REFLEXIVE DIFFERENTIAL FORMS ON SINGULAR SPACES – GEOMETRY AND COHOMOLOGY

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ABSTRACT. Based on a recent extension theorem for reflexive differential forms, that is, regular differential forms defined on the smooth locus of a possibly singular variety, we study the geometry and cohomology of sheaves of reflexive differentials.

First, we generalise the extension theorem to holomorphic forms on locally algebraic complex spaces. We investigate the (non-)existence of reflexive pluridifferentials on singular rationally connected varieties, using a semistability analysis with respect to movable curve classes. The necessary foundational material concerning this stability notion is developed in an appendix to the paper. Moreover, we prove that Kodaira–Akizuki–Nakano vanishing for sheaves of reflexive differentials holds in certain extreme cases, and that it fails in general. Finally, topological and Hodge-theoretic properties of reflexive differentials are explored.

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1. INTRODUCTION

Holomorphic differential forms are an indispensable tool to study the global geometry of non-singular projective varieties and compact Kähler manifolds. In the presence of singularities, with the exception of forms of degree one and forms of top degree, the influence of differential forms on the geometry of a variety is much less explored. At the same time, in higher-dimensional algebraic geometry one is naturally led to consider singular varieties in a given birational equivalence...
class, even if one is primarily interested in projective manifolds. Hence, the need for a comprehensive geometric and cohomological theory of differential forms on singular spaces arises.

In this paper we study the geometry and cohomology of sheaves of reflexive differential forms, that is, regular forms defined on the smooth locus of a possibly singular variety. Motivated by and using the techniques of the Minimal Model Program we focus on the natural classes of singularities in Mori Theory: log canonical, kawamata log terminal, canonical, and terminal. For definitions and a thorough discussion of these classes of singularities we refer the reader to [KM98, Chap. 2.3] and [Kol97].

The basic tool for our investigation is an extension theorem for reflexive differentials, which we state here in a simplified version.

**Theorem 1.1 (Extension Theorem).** Let $X$ be a quasi-projective variety and $D$ an effective $\mathbb{Q}$-divisor such that the pair $(X, D)$ is Kawamata log terminal (klt). Let $\pi : \tilde{X} \to X$ be log resolution of the pair $(X, D)$. Then, the sheaves $\pi_* (\Omega^p_{\tilde{X}})$ are reflexive for $1 \leq p \leq n$.

In fact, a much more general statement holds, for which we refer the reader to [GKKP11, Thm. 1.5] or to the survey [Keb11].

Introducing the sheaves $\Omega^p_X = (\bigwedge^p \Omega^1_X)^{**}$ of reflexive differentials, Theorem 1.1 can be rephrased by saying that

$$\pi_* (\Omega^p_X) = \Omega^p_X \quad \forall p \in \{1, \ldots, n\},$$

or in yet another way by saying that any regular differential form defined on the smooth part of a pair $(X, D)$ with at worst klt singularities defines a regular differential form on any log resolution of $(X, D)$.

We will use this result (as well as a generalisation to holomorphic differential forms on locally algebraic varieties established here) to study a number of problems about existence of differential forms, vanishing theorems, and Hodge-theoretic properties of reflexive differentials.

**Outline of the paper.** We conclude this introduction by a summary of the contents of the paper.

*Extension theorems in the holomorphic setting.* The Extension Theorem 1.1 is established in the algebraic category. Although some of the methods employed in [GKKP11] to prove Theorem 1.1, especially the use of the Minimal Model Program and certain vanishing theorems, are not at one’s disposal in the analytic context, one should nevertheless expect the extension property to hold also in the holomorphic setting. In Section 2 we prove this for klt complex spaces $X$ that locally algebraic:

**Theorem 1.2 (Extension Theorem for holomorphic forms).** Let $X$ be a normal complex space and $D$ an effective $\mathbb{Q}$-divisor such that the pair $(X, D)$ is analytically klt. Suppose $(X, D)$ is locally algebraic. Let $\pi : \tilde{X} \to X$ be log resolution of the pair $(X, D)$. Then, the sheaves $\pi_* (\Omega^p_X)$ are reflexive for $1 \leq p \leq n$.

In fact, analogous to [GKKP11, Thm. 1.5] we prove a much more general result for locally algebraic log canonical pairs, see Theorem 2.11. Using Artin’s Algebrasation Theorem, an extension theorem for normal complex spaces with isolated
singularities is derived as an immediate consequence of Theorem 1.2, see Corollary 2.16.

Reflexive differentials on rationally chain connected spaces. Rationally connected and, more generally, rationally chain connected projective varieties play a key role in the classification and structure theory of algebraic varieties. It is a fundamental fact of higher-dimensional algebraic geometry that a rationally connected projective manifold $X$ does not carry any non-trivial differential forms. This statement also holds for reflexive differential forms on rationally connected varieties with at worst klt singularities, see [GKKP11, Thm. 5.1]. In the smooth case, considering general tensor powers instead of exterior powers of $\Omega^1_X$, it can even be shown that a rationally connected projective manifold does not carry any pluri-form:

$$H^0(X, (\Omega^1_X)^{\otimes m}) = 0 \quad \forall m \in \mathbb{N}^{\geq 1}.$$ (1.2.1)

Exploring the singular setup, in Section 3 we derive an analogous vanishing result for reflexive pluri-forms on factorial klt spaces:

**Theorem 1.3** (cf. Theorem 3.3). Let $X$ be a rationally chain-connected factorial klt space. Then

$$H^0(X, ((\Omega^1_X)^{\otimes m})^{**}) = 0.$$ (1.3.1)

Note here that factorial klt spaces automatically have canonical singularities. Moreover, they are rationally connected by a result of Hacon and McKernan, cf. Remark 3.2. The above generalisation of (1.2.1) to the singular case is not at all obvious and, as a matter of fact, surprisingly fails in the non-factorial canonical case. A corresponding two-dimensional example is provided in Section 3.B. The proof of Theorem 1.3 relies on a semistability analysis of sheaves of reflexive differentials with respect to movable curves on klt spaces. The necessary foundational material concerning this semistability notion is developed in detail in Appendix A; see below for an introduction.

Vanishing theorems. Section 4 of the paper discusses vanishing theorems for reflexive differentials on klt spaces. In the classical smooth setting, the Kodaira-Akizuki-Nakano theorem states that for any ample line bundle $\mathscr{L}$ on a projective manifold $X$ we have

$$H^q(X, \Omega^p_X \otimes \mathscr{L}) = 0 \quad \text{for } p + q > n,$$

(1.3.1) $$H^q(X, \Omega^p_X \otimes \mathscr{L}^{-1}) = 0 \quad \text{for } p + q < n.$$ (1.3.2)

We will prove in Section 4.A that these vanishings still hold on klt spaces in certain extremal ranges, again with $\Omega^p_X$ replaced by $\Omega^{[p]}_X$. At the same time we show that the generalisation of (1.3.1) and (1.3.2) to sheaves of reflexive differentials on klt spaces fails in general. In fact, in Section 4.B we exhibit a projective 4-fold $X$ with a single isolated terminal Gorenstein singularity carrying an ample line bundle $\mathscr{L}$ such that

$$H^2(X, \Omega^3_X \otimes \mathscr{L}) \simeq H^2(X, \Omega^1_X \otimes \mathscr{L}^{-1}) \simeq \mathbb{C},$$

We do not know at the moment whether a similar counterexample also exists in dimension 3.
Hodge-Theory and the Poincaré-Lemma. It follows from standard Hodge theory for compact Kähler manifolds that every regular differential form on a projective manifold is closed. While the existence of a suitable Hodge theory for reflexive differentials is still an open question, the Extension Theorem [GKKP11, Thm. 1.5] implies the following generalisation of this particular Hodge-theoretic result to log canonical pairs, which we show in Section 5.

**Proposition 1.4** (Closedness of global logarithmic forms). Let \( X \) be a projective variety, and let \( D \) be a \( \mathbb{Q} \)-divisor such that the pair \((X, D)\) is log-canonical. Then, any reflexive logarithmic \( p \)-form \( \sigma \in H^0(X, \Omega^p_X(\log \lfloor D \rfloor)) \) is closed.

Next, it is shown that reflexive one-forms on klt spaces satisfy a Poincaré lemma. As a corollary, we obtain that every globally defined closed reflexive one-form is represented in a canonical manner by a Kähler differential.

**Semistability notions on singular spaces.** In Appendix A we develop a semistability theory for torsion-free sheaves with respect to a movable class on a klt space, generalizing the results presented in [CP11]. Specifically, we establish the existence of a maximal destabilizing subsheaf and prove that semistability is preserved under tensor operations.

**General Remark.** Throughout the paper we will use fundamental facts about reflexive differentials on singular varieties, as well as basic definitions concerning logarithmic pairs and resolution of singularities, for which we refer the reader to [GKKP11, Sect. 2] and [GKK10, Sect. 2].

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## 2. Extension of Holomorphic Differential Forms

In this section we discuss the extension properties of holomorphic differential forms on spaces with at worst log canonical singularities. In order to distinguish the euclidean topology of a given complex space from its Zariski topology, an open subset in the earlier topology will be called “open”, while open subsets in the latter topology will be called “Zariski-open”. A corresponding convention will be used for closed subsets.

### 2.A. Analytically klt and log canonical singularities.** We first extend the notion of klt pair and log canonical pair from the algebraic to the analytic category.

**Definition 2.1** (Analytically klt and log canonical pairs). Let \( X \) be a normal complex space and \( D \) an effective \( \mathbb{Q} \)-divisor on \( X \). Let \( p \in X \). We call the pair \((X, D)\) analytically log canonical at \( p \), respectively analytically klt at \( p \) if there exists an open neighbourhood \( U = U(p) \) such that

\[
(2.1.1) \quad K_X + D \text{ is } \mathbb{Q}\text{-Cartier on } U,
\]
(2.1.2) if $\pi : \tilde{U} \to U$ is any log resolution of $(U, D|_U)$, and $m \in \mathbb{N}$ is an integer such that $m(K_X + D)|_U = m(K_U + D|_U)$ is Cartier on $U$, then writing

$$m(K_U + \pi^{-1}_*(D|_U)) \sim_{\text{Q-lin.}} \pi^* (m(K_U + D|_U)) + \sum_{E \in \pi-\text{except. divisor}} m \cdot a(E, U, D|_U) \cdot E$$

with rational numbers $a(E, U, D|_U) \in \mathbb{Q}$, we have $a(E, U, D|_U) \geq -1$, respectively $a(E, U, D|_U) > -1$ for all $\pi$-exceptional divisors $E$.

(2.1.3) for the “klt” case, $|D|_U| = 0$.

A pair $(X, D)$ is called analytically log canonical, respectively analytically klt if it is analytically log canonical, respectively analytically klt at every point $p \in X$.

Remark 2.2. Analogously, the notion of “analytically terminal” pair is defined. A pair $(X, D)$ is analytically klt at a point $p \in X$, respectively analytically log canonical at a point $p \in X$ if and only if the triple $(X, \{p\}, D)$ is klt, respectively log canonical in the sense of [Kaw88, p. 104].

Remark 2.3. As in the algebraic case, the discrepancy condition can be checked on a single resolution. More precisely, a pair $(X, D)$ is analytically klt at a point $p \in X$ if and only if there exists an open neighbourhood $U$ of $p$ such that (2.1.1) and (2.1.3) hold, and the discrepancy inequalities required in (2.1.2) are fulfilled for a single log resolution $\pi : \tilde{U} \to U$.

Definition 2.4 (Non-klt locus). Let $(X, D)$ be an analytically log canonical pair. Then we define the non-klt locus of $(X, D)$ to be

$$\text{nklt}(X, D) := \{p \in X \mid (X, D) \text{ is not klt at } p\}.$$ 

Remark 2.5. Let $(X, D)$ be an analytically log canonical pair. Then, $\text{nklt}(X, D)$ is a Zariski-closed subset of $X$.

Next, we compare the newly introduced classes of analytic singularities to the classical notions in the algebraic category. In the following, given an algebraic object $O$ the corresponding analytic object will be denoted by $O^\text{an}$.

Lemma 2.6. Let $X$ be a normal algebraic variety and $D$ an effective $\mathbb{Q}$-divisor on $X$. Then

(2.6.1) $(X, D)$ is log canonical if and only if $(X^\text{an}, D^\text{an})$ is analytically log canonical.

(2.6.2) $(X, D)$ is klt if and only if $(X^\text{an}, D^\text{an})$ is analytically klt.

(2.6.3) If $(X, D)$ is log canonical, we have

$$\text{nklt}(X^\text{an}, D^\text{an}) = (\text{nklt}(X, D))^\text{an}.$$ 

Proof. This follows from the fact that whether or not $(X, D)$ is log canonical or klt can be checked on open subsets of $(X^\text{an}, D^\text{an})$, see [Kaw88, p. 104].

Definition 2.7. Let $X$ be a normal complex space, and $D$ an effective $\mathbb{Q}$-divisor on $X$. We call the pair $(X, D)$ locally algebraic if there exists a cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of $X$ by open subsets $U_\lambda$ such that for every $\lambda \in \Lambda$ there exists

- a normal quasi-projective variety $Y_\lambda$,
- an effective $\mathbb{Q}$-divisor $D_\lambda$ on $Y_\lambda$,
- an open subset $V_\lambda \subset (Y_\lambda)^\text{an}$, and
- a biholomorphic map of pairs $\phi_\lambda : (U_\lambda, D|_{U_\lambda}) \to (V_\lambda, (D_\lambda)^\text{an}|_{V_\lambda})$. 

Example 2.8. Every complex manifold and every complex space with finite quotient singularities is locally algebraic.

Example 2.9. It follows from Artin’s Approximation Theorem [Art69, Thm. 3.8] that any complex space with isolated singularities is locally algebraic.

Example 2.10. Let X be a Moishezon space. Then, X is the complex space associated with an algebraic space Z in the sense of Artin, see [Art70, Theorem 7.3]. Consequently, every point \( p \in X = Z^{an} \) admits an étale neighbourhood \( \varphi : U \to Z \), where \( U \) is an affine algebraic variety. It follows that X is locally algebraic in the sense of Definition 2.7 above.

2.B. The extension theorem for holomorphic forms. If Z is any non-singular complex space and \( D \) is a normal crossing divisor on \( Z \), then the sheaf of holomorphic (logarithmic) \( p \)-forms will be denoted by \( \Omega^p_Z(\log D) \).

The following is the main result of this section.

Theorem 2.11 (Extension theorem for holomorphic forms). Let \( X \) be a locally algebraic normal complex space and \( D \) an effective \( \mathbb{Q} \)-divisor on \( X \) such that \((X, D)\) is analytically log canonical. Let \( \pi : \tilde{X} \to X \) be a log resolution of \((X, D)\) with \( \pi \)-exceptional set \( E \) and

\[
\tilde{D} := \text{largest reduced divisor contained in } \text{supp } \pi^{-1}(\text{nkl}(X, D)).
\]

Then, the sheaves \( \pi_* \Omega^p_X(\log \tilde{D}) \) are reflexive for all \( p \leq n \).

Remark 2.12. As in the algebraic case, the name “Extension Theorem” is justified by the following observation: the sheaves \( \pi_* \Omega^p_X(\log \tilde{D}) \) are reflexive if and only if for any open set \( U \subseteq X \) and any number \( p \), the restriction morphism

\[
H^0(U, \pi_* \Omega^p_X(\log \tilde{D})) \to H^0(U \setminus \pi(E), \Omega^p_X(\log |D|))
\]

is surjective. In other words, Theorem 2.11 states that any holomorphic logarithmic \( p \)-form defined on the non-singular part of \((X, D)\) can be extended to any resolution of singularities, possibly with logarithmic poles along the divisor \( \tilde{D} \).

Remark 2.13. Note that Theorem 1.2 as stated in the introduction (Section 1) of this paper is an immediate consequence of Theorem 2.11 (\( \text{nkl}(X, \emptyset) \) being empty in this case).

2.C. Proof of Theorem 2.11. We begin with two preparatory lemmas.

Lemma 2.14. Let \( X \) be a normal complex space and \( D \) an effective \( \mathbb{Q} \)-divisor on \( X \) such that \((X, D)\) is analytically log canonical. Let \( \pi : \tilde{X} \to X \) be a log resolution of \((X, D)\) and

\[
\tilde{D} := \text{largest reduced divisor contained in } \text{supp } \pi^{-1}(\text{nkl}(X, D)).
\]

Then, the sheaf \( \pi_* \Omega^p_X(\log \tilde{D}) \) is reflexive if and only if there exists an open cover \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( X \) as well as log resolutions \( \pi_\lambda : \tilde{U}_\lambda \to U_\lambda \) such that for every \( \lambda \in \Lambda \) the sheaf \( \pi_* \Omega^p_{\tilde{U}_\lambda}(\log \tilde{D}_\lambda) \) is reflexive, where

\[
\tilde{D}_\lambda := \text{largest reduced divisor contained in } \text{supp}(\pi_\lambda)^{-1}(\text{nkl}(U_\lambda, D|_{U_\lambda})).
\]

Proof. The claim follows from the fact that any two log resolutions of an analytic pair \((Z, \Delta)\) can be dominated by a third and by arguments analogous to the algebraic case proven for example in [GKK10, Lem. 2.13].
Lemma 2.15. Let \( \pi : Y \to X \) be a projective morphism of normal algebraic varieties, and let \( \mathcal{F} \) be a coherent algebraic sheaf on \( Y \). If \( \pi_*\mathcal{F} \) is a reflexive sheaf on \( X \), then \( (\pi^an)_*(\mathcal{F}^an) \) is an (analytically) reflexive sheaf on \( X^an \).

Proof. Relative GAGA [Gro71, Ch. XII, Thm. 4.2] applied to the projective morphism \( \pi \) yields

\[
(\pi^an)_*(\mathcal{F}^an) \cong (\pi_*\mathcal{F})^an.
\]

Moreover, for any coherent algebraic sheaf \( \mathcal{G} \) on \( X \) we have

\[
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)^an \cong \mathcal{H}om_{\mathcal{O}_{X^an}}(\mathcal{G}^an, \mathcal{O}_{X^an}),
\]

see [Gro71, Ch. XII, part (1) of Thm. 4.4]. As a consequence, applying the functor \( \mathcal{H}om \) twice, the analytification of any reflexive coherent algebraic sheaf on \( X \) is an (analytically) reflexive coherent analytic sheaf on \( X^an \). Since \( \pi_*\mathcal{F} \) is reflexive by assumption, the claim hence follows from the isomorphism (2.15.1).

Proof of Theorem 2.11. Since \( X \) is locally algebraic, using Lemma 2.14 we may assume that there exists a quasi-projective variety \( Z \) and an effective \( \mathbb{Q} \)-divisor \( \Delta \), a log resolution \( q : \tilde{Z} \to Z \) of \((Z, \Delta)\), as well as open holomorphic embeddings \( \varphi : (X, D) \to (\tilde{Z}^an, \tilde{\Delta}^an) \) and \( \psi : \tilde{X} \to \tilde{Z}^an \) such that the following diagram commutes

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\varphi} & \tilde{Z}^an \\
\downarrow{\pi} & & \downarrow{\psi} \\
X & \xrightarrow{\psi} & Z^an.
\end{array}
\]

Lemma 2.6.(1) implies that \((Z, \Delta)\) is log canonical at every point of \( \varphi(X) \). Hence, removing a closed algebraic subset from \( Z \) if necessary, we may assume that \((Z, \Delta)\) is log canonical. Furthermore, Lemma 2.6.(2) implies that

\[
nklt(X, D) = \varphi^{-1}(nklt(Z, \Delta)^an).
\]

As a consequence, setting

\[
\tilde{\Delta} := \text{largest reduced divisor contained in } \supp \pi^{-1}(nklt(Z, \Delta)),
\]

we have \( \tilde{D} = \psi^{-1}(\tilde{\Delta}^an) \). In order to establish the claim it therefore suffices to show that \( (q^an)_* \Omega^p_{\tilde{Z}^an}(\log \tilde{\Delta}^an) \) is reflexive.

Elementary considerations show that for all \( p \leq n \)

\[
(\Omega^p_{\tilde{Z}^an}(\log \tilde{\Delta}^an)) = (\Omega^p_{\tilde{Z}}(\log \tilde{\Delta}))^an.
\]

Since \( q_*\Omega^p_{\tilde{Z}}(\log \tilde{\Delta}) \) is reflexive by the Extension Theorem, [GKKP11, Thm. 1.5], Lemma 2.15 together with equation (2.15.3) yields reflexivity of \( (q^an)_* \Omega^p_{\tilde{Z}^an}(\log \tilde{\Delta}^an) \). This finishes the proof.

The observation made in Example 2.9 immediately yields the following consequence of Theorem 2.11.

Corollary 2.16 (Holomorphic extension over isolated singularities). Let \( X \) be a normal complex space, \( D \) a \( \mathbb{Q} \)-divisor on \( X \) such that \((X, D)\) has only isolated log canonical singularities. Let \( \pi : \tilde{X} \to X \) be a log resolution of \((X, D)\) and

\[
\tilde{D} := \text{largest reduced divisor contained in } \supp \pi^{-1}(nklt(X, D)).
\]
Then, the sheaves $\pi_*(\Omega^p_X(\log D))$ are reflexive for all $p \leq n$.

Remark 2.17. In particular, Corollary 2.16 implies that any holomorphic differential forms defined on the smooth locus of a three-dimensional complex space $X$ such that $(X, \emptyset)$ is analytically terminal extends to any resolution of singularities.

3. Reflexive Differential Forms on Rationally Connected Varieties

3.A. Non-existence of reflexive pluri-forms. Rationally connected and, more generally, rationally chain-connected projective varieties play a key role in the classification and structure theory of algebraic varieties. It is a fundamental fact of higher-dimensional algebraic geometry that a rationally connected projective manifold $X$ does not carry any pluri-forms, that is

$$H^0(X, (\Omega^1_X)^{\otimes m}) = 0 \quad \forall m \in \mathbb{N}^+.$$  

We refer to [Kol96, IV.3.8] for a thorough review of this result. At least conjecturally, (3.0.1) is also sufficient to conclude that a projective manifold $X$ is projective. This has been proven in dimension three by Kollár–Miyaoka–Mori, see [KMM92, Thm. 3.2].

For reflexive $p$-forms, the vanishing result has been generalized to spaces which support klt pairs.

**Theorem 3.1** (Reflexive differentials on rcc spaces, cf. [GKKP11, Thm. 5.1]). Let $X$ be a normal, rationally chain-connected projective variety. If there exists a $Q$-divisor $D$ on $X$ such that $(X, D)$ is klt, then $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for all $1 \leq p \leq \dim X$.  

**Remark 3.2** (Rational chain-connectedness vs. rational connectedness). Let $X$ be a normal, rationally chain-connected projective variety. If there exists a $Q$-divisor $D$ on $X$ such that $(X, D)$ is klt, then $X$ is in fact rationally connected, cf. [HM07, Cor. 1.5].

In this section we investigate whether a similar vanishing result also holds for reflexive pluri-differentials. Somewhat surprisingly, the answer is mixed: vanishing holds if $X$ is factorial, but fails in general.

**Theorem 3.3** (Reflexive pluri-forms on factorial rcc spaces). Let $X$ be a normal rationally chain-connected projective variety. If $X$ is factorial and has canonical singularities, then

$$H^0(X, (\Omega^1_X)^{\otimes m}) = 0 \quad \forall m \in \mathbb{N}^+, \text{ where } (\Omega^1_X)^{\otimes m} := (\Omega^1_X)^{\otimes m}.$$  

**Remark 3.4** (Relation between Theorems 3.1 and 3.3). Let $X$ be a normal space. Assume that there exists a $Q$-divisor $D$ on $X$ such that $(X, D)$ is klt. If $X$ is factorial, then $X$ has canonical singularities, cf. [KM98, Cor. 2.35].

**Remark 3.5** (Necessity of the assumption that $X$ is canonical). There are examples of rational surfaces $X$ with log terminal singularities whose canonical bundle is torsion or even ample, cf. [Tot10, Example 10] or [Kol08, Example 43]. Since $H^0(X, \mathcal{O}_X(mK_X)) \subset H^0(X, (\Omega^1_X)^{\otimes m})$, these examples show that the assumption that $X$ has canonical singularities cannot be omitted in Theorem 3.3.

**Remark 3.6** (Sharpness of Theorem 3.3). In the class of varieties with canonical singularities Theorem 3.3 is sharp: Section 3.B contains an example of a rationally connected $Q$-factorial (but non-factorial) surface with canonical singularities which does carry non-trivial reflexive pluri-forms.
The proof of Theorem 3.3 uses the notion of semistable sheaves on singular spaces, where semistability is defined respect to a movable curve class. Appendix A contains detailed proofs of the necessary foundational results used here.

**Proof of Theorem 3.3.** We argue by contradiction and assume that there exists a positive number $m \in \mathbb{N}$ and a non-trivial element

$$
\sigma \in H^0 \left( X, (\Omega^1_X)^{[m]} \right) \not\{0\}.
$$

Let $\pi : \tilde{X} \to X$ be a log resolution of $X$ with exceptional divisor $E$. By Remark 3.2 the smooth variety $\tilde{X}$ is rationally connected. As explained in [Kol96, IV.3.9.4], we can therefore choose a dominating family of smooth rational curves in $\tilde{X}$, and a general member $C \cong \mathbb{P}^1$ such that the following holds.

(3.6.1) The restriction of $\mathcal{F}_X$ to the smooth curve $C$ is an ample vector bundle.

By general choice, the curve $C$ will not be contained in the $\pi$-exceptional set $E$. We consider the image curve $C := \pi(C)$ with its reduced structure, and write $a = [C]$ for the corresponding class in the Mori-cone $\overline{NE}(X)$. Since $[C]$ is movable, that is, $[C]$ has non-negative intersection with all effective divisors, the class $[C]$ is movable as well.

The section $\sigma$ defines an inclusion $\mathcal{O}_X \subseteq (\Omega^1_X)^{[m]}$. Using the notation $\mu^\text{max}_a$ as introduced in Definition A.1, we see that $\mu^\text{max}_a((\Omega^1_X)^{[m]}) \geq 0$. It follows from Item (A.16.1) of Proposition A.16 that $\mu^\text{max}_a(\Omega^1_X) \geq 0$. Proposition A.2 therefore allow us to find a subsheaf $\mathcal{F} \subseteq \Omega^1_X$ of rank $r \geq 1$ satisfying

(3.6.2)

$$
\mu_a(\mathcal{F}) = \mu^\text{max}_a(\Omega^1_X) \geq 0.
$$

Since $X$ is supposed to be factorial, the determinant $\det \mathcal{F}$ is an invertible subsheaf of $\Omega^1_X$. Inequality (3.6.2) implies that the restriction $(\det \mathcal{F})|_C$ is a nef line bundle on the (possibly singular) curve $C$. It follows that $\pi^*(\det \mathcal{F})|_C = (\pi|_C)^*(\det \mathcal{F})|_C$ is a nef line bundle on $\tilde{C}$.

Finally, recall from [GKKP11, Thm. 4.3] that there exists a pull-back map $\phi : \pi^*\Omega^1_X \to \Omega^1_{\tilde{X}'}$ isomorphic away from $E$. Its restriction to $\tilde{C}$ induces the following sequence of sheaf morphisms

(3.6.3)

$$
\pi^*(\det \mathcal{F})|_C \xrightarrow{\phi} \pi^*\Omega^1_X|_C \xrightarrow{\psi} \Omega^1_{\tilde{X}'}|_C.
$$

Since $\tilde{C}$ is not contained in $E$, all arrows in (3.6.3) are generically injective; in particular, $\psi$ is not the trivial map.

On the other hand, note that the restriction $\Omega^1_{\tilde{X}'}|_C$ is negative owing to (3.6.1). Since both $\pi^*(\det \mathcal{F})|_C$ and $\Omega^1_{\tilde{X}'}|_C$ are locally free, and $\psi$ is non-trivial, it is an injection of a nef line bundle into a negative vector bundle, which is absurd. This contraction concludes the proof. $\square$
3.B. A counterexample in the $\mathbb{Q}$-factorial setup. If $X$ is not factorial, the arguments used in the proof of Theorem 3.3 break down: with the notation used in the proof, we can write

$$(\pi|_{\tilde{C}})^* (\det \mathcal{S}) \cong \mathcal{T} \oplus \mathcal{A}$$

where $\mathcal{T}$ is torsion and $\mathcal{A}$ locally free of rank one. The argument fails because $\mathcal{A}$ might be negative. The following example shows that Theorem 3.3 is in fact no longer true in the non-factorial setting, even if $X$ has the mildest form of $\mathbb{Q}$-factorial singularities.

**Example 3.7 (A rationally connected surface supporting pluri-differential forms).** Let $\pi' : X' \to \mathbb{P}^1$ be any rational ruled surface. Choose four distinct points $q_1, \ldots, q_4$ in $\mathbb{P}^1$. For each point $q_i$, perform the following sequence of birational transformations of the ruled surface, outlined also in Figure 3.1.

1. Choose a point $x$, contained in the fibre over $q_i$, and blow up this point. The result is a surface with a map to $\mathbb{P}^1$ such that the fibre over $q_i$ is the union of two reduced rational curves each with self-intersection number $(-1)$, meeting transversely in a point $x'$.
2. Blow up the point $x'$. The result is a surface with a map to $\mathbb{P}^1$ such that the fibre over $q_i$ is the union of two reduced rational curves each with self-intersection number $(-2)$, and one rational curve with self-intersection $(-1)$. The $(-2)$-curves are disjoint, the $(-1)$-curve appears in the fibre with multiplicity two.
3. Blow down the $(-2)$-curves contained in the fibre over $q_i$. The result is a normal surface with a map to $\mathbb{P}^1$ such that the set-theoretic fibre $F_i$ over $q_i$ is a smooth rational curve. The curve $F_i$ appears in the cycle theoretic fibre over $q_i$ with multiplicity two. The surface has two singular points of type $A_1$ on $F_i$. Seen as a divisor, the curve $F_i$ is not Cartier, but $2 \cdot F_i$ is.

This way, we obtain a rational, rationally connected surface $\pi : X \to \mathbb{P}^1$ containing eight singular points, two on each of the fibres $F_1, \ldots, F_4$.

We claim that there exist sections in the second reflexive product $(\Omega^1_X)^2$. To this end, let $X^0 \subset X$ be the smooth locus of $X$, and set $F_i^0 := X^0 \cap F_i$, for each $1 \leq i \leq 4$. Finally, choose any point $p \in \mathbb{P}^1 \setminus \{q_1, \ldots, q_4\}$ and let $F := \pi^{-1}(p)$ denote the associated fibre. Since the fibres over the $q_i$ all have multiplicity two, it
is clear that the pull-back map of differentials,
\[ d(\pi|_{X^o}) : \pi^*(\Omega^1_{\mathbb{P}^1})|_{X^o} \to \Omega^1_{X^o}, \]
has single zeros along the \( F_i^o \). Accordingly, there exists a factorization
\[ \pi^*(\Omega^1_{\mathbb{P}^1})|_{X^o} \to \mathcal{O}_{X^o}(-2 \cdot F + \sum F_i^o) \to \Omega^1_{X^o}. \]
Setting \( D := -2 \cdot F + \sum F_i \), Factorization (3.7.1) then gives a non-trivial morphism of reflexive sheaves
\[ \alpha^1 : \mathcal{O}_X(D) \to \Omega^1_X, \]
hence also \( \alpha^2 : \mathcal{O}_X(2 \cdot D) \to (\Omega^1_X)^2 = (\Omega^1_X \otimes \Omega^1_X)^{**}. \)
To finish the construction, observe that the divisor \( D \) is \( \mathbb{Q} \)-linearly equivalent to zero, but not linearly equivalent to zero. The divisor \( 2 \cdot D \), however, is linearly equivalent to zero, giving a non-trivial map
\[ H^0(X, \mathcal{O}_X(2 \cdot D)) \to H^0(X, (\Omega^1_X)^2) \not\equiv 0, \]
that corresponds to a non-trivial section \( \tau \in H^0(X, (\Omega^1_X)^2) \).

**Remark and Question 3.8.** There are other ways to see that the surface \( X \) constructed in Example 3.7 admits a pluri-form. Semistable reduction yields a diagram

\[
\tilde{X} \quad \xrightarrow{\text{2:1 cover}} \quad X \quad \xrightarrow{\text{2:1 cover}} \quad \mathbb{P}^1
\]

\[ \text{branched over the singularities} \quad \text{branched over } q_1, \ldots, q_4 \]

where \( \tilde{X} \) is smooth, \( E \) is an elliptic curve, and where the vertical arrows are quotients by the associated action of \( G := \mathbb{Z}/2\mathbb{Z} \). It is not difficult to construct a \( G \)-invariant form
\[ \tilde{\tau} \in H^0(\tilde{X}, (\tilde{\pi}^*\Omega^1_E)^\otimes 2) \subset H^0(\tilde{X}, (\Omega^1_{\tilde{X}})^\otimes 2) \]
that corresponds to the form \( \tau \) constructed above. One may ask if any pluri-form on a rationally connected space arises in one of the two ways indicated in our specific example.

**Remark 3.9.** Example 3.7 underlines the fact that the extension theorem is not valid for pluri-forms, cf. [GKK10, Example 3.1.3].
**Theorem 4.1** (Kodaira-Akizuki-Nakano Vanishing Theorem, [AN54]). Let $X$ be a smooth projective variety and let $L$ be an ample line bundle on $X$. Then

\begin{align*}
H^q(X, \Omega^p_X \otimes L) &= 0 \quad \text{for } p + q > n, \text{ and} \\
H^q(X, \Omega^p_X \otimes L^{-1}) &= 0 \quad \text{for } p + q < n.
\end{align*}

**Remark 4.2.** Assertions (4.1.1) and (4.1.2) are equivalent to one another by Serre duality. Ramanujam [Ram72] gave a simplified proof of Theorem 4.1 and showed that it does not hold if one only requires $L$ to be semi-ample and big.

Esnault and Viehweg generalised Theorem 4.1 to logarithmic differentials, [EV86]. Moreover, Kodaira-Akizuki-Nakano vanishing has been shown to hold for sheaves of reflexive differentials on varieties with quotient singularities, see [Ara88], as well as on toric varieties, see [CLS11, Thm. 9.3.1].

In this section, we prove similar vanishing results for reflexive differentials on varieties with more general singularities. However, these vanishing results are restricted to special values of $p$ and $q$. It turns out that even for spaces with isolated terminal Gorenstein singularities Theorem 4.1 does not hold for arbitrary $p + q > n$, respectively $p + q < n$. A corresponding example is provided in Section 4.B.

**4.A. Partial vanishing results for lc and klt pairs.** In this section we prove some partial generalisations of Theorem 4.1 to lc and klt pairs.

**Proposition 4.3** (KAN-type vanishing for lc pairs, analogue of (4.1.2)). Let $X$ be a normal projective variety of dimension $n$, let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is log canonical, and let $L \in \text{Pic}(X)$ be an ample line bundle.

\begin{align*}
H^0(X, \Omega^p_X(\log \lfloor D \rfloor) \otimes L^{-1}) &= 0 \quad \text{for all } p < n, \quad \text{and} \\
H^1(X, \Omega^p_X(\log \lfloor D \rfloor) \otimes L^{-1}) &= 0 \quad \text{for all } p < n - 1.
\end{align*}

**Remark 4.4** (Vanishing for dlt pairs). If $(X, D)$ is not only lc but dlt, then additionally $H^q(X, L^{-1}) = 0$ holds for all $q < n$. This follows by observing that $X$ is Cohen-Macaulay [KM98, Thm. 5.22] and by using the more general result, [KSS10, Cor. 6.6], that vanishing holds already if $(X, D)$ is lc and Cohen-Macaulay.

**Proof of Proposition 4.3.** First note that (4.3.1) is a special case of the Bogomolov-Sommese vanishing theorem for log canonical pairs, [GKKP11, Thm. 7.2].

For the other case, choose a log resolution $\pi : \tilde{X} \to X$, consider the set

$$E := \text{supp } \pi^*((\lfloor D \rfloor)) \cup (\pi\text{-exceptional set}).$$

and let $\mathcal{F}_p := \Omega^p_X(\log E) \otimes \pi^*L^{-1}$. The projection formula and the extension theorem [GKKP11, Thm. 1.5] together imply that

$$\pi_*\mathcal{F}_p = \Omega^p_X(\log \lfloor D \rfloor) \otimes L^{-1}.$$ 

In order to prove (4.3.2) we need to show that $H^1(X, \pi_*\mathcal{F}_p) = 0$ if $p < n - 1$. For this, we will use the Leray spectral sequence, and Steenbrink vanishing, [Ste85, Thm. 2(a')]. The latter asserts that

\begin{align*}
H^q(\tilde{X}, \mathcal{F}_p) &= 0 \quad \text{for } p + q < n.
\end{align*}
Now consider the beginning of the five term exact sequence associated with the Leray spectral sequence,
\[ 0 \to H^1(X, \pi_* F_p) \to H^1(\tilde{X}, F_p) \to H^0(X, R^1 \pi_* F_p) \to \cdots \]
\[ = E_2^{1,0} = E_1 = E_0^{1,2} \to \cdots \]

Steenbrink vanishing \((4.4.1)\) gives that \(E_1 = 0\), hence \(E_2^{1,0} = 0\) and therefore \((4.3.2)\). □

**Proposition 4.5** (KAN-type vanishing for klt pairs, analogue of \((4.1.1)\)). Let \(X\) be a normal projective variety of dimension \(n\), let \(D\) be an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X,D)\) is klt, and let \(\mathcal{L} \in \text{Pic}(X)\) be an ample line bundle. Then
\[(4.5.1) \quad H^q(X, \omega_X \otimes \mathcal{L}) = 0 \quad \text{for all } q > 0, \text{ and} \]
\[(4.5.2) \quad H^n(X, \Omega_X^p \otimes \mathcal{L}) = 0 \quad \text{for all } p > 0.\]

**Proof.** To prove \((4.5.1)\), choose a log resolution \(\pi : \tilde{X} \to X\). The extension theorem for differential forms on log canonical pairs, \([GKKP11, \text{Thm. 1.5}]\) then asserts that \(\omega_X = \pi^* \omega_{\tilde{X}}\), and the assertion of \((4.5.1)\) is just the Grauert-Riemenschneider vanishing theorem \([GR70, \text{p. 263}]\).

To prove \((4.5.2)\), consider the chain of isomorphisms,
\[ H^n(X, \Omega_X^p \otimes \mathcal{L}) \cong \text{Hom}(\Omega_X^p \otimes \mathcal{L}, \omega_X) \quad \text{since } \omega_X \text{ is dualising, [Har77, p. 241]} \]
\[ \cong \text{Hom}(\mathcal{L}, \Omega_X^{n-p}) \quad \text{pairing assoc. with wedge product.} \]

The Bogomolov-Sommese vanishing theorem for log canonical pairs, \([GKKP11, \text{Thm. 7.2}]\), asserts that the last space is zero. □

### 4.B. A counterexample to Kodaira-Akizuki-Nakano vanishing for klt spaces.

We will show by way of example that the Kodaira-Akizuki-Nakano type vanishing theorems of Propositions 4.3 and 4.5 does not hold for all values of \(p\) and \(q\), not even for Gorenstein spaces with isolated terminal singularities.

**Example 4.6** (A fourfold with terminal singularities violating KAN vanishing). We construct a 4-dimensional terminal variety \(X\) following the steps outlined in the following diagram.

\[ \begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi, \mathbb{P}^1\text{-bundle}} & Y \\
\pi \ \text{contraction map} & & \\
X & \xrightarrow{\phi, \mathbb{P}^1\text{-bundle}} & \mathbb{P}^2 \\
\end{array} \]

To describe the construction in detail, consider the 3-dimensional smooth variety \(Y := \mathbb{P}_{\mathbb{P}^2}(\mathcal{I}_{\mathbb{P}^2})\). The tangent bundle \(\mathcal{I}_{\mathbb{P}^2}\) of the projective plane being ample, by definition the tautological bundle \(\mathcal{O}_{\mathbb{P}^2}(1) \in \text{Pic}(Y)\) is ample. Better still, using the Euler sequence to present \(\mathcal{I}_{\mathbb{P}^2}\) as a quotient of \(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}\), one shows that the ample bundle \(\mathcal{O}_{\mathbb{P}^2}(1)\) is in fact very ample.

The bundle \(\mathcal{O}_{\mathbb{P}^2}(1)\) induces an embedding \(Y \to \mathbb{P}^N\), with projectively normal image. Let \(X \subset \mathbb{P}^{N+1}\) be the cone over \(Y\). The variety \(X\) is then normal and

\[1\text{or the fact that } X \text{ has rational singularities, [KM98, Thm. 5.22]} \]
has a single isolated singularity, the vertex $x \in X$. Blowing up $x$, we obtain a resolution of singularities, $\pi : \tilde{X} \to X$. The variety $\tilde{X}$ is isomorphic to the $\mathbb{P}^1$-bundle $\psi : \mathbb{P}_Y(\mathcal{O}_Y(1) \oplus \mathcal{O}_Y) \to Y$. The $\pi$-exceptional set $E$ is canonically identified with $\mathbb{P}_Y(\mathcal{O}_Y) \subseteq \tilde{X}$. The divisor $E$ is thus a section of $\psi$ and naturally isomorphic to $Y$. Its normal bundle is $N_{E/\tilde{X}} \cong \mathcal{O}_Y(-1)$. Finally, consider the ample bundle $L := \mathcal{O}_X(1) \in \text{Pic}(X)$.

The following two remarks summarise the main properties of $X$.

Remark 4.7 (Dualising sheaves of $Y$, $X$ and $\tilde{X}$). An elementary computation shows that the canonical bundle of $Y$ is given as

\[(4.7.1) \quad \omega_Y \cong \mathcal{O}_Y(-2)\]

Using the bundle structure of $\tilde{X}$, Equation (4.7.1) and the adjunction formula, one computes the canonical bundle of $\tilde{X}$ as

\[(4.7.2) \quad \omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(E) \otimes \pi^* \mathcal{O}_X(-3).\]

Equation (4.7.2) has two consequences:

\[(4.7.3) \quad \text{The dualising sheaf of } X \text{ is invertible, } \omega_X \cong \mathcal{O}_X(-3) = \mathcal{O}_{\mathbb{P}^N}(-3)|_X.\]

\[(4.7.4) \quad \text{The discrepancy formula for } \pi \text{ reads } K_{\tilde{X}} = \pi^*(K_X) + E.\]

In particular, we obtain that the isolated singularity $x \in X$ is terminal. Recall from [KM98, Thm. 5.22] that terminal singularities are rational, hence Cohen-Macaulay. Assertion (4.7.3) thus implies that $X$ is in fact Gorenstein.

We are now ready to formulate the main results of this section. The following two propositions show that both versions of the Kodaira-Akizuki-Nakano vanishing theorem fail for the variety constructed in Example 4.6.

Proposition 4.8 (Generalisation of Proposition 4.3 does not hold). In the setup of Example 4.6, we have $H^2(X, \Omega_X^{[3]} \otimes L^{-1}) \neq 0$.

Proposition 4.9 (Generalisation of Proposition 4.5 does not hold). In the setup of Example 4.6, we have $H^2(X, \Omega_X^{[3]} \otimes L) \neq 0$.

The proofs of Proposition 4.8 and 4.9, both of which rely on somewhat lengthy cohomology computations, are given in Subsections 4.B.2 and 4.B.3 below.

4.B.1. KAN-vanishing and the DuBois complex. Terminal singularities are rational, and therefore DuBois, see [Kov99]. By definition, this means that the zeroth graded piece of the filtered de Rham (or DuBois) complex $\Omega_X^p$ is quasi-isomorphic to $\mathcal{O}_X$. One might wonder whether on terminal or more generally log canonical spaces this remains true for higher degrees, that is, whether or not the $p$-th graded piece of the DuBois complex is quasi-isomorphic to $\Omega_X^p[-p]$, the complex having the single sheaf $\Omega_X^p$ in the $p$-th place, for all values of $p$, cf. [PS08, Rem. on p. 180]. The Example 4.6 constructed above shows that this question has to be answered negatively.

Proposition 4.10. Let $X$ be the variety constructed in Example 4.6. Then, the 3rd graded piece of the filtered de Rham complex of $X$ is not quasi-isomorphic to $\Omega_X^{[3]}[-3]$. 
Proof. Denoting the filtered de Rham complex of $X$ by $(\Omega_X^\bullet, F)$, the vanishing theorem of Guillen-Navarro Aznar-Puerta-Steenbrink [PS08, Thm. 7.29] states that for any ample line bundle $\mathcal{L}$ on $X$ we have

$$H^m(X, \text{Gr}^p\Omega_X^\bullet \otimes \mathcal{L}) = 0 \quad \forall m > n. \tag{4.10.1}$$

Suppose that $\text{Gr}^2\Omega_X^\bullet$ is quasi-isomorphic to $\Omega_X^{[3]}[-3]$. Then (4.10.1) would imply that

$$H^2(X, \Omega_X^{[3]} \otimes \mathcal{L}) \cong H^3(X, \Omega_X^{[3]}[-3] \otimes \mathcal{L}) \cong H^3(X, \text{Gr}^2\Omega_X^\bullet \otimes \mathcal{L}) = 0,$$

contradicting Proposition 4.9 above. \hfill \Box

4.B.2. Proof of Proposition 4.8. Proposition 4.8 follows essentially from the Leray spectral sequence. The following lemma summarises the relevant statements in our setting.

**Lemma 4.11.** Let $X$ be a normal variety such that the pair $(X, \mathcal{O})$ is log canonical with isolated singularities. Assume furthermore that $\dim X \geq 4$. Let $\pi : \tilde{X} \to X$ be a log resolution of singularities, with $\pi$-exceptional divisor $E \subset \tilde{X}$. If $\mathcal{L} \in \text{Pic}(X)$ is any ample line bundle, then

$$H^2(X, \Omega_X^{[1]} \otimes \mathcal{L}^{-1}) \simeq H^0(X, R^1\pi_* \Omega_X^1(\log E)). \tag{4.11.1}$$

In particular, it follows that the left hand side of (4.11.1) is independent of the ample line bundle $\mathcal{L}$.

**Proof.** As in the proof of Proposition 4.3, set $\mathcal{F}_p := \Omega_X^{[1]}(\log E) \otimes \pi^*\mathcal{L}^{-1}$ and consider the following excerpt from the exact five term exact sequence associated with the Leray spectral sequence,

$$(4.11.2) \quad H^1(\tilde{X}, \mathcal{F}_p) \to H^0(X, R^1\pi_* \mathcal{F}_p) \to H^2(\tilde{X}, \mathcal{F}_p) \to H^2(X, \mathcal{F}_p) \to H^2(\tilde{X}, \mathcal{F}_p)$$

As before, Steenbrink vanishing [Ste85, Thm. 2(a')] and the assumption that $\dim X \geq 4$ give that $E^1 = E^2 = 0$. Using the projection formula and the extension theorem [GKKP11, Thm. 1.5] to identify $\pi_*\mathcal{F}_p$ with $\Omega_X^{[1]} \otimes \mathcal{L}^{-1}$, Sequence (4.11.2) reads

$$H^2(X, \Omega_X^{[1]} \otimes \mathcal{L}^{-1}) \cong E^{0,1}_2 \cong E^{0,1}_2 = H^0(X, R^1\pi_* \Omega_X^1(\log E) \otimes \mathcal{L}^{-1}).$$

Since $X$ has only isolated singularities, the push-forward sheaf $R^1\pi_* \Omega_X^1(\log E)$ has finite support, and

$$H^0(X, R^1\pi_* \Omega_X^1(\log E) \otimes \mathcal{L}^{-1}) \simeq H^0(X, R^1\pi_* \Omega_X^1(\log E)),$$

finishing the proof of Lemma 4.11. \hfill \Box

**Proof of Proposition 4.8.** Lemma 4.11 asserts that it suffices to verify that the push-forward $R^1\pi_* \Omega_X^1(\log E)$, which is a skyscraper sheaf whose support equals the vertex $x \in X$, is not the zero sheaf. To this end, consider the residue sequence for logarithmic differentials,

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log E) \longrightarrow \mathcal{O}_E \longrightarrow 0.$$
Recalling from the extension theorem that the inclusion $\pi_*\Omega^1_X \to \pi_*\Omega^1_X (\log E)$ is an isomorphism, the push-down yields an exact sequence
\begin{equation}
0 \to \pi_* \mathcal{O}_E \to \pi^* \Omega^1_X \to \pi^* \pi_* \Omega^1_X (\log E).
\end{equation}

In (4.11.3), the symbol $\mathcal{C}_X$ denotes the skyscraper sheaf supported on $x$ with stalk $\mathcal{C}$.

Next, observe that for any open neighborhood $U = U(x) \subseteq X$, the Chern classes
\[ c_1(\psi^* \mathcal{O}_Y(1)) \quad \text{and} \quad c_1((\phi \circ \psi)^* \mathcal{O}_\mathbb{P}^2(1)) \]
yield two linearly independent elements of $H^1(\pi^{-1}(U), \Omega^1_X)$, since their restrictions to $E \cong Y$ are linearly independent. The stalk of $\pi^* \pi_* \Omega^1_X$ is therefore at least two-dimensional. Consequently, $\pi^* \pi_* \Omega^1_X (\log E) \neq 0$ by the exact sequence (4.11.3). This concludes the proof of Proposition 4.8. \hfill \qed

4.B.3. Proof of Proposition 4.9. The proof of Proposition 4.9 is similar to, but more involved than the proof of Proposition 4.8. As in Section 4.B.2, we start giving a number of remarks and lemmas computing cohomology groups relevant for our line of reasoning. Once these results are established, the proof of Proposition 4.9 follows on Page 19.

Remark 4.12 (Hodge numbers of $Y$ and $\tilde{X}$). It follows immediately from the construction of $Y$ and $\tilde{X}$ as $\mathbb{P}^1$-bundles that both spaces can be written as the disjoint union of affine subvarieties,
\[ Y = \bigcup_i Y_i \quad \text{and} \quad \tilde{X} = \bigcup_j \tilde{X}_j, \]
such that the following holds.

(4.12.1) All $Y_i$, respectively $\tilde{X}_j$, are algebraically isomorphic to affine spaces $\mathbb{C}^{n_i}$, respectively $\mathbb{C}^{n_j}$ where $n_i, n_j \in \mathbb{N}$ are suitable numbers.

(4.12.2) For all $i, j$, the irreducible components of $Y \setminus Y_i$ and $\tilde{X} \setminus \tilde{X}_j$ are smooth subvarieties of $Y$ and $\tilde{X}$, respectively.

Assertion (4.12.1) implies that the topological spaces $Y$ and $\tilde{X}$ admit CW-structures without odd-dimensional cells. Using these CW-structures to compute (co)homology, cf. [Hat02, Sect. 2.2], it follows that
\begin{equation}
H^k(Y, \mathbb{C}) = H^k(\tilde{X}, \mathbb{C}) = 0 \quad \text{if } k \text{ is odd},
\end{equation}

In fact, more is true. Property (4.12.2) implies that all topological cohomology groups $H^{2k}(Y, \mathbb{C})$ are generated by cohomology classes of smooth algebraic cycles, that is, by elements of $H^{k,k}(Y)$, cf. [Hat02, Rem. (iii) on p. 140] and [Voi07, Prop. 11.20]. The same being true for $\tilde{X}$, we find that
\begin{equation}
H^{k,l}(Y) = 0 \quad \text{and} \quad H^{k,l}(\tilde{X}) = 0 \quad \text{if } k \neq l.
\end{equation}

Lemma 4.13. In the setting of Example 4.6, we have

(4.13.1) $H^1(Y, \Omega^2_Y \otimes \mathcal{O}_Y(1)) \neq 0$, and
(4.13.2) $H^2(Y, \Omega^2_Y \otimes \mathcal{O}_Y(v)) = 0$ for all $v > 0$. 

Proof. Let \( \nu > 0 \) be any number. We consider the sequence of relative differentials for the smooth morphism \( \phi \),

\[
0 \to \phi^*\Omega^1_{\mathbb{P}^2} \to \Omega^1_Y \to \Omega^1_{Y/\mathbb{P}^2} \to 0.
\]

Twisting the second exterior power of this sequence with \( \mathcal{O}_Y(\nu) \), one obtains

\[
(4.13.3) \quad 0 \to \mathcal{O}_Y(\nu) \otimes \phi^*\omega_{\mathbb{P}^2} \to \mathcal{O}_Y(\nu) \otimes \Omega^2_Y \to \phi^*\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_Y(\nu) \otimes \omega_{Y/\mathbb{P}^2} \to 0.
\]

Using Equation (4.7.1) of Remark 4.7 to expand the definition of \( \omega_{Y/\mathbb{P}^2} \), we obtain that

\[
(4.13.4) \quad \mathcal{A}_\nu = \phi^*\left(\Omega^1_{\mathbb{P}^2} \otimes \phi^*\omega_{\mathbb{P}^2}^{-1}\right) \otimes \mathcal{O}_Y(\nu - 2).
\]

The following observations are crucial for cohomology computations.

(4.13.5) If \( F \) is any fibre of \( \phi \), then \( F \cong \mathbb{P}^1 \), the restriction \( \mathcal{A}_\nu|_F \) is ample, and
\[
H^1(F, \mathcal{A}_\nu|_F) = 0 \text{ for all } \nu > 0.
\]

The Leray spectral sequence thus gives
\[
H^i(Y, \mathcal{A}_\nu) = H^i(\mathbb{P}^2, \phi_*\mathcal{A}_\nu) \text{ for all positive } i \text{ and } \nu.
\]

(4.13.6) Likewise, the Leray spectral sequence gives
\[
H^i(Y, \mathcal{A}_\nu) = H^i(\mathbb{P}^2, \phi_*\mathcal{A}_\nu) \text{ for all positive } i, \nu.
\]

(4.13.7) The negativity of \( \mathcal{B}_1 \) on \( \phi \)-fibres implies that \( \phi_*\mathcal{B}_1 = 0 \). In particular, we obtain that \( H^i(Y, \mathcal{B}_1) = 0 \) for all \( i \geq 0 \). This already shows Claim (4.13.2) in case \( \nu = 1 \).

To prove Claim (4.13.1), compute cohomology groups in case \( \nu = 1 \). We obtain that

\[
H^1(Y, \mathcal{A}_1) = H^1(\mathbb{P}^2, \phi_*\mathcal{A}_1) \quad \text{by (4.13.5)}
\]
\[
= H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2} \otimes \omega_{\mathbb{P}^2}) \quad \text{since } Y = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2}), \text{ so } \phi_*\mathcal{O}_Y(\nu) = \text{Sym}^\nu \mathcal{T}_{\mathbb{P}^2}
\]
\[
\cong H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \cong \mathbb{C} \quad \text{Serre duality.}
\]

The long exact cohomology sequence for (4.13.3) in case \( \nu = 1 \),

\[
\cdots \to H^0(Y, \mathcal{A}_1) \to H^1(Y, \mathcal{A}_1) \to H^1(Y, \mathcal{O}_Y(1) \otimes \Omega^2_Y) \to \cdots
\]
\[
=0 \text{ by (4.13.7)} \quad \cong \mathbb{C}
\]
then shows Claim (4.13.1).

It remains to show Claim (4.13.2) for \( \nu > 1 \). We identify the following cohomology groups.

\[
H^2(Y, \mathcal{A}_\nu) = H^2(\mathbb{P}^2, \phi_*\mathcal{A}_\nu) \quad \text{by (4.13.5)}
\]
\[
= H^2(\mathbb{P}^2, \text{Sym}^\nu \mathcal{T}_{\mathbb{P}^2} \otimes \omega_{\mathbb{P}^2}) \quad \text{since } Y = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})
\]
\[
\cong H^0(\mathbb{P}^2, \text{Sym}^\nu \Omega^1_{\mathbb{P}^2}) \quad \text{Serre duality}
\]
\[
= 0 \quad \text{since } \Omega^1_{\mathbb{P}^2} \text{ is anti-ample}
\]
The divisor $E$ bundle sequence of $\mathcal{C}$ the last equality holds because the restriction $\mathcal{C}_l$ to any line $l$ is obviously negative. The long exact cohomology sequence for (4.13.3) will then immediately give Claim (4.13.2) for $\nu > 1$, as required. 

\textbf{Corollary 4.14.} In the setting of Example 4.6, we have $H^1(E, \Omega^3_X[E]) \neq 0$.

\textbf{Proof.} The divisor $E \cong Y$ being a section of $\psi : \tilde{X} \to Y$, we see that the normal bundle sequence of $E$ splits, that is, $\Omega^1_X|E = \Omega^1_E \oplus N^*_E/\tilde{X} \cong \Omega^1_Y \oplus \mathcal{O}_Y(1)$. Taking cohomology of the third wedge-power, we obtain that

$$H^1(E, \Omega^3_X|E) \cong H^1(Y, \Omega^3_Y) \oplus H^1(Y, \Omega^3_Y \otimes \mathcal{O}_Y(1)).$$

\neq 0 \text{ by Assertion (4.13.1) of Lemma 4.13}.

\textbf{Corollary 4.15.} In the setting of Example 4.6, let $\hat{E}$ be the formal completion of the $\pi$-exceptional divisor $E$ in $\tilde{X}$. Then $H^1(\hat{E}, \Omega^3_X|\hat{E}) \neq 0$.

\textbf{Proof.} We follow the standard approach and compute cohomology on $\hat{E}$ as an inverse limit of cohomology on higher-order infinitesimal neighborhoods of $E$. Denoting by $E_n \subset \tilde{X}$ the subscheme defined by $\mathcal{I}_n^{n+1}$, recall from [Har68, Prop. 4.1] that the cohomology on the formal completion is computed as

$$H^1(\hat{E}, \Omega^3_X|\hat{E}) = \lim_{\longrightarrow} H^1(E_n, \Omega^3_X|E_n),$$

where the limit is taken over the inverse system given by restriction maps

$$\cdots \longrightarrow H^1(E_3, \Omega^3_X|E_3) \longrightarrow H^1(E_2, \Omega^3_X|E_2) \longrightarrow H^1(E_1, \Omega^3_X|E_1) \longrightarrow H^1(E, \Omega^3_X|E).$$

\neq 0 \text{ by Corollary 4.14}.

Corollary 4.15 will thus follow once we show that all restriction morphisms $r_i$ are surjective. To this end, fix a number $n > 1$ and consider the sequence of coherent $\mathcal{O}_X$-modules,

$$0 \longrightarrow \mathcal{I}_E^n/\mathcal{I}_E^{n+1} \longrightarrow \mathcal{O}_X/\mathcal{I}_E^n \longrightarrow \mathcal{O}_X/\mathcal{I}_E \longrightarrow 0.$$

Tensoring over $\mathcal{O}_X$ with $\Omega^3_X$, the associated long exact cohomology sequence reads

$$H^1(E_n, \Omega^3_X|E_n) \longrightarrow H^1(E_{n-1}, \Omega^3_X|E_{n-1}) \longrightarrow H^2(E, \Omega^3_X|\mathcal{I}_E(n)),
$$

\neq 0 \text{ by Assertion 4.13.2 of Lemma 4.13}.

giving the surjectivity we needed to show. 

\textbf{Lemma 4.16.} In the setting of Example 4.6, we have $H^1(\tilde{X}, \Omega^3_X \otimes \mathcal{L}) = 0.$
Proof. Let \( \Sigma \in |\pi^*\mathcal{L}| \) be a general element. The divisor \( \Sigma \subset \widetilde{X} \) is then a section for the map \( \psi : \widetilde{X} \to Y \). In particular, \( \Sigma \cong Y \) and \( N_{\Sigma/X} \cong \mathcal{L}|_\Sigma \cong \mathcal{O}_Y(1) \). Twisting the ideal sheaf sequence for \( \Sigma \subset \widetilde{X} \) with \( \Omega^3_X \otimes \mathcal{L} \), we obtain
\[
0 \to \Omega^3_X \to \Omega^3_X \otimes \mathcal{L} \to (\Omega^3_X \otimes \mathcal{L})|_\Sigma \to 0.
\]
Using the long exact cohomology sequence of (4.16.1), Lemma 4.16 follows once we show that
\[
(4.16.2) \quad H^1(\widetilde{X}, \Omega^3_X) = H^1(\widetilde{X}) = 0
\]
and
\[
(4.16.3) \quad H^1(\Sigma, \Omega^3_X \otimes \mathcal{L}|_\Sigma) = 0.
\]
Vanishing (4.16.2) has already been shown in Remark 4.12 above. For (4.16.3), use the mapping \( \psi \) to write \( \Omega^1_X(\mathcal{L}) \cong \Omega^1_X(\mathcal{L}) \otimes N^*_\Sigma/X \) so that
\[
(\Omega^3_X \otimes \mathcal{L})|_\Sigma \cong (\omega_\Sigma \otimes \mathcal{L}|_\Sigma) \oplus (\Omega^2_X \otimes N^*_\Sigma/X \otimes \mathcal{L}|_\Sigma) \cong (\omega_\Sigma \otimes \mathcal{L}|_\Sigma) \oplus \Omega^2_X
\]
and
\[
H^1(\Sigma, (\Omega^3_X \otimes \mathcal{L})|_\Sigma) \cong H^1(Y, \omega_Y \otimes \mathcal{O}_Y(1)) \oplus H^1(Y, \Omega^2_Y).
\]
This finishes the proof of Lemma 4.16. \( \square \)

Proof of Proposition 4.9. Consider the following excerpt of the 5-term exact sequence associated with the Leray spectral sequence for the sheaf \( \Omega^3_X \otimes \pi^*\mathcal{L} \),
\[
\cdots \to H^1(\widetilde{X}, \Omega^3_X \otimes \pi^*\mathcal{L}) \to H^0(X, R^1\pi_*\Omega^3_X \otimes \mathcal{L}) \to H^2(X, \pi_*\Omega^3_X \otimes \mathcal{L}) \to \cdots
\]
To show that \( H^2(X, \Omega^3_X \otimes \mathcal{L}) \neq 0 \), it suffices to show that \( E^{0,1}_2 \neq 0 \). Since the higher direct image sheaf vanishes outside of the singular point \( x \in X \), this will follow from \( R^1\pi_*\Omega^3_X \neq 0 \). Non-vanishing of \( R^1\pi_*\Omega^3_X \) follows from Corollary 4.15 and from the theorem on formal functions, in the form of [Gro61, Chapt. III, Cor. 4.1.7]. This finishes the proof of Proposition 4.9. \( \square \)

5. Closedness of reflexive forms, Poincaré’s lemma

5.A. Closedness of reflexive forms. It is a standard result of Hodge theory that logarithmic differential forms on projective snc pairs are closed. Here, we show that similar results also hold for reflexive differentials, in the log canonical setting. The following notation is fundamental in the discussion.

Definition 5.1 (Closedness for reflexive forms). Let \( X \) be a normal algebraic variety (or complex space), \( E \) a reduced effective divisor on \( X \) and \( \sigma \in H^0(X, \Omega^p_X(\log E)) \) a reflexive \( p \)-form. We call \( \sigma \) closed if \( d(\sigma|_{X_{\text{reg}}} \cap E) = 0 \).

While a Hodge theory for reflexive differentials is still missing, we prove the following generalisation of the particular Hodge-theoretic result mentioned above.

Theorem 5.2 (Closedness of global logarithmic forms). Let \( X \) be a projective variety, and let \( D \) be an effective \( \mathbb{Q} \)-divisor such that the pair \( (X, D) \) is log canonical. Then, any reflexive logarithmic \( p \)-form \( \sigma \in H^0(X, \Omega^p_X(\log|D|)) \) is closed.
Proof. Let \( \pi : \tilde{X} \to X \) be a strong log-resolution of the pair \((X, D)\) and set
\[
\tilde{D} := \text{the largest reduced divisor contained in } \text{supp}\, \pi^{-1}(\text{nklt}(X, D)).
\]
Now, if \( \sigma \in H^0(X, \Omega^p_X(\log|D|)) \) is any reflexive \( p \)-form on \( X \), it follows from the Extension Theorem, \cite[Thm. 1.5]{GKKP11}, that the pull back of \( \sigma \) via \( \pi \) extends to an element \( \tilde{\sigma} \in H^0(\tilde{X}, \Omega^p_{\tilde{X}}(\log \tilde{D})) \). However, global logarithmic forms on the projective snc pair \((\tilde{X}, \tilde{D})\) are closed by \cite[(3.2.14)]{Del71}. In particular,
\[
d(\tilde{\sigma}|_{\tilde{X} \setminus \tilde{D}}) = 0, \quad \text{where } \tilde{D} := (\pi\text{-exceptional set}) \cup \text{supp} \tilde{D}.
\]
Since \( \pi \) identifies \( \tilde{X} \setminus \tilde{D} \) with \( X_{\text{reg}} \setminus \text{supp}[D] \), this shows that \( \sigma \) is closed in the sense of Definition 5.1. \( \Box \)

Remark 5.3. Using the Holomorphic Extension Theorem 2.11, an analogous result can be shown to hold for (logarithmic) differential forms on log canonical locally algebraic compact complex spaces in class \( \mathcal{C} \). In particular, this applies to Moishezon spaces with log canonical singularities.

5.B. The Poincaré Lemma in the reflexive setting. Poincaré’s lemma is one of the cornerstones in the theory of differential forms on complex manifolds. For (analytically) klt spaces, the lemma still holds for reflexive one-forms. It is currently unclear to what extent a Poincaré lemma can be expected to hold for reflexive forms of higher degree.

Theorem 5.4 (Poincaré Lemma for reflexive one-forms). Let \( X \) be a normal complex space and \( D \) an effective \( \mathbb{Q} \)-divisor on \( X \) such that \((X, D)\) is analytically klt and locally algebraic. Let \( \sigma \in H^0(X, \Omega^1_X) \) be a closed holomorphic reflexive one-form on \( X \). Then there exists a covering of \( X \) by subsets \((U_a)_{a \in A}\) that are open in the euclidean topology, and holomorphic functions \( f_a \in \mathcal{O}_X(U_a) \) such that \( \sigma|_{U_a, \text{reg}} = df_a|_{U_a, \text{reg}} \).

Remark 5.4.1. For isolated rational singularities, slightly more general results have been obtained in \cite[Prop. 2.5]{CF02}.

Remark 5.4.2. If the pair \((X, D)\) is only assumed to be log-canonical, the Poincaré lemma fails to be true in general. As an example, consider the affine cone \( X \) over a smooth plane cubic curve \( C \subset \mathbb{P}^2 \), together with its minimal resolution \( \tilde{X} \to X \).

The variety \( \tilde{X} \) is the total space of the line bundle \( \mathcal{O}_C(-1) \), hence it fibres over the exceptional set \( E \subset \tilde{X} \). Pulling-back a nowhere vanishing global regular one-form from \( E \cong C \) to \( \tilde{X} \) yields a closed regular one-form which does not have a primitive in any euclidean neighbourhood of \( E \) in \( \tilde{X} \).

Proof of Theorem 5.4. Let \( x \in X \) be any point. We aim to construct an open neighborhood \( U = U(x) \) and a function \( f \in \mathcal{O}_X(U) \) satisfying the requirements of the theorem. To this end, consider a log resolution \( \pi : \tilde{X} \to X \) such that both the \( \pi\)-exceptional set \( E \) and the fiber \( F := \pi^{-1}(x) \) are divisors with simple normal crossing support. Let \( F = \bigcup_i F_i \) be the decomposition into irreducible components.

In this setting, the holomorphic Extension Theorem 2.11 guarantees that there exists a differential form \( \tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}) \) which agrees over the smooth part \( X_{\text{reg}} \) with the pull-back of \( \sigma \). Using the classical Poincaré Lemma and elementary topology, we find a finite number of sets \((V_b)_{b \in B} \subset \tilde{X} \), open in the euclidean topology, which cover \( F \) and satisfy the following additional properties.
(5.4.3) For each index $\beta \in B$, the intersection $V_\beta \cap F$ is not empty and connected.

(5.4.4) Given indices $\beta_1, \beta_2 \in B$ with $V_{\beta_1} \cap V_{\beta_2} \neq \emptyset$, then $V_{\beta_1} \cap V_{\beta_2}$ is connected and $V_{\beta_1} \cap V_{\beta_2} \cap F \neq \emptyset$.

(5.4.5) For each index $\beta \in B$, there exists a holomorphic function $g_\beta \in \mathcal{O}_X(V_\beta)$ such that $\partial |_{V_\beta} = dg_\beta$.

To continue, we recall a result of Namikawa [Nam01, Lem. 1.2], which asserts in our setup that $H^0(F, \Omega^1_F / \text{tor}) = 0$. In particular, if $i$ is any index and $i_i : F_i \to X$ the inclusion map, then

$$d(i_i|_{V_\beta \cap F_i})(dg_\beta) = 0 \in H^0(F_i, \Omega^1_{F_i})$$

Since $d(i_i|_{V_\beta \cap F_i})(dg_\beta) = d(g_\beta|_{V_\beta \cap F_i}) = 0$, Equation (5.4.6) implies that the functions $g_\beta$ are locally constant along $V_\beta \cap F$. Using (5.4.3), we can thus assume the following.

(5.4.7) For each index $\beta \in B$, the function $g_\beta$ of (5.4.5) vanishes along $F$, that is, $g_\beta \in \mathcal{F}_F(V_\beta) \subset \mathcal{O}_X(V_\beta)$.

Now, given any two indices $\beta_1, \beta_2 \in B$ with $V_{\beta_1} \cap V_{\beta_2} \neq \emptyset$, it follows from (5.4.5) that the difference $g_{\beta_1}|_{V_{\beta_1} \cap V_{\beta_2}} - g_{\beta_2}|_{V_{\beta_1} \cap V_{\beta_2}}$ is locally constant, hence constant by (5.4.4). Using (5.4.7), we see that this difference is actually zero along the non-empty set $V_{\beta_1} \cap V_{\beta_2} \cap F$. In summary, the functions $g_\beta$ glue to give a globally defined holomorphic function $g \in \mathcal{O}_X(V_\beta)$. Since $\pi$ is proper, we find a neighborhood $U = U(x)$, open in the euclidean topology, such that $\pi^{-1}(U) \subset \cup V_\beta$. The existence of a holomorphic function $f \in \mathcal{O}_X(U)$ satisfying $g = f \circ \pi$ is then immediate.

Remark 5.5. In the setting of the proof, Namikawa’s vanishing $H^0(F, \Omega^1_F / \text{tor}) = 0$ can also be shown by elementary methods, using that $F$ is rationally chain connected.

5.C. Representing reflexive forms by Kähler differentials. Given a normal variety or a normal complex space $X$, there exists a natural morphism $b : \Omega^p_X \to \Omega^1_X$ from the sheaf of Kähler differentials to its double dual. In general, the morphism $b$ is neither injective nor surjective, even if $X$ is assumed to have the mildest possible singularities considered in the Minimal Model Program. A corresponding series of examples is worked out in detail in [GR11].

The kernel of $b$ is exactly the subsheaf of torsion elements in $\Omega^1_X$, so that there is an exact sequence

$$0 \longrightarrow \text{tor} \Omega^1_X \overset{a}{\longrightarrow} \Omega^1_X \overset{b}{\longrightarrow} \Omega^1_X \overset{\text{tor}}{\longrightarrow}.$$  

Theorem 5.6 (Representation of closed forms by Kähler differentials). Let $X$ be an irreducible normal complex space and $D$ an effective $\mathbb{Q}$–divisor such that $(X, D)$ is analytically klt and locally algebraic.

(5.6.1) If $\sigma \in H^0(X, \Omega^1_X)$ is any closed reflexive one-form, then there exists a canonically defined Kähler form $\sigma_K \in H^0(X, \Omega^1_X)$ such that $\sigma = b(\sigma_K)$. 
If $X$ is a projective algebraic variety with at worst klt singularities, then the sequence
\[ 0 \to H^0(X, \text{tor} \Omega^1_X) \to H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X) \to 0 \]
is exact and canonically split.

**Proof.** Assertion (5.6.1) follows immediately from the Poincaré-Lemma 5.4 above. In order to prove Assertion (5.6.2), let $X$ be a projective algebraic variety with at worst klt singularities. By Theorem 5.2, Remark 5.3, and Assertion (5.6.1), the torsion sequence for analytic Kähler differentials yields the following exact and canonically split sequence,
\[ (5.6.3) \quad \xymatrix{ H^0(X^{an}, \text{tor} \Omega^1_{X^{an}}) \ar[r] & H^0(X^{an}, \Omega^1_{X^{an}}) \ar[r] & H^0(X^{an}, \Omega^1_{X^{an}}) } \]

To compare the torsion sequences for analytic and algebraic Kähler differentials we consider the following commutative diagram, both rows of which are exact.
\[ \xymatrix{ 0 \ar[r] & \text{tor} \Omega^1_{X^{an}} \ar[r] \ar@{->>}[d]^a & \Omega^1_{X^{an}} \ar[r] \ar@{->>}[d]^{\beta} & \Omega^1_{X^{an}} \ar[r] \ar@{->>}[d]^{\gamma} & 0 \\ 0 \ar[r] & (\text{tor} \Omega^1_X)^{an} \ar[r] & (\Omega^1_X)^{an} \ar[r] & (\Omega^1_X)^{an} \ar[r] & 0 } \]

By the functoriality properties of the sheaf of Kähler differentials $\beta$ is an isomorphism. Additionally, it follows from equation (2.15.2) that $\gamma$ is isomorphic. Consequently, $\alpha$ is likewise an isomorphism.

It hence follows from GAGA [Ser56, Thm. 1] that in the following commutative diagram the vertical maps are isomorphic.
\[ \xymatrix{ 0 \ar[r] & H^0(X^{an}, \text{tor} \Omega^1_{X^{an}}) \ar[r] \ar[d]^\pi & H^0(X^{an}, \Omega^1_{X^{an}}) \ar[r] \ar[d]^\pi & H^0(X^{an}, \Omega^1_{X^{an}}) \ar[r] \ar[d]^\gamma & 0 \\ 0 \ar[r] & H^0(X, \text{tor} \Omega^1_X) \ar[r] & H^0(X, \Omega^1_X) \ar[r] & H^0(X, \Omega^1_X) \ar[r] & 0 } \]
The upper row coincides with (5.6.3) and is therefore exact and canonically split. Consequently, the same holds for the lower row, which coincides with the sequence of (5.6.2). This concludes the proof.

**Remark 5.7.** For projective varieties, exactness and splitting of Sequence (5.6.2) can also be concluded from the fact that the Albanese map of any desingularisation factors via $X$, and that any 1-form is a pull-back from the Albanese torus.

**Questions 5.8.** Are there similar results for reflexive $p$-forms, for $p > 1$? If in the setup of Theorem 5.6 both the pair $(X, D)$ and the form $\sigma$ are algebraic, is $\sigma_K$ also algebraic?
—The last question has a positive answer in the case of isolated singularities.

**Appendix A. Stability notions on singular spaces**

The proof of Theorem 3.3 uses the notion of semistable sheaves on singular spaces, where semistability is defined with respect to a fixed movable curve class. As we are not aware of any reference that discusses these matters in detail, we
chose to include a short and self-contained introduction here. We feel that these results might be of independent interest.

A.A. Semistability with respect to a movable class. On a polarized complex manifold, it is well-understood that the tensor product of any two semistable locally free sheaves is again semistable. We will show in this appendix that the reflexive tensor product of two semistable sheaves on a singular, $\mathbb{Q}$-factorial space is again semistable, even if the polarization is only given by a movable curve class, see Proposition A.16 below. To start, we recall the relevant definition of semistability with respect to movable classes.

**Definition A.1** (Semistability with respect to a movable class). Let $X$ be a normal $\mathbb{Q}$-factorial projective variety and $\alpha \in N_1(X)_{\mathbb{Q}}$ a numerical curve class\(^2\). Assume that $\alpha$ is movable, that is, that $\alpha$ intersects any effective Cartier divisor non-negatively.

If $\mathcal{F}$ is any coherent sheaf of $\mathcal{O}_X$-modules that is torsion free in codimension one, the determinant $\det \mathcal{F} := (\wedge^{\text{rank} \mathcal{F}})^{**}$ is a $\mathbb{Q}$-Cartier Weil divisorial sheaf and therefore defines a class $[\det \mathcal{F}] \in N_1(X)_{\mathbb{Q}}$. Using the non-degenerate bilinear pairing

$$N_1(X)_{\mathbb{Q}} \times N_1(X)_{\mathbb{Q}} \rightarrow \mathbb{Q},$$

one defines the slope of $\mathcal{F}$ with respect to $\alpha$ as the rational number

$$\mu_\alpha(\mathcal{F}) := \frac{[\det \mathcal{F}] \alpha}{\text{rank}(\mathcal{F})}.$$

Recalling that subsheaves of $\mathcal{F}$ are again torsion free in codimension one, we set

$$\mu^\text{max}_\alpha(\mathcal{F}) := \sup \{ \mu_\alpha(\mathcal{G}) \mid \mathcal{G} \subseteq \mathcal{F} \text{ a coherent subsheaf} \}$$

We say that $\mathcal{F}$ is $\alpha$-semistable if $\mu^\text{max}_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{F})$.

The following proposition asserts that $\mu^\text{max}_\alpha$ is never infinite. The supremum used in its definition is actually a maximum.

**Proposition A.2.** In the setting of Definition A.1, there exists a coherent subsheaf $\mathcal{G} \subseteq \mathcal{F}$ such that $\mu^\text{max}_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$. In particular, $\mu^\text{max}_\alpha(\mathcal{F})$ is a rational number.

The remainder of the present Section A.A is devoted to the proof of Proposition A.2. We have subdivided the proof into several relatively independent steps, given in Sections A.A.1–A.A.6, respectively. The subsequent Sections A.B–A.D establish semistability of tensor products, using invariance properties of slopes to reduce the problem to known cases.

A.A.1. Proof of Proposition A.2, invariance of $\mu^\text{max}_\alpha$ under removal of torsion. We maintain notation and assumptions of Proposition A.2 throughout the proof. The following observation will help to simplify the setting.

**Observation A.3.** Consider the natural quotient map

$$p : \mathcal{F} \rightarrow \mathcal{F} : = \mathcal{F} / \text{tor}.$$

If $\mathcal{G} \subseteq \mathcal{F}$ is any coherent subsheaf, it is clear that $\mathcal{G}$ and $\mathcal{G} : = p(\mathcal{G})$ differ only along a set of codimension two, if at all. It follows that $\mu_\alpha(\mathcal{G}) = \mu_\alpha(\mathcal{G})$. Likewise, given any coherent subsheaf $\mathcal{H} \subseteq \mathcal{F}$, set $\mathcal{G} : = p^{-1}(\mathcal{H})$. Again, $\mathcal{G}$ and $\mathcal{G}$ differ only along a small set, and $\mu_\alpha(\mathcal{G}) = \mu_\alpha(\mathcal{G})$.

---

\(^2\)See [Kol96, Sect. II.4] for the definitions of the spaces $N_1(X)_{\mathbb{Q}}$ and $N_1(X)_{\mathbb{Q}}$, and for all relevant facts used here.
In summary, Observation A.3 shows that \( \mu_{\alpha}^{\text{max}}(F) = \mu_{\alpha}^{\text{max}}(\mathcal{F}) \). Better still, there exists a sheaf \( \mathcal{G} \subset \mathcal{F} \) such that \( \mu_{\alpha}(\mathcal{G}) = \mu_{\alpha}^{\text{max}}(\mathcal{F}) \) if and only if there exists a sheaf \( \mathcal{G} \subset \mathcal{F} \) such that \( \mu_{\alpha}(\mathcal{G}) = \mu_{\alpha}^{\text{max}}(\mathcal{F}) \). We will therefore make the following assumption for the remainder of the proof.

**Assumption without loss of generality A.4.** The sheaf \( \mathcal{F} \) is torsion-free.

**A.A.2. Proof of Proposition A.2, bounding \( \mu_{\alpha}^{\text{max}}(F) \).** As a first step towards a proof of Proposition A.2, the following Lemma A.5 asserts that the slopes \( \mu_{\alpha}(G) \) of sub-sheaves \( G \subset F \) are uniformly bounded from above. Its proof uses only fairly standard arguments, see for instance [MP97, p. 62].

**Lemma A.5.** Setting as above. Then \( \mu_{\alpha}^{\text{max}}(F) < \infty \).

**Proof.** Fix a very ample Cartier divisor \( H \). Since \( F \) is torsion-free, we find positive numbers \( m \) and \( N \) and an inclusion \( F \subset \mathcal{O}_X(mH)^{\oplus N} \). Since \( \mu_{\alpha}^{\text{max}}(F) \leq \mu_{\alpha}^{\text{max}}(\mathcal{O}_X(mH)^{\oplus N}) \), we may assume without loss of generality for the remainder of the proof that \( F = \mathcal{O}_X(mH)^{\oplus N} \).

We claim that the numerical class of any coherent subsheaf \( G \subset F \) satisfies the inequality

\[
\left[ \det G \right] \cdot \alpha \leq Nm[H] \cdot \alpha.
\]

In fact, given any \( G \subset F = \mathcal{O}_X(mH)^{\oplus N} \), consider the following commutative diagram with exact rows,

\[
\begin{array}{c}
\text{(A.5.2)} & 0 & \mathcal{G} & \mathcal{O}_X(mH)^{\oplus N} & \mathcal{D} & 0 \\
& & \downarrow & \downarrow & \downarrow & \\
& 0 & \mathcal{G}' & \mathcal{O}_X(mH)^{\oplus N} & \mathcal{D}' & 0.
\end{array}
\]

where \( D := \mathcal{F}/\mathcal{G} \)

The sheaf \( \mathcal{G}' \) is called “saturation of \( \mathcal{G} \) in \( \mathcal{O}_X(mH)^{\oplus N} \). The sheaf \( \mathcal{G}' \) and its subsheaf \( G \) have the same rank. Consequently, there exists an effective divisor \( D \) and an equality of numerical classes

\[
\left[ \det \mathcal{G}' \right] = \left[ \det \mathcal{G} \right] + [D].
\]

Since the curve class \( \alpha \) is movable, this implies that \( [\det \mathcal{G}] \cdot \alpha \leq [\det \mathcal{G}'] \cdot \alpha \).

Since \( H \) is assumed to be very ample, the sheaf (\( \det \mathcal{D} \)) has a non-trivial section, and is therefore represented by an effective divisor. It follows that \( \text{Nm} \cdot \alpha \geq 0 \). Using that \( \mathcal{D}' \) is torsion-free, we obtain an equality of numerical divisor classes,

\[
\left[ \det \mathcal{D}' \right] = \text{Nm}[H] - \left[ \det \mathcal{D} \right] \in N^1(X)_{\mathbb{Q}}
\]

which yields the desired inequality of intersection numbers,

\[
\left[ \det \mathcal{G} \right] \cdot \alpha \leq [\det \mathcal{G}'] \cdot \alpha \leq \text{Nm}[H] \cdot \alpha.
\]

This shows Inequality (A.5.1) and finishes the proof of Lemma A.5. \( \square \)
A.A.3. Proof of Proposition A.2, setup for Reducio Ad Absurdum. It remains to show that \( \mu_{\alpha}^{\text{max}}(\mathcal{F}) = \mu_{\alpha}(\mathcal{G}) \), for a suitable sheaf \( \mathcal{G} \subseteq \mathcal{F} \). We argue by contradiction and assume that this is not the case.

**Assumption A.6.** There is no subsheaf \( \mathcal{G} \subseteq \mathcal{F} \) such that \( \mu_{\alpha}(\mathcal{G}) = \mu_{\alpha}^{\text{max}}(\mathcal{F}) \). In particular, \( \mu_{\alpha}(\mathcal{F}) < \mu_{\alpha}^{\text{max}}(\mathcal{F}) \).

As an immediate consequence of this assumption, there exists a sequence \((\mathcal{G}_j)_{j \in \mathbb{N}^+}\) of subsheaves \( \mathcal{G}_j \subseteq \mathcal{F} \) such that the associated sequence of slopes is increasing and converges,

\[
\lim_{j \to \infty} \mu_{\alpha}(\mathcal{G}_j) = \mu_{\alpha}^{\text{max}}(\mathcal{F}) \in \mathbb{R}.
\]

Given any coherent subsheaf \( \mathcal{G} \subseteq \mathcal{F} \), its rank will come from the bounded set \( \{0, \ldots, \text{rank } \mathcal{F} \} \). As a consequence, we may assume that the ranks of the \( \mathcal{G}_i \) are maximal.

**Assumption without loss of generality A.7** (Maximality of rank). There exists a number \( r \in \mathbb{N}^+ \) such that the following two conditions hold.

(A.7.1) For all \( i \in \mathbb{N}^+ \), \( \text{rank } \mathcal{G}_i = r \).

(A.7.2) Given any number \( r' \) and sequence of coherent subsheaves \((\mathcal{H}_i)_{i \in \mathbb{N}^+}\) such that \( \lim_{j \to \infty} \mu_{\alpha}(\mathcal{H}_j) = \mu_{\alpha}^{\text{max}}(\mathcal{F}) \) and \( \text{rank } \mathcal{H}_i = r' \) for all \( i \in \mathbb{N}^+ \), then \( r' \leq r \).

Replacing the subsheaves \( \mathcal{G}_i \subseteq \mathcal{F} \) with their saturations, we obtain a sequence of subsheaves of the same rank \( r \), but possibly larger slopes. We can therefore assume the following.

**Assumption without loss of generality A.8** (Saturatedness). The sheaves \( \mathcal{G}_i \) are saturated subsheaves of \( \mathcal{F} \) for all \( i \in \mathbb{N}^+ \). In other words, the quotients \( \mathcal{F}/\mathcal{G}_i \) are torsion-free.

None of the assumptions made so far is affected when we replace the sequence \((\mathcal{G}_i)_{i \in \mathbb{N}}\) by subsequence. This allows to assume the following.

**Assumption without loss of generality A.9** (Sequence of slopes). The sequence of slopes, \((\mu_{\alpha}(\mathcal{G}_i))_{i \in \mathbb{N}^+}\), is strictly increasing. Given any number \( i \in \mathbb{N} \), then

\[
\mu_{\alpha}(\mathcal{G}_i) > \mu_{\alpha}^{\text{max}}(\mathcal{F}) - \frac{1}{i}.
\]

A.A.4. Proof of Proposition A.2, mutual containment. In this subsection, we aim to show that the sequence \( \mathcal{G}_i \) can be replaced by a subsequence with the additional property that none of the \( \mathcal{G}_i \) is contained in any one of the \( \mathcal{G}_j \), for \( j > i \). The following lemma, which essentially relies on Noetherian induction for the \( \mathcal{O}_X \)-module \( \mathcal{F} \), is a first step in the direction.

**Lemma A.10.** There exists a strictly increasing function \( \phi : \mathbb{N}^+ \to \mathbb{N}^+ \) such that the associated subsequence \( \mathcal{G}_{\phi(i)} \) satisfies the following property: Given any number \( i \in \mathbb{N}^+ \), there exist only finitely many \( j > i \) such that \( \mathcal{G}_{\phi(i)} \subseteq \mathcal{G}_{\phi(j)} \).

**Proof.** We argue by contradiction and assume that for any subsequence \( \phi \), there exists a number \( i \) and infinitely many \( j > i \) such that \( \mathcal{G}_{\phi(i)} \subseteq \mathcal{G}_{\phi(j)} \). As we will see, this assumption will allow to construct an ascending subsequence of sheaves violating the Noetherian condition. Figure A.1 illustrates the somewhat technical definitions involved.
The diagram shows a sequence of subsheaves \( G_i \subseteq F \). Arrows indicate inclusion: \( G_2 \subseteq G_7 \), but \( G_4 \nsubseteq G_5 \). Assuming that only the highlighted sheaves are contained in infinitely many of the \( G_i \), the first sequences and numbers constructed in Lemma A.10 read as follows.

\[
\begin{align*}
  i_2 &= 2 \\
  j_2 &= 5 \\
  \phi_1 &= 4, 5, 7, 8, 9, 10, \ldots \\
  \phi_2 &= 7, 8, 9, 10, \ldots \\
  \phi_3 &= 10, \ldots
\end{align*}
\]

**Figure A.1. Illustration of sequences constructed in the proof of Lemma A.10**

Applying the assumption to the identity map, let \( j_1 \) be the smallest number for which there are infinitely many \( j > j_1 \) satisfying \( G_{j_1} \subseteq G_j \). We define a subsequence, that is, a strictly increasing function \( \phi_1 \) as follows.

\[ 
\phi_1(n) := \begin{cases} 
\text{smallest number } j > j_1 \text{ such that } G_{j_1} \subseteq G_j & \text{if } n = 1 \\
\text{smallest number } j > \phi_1(n-1) \text{ such that } G_{j_1} \subseteq G_j & \text{if } n > 1
\end{cases}
\]

By construction \( \phi_1(1) > j_1 \) and \( G_{j_1} \subseteq G_{\phi_1(i)} \) for all \( i \in \mathbb{N}^+ \). Applying the assumption to the subsequence \( \phi_1 \) we can then define numbers

\[
\begin{align*}
  i_2 &= \text{smallest } i \in \mathbb{N}^+ \text{ such that } \# \{ j > i \mid G_{\phi_1(i)} \subseteq G_{\phi_1(j)} \} = \infty \\
  j_2 &= \phi_1(i_2).
\end{align*}
\]

Observe that \( j_1 < j_2 \), that \( G_{j_1} \subseteq G_{j_2} \), and that there are infinitely many sheaves \( G_j \) containing \( G_{j_2} \). Proceeding by induction, we define and infinite sequence of sequences and numbers, for \( m \geq 2 \):

\[
\begin{align*}
  \phi_m(n) := \begin{cases} 
\text{smallest number } j > j_m \text{ such that } G_{j_m} \subseteq G_j & \text{if } n = 1 \\
\text{smallest number } j > \phi_m(n-1) \text{ such that } G_{j_m} \subseteq G_j & \text{if } n > 1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
  i_{m+1} &= \text{smallest } i \in \mathbb{N}^+ \text{ such that } \# \{ j > i \mid G_{\phi_m(i)} \subseteq G_{\phi_m(j)} \} = \infty \\
  j_{m+1} &= \phi_m(i_{m+1}).
\end{align*}
\]

Again, we have that \( \phi_m(1) > j_m \) and \( G_{j_m} \subseteq G_{\phi_m(i)} \) for all \( i, m \in \mathbb{N}^+ \). In summary, we have constructed an infinite ascending sequence of sheaves,

\[ G_{j_1} \subseteq G_{j_2} \subseteq G_{j_3} \subseteq \cdots \subseteq F. \]

We claim that such a sequence of sheaves cannot exist. On the one hand, since \( F \) is a coherent sheaf of \( O_X \)-modules over a Noetherian scheme, hence itself Noetherian by [AM69, Prop. 6.5], this sequence must eventually stabilise. On the other hand,

\[
\lim_{i \to \infty} \mu_n(G_{j_i}) = \mu_n^{\max}(F).
\]
This contradicts both Assumption A.6 and A.9, and finishes the proof of Lemma A.10.

Corollary A.11. There exists a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that the associated subsequence $\mathcal{G}_{\phi(i)}$ satisfies the following property: Given any two numbers $i < j$, then $\mathcal{G}_{\phi(i)}$ is not contained in $\mathcal{G}_{\phi(j)}$.

Proof. We aim to define $\phi$ inductively. To this end, set

$$
\phi(1) := 1 \\
I_1 := \mathbb{N}^+ \setminus \{j \in \mathbb{N}^+ | \mathcal{G}_1 \subseteq \mathcal{G}_j\}.
$$

Observe that $1 \not\in I_1$, and that $I_1$ is infinite by Lemma A.10. Proceeding by induction, we define a strictly increasing sequence of numbers, and a sequence of sets,

$$
\phi(m + 1) := \min I_m \\
I_{m+1} := I_m \setminus \{j \in I_m | \mathcal{G}_{\phi(m+1)} \subseteq \mathcal{G}_j\}.
$$

The desired property now holds by construction.

Again, since none of the assumptions made so far is affected when passing to a subsequence, we can therefore assume the following.

Assumption without loss of generality A.12 (Mutual containment). Given any two numbers $i < j$, then the sheaf $\mathcal{G}_i$ is not contained in $\mathcal{G}_j$.

By Assumption A.8, the sheaves $\mathcal{G}_i$ are saturated as subsheaves of $\mathcal{F}$. Assumption A.12 therefore has the following immediate consequence.

Consequence A.13. Given any two numbers $i < j$, let $\mathcal{G}_i + \mathcal{G}_j \subseteq \mathcal{F}$ be the coherent subsheaf generated by $\mathcal{G}_i$ and $\mathcal{G}_j$. Then $\text{rank}(\mathcal{G}_i + \mathcal{G}_j) > \text{rank} \mathcal{G}_i = \text{rank} \mathcal{G}_j = r$.

A.A.5. Proof of Proposition A.2, slope computations. We have seen in Consequence A.13 that sheaves of the form $\mathcal{G}_i + \mathcal{G}_j$ have rank larger than $r$. The following lemma shows that $\mu^\text{max}_{a}(\mathcal{F})$ can be approximated by $\mathcal{G}_{ij}$, for $i, j$ sufficiently large. As we will point out in Section A.A.6 below, this violates the Maximality Assumption A.7, thus finishing the proof.

Lemma A.14. Given any two numbers $i < j$, then

$$
\mu_a(\mathcal{G}_i + \mathcal{G}_j) > \mu^\text{max}_{a}(\mathcal{F}) - \left(\frac{1}{i} + \frac{1}{j}\right).
$$

Proof. Consider the exact sequence

$$
0 \to \mathcal{G}_i \cap \mathcal{G}_j \to \mathcal{G}_i \oplus \mathcal{G}_j \to \mathcal{G}_i + \mathcal{G}_j \to 0.
$$

Since all sheaves involved are torsion-free, hence in particular locally free in codimension one, it follows that

$$
[\det(\mathcal{G}_i + \mathcal{G}_j)] = [\det \mathcal{G}_i] + [\det \mathcal{G}_j] - [\det(\mathcal{G}_i \cap \mathcal{G}_j)], \\
\text{rank}(\mathcal{G}_i + \mathcal{G}_j) = \text{rank}(\mathcal{G}_i) + \text{rank}(\mathcal{G}_j) - \text{rank}(\mathcal{G}_i \cap \mathcal{G}_j).
$$
In other words,
\[
\begin{align*}
\text{rank}(\mathcal{G}_i + \mathcal{G}_j) & \cdot \mu_a(\mathcal{G}_i + \mathcal{G}_j) \\
& = r \cdot \mu_a(\mathcal{G}_i) + r \cdot \mu_a(\mathcal{G}_j) - \text{rank}(\mathcal{G}_i \cap \mathcal{G}_j) \cdot \mu_a(\mathcal{G}_i \cap \mathcal{G}_j) \quad \text{by (A.14.2), (A.1.2)} \\
& \geq r \left( \mu_a(\mathcal{G}_i) + \mu_a(\mathcal{G}_j) \right) - \text{rank}(\mathcal{G}_i \cap \mathcal{G}_j) \cdot \mu_a^\text{max}(\mathcal{F}) \quad \text{by (A.1.3)} \\
& \geq r \left( 2\mu_a^\text{max}(\mathcal{F}) - \frac{1}{r} - \frac{1}{2} \right) - \text{rank}(\mathcal{G}_i \cap \mathcal{G}_j) \cdot \mu_a^\text{max}(\mathcal{F}) \quad \text{by Assumption A.9} \\
& = -r \left( \frac{4}{r^2} + \frac{1}{r} \right) + \text{rank}(\mathcal{G}_i \cap \mathcal{G}_j) \cdot \mu_a^\text{max}(\mathcal{F}) \quad \text{by (A.14.3)}
\end{align*}
\]
With this computation in place, Equation (A.14.1) follows from Consequence A.13 when we divide by \(\text{rank}(\mathcal{G}_i + \mathcal{G}_j)\).

A.A.6. Proof of Proposition A.2, end of proof. Consider the sequence of coherent sheaves \(\mathcal{H}_j := \mathcal{G}_j + \mathcal{G}_{j+1} \subseteq \mathcal{F}\). With this definition, Lemma A.14 asserts that
\[
\lim_{j \to \infty} \mu_a(\mathcal{H}_j) = \mu_a^\text{max}(\mathcal{F}).
\]
Passing to a suitable subsequence, we can assume that there exists a number \(r'\) such that \(\text{rank} \mathcal{H}_j = r', \) for all \(j \in \mathbb{N}^+\). We have seen in Consequence A.13 that \(r' > r\), clearly contradicting the Maximality Assumption A.7. The contradiction obtained concludes the proof of Proposition A.2. \(\square\)

A.B. Behaviour of semistability under tensor products. In the setting of Definition A.1, if \(\mathcal{L} \in \text{Pic}(X)\) is a very ample line bundle, if \((H_i)_{1 \leq i < \dim X} \in |\mathcal{L}|\) are general elements and \(\alpha\) is the class of the intersection curve,
\[
\alpha = [H_1 \cap \cdots \cap H_{\dim X-1}],
\]
a classical theorem specific to characteristic zero asserts that the tensor product of any two semistable locally free sheaves is again semistable, cf. [HL97, Thm. 3.1.4]. This result has been generalized to the case where \(X\) is smooth and \(\alpha \in N_1(X)\) an arbitrary movable class.

Fact A.15 (Reflexive product preserves semistability on manifolds, [CP11, Thm. 5.1 and Cor. 5.5]). In the setting of Definition A.1, assume additionally that \(X\) is smooth. If \(\mathcal{F}\) and \(\mathcal{G}\) are two torsion free coherent sheaves of \(\mathcal{O}_X\)-modules, then the following holds
\[
\mu_a^\text{max}(\mathcal{F} \otimes \mathcal{G})^{**} = \mu_a^\text{max}(\mathcal{F}) + \mu_a^\text{max}(\mathcal{G})^{**}.
\]
Fact A.16 (Reflexive tensor operations preserve semistability on \(\mathbb{Q}\)-factorial spaces). In the setting of Definition A.1, let \(\mathcal{F}\) and \(\mathcal{G}\) be two coherent sheaves of \(\mathcal{O}_X\)-modules that are torsion free in codimension one. Then the following holds.
\[
\mu_a^\text{max}(\mathcal{F} \otimes \mathcal{G})^{**} = \mu_a^\text{max}(\mathcal{F}) + \mu_a^\text{max}(\mathcal{G})^{**}.
\]
Proposition A.16. If \(\mathcal{F}\) and \(\mathcal{G}\) are \(\alpha\)-semistable, then \((\mathcal{F} \otimes \mathcal{G})^{**}\) is likewise \(\alpha\)-semistable. \(\square\)

We generalize Fact A.15 to the singular case.

Proposition A.16. (Reflexive tensor operations preserve semistability on \(\mathbb{Q}\)-factorial spaces). In the setting of Definition A.1, let \(\mathcal{F}\) and \(\mathcal{G}\) be two coherent sheaves of \(\mathcal{O}_X\)-modules that are torsion free in codimension one. Then the following holds.
\[
\mu_a^\text{max}(\mathcal{F} \otimes \mathcal{G})^{**} = \mu_a^\text{max}(\mathcal{F}) + \mu_a^\text{max}(\mathcal{G})^{**}.
\]
Proposition A.16. (Reflexive tensor operations preserve semistability on \(\mathbb{Q}\)-factorial spaces). In the setting of Definition A.1, let \(\mathcal{F}\) and \(\mathcal{G}\) be two coherent sheaves of \(\mathcal{O}_X\)-modules that are torsion free in codimension one. Then the following holds.
\[
\mu_a^\text{max}(\mathcal{F} \otimes \mathcal{G})^{**} = \mu_a^\text{max}(\mathcal{F}) + \mu_a^\text{max}(\mathcal{G})^{**}.
\]

To prove Proposition A.16, we choose a resolution of singularities, \(\pi : \tilde{X} \to X\), and compare semistability of sheaves on \(X\) with semistability of their reflexive pull-back sheaves. The proof of Proposition A.16, given on page 30 below, is essentially a combination of the invariance lemmas shown in the next section.
A.C. Invariance properties. Movable curve classes are more flexible than complete intersection curves: given a singular space $X$ and a resolution of singularities, $\pi : \tilde{X} \to X$, there exists a meaningful notion of pull-back that maps a movable curve class on $X$ to one on $\tilde{X}$. The results of this section discuss stability with respect to pull-back sheaves and relate stability on $X$ with that on $\tilde{X}$. The following construction is crucial.

Construction A.17. Let $X$ be a normal, $\mathbb{Q}$-factorial projective variety and $\pi : \tilde{X} \to X$ a resolution of singularities. The push-forward map of divisors respects $\mathbb{Q}$-linear and numerical equivalence and therefore induces a surjective $\mathbb{Q}$-linear map $\pi_* : N^1(\tilde{X})_\mathbb{Q} \to N^1(X)_\mathbb{Q}$.

Using the non-degenerate pairing (A.1.1), the dual of $\pi_*$ gives rise to an injective map,

$$(\text{A.17.1}) \quad \pi^* : N^1(X)_\mathbb{Q} \to N^1(\tilde{X})_\mathbb{Q},$$

which clearly satisfies the projection formula,

$$(\text{A.17.2}) \quad \pi^*(\alpha) \cdot \beta = \alpha \cdot \pi_*(\beta), \quad \text{for all } \alpha \in N^1(X)_\mathbb{Q} \text{ and } \beta \in N^1(\tilde{X})_\mathbb{Q}.$$

Remark A.18. The map $\pi^*$ of Construction A.17 appears in the literature under the name “numerical pull-back”.

Lemma A.19. In the setup of Construction A.17, if $\alpha \in N^1(X)_\mathbb{Q}$ is any class, then $\alpha$ is movable if and only if $\pi^*(\alpha)$ is movable.

Proof. The statement follows immediately from the fact that the push-forward and strict transform of any effective divisor is always effective. □

We maintain the following setting throughout the remainder of the present Section A.C.

Setting A.20. In the setup of Definition A.1, let $\pi : \tilde{X} \to X$ be a resolution of singularities. Let $E \subset \tilde{X}$ be the $\pi$-exceptional divisor, that is, the codimension-one part of the $\pi$-exceptional locus. Note that $E$ will be zero if the resolution map is small. Finally, set $\tilde{\alpha} := \pi^*(\alpha)$, where $\pi^*$ is the pull-back map (A.17.1).

Lemma A.21 (Invariance of slope and semi-stability under modifications along the exceptional set). In Setting A.20, let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves of $\mathcal{O}_{\tilde{X}}$-modules which are torsion free in codimension one. Assume that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic outside of the $\pi$-exceptional set. Then $\mu_{\tilde{\alpha}}(\mathcal{F}) = \mu_{\alpha}(\mathcal{G})$.

Proof. Observe that the line bundles $\det \mathcal{F}$ and $\det \mathcal{G}$ agree outside of $E$. Denoting the irreducible components of $E$ by $E_i$, we can therefore write

$$\det \mathcal{F} \cong (\det \mathcal{G}) \otimes \mathcal{O}_{\tilde{X}}(\sum_i \lambda_i E_i), \quad \text{for suitable } \lambda_i \in \mathbb{Z}.$$

Since $\pi_*(E_i) = 0$ for all $i$, the equality of slopes then follows from the projection formula (A.17.2). □

Lemma A.22 (Invariance of slope under push-forward). In Setting A.20, let $\mathcal{F}$ be any coherent sheaf on $\tilde{X}$ which is torsion free in codimension one.

(A.22.1) The push-forward $\pi_* \mathcal{F}$ is again torsion free in codimension one.

(A.22.2) We have equality of slopes, $\mu_{\tilde{\alpha}}(\mathcal{F}) = \mu_{\alpha}(\pi_* \mathcal{F})$. 
Proof. The push-forward of any torsion free sheaf under a surjective map is again torsion free. Claim (A.22.1) thus follows from the observation that the image of any codimension-two set is again of codimension two.

For the second claim, observe that the sheaves $\pi_* \det \mathcal{F}$ and $\det \pi_* \mathcal{F}$ are both torsion free of rank one and agree outside the singular set of $X$. Consequently, we have an equality of numerical classes on $X$,

$$\pi_* [\det \mathcal{F}] = [\pi_* \det \mathcal{F}] = [\det \pi_* \mathcal{F}] \in N^1 (X)_{\mathbb{Q}}.$$  

Equality of (A.22.2) thus follows from the projection formula (A.17.2).

Lemma A.23 (Invariance of slope under pull-back). In Setting A.20, let $\mathcal{F}$ be any coherent sheaf on $X$ that is torsion free in codimension one, and consider its reflexive pull-back $\pi^{[i]} \mathcal{F} := (\pi^* \mathcal{F})^{**}$. Then we have equality of slopes, $\mu_{\tilde{\alpha}} (\pi^{[i]} \mathcal{F}) = \mu_{\alpha} (\mathcal{F})$.

Proof. We know from Claim (A.22.2) of Lemma A.22 that $\mu_{\tilde{\alpha}} (\pi^{[i]} \mathcal{F}) = \mu_{\alpha} (\pi_* \pi^{[i]} \mathcal{F})$. Since $\mathcal{F}$ and $\pi_* \pi^{[i]} \mathcal{F}$ agree in codimension one, we have $\det \mathcal{F} \cong \det \pi_* \pi^{[i]} \mathcal{F}$. This immediately implies the claim.

Corollary A.24 (Invariance of $\mu^\max$ and semistability under pull-back). In Setting A.20, let $\mathcal{F}$ and $\mathcal{F}^{\sim}$ be sheaves on $X$ and $\tilde{X}$, respectively, both torsion free in codimension one. Assume that $\mathcal{F}^{\sim}$ is isomorphic to $\pi^* (\mathcal{F})$ away from the $\pi$-exceptional set.

(A.24.1) Given numbers $(r, \mu) \in \mathbb{N} \times \mathbb{Q}$, then $\mathcal{F}$ contains a subsheaf $\mathcal{G}$ of rank $r$ and slope $\mu_{\tilde{\alpha}} (\mathcal{G}) = q$ if and only if $\mathcal{F}^{\sim}$ contains a subsheaf $\mathcal{G}^{\sim}$ of rank $r$ and slope $\mu_{\tilde{\alpha}} (\mathcal{G}^{\sim}) = q$.

In particular,

(A.24.2) we have equality $\mu^\max_{\tilde{\alpha}} (\mathcal{F}) = \mu^\max_{\alpha} (\mathcal{F}^{\sim})$, and

(A.24.3) the sheaf $\mathcal{F}$ is semistable with respect to $\alpha$ if and only if $\mathcal{F}^{\sim}$ is semistable with respect to $\tilde{\alpha}$.

Proof. Items (A.24.2) and (A.24.3) are immediate consequences of (A.24.1). To prove the latter, let $\mathcal{G} \subseteq \mathcal{F}$ be any subsheaf with rank $\mathcal{G} = r$ and $\mu_{\tilde{\alpha}} (\mathcal{G}) = q$. Lemma A.23 asserts that $\mu_{\tilde{\alpha}} (\pi^{[i]} \mathcal{G}) = q$. By Grothendieck’s extension theorem for coherent subsheaves, [Gro60, I.Thm. 9.4.7 and 0.Sect. 5.3.2], there exists a subsheaf $\mathcal{G}^{\sim} \subseteq \mathcal{F}^{\sim}$ which agrees with $\pi^{[i]} \mathcal{G}$ wherever $\pi$ is an isomorphism. It is clear that rank $\mathcal{G}^{\sim} = r$. Lemma A.21 shows that $\mu_{\tilde{\alpha}} (\mathcal{G}^{\sim}) = q$.

Conversely, assume there exists a subsheaf $\mathcal{G}^{\sim} \subseteq \mathcal{F}^{\sim}$ with rank $\mathcal{G}^{\sim} = r$ and $\mu_{\tilde{\alpha}} (\mathcal{G}^{\sim}) = q$. Lemma A.22 shows that $\mu_{\tilde{\alpha}} (\pi_* \mathcal{G}^{\sim}) = q$, and another application of the extension theorem for coherent subsheaves gives the existence of a subsheaf $\mathcal{G} \subseteq \mathcal{F}$ which agrees with $\pi_* \mathcal{G}^{\sim}$ in codimension one. It is therefore clear that $\det \pi_* \mathcal{G}^{\sim} = \det \mathcal{G}$, so that $\mu_{\alpha} (\mathcal{G}) = q$. Since rank $\mathcal{G} = r$, this finishes the proof of (A.24.1) and hence of Corollary A.24.

A.D. Proof of Proposition A.16. We maintain notation and assumptions of Proposition A.16. Let $\pi : \tilde{X} \to X$ be a resolution of singularities and set $\pi^* (\tilde{\alpha})$. Set

$$\mathcal{A} := (\mathcal{F} \otimes \mathcal{G})^{**} \text{ and } \mathcal{A}^{\sim} := (\pi^{[i]} (\mathcal{F}) \otimes \pi^{[i]} (\mathcal{G}))^{**}.$$
It follows immediately from the definition that \( \mathcal{A} \) is isomorphic to \( \tilde{\pi}^*[\mathcal{A}] \) whenever \( \pi \) is an isomorphism.

Proof of Assertion (A.16.1). Immediate from Fact (A.15.1) and Corollary (A.24.2).

Proof of Assertion (A.16.2). Assume that \( \mathcal{F} \) and \( \mathcal{G} \) are \( \alpha \)-semistable. In this setup, Item (A.24.3) of Corollary A.24 says that

- the sheaves \( \pi^*[\mathcal{F}] \) and \( \pi^*[\mathcal{G}] \) are semistable with respect to \( \tilde{\alpha} \), and
- \( \mathcal{A} \) is \( \alpha \)-semistable if and only if \( \tilde{\mathcal{A}} \) is \( \tilde{\alpha} \)-semistable.

Semistability of \( \tilde{\mathcal{A}} \) being guaranteed by Fact (A.15.2), the claim thus follows.

Proof of Assertion (A.16.3). Since the arguments used to prove semistability for symmetric and exterior powers are the same, we consider the case of symmetric powers only. Recall from [OSS80, p. 148] that there exists a Zariski-open subset \( X^\diamond \subseteq X \) with \( \text{codim}_X X \setminus X^\diamond \geq 2 \), such that \( \mathcal{F}|_{X^\diamond} \) is locally free. In particular, there exists a direct sum decomposition,

\[
\mathcal{F}|_{X^\diamond} = \text{Sym}^q \mathcal{F}|_{X^\diamond} \oplus \mathcal{F}|_{X^\diamond} \bigg/ \text{Sym}^q \mathcal{F}|_{X^\diamond}.
\]

Using that the complement of \( X^\diamond \) is small, we obtain a direct sum decomposition of reflexive sheaves,

\[
(\mathcal{F}^\otimes q)^{**} = \text{Sym}^q[\mathcal{F}] \oplus (\mathcal{F}^\otimes q / \text{Sym}^q \mathcal{F})^{**}.
\]

Semistability of \( (\mathcal{F}^\otimes q)^{**} \) as asserted in (A.16.2) then shows the last remaining claim, finishing the proof of Proposition A.16.

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