The Stochastic Score Classification Problem

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Abstract

Consider the following Stochastic Score Classification Problem. A doctor is assessing a patient’s risk of developing a certain disease, and can perform \( n \) tests on the patient. Each test has a binary outcome, positive or negative. A positive test result is an indication of risk, and a patient’s score is the total number of positive test results. The doctor needs to classify the patient into one of \( B \) risk classes, depending on the score (e.g., LOW, MEDIUM, and HIGH risk). Each of these classes corresponds to a contiguous range of scores. Test \( i \) has probability \( p_i \) of being positive, and it costs \( c_i \) to perform the test. To reduce costs, instead of performing all tests, the doctor will perform them sequentially and stop testing when it is possible to determine the risk category for the patient. The problem is to determine the order in which the doctor should perform the tests, so as to minimize the expected testing cost. We provide approximation algorithms for adaptive and non-adaptive versions of this problem, and pose a number of open questions.

1 Introduction

We consider the following Stochastic Score Classification (SSClass) problem. A doctor can perform \( n \) tests on a patient, each of which has a positive or negative outcome. Test \( i \) has known probability \( p_i \) of having a positive outcome, and costs \( c_i \) to perform. A positive test is indicative of the disease. The professor

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needs to assign the patient to a risk class (e.g., LOW, MEDIUM, HIGH) based on how many of the \( n \) tests are positive. Each class corresponds to a contiguous range of scores.

To reduce costs, instead of performing all tests and computing an exact score, the doctor will perform them one by one, stopping when the class becomes a foregone conclusion. For example, suppose there are 10 tests and the MEDIUM class corresponds to a score between 4 and 7 inclusive. If the doctor performed 8 tests, of which 5 were positive, the doctor would not perform the remaining 2 tests, because the final score will be between 5 and 7, meaning that the risk class will be MEDIUM regardless of the outcome of the 2 remaining tests. The problem is to compute the optimal (adaptive or non-adaptive) order in which to perform the tests, so as to minimize expected testing cost.

Formally, the Stochastic Score Classification problem is as follows. Given \( B + 1 \) integers \( 0 = \alpha_1 < \alpha_2 < \ldots < \alpha_B < \alpha_{B+1} = n + 1 \), let class \( j \) correspond to the scoring interval \( \{ \alpha_j, \alpha_j + 1, \ldots, \alpha_j+1 - 1 \} \). The \( \alpha_j \) define an associated pseudo-Boolean score classification function \( f : \{0,1\}^n \rightarrow \{1,\ldots,B\} \), such that \( f(X_1,\ldots,X_n) \) is the class whose scoring interval contains the score \( r(X) = \sum X_i \). Note that \( B \) is the number of classes. Each input variable \( X_i \) is independently 1 with given probability \( p_i \), where \( 0 < p_i < 1 \), and is 0 otherwise. The value of \( X_i \) can only be determined by asking a query (or performing a test), which incurs a given non-zero, real-valued cost \( c_i \).

An evaluation strategy for \( f \) is a sequential adaptive or non-adaptive ordering in which to ask the \( n \) possible queries. Each query can only be asked once. Querying must continue until the value of \( f \) can be determined, i.e., until the value of \( f \) would be the same, no matter how the remainder of the \( n \) queries were answered. The goal is to design an evaluation strategy for \( f \) with minimum expected total query cost.

We consider both adaptive and non-adaptive versions of the problem. In the adaptive version, we seek an adaptive strategy, where the choice of the next query can depend on the outcomes of previous queries. An adaptive strategy corresponds to a decision tree, although we do not require the tree to be output explicitly (it may have exponential size). In the non-adaptive version, we seek a non-adaptive strategy, which is a permutation of the queries. With a non-adaptive strategy, querying proceeds in the order specified by the permutation until the value of \( f \) can be determined from the queries performed so far.

We also consider a weighted variant of the problem, where query \( i \) has given integer weight \( a_i \), the score is \( \sum X_i \), and \( \alpha_1 < \alpha_2 < \ldots < \alpha_B < \alpha_{B+1} \) where \( \alpha_1 \) equals the minimum possible value of the score \( \sum X_i \), and \( \alpha_{B+1} - 1 \) equals the maximum possible score.

While we have described the problem above in the context of assessing disease risk, such classification is also used in other contexts, such as assigning letter grades to students, giving a quality rating to a product, and deciding whether or not a person charged with a crime should be released on bail. In Machine Learning, the focus is on learning the score classification function \[22, 20, 13, 24, 23\]. In contrast, here our focus is on reducing the cost of evaluating the classification function.
Restricted versions of the weighted and unweighted SSClass problem have
been studied previously. In the algorithms literature, Deshpande et al. presented
two approximation algorithms solving the Stochastic Boolean Function Evaluation
(SBFE) problem for linear threshold functions \[8\]. The general SBFE problem
is similar to the adaptive SSClass problem, but instead of evaluating a given
score classification function \(f\) defined by inputs \(\alpha_j\), you need to evaluate a
given Boolean function \(f\). When \(f\) is a linear threshold function, the problem is
equivalent to the weighted adaptive SSClass problem. One of the two algorithms
of Deshpande et al. achieves an \(O(\log W)\)-approximation factor for this problem
using the submodular goal value approach; it involves construction of a goal
utility function and application of the Adaptive Greedy algorithm of Golovin
and Krause to that function \[9\]. Here \(W\) is the sum of the magnitudes of the
weights \(a_i\). The other algorithm achieves a 3-approximation by applying a dual
greedy algorithm to the same goal utility function.

A \(k\)-of-\(n\) function is a Boolean function \(f\) such that \(f(x) = 1\) iff \(x_1 + \ldots + x_n \geq k\). The SBFE problem for evaluating \(k\)-of-\(n\) functions is equivalent to the
unweighted adaptive SSClass problem, with only two classes \(B = 2\). It has
been studied previously in the VLSI testing literature. There is an elegant
algorithm for the problem that computes an optimal strategy \[17, 4, 18, 6\].

The unweighted adaptive SSClass problem for arbitrary numbers of classes
was studied in the information theory literature \[7, 1, 15\], but only for unit
costs. The main novel contribution there was to establish an equivalence between
verification and evaluation, which we discuss below.

## 2 Results and open questions

We give approximation results for adaptive and non-adaptive versions of the
SSClass problem. We describe most of our results here, but leave description
of some others and some of the proofs to the appendix. A table with all our
bounds can be found in the next section.

We begin by using the submodular goal value approach of Deshpande et
al. to obtain an \(O(\log W)\) approximation algorithm for the weighted adaptive
SSClass problem. This immediately gives an \(O(\log n)\) approximation for the
unweighted adaptive problem. We also present a simple alternative algorithm
achieving a \(B - 1\) approximation for the unweighted adaptive problem, and a
3\((B - 1)\)-approximation algorithm for the weighted adaptive problem again using
an algorithm of Deshpande et al.

We then present our two main results, which are both for the case of unit
costs. The first is a 4-approximation algorithm for the adaptive and non-adaptive
versions of the unweighted SSClass problem. The second is a \(\varphi\)-approximation
for a special case of the non-adaptive unweighted version, where the problem is
to evaluate what we call the Unanimous Vote Function. Here \(\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618\)
is the golden ratio. The Unanimous Vote Function outputs POSITIVE if \(X_1 = \ldots = X_n = 1\), NEGATIVE if \(X_1 = \ldots = X_n = 0\), and UNCERTAIN
otherwise. Equivalently, it is a score classification function with \(B = 3\) and
scoring intervals \(\{0\}, \{1, \ldots, n - 1\} \) and \(\{n\}\). The proofs of our two main results imply upper bounds of 4 and \(\varphi\) for the adaptivity gaps of the corresponding problems.

We use both existing techniques and new ideas in our algorithms. We use the submodular goal value approach of Deshpande et al. to get our \(O(\log W)\) bound for the weighted adaptive SSClass problem. This approach cannot yield a bound better than \(O(\log n)\) for SSClass problems, since they involve evaluating a function of \(n\) relevant Boolean variables \([3]\).

For our other bounds, we exploit the exact algorithm for \(k\)-of-\(n\) evaluation, and the ideas used in its analysis. To obtain non-adaptive algorithms for the unit-cost case, we perform a round robin between 2 subroutines, one performing queries in increasing order of \(c_i/p_i\), while the second performs them in increasing order of \(c_i/(1 - p_i)\). For arbitrary costs, instead of standard round robin, we use the modified round robin approach of Allen et al \([2]\). As has been repeatedly shown, the \(c_i/p_i\) ordering and the \(c_i/(1 - p_i)\) ordering are optimal for evaluation of the Boolean OR (1-of-\(n\)) and AND (\(n\)-of-\(n\)) functions respectively (cf. \([21]\)). Intuitively, the first ordering (for OR) favors queries with low cost and high probability of producing the value 1, while the second (for AND) favors queries with low cost and high probability of producing the value 0. The proof of optimality follows from the fact that given any ordering, swapping two adjacent queries that do not follow the designated increasing order will decrease expected evaluation cost.

While the algorithm for our first main result is simple, the proof of its 4-approximation bound is not. It uses ideas from the existing analysis of the \(k\)-of-\(n\) algorithm, which is an easier problem because \(B = 2\). To obtain our 4-approximation result we perform a new, careful analysis. Unlike the analysis of the \(k\)-of-\(n\) algorithm, this analysis only works for unit costs.

To develop our \(\varphi\)-approximation for the unanimous vote function, we first note that for such a function, if you perform the first query and observe its outcome, the optimal ordering of the remaining queries can be determined by evaluating a Boolean OR function, or the complement of an AND function. We then address the problem of determining an approximately optimal permutation, given the first query. A standard round robin between the \(c_i/p_i = 1/p_i\) ordering, and the \(1/(1 - p_i)\) ordering, yields a factor of 2 approximation. To obtain the \(\varphi\) factor, we stop the round robin at a carefully chosen point and commit to one of the two subroutines, abandoning the other. Our full algorithm for the unanimous vote function works by trying all \(n\) possible first queries. For each, we generate the approximately optimal permutation, and algebraically compute its expected cost. Finally, out of these \(n\) permutations, we choose the one with lowest expected cost.

We note that although our algorithms are designed to minimize expected cost for independent queries, the goal value function used to achieve the \(O(\log W)\) approximation result can also be used to achieve a worst-case bound, and a related bound in the Scenario model \([9, 11, 14]\).

A recurring theme in work on SSClass problems has been the relationship between these evaluation problems and their associated verification problems. In
the verification problem, you are given the output class (i.e., the value of the score classification function) before querying, and just need to perform enough tests to certify (verify) that the given output class is correct. Thus optimal expected verification cost lower bounds optimal expected evaluation cost. Surprisingly, the result of Das et al. [7] showed that for the adaptive SSClass problem in the unit-cost case, optimal expected verification cost equals optimal expected evaluation cost. Prior work already implied this was true for evaluating \( k \)-of-\( n \) functions, even for arbitrary costs (cf. [5]). We give a counterexample in the full paper [] showing that this relationship does not hold for the adaptive SSClass problem with arbitrary costs. Thus algorithmic approaches based on optimal verification strategies may not be effective for these problems.

There remain many intriguing open questions related to SSClass problems. The first, and most fundamental, is whether the (adaptive or non-adaptive) SSClass problem is NP-hard. This is open even in the unit-cost case. It is unclear whether this problem will be easy to resolve. It is easy to show that the weighted variants are NP-hard: this follows from the NP-hardness of the SBFE problem for linear threshold functions, which is proved by a simple reduction from knapsack [8]. However, the approach used in that proof is to show that the deterministic version of the problem (where query answers are known a-priori) is NP-hard, which is not the case in the SSClass problem. Further, NP-hardness of evaluation problems is not always easy to determine. The question of whether the SBFE problem for read-once formulas is NP-hard has been open since the 1970’s (cf. [12]).

Another main open question is whether there is a constant-factor approximation algorithm for the weighted SSClass problem. Our bounds depend on \( n \) or \( B \). Other open questions concern lower bounds on approximation factors, and bounds on adaptivity gaps.

## 3 Table of Results

|                         | unit costs                                      | arbitrary costs                          |
|-------------------------|------------------------------------------------|------------------------------------------|
| weighted                | \( O(\log W) \)-approx [Sec. 5]; 3\((B - 1)\) [Sec. 5] | \( O(\log W) \)-approx [Sec. 5]; 3\((B - 1)\) [Sec. 5] |
| unweighted              | 4-approx [Sec. 6.3, C.3]                         | \( O(\log n) \)-approx; (\(B - 1\))-approx [Sec. 5, C.1] |
| \( k \)-of-\( n \) function | exact algorithm [known]                       | exact algorithm [known]                  |
| unanimous vote function | exact algorithm [Sec. 6.4]                      | exact algorithm [Sec. 6.4]               |
Table 2: Results for the non-adaptive SSClass problem

|                  | unit costs                  | arbitrary costs                  |
|------------------|-----------------------------|----------------------------------|
| weighted         | open                        | open                             |
| unweighted       | 4-approx [Sec. 6.3, C.3]    | 2(B − 1)-approx [Sec. 6.3, C.2] |
| \(k\)-of-\(n\) function | 2-approx [Sec. 6.3]      | 2-approx [Sec. 6.3]             |
| unanimous vote function | \(\varphi\)-approx [Sec. 6.5] | 2-approx [Sec. 6.5]           |

4 Further definitions and background

A partial assignment is a vector \(b \in \{0, 1, \ast\}^n\). We use \(f^b\) to denote the restriction of function \(f(x_1, \ldots, x_n)\) to the bits \(i\) with \(b_i = \ast\), produced by fixing the remaining bits \(i\) according to their values \(b_i\). We call \(f^b\) the function induced from \(f\) by partial assignment \(b\). We use \(N_0(b)\) to denote \(|\{i|b_i = 0\}|\), and \(N_1(b)\) to denote \(|\{i|b_i = 1\}|\).

A partial assignment \(b' \in \{0, 1, \ast\}^n\) is an extension of \(b\), written \(b' \supseteq b\), if \(b'_i = b_i\) for all \(i\) such that \(b_i \neq \ast\). We use \(b' \succ b\) to denote that \(b' \supseteq b\) and \(b' \neq b\).

A partial assignment encodes what information is known at a given point in a sequential querying (testing) environment. Specifically, for partial assignment \(b \in \{0, 1, \ast\}^n\), \(b_i = \ast\) indicates that query \(i\) has not yet been asked, otherwise \(b_i\) equals the answer to query \(i\). We may also refer to query \(i\) as test \(i\), and to asking query \(i\) as testing or querying bit \(x_i\).

Suppose the costs \(c_i\) and probabilities \(p_i\) for the \(n\) queries are fixed. We define the expected costs of adaptive evaluation and verification strategies for \(f: \{0, 1\}^n \to \{0, 1\}\) or \(f: \{0, 1\}^n \to \{1, \ldots, B\}\) as follows. (The definitions for non-adaptive strategies are analogous.) Given an adaptive evaluation strategy \(A\) for \(f\), and an assignment \(x \in \{0, 1\}^n\), we use \(C(A, x)\) to denote the sum of the costs of the tests performed in using \(A\) on \(x\). The expected cost of \(A\) is \(\sum_{x \in \{0, 1\}^n} C(A, x)p(x)\), where \(p(x) = \prod_{i=1}^n p_i^{x_i}(1 - p_i)^{1-x_i}\). We say that \(A\) is an optimal adaptive evaluation strategy for \(f\) if it has minimum possible expected cost.

Let \(L\) denote the range of \(f\), and for \(\ell \in L\), let \(X_\ell = \{x \in \{0, 1\}^n : f(x) = \ell\}\). An adaptive verification strategy for \(f\) consists of \(|L|\) adaptive evaluation strategies \(A_\ell\) for \(f\), one for each \(\ell \in L\). The expected cost of the verification strategy is \(\sum_{\ell \in L} (\sum_{x \in X_\ell} C(A_\ell, x)p(x))\) and it is optimal if it minimizes this expected cost.

If \(A\) is an evaluation strategy for \(f\), we call \(\sum_{x \in X_\ell} C(A, x)p(x)\) the \(\ell\)-cost of \(A\). For \(\ell \in L\), we say that \(A\) is \(\ell\)-optimal if it has minimum possible \(\ell\)-cost. In an optimal verification strategy for \(f\), each component evaluation strategy \(A_\ell\) must be \(\ell\)-optimal.

A Boolean function \(f: \{0, 1\}^n \to \{0, 1\}\) is symmetric if its output on \(x \in \{0, 1\}^n\) depends only on \(N_1(x)\). Let \(f\) be a symmetric Boolean function
$f: \{0,1\}^n \rightarrow \{0,1\}$, or an unweighted score classification function $f: \{0,1\}^n \rightarrow \{1,\ldots,B\}$. The value vector for $f$ is the $n+1$ dimensional vector $v^f$, indexed from 0 to $n$, whose $j$th entry $v^f_j$ is the value of $f$ on inputs $x$ where $N_1(x) = j$. We partition value vector $v^f$ into blocks. A block is a maximal subvector of $v^f$ such that entries of the subvector have the same value. If $f$ is a score classification function, the blocks correspond to the score intervals, and block $i$ is the subvector of $v^f$ containing the entries in $[\alpha_i, \alpha_{i+1})$. For a Boolean function, we define the $\alpha_i$ so that $0 = \alpha_1 < \alpha_2 \leq \cdots \leq \alpha_{B+1} = n + 1$ and block $i$ is the subvector containing the indices in the interval $[\alpha_i, \alpha_{i+1})$.

We say that assignment $x$ is in the $i$th block if $N_1(x)$ is in the interval $[\alpha_i, \alpha_{i+1})$.

With each block $i$ of $v^f$, we associate a function $f^i$, where $f^i(x) = 1$ if $x$ is in block $i$, and $f^i(x) = 0$ otherwise. A verification strategy for block $i$ is an evaluation strategy for $f^i$. An optimal verification strategy for block $i$ is an evaluation strategy for $f^i$ with minimum 1-cost.

A function $g: \{0,1,\ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$ is monotone if $g(b') \geq g(b)$ whenever $b' \succeq b$. It is submodular if for $b' \succeq b$, $i$ such that $b'_i = b_i = \ast$, and $k \in \{0,1\}$, we have $g(b'_{i+k}) - g(b') \leq g(b_{i+k}) - g(b)$. Here $b_{i+k}$ denotes the partial assignment produced from $b$ by setting $b_i$ to $k$, and similarly for $b'_{i+k}$.

## 5 Algorithms for the weighted adaptive SSClass problem

Our first algorithm solves the weighted adaptive SSClass Problem using the goal value approach of Deshpande et al., a method of designing approximation algorithms for SBFE problems [8]. The approach can easily be extended to the weighted adaptive SSClass problem. It requires construction of a utility function $g: \{0,1,\ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$, called a goal function, associated with the function $f$ being evaluated. Function $g$ must be monotone and submodular. The maximum value of $g$ must be an integer $Q \geq 0$ such that $g(b) = Q$ iff $f(x)$ has the same value for all $x \in \{0,1\}^n$ such that $x \succeq b$. We call $Q$ the goal value of $g$.

An adaptive strategy for evaluating $f$ can then be obtained by applying the Adaptive Greedy algorithm of Golovin and Krause to solve the Stochastic Submodular Cover problem on goal function $g$ [9]. This algorithm greedily chooses the query with highest expected increase in utility, as measured by $g$, per unit cost. It follows from the bound of Deshpande et al. on Adaptive Greedy for Stochastic Submodular Cover, that this strategy is an $O(\log Q)$-approximation to the optimal adaptive strategy for evaluating $f$ [8].

We construct $g$ as follows. Let $r(x) = a_1x_1 + \cdots + a_nx_n$. Consider an associated score classification function $f$ defined by $\alpha_1, \ldots, \alpha_{B+1}$ and the $a_i$.\footnote{Golovin and Krause originally claimed an $O(\log Q)$ bound for Stochastic Submodular Cover [9], but the proof was recently found to have an error [10]. They have since posted a new proof with an $O(\log^2 Q)$ bound [11]. Deshpande et al. proved an $O(\log Q)$ bound using a different proof technique [8].}
There is a polynomial-time algorithm that achieves a approximation factors of $O(\log W)$ and $3(B - 1)$ respectively for the weighted adaptive SSClass problem. There is a polynomial-time algorithm that achieves a $B - 1$-approximation for the unweighted adaptive SSClass problem.
6 Constant-factor approximations for unit-cost problems

We begin by reviewing relevant existing techniques.

6.1 Adaptive Evaluation of k-of-n Functions

An optimal adaptive strategy, when \( f \) is a \( k \)-of-\( n \) function, was given by Salloum, Ben-Dov, and Breuer [17, 4, 18, 6, 19]. The difficulty in finding an optimal strategy is that you do not know a-priori whether the value of \( f \) will be 1 or 0. If 1, then (ignoring cost) it seems it would be better to choose queries with high \( p_i \), since you want to get \( k \) 1-answers. Similarly, if 0, it seems it would be better to choose queries with low \( p_i \). The algorithm of Salloum et al. is based on showing that when \( f \) is a \( k \)-of-\( n \) function, a 1-optimal strategy is to query the bits in increasing order of \( c_i/p_i \) until getting \( k \) 1's, while a 0-optimal strategy is to query them in increasing order of \( c_i/(1-p_i) \) until getting \( n-k+1 \) 0's.

Since the 1-optimal strategy must perform at least the first \( k \) tests before terminating, these can be reordered within this strategy without affecting its optimality. Similarly, the first \( n-k+1 \) queries of the 0-optimal strategy can be reordered without affecting optimality. The strategy of Salloum et al. is as follows. If \( n=1 \), test the one bit. Else let \( S_1 \) denote the set of the \( k \) bits with smallest \( c_i/p_i \) values. Let \( S_0 \) denote the set of the \( n-k+1 \) bits with smallest \( c_i/(1-p_i) \) values. Since \( |S_0| + |S_1| = n+1 \), by pigeonhole \( S_0 \cap S_1 \neq \emptyset \). Test a bit in \( S_0 \cap S_1 \). If it is 1, the problem is reduced to evaluating the function \( f^1: \{0,1\}^{n-1} \rightarrow \{0,1\} \) where \( f^1(x) = 1 \) iff \( N_1(x) \geq k-1 \). If it is 0, the problem is reduced to evaluating \( f^0: \{0,1\}^{n-1} \rightarrow \{0,1\} \) where \( f^0(x) = 1 \) iff \( N_1(x) \geq k \). Recursively evaluate \( f^1 \) or \( f^0 \) as appropriate. Optimality follows from the fact that the chosen bit is an optimal first bit to test in both 0-optimal and 1-optimal strategies.

6.2 Modified Round Robin

Allen et al. [2] presented a modified round robin protocol, which is useful in designing non-adaptive strategies when test costs are not all equal. Suppose that in a sequential testing environment with \( n \) tests, we have \( M \) conditions on test outcomes, corresponding to \( M \) predicates on the partial assignments in \( \{0,1,*\}^n \). For example, in the \( k \)-of-\( n \) testing problem, we are interested in the following \( M = 2 \) predicates on partial assignments: (1) having at least \( k \) ones and (2) having at least \( n-k+1 \) zeros. Suppose we are given a testing strategy for each of the \( M \) predicates; a strategy stops testing when its predicate is satisfied (by the partial assignment representing test outcomes), or all tests have been performed. Let Alg\(_1\),..., Alg\(_M\) denote those \( M \) strategies. The modified round robin algorithm of Allen et al. interleaves execution of these strategies. We present a modified version of their algorithm in Algorithm 1; the difference is that their algorithm terminates as soon as one of the predicates is satisfied, while Algorithm 1 terminates when all are satisfied.
Algorithm 1 Modified Round Robin of M Strategies

Let $C_i \leftarrow 0$ for $i = 1, \ldots, M$; let $d \leftarrow (**n)$

while at least one of the M testing strategies has not terminated do

Let $j_1, \ldots, j_M$ be the next tests of Alg$_1$, $\ldots$, Alg$_M$ respectively

Let $i^* \leftarrow \arg \min_{i \in \{1, \ldots, M\}} (C_i + c_{j_i})$

Let $t \leftarrow j_{i^*}$; let $C_{i^*} \leftarrow C_{i^*} + c_t$

Perform test $t$ and set $d_t$ to the newly determined value of bit $t$

end while

Allen et al. showed that the modified round robin incurs a cost on $x$ that is at most $M$ times the cost incurred by Alg$_j$ on $x$. We will use variations on this algorithm and this bound to derive approximation factors for our SSClass problems.

6.3 A Round Robin Approach to Non-adaptive Evaluation

We now present an algorithm for the unit-cost case of the non-adaptive, unweighted SSClass problem. The pseudocode is presented in Algorithm 2, with Alg$_1$ denoting the strategy performing tests in increasing order of $c_i/p_i$ and Alg$_0$ denoting the strategy performing tests in increasing order of $c_i/(1 - p_i)$. We prove the following theorem.

Algorithm 2 Non-adaptive Round Robin Algorithm for SSClass

Let $C_0 \leftarrow 0, C_1 \leftarrow 0$

Let $d \leftarrow *n$

repeat

Let $j_0 \leftarrow$ next bit from Alg$_0$

Let $j_1 \leftarrow$ next bit from Alg$_1$

Let $j^* \leftarrow \arg \min_{i \in \{0, 1\}} C_i + c_{j_i}$

Query bit $j^*$ and set $d_{j^*}$ to the discovered value

until induced function $f^d$ is a constant function

return The constant value of $f^d$

Theorem 2. When all tests have unit cost, the expected cost incurred by the non-adaptive Algorithm 2 is at most 4 times the expected cost of an optimal adaptive strategy for the unweighted adaptive SSClass problem.

By Theorem 2, Algorithm 2 is a 4-approximation for the adaptive and non-adaptive versions of the unit-cost unweighted SSClass problem. The theorem also implies an upper bound of 4 on the adaptivity gap for this problem. A simpler analysis shows that for arbitrary costs, Algorithm 2 achieves an approximation factor of $2(B - 1)$ for the non-adaptive version of the problem. Since the $k$-of-$n$ functions are essentially equivalent to score classification functions with $B = 2$, since $k$-of-$n$ functions are essentially equivalent to score classification functions with $B = 2$,
the $2(B-1)$-approximation is a 2-approximation for non-adaptive $k$-of-$n$ function evaluation.

6.4 The Unanimous Vote Function: Adaptive Setting

Adaptive evaluation of the Unanimous Vote function function can be done optimally using the following simple idea. Recall that querying the bits in increasing $c_i/p_i$ order is optimal for evaluating OR, while querying in increasing $c_i/(1-p_i)$ is optimal for AND. Now consider the problem of adaptively evaluating the unanimous vote function. Suppose we know the optimal choice for the first test. After the first test, we have an induced SSClass problem on the remaining bits. If the first test has value 0, the induced function is equivalent to Boolean OR (mapping UNCERTAIN to 1, and NEGATIVE to 0). The subtree rooted at the root node’s 0-child should be the optimal tree for evaluating OR. Specifically, the remaining bits should be tested in increasing order of $c_i/p_i$. If, instead, the first bit is 1, the induced function is equivalent to AND (mapping UNCERTAIN to 0 and POSITIVE to 1) and the remaining bits should be queried in increasing order of $c_i/(1-p_i)$.

Since we don’t actually know the first bit, we can just try each bit as the root and build the rest of the tree according to the optimal OR and AND strategies. We can then calculate the expected cost of each tree, and output the tree with minimum expected cost.

For succinctness, the optimal OR and AND strategies can be represented by paths, because each performs tests in a fixed order. Figure 1 shows an example of the strategy computed by the algorithm, where the root is labeled $x_0$ and the OR permutation is the reversal of the AND permutation (which occurs, for example, with unit costs).

6.5 A Non-adaptive $\varphi$-approximation for the Unanimous Vote Function

![Decision tree](image)

Figure 1: Decision tree $T$ representing optimal adaptive strategy with root $x_0$
A simple modification of the round robin makes the algorithm from the previous section non-adaptive, yielding a 2-approximation. But we now show how to achieve a non-adaptive $\phi$-approximation in the unit-cost case, where $\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$ is the golden ratio. We call the algorithm Truncated Round Robin. We describe the algorithm by describing a subroutine which generates a permutation of input bits to query, given an initial (root) bit. The algorithm then tries all possible bits for the root and chooses the resulting permutation that achieves the lowest expected cost.

Without loss of generality, assume the first bit (the root node) is $x_0$, and the rest are $x_1, \ldots, x_{n-1}$, and $1 > p_1 \geq p_2 \geq \cdots \geq p_{n-1} > 0$. Fix $c$ to be a constant such that $0 < c < \frac{1}{2}$.

**Algorithm 3 Truncated Round Robin Subroutine for Unanimous Vote Fn**

Require: $1 > p_1 \geq p_2 \geq \cdots \geq p_{n-1}$

Query bit $x_0$

Let level $l \leftarrow 1$

while $p_{n-l} < 1 - c$ and $p_l > c$ and evaluation unknown do

if $|p_l - 0.5| < |p_{n-l} - 0.5|$ then

Query $x_l$ followed by $x_{n-l}$

else

Query $x_{n-l}$ followed by $x_l$

end if

$l \leftarrow l + 1$

end while

{first phase: alternate branches of tree}

while evaluation unknown do

if $p_l \geq p_{n-l} \geq 1 - c$ then

Query $x_{n-l}$

else if $c \geq p_l \geq p_{n-l}$ then

Query $x_l$

end if

$l \leftarrow l + 1$

end while

{second phase: single branch in tree}

The subroutine is shown in Algorithm 3. “Evaluation unknown” means tests so far were insufficient to determine the output of the Unanimous Vote function. (The output, POSITIVE, NEGATIVE, or UNCERTAIN, is not shown.)

Given $x_0$ as the root, the optimal adaptive strategy continues with the OR strategy (increasing $1/p_i$) when $x_0 = 0$, and the AND strategy (increasing $1/(1 - p_i)$) when $x_0 = 1$. This is shown in Figure 1 where $x_0 = 0$ is the left branch and $x_0 = 1$ is the right. On the left, we stop querying when we find a bit with value 1 (or all bits are queried). On the right, we stop when we find a bit with value 0.

Let “level $l$” refer to the tree nodes at distance $l$ from the root; namely, $x_l$ and $x_{n-l}$. When all costs are 1, the standard round robin technique of the previous section in effect tests, for $l = 1 \ldots \left\lfloor \frac{n-1}{2} \right\rfloor$, the bit $x_l$ followed by $x_{n-l}$.
Note that the algorithm will terminate by level \( \left\lceil \frac{n-1}{2} \right\rceil \) because at this point all bits will have been queried.

In the Truncated Round Robin, we proceed level by level, in two phases. The first phase concludes once we reach a level \( l \) where \( p_l > p_{n-l} \geq 1 - c \) or \( c \geq p_l > p_{n-l} \). Let \( \ell \) denote this level. In the first phase, we test both \( x_l \) and \( x_{n-l} \), testing first the variable whose probability is closest to \( \frac{1}{2} \). In the second phase, we abandon the round robin and instead continue down a single branch in the adaptive tree. Specifically, in the second phase, if \( p_l > p_{n-l} \geq 1 - c \), then we continue down the right branch, testing the remaining variables in increasing order of \( p_i \). If \( c \geq p_l > p_{n-l} \), then we continue down the left branch, testing the remaining variables in decreasing order of \( p_i \). Fixing \( c = \frac{3 - \sqrt{5}}{2} \approx 0.381966 \) in the algorithm, the following holds.

**Theorem 3.** When all tests have unit cost, the Truncated Round Robin Algorithm achieves an approximation factor of \( \varphi \) for non-adaptive evaluation of the Unanimous Vote function.

**Proof.** Consider the optimal adaptive strategy \( T \). It tests a bit \( x_0 \) and then follows the optimal AND or OR strategy depending on whether \( x_0 = 1 \) or \( x_0 = 0 \). Assume the other bits are indexed so \( p_1 \geq p_2 \geq \ldots \geq p_{n-1} \). Thus \( T \) is the tree in Figure 1. Let \( C^*_{\text{adapt}} \) be the expected cost of \( T \). Let \( C^*_{\text{non-adapt}} \) be the expected cost of the optimal non-adaptive strategy. Let \( C_{i,\text{TRR}} \) be the cost of running the TRR subroutine in (Algorithm 3) with root \( x_i \). We use \( x_0 \) to denote the root of \( T \). Since the TRR algorithm tries all possible roots, its output strategy has expected cost \( \min_i C_{i,\text{TRR}} \). We will prove the following claim: \( C_{0,\text{TRR}} \leq \varphi C^*_{\text{adapt}} \). Since the expected cost of the optimal adaptive strategy is bounded above by the expected cost of the optimal non-adaptive strategy, the claim implies that \( \min_i C_{i,\text{TRR}} \leq C_{0,\text{TRR}} \leq \varphi C^*_{\text{adapt}} \). Further, \( C^*_{\text{adapt}} \leq \varphi C^*_{\text{non-adapt}} \), which proves the theorem.

We now prove the claim. We will write the expected cost of the TRR (with root \( x_0 \)) as \( C_{0,\text{TRR}} = 1 + E_1 + (1 - P_1)E_2 \). Here, \( E_1 \) is the expected number of bits tested in \( T \) in the first phase (i.e. in levels \( l < \ell \)), \( E_2 \) is the expected number of variables tested among levels in \( T \) in the second phase (levels \( l \geq \ell \)), given that the second phase is reached, and \( P_1 \) is the probability of ending during the first phase. Note that the value of \( \ell \) is determined only by the values of the \( p_i \), and it is independent of the test outcomes.

We will write the expected cost of \( T \) (the adaptive tree which is optimal w.r.t all trees with root \( x_0 \)) as \( C^*_{\text{adapt}} = 1 + E'_1 + (1 - P'_1)E'_2 \) where \( E'_1 \) is the expected number of bits queried in \( T \) before level \( \ell \), \( P'_1 \) is the probability of ending before level \( \ell \), and \( E'_2 \) is the expected number of bits queried in levels \( \ell \) and higher, given that \( \ell \) was reached.

To prove our claim, we will upper bound the ratio \( \alpha := \frac{1 + E_1 + (1 - P_1)E_2}{1 + E'_1 + (1 - P'_1)E'_2} \). Recall that since \( c < 1/2 \), we have \( c < 1 - c \). Also, the first phase ends if all bits have been tested, which implies that for all \( l \) in the first phase, \( l \leq \left\lceil (n-1)/2 \right\rceil \) so \( p_{n-l} \leq 1 - c \). We break the first phase into two parts: (1) The first part consists of all levels \( l \) where \( p_{n-l} \leq c < 1 - c \leq 1 - p_l \). (2) The second part consists of all
levels \( l \) where \( p_l \in (c, 1 - c) \) or \( p_{n-1} \in (c, 1 - c) \), or both.

Let us rewrite the expected cost \( E_1 \) as \( E_1 = E_{1,1} + (1 - P_{1,1})E_{1,2} \), where \( E_{1,1} \) is the expected cost of the first part of phase 1, \( E_{1,2} \) is the expected cost of the second part of phase 1, and \( P_{1,1} \) is the probability of terminating during the first part of phase 1. Analogously for the cost on tree \( T \), we can rewrite \( E'_1 = E'_{1,1} + (1 - P'_{1,1})E'_{1,2} \). Then, the ratio we wish to upper bound becomes

\[
\alpha = \frac{1+\frac{E_{1,1}}{1+\frac{E_{1,1}}{P_{1,1}}}}{1+\frac{E'_{1,1}}{1+\frac{E'_{1,1}}{P'_{1,1}}}} \frac{E_{1,2} + (1 - P_{1,1})E_{1,2}}{E'_{1,2} + (1 - P'_{1,1})E'_{1,2}}
\]

which we will upper bound by examining the three ratios \( \theta_1 := \frac{1+\frac{E_{1,1}}{1+\frac{E_{1,1}}{P_{1,1}}}}{1+\frac{E'_{1,1}}{1+\frac{E'_{1,1}}{P'_{1,1}}}} \frac{E_{1,2} + (1 - P_{1,1})E_{1,2}}{E'_{1,2} + (1 - P'_{1,1})E'_{1,2}} \) and \( \theta_2 := \frac{(1-P_{1,1})E_{1,2}}{(1-P'_{1,1})E'_{1,2}} \).

For ratio \( \theta_1 \), notice that the TRR does at most two tests for every tree level, so \( E_{1,1} \leq 2E'_{1,1} \), and thus \( \frac{1+\frac{E_{1,1}}{1+\frac{E_{1,1}}{P_{1,1}}}}{1+\frac{E'_{1,1}}{1+\frac{E'_{1,1}}{P'_{1,1}}}} \leq \frac{1+2E'_{1,1}}{1+2E'_{1,1}} \). Also, \( \frac{d}{dx} \left( \frac{1+2x}{1+x} \right) = \frac{1}{(1+x)^2} > 0 \) for \( x > 0 \). For each path in tree \( T \), for the levels in the first part of the first phase, the probability of getting a result that causes termination is at least \( 1 - c \). This is because in the first part, \( p_l \geq 1 - c > c \geq p_{n-1} \). If we are taking the left branch (because \( x_0 = 0 \)) we terminate when we get a test outcome of 1, and on the right (\( x_0 = 1 \)), we terminate when we get a test outcome of 0. Each bit queried is an independent Bernoulli trial, so \( E'_{1,1} \leq \frac{1}{1-c} \). Because \( \frac{1+2e}{1+e} \) is increasing, we can assert that \( \theta_1 = \frac{1+\frac{E_{1,1}}{1+\frac{E_{1,1}}{P_{1,1}}}}{1+\frac{E'_{1,1}}{1+\frac{E'_{1,1}}{P'_{1,1}}}} \leq \frac{1+2(1-c)^{-1}}{1+2(1-c)^{-1}} = \frac{3-c}{2-c} \).

Next we will upper bound the second ratio \( \theta_2 \). Let \( P(l) \) represent the probability of reaching level \( l \) in the TRR. Further, let \( q_l \) represent the probability of querying the second bit in level \( l \) given that we have reached level \( l \). Then, observe that \( (1 - P_{1,1})E_{1,2} \) can be written as the sum over all levels \( l \) in phase 1, part 2 of \( P(l)(1 + q_l) \). Note that in phase 1, the first bit queried is the bit \( x_i \) such that \( p_{i} \leq 0.5 \). Notice also that in the second part of the first phase, each level has at least one variable \( x_i \) such that \( p_{i} \in (c, 1 - c) \). This also means that \( 1 - p_{i} \leq (c, 1 - c) \). This means that the first test performed in any given level in phase 1, part 2 will cause the TRR to terminate with probability at least \( c \). This means that for each level \( l \) in this part of the TRR, we will have \( q_{l} \leq 1 - c \).

Similarly, \( (1 - P'_{1,1})E'_{1,2} \) is the sum over all levels \( l \) which comprise phase 1, part 2 in the TRR of \( P'(l) \). Here, \( P'(l) \) is defined as the probability of reaching level \( l \) in tree \( T \). We do not multiply by \( 1 + q_{l} \) since in the evaluation of \( T \) we only perform one test at each level.

Consider the evaluation of tree \( T \) on an assignment. If the evaluation terminates upon reaching level \( l \) in the tree, for \( l < l' \), then the evaluation using the TRR must terminate at a level \( l' \leq l \). That is, the TRR will terminate at level \( l \) or earlier for the same assignment. Thus, we get that \( P(l) \leq P'(l) \). Using this, we can achieve the following bound on the second ratio (letting \( S_2 \) denote the set of all levels included in the second part of phase 1): \( \theta_2 = \frac{(1-P_{1,1})E_{1,2}}{(1-P'_{1,1})E'_{1,2}} = \frac{\sum_{l \in S_2} P(l)(1+q_l)}{\sum_{l \in S_2} P(l)} = \frac{2 - c}{1+\frac{E_{1,2}}{1+\frac{E_{1,2}}{P_{1,1}}} \frac{E_{1,2} + (1 - P_{1,1})E_{1,2}}{E'_{1,2} + (1 - P'_{1,1})E'_{1,2}}} = 2 - c \).

Finally, we wish to upper bound the last ratio, \( \theta_3 = \frac{(1-P_{1,1})E_{1,2}}{(1-P'_{1,1})E'_{1,2}} \). Let \( l^* = l \) denote the first level included in the second phase of the TRR. Without loss of generality, assume that \( c \geq p_{l^*} \geq p_{n-1} \), so that in the TRR, the second phase
queries the remaining bits in decreasing order of \( p_i \). Thus, all bits \( x_i \) queried in the second phase satisfy \( p_i \leq c \). (The argument is symmetric for the case where \( p_{n-i} \geq p_{n-i-1} \geq 1-c \).

In this case, any assignments that do not cause termination in the TRR during the first phase, and that have \( x_0 = 0 \) (i.e., they would go down the left branch of \( T \)), will follow the same path through the nodes in left branch, for levels \( l^* \) and higher, that they would have followed in the optimal strategy \( T \). (In fact, tests from the right branch of the tree that were previously performed in phase 1 of the TRR do not have to be repeated.)

The numerator of the third ratio \( \theta_3 \) is equal to the sum, over all assignments \( x \) reaching level \( l^* \) in the TRR, of \( \Pr(x)C_2(x) \), where \( C_2(x) \) is the total cost of all bits queried in phase 2 for assignment \( x \). Let \( Q_0 \) be the subset of assignments reaching level \( l^* \) in the TRR which have \( x_0 = 0 \) and let \( Q_1 \) be the subset of assignments reaching level \( l^* \) in the TRR which have \( x_0 = 1 \). Let \( D_0 \) represent the sum over all assignments in \( Q_0 \) of \( \Pr(x)C_2(x) \) and let \( D_1 \) represent the sum over all assignments in \( Q_1 \) of \( \Pr(x)C_2(x) \). Then, letting \( S_l \) represent the set of assignments reaching level \( l^* \) in the TRR, we can rewrite the numerator of the third ratio as \( \sum_{x \in S_l} \Pr(x)C_2(x) = \sum_{x \in Q_0} \Pr(x)C_2(x) + \sum_{x \in Q_1} \Pr(x)C_2(x) = D_0 + D_1 \).

The denominator of the third ratio is the sum, over all assignments \( x \) reaching level \( l^* \) in the tree, of \( \Pr(x)C_2'(x) \), where \( C_2'(x) \) is the total cost of all bits queried in tree \( T \) at level \( l^* \) and below. Let \( S_l^* \) denote the set of assignments \( x \) reaching level \( l^* \) in tree \( T \). Next, observe that \( S_l^* \subseteq S_l \), since any assignment that reaches level \( l^* \) in the TRR must also reach level \( l^* \) in the tree. We can again rewrite the denominator as \( \sum_{x \in S_l^*} \Pr(x)C_2'(x) \geq \sum_{x \in S_l} \Pr(x)C_2'(x) = B_0 + B_1 \) where \( B_0 = \sum_{x \in Q_0} \Pr(x)C_2'(x) \) and \( B_1 = \sum_{x \in Q_1} \Pr(x)C_2'(x) \). The third ratio \( \theta_3 \) is thus upper bounded by \( \frac{(1-p_0)E_2}{(1-p_1)E_2} \leq \frac{D_0 + D_1}{B_0 + B_1} \).

For any \( x \in Q_0 \), the number of bits queried in level \( l^* \) or below in the TRR is less than or equal to the number of bits queried on \( x \) in level \( l^* \) or below in the tree. Thus \( D_0 \leq B_0 \).

For \( x \in Q_1 \), the number of bits queried at level \( l^* \) or below is at least one. Thus \( B_1 \geq J_1 \), where \( J_1 \) is the probability that a random assignment \( x \) has \( x_0 = 1 \) and reaches level \( l^* \).

Note that TRR will terminate on an assignment with \( x_0 = 1 \) when it first tests a bit that has value 0. Also note that each bit \( x_i \) in level \( l^* \) and below has probability \( p_i \leq c \) of having value 1 and thus probability \( 1 - p_i \geq 1 - c \) of having value 0 and ending the TRR. Since each bit queried is an independent trial, the expected number of bits queried before termination is at most \( (1-c)^{-1} \). Thus, \( D_1 \leq (1-c)^{-1}J_1 \). Together with the fact that \( D_0 \leq B_0 \), we get \( \frac{D_0 + D_1}{B_0 + B_1} \leq \frac{B_0 + (1-c)^{-1}J_1}{B_0 + J_1} \). Finally, we observe that since \( \frac{B_0}{B_0 + J_1} = 1 \) and \( \frac{(1-c)^{-1}J_1}{J_1} \leq \frac{1}{1-c} \), it follows from our earlier upper bound on \( \theta_3 \), namely \( \theta_3 \leq \frac{D_0 + D_1}{B_0 + B_1} \), that \( \theta_3 \leq \frac{B_0 + D_1}{B_0 + B_1} \leq \frac{1}{1-c} \).

Thus, we have three upper bounds: (1) \( \theta_1 \leq \frac{3-c}{2c} \), (2) \( \theta_2 \leq 2 - c \), and (3) \( \theta_3 \leq \frac{1}{1-c} \). This gives us an upper bound on the ratio of the expected cost of the
TRR to the tree $T$, and thus an upper bound on the approximation factor. This bound is simply the maximum of the three upper bounds:

$$\frac{1+P_1+(1-P_2)E_2}{1+P_1+(1-P_1)E_2} \leq \max \left\{\frac{3-c}{2-c} \frac{2-c}{1-c} \right\}.$$  Setting $c = \frac{3-\sqrt{5}}{2} \approx 0.381966$ causes all three upper bounds to equal $\phi$. Thus, running the TRR algorithm with $c = \frac{3-\sqrt{5}}{2}$ produces an expected cost of no more than $\phi$ times the expected cost of an optimal strategy.

\[
\square
\]

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A Verification vs. Evaluation

Let $f$ be a symmetric Boolean function. Let $g$ be the corresponding block identification function.

We use the following terminology, based on [7].

- $V^c(f)$ optimal expected verification cost of $f$ with respect to cost vector $c$
- $C^c(f)$ optimal expected evaluation cost of $f$ with respect to cost vector $c$
- $V^c(g)$ optimal expected verification cost of $g$ with respect to cost vector $c$
- $C^c(g)$ optimal expected evaluation cost of $g$ with respect to cost vector $c$

It is obvious that $V^c(g) \leq V^c(f) \leq C^c(f) \leq C^c(g)$.

Das et al. [7] proved that for symmetric Boolean functions under unit costs, $V^c(g) = V^c(f) = C^c(f) = C^c(g)$. We show that that does not hold under arbitrary costs. Namely, we show that there exist symmetric Boolean functions for which cost of evaluation exceeds the cost of verification.

**Theorem 4.** There exists a symmetric Boolean function $f$ and cost vector $c$ such that $V^c(g) < C^c(f)$.

**Proof.** We give a symmetric function $f$ on $n = 4$ bits that is defined by value vector $v^f = 01100$. That is, for all $x \in \{0, 1\}^n$ with $N_1(x) = j$, then $f(x) = v^f_j$.

The blocks of this vector are $B_1 = 0$, $B_2 = 11$, and $B_3 = 00$. The costs and probabilities for the variables are given in Table 3.

| variable | $p_i$ | cost |
|----------|-------|------|
| $x_0$    | 0.1   | 5000 |
| $x_1$    | 0.3   | 6000 |
| $x_2$    | 0.9   | 3000 |
| $x_3$    | 0.8   | 5000 |

Table 3: Table of variables

The optimal evaluation tree for $f$ is given in Figure 2, we denote it as $T$. (Following convention, left edges are implicitly labeled with 0s and right edges with 1s.) It has an expected evaluation cost $C(f) = 14,618$. Note that for any given root and its left child, the structure of the optimal evaluation tree for $f$ can be determined through a series of $k$-of-$n$ evaluations. Hence, the optimal evaluation tree for $f$ can be found by trying all root-left child combinations and choosing the optimal. Those combinations and the expected tree cost are given in Table 4, the optimal tree cost is bolded.

The expected cost of verifying that an assignment is in $B_2$ using $T$ is 10,248.8.

But the optimal verification cost for $B_2$ is actually 10,241.8. That cost is achieved in the tree in Figure 2. (The leaf nodes labeled X are nodes that the verification tree can never reach; they correspond to assignments not in $B_2$.) Hence, $C^c(g) \neq V^c(g)$.
The construction of this counterexample was based on the following observations. The optimal verification tree for $B_1$ is obvious since it must test all four variables on assignments in $B_1$ (in any order). The optimal verification tree for $B_3$ is obvious as well; since it must verify the block by finding at least three 1’s, it tests the variables in increasing order of $c_i p_i$ and terminates as soon as three 1’s are found. However, since at least three variables must be tested, any tree that tests the three cheapest $c_i p_i$ variables first, in any order, has the same (optimal) cost. We call the set of all trees that test those variables first $S(T_{B_3})$; it is the set of all optimal verification trees for $B_3$.

The optimal verification tree for $B_2$ is less obvious; however, given variables for the root and its left child, the rest of the tree follows from a series of $k$-of-$n$ evaluations, just like $T$. We give the structure for the tree in Figure 4 and denote it as $T_{B_2}$.

Specifically, the rules for the nodes of $T_{B_2}$ are as follows:

Table 4: Possible trees for $f$ and their cost

| root | left child | expected cost of tree |
|------|------------|-----------------------|
| $x_0$ | $x_1$      | 15,529                |
| $x_0$ | $x_2$      | 15,259                |
| $x_0$ | $x_3$      | 16,042                |
| $x_1$ | $x_0$      | 14,881                |
| $x_1$ | $x_2$      | 14,643                |
| $x_1$ | $x_3$      | 15,616                |
| $x_2$ | $x_0$      | **14,618**            |
| $x_2$ | $x_1$      | 14,670                |
| $x_2$ | $x_3$      | 14,623                |
| $x_3$ | $x_0$      | 15,394                |
| $x_3$ | $x_1$      | 15,616                |
| $x_3$ | $x_2$      | 15,406                |
Nodes a, b, and c are the remaining variables on the right-hand side after the root is chosen, ordered in increasing order of \( \frac{c_i}{1 - p_i} \). This is due to the fact that once the root node is tested and has the value of 1, the goal is to find 0’s as cheaply as possible.

Node d is chosen to be the variable with the maximum \( \frac{c_i}{p_i} \); since two 0’s have already been found, the goal is to find cheap 1’s.

Finally, node e is again chosen to be the variable with low \( \frac{c_i}{1 - p_i} \), reflecting once again that once one 1 has been found, the goal is to find cheap 0’s.

If the root of the optimal verification tree for \( B_2 \) has the maximum value for \( \frac{c_i}{p_i} \), and furthermore, the variable tested in node child differs from the variables tested in a and b, \( T_{B_2} \) will differ from all of the trees in \( S\{T_{B_3}\} \). This is in fact the case for \( f \) under the cost and probabilities given in Table 3. Hence the optimal evaluation tree for function \( f, T \), must achieve a non-optimal verification cost on either block \( B_2 \) or \( B_3 \).

(We note that the particular variables given here are far from the only choice.
of variables that satisfy these conditions and prove the theorem. They were chosen as an illustrative example.)

A corollary follows:

**Corollary 1.** For all interval functions $f$ and cost vectors $c$, $\mathcal{V}^c(g) = \mathcal{V}^c(f)$.

**Proof.** For the particular function $f$ given above, defined by value vector 01100, verifying the value of the function when it is 1 is equivalent to verifying block $B_2$. Verifying the value of the function when it is 0 requires verifying either block $B_1$ or $B_3$; however, since the optimal verification strategy for $B_1$ is to test every bit (in any order), the optimal verification tree for $B_3$ is the optimal verification tree for $f = 0$. Hence, $\mathcal{V}^c(f) = \mathcal{V}^c(g)$ for any cost vector $c$.

More generally, for any three-blocked value vector, the verification tree for the value of the function will either be the verification tree for the middle block, or a verification tree for blocks 1 and 3. Whenever it is the latter, there will always exist at least one bit in the intersection of the optimal verification strategies for blocks 1 and 3. Then we can use a strategy similar to the one in Section 6.1 to continuously choose the bit in the intersection of the strategies to form the optimal verification tree. In doing so, we replace the verification trees for the first and last blocks with a tree of equal expected cost.

Hence, for any three-block value vector, $\mathcal{V}^c(g) = \mathcal{V}^c(f)$. 

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**B Background: Optimality of the k-of-n Algorithm**

In Section 6.1, we described the known algorithm for evaluating $k$-of-$n$ functions [17, 4, 18, 6, 19]. It is helpful to understand why this algorithm is, in fact, an optimal adaptive evaluation strategy. Here we review a version of the proof that is given in [6].

The proof relies on the fact that evaluating the bits in increasing $c_i/p_i$ order is a 1-optimal strategy, and evaluating them in increasing $c_i/(1 - p_i)$ ordering is a 0-optimal strategy. (We omit the proof of this fact here.) Thus these two strategies constitute an optimal verification strategy.

The expected cost of this optimal verification strategy is a lower bound on the expected cost of an optimal evaluation strategy. If $f(x) = 1$, the 1-optimal strategy cannot terminate on $x$ before it has tested all $k$ bits in $S_1$. Thus the strategy is still 1-optimal if those bits are permuted. Similarly, if $f(x) = 0$, the 0-optimal strategy cannot terminate before it has tested all bits in $S_0$, and those can be permuted. If $x_i \in S_1 \cap S_0$, it there is both a 1-optimal strategy and a 0-optimal strategy that tests $x_i$ first. Inductively, it follows that the above $k$-of-$n$ evaluation strategy is both 1-optimal and 0-optimal. Since its expected cost is equal to the optimal expected verification cost, it is an optimal evaluation strategy.
C Omitted Proofs and Related Material

C.1 Details of the $B - 1$ approximation

Let $f : \{0,1\}^n \to \{1,\ldots,B\}$ be the unweighted score classification function associated with the values $0 = \alpha_1 < \ldots < \alpha_B < \alpha_{B+1} = n + 1$. Let $v = v^f$ be its value vector. An assignment $x$ belongs to block $j$ if $\alpha_j \leq N_1(x) < \alpha_{j+1}$.

We present Algorithm 4 and show it achieves a $(B - 1)$-approximation for the Symmetric SLSC problem. In the algorithm, we denote as $f_i$ the $k$-of-$n$ function with $k = \alpha_i$. We note that in different iterations of the for loop, the strategy that is executed in the body may choose a test that was already performed in a previous iteration. The test does not actually have to be repeated, as the outcome can be stored after the first time the test is performed, and accessed whenever the test is chosen again.

Algorithm 4 Adaptive Algorithm for Evaluating Score Classification Function $f$

for $i \leftarrow 2$ to $B$
    Run the optimal adaptive $k$-of-$n$ strategy to evaluate $f_i(x)$
end for

Let $i^* \leftarrow \max\{i \mid f_i(x) = 1\}$ // $i^* = \alpha_1 = 0$ if $f_i(x) = 0$

for all $i > 1$

return $v_{\alpha_{i^*}}$

The correctness of the algorithm follows easily from the fact that $\alpha_{i^*} \leq N_1(x) < \alpha_{i^*+1}$, and so $f(x) = v^f_{\alpha_{i^*}}$.

We now examine the expected cost of the strategy computed in Algorithm 4.

Let $C(f_i)$ denote the expected cost of evaluating $f_i$ using the optimal $k$-of-$n$ strategy. Let $\text{OPT}$ be expected cost of the optimal adaptive strategy for $f$.

Lemma 1. $C(f_i) \leq \text{OPT}$ for $i \in \{1,\ldots,B-1\}$.

Proof. Let $T$ be an optimal adaptive strategy for evaluating $f$. Consider using $T$ to evaluate $f$ on an initially unknown input $x$. When a leaf of $T$ is reached, we have discovered the values of some of the bits of $x$. Let $d$ be the partial assignment representing that knowledge. Recall that $f^d$ is the function induced from $f$ by $d$. The value vector of $f^d$ is a subvector of $v^f$, the value vector of $f$. More particularly, it is the subvector stretching from index $N_1(d)$ of $v^f$ to index $n - N_0(d)$. Since $T$ is an evaluation strategy for $f$, reaching a leaf of $T$ means that we have enough information to determine $f(x)$. Thus all entries of the subvector must be equal, implying that it is contained within a single block of $v^f$. We call this the block associated with the leaf.

For each block $i$, we can create a new tree $T'_i$ from $T$ which evaluates the function $f_i$. We do this by relabeling the leaves of $T$: if the leaf is associated with block $i'$, then we label the leaf with output value 1 if $i' > i$, and with 0 otherwise. $T'_i$ is an adaptive strategy for evaluating $f_i$. 

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The expected cost of evaluating \( f_i \) using \( T_i' \) is equal to \( \text{OPT} \), since the structure of the tree is unchanged from \( T \) (we’ve only changed the labels). Since \( T_i' \) cannot do better than the optimal \( k \)-of-\( n \) strategy, \( C(f_i) \leq \text{OPT} \).

This yields an approximation bound for Algorithm 4.

**Theorem 5.** Algorithm 4 is a \((B - 1)\)-approximation algorithm for the unweighted adaptive SSClass problem.

**Proof.** The total cost incurred by the algorithm is no greater than the sum of the costs incurred by the \( B - 1 \) runs of the \( k \)-of-\( n \) algorithm. Thus by Lemma 1, \( ALG \leq \sum_{i=1}^{B-1} C(f_i) \leq \sum_{i=1}^{B-1} \text{OPT} \).

**C.2 The \( 2(B-1) \) approximation for non-adaptive unweighted SSClass, arbitrary costs**

We briefly mentioned the result in Section C.3. Note that we already have a simple \( B - 1 \) approximation algorithm for the adaptive case.

**Theorem 6.** Algorithm 2 is a \( 2(B - 1) \)-approximation for the non-adaptive unweighted SSClass problem.

**Proof.** Let \( f : \{0, 1\}^n \rightarrow \{1, \ldots, B\} \) be the score classification function associated with an instance of the problem. Let \( \mathcal{A} \) be an optimal non-adaptive algorithm for evaluating \( f \) and let \( \text{OPT} \) be its expected cost.

Consider running Algorithm 2 to evaluate \( f \). For each assignment \( a \in \{0, 1\}^n \), there is some block boundary \( \alpha_i \) that is the final block boundary “crossed” before execution of Algorithm 2 terminates. In other words, immediately before the final test is chosen, the value vector of the pseudo-Boolean function induced by the prior test results contains entries \( \alpha_i - 1 \) and \( \alpha_i \) of the original value vector, where \( i \) is the index of a block of that vector. The final test will cause the induced value vector to contain only one of these entries, thereby determining whether \( x \) is in block \( i - 1 \) or block \( i \). Either way, we say that \( \alpha_i \) was the final block boundary crossed.

There are \( B - 1 \) possible final block boundaries, \( \alpha_2, \ldots, \alpha_B \). We will partition the assignments \( x \in \{0, 1\}^n \) into sets \( S_i \) for \( i \in \{2, 3, \ldots, B\} \) where each set \( S_i \) contains all assignments on which execution of Algorithm 2 terminates after crossing block boundary \( \alpha_i \). Let \( RR \) denote the strategy of Algorithm 2.

Quantity \( C(\text{RR}, a) \) is the cost incurred by \( RR \) on assignment \( a \). Let \( C^{RR}(\text{Alg}_0, a) \) and \( C^{RR}(\text{Alg}_1, a) \) represent the cost incurred on assignment \( a \) during the execution of \( RR \) by \( \text{Alg}_0 \) and \( \text{Alg}_1 \) respectively, so \( C(\text{RR}, a) = C^{RR}(\text{Alg}_0, a) + C^{RR}(\text{Alg}_1, a) \). Let \( Q_0 \) and \( Q_1 \) be the sets of assignments \( a \) for which the final bit queried in Algorithm 2 was determined by \( \text{Alg}_0 \) and \( \text{Alg}_1 \), respectively.

Let \( f_i \) denote the \( k \)-of-\( n \) function with \( k = \alpha_i \). Let \( \text{Alg}_0 \) denote the 0-optimal strategy for evaluating \( f_i \), which queries bits in increasing order of \( c_i / (1 - p_i) \) until \( n - \alpha_i + 1 \) 0’s are obtained, or all bits are queried. Similarly, let \( \text{Alg}_1 \) denote the 1-optimal strategy for evaluating \( f_i \), which queries bits \( j \) in increasing order.
of $c_j/p_j$ until $\alpha_i$ 1’s are obtained, or all bits are queried. We have the following two inequalities, one each for $Q_0$ and $Q_1$.

\[ \sum_{a \in S_i \cap Q_1} C(RR, a)p(a) \leq \sum_{a \in S_i \cap Q_0} 2C^{RR}(Alg_1, a)p(a) \leq \sum_{a \in \{0,1\}^n} 2C(Alg_1^i, a)p(a) \]

\[ \sum_{a \in S_i \cap Q_0} C(RR, a)p(a) \leq \sum_{a \in S_i \cap Q_0} 2C^{RR}(Alg_0, a)p(a) \leq \sum_{a \in \{0,1\}^n} 2C(Alg_0^i, a)p(a) \]

For each, the first inequality holds because $C(RR, a) = C^{RR}(Alg_0, a) + C^{RR}(Alg_1, a)$. Further, it holds that $C^{RR}(Alg_0, a) \leq C^{RR}(Alg_1, a)$ for assignments in $Q_1$ (and similarly for assignments in $Q_0$).

As in the proof of Lemma 1, the strategy $A$ for evaluating $f$ could be turned into a strategy for evaluating $f_i$ by relabeling the leaves of $A$, without changing the cost incurred by the strategy on any assignment. Since $Alg_0^i$ is a 0-optimal strategy for $f_i$, $f_i^0(a) = 1$ iff $N_1(a) \geq b_i$, and $f_i^0(a) = 0$ iff $N_1(a) < \alpha_i$ (equivalently, $N_0(a) \geq n - \alpha_i + 1$).

\[ \sum_{a \in \{0,1\}^n} C^{RR}(Alg_0, a)p(a) \leq \sum_{a \in \{0,1\}^n} C(A, a)p(a) \]

and

\[ \sum_{a \in \{0,1\}^n} C^{RR}(Alg_0, a)p(a) \leq \sum_{a \in \{0,1\}^n} C(A, a)p(a) \]

Using this, we sum the two quantities of (2) and (5) to get the following inequality representing the cost incurred for assignments in $S_i$.

\[ \sum_{a \in S_i} C(RR, a)p(a) \leq 2 \sum_{a \in \{0,1\}^n} C^{RR}(Alg_1, a)p(a) + 2 \sum_{a \in \{0,1\}^n} C^{RR}(Alg_0, a)p(a) \leq 2 \sum_{a \in \{0,1\}^n} C(A, a)p(a) = 2OPT \]
Summing over all block boundaries we get

$$\sum_{a \in \{0,1\}^n} C(RR,a)p(a) = \sum_{i=1}^{B-1} \sum_{a \in S_i} C(RR,a)p(a) \leq 2(B-1)OPT$$  \hspace{1cm} (9)$$
as desired.

### C.3 Proof of the 4-approximation for Unweighted SSClass with Unit Costs

Before proving Theorem 2 we first prove some claims. Consider applying Algorithm 2 to evaluate the pseudo-Boolean function \( f \) associated with a symmetric SLSC function \( f \). Assume further that the costs \( c_i \) are all equal to 1. Let \( \beta_j = \alpha_{j+1} \). Consider block \( j \) of \( v^l \), represented by \([\alpha_j, \beta_j)\). Let \( M^j = \{ a \in \{0,1\}^n \mid \alpha_j \leq N_1(a) < \beta_j \} \). That is, \( M^j \) is the set of assignments in the \( j \)th block.

For a permutation \( \sigma \) and an assignment \( a \in M^j \), let \( c^1_1(\sigma, a) \) denote the total cost incurred when bits are queried in the order specified by \( \sigma \), until it is verified that \( N_1(a) \geq \alpha_j \) (i.e., until \( \alpha_j \)'s are seen). Similarly, let \( c^0_0(\sigma, a) \) denote the total cost incurred until it is verified that \( N_1(a) < \beta_j \) (equivalently, \( n - \beta_j + 1 \) 0's are seen). Since we are assuming unit cost tests, total cost incurred is equal to the number of bits queried.

Let \( C^1(\sigma) = \sum_{a \in M^j} [c^1_1(\sigma, a)p(a)] \) and similarly \( C^0(\sigma) = \sum_{a \in M^j} [c^0_0(\sigma, a)p(a)] \).

Let \( \sigma^1 \) be the permutation that orders bits in increasing order of \( 1/p_i \) (equivalently, decreasing order of \( p_i \)), and let \( \sigma^0 \) be the permutation that orders bits in increasing order of \( 1/(1 - p_i) \) (equivalently, increasing order of \( p_i \)). For simplicity, we assume in what follows that the \( p_i \) are all different; the arguments can be easily extended if this is not the case.

**Claim 1.** \( C^1(\sigma^1) \leq C^1(\sigma) \) for all permutations \( \sigma \). Similarly, \( C^0(\sigma^0) \leq C^0(\sigma) \) for all permutations \( \sigma \).

**Proof.** We give the proof for \( C^1_1 \). The proof for \( C^0_1 \) is analogous.

Suppose there exists a permutation \( \pi \) such that \( C^1_1(\pi) < C^1_1(\sigma^1) \). Let \( \pi \) be an optimal such permutation, so \( C^1_1(\pi) \leq C^1_1(\sigma) \) for all permutations \( \sigma \). Re-number the bits so that \( \pi(i) = i \) for all \( i \).

Since the \( p_i \)’s are distinct and \( \pi \neq \sigma^1 \), there exists a bit \( 1 \leq l \leq n - 1 \), such that \( p_l < p_{l+1} \). Consider the permutation \( \pi' \) produced from \( \pi \) by swapping the elements in positions \( l \) and \( l+1 \).

We will obtain a contradiction by showing that \( C^1_1(\pi') < C^1_1(\pi) \). Consider the four possible values of \( x_l \) and \( x_{l+1} \):

- \( x_l = 0 \) and \( x_{l+1} = 0 \)
- \( x_l = 1 \) and \( x_{l+1} = 1 \)
- \( x_l = 0 \) and \( x_{l+1} = 1 \)
- \( x_l = 1 \) and \( x_{l+1} = 0 \)

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• \( x_l = 1 \) and \( x_{l+1} = 0 \)

Consider the difference

\[
C^1_1(\pi) - C^1_1(\pi') = \sum_{a \in M^j} \left[ c^1_i(\pi, a) - c^1_i(\pi', a) \right] p(a)
\]

and consider a specific assignment, \( a \in M^j \). Let \( d \) represent the partial assignment where \( d_i = a_i \) for all \( i \) such that \( i < l \) and \( d_i = * \) otherwise. That is, \( d \) contains the values of the variables which appear before \( x_l \) in permutation \( \pi \) (and before \( x_{l+1} \) in permutation \( \pi' \)).

If \( N_1(d) < \alpha_j - 1 \), then verifying \( N_1(a) \geq \alpha_j \) using \( \pi \) results in querying both \( x_l \) and \( x_{l+1} \), so \( c^1_i(\pi, a) = c^1_i(\pi', a) \). If \( N_1(d) \geq \alpha_j \), then verifying \( N_1(a) \geq \alpha_j \) using \( \pi \) does not involve querying either \( x_l \) or \( x_{l+1} \), so \( c^1_i(\pi, a) = c^1_i(\pi', a) \).

Suppose \( N_1(d) = \alpha_j - 1 \). In this case, if \( a_l = a_{l+1} = 0 \), then \( \pi \) and \( \pi' \) will query both \( x_l \) and \( x_{l+1} \) and incur the same total cost. If \( a_l = a_{l+1} = 1 \), then \( \pi \) and \( \pi' \) will each query exactly one of \( x_l \) and \( x_{l+1} \) before terminating. Since both queries have unit cost, \( \pi \) and \( \pi' \) will incur the same total cost on \( a \).

We are left with the assignments \( a \in M^j \) where for the corresponding \( d \), \( N_1(d) = \alpha_j - 1 \) and \( a_l \neq a_{l+1} \). Let \( A \) represent the set of such assignments. It follows that

\[
C^1_1(\pi) - C^1_1(\pi') = \sum_{a \in A} \left[ c^1_i(\pi, a) - c^1_i(\pi', a) \right] p(a)
\]

Let \( p(a, i) = (p_l)^{a_l}(1 - p_l)^{(1 - a_l)} \). Then \( p(a) = \prod_{i=1}^n p(a, i) \). Let \( p'(a) = p(a)/[p(a, l) \cdot p(a, l + 1)] \). Observe that for \( a \in A \), both permutation \( \pi \) and \( \pi' \) will result in terminating after querying \( l \) or \( l + 1 \) bits (which of the two depends on the values of \( a_l \) and \( a_{l+1} \)). There are two cases to consider:

1. \( a_l = 1 \) and \( a_{l+1} = 0 \). In this case, \( c^1_i(\pi, a) = l \) and \( c^1_i(\pi', a) = l + 1 \).
   
   \[
p(a) = p'(a) \cdot p_l(1 - p_{l+1}).
   \]

2. \( a_l = 0 \) and \( a_{l+1} = 1 \). In this case, \( c^1_i(\pi, a) = l + 1 \) and \( c^1_i(\pi', a) = l \).
   
   \[
p(a) = p'(a) \cdot (1 - p_l)p_{l+1}.
   \]

In the first case, we get

\[
\left[ c^1_i(\pi, a) - c^1_i(\pi', a) \right] p(a) = \left[ p'(a) \cdot p_l(1 - p_{l+1}) \right] \left[ l - (l + 1) \right]
\]

\[
= -p'(a) \cdot p_l(1 - p_{l+1})
\]

and in the second case, we get

\[
\left[ c^1_i(\pi, a) - c^1_i(\pi', a) \right] p(a) = p'(a) \cdot (1 - p_l)p_{l+1}.
\]

Let \( Q_{10} \) represent the set of assignments which fall in the first case, and \( Q_{01} \) the set of assignments which fall in the second case. Note that each assignment
Proof. We will prove this by arguing that the optimal adaptive strategy (with go in increasing order of 1

```latex
\begin{align*}
C_1^j(\pi) - C_1^j(\pi') &= \sum_{a \in Q_{01}} p'(a) \cdot (1 - p_l) p_{l+1} - \sum_{a \in Q_{10}} p'(a) \cdot p_l (1 - p_{l+1}) \\
&= \sum_{a \in Q_{01}} [p'(a) \cdot (1 - p_l) p_{l+1} - p'(\hat{a}) \cdot p_l (1 - p_{l+1})] \\
&= \sum_{a \in Q_{01}} p'(a) (p_{l+1} - p_l) \quad \text{since } p'(a) = p'(\hat{a}) \\
&= (p_{l+1} - p_l) \sum_{a \in Q_{01}} p'(a)
\end{align*}
```

(10)

But since $p_l < p_{l+1}$ and the $p'(a)$ are non-negative, $C_1^j(\pi) > C_1^j(\pi')$. This contradicts the optimality of $\pi$.

A symmetric argument shows that $\sigma^0$ minimizes $C_0^j$. \qed

Let $T$ be a decision tree representing an adaptive testing strategy. For an assignment $a \in M^j$, let $C_1^j(T, a)$ denote the total cost incurred when bits are queried as specified by $T$, until it is verified that $N_1(a) \geq \alpha_j$. Similarly, define $C_0^j(T, a)$ as the total cost incurred by the adaptive strategy $T$ when querying bits until it is verified that $N_1(a) < \beta_j$. We similarly define $C_1^j(T) = \sum_{a \in M^j} [c_1^j(T, a)p(a)]$ and $C_0^j(T) = \sum_{a \in M^j} [c_0^j(T, a)p(a)]$. We can further claim that not only are $\sigma^1$ and $\sigma^0$ better than any other permutation (in terms of $C_1^j$ and $C_0^j$) but also that they are optimal with respect to adaptive strategies. That is:

**Remark 1.** For any $j$, and for all adaptive strategies $T$: $C_1^j(\sigma^1) \leq C_1^j(T)$ and $C_0^j(\sigma^0) \leq C_1^j(T)$.

**Proof.** We will prove this by arguing that the optimal adaptive strategy (with respect to $C_1^j$ or $C_0^j$) is in fact a permutation (i.e., is nonadaptive). Then, it must follow from Claim 1 that this adaptive strategy is $\sigma^1$ (respectively, $\sigma^0$).

We do this by induction on $n$. For $n = 1$, the optimal adaptive strategy is to query the single bit. Then, assume that for any function on $n$ bits, the adaptive strategy which minimizes $C_1^j$ (resp. $C_0^j$) is the permutation $\sigma^1$ (resp. $\sigma^0$). Then, for a function on $n + 1$ bits, the optimal adaptive strategy is a decision tree with some bit at the root. Whether this first bit is a 0 or a 1, the result induces a new function on $n$ variables (the same $n$ variables for either outcome), and the optimal strategy in this case is the permutation that orders bits by increasing order of $1/p_i$ (resp. $1/(1 - p_i)$). Thus the subtrees rooted at the 0-child and 1-child of the root are in fact the same permutation, and thus the entire strategy can be expressed as a permutation of the $n + 1$ bits: Choose the root first, then go in increasing order of $1/p_i$ (resp. $1/(1 - p_i)$). Since the strategy minimizing $C_1^j$
(resp. \(C_0^j\)) for \(n + 1\) bits is a permutation, by Claim 1 it must be the permutation \(\sigma^1\) (resp. \(\sigma^0\)).

For a strategy \(A\) and assignment \(a\), let \(C(A, a)\) denote the cost incurred evaluating \(a\) using strategy \(A\). Thus, the expected cost of strategy \(A\) is \(\sum_{a \in \{0, 1\}^n} C(A, a)p(a)\).

Now let \(A_{OPT}\) be an adaptive strategy that minimizes the expected cost of evaluating \(f\). Let \(T_{OPT}\) be the corresponding decision tree of this adaptive strategy.

Claim 2. \(C_0^j(\sigma^0) \leq \sum_{a \in M_j} C(A_{OPT}, a)p(a)\) and \(C_1^j(\sigma^1) \leq \sum_{a \in M_j} C(A_{OPT}, a)p(a)\).

**Proof.** In evaluating \(f\) on some input \(a \in M_j\), we cannot terminate until we have seen at least \(\alpha_j\) ones and at least \(n - \beta_j + 1\) zeros. Thus if we perform tests on \(a\) in the order indicated by \(T_{OPT}\), and terminate as soon as we see \(\alpha_j\) ones, the resulting cost will be at most \(C(A_{OPT}, a)\). Thus \(\sum_{a \in M_j} c_j^1(T_{OPT}, a)p(a) \leq \sum_{a \in M_j} C(A_{OPT}, a)p(a)\). Since \(\sigma^1\) minimizes \(C_1^j\), \(\sum_{a \in M_j} c_j^1(\sigma^1, a)p(a) \leq \sum_{a \in M_j} C(A_{OPT}, a)p(a)\). This implies the statement for \(\sigma^1\), and an analogous argument with \(n - \beta_j + 1\) zeros yields the statement for \(\sigma^0\).

Below we use Claims 1 and 2 in order to prove Theorem 2.

**Proof of Theorem 2.** Let \(\text{OPT}\) be the expected cost incurred by an optimal strategy. We partition the set of all possible assignments \(a \in \{0, 1\}^n\) into two groups, \(Q_0\) and \(Q_1\), depending on whether running Algorithm 2 on \(a\) causes it to terminate after querying a bit chosen by \(A_{OPT}\) or a bit chosen by \(A_{G}\) (respectively).

For \(l \in \{0, 1\}\), let \(C_{RR}(A_{G}, a)\) represent the cost incurred by \(A_{G}\) during execution of Algorithm 2 on assignment \(a\). As in Section 6.2 it holds that for \(a \in Q_0\), \(C_{RR}(A_{G}, a) \geq C_{RR}(A_{G}, a)\) and for \(a \in Q_1\), \(C_{RR}(A_{G}, a) \geq C_{RR}(A_{G}, a)\).

Suppose \(a \in M_j \cap Q_1\). Algorithm 2 terminates on input \(a\) as soon as it has seen at least \(\alpha_j\) ones and at least \(n - \beta_j + 1\) zeros. Since \(a \in Q_1\), Algorithm 2 terminated as soon as it saw its \(\alpha_j\)th 1. It follows that \(C_{RR}(A_{G}, a) \leq c_j^1(\sigma^1, a)\). Similarly, for \(a \in M_j \cap Q_0\), \(C_{RR}(A_{G}, a) \leq c_j^0(\sigma^0, a)\). Letting \(B\) be the total number of blocks, so blocks are numbered from 1 to \(B\), Claim 2 implies that for \(l \in \{0, 1\}\)

\[
\sum_{j=1}^{B} \sum_{a \in M_j \cap Q_l} C_{RR}(A_{G}, a)p(a) \leq \sum_{j=1}^{B} \sum_{a \in M_j \cap Q_l} c_j^l(\sigma^l, a)p(a) \\
\leq \sum_{j=1}^{B} C_j^l(\sigma^l) \leq \sum_{j=1}^{B} \sum_{a \in M_j} C(A_{OPT}, a)p(a) = \text{OPT}
\]

\[(11)\]
Thus, letting $EC$ be the expected cost of the Algorithm 2, it follows from (11) that we have

$$
EC = \sum_{a \in Q_0} \left[ C^{RR}(Alg_0, a) + C^{RR}(Alg_1, a) \right] p(a)
+ \sum_{a \in Q_1} \left[ C^{RR}(Alg_0, a) + C^{RR}(Alg_1, a) \right] p(a)
\leq 2 \sum_{a \in Q_0} C^{RR}(Alg_0, a)p(a) + 2 \sum_{a \in Q_1} C^{RR}(Alg_1, a)p(a)
\leq 2 \sum_{j=1}^B \sum_{a \in M_j \cap Q_0} C^{RR}(Alg_0, a)p(a) + 2 \sum_{j=1}^B \sum_{a \in M_j \cap Q_1} C^{RR}(Alg_1, a)p(a)
\leq 2OPT + 2OPT = 4OPT
$$

(12)