UNIMODALITY OF ORDINARY MULTINOMIALS AND MAXIMAL PROBABILITIES OF CONVOLUTION POWERS OF DISCRETE UNIFORM DISTRIBUTION

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Abstract

We establish the unimodality and the asymptotic strong unimodality of the ordinary multinomials and give their smallest mode leading to the expression of the maximal probability of convolution powers of the discrete uniform distribution. We conclude giving the generating functions of the sequence of generalized ordinary multinomials and for an extension of the sequence of maximal probabilities for convolution power of discrete uniform distribution.

Since the eighteenth century, the expression of the convolution power of the discrete uniform distribution has been very well known (e.g. de Moivre in 1711, see [11] 3rd ed., 1756] or [12, 1731]). This probability distribution arises in many practical situations including, in particular, games with equal chance, random affectation of tasks for many servers, and random walks. It is well known, see Dharmadhikari & Joak-Dev [7, 1988, p. 108-109.], that the convolution of two discrete unimodal distributions may be non unimodal. However, if these distributions are symmetric, we obtain a symmetric unimodal distribution. It is a discrete analog of Wintner’s Theorem [19, 1938]. Knowing that the convolution power of the discrete uniform distribution is symmetric unimodal, the determination of the maximal probability (mode) of such a distribution and its argument remains a question for consideration. As a recent work on the problem, one can see the article by Mattner & Roos [9, 2007] where they establish the upper bound for the maximal probability $c_{q,L} < \sqrt{6/\pi q(q + 2)} L$ ($c_{q,L}$ being the maximal probability of the $L$-th convolution power of the discrete uniform distribution on $\{0, 1, \ldots, q\}$). Before them, there were several works aiming at finding such a bound. For example, Siegmund-Schultze & von Weizsäcker [15, 2007] proved the existence of a constant $A$ such that $c_{q,L} < A/(q + 1) \sqrt{L}$ and gave an application of these upper bounds in the construction of a polygonal recurrence of a two-dimensional random walk.

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Alternatively, our aim is to give an explicit expression of the mode of the $L$-th convolution powers of the discrete uniform distribution (section two) by means of the unimodality of the ordinary multinomials for which we study also the strong unimodality (section one), we end the paper (section three) by giving the generating functions for the two sequences of generalized ordinary multinomials: $\{(\alpha)^q\}_n$ and $\{(\alpha^2)^q\}_n$, $\alpha \in \mathbb{C}$, and thus of $\{c_{q,2n/q}\}_n$, when $q$ is even.

1 Unimodality of ordinary multinomials

The ordinary multinomials are a natural extension of binomial coefficients (see [1, 2007] for a recent overview on ordinary multinomials). Letting $q, L \in \mathbb{N}$, for an integer $k = 0, 1, \ldots, qL$, the ordinary multinomial $\binom{L}{k}_q$ is the coefficient of the $k$-th term of the following multinomial expansion

$$\left(1 + x + x^2 + \cdots + x^q\right)^L = \sum_{k \geq 0} \binom{L}{k}_q x^k.$$  \hspace{1cm} (1)

with $\binom{L}{k}_1 = \binom{L}{k}$ (being the usual binomial coefficient) and $\binom{L}{k}_q = 0$ for $k > qL$. Using the classical binomial coefficient, one has

$$\binom{L}{k}_q = \sum_{j_1 + j_2 + \cdots + j_q = a} \binom{L}{j_1}_1 \binom{j_1}{j_2} \cdots \binom{j_q - 1}{j_q}.$$  \hspace{1cm} (2)

Readily established properties are the symmetry relation

$$\binom{L}{k}_q = \binom{L}{qL - k}_q.$$  \hspace{1cm} (3)

and the recurrence relation

$$\binom{L}{k}_q = \sum_{m=0}^{q} \binom{L-1}{k-m}_q.$$  \hspace{1cm} (4)

As an illustration of the latter recurrence relation, we give the triangles of pentanomial and hexanomial coefficients which are just an extension, well known in the combinatorial literature, of the standard Pascal triangle.

Table 1: Triangle of pentanomial coefficients: $\{\binom{L}{k}_4\}_q$

| $L \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0               | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1               | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |
| 2               | 1  | 2  | 3  | 4  | 5  | 4  | 3  | 2  | 1  |    |    |    |    |    |
| 3               | 1  | 3  | 6  | 10 | 15 | 18 | 19 | 18 | 15 | 10 | 6  | 3  | 1  |    |
| 4               | 1  | 4  | 10 | 20 | 35 | 52 | 68 | 80 | 85 | 80 | 68 | 52 | 35 | 20 |
| 5               | 1  | 5  | 15 | 35 | 70 | 121| 185| 255| 320| 365| 381| 365| 320| 255|...|
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Table 2: Triangle of hexanomial coefficients: \( \binom{L}{k}_5 \)

| L \( \text{mod} \) 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| 0                     | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   | 1   | 1   | 1   |
| 1                     | 1 | 3 | 6 | 10 | 15 | 21 | 25 | 27 | 27 | 25 | 21 | 15 | 10 | 6   |
| 2                     | 1 | 4 | 10 | 20 | 35 | 56 | 80 | 104 | 125 | 140 | 146 | 140 | 125 | 104 | 80 |
| 3                     | 1 | 5 | 15 | 35 | 70 | 126 | 205 | 305 | 420 | 540 | 651 | 735 | 780 | 780 | 735 |

Let us investigate the unimodality of the sequence \( \{ L_k^q \}_{k=0}^\infty \). A finite sequence of real numbers \( \{ a_k \}_{k=0}^m \) is called unimodal if there exists an integer \( l \in \{ 0, \ldots, m \} \) such that the subsequence \( \{ a_k \}_{k=0}^l \) increases, while \( \{ a_k \}_{k=0}^m \) decreases. If \( a_0 \leq a_1 \leq \cdots \leq a_l \leq \cdots \leq a_m \) then the integers \( l_0, \ldots, l_1 \) are the modes of \( \{ a_k \}_{k=0}^m \). In the case where \( l_0 = l_1 \) we talk about a peak, otherwise the set of the values of the mode is called plateau. For positive non increasing and non decreasing sequences, unimodality is implied by log-concavity. A sequence \( \{ a_k \}_{k=0}^m \) is said to be logarithmically concave (log-concave for short) or strongly unimodal if \( a_{2l} \geq a_{l-1}a_{l+1} \), \( 1 \leq l \leq m-1 \). Also if the sequence is strictly log-concave (SLC for short), i.e. if the previous inequalities are strict, then the sequences have at most two consecutive modes (a peak or a plateau). For these notions, one can see Belbachir & Bencherif [3, 2007], Belbachir & al [2, 2007], Bertin & Theodorescu [4, 1984], Brenti [5, 1994], Comtet [6, 1970], Dharmadhikari & Joak-Dev [7, 1988], Keilson & Gerber [8, 1971], Medgyessy [10, 1972], Sagan [14, 2007], Stanley [17, 1986] and [16, 1989] and Tanny & Zuker [18, 1974]. In the following, \( \lfloor a \rfloor \) denotes the greatest integer in \( a \).

The first main result of this article is the following.

**Theorem 1** Let \( q \geq 1 \) and \( L \geq 0 \) be integers. Then the sequence \( \{ \binom{L}{k}_q \}_{k=0}^L \) is unimodal and its smallest mode is given by

\[
k_L := \arg \max_k \binom{L}{k}_q = \lfloor (qL + 1)/2 \rfloor.
\]

Furthermore, we have the following recurrence relation

\[
\binom{L}{k}_q = \sum_{i \in I_q} \binom{L-1}{k_{L-1} + i}_q,
\]

where

\[
I_q = \begin{cases} 
\{ -q/2, \ldots, q/2 \} & \text{if } q \text{ is even}, \\
\{ -(q+1)/2, \ldots, (q-1)/2 \} & \text{if } q \text{ and } L \text{ are odd}, \\
\{ -(q-1)/2, \ldots, (q+1)/2 \} & \text{otherwise}.
\end{cases}
\]

**Proof.** It suffices, for each one of the two cases: \( q \) odd and \( q \) even, to proceed by induction over \( L \) using the recurrence relation (4). \( \square \)

**Remark 2** For odd \( qL \) we have a plateau of two modes: \( qL/2 \) and \( qL/2 + 1 \). Otherwise we have a peak: \( (qL + 1)/2 \).
2 Determining the maximal probability for convolution powers of discrete uniform distribution

We are now able to achieve our purpose: the expression of the maximal probability of the \( L \)-th convolution powers of the discrete uniform distribution. Let \( U_q \) be the random variable of the discrete uniform distribution on \( \{0, 1, \ldots, q\} \) and let \( U_q^{*L} \) be its \( L \)-th convolution powers:

\[
U_q := \frac{1}{q + 1} (\delta_0 + \delta_1 + \cdots + \delta_q) \quad (\delta_n \text{ is the Dirac measure}).
\]

In [1, 2007], Belbachir and al. established a link between the ordinary multinomials and the density probability of convolution powers of discrete uniform distribution. With respect to the counting measure, such a density is given by

\[
P(U_q^{*L} = k) = \binom{L}{k}_q \frac{1}{(q + 1)^L}, \quad k = 0, 1, \ldots, qL.
\]

**Remark 3** From Odlyzko and Richmond [13, 1985] we know that for \( L \) sufficiently large, the sequence of probabilities \( \{P(U_q^{*L} = k)\}_k \) is strongly unimodal, from which we easily deduce that the sequence \( \{(\binom{L}{k}_q)\}_k \) is also asymptotically strongly unimodal.

**Conjecture 4** For each positive integer \( q \), the sequence \( \{(\binom{L}{k}_q)\}_k \) is SLC.

From Theorem 1 as second main result, we give the values of \( c_{q,L} := \max_k \binom{L}{k}_q / (q + 1)^L \).

**Theorem 5** The maximal probability of the \( L \)-th convolution power of the discrete uniform distribution over \( \{0, 1, \ldots, q\} \) is

\[
c_{q,L} = \frac{1}{(q + 1)^L} \left( \binom{L}{\lfloor (qL + 1)/2 \rfloor}_q \right).
\]

3 Some generating functions

As a third main result, we give the generating functions for the sequence of generalized ordinary multinomials, the sequences \( \{\binom{z}{n}_q\}_n \) and \( \{\binom{nz}{n}_q\}_n \), \( z \in \mathbb{C} \), and the extended sequence of maximal probabilities for convolution power of discrete uniform distribution: \( \{c_{q,2n/q}\}_n \).

**Definition 6** For \( z \in \mathbb{C} \), we define the generalized ordinary multinomials, as follows

\[
\binom{z}{k}_q := \sum_{k_1+k_2+\cdots+k_q=k} \frac{z(z-1)\cdots(z-k_1+1)}{(k_1-k_2)! (k_2-k_3)! \cdots (k_{q-1}-k_q)! k_q!}.
\]

This definition is motivated by the relation (5).
Lemma 7  We have the following inequality
\[
\sum_{k_1+\cdots+k_q=k} \frac{z}{(1-k_2)!\cdots(k_q-1)!k_q!} z(n-1)\cdots(z-k_1+1) = \sum_{h_1+2h_2+\cdots+qh_q=k} \frac{z}{h_1!h_2!\cdots h_q!} \frac{z}{k_1!k_2!\cdots k_q!}.
\]

Theorem 8  Let \( z \in \mathbb{C} \), the generating function for generalized ordinary multinomials is given by
\[
\sum_{n \geq 0} \binom{z}{n} t^n = (1 + t + t^2 + \cdots + t^q)^n.
\]

Proof.  Using the Lemma, we have \( \sum_{n \geq 0} \binom{z}{n} t^n = \sum_{h_1+2h_2+\cdots+qh_q=n} \binom{z}{h_1} \frac{m!}{h_1!h_2!\cdots h_q!} t^n. \)

On the other hand
\[
(1 + t + t^2 + \cdots + t^q)^z = \sum_{m \geq 0} \binom{z}{m} (1 + t + t^2 + \cdots + t^q)^m
= \sum_{m \geq 0} \binom{z}{m} \sum_{l_1+l_2+\cdots+l_q=m} \frac{m!}{l_1!l_2!\cdots l_q!} t^{l_1+2l_2+\cdots+ql_q}.
\]

We conclude by summation over \( n \geq 0 \) is equivalent to summation over \( m \geq 0 \). \( \square \)

Remark 9  Problem 19 of Comtet [6], Vol.1, p. 172, states that
\[
\sum_{n \geq 0} \binom{z}{n} (1 + t + t^2)^n = (1 - 2x - 3x^2)^{-\frac{1}{2}},
\]
using the fact that the coefficient of \( t^n \) in the development of \((1 + t + t^2)^n : \binom{z}{n} (1 + t + t^2)^n = \binom{n}{2} \) is \( \max_k \binom{k}{2} \), we obtain the following combinatorial identity
\[
G_2(t) := \sum_{n \geq 0} c_{2,n} t^n = \left(1 + t^3\right)^{-1/2} (1 - t)^{-1/2}.
\]

This last identity can be shown as the generating function of the sequence \( \{c_{2,n}\}_n \).

Theorem 10  Let \( z \in \mathbb{C} \), the generating function of the sequence \( \{\binom{n}{q}\}_{n} \) is given by
\[
\sum_{n \geq 0} \binom{n}{q} \frac{z^n}{n!} = u \left(1 - z \frac{u + 2u^2 + \cdots + qu^q}{1 + u + u^2 + \cdots + u^q}\right)^{-1},
\]
where \( u \) is a solution of the equation \( t = u \left(1 + u + u^2 + \cdots + u^q\right)^{-z} \).

Proof.  Use Hermite’s Theorem [6] for the function \( t \mapsto t \left(1 + t + \cdots + t^q\right)^{-z} \). \( \square \)
Theorem 11 For \( q \) even, the generating function of the sequence \( \{c_{q,2n/q}\}_n \) is given by

\[
G_q(t) := \sum_{n \geq 0} t^n c_{q,2n/q} = \left(1 - \frac{2}{q} \frac{u + 2u^2 + \cdots + qu^q}{1 + u + u^2 + \cdots + u^q}\right)^{-1} = \frac{\left(1 + \sum_{k=1}^{q/2} (u^{-k} + u^k)\right)}{\sum_{k=1}^{q/2} k (u^{-k} - u^k)},
\]

where \( u \) is a solution of the equation

\[
t = u \left(\frac{q + 1}{1 + u + u^2 + \cdots + u^q}\right)^{2/q} = \left(\frac{q + 1}{1 + \sum_{k=1}^{q/2} (u^{-k} + u^k)}\right)^{2/q}.
\]

Proof. Use the above Theorem for \( z = 2/q \), and the change of variable \( t \to (q + 1) t \). \( \square \)

Remark 12 The sequence \( \{c_{q,2n/q}\}_n \) contains strictly the subsequence \( \{c_{q,L}\}_L \).

Corollary 13 For \( q = 4 \), the generating function of \( \{c_{4,n/2}\}_n \) is given for \( t \in [-\sqrt{5}, 1] \) by

\[
G_4(t) := \sum_{n \geq 0} t^n c_{4,n/2} = \left(1 - \frac{1}{4} t^2 - 1 - \frac{1}{200} t (5t^2 + 20)\right)^{-1/2}.
\]

Corollary 14 We have the following identities

\[
\sum_{n \geq 0} (-5)^n \left(\frac{n/2}{n}\right)_4 = 2 \quad \text{and} \quad \sum_{n \geq 0} (-1)^n c_{4, n/2} = 2/\sqrt{5}.
\]

Remark 15 The generating function of the sequence \( \{c_{4,n}\}_n \) is given for \( t \in [-1, 1] \) by

\[
\sum_{n \geq 0} t^n c_{4,n} = (G_4(\sqrt{|t|}) + G_4(-\sqrt{|t|}))/2.
\]

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