SOLUTION OF A PROBLEM OF PELLER CONCERNING SIMILARITY

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Abstract. We answer a question of Peller by showing that for any $c > 1$ there exists a power-bounded operator $T$ on a Hilbert space with the property that any operator $S$ similar to $T$ satisfies $\sup_n \|S^n\| > c$.

1. Introduction

In this note we answer a question due to Peller [12] which has also recently been raised by Pisier [13] p.114. Peller’s question is whether, for any $\epsilon > 0$, every power-bounded operator $T$ is similar to an operator $S$ with $\sup_n \|S^n\| < 1 + \epsilon$.

It was shown by Foguel [5] in 1964 that there is a power-bounded operator $T$ on a Hilbert space $\mathcal{H}$ which is not similar to a contraction. It was later shown by Lebow that this example is not polynomially bounded [11]; for other examples see [2] and [13], Chapter 2. Recently Pisier [14] answered a problem raised by Halmos by constructing an operator which is polynomially bounded and not similar to a contraction.

We shall construct a family of counter-examples to Peller’s question. These counter-examples have a rather simple structure. Let $w$ be an $A_2$-weight on the circle $\mathbb{T}$ and let $H^2(w)$ be the closed linear span of $\{e^{in\theta} : n \geq 0\}$ in $L^2(w)$. We consider an operator

$$T(\sum_{n=0}^{\infty} a_n e^{in\theta}) = \sum_{n=0}^{\infty} \lambda_n a_n e^{in\theta}$$

where $(\lambda_n)_{n=0}^{\infty}$ is a monotone increasing sequence of positive reals with $\lambda_n \uparrow 1$ and $\lambda_n < 1$ with

$$\lim_{n \to \infty} \frac{1 - \lambda_{n+1}}{1 - \lambda_n} = 0.$$ 

For such operators we can prove a rather precise result (Theorem 3.4):

$$\inf_n \{ \sup \|(A^{-1}TA)^n\| : A \text{ invertible} \} = \sec \left( \frac{\pi}{2p} \right)$$

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where \( p = \sup \{ a : w^a \in A_2 \} \). By taking simple choices of \( A_2 \)-weights where \( p < \infty \) we can create a family of counter-examples.

The proof of Theorem 3.4 depends heavily on estimates for the norm of the Riesz projection in Section 2 particularly Theorem 2.4. These results can be obtained by a careful reading of the classical work of Helson and Szegö \[8\] on \( A_2 \)-weights (cf. \[4\]). However, we present a self-contained argument, in which the reader will recognize many similarities with the Helson-Szegö theory.

We also show that our examples can only be polynomially bounded in the trivial situation when \( w \) is equivalent to the constant function and then \( T \) is similar to contraction. We also note that the case \( p = \infty \) in (1.1) (when Peller’s conjecture holds for \( T \)) corresponds to the case when \( \log w \) is in the closure of \( L^\infty(\mathbb{T}) \) in \( BMO(\mathbb{T}) \).

2. The norm of the Riesz projection on weighted \( L^2 \)-spaces

We start by recalling an easy lemma concerning projections on a Hilbert space.

**Lemma 2.1.** Let \( E \) and \( F \) be closed subspaces of a Hilbert space \( \mathcal{H} \) so that \( E + F \) is dense in \( \mathcal{H} \). Suppose \( 0 \leq \varphi < \pi/2 \). In order that there is a projection \( P \) of \( \mathcal{H} \) onto \( E \) with \( F = \ker P \) with \( \|P\| \leq \sec \varphi \) it is necessary and sufficient that

\[
|\langle e, f \rangle| \leq \sin \varphi \|e\| \|f\| \quad e \in E, \quad f \in F.
\]

**Remark.** Note that a consequence of Lemma 2.1 is that if \( P \) is any non-trivial projection on a Hilbert space then \( \|P\| = \|I - P\| \).

Now let \( \mathbb{T} \) be the unit circle (which we identify with \((\pi, \pi)\) in the usual way) equipped with the standard Haar measure \( d\theta/2\pi \). Let \( \mu \) be any finite positive Borel measure on \( \mathbb{T} \). We denote by \( L^2(\mu) = L^2(\mathbb{T}; \mu) \) the corresponding weighted \( L^2 \)-space; if \( \mu \) is absolutely continuous with respect to Haar measure so that \( d\mu = (2\pi)^{-1} w(\theta) d\theta \) then we write \( L^2(w) \). We refer to any nonnegative \( w \in L^1(\mathbb{T}) \) so that \( w > 0 \) on a set of positive measure as a weight.

Suppose \( w \) is a weight. We recall that \( H^2(w) \) is the closed subspace of \( L^2(w) \) generated by the functions \( \{ e^{in\theta} : n \geq 0 \} \). We recall that \( w \) is an \( A_2 \)-weight if there is a bounded projection \( R \) of \( L^2(w) \) onto \( H^2(\mu) \) with \( R(e^{in\theta}) = 0 \) if \( n < 0 \). In this case we always have that \( w > 0 \) a.e., \( w^{-1} \) is an \( A_2 \)-weight and \( L^2(w) \subset L^1 \); the operator \( R \) must coincide with the Riesz projection \( Rf \sim \sum_{n \geq 0} \hat{f}(n) e^{in\theta} \). Let us denote by \( \|R\|_w \) the norm of the Riesz projection on \( L^2(w) \). Note that for an \( A_2 \)-weight \( H^2(w) = H^1 \cap L^2(w) \). In particular we can define \( f(z) = \sum_{n \geq 0} \hat{f}(n) z^n \) for \(|z| < 1 \).

The following Proposition can be derived from the classical work of Helson-Szegö \[8\] or \[4\]. However, we give a self-contained direct proof.
Proposition 2.2. Let $w$ be a weight function on $\mathbb{T}$. Assume $0 \leq \varphi < \frac{\pi}{2}$. The following conditions are equivalent:
(1) $w$ is an $A_2$-weight and $\|R\|_w \leq \sec \varphi$.
(2) There exists $h \in H^1$ so that $|w - h| \leq w \sin \varphi$ a.e.

Proof. First note that by Lemma 2.1 (1) is equivalent to
\begin{equation}
\left| \int_{-\pi}^{\pi} f(\theta)g(\theta)w(\theta) \frac{d\theta}{2\pi} \right| \leq \sin \varphi \left( \int_{-\pi}^{\pi} |f(\theta)|^2 w(\theta) \frac{d\theta}{2\pi} \right)^{1/2} \left( \int_{-\pi}^{\pi} |g(\theta)|^2 w(\theta) \frac{d\theta}{2\pi} \right)^{1/2},
\end{equation}
for all $f, g \in H^2(w)$ with $g(0) = 0$.

To prove (1) implies (2) we note that if $w$ is an $A_2$-weight so that $\log w \in L^1$ we can find an outer function $F \in H^2$ so that $w = |F|^2$ a.e.. Then (2.1) gives
\begin{equation}
\left| \int_{-\pi}^{\pi} f g w F^{-2} \frac{d\theta}{2\pi} \right| \leq \sin \varphi \left( \int_{-\pi}^{\pi} |f|^2 \frac{d\theta}{2\pi} \right)^{1/2} \left( \int_{-\pi}^{\pi} |g|^2 \frac{d\theta}{2\pi} \right)^{1/2},
\end{equation}
for $f, g \in H^2$ with $g(0) = 0$. This in turn implies that
\begin{equation}
\left| \int_{-\pi}^{\pi} f w F^{-2} \frac{d\theta}{2\pi} \right| \leq \sin \varphi \|f\|_1
\end{equation}
for all $f \in H^1$, with $f(0) = 0$. By the Hahn-Banach Theorem this implies there exists $G \in H^\infty$ so that $\|w F^{-2} - G\|_\infty \leq \sin \varphi$ or $|w - h| \leq w \sin \varphi$ where $h = F^2 G \in H^1$.

For the reverse direction just note that if $f, g \in H^2(w)$ with $g(0) = 0$ then
\begin{equation}
\int_{-\pi}^{\pi} f g w \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} f g (w - h) \frac{d\theta}{2\pi}
\end{equation}
so that (2.1) follows from the Cauchy-Schwartz inequality.

Let us isolate a simple special case of the above proposition.

Proposition 2.3. Let $0 \neq f \in H^1$ be such that $\arg f(\theta) \leq \varphi < \frac{\pi}{2}$ almost everywhere. If $f$ is not identally zero then $w = \Re f$ is an $A_2$-weight for which $\|R\|_w \leq \sec \varphi$.

Proof. In this case $w = \Re f \geq 0$ a.e. and $|\Re f| \leq \tan \varphi w$ a.e. Furthermore:
\begin{equation}
|w - \cos^2 \varphi f|^2 \leq (\sin^4 \varphi + \cos^4 \varphi \tan^2 \varphi) w^2 \leq \sin^2 \varphi w^2
\end{equation}
a.e., so that we obtain the result from Proposition 2.2.

Remark. Suppose $0 < \alpha < 1$ and $f \in H^1(\mathbb{D})$ is given by
\begin{equation}
f(z) = \frac{(z - 1)^\alpha}{(z + 1)^\alpha}
\end{equation}
(taking the usual branch of \( w \mapsto w^\alpha \).) Then
\[
w = \Re f = \cos \frac{\alpha \pi}{2} \tan \alpha \frac{\theta}{2}.
\]
It follows that
\[
\|R\|_{\tan^{\alpha}(\theta/2)} \leq \sec \frac{\alpha \pi}{2}.
\]
In fact (2.2) is well-known (see [10], for example). We are grateful to Igor
Verbitsky for bringing this reference to our attention.

We will say that two weights \( v, w \) are equivalent (\( v \sim w \)) if
\( v/w, w/v \in L^\infty \).

Theorem 2.4. Suppose \( w \) is an \( A_2 \)−weight on \( \mathbb{T} \). Then
\[
\inf \{ \|R\|_v : v \sim w \} = \sec \frac{\pi}{2p}
\]
where
\[
p = \sup \{ a > 0 : w^a \in A_2 \}.
\]
Proof. First suppose \( v \sim w \) and \( \|R\|_v = \sec \psi \) where \( 0 \leq \psi < \pi/2 \). Then there
exists \( h \in H^1 \) with \( |v - h| \leq v \sin \psi \) a.e. In particular, \( |\arg h| \leq \psi \) a.e. and
so \( h \) maps \( \mathbb{D} \) into the same sector. It follows that we can define \( h^r \in H^{1/r} \)
for all \( r > 0 \). Choose \( r \) so that \( r \psi < \pi/2 \), and let \( g = h^r \). Then \( \Re g \geq 0 \) and
\( |\Re g| \leq \tan(r\psi)\Re g \) so that \( g \in H^1 \). Now by Proposition 2.3 we have that \( \Re g \)
is an \( A_2 \)−weight. However \( \Re g \sim |h|^r \sim w^r \) so that \( r \leq p \). We deduce that
\( \psi \geq \pi/(2p) \).

For the converse direction assume that \( w^r \) is an \( A_2 \)−weight. Then there
exists \( h \in H^1 \) so that \( |w^r - h| \leq w^r \sin \psi \) where \( 0 \leq \psi < \pi/2 \). Arguing as
above we have \( g = h^{1/r} \in H^1 \) and \( \Re g \) is an \( A_2 \)−weight with \( \|R\|_{\Re g} \leq \sec(\psi/r) \).
Note that \( \Re g \sim w \), and this establishes the other direction.

Remark. If we now let \( w(\theta) = |\tan \frac{\theta}{2/\alpha} \) where \( 0 < \alpha < 1 \) then we can apply
(2.2) to deduce that, for this particular weight the infimum is attained, i.e.
\[
\inf \{ \|R\|_v : v \sim w \} = \|R\|_{\tan^{\alpha}(\theta/2)} = \sec \frac{\alpha \pi}{2}.
\]

3. Multipliers

Suppose \( (e_n)_{n=0}^\infty \) be any Schauder basis of a Hilbert space \( \mathcal{H} \); note that we
do not assume \( (e_n) \) to be orthonormal or even unconditional. Let \( (P_n) \) be the
associated partial sum operators \( P_n(\sum_{k=0}^\infty a_k e_k) = \sum_{k=0}^n a_k e_k \). Let \( Q_n = I - P_n \)
and note that \( \|Q_n\| = \|P_n\| \) for all \( n \geq 0 \). Since \( (e_n) \) is a basis we have
that \( \sup_n \|P_n\| = b < \infty \) where \( b \) is the \textit{basis constant}. We call an operator
T: \mathcal{H} \to \mathcal{H} a monotone multiplier (with respect to the given basis) if there is an increasing sequence \((\lambda_k)_{k=0}^{\infty}\) in \(\mathbb{R}\) so that \(0 \leq \lambda_k \leq 1\) so that
\[
T(\sum_{k=0}^{\infty} a_k e_k) = \sum_{k=0}^{\infty} \lambda_k a_k e_k.
\]

**Lemma 3.1.** If \(T\) is defined as above then \(T\) is (well-defined and) bounded and \(\sup_n ||T^n|| \leq b\).

**Proof.** It is enough to show \(T\) is bounded and \(||T|| \leq b\) since \(T^n\) is also a monotone multiplier. To see this note that if \((a_k)_{k=0}^{\infty}\) is finitely nonzero and \(x = \sum_{k=0}^{\infty} a_k e_k\), then
\[
Tx = \lambda_0 x + \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k-1}) Q_k x
\]
so that \(||Tx|| \leq \sup_n ||Q_n|| = b. \)

We shall say that \(T\) is a fast monotone multiplier if in addition, \(\lambda_k < 1\) for all \(k\) and
\[
(3.1) \quad \lim_{k \to \infty} \frac{1 - \lambda_k}{1 - \lambda_{k-1}} = 0.
\]

**Lemma 3.2.** Suppose \(T\) is a fast monotone multiplier. Then there is an increasing sequence of integers \((N_n)_{n=0}^{\infty}\) so that \(\lim_{n \to \infty} ||T^{N_n} - Q_n|| = 0\).

**Proof.** Note that if \(x = \sum_{k=0}^{\infty} a_k e_k\) then
\[
T^{N_n} x - Q_n x = \sum_{k=0}^{n} \lambda_k N_n a_k e_k - (1 - \lambda_{N_n}^{N_n+1}) Q_n x + \sum_{k=n+1}^{\infty} (\lambda_k^{N_n} - \lambda_{N_n}^{N_n+1}) a_k e_k
\]
whence a calculation as in Lemma [3.3] gives
\[
||T^{N_n} x - Q_n x|| \leq b \lambda^{N_n}_n ||P_n x|| + (b + 1)(1 - \lambda^{N_n}_n) ||Q_n x||.
\]

It follows that
\[
||T^{N_n} - Q_n|| \leq b(b \lambda^{N_n}_n + (b + 1)(1 - \lambda^{N_n}_n))
\]
It remains therefore only to select \(N_n\) so that \(\lim_{n \to \infty} \lambda^{N_n}_n = 0\) and \(\lim_{n \to \infty} \lambda^{N_n+1}_{N_n+1} = 1\).

For convenience we write \(\lambda_n = e^{-\nu_n}\) where \(\nu_n/\nu_{n+1} = \kappa^2_n\) and \(\kappa_n \to \infty\). For any \(n \geq 0\), pick \(N_n\) to be the greatest integer so that \(N_n \nu_n^{1/2} \nu_{n+1}^{1/2} \leq 1\). Then
\[
N_n \nu_n^{1/2} \nu_{n+1}^{1/2} \geq \frac{N_n}{N_n + 1}
\]
and \(\lim N_n = \infty.\)
Now
\[ N_n \nu_n \geq \frac{N_n \kappa_n}{N_n + 1} \]
and
\[ N_n \nu_{n+1} \leq \kappa_n^{-1}. \]
This yields the desired result.

We now turn to the case when \( \mathcal{H} = H^2(w) \) where \( w \) is an \( A_2 \)-weight and \( e_k(\theta) = e^{ik\theta} \) for \( k \geq 0 \).

**Lemma 3.3.** The basis constant of \( (e_k)_{k=0}^{\infty} \) in \( H^2(w) \) is given by \( b = \|R\|_w \).

**Proof.** In fact \( Q_n-1 f = e_n R(e_{-n} f) \) so it is clear that \( \|Q_n-1\| \leq \|R\|_w \). For the other direction suppose \( f \) is a trigonometric polynomial in \( L^2(w) \). Then for large enough \( n \) we have \( e_n f \in H^2(w) \) and then \( Rf = e_{-n} Q_n-1(e_n f) \). This quickly yields \( \|R\|_w \leq \sigma \).

**Theorem 3.4.** Let \( w \) be an \( A_2 \)-weight on \( \mathbb{T} \) and let \( T : H^2(w) \to H^2(w) \) be a fast monotone multiplier corresponding to the sequence \( (\lambda_n) \). Then
\[
\inf \left\{ \sup_n \|(A^{-1}TA)^n\| : A \text{ invertible} \right\} = \sec \frac{\pi}{2p}
\]
where
\[ p = \sup \{ a > 0 : w^a \in A_2 \}. \]

**Proof.** We shall prove that if \( \sigma \geq 1 \) then the existence of an invertible \( A \) so that \( \sup_n \|(A^{-1}TA)^n\| \leq \sigma \) is equivalent to the existence of a weight \( v \) equivalent to \( w \) so that \( \|R\|_v \leq \sigma \). Once this is done, the result follows from Theorem 2.4.

In one direction this is easy. Assume \( v \) equivalent to \( w \) and \( \|R\|_v \leq \sigma \). This means that there is an equivalent inner-product norm on \( H^2(w) \) in which the basis constant of \( (e_k)_{k=0}^{\infty} \) bounded by \( \sigma \). It follows from Lemma 3.1 that in this equivalent norm we have \( \sup_n \|T^n\|_v \leq \sigma \). Hence \( T \) is similar to an operator \( A^{-1}TA \) such that \( \sup \|(A^{-1}TA)^n\| \leq \sigma \).

We now consider the converse. Let \( S : H^2(w) \to H^2(w) \) be the operator \( Sf = e_1 f \). Suppose \( A \) is an invertible operator such that \( \|(A^{-1}TA)^n\| \leq \sigma \). We will define a new inner-product on \( H^2(w) \) by
\[
\langle f, g \rangle = \text{LIM} \ (AS^n f, AS^n g)
\]
where LIM denotes any Banach limit (see e.g. [4] p. 85). Since \( S \) is an isometry on \( H^2(w) \) and \( A \) is invertible this defines an equivalent inner-product \( \| \cdot \| \) norm on \( H^2(w) \). Now for any \( f \in H^2(w) \) and fixed \( m \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} \|AQ_{m+n} S^n f - AT^{N_{m+n}} S^n f\| = 0
\]
where \((N_n)\) is given in Lemma \([3.2]\). Hence
\[
\limsup_{n \to \infty} \left( \|AQ_{m+n}S^n f\|^2 - \sigma^2 \|AS^n f\|^2 \right) \leq 0.
\]

Now
\[
|Q_m f|^2 = \lim \|AS^n Q_m f\|^2 = \lim \|AQ_{m+n}S^n f\|^2 \leq \sigma^2 |f|^2.
\]

Thus with respect to the new norm \(| \cdot |\) the basis constant is at most \(\sigma\).

Now let \(c_k = \langle e_0, e_k \rangle\) for \(k \geq 0\) and let \(c_k = \overline{c}_{-k}\) when \(k < 0\). Then it follows easily that \(\langle e_k, e_l \rangle = c_{l-k}\) for all \(k, l\) and that for all finitely nonzero sequences \((a_k)\) of complex numbers we have that
\[
\sum_{k,l} a_k \overline{a_l} c_{k-l} \geq 0.
\]

This implies (see \([3]\) p. 38) that there is a finite positive measure \(\mu\) on \(\mathbb{T}\) so that
\[
\int e^{-ik\theta} d\mu(\theta) = c_k.
\]

Thus
\[
\langle f, g \rangle = \int f \overline{g} d\mu.
\]

However this norm is equivalent to the original norm so that \(\mu\) is absolutely continuous with respect to Lebesgue measure and of the form \((2\pi)^{-1} v(\theta) d\theta\) where \(v \sim \omega\).

It follows that in \(H^2(v)\) the basis constant of the exponential basis is at most \(\sigma\) and so by Lemma \([3.3]\) we have \(\|R\|_v \leq \sigma\) and the proof is complete.

We can now give explicit examples by taking the weights \(w(t) = |\theta|^\alpha\) where \(0 < \alpha < 1\). It is clear that in Theorem \([3.4]\) we have \(p = \alpha^{-1}\) and so for any fast monotone multiplier we have
\[
\inf \left\{ \sup_{n} \| (A^{-1}TA)^n \| : \ A \text{ invertible} \right\} = \sec \frac{\pi \alpha}{2} > 1.
\]

Note that we are essentially using here the original example of a conditional basis for Hilbert space due to Babenko \([1]\). We can also utilize \((2.3)\) to show that for this example the infimum in \((3.2)\) is actually attained. In general the infimum in \((3.2)\) need not be attained; this will be seen easily from Theorem \([3.6]\) below.

**Theorem 3.5.** Let \(w\) be an \(A_2\)–weight and suppose \(T : H^2(w) \to H^2(w)\) is a fast monotone multiplier, corresponding to the sequence \((\lambda_n)\). Then the following are equivalent:

(i) \(T\) is similar to a contraction.

(ii) \(T\) is polynomially bounded.

(iii) \(w \sim 1\).
Proof. That (i) implies (ii) is a consequence of von Neumann’s inequality (see [13]). Similarly (iii) implies (i) is trivial. It therefore remains to prove that (ii) implies (iii). We shall treat the case when the \( \lambda_k \) are distinct; small modifications are necessary in the other cases. We shall also suppose the measure \( d\mu = (2\pi)^{−1}w(\theta)d\theta \) is a probability measure so that \( \|e_k\| = 1 \) for all \( k \).

First note that if \( f \in H^\infty(\mathbb{D}) \) then for any \( r < 1 \), then \( f_r(T) \) is well-defined where \( f_r(z) = f(rz) \) and if \( T \) is polynomially bounded we have an estimate

\[
\|f_r(T)\| \leq C\|f\|_{H^\infty(\mathbb{D})},
\]

or equivalently

\[
\|\sum_{k=0}^{\infty} f(r\lambda_k)a_k e_k\| \leq C\|f\|_{H^\infty(\mathbb{D})}\|\sum_{k=0}^{\infty} a_k e_k\|
\]

whenever \( (a_k) \) is finitely non-zero. Letting \( r \rightarrow 1 \) we obtain

\[
\|\sum_{k=0}^{\infty} f(\lambda_k)a_k e_k\| \leq C\|f\|_{H^\infty(\mathbb{D})}\|\sum_{k=0}^{\infty} a_k e_k\|
\]

Recall that by Carleson’s theorem \([3]\) the sequence \( (\lambda_n) \) is interpolating (cf. \([6]\) p. 287-288) so that there is a constant \( B \) such that for any sequence \( \epsilon_k = \pm 1 \) there exists \( f \in H^\infty(\mathbb{D}) \) with \( \|f\|_{H^\infty(\mathbb{D})} \leq B \) and \( f(\lambda_k) = \epsilon_k \) for all \( k \geq 0 \). Hence

\[
\|\sum_{k=0}^{\infty} \epsilon_k a_k e_k\| \leq BC\|\sum_{k=0}^{\infty} a_k e_k\|
\]

for all finitely non-zero sequences \( (a_k) \). Hence by the parallelogram law we have

\[
(BC)^{-1}(\sum_{k=0}^{\infty} |a_k|^2)^{1/2} \leq \|\sum_{k=0}^{\infty} a_k e_k\| \leq BC(\sum_{k=0}^{\infty} |a_k|^2)^{1/2}
\]

from which it follows that \( w \sim 1 \).

We conclude by considering the cases when

\[
\inf\{\sup_n \|(A^{-1}TA)^n\| : A \text{ invertible}\} = 1.
\]

**Theorem 3.6.** Let \( w \) be an \( A_2 \)-weight and suppose \( T : H^2(w) \rightarrow H^2(w) \) is a fast monotone multiplier, corresponding to the sequence \( (\lambda_n) \). Then the following are equivalent:

(i) For any \( \epsilon > 0 \), \( T \) is similar to an operator \( S \) with \( \sup_n \|S^n\| < 1 + \epsilon \).

(ii) \( \log w \) is in the closure of \( L^\infty \) in \( BMO \).

(iii) \( w^a \in A_2 \) for every \( a > 0 \).
Proof. The equivalence of (i) and (iii) is proved in Theorem 3.4. The equivalence of (ii) and (iii) is due to Garnett and Jones [7] or [8], Corollary 6.6 and its proof (p.258-9).

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