Research Article

Infinitely Many High Energy Solutions for the Generalized Chern-Simons-Schrödinger System

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In this paper, by virtue of the critical point theory, we are pleased to investigate the existence of infinitely many high energy solutions for the generalized Chern-Simons-Schrödinger system with a perturbation.

1. Introduction

In this work, we are concerned with the following generalized Chern-Simons-Schrödinger system with a perturbation:

\[
\begin{align*}
\Delta u + \lambda V(x)u + A_0(u(x))u + \sum_{j=1}^{2} A_j^2(u(x))u &= f(u) + \mu g(x)|u|^{q-2}u, \\
\partial_1 A_2(u(x)) - \partial_2 A_1(u(x)) &= -\frac{1}{2} u, \\
\partial_1 A_1(u(x)) + \partial_2 A_2(u(x)) &= 0, \\
\Delta A_0(u(x)) &= \delta (A_2(u(x))|u|^2) - \partial_2 (A_1(u(x))|u|^2),
\end{align*}
\]

where \(\lambda, \mu\) are positive parameters and \(V, f, g\) satisfy the following conditions:

(V1) \(V(x) \in C(\mathbb{R}^2), V(x) = V(|x|)\), and \(V(x) \geq 0\) on \(\mathbb{R}^2\);

(V2) There exists \(b > 0\) such that \(V_b = \{x \in \mathbb{R}^2 : V(x) < b\}\) is nonempty and has finite measure;

(V3) There exists \(R > 0\) such that \(B(0, R) = \text{int } V^{-1}(0)\) and \(B(0, R) = V^{-1}(0)\), where \(B(0, R)\) denotes the ball of radius \(R\) centered at \(0\);

(H1) \(f \in C(\mathbb{R}, \mathbb{R})\), and \(f(u) = o(|u|)\) as \(|u| \to 0\);

(H2) There exists \(R_0 > 0\) such that \(F(u) = \int_0^u f(t) \, dt \geq 0\) and \(\mathcal{F}(u) = (1/6)f(u)u - F(u) \geq 0\) for \(|u| \geq R_0\);

(H3) \(f(u)u/|u|^6 \to +\infty\) as \(|u| \to +\infty\);

(H4) There exist \(a_0, R_1 > 0\) and \(\tau \in (1, +\infty)\) such that \(|f(u)|^\tau \leq a_0 \mathcal{F}(u)|u|^\tau\), for \(|u| \geq R_1\);

(H5) \(f(-u) = -f(u)\) for \(u \in \mathbb{R}\);

(g) \(g \in L^{q'}(\mathbb{R}^2)\), and \((x) \geq 0(\equiv 0)\), for \(x \in \mathbb{R}^2\), where \(q' \in (1, 2/(2 - q))\), \(q \in (1, 2)\).

Recently, the Chern-Simons-Schrödinger system

\[
\begin{align*}
iD_0 u + (D_1D_1 + D_2D_2)u &= -f(u), \\
\partial_0 A_1 - \partial_1 A_0 &= -\operatorname{Im}(\bar{u}D_2 u), \\
\partial_0 A_2 - \partial_2 A_0 &= -\operatorname{Im}(\bar{u}D_1 u), \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |u|^2
\end{align*}
\]

has been paid more attention by many researchers (for example, see [1–10]), where \(i\) denotes the imaginary unit, \(\partial_0 = \partial/\partial t, \partial_1 = \partial/\partial x_1, \text{ and } \partial_2 = \partial/\partial x_2\) for \((t, x_1, x_2) \in \mathbb{R}^{1+2}, u : \mathbb{R}^{1+2} \to \mathbb{C}\) is the complex scalar field, \(A_\mu\)
$\mathbb{R}^{1+2} \rightarrow \mathbb{R}$ is the gauge field, and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative for $\mu = 0, 1, 2$.

In [1], the authors studied the nonlocal semilinear Schrödinger equation with the gauge field

$$\Delta u + \lambda V(|x|)u + \left(\frac{k^2(|x|)}{|x|^2} + \int_0^{|s|} \frac{m(s)}{s} \, du(s) \, ds\right) u = f(|x|, u), \quad u \in \mathbb{R}^2,$$

(3)

where the potential $V$ satisfies (V1)-(V3) (this potential can also be found in [11, 12]). When $f$ satisfies more general 6 superlinear conditions at infinity, they obtained some existence theorems of nontrivial solutions for (3). Some similar results can also be found in [2, 5] with a constant external potential. In [4], the authors improved the results in [1, 2, 5] and used the concentration-compactness principle to obtain two bound state solutions for the generalized Chern-Simons-Schrödinger system

$$-\Delta u + A_0 u + \sum_{j=1}^{2} A_j^2 u = |u|^{p-2} u + \lambda u,$$

(4)

$$\partial_t A_2 - D_2 A_1 = -\frac{1}{2}u^2,$$

$$\partial_t A_1 + D_2 A_2 = 0,$$

$$\Delta A_0 = \delta_1 (A_2 u^2) - D_2 (A_1 u^2),$$

where $p > 4$. In [3], the authors used some new techniques joined with the manifold of Pohožaev-Nehari type to study the existence of a semiclassical ground state solution for the generalized Chern-Simons-Schrödinger system

$$\varepsilon^2 \Delta u + V(x)u + A_0(u(x))u + \sum_{j=1}^{2} A_j^2 (u(x))u = f(u),$$

$$\varepsilon (D_1 A_2 (u(x)) - D_2 A_1 (u(x))) = -\frac{1}{2}u^2,$$

$$\partial_t A_1 (u(x)) + D_2 A_2 (u(x)) = 0,$$

$$\varepsilon \Delta A_0 (u) = \delta_1 (A_2 (u(x)) |u|^2) - \delta_2 (A_1 (u(x)) |u|^2),$$

(5)

where their results are available to the nonlinearity $f(u) \sim |u|^{p-2} u$ for $s \in (4, 6]$.

There also are some papers in the literature which consider perturbation terms (see [12–21]) and the references therein (also refer to [22–27]). For example, in [13, 14], the authors used the famous Ambrosetti-Rabinowitz condition (see also [12, 15]) to study the existence of solutions for the following Schrödinger equations:

$$\left\{ \begin{array}{c}
-\Delta u + V(x)u + \phi u = f(x, u) + g(x), \quad x \in \mathbb{R}^3, \\
-\Delta \phi = u^2, \quad x \in \mathbb{R}^3,
\end{array} \right.$$

$$-\Delta \phi + V(x)|u|^{p-2} u - \lambda |u|^{p-2} u = f(x, u) + g(x)|u|^p u, \quad x \in \mathbb{R}^3,$$

(6)

where $(-\Delta)^n_p$ is the fractional $p$-Laplace operator. It is generally known that our conditions (H2) and (H4) are weaker than the corresponding (AR) condition: there exists $\mu > 6$ such that

$$0 < \mu F(u) \leq f(u)u \quad \text{for } u \in \mathbb{R} \setminus \{0\}. \quad (7)$$

So, our results here can be viewed as an extension to the ones in [12–15].

In [16], the authors studied the following nonhomogeneous Schrödinger-Kirchhoff-type fourth-order Elliptic equations in $\mathbb{R}^N$:

$$\left\{ \begin{array}{c}
\Delta^2 u + \left( a + h \varepsilon^p \right) \Delta u + V(x)u = f(x, u) + h(x)u \quad \text{in } \mathbb{R}^N, \\
\Delta u \in H^2(\mathbb{R}^N).
\end{array} \right.$$

(8)

They obtained the existence of infinitely many solutions for this system by means of the symmetry mountain pass theorem and the fountain theorem.

Now, we state the main result:

**Theorem 1.** Suppose that (V1)-(V3), (H1)-(H5), and (g) hold. Then, for arbitrarily small $\mu > 0$, there exists $\mu_0 > 0$ such that system (1) possesses infinitely many high energy solutions when $\lambda \geq \lambda_{\mu_0}$.

**Remark 2.** From (H1), (H2), and (H4), we can get a growth condition for $f$. Using (H2) and (H4), for $|u| \geq R_1 := \max \{ R_0, R_1 \}$, we have $|f(u)| \leq \alpha_0 \phi(u)|u|^r = \alpha_0 \phi(u) |u|^{r+1} |u|^{r+1}$

$$|u|^2 \leq \alpha_0 \phi(u) |u|^{r+1} \quad \text{and} \quad |f(u)| \leq \sqrt{\alpha_0 \phi(u)} |u|^{r+1}. \quad \text{2 

Let $p = \frac{r+1}{r-1}$, then $f(u) \leq \sqrt{\alpha_0 \phi(u)} |u|^{r+1}$, for $|u| \geq R_2$. Therefore, we obtain

$$|f(u)| \leq \frac{e}{2} |u|^2 + c_r |u|^p, \quad u \in \mathbb{R}, \quad c_r = \sqrt{\frac{\alpha_0}{6}} \quad (9)$$

and thus

$$|F(u)| \leq \frac{e}{2} |u|^2 + \frac{c_r}{p} |u|^p, \quad u \in \mathbb{R}. \quad (10)$$

2. Preliminaries

Let $\| \cdot \|_{s}$ be the usual $L^s$-norm for $s \in [1, +\infty)$, and $\ell_i (i \in \mathbb{R}^n)$ stand for different positive constants. We use $H^s_0(\mathbb{R}^2)$ to denote a Sobolev space with the norm

$$\| u \|_{H^s_0(\mathbb{R}^2)}^2 = \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx \right)^{1/2}. \quad (11)$$

Define the space

$$E := \left\{ u \in H^s_0(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) \, dx < \infty \right\}. \quad (12)$$
with the inner product and norm

\[(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x)uv)dx,\]

\[\|u\| = \sqrt{(u, u)}. \quad (13)\]

Note the large parameter \(\lambda\) in Theorem 1, so we need the following inner product and norm:

\[(u, v)_{\lambda} = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \lambda V(x)uv)dx,\]

\[\|u\|_{\lambda} = \sqrt{(u, u)_{\lambda}}, \quad u, v \in E. \quad (14)\]

Define \(E_1 = (E_2 \cdot \cdot \cdot )\); then, we have \(E_1\) which is a Hilbert space. Using (V1)-(V3), there exist positive constants \(\lambda_0, l_0\) (independent of \(\lambda\)) such that

\[\|u\|_{H_1^\infty} \leq l_0 \|u\|_{\lambda}, \quad \text{for all } u \in E_1, \lambda \geq \lambda_0. \quad (15)\]

Moreover, by [28], the embedding \(E_1^p \hookrightarrow L^p(\mathbb{R}^2)\) is continuous for \(p \in [2, +\infty)\), and \(E_1 \hookrightarrow L^p(\mathbb{R}^2)\) is compact for \(p \in (2, +\infty)\), i.e., there exists \(l_\infty > 0\) such that

\[\|u\| \leq l_\infty \|u\|_{H_1^\infty} \leq l_0 \|u\|_{\lambda}, \quad \text{for all } u \in E_1, \lambda \geq \lambda_0, 2 \leq p < +\infty. \quad (16)\]

For convenience, let \(l_0 = \tilde{l}_0\).

Now, on \(E_\lambda\), we define the following energy functional:

\[\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + \lambda V(x)|u|^2 + A_1^2(u)|u|^2 + A_2^2(u)|u|^2 \right] dx - \int_{\mathbb{R}^2} F(u)dx - \mu \int_{\mathbb{R}^2} g(x)|u|^qdx. \quad (17)\]

By (V1)-(V3), (9), (10), and [8], \(\mathcal{J}\) is of class \(C^1(E_\lambda, \mathbb{R})\), and

\[\left\langle \mathcal{J}'(u), \phi \right\rangle = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla \phi + \lambda V(x)u\phi + (A_1^2(u) + A_2^2(u))u\phi + A_0(u)\phi]dx - \int_{\mathbb{R}^2} f(u)\phi dx - \mu \int_{\mathbb{R}^2} g(x)|u|^{q-2}u\phi dx, \quad \text{for all } \phi \in E_\lambda. \quad (18)\]

Note that (16) in [3], we have

\[\int_{\mathbb{R}^2} A_0(u)|u|^2dx = 2 \int_{\mathbb{R}^2} \left( A_1^2(u) + A_2^2(u) \right) |u|^2dx, \quad \text{for all } u \in H_1^\infty(\mathbb{R}^2). \quad (19)\]

Consequently, we have

\[\left\langle \mathcal{J}'(u), u \right\rangle = \int_{\mathbb{R}^2} [\nabla u |u|^2 + \lambda V(x)|u|^2 + 3 (A_1^2(u) + A_2^2(u))|u|^2] \cdot dx - \int_{\mathbb{R}^2} f(u)udy - \mu \int_{\mathbb{R}^2} g(x)|u|^qdx. \quad (20)\]

**Lemma 3** (see [4, 7, 8]). Suppose that \(u_n\) converges to \(u\) a.e. in \(\mathbb{R}^2\) and \(u_n\) converges weakly to \(u\) in \(H_1^\infty(\mathbb{R}^2)\). Let \(A_{\alpha,n} := A_\alpha(u_n(x)), \alpha = 0, 1, 2\). Then, 
\[\int_{\mathbb{R}^2} A_{\alpha,n}^2u_nudy, \int_{\mathbb{R}^2} A_{\alpha,n}^2|u|^2dx, \quad \text{and} \quad \int_{\mathbb{R}^2} A_{\alpha,n}^2|u|^2dx \]
converge to \(\int_{\mathbb{R}^2} A_{\alpha}^2u|u|^2dx, \quad \text{for } i = 1, 2; \]
\[\int_{\mathbb{R}^2} A_{0,n}u_nudy \quad \text{and} \quad \int_{\mathbb{R}^2} A_{0,n}|u|^2dx \]
converge to \(\int_{\mathbb{R}^2} A_0u|u|^2dx\).

We say that \(\mathcal{J} \in C^1(\mathbb{X}, \mathbb{R})\) satisfies \((C)\)-condition if any sequence \(\{u_n\}\) such that

\[\mathcal{J}(u_n) \rightarrow c, \quad (1 + \|u_n\|_\lambda)\mathcal{J}'(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (21)\]

has a convergent subsequence.

**Lemma 4** (see [29]). Suppose that \(X\) is an infinite dimensional Banach space, and \(Y, Z\) are two subspaces of \(X\) with \(X = Y \oplus Z\), where \(Y\) is finite dimensional. If \(\mathcal{J} \in C^1(X, \mathbb{R})\) satisfies the \((C)\)-condition for all \(c > 0\) and

(C1). \(\mathcal{J}(0) = 0\) and \(\mathcal{J}(-u) = \mathcal{J}(u)\) for all \(u \in X\);

(C2). there exist constants \(\rho, \alpha > 0\) such that \(\mathcal{J}|_{\mathbb{B} \cap X} \geq \alpha\);

(C3). for any finite dimensional subspaces \(X \subset X\), there exists \(R = R(\tilde{X}) > 0\) such that \(\mathcal{J}(u) \leq 0\) on \(\tilde{X} \backslash B_{R}\),

then \(\mathcal{J}\) possesses an unbounded sequence of critical values.

3. Main Results

In order to prove Theorem 1, we provide some lemmas.

**Lemma 5.** Under assumptions (V1)-(V3), (H1)-(H5), and (g), any sequence \(\{u_n\} \subset E_\lambda\) satisfying

\[\mathcal{J}(u_n) \rightarrow c > 0, \quad (22)\]

\[\left\langle \mathcal{J}'(u_n), u_n \right\rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (23)\]

is bounded in \(E_\lambda\).

Proof. To prove the boundedness of \(\{u_n\}\), argument by contrary, assume that \(\|u_n\|_\lambda \rightarrow \infty\). Let \(v_n = u_n/\|u_n\|_\lambda\).

Then, \(\|v_n\|_{\lambda} = 1\), and \(\|v_n\|_p \leq l_p \|v_n\|_{\lambda} = l_p, p \geq 2\). Note that
\[ q^q / (q^q - 1) > 2 \]  by (g); for large \( n \), from (16), we have

\[ c + 1 \geq \mathcal{F}(u_n) - \frac{1}{6} \int \mathcal{F}(u_n, u_n) \]

\[ = \frac{1}{2} \| u_n \|_A^2 + \frac{1}{2} \int \left[ A^2(u_n) |u_n|^2 + A^2(u_n) |u_n|^2 \right] \]

\[ \cdot dx = \int \frac{1}{2} f(u_n) |u_n|^2 dx + \frac{1}{2} \int g(x) |u_n|^q \]

\[ \cdot dx = \frac{1}{6} \| u_n \|_A^2 - \frac{1}{2} \int \left[ A^2(u_n) |u_n|^2 + A^2(u_n) |u_n|^2 \right] \]

\[ \cdot dx + \frac{1}{2} \int \frac{1}{2} f(u_n) |u_n|^2 dx + \frac{1}{6} \int g(x) |u_n|^q \]

\[ \cdot dx \geq \frac{1}{6} \| u_n \|_A^2 + \int \mathcal{F}(u_n) \]

\[ \cdot dx - \frac{1}{3} \int \mathcal{F}(u_n) dx, \]

(24)

for the fact that \( q \in (1, 2) \) and \( \mu > 0 \) is an arbitrarily small parameter.

In view of (20), we have

\[ 1 = \frac{\| u_n \|_A^2}{\| u_n \|_A^2} = \frac{3}{\| u_n \|_A^2} \int \mathcal{F}(u_n, u_n) \]

\[ \cdot dx + \frac{1}{\| u_n \|_A^2} \int f(u_n) u_n dx + \frac{1}{\| u_n \|_A^2} \int g(x) |u_n|^q \]

\[ \cdot dx \leq \limsup_{n \to \infty} \left[ \frac{\| u_n \|_A^2}{\| u_n \|_A^2} \int f(u_n) u_n dx \right] \]

\[ \cdot dx + \frac{\| g \|_{\mathcal{L}^q(\mathcal{O})}^q |u_n|^q}{\| u_n \|_A^2} \]

\[ \leq \limsup_{n \to \infty} \int f(u_n) u_n dx. \]

(25)

Recall that \( \| u_n \|_A = 1 \), and there exists a function \( v \in E_\lambda \) such that \( v_n \to v \) weakly in \( E_\lambda \), \( v_n \to v \) strongly in \( L^r(\mathbb{R}^2) \) with \( r \in (2, \infty) \) and \( v_n(x) \to v(x) \) for a.e. \( x \in \mathbb{R}^2 \). Define a set \( \Omega \) by \( \{ x \in \mathbb{R}^2 : a \leq |x| < b \} \) with \( 0 \leq a < b \), and we consider the following two possible cases.

**Case 1.** \( v = 0 \), and \( v_n \to 0 \) weakly in \( E_\lambda \), \( v_n(x) \to 0 \) for a.e. \( x \in \mathbb{R}^2 \). From (9), we have

\[ \int_{\Omega_n(\mathbb{R})} \frac{f(u_n) u_n}{\| u_n \|_A^2} dx = \int_{\Omega_n(\mathbb{R})} \frac{f(u_n) u_n}{|u_n|^2} |v_n|^2 \]

\[ \cdot dx \leq \left( \epsilon + c R_{n, R}^{r-2} \right) \int_{\Omega_n(\mathbb{R})} |v_n|^2 \]

\[ \cdot dx \leq \left( \epsilon + c R_{n, R}^{r-2} \right) \int_{\mathbb{R}^2} |v_n|^2 dx \to 0. \]

(26)

On the other hand, by the Hölder inequality, (24), and (H4), we obtain

\[ \int_{\Omega_n(\mathbb{R})} f(u_n) u_n dx = \int_{\Omega_n(\mathbb{R})} \frac{f(u_n) u_n}{|u_n|^2} |v_n|^2 \]

\[ \cdot dx \leq \left( \int_{\Omega_n(\mathbb{R})} \frac{f(u_n) u_n}{|u_n|^2} \right)^{\tau} \]

\[ \cdot \left( \int_{\Omega_n(\mathbb{R})} |v_n|^2 \right)^{(r-1)/\tau} \]

\[ \leq \left( \int_{\Omega_n(\mathbb{R})} \frac{f(u_n)}{|u_n|^2} dx \right)^{1/\tau} \]

\[ \cdot \left( \int_{\Omega_n(\mathbb{R})} |v_n|^2 dx \right)^{(r-1)/\tau} \]

\[ \leq \left( \int_{\Omega_n(\mathbb{R})} \frac{f(u_n)}{|u_n|^2} dx \right)^{1/\tau} \]

\[ \cdot \left( \int_{\Omega_n(\mathbb{R})} |v_n|^2 dx \right)^{(r-1)/\tau} \]

\[ \leq \left[ \alpha_0 \mathcal{F}(u_n) \right]^{1/\tau} \]

\[ \leq [\alpha_0 (c + 1)]^{1/\tau} \| v_n \|_p \to 0. \]

(27)

Combining (26) and (27), we have

\[ \int_{\mathbb{R}^2} f(u_n) u_n dx = \int_{\Omega_n(\mathbb{R})} f(u_n) u_n dx + \int_{\Omega_n(\mathbb{R})} f(u_n) u_n dx \to 0, \]

(28)

which contradicts (25).

**Case 2.** \( v(x) \neq 0, x \in \mathbb{R}^2 \). Hence, let \( A = \{ x \in \mathbb{R}^2 : v(x) \neq 0 \} \) and then, \( \text{meas}(A) > 0 \). For \( x \in A \), we have \( \lim_{n \to \infty} |u_n(x)| = \infty \) and hence, \( A \subset \Omega_n(\mathbb{R}) \) for large \( n \). Note that from Proposition 2.1 in [8] and (2.15) in [1], there exists a constant \( \mathcal{C}_0 > 0 \) such that

\[ \int_{\mathbb{R}^2} (A^2(u) + A^2(u)) |u|^2 dx \leq \mathcal{C}_0 \| u \|^4_{\mathcal{H}_1}, \quad \text{for all } u \in \mathcal{H}_1(\mathbb{R}^2). \]

(29)

Therefore, by (H3), (29), and (9), note the nonnegativity of \( f(u) \) and \( q \in (1, 2) \), Fatou’s lemma enables us to obtain

\[ 0 = \lim_{n \to \infty} \frac{\mathcal{F}(u_n)}{\| u_n \|_A^2} \]

\[ = \lim_{n \to \infty} \left[ \| u_n \|_A^2 + \frac{3}{\| u_n \|_A^2} \int_{\mathbb{R}^2} (A^2(u_n) + A^2(u_n)) |u_n|^2 dx \right] \]

\[ - \left( \int_{\mathbb{R}^2} f(u_n) u_n dx \right) \]

\[ \cdot \left[ \frac{\mu g(x) |u_n|^q dx}{\| u_n \|_A^2} \right] \leq \lim_{n \to \infty} \]
that any sequence \( f \in \text{sequence in } \mathbb{R}^n \).

Proof. From Lemma 5 and the compactness of \( \Omega \), we have
\[
\lim_{n \to \infty} \left( \int_{\Omega_n(0,R_1)} \left( \frac{f(u_n)}{u_n^0} \right) \left| \nabla u_n \right|^6 \right) dx = 0.
\]

Note that from Lemma 1 in [30], there exists \( C_q > 0 \) such that \( \left| u_n \right|^{q+2} u_n - \left| u \right|^{q+2} u \leq C_q \left| u_n - u \right|^{q+1} \). Therefore, from (g)

and the Hölder inequality, we have
\[
\int_{\mathbb{R}^2} g(x) \left( \left| u_n \right|^{q+2} u_n - \left| u \right|^{q+2} u \right) (u_n - u) \cdot dx \leq C_q \int_{\mathbb{R}^2} g(x)(u_n - u)^q \cdot dx.
\]

By a simple calculation, we have
\[
\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), u_n - u \rangle = \left\| u_n - u \right\|_1^2.
\]

It is clear that \( \langle \mathcal{F}'(u_n) - \mathcal{F}'(u), u_n - u \rangle \to 0 \). As a result, from (32)-(34), we have
\[
\left\| u_n - u \right\|_1 \to 0 \quad \text{as } n \to \infty.
\]

Lemma 6. Under assumptions (V1)-(V3), (H1)-(H5), and (g), any sequence \( \{u_n\} \subset E_\lambda \) satisfying (22) has a convergent subsequence in \( E_\lambda \).

Proof. From Lemma 5 and the compactness of \( E_\lambda ' L'(\mathbb{R}^2) \) for \( r \in (2,\infty) \), we have
\[
\begin{align*}
& \begin{cases}
  u_n \to u \text{ weakly in } E_\lambda, \\
  u_n \to u \text{ strongly in } L^r(\mathbb{R}^2) \quad \text{for } r \in (2,\infty), \\
  u_n \to u \quad \text{for a.e. } x \in \mathbb{R}^2.
\end{cases}
\end{align*}
\]

Note that \( \{u_n\} \subset E_\lambda \) and from Lemma A.1 of [28], there exists \( \sigma(x) \in L^r(\mathbb{R}^2) \) such that
\[
|u_n(x)| \leq \sigma(x), \quad |u(x)| \leq \sigma(x), \quad \text{for } x \in \mathbb{R}^2, \quad n \in \mathbb{N}. \quad \text{(38)}
\]

From this and (2.12), for \( \sigma_1 \in L^2(\mathbb{R}^3) \) and \( \sigma_2 \in L^p(\mathbb{R}^3) \) with \( p \in (4,2^* \mathbb{A}) \), we have
\[
\left( F(x,u_n) - F(x,u) \right) \leq c_1 \frac{1}{2} \left( |u_n|^2 + |u|^2 \right) + c_2 \frac{2}{p} \left( |u_n|^p + |u|^p \right) \leq c_4 \sigma_1^2(x) + \frac{2c_2}{p} \sigma_2^p(x) \in L^1(\mathbb{R}^3).
\]
Recall $w_n = u_n - u$. From (2.11), (2.12), and (29), we have
\[
\int_{\mathbb{R}^2} \tilde{F}(x, w_n)\,dx = \int_{\mathbb{R}^2} \left( \frac{1}{4} f(x, w_n) w_n - \tilde{F}(x, w_n) \right) \cdot dx \leq \int_{\mathbb{R}^2} \left( \frac{3}{4} c_1 |w_n|^2 + \frac{p + 4}{4p} c_2 |w_n|^p \right) \cdot dx \leq \int_{\mathbb{R}^2} \left( 3c_1 \sigma^2(x) + \frac{p + 4}{2p - 2} c_2 \sigma^p_2(x) \right) \,dM \leq \tilde{M},
\]
where $\tilde{M} > 0$.

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} \frac{F(u_n)}{||u_n||^6} \,dx = \lim_{n \to \infty} \frac{1/2 ||u_n||^2 + 1/2 \int_{\mathbb{R}^2} \left( A_1^2(u_n) + A_2^2(u_n) \right) |u_n|^2 \,dx - \mathcal{J}(u_n) - (\mu q) \int_{\Omega} g(x) |u_n|^q \,dx}{||u_n||^6} \leq \lim_{n \to \infty} \frac{(1/2 ||u_n||^2 + (C_0/2) ||u_n||^6 - \mathcal{J}(u_n) + (\mu q) g_{q}^{\prime \prime}(\eta \mathbb{1}_q - 1) ||u_n||^q}{||u_n||^6} = \frac{C_0}{2}.
\]

Note that from the L’Hospital rule and (H3), we have
\[
\lim_{|u| \to \infty} \frac{F(u)}{|u|^q} = +\text{couniformly in } x \in \mathbb{R}^2.
\]

Fatou’s lemma implies that
\[
\lim_{n \to \infty} \int_{\Omega} \frac{F(u_n)}{||u_n||^6} \,dx \geq \lim_{n \to \infty} \int_{\Omega} \frac{F(u_n)}{||u_n||^6} \,dx \geq \lim_{n \to \infty} \int_{\Omega} \frac{F(u_n)}{||u_n||^6} \,dx \cdot \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^6 \,dx \geq \lim_{n \to \infty} \int_{\Omega} \frac{F(u_n)}{||u_n||^6} \,dx \cdot |\nabla \omega(X)| |\nabla u_n|^6 \,dx = +\infty.
\]

This contradicts ((43)), and thus, ((41)) holds.

Proof of Theorem 1. Note that $E_\lambda$ is a Hilbert space, and let \{c_j\} be a total orthonormal basis of $E_\lambda$, and define $X_j = \text{Re}_j$, $Y_j = \sigma_j X_j$, $Z_k = \sigma_j \infty X_j$, $k \in \mathbb{Z}$. From the compact embedding $E_\lambda \subset L^{r}(\mathbb{R}^2)$ with $r \in (2, +\infty)$ and Lemma 3.8 in [28], we have
\[
\beta_k(r) = \sup_{u \in Z_k, ||u||_1 = 1} |u|_r = 0, \quad k \to \infty.
\]
satisfied. Thus, $\mathcal{F}$ possesses an unbounded sequence of critical values $\{u_n\}$ and then (1) possesses infinitely many high energy solutions, i.e.,

$$
\mathcal{F}(u_n) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u_n|^2 + A V(x) u_n^2 + A_1^2(u_n)|u_n|^2 \right) dx + A_2^2(u_n)|u_n|^2 dx - \int_{\mathbb{R}^2} F(u_n) \cdot dx - \frac{\mu}{q} \int_{\mathbb{R}^2} g(x)|u_n|^q dx \to \infty.
$$

(49)

4. Conclusions

In this paper, we use a variational method and critical point theory to study the existence of infinitely many high energy solutions for the generalized Chern-Simons-Schrödinger system (1) with a perturbation. The conditions used in this paper are weaker than the famous Ambrosetti-Rabinowitz condition. Moreover, we consider the effect of the parameters $\lambda, \mu$ on the existence of solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration with all authors. All authors read and approved the final manuscript.

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