Improvement on a Central Theory of PDEs

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Abstract

One of the two basic theorems in [5] on the existence of solutions of PDEs is improved with the use of a group analysis type argument.

1. Preliminaries

Recently, two rather different and general, that is, type independent solution methods have been developed for very large classes of linear and nonlinear systems of PDEs with possibly associated initial and/or boundary conditions. One of them, [5], is using standard Functional Analytic methods, while the other, [6,1,2,7-11], is based on a new idea in the realms of PDEs, namely, the Dedekind order completion of spaces of smooth functions.

Contrary to widespread perceptions, it thus proves to be possible to implement no less than two powerful solution methods for a very large variety of linear and nonlinear PDEs. These two methods are type independent in the sense that they are no longer dependent on specifics of one or another of the countless particular types of PDEs.

In fact, the essence of both methods is that, each in its own way is able to solve far more general equations than PDEs. And it is precisely in this lifting to a higher level of generality, one beyond PDEs, that the two methods attain their respective type independent power.
These two solution methods have rather complementary strong, respectively, weak points. The one in [5] does in fact deliver not only the existence of solutions, but also efficient numerical methods for approximating them. On the other hand, the method in [6,1,2,7-11] can deal with considerably more general equations, and among them linear and nonlinear systems of PDEs with possibly associated initial and/or boundary conditions.

In this paper one of the basic theorems in [5] is improved, thus allowing for its application to larger classes of PDEs. This theorem assumes two inequalities which prove to be sufficient for the existence of solutions.

Applying group theoretic ideas, in this paper the respective to inequalities as significantly relaxed, thus they lead to weaker sufficient conditions for the existence of solutions.

As for the mentioned general mentality regarding ways of solving PDEs, we cite here two rather typical views:

The 2004 edition of the Springer Universitext book ”Lectures on PDEs” by V I Arnold, [3], starts on page 1 with the statement:

"In contrast to ordinary differential equations, there is no unified theory of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry ..." (italics added)

The 1998 edition of the book ”Partial Differential Equations” by L C Evans, [4], starts his Examples on page 3 with the statement:

"There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modelled by PDEs. Instead, research focuses on various particular partial differential equations ...” (italics added)
What other comment could one possibly make on such views, except to recall the ancient Latin one:

"Sic transit gloria mundi ..."

2. The Basic Theorem

For convenience, we recall here the mentioned basic theorem, see [5, p. 50, Theorem 7]. The respective setup is as follows. We have a mapping

\[ F : H \rightarrow K, \quad F \in \mathcal{C}^1 \]

where \( H \) and \( K \) are two Hilbert spaces. This mapping has the property that the linear or nonlinear system of PDEs, with possibly associated initial and/or boundary conditions, which is to be solved can be reduced to solving in \( u \in H \) the equation

\[ F(u) = 0 \]

and obviously, such a reduction can cover a very large variety of PDEs.

Now in order to solve (2.2), we associate with it the mapping \( \phi : H \rightarrow \mathbb{R} \) given by

\[ \phi(x) = \| F(x) \|_K^2 / 2, \quad x \in H \]

and we take the square of the norm, in order not to affect the smoothness of \( \phi \).

The aim is then to solve (2.2) by a least square method applied to \( \phi \). The respective result, [5, p. 50, Theorem 7], is as follows:

Theorem

Suppose \( \nabla \phi \) is locally Lipschitz and for certain \( c, r > 0 \) and \( x \in H \), we have
(2.4) \[ \| (\nabla \phi)(v) \|_H \geq c \| F(v) \|_K, \quad v \in B_r(x) \]
and
(2.5) \[ \| F(x) \| \leq r c \]
where as usual, \( B_r(x) \) denotes the closed ball of radius \( r \) centered at \( x \).

Then there is \( u \in B_r(x) \), such that

(2.6) \[ F(u) = 0 \]

**Remark**

There is an obvious conflict between the two necessary conditions (2.4) and (2.5). Indeed, the interest in (2.4) is in a small constant \( c > 0 \), while that is clearly not convenient in (2.5).

Therefore, there is an interest in properly dealing with this conflict, and in this paper a group analysis type argument is employed in this regard.

### 3. Group Analysis Type Argument

Obviously, the equation (2.2) which is of our concern can be written in a large variety of equivalent forms by subjecting it to suitable transformations of the dependent and/or independent variables involved. And under such transformations the sufficient conditions (2.4), (2.5) securing the existence of a solution for (2.2) may take different forms, and specifically, the constants \( c \) and \( r \) involved can obtain various values. Consequently, one or the other of these two conditions may become weaker, or possibly stronger. And if both of them happen to become weaker, that naturally leads to a convenient situation.

Our aim, therefore, is to identify transformations of the equation (2.2) which give weaker forms of conditions (2.4), (2.5).
Dependent Variable Transforms. Let us consider the case of transformations of the \textit{dependent} variable, namely, $F$. This means considering all the mappings

$$ (3.1) \quad A : K \rightarrow K, \quad A \in C^1, \quad A(0) = 0 $$

By composition with $F$ in (2.1) they give $C^1$ mappings

$$ (3.2) \quad AF = A \circ F : H \rightarrow K $$

with the property

$$ (3.4) \quad u \in H, \quad F(u) = 0 \quad \implies \quad AF(u) = 0 $$

Now, for the given $F$ in (2.1), let us suppose that it is of the form

$$ (3.5) \quad F = AG = A \circ G, \quad \text{with} \quad G : H \rightarrow K, \quad G \in C^1 $$

and the issue is to see what will the conditions (2.4), (2.5) become in terms of $G$.

Clearly, in order to find $G$ in (3.5) for a given $F$, it is sufficient to assume that $A$ in (3.1) has an inverse which satisfies

$$ (3.6) \quad A^{-1} : K \rightarrow K, \quad A^{-1} \in C^1 $$

In this way, the \textit{group} of transformations we are interested in is given by

$$ (3.7) \quad A_K = \{ A : K \rightarrow K \mid A \text{ satisfies } (3.1), (3.6) \} $$

And then, for $A \in A_K$ and $F$ in (3.5) we have

$$ (3.8) \quad u \in H, \quad F(u) = 0 \quad \implies \quad G(u) = 0 $$

Independent Variable Transforms. Alternatively, we can consider all the \textit{surjective} mappings
(3.9) \( B : H \rightarrow H, \ B \in C^1 \)

By composition with \( F \) in (2.1) they give \( C^1 \) mappings 

(3.10) \( F_B = F \circ B : H \rightarrow K \)

with the property 

(3.11) \( u, v \in H, \ F(u) = 0, \ B(v) = u \implies F_B(v) = 0 \)

And now, alternatively, for the given \( F \) in (2.1), let us suppose that it is of the form 

(3.12) \( F = _BG = G \circ B, \ \text{with} \ B : H \rightarrow H, \ B \in C^1 \)

and then the issue is to see what will the conditions (2.4), (2.5) become in terms of \( G \).

Here, in order to find \( G \) in (3.12) for a given \( F \), it is sufficient to assume that \( B \) in (3.9) has an inverse which satisfies 

(3.13) \( B^{-1} : H \rightarrow H, \ B^{-1} \in C^1 \)

Therefore, this time we are interested in the group of transformations 

(3.14) \( \mathcal{B}_H = \{ B : H \rightarrow H \mid B \text{ satisfies (3.9), (3.13)} \} \)

And thus for \( B \in \mathcal{B}_H \) and \( F \) in (3.12) we have 

(3.15) \( u \in H, \ F(u) = 0 \implies G(B(u)) = 0 \)

4. A Particular Case

For a clarification of what is involved, let us consider the simplest case when \( H = K = \mathbb{R} \). Then for \( x \in \mathbb{R}, \) we have, see (2.3)
\[ \phi(x) = \frac{(F(x))^2}{2}, \quad \nabla \phi(x) = F'(x) \]

hence (2.4), (2.5) become

(4.1) \[ |F'(v)| \geq c |F(v)|, \quad v \in B_r(x) \]

and

(4.2) \[ |F(x)| \leq rc \]

**Dependent Variable Transform.** Clearly \( A \in A_K \), if and only if \( A \in C^1(\mathbb{R}, \mathbb{R}), A(0) = 0 \) and \( A \) is strictly monotonous.

Now let us assume (3.5) for some \( A \in A_K \), then (4.1), (4.2) become

(4.3) \[ |A'(G(v))G'(v)| \geq c |A(G(v))|, \quad v \in B_r(x) \]

(4.4) \[ |A(G(x))| \leq rc \]

On the other hand, the version of (2.4), (2.5) for \( G \) would be

(4.5) \[ |G'(v)| \geq c_G |G(v)|, \quad v \in B_{r_G}(x) \]

(4.6) \[ |G(x)| \leq r_G c_G \]

for suitable \( c_G, r_G > 0 \).

And then the problem is to find \( A \) and \( G \) in (3.5), such that

(4.7) \[ (4.3), (4.4) \implies (4.5), (4.6) \]

We note that (4.3) is equivalent with

\[ |G'(v)| \geq c |A(G(v)) / A'(G(v))|, \quad v \in B_r(x) \]

hence if
(4.8) \[ r_G \leq r \]

and

(4.9) \[ |A(G(v)) / (G(v) A'(G(v)))| \geq c_G / c, \ v \in B_r(x) \]

then (4.3) implies (4.5).

**Independent Variable Transform.** Obviously \( B \in \mathcal{B}_H \), if and only if \( B \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \) and \( B \) is strictly monotonous.

Let now assume (3.12) for some \( B \in \mathcal{B}_H \), then (4.1), (4.2) take the form

(4.10) \[ |G'(B(v)) B'(v)| \geq c |G(B(v))|, \ v \in B_r(x) \]

(4.11) \[ |G(B(x))| \leq r c \]

On the other hand, the version of (2.4), (2.5) for \( G \) would be

(4.12) \[ |G'(v)| \geq c_G |G(v)|, \ v \in B_{r_G}(x) \]

(4.13) \[ |G(x)| \leq r_G c_G \]

for suitable \( c_G, r_G > 0 \).

Thus the problem is to find \( B \) and \( G \) in (3.12), such that

(4.14) \[ (4.10), (4.11) \implies (4.12), (4.13) \]

However

\[ B'(v) = 1/((B^{-1})'(B(v))), \ v \in H \]

hence (4.10) is equivalent with

\[ |G'(B(v))| \geq (c (B^{-1})'(B(v))) |G(B(v))|, \ v \in B_r(x) \]
thus if

\[(4.15) \quad r_G \leq r\]

and

\[(4.16) \quad (B^{-1})'(B(v)) \geq c_G/c, \quad v \in B_r(x)\]

then (4.10) implies (4.12).

Here we note that it is far more easy to obtain $B$ from condition (4.16), than it is to obtain $A$ from condition (4.9).

5. A Simple Example

Let us illustrate the above in the case of the equation, see (2.1), (2.2)

\[(5.1) \quad F(u) = \lambda u^2 - 1 = 0, \quad u \in \mathbb{R}\]

where $\lambda \in \mathbb{R}$ is given, and thus $F : \mathbb{R} \to \mathbb{R}$. Then (2.3) becomes

\[(5.2) \quad \phi(x) = (\lambda x^2 - 1)^2/2, \quad x \in \mathbb{R}\]

hence (4.1), (4.2) take the form

\[(5.3) \quad |2\lambda v(\lambda v^2 - 1)| \geq c |\lambda v^2 - 1|, \quad v \in B_r(x)\]

\[(5.4) \quad |\lambda x^2 - 1| \leq r c\]

We note that, if $\lambda v^2 - 1 \neq 0$, then (5.3) is equivalent with

\[2 |\lambda v| \geq c\]

thus for given $x \in \mathbb{R}$ and $r > 0$, the largest possible $c$ in (5.3) is
It follows that in such a case (5.4) becomes

\begin{equation}
| \lambda x^2 - 1 | \leq r c_{\lambda, x, r}
\end{equation}

which is a rather difficult *implicit* relation in all its variables $\lambda, x$ and $r$.

Let us now take $B \in \mathcal{B}_H$ as

\begin{equation}
B(v) = \mu v, \quad v \in \mathbb{R}
\end{equation}

where $\mu \in \mathbb{R}$, $\mu \neq 0$ is fixed, and assume (3.12). Then

\begin{equation}
G(v) = (\lambda/\mu^2)v^2 - 1, \quad v \in \mathbb{R}
\end{equation}

hence the largest possible $c$ for the version of (5.3) corresponding to $G$ is, see (5.5)

\begin{equation}
c_{\mu, \lambda, x, r} = 2 | \lambda / \mu^2 | \begin{cases} 
0 & \text{if } x - r \leq 0 \leq x + r \\
x - r & \text{if } 0 \leq x - r \\
-x - r & \text{if } x + r \leq 0
\end{cases}
\end{equation}

while (5.6) turns to

\begin{equation}
| (\lambda / \mu^2)x^2 - 1 | \leq r c_{\mu, \lambda, x, r}
\end{equation}

or in view of (5.5) and (5.9), to the equivalent relation

\begin{equation}
| \lambda x^2 - \mu^2 | \leq r c_{\lambda, x, r}
\end{equation}

And now we can recapitulate.
With $F$ in (5.1), we had the sufficient conditions (2.4), (2.5) expressed in (5.3), (5.4), and they came down to (5.6), with the largest possible $c_{\lambda, x, r}$ given in (5.5).

Here we have to note the following conflict, see Remark in section 2:

- in satisfying (5.3), there is an interest in a small $c$,

while on the other hand,

- in satisfying (5.4), a contrary interest appears.

Applying now the group transformation (5.7), instead of $F$ in (5.1), we obtain $G$ in (5.8). And then the corresponding transformed version of (5.3), (5.4) comes down to, see (5.11)

\[(5.12) \quad |\lambda x^2 - \mu^2| \leq r c_{\lambda, x, r}\]

In this way, the mentioned conflict in optimally handling the two basic necessary conditions (5.3), (5.4) can now be approached through the latitude obtained in (5.12) by the possibility of choosing $\mu \in \mathbb{R}$, $\mu \neq 0$ arbitrarily, when compared with (5.6), where $\mu$ does not appear.

**Conclusion**

The above difference between the following two relations, see (5.6), (5.12)

\[ |\lambda x^2 - 1| \leq r c_{\lambda, x, r} \]

\[ |\lambda x^2 - \mu^2| \leq r c_{\lambda, x, r} \]

where $\mu \in \mathbb{R}$, $\mu \neq 0$ is the group parameter, illustrates the existence of possibilities for a group analysis type argument even in that simple example of equation (5.1) and of that simplest group transformation of the independent variable in (5.7).
Within the general case of the Theorem in section 2, [5, p. 50, Theorem 7], and the consequent attempt for an *optimal* approach to the *conflict* inherent between the two sufficient conditions (2.4), (2.5) for the existence of solutions of very large classes of linear and nonlinear systems of PDEs with possibly associated initial and/or boundary conditions, the simple result in this regard in section 5 indicates the possibilities which may exist in general.

The more detailed exploration of such possibilities of group analysis, applied both to dependent and independent variables, in order to properly deal with the sufficient conditions in the mentioned theorem, is to be presented in subsequent papers.

**References**

[1] Anguelov R, Rosinger E E : Hausdorff continuous solutions of nonlinear PDEs through the order completion method. Quaestiones Mathematicae, Vol. 28, 2005, 1-15, arXiv : math.AP/0406517

[2] Anguelov R, Rosinger E E : Solving large classes of nonlinear systems of PDEs. Computers and Mathematics with Applications (to appear). arXiv:math.AP/0505674

[3] Arnold V I : Lectures on PDEs. Springer Universitext, 2004

[4] Evans L C : Partial Differential Equations. AMS Graduate Studies in Mathematics, Vol. 19, 1998

[5] Neuberger J W : Prospects for a central theory of partial differential equations. The Mathematical Intelligencer, Vol. 27, No. 3, Summer 2005, 47-55

[6] Oberguggenberger M B, Rosinger E E : Solution of Continuous Nonlinear PDEs through Order Completion. Mathematics Studies Vol. 181, North-Holland, Amsterdam, 1994, see MR95k:35002.
[7] Rosinger E E: New Method for Solving Large Classes of Nonlinear Systems of PDEs. \url{arXiv:math.GM/0610279}

[8] Rosinger E E: Extending Mappings between Posets. \url{arXiv:math.GM/0609234}

[9] Rosinger E E: Solving General Equations by Order Completion. \url{arXiv:math.AP/0608450}

[10] Rosinger E E: Can there be a general nonlinear PDE theory for the existence of solutions? \url{arXiv:math.AP/0407026}

[11] Rosinger E E: Hausdorff continuous solutions of arbitrary continuous nonlinear PDEs through the order completion method. \url{arXiv:math.AP/0405546}