BRAID GROUP ACTIONS FOR QUANTUM SYMMETRIC PAIRS OF TYPE AIII/AIV

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ABSTRACT. In the present paper we construct braid group actions on quantum symmetric pair coideal subalgebras of type AIII/AIV. This completes the proof of a conjecture by Kolb and Pellegrini in the case where the underlying Lie algebra is \( \mathfrak{sl}_n \). The braid group actions are defined on the generators of the coideal subalgebras and the defining relations and braid relations are verified by explicit calculations.

1. Introduction

1.1. Background. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra and \( U_q(\mathfrak{g}) \) the corresponding Drinfeld-Jimbo quantised enveloping algebra. In the theory of quantum groups, a crucial role is played by Lusztig's braid group action on \( U_q(\mathfrak{g}) \), see [Lus94]. This braid group action provides an algebra automorphism \( T_w \) of \( U_q(\mathfrak{g}) \) for each element \( w \) in the Weyl group \( W \) of \( \mathfrak{g} \).

Let \( \theta : \mathfrak{g} \to \mathfrak{g} \) be an involutive Lie algebra automorphism and let \( \mathfrak{k} = \{ x \in \mathfrak{g} \mid \theta(x) = x \} \) denote the fixed Lie subalgebra. Recall from [Ara62] that involutive automorphisms of \( \mathfrak{g} \) are parameterised up to conjugation by combinatorial data \( (X, \tau) \) attached to the Dynkin diagram of \( \mathfrak{g} \). Here \( X \subset I \) where \( I \) denotes an index set for the nodes of the Dynkin diagram of \( \mathfrak{g} \), and \( \tau : I \to I \) is a diagram automorphism.

In a series of papers, G. Letzter constructed and investigated quantum group analogues of \( \mathfrak{k} \), see [Let99, Let02, Let03]. More precisely, she defined families of coideal subalgebras \( B_{c,s} = B_{c,s}(X, \tau) \subset U_q(\mathfrak{g}) \) which are quantum group analogues of \( U(\mathfrak{k}) \) depending on parameters \( c \) and \( s \). The algebras \( B_{c,s} \) can be described explicitly in terms of generators and relations. We call \( (U_q(\mathfrak{g}), B_{c,s}) \) a quantum symmetric pair and we refer to \( B_{c,s} \) as a quantum symmetric pair coideal subalgebra.

There exists a braid group action on the fixed Lie subalgebra \( \mathfrak{k} \) by Lie algebra automorphisms. Let \( W \) denote the Weyl group of \( \mathfrak{g} \) with corresponding braid group \( Br(\mathfrak{g}) \), generated by elements \( \{ \varsigma_i \mid i \in I \} \). We write \( Br(W_X) \) to denote the subgroup of \( Br(\mathfrak{g}) \) generated by \( \{ \varsigma_i \mid i \in X \} \); this is the braid group corresponding to the parabolic subgroup \( W_X \subset W \). Further, associated to the pair \( (X, \tau) \) is a restricted root system \( \Sigma \) with Weyl group \( \tilde{W} \) generated by elements \( \tilde{\sigma}_i \) parameterised by the \( \tau \)-orbits in \( I \setminus X \). The group \( \tilde{W} \) can be considered as a subgroup of \( W \). Let \( Br(\tilde{W}) \subset Br(\mathfrak{g}) \) denote the corresponging braid group, generated by elements \( \tilde{\varsigma}_i \). Then the semidirect product \( Br(W_X) \rtimes Br(\tilde{W}) \subset Br(\mathfrak{g}) \) acts on \( \mathfrak{k} \) by Lie algebra automorphisms.

It was conjectured by Kolb and Pellegrini that there exists a quantum group analogue of this action on \( B_{c,s} \) by algebra automorphisms [KPT1]. Conjecture 1.2.

2010 Mathematics Subject Classification. 17B37; 81R50.

Key words and phrases. Quantum groups, quantum symmetric pairs, braid groups.
This conjecture has been proved in type AII and in all cases where \( X = \emptyset \) with the help of computer calculations [KP11]. In the case \( \mathfrak{g} = \mathfrak{sl}_n \), this leaves one substantial case in Araki’s list [Ara62, p.32], namely the type AIII/AIV with \( X \neq \emptyset \). This case is shown in Figure 1.

1.2. Results. In the present paper we construct an action of \( Br(W_X) \times Br(\tilde{W}) \) on \( B_{c,s} \) by algebra automorphisms in type AIII/AIV, hence completing the proof of Kolb and Pellegrini’s conjecture for \( \mathfrak{g} = \mathfrak{sl}_n \). In this case \( Br(W_X) \) is a classical braid group in \( n - 2r \) strands and \( Br(\tilde{W}) \) is isomorphic to an annular braid group in \( r \) strands. The subgroups \( Br(W_X) \) and \( Br(\tilde{W}) \) of \( Br(\mathfrak{g}) \) commute and hence their semidirect product is just a direct product. Moreover, the parameters \( s \) satisfy \( s = (0,0,\ldots,0) \) and hence we write \( B_{c} = B_{c,0} \). The following is the main result of this paper.

**Theorem A.** (Theorem 3.9) Let \((X, \tau)\) be a Satake diagram of type AIII/AIV with \( X \neq \emptyset \). Then there exists an action of \( Br(W_X) \times Br(\tilde{W}) \) on \( B_{c} \) by algebra automorphisms. The action of \( Br(\tilde{W}) \) on \( B_{c} \) is given by algebra automorphisms \( T_i \) defined by Equations (3.7) and (3.9). The action of \( Br(W_X) \) on \( B_{c} \) coincides with the Lusztig action, see [BW18, Section 4.1]. However, the action of \( Br(\tilde{W}) \subset Br(\mathfrak{g}) \) on \( U_q(\mathfrak{g}) \) does not restrict to an action on the coideal subalgebra \( B_{c} \). Taking guidance from [KP11], the Lusztig automorphisms corresponding to elements of \( \tilde{W} \subset W \) are used in order to construct algebra automorphisms \( T_i \) for \( 1 \leq i \leq r \). In particular, for \( 1 \leq i \leq r \) and \( j \in I \setminus X \), we construct elements \( T_i(B_j) \) such that \( T_i(B_j) \) and \( T_{\sigma_i}(B_j) \) have identical terms containing maximal powers of the generators \( F_k \) of \( U_q(\mathfrak{g}) \) for \( k \in I \), up to a factor. For \( 1 \leq i \leq r - 1 \) the results of [KP11] imply that \( T_i \) defines an algebra automorphism of \( B_{c} \). In this case, our definition of \( T_i(B_j) \) coincides with the definition in [KP11, Equation 4.11] up to a factor.

The proof of Theorem A proceeds in three steps. First we explicitly verify that the formulae for \( T_r \) given in (3.9) define an algebra automorphism of \( B_{c} \). This reduces to checking a limited number of relations. Secondly, we show that the algebra automorphisms \( T_i \) for \( 1 \leq i \leq r \) satisfy the braid relations for \( Br(\tilde{W}) \). By [KP11, Section 4.2] we only need to show that \( T_i T_{r-1} T_{r-1} T_{r-1} = T_{r-1} T_{r-1} T_{r-1} T_r \) and \( T_i T_r = T_r T_i \) for \( 1 \leq i \leq r - 2 \) holds on each generator of \( B_{c} \). This gives an action of \( Br(\tilde{W}) \) on \( B_{c} \) by algebra automorphisms. Finally, we show directly that the actions of \( Br(W_X) \) and \( Br(\tilde{W}) \) commute by case-by-case checks.

![Figure 1](image-url)
We emphasise that all of the original results of the present paper are established without the use of computer calculations. The fact that the maps $T_i$ for $1 \leq i \leq r-1$ define algebra automorphisms of $B_c$ translates from [KP11] where it was verified by computer calculations using the package QUAGROUP of the computer algebra program GAP. We do not reprove this fact. However, the calculations in the present paper suggest that one can prove Theorem 3.2 ([KP11, Theorem 4.6]) without the use of computer calculations.

In [LW19a], M. Lu and W. Wang developed a Hall algebra approach to the construction of quantum symmetric pairs with $X = \emptyset$ for $g$ of type ADE (also excluding type $A_n$ for $n$ even if $\tau \neq id$). In this setting they subsequently constructed Bernstein-Gelfand-Ponomarev type reflection functors in [LW19b] which recover the corresponding braid group action in [KP11]. At the end of [LW19b, Section 1.5], they express great interest to develop this approach fully to cover general Satake diagrams with $X \neq \emptyset$. The braid group action for quantum symmetric pairs of type AIII/AIV with $X \neq \emptyset$ constructed in the present paper provides a crucial test case for any such generalisations. Formula (3.9) indicates that the general setting will be substantially more complicated.

1.3. Organisation. In Section 2 we recall fixed Lie subalgebras of type AIII/AIV and their corresponding quantum symmetric pairs. We further recall in Section 2.2 the construction of a braid group action of $Br(W_X) \times Br(\tilde{W})$ on the fixed Lie subalgebra in this case.

In Section 3 we establish a quantum analogue of the action on the fixed Lie subalgebra. In Section 3.1 we recall the action of $Br(W_X)$ on $B_c$ and define the algebra automorphisms $T_i$, giving an action of $Br(\tilde{W})$ on $B_c$. We also show that the two actions commute. In Sections 3.2 and 3.3 we prove that $T_r$ is an algebra automorphism of $B_c$ and that the automorphisms $T_i$ satisfy the braid relations for $Br(\tilde{W})$, respectively. This requires the use of many involved relations in $B_c$, which are given in Appendix A.

Acknowledgement. The author is grateful to Stefan Kolb for useful comments and advice.

2. Preliminaries

2.1. Fixed Lie subalgebras of type AIII/AIV. Let $g = sl_{n+1}(\mathbb{C})$ for $n \in \mathbb{N}$ with Cartan subalgebra $\mathfrak{h}$ consisting of traceless diagonal $(n+1) \times (n+1)$ matrices. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding root system. Choose a set $\Pi = \{\alpha_i \mid i \in I\}$ of simple roots where $I = \{1, 2, \ldots, n\}$ denotes an index set for the nodes of the Dynkin diagram of $g$.

![Dynkin diagram for $sl_{n+1}(\mathbb{C})$]

Let $Q = \mathbb{Z}\Pi$ denote the root lattice of $g$. Let $\varpi_i \in \mathfrak{h}^*$ denote the $i$th fundamental weight and let $P = \sum_{i \in I} \mathbb{Z}\varpi_i$ denote the weight lattice of $g$. Write $W$ to denote the Weyl group of $g$, generated by reflections $\sigma_i$ for $i \in I$. Fix a $W$-invariant scalar product $(\cdot, \cdot)$ on the real vector space spanned by $\Phi$ such that $(\alpha, \alpha) = 2$ for all
roots $\alpha$. For $i, j \in I$ let
\[
a_{ij} = \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } |i - j| = 1, \\
0 & \text{otherwise}
\end{cases}
\] (2.1)
denote the entries of the Cartan matrix of $\mathfrak{g}$. Let $Br(\mathfrak{g})$ denote the classical braid group corresponding to $\mathfrak{g}$. This is the group generated by elements $\{\varsigma_i \mid i \in I\}$ subject to relations
\[
\varsigma_i \varsigma_j = \varsigma_j \varsigma_i \quad \text{if } a_{ij} = 0, \quad (2.2)
\]
\[
\varsigma_i \varsigma_j \varsigma_i = \varsigma_j \varsigma_i \varsigma_j \quad \text{if } a_{ij} = -1. \quad (2.3)
\]

We recall the action of $Br(\mathfrak{g})$ on $\mathfrak{g}$ by Lie algebra automorphisms, as indicated in [Ste67]. In particular, let $\{e_i, f_i, h_i \mid i \in I\}$ denote a set of Chevalley generators of $\mathfrak{g}$ and define
\[
\text{Ad}(\varsigma_i) = \exp(\text{ad}(e_i)) \exp(-\text{ad}(f_i)) \exp(\text{ad}(e_i)) \quad (2.4)
\]
for $i \in I$. Here $\exp : \mathfrak{gl}(\mathfrak{g}) \to \text{Aut}(\mathfrak{g})$ denotes the exponential series and $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denotes the adjoint action. Note that $\exp$ is well defined on nilpotent elements and hence $\text{Ad}(\varsigma_i)$ is well defined for all $i \in I$.

**Lemma 2.1** ([Ste67] Lemma 56). There exists a group homomorphism
\[
\text{Ad} : Br(\mathfrak{g}) \to \text{Aut}(\mathfrak{g}) \quad (2.5)
\]
such that $\text{Ad}(\varsigma_i)$ is given by Equation (2.4).

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism and let $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ denote the $+1$-eigenspace. Recall from [Let03] Section 7 and [Kol14] Section 2.4 that involutive automorphisms of $\mathfrak{g}$ are classified up to conjugation via Satake diagrams $(X, \tau)$ where $X \subset I$ and $\tau : I \to I$ is a diagram automorphism.

Throughout this paper, we consider Satake diagrams of type AIII/AIV, as indicated by [Ara62, Table 1]. In particular, let $r \in \mathbb{N}$ such that $1 \leq r \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and let $X = \{r+1, r+2, \ldots, n-r\} \neq \emptyset$. The diagram automorphism $\tau$ is given by
\[
\tau(i) = n - i + 1 \quad (2.6)
\]
for each $i \in I$. This information is recorded graphically in Figure 1. To this Satake diagram we associate an involutive automorphism $\theta$. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ denote the Chevalley involution given by
\[
\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h_i) = -h_i. \quad (2.7)
\]
The diagram automorphism $\tau$ can be lifted to a Lie algebra automorphism, also denoted by $\tau$. Let $W_X$ denote the parabolic subgroup of $W$ generated by $\{\sigma_i \mid i \in X\}$ and let $Br(W_X)$ denote the associated braid group, generated by $\{\varsigma_i \mid i \in X\}$. The element
\[
w_X = \sigma_{r+1} \sigma_{r+2} \cdots (\sigma_{n-r} \sigma_{n-r-1} \cdots \sigma_{r+1}) \quad (2.8)
\]
is the longest element in $W_X$. Let $m_X$ denote the corresponding element of $Br(W_X)$, given by
\[
m_X = \varsigma_{r+1} \varsigma_{r+2} \cdots (\varsigma_{n-r} \varsigma_{n-r-1} \cdots \varsigma_{r+1}) \quad (2.9)
\]
Let $s : I \to \mathbb{C}^\times$ be a function such that
\begin{align}
    s(i) &= 1 & \text{if } i \in X, \\
    s(i) &= s(\tau(i)) & \text{if } i \not\in X \cup \{r, \tau(r)\}, \\
    \frac{s(i)}{s(\tau(i))} &= (-1)^{|X|} & \text{if } i = r, \tau(r),
\end{align}

see [BK19] Equations (5.1) and (5.2)]. This map extends to a group homomorphism $s_Q : Q \to \mathbb{C}^\times$ such that $s_Q(\alpha_i) = s(i)$ for all $i \in I$. Let $\Ad(s) : \mathfrak{g} \to \mathfrak{g}$ denote the Lie algebra automorphism such that
\[
    \Ad(s)|_{\mathfrak{h}} = \text{id}|_{\mathfrak{h}}, \quad \Ad(s)(x) = s_Q(\alpha)(x) \text{ for all } x \in \mathfrak{g}_\alpha
\]
where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denotes the root space corresponding to $\alpha$. The involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ is then generated by
\[
    \theta = \Ad(s) \circ \Ad(m_X) \circ \omega \circ \tau,
\]
see [Kol14] Theorem 2.5]. By [Kol14] Lemma 2.8] the fixed Lie subalgebra $\mathfrak{k}$ corresponding to $\theta$ is generated by the elements
\begin{align}
    e_i, f_i, h_i & \quad \text{for } i \in X, \\
    h_i - h_{\tau(i)} & \quad \text{for } 1 \leq i \leq r, \\
    b_i & := f_i + \theta(f_i) & \text{for } i \in I \setminus X.
\end{align}

We describe the elements $b_i \in \mathfrak{k}$ explicitly. For any subset $J = \{a, a + 1, \ldots, b - 1, b\} \subset I$ with $a < b$ let
\begin{align}
    e_J^+ & := [e_a, [e_{a+1}, \ldots, [e_{b-1}, e_b] \ldots]], \\
    e_J^- & := [e_b, [e_{b-1}, \ldots, [e_{a+1}, e_a] \ldots]].
\end{align}

If $J = \{a\} \subset I$, then we write
\[
    e_J^+ = e_J^- = e_a.
\]

Then for $i \in I \setminus X$ the elements $b_i$ are given by
\[
    b_i = \begin{cases} 
    f_i - s(\tau(i))e_{\tau(i)} & \text{if } i \neq r, \tau(r), \\
    f_r - s(\tau(r))[e_X^+, e_r] & \text{if } i = r, \\
    f_{\tau(r)} - s(r)[e_X^-, e_r] & \text{if } i = \tau(r).
\end{cases}
\]

### 2.2. Braid group action on $\mathfrak{k}$

Recall from Lemma 2.4 that there exists an action of $Br(\mathfrak{g})$ on $\mathfrak{g}$ by Lie algebra automorphisms given by $\Ad : Br(\mathfrak{g}) \to \text{Aut}(\mathfrak{g})$. Generally, we have $\Ad(b)(\mathfrak{k}) \neq \mathfrak{k}$ for $b \in Br(\mathfrak{g})$ which implies that we cannot restrict the action on $\mathfrak{g}$ to an action on $\mathfrak{k}$. For instance, for $1 \leq i < r$ we have
\[
    \Ad(\sigma_{i+1})(b_i) = \Ad(\sigma_{i+1})(f_i - s(\tau(i))e_{\tau(i)}) = [f_i, f_{i+1}] - s(\tau(i))e_{\tau(i)} \notin \mathfrak{k}.
\]

Instead, we consider a suitable subgroup of $Br(\mathfrak{g})$ that depends on $X \subset I$ and $\tau : I \to I$. For any $J \subset I$ let $w_J$ denote the longest element in the parabolic subgroup $W_J$ of $W$ and let $m_J$ denote the corresponding element in the braid group $Br(\mathfrak{g})$. For any $1 \leq i \leq r$ let
\[
    \bar{\sigma}_i = w_{(i, \tau(i)) \cup X}w_X^{-1} = \begin{cases} 
    \sigma_i \sigma_{\tau(i)} & \text{if } 1 \leq i \leq r - 1, \\
    \sigma_r \sigma_{r+1} \cdots \sigma_{n-r} \cdots \sigma_{r+1} \sigma_r & \text{if } i = r.
\end{cases}
\]
and denote by $\tilde{W}$ the subgroup of $W$ generated by $\{\tilde{\sigma}_i \mid 1 \leq i \leq r\}$. The subgroup $\tilde{W}$ can be interpreted as the Weyl group of the restricted root system $\Sigma$ of the symmetric Lie algebra $(\mathfrak{g},\theta)$, see [DK18, Section 2.2]. Let $Br(\tilde{W})$ denote the subgroup of $Br(\mathfrak{g})$ generated by the elements

$$\tilde{\varsigma}_i = m_{(i,\tau(i))} \cdot \tau^{-1} X^1 = \begin{cases} \varsigma_i \varsigma_{\tau(i)} & \text{if } 1 \leq i \leq r-1, \\ \varsigma_{r-1} \cdot \varsigma_{r-2} \cdot \varsigma_{r-3} & \text{if } i = r. \end{cases}$$  \tag{2.23}

The elements $\tilde{\varsigma}_i$ satisfy the relations

$$\tilde{\varsigma}_i \tilde{\varsigma}_j = \tilde{\varsigma}_j \tilde{\varsigma}_i \quad \text{if } a_{ij} = 0 \text{ and } 1 \leq i,j \leq r,$$

$$\tilde{\varsigma}_i \tilde{\varsigma}_j \tilde{\varsigma}_i = \tilde{\varsigma}_j \tilde{\varsigma}_i \tilde{\varsigma}_j \quad \text{if } a_{ij} = -1 \text{ and } 1 \leq i < j < r,$$

$$\tilde{\varsigma}_i \tilde{\varsigma}_j \tilde{\varsigma}_i \tilde{\varsigma}_j \tilde{\varsigma}_i = \tilde{\varsigma}_j \tilde{\varsigma}_i \tilde{\varsigma}_j \tilde{\varsigma}_i \tilde{\varsigma}_j \tilde{\varsigma}_i \quad \text{if } i = r, j = r-1.$$

By [D94] Theorem 9.1, $Br(\tilde{W})$ is isomorphic to the braid group $Br(\mathfrak{b}_r)$, that is, the annular braid group.

**Remark 2.2.** We describe the elements $\tilde{\varsigma}_i$ for $1 \leq i \leq r$ geometrically. Recall that elements of $Br(\mathfrak{g})$ are realised as subsets of $\mathbb{R}^2 \times [0,1]$ that are formed by $n+1$ strings oriented vertically from top to bottom. Elements of $Br(\tilde{W})$ have the following two properties. First, the $i^{th}$ string is fixed for $i \in \{r+1, \ldots, \tau(r)\}$. Additionally, any string $j \not\in \{r+1, \ldots, \tau(r)\}$ that wraps around any of these fixed strings must wrap around all of them collectively. We may hence consider elements of $Br(\tilde{W})$ as braids in $2r$ strings with a pole in the center. We denote this pole by a vertical dashed line. The elements $\tilde{\varsigma}_i$ for $1 \leq i \leq r-1$ are realised geometrically by the following diagram.

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In addition, the element $\tilde{\varsigma}_r$ is realised by

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where the $r^{th}$ and $\tau(r-1)^{th}$ strands wrap around the fixed axis. In this way, $Br(\tilde{W})$ is realised as an annular braid group.

Since $Br(W_X)$ and $Br(\tilde{W})$ commute, we consider the subgroup $Br(W_X) \times Br(\tilde{W})$. Observe that any $b \in Br(W_X) \times Br(\tilde{W})$ satisfies

$$m_X \tau(b) m_{X^{-1}} = b.$$ \tag{2.24}

The following lemma shows that the action $Ad$ from Lemma 2.1 restricted to $Br(W_X) \times Br(\tilde{W})$ almost commutes with $\theta$.

**Lemma 2.3.** For any $b \in Br(W_X) \times Br(\tilde{W})$ the relation

$$Ad(b) \circ Ad(m_X) \circ \omega \circ \tau = Ad(m_X) \circ \omega \circ \tau \circ Ad(b)$$ \tag{2.25}
holds in $\operatorname{Aut}(\mathfrak{g})$.

Proof. By [Kol14, Proof of Proposition 2.2 (3)] it follows that

$$\operatorname{Ad}(\varsigma_i) \circ \tau \circ \omega = \tau \circ \omega \circ \operatorname{Ad}(\varsigma_{r(i)})$$

for all $i \in I$. Hence for any $b \in Br(W_X) \times Br(\tilde{W})$ and $x \in \mathfrak{g}$ we have

$$\operatorname{Ad}(m_X) \circ \tau \circ \omega \circ \operatorname{Ad}(b)(x) = \operatorname{Ad}(m_X) \circ \operatorname{Ad}(\tau(b)) \circ \tau \circ \omega(x)$$

$$= \operatorname{Ad}(m_X \tau(b)m_X^{-1}) \circ \operatorname{Ad}(m_X) \circ \tau \circ \omega(x)$$

$$= \operatorname{Ad}(b) \circ \operatorname{Ad}(m_X) \circ \tau \circ \omega(x)$$

where the last equality follows from Equation (2.24). \hfill \square

Generally $\operatorname{Ad}(s)$ as given in Equation (2.13) does not commute with $\operatorname{Ad}(b)$ for $b \in Br(W_X) \times Br(\tilde{W})$. However, we can construct an explicit map $s' : I \to \mathbb{C}^\times$ such that the resulting involution $\theta' = \operatorname{Ad}(s') \circ \operatorname{Ad}(m_X) \circ \tau \circ \omega$ does commute with $\operatorname{Ad}(b)$ for any $b \in Br(W_X) \times Br(\tilde{W})$. Fix a total order $<$ on the set $I = \{1, \ldots, n\}$ in the natural way. With this total order, let $s' : I \to \mathbb{C}^\times$ be the function given by

$$s'(j) = \begin{cases} i |X| & \text{if } j = r, \\ (-i)^{|X|} & \text{if } j = \tau(r), \\ 1 & \text{otherwise} \end{cases}$$

(2.26)

where $i \in \mathbb{C}$ denotes the square root of $-1$. This map appears in [Kol14, Equation (2.2)] and has the advantage of commuting with $\theta$, see [Kol14, Theorem 2.5]. Moreover, $\operatorname{Ad}(s')$ commutes with $\operatorname{Ad}(b)$ for any $b \in Br(W_X) \times Br(\tilde{W})$. Let $\mathfrak{t}$ denote the fixed Lie subalgebra corresponding to the involution $\theta'$. Then Lemma 2.3 implies that $Br(W_X) \times Br(\tilde{W})$ maps $\mathfrak{t}$ to itself under the action $\operatorname{Ad}$. Given any involutive automorphism $\theta = \operatorname{Ad}(s) \circ \operatorname{Ad}(m_X) \circ \tau \circ \omega$ we can find a Lie algebra automorphism $\psi_s$ such that

$$\theta = \psi_s \circ \theta' \circ \psi_s^{-1}$$

(2.27)

where we make the dependence on $s : I \to \mathbb{C}^\times$ explicit. Since $\theta$ and $\theta'$ only differ by a scalar on each root space, we can choose $\psi_s$ to act by rescaling on each root space. In particular we may take $\psi_s = \operatorname{Ad}(\overline{\pi})$ where $\overline{\pi} : I \to \mathbb{C}^\times$ satisfies Equations (2.10) to (2.12). Then for any $x \in \mathfrak{g}_\alpha$, we have

$$\psi_s \circ \theta' \circ \psi_s^{-1}(x) = \frac{s'(j)}{\overline{s}(j)\overline{\pi}(\tau(j))} \cdot \operatorname{Ad}(m_X) \circ \omega \circ \tau$$

$$= \frac{s'(j)s(\tau(j))}{(-1)^{|X|}|\underline{s}(j)|^2} \theta(x).$$

If $j \notin X$ then we need $\overline{\pi}(j)^2 = s'(j)s(j)$ and hence we choose

$$\overline{\pi}(j) = \begin{cases} 1 & \text{if } j \in X, \\ s(j)^{1/2} & \text{if } j \notin X \cup \{r, \tau(r)\}, \\ (s(j)s'(j))^{1/2} & \text{if } j = r, \tau(r). \end{cases}$$

(2.28)

By Remark 2.3 we can take either the positive or negative square root. The following lemma is the version of [KPT11, Lemma 2.1] in the current setting.

**Lemma 2.4.** Under the action $\psi_s \circ \operatorname{Ad} \circ \psi_s^{-1}$ the subgroup $Br(W_X) \times Br(\tilde{W})$ maps $\mathfrak{t}$ to itself.
Proof. By Lemma 2.3 we have
\[ \theta' \circ \text{Ad}(b) = \text{Ad}(b) \circ \theta' \]
for any \( b \in Br(W_X) \times Br(\widetilde{W}) \). Using Equation (2.27) we see that
\[ \psi_s^{-1} \circ \theta \circ \psi_s \circ \text{Ad}(b) = \text{Ad}(b) \circ \psi_s^{-1} \circ \theta \circ \psi_s. \]
Applying \( \psi_s \) on the left and \( \psi_s^{-1} \) on the right gives
\[ \theta \circ \psi_s \circ \text{Ad}(b) \circ \psi_s^{-1} = \psi_s \circ \text{Ad}(b) \circ \psi_s^{-1} \circ \theta. \]
This implies that \( \psi_s \circ \text{Ad}(b) \circ \psi_s^{-1}(f) = f \) as required. \( \square \)

Remark 2.5. In [KP11, Lemma 2.1], the Lie algebra automorphism \( \psi_s \) does not appear. The reason for this is that Kolb and Pellegrini take \( s(j) = 1 \) for all \( j \in I \) since they only consider cases where either \( X = \emptyset \) or \( \tau = \text{id} \).

To shorten notation let
\[ \text{Ad}_i = \psi_s \circ \text{Ad}(\zeta_i) \circ \psi_s^{-1} \tag{2.29} \]
for \( 1 \leq i \leq r \). For any subset \( J = \{a, a+1, \ldots, b-1, b\} \subseteq I \) with \( a < b \) let
\[
\begin{align*}
J^+ &= [f_a, [f_{a+1}, \ldots, [f_{b-1}, f_b] \ldots]], \\
J^- &= [f_b, [f_{b-1}, \ldots, [f_{a+1}, f_a] \ldots]].
\end{align*}
\]
If \( J = \{a\} \subseteq I \) we write
\[ f_J^+ = f_J^- = f_a. \]

The following lemmas are given without proof and show how \( \text{Ad}_i \) acts on the element \( b_j \) for \( 1 \leq i \leq r \) and \( j \in I \setminus X \).

Lemma 2.6. Suppose that \((X, \tau)\) is a Satake diagram of type AIII. For \( 1 \leq i < r \) and \( j \in I \setminus X \) we have
\[
\text{Ad}_i(b_j) = \begin{cases} 
  b_j & \text{if } a_{ij} = 0 \text{ and } a_{i \tau(j)} = 0, \\
  b_{\tau(i)} & \text{if } a_{ij} = 2 \text{ or } a_{i \tau(j)} = 2, \\
  s(i)^{-1/2}b_{ji} & \text{if } a_{ij} = -1, \\
  s(\tau(i))^{-1/2}b_{ji} & \text{if } a_{i \tau(j)} = -1.
\end{cases} \tag{2.30}
\]

Lemma 2.7. Suppose that \((X, \tau)\) is a Satake diagram of type AIII/AIV and let \( c = (s(r)s(\tau(r)))^{-1/2} \). Then for \( j \in I \setminus X \) we have
\[
\text{Ad}_r(b_j) = \begin{cases} 
  b_j & \text{if } a_{rj} = 0 \text{ and } a_{r \tau(j)} = 0, \\
  s'(\tau(j))b_j & \text{if } a_{rj} = 2 \text{ or } a_{r \tau(j)} = 2, \\
  c([b_{r-1}, [b_r, [f_X, b_{\tau(r)}]]]) + s(r)b_{r-1} & \text{if } j = r-1, \\
  c([b_{(r-1)}, [b_{\tau(r)}, [f_X, b_r]]]) + s(\tau(r))b_{r-1} & \text{if } j = \tau(r-1).
\end{cases} \tag{2.31}
\]

Remark 2.8. The coefficients that appear in Equations (2.30) and (2.31) are independent of how \( \tau \) is chosen in Equation (2.28), as long as \( \tau \) satisfies Equations (2.10) to (2.12). Moreover, even if \( s : I \to \mathbb{C}^\times \) takes rational values, the braid group action may involve the imaginary factors \( s'(r) \) or \( s'(\tau(r)) \), and \( c \).
2.3. Quantum symmetric pairs of type AIII/AIV. Let $K$ be a field of characteristic zero and $q$ an indeterminate. Denote by $\mathbb{K}(q^{1/2})$ be the field of rational functions in $q^{1/2}$ with coefficients in $K$. Following [Jan96, Section 4.3] and [Lus76] the quantised enveloping algebra $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_{n+1}(\mathbb{C}))$ is the associative $\mathbb{K}(q^{1/2})$-algebra generated by elements $E_i, F_i, K_\mu$ for $i \in I$ and $\mu \in P$ satisfying the following relations:

1. $K_0 = 1, K_\mu K_\lambda = K_{\mu + \lambda}$ for all $\mu, \lambda \in P$.
2. $K_\mu E_i = q^{(\alpha_i, \mu)} E_i K_\mu$ for all $i \in I, \mu \in P$.
3. $K_\mu F_i = q^{-(\alpha_i, \mu)} F_i K_\mu$ for all $i \in I, \mu \in P$.
4. $E_i F_j - F_j E_i = \delta_{ij} \frac{K_j - K_j^{-1}}{q^{1/2} - q^{-1/2}}$ for all $i, j \in I$.
5. Quantum Serre relations.

We use the notation $K_i = K_{\alpha_i}$ for $i \in I$ and $K_\mu^{-1} = K_-\mu$ for $\mu \in P$ throughout. We make the quantum Serre relations (5) more explicit. Let $p$ denote the non-commutative polynomial in two variables given by

$$p(x, y) = x^2 y - (q + q^{-1})xyx + yx^2.$$  

Then the quantum Serre relations can be written as

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \quad \text{if} \quad a_{ij} = 0,$$

$$E_i E_j = p(E_i, E_j) = p(F_i, F_j) = 0 \quad \text{if} \quad a_{ij} = -1.$$  

The quantised enveloping algebra has the structure of a Hopf algebra, with coproduct $\Delta$ given explicitly by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_\mu) = K_\mu \otimes K_\mu$$

for all $i \in I, \mu \in P$ see [Jan96, Proposition 4.11].

Analogously to Lemma 2.1, there exists an action of the braid group $Br(\mathfrak{g})$ on $U_q(\mathfrak{g})$ by algebra automorphisms, see [Lus94, 39.4.3]. Under this action the generator $\sigma_i \in Br(\mathfrak{g})$ is mapped to the Lusztig automorphism $T_i$ as in [Jan96, Section 8.14]. We recall explicitly how $T_i$ acts on the generators of $U_q(\mathfrak{g})$. For any $a, b \in U_q(\mathfrak{g}), c \in \mathbb{K}(q^{1/2})$ let

$$[a, b]_c = ab - cba.$$  

For any $i, j \in I$ and $\mu \in P$ we have

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$$

and

$$T_i(K_\mu) = K_{\sigma_i(\mu)},$$

$$T_i(E_j) = \begin{cases} E_j & \text{if} \quad a_{ij} = 0, \\ [E_i, E_j]_{q^{-1}} & \text{if} \quad a_{ij} = -1, \end{cases}$$

$$T_i(F_j) = \begin{cases} F_j & \text{if} \quad a_{ij} = 0, \\ [F_j, F_i]_q & \text{if} \quad a_{ij} = -1, \end{cases}$$

For any $w \in W$ with reduced expression $w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r}$, we write

$$T_w := T_{i_r} T_{i_2} \cdots T_{i_1}.$$  

Following [Let99] in the conventions of [Kol14] we now recall the definition of quantum symmetric pair coideal subalgebras for Satake diagrams $(X, \tau)$ of type AIII/AIV. Let $s : I \rightarrow \mathbb{K}^\times$ be a function satisfying Equations (2.10) to (2.12). Let
\[ \mathcal{M}_X = U_q(g_X) \] denote the subalgebra of \( U_q(g) \) generated by \( \{ E_i, F_i, K_i^{\pm 1} \mid i \in X \} \).

Let \( U^0_\mu = \mathbb{K}(q^{1/2})(K_\mu) \mid \mu \in P, -w_\mu \circ \tau(\mu) = \mu \). By construction, \( K_i \in U^0_\mu \) for \( i \in X \) and \( K_{w_i - w_{\tau(i)}} \in U^0_\mu \) for \( i \in I \). We use the notation
\[ \varpi_i' = \varpi_i - \varpi_{\tau(i)} \quad \text{for} \ i \in I. \quad (2.41) \]

Quantum symmetric pair coideal subalgebras of type AIII/AIV depend on a choice of parameters \( c = (c_i)_{i \in I \setminus X} \in (\mathbb{K}(q^{1/2}))^{I \setminus X} \) satisfying additional constraints. We assume for the remainder of this paper that
\[ c_i = c_{\tau(i)} \quad \text{for} \ i \in I \setminus X \cup \{ r, \tau(r) \}, \quad (2.42) \]

compare with [Kol14, Section 5.1]. Analogously to Equations (2.18) and (2.19), for any \( J = \{ a, a+1, \ldots, b-1, b \} \subset I \) with \( a < b \) we define elements
\[ E_J^+ := [E_a, [E_{a+1}, \ldots, [E_{b-1}, E_b]_{q^{-1}}]_{q^{-1}}]_{q^{-1}}, \quad (2.43) \]
\[ E_J^- := [E_b, [E_{b-1}, \ldots, [E_{a+1}, E_a]_{q^{-1}}]_{q^{-1}}]_{q^{-1}} \quad (2.44) \]

and similarly we write
\[ F_J^+ := [F_a, [F_{a+1}, \ldots, [F_{b-1}, F_b]_{q}]_{q}]_{q}, \quad (2.45) \]
\[ F_J^- := [F_b, [F_{b-1}, \ldots, [F_{a+1}, F_a]_{q}]_{q}]_{q} \quad (2.46) \]

Additionally, let
\[ K_J = K_a K_{a+1} \cdots K_{b-1} K_b. \quad (2.47) \]

If \( J = \{ a \} \subset I \) then we write
\[ E_J^+ = E_J^- = E_a, \quad F_J^+ = F_J^- = F_a. \quad (2.48) \]

For later use, we note the following formulæ, which follow from [Kol14, Lemma 3.4]. We have
\[ T_{wx}(F_X^+) = -K_X^{-1}E_X^+, \quad T_{wx}(F_X^-) = -K_X^{-1}E_X^-, \quad (2.49) \]
\[ T_{wx}(E_X^+) = -F_X^+ K_X, \quad T_{wx}(E_X^-) = -F_X^- K_X. \quad (2.50) \]

Following [Kol14, Definition 5.1, 5.6] we denote by \( B_c = B_c(X, \tau) \) the subalgebra of \( U_q(g) \) generated by \( \mathcal{M}_X, U^0_\mu \) and the elements
\[ B_i = \begin{cases} F_i - c_i s(\tau(i)) E_{\tau(i)} K_{\tau(i)}^{-1} & \text{if} \ i \neq r, \tau(r), \\ F_r - c_r s(\tau(r)) [E_X^+, E_{\tau(r)}]_{q^{-1}} K_{\tau(r)}^{-1} & \text{if} \ i = r, \\ F_{\tau(r)} - c_{\tau(r)} s(r) [E_X^+, E_r]_{q^{-1}} K_{\tau(r)}^{-1} & \text{if} \ i = \tau(r). \end{cases} \quad (2.51) \]

for all \( i \in I \setminus X \).

For consistency, we set \( B_i = F_i \) and \( c_i = 0 \) for \( i \in X \). Additionally the parameters \( c_i \) and \( s(\tau(i)) \) for \( i \in I \) only appear together so we abbreviate
\[ \epsilon_i = c_i s(\tau(i)) \quad \text{for} \ i \in I \quad (2.52) \]

for the remainder of this paper. The subalgebras \( B_c \) are right coideal subalgebras of \( U_q(g) \), i.e.
\[ \Delta(B_c) \subseteq B_c \otimes U_q(g). \quad (2.53) \]

We recall now the defining relations of \( B_c \), following [Le tolerant 3] and [Kol14, Section 7]. For \( i \in I \setminus X \) let
\[ L_i = K_i K_{\tau(i)}^{-1} \quad (2.54) \]
and define
\[ Z_i = \begin{cases} 
-(1 - q^{-2})E_X^+L_{(r)} & \text{if } i = r, \\
-(1 - q^{-2})E_X^-L_r & \text{if } i = \tau(r), \\
-L_{\tau(i)} & \text{otherwise}. 
\end{cases} \] (2.55)
Further, let
\[ \Gamma_i := \epsilon_i Z_i - \epsilon_{\tau(i)}Z_{\tau(i)} \] (2.56)
for \( i \in I \setminus X \). Then the algebra \( B_c \) is generated over \( \mathcal{M}_X U_0^0 \) by the elements \( B_i \) for \( i \in I \setminus X \), subject to the relations
\[ B_i K_\mu = q^{(\mu,\alpha_i)} K_\mu B_i \quad \text{for } i \in I \setminus X, \ K_\mu \in U_0^0, \] (2.57)
\[ B_i E_j = E_j B_i \quad \text{for } i \in I \setminus X, \ j \in X, \] (2.58)
\[ B_i B_j - B_j B_i = \delta_{i,\tau(j)}(q - q^{-1})^{-1} \Gamma_i \quad \text{for } i \in I \setminus X, \ j \in I, \ a_{ij} = 0, \] (2.59)
\[ p(B_i, B_j) = 0 \quad \text{for } i, j \in I, \ a_{ij} = -1. \] (2.60)

3. Braid group action on \( B_c \)

3.1. Actions of \( Br(W_X) \times Br(\tilde{W}) \) on \( B_c \). By Lemma 2.4 an action of \( Br(W_X) \times Br(\tilde{W}) \) on \( \mathfrak{t} \) by Lie algebra automorphisms is obtained by restriction of the action of \( Br(\mathfrak{g}) \) on \( \mathfrak{g} \). We now construct an analogous braid group action in the setting of quantum symmetric pairs of type AIII/AIV. Recall that the algebra automorphisms \( T_i \) for \( i \in X \) give rise to a representation of \( Br(W_X) \) on \( U_q(\mathfrak{g}) \). In \( \text{BW18 Section 4.1} \) it was shown that \( B_c \) is invariant under the automorphisms \( T_i \) for \( i \in X \).

**Theorem 3.1 (BW18 Section 4.1).** There exists an action of \( Br(W_X) \) on \( B_c \) by algebra automorphisms such that the generator \( \varsigma_i \in Br(W_X) \) is mapped to the Lusztig automorphism \( T_i \).

We give this action explicitly on the elements \( B_i \) for \( i \in I \setminus X \). For \( i \in X \) and \( j \in I \setminus X \) we have
\[ T_i(B_j) = \begin{cases} 
B_j & \text{if } a_{ij} = 0, \\
[B_j, F_i]_q & \text{if } a_{ij} = -1. 
\end{cases} \]
(3.1)
It follows from this that
\[ T_{wx}^{-1}(B_r) = [B_r, F_X^+]_q, \] (3.2)
\[ T_{wx}(B_{r(r)}) = [B_{r(r)}, F_X^-]_q. \] (3.3)
Similarly, one also obtains
\[ T_{wx}^{-1}(B_r) = [F^{-}_X, B_r]_q, \] (3.4)
\[ T_{wx}(B_{r(r)}) = [F^+_X, B_{r(r)}]_q. \] (3.5)

We now construct the action of \( Br(\tilde{W}) \) on \( B_c \) by algebra automorphisms. For reasons observed in Equations (3.7) and (3.9) we now consider an extension \( \mathbb{K}' \) of the field \( \mathbb{K}(q^{1/2}) \) that contains \( \sqrt{q_i} \) for \( i \in I \setminus X \). For \( 1 \leq i \leq r \) the algebra automorphisms
\[ \tilde{T}_i := T_{\tilde{a}_i} = \begin{cases} 
T_i T_{r(i)} & \text{if } 1 \leq i \leq r, \\
T_r T_{r+1} \cdots T_{r(r)} \cdots T_{r+1} T_r & \text{if } i = r. \end{cases} \] (3.6)
do not leave $B_c$ invariant. However, they are still used as a guide to the construction of a braid group action on $B_c$. The general strategy is similar to that of [KP11].

We first define the action of the generators $\zeta_i$ for $1 \leq i \leq r - 1$. For $1 \leq i \leq r - 1$ and $j \in I \setminus X$ define

$$
T_i(B_j) = \begin{cases}
q^{-1}B_{\tau(j)}L_{\tau(j)} & \text{if } j = i \text{ or } j = \tau(i), \\
(qe_i)^{-1/2}[B_j, B_i] & \text{if } a_{ij} = -1, \\
(qe_{\tau(i)})^{-1/2}[B_{\tau(i)}, B_j] & \text{if } a_{\tau(i)j} = -1, \\
B_j & \text{if } a_{ij} = 0 \text{ and } a_{\tau(i)j} = 0.
\end{cases}
$$

(3.7)

**Theorem 3.2.** Suppose $(X, \tau)$ is a Satake diagram of type AIII with $X = \{ r + 1, \ldots, r(r + 1) \}$ and $1 \leq r \leq \lfloor \frac{3}{2} \rfloor - 1$. Let $1 \leq i \leq r - 1$.

1. There exists a unique algebra automorphism $T_i$ of $B_c$ such that $T_i(B_j)$ is given by Equation (3.8) for $j \in I \setminus X$ and $T_i|_{M_XU^0} = \tilde{T}_i|_{M_XU^0}$.

2. The inverse automorphism $T_i^{-1}$ is given by

$$
T_i^{-1}(B_j) = \begin{cases}
qB_{\tau(j)}L_j & \text{if } j = i \text{ or } j = \tau(j), \\
(qe_i)^{-1/2}[B_j, B_i] & \text{if } a_{ij} = -1, \\
(qe_{\tau(i)})^{-1/2}[B_{\tau(i)}, B_j] & \text{if } a_{\tau(i)j} = -1, \\
B_j & \text{if } a_{ij} = 0 \text{ and } a_{\tau(i)j} = 0.
\end{cases}
$$

(3.8)

and $T_i^{-1}|_{M_XU^0} = \tilde{T}_i|_{M_XU^0}$.

3. The relation $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ holds for $1 \leq i < r - 1$. Further the relation $T_iT_j = T_jT_i$ holds for $a_{ij} = 0$ with $1 \leq i, j < r - 1$.

**Proof.** The result follows from [KP11] Theorems 4.3 and 4.6 where the only difference occurs in $T_i(B_j)$ when $a_{ij} = -1$ or $a_{\tau(i)j} = -1$. Here, one checks that

$$
[B_j, B_i][B_{\tau(j)}, B_{\tau(i)}] - [B_{\tau(j)}, B_{\tau(i)}][B_j, B_i]q^{e_{\tau(i)}} = \frac{q - q^{-1}}{q - q^{-1}}(\epsilon_j T_i(Z_j) - \epsilon_{\tau(j)} T_i(Z_{\tau(i)})).
$$

Hence for symmetry reasons and the fact that $\epsilon_i = \epsilon_{\tau(i)}$ for $1 \leq i \leq r - 1$, we choose $T_i(B_j)$ and $T_i(B_{\tau(i)})$ as in Equation (3.7). \qed

**Remark 3.3.** In Equations (3.7) and (3.8) the coefficients $\epsilon_i$ appear whereas they did not in [KP11]. This is because Kolb and Pellegrini took $\epsilon_i = 1$ for all $i \in I$ in their paper.

It remains to construct the algebra automorphism $T_r$. For ease of notation, let $C = (qe_{\tau(r)})^{-1/2}$. Recall from Equation (2.11) that we set $\varpi'_i = \varpi_i - \varpi_{\tau(i)}$ for $i \in I$. Define

$$
T_r(B_j) = \begin{cases}
q^{-1}B_rL_rK_{\varpi'_r + 1} & \text{if } j = r, \\
q^{-1}B_{\tau(r)}L_{\tau(r)}K_{\varpi'_{\tau(r) + 1}} & \text{if } j = \tau(r), \\
C([B_{\tau(r-1)}, B_{\tau(r)}, F^+_{\tau(r)}]_q) & \text{if } j = r - 1, \\
q\epsilon_{\tau(r)}B_{\tau(r-1)}L_{\tau(r)}K_{\varpi'_r} & \text{if } j = \tau(r - 1), \\
B_j & \text{otherwise}.
\end{cases}
$$

(3.9)
for \( j \in I \setminus X \). The following theorem establishes that Equation \((3.9)\) defines an algebra automorphism \( \mathcal{T}_r : B_c \to B_c \).

**Theorem 3.4.** Suppose \((X, \tau)\) is a Satake diagram of type AIII/AIV with \( X = \{r + 1, \ldots, \tau(r + 1)\} \) and \( 1 \leq r \leq \left\lceil \frac{n}{2} \right\rceil - 1 \).

1. There exists a unique algebra automorphism \( \mathcal{T}_r \) of \( B_c \) such that \( \mathcal{T}_r(B_j) \) is given by Equation \((3.9)\) and \( \mathcal{T}_r|_{M_Xv_0^\mathcal{S}} = \tilde{T}_r|_{M_Xv_0^\mathcal{S}} \).
2. The inverse automorphism \( \mathcal{T}^{-1}_r \) is given by

\[
\mathcal{T}^{-1}_r(B_j) = \begin{cases} 
q B_r L_{\tau(r)} K_{w_{\tau(r+1)}} & \text{if } j = r, \\
q B_{\tau(r)} L_{\tau(r)} K_{w_{\tau(r+1)}} & \text{if } j = \tau(r), \\
C([B_{\tau(r)} F_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q) & \text{if } j = r - 1, \\
C([B_r, F_X, [B_{\tau(r)} B_{\tau(r-1)}] q]_q) & \text{if } j = \tau(r - 1), \\
B_j & \text{otherwise}
\end{cases}
\]

and \( \mathcal{T}^{-1}_r|_{M_Xv_0^\mathcal{S}} = \tilde{T}_r^{-1}|_{M_Xv_0^\mathcal{S}} \).

**Remark 3.5.** Comparing Equations \((2.30)\) and \((2.31)\) with Equations \((3.7)\) and \((3.9)\), we see that \( \mathcal{T}_r \) specialises to \( \text{Ad}_i \) for \( 1 \leq i \leq r \) if \( \epsilon_i \to s(\tau(i)) \) for all \( i = 1, \ldots, r \).

**Remark 3.6.** A desirable property of the algebra automorphism \( \mathcal{T}_r \) is that it is local, meaning \( \mathcal{T}_r(B_i) = B_i \) is satisfied for \( i \in I \setminus X \) and \( a_{ir} = a_{\tau(i)r} = 0 \). With this, it is not possible to omit the elements \( K_{w_{\tau(i)}} \) for \( i \in I \setminus X \) from our constructions. In particular, \( K_{w_{\tau(i)}} \) appears so that the relation

\[
B_i E_j - E_j B_i = 0
\]

for \( i \in I \setminus X, j \in X \) is preserved under \( \mathcal{T}_r \).

The proof of Theorem 3.4 requires non-trivial calculations which are postponed to Section 3.2. Crucially, the algebra automorphisms \( \mathcal{T}_1, \ldots, \mathcal{T}_r \) satisfy type \( B_r \) braid relations.

**Theorem 3.7.** Suppose \((X, \tau)\) is a Satake diagram of type AIII with \( X = \{r + 1, \ldots, \tau(r + 1)\} \) and \( 1 \leq r \leq \left\lceil \frac{n}{2} \right\rceil - 1 \). Then the relation

\[
\mathcal{T}_r \mathcal{T}_{r-1} \mathcal{T}_r \mathcal{T}_{r-1} = \mathcal{T}_{r-1} \mathcal{T}_r \mathcal{T}_{r-1} \mathcal{T}_r
\]

holds. Further, the relations \( \mathcal{T}_r \mathcal{T}_i = \mathcal{T}_i \mathcal{T}_r \) hold for any \( 1 \leq i < r - 1 \).

Similarly to Theorem 3.4, the proof of Theorem 3.7 requires a series of calculations which are given in Section 3.3. As a result of Theorems 3.2, 3.4 and 3.7, a braid group action of \( Br(\tilde{W}) \) on \( B_c \) by algebra automorphisms is established.

**Corollary 3.8.** Suppose \((X, \tau)\) is a Satake diagram of type AIII/AIV with \( X = \{r + 1, \ldots, \tau(r + 1)\} \) and \( 1 \leq r \leq \left\lceil \frac{n}{2} \right\rceil - 1 \). Then there exists an action of \( Br(\tilde{W}) \) on \( B_c \) by algebra automorphisms. Under this action the generator \( \tilde{c}_i \in Br(\tilde{W}) \) is mapped to the algebra automorphism \( \mathcal{T}_i \) for \( 1 \leq i \leq r \).

Since the subgroups \( Br(W_X) \) and \( Br(\tilde{W}) \) of \( Br(\mathfrak{g}) \) commute, we now combine Theorem 3.7 and Corollary 3.8 to give an action of \( Br(W_X) \times Br(\tilde{W}) \) on \( B_c \) by algebra automorphisms.
Theorem 3.9. Let $(X, \tau)$ be a Satake diagram of type $AIII/AIV$ with $X = \{r + 1, \ldots, \tau(r + 1)\}$ and $1 \leq r \leq \left\lceil \frac{r}{2} \right\rceil - 1$. Then there exists an action of $\text{Br}(W_X) \times \text{Br}(\tilde{W})$ on $B_e$ by algebra automorphisms. The action of $\text{Br}(W_X)$ on $B_e$ is given by the Lusztig automorphisms $T_i$ for $i \in X$ and the action of $\text{Br}(\tilde{W})$ on $B_e$ is given by the algebra automorphisms $\tilde{T}_i$ for $1 \leq i \leq r$ given by Equations (3.1) and (3.7).

In order to prove Theorem 3.9 it suffices to show that the actions of $\text{Br}(W_X)$ and $\text{Br}(\tilde{W})$ on $B_e$ commute. For the remainder of this section, we show this by casework.

Lemma 3.10. If $x \in M_XU^\emptyset$, then $T_iT_j(x) = T_jT_i(x)$ for all $j \in X$ and $1 \leq i \leq r$.

Proof. Recall that $T_i|_{M_XU^\emptyset} = \tilde{T}_i|_{M_XU^\emptyset}$ for all $1 \leq i \leq r$. If $i \leq r - 1$ then $\tilde{T}_i = T_iT_{r(i)}$ commutes with $T_j$ for any $j \in X$. If $i = r$ then $\tilde{T}_r|_{M_XU^\emptyset} = \text{id}|_{M_XU^\emptyset}$ and hence there is nothing to show in this case. By symmetry, it is enough to show that

$$T_iT_j(B_k) = T_jT_i(B_k)$$

for $1 \leq i \leq r$, $j \in X$ and $1 \leq k \leq r$.

Lemma 3.11. If $1 \leq i \leq r - 1$ and $j \in X \setminus \{r + 1\}$ then the relation

$$T_iT_j(B_k) = T_jT_i(B_k)$$

holds for all $1 \leq k \leq r$.

Proof. By Equations (3.1) and (3.7) it follows that $T_i(B_k)$ is invariant under $T_j$ for all $j \in X \setminus \{r + 1\}$ and $1 \leq k \leq r$. The result follows from this.

Lemma 3.12. If $1 \leq i \leq r - 1$ then the relation

$$T_iT_{r+1}(B_k) = T_{r+1}T_i(B_k)$$

holds for all $1 \leq k \leq r$.

Proof. Recall from Equation (3.1) that for $1 \leq k \leq r$ we have

$$T_{r+1}(B_k) = \begin{cases} B_k & \text{if } 1 \leq k \leq r - 1, \\ [B_r, F_{r+1}]_q & \text{if } k = r. \end{cases}$$

There are three cases to consider, depending on the value of $a_{ik}$. If $a_{ik} = 0$ then $T_i(B_k) = B_k$ and $T_i \circ T_{r+1}(B_k) = T_{r+1}(B_k)$. The claim follows from this. If $a_{ik} = -1$ then $T_i(B_k) = (q^e_i)^{-1/2} [B_k, B_i]_q$. Since $1 \leq i \leq r - 1$, Equation (3.1) implies that we need only check the claim when $k = r$ and $i = r - 1$. We obtain

$$T_{r+1}T_{r-1}(B_r) = (q^e_i)^{-1/2}T_{r+1}([B_r, B_{r-1}]_q)$$

$$= (q^e_i)^{-1/2}[B_r, F_{r+1}]_q, B_{r-1}]_q$$

$$= (q^e_i)^{-1/2}[B_r, B_{r-1}]_q, F_{r+1}]_q = T_{r-1}T_{r+1}(B_r)$$

as required. Finally, if $a_{ik} = 2$ then the claim follows since $1 \leq i \leq r - 1$ and hence $T_i(B_i) = q^{-1}B_{r(i)}L_{r(i)}$ is invariant under $T_{r+1}$.
Lemmas [3.11] and [3.12] imply that

\[ T_i T_j(B_k) = T_j T_i(B_k) \]

for all \( 1 \leq i \leq r - 1, j \in X \) and \( 1 \leq k \leq r \). All that remains is to check the case \( i = r \).

**Lemma 3.13.** Suppose that \( j \in X \) and \( 1 \leq k \leq r \) with \( k \neq r - 1 \). Then the relation

\[ T_i T_j(B_k) = T_j T_i(B_k) \]

holds.

*Proof.* Suppose that \( 1 \leq k \leq r - 2 \). Then both \( T_j \) and \( T_i \) act as the identity on \( B_k \) so the claim follows. Hence assume that \( k = r \). Then recall from Equation (3.1) that

\[ T_r(B_r) = q^{-1}B_r L_r K_{\tau r+1} \]

where \( \omega'_{r+1} = \omega_{r+1} - \omega_{\tau(r+1)} \). Let \( \lambda = \alpha_r - \alpha_{\tau(r)} + \omega'_{r+1} \). Since \( \alpha_r = -\omega_{r+1} + 2\omega_r - \omega_{r-1} \) it follows that \( (\alpha_j, \lambda) = 0 \) for all \( j \in X \). This implies that \( \sigma_j(\lambda) = \lambda \) for all \( j \in X \) and hence \( T_j(L_r K_{\omega_{r+1}}) = L_r K_{\omega'_{r+1}} \).

If \( j \neq r + 1 \) then \( T_j(B_r) = B_r \) and the result follows. Otherwise by Equation (3.1) we have

\[ T_{r+1} T_r(B_r) = q^{-1} T_{r+1}(B_r) K_{\lambda} \]

\[ = q^{-1} [B_r, F_{r+1}]_q K_{\lambda} = T_r T_{r+1}(B_r) \]

where we use the fact that \( K_{\lambda} \) commutes with \( F_{r+1} \). This completes the proof. \( \square \)

**Lemma 3.14.** For all \( j \in X \setminus \{ r + 1 \} \) the relation

\[ T_r T_j(B_{r-1}) = T_j T_r(B_{r-1}) \]

holds.

*Proof.* By Lemma [A.6] the result is clear for \( j \neq \tau(r + 1) \) since \( T_j \) acts as the identity on the elements \( F_X^+, L_r, K_X^{-1} \) and \( B_k \) for \( k \in I \setminus X \). On the other hand if \( j = \tau(r + 1) \) then by Equation (3.1) we have

\[ T_{r(r+1)}(F_X^+, T_{\tau(r)} B_{\tau(r)} q) = T_{r(r+1)}([F_X^{\tau(r+1)}, B_{\tau(r)}]_q) \]

\[ = [F_X^+, B_{\tau(r)}]_q. \]

Hence \( T_r(B_{r-1}) \) is invariant under \( T_{r(r+1)} \) and the result follows. \( \square \)

**Lemma 3.15.** The relation

\[ T_r T_{r+1}(B_{r-1}) = T_{r+1} T_r(B_{r-1}) \]

holds.

*Proof.* Recall from Equation (3.9) that

\[ T_r(B_{r-1}) = C([B_{r-1}, [B_r, F_X^+, T_{\tau(r)} B_{\tau(r)} q]_q] + q \epsilon_{\tau(r)} B_{r-1} L_r K_X^{-1}). \]

We are done if we show that \( T_r(B_{r-1}) \) is invariant under \( T_{r+1} \). Using Lemma [A.7] we obtain

\[ T_{r+1}([B_{r-1}, [B_r, F_X^+, T_{\tau(r)} B_{\tau(r)} q]_q] q) = [B_{r-1}, [B_r, F_X^+, T_{\tau(r)} B_{\tau(r)} q] q] q \]

\[ + q \epsilon_{\tau(r)} B_{r-1} L_r (K_{r+1}^{-1} - K_{r+1}) K_X^{-1}_{X \setminus \{r+1\}}. \]
Lemma 3.16. For any invariance for \( x \in U \), we first show that (3.12) is invariant under \( T \). We use Equations (3.20) and (3.21) to establish many of the results of this section. Then we have

\[
T_r(B_{r-1}L_rK_X^{-1}) = B_{r-1}L_rK_{r+1}K_X^{-1}. \tag{r+1}
\]

Combining these we obtain

\[
T_r + 1 T_r(B_{r-1}) = C [B_{r-1}, [B_r, [F_X, B_{r(r)}]_q]]_q + {C_q} \epsilon_{r(r)} B_{r-1}L_rK_{r+1}K_X^{-1}
\]

\[
+ C q \epsilon_{r(r)} B_{r-1}L_r(K_{r+1} - K_{r+1})K_{r+1}^{-1}
\]

\[
= T_r(B_{r-1})
\]

as required.

\[\square\]

Lemmas 3.10 to 3.15 together prove Theorem 3.9.

3.2. Proof of Theorem 3.4. Recall from Equations (2.32) and (2.56) the polynomial \( p : U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \) and the elements \( \Gamma_i = \epsilon_i Z_i - \epsilon_i(\iota) Z_{\iota(i)} \). In view of relations (2.57) to (2.60) and Equation (3.9) we show that the relations

\[
X_{r-1}x - xB_{r-1} = 0 \quad \text{for} \quad x \in \mathcal{M}_X, \tag{3.12}
\]

\[
B_{r-1}B_{r(r-1)} - B_{r(r-1)}B_{r-1} = \frac{1}{q - q^{-1}} \Gamma_{r-1}, \tag{3.13}
\]

\[
p(B_r, B_{r-1}) = 0, \tag{3.14}
\]

\[
p(B_{r-1}, B_r) = 0 \tag{3.15}
\]

are preserved under the map \( T_r \). The remaining relations either follow from the above by symmetry, or can be verified by short calculations. Such checks are not shown here. Using Lemma A.9 the elements \( T_r(B_{r-1}) \) and \( T_r(B_{r(r-1)}) \) can be expressed in the following way. Let

\[
S = [B_{r-1}, [B_r, F_X^\dagger]_q], \tag{3.16}
\]

\[
S^\dagger = [B_{r(r-1)}, [B_{r(r)}, F_X^-]_q]. \tag{3.17}
\]

and let

\[
\Delta = q \epsilon_{r(r)} B_{r-1}L_rK_X, \tag{3.18}
\]

\[
\Delta^\dagger = q \epsilon_{r(r)} B_{r(r-1)}L_rK_X. \tag{3.19}
\]

Then we have

\[
T_r(B_{r-1}) = C([S, B_{r(r)}] + \Delta), \tag{3.20}
\]

\[
T_r(B_{r(r-1)}) = C([S^\dagger, B_r] + \Delta^\dagger). \tag{3.21}
\]

We use Equations (3.20) and (3.21) to establish many of the results of this section. We first show that (3.12) is invariant under \( T_r \). The following lemma establishes invariance for \( x \in \{ E_i, F_i \mid i \in X, i \neq r + 1, \tau(r + 1) \} \).

Lemma 3.16. For any \( i \in X \setminus \{ r + 1, \tau(r + 1) \} \) the relations

\[
F_X^\dagger E_i - E_i F_X^\dagger = 0,
\]

\[
F_X^\dagger F_i - F_i F_X^\dagger = 0
\]

hold in \( U_q(\mathfrak{g}) \).
Proof. For any $i \in X \setminus \{r + 1, \tau(r + 1)\}$ let $W_i = \{r + 1, r + 2, \ldots, i - 1\}$ and $Y_i = \{i + 1, i + 2, \ldots, \tau(r + 1)\}$. Since $E_i F_j - F_j E_i = \delta_{ij} (q - q^{-1})^{-1} (K_i - K_i^{-1})$ for $i, j \in I$, it follows that

$$F_X^i E_i = E_i F_X^i - \frac{1}{q - q^{-1}} [F_{W_i}^i, [K_i, K_i^{-1}], F_{Y_i}^i]_q.$$  

Since $[K_i, F_{i+1}]_q = 0$ and $[F_{i-1}, K_i^{-1}]_q = 0$, it follows that $F_X^i E_i = E_i F_X^i$. Now, using Equation (2.39) we have

$$F_i = T_i^{-1} T_i^{-1}(F_{i-1}),$$

$$[F_{i-1}, [F_i, F_{i+1}]_q]_q = T_i^{-1} T_i^{-1}(F_{i+1}).$$

This implies that

$$[F_{i-1}, [F_i, F_{i+1}]_q]_q F_i = F_i [F_{i-1}, [F_i, F_{i+1}]_q],$$

and hence $F_X^i F_i = F_i F_X^i$ as required. \hfill \Box

We now consider (3.12) for $x \in \{E_{r+1}, E_{\tau(r+1)}, F_{r+1}, F_{\tau(r+1)}\}$.

**Lemma 3.17.** The relations

$$T_r(B_{r-1}) E_{r+1} - E_{r+1} T_r(B_{r-1}) = 0,$$

$$T_r(B_{r-1}) E_{\tau(r+1)} - E_{\tau(r+1)} T_r(B_{r-1}) = 0$$

hold in $B_c$.

**Proof.** Suppose first that $X = \{r + 1\}$. Since $[B_r, K_{r+1}]_q = 0$ we have

$$SE_{r+1} = [B_{r-1}, [B_r, F_{r+1} E_{r+1}]_q],$$

$$= E_{r+1} S - \frac{1}{q - q^{-1}} [B_{r-1}, [B_r, K_{r+1}]_q]_q,$$

$$= E_{r+1} S + q[B_{r-1}, B_r]_q K_{r+1}.$$

It follows that

$$[S, B_{r+2}]_q E_{r+1} = E_{r+1} [S, B_{r+2}]_q + q[[B_{r-1}, B_r]_q K_{r+1}, B_{r+2}]_q,$$

$$= E_{r+1} [S, B_{r+2}]_q + q^2 [B_{r-1}, [B_r, B_{r+2}]_q]_q K_{r+1},$$

$$= E_{r+1} [S, B_{r+2}]_q - \frac{q^2 \epsilon_{r+2}}{q - q^{-1}} [B_{r-1}, B_{r+2}]_q K_{r+1}.$$

Recalling that $Z_{r+2} = -(1 - q^{-2}) E_{r+1} L_r$, we obtain using (2.57) and (2.58)

$$[S, B_{r+2}]_q E_{r+1} = E_{r+1} [S, B_{r+2}]_q + q(1 - q^2) \epsilon_{r+2} E_{r+1} B_{r-1} L_r K_{r+1}.$$

This and Equation (5.19) imply that

$$\frac{1}{C} [T_r(B_{r-1}), E_{r+1}] = [S, B_{r+2}]_q E_{r+1} + \Delta E_{r+1} - E_{r+1} [S, B_{r+2}]_q - E_{r+1} \Delta$$

$$= (1 - q^2) E_{r+1} \Delta + \Delta E_{r+1} - E_{r+1} \Delta$$

$$= 0$$

where the last equality follows since $E_{r+1} \Delta = q^{-2} \Delta E_{r+1}$. This shows that $E_{r+1}$ commutes with $T_r(B_{r-1})$ when $|X| = 1$. 
Suppose now that $|X| > 1$. Let $Y = X \setminus \{r + 1\}$. Then

$$F_X^+E_{r+1} = E_{r+1}F_X^+ - \frac{1}{q - q^{-1}}[K_{r+1} - K_{r+1}^{-1}, F_Y^+]_q$$

$$= E_{r+1}F_X^+ - F_Y^+K_{r+1}.$$  

It follows from this and the relation $[B_r, K_{r+1}^{-1}]_q = 0$ that

$$SE_{r+1} = [B_{r-1}, [B_r, F_X^+E_{r+1}]_q]_q$$

$$= [B_{r-1}, [B_r, E_{r+1}F_X^+ - F_Y^+K_{r+1}]_q]_q$$

$$= E_{r+1}S - F_Y^+[B_{r-1}, [B_r, K_{r+1}^{-1}]_q]_q$$

$$= E_{r+1}S.$$  

Further, we have $\Delta E_{r+1} = E_{r+1}\Delta$ since $E_{r+1}$ commutes with $L_rK_X$. This implies that $T_r(B_{r-1})$ commutes with $E_{r+1}$. To show that $T_r(B_{r-1})$ commutes with $E_{r(r+1)}$, one proceeds similarly but instead using the form of $T_r(B_{r-1})$ given in Equation (3.19) and the relation $[K_{r(r+1)}, B_{r(r)}]_q = 0$.  

**Lemma 3.18.** The relations

$$T_r(B_{r-1})F_{r+1} - F_{r+1}T_r(B_{r-1}) = 0,$$

$$T_r(B_{r-1})F_{r(r+1)} - F_{r(r+1)}T_r(B_{r-1}) = 0$$

hold in $B_e$.

**Proof.** Suppose first that $X = \{r + 1\}$. Since $B_{r-1}$ commutes with $B_{r+2}$ we have

$$[S, B_{r+2}]_q = [B_{r-1}, [B_r, F_{r+1}]_q, B_{r+2}]_q.$$  

We commute $F_{r+1}$ through $[S, B_{r+2}]_q$ using the algebra automorphism $T_{wX} = T_{r+1}$. In particular, by Equations (2.36) and (3.1) we have

$$F_{r+1} = -T_{r+1}(E_{r+1}K_{r+1}),$$

$$[B_r, F_{r+1}]_q = T_{r+1}(B_r),$$

$$B_{r+2} = T_{r+1}([F_r, B_{r+2}]_q).$$  

It hence follows that

$$[S, B_{r+2}]_qF_{r+1} = -T_{r+1}([B_{r-1}, [B_r, F_{r+1}, B_{r+2}]_q]_q,E_{r+1}K_{r+1}).$$

We consider the right hand side of Equation (3.22). We have

$$[B_r, [F_{r+1}, B_{r+2}]_q]_qE_{r+1}K_{r+1} - E_{r+1}K_{r+1}B_r, [F_{r+1}, B_{r+2}]_q]_q$$

$$= \frac{1}{q - q^{-1}}[B_r, [K_{r+1}^{-1}, B_{r+2}]_q]_qK_{r+1}$$

$$= -\frac{1}{q - q^{-1}}(\epsilon_rZ_r - \epsilon_{r+2}Z_{r+2}).$$

Substituting this into Equation (3.22) we obtain

$$[S, B_{r+2}]_qF_{r+1} = F_{r+1}[S, B_{r+2}]_q + \frac{1}{q - q^{-1}}T_{r+1}([B_{r-1}, \epsilon_rZ_r - \epsilon_{r+2}Z_{r+2})]_q$$

$$= F_{r+1}[S, B_{r+2}]_q + q\epsilon_{r+2}B_{r-1}T_{r+1}(Z_{r+2}).$$

Since $Z_{r+2} = -(1 - q^{-2})E_{r+1}L_r$ we have

$$T_{r+1}(Z_{r+2}) = (1 - q^{-2})F_{r+1}K_{r+1}L_r.$$
This implies that
\[
[S, B_{r+2}]_q F_{r+1} = F_{r+1} [S, B_{r+2}]_q + q (1 - q^{-2}) \epsilon_{r+2} F_{r+1} B_{r-1} K_{r+1} L_r
\]
\[
= F_{r+1} [S, B_{r+2}]_q + (1 - q^{-2}) F_{r+1} \Delta.
\]
Since \(F_{r+1} \Delta = q^2 \Delta F_{r+1}\) we have
\[
\frac{1}{C} [\mathcal{T}_r(B_{r-1}), F_{r+1}] = [S, B_{r+2}]_q F_{r+1} + \Delta F_{r+1} - F_{r+1} [S, B_{r+2}]_q - F_{r+1} \Delta
\]
\[
= (1 - q^{-2}) F_{r+1} + \Delta F_{r+1} - F_{r+1} \Delta
\]
\[
= 0.
\]
This shows that \(F_{r+1}\) commutes with \(\mathcal{T}_r(B_{r-1})\) when \(|X| = 1\).

Suppose now that \(|X| > 1\). The relation \(F_{r+1} = - T_w X (E_{r(r+1)} K_{r(r+1)}^1)\) and Equation (3.2) imply
\[
[F_{r+1}, [B_r, F_X^+]_q] = - T_w X ([E_{r(r+1)} K_{r(r+1)}, B_r])
\]
\[
= 0
\]
and hence \(S\) commutes with \(F_{r+1}\). This, paired with the relation
\[
F_{r+1} L_r K_X = L_r K_X F_{r+1},
\]
shows that \(F_{r+1}\) commutes with \(\mathcal{T}_r(B_{r-1})\). In order to verify that \(\mathcal{T}_r(B_{r-1})\) commutes with \(F_{r(r+1)}\), one shows that \(F_{r(r+1)}\) commutes with \([F_X^+, B_{r(r)}]_q\) similarly to the above. The result then follows by considering Equation (3.9).

This completes the proof that Equation (3.12) is preserved under \(\mathcal{T}_r\). We now show that (3.13) is invariant under \(\mathcal{T}_r\).

**Proposition 3.19.** The relation
\[
[\mathcal{T}_r(B_{r-1}), \mathcal{T}_r(B_{r(r-1)})] = \frac{1}{q - q^{-1}} \Gamma_i
\]
holds in \(B_c\).

**Proof.** Using the expressions for \(\mathcal{T}_r(B_{r-1})\) and \(\mathcal{T}_r(B_{r(r-1)})\) given in Equations (3.20) and (3.21) we have
\[
\frac{1}{C^2} [\mathcal{T}_r(B_{r-1}), \mathcal{T}_r(B_{r(r-1)})] = [[S, B_{r(r)}]_q, [S^+, B_r]_q] + [\Delta, [S^+, B_r]_q]
\]
\[
+ [[S, B_{r(r)}]_q, \Delta^+] + [\Delta, \Delta^+].
\]
By Lemma (3.14) we have
\[
[[S, B_{r(r)}]_q, [S^+, B_r]_q] = - [\Delta, [S^+, B_r]_q] - [[S, B_{r(r)}]_q, \Delta^+]
\]
\[
- \frac{q \epsilon_{r(r)}(K_X - K_X^{-1}) K_X \Gamma_{r-1}}{q - q^{-1}}.
\]
The result follows by recalling from (3.9) that \(C^2 = (q \epsilon_{r(r)})^{-1}\) and noting that
\[
[\Delta, \Delta^+] = \frac{q \epsilon_{r(r)}(K_X - K_X^{-1}) K_X \Gamma_{r-1}}{q - q^{-1}}.
\]
\[
\square
\]
It remains to show that Equations (3.14) and (3.15) are invariant under \(\mathcal{T}_r\).
Proposition 3.20. The relation
\[ p(T_r(B_r), T_r(B_{r-1})) = 0 \] (3.24)
holds in \( B_r \).

Proof. Using the expression for \( T_r(B_{r-1}) \) from Equation (A.20) and recalling that
\[ T_r(B_r) = q^{-1}B_rL_rK_{\pi_r} \]
we have
\[ \frac{q^2}{C}p(T_r(B_r), T_r(B_{r-1})) \]
\[ = (B_rL_rK_{\pi_r})^2([S, B_{\tau(r)}]q + \Delta) + ([S, B_{\tau(r)}]q + \Delta)(B_rL_rK_{\pi'_r})^2 \]
\[ - (q + q^{-1})(B_rL_rK_{\pi'_r})([S, B_{\tau(r)}]q + \Delta)(B_rL_rK_{\pi'_r}). \]

By taking \((L_rK_{\pi'_r})^2\) out as a factor, we obtain
\[ \frac{q^2}{C}p(T_r(B_r), T_r(B_{r-1}))(L_rK_{\pi'_r})^{-2} \]
\[ = B_r^2[S, B_{\tau(r)}]q - q^{-1}(q + q^{-1})B_r[S, B_{\tau(r)}]qB_r + q^{-2}[S, B_{\tau(r)}]qB_r^2 \] (3.25)
\[ + B_r^2\Delta - q^{-1}(q + q^{-1})B_r\Delta B_r + q^{-2}\Delta B_r^2. \]

By Equation (A.22) the element \( B_r \) commutes with \( S \) which implies
\[ B_r[S, B_{\tau(r)}]q = [S, B_{\tau(r)}]qB_r + \frac{1}{q - q^{-1}}[S, \Gamma_r]q. \] (3.26)

It follows from this that
\[ B_r^2[S, B_{\tau(r)}]q = B_r[S, B_{\tau(r)}]qB_r + \frac{1}{q - q^{-1}}B_r[S, \Gamma_r]q, \]
\[ [S, B_{\tau(r)}]qB_r^2 = B_r[S, B_{\tau(r)}]qB_r + \frac{1}{q - q^{-1}}[S, \Gamma_r]qB_r. \]

Substituting these two expressions into Equation (3.25) we obtain
\[ \frac{q^2}{C}p(T_r(B_r), T_r(B_{r-1}))(L_rK_{\pi'_r})^{-2} = \frac{1}{q - q^{-1}}[S, [B_r, \Gamma_r]q^{-2}]q + B_r^2\Delta \]
\[ - q^{-1}(q + q^{-1})B_r\Delta B_r + q^{-2}\Delta B_r^2. \] (3.27)

Recall from Lemma (A.10) that
\[ SZ_{\tau(r)} = qZ_{\tau(r)}S - q(q - q^{-1})[B_{r-1}L_rK_X, B_r]. \] (3.28)

Since
\[ [B_r, \Gamma_r]q^{-2} = -(q^2 - q^{-2})\epsilon_{\tau(r)}Z_{\tau(r)}B_r \]
one calculates that
\[ \frac{1}{q - q^{-1}}[S, [B_r, \Gamma_r]q^{-2}]q = -(q + q^{-1})\epsilon_{\tau(r)}[S, Z_{\tau(r)}]qB_r \]
\[ = (q^2 - q^{-2})[\Delta, B_r]B_r. \] (3.26)

The result follows by substituting this expression into Equation (3.27) and observing that
\[ [\Delta, B_r]B_r = q^{-2}B_r[\Delta, B_r] \]
holds since \( p(B_r, B_{r-1}) = 0 \). \( \Box \)
Proposition 3.21. The relation
\[ p(\mathcal{T}_r(B_{r-1}), \mathcal{T}_r(B_r)) = 0 \]
holds in \( \mathcal{B} \).

Proof. Using Equations (3.9) and (3.20) we have
\[
\frac{q}{C^2} p(\mathcal{T}_r(B_{r-1}), \mathcal{T}_r(B_r))(L, K_{\mathcal{B}_r})^{-1} = ([S,B_{\tau(r)}]_q + \Delta)^2 B_r + q^2 B_r ([S,B_{\tau(r)}]_q + \Delta)^2
\]
\[
- q(q + q^{-1}) ([S,B_{\tau(r)}]_q + \Delta) B_r ([S,B_{\tau(r)}]_q + \Delta). \tag{3.29}
\]
Since \( B_{r-1}[B_{r-1}, B_r]_q = q^{-1}[B_{r-1}, B_r]_q B_{r-1} \) and \( B_{r-1} \) commutes with \( F_X^+ \) and \( B_{\tau(r)} \) it follows that
\[
[S,B_{\tau(r)}]_q \Delta = \Delta[S,B_{\tau(r)}]_q.
\]
This implies
\[
([S,B_{\tau(r)}]_q + \Delta)^2 = [S,B_{\tau(r)}]_q^2 + 2[S,B_{\tau(r)}]_q \Delta + \Delta^2.
\]
We consider terms involving different powers of \( \Delta \) in (3.29) separately. First, we consider the expression
\[
[S,B_{\tau(r)}]_q^2 B_r - q(q + q^{-1})[S,B_{\tau(r)}]_q B_r [S,B_{\tau(r)}]_q + q^2 B_r [S,B_{\tau(r)}]_q^2. \tag{3.30}
\]
Using the relation
\[
[S,B_{\tau(r)}]_q B_r = B_r[S,B_{\tau(r)}]_q - \frac{1}{q - q^{-1}}[S,\Gamma_r]_q
\]
from Equation (3.20) it follows that
\[
[S,B_{\tau(r)}]_q^2 B_r - q(q + q^{-1})[S,B_{\tau(r)}]_q B_r [S,B_{\tau(r)}]_q + q^2 B_r [S,B_{\tau(r)}]_q^2
\]
\[
= \frac{q^2}{q - q^{-1}}[S,\Gamma_r]_q [S,B_{\tau(r)}]_q - \frac{1}{q - q^{-1}}[S,B_{\tau(r)}]_q [S,\Gamma_r]_q. \tag{3.31}
\]
By Lemma A.10 we have
\[
\tau_{\tau(r)}[S,Z_{\tau(r)}]_q = -(q - q^{-1})[\Delta, B_r]
\]
which implies
\[
[S,\Gamma_r]_q [S,B_{\tau(r)}]_q = \epsilon_r[S,Z_r]_q [S,B_{\tau(r)}]_q + (q - q^{-1})[\Delta, B_r][S,B_{\tau(r)}]_q, \tag{3.32}
\]
\[
[S,B_{\tau(r)}]_q [S,\Gamma_r]_q = \epsilon_r[S,B_{\tau(r)}]_q [S,Z_r]_q + (q - q^{-1})[S,B_{\tau(r)}]_q [\Delta, B_r]. \tag{3.33}
\]
Using Lemmas A.10 and A.18 we commute \([S,Z_r]_q\) through \([S,B_{\tau(r)}]_q\). In particular we have
\[
Z_r[S,B_{\tau(r)}]_q = q^{-1}[S,B_{\tau(r)}]_q Z_r + q^{-2}(q - q^{-1}) \Delta Z_r,
\]
\[
S[S,B_{\tau(r)}]_q = q^{-1}[S,B_{\tau(r)}]_q S - q^2(q - q^{-2}) S \Delta.
\]
Combining this with the relations
\[
S \Delta = q^3 \Delta S,
\]
\[
Z_r \Delta = q^3 \Delta Z_r
\]
we obtain
\[
[S,Z_r]_q [S,B_{\tau(r)}]_q = q^{-2}[S,B_{\tau(r)}]_q [S,Z_r]_q - (1 - q^{-2}) \Delta [S,Z_r]_q \tag{3.34}
\]
Again by Equation (3.28) we have
\[
[\Delta, B_r][S, B_{\tau(r)}]_q = \Delta[S, B_{\tau(r)}]_q B_r - B_r[S, B_{\tau(r)}]_q \Delta + \frac{1}{q - q^{-1}} \Delta[S, \Gamma_r]_q, \tag{3.35}
\]
\[
[S, B_{\tau(r)}]_q[\Delta, B_r] = -B_r[S, B_{\tau(r)}]_q \Delta + \Delta[S, B_{\tau(r)}]_q B_r + \frac{1}{q - q^{-1}} [S, \Gamma_r]_q \Delta. \tag{3.36}
\]
By Equation (3.28) we have
\[
\epsilon_{\tau(r)}[S, Z_{\tau(r)}]_q = -(q - q^{-1})[\Delta, B_r]. \tag{3.37}
\]
Substituting Equations (3.34) and (3.35) into (3.32), Equation (3.36) into (3.33) and using (3.37) we obtain
\[
[S, \Gamma_r]_q[S, B_{\tau(r)}]_q = q^{-2} \epsilon_r[S, B_{\tau(r)}]_q[S, Z_r]_q + q^{-2} \epsilon_r \Delta[S, Z_r]_q + (q - q^{-1}) \Delta[\Delta, B_r] + \epsilon_r[S, Z_r]_q \Delta + (q - q^{-1}) B_r[S, B_{\tau(r)}]_q \Delta,
\]
\[
[S, B_{\tau(r)}]_q[S, \Gamma_r]_q = \epsilon_r[S, B_{\tau(r)}]_q[S, Z_r]_q + \epsilon_r[S, Z_r]_q \Delta + (q - q^{-1}) \Delta[S, B_{\tau(r)}]_q B_r - (q - q^{-1}) B_r[S, B_{\tau(r)}]_q \Delta + (q - q^{-1})[\Delta, B_r] \Delta.
\]
Hence Equation (3.31) implies that
\[
[S, B_{\tau(r)}]_q^2 B_r - q(q + 1)[S, B_{\tau(r)}]_q B_r[S, B_{\tau(r)}]_q + q^2 B_r[S, B_{\tau(r)}]_q^2 = (q^2 - 1) \Delta[S, B_{\tau(r)}]_q B_r - (q^2 - 1) B_r[S, B_{\tau(r)}]_q \Delta - [\Delta, B_r] \Delta. \tag{3.38}
\]
We next consider the expression
\[
2 \Delta[S, B_{\tau(r)}]_q B_r - (1 + q^2)[S, B_{\tau(r)}]_q B_r \Delta + \Delta B_r[S, B_{\tau(r)}]_q = 2q^2 B_r[S, B_{\tau(r)}]_q \Delta.
\]
By Equations (3.26) and (3.18) we have
\[
[S, B_{\tau(r)}]_q B_r \Delta = B_r[S, B_{\tau(r)}]_q \Delta - \frac{1}{q - q^{-1}} \epsilon_r[S, Z_r]_q \Delta - [\Delta, B_r] \Delta,
\]
\[
\Delta B_r[S, B_{\tau(r)}]_q = \Delta[S, B_{\tau(r)}]_q B_r + \frac{1}{q - q^{-1}} \epsilon_r \Delta[S, Z_r]_q + \Delta[\Delta, B_r]
\]
from which it follows that
\[
2 \Delta[S, B_{\tau(r)}]_q B_r - (1 + q^2)[S, B_{\tau(r)}]_q B_r \Delta + \Delta B_r[S, B_{\tau(r)}]_q = 2q^2 B_r[S, B_{\tau(r)}]_q \Delta
\]
\[
= (q^2 - 1) B_r[S, B_{\tau(r)}]_q \Delta - (q^2 - 1) \Delta[S, B_{\tau(r)}]_q B_r + (q^2 + 1)[[\Delta, B_r], \Delta]. \tag{3.39}
\]
Combining Equations (3.38) and (3.39) and substituting into Equation (3.20) gives
\[
\frac{q}{C_2} b(T_r(B_{r-1}), T_r(B_r)) (L_r K_{r-1}^{-1}) = -[[\Delta, B_r], \Delta]_q + (q^2 + 1)[[\Delta, B_r], \Delta] + \Delta^2 B_r - q(q + 1) \Delta B_r \Delta
\]
\[
+ q^2 B_r \Delta^2
\]
\[
= 0
\]
as required. \qed

Lemmas 3.16 to 3.18 and Propositions 3.19 to 3.21 together show that $T_r$ is an algebra endomorphism of $B_c$. We now repeat this procedure to show that $T_r^{-1}$ as
The relations underlying symmetry we have for any \( i \)

\[
T = [F^X, [B_r, B_{r-1}]_q]_q, \\
T^\tau = [F^X, [B_{r(r)}, B_{r(r-1)}]_q]_q
\]

and

\[
\Lambda = \epsilon_r B_{r-1} L_{r(r)} K^{-1}_X, \\
\Lambda^\tau = \epsilon_{\tau(r)} B_{\tau(r-1)} L_r K^{-1}_X.
\]

With this, we may write

\[
T^{-1}(B_{r-1}) = C([B_{r(r)}, T]_q + \Lambda), \\
T^{-1}(B_{r(r-1)}) = C([B_r, T^\tau]_q + \Lambda^\tau).
\]

We first show that Equation (3.12) is preserved by \( T^{-1} \). By Lemma 3.16 and the underlying symmetry we have

\[
F^X E_i - E_i F^X = 0, \\
F^X F_i - F_i F^X = 0
\]

for any \( i \in X \setminus \{ r + 1, \tau(r + 1) \} \). This implies that Equation (3.12) is preserved by \( T^{-1} \) for any \( x \in \{ E_i, F_i \mid i \in X \setminus \{ r + 1, \tau(r + 1) \} \} \).

**Lemma 3.22.** The relations

\[
T^{-1}(B_{r-1}) E_{r+1} - E_{r+1} T^{-1}(B_{r-1}) = 0, \\
T^{-1}(B_{r-1}) E_{\tau(r-1)} - E_{\tau(r-1)} T^{-1}(B_{\tau(r-1)}) = 0
\]

hold in \( B_c \).

**Proof.** Suppose that \( X = \{ r + 1 \} \). Then

\[
T E_{r+1} = [F_{r+1}, [B_r, B_{r-1}]_q]_q E_{r+1}
= E_{r+1} T - [B_r, B_{r-1}]_q K^{-1}_{r+1}.
\]

It hence follows that

\[
[B_{r+2}, T]_q E_{r+1} - E_{r+1} B_{r+2}, T]_q = -[B_{r+2}, [B_r, B_{r-1}]_q K^{-1}_{r+1}]_q
= -[[B_{r+2}, B_r], B_{r-1}]_q K^{-1}_{r+1}
= -\epsilon_r B_{r-1} Z_r K^{-1}_{r+1}
= (1 - q^{-2}) E_{r+1} \Lambda.
\]

This implies that

\[
\frac{1}{C} [T^{-1}(B_{r-1}), E_{r+1}] = [B_{r+2}, T]_q E_{r+1} + \Lambda E_{r+1} - E_{r+1} [B_{r+2}, T]_q - E_{r+1} \Lambda
= (1 - q^{-2}) E_{r+1} \Lambda + \Lambda E_{r+1} - E_{r+1} \Lambda
= 0
\]

where the last equality follows since \( E_{r+1} \Lambda = q^2 \Lambda E_{r+1} \). This shows that \( E_{r+1} \) commutes with \( T^{-1}(B_{r-1}) \) if \( |X| = 1 \).

Suppose now that \( |X| > 1 \). Let \( Y = X \setminus \{ r + 1 \} \). Then

\[
F^X E_{r+1} = E_{r+1} F^X - q F^Y K_{r+1}.
\]
Substituting this into Equation (3.46) we obtain
\[ TE_{r+1} = E_{r+1}T + q[F, K_{r+1}, [B_r, B_{r-1}]]_q \]
\[ = E_{r+1}T. \]

Further, since \( E_{r+1} \) commutes with \( L_r(K_{X}) \) we have \( \Lambda E_{r+1} = E_{r+1}\Lambda \). This implies that \( T_r^{-1}(B_{r-1}) \) commutes with \( E_{r+1} \) also for \( |X| > 1 \). A similar argument shows that \( T_r^{-1}(B_{r-1}) \) commutes with \( E_{r(r+1)} \).

**Lemma 3.23.** The relations
\[ T_r^{-1}(B_{r-1})F_{r+1} - F_{r+1}T_r^{-1}(B_{r-1}) = 0, \]
\[ T_r^{-1}(B_{r-1})F_{r(r+1)} - F_{r(r+1)}T_r^{-1}(B_{r-1}) = 0 \]
hold in \( B_c \).

**Proof.** Suppose that \( X = \{ r + 1 \} \). Since \( B_{r-1} \) commutes with \( B_{r+2} \) we have
\[ [B_{r+2}, T]_q = \left[ [B_{r+2}, [F_{r+1}, B_r]]_q, B_{r-1} \right]_q. \]

We commute \( F_{r+1} \) through \( [B_{r+2}, T]_q \) using the algebra automorphism \( T_{wX} = T_{r+1}^{-1}. \)

In particular by (3.3) and (3.4) we have
\[ F_{r+1} = -T_{wX}^{-1}(K_{r+1}^{-1}E_{r+1}), \]
\[ [F_{r+1}, B_r]_q = T_{wX}^{-1}(B_r), \]
\[ B_{r+2} = T_{wX}^{-1}([B_{r+2}, F_{r+1}]_q). \]

It hence follows that
\[ [B_{r+2}, T]_q F_{r+1} = -T_{r+1}^{-1}([[[B_{r+2}, F_{r+1}]_q, B_r]_q, B_{r-1}]_q) K_{r+1}^{-1}E_{r+1}). \] (3.46)

Considering the right hand side of (3.46) we have
\[ [[[B_{r+2}, F_{r+1}]_q, B_r]_q K_{r+1}^{-1}E_{r+1} \]
\[ = K_{r+1}^{-1}E_{r+1}[[[B_{r+2}, F_{r+1}]_q, B_r]_q - \frac{1}{q-q^{-1}} K_{r+1}^{-1}[[B_{r+2}, K_{r+1}]_q, B_r]_q \]
\[ = K_{r+1}^{-1}E_{r+1}[[[B_{r+2}, F_{r+1}]_q, B_r]_q - \frac{1}{q-q^{-1}} (\epsilon_r Z_r - \epsilon_{r+2} Z_{r+2}). \]

Substituting this into Equation (3.46) we obtain
\[ [B_{r+2}, T]_q F_{r+1} = F_{r+1}[B_{r+2}, T]_q + \frac{1}{q-q^{-1}} T_{r+1}^{-1}([\epsilon_r Z_r - \epsilon_{r+2} Z_{r+2}, B_{r-1}]_q) \]
\[ = F_{r+1}[B_{r+2}, T]_q - \epsilon_r B_{r-1} T_{r+1}^{-1}(Z_r). \]

Recalling that \( Z_r = -(1 - q^{-2})E_{r+1}L_{r+2} \) we have
\[ T_{r+1}^{-1}(Z_r) = (1 - q^{-2})K_{r+1}^{-1}F_{r+1}L_{r+2}. \]

This implies that
\[ [B_{r+2}, T]_q F_{r+1} = F_{r+1}[B_{r+2}, T]_q - (1 - q^{-2})\epsilon_r B_{r-1} K_{r+1}^{-1}F_{r+1}L_{r+2} \]
\[ = F_{r+1}[B_{r+2}, T]_q - (1 - q^{-2})\Lambda F_{r+1}. \]
Since $F_{r+1} \Lambda = q^{-2} \Lambda F_{r+1}$ we obtain

\[
\frac{1}{C}[T_{r}^{-1}(B_{r-1}), F_{r+1}] = [B_{r+2}, T_q] F_{r+1} + \Lambda F_{r+1} - F_{r+1}[B_{r+2}, T_q] - F_{r+1} \Lambda \\
= -(1 - q^{-2}) \Lambda F_{r+1} + \Lambda F_{r+1} - F_{r+1} \Lambda \\
= 0.
\]

This shows that $T_{r}^{-1}(B_{r-1})$ commutes with $F_{r+1}$ if $|X| = 1$.

Suppose now that $|X| > 1$. Noting that $[F_X^{-}, B_r] \equiv T_{w_X}^{-1}(B_r)$ and $F_{r+1} = -T_{w_X}^{-1}(K_{\tau(r+1)}^{-1})E_{\tau(r+1)}$ it follows that

\[
[F_{r+1}, [F_X^{-}, B_r]] = -T_{w_X}^{-1}([K_{\tau(r+1)}^{-1}]E_{\tau(r+1)}, B_r]) = 0.
\]

This implies that $T$ commutes with $F_{r+1}$. This, and the relation $F_{r+1} \Lambda = \Lambda F_{r+1}$ shows that $F_{r+1} \Lambda = \Lambda F_{r+1}$ also for $|X| > 1$. In order to show that $T_{r}^{-1}(B_{r-1})$ commutes with $F_{r(r+1)}$, one shows that $F_{r(r+1)}$ commutes with $[B_{r(r)}, F_X^{-}]$ as above. By a similar argument to Lemma A.9, we write

\[
\frac{1}{C}[T_{r}^{-1}(B_{r-1}), T_{r}^{-1}(B_{r(r-1)})] = \frac{1}{q - q^{-1}} \Gamma_{r-1}
\]

The result follows by considering this expression.

This completes the proof that Equation (3.12) is preserved under $T_{r}^{-1}$. We now show that Equation (3.13) is invariant under $T_{r}^{-1}$.

**Proposition 3.24.** The relation

\[
[T_{r}^{-1}(B_{r-1}), T_{r}^{-1}(B_{r(r-1)})] = \frac{1}{q - q^{-1}} \Gamma_{r-1}
\]

holds in $B_c$.

**Proof.** By Equations (3.44) and (3.45) we have

\[
\frac{1}{C^2}[T_{r}^{-1}(B_{r-1}), T_{r}^{-1}(B_{r(r-1)})] = [[B_{r(r)}, T_q][B_r, T_r]_q] + [\Lambda, [B_r, T_r]_q] \\
+ [[B_{r(r)}, T_q, \Lambda^r] + [\Lambda, \Lambda^r].
\]

(3.47)

By Lemma A.13, we have

\[
[[B_{r(r)}, T_q][B_r, T_r]_q] = -[[\Lambda, [B_r, T_r]_q] + [[B_{r(r)}, T_q, \Lambda^r] \\
+ \frac{q \varepsilon_{r(r)}}{q - q^{-1}}(K_{X} - K_{X}^{-1})K_{X}^{-1} \Gamma_{r-1}.
\]

Further we have

\[
[\Lambda, \Lambda^r] = \frac{q \varepsilon_{r(r)}}{q - q^{-1}} K_{X}^{-2} \Gamma_{r-1}.
\]

Substituting the above into (3.47) we obtain

\[
\frac{1}{C^2}[T_{r}^{-1}(B_{r-1}), T_{r}^{-1}(B_{r(r-1)})] = \frac{q \varepsilon_{r(r)}}{q - q^{-1}} \Gamma_{r-1}
\]

as required. □

We now show that Equations (3.14) and (3.15) are preserved under $T_{r}^{-1}$.
Proposition 3.25. The relation
\[ p(\mathcal{T}_r^{-1}(B_r), \mathcal{T}_r^{-1}(B_{r-1})) = 0 \]
holds in \( B_e \).

Proof. Recall from Equation (A.10) that
\[ \mathcal{T}_r^{-1}(B_r) = qB_rL\tau(r)K\omega^e_{(r+1)} \]
We have
\[
\frac{q^{-2}}{C} p(\mathcal{T}_r^{-1}(B_r), \mathcal{T}_r^{-1}(B_{r-1}))
\]
\[
= (B_rL\tau(r)K\omega^e_{(r+1)})^2 ([B_r, T]_q + \Lambda) + ([B_r, T]_q + \Lambda) (B_rL\tau(r)K\omega^e_{(r+1)})^2
\]
\[
- (q + q^{-1})(B_rL\tau(r)K\omega^e_{(r+1)}) ([B_r, T]_q + \Lambda) (B_rL\tau(r)K\omega^e_{(r+1)}).
\]
By taking \((L\tau(r)K\omega^e_{(r+1)})^2\) out as a factor we obtain
\[
\frac{q^{-2}}{C} p(\mathcal{T}_r^{-1}(B_r), \mathcal{T}_r^{-1}(B_{r-1})) (L\tau(r)K\omega^e_{(r+1)})^{-2}
\]
\[
= B_r^2[B_r, T]_q - q(q + q^{-1})B_r[B_r, T]_q B_r + q^2[B_r, T]_q B_r^2
\]
\[
+ B_r^2\Lambda - q(q + q^{-1})B_r\Lambda B_r - r + q^2\Lambda B_r^2.
\]
By Equation (A.21), the element \( B_r \) commutes with \( T \), which implies
\[
B_r[B_r, T]_q = [B_r, T]_q B_r + \frac{1}{q - q^{-1}}[\Gamma_r, T]_q.
\]
(3.49)
It follows from this that
\[
B_r^2[B_r, T]_q = B_r[B_r, T]_q B_r + \frac{1}{q - q^{-1}}B_r[\Gamma_r, T]_q,
\]
\[
[B_r, T]_q B_r^2 = B - r[B_r, T]_q B_r + \frac{1}{q - q^{-1}}[\Gamma_r, T]_q B_r.
\]
Substituting these two expressions into Equation (3.48) we obtain
\[
\frac{q^{-2}}{C} p(\mathcal{T}_r^{-1}(B_r), \mathcal{T}_r^{-1}(B_{r-1})) (L\tau(r)K\omega^e_{(r+1)})^{-2}
\]
\[
= \frac{1}{q - q^{-1}}[[B_r, \Gamma_r]_q^2, T]_q + B_r^2\Lambda - q(q + q^{-1})B_r\Lambda B_r + q^2\Lambda B_r^2.
\]
(3.50)
Recall from Equation (A.20) that
\[
Z_r T = qT Z_r - (q - q^{-1})[B_r, \xi_{r-1} L\tau(r)K_X^{-1}].
\]
(3.51)
Since
\[
[B_r, \Gamma_r]_q^2 = -(q^2 - q^{-2})\epsilon_r Z_r B_r
\]
we have
\[
\frac{1}{q - q^{-1}}[[B_r, \Gamma_r]_q, T]_q = -(q + q^{-1})\epsilon_r [Z_r, T]_q B_r
\]
(3.50)
Substituting this into Equation (3.51) and using the relation
\[
[B_r, \Lambda] B_r = q^2 B_r [B_r, \Lambda]
\]
it follows that
\[
\frac{q^{-2}}{C} p(T_r^{-1}(B_r), T_r^{-1}(B_{r-1}))(L_{\tau(r)}K_{\tau(r)+1}^{-1})^{-2} = (q^2 - q^{-2})[B_r, \Lambda]B_r + B_r^2\Lambda - q(q + q^{-1})B_r\Lambda B_r + 1^2\Lambda B_r^2 = 0
\]
as required. \qed

**Proposition 3.26.** The relation
\[
p(T_r^{-1}(B_{r-1}), T_r^{-1}(B_r)) = 0
\]
holds in $B_r$.

**Proof.** By Equations (3.10) and (3.44) we have
\[
\frac{q^{-1}}{C^2} p(T_r^{-1}(B_{r-1}), T_r^{-1}(B_r))(L_{\tau(r)}K_{\tau(r)+1}^{-1})^{-1} = ([B_r, T]_q + \Lambda)^2B_r - q^{-2}B_r([B_r, T]_q + \Lambda)^2
\]
\[
- q^{-1}(q + q^{-1})([B_r, T]_q + \Lambda)B_r([B_r, T]_q + \Lambda).
\]
Since $B_{r-1}[B_r, B_{r-1}] = q[B_r, B_{r-1}]B_{r-1}$ and $B_{r-1}$ commutes with $F_X$ and $B_{\tau(r)}$ it follows that
\[
[B_{\tau(r)}, T]_q\Lambda = \Lambda[B_{\tau(r)}, T]_q.
\]
This implies that
\[
([B_{\tau(r)}, T]_q + \Lambda)^2 = [B_{\tau(r)}, T]_q^2 + 2[B_{\tau(r), T}]_q\Lambda + \Lambda^2.
\]
We consider terms involving different powers of $\Lambda$ in Equation (3.52) separately. First, we consider the expression
\[
[B_{\tau(r)}, T]_q^2B_r - q^{-1}(q + q^{-1})[B_{\tau(r), T}]_qB_r[B_{\tau(r), T}]_q + q^{-2}B_r[B_{\tau(r), T}]_q^2.
\]
Using the relation
\[
B_r[B_{\tau(r), T}]_q = [B_{\tau(r), T}]_q B_r + \frac{1}{q - q^{-1}}[\Gamma_r, T]_q
\]
from Equation (3.49) it follows that
\[
[B_{\tau(r)}, T]_q^2B_r - q^{-1}(q + q^{-1})[B_{\tau(r), T}]_qB_r[B_{\tau(r), T}]_q + q^{-2}B_r[B_{\tau(r), T}]_q^2 = \frac{q^{-2}}{q - q^{-1}}[\Gamma_r, T]_q[B_{\tau(r), T}]_q - \frac{1}{q - q^{-1}}[B_{\tau(r), T}]_q[\Gamma_r, T]_q.
\]
By Equation (3.51) we have
\[
\epsilon_r[Z_r, T]_q = -(q - q^{-1})[B_r, \Lambda]
\]
which implies
\[
[\Gamma_r, T]_q[B_{\tau(r), T}]_q = -(q - q^{-1})[B_r, \Lambda][B_{\tau(r), T}]_q - \epsilon_{\tau(r)} Z_{\tau(r), T]_q[B_{\tau(r), T}]_q.
\]
\[
[B_{\tau(r), T}]_q[\Gamma_r, T]_q = -(q - q^{-1})[B_{\tau(r), T}]_q[B_r, \Lambda] - \epsilon_{\tau(r)}[B_{\tau(r), T}]_q Z_{\tau(r), T]_q.
\]

\[3.54\]
\[3.55\]
Using Equations (A.46) and (A.51) we commute $[Z_{\tau(r)}, T]_q$ through $[B_{\tau(r)}, T]_q$. In particular, we have

$$Z_{\tau(r)}[B_{\tau(r)}, T]_q = q[B_{\tau(r)}, T]_q Z_{\tau(r)} - q^2 (q - q^{-1}) \Lambda Z_{\tau(r)},$$

$$T[B_{\tau(r)}, T]_q = q[B_{\tau(r)}, T]_q T + q^2 (q^2 - q^{-2}) T \Lambda.$$

Combining this with the relations

$$T \Lambda = q^{-3} AT,$$

$$Z_{\tau(r)} \Lambda = q^3 \Lambda Z_{\tau(r)}$$

we obtain

$$[Z_{\tau(r)}, T]_q[B_{\tau(r)}, T]_q = q^2 [B_{\tau(r)}, T]_q[Z_{\tau(r)}, T]_q + (q^2 - 1) \Lambda[Z_{\tau(r)}, T]_q.$$

(3.56)

Further, using Equation (3.49) we have

$$[B_r, \Lambda][B_{\tau(r)}, T]_q = B_r[B_{\tau(r)}, T]_q \Lambda - \Lambda[B_{\tau(r)}, T]_q B_r - \frac{1}{q - q^{-1}} \Lambda[\Gamma_r, T]_q,$$

(3.57)

$$[B_{\tau(r)}, T]_q[B_r, \Lambda] = B_r[B_{\tau(r)}, T]_q \Lambda - \Lambda[B_{\tau(r)}, T]_q B_r - \frac{1}{q - q^{-1}} [\Gamma_r, T]_q \Lambda.$$  

(3.58)

Substituting Equations (3.56) and (3.57) into Equation (3.54) we obtain

$$[\Gamma_r, T]_q[B_{\tau(r)}, T]_q = -q^2 \epsilon_{\tau(r)}[B_{\tau(r)}, T]_q[Z_{\tau(r)}, T]_q - q^2 \epsilon_{\tau(r)} \Lambda[Z_{\tau(r)}, T]_q$$

$$- (q - q^{-1}) B_r[B_{\tau(r)}, T]_q \Lambda + (q - q^{-1}) \Lambda[B_{\tau(r)}, T]_q B_r$$

(3.59)

$$- (q - q^{-1}) \Lambda[B_r, \Lambda].$$

Similarly, substituting Equation (3.58) into Equation (3.55) we have

$$[B_{\tau(r)}, T]_q[\Gamma_r, T]_q = -\epsilon_{\tau(r)}[B_{\tau(r)}, T]_q[Z_{\tau(r)}, T]_q - (q - q^{-1}) B_r[B_{\tau(r)}, T]_q \Lambda$$

$$+ (q - q^{-1}) \Lambda[B_{\tau(r)}, T]_q B_r - (q - q^{-1})[B_r, \Lambda] \Lambda$$

(3.60)

Now substituting Equations (3.59) and (3.60) into Equation (3.58) gives

$$[B_{\tau(r)}, T]_q^2 B_r - q^{-1}(q + q^{-1})[B_{\tau(r)}, T]_q B_r[B_{\tau(r)}, T]_q + q^{-2} B_r[B_{\tau(r)}, T]_q^2$$

$$= (1 - q^{-2}) B_r[B_{\tau(r)}, T]_q \Lambda - (1 - q^{-2}) \Lambda[B_{\tau(r)}, T]_q B_r$$

(3.61)

$$- q^{-2} \Lambda[B_r, \Lambda] |q^2|.$$

We now consider the expression

$$2[B_{\tau(r)}, T]_q \Lambda B_r - (1 + q^{-2}) ([B_{\tau(r)}, T]_q B_r \Lambda + \Lambda B_r[B_{\tau(r)}, T]_q) + 2q^{-2} B_r[B_{\tau(r)}, T]_q \Lambda$$

which appears in Equation (3.62). By Equations (A.49) and (A.51) we have

$$[B_{\tau(r)}, T]_q B_r \Lambda = B_r[B_{\tau(r)}, T]_q \Lambda - \frac{1}{q - q^{-1}} [\Gamma_r, T]_q \Lambda$$

$$= B_r[B_{\tau(r)}, T]_q \Lambda + [B_r, \Lambda] \Lambda + \frac{1}{q - q^{-1}} \epsilon_{\tau(r)}[Z_{\tau(r)}, T]_q \Lambda,$$

$$\Lambda B_r[B_{\tau(r)}, T]_q = \Lambda[B_{\tau(r)}, T]_q B_r + \frac{1}{q - q^{-1}} \Lambda[\Gamma_r, T]_q$$

$$= \Lambda[B_{\tau(r)}, T]_q B_r - \Lambda[B_r, \Lambda] - \frac{1}{q - q^{-1}} \epsilon_{\tau(r)} \Lambda[Z_{\tau(r)}, T]_q.$$
Recalling that \( \Lambda \) commutes with \([B_{r(\tau)}, T]_q\) it hence follows that
\[
2[B_{r(\tau)}, T]_q \Lambda B_r - (1 + q^{-2}) ([B_{r(\tau)}, T]_q B_r + \Lambda B_r [B_{r(\tau)} T]_q) + 2q^{-2} B_r [B_{r(\tau)}, T]_q \Lambda
= (1 - q^{-2}) \Lambda [B_{r(\tau)}, T]_q B_r - (1 + q^{-2}) B_r [B_{r(\tau)}, T]_q \Lambda + (1 + q^{-2}) \Lambda [B_r, \Lambda].
\]
(3.62)

Combining Equations (3.61) and (3.62) and substituting into Equation (3.52) gives
\[
\text{as required.}
\]
\[\Box\]

**Proposition 3.27.**

Now show that
\[
B \text{ an algebra endomorphism of } B_i.
\]

It follows from this that
\[
\text{Proof.}
\]
\[
B \text{ commute this with } F_{r(\tau)}.
\]

We consider the case
\[
\text{consideration (A.22), the element } B_r \text{ commutes with } S. \text{ It follows that}
\]
\[
[q^{-1} B_r L_r, [S, B_{r(\tau)}]_q + \Delta]_q = [B_r, [S, B_{r(\tau)}]_q] L_r + [B_r, \Delta] L_r
= [S, [B_r, B_{r(\tau)}]_q] L_r + [B_r, \Delta] L_r
= \frac{1}{q - q^{-1}} [S, \Gamma_r]_q L_r + [B_r, \Delta] L_r.
\]

We now commute this with \( F_X^- \). Using Equation (A.18) and Equation (A.54) we obtain
\[
\left[F_X^-, [S, \Gamma_r]_q \right]_q = \epsilon_r \left[F_X^-, [S, \Gamma_r]_q \right]_q - \epsilon_{r(\tau)} \left[F_X^-, [S, \Gamma_{r(\tau)}]_q \right]_q
= q(q - q^{-1}) \epsilon_r S L_{r(\tau)} K_X^{-1} + (q - q^{-1}) [F_X^-, [\Delta, B_r]]_q.
\]

It follows from this that
\[
\left[F_X^-, [q^{-1} B_r L_r, [S, B_{r(\tau)}]_q + \Delta]_q \right]_q = \frac{1}{q - q^{-1}} \left[F_X^-, [S, \Gamma_r]_q \right]_q L_r + \left[F_X^-, [B_r, \Delta] \right] L_r
= q \epsilon_r SK_X^{-1}
\]

**Lemmas 3.22 and 3.23** and **Propositions 3.24 to 3.26** together show that \( T_r^{-1} \) is an algebra endomorphism of \( B_i \). In order to complete the proof of Theorem 3.4 we now show that \( T_r \circ T_r^{-1} = T_r^{-1} \circ T_r = \text{id} \). It suffices to check this on the generators \( B_i \) for \( i \in I \setminus X \). The relation
\[
T_r \circ T_r^{-1}(B_i) = B_i = T_r^{-1} \circ T_r(B_i)
\]
is straightforward for all \( i \in I \setminus X \) except \( i = r - 1 \) and \( i = \tau(r - 1) \). Here we only consider the case \( i = r - 1 \) since the case \( i = \tau(r - 1) \) is analogous.

**Proposition 3.27.**

The relation
\[
T_r \circ T_r^{-1}(B_{r-1}) = B_{r-1}
\]
(3.63)

**Proof.**

By Equations (3.9) and (3.10) we have
\[
\frac{1}{C^2} T_r \circ T_r^{-1}(B_{r-1}) = [q^{-1} B_{r(\tau)} L_{r(\tau)}, [F_X^-, [q^{-1} B_r L_r, [S, B_{r(\tau)}]_q + \Delta]]_q]_q
+ \epsilon_r ([S, B_{r(\tau)}]_q + \Delta) L_{r(\tau)} K_X^{-1}.
\]

We consider the first summand of the above expression and simplify it. By Equation (A.22), the element \( B_r \) commutes with \( S \). It follows that
\[
[q^{-1} B_r L_r, [S, B_{r(\tau)}]_q + \Delta]_q = [B_r, [S, B_{r(\tau)}]_q] L_r + [B_r, \Delta] L_r
= [S, [B_r, B_{r(\tau)}]_q] L_r + [B_r, \Delta] L_r
= \frac{1}{q - q^{-1}} [S, \Gamma_r]_q L_r + [B_r, \Delta] L_r.
\]

We now commute this with \( F_X^- \). Using Equation (A.18) and Equation (A.54) we obtain
\[
\left[F_X^-, [S, \Gamma_r]_q \right]_q = \epsilon_r \left[F_X^-, [S, \Gamma_r]_q \right]_q - \epsilon_{r(\tau)} \left[F_X^-, [S, \Gamma_{r(\tau)}]_q \right]_q
= q(q - q^{-1}) \epsilon_r S L_{r(\tau)} K_X^{-1} + (q - q^{-1}) [F_X^-, [\Delta, B_r]]_q.
\]

It follows from this that
\[
\left[F_X^-, [q^{-1} B_r L_r, [S, B_{r(\tau)}]_q + \Delta]_q \right]_q = \frac{1}{q - q^{-1}} \left[F_X^-, [S, \Gamma_r]_q \right]_q L_r + \left[F_X^-, [B_r, \Delta] \right] L_r
= q \epsilon_r SK_X^{-1}
\]
We now $q$-commute $q^{-1}B_{T(r)}L_{T(r)}$ and $q\epsilon_rSK_X^{-1}$ which gives
\[
[q^{-1}B_{T(r)}L_{T(r)}, q\epsilon_rSK_X^{-1}]_q = q\epsilon_r[B_{T(r)}, S]q^{-1}L_{T(r)}K_X^{-1} = -\epsilon_r[S, B_{T(r)}]qL_{T(r)}K_X^{-1}.
\]
Substituting the above into Equation (3.64) we obtain
\[
\frac{1}{C^2}T_r \circ T_r^{-1}(B_{r-1}) = \epsilon_r\Delta L_{T(r)}K_X^{-1} = q\epsilon_r\epsilon_{T(r)}B_{r-1} = \frac{1}{C^2}B_{r-1}
\]
and hence we have $T_r \circ T_r^{-1}(B_{r-1}) = B_{r-1}$ as required.

**Proposition 3.28.** The relation
\[
T_r^{-1} \circ T_r(B_{r-1}) = B_{r-1}
\]
holds in $B_c$.

**Proof.** By Equations (3.10) and (3.20) we have
\[
\frac{1}{C^2}T_r^{-1} \circ T_r(B_{r-1}) = [[[B_{T(r)}, T]_q + \Lambda, qB_rL_{T(r)}]_q, qB_{T(r)}L_{T(r)}]_q
\]
\[
q\epsilon_{T(r)}[[B_{T(r)}, T]_q + \Lambda]L_{T(r)}K_X.
\]
By a similar proof to Lemma A.11 the elements $B_r$ and $T$ commute. It follows that
\[
[[B_{T(r)}, T]_q + \Lambda, qB_rL_{T(r)}]_q = q[[B_{T(r)}, T]_q, B_rL_{T(r)}]_q + q[\Lambda, B_rL_{T(r)}]
\]
\[
= q[[B_{T(r)}, B_r]_q, T]_qL_{T(r)} + q[\Lambda, B_rL_{T(r)}]
\]
\[
= -\frac{q}{q - q^{-1}}[\Gamma_r, T]_qL_{T(r)} + q[\Lambda, B_rL_{T(r)}].
\]
By Equation (A.20) we have
\[
-\frac{q}{q - q^{-1}}\epsilon_{\tau}[[Z_{\tau}, T]_q, F_X^+]_qL_{\tau(r)} = \epsilon_r[[B_r, B_{T-1}]_qL_{T(r)}K_X^{-1}, F_X^+]_qL_{\tau(r)}
\]
\[
= q[[B_r, \epsilon_{\tau}B_{T-1}L_{T(r)}K_X^{-1}], F_X^+]_qL_{\tau(r)}
\]
\[
= -q[[\Lambda, B_r], F_X^+]_qL_{\tau(r)}.
\]
Hence Equation (3.67) and Equation (A.55) imply that
\[
[[B_{T(r)}, T]_q + \Lambda, qB_rL_{T(r)}]_q, F_X^+]_q
\]
\[
= \frac{q}{q - q^{-1}}\epsilon_{\tau}[[Z_{\tau(r)}, T]_q, F_X^+]_qL_{\tau(r)} - \frac{q}{q - q^{-1}}\epsilon_{\tau}[[Z_{\tau}, T]_q, F_X^+]_qL_{\tau(r)}
\]
\[
+ q[[\Lambda, B_r], F_X^+]_qL_{\tau(r)}
\]
\[
= \epsilon_{\tau(r)}TK_X.
\]
We substitute this into Equation (3.66) to obtain the Lusztig automorphism:

\[
\tilde{\sigma} = \frac{1}{C^2} \tau_r^{\lambda_1} \circ \tau_r(B_{r-1}) = \left[ \epsilon_{\tau(r)} T K_X, qB_{\tau(r)} L_r \right]_q + \epsilon_{\tau(r)} ([B_{\tau(r)}, T]_q + \Lambda)L_r K_X \\
= q^2 \epsilon_{\tau(r)} [T, B_{\tau(r)}]_q^{-1} L_r K_X + \epsilon_{\tau(r)} ([B_{\tau(r)}, T]_q + \Lambda)L_r K_X \\
= \frac{1}{C^2} B_{r-1}
\]

as required. \(\Box\)

3.3. Proof of Theorem 3.7 Restricted to \(M X U_0^\Theta\) the automorphisms \(\tau_i\) act as the Lusztig automorphism \(\tilde{T}_i = T_{\tilde{a}_i}\) for \(i \in I \setminus X\). As a result, the braid relations of Theorem 3.7 hold on elements of \(M X U_0^\Theta\). Hence it suffices to verify Theorem 3.7 on the elements \(B_i\) for \(i \in I \setminus X\).

We first check that the relation \(\tau_r \tau_i = \tau_i \tau_r\) holds for all \(1 \leq i \leq r - 2\).

**Proposition 3.29.** For \(1 \leq i \leq r - 2\) and \(j \in I \setminus X\) the relation

\[
\tau_r \tau_i(B_j) = \tau_i \tau_r(B_j)
\]

holds.

**Proof.** By symmetry, we only check Equation (3.68) for \(1 \leq j \leq r\). This is done by a case-by-case analysis.

**Case 1.** \(a_{ij} = 0, a_{jr} = 2\).
In this case we have \(j = r\) and hence \(T_i(B_j) = B_j\). This implies

\[
\tau_r \tau_i(B_j) = \tau_r(B_j) = q^{-1} B_r K_j K_{\tau(r)} r^{-1} K_{\tau(r)} = \tau_i \tau_r(B_j)
\]

as required.

**Case 2.** \(a_{ij} = 0, a_{jr} = -1\). Then \(j = r - 1\) and \(T_i(B_j) = B_j\) hence

\[
\tau_r \tau_i(B_j) = \tau_r(B_j) = \tau_i \tau_r(B_j)
\]

as required.

**Case 3.** \(a_{ij} = 0, a_{jr} = 0\).
In this case, we have \(\tau_r(B_j) = B_j\) and \(\tau_i(B_j) = B_j\) so the statement of the proposition holds.

**Case 4.** \(a_{ij} = -1, a_{jr} = 0\).
Here, we have \(\tau_r(B_j) = B_j\) and \(\tau_i(B_j) = (q \epsilon_i)^{-1/2} [B_j, B_i]_q\). Hence

\[
\tau_r \tau_i(B_j) = (q \epsilon_i)^{-1/2} [\tau_r(B_j), \tau_i(B_j)]_q \\
= (q \epsilon_i)^{-1/2} [B_j, B_i]_q = \tau_i \tau_r(B_j).
\]

**Case 5.** \(a_{ij} = -1, a_{jr} = -1\).
This case can only occur if \(i = r - 2\) and \(j = r - 1\). Then by Equation (3.9) we
have
\[
T_{r-2}T_r(B_{r-1}) = CT_{r-2}\left([B_{r-1}, [B_{r}, [F_X^+, B_{\tau(r)}]_q]]_q + q\epsilon_{r(r)}B_{r-1}K_{\tau(r)}^{-1}K_X^{-1}\right)
\]
\[
= q^{-1/2}C\left(\left([B_{r-1}, [B_{r}, [F_X^+, B_{\tau(r)}]_q]]_q + q\epsilon_{r(r)}B_{r-1}K_{\tau(r)}^{-1}K_X^{-1}\right)
\]
\[
= q^{-1/2}C\left(\left([B_{r-1}, [B_{r}, [F_X^+, B_{\tau(r)}]_q]]_q + q\epsilon_{r(r)}B_{r-1}K_{\tau(r)}^{-1}K_X^{-1}\right)
\]
\[
= q^{-1/2}C\left(\left(\left([B_{r-1}, [B_{r}, [F_X^+, B_{\tau(r)}]_q]]_q - 1\right)\right)\right)
\]
\[
= T_rT_{r-2}(B_{r-1})
\]
as required.

**Case 6.** \(a_{ij} = 2\)

Then we have \(T_r(B_i) = B_i\) and \(T_r(T_i(B_i)) = T_i(B_i)\) which implies the result in this case. This completes the proof. \(\square\)

We now check that the relation
\[
T_rT_{r-1}T_rT_{r-1}(B_j) = T_{r-1}T_rT_{r-1}T_r(B_j)
\]
holds for all \(j \in I \setminus X\). Again for symmetry reasons it is enough to only consider \(1 \leq j \leq r\). Many of the remaining proofs in this section require the use of relations that are proven in Appendix \(A\). Since \(T_r(B_j) = B_j\) and \(T_{r-1}(B_j) = B_j\) for \(1 \leq j < r - 2\) the following lemma is immediate.

**Lemma 3.30.** For \(1 \leq j < r - 2\) the relation \(3.69\) holds.

As a result of the above lemma, it remains to verify Equation \(3.69\) for \(j \in \{r - 2, r - 1, r\}\). For the next result, we use the relation
\[
T_{r-1}T_r(B_{r-1}) = T_{r-1}(B_{(r-1)})
\]
which appears in the proof of Lemma \(A.3\).

**Proposition 3.31.** For \(j = r - 1\) the relation \(3.69\) holds.

**Proof.** Using Equation \(3.70\) we have
\[
T_rT_{r-1}T_rT_{r-1}(B_{r-1}) = T_{r-1}T_r\left(q^{-1}B_{\tau(r-1)}K_{\tau(r-1)}^{-1}\right)
\]
\[
= q^{-1}B_{r-1}K_{\tau(r-1)}^{-1}
\]
\[
= T_{r-1}(B_{(r-1)}).
\]
The result follows from Lemma \(A.3\). \(\square\)

**Proposition 3.32.** For \(j = r - 2\) the relation \(3.69\) holds.

**Proof.** On one hand we have
\[
T_rT_{r-1}T_rT_{r-1}(B_{r-2}) = q^{-1/2}T_rT_{r-1}T_r([B_{r-2}, B_{r-1}]_q)
\]
\[
= q^{-1/2}T_rT_{r-1}([B_{r-2}, T_r(B_{r-1})]_q)
\]
\[
= q^{-1}T_r([B_{r-2}, B_{r-1}]_q, T_rT_r(B_{r-1})]_q)
\]
Again by Equation (3.70) it follows that

$$T_r T_{r-1} T_{r-1} T_r (B_{r-2}) = q^{-1} [[B_{r-2}, T_r(B_{r-1})]_q, B_{r(r-1)}]_q$$

$$= q^{-1} [B_{r-2}, [T_r(B_{r-1}), B_{r(r-1)}]_q]_q$$

where the last equality follows from Equation (2.57) and noting that

Proof.

and hence we obtain

$$T_r T_{r-1} T_{r-1} T_r (B_{r-2}) = T_{r-1} T_r T_{r-1} (B_{r-2})$$

$$= q^{-1} [[B_{r-2}, B_{r-1}]_q, T_{r-1} (B_{r(r-1)})]_q.$$ 

Since $B_{r(r-1)}$ commutes with $T_r(B_{r-2})$ it follows that

$$B_{r-2} T_{r-1}^{-1} (B_{r(r-1)}) = T_r^{-1} (B_{r(r-1)}) B_{r-2}. \quad (3.71)$$

By Corollary A.5 the element $[B_{r-1}, T_r^{-1} (B_{r(r-1)})]_q$ is invariant under $T_r$. This and Equation (3.71) imply

$$T_{r-1} T_r T_{r-1} T_r (B_{r-2}) = q^{-1} [B_{r-2}, [B_{r-1}, T_r^{-1} (B_{r(r-1)})]_q]_q$$

$$= q^{-1} [B_{r-2}, [T_r(B_{r-1}), B_{r(r-1)}]_q]$$

$$= T_r T_{r-1} T_r T_{r-1} (B_{r-2})$$

as required. \qed

**Proposition 3.33.** For $j = r$ the relation (3.69) holds.

**Proof.** Consider first the term $T_{r-1} T_r T_{r-1} T_r (B_r)$. By Equations (3.70) and (3.59), Equation (3.71) and $T_i |_{M_X U_0^a} = T_i |_{M_X U_0^a}$ for $1 \leq i \leq r$ we obtain

$$T_{r-1} T_r T_{r-1} T_r (B_r)$$

$$= q^{-3} [[B_r, B_{r-1}]_q L_r L_{r-1} K_{\varphi'_r + 1}, T_r^{-1} (B_{r(r-1)})]_q L_r K_{\varphi'_r + 1}$$

$$= q^{-2} [[B_r, B_{r-1}]_q, T_r^{-1} (B_{r(r-1)})]_q L_r^2 L_{r-1} K^2_{\varphi'_r + 1},$$

where the second equality follows from Equation (2.57) and noting that $K_{\varphi'_r + 1}$ commutes with $T_r^{-1} (B_{r(r-1)})$. Since $[T_r(B_r), B_{r(r-1)}]_{q^{-1}} = 0$, it follows that $[B_r, T_r^{-1} (B_{r(r-1)})]_{q^{-1}} = 0$. Using this and Corollary A.5 it follows that

$$[[B_r, B_{r-1}]_q, T_r^{-1} (B_{r(r-1)})]_q = [B_r, [B_{r-1}, T_r^{-1} (B_{r(r-1)})]_q]$$

$$= [B_r, [T_r(B_{r-1}), B_{r(r-1)}]_q]$$

and hence we obtain

$$T_{r-1} T_r T_{r-1} T_r (B_r) = q^{-2} [B_r, [T_r(B_{r-1}), B_{r(r-1)}]_q] L_r^2 L_{r-1} K^2_{\varphi'_r + 1}. $$
Considering now the term $T_r T_{r-1} T_r T_{r-1}(B_r)$ we obtain

\[
T_r T_{r-1} T_r T_{r-1}(B_r) \\
= q^{-1/2} T_r T_{r-1} T_r ([B_r, B_{r-1}]_q) \\
= q^{-3/2} T_r T_{r-1} ([B_r, L_r, K_{w_{r+1}}]_q, T_r(\tau_{B_{r-1}})]_q) \\
= q^{-2} T_r ([B_r, B_{r-1}]_q L_r L_{r-1} K_{w_{r+1}}, T_r(\tau_{B_{r-1}})]_q) \\
= q^{-2} [B_r, T_r(B_{r-1})]_q L_r L_{r-1} K_{w_{r+1}} \\
= q^{-2} [B_r, T_r(B_{r-1}), B_{r(r-1)}]_q L_r^2 L_{r-1} K_{w_{r+1}} \\
= T_r-1 T_r T_{r-1} T_r(B_r)
\]

as required. \(\square\)

\section*{Appendix A. Relations in \(B_e\)}

Many of the results in Sections 3.1 to 3.3 require the use of additional relations which we provide here. Recall from Equations (2.43) to (2.45) the elements $E_j^+, E_j^-, F_j^+, F_j^-$ and $K_j$ where $J \subseteq I$ is a subset of the form $J = \{1, a+1, \ldots, b-1, b\}$ with $a \leq b$. Rewriting these elements using the Lusztig automorphisms, one sees that

\[
E_j^+ F_j^- - F_j^+ E_j^- = \frac{K_j - K_j^{-1}}{q - q^{-1}} = E_j^- F_j^+ - F_j^+ E_j^- \tag{A.1}
\]

holds in $U_q(\mathfrak{g})$. Additionally, the $q$-commutator satisfies

\[
[[x, y], z]_q - [x, [y, z]]_q = q[[x, z], y] \tag{A.2}
\]

for all $x, y, z \in U_q(\mathfrak{g})$.

Recall from Theorem 3.2 that for $1 \leq i \leq r - 1$ and $k \in I \setminus X$ the elements $T_i(B_k)$ define algebra automorphisms of $B_e$, denoted by $T_i$.

\begin{lemma}
The relation

\[
[T_{r-1}(B_{r-1}), [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]]_q] = 0 \tag{A.3}
\]

holds in $B_e$.
\end{lemma}

\textbf{Proof.} Since $T_{r-1}(B_{r-1}) = q^{-1}B_{\tau(r-1)}L_{\tau(r-1)}$ it follows that $T_{r-1}(B_{r-1})$ commutes with $F_j$ for $j \in X$. Further, Equation (2.60) implies that

\[
B_{\tau(r-1)}[B_{\tau(r)}, B_{\tau(r-1)}]_q = q[B_{\tau(r)}, B_{\tau(r-1)}]_q B_{\tau(r-1)}
\]

and hence $T_{r-1}(B_{r-1})$ commutes with $[B_{\tau(r)}, B_{\tau(r-1)}]_q$. The result follows from this. \(\square\)

\begin{lemma}
For any $i \in I \setminus \{r, \tau(r)\}$ the relations

\[
[B_{\tau(i)} L_{\tau(i)}, [B_{i+1}, B_i]]_q = q^2 \epsilon_i B_{i+1}, \tag{A.4}
\]

\[
[[B_i, [B_{i+1}, B_i]], B_{\tau(i)}]_q = \epsilon_i B_{i+1} \tag{A.5}
\]

hold in $B_e$.
\end{lemma}
Lemma A.3. The relation
\[ T_{r-1}T_rT_{r-1}(B_{r-1}) = T_{r-1}(B_{\tau(r-1)}) \]  
(A.6)
holds in \( B_e \).

Proof. Calculating directly we have
\[
T_{r-1}T_r(B_{r-1}) = C\left( [T_{r-1}(B_{r-1}), [T_r(B_{r}), [F_X^+, T_{r-1}(B_{\tau(r)})]_q]_q, \right. \\
+ q\epsilon_rT_{r-1}(B_{r-1})L_rK_X^{-1}) \\
= C\left( q^{-2}\epsilon_r^{-1}[B_{\tau(r-1)}L_{\tau(r-1)}, [B_{r}, B_{r-1}]_q, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q, \right.
\]
\[
+ \epsilon_rB_{\tau(r-1)}L_{\tau(r-1)}K_X^{-1}) \\
= T_{r-1}(B_{\tau(r-1)}),
\]
where the last equality is obtained using Equations (A.3) and (A.4). This implies that
\[ T_{r-1}T_rT_{r-1}(B_{r-1}) = T_{r-1}(B_{\tau(r-1)}) \]
as required.

Lemma A.4. The relation
\[ [B_{r-1}, [B_r, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q]_q = [[B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q, B_{\tau(r-1)}]]_q \]  
(A.7)
holds in \( B_e \).

Proof. First observe that since \( B_{\tau(r-1)} \) commutes with \( B_r \) and \( F_X^+ \) we have
\[
[B_r, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q = [[B_r, [F_X^+, B_{\tau(r)}]_q]_q, B_{\tau(r-1)}]]_q.
\]
To shorten notation, let \( Y = [B_r, [F_X^+, B_{\tau(r)}]_q]_q \). Recall from Equation (2.59) that
\[ B_{\tau-1}B_{\tau(r-1)} - B_{\tau(r-1)}B_{\tau-1} = (q - q^{-1})^{-1}(\epsilon_{\tau-1}Z_{\tau-1} - \epsilon_{\tau(r-1)}Z_{\tau(r-1)}) \]
where \( Z_{\tau-1} = -L_{\tau-1} \) and \( Z_{\tau(r-1)} = -L_{\tau-1} \). Then \( Y \) commutes with both \( Z_{\tau-1} \) and \( Z_{\tau(r-1)} \). Recalling the notation \( \Gamma_i \) from Equation (2.56) we hence have
\[
[B_{\tau-1}, Y, B_{\tau(r-1)}]_q]_q = B_{\tau-1}YB_{\tau(r-1)} - qB_{\tau-1}B_{\tau(r-1)}Y - qYB_{\tau(r-1)}B_{\tau-1} + q^2B_{\tau(r-1)}B_{\tau-1}YB_{\tau-1} \\
= B_{\tau-1}YB_{\tau(r-1)} - q(B_{\tau(r-1)}B_{\tau-1} + (q - q^{-1})^{-1}\Gamma_{\tau-1}Y) \\
- qY(B_{\tau-1}B_{\tau(r-1)} - (q - q^{-1})^{-1}\Gamma_{\tau-1}) + q^2B_{\tau(r-1)}YB_{\tau-1} \\
= [[B_{\tau-1}, Y]_q, B_{\tau(r-1)}]_q
\]
as required.
Corollary A.5. The element $[B_{r-1}, T_r^{-1}(B_{r(r-1)})]_q$ is $T_r$-invariant i.e.

$$[B_{r-1}, T_r^{-1}(B_{r(r-1)})]_q = [T_r(B_{r-1}), B_{r(r-1)}]_q. \quad \text{(A.8)}$$

Proof. The result follows immediately from Lemma A.4 and the fact that

$$[B_{r-1}, B_{r(r-1)}L_rK_X^{-1}]_q = q[B_{r-1}L_rK_X^{-1}, B_{r(r-1)}]_q.
\square
$$

Lemma A.6. For any $j \in X \setminus \{r+1, \tau(r+1)\}$ the relation

$$T_j(F_X^+) = F_X^+ \quad \text{(A.9)}$$

holds.

Proof. Since $j \in X \setminus \{r+1, \tau(r+1)\}$ we assume that $|X| \geq 3$. Recalling that

$$F_X^+ = [F_{r+1}, [F_{r+2}, \ldots, [F_{r(r+2)}, F_{r(r+1)}]_q \ldots]_q
$$

the result follows since for any $j \in X \setminus \{r+1, \tau(r+1)\}$ we have

$$T_j([F_{j-1}, [F_j, F_{j+1}]_q]_q) = T_j([F_{j-1}, T_j^{-1}(F_{j+1})]_q)
= [[F_{j-1}, F_j]_q, F_{j+1}]_q
= [F_{j-1}, [F_j, F_{j+1}]_q]_q.
\square
$$

Lemma A.7. The relation

$$T_{r+1}([B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q]_q = [B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q]_q
+ q\epsilon_{\tau(r)}B_{r-1}L_r(K_{r+1}^{-1} - K_{r+1})K_{X\setminus\{r+1\}}^{-1}\quad \text{(A.10)}$$

holds in $B_e$.

Proof. First suppose that $X = \{r+1\}$. By Equation (3.1) we have $[F_{r+1}, B_{r+2}]_q = T_{r+1}^{-1}(B_{r+2})$ and $T_{r+1}(B_r) = [B_r, F_{r+1}]_q$. Hence

$$T_{r+1}([B_{r-1}, [B_r, [F_{r+1}, B_{r+2}]_q]_q]_q = [B_{r-1}, [B_r, [F_{r+1}, B_{r+2}]_q]_q]_q.
\text{(A.11)}$$

By Equation (A.2) we have

$$[[B_r, F_{r+1}]_q, B_{r+2}]_q = [B_r, [F_{r+1}, B_{r+2}]_q]_q + q[[B_r, B_{r+2}], F_{r+1}]
= [B_r, [F_{r+1}, B_{r+2}]_q]_q + \frac{q}{q^{-1}}[\epsilon_r Z_r - \epsilon_{r+2}Z_{r+2}, F_{r+1}].$$

Since $[B_{r-1}, B_{r+2}]_q = 0$ it follows that $[B_{r-1}, [Z_r, F_{r+1}]_q]_q = 0$. On the other hand we have

$$[Z_{r+2}, F_{r+1}] = -(1 - q^{-2})[E_{r+1}L_r, F_{r+1}]
= -(1 - q^{-2})L_r[E_{r+1}, F_{r+1}]
= -q^{-1}L_r(K_{r+1}^{-1} - K_{r+1}^{-1})$$

and hence

$$[B_{r-1}, [Z_{r+2}, F_{r+1}]_q]_q = -q^{-1}[B_{r-1}, L_r]_q(K_{r+1}^{-1} - K_{r+1}^{-1})
= (q - q^{-1})B_{r-1}L_r(K_{r+1}^{-1} - K_{r+1}^{-1}).$$
It follows that
\[ [B_{r-1}, [B_r, F_{r+1}]_q, B_{r+2}]_q = [B_{r-1}, [B_r, [F_{r+1}, B_{r+2}]_q]_q + qe_{r+2}B_{r-1}L_r(K_{r+1}^{-1} - K_{r+1}). \]

The result follows by substituting this into (A.11).

We now consider the case \(|X| > 1\). Let \(Y = X \setminus \{r + 1\}\). Observing that
\[ [F_X^+, B_{\tau(r)}]_q = T_{\tau(r)}^{-1}([F_X^+, B_{\tau(r)}]_q) \]
we have
\[ T_{\tau+1}([B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q] = [B_{r-1}, [B_r, F_{r+1}]_q, [F_Y^+, B_{\tau(r)}]_q]_q. \]  
(A.12)

Since \(F_Y^+\) commutes with \(B_r\), Equation (A.2) implies
\[ \left[[B_r, F_{r+1}]_q, [F_Y^+, B_{\tau(r)}]_q - [B_r, [F_X^+, B_{\tau(r)}]_q]_q \right] \]
\[ = \frac{q}{q - q^{-1}}[[F_Y^+, \epsilon_r Z_r - \epsilon_{r(r)} Z_{\tau(r)}]_q, F_{r+1}]. \]

Since \([B_{r-1}, L_{\tau(r)}]_q = 0\) it follows that
\[ [B_{r-1}, [F_Y^+, Z_r]_q, F_{r+1}]_q = 0. \]

On the other hand, using Equation (A.1) it follows that
\[ [F_Y^+, E_X^-] = [[F_Y^+, E_X^-], E_{r+1}]_q^{-1} \]
\[ = \frac{1}{q - q^{-1}}[K_Y^{-1} - K_Y, E_{r+1}]_q^{-1} \]
\[ = q^{-1} K_Y^{-1} E_{r+1}. \]

This implies that
\[ [[F_{r+1}^+, Z_{\tau(r)}]_q, F_{r+1}] = -(q - q^{-1})[[F_Y^+, E_X^-], F_{r+1}]_q^{-1} L_r \]
\[ = -(1 - q^{-2})[K_{r+1}^{-1} E_{r+1}, F_{r+1}]_q^{-1} L_r \]
\[ = q^{-1} L_r(K_{r+1}^{-1} - K_{r+1}) K_Y^{-1}. \]

As a result we obtain
\[ [B_{r-1}, [B_r, F_{r+1}]_q, [F_{r+1}, B_{\tau(r)}]_q]_q - [B_{r-1}, [B_r, [F_{r+1}, B_{\tau(r)}]_q]_q] \]
\[ = q(q - q^{-1})^{-1} \epsilon_{r(r)} [B_{r-1}, [[F_{r+1}, Z_{\tau(r)}]_q, F_{r+1}]_q] \]
\[ = (q - q^{-1})^{-1} \epsilon_{r(r)} [B_{r-1}, L_r(K_{r+1}^{-1} - K_{r+1}) K_Y^{-1}]_q \]
\[ = qe_{r(r)} B_{r-1} L_r(K_{r+1}^{-1} - K_{r+1}) K_Y^{-1}. \]

By substituting this into (A.12), we obtain the required result. \(\square\)

**Lemma A.8.** The relations
\[ [B_{r-1}, [F_{r+1}^+, Z_r]]_q = 0, \]  
(A.13)
\[ [B_{r-1}, [F_{r+1}^+, Z_{\tau(r)}]]_q = -(q - q^{-1})B_{r-1} L_r(K_X - K_X^{-1}) \]  
(A.14)
hold in \(B_c\).

**Proof.** Since \([B_{r-1}, F_{r+1}^+]_q = 0\) and \([B_{r-1}, Z_r]_q = 0\) it follows that Equation (A.13) holds. On the other hand, making use of the relation
\[ [F_X^+, Z_{\tau(r)}]_q = q^{-1}(K_X - K_X^{-1})L_r, \]  
(A.15)
which follows from (A.1), we obtain
\[
[B_{r-1}, [F^+_X, Z_{\tau(r)}]]_q = q^{-1}[B_{r-1}, (K_X - K_X^{-1})L_r]_q \\
= -(q - q^{-1})B_{r-1}(K_X - K_X^{-1})L_r
\]
as required.

Lemma A.9. The relations
\[
[B_{r-1}, [B_r, [F^+_X, B_{\tau(r)}]_q]]_q = [[B_{r-1}, [B_r, F^+_X]_q]_q, B_{\tau(r)}]_q \\
+ q\epsilon_{\tau(r)}B_{r-1}L_r(K_X - K_X^{-1}), \quad (A.16)
\]
\[
[B_{\tau(r-1)}, [B_{\tau(r)}, [F^-_X, B_r]_q]]_q = [[B_{\tau(r-1)}, [B_{\tau(r)}, F^-_X]_q]_q, B_r]_q \\
+ q\epsilon_rB_{\tau(r-1)}L_{\tau(r)}(K_X - K_X^{-1}) \quad (A.17)
\]
hold in \(B_c\).

Proof. By symmetry we only verify Equation (A.16). By Equation (A.2) we have
\[
[B_r, [F^+_X, B_{\tau(r)}]_q]_q = [[B_r, F^+_X]_q, B_{\tau(r)}]_q + q[F^+_X, [B_r, B_{\tau(r)}]]_q \\
= [[B_r, F^+_X]_q, B_{\tau(r)}]_q - \frac{q\epsilon_{\tau(r)}}{q - q^{-1}}[F^+_X, Z_{\tau(r)}]_q + \frac{q\epsilon_r}{q - q^{-1}}[F^+_X, Z_r].
\]
Since \(B_{r-1}\) commutes with \(B_{\tau(r)}\), Lemma A.8 implies
\[
[B_{r-1}, [B_r, [F^+_X, B_{\tau(r)}]_q]]_q \\
= [B_{r-1}, [[B_r, F^+_X]_q, B_{\tau(r)}]_q] - \frac{q\epsilon_{\tau(r)}}{q - q^{-1}}[B_{r-1}, [F^+_X, Z_{\tau(r)}]]_q \\
= [[B_{r-1}, [B_r, F^+_X]_q]_q, B_{\tau(r)}]_q + q\epsilon_{\tau(r)}B_{r-1}L_r(K_X - K_X^{-1})
\]
as required.

Recall from Section 3.2 the elements
\[
S = [B_{r-1}, [B_r, F^+_X]_q]_q, \\
S^* = [B_{\tau(r-1)}, [B_{\tau(r)}, F^+_X]_q]_q, \\
\Delta = q\epsilon_{\tau(r)}B_{r-1}L_rK_X, \\
\Delta^* = q\epsilon_rB_{\tau(r-1)}L_{\tau(r)}K_X.
\]
and
\[
T = [F^-_X, [B_r, B_{r-1}]_q]_q, \\
T^* = [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q, \\
\Lambda = \epsilon_rB_{r-1}L_{\tau(r)}K_X^{-1}, \\
\Lambda^* = \epsilon_{\tau(r)}B_{\tau(r-1)}L_rK_X^{-1}.
\]
For the remainder of this section, we provide relations that include the terms \(S, S^*, T\) and \(T^*\). These are required throughout Section 3.2.
Lemma A.10. The relations
\begin{align}
Z_{\tau(r)}S &= q^{-1}S Z_{\tau(r)} + (1 - q^{-2})[B_{r-1}, B_r]_q L_r K_X, & (A.18) \\
Z_r S^r &= q^{-1}S^r Z_r + (1 - q^{-2})[B_{\tau(r)}, B_{\tau(r)}]_q L_{\tau(r)} K_X, & (A.19) \\
Z_r T &= qT Z_r - (q - q^{-1})[B_r, B_{\tau(r)-1}]_q L_{\tau(r)} K_X^{-1}, & (A.20) \\
Z_{\tau(r)} T^r &= qT^r Z_{\tau(r)} - (q - q^{-1})[B_{\tau(r)}, B_{\tau(r)-1}]_q L_r K_X^{-1} & (A.21)
\end{align}
hold in $B_c$.

Proof. We only prove that Equation (A.18) holds since the remaining checks are similar. By Equation (A.1) and the relation $Z_{\tau(r)} B_i = q^{(\alpha_i, \alpha_{\tau(r)} - \alpha_r)} B_i Z_{\tau(r)}$ for any $i \in I \setminus X$ we have
\begin{align}
Z_{\tau(r)} S &= Z_{\tau(r)} [B_{r-1}, [B_r, F_X^+]]_q \\
&= q^{-1}[B_{r-1}, [B_r, Z_{\tau(r)} F_X^+]]_q \\
&= q^{-1}[B_{r-1}, [B_r, F_X^+ Z_{\tau(r)} - q^{-1}(K_X - K_X^{-1}) L_r]]_q \\
&= q^{-1}S Z_{\tau(r)} - q^{-2}[B_{r-1}, [B_r, (K_X - K_X^{-1}) L_r]]_q \\
&= q^{-1}S Z_{\tau(r)} + [B_{r-1}, B_r L_r K_X] \\
&= q^{-1}S Z_{\tau(r)} + (1 - q^{-2})[B_{r-1}, B_r]_q L_r K_X
\end{align}
as required.

Lemma A.11. The relations
\begin{align}
B_r S &= SB_r, & (A.22) \\
B_{\tau(r)} S^r &= S^r B_{\tau(r)}, & (A.23) \\
B_r T &= T B_r, & (A.24) \\
B_{\tau(r)} T^r &= T^r B_{\tau(r)} & (A.25)
\end{align}
hold in $B_c$.

Proof. By symmetry we only verify Equation (A.22). Using the relations 
$p(B_r, B_{r-1}) = p(B_r, F_X^+) = 0$
we obtain
\begin{align}
B_r S &= B_r [[B_{r-1}, B_r]_q, F_X^+]_q \\
&= B_r B_{r-1} B_r F_X^+ - q B_r^2 B_{r-1} F_X^+ - q B_r F_X^+ B_{r-1} B_r + q^2 B_r F_X^+ B_r B_{r-1} \\
&= \frac{1}{q + q} (B_r^2 B_{r-1} + B_{r-1} B_r^2) F_X^+ - q B_r^2 B_{r-1} F_X^+ - q B_r F_X^+ B_r B_{r-1} \\
&+ \frac{q^2}{q + q} (B_r^2 F_X^+ + F_X^+ B_r^2) B_{r-1} \\
&= \frac{1}{q + q} B_r B_{r-1} B_r F_X^+ - q B_r F_X^+ B_{r-1} B_r + \frac{q^2}{q + q} F_X^+ B_r B_{r-1} \\
&= \frac{1}{q + q} B_r (q + q^{-1}) B_r F_X^+ B_r - F_X^+ B_r^2 - q B_r F_X^+ B_{r-1} B_r \\
&+ \frac{q^2}{q + q} F_X^+ (q + q^{-1}) B_r B_{r-1} B_r - B_r B_{r-1} B_r^2 \\
&= [[B_{r-1}, B_r]_q, F_X^+]_q B_r \\
&= SB_r
\end{align}
as required.

\[ \square \]

**Lemma A.12.** The relations

\[
SB_{\tau(r-1)} = B_{\tau(r-1)}S - q\epsilon_{r-1}Z_{\tau-1}[B_r, F_X^+]q, \tag{A.26}
\]

\[
S^* B_{\tau-1} = B_{\tau-1}S^* - q\epsilon_{r(r-1)}Z_{\tau(r-1)}[B_{\tau(r)}, F_X^+]q, \tag{A.27}
\]

\[
TB_{\tau(r-1)} = B_{\tau(r-1)}T + \epsilon_{r-1}Z_{\tau-1}[F_X^-, B_r]q, \tag{A.28}
\]

\[
T^* B_{\tau-1} = B_{\tau-1}T^* + \epsilon_{r-1}Z_{\tau-1}[F_X^+, B_{\tau(r)}]q \tag{A.29}
\]

hold in \( B_c \).

\[ \square \]

**Proof.** We have

\[
SB_{\tau(r-1)} = [B_{\tau-1}, [B_r, F_X^+]q]_q B_{\tau(r-1)}
\]

\[
= [B_{\tau-1}B_{\tau(r-1)}, [B_r, F_X^+]q]
\]

\[
= B_{\tau(r-1)}S + \frac{1}{q-q^{-1}}[[T_{\tau-1}, B_r]q, F_X^+]q
\]

\[
= B_{\tau(r-1)}S - q\epsilon_{r-1}Z_{\tau-1}[B_r, F_X^+]q
\]

as required. Equation \( \text{(A.27)} \) is verified similarly.

In the following lemma, we introduce the terms

\[
\Omega^- = \epsilon_{\tau(r)}\epsilon_{r-1}F_X^+KXL_{\tau(r)}Z_{\tau(r-1)}, \tag{A.30}
\]

\[
\Omega^+ = \epsilon_{r-1}F_X^+KXL_{\tau(r)}Z_{\tau(r-1)}, \tag{A.31}
\]

\[
\nabla^- = \epsilon_{\tau(r)}\epsilon_{r-1}K_X^{-1}F_X^-L_{\tau(r)}Z_{\tau(r-1)}, \tag{A.32}
\]

\[
\nabla^+ = \epsilon_{r-1}K_X^{-1}F_X^+L_{\tau(r)}Z_{\tau(r-1)}. \tag{A.33}
\]

**Lemma A.13.** The relations

\[
SS^* - S^*S = \Omega^- - \Omega^+, \tag{A.34}
\]

\[
TT^* - T^*T = \nabla^- - \nabla^+ \tag{A.35}
\]

hold in \( B_c \).

\[ \square \]

**Proof.** Recall from Theorem \( \text{3.32} \) that \( T_{\tau-1} \) is an algebra automorphism of \( B_c \) with inverse \( T_{\tau-1}^{-1} \) given by Equation \( \text{(3.8)} \). We express \( S \) and \( S^* \) using the algebra automorphisms \( T_{wx} \) and \( T_{\tau-1}^{-1} \). In particular using Equations \( \text{(3.2)} \), \( \text{(3.3)} \) and \( \text{(3.8)} \) we have

\[
S = (q\epsilon_{r-1})^{1/2}T_{\tau-1}^{-1} \circ T_{wx}(B_r),
\]

\[
S^* = (q\epsilon_{r-1})^{1/2}T_{\tau-1}^{-1} \circ T_{wx}(B_{\tau(r)}).
\]

Since \( \epsilon_{\tau-1} = \epsilon_{r(r-1)} \) we obtain

\[
SS^* - S^*S = q\epsilon_{r-1}T_{\tau-1}^{-1} \circ T_{wx}(B_r B_{\tau(r)} - B_{\tau(r)}B_r)
\]

\[
= \frac{q}{q - q^{-1}}T_{\tau-1}^{-1} \circ T_{wx}(\epsilon_{\tau-1}\epsilon_{\tau(r)}Z_r - \epsilon_{r(r-1)}Z_{\tau(r)}).
\]

Equation \( \text{(2.50)} \) implies that

\[
T_{wx}(Z_\tau) = (1 - q^{-2})F_X^+K_XL_{\tau(r)},
\]

\[
T_{wx}(Z_{\tau(r)}) = (1 - q^{-2})F_X^+K_XL_{\tau(r)}.
\]
Hence we obtain
\[
SS^r - S^r S = T_{r-1}^{-1}(\epsilon_r\epsilon_{r-1}F^+_{r}K_XL_T - \epsilon_{(r-1)}F^+_{r}K_XL_{r-1}) \\
= \epsilon_{(r)}\epsilon_{(r-1)}F^+_{r}K_XL_T - \epsilon_{(r-1)}\epsilon_{r}F^+_{r}K_XL_{r-1} \\
= \Omega^- - \Omega^+ 
\]
as required. Equation (A.35) holds similarly, instead using the algebra automorphisms \(T_{w_{X}}^{-1}\) and \(T_{r-1}\).

**Lemma A.14.** The relation
\[
[[S, B_{r}]_{q}, [S^r, B_{r}]_{q}] = -[\Delta, [S^r, B_{r}]_{q}] - [[S, B_{r}]_{q}, \Delta^r] \\
- \frac{q\epsilon_r\epsilon_{r-1}}{q - q^{-1}}(K_X - K_X^{-1})K_X\Gamma_{r-1} 
\]
holds in \(B_c\).

**Proof.** We first use Equations (A.22), (A.23) and (A.34) to rewrite each term of \([S, B_{r}]_{q}, [S^r, B_{r}]_{q}\). In particular we have
\[
SB_{r}S^r B_{r} = SS^r B_{r}S^r B_{r} \\
= (S^r + \Omega^- - \Omega^+)(B_{r}B_{r} - \frac{1}{q - q^{-1}}\Gamma_{r}) \\
= S^rSB_{r}B_{r} - \frac{1}{q - q^{-1}}S^r\Gamma_{r} + (\Omega^- - \Omega^+)B_{r}B_{r} \\
- \frac{1}{q - q^{-1}}(\Omega^- - \Omega^+)(\Gamma_{r} - S_{r}^r).
\]

Similarly, one finds that
\[
SB_{r}B_{r}S^r = B_{r}S^r SB_{r} + B_{r}(\Omega^- - \Omega^+)B_{r} - \frac{1}{q - q^{-1}}S^r\Gamma_{r}S^r, \\
B_{r}S^r B_{r}S = S^r B_{r}B_{r}S + B_{r}(\Omega^- - \Omega^+)B_{r} - \frac{1}{q - q^{-1}}S^r\Gamma_{r}S^r, \\
B_{r}S^r B_{r}S = B_{r}S^r B_{r}S + B_{r}(\Omega^- - \Omega^+)B_{r} - \frac{1}{q - q^{-1}}\Gamma_{r}SS^r \\
+ \frac{1}{q - q^{-1}}(\Omega^- - \Omega^+).
\]

Combining these four expressions we obtain
\[
[[S, B_{r}]_{q}, [S^r, B_{r}]_{q}] = \frac{-1}{q - q^{-1}}[[S, S^r\Gamma_{r} S], [S, S^r\Gamma_{r} S]] - \frac{q\epsilon_r\epsilon_{r-1}}{q - q^{-1}}[[\Omega^- - \Omega^+, B_{r}], [\Omega^- - \Omega^+, \Gamma_{r}]] \tag{A.37}
\]

We now consider the term \([S^r, [S, \Gamma_{r}]]_{q}\) in more detail. Using Equations (A.18) and (A.19) we have
\[
S^r\Gamma_{r} S = \epsilon_rS^r\Gamma_{r}S - \epsilon_{r}(S^r\Gamma_{r}S)^r S \\
= \epsilon_r(q\Gamma_{r}S - (q - q^{-1})[B_{r-1}, \Gamma_{r}]_{q}K_XL_T)S \\
- \epsilon_r(S^r(q^{-1}S\Gamma_{r}S + (1 - q^{-2})[B_{r-1}, \Gamma_{r}]_{q}K_XL_T)) \\
= q\epsilon_r\Gamma_{r}S - q^{-1}\epsilon_{r}(S^r\Gamma_{r}S)S - (q - q^{-1})[\Delta^r, B_{r}]_{q}S \\
- (1 - q^{-2})S^r[\Delta, B_{r}].
\]

We similarly obtain
\[
S^r\Gamma_{r} S = q^{-1}\epsilon_rS^r\Gamma_{r}S - q\epsilon_{r}(S^r\Gamma_{r}S)S + (1 - q^{-2})S[\Delta^r, B_{r}]_{q} \\
+ (q - q^{-1})[\Delta, B_{r}]S^r.
\]
It hence follows from this and Equation (A.34) that
\[ [S^r, [S, \Gamma_r]_q]_q = (q - q^{-1})[S^r, [\Delta, B_r]_q] - (q - q^{-1})[S^r, [\Delta^r, B_{r(r)}]_q] \]
\[ - c_r(\Omega^- - \Omega^+)Z_r + q^2 c_r Z_r(\Omega^- - \Omega^+). \]  \hspace{1cm} (A.38)

We now consider the elements \([S^r, [\Delta, B_r]_q]\) and \([S, [\Delta^r, B_{r(r)}]_q]\) and write them in the form that appears in Equation (A.30). By Equation (A.27) it follows that
\[ [S^r, [\Delta, B_r]_q] = S^r \Delta B_r - S^r B_r \Delta - q \Delta B_r S^r + q B_r \Delta S^r \]
\[ = (\Delta S^r + [\Omega^-, B_{r(r)}]_q) B_r - S^r B_r \Delta - q \Delta B_r S^r \]
\[ + q B_r (S^r \Delta - [\Omega^-, B_{r(r)}]_q) \]
\[ = [\Delta, [S^r, B_r]_q] + [[\Omega^-, B_{r(r)}]_q, B_r]_q. \]

Similarly, by Equation (A.26) we have
\[ [S, [\Delta^r, B_{r(r)}]_q] = -[[S, B_{r(r)}]_q, \Delta^r] + [[\Omega^+, B_r]_q, B_{r(r)}]_q. \]

Substituting these two expressions into Equation (A.38) gives
\[ [S^r, [S, \Gamma_r]_q]_q = (q - q^{-1})[\Delta, [S^r, B_r]_q] + (q - q^{-1})[[S, B_{r(r)}]_q, \Delta^r] \]
\[ + (q - q^{-1})[[\Omega^-, B_{r(r)}]_q, B_r]_q - (q - q^{-1})[[\Omega^+, B_r]_q, B_{r(r)}]_q \]
\[ - \epsilon_r(\Omega^- - \Omega^+)Z_r + q^2 \epsilon_r Z_r(\Omega^- - \Omega^+). \]  \hspace{1cm} (A.39)

We substitute Equation (A.39) into Equation (A.37). Noting that
\[ [[\Omega^-, B_r]_q, B_{r(r)}]_q - [[\Omega^-, B_{r(r)}]_q, B_r]_q = \frac{1}{q - q^{-1}}[[\Omega^-, \Gamma_r]_q q^2, \]
we obtain
\[ [[S, B_{r(r)}]_q, [S^r, B_r]_q] = -[\Delta, [S^r, B_r]_q] - [[S, B_{r(r)}]_q, \Delta^r] \]
\[ + \frac{1}{q - q^{-1}}(c_r[\Omega^-, Z_r]_q q^2 - c_{r(r)}[\Omega^+, Z_{r(r)}]_q q^2). \]

Using Equation (A.1) we compute \([\Omega^-, Z_r]_q q^2\) and \([\Omega^+, Z_{r(r)}]_q q^2\). This gives
\[ [\Omega^-, Z_r]_q q^2 = q \epsilon_{r(r)} \epsilon_{r(r-1)} (K_X - K_X^{-1}) K_X Z_{r(r-1)}, \]
\[ [\Omega^+, Z_{r(r)}]_q q^2 = q \epsilon_r \epsilon_{r-1} (K_X - K_X^{-1}) K_X Z_{r-1}. \]

It hence follows that
\[ [[S, B_{r(r)}]_q, [S^r, B_r]_q] = -[\Delta, [S^r, B_r]_q] - [[S, B_{r(r)}]_q, \Delta^r] \]
\[ - \frac{q \epsilon_{r(r)} \epsilon_{r(r-1)}}{q - q^{-1}} (K_X - K_X^{-1}) K_X \Gamma_{r-1} \]
as required.

\[ \square \]

**Lemma A.15.** The relation

\[ [[B_{r(r)}, T]_q, [B_r, T^r]_q] = -[[\Delta, [B_r, T^r]_q] - [[B_{r(r)}, T]_q, \Lambda^r] \]
\[ + \frac{q \epsilon_{r(r)} \epsilon_{r(r-1)}}{q - q^{-1}} (K_X - K_X^{-1}) K_X^{-1} \Gamma_{r-1} \]  \hspace{1cm} (A.40)

holds in \(B_c\).
Proof. The proof proceeds similarly to the proof of Lemma A.14. Using Equations (A.24), (A.25) and (A.35) we have

\begin{align*}
B_{\tau(r)} T B_r T^r &= B_{\tau(r)} B_r T T^r \\
&= (B_r B_{\tau(r)} - \frac{1}{q-q^{-1}} \Gamma_r) (T^r T + \nabla^- - \nabla^+) \\
&= B_r T^r B_{\tau(r)} T + B_r B_{\tau(r)} (\nabla^- - \nabla^+) - \frac{1}{q-q^{-1}} \Gamma_r T^r T \\
&\quad - \frac{1}{q-q^{-1}} \Gamma_r (\nabla^- - \nabla^+) \\
&= B_r T^r B_{\tau(r)} T + B_r (\nabla^- - \nabla^+) - \frac{1}{q-q^{-1}} \Gamma_r T T^r \\
&\quad + \frac{1}{q-q^{-1}} \Gamma_r (\nabla^- - \nabla^+).
\end{align*}

Similarly, we also have

\begin{align*}
B_{\tau(r)} T T^r B_r &= T^r B_r B_{\tau(r)} T - \frac{1}{q-q^{-1}} T^r \Gamma_r T + B_{\tau(r)} (\nabla^- - \nabla^+) B_r, \\
T B_{\tau(r)} B_r T^r &= B_r T^r T B_{\tau(r)} + B_r (\nabla^- - \nabla^+) B_{\tau(r)} - \frac{1}{q-q^{-1}} T T^r T, \\
T B_{\tau(r)} T^r B_r &= T^r B_r T B_{\tau(r)} - \frac{1}{q-q^{-1}} T^r T T^r + (\nabla^- - \nabla^+) B_r B_{\tau(r)} \\
&\quad - \frac{1}{q-q^{-1}} (\nabla^- - \nabla^+) \Gamma_r.
\end{align*}

Combining these four expressions, we obtain

\begin{align*}
[[B_{\tau(r)}, T], [B_r, T^r]]_q &= -\frac{1}{q-q^{-1}} [[\Gamma_r, T], T^r]_q + [B_{\tau(r)}, [B_r, \nabla^- - \nabla^+]]_q \\
&\quad + [[\Gamma_r, \nabla^- - \nabla^+], T^r]_q. \tag{A.41}
\end{align*}

We now consider the term \([[[\Gamma_r, T], T^r]]_q\) in more detail. Using Equations (A.20) and (A.21) we have

\begin{align*}
T T^r \Gamma_r &= \epsilon_r T Z_T, T^r - \epsilon_r T Z_{T^r} T^r \\
&= \epsilon_r (q^{-1} Z_T + (1 - q^{-2}) [B_r, B_{\tau(r-1)}] L_{\tau(r-1)} K_X^{-1}) T^r \\
&\quad - \epsilon_r (T (q T^r Z_{\tau(r)} - (q - q^{-1}) [B_{\tau(r)}, B_{\tau(r-1)}] L_r K_X^{-1}) \\
&= q^{-1} \epsilon_r Z_T T^r - q \epsilon_{\tau(r)} T T^r Z_{\tau(r)} + (1 - q^{-2}) [B_r, \Lambda] T^r \\
&\quad + (q - q^{-1}) T [B_{\tau(r)}, \Lambda^r].
\end{align*}

We similarly obtain

\begin{align*}
T^r \Gamma_r T &= q \epsilon_r T T^r Z_T - q^{-1} \epsilon_{\tau(r)} T^r T - (q - q^{-1}) T^r [B_r, \Lambda] - (1 - q^{-2}) [B_{\tau(r)}, \Lambda^r] T.
\end{align*}

It follows from this and Equation (A.35) that

\begin{align*}
[[\Gamma_r, T], T^r]_q &= -\epsilon_{\tau(r)} [Z_{\tau(r)}, \nabla^- - \nabla^+]_q - (q - q^{-1}) [[B_r, \Lambda], T^r]_q \\
&\quad + (q - q^{-1}) [[[B_{\tau(r)}, \Lambda^r], T]]_q. \tag{A.42}
\end{align*}

Consider the terms \([[[B_r, \Lambda], T^r]]_q\) and \([[B_{\tau(r)}, \Lambda^r], T]]_q\) appearing in (A.42). By Equations (A.28) and (A.29) we have

\begin{align*}
T^r \Lambda &= \Lambda T^r - [B_{\tau(r)}, \nabla^+]_q, \\
T \Lambda^r &= \Lambda^r T - [B_r, \nabla^-]_q.
\end{align*}
It hence follows that
\[ [[B_r, \Lambda], T^\tau]_q = B_r \Lambda T^\tau - \Lambda B_r T^\tau - q T^\tau B_r \Lambda + q T^\tau \Lambda B_r \]
\[ = B_r \left(T^\tau \Lambda + [B_{\tau(r)}, \nabla^+]_q\right) - \Lambda B_r T^\tau - q T^\tau B_r \Lambda \]
\[ + q \left(\Lambda T^\tau - [B_{\tau(r)}, \nabla^+]_q\right) B_r \]
\[ = - [\Lambda, [B_r, T^\tau]_q] + [B_r, [B_{\tau(r)}, \nabla^+]_q]_q. \]

Similarly we obtain
\[ [[B_{\tau(r)}, \Lambda^\tau], T]_q = [[B_{\tau(r)}, T]_q, \Lambda^\tau] + [B_{\tau(r)}, [B_r, \nabla^-]_q]_q. \]

Substituting these two expressions into Equation (A.42) gives
\[ [[\Gamma_r, T]_q, T^\tau]_q = (q - q^{-1}) [\Lambda, [B_r, T^\tau]_q] + (q - q^{-1}) [[B_{\tau(r)}, T]_q, \Lambda^\tau] \]
\[ - (q - q^{-1}) [B_r, [B_{\tau(r)}, \nabla^+]_q]_q + (q - q^{-1}) [B_{\tau(r)}, [B_r, \nabla^-]_q]_q \]
\[ - \epsilon_{\tau(r)} [Z_{\tau(r)}, \nabla^- - \nabla^+]_q. \quad \text{(A.43)} \]

We now return to Equation (A.41). Noting that
\[ [B_r, [B_{\tau(r)}, \nabla^+]_q] - [B_{\tau(r)}, [B_r, \nabla^+]_q] = [[B_r, B_{\tau(r)}], \nabla^+]_q \]
\[ = \frac{1}{q - q^{-1}} [\Gamma_r, \nabla^+]_q \]
and substituting Equation (A.43) into Equation (A.41) we obtain
\[ [[B_{\tau(r)}, T]_q, [B_r, T^\tau]_q] = - [\Lambda, [B_r, T^\tau]_q] - [[B_{\tau(r)}, T]_q, \Lambda^\tau] \]
\[ + \frac{1}{q - q^{-1}} (\epsilon_{\tau(r)} [Z_{\tau}, \nabla^-]_q - \epsilon_{\tau(r)} [Z_{\tau(r)}, \nabla^+]_q). \]

Computing directly we have
\[ [Z_{\tau}, \nabla^-]_q = -q \epsilon_{\tau(r)} \epsilon_{\tau(r-1)} (K_{\tau X} - K_{\tau X}^{-1}) K_{\tau X}^{-1} Z_{\tau(r-1)}, \]
\[ [Z_{\tau(r)}, \nabla^+]_q = -q \epsilon_{\tau(r)} (K_{\tau X} - K_{\tau X}^{-1}) K_{\tau X}^{-1} Z_{\tau(r-1)}. \]

It hence follows that
\[ [[B_{\tau(r)}, T]_q, [B_r, T^\tau]_q] = - [\Lambda, [B_r, T^\tau]_q] - [[B_{\tau(r)}, T]_q, \Lambda^\tau] \]
\[ + \frac{q \epsilon_{\tau(r)}}{q - q^{-1}} (K_{\tau X} - K_{\tau X}^{-1}) K_{\tau X}^{-1} \Gamma_{r-1} \]
as required.

\[ \square \]

**Lemma A.16.** The relation
\[ Z_r[S, B_{\tau(r)}]_q = q^{-1} [S, B_{\tau(r)}]_q Z_r + q^{-2} (q - q^{-1}) \Delta Z_r \quad \text{(A.44)} \]
holds in $B_e$.

**Proof.** The difficulty in the proof comes from that fact that the element $Z_r$ contains $E_{\tau X}^+$ as a factor, and there is generally no simple way to commute $E_{\tau X}^+$ with $F_{\tau X}$. The
idea is to verify that Equation \((A.44)\) holds if the algebra automorphism \(T_{wX}^{-1}\) is applied to both sides. More precisely, using \((2.49), (2.50), (3.2)\) and \((3.5)\) we have

\[
[S, B_{\tau(r)}]_q = T_{wX} \left( [B_{r-1}, B_{\tau(r)} [F^+_X, B_{\tau(r)}]_q]_q \right),
\]

\[
Z_r = q^2 (1 - q^{-2}) T_{wX} \left( F^+_X K^{-1}_X L_{\tau(r)} \right),
\]

\[
\Delta = q \epsilon_{\tau(r)} T_{wX} \left( B_{r-1} L_r K^{-1}_X \right)
\]

and hence verifying Equation \((A.44)\) is equivalent to showing that

\[
F^+_X [B_{r-1}, [B_r, [F^+_X, B_{\tau(r)}]_q]_q]_q = [B_{r-1}, [B_r, [F^+_X, B_{\tau(r)}]_q]_q]_q F^+_X
\]

+ \((q - q^{-1}) \epsilon_{\tau(r)} B_{r-1} L_r K^{-1}_X F^+_X \) \hspace{1cm} (A.45)

holds in \(B_c\). Using the relations

\[
p(F^+_X, B_r) = p(F^+_X, B_{\tau(r)}) = 0
\]

we can commute \(F^+_X\) through \([B_r, [F^+_X, B_{\tau(r)}]_q]_q\). This gives

\[
F^+_X [B_r, [F^+_X, B_{\tau(r)}]_q]_q = \frac{q}{q + q^{-1} + q^{-2}} F^+_X B_r, B_{\tau(r)} [F^+_X, B_{\tau(r)}]_q = q q^{-1} F^+_X B_r, B_{\tau(r)} F^+_X - q q^{-1} F^+_X B_r, B_{\tau(r)} F^+_X
\]

+ \(q q^{-1} F^+_X (F^+_X B_r + B_r F^+_X) B_{\tau(r)} = (1 + q^2) F^+_X B_{\tau(r)} F^+_X - q q^{-1} F^+_X B_r, B_{\tau(r)} F^+_X
\]

where the third equality follows from observing that terms beginning with \(F^+_X^2\) simplify. Since \(B_{r-1}\) commutes with \(F^+_X\), and

\[
[B_{r-1}, \Gamma_r]_q = (q^2 - 1) \epsilon_{\tau(r)} B_{r-1} Z_{\tau(r)}
\]

it follows that

\[
F^+_X \left[ B_{r-1}, [B_r, [F^+_X, B_{\tau(r)}]_q]_q \right] - [B_{r-1}, [B_r, [F^+_X, B_{\tau(r)}]_q]_q]_q F^+_X
\]

\[
= \frac{q}{q + q^{-1} + q^{-2}} \epsilon_{\tau(r)} B_{r-1} (F^+_X^2 Z_{\tau(r)} - (1 + q^2) F^+_X Z_{\tau(r)} F^+_X + q^2 Z_{\tau(r)} F^+_X^2)
\]

The relation

\[
F^+_X Z_{\tau(r)} = Z_{\tau(r)} F^+_X + q^{-1} (K_X - K^{-1}_X) L_r
\]

implies that

\[
F^+_X Z_{\tau(r)} - (1 + q^2) F^+_X Z_{\tau(r)} F^+_X + q^2 Z_{\tau(r)} F^+_X^2
\]

\[
= q^{-1} F^+_X (K_X - K^{-1}_X) L_r - q (K_X - K^{-1}_X) L_r F^+_X
\]

\[
= q^{-1} (q^2 - q^{-2}) K^{-1}_X L_r F^+_X.
\]

Hence \((A.45)\) holds as required. \(\Box\)

**Lemma A.17.** The relation

\[
Z_{\tau(r)} [B_{\tau(r)}, T]_q = q [B_{\tau(r)}, T]_q Z_{\tau(r)} - q^2 (q - q^{-1}) \Delta Z_{\tau(r)} \) \hspace{1cm} (A.46)

holds in \(B_c\).
Proof: We proceed similarly as in the proof of Lemma A.16. In particular, by (2.40), (2.50), (3.4) and (3.3) we have
\[ [B_{\tau(r)}T]_q = T^{-1}_{\text{w,}X} \left( [ [B_{\tau(r)}, F^-_X]_q, B_r]_q, B_{r-1} \right)_q, \]
\[ Z_{\tau(r)} = q^2 (1 - q^{-2}) T^{-1}_{\text{w,}X} (KX F^-_X L_{\tau(r)}), \]
\[ \Lambda = \epsilon_r T^{-1}_{\text{w,}X} (B_{r-1} L_{\tau(r)} K_X). \]
Thus, verifying Equation (A.46) is equivalent to showing that
\[ F^-_X [[B_{\tau(r)}, F^-_X]_q, B_r]_q, B_{r-1} = \left( [ [B_{\tau(r)}, F^-_X]_q, B_r]_q, B_{r-1} \right)_q F^-_X \]
\[ - q(q - q^{-1}) \epsilon_r B_{r-1} L_{\tau(r)} K_X F^-_X \] (A.47)
holds in \( B_c \). Using the relations
\[ p(F^-_X, B_r) = 0 = p(F^-_X, B_{\tau(r)}) \]
we can commute \( F^-_X \) through \([ [B_{\tau(r)}, F^-_X]_q, B_r]_q \). We have
\[ F^-_X [[B_{\tau(r)}, F^-_X]_q, B_r]_q \]
\[ = F^-_X B_{\tau(r)} F^-_X B_r - qF^-_X B_r B_{\tau(r)} F^-_X - qF^-_X^2 B_{\tau(r)} B_r + q^2 F^-_X B_r F^-_X B_{\tau(r)} \]
\[ = \frac{1}{q^2 - q^{-2}} (F^-_X^2 B_{\tau(r)} + B_{\tau(r)} F^-_X^2) B_r - qF^-_X B_r B_{\tau(r)} F^-_X - qF^-_X^2 B_{\tau(r)} B_r \]
\[ + \frac{q^2}{q^2 + 2} (F^-_X^2 B_r + B_r F^-_X^2) B_{\tau(r)}. \]
Observing that the terms beginning with \( F^-_X^2 \) simplify, we obtain
\[ F^-_X [[B_{\tau(r)}, F^-_X]_q, B_r]_q \]
\[ = \frac{q^2}{q^2 - q^{-2}} F^-_X^2 \Gamma_r + \frac{1}{q^2 + 2} B_{\tau(r)} ((q + q^{-1}) F^-_X B_r F^-_X - B_r F^-_X^2) \]
\[ - qF^-_X B_r B_{\tau(r)} F^-_X + \frac{q^2}{q^2 + 2} B_r ((q + q^{-1}) F^-_X B_{\tau(r)} F^-_X - B_{\tau(r)} F^-_X^2) \]
\[ = [[B_{\tau(r)}, F^-_X]_q, B_r]_q F^-_X + \frac{1}{q^2 - q^{-2}} (q^2 F^-_X^2 \Gamma_r + \Gamma_r F^-_X^2 - (1 + q^2) F^-_X \Gamma_r, F^-_X). \]
Since \( B_{r-1} \) commutes with \( F^-_X \) and
\[ [\Gamma_r, B_{r-1}] = -(q^2 - 1) \epsilon_r Z_r B_{r-1} \]
it follows that
\[ F^-_X [[B_{\tau(r)}, F^-_X]_q, B_r]_q, B_{r-1} = - q^{2r} (KX - K_X^{-1}) L_{\tau(r)} \]
\[ = q^{-1} (KX - K_X^{-1}) L_{\tau(r)} F^-_X \]
implies that
\[ Z_r F^-_X^2 - (1 + q^2) F^-_X Z_r F^-_X + q^2 F^-_X^2 Z_r \]
\[ = - q^{-1} (KX - K_X^{-1}) L_{\tau(r)} F^-_X + q F^-_X (KX - K_X^{-1}) L_{\tau(r)} \]
\[ = q(q^2 - q^{-2}) K_X L_{\tau(r)} F^-_X \]
Hence Equation (A.47) holds as required.
Lemma A.18. The relation

\[ S[S, B_{\tau(r)}]_q = q^{-1}[S, B_{\tau(r)}]_q S - q^{-2}(q^2 - q^{-2})S\Delta \quad (A.49) \]

holds in \( B_q \).

Proof. As in the proof of Lemma A.16 we verify a relation that is equivalent to Equation (A.49). The difference here is that we additionally use the algebra automorphism \( T_{r-1} \) from Equation (4.7). In particular using (5.2), (5.3) and (5.8) we have

\[
S = (q\epsilon_{r-1})^{1/2}T_{w_X} \circ T_{r-1}^{-1}(B_r),
\]

\[
B_{\tau(r)} = (q\epsilon_{r(r-1)})^{-1/2}T_{w_X} \circ T_{r-1}^{-1}\left( [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q \right),
\]

\[
\Delta = \epsilon_{\tau(r)}T_{w_X} \circ T_{r-1}^{-1}(B_{\tau(r-1)}L_rK_X^{-1})
\]

and hence we are done if we show that

\[
B_r[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q = q^{-1}[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q B_r
- q^{-2}(q^2 - q^{-2})\epsilon_{\tau(r)}B_rB_{\tau(r-1)}L_rK_X^{-1}.
\]

Noting that

\[
B_r[B_{\tau(r)}, B_{\tau(r-1)}]_q = [B_{\tau(r)}, B_{\tau(r-1)}]_q B_r + \frac{1}{q-q^{-1}}[\Gamma_r, B_{\tau(r-1)}]_q
\]

\[
= [B_{\tau(r)}, B_{\tau(r-1)}]_q B_r + q\epsilon_{\tau(r)}Z_{\tau(r)}B_{\tau(r-1)}
\]

(A.50)

one calculates

\[
[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q = [[B_r, F^+_X]_q, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q + q^2\epsilon_{\tau(r)}[F^+_X, Z_{\tau(r)}]B_{\tau(r-1)}.
\]

Using this, the relation \( B_r[B_r, F^+_X]_q = q^{-1}[B_r, F^+_X]_q B_r \) and Equation (A.50) we have

\[
B_r[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q
= B_r[[B_r, F^+_X]_q, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q + q^2\epsilon_{\tau(r)}B_r[F^+_X, Z_{\tau(r)}]B_{\tau(r-1)}
= q^{-1}[[B_r, F^+_X]_q, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q B_r + \epsilon_{\tau(r)}B_r[F^+_X, Z_{\tau(r)}]B_{\tau(r-1)}
- q^3\epsilon_{\tau(r)}[F^+_X, Z_{\tau(r)}]B_rB_{\tau(r-1)} + q^2\epsilon_{\tau(r)}B_r[F^+_X, Z_{\tau(r)}]B_{\tau(r-1)}
= q^{-1}[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q B_r + (1+q^2)\epsilon_{\tau(r)}[B_r, [F^+_X, Z_{\tau(r)}]_q]_q B_{\tau(r-1)}.
\]

Since \([F^+_X, Z_{\tau(r)}] = q^{-1}(K_X - K_X^{-1})L_r \) we have

\[
[B_r, [F^+_X, Z_{\tau(r)}]_q] = q^{-1}[B_r, (K_X - K_X^{-1})L_r]_q
= -q^{-1}(1 - q^{-2})B_rK_X^{-1}L_r
\]

and hence we obtain

\[
B_r[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q = q^{-1}[B_r, [F^+_X, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q B_r
- q^{-2}(q^2 - q^{-2})\epsilon_{\tau(r)}B_rB_{\tau(r-1)}K_X^{-1}L_r
\]

as required. \( \square \)
Lemma A.19. The relation
\[ T[B_{\tau(r)}, T]_q = q[B_{\tau(r)}, T]_q T + q^2(q^2 - q^{-2})T \Lambda \quad (A.51) \]
holds in $B_r$.

Proof. Proceeding as in the proof of Lemma A.18 we have
\[
T = (q\epsilon_{\tau^{-1}})^{1/2}T^{-1}_{w_X} \circ T_{\tau^{-1}}(B_r),
\]
\[
B_{\tau(r)} = (q\epsilon_{\tau^{-1}})^{1/2}T^{-1}_{w_X} \circ T_{\tau^{-1}}([B_{\tau(r)}, B_{\tau(r)}]|_q, F_{X^{-1}})_q
\]
\[
\Lambda = q\epsilon_{\tau^{-1}}T^{-1}_{w_X} \circ T_{\tau^{-1}}(B_{\tau(r)}L_{\tau(r)}K_X).
\]

Then showing that Equation (A.51) holds is equivalent to showing that
\[
B_r[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}})_q, B_r]_q = q[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q B_r + q^3(q^2 - q^{-2})\epsilon_r B_r B_{\tau(r-1)}L_{\tau(r)}K_X
\]
holds. Noting that
\[
B_r[B_{\tau(r-1)}, B_{\tau(r)}]_q = [B_{\tau(r-1)}, B_{\tau(r)}]_q B_r - q\epsilon_r B_{\tau(r-1)}Z_r \quad (A.52)
\]
we have
\[
[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}})_q, B_r]_q = [[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q - q^2\epsilon_r B_{\tau(r-1)}[F_{X^{-1}}, Z_r] \quad (A.53)
\]
Using this, Equation (A.52) and the relation $B_r[F_{X^{-1}}, B_r]_q = q[F_{X^{-1}}, B_r]_q B_r$ we hence obtain
\[
B_r[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}})_q, B_r]_q
\]
\[
= B_r[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q - q^2\epsilon_r B_{\tau(r-1)}B_r[F_{X^{-1}}, Z_r]

\textit{A.52}
\]
\[
q[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q B_r - q\epsilon_r B_{\tau(r-1)}Z_r[F_{X^{-1}}, B_r]_q
\]
\[
+ q^3\epsilon_r B_{\tau(r-1)}[F_{X^{-1}}, B_r]_q Z_r - q^2\epsilon_r B_{\tau(r-1)}B_r[F_{X^{-1}}, Z_r]

\textit{A.53}
\]
\[
q[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q B_r - q\epsilon_r B_{\tau(r-1)}[F_{X^{-1}}, Z_r]_q
\]
\[
- q^2\epsilon_r B_{\tau(r-1)}B_r[F_{X^{-1}}, Z_r]_q + q^3\epsilon_r B_{\tau(r-1)}[F_{X^{-1}}, Z_r]_q B_r
\]
\[
= q[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q B_r + (q^3 + q)\epsilon_r B_{\tau(r-1)}[[F_{X^{-1}}, Z_r], B_r]_q.
\]
Since $[F_{X^{-1}}, Z_r]_q = q^{-1}(K_X - K_X^{-1})L_{\tau(r)}$ it follows that
\[
[[F_{X^{-1}}, Z_r], B_r]_q = q^{-1}(1 - q^{-2})K_X L_{\tau(r)}B_r
\]
which implies that
\[
B_r[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}})_q, B_r]_q = q[[B_{\tau(r-1)}, B_{\tau(r)}]|_q, F_{X^{-1}}, B_r]_q B_r + q^3(q^2 - q^{-2})\epsilon_r B_r B_{\tau(r-1)}L_{\tau(r)}K_X
\]
as required. ■
Lemma A.20. The relations

\[ [F^-_X, [S, Z_r]_q]_q = (q^2 - 1)SL_{\tau(r)}K_X^{-1}, \quad (A.54) \]
\[ [[Z_{\tau(r)}, T], F^+_X]_q = (1 - q^{-2})TL_r K_X \quad (A.55) \]

hold in \( B_c \).

Proof. We only prove Equation (A.54); the proof for Equation (A.55) is similar.

Observe that

\[ [F^-_X, [S, Z_r]_q]_q = [[F^-_X, S]_q, Z_r]_q + q[S, [F^-_X, Z_r]] \]

holds by Equation (A.2). Since \( S = T_{w_X}([B_{r-1}, B_r]_q) \) and \( F^-_X = -T_{w_X}(E^-_X K_X) \) it follows that

\[ [F^-_X, S]_q = -T_{w_X}([[E^-_X K_X, [B_{r-1}, B_r]_q]]) \]
\[ = 0 \]

since \( [E^-_X K_X, B_r]_q = 0 \). Further we have

\[ [F^-_X, Z_r] = -(1 - q^{-2})[F^-_X, E^+_X L_{\tau(r)}] \]
\[ = -(1 - q^{-2})[F^-_X, E^+_X L_{\tau(r)}] \]
\[ = q^{-1}(K_X - K_X^{-1})L_{\tau(r)}. \]

Hence we obtain

\[ [F^-_X, [S, Z_r]_q]_q = q[S, [F^-_X, Z_r]] \]
\[ = [S, K_X - K_X^{-1}]_q L_{\tau(r)} \]
\[ = (q^2 - 1)SL_{\tau(r)}K_X^{-1} \]

as required. \( \square \)

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