell < 0 \end{cases} \quad \text{and} \quad \langle \ell | = \begin{cases} \langle 0 | \psi_{\ell-1}^* \cdots \psi_0^* & \ell > 0 \\ \langle 0 | \psi_{\ell}^* \cdots \psi_{\ell-1}^* & \ell < 0 \end{cases}. \]  

(1.8)

General states of this basis are built as equal in number particle–hole excitations over the vaccua \(|\ell\rangle\), resp. their dual vaccua \(\langle \ell |\). Let

\[ Y = \{ \{ \alpha_i \}^n_{i=1} : \{ \beta_i \}^n_{i=1} \} \quad \text{with} \quad \begin{cases} 1 \leq \alpha_1 < \cdots < \alpha_n \\ 1 \leq \beta_1 < \cdots < \beta_n \end{cases} \]  

be the Frobenius coordinates of the Young diagram \(Y\). The second basis we mentioned above takes the form

\[ |Y; \ell\rangle = \psi_{\ell-\beta_1}^* \cdots \psi_{\ell-\beta_n}^* \psi_{\ell+\alpha_1-1} \cdots \psi_{\ell+\alpha_n-1} |\ell\rangle, \]

\[ \langle \ell; Y | = \langle \ell | \psi_{\ell+\alpha_1-1} \cdots \psi_{\ell+\alpha_n-1}^* \psi_{\ell-\beta_1} \cdots \psi_{\ell-\beta_n}^*. \]  

(1.10)

In fact, one can even consider mixtures of the basis \(|Y; \ell\rangle\) and \(|\mathcal{J}_{n_p n_h}\rangle\), namely the basis

\[ |\mathcal{J}_{n_p n_h}; \ell\rangle = \psi_{\ell-h_1} \cdots \psi_{\ell-h_n} \cdot \psi_{p_1+t-1} \cdots \psi_{p_1+t-1} |\ell\rangle. \]  

(1.11)

It is easy to convince oneself that

\[ \text{Span}\{|Y; \ell + r\rangle : Y \text{ Young diagram} \} = \text{Span}\{|\mathcal{J}_{n_p n_h}\rangle : \text{ sets } \mathcal{J}_{n_p n_h} n_p - n_h = \ell + r \}
\]

\[ = \text{Span}\{|\mathcal{J}_{n_p n_h}'; \ell\rangle : \text{ sets } \mathcal{J}_{n_p n_h} n_p - n_h = r \}. \]  

(1.12)

1.1.2. The space of operators. The fermionic operators \(\psi_j\) and \(\psi_j^*\) can be thought of as building bricks allowing one to construct more general operators on the Hilbert space \(\mathcal{H}\). The simplest class of operators is obtained as a linear combination in the fermions:

\[ v = \sum_m V_m \psi_m \quad \text{resp.} \quad v^* = \sum_m W_m \psi_m^*, \]  

(1.13)

where \(V_m, W_m\) are bounded sequences. The field and conjugated field operators

\[ \Psi(z) = \sum_{j \in \mathbb{Z}} \psi_j z^j \quad \text{and} \quad \Psi^*(z) = \sum_{j \in \mathbb{Z}} \psi_j^* z^{-j} \]  

are an archetype of such operators.

The second important class of operators is obtained by means of a bilinear pairing of the fermions realized by an infinite matrix \(A\) having bounded entries:

\[ \sum_{i,j \in \mathbb{Z}} A_{ij} \psi_i \psi_j^*. \]  

(1.15)

Expressions of the type (1.13) or (1.15) may look a bit formal in particular due to convergence issues of the sums. However, all the sums of interest truncate as soon as one computes matrix elements taken between the fundamental system of basis vectors labelled by the sets \(\mathcal{J}_{n_p n_h}\). It is in this sense that all of the above expressions should be understood.

The most fundamental example of an operator built through a bilinear pairing is the charge operator which takes the explicit form

\[ J_0 = \sum_{k>0} \psi_k \psi_k^* - \sum_{k<0} \psi_k^* \psi_k. \]  

(1.16)
It is readily seen that the vector \( \phi_{n_p - n_h} \) is associated with the eigenvalue \( n_p - n_h \) of the charge operator \( J_0 \). We will sometimes say that the vector \( \phi_{n_p - n_h} \) has charge \( n_p - n_h \).

In other words, the charge operator induces a grading of the Hilbert space \( \mathcal{H} \) into the direct sum of spaces \( \ell \mathcal{H} \) having a fixed charge \( \ell \) with \( \phi_{n_p - n_h} \).

The current operators provide one with another example of important operators that are bilinear in the fermions. To each half-infinite sequence \( \{ t_k \} \) one associates the current operators

\[
J_k(t) = \sum_{k \geq 1} t_k \cdot J_{-k}, \quad J_k = \sum_{j \in \mathbb{Z}} \psi_j \bar{\psi}_{j+k} \quad \text{for} \quad k \neq 0.
\]

It is readily checked that the components of the current operator satisfy the algebra

\[
\left[ J_k, J_l \right] = k \delta_{k,-l}.
\]

These commutation relations allow one to interpret the currents \( J_k \) as bosonic operator modes (see e.g. [1]). Furthermore since, for \( k \in \mathbb{N}^* \), one has

\[
\left( J_k \right) |0\rangle = 0 \quad \text{and} \quad \langle 0 | J_{-k} = 0
\]

it follows that one can interpret \( \{ J_k \} \) as bosonic creation operators while \( \{ J_{-k} \} \) as the bosonic annihilation operators. It is on the basis of such an observation that the construction of the free boson field theory is made in the free fermion model we have introduced so far.

It is also straightforward to check the commutation relations

\[
\left[ J_k, \psi_{-l} \right] = \psi_{l-k}, \quad \left[ J_k, \bar{\psi}_{l} \right] = -\bar{\psi}_{l+k}.
\]

The above commutation relations can be conveniently encoded on the level of the field and conjugated field operators. As a matter of fact, then, they take the form of the following exchange relations

\[
\Psi(z) \cdot e^{J_1(t)} = e^{-\xi(t, z)} \cdot e^{J_1(t)} \cdot \Psi(z) \quad \text{and} \quad \Psi^*(z) \cdot e^{J_1(t)} = e^{\xi(t, z)} \cdot e^{J_1(t)} \cdot \Psi^*(z)
\]

where \( \xi(t, z) = \sum_{k \geq 1} t_k \cdot z^k \). Given a generic sequence \( t \), this is the most compact form of the exchange relation since the sum defining \( \xi \) cannot be computed explicitly. However, for the specific choice \( t = t_\pm \) where

\[
(t_+)_k = -\frac{\nu \omega^{-k}}{k} \quad \text{and} \quad (t_-)_k = \frac{\nu \omega^k}{k}
\]

the sum reduces to the Taylor expansion of the logarithm. It is the \( t = t_\pm \) choice that plays a particularly important role in our analysis in that it is associated with the construction of vertex operators. We therefore introduce a special notation for the current operators associated with the parameters \( t_\pm \), namely

\[
\mathcal{J}_{\pm}(t, \omega) \equiv J_k(t_\pm) = \mp \nu \sum_{k \geq 1} \frac{\omega^{-k}}{k} \cdot J_{-k}.
\]
In the case of the operators $\mathcal{J}_J(\nu, \omega)$, the exchange relations (1.22) particularize to

$$\Psi(z) \cdot e^{\mathcal{J}_J(\nu, \omega)} = \left( 1 - \left( \frac{e^{-i\nu \omega}}{\omega} \right)^{\pm \nu} \right) e^{\mathcal{J}_J(\nu, \omega)} \cdot \Psi(z) \quad \text{and}$$

$$\Psi^\ast(z) \cdot e^{\mathcal{J}_J(\nu, \omega)} = \left( 1 - \left( \frac{e^{i\nu \omega}}{\omega} \right)^{\pm \nu} \right) e^{\mathcal{J}_J(\nu, \omega)} \cdot \Psi^\ast(z). \quad (1.25)$$

There is also another operator that plays a role in our setting, the so-called shift operator which we shall denote as $e^P$. The shift operator maps $h_\ell$ onto $h_{\ell'}$. In fact, the operator $P$ arising in the exponent is the conjugate operator to $J_0$. Although the shift operator has no simple expression in terms of the fermions, it takes a particularly simple form in the basis of $h$ subordinate to Young diagrams, see (1.10). Indeed, any integer power $e^P$ of the shift operator satisfies $e^P |\Psi; \ell\rangle = |\Psi; \ell' + r\rangle$, viz

$$e^P = \sum_{\Psi, \ell} |\Psi; \ell + r\rangle \langle \ell; \Psi|, \quad (1.26)$$

where the sum runs over all integers $\ell$ and all Young diagrams $\Psi$ (see [1] for more details).

We end this sub-section by introducing the $r$-shifted bosonic vertex operators which play a central role in the correspondence:

$$\mathcal{V}(\nu, r|\omega) = e^{\mathcal{J}_J(\nu + r, \omega)} \cdot e^{\mathcal{J}_J(\nu, \omega)} \cdot e^{rP}. \quad (1.27)$$

**1.1.3. The Wick theorem.** In this sub-section we review the statement of Wick’s theorem in the case of an insertion of group-like elements. This name refers to a class of operators $G$ satisfying to the so-called basic bilinear condition:

$$\sum_{j \in \mathbb{Z}} \left( G \cdot \psi_j \right) \otimes \left( G \cdot \psi_j^\ast \right) = \sum_{j \in \mathbb{Z}} \left( \psi_j \cdot G \right) \otimes \left( \psi_j^\ast \cdot G \right). \quad (1.28)$$

Group-like elements single out, among other things, because they allow one for a powerful generalization of the Wick theorem which we shall recall at the end of this sub-section.

The operators we have introduced so far provide several examples of group-like elements. For instance, it is easily checked that the operators $e^{\mathcal{J}_J(\nu)}$ in general, $e^{\mathcal{J}_J(\nu, \omega)}$ in particular, and $e^P$ all satisfy to the basic bilinear condition. It is likewise trivial to establish that, if $G, G'$ are group-like elements, then so is their product $G \cdot G'$. These facts ensure that that the $r$-shifted vertex operators $\mathcal{V}(\nu, r|\omega)$ are group-like elements.

We stress that group-like elements need not to be invertible. For instance, given any group element $G$ and a collection $v_1, \ldots, v_n$, resp. $m_1^\ast, \ldots, m_n^\ast$, of linear combinations of the basic fermion operators $\psi_j$, resp. $\psi_j^\ast$, viz

$$v_\ell = \sum_m W_{km} \psi_m \quad \text{resp.} \quad m_\ell^\ast = \sum_m W_{km} \psi_m^\ast \quad (1.29)$$

the operators

$$G_{v,m} = m_1^\ast \cdots m_n^\ast v_n \cdots v_1 \cdot G \quad \text{and} \quad \tilde{G}_{v,m} = G \cdot m_1^\ast \cdots m_n^\ast v_n \cdots v_1 \quad (1.30)$$

are also a group-like. We shall not enter deeper into the detail here and refer the reader to [1] for more details.

Group-like elements allow for the following generalization of Wick’s theorem.
Theorem 1.1. Let $v_j$, resp. $w_j$, $j = 1, ..., n$ be any operators linear in the fermions and $G, G'$ group-like elements. Then the associated expectation value admits a determinant representation:

$$
\frac{\langle 0 | G w_1^j \cdots w_n^j v_n \cdots v_1 | G' | 0 \rangle}{\langle 0 | G \cdot G' | 0 \rangle} = \text{det}_\nu \left[ \frac{\langle 0 | G w_1^j \cdot v_j G' | 0 \rangle}{\langle 0 | G \cdot G' | 0 \rangle} \right].
$$

(1.31)

1.2. Expectation values of exponents of current operators

In the present sub-section, we compute the expectation value of the operators $e^{\mathcal{J} \nu, \omega}$ between any two states $|\mathcal{J}_{n_p, n_k}\rangle$, $|\mathcal{J}_{n_r, n_l}\rangle$ of the basis. The corresponding result will provide a first brick towards the correspondence with the free boson model that will be built in section 3.

Prior to stating the result, we first need to introduce a few handy notations. Given two sets of integers $\mathcal{J}_{n_p, n_k}$, $\mathcal{J}_{n_r, n_l}$

$$
\mathcal{J}_{n_p, n_k} = \{ p_a^k \}; \quad \mathcal{J}_{n_r, n_l} = \{ r_a^l \}
$$

one defines the set functions

$$
\varpi \left( \mathcal{J}_{n_p, n_k}; \mathcal{J}_{n_r, n_l} | \nu \right) = \prod_{a=1}^{n_p} \left\{ \prod_{b=1}^{n_k} \left( \frac{1 - k_b - h_a + \nu}{t_b - h_a + \nu} \right) \right\} \times \prod_{a=1}^{n_r} \left\{ \prod_{b=1}^{n_l} \left( \frac{p_b + t_b + \nu - 1}{p_b - k_b + \nu} \right) \right\}
$$

(1.33)

and

$$
\mathcal{D} \left( \mathcal{J}_{n_p, n_k} | \nu, \omega \right) = \left( \frac{\sin [\pi \nu]}{\pi} \right)^{n_k} \cdot \prod_{a=1}^{n_p} \left\{ \omega^{p_a - h_a} \Gamma \left( \frac{p_a + \nu}{p_a} \right) \right\} \cdot \prod_{a=1}^{n_r} \left\{ \omega^{h_a - h_a} \Gamma \left( \frac{h_a - \nu}{h_a} \right) \right\} \times \prod_{a > b}^{n_p} \left( p_b - p_a \right) \cdot \prod_{a > b}^{n_r} \left( h_b - h_a \right) \cdot \prod_{a=1}^{n_p} \prod_{b=1}^{n_r} \left( p_a + h_b - 1 \right).
$$

(1.34)

These functions appear as the building blocks of the so-called discrete form factor

$$
\mathcal{F} \left( \mathcal{J}_{n_p, n_k}; \mathcal{J}_{n_r, n_l} | \nu, \omega \right) = \left( -1 \right)^{n_p + n_k} \cdot \left( -1 \right)^{n_p - n_k} \cdot \sin [\pi \nu] \cdot \mathcal{D} \left( \mathcal{J}_{n_p, n_k} | \nu, \omega \right) \times \mathcal{D} \left( \mathcal{J}_{n_r, n_l} | \nu, \omega^{-1} \right) \cdot \varpi \left( \mathcal{J}_{n_p, n_k}; \mathcal{J}_{n_r, n_l} | \nu \right).
$$

(1.35)

In (1.35) we made use of the hypergeometric-like notations for ratios of $\Gamma$-functions, see appendix A.

It is the above discrete form factor that enters in the description of the expectation values of the current operators $e^{\mathcal{J} \nu, \omega} e^{\mathcal{J} \nu, \omega}$ taken between the states $|\mathcal{J}_{n_p, n_k}\rangle$ and $|\mathcal{J}_{n_r, n_l}\rangle$.

Proposition 1.1. Let $\mathcal{J}_{n_p, n_k}$ and $\mathcal{J}_{n_r, n_l}$ be as given in (1.32). Then form factor of the product of exponents of current operators admits the following representation

$$
\langle \mathcal{J}_{n_p, n_k} | e^{\mathcal{J} \nu, \omega} e^{\mathcal{J} \nu, \omega} | \mathcal{J}_{n_r, n_l} \rangle = \delta_{n_p - n_k - n_l} \cdot \mathcal{F} \left( \mathcal{J}_{n_p, n_k}; \mathcal{J}_{n_r, n_l} | \nu, \omega \right).
$$

(1.36)
Note that the Kronecker symbol in (1.36) is there to emphasize that the form factor is zero unless \( n_p - n_q = n_k - n_l \). This is due to the fact that the exponents of current operators preserve the charge of a state, viz can only connect states belonging to the same charge sector.

The above statement is well known in the case \( n_k = 0 \). Indeed, then, the representation (1.36) follows, for instance, from the Giambrelli determinant representation for Schur functions along with the possibility to compute explicitly the Schur functions \( s_{\alpha, \beta}(t_{\omega}) \) associated with hook diagrams \((\alpha, \beta)\) in Frobenius notations (see appendix A of [1] for more details). We do however stress that the general case given above is, to the best of our knowledge, new. In particular, the proof of the general case relies on new ideas not related to the theory of Schur functions. The proof, being slightly technical, is postponed to appendix C.

We would also like to point out that proposition 1.1 yields a new type of explicit representation for the skew Schur functions associated with the parameters \( t_{\omega} \).

In order to identify the emergence of an effective field theory description in the large-distance spacing between the operators, one needs to determine the matrix elements of the general \( r \)-shifted vertex operators that have been defined in (1.27). These will be given in theorem 1.2 below and can be deduced from (1.36).

**Theorem 1.2.** The form factor of the \( r \)-shifted vertex operator \( \mathcal{V}(\nu, r | \omega) \) reads

\[
\langle \mathcal{J}_{n; \omega} \rangle \mathcal{V}(\nu, r | \omega) \rangle \mathcal{J}_{n; \omega} = (-1)^{\frac{r(r+1)}{2}} \cdot \frac{\delta_{n_p-n_q-n_l-n_k+r}}{\delta_{(\omega)^{r(r+1)}}} \cdot G \left( \frac{1 - \nu}{1 - \nu - r} \right) \cdot \mathcal{F}(\mathcal{J}_{n; \omega} | \mathcal{J}_{n; \omega} | \nu, \omega). \tag{1.37}
\]

The symbol \( G \) appearing above stands for the hypergeometric-like notation for the ratio of two Barnes functions, see appendix A.

The proof of the theorem builds, in particular, on the form of the action of the \( r \)-shifted shift operator \( e^{i \theta} \) on the states \( \langle \mathcal{J}_{n; \omega} \rangle \). We shall determine the form of this action in lemma 1.1 given below. Still, first of all, we need to introduce some notations relative to so-called \( r \)-translates \( \mathcal{J}(r)_{n; \omega} \) of the set \( \mathcal{J}_{n; \omega} \).

Given a set \( \mathcal{J}_{n; \omega} \), we define \( q, 0 \leq q \leq r \), to be the unique integer such that

\[
1 \leq t_1 < \cdots < t_q < r < t_{q+1} < \cdots < t_{n}. \tag{1.38}
\]

Then, the set \( \{ \tilde{t}_a \}_{a=1}^{q} \) stands for the complement of the set \( \{ r + 1 - t_a \}_{a=1}^{q} \) in \( \| n \| \), namely

\[
1 \leq \tilde{t}_1 < \cdots < \tilde{t}_{r-q} < r \quad \text{and} \quad \{ r + 1 - t_a \}_{a=1}^{q} \cup \{ \tilde{t}_a \}_{a=1}^{r-q} = \| n \|. \tag{1.39}
\]

The \( r \)-translate \( \mathcal{J}(r)_{n; \omega} \) of the set \( \mathcal{J}_{n; \omega} \) corresponds to the set

\[
\mathcal{J}(r)_{n; \omega} = \{ \{ k_a \}_{a=1}^{n} : \{ t_a \}_{a=1}^{n} \}
\]

The sequence \( k_a \) appearing in (1.40) reads

\[
k_a = \tilde{t}_a \quad \text{for} \quad a = 1 \ldots r - q \quad \text{and} \quad k_{a+r-q} = k_a + r \quad \text{for} \quad a = 1 \ldots n_k \tag{1.41}
\]
and has a total of \( n_i' = n_k + r - q \) elements while the sequence \( t_i' \) is defined as
\[
t_i' = t_{a+q} - r \quad \text{for} \quad a = 1, \ldots, n_i - q
\]
and has a total of \( n_i' = n_i - q \) elements.

**Proof.** (Proof of theorem 1.2) Let \( r \geq 0 \). Then the representation (1.37) follows from the form taken by the action of the shift operator (1.45) on the states \( |J_{n,n} \rangle \) and from the identity
\[
(-1)^{n_i(t_i)} \cdot |J_{n,n} \rangle \cdot \mathcal{F}(J_{n,n}, J_{n,n}^{(r)}) \cdot \mathcal{F}(J_{n,n}, |r, \omega \rangle) = G(1 + r - \nu) \cdot (-1)^{n_i(t_i)} \cdot |r, \omega \rangle \cdot \mathcal{F}(J_{n,n}, |r - r - n_i, \omega \rangle).
\]

There \( q \) is as defined in (1.38) while \((-1)^{n_i(t_i)}\) refers to the signature of the permutation introduced in (1.46).

The formula (1.37) holds, in fact, for \( r < 0 \) as well. To establish this fact, observe that the whole construction is invariant under the transformation \( |J_{n,n} \rangle \leftrightarrow |J_{n,n}^{(r)}| \). It is easy to see by using the properties of the states and operators involved that, under this transformation, one has
\[
\langle J_{n,n} | \mathcal{F}(r, |J_{n,n} \rangle) \leftrightarrow (-1)^{n_i(t_i)} \cdot \mathcal{F}(J_{n,n}, |r, \omega \rangle) \cdot \langle J_{n,n} | \mathcal{F}(r, -r, |J_{n,n} \rangle).
\]

After some algebra, one can convince oneself that the rhs of (1.37) satisfies to the same transformations.

**Lemma 1.1.** For any \( r \geq 0 \), it holds
\[
e^{i \varphi} \cdot |J_{n,n} \rangle = (-1)^{n_i(t_i)} \cdot (-1)^{n_i(t_i)} \cdot (-1)^{\frac{r(r+1)}{2}} \cdot |J_{n,n} \rangle.
\]

Above, \( \sigma_\ell(t_i) \) corresponds to the permutation
\[
\sigma_\ell(t_i) \cup \{0, \ldots, r \} \rightarrow r - \ell + 1, \ldots, r - 1, 1, \ell, \ldots.
\]

\((-1)^{n_i(t_i)}\) stands for its signature, while \( q \) and \( \{t_i\} \) are as defined in (1.38) and (1.39).

**Proof.** We shall prove the lemma in the case when \( n_k - n_i \equiv \ell \geq 0 \). The \( \ell < 0 \) case follows from a similar reasoning so that its proof is left as an exercise. Let \( s \in \mathbb{N} \); \( \ell \) be the unique integer such that
\[
1 \leq k_1 < \cdots < k_s \leq \ell < k_{s+1} \leq \cdots < k_n.
\]

We then introduce the complementary sequence \( \{k_s\} \) to \( \{k_i\} \). The sequence \( \{k_s\} \) is such that
\[
1 \leq k_s < \cdots < k_{s-s} \leq \ell \quad \text{and} \quad \{k_s\} \cup \{\ell + 1 - k_s\} = \mathbb{N} \cup \{1, \ell \}.
\]
The sequence \( \{ \alpha_a \}_{n}^{n-s} \) is then defined as
\[
\alpha_a = k_{t+a} - \ell \quad \text{for} \quad a = 1, \ldots, \ell - s
\]
and the sequence \( \{ \beta_a \}_{n}^{n-s} \) as
\[
\beta_a = k_a \quad \text{for} \quad a = 1, \ldots, \ell - s \quad \text{and} \quad \beta_{t-a} = t_a + \ell \quad \text{for} \quad a = 1, \ldots, n_t.
\]

Note that there is, in total, \( n_s = n_k - s \) labels for the \( \beta \)'s due to the condition \( n_t = n_k - \ell \).

Finally, for convenience, we introduce the sequence \( \{ \kappa_a \}_{n}^{n-s} \) which we define as
\[
\kappa_a = \ell + 1 - \tilde{k}_{t-a+1-a}, \quad \text{so that one has the ordering}
\]
\[
1 \leq k_1^s < \cdots < k_{\ell-s}^s \leq \ell.
\]

Defining \( \Pi(\{ \kappa_a \}) \) as the permutation
\[
\Pi(\{ \kappa_a \}): 1, \ldots, \ell \mapsto k_1, \ldots, k_{\ell-s}, k_{\ell-s+1}, \ldots, k_{\ell},
\]
one has
\[
\Psi_{-1} \cdots \Psi_0 \, |0\rangle = (-1)^{\frac{\ell(\ell-1)}{2}} \Pi(\{ \kappa_a \}) \cdot \Psi_{k_{t-1}} \cdots \Psi_{k_{t-1}} \cdot \Psi_{k_{t-1}} \cdots \Psi_{k_{t-1}} |0\rangle
\]
\[
= (-1)^{\frac{\ell(\ell-1)}{2}} (-1)^{\ell(n_1+n_0)(\ell-s)} \cdot |J_{n_1,n_0}\rangle.
\]

We now recast the vector \( |\Upsilon; \ell + r\rangle \) where the partition \( \Upsilon \) is defined in terms of exactly the same Frobenius coordinates. Permuting the fermionic operators labelled by the indices \( \{ r + 1, \ldots, r + \ell \} \) and \( \{ 1, \ldots, r \} \) we are led to
\[
|\Upsilon; \ell + r\rangle = (-1)^{\Pi(\{ \kappa_a \})} (-1)^{\frac{\ell(\ell-1)}{2}} (-1)^{\frac{r(r-1)}{2} + \sigma_r(\{ t_a \})}
\]
\[
\times \Psi_{r+k_{t-1}} \cdots \Psi_{r+k_{t-1}} \cdot \Psi_{r+k_{t-1}} \cdots \Psi_{r+k_{t-1}}
\]
\[
\times \Psi_{r+k_{t-1}} \cdots \Psi_{r+k_{t-1}} \cdot \Psi_{r+k_{t-1}} \cdots \Psi_{r+k_{t-1}}
\]
\[
= (-1)^{\Pi(\{ \kappa_a \})} (-1)^{\frac{\ell(\ell-1)}{2}} (-1)^{\frac{r(r-1)}{2} + \sigma_r(\{ t_a \})}
\]
\[
\times (-1)^{n_1+n_0(\ell-s)+q(n_1+n_0-q)} \cdot |J_{n_1,n_0}^{(r)}\rangle.
\]

The action (1.45) follows straightforwardly from the above and from the representation (1.26) of the shift operator.
2. The model

In the present section, we describe the setting in which we assume the one-dimensional quantum Hamiltonian to fit in. More precisely, we describe the assumptions we make on its spectrum, its states and on certain general features satisfied by the form factors of the model’s local operators. Such a description is standard for quantum integrable systems in one-dimension, see e.g. [8] in what concerns the spectrum part and the papers [24, 26] in what concerns the form factors. The validity of the picture has been also established, on a perturbative level, in [47]. We thus believe that this setting is common, in particular, to all models belonging to the universality class of the Luttinger model.

2.1. General hypothesis on the spectrum

We consider a one-dimensional quantum Hamiltonian $\mathcal{H}$ representing a physical system in finite volume $L$. The volume $L$ represents, for instance, the number of sites in the case of a lattice model or the overall volume of the occupied space in the case of a model already formulated in the continuum.

We shall assume that the eigenstates of the Hamiltonian can be organized into sectors with a fixed bare-particle number $N'$. In practical situations, the integer $N'$ may be related to the total longitudinal spin of a state (in the case of spin chains with total longitudinal spin conservation) or may simply correspond to the number of bare particles building up the many body eigenstate of $\mathcal{H}$ (in the case of models enjoying a conservation of the total number of bare particles).

In such a setting, the ground state of the model $|\Psi_{L,N}\rangle$ is located within the sector with $N$ bare-particles. This number does depend on $L$ and is such that, in the thermodynamic limit $L \to +\infty$, one has $\lim_{L \to +\infty} (N/L) = D > 0$. Furthermore, taking the thermodynamic limit restricts the space of states to the sector corresponding to excitations having a finite, when $L \to +\infty$, energy relatively to the ground state. We assume that the eigenstates having this property are located in sectors with $N'$ bare particles where $N'$ is such that the difference $s = N' - N$ remains finite in the thermodynamic limit. Having in mind the CFT-based interpretation of the integer $s$, just as the spin-chain setting, we shall refer to $s$ as the spin of the excited state.

We further assume that the excited states are only built up from particle–hole excitations. This means that we assume that it is possible to label the eigenstates, within each sector built up from $N + s$ bare-particles, by a set of integers

\[
I_n^{(s)} = \{p_n^{(s)}\}_{1}^{\ell}; \quad \{h_n^{(s)}\}_{1}^{\ell}
\]

containing two collections of integers which label the so-called particle $\{p_n^{(s)}\}_{1}^{\ell}$ and hole $\{h_n^{(s)}\}_{1}^{\ell}$ excitations. In this parametrization, the integer $n$ may run through $0, 1, \ldots, N + s$ while:

\[
p_1^{(s)} < \cdots < p_n^{(s)} \quad \text{and} \quad h_1^{(s)} < \cdots < h_n^{(s)}
\]

\[
\left\{ \begin{array}{l}
p_n^{(s)} \in [ -M_{1}^{(a)}; M_{2}^{(a)} ] \setminus [ 1; N + s ] \\
h_n^{(s)} \in [ 1; N + s ]
\end{array} \right.
\]

The precise values of the integers $M_{1}^{(a)}$ defining the range of the $p_n$’s vary from one model to another. Typically for models having no upper bound on their energy, one has $M_{1}^{(a)} = +\infty$ while for model having an upper bound, $M_{1}^{(1)}, M_{1}^{(2)}$ are both finite but such that $M_{1}^{(a)} - N$, $a = 1, 2$, both go to $+\infty$ sufficiently fast with $L$. According to this setting, we shall denote the eigenstates of the model as $|I_n^{(s)}\rangle$. 
The vectors \( |I_n^{(i)}\rangle \) provide one with the so-called microscopic description of the model, namely a complete parametrization of the space of states in terms of discrete integers. However, it is the macroscopic description that is pertinent for describing the thermodynamic limit of the observables in the model. This macroscopic description arises by means of the so-called counting function \( n_s() \), associated with each given excited state \( |I_n^{(i)}\rangle \). More precisely, the particle, resp. the hole-like excitations in a given eigenstate are described directly by a set of rapidities \( \mu_{n_s}^{(i)} \), resp. \( \mu_{h_s}^{(i)} \). These rapidities are defined as the unique solutions to

\[
\xi^{(i)}(\mu_{n_s}) = \frac{\mu_{n_s}^{(i)}}{L} \quad \text{and} \quad \xi^{(i)}(\mu_{h_s}) = \frac{\mu_{h_s}^{(i)}}{L}.
\]

Since, as mentioned, the counting function does depend, a priori on the set of integers labelling the excited state \( |I_n^{(i)}\rangle \), the system of equations (2.3) is, in fact, extremely involved.

In the following, we shall build on the assumption that any counting function admits, in the \( L \to +\infty \) limit, the asymptotic expansions

\[
\xi^{(i)}(\omega) = \frac{\xi(\omega)}{L} + \frac{1}{L} F(\xi^{(i)}(\omega)) + O\left(\frac{1}{L^2}\right).
\]

The asymptotic expansion of the counting functions involves three ‘macroscopic’ functions.

- The function \( \xi \) is the asymptotic counting function. It is the same for all excited states and we assume that it is strictly increasing. This function defines a set of ‘macroscopic’ rapidities \( \mu_{n_s} \in \mathbb{Z} \)

\[
\xi(\mu_{n_s}) = \frac{a}{L}.
\]

These macroscopic rapidities provide one with the leading order in \( L \) approximation of the rapidities \( \mu_{n_s}^{(i)} \) and \( \mu_{h_s}^{(i)} \):

\[
\tilde{\mu}_{n_s}^{(i)} \simeq \mu_{n_s}^{(i)} \quad \text{and} \quad \tilde{\mu}_{h_s}^{(i)} \simeq \mu_{h_s}^{(i)}.
\]

- The function \( F^{(i)}_n \) stands for the shift function (of the given excited state in respect to the model’s ground state). It is a function of the spin \( s \) and of the set macroscopic rapidities

\[
\mathcal{R}_n^{(i)} = \left\{ \mu_{n_s}^{(i)} \right\}; \quad \left\{ \mu_{h_s}^{(i)} \right\}.
\]

This function measures the small \( O(L^{-1}) \) drift in the position of a rapidity in the Fermi sea under the effect of interactions.

- Finally, the function \( \xi_{-1} \) represents the \( 1/L \) corrections to the ground state’s counting functions. We stress that our way of decomposing the \( 1/L \) corrections to the counting function is such that the ground state’s shift function vanishes, i.e. \( F^{(0)}_n = 0 \).

In the large-\( L \) limit and within such a setting, the rapidities for the ground state form a dense distribution on \([q, -q]\)—the so-called Fermi zone of the model—with density \( \xi' \). The endpoints \( \pm q \) are called the Fermi boundaries. The particle and hole excitations in the model carry a dressed momentum \( p \) and a dressed energy \( \varepsilon \). The dressed energy and momentum are smooth and satisfy to the general properties

7 Typically, for quantum integrable models, these functions are given as solutions to linear integral equations [13]. Their description, for more complex, non-integrable models, is in principle much more complicated.
The relative momentum and energy of an excited state are expressed in terms of the dressed momentum and energy as

\[ \Delta E(T^{(s)}_n) \equiv E(T^{(s)}_n) - E(T^{(0)}_n) = \sum_{a=1}^{\infty} (\varepsilon(\mu_{p_a}^{\alpha}) - \varepsilon(\mu_{p_a}^{\beta})) + O\left(\frac{1}{L}\right), \]  
\[ \Delta P(T^{(s)}_n) \equiv P(T^{(s)}_n) - P(T^{(0)}_n) = \sum_{a=1}^{\infty} (p(\mu_{p_a}^{\alpha}) - p(\mu_{p_a}^{\beta})) + O\left(\frac{1}{L}\right). \]

Above, \(T^{(0)}_n = \{e; \bar{e}\}\) refers to the set of integers which parametrizes the ground state of the model and \(P(T^{(s)}_n)\) and \(E(T^{(s)}_n)\) are respectively the momentum and energy of the state parametrized by the set of integers \(T^{(s)}_n\). We remind that the superscript \((s)\) refers to the fact that the excitation labelled by \(T^{(s)}_n\) takes place above the lowest lying energy level in the spin \(s\) sector.

The excitation momentum and energy are probably the best examples allowing one to discuss a loss of information due to parametrizing the particles and holes in terms of their ‘macroscopic’ momenta. Indeed, on the level of (2.9) and (2.10), there arises a huge degeneracy in what concerns the \(O(1)\) finite parts of the excitation energy and momentum in that numerous choices of the set \(T^{(s)}_n\) will lead to the same value of the finite part. Indeed, the very definition (2.5) of the macroscopic rapidities ensures that

\[ \mu_k - \mu_{k+L} = O\left(\frac{1}{L}\right) \] (2.11)

Thus, a direct computation shows that for any fixed collection \(\{k_0, t_a\}\) of \(L\)-independent integers, the two sets of integers

\[ T^{(s)}_n = \{\{p_0\}^n; \{h_0\}^n\} \quad \text{and} \quad \tilde{T}^{(s)}_n = \{\{p_0 + k_a\}^n; \{h_0 + t_a\}^n\} \] (2.12)

will generate the same finite part of \(\Delta E\) or \(\Delta P\), namely

\[ \Delta E(T^{(s)}_n) = \Delta E(\tilde{T}^{(s)}_n) = O\left(\frac{1}{L}\right) \quad \text{and} \quad \Delta P(T^{(s)}_n) = \Delta P(\tilde{T}^{(s)}_n) = O\left(\frac{1}{L}\right). \] (2.13)

Thus, the so-called microscopic variables allow one to distinguish between all of the states of the Hilbert space whereas the macroscopic variables, at least in what concerns the leading in \(L\) order, naturally give rise to a huge indeterminacy in identifying a given state in the model—many microscopically distinct states give rise to the same macroscopic variables. Hence, in the large \(L\) limit, there is a loss of information when passing from the microscopic to the macroscopic description.

2.2. The operators and their form factors

We assume that the model is endowed with a collection of local operators \(O_r\). These operators are best characterized in terms of their form factor, namely their expectation values taken between two excited states. We shall assume that the operator \(O_r\) only connects those eigenstates which differ by \(o_r\) in their spin, namely

\[ \mathcal{F}_{O_r}\left(T^{(s)}_m; T^{(s)}_n | x\right) = e^{ix\left[p(T^{(s)}_m) - p(T^{(s)}_n)\right]}, \langle T^{(s)}_m | O_r(0) | T^{(s)}_n \rangle \neq 0 \] (2.14)
only if $s - s' = o_r$. Scalar observables introduced in the last subsection were parametrized, in the large-$L$ limit, solely by the macroscopic set of rapidities $\mathcal{R}_{m}^{(s)}$ subordinate to the set of multi-indices $\mathcal{I}_{m}^{(s)}$. This is no longer the case for form factors as we demonstrated in our previous work [26]. Within our setting, the latter are parametrized, in the large-$L$ limit, not only by the sets of macroscopic rapidities $\mathcal{R}_{n}^{(s + o_r)}$, $\mathcal{R}_{n}^{(s)}$ but also by the sets of discrete integers $\mathcal{I}_{n}^{(s + o_r)}$, $\mathcal{I}_{n}^{(s)}$. Namely, for properly normalized states $|\Psi(\mathcal{I}_{n}^{(s + o_r)})\rangle$ and their duals $\langle \Psi(\mathcal{I}_{n}^{(s)}) |$, the form factors take the form

$$
\mathcal{F}_{O}(\mathcal{I}_{m}^{(s)}, \mathcal{I}_{n}^{(s + o_r)} | 0) = \mathcal{S}^{(C)}(\mathcal{R}_{m}^{(s)}, \mathcal{R}_{n}^{(s + o_r)}) \cdot \mathcal{D}^{(s)}(\mathcal{R}_{m}^{(s)}, \mathcal{R}_{n}^{(s + o_r)} | \mathcal{I}_{m}^{(s)}, \mathcal{I}_{n}^{(s + o_r)}) \cdot \left( 1 + O\left(\frac{\ln L}{L}\right) \right)
$$

(2.15)

In this decomposition

• $\mathcal{S}^{(C)}$ is called the smooth part: it solely depends on the macroscopic rapidities $\mathcal{R}_{n}^{(s + o_r)}$, $\mathcal{R}_{m}^{(s)}$. Furthermore, this dependence is smooth what ensures that a change in the value of the integers in the spirit of (2.12) will only affect the value of the smooth part in the $1/L$ corrections.

• The smooth part is a set function so that it is invariant under permutation of the particle or hole rapidities. It satisfies to particle–hole reduction properties, meaning that

$$
\mathcal{S}^{(C)}(\mathcal{R}_{m}^{(s)}, \mathcal{R}_{n}^{(s + o_r)})_{|\mu_{\mathcal{I}_{m}^{(s)}}, \mu_{\mathcal{I}_{n}^{(s + o_r)}}} = \mathcal{S}^{(C)}(\mathcal{R}_{m}^{(s)}, \mathcal{R}_{n}^{(s + o_r)})
$$

with

$$
\mathcal{R}_{m}^{(s)} = \{ \{ \mu_{\mathcal{I}_{m}^{(s)}} \}^{n}_{i \neq k} ; \{ \mu_{\mathcal{I}_{n}^{(s)}} \}^{n}_{i \neq \ell} \}.
$$

The same type of reduction holds as well for the particle–hole rapidities that belong to the set $\mathcal{R}_{n}^{(s + o_r)}$.

• $\mathcal{D}^{(s)}$ is called the discrete part because it depends explicitly on both types of parametrizations: the macroscopic rapidities $\mathcal{R}_{n}^{(s + o_r)}$, $\mathcal{R}_{m}^{(s)}$ and the sets of integers labelling the excited states $\mathcal{I}_{n}^{(s + o_r)}$, $\mathcal{I}_{m}^{(s)}$. We do stress that this last dependence is explicit in the sense that it goes beyond a sole dependence of the integers through the parametrization by the macroscopic rapidities. The main effect of such a dependence is that a change in the value of the integers in the spirit of (2.12) will result in a significant change (of the order of $O(1)$) in the value of $\mathcal{D}^{(s)}$. The discrete part thus keeps track of the microscopic details of the different excited states.

The smooth part represents, in fact, a non-universal part of the model’s form factors. Its explicit expressions not only depends on the operator $\mathcal{O}_{l}$ but also varies strongly from one model to another, see [26, 34] for concrete examples of the form taken by $\mathcal{S}^{(C)}$ for different quantum integrable models. The part $\mathcal{D}^{(s)}$ is, however, entirely universal within the present setting of the description of the model’s spectrum. It solely depends on the values of the spin $o_r$ of the operator $\mathcal{O}_{r}$. Its general explicit expression plays no role in our analysis, in the sense that we shall only need the expression for specific excited states, namely the ones belonging to the so-called $\ell_{t}$-critical classes that we define below.

We refer the interested reader to [25] for a thorough discussion relative to the origin of the discrete part $\mathcal{D}^{(s)}$ within the framework of Bethe Ansatz solvable models.
2.3. The critical $\ell_s$ class

Observe that, in virtue of (2.8), one has $\varepsilon(q) = 0$. As a consequence, by (2.9), there exists a possibility to realize zero energy excitations in the thermodynamic limit $L \to +\infty$ by forming holes and particles whose rapidities scale down to $\pm q$ when $L \to +\infty$. This singles out a class of excited states of the model which we shall refer to as critical states. An excited state is said to belong to the critical class if, in the $L \to +\infty$ limit, all the macroscopic rapidities of the particles and holes associated with this state ‘collapse’ on the Fermi boundary:

$$\mu_{p_s^{(a)}} \simeq \pm q \quad \text{and} \quad \mu_{h_s^{(a)}} \simeq \pm q.$$  \hspace{1cm} (2.16)

There, the $\pm$ sign depends on whether the particle or hole’s rapidity collapses on the right or the left Fermi boundary.

A set of integers $I_n^{(s)}$ is said to parametrize a critical excited state if the associated particle–hole integers $[p_{p_s^{(a)}}]_n^{(a)}$ and $[h_{h_s^{(a)}}]_n^{(a)}$ can be represented as

$$[p_{p_s^{(a)}}]_n^{(a)} = [N + s + p_{p_s^{(a)}}]_1^{(a)} \cup \{1 - p_{p_s^{(a)}} - h_{h_s^{(a)}}\}_1^{(a)},$$
$$[h_{h_s^{(a)}}]_n^{(a)} = [1 + N + s - h_{h_s^{(a)}}]_1^{(a)} \cup \{h_{h_s^{(a)}} - p_{p_s^{(a)}}\}_1^{(a)},$$ \hspace{1cm} (2.17)

where the integers $p_{p_s^{(a)}}^{(a)} , h_{h_s^{(a)}}^{(a)} \in \mathbb{N}^*$ are ‘small’ compared to $L$, i.e.

$$\lim_{L \to +\infty} \frac{p_{p_s^{(a)}}^{(a)}}{L} = \lim_{L \to +\infty} \frac{h_{h_s^{(a)}}^{(a)}}{L} = 0,$$  \hspace{1cm} (2.18)

and the integers $n_{p^{(s)}}^{(s)}$ satisfy to the constraint

$$n_{p^{(s)}}^{(s)} + n_{h^{(s)}}^{(s)} = n_{p^{(s)}}^{(s)} = n.$$  \hspace{1cm} (2.19)

Within this setting, one can readily check that the critical excited state described above will have $n_{p^{(s)}}^{(s)}$ particles, resp. $n_{h^{(s)}}^{(s)}$ holes, on the right/left end of the Fermi zone $[-q; q]$ associated with the spin $s$ sector.

In fact, one can distinguish between various critical states by organizing them into so-called $\ell_s$-critical classes. This classification takes its origin in the fact that all such states have a vanishing excitation energy (up to $O(1/L)$ corrections) but can be gathered into classes depending on the value of their macroscopic momenta $2\ell_s p_F$, where $p_F = p(q)$ is the so-called Fermi momentum and

$$\ell_s = n_{p^{(s)}}^{(s)} - n_{h^{(s)}}^{(s)} = n_{h^{(s)}}^{(s)} - n_{p^{(s)}}^{(s)}.$$  \hspace{1cm} (2.20)

Note that the subscript $s$ in $\ell_s$ allows one to localize the given critical class in a specific spin $s$ sector.

The finite part of the relative excitation momentum associated with an excited state belonging to the $\ell_s$ critical class only depends on $\ell_s$, while the first dependence on the specific representative of the class arises on the level of the $L^{-1}$ corrections. More precisely, one has:
\[ \Delta P (\mathcal{T}^{(s)}_{n}) = 2\ell_s p_F + \frac{2\pi}{L} \alpha^s \left\{ \sum_{a=1}^{n_{a^+}} (p_{a^+}^{(s)} - 1) + \sum_{a=1}^{n_{a^+}} h_{a^+}^{(s)} \right\} \]
\[ - \frac{2\pi}{L} \alpha^s \left\{ \sum_{a=1}^{n_{a^-}} (p_{a^-}^{(s)} - 1) + \sum_{a=1}^{n_{a^-}} h_{a^-}^{(s)} \right\} \]
\[ + \frac{2\pi}{L} \left( \alpha^s \ell_s + \frac{1}{2} \right) - \alpha^s \ell_s \left( \ell_s - 1 \right) \] + ..., \hspace{1cm} (2.21)

where we have set
\[ \alpha^\pm = \frac{1}{2\pi} \frac{p'_\pm (\pm q)}{\xi_\pm (\pm q)} \] \hspace{1cm} (2.22)

We *insist* that all the terms included in the dots either

- are of the order of \( O(1/L) \) but do not depend on the integers \( p_{a^\pm} \) and \( h_{a^\pm} \) nor on \( n_F/h^\pm \) (they can, nonetheless depend on \( \ell_s \));
- depend on these integers but are of the order of \( O(1/L^2) \).

Note that, in principle, the terms present in the second line of (2.21) could have been simply included in the \( \cdots \). We chose to write them down for normalization purposes. Namely, the formula (2.21) in its present form allows one for a more straightforward correspondence with the free boson model.

The partitioning of the particle and hole integers of a given state belonging to the \( \ell_s \) critical class into two collections of integers of \( \pm \) types (2.17) suggests to parametrize the excited states within the \( \ell_s \) critical class directly in terms of the two subsets \( \mathcal{J}_{n_F^+, n_{h^+}^s}^{(s)} \cup \mathcal{J}_{n_F^-, n_{h^-}^s}^{(s)} \), where
\[ \mathcal{J}_{n_{a,m}}^{(s)} = \{ (p_{a}^{(s)})^{n_{a}} \mid (h_{a}^{(s)})^{m_{a}} \}. \] \hspace{1cm} (2.23)

Hence, from now on, for any state belonging to a critical \( \ell_s \) class, we shall identify the sets
\[ \mathcal{T}^{(s)}_n \quad \text{and} \quad \mathcal{J}_{n_F^+, n_{h^+}^s}^{(s)} \cup \mathcal{J}_{n_F^-, n_{h^-}^s}^{(s)}. \] \hspace{1cm} (2.24)

We do stress that, due to (2.17), the value of the spin \( s \) does play a role in the correspondence between \( \mathcal{T}^{(s)}_n \) and \( \mathcal{J}_{n_F^+, n_{h^+}^s}^{(s)} \cup \mathcal{J}_{n_F^-, n_{h^-}^s}^{(s)} \). We also draw the reader’s attention to the fact that the value of \( \ell_s \) is encoded in the very notation \( \mathcal{J}_{n_F^+, n_{h^+}^s}^{(s)} \cup \mathcal{J}_{n_F^-, n_{h^-}^s}^{(s)} \), see (2.20).

### 2.4. Large-\( L \) expansion of form factors connecting critical states

We are finally able to discuss the expression for the form factors of local operators \( \mathcal{O}_r \) taken between two eigenstates belonging to critical classes. It is precisely these form factors that are responsible for the emergence of an effective field theory description at large distances of separation between the operators. Let
be two sets of integers\(^8\) parametrizing excited states belonging to the critical classes.

Within the framework we impose on the model, the form factors of local operators taken between two excited states belonging to the critical classes introduced above take the form:

\[
\mathcal{F}_{\mathcal{O}}\left(\mathcal{T}^{(s)}_{m}, \mathcal{T}^{(s')}_{n}\right) = e^{2i\pi q} \mathcal{C}(\ell_{\text{out}} - \ell_{\text{in}}) \mathcal{O}_{r}(\ell_{\text{out}} - \ell_{\text{in}}) \mathcal{F}_{\ell_{\text{out}} - \ell_{\text{in}}} (\mathcal{O}_{r}) \times \left(\frac{2\pi}{L}\right)^{p(\nu_{r}^{+} + \nu_{r}^{-}) + p(\nu_{r}^{+} + \nu_{r}^{-})} \mathcal{F}_{J_{m}, J_{n}} (\nu_{r}^{+}, \nu_{r}^{-}) \times \mathcal{F}_{J_{m}, J_{n}} (\nu_{r}^{+}, \nu_{r}^{-}) \times (\omega_{r}^{+})^{\frac{\ell_{\text{out}} - \ell_{\text{in}} - 1}{2}} \times \left(1 + \mathcal{O}\left(\frac{\ln L}{L}\right)\right).
\]

(2.27)

The constituents of the above formula are parametrized by the values

\[
\nu_{r}^{+} = \nu_{r}(q) - \alpha_{r} \quad \text{and} \quad \nu_{r}^{-} = \nu_{r}(-q)
\]

(2.28)

that the relative shift function between the \(\ell_{\text{in}}, \ell_{\text{out}}\) critical states

\[
\nu_{r}(\lambda) = E(\lambda) - E_{r+}(\lambda)
\]

(2.29)

takes on the right/left endpoints of the Fermi zone, up to subtracting the level \(\alpha_{r}\) of the operator \(\mathcal{O}_{r}\) in the case of the right endpoint. The also depend on

\[
\omega_{r}^{+} = e^{2\pi\alpha_{r}^{+} \frac{L}{\tau}} \quad \text{and} \quad \omega_{r}^{-} = e^{-2\pi\alpha_{r}^{-} \frac{L}{\tau}}
\]

(2.30)

representing the exponent of the individual momentum brought by a particle excitation on the left or right Fermi boundary. In the above large-\(L\) asymptotics, the quantity \(\mathcal{F}_{\ell_{\text{out}} - \ell_{\text{in}}} (\mathcal{O}_{r})\) represents the properly normalized finite and non-universal (i.e. model and operator dependent) part of the large-\(L\) behaviour of the form factor of the operator \(\mathcal{O}_{r}\) taken between fundamental representatives of the \(\ell_{\text{in}}\) and \(\ell_{\text{out}}\) critical classes. More precisely, it is defined as

\[
\mathcal{F}_{\ell_{\text{out}} - \ell_{\text{in}}} (\mathcal{O}_{r}) = \lim_{L \to +\infty} \left\{ \left(\frac{L}{2\pi}\right)^{p(\nu_{r}^{+} + \nu_{r}^{-}) + p(\nu_{r}^{+} + \nu_{r}^{-})} \langle \mathcal{L}^{(s)}_{\ell_{\text{out}}} | \mathcal{O}_{r}(0) | \mathcal{L}^{(s')}_{\ell_{\text{in}}} \rangle \right\}
\]

(2.31)

in which the sets of integers \(\mathcal{L}^{(s)}_{\ell_{\text{out}}} \) and \(\mathcal{L}^{(s')}_{\ell_{\text{in}}} \) parametrizing the excited states correspond to the fundamental representatives of the \(\ell_{\text{out}}\) and \(\ell_{\text{in}}\) critical classes. Namely, \(\mathcal{L}^{(s)}_{\ell}\) is the set of particle–hole integers living on the Fermi boundary in the spin \(s\) sector and such that

\[
\mathcal{L}^{(s)}_{\ell} = \left\{ \{ p^{\ell}_{a} = a \}_{\ell}; \{ \emptyset \}_{\ell}; \{ h^{\ell}_{a} = a \}_{\ell}; \{ \emptyset \}_{\ell}; \{ p^{\ell}_{a} = a \}_{\ell}; \{ \emptyset \}_{\ell}; \{ h^{\ell}_{a} = a \}_{\ell} \right\}
\]

(2.32)

Note that this form factor solely depends on the difference \(\ell_{\text{out}} - \ell_{\text{in}}\). This issues from the fact that, \textit{a priori}, in the thermodynamic limit, any fundamental representative can be chosen to be

\(8\) In this writing we chose not to omit the spin sector label so as to lighten the formulae. Since the context of the setting is clear, this ought not lead to any confusion.
the reference ground state. Yet, relatively to the fundamental representative of the \( \ell_{\text{out}} \)-class, the other excited state corresponds to the fundamental representative of the \( \ell_{\text{out}} - \ell_{\text{in}} \) class.

The power of the volume \( L \) arising in (2.27) and (2.31) involves the right \( \rho(\nu_{\ell}^+) \) and left \( \rho(\nu_{\ell}^-) \) scaling dimensions whose generic expression reads

\[
\rho(\nu) = \frac{\nu^2}{2}.
\]  

(2.33)

The factors \( F \) have been defined in (1.35) and contain all the ‘microscopic’ contributions issuing from excitations on the right or left Fermi boundary. These local microscopic form factors depend on the value taken on the right or left Fermi boundary by the relative shift function \( \nu_{\ell} \) associated with the critical excited states of interest and on the position of the operator. These local microscopic form factors also depend on the sets of integers \( m, n, \ldots \) parametrizing the excitations on the boundary of the Fermi zone to which they are associated. Those microscopic form factors are as defined in (1.35).

Finally, the normalization constant \( C(\ell_{\text{out}} - \ell_{\text{in}})(\nu_{\ell}^+, \nu_{\ell}^-) \) is chosen in such a way that it cancels out the \( L \)-independent contributions of the right and left critical form factors when focusing on the fundamental representative of the \( \ell_{\text{out}} \) and \( \ell_{\text{in}} \) critical classes. Its explicit form can be computed in terms of the Barnes \( G \)-function \([2]\) and reads:

\[
C(\ell_{\text{out}} - \ell_{\text{in}})(\nu_{\ell}^+, \nu_{\ell}^-) = G\left( \frac{1 + \nu_{\ell}^+ - \nu_{\ell}^-}{1 + \nu_{\ell}^+ - \nu_{\ell}^- - \ell_{\text{out}} - \ell_{\text{in}}} \right). 
\]  

(2.34)

Above we have used the hypergeometric-like notations for ratios. These are explained in (A.5).

2.5. General remarks on the scope of applicability of the model

The structure of the spectrum and form factors that we discussed throughout this section already appeared in the paper \([28]\).

The formula for the form factors of local operators taken between excited states belonging to critical classes can be proven within the framework of the algebraic Bethe Ansatz for various quantum integrable models on the basis of determinant representations for their form factors \([29, 33, 34, 41, 48]\). The corresponding calculations are a straightforward generalization of the method developed in \([24, 26, 48]\). However, we strongly believe that the decomposition (2.27) is, in fact, universal. More precisely, the properly normalized form factor \( F_{\ell_{\text{out}} - \ell_{\text{in}}}(\mathcal{O}) \) is definitely model dependent and thus can only be obtained on the basis of exact computations. Its explicit expression is available for many quantum integrable models. However, we do trust that the local microscopic pre-factors and the leading power-law behaviour in \( L \) is universal; namely that they take the same form for models belonging to the Luttinger liquid universality class. Parts of this structure have been confirmed by perturbative calculations around a free model in \([47]\).

The attentive reader might observe a slight difference between the expression for the form factors taken between critical excited states as given in section 2.4 and section 2.4 of paper \([28]\). More precisely, the formulae differ by

(i) an overall sign

\[
(-1)^{p_{\ell}^+ + p_{\ell}^- + m_{\ell}^+ + m_{\ell}^-} \cdot (-1)^{\frac{n_{\ell}^+}{2} (n_{\ell}^- - 1)} \cdot (-1)^{\frac{n_{\ell}^-}{2} (n_{\ell}^+ - 1)}.
\]  

(2.35)
in [28], the expression for the normalization constant \( C_{\text{original}} (\nu^+, \nu^-) \) took the form
\[
C_{\text{original}} (\nu^+, \nu^-) = \frac{1}{\sin(\pi \nu^+) \sin(\pi \nu^-)} \left( 1 + \nu^+, 1 - \nu^+, 1 + \ell_\text{in} - \nu^+, 1 - \ell_\text{in} + \nu^+ \right) \left( 1 + \nu^-, 1 - \nu^-, 1 + \ell_\text{out} - \nu^-, 1 - \ell_\text{out} + \nu^- \right).
\] (2.36)

(iii) the presence, in [28], of the factor
\[
C_{\text{extra}} (\nu^+, \nu^-) = \frac{\sin(\pi \nu^+)}{\sin(\pi \nu^-)}
\] (2.37)
in the definition of the local microscopic form factors as compared to the definition (2.27) used in the present paper.

The difference of sign pre-factors is irrelevant in that the sign can be readily absorbed into the normalization of the eigenstates. The present choice of the sign was more natural here since it makes the comparison with the free boson model more direct. Finally, in what concerns the normalization constants, one can check, on the basis of identity (A.4), that one has
\[
C_{\text{extra}} (\nu^+, \nu^-) \cdot C_{\text{original}} (\nu^+, \nu^-) = C_{\ell_\text{out} \ell_\text{in}} (\nu^+, \nu^-).
\] (2.38)

3. The resurgence of an effective free boson field theory in the large-distance regime

3.1. An effective free boson field theory model

The effective Hilbert space is defined as the tensor product of two copies \( h_L \) and \( h_R \) of the free boson Hilbert space \( h \) introduced previously:
\[
h_{\text{eff}} = h_L \otimes h_R.
\] (3.1)
The first space (resp. second) arising in the tensor product will be called left (resp. right) space and will be associated to modelling what happens on the left (resp. right) Fermi boundaries.

To a local operator \( \mathcal{O}_j(x) \), we associate the below operator on \( h_{\text{eff}} \)
\[
\mathcal{O}_j (\omega) = \sum_{n \in \mathbb{Z}} \mathcal{F}_n (\mathcal{O}_j) \left( \frac{2\pi}{L} \right)^{(\nu_j(q) + \omega_j + \kappa_j + \omega_j)} \cdot e^{2i\nu_j \kappa_j} \times \mathcal{F}_L (\nu_j(q) - \kappa_j, \kappa_j - \omega_j) \cdot \mathcal{F}_R (\nu_j(q) - \omega_j).
\] (3.2)

In this formula,
- \( \mathcal{F}_L/R (\nu, \kappa; \omega) \) stands for the operator acting non-trivially on the L/R copy of the original Hilbert space as the vertex operator \( \mathcal{V} (\nu, \kappa; \omega) \) defined in (1.27), viz
- \( \mathcal{F}_L (\nu, \kappa | \omega) = \mathcal{F} (\nu, \kappa | \omega) \otimes \text{id} \) and \( \mathcal{F}_R (\nu, \kappa | \omega) = \text{id} \otimes \mathcal{F} (\nu, \kappa | \omega) \).
- \( \nu_j \) are the scaling dimensions introduced in (2.33);
- \( \mathcal{F}_n (\mathcal{O}_j) \) is the properly normalized in the volume thermodynamic amplitude defined in (2.31) and taken between excited states satisfying \( \ell_\text{out} - \ell_\text{in} = \kappa \).
• $\omega_s^\pm$ is a phase factor that reads
  \[ \omega_s^\pm = e^{\pm 2i\pi \frac{\xi_s}{L}}. \]  

It follows readily from the discussions carried so far that given two sets

\[ \mathcal{I}^{(s)}_m = \mathcal{J}_{m_p,...,m_{n_+}} \cup \mathcal{J}_{m_p,...,m_{n_-}} \quad \text{and} \quad \mathcal{I}^{(s+\alpha)}_n = \mathcal{J}_{n_p,...,n_{m_{n_+}}} \cup \mathcal{J}_{n_p,...,n_{m_{n_-}}} \]  

parametrizing critical excited states in the physical Hilbert space $\mathcal{H}_{\text{phys}}$, one has an equality, up to $O(\ln L/L)$ corrections between matrix elements:

\[ \mathcal{F}_\mathcal{C}\left(\mathcal{I}^{(s)}, \mathcal{I}^{(s+\alpha)}\right) = (-1)^{s+\alpha} \left\{ \langle s ; \mathcal{J}_{m_p,...,m_{n_m}} | \otimes | s ; \mathcal{J}_{m_p,...,m_{n_m}} \rangle \right\} \mathcal{C}_\varepsilon(\omega_s) 
\times \left\{ |\mathcal{J}_{n_p,...,n_{m_n}}; s \rangle \otimes |\mathcal{J}_{n_p,...,n_{m_n}}; s + \alpha \rangle \right\} \cdot \left( 1 + O\left(\frac{\ln L}{L}\right) \right), \]  

where

\[ \varepsilon = m_{p,+} - m_{n,+} - n_{p,+} + n_{n,+}. \]  

Note that, in order to establish (3.6), one should use that

\[ \langle s ; \mathcal{J}_{m_p,...,m_{n_m}} | = (\mathcal{J}_{m_p,...,m_{n_m}}) e^{-\alpha_p} \quad \text{and} \quad | s ; \mathcal{J}_{m_p,...,m_{n_m}} \rangle = e^{\alpha_p} |\mathcal{J}_{m_p,...,m_{n_m}} \rangle \]  

as readily inferred from lemma 1.1.

Note also that (3.5) and (3.6) gives rise to a correspondence between states belonging to $\ell$ -classes in the original Hilbert space and general states in the effective Hilbert space:

\[ |\mathcal{I}^{(s)}_m \rangle = |\mathcal{J}_{n_p,...,n_{m_n}}; s \rangle \otimes |\mathcal{J}_{n_p,...,n_{m_n}}; s + \alpha \rangle \]  

3.2. The effective truncation of $\mathcal{H}_{\text{phys}}$ and the correspondence

Consider the $r$-point ground-to-ground state expectation value $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_r(x_r) \rangle_{\mathcal{H}_{\text{phys}}}$. One way to express this quantity is to write its form factor expansion, viz insert the closure relation between each operator. According to our hypothesis on the space of states, this recast the correlator as

\[ \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_r(x_r) \rangle_{\mathcal{H}_{\text{phys}}} = \sum_{\mathcal{I}^{(s)}} \prod_{\ell=1}^r \left\{ \langle \mathcal{I}^{(s_\ell-1)}_n | \mathcal{O}_\ell(0) | \mathcal{I}^{(s_\ell)}_n \rangle \right\} \cdot \prod_{\ell=1}^{r-1} \left\{ \exp \left[ i(x_{\ell+1} - x_\ell) \Delta \mathcal{P} \left( \mathcal{I}^{(s_\ell)}_n \right) \right] \right\}. \]

Above we have introduced the sequence of spin $\xi_r = \sum_{p=1}^r \alpha_p$ and the summation runs over all choices of $n^{(s)} \in \mathbb{N}$ and sets $\mathcal{I}^{(s)}_n$, with $s = 1, ..., r$. Note that we have adopted the convention

\[ s_0 = s_r = 0 \quad \text{and} \quad \mathcal{I}^{(s_0)}_{n^{(s_0)}} = \mathcal{I}^{(s_r)}_{n^{(s_r)}} = \{ \{ \emptyset \}; \emptyset \}. \]  

The above sum is a highly oscillatory sum in the large-distance regime $|x_{\ell+1} - x_\ell| p_\ell \gg 1$. Therefore, it should localize, in this regime, either around the saddle-points of the relative excitation momentum $\Delta \mathcal{P}(\mathcal{I}^{(s)})$ defined in (2.10) or around the boundaries of the summation region. In the present setting, due to the hypothesis (2.8), there are no saddle points, so that the leading in $|x_{\ell+1} - x_\ell| p_\ell \gg 1$ contribution to the sums will
issue from a vicinity of the boundaries of summation. This corresponds precisely to the case of the states belonging to critical classes. Thus, this reasoning leads to

\[
\{\mathcal{O}(x_1)\cdots \mathcal{O}(x_r)\}_{\mathcal{F}_{\text{phys}}} \simeq \sum_{\{\mathcal{O}(x_k)\}_{\mathcal{F}_{\text{phys}}}^{r}} \prod_{r=1}^{r} \mathcal{F}_{\mathcal{O}_r} \left( \mathcal{F}^{(r)}_{\text{eff}} \left( \mathcal{O}(x_1) \cdots \mathcal{O}(x_r) \right) \right).
\]

(3.12)

Note that the \( \simeq \) symbol refers to an equality up to sub-leading corrections to each oscillating factor \( e^{2\ell_1(x_r-x_r)_{\mathcal{F}_{\text{phys}}}} \) with \( r = 1, \ldots, r-1 \) and \( \ell_r \in \mathbb{Z} \). Hence invoking the identity (3.6), using that

\[
\prod_{a=1}^{r} (-1)^{\varepsilon_a} = 1 \quad \text{with} \quad \varepsilon_a = m_{1:;}^{(\omega_1)} - m_{2:;}^{(\omega_2)} - n_{1:;}^{(\omega_1)} + n_{2:;}^{(\omega_2)} \quad (3.13)
\]

and ignoring the effect of the \( O(\ln L/L) \) corrections that should vanish in the thermodynamic limit, we re-absorb the sums over the intermediate states of \( \mathcal{F}_{\text{eff}} \), hence obtaining

\[
\{\mathcal{O}(x_1)\cdots \mathcal{O}(x_r)\}_{\mathcal{F}_{\text{phys}}} \simeq \langle \mathcal{O}_1(\omega_1) \cdots \mathcal{O}_r(\omega_r) \rangle_{\mathcal{F}_{\text{eff}}} .
\]

(3.14)

Thus the claimed correspondence holds. Above, we have added the subscripts \( \mathcal{F}_{\text{phys}} \) and \( \mathcal{F}_{\text{eff}} \) so as to insist in which Hilbert space the expectation values are computed.

3.3. Some examples

3.3.1. The XXZ spin-1/2 chain. We shall now illustrate the general framework described above on the example of the XXZ spin-1/2 chain embedded in an external magnetic field \( h > 0 \). This model corresponds to the Hamiltonian

\[
\mathcal{H}_{\text{XXZ}} = \sum_{k=1}^{L} \left( \sigma^x_k \sigma^x_{k+1} + \sigma^y_k \sigma^y_{k+1} + \Delta \left( \sigma^z_k \sigma^z_{k+1} - 1 \right) \right) - \frac{h}{2} S^z \quad \text{with} \quad S^z = \sum_{k=1}^{L} \sigma^z_k .
\]

(3.15)

Above \( \sigma^{x,y,z} \) are the spin operators (Pauli matrices) acting on the \( k \)-th site of the chain, \( h \) is an external magnetic field and the model is subject to periodic boundary conditions. It is well known that \( \mathcal{H}_{\text{XXZ}} \) possesses different phases depending on the value of the anisotropy parameter \( \Delta \). When \( 1 < \Delta < 1 \), \( \mathcal{H}_{\text{XXZ}} \) has a massless spectrum in the \( L \to +\infty \) limit. Below, we shall only consider this regime and adopt the parametrization \( \Delta = \cos \zeta \). It is known that, in its massless phase at \( h > 0 \), the excitations in the XXZ-chain can be either of bound state nature (so-called string solutions) or be of particle–hole type. The particle–hole spectrum enjoys of the structure described in the earlier part of this section. The bound states, however, are described differently. None-the-less, one can argue that, in the long-distance regime, they only produce corrections that are exponentially small in the distance; they can thus be disregarded when studying the correspondence with the free boson model in that it solely involve the algebraically decaying terms.

Owing to \( \{\mathcal{H}_{\text{XXZ}}, S^z\} = 0 \), the role of bare-particles is played by the number of down spins building up a given eigenstate. The local operators of the model consist in products of elementary matrices

\[
\mathcal{O}(m) = E_{m_0}^{(1)} E_{m_1}^{(2)} \cdots E_{m_{d_k}}^{(d_k)} .
\]

(3.16)

The length \( d_k \) of the string of elementary matrices is assumed fixed, viz \( m_k \) and \( L \) independent. There \( E_{p}^{(d)} \) is the elementary matrix \( E^{(d)} \) acting non-trivially on the \( p \)-th site of the chain. The
operator defined by (3.16) carries the spin
\[ \alpha_k = \# \left\{ a : (e_{k,2}, e_{k,1}^\alpha) = (2, 1) \right\} - \# \left\{ a : (e_{k,2}, e_{k,1}^\alpha) = (1, 2) \right\}. \] (3.17)

One can show using Bethe Ansatz methods that the shift functions—in the sense of (2.4)—take the form
\[
F_{\mathcal{R}^\omega_{\alpha}}(\lambda) = s \left[ \phi(\lambda, q) - \frac{Z(\lambda)}{2} \right] + \sum_{a=1}^n \left[ \phi(\lambda, \mu_a) - \phi(\lambda, \mu_{b_a}) \right],
\] (3.18)
in which \( \phi \) is the dressed phase, \( Z \) the dressed charge and the rapidities \( \{\mu_a\}_{a \in \mathbb{Z}} \) are the unique solutions to
\[
\xi(\mu_a) = \frac{a}{L}, \quad \text{with} \quad \xi(\omega) = \frac{D(\omega)}{2\pi} + \frac{D}{2}.
\] (3.19)
The function \( p \) arising in the expression of the asymptotic counting function is the so-called dressed momentum and \( D = \lim_{N \to +\infty} (N/L) \), \( N \) being the number of down spins in the model’s ground state. The function \( p \) corresponds to the unique odd solution to the linear integro-differential equation:
\[
p(\lambda) + \int_{-q}^q \theta'(\lambda - \mu) \cdot p'(\mu) \cdot \frac{d\mu}{2\pi} = \frac{i}{\pi} \ln \left( \frac{\sinh(\zeta/2 + \lambda)}{\sinh(\zeta/2 - \lambda)} \right).
\] (3.20)
The functions \( Z \) and \( \phi \) solve the Lieb integral equations
\[
Z(\lambda) + \int_{-q}^q \theta'(\lambda - \mu) Z(\mu) \cdot \frac{d\mu}{2\pi} = 1 \quad \phi(\lambda, \nu)
\] \[+ \int_{-q}^q \theta'(\lambda - \mu) \phi(\mu, \nu) \cdot \frac{d\mu}{2\pi} = \frac{i}{\pi} \ln \left( \frac{\sinh(\zeta + \lambda - \nu)}{\sinh(\zeta - \lambda + \nu)} \right).
\] (3.21)
The above integral equation depend on the endpoint \( q \) of the model’s Fermi zone, which is fixed once that \( D \) is given.

Finally, the relative shift function \( \nu \) associated to critical excited states belonging to the \( \ell_s \) and \( \ell_{s-1} \) classes and differing in \( \ell_s \) in their bare-particle numbers, takes the form
\[
\nu_s(\lambda) = \alpha_s \left[ \frac{Z(\lambda)}{2} - \phi(\lambda, q) \right] + (\ell_{s-1} - \ell_s)(Z(\lambda) - 1).
\] (3.22)

One can then show using certain identities satisfied by \( Z(q) \) and \( \phi(q, q) \) that
\[
\nu_s(q) + \kappa = \alpha_s = sZ(q) - \frac{\alpha_s}{2Z(q)} \quad \text{and} \quad \nu_s(-q) + \kappa = sZ(q) - \frac{\alpha_s}{2Z(q)}.
\] (3.23)

We do stress that numerous lattice operators \( O_k(m_k) \) will carry the same spin \( \alpha_k \) and thus the same scaling dimensions since these are parametrized by \( \nu_s(q) + \kappa = \alpha_s + \nu_s(-q) + \kappa \).

Finally, the behaviour of the form factors of the model follows exactly the form provided in section 2. We do not reproduce the corresponding formulae here but refer the interested reader to the aforementioned literature.

3.3.2. The Luttinger structure recovered. One can reproduce the structure of the Luttinger liquid-like critical exponents by making additional assumptions on certain ingredients of our general framework. We first assume that the shift function \( F_{\mathcal{R}^\omega_{\alpha}}(\lambda) \) satisfies the reflection property:
This property can be argued, on the heuristic level, as follows. The physics should be invariant in respect to reflections of the Fermi surface. When taking the reflected of an excited state belonging to the $N_1 + s$ bare-particle excitation sector above the ground state, one should reflect all the rapidities of the particles and holes but also take into account that the $s$ additional bare-particles that are used to construct this excited state which are initially located on the right boundary are now located on the left one. This induces the shift by $-s$. These properties can be directly verified on the example of the XXZ chain introduced earlier on.

Now, we shall make the additional assumption of a linear response, namely assume that if the macroscopic variables $\mathcal{R}_{n+n'}^{(s)}$ can be partitioned as

$$\mathcal{R}_{n+n'}^{(s)} = \mathcal{R}_n^{(s)} \cup \mathcal{R}_{n'}^{(s)}$$

then one also has

$$F_{\mathcal{R}_{n+n'}^{(s)}}(\lambda) = F_{\mathcal{R}_n^{(s)}}(\lambda) + F_{\mathcal{R}_{n'}^{(s)}}(\lambda).$$

It is then enough to observe that the set of macroscopic rapidities $\tilde{\mathcal{R}}_\ell^{(s)}$ of an $\ell$ class, $\ell \geq 0$ can be decomposed as

$$\tilde{\mathcal{R}}_\ell^{(s)} = \{q\}^1_1; \{q\}^1_s; \{q\}^1_0\{q\}^1_{-s}; \{q\}^1_{-s}; \{q\}^1_{-s}; \{q\}^1_{-s}; \{q\}^1_{-s}$$

so as to get

$$F_{\tilde{\mathcal{R}}_\ell^{(s)}}(\lambda) = \ell F_{\mathcal{R}_{n+k}}(\lambda) + s F_{\mathcal{R}_{n-k}}(\lambda).$$

Then the shift function arising in our estimates is recast as

$$\nu_l(\lambda) = F_{\mathcal{R}_{n+k}}(\lambda) - F_{\mathcal{R}_{n-k}}(\lambda) = \kappa F_{\mathcal{R}_{n+k}}(\lambda) - \alpha_l F_{\mathcal{R}_{n-k}}(\lambda),$$

where we have set $\kappa = \ell_{\text{out}} - \ell_{\text{in}}$. We do stress that all the above properties are verified in quantum integrable models, the XXZ chain in particular.

Our relations lead to

$$\nu_l(-q) = v \cdot \left\{ kK - \frac{\alpha_l}{K} \right\} \quad \text{with} \quad vK = F_{\mathcal{R}_{n+k}}(-q) \quad \text{and} \quad \frac{v}{K} = F_{\mathcal{R}_{n-k}}(-q).$$

Then, the reflection property implies that $\nu_l(q) = \alpha_l - \nu_l(-q)$. All-in-all, this reproduces the scaling dimensions of the Luttinger liquid model.

**Conclusion**

In the present paper we have provided a first principle-based derivation of the emergence, in the large-separation regime between the operators, of an effective description of the model’s operators in terms of vertex operators associated with the free boson model. Our construction naturally allows one to treat as well the case of time dependent correlations and allows one to recover all the features of an effective field theoretic description which are at the very base of the nonlinear Luttinger liquid model [16]. In this respect, our method, allows one to justify, starting from the first principles, the use of the nonlinear Luttinger liquid model.
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Appendix A. Special functions

In this section we recall the definition and main properties of the special functions that are of use to our analysis. The Euler $\Gamma$-function is defined as

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t} \, dt$$

and satisfies $\Gamma(z + 1) = z \Gamma(z)$. (A.1)

The Barnes $G$-function is a generalization of the $\Gamma$ function, in the sense that it satisfies the functional relation $G(z + 1) = \Gamma(z)G(z)$. It admits the integral representation

$$G(1 + z) = \left(\sqrt{2\pi} \cdot \Gamma(z)\right)^z \cdot \exp \left\{ \frac{z(1 - z)}{2} - \int_0^z \ln \Gamma(s) \cdot ds \right\}$$

from which one can deduce the reflection formula

$$\frac{G(1 - z)}{G(1 + z)} = (2\pi)^{-z} \cdot \exp \left\{ \int_0^z \pi s \cot(\pi s) \cdot ds \right\}.$$  (A.3)

The latter readily implies the relation

$$G\left(1 + z, 1 - z - \ell, 1 - z + \ell\right) = \left(\frac{\sin[\pi z]}{\pi}\right)^\ell \cdot (-1)^{(\ell+1)/2}.$$  (A.4)

Above we have introduced the so-called hypergeometric like notation which will be used for products of ratios of Euler $\Gamma$ or Barnes $G$-function, e.g.

$$\Gamma\left(\begin{array}{c} \{ v_i \}_1^m \\ \{ w_k \}_1^m \end{array} \right) = \frac{\Gamma(v_1, \ldots, v_n)}{\Gamma(w_1, \ldots, w_m)}$$

and

$$G\left(\begin{array}{c} \{ v_i \}_1^m \\ \{ w_k \}_1^m \end{array} \right) = \frac{G(v_1, \ldots, v_n)}{G(w_1, \ldots, w_m)}.$$  (A.5)

Appendix B. Some integrals of interest

In this appendix, we compute some integrals that are of direct application to our study. Namely, given $p, h, t \in \mathbb{N}^n$, define

$$I_{hl}^{(1)}(\nu|\omega) = \oint_{|z|>|\omega|>|t|} \frac{d\zeta d\tau}{(2\pi)^2} \frac{\tau^{-p} \zeta^{h-1} (1 - \omega/\zeta)^\nu}{(1 - \tau/\omega)^\nu}$$  (B.1)

and

$$I_{hp}^{(2)}(\nu|\omega) = \oint_{|z|>|\omega|>|t|} \frac{d\zeta d\tau}{(2\pi)^2} \frac{\tau^{-p} \zeta^{-h} (1 - \tau/\omega)^\nu}{(1 - \zeta/\omega)^\nu}.$$  (B.2)
Two other integrals appear in the course of the analysis, namely
\[
\tilde{I}_{ph}^{(1)}(\nu|\omega) = \oint_{|\tau|>1,|\zeta|>1} d\zeta d\tau \frac{\tau^{-k}\zeta^{p-1}(1-\tau/\zeta)^\nu}{(2\pi)^2}, \\
\tilde{I}_{ph}^{(2)}(\nu|\omega) = \oint_{|\tau|>1,|\zeta|>1} d\zeta d\tau \frac{\tau^{-p+k-1}(1-\omega/\zeta)^\nu}{(2\pi)^2}.
\] (B.3)

These two new integrals are however related to the first two introduced above. Indeed, one has
\[
\tilde{I}_{ph}^{(1)}(\nu|\omega) = \tilde{I}_{ph}^{(1)}(-\nu|\omega) \quad \text{and} \quad \tilde{I}_{ph}^{(2)}(\nu|\omega) = \omega^{2(p+h-1)}\tilde{I}_{ph}^{(2)}(-\nu|\omega).
\] (B.4)

The first equality is straightforward whereas the second follows from the change of variables \((\tau, \zeta) \mapsto (\omega^2/x, \omega^2/y)\).

**Lemma B.1.** The double integrals \(\tilde{I}_{ph}^{(1)}(\nu|\omega)\) and \(\tilde{I}_{ph}^{(2)}(\nu|\omega)\) can be explicitly computed as
\[
\tilde{I}_{ph}^{(1)}(\nu|\omega) = \frac{\sin[\pi \nu]}{\pi (t - h + \nu)} \cdot \Gamma \left( h - \nu, \frac{t + \nu}{h} \right),
\] (B.5)
and
\[
\tilde{I}_{ph}^{(2)}(\nu|\omega) = \frac{\sin[\pi \nu]}{\pi (h + p - 1)} \cdot \Gamma \left( h + \nu, \frac{p - \nu}{h} \right).
\] (B.6)

**Proof.** The calculation can be done by means of the method proposed in appendix A of [1] which allows for an effective separation of the integrals. First of all, one observes that
\[
\left( \zeta \partial_\zeta + \tau \partial_\tau \right) \left\{ \frac{\zeta}{\zeta - \tau} \cdot \zeta^\nu \cdot \frac{(1 - \zeta^{-1})^\nu}{(1 - \tau)^\nu} \right\} = \nu \zeta^\nu \cdot \frac{(1 - \zeta^{-1})^{\nu-1}}{(1 - \tau)^{\nu+1}}.
\] (B.7)
Then, upon a rescaling of the integration variables, one gets
\[
\tilde{I}_{ph}^{(1)}(\nu|\omega) = \frac{\omega^{h-\nu}}{h - t - \nu} \oint_{|\tau|>1,|\zeta|>1} \frac{d\zeta}{2\pi} \frac{d\tau}{2\pi} \left\{ \frac{\zeta}{\zeta - \tau} \cdot \zeta^\nu \cdot \frac{(1 - \zeta^{-1})^\nu}{(1 - \tau)^\nu} \right\} \times \left( \zeta \partial_\zeta + \tau \partial_\tau \right) \left\{ \tau^{-1}(\zeta^{h-1-\nu}) \right\}.
\] (B.8)
An integration by parts then leads to
\[
\tilde{I}_{ph}^{(1)}(\nu|\omega) = \frac{-\omega^{h-\nu}}{h - t - \nu} \oint_{|\zeta|>1} \frac{d\zeta}{2\pi} \frac{d\tau}{2\pi} \tau^{-1}(\zeta^{h-2}(1 - \zeta^{-1})^{\nu-1}) \frac{(1 - \zeta^{-1})^{\nu-1}}{(1 - \tau)^{\nu+1}} \times \left( \zeta \partial_\zeta + \tau \partial_\tau \right) \left\{ \tau^{-1}(\zeta^{h-1-\nu}) \right\}.
\] (B.9)
The two integrals can then be evaluated by means of the series expansion
\[ (1 - z)^{\alpha} = \frac{\sin(\pi \alpha)}{\pi} \sum_{n \geq 0} z^n \cdot \Gamma\left(\alpha + 1, \frac{1}{n + 1}\right) \] (B.10)

hence leading to (B.5).

The integral \( \mathcal{I}_{\text{ph}}^{(2)}(\nu, \omega) \) is computed along the same lines on the basis of the identity
\[ \left( \zeta \partial_t + \tau \partial_v \right) \left( \frac{1 - \tau r^p}{1 - \zeta r^p} \right) = \nu \tau \cdot \frac{(1 - \tau r^{p-1})}{(1 - \zeta r^{p+1})}. \] (B.11)

The details are left to the reader. \( \square \)

**Appendix C. Proof of proposition 1.1**

The exponents of current operators \( e^{J_{\nu,\omega}} \) preserve the charge of a state. Therefore, the expectation value \( \langle J_{n_0, n_1}\rangle \langle e^{J_{\nu,\omega}} e^{J_{\nu,\omega}} | J_{n_0, n_1}\rangle \) is non-zero only if
\[ n_p = n_h = n_k - n_l. \] (C.1)

This explains the occurrence of the Kronecker symbol in (1.36).

Observe that the role played by \( \psi_0 \) and \( \psi_0^* \) in the whole construction of the Hilbert space \( \mathcal{H} \) is symmetric. In particular, this ensures that the result will be invariant under the transformation \( (\psi_0, \psi_0^*) \mapsto (\psi_{-1-k}, \psi_{-1-k}) \). We stress that this transformation leaves the vacuum \( |0\rangle \) unchanged. Since this transform maps the local current operators as \( J_k \mapsto - J_k \), we get the symmetry
\[ \langle J_{n_0, n_1} | e^{J_{\nu,\omega}} e^{J_{\nu,\omega}} | J_{n_0, n_1}\rangle = (-1)^{n_0 + n_1} \langle J_{n_0, n_1} | e^{-J_{\nu,\omega}} e^{-J_{\nu,\omega}} | J_{n_0, n_1}\rangle, \] (C.2)

where the sign prefactor issues from the change in the order of products of the \( \psi \)'s and \( \psi^* \)'s. It is easy to check that indeed, as soon as the condition \( n_p - n_h = n_k - n_l \) is imposed, the rhs of (1.35) is indeed invariant under the simultaneous transformations
\[ \left( J_{n_0, n_1}, J_{n_0, n_1}, \nu \right) \mapsto \left( J_{n_0, n_1}, J_{n_0, n_1} - \nu \right) \] (C.3)

followed by a multiplication by \((-1)^{n_0 + n_1}\). The above thus ensures that it is enough to establish the validity of the representation (1.35) solely in the case when \( n_p - n_h = n_k - n_l \geq 0 \). Hence, from now on, we focus on this case only and, so as to lighten the notation, we introduce the shorthand notations
\[ n_h = n \quad n_p = n + \ell \quad n_l = d \quad \text{and} \quad n_k = d + \ell. \] (C.4)

In order to compute the expectation value, we first reduce the problem to a simpler one where the application of Wick’s theorem is straightforward. This boils down the problem to the computation of determinants whose entries are given by bi- and quadrilinear expectation values in the fermions. The expectation values can be expressed in terms of the integrals computed in appendix B. Building on these results, we compute the various determinants by using their relation to Cauchy determinants.

*Reduction to a simpler problem*

Observe that, for any \( r \geq 0 \), one has
\[ \langle 0 | \psi_{-r}, \psi^*_{-r} = \langle 0 \rangle \] (C.5)
Let \( r_1, \ldots, r_\ell \) be distinct integers satisfying to the constraints
\[
  r_b \geq \max \left\{ h_a : a = 1, \ldots, n \right\}
\]
for any \( b = 1, \ldots, \ell \).
\[\text{(C.6)}\]

Then, in virtue of \((\text{C.5})\), we get
\[
  \langle \mathcal{J}_{n,b} \rangle = \left\langle 0 \right| \prod_{a=1}^{\ell} \left\{ \psi_{r_{a-1}}^* \right\} \cdot \psi_{r_1}^* \cdots \psi_{r_{\ell-1}}^* |\mathcal{J}_{n,b}\rangle
\]
\[
  \times ( -1 )^{n-b} |0\rangle \psi_{r_{b-1}}^* \cdots \psi_{r_{n-1}}^* |\mathcal{J}_{n,b}\rangle
\]
\[
  \times \psi_{r_{b-1}}^* \cdots \psi_{r_{n-1}}^* e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} |\mathcal{J}_{n,b}\rangle
\]
\[\text{(C.7)}\]

We now move the operators \( \psi_{r_{a-1}}^* \) through the exponents of current operators. For this purpose, we first represent \( \psi_{r_{a-1}}^* \) in terms of the conjugate field \( \Psi^a(\tau) \) by means of a contour integral around 0 and then we apply the exchange relation \((1.25)\) with:
\[
  \psi_{r_{a-1}}^* \cdots \psi_{r_{b-1}}^* e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} = \oint_{\gamma} \frac{d\tau}{2\pi i} \prod_{a=1}^{\ell} \left\{ \tau_{r_a}^{-1} \right\} \cdot \Psi^a(\tau) \cdots \Psi^a(\tau) \cdot e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)}
\]
\[
  = e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} \cdot \Phi^a_{r_{a-1}} \cdots \Phi^a_{r_{b-1}}
\]
\[
  \Phi^a_{r_{a-1}} = \oint_{\gamma} \frac{d\tau}{2\pi i} \left\{ \frac{1}{1 - \tau/\omega} \right\} \Psi^a(\tau) \cdot \frac{d\tau}{1 - \omega/\tau}
\]
\[\text{(C.8)}\]

Thus, upon extending the sequence \( h_a \) as \( h_{a+b} = r_b \) for \( b = 1, \ldots, \ell \) and after setting
\[
  w_a^* = \psi_{r_{a-1}}^* \quad a = 1, \ldots, d \quad w_{a+d}^* = \Phi^a_{r_{a-1}} \quad a = 1, \ldots, \ell
\]
\[\text{(C.9)}\]

the expectation value we started with reduces to
\[
  \langle \mathcal{J}_{n,b} \rangle e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} |\mathcal{J}_{n,b}\rangle = \left\langle 0 \right| e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} w_1^* \cdots w_{d+c}^* \cdot w_{d+c+1}^* \cdots w_1^* |0\rangle
\]
\[
  = (-1)^{f(n+d)} \left\langle 0 \right| e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} |0\rangle \cdot \det_{d+c+1} \left\{ M_{ab} \right\}
\]
\[\text{(C.10)}\]

The set \( \mathcal{J}_{n,\ell+n+\ell} \) appearing in \( \text{terms of the extended sequence} \{ h_a \}_{n+\ell}^{n+\ell} \) as
\[
  \mathcal{J}_{n,\ell+n+\ell} = \left\{ \left. \left( h_a \right)_{n+\ell} \cup \{ h_{a+b} \} \right\} \right\}
\]
\[\text{(C.11)}\]

We also specify that we have obtained the second line in \((\text{C.10})\) by applying Wick’s theorem. Finally, the matrix \( M \) arising in \((\text{C.10})\) reads
\[
  M_{ab} = \left\langle \mathcal{J}_{n,\ell+n+\ell} \right| e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} \psi_{r_{a-1}}^* \psi_{r_{b-1}}^* |0\rangle
\]
\[
  \left\langle \mathcal{J}_{n,\ell+n+\ell} \right| e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} |0\rangle
\]
\[\text{(C.12)}\]

for \( a = 1, \ldots, d \) and
\[
  M_{a+d,b} = \oint_{\gamma} \frac{d\tau}{2\pi i} \left\{ \frac{1}{1 - \tau/\omega} \right\} \left\langle \mathcal{J}_{n,\ell+n+\ell} \right| e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} \psi_{r_{b-1}}^* |0\rangle
\]
\[
  \left\langle \mathcal{J}_{n,\ell+n+\ell} \right| e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} |0\rangle
\]
\[\text{(C.13)}\]

for \( a = 1, \ldots, \ell \).

* Evaluation of \( \langle \mathcal{J}_{n,\ell+n+\ell} \rangle e^{\mathcal{F}(\nu,\omega)} e^{\mathcal{F}(\nu,\omega)} |0\rangle \)
The expectation value \( \langle J_{n+t,n+t+|} e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle \) can be computed by applying, again, Wick’s theorem:
\[
\langle J_{n+t,n+t+|} e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle = \text{det}_{n+t}\left[ \langle 0 | \psi_{p-1}^* \psi_{n+k} e^{J_{(ρ,ω)}} |0 \rangle \right].
\]
(C.14)
where we have used that \( e_{0} e_{0} |0 \rangle = |0 \rangle \) and \( 0| e_{0} e_{0} = \langle 0 |. \) The entries of the matrix arising in the determinant coincide with one of the integrals studied in appendix B. Indeed, one has
\[
\langle 0 | e_{0} e_{0} \rangle = \int_{|τ|>|ζ|} \frac{τ^{−2}ζ^{−1}}{τ − ζ} |0 \rangle \Psi(τ) \Psi(ζ) \cdot e^{J_{(ρ,ω)}} \cdot \frac{dτ dζ}{(2π)^{2}} = \mathcal{I}^{(2)}_{ph}(ω).
\]
(C.15)
In the intermediate calculations, we have used the exchange relation (1.25) along with
\[
\langle 0 | \Psi(τ) \Psi(ζ) |0 \rangle = \frac{τ}{τ − ζ} \quad \text{for} \quad |τ| > |ζ|.
\]
(C.16)
Hence, we arrive to the determinant representation
\[
\langle J_{n+t,n+t+|} e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle = \text{det}_{n+t}\left[ \mathcal{I}^{(2)}_{ph}(ω) \right].
\]
(C.17)
* Evaluation of \( \langle J_{n+t,n+t+|} e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle \)
Since the operator \( e_{0} e_{0} |0 \rangle \) is a group-like element, Wick’s theorem ensures that
\[
\langle J_{n+t,n+t+|} e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle = \text{det}_{n+t}\left[ \mathcal{I}^{(2)}_{ph}(ω) \right].
\]
(C.18)
The two-fermion expectation value can be computed analogously to (C.15)
\[
\langle 0 | e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle = \int_{|τ|>|ζ|} \frac{τ^{−2}ζ^{−1}}{τ − ζ} |0 \rangle \Psi(τ) \Psi(ζ) \cdot e^{J_{(ρ,ω)}} \cdot \frac{dτ dζ}{(2π)^{2}} = \mathcal{I}^{(2)}_{ph}(ω).
\]
(C.19)
The four point function is recast as
\[
\langle 0 | \psi_{p-1}^* \psi_{n+k} e^{J_{(ρ,ω)} e^{J_{(ρ,ω)}}} |0 \rangle = \int_{|τ|>|ζ|} \frac{τ^{−2}ζ^{−1}}{τ − ζ} |0 \rangle \Psi^2(τ) \Psi^2(ζ) \cdot e^{J_{(ρ,ω)}} \cdot \frac{dτ dζ}{(2π)^{2}}.
\]
(C.20)
A straightforward computation shows that
\[
\langle 0 | \Psi^2(τ) \Psi^2(ζ) |0 \rangle = \frac{τ • z}{(τ − ζ) • (τ − ζ)} + \frac{τ • z}{(τ − ζ) • (τ − ζ)}
\]
(C.21)
provided that the variables satisfy to the ordering $|c| > |\zeta| > |\tau| > |\xi|$. After inserting the above expectation value in (C.20), we are led to

$$
\langle \mathcal{J}_{n+t+n+\ell+1} \rangle_{\nu, \tau, \omega} = \langle \mathcal{J}_{n+1} \rangle_{\nu, \tau, \omega} \cdot \tilde{\mathcal{T}}_{n+1}^{(1)}(\nu|\omega) \cdot \tilde{\mathcal{T}}_{n+1}^{(2)(\nu)|\omega} \cdot \tilde{\mathcal{I}}_{n+1}^{(3)}(\nu|\omega).
$$

(C.22)

All this allows us to represent the expectation value of interest as

$$
\langle \mathcal{J}_{n+t+n+\ell} \rangle_{\nu, \tau, \omega} e^{\mathcal{J}_{n+t+n+\ell}} \cdot \tilde{\mathcal{T}}_{n+1}^{(1)}(\nu|\omega) \cdot \tilde{\mathcal{T}}_{n+1}^{(2)(\nu)|\omega} \cdot \tilde{\mathcal{I}}_{n+1}^{(3)}(\nu|\omega).
$$

This representation involves the determinant of the matrix $\tilde{\mathcal{N}} = \tilde{\mathcal{T}}_{n+1}^{(2)(\nu)|\omega}$ perturbed by a rank 1 matrix. Since the matrix $\mathcal{N}$ is of Cauchy type, its inverse is explicitly computable and takes the form

$$
(N^{-1})_{ab} = -\frac{\pi \omega^1 p - h_a}{\sin(\pi \nu)} \cdot \Gamma\left(\begin{array}{c} p_b, \\ h_a \\ - \frac{h_a}{h_a + p_b - 1} \end{array} \right) \cdot \frac{1}{h_a + p_b - 1} 
\times \prod_{c=1}^{n+\ell} \left( (h_c + p_b - 1) \cdot (h_a + p_b - 1) \right). 
$$

(C.24)

Thus, recalling the representation (C.17) we are led to

$$
\langle \mathcal{J}_{n+t+n+\ell} \rangle_{\nu, \tau, \omega} e^{\mathcal{J}_{n+t+n+\ell}} \cdot \tilde{\mathcal{T}}_{n+1}^{(1)}(\nu|\omega) \cdot \langle \mathcal{J}_{n+1} \rangle_{\nu, \tau, \omega} \cdot \mathcal{T}_{n+1}^{(2)(\nu)|\omega} \cdot \mathcal{S}_{n+1}^{(3)}(\mathcal{J}_{n+1+t+n+\ell}).
$$

(C.25)

The set function $S_{n+1+t+n+\ell}$ takes into account the contribution issuing from the rank 1 perturbation of $\mathcal{N}$ and takes the form:

$$
S_{n+1+t+n+\ell} = 1 + \sum_{a,b=1}^{n+\ell} \frac{T_{n+1}^{(1)}(\nu|\omega) \cdot (N^{-1})_{ab} \cdot \tilde{\mathcal{T}}_{n+1}^{(1)}(\nu|\omega)}{	ilde{\mathcal{T}}_{n+1}^{(2)(\nu)|\omega}}
$$

$$
= 1 + \sum_{a,b=1}^{n+\ell} \left( \prod_{c=1}^{n+\ell} \left( (h_c + p_a - 1) \cdot (h_b + p_a - 1) \right) \right) \prod_{c\neq a}^{n+\ell} \left( (h_c - h_b) \right)
$$

$$
\times \left[ \prod_{c=1}^{n+\ell} \left( (h_c + p_b - 1) \cdot (h_a + p_b - 1) \right) \right] \prod_{c\neq a}^{n+\ell} \left( (h_c - h_b) \right)
$$

$$
= 1 + \oint_{\Gamma_1} \frac{ds}{2\pi i} \oint_{\Gamma_2} \frac{dz}{2\pi i} \left( \prod_{c=1}^{n+\ell} \left( \frac{(h_c + s - 1) \cdot (z + p_b - 1)}{(p_b - s) \cdot (h_c - z)} \right) \right).
$$

(C.26)

Above, $\Gamma_1(\{p_b\})$, resp. $\Gamma_2(\{h_a\})$, stands for a small counterclockwise loop around $p_1, ..., p_{n+\ell}$, resp. $h_1, ..., h_{n+\ell}$, that avoids all other singularities of the integrand. Note that the contour integral should be understood as an encased integral. As a consequence, the $z$-integral does not contain poles at $z = 1 - s$. The integral can be evaluated by taking the residues located
outside of the contours of integration:

\[
S_{t,k}(\mathcal{J}_{n+t,n+\ell}) = 1 + \oint_{(t,k)} \frac{dx}{2\pi i} \frac{t + k - 1}{(s + t + \nu - 1)(k - s - \nu)} \times \prod_{c=1}^{n+\ell} \left\{ \left( h_c + s - 1 \right) \cdot \left( t + \nu + p_c - 1 \right) \right. \\
\left. \left( p_c - s \right) \cdot \left( h_c - t - \nu \right) \right\} \\
= \prod_{c=1}^{n+\ell} \left\{ \left( h_c + k - \nu - 1 \right) \cdot \left( t + \nu + p_c - 1 \right) \right. \\
\left. \left( p_c - k + \nu \right) \cdot \left( h_c - t - \nu \right) \right\}. \tag{C.27}
\]

All-in-all, we are thus led to the representation

\[
\langle \mathcal{J}_{n+t,n+\ell} | e^{\mathcal{F}(t,\omega)} e^{\mathcal{F}(\nu,\omega)} | \psi_{k-1} \rangle = \mathcal{I}^{(2)}_{t,k}(\nu|\omega) \prod_{c=1}^{n+\ell} \left\{ \left( h_c + k - \nu - 1 \right) \cdot \left( t + \nu + p_c - 1 \right) \right. \\
\left. \left( p_c - k + \nu \right) \cdot \left( h_c - t - \nu \right) \right\}. \tag{C.28}
\]

* Evaluation of $M_{a+d,b}^a$ for $a = 1, \ldots, \ell$

It follows from Wick’s theorem that one has the representation

\[
\langle \mathcal{J}_{n+t,n+\ell} | e^{\mathcal{F}(t,\omega)} e^{\mathcal{F}(\nu,\omega)} | \psi_{k-1} \rangle \cdot M_{a+d,b}^a = \oint \frac{(1 - \tau/\omega)^{\nu} g_k^{(2)}(\tau)}{(1 - \omega/\tau)^{\nu}} \tau^{n+1} \\
\times \det_{n+\ell} \left[ \langle 0 | \psi_{R-1} | \psi_{R} \rangle e^{\mathcal{F}(t,\omega)} e^{\mathcal{F}(\nu,\omega)} \Psi^*(\tau) | \psi_{k-1} \rangle \right] \frac{d\tau}{2\pi}. \tag{C.29}
\]

where $q$, $m$ label the entries of the matrix whose determinant is to be evaluated while the function $g_k^{(2)}(\tau)$ is defined by means of a contour integral:

\[
g_k^{(2)}(\tau) = \langle 0 | e^{\mathcal{F}(t,\omega)} \Psi^*(\tau) | \psi_{k-1} \rangle = \oint_{|\tau| > \ldots > |\zeta| > 1} \frac{\tau \cdot \zeta^{-k}}{\tau - \zeta} \frac{(1 - \zeta/\omega)^{\nu}}{(1 - \tau/\omega)^{\nu}} \frac{d\zeta}{2\pi}. \tag{C.30}
\]

The expectation value in the determinant is computed along the lines already discussed at length and reads:

\[
\langle 0 | \psi_{R-1} | \psi_{R} e^{\mathcal{F}(t,\omega)} e^{\mathcal{F}(\nu,\omega)} \Psi^*(\tau) | \psi_{k-1} \rangle = \mathcal{I}_{k+1}^{(1)}(\nu|\omega) \cdot g_k^{(1)}(\tau) + \mathcal{I}_{k+1}^{(2)}(\nu|\omega) \cdot g_k^{(2)}(\tau), \tag{C.31}
\]

where the function $g_k^{(1)}(\tau)$ is defined as

\[
g_k^{(1)}(\tau) = \oint_{|\tau| > \ldots > |\omega| > 1} \frac{\tau \cdot \zeta^{-k}}{\zeta - \tau} \frac{(1 - \omega/\zeta)^{\nu}}{(1 - \tau/\zeta)^{\nu}} \frac{d\zeta}{2\pi}. \tag{C.32}
\]

Upon factorizing the matrix $(\mathcal{M})_{ab} = \mathcal{I}_{n+\ell}(\nu|\omega)$ from the determinant in (C.29) we are led to the representation
M_{a+d b} = \oint_{|\tau|>|\omega|>1} \frac{\tau^{-s_a} \cdot \zeta^{-s_b} \cdot (1 - \zeta/\omega)^{\nu} \cdot (1 - \omega/\tau)^{\nu}}{\tau - \zeta} \cdot \oint_{|\tau|>|\omega|>1} \frac{\tau^{-s_a} \cdot \zeta^{-s_b} \cdot (1 - \omega/\tau^\nu)}{\tau - \zeta} \cdot \frac{d\zeta d\tau}{(2\pi)^2} \
+ \sum_{m,q=1}^{n+\ell} (N_{mq})_{\nu} \oint_{|\tau|>|\omega|>1} \frac{\tau^{-s_a} \cdot \zeta^{-s_b} \cdot (1 - \zeta/\omega)^{\nu} \cdot (1 - \omega/\tau)^{\nu}}{\tau - \zeta} \cdot \oint_{|\tau|>|\omega|>1} \frac{\tau^{-s_a} \cdot \zeta^{-s_b} \cdot (1 - \omega/\tau^\nu)}{\tau - \zeta} \cdot \frac{d\zeta d\tau}{(2\pi)^2}.

(C.33)

The first integral vanishes as can be seen by deforming the \( \tau \)-contour to \( \infty \). Likewise, by deforming the \( \tau \)-contour to \( \infty \) in the integral arising under the double summation symbol, one solely gets the contribution of the pole at \( \tau = \zeta \). The latter yields the Kronecker symbol \( \delta_{m,n+a} \) which is there so as to enforce the constraint that \( m \) should be such that \( h_m = \ell_c \). After some algebra, one recasts \( M_{a+d b} \) in the form

\[
M_{a+d b} = \omega^{1-k_a-h_{a+b}} \cdot \Gamma \left( \frac{h_{n+a} + p_c - 1}{h_{n+a} - h_{n+a}} \right) \cdot \frac{\prod_{c=1}^{n+\ell} (h_c + p_c - 1)}{\prod_{c=1}^{n+\ell} (p_c - p_q)}.
\]

(C.34)

where

\[
\mathcal{G}_{a+b} = \sum_{q=1}^{n+\ell} \frac{1}{(k_b - p_q - \nu) \cdot (h_{n+a} + p_q - 1)} \cdot \prod_{c=1}^{n+\ell} \left( h_c + p_c - 1 \right) \cdot \prod_{c=1}^{n+\ell} (p_c - p_q) = \oint \left( \frac{1}{h_{n+a} + k_b - \nu} \right) \cdot \prod_{c=1}^{n+\ell} \frac{h_c + k_b - \nu - 1}{p_c - k_b + \nu} \cdot \frac{d\nu}{2\pi i}.
\]

(C.35)

* Calculation of \( \det_{d+\ell}[M_{a b}] \)

Summarising the results obtained so far, we get

\[
M_{a b} = \sin[\pi \nu] \cdot \omega^{1-k_a-\ell_a} \cdot \Gamma \left( \frac{k_b - \nu, \ell_a + \nu}{k_b, \ell_a} \right) \times \prod_{c=1}^{n+\ell} \left\{ \frac{(p_c + \ell_a + \nu - 1) \cdot (h_c + k_b - \nu - 1)}{(h_c - \ell_a - \nu)(p_c - k_b + \nu)} \right\}
\]

(C.36)

for \( a = 1, \ldots, d \) and \( b = 1, \ldots, d + \ell \) while

\[
M_{a+d b} = \frac{-\omega^{1-k_a-h_{n+a}}}{h_{n+a} + k_b - \nu - 1} \cdot \Gamma \left( \frac{h_{n+a}, k_b - \nu}{h_{n+a} - \nu, k_b} \right) \times \prod_{c=1}^{n+\ell} \left\{ \frac{(h_c + k_b - \nu - 1)(h_{n+a} + p_c - 1)}{(p_c - k_b + \nu) \cdot \prod_{c=1}^{n+\ell} (h_c - h_{n+a})} \right\}
\]

(C.37)

We now recast the matrix \( M_{a b} \) into a uniform representation. Having this in mind, we introduce two sequences \( \alpha_a \), and \( \beta_a \) with \( a = 1, \ldots, d + \ell \). The sequences \( \alpha_a \) reads

\[
\alpha_a = \sin[\pi \nu] \cdot \prod_{c=1}^{n+\ell} \left\{ \frac{(p_c + \ell_a + \nu - 1)}{(h_c - \ell_a - \nu)} \right\}
\]

(C.38)
and
\[
\alpha_{d+\nu} = -\omega^{h_{n+\nu}} \cdot \Gamma \left( \frac{h_{n+\nu}}{h_{n+\nu} - \nu} \right) \cdot \prod_{c=1}^{\nu+\ell} \frac{(h_{c} + p_{c} - 1)}{h_{c} - h_{n+\nu}} \tag{C.39}
\]
where the index \( a \) runs over \( \{1, \ldots, d\} \) while the index \( s \) runs over \( \{1, \ldots, \ell\} \). Furthermore, let
\[
\beta_{b} = \omega^{k_{b} - \nu} \cdot \Gamma \left( \frac{k_{b} - \nu}{k_{b}} \right) \cdot \prod_{c=1}^{\nu+\ell} \left\{ \frac{h_{c} + k_{b} - \nu - 1}{(p_{c} - k_{b} + \nu)} \right\} \quad \text{for} \quad b = 1, \ldots, d + \ell. \tag{C.40}
\]
Then, upon extending the sequence \( t_{a} \), for \( a = d + 1, \ldots, d + \ell \), as
\[
t_{a+d} = h_{n+a} - \nu \tag{C.41}
\]
we see that \( M \) is closely related to a Cauchy matrix
\[
M_{ab} = \frac{\alpha_{a} \cdot \beta_{b}}{k_{b} + t_{a} - 1} \quad \text{for} \quad a, b \in \{1, \ldots, d + \ell\}. \tag{C.42}
\]

* Synthesis of the calculations

By gathering all together the results obtained in the course of the proof, one arrives at the representation
\[
(J_{n,z}) e^{i\nu,\omega} e^{i\nu,\omega} (J_{n,z}) = (-1)^{d+\nu+\ell} \det \left[ \frac{-\sin \pi \nu}{\pi (p_{c} + h_{b} - 1)} \cdot \Gamma \left( \frac{h_{b} - \nu}{h_{b}}, \frac{p_{c} + \nu}{p_{c}} \right) \right]
\times \prod_{a=1}^{d+\ell} \left\{ \frac{\alpha_{a} \cdot \beta_{a}}{p_{a} + h_{b} - 1} \cdot \det_{a+\ell} \left[ \frac{1}{p_{a} + h_{b} - 1} \right] \cdot \det_{d+\ell} \left[ \frac{1}{t_{a} + k_{b} - 1} \right] \right\}. \tag{C.43}
\]
It is then a matter of straightforward calculations to obtain the representation \((1.35)\). In particular, all the dependence on the auxiliary integers \( \{t_{a}\}_{d}^{\ell} \) completely disappears. \( \square \)

Appendix D. The multi-point restricted sums

The multi-point restricted sums were first introduced in \([28]\). This name refers to a multidimensional summation formula that is necessary for the computation of the localized form factor series expansion. More precisely, given \( |z_{1}| > \ldots > |z_{e}| \) and generic complex numbers \( \nu_{1}, \ldots, \nu_{e} \) it holds
\[ \mathcal{F}(\nu_a; z_a) = \prod_{x=1}^{r-1} \left\{ \sum_{n^0_k, n^0_h=0}^{+\infty} \sum_{n^0_k - n^0_h = L_x}^* \prod_{s=1}^{r-1} \mathcal{R} \left( \mathcal{J}_{n_k, n_h}^{s, s_0} \left| \nu_s, \nu_{s+1}; \frac{z_s}{z_{s+1}} \right) \right\} \times \prod_{x=2}^{r-1} \mathcal{R} \left( \mathcal{J}_{n_k, n_h}^{s-1, s_0} \left| \nu_s, \nu_{s+1}; \pm \nu_s \right) \right. \]

\[ = \prod_{x=1}^{r-1} \left\{ \frac{z_{x+1}}{z_x} \frac{\ell_x (\ell_{x+1})}{2} G \left( \frac{1 + \ell_x - \nu_s, 1 + \ell_x + \nu_{x+1}}{1 - \nu_s, 1 + \nu_{x+1}} \right) \right\} \times \prod_{s=2}^{r-1} G \left( 1 + \nu_s, 1 + \ell_{s-1} - \ell_s + \nu_s, 1 - \ell_s + \nu_s, 1 + \ell_{s-1} - \nu_s \right) \]

\[ \times \prod_{b > a} \left( 1 - \frac{z_b}{z_a} \right)^{\kappa_b - \kappa_a (\nu_b + \kappa_b)} \]  

The function \( \mathcal{R} \) appearing above takes the form

\[ \mathcal{R}(\mathcal{J}_{n_k, n_h}^{s, s_0}; \nu, \eta; z) = \left( -\frac{\sin[\pi \nu]}{\pi} \cdot \frac{\sin[\pi \eta]}{\pi} \right)^{n_s} \cdot \frac{\prod_{a \neq b}^{n} (p_a - p_b)^2 \cdot \prod_{a=1}^{n} (h_a - h_b)^2}{\prod_{a=1}^{n} (p_a + h_a - 1)^2} \]

\[ \times \prod_{a=1}^{n} \left\{ z^a \Gamma \left( \frac{p_a - \nu, p_a + \eta}{p_a, p_a} \right) \right\} \cdot \prod_{a=1}^{n} \left\{ z^{h_a-1} \Gamma \left( \frac{h_a + \nu, h_a - \eta}{h_a, h_a} \right) \right\}. \]

This summation formula has been argued in [28] on the basis of comparing the large-\( N \) asymptotic expansion of a Toeplitz determinant generated by a symbol having Fisher–Hartwig singularities obtained by two different means. In this section of the appendix, we derive the above formula by using the free fermion formalism that we have developed. For this purpose we compute the expectation value

\[ \langle 0 | \mathcal{F}(\nu_1, \kappa_1 | z_1) \cdots \mathcal{F}(\nu_r, \kappa_r | z_r) | 0 \rangle \]  

in two ways. First of all, by inserting the decomposition of the identity

\[ \sum_{n_k, n_h} | \mathcal{J}_{n_k, n_h} \rangle \langle \mathcal{J}_{n_k, n_h} | = \text{id} \]  

between each vertex operator and this, for the specific choice of the sequence \( \kappa_a; \)

\[ \kappa_a = \ell_{a-1} - \ell_a, \quad \text{where} \quad \ell_0 = \ell_r = 0. \]

Second, by applying repeatedly the exchange relation

\[ e^{\mathcal{F}(\nu, \omega)} \cdot e^{\mathcal{F}(\mu, \omega)} = \left( 1 - \frac{z}{\omega} \right)^{\mu \nu} \cdot e^{\mathcal{F}(\mu, \omega)} \cdot e^{\mathcal{F}(\nu, \omega)}, \]

satisfied by the current operators. The second method yields

\[ \langle 0 | \mathcal{F}(\nu_1, \kappa_1 | z_1) \cdots \mathcal{F}(\nu_r, \kappa_r | z_r) | 0 \rangle = \prod_{b > a} \left( 1 - \frac{z_b}{z_a} \right)^{\kappa_b - \kappa_a (\nu_b + \kappa_b)} \]
while the first one leads to
\[
\langle 0 | \mathcal{F}(\nu_1, \kappa_1|z_1) \cdots \mathcal{F}(\nu_r, \kappa_r|z_r)|0 \rangle = \sum_{\nu_p^{(i)}, n_p^{(i)}=\nu_p^{(i)} \in J_n^{(i)}, n_p^{(i)}=\nu_p^{(i)} \in J_n^{(i)}} \times \prod_{s=1}^{r} \left\{ \langle J_n^{(i+1)-n_s^{(i-1)}} \mathcal{F}(\nu_s, \kappa_s|z_s)|J_n^{(i+1)-n_s^{(i-1)}} \rangle \right\} \tag{D.7}
\]

Above, we have adopted the convention that
\[
n_p^{(0)} = n_h^{(0)} = n_p^{(r)} = n_h^{(r)} = 0 \quad \text{viz} \quad |J_n^{(0)-n_s^{(0)}} = |J_n^{(r)-n_s^{(r)}} = |0 \rangle. \tag{D.8}
\]

The constraints on the differences in the numbers of particles and holes in the excitations of the intermediary states issue from charge conservation requirements and can be recast, due to the parametrization (D.4),
\[
n_p^{(s)} - n_h^{(s)} = \ell_s \quad s = 1, \ldots, r - 1. \tag{D.9}
\]

Then a straightforward calculation recasts the sum as
\[
\langle 0 | \mathcal{F}(\nu_1, \kappa_1|z_1) \cdots \mathcal{F}(\nu_r, \kappa_r|z_r)|0 \rangle \cdot C \left( \{ \nu_a \}, \{ z_a \} \right) = \mathcal{F} \left( \{ \nu_a \}, \{ z_a \} \right), \tag{D.10}
\]

where the constant \( C \left( \{ \nu_a \}, \{ z_a \} \right) \) takes the form
\[
C \left( \{ \nu_a \}, \{ z_a \} \right) = \prod_{s=1}^{r} \left\{ G \left( 1 - \nu_s - \kappa_s \right), \left( -\sin[\pi \nu_s] / \pi \right)^{-\ell_{s-1}} \right\} \times \frac{1}{(-1)^{\ell_{s-1}(\ell_{s-1}+1)/2 + \kappa_s(\kappa_s+1)/2}} \prod_{s=1}^{r-1} \left\{ \left( z_{s+1} / z_s \right)^{\ell_s} \right\}. \tag{D.11}
\]

By using the reduction property of the Barnes function (A.4) and the parametrization of \( \kappa_s \) in terms of \( \ell_s \), one gets that
\[
\prod_{s=1}^{r} \left\{ G \left( 1 - \nu_s - \kappa_s \right), \left( -\sin[\pi \nu_s] / \pi \right)^{-\ell_{s-1}} \right\} \times \frac{1}{(-1)^{\ell_{s-1}(\ell_{s-1}+1)/2 + \kappa_s(\kappa_s+1)/2}} \prod_{s=1}^{r-1} \left\{ \left( z_{s+1} / z_s \right)^{\ell_s} \right\}
\]

and
\[
\prod_{s=1}^{r} \left\{ \frac{1}{z_s^{\kappa_s(\kappa_s+1)/2}}, \prod_{s=1}^{r-1} \left\{ \left( z_{s+1} / z_s \right)^{\ell_s} \right\} \right\} = \prod_{s=1}^{r-1} \left\{ \left( z_{s+1} / z_s \right)^{\ell_s} \right\}. \tag{D.12}
\]

Putting all of this together leads to (D.1).
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