FUNCTIONS OF CLASSES $\mathcal{N}_\kappa^+$

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ABSTRACT. In the present note we give an elementary proof of the necessary and sufficient condition for a univariate function to belong the class $\mathcal{N}_\kappa^+$. Although this class was introduced mainly to deal with the indefinite version of the Stieltjes moment problem (and corresponding $\pi$-Hermitian operators), it can be useful far beyond the original scope. Our result elaborates the criterion given by Krein and Langer in their joint paper of 1977; they overlooked one attainable case. The correct condition was stated by Langer and Winkler in 1998, although they provided no proper reasoning.

1. INTRODUCTION

The function classes $\mathcal{N}_\kappa$ with $\kappa = 0, 1, \ldots$ were introduced in the prominent paper [3] of M. Krein and H. Langer. They serve as a natural generalisation of the Nevanlinna class $\mathcal{N} := \mathcal{N}_0$ of all holomorphic mappings $\mathbb{C}_+ \to \mathbb{C}_+$, which are also known as $\mathcal{R}$-functions (here $\mathbb{C}_+ := \{ z \in \mathbb{C} : \Re z > 0 \}$ is the upper half of the complex plane). A function $\varphi(z)$ belongs to $\mathcal{N}_\kappa$ whenever it is meromorphic in $\mathbb{C}_+$, for any set of non-real points $z_1, z_2, \ldots, z_k$ the Hermitian form

$$ h_\varphi(\xi_1, \ldots, \xi_k | z_1, \ldots, z_k) := \sum_{n,m=0}^{k} \frac{\varphi(z_m) - \varphi(z_n)}{z_m - z_n} \xi_n \xi_n $$

has at most $\kappa$ negative squares and for some set of points there are exactly $\kappa$ negative squares. It is convenient (and generally accepted) to define $\mathcal{N}_\kappa$-functions in the lower half of the complex plane by complex conjugation, i.e. $\overline{\varphi(z)} = \varphi(\overline{z})$. A significant particular case is presented by the classes $\mathcal{N}_0^+$, which are considered here. They contain all $\mathcal{N}_\kappa$-functions $\varphi(z)$ such that $z\varphi(z)$ belongs to $\mathcal{N}$. Among various applications, $\mathcal{N}^+$, $\mathcal{N}_\kappa^+$ and $\mathcal{N}_\kappa^{++}$ appear in the moment problems and have connections to the spectral theory of operators. However, the classes $\mathcal{N}_\kappa^+$ can find even more applications as a foremost generalisation of the Stieltjes functions $\mathcal{N}_0^+$.

This short note aims at proving the necessary and sufficient condition for a function to be in the class $\mathcal{N}_\kappa^+$ by methods of complex analysis. Our approach rests on the asymptotic analysis of the corresponding Hermitian forms. As a main tool, we use the basic Nevanlinna-Pick theory for the halfplane within the framework presented, for example, in [1] Chapter 3 or [2] Chapters II–III. We show that, roughly speaking, $\mathcal{N}_\kappa^+$ differs from the Stieltjes class $\mathcal{N}_0^+$ in having $\kappa$ simple negative poles, one of which can reach the origin and merge there into another singularity. More precisely, a function $\varphi(z)$ belongs to the class $\mathcal{N}_\kappa^+$ if and only if it has one the forms

$$ \varphi(z) = s_0 + \sum_{j=1}^{\kappa} \frac{\gamma_j}{\alpha_j - z} + \int_0^{\infty} \frac{d\nu(t)}{t-z}; \quad (A) $$

$$ \varphi(z) = s_0 + s_1 z^{-1} - \frac{s_2}{z^2} + \sum_{j=1}^{\kappa-1} \frac{\gamma_j}{\alpha_j - z} + \int_0^{\infty} \frac{d\nu(t)}{t-z}, \quad \text{where } \max\{s_1, s_2\} > 0, \nu(+0) = 0; \quad (B) $$

$$ \varphi(z) = s_0 + s_1 z^{-1} - \frac{s_2}{z^2} + \sum_{j=1}^{\kappa-1} \frac{\gamma_j}{\alpha_j - z} + \frac{1}{z} \int_0^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t), $$

$$ \text{where } \int_0^{1} \frac{d\sigma(t)}{t} = \infty, \sigma(+0) = 0, \int_0^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty. \quad (C) $$

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Here $s_0, s_2 \geq 0$, $s_1 \in \mathbb{R}$, $\gamma_j, \alpha_j < 0$ for $j = 1, 2, \ldots, \infty$ and $\nu(t)$, $\sigma(t)$ are nondecreasing left-continuous functions such that $\nu(0) = \sigma(0) = 0$ and $\int_0^\infty \frac{d\nu(t)}{t + \varepsilon^2} < \infty$. The function $\sigma(t)$ is intentionally denoted by a distinct letter to emphasize that its constraints at infinity are weaker. For our purposes, it is more convenient to formulate this criterion as Theorem 2 with respect to $z\varphi(z)$, to include poles at the origin into the Stieltjes integrals and to merge the cases (A) and (C) together.

This result corrects Theorem 3.8 of [3]: the authors put the condition $\int_0^\infty \frac{d\sigma(t)}{t + \varepsilon^2} < +\infty$ in the case (C), which is excessively strict. As a result, Theorem 3.8 fails to address $\mathcal{N}_{\infty}$-functions like
\[
\psi(z) = \frac{1}{\sqrt{z}} \cot \frac{1}{\sqrt{z}} = \sqrt{z} \cot \sqrt{z}
\]
with $\varepsilon = 1$. More than likely, this mistake is just an oversight: for proving the representations (A)–(B) authors use, in fact, the measure $d\nu(t)$; then they put $\nu(t)$ instead of $\sigma(t)$ in (C). Furthermore, their proof involves operator theory, which makes it less transparent.

Lemma 5.3 from [4, p. 421] (see Theorem 2 herein) has a proper statement, and the function $\psi(z)$ given above is allowed as an entry of $\mathcal{N}_{\infty}^{-}$. At the same time, the proof in [3] relies on the aforementioned Theorem 3.8 from [3], and the relevant piece of the proof is omitted as “similar” to another part. Our approach does not depend on the results of the works [3][4].

2. PRELIMINARIES

Each $\mathcal{N}$-function $\Phi$ has the following integral representation (see e.g. [11, p. 92], [2, p. 20]):
\[
\Phi(z) = bz + a + \int_{-\infty}^\infty \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t)
\]
where $a$ is real, $b \geq 0$ and $\sigma(t)$ is a real non-decreasing function satisfying $\int_{-\infty}^\infty \frac{d\sigma(t)}{1 + t^2} < \infty$. The converse is also true: all functions of the form $\Phi(z)$ belong $\mathcal{N}$.

To be definite, we assume that the function $\sigma(t)$ is left-continuous, that is $\sigma(t) = \sigma(t - 0)$ for all $t \in \mathbb{R}$. Accordingly, the notation for integrals with respect to $d\sigma(t)$ is as in the formula $\int_0^\infty f(t) d\sigma(t) := \int_{(a, b)} f(t) d\sigma(t)$ for arbitrarily taken real numbers $a, b$ and function $f(t)$.

Remark 1. A function $\Phi$ given by the formula (2) is holomorphic outside the real line. Furthermore, it has an analytic continuation through the intervals outside the support of $d\sigma$. The function $\varphi(z) := \Phi(z)/z$ has the same singularities with the exception of the origin (generally speaking). We can additionally note that,

if $z_1 < z_2 < t$ or $t < z_1 < z_2$, then $1 \frac{z_2 - z_1}{t - z_2} + \frac{1}{t - z_1} = \frac{z_2 - z_1}{(t - z_1)(t - z_2)} > 0$.

Consequently, given a real interval $(\alpha, \beta)$ that has no common points with the support of $d\sigma$, the condition $\alpha < z_1 < z_2 < \beta$ implies $\Phi(z_1) < \Phi(z_2)$ unless $\Phi(z) \equiv a$, which is seen from the representation (2). (This fact is also seen immediately from the definition of $\mathcal{N}$: see [2, p. 18]). Put in other words, the function $\Phi(z)$ increases in the interval $(\alpha, \beta)$ unless it is identically constant.

Theorem 2 (Coincides with Lemma 5.3 from [4, p. 421]). Let a function $\Phi \in \mathcal{N}$. The function $\varphi(z) := \Phi(z)/z$ belongs to $\mathcal{N}_{\infty}$ if and only if the representation (2) of $\Phi(z)$ is either of the form
\[
\Phi(z) = bz + a + \sum_{n=1}^{\infty} \frac{\sigma_n}{\lambda_n - z} + \int_0^{\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t)
\]
where $\lambda_n < 0$, $n = 1, 2, \ldots, \infty - 1$ and
\[
0 < \Phi(-0) = a + \sum_{n=1}^{\infty} \frac{\sigma_n}{\lambda_n} + \int_0^{\infty} \frac{d\sigma(t)}{t + \varepsilon^2} \leq \infty,
\]
or of the form
\[
\Phi(z) = bz + a + \sum_{n=1}^{\infty} \frac{\sigma_n}{\lambda_n - z} + \int_0^{\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t),
\]
where \( \lambda_n < 0, \ n = 1, 2, \ldots, \kappa \) and

\[
\Phi(-\varepsilon) = a + \sum_{n=1}^{\kappa} \frac{\sigma_n}{\lambda_n} + \int_{0}^{\infty} \frac{d\sigma(t)}{t + t^3} \leq 0.
\] (4b)

Note that \( \mathcal{N} \)-functions of the forms (3a) and (4a) have the corresponding limit \( \Phi(-\varepsilon) \) defined, because (when non-constant) they grow monotonically (see Remark 1) outside the support of corresponding measure \( d\sigma(t) \). Moreover, all numbers \( \sigma_n \) in (3)–(4) are positive due to the condition \( \Phi \in \mathcal{N} \).

### 3. Proofs

**Definition 3.** A real point \( \lambda \) is called a point of increase of a function \( \sigma(t) \) if \( \sigma(\lambda + \varepsilon) > \sigma(\lambda - \varepsilon) \) for every \( \varepsilon > 0 \). In particular, the set of all points of increase of a non-decreasing function \( \sigma(t) \) is the support of \( d\sigma(t) \).

In each punctured neighbourhood of the point of increase \( \lambda \) exists \( \lambda' \) such that the limit

\[
\sigma'((\lambda') = \lim_{\varepsilon \to 0} \frac{\sigma((\lambda') + \varepsilon) - \sigma((\lambda') - \varepsilon)}{2\varepsilon}
\]

is positive or nonexistent. Indeed, otherwise \( \sigma'(t) \leq 0 \) in the closed interval \(-\varepsilon \leq t \leq \varepsilon\) for some \( \varepsilon \) small enough, thus integrating \( \sigma'(t) \) over this interval leads us to a contradiction. Consequently, if we know that the function \( \sigma(t) \) has at least \( \kappa \) negative points of increase, then we always can select \( \kappa \) points of increase \( \lambda_1 < \lambda_{\kappa-1} < \cdots < \lambda_1 < 0 \), in which the derivative \( \sigma' \) is nonexistent or positive. Given such a set of points denote

\[
\delta := \frac{1}{3} \min\left\{ -\lambda_1, \min_{1 \leq n \leq \kappa-1} \left( \lambda_n - \lambda_{n+1} \right) \right\},
\]

put \( U_n := (\lambda_n - \delta, \lambda_n + \delta) \), where \( n = 1, \ldots, \kappa, \) and \( U_0 := \mathbb{R} \setminus \left( \bigcup_{n=1}^{\kappa} U_n \right) \).

**Proposition 4.** Consider a function \( \Phi(z) = z\varphi(z) \) of the form (2). Let the function \( \sigma(t) \) have at least \( \kappa \) negative points of increase \( \lambda_\kappa < \cdots < \lambda_1 \) in which the derivative \( \sigma' \) is nonexistent or positive. Then the Hermitian form \( h_\eta(\xi_1, \xi_\kappa | \lambda_1 + \iota \eta, \cdots, \lambda_\kappa + \iota \eta) \) defined in (1) has \( \kappa \) negative squares for some small values of \( \eta > 0 \).

**Proof.** Since

\[
\frac{1}{t - z} = \frac{z + t - z}{t(t - z)} = \frac{z}{t(t - z)} + \frac{1}{t} \quad \text{and} \quad \frac{1}{t} - \frac{t}{1 + t^2} = \frac{1 + t^2 - t^2}{1(t + t^2)} = \frac{1}{t(t + t^2)},
\] (5)

from the expression (2) we obtain

\[
\varphi(z) = b + \frac{a}{z} + \frac{1}{z} \int_{-\infty}^{\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t) = b + \frac{a}{z} + \int_{-\infty}^{\infty} \left( \frac{1}{t - z} + \frac{1}{z(1 + t^2)} \right) \frac{d\sigma(t)}{t},
\] (6)

where

\[
\varphi_n(z) = \int_{U_n} \frac{1}{t - z} \frac{d\sigma(t)}{t}, \quad n = 1, \ldots, \kappa,
\]

are the terms principal on the intervals \( U_n \), and

\[
\varphi_0(z) := b + \frac{\tilde{a}}{z} + \frac{1}{z} \int_{U_0} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t) \quad \text{with} \quad \tilde{a} := a + \sum_{n=1}^{\kappa} \int_{U_n} \frac{d\sigma(t)}{t + t^3}
\]

contains the remainder term. Hereinafter we additionally assume \( z_n := \lambda_n + \iota \eta \).

Each function \( \varphi_n \) is holomorphic outside \( U_n \), therefore when \( m, k \neq n \) we have that the limits

\[
\lim_{\eta \to 0} \frac{\varphi_n(z_m) - \varphi_n(z_k)}{z_m - z_k} = \begin{cases} \frac{\varphi_n(\lambda_m) - \varphi_n(\lambda_k)}{\lambda_m - \lambda_k}, & \text{if } k \neq m, \\ \varphi_n'(\lambda_m), & \text{if } k = m \end{cases}
\] (7)
are finite as $n = 0, \ldots, \infty$. In opposite, for $n \neq 0$ with the notation $\sigma(t) = -\int_{-\lambda_n - \delta}^t s^{-1} \, ds$ we have
\begin{equation}
\rho_n^2(\eta) := -\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n} = \int_{U_n} \frac{t - z_n - t + z_n}{(t - z_n)(t - \overline{z}_n)} \, d\sigma(t) = \int_{U_n} \frac{d\sigma(t)}{(t - z_n)^2},
\end{equation}
consequently $\limsup_{\eta \to +0} (\eta \cdot \rho_n^2(\eta))$ is positive or $+\infty$ because $\lambda_n$ is a point of increase of $\sigma(t)$. Moreover, we fix $\rho_n(\eta) > 0$ for definiteness. In terms of big $O$ notation, there exists a sequence of positive numbers $\eta_1, \eta_2, \ldots$ tending to zero such that
\begin{equation}
\frac{1}{\rho_n(\eta_k)} \leq O\left(\sqrt{\eta_k}\right) \quad \text{as} \quad k \to +\infty \quad \text{and} \quad n = 1, \ldots, \infty.
\end{equation}
This fact compliments that, according to [7],
\begin{equation}
-\frac{\varphi(z_n) - \varphi(\overline{z}_n)}{z_n - \overline{z}_n} = -\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n} - \sum_{m \neq n} \frac{\varphi_m(z_n) - \varphi_m(\overline{z}_n)}{z_n - \overline{z}_n} = \rho_n^2(\eta) + O(1)
\end{equation}
when $\eta$ is considered as small. Furthermore,
\begin{equation}
-\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n} = \int_{U_n} \frac{t - z_n - t + z_n}{(t - z_n)(t - \overline{z}_n)} \, d\sigma(t) = \int_{U_n} \frac{d\sigma(t)}{(t - z_n)^2},
\end{equation}
which implies (with the help of the elementary inequality $2\alpha\beta \leq \frac{\alpha^2}{c} + c\beta^2$ valid for any positive numbers)
\begin{equation}
\left|\frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n}\right| \leq \int_{U_n} \frac{d\sigma(t)}{|t - z_n||t - \overline{z}_n|} \leq \int_{U_n} \frac{d\sigma(t)}{2|t - z_n|} \leq \int_{U_n} \frac{d\sigma(t)}{2|t - z_m|} = \frac{1}{2} \rho_n(\eta) + \frac{\rho_n(\eta)}{2} \int_{U_n} \frac{d\sigma(t)}{|t - z_m|} \leq C(n, m, \delta) \rho_n(\eta)
\end{equation}
provided that the distance between $U_n$ and $z_m$ is more than $\delta$. The factor $C(n, m, \delta) > 0$ in [10] is independent of $\eta$. The finiteness of [7] gives
\begin{equation}
\frac{\varphi(z_n) - \varphi(\overline{z}_n)}{z_n - \overline{z}_n} = \frac{\varphi_n(z_n) - \varphi_n(\overline{z}_n)}{z_n - \overline{z}_n} + O(1) \quad \text{as} \quad \eta \to +0.
\end{equation}
This can be combined with the estimate [10] thus giving us for small $\eta$
\begin{equation}
\left\|\frac{\varphi(z_n) - \varphi(\overline{z}_n)}{(z_n - \overline{z}_n)\rho_n(\eta)\rho_n(\eta)}\right\| \leq \frac{C(n, m, \delta)}{\rho_n(\eta)} + \frac{C(m, n, \delta)}{\rho_n(\eta)} + O\left(\frac{1}{\rho_n(\eta)\rho_n(\eta)}\right).
\end{equation}
The relations [9] and [11] allow us to make the final step in the proof. The substitution $z_n \mapsto \zeta_n/\rho_n(\eta)$ gives us
\begin{equation}
h_\varphi\left(\frac{\zeta_1}{\rho_n(\eta)}, \ldots, \frac{\zeta_\kappa}{\rho_n(\eta)}\right) z_1, \ldots, z_\kappa = \sum_{n,m=1}^\infty \frac{\varphi(z_n) - \varphi(\overline{z}_n)}{z_n - \overline{z}_n} \frac{\zeta_n\zeta_m}{\rho_n(\eta)\rho_n(\eta)} = R(\eta) - \sum_{n=1}^\kappa |\zeta_n|^2,
\end{equation}
where \(|R(\eta)| \leq \sum_{n=1}^\kappa |\zeta_n|^2 O\left(\frac{1}{\rho_n(\eta)}\right) + \sum_{m \neq n} \zeta_n\zeta_m O\left(\frac{1}{\rho_n(\eta)} + \frac{1}{\rho_m(\eta)}\right)\).
According to [9], there exist arbitrarily small values of $\eta \in \{\eta_k\}_{k=1}^\infty$ for which the last inequality implies $|R(\eta)| \leq M\sqrt{\eta} \cdot \sum_{n=1}^\kappa |\zeta_n|^2$ with a fixed constant $M$. Thus, if we fix such $\eta$ small enough, the sign of the Hermitian form [12] will be determined by the last term $-\sum_{n=1}^\kappa |\zeta_n|^2$ alone for every set of complex numbers $\{\zeta_1, \ldots, \zeta_\kappa\}$.

**Proposition 5.** Under the conditions of Proposition 4, assume that $\Phi(\zeta)$ is regular in the interval $(-\varepsilon, 0)$ and $0 < \Phi(-0) \leq \infty$. Then the Hermitian form $h_\varphi(\zeta_1, \ldots, \zeta_\kappa) z_0, \ldots, z_\kappa$, where $z_m = \lambda_m + i\eta_m$ for $m = 1, \ldots, \kappa$ and $z_0 = -\sqrt{\eta} + i\mu$, is negative definite when the numbers $\eta > 0$ and $\mu > 0$ are chosen appropriately.
**Proof.** Split the function \( \varphi(z) \) into two parts \( \psi_0(z) \) and \( \psi_1(z) \) such that \( \varphi(z) = \psi_0(z) + \psi_1(z) \) and

\[
\psi_1(z) := b + \frac{1}{z} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} - \frac{1}{t + t^3} \right) \, d\sigma(t).
\]

The integral here is analytic for \( |z| < \varepsilon \) and vanishes at the origin (see (5)). The function \( \psi_1(z) \) is also analytic for \( |z| < \varepsilon \). The part \( \psi_0(z) \) has the form

\[
\psi_0(z) := \frac{a}{z} + \frac{1}{z} \int_{(-\varepsilon, \varepsilon)} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) \, d\sigma(t) + \frac{1}{z} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{d\sigma(t)}{t + t^3}
\]

\[
= A + \frac{1}{z} \int_0^\varepsilon \frac{d\sigma(t)}{t - z}, \quad \text{where we put} \quad A := a + \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{d\sigma(t)}{t + t^3} - \int_0^\varepsilon \frac{d\sigma(t)}{1 + t^2},
\]

i.e. \( A \) is a finite real constant. The integral over \((-\varepsilon, 0)\) is zero in the representation of \( \psi_0 \), because the function \( \Phi(z) \) is regular in this interval, and thus \( \sigma(t) \) is constant for \(-\varepsilon < t < 0\).

First assume that \( x \) varies on \((-\varepsilon, 0)\) close enough to 0, so that \( \Phi(x) > 3M > 0 \). On the one hand, one of the Cauchy-Riemann equations and the condition \( \Phi(x) \geq 0 \) (see Remark [1]) imply

\[
\frac{\partial^2 \varphi(x + iy)}{\partial y^2} \bigg|_{y=0} - \frac{\varphi(x + i\mu) - \varphi(x - i\mu)}{2i\mu} \leq \frac{M}{x^2},
\]

(13)

relying on the fact that \( \varphi(x + iy) \) is smooth for real \( y \). The last two inequalities together imply that

\[
\frac{\varphi(z_0) - \varphi(z_0)}{z_0 - z_0} = \frac{\varphi(x + i\mu) - \varphi(x - i\mu)}{(x + i\mu) - (x - i\mu)} < -\frac{3M}{x^2} + \frac{M}{x^2} = -\frac{2M}{x^2}
\]

for \( z_0 = x + i\mu \). Therefore,

\[
\rho_0^2(x^2) := -\frac{\psi_0(z_0) - \psi_0(z_0)}{z_0 - z_0} + \frac{\psi_1(z_0) - \psi_1(z_0)}{z_0 - z_0} \geq \frac{M}{x^2}
\]

(14)

for small enough \( |x| \) on account of the smoothness of \( \psi_1(z) \). We assume \( \rho_0(x^2) > 0 \) for definiteness.

On the other hand,

\[
\frac{\psi_0(z_0) - \psi_0(z_m)}{z_0 - z_m} = \frac{A}{z_0 - z_m} + \frac{1}{z_0 - z_m} \int_0^\mu \left( \frac{z_m}{t - z_0} - \frac{z_0}{t - z_m} \right) \, d\sigma(t)
\]

\[
= \frac{1}{z_0 - z_m} \left( -A + \int_0^\mu \frac{t - z_m}{t - z_0} \, d\sigma(t) \right)
\]

\[
= -\frac{1}{z_0 - z_m} \left( A + \int_0^\mu \frac{t - z_0}{t - z_m} \, d\sigma(t) \right), \quad m = 0, \ldots, x.
\]

Therefore, with the help of the inequality \( \frac{|z_m|}{|t - z_0|} \leq 1 \) valid for \( t \geq 0 \) we obtain

\[
\left| \frac{\psi_0(z_0) - \psi_0(z_m)}{z_0 - z_m} \right| \leq \frac{1}{|z_0 - z_m|} \left( |A| + \int_0^\mu \frac{-z_0}{t - z_0} + \frac{1}{|t - z_0|} \, d\sigma(t) \right)
\]

\[
\leq \frac{|A|}{|z_0 - z_m|} + \frac{1}{|z_0 - z_m|} \int_0^\mu \frac{d\sigma(t)}{|t - z_0|}.
\]

In particular, putting \( m = 0 \) in (14) gives us

\[
\rho_0^2(x^2) = \left| \frac{\psi_0(z_0) - \psi_0(z_0)}{z_0 - z_0} \right| \leq \frac{2 |A|}{|z_0 - z_0|} \int_0^\varepsilon d\sigma(t) + \frac{|A|}{|z_0 - z_0|} \leq \frac{2 \sigma(e - 0) - \sigma(0)}{|x|} + \frac{|A|}{|x|^3}
\]

(17)

which complements

\[
\left| \frac{\varphi(z_0) - \varphi(z_0)}{z_0 - z_0} + \rho_0^2(x^2) \right| = \left| \frac{\psi_1(z_0) - \psi_1(z_0)}{z_0 - z_0} \right| \leq O(1) \quad \text{as} \quad x \to 0,
\]

(18)
where $O(1)$ in the right-hand side does not depend on $\mu \in (0, \varepsilon)$. Now recall that $\Re z_\kappa = \lambda_\kappa < \cdots < \Re z_1 = \lambda_1 < -\varepsilon$. Since $|t - z_0| = t - x + \mu < t - 2\varepsilon$ provided that $t \geq 0$ and $\mu < |x|$, from (16) we obtain

$$\left| \frac{\psi_0(z_0) - \psi_0(\pi_m)}{z_0 - \pi_m} \right| \leq \frac{1}{|z_0 - \pi_m|^2} \int_0^\varepsilon \frac{d\sigma(t)}{|z_0 - \pi_m|^2} + \frac{1}{|z_0 - \pi_m|^2} \int_0^\varepsilon \frac{\frac{t - z_0}{2} - \frac{t - z_0}{2}|^2 d\sigma(t)}{|z_0 - \pi_m|^2} \leq 1 \left( \frac{A}{|z_0 - \pi_m|^2} + \frac{|z_0|}{|z_0 - \pi_m|^2} \int_0^\varepsilon \frac{t - 2\varepsilon}{|t - z_0|^2} d\sigma(t) \right)$$

(19)

where $x$ is tending to zero and $m = 1, \ldots, \infty$.

To implement the same technique as in the proof of Proposition 4 it is enough to put $x := -\sqrt{\eta}$ and to study the order of summands in the form $\Phi_h(\xi_0, \ldots, \xi_\kappa | z_0, \ldots, z_\kappa)$. Our supposition $x > -\varepsilon$ we supplement with $x > -\delta$, which automatically induces $\eta < \min\{\varepsilon^2, \delta^2\}$. We regard $\eta$ as tending to zero, so the conditions (14) and (17)–(18) imply that

$$M \leq \rho_0(\eta) \leq O \left( \frac{1}{\sqrt{\eta}} \right)$$

and

$$\left| \frac{\varphi(z_0) - \varphi(\pi_0)}{(z_0 - \pi_0)^2 \rho_0(\eta)} + 1 \right| \leq O(\eta). \quad (20)$$

Now we make use of the same notation as in the proof of Proposition 4. If $m, n \neq 0$ and $m \neq n$, then the estimates (9) and (11) concerning $\varphi(z_1), \ldots, \varphi(z_\kappa)$ are valid. Since the distance between $\mathcal{U}_n$ and $z_0$ is more than $\delta$, the inequality (10) is satisfied on condition that $m = 0 \neq n$. Then (10) and (19) give us the following:

$$\left| \frac{\varphi(z_0) - \varphi(\pi_0)}{(z_0 - \pi_0)^2 \rho_0(\eta)} \right| = \left| \frac{\varphi(z_0) - \varphi(\pi_0)}{(z_0 - \pi_0)^2 \rho_0(\eta) \rho_0(\eta)} \right| + \left| \frac{\psi_0(z_0) - \psi_0(\pi_0)}{(z_0 - \pi_0)^2 \rho_0(\eta) \rho_0(\eta)} \right| \leq C(m, n, \delta)$$

(22)

Assume that $\eta$ is taken from the sequence $\left\{ \eta_k \right\}_{k=1}^\infty$ corresponding to $\left( \delta_0 \right)$ and that the choice of $\mu \in (0, \sqrt{\eta})$ satisfies the condition (13). Then (20) implies

$$\left| \frac{\varphi(z_0) - \varphi(\pi_0)}{(z_0 - \pi_0)^2 \rho_0(\eta) \rho_0(\eta)} \right| = \left| \frac{\varphi(z_0) - \varphi(\pi_0)}{(z_0 - \pi_0)^2 \rho_0(\eta) \rho_0(\eta)} \right| \leq O(\sqrt{\eta}) + O \left( \eta^{-\frac{1}{2} + \frac{1}{4} + \frac{1}{2}} \right) + O \left( \eta^{-\frac{1}{2} + \frac{1}{4} + \frac{1}{2}} \right) + O \left( \eta^{-\frac{1}{2} + \frac{1}{4} + \frac{1}{2}} \right) = O \left( \sqrt{\eta} \right).$$

This estimate together with (20), (9), (11) and (8) yields that

$$h \left( \frac{\xi_0}{\rho_0(\eta)}, \ldots, \frac{\xi_\kappa}{\rho_\kappa(\eta)} | z_0, \ldots, z_\kappa \right) = - \sum_{m=0}^\kappa |c_m|^2 + O \left( \sqrt{\eta} \right) \sum_{n,m=0}^\kappa \zeta_m \zeta_n,$$

where $O \left( \sqrt{\eta} \right)$ does not depend on $\zeta_0, \ldots, \zeta_\kappa$. That is, this Hermitian form is negative definite provided that the value of $\eta \in \left\{ \eta_k \right\}_{k=1}^\infty$ is small enough.

**Proof of Theorem 2** Suppose that $\Phi(z) = z\varphi(z)$ can be represented as in (4). Then for specially chosen numbers $z_1, \ldots, z_\kappa \not\in \Re$ the Hermitian form $h_\varphi(\xi_1, \ldots, \xi_\kappa | z_1, \ldots, z_\kappa)$ has $\kappa$ negative squares by Proposition 4. Let us show that this is the greatest possible number of negative squares in the form $h_\varphi(\xi_1, \ldots, \xi_\kappa | z_1, \ldots, z_\kappa)$. 

Denote $\tilde{\sigma}(t) = - \int_0^t s^{-1} d\sigma(s)$. Since the integral
\[
0 \leq \int_0^\infty \frac{d\tilde{\sigma}(t)}{t(1 + t^2)} = \int_0^\infty \frac{d\tilde{\sigma}(t)}{t + 1} \leq a - \sum_{i=1}^\infty \frac{\sigma_i}{\lambda_i} < \infty
\]
is finite, we can split the last term of (4b) divided by $z$ into two parts to obtain (cf. (6))
\[
\varphi(z) = b + \frac{a}{z} + \sum_{i=1}^\infty \left( \frac{\sigma_i/\lambda_i}{\lambda_i - z} + \frac{\sigma_i/\lambda_i}{z} \right) + \int_0^\infty \frac{d\sigma(t)}{t(t - z)} + \frac{1}{z} \int_0^\infty \frac{d\sigma(t)}{t(1 + t^2)} = \tilde{\varphi}(z) + \varphi_0(z),
\]
where $\tilde{\varphi}(z) := \sum_{i=1}^\infty \frac{\sigma_i/\lambda_i}{\lambda_i - z}$ and $\varphi_0(z) := b + \frac{\Phi(-0)}{z} + \int_0^\infty \frac{d\tilde{\sigma}(t)}{t - z}$.

The functions $\varphi_0(z)$ and $-\tilde{\varphi}(z)$ have the form (2), i.e. belong to the class $\mathcal{N}$. For $\mathcal{N}$-functions and any set of numbers $\{z_1, \ldots, z_k\}$ the Hermitian form as in (1) is nonnegative definite. That is, the conditions
\[
h[\varphi] := h\varphi(\xi_1, \ldots, \xi_k|z_1, \ldots, z_k) \geq 0 \quad \text{and} \quad h[\tilde{\varphi}] := h\tilde{\varphi}(\xi_1, \ldots, \xi_k|z_1, \ldots, z_k) \leq 0
\]
holds true. Moreover, since $\tilde{\varphi}(z)$ is a rational function with $\varkappa$ poles, which is bounded at infinity, the rank of $h[\tilde{\varphi}]$ can be at most $\varkappa$ (see Theorem 3.3.3 and its proof in [1] pp. 105–108) or Theorem 1 in [2] p. 34). Consequently, the form $h[\varphi] = h\varphi(\xi_1, \ldots, \xi_k|z_1, \ldots, z_k)$ has at most $\varkappa$ negative squares as a sum of $h[\varphi_0]$ and $h[\tilde{\varphi}]$. (This become evident after the reduction of the Hermitian form $h[\varphi]$ to principal axes since $h[\varphi_0]$ is nonnegative definite irrespectively of coordinates.)

Suppose that $\Phi(z)$ can be expressed as in (3), and let $\varepsilon_0 > 0$ be such that $\Phi(-\varepsilon) > 0$ provided that $0 < \varepsilon < \varepsilon_0$. In particular, it implies $\max \lambda_i, \varkappa < -\varepsilon_0$ since $\Phi$ changes the sign near its poles. Proposition [5] provides a set of points $\{z_0, \ldots, z_{\varepsilon-1}\}$ such that the corresponding Hermitian form $h[\varphi]$ has $\varkappa$ negative squares. Let us prove that $h[\varphi]$ has at most $\varkappa$ squares negative. Consider the function
\[
\varphi_\varepsilon(z) := \frac{\Phi(z) - \Phi(-\varepsilon)}{z + \varepsilon} + \frac{\Phi(-\varepsilon)}{z + \varepsilon} = b + \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i - z} + \int_0^\infty \frac{1}{t - z} \cdot \frac{d\sigma(t)}{t + \varepsilon} + \frac{\Phi(-\varepsilon)}{z + \varepsilon}.
\]

Denote $\Phi_\varepsilon(z) := z\varphi(z)$ and $A_i := \frac{\sigma_i/\lambda_i}{\lambda_i + \varepsilon}$. Then, on account of the identities (5),
\[
\Phi_\varepsilon(z) = bz + \Phi(-\varepsilon) - \frac{z\Phi(-\varepsilon)}{z + \varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i - z} - \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i} + \int_0^\infty \frac{1}{t - z} \cdot \frac{d\sigma(t)}{t + \varepsilon}
\]
\[
= bz + \Phi(-\varepsilon) - \frac{z\Phi(-\varepsilon)}{z + \varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i - z} - \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i} + \int_0^\infty \frac{1}{t - z} \cdot \frac{d\sigma(t)}{t + \varepsilon}
\]
\[
= bz + \left[ \Phi(-\varepsilon) - \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i} \right] - \int_0^\infty \frac{d\sigma(t)}{t(\varepsilon + 1)}
\]
\[
= -\frac{z\Phi(-\varepsilon)}{z + \varepsilon} + \sum_{i=1}^{\varkappa-1} \frac{A_i}{\lambda_i - z} + \int_0^\infty \frac{1}{t - z} \cdot \frac{d\sigma(t)}{t + \varepsilon},
\]
i.e. $\Phi_\varepsilon(z) \in \mathcal{N}$. Moreover, $\Phi_\varepsilon(z)$ is an increasing function when $-\varepsilon < z < 0$ (see Remark [1] which implies
\[
\Phi_\varepsilon(-0) = \lim_{z \to -0} \int_0^\infty \frac{z}{t - z} \cdot \frac{d\sigma(t)}{t + \varepsilon} \leq 0,
\]
since the integrand is negative. That is, the function $\Phi_\varepsilon(z)$ has the form (4). As it is shown above, we have $\varphi_\varepsilon \in \mathcal{N}_\varepsilon$ for each $\varepsilon$ between $0$ and $\varepsilon_0$.

There exists some positive $\varepsilon_1 < \varepsilon_0$ such that the form $h[\varphi_{\varepsilon}] := h\varphi_{\varepsilon}(\xi_1, \ldots, \xi_k|z_1, \ldots, z_k)$ for $0 < \varepsilon < \varepsilon_1$ and a fixed set of points $\{z_1, \ldots, z_k\}$ has at least the same number of negative squares as the form $h[\varphi]$ does. (Indeed: the characteristic numbers of $h[\varphi]$ depend continuously on its coefficients.) Suppose that the Hermitian form $h[\varphi]$ has more than $\varkappa$ negative squares. Then $h[\varphi_{\varepsilon}]$ must have more than $\varkappa$ negative squares as well, which is impossible. Thus, the form $h[\varphi]$ has at most $\varkappa$ negative squares.
Suppose that $\varphi \in \mathcal{N}_\kappa^+$. Then the function $\Phi(z) = z^\varphi(z)$ can be represented as in (2) and the form $\Phi(z)$ for any set of numbers $\{z_1, \ldots, z_k\}$ has at most $\kappa$ negative squares (as stated in the definition of $\mathcal{N}_\kappa^+$). By Proposition 4, the function $\sigma(t)$ appearing in (3) can have at most $\kappa$ negative points of increase. These points are isolated, and therefore (see Definition 3) for negative $t$ the function $\sigma(t)$ is a step function with at most $\kappa$ steps. That is, all negative singular points of $\Phi(z)$ are simple poles; they have negative residues since $\Phi \in \mathcal{N}$, i.e. $\sigma_i > 0$ for all $i$. Here we have two mutually exclusive options: $\Phi(0) \leq 0$, then $\Phi(z)$ has the form (4) corresponding to some $\kappa_0 \leq \kappa$, and $0 < \Phi(0) \leq \infty$, i.e. $\Phi(z)$ has the form (3) corresponding to $\kappa_0 \leq \kappa + 1$. The sufficiency (first) part of the current proof shows that $\varphi \in \mathcal{N}_{\kappa_0}^+$ in both cases. Since the classes $\mathcal{N}_{\kappa_0}^+$ and $\mathcal{N}_{\kappa_0}^+$ are disjoint by definition, we necessarily have $\kappa_0 = \kappa$. □

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