A PRIMAL-DUAL WEAK GALERKIN METHOD FOR DIV-CURL SYSTEMS WITH LOW-REGULARITY SOLUTIONS

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Abstract. This article presents a new primal-dual weak Galerkin finite element method for the div-curl system with tangential boundary conditions and low-regularity assumptions on the solution. The numerical scheme is based on a weak variational form involving no partial derivatives of the exact solution supplemented by a dual or adjoint problem in the general context of the weak Galerkin finite element method. Optimal order error estimates in $L^2$ are established for solution vector fields in $H^{\theta}(\Omega)$, $\theta > \frac{1}{2}$. The mathematical theory was derived on connected domains with general topological properties (namely, arbitrary first and second Betti numbers). Numerical results are reported to confirm the theoretical convergence.

Key words. primal-dual weak Galerkin, finite element methods, div-curl system, tangential boundary conditions, low-regularity.

AMS subject classifications. Primary 65N30, 65N12, 65N15; Secondary 35Q60, 35B45

1. Introduction. This paper is concerned with the development of a primal-dual weak Galerkin finite element method for div-curl systems equipped with tangential boundary conditions. For simplicity, consider the problem of seeking a vector field $u$ satisfying

\[
\begin{align*}
\nabla \cdot (\varepsilon u) &= f, &\text{in } \Omega, \\
\nabla \times u &= g, &\text{in } \Omega, \\
\n\nabla \times (u \cdot n) &= \chi, &\text{on } \Gamma, \\
\langle \varepsilon u \cdot n_i, 1 \rangle_{\Gamma_i} &= \alpha_i, &i = 1, \ldots, L,
\end{align*}
\]

where $\Omega$ is an open bounded and connected domain in $\mathbb{R}^3$, with Lipschitz continuous boundary $\Gamma = \partial \Omega$ which consists of a finite number of disjoint surfaces $\Gamma = \bigcup_{i=0}^{L} \Gamma_i$, where each component $\Gamma_i$ is connected, Lipschitz continuous, and has finite surface area. $\Gamma_0$ is the exterior boundary of the domain, and each $\Gamma_i$ corresponds to a "hole" so that $L$ geometrically counts the number of holes in the region $\Omega$. The number $L$ is known as the second Betti number of $\Omega$ or the dimension of the second de Rham cohomology group of $\Omega$. In the equation (1.1a), $\varepsilon = \{ \varepsilon_{ij}(x) \}_{3 \times 3}$ is a symmetric and uniformly positive definite matrix in $\Omega$ with entries in $L^\infty(\Omega)$. The load function $f = f(x)$ is Lebesgue-integrable and real-valued, and the vector field $g = g(x)$ are given in the domain $\Omega$. The tangential boundary condition (1.1d) corresponds to a given value for the tangential component of the vector field $u$, where $n_i$ is the unit outward normal direction on $\Gamma_i$, $\chi \in [L^2(\Gamma)]^3$ is a given vector field on the boundary.

The div-curl system has many important applications in computational fluid and electromagnetic problems \cite{5}. For this reason, various numerical methods have been

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developed for solving this system in the last several decades. A control volume method was proposed by Nicolaides [18] for the planar div-curl problems in 1992, thereafter a co-finite volume method was developed by Nicolaides and Wu [19] for three-dimensional div-curl problems. Delcourte et al. [13] proposed a discrete duality finite volume method for the div-curl problems on almost arbitrary polygonal meshes. Bramble and Pasciak [6] developed a finite element formulation for the div-curl systems under a very weak formulation where the solution space was $L^2(\Omega)^3$. In [12], Copeland et al. presented a mixed finite element method for 3D axisymmetric div-curl systems, which reduces the computational domain from 3D to 2D via cylindrical coordinates in simply connected axisymmetric domains. A least-squares method based on discontinuous elements was propose by Bensow and Larson in [4]. Bochev et al. [3] proposed a least-squares finite element methods for two div-curl elliptic boundary value problems. The mimetic finite difference method was proposed by Brezzi et al. [8, 16] and applied to the 3D magnetostatic problems on general polyhedral meshes. Most recently, Wang et al. [22] developed a weak Galerkin method for the div-curl problems with either normal or tangential boundary conditions. Ye et al. proposed least-squares methods [15, 20] for the div-curl problems, which result in symmetric positive definite linear systems.

For the div-curl system (1.1a)-(1.1d) to be well-posed, the functional data in the system must satisfy certain compatibility conditions (e.g., the equations (2.5), (2.8), and (2.10)). In fact, the tangential boundary value problem (1.1a)-(1.1d) has one and only one solution if all the desired compatibility conditions are met. One of the main challenges in the design of numerical methods for (1.1a)-(1.1d) is the low-regularity nature of the exact solution $u$. The goal of this paper is to address this challenge by devising a primal-dual weak Galerkin (PDWG) scheme which provides reliable numerical approximations for solutions with low-regularity. In particular, the new PDWG finite element method will approximate the vector field $u$ by using piecewise polynomials in $L^2(\Omega)^3$, and an optimal order error estimate shall be derived in $L^2$ for solutions in $H^\theta(\Omega), \theta > \frac{1}{2}$. The primal-dual idea for solving PDEs was also developed by Burman [9, 10] in other finite element contexts. The PDWG method has been successfully applied to several challenging problems including the second order elliptic equation in non-divergence form [23], the Fokker-Planck equation [24], the elliptic Cauchy problem [25], and linear transport problems [26].

Throughout the paper, we follow the usual notation for Sobolev spaces and norms as in [11, 14]. For any open bounded domain $D \subset \mathbb{R}^3$ with Lipschitz continuous boundary, we use $\| \cdot \|_{s,D}$ and $| \cdot |_{s,D}$ to denote the norm and seminorm in the Sobolev space $H^s(D)$ for any $s \geq 0$, respectively. The inner product in $H^s(D)$ is denoted by $(\cdot, \cdot)_{s,D}$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\| \cdot \|_D$ and $(\cdot, \cdot)_D$, respectively. We use $H(div; D)$ to denote the closed subspace of $[L^2(D)]^3$ so that $\nabla \cdot (\varepsilon \v) \in L^2(D)$. The space $H(div; D)$ corresponds to the case of the identity matrix $\varepsilon = I$. Analogously, we use $H(curl; D)$ to denote the closed subspace of $[L^2(D)]^3$ so that $\nabla \times \v \in [L^2(D)]^3$. The space of normal $\v$-harmonic vector fields, denoted by $H_{\v,0}(\Omega)$, consists of all $\v$-harmonic vector fields satisfying the zero normal boundary condition; i.e.,

$$H_{\v,0}(\Omega) = \{ \v \in [L^2(\Omega)]^3 : \nabla \times \v = 0, \nabla \cdot (\v \v) = 0, \v \v \cdot n = 0 \text{ on } \Gamma \}.$$

When $\v = I$ is the identity matrix, the spaces $H_{\v,0}(\Omega)$ shall be denoted as $H_{0,0}(\Omega)$. 
2. A Weak Formulation. Denote by \( \bar{H}^1(\Omega) = \frac{H^1(\Omega)}{\mathbb{R}} \) the quotient space and
\[
H^1_{0c}(\Omega) = \{ w \in H^1(\Omega) : w|_{\Gamma_0} = 0, \ w|_{\Gamma_i} = c_i, \ i = 1, \ldots, L, \ c_i \in \mathbb{R} \}
\]
the closed subspace of \( H^1(\Omega) \) with vanishing value on \( \Gamma_0 \) and constant value on each \( \Gamma_i, \ i = 1, \ldots, L \).

A weak formulation for the tangential boundary value problem \((1.1a)-(1.1d)\) seeks \( u \in [L^2(\Omega)]^3 \) and \( s \in \bar{H}^1(\Omega) \) such that
\[
(2.1) \quad b(u, s; \varphi, \psi) = G(\varphi, \psi), \quad \forall \varphi \in H^1_{0c}(\Omega), \ \psi \in H(\text{curl}; \Omega),
\]
where
\[
(2.2) \quad b(u, s; \varphi, \psi) := (u, \varepsilon \nabla \varphi + \nabla \times \psi) + (\psi, \nabla s),
\]
\[
(2.3) \quad G(\varphi, \psi) := (g, \psi) - (f, \varphi) + \langle \chi \times n, \psi \times n \rangle + \sum_{i=1}^{L} \alpha_i \varphi|_{\Gamma_i}.
\]
The corresponding homogeneous dual or adjoint problem for \((2.1)\) seeks \( \lambda \in H^1_{0c}(\Omega) \) and \( q \in H(\text{curl}; \Omega) \) such that
\[
(2.4) \quad b(v, r; \lambda, q) = 0, \quad \forall v \in [L^2(\Omega)]^3, \ r \in \bar{H}^1(\Omega).
\]

For \((1.1a)-(1.1d)\) to be well-imposed, certain compatibility conditions must be satisfied for the boundary value and the load functions \( f \) and \( g \). First, from the curl equation \((1.1b)\) we have
\[
(2.5) \quad \nabla \cdot g = 0.
\]
Next, as \( \chi = u \times n \) is orthogonal to the normal direction \( n \), hence
\[
(2.6) \quad \chi = n \times (\chi \times n).
\]
By testing the equation \((1.1b)\) against any \( \psi \in H(\text{curl}; \Omega) \) we have from the Green’s formula and the tangential boundary condition \((1.1c)\) that
\[
(2.7) \quad (u, \nabla \times \psi) = (g, \psi) + \langle \chi, \psi \rangle, \quad \forall \psi \in H(\text{curl}; \Omega).
\]
By letting \( \psi = \nabla \rho \) in \((2.7)\) and then using \((2.5)\) and \((2.6)\), we arrive at the following compatibility condition:
\[
(2.8) \quad (g \cdot n, \rho) + \langle \chi \times n, \nabla \rho \times n \rangle = 0, \quad \forall \rho \in H^1(\Omega).
\]
Analogously, by letting \( \psi = \eta \in H_{n,0}(\Omega) \) in \((2.7)\), we have
\[
(2.9) \quad (g, \eta) + \langle \chi, \eta \rangle = 0, \quad \forall \eta \in H_{n,0}(\Omega),
\]
which, together with \((2.6)\), leads to
\[
(2.10) \quad (g, \eta) + \langle \chi \times n, \eta \times n \rangle = 0, \quad \forall \eta \in H_{n,0}(\Omega).
\]

**Theorem 2.1.** The solution \( (u, s) \in [L^2(\Omega)]^3 \times \bar{H}^1(\Omega) \) for the primal problem \((2.1)\) is unique.
Proof. It suffices to show that solutions to the homogeneous problem must be trivial. To this end, let \((u, s) \in [L^2(\Omega)]^3 \times H^1(\Omega)\) be a solution of (2.1) with homogeneous data, i.e.,

\[(u, \varepsilon \nabla \varphi + \nabla \times \psi) + (\psi, \nabla s) = 0\]  

for all \(\varphi \in H^1_{0c}(\Omega)\) and \(\psi \in H(\text{curl}; \Omega)\). From the Helmholtz decomposition (A.1), we may choose \(\varphi \) and \(\psi \perp \mathbb{H}_{n,0}(\Omega)\) such that

\[u = \nabla \varphi + \varepsilon^{-1} \nabla \times \psi, \quad \text{in } \Omega,\]

\[\nabla \cdot \psi = 0, \quad \text{in } \Omega,\]

\[\psi \cdot n = 0, \quad \text{on } \partial \Omega.\]

Substituting the above into (2.11) yields \((\varepsilon u, u) = 0\) so that \(u \equiv 0\). It follows that \((\psi, \nabla s) = 0\) for all \(\psi \in H(\text{curl}; \Omega)\) so that \(\nabla s = 0\), and hence \(s = \text{const} = 0\).

Theorem 2.2. Assume the compatibility conditions (2.5), (2.6), and (2.8) hold true. For any weak solution \((u, s) \in [L^2(\Omega)]^3 \times H^1(\Omega)\) of the variational problem (2.1), there holds \(s = 0\) and that \(u\) satisfies the div-curl system (1.1a)-(1.1d) in the strong form.

Proof. By letting \(\psi = 0\) in (2.1) we have

\[(u, \varepsilon \nabla \varphi) = -(f, \varphi) + \sum_{i=1}^{L} \alpha_i \varphi|_{\Gamma_i}, \quad \forall \varphi \in H^1_{0c}(\Omega),\]

which, with the integration by parts, leads to

\[\nabla \cdot (\varepsilon u) = f, \quad \text{in } \Omega,\]

\[\langle \varepsilon u \cdot n_i, 1 \rangle_{\Gamma_i} = \alpha_i, \quad i = 1, 2, \ldots, L,\]

so that (1.1a) and (1.1d) are satisfied. Next, by choosing \(\varphi = 0\) and \(\psi = \nabla s\) in (2.1) we obtain

\[\langle \nabla s, \nabla s \rangle = (g, \nabla s) + \langle \chi \times n, \nabla s \times n \rangle,\]

which, together with the compatibility conditions (2.5) and (2.8), leads to \(\nabla s = 0\) and hence \(s \equiv 0\).

Now, by letting \(\varphi = 0\) in (2.1), we have for any \(\psi \in H(\text{curl}; \Omega)\)

\[(u, \nabla \times \psi) = (g, \psi) + \langle \chi \times n, \psi \times n \rangle,\]

which leads to

\[\langle \nabla \times u, \psi \rangle + \langle u \times n, \psi \rangle = (g, \psi) + \langle \chi \times n, \psi \times n \rangle,\]

and hence

\[(\nabla \times u, \psi) + \langle u \times n, \psi \rangle = (g, \psi) + \langle \chi \times n, \psi \times n \rangle,\]

(2.12)

\[\nabla \times u = g, \quad \text{in } \Omega,\]

(2.13)

\[u \times n = n \times (\chi \times n) = \chi, \quad \text{on } \Gamma.\]

Equation (2.13) verifies the tangential boundary condition (1.1c), and the curl equation (1.1b) is seen from (2.12). This completes the proof of the theorem. \(\square\)
Theorem 2.3. The homogeneous dual problem \[ 2.4 \] has only “trivial” solutions for \( \lambda \): i.e., if \( \lambda \in H^1_{0c}(\Omega) \) and \( q \in H(\text{curl}; \Omega) \) satisfy the weak form \[ 2.4 \], then we must have \( \lambda = 0 \) and that \( q \in H_{n,0}(\Omega) \) is a harmonic field.

Proof. Let \( \lambda \in H^1_{0c}(\Omega) \) and \( q \in H(\text{curl}; \Omega) \) be the solution of the homogeneous dual problem \[ 2.4 \]. Then,

\[
(v, \varepsilon \nabla \lambda + \nabla \times q) + (q, \nabla r) = 0, \quad \forall (v, r) \in [L^2(\Omega)]^3 \times \tilde{H}^1(\Omega).
\]

Observe that the test against \( r \in \tilde{H}^1(\Omega) \) ensures \( \nabla \cdot q = 0 \) and \( q \cdot n = 0 \) on \( \Gamma \). It follows that

\[
\varepsilon \nabla \lambda + \nabla \times q = 0, \quad \nabla \cdot q = 0, \quad q \cdot n = 0 \text{ on } \partial \Omega.
\]

In other words, \( q \in H_{n,0}(\Omega) \) is a harmonic field. This completes the proof of the theorem. \( \square \)

It is known that the dimension of the harmonic space \( \mathbb{H}_{n,0}(\Omega) \) is the first Betti number of the domain \( \Omega \). The first Betti number is the rank of the first homology group of \( \Omega \). It is the number of elements of a maximal set of homologically independent non-bounding cycles in the domain. It is also the dimension of the first de Rham cohomology group of \( \Omega \). The dimension of \( \mathbb{H}_{n,0}(\Omega) \) is clearly zero if the domain \( \Omega \) is simply connected.

3. Discrete Weak Differential Operators. The variational problems \[ 2.1 \] and \[ 2.4 \] are formulated with two principal differential operators: gradient and curl. This section shall introduce the notion of weak differential operators. These weak differential operators shall be discretized by using piecewise polynomials which lead to discretization schemes for the variational problems.

Let \( T \) be a polyhedral domain with boundary \( \partial T \). Denote by \( n \) the unit outward normal direction on \( \partial T \). The space of weak functions in \( T \) is defined as

\[
W(T) = \{ v = \{ v_0, v_b \} : v_0 \in L^2(T), v_b \in L^2(\partial T) \},
\]

where \( v_0 \) represents the value of \( v \) in the interior of \( T \), and \( v_b \) represents certain information of \( v \) on the boundary \( \partial T \). Similarly, we define \( V(T) \) the space of vector-valued weak functions in \( T \) given by:

\[
V(T) = \{ v = \{ v_0, v_b \} : v_0 \in [L^2(T)]^3, v_b \in [L^2(\partial T)]^3 \}.
\]

3.1. Weak gradient. The weak gradient of \( v \in W(T) \), denoted by \( \nabla_w v \), is defined as a continuous linear functional in the Sobolev space \( [H^1(T)]^3 \) with the following action

\[
(\nabla_w v, \varphi)_T = -(v_0, \nabla \cdot \varphi)_T + \langle v_b, \varphi \cdot n \rangle_{\partial T}, \quad \forall \varphi \in [H^1(T)]^3.
\]
Denote by $P_r(T)$ the space of polynomials on $T$ with total degree $r$ and less. The discrete weak gradient operator, denoted by $\nabla_{w,r,T}v$, is defined as the unique vector-valued polynomial in $[P_r(T)]^3$ satisfying

$$\nabla_{w,r,T}v(\varphi)_T = -(v_0, \nabla \cdot \varphi)_T + (v_b, \varphi \cdot n)|_{\partial T}, \quad \forall \varphi \in [P_r(T)]^3.$$  

For smooth $v_0 \in H^1(T)$, we have from the usual integration by parts that

$$\nabla_{w,r,T}v(\varphi)_T = (\nabla v_0, \varphi)_T + (v_b - v_0, \varphi \cdot n)|_{\partial T}, \quad \forall \varphi \in [P_r(T)]^3.$$  

### 3.2. Weak curl

The weak curl of $v \in V(T)$ (see [22]), denoted by $\nabla \times v$, is defined as a bounded linear functional in the Sobolev space $[H^1(T)]^3$ with actions given by

$$\langle (\nabla \times v, \varphi) \rangle_T = (v_0, \nabla \times \varphi)_T - \langle v_b \times n, \varphi \rangle_{\partial T}, \quad \forall \varphi \in [H^1(T)]^3.$$  

The discrete weak curl of $v \in V(T)$, denoted by $\nabla_{w,r,T} \times v$, is defined as the unique vector-valued polynomial in $[P_r(T)]^3$, such that

$$\nabla_{w,r,T} \times v(\varphi)_T = (\nabla v_0, \varphi)_T - \langle (v_b - v_0) \times n, \varphi \rangle_{\partial T}, \quad \forall \varphi \in [P_r(T)]^3.$$  

For sufficiently smooth $v_0$ such that $\nabla \times v_0 \in [L_2(T)]^3$, we have from the integration by parts that

$$\nabla_{w,r,T} \times v(\varphi)_T = (\nabla v_0, \varphi)_T - \langle (v_b - v_0) \times n, \varphi \rangle_{\partial T}, \quad \forall \varphi \in [P_r(T)]^3.$$  

### 4. A Primal-Dual Weak Galerkin Method

Assume that the domain $\Omega$ is of polyhedral type, and $\mathcal{T}_h = \{T\}$ is a finite element partition of $\Omega$ that is shape regular as described in [23, 27]. Denote by $h_T = \text{diam}(T)$ the diameter of the element $T$, and $h = \max_T h_T$ the meshsize of the partition $\mathcal{T}_h = \{T\}$. Denote by $\mathcal{E}_h$ the set of all faces in $\mathcal{T}_h$ so that each $\sigma \in \mathcal{E}_h$ is either on the boundary of $\Omega$ or shared by two elements $T_1$ and $T_2$. Denote by $\mathcal{E}^0_h = \mathcal{E}_h \setminus \partial \Omega$ the set of all interior faces in $\mathcal{E}_h$. Let $k \geq 0$ be a given integer. On each $T \in \mathcal{T}_h$, we introduce two local weak finite element spaces as follows:

- $W(k, T) = \{v = \{v_0, v_b\} : v_0 \in P_k(T), v_b|_{\sigma} \in P_k(\sigma), \sigma \in (\partial T \cap \mathcal{E}_h)\}$
- $V(k, T) = \{v = \{v_0, v_b\} : v_0 \in [P_k(T)]^3, v_b|_{\sigma} \in [P_k(\sigma)]^3 \times n_{\sigma}, \sigma \in (\partial T \cap \mathcal{E}_h)\}$

where $n_{\sigma}$ is a unit normal vector to the face $\sigma$. Note that $v_b|_{\sigma}$ effectively a vector-valued polynomial of degree $k$ in the tangent space of $\sigma$. The global weak finite element space is constructed by patching all the local elements $W(k, T)$ (or $V(k, T)$) through a common value on the interior faces:

- $W^k_h = \{v = \{v_0, v_b\} : v|_T \in W(k, T), v_b|_{\partial T_1 \cap \sigma} = v_b|_{\partial T_2 \cap \sigma}, T \in \mathcal{T}_h, \sigma \in \mathcal{E}^0_h\}$
- $V^k_h = \{v = \{v_0, v_b\} : v|_T \in V(k, T), v_b|_{\partial T_1 \cap \sigma} = v_b|_{\partial T_2 \cap \sigma}, T \in \mathcal{T}_h, \sigma \in \mathcal{E}^0_h\}$

where $v_b|_{\partial T_1 \cap \sigma}$ is the value of $v_b$ on the face $\sigma$ as seen from the element $T_1$. A third finite element space consists of piecewise vector-valued polynomials of degree $k$:

$$U^k_h = \{u : u \in [L_2(\Omega)]^3, u|_T \in [P_k(T)]^3, T \in \mathcal{T}_h\}.$$
Next, we introduce the following finite element spaces:

\[ U_h = U_h^k, \]
\[ M_h = \{ s = \{ s_0, s_b \} \in W_h^k : (s_0, 1) = 0 \}, \]
\[ S_h = \{ \lambda = \{ \lambda_0, \lambda_b \} \in W_h^k : \lambda_0|_{\Gamma_0} = 0, \lambda_b|_{\Gamma_i} = \text{const, } i = 1, \ldots, L \}, \]
\[ V_h = V_h^k. \]

For functions in \( S_h \) and \( M_h \), the discrete weak gradient is defined by using (3.1) with \( r = k \) on each element \( T \). Likewise, the discrete weak curl is defined for functions in \( V_h \) by using (3.2) with \( r = k \); i.e.,

\[ (\nabla_{w,k} v)|_T = \nabla_{w,k,T}(v|_T), \quad v \in W_h^k, \]
\[ (\nabla_{w,k} \times v)|_T = \nabla_{w,k,T} \times (v|_T), \quad v \in V_h^k. \]

Note that the weak gradient and the weak curl operators \( \nabla_{w,k} \) and \( \nabla_{w,k} \times \) are defined by using the same degree of polynomials as the function themselves on each element, as opposed to using polynomials of lower degree in [23]. For simplicity of notation, we shall drop the subscript \( k \) from the notations \( \nabla_{w,k} \) and \( \nabla_{w,k} \times \) in the rest of the paper.

Introduce an approximate bilinear form as follows:

\[ B_h(v, r; \varphi, \psi) := (v, \varepsilon \nabla \varphi + \nabla \psi) + (\psi_0, \nabla vr) \]
for \( v \in U_h, \ r \in M_h, \ \varphi \in S_h, \ \psi \in V_h. \)

**Algorithm 4.1** (PDWG for the div-curl system with tangential BV). Find \( u_h, s_h \in U_h \times M_h \) and \( \lambda_h, q_h \in S_h \times V_h \) such that

\[ \begin{cases} S_1(\lambda_h, q_h; \varphi, \psi) + B_h(u_h, s_h; \varphi, \psi) = G(\varphi, \psi), \\ -S_2(s_h, r) + B_h(v, r; \lambda_h, q_h) = 0, \end{cases} \]
for all \( (v, r) \in U_h \times M_h \) and \( \varphi, \psi \in S_h \times V_h, \) where

\[ \begin{align*}
S_1(\lambda_h, q_h; \varphi, \psi) & = \rho_1 \sum_T h^{-1}_T (\lambda_0 - \lambda_b, \varphi_0 - \varphi_b)_{\partial T} \\
& \quad + \rho_2 \sum_T h^{-1}_T (q_0 \times n - q_b \times n, \psi_0 \times n - \psi_b \times n)_{\partial T}, \\
S_2(s_h, r) & = \rho_3 \sum_T h^{-1}_T (s_0 - s_b, r_0 - r_b)_{\partial T}, \\
G(\varphi, \psi) & = (g, \psi_0) + (\chi, \psi_b) - (f, \varphi_0) + \sum_{i=1}^L \alpha_i \varphi|_{\Gamma_i}. \end{align*} \]

Here \( \rho_i > 0 \) are parameters with prescribed values at user’s discretion. The default value for these parameters is \( \rho_i = 1. \)

5. **Element Stiffness Matrix and Load Vector**. In this section, we shall present a formula for the computation of the element stiffness matrix and the element load vector on general 3D polyhedral elements as illustrated in Fig. 5.1 for the PDWG finite element scheme (4.2) with the lowest order element (i.e., \( k = 0 \)), extensions to higher order elements are straightforward.
Let \( T \in T_h \) be a polyhedral element with \( N \) lateral faces; i.e., \( \partial T = \bigcup_{i=1}^N \sigma_i \). The finite elements for the primal variable \( u_h \) and the Lagrangian multipliers \( s_h, \lambda_h, q_h \) on \( T \) are given respectively as follows:

\[
\begin{align*}
  u_h|_T & \in [P_0(T)]^3, \\
  s_h|_T & = \{s_0, s_b\} \in \{P_0(T), P_0(\partial T)\}, \\
  \lambda_h|_T & = \{\lambda_0, \lambda_b\} \in \{P_0(T), P_0(\partial T)\}, \\
  q_h|_T & = \{q_0, q_b\} \in \{[P_0(T)]^3, [P_0(\partial T)]^3\}.
\end{align*}
\]

We use the following representation for each of them:

\[
\begin{align*}
  u_h & = u_1 e_1 + u_2 e_2 + u_3 e_3, \\
  s_0 & = s_0 \cdot 1, \quad s_b|_{\sigma_i} = s_{b,i}, \quad i = 1, \ldots, N, \\
  \lambda_0 & = \lambda_0 \cdot 1, \quad \lambda_b|_{\sigma_i} = \lambda_{b,i}, \quad i = 1, \ldots, N, \\
  q_0 & = q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_b = \sum_{i=1}^N \sum_{k=1}^2 q_{b,i} e_{b,i}. \\
\end{align*}
\]

Note that \( q_b \) is in fact a vector in the tangent plane of \( \sigma_i \), and thus has 2 Dofs. The \( P_0 \) vector basis \( e^k \) on \( T \) are as follows:

\[
(5.1) \quad e^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_T, \quad e^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_T, \quad e^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_T.
\]

The tangential basis \( \{e_{b,1}^1, e_{b,1}^2\} \) on face \( \sigma_i \) is computed as follows. First we fix a unit normal direction \( \tilde{n}_i \) to \( \sigma_i \), then take an arbitrary vector \( r \) that is not normal to \( \sigma_i \), and compute

\[
\begin{align*}
  v_1 &= r \times \tilde{n}_i, \\
  v_2 &= (r \times \tilde{n}_i) \times \tilde{n}_i.
\end{align*}
\]

The tangential basis \( \{e_{b,1}^1, e_{b,1}^2\} \) is chosen as the normalizations:

\[
\begin{align*}
  e_{b,1}^1 &= \frac{v_1}{|v_1|}, \quad e_{b,1}^2 = \frac{v_2}{|v_2|}.
\end{align*}
\]

Denote by

\[
\begin{align*}
  \{u^k\}_{k=1,2,3}, \\
  \{s_0\}, \quad \{s_{b,i}\}_{i=1,...,N}, \\
  \{\lambda_0\}, \quad \{\lambda_{b,i}\}_{i=1,...,N}, \\
  \{q^k\}_{k=1,2,3}, \quad \{q_{b,i}^k\}_{k=1,2,i=1,...,N}
\end{align*}
\]

the degree of freedoms on element \( T \) for the corresponding variables. For the numerical scheme \( (4.2) \), we have the following formula for the element stiffness matrix and the load vector:
where the block components in (5.2) are given explicitly as follows when \( \rho_i = 1 \):

\[
A = \{a\}_{1 \times 1}, \quad a = h_T^{-1} \sum_{i=1}^{N} |\sigma_i|, \quad B = \{b_i\}_{1 \times N}, \quad b_i = -h_T^{-1} |\sigma_i|,
\]

\[
C = \{c_{i,j}\}_{N \times N}, \quad C = \text{diag}(B), \quad D = \{d_{i,j}\}_{N \times d}, \quad d_{i,j} = e^j \cdot (\varepsilon \mathbf{n}_i) |\sigma_i|,
\]

\[
E = \{x_{k,j}\}_{d \times d}, \quad x_{k,j} = h_T^{-1} \sum_{i=1}^{N} (e^k \times \mathbf{n}_i) \cdot (e^j \times \mathbf{n}_i) |\sigma_i|,
\]

\[
F = \{f_{i,k}\}_{2N \times d}, \quad f_{i,k} = -h_T^{-1} e^{k}_{bn,i} \cdot (e^j \times \mathbf{n}_i) |\sigma_i|, \quad G = \{g_{j,i}\}_{d \times N}, \quad g_{j,i} = e^j \cdot \mathbf{n}_i |\sigma_i|,
\]

\[
I = \{x_{i,k}\}_{2N \times d}, \quad x_{i,k} = e^i \cdot e^{k}_{bn,i} |\sigma_i|, \quad J = -A, \quad K = -B, \quad L = -C,
\]

\[
H = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}_{2N \times 2N},
\]

where \( H_{k,j}, \ k, j = 1, 2, \) are diagonal matrices of size \( N \times N \) given as follows:

\[
H_{k,j} = \text{diag}(V^{kj}), \quad V^{kj} = \{x_{i,k}^{kj}\}_{1 \times N}, \quad x_{i,k}^{kj} = h_T^{-1} e^{k}_{bn,i} \cdot e^{j}_{bn,i} |\sigma_i|.
\]

Here \( e^{k}_{bn,i} = e^{k}_i \times \mathbf{n}_i, \ k = 1, 2, \mathbf{n}_i \) is the unit outward normal vector of \( \sigma_i \), \( d = 3 \) is the space dimension. Please note the difference between \( \mathbf{n}_i \) and \( \hat{\mathbf{n}}_i \) in their directions; \( \mathbf{n}_i \) is outward normal to \( \sigma_i \) and \( \hat{\mathbf{n}}_i \) is a prescribed orientation of \( \sigma_i \).

**Proof.** From the definition of the weak gradient (3.1), we have

\[
\langle \nabla_w \{1, 0\}, \phi \rangle = -\langle 1, \nabla \cdot \phi \rangle + \langle 0, \phi \cdot \mathbf{n}\rangle_{\partial T},
\]

\[
\langle \nabla_w \{0, 1_b,i\}, \phi \rangle = \langle 1_b,i, \phi \cdot \mathbf{n}\rangle_{\partial T},
\]

\[
\langle \nabla_w s_{b,h}, \phi \rangle = -\langle s_{b,h}, \phi \cdot \mathbf{n}\rangle_{\partial T}, \quad \forall \phi \in [P_0]^d.
\]

Since \( \phi \in [P_0]^d \), then we have \( \nabla \cdot \phi = 0 \) and

\[
\nabla_w \{1, 0\} = 0,
\]

\[
\nabla_w \{0, 1_b,i\} = |\sigma_i| \mathbf{n}_i / |T|,
\]

\[
\nabla_w s_{b,h} = \sum_{i=1}^{N} s_{b,i} |\sigma_i| \mathbf{n}_i / |T|, \quad \nabla_w \lambda_h = \sum_{i=1}^{N} \lambda_{b,i} |\sigma_i| \mathbf{n}_i / |T|.
\]

Similarly, by using the definition of the weak curl (3.2), we have

\[
\nabla_w \times \{e^k, 0\} = 0,
\]

\[
\nabla_w \times \{0, e^k_{b,i}\} = -e^k_{b,i} \times \mathbf{n}_i |\sigma_i| / |T| = -e^{k}_{bn,i} |\sigma_i| / |T|,
\]

\[
\nabla_w \times q_h = -\sum_{k=1}^{2} \sum_{i=1}^{N} q^{k}_{b,i} e^k_{bn,i} |\sigma_i| / |T|.
\]
To derive a formula for the element stiffness matrix and the load vector, we may consider \[ \[4.2\] \] with a finite element partition consisting of only one element \( T \). By testing this equation with test functions \( v = e^j, r = \{1,0\}, \{0,1_i\}, \varphi = \{1,0\}, \{0,1_i\}, \psi = \{e^j,0\}, \{0,e^j_i\} \), we easily arrive at the following discrete equations:

\[
\begin{aligned}
& h^{-1}_T \sum_{i=1}^N (\lambda_0 - \lambda_{b,i})|\sigma_i| = -(f,1), \\
& h^{-1}_T (\lambda_0 - \lambda_{b,i})|\sigma_i| + \left( \sum_{k=1}^d u_k e^k \varepsilon n|\sigma_i|/|T| \right) = \sum_{i=1}^N \alpha_i 1_{\Gamma^c|\sigma_i}, \\
& h^{-1}_T \sum_{i=1}^N \sum_{k=1}^d q_k (e^k \times n_i) \cdot (e^j \times n_i)|\sigma_i| - h^{-1}_T \sum_{k=1}^d q_k e^k_i \cdot (e^j \times n_i)|\sigma_i| \\
& \quad + \sum_{i=1}^N s_{b,i} |\sigma_i| e^j \cdot n_i = \int_T g^j, \\
& -h^{-1}_T \sum_{k=1}^d q_k (e^k \times n_i) \cdot e^j_{b_{n,i}} |\sigma_i| + h^{-1}_T \sum_{k=1}^d q_k e^k_i \cdot e^j_{b_{n,i}} |\sigma_i| \\
& \quad - \sum_{k=1}^d u_k e^k \cdot e^j_{b_{n,i}} |\sigma_i| = \langle \chi, e^j_{b_{n,i}} \rangle_{\sigma_i}, \\
& h^{-1}_T \sum_{i=1}^N (s_0 - s_{b,i}) |\sigma_i| = 0, \\
& h^{-1}_T (s_0 - s_{b,i}) |\sigma_i| + \sum_{k=1}^d q_k e^k \cdot n_i |\sigma_i| = 0, \\
& \sum_{i=1}^N \lambda_{b,i} e^j \cdot \varepsilon n_i |\sigma_i| - \sum_{k=1}^2 \sum_{i=1}^N q_k e^k_i \cdot e^k_{b_{n,i}} |\sigma_i| = 0.
\end{aligned}
\]

A matrix version for the above discrete equations gives rise to the formula \[ \[5.2\] \]. □

Remark 5.1. It is not hard to see that the element stiffness matrix is of size \( 2 + 2N + 2d + N(d - 1) \). Therefore, a cubic element would have 32 dofs in total and a tetrahedral element has 24 dofs. In general, for a finite element partition of \( N_T \) elements with \( N_\sigma \) faces for each element, the corresponding linear system has dofs no more than \( 2N_T + 2N_\sigma + 2N_T^* d + N_\sigma^*(d - 1) \). While the scheme \[ \[4.2\] \] appears to have a lot of dofs with piecewise constant approximations, the element stiffness matrix is in fact quite easy to compute. This numerical scheme can be further simplified through condensation or hybridization techniques for fast and parallel computing, which will be addressed in forthcoming papers.

6. Solution Existence and Uniqueness. In this section we show that the PDWG scheme \[ \[4.2\] \] has solutions, and the solution is unique for the component \( u_k \).

Denote by \( Q_0 \) the \( L^2 \) projection operator onto \( P_h(T) \). On each face \( \sigma \in \partial T \), we use \( Q_h \) to denote the \( L^2 \) projection operator onto \( P_h(\sigma) \). Denote by \( Q_h \) the projection operator onto the weak finite element space \( W(k,T) \) such that

\[
(Q_h w)|_T = \{Q_0 w|_T, Q_0 w|_{\partial T}\}.
\]

Analogously, we use \( Q_0, Q_h \) and \( Q_h \) to denote the \( L^2 \) projection operators onto the vector-valued finite element spaces \( [P_h(T)]^3, [P_h(\sigma)]^3 \), and \( V(k,T) \), respectively.
Theorem 6.1. For the finite element spaces $U_h, M_h, S_h, V_h$ constructed in (4.1), the solution $(u_h, s_h, \lambda_h, q_h)$ of the primal-dual weak Galerkin finite element scheme (4.2) is unique for all the components except $q_h$. The solution for $q_h$ is unique up to a continuous piecewise $[P_k]^{13}$ harmonic field in $H^n_0(\Omega)$.

Proof. For any solution $(u_h, s_h, \lambda_h, q_h)$ of (4.2) arising from the finite element spaces $U_h, M_h, S_h, V_h$ with homogeneous data, the following clearly holds true:

\begin{align}
S_1(\lambda_h, q_h; \lambda_h, q_h) &= 0, \quad S_2(s_h, s_h) = 0, \\
(u_h, \varepsilon \nabla w \varphi + \nabla w \times \psi) + (\psi_0, \nabla w s_h) &= 0, \quad \forall \varphi, \psi \in S_h \times V_h, \\
(q_0, \nabla w r) + (v, \varepsilon \nabla w \lambda_h + \nabla w \times q_h) &= 0, \quad \forall (v, r) \in U_h \times M_h.
\end{align}

From (6.1), we have

\begin{align}
\lambda_0 &= \lambda_b, \quad q_0 \times n = q_b \times n, \quad s_0 = s_b \text{ on } \partial T, \quad \forall T \in \mathcal{T}_h
\end{align}

so that $\lambda_0 \in H^1_0(\Omega), q_0 \in H(\text{curl}; \Omega), s_0 \in H^1(\Omega)$ and hence

\begin{align}
\nabla \lambda_0 &= \nabla w \lambda_b, \quad \nabla \times q_0 = \nabla w \times q_b, \quad \nabla s_0 = \nabla w s_b.
\end{align}

By letting $r = 0$ in (6.3) we obtain

\begin{align}
\varepsilon \nabla \lambda_0 + \nabla \times q_0 &= 0 \quad \text{in } \Omega.
\end{align}

It follows from $\lambda_0 \in H^1_0(\Omega)$ that

\begin{align}
(\varepsilon \nabla \lambda_0 + \nabla \times q_0, \nabla \lambda_0) &= (\varepsilon \nabla \lambda_0, \nabla \lambda_0) + (\nabla \times q_0, \nabla \lambda_0) \\
&= (\varepsilon \nabla \lambda_0, \nabla \lambda_0) + (q_0, \nabla \lambda_0 \times n) \\
&= (\varepsilon \nabla \lambda_0, \nabla \lambda_0).
\end{align}

Substituting (6.6) into the above identity yields

\begin{align}
(\varepsilon \nabla \lambda_0, \nabla \lambda_0) &= 0,
\end{align}

which leads to

\begin{align}
\nabla \lambda_0 &= 0,
\end{align}

so that $\lambda_0 \equiv 0$ as $\lambda_0 \in H^1_0(\Omega)$. This further implies that $\lambda_b \equiv 0$ and $\nabla \times q_0 = 0$ in $\Omega$.

Next, from (6.3), the Lagrangian multiplier $q_0$ is seen to satisfy the following equation:

\begin{align}
(q_0, \nabla w r) &= 0, \quad \forall r \in M_h,
\end{align}

which implies $q_0 \in H(\text{div}; \Omega), \nabla \cdot q_0 = 0$, and $q_0 \cdot n = 0$ on the domain boundary so that $q_0 \in H^n_0(\Omega)$.

Finally, from the Helmholtz decomposition (A.1) in Theorem A.1, there exist $\tilde{\varphi} \in H^1_0(\Omega)$ and $\tilde{\psi} \in H(\text{curl}; \Omega)$ such that

\begin{align}
u_h &= \nabla \tilde{\varphi} + \varepsilon^{-1} \nabla \times \tilde{\psi}, \quad \nabla \cdot \tilde{\psi} = 0, \quad \tilde{\psi} \cdot n = 0 \text{ on } \partial \Omega.
\end{align}

By letting $\varphi = Q_h \tilde{\varphi} \in S_h$ and $\psi = Q_h \tilde{\psi} \in V_h$, from the above equation we obtain

\begin{align}u_h &= \nabla w \varphi + \varepsilon^{-1} \nabla w \times \psi.
\end{align}
As $\nabla w s_h = \nabla s_0$, from the above equation and (6.2) we have
\[
0 = (\varepsilon u_h, \nabla w \varphi + \varepsilon^{-1} \nabla w \times \psi) + (\psi_0, \nabla w s_h)
= (\varepsilon u_h, u_h) + (Q_0 \tilde{\psi}, \nabla s_0)
= (\varepsilon u_h, u_h) + (\tilde{\psi}, \nabla s_0)
= (\varepsilon u_h, u_h),
\]
where we have also used the fact that $\nabla \cdot \tilde{\psi} = 0$ and $\tilde{\psi} \cdot n = 0$ on $\partial \Omega$. It follows that $u_h \equiv 0$.

Going back to (6.2), from $u_h = 0$ we obtain
\[
(\psi_0, \nabla w s_h) = 0 \quad \forall \psi \in V_h,
\]
which leads to $\nabla s_0 = \nabla w s_h = 0$ so that $s_0 \equiv 0$ and hence $s_h \equiv 0$. This completes the proof of the solution uniqueness for $u_h, s_h$, and $\lambda_h$. The solution for the Lagrangian multiplier $q_h$ is unique up to a continuous piecewise $[P_k]^3$ polynomial in the harmonic space $H_{0,0}(\Omega)$.

The proof of the solution uniqueness indicates that the kernel of the matrix for the PDWG finite element scheme (4.2) consisting of functions in the following form:
\[
(u_h, s_h, \lambda_h, q_h) = (0, 0, 0, \eta_h) \in U_h \times M_h \times S_h \times V_h,
\]
where $\eta_h \in H_{n,0}(\Omega)$ is continuous piecewise polynomials in $[P_k]^3$. For simplicity, we denote this kernel space by $H_h \subset H_{n,0}(\Omega)$. For the case of $k = 0$ (i.e., piecewise constant approximating functions), the kernel space $H_h$ would consist of a constant vector in $\mathbb{R}^3$ satisfying the homogeneous normal boundary condition on $\partial \Omega$. Thus, we have $H_h = \{0\}$ in nearly all the applications, so that the solution for $q_h$ is in fact unique in the usual sense.

**Theorem 6.2.** The primal-dual weak Galerkin finite element scheme (4.2) has at least one solution $(u_h, s_h, \lambda_h, q_h)$ in the finite element spaces $U_h, M_h, S_h, V_h$ given in (4.1).

**Proof.** The linear system (4.2) has solutions as long as the following compatibility condition is satisfied:
\[
G(\varphi, \psi) = 0 \quad \forall \psi \in H_h, \quad \varphi = 0.
\]
In fact, from (4.5) and the compatibility condition (2.9), we have
\[
G(0, \eta_h) = (g, \eta_h) + (\chi, \eta_h) = 0 \quad \forall \eta_h \in H_h,
\]
which completes the proof of the theorem. $\Box$

**7. Error Equations.** For the numerical approximation $(u_h, s_h, \lambda_h, q_h) \in U_h \times M_h \times S_h \times V_h$ of the div-curl system with tangential boundary condition arising from the PDWG scheme (4.2), we introduce the following error functions:
\[
e_{u} = Q_0 u - u_h, \quad e_{s} = Q_h s - s_h, \quad e_{\lambda} = Q_h \lambda - \lambda_h, \quad e_{q} = Q_h q - q_h,
\]
where $(u, s)$ is the exact solution of the variational problem (2.1)-(2.3), and $(\lambda, q)$ is the exact solution of the dual problem (2.4). Recall that we have $s = 0$, $\lambda = 0$, and shall take $q = 0$ (note that the solution for $q$ is non-unique). It is clear that $(e_{u}, e_{s}, e_{\lambda}, e_{q}) \in U_h \times M_h \times S_h \times V_h$. 
Lemma 7.1. The following equations are satisfied by \((e_u, e_s, e_\lambda, e_q) \in U_h \times M_h \times S_h \times V_h:\)

\[
\begin{align*}
S_1(e_\lambda, e_q; \varphi, \psi) + B_h(e_u, e_s; \varphi, \psi) &= \ell_u(\varphi, \psi) \quad \forall \varphi \in S_h, \psi \in V_h, \quad (7.1) \\
-S_2(e_s, r) + B_h(v, r; e_\lambda, e_q) &= 0 \quad \forall v \in U_h, r \in M_h, \quad (7.2)
\end{align*}
\]

where

\[
\ell_u(\varphi, \psi) := \langle e_u, e_n(\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h}.
\]

The equations \((7.1)\) and \((7.2)\) are called error equations. Here

\[
\langle e_u, e_n(\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h} := \sum_{T \in T_h} \langle e_u, e_n(\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T}.
\]

Proof. We first derive an equation for the \(L^2\) projections of the exact solution. To this end, for the exact solution \((u, s = 0)\), we have

\[
B_h(Q_0 u, Q_h s; \varphi, \psi) = \langle Q_0 u, e\nabla w(\varphi + \nabla w \times \psi) + (\psi_0, \nabla w Q_h s) \\
= \langle Q_0 u, e\nabla \varphi_0 + \nabla \times \psi_0 \rangle \\
+ \langle Q_0 u, e\nabla (\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h} \\
= \langle u, e\nabla \varphi_0 + \nabla \times \psi_0 \rangle \\
+ \langle Q_0 u, e\nabla (\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h} \\
= \langle u, e\nabla \varphi_0 + \nabla \times \psi_0 \rangle \\
+ \langle Q_0 u, e\nabla (\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h}
\]

\[
(7.4)
\]

\[
= \langle u, \psi_b \times n \rangle_{\partial \Omega} + \sum_{i=1}^L \alpha_i \varphi_b |_{\Gamma_i} \\
= \langle f, \varphi_0 \rangle + \langle g, \psi_0 \rangle \\
+ \langle u - Q_h u, e\nabla (\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h}
\]

\[
+ \langle \psi_b \times n \rangle_{\partial \Omega} - \langle f, \varphi_0 \rangle + \langle g, \psi_0 \rangle + \sum_{i=1}^L \alpha_i \varphi_b |_{\Gamma_i} \\
= \langle \chi \times n, \psi_b \times n \rangle_{\partial \Omega} + \sum_{i=1}^L \alpha_i \varphi_b |_{\Gamma_i}
\]

where we have used the usual integration by parts and the fact that \(u\) satisfies the div-curl system \((1.1a)-(1.1d)\), plus \langle u, e\nabla \varphi_b \rangle_{\partial \Omega} = \sum_i \alpha_i \varphi_b |_{\Gamma_i}\), and \(u \times n = \chi\) on \(\partial \Omega\).

Thus, from \((7.4)\) and the fact that \(\lambda = 0\) and \(q = 0\) we arrive at

\[
S_1(Q_h \lambda - \lambda_h, Q_h q - q_h; \varphi, \psi) + B_h(Q_0 u - u_h, Q_h s - s_h; \varphi, \psi) = \langle u - Q_0 u, e\nabla (\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n \rangle_{\partial T_h}, \forall \varphi, \psi, \quad (7.5)
\]

The second error equation can be easily seen as follows:

\[
-S_2(Q_h s - s_h, r) + B_h(v, r; Q_h \lambda - \lambda_h, Q_h q - q_h) = 0, \forall v, r, \quad (7.6)
\]

where we have used the fact that \(s = 0, q = 0, \) and \(\lambda = 0\). The equations \((7.5)-(7.6)\) lead to \((7.1)-(7.2)\).
8. Error Estimates. In the space $M_h$ and $S_h \times V_h$, we introduce the following semi-norms

\[(8.1) \quad \|s\|^2 = S_2(s, s), \]
\[(8.2) \quad \|(\lambda, q)\|^2 = S_1(\lambda, q; \lambda, q). \]

**Lemma 8.1.** The error functions $e_s$ and $(e_\lambda, e_q)$ have the following error estimates

\[(8.3) \quad \|(e_\lambda, e_q)\| + \|e_s\| \leq C h^{k+\theta} \|u\|_{k+\theta}, \]

where $\theta \in (1/2, 1]$ and $k$ is the order of polynomials in the finite element space $U_h$.

**Proof.** By choosing $\varphi = e_\lambda, \psi = e_q$ in (7.1), and $v = e_u, r = e_s$ in (7.2), the two resulting equations give rise to

\[S_1(e_\lambda, e_q; e_\lambda, e_q) + S_2(e_s, e_s) = (u - Q_0 u, \varepsilon n (\varepsilon_\lambda, 0) - e_\lambda, b) + (e_q, b - e_q, 0) \times n) \partial \Omega_h, \]

which, from the Cauchy-Schwarz inequality, leads to

\[(8.4) \quad S_1(e_\lambda, e_q; e_\lambda, e_q) + S_2(e_s, e_s) \leq C \sum_{T \in \Omega_h} h_T \|u - Q_0 u\|_{\partial T}^2 \]
\[\leq C h^{2k+2\theta} \|u\|_{k+\theta}^2 \]

which verifies the error estimate (8.3). $\Box$

Next we derive an estimate for the error function $e_u$. To this end, from the Helmholtz decomposition (A.1), there exist two functions $\tilde{\phi} \in H^k(\Omega)$ and $\tilde{\psi} \in H(curl; \Omega) \cap H(div; \Omega)$ such that

\[e_u = \varepsilon^{-1} \nabla \times \tilde{\psi} + \nabla \tilde{\phi}, \quad \nabla \cdot \tilde{\psi} = 0, \quad \tilde{\psi} \cdot n = 0 \quad \text{on} \ \partial \Omega. \]

Assume that the following $H^\alpha$-regularity holds true for this Helmholtz decomposition:

\[(8.5) \quad \|\tilde{\psi}\|^\alpha + \|\tilde{\phi}\|^\alpha \leq C \|e_u\|_0, \]

with some $\alpha \in (1/2, 1]$.

**Theorem 8.2.** Let $u \in [L^2(\Omega)]^3$ be the solution of (1.1a)-(1.1d), and $u_h \in U_h$ be the solution of the PDWG scheme (4.2). Then, the following error estimate holds true:

\[(8.6) \quad \|\varepsilon^{-1/2} (Q_0 u - u_h)\| \leq C h^{k+\theta+\alpha-1} \|u\|_{k+\theta}, \]

where $\alpha \in (1/2, 1]$ is the regularity parameter in (8.5), $k+\theta$ is the regularity of $u$ with some $\theta \in (1/2, 1]$ and $k$ is the order of polynomials for the finite element space $U_h$.

**Proof.** By choosing $\psi = Q_h \tilde{\psi}$ and $\varphi = Q_h \tilde{\phi}$ in equation (7.1), we obtain

\[(8.7) \quad B_h(e_u, e_s; \varphi, \psi) = \langle u - Q_0 u, \varepsilon n (\varphi_0 - \varphi_b) + (\psi_0 - \psi_b) \times n) \partial \Omega_h, \]
\[\quad -S_1(e_\lambda, e_q; \varphi, \psi). \]

On the other hand, we have

\[(8.8) \quad B_h(e_u, e_s; \varphi, \psi) = (e_u, \varepsilon \nabla w Q_h \tilde{\phi} + \nabla w \times Q_h \tilde{\psi}) + (Q_0 \tilde{\psi}, \nabla w e_s) \]
\[= (e_u, \varepsilon \nabla \tilde{\phi} + \nabla \times \tilde{\psi}) + (Q_0 \tilde{\psi}, \nabla w e_s) \]
\[= (e_u, \varepsilon u) + (Q_0 \tilde{\psi}, \nabla w e_s), \]
and from the definition of the weak gradient

\[(Q_0 \tilde{\psi}, \nabla_u e_u) = (Q_0 \tilde{\psi}, \nabla e_{s,0}) + (Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0})_{\partial T_h}\]

\[= (\tilde{\psi}, \nabla e_{s,0}) + (Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0})_{\partial T_h}\]

\[= - (\nabla \cdot \tilde{\psi}, \nabla e_{s,0}) + (\tilde{\psi} \cdot n, e_{s,0})_{\partial T_h} + (Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0})_{\partial T_h}\]

\[= (\tilde{\psi} \cdot n, e_{s,0} - e_{s,b})_{\partial T_h} + (Q_0 \tilde{\psi} \cdot n, e_{s,b} - e_{s,0})_{\partial T_h}\]

\[= (\tilde{\psi} - Q_0 \tilde{\psi}) \cdot n, e_{s,0} - e_{s,b})_{\partial T_h}.\]

Substituting the above into (8.8) then (8.7) yields

\[\|\varepsilon \frac{1}{2} e_u\|^2 = B_h(e_u, e_s; \varphi, \psi) - (\tilde{\psi} - Q_0 \tilde{\psi}) \cdot n, e_{s,0} - e_{s,b})_{\partial T_h}\]

\[= (u - Q_0 u, \varepsilon n(\varphi_0 - \varphi_b) + (\psi_b - \psi_0) \times n)_{\partial T_h} - S_1(e_{\lambda}, e_{q}; \varphi, \psi)\]

\[- (\tilde{\psi} - Q_0 \tilde{\psi}) \cdot n, e_{s,0} - e_{s,b})_{\partial T_h},\]

which gives

\[\|\varepsilon \frac{1}{2} e_u\|^2 \leq C h^{n-1} \|u - Q_0 u\| + h^\theta\|u - Q_0 u\| + \|(e_{\lambda}, e_{q})\| \|\tilde{\psi}\|_{\alpha} + \|\tilde{\psi}\|_{\alpha}\]

\[\leq C h^{n} \|e_u\|_{\alpha},\]

which, by using the regularity assumption (8.5), leads to

\[\|\varepsilon \frac{1}{2} e_u\| \leq C h^{n-1} \|u - Q_0 u\| + h^\theta\|u - Q_0 u\| + \|(e_{\lambda}, e_{q})\|\]

\[\leq C h^n \|e_u\|_{\alpha}.\]

Substituting (8.3) into the above estimate gives

\[\|\varepsilon \frac{1}{2} e_u\| \leq C h^{k+\theta + n-1} \|u\|_{k+\theta}.\]

This completes the proof of the theorem. □

9. Numerical Experiments. The goal of this section is to numerically demonstrating the performance of the PDWG finite element method [4.2]. Various numerical examples are employed in the numerical experiments; some are defined on convex domains and others are on non-convex polyhedral domains with various topological properties. In the case of convex domain, we use a test problem defined on the unit cube \(\Omega = (0, 1)^3\). The non-convex domains include domains with single or multiple holes. The PDWG scheme [4.2] was implemented by using the lowest order element; i.e., \(k = 0\), so that the vector field \(u\) is approximated by piecewise constant functions.

9.1. Tests on the unit cubic domain. The computational domain is given by \(\Omega = (0, 1)^3\), which is partitioned into cubic elements with different meshsize \(h\). The div-curl system with \(\varepsilon = I\) was considered. Our test examples assumed the following exact solutions:

\[u_1 = \begin{bmatrix} y(1 - y)z(1 - z) \\ x(1 - x)z(1 - z) \\ x(1 - x)y(1 - y) \end{bmatrix}, \quad u_2 = \begin{bmatrix} \sin(\pi x) \sin(\pi y) \sin(\pi z) \\ xyz \\ (x + 1)(y + 1)(z + 1) \end{bmatrix},\]

\[u_3 = \begin{bmatrix} y(1 - y)z(1 - z) \\ x(1 - x)z(1 - z) \\ r^2 \sin(\theta)(1-x)(1-y) \end{bmatrix}, \quad u_4 = \begin{bmatrix} \nabla(r \hat{z} \sin(\frac{\theta}{2})) \end{bmatrix},\]
where the cylindrical coordinates are used in the third and fourth test cases; i.e., \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1}(y/x) \). Note that the vector field \( \mathbf{u}_3 \) is in \( H^{1+\frac{2}{3}-\epsilon}(\Omega) \) and \( \mathbf{u}_4 \) is in \( H^{3-\epsilon}(\Omega) \) with \( \epsilon > 0 \). The test examples with \( \mathbf{u}_1, \mathbf{u}_3, \) and \( \mathbf{u}_4 \) as exact solutions have been considered in [13, 20]. The right-hand side functions \( f \) and \( g \) are chosen to match the exact solution for each test example. The tangential boundary condition was imposed on the boundary \( \Gamma = \partial \Omega \).

The approximation error and convergence rates for the lowest order PDWG scheme (4.2) are reported in Table 9.1. A super-convergence of order 2 was clearly seen for the test case with exact solution \( \mathbf{u}_1 \). Observe that the tangential boundary condition is of homogenous for the case of \( \mathbf{u}_1 \). For the case of \( \mathbf{u}_2 \) and \( \mathbf{u}_3 \), the numerical convergence has an order higher than the optimal order of \( r = 1 \), which outperforms the convergence theory developed in previous sections. For \( \mathbf{u}_4 \), the numerical results show that the PDWG scheme performs better than the theoretical rate of convergence \( r = \frac{2}{3} \).

Table 9.1
Error and convergence performance of the PDWG scheme for the div-curl systems on cubic meshes. \( r \) refers to the order of convergence in \( O(h^r) \).

| \( n \) | \( \| \mathbf{e}_u \|_0 \) | \( r = \) | \( \| \mathbf{e}_u \|_0 \) | \( r = \) |
|-----|----------------|-----|----------------|-----|
| 2   | 2.48e-02       | 1.57e-01 | 5.34e-03 | 2.11 |
| 4   | 5.34e-03       | 2.22  | 7.64e-02 | 1.04 |
| 8   | 1.24e-03       | 2.11  | 2.75e-02 | 1.47 |
| 16  | 3.03e-04       | 2.03  | 8.25e-03 | 1.74 |

9.2. Numerical tests on domains with complex topology. The test problems involve three different type of domains, including (a) a toroidal domain given specifically by \( \Omega_A = (-2, 2)^3/H_A \), with \( H_A = [-1, 1] \times [-1, 1] \times [-2, 2] \); (b) a cubic domain with a small hole inside, given by \( \Omega_B = (-2, 2)^3/H_B \), with \( H_B = [-1, 1]^3 \); and (c) a domain with two holes given by \( \Omega_C = (-2, 2) \times (-2, 6) \times (0, 1)/(H_C \cup H_D) \), with \( H_C = [-1.5, 1.5] \times [-1.5, 1.5] \times [0, 1] \) and \( H_D = [-1.5, 1.5] \times [2.5, 5.5] \times [0, 1] \). The domains are illustrated in Figure 9.1.

The exact solutions of the test problems are given as follows:

\[
\mathbf{u}_5 = \begin{bmatrix} x + y + z \\ x - z \\ x + 3y \end{bmatrix}, \quad \mathbf{u}_6 = \begin{bmatrix} \sin(x) \sin(y) \sin(z) \\ xyz \\ (x+1)(y+1)(z+1) \end{bmatrix}.
\]

The right-hand side functions \( f \) and \( g \) are computed to match the exact solutions for each test case. The tangential boundary condition is imposed on the boundary \( \Gamma = \partial \Omega \).

The numerical errors and convergence rates for the scheme (4.2) are reported in Table 9.2. It can be seen that the numerical convergence has rates higher than
Table 9.2

Numerical error and convergence performance of the PDWG scheme for the div-curl system on cubic partitions. \( r \) refers to the order of convergence in \( O(h^r) \).

| domain a |  | domain b |  | domain c |  |
|----------|---|----------|---|----------|---|
| \( u_5 \) | \( u_6 \) | \( u_5 \) | \( u_6 \) | \( u_5 \) | \( u_6 \) |
| \( n \) | \( ||e_u||_0 \) | \( r = \) | \( ||e_u||_0 \) | \( r = \) | \( ||e_u||_0 \) | \( r = \) |
| 2       | 1.13e-01 | -       | 3.05e-01 | -       | 1.19e-01 | 1.36 |
| 4       | 4.34e-02 | 1.38    | 1.19e-01 | 1.36    | 4.20e-02 | 1.50 |
| 8       | 1.43e-02 | 1.60    | 4.20e-02 | 1.50    | 1.19e-01 | 1.36 |

the optimal rate of convergence of \( r = 1 \) in all test cases. The computation thus outperforms the theory for the PDWG scheme (4.2). The numerical solutions are plotted in Figure 9.2 for each test, which clearly indicate an excellent performance of the PDWG scheme (4.2).

Appendix A. Helmholtz Decomposition. The following Helmholtz decomposition holds the key to the derivation of a suitable variational form for the div-curl problem (1.1a)-(1.1d).

Theorem A.1. For any vector-valued function \( u \in [L^2(\Omega)]^3 \), there exists a unique \( \phi \in H^1_0(\Omega) \) and a vector field \( \psi \in H(\text{curl}; \Omega) \) such that

\[
(A.1) \quad u = \varepsilon^{-1} \nabla \times \psi + \nabla \phi,
\]

where \( \psi \) additionally satisfies

\[
(A.2) \quad \nabla \cdot \psi = 0, \quad \psi \cdot n = 0 \quad \text{on} \ \partial \Omega.
\]
Proof. A proof of the decomposition (A.1) has been given in [22]. Below is an outline of the proof.

For any vector field $\mathbf{u} \in [L^2(\Omega)]^3$, let $\phi \in H^1_{0c}(\Omega)$ be the unique solution of the following problem:

$$
(\varepsilon \nabla \phi, \nabla s) = (\varepsilon \mathbf{u}, \nabla s) \quad \forall \ s \in H^1_{0c}(\Omega).
$$

By letting $\mathbf{v} = \mathbf{u} - \nabla \phi$, it is not hard to see that

$$
\nabla \cdot (\varepsilon \mathbf{v}) = 0, \quad (\varepsilon \mathbf{v} \cdot \mathbf{n})_{\Gamma_i} = 0
$$
for \( i = 0, 1, \cdots, L \). Thus, from Theorem 3.4 of [14], there exists a vector potential field \( \psi \in [H^1(\Omega)]^3 \) such that

\begin{align}
&A.3 \quad \varepsilon v = \nabla \times \psi, \ \nabla \cdot \psi = 0, \\
&A.4 \quad \|\psi\|_1 \lesssim (\varepsilon v, v)^{\frac{1}{2}}.
\end{align}

Furthermore, from Theorem 3.5 of [14], among all the vector fields \( \psi \) satisfying (A.3), we may choose \( \psi \in H(\text{curl}; \Omega) \) such that

\[ \psi \cdot n = 0 \quad \text{on } \partial \Omega. \]

It should be pointed out that the vector field \( \psi \) in the Helmholtz decomposition (A.1) is not uniquely determined by the condition (A.2), as nothing will change when \( \psi \) is altered by any harmonic function \( H_{n,0}(\Omega) \). But the decomposition (A.1) would be unique when \( \psi \) is restricted to the \( L^2 \)-orthogonal complement of the harmonic space \( H_{n,0}(\Omega) \).

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