APPLICATION OF HARMONIC MAPS $CP^{(N-1)}$ ON $SU(N)$ BOGOMOLNY EQUATION FOR BPS MAGNETIC MONOPOLES

THESIS

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in memory of Hans Jacobus Wospakrik...
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Abstract

In this thesis we study dynamic of magnetic monopoles from Lagrangian density in Yang-Mills-Higgs field theory. In particular, we discuss BPS (Bogomolny Prasad Sommerfield) magnetic monopoles, described by $SU(N)$ Bogomolny equations, which has field equations in form of non-linear coupled matrix field equations. One of the methods to simplify $SU(N)$ Bogomolny equations is by using harmonic maps $\mathbb{CP}^{(N-1)}$. This method has relation with $Gr(n,N)$ $\sigma$-model and can transform $SU(N)$ Bogomolny equation into more simple scalar field equations that depends only on one variable. As an example, we consider the case of $SU(2)$ Bogomolny equation.
4.2 $SU(2)$ Bogomolny Equation

5 Conclusions

A Riemann Sphere

B Derivation of Equations (3.40) and (3.41)

C Derivation of Equations (3.49) and (3.50)

D Derivation of Properties (3.59) and (3.60)

E Derivation of Equations (3.82) and (3.84)
Chapter 1

Introduction

Magnetic monopoles concept was first introduced by P.A.M. Dirac while he tried to search for explanation about the unit of electronic charge $e$ \cite{1}. His research was based on the fact that electric charge is always observed in integral multiples of the electronic charge $e$. This electronic charge $e$ has made $\frac{\hbar}{2e}$ approximately 137 which then became his focus of research and some others physicist, one of them was A.S. Eddington. In his research development, Dirac considered some arguments but unfortunately did not lead him to any value of 137 and for that reason he felt that his arguments were a failure. Instead, the result of his research, which he wrote it in a paper titled by Quantized Singularities in The Electromagnetic Field, born the new idea of magnetic monopoles. It was the concept that for years later interest many scientists with capable of wide generalizations.

In the following years, many papers have been published in the topic of magnetic monopoles. Between year 1973 through 1976, there were more than 300 research papers on the subject of magnetic monopoles \cite{2}. A primary contribution for the theoretical investigations was provided by ’t Hooft and Polyakov in 1974 that they discovered a magnetic monopoles solution in a spontaneously broken non-Abelian gauge theory \cite{3}. Their work pointed out the natural manner in which magnetic monopoles make their appearance in these theories and encouraged further exploration of this phenomenon.

Many studies about magnetic monopoles in non-Abelian gauge theory were motivated by two sources. First, if magnetic monopoles solitons occurs in a unified gauge theory of the weak and electromagnetic interactions, which then the theory describes the real world proved to be correct, then they will be experimentally accessible even though at extremely high energies. In this case, there must be available information about these soliton of magnetic monopoles field configurations. Second, there
may be a connection between magnetic monopoles and the quark confinement mechanism. These magnetic monopoles would probably be counterparts of some charge other than electric charge and therefore be expected to have little connection with the magnetic monopoles sought in the experiments [4].

In this thesis, we describe how to develop $SU(N)$ Bogomolny equations for BPS magnetic monopoles from Yang-Mills-Higgs field theory and applying a mathematical device, harmonic maps $CP^{(N-1)}$, to transform the equations in the more simple form that can be used for further methods to find the solutions. In the first chapter, we discuss the Yang-Mills-Higgs field in Riemann sphere and derive its dynamics equations. Then by using its energy equation, we take some conditions and Bogomolny analysis to get $SU(N)$ Bogomolny equations for BPS magnetic monopoles. For Chapter two, we investigate on harmonic maps and how its connection with $Gr(n,N)$ $\sigma$-model. We take special case for $CP^{(N-1)}$ space of $Gr(n,N)$ $\sigma$-model and derive many of its properties and use also a Veronese map as we will used for next chapter to obtain the solutions. Chapter three is the major work of this thesis where we use ansatz for $SU(N)$ Bogomolny equations in chapter one to connect with harmonic maps method in chapter two. With that ansatz, we can transform the $SU(N)$ Bogomolny equations form its matrix form into scalar equations which are much more simple to be discussed. We also take an example for $SU(2)$ Bogomolny equations and find its solutions. The last chapter is the conclusion about the results that we get in the previous chapters.
Chapter 2

Magnetic Monopoles

2.1 Dynamics of Yang-Mills-Higgs Field

In searching for quantization of a unit of electric charge $e$, Dirac starts his attempt by writing Maxwell equations in matter (with electric charge and current source) into symmetrical form between electric and magnetic field. This is based on the fact that in vacuum conditions, Maxwell equations has symmetrical form between its fields. So, why it does not occur in the same way as we move into Maxwell equations in matter. For that reason, Dirac introduces a magnetic charge and current source in his version of Maxwell equations \[1\]. Next, this idea was extended in more general because as we know that Maxwell equations in vacuum basically can be derived from pure $U(1)$ Yang-Mills-Higgs field theory.

In this section, we discuss about magnetic monopoles which is derived from Yang-Mills-Higgs field theory with Lagrangian density given by \[5\]

$$
\mathcal{L} = \frac{1}{8} \text{tr} (F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} \text{tr} (D_\mu \Phi D^\mu \Phi) - \frac{\lambda}{8} (\|\Phi\|^2 - 1)^2 \tag{2.1}
$$

where $A_\mu$ is the gauge field and $\Phi$ is a Higgs field, as sources for magnetic monopoles, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and covariant derivative $D_\mu = \partial_\mu + [A_\mu]$. We also write the Higgs field in $\Phi = i \Phi^a T_a$ which is a skew Hermitian form so that $\|\Phi\|^2 = -\frac{1}{2} \text{tr} (\Phi\Phi)$ and with $\text{tr} (T_a T_b) = 2 \delta_{ab}$ ($T_a$ is generator of Lie group $SU(N)$). For this chapter, we use index convention $\mu, \nu = 1, 2, 3$ and $a, b = 1, \ldots, \text{dim}[SU(N)]$. If we look at the equation (2.1), we recognize that on the first part of equation (2.1) is for electric and magnetic field while the second part is for the sources of magnetic monopoles and then the third part is a Lagrange multiplier that comes from constraint condition $\|\Phi\|^2 = 1$. In order to write dynamics equations from Lagrangian density
above, we have to take variation of action to each of its fields. As variation of action is wrote by

\[ \delta S = \int \delta \mathcal{L} \, d^4x \]  

(2.2)

If we look for dynamics equations of \( A_\mu \) field then we take variation of \( A_\mu \) to field to the equation (2.2) and action on Lagrangian becomes

\[ \delta \mathcal{L} = \frac{1}{4} tr(\partial_\mu (F^\mu_\nu \delta A_\nu) - \partial_\nu (F^\nu_\mu \delta A_\mu) - \partial_\mu F^\mu_\nu \delta A_\nu + \partial_\nu F^\nu_\mu \delta A_\mu - F^\mu_\nu [A_\nu, \delta A_\mu] 
+ F^\mu_\nu [A_\mu, \delta A_\nu]) + \frac{1}{2} tr(D^\mu \Phi [\Phi, \delta A_\mu]) \]  

(2.3)

Because we have boundary condition for integral on surface \( \delta A_\mu = 0 \), then the second part of equation (2.3) can be ignored so that

\[ \delta \mathcal{L} = \frac{1}{2} tr((D_\mu F^\mu_\nu + [D^\mu \Phi, \Phi])\delta A_\mu) \]  

(2.4)

from least action principal \( \delta S = 0 \), then dynamics equation for fields \( A_\mu \) are

\[ D_\mu F^\mu_\nu + [D^\mu \Phi, \Phi] = 0 \]  

(2.5)

While for \( \Phi \) field, we take variation of \( \Phi \) field to the equation (2.2) and the action on Lagrangian becomes

\[ \delta \mathcal{L} = -\frac{1}{2} tr(\partial_\mu (D^\mu \Phi \delta \Phi) - \partial_\mu D^\mu \Phi \delta \Phi + (D^\mu \Phi A_\mu \delta \Phi - D^\mu \Phi \delta \Phi A_\mu)) 
+ \frac{\lambda}{4}(||\Phi||^2 - 1)tr(\Phi \delta \Phi) \]  

(2.6)

and also for integral on surface we have boundary condition \( \delta \Phi = 0 \), then the second part of equation (2.6) can be ignored so that

\[ \delta \mathcal{L} = \frac{1}{2} tr((D_\mu D^\mu \Phi + \frac{\lambda}{2}(||\Phi||^2 - 1)\Phi)\delta \Phi) \]  

(2.7)

From least action principal \( \delta S = 0 \), then dynamics equation of \( \Phi \) field is

\[ D_\mu D^\mu \Phi + \frac{\lambda}{2}(||\Phi||^2 - 1)\Phi = 0 \]  

(2.8)

As we can see in equation (2.5) which is similar to Maxwell equations in matter with current sources is in form of field \( \Phi \).
2.2 Energy Equation for Yang-Mills-Higgs Field

Now, let we find energy equation for Lagrangian (2.1) and then use it to get $SU(N)$ Bogomolny equations. To do that, we have to count the tensor energy-momentum for Yang-Mill-Higgs field by rewriting the action of Lagrangian (2.1) for arbitrary metric

\[
S = \int \sqrt{-g} \mathcal{L} \, d^4x
\]  

(2.9)

where $g = det(g_{\mu\nu}) = e^{tr(ln g_{\mu\nu})}$. Then, we take variation over $g_{\mu\nu}$ to the action (2.9) so that

\[
\delta S = \int (\delta(\sqrt{-g})\mathcal{L} + \sqrt{-g}\delta\mathcal{L}) d^4x
\]  

(2.10)

with

\[
\delta \sqrt{-g} = \frac{1}{2} (-g)^{-\frac{1}{2}} (-\delta g)
\]

\[
\delta g = e^{tr(ln g_{\mu\nu})} tr \left( \frac{1}{g_{\mu\nu}} \delta g_{\mu\nu} \right) = g \ tr (g^{\mu\nu} \delta g_{\mu\nu})
\]  

(2.11)

If we write in its components $\delta g = g^{\mu\nu} \delta g_{\mu\nu}$, so that

\[
\delta \sqrt{-g} = \frac{1}{2} (-g)^{-\frac{1}{2}} g^{\mu\nu} \delta g_{\mu\nu}
\]  

(2.12)

While

\[
\delta \mathcal{L} = \frac{1}{4} g^{\nu\sigma} tr (F_{\rho\sigma} F_{\mu\nu}) \delta g^{\mu\rho} - \frac{1}{4} tr (D_\rho \Phi D_\mu \Phi) g^{\mu\rho}
\]  

(2.13)

and using $\delta g^{\mu\rho} = -g^{\mu\alpha} \delta g_{\alpha\beta} g^{\beta\rho}$, then

\[
\delta \mathcal{L} = \left( -\frac{1}{4} g^{\nu\sigma} tr (F_{\rho\sigma} F_{\mu\nu}) + \frac{1}{4} tr (D_\rho \Phi D_\mu \Phi) \right) g^{\mu\alpha} g^{\beta\rho} \delta g_{\alpha\beta}
\]  

(2.14)

Substitute equations (2.12) and (2.14) into equation (2.10), then

\[
\delta S = -\frac{1}{2} \sqrt{-g} \int (-g^{\alpha\beta} \mathcal{L} + \frac{1}{2} (g^{\nu\sigma} tr (F_{\rho\sigma} F_{\mu\nu}) - tr (D_\rho \Phi D_\mu \Phi)) g^{\mu\alpha} g^{\beta\rho} \delta g_{\alpha\beta} d^4x
\]  

(2.15)

From general relativity for action of matter is $\delta S = -\frac{1}{2} \sqrt{-g} \int T^{\alpha\beta} \delta g_{\alpha\beta} d^4x$ so we have tensor energy-momentum

\[
T^{\alpha\beta} = -g^{\alpha\beta} \mathcal{L} + \frac{1}{2} (g^{\nu\sigma} tr (F_{\rho\sigma} F_{\mu\nu}) - tr (D_\rho \Phi D_\mu \Phi)) g^{\mu\alpha} g^{\beta\rho}
\]  

(2.16)

and take its energy part $T^{00} = -g^{00} \mathcal{L} + \frac{1}{2} (g^{\nu\sigma} tr (F_{\rho\sigma} F_{\mu\nu}) - tr (D_\rho \Phi D_\mu \Phi)) g^{\mu0} g^{00}$. From definition, we know that the energy is written by $E = \int T^{00} \sqrt{g} d^3x$ for arbitrary metric. Below, we write some metrics where $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$:
1. Standard Minkowskian metric
\[ ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \] (2.17)

2. Minkowskian metric with spherical coordinates
\[ ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \] (2.18)

3. Minkowskian metric with Riemann sphere coordinates
\[ ds^2 = dt^2 - dr^2 - \frac{2r^2}{(1 + |\xi|^2)^2} (d\xi d\bar{\xi} + d\bar{\xi} d\xi) \]
\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2r^2}{(1 + |\xi|^2)^2} \\ 0 & 0 & -\frac{2r^2}{(1 + |\xi|^2)^2} & 0 \end{pmatrix} \] (2.19)

derivation of metric (2.19) is in Appendix A.

So, we have the energy part \( T^{00} \) with metric (2.19) is
\[ T^{00} = -\mathcal{L} + \frac{1}{2} g^{i\sigma} tr(F_{0\sigma}F_{0\nu}) - \frac{1}{2} tr(D_0 \Phi D_0 \Phi) \] (2.20)

### 2.2.1 Static field and BPS condition

For this thesis, we will only look for solutions of static fields and in BPS limit. In that case, we have for static field \( (\partial_0 \Phi = 0, A_0 = 0) \) and with BPS condition on \( \lambda = 0 \), then equation (2.20) becomes
\[ T^{00} = -\mathcal{L} + \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} tr(F_{\rho\sigma}F_{\mu\nu}) + \frac{1}{2} g^{ij} tr(F_{0i}F_{0j}) + \frac{1}{4} g^{ij} tr(D_i \Phi D_j \Phi) \] (2.21)

where we use index \( i \) and \( j = 1, 2, 3 = r, \bar{\xi}, \xi \) and for index \( \mu, \nu, \rho \) and \( \sigma = 0, 1, 2, 3 = t, r, \xi \bar{\xi} \). We can calculate equation (2.21) separately for each field. For the first one, we calculate the part that contains \( \Phi \) field which is written as below
\[ \frac{1}{4} g^{ij} tr(D_i \Phi D_j \Phi) = -\frac{1}{4} tr(D_i \Phi D_i \Phi + \frac{(1 + |\xi|^2)^2}{r^2} D_\xi \Phi D_\bar{\xi} \Phi) \] (2.22)

While for the others part that bear \( A_\mu \) field which are
\[ -\frac{1}{8} g^{\mu\rho} g^{\nu\sigma} tr(F_{\rho\sigma}F_{\mu\nu}) + \frac{1}{2} g^{ij} tr(F_{0i}F_{0j}) \] (2.23)

we can separate it again as below
\[ T^{00} = -\frac{1}{4} tr \left( D_r \Phi D_r \Phi + \frac{(1 + |\xi|^2)^2}{r^2} D_\xi \Phi D_\xi \Phi \right) + \frac{1}{4} g^{ij} tr(F_{0i} F_{0j}) - \frac{1}{8} g^{ik} g^{jl} tr(F_{kl} F_{ij}) \]

(2.24)

### 2.2.2 Magnetic monopoles conditions

For magnetic monopoles it means that the sources only produces magnetic field or we may write \( F_{0i} = 0 \) (if \( F_{0i} \neq 0 \) then it is known as dyons, it means the sources produces two fields which are magnetic and electric field). With this condition then equation (2.24) becomes

\[ T^{00} = -\frac{1}{4} tr \left( D_r \Phi D_r \Phi + \frac{2r^2}{(1 + |\xi|^2)^2} D_\xi \Phi D_\xi \Phi + 2 D_\xi \Phi D_\xi \Phi + 2 F_{r\xi} F_{r\xi} + \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\xi} F_{\xi\xi} \right) \]

(2.25)

So, we have equation of energy for BPS magnetic monopoles as below

\[ E = -\frac{1}{4} \int tr \left( \frac{2r^2}{(1 + |\xi|^2)^2} D_r \Phi D_r \Phi + 2 D_\xi \Phi D_\xi \Phi + 2 F_{r\xi} F_{r\xi} + \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\xi} F_{\xi\xi} \right) d^3 x \]

(2.26)

### 2.3 Magnetic Monopoles Equations of Motion

From the energy (2.26), we can derive equations of motion for each field by using variation of energy about each field and least action principal \( \delta E = 0 \). So, for \( \Phi \) field, the variation of the energy (2.26) is

\[ \delta E = -\frac{1}{4} \int tr \left( \frac{2r^2}{(1 + |\xi|^2)^2} D_r \Phi \delta \Phi \right) - \partial_r \left( \frac{2r^2}{(1 + |\xi|^2)^2} D_r \Phi \right) \delta \Phi \]
+ \frac{2r^2}{(1 + |\xi|^2)^2} (D_r \Phi, A_r) \delta \Phi + \partial_\xi (D_\xi \Phi \delta \Phi) - \partial_\xi (D_\xi \Phi) \delta \Phi + [D_\xi \Phi, A_\xi] \delta \Phi \\
+ \partial_\xi (D_\xi \Phi \delta \Phi) - \partial_\xi (D_\xi \Phi) \delta \Phi + [D_\xi \Phi, A_\xi] \delta \Phi \right) d^3 x 
(2.27)

Because we have boundary condition for surface integral \( \delta \phi = 0 \), then

\[ \delta E = -\frac{1}{4} \int tr \left( -D_r \left( \frac{2r^2}{(1 + |\xi|^2)^2} D_r \Phi \right) \right. \delta \Phi - D_\xi (D_\xi \Phi) \delta \Phi - D_\xi (D_\xi \Phi) \delta \Phi \right) d^3 x 
(2.28)

and equation of motion of \( \Phi \) field is

\[ -D_r \left( \frac{2r^2}{(1 + |\xi|^2)^2} D_r \Phi \right) - D_\xi (D_\xi \Phi) - D_\xi (D_\xi \Phi) = 0 
(2.29)

\[ D_r (r^2 D_r \Phi) = -\frac{(1 + |\xi|^2)^2}{2} (D_\xi (D_\xi \Phi) + D_\xi (D_\xi \Phi)) 
(2.30)

With the same manner for \( A_\xi \) field, then variation of the energy becomes

\[ \delta E = -\frac{1}{4} \int tr \left( -[D_\xi \Phi, \Phi] \delta A + \partial_\xi (F_{r\xi} \delta A) - \partial_r (F_{r\xi} \delta A) + [F_{r\xi}, A_r] \delta A \right. \\
- \partial_\xi \left( \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi \xi} \delta A \right) + \partial_\xi \left( \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi \xi} \right) \delta A \\
- \left[ \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi \xi}, A_\xi \right] \delta A \right) d^3 x 
(2.31)

and using boundary condition for surface integral \( \delta A = 0 \), then

\[ \delta E = -\frac{1}{4} \int tr \left( -[D_\xi \Phi, \Phi] \delta A - D_r F_{r\xi} \delta A + D_\xi \left( \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi \xi} \right) \delta A \right) d^3 x 
(2.32)

So, equation of motion for \( A_\xi \) field is

\[ - [D_\xi \Phi, \Phi] - D_r F_{r\xi} + D_\xi \left( \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi \xi} \right) = 0 
(2.33)

\[ [D_\xi \Phi, \Phi] + D_r F_{r\xi} = \frac{1}{2r^2} D_\xi \left( (1 + |\xi|^2)^2 F_{\xi \xi} \right) 
(2.34)

If we do for \( A_\xi \) field then we have the same equation of motion above just by changing the index \( \xi \leftrightarrow \bar{\xi} \) from equation (2.34), so the equation of motion for field \( A_\xi \) is

\[ [D_\xi \Phi, \Phi] + D_r F_{r\xi} = \frac{1}{2r^2} D_\xi \left( (1 + |\xi|^2)^2 F_{\xi \xi} \right) 
(2.35)\]
While for $A_r$ field, if we take its variation on the energy, then

$$\delta E = -\frac{1}{4} \int tr \left( -\frac{2r^2}{(1+|\xi|^2)^2} [D_r \Phi, \Phi] \delta A + \partial_\xi (F_{\xi r} \delta A) + \partial_\xi F_{\xi r} \delta A + [F_{\xi r}, A_\xi] \delta A \
+ [F_{\xi r}, A_r] \delta A \right) d^3 x \tag{2.36}$$

and with boundary condition for surface integral $\delta A_r$, then equation (2.36) becomes

$$\delta E = -\frac{1}{4} \int tr \left( -\frac{2r^2}{(1+|\xi|^2)^2} [D_r \Phi, \Phi] \delta A - D_\xi F_{\xi r} \delta A - D_\xi F_{\xi r} \delta A \right) d^3 x \tag{2.37}$$

So, equation of motion for $A_r$ field is

$$-\frac{2r^2}{(1+|\xi|^2)^2} [D_r \Phi, \Phi] - D_\xi F_{\xi r} - D_\xi F_{\xi r} = 0 \tag{2.38}$$

$$[D_r \Phi, \Phi] = \frac{(1+|\xi|^2)^2}{2r^2} (D_\xi F_{\xi r} + D_\xi F_{\xi r}) \tag{2.39}$$

### 2.3.1 SU($N$) Bogomolny equation

In the previous, we have derive equations of motion (2.30), (2.34), (2.35), and (2.39) which are known as SU($N$) non-Bogomolny equations for BPS magnetic monopoles. Those equations are in second order differential and have properties of non-linear and coupled fields, and it is hard to find solutions for that kind of equations. Next, we will derive SU($N$) Bogomolny equations for BPS magnetic monopoles which have equations in form of first order differential. Those kind of equations are more simple then SU($N$) non-Bogomolny equations and it is possible to find the solutions. In deriving SU($N$) Bogomolny equations, we use Bogomolny analysis for BPS magnetic monopoles to the energy equation (2.26). In Bogomolny analysis, then energy equation (2.26) is transformed into equation that contains square sum of the fields. We must consider that the energy has to have real value $E \geq 0$, so that :

1. For first part of the energy

$$- D_\xi \Phi = (D_\xi \Phi)^\dagger = -\partial_\xi \Phi + [A_\xi^\dagger, \Phi] \tag{2.40}$$

then $A_\xi^\dagger = -A_r$. 

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2. and second part

\[- D_\xi \Phi = (D_\xi \Phi)^\dagger \]
\[= - \left( \partial_\xi \Phi - [A_\xi^\dagger, \Phi] \right) \]  
(2.41)

then \(A_\xi^\dagger = -A_\xi\).

3. and third part is correct by using the result from 1) and 2)

\[- F_{r\xi} = (F_{r\xi})^\dagger \]
\[= - (\partial_r A_\xi - \partial_\xi A_r + [A_r, A_\xi]) \]  
(2.42)

4. also for fourth part is correct if we use the result from 1) and 2)

\[- F_{\xi\bar{\xi}} = (F_{\xi\bar{\xi}})^\dagger \]
\[= - (\partial_\xi A_{\bar{\xi}} - \partial_{\bar{\xi}} A_\xi + [A_\xi, A_{\bar{\xi}}]) \]  
(2.43)

Let we look for some form of equations below:

\[(i D_\xi \Phi - F_{r\xi})(i D_\xi \Phi - F_{r\xi})^\dagger = -D_\xi \Phi D_\xi \Phi - F_{r\xi} F_{r\xi} + i D_\xi \Phi F_{r\xi} - i F_{r\xi} D_\xi \Phi \]  
(2.44)

\[
\left( i D_r \Phi - \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\bar{\xi}} \right) \left( i D_r \Phi - \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\bar{\xi}} \right)^\dagger = -D_r \Phi D_r \Phi - \frac{(1 + |\xi|^2)^4}{4r^4} F_{\xi\bar{\xi}} F_{\xi\bar{\xi}} \\
+ i \frac{(1 + |\xi|^2)^2}{2r^2} D_r \Phi F_{\xi\bar{\xi}} \\
- i \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\bar{\xi}} D_r \Phi \]  
(2.45)

then we can transform energy equation (2.26) so that it contains equations (2.44) and (2.45), and we may write it as

\[E = \frac{1}{4} \int tr \left( \frac{(1 + |\xi|^2)^2}{r^2} (i D_\xi \Phi - F_{r\xi})(i D_\xi \Phi + F_{r\xi}) \right) \\
+ \left( i D_r \Phi - \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\bar{\xi}} \right) \left( i D_r \Phi + \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\bar{\xi}} \right) \\
+ \left( i \frac{(1 + |\xi|^2)^2}{r^2} F_{r\xi} D_\xi \Phi \right)^\dagger + i \frac{(1 + |\xi|^2)^2}{r^2} F_{r\xi} D_\xi \Phi \\
+ \left( i \frac{(1 + |\xi|^2)^2}{2r^2} D_r \Phi F_{\xi\bar{\xi}} \right)^\dagger + i \frac{(1 + |\xi|^2)^2}{2r^2} D_r \Phi F_{\xi\bar{\xi}} \frac{2r^2}{(1 + |\xi|^2)^2} d^3x \]  
(2.46)
This energy (2.46) has minimum a value if

\[ iD_{\xi}\Phi = F_{r\xi} \]  \hspace{1cm} (2.47)
\[ iD_{r}\Phi = \frac{(1 + |\xi|^2)^2}{2r^2} F_{\xi\xi} \]  \hspace{1cm} (2.48)

and those equations are known as SU(N) Bogomolny equations for BPS magnetic monopoles in Riemann sphere metric (2.19).
Chapter 3
Harmonic Maps

3.1 Harmonic Maps Definition

Basically, harmonic map is a map between Riemannian manifolds. The theory of harmonic maps was introduced in 1945 by B.F. Fuller and developed by J. Eeels and J. M. Sampson ten years later [7]. Its role in physics was shown firstly by C. W. Misner [8] in 1978, who formulated non-linear $\sigma$-model field theory in geometrical description. The non-linear $\sigma$-models in two dimensions are special interest because they bear many similarities to the non-abelian gauge theory in four dimensions and have a property of being an integrable system. Later in this thesis, the non-linear $\sigma$-model is shortly named $\sigma$-model.

Next, we explain some basic definitions of harmonic maps theory. Let $M_0$ is a Riemannian manifold (source manifold) with local coordinates $u^i$ and metric

$$ ds^2 = g_{ij}du^i du^j $$

(3.1)

with $i, j = 1, 2, \ldots, \text{dim}[M_0]$ and $M$ is another Riemannian manifold (target manifold) with local coordinates $f^A$ with metric

$$ ds^2 = h_{AB}df^A df^B $$

(3.2)

with $A, B = 1, 2, \ldots, \text{dim}[M]$.

Then, a map

$$ f : M_0 \rightarrow M $$

$$ f = f(u) $$

(3.3)
is called a harmonic maps if it extremizes the action

\[
S = \int_{\mathcal{M}_0} d^n u \sqrt{g} \mathcal{L}
\]  

(3.4)

such that \(\delta_i S = 0\) with \(n = \text{dim}[\mathcal{M}_0]\) and \(g = \det(g_{ij})\) where

\[
\mathcal{L} = \frac{1}{2} h_{AB} \frac{\partial f^A}{\partial u^i} \frac{\partial f^B}{\partial u^j} g^{ij}
\]  

(3.5)

is the Lagrangian density.

### 3.2 \(\sigma\)-Model

The theory of harmonic maps in physics literature is related to \(\sigma\)-model. The \(\sigma\)-model is a field theory with the following properties [9]:

1. Fields \(\phi(u)\) of \(\sigma\)-model are constraints for all points \(\phi(u), \forall u \in \mathcal{M}_0\).

2. The constraints and Lagrangian density are invariant under action of global symmetry group \(G\) on \(\phi(u)\).

#### 3.2.1 \(O(N)\) \(\sigma\)-model

As an example, we look on \(O(N)\) \(\sigma\)-model which consist of \(N\)-real scalar fields \(\phi^A\) with \(A = 1, \ldots, N\) and has Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \frac{\partial \phi^A}{\partial u^i} \frac{\partial \phi^A}{\partial u^j} g^{ij}
\]  

(3.6)

where scalar fields \(\phi^A\) satisfy the constraints

\[
\phi^A \phi^A = 1
\]  

(3.7)

with repeated index means the sum of all its value. Lagrangian density (3.6) is invariant under global transformation group \(O(N)\)

\[
\phi^A \rightarrow (\phi^A)' = O_B^A \phi^B
\]  

(3.8)
Geometrically, the constraints (3.7) define a sphere \((N - 1)\) dimension \(S^{N-1}\) in \(N\) dimension Euclidean space \(\mathbb{R}^N\) of fields manifold \(\phi^A\). This constraints can be solved by introducing parametrization below:

\[
\phi^A = f^A, \quad \phi^N = \pm \sqrt{1 - |f|^2}, \quad A = 1, 2, \ldots, (N - 1) \tag{3.9}
\]

where \(|f|^2 = f^Af^A\) and the range of \(|f|\) is limited to \(-1 \leq |f| \leq 1\). The choice of sign \(+\) or \(-\) in equations (3.9) determines parametrization whether we use upper or lower hemisphere of \(S^{N-1}\). By parametrization in (3.9), we can write Lagrangian density (3.6) as

\[
\mathcal{L} = \frac{1}{2} \frac{\partial \phi^A}{\partial u^i} \frac{\partial \phi^A}{\partial u^j} g^{ij} \tag{3.10}
\]

with \(h_{AB}\) is tensor metric of target manifold \(S^{(N - 1)}\). The tensor metric \(h_{AB}\) is defined by substituting parametrization (3.9) into metric

\[
d^2\sigma = d\phi^Ad\phi^A = \left( \delta_{AB} + \frac{f_Af_B}{1 - |f|^2} \right) df^A df^B \tag{3.11}
\]

so that

\[
h_{AB} = \delta_{AB} + \frac{f_Af_B}{1 - |f|^2} \tag{3.12}
\]

where \(\delta_{AB}\) is the Kronecker delta. As the result, the fields \(f\) are free of constraints and are solution of \(O(N)\) \(\sigma\)-model field, so its defines a harmonic maps

\[
f : \mathcal{M}_0 \to S^{N-1}. \tag{3.13}
\]

### 3.2.2 Group formulation in \(\sigma\)-model

In group formulation, target manifold \(\mathcal{M}\) of a \(\sigma\)-model, with \(G\) is the invariant global group, are homogeneous space of \(G\). It means that by applying transformation group \(G\) over a field \(\phi_p \in \mathcal{M}\) then we can get into all the fields in target manifold \(\mathcal{M}\). In another words, for an arbitrary field \(\phi_q \neq \phi_p \in \mathcal{M}\) at least there is an element \(g \in G\) such that \(\phi_q = g\phi_p\).

If there is any stability or isotropy group \(H \subset G\) on the fields \(\phi_p \in \mathcal{M}\)

\[
H = \{ h \in G | h\phi_p = \phi_p \} \tag{3.14}
\]
then target Manifold $\mathcal{M}$ is a coset space of $G/H = \{gh|g \in G\}$ that works on the field $\phi_p$. If the identity $I_G$ is the only subgroup of $G$, then $\mathcal{M} = G/H = G$ is manifold of group $G$, then we call it as chiral model.

In general, the target manifold $\mathcal{M}$ of $\sigma$-model is a manifold of group $G$. In that case, the $\sigma$-model can be represented in group $G$ by writing parameters of group $G$ as the fields $f^A$, $A = 1, \ldots, (n = \text{dim}[G])$, with $\text{dim}[G]$ is dimension of group $G$ or number of generator in group $G$. Hence, the tensor metric for target manifold in $\sigma$-model is written by

$$h_{AB} = -2\text{tr}\left(G^{-1} \frac{\partial G}{\partial f^A} G^{-1} \frac{\partial G}{\partial f^B}\right)$$

and the Lagrangian density is

$$\mathcal{L} = -\text{tr}\left(G^{-1} \frac{\partial G}{\partial f^A} G^{-1} \frac{\partial G}{\partial f^B}\right)$$

### 3.3 Grassmannian $\sigma$-Model

Grassmannian manifold $Gr(n, N)$, $1 \leq n < N$, is the manifold of $n$ dimensional planes passing through the origin in the $N$ dimensional complex space $C^N$, where $Gr(1, N) = CP^{N-1}$. So, the $CP^{N-1}$ is a set of lines passing through the origin in $N$ dimensional complex space $C^N$ such that a point in $CP^{N-1}$ is a line on $C^N$. In this thesis, we use Grassmannian manifold in complex coordinates.

In coset space formulation, the Grassmannian manifold is

$$Gr(n, N) = \frac{U(N)}{U(N-n) \times U(n)}$$

and if we take into account the orientation of the planes, then

$$Gr(n, N) = \frac{SU(N)}{SU(N-n) \times SU(n)}$$

$Gr(n, N)$ $\sigma$-Model consists of $(N \times n)$ complex matrix fields $Z = (Z^{\hat{A}a})$ with $\hat{A} = 1, \ldots, N$ and $a = 1, \ldots, n$. Those fields satisfy the constraint:

$$Z^\dagger Z = I_n$$
The Lagrangian density of the model is required to be invariant under global unitary transformations $G \in U(N)$ that work on $Z$ field from the left

$$Z \rightarrow Z' = GZ$$

(3.20)

and also invariant under local gauge transformations $H(x) \in U(n)$ from the right of $Z$ field

$$Z \rightarrow Z' = ZH(x)$$

(3.21)

In this thesis, we use Lagrangian density given by W. J. Zakrzewski [10]

$$\mathcal{L} = \text{tr} \left( (D^\mu Z)^\dagger D_\mu Z \right)$$

(3.22)

with

$$D_\mu Z = \partial_\mu Z - ZZ^\dagger \partial_\mu Z$$

(3.23)

Next, we define $w^a, a = 1, \ldots, n$ to be a set of $N$ components orthonormal vectors in $C^N$, then the matrix field $Z$ can be represented as

$$Z = (w^1, \ldots, w^n)$$

(3.24)

In this representation, $Z$ defines an orthonormal $n$-frame in $C^N$.

Let we choose $Z$ as the last $n$-columns of matrix $G$

$$G = \left( Y^{AB} \ Z^{Aa} \right), \ B = 1, \ldots, (N - n)$$

(3.25)

where $Y$ is an $(N \times (N - n))$ matrix and $Z$ is an $(N \times n)$ matrix. Since we have unitary condition for $G \in U(N)$ which is $G^\dagger G = GG^\dagger = I_N$, then we have

$$Y^\dagger Y = I_{N-n}, \ Y^\dagger Z = 0, \ Z^\dagger Z = I_n, \ YY^\dagger + ZZ^\dagger = I_N$$

(3.26)

that makes $Z$ satisfies the constraint (3.19). For our purpose, we define field

$$\Phi = (Y \ Z) \begin{pmatrix} Y^\dagger \\ Z^\dagger \end{pmatrix} = \left\{ (Y \ Z) - (0 \ 2Z) \right\} \begin{pmatrix} Y^\dagger \\ Z^\dagger \end{pmatrix}$$

(3.27)

and by using the property in equation (3.24), then we can write as

$$\Phi = (I - 2P)$$

(3.28)

where $P = ZZ^\dagger$. A complete discussion of $Gr(n, N) \sigma$-model and for interested reader can study the reference on dissertation of Hans Jacobus Wospakrik [11].
3.4 Gr(n, N) σ-Model in Projection Space

The Lagrangian density for the Gr(n, N) σ-model is simply given by W. J. Zakrzewski [10]

\[ \mathcal{L} = \frac{1}{8} \text{tr} (\partial_\mu \Phi \partial^\mu \Phi) \]  

(3.29)

with \( \Phi = I - 2P \) and \( P \) is the projection operator, then

\[ \mathcal{L} = \frac{1}{2} \text{tr} (\partial_\mu P \partial^\mu P) \]  

(3.30)

Taking into account the constraint below:

\[ P^2 = P = P^\dagger \]
\[ (P^2 - P) = 0 \]  

(3.31)

then we can write Lagrangian density as

\[ \mathcal{L} = \frac{1}{2} \text{tr} (\partial_\mu P \partial^\mu P) + \lambda \text{tr} (P^2 - P) \]  

(3.32)

So, action on Lagrangian density (3.32) is

\[ S = \int \left( \frac{1}{2} \text{tr} (\partial_\mu P \partial^\mu P) + \lambda \text{tr} (P^2 - P) \right) d^4x \]  

(3.33)

3.4.1 Equations of Motion for Gr(n, N) σ-model

Equations of motion of Lagrangian density (3.32) is searched by varying the action (3.33) about \( P \) and \( \lambda \), then we get

\[ \delta S = \int \text{tr} \left( \partial^\mu (\partial_\mu P \delta P) - \partial_\mu \partial^\mu P \delta P + \lambda (2P - I) \delta P + (P^2 - P) \delta \lambda \right) d^4x \]  

(3.34)

Because of boundary condition \( \delta P = 0 \) on surface integral, we can discard the first part of equation (3.34). And then, we get the equations of motion as below

\[ \partial^\mu \partial_\mu P - \lambda (2P - I) = 0 \]  

(3.35)

\[ P^2 - P = 0 \]  

(3.36)
We can see that the equation (3.36) returns to the constraint (3.31). Next, if we multiply the equation (3.35) by $P$ on the right and left separately, then we have two equations
\begin{align*}
\partial^\mu \partial_\mu P P - \lambda (2P - I)P &= 0 \tag{3.37} \\
P \partial^\mu \partial_\mu P - \lambda P (2P - I) &= 0 \tag{3.38}
\end{align*}
If we count the difference between equation (3.37) and equation (3.38) then use property (3.31), then we get
\[ [P, \partial^\mu \partial_\mu P] = 0 \tag{3.39} \]

### 3.4.2 $Gr(n, N)$ $\sigma$-model solution

In this section, we discuss the construction of solutions of $Gr(n, N)$ $\sigma$-model from equation (3.39) in two dimensional Euclidean space $R^2$ or the complex plane $C$. Later, we compactify $R^2$ by including points at $\infty$ to obtain the Riemann sphere $S^2 = R^2 \cup \infty$ and consider the harmonic maps: $S^2 \rightarrow Gr(n, N)$. For next discussion, we use a $Gr(n, N)$ $\sigma$-model that has source manifold in two dimensional Euclidean space.

In searching for the solutions, we use the complex coordinates $(\xi, \bar{\xi})$ for $Gr(n, N)$ $\sigma$-model therefore the equation (3.39) becomes
\[ [P, \partial_\xi \partial_{\bar{\xi}} P] = 0 \tag{3.40} \]
Derivation of equation (3.40) is in Appendix B. We also can write equation (3.40) in form of
\[ \partial_\xi [P, \partial_{\bar{\xi}} P] + \partial_{\bar{\xi}} [P, \partial_\xi P] = 0 \tag{3.41} \]
Complete derivation of this equation in Appendix B.

### 3.4.3 Instanton solution

Next, we look at differentiation of operator property $P^2 = P$ on $\partial_\xi (P^2 = P)$ and $\partial_{\bar{\xi}} (P^2 = P)$ then we get
\begin{align*}
\partial_\xi P P &= \partial_\xi P - P \partial_\xi P \tag{3.42} \\
P \partial_{\bar{\xi}} P &= \partial_{\bar{\xi}} P - \partial_{\bar{\xi}} PP \tag{3.43}
\end{align*}
while the equation (3.41) is written by
\[ \partial_\xi \left( P \partial_{\bar{\xi}} P - \partial_{\bar{\xi}} PP \right) + \partial_{\bar{\xi}} \left( P \partial_\xi P - \partial_\xi PP \right) = 0 \tag{3.44} \]
Substitute equations (3.42) and (3.43) to equation (3.44), then we get

$$\partial_{\bar{\xi}} (P \partial_{\xi} P) - \partial_{\xi} \left( \partial_{\bar{\xi}} PP \right) = 0 \quad (3.45)$$

Special class of solutions for equation (3.45) is satisfied when $P \partial_{\xi} P = 0$ such that $(P \partial_{\xi} P)^\dagger = \partial_{\bar{\xi}} PP = 0$, so $P \partial_{\xi} P = 0$ is the solution of field equation which is called selfdual equation. If we do the same way by substituting $P \partial_{\xi} P$ and $\partial_{\bar{\xi}} PP$ in equations (3.42) and (3.423) into equation (3.44), then it will produce a solution $P \partial_{\xi} P = 0$ which is called anti-selfdual equation.

In this thesis, we focus on solution from selfdual equation $P \partial_{\xi} P = 0$. For this purpose, we consider an un-normalized $(N \times n)$ matrix field $M = M(\xi, \bar{\xi})$ for which the $|M|^2 = M^\dagger M$ is assumed to be non-singular. As $|M|^2$ is Hermitian, its eigenvalues are real and there exist an unitary matrix $U$ such that

$$|M|^2 = U^\dagger \Lambda^2 U \quad (3.46)$$

where $\Lambda^2$ is a diagonal matrix with eigenvalues $(\lambda_1^2, \ldots, \lambda_N^2)$. We define $\Lambda$ to be the square root matrix of $\Lambda^2$, hence $\Lambda = (\Lambda^2)^{1/2} = (\lambda_1, \ldots, \lambda_N)$. So, matrix $|M|^2 = U^\dagger \Lambda^2 U$ can be denoted by $|M|^2 = (U^\dagger \Lambda U) (U^\dagger \Lambda U)$ such that $|M| = (|M|^2)^{1/2} = U^\dagger \Lambda U$.

In terms of $M$, the matrix field $Z$ of $Gr(n, N) \sigma$-model is given by

$$Z = M|M|^{-1} \quad (3.47)$$

and so the $P$ projection operator is

$$P = ZZ^\dagger = \left( M|M|^{-1} \right) \left( |M|^{-1} M^\dagger \right) = M|M|^2 M^\dagger \quad (3.48)$$

where we have use property of matrix $(|M|^{-1})^\dagger = |M|^{-1}$

Using property of

$$(I - P)M = 0 \quad (3.49)$$

then the selfdual equation is given by

$$M|M|^{-2} \left( \partial_{\bar{\xi}} M \right)^\dagger (I - P) = 0 \quad (3.50)$$

Complete derivation of equations (3.49) and (3.50) is available in Appendix C.

The solution of equation (3.50) is $\partial_{\bar{\xi}} M = 0$ or $M = M_0(\bar{\xi})$ which is called a holomorphic matrix field. This solution of selfdual equation is named instanton solution. With the same analogy, if we discuss about anti-selfdual equation then we will get anti-holomorphic matrix field $M = \bar{M_0}(\bar{\xi})$ which is named anti-instanton solution.

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3.5 Full Solutions of $Gr(n, N)$ $\sigma$-Model

In this section, we will discuss the method of generating more general exact solutions of the two dimensional $Gr(n, N)$ $\sigma$-model starting from an instanton solution. The method was originally introduced by A. Din and W. J Zakrzewski [12, 10].

Let $M_k = M_k(\xi, \bar{\xi})$, $k = 1, \ldots, \lambda$ where $\lambda \leq (N - 1)$ be a set of $(\lambda + 1)$ mutually orthogonal $(N \times n)$ matrices, with $n < N$.

\[ M_k \dagger M_l = |M_k|^2 \delta_{kl} \quad (3.51) \]

where \[ |M_k|^2 = M_k \dagger M_k \quad (3.52) \]

are $(n \times n)$ non-singular matrices. Then the corresponding projection operator for each matrix $M_k$ is given by

\[ P_k(n) = M_k |M_k|^{-2} M_k \dagger \quad (3.53) \]

Clearly, $tr(P_k(n)) = tr(I_n)$, which means that each projection operator has rank-$n$. From equation (3.53), we can show that the projection operators are mutually orthogonal and also Hermitian

\[ P_k(n) P_l(n) = \delta_{kl} P_l(n) \quad (3.54) \]

\[ P_k(n) \dagger = P_k(n) \quad (3.55) \]

where $|M|^2$ are Hermitian. In the following we want to present a generalized harmonic maps ansatz. To do this we use a sequence of mutually orthogonal matrices $(M_0, M_1, \ldots, M_\lambda)$ obtained from a sequence of holomorphic (analytic) matrices $(M, \partial_\xi M, \ldots, \partial_\lambda^\xi M)$, $\partial_\xi M = 0$ through the process of Gram-Schmidt orthogonalization.

We can do this using the operator $P_+$ which is defined by its action on any matrix $M \in C^{N \times n}$ [12, 10]

\[ P_+ M = \partial_\xi M - M |M|^2 M \dagger \partial_\xi M \quad (3.56) \]

Then we have $M_0 = M, M_1 = P_+ M, \ldots, M_k = P_+^k M, \ldots, M_\lambda = P_+^\lambda M$ or simply

\[ M_0 = M, \quad M_k = (I - P_{k-1}) \partial_\xi M_{k-1}, \quad k = 1, \ldots, \lambda \quad (3.57) \]

where $P_{k-1}$ is the projection operator (3.53). An equivalent formulation for the sequence $M_k$, in terms of the projection operators $P_k$, is given by

\[ M_k = (1 - P_0 - \cdots - P_{k-1}) \partial_\xi^k M_0 \quad (3.58) \]
With either one of these constructions in equations (3.57) and (3.58), the following properties of the matrices \(M_k\) hold when \(M_0\) is holomorphic

\[
\partial \bar{\xi} M_k = -M_{k-1} |M_{k-1}|^{-2} |M_k|^2 \tag{3.59}
\]

\[
\partial \left( M_k |M_k|^{-2} \right) = M_{k+1} |M_k|^{-2} \tag{3.60}
\]

Detail derivation is in Appendix D.

Notice also that, for the \(CP^{(N-1)}\) case, projection operators \(P_k\) with \(k = 0, \ldots, \lambda\) are complete

\[
P_0 + P_1 + \cdots + P_{N-1} = I \tag{3.61}
\]

and according to the construction \(M_k\) in equation (3.58), then

\[
M_N = 0 \tag{3.62}
\]

With the projection operators \(P_k\) that we have constructed above, we have the following result that was originally proved by A. Din and W.J. Zakrzewski [12] using the \(Z\) fields formalism.

**Theorem:** Each \((N \times N)\) projection operator \(P_k(n) = M_k |M_k|^{-2} M_k^\dagger, k = 0, 1, \ldots, \lambda,\) where \(M_k = P_k^+ M_0\) and \(M_0 = M_0(\xi)\) is a holomorphic \((N \times n)\) matrix field, solves the \(Gr(n, N)\) \(\sigma\)-model (2.19) for its equation of motion (3.40).

As a proof, we use the method that was developed by Sasaki [13], where we shall be using the projection operator in term of \(M_k\) matrix field as follows:

From properties (3.58) and (3.59), we derive:

\[
\partial \bar{\xi} P_0 = \partial \bar{\xi} (M_0 |M_0|^{-2}) M_0^\dagger + M_0 |M_0|^{-2} \partial \bar{\xi} M_0^\dagger = M_1 |M_0|^{-2} M_0^\dagger \tag{3.63}
\]

\[
\partial \bar{\xi} P_k = \partial \bar{\xi} (M_k |M_k|^{-2}) M_k^\dagger + M_k |M_k|^{-2} (\partial \bar{\xi} M_k)^\dagger = M_{k+1} |M_k|^{-2} M_k^\dagger - M_k |M_{k-1}|^{-2} M_{k-1}^\dagger \tag{3.64}
\]

with \(k = 1, \ldots, \lambda.\)

Define

\[
Q_k = \sum_{l=0}^{k-1} P_l, \quad Q_0 = 0 \tag{3.65}
\]

and take its differential about \(\xi\) then use equations (3.63) and (3.64)

\[
\partial \bar{\xi} Q_k = \sum_{l=0}^{k-1} \partial \bar{\xi} P_l = M_k |M_{k-1}|^{-2} M_{k-1}^\dagger \tag{3.66}
\]
Using orthogonality property of $M_k$ we obtain

$$Q_k \partial_\xi Q_k = 0$$  \hspace{2cm} (3.67)

where $P_k$ satisfy

$$Q_k \partial_\xi P_k = 0, \ P_k \partial_\xi Q_k = \partial_\xi Q_k$$  \hspace{2cm} (3.68)

Then define operator

$$R_k = Q_k + P_k$$  \hspace{2cm} (3.69)

which satisfy _self-dual_ equation

$$R_k \partial_x iK = \partial_\xi Q_k + P_k \partial_\xi P_k = 0$$  \hspace{2cm} (3.70)

and take the Hermitian conjugate of (3.70) gives

$$\partial_\xi Q_k + \partial_\xi P_k P_k = 0$$  \hspace{2cm} (3.71)

If we take differential of equation (3.70) about $\xi$ and of equation (3.71) about $\overline{\xi}$, then we get

$$\partial_\xi \partial_{\overline{\xi}} Q_k + \partial_\xi P_k \partial_{\overline{\xi}} P_k + P_k \partial_\xi \partial_{\overline{\xi}} P_k = 0$$  \hspace{2cm} (3.72)

and

$$\partial_\xi \partial_{\overline{\xi}} Q_k + \partial_\xi P_k \partial_{\overline{\xi}} P_k + \partial_\xi \partial_{\overline{\xi}} P_k P_k = 0$$  \hspace{2cm} (3.73)

The difference between equation (3.72) and (3.73) gives us

$$[P_k, \partial_\xi \partial_{\overline{\xi}} P_k] = 0$$  \hspace{2cm} (3.74)

which completes the proof.

### 3.6 Action of Full Solutions and Non-Abelian Toda Equations

As each projection operator $P_k(n)$ solves the $Gr(n, N)$ $\sigma$-model as stated in previous theorem (3.74), each projection operator $P_k(n)$ describes a specific field configuration having the action

$$S_k = i \int d\xi d\overline{\xi} \ tr \left( \partial_\xi P_k \partial_{\overline{\xi}} P_k \right)$$  \hspace{2cm} (3.75)

where $S_0$ corresponds to instanton(or anti-instanton) configuration.
Using relation relation in equations (3.59) and (3.60), we have

\[
\partial_\xi P_k = \partial_\xi M_k |M_k|^{-2} M_k^\dagger + M_k \partial_\xi \left( |M_k|^{-2} M_k^\dagger \right)
\]

\[
= -M_{k-1} |M_{k-1}|^{-2} M_k^\dagger + M_k |M_k|^{-2} M_{k+1}^\dagger
\]

(3.76)

If we substitute equations (3.64) and (3.76) into equation (3.75), then the action becomes

\[
S_k = i \int d\xi d\bar{\xi} \; \text{tr} \left( |M_k|^2 |M_{k-1}|^{-2} + |M_{k+1}|^2 |M_k|^{-2} \right)
\]

\[
= 2\pi (N_k + N_{k-1})
\]

(3.77)

where

\[
N_k = \frac{i}{2\pi} \int d\xi d\bar{\xi} \; \text{tr} \left( |M_{k+1}|^2 |M_k|^{-2} \right)
\]

(3.78)

and by definition \(N_{-1} = 0\) because \(\partial_\xi M_0 = 0\).

In the following, we derive recurrence relations for \(\text{tr} \left( |M_k|^2 |M_{k-1}|^{-2} \right)\) appearing in the integral (3.78). To do this we rewrite the definition of \(M_{k+1}\) in (3.57) as below:

\[
\partial_\xi M_k = M_{k+1} + P_k \partial_\xi M_k
\]

(3.79)

By using \(\partial_\xi M_k\) in equation (3.59) and property

\[
M_k^\dagger \partial_\xi M_k = -M_k^\dagger M_{k-1} |M_{k-1}|^{-2} |M_k|^2
\]

\[
= 0
\]

(3.80)

then we take differential of (3.79) about \(\bar{\xi}\) and multiply on the left by \(M_k^\dagger\) and on the right by \(|M_k|^{-2}\), and becomes

\[
M_k^\dagger \partial_\xi \partial_\xi M_k |M_k|^{-2} = M_k^\dagger \partial_\xi M_{k+1} |M_k|^{-2} + M_k^\dagger \partial_\xi (P_k \partial_\xi M_k) |M_k|^{-2}
\]

(3.81)

The left side of equation (3.81) gives

\[
M_k^\dagger \partial_\xi \partial_\xi M_k |M_k|^{-2} = -|M_k|^2 |M_{k-1}|^{-2}
\]

(3.82)

From equation (3.59), the first part of the right side gives

\[
M_k^\dagger \partial_\xi M_{k+1} |M_k|^{-2} = -M_k^\dagger M_k |M_k|^{-2} |M_{k+1}|^2 |M_k|^{-2}
\]

\[
= -|M_{k+1}|^2 |M_k|^{-2}
\]

(3.83)
and the second part becomes

\[ M_k^\dagger \partial_\xi (P_k \partial_\xi M_k) |M_k|^{-2} = \partial_\xi \left( \partial_\xi |M_k|^2 |M_k|^{-2} \right) \]  

(3.84)

The detail of equations (3.82) and (3.84) is in Appendix E.

Substitute equations (3.82), (3.83), and (3.84) into equation (3.81) that gives

\[ \partial_\xi \left( \partial_\xi |M_k|^2 |M_k|^{-2} \right) = |M_{k+1}|^2 |M_k|^{-2} - |M_k|^2 |M_{k-1}|^{-2} \]  

(3.85)

For the \( n = 1 \) or \( CP(N-1) \) case, when \( M_k \) are a sequence of \( N \)-component vector fields, the equation (3.85) gives the Toda equation [14]. If we take the trace of (3.85) then we obtain

\[ \text{tr} \left( \partial_\xi \left( \partial_\xi |M_k|^2 |M_k|^{-2} \right) \right) = \partial_\xi \partial_\xi \text{tr} \left( \ln |M_k|^2 \right) \]  

(3.86)

Because \( \text{det}|M_k|^2 = e^{\text{tr}(\ln |M_k|^2)} \), we may write equation (3.86) as

\[ \text{tr} \left( \partial_\xi \left( \partial_\xi |M_k|^2 |M_k|^{-2} \right) \right) = \partial_\xi \partial_\xi \ln \left( \text{det}|M_k|^2 \right) \]  

(3.87)

So, the trace of equation (3.85) is

\[ \partial_\xi \partial_\xi \ln \left( \text{det}|M_k|^2 \right) = \text{tr} \left( |M_{k+1}|^2 |M_k|^{-2} \right) - \text{tr} \left( |M_k|^2 |M_{k-1}|^{-2} \right) \]  

(3.88)

Note that, for \( Gr(1, N) = CP^{N-1} \) case, the equation (3.79) becomes

\[ \partial_\xi \partial_\xi \ln \left( |M_k|^2 \right) = |M_{k+1}|^2 |M_k|^{-2} - |M_k|^2 |M_{k-1}|^{-2} \]  

(3.89)

### 3.7 Veronese Map

In this subsection, we describe how to construct the full solutions of the two dimensional Grassmanian \( \sigma \)-model using Veronese map. Here, we consider only for \( Gr(1, N) = CP^{N-1} \) case with \( M_0 \in C^{N \times 1} \) is a \( N \)-component vector field in complex space

\[ M_0 : C \rightarrow C^N \]  

\[ \xi \rightarrow M_0 = (f_0, \ldots, f_p, \ldots, f_{N-1})^T \]  

(3.90)

In the Veronese map [15], we choose \( f_p \) as function of \( \xi \) in order \( p \) such that

\[ |M_0|^2 = \left( 1 + |\xi|^2 \right)^{N-1} \]  

(3.91)
This condition restricts the component of $f_p$ to have the form $f_p = \sqrt{C_p^{N-1}}\xi^p$ where $C_p^{N-1}$ is the combinatorial factor and so the corresponding $CP^{N-1}$ field is

$$Z_0 = \frac{M_0}{|M_0|} = \left(\frac{1, \ldots, \sqrt{C_p^{N-1}}\xi^p, \ldots, \xi^{N-1}}{\sqrt{(1 + |\xi|^2)^{N-1}}}\right)^T$$

(3.92)

For general case $N$ in the sequence of $M_k$, we use proof that was first given by Ioannidou et. al [16]:

**Proposition:** For the Veronese map $M_0$ in (3.92), the sequence $M_k$, ($k = 0, \ldots, N-1$), constructed by equations (3.57) or (3.58), called Veronese sequence [15], satisfy

$$\frac{|M_{k+1}|^2}{|M_k|^2} = \frac{(k + 1)(N - k - 1)}{(1 + |\xi|^2)^2}$$

(3.93)

To prove it, we make use of the recurrence relations (3.85) or (3.88). As $|M_0|^2 = (1 + |\xi|^2)^{N-1}$, then

$$\partial_\xi \partial_{\xi^*} \ln |M_0|^2 = (N - 1) \left(1 + |\xi|^2\right)^{-2}$$

(3.94)

and by using relation (3.89) for $k = 0$, we obtain

$$\partial_\xi \partial_{\xi^*} \ln |M_0|^2 = \frac{|M_1|^2 |M_0|^{-2} - |M_0|^2 |M_1|^{-2}}{|M_0|^2}$$

(3.95)

where by definition $M_{1} = 0$. From equations (3.94) and (3.95), we have

$$\frac{|M_1|^2}{|M_0|^2} = \frac{(N - 1)}{\left(1 + |\xi|^2\right)^2}$$

(3.96)

For general $k$, $1 < k < (N - 1)$, we use inductive proof by assuming

$$\frac{|M_k|^2}{|M_{k-1}|^2} = \frac{k(N - k)}{(1 + |\xi|^2)^2}$$

(3.97)

which is already true for $k = 1$. We can write $|M_k|^2$ using equation (3.97) as

$$|M_k|^2 = \frac{|M_k|^2}{|M_{k-1}|^2} \frac{|M_{k-1}|^2}{|M_{k-2}|^2} \cdots \frac{|M_1|^2}{|M_0|^2} |M_0|^2$$

$$= \frac{k!(N - 1)!}{(N - k - 1)!} \left(1 + |\xi|^2\right)^{N-2k-1}$$

(3.98)
then if we take differential over $\xi$ and $\bar{\xi}$ of equation (3.98), we obtain

$$\frac{\partial \xi}{\partial \bar{\xi}} \ln |M_0|^2 = \frac{(N - 2k - 1)}{(1 + |\xi|^2)^2}$$  \hspace{1cm} (3.99)

Substitute equations (3.97) and (3.99) into (3.89), gives

$$\frac{(N - 2k - 1)}{(1 + |\xi|^2)^2} = \frac{|M_{k+1}|^2}{|M_k|^2} - \frac{k(N - k)}{(1 + |\xi|^2)^2}$$  \hspace{1cm} (3.100)

therefore

$$\frac{|M_{k+1}|^2}{|M_k|^2} = \frac{(k + 1)(N - k - 1)}{(1 + |\xi|^2)^2}$$  \hspace{1cm} (3.101)

is true.

**Corollary:** Each configuration $M_k$, $(k = 0, \ldots, N - 1)$, of the Veronese map (3.92), has

$$N_k = (k + 1)(N - k - 1)$$  \hspace{1cm} (3.102)

We prove it using the integral formula on surface

$$i \int \frac{d\xi d\bar{\xi}}{(1 + |\xi|^2)^2} = i \int \det \begin{vmatrix} e^{i\phi} & i\rho e^{i\phi} \\ e^{-i\phi} & -i\rho e^{i\phi} \end{vmatrix} \frac{d\rho d\phi}{(1 + \rho^2)^2}$$  \hspace{1cm} (3.103)

with $\rho = \tan(\frac{\theta}{2})$. Substitute equations (3.103) and (3.101) into (3.78) for $CP^{N-1}$ case, we obtain

$$N_k = (k + 1)(N - k - 1)$$  \hspace{1cm} (3.104)
Chapter 4

\textit{SU}(N) Bogomolny Solutions

4.1 Harmonic Maps Ansatz

In this chapter, we discuss the general construction of the solutions for \textit{SU}(N) Bogomolny equations for BPS magnetic monopoles using previous description in harmonic maps. We also give an example for \textit{SU}(2) Bogomolny equations that give the same solutions with one was proved by M.K. Prasad and C.M. Sommerfield \[17].

For this purpose, we take an ansatz for \textit{SU}(N) Bogomolny equations given by T. Ioannidou and P.M. Sutcliffe \[18]

\begin{align*}
\Phi &= i \sum_{j=0}^{N-2} h_j(r) \left( P_j - \frac{1}{N} \right), \quad A_\xi = \sum_{j=0}^{N-2} g_j(r) [P_j, \partial_\xi P_j], \quad A_r = 0 \quad (4.1)
\end{align*}

We substitute the ansatz into \textit{SU}(N) Bogomolny equations (2.47) and (2.48) then multiply from the right with vectors field \( M_l \) as follows

\begin{align*}
(i D_\xi \Phi - F_\xi) M_l &= 0 \quad (4.2) \\
\left( i D_\xi \Phi - \frac{(1 + |\xi|^2)^2}{2r^2} F_\xi \right) M_l &= 0 \quad (4.3)
\end{align*}

Next, the equation (4.2) is expressed in term of projection operators \( P_k \). In that case, we need to describe the properties of projection operators \( P_k \) and their derivatives applied to \( M_l \):

\begin{align*}
P_k M_l &= \delta_{kl} M_l \quad (4.4)
\end{align*}

from equations (3.59) and (3.60), we obtain

\begin{align*}
\partial_\xi P_k M_l &= \partial_\xi \left( M_k |M_k|^{-2} \right) M_k^l M_l + M_k |M_k|^{-2} \partial_\xi M_k^1 M_l
\end{align*}
\[ (\delta_{kl} - \delta_{k,l+1}) M_{l+1} \tag{4.5} \]

\[
\begin{align*}
\partial_{\xi} P_k M_l &= \partial_{\xi} (P_k M_l) - P_k \partial_{\xi} M_l \\
&= (\delta_{k,l-1} - \delta_{kl}) M_{l-1} K_{l-1} \tag{4.6}
\end{align*}
\]

where \( K_{l-1} = |M_{l-1}|^{-2} |M_l|^2 \). For simplicity, we use a convention for index \( j \) which means the sum of \( j = 0, \ldots, N - 2 \).

Therefore, we can take the action of the fields, as given by the ansatz (4.1), on the vectors field \( M_l \) by using properties (4.4)-(4.6)

\[ A_{\xi} M_l = -G_l M_{l+1} \tag{4.7} \]
\[ A_{\bar{\xi}} M_l = G_{l-1} M_{l-1} K_{l-1} \tag{4.8} \]

where \( G_l = g_l + g_{l-1} \). Then, we derive that

\[ F_{r\xi} M_l = -\partial_r G_l M_{l+1} \tag{4.9} \]
\[ F_{\xi\bar{\xi}} M_l = \left( \partial_{\xi} A_{\bar{\xi}} - \partial_{\bar{\xi}} A_{\xi} + [A_{\xi}, A_{\bar{\xi}}] \right) M_l \tag{4.10} \]

For equation (4.10), we calculate separately for its parts as follows

\[ \partial_{\xi} A_{\bar{\xi}} M_l = M_l (K_{l-1} G_{l-1} - K_l G_l) \tag{4.11} \]
\[ \partial_{\bar{\xi}} A_{\xi} M_l = M_l (K_l G_l - K_{l-1} G_{l-1}) \tag{4.12} \]
\[ A_{\xi} A_{\bar{\xi}} M_l = -G_{l-1}^2 M_l K_{l-1} \tag{4.13} \]
\[ A_{\bar{\xi}} A_{\xi} M_l = -G_l^2 M_l K_l \tag{4.14} \]

Substitute equations (4.11)-(4.14) into equation (4.10), brings

\[ F_{\xi\bar{\xi}} M_l = M_l \left( K_{l-1} \bar{Q}_{l-1} - K_l \bar{Q}_l \right) \tag{4.15} \]

where \( \bar{Q}_l = G_l (2 - G_l) \), then

\[ \Phi M_l = i \left( h_l - \frac{h_j}{N} \right) M_l \tag{4.16} \]
\[ \partial_r \Phi M_l = i \left( \partial_r h_l - \frac{\partial_r h_j}{N} \right) M_l \tag{4.17} \]
\[ \partial_{\xi} \Phi M_l = i (h_l - h_{l+1}) M_{l+1} \tag{4.18} \]
Substitute all the previous result into equation (4.2), gives

\[
\{i(\partial_\xi \Phi + [A_\xi, \Phi]) - F_{r\xi}\} M_l = 0
\]

\[
\{(h_{l+1} - h_l) - (h_{l+1} - h_l)G_l + \partial_r G_l\} M_{l+1} = 0
\]

(4.19)

because \(M_{l+1} \neq 0\), therefore

\[
(h_{l+1} - h_l)(1 - G_l) + \partial_r G_l = 0
\]

(4.20)

If we do the same way to the equation (4.3), then we have

\[
\left( i\partial_r \Phi - \left( \frac{1 + |\xi|^2}{2r^2} F_{r\xi} \right) \right) M_l = 0
\]

\[
- \left( \frac{\partial_r h_l - \partial_r h_j}{N} \right) M_l - \left( \frac{1 + |\xi|^2}{2r^2} M_l(K_{l-1} \bar{Q}_{l-1} - K_l \bar{Q}_l) \right) = 0
\]

(4.21)

For \(CP^{(N-1)}\) case, we take vector fields \(M_l\) as given by the Veronese map from the previous chapter

\[
M_0 = \left[ 1, \sqrt{C_1^{N-1}} \xi, \ldots, \sqrt{C_k^{N-1}} \xi^k, \ldots, \xi^{N-1} \right]^T
\]

(4.22)

then we may write the factors \((1 + |\xi|^2)^2K_l\) as constant \(K_l\), such that

\[
K_l = \frac{K_l}{(1 + |\xi|^2)^2}
\]

(4.23)

where we find that \(K_l = N_l = (l+1)(N-l-1)\) are constants which depend on index \(l\) as shown from the equation (3.100). Hence, the equation (4.21) becomes

\[
\left\{ \left( \frac{\partial_r h_j}{N} - \partial_r h_l \right) - \frac{1}{2r^2} \left( K_{l-1} \bar{Q}_{l-1} - K_l \bar{Q}_l \right) \right\} M_l = 0
\]

(4.24)

because \(M_l \neq 0\), then we obtain

\[
\left( \frac{\partial_r h_j}{N} - \partial_r h_l \right) - \frac{1}{2r^2} \left( K_{l-1} \bar{Q}_{l-1} - K_l \bar{Q}_l \right) = 0
\]

(4.25)

or in complete

\[
\left( \sum_{j=0}^{N-2} \frac{\partial_r h_j}{N} - \partial_r h_l \right) - \frac{1}{2r^2} \left( K_{l-1} \bar{Q}_{l-1} - K_l \bar{Q}_l \right) = 0
\]

(4.26)
Note that, by definition \( h_l, g_l = 0 \) if \( l \notin \{0, 1, \ldots, (N-2)\} \).

The equations (4.20) and (4.26) are called \( SU(N) \) Bogomolny equations for BPS magnetic monopoles. Clearly, those equations are in of scalar and non-linear coupled fields. Compared to the form of the \( SU(N) \) Bogomolny equations before we apply harmonic maps method, the previous \( SU(N) \) Bogomolny equations are in form of matrix and also non-linear coupled fields. It is much easy to work with scalar field equations than in form of matrix field equations. But, to find the exact solutions of \( SU(N) \) Bogomolny equations, even in scalar field equations, is another complex problem since the fields in form of non-linear coupled. In this thesis, we do not discuss about the exact solutions since it needs others advance mathematical techniques to find the soliton solutions.

### 4.2 \( SU(2) \) Bogomolny Equation

In this section, we show a simple example for \( SU(2) \) Bogomolny case for which the equations (4.20) and (4.26) has range of value \( l = 0, \ldots, (2-2) = 0 \), so by definition \( h_l, g_l = 0 \) if \( l \neq 0 \), then equations (4.20) and (4.26) become

\[
\begin{align*}
- h_0 (1 - g_0) + \dot{g}_0 &= 0 \\
- \frac{1}{2} \dot{h}_0 + \frac{1}{2r^2} g_0 (2 - g_0) &= 0
\end{align*}
\]  

with \( \dot{g}_0 = \partial_r g_0 \) and \( \dot{h}_0 = \partial_r h_0 \)

Now, define a function \( K = 1 - g_0 \) and substitute it into equations (4.27) and (4.28), such that

\[
\begin{align*}
- h_0 K - \dot{K} &= 0 \\
- \dot{h}_0 + \frac{1}{r^2} (1 - K^2) &= 0
\end{align*}
\]  

From equation (4.30), we have

\[
\dot{h}_0 = \frac{1}{r^2} (1 - K^2)
\]  

Take the differential of equation (4.29) about \( r \) and substitute it into equation (4.31)

\[
\begin{align*}
- \dot{h}_0 K - h_0 \dot{K} - \dot{K} &= 0 \\
\dot{K} &= -\frac{1}{r^2} K (1 - K^2) - h_0 \dot{K}
\end{align*}
\]
from equation (4.29), gives

\[ r^2 \ddot{K} = K(K^2 - 1) + r^2 h_0^2 K \]  

(4.33)

Take the differential of equation (4.30) about \( r \) and substitute it into equation (4.29), then

\[- \ddot{h}_0 + \frac{1}{r^2} (-2 K \dot{K}) - \frac{2}{r^3} (1 - K) = 0 \]

\[ r^2 \dot{h} = -2 \frac{1}{r} (1 - K) + 2h_0 K^2 \]

(4.34)

Next, we define a function \( H = rh_0 \) with

\[ \dot{h}_0 = - \frac{H}{r^2} + \frac{\dot{H}}{r} \]

(4.35)

\[ \ddot{h}_0 = \frac{2H}{r^3} - \frac{2\dot{H}}{r^2} + \frac{\ddot{H}}{r} \]

(4.36)

and substitute into equations (4.33) and (4.34)

\[ r^2 \ddot{K} = K(K^2 - 1) + H^2 K \]

(4.37)

\[ r^2 \ddot{H} = 2K^2 H \]

(4.38)

The result in equations (4.37) and (4.38) are same as the one that was obtained by M. Prasad and C. Sommerfield \[17\] which gives

\[ K = \frac{C r}{\sinh (C r)} \]

(4.39)

\[ H = C r \coth (C r) - 1 \]

(4.40)

where \( C \) is a constant.
Chapter 5

Conclusions

In this thesis, we have described how to use harmonic maps in $CP^{(N-1)}$ space to simplify the $SU(N)$ Bogomolny equations for BPS magnetic monopoles. As the result obtained in chapter IV equations (4.20) and (4.26), we just need to solve the non-linear coupled equations for scalar fields that depend on variable $r$, as follows:

\begin{align}
(h_{l+1} - h_l)(1 - G_l) + \partial_r G_l &= 0 \quad (5.1) \\
\left(\frac{\sum_{j=0}^{N-2} \partial_r h_j}{N} - \partial_r h_l\right) - \frac{1}{2r^2}(K_{l-1} \tilde{Q}_{l-1} - K_l \tilde{Q}_l) &= 0 \quad (5.2)
\end{align}

with $l = 0, 1, \ldots, (N-2)$.

Even it is not a simple problem to solve the non-linear coupled equations, but at least we have more simple form of the equations in scalar than before. As an example, we also have shown for $SU(2)$ Bogomolny equations and verified the result with the one obtained by M. Prasad and C. Sommerfield.
Appendix A

Riemann Sphere

In deriving metric (2.19), we introduce equation \( \xi = \tan \left( \frac{\theta}{2} \right) e^{i\phi} \) and transform variables of the metric (2.18) into complex variables \((\theta, \phi) \rightarrow (\xi, \bar{\xi})\). By using other forms of the transformation equation which are

\[
\tan \left( \frac{\theta}{2} \right) = \sqrt{\xi \bar{\xi}} = |\xi|, \quad \sin \left( \frac{\theta}{2} \right) = \frac{|\xi|}{\sqrt{1 + |\xi|^2}}, \quad \cos \left( \frac{\theta}{2} \right) = \frac{1}{\sqrt{1 + |\xi|^2}} \quad (A.1)
\]

so that

\[
\partial_\xi \theta = \frac{|\xi|}{(1 + |\xi|^2) \xi}, \quad \partial_\bar{\xi} \theta = \frac{|\xi|}{(1 + |\xi|^2) \bar{\xi}} \nonumber
\]

\[
\partial_\xi \phi = -\frac{i}{2\xi}, \quad \partial_\bar{\xi} \phi = \frac{i}{2\xi} \quad (A.2)
\]

with the result of equations (A.1) and (A.2), then \(d\theta^2\) and \(d\phi^2\) are written as

\[
\left( \partial_\xi \theta \ d\xi + \partial_\bar{\xi} \theta \ d\bar{\xi} \right)^2 = \frac{|\xi|^2}{(1 + |\xi|^2)^2} \left( \frac{1}{\xi^2} d\xi^2 + \frac{1}{\bar{\xi}^2} d\bar{\xi}^2 \right) + \frac{1}{(1 + |\xi|^2)^2} (d\xi d\bar{\xi} + d\bar{\xi} d\xi)
\]

\[
\left( \partial_\xi \phi \ d\xi + \partial_\bar{\xi} \phi \ d\bar{\xi} \right)^2 = -\frac{1}{4} \left( \frac{1}{\xi^2} d\xi^2 + \frac{1}{\bar{\xi}^2} d\bar{\xi}^2 - \frac{1}{|\xi|^2} (d\xi d\bar{\xi} + d\bar{\xi} d\xi) \right) \quad (A.3)
\]

and

\[
\sin^2 \theta = 4 \sin^2 \left( \frac{\theta}{2} \right) \cos^2 \left( \frac{\theta}{2} \right)
\]

\[
= \frac{4|\xi|^2}{(1 + |\xi|^2)^2} \quad (A.4)
\]
Substitute all the result from (A.3) and (A.4) into equations (2.18) then we have metric (2.19)

\[ ds^2 = dt^2 - dr^2 - \frac{2r^2}{(1 + |\xi|^2)^2}(d\xi d\bar{\xi} + d\bar{\xi} d\xi) \]  

(A.5)

So, we can write its tensor metric by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -\frac{2r^2}{(1+|\xi|^2)^2} \\
0 & 0 & -\frac{2r^2}{(1+|\xi|^2)^2} & 0 \\
\end{pmatrix}
\]  

(A.6)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -\frac{(1+|\xi|^2)^2}{2r^2} \\
0 & 0 & -\frac{(1+|\xi|^2)^2}{2r^2} & 0 \\
\end{pmatrix}
\]  

(A.7)
Appendix B

Derivation of Equations (3.40) and (3.41)

To derive equation (3.40), we need to use tensor metric (A.7), such that equation (3.39) becomes

\[ [P, \partial^\mu \partial_\mu P] = [P, g^{\mu\nu} \partial_\mu \partial_\nu P] \]  \hspace{1cm} (B.1)

As stated in section 3.4.2 that the source manifold is a two dimensional Euclidean space which is wrote is Riemann sphere coordinates. It means that tensor metric (A.7) only use variables of Riemann sphere \((\xi, \bar{\xi})\), then the equation (B.1) becomes

\[ [P, \partial^\mu \partial_\mu P] = [P, g^{\xi\bar{\xi}} \partial_\xi \partial_{\bar{\xi}} P + g^{\bar{\xi}\xi} \partial_{\bar{\xi}} \partial_\xi P] \]

\[ = [P, -(1 + |\xi|^2)^2 \frac{1}{r^2} \partial_\xi \partial_{\bar{\xi}} P] \]

\[ = -(1 + |\xi|^2)^2 \frac{1}{r^2} [P, \partial_\xi \partial_{\bar{\xi}} P] \]  \hspace{1cm} (B.2)

and from equation (3.39) then

\[ [P, \partial_\xi \partial_{\bar{\xi}} P] = 0 \]  \hspace{1cm} (B.3)

Next, for equation (3.41), we write equation (3.40) as follows:

\[ [P, \partial_\xi \partial_{\bar{\xi}} P] + [P, \partial_{\bar{\xi}} \partial_\xi P] = 0 \]

\[ \partial_\xi (P \partial_{\bar{\xi}} P) - \partial_{\bar{\xi}} (\partial_\xi PP) + \partial_{\bar{\xi}} (P \partial_\xi P) - \partial_\xi (\partial_{\bar{\xi}} PP) = 0 \]

\[ \partial_\xi [P, \partial_\xi P] + \partial_{\bar{\xi}} [P, \partial_{\bar{\xi}} P] = 0 \]  \hspace{1cm} (B.4)
Appendix C

Derivation of Equations (3.49) and (3.50)

Equation (3.49) is derived by writing it in form of matrix field $M$:

$$\begin{align*}
(I - P)M &= M - M|M|^{-2}M^\dagger M \\
&= 0 \quad \text{(C.1)}
\end{align*}$$

and if we take its Hermitian conjugate, then

$$\begin{align*}
((I - P)M)^\dagger &= 0 \\
M^\dagger(I - P) &= 0 \quad \text{(C.2)}
\end{align*}$$

While for equation (3.49), we can derive it from instanton solution as follows:

$$\begin{align*}
P\partial_\xi P &= 0 \\
P\partial_\xi (I - P) &= 0 \\
\partial_\xi (P(I - P)) - \partial_\xi P (I - P) &= 0 \\
\partial_\xi (M|M|^{-2}M^\dagger(I - P)) - \partial_\xi (M|M|^{-2}M^\dagger)(I - P) &= 0 \\
-\partial_\xi (M|M|^{-2})M^\dagger(I - P) - M|M|^{-2}\partial_\xi M^\dagger(I - P) &= 0 \\
M|M|^{-2}\partial_\xi M^\dagger(I - P) &= 0 \\
M|M|^{-2}(\partial_\xi M)^\dagger(I - P) &= 0 \quad \text{(C.3)}
\end{align*}$$
Appendix D

Derivation of Properties (3.59) and (3.60)

In this Appendix, we present derivation of the properties of equations (3.59) and (3.60). For equation (3.59), we use the fact that the sequence of $M_k$ are independent, so we have the expansion:

$$\partial_\xi M_l = \sum_{k=0}^{\lambda} M_k a_{kl}, \ l = 1, \ldots, \lambda \quad (D.1)$$

where $A_{kl}$ are the $(n \times n)$ matrices. We have an assumption that $M_0$ is holomorphic $\partial_\xi M_0 = 0$.

Multiplying (D.1) from the left by $M_m^\dagger$ then we have

$$M_m^\dagger \partial_\xi M_l = |M_m|^2 a_{ml} \quad (D.2)$$

From recurrence relation (3.56), gives

$$M_{m+1} = (I - P_m) \partial_x i M_m$$
$$M_{m+1}^\dagger = (\partial_\xi M_m)^\dagger (I - P_m)$$
$$(\partial_\xi M_m)^\dagger = M_{m+1}^\dagger + (\partial_\xi M_m)^\dagger P_m \quad (D.3)$$

and the left side of equation (D.2) becomes

$$M_m^\dagger \partial_\xi M_l = \partial_\xi (M_m^\dagger M_l) - \partial_\xi M_m^\dagger M_l$$
$$= \partial_\xi (M_l^\dagger M_l \delta_{lm}) - (\partial_\xi M_m)^\dagger M_l$$
$$= \delta_{lm} M_l^\dagger \partial_\xi M_l + \delta_{lm} \partial_\xi M_l^\dagger M_l - (\partial_\xi M_m)^\dagger M_l \quad (D.4)$$
Substitute equation (D.3) into (D.4), such that
\[ M_m^l \partial M_l = \delta_{lm} M_l^l \partial M_l M_l + \delta_{lm} \partial M_l M_l - (M_{l+1}^l + (\partial M_m^l) P_m^l) M_l \]
\[ = \delta_{lm} M_l^l \partial M_l M_l + \delta_{lm} \partial M_l M_l - \delta_{l,m+1} |M_l|^2 - \partial M_l M_l M_l \delta_{lm} \]
\[ = \delta_{lm} M_l^l \partial M_l M_l + \delta_{lm} \partial M_l M_l - \delta_{l,m+1} |M_l|^2 - \partial M_l M_l M_l \delta_{lm} \]
\[ = \delta_{lm} M_l^l \partial M_l M_l - \delta_{l,m+1} |M_l|^2 \]  
(D.5)
and substitute the result into equation (D.2) as
\[ M_m^l \partial M_l = |M_m|^2 a_{ml} \]
\[ = \delta_{lm} M_l^l \partial M_l M_l - \delta_{l,m+1} |M_l|^2 \]
\[ a_{ml} = \delta_{lm} |M_m|^2 M_l^l \partial M_l M_l - \delta_{l,m+1} |M_m|^2 |M_l|^2 \]  
(D.6)
If we substitute equation (D.6) into (D.1), we obtain
\[ \partial M_l = \sum_{k=0}^{\lambda} M_k a_{kl} \]
\[ = \sum_{k=0}^{\lambda} M_k \left( \delta_{lk} |M_k|^{-2} M_l^l \partial M_l M_l - \delta_{l,k+1} |M_k|^{-2} |M_l|^2 \right) \]
\[ = M_l |M_l|^{-2} M_l^l \partial M_l M_l - M_{l-1} |M_{l-1}|^{-2} |M_l|^2 \]  
(D.7)
Because \( a_{ll} = |M_l|^{-2} M_l^l \partial M_l M_l \) and \( a_{l-1,l} = -|M_{l-1}|^{-2} |M_l|^2 \), then
\[ \partial M_l = M_l a_{ll} + M_{l-1} a_{l-1,l} \]  
(D.8)
Next, we prove that \( a_{ll} = 0 \), \( l = 1, \ldots, \lambda \) or equivalent to \( M_l^l \partial M_l M_l = 0 \). For the case \( l = 1 \), using the construction (3.57) or (3.58), we obtain
\[ M_1 = (I - P_0) \partial M_0, \ \partial \partial M_0 = 0 \]
\[ \partial \partial M_l = \partial \partial (I - P_0) \partial M_0 + (I - P_0) \partial \partial \partial M_0 \]
\[ = -\partial \partial P_0 \partial M_0 \]
\[ = -\partial \partial (M_0 |M_0|^{-2} M_0^l) \partial M_0 \]
\[ = -M_0 \partial \partial (|M_0|^{-2} M_0^l) \partial M_0 \]
\[ M_l^l \partial \partial M_l = 0 \]  
(D.9)
For general case with \( 1 < k < \lambda \), we use the inductive argument by assuming that \( a_{kk} = 0 \) such that \( \partial \partial M_k = M_{k-1} a_{k-1,k} \).
As
\[
\partial_\xi P_k = \partial_\xi M_k |M_k|^{-2} M_k^\dagger + M_k \partial_\xi (|M_k|^{-2} M_k^\dagger) \\
= (\partial_\xi M_k + M_k \partial_\xi) (|M_k|^{-2} M_k^\dagger) \\
= (M_{k-1} a_{k-1,k} + M_k \partial_\xi) (|M_k|^{-2} M_k^\dagger)
\] (D.10)
using the construction (3.59) and orthogonality property (3.51), it follows that
\[
M_{k+1} = (I - P_0 - \cdots - P_k) \partial_\xi^{k+1} M_0 \\
\partial_\xi M_{k+1} = \partial_\xi (I - P_0 - \cdots - P_k) \partial_\xi^{k+1} M_0 + (I - P_0 - \cdots - P_k) \partial_\xi^{k+1} \partial_\xi M_0 \\
= - \partial_\xi (P_0 + \cdots + P_k) \partial_\xi^{k+1} M_0 \\
= - \sum_{l=0}^k \partial_\xi P_l \partial_\xi^{k+1} M_0
\] (D.11)
If we multiply equation (D.11) from the left by \( M_{k+1}^\dagger \)
\[
M_{k+1}^\dagger M_{k+1} = -M_{k+1}^\dagger \sum_{l=0}^k \partial_\xi P_l \partial_\xi^{k+1} M_0 \\
= - \sum_{l=0}^k M_{k+1}^\dagger (M_{l-1} a_{l-1,l} + M_l \partial_\xi) (|M_l|^{-2} M_l^\dagger) \partial_\xi^{k+1} M_0 \\
= 0
\] (D.12)
then it shows that \( a_{k+1,k+1} = 0 \). Therefore, equation (D.8) becomes
\[
\partial_\xi M_l = M_l a_{ll} + M_{l-1} a_{l-1,l} \\
= M_{l-1} a_{l-1,l} \\
= -M_{l-1} |M_{l-1}|^{-2} |M_l|^2
\] (D.13)
which is proof of equation (3.59).
We may write equation (D.13) as follows:
\[
\partial_\xi M_k = (a_{k-1,k})^\dagger M_{k-1}^\dagger
\] (D.14)
and use the orthogonality property (3.50), then
\[
\partial_\xi |M_k|^2 = M_k^\dagger \partial_\xi M_k + \partial_\xi M_k^\dagger M_k \\
= M_k^\dagger \partial_\xi M_k + (a_{k-1,k})^\dagger M_{k-1}^\dagger M_k \\
= M_k^\dagger \partial_\xi M_k
\] (D.15)
such that
\[ \partial_\xi (M_k|M_k|^{-2}) = \partial_\xi M_k|M_k|^{-2} + M_k \partial_\xi |M_k|^{-2} \]  \hspace{1cm} (D.16)
We have
\[
\partial_\xi (|M_k|^2|M_k|^{-2}) = 0 \\
|M_k|^2 \partial_\xi |M_k|^{-2} + \partial_\xi |M_k|^2|M_k|^{-2} = 0 \\
\partial_\xi |M_k|^{-2} = -|M_k|^{-2} \partial_\xi |M_k|^2|M_k|^{-2} \]  \hspace{1cm} (D.17)
and substitute it into equation (D.16), so that
\[
\partial_\xi (M_k|M_k|^{-2}) = \partial_\xi M_k|M_k|^{-2} - M_k|M_k|^{-2} \partial_\xi |M_k|^2|M_k|^{-2} \\
= \partial_\xi M_k|M_k|^{-2} - M_k|M_k|^{-2} M_k^\dagger \partial_\xi M_k|M_k|^{-2} \\
= (I - P_k) \partial_\xi M_k|M_k|^{-2} \\
= M_{k+1}|M_k|^{-2} \]  \hspace{1cm} (D.18)
which is proof of equation (3.60).
Appendix E

Derivation of Equations (3.82) and (3.84)

Derivation of equation (3.82) works as follows:

\[ |M_k\dagger \partial_\xi \partial_\xi M_k|M_k|^{-2} = M_k\dagger \partial_\xi \left(-M_{k-1}|M_{k-1}|^{-2}|M_k|^2\right) |M_k|^{-2} \]
\[ = -\partial_\xi \left(M_k\dagger M_{k-1}|M_{k-1}|^{-2}|M_k|^2\right) |M_k|^{-2} \]
\[ + (\partial_\xi M_k)\dagger M_{k-1}|M_{k-1}|^{-2}|M_k|^2 |M_k|^{-2} \]
\[ = -|M_k|^2|M_{k-1}|^{-2} M_k\dagger M_{k-1} |M_{k-1}|^{-2} \]
\[ = -|M_k|^2|M_{k-1}|^{-2} \quad (E.1) \]

While for equation (3.84), we derive as follows:

\[ M_k\dagger \partial_\xi (P_k \partial_\xi M_k)|M_k|^{-2} = \partial_\xi (M_k\dagger P_k \partial_\xi M_k)|M_k|^{-2} - \partial_\xi M_k\dagger P_k \partial_\xi M_k |M_k|^{-2} \]
\[ = \partial_\xi (M_k\dagger \partial_\xi M_k)|M_k|^{-2} - \partial_\xi M_k\dagger M_k |M_k|^{-2} M_k\dagger \partial_\xi M_k |M_k|^{-2} \]
\[ = \partial_\xi \left(\partial_\xi (M_k\dagger M_k) - \partial_\xi M_k\dagger M_k \right) |M_k|^{-2} \]
\[ = \partial_\xi \partial_\xi (M_k\dagger M_k)|M_k|^{-2} \]
\[ = \partial_\xi \left(\partial_\xi |M_k|^2|M_k|^{-2}\right) - \partial_\xi (M_k\dagger M_k) \partial_\xi |M_k|^{-2} \]
\[ = \partial_\xi \left(\partial_\xi |M_k|^2|M_k|^{-2}\right) - \partial_\xi M_k\dagger M_k \partial_\xi |M_k|^{-2} - M_k\dagger \partial_\xi M_k \partial_\xi |M_k|^{-2} \]
\[ = \partial_\xi \left(\partial_\xi |M_k|^2|M_k|^{-2}\right) \quad (E.2) \]
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