Maximal \( L^p - L^q \) regularity is proved for the strong, weak and very weak solutions of the inhomogeneous Stokes problem with Navier-type boundary conditions in a bounded domain \( \Omega \), not necessarily simply connected. This extends previous results of the authors (2017).

Keyword. inhomogeneous Stokes Problem, Navier-type boundary conditions, Maximal regularity.

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1. Introduction

We consider in this paper the maximal \( L^p - L^q \) regularity for the following Stokes problem with slip frictionless boundary conditions involving the tangential component of the velocity vortex instead of the stress tensor:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \nabla \pi &= f, & \text{div } u &= 0 & \text{in } \Omega \times (0, T), \\
u \cdot n &= 0, & \text{curl } u \times n &= 0 & \text{on } \Gamma \times (0, T), \\
u(0) &= 0 & \text{in } \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^3 \) of class \( C^{2,1} \) not necessarily simply-connected, \( \Gamma \) is its boundary and \( n \) is the exterior unit normal vector on \( \Gamma \). The unknowns \( u \) and \( \pi \) denote respectively the velocity field and the pressure of a fluid occupying the domain \( \Omega \), while \( u(0) \) and \( f \) represent respectively the given initial velocity and the external force. Under these conditions, the Stokes problem has a non trivial kernel \( K_\tau(\Omega) \) (see (11) below).

Given a Cauchy-Problem of the form:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + \mathcal{A} u(t) &= f(t) \quad 0 \leq t \leq T \\
u(0) &= 0,
\end{array} \right.
\end{align*}
\]

...
where $-\mathcal{A}$ is the infinitesimal generator of a semi-group $e^{-t\mathcal{A}}$ on a Banach space $X$ and $f \in L^p(0, T; X)$, we say that a solution $u$ satisfies the maximal $L^p$-$L^q$ regularity if
\[ u \in W^{1,p}(0, T; X) \cap L^p(0, T; D(\mathcal{A})). \] (5)

It is known that the analyticity of $e^{-t\mathcal{A}}$ is not enough to ensure that property to be satisfied, although it is enough when $X$ is a Hilbert space (cf. [7], [8]).

When $1 < p, q < \infty$, the maximal $L^p$-$L^q$ regularity has been proved by the authors in [2] for solutions to (1)-(3) lying in the orthogonal of the kernel $\mathbf{K}_\tau(\Omega)$. In terms of the abstract example (4), the main argument of the proof, based on the use of the results of [9], was to show that the pure imaginary powers of $(I + \mathcal{A})$ are suitably bounded operators, and deduce that so where the imaginary powers of $\mathcal{A}$. That could only be done assuming the operator $\mathcal{A}$ to be invertible, but that is not the case of the Stokes operator on a non simply-connected domain, with boundary conditions (2). The maximal regularity result was then proved only for the restriction of the Stokes operator to the kernel’s orthogonal, where it was of course invertible. The purpose of the present work is to extend that result to the solutions of (1)-(3) that do not necessarily lie in the orthogonal of $\mathbf{K}_\tau(\Omega)$. The idea is to decompose the solution as an element of the kernel and an element of its orthogonal and to apply the result of [2].

We are interested in three different types of solutions for (1)-(3). The first, that we call strong solutions, are solutions $u$ that belong to $L^p(0, T; L^q(\Omega))$ type spaces. The second, called weak solutions, are solutions (in a suitable sense) $u(t)$ that may be written for a.e. $t > 0$, as $u(t) = v(t) + \nabla w(t)$ where $v(t) \in L^p(0, T; L^q(\Omega))$ and $w \in L^p(0, T; L^q(\Omega))$. The third and last, called very weak, are solutions $u(t)$ that may be decomposed as before but where now $w \in L^p(0, T; W^{-1,q}(\Omega))$ (cf [2] for more details). Of course, these different types of solutions correspond to data $u(0)$ and $f$ with different regularity properties.

There is a wide literature on the maximal regularity for the Stokes problem with different type of boundary conditions and different domains. Among the firsts articles on this problem we may mention [17] by V. A. Solonnikov. The works by Y. Giga and H. Sohr [10, 11] consider that question for the Stokes problem with Dirichlet boundary conditions in bounded and unbounded domains; J. Saal [14] for the Stokes problem with homogeneous Robin boundary conditions in the half space $\mathbb{R}^3_+$; R. Shimada [15] for the Stokes problem with non-homogeneous Robin boundary conditions. The maximal regularity for
general parabolic problems is treated in detail in the long report [12]
by P. C. Kunstmann and L. Weisand.

In the next Section we introduce some notations and recall several
results, already known, that are needed thereafter. Our main results
are stated, and their proofs given in Section 3.

2. Stokes operator

In order to obtain strong, weak and very weak solutions to our pro-
blem (1)-(3), we introduced in [2] three different extensions $A_p, B_p, C_p$, of
the Stokes operators with boundary conditions (2), defined in different
spaces of distributions with different regularity properties. Throughout
this paper, if not stated otherwise, $p$ will be a real number such that
$1 < p < \infty$.

We first consider $A_p$, the Stokes operator with the boundary condi-
tions (2) on the space $L^p_{\sigma,\tau}(\Omega)$ given by

\[
L^p_{\sigma,\tau}(\Omega) = \{f \in L^p(\Omega); \text{div} f = 0 \text{ in } \Omega, \ f \cdot n = 0 \text{ on } \Gamma\}.
\]

By [2, Corollary 3.7], this is a well defined subspace of

\[
H^p(\text{div}, \Omega) = \{v \in L^p(\Omega); \ \text{div} v \in L^p(\Omega)\},
\]

equipped with the graph norm. As described in [1, Section 3], $A_p$ is a
closed linear densely defined operator on $L^p_{\sigma,\tau}(\Omega)$ defined as follows

\[
D(A_p) = \{u \in W^{2,p}(\Omega); \text{div} u = 0 \text{ in } \Omega, u \cdot n = 0, \text{curl} u \times n = 0 \text{ on } \Gamma\}
\]

\[
\forall u \in D(A_p), \quad A_p u = -\Delta u \quad \text{in } \Omega.
\]

The operator $P$ in (8), is the Helmholtz projection defined as follows:

\[
P : L^p(\Omega) \mapsto L^p_{\sigma,\tau}(\Omega); \quad \forall f \in L^p(\Omega) : \quad Pf = f - \text{grad} \pi,
\]

where $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ is the unique solution of the following weak
Neuman Problem (cf. [16]):

\[
\text{div} (\text{grad} \pi - f) = 0 \text{ in } \Omega, \quad (\text{grad} \pi - f) \cdot n = 0, \text{ on } \Gamma.
\]

It is known that, due to the slipping frictionless boundary condition
(2), the pressure gradient disappears in the Stokes operator (cf. [1,
Proposition 3.1]). As a result the Stokes problem with the boundary
condition (2) is reduced to the study of a vectorial Laplace like problem
under a free-divergence condition and the boundary conditions (2).

\[
\forall u \in D(A_p), \quad A_p u = -\Delta u \quad \text{in } \Omega.
\]

We also recall that the operator $-A_p$ is sectorial and generates a
bounded analytic semi-group on $L^p_{\sigma,\tau}(\Omega)$, for all $1 < p < \infty$ (cf. [1,
Theorem 4.12). We denote by \( e^{-tA_p} \) the analytic semi-group associated to the operator \( A_p \) in \( L^p_{\sigma,\tau}(\Omega) \).

When \( \Omega \) is not simply-connected, the Stokes operator with boundary condition (2) has a non trivial kernel included in all the \( L^p \) spaces for \( p \in (1, \infty) \). It may be characterized as follows (see [5])

\[
K_\tau(\Omega) = \{ v \in L^p_{\sigma,\tau}(\Omega); \, \text{div} v = 0, \, \text{curl} v = 0 \text{ in } \Omega \}. \tag{11}
\]

The restriction of the Stokes operator \( A_p \) to the subspace

\[
X_p = \{ f \in L^p_{\sigma,\tau}(\Omega); \, \int_{\Omega} f \cdot \nabla \, dx = 0, \, \forall \, v \in K_\tau(\Omega) \}, \tag{12}
\]
gives a sectorial operator which is invertible, with bounded inverse. Notice that

\[
L^p_{\sigma,\tau}(\Omega) = K_\tau(\Omega) \oplus X_p. \tag{13}
\]

We consider now the extension of \( A_p \) to the following subspace of \( [H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}' \) (the dual space of \( H_0^p(\text{div},\Omega) \)):

\[
[H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}' = \{ f \in [H^\prime_0(\text{div},\Omega)]'; \, \text{div} f = 0 \text{ in } \Omega, \, f \cdot n = 0 \text{ on } \Gamma \}. \tag{14}
\]

By [2, Corollary 3.7], that space is well defined, and the extended operator, denoted \( B_p \), is a closed linear densely defined operator such as:

\[
D(B_p) \subset [H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}' \mapsto [H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}',
\]

\[
D(B_p) = \{ u \in W^{1,p}(\Omega); \, \text{div} u = 0 \text{ in } \Omega, \, u \cdot n = 0, \, \text{curl} u \times n = 0 \text{ on } \Gamma \}
\]

and

\[
\forall \, u \in D(B_p), \quad B_p u = -\Delta u \quad \text{in } \Omega. \tag{15}
\]

By [6, Corollary 4.2], the domain \( D(B_p) \) is well defined and, by [2, Theorem 4.15] the operator \(-B_p^*\) generates a bounded analytic semi-group on \([H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}'\), for all \( 1 < p < \infty \), whose restriction to

\[
Y_p = \{ f \in [H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}'; \, \forall \, v \in K_\tau(\Omega), \, \langle f, v \rangle_{\Omega} = 0 \}, \tag{16}
\]

where \( \langle \cdot, \cdot \rangle_{\Omega} = \langle \cdot, \cdot \rangle_{[H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}' \times [H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}'} \), is a sectorial operator, invertible with bounded inverse. Notice also that:

\[
[H^\prime_0(\text{div},\Omega)]_{\sigma,\tau}' = K_\tau(\Omega) \oplus Y_p. \tag{17}
\]

In order to introduce our third operator we first need the following space:

\[
T^p(\Omega) = \{ v \in H^0_0(\text{div},\Omega); \, \text{div} v \in W^{1,p}_0(\Omega) \} \tag{18}
\]

and consider the following subspace

\[
[T^p(\Omega)]_{\sigma,\tau}' = \{ f \in (T^p(\Omega))'; \, \text{div} f = 0 \text{ in } \Omega \quad \text{and} \quad f \cdot n = 0 \text{ on } \Gamma \}.
\]
that is well defined by \[2, \text{Corollary } 3.12\].

The Stokes operator \(A_p\) can be extended to the space \([T^{p'}(\Omega)]_{\sigma,\tau}\) (cf. \[2, \text{Section } 3.2.3\]). This extension is a densely defined closed linear operator, denoted \(C_p\):

\[
D(C_p) \subset [T^{p'}(\Omega)]_{\sigma,\tau} \hookrightarrow [T^{p'}(\Omega)]_{\sigma,\tau},
\]

where

\[
D(C_p) = \{ u \in L^p(\Omega); \text{div } u = 0 \text{ in } \Omega, \ u \cdot n = 0, \ \text{curl } u \times n = 0 \text{ on } \Gamma \}
\]

and for all \(u \in D(C_p)\), \(C_p u = -\Delta u \) in \(\Omega\). The domain \(D(C_p)\) is well defined by \[6, \text{Lemma } 4.14\]. The operator \(-C_p\) generates a bounded analytic semi-group on \([T^{p'}(\Omega)]_{\sigma,\tau}\) for all \(1 < p < \infty\) (see \[2, \text{Theorem } 4.18\]). If we define now

\[
Z_p = \left\{ f \in [T^{p'}(\Omega)]_{\sigma,\tau} \; \forall v \in K_\tau(\Omega), \; \langle f, v \rangle_\Omega = 0 \right\}, \tag{19}
\]

where \(\langle ., . \rangle_\Omega = \langle ., . \rangle_{[T^{p'}(\Omega)] \times [T^{p'}(\Omega)]}\), then

\[
[T^{p'}(\Omega)]_{\sigma,\tau} = K_\tau(\Omega) \oplus Z_p \tag{20}
\]

and the restriction of the Stokes operator to the space \(Z_p\), gives a sectorial operator, invertible with bounded inverse.

3. Maximal Regularity: our main results.

We consider in this Section the problem \((1)-(3)\) under different conditions of the external force \(f\). In our first result we assume \(f \in L^q(0, T; L^p_{\sigma,\tau}(\Omega))\) and \(1 < p, q < \infty\).

**Theorem 3.1.** Let \(1 < p, q < \infty\) and \(0 < T \leq \infty\). Then for every \(f \in L^q(0, T; L^p_{\sigma,\tau}(\Omega))\) there exists a unique solution \(u\) of \((1)-(3)\) satisfying

\[
u \in L^q(0, T_0; W^{2,p}(\Omega)), T_0 \leq T, \text{ if } T < \infty \text{ and } T_0 < T, \text{ if } T = \infty \tag{21}
\]

\[
\frac{\partial u}{\partial t} \in L^q(0, T; L^p_{\sigma,\tau}(\Omega)) \tag{22}
\]

and

\[
\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^q(\Omega)}^q \, dt + \int_0^T \left\| \Delta u(t) \right\|_{L^q(\Omega)}^q \, dt \leq C(p, q, \Omega) \int_0^T \left\| f(t) \right\|_{L^q(\Omega)}^q \, dt. \tag{23}
\]

**Proof.** Since the operator \(-A_p\) generates a bounded analytic semi-group in \(L^p_{\sigma,\tau}(\Omega)\), and \(f \in L^q(0, T; L^p_{\sigma,\tau}(\Omega))\), problem \((1)-(3)\) has a unique solution \(u \in C(0, T; L^p_{\sigma,\tau}(\Omega))\). To prove the maximal \(L^p-L^q\) regularity \((21)-(23)\) we proceed as follows.

By \((13)\) we may write \(f\) in the form, \(f = f_1 + f_2\) where \(f_1 \in L^q(0, T; X_p)\) and \(f_2 \in L^q(0, T; K_\tau(\Omega))\). Thus the solution \(u\) to \((1)-(3)\) is such that \(u = u_1 + u_2\), where \(u_1\) and \(u_2\) satisfy
\[
\begin{aligned}
\begin{cases}
\frac{\partial u_1}{\partial t} - \Delta u_1 = f_1, & \text{div} u_1 = 0 \quad \text{in} \quad \Omega \times (0, T), \\
u_1 \cdot n = 0, & \text{curl} u_1 \times n = 0 \quad \text{on} \quad \Gamma \times (0, T), \\
u_1(0) = 0 & \quad \text{in} \quad \Omega
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
\frac{\partial u_2}{\partial t} - \Delta u_2 = f_2, & \text{div} u_2 = 0 \quad \text{in} \quad \Omega \times (0, T), \\
u_2 \cdot n = 0, & \text{curl} u_2 \times n = 0 \quad \text{on} \quad \Gamma \times (0, T), \\
u_2(0) = 0 & \quad \text{in} \quad \Omega
\end{cases}
\end{aligned}
\]

respectively.

By [2, Theorem 1.2] we know that \( u_1 \) satisfies

\[
\int_0^T \| \frac{\partial u_1}{\partial t} \|^q_{L^p(\Omega)} dt + \int_0^T \| \Delta u_1(t) \|^q_{L^p(\Omega)} dt \leq C(p, q, \Omega) \int_0^T \| f_1(t) \|^q_{L^p(\Omega)} dt.
\]

(27)

Set \( z_2 = \text{curl} u_2 \). Then \( z_2 \) is a solution of the problem

\[
\begin{aligned}
\begin{cases}
\frac{\partial z_2}{\partial t} - \Delta z_2 = 0, & \text{div} z_2 = 0 \quad \text{in} \quad \Omega \times (0, T), \\
z_2 \times n = 0, & \quad \text{on} \quad \Gamma \times (0, T), \\
z_2(0) = 0 & \quad \text{in} \quad \Omega
\end{cases}
\end{aligned}
\]

(28)

Thus, using [3, Theorem 4.1] we deduce that \( \text{curl} u_2 = z_2 = 0 \) in \( \Omega \).

This means that \( u_2 \in K_\tau(\Omega) \) and then

\[
\forall t \geq 0, \quad \frac{\partial u_2(t)}{\partial t} = f_2(t) \quad \text{in} \quad \Omega.
\]

(29)

As a result \( u_2 \) satisfies

\[
u_2 \in L^q(0, T_0; D(A_p)) \cap W^{1,q}(0, T; L^p_{\sigma,\tau}(\Omega))
\]

(30)

and

\[
\int_0^T \| \frac{\partial u_2}{\partial t} \|^q_{L^p(\Omega)} dt = \int_0^T \| f_2(t) \|^q_{L^p(\Omega)} dt \leq C(p, q, \Omega) \int_0^T \| f(t) \|^q_{L^p(\Omega)} dt.
\]

(31)

Thus putting together (26)-(27) and (30)-(31) we deduce our result. □

We now extend the previous result to the more general case where the external force \( f \in L^q(0, T; L^p(\Omega)) \) is not necessarily divergence free. It is used that the pressure can be decoupled, using the weak Neumann Problem (10).
Theorem 3.2 (Strong Solutions for the inhomogeneous Stokes Problem). Let $T \in (0, \infty]$, $1 < p, q < \infty$, $f \in L^q(0,T; L^p(\Omega))$ and $u_0 = 0$. The Problem (1)-(3) has a unique solution $(u, \pi)$ such that
\[ u \in L^q(0,T; W^{2,p}(\Omega)), \quad T_0 \leq T \text{ if } T < \infty \text{ and } T_0 < T \text{ if } T = \infty, \]
\[ \pi \in L^q(0,T; W^{1,p}(\Omega)/\mathbb{R}), \quad \frac{\partial u}{\partial t} \in L^q(0,T; L^p(\Omega)) \]
and
\[ \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^p(\Omega)}^q \, dt + \int_0^T \| \Delta u(t) \|_{L^p(\Omega)}^q \, dt + \int_0^T \| \pi(t) \|_{W^{1,p}(\Omega)/\mathbb{R}}^q \, dt \leq C(p, q, \Omega) \int_0^T \| f(t) \|_{L^p(\Omega)}^q \, dt. \]

Proof. As we saw in Section 2 when defining the Helmholtz projection $P$, for every $f \in L^q(0,T; L^p(\Omega))$, and almost every $0 < t < T$, the problem
\[ \text{div} (\text{grad} \, \pi(t) - f(t)) = 0 \text{ in } \Omega, \quad (\text{grad} \, \pi(t) - f(t)) \cdot \mathbf{n} = 0 \text{ on } \Gamma, \]
has a unique solution $\pi(t) \in W^{1,p}(\Omega)/\mathbb{R}$ that satisfies the estimate
\[ \| \pi(t) \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega) \| f(t) \|_{L^p(\Omega)}. \]
It follows that:
\[ \pi \in L^q(0,T; W^{1,p}(\Omega)/\mathbb{R}), \quad (f - \text{grad} \, \pi) \in L^q(0,T; L^p_{\sigma,\tau}(\Omega)). \]
As a result, thanks to Theorem 3.1, Problem (1)-(3) has a unique solution $(u, \pi)$ satisfying (32)-(34). \qed

Similar results hold for weak and very weak solutions.

Theorem 3.3 (Weak Solutions for the inhomogeneous Stokes Problem). Let $1 < p, q < \infty$, $u_0 = 0$ and let $f \in L^q(0,T; [H_0^p(\text{div},\Omega)]')$, $0 < T \leq \infty$. The Problem (1)-(3) has a unique solution $(u, \pi)$ satisfying
\[ u \in L^q(0,T_0; W^{1,p}(\Omega)), \quad T_0 \leq T \text{ if } T < \infty \text{ and } T_0 < T \text{ if } T = \infty, \]
\[ \pi \in L^q(0,T; L^p(\Omega)/\mathbb{R}), \quad \frac{\partial u}{\partial t} \in L^q(0,T; [H_0^p(\text{div},\Omega)]') \]
and
\[ \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{[H_0^p(\text{div},\Omega)]'}^q \, dt + \int_0^T \| \Delta u(t) \|_{[H_0^p(\text{div},\Omega)]'}^q \, dt + \int_0^T \| \pi(t) \|_{L^p(\Omega)/\mathbb{R}}^q \, dt \leq C(p, q, \Omega) \int_0^T \| f(t) \|_{[H_0^p(\text{div},\Omega)]'}^q \, dt. \]
Proof. Suppose first \( f \in L^q(0, T; [H^p_0(\text{div}, \Omega)]_{\sigma, \tau}) \). Using that \(-B_p\) generates a bounded analytic semigroup in \([H^p_0(\text{div}, \Omega)]_{\sigma, \tau}\) we deduce the existence of a unique weak solution \( u \in C(0, T; [H^p_0(\text{div}, \Omega)]_{\sigma, \tau}) \) of (1)–(3).

By (17) we may write now \( f \) as, \( f = f_1 + f_2 \) where \( f_1 \in L^q(0, T; Y_p) \) and \( f_2 \in L^q(0, T; K_\sigma(\Omega)) \). Proceeding as in the proof of Theorem 3.1 we deduce that the solution \( u \) to problem (1)–(3) is such that \( u = u_1 + u_2 \), where \( u_1 \) and \( u_2 \) are weak solutions of (24) and (25) respectively and that \( u_2 \in K_\sigma(\Omega) \) for almost all \( 0 < t \leq T \). Using [2, Proposition 6.4, Remark 7.15] we deduce that the solution \( u \) satisfies the maximal regularity (38)–(40). Suppose now \( f \in L^q(0, T; [H^p_0(\text{div}, \Omega)]') \). Then, for almost every \( t \in (0, T) \), there exists a unique solution \( \pi(t) \in L^p(\Omega)/\mathbb{R} \) such that:

\[
\|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C_2(\Omega, p) \|f\|_{[H^p_0(\text{div}, \Omega)]'}, \tag{41}
\]

(cf. [4]), and then also:

\[
\pi \in L^q(0, T; L^p(\Omega)/\mathbb{R}) \quad \text{and} \quad (f - \text{grad } \pi) \in L^q(0, T; [H^p_0(\text{div}, \Omega)]').
\]

We deduce from the previous step that \((u, \pi)\) satisfies (38)–(40).

\[\square\]

**Theorem 3.4** (Very weak solutions for the inhomegeneous Stokes Problem). Let \( T \in (0, \infty), 1 < p, q < \infty, u_0 = 0 \) and \( f \in L^q(0, T; [T'(\Omega)]') \). Then the time dependent Stokes Problem (1)–(3) has a unique solution \((u, \pi)\) satisfying

\[
u \in L^q(0, T_0; L^p(\Omega)), \quad T_0 \leq T \quad \text{if} \quad T < \infty \quad \text{and} \quad T_0 < T \quad \text{if} \quad T = \infty,
\]

\[
\pi \in L^q(0, T; W^{-1,p}(\Omega)/\mathbb{R}), \quad \frac{\partial u}{\partial t} \in L^q(0, T; ([T'(\Omega)]')_{\sigma, \tau}) \tag{43}
\]

and

\[
\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{[T'(\Omega)]'}^q dt + \int_0^T \left\| \Delta u(t) \right\|_{[T'(\Omega)]'}^q dt + \int_0^T \left\| \pi(t) \right\|_{W^{-1,p}(\Omega)/\mathbb{R}}^q dt \leq C(p, q, \Omega) \int_0^T \left\| f(t) \right\|_{[T'(\Omega)]'}^q dt. \tag{44}
\]

Proof. The proof follows the same arguments as those in the proof of Theorem 3.3. In a first step one uses \( C_p \), the analytic semigroup on \([T'(\Omega)]_{\sigma, \tau}\) and (20) to prove that (42)–(44) are satisfied when \( f \in [T'(\Omega)]_{\sigma, \tau} \). In the general case, one uses the results in [4] to obtain \( \pi \in L^q(0, T; W^{-1,p}(\Omega)/\mathbb{R}) \) such that \((f - \text{grad } \pi) \in L^q(0, T; [T'(\Omega)]_{\sigma, \tau})\), and the result follows using the first step. \[\square\]
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