Higher Derivatives
and
Canonical Formalisms

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Abstract

Path integral expressions for three canonical formalisms – Ostrogradski’s one, constrained one and generalized one – of higher-derivative theories are given. For each formalism we consider both nonsingular and singular cases. It is shown that three formalisms share the same path integral expressions. In particular it is pointed out that the generalized canonical formalism is connected with the constrained one by a canonical transformation.
1 Introduction

Higher-derivative theories appear in various scenes of physics,$^{1,2}$ Higher-derivate terms occur as quantum corrections; nonlocal theories, e.g. string theories, are essentially higher-derivative theories; Einstein gravity supplemented by curvature squared terms has attracted attention because of its renormalizability.$^3$

A canonical formalism for higher-derivative theories was first developed by Ostrogradski about one and a half centuries ago.$^4$ He treated only nonsingular cases, where the Hessian matrices of Lagrangians with respect to highest derivatives are nonsingular. For singular cases, Dirac's algorithm$^5$ for constrained Hamiltonian systems was shown to be applicable.$^6,7$ Though being self-consistent, these formulations for nonsingular and singular cases look different from the conventional canonical formalism: highest derivatives are discriminated from lower ones, only the highest ones enjoying Legendre transformations. If we regard the original higher-derivative systems as equivalent first-derivative systems with constraints and apply the Dirac’s algorithm to the latter ones, we could give the foundation of the ordinary canonical formalism to the Ostrogradskii’s canonical one. This program, constrained canonical formulation of higher-derivative theories, has been carried out in Refs. $6$ and $8$ for both nonsingular and singular cases. A generalization of the constrained canonical formalism has been discussed in Ref. $9$.

In all these approaches the sets of canonical equations provided by the respective formalisms have mainly been considered, and their equivalence to the set of Euler-Lagrange equations has been shown. To go to quantum theory, however, the equivalence of the sets of equations of motion is not enough. We have to confirm the equivalence of off-shell information. That is, comparing path integral expressions of the respective formalisms is essentially important. This is the subject of the present paper. We give path integral expressions for each formalism and show they are equivalent to one another. In particular it is pointed out that the generalized canonical formalism is connected with the constrained canonical one by a canonical transformation.

In §2, path integral expressions of the Ostrogradski’s canonical formalism are given for both singular and nonsingular cases. In §3, path integral expressions of the constrained canonical formalism are given and it is shown that the constrained one is equivalent to the Ostrogradski’s one. In §4, path integral expressions of the generalized canonical formalism are given. A further generalization of the formalism described in Ref. $9$ is developed. It is shown by doing a canonical transformation that the generalized one is equivalent to the Ostrogradski’s. Section 5 gives summary and discussion.
2 Ostrogradski’s canonical formalism

We consider a system described by a generic Lagrangian which contains up to \( n \)-th derivative of \( x_a(t) \) (\( a = 1, \cdots, N \))

\[
L = L(x_a, \dot{x}_a, \ddot{x}_a, \cdots, x_a^{(n_a)}),
\]

where

\[
x_a^{(r_a)} \equiv \frac{d^{r_a} x_a}{dt^{r_a}}, \quad (r_a = 1, \cdots, n_a)
\]

The canonical formalism of Ostrogradski regards \( x_a^{(s_a)}(s_a = 1, \cdots, n_a - 1) \) as independent coordinates \( q_a^{s_a+1} \):

\[
x_a^{(s_a)} \rightarrow q_a^{s_a+1},
\]

\[
L(x_a, \dot{x}_a, \cdots, x_a^{(n_a)}) \rightarrow L(q_1^1, \cdots, q_n^{n_a}, \dot{q}_a^{n_a}).
\]

The momenta conjugate to \( q_a^{n_a} \) is defined as usual by

\[
p_a^{n_a} \equiv \frac{\partial L}{\partial \dot{q}_a^{n_a}}.
\]

The Hessian matrix of \( L_q \) is defined by

\[
A_{ab} \equiv \frac{\partial^2 L}{\partial q_a^{n_a} \partial q_b^{n_b}}.
\]

We say that the system is nonsingular if \( \det A_{ab} \neq 0 \), while singular if \( \det A_{ab} = 0 \).

**Nonsingular case** (\( \det A_{ab} \neq 0 \))

In this case, the relation \(1\) can be inverted to give \( \dot{q}_a^{n_a} \) as functions of \( q^r (r = 1, \cdots, n) \) and \( p^n \):

\[
\dot{q}_a^{n_a} = \dot{q}_a^{n_a} (q^r, p^n).
\]

The Hamiltonian is defined by

\[
H_0 \equiv p_a^{n_a} q_a^{n_a+1} + p_a^{n_a} \dot{q}_a^{n_a} (q^r, p^n) - L_q (q^r, \dot{q}^r (q^r, p^n)).
\]

It is seen that this construction of the Hamiltonian has several peculiarities from the view point of the ordinary Legendre transformation:

1. What appears in Eq.\(1\) is just a function \( L_q(q^1, \cdots, q^n, \dot{q}^n) \) whose Euler derivatives do not produce any meaningful equations of motion.
2. The momenta \( p^s \) (\( s = 1, \cdots, n - 1 \)) are multiplied by \( q^{s+1} \) not by \( \dot{q}^s \).
3. The momenta \( p^n \) (\( s = 1, \cdots, n - 1 \)) are not defined from the Lagrangian through relations like \( \frac{\partial L}{\partial \dot{q}^n} \), but just introduced as independent canonical variables; only \( p^n \)’s enjoy special treatment, defined by Eq.\(1\) as usual.
where

obtaining

\( \prod_a \) factor

Singular case

\( Z \)

can be well-defined. Integrations with respect to

At this stage we do not enter into the problem whether or not this expression can be well-defined. Integrations with respect to \( p_a \) offer a factor \( \prod_{a=1}^{n_a-1} \delta(q_a^{s_a} - q_a^{s_a+1}) \). We can further integrate with respect to \( q_a^{s_a+1} \), obtaining

\[
Z_O = \int \mathcal{D}q_a^{1s_a} \mathcal{D}p_a^{ns_a} \exp i \int dt [p_a^{ns_a} \dot{q}_a^{s_a} - H_O(p^r, q^r)].
\]  

(9)

At this stage we do not enter into the problem whether or not this expression can be well-defined. Integrations with respect to \( p_a^{ns_a} \) offer a factor \( \prod_{a=1}^{n_a-1} \delta(q_a^{s_a} - q_a^{s_a+1}) \). We can further integrate with respect to \( q_a^{s_a+1} \), obtaining

\[
Z_O = \int \mathcal{D}q_a^{1s_a} \mathcal{D}p_a^{ns_a} \exp i \int dt [p_a^{ns_a} q_a^{1(n_a)} - \tilde{H}_O(q^1, q^{1(s)}, p^n)],
\]  

(10)

where

\[
\tilde{H}_O(q^1, q^{1(s)}, p^n) \equiv p_a^{ns_a} q_a^{1ns_a}(q^1, q^{1(s)}, p^n) - L_q(q^1, q^{1(s)}, q^n(q^1, q^{1(s)}, p^n)).
\]  

(11)

\[
q_a^{1(s_a)} \equiv \frac{d^{ns_a} q_a^{1}}{dt^{ns_a}}.
\]  

(12)

**Singular case** (\( \det A_{ab} = 0, \ \text{rank} A_{ab} = N - \rho \))

In this case, the relation (5) can not be inverted. We have \( \rho \) primary constraints:

\[
\phi_A(q^r, p^n) \approx 0, \quad (A = 1, \cdots, \rho)
\]  

(13)

such that

\[
\det \{ \phi_A, \phi_B \}_V \neq 0.
\]  

(14)

By using Lagrange multipliers \( \lambda_A \), we define the Hamiltonian as usual:

\[
\tilde{H}_S(q^r, p^r) = H_S(q^r, p^r) + \lambda_A \phi_A(q^r, p^n),
\]  

(15)

where

\[
H_S(q^r, p^r) \equiv p_a^{ns_a} q_a^{ns_a} + p_a^{ns_a} \dot{q}_a^{ns_a} - L_q(q^r, \dot{q}^r).
\]  

(16)

Since \( \det \{ \phi_A, \phi_B \}_V \neq 0 \), the primary constraints (13) are second-class ones. The consistency of the primary constraints (13) under their time developments determines all the Lagrange multipliers \( \lambda_A \). The path integral is

\[
Z_{Os} = \int \mathcal{D}q_a^{1s_a} \mathcal{D}p_a^{ns_a} \exp \frac{i}{\hbar} \left\{ \phi_A, \phi_B \right\}_V \delta (\phi_A(q^r, p^n)) \exp i \int dt [p_a^{ns_a} \dot{q}_a^{s_a} - H_S].
\]  

(17)

Integrations with respect to \( p_a^{ns_a} \) and \( q_a^{s_a+1} \) give

\[
Z_{Os} = \int \mathcal{D}q_a^{1s_a} \mathcal{D}p_a^{ns_a} \exp \frac{i}{\hbar} \left\{ \phi_A, \phi_B \right\}_V \delta (\phi_A(q^r, p^n)) \exp i \int dt [p_a^{ns_a} q_a^{1(n_a)} - \tilde{H}_S(q^1, q^{1(s)}, p^n)],
\]  

(18)

where

\[
\tilde{H}_S(q^1, q^{1(s)}, p^n) \equiv p_a^{ns_a} \dot{q}_a^{ns_a} - L_q(q^1, q^{1(s)}, \dot{q}^n).
\]  

(19)
3 Constrained canonical formalism

It has been seen that the Ostrogradski’s formalism gives special treatment to the highest derivatives \( q_a^{n_a} \). To treat all the derivatives equally, we introduce Lagrangian multipliers \( \mu_a^{n_a} \) and start with the following Lagrangian:

\[
L_D(q^r, \dot{q}^r, \mu^s) \overset{\text{def}}{=} L_q(q^r, \dot{q}^n) + \mu_a^{n_a}(q_a^{n_a} - q_a^{n_a + 1}).
\] (20)

The conjugate momenta

\[
\pi_a^{n_a} \overset{\text{def}}{=} \frac{\partial L_D}{\partial \dot{\mu}_a^{n_a}} = 0,
\] (21)

\[
p_a^{n_a} \overset{\text{def}}{=} \frac{\partial L_D}{\partial \dot{q}_a^{n_a}} = \mu_a^{n_a},
\] (22)

\[
p_a^{n_a} \overset{\text{def}}{=} \frac{\partial L_D}{\partial q_a^{n_a}} - \frac{\partial L_q}{\partial q_a^{n_a}}
\] (23)

provide the following primary constraints:

\[
\pi_a^{n_a} \approx 0,
\] (24)

\[
\psi_a^{n_a} \overset{\text{def}}{=} p_a^{n_a} - \mu_a^{n_a} \approx 0. \quad (25)
\]

**Nonsingular case** \((\det A_{ab} \neq 0)\)

In this case, the relation (23) can be inverted to give \( \dot{q}_a^{n_a} \) as functions of \( q^r \) and \( p^n \):

\[
\dot{q}_a^{n_a} = q_a^{n_a}(q^r, p^n).
\] (26)

By introducing Lagrange multipliers \( \lambda_a^{(1)s_a} \) and \( \lambda_a^{(2)s_a} \), the Hamiltonian is defined by

\[
\bar{H}_D(q^r, p^r) = \pi_a^{n_a} \dot{\mu}_a^{n_a} + p_a^{n_a} \dot{q}_a^{n_a} - L_D + \lambda_a^{(1)s_a} \pi_a^{n_a} + \lambda_a^{(2)s_a} \psi_a^{n_a}.
\] (27)

This can be rewritten as

\[
\bar{H}_D(q^r, p^r) = H_D(q^r, p^r) + \lambda_a^{(1)s_a} \pi_a^{n_a} + \lambda_a^{(2)s_a} \psi_a^{n_a},
\] (28)

where

\[
H_D(q^r, p^r) \overset{\text{def}}{=} p_a^{n_a} q_a^{n_a+1} + p_a^{n_a} \dot{q}_a^{n_a} - L_q(q^r, \dot{q}^n),
\] (29)

\[
\lambda_a^{(1)s_a} \overset{\text{def}}{=} \bar{\lambda}_a^{(1)s_a} + \dot{\mu}_a^{n_a},
\] (30)

\[
\lambda_a^{(2)s_a} \overset{\text{def}}{=} \bar{\lambda}_a^{(2)s_a} + \dot{q}_a^{n_a} - q_a^{n_a+1}.
\] (31)

The Poisson brackets between the primary constraints (24) and (25) are

\[
\{ \pi_a^{n_a}, \psi_b^{n_b} \}_P = \delta_{ab} \delta_{n_a n_b},
\] (32)

otherwise = 0.
Thus, these primary constraints are of the second class. The path integral is

$$Z_D = \int D\pi_a^s \, D\mu_a^s \, D\pi_a^s \, \delta(\pi^s) \delta(\psi^s) \exp i \int dt [p_a^s \dot{q}_a^s + \pi_a^s \dot{\mu}_a^s - H_D]. \quad (33)$$

Integrations with respect to $\pi_a^s$ and $\mu_a^s$ give

$$Z_D = \int Dq_a^s \, Dp_a^s \exp i \int dt [p_a^s (\dot{q}_a^s - q_a^{s+1}) + p_a^s (\dot{\mu}_a^s - \dot{q}_a^s (q^r, p^n)) - L_q]. \quad (34)$$

We can further integrate with respect to $p_a^s$ and $q_a^{s+1}$, obtaining

$$Z_D = \int Dq_a^1 \, Dp_a^s \exp i \int dt [\rho_a^s q_a^{1(n_a)} - \hat{H}_D(q^1, q^{1(s)}, p^n)], \quad (35)$$

where

$$\hat{H}_D(q^1, q^{1(s)}, p^n) = p_a^s q_a^{1(n_a)} (q^1, q^{1(s)}, p^n) - L_q (q^1, q^{1(s)}, q^n (q^1, q^{1(s)}, p^n)). \quad (36)$$

This shows that the path integral $Z_D$ is the same as $Z_O$ given by Eq.(11).

**Singular case** (det $A_{ab} = 0$, rank $A_{ab} = N - \rho$)

In this case, the relation (24) provides $\rho$ additional constraints besides (23) and (25):

$$\phi_A(q^r, p^n) \approx 0 \quad (A = 1, \cdots, \rho) \quad \text{(37)}$$

such that

$$\text{det}\{\phi_A, \phi_B\}_P \neq 0. \quad (38)$$

By using Lagrange multipliers $\lambda_A, \lambda_{a}^{(1)s_a}$ and $\lambda_{a}^{(2)s_a}$, the Hamiltonian is defined by

$$\hat{H}_{D_N}(q^r, p^r) = H_D(q^r, p^r) + \lambda_{a}^{(1)s_a} \pi_a^s + \lambda_{a}^{(2)s_a} \psi_a^s - \lambda_A \phi_A, \quad (39)$$

where

$$H_{D_N}(q^r, p^r) \overset{\text{def}}{=} p_a^s \dot{q}_a^s + \dot{p}_a^s \dot{q}_a^s - L_q(q^r, q^n). \quad (40)$$

The Poisson brackets between the primary constraints are

$$\{\pi_a^s, \psi_b^s\}_P = \delta_{ab} \delta_{s_a s_b}, \quad (41)$$

$$\{\psi_a^s, \phi_B\}_P = - \frac{\partial \phi_B}{\partial \dot{q}_a^s}, \quad (42)$$

$$\{\phi_A, \phi_B\}_P \overset{\text{def}}{=} c_{AB}, \quad (43)$$

otherwise $= 0. \quad (43)$

All the constraints $\Phi_\alpha \overset{\text{def}}{=} (\pi_a^s, \psi_a^s, \phi_A)$ form a set of second-class constraints because the determinant of the matrix $\{\Phi_\alpha, \Phi_\beta\}_P$ is non-zero:

$$\text{det}\{\Phi_\alpha, \Phi_\beta\}_P = \text{det} c_{AB} \neq 0. \quad (44)$$
The consistency of these constraints under their time developments fixes all the Lagrange multipliers. The path integral is

\[ Z_{Ds} = \int Dq_a^s Dp_a^s D\mu_a^s D\pi_a^s \det \frac{1}{2} c_{AB} \delta(\pi_a^s) \delta(\psi_a^s) \delta(\phi_A) \exp \left( \int dt \left[ p_a^s \dot{q}_a^s + \pi_a^s \dot{\mu}_a^s - H_{Ds} \right] \right). \]

(45)

Integrations with respect to \( \mu_a^s, \pi_a^s, p_a^s \) and \( q_a^{s+1} \) give

\[ Z_{Ds} = \int Dq_1^s Dp_1^s \det \frac{1}{2} c_{AB} \delta(\phi_A) \exp \left( \int dt \left[ p_1^s q_1^s - \hat{H}_{Ds}(q_1^s, q_1^{1(s)}, p^n) \right] \right), \]

where

\[ \hat{H}_{Ds}(q_1^s, q_1^{1(s)}, p^n) \equiv p_a^n q_a^n - L_1(q_1^s, q_1^{1(s)}, q^n). \]

(46)

(47)

This shows that the path integral \( Z_{Ds} \) is the same as \( Z_{Os} \) given by (18).

4 Generalized canonical formalism

In this section we consider a further generalization of the formalism described in Ref. 9.

We regard \( x_a^{(s_a)} \) and \( x_a^{(n_a)} \) as independent coordinates \( q_a^{s+1} \) and \( v_a \) respectively:

\[ x_a^{(s_a)} \rightarrow q_a^{s+1}, \]

\[ x_a^{(n_a)} \rightarrow v_a, \]

\[ L(x, \dot{x}, \ddot{x}, \cdots, x^{(n)}) \rightarrow L_1(q_1^s, \cdots, q^n, v). \]

(48)

(49)

(50)

The other generalized coordinates \( Q_a^r \) are introduced as arbitrary functions of \( q^r \)

\[ Q_a^r = Q_a^r(q^r) \]

(51)

such that

\[ \det \frac{\partial Q_a^r}{\partial q_a^s} \neq 0. \]

(52)

Eq. (51) can be inverted to give \( q^r \) as functions of \( Q^r \):

\[ q_a^r = q_a^r(Q^r). \]

(53)

Differentiating Eq. (51) and (53) with respect to time gives

\[ \dot{q}_a^r = \dot{Q}_b^s \frac{\partial q_a^r(Q^r)}{\partial Q_b^s}, \]

\[ \dot{Q}_a^r = \dot{q}_b^s \frac{\partial Q_a^r(q^r)}{\partial q_b^s}. \]

(54)

(55)
We introduce new variables defined by
\[ V_a \overset{\text{def}}{=} q_b^{s_b+1} \frac{\partial Q_a^{n_a}}{\partial q_b^{s_b}} + v_b \frac{\partial Q_a^{n_a}}{\partial q_b^{n_a}}, \tag{56} \]
where we assume that \( Q_a^{n_a} \)'s satisfy
\[ \text{det} \frac{\partial Q_a^{n_a}}{\partial q_b^{n_a}} \neq 0. \tag{57} \]

Eq. (56) can be inverted with respect to \( v \) as
\[ v_a = \left( \frac{\partial Q_a^{n_a}}{\partial q_a^{n_a}} \right)^{-1} \left( V_b - q_b^{s_a+1} \frac{\partial Q_a^{n_a}}{\partial q_c^{n_a}} \right). \tag{58} \]

Functions \( \bar{Q}_a^{s_a} \) are defined by
\[ \bar{Q}_a^{s_a} \overset{\text{def}}{=} \left( q_b^{s_b+1} \frac{\partial Q_a^{s_a}}{\partial q_b^{s_b}} + v_b \frac{\partial Q_a^{s_a}}{\partial q_c^{s_a}} \right) \big|_{q = q(Q), v = v(Q, V)}. \tag{59} \]

We introduce Lagrange multipliers \( M_a^{r_a} \) and start from the following generalized Lagrangian:
\[ L_G(Q^r, \dot{Q}^r, V, M^r) \overset{\text{def}}{=} L_Q(Q_r, V) + M_a^{s_a}(\dot{Q}_a^{s_a} - \bar{Q}_a^{s_a}) + M_a^{n_a}(\dot{Q}_a^{n_a} - V_a), \tag{60} \]
where
\[ L_Q(Q_r, V) \overset{\text{def}}{=} L_q(q_r, v)\big|_{q = q(Q), v = v(Q, V)}. \tag{61} \]

Here it is interesting to consider a special case of the generalized Lagrangian. Choose
\[ Q^r = q^r, \quad V = v. \tag{62} \]

Then the Lagrangian (60) reduces to
\[ L_g(q^r, \dot{q}^r, v, \mu^r) = L_q(q^r, v) + \mu_a^{s_a}(\dot{q}_a^{s_a} - q_a^{s_a+1}) + \mu_a^{n_a}(\dot{q}_a^{n_a} - v_a). \tag{63} \]

This Lagrangian is similar to the Lagrangian (20), except for term containing the variables \( v \). The equivalence between the two Lagrangians is proved later.

For the Lagrangian (60) the conjugate momenta
\[ \Pi_a^{s_a} \overset{\text{def}}{=} \frac{\partial L_G}{\partial M_a^{s_a}} = 0, \tag{64} \]
\[ P_a^{r_a} \overset{\text{def}}{=} \frac{\partial L_G}{\partial Q_a^{r_a}} = M_a^{r_a}, \tag{65} \]
\[ \Theta_a \overset{\text{def}}{=} \frac{\partial L_G}{\partial V_a} = 0. \tag{66} \]
provide the following primary constraints:

\[ \Pi_a^{ra} \approx 0, \quad (67) \]
\[ \Psi_a^{ra} \equiv P_a^{ra} - M_a^{ra} \approx 0, \quad (68) \]
\[ \Theta_a \approx 0. \quad (69) \]

The consistency of the primary constraints under their time developments produces a secondary constraint:

\[ \Gamma_a \equiv -P_b^{sa} \frac{\partial \bar{Q}_b^{sa}}{\partial V_a} - P_a^{na} V_a + \frac{\partial L_Q}{\partial Q_a}. \quad (70) \]

By introducing Lagrange multipliers \( \Lambda_1^{ra}, \Lambda_2^{ra}, \Lambda_3^{ra}, \) and \( \Lambda_4^{ra} \), the Hamiltonian is given by

\[ H_G = H_G(Q^r, P^r, V) + \Lambda_1^{ra} \Pi_a^{ra} + \Lambda_2^{ra} \Psi_a^{ra} + \Lambda_3^{ra} \Theta_a + \Lambda_4^{ra} \Gamma_a, \quad (71) \]

where

\[ H_G(Q^r, P^r, V) \equiv P_a^{sa} \bar{Q}_a^{sa} + P_a^{na} V_a - L_Q(Q^r, V). \quad (72) \]

The Poisson brackets between the constraints are

\[ \{\Pi_a^{ra}, \Psi_b^{rb}\}_P = \delta_{ab} \delta_{ra, rb}, \quad (73) \]
\[ \{\Psi_a^{ra}, \Gamma_b\}_P = P_c^{sc} \frac{\partial^2 \bar{Q}_c^{sa}}{\partial Q_a^{sa} \partial V_b} - \frac{\partial^2 L_Q}{\partial Q_a^{sa} \partial V_b}, \quad (74) \]
\[ \{\Theta_a, \Gamma_b\}_P = -\frac{\partial^2 L_Q}{\partial V_a \partial V_b}, \quad (75) \]
\[ \{\Gamma_a, \Gamma_b\}_P \equiv C_{ab}, \quad (76) \]
\[ \text{otherwise} = 0. \]

All the constraints \( \Sigma_\alpha \equiv (\Theta_a, \Psi_a^{ra}, \Pi_a^{ra}, \Gamma_a) \) give for the determinant of the matrix \( (\{\Sigma_\alpha, \Sigma_\beta\}_P) \)

\[ \det\{\Sigma_\alpha, \Sigma_\beta\}_P = -\det^2 \frac{\partial^2 L_Q}{\partial V_a \partial V_b}. \quad (77) \]

Therefore we find that if

\[ \det \frac{\partial^2 L_Q}{\partial V_a \partial V_b} \neq 0, \quad (78) \]

then the system is nonsingular; on the other hand if

\[ \det \frac{\partial^2 L_Q}{\partial V_a \partial V_b} = 0, \quad (79) \]

then it is singular.
**Nonsingular case**

In this case, the constraints \((67) \sim (70)\) are second-class ones. Thus the consistency of the constraints under their time developments fixes all the Lagrange multipliers. The path integral is

\[
Z_G = \int DQ^a_r DP^a_r dM^a_r d\Pi^a_r dV_d \Theta_a \delta(\Pi^r) \delta(\Psi^r) \delta(\Theta) \delta(\Gamma) \det \frac{\delta^2 L_Q}{\partial V_a \partial V_b} \times \exp i \int dt \left[ P^a_r \dot{Q}^a_r + \Pi^a_r \dot{M}^a_r + \Theta_a \dot{V}_a - H_G \right]. \tag{80}
\]

Integrations with respect to \(\Pi^r, \Theta, M^r\) give

\[
Z_G = \int DQ^a_r DP^a_r dV_d \Theta_a \delta(\Gamma(Q^r, P^r, V)) \det \frac{\delta^2 L_Q}{\partial V_a \partial V_b} \times \exp i \int dt \left[ P^a_r (\dot{Q}^a_r - \bar{Q}^a_r) + P^a_r (\dot{Q}^a_r - V_a) + L_Q \right]. \tag{81}
\]

**Singular case**

In this case, we have extra constraints in addition to \((67) \sim (70)\):

\[
\Omega_A(Q^r, P^s, V) \approx 0. \tag{82}
\]

Then by introducing Lagrange multipliers \(\Lambda^{(1)}_a, \Lambda^{(2)}_a, \Lambda^{(3)}_a, \Lambda^{(4)}_a\) and \(\Lambda^{(5)}_A\), the Hamiltonian is given by

\[
\bar{H}_{Gs} = H_G(Q^r, P^r, V) + \Lambda^{(1)}_a \Pi^a_r + \Lambda^{(2)}_a \Psi^a_r + \Lambda^{(3)}_a \Theta_a + \Lambda^{(4)}_a \Gamma_a + \Lambda^{(5)}_A \Omega_A. \tag{83}
\]

The Poisson brackets between the constrains are

\[
\{\Pi^a_r, \Psi^b_r\}_P = \delta_{ab} \delta_{r_a r_b}. \tag{84}
\]

\[
\{\Psi^a_r, \Gamma_b\}_P = P^c_r \frac{\partial^2 \bar{Q}^c_r}{\partial Q^a_r \partial V_b} - \frac{\partial^2 L_Q}{\partial Q^a_r \partial V_b}, \tag{85}
\]

\[
\{\Theta_a, \Gamma_b\}_P = -\frac{\partial^2 L_Q}{\partial V_a \partial V_b}, \tag{86}
\]

\[
\{G_a, \Gamma_b\}_P \overset{\text{def}}{=} C_{ab}, \tag{87}
\]

\[
\{\Psi^a_r, \Omega_A\}_P = -\frac{\partial \Omega_A}{\partial Q^a_r}, \tag{88}
\]

\[
\{\Theta_a, \Omega_A\}_P = -\frac{\partial \Omega_A}{\partial V_a}, \tag{89}
\]

\[
\{G_a, \Omega_B\}_P \overset{\text{def}}{=} D_{AB}, \tag{90}
\]

\[
\{\Omega_A, \Omega_B\}_P = 0. \tag{91}
\]
For all the constraints $\Sigma^{(s)}_\alpha \equiv (\Theta_a, \Psi^r_a, \Pi^{rs}_a, \Gamma_a, \Omega_A)$, the determinant of the matrix $\left\{ \Sigma^{(s)}_\alpha, \Sigma^{(s)}_\beta \right\}_P$ is

$$\det\{\Sigma^{(s)}_\alpha, \Sigma^{(s)}_\beta \}_P = \det \begin{pmatrix} 0 & -\frac{\partial^2 L_Q}{\partial V_a \partial V_b} & -\frac{\partial \Omega_B}{\partial V_a} \\ -\frac{\partial^2 L_Q}{\partial V_a \partial V_b} & C_{ab} & \{\Gamma_a, \Omega_B\}_P \\ -\frac{\partial \Omega_B}{\partial V_b} & \{\Omega_A, \Gamma_B\}_P & D_{AB} \end{pmatrix}. \quad (92)$$

If this determinant is nonzero, we assume this is the case, then all the constraints are of the second class and all the Lagrange multipliers are fixed. The path integral is

$$Z_{Gs} = \int DQ^r D\Pi^r D\Pi^r D\Omega \delta(\Pi^r) \delta(\Gamma) \delta(\Theta) \det\{\Sigma^{(s)}_\alpha, \Sigma^{(s)}_\beta \}_P \times \exp i \int dt [P^r_a \dot{Q}^r_a + \Pi^{rs}_a \dot{\Pi}^{rs}_a + \Theta_a \dot{V}_a - H_G]. \quad (93)$$

Integrations with respect to $M^{rs}_a, \Pi^{rs}_a$ and $\Theta_a$ give

$$Z_{Gs} = \int DQ^r D\Pi^r D\Omega \delta(\Omega_A) \det\{\Sigma^{(s)}_\alpha, \Sigma^{(s)}_\beta \}_P \times \exp i \int dt [P^r_a (\dot{Q}^r_a - \bar{Q}^r_a) + P^{rs}_a (\dot{Q}^{rs}_a - \bar{Q}^{rs}_a) + L_Q(Q, V)]. \quad (94)$$

Next, we consider the relations between the path integral expressions $Z_D$ (85) and $Z_G$ (81) (or $Z_{Ds}$ (46) and $Z_G$ (81)). In fact, these are shown to be connected with each other through a canonical transformation.

Consider a canonical transformation $(q, p) \rightarrow (Q, P)$. The generating function has the form

$$F(Q, p) = p^{rs}_a q^{rs}_a(Q'), \quad (95)$$

and gives

$$q^{rs}_a = \frac{\partial F}{\partial p^{rs}_a} = q^{rs}_a(Q'), \quad (96)$$

$$P^{rs}_a = \frac{\partial F}{\partial Q^{rs}_a} = p^{rs}_b \frac{\partial q^{rs}_b(Q')}{{\partial Q^{rs}_a}}. \quad (97)$$

Eqs. (96) and (97) can be inverted to give

$$Q^{rs}_a = Q^{rs}_a(q'), \quad (98)$$

$$p^{rs}_a = P^{rs}_b \frac{\partial Q^{rs}_b(q')}{\partial q^{rs}_a}. \quad (99)$$
Nonsingular case

We start with the Lagrangian \( L_g \) (63). The conjugate momenta

\[
\pi_r^a \overset{\text{def}}{=} \frac{\partial L_g}{\partial \dot{\mu}^a} = 0, \quad (100)
\]

\[
p_r^a \overset{\text{def}}{=} \frac{\partial L_g}{\partial \dot{q}^a} = p_r^a , \quad (101)
\]

\[
\theta_a \overset{\text{def}}{=} \frac{\partial L_g}{\partial \dot{v}_a} = 0 \quad (102)
\]

provide the following primary constraints:

\[
\pi_r^a \approx 0, \quad (103)
\]

\[
\psi_r^a \overset{\text{def}}{=} p_r^a - \mu_r^a \approx 0, \quad (104)
\]

\[
\theta_a \approx 0. \quad (105)
\]

We get the following secondary constraints:

\[
\gamma_a \overset{\text{def}}{=} p_a^{n^a} - \frac{\partial L_g}{\partial v_a}, \quad (106)
\]

By introducing Lagrange multipliers \( \lambda^{(1)}_a, \lambda^{(2)}_a, \lambda^{(3)}_a \) and \( \lambda^{(4)}_a \), the Hamiltonian is given by

\[
\bar{H}_g = H_g(q^r, p^r) + \lambda^{(1)}_a \pi_r^a + \lambda^{(2)}_a \psi_r^a + \lambda^{(3)}_a \theta_a + \lambda^{(4)}_a \gamma_a , \quad (107)
\]

where

\[
H_g(q^r, p^r) \overset{\text{def}}{=} p^s_a q^{s_a + 1} + p_a^{n^a} v_a - L(q,v). \quad (108)
\]

For all the constraints are \( \sigma_a \overset{\text{def}}{=} (\theta_a, \psi_r^a, \pi_r^a, \gamma_a) \), the determinant of the matrix \( (\{\sigma_a, \sigma_b\}) \) is

\[
\det(\{\sigma_a, \sigma_b\}) = -\det^2 \frac{\partial^2 L_{\eta}}{\partial v_a \partial v_b}. \quad (109)
\]

If this determinant is nonzero, then all the Lagrange multipliers are determined.

The path integral is

\[
Z_g = \int \mathcal{D}q_a^r \mathcal{D}p_a^r \mathcal{D}\mu_a^r \mathcal{D}\pi_a^r \mathcal{D}v_a \mathcal{D}\theta_a \delta(\pi^r) \delta(\psi^r) \delta(\theta) \delta(\gamma) \det \frac{\partial^2 L_{\eta}}{\partial v_a \partial v_b} \times \exp i \int dt [p_a^r \dot{p}^r_a + \pi_a^r \dot{\pi}^r_a + \theta_a \dot{\theta}_a + \dot{\bar{H}}_g]. \quad (110)
\]

Integrations with respect to \( \mu^r, \pi^r \) and \( \theta \) give

\[
Z_g = \int \mathcal{D}q_a^r \mathcal{D}p_a^r \mathcal{D}v_a \delta(\gamma_a) \det \frac{\partial^2 L_{\eta}}{\partial v_a \partial v_b} \times \exp i \int dt [p_a^r (\dot{q}^a + q^{s_a + 1}_a) + p_a^{n^a} (\dot{q}^a - q_a^s - v_a) + L_a]. \quad (111)
\]
We can further integrate with respect to $p^{s_a}_{a}, q^{s_a+1}_{a}$ and $v_a$, obtaining

$$Z_g = \int Dq^1_a Dp_{a}^{s_{a}} \exp i \int dt [p_{a}^{s_{a}}q^{1(n_{a})}_{a} - \hat{H}_{g}(q^{1}, q^{1(s)}, p^n)],$$  

(112)

where

$$\hat{H}_{g}(q^{1}, q^{1(s)}, p^n) \overset{\text{def}}{=} p_{a}^{s_{a}}v_{a}(q^{1}, q^{1(s)}, p^n) - L_{q}(q^{1}, q^{1(s)}, v(q^{1(s)}, p^n)).$$  

(113)

Putting $v_a = \dot{q}^{s_{a}}_{a}$ in this equation shows that the path integral $Z_g$ is the same as $Z_O$ given by (11) (and also $Z_D$ in (13)).

Next, by doing the canonical transformation generated by $F$ in (13), we show that the path integral $Z_g$ is equivalent to $Z_G$ given by (81). Referring to Eqs. (60) ~ (63) and (58), the following relation is inserted into $Z_g$ in Eq. (111):

$$\int Dq^r_aDP^{r_{a}}aDVa \delta (q^{r}_{a} - q^{r}_{a} (Q)) \delta \left(p^{r_{a}}_{a} - P^{r_{a}}b \frac{\partial Q^{r_{b}}}_{a}}{\partial q^{r}_{a}}\right) \times \det \left(\frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}}\right)^{-1} \delta \left(v_{b} - \frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}}\right)^{-1} \left(V_{a} - q_{a}^{s_{a}+1} \frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}}\right) = 1 \text{(114)}$$

Then we have

$$Z_g = \int Dq^{r}aDP^{r_{a}}aDVa \left[ \delta \left(\frac{\partial L_{q}}{\partial v_{a}} - P^{r_{a}}b \frac{\partial Q^{r_{b}}}_{a}}{\partial q^{r}_{a}}\right) \det \frac{\partial^{2}L_{q}}{\partial v_{a} \partial v_{b}} \exp i \int dt [p_{a}^{s_{a}}(\dot{q}^{s_{a}}_{a} - q^{s_{a}+1}_{a}) + p_{a}^{s_{a}}(\dot{q}^{s_{a}}_{a} - v_{a}) + L_{q}]] \text{(115)}$$

Integrations with respect to $q^{r}, p^{r}$ and $v$ give

$$Z_g = \int Dq^{r_{a}}aDP^{r_{a}}aDVa \left[ \delta \left(\frac{\partial L_{q}}{\partial v_{a}} - P^{r_{a}}b \frac{\partial Q^{r_{b}}}_{a}}{\partial q^{r}_{a}}\right) \det \frac{\partial^{2}L_{q}}{\partial v_{a} \partial v_{b}} \det \left(\frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}}\right)^{-1} \right]_{q = v} \exp i \int dt [P^{r_{a}}b \frac{\partial Q^{r_{b}}}{\partial q^{r}_{a}}\dot{Q}^{r_{a}}_{a} - P^{r_{a}}c \frac{\partial Q^{r_{b}}}{\partial q^{r}_{a}}\dot{Q}^{r_{a}}_{a} + (Q) - P^{r_{a}}c \frac{\partial Q^{r_{b}}}{\partial q^{r}_{a}}v_{b}(Q, V) + L_{Q}]] \text{ (116)}$$

By using (57), (59) and the relations

$$\delta (\gamma_{a}(q^{r}, p^{n}, v))_{q_1(q), p_1(Q, P), v_1(v, Q, V)} \overset{\text{def}}{=} \det \left(\frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}}\right)^{-1} \delta (\Gamma_{a}), \text{ (117)}$$

$$\det \frac{\partial^{2}L_{q}}{\partial v_{a} \partial v_{b}} \frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}} = \det \left(\frac{\partial Q^{r_{a}}_{a}}{\partial q^{r}_{a}}\right) \det \frac{\partial^{2}L_{q}}{\partial v_{a} \partial v_{b}} \text{ (118)}$$

we get

$$Z_g = \int Dq^{r_{a}}aDP^{r_{a}}aDVa \delta (\Gamma_{a}) \det \frac{\partial^{2}L_{Q}}{\partial v_{a} \partial v_{b}}$$

$$\times \exp i \int dt [P^{s_{a}}(\dot{Q}^{s_{a}}_{a} - \dot{Q}^{s_{a}}_{a}) + P^{s_{a}}(\dot{Q}^{s_{a}}_{a} - V_{a}) + L_{Q}]. \text{ (119)}$$
This shows that

\[ Z_G = Z_{\Omega} = Z_D = Z_G. \]  

(120)

We have found that the generalized canonical formalism is equivalent to the Ostrogradski’s one and these two formalisms are connected by a canonical transformation.

**Singular case**

First, we show the equivalence between the path integrals \( Z_{Dn} \) given by (108) and \( Z_{gs} \) constructed from the Lagrangian \( L_g \) in (121). In this case, we choose, without loss of generality, for extra constraints the following form:

\[
\omega_A(q^r, p^{n-1}, v) \equiv p^{n-1}_A - \frac{\partial L_A}{\partial q^n_A} + \frac{\partial^2 L_A}{\partial v_A \partial q^n_A} v + \frac{\partial^2 L_A}{\partial v_A \partial q^n_B} \approx 0.
\]  

(121)

By introducing additional multipliers \( \lambda^{(a)}_A \), the Hamiltonian is given by

\[
\tilde{H}_{gs} = H_g(q^r, p^r) + \lambda^{(1)}_a \pi^r_a + \lambda^{(2)}_a \psi^r_a + \lambda^{(3)}_a \theta_a + \lambda^{(4)}_a \gamma_a + \lambda^{(5)}_A \omega_A.
\]  

(122)

All the constraints \( \sigma^{(s)}_A \) defined by \( (\theta_a, \psi^r_a, \pi^r_a, \gamma_a, \omega_A) \) give for the determinant of the matrix \( \left\{ \{\sigma^{(s)}_A, \sigma^{(s)}_B\} \right\}_P \)

\[
\det\{\sigma^{(s)}_A, \sigma^{(s)}_B\}_P = \det \begin{pmatrix}
0 & -\frac{\partial^2 L_A}{\partial v_a \partial v_b} & -\frac{\partial \omega_A}{\partial v_a} \\
\frac{\partial^2 L_A}{\partial v_a \partial v_b} & \{\gamma_a, \gamma_b\}_P & \{\gamma_a, \omega_B\}_P \\
\frac{\partial \omega_A}{\partial v_b} & \{\omega_A, \gamma_b\}_P & \{\omega_A, \omega_B\}_P
\end{pmatrix}.
\]  

(123)

If this is nonzero, all the Lagrange multipliers are determined. The path integral is given by

\[
Z_{gs} = \int Dq^r_a Dp^r_a D\mu^r_a D\pi^r_a D\psi^r_a D\theta_a D\delta(\pi^r) \delta(\psi^r) \delta(\theta) \delta(\omega_A) \det \frac{1}{2} \{\sigma^{(s)}_A, \sigma^{(s)}_B\}_P 
\times \exp i \int dt [p^r_a \dot{q}^r_a + \pi^r_a \dot{\mu}^r_a + \theta_a \dot{\psi}^r_a - H_{gs}].
\]  

(124)

Integrations with respect to \( \mu^r_a, \pi^r_a \) and \( \theta_a \) give

\[
Z_{gs} = \int Dq^r_a Dp^r_a Dv_a \delta(\gamma_a(q^r, p^{n-1}, v)) \frac{1}{2} \{\omega_A(q^r, p^{n-1}, v)\} \det \frac{1}{2} \{\sigma^{(s)}_A, \sigma^{(s)}_B\}_P 
\times \exp i \int dt [p^r_a (\dot{q}^r_a - q^{a+1}_a) + p^{n-1}_a (v_a + L_a(q, v))] = Z_{gs}(q, v).
\]  

(125)

Here, we consider the matrix \( \left\{ \{\sigma^{(s)}_A, \sigma^{(s)}_B\}_P \right\} \). We change this into a form which can be integrated with respect to \( v_a \). The assumption that the determinant of
this matrix is nonzero means

\[
\text{rank} \frac{\partial \omega_B}{\partial v_a} = \rho. \quad (126)
\]

In the matrix

\[
\left( \frac{\partial^2 L_q}{\partial v_a \partial v_b} \frac{\partial \omega_A}{\partial v_a} \right) = \frac{\partial}{\partial v_a} \begin{pmatrix} \gamma_b & \omega_A \end{pmatrix},
\]

we select \( \gamma_\xi (\xi = \rho + 1, \cdots, N) \) which satisfy

\[
\det \left( \frac{\partial (\gamma_\xi, \omega_A; \varepsilon)}{\partial v_a} \right) \neq 0, \quad (128)
\]

to define as \( \Xi_a(q^r, p^n, p^{n-1}) \) which satisfy

\[
\frac{\partial}{\partial v_a} \left( \gamma_\xi, \omega_A \right) \neq 0,
\]

(127)

The determinant of the matrix \([123]\) reduces to

\[
\det \left\{ \sigma^{(s)}_\alpha, \sigma^{(s)}_\beta \right\}_P = \det^2 \left( \frac{\partial \Xi_a}{\partial v_a} \right) \det \left\{ \gamma_A, \gamma_B \right\}_P. \quad (129)
\]

Then the path integral \((125)\) is given by

\[
Z_{gs} = \int Dq^r_a Dp^n_a Dv_a \delta(\gamma_A) \delta(\Xi_a) \det \left( \frac{\partial \Xi_a}{\partial v_b} \right) \det^2 \left\{ \gamma_A, \gamma_B \right\}_P
\]

\[
\times \exp i \int dt [p^n_a (\dot{q}^r_a - q^{a+1}_a) + p^n_a (\dot{q}^{n_a}_a - v_a) + L_q]. \quad (130)
\]

Integrations with respect to \( v_a, p^n_a \) and \( q^{a+1}_a \) give

\[
Z_{gs} = \int Dq^r_a Dp^n_a \delta(\gamma_A) \det^2 \left\{ \gamma_A, \gamma_B \right\}_P \exp i \int dt [p^n_a (\dot{q}^r_a - q^{a+1}_a) + p^n_a (\dot{q}^{n_a}_a - v_a) + L_q]. \quad (131)
\]

where

\[
\hat{H}_{gs} \equiv p^n_a v_a - L_q \left( q^1, q^{1(s)}, v \right). \quad (132)
\]

Putting \( \gamma_A = \phi_A \), we have arrived at the same expression as \( Z_{Ds} \) in \([16]\).

Next task is canonical transformation. Since the exponent in \([94]\) is the same as in Eq. \([81]\), we insert Eq. \([114]\) into the expression \([125]\) and integrate with respect to \( q^r, p^r \) and \( v \) to obtain

\[
Z_{gs} = \int DQ^r Dp^r DV \delta(\gamma_A) \delta(\omega_A) \det^2 \left\{ \sigma^{(s)}_\alpha, \sigma^{(s)}_\beta \right\}_P
\]

\[
\times \det \left( \frac{\partial Q^{n_b}_a}{\partial q^{n_a}_a} \right)^{-1} \mid_{q=\eta(Q), p=p(Q, P), v=v(Q, V) \quad \times \exp i \int dt [p^n_a (\dot{Q}^r_a - \dot{Q}^{n_a}_a) + p^n_a (\dot{Q}^{n_a}_a - V_a) + L_Q]. \quad (133)
\]

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By using the relations
\[
\delta(\gamma_a)|_{q=q(Q),p=p(Q,P),v=v(Q,V)} = \delta(\Gamma_a)\det \left( \frac{\partial Q^n}{\partial q^a} \right)^{-1} |_{q=q(Q)}, \quad (134)
\]
\[
\delta(\omega_A)|_{q=q(Q),p=p(Q,P),v=v(Q,V)} = \delta(\Omega_A)\det \left( \frac{\partial P^n}{\partial q^a} \right)^{-1} |_{q=q(Q)}, \quad (135)
\]
\[
\det\{\sigma^{(s)}_a, \sigma^{(s)}_b\} = \det 4 \left( \frac{\partial Q^n_a}{\partial q^a} \right) \det 2 \left( \frac{\partial Q^n_b}{\partial q^a} \right) \det \{\Sigma^a_{\alpha}, \Sigma^a_{\beta}\} P, \quad (136)
\]
we obtain
\[
Z_{gs} = \int DQ^r DP^r DV [\delta(\Gamma_a)\delta(\Omega_A)\det \{\Sigma^a_{\alpha}, \Sigma^a_{\beta}\} P] 
\times \exp i \int dt [P^s_a (Q^s_a - \dot{Q}^s_a) + P^n_a (\dot{Q}^n_a - V_a) + L_Q]. \quad (137)
\]
This shows
\[
Z_{gs} = Z_{Os} = Z_{Ds} = Z_{Gs}. \quad (138)
\]
The path integrals $Z_{gs}$ and $Z_{Gs}$ are connected with each other by the canonical transformation generated by $F$ in (95).

5 Summary and Discussion

In the present paper we have given path integral expressions for three canonical formalisms of higher-derivative theories. For each formalism we have considered both nonsingular and singular cases. It has been shown that these formalisms share the same path integral expressions. In particular, it has been pointed out that the generalized canonical formalism is canonically transmuted from the constrained canonical one.

Here we have to mention some crucial properties involved in higher-derivative theories. The Hamiltonian is unbounded from below in general; unitarity is violated in general; whether or not stable vacuum can be well defined is problematic. That means we should worry about how to define path integral. Leaving these problems to the future investigation, we have just assumed in this paper that stable lowest state can be defined, and the path integral can be written down as usual by the use of a time development operator, the Hamiltonian.

Acknowledgements

The authors would like to thank C. Darieescu, M.-A. Darieescu, M. Hirayama, T. Kurimoto, K. Matumoto and H. Yamakoshi for discussions.
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