Symplectic geometry on moduli spaces of $J$-holomorphic curves

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Abstract

Let $(M, \omega)$ be a symplectic manifold, and $\Sigma$ a compact Riemann surface. We define a 2-form $\omega_{S_i(\Sigma)}$ on the space $S_i(\Sigma)$ of immersed symplectic surfaces in $M$, and show that the form is closed and non-degenerate, up to reparametrizations. Then we give conditions on a compatible almost complex structure $J$ on $(M, \omega)$ that ensure that the restriction of $\omega_{S_i(\Sigma)}$ to the moduli space of simple immersed $J$-holomorphic $\Sigma$-curves in a homology class $A \in H_2(M, \mathbb{Z})$ is a symplectic form, and show applications and examples. In particular, we deduce sufficient conditions for the existence of $J$-holomorphic $\Sigma$-curves in a given homology class for a generic $J$.

1 Introduction

In this paper we define and study geometric structures induced on the moduli space of $J$-holomorphic curves in a symplectic manifold with a compatible almost complex structure $J$. These constructions yield symplectic invariants of the original manifold.

Let $(M, \omega)$ be a finite-dimensional symplectic manifold, and $\Sigma$ a closed 2-manifold. Denote by $\text{ev} : C^\infty(\Sigma, M) \times \Sigma \to M$ the evaluation map $\text{ev}(f, x) := f(x)$.

Definition 1.1 We define a 2-form on $C^\infty(\Sigma, M)$ as the push-forward of the 4-form $\text{ev}^*(\omega \wedge \omega)$ along the coordinate-projection $\pi_{C^\infty(\Sigma, M)} : C^\infty(\Sigma, M) \times \Sigma \to C^\infty(\Sigma, M)$:

$$(\omega_{C^\infty(\Sigma, M)})_f(\tau_1, \tau_2) := \int_{\{f\} \times \Sigma} \iota_{(\ell_1 \wedge \ell_2)} \text{ev}^*(\omega \wedge \omega).$$

(1.1)

Here $\ell_i \in T(C^\infty(\Sigma, M) \times \Sigma)$ is a lifting of $\tau_i \in T_f(C^\infty(\Sigma, M))$, i.e.,

$$d(\pi_{C^\infty(\Sigma, M)})\ell_i(f, x) = \tau_i \text{ at each point } (f, x) \in \pi_{C^\infty(\Sigma, M)}^{-1}(f).$$

Denote

$S_i(\Sigma) := \{f : \Sigma \to M \mid f \text{ is an immersion, } f^*\omega \text{ is a symplectic form on } \Sigma\}.$

The space $S_i(\Sigma)$ is an open subset of the Fréchet manifold $C^\infty(\Sigma, M)$. We identify the tangent space with the space $\Omega^0(\Sigma, f^*(TM))$ of smooth vector fields $\tau : \Sigma \to f^*(TM)$. We say that a vector field
\[\tau: \Sigma \to f^*(TM) \text{ is tangent to } f(\Sigma) \text{ at } x \text{ if } \tau(x) \in df_x(T_x \Sigma).\] We say that \(\tau\) is everywhere tangent to \(f(\Sigma)\) if \(\tau\) is tangent to \(f(\Sigma)\) at \(x\) for every \(x \in \Sigma\). Let

\[\omega_{\mathcal{S}(\Sigma)}\]

be the 2-form on \(\mathcal{S}(\Sigma)\) given by the restriction of \(\omega_{C^\infty(\Sigma, M)}\).

**Theorem 1.2.** The 2-form \(\omega_{C^\infty(\Sigma, M)}\) on \(C^\infty(\Sigma, M)\) is well defined and closed, and \(\omega_{C^\infty(\Sigma, M)}(\tau, \cdot)\) vanishes at \(f\) if \(\tau\) is everywhere tangent to \(f(\Sigma)\).

The 2-form \(\omega_{\mathcal{S}(\Sigma)}\) on \(\mathcal{S}(\Sigma)\) is closed, and \(\omega_{\mathcal{S}(\Sigma)}(\tau, \cdot)\) vanishes at \(f\) if and only if \(\tau\) is everywhere tangent to \(f(\Sigma)\).

Heuristically, the theorem says that the form \(\omega_{\mathcal{S}(\Sigma)}\) descends to a non-degenerate closed 2-form on the quotient space of unparametrized \(\Sigma\)-curves. We prove the theorem in Section 2.

Consider the space of \(\omega\)-compatible almost complex structures \(\mathcal{J} = \mathcal{J}(M, \omega)\) on \((M, \omega)\). Fix \(\Sigma = (\Sigma, j)\), where \(j\) is a complex structure on \(\Sigma\). The moduli space \(\mathcal{M}(A, \Sigma, J)\) is defined as the intersection of \(\mathcal{S}_1(\Sigma)\) with the moduli space \(\mathcal{M}(A, \Sigma, J)\) of simple \(J\)-holomorphic \(\Sigma\)-curves in a homology class \(A \in H_2(M, \mathbb{Z})\). The universal moduli space \(\mathcal{M}(A, \Sigma, \mathcal{J})\) is defined as the space of pairs \(\{(f, J) \mid f \in \mathcal{M}(A, \Sigma, J)\}\). We look at almost complex structures that are regular for the projection map

\[p_A: \mathcal{M}(A, \Sigma, \mathcal{J}) \to \mathcal{J};\]

for such a \(J\), the space \(\mathcal{M}(A, \Sigma, J)\) is a finite-dimensional manifold. Denote the set of \(p_A\)-regular \(\omega\)-compatible almost complex structures by \(\mathcal{J}_{\text{reg}}(A)\). This set is of the second category in \(\mathcal{J}\). A class \(A \in H_2(M, \mathbb{Z})\) is \(J\)-indecomposable if it does not split as a sum \(A_1 + \ldots + A_k\) of classes all of which can be represented by non-constant \(J\)-holomorphic curves. We give the necessary background on \(J\)-holomorphic curves in Section 3.

If \(J_\ast \in \mathcal{J}_{\text{reg}}(A)\) is integrable, then the restriction of the form \(\omega_{\mathcal{S}(\Sigma)}\) to \(\mathcal{M}(A, \Sigma, J_\ast)\) is non-degenerate, up to reparametrizations; see Proposition 4.4. Therefore under some conditions on a path starting from \(J_\ast\), there is a neighborhood \(U\) of \(J_\ast\) in the path such that for every \(J \in U\), the restriction of the form \(\omega_{\mathcal{S}(\Sigma)}\) to the finite-dimensional manifold \(\mathcal{M}(A, \Sigma, J)\) is non-degenerate up to reparametrizations; see Lemma 4.9. This form descends to a symplectic 2-form \(\tilde{\omega}_{\mathcal{S}(\Sigma)}\) on the quotient space \(\tilde{\mathcal{M}}(A, \Sigma, J)\) of \(\mathcal{M}(A, \Sigma, J)\) by the proper action of the group \(\text{Aut}(\Sigma, j)\) of bi-holomorphisms of \(\Sigma\); see Remark 4.7. Similarly, the form \(\omega_{C^\infty(\Sigma, M)}\) descends to a closed 2-form \(\tilde{\omega}_{C^\infty(\Sigma, M)}\) on the quotient space \(\tilde{\mathcal{M}}(A, \Sigma, J)\) by the proper action of \(\text{Aut}(\Sigma, j)\).

The form \(\omega_{\mathcal{S}(\Sigma)}\) yields symplectic invariants of \((M, \omega, A)\). In Section 4 we apply Theorem 1.2 Gro-mov’s compactness theorem, and Stokes’ theorem to deduce the following corollary.

**Corollary 1.3.** Assume that \(M\) is compact. Let \(S\) be the subset of \(p_A\)-regular \(\omega\)-compatible \(J\)-s for which the class \(A\) is \(J\)-indecomposable. Then

\[\int_{\tilde{\mathcal{M}}(A, \Sigma, J)} \wedge^n \tilde{\omega}_{C^\infty(\Sigma, M)} \]

(1.2)

is well defined and does not depend on \(J \in S\).

If there is an integrable \(J_\ast \in S\) such that \(\mathcal{M}(A, \Sigma, J_\ast) \neq \emptyset\) then

\[\int_{\tilde{\mathcal{M}}(A, \Sigma, J)} \wedge^n \tilde{\omega}_{C^\infty(\Sigma, M)} \neq 0 \text{ for every } J \in S.\]

In particular, for every \(J \in S\) there exists a \(J\)-holomorphic curve in \(A\).
When $S$ is of the second category, the corollary implies that the integral (1.2) does not depend on $J$ for a generic $J$; in some cases, the corollary implies the existence of a $J$-holomorphic $\Sigma$-curve in $A$ for a generic $J$. In Section 5, we give examples in which Corollary 1.3 and Lemma 4.9 apply.

We observe that for $J \in \mathcal{J}(A)$ such that $\tilde{\omega}_{S_1}(\Sigma)$ on $\mathcal{M}_1(A, \Sigma, J)$ is symplectic, we obtain a canonical almost complex structure on $\mathcal{M}_1(A, \Sigma, J)$ that is compatible with the form; see Corollary 4.11. Thus this paper provides a setting to understand a symplectic manifold with a compatible almost complex structure by studying curves on the moduli spaces of $J$-holomorphic curves, a technique that has been proven to be far reaching in algebraic geometry.

2 Properties of the 2-forms $\omega^\infty(\Sigma, M)$ and $\omega_{S_i}(\Sigma)$

We first observe the following fact, that we will use throughout this section.

**Lemma 2.1.** For $(\nu, \nu_\Sigma) \in T(f, x)(C^\infty(\Sigma, M) \times \Sigma),
\quad d(ev)(f, x)(\nu, \nu_\Sigma) = \nu_f(x) + df_x(\nu_\Sigma) \quad
In particular, $d(ev)|_{T(f, x) \times \Sigma} = df$ and $d(ev)(f, x)(\nu, 0) = \nu_f(x)$.

**The form is well defined**

**Claim 2.2.** Let $f: \Sigma \to M$ and $\tau_1, \tau_2 \in T(f, x)(C^\infty(\Sigma, M)).$ The value of $\int_{\{f\} \times \Sigma} t(\ell_1 \wedge \ell_2)ev^*(\omega \wedge \omega)$ does not depend on the choice of liftings $\ell_i$ of $\tau_i$.

**Proof.** Let $\ell_i, \ell'_i$ be two pairs of liftings of $\tau_i$. Let
\[ v_i(f, x) := \ell'_i(f, x) - \ell_i(f, x). \]
Then
\[ v_i(f, x) = (0, \nu_1^\Sigma) \in T(C^\infty(\Sigma, M) \times \Sigma), \]
and so by Lemma 2.1
\[ d(ev)(v_i(f, x)) = df_x(\nu_1^\Sigma) \in df_x(T_x \Sigma). \]

On the other hand we have that
\[ \int_{\{f\} \times \Sigma} t(\ell_1 \wedge \ell_2)ev^*(\omega \wedge \omega) \]
is equal to
\[ \int_{\{f\} \times \Sigma} t(\ell_1 \wedge \ell_2)ev^*(\omega \wedge \omega) + \int_{\{f\} \times \Sigma} t(\tau_1 \wedge \ell_2)ev^*(\omega \wedge \omega) + \int_{\{f\} \times \Sigma} t(\ell_1 \wedge \tau_2)ev^*(\omega \wedge \omega) + \int_{\{f\} \times \Sigma} t(\tau_1 \wedge \tau_2)ev^*(\omega \wedge \omega). \]

To complete the proof it is enough to show that the last three terms vanish. We will show that their integrands are identically zero when restricted to $T(\{f\} \times \Sigma)$. Let $z_1, z_2$ be any pair of vectors in $T(\{f\} \times \Sigma)$. Then
\[ t(v_1 \wedge v_2)(\omega \wedge \omega)(z_1, z_2) = \left( t(d(ev)(v_1) \wedge d(ev)(v_2)) \right) \left( d(ev)(z_1), d(ev)(z_2) \right) \]
\[ = \left( t(df(v_1^\Sigma) \wedge d(ev)(v_2)) \right) \left( df(z_1), df(z_2) \right). \]
Similarly we obtain that
\[
\left( \iota_{(\ell_1 \wedge \ell_2)} \mathrm{ev}^* (\omega \wedge \omega) \right)(z_1, z_2) = \left( \iota_d(\mathrm{ev})(\ell_1) \wedge d(\mathrm{ev})(\ell_2) \right) (\omega \wedge \omega) (d(\mathrm{ev})(z_1), d(\mathrm{ev})(z_2))
\]
and that
\[
\left( \iota_{(\ell_1 \wedge \ell_2)} \mathrm{ev}^* (\omega \wedge \omega) \right)(z_1, z_2) = \left( \iota_{(d(\mathrm{ev})(\ell_1) \wedge d(\mathrm{ev})(\ell_2))} (\omega \wedge \omega) \right) (d(\mathrm{ev})(z_1), d(\mathrm{ev})(z_2))
\]
(2.4)
The terms
\[
\iota_{(d(f(v_1^2) \wedge d(\mathrm{ev})(\ell_2))) (\omega \wedge \omega)}, \quad \iota_{(d(\mathrm{ev})(\ell_1) \wedge d(f(v_2^2))) (\omega \wedge \omega)}, \quad \iota_{(d(f(v_2^2) \wedge d(\mathrm{ev})(\ell_2))) (\omega \wedge \omega)}
\]
each vanish when restricted to the 2-dimensional subspace \( d_f(x) \subseteq T_{f(x)} \), by Lemma 2.3. It follows that the right hand side in expressions (2.3), (2.4) and (2.5) identically vanishes. Hence
\[
\int_{\{f\} \times \Sigma} \iota_{(\ell_1 \wedge \ell_2)} \mathrm{ev}^* (\omega \wedge \omega) = \int_{\{f\} \times \Sigma} \iota_{(\ell_1 \wedge \ell_2)} \mathrm{ev}^* (\omega \wedge \omega).
\]

\[\square\]

**Lemma 2.3.** Let \( W \) be a vector space, and let \( \alpha \) be a 4-form: \( \wedge^4 W \to \mathbb{R} \). Let \( V \subseteq W \) be a subspace of dimension \( \leq 2 \). Then \( \left( \iota_{(\ell v \wedge \omega)} \alpha \right)|_V = 0 \) for all \( v \in V, w \in W \).

This is the case since any three vectors in \( V \) are linearly dependent.

**The form is closed**

*Proof of the closedness part of Theorem 7.2.* The 2-form \( \omega^{C^\infty}(\Sigma, M) \) is closed if and only if for any two surfaces \( R_1 \) and \( R_2 \) in \( C^\infty(\Sigma, M) \), that are homologous relative to a common boundary \( \partial R \),
\[
\int_{R_1} \omega^{C^\infty}(\Sigma, M) = \int_{R_2} \omega^{C^\infty}(\Sigma, M).
\]
(2.6)

Now (by considering the liftings that are zero along \( \Sigma \)),
\[
\int_{R_1} \omega^{C^\infty}(\Sigma, M) = \int_{R_1 \times \{x\}} \left( \int_{\{f\} \times \Sigma} \mathrm{ev}^* (\omega \wedge \omega) \right) = \int_{R_1 \times \Sigma} \mathrm{ev}^* (\omega \wedge \omega).
\]
Since \( R_1 \times \Sigma \) is homologous to \( R_2 \times \Sigma \) relative to the boundary \( \partial R \times \Sigma \), and \( \mathrm{ev}^* (\omega \wedge \omega) \) is closed, we have that:
\[
\int_{R_1 \times \Sigma} \mathrm{ev}^* (\omega \wedge \omega) = \int_{R_2 \times \Sigma} \mathrm{ev}^* (\omega \wedge \omega).
\]

Therefore we get (2.6). \[\square\]
Directions of degeneracy

Proof of the domain of non-degeneracy in Theorem 1.2.

Case 1. Suppose that \( f : \Sigma \to M \), and \( \tau : \Sigma \to f^*TM \) is everywhere tangent to \( f(\Sigma) \).

We claim that \( \omega_{\infty(\Sigma, M)} f(\tau, \cdot) = 0 \). To see this, lift \( \tau \) to a vector field \( \ell = (\tau, 0) \) along \( C^\infty(\Sigma, M) \times \Sigma \); let \( \tau_2 \in T_f(C^\infty(\Sigma, M)) \) and \( \ell_2 \) a lifting of \( \tau_2 \). We show that the integrand \( \iota_{\ell \wedge \ell_2} \ev^*(\omega \wedge \omega) \) vanishes when restricted to \( T(\{f\} \times \Sigma) \). Indeed, for \( z_1, z_2 \in T_x(\{f\} \times \Sigma) \),

\[
\iota_{\ell \wedge \ell_2} \ev^*(\omega \wedge \omega)(z_1, z_2) = \iota_{\tau \wedge \ev(\ell_2)}(\omega \wedge \omega)_{f(\cdot)}(d(f(z_1)), d(f(z_2))).
\]

So it is enough to show that

\[
\iota_{\tau \wedge \ev(\ell_2)}(\omega \wedge \omega)|_{df(T_x \Sigma)}
\]

vanishes. This follows from Lemma 2.3 since, by assumption \( \tau(x) \in df_x(T_x \Sigma) \) and \( df_x(T_x \Sigma) \subset T_f(x) M \) is a 2-dimensional subspace.

Case 2. Suppose that \( f \in \mathcal{S}(\Sigma), \) and \( \tau \in T_f(\mathcal{S}(\Sigma)) \) is not tangent to \( f(\Sigma) \) at \( x \in \Sigma \).

Let \( \tau_\perp \in (f^*TM)_x \) denote (a representative of) the orthogonal projection of \( \tau \) to the normal bundle to \( df(T \Sigma) \). In particular, \( \tau_\perp(x) \neq 0 \). Let \( \tau_1 \) be a vector in \((f^*TM)_x\) such that

\[
\omega(\tau_\perp(x), \tau_1) = \omega_{f(x)}(\tau_\perp(x), \tau_1) > 0,
\]

and \( \tau_1 \) is symplectically orthogonal to \( df_x(T_x \Sigma) \).

Now extend \( \tau_1 \) to a section \( \tau_1 : \Sigma \to f^*TM \) such that \( \omega(\tau_\perp(y), \tau_1(y)) > 0 \) and \( \tau_1(y) \) is symplectically orthogonal to \( df_y(T_y \Sigma) \) for \( y \) in a small neighborhood of \( x \), and vanishing outside it. Let \( \ell \) and \( \ell_1 \) be liftings of \( \tau_\perp \) and \( \tau_1 \) that are zero along \( \Sigma \). We claim that \( \omega_{\mathcal{S}(\Sigma)} f(\tau_\perp, \tau_1) = 0 \).

Notice that, in general for vectors \( \mu_1, \mu_2 : \Sigma \to f^*TM \) and their zero-liftings \( k_1^0 = (\mu_1, 0), k_2^0 = (\mu_2, 0) \), we have that for \( z_1, z_2 \in T(\{f\} \times \Sigma) \),

\[
\iota_{k_1^0 \wedge k_2^0} \ev^*(\omega \wedge \omega)(z_1, z_2) = \left( \iota_{(d \ev)(k_1^0) \wedge d \ev(k_2^0)}(\omega \wedge \omega)(d \ev(z_1), d \ev(z_2)) \right) = \iota_{\mu_1 \wedge \mu_2}(\omega \wedge \omega)(d \ev(z_1), d \ev(z_2)) = 2 \omega(\mu_1, \mu_2)(d \ev(z_1), d \ev(z_2)) + 2 \omega(\mu_1, d \ev(\cdot)) \wedge \omega(\mu_2, d \ev(\cdot))(z_1, z_2).
\]

In particular, and since \( \tau_1(y) \) is symplectically orthogonal to \( df_y(T_y \Sigma) \) for every \( y \in \Sigma \) (hence the second summand in the last term vanishes), we get that

\[
\int_{(f \times \Sigma)} \iota_{(\ell_1 \wedge \ell_2)} \ev^*(\omega \wedge \omega) = \int_{\Sigma} \omega(\tau_\perp, \tau_1) f^*\omega \neq 0,
\]

where the last inequality follows from the choice of \( \tau_1 \), and the fact that, by definition of \( \mathcal{S}(\Sigma) \), the form \( f^*\omega \) is a symplectic 2-form on a surface, hence an area-form.

Since, (by Case 1), \( \omega_{\mathcal{S}(\Sigma)} f(\tau, \tau_1) = \omega_{\mathcal{S}(\Sigma)} f(\tau_\perp, \tau_1) \), we deduce that

\[
\omega_{\mathcal{S}(\Sigma)} f(\tau, \tau_1) \neq 0.
\]
Compatible almost complex structures on $S_i(\Sigma)$

An *almost complex structure* on a manifold $M$ is an automorphism of the tangent bundle,

$$J: TM \to TM,$$

such that $J^2 = -\text{Id}$. The pair $(M, J)$ is called an *almost complex manifold*.

An almost complex structure is *integrable* if it is induced from a complex manifold structure. In dimension two any almost complex manifold is integrable (see, e.g., [9, Th. 4.16]). In higher dimensions this is not true [3].

An almost complex structure $J$ on $M$ is *tamed* by a symplectic form $\omega$ if $\omega_x(v, Jv) > 0$ for all non-zero $v \in T_x M$. If, in addition, $\omega_x(Jv, Jw) = \omega_x(v, w)$ for all $v, w \in T_x M$, we say that $J$ is $\omega$-compatible. The space $\mathcal{J}(M, \omega)$ of $\omega$-compatible almost complex structures is non-empty and contractible, in particular path-connected [9, Prop. 4.1].

**Definition 2.4** Let $J$ be an almost complex structure on $M$. We define a map

$$\tilde{J} := J_{S_i(\Sigma)} : TS_i(\Sigma) \to TS_i(\Sigma)$$

as follows: for $\tau: \Sigma \to f^*(TM)$, the vector $\tilde{J}(\tau)$ is the section $\hat{J} \circ \tau$, where $\hat{J}$ is the map defined by the commutative diagram

$$
\begin{array}{ccc}
 f^*(TM) & \xrightarrow{\hat{J}} & f^*(TM) \\
 \downarrow & & \downarrow \\
 TM & \xrightarrow{J} & TM
\end{array}
$$

Since $S_i(\Sigma)$ is an open subset of $C^\infty(\Sigma, M)$, we get that $TS_i(\Sigma)$ is indeed closed under $\tilde{J}$. Due to the properties of the almost complex structure $J$, the map $\tilde{J}$ is an automorphism and $\tilde{J}^2 = -\text{Id}$.

**Claim 2.5.** Let $J$ be an almost complex structure on $M$. Then $\tilde{J}$ is an almost complex structure on $S_i(\Sigma)$.

A smooth ($C^\infty$) curve $f: \Sigma \to M$ from a compact Riemann surface $(\Sigma, j)$ to an almost complex manifold $(M, J)$ is called $(j, J)$-holomorphic if the differential $df$ is a complex linear map between the fibers $T_p(\Sigma) \to T_{f(p)}(M)$ for all $p \in \Sigma$, i.e.,

$$df_p \circ j_p = J_{f(p)} \circ df_p.$$  

(When $j$ is clear from the context, we call $f$ a $J$-holomorphic curve.)

Fix $\Sigma = (\Sigma, j)$.

**Claim 2.6.** Let $J$ be an almost complex structure on $M$. Assume that $f: \Sigma \to M$ is $J$-holomorphic. Then, at $x \in \Sigma$,

1. if $\tau(x) \in df_x(T_x \Sigma)$ then $J_{f(x)}(\tau_x) \in df_x(T_x \Sigma)$;

2. if $J$ is $\omega$-compatible and $\phi_x$ is symplectically orthogonal with respect to $df_x(T_x \Sigma)$, then so is $J_{f(x)}(\phi_x)$. 

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Proof. 1. By assumption $\tau_x = df_x(\alpha)$ for $x \in T_\Sigma$. Hence, since $f$ is $J$-holomorphic,

$$J_{f(x)}(\tau_x) = J_{f(x)}(df_x(\alpha)) = df_x(j_x \alpha).$$

2. By the previous item, $J_{f(x)}(df_x(T_\Sigma)) \subseteq df_x(T_\Sigma)$, hence, since $J^2 = -\text{Id}$,

$$J_{f(x)}(df_x(T_\Sigma)) = df_x(T_\Sigma).$$

Let $\tau_x \in df_x(T_\Sigma)$, then there exists $\tau'_x \in df_x(T_\Sigma)$ such that $\tau_x = J_{f(x)}(\tau'_x)$. By assumption, $\omega(\phi_x, \tau_x'') = 0$. Thus

$$\omega(J_{f(x)}(\phi_x), \tau_x) = \omega(J_{f(x)}(\phi_x), J_{f(x)}(\tau_x'')) = \omega(\phi_x, \tau_x'') = 0.$$

\[ \square \]

**Corollary 2.7.** Let $J$ an $\omega$-tamed almost complex structure on $M$. Assume that $f : \Sigma \to M$ is $J$-holomorphic. Then, at $x \in \Sigma$, for $W_x = df_x(T_\Sigma)$, $W_x \cap W_x^\omega = \{0\}$.

**Proof.** By part (1) of Claim 2.6, if $v \in W_x$, then $J(v) \in W_x$; since $J$ is $\omega$-tamed, if $v \neq 0$, $\omega(v, J(v)) > 0$, hence $0 \neq v \in W_x$.

Since $\dim W_x^\omega = 2 \dim M - \dim W_x$, this implies the following corollary.

**Corollary 2.8.** In the assumptions and notations of Corollary 2.7,

$$T_{f(x)} M = W_x + W_x^\omega.$$

Recall that if a bundle $E \to B$ equals the direct sum of sub-bundles $E_1 \to B$ and $E_2 \to B$, then the space of sections of $E$ equals the direct sum of the space of sections of $E_1$ and the space of sections of $E_2$. Thus, Corollary 2.8 implies the following corollary.

**Corollary 2.9.** Let $J$ an $\omega$-tamed almost complex structure on $M$. Assume that $f : \Sigma \to M$ is $J$-holomorphic. Then every $\mu \in T_f(S_\Sigma)$ can be uniquely decomposed as

$$\mu = \mu' + \mu'',$$

where $\mu'$ is everywhere tangent to $f(\Sigma)$ (i.e. $\mu'(x) \in df_x(T_\Sigma)$ at every $x \in \Sigma$), and $\mu''(x)$ is symplectically orthogonal to $df_x(T_\Sigma)$ at every $x \in \Sigma$.

**Proposition 2.10.** Let $J$ be an almost complex structure on $M$, and $f : \Sigma \to M$ a $J$-holomorphic map.

1. Assume that $J$ is $\omega$-tamed. Then for $\tau$ that is not everywhere tangent to $f(\Sigma)$, $\omega_{S_\Sigma} f(\tau, J(\tau)) > 0$.

2. Assume that $J$ is $\omega$-compatible. Then $\tilde{J}$ is compatible with $\omega_{S_\Sigma}$.

**Proof.** (1) By Corollary 2.9 part (1) of Claim 2.6 and Theorem 1.2 it is enough to show that $\omega_{S_\Sigma} f(\phi, J(\phi)) \neq 0$ for $\phi$ such that $\phi(x)$ is symplectically orthogonal to $df_x(T_\Sigma)$ at every $x \in \Sigma$. Indeed, for such $\phi$, for $z_1, z_2 \in T\{(f \times \Sigma)$,

$$t_{(\phi, 0) \wedge (\tilde{J}(\phi), 0)}(\text{ev}^*(\omega \wedge \omega))(z_1, z_2)$$

$$= \left(t_{(d(\text{ev})(\phi, 0) \wedge d(\text{ev})(\tilde{J}(\phi), 0))}(\omega \wedge \omega)\right)(d(\text{ev})(z_1), d(\text{ev})(z_2))$$

$$= \left(t_{\phi \wedge (\tilde{J}(\phi))(\omega \wedge \omega)}\right)(df(z_1), df(z_2))$$

$$= 2 \omega(\phi, J(\phi)) \omega(df(z_1), df(z_2)) + 2 \omega(\phi, df(\cdot)) \wedge \omega(\tilde{J}(\phi), df(\cdot))(z_1, z_2).$$
By assumption on \( \phi \), the second summand in the last term equals zero, so
\[
\int_{\{f\} \times \Sigma} \iota_{(\phi, 0) \wedge (\bar{J}(\phi), 0)} (\ev^* (\omega \wedge \omega)) = \int_{\Sigma} \omega(\phi, \bar{J}(\phi)) f^* \omega \neq 0,
\]
where the last inequality follows from the fact that \( J \) is \( \omega \)-tamed and \( f^* \omega \) is a symplectic 2-form on a surface, hence an area-form.

(2) By Corollary 2.9 part (1) of Claim 2.6 and Theorem 1.2, it is enough to show that
\[
\omega_{S_{(\Sigma)} f}(\phi_1, \phi_2) = \omega_{S_{(\Sigma)} f}(\bar{J}(\phi_1), \bar{J}(\phi_2))
\]
for \( \phi_1, \phi_2 \) such that \( \phi_i(x) \) is symplectically orthogonal to \( df_x(T_x \Sigma) \) at every \( x \in \Sigma \). Indeed the above calculation shows that for such \( \phi_i \),
\[
\iota_{(\bar{J}(\phi_1), 0) \wedge (\bar{J}(\phi_2), 0)} (\ev^* (\omega \wedge \omega))(z_1, z_2)
= 2 \omega(\bar{J}(\phi_1), \bar{J}(\phi_2)) \omega(df(z_1), df(z_2)) + 2 \omega(\bar{J}(\phi_1), df(\cdot)) \wedge \omega(\bar{J}(\phi_2), df(\cdot))(z_1, z_2).
\]
Since \( J \) is \( \omega \)-compatible, the first summand in the last term equals \( 2 \omega(\phi_1, \phi_2) \omega(df(z_1), df(z_2)) \). By assumption on \( \phi_i \), and since \( J \) is \( \omega \) compatible, we deduce by part (2) of Claim 2.6 that \( \bar{J}(\phi_1)(x) \) is symplectically orthogonal to \( df_x(T_x \Sigma) \) at every \( x \in \Sigma \), therefore the second summand in the last term equals zero. Thus
\[
\iota_{(\bar{J}(\phi_1), 0) \wedge (\bar{J}(\phi_2), 0)} (\ev^* (\omega \wedge \omega))(z_1, z_2) = 2 \omega(\phi_1, \phi_2) \omega(df(z_1), df(z_2))
= \iota_{(\phi_1, 0) \wedge (\phi_2, 0)} (\ev^* (\omega \wedge \omega))(z_1, z_2)
\]
(The last equality follows again from the assumption on \( \phi_i \).)

\[\square\]

3 Moduli spaces of \( J \)-holomorphic curves

We review here the ingredients of the theory of \( J \)-holomorphic curves that we need in this study, as we learned from [4] and [10], and deduce a few facts that we use later in the paper.

Let \((M, J)\) be an almost complex \(2n\)-manifold. Fix a compact Riemann surface \( \Sigma = (\Sigma, j) \).

Simple \( J \)-holomorphic curves

We say that a \( J \)-holomorphic curve is \textit{simple} if it is not the composite of a holomorphic branched covering map \((\Sigma, j) \rightarrow (\Sigma', j')\) of degree greater than one with a \( J \)-holomorphic map \((\Sigma', j') \rightarrow (M, J)\).

The operators \( \bar{\partial}_J \) and \( D_f \)

The \( J \)-holomorphic maps from \((\Sigma, j)\) to \((M, J)\) are the maps satisfying \( \bar{\partial}_J(f) = 0 \), where
\[
\bar{\partial}_J(f) := \frac{1}{2} (df + J \circ df \circ j).
\]
Let \( A \in H_2(M, \mathbb{Z}) \) be a homology class. The \( \bar{\partial}_J \) operator defines a section \( S: B \rightarrow E \),
\[
S(f) := (f, \bar{\partial}_J(f)), \quad (3.7)
\]
where $B \subset C^\infty(\Sigma, M)$ denotes the space of all smooth maps $f: \Sigma \to M$ that represent the homology class $A$, and the bundle $\mathcal{E} \to B$ is the infinite dimensional vector bundle whose fiber at $f$ is the space $\mathcal{E}_f := \Omega^{0,1}(\Sigma, f^*(TM))$ of smooth $J$-antilinear 1-forms on $\Sigma$ with values in $f^*(TM)$.

Given a $J$-holomorphic map $f: (\Sigma, j) \to (M, J)$ in the class $A$, the *vertical differentiation* of the section $S$ at $f$

$$D_f: \Omega^0(\Sigma, f^*(TM)) \to \Omega^{0,1}(\Sigma, f^*(TM))$$

is the composition of the differential $D_S(f): T_f B \to T_{(f,0)} \mathcal{E}$ with the projection map $\pi_f: T_{(f,0)} \mathcal{E} = T_f B \oplus \mathcal{E}_f \to \mathcal{E}_f$.

Let $(M, \omega)$ be a symplectic manifold and $J$ an $\omega$-compatible almost complex structure on $M$. The operator $D_f$ can be expressed as a sum

$$D_f(\xi) = D^I_f(\xi) + D^J_f(\xi),$$

where $D^I_f(\xi)$ is complex linear (meaning that $D^I_f(J\xi) = J D^I_f(\xi)$) and

$$D^J_f(\xi) = \frac{1}{4} N_J(\xi, \partial_J(f)). \quad (3.8)$$

is complex antilinear (meaning that $D^J_f(J\xi) = -J D^J_f(\xi)$). See [10] Remark 3.1.2. Recall that the Nijenhuis tensor $N_J$ is defined by

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

where $X, Y: M \to TM$ are vector fields on $M$. It is a result of Newlander-Nirenberg that $J$ is integrable if and only if $N_J = 0$, c.f. [11] or [9, Thm. 4.12].

**Lemma 3.1.** If $\tau \in \Omega^0(\Sigma, f^*(TM))$ is such that $D_f(\tau) = 0$, then

$$D_f(J \tau) = -2J D^J_f(\tau). \quad (3.9)$$

**Proof.** If $D_f(\tau) = 0$, then $D^I_f(\tau) = -D^J_f(\tau)$, and therefore we have that

$$D_f(J \tau) = J D^I_f(\tau) - J D^J_f(\tau) = -J D^J_f(\tau) - J D^J_f(\tau) = -2J D^J_f(\tau).$$

\qed

**Moduli spaces of simple $J$-holomorphic curves**

**Definition 3.2** The *moduli space* of $J$-holomorphic maps from $(\Sigma, j)$ to $(M, J)$ in the class $A$ is the set $\mathcal{M}(A, \Sigma, J)$ given as the intersection of the zero set of the section $S$ in $(3.7)$, and the open set of all simple maps $\Sigma \to M$ which represent the class $A$.

Explicitly,

$$\mathcal{M}(A, \Sigma, J) := \{ f \in C^\infty(\Sigma, M) \mid f \text{ is a simple } (j, J)\text{-holomorphic map in } A \}.$$

The tangent space to $\mathcal{M}(A, \Sigma, J)$ at $f$ is the zero set of $D_f$.

**Lemma 3.3.** If $J$ is an integrable almost complex structure on $M$, and $f: \Sigma \to M$ is a $J$-holomorphic map, then for $\tau \in T_f \mathcal{M}(A, \Sigma, J)$, the vector $J \circ \tau$ is also in $T_f \mathcal{M}(A, \Sigma, J)$.

**Proof.** Since $J$ is integrable $N_J = 0$, hence by $(3.8)$ and $(3.9)$, $D_f(J \circ \tau) = 0$. \qed
The universal moduli space

Let $\mathcal{J} := \mathcal{J}(M, \omega)$.

**Definition 3.4** The universal moduli space of holomorphic maps from $\Sigma$ to $M$, in the class $A$, is the set

$$\mathcal{M}(A, \Sigma, \mathcal{J}) := \{(f, J) \mid J \in \mathcal{J}, f \in \mathcal{M}(A, \Sigma, J)\}.$$ 

Regular value and regular path

We have the following consequences of the Sard-Smale theorem, the infinite dimensional inverse mapping theorem, and the ellipticity of the Cauchy-Riemann equations.

**Lemma 3.5.** (a) The universal moduli space $\mathcal{M}(A, \Sigma, \mathcal{J})$ and the space $J$ are Fréchet manifolds.

(b) Consider the projection map

$$p_A: \mathcal{M}(A, \Sigma, \mathcal{J}) \to \mathcal{J}$$

for any $J$ in the set of $\omega$-compatible $J$’s that are regular for the map $p_A$. The moduli space $\mathcal{M}(A, \Sigma, J)$ is a smooth manifold of dimension $2c_1(A) + n(2 - 2g)$.

(c) The set of $\omega$-compatible $J$’s that are regular for the map $p_A$ is of the second category in $J$.

(d) If $(u, J)$ is a regular value for $p_A$, then for any neighborhood $U$ of $(u, J)$ in $\mathcal{M}(A, \Sigma, \mathcal{J})$, its image, $p_A(U)$, contains a neighborhood of $J$ in $\mathcal{J}$.

(e) Let $J_0, J_1 \in \mathcal{J}$, assume that $J_0, J_1$ are regular for $p_A$. If a path $\lambda \to J_\lambda$ is transversal to $p_A$, then

$$\mathcal{W}(A, \Sigma, \{J_\lambda\}_\lambda) := \{(\lambda, f) \mid 0 \leq \lambda \leq 1, f \in \mathcal{M}(A, \Sigma, J_\lambda)\}$$

is a smooth oriented manifold with boundary $\partial\mathcal{W}(A, \Sigma, \{J_\lambda\}_\lambda) = \mathcal{M}(A, \Sigma, J_0) \cup \mathcal{M}(A, \Sigma, J_1)$. The boundary orientation agrees with the orientation of $\mathcal{M}(A, \Sigma, J_1)$ and is opposite to the orientation of $\mathcal{M}(A, \Sigma, J_0)$.

(f) Let $\{J_t\}_{t \in [0,1]}$ be a $C^1$ simple path in $\mathcal{J}$ whose endpoints are regular values for $p_A$. Then there exists a $C^1$ perturbation $\{\tilde{J}_t\}$ of $J_t$ with the same endpoints which is transversal to $p_A$.

For items (a), (d) and (f) see, e.g., [7] and [6] where they are derived using the results of [10, chapter 3]. For items (b) and (c) see [10, Theorem 3.1.5]. For item (e) see [10, Theorem 3.1.7].

4 Symplectic structure on moduli spaces of simple immersed $J$-holomorphic curves

Consider the moduli subspaces of $\mathcal{S}_i(\Sigma)$:

$$\mathcal{M}_i(A, \Sigma, J) := \mathcal{M}(A, \Sigma, J) \cap \mathcal{S}_i(\Sigma) \quad \text{and} \quad \mathcal{M}_i(A, \Sigma, \mathcal{J}) := \mathcal{M}(A, \Sigma, \mathcal{J}) \cap \mathcal{S}_i(\Sigma).$$
Remark 4.1 For $J \in \mathcal{J}$, if $f$ is a $J$-holomorphic curve and an immersion then $f^*\omega$ is a symplectic form. Closedness is immediate since $\omega$ is closed and the differential commutes with the pullback. The fact that $f^*\omega$ is not degenerate is by the following argument. For $u \neq 0$,

$$f^*\omega(u, j(u)) = \omega(df(u), df(j(u))) = \omega(df(u), J df(u)) > 0,$$

the last strict inequality is since $J$ is $\omega$-compatible and $df(u) \neq 0$ (since $u \neq 0$ and $df$ is an immersion). Thus $\mathcal{M}_i(A, \Sigma, \mathcal{J})$ is the subset of immersions in $\mathcal{M}(A, \Sigma, \mathcal{J})$.

Lemma 4.2. The moduli space $\mathcal{M}_i(A, \Sigma, \mathcal{J})$ is a Fréchet submanifold of the Fréchet manifold $\mathcal{M}(A, \Sigma, \mathcal{J})$. If $J \in \mathcal{J}$ is regular for $p_A|_{\mathcal{M}_i(A, \Sigma, \mathcal{J})}$, the moduli space $\mathcal{M}_i(A, \Sigma, J)$ is a finite dimensional manifold. Moreover, the statements of Lemma 3.5 hold true for $\mathcal{M}_i(A, \Sigma, \mathcal{J})$, $\mathcal{M}_i(A, \Sigma, J)$, and $p_A|_{\mathcal{M}_i(A, \Sigma, \mathcal{J})}$.

Corollary 4.3. Assume that $S$ is an open and path-connected subset of $\mathcal{J}$, and that $S \subset \mathcal{J}_{\text{reg}}(A)$. Then for every $J_0, J_1 \in S$ there is an oriented cobordism between the manifolds $\mathcal{M}_i(A, \Sigma, J_0)$ and $\mathcal{M}_i(A, \Sigma, J_1)$ that is the $p_A$-preimage of a $p_A$-transversal path from $J_0$ to $J_1$ in $\mathcal{J}$.

The form restricted to the moduli space of simple immersed $J$-holomorphic curves is symplectic up to reparametrizations when $J$ is integrable or close to integrable on a path

As a result of Lemma 3.3 and Proposition 2.10 we get the following proposition.

Proposition 4.4. If $J$ is an integrable almost complex structure on $M$ that is compatible with $\omega$ and regular for $A$, then $\omega_{S_i(\Sigma)}$ restricted to $\mathcal{M}_i(A, \Sigma, J)$ is symplectic up to reparametrizations.

Remark 4.5 For examples of regular integrable compatible almost complex structures, we look at Kähler manifolds whose automorphism groups act transitively. By [10, Proposition 7.4.3], if $(M, \omega_0, J_0)$ is a compact Kähler manifold and $G$ is a Lie group that acts transitively on $M$ by holomorphic diffeomorphisms, then $J_0$ is regular for every $\omega \in H^2(M, \mathbb{Z})$. This applies, e.g., when $M = \mathbb{C}P^n, \omega_0$ the Fubini-Study form, $J_0$ the standard complex structure on $\mathbb{C}P^n$, and $G$ is the automorphism group $\text{PSL}(n+1)$.

Remark 4.6 Notice that if $J$ is not integrable, then $T_f \mathcal{M}_i(A, \Sigma, J)$ is not necessarily closed under $\tilde{\sigma}$, so non-degeneracy is harder to witness.

Remark 4.7 If the compact Riemann surface $\Sigma = (\Sigma, j)$ is of genus 0, its group of automorphisms $\text{Aut}(\Sigma, j)$ is $\text{PSL}(2, \mathbb{C})$ and its action on $\mathcal{M}(A, \Sigma, \mathcal{J})$ is proper, assuming that $0 \neq A \in H_2(M, \mathbb{Z})$ (see, e.g., Lemma 3.1 in [8]). If $\Sigma$ is of genus 1, it is a torus $\mathbb{C}/(\mathbb{Z} + \alpha \mathbb{Z})$, where the imaginary part of $\alpha$ is nonzero; the group $\text{Aut}(\Sigma, j)$ contains the torus itself (as left translations). When the genus is $\geq 2$, the automorphism group is finite, by Hurwitz’s automorphism Theorem. So the action of $\text{Aut}(\Sigma, j)$ on $\mathcal{M}_i(A, \Sigma, J)$ is proper; therefore if the form $\omega_{S_i(\Sigma)}$ on $\mathcal{M}_i(A, \Sigma, J)$ is symplectic up to reparametrizations, it descends to a symplectic 2-form $\tilde{\omega}_{S_i(\Sigma)}$ on the quotient space $\tilde{\mathcal{M}}_i(A, \Sigma, J)$.

4.8 A class $A \in H_2(M, \mathbb{Z})$ is $J$-indecomposable if it does not split as a sum $A_1 + \ldots + A_k$ of classes all of which can be represented by non-constant $J$-holomorphic curves. The class $A$ is called indecomposable if it is $J$-indecomposable for all $\omega$-compatible $J$. Notice that if $A$ cannot be written as a sum $A = A_1 + A_2$ where $A_i \in H_2(M, \mathbb{Z})$ and $\int_{A_i} \omega > 0$, then it is indecomposable.
Assume that $M$ is compact. If $A$ is indecomposable, then Gromov’s compactness theorem \cite[1.5.B.]{gromov} implies that if $J_n$ converges in $J$, then, modulo parametrizations, every sequence $(f_n, J_n)$ in $\mathcal{M}(A, \Sigma, J)$ has a $(C^\infty -)$convergent subsequence. Therefore the map $p_A: \mathcal{M}(A, \Sigma, J)/\text{Aut}(\Sigma) \to J$ induced by $p_A$ is proper; in particular every quotient space $\tilde{M}^H(A, \Sigma, J)$ is compact.

Similarly, if $A$ is $J$-indecomposable for all $J$ in a set $S \subset J$, then the map $p_A^{-1}S/\text{Aut}(\Sigma) \to S$ induced by $p_A$ is proper; in particular, its image is closed in $S$.

We can now prove Corollary \ref{corollary:1.3}.

**Proof of Corollary \ref{corollary:1.3}** By part (f) of Lemma \ref{lemma:3.5}, a path in $\mathcal{J}$ between $J_0, J_1 \in S$ can be perturbed to a $p_A$-transversal path with the same endpoints. By part (e) of Lemma \ref{lemma:3.5} the $p_A$-preimage of the path, $\tilde{W}(A, \Sigma, \{J_\lambda\}_\lambda)$, is a smooth oriented manifold with boundary $\partial \tilde{W}(A, \Sigma, \{J_\lambda\}_\lambda) = \mathcal{M}(A, \Sigma, J_0) \cup \mathcal{M}(A, \Sigma, J_1)$, and the boundary orientation agrees with the orientation of $\mathcal{M}(A, \Sigma, J_1)$ and is opposite to the orientation of $\mathcal{M}(A, \Sigma, J_0)$. By Gromov’s compactness theorem the quotient $\tilde{W}(A, \Sigma, \{J_\lambda\}_\lambda)$, (i.e., the preimage of the path under the map $p_A: \mathcal{M}(A, \Sigma, J)/\text{Aut}(\Sigma) \to J$ induced by $p_A$), is compact (see \ref{4.8}).

Thus, using Theorem \ref{theorem:1.2} the integration of $\wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M)$ along $\tilde{M}(A, \Sigma, J_i)$ (for $i = 0, 1$) is well defined, and by Stokes’ Theorem (and the fact that $\tilde{\omega}^{C^\infty}(\Sigma, M)$ is closed), we get that

\begin{align*}
0 &= \int_{\tilde{W}(A, \Sigma, \{J_\lambda\}_\lambda)} d(\wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M)) = \int_{\partial \tilde{W}(A, \Sigma, \{J_\lambda\}_\lambda)} (\wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M)) \\
&= \int_{\tilde{M}(A, \Sigma, J_0)} \wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M) - \int_{\tilde{M}(A, \Sigma, J_1)} \wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M).
\end{align*}

If there is an integrable $J_i \in S$ such that $\mathcal{M}_4(A, \Sigma, J_i) \neq \emptyset$, then, by Proposition \ref{proposition:4.4}, $\int_{\tilde{M}(A, \Sigma, J_i)} \wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M) \neq 0$ hence, by the above, $\int_{\tilde{M}(A, \Sigma, J_i)} \wedge^i \tilde{\omega}^{C^\infty}(\Sigma, M) \neq 0$ for every $J \in S$.

\[\square\]

**Lemma 4.9.** Let $(M, \omega)$ be a symplectic manifold, $A \in H_2(M, \mathbb{Z})$, and $\Sigma = (\Sigma, j)$ a compact Riemann surface. Assume that $J_*$ is an integrable and $A$-regular $\omega$-compatible almost complex structure. Let \[\{J_\lambda\}_{0 \leq \lambda \leq 1},\] with $J_0 = J_*$, be a path in $J = J(M, \omega)$ that is $A$-regular, i.e., the path is transversal to $p_A$ and $J_\lambda \in J_{\text{reg}}(A)$ for all $0 \leq \lambda \leq 1$. Assume that the map \[p_A^{-1}(\{J_\lambda\}_{0 \leq \lambda \leq 1})/\text{Aut}(\Sigma, j) \to \{J_\lambda\}_{0 \leq \lambda \leq 1}\] (4.10)

that is induced from $p_A$ is proper.

Then there is an open neighbourhood $I$ of $J_*$ in $\{J_\lambda\}_{0 \leq \lambda \leq 1}$, such that for every $J \in I$, the restriction of the form $\omega_{S}(\Sigma)$ to the moduli space $\mathcal{M}_4(A, \Sigma, J)$ is symplectic up to reparametrizations.

**Proof.** By part (e) of Lemma \ref{lemma:3.5} and Lemma \ref{lemma:4.2}, the universal moduli space over the path, \[p_A^{-1}(\{J_\lambda\}) = W(A, \Sigma, \{J_\lambda\}_\lambda)\]
is a finite dimensional manifold; for each $0 \leq \lambda \leq 1$, since $J_\lambda \in J_{\text{reg}}(A)$, the moduli space $\mathcal{M}_4(A, \Sigma, J_\lambda)$ is a finite dimensional manifold. Moreover, since (4.10) is proper, we get that the quotient $\tilde{W}(A, \Sigma, \{J_\lambda\}_\lambda)$
is compact, and each of the quotient spaces $\tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ is compact. By Proposition 4.4, the restriction of the form $\tilde{\omega}_{\Sigma}(\cdot)$ to $\tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ is non-degenerate.

Since non-degeneracy is an open condition, for every $f \in \tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ there is an open neighbourhood $$f \in V_f \subseteq \tilde{\mathcal{V}}(A, \Sigma, \{J_\lambda\}_\lambda),$$ such that for all $(h, J) \in V_f$, for every $\tau_h \in T_h \tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$, the map $\tilde{\omega}_{\Sigma}(\tau_h, \cdot)$ on $T_h \tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ is not equal to zero. ($V_f$ should be small enough such that $h$ and $T_h \tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ are close to $f$ and to $T_f \mathcal{M}_{\tau}(A, \Sigma, J)$, respectively.)

Let $$U_f = V_f \cap \tilde{\mathcal{M}}_{\tau}(A, \Sigma, J).$$

Since $\tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ is compact, it is covered by finitely many $U_f$-s. Let $V$ be the intersection of the corresponding $V_f$-s, then $V$ is an open neighbourhood of $\tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ in $\tilde{\mathcal{V}}(A, \Sigma, \{J_\lambda\}_\lambda)$. By properness of the map (4.10), there is an open neighborhood $J_\lambda \in I \subset \{J_\lambda\}_{0 \leq \lambda \leq 1}$, such that the preimage of $I$ under (4.10) is contained in $V$. This completes the proof.

\[\square\]

Compatible almost complex structures on moduli spaces of simple immersed $J$-holomorphic curves

We apply the following well known fact, c.f. [2, Prop. 12.6]

**Proposition 4.10.** Let $(N, \omega)$ be a symplectic manifold, and $g$ a Riemannian metric on $N$. Then there exists a canonical almost complex structure $J$ on $N$ which is $\omega$-compatible. The structure $J$ equals $\sqrt{AA^*}A$ where $A: \mathbb{T}N \to \mathbb{T}N$ is such that $\omega(u, v) = g(Au, v)$.

**Corollary 4.11.** Let $J \in \mathcal{J}(M, \omega)_{\text{reg}}$ such that the form $\tilde{\omega}_{\Sigma}(\cdot)$ on $\tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ is symplectic. Let $g$ be the metric induced on the quotient space by the $L^2$-norm

$$\sqrt{\tilde{\omega}_{\Sigma}(\cdot, J\cdot)}.$$ 

Then, by Proposition 4.10 we obtain a canonical almost complex structure on $\tilde{\mathcal{M}}_{\tau}(A, \Sigma, J)$ that is compatible with the form.

**5 Examples**

In this section $\Sigma$ is $\mathbb{CP}^1 (\equiv S^2)$ with the standard complex structure.

**Example 5.1** In case $(M, \omega) = (\mathbb{CP}^n, \omega_{FS})$ and $A = L$, the homology class of $\mathbb{CP}^1$ that generates $H_2(M, \mathbb{Z})$, the class $A$ is indecomposable. By [10, Proposition 7.4.3], the standard complex structure on $\mathbb{CP}^n$ is regular (see Remark 4.5). Thus, by Corollary 4.3 for every $A$-regular almost complex structure $J \in \mathcal{J}$ there is a $J$-holomorphic sphere in $A$.

In general, if $M$ is compact, $H_2(M, \mathbb{Z})$ is of dimension one, and $A$ is the generator of $H_2(M, \mathbb{Z})$ of minimal (symplectic) area, then $A$ is indecomposable. If $(M, \omega_0, J_0)$ is a compact Kähler manifold whose automorphism group acts transitively, then $J_0$ is regular for every $C \in H_2(M, \mathbb{Z})$ [10, Proposition 7.4.3].
Thus if $A$ is represented by a $J_0$-holomorphic sphere, then, by Corollary 1.3 for every $p_A$-regular almost complex structure $J \in \mathcal{J}$ there is a $J$-holomorphic sphere in $A$. As an example of such a manifold, consider the Grassmannian $M = G/P$ of oriented 2-planes in $\mathbb{R}^n$: $G = \text{SO}(n)$, $P = S_1 \times \text{SO}(n - 2)$. We thank Yael Karshon for suggesting the Grassmannian example.

5.2 Let $(M, \omega)$ be a symplectic four-manifold. By the adjunction inequality, in a four-dimensional manifold, if $A \in H_2(M, \mathbb{Z})$ is represented by a simple $J$-holomorphic sphere $f$, then

$$A \cdot A - c_1(A) + 2 \geq 0,$$

with equality if and only if $f$ is an embedding; see [10 Cor. E.1.7]. Thus, if there is $J' \in \mathcal{J}$ such that $A = [u]$ for an embedded $J'$-holomorphic sphere $u: \mathbb{C}P^1 \to M$, then for every $(f, J) \in \mathcal{M}(A, \mathbb{C}P^1, \mathcal{J})$, the sphere $f$ is an embedding; in particular,

$$\mathcal{M}(A, \mathbb{C}P^1, \mathcal{J}) = \mathcal{M}_1(A, \mathbb{C}P^1, \mathcal{J}).$$

The existence of such $J' \in \mathcal{J}(M, \omega)$ is guaranteed when $A$ is represented by an embedded symplectic sphere (see [3 Section 2.6]).

The Hofer-Lizan-Sikorav regularity criterion asserts that in a four-dimensional manifold, if $f$ is an immersed $J$-holomorphic sphere, then $(f, J)$ is a regular point for the projection $p_{[f]}$ if and only if $c_1([f]) \geq 1$ and $f$ is an embedding; see [5, Section 2.6].

Therefore, if $A \in H_2(M, \mathbb{Z})$ is such that $c_1(A) \geq 1$, and $A$ is represented by an embedded $J'$-holomorphic sphere for some almost complex structure $J'$ on a four-manifold $M$, then every $(f, J) \in \mathcal{M}(A, \mathbb{C}P^1, \mathcal{J})(= \mathcal{M}_1(A, \mathbb{C}P^1, \mathcal{J}))$ is a regular point for $p_A$, thus, by part (d) of Lemma 5.5 the image of $p_A$ is open in $\mathcal{J}$.

As a result of 4.8 and 5.2 we get the following lemma.

**Lemma 5.3.** Let $(M, \omega)$ be a compact symplectic four-manifold, and $A \in H_2(M, \mathbb{Z})$ with $c_1(A) \geq 1$. Assume that $S \subset \mathcal{J}(M, \omega)$ is such that:

- $A$ is $J$-indecomposable for all $J \in S$;
- $A$ is represented by an embedded $J'$-holomorphic sphere for some $J \in S$;
- $S$ is connected.

Then the map $p_A: p_A^{-1}(S) \to S$ is onto, and its image is open and closed, thus $S = p_A(p_A^{-1}(S)) \subseteq \mathcal{J}_{\text{reg}}(A)$.

In particular if $S$ satisfies the assumptions of Lemma 5.3 and $J_s \in S$ is integrable, then Lemma 4.9 applies to every $p_A$-transversal path $\{J_s\}_{0 \leq s \leq 1}$ in $S$, with $J_0 = J_s$. If $S \subset \mathcal{J}_{\text{reg}}(A)$ is path-connected and open then by Lemma 5.5 there is a $p_A$-transversal path in $S$ between every two elements of $S$. We give here three examples of $(M, \omega, A)$ and $S$, in all of which $S$ satisfies the assumptions of Lemma 5.3 and $S$ is path-connected and open; in addition, there is an integrable $J_s \in S$ such that $\mathcal{M}_1(A, \Sigma, J_s) \neq \emptyset$. Consequently, Corollary 1.3 implies the existence of a $J$-holomorphic sphere in $A$ for every $J$ in the outlined sets $S$; in all of the examples $S$ is dense in $\mathcal{J}$ thus we get the existence of a $J$-holomorphic sphere in $A$ for a generic $J$.

**Example 5.4** In case $(M, \omega) = (\mathbb{C}P^2, \omega_{FS})$ and $A = L$, the homology class of $\mathbb{C}P^1$ that generates $H_2(M, \mathbb{Z})$, the assumptions of Lemma 5.3 are satisfied for the set $S = \mathcal{J}(M, \omega)$.
The same holds in case $(M, \omega) = (S^2 \times S^2, \tau \oplus \tau)$, where $\tau$ is the rotation invariant area form on $S^2$ (with total area equal to 1), and $A = [S^2 \times \{pt\}]$.

In each of these cases, there is a standard integrable compatible complex structures in $S$, and $A$ is represented by a sphere that is holomorphic for the standard structure.

**Example 5.5** Consider $(S^2 \times S^2, (1 + \lambda)\tau \oplus \tau)$. (When $\lambda > 0$, this symplectic manifold has a compatible almost complex structure $J$ for which there is a non-regular sphere, namely the antidiagonal $D = \{(s, -s) \in S^2 \times S^2\}$; see Example 3.3.6.) By Abreu [1, Sec. 1.2], for $0 < \lambda \leq 1$, the subset ${\mathcal J}_0^\lambda$ of $(1 + \lambda)\tau \oplus \tau$-compatible almost complex structures for which the class $[D]$ is represented by a (unique embedded) $J$-holomorphic sphere is a non-empty, closed, codimension 2 submanifold of $\mathcal J = \mathcal J(S^2 \times S^2, (1 + \lambda)\tau \oplus \tau)$. Abreu also shows that for all $J$ in the complement of ${\mathcal J}_0^\lambda$, the class $A = [S^2 \times \{pt\}]$ is $J$-indecomposable. Thus, the assumptions of Lemma 5.3 are satisfied for $S = \mathcal J \setminus {\mathcal J}_0^\lambda$, and $S$ is also open dense and path-connected in $\mathcal J$. Notice that the standard split compatible complex structure $j \oplus j$ is in $S$, and that the class $A = [S^2 \times \{pt\}]$ is represented by (a 2-parameter family of) embedded spheres $S^2 \times \{s\}$ that are holomorphic for $j \oplus j$. Therefore, by Corollary 1.3 there is a $J$-holomorphic sphere in $A$ for every $J$ in the complement of ${\mathcal J}_0^\lambda$.

Abreu [1, Theorem 1.8] shows that $\mathcal J \setminus {\mathcal J}_0^\lambda$ equals the space $\mathcal J_\lambda^0$ of $J$ for which the homology class $[S^2 \times \{pt\}]$ is represented by an embedded $J$-holomorphic sphere.

**Example 5.6** Let $(M, \omega)$ be a compact symplectic four-manifold and $A \in H_2(M, \mathbb{Z})$ a homology class that can be represented by an embedded symplectic sphere and such that $c_1(A) = 1$. Consider the set of $J \in \mathcal J(M, \omega)$ for which there is a non-constant $J$-holomorphic sphere in a homology class $H \in H_2(M, \mathbb{Z})$ with $c_1(H) < 1$ and $\omega(H) < \omega(A)$, and denote its complement by $U_A$. Then, by definition, $A$ is $J$-indecomposable for every $J$ in $S = U_A$. The set $S = U_A$ is an open dense and path-connected subset of $\mathcal J(M, \omega)$, see, e.g., [7, App. A].

In case $(M, \omega)$ is obtained by a sequence of blow ups from $(\mathbb{C}P^2, \omega_{FS})$ or from $(S^2 \times S^2, \tau \oplus \tau)$ and $A$ is the homology class of one of the blow ups, there is an integrable structure $J_*$ in $S = U_A$, and $A$ is represented by a $J_*$-holomorphic sphere. Therefore, by Corollary 1.3 there is a $J$-holomorphic sphere in $A$ for every $J$ in $U_A$. In [7, App. A], it is shown that for every $J \in U_A$, the class $A$ is represented by an embedded $J$-holomorphic sphere.

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**References**

[1] M. Abreu: Topology of symplectomorphism groups of $S^2 \times S^2$, Invent. Math. 131 (1998), 1-23.

[2] A. Cannas da Silva: Lectures in Symplectic Geometry, Springer-Verlag, Berlin (2000).

[3] E. Calabi: Constructions and properties of some 6-dimensional almost complex manifolds, Transactions of the American Mathematical Society 87, 407–438.

[4] M. Gromov: Pseudo holomorphic curves in symplectic manifolds, Invent. Math. 82 (1985), 307–347.

[5] H. Hofer, V. Lizan, and J-C. Sikorav: On genericity for holomorphic curves in four-dimensional almost-complex manifolds, J. Geom. Anal. 7 (1997), no. 1, 149–159.
[6] Y. Karshon, L. Kessler and M. Pinsonnault: A compact symplectic four-manifold admits only finitely many inequivalent toric actions, arXiv:math/0609043 (extended version of [7]).

[7] Y. Karshon, L. Kessler and M. Pinsonnault: A compact symplectic four-manifold admits only finitely many inequivalent toric actions, J. Symplectic Geom. 5 (2007), no. 2, 139-166.

[8] L. Kessler: Torus actions on small blowups of $\mathbb{CP}^2$, Pacific J. of Math. 244 (2010), no. 1, 133–154.

[9] D. McDuff and D. Salamon: Introduction to Symplectic Topology, Sec. Ed., Oxf. Univ. Press, 1998.

[10] D. McDuff and D. Salamon: J-Holomorphic Curves and Symplectic Topology, Amer. Math. Soc. 2004.

[11] A. Newlander and L. Nirenberg: Complex analytic coordinates in almost complex manifolds, Ann. Math. 65, 1967, 391-404.

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