Cosmology Is Not A Renormalization Group Flow

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A critical examination is made of two simple implementations of the idea that cosmology can be viewed as a renormalization group flow. Both implementations are shown to fail when applied to a massless, minimally coupled scalar with a quartic self-interaction on a locally de Sitter background. Cosmological evolution in this model is not driven by any RG screening of couplings but rather by inflationary particle production gradually filling an initially empty universe with a sea of long wavelength scalars.

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INTRODUCTION

The Renormalization Group (RG) describes how the predictions of flat space quantum field theory and statistical mechanics change when all spacetime coordinates are adiabatically scaled up by a constant [1].

Renormalization Group: \[ x^\mu \rightarrow A \times x^\mu. \] (1)

Cosmological evolution can be described by a superficially similar rescaling of infinitesimal conformal coordinate intervals [2].

Cosmological Evolution: \[ dx^\mu \rightarrow a(\eta) \times dx^\mu. \] (2)

It is not clear that the one rescaling relates to the other. However, particle physicists know so much about RG flows, and so little about how quantum field theoretic states evolve in an expanding universe, that many have posited there is a relation and attempted to exploit it [3].

Two simple techniques have been suggested for implementing the RG idea:

- Naive Scaling — setting the parameter \( A \) in (1) to the ratio of the cosmological scale factor \( a(\eta) \) to its initial value; and
- Hubble Scaling — setting \( 1/A \) to the ratio of the instantaneous Hubble parameter, \( H(\eta) \equiv a'/a^2 \), to its initial value.

Both techniques are dubious. For naive scaling the flat space loop amplitudes that determine the various \( \beta \) and \( \gamma \) functions involve integrations over all times. This makes no difference for constant scalings of the metric — and introduces no nonlocality, even in curved space [2] — but it can matter a great deal when the scaling is made time dependent. Which time ought we to pick, and why? Hubble scaling corresponds to taking \( H(\eta) \) as the dimensional regularization mass scale \( \mu \). That is possible in de Sitter background, for which \( H \) is a constant, but it would break general coordinate invariance otherwise.

The burden of this paper is that both of these scalings are provably wrong in a simple theory where their predictions can be checked. No shame should attach to having explored the RG idea. Using the RG to understand cosmology would have effected a vast simplification. And there is still the chance that some more complicated scaling works, or even that one of the simple scalings works for other theories. It should also be noted that curved space in no way precludes using the RG conventionally to relate quantities at different constant scales [3]. Nor do these comments apply to using the RG on the world sheet of a string model [6].

The model for which RG predictions can be checked is a massless, minimally coupled scalar with a quartic self-interaction on a nondynamical, locally de Sitter background. Perturbative results are described in Section 2. These suffice to falsify Hubble Scaling. Section 3 presents the leading logarithm solution for the late time limit of this model which was obtained by Starobinski˘ı and Yokoyama [7]. This falsifies Naive Scaling. Section 4 describes the true origin of evolution in this model.

THE MODEL

The background geometry is the \( D \)-dimensional, conformal coordinate patch of de Sitter space,

\[ ds^2 = a^2 \left[ -d\eta^2 + d\vec{x} \cdot d\vec{x} \right], \quad a(\eta) \equiv -\frac{1}{H\eta} \] (3)

Here and throughout the Hubble constant \( H \) relates to the bare cosmological constant \( \Lambda \) as \( H^2 \equiv \Lambda/(D-1) \). In terms of the renormalized fields and couplings the Lagrangian is,

\[ \mathcal{L} = \frac{1}{2}(1+\delta Z)\partial_\mu \varphi \partial_\nu \varphi \eta^{\mu\nu} \sqrt{-g} - \frac{1}{4!}(\lambda+\delta\lambda)\varphi^4 \sqrt{-g} \\
- \frac{1}{2} \delta\xi \varphi^2 R \sqrt{-g} - \frac{(D-2)\delta\Lambda}{16\pi G} \sqrt{-g}. \] (4)

There is no mass counterterm because the bare mass is zero and mass is multiplicatively renormalized. However, even this minimally coupled scalar requires a conformal counterterm \( \delta\xi \), and there is of course field strength and coupling constant renormalization. That suffices for non-coincident one-particle-irreducible (1PI) functions. Additionally renormalizing the stress tensor requires the \( \delta\Lambda \) counterterm.
The expectation value of the stress tensor has been computed in this model at one and two loop orders [8]. The scalar self-mass-squared has also been computed at one and two loop orders [9] and used to quantum-correct the scalar mode functions [10]. In each case the calculations were performed using a version of the Schwinger-Keldysh formalism [11], in the presence of an initial state [12] which is free Bunch-Davies vacuum (for modes which are initially sub-horizon [13]) at \( \eta = -1/H \).

An unfortunate renormalization scheme was employed involving a mass counterterm, which on de Sitter background [3] cannot be distinguished from the conformal counterterm for 1PI functions. However, it is simple to convert to the convention given in [4]. When this is done, the various counterterms can all be expressed in terms of the dimensionless coupling constant,

\[
\lambda = \frac{\lambda}{16\pi^2} \left( \frac{H}{\sqrt{4\pi}} \right)^{D-4}.
\]

The lowest nontrivial results for \( \delta Z, \delta \lambda \), and \( \delta \xi \) are,

\[
\delta Z = -\frac{\lambda^2}{12} \left( \frac{\Gamma^2(1-\frac{D}{2})}{\pi^2} \right) (1-\frac{D}{2})(1-\frac{D}{2}) + O(\lambda^3),
\]

\[
\delta \lambda = \lambda \times \left\{ 3\lambda \Gamma(1-\frac{D}{2}) \right\} + O(\lambda^2),
\]

\[
\delta \xi = -\frac{\lambda}{2\pi} \left[ \frac{\pi \cot(\frac{\pi}{2})}{\Gamma(1-\frac{D}{2})} \right] + O(\lambda^2).
\]

Here and henceforth we define \( \epsilon \equiv 4 - D \). Of course \( \delta Z \) and \( \delta \lambda \) agree up to finite terms with the well known results of flat space [1]. The one and two loop results for \( \delta \Delta \) are,

\[
\frac{(D-2)\delta \Delta}{16\pi G} = \frac{H^4}{\lambda} \left\{ \frac{3\lambda}{2} \left( 1-\frac{D}{2} \right) \right\} + O(\lambda^2),
\]

\[
\frac{\lambda}{H^4} \times p = \lambda^2 \left[ -2 \ln^2(a) - \frac{7}{2} \ln(a) + \frac{5}{3} - \frac{\pi^2}{3} \right]
\]

\[
-\frac{2}{3} \sum_{n=2}^{\infty} \frac{n(n-3)(n+1)}{n^2 a^n}
\]

\[
O(\lambda^3), \quad (10)
\]

\[
\frac{\lambda}{H^4} \times (\rho+p) = \lambda^2 \left[ -\frac{4}{3} \ln(a) - \frac{13}{18} \right]
\]

\[
+ \frac{8}{9} a^{-3}
\]

\[
-\frac{2}{3} \sum_{n=2}^{\infty} \frac{(n+1)}{n a^n}
\]

\[
O(\lambda^3). \quad (11)
\]

It is straightforward to verify that these results obey stress-energy conservation and that they violate the weak energy condition [8],

\[
\frac{d}{d\eta} \left[ a^3 (\rho + p) \right] = a^3 \frac{d\rho}{d\eta}, \quad \rho + p < 0.
\]

Each of the inverse powers of \( a \) in (10-11) is separately conserved. This and their rapid falloff away from the initial value surface have prompted the speculation that these terms can be absorbed into a perturbative correction of the initial state [3]. Of course the constant part of (10) can be absorbed into a finite shift of the bare cosmological constant. This leaves the secular contributions,

\[
\frac{\lambda}{H^4} \times p_{sec} = \lambda^2 \left[ -2 \ln^2(a) - \frac{7}{2} \ln(a) \right] + O(\lambda^3), \quad (13)
\]

\[
\frac{\lambda}{H^4} \times (\rho+p)_{sec} = \lambda^2 \left[ -\frac{4}{3} \ln(a) - \frac{13}{18} \right] + O(\lambda^3). \quad (14)
\]

We conclude this section by giving the RG predictions for this model. The RG evolution of its flat space analogue is the best understood of all quantum field theories. In fact, it is the paradigm for critical phenomena in systems for which the order parameter is one dimensional [1]. And there is absolutely no doubt that scaling \( \Lambda \) to infinity carries this system to a free theory in \( D = 4 \). Of course the cosmological scale factor \( a(\eta) \) grows without bound, so the prediction of Naive Scaling is that all correlation functions should become Gaussian at late times. On the other hand, the Hubble parameter is constant in de Sitter background, so the prediction of Hubble Scaling is that the expectation values of operators such as the stress tensor should show no change with time.

The fact that two simple implementations of the RG idea give different evolutions is already suspicious, even without explicit results. Of course the time dependence evident even in perturbative results such as (10-11) suffices to falsify the prediction of Hubble Scaling. The next section will show that the prediction of Naive Scaling is also wrong.

**LEADING LOG SOLUTION**

The factors of \( \ln(a) \) in expressions (13-14) are known as infrared logarithms. Any quantum field theory which involves undifferentiated gravitons or massless, minimally coupled scalars will show similar infrared logarithms in some of its Green’s functions. They occur as well in scalar quantum electrodynamics [14], in Yukawa theory [15], in pure quantum gravity [16], and in quantum gravity with fermions [17]. They even contaminate loop corrections to the power spectrum of cosmological perturbations [18] and other fixed-momentum correlators [20].

Infrared logarithms introduce a fascinating secular element into the usual, static results of quantum field theory. It was this secular evolution in perturbative results...
such as \( [13, 14] \) that allowed us to falsify the RG prediction for Hubble Scaling. Achieving a similar falsification for Naive Scaling requires a more powerful analysis because the continued growth of \( \ln(a) \) must eventually overwhelm the loop counting parameter \( \lambda \), which causes perturbation theory to break down. To evolve to late times requires a nonperturbative technique.

For certain models there are natural resummation schemes such as the \( 1/N \) expansion \([21]\). A more general technique is suggested by the form of the expansion for the pressure. The \(-2\ln^2(a)\) in expression \([13]\) is a leading logarithm, while the \(-\frac{n}{2} \ln(a)\) is a sub-leading logarithm. For this model one can show that each extra factor of \( \lambda \) in the perturbative expansion of any quantity brings at most two more powers of \( \ln(a) \) \([22]\). The general expansion for the pressure is therefore \( H^4 \) times,

\[
\sum_{n=1}^{\infty} \lambda^n \left\{ c_{n,0}(\ln(a))^{2n} + c_{n,1}(\ln(a))^{2n-1} + \ldots + c_{n,2n-1} \ln(a) \right\}.
\]

(15)

Here the constants \( c_{n,k} \) are pure numbers. Perturbation theory breaks down when \( \ln(a) \sim 1/\sqrt{\lambda} \), at which time the leading infrared logarithms at each loop order contribute numbers of order one. In contrast, the subleading logarithms are all suppressed by at least one factor of the small parameter \( \sqrt{\lambda} \). So a sensible approximation is to retain only the leading infrared logarithms.

Starobinski\u0161 has developed a simple stochastic formalism \([23]\) that reproduces the leading infrared logarithms at each order \([24]\) for any scalar potential model, including \([3]\). Probabilistic representations of inflation have been studied to understand initial conditions \([25]\), global structure \([26]\) and non-Gaussianity \([27]\). However, the focus here is on recovering the most important secular effects of inflationary quantum field theory \([28]\). It is of particular importance that Starobinski\u0161 and Yokoyama have derived the late time limit whenever the potential is bounded below \([7]\). This is the true analogue of what the RG accomplishes.

\( \langle \Omega | \varphi^2(x) | \Omega_{\text{free}} \rangle = \text{Divergent Constant} + \frac{H^2 \ln(a)}{4\pi^2} \). \hspace{1cm} (18)

This tends to drive the scalar up its \( \varphi^4 \) potential, which induces the negative pressure \( \sim \ln^2(a) \) that is evident at lowest order in expression \([19]\). At next order the classical force associated with being away from the potential minimum tends to decrease the scalar field strength, which makes the pressure less negative, and so on. The resulting pressure is an oscillating series of leading logarithms that approaches a constant at late times,

\[
-\frac{H^4}{8\pi^2} \left[ \lambda \ln^2(a) - \frac{4}{3} \lambda^2 \ln^4(a) + \frac{5936}{3645} \lambda^3 \ln^6(a) - \ldots \right] \rightarrow -\frac{3H^4}{32\pi^2}. \hspace{1cm} (19)
\]

This is just what common sense suggests must happen, although Starobinski\u0161 and Yokoyama deserve enormous credit for having proved it \([7]\). Eventual equilibrium is inevitable because there is no increase in the upward pressure on \( \varphi^2 \) from inflationary particle production, whereas the downward pressure from the classical force grows without bound. So evolution in this model can be understood in even simpler terms than RG flows. And it was, after all, the hope for such enlightenment that motivated the RG approach to cosmology \([3]\).

CONCLUSIONS

We have seen that the evolution of this model is not described by any simple RG flow. It is driven instead by the inflationary expansion ripping long wavelength, virtual scalars out of the vacuum. That process can be derived in very simple terms using the energy-time uncertainty principle and the scalar’s breaking of classical conformal invariance \([29]\). It results in a slow growth of the scalar field strength that appears in the classic result for the coincidence limit of the free scalar propagator \([30]\),

\[
\langle \Omega | \varphi^2(x) | \Omega_{\text{free}} \rangle = \text{Divergent Constant} + \frac{H^2 \ln(a)}{4\pi^2} \].
\]

(18)

It also allows one to compute the late time limits \([7, 22]\),

\[
\lim_{a \rightarrow \infty} \langle \Omega | \varphi^2(x) | \Omega_{\text{leading}} \rangle = \frac{9H^4}{16\pi^2\lambda} \]. \hspace{1cm} (17)

Now recall that the RG prediction for Naive Scaling is that the late time limit becomes Gaussian. Were that correct, the expectation value of \( 2n \) coincident fields would be \((2n-1)!\) times the \( n \)-th power of the coincident 2-point function. From \([17]\) one can see that the actual result for \( n = 2 \) is smaller by a factor of about \(.72948.\) For \( n = 3 \) the ratio is approximately \(.437688;\) for \( n = 4 \) it is about \(.22806;\) and the ratio falls exponentially for large \( n \). Hence Starobinski\u0161’s formalism falsifies the RG prediction for Naive Scaling.

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