Particle-Flux Separation of Electrons in the Half-Filled Landau Level: 
- Chargeon-Fluxon Approach - 

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Abstract 

We have previously studied the phase structure at finite temperatures of the Chern-Simons (CS) gauge theory coupled with fermions by using lattice gauge theory. In this paper, we formulate the “chargeon-fluxon” representation of electrons and use it to reinvestigate the phenomenon of particle-flux separation (PFS) of electrons in the half-filled Landau level. We start with a lattice system of fermions interacting with a CS gauge field, and introduce two slave operators named chargeon and fluxon that carry the CS charge and flux, respectively. The original fermion, the composite fermion of Jain, is a composite of a chargeon and a fluxon. We further rewrite the model by introducing an auxiliary link field, the phase of which behaves as a gauge field gluing chargeons and fluxons. Then we study a confinement-deconfinement transition of that gauge field by using the theory of separation phenomena as in the previous paper. The residual four-fermi interactions play an important role to determine the critical temperature $T_{\text{PFS}}$, below which the PFS takes place. The new representation

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has some advantages: (1) It allows a field-theoretical description also for the flux degrees of freedom. (2) It has a close resemblance to the slave-boson or slave-fermion representations of the t-J model of high-Tc superconductors in which an electron is a composite of a holon and a spinon. This point opens a way to understand the two typical separation phenomena in strongly-correlated electron systems in a general and common setting.
1 Introduction

In the last several years, it has been recognized that the gauge theory plays an essentially important role in some topics of condensed matter physics. Especially, for the fractional quantum Hall effect (FQHE) at the filling factor $\nu = \frac{1}{2n+1} (n = 1, 2, \ldots)$, a Ginzburg-Landau (GL) theory was proposed [1], which is a gauge theory of a Chern-Simons (CS) gauge field interacting with so-called bosonized electrons. A FQH state is characterized as a Bose-condensation of these bosonized electrons.

Another important idea for the FQHE was proposed by Jain [2], that is, the composite-fermion (CF) approach. Jain asserted that the quasi-excitations in the half-filled Landau level are fermions which he called CF’s; A CF is nothing but a composite of an electron and two solenoidal flux quanta. The FQH states observed at a sequence of $\nu = \frac{p}{2p\pm 1} (p = 1, 2, \ldots)$, are understood as a result of the Landau-level quantization of composite fermions, i.e., integer QHE of CF’s. The essential assumption of the CF approach at $\nu = 1/2$ is that two units of fluxes attaching to each CF to form electrons cancel the external magnetic field on the average and fluctuations of fluxes around the mean value behave almost independently of the locations (densities) of CF’s. Various experiments [3] and also numerical calculations [4] support, or at least are consistent with, the idea of CF.

The CS gauge theory is suitable for describing CF’s. In the CS description, the above assumption means that the CS local constraint, which connects fluctuations of fluxes with the density of CF’s at each spatial point, becomes irrelevant at low energies. We called this phenomenon “particle-flux separation (PFS)” [5] because this bears close resemblance to the charge-spin separation (CSS) in the strongly-correlated electron systems for high-Tc superconductivity [6]. It is naturally expected that the repulsive interactions among electrons play an essential role for the stability of CF’s.

In the previous paper [7], we studied the PFS in the framework of lattice formulation of CS gauge theory. We argued that the PFS takes place below some critical temperature $T_{\text{PFS}}$. (In Ref. [7] we wrote it $T_{\text{CD}}$.) We applied the method also to the
quantum spin models and got the gauge-theoretical interpretation of each possible phases.

In the present paper, we shall address the same problem but with a different formalism. The present method is closely related in its spirit with the slave-boson or slave-fermion formalism of the t-J model. There the original electron operator is expressed in terms of a bilinear form of spinon and holon operators which satisfy a local constraint. If the CSS takes place, spinons and holons move almost freely as quasi-excitations. In the present formulation of the CS gauge theory of fermions, we introduce yet another set of two slave operators to express the fermion operator, the original CF operator, as a bilinear form of them. We call them chargeon and fluxon operators since they carry charges and fluxes, respectively. In the CS gauge theory of fermions, fermions move in a statistical magnetic field, and that statistical magnetic field is made of certain amount of flux quanta attaching to each fermion due to the CS constraint. Therefore, a fermion carries a magnetic flux as well as a charge for the CS vector potential. In order to discuss the PFS, it seems natural to introduce corresponding operator for each property, the CS charges and the CS magnetic fluxes, i.e., the chargeon and the fluxon operators. The PFS is understood as a deconfinement phenomenon of chargeons and fluxons. Furthermore, introduction of fluxon operators makes it possible to describe possible excitations of magnetic fluxes as independent quasi-excitations.

This paper is organized as follows. In Sect.2, we introduce a model of the CS gauge theory interacting with fermions, and explain its relationship with electrons in the half-filled Landau level. The chargeons and fluxons are introduced similarly as the spinons and holons in the t-J model. We further rewrite the model by a Hubbard-

1Before our study of PFS, we studied the CSS in the framework of gauge theory, where the gauge field glues spinons and holons. Among other things, we showed that the CSS can be understood as the deconfinement phenomenon of this gauge dynamics, and calculated the critical temperature $T_{CSS}$ below which the CSS takes place.

2These charges and fluxes are of the CS gauge field and not of the usual electromagnetism.
Stratonovich transformation. The auxiliary link field introduced there is regarded as a gauge field which glues a chargeon and a fluxon. In Sect.3, we derive an effective action for that gauge field. It is explained that the deconfinement phase of that gauge field is the phase of PFS. The effective gauge-coupling constant is explicitly calculated as a function of the temperature $T$ and the concentration of fermions. In Sect.4, we study the phase structure of the effective gauge theory, showing that the PFS takes place below certain critical temperature $T_{\text{PFS}}$. The value of $T_{\text{PFS}}$ is slightly different from that given in Ref.\cite{7} reflecting the different representations. We see that the Coulombic repulsion between electromagnetic (EM) charges is very important for PFS, which supports the intuitive physical expectation for the stability problem of CF’s. Section 5 is devoted for conclusion.

2 Electrons in the half-filled Landau level and the chargeon and fluxon operators

2.1 Model

Let us start with a model of CS gauge theory coupled with spinless nonrelativistic fermions $\psi_x$ on a 2-dimensional spatial lattice. These fermions are identified with the CF’s at the half filling. We employ the imaginary-time formalism at finite $T$. Its Lagrangian is given by

\begin{equation}
L = -\sum_x \psi_x^\dagger (\partial_\tau - iA_0^{\text{CS}} - \mu_c) \psi_x + \frac{1}{2m} \sum_{x,j} (\psi_{x+j}^\dagger e^{i(A_{xj}^{\text{CS}} - eA_{xj}^{\text{ex}})} \psi_x + \text{H.c.})
\end{equation}

\begin{equation}
-\frac{i}{2\pi q} \sum_{x,j,\mu,\lambda} \epsilon_{\mu\nu\lambda} A_{x\mu}^{\text{CS}} F_{x\nu\lambda}^{\text{CS}} + L_{\text{int}}(\psi_x^\dagger \psi_x),
\end{equation}

where the Grassmann number $\psi_x(\tau)$ and the CS gauge potential $A_{x\mu}^{\text{CS}}(\tau)$ are functions of the imaginary time $\tau$, where $0 < \tau < \beta \equiv 1/T$. $x$ denotes the lattice sites, $q$ is a parameter of the model, Greek indices $\mu, \nu, \ldots$ take 0, 1, 2 and denote the directions (0; imaginary-time $\tau$, 1, 2; spatial), and $j = 1, 2$ (and $i, k$ appearing later) denotes the
spatial directions. The field strength $F_{\mu\nu}$ and the CS magnetic field $B_{x}^{CS}$ are given by

$$
F_{xij} = \nabla_i A_{xj}^{CS} - \nabla_j A_{xi}^{CS},
$$
$$
B_{x}^{CS} \equiv F_{x12},
$$
$$
F_{x0i} = \partial_\tau A_{xi}^{CS} - \nabla_i A_{x0}^{CS},
$$
(2.2)

where $\nabla_i$ is the lattice difference operator. We have also introduced the external electromagnetic field $A_{x}^{ex}$ in the model. $L_{int}(\psi_x^\dagger \psi_x)$ represents the residual interactions among fermions.

Let us see that the electron system in a uniform magnetic field $B_{x}^{ex}$ at Landau filling factor $\nu = \frac{1}{2}$ (or more generally $\nu = 1/(2n)$ with a positive integer n) is described by the above model. The Hamiltonian of the electron system is given by

$$
H_e = -\frac{1}{2m} \sum \left( C_{x+j}^\dagger e^{-ieA_{x+j}^{ex}} C_x + H.c. \right) - L_{int}\{C_{x}^\dagger C_x\},
$$
(2.3)

where $C_x$ is the polarized electron annihilation operator at site $x$, and the vector potential $A_{x}^{ex}$ describes $B_{x}^{ex}$,

$$
B_{x}^{ex} = \epsilon_{ij} \nabla_i A_{xj}^{ex}.
$$

$L_{int}\{C_{x}^\dagger C_x\}$ represents interactions among electrons. The lattice spacing $a$, which is often set to unity, is identified with the magnetic length.

To this end, we first differentiate $L$ of (2.1) w.r.t. $A_{x0}^{CS}$ to obtain the equation of motion,

$$
B_{x}^{CS} = 2\pi q \hat{\rho}_x,
$$
$$
\hat{\rho}_x \equiv \psi_x^\dagger \psi_x,
$$
(2.4)

which is the well-known CS constraint showing that the CS fluxes of $q$ units are attached to each fermion $\psi_x^\dagger$. Eq.(2.4) can be solved in the transverse gauge $\nabla_i A_{x_i}^{CS} = 0$ as

$$
A_{x_i}^{CS} = 2\pi q \epsilon_{ij} \sum_y \nabla_j G(x, y) \hat{\rho}_y
$$
$$
= \sum_y \nabla_i \theta(x - y) \hat{\rho}_y,
$$
(2.5)
where $G(x, y)$ is the 2-dimensional Green function, and $\theta(x)$ is the multi-valued angle function on a lattice with $\theta(0) = 0$. The corresponding Hamiltonian is given by the standard procedure as

$$H_\psi = -\frac{1}{2m} \sum \left( \psi_{x+y}^\dagger e^{i(A_{x+y}^{\text{CS}}-\epsilon A_{x+y}^{\text{ex}})} \psi_x + \text{H.c.} \right) - L_{\text{int}}(\{\psi_x^\dagger \psi_x\}), \quad (2.6)$$

where $A_{x+y}^{\text{CS}}$ is given by (2.3).

Next, let us introduce the operator $C_x$ as

$$C_x \equiv \exp \left[ i q \sum_y \theta(x-y) \hat{\rho}_y \right] \psi_x, \quad (2.7)$$

where

$$\hat{\rho}_x = C_x^\dagger C_x = \psi_x^\dagger \psi_x. \quad (2.8)$$

For the half-filled Landau level, we put the parameter $q = 2$ in (2.7). Then one can check that $C_x$ satisfy the canonical anticommutation relations (CACR) for fermions. By substituting (2.7) into (2.6), we reach the Hamiltonian of electron system (2.3).

As the filling factor $\nu$ is given by $\nu = 2\pi \rho / eB^{\text{ex}}$, where $\rho$ is average density of electrons or $\psi_x$’s, $\rho = \langle \hat{\rho}_x \rangle$, we have

$$\langle B^{\text{CS}} \rangle = eB^{\text{ex}}, \quad (2.9)$$

for $\nu = 1/q$ from (2.3). Therefore, the external magnetic field is cancelled out on the average by the CS magnetic field.

### 2.2 Chargeon and fluxon

From the Hamiltonian (2.6) and the CS constraint (2.4), it is obvious that a $\psi_x$ quantum carries $q$ magnetic flux quanta, and, at the same time, interacts minimally with the CS magnetic field. It plays a dual role of matter field and source of the force field. As explained in the introduction, the essential assumption of the CF approach to the FQHE is that the CS constraint becomes irrelevant at low energies.
and correlations between charge and flux degrees of freedom become weak, i.e., the PFS takes place for quasi-excitaions.

To describe this possibility, let us introduce the chargeon operator $\eta_x$ and the fluxon operator $\phi_x$, and express the fermion operator $\psi_x$ as

$$
\psi_x = \phi_x \eta_x.
$$

(2.10)

In order that the operator $\psi_x$ satisfies the fermionic CACR, i.e., $\{\psi_x, \psi_y^\dagger\} = \delta_{xy}$, we assign statistics for the chargeon and the fluxon as follows: chargeon $\eta_x$: fermion,

fluxon $\phi_x$: hard-core boson,

(2.11)

and impose that they have to satisfy the following local constraint;

$$
\eta_x^\dagger \eta_x = \phi_x^\dagger \phi_x.
$$

(2.12)

By the term hard-core bosons, we mean that $\phi_x$ and $\phi_y$ satisfy the usual canonical commutation relation (CCR) of bosons, $[\phi_x, \phi_y] = [\phi_x, \phi_y^\dagger] = 0$, for $x \neq y$, and satisfy CACR, $\{\phi_x, \phi_x\} = 0, \{\phi_x, \phi_x^\dagger\} = 1$, for each $x$. We also assign that $\eta_x$'s and $\phi_x$'s commute each other. The physical meaning of the above constraint (2.12) is understood as follows. For each site $x$, there are two physical states, no fermion state $|0\rangle = \psi_x |0\rangle = 0$ and one fermion state $|1\rangle \equiv \psi_x^\dagger |0\rangle$. They are described using $\eta_x$ and $\phi_x$ and their vacuum state $|V\rangle, \eta_x |V\rangle = \phi_x |V\rangle = 0$, as

$$
|0\rangle = |V\rangle,
$$

$$
|1\rangle = \eta_x^\dagger \phi_x^\dagger |V\rangle.
$$

(2.13)

Of course we can assign the alternative statistics; $\eta_x$: hard-core boson, $\phi_x$: fermion. Final result for the PFS does not depends on which assignment is employed. However, this assignment is more suitable as the chargeons behave as CF's in the state of PFS. See later discussion.

One may check easily that still another assignment for $\phi_x$, the CCR (for all $x$ and $y$) of canonical bosons instead of hard-core bosons, together with (2.12), works well ($\eta_x$ is kept fermion). Since the results obtained in later Sections are unchanged, we use (2.11) for definiteness.
Actually, the constraint (2.12) allows for only these states. We should be careful for “defining” operators like the number operator $\psi_x^\dagger \psi_x$. Its consistent definition with respect to (2.12) is given as

$$
\psi_x^\dagger \psi_x = \eta_x^\dagger \eta_x \phi_x^\dagger \phi_x = \eta_x^\dagger \eta_x = \phi_x^\dagger \phi_x.
$$

(2.14)

To implement the meaning of flux annihilation operator to $\phi_x$, we write the CS constraint in the form,

$$
B_{CS}^x = 2\pi q \phi_x^\dagger \phi_x,
$$

(2.15)

which manifestly shows that a fluxon carries $q$ units of CS flux quanta. We respect this equation faithfully irrespective of whether the PFS occurs or not. The very condition for PFS is that the chargeon-fluxon constraint (2.12) becomes irrelevant to quasi-excitations, as we argue just below. Since our strategy is to prepare two operators, each by each for fluxes and CS charges, we assign so that a chargeon carries one unit of CS charge. These properties and other assignment are summerized in Table 1.

In terms of these operators, let us express the Hamiltonian. By substituting (2.10) into (2.7), we have

$$
H_{\eta\phi} = -\frac{1}{2m} \left( \eta_x^\dagger \phi_x + \phi_x^\dagger \eta_x \right) + \lambda \eta_x^\dagger \eta_x - \phi_x^\dagger \phi_x - L_{\text{int}}(\{\eta_x^\dagger \eta_x, \phi_x^\dagger \phi_x\})
$$

(2.16)

where

$$
W_x = \exp \left[ i q \sum_y \theta(x - y)(\phi_y^\dagger \phi_y - \rho) \right].
$$

(2.17)

Here we used the expression,

$$
e^{-ieA_{sj}^{\text{ex}}} = \exp \left[ - i q \sum_y \theta(x - y)\rho \right].
$$

(2.18)

Remark that single operators like $\eta_x$ or $\phi_x$ do not commute with the constraint (2.12).

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5Remark that single operators like $\eta_x$ or $\phi_x$ do not commute with the constraint (2.12).
We have introduced Lagrange multiplier field $\lambda_x$ for the slave-particle constraint (2.12), and the chemical potentials $\mu_\eta$ and $\mu_\phi$ (chargeons and fluxons must have the same average density, $\langle \eta_x^\dagger \eta_x \rangle = \langle \phi_x^\dagger \phi_x \rangle$). This Hamiltonian (2.16) is invariant under the following local gauge transformation;

$$(\eta_x, \phi_x) \rightarrow (e^{i\alpha_x} \eta_x, e^{-i\alpha_x} \phi_x),$$

whose origin is obvious from (2.10).

We shall investigate the possibility of PFS. In the state of PFS, chargeons and fluxons are quasi-excitations and move almost independently. The constraint (2.12) is not faithfully respected by them. We shall start with (2.15) instead of (2.4), and investigate nature of the quasi-excitations. This assertion is motivated by the success of the CF idea. Similar (but more loose) approximation is recently used in the renormaization-group (RG) study of CS gauge theories of nonrelativistic fermions [9]. There, Coulombic-type interaction term between fermions is converted to a kinetic term of the “CS” gauge field through the CS constraint (2.4) or (2.15), and then the CS constraint is neglected in the RG transformation. This approximation gives interesting and physically acceptable results, though there appeared no solid justifications yet. Strictly speaking, there is a problem that there seem to be no a priori principles to fix the most suitable representation of Hamiltonian to study quasi-excitations in the PFS state. There are lots of representations that are all equivalent under the CS constraint, but they become different right after relaxing the constraint. Our chargeon-fluxon representation gives a partial answer to this problem; prepare an operator for each possible degree of freedom (charge, flux,..), and then relax the constraint for these operators. In this sense, the present approach is different from our previous analysis [7] in which we did not introduce fluxon operators explicitly. (We shall explain more on it below.)

6The slave-particle constraint (2.12) is surely satisfied by the original bare operators. However, if the PFS takes place, the same constraint is not satisfied by the asymptotic fields for quasi-particles at low energies. See Subsect.3.1 for more concrete discussion.
Before going into details of the analysis of PFS, it may be instructive to compare the above expressions with those of the t-J model in the slave-boson or fermion formalism. In the slave-boson formalism, the electron operator \( C_{x\sigma} (\sigma = \uparrow, \downarrow) \) is expressed as
\[
C_{x\sigma} = b_x^\dagger f_{x\sigma},
\]
(2.20)
where \( b_x \) is bosonic holon operator that carries charge \(+e\), and \( f_{x\sigma} \) is fermionic spinon operator that is charge-neutral and has a spin. The local constraint is given by
\[
b_x^\dagger b_x + \sum_{\sigma} f_{x\sigma}^\dagger f_{x\sigma} = 1.
\]
(2.21)
In the state of CSS, holons and spinons move almost independently with each other and they interact perturbatively with dynamical gauge fields which are phase degrees of freedom of link mean fields \([8]\). Then the holons and spinons appear as quasi-particles and they do not satisfy the above local constraint. The analogy becomes more striking if one makes a particle-anti-particle transformation, \( \phi_x \rightarrow \phi_x^\dagger \) in (2.10) and (2.12). Then they become
\[
\psi_x = \phi_x^\dagger \eta_x,
\]
\[
\phi_x^\dagger \phi_x + \eta_x^\dagger \eta_x = 1.
\]
(2.22)
This implies the correspondences,
\[
\psi_x \leftrightarrow C_{x\sigma}, \quad \eta_x \leftrightarrow f_{x\sigma}, \quad \phi_x \leftrightarrow b_x.
\]
(2.23)

### 2.3 Auxiliary gauge field, PFS, and CF’s

The partition function \( Z \equiv \text{Tr} \exp(-\beta H_{\eta\phi}) \) for (2.16) is expressed in path-integral formalism as
\[
Z = \int [d\eta][d\phi] \exp(\int_0^\beta d\tau L_{\eta\phi}).
\]
(2.24)
The Lagrangian is
\[
L_{\eta\phi} = -\sum \eta_x^\dagger \partial_\tau \eta_x - \sum \phi_x^\dagger \partial_\tau \phi_x - H_{\eta\phi}.
\]
(2.25)
Here \( \eta_x(\tau) \) is a Grassmann number. The integration over the hard-core bosons \( \phi_x \) needs a special treatment. We treat \( \phi_x(\tau) \) as a complex number. Its hard-core nature can be incorporated faithfully in the hopping expansion carried out below. Now let us perform a Hubbard-Stratonovich transformation to introduce an auxiliary complex field \( V_{xj} \) on the link \((x, x + j)\). Then we have

\[
Z = \int [d\eta][d\phi][dV] \exp(\int^\beta_0 d\tau L_{\eta\phi V}), \quad (2.26)
\]

and then the Lagrangian \( L_{\eta\phi V} \) is given by

\[
L_{\eta\phi V} = -\sum \eta^\dagger_x(\partial_\tau + i\lambda_x - \mu_\eta)\eta_x - \sum \phi^\dagger_x(\partial_\tau - i\lambda_x - \mu_\phi)\phi_x \\
+ \sum \frac{1}{2m} \left[ V_{xj} (\phi_{x+j}\phi^\dagger_x + \eta^\dagger_{x+j} W_{x+j} W^\dagger_x \eta_x) + H.c. \right] \\
- \sum \frac{1}{2m} (\phi^\dagger_{x+j} \phi_x \phi^\dagger_x \phi_{x+j}) - \sum \frac{1}{2m} (\eta^\dagger_{x+j} \eta_x \eta^\dagger_{x+j} \eta_x) \\
- \sum \frac{1}{2m} |V_{xj}|^2 + L_{\text{int}}(\{\eta^\dagger_x \eta_x, \phi^\dagger_x \phi_x\}). \quad (2.27)
\]

This Lagrangian \((2.27)\) clearly shows that a chargeon \( \eta_x \) moves in the statistical magnetic field that is generated by fluxons \((2.17)\), as we expected. This is welcome since we are now free from facing a complicated problem that a single field hops through an effective field generated via itself nonlocally. Furthermore, both chargeons and fluxons couple minimally to the link field \( V_{xj} \), which gives rise to attractive interactions between a chargeon \( \eta_x \) and a fluxon \( \phi_x \). It induces attractions also in the \( \eta^\dagger_{x+j} - \eta_x \) and \( \phi^\dagger_{x+j} - \phi_x \) channels. The four-Fermi \( \eta^4 \) term and the \( \phi^4 \) term in \((2.27)\) appear to cancel these residual “gauge” interactions. See Table 1 for the charges that couple to the gauge field \( V_{xj} \), which we denoted “V charge”. In Fig.1 we illustrate the key concepts and objects appeared in each step to reach \( L_{\eta\phi V} \) of \((2.27)\), such as \( C^\dagger_x, \psi^\dagger_x, \tilde{W}_x, \eta^\dagger_x, \phi^\dagger_x, W_x \).

As a model of the half-filled Landau level, the interaction term of chargeons and fluxons, \( L_{\text{int}}(\{\eta^\dagger_x \eta_x, \phi^\dagger_x \phi_x\}) \) in \((2.27)\), is obtained from the interactions between electrons in the following way. For example, let us assume the following nearest-neighbor
interactions;

\[ L_{\text{int}}(\{C_x^\dagger C_x\}) = L_{\text{int}}(\{\psi_x^\dagger \psi_x\}) = g \sum \psi_{x+j}^\dagger \psi_{x+j} \psi_x^\dagger \psi_x, \quad (2.28) \]

where \( g \) is the coupling constant. Then from the chargeon-fluxon constraint (2.12) and the Fermi statistics of \( \eta_x \), it can be written as

\[ \psi_{x+j}^\dagger \psi_{x+j} \psi_x^\dagger \psi_x = (\phi_{x+j}^\dagger \phi_{x+j} \eta_{x+j}^\dagger \eta_{x+j}) (\phi_x^\dagger \phi_x \eta_x^\dagger \eta_x) \]

\[ = \eta_{x+j}^\dagger \eta_{x+j} \eta_x \]

\[ = \phi_{x+j}^\dagger \phi_{x+j} \phi_x^\dagger \phi_x. \quad (2.29) \]

This leads us to generalize the model by introducing two coupling constants \( g_1 \) and \( g_2 \) to write

\[ L_{\text{int}}(\{\eta_x^\dagger \eta_x, \phi_x^\dagger \phi_x\}) = g_1 \sum \eta_{x+j}^\dagger \eta_{x+j} \eta_x \eta_x + g_2 \sum \phi_{x+j}^\dagger \phi_{x+j} \phi_x^\dagger \phi_x. \quad (2.30) \]

For example, if the interaction (2.28) represents Coulombic (but short-range) repulsion between the EM charges, the parameters are \( g_1 < 0 \) and \( g_2 = 0 \) since we assign that \( \eta_x \) and \( \phi_x \) carry the EM charge \(-e\) and 0, respectively (See Table 1).

It is obvious that if the amplitude of link field \( V_{xj} \) is nonvanishing, its phase degrees of freedom exist and behave as a gauge field, because under the gauge transformation (2.19), \( V_{xj} \) transforms as

\[ V_{xj} \rightarrow e^{i\alpha_{x+j}} V_{xj} e^{-i\alpha_x}. \quad (2.31) \]

If the gauge dynamics of \( V_{xj} \) is in the confinement phase, the gauge field fluctuates largely, giving rise to the vanishing expectation value of \( V_{xj} \). The only charge-neutral compounds with respect to (2.19) and (2.31) appear as physical excitations, such as \( \eta_x^\dagger \phi_x^\dagger \), which are nothing but the original fermions \( \psi_x^\dagger \). Due to the CS constraint (2.4), each of these fermions necessarily accompany \( q \) units of fluxes, forming the electrons for \( q = 2 \). Therefore we conclude that the quasi-excitations at half filling are the original electrons when the gauge dynamics is realized in the confinement phase. On
the other hand, if it is in the deconfinement phase, gauge fluctuations are small and $V_{xj}$ develops a quasi-long-range order. Therefore chargeons and fluxons acquire their own hopping amplitudes which are proportional to $\langle V_{xj} \rangle \neq 0$, so behave as quasi-excitations. Especially, the chargeons $\eta_x$ describe nothing but the weakly interacting CF’s proposed by Jain. This last point deserves more explanation. In the literatures the word “composite fermion” is used for almost free fermions which appear as a result of the cancellation between the external magnetic field and the average of the CS field with neglecting the CS constraint (2.4). In this sense, the chargeons in our case can be regarded as CF’s. However, there is another possible definition of CF’s, that is, the field $\psi_x$ itself may describe CF’s. This interpretation has been advocated in Ref.[7]. There we worked directly with $\psi_x$ (without further decomposition into chargeon and fluxon) and introduced the same auxiliary link field $V_{xj}$. The hopping term in the action thus reads like

$$\sum \left[ V_{xj} (\psi^\dagger_{x+j} \psi_x + W^\dagger_{x+j} W_x) + \text{H.c.} \right],$$

(2.32)

where the nonlocal operator $W_x$ is given by the same expression (2.17) but with $\psi^\dagger_x \psi_x$ for its source instead of $\phi^\dagger_x \phi_x$. We argued that the PFS takes place if the gauge dynamics is in the deconfinement phase, and the quasi-excitations are $\psi_x$ particles, which, in turn, we interpreted as CF’s. The difference between these two candidates for CF’s, $\eta_x$ and $\psi_x$, lies whether they have own CS fluxes or not. (See Table 1.) However, this difference is subtle. Actually, at $T = 0$, one may expect that the boson field $\phi_x$ may Bose condense. Then, in the leading treatment, one can set $\phi_x$ as a constant $\phi_x = \sqrt{\rho}$, which washes out the difference of two operators since $\psi_x = \phi_x \eta_x \propto \eta_x$. The difference may appear in the next order in the small fluctuations $\phi_x = \sqrt{\rho} + \delta \phi_x$. In Sect.4 we mention the perturbative analyses in the literature in connection with this point. However, we find no strong reasons why one interpretation is better than the other. They are physically equivalent in the leading (mean-field) treatment, and there appear no explicit quantitative comparisons of higher-order corrections. What we can say is that the chargeon-fluxon approach manifests the
separation of degrees of freedom, and opens a possibility to describe the dynamics of charges and fluxes on an equal footing. We expect that it is certainly superior if the system supports nontrivial but quasi-local flux excitations as quasi-excitations.

In the following section, we shall obtain an effective gauge theory of $V_{xj}$ from (2.27) by integrating out the fields $\eta_x$ and $\phi_x$. To this end, we use the hopping expansion as in the previous studies of the CSS and the PFS [7, 8].

3 Effective gauge theory

In this section, we derive an effective gauge theory of $V_{xj}$. We first decompose $V_{xj}$ into its amplitude and phase variable,

$$V_{xj} = V_0 \ U_{xj}, \ U_{xj} \in U(1).$$

As explained in the previous section, $U_{xj}$ behaves as a gauge field and plays an important role to determine the spectrum of low-energy excitations. In Sect.3.1 we start with a brief discussion on the relationship between the local constraint and the gauge dynamics. Then we study the behavior of the amplitude $V_0$ as a function of $T$, by using the hopping expansion w.r.t. $\eta_x$ and $\phi_x$ in Sect.3.2, and by using the mean-field type calculation in Sect.3.3. In Sect.3.4, we derive the effective action of $U_{xj}$ by the hopping expansion. The usefulness of the hopping expansion in such a situation has been explained in Ref.[7, 8].

3.1 Local constraint and gauge dynamics

The effective action $A[V]$ is defined as:

$$e^{A[V]} = \int [d\eta][d\phi][d\lambda] \exp \left[ \int_0^\beta d\tau L_{\eta\phi V} \right].$$

As stated before, this path-integral expression is rather formal for the hard-core boson $\phi_x$. We shall use the knowledge of operator formalism for deriving propagators and calculating matrix elements.
In the hopping expansion to evaluate $A[V]$, one needs the propagators of $\eta_x$ and $\phi_x$, which are obtained from (2.27) as

$$\langle \eta_x(\tau_1)\eta^\dagger_y(\tau_2) \rangle = \delta_{xy} G_\eta(\tau_1 - \tau_2),$$

$$G_\eta(\tau) = \frac{e^{\beta \mu \tau}}{1 + e^{\beta \mu \eta}} [\theta(\tau) - e^{\beta \mu \eta} \theta(-\tau)],$$

$$\rho = \frac{e^{\beta \mu \eta}}{1 + e^{\beta \mu \eta}},$$

(3.3)

$$\langle \phi_x(\tau_1)\phi^\dagger_y(\tau_2) \rangle = \delta_{xy} G_\phi(\tau_1 - \tau_2),$$

$$G_\phi(\tau) = \frac{e^{\beta \mu \tau}}{1 + e^{\beta \mu \phi}} [\theta(\tau) - e^{\beta \mu \phi} \theta(-\tau)],$$

$$\rho = \frac{e^{\beta \mu \phi}}{1 + e^{\beta \mu \phi}}.$$  

(3.4)

In order to see how the chargeon-fluxon constraint (2.12) is affected by the gauge dynamics, let us calculate the quadratic term of $\lambda_x$ in the leading order. From (2.27), the contribution from $\phi_x$ is estimated as follows;

$$-\int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \lambda_x(\tau_1)\lambda_{x+j}(\tau_3)$$

$$\times \langle \phi^\dagger_x(\tau_1)\phi_x(\tau_1)\phi^\dagger_{x+j}(\tau_2)\phi_x(\tau_2)\phi^\dagger_{x+j}(\tau_3)\phi_x(\tau_3)\phi^\dagger_{x+j}(\tau_4)\phi_x(\tau_4) \rangle$$

$$\times \langle V^\dagger_{xj}(\tau_4)V_{xj}(\tau_2) \rangle$$

$$= -\int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \lambda_x(\tau_1)\lambda_{x+j}(\tau_3) \prod_i G_\phi(\tau_{i+1} - \tau_i) \cdot \langle V^\dagger_{xj}(\tau_4)V_{xj}(\tau_2) \rangle,$$  

(3.5)

where $\tau_{i+4} = \tau_i$. From (3.3) it is verified that the $\tau_i$-dependence of $\prod_i G_\phi(\tau_{i+1} - \tau_i)$ (especially $\tau_2$ and $\tau_4$-dependence) cancels with each other and only the $\theta$-functions remain. Therefore, the relevant $\tau_i$-dependence of the integrand in (3.3) may stem only from the correlation function of $V_{xj}$.

It is convenient to introduce Fourier decomposition of the gauge field $U_{xj}$,

$$U_{xj}(\tau) = \sum_n e^{i \omega_n \tau} U_{xj,n},$$

$$\sum_n U^\dagger_{xj,n} U_{xj,n+m} = \delta_{m0},$$

(3.6)

where $\omega_n = 2\pi n/\beta$, $n = 0, \pm 1, \pm 2, \cdots$. Then the above correlator is given as

$$\langle V^\dagger_{xj}(\tau_4)V_{xj}(\tau_2) \rangle = |V_0|^2 \sum_{n,m} \langle U^\dagger_{xj,n} U_{xj,m} \rangle \cdot e^{i (\omega_n \tau_2 - \omega_m \tau_4)}.$$  

(3.7)
From (3.7), it is obvious that if and only if $V_0 \neq 0$ and the static mode of $U_{xj}(\tau)$, i.e., $U_{xj,n=0}$, dominates over all the other oscillating modes, the quadratic term (3.5) gives nontrivial contribution with the coefficient

$$\langle V_{xj}^\dagger(\tau_4)V_{xj}(\tau_2) \rangle \sim |V_0|^2 \langle U_{xj,n=0}^\dagger U_{xj,n=0} \rangle.$$ (3.8)

Similar term appears from the $\eta_x$-hopping term. Then the term (3.3) behaves as a mass term, and because of that, the slave-particle constraint becomes less strict at low energies. If the above condition for $V_{xj}(\tau)$ is satisfied, coherent movement of fluxons $\phi_x$ and/or chargeons $\eta_x$ occur. As the field $\lambda_x$ can be regarded as a scalar component of the vector potential, the above conclusion implies nothing but the shielding of the static potential, a phenomena being observed quite often in many-body systems.

What does the above condition mean for the gauge dynamics? In the following sections, we shall show that the above condition is satisfied if the gauge dynamics of $U_{xj}$ is in the deconfinement phase. Therefore the above result means that in the deconfinement phase the slave-particle constraint is not faithfully respected by the quasi-excitations. It is also expected that near the confinement-deconfinement (CD) phase transition the slave-particle constraint is less effective for quasi-excitations compared with the original variables\(^8\) In the following sections, we shall study the possibility of the PFS, and then we shall ignore the slave-particle constraint in most of the discussion.

The above conclusion is consistent with the fact that $\lambda_x$ can be regarded as a time-component of the transverse gauge field $U_{xj}$. Actually, the Lagrangian (2.27) is invariant under time-dependent local gauge transformation with $\lambda_x \rightarrow \lambda_x - \partial_{\tau} \alpha_x$. Then we can take the temporal gauge, $\lambda_x = \text{constant}$, and in this gauge solely the dynamics of the transverse gauge field $U_{xj}$ determines which phase the system is

\(^8\)Here one should distinguish the variables for quasi-excitations which are asymptotic fields from the original variables in the Heisenberg picture, though we often use the same notation for both of them. The operator in the Heisenberg picture always satisfies the constraint, but the asymptotic field does not. Somewhat detailed discussion on this point is given in the previous papers \(\)\(\)\(\).
in. This consideration also supports our treatment of the local constraint in the subsequent discussions on the phase structure of the present model.

In the following subsection, we shall study behavior of the amplitude $V_0$ as a function of $T$.

### 3.2 Amplitude: the hopping expansion

In this and subsequent subsections, we study behavior of the amplitude $V_0$. We shall use both the hopping expansion and the mean-field type calculation, which are reliable at intermediate and low $T$, respectively.

Effective potential of $V_0$ is obtained by setting the all link fields as $V_{xj} = V_0$ in (3.2), $2\beta N P(V_0) = -A[V_{xj} = V_0]$, where $N$ is the total number of sites in the system. We assume that $V_0$ is a real variable, though this assumption is not essential for the following calculation. In this subsection, we shall focus on the single-link potential (SLP) of $V_0$, and calculate it by the hopping expansion. From the SLP, we can determine behavior of $V_0$ as a function of $T$. As we show, the amplitude develops a nonvanishing value at low $T$.

From (2.27), at the tree level, we have

$$ P_{\text{tree}}(V_0) = \frac{1}{2m} V_0^2. \quad (3.9) $$

In the second order of the hopping expansion, the $\phi_x$ contributes to the SLP as

$$ \Delta P^{(2)}_{\phi} = -\frac{1}{(2m)^2} \beta^{-1} V_0^2 \int d\tau_1 d\tau_2 \langle \phi_x^\dagger(\tau_1) \phi_{x+j}(\tau_2) \phi_x^\dagger(\tau_1) \phi_{x+j}(\tau_2) \rangle $$

$$ = -\frac{1}{(2m)^2} \beta \rho (1 - \rho) V_0^2, \quad (3.10) $$

where we have used the the propagator of $\phi_x$ (3.4). Similarly, the $\eta_x$ hopping gives contribution as

$$ \Delta P^{(2)}_{\eta} = -\frac{1}{(2m)^2} \beta^{-1} V_0^2 \int d\tau_1 d\tau_2 \langle \eta_x^\dagger W_x^\dagger(\eta_{x+j}(\tau_1) \eta_{x+j}(\tau_2)) \eta_x^\dagger W_x^\dagger(\eta_{x+j}(\tau_1) \eta_{x+j}(\tau_2)) \rangle $$

$$ = -\rho(1 - \rho) \frac{1}{(2m)^2} \beta^{-1} V_0^2 \int d\tau_1 d\tau_2 \langle W_x^\dagger(\tau_1) W_{x+j}^\dagger(\tau_1) \rangle. \quad (3.11) $$
As the fluxon is the hard-core boson, the expectation value $\langle WW^\dagger WW^\dagger \rangle$ in (3.11) is evaluated by using the following identity which is satisfied for an arbitrary c-number $\alpha$

$$e^{\alpha\phi^\dagger\phi} = 1 + (e^\alpha - 1)\phi^\dagger\phi.$$  \hfill (3.12)

In the leading order of the $\phi$-hopping $\langle WW^\dagger WW^\dagger \rangle = 1$, and therefore

$$\Delta P^{(2)} = -\frac{\rho(1-\rho)}{(2m)^2}\beta V_0^2.$$  \hfill (3.13)

Collecting (3.9), (3.10) and (3.13), the quadratic term of $V_0$ in the SLP is given by

$$P^{(2)}(V_0) = \frac{1}{2m} \left[ 1 - \frac{\beta \rho(1-\rho)}{m} \right] V_0^2.$$  \hfill (3.14)

From (3.14), it is obvious that, at $T < T_V \equiv \rho(1-\rho)/m$, the amplitude $V_0$ develops a nonvanishing expectation value.

Higher-order terms of $V_0$ in the SLP is evaluated in a similar way. In the previous paper [7], the quartic term is calculated and $V_0$ is obtained as a function of $T$. In the present case, qualitatively same result is obtained, which behaves near $T_V$ as

$$V_0 \sim C_V \sqrt{(T_V - T)/T},$$  \hfill (3.15)

where $C_V$ is some positive constant (see Ref.[7] for detailed calculation).

### 3.3 Amplitude: mean-field calculation at low $T$

In this subsection, we consider the mean-field type calculation, which is complementary to the hopping expansion in Sect.3.2. It also shows that $V_0$ develops a nonvanishing expectation value at low $T$.

Let us start with the observation that the fluxon $\phi_x$ field should Bose condense at $T = 0$, $\langle \phi_x \rangle \sim \sqrt{\rho}\exp(i\chi_x)$, due to its Bose statistics. ($\chi_x$ is the phase of $\phi_x$.) Because the system is just two-dimensional, the genuine long-range order disappear at $T > 0$, but the short-range orders should survive well at low $T$. This leads us to a simplification of the Lagrangian (2.27) by replacing the nearest-neighbor term
\( \phi_{x+j} \phi_x^\dagger \rightarrow \rho \exp(\im \chi_{x+j} - \im \chi_x) \). This also allows us to replace \( \phi_x^\dagger \phi_x \) in \( W_x \) by \( \rho \), which implies \( W_x \rightarrow 1 \). These simplifications give rise to the mean-field Lagrangian \( L_{MF} \),

\[
L_{MF} = - \sum \eta_x^\dagger (\partial_\tau - \mu_\eta) \eta_x + \sum \frac{V_0}{2m} \left[ (\rho + \eta_{x+j}^\dagger \eta_x) + \text{H.c.} \right] - 2N \frac{\rho^2}{2m} - \sum \frac{1}{2m} \left( \eta_{x+j}^\dagger \eta_x^\dagger \eta_x \eta_{x+j} \right)
- 2N \frac{V_0^2}{2m} + \text{Int}(\{ \eta_x^\dagger \eta_x \}).
\]

(3.16)

Here we used the unitary gauge \( \chi_x = 0 \). This represents a system of fermions \( \eta_x \) moving with a hopping amplitude \( V_0/(2m) \). To obtain the MF equation that determines the value of \( V_0 \), let us assume that the sum of two four-fermi terms is negligibly small. This assumption will be explained more in Sect.3.3. Then, in the leading order, the system is a collection of free fermions, and its free energy \( F_{MF} \) per site can be calculated in a straightforward manner (See Ref.[7] for details) as

\[
\frac{F_{MF}}{N} = \frac{(V_0 - \rho)^2}{m} - \frac{1}{\beta N} \ln \left[ 1 + \exp\left( -\beta \left[ \frac{V_0}{m} \sum \cos k_i - \mu_\eta \right] \right) \right].
\]

(3.17)

where \( k_i \) is the lattice momentum. The value of \( V_0 \) is determined by minimizing \( F_{MF}/N \),

\[
\frac{\partial}{\partial V_0} \frac{F_{MF}}{N} = \frac{2(V_0 - \rho)}{m} + \frac{1}{mN} \sum_k (\sum_i \cos k_i) f(k)
= 0,
\]

(3.18)

where \( f(k) \) is the Fermi distribution function,

\[
f(k) = \frac{\exp\left\{ -\beta \left[ \frac{1}{m} \sum \cos k_i - \mu_\eta \right] \right\}}{1 + \exp\left\{ -\beta \left[ \frac{1}{m} \sum \cos k_i - \mu_\eta \right] \right\}}.
\]

(3.19)

\( \mu_\eta \) is determined by the condition for \( \rho \),

\[
\rho = \frac{1}{N} \sum_k f(k).
\]

(3.20)

The set of mean-field equations (3.18) and (3.20) can be solved numerically. Such an analyses has been done in Ref.[7]. In particular, there is a nonvanishing solution for \( V_0 \) at low \( T \). For example, \( V_0 = \frac{1}{2} + \frac{2}{\pi^2} \) at \( T = 0 \) and \( \rho = 1/2 \).

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3.4 Effective action of the gauge field $U_{xj}$

In this subsection, we shall calculate the effective action of the gauge field $U_{xj}$, $A[U] \equiv A[V_{xj} = V_0 U_{xj}]$, and in the following section we shall study its phase structure. We shall employ the hopping expansion, and it gives (approximately) the following canonical form of the electric and the magnetic terms of the lattice gauge theory:

$$A[U] = A_e + A_m,$$

$$A_e = -\frac{1}{g_e^2} \int d\tau \sum_{x,j} \left[ \partial_\tau U^\dagger_{xj} \partial_\tau U_{xj} + \cdots \right],$$

$$A_m = \frac{1}{g_m^2} \int d\tau \sum_x \left[ U_{x,2} U_{x+2,1} U^\dagger_{x+1,2} U^\dagger_{x,1} + \text{H.c.} + \cdots \right],$$

where $g_e^2$ and $g_m^2$ are the effective electric and magnetic gauge couplings, respectively. The reader who is not interested in the detailed derivations given below may skip them; just note the main result given in Eqs. (3.30) and below, and go to Sect. 4 to find the discussion on the physical results like the phase structure, etc.

The $\phi -$ and $\eta -$ hopping terms and the $\phi^4$ and the $\eta^4$ interactions in (2.27) give each contribution to the effective action, which we write as

$$A_e = A_{e,\phi} + A_{e,\eta} + A_{e,\phi^4} + A_{e,\eta^4},$$

$$A_m = A_{m,\phi} + A_{m,\eta} + A_{m,\phi^4} + A_{m,\eta^4}.$$ (3.23)

Let us start with $A_{e,\phi}$. As in the calculation of the SLP in Sect. 3.2, in the second-order of the hopping expansion of $\phi_x$, we have

$$A_{e,\phi} = |V_0|^2 \left( \frac{1}{(2m)^2} \sum \int d\tau_1 d\tau_2 U^\dagger_{xj}(\tau_1) U_{xj}(\tau_2) \langle \phi^\dagger_{x} \phi_{x+j}(\tau_1) \phi^\dagger_{x+j} \phi_{x}(\tau_2) \rangle \right)$$

$$= |V_0|^2 \left( \frac{1}{(2m)^2} \rho(1-\rho) \beta^2 \sum U^\dagger_{xj,0} U_{xj,0} \right),$$

where $U_{xj,0}$ is the static component of the Fourier decomposition of $U_{xj}$ (3.6).

Similarly, the $\eta_x$ hopping gives rise to $A_{e,\eta}$ as

$$A_{e,\eta} = |V_0|^2 \left( \frac{1}{(2m)^2} \sum \int d\tau_1 d\tau_2 U^\dagger_{xj}(\tau_1) U_{xj}(\tau_2) \langle \eta^\dagger_{x} W^\dagger_{x+j} \eta_{x+j}(\tau_1) \eta^\dagger_{x+j} W_{x+j} \eta_{x}(\tau_2) \rangle \right)$$

$$= |V_0|^2 \left( \frac{1}{(2m)^2} \rho(1-\rho) \beta^2 \sum U^\dagger_{xj,0} U_{xj,0} \right).$$

(3.25)
Both $A_{e,\phi}$ and $A_{e,\eta}$ above have the form $\sum U_{xj}^\dagger U_{xj,0}$. From the factor $\beta^2$ in (3.24) and (3.25), it is obvious that, at high $T$, the coefficients of these terms are small and almost no significant enhancement or depression appear for $U_{xj}$'s; All the modes $U_{xj,n}$ fluctuate randomly. On the other hand, at low $T$, the coefficients develop and the static mode $U_{xj,n=0}$ dominates over all the other oscillating modes. By using the unitarity condition (3.6), these electric terms are rewritten as the following canonical form effectively,

$$
\beta^2 U_{xj,0}^\dagger U_{xj,0} = \beta^2 \left( 1 - \sum_{n \neq 0} U_{xj,n}^\dagger U_{xj,n} \right)
\sim \beta^2 - \frac{2\beta^3}{(2\pi)^2} \int_0^\beta d\tau \partial_\tau U_{xj}^\dagger \partial_\tau U_{xj}.
$$

(3.26)

Therefore, the effective gauge coupling $g_{e}^2$ in this system has strong $T$ dependence. Because of this fact, the present gauge system exhibits a rather nontrivial phase structure as $T$ changes.

Let us turn to the contributions from the interaction terms in $L_{\eta \phi V}$ of (2.27) with $L_{\text{int}}$ of (2.30). It is not so easy to evaluate their effects for general coupling constants $g_1$ and $g_2$. Therefore, we assume that the interaction between fermions almost cancels the $\phi^4$ and the $\eta^4$ terms in $L_{\eta \phi}$ which appear as a result of the introduction of the "gauge field" $V_{xj}$. That is, we adjust the parameters $g_1$ and $g_2$ such that the total results,

$$
\frac{1}{2m} \left( \phi_{x+j}^\dagger \phi_{x+j} \phi_x^\dagger \phi_x \right) + \frac{1}{2m} \left( \eta_{x+j}^\dagger \eta_{x+j} \eta_x^\dagger \eta_x \right) + L_{\text{int}}(\{\eta_x^\dagger \eta_x, \phi_x^\dagger \phi_x\})
= \lambda_1 \left( \eta_{x+j}^\dagger \eta_{x+j} \eta_x^\dagger \eta_x \right) + \lambda_2 \left( \phi_{x+j}^\dagger \phi_{x+j} \phi_x^\dagger \phi_x \right),
$$

(3.27)

have small constants $\lambda_1 (= (2m)^{-1} + g_1)$ and $\lambda_2 (= (2m)^{-1} + g_2)$. We shall use the perturbative calculation in powers of these $\lambda_1$ and $\lambda_2$. This calculation still give some important results, as we shall see. This assumption implies $g_1 < 0$ and $g_2 < 0$, that is, there exist repulsions between fluxons and also between chargeons. For electrons in the half-filled Landau level, only chargeons have EM charge, $g_1 < 0$ and $g_2 = 0$, so $\lambda_2 = (2m)^{-1}$. 

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Explicitly, for \( A_{e,\phi^4} \), the \( \phi^4 \) term is contracted with the fluxon hopping, giving rise to

\[
A_{e,\phi^4} = \frac{\lambda_2}{(2m)^2} |V_0|^2 \int \prod_{i=1}^{3} d\tau_i U_{x,j}^\dagger(\tau_1)U_{x,j}(\tau_2)(\phi_{x,j}(\tau_1)\phi_{x,j}(\tau_2)
\times (\phi_{x,j}(\tau_3)\phi_{x,j}(\tau_3) - \rho)(\phi_{x,j}(\tau_3)\phi_{x,j}(\tau_3) - \rho))
\]

\[
= \frac{\lambda_2}{(2m)^2} |V_0|^2 \beta^3 \rho^2 (1 - \rho)^2 \sum U_{x,j,0}^\dagger U_{x,j,0}. \tag{3.28}
\]

Here we have replaced the \( \phi^4 \) term (and \( \eta^4 \) term) by its “normal-ordered” form, 
\((\phi_{x,j}^\dagger \phi_{x,j} - \rho)(\phi_{x,j}^\dagger \phi_{x,j} - \rho)\) for convenience. This brings an irrelevant shift of chemical potential and addition of an irrelevant constant to the Lagrangian.

Similarly, for \( A_{e,\eta^4} \) we have

\[
A_{e,\eta^4} = \frac{\lambda_1}{(2m)^2} |V_0|^2 \int \prod_{i=1}^{3} d\tau_i U_{x,j}^\dagger(\tau_1)U_{x,j}(\tau_2)
\langle \eta_{x,j}(\tau_1)W_{x,j}^\dagger \eta_{x,j}(\tau_2)W_{x,j}^\dagger \eta_{x,j}(\tau_2)
\times (\eta_{x,j}^\dagger(\tau_3)\eta_{x,j}(\tau_3) - \rho)(\eta_{x,j}^\dagger(\tau_3)\eta_{x,j}(\tau_3) - \rho))
\]

\[
= -\frac{\lambda_1}{(2m)^2} |V_0|^2 \beta^3 \rho^2 (1 - \rho)^2 \sum U_{x,j,0}^\dagger U_{x,j,0}. \tag{3.29}
\]

Collecting these terms, we obtain the result for \( A_e \),

\[
A_{e,\phi} + A_{e,\eta} + A_{e,\phi^4} + A_{e,\eta^4}
\sim -\frac{1}{(2m)^2} |V_0|^2 \rho(1 - \rho) \left( \frac{\beta^3}{(2\pi)^2} \right)^2 \left( 2 + (\lambda_2 - \lambda_1)\rho(1 - \rho)\beta \right)
\times \sum \int d\tau \partial_x U_{x,i}^\dagger \partial_x U_{x,i}. \tag{3.30}
\]

Thus the effective electric gauge-coupling constant \( g_e^2 \) is given by

\[
g_e^2 = \left( |V_0|^2 \rho(1 - \rho) \left( \frac{\beta^3}{(2m)^2} \right)^2 \left( 2 + (\lambda_2 - \lambda_1)\rho(1 - \rho)\beta \right) \right)^{-1}. \tag{3.31}
\]

The magnetic terms are calculated in a similar way. They determine spatial configuration of the gauge field \( U_{x,j} \). From the fluxon hopping, we have

\[
A_{m,\phi} = \left( \frac{V_0}{2m} \right)^4 \sum_{x} \prod_{i=1}^{4} \int d\tau_i (U_{x,2}(\tau_4)U_{x,1,2}(\tau_3)U_{x,1}^\dagger(\tau_2)U_{x,1}^\dagger(\tau_1) + \text{H.c.})
\]

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\[
\times \prod_{i=1}^{4} G_{\phi}(\tau_i - \tau_{i+1}) \\
\simeq C_{\phi} \left( \frac{V_0}{2m} \right)^4 \beta^4 \sum_x \left( U_{x,2,0}(\tau_4) U_{x+2,1,0}(\tau_3) U_{x+1,2,0}^{\dagger}(\tau_2) U_{x+1,0}^{\dagger}(\tau_1) + \text{H.c.} \right) \tag{3.32}
\]
where \( C_{\phi} \) is given by
\[
C_{\phi} = \frac{1}{4!} \left\{ 4\rho(1 - \rho)^3 + 12\rho^2(1 - \rho)^2 + 4\rho^3(1 - \rho) \right\}. \tag{3.33}
\]
In the last line of (3.32), we have retained only the terms of product of four zero modes \( U_{x,j} \). Similarly, from the \( \eta_x \)-hopping, we get
\[
A_{m,\eta} = - \left( \frac{V_0}{2m} \right)^4 \sum_x \prod_{i=1}^{4} \int d\tau_i \left( U_{x,2,0}(\tau_4) U_{x+2,1,0}(\tau_3) U_{x+1,2,0}^{\dagger}(\tau_2) U_{x+1,0}^{\dagger}(\tau_1) \right) \\
\times \prod_{i=1}^{4} G_{\eta}(\tau_i - \tau_{i+1}) \\
\times \langle W_{x}^{\dagger} W_{x+1} W_{x+1}^{\dagger} W_{x+2} W_{x+2}^{\dagger} W_{x} \rangle + \text{H.c.} \\
\simeq - C_{\eta} \left( \frac{V_0}{2m} \right)^4 \beta^4 \sum_x \left( U_{x,2,0} U_{x+2,1,0} U_{x+1,2,0}^{\dagger} U_{x+1,0}^{\dagger} \right) \\
\times e^{-2\pi q i \rho} \langle e^{2\pi q i \phi_x} \rangle + \text{H.c.},
\]
\[
C_{\eta} = \frac{1}{4!} \left\{ -4\rho(1 - \rho)^3 + 12\rho^2(1 - \rho)^2 - 4\rho^3(1 - \rho) \right\}. \tag{3.34}
\]
where we have used \( \nabla_x \nabla_x G(x, x') = \delta_{xx'} \).

For the case of the half-filled Landau level, one sets \( \rho = 1/2 \) and \( q = 2 \), which leads to \( C_{\phi} > C_{\eta} \). Therefore, the lowest-energy state is realized by the fluxless and uniform configuration of \( U_{x,j} \)’s; \( \langle U_{x,j} \rangle = \text{constant} \). There are also contributions from the interaction terms \( A_{m,\phi^4} \) and \( A_{m,\eta^4} \). Anyway, it is true that the magnetic term \( A_m \) also prefers the static modes of the gauge field at low \( T \).
4 Phase structure of the effective gauge theory and quasi-particles

Before discussing the PFS in our effective gauge model derived in Sect.3.4, let us recall some general arguments on the CD transition of the canonical lattice gauge theory, whose effective gauge coupling constant $g_e^2 = g_m^2 = g_{can}^2$ is $T$-independent. From the work of Polyakov and Susskind [10] and the explicit Monte Carlo simulations, it is well-known that the CD phase transition takes place at finite $T$ for such a canonical lattice gauge theory. In Ref.[10], this CD phase transition is observed by mapping the strongly-coupled gauge system to an effective classical spin model [10]; the CD transition is identified with the order-disorder transition of the spin dynamics. For compact U(1) gauge theory in $(2 + 1)$ dimensions, the mapped spin model is the XY model in two dimensions which exhibits the Kosterlitz-Thouless (KT) phase transition. According to Ref.[10], for the canonical gauge system, the transition temperature $T_{CD}^*$ is estimated as

$$T_{CD}^* \simeq g_{can}^2.$$  (4.1)

It is concluded that the deconfinement phase appears at $T > T_{CD}^*$, while the confinement phase appears at $T < T_{CD}^*$. This result is easily seen in the Lagrangian formalism. In terms of the Fourier components of the gauge field (3.6), the action of the canonical gauge system is written as

$$A_{can} = -\frac{1}{g_e^2} \int d\tau \sum_{x,j} \partial_\tau U_{xj}^\dagger(\tau) \partial_\tau U_{xj}(\tau)$$

$$= -\frac{\beta}{g_e^2} \sum_{x,j} \sum_n \omega_n^2 U_{xj,n}^\dagger U_{xj,n},$$  (4.2)

9 For a more complicated effective gauge theory derived for the t-J model, the mapping is performed in Ref.[8].
where $\omega_n = 2\pi n/\beta$. Then it is obvious that at very high $T$, fluctuations of all the oscillating modes $U_{xj,n \neq 0}$ are suppressed and only the static mode $U_{xj,n = 0}$ develops its amplitude. On the other hand, at very low $T$, the oscillating modes as well as the static mode fluctuate randomly. There should be a CD phase transition at some intermediate $T$, that is at $T = T_{CD}^*$. 

Now let us consider the effective gauge model derived in Sect.3.4 for the CS gauge theory coupled with nonrelativistic fermions. The effective gauge couplings $g_e^2$ and $g_m^2$ have strong-$T$ dependence as given by (3.31), in contrast with the usual canonical lattice gauge theory. This is because the present gauge field $U_{xj}$ is not a genuine gauge field, but is a composite, or “bound state” of the “elementary fields” $\phi_x$ and $\eta_x$. It is obvious that at low $T$, $g_e^2 \to 0$ very rapidly, and at high $T$, $g_e^2 \to \infty$. Therefore we expect again a CD transition. Actually, from the explicit form of (3.31) and (4.1), we conclude that the CD phase transition occurs at

$$T_{PFS} \simeq \left( \frac{\rho (1 - \rho)}{2m^2} \right)^{1/2} V_0 + \frac{1}{4} \rho (1 - \rho) (\lambda_2 - \lambda_1).$$

(4.3)

Due to the strong $T$-dependence of $g_e^2$, the confinement phase takes place at $T$ above $T_{PFS}$, and the deconfinement phase, i.e., the PFS, takes place at $T$ lower than $T_{PFS}$. To be able to neglect the higher-order terms, the parameter $\lambda_1 = (2m)^{-1} + g_1$ in (3.27) must be small, which implies that the coefficient $g_1$ of the repulsion between EM charges should be about $g_1 \sim -(2m)^{-1}$. When the repulsion become stronger, $\lambda_1$ become negative and $T_{PFS}$ rises, that is, the PFS is more enhanced. As explained before, this result supports the intuitive physical expectation for the stability of CF’s in the half-filled Landau level. This point can be rephrased as follows; If there were no repulsions between EM charges at all, the assumption of smallness of $\lambda$’s would lose its support. Then there might be no convincing calculations showing that the PFS still takes place. Actually it is quite possible that the PFS disappears at certain

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10Of course the action contains also the magnetic term. In the usual consideration [10] its effect is neglected preferring a simple treatment. It is shown that the magnetic term enhances the deconfinement phase, hence the existence of the CD transition itself remains true.
point as the strength of the repulsion is decreased.\footnote{The extreme case of this possibility is the system of free electrons without repulsions. One may rewrite the system by some constituent operators. But we know that the system cannot exhibit any separation phenomena at all.}

In (4.3), the amplitude $V_0$ itself depends on $T$. Actually, it is a decreasing function of $T$ and so Eq. (4.3) has a unique solution for $T_{\text{PFS}}$. We note that the expression (4.3) is different from the corresponding expression calculated in Ref. [7], reflecting the different treatments of the flux degrees of freedom. For the t-J model based on the slave-boson formalism, we performed systematic numerical calculations for the mean-field amplitudes and determined the transition $T$ of CSS, $T_{\text{CSS}}$, at each hole concentration \footnote{Of course, only the low-lying energy states of linear combinations of these bound states appear as quasi-particles.}. The result shows that the effect of gauge-field fluctuations is so large that $T_{\text{CSS}}$ is reduced to about 10 \% of the mean-field critical temperature $T_V$ which is defined as the $T$ at which the mean field $V_0$ vanishes as in (3.13); thus we have $T_{\text{CSS}} \sim \frac{1}{10} T_V$. We can expect similar behavior for the PFS. Practical calculations of $T_{\text{PFS}}$ and the effective mass of the CF is under study and results will be reported in a forthcoming paper. There we shall also compare $T_{\text{PFS}}$ numerically with that of Ref. [7].

Let us discuss the nature of each phase in some detail. In the confinement phase, as explained in Sect.2, the gauge field fluctuates so strongly that only charge-neutral compounds with respect to (2.19) and (2.31) appear as quasi-excitations. They are bound states of the chargeon and the fluxon like $\eta_x \phi_x$, $\phi_{x+j} U_{xj} \eta_x$, etc. In the system of the half-filled Landau level, they are nothing but the original electrons, which are bound state of the CF and flux quanta. In this phase, the CF’s are not quasi-excitations.

On the other hand, in the deconfinement phase of $U_{xj}$’s, the fluctuations of gauge field are small, so the quasi-particles carry the same quantum numbers as the “elementary” fields that appear in the Hamiltonian and couple with the gauge field.
Therefore in the present system, the quasi-particles are the chargeons, the fluxons and the gauge bosons. We stress that the gauge field itself is an independent degree of freedom and appears as quasi-excitations. Actually, the transverse components of the gauge field is not shielded in contrast with the longitudinal part which is shielded as we have seen in Sect.3.1. The transverse gauge field produces nontrivial effects on the chargeons and fluxons at low energies.

To understand this mechanism, it is convenient to introduce the following variable $\tilde{V}_{xj}$ instead of $V_{xj}$,

$$
\tilde{V}_{xj} = V_{xj} W_{x+j} W_x^\dagger.
$$

(4.4)

Then the hopping terms of chargeons and fluxons in (2.27) are written as

$$
\phi_{x+j}^\dagger \tilde{V}_{xj} W_{x+j}^\dagger W_x \phi_x + \eta_{x+j}^\dagger \tilde{V}_{xj} \eta_x + \text{H.c.}
$$

(4.5)

The field $\tilde{V}_{xj}$ as well as $V_{xj}$ can be regarded as a dynamical gauge field, which is generated as a result of the PFS. If there were no $\phi$-hopping term, the generated gauge field $\tilde{V}_{xj}$ had no correlations with the CS gauge field which represents fluxes attaching fermions; The chargeons $\eta_x$ would be free from any constraints and just move interacting with that “gauge field” whose kinetic term does not exist at the tree level. In the random-phase approximation (RPA) or the renormalization-group study (RGS), a kinetic term of this gauge field appears from the loop effects of matter fields, i.e., chargeon in the present case. Recently, related models of gauge theory coupled with nonrelativistic fermions are studied [9]. It has been shown that fermions exhibit non-Fermi-liquid behavior like the Luttinger liquid in (1+1) dimensions or the marginal Fermi liquid in high-$T_C$ cuprates.

However, in the present case, there exists the first term in (4.5); the gauge field $\tilde{V}_{xj}$ does interact also with the fluxons, and this flux-hopping term gives rise to the correlation between the dynamical gauge field $\tilde{V}_{xj}$ and the CS gauge field, i.e.,

$$
\tilde{V}_{xj} \sim W_x^\dagger W_{x+j}.
$$

(4.6)
In the recent studies of the gauge theory of fermions in the half-filled Landau level \([9, 11, 5]\), one first starts with the CS gauge theory. However, in the RPA or the RGS, the CS constraint is totally ignored. This is essentially due to the technical difficulty in handling the CS constraint. It should be noted that even in the PFS state, the CS constraint is to be partially respected by the real quasi-excitations; the chargeons and fluxons necessarily interact via the gauge field. Such effects are important for quantitative analyses. In the present system, just the fluxons are in charge of the correlations between the dynamical gauge field and the CS fluxes. Then, it is very interesting to study the present gauge theory of chargeons and fluxons by the RPA and/or the RGS (in the continuum) at low or zero \(T\). The local constraint (2.12) can be partially taken into account through the massive Lagrange multiplier \(\lambda_x\). Such analyses are complementary to the studies in this paper by the hopping expansion on the lattice. The hopping expansion assume the smallness of \(\langle U_{xj} \rangle\), the order parameter of PFS, hence reliable at \(T\) near \(T_{\text{PFS}}\). As an example of the importance of such a residual correlation effect, we have a mass of CF. As explained before, the chargeon is the CF in the case of half-filled Landau level. Its mass read off from the Lagrangian is given by

\[
m_{\text{CF}} = \frac{m}{V_0},
\]

and we estimated as \(V_0 = \text{constant of } O(1)\) at low \(T\) in Sect.3.3. There are various kinds of experiments \([3, 12]\) measuring \(m_{\text{CF}}\) through different physical quantities. The estimated values of \(m_{\text{CF}}\) are scattered in a spectrum, \(m_{\text{CF}}/m = O(1) \sim O(10)\). These results should be coherently explained by the residual correlation effects.

5 Conclusion

In this paper, we have studied the CS gauge theory of nonrelativistic fermions. Especially, we are interested in the phenomenon of the PFS. This is very important for the CF approach to the electrons in the half-filled Landau level and also for the
FQHE near the half filling. The main problem of the CS gauge theory is how to treat the CS constraint. To this end, we have introduced the chargeon and the fluxon operators, and expressed the original fermion operator by a bilinear form of them.

To discuss the PFS, we have rewritten the model by using the gauge field, which glues the chargeon and the fluxon. It is shown that the possibility of the PFS or (ir)relevance of the CS constraint is reduced to the possibility of the CD phase transition of that gauge field. We showed by using the hopping expansion that the PFS takes place at low $T < T_{\text{PFS}}$. The repulsion between charges plays a very important role for PFS. This result supports the intuitive physical expectation for the stability of CF’s.

In contrast to the previous approach, the present approach has a new operator, the fluxon $\phi_x$, which opens a possibility to describe the higher-order effects of CS constraint, such as the mass of CF near the half filling. More generally, it is certainly an improvement that a field-theoretical description is now possible for the dynamics of CS flux degrees of freedom as well as of CS charge degrees of freedom. Also, as pointed out, the chargeon-fluxon approach reveals the strong resemblance to the slave-boson or fermion approach to the t-J model. This opens a possibility of a more coherent and universal understanding of these separation phenomena. We will return to these topics in future.
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Table 1. Quantum numbers carried by various elementary fields, $C_x^\dagger$, $\psi_x^\dagger (= \eta_x^\dagger \phi_x^\dagger)$, $\eta_x^\dagger$, $\phi_x^\dagger$. In the confinement phase of the gauge dynamics of $V_{xj}$ at $T > T_{\text{PFS}}$, only the neutral objects of $V$ charge, like $\psi_x$ ($C_x$), appear as physical excitations. In the deconfinement phase at $T < T_{\text{PFS}}$, the PFS is realized and $V$-charged objects, like $\eta_x$, $\phi_x$, can apper. When fluxons $\phi_x$ Bose-condense, $\phi_x \simeq \sqrt{\rho}$ (at $T = 0$), two candidates for CF’s, $\psi_x^\dagger$ and $\eta_x^\dagger$, become indistinguishable in the leading order.
Figure Captions.

**Fig.1**: Illustrations of the key concepts and objects appeared in each step to reach $L_{\eta\phi V}$ of (2.27).

**Fig.1a**: Illustration of a system of electrons under an external magnetic field $B^{ex}$ in the $z$-direction. Each black bullet represents an electron, and straight lines with arrows represent $B^{ex}$.

**Fig.1b**: Illustration of Eq.(2.7). Each electron $C_x^\dagger$ is represented as a product of a fermion $\psi_x^\dagger$ and the operator $\tilde{W}_x^\dagger \equiv \exp[-iq\sum \theta(x - y)\hat{\rho}_y]$. The latter represents $q$ units of CS fluxes in the *negative* $z$-direction, which are represented by wavy lines with arrows. The sources of these fluxes are the fermions themselves as shown in Eq.(2.4).

**Fig.1c**: Illustration of Eq.(2.10). Each fermion $\psi_x^\dagger$ is a composite of a fluxon $\phi_x^\dagger$ and a chargeon $\eta_x^\dagger$. A broad arrow represents a fluxon which carries $q$ units of CS fluxes in the $z$-direction, the sources of which are nothing but the fluxons themselves as shown in Eq.(2.13). An open circle represents a chargeon which carries CS and EM charges. When the PFS phenomenon takes place, $\psi_x^\dagger$ dissociates into $\phi_x^\dagger$ and $\eta_x^\dagger$.

**Fig.1d**: Illustration of the system of Fig.1a in the chargeon-fluxon representation. As seen from Eq.(2.27), the chargeons move in the statistical potential $\Delta A_{xj} \equiv eA_{xj}^{ex} - A_{xj}^{CS}$ described by $W_{xj}$ and, at the same time, in the gauge field $V_{xj}$. ($V_{xj}$ has not been drawn explicitly.) We depicted the magnetic field $\Delta B$ corresponding to this $\Delta A_{xj}$ by black broad arrows, and put them on each fluxon since its local part $B^{CS}$ sits at each fluxon as shown by Eq.(2.13). The fluxons have minimal interactions with $V_{xj}$. In the PFS state, cancellation between $eA^{ex}$ and $A^{CS}$ works well (e.g., perfectly in the leading order of Bose condensation, $\phi_x^\dagger \phi_x = \rho$), and the fluctuation effects by $\Delta A_{xj}$ can be treated legitimately as a perturbation.