Abstract: The quantum version of the Bernstein–Gelfand–Gelfand resolution is used to construct a Dolbeault–Dirac operator on the anti-holomorphic forms of the Heckenberger–Kolb calculus for the $B_2$-irreducible quantum flag manifold. The spectrum and the multiplicities of the eigenvalues of the Dolbeault–Dirac operator are computed. It is shown that this construction yields an equivariant, even, $0^+$-summable spectral triple.

1. Introduction

Soon after the simultaneous and independent emergence of quantum groups and noncommutative geometry in the 1980’s, mathematicians have tried to relate these two theories to each other. In fact, the question of how to reconcile the theory of spectral triples with Drinfeld–Jimbo quantum groups is one of the major open problems in noncommutative geometry. The best-studied example is the spectral triple on the standard Podleś sphere constructed by Dąbrowski and Sitarz [6]. Later it was shown in [26] that this Dirac operator yields a commutator representation of Podleś’ distinguished 2-dimensional covariant first order differential calculus [24]. Furthermore, this spectral triple satisfies most of Connes’ conditions for a noncommutative spin geometry, albeit in a modified manner [25,28]. Most notably, the eigenvalues of the Dirac operator grow exponentially so that the spectral triple has spectral dimension 0.

Other attempts to reconcile Connes’ noncommutative geometry with quantum groups led to equivariant isospectral Dirac operators on all Podleś spheres [4], on quantum SU(2) [5], on the 4-dimensional orthogonal quantum sphere [3], and on all compact quantum groups [22], but none of these equivariant spectral triples has been related to a finite-dimensional covariant differential calculus. Moreover, basic conditions on a noncommutative spin geometry, like the first order condition and the real structure, fail
or can be satisfied only up to smoothing operators, which makes the standard Podleś sphere even more outstanding.

What has become increasingly clear in recent years is that quantum flag manifolds have a major role to play in resolving this problem. These quantum homogeneous spaces, which \( q \)-deform the coordinate rings \( \mathcal{O}(G/L_S) \) of the classical flag manifolds, have a noncommutative geometric structure much closer to the classical situation than quantum groups themselves. For those quantum flag manifolds of irreducible type, Heckenberger and Kolb showed that \( \mathcal{O}_q(G/L_S) \) comes endowed with an essentially unique \( q \)-deformed de Rham complex \([10,11]\) such that the homogeneous components have the same dimension as in the classical case, thus directly extending Podleś’ calculus construction for the standard Podleś sphere.

The existence of such a canonical deformation is one of the most important results in the noncommutative geometry of quantum groups, and has served as a solid base for the construction of Dirac operators for these quantum spaces. The first to notice it was Krähmer. In \([16]\), he introduced an influential algebraic Dirac operator for the irreducible quantum flag manifolds giving a commutator realization of their Heckenberger–Kolb calculi. However, the question of a compact resolvent of the Dirac operator remained unanswered as the computation of the spectrum is a difficult task.

The problem of computing the spectrum was solved only in the particular case of \( \mathcal{O}_q(\mathbb{C}P^2) \) by Dąbrowski, D’Andrea and Landi \([2]\). As for the standard Podleś sphere \( \mathcal{O}_q(\mathbb{C}P^1) \), the spectrum had exponential growth. Then Dąbrowski and D’Andrea proved in \([1]\) that the eigenvalues of the Dolbeault–Dirac operator for any quantum projective space grow exponentially. This was achieved by relating the square of the Dolbeault–Dirac operator to a Casimir operator of the corresponding quantized universal enveloping algebra \( \mathcal{U}_q(g) \) and without explicitly computing the spectrum. Later Krähmer and Tucker-Simmons gave a new procedure for the construction of quantum Clifford algebras and Dolbeault–Dirac operators on quantized symmetric spaces \([17]\). However, Matassa \([18]\) showed that there does not hold in general a Parthasarathy-type formula which relates the square of these operators to a quadratic Casimir. Despite that, Matassa obtained an asymptotic (up to bounded operators) Parthasarathy-type formula for the quantum projective spaces \([19]\) and for the quantum Lagrangian Grassmannian of rank 2 \([21]\), following the construction of Krähmer and Tucker-Simmons and verifying in that way the compact resolvent condition.

In recent years, an alternative approach to the construction of Dolbeault–Dirac operators was initiated in \([23]\), where the notion of a noncommutative Kähler structure was introduced. A covariant Kähler structure for quantum projective space was constructed and shown to be the unique such structure. Later Matassa \([20]\) extended this construction to all the irreducible quantum flag manifolds, for all but a finite number of values of \( q \). The rich structure associated to Kähler structures allows one to construct associated metrics, Hilbert space completions, and to verify all the requirements of a spectral triple with the crucial exception of the compact resolvent condition, the most challenging spectral triple axiom. This question was answered for the case of quantum projective space \( \mathcal{O}_q(\mathbb{C}P^n) \) in \([7]\). However, the proof drew heavily on the multiplicity free nature of the anti-holomorphic forms of quantum projective space and has proved difficult to extend to other examples.

In this paper we adopt yet another approach for the construction of a Dolbeault–Dirac operator related to the Heckenberger–Kolb calculus. It is based on a quantum group version of the Bernstein–Gelfand–Gelfand resolution for quantized irreducible flag manifolds established by Heckenberger and Kolb in \([12,13]\). Taking the locally finite
exponential growth.

of the Dolbeault–Dirac operators of all the irreducible quantum flag manifolds have a Dirac operator. It turns out that, similar to the quantum projective spaces, the eigenvalues of given highest weight occurs as a subrepresentation of the Dolbeault–Dirac operator \( D = \overline{\partial} + \overline{\partial}^\dagger \). The bounded commutator condition follows from the fact that the quantum tangent space acts on the invariant algebra \( \mathcal{O}_q(G/L_S) \) by derivations satisfying the (non-twisted) Leibniz rule.

Computing the spectrum of the \( q \)-deformed Dolbeault–Dirac operator in the general case remains a difficult task. In this paper, we accomplish it for the lowest dimensional irreducible quantum flag manifold of the B-series, i.e., for a quantum version of \( \text{SO}(5)/(\text{SO}(2) \times \text{SO}(3)) \). The difficulties are due to the lack of an explicit description of the Haar state and of a convenient orthonormal basis, e.g. a quantum version of the Gelfand–Tsetlin basis, which only exists for the A-series. We circumvent these problems by expressing the actions of \( \overline{\partial} \) and \( \overline{\partial}^\dagger \) in purely algebraic terms by elements from \( \mathcal{U}_q(\text{so}(5)) \) and working solely on highest weight vectors. By equivariance, these computation determine completely the eigenvalues and multiplicities of the Dolbeault–Dirac operator. It turns out that, similar to the quantum projective spaces, the eigenvalues grow exponentially. This offers strong support for the conjecture that the eigenvalues of the Dolbeault–Dirac operators of all the irreducible quantum flag manifolds have exponential growth.

The paper is organized as follows. In the Sect. 2, we describe the Hopf *-algebras \( \mathcal{U}_q(\text{so}(5)) \) and \( \mathcal{O}(\text{SO}_q(5)) \), the Levi factor \( \mathcal{U}_q(l) \subset \mathcal{U}_q(\text{so}(5)) \), the invariant algebra \( B := \mathcal{O}(\text{SO}_q(5))^{\text{inv} (\mathcal{U}_q(l))} \), the representation theory of \( \mathcal{U}_q(\text{so}(5)) \) and \( \mathcal{U}_q(l) \), and some useful relations of the generators of \( \mathcal{O}(\text{SO}_q(5)) \). In the Sect. 3, we give an explicit description of the anti-holomorphic \( k \)-forms and the \( \overline{\partial} \)-operator by determining the Bernstein–Gelfand–Gelfand resolution for our example. Our strategy to compute the spectrum of the equivariant Dolbeault–Dirac operator is to describe its action on highest weight vectors. As a prerequisite, we study in Sect. 4 the branching rules, i.e., the problem of determining the multiplicity with which an irreducible representation of \( \mathcal{U}_q(l) \) of given highest weight occurs as a subrepresentation of \( \mathcal{U}_q(\text{so}(5)) \). Since in our case the highest weight vectors of \( \mathcal{U}_q(l) \) are given by the kernel of a unique ladder operator, we solve this problem by applying Kostant’s multiplicity formula. Then we present an explicit description of these highest weight vectors in terms of the generators of \( \mathcal{O}(\text{SO}_q(5)) \).

The \( q \)-deformed Dolbeault–Dirac operator \( D = \overline{\partial} + \overline{\partial}^\dagger \) is introduced in Sect. 5. For the definition of the Hilbert space adjoint \( \overline{\partial}^\dagger \) of \( \overline{\partial} \), we need to equip the space of \( \text{anti-holomorphic} \ k \text{-forms} \ \Omega^{(0,k)} \) with an inner product. This is done by using the Haar state to define an inner product on \( \mathcal{O}(\text{SO}_q(5)) \) and considering \( \Omega^{(0,k)}(\text{SO}_q(5)) \) as a subspace of \( \mathcal{O}(\text{SO}_q(5)) \) for \( k = 0, 3 \), and as a subspace of \( \mathcal{O}(\text{SO}_q(5)) \otimes \mathbb{C}^5 \) for \( k = 1, 2 \). Since \( \overline{\partial} \) can be expressed by the left action of elements from the quantum tangent space, \( D \) is equivariant with respect to the right \( \mathcal{U}_q(\text{so}(5)) \)-action on \( \mathcal{O}(\text{SO}_q(5)) \). This allows us to reduce the computation of the spectrum to the action of \( D \) on highest weight vectors. To simplify the computations even more, we establish in Lemma 6 a unitary isomorphism between the space the highest weight vectors in \( \Omega^{(0,k)} \) and the corresponding highest weight vectors in \( \mathcal{O}(\text{SO}_q(5)) \), and consider then the unitarily equivalent action of \( \overline{\partial} \) between subspaces of \( \mathcal{O}(\text{SO}_q(5)) \) with dimension at most 2.
Although taking Hilbert space adjoints with respect to the inner product defined by the Haar state coincides with the involution of $\mathcal{U}_q(\mathfrak{so}(5))$, we cannot simply apply the involution to obtain an expression for $\partial^\dagger$ because these operators don’t leave the subspace in $\mathcal{O}(\mathfrak{SO}_q(5))$ corresponding to $\Omega^{(0,k)}$ invariant. Therefore it is necessary to apply subsequently an orthogonal projection onto the correct subspaces. In order to avoid the computation of inner products with respect to the Haar state, we use the fact that the positive and negative simple root vectors act as ladder operators to give an algebraic description of these projections. From the equivariance of $\partial$ and $\partial^\dagger$, it follows immediately that the highest weight vectors belonging to weight spaces with multiplicity 1 are eigenvectors of the Laplacians $\partial^\dagger_k \partial_k$ and $\partial_k^\dagger \partial_k$. Now the corresponding eigenvalues can be computed directly by applying the unitarily equivalent operators, expressed in terms of elements from $\mathcal{U}_q(\mathfrak{so}(5))$, on the highest weight vectors given by polynomials in the generators of $\mathcal{O}(\mathfrak{SO}_q(5))$. We present a few illustrative cases of these calculations to show our strategy of avoiding the use of complicated commutation rules in $\mathfrak{so}_q(5)$, the other calculations can be found in [8]. For the calculation of the eigenvalues on the 0-forms and on the top forms, we relate the Laplacians to a Casimir operator and employ a Parthasarathy-type formula. The knowledge of the eigenvalues of the Laplacians on the spaces of highest weight vectors with multiplicity 1 suffices to deduce the complete spectrum and the multiplicities of the eigenvalues of Dolbeault–Dirac operator.

Our main result is the last theorem, where we prove that the Dolbeault–Dirac operator on the anti-holomorphic forms yields an equivariant, even, $q^+$-summable spectral triple. We are confident that the techniques and strategies presented in this paper can be used to compute further examples of spectral triples on irreducible quantum flag manifolds. Of course, the long-standing goal is to understand quantization in physics.

2. Preliminaries

We begin by recalling some general facts about the complex simple Lie algebra $\mathfrak{so}(5)$. Since $\mathfrak{so}(5)$ has rank 2, a chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(5)$ is 2-dimensional. Let $\{\alpha_1, \alpha_2\}$ be a set of simple roots corresponding to $\mathfrak{h}$ such that $(\alpha_1, \alpha_1) = 2$ and $(\alpha_2, \alpha_2) = 1$, where $(\cdot, \cdot)$ denotes the symmetric bilinear form on the dual $\mathfrak{h}'$ of $\mathfrak{h}$ induced by the Killing form. This symmetric bilinear form can be deduced from the Cartan matrix of $\mathfrak{so}(5)$:

\[
\begin{pmatrix}
2(\alpha_i, \alpha_j) \\
(\alpha_j, \alpha_j)
\end{pmatrix}
_{i,j=1,2} = \begin{pmatrix}
2 & -2 \\
-1 & 2
\end{pmatrix}.
\]

A basis of fundamental weights $\{\omega_1, \omega_2\} \subset \mathfrak{h}'$ satisfying $\frac{2(\omega_i, \alpha_j)}{\omega_j, \alpha_j} = \delta_{ij}$ (Kronecker delta) is given by $\omega_1 = \alpha_1 + \alpha_2$ and $\omega_2 = \frac{1}{2}(\alpha_1 + 2\alpha_2)$. We will use Cartesian coordinates to describe the weight lattice $P := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, i.e., $(n_1, n_2) \in \mathbb{Z}^2$ stands for $n_1 \omega_1 + n_2 \omega_2 \in P$. In these coordinates, the set of positive roots reads as follows:

\[
R^+ := \{(\alpha_1 = (2, -2), \alpha_2 = (-1, 2), \alpha_1 + \alpha_2 = (1, 0), \alpha_1 + 2\alpha_2 = (0, 2))\}.
\]

We write $P^+ := \mathbb{N}_0 \omega_1 + \mathbb{N}_0 \omega_2 \cong \mathbb{N}_0^2$ for the set of dominant weights. It contains the fundamental Weyl chamber $P^{++} := \mathbb{N}_0 \omega_1 + \mathbb{N}_0 \omega_2$. A weight $\mu \in P$ is said to be higher than $\nu \in P$ if $\mu - \nu \in \mathbb{N}_0 \alpha_1 + \mathbb{N}_0 \alpha_2$.

The Weyl group $W$ is generated the by the reflections $w_\alpha : P \to P$, $\alpha \in R^+$, where $w_\alpha(\mu) := \mu - \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \alpha$. For instance, the reflection $w_{\alpha_2}$ is given by

\[
w_{\alpha_2}(x, y) = (x + y, -y), \quad (x, y) \in P.
\]
Throughout the paper, \( q \) stands for a real number from the interval \((0, 1)\). Set \( q_1 := q \), \( q_2 := \sqrt{q} \), and
\[
[n]_1 := \frac{q^n - q^{-n}}{q_1 - q_1^{-1}} = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_2 := \frac{q^n - q_2^{-n}}{q_2 - q_2^{-1}} = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad n \in \mathbb{R}. \tag{4}
\]
The quantized enveloping algebra \( \mathcal{U}_q(\mathfrak{so}(5)) \) is defined as the complex \(*\)-algebra generated by \( K_1^\pm, K_2^\pm, E_1, E_2, F_1, F_2 \) with relations [15]
\[
K_1 E_1 = q_1^2 E_1 K_1, \quad K_1 E_2 = q_1^{-1} E_2 K_1, \quad K_1 F_1 = q_1^{-2} F_1 K_1, \quad K_1 F_2 = q_1 F_2 K_1.
\]
\[
K_2 E_1 = q_2^{-2} E_1 K_2, \quad K_2 E_2 = q_2^2 E_2 K_2, \quad K_2 F_1 = q_2^2 F_1 K_2, \quad K_2 F_2 = q_2^{-2} F_2 K_2,
\]
\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad E_1 F_2 = F_2 E_1, \quad E_2 F_1 = F_1 E_2, \quad i, j = 1, 2,
\]
\[
E_1 F_1 - F_1 E_1 = \frac{1}{q_1 - q_1^{-1}} (K_1 - K_1^{-1}), \quad E_2 F_2 - F_2 E_2 = \frac{1}{q_2 - q_2^{-1}} (K_2 - K_2^{-1}), \tag{5}
\]
the quantum Serre relations
\[
0 = E_2^3 E_1 - [3]_2 E_2^2 E_1 E_2 + [3]_2 E_2 E_1 E_2^2 - E_1 E_2^3, \quad 0 = E_2^3 E_2 - [2]_1 E_1 E_2 E_1 + E_2 E_1^2,
\]
\[
0 = F_1^3 F_2 - [3]_2 F_1^2 F_2 F_1 + [3]_2 F_1 F_2 F_1^2 - F_1 F_2^3, \quad 0 = F_1^3 F_2 - [2]_1 F_1 F_2 F_1 + F_2 F_1^2, \tag{6}
\]
and involution
\[
K_1^* = K_1, \quad E_1^* = q_1 K_1 F_1, \quad F_1^* = q_1^{-1} E_1 K_1^{-1},
\]
\[
K_2^* = K_2, \quad E_2^* = q_2 K_2 F_2, \quad F_2^* = q_1^{-1} E_2 K_2^{-1}. \tag{7}
\]
It is a Hopf \(*\)-algebra with coproduct \( \Delta \), counit \( \varepsilon \) and antipode \( S \) determined by
\[
\Delta(K_1^\pm) = K_i^\pm \otimes K_1^\pm, \quad \Delta(E_i) = E_i \otimes K_1 + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,
\]
\[
\varepsilon(K_1) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,
\]
\[
S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad i = 1, 2. \tag{8}
\]
Note that \( q_1 \) and \( q_2 \) were chosen in such a way that the powers in the commutation relations of \( K_1 \) and \( E_j \) in (5) coincide with the Cartan matrix (1). Since the Cartan matrix of \( \mathfrak{so}(5) \) is the transpose of the Cartan matrix of \( \mathfrak{sp}(4) \), it follows from the definitions in [15, Section 6] that the \( q \)-deformations of \( \mathcal{U}(\mathfrak{so}(5)) \) and \( \mathcal{U}(\mathfrak{sp}(4)) \) are isomorphic. Applying the \(*\)-isomorphism \( \Phi : \mathcal{U}_q(\mathfrak{sp}(4)) \to \mathcal{U}_q(\mathfrak{so}(5)) \) given by
\[
\Phi(K_1) = K_2^{-1}, \quad \Phi(E_1) = q_2 F_2, \quad \Phi(F_1) = q_2^{-1} E_2,
\]
\[
\Phi(K_2) = K_1^{-1}, \quad \Phi(E_2) = q_2 F_1, \quad \Phi(F_2) = q_2^{-1} E_1,
\]
a central element \( C \) in \( \mathcal{U}_q(\mathfrak{so}(5)) \) can be read off from the quadratic Casimir operator in [21, Proposition 6.6]. Defining
\[
\mathcal{K}_1 := K_2^{-2} K_1^{-1}, \quad \mathcal{K}_2 := K_1^{-1}, \quad \mathcal{F}_1 := F_2, \quad \mathcal{F}_4 := F_1,
\]
\[
\mathcal{F}_2 := F_2^2 F_1 - (1 + q^{-1}) F_2 F_1 F_2 + q^{-1} F_1 F_2^2, \quad \mathcal{F}_3 := F_2 F_1 - q^{-1} F_1 F_2, \tag{9}
\]
we get
\[ C = \frac{1}{(q_2 - q_1^{-1})^2}(q^{-2}K_1^{-1} + q^{-1}K_2^{-1} + qK_2 + q^2K_1) + (q_2^{-3}K_1^{-1} + q_2^2K_2)F_1^*S_1F_1 \\
+ K_1^{-1}F_2^*F_2 + (q_2^{-3}K_1^{-1} + q_2^3K_2^{-1})F_3^*F_3 + q^{-1}[2]_2K_1^{-1}F_4^*F_4 \\
+ (1-q)[2]_2K_1^{-1}(F_1^*F_2 + F_2^*F_3F_1) + (1-q)^2[q]_2^{-5}[2]_2K_1^{-1}F_4^*F_4F_1 \\
+ (q^{-2}q^{-1})[2]_2K_1^{-1}(F_1^*F_2^*F_3 + F_3^*F_4F_1) + (1-q)^2K_1^{-1}F_1^*F_2^*F_3F_1. \] (10)

Another version of a quadratic Casimir operator is obtained by applying the antipode to \( C \). Setting
\[ \mathcal{K}_1 = K_2^{-2}K_1^{-1}, \quad \mathcal{K}_2 = K_1^{-1}, \quad \mathcal{E}_1 = E_2, \quad \mathcal{E}_4 = E_1, \]
\[ \mathcal{E}_2 = E_1E_2^2 - (1 + q)E_2E_1E_2 + qE_2^2E_1, \quad \mathcal{E}_3 = E_1E_2 - qE_2E_1. \] (11)

the central element \( S(C) \) can be written as
\[ S(C) = \frac{1}{(q_2 - q_1^{-1})^2}(q^{-2}K_1 + q^{-1}K_2 + qK_2 + q^2K_1^{-1}) + q^{-1}(q_2^{-3}K_1^{-1} + q_2^3K_2^{-1})E_1E_1 \\
+ q^{-4}K_2E_2 + q^{-3}(q_2^{-3}K_1^{-1} + q_2^3K_2^{-1})E_3 + q^{-1}[2]_2K_3E_4E_4 \\
- (q^{-4} - q^{-3})[2]_2K_1(e_1^*e_2^* + e_2^*e_3^*e_3^*e_1) + (q^{-2} - q^{-1})^2q_2[2]_2K_1e_1^*e_4^*e_4e_1 \\
- (q^{-4} - q^{-3})[2]_2K_1(e_1^*e_2^* + e_2^*e_3^*e_4^*e_1) + (q^{-2} - q^{-1})^2K_2^*e_3^*e_3^*e_3^*e_1. \] (12)

Let \( \mathcal{O}(\mathfrak{so}(5)) \) denote the dual Hopf algebra of \( \mathcal{O}_q(\mathfrak{so}(5)) \) [15, Section 1.2.8]. Then \( \mathcal{O}(\mathfrak{so}(5)) \subset \mathcal{O}_q(\mathfrak{so}(5)) \) is the Hopf subalgebra generated by the matrix coefficients of the 5-dimensional irreducible \( \mathcal{O}_q(\mathfrak{so}(5)) \)-representation of highest weight \( (1, 0) \). Denoting the matrix coefficients by \( u_{ij}^k, i, j = 1, \ldots, 5 \), they satisfy the relations
\[ \sum_{k,l=1,\ldots,5} R_{kl}^{ij} u_{jm}^k u_{in}^j = \sum_{k,l=1,\ldots,5} u_{im}^k u_{jn}^l R_{mn}^{lk}, \quad i, j, m, n = 1, \ldots, 5, \quad D_q = 1, \]
\[ \sum_{i,j,k=1,\ldots,5} C_{ij}^k (C^{-1})_m^j u_{im}^k u_{jn}^j = \sum_{i,j,k=1,\ldots,5} C_{ij}^k (C^{-1})_m^j u_{im}^k u_{jn}^j = \delta_{nm}, \quad m, n = 1, \ldots, 5, \] (13)

with the R-matrix coefficients \( R_{kl}^{ij} \in \mathbb{R} \), the C-matrix coefficients \( C_{ij}^k \in \mathbb{R} \) and the quantum determinant \( D_q \in \text{alg}[u_{ij}^k : j, k = 1, \ldots, 5] \) given in [15, Section 9.3], and \( \delta_{nm} \) denotes the Kronecker delta. \( \mathcal{O}(\mathfrak{so}(5)) \) is a Hopf \(*\)-algebra, where the structure maps are determined by
\[ \Delta(u_{ij}^k) = \sum_i u_{ij}^k \otimes u_{ij}^l, \quad \varepsilon(u_{ij}^k) = \delta_{kl}, \quad S(u_{ij}^k) = \sum_i C_{ij}^k (C^{-1})_l^j u_{im}^k, \quad u_{ij}^{kk*} = S(u_{ij}^k). \]

There is a left action \( \mathcal{O}_q(\mathfrak{so}(5)) \otimes \mathcal{O}(\mathfrak{so}(5)) \ni X \otimes a \mapsto X \triangleright a \in \mathcal{O}(\mathfrak{so}(5)) \) and a right action \( \mathcal{O}(\mathfrak{so}(5)) \otimes \mathcal{O}_q(\mathfrak{so}(5)) \ni a \otimes X \mapsto a \triangleleft X \in \mathcal{O}(\mathfrak{so}(5)) \) such that \( \mathcal{O}(\mathfrak{so}(5)) \) becomes a left and a right \( \mathcal{O}_q(\mathfrak{so}(5)) \)-module \(*\)-algebra. Here, module \(*\)-algebra means that these actions satisfy
\[ X \triangleright (ab) = (X(1) \triangleright a)(X(2) \triangleright b), \quad (X \triangleright a)^* = S(X)^* \triangleright a^*, \]
\[ (ab) \triangleleft X = (a \triangleleft X(1))(b \triangleleft X(2)), \quad (a \triangleleft X)^* = a^* \triangleleft S(X)^* \] (14)
for all \( a, b \in \mathcal{O}(SO_q(5)) \) and \( X \in \mathcal{U}_q(so(5)) \). We employ Sweedler’s notation for co-products, i.e., \( \Delta(X) = X_{(1)} \otimes X_{(2)} \). As a consequence of (14), it suffices to specify the actions on generators, which are given as follows:

\[
\begin{align*}
K_1 &\triangleright u_1^k = q^{-1}u_1^k, \quad K_1 \triangleright u_2^k = q u_2^k, \quad K_1 \triangleright u_3^k = u_3^k, \quad K_1 \triangleright u_4^k = q^{-1}u_4^k, \quad K_1 \triangleright u_5^k = qu_5^k, \\
E_1 &\triangleright u_1^k = u_1^k, \quad E_1 \triangleright u_4^k = -u_4^k, \quad F_1 \triangleright u_2^k = u_1^k, \quad F_1 \triangleright u_5^k = -u_4^k, \\
K_2 &\triangleright u_1^k = u_1^k, \quad K_2 \triangleright u_2^k = q^{-1}u_2^k, \quad K_2 \triangleright u_3^k = u_3^k, \quad K_2 \triangleright u_4^k = qu_4^k, \quad K_2 \triangleright u_5^k = u_5^k, \\
E_2 &\triangleright u_2^k = [2]^{1/2}u_2^k, \quad E_2 \triangleright u_5^k = -q_2[2]^{1/2}u_4^k, \\
F_2 &\triangleright u_3^k = [2]^{1/2}u_2^k, \quad F_2 \triangleright u_4^k = -q_2^{-1}[2]^{1/2}u_5^k,
\end{align*}
\]

(15)

for \( k = 1, \ldots, 5 \), and zero in all other cases.

The Levi factor \( \mathcal{U}_q(1) \subset \mathcal{U}_q(so(5)) \) is the Hopf-*-subalgebra generated by \( K_1^{\pm 1}, K_2^{\pm 1}, E_2, F_2 \). The subalgebra \( B \subset \mathcal{O}(SO_q(5)) \) of \( \mathcal{U}_q(1) \)-invariant elements, i.e.,

\[
B := \mathcal{O}(SO_q(5))^{inv(\mathcal{U}_q(1))} = \{ a \in \mathcal{O}(SO_q(5)) : X \triangleright a = \varepsilon(X) a \text{ for all } X \in \mathcal{U}_q(1) \},
\]

(17)

may be viewed as the polynomial coordinate ring of the quantum quadratic \( SO(5)/(SO(2) \times SO(3)) \).

It is known that \( \mathcal{O}(SO_q(5)) \) admits a Haar state, that is, a positive, faithful and \( \mathcal{U}_q(so(5)) \)-invariant state

\[
h : \mathcal{O}(SO_q(5)) \rightarrow \mathbb{C},
\]

(18)

where \( \mathcal{U}_q(so(5)) \)-invariance means that

\[
h(X \triangleright a) = h(a \triangleright X) = \varepsilon(X) h(a), \quad X \in \mathcal{U}_q(so(5)), \quad a \in \mathcal{O}(SO_q(5)).
\]

(19)

As \( h \) is positive and faithful, it equips \( \mathcal{O}(SO_q(5)) \) with an inner product by setting

\[
\langle a, b \rangle_h := h(a^* b), \quad a, b \in \mathcal{O}(SO_q(5)).
\]

(20)

The left regular representation \( \pi_L \) and the right regular representation \( \pi_R \) of \( \mathcal{U}_q(so(5)) \) on \( \mathcal{O}(SO_q(5)) \) are defined by

\[
\pi_L(X)(a) := X \triangleright a, \quad \pi_R(X)(a) := a \triangleleft S^{-1}(X), \quad X \in \mathcal{U}_q(so(5)), \quad a \in \mathcal{O}(SO_q(5)).
\]

(21)

These are *-representations with respect to the inner product defined in (20). To see this, let us show the compatibility of \( \pi_R \) with the involution. Using (14) and (19), we compute for \( a, b \in \mathcal{O}(SO_q(5)) \) and \( X \in \mathcal{U}_q(so(5)) \) that

\[
\langle \pi_R(X)(a), b \rangle_h
= h((a^* \triangleright X^*)b) = \varepsilon(S^{-1}(X^*)) h((a^* \triangleright X^*)b)
\]

\[
= h( ((a^* \triangleright X^*)b) \triangleleft S^{-1}(X^*) ) = h((a^* \triangleright X^*) S^{-1}(X^*) (b \triangleleft S^{-1}(X^*)))
\]

\[
= h(a^* (b \triangleleft S^{-1}(X^* \varepsilon(X^*)))) = h(a^* (b \triangleleft S^{-1}(X^*))) = \langle a, \pi_R(X)(b) \rangle_h.
\]

(22)
By a slight abuse of notation, we will omit the symbol $\pi_L$ and either write $X(a)$ or continue to use the notation $X \triangleright a$ for the left regular representation.

A weight $(n, l) \in \mathbb{Z} \times \mathbb{N}_0$ defines an irreducible representation of the Levi factor $U_q(l)$ on a space $V^{(n,l)} := \text{span}\{v_{(n,l) - j(-1,2)} : j = 0, 1, \ldots, l\}$ of dimension $l + 1$ by setting

$$K_1 v_{(\mu_1, \mu_2)} = q_1^{\mu_1} v_{(\mu_1, \mu_2)}, \quad K_2 v_{(\mu_1, \mu_2)} = q_2^{\mu_2} v_{(\mu_1, \mu_2)}, \quad E_2 v_{(\mu_1, \mu_2)} = v_{(\mu_1 - 1, \mu_2 + 2)},$$

$$F_2 v_{(\mu_1, \mu_2)} = \left[\frac{1}{2} (l + \mu_2) \right]_2 \left[\frac{1}{2} (l - \mu_2) + 1\right]_2 v_{(\mu_1 + 1, \mu_2 - 2)}. \quad (23)$$

For instance, on $V^{(n,2)} = \text{span}\{v^{(-1)} := v_{(n+2, -2)}, \ v^0 := v_{(n+1, 0)}, \ v^1 := v_{(n, 2)}\}$, we have

$$K_1 v^j = q^{n-j+1} v^j, \quad K_2 v^j = q^j v^j, \quad E_2 v^j = v^{j+1}, \quad F_2 v^j = [2]_2 v^{j-1}, \quad (24)$$

and on $V^{(n,0)} = \text{span}\{u^0 := v_{(n, 0)}\}$, we get

$$K_1 u^0 = q^n u^0, \quad K_2 u^0 = u^0, \quad E_2 u^0 = 0, \quad F_2 u^0 = 0. \quad (25)$$

For $(n, l) \in \mathbb{Z} \times \mathbb{N}_0$ and $(\mu_1, \mu_2) \in \mathbb{Z} \times \mathbb{Z}$, let $V_{(\mu_1, \mu_2)}^{(n,l)}$ denote the subspace of vectors of weight $(\mu_1, \mu_2)$ in $V^{(n,l)}$, i.e.,

$$V_{(\mu_1, \mu_2)}^{(n,l)} := \{ v \in V^{(n,l)} : K_i v = q_i^{\mu_i} v, \ i = 1, 2\}.$$ 

Suppose that $v_{(n,2)}^{(n-1,4)} \in V_{(n,2)}^{(n-1,4)} \subseteq V^{(n,2)}$. Then $F_2 E_2 (v_{(n,2)}^{(n-1,4)}) = 0$ and $F_2 E_2 (v_{(n,2)}^{(n-1,4)}) = [4]_2 v_{(n,2)}^{(n-1,4)}$ by (23), thus

$$\text{pr}_{(n,2)} := (1 - \frac{1}{[4]_2})_2 F_2 E_2 : V_{(n,2)}^{(n,2)} \oplus V_{(n,2)}^{(n-1,4)} \rightarrow V_{(n,2)}^{(n,2)}, \quad (26)$$

$$\text{pr}_{(n,2)}(v_{(n,2)}^{(n,2)} \oplus v_{(n,2)}^{(n-1,4)}) = v_{(n,2)}^{(n,2)},$$

agrees with the (orthogonal) projection onto the first component.

Similarly, elementary calculations using (23) show that the orthogonal projection

$$\text{pr}_{(n,0)} : V_{(n,0)}^{(n,0)} \oplus V_{(n,0)}^{(n-1,2)} \oplus V_{(n,0)}^{(n-2,4)} \rightarrow V_{(n,0)}^{(n,0)}$$

satisfying

$$\text{pr}_{(n,0)}(v_{(n,0)}^{(n,0)} \oplus v_{(n,0)}^{(n-1,2)} \oplus v_{(n,0)}^{(n-2,4)}) = v_{(n,0)}^{(n,0)}$$

can be given by

$$\text{pr}_{(n,0)} = (1 - \frac{1}{[2]_2})_2 F_2 E_2 + \frac{1}{[2]_2^3[3]_2} F_2^2 E_2^2 = (1 - \frac{1}{[2]_2})_2 E_2 F_2 + \frac{1}{[2]_2^3[3]_2} E_2^2 F_2^2. \quad (27)$$

If $a \in B$ and $x \in \mathcal{O}(SO_q(5))$ belongs to a highest weight representation $V_\mu^\lambda$ with respect to $\pi_L$ from (21), then it follows from $E_2(a) = F_2(a) = 0$ and $K_1(a) = K_2(a) = a$ that $ax$ and $xa$ belong to a representation space of the same weights. Moreover, using the explicit expression in (26) for $\text{pr}_{(n,2)}$, a straightforward computation shows that

$$\text{pr}_{(n,2)}(avb) = a \text{pr}_{(n,2)}(v) b, \quad v \in V_{(n,2)}^{(n,2)} \oplus V_{(n,2)}^{(n-1,4)} \subset \mathcal{O}(SO_q(5)), \quad a, b \in B. \quad (28)$$

Here, $v$ can be any element in $\mathcal{O}(SO_q(5))$ of weight $(n, 2)$ which lies in the direct sum of (at most) two representations of the Levi factor $U_q(l)$ with corresponding highest weights $(n, 2)$ and $(n - 1, 4)$, where the action is given by the first equation in (21).

For the convenience of the reader, we finish this section with a lemma that collects those relations of the generators of $\mathcal{O}(SO_q(5))$ which are most frequently used in this paper.
Lemma 1. The generators $u^i_j$, $i, j = 1, \ldots, 5$, of $O(\text{SO}_q(5))$ satisfy the following relations:

\begin{align*}
u^i_1 u^j_5 &= q^2 u^i_2 u^j_1, \quad i \neq 3, \quad u^i_1 u^j_k &= q u^i_k u^j_1, \quad l < k, l \neq k', i \neq 3, \quad (29) \\
u^i_1 u^j_k &= u^i_k u^j_1, \quad i < j, \quad l < k, \quad i \neq j', \quad k \neq l', \quad (30) \\
u^i_2 u^j_1 &= q^{-1} u^i_4 u^j_5, \quad u^i_1 u^j_5 = q u^i_2 u^j_1 + (q^2 - 1) u^i_2 u^j_1 i < j, \quad i \neq j', \quad (31) \\
u^i_1 u^j_1 &= u^i_1 u^j_1 - (q^{-1} - q) u^i_k u^j_1, \quad i < j, \quad k < l, \quad i \neq j', \quad k \neq l', \quad (32) \\
u^i_1 u^j_2 &= q^{-2} u^i_2 u^j_4 + (q^{-1} - q^{-1}) u^i_1 u^j_5, \quad i \neq 3, \quad (33) \\
u^i_1 u^j_3 &= -2 q^{-1} u^i_2 u^j_4 - 2 q^{-2} u^i_1 u^j_5, \quad i \neq 3, \quad (34) \\
u^i_1 u^j_3 &= u^i_3 u^j_1 + (q^{-1} - q) q^{-1/2} u^i_2 u^j_4 + (q^{-1} - q) q^{-1/2} u^i_1 u^j_5, \quad (35) \\
u^i_1 u^j_1 + (q - q^{-1}) u^i_3 u^j_5 &= u^i_1 u^j_3 + (q^{-1} - q) q^{-1/2} u^i_2 u^j_4 + (q^{-1} - q) q^{-1/2} u^i_1 u^j_5, \quad (36) \\
u^i_1 u^j_2 - q u^i_2 u^j_4 &= (q^{1/2} - q^{-1/2})(u^i_2 u^j_4 - q u^i_2 u^j_4) + (q^{-1/2} - q^{1/2})(u^i_1 u^j_5 - q u^i_1 u^j_5), \quad (37) \\
u^i_1 u^j_2 &= q u^i_1 u^j_k \quad k \neq 3, \quad i < j, \quad i \neq j', \quad (38) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (39) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^{-1} (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (40) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (41) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (42) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (43) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^{-2} (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (44) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^{-2} (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (45) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (46) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (47) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (48) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3 - q u^i_2 u^2_3) (u^i_1 u^2_3), \quad (49) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3) (u^i_1 u^2_3), \quad (50) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^{-2} (u^i_1 u^2_3 - q u^i_2 u^2_3) (u^i_1 u^2_3), \quad (51) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (52) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (53) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (54) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= q^2 (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (55) \\
u^i_1 (u^1_1 u^2_3 - q u^1_2 u^2_3) &= (u^i_1 u^2_3 - q u^i_2 u^2_3) u^i_1, \quad (56)
\end{align*}

where $a = 2, 3, 4$.

Proof. Equations (29)–(36) follow directly from the R-matrix relations (13). Equation (37) is obtained by subtracting $q$ times (35) from (36) and then dividing both sides by $-(q^{-1} + 1)$. Choosing $k = l + 1 \in \{2, 5\}$ in (30) and acting on both sides by $E_1$ or
$F_1$ from the left shows (38). The proven relations imply (39), (40), (49) and (50) by elementary computations. For instance, (29), (31) and (38) give
\[
 u^1_1(u^1_1u^2_2 - qu^7_1u^2_2) = u^1_1(qu^2_2u^1_1 - q^{-1}u^7_1u^1_1) = q(u^1_1u^2_2 - qu^7_1u^2_2)u^1_1,
\]
which yields (39).

Note that, if we prove one of the relations (41)–(47) and (49)–(56) for $a = 2$, then the validity of this equation for $a = 3$ and $a = 4$ follows by acting on both sides with $E_2$ from the left. Similarly, if one of these relations holds for $a = 4$, then acting on both sides with $F_2$ from the left yields the result for $a = 3$ and $a = 2$.

From (39), we obtain (41) for $a = 4$ by acting on both sides from the left by $F_1$. Analogously, we get (42) for $a = 2$ from (40) by acting on both sides by $E_1$. Next, using (29)–(32), we compute that
\[
u^2_3(u^1_1u^2_2 - qu^7_1u^2_2) = q^{-1}(u^1_1u^2_2 + (q - q^{-1})u^7_1u^2_2)u^1_3 - q^{-1}u^2_2u^2_2u^1_4
= q^{-1}(u^1_1u^2_2 + (q - q^{-1})u^7_1u^2_2)u^1_3 - q^{-1}u^2_2(u^4_2u^2_2 + (q - q^{-1})u^5_2u^2_2)
= q^{-2}(u^1_1u^2_2 - qu^7_1u^2_2)u^2_3.
\]
This shows (45) for $a = 4$. The proofs of (43), (44) and (46) are similar. Much in the same way, (47) follows from
\[
u^2_3(u^1_1u^2_2 - qu^7_1u^2_2) = \nu^1_4u^2_5 + (q^{-1} - q)u^1_1u^2_2u^1_3 - u^2_2u^2_2u^1_4
= u^1_4(u^2_3u^2_3 - qu^7_3u^2_3) + q^{-1}(u^1_1u^2_2 - qu^7_1u^2_2)u^1_5
= q^2(u^1_4u^2_3 - qu^7_4u^2_3)u^1_1 + q^{-1}(u^1_1u^2_2 - qu^7_1u^2_2)u^1_5,
\]
where we applied (29) and (32) in first equality, and (43) in the third.

Equations (39)–(44) imply immediately (48), (51) and (52). The commutation relations of $u^4_2$ with $u^1_1u^2_2 - qu^7_1u^2_2$ and $u^1_1u^2_2 - qu^7_1u^2_2$ can easily be deduced from (29)–(31) and (38), and the commutation relations of $u^1_1$, $u^1_3$ and $u^2_3$ with $u^1_1u^2_2 - qu^7_1u^2_2$ and $u^1_1u^2_2 - qu^7_1u^2_2$ are shown in (41)–(46). Combining these results yields (53) and (54).

Acting on both sides of (40) from the left by $F_1$ gives
\[
u^1_4(u^1_1u^2_2 - qu^7_2u^2_2) + u^1_5(u^1_1u^2_2 - qu^7_1u^2_2) = (u^1_1u^2_2 - qu^7_1u^2_2)u^1_5 + q^{-1}(u^1_1u^2_2 - qu^7_1u^2_2)u^1_5.
\]
Therefore, by (44),
\[
u^1_4(u^1_1u^2_2 - qu^7_2u^2_2) = q^{-1}(u^1_1u^2_2 - qu^7_1u^2_2)u^1_4 - (1 - q^2)u^2_2(u^1_1u^2_2 - qu^7_1u^2_2).
\]
Multiplying this relation by $u^1_1$ from the left and applying (39) yields (55) for $a = 4$. Equation (56) can be proven in the same way by acting with $E_1$ on (39). $\square$

3. Differential Calculus from a Quantum Bernstein–Gelfand–Gelfand Resolution

Let $M^{(0, 0)} := \mathbb{C}$ and
\[
M^{(0, 2)} := \text{span}\{\nu^{-1}, \nu^0, \nu^1\}, \quad M^{(1, 2)} := \text{span}\{\omega^{-1}, \omega^0, \omega^1\}, \quad M^{(3, 0)} := \mathbb{C} \nu^0
\]
be left $\mathcal{U}_q(l)$-modules with the left actions described in (24) and (25). From [12] and [13], we conclude that there exists an exact sequence of left $\mathcal{U}_q(so(5))$-modules

$$0 \leftarrow \mathbb{C} \leftarrow E_0 \leftarrow E_1 \leftarrow E_2 \leftarrow E_3 \leftarrow \mathbb{C}$$

and that the $\mathcal{U}_q(so(5))$-module maps $\varphi_0, \varphi_1$ and $\varphi_2$ are unique up to a non-zero scaling factor. Our aim is to find explicit expressions for these maps, which en passant proves the existence of the exact sequence (57) in our setting.

For shortness of notation, we write $x \otimes y$ for $x \otimes \mathcal{U}_q(l) y$ and set

$$X_{-1} := E_1, \quad X_0 := E_2 E_1, \quad X_1 := E_2^2 E_1. \quad (58)$$

These elements are known to generate the (anti-)holomorphic quantum tangent space of the invariant algebra $B$, see [11]. By equivariance and preservation of weights, we deduce that

$$\varphi_2(1 \otimes v^0) = \alpha_{-1} X_1 \otimes \omega^{-1} + \alpha_0 X_0 \otimes \omega^0 + \alpha_1 X_{-1} \otimes \omega^1.$$ 

To determine the constants, we apply $E_2$ to the highest weight vector and obtain

$$0 = \varphi_2(1 \otimes E_2 v^0) = E_2 \varphi_2(1 \otimes v^0) = \alpha_{-1} E_2^3 E_1 \otimes \omega^{-1} + \alpha_0 E_2^2 E_1 \otimes \omega^0 + \alpha_1 E_2 E_1 \otimes \omega^1$$

$$= \alpha_{-1} ([3]_2 E_2^3 E_1 \otimes E_2^2 \omega^{-1} - [3]_2 E_2 E_1 \otimes E_2^2 \omega^{-1}) + \alpha_0 E_2^2 E_1 \otimes \omega^0 + \alpha_1 E_2 E_1 \otimes \omega^1$$

$$= (\alpha_0 + \alpha_{-1} [3]_2) E_2^3 E_1 \otimes \omega^0 + (\alpha_1 - \alpha_{-1} [3]_2) E_2 E_1 \otimes \omega^1,$$

where we used the quantum Serre relation (6) and the third equation in (24) together with $E_2^3 \omega^{-1} = 0$. Setting $\alpha_1 := 1$, we get $\alpha_0 = -1$ and $\alpha_{-1} = [3]_2^{-1}$, thus

$$\varphi_2(Z \otimes v^0) = \frac{1}{[3]_2} ZX_{-1} \otimes \omega^1 - ZX_0 \otimes \omega^0 + ZX_{-1} \otimes \omega^1, \quad Z \in \mathcal{U}_q(so(5)). \quad (59)$$

Similarly, to find an expression for $\varphi_1$, we make the ansatz

$$\varphi_1(1 \otimes \omega^1) = \beta_0 X_1 \otimes v^0 + \beta_1 X_0 \otimes v^1$$

and apply $E_2$ to the highest weight vector $\omega^1$ obtaining

$$0 = \varphi_1(1 \otimes E_2 \omega^1) = \beta_0 E_2^3 E_1 \otimes \omega^0 + \beta_1 E_2^2 E_1 \otimes \omega^1 = (\beta_1 + [3]_2 \beta_0) E_2^2 E_1 \otimes \omega^1.$$ 

Setting $\beta_1 := [3]_2$ and $\beta_0 := -1$ gives

$$\varphi_1(Z \otimes \omega^1) = -ZX_{-1} \otimes v^0 + [3]_2 ZX_0 \otimes v^1, \quad Z \in \mathcal{U}_q(so(5)). \quad (60)$$
Using the equivariance of $\varphi_1$ and the defining relations of $U_q(so(5))$, we compute

\[
\varphi_1(1 \otimes \omega^0) = \frac{1}{[2]_2} \varphi_1(1 \otimes F_2 \omega^1) = \frac{-1}{[2]_2} F_2 E_2^2 E_1 \otimes \omega^0 + \frac{[3]_2}{[2]_2} F_2 E_2 E_1 \otimes \nu^1
\]

\[
= \frac{-1}{[2]_2} \left( E_2^2 E_1 \otimes F_2 \omega^0 - E_2 E_1 \otimes \frac{K_2 - K_2^{-1}}{q_2 - q_2^{-1}} \nu^0 - E_2 E_1 \otimes \frac{q^{-1} K_2 - q K_2^{-1}}{q_2 - q_2^{-1}} \nu^1 \right)
\]

\[
+ \frac{[3]_2}{[2]_2} \left( E_2 E_1 \otimes F_2 \nu^1 - E_1 \otimes \frac{q^{-1} K_2 - q K_2^{-1}}{q_2 - q_2^{-1}} \nu^1 \right)
\]

\[
= -E_2^2 E_1 \otimes \nu^{-1} + [2]_1 E_2 E_1 \otimes \nu^0,
\]

where we used (24) and $[3]_2 - 1 = q + q^{-1} = [2]_1$. Hence

\[
\varphi_1(Z \otimes \omega^0) = -ZX_1 \otimes \nu^{-1} + [2]_1 Z X_0 \otimes \nu^0, \quad Z \in U_q(so(5)). \tag{61}
\]

Analogously,

\[
\varphi_1(1 \otimes \omega^{-1}) = \frac{1}{[2]_2} \varphi_1(1 \otimes F_2 \omega^0) = -\frac{1}{[2]_2} F_2 E_2^2 E_1 \otimes \nu^{-1} + \frac{[2]_1}{[2]_2} F_2 E_2 E_1 \otimes \nu^0
\]

\[= (-1 - \frac{[4]_2}{[2]_2} + [2]_1) E_2 E_1 \otimes \nu^{-1} + [2]_1 E_1 \otimes \nu^0.
\]

Since $[2]_1 - \frac{[4]_2}{[2]_2} = q + q^{-1} - \frac{q^2 - q^{-2}}{q - q^{-1}} = 0$, we obtain

\[
\varphi_1(Z \otimes \omega^{-1}) = -ZX_0 \otimes \nu^{-1} + [2]_1 Z X_1 \otimes \nu^0, \quad Z \in U_q(so(5)). \tag{62}
\]

Finally, we set

\[
\varphi_0(1 \otimes \nu^1) := X_1 \otimes 1
\]

since $1 \otimes \nu^1 \in U_q(so(5)) \otimes_{U_q(i)} M^{(0,2)}(\mathcal{O}, \mathcal{O})$ and $1 \otimes 1 \in U_q(so(5)) \otimes_{U_q(i)} M^{(0,0)}(\mathcal{O}, \mathcal{O})$ are both highest weight vectors of irreducible $U_q(i)$-modules with the same weights. In the latter case, this follows from the quantum Serre relation (6). Furthermore,

\[
E_2(X_0 \otimes 1) = X_1 \otimes 1 = \varphi_0(1 \otimes \nu^1) = \varphi_0(1 \otimes E_2 \nu^0) = E_2 \varphi_0(1 \otimes \nu^0),
\]

\[
E_2(X_{-1} \otimes 1) = X_0 \otimes 1 = \varphi_0(1 \otimes \nu^0) = \varphi_0(1 \otimes E_2 \nu^{-1}) = E_2 \varphi_0(1 \otimes \nu^{-1}),
\]

from which we conclude that, for all $Z_{-1}, Z_0, Z_1 \in U_q(so(5))$,

\[
\varphi_0(Z_{-1} \otimes \nu^{-1} + Z_0 \otimes \nu^0 + Z_1 \otimes \nu^1) = (Z_{-1} X_{-1} + Z_0 X_0 + Z_1 X_1) \otimes 1. \tag{63}
\]

One can now verify directly that $\varphi_{j-1} \circ \varphi_j = 0$. Moreover, $(\epsilon \otimes \text{id}) \circ \varphi_0 = 0$ which proves that (57) is a complex. In fact, by [12, Section 3.4], it is exact.

As in [13], we obtain a differential calculus over $B = \mathcal{O}(SO_q(5))^{inv(U_q(i))}$ by considering the locally finite dual of the complex (57). First recall that the dual $M'$ of a left $U$-module $M$ becomes a right $U$-module and the dual $N'$ of a right $U$-module $N$ becomes a left $U$-module by setting

\[
(f \cdot \circ X)(m) := f(Xm) \quad \text{and} \quad (X \cdot \circ g)(n) := g(nX) \tag{64}
\]
respectively, where \( m \in M, \ f \in M', n \in N, g \in N' \) and \( X \in \mathcal{U} \). For \( k = 0, \ldots, 3 \), define
\[
\Omega^{(0,k)} := \left\{ f \in (\mathcal{U}_q(\mathfrak{so}(5)) \otimes \mathcal{U}_q(1)) M^\lambda \right\} : \dim (f \ll \mathcal{U}_q(\mathfrak{so}(5))) < \infty, \]
where
\[
\lambda_0 := (0, 0), \quad \lambda_1 := (0, 2), \quad \lambda_2 := (1, 2), \quad \lambda_3 := (3, 0). \tag{66}
\]
The finiteness condition in (65) is what is referred to as the locally finite part. From [13, Lemma 6.4], it follows that
\[
\Omega^{(0,k)} \subset \mathcal{O}(\mathfrak{so}(5)) \otimes M^{\lambda_k'}, \quad k = 0, 1, 2, 3, \tag{67}
\]
and
\[
\Omega^{(0,0)} \cong \{ b \in \mathcal{O}(\mathfrak{so}(5)) : X \triangleright b = \varepsilon(X) b \text{ for all } X \in \mathcal{U}_q(1) = B \}. \tag{68}
\]
Let \( \{v_0\}, \{\omega_1, \omega_0, \omega_1\} \) and \( \{u_1, v_0, v_1\} \) denote the dual bases of \( \{v^0\}, \{\omega^{-1}, \omega^0, \omega^1\} \) and \( \{u^{-1}, v^0, v^1\} \), respectively. Then, with the left and right action from (64),
\[
\sum_{j=-1,0,1} a_j \otimes v_j \in \mathcal{O}(\mathfrak{so}(5)) \otimes M^{\lambda_k'} \text{ belongs to } \Omega^{(0,k)} \text{ if and only if the equation}
\]
\[
\sum_{j=-1,0,1} (X \triangleright a_j) \otimes v_j = \sum_{j=-1,0,1} a_j \otimes (v_j \ll X) \text{ holds. From (24) and (64), we get}
\]
\[
v_j \ll K_1 = q^{-1} v_j, \quad v_j \ll K_2 = q^j v_j, \quad v_j \ll E_2 = v_{j-1} \text{ and } v_j \ll F_2 = [2]_2 v_{j+1}, \text{ therefore}
\]
\[
\Omega^{(0,1)} = \left\{ \sum_{j=-1}^1 a_j \otimes v_j \in \mathcal{O}(\mathfrak{so}(5)) \otimes M^{(0,2)'} : \begin{aligned}
K_1 \triangleright a_j &= q^{-1} a_j, \quad K_2 \triangleright a_j = q^j a_j, \quad K_2 \triangleright a_j = a_{j+1}, \quad F_2 \triangleright a_j = [2]_2 a_{j-1}
\end{aligned} \right\}. \tag{69}
\]
Analogously,
\[
\Omega^{(0,2)} = \left\{ \sum_{j=-1}^1 b_j \otimes \omega_j \in \mathcal{O}(\mathfrak{so}(5)) \otimes M^{(1,2)'} : \begin{aligned}
K_1 \triangleright b_j &= q^{2-j} b_j, \quad K_2 \triangleright b_j = q^j b_j, \quad K_2 \triangleright b_j = b_{j+1}, \quad F_2 \triangleright b_j = [2]_2 b_{j-1}
\end{aligned} \right\}. \tag{70}
\]
and, under the isomorphism \( a \otimes v_0 \mapsto a \),
\[
\Omega^{(0,3)} \cong \{ a \in \mathcal{O}(\mathfrak{so}(5)) : K_1 \triangleright a = q^3 a, \quad K_2 \triangleright a = a, \quad E_2 \triangleright a = F_2 \triangleright a = 0 \}. \tag{71}
\]
Note that \( \Omega^{(0,k)}, k = 1, 2, 3 \), are \( B \)-bimodules with the left and the right \( B \)-module structure given by left and right multiplication on the left tensor factor, respectively. Since \( K_i \triangleright (b_1 a b_2) = b_1 (K_i \triangleright a) b_2, E_2 \triangleright (b_1 a b_2) = b_1 (E_2 \triangleright a) b_2, F_2 \triangleright (b_1 a b_2) = b_1 (F_2 \triangleright a) b_2 \) for all \( a \in \mathcal{O}(\mathfrak{so}(5)), b_1, b_2 \in B \) and \( i = 1, 2 \), which follows from (14) and \( K_i \triangleright b = b, E_2 \triangleright b = 0 = F_2 \triangleright b \) for all \( b \in B \), the \( B \)-module structure is well defined.
Of course, we could have described $\Omega^{(0,k)}$ more concisely by a cotensor product as in [13]. However, the explicit description using a fixed basis of $M^k$ will be useful for the computation of the spectrum of the Dirac operator.

Next, consider the pull-backs
\[
\tilde{\delta}_k := \varphi_k^* : \Omega^{(0,k)} \longrightarrow (\mathcal{U}_q(\text{so}(5)) \otimes \mathcal{U}_q(1))^{\prime}, \quad \tilde{\delta}_k(f)(x) = f(\varphi_k(x)).
\]
Since we are interested in explicit formulas, we compute for all $Z_{-1}, Z_0, Z_1 \in \mathcal{U}_q(\text{so}(5))$
\[
\tilde{\delta}_0(b)(Z_{-1} \otimes v^{-1} + Z_0 \otimes v^0 + Z_1 \otimes v^1) = (X_{-1} \triangleright b \otimes v_{-1} + X_0 \triangleright b \otimes v_0 + X_1 \triangleright b \otimes v_1)(Z_{-1} \otimes v^{-1} + Z_0 \otimes v^0 + Z_1 \otimes v^1),
\]
hence, for all $b \in B$,
\[
\tilde{\delta}_0(b) = X_{-1} \triangleright b \otimes v_{-1} + X_0 \triangleright b \otimes v_0 + X_1 \triangleright b \otimes v_1. \tag{72}
\]

Similar computations show that
\[
\tilde{\delta}_1(a_{-1} \otimes v_{-1} + a_0 \otimes v_0 + a_1 \otimes v_1) = (-1)^{[3]} X_0 \triangleright a_{-1} + \frac{[2]}{[3]} X_1 \triangleright a_0) \otimes \omega_{-1}
\]
\[+ (-1)^{[3]} X_1 \triangleright a_{-1} + \frac{[2]}{[3]} X_0 \triangleright a_0) \otimes \omega_0 + (-1)^{[3]} X_1 \triangleright a_0 + X_0 \triangleright a_1) \otimes \omega_1], \tag{73}
\]
and, after composing it with the isomorphism $a \otimes v_0 \mapsto a$,
\[
\tilde{\delta}_2(b_{-1} \otimes v_{-1} + b_0 \otimes v_0 + b_1 \otimes v_1) = \frac{1}{[3]} X_1 \triangleright b_{-1} - X_0 \triangleright b_0 + X_{-1} \triangleright b_1. \tag{74}
\]

In particular, since the left and the right $\mathcal{U}_q(\text{so}(5))$-action on $\mathcal{O}(\text{SO}_q(5))$ commute, it follows from (65) and (72)–(74) that $\tilde{\delta}_k$ maps $\Omega^{(0,k)}$ into $\Omega^{(0,k+1)}$. Therefore we obtain a complex
\[
0 \longrightarrow \Omega_{(0,0)} \longrightarrow \Omega_{(0,1)} \longrightarrow \Omega_{(0,2)} \longrightarrow \Omega_{(0,3)} \longrightarrow 0. \tag{75}
\]

Moreover, the analogue of [13, Lemma 7.4] shows that
\[
\Omega^{(0,k)} = \text{span}\{a_0 \tilde{\delta}_0(a_1) \wedge \ldots \wedge \tilde{\delta}_0(a_k) : a_0, \ldots, a_k \in B\}, \tag{76}
\]
where the wedge product is determined by
\[
a \tilde{\delta}_0(b) \wedge \omega_k = a \tilde{\delta}_k(b \omega_k) - a b \tilde{\delta}_k(\omega_k), \quad \omega_k \in \Omega^{(0,k)}, \quad k = 0, 1, 2,
\]
\[
(a_0 \tilde{\delta}_0(a_1) \wedge \tilde{\delta}_0(a_2)) \wedge a_3 \tilde{\delta}_0(a_4) = a_0 \tilde{\delta}_0(a_1) \wedge (\tilde{\delta}_0(a_2) \wedge a_3 \tilde{\delta}_0(a_4)), \quad a_0, \ldots, a_4 \in B,
\]
and $\omega_k \wedge \omega_j = 0$ for $\omega_k \in \Omega^{(0,k)}$ and $\omega_j \in \Omega^{(0,j)}$ with $k + j > 3$. It follows now from [13, Section 7.3], that the complex (75) is isomorphic to the differential calculus $\Gamma_{a, u}^{\triangleright}$ from [11, Section 3.3.2].
4. Branching Rules

To distinguish between irreducible highest weight representations of \( U_\theta(\mathfrak{so}(5)) \) and \( U_\theta(l) \), we will use double parentheses to designate the irreducible highest weight representations of \( U_\theta(\mathfrak{so}(5)) \). In what follows, \( (m,j)^V \) denotes a linear subspace of \( \mathcal{O}(\mathfrak{so}(5)) \) carrying an irreducible \( U_\theta(\mathfrak{so}(5)) \)-representation of weight \((m,j)\) with respect to the right regular representation \( \pi_R \), and \( V^{(m,j)} \subset \mathcal{O}(\mathfrak{so}(5)) \) denotes a subspace of \( \mathcal{O}(\mathfrak{so}(5)) \) carrying an irreducible \( U_\theta(\mathfrak{so}(5)) \)-representations of weight \((m,j)\) with respect to the left regular representation \( \pi_L \), see (21). Let \( t^{(m,j)}_{\lambda,\mu} \) denote the matrix coefficients of an irreducible \( \mathcal{O}(\mathfrak{so}(5)) \)-corepresentation of weight \((m,j)\) with respect to the coproduct of \( \mathcal{O}(\mathfrak{so}(5)) \), and set \( d^{(m,j)} := \dim(\langle m,j \rangle V) \). Then

\[
\text{span}\{t^{(m,j)}_{\lambda,\mu} : \lambda = 1, \ldots, d^{(m,j)}\} \cong \langle m,j \rangle V \quad \text{for any fixed } \mu,
\]

\[
\text{span}\{t^{(m,j)}_{\lambda,\mu} : \mu = 1, \ldots, d^{(m,j)}\} \cong V^{\langle m,j \rangle} \quad \text{for any fixed } \lambda,
\]

and \( \text{span}\{t^{(m,j)}_{\lambda,\mu} : \lambda, \mu = 1, \ldots, d^{(m,j)}\} \cong \langle m,j \rangle V \otimes V^{\langle m,j \rangle} \). From the Peter-Weyl theorem for compact quantum groups, it is follows that

\[
\mathcal{O}(\mathfrak{so}(5)) \cong \bigoplus_{(m,j) \in \mathbb{N}_0 \otimes 2\mathbb{N}_0} \langle m,j \rangle V \otimes V^{\langle m,j \rangle}. \tag{77}
\]

The descriptions of \( \Omega^{(0,0)}, \ldots, \Omega^{(0,3)} \) in Equations (68)–(71) show that, whenever the representation of \( U_\theta(l) \) on \( V^{\langle m,j \rangle} \) contains an (irreducible) subrepresentation determined by the corresponding highest weight \( \lambda_k \) in (66), we get a 1-dimensional subspace of \( \Omega^{(0,k)} \) from a chosen highest weight vector. Moreover, by (67), all \((0,k)\)-forms are given by linear combinations of such elements. This leads us to the so-called branching rule, i.e., the problem of determining if a certain irreducible representation of \( U_\theta(l) \) occurs as a subrepresentation of \( U_\theta(\mathfrak{so}(5)) \) on \( V^{\langle m,j \rangle} \) and with which multiplicity.

Let \( V^{\langle m,j \rangle}_\mu \subset V^{\langle m,j \rangle} \) denote the subspace of vectors of weight \( \mu = (\mu_1, \mu_2) \), i.e., \( v_\mu \in V^{\langle m,j \rangle}_\mu \) if and only if \( v_\mu \in V^{\langle m,j \rangle} \) and \( K_i \triangleright v_\mu = q^{\mu_i}v_\mu, i = 1, 2 \). Then the branching rule for \( \lambda \in \mathbb{Z} \times \mathbb{N}_0 \) can be solved by determining the space of highest weight vectors in \( V^{\langle m,j \rangle}_\lambda \) for the \( U_\theta(l) \)-representation, that is, the vectors \( v_\lambda \in V^{\langle m,j \rangle}_\lambda \) such that \( E_2 \triangleright v_\lambda = 0 \). In particular, the multiplicities are given by \( \dim(\ker(E_2 V^{\langle m,j \rangle}_\lambda)) \).

Equation (23) shows that \( F_2 v_{(\mu_1,\mu_2)} \neq 0 \) for \( v_{(\mu_1,\mu_2)} \in V^{\langle m,j \rangle}_{(\mu_1,\mu_2)} \setminus \{0\} \) whenever \( \mu_2 > 0 \) and that, in this case, \( E_2 F_2 v_{(\mu_1,\mu_2)} \neq 0 \). Therefore \( E_2 : F_2 V^{\langle m,j \rangle}_\lambda \to V^{\langle m,j \rangle}_{\lambda+\alpha} \) is an isomorphism, hence the multiplicities are equal to \( \dim(V^{\langle m,j \rangle}_{\lambda}) - \dim(V^{\langle m,j \rangle}_{\lambda+\alpha}) \). This difference can easily be computed by Kostant’s multiplicity formula (see e.g. [14])

\[
\dim(V^{\mu}_{\lambda}) = \sum_{w \in W} (-1)^{\ell(w)} \#\Pi(w(\mu + \delta) - (\lambda + \delta)),
\]

where \( \delta \) denotes the half-sum of positive roots, \( W \) the Weyl group, \( \ell(w) \) the length of the Weyl group element \( w \in W \), and \( \mathbb{Z}^2 \ni v \mapsto \#\Pi(v) \) corresponds to the partition function, i.e. the number of all possible ways of writing \( v \) as a sum of positive roots. In our situation, \( \delta = (1,1) \) and \( \#\Pi(v) \) equals the number of elements of the set

\[
\Pi(v) := \{(n_1, n_2, n_3, n_4) \in \mathbb{N}_0^4 : v = n_1\alpha_1 + n_2\alpha_2 + n_3(\alpha_1 + \alpha_2) + n_4(\alpha_1 + 2\alpha_2)\}. \tag{78}
\]
see (2). Setting
\[ d(v) := \#\Pi(v) - \#\Pi(v - \alpha_2), \quad v \in \mathbb{Z}^2, \]
we obtain
\[ \dim(V_{\lambda}(\mu)) - \dim(V_{\lambda+\alpha_2}(\mu)) = \sum_{w \in W} (-1)^{\ell(w)} d(w(\mu + \delta) - (\lambda + \delta)), \quad (79) \]
where the parenthesis in the expression (\( \mu \)) indicate that \( V^{(\mu)} \) denotes the vector space of an irreducible \( U_q(\mathfrak{so}(5)) \)-representation of weight \( \mu \). The value of \( d(v) \) is easily derived from combinatorial considerations. In the following lemma, we compute those cases that are of interest to us.

**Lemma 2.** Let \( \lfloor x \rfloor \) denote the integer part of \( x \geq 0 \). Then
(i) \( d(k, 2l) = 1 + \lfloor \frac{k}{2} \rfloor \) for \( (k, 2l) \in \mathbb{N}_0 \times 2\mathbb{N}_0 \).
(ii) \( d(k, -2l) = 1 + \lfloor \frac{k}{2} \rfloor - 1 \) for \( (k, 2l) \in \mathbb{N}_0 \times 2\mathbb{N}_0 \) and \( k \geq 2l \).
(iii) \( d(k, 2l) = 0 \) if \( (k, 2l) \in \mathbb{Z} \times 2\mathbb{Z} \) such that either \( k < 0 \), or \( k \geq 0 \) and \( -2l > k \).

**Proof.** Obviously, for all \( \xi \in \Pi(v - \alpha_2) \), we have \( \xi + \alpha_2 \in \Pi(v) \). On the other hand, if \( \kappa = (n_1, n_2, n_3, n_4) \in \Pi(v) \) such that \( \kappa - \alpha_2 \in \Pi(v - \alpha_2) \), then \( n_2 > 0 \). Therefore,
\[ d(v) = \#\Pi(v) - \#\Pi(v - \alpha_2) = \#\{(n_1, n_2, n_3, n_4) \in \Pi(v) : n_2 = 0\}. \]
Hence \( d(v) \) equals the number of solutions of \( v = n_1 \alpha_1 + n_3(\alpha_1 + \alpha_2) + n_4(\alpha_1 + 2\alpha_2) \) with \( n_1, n_2, n_3 \in \mathbb{N}_0 \). For \( v = (x, y) \in \mathbb{Z} \times 2\mathbb{Z} \), \( \alpha_1 = (2, -2) \), \( \alpha_1 + \alpha_2 = (1, 0) \) and \( \alpha_1 + 2\alpha_2 = (0, 2) \), we need to determine the number of solutions of
\[ x = 2n_1 + n_3, \quad y = -n_1 + n_4, \quad n_1, n_2, n_3 \in \mathbb{N}_0. \]
Clearly, there is no solution if \( x < 0 \) or if \( x + 2y < 0 \). This implies (iii).

If \( x \geq 0 \), then \( n_3 = x - 2n_1 \) and \( n_4 = y + n_1 \), so that \( n_3 \) and \( n_4 \) are uniquely determined by \( n_1 \). Thus it now suffices to count the number of possible \( n_1 \in \mathbb{N}_0 \). For \( x \in \mathbb{N}_0 \) and \( y \in \mathbb{N}_0 \), there is a unique solution if and only if \( n_1 \leq \lfloor \frac{x}{2} \rfloor \). This proves (i).

Given \( x \in \mathbb{N}_0 \) and \( y \in \mathbb{Z} \setminus \mathbb{N}_0 \), there is a unique solution if and only if \( |y| \leq n_1 \leq \lfloor \frac{x}{2} \rfloor \), which implies (ii).

The multiplicities of irreducible representations of the Levi factor on \( V^{(m, j)} \) can now be computed by applying Lemma 2 to (79).

**Proposition 3.** For \( (n, l) \in \mathbb{N}_0^2 \), let \( V^{(n, l)} \) be a vector space carrying an irreducible \( U_q(\mathfrak{so}(5)) \)-representation of highest weight \( (n, l) \), and let \( V^{(n, l)} \) denote the vector space of an irreducible \( U_q(l) \)-representation as described in (23).

1. The trivial representation \( V^{(0, 0)} \) of the Levi factor \( U_q(l) \) occurs in \( V^{(n, l)} \) with the following multiplicities:
   (i) multiplicity 1 in \( V^{(2n, 2l)} \),
   (ii) multiplicity 0 in all other cases.
2. The irreducible representation \( V^{(0, 2)} \) of the Levi factor \( U_q(l) \) occurs in \( V^{(n, l)} \) with the following multiplicities:
   (i) multiplicity 2 in \( V^{(2n+2, 2l+2)} \),
   (ii) multiplicity 1 in \( V^{(2n+2, 0)} \),
   (iii) multiplicity 1 in \( V^{(0, 2l+2)} \).
(iv) multiplicity 1 in \( V^{(2n+1, 2l+2)} \),
(v) multiplicity 0 in all other cases.

3. The irreducible representation \( V^{(1, 2)} \) of the Levi factor \( U_q(I) \) occurs in \( V^{(n, l)} \) with the following multiplicities:
  (i) multiplicity 1 in \( V^{(2n+2, 2l+2)} \),
  (ii) multiplicity 1 in \( V^{(2n+3, 0)} \),
  (iii) multiplicity 1 in \( V^{(1, 2l+2)} \),
  (iv) multiplicity 2 in \( V^{(2n+3, 2l+2)} \),
  (v) multiplicity 0 in all other cases.

4. The irreducible representation \( V^{(3, 0)} \) of the Levi factor \( U_q(I) \) occurs in \( V^{(n, l)} \) with the following multiplicities:
  (i) multiplicity 1 in \( V^{(2n+3, 2l)} \),
  (ii) multiplicity 0 in all other cases.

**Proof.** Clearly, in order that a non-zero multiplicity of \( V^\lambda \) in \( V^{(m, j)} \) exists, \((m, j)\) must be higher than \( \lambda \). Moreover, since \( \lambda_k \in \mathbb{N}_0 \times 2\mathbb{N}_0 \) for all \( \lambda_k \) from (66), the multiplicity of \( V^{\lambda_k} \) in \( V^{(m, j)} \) is 0 whenever \( j \neq 2\mathbb{N}_0 \).

Let \( P^+, P^{++} \) and \( w_{\alpha_2} \in W \) be as defined in Sect. 2. Note that the sets in Lemma 2(i) and (ii) are contained in \( P^+ \) and \( w_{\alpha_2}(P^+) \), respectively. It thus follows from Lemma 2 that, for \( v \in \mathbb{Z} \times 2\mathbb{Z} \), \( d(v) = 0 \) if \( v \notin P^+ \cup w_{\alpha_2}(P^+) \). As \( \delta \in P^{++} \), we have \( \mu + \delta \in P^{++} \) for all \( \mu \in P^+ \) and \( \lambda_k + \delta \in P^{++} \) for \( k = 0, \ldots, 3 \). By Lemma 2(iii), the latter implies that \( d(v - (\lambda_k + \delta)) = 0 \) if \( d(v) = 0 \). Therefore the formula in (79) reduces to

\[
\dim(V_{\mu_k}) - \dim(V_{\lambda_k + \alpha_2}) = d(\mu - \lambda_k) - d(w_{\alpha_2}(\mu + \delta) - (\lambda_k + \delta)).
\]

As explained before Lemma 2, this difference computes the multiplicity of the irreducible \( U_q(I) \)-representation of highest weight \( \lambda_k \) in an irreducible \( U_q(\text{so}(5)) \)-representation of with highest weight \( \mu \in \mathbb{N}_0 \times 2\mathbb{N}_0 \). These numbers are easily computed by applying Equation (3) and Lemma 2. We present the results in the following tables.

|   |                          |                           |
|---|--------------------------|---------------------------|
| 1(i) | \( d((2n, 2l)) - d((2n+2l+1, -2l-2)) = n - (n+l)+(l+1) = 1 \) |
| 1(ii) | \( d((2n+1, 2l+1) - d((2n+2l+2, -2l-2)) = n - (n+l+1)+(l+1) = 0 \) |
| 2(i) | \( d((2n-1, 2l-2)) - d((2n+2l+1, -2l-4)) = n - (n+l)+(l+2) = 2 \) |
| 2(ii) | \( d((2n, 2l-2)) - d((2n+1, -2l-4)) = n - 1 - n + 2 = 1 \) |
| 2(iii) | \( d((0, 2l-2)) - d((2l+1, -2l-4)) = 1 - 0 = 1 \) |
| 2(iv) | \( d((2n+1, 2l-2)) - d((2n+2l+2, -2l-4)) = n - (n+l+1)+(l+2) = 1 \) |
| 3(i) | \( d((2n-1, 2l-2)) - d((2n+2l, -2l-4)) = n - 1 - (n+l)+(l+2) = 1 \) |
| 3(ii) | \( d((2n, 2l-2)) - d((2n+1, -2l-4)) = n - 1 - n + 2 = 1 \) |
| 3(iii) | \( d((0, 2l-2)) - d((2l+1, -2l-4)) = 1 - 0 = 1 \) |
| 3(iv) | \( d((2n, 2l-2)) - d((2n+2l+1, -2l-4)) = n - (n+l)+(l+2) = 2 \) |
| 4(i) | \( d((2n-2, 2l)) - d((2n+2l-1, -2l-2)) = n - 1 - (n+l)+(l+1) = 1 \) |
| 4(ii) | \( d((2n-3, 2l)) - d((2n+2l-2, -2l-2)) = n - 2 - (n+l-1)+(l+1) = 0 \) |

The formula in 3(ii) assumes \( n > 1 \). For \( n = 1 \), we have \((3, -4) \notin P^+ \cup w_{\alpha_2}(P^+) \) so that \( d((2, -2)) - d((3, -4)) = d((2, -2)) = 1 \). This completes the proof. \( \Box \)

Our next aim is to provide an explicit description of the elements in \( \mathcal{O}(\text{SO}_q(5)) \) which generate the subspaces that satisfy the branching rules of Proposition 3. Since the vector space of an irreducible highest weight representation of \( U_q(\text{so}(5)) \) (resp. \( U_q(I) \))
is generated by acting with elements from $\mathcal{U}_q\left(\mathfrak{so}(5)\right)$ (resp. $\mathcal{U}_q\left(\mathfrak{l}\right)$) on a non-zero highest weight vector, it suffices to restrict our attention to highest weight vectors.

By a slight abuse of notation, which amounts to identifying elements under the isomorphism in (77), let

$$\langle(n,l)\rangle V^{(m,j)} \subset \langle(n,l)\rangle V \otimes V^{\langle(n,l)\rangle} \subset \mathcal{O}(\mathfrak{so}_q(5))$$

denote the vector space of all elements in $\mathcal{O}(\mathfrak{so}_q(5))$ belonging to a representation of highest weight $(n, l)$ with respect to the right and left regular representations of $\mathcal{U}_q\left(\mathfrak{so}(5)\right)$ on $\mathcal{O}(\mathfrak{so}_q(5))$, and to a representation of highest weight $(m, j)$ with respect to the left $\mathcal{U}_q\left(\mathfrak{l}\right)$-action on $\mathcal{O}(\mathfrak{so}_q(5))$. Note that we do not assume that the representation of $\mathcal{U}_q\left(\mathfrak{l}\right)$ on $\langle(n,l)\rangle V^{(m,j)}$ is irreducible. Furthermore, let

$$\langle(n,l)\rangle V^{(m,j)}_{(s_1,s_2)} := \{v \in \langle(n,l)\rangle V^{(m,j)} : K_i \triangleright v = q_i^{s_i} v, \ i = 1, 2\}, \quad (80)$$

$$\langle(n,l)\rangle V^{(m,j)}_{(r_1,r_2)} := \{v \in \langle(n,l)\rangle V^{(m,j)} : \pi_R(K_i)(v) = q_i^{r_i} v, \ K_i \triangleright v = q_i^{s_i} v, \ i = 1, 2\} \quad (81)$$

denote the vector spaces of weight vectors of weight $(s_1, s_2)$ with respect to the left $\mathcal{U}_q\left(\mathfrak{l}\right)$-action on $\mathcal{O}(\mathfrak{so}_q(5))$ and, in the second case, also of weight $(r_1, r_2)$ with respect to the right regular representation of $\mathcal{U}_q(\mathfrak{so}(5))$. In particular, $\langle(n,l)\rangle V^{(m,j)}_{(n,l)}$ is the vector space of highest weight vectors with respect to both representations. These highest weight vectors determine the whole representation space $\langle(n,l)\rangle V^{(m,j)}$ since

$$\langle(n,l)\rangle V^{(m,j)} = \pi_R(\mathcal{U}_q(\mathfrak{so}(5)))\left(\mathcal{U}_q\left(\mathfrak{l}\right) \triangleright \langle(n,l)\rangle V^{(m,j)}_{(n,l)}\right)$$

$$= \mathcal{U}_q\left(\mathfrak{l}\right) \triangleright \left(\pi_R(\mathcal{U}_q(\mathfrak{so}(5)))\langle(n,l)\rangle V^{(m,j)}_{(n,l)}\right).$$

A complete list of highest weight vectors in the cases of interest will be given in the next proposition.

**Proposition 4.** Let $u_j^i$, $i, j = 1, \ldots, 5$, denote the generators of $\mathcal{O}(\mathfrak{so}_q(5))$. Set

$$z_1 := u_1^1 u_3^1, \quad z_2 := u_1^1 u_3^2 - q u_1^2 u_3^1, \quad (82)$$

Then, for $n, l \in \mathbb{N}_0$,

1. (i) $\langle 2n,2l\rangle V^{(0,0)}_{(2n,2l)} = \text{span}\{z_2^{l}z_1^n\}$.
2. (i) $\langle 2n+2,2l+2\rangle V^{(0,2)}_{(2n+2,2l+2)} = \text{span}\{z_2^{l+1}u_4^1 u_5^1 z_1^n, \ z_2^{l} (u_4^1 u_5^2 - q u_4^2 u_5^1) z_1^{n+1}\}$.
3. (i) $\langle 2n+2,2l+2\rangle V^{(0,2)}_{(2n+2,2l+2)} = \text{span}\{u_4^1 u_5^1 z_1^n\}$.
4. (i) $\langle 2n+3,2l\rangle V^{(0,1)}_{(2n+3,2l)} = \text{span}\{z_2^{l} (u_4^1 u_5^2 - q u_4^2 u_5^1) u_5^1 z_1^n\}$.

5. (i) $\langle 2n+2,2l+2\rangle V^{(1,2)}_{(2n+2,2l+2)} = \text{span}\{z_2^{l} (u_3^1 u_4^2 - q u_3^2 u_4^1) u_5^1 z_1^n\}$.

6. (i) $\langle 2n+3,2l+3\rangle V^{(0,0)}_{(2n+3,2l+3)} = \text{span}\{z_2^{l} (u_4^1 u_5^2 - q u_4^2 u_5^1) u_5^1 z_1^n\}$.

7. (i) $\langle 2n+3,2l+3\rangle V^{(0,0)}_{(2n+3,2l+3)} = \text{span}\{u_4^1 u_5^1 z_1^n\}$.

8. (i) $\langle 2n+3,2l+3\rangle V^{(0,0)}_{(2n+3,2l+3)} = \text{span}\{z_2^{l} (u_4^1 u_5^2 - q u_4^2 u_5^1) u_5^1 z_1^n\}$.
Proof. First note that the numbers of generating vectors in Proposition 4 agree with
the multiplicities in Proposition 3. Therefore it suffices to show that the sets of generating
vectors are highest weight vectors of the corresponding weights and are linearly
independent.

Using the formulas given in (15) and (16), elementary computations show that the
following elements are highest weight vectors for the right regular representation of
\( \mathcal{U}_q(\text{so}(5)) \) and the left action of \( \mathcal{U}_q(1) \):

\[
\begin{align*}
\alpha(z_1) &= u_1^1u_5^1, \quad \alpha(z_2) = u_1^1u_5^2 - qu_1^2u_5^1, \quad \alpha(u_4^1), \quad \alpha(u_5^1), \quad \alpha(u_4^2u_5^2 - qu_1^2u_5^1), \quad \alpha(u_3^1u_4^2 - qu_1^2u_5^1).
\end{align*}
\]

Since, by (8) and (14), the product of highest weight vectors is a highest weight vector,
where the resulting weight is the sum of the individual weights, it follows that all elements
listed in Proposition 4 are highest vectors belonging to the corresponding weight spaces.

We need to prove that none of the highest weight vectors is zero. Since it is known
that \( \mathcal{O}(\text{SO}_q(5)) \) is a domain, it suffices to verify this for \( u_1^1u_5^2 - qu_1^2u_5^1, u_4^2u_5^2 - qu_1^2u_5^1 \)
and \( u_3^1u_4^2 - qu_1^2u_5^1 \).

Elementary considerations show that \( u_4^2u_5^2 - qu_1^2u_5^1 \) is the unique (up to a scaling
factor) highest weight vector of the representation \( \mathcal{O}(0,2) \otimes \mathcal{V}(0,2) \) from the Peter-Weyl
decomposition (77), so it cannot be zero. Since \( E_2^2E_1(u_1^1u_5^2 - qu_1^2u_5^1) \) yields a non-zero
multiple of \( u_4^1u_5^2 - qu_1^2u_5^1 \), we have necessarily \( z_2 = u_1^1u_5^2 - qu_1^2u_5^1 \neq 0 \). Analogously,
\( E_2^2E_1(u_4^2u_5^2 - qu_1^2u_5^1) \) is a non-zero multiple of \( u_4^1u_5^2 - qu_1^2u_5^1 \), hence \( u_3^1u_4^2 - qu_1^2u_5^1 \neq 0 \).

It remains to prove that the two generating vectors are linearly independent when the
multiplicity is equal to 2. Let \( \alpha, \beta \in \mathbb{C} \) such that

\[
\begin{align*}
0 &= \alpha z_2 \alpha(z_2) + \beta(z_2)^2 u_5^1u_3^1 \alpha(z_2)^n + 1)
\end{align*}
\]

Then \( \alpha(u_1^1u_5^2 - qu_1^2u_5^1)u_4^1 + \beta(u_4^1u_5^2 - qu_1^2u_5^1)u_1^1 = 0 \) since \( \mathcal{O}(\text{SO}_q(5)) \) is a domain.
Acting on this expression with \( F_1 \) from the left yields

\[
0 = -\alpha q(u_1^1u_4^2 - qu_1^2u_4^1)u_1^1 + \beta q(u_4^1u_5^2 - qu_1^2u_5^1)u_1^1 = -\alpha q(u_1^1u_4^2 - qu_1^2u_4^1)u_1^1
\]
as \( u_4^1u_5^2 - qu_1^2u_4^1 = 0 \) by (38). On the other hand, the unique element (up to a constant)
in \( \mathcal{O}(0,2) \otimes \mathcal{V}(0,2) \), which is a highest weight vector with respect to right regular represen-
tation and a lowest weight vector with respect to left regular representation, is given
by \( u_1^1u_5^2 - qu_1^2u_4^1 = -q_2[2]^{-1}F_2^2(u_1^1u_4^2 - qu_1^2u_4^1) \). Therefore \( u_1^1u_5^2 - qu_1^2u_4^1 \neq 0 \), hence
\( \alpha = 0 \) and consequently also \( \beta = 0 \). This proves that the two vectors on the right-hand
side of 2.(i) are linearly independent.

In Case 3.(iv), if \( \alpha(z_2)^2u_4^1(u_5^2)^2z_1^n + \beta(z_2)^2(u_4^2u_5^2 - qu_1^2u_5^1)u_4^1z_1^{n+1} = 0 \), then

\[
0 = z_2^2((\alpha(u_4^1u_5^2 - qu_1^2u_5^1)u_4^1 + q^{-2}\beta(u_4^1u_5^2 - qu_1^2u_5^1)u_1^1)(u_5^2)^2z_1^n,
\]
and therefore \( \alpha = \beta = 0 \) by the same reasoning, see (83). \qed
5. Dobeault–Dirac Operator

The aim of this section is to study the Dobeault–Dirac operator $\tilde{\partial} + \tilde{\partial}^\dagger$ on the Hilbert space closure of $\Omega^{(0,0)} \oplus \Omega^{(0,1)} \oplus \Omega^{(0,2)} \oplus \Omega^{(0,3)}$, where $\tilde{\partial} = \partial_0 \oplus \partial_1 \oplus \partial_2$ maps the first three components into the last three, and $\tilde{\partial}^\dagger$ denotes the Hilbert space adjoint.

The first step is to assign a “natural” inner product to the space of $(0, k)$-forms. By (68) and (71), we may view $\Omega^{(0,0)}$ and $\Omega^{(0,3)}$ as subspace of $\mathcal{O}(SO_q(5))$. Therefore it is natural to define

$$\langle \cdot, \cdot \rangle_0 : \Omega^{(0,0)} \times \Omega^{(0,0)} \to \mathbb{C}; \quad \langle b_1, b_2 \rangle_0 := c_0 h(b_1^* b_2),$$

(84)

$$\langle \cdot, \cdot \rangle_3 : \Omega^{(0,3)} \times \Omega^{(0,3)} \to \mathbb{C}; \quad \langle a_1, a_2 \rangle_3 := c_3 h(a_1^* a_2),$$

(85)

where $h$ denotes the Haar state from (18) and $c_0, c_3 \in (0, \infty)$. Furthermore, by (69) and (70), we have $\Omega^{(0,k)} \subset \mathcal{O}(SO_q(5)) \otimes M_2^{k'} \cong \mathcal{O}(SO_q(5)) \otimes \mathbb{C}^3$, $k = 1, 2$. Taking into account that $K_1$ and $K_2$ should act as self-adjoint operators, we require that $\text{span}\{\nu_1, \nu_0, \nu_1\} \subset M^{(0,2)'}$ and $\text{span}\{\omega_1, \omega_0, \omega_1\} \subset M^{(1,2)'}$ are orthogonal bases.

For this reason, we first set

$$\langle \nu_1, \nu_j \rangle_{\lambda_1} := c_{1j} \delta_{ij}, \quad \langle \omega_i, \omega_j \rangle_{\lambda_2} := c_{ij} \delta_{ij},$$

(86)

where $c_{1j}, c_{2j} \in (0, \infty)$ and $\delta_{ij}$ denotes the Kronecker delta, and then consider the tensor product inner product on $\mathcal{O}(SO_q(5)) \otimes M_2^{k'}$, $k = 1, 2$, given by

$$\left\langle \sum_{i=-1}^1 a_i \otimes \nu_i, \sum_{j=-1}^1 b_j \otimes \nu_j \right\rangle_1 = \sum_{j=-1}^1 c_{1j} h(a_j^* b_j),$$

(87)

$$\left\langle \sum_{i=-1}^1 x_i \otimes \omega_i, \sum_{j=-1}^1 y_j \otimes \omega_j \right\rangle_2 = \sum_{j=-1}^1 c_{2j} h(x_j^* y_j).$$

(88)

The inner product on $(0, 1)$- and $(0, 2)$-forms will be given by the restriction of $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ to $\Omega^{(0,1)} \subset \mathcal{O}(SO_q(5)) \otimes M^{(0,2)'}$ and $\Omega^{(0,2)} \subset \mathcal{O}(SO_q(5)) \otimes M^{(1,2)'}$, respectively.

Of course, the spectrum of the Dirac operator may depend on the positive real numbers $c_0, c_{1j}, c_{2j}$ and $c_3$. A similar dependence on a scaling factor was observed in the definition of the differentials $\tilde{\partial}_q$. The dependence of the spectrum of the Dirac operator on these parameters will be discussed in Remark 2.

By the definition of $\Omega^{(0,k)}$ as a subspace of $\mathcal{O}(SO_q(5)) \otimes M_2^{k'}$, the right regular representation of $\mathcal{U}_q(\text{so}(5))$ on the first leg of $\mathcal{O}(SO_q(5)) \otimes M_2^{k'}$ restricts to $\Omega^{(0,k)}$ since it acts only on the left tensor factor in the Peter-Weyl decomposition (77). The next lemma shows that the differential complex (75) is equivariant with respect to the right regular representation.

**Lemma 5.** On $\Omega^{(0,k)} \subset \mathcal{O}(SO_q(5)) \otimes M_2^{k'}$, consider the right regular representation of $\mathcal{U}_q(\text{so}(5))$ given by $\pi_R(X)(a \otimes v) := a \cdot S^{-1}(X) \otimes v$ for $a \otimes v \in \mathcal{O}(SO_q(5)) \otimes M_2^{k'}$ and $X \in \mathcal{U}_q(\text{so}(5))$. Then $\pi_R$ defines a $*$-representation of $\mathcal{U}_q(\text{so}(5))$ on $\Omega^{(0,k)}$ such that $\pi_R(X)\tilde{\partial}_q(\omega_k) = \tilde{\partial}_k(\pi_R(X)(\omega_k))$ for all $\omega_k \in \Omega^{(0,k)}$.

On the subalgebra $B = \mathcal{O}(SO_q(5))^{\text{inv}(U_q(1))}$ of $\mathcal{O}(SO_q(5))$, consider the left action $\triangleright : \mathcal{U}_q(\text{so}(5)) \times B \to B, X \triangleright b := b \cdot S^{-1}(X)$. Then

$$\pi_R(X)(b \omega_k) = (X(2) \triangleright b)\pi_R(X(1))(\omega_k)$$

for all $X \in \mathcal{U}_q(\text{so}(5)), b \in B$ and $\omega_k \in \Omega^{(0,k)}$.  

Proof. Since, for all \( a \in \mathcal{O}(SO_q(5)) \) and \( X, Y \in \mathcal{U}_q(\text{so}(5)) \),

\[
Y \triangleright (\pi_R(X)(a)) = Y \triangleright a \triangleleft S^{-1}(X) = \pi_R(X)(Y \triangleright a),
\]

it follows from (68)–(71) that \( \pi_R(X) : \Omega^{(0,k)} \to \Omega^{(0,k)} \) is well defined. As mentioned before (22), \( \pi_R \) is a \(*\)-representation with respect to the inner product on \( \mathcal{O}(SO_q(5)) \) defined by the Haar state. As a consequence, by the definitions in (84)–(88), \( \pi_R \) yields a \(*\)-representation on \( \Omega^{(0,k)} \). Note that the definition of \( \tilde{\partial}_h \) involves only the left \( \mathcal{U}_q(\text{so}(5)) \)-action. Therefore (89) implies that \( \pi_R(X) \) and \( \tilde{\partial}_h \) commute for all \( X \in \mathcal{U}_q(\text{so}(5)) \) and \( k = 0, 1, 2 \).

As \( B = \Omega^{(0,0)} \) and \( \pi_R(X) : \Omega^{(0,0)} \to \Omega^{(0,0)} \), we know already that the action \( \triangleright \) leaves \( B \) invariant. Furthermore, for all \( \sum_i a_i \otimes v_i \in \Omega^{(0,k)} \subset \mathcal{O}(SO_q(5)) \otimes M^k \) and all \( X \in \mathcal{U}_q(\text{so}(5)) \), an application of (14) gives

\[
\pi_R(X)(\sum_i b a_i \otimes v_i) = \sum_i (b a_i) \triangleleft S^{-1}(X) \otimes v_i
\]

\[
= \sum_i (b \triangleleft S^{-1}(X_2))(a_i \triangleleft S^{-1}(X_1)) \otimes v_i
\]

\[
= (X_2 \triangleright b)\pi_R(X_1)(\sum_i a_i \otimes v_i),
\]

which proves the second part of the lemma. \( \square \)

As in Sect. 4, let \( \Omega^{(n,l)}_{(n,l)} \) denote the space of vectors belonging to an irreducible representation of highest weight \((n, l)\) with respect to the right regular representation and let \( \Omega^{(n,l)}_{(r,s)} \subset \Omega^{(n,l)}_{(r,s)} \) denote the subspace of weight vectors of weight \((r, s)\). By the Peter-Weyl decomposition (77), elements of \( \mathcal{O}(SO_q(5)) \) belonging to irreducible representations of different highest weights are orthogonal with respect to the inner product given by the Haar state. Therefore, from Equations (68)–(71), (81) and Proposition 4, we obtain the orthogonal decompositions

\[
\Omega^{(0,k)} = \bigoplus \{ \Omega^{(n,l)}_{(n,l)} : n \in \mathbb{N}_0, \ l \in 2\mathbb{N}_0, \ \dim \langle \Omega^{(n,l)}_{(n,l)} \rangle V_{\lambda_k} \neq 0 \},
\]

(90)

where

\[
\Omega^{(n,l)}_{(n,l)} = \pi_R(\mathcal{U}_q(\text{so}(5))).
\]

(91)

and, for \( n, m \in \mathbb{N}_0 \) and \( l \in 2\mathbb{N}_0 \),

\[
\Omega^{(2n,2m)}_{(2n,2m)} = \Omega^{(2n,2m)}_{(2n,2m)}, \quad (2n+3,2m) \Omega^{(0,3)}_{(2n+3,2m)} = (2n+3,2m) \Omega^{(3,0)}_{(2n+3,2m)} \otimes v^0,
\]

(92)

\[
\Omega^{(n,l)}_{(n,l)} = \left\{ \frac{1}{[2]^2} F_2 \triangleright a_1 \otimes v \otimes a_1 + \frac{1}{[2]^2} F_2 \triangleright a_1 \otimes a_0 + a_1 \otimes v_1 : a_1 \in \Omega^{(n,l)}_{(n,l)} V_{(0,2)} \right\}
\]

(93)

\[
\Omega^{(n,l)}_{(n,l)} = \left\{ \frac{1}{[2]^2} F_2 \triangleright b_1 \otimes v_0 + F_2 \triangleright b_1 \otimes a_0 + b_1 \otimes v_0 : b_1 \in \Omega^{(n,l)}_{(n,l)} V_{(1,2)} \right\}
\]

(94)

Note that \( \Omega^{(n,l)}_{(n,l)} \) is uniquely determined by \( \Omega^{(n,l)}_{(n,l)} V_{\lambda_k} \). The following lemma establishes an explicit unitary isomorphisms between these spaces.
Lemma 6. On the linear subspaces \( \mu V_{\lambda_k} \subset \mathcal{O}(SO_q(5)) \) from Proposition 4, consider the inner product (20) given by the Haar state on \( \mathcal{O}(SO_q(5)) \). Let \( c_0 \) and \( c_3 \) denote the constants from (84) and (85), respectively. With the constants \( c_{kj} \) from (86), set

\[
c_k := \sqrt{\frac{c_{k,0}}{q[2]_2^2} + \frac{c_{k,1}}{q[2]_2}} + \frac{c_{k,2}}{q[2]_2}, \quad k = 1, 2, \tag{95}
\]

Then the linear operators

\[
J_0 : (\mu \nu V_{(0,0)}) \rightarrow (\mu \nu \Omega(0,0)), \quad J_0(a) := \frac{1}{c_0} a,
\]

\[
J_1 : (\mu \nu V_{(0,2)}) \rightarrow (\mu \nu \Omega(0,1)), \quad J_1(a) := \frac{1}{c_1} \left( \frac{1}{[2]_2} F_2^2 \triangleright a \otimes \nu_{-1} + \frac{1}{[2]_2} F_2 \triangleright a \otimes \nu_0 + a \otimes \nu_1 \right),
\]

\[
J_2 : (\mu \nu V_{(1,2)}) \rightarrow (\mu \nu \Omega(0,2)), \quad J_2(a) := \frac{1}{c_2} \left( \frac{1}{[2]_2} F_2^2 \triangleright a \otimes \nu_{-1} + \frac{1}{[2]_2} F_2 \triangleright a \otimes \nu_0 + a \otimes \nu_1 \right),
\]

\[
J_3 : (\mu \nu V_{(3,0)}) \rightarrow (\mu \nu \Omega(0,3)), \quad J_3(a) := \frac{1}{c_3} a \otimes \nu_0.
\]

are unitary isomorphisms.

Proof. We prove the lemma for \( J_1 \). For \( J_2 \), the proof is similar and for \( J_0 \) and \( J_3 \), the proof is even more elementary. From (93), it follows that \( J_1 \) defines a linear isomorphism. Note that, by (24), \( \frac{1}{[2]_2} E_2^2 F_2^2 \triangleright a = \frac{1}{[2]_2} E_2^2 F_2 \triangleright a = a \) for all \( a \in (\mu \nu V_{(0,2)}) \). Let \( a_1, a_2 \in (\mu \nu V_{(0,2)}) \). Then

\[
\langle J_1(a_1), J_1(a_2) \rangle = \frac{1}{c_1^2} \left( \frac{c_{1,1}}{[2]_2^2} h \left( (F_2^2 \triangleright a_1)^* (F_2^2 \triangleright a_2) \right) + \frac{c_{1,0}}{[2]_2^2} h \left( (F_2 \triangleright a_1)^* (F_2 \triangleright a_2) \right) + c_{1,1} h(\nu a_1 a_2) \right)
\]

\[
= \frac{1}{c_1^2} \left( \frac{c_{1,1}}{[2]_2^2} h \left( a_1^* (K_2^{-1} E_2^2 F_2^2 \triangleright a_2) \right) + \frac{c_{1,0}}{[2]_2^2} h \left( a_1^* (K_2^{-1} E_2^2 F_2 \triangleright a_2) \right) + c_{1,1} h(a_1 a_2) \right)
\]

\[
= \frac{1}{c_1^2} \left( \frac{c_{1,1}}{[2]_2^2} h(a_1^* a_2) + \frac{c_{1,0}}{q[2]_2} + c_{1,1} \right) h(a_1^* a_2) = h(a_1^* a_2),
\]

where we used the definition of \( J_1 \) and (87) in the first equality, the fact that \( \pi_L \) from (21) is a *-representation in the second equality, the defining relations of \( \mathcal{U}_q(so(5)) \) and (24) in the third equality, and (95) in the last equality. Hence the linear isomorphism \( J_1 \) is also isometric which proves its unitarity. \( \square \)

Lemma 6 and Eqs. (90)–(94) show that the operator \( \tilde{\partial}_k \) is uniquely determined by the unitarily equivalent action on the spaces of highest weight vectors \( (\mu \nu V_{\lambda_k}) \). The next corollary gives explicit formulas.

Corollary 7. The restrictions of \( \tilde{\partial}_k \) to the spaces of highest weight vectors,

\[
\tilde{\partial}_k : (\mu \nu \Omega^{(0,k)}) \longrightarrow (\mu \nu \Omega^{(0,k+1)}), \quad \mu \in \mathbb{N}_0 \times 2\mathbb{N}_0, \quad k = 0, 1, 2, \tag{96}
\]
are unitarily equivalent to
\[
\delta_0 := J_1^* \circ \tilde{\delta}_0 \circ J_0 : (\mu) V_{(0,0)}^{(0)} \rightarrow (\mu) V_{(0,2)}^{(0,2)}, \quad \delta_0(v) = \frac{c_1}{c_0} X_1(v), \tag{97}
\]
\[
\delta_1 := J_2^* \circ \tilde{\delta}_1 \circ J_1 : (\mu) V_{(0,2)}^{(0,2)} \rightarrow (\mu) V_{(1,2)}^{(1,2)}, \quad \delta_1(v) = \frac{c_2}{c_1} \left[ \frac{3}{2} - 1 \right] \frac{1}{[3,2]} \pr_{(1,2)} \circ X_0(v), \tag{98}
\]
\[
\delta_2 := J_3^* \circ \tilde{\delta}_2 \circ J_2 : (\mu) V_{(1,2)}^{(1,2)} \rightarrow (\mu) V_{(3,0)}^{(3,0)}, \quad \delta_2(v) = \frac{c_3}{c_2} \pr_{(3,0)} \circ X_{-1}(v), \tag{99}
\]
where \( \pr_{(1,2)} \) and \( \pr_{(3,0)} \) are defined in (26) and (27), respectively.

**Proof.** By the equivariance property shown in Lemma 5, the operator \( \tilde{\delta}_k \) maps highest weight vectors of weight \( \mu \in \mathbb{N}_0 \times 2\mathbb{N}_0 \) into highest weight vectors of the same weight, thus (96) holds. From Lemma 6 and (72), it follows that, for all \( v \in (\mu) V_{(0,0)}^{(0)} \),
\[
\delta_0(v) = \frac{1}{c_0} J_1^{-1}(X_{-1} \triangleright v \otimes v_{-1} + X_0 \triangleright v \otimes v_0 + X_1 \triangleright v \otimes v_1) = \frac{c_1}{c_0} X_1 \triangleright v.
\]
Similarly, Lemma 6 and (73) give
\[
\delta_1(v) = \frac{c_2}{c_1} \left[ \frac{3}{2} - 1 \right] \frac{1}{[3,2]} X_1 F_2 \triangleright v + X_0 \triangleright v = \frac{c_2}{c_1} \left[ \frac{3}{2} - 1 \right] \frac{1}{[3,2]} E_2 F_1 \triangleright v + E_2 E_1 \triangleright v,
\]
where we applied the defining relations of \( \mathcal{U}_q(\text{so}(5)), K_{2v} = q_2^2 v, [2,2]([3,2] - 1) = [4,2] \) and (26). Before using (26), it is important to note that \( F_3^3 E_1 \triangleright v = E_1 F_3^3 \triangleright v = 0 \) for all \( v \in (\mu) V_{(0,2)}^{(0,2)} \), so that \( E_1 \triangleright v \in (\mu) V_{(2,0)}^{(2,0)} \oplus (\mu) V_{(2,0)}^{(1,2)} \oplus (\mu) V_{(2,0)}^{(0,4)} \), thus \( E_2 E_1 \triangleright v \in (\mu) V_{(1,2)}^{(1,2)} \oplus (\mu) V_{(1,2)}^{(0,4)} \).

Much in the same way, by Lemma 6 and (74),
\[
\delta_2(v) = \frac{c_3}{c_2} \left[ \frac{3}{2} - 1 \right] \frac{1}{[3,2]} X_1 F_2 \triangleright v - \frac{1}{[2,2]} X_0 F_2 \triangleright v + X_{-1} \triangleright v,
\]
where we used the fact that \( F_2^3 E_1 \triangleright v = 0 \) for \( v \in (\mu) V_{(1,2)}^{(1,2)} \) which implies that \( E_1 \triangleright v \) belongs to \( (\mu) V_{(3,0)}^{(3,0)} \oplus (\mu) V_{(2,0)}^{(2,2)} \oplus (\mu) V_{(2,0)}^{(1,4)} \) so that (27) can be applied. \( \square \)
Remark 1. The reason why the projection $\text{pr}_{(0,2)}$ does not appear in (97) is because $F_2 E_1 \triangleright v = E_1 F_2 \triangleright v = 0$ which implies that $E_1 \triangleright v \in (\mu)_V^{02(0,2)}$ is a lowest weight vector for $U_q(I)$, thus $X_1 \triangleright v = E_2^2 E_1 \triangleright v$ belongs already to $(\mu)_V^{02(0,2)}$.

To determine the spectrum of the Dirac operator defined as the operator closure of $\partial + \partial^\dagger$, we begin with computing the eigenvalues of $\partial_j \partial^\dagger_j$ and $\partial_j \partial^\dagger_k$. For this, we need an explicit description of the Hilbert space adjoint $\partial^\dagger_j$ of $\partial_j$. Recall that the operators $\partial_j$ in Corollary 7 are described by actions of elements from $U_q(\mathfrak{so}(5))$ and that the inner product on $(\mu)_V^{02(0,2)}$ is given by the Haar state. As mentioned in Sect. 2, the Hilbert space adjoint with respect to the left regular representation of $U_q(\mathfrak{so}(5))$ coincides with the conjugate element in the $^*$-algebra $U_q(\mathfrak{so}(5))$. However, we cannot simply apply the involution of $U_q(\mathfrak{so}(5))$ to $\partial_j$ since, in general, these adjoint elements from $U_q(\mathfrak{so}(5))$ do not leave the orthogonal complement $(\mu)_V^{02(0,2)} \subset O(SO_q(5))$ invariant. In this case, it will be necessary to project onto the correct weight space. The following lemma will state explicit formulas for the Hilbert space adjoints of $\partial_j$, $j = 0, 1, 2$, using the projection from (26) and (27).

**Lemma 8.** The Hilbert space adjoints of the operators $\partial_0$, $\partial_1$ and $\partial_2$ from Corollary 7 are given by

\[
\begin{align*}
\partial^\dagger_0 : (\mu)_V^{02(0,2)} &\longrightarrow (\mu)_V^{00(0,0)}, \\
\partial^\dagger_1 : (\mu)_V^{12(1,2)} &\longrightarrow (\mu)_V^{02(0,2)}, \\
\partial^\dagger_2 : (\mu)_V^{30(3,0)} &\longrightarrow (\mu)_V^{12(1,2)},
\end{align*}
\]

(100)

\[
\begin{align*}
\partial^\dagger_0(v) &= q^2 \frac{c_1}{c_0} \text{pr}_{(0,0)} \circ F_1 F_2^2 \triangleright v, \\
\partial^\dagger_1(v) &= q^2 \frac{c_2}{c_1} \frac{[3]_2 - 1}{[3]_2} \text{pr}_{(0,2)} \circ F_1 F_2 \triangleright v, \\
\partial^\dagger_2(v) &= q^2 \frac{c_3}{c_2} F_1 \triangleright v.
\end{align*}
\]

(101)

(102)

**Proof.** From the fact that the left $U_q(\mathfrak{so}(5))$-action on $O(SO_q(5))$ defines a $^*$-representation of $U_q(\mathfrak{so}(5))$ with respect to the inner product (20), the defining relations of $U_q(\mathfrak{so}(5))$ and (24), we obtain for all $v_0 \in (\mu)_V^{0,0}$ and $v_1 \in (\mu)_V^{0,2}$

\[
\langle v_1, X_1 \triangleright v_0 \rangle_h = \langle X_1^\dagger \triangleright v_1, v_0 \rangle_h = q^{-1} \langle F_1 K_1 (F_2 K_2)^2 \triangleright v_1, v_0 \rangle_h = q^2 \langle F_1 F_2^2 \triangleright v_1, v_0 \rangle_h.
\]

(103)

Since $E_2^3 F_1 F_2^2 \triangleright v_1 = F_1 E_2^3 F_2^2 \triangleright v_1 = [2]_2^2 F_1 E_2^2 \triangleright v_1 = 0$ by (5) and (24), it follows that $F_1 F_2^2 \triangleright v_1 \in (\mu)_V^{0,0} \oplus (\mu)_V^{1,-2} \oplus (\mu)_V^{2,-4}$, where the orthogonality of the decomposition results from the Peter-Weyl theorem. To pick the element in $(\mu)_V^{0,0}$ without changing the inner product (103), we apply the orthogonal projection $\text{pr}_{(0,0)}$ from (27) and obtain

\[
\langle v_1, \partial_0 v_0 \rangle_h = \frac{c_1}{c_0} q^2 \langle \text{pr}_{(0,0)} \circ F_1 F_2^2 \triangleright v_1, v_0 \rangle_h
\]

which proves (100).

From the orthogonality of the Peter-Weyl decomposition and the definition of $\text{pr}_{(1,2)}$ below Equation (26), it follows that $\langle v_2, \text{pr}_{(1,2)} \circ X_0 \triangleright v_1 \rangle_h = \langle v_2, X_0 \triangleright v_1 \rangle_h$ for all $v_1 \in (\mu)_V^{0,2}$ and $v_2 \in (\mu)_V^{1,2}$. As in the previous paragraph, we need to combine the action of $X_0^\dagger = F_1 F_2 K_1 K_2$ on $v_2$ with the orthogonal projection onto $(\mu)_V^{0,2}$. Since $E_2^2 F_1 F_2 \triangleright v_2 = [2]_2^2 F_1 E_2 v_2 = 0$, we conclude that $F_1 F_2 \triangleright v_2 \in (\mu)_V^{0,2} \oplus (\mu)_V^{1,-4}$ so that we can apply the orthogonal projection $\text{pr}_{(0,2)}$ from (26). This yields
so that (101) holds.

Analogously, we have for all \( v_2 \in (\mu^1 V_{(1,2)})_\lambda \) and \( v_3 \in (\mu^3 V_{(3,0)})_\lambda \),

\[
\langle v_2, \partial_1 v_1 \rangle_h = \frac{c_2}{c_1} \frac{[3]_2 - 1}{[3]_2} q^2 \langle \text{pr}_{(0,2)} \circ F_1 F_2 \triangleright v_2, v_1 \rangle_h
\]

(104)

Since \( E_2 F_1 \triangleright v_3 = F_1 E_2 \triangleright v_3 = 0 \), it follows that \( F_1 \triangleright v_3 \in (\mu^1 V_{(1,2)}) \), so we do not need to apply an orthogonal projection and consequently (104) yields (102).

Our next aim is to compute the eigenvalues of self-adjoint operators \( \partial_k^\dagger \partial_k \) and \( \partial_k \partial_k^\dagger \) acting on the 1 or 2 dimensional Hilbert spaces of highest weight vectors. Since the projections, and therefore all operators appearing in Corollary 7 and Lemma 8, can be expressed by elements from \( \mathcal{U}_q(\mathfrak{so}(5)) \), we can compute the eigenvalues directly by acting with these operators on the highest weight vectors from Proposition 4. The next lemma shows that it suffices to compute the eigenvalues of \( \partial_j \partial_j^\dagger \) and \( \partial_j^\dagger \partial_j \) on a certain collection of 1-dimensional spaces from Proposition 4, the other eigenvalues can be deduced from these.

**Lemma 9.** (a) In the notation of Proposition 4 and with \( \lambda_k \) defined in (66), let \( \mu \in \mathbb{N}_0 \times 2\mathbb{N}_0 \) such that \( \dim (\mu^1 V_{(1,2)})_\lambda = \dim (\mu^3 V_{(3,0)})_\lambda = 1 \). Then the 1-dimensional operators \( \partial_k^\dagger \partial_k : (\mu^1 V_{(1,2)})_\lambda \to (\mu^1 V_{(1,2)})_\lambda \) and \( \partial_k \partial_k^\dagger : (\mu^3 V_{(3,0)})_\lambda \to (\mu^3 V_{(3,0)})_\lambda \) have the same eigenvalue.

(b) Let \( \mu \in \mathbb{N}_0 \times 2\mathbb{N}_0 \) and \( k \in \{1, 2\} \) such that \( \dim (\mu^1 V_{(1,2)})_\lambda = 2 \). Assume that \( \partial_{k-1}^\dagger \partial_{k-1} \) and \( \partial_k \partial_k^\dagger \) have the non-zero eigenvalues \( c_{\mu}^{k-1} = d_{\mu}^{k+1} \) and \( c_{\mu}^{k+1} = d_{\mu}^{k+1} \), respectively. Then there exists an orthogonal basis \( \{a, b\} \subset (\mu^1 V_{(1,2)})_\lambda \) such that

\[
\partial_{k-1}^\dagger \partial_{k-1}(a) = c_{\mu}^{k-1} a, \quad \partial_{k-1}^\dagger \partial_{k-1}(b) = 0, \quad \partial_k \partial_k^\dagger \partial_k(a) = 0, \quad \partial_k \partial_k^\dagger \partial_k(b) = d_{\mu}^{k+1} b.
\]

**Proof.** (a) It is known that \( \text{spec}(T^* T) \setminus \{0\} = \text{spec}(T T^*) \setminus \{0\} \) for all operators \( T \) from the C*-algebra \( B(\mathcal{H}) \). By considering the Hilbert space \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), it is readily seen that the same holds true for operators \( T \in B(\mathcal{H}_1, \mathcal{H}_2) \). Since \( \dim (\mu^1 V_{(1,2)})_\lambda = \dim (\mu^3 V_{(3,0)})_\lambda = 1 \), the eigenvalues of \( \partial_k^\dagger \partial_k \) and \( \partial_k \partial_k^\dagger \) are either both zero, or both are non-zero and equal.

(b) If \( \dim (\mu^1 V_{(1,2)})_\lambda = 2 \), then \( \dim (\mu^1 V_{(1,2)})_\lambda = \dim (\mu^3 V_{(3,0)})_\lambda = 1 \) by Proposition 4. Considering the 1-dimensional operators \( \partial_{k-1}^\dagger : (\mu^1 V_{(1,2)})_\lambda \to \text{im}(\partial_{k-1}) \subset (\mu^1 V_{(1,2)})_\lambda \) and \( \partial_k^\dagger : (\mu^1 V_{(1,2)})_\lambda \to \text{im}(\partial_k) \subset (\mu^1 V_{(1,2)})_\lambda \) it follows from the same arguments as in the proof of part a) that \( \partial_{k-1}^\dagger \partial_{k-1}(a) = c_{\mu}^{k+1} a \) for all \( a \in \text{im}(\partial_{k-1}) \) and \( \partial_k^\dagger \partial_k(b) = d_{\mu}^{k+1} b \) for all \( b \in \text{im}(\partial_k) \). Moreover, \( \text{im}(\partial_{k-1}) \neq \{0\} \) and \( \text{im}(\partial_k) \neq \{0\} \) since the eigenvalues \( c_{\mu}^{k-1} \) and \( d_{\mu}^{k+1} \) are non-zero. As (75) is a complex, we have \( \text{im}(\partial_{k-1}) \subset \ker(\partial_k) = \text{im}(\partial_k)^\perp \). Therefore any pair of non-zero elements \( a \in \text{im}(\partial_{k-1}) \) and \( b \in \text{im}(\partial_k) \) yields the required orthogonal basis. \( \square \)
The following lemma shows how the eigenvalues and eigenvectors of the “Laplacians” $\delta_k^\dagger \delta_k$ and $\delta_k \delta_k^\dagger$ determine the eigenvalues and eigenvectors of the Dolbeault–Dirac operator $D := \bar{\delta} + \bar{\delta}^\dagger$.

**Lemma 10.** (a) Let $\mu \in \mathbb{N}_0 \times 2\mathbb{N}_0$ and $k \in \{0, 1, 2\}$ be given such that $\dim((\mu)_\mu \Omega_{\lambda_k}) = 1$, and let $\delta^\dagger_k \delta_k(u) = c^k_\mu u$ for all $u \in (\mu)_\mu \Omega_{\lambda_k}$, where $c^k_\mu > 0$. Then there exists an orthonormal basis $\{u_-, u_+\} \subset (\mu)_\mu \Omega_{\lambda_k}$ such that $Du_- = -\sqrt{c^k_\mu} u_-$ and $Du_+ = \sqrt{c^k_\mu} u_+$.

(b) Let $\mu \in \mathbb{N}_0 \times 2\mathbb{N}_0$ and $k \in \{1, 2, 3\}$ be given such that $\dim((\mu)_\mu \Omega_{\lambda_k}) = 1$, and let $\delta_k \delta_k^\dagger(\nu) = d^k_\mu \nu$ for all $\nu \in (\mu)_\mu \Omega_{\lambda_k}$, where $d^k_\mu > 0$. Then there exists an orthonormal basis $\{v_-, v_+\}$ in the orthogonal sum $\dim((\mu)_\mu \Omega_{\lambda_k}) = 1$ such that $Dv_- = -\sqrt{d^k_\mu} v_-$ and $Dv_+ = \sqrt{d^k_\mu} v_+$.

(c) Let $\mu \in \mathbb{N}_0 \times 2\mathbb{N}_0$ and $k \in \{1, 2\}$ be given such that $\dim((\mu)_\mu \Omega_{\lambda_k}) = 2$ and assume that $\delta^\dagger_{k-1} \delta_{k-1}(u) = c^{k-1}_\mu u$ for all $u \in (\mu)_\mu \Omega_{\lambda_{k-1}}$ and $\delta_k \delta_k^\dagger(\nu) = d^{k+1}_\mu \nu$ for all $\nu \in (\mu)_\mu \Omega_{\lambda_{k+1}}$. Then there exists an orthonormal basis $\{u_-, u_+, v_-, v_+\}$ such that $Du_\pm = \pm \sqrt{c^{k-1}_\mu} u_\pm$ and $Dv_\pm = \pm \sqrt{d^{k+1}_\mu} v_\pm$.

**Proof.** (a) Let $u \in (\mu)_\mu \Omega_{\lambda_k} \setminus \{0\}$ be given. With the unitary operator $J_k$ from Lemma 6, set $\tilde{u} := J_k(u) \in (\mu)_\mu \Omega_{\lambda_{k+1}}$ and $\tilde{\tilde{u}} := \tilde{\delta}_k(\tilde{u}) \in (\mu)_\mu \Omega_{\lambda_{k+1}}$. Then $\tilde{\delta}_k(\tilde{u}) = J_k J_k^\dagger \tilde{\delta}_k \delta_k(\tilde{u}) = J_k \delta^\dagger_k \delta_k(u) = c^k_\mu J_k(u) = c^k_\mu \tilde{u}$.

In particular, $\tilde{\tilde{u}} \neq 0$ since $c^k_\mu \neq 0$ and $\tilde{u} \neq 0$. Moreover, $\pm \sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}} \neq 0$ since $\tilde{u} \perp \tilde{\tilde{u}}$, and $\tilde{\delta}_{k+1}(\tilde{u}) = \tilde{\delta}_{k+1} \tilde{\delta}_k(\tilde{u}) = 0 = \frac{1}{c^k_\mu} \tilde{\delta}_{k-1}^\dagger \tilde{\delta}_k(\tilde{u}) = \tilde{\delta}_{k-1}^\dagger(\tilde{u})$ since (75) is a complex and $\tilde{\delta}_{k-1}^\dagger \tilde{\delta}_k^\dagger = (\tilde{\delta}_k \tilde{\delta}_{k-1})^\dagger$. Therefore,

$$D(\pm \sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}}) = \pm \sqrt{c^k_\mu} \tilde{\delta}_k(\tilde{u}) + \tilde{\delta}_{k-1}^\dagger(\tilde{u}) = \pm \sqrt{c^k_\mu} \tilde{\tilde{u}} + c^k_\mu \tilde{u} = \pm \sqrt{c^k_\mu}(\pm \sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}}).$$

In particular, $\sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}}$ and $-\sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}}$ are orthogonal because they are eigenvectors corresponding to distinct eigenvalues of the symmetric operator $D = \tilde{\delta} + \tilde{\delta}^\dagger$. Finally, setting $u_\pm := \frac{1}{\|\pm \sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}}\|}(\pm \sqrt{c^k_\mu} \tilde{u} + \tilde{\tilde{u}})$ gives the required orthonormal basis.

(b) Analogously, set $v_\pm := \frac{1}{\|\tilde{\delta}_{k-1}(J_k(v)) \mp \sqrt{d^k_\mu} J_k(v)\|}(\tilde{\delta}_{k-1}^\dagger J_k(v)) \pm \sqrt{d^k_\mu} J_k(v)$ for $v \in (\mu)_\mu \Omega_{\lambda_k} \setminus \{0\}$. By similar arguments as in a), it can be shown that $\{v_-, v_+\}$ is an orthonormal basis of $\tilde{\delta}^\dagger((\mu)_\mu \Omega_{\lambda_{k-1}}) \subset (\mu)_\mu \Omega_{\lambda_{k-1}}$ such that $Dv_\pm = \pm \sqrt{d^k_\mu} v_\pm$. 
(c) Finally, if \( \dim((\mu) V_{\lambda_k}^\lambda) = 2 \), then \( \dim((\mu) V_{\lambda_{k-1}}^{\lambda_{k-1}}) = \dim((\mu) V_{\lambda_{k+1}}^{\lambda_{k+1}}) = 1 \) by Proposition 4. Applying a) and b) to \( u \in (\mu) V_{\lambda_{k-1}}^{\lambda_{k-1}} \setminus \{0\} \) and \( v \in (\mu) V_{\lambda_{k+1}}^{\lambda_{k+1}} \setminus \{0\} \) gives thus an orthonormal bases \( \{u_-, u_+\} \subset (\mu) \Omega(0,k-1) \oplus \delta_{k-1}^- (\mu) \Omega(0,k-1) \) corresponding to the eigenvalues \( \pm \sqrt{c_{\mu}^k} \) and also an orthonormal bases \( \{v_-, v_+\} \subset \delta_{k}^+ (\mu) \Omega(0,k+1) \oplus (\mu) \Omega(0,k+1) \) corresponding to the eigenvalues \( \pm \sqrt{d_{\mu}^{k+1}} \).

The previous lemma allows us to deduce a complete list of eigenvalues and their multiplicities for the Dolbeault–Dirac operator from the eigenvalues of the “Laplacians” \( \delta_k^\dagger \delta_k \) and \( \delta_k \delta_k^\dagger \). Moreover, by Lemma 9 and Lemma 10,(c), it suffices to compute the eigenvalues on 1-dimensional highest weight spaces. This can be done directly and without the need to compute inner products by applying the algebraic expressions of \( \delta_j^\dagger \) and \( \delta_j \) from Corollary 7 and Lemma 8 to the generating vectors in Proposition 4. We pursue this strategy in Proposition 13 to compute the eigenvalues of \( \delta_0^\dagger \delta_0 \) and \( \delta_1 \delta_1^\dagger \) on 1-dimensional highest weight spaces. The eigenvalues of \( \delta_0^\dagger \delta_0 \) and \( \delta_1 \delta_1^\dagger \) will be obtained in the next proposition by using the Casimir operators from Sect. 2.

**Proposition 11.** For \( n, l \in \mathbb{N}_0 \),

\[
\delta_0^\dagger \delta_0(u) = \frac{c_0^2}{c_0^2} q^2 [2] \frac{[n + l + 2][n + l + 1] [n + 1][n]}{[3]_2} u, \quad u \in \langle (2n,2l) \rangle V(0,0),
\]

\[
\delta_2 \delta_2^\dagger(v) = \frac{c_2^2}{c_2^2} q^2 [n + l + 3][n + l + 2][n + 3][n + 1] [n + 1][n] \frac{[n + 1][n]}{[3]_2} v, \quad v \in \langle (2n+3,2l) \rangle V(0,0),
\]

**Proof.** Let \( u \in \langle (2n,2l) \rangle V(0,0) \) with \( \|u\| = 1 \). As \( \delta_0^\dagger \delta_0(u) \in \langle (2n,2l) \rangle V(0,0) = \text{span}\{u\} \), it is clear that \( \delta_0^\dagger \delta_0(u) = \lambda u \) with \( \lambda = \langle u, \delta_0^\dagger \delta_0(u) \rangle \). Since \( \text{pr}(0,0) \) is an orthogonal projection, we have, for all \( v \in \langle (2n,2l) \rangle V(0,2)_2 \),

\[
\frac{c_0^2}{q^2 c_1} \langle u, \delta_0^\dagger \delta_0(v) \rangle = \langle u, \text{pr}(0,0)(F_1 F_2^2(v)) \rangle = \langle \text{pr}(0,0)(u), F_1 F_2^2(v) \rangle = \langle u, F_1 F_2^2(v) \rangle.
\]

By (5) and (25), \( F_2 E_1(u) = E_1 F_2(u) = 0 \), hence \( E_1(u) \) is a lowest weight vector of weight \( \alpha_1 = (2, -2) \) and consequently \( E_1(u) \in \langle (2n,2l) \rangle V(0,2)_{(2,2)} \). Applying now (24) and (107), we get

\[
\frac{c_0^2}{q^2 c_1} \langle u, \delta_0^\dagger \delta_0(u) \rangle = \langle u, F_1 F_2^2 E_2^2 E_1(u) \rangle = [2]_2 \langle u, F_1 F_2 E_2 E_1(u) \rangle = [2]_2 \langle u, F_1 E_1(u) \rangle.
\]

Let \( K_1, K_2 \) and \( E_1, \ldots, E_4 \) be given as in (11). Again by (25), \( K_i^+(u) = K_i(u) = u \) for \( i = 1, 2 \) and \( E_i(u) = E_2(u) = F_2^*(u) = 0 \). Hence

\[
\langle u, K_i^\pm Z(u) \rangle = \langle u, K_i^\pm Z(u) \rangle = \langle u, Z(u) \rangle, \quad i = 1, 2.
\]
and

\[ \langle u, Z\mathcal{E}_1(u) \rangle = \langle u, \mathcal{E}_1^* Z(u) \rangle = \langle u, Z\mathcal{E}_2(u) \rangle = \langle u, F_2 Z(u) \rangle = 0 \]  

(110)

for all \( Z \in \mathcal{U}_q(\text{so}(5)) \). Equations (5), (7), (11), (108), (109) and (110) imply

\[
\langle u, \mathcal{E}_2^* \mathcal{E}_2(u) \rangle = q^4 \langle u, F_1 F_2^2 E_2^2 E_1(u) \rangle = \frac{q^2 c_0^2}{c_1^2} \langle u, \mathcal{E}_0^* \mathcal{E}_0(u) \rangle, \\
\langle u, \mathcal{E}_3^* \mathcal{E}_3(u) \rangle = q^3 \langle u, F_1 F_2 E_2 E_1(u) \rangle = \frac{q c_0^2}{[2]_2 c_1^2} \langle u, \mathcal{E}_0^* \mathcal{E}_0(u) \rangle, \\
\langle u, \mathcal{E}_4^* \mathcal{E}_4(u) \rangle = q \langle u, F_1 E_1(u) \rangle = \frac{c_0^2}{q[2]_2 c_1^2} \langle u, \mathcal{E}_0^* \mathcal{E}_0(u) \rangle. 
\]

Inserting (12) into \( \langle u, S(C)(u) \rangle \) and using (109)–(113) yields

\[
\langle u, S(C)(u) \rangle = \frac{q^{-2} + q^{-1} + q + q^2}{(q_2 - q_2^{-1})^2} + \frac{c_0^2}{q^2[2]_2 c_1^2} \left( [2]_2 + (q_2^{-3} + q_2^3) + [2]_2 \right) \langle u, \mathcal{E}_0^* \mathcal{E}_0(u) \rangle. 
\]

(114)

On the other hand, \( S(C) = \lambda_{(2n,2l)} \text{id} \) on \( \langle (2n,2l) \rangle V(0,0) \subset \langle (2n,2l) \rangle V \otimes \langle (2n,2l) \rangle \) for some scalar \( \lambda_{(2n,2l)} \). Choosing a unit vector \( u_0 \in \langle (2n,2l) \rangle V \otimes \langle (2n,2l) \rangle \), which is a highest weight vector of weight \( (2n,2l) \) for the left regular representation of \( \mathcal{U}_q(\text{so}(5)) \), we have \( \mathcal{K}_1(u_0) = q^{-2n-2l} u_0, \mathcal{K}_2(u_0) = q^{-2n} u_0 \) and \( E_1(u_0) = E_2(u_0) = 0 \), so that

\[
\lambda_{(2n,2l)} = \langle u, S(C)(u) \rangle = \langle u_0, S(C)(u_0) \rangle = \frac{q^{-2n-2l-2} + q^{-2n-1} + q^{2n+1} + q^{2n+2l+2}}{(q_2 - q_2^{-1})^2}. 
\]

(115)

Straightforward calculations show that

\[
[2]_2 + (q_2^{-3} + q_2^3) + [2]_2 = [4]_2 + [2]_2 = [3]_2 [2]_2, \\
\frac{q^{-2n-2l-2} + q^{-2n-1} + q^{2n+1} + q^{2n+2l+2}}{(q_2 - q_2^{-1})^2} = [2]_2^2 ([n+l+2][n+l+1] + [n+1][n]_1). 
\]

(116)

(117)

From (114)–(117), we finally obtain

\[
\lambda = \langle u, \mathcal{E}_0^* \mathcal{E}_0(u) \rangle = \frac{q^2 c_1^2}{[3]_2 c_0^2} \left( \langle u, S(C)(u) \rangle - \frac{q^{-2} + q^{-1} + q + q^2}{(q_2 - q_2^{-1})^2} \right) = \frac{q^2 [2]_2^2 ([n+l+2][n+l+1] + [n+1][n]_1)}{[3]_2} c_1^2 c_0^2. 
\]

(118)

The eigenvalues of the operator \( \mathcal{E}_2^* \mathcal{E}_2 \) can be determined in the same way. Let \( v \in \langle (2n+3,2l) \rangle V \langle (3,0) \rangle \) such that \( \|v\| = 1 \). Since \( \langle (2n+3,2l) \rangle V \langle (3,0) \rangle \) is 1-dimensional, we have \( \mathcal{E}_2^* \mathcal{E}_2(v) = \mu v \) with \( \mu = \langle v, \mathcal{E}_2^* \mathcal{E}_2(v) \rangle \). As in (107),

\[
\frac{c_2}{c_3} \langle v, \mathcal{E}_2(w) \rangle = \langle v, \text{pr}_{(3,0)}(E_1(w)) \rangle = \langle \text{pr}_{(3,0)}(v), E_1(w) \rangle = \langle v, E_1(w) \rangle 
\]

(119)
for all $w \in \mathcal{V}^{(1,2)}(\mathbb{C}^{2n+2l})$. By (102), $\bar{\partial}^+_2(v) = q^2 c_3^2 F_1(v) \in \mathcal{V}^{(1,2)}(\mathbb{C}^{2n+2l})$ is a highest weight vector of weight $(1, 2)$, therefore

$$\frac{c_2^2}{q^2 c_3^2} \langle v, \bar{\partial}^+_2(v) \rangle = \langle v, E_1 F_1(v) \rangle = \frac{1}{[2]^2} \langle v, E_1 E_2 F_2 F_1(v) \rangle = \frac{1}{[2]^2} \langle v, E_1 E_2^2 F_2^2 F_1(v) \rangle. \quad (120)$$

Next, $K_1(v) = q^3 v$, $K_2(v) = v$ and $K_i^* = K_i$ imply

$$\langle v, K_i^\pm Z(v) \rangle = q^{\pm 3} \langle v, Z(v) \rangle, \quad \langle v, K_2^\pm Z(v) \rangle = \langle v, Z(v) \rangle, \quad (121)$$

$$\langle v, K_i^\pm Z(v) \rangle = \langle v, K_2^\pm Z(v) \rangle = q^{\mp 3} \langle v, Z(v) \rangle. \quad (122)$$

Recall the definitions of $\mathcal{F}_1, \ldots, \mathcal{F}_4$ in (9). Then $\mathcal{F}_1(v) = F_2(v) = E_2^*(v) = 0$ yields

$$\langle v, Z \mathcal{F}_1(v) \rangle = \langle v, \mathcal{F}_1^* Z(v) \rangle = \langle v, Z F_2(v) \rangle = \langle v, E_2 Z(v) \rangle = 0 \quad (123)$$

for all $Z \in \mathcal{U}_q(\text{so}(5))$. Furthermore, analogous to (111)–(113), we deduce from the previous relations that

$$\langle v, \mathcal{F}_2^* \mathcal{F}_2(v) \rangle = \langle v, K_1^{-1} K_2^{-2} E_1 E_2^2 F_2^2 F_1(v) \rangle = \frac{[2]^2 c_2^2}{q^2 c_3^2} \langle v, \bar{\partial}_2 \bar{\partial}_2^\dagger(v) \rangle, \quad (124)$$

$$\langle v, \mathcal{F}_3^* \mathcal{F}_3(v) \rangle = \langle v, K_1^{-1} K_2^{-1} E_1 E_2 F_2 F_1(v) \rangle = \frac{[2]^2 c_2^2}{q^2 c_3^2} \langle v, \bar{\partial}_2 \bar{\partial}_2^\dagger(v) \rangle, \quad (125)$$

$$\langle v, \mathcal{F}_4^* \mathcal{F}_4(v) \rangle = \langle v, q K_1^{-1} E_1 F_1(v) \rangle = \frac{c_2^2}{q^4 c_3^2} \langle v, \bar{\partial}_2 \bar{\partial}_2^\dagger(v) \rangle. \quad (126)$$

From (122)–(126) and (116), it follows that

$$\langle v, \mathcal{C}(v) \rangle = \frac{q + q^2 + q^{-2} + q^{-1}}{(q_2 - q_2^{-1})^2} + \frac{[3]_2 [2]_2 c_2^2}{q^2 c_3^2} \langle v, \bar{\partial}_2 \bar{\partial}_2^\dagger(v) \rangle. \quad (127)$$

To determine $\langle v, \mathcal{C}(v) \rangle$, we use again the fact that $\mathcal{C}$ is a multiple of the identity on the whole space $(\mathbb{C}^{2n+2l})^* \otimes (\mathbb{C}^{2n+2l})^*$ and evaluate $\mathcal{C} = \lambda_{(2n+3,2l)} \text{id}$ on a lowest weight vector $v_0 \in \mathcal{V}^{(2n+3,2l)}(\mathbb{C}^{2n+2l}) \otimes \mathcal{V}^{(-2n+3,-2l)}(\mathbb{C}^{2n+2l})$ for which $F_1(v_0) = F_2(v_0) = 0$. Assuming $\|v_0\| = 1$ and inserting (10) into $\langle v_0, \mathcal{C}(v_0) \rangle$ gives

$$\lambda_{(2n+3,2l)} = \langle v_0, \mathcal{C}(v_0) \rangle = \langle v_0, \mathcal{C}(v_0) \rangle = \frac{q^{-2n-2l-5} + q^{-2n-4} + q^{2n+4} + q^{2n+2l+5}}{(q_2 - q_2^{-1})^2}. \quad (128)$$

Similarly to (117), we have

$$q^{-2n-2l-5} + q^{-2n-4} + q^{2n+4} + q^{2n+2l+5} - q - q^{-2} - q^{-2} - q^{-1} \quad (129)$$

$$= [2]_2^5 ([n + l + 3]_1 [n + l + 2]_1 + [n + 3]_1 [n + 1]_1).$$
Combining (127)–(129) yields
\[
\langle v, \partial_2 \partial_2^+(v) \rangle = \frac{q^2([n + l + 3]_1[n + l + 2]_1 + [n + 3][n + 1]_1)}{[3]_2} \frac{c_3^2}{c_2^2},
\]
which proves (106).

Before determining in Proposition 13 the eigenvalues of $\partial_1^+\partial_1$ and $\partial_1 \partial_1^+$ on 1-dimensional weight spaces by computing directly the action on highest weight vectors, we collect some auxiliary results regarding the projections $\text{pr}_{(0,2)}$ and $\text{pr}_{(1,2)}$ from (26) in the following lemma.

**Lemma 12.** Let $z_1$ and $z_2$ be given as in (82) and set $b := (u_1^1 u_4^2 - q u_2^2 u_4^1)u^1_2$. Then, with $X_0 := E_2 E_1$,
\[
\begin{align*}
\text{pr}_{(1,2)}(bX_0(z_1)) &= \frac{[2]_2^{1/2}}{[2]_1} (q^4[2]_2(u_4^1 u_2^2 - q u_2^2 u_4^1)u^1_5 z_1 - q^{-1/2} z_2 u_4^1 (u^1_5)^2), \\
\text{pr}_{(1,2)}(X_0(z_2)b) &= \frac{[2]_2^{1/2}}{q^{5/2}[2]_1} z_2 (u_4^1 u_2^2 - q u_2^2 u_4^1)u^1_5, \\
\text{pr}_{(0,2)}(F_1(z_2)u_3^1 (u_5^1)^2) &= -\frac{q^5}{[2]_1} b z_1, \\
\text{pr}_{(0,2)}(F_1(z_2)(u_4^1 u_2^2 - q u_2^2 u_4^1)u_5^1) &= -\frac{q^3}{[2]_1} z_2 b, \\
\text{pr}_{(0,2)}((u_3^1 u_5^2 - q u_2^2 u_4^1)u_4^1) &= \frac{q^{-2}}{[2]_1} b.
\end{align*}
\]
Let $e := (u_3^1 u_4^2 - q u_2^2 u_4^1)(u_5^1)^2$. Then
\[
\begin{align*}
\text{pr}_{(0,2)}(F_1(z_2) F_2(e)) &= q^4[2]_2^{1/2} (z_2^2 u_4^1 u_3^1 - (u_4^1 u_2^2 - q u_2^2 u_4^1)z_1), \\
\text{pr}_{(0,2)}(F_2(e) F_1(z_1)) &= \frac{q^{-4}}{[2]_1} (z_2 (u_4^1 u_2^2 - q u_2^2 u_4^1)z_1 - (q^2 + q^3) (u_4^1 u_2^2 - q u_2^2 u_4^1)z_1^2), \\
\text{pr}_{(0,2)}(F_2(F_1(e))) &= [2]_2^{1/2} ((q + q^{-3}) z_2 u_4^1 u_5^1 - (1 + q^{-1} + q^3) (u_4^1 u_2^2 - q u_2^2 u_4^1)z_1), \\
\text{pr}_{(1,2)}(X_0(z_2) u_4^1 u_5^1) &= \frac{q^{-1} [2]_2^{1/2}}{[2]_1} e, \\
\text{pr}_{(1,2)}((u_3^1 u_5^2 - q u_2^2 u_4^1) X_0(z_1)) &= -\frac{[2]_2^{1/2}}{[2]_1} e.
\end{align*}
\]

**Proof.** The lemma is proven by direct computation using the action (15) and the relations in Lemma 1. For the convenience of the reader, we show (130). The proof of the other equations is similar and can be found in [8].

First note that $b \in \langle (1,2) \rangle V^{(0,2)}(2,0)$ and $X_0(z_1) \in \langle (2,0) \rangle V^{(0,0)}(0,0)$ by Proposition 4. Thus, by (14) and (24), $E_2^2(bX_0(z_1)) = 0$, $K_1(b X_0(z_1)) = q b X_0(z_1)$ and $K_2(b X_0(z_1)) = q b X_0(z_1)$, which shows that $b X_0(z_1) \in \langle (3,2) \rangle V^{(1,2)}(3,2) \oplus \langle (3,2) \rangle V^{(0,4)}(3,2)$. As a consequence, the orthogonal projection onto $\langle (3,2) \rangle V^{(1,2)}(3,2)$ is given by (26). From (8),
Inserting \( h \) hence there exist

\[
\left( q^2 \right) \left( q^2 \right) = \left( q^2 \right) \left( q^2 \right) = \left( q^2 \right) \left( q^2 \right) = \left( q^2 \right) \left( q^2 \right) = \left( q^2 \right) \left( q^2 \right).
\]

where we applied (37) in the last equation, and \( u_3^j u_j^1 = q^{-1} u_j^1 u_3^j \), \( j = 3, 4 \), we obtain

\[
\text{pr}_{(1,2)}(b X_0(z_1)) = \frac{q[2]_{1/2}^1}{[2]_1} \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 (u_3^1)\right)^2 \\
+ \frac{q^{-1/2}[2]_{1/2}^2}{[2]_1} \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 (u_3^1)\right)^2 + \frac{\left( q^{-3/2} - q^{-1/2}\right)[2]_{1/2}^2}{[2]_1} \left( z u_3^1 (u_3^1)\right)^2.
\]

(140)

As \( \text{pr}_{(1,2)}(b X_0(z_1)), z u_3^1 (u_3^1) \right)^2 \in \left( \mathcal{O}(\text{SO}(3)), \mathcal{V}_{(1,2)} \right) \) span \((u_3^1 u_4^1 - q u_3^1 u_4^1) u_3^1 z_1, z u_3^1 (u_3^1)\), there exist \( \alpha, \beta \in \mathbb{C} \) such that

\[
q \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 (u_3^1)\right)^2 + q^{-1/2} \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 (u_3^1)\right)^2 \\
= \alpha \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 z_1 + \beta z u_3^1 (u_3^1)\right)^2.
\]

(141)

Since \( u_3^1 z_1 = q^{-2} u_3^1 (u_3^1) \) and \( \mathcal{O}(\text{SO}(3)) \) has no zero divisors, (141) is equivalent to

\[
q \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 + q^{-1/2} \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 \right)^2 \right) \\
= \alpha q^{-2} \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 + \beta (u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1) u_3^1 \right).
\]

(142)

Acting on both sides of (142) with \( F_1 \) yields

\[
q^{-1/2} \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 \right) u_3^1 = -\beta q \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 \right) u_3^1,
\]

hence \( \beta = -q^{-3/2} \). Similarly, acting on both sides of (142) with \( -q^{-1/2} E_2^2 E_1 \) gives

\[
\left( q^3/2 + q^{5/2} - q^{1/2}\right) \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 \right) u_3^1 = \left( q^{-2} \alpha + q^2 \beta\right) \left( u_3^1 u_4^1 - q u_3^1 u_4^1 u_3^1 \right) u_3^1.
\]

Thus, with \( \beta = -q^{-3/2} \), we obtain \( \alpha = q^4 [2]_2 \). Inserting first \( \alpha \) and \( \beta \) into (141) and then (141) into (140) proves (130).

\[\square\]

**Proposition 13.** For \( n, l \in \mathbb{N}_0 \),

\[
\partial_1^\dagger \partial_1^\dagger (u) = \frac{c_2^2}{c_1^2} q^2 \left[ 2 \right]_2 \left( \left[ 3 \right]_2 - 1 \right) \left( \left[ n + l + 3 \right]_1 \left[ n + l + 2 \right]_1 + \left[ n + 2 \right]_1 \left[ n \right]_1 \right) u,
\]

\[ u \in \left( \left( 2n + 1, 2l + 2 \right) \right) \mathcal{V} \left( 0, 2 \right), \quad \left( 2n + 1, 2l + 2 \right) \mathcal{V} \left( 0, 2 \right). \tag{143} \]

\[
\partial_1 \partial_1^\dagger (v) = \frac{c_2^2}{c_1^2} q^2 \left[ 2 \right]_2 \left( \left[ 3 \right]_2 - 1 \right) \left( \left[ n + l + 4 \right]_1 \left[ n + l + 2 \right]_1 + \left[ n + 2 \right]_1 \left[ n + 1 \right]_1 \right) v,
\]

\[ v \in \left( \left( 2n + 2, 2l + 2 \right) \right) \mathcal{V} \left( 1, 2 \right), \quad \left( 2n + 2, 2l + 2 \right) \mathcal{V} \left( 1, 2 \right). \tag{144} \]
Proof. Recall that \( X_0 = E_2 E_1, z_1 = u_1^1 u_3^1 \) and \( z_2 = u_1^1 u_3^2 - q u_3^2 u_5^1 \). Applying the formulas of the left action from Sect. 2 gives

\[
X_0(z_1) = q[2]^{1/2} u_3^1 u_3^1, \quad X_0(z_2) = q[2]^{1/2} (u_3^1 u_3^2 - q u_3^2 u_5^1),
\]
\[
F_1(z_1) = -qu_1^1 u_4^1, \quad F_1(z_2) = -(u_1^1 u_4^2 - q u_3^2 u_4^1).
\]  

(145)

From Lemma 1, it follows that

\[
z_1 X_0(z_1) = q^2 X_0(z_1) z_1, \quad z_2 X_0(z_2) = q^2 X_0(z_2) z_2,
\]
\[
F_1(z_1) z_1 = q^2 z_1 F_1(z_1), \quad F_1(z_2) z_2 = q^2 z_2 F_1(z_2).
\]  

(146)

Note that, by (14) and (17), \( X_0(ab) = X_0(a) b + a X_0(b) \) and \( F_1(ab) = F_1(a) b + a F_1(b) \) for all \( a, b \in B \). Therefore (146) gives for \( n, l \in \mathbb{N} \)

\[
X_0(z_1^n) = \sum_{k=0}^{n-1} z_1^k X_0(z_1) z_1^{n-k-1} = \sum_{k=0}^{n-1} q^{2k} X_0(z_1) z_1^{n-1} = q^{n-1} [n]_1 X_0(z_1) z_1^{n-1},
\]
\[
X_0(z_2^l) = \sum_{k=0}^{l-1} z_2^k X_0(z_2) z_2^{l-k-1} = \sum_{k=0}^{l-1} q^{-2k} z_2^{l-1} X_0(z_2) = q^{-l+1} [l]_1 z_2^{l-1} X_0(z_2),
\]  

(147)

and analogously

\[
F_1(z_1^n) = q^{-n+1} [n]_1 F_1(z_1) z_1^{n-1}, \quad F_1(z_2^l) = q^{-l} [l]_1 z_2^{l-1} F_1(z_2).
\]  

(148)

Let \( b := (u_1^1 u_3^2 - q u_3^2 u_4^1) u_3^1 \). Then \( z_1^b b z_1^n \) spans \( (2n+1,2l+2) V_{(0,2)} \) and \( E_2(b) = 0 \) by Proposition 4. Using again (14) and (17), we get

\[
pr_{(1,2)} \circ X_0(z_1^b z_1^n) = pr_{(1,2)}(X_0(z_1^b) z_1^n + z_1^b X_0(b) z_1^n + z_1^b b X_0(z_1^n)) = pr_{(1,2)}(q^{-l+2} [l]_1 z_1^{l-1} X_0(z_2) z_2^n + z_1^b X_0(b) z_2^n + q^{n-1} [n]_1 z_1^b b X_0(z_1) z_1^{n-1}) = q^{-l+2} [l]_1 z_1^{l-1} pr_{(1,2)}(X_0(z_2) b z_1^n + z_1^b pr_{(1,2)}(X_0(b) z_1^n)) + q^{n-1} [n]_1 z_1^b pr_{(1,2)}(b X_0(z_1)) z_1^{n-1}.
\]  

(149)

where we applied (147) in the second equality and (28) in the third. By the formulas in (15), \( X_0(b) = E_2 E_1((u_1^1 u_3^2 - q u_3^2 u_4^1) u_3^1) = q^{3/2} [2]^{1/2} (u_4^1 u_5^2 - q u_3^1 u_5^1) u_3^1 \in (1,2) V_{(1,2)} \), so that \( pr_{(1,2)}(X_0(b)) = X_0(b) \). Inserting \( pr_{(1,2)}(b X_0(z_1)) \) and \( pr_{(1,2)}(X_0(z_2) b) \) from Lemma 12 gives

\[
pr_{(1,2)} \circ X_0(z_1^b z_1^n) = \left( (q^{3/2} [2]^{1/2} + q^{-l} [l]_1 [2]^{1/2} q^{-l} [2]_1 + q^{n+3} [n]_1 [2]^{3/2} \right)
\times z_1^b (u_4^1 u_5^2 - q u_3^1 u_5^1) u_3^1) - q^{n-1} [n]_1 [2]^{1/2} [2]_1 z_2^{l+1} u_4^1 (u_5^2)^2 z_1^{n-1}.
\]  

(150)
Our next aim is to compute the action of $\text{pr}_{(0,2)} \circ F_1 F_2$ on (150). Acting by $F_2$ on the last equation yields

$$
F_2(\text{pr}_{(1,2)} \circ X_0(z_2^l b z_1^n)) = q^{n-2}[n_1] \frac{[2]}{[2]} l z_2^{l+1} u_3^1 u_5^1 z_1^{n-1}
- (q[2]_2 + q^{-l-1}[l_1] \frac{[2]}{[2]} + q^{n+3}[n_1] \frac{[2]}{q^{l/2}[2]} l z_2^{l}(u_3^2 u_5^2 - qu_3^2 u_5^1)u_5^1 z_1^n. \quad (151)
$$

Using (148) and $u_5^1 F_1(z_1) = -q^{-1} u_4^1 z_1$, we compute that

$$
F_1(z_2^l u_3^1 u_5^1 z_1^{n-1}) = q^{-l}[l_1] z_2^{l-1} F_1(z_2) (u_3^1 u_5^2 - qu_3^2 u_5^1) u_5^1 z_1^n
- z_2^{l}(u_3^1 u_4^2 - qu_3^2 u_4^1) u_5^1 z_1^n - (q^{-l} + q^{-n-2}[n_1]) z_2^{l}(u_3^1 u_4^2 - qu_3^2 u_4^1) u_4^1 z_1^n \quad (152)
$$

and similarly

$$
F_1(z_2^{l+1} u_3^1(u_5^1)^2 z_1^{n-1}) = q^{l+1}[l+1] z_2^{l+1} F_1(z_2) u_3^1(u_5^1)^2 z_1^{n-1}
- (q^{-2}[2]_1 + q^{-n-2}[n-1]) z_2^{l+1} u_3^1 u_4^1 u_5^1 z_1^{n-1}. \quad (153)
$$

As $u_3^1 u_4^1 u_5^1 = \alpha F_2((u_3^1)^2 u_5^1)$ with $\alpha \in \mathbb{R} \setminus \{0\}$, it follows that $u_3^1 u_4^1 u_5^1 \in \mathbb{R} \setminus \{0\} V_{(3,0)}^{(-1,4)} \oplus \mathbb{R} \setminus \{0\} V_{(0,2)}$. This and the definition of $\text{pr}_{(0,2)}$ in (26) imply $\text{pr}_{(0,2)}(u_3^1 u_4^1 u_5^1) = 0$. By applying $\text{pr}_{(0,2)}$ to (153), using (28), and inserting (132) and the last relation, we obtain

$$
\text{pr}_{(0,2)} \circ F_1(z_2^{l+1} u_3^1(u_5^1)^2 z_1^{n-1}) = -q^{l+5}[l+1] \frac{[2]}{[2]} l z_2^l b z_1^n. \quad (154)
$$

To compute the action of $\text{pr}_{(0,2)}$ on (152), recall that $z_2^{l}(u_3^1 u_4^2 - qu_3^2 u_4^1) u_5^1 z_1^n$ belongs to $\mathbb{R} \setminus \{0\} V_{(0,2)}$ and that $\text{pr}_{(0,2)}$ acts on this space as the identity. From this and Equations (133) and (134), it follows that

$$
\text{pr}_{(0,2)}(F_1(z_2^{l}(u_3^1 u_5^2 - qu_3^2 u_5^1) u_5^1 z_1^n)) = -q^{l+2}[l_1] + [2]_1 + q^{-3} + q^{-n-4}[n_1] \frac{[2]}{[2]} l z_2^l b z_1^n. \quad (155)
$$

Combining (151), (154) and (155) gives

$$
\frac{c_2^l}{c_2^2} \frac{q^{-2}[3]_2}{[(3)_2 - 1]^2} \delta_2^i \tilde{\phi}_1(z_2^l b z_1^n) = \text{pr}_{(0,2)} \circ F_1 F_2 \circ \text{pr}_{(1,2)} \circ X_0(z_2^l b z_1^n)
= \frac{[2]_2}{[2]} \bigg(- q^{n+4}[n_1][l+1]
+ \big(q[2]_1 + q^{-l-1}[l_1] + q^{n+5/2}[2]_2[n_1]) \big(q^{l+2}[l_1] + [2]_1 + q^{-3} + q^{-n-4}[n_1]\big) \bigg) \frac{z_2^l b z_1^n}{[2]_1^2}
= \frac{[2]_2}{[2]} \bigg([n+l+3]_1[n+l+2]_1 + [n+2]_1[n_1]\bigg) \frac{z_2^l b z_1^n}{[2]_1^2},
$$

where the last equation follows by elementary calculations, see [8]. This shows (143).
Equation (144) can be proven in the same way. Let $e := (u_1^2 u_2^2 - q u_2^2 u_4^2)(u_1^2)^{-2}$. By Proposition 4, $(2n^{-1} u_2)^{(1, 2)} = \text{span}[z_2^l e z_1^n]$. Since $F_2(z_1^n) = F_2(z_1^n) = 0$, we obtain from (14), (15) and (148) that

$$F_1 F_2(z_2^l e z_1^n) = q^{-l-1}[l]_1 z_2^l F_1(z_2) F_2(e) z_1^n + z_2^l F_1 F_2(e) z_1^n + q^{-n-1}[n]_1 z_2^l F_2 F_1(z_1) z_1^n - 1. $$

Applying (28) and inserting (135)–(137) gives

$$\text{pr}(0, 2) \circ F_1 F_2(z_2^l e z_1^n) = \frac{[2]^{1/2}_2 \left(q^{l+3}[l] + (q + q^{-3})[2]_1 + q^{-n-5}[n]_1\right) z_2^l u_4^l u_5^l z_1^n}{[2]_1} - \frac{[2]^{1/2}_2 \left(q^{l+5}[l] + (1 + q^{-1} + q^3)[2]_1 + (1 + q)q^{-n-3}[n]_1\right)}{[2]_1} \times z_2^l (u_4^l u_5^l - q u_2^2 u_3^2) z_1^{n+1}. $$

(156)

Next, using (14), (15), (17), (147) and $E_2(u_4^l u_5^l) = 0$, we compute that

$$X_0(z_2^l u_4^l u_5^l z_1^n) = X_0(z_2^l u_4^l u_5^l z_1^n) + z_2^l u_4^l u_5^l X_0(z_1^n) = q^{-l-1}[l]_1 z_2^l X_0(z_2) u_4^l u_5^l z_1^n + q^{-n-1}[n]_1 u_4^l u_5^l X_0(z_1) z_1^n. $$

As $u_4^l u_5^l X_0(z_1) = \beta u_4^l u_5^l u_5^l z_1^n = \overset{\beta}{\beta} F_2 F_1^2(u_4^l)^4 \in \langle 0, 4 \rangle V \otimes V^{(0, 4)}$ with $\beta, \overset{\beta}{\beta} \in \mathbb{R} \setminus \{0\}$, and $(\langle 0, 4 \rangle V^{(1, 2)}) = \{0\}$ by Proposition 3, we have $\text{pr}(1, 2)(u_4^l u_5^l X_0(z_1)) = 0$. Hence, by (28) and (138),

$$\text{pr}(1, 2) \circ X_0(z_2^l u_4^l u_5^l z_1^n) = \frac{q^{n-1}[2]^{1/2}[l_1]_1}{[2]_1} z_2^l e z_1^n. $$

(157)

Similarly, (14), (15), (17), (147) and $E_2(u_4^l u_5^l - q u_2^2 u_3^2) = 0$, we imply

$$X_0(z_2^l u_4^l u_5^l - q u_2^2 u_3^2) z_1^{n+1} = q X_0(z_2^l u_4^l u_5^l - q u_2^2 u_3^2) z_1^{n+1} + z_2^l (u_4^l u_5^l - q u_2^2 u_3^2) X_0(z_1) z_1^n. $$

As $X_0(z_2^l u_4^l u_5^l - q u_2^2 u_3^2) = \gamma (u_4^l u_5^l - q u_2^2 u_3^2)(u_4^l u_5^l - q u_2^2 u_3^2) = \gamma F_2 ((u_4^l u_5^l - q u_2^2 u_3^2)^2)$ for $\gamma \in \mathbb{R} \setminus \{0\}$, it belongs to $\langle 0, 4 \rangle V \otimes V^{(0, 4)}$. To verify above equations, use the $q$-commutation relation between the weight vectors $(u_4^l u_5^l - q u_2^2 u_3^2)$ and $(u_4^l u_5^l - q u_2^2 u_3^2)$ obtained by setting $a = 4$ in (54) and by acting on both sides of (54) by $E_2 E_1$. Since $(\langle 0, 4 \rangle V^{(1, 2)}) = \{0\}$ by Proposition 3, we get $\text{pr}(1, 2)(X_0(z_2^l u_4^l u_5^l - q u_2^2 u_3^2)) = 0$. Thus, by (28) and (139),

$$\text{pr}(1, 2) \circ X_0(z_2^l u_4^l u_5^l - q u_2^2 u_3^2) z_1^{n+1} = \frac{q^{n-1}[2]^{1/2}[l_1]_1}{[2]_1} z_2^l e z_1^n. $$

(158)

Combining (156)–(158) yields

$$\frac{c^2}{c_2^2 (3 - 1)} \partial_1 \partial_2 (z_2^l e z_1^n) = \text{pr}(1, 2) \circ X_0 \circ \text{pr}(0, 2) \circ F_1 F_2(z_2^l e z_1^n) = \lambda z_2^l e z_1^n. $$
with 
\[
\lambda = \frac{[2]^2}{[2]^2} \left( q^{-l}[l+1]_1(q^{|l+3}|[l]_1 + (q + q^{-3})[2]_1 + q^{-n-5}[n]_1) + q^n[n+1]_1(q^{|l+5}|[l]_1 + (1 + q^{-1} + q^3)[2]_1 + (1 + q)q^{-n-3}[n]_1) \right) \]
\[= \frac{[2]^2([n+l+4]_1[n+l+2]_1 + [n+2]_1[n+1]_1)}{[2]^2},\]
where the last equation can easily be shown by elementary computations, see [8]. This finishes the proof. \(\Box\)

We are now in a position to give a complete set of eigenvalues of the Dirac operator together with the corresponding multiplicities.

**Corollary 14.** Let \(D\) denote the operator closure of the densely defined, symmetric operator \(\bar{\partial} + \bar{\partial}^\dagger\) on the domain \(\Omega^{(0,\bullet)} := \Omega_2^{(0,0)} \oplus \Omega_2^{(0,1)} \oplus \Omega_2^{(0,2)} \oplus \Omega_2^{(0,3)}\). Let

\[
c^0_{(2n,2l)} := \frac{c_0^2}{\sqrt{[3]^2}} q^2[2]^2 \frac{([n+l+2]_1[n+l]_1 + [n+1]_1[n]_1)}{[3]^2}, \quad n, l \in \mathbb{N}_0,
\]
\[
c^1_{(2n+1,2l)} := \frac{c_1^2}{\sqrt{[3]^2}} q^2[2]^2([3]^2 - 1)^2 \frac{([n+l+2]_1[n+l+1]_1 + [n+2]_1[n+1]_1)}{[3]^2}, \quad n \in \mathbb{N}_0, l \in \mathbb{N},
\]
\[
d^2_{(2n,2l)} := \frac{c_2^2}{\sqrt{[3]^2}} q^2[2]^2([3]^2 - 1)^2 \frac{([n+l+2]_1[n+l+1]_1 + [n+1]_1[n]_1)}{[3]^2}, \quad n, l \in \mathbb{N},
\]
\[
d^3_{(2n+1,2l)} := \frac{c_3^2}{\sqrt{[3]^2}} q^2([n+l+2]_1[n+l+1]_1 + [n+2]_1[n+1]_1) \frac{([n+l+2]_1[n+l+1]_1 + [n+2]_1[n+1]_1)}{[3]^2}, \quad n \in \mathbb{N}, l \in \mathbb{N}_0,
\]

denote the eigenvalues of the operators \(\bar{\partial}^\dagger \bar{\partial}_0 \bar{\partial}_0, \bar{\partial}^\dagger \bar{\partial}_1, \bar{\partial}^\dagger \bar{\partial}_2 \bar{\partial}_2, \bar{\partial}^\dagger \bar{\partial}^\dagger \bar{\partial}^\dagger \) on \(\Omega^{(0,0)}\), \(\Omega^{(0,2)}\), \(\Omega^{(1,2)}\), \(\Omega^{(2,1)}\) and \(\Omega^{(3,0)}\), respectively. Then \(D\) is self-adjoint and has spectrum

\[
\text{spec}(D) = \left\{ 0 \right\} \cup \left\{ \pm \sqrt{c^0_{(2n,2l)}} : n, l \in \mathbb{N}_0, n+l > 0 \right\} \cup \left\{ \pm \sqrt{d^2_{(2n,2l)}} : n, l \in \mathbb{N} \right\}
\]
\[
\cup \left\{ \pm \sqrt{c^1_{(2n+1,2l)}} : n \in \mathbb{N}_0, l \in \mathbb{N} \right\} \cup \left\{ \pm \sqrt{d^3_{(2n+1,2l)}} : n \in \mathbb{N}, l \in \mathbb{N}_0 \right\} \quad (159)
\]

with the following multiplicities: 1 for the eigenvalue 0 and

\[
\frac{(n+1)(2l+1)(n+l+1)(4n+2l+3)}{3} \quad \text{for the eigenvalues} \quad \pm \sqrt{c^0_{(2n,2l)}} \quad \text{and} \quad \pm \sqrt{d^2_{(2n,2l)}},
\]
\[
\frac{(n+1)(2l+1)(2n+2l+3)(4n+2l+5)}{3} \quad \text{for the eigenvalues} \quad \pm \sqrt{c^1_{(2n+1,2l)}} \quad \text{and} \quad \pm \sqrt{d^3_{(2n+1,2l)}},
\]

where \(n, l \in \mathbb{N}_0\) such that the corresponding eigenvalues are listed in (159).

**Proof.** From (90), (91), Lemma 6, Corollary 7 and Lemma 8, we conclude that \(\Omega^{(0,\bullet)}\) belongs to the domain of \(\bar{\partial}^\dagger\). Hence \(\bar{\partial} + \bar{\partial}^\dagger\) is densely defined and symmetric.

Let \(n, l \in \mathbb{N}\). From Proposition 3, Equations (92)–(94) and Lemma 10, it follows that

\[
\langle \Omega^{(0,0)} \rangle \subset \Omega^{(0,0)} \oplus \Omega^{(0,1)} \oplus \Omega^{(0,2)} \oplus \Omega^{(0,3)}
\]
\[
\langle \Omega^{(0,2)} \rangle \subset \Omega^{(0,0)} \oplus \Omega^{(0,2)} \oplus \Omega^{(1,2)} \oplus \Omega^{(2,1)}
\]
\[
\langle \Omega^{(1,2)} \rangle \subset \Omega^{(0,0)} \oplus \Omega^{(0,1)} \oplus \Omega^{(1,2)} \oplus \Omega^{(3,0)}
\]
\[
\langle \Omega^{(2,1)} \rangle \subset \Omega^{(0,0)} \oplus \Omega^{(0,2)} \oplus \Omega^{(1,2)} \oplus \Omega^{(3,0)}
\]
has an orthonormal basis given by 4 eigenvectors of $D$ corresponding to the eigenvalues $\pm \sqrt{c^0_{(2n, 2l)}}$ and $\pm \sqrt{d^2_{(2n, 2l)}}$. By Lemma 5 and Equation (91), each of these eigenvectors generates an irreducible $U_q(\mathfrak{so}(5))$-representation of highest weight $(2n, 2l)$ consisting of eigenvectors corresponding to the same eigenvalue. Hence the multiplicity of this eigenvalue is given by $\dim((\mathbb{S}_{2n, 2l})^V)$. According to Weyl’s dimension formula, $\dim((\mathbb{S}_{m, k})^V) = \frac{1}{6}(m + 1)(k + 1)(m + k + 2)(2m + k + 3)$. Inserting in this formula $(m, k) = (2n, 2l)$ yields the stated multiplicities for the eigenvalues $\pm \sqrt{c^0_{(2n, 2l)}}$ and $\pm \sqrt{d^2_{(2n, 2l)}}$. Moreover, all these eigenvectors span the vector space $(\mathbb{S}_{2n, 2l})((\Omega^{(0, 0)} \oplus \Omega^{(0, 1)} \oplus \Omega^{(0, 2)} \oplus \Omega^{(0, 3)}))$ by (91).

Analogously, for $n, l \in \mathbb{N}$,

\[
(\mathbb{S}_{2n+1, 2l})((\Omega^{(0, 0)} \oplus \Omega^{(0, 1)} \oplus \Omega^{(0, 2)} \oplus \Omega^{(0, 3)})) = (\mathbb{S}_{2n+1, 2l})\Omega^{(0, 1)} \oplus (\mathbb{S}_{2n+1, 2l})\Omega^{(0, 2)} \oplus (\mathbb{S}_{2n+1, 2l})\Omega^{(0, 3)}
\]

is spanned by exactly 4 orthogonal eigenvectors corresponding to the eigenvalues $\pm \sqrt{c^1_{(2n+1, 2l)}}$ and $\pm \sqrt{d^2_{(2n+1, 2l)}}$. By the same reasoning as above, each eigenvalue has the multiplicity $\dim((\mathbb{S}_{2n+1, 2l})^V)$ in $(\mathbb{S}_{2n+1, 2l})((\Omega^{(0, 0)} \oplus \Omega^{(0, 1)} \oplus \Omega^{(0, 2)} \oplus \Omega^{(0, 3)}))$, and applying Weyl’s dimension formula gives the result.

The cases $n = 0$ or $l = 0$ are treated similarly, the difference being that certain eigenvalues do not appear if the corresponding weight spaces have dimension 0 according to Proposition 4. For $n = l = 0$, we have $(\mathbb{S}_0^0)^\oplus = \text{span}[1] = \ker(D)$, so that the eigenvalue $c^0_{(0, 0)} = 0$ also satisfies the stated multiplicity.

Finally, Proposition 3 and Equation (90) show that

\[
(\Omega^{(0, 0)} \oplus \Omega^{(0, 1)} \oplus \Omega^{(0, 2)} \oplus \Omega^{(0, 3)}) = \bigoplus_{(m, k) \in \mathbb{N}_0 \times 2\mathbb{N}_0} ((\mathbb{S}_{m, k})((\Omega^{(0, 0)} \oplus \Omega^{(0, 1)} \oplus \Omega^{(0, 2)} \oplus \Omega^{(0, 3)})).
\]

Hence the considered eigenvectors of $D$ form a complete orthonormal basis. This proves first that the restriction of $D$ to $\Omega^{(0, 0)} \oplus \Omega^{(0, 1)} \oplus \Omega^{(0, 2)} \oplus \Omega^{(0, 3)}$ is essentially self-adjoint and then that its spectrum is given by the set described in (159) since the eigenvalues tend to infinity for $n, l \rightarrow \infty$ and therefore have no finite accumulation point.

In our final theorem, which presents the main results of this paper, we show that the Dolbeault–Dirac operator $\overline{D} := \bar{\partial} + \bar{\partial}^\dagger$ defines an even spectral triple and summarize some of its properties. Recall that an even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is given by a $*$-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$, a faithful $*$-representation of $\mathcal{A}$ as bounded operators on $\mathcal{H}$, a self-adjoint operator $D$ on $\mathcal{H}$ with compact resolvent and a self-adjoint grading operator $\gamma$ satisfying the following conditions: $\gamma^2 = \text{id}$, $\gamma D = -D \gamma$, $\gamma a = a \gamma$, $[D, a]$ is densely defined and bounded for all $a \in \mathcal{A}$, where we identify (by a slight abuse of notation) an element of $\mathcal{A}$ with the Hilbert space operator representing it, see e.g. [9]. The spectral triple is called $0^+$-summable if $(1 + |D|)^{-t}$ yields a trace class operator for all $t > 0$. If $\mathcal{U}$ is a Hopf $*$-algebra and $\mathcal{A}$ is a left $\mathcal{U}$-module algebra with left action $\triangleright$, then we say that the spectral triple is $\mathcal{U}$-equivariant, if there exists a $*$-representation $\pi$ of $\mathcal{U}$ on $\mathcal{H}$ such that $\pi(X)D(h) = D\pi(X)(h)$ and $\pi(X)(ah) = (X(1) \triangleright a)\pi(X(2))(h)$ for all $X \in \mathcal{U}$, $a \in \mathcal{A}$ and $h$ from a dense domain in $\mathcal{H}$ [27].

Let us also recall that, for a Hopf algebra $\mathcal{U}$ with invertible antipode, $\mathcal{U}^{\text{cop}}$ stands for the co-opposite Hopf algebra with the opposite coproduct $\Delta^{\text{cop}}(X) := X(2) \otimes X(1)$. 

Theorem 15. Let $\Omega^{(0,k)}$ and $\overline{\partial}_j$ be given as in Sect. 3. With the inner product on $\Omega^{(0,k)}$, $k = 0, \ldots, 3$, defined in (84)–(88), let $\mathcal{H}$ denote the Hilbert space closure of $\Omega^{(0,0)} \oplus \Omega^{(0,1)} \oplus \Omega^{(0,2)} \oplus \Omega^{(0,3)}$ and extend the left module multiplication of the $*-$algebra $B := \mathcal{O}(SO_q(5))$ inv$(\mathcal{U}_q(1))$ to a Hilbert space $*-$representation on $\mathcal{H}$. Consider the operator $\overline{\partial} := \overline{\partial}_0 \oplus \overline{\partial}_1 \oplus \overline{\partial}_2$ on the domain $\Omega^{(0,\bullet)} := \Omega^{(0,0)} \oplus \Omega^{(0,1)} \oplus \Omega^{(0,2)} \oplus \Omega^{(0,3)}$ and let $D$ denote the closure of the densely defined, symmetric operator $\overline{\partial} + \overline{\partial}^* \gamma$ on $\Omega^{(0,\bullet)}$.

Then $D$ is a self-adjoint operator with spectrum and multiplicities of eigenvalues given in Corollary 14, and $(B, \mathcal{H}, D, \gamma)$ defines a $\mathcal{U}_q(\text{so}(5))$-equivariant, $0^+$-summable, even spectral triple, where the grading operator $\gamma$ acts on $\Omega^{(0,k)}$ by multiplication by $(-1)^k$ and the $*-$representation of $\mathcal{U}_q(\text{so}(5))$ on $\mathcal{H}$ has been described in Lemma 5.

Proof. The fact that the left module multiplication of $B$ defines a $*$-representation follows from the definition of the inner product and $h((bx)\gamma y) = h(x\gamma (b\gamma y))$ for all $b \in B$ and $x, y \in \mathcal{O}(SO_q(5))$. The $\mathcal{U}_q(\text{so}(5))$-equivariance on the domain $\Omega^{(0,\bullet)}$ can be derived from Lemma 5 and $\overline{\partial}^* \pi_R(X)(\omega) = (\pi_R(X)\overline{\partial})^*(\omega) = (\overline{\partial} \pi_R(X))\overline{\partial}^*(\omega)$ for all $\omega \in \Omega^{(0,\bullet)}$. The statements about self-adjointness of $D$, its spectrum and multiplicities of eigenvalues are the content of Corollary 14.

To prove that $D$ has bounded commutators with the elements of $B$, we consider the embeddings $\Omega^{(0,k)} \subset \mathcal{O}(SO_q(5)) \otimes M^{k \ell}$ with the inner products given by (84)–(88) and view $\overline{\partial}_k$ as an operator from $\Omega^{(0,k)}$ to $\mathcal{O}(SO_q(5)) \otimes M^{k \ell+1}$. Observing that

$$\mathcal{O}(SO_q(5)) \otimes M^{k \ell} \cong \mathcal{O}(SO_q(5)) \otimes \mathbb{C} \cong \mathcal{O}(SO_q(5)), \quad k = 0, 3,$$

$$\mathcal{O}(SO_q(5)) \otimes M^{k \ell} \cong \mathcal{O}(SO_q(5)) \otimes \mathbb{C}^3 \cong \bigoplus_{i=-1}^1 \mathcal{O}(SO_q(5)), \quad k = 1, 2,$$

we can express $\overline{\partial}_k$ in the following matrix notation:

$$\overline{\partial}_0 = \begin{pmatrix} X_{-1} \\ X_0 \\ X_1 \end{pmatrix}, \quad \overline{\partial}_1 = \frac{1}{[3]_2} \begin{pmatrix} -X_0 [2]_1 X_{-1} & 0 \\ -X_1 [2]_1 X_0 & 0 \\ 0 & -X_1 [3]_2 X_0 \end{pmatrix}, \quad \overline{\partial}_2 = \frac{1}{[3]_2} X_1, -X_0, X_{-1}. \quad (160)$$

Furthermore, the action of $b \in B$ is represented by a diagonal matrix with $b$ in each diagonal entry. Viewing the operators in (160) as a finite sum of matrix operators with exactly one non-zero matrix entry, it suffices to show the boundedness for commutators with matrices having at one place an $X_j$ and 0 in all the others. We will show this for

$$\begin{bmatrix} 0 & 0 & 0 \\ X_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & a_0 \\ b & a_0 \end{bmatrix} = \begin{bmatrix} X_1 \triangleright (ba_{-1}) - b(X_1 \triangleright a_{-1}) \\ 0 \\ 0 \end{bmatrix}, \quad (161)$$

where $a := \sum_{i=-1}^1 \overline{\partial}_k$ and $v_i \in \Omega^{(0,1)} \subset \mathcal{O}(SO_q(5)) \otimes M^{(0,2)} \cong \bigoplus_{i=-1}^1 \mathcal{O}(SO_q(5))$. The other cases are similar (and simpler).

As $a \in \Omega^{(0,1)}$, we have $K_1 \triangleright a_j = q^{1-i} a_j$, $K_2 \triangleright a_j = q^i a_j$ and $E_2 \triangleright a_j = a_{j+1}$ by (24) and (69). Moreover, $K_1 \triangleright b = b = K_2 \triangleright b$ and $E_2 \triangleright b = 0$ for all $b \in B$. From this and (14), it follows that

$$X_1 \triangleright (ba_{-1}) - b(X_1 \triangleright a_{-1}) = (E_2^2 E_1 \triangleright b) a_{-1} + (q + q^2)(E_2 E_1 \triangleright b) a_0 + q^2 (E_1 \triangleright b) a_1.$$
Set $\| \cdot \|_h := \sqrt{\langle \cdot, \cdot \rangle_h}$ and let $\| \cdot \|_{\text{GNS}}$ denote the operator norm of the GNS-representation of the compact quantum group $\mathcal{O}(SO_q(5))$ with respect to the Haar state $h$. Then we obtain in view of (86) that

$$
\|X_1 \circ (ba_{-1}) - b(X_1 \circ a_{-1})\|_h 
\leq \sqrt{\frac{\|E_1^2 E_1 \circ b\|^2_{\text{GNS}}}{c_{1, -1}} + \frac{(q + q^2)^2 \|E_2 E_1 \circ b\|^2_{\text{GNS}}}{c_{1, 0}} + \frac{q^2 \|E_1 \circ b\|^2_{\text{GNS}}}{c_{1, 1}}} \|a\|.
$$

Consequently, the commutator in (161) is bounded. Furthermore, the boundedness of $[\hat{\theta}, b^*]$ implies the boundedness of $[\hat{\theta}^\dagger, b] \subset -[\hat{\theta}, b^*]^\dagger$, and combining both results, we conclude that $[D, b] = [\hat{\theta}, b] + [\hat{\theta}^\dagger, b]$ is bounded for all $b \in B$.

Since $[n + l + k]_1 = q^{-(n+l+k)} \frac{1-q^{2n+2l+2k}}{1-q}$ with $q^{2n+2l+2k} \to 0$ for $n, l \to \infty$ and fixed $k \in \mathbb{N}_0$, we see that the eigenvalues grow exponentially fast, and since the multiplicities grow asymptotically no faster than $n^3 l^3$, it follows that the trace of the positive operator $(1 + |D|)^{-t}$ exists for all $t > 0$. As a consequence, $D$ has compact resolvent and the spectral triple is $0^+$-summable.

It is obvious that $\gamma$ defined by $\gamma(\omega_k) = (-1)^k \omega_k$ for all $\omega_k \in \Omega^{(0,k)}$ satisfies $\gamma^\dagger = \gamma$, $\gamma^2 = 1$, $\gamma b = b \gamma$ and $\gamma D = -D \gamma$, hence $(B, \mathcal{H}, D, \gamma)$ yields an even spectral triple. \hfill \Box

Remark 2. The spectrum of the Dolbeault–Dirac operator $D$ in Theorem 15 depends on the positive scaling factors $\frac{c_1}{c_0}$, $\frac{c_1}{c_1}$ and $\frac{c_1}{c_2}$, where the constants $c_k$ are related to the inner product on $\Omega^{(0,k)}$, see Lemma 6. On the other hand, the Bernstein–Gelfand–Gelfand resolution (57) is unique only up to a non-zero scaling factor. Corollaries 7 and 14 show that a rescaling of the Dolbeault operators $\hat{\theta}_k$ by nonzero constants will have the same effect on the spectrum of $D$, i.e., the same sets of eigenvalues will be rescaled by a positive constant. Therefore, the appearance of the scaling factors can be viewed as a rescaling of the inner products on $(0, k)$-forms, or as a rescaling of the Dolbeault operators $\hat{\theta}_k$.

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