Quantum backreaction in dilute Bose-Einstein condensates

Ralf Schützhold, Michael Uhlmann, and Yan Xu
Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany

Uwe R. Fischer
Eberhard-Karls-Universität Tübingen, Institut für Theoretische Physik
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

For many physical systems which can be approximated by a classical background field plus small (linearized) quantum fluctuations, a fundamental question concerns the correct description of the backreaction of the quantum fluctuations onto the dynamics of the classical background. We investigate this problem for the example of dilute atomic/molecular Bose-Einstein condensates, for which the microscopic dynamical behavior is under control. It turns out that the effective-action technique does not yield the correct result in general and that the knowledge of the pseudo-energy-momentum tensor \( \langle T_{\mu \nu} \rangle \) is not sufficient to describe quantum backreaction.

PACS numbers: 03.75.Kk, 04.62.+v, 43.35.+d, 04.60.-m

I. INTRODUCTION

Many phenomena in physics can be described, to a sufficient degree of accuracy, by means of the background-field method, wherein one considers linearized quantum fluctuations on top of an approximately classical background. For a scalar gap-less (e.g., Goldstone) mode \( \phi \), propagating in an arbitrary background (e.g., zeroth sound in inhomogeneous superfluids), the low-energy effective action can be cast into the pseudocovariant form

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial_\mu \phi)^* G^{\mu \nu} (\partial_\nu \phi),
\]

(1)

with \( \partial_\mu = (\partial / \partial t, \nabla) \) denoting the space-time derivatives (we employ the summation convention), and the space-time dependent tensor \( G^{\mu \nu} \) characterizing the background \[1, 2\]. Hence the propagation of \( \phi \) is completely analogous to that of a scalar field in a curved background \[1, 2\]. However, in calculating the quantum backreaction in that way, one is implicitly making two essential assumptions: firstly, that the leading contributions to the backreaction are completely determined by the effective action in Eq. (1), and, secondly, that deviations from the low-energy effective action at high energies do not affect the (renormalized) expectation value of the pseudo-energy-momentum tensor, \( \langle T_{\mu \nu} \rangle \), which is difficult to grasp in general, due to the nonuniqueness of the vacuum state in a complicated curved space-time background and the ultraviolet (UV) renormalization procedure. Adopting a covariant renormalization scheme, the results for \( \langle T_{\mu \nu} \rangle \) can be classified in terms of geometrical quantities (cf. the trace anomaly \[3\]).

The precise meaning of the expectation value of the pseudo-energy-momentum tensor, \( \langle T_{\mu \nu} \rangle \), is difficult to grasp in general, due to the nonuniqueness of the vacuum state in a complicated curved space-time background and the ultraviolet (UV) renormalization procedure. Adopting a covariant renormalization scheme, the results for \( \langle T_{\mu \nu} \rangle \) can be classified in terms of geometrical quantities (cf. the trace anomaly \[3\]).

However, in calculating the quantum backreaction in that way, one is implicitly making two essential assumptions: firstly, that the leading contributions to the backreaction are completely determined by the effective action in Eq. (1), and, secondly, that deviations from the low-energy effective action at high energies do not affect the (renormalized) expectation value of the pseudo-energy-momentum tensor, \( \langle T_{\mu \nu} \rangle \). Since the effective covariance in Eq. (1) is only a low-energy property, the applicability of a covariant renormalization scheme is not obvious in general. In the following, we critically examine the question of whether the two assumptions mentioned above are justified, e.g., whether \( \langle T_{\mu \nu} \rangle \) completely determines the backreaction of the linearized quantum fluctuations. In order to address this question, we shall consider phonon modes in a well understood superfluid, a dilute Bose-Einstein condensate.

This paper is organized as follows. In Section II, we give a brief introduction to Bose-Einstein condensates, and introduce a particle-number-conserving ansatz for the field operator. In the subsequent Section III based on this ansatz, the backreaction to the full current will be calculated, yielding a different result than that ob-

[3]
tained by the effective action method. The differences of the two expressions will be investigated in more detail in Sec. III. Since the choice of the (classical) background is of particular importance, we will demonstrate its influence in Sec. IV where we consider two different choices for the background yielding two different expressions for the backreaction. Afterwards, the failure of the effective action technique is discussed in more detail in Sec. VII.

The cutoff dependence of the pseudo-energy-momentum tensor \[ \mathcal{T} \] is addressed in Sec. VII. As a simple example, we consider the influence of the backreaction contribution to a static 1D condensate in Sec. VIII.

II. BOSE-EINSTEIN CONDENSATES

In the usual \( s \)-wave scattering approximation, the many-particle system is described by the field operator in the Heisenberg picture (we set \( \hbar = m = 1 \))

\[
i \frac{\partial}{\partial t} \hat{\Psi} = \left( -\frac{1}{2} \nabla^2 + V + g \hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi},
\]

where \( V \) denotes the one-particle trapping potential, and the coupling constant \( g \) is related to the \( s \)-wave scattering length \( a_s \) via \( g = 4\pi a_s \) (in three spatial dimensions). In the limit of many particles \( N \gg 1 \), in a finite trap at zero temperature with almost complete condensation, the full field operator \( \hat{\Psi} \) can be represented in terms of the particle-number-conserving mean-field ansatz [6,7]

\[
\hat{\Psi} = \left( \psi_c + \hat{\chi} + \hat{\zeta} \right) \hat{A}/\sqrt{N},
\]

with the order parameter \( \psi_c = \mathcal{O}(\sqrt{N}) \), the one-particle excitations \( \hat{\chi} = \mathcal{O}(N^0) \), and the remaining higher-order corrections \( \hat{\zeta} = \mathcal{O}(1/\sqrt{N}) \). The above mean-field ansatz can be derived in the dilute-gas limit by formally setting \( g = \mathcal{O}(1/N) \) [7,8,9,10]: we shall use this formal definition of the dilute-gas limit in what follows. Insertion of Eq. (3) into (4) yields the Gross-Pitaevskii equation [11] for the order parameter \( \psi_c \)

\[
i \frac{\partial}{\partial t} \psi_c = \left( \frac{1}{2} \nabla^2 + V + g|\psi_c|^2 + 2g \langle \hat{\chi}^\dagger \hat{\chi} \rangle \right) \psi_c + g \langle \hat{\chi}^2 \rangle \psi_c^*,
\]

the Bogoliubov-de Gennes equations [12] for \( \hat{\chi} \)

\[
i \frac{\partial}{\partial t} \hat{\chi} = \left( \frac{1}{2} \nabla^2 + V + 2g|\psi_c|^2 \right) \hat{\chi} + g\psi_c^2 \hat{\chi}^\dagger,
\]

and for the remaining higher-order corrections \( \zeta = \mathcal{O}(1/\sqrt{N}) \), neglecting \( \mathcal{O}(1/N) \) terms:

\[
i \frac{\partial}{\partial t} \hat{\zeta} \approx \left( \frac{1}{2} \nabla^2 + V + 2g|\psi_c|^2 \right) \hat{\zeta} + g\psi_c^2 \hat{\chi}^\dagger + 2g(\hat{\chi}^\dagger \hat{\chi} - \langle \hat{\chi}^\dagger \hat{\chi} \rangle) \psi_c + g(\hat{\chi}^2 - \langle \hat{\chi}^2 \rangle) \psi_c^*. \]

Note that the additional terms \( 2g(\hat{\chi}^\dagger \hat{\chi}) \) and \( g(\hat{\chi}^2) \) in the Gross-Pitaevskii equation ensure that the expectation value of \( \hat{\zeta} = \mathcal{O}(1/\sqrt{N}) \) vanishes in leading order, \( \langle \hat{\zeta} \rangle = \mathcal{O}(1/N) \), see also [11,12]. Without these additional terms, the mean-field expansion would still be valid with \( \hat{\zeta} = \mathcal{O}(1/\sqrt{N}) \), but without \( \langle \hat{\zeta} \rangle = \mathcal{O}(1/N) \).

III. BACKREACTION

The observation that the Gross-Pitaevskii equation yields an equation correct to leading order \( \mathcal{O}(\sqrt{N}) \), using either \( |\psi_c|^2 \) or \( (\hat{\chi}^\dagger \hat{\chi}) \) in the first line of (6), hints at the fact that quantum backreaction effects correspond to next-to-leading order terms and cannot be derived \textit{ab initio} in the above manner without additional assumptions. Therefore, we shall employ an alternative method: In terms of the exact density and current given by

\[
\rho = \langle \hat{\Psi}^\dagger \hat{\Psi} \rangle, \quad j = \frac{1}{2\hbar} \langle \hat{\Psi}^\dagger \nabla \hat{\Psi} - \text{H.c.} \rangle,
\]

the time-evolution is governed by the equation of continuity for \( \rho \) and an Euler type equation for the current \( j \). After insertion of Eq. (4), we find that the equation of continuity is not modified by the quantum fluctuations but satisfied exactly (i.e., to all orders in \( 1/N \) or \( \hbar \))

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot j = 0,
\]

in accordance with the Noether theorem and the \( U(1) \) invariance of the Hamiltonian, cf. [8]. However, if we insert the mean-field expansion and write the full density as a sum of condensed and noncondensed parts

\[
\rho = \rho_c + \langle \hat{\chi}^\dagger \hat{\chi} \rangle + \mathcal{O}(1/\sqrt{N}),
\]

with \( \rho_c = |\psi_c|^2 \), we find that neither part is conserved separately in general. Note that this split requires \( \langle \hat{\zeta} \rangle = \mathcal{O}(1/N) \), i.e., the modifications to the Gross-Pitaevskii equation discussed above. Similarly, we may split up the full current [with \( \rho_c \mathbf{v}_c = \hat{3}(\psi_c^\dagger \nabla \psi_c) \)]

\[
\mathbf{j} = \rho_c \mathbf{v}_c + \frac{1}{2\hbar} \langle \hat{\chi}^\dagger \hat{\chi} \nabla - \text{H.c.} \rangle + \mathcal{O}(1/\sqrt{N}),
\]

and introduce an average velocity \( \mathbf{v} \) via \( j = \rho \mathbf{v} \). This enables us to \textit{unambiguously} define the quantum backreaction \( \mathcal{Q} \) as the following additional contribution in an equation of motion for \( \mathbf{j} \) analogous to the Euler equation:

\[
\frac{\partial}{\partial t} \mathbf{j} = \mathbf{f}(j, \rho) + \mathcal{Q} + \mathcal{O}(1/\sqrt{N}),
\]

where \( (j = \rho \mathbf{v}) \)

\[
\mathbf{f}(j, \rho) = -\mathbf{v} [\nabla \cdot (\rho \mathbf{v})] - \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \rho \nabla \left[ \frac{1}{2} \frac{\nabla^2 \mathbf{v}}{\sqrt{\rho}} - V_{\text{ext}} - \rho \mathbf{g} \right] = \left[ \frac{1}{2} \frac{\nabla^2 \mathbf{v}}{\sqrt{\rho}} - V_{\text{ext}} - \rho \mathbf{g} \right] \rho \mathbf{v}.
\]

(14)
denotes the usual classical force density term. Formulation in terms of the conventional Euler equation yields
\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v = -\nabla \left( V + g_\phi - \frac{1}{2} \nabla^2 \sqrt{\phi} \right) + Q/\phi + O(1/\sqrt{N^3}). \tag{15}
\]
The quantum backreaction \( Q \) can now be calculated by comparing the two equations above and expressing \( \partial j/\partial t \) in terms of the field operators via Eqs. (4) and (9).

After insertion of the mean-field expansion, we obtain the leading contributions in the Thomas-Fermi limit
\[
Q = \nabla \cdot (v \otimes j_\chi + j_\chi \otimes v - \varphi v \otimes v)
- \frac{1}{2g} \nabla \left( g^2 \langle 2|\psi_c^2 \chi^1 \chi^1 + \psi_c^2 \chi^1 \chi^1 \rangle \right) + \hat{\chi} \psi \hat{\chi} + H.c.
\]
\[
- \frac{1}{2} \nabla \cdot \langle (\nabla \chi^1) \otimes \nabla \chi^1 + H.c. \rangle,
\tag{17}
\]
with \( \varphi = \langle \chi^1 \chi^1 \rangle \) and \( j_\chi = \Im \langle \chi^1 \nabla \chi \rangle \). Under the assumption that the relevant length scales \( \lambda \) for variations of, e.g., \( \varphi \) and \( g \) are much larger than the healing length \( \xi = (\varphi_0 g)^{-1/2} \), we have neglected terms containing quantum pressure contributions \( \nabla^2 \varphi \) and \( \nabla \varphi \) (Thomas-Fermi or local-density approximation). These contributions would, in particular, spoil the effective (local) geometry in Eq. (1).

IV. COMPARISON WITH EFFECTIVE-ACTION TECHNIQUE

In order to compare the expression (17) with Eq. (3), we have to identify \( \phi \) and \( G^{\mu\nu} \). Phases modes with wavelength \( \lambda > \xi \) are described by the action in Eq. (11) in terms of the phase fluctuations \( \phi \) provided that \( G^{\mu\nu} \) is given by
\[
G^{\mu\nu} = \frac{1}{g} \left( \frac{1}{v} v \otimes v - c^2 1 \right), \tag{18}
\]
where \( c^2 = g_\phi \) is the speed of sound, cf. (11,2). The density fluctuations \( \delta \phi = -g^{-1} (\partial/\partial t + v \cdot \nabla) \phi \).

If we now calculate the quantum corrections to the Bernoulli equation using the effective-action method in Eq. (3) by inserting \( \partial G^{\mu\nu}/\partial \phi \) or \( \partial g^{\mu\nu}/\partial \phi \), we obtain
\[
\frac{\delta A_{\text{eff}}}{\delta \phi} = \frac{1}{2} \langle (\nabla \phi)^2 \rangle. \tag{19}
\]
Clearly, this result differs from the expression (17) derived in the previous Section. Moreover, it turns out that Eq. (17) contains contributions which are not part of the pseudo-energy-momentum tensor \( \langle T_{\mu\nu} \rangle \). Even though the phonon flux \( j_\chi \) in the first line of Eq. (17) corresponds to \( \langle \delta \phi \nabla \phi + H.c. \rangle \), which indeed reproduces the mixed space-time components of the pseudo-energy-momentum tensor \( \langle T_{\mu\nu} \rangle \), the phonon density \( \varphi_\chi \) contains \( \langle (\delta \phi)^2 \rangle \) (where \( \langle \ldots \rangle \) means that the divergent c-number \( \chi^1 \chi^1 \) has been subtracted already) which is part of \( \langle T_{\mu\nu} \rangle \) but also \( \langle \phi^2 \rangle \) which is not. (Note that \( \langle \phi^2 \rangle \) cannot be cancelled by the other contributions.) The expectation value in the second line of Eq. (17) is basically \( \langle (\delta \phi)^2 \rangle \) and thus part of \( \langle T_{\mu\nu} \rangle \). Finally, the expression \( \langle (\nabla \chi^1) \otimes \nabla \chi^1 + H.c. \rangle \) in the last line of Eq. (17) contains \( \langle (\nabla \phi \nabla \hat{\phi}) \rangle \) which does occur in \( \langle T_{\mu\nu} \rangle \), but also \( \langle \nabla \delta \phi \nabla \hat{\phi} \rangle \), which does not. One could argue that the latter term ought to be neglected in the Thomas-Fermi or local-density approximation since it is on the same footing as the quantum pressure contributions \( \nabla^2 \varphi \) and \( \nabla \varphi \), but it turns out that this expectation value yields cutoff dependent contributions of the same order of magnitude as the other terms, see Section IV below.

V. CHOICE OF CLASSICAL BACKGROUND

In order to understand the disagreement between expressions (17) and (19), and to demonstrate the dependence of the explicit expression for the quantum backreaction on the choice of the classical background (see also Appendix A), we will consider two particular alternative backgrounds. Firstly, the background is prescribed by the density and current generated by the condensed part \( \psi_c \) of the field operator \( \Psi \), cf. Eqs. (11) and (12). And secondly, we consider (the expectation values of) the full density and the velocity potential as a (classical) background.

In the first case, the background current is given by
\[
j_c = g_c v_c = \frac{1}{2} \langle \psi_c^\dagger \nabla \psi_c - H.c. \rangle = j - j_\chi. \tag{20}
\]
Note that the exact splitting \( j = j_c + j_\chi \) is not unique and thus the classical background \( j_\chi \) and the fluctuations \( j_\chi \) (in contrast to \( j \)) are not directly measurable, cf. the remark on the Gross-Pitaevskii equation (20). The evolution of the phonon flux can be inferred from the Bogoliubov-de Gennes equation (17)
\[
\frac{\partial}{\partial t} j_\chi = \frac{1}{4} \nabla^3 \varphi_\chi - \frac{1}{2} \nabla \langle (\nabla \chi^1) \otimes \nabla \chi^1 + H.c. \rangle
- \varphi_\chi \nabla \left( V + 2g_\phi \right)
- \frac{1}{2} \langle (\chi^2) \nabla \left( g_\phi \chi^2 \right) + H.c. \rangle, \tag{21}
\]
which enables us to derive the evolution equation for the
condensate part via the subtraction $j_c = j - j_x$

$$\frac{\partial}{\partial t} j_c = f(\rho_c, v_c) - 2\rho_c \nabla (g\rho_c) - \frac{1}{2} \left[ (\psi_c^*)^2 \nabla (g \langle \chi^2 \rangle) + \text{H.c.} \right],$$  \hspace{1cm} (22)

which is the same result as obtained from the modified Gross-Pitaevskii equation [14]. However, the interpretation of the r.h.s. of the equation above as a force density is not straightforward, since the equation of continuity is not satisfied

$$\frac{\partial}{\partial t} \rho_c + \nabla \cdot j_c = ig \left( \psi_c^2 \langle \chi^1 \rangle^2 - (\psi_c^*)^2 \chi^2 \right).$$  \hspace{1cm} (23)

I.e., an acceleration of the atoms of the condensate caused by a real force could make them leave the condensate or induce a change in $j_c$ and it is difficult to disentangle these two effects. Roughly speaking, choosing the condensate part $j_c$ (which is not measurable and not conserved) as the classical background would correspond to the artificial variable $\hat{\Phi}$, is not [14]. It can be introduced via the following ansatz for the full field operator

$$\Psi = e^{\hat{\Phi}} \sqrt{\rho}.$$  \hspace{1cm} (25)

Since $\hat{\Phi}$ and $\hat{\phi}$ do not commute, other forms such as $\Psi = \sqrt{\rho} e^{i\hat{\Phi}}$ would not generate a self-adjoint $\hat{\Phi}$ (and simultaneously satisfy $\Psi^\dagger \Psi = \hat{\rho}$). Insertion of the above ansatz into the expression for the full current yields

$$j = \left\langle \sqrt{\rho} \nabla \hat{\Phi} \sqrt{\rho} \right\rangle.$$  \hspace{1cm} (26)

The quantum corrections to the equation of continuity in this formulation can be obtained by inserting the split in Eq. (22) and neglecting terms of third or higher order

$$j = \rho_b \nabla \phi_b + \frac{1}{2} (\delta \hat{\phi} \nabla \hat{\phi} + \text{H.c.}) = \rho_b \nabla \phi_b + j_x,$$  \hspace{1cm} (27)

and are in perfect agreement with the effective-action method $\delta A_{\text{eff}} / \delta \phi_b$.

In order to relate the background velocity $\nabla \phi_b = v_b$ to known quantities (the background density of course equals the full density $\rho_b = \rho$), we remember $j = j_c + j_x$ and deduce

$$\rho_b \nabla \phi_b = \rho_b v_b = \rho_c v_c,$$  \hspace{1cm} (28)

i.e., the background velocity is (again up to terms of third or higher order) proportional to the condensate velocity. Now, considering a condensate which is initially at rest $v_c(t = 0) = 0$ and calculating

$$\frac{\partial}{\partial t} v_b = \frac{\partial}{\partial \rho_b} \frac{\partial}{\partial \rho_c} = \frac{\partial}{\partial \rho_c} - 2 \nabla (g\rho_c) + \frac{1}{2} \left[ (\psi_c^*)^2 \nabla (g \langle \chi^2 \rangle) + \text{H.c.} \right],$$  \hspace{1cm} (29)

we observe that $\partial_t v_b$ contains the terms $\rho_c \chi, \langle \chi^2 \rangle$, and $\langle (\chi^1)^2 \rangle$, but not $\langle (\nabla \phi)^2 \rangle$. As a result, the quantum corrections to the Bernoulli equation obtained in the above direct manner are in contradiction to the effective-action method result $\delta A_{\text{eff}} / \delta \rho_b$.

VI. FAILURE OF EFFECTIVE-ACTION TECHNIQUE

After having demonstrated the failure of the effective-action method for deducing the quantum backreaction, let us study the reasons for this failure in more detail. The full action governing the dynamics of the fundamental fields $\Psi^*$ and $\Psi$ reads

$$\mathcal{L} = i \Psi^* \frac{\partial}{\partial t} \Psi - \Psi^* \left( -\frac{1}{2} \nabla^2 + V + \frac{g}{2} \Psi^* \Psi \right) \Psi.$$  \hspace{1cm} (30)

Linearization $\Psi = \psi_c + \chi$ yields the effective second-order action generating the Bogoliubov-de Gennes equations [7]

$$\mathcal{L}_{\text{eff}}^\chi = i \chi^* \frac{\partial}{\partial t} \chi - \chi^* \left( -\frac{1}{2} \nabla^2 + V + 2g \left| \psi_c \right|^2 \right) \chi - \left[ \frac{g}{2} \left( \psi_c^2 \right)^2 \chi^2 + \text{H.c.} \right].$$  \hspace{1cm} (31)

Now $\delta A_{\text{eff}} / \delta \rho_b$ indeed yields the correct backreaction [24] in the condensate formulation. However, if we start with the action in terms of the nonfundamental variable $\Phi$

$$\mathcal{L} = -\frac{\epsilon}{2} \left( \frac{\partial}{\partial t} \Phi + \frac{1}{2} (\nabla \Phi)^2 \right) - \epsilon \phi - V \phi,$$  \hspace{1cm} (32)

with $\epsilon \phi$ denoting the internal energy density, the quantum corrections to the equation of continuity $\delta A_{\text{eff}} / \delta \phi_b$ are reproduced correctly but the derived quantum backreaction contribution to the Bernoulli equation $\delta A_{\text{eff}} / \delta \rho_b$ is wrong.
Why is the effective-action method working for the fundamental field $\Psi$, but not for the nonfundamental variable $\Phi$? The quantized fundamental field $\Psi$ satisfies the equation of motion $\partial_t \hat{\Phi} + \frac{1}{2} (\nabla \hat{\Phi})^2 + h[\hat{\phi}] = 0$, (33) where $h = d\varepsilon/d\phi + V$. The problem is that the commutator of $\hat{\phi}$ and $\hat{\Phi}$ at the same position diverges and hence the quantum Madelung ansatz in Eq. (24) is singular. As a result, the above quantum Bernoulli equation is not well-defined (in contrast to the equation of continuity), i.e., insertion of the quantum Madelung ansatz in Eq. (24) into Eq. (33) generates divergences.

In order to study these divergences by means of a simple example, let us consider the generalized Bose-Hubbard Hamiltonian

$$\hat{H} = \alpha \sum_{<ij>} (\hat{\Psi}_i^\dagger \hat{\Psi}_j + \text{H.c.}) + \sum_i (\beta \hat{n}_i + \gamma \hat{n}_i^2),$$

(34)

where $<ij>$ denote nearest neighbors and $\hat{n}_i = \hat{\Psi}_i^\dagger \hat{\Psi}_i$ is the filling factor of one lattice site $i$. In the usual continuum limit, this Hamiltonian generates Eq. (11) where $\alpha$ is related to the lattice spacing and the effective mass, $\gamma$ determines $g$, and $V$ is governed by $\alpha$, $\beta$, and $\gamma$. Inserting the quantum Madelung ansatz in Eq. (24), however, the problem of operator ordering arises and the (for the Bernoulli equation) relevant term reads

$$\hat{H}_\Phi = \frac{1}{4} \sum_i \sqrt{\hat{n}_i (\hat{n}_i + 1)} (\nabla \hat{\Phi})_i^2 + \text{H.c.},$$

(35)

with the replacement $\hat{n}_i + 1$ instead of $\hat{n}_i$ being one effect of the noncommutativity. In the superfluid phase with large filling $n \gg 1$, we obtain the following leading correction to the equation of motion

$$\frac{\partial}{\partial t} \hat{\Phi} + \frac{1}{2} (\nabla \hat{\Phi})^2 + h[\hat{\phi}] + \frac{1}{\hat{n}} (\nabla \hat{\Phi})^2 \frac{1}{\hat{n}} = \mathcal{O} \left( \frac{1}{n^4} \right),$$

(36)

which depends on microscopic details (such as the filling).

By means of this simple example, we already see that the various limiting procedures such as the quantization and subsequent mean-field expansion, the variable transformation $(\hat{\Psi}^*, \hat{\Psi}) \leftrightarrow (\hat{\phi}, \hat{\Phi})$, and the linearization for small fluctuations, as well as continuum limit do not commute in general – which explains the failure of the effective-action method for deducing the quantum backreaction. The variable transformation $(\hat{\Psi}^*, \hat{\Psi}) \leftrightarrow (\hat{\phi}, \hat{\Phi})$ is applicable to the zero-order equations of motion for the classical background as well as to the first-order dynamics of the linearized fluctuations – but the quantum backreaction is a second-order effect, where the aforementioned difficulties arise.

\section{Cutoff Dependence}

As mentioned in the Introduction, another critical issue is the UV divergence of $\langle T_{\mu \nu} \rangle$. Extrapolating the low-energy effective action in Eq. (11) to large momenta $k$, the expectation values $\langle \delta \hat{\phi}^2 \rangle$ and $\langle \delta \hat{\phi}^2 \rangle$ entering $g(\lambda)$ would diverge. For Bose-Einstein condensates, we may infer the deviations from Eq. (11) at large $k$ from the Bogoliubov-de Gennes equations (7). Assuming a static and homogeneous background (which should be a good approximation for large $k$), a normal-mode expansion yields

$$\hat{\chi}_k = \sqrt{\frac{k^2}{2 \omega_k}} \left[ \left( \frac{\omega_k}{k^2} - \frac{1}{2} \right) \hat{a}_k^\dagger + \left( \frac{\omega_k}{k^2} + \frac{1}{2} \right) \hat{a}_k \right],$$

(37)

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the creation and annihilation operators of the phonons, respectively, and the frequency $\omega_k$ is determined by the Bogoliubov dispersion relation $\omega_k^2 = g g(k^2 + k^4)/4$. Using a linear dispersion $\omega_k^0 \propto k^2$ instead, expectation values such as $\langle \chi_1^2 \rangle$ would be UV divergent, but the correct dispersion relation implies $\chi_k \sim \hat{a}_k^\dagger g k^2 + \hat{a}_k$ for large $k^2$, and hence $\langle \chi_1^2 \rangle$ is UV finite in three and lower spatial dimensions. Thus the healing length $\xi$ acts as an effective UV cutoff $k_{cut}$.

Unfortunately, the quadratic decrease for large $k$ in Eq. (37), $\hat{\chi}_k \sim \hat{a}_k^\dagger g k^2 + \hat{a}_k$, is not sufficient for rendering the other expectation values (i.e., apart from $\langle \chi_1 \rangle$) in Eq. (11) UV finite in three spatial dimensions. This UV divergence indicates a failure of the s-wave pseudo-potential $g \delta^3(r-r')$ in Eq. (1) at large wave-numbers $k$ and can be eliminated by replacing $g \delta^3(r-r')$ by a more appropriate two-particle interaction potential $V_{\text{int}}(r-r')$, see (12). Introducing another UV cutoff wavenumber $k_{cut}$ related to the breakdown of the s-wave pseudo-potential, we obtain $\langle \nabla \chi_1^2 \rangle \otimes \nabla \chi_1 \text{H.c.} \sim \delta^3 [U \otimes U]_{\text{cut}}$ and $\langle \chi_1^2 \rangle \sim g_{\text{cut}}^2 k_{\text{cut}}^4$.

In summary, there are two different cutoff wavenumbers: The first one, $k_{cut}$, is associated to the breakdown of the effective Lorentz invariance (change of dispersion relation from linear to quadratic) and renders some – but not all – of the naively divergent expectation values finite. The second wavenumber, $k_{s}$, describes the cutoff for all (remaining) UV divergences. In dilute Bose-Einstein condensates, these two scales are vastly different by definition; because the system is dilute, the inverse range of the true potential, which is of order $k_{cut}$, must be much larger than the inverse healing length:

$$k_{cut} \gg k_{s} \gg k_{\xi} \gg k_{\text{Lorentz}}.$$

(38)

Note the opposite relation $k_{cut}^L \gg k_{cut} \gg k_{UV}$ is very unnatural since every quantum field theory which has the usual properties such as locality and Lorentz invariance etc., must have UV divergences (e.g., in the two-point function).

The renormalization of the cutoff-dependent terms is the same for the two cases: The $k_{cut}$-contributions can be
absorbed by a $\rho$-independent renormalization of the coupling $g$, whereas the $k^2$-contributions depend on the density in a nontrivial way and thus lead to a renormalization of the pressure and the chemical potential etc., see the next Section.

**VIII. SIMPLE EXAMPLE**

In order to provide an explicit example for the quantum backreaction term in Eq. (17), without facing the UV problem, let us consider a quasi-one-dimensional (quasi-1D) condensate $\rho_{1D}$, where all the involved quantities are UV finite. In accordance with general considerations $[18]$, the phonon density $\rho_{\chi}$ is infrared (IR) divergent in one spatial dimension, therefore inducing finite-size effects. Nevertheless, in certain situations, we are able to derive a closed local expression for the quantum backreaction term $Q$: Let us assume a completely static condensate $v = 0$ in effectively one spatial dimension, still allowing for a spatially varying density $\rho$ and possibly also coupling $g$. Furthermore, since spatial variations of $\rho$ and $g$ occur on length scales $\lambda$ much larger than the healing length, we keep only the leading terms in $\xi/\lambda \ll 1$, i.e., the variation of $\rho$ and $g$ will be neglected in the calculation of the expectation values. In this case, the quantum backreaction term $Q$ simplifies considerably and yields (in effectively one spatial dimension, where $g \equiv g_{1D}$ and $\rho \equiv \rho_{1D}$ now both refer to the 1D quantities)

$$Q = -\nabla \langle (\hat{\rho}^2) \hat{\chi} \rangle - \frac{1}{2g} \nabla \langle g^2 (2\hat{\chi}^2 + (\hat{\chi})^2) \rangle$$
$$= -\nabla \left( \frac{1}{3\pi} (g\rho)^{3/2} \right) + \frac{1}{2\pi g} \nabla \left( g^{5/2} \rho^{3/2} \right) + O(\xi^2/\lambda^2)$$
$$= \frac{\rho}{2\pi} \nabla g^3 \rho + O(\xi^2/\lambda^2).$$

(39)

It turns out that the IR divergences of $2\langle \hat{\chi}^2 \rangle$ and $\langle (\hat{\chi})^2 \rangle$ cancel each other such that the resulting expression is not only UV but also IR finite. Note that the sign of $Q$ is positive and hence opposite to the contribution of the pure phonon density $\langle \hat{\chi}^2 \chi \rangle$, which again illustrates the importance of the term $\langle (\hat{\chi})^2 \rangle$.

A possible experimental signature of the quantum backreaction term $Q$ calculated above, is the change incurred on the static Thomas-Fermi solution of the Euler equation $[15]$ for the density distribution (cf. $[11, 13]$)

$$g_{1D} = \frac{\mu - V}{g_{1D}} + \sqrt{\frac{\mu - V}{2\pi}} + O(1/\sqrt{N}),$$

(40)

with $\mu$ denoting the (constant) chemical potential. The classical $[O(N)]$ density profile $g_{cl} = (\mu - V)/g_{1D}$ acquires nontrivial quantum $[O(N^0)]$ corrections $g_Q = \sqrt{\mu - V}/2\pi$ where the small parameter is the ratio of the interparticle distance $1/\rho = O(1/N)$ over the healing length $\xi = O(N^0)$. Note that the quantum backreaction term $g_Q$ in the above split $\rho = g_{cl} + g_Q$ should neither be confused with the phonon density $\rho_{\chi}$ in $g = g_{cl} + g_{\chi}$ (remember that $\rho_{\chi}$ is IR divergent and hence contains finite-size effects) nor with the quantum pressure contribution $\propto \nabla^2 \sqrt{\rho}$ in Eq. (19).

Evaluating explicitly the change $\Delta R$ of the Thomas-Fermi size (half the full length), where $\mu = V$, of a quasi-1D Bose-Einstein condensate induced by backreaction, from Eq. (10), we get $\Delta R = -2^{-5/2}(\omega_{1D}/\omega_2)\alpha_s$. (The quasi-1D coupling constant $g_{1D}$ is related to the 3D s-wave scattering length $a_s$ and the perpendicular harmonic trapping $\omega_2$ by $g_{1D} = 2a_s\omega_2^{1/3}$.) In units of the classical size $R_{cl} = (3a_sN\omega_{1D}/\omega_2^{3/5})$, we have

$$\frac{\Delta R}{R_{cl}} = \frac{1}{4\sqrt{2}} \left( \frac{1}{3N} \right)^{1/3} \left( \frac{\omega_1}{\omega_2} \right)^{2/3} a_s^{2/3},$$

(41)

where $a_s = 1/\sqrt{\omega_2}$ describes the longitudinal harmonic trapping. In quasi-1D condensates, backreaction thus leads to a shrinking of the cloud relative to the classical expectation – whereas in three spatial dimensions we have the opposite effect $[11, 13]$. For reasonably realistic experimental parameters, the effect of backreaction should be measurable: for $N \approx 10^3$, $\omega_1/\omega_2 \approx 10^3$, and $a_s/a_z \approx 10^{-3}$, we obtain $|\Delta R/R_{cl}| \approx 1\%$.

**IX. CONCLUSIONS**

Even though the explicit form of the quantum backreaction terms depends on the definition of the classical background, the effective-action method does not yield the correct result in the general case (i.e., independent of the choice of variable etc.). The knowledge of the classical (macroscopic) equation of motion – such as the Bernoulli equation – may be sufficient for deriving the first-order dynamics of the linearized quantum fluctuations (phonons), but the quantum backreaction as a second-order effect cannot be obtained without further knowledge of the microscopic structure (e.g., operator ordering). It is tempting to compare these findings to gravity, where we also know the classical equations of motion only

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},$$

(42)

which – in analogy to the Bernoulli equation – might yield the correct first-order equations of motion for the linearized gravitons, but perhaps not their (second-order) quantum backreaction. Another potentially interesting point of comparison is the existence of two different high-energy scales – one associated to the breakdown of Lorentz invariance $k^2_{cut} = k^2_{cut\, Lorentz}$ and the other $k_{UV}^2 = k^2_{cut\, UV}$ to the UV cutoff as well as the question of whether one of the two (or both) correspond to the Planck scale in gravity.

The dominant $O(\xi/\lambda)$ quantum backreaction contributions like those in Eq. (18) depend on the healing length as the lower UV cutoff and hence cannot be derived from the low-energy effective action in Eq. (1) using a covariant
(i.e., cutoff independent) regularization scheme, which does not take into account details of microscopic physics (represented, for example, in the quasiparticle dispersion relation). Note that the leading $O(\xi/\lambda)$ quantum correction to the pressure could be identified with a cosmological term, $\langle T_{\mu\nu} \rangle = \Lambda g_{\mu\nu}$ in Eq. (3), provided that $\Lambda$ is not constant but depends on $g$ and $g$. (In general relativity, the Einstein equations demand $\Lambda$ to be constant.)

As became evident from the remarks after Eq. (15), the knowledge of the expectation value of the pseudo-energy-momentum tensor $\langle T_{\mu\nu} \rangle$ is not sufficient for determining the quantum backreaction effects in general. Even though $\langle T_{\mu\nu} \rangle$ is a useful concept for describing the phonon kinematics (at low energies), and one may identify certain contributions in Eq. (17) with terms occurring in $\langle T_{\mu\nu} \rangle$ in curved space-times (e.g., next-to-leading-order terms in $\xi/\lambda$ in the gradient expansion), we have seen that it does not represent the full dynamics. Related limitations of the classical pseudo-energy-momentum tensor have been discussed in [4].

In general, the quantum backreaction corrections to the Euler equation in Eq. (15) cannot be represented as the gradient of some local potential, cf. Eq. (17). Hence they may effectively generate vorticity and might serve as the seeds for vortex nucleation. Note that this effect cannot be observed in the “force” density in the condensate formulation [22] since $j_{\chi} = \phi_{\chi} \nabla \phi_{\chi}$ holds and thus introducing vorticity into the condensate requires creating condensate holes with $\phi_{\chi} = 0$ (vortices), which is beyond the regime of applicability of our linearized analysis. Therefore, vorticity (in the full-j formulation) can only be generated by extracting atoms from the condensate.

In contrast to the three-dimensional case (see, e.g., [11, 15]), the quantum backreaction corrections given by Eq. (39) diminish the pressure in condensates that can be described by Eq. (4) in one spatial dimension (quasi-1D case). This is a direct consequence of the so-called “anomalous” term $\langle (\dot{\chi}^2 + \chi^2) \rangle$ in Eq. (39), which – together with the cancellation of the IR divergence – clearly demonstrates that it cannot be neglected in general. We emphasize that even though Eqs. (39) – (41) describe the static quantum backreaction corrections to the ground state, which can be calculated by an alternative method [15] as well, the expression in Eq. (14) is valid for more general dynamical situations, such as expanding condensates. The static quantum backreaction corrections to the ground state can be absorbed by a redefinition of the chemical potential $\mu(\rho)$ determining a (barotropic) equation of state $p(\rho)$; this is however not possible for the other terms in Eq. (14), such as the quantum friction-type terms $j \otimes \nabla \rho$ etc., occurring in more general dynamical situations.

We have derived, from the microscopic physics of dilute Bose-Einstein condensates, the backreaction of quantum fluctuations onto the motion of the full fluid and found possible experimentally observable consequences. We observed a failure of the effective-action technique, Eq. (8), and a cutoff dependence of the backreaction term due to the breakdown of covariance at high energies. Whether similar problems beset “real” (quantum) gravity remains an interesting open question.

Acknowledgments

R.S., M.U., and Y.X. gratefully acknowledge financial support by the Emmy Noether Programme of the German Research Foundation (DFG) under grant No. SCHU 1557/1-1/2. The authors acknowledge support by the COSLAB programme of the ESF.

APPENDIX A: TOY MODEL

To give an illustration of the influence of the choice of coordinates describing the classical background, let us consider a very simple quantum system, the one-dimensional harmonic oscillator with the frequency $\Omega$. The equation of motion for the position operator $\hat{X}$ reads

$$\frac{\partial^2}{\partial t^2} \hat{X} + \Omega^2 \hat{X} = 0. \quad (A1)$$

Of course (as is well known from the theory of coherent states), splitting up this position operator $\hat{X}$ into a classical (background) part $X_b = \langle \hat{X} \rangle$ and a quantum fluctuation part $\delta \hat{X}$ with $\langle \delta \hat{X} \rangle = 0$ via $\hat{X} = X_b + \delta \hat{X}$ yields the same equation of motion for both parts separately, i.e., there is no backreaction at all.

However, if we artificially introduce another variable $Y$ via $X = Y^2/2$, its (classical) equation of motion reads

$$\frac{\partial^2}{\partial Y^2} Y + \frac{1}{Y} \left( \frac{\partial}{\partial Y} Y \right)^2 + \frac{1}{2} \Omega^2 Y = 0. \quad (A2)$$

Now the same procedure $\hat{Y} = Y_b + \delta \hat{Y}$ would yield a non-vanishing quantum backreaction term. Therefore, one must be careful when comparing different expressions for the quantum backreaction, since the explicit form may depend on the choice of splitting the system into a classical background and quantum fluctuations (e.g., $X = X_b + \delta \hat{X}$ versus $Y = Y_b + \delta \hat{Y}$). This illustrates the importance of working with measurable quantities such as $\hat{X}$ or $j$ and $\rho$. Exploiting the simple example a bit further, an an-harmonic oscillator $\frac{\partial^2}{\partial t^2} \hat{X} + \Omega^2 \hat{X} = g\hat{X}^{\hat{X}}$ yields

$$\frac{\partial^2}{\partial \hat{X}} \langle \hat{X} \rangle + \Omega^2 \langle \hat{X} \rangle - g\langle \hat{X} \rangle^2 = g(\delta \hat{X}^2), \quad (A3)$$

i.e., a real quantum backreaction term.
[1] W. G. Unruh, Phys. Rev. Lett. 46, 1351 (1981); M. Visser, Class. Quantum Grav. 15, 1767 (1998).
[2] C. Barceló, S. Liberati, and M. Visser, Class. Quantum Grav. 18, 3595 (2001); ibid. 18, 1137 (2001); L. J. Garay et al., Phys. Rev. Lett. 85, 4643 (2000).
[3] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge, England, 1982).
[4] M. Stone, Phys. Rev. E 62, 1341 (2000); see also pp. 335 in M. Novello, M. Visser, and G. Volovik (editors), Artificial Black Holes (World Scientific, Singapore, 2002).
[5] R. Balbinot, S. Fagnocchi, and A. Fabbri, Phys. Rev. D 71, 064019 (2005); R. Balbinot, S. Fagnocchi, A. Fabbri, and G. P. Procopio, Phys. Rev. Lett. 94, 161302 (2005).
[6] M. Girardeau and R. Arnowitt, Phys. Rev. 113, 755 (1959); C. W. Gardiner, Phys. Rev. A 56, 1414 (1997); M. D. Girardeau, ibid. 58, 775 (1998).
[7] Y. Castin and R. Dum, Phys. Rev. A 57, 3008 (1998).
[8] E. H. Lieb, R. Seiringer, and J. Yngvason, Phys. Rev. A 61, 043602 (2000).
[9] The total particle number operator $\hat{N} = \hat{A}^{\dagger}\hat{A}$ and the corresponding creation and annihilation operators satisfy $[\hat{A}, \hat{A}^{\dagger}] = 1$ and $[\hat{\chi}, \hat{A}\hat{N}^{-1/2}] = [\hat{\zeta}, \hat{A}\hat{N}^{-1/2}] = 0$, i.e., the excitations $\chi$ and $\zeta$ are particle-number-conserving, cf. [11]. The mean-field ansatz in Eq. (5) can be motivated by starting with $N$ free particles, $g = 0$, in the same single-particle state $\psi_c$ with $\zeta = 0$ and subsequently switching on the coupling $g > 0$ by following the evolution in Eqs. (6)-(8) such that the corrections $\chi = O(1/\sqrt{N})$ remain small [10].
[10] R. Schützhold, M. Uhlmann, Y. Xu, and U.R. Fischer, preprint cond-mat/0505074.
[11] E. P. Gross, Nuovo Cimento 20, 454 (1961); J. Math. Phys. 4, 195 (1963); L. P. Pitaevskii, Sov. Phys. JETP 13, 451 (1961); F. Dalfovo et al., Rev. Mod. Phys. 71, 463 (1999); A. J. Leggett, ibid. 73, 307 (2001).
[12] N. N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947); P.G. de Gennes, Superconductivity of Metals and Alloys (W.A. Benjamin, New York, 1966).
[13] A. Griffin, Phys. Rev. B 53, 9341 (1996); E. Zaremba, A. Griffin, and T. Nikuni, Phys. Rev. A 57, 4695 (1998).
[14] We note that the problem that the commutator between density and phase operators, $[\hat{\rho}(r), \hat{\Phi}(r')] = i\delta(r - r')$, leads to fundamental inconsistencies if one takes matrix elements of the space integral of both sides in the number basis was first pointed out in the context of superfluid hydrodynamics by H. Fröhlich, Physica (Amsterdam) 34, 47 (1967). The commutator makes proper sense only if coarse-grained over a volume with large enough number of particles, and not locally, i.e., in arbitrarily small volumes, see also Y. Castin, J. Phys. IV France 116, 89 (2004).
[15] T. D. Lee and C. N. Yang, Phys. Rev. 105, 1119 (1957); T. D. Lee, K. Huang, and C. N. Yang, ibid. 106, 1135 (1957); E. Timmermans, P. Tommasini, and K. Huang, Phys. Rev. A 55, 3645 (1997).
[16] M. Olshanii, Phys. Rev. Lett. 81, 938 (1998).
[17] A. Görlitz et al., Phys. Rev. Lett. 87, 130402 (2001).
[18] L. Pitaevskii and S. Stringari, Phys. Rev. B 47, 10915 (1993).