POLYNOMIALS DEFINED BY TABLEAUX AND LINEAR RECURRENCES

PER ALEXANDERSSON

Abstract. We show that several families of polynomials defined via fillings of diagrams satisfy linear recurrences under a natural operation on the shape of the diagram. We focus on key polynomials, (also known as Demazure characters), and Demazure atoms. The same technique can be applied to Hall–Littlewood polynomials and dual Grothendieck polynomials.

The motivation behind this is that such recurrences are strongly connected with other nice properties, such as interpretations in terms of lattice points in polytopes and divided difference operators.

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1. Introduction

Using a similar technique as in [Ale14], we provide a framework for showing that under certain conditions, polynomials encoding statistics on certain tableaux, or fillings of diagrams, satisfy a linear recurrence. We prove that several of the classical polynomials from representation theory fall into this category, such as (skew) Schur polynomials and Hall–Littlewood polynomials.

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The main concern in this paper are the so called key polynomials, indexed by integer partitions, and atoms. The key polynomials are natural, non-symmetric generalizations of Schur polynomials and are specializations of the non-symmetric integer form Macdonald polynomials, see [Mas09] for details.

Let $\lambda$ be a fixed diagram shape, (a partition shape, skew shape, etc.) and let $P_k(\lambda)(x)$, $k = 1, 2, \ldots$, be a sequence of polynomials which are generating functions of fillings of shape $k\lambda$. For partitions, $k\lambda$ is simply elementwise multiplication by $k$. There are several reasons why one would be interested in showing that a such sequence satisfies a linear recurrence:

1. To obtain hints about the existence or non-existence of formulas of certain type. For example, the Weyl determinant formula for Schur polynomials implies that the ordinary Schur polynomials satisfy a linear recurrence.
2. To obtain evidence for alternative combinatorial interpretations of the tableaux involved. For example, the skew Schur polynomials can be obtained as lattice points in certain marked order polytopes, called Gelfand–Tsetlin polytopes. Such a polytope interpretation implies the existence of a linear recurrence relation.
3. To prove polynomiality in $k$ of the number of fillings of shape $k\lambda$.
4. To obtain results about asymptotics. For example, in [Ale12] we used such recurrences to give a new proof of a classical result on asymptotics of eigenvalues of Toeplitz matrices.

In the last section, we provide several examples of polynomials that satisfy such linear recurrences. We also sketch two additional proofs in the case of key polynomials, to illustrate that several nice properties imply the existence of a linear recurrence relation. These methods are based on a lattice-point representation and an operator characterization of the key polynomials. There is no straightforward way to check if a family of polynomials have such characterizations, but it is easy to generate computer evidence that a sequence of polynomials satisfy a linear recurrence. Thus, proving the existence or non-existence of linear recurrence relations is an informative step towards alternative combinatorial descriptions of the family of polynomials.

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2. Diagrams and fillings

A diagram $D$ is a subset of $\{(i, j) : i, j \geq 1\}$ which realized as an arrangement of boxes, with a box at $(i, j)$ for every $(i, j)$ in $D$. Here, $i$ refers to the row and $j$ is the column of box $(i, j)$ and we draw diagrams in the English notation. For example, $D = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 2)\}$ is shown as

```
  +---+
  |   |
  +---+---+
  |   |   |
  +---+---+---+
  |   |   |   |
  +---+---+---+---+
```

.
A filling is said to have \( l \) rows, if row \( l \) contains a box, but every row below row \( l \) is box-free.

Given an integer composition \( \alpha = (\alpha_1, \ldots, \alpha_l) \), the diagram of shape \( \alpha \), \( D_\alpha \), is given by \( D_\alpha = \{(i, j) : 1 \leq j \leq \alpha_i, 1 \leq i \leq l\} \). If \( \beta = (\beta_1, \ldots, \beta_l) \) is another integer composition such that \( \alpha \supseteq \beta \), that is, \( \alpha_i \geq \beta_i \) for all \( i \), then the diagram of shape \( \alpha/\beta \) is given by the set-theoretical difference \( D_\alpha \setminus D_\beta \), and is denoted \( D_{\alpha/\beta} \). Finally, if \( D \) is a diagram, let \( kD \) be the diagram obtained from \( D \) by repeating each column in \( D \) \( k \) times, that is,

\[
kD = \bigcup_{(i, j) \in D} \{(i, kj - k + 1), (i, kj - k + 2), \ldots, (i, kj)\}.
\]

Note that \( kD_{\alpha/\beta} = D_{k\alpha/k\beta} \).

2.1. Fillings. A filling of a diagram is a map \( T : D \rightarrow \mathbb{N} \), which we represent by writing \( T(i, j) \) in the box \((i, j)\). For example,

\[
\begin{array}{c}
1 & 8 \\
\times \times & 9 & 1 \\
\times \times & 5 \\
\times \times \times & 9 & 2 & 7 \\
\end{array}
\]

is a filling of the diagram with shape \((2, 4, 3, 1, 6)/(0, 2, 1, 3)\), where the places marked \( \times \) correspond to boxes in \( D_\beta \). The shape of a filling refers to the shape of the underlying diagram.

The \( j \)th column in a diagram \( D \) with \( l \) rows has a shape defined by the integer composition \((s_1, \ldots, s_l)\), where \( s_i = 1 \) if \((i, j) \in D\) and 0 otherwise. Thus, if \( \alpha \) is an integer composition with only 0 and 1 as parts, then the first column of \( D_\alpha \) has shape \( \alpha \). Whenever \( \alpha \) is an integer partition, \( D_\alpha \) is called a Young diagram and any filling of a Young diagram is called a tableau. A filling with shape \( \alpha/\beta \) where both \( \alpha \) and \( \beta \) are partitions, is called a skew tableau.

Given a diagram or a filling, we can duplicate or delete columns. For example, deleting the fourth column and duplicating the third column in the filling in Eq. (1) results in the filling

\[
\begin{array}{c}
1 & 8 \\
\times \times & 9 & 9 & 9 \\
\times \times & 5 & 5 & 5 \\
\times \times \times \times \times & 2 & 7 \\
\end{array}
\]

Note that if the original filling \( T \) has shape \( D_{\alpha/\beta} \), then duplication and deletion on \( T \) will result in some \( T' \) of shape \( D_{\alpha'/\beta'} \). This is straightforward to prove.

2.2. Column-closed families of fillings. In most applications, one is interested in a restricted family of fillings, perhaps tableaux or skew tableaux, together with some conditions on the numbers that appear in the boxes. Note that a filling \( T \) can be viewed as a concatenation of its columns — some of which might be empty. Obviously, \( T \) can only be expressed in one such way if we require that the last (rightmost) column is non-empty.
Let \((C_1, \ldots, C_l)\) be a filling with columns \(C_1, \ldots, C_l\) and let \((m_1C_1, \ldots, m_lC_l)\) denote the filling with \(m_1\) copies of the column \(C_1\), followed by \(m_2\) copies of \(C_2\) and so on. Finally, the concatenation, \(\sim\), of two fillings \((C_1, \ldots, C_l)\) and \((C'_1, \ldots, C'_l)\) is simply given by
\[
(C_1, \ldots, C_l) \sim (C'_1, \ldots, C'_l) = (C_1, \ldots, C_l, C'_1, \ldots, C'_l).
\]

**Definition 1.** A family of fillings, \(\mathcal{T}\), is said to be weakly column-closed if
\[
(C_1, \ldots, C_i, \ldots, C_l) \in \mathcal{T} \text{ if and only if } (m_1C_1, \ldots, m_lC_l) \in \mathcal{T}
\]
holds for every combination of integers \(m_i\) where \(m_i \geq 1\). The family \(\mathcal{T}\) is said to be strictly column-closed if Eq. (3) holds for every combination where \(m_i \geq 0\). That is, the family is closed under deletion of any column.

Less formally, \(\mathcal{T}\) is weakly column-closed if it is closed under column duplication, and reduction of duplicate columns. The family is strictly column closed if it, in addition, is closed under removal of any column.

Combinatorial objects would not be interested if it weren’t for combinatorial statistics. A combinatorial statistic on a family \(\mathcal{T}\) is a map \(\sigma : \mathcal{T} \to \mathbb{N}^s\). We will study a special type of statistics on fillings:

**Definition 2.** A combinatorial statistic \(\sigma\) on a weakly column-closed family \(\mathcal{T}\) is affine if
\[
\sigma(m_1C_1, \ldots, m_lC_l) = A + S_1m_1 + S_2m_2 + \cdots + S_lm_l
\]
for all choices of \(m_i \geq 1\), where \(A\) and \(S_i\) are vectors in \(\mathbb{N}^s\). Similarly, \(\sigma\) defined on a strictly column-closed family \(\mathcal{T}\) is linear if
\[
\sigma(m_1C_1, \ldots, m_lC_l) = S_1m_1 + S_2m_2 + \cdots + S_lm_l
\]
for all choices of \(m_i \geq 0\). Note that this is equivalent with the statement that \(\sigma(T_1 \sim T_2) = \sigma(T_1) + \sigma(T_2)\) for every pair \(T_1, T_2\) of fillings such that \(T_1 \sim T_2\) is in \(\mathcal{T}\).

Note that the statistic given by \(w(T) = (w_1, w_2, \ldots, w_n)\) where \(w_i\) are the number of boxes filled with \(i\) in \(T\) is a linear statistic. This is usually called the weight of \(T\). Finally, two statistics \(\sigma_1 : \mathcal{T} \to \mathbb{N}^{s_1}\) and \(\sigma_2 : \mathcal{T} \to \mathbb{N}^{s_2}\) can be combined into a new statistic \(\sigma\) in the obvious manner as \(\sigma(T) = (\sigma_1(T), \sigma_2(T))\), which map to \(\mathbb{N}^{s_1+s_2}\).

## 3. Properties of linear recurrences

We first recall some basic notions about linear recurrences. This can be seen as analogous to the theory of linear differential equations.

A sequence \(\{a_k(x)\}_{k=0}^\infty\) of functions are said to satisfy a linear recurrence of length \(r\) if there are functions \(c_1(x), \ldots, c_r(x)\) such that
\[
a_k(x) + c_1(x)a_{k-1}(x) + \cdots + c_r(x)a_{k-r}(x) = 0
\]
for all integers \(k \geq r\). The polynomial (in \(t\))
\[
\chi(t) = t^k + c_1(x)t^{k-1} + \cdots + c_{r-1}(x)t + c_r(x)
\]
is called the characteristic polynomial of the recursion. If the characteristic polynomial factors as \((t - \rho_1)^{m_1} \cdots (t - \rho_r)^{m_r}\), where all \(\rho_i(x)\) are different, then one can express \(a_k(x)\) as
\[
a_k(x) = \sum_{i=1}^{r} (\rho_i(x))^k \sum_{j=0}^{m_j-1} g_{ij}(x)k^j
\]
for some functions \(g_{ij}(x)\), that only depend on the initial conditions, that is, the functions \(a_0(x)\) to \(a_{-1}(x)\). In the other direction, any sequence of functions which are of the form given in Eq. (5) satisfy a linear recurrence with \(\chi(t)\) as characteristic polynomial. Notice that the \(c_i\) are elementary symmetric polynomials in the \(\rho_i\), with some signs.

From now on, we are only concerned about sequences where the \(a_k(x)\) and \(\rho_j(x)\) are polynomials, which implies that the \(c_i(x)\) are polynomials and the \(g_{ij}(x)\) are rational functions. Let \(a_k(x)\) and \(b_k(x)\) be sequences of polynomials with characteristic polynomials given by \(\prod_i (t - \rho_i(x))^{p_i}\) and \(\prod_i (t - \rho_i(x))^{q_i}\) respectively, where some of the \(p_i\) or \(q_i\) may be zero. Then, as sequences for \(k = 0, 1, \ldots\),
- \(h(x)a_k(x)\) satisfy the same linear recurrence as \(a_k(x)\), where \(h(x)\) is any polynomial,
- \(a_k(x) + b_k(x)\) satisfy a linear recurrence with characteristic polynomial given by
  \[
  \prod_i (t - \rho_i(x))^{\max(p_i, q_i)}
  \]
- \(a_k(x) \cdot b_k(x)\) satisfy a linear recurrence with characteristic polynomial given by
  \[
  \prod_{p_i \geq 1, q_i \geq 1} (t - \rho_i(x)\rho_j(x))^{p_i+q_i-1}.
  \]

However, if \(\rho_{i_1}(x)\rho_{j_1}(x) = \rho_{i_2}(x)\rho_{j_2}(x)\) for some \((i_1, j_1) \neq (i_2, j_2)\), some roots of the characteristic equation can be removed — details are left as an exercise. As an example:
\[
a_k(x) = (1 + k^3)(5x)^k, \quad b_k(x) = (2 + k^2 - k^4)(2x - 1)^k
\]
satisfy linear recurrences with characteristic polynomials \((t - 5x)^4\) and \((t - (2x - 1))^5\) respectively. The product, \(a_k(x)b_k(x) = (1 + k^3)(2 + k^2 - k^4)(10x^2 - 5x)^k\) satisfy a linear recurrence with characteristic polynomial \((t - (10x^2 - 5x))^6\).
- \(a_{sk}(x)\) with \(s\) a fixed positive integer satisfy a linear recurrence with characteristic polynomial given by
  \[
  \prod_i (t - \rho_i(x)^s)^{p_i}.
  \]

The proofs for these statements follows from writing \(a_k(x)\) and \(b_k(x)\) in the form Eq. (5) and examining the expressions above. Note that if \(a_k(x)\) and \(b_k(x)\) have characteristic polynomials with simple roots, then so does \(h(x) \cdot a_k(x)\), \(a_k(x) + b_k(x)\), \(a_k(x) \cdot b_k(x)\) and \(a_{sk}(x)\).

Finally, the definition of a sequence satisfying a linear recurrence in Eq. (4) does not provide an easy method to check for a linear recurrence if the \(c_i\) are unknown.
A useful shortcut might then be the following observation: a sequence \( \{a_k(x)\}_{k=0}^{\infty} \) satisfy a linear recurrence of length \( r \) if and only if the following \( r \times r \)-determinant vanish for all \( k \geq r - 1 \):

\[
\begin{vmatrix}
  a_k & a_{k-1} & \cdots & a_{k-r+1} \\
  a_{k+1} & a_k & \cdots & a_{k-r+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k+r-1} & a_{k+r-2} & \cdots & a_k
\end{vmatrix}
\]

This classical trick can be found in e.g. [Lyn57].

### 3.1. Tableaux and linear recurrences.

**Lemma 3.** Let \( \mathcal{T} \) be a weakly column-closed family of fillings and \( T = (C_1, \ldots, C_l) \) is a fixed filling in \( \mathcal{T} \), where no adjacent columns are equal. Let \( \sigma : \mathcal{T} \rightarrow \mathbb{N}^n \) be a linear combinatorial statistic such that

\[
\sigma(a_1C_1, \ldots, a_lC_l) = a_1S_1 + a_2S_2 + \cdots + a_lS_l.
\]

Define the sequence of polynomials

\[
F_k(z) = \sum_{a_1, a_2, \ldots, a_l = k} z^{\sigma(a_1C_1, \ldots, a_lC_l)} \quad \text{and} \quad F_0(z) = (-1)^{l+1}.
\]

Then \( \{F_k(z)\}_{k=0}^{\infty} \) satisfy a linear recurrence, with characteristic polynomial

\[
(t - z^{S_1})(t - z^{S_2}) \cdots (t - z^{S_l}). \tag{6}
\]

**Proof.** Note that the definition of \( F_k(z) \) implies that \( F_k(z) \equiv 0 \) whenever \( 1 \leq k < l \), and that \( F_l(x) = z^{S_1 + \cdots + S_l} \). These are \( l \) conditions and it is easy to see that if we have a characteristic polynomial of the form (6), then \( F_0(z) \) must be equal to \((-1)^{l+1}\).

Any tableau of the form \((a_1C_1, \ldots, a_lC_l)\) where \( a_i \geq 1 \) and \( a_1 + a_2 + \cdots + a_l > l \), must have some \( a_i \geq 2 \). Thus, this tableau can be constructed from some \((a_1C_1, \ldots, (a_i-1)C_i, \ldots, a_lC_l)\) by duplicating column \( C_i \). However, there might be several ways to do this. By using an inclusion-exclusion argument, it is straightforward to show that

\[
F_{k+l}(z) - (z^{S_1} + \cdots + z^{S_l})F_{k+l-1}(z) + \cdots + (-1)^{l}(z^{S_1} \cdots z^{S_l})F_k(z) \equiv 0
\]

for all \( k \geq 0 \). Note that the coefficients are the elementary symmetric polynomials, evaluated at \( z^{S_1}, \ldots, z^{S_l} \), so factoring the characteristic polynomial gives exactly the expression in (6). \( \square \)

**Lemma 4.** Let \( \mathcal{T} \) be a weakly column-closed family of fillings, and let the \( T(a) \in \mathcal{T} \) be given as

\[
T(a) = T_1 \sim (a_1C_{1_1}, \ldots, a_{l_1}C_{1_{l_1}}) \sim T_2 \sim (a_2C_{2_1}, \ldots, a_{2l_2}C_{2l_2}) \sim \cdots
\]

\[
\sim T_m \sim (a_mC_{m_1}, \ldots, a_{m,l_m}C_{m,l_m}) \sim T_{m+1}
\]

where each \( T_i \) is some fixed (possibly empty) filling and no adjacent columns in each \((C_{1_1}, \ldots, C_{l_i})\) are equal. Furthermore, let \( \sigma : \mathcal{T} \rightarrow \mathbb{N}^n \) be an affine combinatorial statistic, such that

\[
\sigma(T(a)) = A + a_1S_{11} + \cdots + a_{m,l_m}S_{m,l_m}.
\]
Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a fixed integer composition and define the polynomial

$$G_\alpha(z) = \sum_{a_{ij} \geq 1 \atop a_{11} + a_{12} + \cdots + a_{1i} = \alpha_1} z^{a(T(\alpha))}$$

where the sum is over all $a_{ij} \geq 1$, $1 \leq i \leq m$, and $1 \leq j \leq \alpha_i$ such that $a_{i1} + \cdots + a_{il_i} = \alpha_i$. Then

$$G_{k\alpha}(z) = z^A \prod_{i=1}^m F_{k\alpha_i}^i(z)$$

where $F_{k\alpha_i}^i(z) = \sum_{c_{il} \geq 1 \atop c_{1i} + \cdots + c_{il_i} = k} z^{c_1 S_1 + \cdots + c_{il_i} S_{il_i}}$, $F_0(z) = -(-1)^{l_1}$. 

Proof. This is immediate from the definition of $\sigma$ and the $F_{k\alpha_i}^i(z)$, by simply substituting the definition of $F_{k\alpha_i}^i(z)$ in the product and recognizing the expression for $\sigma$. \qed

Note that the integer composition $\alpha$ should not be confused with some shape of a tableau. The composition $\alpha$ rather serves as the number of columns there are in each of the $m$ “blocks” of columns in $T(\alpha)$. The functions $G_{k\alpha}(x)$ can now be seen as the generating functions of $\sigma$, as the block sizes grows linearly with $k$, and each block $i$ consists of column fillings with columns from $\{C_{il_1}, \ldots, C_{il_i}\}$, each present at least once. However, note that if all $T_i$ are empty fillings (no columns), then

$$G_{k\alpha}(x)$$

can be seen as generating function for fillings of shape $kD$ for some fixed diagram $D$ as in Fig. 1. In the proof of Proposition 6, the relation between $\alpha$ and $D$ is explained in more detail.

Corollary 5. \{G_{k\alpha}(z)\}_{k=0}^\infty satisfy a linear recurrence with characteristic polynomial given by

$$\prod_{1 \leq j_1 \leq l_1 \atop 1 \leq j_2 \leq l_2} (t - z^{\alpha_1 S_{1j_1} \alpha_2 S_{2j_2} \cdots \alpha_m S_{mj_m}}).$$

Furthermore, if $\sigma$ is linear, then Eq. (7) can be expressed as

$$\prod_{1 \leq j_1 \leq l_1 \atop 1 \leq j_2 \leq l_2} (t - z^{\sigma(C_{1j_1}, C_{2j_2}, \ldots, C_{mj_m})}).$$

Multiple roots can be disregarded if $\{S_{1l}, \ldots, S_{il_i}\}$ are all distinct for every $i$. 

Figure 1. The role of $\alpha$. Here, all $T_i$ are empty.
Proof. Each \( F_i^k(z) \) can be seen as generated by a linear statistic \( \sigma'(T) = \sigma(T) - A \). Therefore, each of these satisfy a linear recurrence, according to Lemma 3. The theory of linear recurrences now imply that the \( G_{k\alpha}(z) \) also does, with a characteristic polynomial as described above, since \( G_\alpha \) is essentially a product of the \( F_i^k \).

Eq. (8) follows from linearity of \( \sigma \) together with the definition of \( \sigma \). Note that the value of \( \sigma(\alpha_1 C_{1j_1}, \alpha_2 C_{2j_2}, \ldots, \alpha_m C_{mj_m}) \) is defined by linearity \( \sigma \), but that the tableau we evaluate on might not be in \( T \) (if some \( \alpha_i = 0 \)) unless this family is strictly column closed. The statement about simple roots follows immediately from the theory of linear recurrences. \( \square \)

So far, we have only treated generating functions of subsets of tableaux where the columns are from a specified subset and each column appear at least once. We will now treat the case where only the family of fillings and the diagram shape defines the generating function. To do that, it is natural to restrict ourself to a special type of families of fillings.

A family \( T \) is said to be well-behaved if every filling \( T \in T \) satisfies the following two properties:

- if two columns in \( T \) are identical, then all columns in between are also identical to these two.
- if two columns \( C_1 \) and \( C_2 \) are different and \( C_1 \) appears to somewhere the left \( C_2 \), then \( C_1 \) never appears to the right of \( C_2 \) in some other filling \( T' \in T \).

For example, fillings such that every row is weakly decreasing (or increasing) are well-behaved.

**Proposition 6.** Let \( T \) be well-behaved weakly column-closed family of fillings and let \( \sigma \) be an affine statistic defined on \( T \). Let \( D \) be a fixed diagram and define the polynomials \( H_D(z) \) as

\[
H_D(z) = \sum_{T \in T(D, n)} z^{\sigma(T)}
\]

where \( T(D, n) \) is the set of all fillings in \( T \) with shape \( D \) and for every box \((i, j)\) in such a filling, we have \( 1 \leq T(i, j) \leq n \). Then \( \{H_{kD}(z)\}_{k=1}^\infty \) satisfy a linear recurrence. Furthermore, if \( \sigma \) is linear, then the characteristic polynomial of the recurrence is given by

\[
\prod_T \left( t - z^{\sigma(T)} \right)
\]

where \( T \) runs over all tableaux of shape \( D \) such that any adjacent columns of same shape in \( T \) are identical, and each \( T \) can be obtained from some \( T(kD, n) \) by deleting some columns. Note that \( T \) might not itself be an element in \( T \). However, if \( T \) is strictly column closed, then each such \( T \) is in \( T(D, n) \).

**Proof.** Note that every column in \( kD \) has the same shape as some column in \( D \). Since we may only fill boxes with entries from \([n]\), there is a finite number of columns that can appear in \( T(kD, n) \). Furthermore, since \( T \) is well-behaved, there is a finite number of lists of columns, \( (C_1, C_2, \ldots, C_l) \), such that all \( C_i \) are different and \( C_i \) never appears to the right of \( C_j \) in some filling in \( T \), whenever \( i < j \). Thus, for every \( k \), every filling in \( T(kD, n) \) can be obtained in a unique way from such a list,
by duplicating some columns in that list. Hence, $H_{kD}(z)$ can be expressed as a finite sum over such lists $(C_1, C_2, \ldots, C_l)$, where each term corresponds to fillings $T$ of shape $kD$ where each column in $T$ is in the list and every column in the list appears at least once in $T$.

More specifically, let the diagram $D$ be expressed as the concatenation $D = (\alpha_1 s_1, \alpha_2 s_2, \ldots, \alpha_m s_m)$ where the $s_i$ are column shapes and $s_i \neq s_{i+1}$ (here, we use the same notation as for filled columns). Then every filling in $T(kD, n)$ can be obtained in a unique way as

\[(a_{i1} C_{i1}, \ldots, a_{il_i} C_{il_i}) \sim (a_{21} C_{21}, \ldots, a_{2l_2} C_{2l_2}) \sim \cdots \sim (a_{m1} C_{m1}, \ldots, a_{ml_m} C_{ml_m})\]

where each $C_{ij}$ has shape $s_i$ and $a_{i1} + \cdots + a_{il_i} = \alpha_i$. Hence, $H_{kD}(x, t)$ can be expressed as a sum over polynomials of the same form as $G_{k\alpha}$ in Lemma 4. Corollary 5 tells us that each $G_{k\alpha}$ satisfy a linear recurrence, so the sum of such sequences must too. This proves the first statement in the proposition.

The second statement follows from Corollary 5 and observing that the $S_{ij}$ in Eq. (7) can be replaced by $\sigma(C_{ij})$, since $\sigma$ is linear. The observation that it is enough to only consider tableaux where adjacent columns of same shape have identical fillings is a consequence of the combinatorial interpretation of the $F_{kD}(z)$ in Lemma 4 — a block of size $k\alpha$ must have $\alpha_i$ copies of some column if $k$ is sufficiently large and now a similar inclusion-exclusion reasoning apply as in Lemma 3.

\[\square\]

**Corollary 7.** Let $(\sigma, \tau)$ be an affine statistic, such that the restriction to $\sigma$ is linear and $\sigma(C_1) \neq \sigma(C_2) \Rightarrow C_1 \neq C_2$ for any pair of columns that appear in a filling in $T$. Then the characteristic polynomial in (9) can be taken to have only simple roots.

**Proof.** The fact that $\sigma(C_1) \neq \sigma(C_2) \Rightarrow C_1 \neq C_2$ implies that the values of the $S_{ij}$ in (7) are all distinct. Since $\{F_{kD}(z)\}_{k=1}^\infty$ can be expressed as a sum of sequences, each of which has simple roots in its characteristic polynomial, the statement follows by using the theory in Section 3. \[\square\]

### 4. Augmented Fillings

This section introduce the diagram fillings that are responsible for key polynomials, Demazure atoms and Hall–Littlewood polynomials. We follow the terminology in [HLMvW11, Mas09], with a few minor modifications.

Let $\beta = (\beta_1, \ldots, \beta_n)$ be a list of $n$ different positive integers and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a weak integer composition, that is, a vector with non-negative integer entries. An augmented filling of shape $\alpha$ and basement $\beta$ is a filling of a Young diagram of shape $(\alpha_1, \ldots, \alpha_n)$ with positive integers, augmented with a zeroth column filled from top to bottom with $\beta_1, \ldots, \beta_n$.

**Definition 8.** Let $T$ be an augmented filling. Two boxes $a, b$ are attacking if $T(a) = T(b)$ and the boxes are either in the same column, or they are in adjacent columns, with the rightmost box in a row strictly below the other box.

\[
\begin{align*}
&\begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array} & \text{or} & \begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
a \\
b
\end{array}
\]

\[
\begin{array}{c}
a \\
b
\end{array}
\]

\[
\begin{array}{c}
a \\
b
\end{array}
\]
A filling is *non-attacking* if there are no attacking pairs of boxes.

**Definition 9.** Let $T$ be an augmented filling with weakly decreasing rows. An *inversion triple of type A* is an arrangement of boxes, $a$, $b$, $c$, located such that $a$ is immediately to the left of $b$, and $c$ is somewhere below $b$, and the row containing $a$ and $b$ is at least as long as the row containing $c$ and $T(a) \geq T(c) \geq T(b)$.

Similarly, an *inversion triple of type B* is an arrangement of boxes, $a$, $b$, $c$, located such that $a$ is immediately to the left of $b$, and $c$ is somewhere above $a$, and the row containing $a$ and $b$ is *strictly* longer than the row containing $c$ and $T(a) \geq T(c) \geq T(b)$.

![Type A: $\begin{array}{c} a \\ b \\ c \end{array}$ Type B: $\begin{array}{c} c \\ a \\ b \end{array}$](Equation (10))

**Warning!** This definition slightly differs from what is stated in [HLMvW11]. However, the definitions coincide whenever the rows in the filling are weakly decreasing and we are only concerned with that special case.

**Definition 10.** A *semi-standard augmented filling* (SSAF) of shape $\alpha$ and basement $\beta$ is an augmented filling of shape $\alpha$ and basement $\beta$ with weakly decreasing rows and without any inversion triples.

Note that this definition implies that there are no attacking boxes in an SSAF. In particular, all entries in every column are different.

**Example 11.** Here is an example of a semi-standard augmented filling, with basement $(1, 3, 2, 5, 4)$.

\[
\begin{array}{cccc}
1 \\
3 & 3 & 1 & 1 \\
2 & 2 \\
5 & 5 & 5 & 5 \\
4 & 4 & 3 & 2 \\
\end{array}
\]

We can for example check the underlined entries for the type $B$ inversion triple condition — since $4 \leq 1 \leq 3$ is *not* true, they do not form such a triple. It is left as an exercise to check that no other triples are inversion triples.

**Lemma 12.** The semi-standard augmented fillings is a weakly column-closed, well-behaved family.

**Proof.** It suffices to show that duplication of a column in a SSAF $T$ do not introduce any inversion triples and it is enough to check that there are no inversion triples in adjacent and identical columns.

Assume that $a$, $b$, $c$ form an inversion triple, as in Eq. (10) (either type). Since the columns are identical, $T(a) = T(b)$ which implies $T(a) = T(c) = T(b)$. However, this means that two boxes in the same column are identical, so they are attacking. This contradicts the fact that the filling is a SSAF.

Let SSAF($\beta$, $\alpha$) be the set of all semi-standard augmented fillings with basement $\beta$ and shape $\alpha$. Note that SSAF($\beta$, $\alpha$) is a finite set. Given an augmented filling $T$,
let \( w(T) = (w_1, \ldots, w_n) \) where \( w_i \) count the number of boxes with content \( i \) not including the basement. The generalized Demazure atoms are defined as

\[
A_{\beta,\alpha}(x) = \sum_{T \in \text{SSAF}(\beta,\alpha)} x^{w(T)}.
\]

The special case when \( \beta_i = i \) corresponds to the ordinary Demazure atoms, introduced by Lascoux and Schützenberger in [LS90] under the name standard bases.

Let \( \text{NAWF}(\alpha) \) denote the set of all non-attacking augmented fillings of shape \( \alpha \) with weakly decreasing rows and basement given by \( \beta_i = i \). The non-symmetric, integral form Hall–Littlewood polynomials, \( E_{\alpha}(x_1, \ldots, x_n) \), may be defined as

\[
E_{\alpha}(x; t) = \sum_{T \in \text{NAWF}(\alpha)} x^{w(T)} t^{\text{coinv}(T)} (1 - t)^{\text{dn}(T)}
\]

where \( \text{coinv}(T) \) is the number of triples in \( T \) which are not inversion triples, and \( \text{dn}(T) \) is the number of pairs of adjacent boxes, \( (i, j) \) and \( (i, j+1) \) such that \( T(i, j) \neq T(i, j+1) \) (different neighbors). This formula was first given in [HLMvW11]. It is straightforward to show that the \( \text{NAWF} \) form a weakly column-closed and well-behaved family. They show that the ordinary Hall–Littlewood polynomials \( P_{\mu}(x; t) \) can be expressed as

\[
P_{\mu}(x; t) = \sum_{\gamma \rightarrow \lambda(\gamma)} E_{\gamma}(x; t)
\]

where \( \lambda(\gamma) \) is the unique integer partition that is obtained from the weak integer composition \( \gamma \) by sorting the parts in decreasing order.

**Lemma 13.** The statistics \( \text{dn} \) and \( \text{coinv} \) are affine statistics.

**Proof.** It follows immediately from the definition of \( \text{dn} \) that if \( \text{dn}(C_1, \ldots, C_l) = A \), then \( \text{dn}(m_1 C_1, \ldots, m_l C_l) = A \) for all \( m_i \geq 1 \), so this is affine. The fact that \( \text{coinv} \) is affine is also quite straightforward and is left as an exercise to the reader.

Using Proposition 6, Lemma 12 and Corollary 7, we have the following result:

**Corollary 14.** The sequences \( A_{\beta,\alpha}(x) \) and \( E_{\alpha}(x; t) \) for \( k = 1, 2, \ldots \) satisfy linear recurrences with simple roots.

Note that Eq. (13) implies that the ordinary Hall–Littlewood \( P \) polynomials satisfy a linear recurrence. These are usually (see [Mac95]) defined as

\[
P_{\lambda}(x; t) = \left( \prod_{i \geq 0} \prod_{j=1}^{m_\lambda(i)} \frac{1 - t x_j}{1 - t^i} \right) \sum_{\sigma \in S_n} \sigma \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} \frac{x_i - t x_j}{x_i - x_j} \right),
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \), some parts might be zero and \( m_\lambda(i) \) denotes the number of parts of \( \lambda \) equal to \( i \), and \( \sigma \) act on the indexing of the variables.

Observe that from this definition, it is quite clear that \( \{ P_{k\lambda}(x; t) \}_{k=1}^{\infty} \) satisfies a linear recurrence, since for a fixed \( \sigma \) in (14), the expression is of the form \( g(x; t) \sigma(x)^{\lambda} \) where \( g \) is independent under \( \lambda \mapsto k\lambda \). Now compare with Eq. (5) above.
4.1. Key tableaux and key polynomials. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a weak integer composition. To any such composition, construct a composition with unique entries, \( \beta \), and a partition \( \lambda \) as follows: Create an augmented Young diagram with shape \( \alpha \) and fill the zeroth column with the numbers 1, \ldots, n in decreasing order. Remove all rows for which \( \alpha_i = 0 \) and sort the remaining rows according to the number of boxes, in a decreasing manner. If two rows has the same number of boxes, preserve the relative order. The resulting diagram has the shape of a partition, \( \lambda \), which we denote \( \lambda(\alpha) \), and the zeroth column will be our basement \( \beta(\alpha) \). It is easy to show that this process can be reversed, that is, to any pair \( (\beta, \lambda) \), there is a corresponding \( \alpha \). Finally, note that \( \beta(k\alpha) = \beta(\alpha) \) and \( \lambda(k\alpha) = k\lambda(\alpha) \) for non-negative integers \( k \).

This correspondence is illustrated in Eq. (15), for \( \alpha = (0, 2, 3, 4, 2, 0, 1) \) and the tuple \( \beta = (4, 5, 6, 3, 1) \), \( \lambda = (4, 3, 2, 2, 1) \).

\[
\begin{array}{cccc}
7 & 6 & 5 & 4 \\
4 & 3 & 2 & 1 \\
\end{array}
\quad \leftrightarrow \quad
\begin{array}{cccc}
4 & 5 & 6 & 3 \\
3 & 1 &  &  \\
\end{array}
\] (15)

The key polynomials generalize the Schur polynomials and are indexed by integer compositions. They can be defined as

\[
K_\alpha(x) = \sum_{T \in \text{ssaf}(\beta(\alpha), \lambda(\alpha))} x^{w(T)}. \tag{16}
\]

Note that the key polynomials are a subset of the generalized Demazure atoms. This motivates the definition of a key tableau as a semi-standard augmented filling of partition shape, and we let \( k\text{tab}(\beta, \lambda) = \text{ssaf}(\beta, \lambda) \) to emphasize that this subset is of special interest. Note that we only need to be concerned about inversion triples of type A since we are dealing with partition shapes.

Given a column \( (\beta_1, \ldots, \beta_n) \) and a set of entries \( \{c_1, \ldots, c_n\} \), there is at most one way to arrange the entries \( c_1, \ldots, c_n \) in a column next to \( \beta \) such that the result fulfills all properties of a key tableau. First of all, it is easy to see that a necessary condition is that \( \beta_i \geq c_i \) for all \( i \), for some enumeration of the \( c_i \), in order to have decreasing rows in the result.

Secondly, the lack of inversion triples in a key tableau implies that the order of the entries in the second column is unique; if \( (a, b, c) \) is an inversion triple of type A, then transposing the entries in box \( b \) and \( c \) yield a non-inversion triple. This defines a total order among the elements in the second column, so there can be at most one filling where the second column (as a set) consists of the entries \( c_1, \ldots, c_n \).

The following lemma shows that under the obvious condition that if \( \beta_i \leq c_i \) for all \( i \), then the \( c_i \) can be arranged in such a way that the columns form a proper key tableau with basement \( \beta \). We prove a stronger statement:

\footnote{Although, in this paper, this convention will not be important.}
Lemma 15. Let $T$ be a Young diagram of partition shape $1 + \lambda$, filled with positive integers in such a way that each row is weakly decreasing, and each column contains unique entries, and the first column is given by $\beta$. Then each column in $T$ can be sorted in a unique way such that the result is a key tableau of with shape $\lambda$ and basement $\beta$.

Proof. We do the proof in several steps. The first case we cover is the case when all parts in $\lambda$ has the same size, that is, all columns of $T$ has the same height. It is easy to see that in this case, we only need to show the statement for two columns. Thus, assume $(\beta_1, \ldots, \beta_n)$ and $(c_1, \ldots, c_n)$ are given, with $\beta_i \geq c_i$ for all $i$.

We now perform the following sorting procedure on the second column. Let $c_i$ be the largest entry in the second column such that $\beta_1 \geq c_i$, and transpose $c_1$ and $c_i$. Since $c_i \geq c_1$, the rows are still weakly decreasing after this transposition. Note that $\beta_1$ and $c_i$ cannot be involved in an inversion triple later on: if there is some $c_j$ such that $\beta_1 \geq c_j > c_i$, this would violate the maximality of the choice of $c_i$.

We now proceed recursively on the remaining entries of the two columns, $(\beta_2, \ldots, \beta_n)$ and $(c_2, \ldots, c_n)$ where we have performed a transposition in the second column.

To handle tableaux with more than two columns, simply apply the permutation that takes the original column $c$ to the result on all subsequent columns. The result will now still be a tableau with weakly decreasing rows, but the first two columns do not contain any inversion triples. Proceed with the same method on column 2 and 3, then 3 and 4, and so on.

Note that if $c_1 < c_2 < \cdots < c_r \leq \beta_i$ for all $i$, and $c_r < c_j$ for all $j > r$, then the second column after the above procedure will end in the sequence $c_r, c_{r-1}, \ldots, c_1$, reading from top to bottom. Thus, to turn an arbitrarily shaped tableau $T$ into a key tableau, we first augment each column with negative integers, such that all columns have the same height, and a new entry on row $i$ will get the value $-i$. After performing the sorting procedure, the above observation implies that we can remove all boxes with negative entries from the result and recover a key tableau with the same shape as $T$.

Note that Lemma 15 implies that if $T$ is a key tableau, then one can remove any column from it, reorder the entries in each column and obtain another key tableau. In some sense, key tableaux behave similarly to a strictly column-closed family of tableaux.

Example 16. Here we illustrate the sorting procedure described in Lemma 15. We start with the tableau $T$ which is then augmented with negative integers.

\[
T = \begin{array}{cccc}
8 & 5 & 4 & 1 \\
| & 4 & 3 & 2 & 2 \\
6 & 6 & 5 \\
7 & 4 \\
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
8 & 5 & 4 & 1 \\
| & 4 & 3 & 2 & 2 \\
6 & 6 & 5 & -3 \\
7 & 4 & -4 & -4 \\
\end{array}
\]

The second column is then sorted and the entries in the other columns are permuted in the same fashion. Two more steps are performed to sort the remaining two
columns.

\[
\begin{array}{cccc}
8 & 6 & 5 & -3 \\
4 & 4 & 4 & -4 \\
6 & 5 & 4 & 1 \\
7 & 3 & 2 & 2 \\
\end{array} \rightarrow \quad
\begin{array}{cccc}
8 & 6 & 5 & -3 \\
4 & 4 & 4 & 1 \\
6 & 5 & 2 & 2 \\
7 & 3 & -4 & -4 \\
\end{array}
\]

Removing the boxes with negative entries now yield a proper key tableau with the same shape and basement as \( T \).

**Remark 17.** Note that Lemma 15 does not generalize to arbitrary semi-standard augmented fillings. For example, it is impossible to remove the first column in Example 11 and reorder the entries in the remaining non-basement columns into a valid SSAF — the 1s always appear in some attacking configuration.

4.2. **Key polynomial recurrence.** We are now ready to state one of the main results of this paper.

**Theorem 18.** The sequence of polynomials \( \{K_{kn}(\mathbf{x})\}_{k=1}^{\infty} \) satisfy a linear recurrence with

\[
\prod_{T} (t - x^T)
\]

as characteristic polynomial, where the product is taken over all key tableaux of shape \( \alpha \) such that columns of equal height have the same filling and multiple roots in the product are ignored.

**Proof.** This follows almost immediately from Proposition 6, except that \( \text{ktab} \) is not a strictly column-closed family. However, Lemma 15 implies that the tableaux that appear in the product Eq. (9), can be rearranged to key tableaux, while preserving the weight of the tableau. \(\square\)

5. **The polytope side**

In this section, we show that the integer point transform of polytopes with the integer decomposition property, (IDP), satisfy a linear recurrence. In particular, this can be used to give an alternate proof of Theorem 18.

An integral polytope is the convex hull of a finite set of integer points in \( \mathbb{R}^d \). The \( k \)-dilation of a polytope \( \mathcal{P} \) is defined as \( k\mathcal{P} = \{k\mathbf{x} : \mathbf{x} \in \mathcal{P}\} \) where \( k \) is a non-negative integer, and it is easy to see that this is an integral polytope if \( \mathcal{P} \) is. Furthermore, a polytope \( \mathcal{P} \) is said to have the integer decomposition property if for every integer \( k \geq 1 \), every lattice point \( \mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d \) can be expressed as \( \mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k \) with \( \mathbf{x}_i \in \mathcal{P} \). Note that only integral polytopes can have the integer decomposition property and that every face of a polytope with IDP is also a polytope with the IDP.

The following proposition shows that certain polynomials obtained from polytopes satisfies a linear recurrence. The argument is very similar to that in Lemma 3.

**Proposition 19.** Let \( \mathcal{P} \) be an integrally closed polytope in \( \mathbb{R}^d \) and let \( p_k(\mathbf{z}) \) be the polynomial defined as

\[
p_k(\mathbf{z}) = \sum_{\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^\mathbf{x}.
\]  

(17)
Then the sequence $p_j(z)$ for $j = 0, 1, \ldots$ satisfies a linear recurrence with characteristic polynomial given by

$$\prod_{x \in \mathcal{P} \cap \mathbb{Z}^d} (t - z^x).$$

**Proof.** Since $\mathcal{P}$ has the IDP, one can easily show that every lattice point in $k\mathcal{P}$ can be expressed as a sum of a lattice point in $(k-1)\mathcal{P}$ plus a lattice point in $\mathcal{P}$. Therefore

$$p_k(z) - \left( \sum_{x \in \mathcal{P} \cap \mathbb{Z}^d} z^x \right) p_{k-1}(t)$$

is a polynomial with only negative coefficients corresponding to points in $k\mathcal{P}$ that are expressible in more than one way as $x + y$ with $x$ in $(k-1)\mathcal{P} \cap \mathbb{Z}^d$ and $y$ in $\mathcal{P} \cap \mathbb{Z}^d$. Hence

$$p_k(z) - \left( \sum_{x \in \mathcal{P} \cap \mathbb{Z}^d} z^x \right) p_{k-1}(z) + \left( \sum_{x \neq y \in \mathcal{P} \cap \mathbb{Z}^d} z^x \cdot z^y \right) p_{k-2}(z)$$

is again a polynomial with positive coefficient corresponding to lattice points in $k\mathcal{P}$ expressible in at least three different ways. Repeating this argument using the principle of inclusion-exclusion then yield the desired formula. $\square$

The polynomial defined in Eq. (17) for $k = 1$ is commonly known as the integer-point transform of $\mathcal{P}$.

The intersection of two faces of a polytope is also a face (of possibly lower dimension) of the polytope. This enables us to generalize Proposition 19 slightly:

**Corollary 20.** Let $\mathcal{P}$ be a polytope with the integer decomposition property and let $F_1, \ldots, F_l$ be faces of $\mathcal{P}$. Define the sequence of polynomials

$$p_k(z) = \sum_{x \in k(\cup_i F_i) \cap \mathbb{Z}^d} z^x. \quad (18)$$

Then the sequence $p_j(z)$ for $j = 0, 1, \ldots$ satisfies a linear recurrence with characteristic polynomial given by

$$\prod_{x \in (\cup_i F_i) \cap \mathbb{Z}^d} (t - z^x).$$

**Proof.** The integer point transform of each face $F_i$ satisfy a linear recurrence under dilation by $k$, and the polynomials in Eq. (18) can be expressed (via inclusion-exclusion) as a linear combination of such integer point transforms of faces. $\square$

In [KST10], it was proven that key polynomials (Demazure characters) can be expressed as (a certain specialization of) the integer point transform of a union of faces of a Gelfand–Tsetlin polytope. Such polytopes are known to have the integer decomposition property, see e.g. [Ale14], so Corollary 20 implies a weaker version of Theorem 18. Note that the linear recurrences allow us to define $K_{\alpha}(x)$ and it follows from the polyhedral complex interpretation that this always evaluates to 1 (there is exactly one lattice point in the union of faces with dilation 0, namely the
origin). It would be interesting to see a direct proof of this fact without using the polytope interpretation.

After extensive computer experimentation, it is hard not to ask the following question:

**Question 21.** Does the polynomial \( k \mapsto K_{k\alpha}(1^n) \) always have non-negative coefficients?

The case when \( \alpha \) is a partition corresponds to a Schur polynomial and it is known that

\[
  s_\lambda(1^n) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}
\]

where \( n \) is the number of variables. This gives a positive answer to the question in this case.

6. **The operator side**

Some families of polynomials, such as the Schubert and key polynomials can be defined via *divided difference operators*. Let \( s_i \) denote the transposition \((i, i+1)\) and let such transpositions act on \( \mathbb{Z}[x_1, \ldots, x_n] \) by permuting the indices of the variables. Define the divided difference operators

\[
  \partial_i = 1 - s_i, \quad \pi_i = \frac{x_i}{x_i - x_{i+1}}.
\]

Given a permutation \( \omega \in S_n \), it can be expressed as a product of transpositions, \( \omega = s_{i_1} \cdots s_{i_l} \). When the length \( l \) is minimal, we say that \( i_1 i_2 \ldots i_l \) is a *reduced word* of \( \omega \). Then, let \( \partial_\omega = \partial_{i_1} \cdots \partial_{i_l} \) and \( \pi_\omega = \pi_{i_1} \cdots \pi_{i_l} \). It can be shown that these operators does not depend on the choice of the reduced word.

The key polynomials may now be defined [RS95] as \( K_\alpha(x) = \pi_{u(\alpha)} x^{\lambda(\alpha)} \), where \( \lambda(\alpha) \) is the partition obtained by sorting the parts of \( \alpha \) in decreasing order and \( u(\alpha) \) is a permutation that sorts \( \alpha \) into a partition shape. That this indeed is equivalent to the definition above was proved in [Mas09]. We will now give yet another proof that the key polynomials satisfy linear recurrences. First, note that \( x^\lambda \) is a geometric series as \( k = 0, 1, \ldots \) and thus satisfy a linear recurrence with characteristic polynomial \( t - x^\lambda \). Now note that if \( \{f_k(x)\}_{k=0}^\infty \) satisfy a linear recurrence, then so does \( \{\partial_i f_k(x)\}_{k=0}^\infty \) and \( \{\pi_i f_k(x)\}_{k=0}^\infty \). The result is now a consequence of induction.

The *Schubert polynomials*, \( \mathcal{S}_\omega(x) \), indexed by permutations in \( S_n \), are defined in a similar fashion,

\[
  \mathcal{S}_\omega(x) = \partial_{(w-w_0)} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1
\]

where \( w_0 \) is the longest permutation in \( S_n \), namely \((n, n-1, \ldots, 1)\) in one-line notation. Using a similar reasoning as for the key polynomials, one can produce sequences of Schubert polynomials that satisfies linear recurrences.
7. Appendix: Some families of column-closed fillings

In this section, we review some common families of column-closed fillings, related combinatorial statistics and generating functions over subsets of such families. Some statements here are well-known or very easy to show, so we present them without proof. Proposition 6 implies that all these polynomials we define below satisfy linear recurrences.

7.1. Flagged skew semi-standard Young tableaux. Let $\lambda$ and $\mu$ be partitions with at most $l$ parts, such that $\lambda \supseteq \mu$. Let $\text{ssyt}(\lambda/\mu, n)$ be the set of fillings of $D_{\lambda/\mu}$ with entries in $[n]$, such that each row is weakly increasing and each column is strictly increasing. Then for every $l$ and $n$, the families

$$\bigcup_{\lambda \supseteq \mu} \text{ssyt}(\lambda/\mu, n) \text{ and } \bigcup_{\lambda} \text{ssyt}(\lambda, n)$$

are strictly column-closed families, where the unions are taken over shapes with at most $l$ rows. On any filling $T$ with entries in $[n]$, we define the statistic $w(T) : T \to \mathbb{N}^n$ such that if $w_i$ is the number of boxes in $T$ filled with $i$. It is evident that $w$ is a linear statistic.

Finally, the skew Schur polynomials in $n$ variables, indexed by skew partition shapes $\lambda/\mu$, are defined as

$$s_{\lambda/\mu}(x) = \sum_{T \in \text{ssyt}(\lambda/\mu, n)} x^{w(T)}.$$

Even more general, let $\lambda \supseteq \mu$ be a shapes with at most $l$ rows and let $a$ and $b$ be increasing sequences of integers of length $l$, such that $a_i \leq b_i$ for all $i$. Let $\text{ssyt}(\lambda/\mu, a, b, n) \subseteq \text{ssyt}(\lambda/\mu, n)$ be the subset of fillings $T$, such that $a_i \leq T(i,j) \leq b_i$ for every box $(i,j) \in D_{\lambda/\mu}$. Then for each $n$,

$$\bigcup_{\lambda \supseteq \mu} \text{ssyt}(\lambda/\mu, a, b, n)$$

is a strictly column-closed family, where the union is taken over all $\lambda \supseteq \mu$ with at most $l$ rows. The row-flagged Schur polynomials, $s_{\lambda/\mu,a,b}(x)$ in $n$ variables are defined as

$$s_{\lambda/\mu,a,b}(x) = \sum_{T \in \text{ssyt}(\lambda/\mu,a,b,n)} x^{w(T)}$$

see e.g. [Wac85] as a reference.

7.2. Symplectic fillings. The following definition is taken from [Kin76] and these polynomials are related to representations of $Sp(2n)$. The symplectic Schur polynomials, $sp_{\lambda}(x)$, in the variables $x_1, x_2, \ldots, x_n$ are defined via fillings of the Young diagram $\lambda$ using the alphabet $1 < T < 2 < T < \cdots < n < n$ such that rows are weakly increasing, columns are strictly increasing, and entries in row $i$ are greater than or equal to $i$. Then, for a partition $\lambda$ with at most $n$ parts,

$$sp_{\lambda}(x) = \sum_{T \in \text{syt}(\lambda)} x^{w(T)} x^{-w(T)}$$
where $w(T)$ is the weight only counting unbarred entries and $\overline{w}(T)$ only counts the barred entries. It is quite clear that the symplectic Young tableaux form a strictly column-closed family, and that the statistics $w$ and $\overline{w}$ are linear. Consequently, $\{\text{sp}_{k\lambda}\}_{k=1}^{\infty}$ satisfies a linear recurrence for every fixed partition $\lambda$.

7.3. Set-valued tableaux and reverse plane partitions. The Grothendieck polynomials $G_\lambda(x)$ can be defined (see [Buc02]) as

$$G_\lambda(x) = \sum_{T \in \text{svt}(\lambda)} (-1)^{|T| - |\lambda|} x^{w(T)}$$

where the sum is taken over set-valued Young tableaux. These are defined as fillings of a diagram of shape $\lambda$, but now each box contains a set of natural numbers. For two such sets $A$, $B$ we have $A < B$ if $\max A < \min B$ and similar for $A \leq B$. With this notation, $\text{svt}(\lambda)$ is the set of all set-valued tableaux (subsets of $[n]$) such that rows are weakly increasing, and columns are strictly increasing. Here, the $i$th component of $w(T)$ is now the total number of sets where $i$ appears, and $|T|$ is the sum over all cardinalities of the sets in the boxes. Note that the lowest-degree part of $G_\lambda(x)$ is the usual Schur polynomial $s_\lambda(x)$.

There is also an operator definition of the more general Grothendieck polynomials which are indexed by permutations and similar to the Schubert polynomials and introduced by Lascoux and Schützenberger in 1982.

To show that $G_\lambda(x)$ satisfy a linear recurrence, one needs to use the more general version of Lemma 4, since the family of set-valued Young tableaux is not a weakly column-closed family; only columns where each set is a singleton can be duplicated. However, we note that the family is well-behaved and that every tableau $T \in \text{svt}(k\lambda)$ contains duplicate columns for every $k$ sufficiently large. These observations together with Lemma 4 allows us to deduce that the Grothendieck polynomials also satisfy a linear recurrence. We leave the details as an exercise to the reader. This can also be proved using the divided difference operator definition, similar to the Schubert polynomials.

Lam and Pylyavskyy [LP07] proved that the dual stable Grothendieck polynomials, $g_\lambda(x)$ in $n$ variables can be defined as

$$g_\lambda(x) = \sum_{T \in \text{rpp}(\lambda)} x^{ev(T)}$$

where $\text{rpp}(\lambda)$ is the set of reverse plane partitions of shape $\lambda$, that is, fillings of $\lambda$ with numbers in $[n]$ such that rows and columns are weakly decreasing. The statistic $ev(T)$ is the total number of columns where $i$ appears. Evidently, $ev$ is a linear statistic and reverse plane partitions is a well-behaved, strictly column-closed family of fillings. Consequently, we get a linear recurrence in this case.

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These are called the stable Grothendieck polynomials.
7.4. A note on Jack and Macdonald polynomials. A consequence of satisfying a linear recurrence, is that the sequence of polynomials must satisfy a linear recurrence under every specialization of the variables. In particular, if we pick a specialization such that all roots of the characteristic polynomial become equal to 1, then the resulting sequence is a polynomial. For example, \( k \mapsto K_{k\alpha}(1^n) \) is a polynomial in \( k \).

This observation allows us to deduce that the Jack polynomials \( J_{\lambda}(x, a) \) do not satisfy a linear recurrence for general values of \( a \), the sequences \( J_{k\lambda}(1^n, a) \) are not of the form given in Eq. (5). This observation holds for both standard normalizations of Jack polynomials.

It follows that there are no linear recursions for Macdonald polynomials either, since the Jack polynomials are a specialization of the Macdonald polynomials.

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