Local super antimagic total vertex coloring of some wheel related graphs

S A Pratama\textsuperscript{1,2}, S Setiawani\textsuperscript{2}, Slamin\textsuperscript{1,3,∗}

\textsuperscript{1}CGANT Research Group, University of Jember, Indonesia
\textsuperscript{2}Department of Mathematics Education, University of Jember, Indonesia
\textsuperscript{3}Study Program of Informatics, University of Jember, Indonesia
∗Corresponding Author
E-mail: asaukedo261014@gmail.com; susisetiawani.fkip@unej.ac.id; slamin@unej.ac.id

Abstract. Let $G$ be a simple and finite graph with vertex set $V$ and edge set $E$. The local antimagic total labeling of $G$ is a map from $V \cup E$ to the set of positive integers from 1 to $|V| + |E|$ such that the weight of two adjacent vertices are distinct. The weight of a vertex $v$ is calculated by the sum of the label of the vertex $v$ and the labels of all edges incident to it. If the vertices of $G$ is labelled by the smallest labels, that is, $\{1, 2, \ldots, |V|\}$, then the such labeling is called local super antimagic total labeling. Thus, the local super antimagic total labeling induces a proper vertex coloring of $G$ where the vertex $v$ is colored by the weight of vertex $v$. The minimum number of colors taken over all colorings induced by super local antimagic total labeling of $G$, is called local super antimagic total chromatic number of graph $G$, and denoted by $\chi_{lsat}(G)$. In this paper, we consider the local super antimagic total chromatic number of some wheel related graphs such as fans, even gear graphs, and sun flower graphs.

1. Introduction

Graph labeling is one of topics in Graph Theory that has grown very rapidly. There are many types of graph labeling with various applications both in real life and other subjects including Graph Theory itself [4]. The application of graph labeling for other topic in Graph Theory is for coloring vertices of graph. The vertex coloring in a graph $G$ is defined as a $k$-coloring $c : V \rightarrow \{1, 2, \ldots, k\}$ where $V$ is the set of vertices of $G$ and $c(v)$ is a color of vertex $v$ such that $c(u) \neq c(v)$ if $u$ and $v$ are adjacent vertices. The chromatic number of the graph $G$, denoted by $\chi(G)$, is the smallest positive integer $k$ such that $G$ satisfy the $k$-coloring [3].

The type of graph labeling that is applied for vertex coloring of graph is the local antimagic labeling. The local antimagic labeling of a graph $G$ with a vertex set $V$ and an edge set $E$ is defined as an assignment $f : E \rightarrow \{1, 2, \cdots, |E|\}$ such that the weights of two adjacent vertices are different. In other word, for two adjacent vertices $u$ and $v$, $w(u) \neq w(v)$ where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to $u$. This concept was introduced by Arumugam, Premalatha, Baća and Semaničová-Feňovčíková [1]. From the definition, it is clear that any local antimagic labeling induces a proper vertex coloring of $G$ by assigning color $w(u)$ to the vertex $u$. The minimum number of colors (different weights) taken over all colorings induced by local antimagic labelings of $G$ is called the local antimagic chromatic number of $G$ and denoted by $\chi_{la}(G)$.
Arunugam et al. [1] determined the local antimagic chromatic number of some families of graphs such as complete graph, star, path, cycle, friendship graph, complete bipartite graph, and wheel. Nazula, Slamin and Dafik [7] presented the local antimagic chromatic number of uni cyclic graphs such as kite and cycle with two neighbour pendants. The local antimagic chromatic number of corona product of two graph such as path, cycle and complete graph with other graph were discovered by Arumugam, Yi-Chun, Premalatha and Tao-Ming [2].

The concept of this labeling then extended by Putri, Dafik, Agustin and Alfarisi [9, 6] with labeling both the vertices and edges of graph $G$. The local vertex antimagic total labeling on a graph $G$ is defined to be an assignment $f: V \cup E \rightarrow \{1, 2, \ldots, |V|+|E|\}$ so that the weights of any two adjacent vertices $u$ and $v$ are distinct, that is, $w(u) \neq w(v)$ where $w(u) = f(u) + \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to $u$. As the local antimagic labeling, any local vertex antimagic total labeling induces a proper vertex coloring of $G$ where the vertex $u$ is assigned the color $w(u)$. The minimum number of colors taken over all colorings induced by local vertex antimagic total labelings of $G$ is called the local antimagic total chromatic number of $G$ and denoted by $\chi_{lat}(G)$. Clearly $\chi_{lat}(G) \geq \chi_{lat}(G) \geq \chi(G)$ for any graph $G$. The local antimagic total chromatic number for some family of graphs such as prisms and Möbius ladders [5], brooms [8], stars [9], wheels, fans and friendship graphs [10] have been discovered.

If the vertices of $G$ receive the smallest labels, that is, $1, 2, \ldots, |V|$ in the local antimagic total labeling of graph $G$, then it is called the \textit{local super antimagic total labeling}. Again as local antimagic labeling and local antimagic total labeling, the local super antimagic total labeling also induces a proper vertex coloring of $G$. The \textit{local super antimagic total chromatic number}, denoted by $\chi_{lsat}(G)$, is the minimum number of colors taken over all colorings induced by local super antimagic total labelings of $G$. Again it is clear that for any graph $G$, $\chi_{lsat}(G) \geq \chi_{lsat}(G) \geq \chi(G)$.

Slamin, Hasan and Dafik [11] presented local super antimagic total chromatic number for some family of graphs such as tree, path, cycle, helm, wheel, odd gear, sun and regular graphs as well as amalgamation of stars and amalgamation of wheels. In this paper, we consider the local super antimagic total chromatic number of some wheel related graphs such as fans, even gear graphs, and sun flower graphs.

\section{Main Results}

We begin this section by presenting the local super antimagic total chromatic number of a fan as follows. The fan is a graph that is obtained from a wheel by removing one rim edge.

\textbf{Theorem 2.1} If $F_n$ is a fan for $n \geq 3$, then $3 \leq \chi_{lsat}(F_n) \leq 4$.

\textbf{Proof.} Let $F_n$ for $n \geq 3$ be a fan. Then $F_n$ has the vertex set $V(F_n) = \{x_i; 1 \leq i \leq n\} \cup \{c\}$ and the edge set $E(F_n) = \{x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{cx_i; 1 \leq i \leq n\}$. We divide the proof into two cases.

Case i. For even $n$.

Label the vertices and edges of $F_n$ with the formula below.

$$
\begin{align*}
  f(c) &= n + 1 \\
  f(x_i) &= i + 1, & \text{for } 1 \leq i \leq n - 1 \\
  f(x_n) &= 1 \\
  f(cx_1) &= n + 2 \\
  f(cx_i) &= 3n + 2 - i, & \text{for } 2 \leq i \leq n - 1 \\
  f(cx_n) &= \frac{3n + 1}{2} \\
  f(x_ix_{i+1}) &= \begin{cases} 
    \frac{4n - i + 5}{2}, & \text{for odd } 1 \leq i \leq n - 2 \\
    n + 2 + \frac{i}{2}, & \text{for even } 2 \leq i \leq n - 1
  \end{cases}
\end{align*}
$$


This labeling provides different weights for any two adjacent vertices, namely,
\[
\begin{align*}
  w(c) &= \frac{5n^2 + 4}{2} \\
  w(x_1) &= w(x_n) = 3n + 6 \\
  w(x_i) &= \begin{cases} 
    6n + 8, & \text{for even } 2 \leq i \leq n - 2 \\
    6n + 7, & \text{for odd } 3 \leq i \leq n - 1
  \end{cases}
\end{align*}
\]
Thus the labeling also gives 4 different weights, that is, \( \chi_{lsat}(F_n) \leq 4 \). Since \( F_n \) is 3-colorable, then \( 3 \leq \chi_{lsat}(F_n) \leq 4 \) for even \( n \geq 3 \).

Case ii. For odd \( n \).

Label the vertices and edges of \( F_n \) with the formula below.
\[
\begin{align*}
  f(c) &= n + 1 \\
  f(x_1) &= n + 3 \\
  f(x_i) &= \begin{cases} 
    1 + \frac{i}{n + i + 2}, & \text{for even } 2 \leq i \leq n - 1 \\
    \frac{n + i + 2}{2}, & \text{for odd } 3 \leq i \leq n - 2
  \end{cases} \\
  f(x_n) &= 1 \\
  f(cx_1) &= \frac{3n + 5}{2} \\
  f(cx_i) &= \begin{cases} 
    \frac{6n + 2 - i}{2}, & \text{for even } 2 \leq i \leq n - 1 \\
    \frac{5n + 4 - i}{2}, & \text{for odd } 3 \leq i \leq n - 2
  \end{cases} \\
  f(cx_n) &= \frac{3n + 3}{2} \\
  f(x_i x_{i+1}) &= \begin{cases} 
    \frac{2n + 3 + i}{4n + 8 - i}, & \text{for odd } 1 \leq i \leq n - 2 \\
    \frac{4n + 6 - i}{2}, & \text{for even } 2 \leq i \leq n - 1
  \end{cases} \\
  f(x_n x_1) &= \frac{3n + 3}{2}
\end{align*}
\]
This labeling also provides different weights for any two adjacent vertices, namely,
\[
\begin{align*}
  w(c) &= \frac{5n^2 + 4n + 4}{2} \\
  w(x_1) &= w(x_n) = 3n + 6 \\
  w(x_i) &= \begin{cases} 
    6n + 6, & \text{for even } 2 \leq i \leq n - 2 \\
    6n + 8, & \text{for odd } 3 \leq i \leq n - 2
  \end{cases}
\end{align*}
\]
Thus the labeling also gives 4 different weights, that is, \( \chi_{lsat}(F_n) \leq 4 \). Since \( F_n \) is 3-colorable, then \( 3 \leq \chi_{lsat}(F_n) \leq 4 \) for odd \( n \geq 3 \). This completes the proof.

As mentioned before, Slamin, Hasan and Dafik [11] determined the local super antimagic total chromatic number of odd gear. In the following theorem, we consider the case for even gear.

**Theorem 2.2** If \( G_n \) for even \( n \geq 4 \) is an even gear, then \( 2 \leq \chi_{lsat}(G_n) \leq 3 \)
Proof. Let \( G_n \) for even \( n \geq 4 \) be an even gear. Then \( G_n \) has the vertex set \( V(G_n) = \{ x_i, y_i; 1 \leq i \leq n \} \cup \{ c \} \) and the edge set \( E(G_n) = \{ x_i y_i; 1 \leq i \leq n \} \cup \{ y_i x_i + 1; 1 \leq i \leq n - 1 \} \cup \{ c x_i; 1 \leq i \leq n - 1 \} \cup \{ y_n x_1 \} \).

Label the vertices and edges of \( G_n \) with the formula below.

\[
\begin{align*}
    f(c) &= \frac{3n + 2}{2}, \\
    f(x_i) &= \begin{cases} 
        \frac{n+i}{2}, & \text{for even } 2 \leq i \leq n \\
        \frac{i+3}{2}, & \text{for odd } 3 \leq i \leq n - 1 
    \end{cases}, \\
    f(y_i) &= \begin{cases} 
        \frac{4n+3-i}{2}, & \text{for odd } 1 \leq i \leq n - 1 \\
        \frac{3n-2-i}{2}, & \text{for even } 2 \leq i \leq n 
    \end{cases}, \\
    f(cx_i) &= 3n + 2 - i \quad \text{for } 1 \leq i \leq n, \\
    f(x_i y_i) &= 3n + i + 1 \quad \text{for } 1 \leq i \leq n, \\
    f(y_i x_{i+1}) &= \begin{cases} 
        \frac{9n+4-2i}{2}, & \text{for odd } 1 \leq i \leq n - 1 \\
        \frac{10n+4-i}{2}, & \text{for even } 2 \leq i \leq n - 2 
    \end{cases}, \\
    f(y_n x_1) &= \frac{9n + 4}{2}
\end{align*}
\]

This labeling provides different weights for any two adjacent vertices, namely,

\[
\begin{align*}
    w(c) &= \frac{5n^2 + 6n + 2}{2}, \\
    w(x_i) &= \frac{23n + 10}{2}, \quad \text{for } 1 \leq i \leq n, \\
    w(y_i) &= \frac{19n + 8}{2}, \quad \text{for } 1 \leq i \leq n
\end{align*}
\]

Thus the labeling gives 3 different weights, that is, \( \chi_{ls}(G_n) \leq 3 \). Since even gear \( G_n \) is 2-colorable, then we conclude that \( 2 \leq \chi_{ls}(G_n) \leq 3 \) for even \( n \geq 4 \). \( \square \)

We finally consider the local super antimagic total chromatic number of sun flower graph \( SF_n \) for odd \( n \geq 5 \) as presented in the following theorem.

Theorem 2.3 If \( SF_n \) for odd \( n \geq 5 \) is a sun flower graph, then \( 4 \leq \chi_{ls}(SF_n) \leq 5 \).

Proof. Let \( SF_n \) for odd \( n \geq 5 \) be a sun flower graph. Then \( SF_n \) has the vertex set \( V(SF_n) = \{ x_i; 1 \leq i \leq n \} \cup \{ y_i; 1 \leq i \leq n \} \cup \{ c \} \) and the edge set \( E(SF_n) = \{ x_i x_{i+1}; 1 \leq i \leq n - 1 \} \cup \{ x_n x_1 \} \cup \{ y_i y_{i+1}; 1 \leq i \leq n \} \cup \{ x_{i-1} y_{i+2}; 2 \leq i \leq n \} \cup \{ x_n y_1 \} \cup \{ c x_{i}; 1 \leq i \leq n \} \).

Label the vertices and edges of \( SF_n \) with the formula below.

\[
\begin{align*}
    f(c) &= 2n + 1, \\
    f(x_i) &= \begin{cases} 
        2n, & \text{for } i = 1 \\
        2n - i, & \text{for even } 2 \leq i \leq n - 1 \\
        2n + 2 - i, & \text{for odd } 3 \leq i \leq n
    \end{cases}, \\
    f(y_i) &= \begin{cases} 
        n + 2 - 2i, & \text{for } 1 \leq i \leq \frac{n+1}{2} \\
        2n + 2 - i, & \text{for } \frac{n+3}{2} \leq i \leq n
    \end{cases}, \\
    f(x_i x_{i+1}) &= \begin{cases} 
        \frac{8n+3+i}{2}, & \text{for odd } 1 \leq i \leq n - 2 \\
        \frac{9n+3+i}{2}, & \text{for even } 2 \leq i \leq n - 1 
    \end{cases}
\end{align*}
\]
\[
f(x_n x_1) = \frac{9n + 3}{2}
\]
\[
f(x_i y_i) = 2n + 1 + i, \quad \text{for } 1 \leq i \leq n
\]
\[
f(x_i-1 y_i) = \begin{cases} 
\frac{7n+1+2i}{2}, & \text{for } 2 \leq i \leq \frac{n-1}{2} \\
\frac{5n+1+2i}{2}, & \text{for } \frac{n+1}{2} \leq i \leq n
\end{cases}
\]
\[
f(x_n y_1) = \frac{7n + 3}{2}
\]
\[
f(cx_i) = \begin{cases} 
6n + 2 - 2i, & \text{for } 1 \leq i \leq \frac{n-1}{2} \\
7n + 2 - 2i, & \text{for } \frac{n+1}{2} \leq i \leq n
\end{cases}
\]

The labeling provides different weights for any two adjacent vertices, namely,

\[
w(c) = \frac{11n^2 + 7n + 2}{2}
\]
\[
w(x_i) = \begin{cases} 
22n + 8, & \text{for } n = 1 \\
22n + 9, & \text{for even } 2 \leq i \leq n - 1 \\
22n + 7, & \text{for odd } 3 \leq i \leq n
\end{cases}
\]
\[
w(y_i) = \frac{13n + 7}{2}, \quad \text{for } 1 \leq i \leq n
\]

Thus the labeling gives 5 different weights, that is, \(\chi_{lsat}(SF_n) \leq 4\). Since odd sun flower graph \(SF_n\) is 4-colorable, then we conclude that \(4 \leq \chi_{lsat}(SF_n) \leq 5\) for odd \(n \geq 5\).

3. Conclusion

This paper presents the local super antimagic total chromatic number of fans, even gear graphs, and sunflower graphs. However, the local super antimagic total chromatic number of such graphs given in this paper are still in some intervals. Consequently, we propose the following open problem.

Problem 3.1 What is the exact value of the local super antimagic total chromatic number of fans, even gear graphs, and sunflower graphs?

Acknowledgement

This research was funded by DRPM Ditjen Penguatan Riset Kemenristekdikti RI through Penelitian Dasar research grant year 2019 Decree No. 7/E/KPT/2019 and Contract No. 175/SP2H/LT/DRPM/2019.

References

[1] Arumugam S, Premalatha K, Bača M and Semaničová-Feňovčíková A 2017 Local antimagic vertex coloring of a graph Graphs and Combinatorics 33 275-285
[2] Arumugam S, Yi-Chun L, Premalatha K and Tao-Ming W 2018 On local antimagic vertex coloring for corona products of graphs Graph and Combinatorics
[3] Demange M, Ekim T, Ries B and Tanasescu C 2014 On some applications of the selective graph coloring problem Eur. J. Oper. Res. 240 307-314
[4] Hagedorn T R 1990 Magic rectangles revisited Discrete Math 207 65-72
[5] Hasan M A, Slamin and Dafik 2018 Pewarnaan titik total anti-ajaib lokal pada graf prisma dan tangga Mobius Pros. Konf. Nas. Mat. XIX
[6] Kurniawati E Y, Agustin I H, Dafik and Alfarisi R 2018 Super local edge antimagic total coloring of \(P_n \bowtie H\) Journal of Physics: Conference Series
[7] Nazula N, Slamin and Dafik 2018 Local antimagic vertex coloring of unicyclic graphs Indonesian Journal of Combinatorics 2(1) 30-34
[8] Nikmah N, Slamin and Hobri 2018 Pewarnaan titik (total) anti-ajaib lokal pada graf sapu Pros. Konf. Nas. Mat. XIX (2018)

[9] Putri D F, Dafik, Agustin I H and Alfarisi R 2018 On the local vertex antimagic total coloring of some families tree Journal of Physics: Conference Series

[10] Slamin, Dafik and Hasan M A 2018 Pewarnaan titik total anti-ajaib lokal pada keluarga graf roda Pros. Konf. Nas. Mat. XIX

[11] Slamin, Hasan M A and Dafik Local super antimagic total labeling for vertex coloring of graphs preprint