THE ADDITION THEOREM FOR TWO-STEP NILPOTENT TORSION GROUPS

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ABSTRACT. The Addition Theorem for the algebraic entropy of group endomorphisms of torsion abelian groups was proved in [4]. Later, this result was extended to all abelian groups [8] and, recently, to all torsion finitely quasihamiltonian groups [2]. In contrast, when it comes to metabelian groups, the additivity of the algebraic entropy fails [8]. Continuing the research within the class of locally finite groups, we prove that the Addition Theorem holds for two-step nilpotent torsion groups.

1. Introduction

The algebraic entropy was first considered in [1] for endomorphisms of (discrete) abelian groups. Weiss [10] and Peters [9] connected this entropy with the topological entropy via Bridge Theorems. We refer the reader to [4] for an extensive study of the algebraic entropy for endomorphisms of torsion abelian groups.

Following [2] we now give the general definition for the algebraic entropy of endomorphisms of (not necessarily abelian) groups. Let $G$ be a group and $\phi \in \text{End}(G)$. For a finite subset $X$ of $G$ and $n \in \mathbb{N}_+$, the $n$-th $\phi$-trajectory of $X$ is

$$T_n(\phi, X) = X \cdot \phi(X) \cdot \phi^{n-1}(X).$$

The algebraic entropy of $\phi$ with respect to $X$ is

$$H(\phi, X) = \lim_{n \to \infty} \frac{\ell(T_n(\phi, X))}{n},$$

where $\ell(T_n(\phi, X)) = \log |T_n(\phi, X)|$. The algebraic entropy of $\phi$ is $h(\phi) = \sup \{H(\phi, X) \mid X \in \mathcal{P}_{\text{fin}}(G)\}$, where $\mathcal{P}_{\text{fin}}(G)$ is the family of all finite subsets of $X$. It is easy to see that $h(\phi) = \sup \{H(\phi, X) \mid X \in \mathcal{C}\}$, where $\mathcal{C}$ is any cofinal subfamily of $\mathcal{P}_{\text{fin}}(G)$. In particular, if $G$ is locally finite, then $\mathcal{C}$ can be chosen to be $\mathcal{F}(G)$ the family of all finite subgroups of $G$.

In this paper, we focus on the following property that is known as the Addition Theorem.

Definition 1.1. Let $\mathcal{X}$ be a class of groups closed under taking subgroups and quotients.

1. We say that $\text{AT}(G, \phi, H)$ holds for a group $G \in \mathcal{X}$, $\phi \in \text{End}(G)$ and a $\phi$-invariant normal subgroup $H$ of $G$ if

$$h(\phi) = h(\phi \mid_H) + h(\hat{\phi}),$$

where $\hat{\phi} = \phi \mid_H \in \text{End}(G/H)$ is the map induced by $\phi$.

2. The Addition Theorem holds in $\mathcal{X}$ for endomorphisms if $\text{AT}(G, \phi, H)$ holds for every $G \in \mathcal{X}$, $\phi \in \text{End}(G)$ and every $\phi$-invariant normal subgroup $H$ of $G$.

3. The Addition Theorem holds in $\mathcal{X}$ for automorphisms if $\text{AT}(G, \phi, H)$ holds for every $G \in \mathcal{X}$, $\phi \in \text{Aut}(G)$ and every $\phi$-stable normal subgroup $H$ of $G$.

Dikranjan, Goldsmith, Salce and Zanardo [1] proved that the Addition Theorem holds for endomorphisms of torsion abelian groups. This result was extended to all abelian groups by Dikranjan and Giordano Bruno [3]. In the non-commutative case, the Addition Theorem was proved for automorphisms of torsion groups which are either FC-groups [4] or quasihamiltonian [11] as well as for endomorphisms of torsion quasihamiltonian FC-groups (see [11]). Extending these results, Giordano Bruno and Salizzoni [7] recently proved the additivity of the algebraic entropy for endomorphisms of torsion finitely quasihamiltonian groups. Indeed, as it was shown in [7], the class of torsion finitely quasihamiltonian groups is a family of locally finite groups that properly contains the class of torsion groups that are either FC-groups or quasihamiltonian.

Nevertheless, the Addition Theorem fails for automorphisms of metabelian groups. By [8], $\text{AT}(G, id_G, H)$ does not hold, where $G = \mathbb{Z}_2^{(\mathbb{Z})} \times \mathbb{Z}$ is the Lamplighter group, $H = \mathbb{Z}_2^{(\mathbb{Z})}$ and $id_G$ is the identity automorphism of $G$.

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1.1. Main results.

(1) Let $\mathcal{X}$ be a class of solvable groups closed under taking subgroups and quotients. We prove that the Addition Theorem holds in $\mathcal{X}$ for endomorphisms if $AT(G,\phi,G')$ holds for every $G \in \mathcal{X}$ and $\phi \in \text{End}(G)$. In case $\mathcal{X}$ consists of nilpotent groups, then the Addition Theorem holds in $\mathcal{X}$ for automorphisms if $AT(G,\phi,Z(G))$ holds for every $G \in \mathcal{X}$ and $\phi \in \text{Aut}(G)$ (see Proposition 3.2).

(2) The reduction for solvable groups, among other things, helps us to prove that the Addition Theorem holds for endomorphisms of two-step nilpotent torsion groups (see Theorem 4.7). This is done by proving the inequality $h(\phi) \geq h(\phi|_{G'}) + h(\phi)$ using Proposition 4.1, while the converse inequality follows from Proposition 4.4.

(3) In Section 5 we provide more results concerning locally finite groups. It is proved in Corollary 5.3 that the Addition Theorem holds for endomorphisms of a locally finite group having a fully characteristic finite index simple subgroup. As a concrete example we may consider the finitary symmetric group $S_{fin}(N_+)$ (see Example 5.4).

1.2. Notation and terminology. The sets of non-negative reals, non-negative integers and positive natural numbers are denoted by $\mathbb{R}_{\geq 0}$, $\mathbb{N}$ and $\mathbb{N}_+$, respectively.

An element $x$ of a group $G$ is a torsion if the subgroup of $G$ generated by $x$, denoted by $\langle x \rangle$, is finite. Moreover, $G$ is torsion if every element of $G$ is torsion. A group is called locally finite if every finitely generated subgroup is finite. Every locally finite group is torsion and for solvable groups the converse also holds.

A group $G$ is solvable if there exist $k \in \mathbb{N}_+$ and a subnormal series

$$G_0 = \{1\} \leq G_1 \leq \cdots \leq G_k = G,$$

where 1 denotes the identity element, such that the quotient group $G_j/G_{j-1}$ is abelian for every $j \in \{1, \ldots, k\}$. In particular, $G$ is metabelian, if $k \leq 2$. The subgroup $Z(G)$ denotes the center of $G$, and we set $Z_0(G) = \{1\}$ and $Z_1(G) = Z(G)$. For $n > 1$, the $n$-th center $Z_n(G)$ is defined as follows:

$$Z_n(G) = \{x \in G : [x,y] \in Z_{n-1}(G) \text{ for every } y \in G\},$$

where $[x,y]$ denotes the commutator $xyx^{-1}y^{-1}$. A group is nilpotent if $Z_n(G) = G$ for some $n \in \mathbb{N}$. In this case, its nilpotency class is the minimum of such $n$. In particular, $G$ is abelian or two-step nilpotent (i.e., nilpotent of class 2) if $G' \subseteq Z(G)$, where $G'$ is the derived subgroup of $G$, namely, the subgroup of $G$ generated by all commutators $[x,y]$, where $x,y \in G$. It is known that every nilpotent group is solvable and every nilpotent group of class at most 2 is metabelian. We denote by $\mathcal{P}(G)$ the power set of $G$, while $\mathcal{L}(G)$ denotes the lattice of all subgroups of $G$.

We denote by $\text{End}(G)$ the set of all endomorphisms of $G$, and $\text{Aut}(G)$ is its subset consisting of all automorphisms. If $\phi \in \text{End}(G)$, then a subgroup $H$ of $G$ is called $\phi$-invariant if $\phi(H) \subseteq H$, and $H$ is $\phi$-stable if $\phi \in \text{Aut}(G)$ and $\phi(H) = H$. A subgroup $H$ of $G$ is characteristic if $H$ is $\phi$-invariant for every $\phi \in \text{Aut}(G)$, and $H$ is fully characteristic if the same holds for every $\phi \in \text{End}(G)$.

2. The functions $\ell(-)$ and $\ell(-,-)$

In this section we collect useful results from [7] concerning the functions $\ell(-)$ and $\ell(-,-)$. For a group $G$ define the function $\ell : \mathcal{P}(G) \times \mathcal{L}(G) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as follows: $\ell(X,B) = \ell(\pi(X))$, where $\pi : G \to \{xB \mid x \in G\}$ is the canonical projection.

**Lemma 2.1.** [7] Lemma 3.2] Let $G$ be a group, $X, X' \in \mathcal{P}(G)$ and $B, B' \in \mathcal{L}(G)$. Then:

(a) the function $\ell(X,B)$ is increasing in $X$ and decreasing in $B$;

(b) $\ell(XB) = \ell(X,B) + \ell(B)$;

(c) $\ell(XX',B) \leq \ell(X,B) + \ell(X',B)$;

(d) if $BB'$ is a subgroup, $\ell(XX',BB') \leq \ell(X,B) + \ell(X',B')$;

(e) for $\phi \in \text{End}(G)$, $\ell(\phi(X),\phi(B)) \leq \ell(X,B)$.

It turns out that the algebraic entropy $H(\phi,X)$ can be computed using a suitable decreasing subsequence of $\left\{\frac{\ell(T_{2^n}(\phi,X))}{2^n}\right\}_{n \in \mathbb{N}^+}$.

**Proposition 2.2.** [7] Proposition 3.1] Let $G$ be a group, $\phi \in \text{End}(G)$ and $X \in \mathcal{P}_{fin}(G)$ with $1 \in X$. Then:

(a) the function

$$n \mapsto \frac{\ell(T_{2^n}(\phi,X))}{2^n}$$

is decreasing;

(b) $H(\phi,X) = \inf_{n \in \mathbb{N}} \frac{\ell(T_{2^n}(\phi,X))}{2^n}$.

The following analogous result will help us to compute the entropy of $h(\phi)$. 

**Proposition 2.3.** [7 Proposition 3.3] Let $G$ be a group, $\phi \in \text{End}(G)$, $H$ a $\phi$-invariant normal subgroup of $G$ and $n : G \to G/H$ the canonical projection. Let $n \in \mathbb{N}$ and $X \in \mathcal{P}_{fn}(G)$ with $1 \in X$. Then:

(a) the function

$$n \mapsto \frac{\ell(T_{2^n}(\phi, X), H)}{2^n}$$

is decreasing;

(b) $H(\phi_{G/H}, \pi(h)) = \inf_{n \in \mathbb{N}} \frac{\ell(T_{2^n}(\phi, X), H)}{2^n}$.

To prove the next lemma one can use the proof of [7 Lemma 3.4] even though we assume here a weaker condition on the finite subgroup $F$.

**Lemma 2.4.** Let $G$ be a group, $\phi \in \text{End}(G)$, $X \in \mathcal{P}_{fn}(G)$ with $1 \in X$. If $F \in F(G)$ such that $T_n(\phi, F) \in \mathcal{L}(G)$ for every $n \in \mathbb{N}_+$, then the function

$$n \mapsto \frac{\ell(T_{2^n}(\phi, X), T_{2^n}(\phi, F))}{2^n}$$

is decreasing.

### 3. Reductions to (fully) characteristic subgroups

A useful property of the algebraic entropy is invariance under conjugation (see [2 Lemma 5.1.7]).

**Lemma 3.1.** Let $G$ and $H$ be groups, $\phi \in \text{End}(G)$ and $\psi \in \text{End}(H)$. If there exists an isomorphism $\xi : G \to H$, then $h(\phi) = h(\psi \xi^{-1})$.

In the sequel we will also use the next property of monotonicity for subgroups and quotients (see [2 Lemma 5.1.6]).

**Lemma 3.2.** Let $G$ be a group, $\phi \in \text{End}(G)$ and $H$ be a $\phi$-invariant subgroup of $G$. Then

(1) $h(\phi) \geq h(\phi |_H)$;

(2) if $H$ is normal and $\phi : G/H \to G/H$ is the endomorphism induced by $\phi$, then $h(\phi) \geq h(\phi)$.

For every group $G$ its derived subgroup $G'$ is fully characteristic, while its center $Z(G)$ is only a characteristic subgroup. The next proposition provides partial answers to Question 5.2.11(c) and Question 5.2.12(c) of [2].

**Proposition 3.3.** Let $X$ be a class of solvable groups closed under taking subgroups and quotients.

(1) if $\mathcal{AT}(G, \phi, G')$ holds for every $G \in X$ and $\phi \in \text{End}(G)$, then the Addition Theorem holds in $X$ for group endomorphisms;

(2) if $\mathcal{AT}(G, \phi, Z(G))$ holds for every nilpotent group $G \in X$ and $\phi \in \text{Aut}(G)$, then $\mathcal{AT}(G, \phi, H)$ holds for every nilpotent group $G \in X$, $\phi \in \text{Aut}(G)$ and $H$ a $\phi$-stable normal subgroup of $G$.

**Proof.** (1) Let $G \in X$ be a solvable group of class $n \in \mathbb{N}_+$. We have to prove that $\mathcal{AT}(G, \phi, H)$ holds for every $\phi \in \text{End}(G)$ and every $\phi$-invariant normal subgroup $H$ of $G$. This will be done using induction on $n$. If $n = 1$, then $G$ is abelian and the assertion follows from the Addition Theorem for abelian groups (see [3 Theorem 1.1]). Using the induction hypothesis and the conjugation of the properties $\mathcal{AT}(G, \phi, G')$, $\mathcal{AT}(H, \phi |_H, H')$ and $\mathcal{AT}((G/H, \phi, (G/H)'$, we deduce that $\mathcal{AT}(G, \phi, H)$ holds by [11 Proposition 5.9]. Note that unlike here, there $G$ is assumed to be metabelian. Nevertheless, $G'$ is a solvable group of class $n - 1$ for which the Addition Theorem holds by the induction hypothesis.

(2) Let $G \in X$ be a nilpotent group of class $n \in \mathbb{N}_+$. We have to prove that $\mathcal{AT}(G, \phi, H)$ holds for every $\phi \in \text{Aut}(G)$ and every $\phi$-stable normal subgroup $H$ of $G$. This will be done using induction on the nilpotency class $n$. If $n = 1$, then $G$ is abelian and the assertion follows from the Addition Theorem for abelian groups. Analogously to the proof of (1), we prove that $\mathcal{AT}(G, \phi, H)$ follows from the conjugation of the properties $\mathcal{AT}(G, \phi, Z(G))$, $\mathcal{AT}(H, \phi |_H, Z(H))$ and $\mathcal{AT}((G/H, \phi, Z(G/H))$. By $\mathcal{AT}(G, \phi, Z(G))$, we deduce that

$$h(\phi) = h(\phi |_{Z(G)}) + h(\tilde{\phi}),$$

where $\tilde{\phi} : G/Z(G) \to G/Z(G)$ is the map induced by $\phi$.

As $Z(G)$ is abelian, and $Z(G) \cap H$ is a $\phi$-stable subgroup of $Z(G)$, we get by the Addition Theorem for abelian groups

$$h(\phi |_{Z(G)}) = h(\phi |_{Z(G) \cap H}) + h(\tilde{\phi} |_{Z(G)}),$$

where $\tilde{\phi} |_{Z(G)} \in \text{Aut}(Z(G)/Z(G) \cap H)$ is the map induced by $\phi |_{Z(G)}$. 
Moreover, since $G/Z(G)$ is nilpotent of class $n-1$, and $HZ(G)/Z(G)$ is a $\tilde{\phi}$-stable subgroup of $G/Z(G)$, we obtain by the induction hypothesis

$$h(\tilde{\phi}) = h(\tilde{\phi} \restriction_{HZ(G)/Z(G)}) + h(\phi),$$

where $\phi \in \text{Aut}((G/Z(G))/(HZ(G)/Z(G)))$ is the map induced by $\tilde{\phi}$.

Hence, by Equations (3.6), (3.4) and (3.5), we have

$$h(\phi) = h(\phi \restriction_{Z(G)/H}) + h(\phi \restriction_{Z(G)}) + h(\phi \restriction_{HZ(G)/Z(G)}) + h(\phi).$$

Claim 1 $h(\phi \restriction_H) = h(\phi \restriction_{Z(G)/H}) + h(\phi \restriction_{Z(G)/Z(G)})$.

Proof. By AT($H, \phi \restriction_H$, $Z(H)$), we deduce that

$$h(\phi \restriction_H) = h(\phi \restriction_{Z(H)}) + h(\phi \restriction_H),$$

where $\phi \restriction_H : H/Z(H) \to H/Z(G)$ is the map induced by $\phi \restriction_H$.

As $Z(H)$ is abelian, and $H \cap Z(G)$ is a $\phi$-stable subgroup of $Z(H)$, we obtain by the Addition Theorem for abelian groups

$$h(\phi \restriction_{Z(H)}) = h(\phi \restriction_{H \cap Z(G)}) + h(\xi),$$

where $\xi : Z(H)/(H \cap Z(G)) \to Z(H)/(H \cap Z(G))$ is the map induced by $\phi \restriction_{Z(H)}$. Hence, to prove Claim 1, it suffices to show that

$$h(\phi \restriction_{HZ(G)/Z(G)}) = h(\xi) + h(\phi \restriction_H).$$

Let $\psi \in \text{Aut}(H/(H \cap Z(G)))$ be the map induced by $\phi \restriction_H$. Then $H/(H \cap Z(G))$ is nilpotent of class less than $n$ having $Z(H)/(H \cap Z(G))$ as a $\psi$-stable subgroup. By the induction hypothesis,

$$\text{AT}(H/(H \cap Z(G)), \psi, Z(H)/(H \cap Z(G)))$$

holds. Moreover, $\xi = \psi \restriction_{Z(H)/(H \cap Z(G))}$ and the automorphisms $\psi, \tilde{\psi}$ are conjugated, respectively, to

$$\phi \restriction_{Z(H)}, \tilde{\phi} \restriction_H,$$

where $\tilde{\psi} \in \text{Aut}(H/(H \cap Z(G)))/(Z(H)/(H \cap Z(G)))$ is the map induced by $\psi$.

Therefore, Equation (3.7) follows from $\text{AT}(H/(H \cap Z(G)), \psi, Z(H)/(H \cap Z(G)))$ and Lemma 3.1.

Claim 2 $h(\tilde{\phi}) = h(\phi \restriction_{Z(G)/H}) + h(\phi)$.

Proof. By $\text{AT}((G/H, \tilde{\phi}, Z(G/H))$ we have,

$$h(\tilde{\phi}) = h(\tilde{\phi} \restriction_{Z(G)/H}) + h(\delta),$$

where $\delta \in \text{Aut}((G/H)/Z(G/H))$ is the map induced by $\tilde{\phi}$. As $Z(G)H/H$ is a $\tilde{\phi}$-stable subgroup of the abelian group $Z(G/H)$, we have

$$h(\tilde{\phi} \restriction_{Z(G)/H}) = h(\tilde{\phi} \restriction_{Z(G)H/H}) + h(\varphi),$$

where $\varphi \in \text{Aut}(Z(G/H)/(Z(G)H/H))$ is the map induced by $\tilde{\phi} \restriction_{Z(G)H/H}$. The map $\tilde{\phi} \restriction_{Z(G)}$ is conjugated to $\tilde{\phi} \restriction_{Z(G)H/H}$. This fact implies that $h(\phi \restriction_{Z(G)}) = h(\tilde{\phi} \restriction_{Z(G)H/H})$. By (3.8) and (3.9) we get

$$h(\tilde{\phi}) = h(\phi \restriction_{Z(G)}) + h(\varphi) + h(\delta).$$

So, to prove Claim 2, it suffices to show that

$$h(\tilde{\phi}) = h(\varphi) + h(\delta).$$

Let $M = (G/H)/(HZ(G)/H)$, $N = Z(G/H)/(Z(G)H/H)$ and $\eta \in \text{Aut}(M)$ be the map induced by $\tilde{\phi}$. As $M$ is nilpotent of class less than $n$, and $N$ is an $\eta$-stable subgroup of $M$, we deduce that $\text{AT}(M, \eta, N)$ holds by the induction hypothesis. We use Invariance under conjugation to conclude that (3.11) is satisfied.

Claim 1, Claim 2 and Equation (3.4) complete the proof of (2), i.e.,

$$h(\phi) = h(\phi \restriction_H) + h(\phi).$$
4. When $G$ is a two-step nilpotent torsion group

We prove in Theorem 4.6 below that the Addition Theorem holds for endomorphisms of two-step nilpotent torsion groups.

**Proposition 4.1.** Let $G$ be a group, $\phi \in \text{End}(G)$ and $N$ be a $\phi$-invariant central torsion subgroup of $G$. Then, 

$$h(\phi) \geq h(\phi \mid N) + h(\tilde{\phi}),$$

where $\tilde{\phi} \in \text{End}(G/N)$ is the map induced by $\phi$.

**Proof.** Let $E$ be a finite subgroup of $N$ and $C = \pi(B)$, where $B$ is a finite subset of $G$ and $\pi : G \to G/N$ is the quotient homomorphism. As $E$ is a finite subgroup of the $\phi$-invariant central subgroup $N$, we deduce that $\phi^k(E)$ is a central subgroup of $G$ for every $k \in \mathbb{N}$. In particular, $T_n(E)$ is central and $T_n(\phi, EB) = T_n(\phi, E) \cdot T_n(\phi, B)$ for every $n \in \mathbb{N}_+$. This implies that $\ell(T_n(\phi, EB)) = \ell(T_n(\phi, E)) + \ell(T_n(\phi, B), T_n(\phi, E))$. By Lemma 2.1(a),

$$\ell(T_n(\phi, E)) + \ell(T_n(\phi, B)) = \ell(T_n(\phi, E)) + \ell(T_n(\phi, B), T_n(\phi, E)) \leq \ell(T_n(\phi, E)) + \ell(T_n(\phi, B), T_n(\phi, E)) = \ell(T_n(\phi, EB)).$$

As $\pi(T_n(\phi, B)) = T_n(\phi, C)$, it follows that $H(\phi \mid N, E) + H(\phi, C) \leq H(\phi, EB) \leq h(\phi)$. Since $E, B$ were chosen arbitrarily, we deduce that

$$h(\phi) \geq h(\phi \mid N) + h(\tilde{\phi}).$$

□

**Remark 4.2.** Another algebraic entropy is defined in [2] Remark 5.1.2 as follows. The $n$-th $\phi$-trajectory of $X$ is

$$T_n^\phi(\phi, X) = \phi^{n-1}(X) \cdot \ldots \cdot \phi(X) \cdot X.$$

The algebraic entropy of $\phi$ with respect to $X$ is $H^\#(\phi, X) = \lim_{n \to \infty} \frac{\ell(T_n^\phi(\phi, X))}{n}$ and the algebraic entropy of $\phi$ is $H^\#(\phi) = \sup\{H^\#(\phi, X) \mid X \in C\}$, where $C$ is any cofinal subfamily of $P_{fin}(G)$. Let us see that $h^\#$ coincides with $h$. To this aim, let $X \in P_{fin}(G)$. Then $X = X' \cup X^{-1}$ is a symmetric set containing $X$ and $(T_n(\phi, X^-1) = T_n^\phi(\phi, X^-1)$. This implies that $H^\#(\phi, X') = H(\phi, X')$ and $h^\#(\phi) = h(\phi)$, as $C = \{X' \mid X \in P_{fin}(G)\}$ is cofinal in $P_{fin}(G)$.

**Lemma 4.3.** Let $G$ be a group, $\phi \in \text{End}(G)$ and $F$ be a subgroup of $G$. Then:

1. $\langle T_n(\phi, F) \rangle = \bigcup_{m \in \mathbb{N}_+} (T_n(\phi, F) \cup T_n(\phi, F)^{-1})^m$. In particular, if $G$ is locally finite and $F$ is finite, then $\langle T_n(\phi, F) \rangle = (T_n(\phi, F) \cup T_n(\phi, F)^{-1})^m$

for some $m \in \mathbb{N}_+$.

2. $\langle T_n(\phi, F) \rangle = T_n(\phi, F) \cdot E_n$, where $E_n = \langle T_n(\phi, F) \rangle \cap G'$, for every $n \in \mathbb{N}_+$.

**Proof.** (1) Since the set $X_n = T_n(\phi, F) \cup T_n(\phi, F)^{-1}$ is symmetric and contains 1, it generates a subgroup of the form $\bigcup_{m \in \mathbb{N}_+} X_n^m$. When $G$ is locally finite and $F$ is finite we may use the finiteness of $\langle T_n(\phi, F) \rangle$ and the containment $X_n^m \subseteq X_{n+1}^m$ to prove the last assertion.

(2) In the notation of Remark 4.2,

$$T_n(\phi, F)^{-1} = T_n(\phi, F) = \phi^{n-1}(F) \cdot \ldots \cdot \phi(F) \cdot F$$

as $F$ is a subgroup of $G$. Let $g_1, \ldots, g_n \in G$, where $n \geq 2$. Then 

$$\pi(g_1 \cdot g_2 \cdot \ldots \cdot g_{n-1} \cdot g_n) = \pi(g_n \cdot g_{n-1} \cdot \ldots \cdot g_2 \cdot g_1),$$

where $\pi : G \to G/G'$ is the quotient map, since $G/G'$ is abelian. It follows that

$$\pi(T_n(\phi, F)) = \pi(T_n^\phi(\phi, F)) = \pi(T_n(\phi, F)^{-1}).$$

Moreover, as $F$ is a subgroup of $G$ and $G/G'$ is abelian we get that 

$$\pi(T_n(\phi, F) \cup T_n(\phi, F)^{-1}) = \pi(F^{m} \cdot \phi(F^{m}) \cdot \ldots \cdot \phi^{n-1}(F^{m})) = \pi(F \cdot \phi(F) \cdot \ldots \cdot \phi^{n-1}(F)) = \pi(T_n(\phi, F)),$$

for every $m \in \mathbb{N}_+$. Using also (1) we deduce that $\pi(T_n(\phi, F)) = \pi(T_n(\phi, F))$. This implies that

$$(T_n(\phi, F)) = T_n(\phi, F) \cdot E_n,$$

where $E_n = \langle T_n(\phi, F) \rangle \cap G'$. □

**Proposition 4.4.** Let $G$ be a torsion metabelian group, $\phi \in \text{End}(G)$. Then, $h(\phi) \leq h(\phi \mid G') + h(\tilde{\phi})$. 

Proof. Since $G$ is a torsion metabelian group it is locally finite by [5] Proposition 1.1.5. Recall that in this case $h(φ) = \sup\{H(φ, F)\mid F \in \mathcal{F}(G)\}$. Let $D$ be a finite subgroup of $G$ and let $C = π(D)$, where $π : G \to G/G′$ is the quotient homomorphism. Fix $ε > 0$. By Proposition [2,3] there exists $M ∈ \mathbb{N}$ such that, for every $n ≥ M$,
\[
\ell(T_*(φ, D), G′) \leq H(φ, C) + ε.
\]
For $T = T_2$ and $E = (T) ∩ G′$ we have $(T) = TE$ and $(T)G′ = TEG′ = TG′$, by Lemma [5]. Note that $E$ and $S = T_2(φ, E)$ are finite abelian as $G$ is torsion metabelian. On the one hand,
\[
\ell(T, E) = \log[(T) : E] = \log[(T) ∩ G′] = \log[(T)G′ : G′] = \log[TG′ : G′] = \ell(T, G′).
\]
On the other hand,
\[
\ell(T, G′) ≤ \ell(T, S) ≤ \ell(T, E)
\]
by Lemma [2,1] a). Hence,
\[
(4.2) \quad \ell(T, G′) = \ell(T, S).
\]
Let $n ≥ M$. By Lemma [2,2] and equations [2,1] and [2,2],
\[
\frac{\ell(T_*(φ, D), T_2(φ, E))}{2^n} ≤ \frac{\ell(T, S)}{2^M} = \frac{\ell(T, G′)}{2^M} ≤ H(φ, C) + ε ≤ h(φ) + ε.
\]
By Proposition [2,2] there exists $N ≥ M$, such that for every $n ≥ N$,
\[
(4.3) \quad \frac{\ell(T_*(φ, D))}{2^n} ≤ H(φ|G′, E) + ε ≤ h(φ|G′) + ε,
\]
and also
\[
(4.4) \quad H(φ, D) ≤ \frac{\ell(T_*(φ, D))}{2^n}.
\]
By Lemma [2,1] b),
\[
(4.5) \quad \ell(T_*(φ, D)) ≤ \ell(T_2(φ, D), T_2(φ, E)) = \ell(T_2(φ, D), T_2(φ, E)) + \ell(T_2(φ, E)).
\]
It follows from (4.3), (4.4), (4.5) and (4.6) that
\[
H(φ, D) ≤ \frac{\ell(T_*(φ, D))}{2^n} ≤ \frac{\ell(T_2(φ, D), T_2(φ, E))}{2^n} + \frac{\ell(T_2(φ, E))}{2^n} ≤ h(φ) + h(φ|G′) + 2ε.
\]
This completes the proof as the latter holds for any finite subgroup $D$ and any $ε > 0$. □

Remark 4.5. It is worth noting that Proposition [4,3] is no longer true in case the metabelian group $G$ is not torsion. Indeed, consider the Lamplighter group $G = \mathbb{Z}_2^2 \times \mathbb{Z}$ which was mentioned in the introduction. Using the arguments appearing in [7] Example 2.7, one can show that for $φ = id_G$ it holds that $h(φ|G′) = h(φ) = 0$ while $h(φ) = ∞$. It follows that
\[
h(φ) > h(φ|G′) + h(φ).
\]

Theorem 4.6. If $G$ is a two-step nilpotent torsion group, $φ ∈ \text{End}(G)$ and $H$ is a $φ$-invariant normal subgroup of $G$, then $AT(G, φ, H)$ holds.

Proof. By Proposition [3,1], it suffices to prove that $AT(G, φ, G′)$ holds. Taking into account that $G′ ⊆ Z(G)$ and using Proposition [4,1] we have,
\[
h(φ) ≥ h(φ|G′) + h(φ).
\]
By Proposition [4,3] the converse inequality also holds and we complete the proof. □

5. More on locally finite groups

Let $G$ be a group and $φ ∈ \text{End}(G)$. In the sequel we say that $AT(G, φ)$ holds if $AT(G, φ, H)$ holds for every $φ$-invariant normal subgroup $H$. If this happens for any $φ ∈ \text{End}(G)$, then we say that $AT(G)$ holds.

Proposition 5.1. Let $G$ be a locally finite group and $φ ∈ \text{End}(G)$. Then the family
\[
L(G, φ) = \{H \mid H \text{ is a } φ\text{-invariant normal subgroup of } G \text{ and } AT(H, φ|H) \text{ holds}\}
\]
contains a maximal element (with respect to inclusion).
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Proof. $\mathcal{L}(G, \phi)$ is not empty as it contains \{1\}. Let $\{H_i\}_{i \in I}$ be a chain in $\mathcal{L}$. By Zorn’s Lemma, it suffices to show that $H = \bigcup_{i \in I} H_i \in \mathcal{L}(G, \phi)$. Clearly, if each $H_i$ is a $\phi$-invariant normal subgroup of $G$, then so is $H$. Let us see that $\text{AT}(H, \phi |_{H \cap N})$ holds, where $N$ is a $\phi$-invariant normal subgroup of $H$. Since $H$ is the direct limit of the $\phi$-invariant subgroups $\{H_i | i \in I\}$ it follows from [2 Proposition 5.1.10] that

$$h(\phi |_{H}) = \sup_{i \in I} h(\phi |_{H_i}).$$

Similarly, we have $h(\phi |_{N}) = \sup_{i \in I} h(\phi |_{H \cap N})$ and

$$h(\phi_{H/N}) = \sup_{i \in I} h(\phi_{H_i/(H_i \cap N)}),$$

where the last equality follows from Lemma 5.1. By our assumption, $\text{AT}(H, \phi |_{H \cap N})$ holds for every $i \in I$. Using this and the previous equalities we obtain

$$h(\phi |_{H}) = \sup_{i \in I} h(\phi |_{H_i}) = \sup_{i \in I} h(\phi |_{H \cap N}) + \sup_{i \in I} h(\phi_{H_i/(H_i \cap N)}) = h(\phi |_{N}) + h(\phi_{H/N}),$$

as needed. \qed

Proposition 5.2. Let $G$ be a locally finite group, $\phi \in \text{End}(G)$ and $H$ be a $\phi$-invariant normal subgroup of $G$.

1. If $h(\phi) = 0$, then $h(\phi) = h(\phi |_{H})$;
2. if, in addition, $\text{AT}(H, \phi |_{H})$ holds, then $\text{AT}(G, \phi)$ holds.

Proof. (1) By Lemma 5.2(1), it suffices to show that $h(\phi) \leq h(\phi |_{H})$, i.e., that $H(\phi, E) \leq h(\phi |_{H})$ for an arbitrary finite subgroup $E$ of $G$. Let $\pi : G \to G/H$ be the canonical homomorphism and let $E_1 = \pi(E)$. Since $\phi$ has zero entropy and $G/H$ is locally finite, it follows from [3 Proposition 4.5] that there exists $m \in \mathbb{N}_+$ such that $T_m(\phi, E_1)$ is $\phi$-invariant. This implies that $\phi^m(E) \subseteq T_m(\phi, E) \cdot H$. As $E$ is finite there exists a finite subset $F$ of $H$ such that $\phi^m(E) \subseteq T_m(\phi, E) \cdot F$. Since $H$ is locally finite, there exists a finite subgroup $E_2$ of $H$ containing $F$, so we have

$$\phi(T_m(\phi, E)) \subseteq T_m(\phi, E) \cdot E_2.$$

We now prove that

$$T_{m+k}(\phi, E) \subseteq T_m(\phi, E)T_k(\phi, E_2)$$

using induction on $k$. For $k = 1$, $T_{m+1}(\phi, E) \subseteq T_m(\phi, E)T_1(\phi, E_2)$ follows from Equation (5.1). Assuming that $T_{m+k}(\phi, E) \subseteq T_m(\phi, E)T_k(\phi, E_2)$, one obtains

$$T_{m+k+1}(\phi, E) = E\phi(T_{m+k}(\phi, E)) \subseteq E\phi(T_m(\phi, E))\phi(T_k(\phi, E_2)) = T_{m+1}(\phi, E)\phi(T_k(\phi, E_2)) \subseteq T_m(\phi, E)E_2\phi(T_k(\phi, E_2)) = T_m(\phi, E)T_{k+1}(\phi, E_2).$$

Thus Equation (5.2) holds, and we have

$$\frac{\log |T_{m+k}(\phi, E)|}{m+k} \cdot \frac{m+k}{k} \leq \frac{\log |T_m(\phi, E)|}{k} + \frac{\log |T_k(\phi, E_2)|}{k}.$$

Since $m$ is fixed, letting $k \to \infty$ (so $k+m \to \infty$ as well), we deduce that

$$H(\phi, E) \leq H(\phi, E_2) \leq h(\phi |_{H}).$$

(2) Let $M$ be a $\phi$-invariant normal subgroup of $G$. By our assumption, $\text{AT}(H, \phi |_{H \cap M})$ holds as $M \cap H$ is $\phi$-invariant normal subgroup of $H$. This means that

$$h(\phi |_{M \cap H}) + h(\phi_{H/H \cap M}) = h(\phi |_{H}).$$

where $\phi_{H/H \cap M}$ is the endomorphism induced by $\phi |_{H \cap M}$. Since $HM/H$ is a subgroup of $G/H$ and $h(\phi_{G/H}) = 0$ it follows from Lemma 5.2(1) that $h(\phi_{H/M}) = 0$. By Lemma 5.1 $\phi_{M/H \cap M}$ has zero entropy as this endomorphism is conjugated to $\phi_{H/M}$. Applying item (1) to the locally finite group $M$ and the endomorphism $\phi |_{M}$ we deduce that

$$h(\phi |_{M}) = h(\phi |_{M\cap H}).$$

As $G/HM \cong (G/M)/(HM/M)$ is a quotient of $G/H$ and $h(\phi_{G/H}) = 0$, we deduce by Lemma 5.1 and Lemma 5.2(2) that $\phi \in \text{End}((G/M)/(HM/M))$ has zero entropy, where $\phi$ is the map induced by $\phi_{G/M}$. By Lemma 5.1
\( h(\tilde{\phi}_{H/H \cap M}) = h(\tilde{\phi}_{HM/M}) \). Applying item (1) to the locally finite group \( G/M \) and the endomorphism \( \tilde{\phi}_{G/M} \) we obtain
\[
(5.5) \quad h(\tilde{\phi}_{G/M}) = h(\tilde{\phi}_{HM/M}) = h(\tilde{\phi}_{H/H \cap M}).
\]
From (1) we have
\[
(5.6) \quad h(\phi) = h(\phi |_H).
\]
It follows from \((5.4), (5.5), (5.5)\) and \((5.6)\) that \(\text{AT}(G, \phi, M)\) holds. \(\square\)

**Corollary 5.3.** If \( G \) is a locally finite group having a fully characteristic finite index simple subgroup \( H \), then \(\text{AT}(G)\) holds.

**Proof.** Let \( \phi \in \text{End}(G) \). Since \( H \) is fully characteristic it is \( \phi \)-invariant normal subgroup of \( G \). Since \([G : H] < \infty\) it follows that \( h(\phi) = 0 \). Clearly, \(\text{AT}(H)\) holds as \( H \) is simple. By Proposition 5.2 \(\text{AT}(G)\) holds. \(\square\)

**Example 5.4.** Let us prove that \(\text{AT}(G)\) holds, where \( G = S_{\text{fin}}(\mathbb{N}_+) \) is the finitary symmetric group which consists of all permutations on \( \mathbb{N}_+ \) of finite support. Note that \( G \) is a locally finite group that is not finitely quasihamiltonian (see [7] Example 2.1(e)). It is known that \(\text{Alt}(\mathbb{N}_+)\), the infinite group of all even permutations, is a fully characteristic simple subgroup of \( G \) of index 2. By Corollary 5.3 \(\text{AT}(G)\) holds.

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