COMPARISON CRITERIA FOR DISCRETE FRACTIONAL STURM-LIOUVILLE EQUATIONS

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Abstract. In this study, we give the Sturm comparison theorems for discrete fractional Sturm-Liouville (DFSL) equations within Riemann-Liouville and Gr¨ unwald-Letnikov sense. The emergence of Sturm-Liouville equations began as one dimensional Schr¨ odinger equation in quantum mechanics and one of the most important results is Sturm comparison theorems [27]. These theorems give information about the properties of zeros of two equations having different potentials.

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1. Introduction

Discrete fractional calculus has made great progress in recent years (see [1, 2, 14]. [8, 22, 10]). It is discrete fractional analogue of ODEs, for this reason some problems in ODEs theory have been started to be applied similarly to discrete fractional problems. Generally, this adaptation firstly was made to difference calculus, secondly to fractional calculus and finally to discrete fractional calculus. Fractional sums and differences were gained firstly in Diaz-Osler [23], Miller-Ross [26] and Gray and Zhang [25] and they obtained various types of discrete fractional integrals and derivatives. Later, several authors started to study on fractional difference calculus Goodrich-Peterson [24], Baleanu et.al. [5], Mohan-Deekshitulu [13]. However, discrete fractional calculus is a rather novel area. The main studies has been done by Atici et.al. [6, 7], Anastassiou [14], Abdeljawad et.al. [11, 2, 3, 4] and Cheng et.al. [22] and so forth.

Discrete fractional calculus has still a lot of open problems when compared to ODEs. One of the most important is Sturm-Liouville problems. Discrete fractional Sturm-Liouville (DFSL) equation was firstly considered in [22], and next considered in our work [11] and we showed the fundamental spectral properties, like self-adjointness of the operator, orthogonality of the eigenfunctions, reality of the eigenvalues in [11]. We considered DFSL equations in two different ways:

i) (nabla right and left) Riemann-Liouville (R-L) fractional operator,

\[ L_1 x(t) = \nabla^\mu_{a} (p(t) \nabla^\mu_{b} x(t)) + q(t) x(t) = \lambda r(t) x(t), \] (1)
The emergence of Sturm-Liouville equations began as one dimensional Schrödinger equation in quantum mechanics and one of the most important results is Sturm comparison theorems \cite{27}. These theorems give information about the properties of zeros of two equations having different potentials. The first theorem asserts that

\begin{align*}
  u'' + g(x)u &= 0, \quad (3) \\
  v'' + h(x)v &= 0, \quad (4)
\end{align*}

with $x \in [a, b]$, if $g(x) < h(x)$ there is at least one zero of each solution of $u$ between any two zeros of any nontrivial solution of $v$.

The second theorem asserts that, let $u(x)$ be the solution of (1) with the initial conditions $u(a) = \sin \alpha, \quad u'(a) = -\cos \alpha$, and let $v(x)$ be the solution of (1) with the same initial conditions. Also, let $g(x) < h(x)$ for $x \in [a, b]$. Then if $u(x)$ has $m$ zeros in $a < x < b$, $v(x)$ has not fewer than $m$ zeros in the same interval and the $k$th zero of $v(x)$ is less than the $k$th zero of $u(x)$.

Various discrete versions of this theorem was considered in detailed in \cite{28} and also was studied for infinite case in \cite{9} as

\begin{align*}
  \Delta^2 u_k + p_k u_{k+1} &= 0, \\
  \Delta^2 v_k + p_k v_{k+1} &= 0.
\end{align*}

Besides, fractional Sturm-Liouville differential operators have been studied by \cite{15, 16, 17, 18}. More recently the fractional version of this theorem has been considered in \cite{19} as, $0 < \alpha < 1$

\begin{align*}
  D^\alpha_{a+} p(r) C D^n_{a+} u(r) + g(r) u(r) &= 0, \\
  D^\alpha_{a+} p(r) C D^n_{a+} u(r) + h(r) u(r) &= 0.
\end{align*}

Additionally to these works, a different version of the comparison theorem was considered in \cite{24}.

In this work, discrete fractional versions of these theorems are discussed by the equations (1) and (2).

\section{Preliminaries}

\textbf{Definition 1.} \cite{20} Falling factorial is defined by, $\alpha \in \mathbb{R}$,

$$ t^{\alpha} = \frac{\Gamma(t + 1)}{\Gamma(t - \alpha + 1)}, $$

where $\Gamma$ is the gamma function.

\textbf{Definition 2.} \cite{20} Rising factorial is defined by, $\alpha \in \mathbb{R}$,
Definition 3. [6, 1, 26] Fractional sum operators are defined by,

(i) The nabla left fractional sum of order \( \mu > 0 \) is defined

\[
\nabla_{a}^{-\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\mu-1} x(s), \quad t \in \mathbb{N}_{a+1},
\]

(5)

(ii) The nabla right fractional sum of order \( \mu > 0 \) is defined

\[
b\nabla_{a}^{-\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a+1}^{b-1} (s - \rho(t))^{\mu-1} x(s), \quad t \in b-1\mathbb{N},
\]

(6)

where \( \rho(t) = t - 1 \) is called backward jump operators, \( \mathbb{N}_{a} = \{a, a+1, \ldots\} \), \( \mathbb{N}_{b} = \{b, b-1, \ldots\} \).

Definition 4. [24, 3] Fractional difference operators in Riemann–Liouville (R-L) sense are defined by,

(i) The nabla left fractional difference of order \( \mu > 0 \) is defined

\[
\nabla_{a}^{\mu} x(t) = \nabla_{n} \nabla_{a}^{(n-\mu)} x(t) = \frac{\nabla^{n}}{\Gamma(n-\mu)} \sum_{s=a+1}^{t} (t - \rho(s))^{n-\mu-1} x(s), \quad t \in \mathbb{N}_{a+1},
\]

(7)

(ii) The nabla right fractional difference of order \( \mu > 0 \) is defined

\[
b\nabla^{\mu} x(t) = (-1)^{n} \nabla_{b}^{n} \nabla_{a}^{(n-\mu)} x(t) = \frac{(-1)^{n} \Delta^{n}}{\Gamma(n-\mu)} \sum_{s=a+1}^{t} (s - \rho(t))^{n-\mu-1} x(s), \quad t \in b-1\mathbb{N}.
\]

(8)

Definition 5. [21, 22, 23] Fractional difference operators in Grünwald–Letnikov (G-L) sense are defined by,

(i) The delta left fractional difference of order \( \mu, 0 < \mu \leq 1 \), is defined

\[
\Delta_{-}^{\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t} (-1)^{s} \frac{\mu(\mu-1)\ldots(\mu-s+1)}{s!} x(t-s), \quad t = 1, \ldots, N.
\]

(9)

(ii) The delta right fractional difference of order \( \mu, 0 < \mu \leq 1 \), is defined

\[
\Delta_{+}^{\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=0}^{N-t} (-1)^{s} \frac{\mu(\mu-1)\ldots(\mu-s+1)}{s!} x(t+s), \quad t = 0, \ldots, N-1.
\]

(10)

Definition 6. [3] Integration by parts formula for R-L nabla fractional difference operator is defined by, \( u \) is defined on \( \mathbb{N}_{b} \) and \( v \) is defined on \( \mathbb{N}_{a} \),

\[
\sum_{s=a+1}^{b-1} u(s) \nabla_{a}^{\mu} v(s) = \sum_{s=a+1}^{b-1} v(s) \nabla_{a}^{\mu} u(s).
\]

(11)

Definition 7. [12, 21] Integration by parts formula for G-L delta fractional difference operator is defined by, \( u, v \) is defined on \( \{0, 1, \ldots, n\} \),

\[
\sum_{s=0}^{n} u(s) \Delta_{+}^{\mu} v(s) = \sum_{s=0}^{n} v(s) \Delta_{+}^{\mu} u(s).
\]

(12)
3. Main Results

Let’s consider the DFSL equation firstly in \((\mathbf{R-L})\) sense; \(0 < \mu < 1\)
\[
\nabla^\mu_a \left( p(t)_b \nabla^\mu u(t) \right) + k(t) u(t) = 0, \quad (13)
\]
\[
\nabla^\mu_a \left( p(t)_b \nabla^\mu u(t) \right) + m(t) v(t) = 0. \quad (14)
\]

**Theorem 3.1.** (Sturm’s 1st comparison theorem) Let \(k(t) < m(t)\) over the entire interval \([a, b]\), then between every two zeros of any nontrivial solution of the equation \((11)\) there is at least one zero of every solution of the equation \((11)\).

**Proof.** If we multiply the equation \((11)\) and \((11)\) by \(u(t)\) and \(v(t)\), respectively and subtract from each other, then we have,
\[
v(t) \nabla^\mu_a \left( p(t)_b \nabla^\mu u(t) \right) - u(t) \nabla^\mu_a \left( p(t)_b \nabla^\mu v(t) \right) = [k(t) - m(t)] u(t) v(t) . \quad (15)
\]
Let \(t_1\) and \(t_2\) are two succesive zeros of \(u\). If we take the sum of \((15)\) from \(t_1\) to \(t_2\), then we have
\[
\sum_{s=t_1}^{t_2} v(s) \nabla^\mu_a \left( p(s)_b \nabla^\mu u(s) \right) - \sum_{s=t_1}^{t_2} u(s) \nabla^\mu_a \left( p(s)_b \nabla^\mu v(s) \right) = \sum_{s=t_1}^{t_2} \left[ k(s) - m(s) \right] u(s) v(s),
\]
and if we apply the integration by parts formula, note that integration by parts is slightly different in here but it is seen by interchanging order of summation, then the left hand side of the last equation is zero, so
\[
\sum_{s=t_1}^{t_2} \left[ k(s) - m(s) \right] u(s) v(s) = 0,
\]
this implies that
\[
k(t) = m(t),
\]
and hence, we arrive at a contradiction. The theorem is proved.

**Theorem 3.2.** (Sturm’s 2nd comparison theorem) Let \(k(t) < m(t)\) over the entire interval \([a, b]\), if \(u(t)\) has \(n\) zeros in the interval \((a, b]\), then \(v(t)\) has not less than \(n\) zeros in the same interval and \(k\)th zero of \(v(t)\) is less than \(k\)th zero of \(u(t)\).

**Proof.** Let \(t_1\) is the zero of \(u(t)\) closest to the point \(a\), note that suppose \(u(a) \neq 0\). Let’s prove that \(v(r)\) has at least one zero in the interval \((a, t_1)\). If we multiply the equation \((11)\) and \((11)\) by \(u(t)\) and \(v(t)\), respectively, subtract from each other and take the sum of the result from \(a\) to \(t_1\), then we have by the help of integration by parts
\[
\sum_{s=a}^{t_1} \left[ k(s) - m(s) \right] u(s) v(s) = 0,
\]
this implies that
Let’s consider the DFSL equation secondly in (G-L) sense; \(0 < \mu < 1\)

\[
\Delta^\mu_- \left( \tilde{p}(t) \Delta^\mu_+ \tilde{u}(t) \right) + \tilde{k}(t) \tilde{u}(t) = 0, \tag{16}
\]

\[
\Delta^\mu_+ \left( \tilde{p}(t) \Delta^\mu_- \tilde{v}(t) \right) + \tilde{m}(t) \tilde{v}(t) = 0. \tag{17}
\]

**Theorem 3.3.** (Sturm’s 1st comparison theorem) Let \(\tilde{k}(t) < \tilde{m}(t)\) over the entire interval \([a, b]\), then between every two zeros of any nontrivial solution of the equation \((1)\) there is at least one zero of every solution of the equation \((1)\).

**Proof.** Proof is similar to the proof of Theorem 3.1.

**Theorem 3.4.** (Sturm’s 2nd comparison theorem) Let \(\tilde{k}(t) < \tilde{m}(t)\) over the entire interval \([a, b]\), if \(\tilde{u}(t)\) has \(n\) zeros in the interval \((a, b]\), then \(\tilde{v}(t)\) has not less than \(n\) zeros in the same interval and \(k\)th zero of \(\tilde{v}(t)\) is less than \(k\)th zero of \(\tilde{u}(t)\).

**Proof.** Proof is similar to the proof of Theorem 3.2.

A symbolic graph is given in what follows to clarify the comparison theorem; the zeros of two linearly independent solutions of the Airy equation \(y'' - xy = 0\) alternate, as asserted by the Sturm comparison theorem

4. Conclusion

In this study, we give the Sturm comparison theorems for discrete fractional Sturm-Liouville (DFSL) equations within Riemann-Liouville and Grünwald-Letnikov sense. One of the most important results is Sturm comparison theorems \([27]\). These theorems give information about the properties of zeros of two equations having different potentials. Besides, these theorems will be the basis of ”oscillation theorems”, which has great importance in spectral theory. This study would be benefit for the theory of DFSL equations.

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