Abstract
We give the solution to the minimum-energy control problem for linear stochastic systems. The problem is as follows: given an exactly controllable system, find the control process with the minimum expected energy that transfers the system from a given initial state to a desired final state. The solution is found in terms of a certain forward-backward stochastic differential equation of Hamiltonian type.

Keywords: Exact controllability, Minimum-energy control, Hamiltonian system.

1. Introduction

The notion of controllability was introduced by Kalman [6] and it characterizes the ability of controls to transfer a system from a given initial state to a desired final state. When the system is completely controllable, there are many controls that can achieve such a transfer of the system state. This naturally leads to the problem of choosing the “best” control that performs this task. Another contribution of Kalman [6] was the solution of this problem for linear deterministic systems and using the quadratic cost as an optimality criterion. These kinds of problems are know as the minimum-energy control problems (see also [7], [8], [3], [10]).

In this paper we formulate and solve the minimum-energy control problem for linear stochastic systems driven by a Brownian motion. The notion of controllability that we adopt is that of exact controllability, as introduced by Peng [13] (see also [11], [12], [5], [14]). This notion of controllability is a

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faithful extension of Kalman’s notion of complete controllability to stochastic systems. The difference between these two definitions is that in the case of exact controllability the terminal state can be a random variable rather than a fixed number. This makes the stochastic minimum-energy control problem considerably harder than in deterministic setting.

The precise formulation of the stochastic minimum-energy control problem is given in the next section. This is followed by the proof of solvability for a Hamiltonian system and its relation with exact controllability. Section 4 contains the solution to the stochastic minimum energy control problem. As an extension of this result, we give the solution to the stochastic linear-quadratic (LQ) regulator problem with a fixed final state in the final section.

2. Problem formulation

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})\) be a given complete filtered probability space on which the scalar standard Brownian motion \((W(t), t \geq 0)\) is defined. We assume that \(\mathcal{F}_t\) is the augmentation of \(\sigma\{W(s) : 0 \leq s \leq t\}\) by all the \(\mathbb{P}\)-null sets of \(\mathcal{F}\). If \(\xi: \Omega \to \mathbb{R}^n\) is an \(\mathcal{F}_T\)-measurable random variable such that \(\mathbb{E}[|\xi|^2] < \infty\), we write \(\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)\). If \(f: [0, T] \times \Omega \to \mathbb{R}^n\) is an \(\{\mathcal{F}_t\}_{t \geq 0}\) adapted process and if \(\mathbb{E}\int_0^T |f(t)|^2 dt < \infty\), we write \(f(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)\); if \(f(\cdot)\) has a.s. continuous sample paths and \(\mathbb{E}\sup_{t \in [0, T]} |f(t)|^2 < \infty\), we write \(f(\cdot) \in L^\infty_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))\); if \(f(\cdot)\) is uniformly bounded (i.e. \(\text{esssup}_{t \in [0, T]} |f(t)| < \infty\)), we write \(f(\cdot) \in L^\infty(0, T; \mathbb{R}^n)\).

Consider the linear stochastic control system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t) \\
x(0) &= x_0 \in \mathbb{R}^n, \quad \text{is given.}
\end{align*}
\]

(2.1)

We assume that \(A(\cdot), C(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n})\), and \(B(\cdot), D(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times m})\). If the control process \(u(\cdot)\) belongs to \(L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)\), then (2.1) has a unique strong solution \(x(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))\) (see, e.g. Theorem 1.6.14 of [19]).

For a given \(\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)\), we are interested in the following subset of control processes:

\[
U_\xi \equiv \{u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) : x(T) = \xi \quad a.s.\}.
\]

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Minimum-energy control problem. Let \( R(\cdot) \in L^\infty(0,T;\mathbb{R}^{m \times m}) \) be a given symmetric matrix such that \( R(t) > 0 \), a.e. \( t \in [0,T] \). For any given \( x_0 \in \mathbb{R}^n \) and \( \xi \in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^n) \) find the control process \( u(\cdot) \in \mathcal{U}_\xi \) that minimizes the cost functional

\[
J(u(\cdot)) = \mathbb{E} \int_0^T u'(t)R(t)u(t)dt.
\] (2.2)

This is clearly the stochastic version of the Kalman’s minimum energy control problem. A related problem was considered by Klamka [9]. However, in [9] only the linear stochastic systems with additive noise are considered, whereas (2.1) has a multiplicative noise. Our approach to solving the stochastic minimum-energy control problem is different from that of [9], where an operator-theoretic method was used, whereas here we base our approach on a forward-backward stochastic differential equation of a Hamiltonian type.

In order to ensure that the set \( \mathcal{U}_\xi \) is not empty, we make some assumptions on the controllability of (2.1). Out of the many possible notions of controllability for stochastic systems, we employ the notion of exact controllability as introduced by Peng [13].

**Definition 1.** System (2.1) is called exactly controllable at time \( T > 0 \) if for any \( x_0 \in \mathbb{R}^n \) and \( \xi \in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^n) \), there exists at least one control \( u(\cdot) \in L^2_T(0,T;\mathbb{R}^m) \), such that the corresponding trajectory \( x(\cdot) \) satisfies the initial condition \( x(0) = x_0 \) and the terminal condition \( x(T) = \xi \), a.s..

We solve the minimum-energy control problem under the following two assumptions.

(A1) The system (2.1) is exactly controllable at time \( T > 0 \).

(A2) There exists an invertible matrix \( M(\cdot) \in L^\infty(0,T;\mathbb{R}^{m \times m}) \) such that \( D(t)M(t) = [I,0] \).

Assumption A1 ensures that the set \( \mathcal{U}_\xi \) is not empty. Assumption A2 implies that \( m \geq n \), i.e. the number of control inputs to the system is at least as large as the number of the states of the system. This may appear as a strong assumption when compared with the minimum-energy control problem of deterministic systems. However, at least when the matrix \( D(\cdot) \) has continuous
coefficients, this assumption is implied by assumption A1. Indeed, by Proposition 2.1. of [13], a necessary condition for exact controllability at time $T$ of the system (2.1) is that rank $D(t) = n$, $\forall t \in [0, T]$. Then from the Doležal’s theorem [4], it follows that there exists the matrix $M(\cdot)$ in assumption A2.

We now reformulate the minimum-energy control problem in a more convenient form. Let the processes $z(\cdot) \in L^2_T((0, T]; \mathbb{R}^n)$ and $v(\cdot) \in L^2_T((0, T]; \mathbb{R}^{m-n})$ be such that

$$u(t) = M(t) \begin{bmatrix} z(t) \\ v(t) \end{bmatrix}. \quad (2.3)$$

Let the matrices $G(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n})$, $F(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times (m-n)})$, $H_1(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n})$, $H_2(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times (m-n)})$, $H_3(\cdot) \in L^\infty(0, T; \mathbb{R}^{(m-n)^2})$, be such that

$$B(t)M(t) = \begin{bmatrix} G(t) & F(t) \end{bmatrix}, \quad M'(t)R(t)M(t) = \begin{bmatrix} H_1(t) & H_2(t) \\ H_2'(t) & H_3(t) \end{bmatrix}. \quad (2.4)$$

Due to the symmetric nature of the matrix $R(\cdot)$, the matrices $H_1(\cdot)$ and $H_3(\cdot)$ are also symmetric. Moreover, due to the positive definiteness of $R(\cdot)$ and the Schur’s lemma, it holds that

$$H_3(t) > 0, \quad a.e. \quad t \in [0, T],$$

$$H_1(t) - H_2'(t)H_3^{-1}(t)H_2(t) > 0, \quad a.e. \quad t \in [0, T].$$

Equation (2.1) and the cost functional (2.2) can now be written as

$$\begin{cases} dx(t) = [A(t)x(t) + F(t)v(t) + G(t)z(t)]dt + [C(t)x(t) + z(t)]dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given}, \end{cases} \quad (2.5)$$

$$J(v(\cdot), z(\cdot)) = \mathbb{E}\int_0^T [z'(t)H_1(t)z(t) + 2v'(t)H_2(t)z(t) + v'(t)H_3(t)v(t)]dt. \quad (2.6)$$

To each element of the set $\mathcal{U}_\xi$ it corresponds a pair of processes $(v(\cdot), z(\cdot))$ from the set

$$\mathcal{A}_\xi \equiv \{ v(\cdot) \in L^2_T((0, T]; \mathbb{R}^{m-n}), z(\cdot) \in L^2_T((0, T]; \mathbb{R}^n) : x(T) = \xi \quad a.s. \}. \quad$$
In this reformulation, the minimum-energy control problem is:

\[
\begin{align*}
\min_{(v(\cdot), z(\cdot)) \in \mathcal{A}} & \quad J(v(\cdot), z(\cdot)), \\
\text{s.t.} & \quad (2.5).
\end{align*}
\]  

Before we proceed to its solution, let us state a useful necessary and sufficient condition for the exact controllability of (2.5). It is a slight modification of the result in [11], and we thus omit the proof.

**Proposition 1.** Let \( E(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times m}) \) be any symmetric matrix such that \( E(t) > 0 \), a.e. \( t \in [0, T] \). Also let \( \Phi(\cdot) \) be the unique solution to the equation

\[
\begin{align*}
  d\Phi(t) & = -\Phi(t)[A(t) - G(t)C(t)] dt - \Phi(t)G(t)dW(t), \\
  \Phi(0) & = I.
\end{align*}
\]

The system (2.5) is exactly controllable at time \( T \) if and only if

\[
\text{rank} \left[ \mathbb{E} \int_0^T \Phi(t)F(t)E(t)F'(t)\Phi'(t)dt \right] = n. \tag{2.8}
\]

3. Hamiltonian system of equations

The following forward-backward stochastic differential equation of Hamiltonian type appears naturally in the next section:

\[
\begin{align*}
  dX(t) & = \{A(t)X(t) + F(t)H_3^{-1}(t)F'(t)Y(t) + [G(t) - F(t)H_3^{-1}(t)H'_3(t)]Z(t)\} dt \\
  & \quad + [C(t)X(t) + Z(t)]dW(t), \\
  dY(t) & = -[A'(t) - C'(t)G'(t) + C'(t)H_2(t)H_3^{-1}(t)F'(t)]Y(t)dt \\
  & \quad -C'(t)[H_1(t) - H_2(t)H_3^{-1}(t)H'_2(t)]Z(t)dt \\
  & \quad + \{-G'(t) + H_2(t)H_3^{-1}(t)F'(t)\}Y(t) + [H_1(t) - H_2(t)H_3^{-1}(t)H'_2(t)]Z(t)\}dW(t), \\
  X(0) & = x_0, \quad X(T) = \xi, \quad Y(0) = K.
\end{align*}
\]
where we can choose the vector $K \in \mathbb{R}^n$. To simplify the notation, we introduce the matrices:

\[
\begin{align*}
\bar{A}(t) & \equiv A(t) - G(t)C(t) + F(t)H_3^{-1}(t)H_2'(t)C(t), \\
\bar{B}(t) & \equiv G(t) - F(t)H_3^{-1}(t)H_2'(t), \\
\bar{H}(t) & \equiv H_1(t) - H_2(t)H_3^{-1}(t)H_2'(t).
\end{align*}
\]

By defining $\bar{X}(t) \equiv -X(t)$ and $\bar{Z}(t) \equiv -[C(t)X(t) + Z(t)]$, we can rewrite (3.1) as

\[
\begin{cases}
\bar{dX}(t) = [\bar{A}(t)\bar{X}(t) - F(t)H_3^{-1}(t)F'(t)Y(t) + \bar{B}(t)\bar{Z}(t)]dt + \bar{Z}(t)dW(t), \\
\bar{dY}(t) = [\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)\bar{X}(t) + C'(t)\bar{H}(t)\bar{Z}(t)]dt \\
\quad + [\bar{B}'(t)Y(t) + \bar{H}(t)C(t)\bar{X}(t) - \bar{H}(t)\bar{Z}(t)]dW(t), \\
\bar{X}(0) = -x_0, \quad \bar{X}(T) = -\xi, \quad Y(0) = K.
\end{cases}
\]

This forward-backward stochastic differential equation is similar to the Hamiltonian system of stochastic LQ control problem [10]. Two main differences are that here the initial value of $\bar{X}(\cdot)$ is fixed and the vector $K$ can be chosen. We thus seek the solution quadruple $(\bar{X}(t), Y(t), \bar{Z}(t), K)$, rather than the solution triple $(\bar{X}(t), Y(t), \bar{Z}(t))$, as is usually the case with Hamiltonian systems.

**Theorem 1.** There exists a unique solution quadruple $(\bar{X}(\cdot), Y(\cdot), \bar{Z}(\cdot), K) \in L^2_T(\bar{X}(0; \mathbb{R}^n) \times L^2_T(0, T; \mathbb{R}^n) \times L^2_T(0, T; \mathbb{R}^n) \times \mathbb{R}^n)$ to (3.2) for any $x_0 \in \mathbb{R}^n$ and any $\xi \in L^2_T(\Omega, \mathbb{P}, \mathcal{F}; \mathbb{R}^n)$, if and only if the system (2.3) is exactly controllable at time $T$. In this case, $X(\cdot)$ and $Z(\cdot)$ in terms of $Y(\cdot)$, and the explicit formula for $K$, are given by

\[
\begin{align*}
\bar{X}(t) &= \bar{P}(t)Y(t) + p(t), \\
\bar{Z}(t) &= [I + \bar{P}(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)\bar{P}(t) - \bar{P}(t)\bar{B}'(t)]Y(t) \\
\quad &\quad + [I + \bar{P}(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)p(t) + q(t)], \\
K &= \bar{P}^{-1}(0) \{x_0 - \mathbb{E}[\mathcal{P}(T)\xi]\}.
\end{align*}
\]
Here $\bar{P}(\cdot)$ is the unique solution of the Riccati equation

$$
\begin{cases}
\dot{\bar{P}}(t) - A(t)\bar{P}(t) - \bar{P}(t)A'(t) + F(t)H_3^{-1}(t)F'(t) + \bar{B}(t)\bar{H}^{-1}(t)\bar{B}'(t) \\
- [\bar{P}(t)C'(t) - \bar{B}\bar{H}^{-1}] [\bar{H}^{-1}(t) + \bar{P}(t)]^{-1} [\bar{P}(t)C'(t) - \bar{B}\bar{H}^{-1}]' = 0,
\end{cases}
$$

(3.6)

$\bar{P}(T) = 0$,

whereas $(p(\cdot), q(\cdot))$ are the unique solution pair of the following linear backward stochastic differential equation

$$
\begin{cases}
dp(t) = [B_1(t)p(t) + B_2(t)q(t)]dt + q(t)dW(t), \\
p(T) = \xi, \\
B_1(t) \equiv \bar{A}(t) + \bar{P}(t)C'(t)\bar{H}(t)C(t) \\
+ [\bar{B}(t) - \bar{P}(t)C'(t)\bar{H}(t)][I + \bar{P}(t)\bar{H}(t)]^{-1} \bar{P}(t)\bar{H}(t)C(t), \\
B_2(t) \equiv [\bar{B}(t) - \bar{P}(t)C'(t)\bar{H}(t)][I + \bar{P}(t)\bar{H}(t)]^{-1}.
\end{cases}
$$

(3.7)

Finally, the process $\mathcal{P}(\cdot)$ is the unique solution to the stochastic differential equation

$$
\begin{cases}
d\mathcal{P}(t) = -\mathcal{P}(t)B_1(t)dt - \mathcal{P}(t)B_2(t)dW(t), \\
\mathcal{P}(0) = I.
\end{cases}
$$

(3.8)

**Proof.** (Positivity of $\bar{P}(0)$) The Riccati differential equation (3.6) has a unique solution $\bar{P}(t) \geq 0, \forall t \in [0, T]$ (see, e.g., Theorem 3.1 of [1]). We show that $\bar{P}(0) > 0$ if and only if (2.5) is exactly controllable at time $T$. Thus consider the stochastic control system

$$
\begin{cases}
d\bar{x}(t) = [-A'(t)\bar{x}(t) + C'(t)\bar{u}(t)]dt - \bar{u}(t)dW(t), \\
\bar{x}(0) = \ddot{x}_0 \neq 0,
\end{cases}
$$

(3.9)

and the associated cost functional

$$
\bar{J}(\bar{u}(\cdot)) = \mathbb{E} \int_0^T \bar{x}'(t)[F(t)H_3^{-1}(t)F'(t) + B(t)\bar{H}^{-1}(t)\bar{B}'(t)]\bar{x}(t)dt
$$

$$
+ \mathbb{E} \int_0^T [-2\bar{x}'(t)B(t)\bar{H}^{-1}(t)\bar{u}(t) + \bar{u}'(t)\bar{H}^{-1}(t)\bar{u}(t)]dt.
$$

(3.10)
From Theorem 2.2 of [1], it follows that $\tilde{x}_0 P(0) \tilde{x}_0 = \min_{\tilde{u}(\cdot)} \tilde{J}(\tilde{u}(\cdot))$. Introducing the new control $\tilde{v}(t) \equiv \tilde{u}(t) - \bar{B}' \tilde{x}(t)$ transforms (3.9) and (3.10) into

\[
\begin{align*}
\left\{ \begin{array}{l}
    d\tilde{x}(t) = \left[ -A'(t) + C'(t)\bar{B}'(t) \right] \tilde{x}(t) dt + C'(t)\tilde{v}(t) dt - [\bar{B}' \tilde{x}(t) + \tilde{v}(t)] dW(t),
    \\
    \tilde{x}(0) = \tilde{x}_0 \neq 0,
\end{array} \right.
\end{align*}
\]

(3.11)

\[
\bar{J}(\tilde{v}(\cdot)) = \mathbb{E} \int_0^T [\tilde{x}'(t) F(t) H_3^{-1}(t) F'(t) \tilde{x}(t) + \tilde{v}'(t) \tilde{H}^{-1}(t) v(t)] dt. 
\]

(3.12)

To prove the sufficiency, let us assume the opposite, i.e. that $\tilde{x}_0 P(0) \tilde{x}_0 = 0$ and the system (2.5) is exactly controllable at time $T$. From (3.12), and the fact that $\tilde{H}^{-1}(t) > 0$, a.e. $t \in [0, T]$, we conclude that in order to minimize (3.12) it is necessary to have $\tilde{v}(t) = 0$, a.e. $t \in [0, T]$ a.s.. For such a $\tilde{v}(\cdot)$, the solution to (3.11) becomes $\tilde{x}(t) = \tilde{\Phi}(t)x_0$, with $\tilde{\Phi}(t)$ being the solution to the stochastic differential equation

\[
\begin{align*}
\left\{ \begin{array}{l}
    d\tilde{\Phi}(t) = \left[ -A'(t) + C'(t)\bar{B}'(t) \right] \tilde{\Phi}(t) dt - \bar{B}' \tilde{\Phi}(t) dW(t),
    \\
    \tilde{\Phi}(0) = I,
\end{array} \right.
\end{align*}
\]

(3.13)

The cost (3.12) now becomes

\[
0 = \tilde{x}_0 \left[ \mathbb{E} \int_0^T \tilde{\Phi}'(t) F(t) H_3^{-1}(t) F'(t) \tilde{\Phi}(t) dt \right] \tilde{x}_0, 
\]

(3.14)

which means condition (2.8) does not hold. Since (2.8) is necessary for the exact controllability of (2.5), we have a contradiction.

To prove the necessity, let us assume that $P(0) > 0$ and the system (2.5) is not exactly controllable at time $T$. Taking $\tilde{v}(t) = 0$ a.e. $t \in [0, T]$ a.s., the cost (3.12) becomes the right-hand side of (3.14), and thus there exists $\tilde{x}_0 \neq 0$ such that equation (3.14) holds. This means that $\min_{\tilde{u}(\cdot)} \tilde{J}(\tilde{u}(\cdot)) = 0$, which contradicts the fact that is should be $\tilde{x}_0 P(0) \tilde{x}_0 > 0$.

(Solvability of (3.2)) Here we follow Yong [17], [18], in seeking the relation

\[
\bar{X}(t) = \bar{P}(t)Y(t) + p(t), 
\]

(3.15)
which implies that the differential of $\bar{X}$ should be

$$d\bar{X}(t) = \dot{P}(t)Y(t) + \bar{P}(t)\dot{Y}(t) + dp(t)$$

$$= \dot{P}(t)Y(t)dt + \bar{P}(t)[-\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)\bar{X}(t) + C'(t)\bar{H}(t)\bar{Z}(t)]dt$$

$$+ [\mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t)]dt$$

$$+ \bar{P}(t)[-\bar{B}'(t)Y(t) + \bar{H}(t)C(t)\bar{X}(t) - \bar{H}(t)\bar{Z}(t)]dW(t) + q(t)dW(t)$$

By comparing this differential with $d\bar{X}(t)$ in (3.2), we conclude that for a.e. $t \in [0, T]$ a.s., we must have

$$\begin{aligned}
\bar{X}(t) - F(t)H_3^{-1}(t)F'(t)Y(t) + \bar{B}(t)\bar{Z}(t) &= \mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t) \\
\bar{P}(t)Y(t) + \bar{P}(t)[-\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)\bar{X}(t) + C'(t)\bar{H}(t)\bar{Z}(t)]
\end{aligned}$$

(3.16)

Since the matrix $[I + \bar{P}(t)\bar{H}(t)]$ is invertible, from (3.17) we obtain (3.4). Substituting such a $\bar{Z}(t)$ into (3.16) gives

$$\begin{aligned}
\bar{A}(t)[\bar{P}(t)Y(t) + p(t)] - F(t)H_3^{-1}(t)F'(t)Y(t)
\end{aligned}$$

$$+ \bar{B}(t)[I + \bar{P}(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)\bar{P}(t) - \bar{P}(t)\bar{B}'(t)]Y(t)$$

$$+ \bar{B}(t)[I + \bar{P}(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)p(t) + q(t)]$$

$$= \mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t)$$

(3.18)

which holds due to our assumptions on $\bar{P}(t)$ and $p(t)$. Substituting (3.15) and (3.4) into the equation for $\dot{Y}(t)$ in (3.2), shows that it is a linear stochastic
differential equation with a unique solution for any \( K \in \mathbb{R}^n \). This proves the existence of a unique solution triple \((\bar{X}(\cdot), Y(\cdot), \bar{Z}(\cdot))\) of (3.2) for any \( \xi \). In order to ensure that \( \bar{X}(0) = x_0 \) for any \( x_0 \in \mathbb{R}^n \), it is necessary and sufficient to have \( x_0 = \bar{P}(0)K + p(0) \). This equation has a unique solution for any \( \xi \) and \( x_0 \) if and only if \( \bar{P}(0) \) is invertible, which we proved is equivalent with the exact controllability of (2.5). □

4. Minimum-energy control

In this section we give the solution to the minimum-energy control problem (2.7). Let us first prove two useful lemmas.

Lemma 1. Let \((v_1(\cdot), z_1(\cdot)) \in A_{\xi}\) and \((v_2(\cdot), z_2(\cdot)) \in A_{\xi}\) be any two pairs of admissible controls. Then

\[
E \int_0^T \Phi(t) F(t) [v_1(t) - v_2(t)] dt = 0. \tag{4.1}
\]

Proof. By Itô’s product rule, for any admissible pair \((v(\cdot), z(\cdot)) \in A_{\xi}\), we obtain

\[
d[\Phi(t)x(t)] = [d\Phi(t)]x(t) + \Phi(t)dx(t) - \Phi(t)G(t)[C(t)x(t) + z(t)] dt
\]

\[
= -\Phi(t)[A(t) - G(t)C(t)] x(t) dt - \Phi(t)G(t)x(t) dW(t)
\]

\[
+ \Phi(t)[A(t)x(t) + F(t)v(t) + G(t)z(t)] dt
\]

\[
+ \Phi(t)[C(t)x(t) + z(t)] dW(t) - \Phi(t)G(t)[C(t)x(t) + z(t)] dt
\]

\[
= \Phi(t) F(t) v(t) dt + \Phi(t) \{ z(t) + [C(t) - G(t)] x(t) \} dW(t). \tag{4.2}
\]

Denoting by \( x^{(1)}(\cdot) \) and \( x^{(2)}(\cdot) \) the solutions to (2.5) corresponding to \((v_1(\cdot), z_1(\cdot))\) and \((v_2(\cdot), z_2(\cdot))\), respectively, we obtain

\[
\Phi(T)\xi - x_0 = \int_0^T \Phi(t) F(t) v_1(t) dt + \int_0^T \Phi(t) \{ z_1(t) + [C(t) - G(t)] x^{(1)}(t) \} dW(t),
\]

\[
\Phi(T)\xi - x_0 = \int_0^T \Phi(t) F(t) v_2(t) dt + \int_0^T \Phi(t) \{ z_2(t) + [C(t) - G(t)] x^{(2)}(t) \} dW(t).
\]
The difference of the above two equations is
\[
0 = \int_0^T \Phi(t)F(t)[v_1(t) - v_2(t)]dt \\
+ \int_0^T \Phi(t)\{z_1(t) - z_2(t) + [C(t) - G(t)][x^{(1)}(t) - x^{(2)}(t)]\}dW(t).
\]
The conclusion follows by taking the expectation of both sides. □

Consider an \(\mathbb{R}^n\)-valued stochastic process \(\Gamma(\cdot)\) defined as the solution to the stochastic differential equation
\[
\begin{aligned}
d\Gamma(t) &= \Gamma_1(t)dt + \Gamma_2(t)dW(t) \\
\Gamma(0) &= 0,
\end{aligned}
\]
where \(\Gamma_2(\cdot)\) is any process in \(L_2^2(0, T; \mathbb{R}^n)\), and
\[
\Gamma_1(t) \equiv -[\Phi^{-1}(t)][C(t) - G(t)]'\Phi'(t)\Gamma_2(t).
\]

**Lemma 2.** Let \((v_1(\cdot), z_1(\cdot)) \in \mathcal{A}_\xi\) and \((v_2(\cdot), z_2(\cdot)) \in \mathcal{A}_\xi\) be any two pairs of admissible controls. Then
\[
\mathbb{E} \int_0^T \Gamma_2'(t)\Phi(t)[z_2(t) - z_1(t)]dt = -\mathbb{E} \int_0^T \Gamma'(t)\Phi(t)F(t)[v_2(t) - v_1(t)]dt.
\]  

**Proof.** For any pair of admissible controls \((v(\cdot), z(\cdot)) \in \mathcal{A}_\xi\), by Itô’s product rule and \((4.2)\), we obtain
\[
d[\Gamma'(t)\Phi(t)x(t)] = [d\Gamma'(t)\Phi(t)x(t) + \Gamma'(t)d[\Phi(t)x(t)] + \Gamma_2'(t)\Phi(t)\{z(t) + [C(t) - G(t)]x(t)\}]dt
\]
\[
= \Gamma_1'(t)\Phi(t)x(t)dt + \Gamma_2'(t)\Phi(t)x(t)dW(t)
\]
\[
+ \Gamma'(t)\Phi(t)F(t)v(t)dt + \Gamma'(t)\Phi(t)\{z(t) + [C(t) - G(t)]x(t)\}dW(t)
\]
\[
+ \Gamma_2'(t)\Phi(t)\{z(t) + [C(t) - G(t)]x(t)\}dt
\]
\[
= [\Gamma'(t)\Phi(t)F(t)v(t) + \Gamma_2'(t)\Phi(t)z(t)]dt
\]
\[
+ \Gamma_2'(t)\Phi(t)x(t)dW(t) + \Gamma'(t)\Phi(t)\{z(t) + [C(t) - G(t)]x(t)\}dW(t).
\]
The rest of the proof proceeds as in the previous lemma.

\[ \square \]

**Theorem 2.** *(Minimum-energy control)* There exists a unique solution to the problem \((2.7)\) given by

\[ v^*(t) = H_3^{-1}(t)[F'(t)Y(t) - H'_2Z(t)], \quad (4.5) \]

\[ z^*(t) = Z(t). \quad (4.6) \]

**Proof.** We first show that \((v^*(\cdot), z^*(\cdot)) \in A_\xi\). By choosing the process \(\Gamma_2(\cdot)\) in \((4.3)\) as

\[ \Gamma_2(t) = [\Phi'(t)]^{-1}[H_1(t)z^*(t) + H_2(t)v^*(t)], \]

the process \(\Gamma_1(\cdot)\) and the equation for \(\Gamma(\cdot)\) become

\[ \Gamma_1(t) = -[\Phi'(t)]^{-1}[C(t) - G(t)][H_1(t)z^*(t) + H_2(t)v^*(t)], \]

\[
\begin{align*}
    d\Gamma(t) &= -[\Phi'(t)]^{-1}[C(t) - G(t)][H_1(t)z^*(t) + H_2(t)v^*(t)]dt \\
    &\quad + [\Phi'(t)]^{-1}[H_1(t)z^*(t) + H_2(t)v^*(t)]dW(t) \\
    \Gamma(0) &= 0,
\end{align*}
\]

Let \(\bar{Y}(\cdot) \equiv \Phi'(\cdot)[\Gamma(\cdot) + K]\). We show that \(\bar{Y}(\cdot)\) is in fact the process \(Y(\cdot)\) of the previous section. Note that \(\bar{Y}(0) = \Phi'(0)[\Gamma(0) + K] = K = Y(0)\). The differential of \(\bar{Y}(\cdot)\) is:

\[ d\bar{Y} = [d\Phi'(t)][\Gamma(t) + K] + \Phi'(t)d\Gamma(t) - G'[H_1z^*(t) + H_2v^*(t)]dt \\
    = -[A(t) - G(t)C(t)]\Phi'(t)[\Gamma(t) + K]dt - G'(t)\Phi'(t)[\Gamma(t) + K]dW(t) \\
    - [C'(t) - G'(t)][H_1(t)z^*(t) + H_2(t)v^*(t)]dt + [H_1(t)z^*(t) + H_2(t)v^*(t)]dW(t) \\
    - G'(t)[H_1(t)z^*(t) + H_2(t)v^*(t)]dt. \]
After substituting the expressions for $v^*(\cdot)$ and $z^*(\cdot)$, this equation becomes
\[
dY(t) = \{-[A'(t) - C'(t)G'(t)]Y(t) - C'(t)H_2(t)H_1^{-1}(t)F'(t)Y(t)\}dt
\]
\[+ \quad C'(t)[H_1(t) - H_2(t)H_1^{-1}H_2'(t)]Z(t)dt
\]
\[+ \quad \{-G'(t)\dot{Y}(t) + H_2(t)H_1^{-1}(t)F'(t)Y(t) + [H_1(t) - H_2(t)H_1^{-1}(t)H_2'(t)]Z(t)\}dW(t),
\]
which is satisfied by the process $Y(\cdot)$. Therefore, $\dot{Y}(t) = Y(t)$. Substituting (4.5) and (4.6) in (2.5) makes it clear that the equations for $x(t)$ and $X(t)$ are the same, and in particular $x(T) = \xi$ a.s. Hence, $(v^*(\cdot), z^*(\cdot)) \in \mathcal{A}_\xi$.

We now focus in proving that $(v^*(\cdot), z^*(\cdot))$ are the unique optimal controls. By using equation (4.4), for any control pair $(v(\cdot), z(\cdot)) \in \mathcal{A}_\xi$, we obtain:
\[
E \int_0^T \{[H_1(t)z^*(t) + H_2(t)v^*(t)]'[z(t) - z^*(t)] + [H_1'(t)z^*(t) + H_2(t)v^*(t)]'[v(t) - v^*(t)]\}dt
\]
\[= E \int_0^T \{-\Gamma'(t)\Phi(t)F(t)[v(t) - v^*(t)] + [H_2'(t)z^*(t) + H_3(t)v^*(t)]'[v(t) - v^*(t)]\}dt
\]
\[= E \int_0^T [H_2'(t)z^*(t) + H_3(t)v^*(t) - F'(t)\Phi'(t)\Gamma(t)]'[v(t) - v^*(t)]dt
\]
\[= E \int_0^T K'\Phi(t)F(t)[v(t) - v^*(t)]dt = 0,
\]
where the last equality is due to (4.1). For any control pair $(v(\cdot), z(\cdot)) \in \mathcal{A}_\xi$ we have
\[
J(v(\cdot), z(\cdot)) = E \int_0^T \left[z'(t)H_1(t)z(t) + 2v'(t)H_2'(t)z(t) + v'(t)H_3(t)v(t)\right]dt
\]
\[= E \int_0^T [z(t) - z^*(t) + z^*(t)]H_1(t)[z(t) - z^*(t) + z^*(t)]dt
\]
\[+ \quad E \int_0^T [v(t) - v^*(t) + v^*(t)]H_2(t)[z(t) - z^*(t) + z^*(t)]dt
\]
\[+ \quad E \int_0^T [v(t) - v^*(t) + v^*(t)]H_3(t)[v(t) - v^*(t) + v^*(t)]dt
\]
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Due to (4.8), for any v regulator problem with a fixed final state. a given symmetric matrix such that v with equality if and only if a.e. t ∈ \text{stochastic minimum-energy control, we give the solution to the following settings. The version of this regulator with a}

\[ J = E \int_0^T \{ [z(t) - z^*(t)]' H_1(t) [z(t) - z^*(t)] + 2[z(t) - z^*(t)]' H_1(t) z^*(t) + (z^*(t))' H_1(t) z^*(t) \} dt \]

\[ + 2E \int_0^T \{ [v(t) - v^*(t)]' H'_1(t) [z(t) - z^*(t)] + [v(t) - v^*(t)]' H'_2(t) z^*(t) \} dt \]

\[ + 2E \int_0^T \{ (v^*(t))' H'_2(t) [z(t) - z^*(t)] + (v^*(t))' H'_2(t) z^*(t) \} dt \]

\[ + E \int_0^T \{ [v(t) - v^*(t)]' H_3(t) [v(t) - v^*(t)] + 2[v(t)]' H_3(t) [v(t) - v^*(t)] + (v^*(t))' H_3(t) v^*(t) \} dt \]

\[ = J(v^*(\cdot), z^*(\cdot)) + E \int_0^T \left[ \frac{z(t) - z^*(t)}{v(t) - v^*(t)} \right] ' \left[ \begin{array}{cc} H_1(t) & H_2(t) \\ H'_2(t) & H_3(t) \end{array} \right] \left[ \begin{array}{c} z(t) - z^*(t) \\ v(t) - v^*(t) \end{array} \right] dt \]

\[ + 2E \int_0^T \{ [z(t) - z^*(t)] [H_1(t) z^*(t) + H_2(t) v^*(t)] + [v(t) - v^*(t)] [H'_2(t) z^*(t) + H_3(t) v^*(t)] \} dt, \]

Due to (4.8), for any (v(\cdot), z(\cdot)) \in A_\xi we have

\[ J(v(\cdot), z(\cdot)) = J(v^*(\cdot), z^*(\cdot)) + E \int_0^T \left[ \frac{z(t) - z^*(t)}{v(t) - v^*(t)} \right] ' \left[ \begin{array}{cc} H_1(t) & H_2(t) \\ H'_2(t) & H_3(t) \end{array} \right] \left[ \begin{array}{c} z(t) - z^*(t) \\ v(t) - v^*(t) \end{array} \right] dt \]

\[ \geq J(v^*(\cdot), z^*(\cdot)), \]

with equality if and only if \( v(t) = v^*(t), a.e. t \in [0, T] \) a.s., and \( z(t) = z^*(t), a.e. t \in [0, T] \) a.s.. \( \square \)

5. Stochastic LQ regulator with a fixed final state

Ever since its introduction by Kalman [6], the LQ regulator has been studied extensively in both deterministic [2, 3], and stochastic [15, 16, 19], settings. The version of this regulator with a fixed state [10] is the minimum-energy control problem with the cost functional that has a penalty on the state as well as on the control. As an extension of our results on minimum-energy control, we give the solution to the following stochastic LQ regulator problem with a fixed final state.

**LQ regulator with a fixed final state.** Let \( Q(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n}) \) be a given symmetric matrix such that \( Q(t) \geq 0, a.e. t \in [0, T] \). For any given
\( x_0 \in \mathbb{R}^n \) and \( \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n) \) find the control process \( u(\cdot) \in \mathcal{U}_\xi \) that minimizes the cost functional

\[
\tilde{J}(u(\cdot)) = E \int_0^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt. \tag{5.1}
\]

We solve this problem by transforming it into an equivalent minimum-energy control problem of the previous sections. Consider the Riccati differential equation

\[
\begin{aligned}
\dot{P}(t) + P(t)A(t) + A'(t)P(t) + C'(t)P(t)C(t) + Q(t) \\
- [P(t)B(t) + C'(t)P(t)D(t)][D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)] = 0,
\end{aligned}
\]

\[
P(T) = 0,
\]

\[
D'(t)P(t)D(t) + R(t) > 0,
\]

which has a unique solution (see, e.g., [13], [16], [19], [1]). We introduce the following matrices for notational convenience:

\[
\begin{aligned}
\hat{A}(t) &\equiv A(t) - B(t)[D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)], \\
\hat{B}(t) &\equiv B(t), \\
\hat{C}(t) &\equiv C(t) - D(t)[D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)], \\
\hat{D}(t) &\equiv D(t), \\
\hat{R}(t) &\equiv D'(t)P(t)D(t) + R(t).
\end{aligned}
\]

By introducing a new control \( \hat{u}(t) \equiv u(t) + [D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)]x(t) \), we can rewrite the state equation (2.1) as

\[
\begin{aligned}
dx(t) = [\hat{A}(t)x(t) + \hat{B}(t)\hat{u}(t)]dt + [\hat{C}(t)x(t) + \hat{D}(t)\hat{u}(t)]dW(t) \\
x(0) = x_0 \in \mathbb{R}^n,
\end{aligned}\tag{5.2}
\]
Using the completion of squares method of stochastic LQ control \cite{19}, we can rewrite the cost functional \((5.1)\) as

\[
\hat{J}(\hat{u}(\cdot)) = x'_0 P(0)x_0 + \mathbb{E} \int_0^T \hat{u}'(t)\hat{R}(t)\hat{u}(t)dt.
\] (5.3)

Apart from the obvious change of notation, the problem of minimizing \((5.3)\) subject to \((5.2)\) and \(\hat{u}(\cdot) \in A_\xi\), is the stochastic minimum-energy control problem. Hence, its solution can be obtained by applying the results of the previous sections.

6. Conclusions

A stochastic version of the classical minimum-energy control problem is formulated. The system is driven by a Brownian motion and all coefficients can be time-varying. By assuming the exact controllability of the system, complete solution is given. The minimum-energy control problem is crucial is solving several optimal control problems involving terminal state constraints and quadratic criteria. One such problem is the stochastic LQ regulator with a fixed final state, and it has been solved as an application of our results on minimum-energy control. We expect that the method proposed in this paper will prove useful in tackling more general minimum-energy control problems.

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