Non-trivial Braiding of Band Nodes in Non-Hermitian Systems

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A prevailing notion in the topological band theory is that the topological charge associated with band degeneracies cannot change upon continuously tuning the Bloch Hamiltonian. Here, we show that this notion is in general incorrect in non-Hermitian systems. In particular, we present a simple three-dimensional two-band model, such that a Weyl point degeneracy flips its chiral charge after encircling an exceptional line degeneracy, upon tuning one parameter. We use the formalism of Abe homotopy to mathematically describe this phenomenon. Our work points to significant richness in the topology of non-Hermitian Hamiltonians that is not shared by their Hermitian counterparts.

Introduction.-- Energy bands inside the momentum ($k$) space of periodic media can exhibit robust degeneracies called band nodes\cite{1}. These are often stabilized by topological charges, such as the $Z$-valued Chern number (chirality) of Weyl points\cite{2–4}, or the $Z_2$-valued quantized Berry phase\cite{5,6} of nodal lines\cite{7,8}. Topological charges of band nodes of Hermitian Bloch Hamiltonians have been classified\cite{9–11} using homotopy theory\cite{12,13}. In this description, the nodes are understood as topological defects of the Bloch Hamiltonian inside the $k$-space. This approach is mathematically analogous to the well understood and older problem of describing topological defects of ordered media inside the coordinate ($r$) space. Topological charges of such order-parameter defects in $r$-space are conventionally derived from homotopy groups of the order-parameter space\cite{14–17}. Similarly, band nodes in $k$-space can be described by homotopy groups $\pi_p(M)$ ($p = 1, 2, \ldots$)\cite{10} where $M$ is the “classifying space” of all symmetry-compatible Bloch Hamiltonians\cite{18}.

When the homotopy groups are abelian (such as $Z_2$ or $Z$), the topological charge of defects is usually a conserved number which cannot be changed as long as discontinuous deformations are suppressed. This simple behavior breaks down in the special\cite{19,20} albeit experimentally relevant\cite{21,22} situations with non-abelian first homotopy group. Astoundingly, there are cases where even abelian topological charges fail to be uniquely defined, resulting in a non-trivial braiding of the defects\cite{23,24}.

A well-known example occurs in uniaxial nematic liquid crystals\cite{16}, where a pair of hedgehog defects with the same charge annihilate each other, if they are brought together along a trajectory enclosing a dislocation line. Mathematically, this phenomenon follows from a non-trivial action induced by $\pi_1(M)$ on $\pi_2(M)$\cite{25}. However, despite the clear mathematical underpinning, the possibility of non-trivial braiding of abelian topological charges has not been previously reported for band nodes.

In this work, we show that non-trivial braiding of band nodes naturally arises inside the $k$-space of non-Hermitian Bloch Hamiltonians, which are approachable in condensed matter, cold atom and photonic systems\cite{26–58}. In three-dimensional Hermitian systems, the generic band nodes are Weyl points\cite{2,3}. Upon adding a non-Hermitian perturbation, a Weyl point generically inflates into an exceptional ring\cite{53–55}, with the $Z$-valued Chern number of the original Weyl point (defined on spheres) still meaningful\cite{59–61}. Importantly, the exceptional ring is further stabilized by an additional $Z$-valued winding number (defined on loops)\cite{62}. These two invariants correspond to the line-gap vs. the point-gap topological classification of Ref.\cite{63}, respectively. Here, we rederive these topological charges from homotopy theory, and we show that they interact non-trivially: the Chern number of an exceptional ring flips sign when the ring is braided around another exceptional line. As a consequence, a pair of nodes carrying the same Chern number can annihilate if they are brought together along a trajectory enclosing an exceptional line.

Global aspects of Chern number.-- Before we present an explicit model that manifests the non-trivial braiding of band nodes, we provide an intuitive understanding of this phenomenon from general considerations. For simplicity, we consider here a special case where the non-

![FIG. 1. (a) Two Weyl points A and B (green dots) near an exceptional line (vertical green). The short blue vs. the long red “sausages” represent two topologically distinct surfaces on which one can compute the total charge of the Weyl points. The total charge signals whether the Weyl points annihilate if brought together along the blue vs. the red trajectory contained inside the two surfaces. (b) Circumnavigating the exceptional line leads to reordering of the energy bands, which effectively flips the Chern number of the transported nodes. Therefore, if the topological charges of the Weyl points add up on the blue surface, they cancel out on the red surface.](https://example.com/figure1.png)
Hermitian effects do not inflate Weyl points into exceptional rings. The presented arguments readily generalize to exceptional rings with a Chern number.

We draw in Fig. 1(a–b) two Weyl points (green dots) near an exceptional line (vertical green). To compute the total charge of the two Weyl points, we have two topologically distinct ways of enclosing them with a two-dimensional surface, represented by the red vs. the blue “sausage” in Fig. 1(a–b). Note that an identical figure could also illustrate a Hermitian systems with certain symmetry that can protect nodal lines, where the green line now represents a nodal line. In Hermitian systems, the two ways of computing the total Chern number of the two Weyl points give identical results. However, from topological principles, there is no guarantee for this to hold true in general, because the blue surface cannot be continuously deformed into the red surface without crossing a band node. Indeed, we find that non-Hermitian Hamiltonians provide an example where the two surfaces exhibit different total charges.

From the physics point of view, the ordering of energy eigenvalues is crucial to define Chern numbers [64]. However, in non-Hermitian systems the energy eigenvalues are complex-valued, hence there is no natural choice of ordering. In fact, around an exceptional line, the band dispersion has a Riemann-sheet structure [65–67], which flips the ordering of the two bands as one circumnavigates the exceptional line. Therefore, if we define an ordering of the two bands near Weyl point A and we apply the same ordering to other momenta by continuing the exceptional line, we can choose one of two inequivalent paths [red vs. blue in Fig. 1(a–b)] to reach Weyl point B. Since the union of these two paths encloses the exceptional line (covering one half of the Riemann sheet), the two ways of continuation give opposite ordering of the two bands near Weyl point B, which results in opposite values of its Chern number. Especially, if the total Chern number of the Weyl points on the blue surface (enclosing the blue path) is 1+1=2, then the Chern number on the red surface is 1–1=0. This suggests that the Weyl points annihilate when brought together along the red trajectory, while avoiding annihilation along the blue trajectory.

Note that the Riemann-sheet structure is also relevant for non-Hermitian band insulators. If we interpret the torus (combination of the red and of the blue sausage) as the Brillouin zone of a 2D non-Hermitian lattice system, then the non-Hermitian Hamiltonian is periodic on the torus. Nevertheless, the Chern number on the torus is ill-defined, unless one considers the double cover [68].

The model.– We introduce a model that exhibits the non-trivial braiding when a single parameter $\alpha$ is varied,

$$
\mathcal{H}(\mathbf{k}; \alpha) = \left[ (k_+ + e^{-i\alpha})(k_- + e^{-i\alpha}) + \frac{1}{2} \sigma_z \right] \sigma_+ + \left[ (k_+ + e^{i\alpha})(k_- + e^{i\alpha}) + \frac{1}{2} \right] k_+ \sigma_- + k_z \sigma_z,
$$

(1)

where we defined $\sigma_\pm = \sigma_x \pm i\sigma_y$ for Pauli matrices and $k_\pm = k_x \pm ik_y$ for momentum coordinates. A roadmap of how we constructed the Hamiltonian appears in the Supplemental Material [69]. (Therein, we also present an alternative lattice model that exhibits a similar behavior.) The evolution of band nodes of the model in Eq. (1) is summarized by Fig. 2(a–f). First, at $\alpha = 0$ there is one exceptional line passing through the $k_z = 0$ plane at $k_x = k_y = 0$. As we increase $\alpha$ to $\pi/4$, the exceptional line ejects two Weyl points of the same chirality. As one further increases $\alpha$, the Weyl points orbit around the exceptional line in opposite directions inside the $k_z = 0$ plane, until they meet on the other side of the
exceptional line at $\alpha = 3\pi/4$. Upon further increment of $\alpha$, the two Weyl points annihilate.

To see that the two Weyl points ejected at $\alpha = \pi/4$ locally have the same chirality, we compute the Chern number on the blue surface displayed in Fig. 3(a). This is achieved by plotting in Fig. 3(b) the Wilson-loop eigenvalues in the corresponding Wilson-loop eigenvalues for paths that sweep along the surface [70]. The observed winding indicates that the total Chern number on the blue surface containing the two Weyl points just after their conception is $+2$. Meanwhile, since the two Weyl points annihilate for $\alpha = 3\pi/4$ on the other side of the exceptional line, the total Chern number on the red surface in Fig. 3(a) must be zero. This is confirmed by plotting the corresponding Wilson-loop eigenvalues in Fig. 3(c). We conclude that the Chern number of Weyl points in non-Hermitian systems exhibits an ambiguity: we are able to flip the Chern number of a Weyl point by moving it around an exceptional line.

**Abe homotopy.**– We describe topological charges of band nodes using homotopy groups [10]. For simplicity, here we only consider two-band models, and we assume the absence of global symmetries (i.e. symmetry class A of Ref. [63]). Recall that we use $M$ to indicate the target space, which in the present application is the classifying space of Hamiltonians [18]. Then the $p^{th}$ based homotopy group $\pi_p(M, \mathfrak{m})$ represents equivalence classes of continuous maps from a $p$-dimensional cube $I^p$ to $M$, such that the boundary $\partial I^p$ is mapped to the base-point $\mathfrak{m} \in M$ [12]. The equivalence $f_1 \sim f_2$ means that $f_1$ can be changed into $f_2$ by continuous deformations. The boundary condition can be understood as identifying $\partial I^p$ into a $p$-sphere, $S^p$. When a homotopy group does not depend on the base-point, one often writes just $\pi_p(M)$.

The mathematical object that governs the observed non-commutative properties of the topological charges is the action of $\pi_1(M)$ on $\pi_2(M)$ [24]. Returning back to Fig. 1, one can imagine continuously transforming the blue surface into two balloons, each containing only one Weyl point. The red surface is obtained by gluing the two balloons on the other side of the line defect (green). Carrying one of the balloons around the line induces an action of $\pi_1(M)$ (characterizing the closed path) on $\pi_2(M)$ (characterizing the transported balloon). For example, in the case of uniaxial nematics [23], the $\pi$-rotation of the order parameter on paths encircling a dislocation line inverts the hedgehog configuration on the balloon [16].

The action can be geometrically visualized using the construction of Abe [71], who considered equivalence classes of maps from a cylinder $S^1 \times [0,1]$ to the target space, such that the boundary, $S^1 \times \{0\} \cup S^1 \times \{1\}$, is mapped to the base-point [Fig. 4(a)]. By further requiring a segment $(x) \times [0,1]$ with a fixed $x \in S^1$ to be mapped to the base-point too, one reproduces $\pi_2(M)$ [Fig. 4(b)]. On the other hand, by limiting attention to “stratified” maps that only depend on the position along $[0,1]$, one reproduces $\pi_1(M)$ [Fig. 4(c)]. By stacking cylinders, we are able to combine elements of $\pi_1(M)$ with elements of $\pi_2(M)$ [Fig. 4(e)]. Especially, one can consider the effect of moving the base-point along a closed path, which corresponds to conjugating an element of $\pi_2(M)$ with an element of $\pi_1(M)$ [Fig. 4(d)]. The conjugation induces a map $\varphi : \pi_1(M) \rightarrow \text{Aut}(\pi_2(M))$, i.e. each element $g \in \pi_1(M)$ is represented by an automorphism $\varphi_g : \pi_2(M) \rightarrow \pi_2(M)$ [72]

**Topological charges revisited.**– To provide a mathematical underpinning of the non-trivial braiding of band nodes, we identify the space $M$ of non-Hermitian two-band Hamiltonians, and we explicitly compute its homotopy groups $\pi_1(M)$, $\pi_2(M)$ and the action $\varphi$. Away from the nodes, the Hamiltonian exhibits two different eigenvalues. We perform spectral flattening: we make the Hamiltonian traceless (we drop the term proportional to the identity matrix), and we normalize the eigenvalues to absolute value $1$. This is achieved with continuous deformations and without producing a band degeneracy, i.e. the procedure does not affect the band topology. To obtain homotopy groups, it is convenient to express $M$ as a coset space $G/H$ with $G$ a simply connected group [14]. Then a mathematical theorem guarantees [73] that $\pi_2(M) = \pi_1(H)$ and $\pi_1(M) = \pi_0(H)$.

To obtain the coset expression, we begin with the spectral decomposition of a generic two-band Hamiltonian $H \in M$. Adopting the biorthogonal normalization of left
and right eigenvectors [74–77],
\[ \mathcal{H} = V^{-1} \cdot (e^{i\pi} \sigma_z) \cdot V, \]  
(2)
where \( e^{i\pi} \sigma_z \) is a diagonal matrix containing the normalized eigenvalues, and \( V \) is the matrix of the left eigenvectors of \( \mathcal{H} \). Here, as a convention, we always rescale the eigenvectors such that \( \det V = 1 \), implying \( V \in SL(2, \mathbb{C}) \). Therefore, the Hamiltonian can be encoded using two pieces of data, \( (V, t) \in SL(2, \mathbb{C}) \times \mathbb{R} \equiv G \), which constitute a simply connected group [78] with composition rule \((V_1, t_1) \circ (V_2, t_2) = (V_1 \cdot V_2, t_1 + t_2)\). However, the decomposition into \((V, t)\) is not unique. On the one hand, the matrix \( \mathcal{H} \) in Eq. (2) is invariant under rescaling the two eigenvectors separately by \((z, z^{-1})\) with \( z \in \mathbb{C}^\times \) (the complex plane without zero), as well as under shifting \( t \) by an even integer. This represents transformations
\[ T_n(z) : (V, t) \rightarrow (R(z) \cdot V, t + n) \]  
(n even),
(3a)
where \( R(z) = \text{diag} (z, z^{-1}) \). On the other hand, we can flip the ordering of the eigenvectors if we appropriately reorder the eigenvalues \( e^{i\pi} \sigma_z \) by shifting \( t \) by an odd integer. This corresponds to transformations
\[ T_n(z) : (V, t) \rightarrow (i\sigma_y \cdot R(z) \cdot V, t + n) \]  
(n odd).
(3b)
Eqs. (3) represent left action on \( G \) by elements \( T_n(z) \) [defined as \((R(z), n)\) for \( n \) even, and as \((i\sigma_y \cdot R(z), n)\) for \( n \) odd], which constitute a subgroup \( H \subset G \). The space of all distinct Hamiltonians is the coset space \( G/H \).

As a topological space, \( H \) is a disjoint union of many copies of \( \mathbb{C}^\times \) (one copy for each \( n \in \mathbb{Z} \)). It follows from Eqs. (3) that \( T_{n_1}(z_1) \circ T_{n_2}(z_2) = T_{n_1+n_2}(z) \) for some \( z \in \mathbb{C}^\times \), implying that connected components of \( H \) have a natural \( \mathbb{Z} \)-group structure. Therefore, \( \pi_1(M) = \pi_0(H) = \mathbb{Z} \), which corresponds to the winding number. Furthermore, each disjoint component supports “looping” of \( z \) around the origin of \( \mathbb{C}^\times \). Therefore, \( \pi_2(M) = \pi_1(H) = \mathbb{Z} \), which corresponds to the Chern number. We are finally ready to compute the action of \( \pi_1(M) \) on \( \pi_2(M) \). According to Fig. 4(d), we should study the conjugation of elements in \( \pi_2(M) \) [looping of the argument of \( T_n(z) \)] by elements in \( \pi_1(M) \) [subscript of \( T_n(z) \)]. We find [69]
\[ T_{n_1}(z_1) \circ T_{n_2}(z_2) \circ T_{n_1}(z_1)^{-1} = T_{n_2} \left( cn \epsilon_2^{P(n_1)} \right) \]  
(4)
where \( P(n_1) = \pm 1 \) is the parity of \( n_1 \), and \( c \in \mathbb{C}^\times \) is an unimportant factor that depends on \( z_1 \) and \( n \). Since \( z_2^{-1} \) loops opposite to \( z_2 \), we conclude that the Chern number of a node flips sign if it is carried along a path with odd winding number. We remark that such a path may exist even in the absence of exceptional lines, provided that the winding number along some direction of the Brillouin zone torus is odd. We present a lattice Hamiltonian with such a property in the Supplemental Material [69].

Conclusions and outlooks.— We have shown that band nodes with a Chern number braid non-trivially around exceptional lines in non-Hermitian systems. While we have explicitly considered only two-band models, both the \( \mathbb{Z} \)-valued Chern number and the \( \mathbb{Z} \)-valued winding number are stable topological invariants [63], therefore the non-trivial braiding of band nodes discussed here persists upon adding more bands. In fact, many-band models provide even richer possibilities. By traversing the Riemann-sheet band-structure near exceptional lines (which may now connect various pairs of bands), we can arbitrarily permute the ordering of the bands, and thus also of their Chern numbers. Especially, this allows us to move a Weyl point in-between a different pair of bands. These phenomena follow from a more general topological structure than the one considered in Ref. [63], which we shall present in a future work [68].

We emphasize that a non-trivial action of \( \pi_1(M) \) on \( \pi_2(M) \) cannot arise for nodes in the stable limit of Hermitian systems. The observation from Ref. [10] is that if both of these homotopy groups are non-trivial, then \( \pi_2(M) = \mathbb{Z}_2 \), which does not support non-trivial automorphisms. There are only handful few-band models, very recently studied in Ref. [24], that enable a non-trivial braiding of monopole charges [9, 79]. However, those examples are unstable against including additional bands. Therefore, a stable non-trivial braiding of band nodes by the action of \( \pi_1(M) \) on \( \pi_2(M) \) constitutes a novel phenomenon enabled by non-Hermitian effects.

Due to the high controllability and tunability, there have been experimental studies of both Weyl points and exceptional lines in photonic systems [27–30]. In fact, the existing experimental techniques readily allow for inferring the topological charges of both exceptional nodes [26] and Weyl points [31] in such systems. Because of the intrinsically non-Hermitian nature of photonic systems, we believe that a two-band model similar to Eq. (1) with one tuning parameter \( \alpha \) (or to the lattice model presented in the Supplemental Material [69]) could be realizable upon considering the gain and loss. Performing experiments analogous to those of Refs. [26–31] on such a model would provide experimental evidence for our predictions. Alternatively, one may consider implementing such a model in a system with synthetic dimension having gain and loss [80].

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[1] C. Herring, Phys. Rev. B 52, 365 (1993).
[2] S. Murakami, New J. Phys. 9, 356 (2007).
[3] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
[4] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. 90, 015001 (2018).
[5] M. V. Berry, Proc. R. Soc. London. A 392, 45 (1984).
[6] J. Zak, Phys. Rev. Lett. 62, 2747 (1989).
[7] A. A. Burkov, M. D. Hook, and L. Balents, Phys. Rev. B 84, 235126 (2011).
[8] C. Fang, H. Weng, X. Dai, and Z. Fang, Chin. Phys. B 25, 117106 (2016).
[9] C. Fang, Y. Chen, H.-Y. Kee, and L. F. Fu, Phys. Rev. B 92, 081201(R) (2015).
[10] T. Bzdušek and M. Sigrist, Phys. Rev. B 96, 155105 (2017).
[11] C. Fang, Y. Chen, H.-Y. Kee, and L. F. Fu, Phys. Rev. B 98, 041124 (2018).
[12] A. Hatcher, Algebraic Topology (Cambridge University Press, Cambridge, 2002).
[13] X.-Q. Sun, S.-C. Zhang, and T. Bzdušek, Phys. Rev. X 9, 041101 (2019).
[14] H. Wang, J. D. Joannopoulos, M. Soljačić, and B. Zhen, Nat. Phys. 13, 1117 (2017).
[15] L. Fu, J. D. Joannopoulos, M. Soljačić, and B. Zhen, Science 349, 136802 (2019).
See Supplemental Material at provided URL for (i) a roadmap for deriving the Hamiltonian in Eq. (1), for (ii) an alternative tight-binding lattice model that exhibits the non-trivial action of $\pi_1(M)$ on $\pi_2(M)$, and for (iii) a detailed derivation of the action of $\pi_1(M)$ on $\pi_2(M)$ for non-Hermitian two-band models.

The compatibility further requires $\Delta g \circ \Delta h = \Delta g \circ h$. The collection $[\pi_1(M), \pi_2(M), \Delta]$, together with an additional piece of data called the Postnikov class, form a mathematical structure called the fundamental 2-group of $M$ [81, 82].

This follows from the long exact sequence of relative homotopy groups for pair $(G, H)$ [11, 12].
I. A ROAD-MAP TO THE $k \cdot p$ MODEL

In this section, we present a $k \cdot p$ Hamiltonian that exhibits a non-trivial braiding of Weyl point as one free parameter is tuned. The discussion is split in two parts. First, in Sec. I A we obtain a Hamiltonian that exhibits a single exceptional nodal line. This is achieved by starting with a simple Weyl Hamiltonian and by including in the Hamiltonian to $-\text{independent non-Hermitian perturbation.}$ In Sec. I B we include in the developed model an additional and more complicated $k$-dependent non-Hermitian perturbation. After properly adjusting the parameters, we find that the resulting model exhibits two Weyl points that are non-trivial braided around the exceptional nodal line.

A. Hamiltonian with a single exceptional nodal line

We begin with the simplest Weyl point Hamiltonian

$$H^{(0)}(k) = k \cdot \sigma,$$  

(1)

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. The spectrum of the Weyl Hamiltonian is $\varepsilon^{(0)}_\pm(k) = \pm |k|$. The superscript of the Hamiltonian indicates the position of the Weyl point inside the complex $k_+ + ik_- \text{plane.}$ This convention will become useful later in Sec. I B.

Non-Hermitian perturbations generically “inflated” a Weyl point into a ring of exceptional nodes. Here, we consider the simple $k$-independent non-Hermitian perturbation $\delta H = i \frac{\alpha}{2} \sigma_y$ with $m \in \mathbb{R}$.

The combined Hamiltonian

$$H^{(0)}(k) + \delta H = H(k) = k_x \sigma_x + (k_y + i \frac{\alpha}{2}) \sigma_y + k_z \sigma_z$$  

(2)

has energy spectrum

$$\varepsilon_\pm(k) = \sqrt{k_x^2 + mk_y^2 - \frac{m^2}{4}},$$  

(3)

which exhibits an exceptional degeneracy at $k_y = 0$ along a ring $4(k_+^2 + k_-^2) = m^2$.

We shift the momentum coordinates as $k = g + \delta k$, where $g = (\frac{\alpha}{2}, 0, 0)$. Furthermore, we find it useful to instead work with matrices $\sigma_\pm = \sigma_x \pm i \sigma_y$ (together with $\sigma_z$) and with momentum-component combinations $k_\pm = \delta k_x \pm i \delta k_y$ (together with $\delta k_z = k_z$). The inverse transformations are

$$\delta k_x = \frac{k_+ + k_-}{2} \quad \text{and} \quad \delta k_y = \frac{k_+ - k_-}{2i},$$  

(4)

and similarly for the Pauli matrices $\sigma_x$ and $\sigma_y$. We will use the fact that the Hamiltonian of the form

$$H = h_+ \sigma_+ + h_- \sigma_- + h_z \sigma_z$$  

(5)

has energy eigenvalues

$$\varepsilon(k) = \pm \sqrt{4h_+ h_- + h_z^2}.$$  

(6)

We also remark that the Weyl Hamiltonian in Eq. (1) can be rewritten as

$$H^{(0)}(k) = \frac{1}{2} (k_+ \sigma_- + k_- \sigma_+) + k_z \sigma_z.$$  

(7)

This decomposition will come in handy later in Sec. I B.

In the basis of $\sigma_\pm$ and using the momentum coordinates $k_\pm$, the perturbed Weyl Hamiltonian from Eq. (2) becomes

$$H(k) = \left( \frac{\alpha}{2} + \delta k_x \right) \sigma_x + \left( \delta k_y + i \frac{\alpha}{2} \right) \sigma_y + \delta k_z \sigma_z$$  

(8)

$$= \frac{1}{2} \left( (m + k_-) \sigma_+ + k_+ \sigma_- + k_z \sigma_z \right).$$  

(9)

Near the origin of coordinates, we can approximate $m + k_- \approx m$, and we further set $m = 1$. This approximates the Hamiltonian to

$$H(k) = \frac{1}{2} (\sigma_+ + k_+ \sigma_-) + k_z \sigma_z$$  

(10)

with spectrum

$$\varepsilon_\pm = \pm \sqrt{k_+^2 + k_z^2},$$  

(11)

which exhibits an exceptional nodal line inside the $k_y = 0$ plane along $k_x = -k_z$. The Hamiltonian in Eq. (10) provides the starting point for the more complicated Hamiltonian discussed in the next subsection. One could make the exceptional nodal line straight, pointing along $k_x = 0 = k_y$, by dropping the $k_z \sigma_z$ term. However, we keep this term in the Hamiltonian, because it is very important for the generation of Weyl points inside the $k_z = 0$ plane in the next section.
B. Hamiltonian with additional Weyl points

Our next goal is to supplement the Hamiltonian in Eq. (10) with a perturbation that produces Weyl points at well-controlled positions. More specifically, we want a pair of Weyl points moving inside the \( k_z = 0 \) plane along semicircles \( k_\pm = re^{\pm i\phi} \) with some radius \( r \) and polar angle \( \phi \in [0, \pi] \). The semicircles “circumnavigate” the exceptional line inside the \( k_z = 0 \) plane. Our strategy to get such a model is to consider \( k_z \)-independent perturbation, and temporarily limit our attention to the spectrum inside the \( k_z = 0 \) plane. The only \( k_z \) dependence of our model would come from the \( k_z \sigma_z \) term that is readily present in Eq. (10), or even in Eq. (1).

Inside the \( k_z = 0 \) plane, both the Weyl points as well as the cross-section of the exceptional nodal line appear as point-like objects. However, they are associated with a different topological structure. While the cross-section of the exceptional nodal line at \( k_x = 0 = k_y \) is associated with a non-trivial winding of the complex-valued band energies [corresponding to the first homotopy group \( \pi_1(\mathcal{M}) \)], the Weyl points do not exhibit such winding. Since we want the perturbed Hamiltonian to exhibit just the single exceptional nodal line at the origin of coordinates, we should keep the scaling of the band energies, as present in Eq. (11).

It follows from shifting the coordinates in Eq. (1) that a Hamiltonian with a single Weyl point at position \( k_+ = re^{i\phi} \) (which by complex conjugation corresponds to \( k_- = re^{-i\phi} \)) is simply

\[
\mathcal{H}^{(\text{re}^{i\phi})}(k) = \frac{1}{2}((k_+ - re^{i\phi})\sigma_- + (k_- - re^{-i\phi})\sigma_+) + k_z\sigma_z. \quad (13)
\]

By referring to Eq. (6), it is clear that the spectrum of the model in Eq. (13) is \( \varepsilon_{\pm} = \pm \sqrt{|k_+ - re^{i\phi}|^2 + k_z^2} \), as expected for a Weyl point at that position. We also emphasize that Eq. (13) explicitly adopts the convention outlined below Eq. (1), namely that the superscript of the Weyl Hamiltonian indicates the position of the Weyl point inside the \( k_z = 0 \) plane.

To get a pair (and potentially even more) of Weyl points inside the \( k_z = 0 \) plane, we just need to keep adding factors \( (k_+ - re^{\pm i\phi^*}) \) to \( \sigma_\pm \). However, there is actually more than one way to do this. For example, there are two ways to get a Hamiltonian with two in-plane Weyl points located at \( r_1e^{i\phi_1} \equiv \tilde{z}_1 \) and \( r_2e^{i\phi_2} \equiv \tilde{z}_2 \), namely

\[
\mathcal{H}^{(\tilde{z}_1, \tilde{z}_2)}_a(k) = \frac{1}{2}((k_+ - \tilde{z}_1)(k_+ - \tilde{z}_2)\sigma_- + \text{h.c.}) + k_z\sigma_z \quad (14)
\]

\[
\mathcal{H}^{(\tilde{z}_1, \tilde{z}_2)}_b(k) = \frac{1}{2}((k_+ - \tilde{z}_1)(k_- - \tilde{z}_2^*)\sigma_- + \text{h.c.}) + k_z\sigma_z \quad (15)
\]

where “h.c.” stands for “Hermitian conjugate”, and the asterisk “*” indicates complex conjugate.

The Hamiltonians in Eqs. (13–15) are all Hermitian, and do not exhibit a non-trivial winding of the complex-valued band energies. Therefore, they do not exhibit the scaling of band energies expressed by Eq. (12) in the limit of large \( |k_\pm| \). Note also that Hamiltonians with two (or more generally with \( n \)) in-plane Weyl points would scale as \( |k_\pm|^2 \) (resp. as \( |k_\pm|^n \)) for large \( |k_\pm| \), i.e. this term will at large momenta take over the exceptional-line Hamiltonian given by Eq. (10), which we intend to perturb. To make sure that the perturbed Hamiltonian would retain the correct band energy winding for \( |k_\pm| \to \infty \), we multiply the term proportional to \( \sigma_- \) by an additional factor of \( k_z \).

We now collect all that we learned from the previous discussion. We opt for the variant \( b \) of including two in-plane Weyl points, corresponding to Eq. (15). Since we want the two Weyl points to be located at polar angles \( \pm\alpha \), we perturb the exceptional-line Hamiltonian in Eq. (10) with

\[
\mathcal{H}'(k; \alpha) = (k_+ + e^{-i\alpha})(k_+ + e^{-i\alpha})\sigma_+ + (k_+ + e^{i\alpha})(k_- + e^{i\alpha})k_z\sigma_- \quad (16)
\]

The combination \( \mathcal{H}(k) + \mathcal{H}'(k; \alpha) \) of the Hamiltonians in Eqs. (10) and (16) corresponds to the model presented in Eq. (1) of the main text. This model exhibits two in-plane Weyl points at complex-conjugated values of \( k_\pm \), and a single exceptional nodal line crossing the \( k_z = 0 \) plane at \( k_z = 0 = k_y \).

It can be analytically derived that the Weyl points of the constructed Hamiltonian move on a circle with radius \( r = 1/\sqrt{2} \), and that their polar coordinates obey \( \cos \phi = -\sqrt{2} \cos \alpha \). It is therefore clear that the Weyl points exist inside the \( k_z = 0 \) plane only for \( \alpha \in (\pi/4, 3\pi/4) \). A more careful analysis reveals that the Weyl points are ejected from the exceptional line touching the \( k_z = 0 \) plane as one increases \( \alpha \) through \( \pi/4 \), and that the two Weyl points pairwise annihilate as one increases \( \alpha \) through \( 3\pi/4 \).

II. THE ALTERNATIVE LATTICE MODEL

As an alternative model that could be more easily implemented in experiments, we consider the lattice model

\[
\mathcal{H}(k; m) = e^{i\frac{T}{2}} \left[ \cos \left( \frac{k_x}{2} - \frac{\pi}{2} \right) \sin k_x \sigma_x + \right. \\
+ \cos \left( \frac{k_y}{2} + \frac{\pi}{2} \right) \sin k_y \sigma_y \\
\left. + \left( \sin k_z \cos \frac{k_x}{2} - 2m \sin \frac{k_y}{2} \right) \sigma_z \right]. \quad (17)
\]

This model respects the periodicity of the momentum space, and the complex-valued determinant \( \det \mathcal{H}(k; m) \) exhibits a non-trivial winding along the \( k_z \) direction of the Brillouin zone torus. More explicitly, the \( e^{i\frac{T}{2}} \) prefactor in Eq. (17) guarantees that \( \arg \det \mathcal{H}(k; m) = k_z \).

The non-Hermitian lattice model in Eq. (17) exhibits several Weyl points. First, for all values of \( m \) there are Weyl points located at \( k = (0, 0, 0), (0, \pi, 0), (\pi, 0, 0), (\pi, \pi, 0) \) inside the \( k_z = 0 \) plane of the Brillouin zone. Furthermore, for \( m \in (0, 1) \), there are additional eight Weyl points located at the same \( k_x \) and \( k_y \) as the previous quadruplet, and with \( \sin \frac{k_x}{2} = \pm \sqrt{1 - m} \). We remark
that this model does not exhibit any exceptional lines. Nevertheless, we find that the chirality of the Weyl points (corresponding to $\pi_1(M)$) interacts non-trivially with the winding number along the $k_z$-direction of the Brillouin zone (corresponding to $\pi_1(M)$).

Here, we shall focus on the Weyl points on the $k_x = k_y = 0$ line, while the analyses of Weyl points on the other lines such as $(k_x, k_y) = (0, \pi), (\pi, 0)$ and $(\pi, \pi)$ are similar. First, we would like expand the Bloch Hamiltonian around the Weyl point $k_x = k_y = k_z = 0$:

$$H(k; m) \approx \frac{1}{\sqrt{2}} k_x \sigma_x + \frac{1}{\sqrt{2}} k_y \sigma_y + (1 - m) k_z \sigma_z$$  \hspace{1cm} (18)

Therefore, for $m < 1$, the Weyl point chirality is positive, while for $m > 1$, the Weyl point chirality is negative.

For $m < 0$ and $m > 1$, there is only one Weyl point on the line at $k_z = 0$. However, this Weyl point is of opposite chirality for the situations with $m < 0$ resp. with $m > 1$. Therefore, as we as we tune $m$ from a negative number to positive number larger than 1, the “total chirality” of Weyl points on the $(k_x, k_y) = (0, 0)$ line changes sign without other band nodes moving onto the line. As we have explained in the main text, this phenomenon corresponds to the fact that the total chirality of Weyl points cannot be well-defined globally in the Brillouin zone.

Here, we would also like to understand the process in more details. As $m$ increases from negative to positive, a pair of Weyl points are created at $k_z = \pi$, which locally exhibit opposite chirality. As $m$ grows from 0 to 1, the two Weyl points move towards $k_z = 0$ from the positive $k_z$ side and from the negative $k_z$ side, respectively. Along the fixed $k_x = 0 = k_y$ line, the phase of the two eigenvalues of the Bloch Hamiltonian are $\pm e^{ik_z/2}$, which means that the two energy bands exchange as one increases $k_z$ by $2\pi$, similar to encircling an exceptional line. Therefore, as the two Weyl points meet at $k_z = 0$, they effectively “encircle an exceptional line” and now have the same chirality, as illustrated in Fig. 1 of the main text. The extra chirality carried by the two Weyl points from $k_z = \pi$ hence flips the chirality of the Weyl point at $k_z = 0$. We illustrate the exchange of Weyl points of the model in Eq. (17) in Fig. 1 of the Supplemental Material.

### III. TOPOLOGICAL INVARIANTS

In this section, we complete the details of computing the action $\triangleright$ of $\pi_1(M)$ on $\pi_2(M)$, which are omitted in the main text. Here, $M$ is the space of $2 \times 2$ Hamiltonians that are traceless and have spectrum normalized to absolute value 1. We shall first review the formalism introduced by the main text. First, we express the space $M$ as a coset space $G/H$, where $G$ is a simply connected Lie group and $H$ is the stabilizer subgroup. Then, we use the coset expression and the computational algorithm described in Ref. [1] to derive $\pi_1(M)$ and $\pi_2(M)$, as well as the action $\triangleright$.

As explained in the main text, we can identify any Hamiltonian in $M$ using $(V, t) \in SL(2, \mathbb{C}) \times \mathbb{R} \equiv G$, as expressed by Eq. (2) of the main text. Furthermore, the main text argues that the stabilizer group $H$ consists of the following elements in $G$: $(R(z), n)$ for $n \in \mathbb{Z}$, as well as $(i\sigma_y \cdot R(z), n)$ for odd $n$, where $R(z) = \text{diag}(z, 1/z)$ with $z$ being any complex number except of zero (which we indicate as $\mathbb{C}/\{0\} \equiv \mathbb{C}^\times$). We then use a mathematical theorem from Ref. [1] to show that $\pi_2(M) = \pi_1(H) = \mathbb{Z}$ [i.e. the Chern number on a 2-sphere corresponds to “looping” of the argument of $R(z)$ around the origin of $\mathbb{C}^\times$], and $\pi_1(M) = \pi_0(H) = \mathbb{Z}$ [i.e. the winding number on a 1-sphere corresponds to the connected component $\pi_0$ of the stabilizer group $H$].

According to Fig. 4(d) in the main text, we should study the conjugation of elements in $\pi_2(M)$ by elements in $\pi_1(M)$. The elements in $\pi_2(M)$ are represented as topologically distinct loops in the space of $H$ $(\pi_1(H))$. The loops can be parameterized by $T_{n_2}(z_2) = (R(z_2), n_2)$ for even $n_2$ and and $T_{n_2}(z_2) = (i\sigma_y \cdot R(z_2), n)$ for odd $n_2$, where $z_2$ is taken along a path that loops around the origin of the complex plane $\mathbb{C}^\times$. On the other hand, the elements in $\pi_1(M)$ are represented as disconnected points in the space of $H$ $(\pi_0(H))$. This corresponds to $T_{n_1}(z_1)$ with some $z_1 \in \mathbb{C}^\times$. Without loss of generality, we set $z_1 = 1$ in our arguments below (we comment on the case of general $z_1$ in the last paragraph below).

We can now compute $T_{n_1}(1) \circ T_{n_2}(z_2) \circ T_{n_1}(1)^{-1}$ explicitly. Recall that the group $G$ is a direct product of Abelian part of integer number addition $\mathbb{R}$ and of a non-Abelian part of $SL(2, \mathbb{C})$. The Abelian part of $T_{n_2}(z_2)$ does not change upon conjugation by $T_{n_1}(1)$. Therefore, we only need to calculate the non-Abelian part of $T_{n_1}(1) \circ T_{n_2}(z_2) \circ T_{n_1}(1)^{-1}$. The calculation has to be split into several cases, corresponding to different parities of $n_1$ and $n_2$.

For even $n_1$, $T_{n_1}(1) = (I_{2 \times 2}, n)$ commute with $T_{n_2}(z_2)$ and therefore

$$T_{n_1}(1) \circ T_{n_2}(z_2) \circ T_{n_1}(1)^{-1} = T_{n_2}(z_2), \hspace{1cm} (19)$$

![FIG. 1. Weyl points of the model in Eq. (17), which are located inside (a part of) the $k_y = 0$ plane for (a) $m < 0$, (b) $m = 0.25$ and (c) $m > 1$. We find that for $m < 0$, there is a Weyl point (red) of positive chirality at $k_x = k_y = 0$. A pair of Weyl points (green) are created at $(k_x, k_y) = (\pi, 0)$ for $m = 0$. After bringing the two green Weyl points to $k_x = 0$, they merge with the Weyl point located at $k_x = k_y = 0$. There remains a Weyl point (blue) with negative chirality at $k_x = k_y = 0$ for $m > 1$.](image-url)
On the other hand, for odd $n_1$, we can use the following commutating relations:
\[
\begin{align*}
    i\sigma_y \cdot R(z_2) &= R(1/z_2) \cdot i\sigma_y \\
    i\sigma_y \cdot (i\sigma_y \cdot R(z_2)) &= (i\sigma_y \cdot R(1/z_2)) \cdot i\sigma_y
\end{align*}
\]
(20)
to derive that for any $n_2$, we have
\[
T_{n_1}(1) \circ T_{n_2}(z_2) = T_{n_2}(1/z_2) \circ T_{n_1}(1).
\]
(21)
It follows that
\[
T_{n_1}(1) \circ T_{n_2}(z_2) \circ T_{n_1}(1)^{-1} = T_{n_2}(1/z_2).
\]
(22)
The results in Eqs. (19) and (22) can be compactly unified into a single equation
\[
T_{n_1}(1) \circ T_{n_2}(z_2) \circ T_{n_1}(1)^{-1} = T_{n_2} \left( z_2^{P(n_1)} \right),
\]
(23)
where $P(n_1) = \pm 1$ is the parity of $n_1$. Note that $1/z_2$ has opposite “looping” around the origin of $\mathbb{C}^\times$ than $z_2$.

It follows that for odd $n_1$ [odd elements of $\pi_1(M)$], the conjugation flips the sign of the $\pi_2(M)$ charge.

For a general $z_1$, one can explicitly compute for the four different combinations of parities of $n_1$ and $n_2$ the following results:

- If $n_1$ is even and $n_2$ is even, then
  \[
  T_{n_1}(z_1) \circ T_{n_2}(z_2) \circ T_{n_1}(z_1)^{-1} = T_{n_2}(z_2).
  \]
  (24)
- If $n_1$ is even and $n_2$ is odd, then
  \[
  T_{n_1}(z_1) \circ T_{n_2}(z_2) \circ T_{n_1}(z_1)^{-1} = T_{n_2}(z_2/z_1^2).
  \]
  (25)
- If $n_1$ is odd and $n_2$ is even, then
  \[
  T_{n_1}(z_1) \circ T_{n_2}(z_2) \circ T_{n_1}(z_1)^{-1} = T_{n_2}(1/z_2).
  \]
  (26)
- If $n_1$ is odd and $n_2$ is odd, then
  \[
  T_{n_1}(z_1) \circ T_{n_2}(z_2) \circ T_{n_1}(z_1)^{-1} = T_{n_2}(z_2^2/z_1^2).
  \]
  (27)

The three equations are compactly summarized in Eq. (4) of the main text.

[1] N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).