A Hilbert Space of Stationary Ergodic Processes

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Abstract—Identifying meaningful signal buried in noise is a problem of interest arising in diverse scenarios of data-driven modeling. We present here a theoretical framework for exploiting intrinsic geometry in data that resists noise corruption, and might be identifiable under severe obfuscation. Our approach is based on uncovering a valid complete inner product on the space of ergodic stationary finite valued processes, providing the latter with the structure of a Hilbert space on the real field. This rigorous construction, based on non-standard generalizations of the notions of sum and scalar multiplication of finite dimensional probability vectors, allows us to meaningfully talk about “angles” between data streams and data sources, and, make precise the notion of orthogonal stochastic processes. In particular, the relative angles appear to be preserved, and identifiable, under severe noise, and will be developed in future as the underlying principle for robust classification, clustering and unsupervised featurization algorithms.

I. Preliminary Concepts

Definition 1 (Inner Product & Inner Product Spaces). An inner product on a real vector space \( X \) is a function \( \langle \cdot, \cdot \rangle : X \times X \to \mathbb{R} \), such that the following conditions are satisfied:

- \( \forall u, v, w \in X, \alpha \in \mathbb{R}, \langle u, \alpha(v + w) \rangle = \alpha \langle u, v \rangle + \langle u, w \rangle \) (Bi-linearity)
- \( \forall u, w \in X, \langle u, w \rangle = \langle w, u \rangle \) (Symmetry)
- \( \forall u, v \in X, \langle u, v \rangle \geq 0 \), where \( \langle u, u \rangle = 0 \Rightarrow u = 0 \) (Positive Definiteness)

A vector space with an inner product is an inner product space. Note that an inner product necessarily induces a norm, which in turn induces a metric [6], [10].

Definition 2 (Complete inner product space or Hilbert Space). A complete inner product space, or a Hilbert space [10], is a Banach space with an inner product, i.e., every Cauchy sequence in the space converges in the space.

Notation 1 (Strictly Positive Probability Vectors). For \( n \in \mathbb{N} \), the space of strictly positive probability vectors is defined as:

\[ \mathcal{P}_n^+ = \left\{ \rho \in \mathbb{R}^n : \forall i, \rho_i > 0, \sum_i \rho_i = 1 \right\} \tag{2} \]

A. An Abelian Group on Probability Vectors

\( \mathcal{P}_n^+ \) can be given the structure of an Abelian group [9], via the following binary operation: \( \oplus : \mathcal{P}_n^+ \times \mathcal{P}_n^+ \to \mathcal{P}_n^+ \) [1]:

\[ \forall \rho, \rho' \in \mathcal{P}_n^+, \forall i \in \{1, \cdots, n\}, \]

\[ (\rho \oplus \rho')_i \triangleq \rho_i \rho'_i \left( \sum_j \rho_j \rho'_j \right)^{-1} \tag{3} \]

We denote \( \oplus \) simply as \( + \) in the sequel if there is no confusion. It is easy to see that we have the following properties (which makes \( \mathcal{P}_n^+ \) into an Abelian group, with \( + \) as the group sum):

\[ \forall \rho, \rho' \in \mathcal{P}_n^+, \rho + \rho' \in \mathcal{P}_n^+ \] \tag{4a}
\[ \rho + \rho' = \rho' + \rho \] \tag{4b}

\[ \exists \rho_n \in \mathcal{P}_n^+, \text{ such that } \forall \rho \in \mathcal{P}_n^+, \rho + \rho_n = \rho \] \tag{4c}
\[ \forall \rho \in \mathcal{P}_n^+, \exists \rho' \in \mathcal{P}_n^+, \text{ such that } \rho + \rho' = \rho_n \] \tag{4d}

It follows that the additive identity \( \rho_n \) is given by the uniform probability vector. In \( \mathcal{P}_n^+ \), it is given by:

\[ \rho_n = (1/n, 1/n, \cdots, 1/n) \tag{5} \]

The “zero element” of the group is the uniform distribution.

B. Closed Scalar Multiplication on Probability Vectors

Since finite dimensional probability vectors reside in \( \mathbb{R}^n \), we already have the usual elementwise multiplication by scalars. However, the result of such elementwise scaling will not be a “probability vector”; the 1-norm will not be unitary. Thus, the latter is considered as a real vector space, with the vector addition and scalar multiplication operations as defined in Eq. (3) and Eq. (4) respectively.

\[ \forall \alpha \in \mathbb{R}, \forall \rho \in \mathcal{P}_n^+, \alpha \rho \in \mathcal{P}_n^+ \] \tag{6a}

In the sequel we denote this scalar multiplication by simple concatenation (dropping the \( \oplus \)) if there is no confusion. It is easy to see that:

\[ \forall \alpha \in \mathbb{R}, \forall \rho \in \mathcal{P}_n^+, \alpha \rho \in \mathcal{P}_n^+ \] \tag{6b}
\[ \forall \rho \in \mathcal{P}_n^+, 0 \rho = \rho_n \] \tag{6c}
\[ \forall \alpha \in \mathbb{R}, \forall \rho, \rho' \in \mathcal{P}_n^+, \alpha (\rho + \rho') = \alpha \rho + \alpha \rho' \] \tag{6d}
\[ \forall \alpha \in \mathbb{R}, \forall \rho \in \mathcal{P}_n^+, \alpha (\rho + (\alpha') \rho) = (\alpha + \alpha') \rho \] \tag{6e}

Thus, \( \mathcal{P}_n^+ \) has the structure of a real vector space, where the group sum is the vector sum, and the above defined product is the scalar product between the vectors and the field elements.

II. Inner Product on Probability Vectors

The usual “dot” product for \( n \)-dimensional vectors quite obviously applies to elements from \( \mathcal{P}_n^+ \). However, this is not the only consistent inner product on \( \mathcal{P}_n^+ \) over the real field.

Definition 3 (Inner product of probability vectors). We define \( \langle \cdot, \cdot \rangle : \mathcal{P}_n^+ \times \mathcal{P}_n^+ \to \mathbb{R} \) as:

\[ \forall \rho, \rho' \in \mathcal{P}_n^+, \langle \rho, \rho' \rangle = \sum_{i=1}^{n-1} \ln (\rho_i/\rho_{i+1}) \ln (\rho'_i/\rho'_{i+1}) \tag{8} \]

Lemma 1. Defn. 3 specifies an inner product on \( \mathcal{P}_n^+ \), when the latter is considered as a real vector space, with the vector addition and scalar multiplication operations as defined in Eq. (3) and Eq. (4) respectively.

Proof: The conditions of Def. 1 are easily verified, which completes the proof.

Notation 2. On account of Lemma 1, we denote the real-valued function introduced in Defn. 3 as the logarithmic inner product.
Next, we claim that \( (\mathcal{P}_n^+, \langle \cdot, \cdot \rangle) \) is indeed a complete inner product space, i.e., a Hilbert space. Note that since \( \mathcal{P}_n^+ \) only considers probability vectors with non-zero entries, it might seem that we lose completeness: a sequence of such strictly elementwise positive probability vectors can very well converge to one that has zero entries, and hence outside \( \mathcal{P}_n^+ \). Nevertheless, we have the following result:

**Lemma 2 (Hilbert space of probability vectors).** \( \mathcal{P}_n^+ \) is complete w.r.t. to the norm induced by the logarithmic inner product.

**Proof:** We need to show that every Cauchy sequence in \( \mathcal{P}_n^+ \) w.r.t. to the norm induced by the logarithmic inner product converges in \( \mathcal{P}_n^+ \).

Let \( \{x_n\} \) be a Cauchy sequence in a normed vector space \( X \), where \( d(\cdot, \cdot) \) denotes the induced metric. We claim:

\[
\forall \varepsilon > 0, \exists k \in \mathbb{N} \text{ such that } \forall n \geq k, \|x_n\| < \varepsilon \quad \text{(Claim A)}
\]

To establish Claim A, we assume if possible:

\[
\forall \varepsilon > 0, \exists k \in \mathbb{N} \text{ such that } \forall n \geq k, \|x_n\| > \varepsilon \quad \text{(Assumption A)}
\]

Now, from definition of Cauchy sequences, we have:

\*

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ such that } \forall m, n > N, d(x_m, x_n) < \varepsilon \quad (9)
\]

Fix some \( \varepsilon \) and a corresponding \( N \). For any \( x_0 \in X \), and \( \forall m, n > N \), we have:

\[
d(x_m, x_0) \leq d(x_m, x_n) + d(x_n, x_0) \quad \text{(triangular inequality)}
\]

\[
\Rightarrow \varepsilon > d(x_m, x_n) \geq d(x_m, x_0) - d(x_n, x_0) \quad (10)
\]

Setting \( x_0 \) to be the vector space zero, we have:

\[
\forall m, n > N, ||x_m|| - ||x_n|| \leq \varepsilon \quad (11)
\]

Clearly, if Assumption A holds, we can pick \( m \) and \( n \) that contradict Eq. (11). Thus, we conclude that the terms of any Cauchy sequence necessarily remains bounded. Since having any zero entry would imply an unbounded induced norm, we conclude that sequences that converge outside \( \mathcal{P}_n^+ \) are not Cauchy. It follows that every Cauchy sequence must converge within \( \mathcal{P}_n^+ \). This completes the proof.

A. Geodesics in the Space of Probability Vectors

A geodesic in a metric space is a path connecting two points, such that no other path has a shorter length. For completeness, we note here the formal definition of path length, and geodesics.

First, we note the following result:

**Lemma 3.** Let \( \varphi_0, \varphi_1 \in \mathcal{P}_n^+ \). Then, for \( \theta \in [0, 1] \),

\[
\varphi_\theta = \theta \varphi_0 \oplus (1 - \theta) \varphi_1 = \varphi_0 + (1 - \theta)\varphi_1 - \delta \theta \|\varphi_0 - \varphi_1\|,
\]

where the norm is induced by the logarithmic inner product.

**Proof:** We note that:

\[
\varphi_\theta = \left[ \cdots \left( \varphi_0 \right)^j \left( \varphi_1 \right)^{1-j} \cdots \right]
\]

\[
\varphi_\theta = \left[ \cdots \left( \varphi_0 \right)^{1-j} \left( \varphi_1 \right)^j - \delta \theta \|\varphi_0 - \varphi_1\| \right]
\]

which implies:

\[
\varphi_\theta - \varphi_0 = \delta \theta \left[ \cdots \left( \varphi_0 \right)^{1-j} \left( \varphi_1 \right)^j - \delta \theta \|\varphi_0 - \varphi_1\| \right]
\]

which completes the proof.

**Definition 4 (Length of a Curve and rectifiable curves).** Let \( (X, d) \) be a metric space, \( I \subset \mathbb{R} \) a non-empty interval, and \( \gamma : I \to X \) a Lipshitz-continuous map, i.e., a curve. We define the length \( L(\gamma) \in [0, \infty) \):

\[
L(\gamma) = \int_I |\gamma(t)| \, dt \quad (18)
\]

and using the fact that for almost all \( t \), the above limsup is a true limit, we can write:

\[
\gamma(t) = \limsup_{h \to 0} \frac{d(\gamma(t + h), \gamma(t))}{h} \quad (17)
\]

**Definition 5 (Length Spaces).** For a metric space \( (X, d) \), the inner or length metric associated with \( d \) is the function \( d^* : X \times X \to [0, \infty] \) defined by:

\[
d^*(x, y) = \inf \{ L(\gamma) ; \gamma \in \text{Lip}[0, 1], X, \gamma(0) = x, \gamma(1) = y \}
\]

where \( \text{Lip}[0, 1], X \) denotes the set of all Lipschitz continuous maps from \([0, 1]\) to \( X \). By triangular inequality, we have \( d^* \geq d \). If \( d = d^* \) for all rectifiable curves, then \( X \) is a length space.

**Definition 6 (Geodesic).** In a metric space \( (X, d) \), a rectifiable curve \( \gamma : I \to X \) is geodesic if \( \gamma \) has constant speed and for all \( t, t' \in I, t \leq t' \):

\[
L(\gamma|_{[t, t']}) = d(\gamma(t), \gamma(t')) \quad (19)
\]

**Remark 1.** It follows immediately that a rectifiable curve...
\( \gamma : I \to X \) is a geodesic if and only if
\[
\forall t, t' \in I, \exists \lambda \in (0, \infty), \quad \left| a(\gamma(t), \gamma(t')) \right| = \lambda |t - t'| \tag{20}
\]

**Proposition 1 (Geodesics in \( \mathcal{P}_n^+ \)).** For any \( \varphi_0, \varphi_1 \in \mathcal{P}_n^+ \), the parametric map \( \gamma : [0, 1] \to \mathcal{P}_n^+ \) is defined as
\[
\gamma(\theta) = \theta \varphi_0 + (1 - \theta) \varphi_1,
\]
1) \( \gamma \) is a geodesic between \( \varphi_0, \varphi_1 \).
2) We have the characterization:
\[
\|\varphi_0 - \varphi_1\| = \inf_{\eta} \sqrt{\int_0^1 |\eta(t)|^2 dt}
\tag{22}
\]
where \( \eta \in \text{Lip}(0, 1, \mathcal{P}_n^+), \eta(0) = \varphi_0, \eta(1) = \varphi_1 \) and
3) \( \gamma \) minimizes the functional on the RHS of Eq. (22).

**Proof:** (1) It follows from Lemma 3 that \( \gamma \) has constant speed equal to \( \|\varphi_0 - \varphi_1\| \), which immediately verifies Eq. (20).
(2) Since \( L(\gamma) \) is equal to \( \|\varphi_0 - \varphi_1\| \), we conclude \( \mathcal{P}_n^+ \) is a length space, which then implies the required result from Eq. (16). (3) By Jensen's inequality [2],
\[
\int_0^1 |\eta(t)| \, dt \leq \int_0^1 |\eta(t)|^2 \, dt
\tag{23}
\]
with equality if and only if \( |\eta(t)| \) is constant for almost all \( t \). Thus, any solution \( \eta \) to the functional is necessarily a constant speed geodesic, implying \( \gamma \) is a minimizer as required. \( \Box \)

1) **Charting Geodesics in \( \mathcal{P}_n^+ \).** We work on the condition for charting normal curves in \( \mathcal{P}_n^+ \). Let two arbitrary curves in \( \mathcal{P}_n^+ \) be denoted as:
\[
\gamma(\theta) = \theta \varphi_0 + (1 - \theta) \varphi_1,
\]
\[
\eta(\theta) = \theta \varphi_0 + (1 - \theta) \varphi_1.
\tag{24}
\tag{25}
\]

The tangent vectors to these curves at \( \theta \) are given by:
\[
\partial \gamma(\theta) = \delta \theta \varphi_0 - \delta \theta \varphi_1,
\]
\[
\partial \eta(\theta) = \delta \theta \varphi_0 - \delta \theta \varphi_1.
\tag{26}
\tag{27}
\]

For the inner product of the tangent vectors to vanish:
\[
(\varphi_0 - \varphi_1) \perp (\varphi_0 - \varphi_1).
\tag{28}
\]

If the curves pass through origin, i.e., if \( \varphi_1 = \varphi_1' = 0 \), then the condition for the curves to intersect orthogonally at the origin is given by \( \varphi_0 \perp \varphi_0' \). On the other hand, if the curves are orthogonal, and do not pass through the origin, then we can calculate the point of intersection as:
\[
\varphi_1 = \varphi_1, \quad \varphi_1 = \varphi_1', \quad \varphi_0 = \varphi_0
\tag{29}
\tag{30}
\]

As a sanity check, if \( \varphi_1 = \varphi_1' = 0 \), we have \( \varphi_0 = 0 \). We map out some of the geodesics for the case of \( |\Sigma| = 3 \) in Fig. 1.

**Remark 2.** We note that for the case of a ternary alphabet, the tangent space of \( \mathcal{P}_n^+ \) at any point is two dimensional – the number of vectors mutually orthogonal at any point for such a scenario is 2. It is clear that in general, the tangent spaces have dimensionality \( |\Sigma| - 1 \).

**III. Modeling Stochastic Processes**

We wish to extend the formalism to stochastic processes. To carry out this extension in a consistent manner, we would require some development. We begin with some notation, and preliminary notions.

**Notation 3 (Sequences over Finite Alphabet).** 1) Let \( \Sigma \) be a finite alphabet, and \( \Sigma^\omega \) be the set of strictly infinite sequences (or strings) over \( \Sigma \) (\( \omega \) is not a variable; it is a shorthand for infinite iteration, and this notation is standard in the context of \( \omega \)-languages [7]).
2) The set of finite but unbounded strings over \( \Sigma \) is denoted by the Kleene closure of \( \Sigma \), namely \( \Sigma^* \) [4].
3) For two two sequences \( x, y \), the concatenation is written simply as \( xy \).
4) The empty word is denoted as \( \lambda \).

We develop a slightly non-standard formalism of modeling stochastic processes, compared to what is generally encountered in the literature. We are interested in processes that take values in a finite set (the specified alphabet), instead of the real line, and our intentional departure from the standard formalism underscores the connection to formal languages arising from the finite valued nature of such processes.

**Definition 7 (Cantor Topology on \( \Omega \)-Languages).** Let \( \mathcal{B}_0 = \{ x \Sigma^\omega : x \in \Sigma \} \) be a family of sets of infinite sequences. Note \( x \Sigma^\omega \) denotes the set of all strictly infinite sequences which have \( x \) as the common prefix. It is easy to check that \( \mathcal{B}_0 \) qualifies as a basis for inducing a topology. In particular, we have:
1) \( \bigcup \mathcal{B}_1 = \Sigma \Sigma^\omega = \Sigma^\omega \)
2) \( \forall B_1, B_2 \in \mathcal{B}_0 \Rightarrow B_1 \cap B_2 = \emptyset \) or \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \), which guarantees that \( \forall z \in B_1 \cap B_2 \Rightarrow \exists B \in \mathcal{B}_0 \) such that \( z \in B_1 \cap B_2 \).
It follows that exists an unique topology for which \( \mathcal{B}_0 \) is a base. We denote this topology as \( \mathcal{U}_0 \). Indeed, this is the Cantor topology induced by the Tychonoff construction [8] on countable product of finite discrete sets [7] (in this case this finite set is the alphabet).

We note that on account of \( \mathcal{B}_0 \) being the base for \( \mathcal{U}_0 \), every open set in \( \mathcal{U}_0 \) may be written as a union of elements of \( \mathcal{B}_0 \). Since \( \mathcal{B}_0 \) is countable, it follows that every open set is of the form \( L \Sigma^\omega, L \subseteq \Sigma^* \).

**Definition 8 (Borel \( \sigma \)-algebra \( \mathcal{F}_\Xi \)).** \( \mathcal{F}_\Xi \) is defined as the smallest \( \sigma \)-algebra containing \( \mathcal{U}_0 \), implying that \( \mathcal{F}_\Xi \) is the Borel \( \sigma \)-algebra wrt \( \mathcal{U}_0 \). It trivially follows that, every measurable set is also of the form \( L \Sigma^\omega, L \subseteq \Sigma^* \).

Using \( \mathcal{F}_\Xi \), we can now define a probability space \( (\Sigma^\omega, \mathcal{F}_\Xi, \mu) \), which models a stochastic process, assigning probabilities to sets of strictly infinite sample paths. Note, in particular, that a strictly infinite single sample path is not measurable (such sets are not included in \( \mathcal{F}_\Xi \)); only sets that are of the form specified before, are; and after a finite length include all possible extensions into future.

We also consider here the map \( T : \Sigma^\omega \to \Sigma^\omega \) defined by:
\[
T(x_1 x_2 x_3 \cdots) = x_2 x_3 \ldots
\tag{31}
\]
It is immediate that \( T \) is measurable wrt \( \mathcal{F}_\Xi \). In going forward, we assume:
\[
\forall A \in \mathcal{F}_\Xi, \mu(T^{-1}A) = \mu(A) \tag{Stationarity}
\]
posing that we are considering only stationary processes. Additionally, we assume:
\[
\forall A \in \mathcal{F}_\Xi, \mu(A \Delta T^{-1}A) = 0 \Rightarrow \mu(A) \in [0, 1] \tag{Ergodicity}
\]
which ensures that our systems of interest are also ergodic.

**Remark 3 (Relationship to Standard Formalism).** There is a quite obvious connection to the standard formalism. Namely, the finite dimensional distributions can be identified as:
\[
\Pr(X_1 X_2 \cdots, X_n = x_1 x_2 \cdots x_n) = \mu(x_1 x_2 \cdots x_n \Sigma^\omega)
\tag{32}
\]
Noting that:

\[ \sum_{x_n \in \Sigma^*} \mu(x_1 x_2 \cdots x_n \Sigma^\nu) = \mu \left( \bigcup_{x_n \in \Sigma^*} x_1 x_2 \cdots x_n \Sigma^\nu \right) = \mu(x_1 x_2 \cdots x_{n-1} \Sigma^\nu) \]

implies that the finite dimensional distributions are Kolmogorov consistent, and hence using Kolmogorov Extension theorem [3], [5], we can go back and forth between the two formalisms.

A. States and Transition Structure

Definition 9 (Probabilistic Nerode Equivalence & Causal States). We define an relation on the set of all finite but unbounded strings, i.e., the set \( \Sigma^* \), as follows:

\[
\forall \omega_1, \omega_2 \in \Sigma^*, \omega_1 \sim \omega_2 \text{ if } \begin{cases} \forall \sigma \in \Sigma \mu(\omega_1 \Sigma^\nu) = \mu(\omega_2 \Sigma^\nu) = 0 \\ \mu(\omega_1 \Sigma^\nu) \neq 0, \text{ and } \mu(\omega_2 \Sigma^\nu) \neq 0, \text{ and } \frac{\mu(\omega_1 \Sigma^\nu)}{\mu(\omega_2 \Sigma^\nu)} \end{cases}
\]

It is easy to see that this is actually a right invariant equivalence relation, i.e.,

\[ x \sim y \Rightarrow \forall z \in \Sigma^*, xz \sim yz \]

and hence intuitions the notion of states. We define the “causal states” of the process, as the equivalence classes of \( \sim \) this relation.

Definition 10 (Symbolic Derivative). For \( x \in \Sigma^* \), with \( \mu(x \Sigma^\nu) > 0 \), the symbolic derivative \( \phi : \Sigma^* \rightarrow [0,1] \) is a probability distribution over the alphabet, defined as:

\[ \phi(x) = \frac{\mu(x \Sigma^\nu)}{\mu(x \Sigma^\nu) : x \in \Sigma^*, \sigma \in \Sigma} \]

Clearly, we have for any \( x \in \Sigma^* \), with \( \mu(x \Sigma^\nu) > 0 \),

\[ \sum_{\sigma \in \Sigma} \phi(x) = 1 \]

We refer to \( \phi(x) \) as the symbolic derivative at \( x \), and denote it as \( \phi_x \).

It is clear that for strings \( x, x' \in \Sigma^* \), we have:

\[ x \sim x' \Rightarrow \forall y \in \Sigma^*, \phi_{xy} = \phi_{x'y} \]

Lemma 4 (Sufficiency of Symbolic Derivatives). The set of symbolic derivatives at all finite strings, i.e., \( \{ \phi_x : x \in \Sigma^* \} \) uniquely specifies a measure \( \mu \) on the measurable space \( (\Sigma^\nu, \mathcal{F}) \).

Proof: \( \mu \) is uniquely specified by the recursions:

\[ \forall \sigma \in \Sigma, \mu(\sigma \Sigma^\nu) = \phi_x |_{\sigma} \]

\[ \forall x \in \Sigma^*, \sigma \in \Sigma, \mu(x \sigma \Sigma^\nu) = \begin{cases} \mu(x \Sigma^\nu) \phi_x |_{\sigma} & \text{if } \mu(x \Sigma^\nu) > 0 \\ 0 & \text{otherwise} \end{cases} \]

This completes the proof.

Remark 4. Another approach to proving the claim in Lemma 4 would be to show that the complete set of symbolic derivatives induces a complete set of finite dimensional distributions (FDD) via:

\[ Pr(X_1) = \phi_x \]

\[ Pr(X_1 \cdots X_n X_{n+1} = x_1 \cdots x_n) = Pr(X_1 \cdots X_n = x_1 \cdots x_n) \phi_x |_{\sigma} \]

which are clearly Kolmogorov consistent, and hence via the Kolmogorov extension theorem [5] induces a stochastic process, which is PDD-equivalent to \( (\Sigma^\nu, F, \mu) \) (See Remark 3).

1) States As A Random Variable: We do not wish to identify any initial state of our processes of interest. Thus, given an observed sequence, we assume that arbitrary sequences could have transpired prior to the observations. This induces the notion of a causal state as a random variable:

\[ [\bullet] : (\Sigma^\nu, F, \mu) \rightarrow (Q, F_Q, Pr) \]

where \( Q \) is the set of equivalence classes (the most countable state space), \( F_Q \) is an appropriate \( \sigma \)-algebra (which generally we will take to be the power set of \( Q \)), and \( Pr \) is the pushforward of the measure \( \mu \). Thus, we have:

\[ \forall q \in Q, Pr(q) = \mu([\bullet]^{-1}(q)) = \mu(x \Sigma^\nu : x \in \Sigma^* \wedge [x] = q) \]

Definition 11 (Conditioning on Observations). Given some observed sequence \( x_0 \in \Sigma^* \), we condition as follows:

\[ Pr(q|x_0) \equiv \text{const.} \times \mu([\bullet]^{-1}(q) | x_0) \]

Lemma 5. Assuming stationarity,

\[ \forall x_0 \in \Sigma^*, \mu(x_0 \Sigma^\nu) > 0 \Rightarrow \]

1) \[ Pr(q|x_0) = \frac{\mu(y x_0 \Sigma^\nu : y \in \Sigma^* \wedge [y x_0] = q)}{\mu(x_0 \Sigma^\nu)} \]

2) \[ \sum_{q \in Q} Pr(q|x_0) = 1 \]

Proof: Denoting the normalizing constant as \( C \),

\[ Pr(q|x_0) = C \mu(y x_0 \Sigma^\nu : y \in \Sigma^* \wedge [y x_0] = q) \]

which implies (invoking stationarity in the last step)

\[ C^{-1} = \mu(y x_0 \Sigma^\nu : y \in \Sigma^*) = \mu(\Sigma^* x_0 \Sigma^\nu) = \mu(x_0 \Sigma^\nu) \]

The second statement is immediate.

Remark 5. In Definition 11 we assume that an observed sequence is the suffix of the complete spatiotemporal sequence; any finite sequence of values could have occurred before the specific observations. Also, note:

\[ \forall q \in Q, Pr(q|\lambda) = Pr(q) \]

2) Probabilistic Automata Generators:

Definition 12 (Probabilistic Automata (PA)). A probabilistic automata is a 4-tuple \((\Sigma, Q, \delta, \pi)\), where \( \Sigma \) is a finite set (the alphabet), \( Q \subseteq N \) is the state space, \( \delta : \Sigma \times \Sigma \rightarrow Q \) is the transition map, and \( \pi : Q \times [0,1] \) specifies the state-specific transition probabilities, satisfying \( \forall q \in Q, \sum_{\sigma \in \Sigma} \pi(q, \sigma) = 1 \).

Definition 13. We use the following terminology:

\[ \tilde{\Pi}_{ij} \equiv \tilde{\pi}(q_i, q_j) \quad \text{(Morph Matrix)} \]

\[ \Pi_{ij} \equiv \sum_{\sigma \in \Sigma} \tilde{\pi}(q_i, \sigma) \quad \text{(Transition Probability Matrix)} \]

\[ \Gamma_{ij} \equiv \begin{cases} \tilde{\pi}(q_i, \sigma) & \text{if } \delta(q_i, \sigma) = q_j \\ 0 & \text{otherwise} \end{cases} \quad \text{(Event-specific Transition Matrix)} \]

Note that, we have \( \sum_{\sigma \in \Sigma} \Gamma_{ij} = \Pi_{ij} \).

We say a probabilistic automata \( G = (\Sigma, Q, \delta, \pi) \) is a probabilistic finite state automata (PFSA) if \( |Q| < \infty \). In that case, we have the morph, transition probability, and the
Proposition 2 (Existence of Canonical Encoders) arises from the following proposition. The importance of probabilistic automata based encodings for stationary ergodic finite-valued stochastic processes. We say that an automaton encodes a process if all finite dimensional distributions (FDD) may be recovered from it, i.e., the model represents the process up to FDD equivalence.

Lemma 6 (Probabilistic Automata to Stochastic Process). \((\Sigma, Q, \delta, \tilde{\pi})\) induces a stationary stochastic process if

\[
\exists \omega' \in [0, 1]^{|Q|}, \text{ with } \sum_{i \in Q} \omega'_i = 1,
\]

s.t. \(\forall i \in Q, \sum_{j \in Q} \omega'_i \delta_{ij} = \varphi_i\) (48)

Proof: We define \(\varphi'_\omega, \varphi_\omega, \omega \in \Sigma^*\) as follows:

\[
\varphi'_\omega = \omega' \quad (49a)
\]

\[
\varphi_\omega = [\omega' \tilde{\pi}] \quad (49b)
\]

\[
\varphi_\omega = \varphi'_\omega \tilde{\pi} \quad (49c)
\]

We then construct a set of Kolmogorov consistent set of finite dimensional distributions recursively as:

\[
Pr(X_1 = \varphi'_\omega) = \varphi'_\omega \quad (50)
\]

\[
Pr(X_1, \ldots, X_n, X_{n+1} = x_1, \ldots, x_n \sigma) = Pr(X_1, \ldots, X_n = x_1, \ldots, x_n) \varphi'_\omega \quad (51)
\]

which, then via invocation of the Kolmogorov Extension Theorem [5] induces a FDD equivalent measure space \((\Sigma', \mathcal{F}_\omega, \mu)\). The recursive construction of the finite dimensional distributions in Eqs. (50),(51) have no dependence on time shifts, and hence guarantee stationarity. This completes the proof.

We use the following notation:

Notation 4. If \((\Sigma, Q, \delta, \tilde{\pi})\) encodes in the sense of Lemma 6 the stationary stochastic process arising from \((\Sigma', \mathcal{F}_\omega, \mu)\) then we write:

\[
(\Sigma, Q, \delta, \tilde{\pi}) \rightarrow (\Sigma', \mathcal{F}_\omega, \mu) \quad (52)
\]

The importanc of probabilistic automata based encodings arises from the following proposition.

Proposition 2 (Existence of Canonical Encoders). For every stationary ergodic process generated by the measure space \((\Sigma', \mathcal{F}_\omega, \mu)\), we have a \((\Sigma, Q, \delta, \tilde{\pi})\), such that:

\[
(\Sigma, Q, \delta, \tilde{\pi}) \rightarrow (\Sigma', \mathcal{F}_\omega, \mu) \quad (53)
\]

Proof: A stationary ergodic process arising from the triple \((\Sigma', \mathcal{F}_\omega, \mu)\) induces a \((\Sigma, Q, \delta, \tilde{\pi})\) as follows (this construction is referred to in the sequel as the canonical encoding):

1) Identify \(Q\) as the set of equivalence classes for \(\sim_{\mathcal{F}}\).
2) Identify the transition structure as:

\[
\forall q \in Q, \text{ choose } x \in \Sigma', \text{ s.t. } [x] = q. \text{ Then } \forall \sigma \in \Sigma, \delta([x], \sigma) = [x\sigma], \tilde{\pi}([x], \sigma) = \varphi_x\sigma\]

We claim that the symbolic derivatives are recoverable from \((\Sigma, Q, \delta, \tilde{\pi})\). To establish this claim, we will construct a set of recursive relationships that would allow us to recover the complete set of symbolic derivatives. We denote \(\varphi \in \mathcal{F}\), and proceed by noting:

\[
\sum_{j \in Q} \mu(\varphi_{\omega} x_i \Pi_{ji} = \sum_{j \in Q} \mu(x \sigma_x^\omega : [x] = j) \frac{\mu(x \sigma_x^\omega : [x] = i \wedge [x] = j)}{\mu(x \sigma_x^\omega : [x] = j)}\]

(where we assume \(\mu(x \sigma_x^\omega) > 0\))

\[
= \sum_{j \in Q} \mu(x \sigma_x^\omega : [x] = i \wedge [x] = j)
\]

\[
= \mu(\sigma_x^\omega : [y] = i) \frac{\mu(\sigma_x^\omega : [y] = j)}{\mu(\sigma_x^\omega : [y] = j)}
\]

which implies that a unique stationary distribution corresponding to \(\Pi\) exists, which is given by \(\varphi_{\omega}\). Next, we observe:

\[
\varphi_{\omega} = \frac{\mu(\sigma_x^\omega : [y] = i)}{\mu(\sigma_x^\omega : [y] = j)}
\]

(Assuming \(\mu(\sigma_x^\omega) > 0\) and \(\mu(\sigma_x^\omega) > 0\))

\[
= \sum_{j \in Q} \mu(\sigma_x^\omega : [y] = j) \frac{\mu(\sigma_x^\omega : [y] = j \wedge [y\sigma_x^\omega] = i)}{\mu(\sigma_x^\omega : [y] = j)}
\]

Finally, we note:

\[
\sum_{q_j \in Q} \varphi_{q_j \sigma}(q_j, \sigma) = \sum_{q_j \in Q} Pr(q_j \sigma x) \varphi_{q_j \sigma}(q_j, \sigma)
\]

(Assuming \(\mu(\sigma_x^\omega) > 0\))

\[
= \sum_{j \in Q} \mu(\sigma_x^\omega : [y] = j) \frac{\mu(\sigma_x^\omega : [y] = j \wedge [y\sigma_x^\omega] = i)}{\mu(\sigma_x^\omega : [y] = j)}
\]

\[
= \sum_{j \in Q} \mu(\sigma_x^\omega : [y] = j) \frac{\mu(\sigma_x^\omega : [y] = j \wedge [y\sigma_x^\omega] = i)}{\mu(\sigma_x^\omega : [y] = j)}
\]

(53)

where stationarity is invoked in Eq. (63). We note that Eqs. (55),(59), and (63), may be summarized as (representing \(\varphi_{\omega}\) as a row vector to use matrix notation):

\[
\varphi_{\omega} \Pi = \varphi_{\omega}
\]

(64a)

And, \(\forall x \in \Sigma, \sigma \in \Sigma, \text{ s.t. } \mu(x \sigma_x^\omega) > 0, \mu(\sigma_x^\omega) > 0, \nu_{\omega} = [\nu_{\omega} \tilde{\pi}]\)

(64b)

\[
\varphi_{\omega} = \varphi_{\omega} \tilde{\pi}
\]

(64c)

which gives us the desired recursions that recover the complete set of symbolic derivatives \(\{\varphi_{\omega} : x \in \Sigma, \mu(x \sigma_x^\omega) > 0\}\). Lemma 4 then guarantees that the measure \(\mu\) may be constructed from \((\Sigma, Q, \delta, \tilde{\pi})\).

Notation 5. The canonical encoding described in Proposition 2 is denoted as \((\Sigma', Q', \delta', \tilde{\pi}')\).

Remark 6. Finiteness of the state space is not invoked in
proving the existence of PA encoders in Proposition 2, and hence \( Q \) in the construction is almost countable.

**Definition 14 (Closed Restriction).** A closed restriction of \((\Sigma, Q, \delta, \pi)\) is a model \((\Sigma', Q', \delta', \pi')\) such that:

\[
\phi \equiv Q' \subseteq Q
\]

\[
\forall \sigma \in \Sigma, q' \in \delta'(q', \sigma) \in Q'
\]

\[
\forall \sigma \in \Sigma, q' \in \delta'(q', \sigma) = \pi'(q', \sigma)
\]

The set of all closed restrictions of a probabilistic automaton \( G = (\Sigma, Q, \delta, \pi) \) is denoted as \( \mathcal{C}(G) \). A closed restriction \( H \in \mathcal{C}(G) \) is a minimal closed restriction if

\[
\mathcal{C}(H) = \{ H \}
\]

The set of all minimal closed restrictions of a probabilistic automaton \( G \) is denoted as \( \mathcal{C}_m(G) \). Note that we have

\[
\mathcal{C}_m(G) \subseteq \mathcal{C}(G)
\]

**Definition 15 (Probability of Closed Restriction).** For a closed restriction \( H \in \mathcal{C}(G) \), and \((\Sigma, Q, \delta, \pi) \models (\Sigma', F, \mu)\), the total probability \( Pr(H) \) is defined as follows:

\[
Pr(H) = \sum_{q \in \delta'} \mu(\Sigma' : [x] = q)
\]

**Lemma 7 (Closed Restriction).** If \((\Sigma', F, \mu)\) is stationary, ergodic with \((\Sigma, Q, \delta, \pi) \models (\Sigma', F, \mu)\) (without loss of generality according to Proposition 2), then:

\[
\exists H \in \mathcal{C}_m((\Sigma, Q, \delta, \pi)), \text{ s.t. } Pr(H) = 1
\]

**Proof:** Indexing elements of \( \mathcal{C}_m((\Sigma, Q, \delta, \pi)) \) as \( G_i = (\Sigma, Q_i, \delta_i, \pi_i) \), it follows immediately:

\[
G_i \neq G_j \Rightarrow Q_i \cap Q_j = \emptyset
\]

Recalling that \( \forall i, Q_i \subseteq Q \), let us define:

\[
L_i = \bigcup_{q \in Q_i} \{ x : x \in \Sigma^* \land [x] = q \}
\]

and we conclude:

\[
G_i \neq G_j \Rightarrow L_i \cap L_j = \emptyset
\]

Since, \( \forall i, G_i \) are minimal closed restrictions, we have (considering the standard shift map \( T \)):

\[
T^{-1}(L_i \Sigma^\omega) = L_i \Sigma^\omega
\]

and then ergodicity of \((\Sigma, Q, \delta, \pi)\) implies:

\[
\forall i, \mu(L_i \Sigma^\omega) \in \{0, 1\}
\]

Finally, \( \bigcup_i L_i \subseteq \Sigma^\omega \), implies that there exists a unique minimal closed restriction with full measure, completing the proof. \( \blacksquare \)

**Notation 6 (Unique Minimal Closed Restriction).** If \((\Sigma', F, \mu)\) is stationary, ergodic with \((\Sigma, Q, \delta, \pi) \models (\Sigma', F, \mu)\), the unique minimal closed restriction \( H \in \mathcal{C}_m((\Sigma, Q, \delta, \pi)) \) with \( Pr(H) = 1 \) is denoted as \((\Sigma, Q, \delta, \pi)\). Note if we denote \((\Sigma, Q', \delta', \pi') = (\Sigma, Q, \delta, \pi)\), then \(\delta', \pi'\) in \((\Sigma, Q', \delta', \pi')\) are appropriate restrictions of the corresponding functions in \((\Sigma, Q, \delta, \pi)\) to \( Q' \).

We show next that the unique minimal closed restriction is sufficient to model the process, and consists of all the non-trivial states in the original model.

**Lemma 8 (Sufficiency of Minimal Closed Restriction).** If \((\Sigma', F, \mu)\) is stationary, ergodic with \((\Sigma, Q, \delta, \pi) \models (\Sigma', F, \mu)\), the unique minimal closed restriction \((\Sigma, Q, \delta, \pi) = (\Sigma, Q', \delta', \pi')\) satisfies:

\[
\forall q \in Q', Pr(q) > 0
\]

\[
\forall q \in Q, Pr(q) > 0 \Rightarrow q \in Q'
\]

\[
(\Sigma, Q, \delta, \pi) = (\Sigma', F, \mu)
\]

**Proof:** Let if possible we have a state \( q_0 \) such that:

\[
Pr(q_0) = 0 \land q_0 \in Q'
\]

Then, recalling that \( q_0 \) is also a state in \((\Sigma, Q, \delta, \pi)\), we have:

\[
\mu(x \Sigma^\omega : x \in \Sigma^* \land [x] = q_0) = 0
\]

Since,

\[
\mu(x \Sigma^\omega) = 0 \Rightarrow \forall y \in \Sigma^*, \mu(xy \Sigma^\omega) = 0
\]

it follows:

\[
[x'] = q_0 \Rightarrow \forall y \in \Sigma^*, [x'y] = q_0
\]

which then implies that \((\Sigma, \{q_0\}, \delta', \pi')\), with \(\delta', \pi'\) appropriate restrictions of \(\delta, \pi\), defines a minimal closed restriction (contradiction). This establishes Eq. (75a). Eq. (75b) follows immediately from \(Pr((\Sigma, \delta, \pi)) = 1\) (Lemma 7).

To establish Eq. (75c), we note that if the stationary probability vector for \((\Sigma, Q, \delta, \pi)\) is denoted as \(\varphi\) (which exists on account of Lemma 7 and Notation 6), then a stationary probability vector \(\varphi^*\) exists for \((\Sigma, Q', \delta', \pi')\), and is given simply as the restriction:

\[
\varphi^* = \varphi|_{Q'}.
\]

Also, note that Eqs. (75a)-(75b) establish that \(\varphi^*\) accounts for all non-zero entries in \(\varphi\). Now, following the construction in Lemma 6, we define:

\[
\varphi_\delta = \varphi
\]

\[
\varphi_{2\delta} = [\varphi_{2\delta} \Gamma_\delta]
\]

\[
\varphi_2^* = \varphi^*
\]

\[
\varphi_{2\delta} = [\varphi_{2\delta} \Gamma_\delta^*]
\]

\[
\varphi_2^* = \varphi_2^* \Gamma_\delta^*
\]

where \(\Gamma_\delta^*\), \(\Gamma_\delta^*\) are the corresponding Event-specific Transition matrix, and the morph matrix (See Definition 12) for \((\Sigma, Q', \delta', \pi')\). We claim that:

\[
\forall x \in \Sigma, \varphi_x = \varphi_0
\]

which follows immediately from noting that since \((\Sigma, Q', \delta', \pi')\) is a minimal closed restriction, no transition from any state in \(Q'\) by any \(\sigma \in \Sigma\) takes us outside the set \(Q'\), implying that since \(\varphi_{2\delta} \Gamma_\delta^*\) is a zero vector, \(\forall x \in \Sigma, \varphi_{2\delta} \Gamma_\delta^*\) is also a zero vector. Hence, the contribution from states outside \(Q'\) to \(\varphi_0\) is zero for all \(x\). Thus, the measure specified on \((\Sigma', F, \mu)\) by \((\Sigma, Q', \delta', \pi')\) coincides with that induced by \((\Sigma, Q, \delta, \pi)\) (Lemma 4). This completes the proof. \( \blacksquare \)

**To paraphrase Lemma 8, given any probabilistic automata that models a finite values stationary ergodic process, the unique minimal closed restriction also models the process.**

And this result holds for almost countable state spaces. And this establishes that the unique minimal closed restriction of the canonical model constructed in Proposition 2 is an actual unique minimal realization of the process.

**Proposition 3 (Existence of Minimal Models).** If an arbitrary probabilistic automata \((\Sigma, Q, \delta, \pi) \models (\Sigma', F, \mu)\), then:
1) \((\Sigma, Q', \delta', \overline{\pi}') = (\Sigma, Q, \delta, \overline{\pi})\) induces an equivalence relation \(\sim\) on \(\Sigma^*\), where there is a one-to-one mapping from the equivalence classes of \(\sim\)' to \(Q'\).

2) \(\sim\)' is a refinement of \(\sim_{\approx}\).

Proof: Since \((\Sigma, Q, \delta, \overline{\pi}) \models (\Sigma^e, \mathcal{F}_e, \mu)\), denoting:
\[
\nu_x = \frac{\mu(y \in \Sigma^* \mid y \in \Sigma^* \land [yx] = x)}{\mu(x \in \Sigma^*)}
\]
we can define an equivalence on \(\Sigma^*\) as follows:
\[
\forall x, y \in \Sigma^*, x \sim' y \text{ if } \forall x \in \Sigma^*, \nu_{x\delta} = \nu_{yx} \quad (89)
\]
We note that there exists a one-to-one map \(\zeta\) from \(Q'\) to the equivalence classes of \(\sim'\):
\[
\forall x \in Q', \zeta(x) = [x]' \quad \forall x \text{ s.t. } \nu_x = 1 \quad (90)
\]
This establishes Statement (1). For Statement (2), we note:
\[
x \sim' y \Rightarrow \forall x \in \Sigma^*, \nu_{x\delta} = \nu_{yx} \quad (91)
\]
\[
\Rightarrow \forall x \in \Sigma^*, \nu_{x\delta} = \nu_{yx} = \phi_{yx} \quad (92)
\]
which completes the proof.

Thus, it follows that unique minimal closed restriction of the canonical encoding, whose states correspond to the non-trivial (consisting of non-zero probability strings) equivalence classes of \(\sim_{\approx}\), represent the unique minimal model, in the sense of representing the coarsest equivalence on \(\Sigma^*\). For probabilistic finite state automata encoders, we have the following result on the state space sizes.

Corollary 1 (To Proposition 3: Minimal Models in Finite State Space Case). Let an arbitrary \((\Sigma, Q, \delta, \overline{\pi}) \models (\Sigma^e, \mathcal{F}_e, \mu)\), and \((\Sigma, Q, \delta, \overline{\pi}) = (\Sigma, Q^0, \delta^0, \overline{\pi}^0)\) be the unique minimal closed restriction of the canonical encoding. If \(|Q| < \infty\), we have:
\[
|Q^0| < \infty \quad (93a)
\]
\[
|Q| \geq |Q^0| \quad (93b)
\]
\[
|Q| = |Q'| \Rightarrow \sim_{\approx} \quad (93c)
\]

Proof: Denote \((\Sigma, Q, \delta, \overline{\pi}) = (\Sigma, Q^0, \delta^0, \overline{\pi})\).

Statement (1): Since \(|Q^0| < \infty\), it follows from the definition of closed restrictions that \(|Q^0| < \infty\). It then follows from Proposition 3 and the definition of canonical encodings that \(|Q^0| < \infty\), as required.

Statement (2): It follows from Proposition 3:
\[
|Q| \geq |Q^0| \geq |Q'| \quad (94)
\]

Statement (3): Follows immediately from Proposition 3.

Thus, the unique minimal closed restriction of the canonical encoding \((\Sigma, Q^0, \delta^0, \overline{\pi})\) is the minimal model up to a renaming of the states.

Remark 7 (Minimal and Non-minimal Realizations of Models). While the minimal realization is unique, it is trivial to generate non-minimal realizations of encoders. In particular, any refinement of the \(\sim_{\approx}\)-equivalence gives us a non-minimal probabilistic automata correctly encoding the same process.

Remark 8. Corollary 1 uses finiteness of the state spaces; the preceding results hold for almost countable states.

B. Synchronization

In the sequel, unless otherwise mentioned, we always consider the unique minimal closed restriction of the canonical embedding by \((\Sigma, Q, \delta, \overline{\pi}) \models (\Sigma^e, \mathcal{F}_e, \mu)\), where \((\Sigma^e, \mathcal{F}_e, \mu)\) is always assumed to be stationary, ergodic. We do not assume finiteness of the state spaces, unless mentioned explicitly.

Fig. 2. Synchronizable & non-synchronizable Models. Proposition 4 establishes that non-synchronizable models are still \(\varepsilon\)-synchronizable.

Lemma 9 (Balance Lemma). For \((\Sigma, Q, \delta, \overline{\pi}) \models (\Sigma^e, \mathcal{F}_e, \mu)\), given some state probability vector \(p\), where as usual \(\forall i, p_i > 0, \sum_i p_i = 1\), we have:
\[
\exists \sigma, \pi \in \Sigma^e, \sum_i p_i \pi(i, \sigma) < 1 \Rightarrow \exists \sigma', \pi' \in \Sigma^e, \sum_i p_i \pi'(i, \sigma') > 1
\]

Proof: Let us assume for some \(\sigma, \pi \in \Sigma^e\),
\[
\sum_i p_i \pi(i, \sigma) < 1
\]
Then, either the claim from left to right is true, or we have for all but some \(\sigma' \in \Sigma^e\):
\[
\forall \sigma \in \Sigma \setminus \{\sigma', \sigma\}, \sum_i p_i \pi(i, \sigma) \leq 1
\]
But then, for \(\sigma'\), we have:
\[
\sum_i p_i \pi(i, \sigma) > 1 - \sum_i p_i \pi(i, \sigma) \geq 1 - \sum_i p_i \pi(i, \sigma') > 1
\]

The converse follows similarly, thus completing the proof.

Proposition 4 (\(\varepsilon\)-Synchronization). For a stationary ergodic system \((\Sigma, Q, \delta, \overline{\pi}) \models (\Sigma^e, \mathcal{F}_e, \mu)\), we have:
\[
\forall \varepsilon > 0, \exists \sigma \in \Sigma^e, \text{ s.t. } \exists q \in Q, P_r(q_{\sigma_0}) \geq 1 - \varepsilon
\]

Proof: Assume, if possible that for some \(\sigma \in \Sigma^e\), \(\rho_{\sigma_0} = ...
Given two ergodic stationary systems, 

\[ \Pr(Q, \pi, \sigma, \delta, \Phi) = (\Sigma, \Omega, \pi, \sigma, \delta, \Phi) \]

we have:

\[ \left( u \leq \sup_{\omega \in Q} \sup_{j \in \Omega} \omega \right) \wedge \left( \forall x \in \Sigma^*, \sup_{j \in \Omega} \omega \leq u \right) \]  

(99)

First, we claim:

\[ \exists j \in Q, \text{ s.t. } \omega \omega \leq u \]  

(100)

i.e., the supremum is achieved by some state. This is trivially true if \( |Q| < \infty \). We claim, it also true in the general countable case. To see this, note that if for some \( i'' \), we have:

\[ \exists i_1, \ldots, i_n, \ldots, \text{ s.t. } \omega \omega \geq \cdots \geq \omega \omega \geq \omega \omega \cdots \]  

(101)

\[ \Rightarrow \forall N \in \mathbb{N}, \sum_{j=1}^{N} \omega \omega \geq \omega \omega N \]  

(102)

implying that for a countably infinite state space, where the supremum is never achieved, we must necessarily have \( \omega \omega = 0 \), resulting in contradiction, thus establishing Eq. (100).

Now, if \( \exists \sigma \in \Sigma \) such that \( \sup_{j \in Q} \omega \omega \omega \omega \leq j \) is reduced below \( u \), then there exists a symbol that increases it as well (Lemma 9). Hence, it follows that we must have:

\[ \omega \omega \omega \omega = \omega \omega \omega \omega = \pi(j, \cdot) \]  

(103)

and since the same argument applies for any extension of \( x' \):

\[ \forall x \in \Sigma^*, \exists i \in Q, \text{ s.t. } \omega \omega \omega \omega \omega \omega \omega = \pi(i, \cdot) \]  

(104)

Let us define:

\[ L_i = \{ x \in \Sigma^* : \arg \max_{j \in \Omega} \omega \omega \omega \omega \omega \omega \omega = i \} \]  

(105)

It follows immediately:

\[ \bigcup_{i \in Q} L_i = \Sigma^* \]  

and \( \forall i \in Q, L_i \cap L_j = \emptyset \)  

(106)

Clearly, we have the following bijections:

\[ \zeta : \{ L_i \} \rightarrow Q, \]  

(107)

\[ \xi : \{ L_i \} \rightarrow N, \text{ s.t. } \xi(L_i) = i \]  

(108)

We define a model \( (\Sigma, Q^\Delta, \delta^\Delta, \pi^\Delta) \), such that:

\[ Q^\Delta = \{ \xi(L_i) \} \]  

(109a)

\[ \forall i \in Q^\Delta, \forall \sigma \in \Sigma, \delta^\Delta(i, \sigma) = \delta(\zeta \circ \xi^{-1}(i), \sigma) \]  

(109b)

\[ \forall i \in Q^\Delta, \forall \sigma \in \Sigma, \pi^\Delta(i, \sigma) = \pi(\zeta \circ \xi^{-1}(i), \sigma) \]  

(109c)

Interpreting \( Q^\Delta \) as a simply renaming of \( Q \), we note that \( Q^\Delta, \delta^\Delta, \pi^\Delta \) is indistinguishable from \( (\Sigma, Q, \delta, \pi) \). Hence, considering the equivalence class of \( x' \) in the identical models:

\[ x' = \xi(L_i) \Rightarrow x' = \xi \circ \xi^{-1} \circ \xi(L_i) = i \Rightarrow u = 1 \]  

(110)

which contradicts Eq. (99). Hence, we have either \( u = 1 \), or

\[ \forall x \in \Sigma^*, \sup_{j \in \Omega} \omega \omega \omega \omega \omega \omega \omega > u \]  

(111)

In either case, we have the desired result.

\[ \Pr(Q, \pi, \sigma, \delta, \Phi) = (\Sigma \times \Sigma, Q', \delta', \Phi') \]

(112)

Proving: We define \( G' \) as \( (\Sigma \times \Sigma, Q', \delta', \Phi') \) such that:

\[ \forall x \in \Sigma^*, \forall \sigma, \delta, \Phi \in \Sigma \]

(113)

\[ \exists x_0 \in \Sigma^*, \text{ such that } (\exists q \in Q, \Pr(q|x_0) \geq 1 - \epsilon) \]

(114)

It is easy to verify that:

\[ \pi''((i, j), (i, \sigma')) = \sum_{\sigma} \pi((i, \sigma)) \pi((j, \sigma')) = 1 \]  

(115)

implying that \( G' \) is a valid model. Now applying Proposition 4, we conclude that:

\[ \forall x \in \Sigma^*, \forall \sigma \in \Sigma, \Phi(x|\delta) \geq 1 - \epsilon \]  

(116)

The absence of any interaction in the dynamics of \( G, G' \) in the construction of \( G' \), then implies that \( x_0 \) jointly \( \epsilon \)-synchronizes both \( G, G' \). This completes the proof.

**C. Vector Space of Ergodic Stationary Processes**

**Definition 16 (Strictly Positive Ergodic Stationary Processes).** A strictly positive process over a finite alphabet \( \Sigma \) is a finite-valued ergodic process such that:

\[ \forall x \in \Sigma^*, \forall \sigma \in \Sigma, \Phi(x|\sigma) > 0 \]  

(117)

\( \Phi^+ \) denotes the space of positive processes over \( \Sigma \).

Note that a finite valued stationary ergodic process is a positive process if and only if every symbolic derivative a strictly positive probability vector on \( \Sigma \).

**Definition 17 (Scalar Product).** For an ergodic stationary process \( (\Sigma, Q, \delta, \Phi) \) \( = (\Sigma^*, \Phi^*, \mu) \), we can construct the scalar product \( \alpha \circ (\Sigma, Q, \delta, \pi) \) as follows:

For an \( \epsilon \)-synchronizing string \( x_0 \),

\[ \forall x \in \Sigma^*, \Phi(x|\sigma) \leq \lim_{\epsilon \to 0^+} \alpha \circ \Phi(x|\sigma) \]  

(118)

We note that:

\[ \exists x_0 \in \Sigma^*, \Phi(x|\sigma) \rightarrow \Phi(x_0) \]  

(119)

where if \( x_0 = \sigma_0 \circ \sigma_\infty \), then \( \Pr(x_0|x_\infty) \rightarrow 1 \)

We define a map \( : \Sigma^* \rightarrow Q \) as:

\[ \zeta(\lambda) = \sigma_0 \]  

(120)

\[ \forall x \in \Sigma^*, \forall \sigma \in \Sigma, \zeta(\sigma x) = \delta(x, \sigma) \]  

(121)

Then, we construct a model \( G' = (\Sigma, Q', \delta', \Phi') \) as:

\[ \forall x \in \Sigma^*, \delta'(x, \sigma) = \zeta(\sigma x) \]  

(122)

\[ \forall x \in \Sigma^*, \Phi'(x, \cdot) = \Phi(x|\sigma) \]  

(123)

Finally, we define:

\[ \alpha \circ (\Sigma, Q, \delta, \Phi) = \sum_{i} \]  

(124)

**Lemma 10 (Scalar Product).** The construction of \( G' = (\Sigma, Q', \delta', \Phi') \) in Definition 17 is consistent.

**Proof:** We only need to establish that \( Q' = Q \) in Eq. (131) is consistent with the definition of \( \delta', \Phi' \) in Eqns. (132) and (133), which follows from noting that \( \sigma_0 \circ \sigma_\infty \) exists some sequence \( x_0 \) such that \( \zeta(\sigma_0) = x_0 \) since \( \Sigma, Q, \delta, \Phi \) is a closed restriction. For the same reason, there exists sequences beginning from \( x_0 \) visiting every state in \( Q \), implying that if we construct \( G' \) using Eq. (132), then we end up with \( Q' = Q \). This completes the proof.

**Definition 18 (Sum).** For ergodic stationary processes \( (\Sigma, Q, \delta, \Phi) \) \( = (\Sigma^*, \Phi^*, \mu) \), \( (\Sigma, Q', \delta, \Phi') \) \( = (\Sigma^*, \Phi', \mu') \), a closed commutative binary operation \( (\Sigma, Q', \delta', \Phi') \) \( = (\Sigma, Q, \delta, \Phi) \oplus (\Sigma, Q, \delta, \Phi) \) may be constructed as follows:

For a jointly \( \epsilon \)-synchronizing string \( x_0 \),

\[ \forall x \in \Sigma^*, x_0 \leq \lim_{\epsilon \to 0^+} x_0 \oplus x_0 \]  

(126)
Denoting state probabilities in \((\Sigma, Q', \delta', \vec{p}''')\) as \(P'\cdot()\), we note:
\[
\exists \alpha_0 \in \Sigma', \phi_0' \rightarrow \phi_0 \oplus \phi_0' \quad (127)
\]
where if \([x_0] = i_0 \in Q, [x_0] = j_0 \in Q'\), then \(P'(i_0|x_0) \rightarrow 1 \sum P'(j_0|x_0) \rightarrow 1\) \( (128)\)

We define a map \(\zeta : \Sigma' \rightarrow Q \times Q'\) as:
\[
\zeta(l) = (i_0, j_0) \quad (129)
\]
\[
\forall x \in \Sigma', \sigma \in \zeta, (\sigma(x_0), \sigma) = \delta(\sigma(x_0), \sigma) \quad (130)
\]

Then, we construct a model \(G'' = (\Sigma, Q', \delta', \vec{p}''')\) as:
\[
Q'' = Q \times Q' \quad (131)
\]
\[
\forall x \in \Sigma', \delta''((\zeta(x_0), \sigma(x_0))) = \zeta(x) \quad (132)
\]
\[
\forall x \in \Sigma', \vec{p}''((\zeta(x_0), \sigma(x_0))) = \phi''_x \quad (133)
\]

Finally, we define:
\[
G \oplus G' = \mathcal{C}(G') \quad (134)
\]

Lemma 11 (Sum). *The construction of \(G'' = (\Sigma, Q', \delta', \vec{p}''')\) in Definition 17 is consistent.*

**Proof:** As in Lemma 10, we only need to establish that \(Q'' = Q \times Q'\) is consistent with the definitions of \(\delta', \vec{p}'\), which follows by beginning with \((i_0, j_0) \in Q'\), and recalling that both \((\Sigma, Q, \delta, \vec{p}), (\Sigma, Q', \delta', \vec{p}')\) are closed restrictions.

Notation 7. *As in the case of probability vectors, we denote \(\oplus\) in the context of processes as simply + and concatenation, if no confusion arises.*

The commutative sum of stochastic processes established above induces an Abelian group on \(P_+^\Sigma\). We note that process equivalence (and uniqueness) is up to equality of finite dimensional distributions (FDD equivalence).

Lemma 12 (Abelian Group on Stochastic Processes).
\[
\forall G, G' \in P_+^\Sigma, G + G' \in P_+^\Sigma \quad (135a)
\]
\[
G + G' = G + G' \quad (135b)
\]
\[
\exists Z \in P_+^\Sigma, \text{ such that } \forall G \in P_+^\Sigma, G + Z = G \quad (135c)
\]
\[
\forall G \in P_+^\Sigma, \exists G' \in P_+^\Sigma, \text{ such that } G + G' = Z \quad (135d)
\]

where uniqueness is assumed up to FDD equivalence.

**Proof:** Eqns. (135a) and (135b) are immediate from Definition 18. Now, using the fact that a complete set of symbolic derivatives uniquely specifies a process up to FDD equivalence (Lemma 4), we define a stationary ergodic process \(W\) as:
\[
\forall x \in \Sigma', \phi''_x = \mathcal{U}_{\Sigma} \quad (136)
\]
where \(\mathcal{U}_{\Sigma} \) is the uniform probability vector over \(\Sigma\). We claim:
\[
\forall G \in P_+^\Sigma, G + W = G \quad \text{(Claim A)}
\]
\[
\forall G \in P_+^\Sigma, G + H = G \Rightarrow H = W \quad \text{(Claim B)}
\]
The first claim follows from noting that for any \(\epsilon\)-synchronizing sequence \(x_\epsilon\) for \(G\), (using \(\phi''_{x_\epsilon}\) to denote the symbolic derivative for \(G\) at \(x_\epsilon\)) we have:
\[
\forall x \in \Sigma', \phi''_{x_\epsilon} \oplus \mathcal{U}_{\Sigma} = \phi''_{x_\epsilon} \quad (137)
\]
For the second claim we begin by noting that if \(G + H = G\) for all \(G \in P_+^\Sigma\), then, for any fixed \(G\), we must have all the finite dimensional distributions for \(G + H\) and \(G\) coincide, i.e.:
\[
\forall x \in \Sigma', \text{ s.t. } \phi''_{x_\epsilon + H} = \phi''_{x_\epsilon} \quad (138)
\]
Now, using the notation used in the construction of the sum \(G + H\) in Definition 18, we have:
\[
\forall x \in \Sigma', \phi''_{x_\epsilon} = \lim_{\epsilon \rightarrow 0} \phi''_{x_{\epsilon_\alpha}} \oplus \phi''_{x_{\epsilon_{\alpha'}}} \quad (139)
\]
where \(\epsilon \geq 0, x_\epsilon\) is a jointly \(\epsilon\)-synchronizing string. If \(\pi^G, \pi^H, \pi^{G + H}\) are the morph matrices, and \(G^G, H^H, G^H + H^H\) are the state sets for \(G, H, G + H\) respectively, it follows that:
\[
\forall q \in G^G + H^H, 3 \exists \epsilon > 0, q \in G^G, h \in H^H, \pi^{G + H}(q, \cdot) = \pi^G(q, \cdot) + \pi^H(q, \cdot) \quad (140)
\]
Since we necessarily have (Proposition 4):
\[
\forall q \in G^G + H^H, \forall \epsilon > 0, \exists x \in \Sigma', P_{G + H}G \cdot (q|x_\epsilon) > 1 - \epsilon \quad (141)
\]
which then implies from Eq. (138):
\[
\exists q \in G^G, h \in H^H, \pi'G \cdot (q, h) = \pi'G(q, h) \quad (142)
\]
where \(\pi'G\) follows from the fact that we assume all models to be minimal closed restrictions. Hence, it follows that:
\[
\forall q \in H^H, \pi'G(q, h) = \mathcal{U}_{\Sigma} \quad (143)
\]
implies that in the process modeled by \(H\), all sequences are equivalent, with the symbolic derivatives as given in Eq. (136). This establishes Claim B, and establishes Eq. (135c), where the required \(Z\) is given by \(W\).

To establish Eq. (135d), given \(G = (\Sigma, Q, \delta, \vec{p})\), we construct \(G' = (\Sigma, Q, \delta, \vec{p})\) as:
\[
\forall x \in \Sigma', \phi''_x = -1 \times \vec{p}(q, \cdot) \quad (144)
\]
and claim that:
\[
G + G' = Z \quad (145)
\]
which follows from noting that (using the notation of Definition 18), we have:
\[
\forall x \in \Sigma', \phi''_x = \lim_{\epsilon \rightarrow 0} \phi''_{x_{\epsilon_\alpha}} \oplus \phi''_{x_{\epsilon_{\alpha'}}} = \mathcal{U}_{\Sigma} \quad (146)
\]
Uniqueness of \(G'\) follows from:
\[
G + G' = G + G' = Z \Rightarrow G' = G'' \quad (147)
\]
This completes the proof.

Lemma 13 (Vector Space). \(P_+^\Sigma\) satisfies the following:
\[
\forall x \in R, \forall G \in P_+^\Sigma, 0G = \mathcal{U}_R \quad (151a)
\]
\[
\forall G \in P_+^\Sigma, 0G = \mathcal{U}_R \quad (151b)
\]
\[
\forall x \in R, \forall G, G' \in P_+^\Sigma, G + G' = \mathcal{U}_R \quad (151c)
\]
\[
\forall x \in R, \forall G \in P_+^\Sigma, G + (\alpha)G = Z \quad (151d)
\]
\[
\forall x, \alpha \in R, \forall G \in P_+^\Sigma, \alpha(G + \alpha')G = \alpha(\alpha'G) = \alpha'\alpha(G) \quad (151e)
\]
**Proof:** Immediate from Definition 17, and corresponding definitions for probability vectors.

IV. INNER PRODUCT OF ERODING STATIONARY PROCESSES

Definition 19 (Inner Product of Stochastic Processes). *For a strictly positive ergodic stationary processes \((\Sigma, Q, \delta, \vec{p})\) =
\((\Sigma', \mathcal{F}_a, \mu'), (\Sigma, Q, \delta, \pi') \models (\Sigma', \mathcal{F}_a, \mu')\),
\[
\langle G, G' \rangle \doteq \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \langle \phi_{x,x_i}^G, \phi_{x,x_i}^{G'} \rangle
\]  
(152)

where \(\forall \epsilon > 0, x_\epsilon \) is a jointly \(\epsilon\)-synchronizing sequence, and \(\forall \epsilon \in N, x_\epsilon = \lambda, x_1 = x_{\epsilon-1}\) or with \(\sigma\) drawn uniformly from \(\Sigma\).

Note that if \(|Q| = 1, |Q'| = 1\), then \(\langle G, G' \rangle\) is indeed a valid inner product, based on the formulation of inner products on finite dimensional probability vectors in Section II. Thus, for strictly positive i.i.d. processes taking values over a finite alphabet, we have a valid inner product. In general, we have:

Lemma 14 (Complete Inner Product). Definition 19 defines a complete inner product on the space of strictly positive stationary ergodic finite-valued processes.

Proof: (Sketch, details omitted.) Since \(\langle \phi_{x,x_i}, \phi_{x,x_i} \rangle\) is a valid inner product on \(\mathcal{F}_\Sigma\) and noting that joint synchronization extends to a finite number of sequences (and hence we can find a jointly \(\epsilon\)-synchronizing sequence for any triplet of ergodic stationary processes \(G, G', G''\)), we conclude that:

\[
\forall \alpha \in R, (G, \alpha(G' + G'')) = \alpha(G, G' + G'')
\]

Symmetry and non-negativity is also immediate. To prove completeness, we need to show that any Cauchy sequence in the space of our class of stochastic processes converges within our class. This is immediate since, if any sequence of processes converges outside our class, then the norm of the limiting process increases without bound, implying the sequence is not Cauchy (by the same argument used in Lemma 2).

V. Example

We consider a simple example of the resilience of the inner product to noise corruption. We consider two processes generated by two state PFSAs, and hence are imperfect ergodic and stationary (See Fig. 3). The noise corrupted versions are shown as well. The uncorrupted processes are easy to distinguish, while post-corruption it becomes a difficult problem to discriminate them from each other, as well as from the average iid approximation. A simple calculation shows that the relative angles remain mostly unchanged, which suggests a new approach to process classification/discovery in high noise scenarios. Plate F in Fig. 3 shows the separation achieved using computation of relative angles from corrupted data-streams (the generated binary data streams have means 0.499 and 0.5004, and standard deviations of 0.5, suggesting that they are indeed very close to flat white noise. Detailed comparison with standard techniques is being carried out at present.

VI. Summary, Conclusion & Future Work

We developed a Hilbert space for ergodic stationary processes, which would potentially allow us to investigate intrinsic structure of data in high noise scenarios. Future work will pursue detailed comparison with state of the art, and explore classification and clustering strategies based on the theoretical foundation developed here.

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