Abstract

We express the equivalent resistance between the origin \((0,0,0)\) and any other lattice site \((n_1,n_2,n_3)\) in an infinite Body Centered Cubic (BCC) network consisting of identical resistors each of resistance \(R\) rationally in terms of known values \(b_o\) and \(\pi\). The equivalent resistance is then calculated. Finally, for large separation between the origin \((0,0,0)\) and the lattice site \((n_1,n_2,n_3)\) two asymptotic formulas for the resistance are presented and some numerical results with analysis are given.

Keywords: Lattice Green’s Function, Infinite network, BCC lattices, Identical Resistors.
1. Introduction

The Lattice Green’s Function (LGF) is a basic concept in physics. Many quantities of interest in solid-state physics can be expressed in terms of it. For example, statistical model of ferromagnetism such as Ising model [1], Heisenberg model [2], spherical model [3], lattice dynamics [4], random walk theory [5, 6], and band structure [7, 8]. In Economou’s book [9] one can find an excellent introduction to the LGF, where a review of the LGF of the so-called tight-binding Hamiltonian (TBH) used for describing the electronic band structures of crystal lattices is presented. The LGF defined in this paper is related to the GF of the TBH. Many efforts have been paid on studying the LGF of cubic lattices [10-25].

The LGF for the BCC lattice has been expressed as a sum of simple integrals of the complete elliptic integral of the first kind [10], Morita and Horiguci [11] presented formulas which are convenient for the evaluation of the LGF for the Face Centered Cubic (FCC), BCC and rectangular lattices. These formulas involve the complete elliptic integral of the first kind with complex modulus. Morita [12] derived a recurrence relation, which gives the values of the LGF along the diagonal direction from a couple of the elliptic integrals of the first and second kind for the square lattice with discussions of how to apply the result to the BCC lattice. Finally, Glasser and Boersma [23] expressed the values of the LGF of the BCC lattice rationally. One can find more works in these works and references within them.

The calculation of the equivalent resistance in infinite networks is a basic problem in the electric circuit theory. It is of so interest for physicists and electrical engineering. There are mainly three approaches used to solve such a problem. The superposition of current distribution has been used to calculate the effective resistance between adjacent sites on infinite networks [26-28].

A mapping between random walk problems and resistor networks problems have been used by Monwhea Jeng [29], where his method was used to calculate the effective resistance between any two sites in an infinite two-dimensional square lattice of unit resistors.

A third educational important method based on the Lattice Green’s Function (LGF) of the lattices has been used in calculating the equivalent resistance [30-38]. This method has been applied to both perfect and perturbed square, simple cubic (SC) networks and recently to the FCC network.

The present work is organized as follows:
In Sec. 2 we briefly introduced the basic formulas of interest for the LGF of the BCC network. In Sec. 3 an application to the LGF of the BCC network has been applied to express the equivalent resistance between the origin and the lattice site \((n_1, n_2, n_3)\) in the infinite BCC network rationally in terms of some constants, and the asymptotic behavior for the resistance is also investigated as the separation between the two sites goes to infinity. Finally, we close this paper (Sec. 4) with a discussion for the results obtained.

2. Basic Definitions and Preliminaries

The LGF for the BCC lattice appears in many areas in physics (e.g., Ising model \([1, 39-40]\), Heisenberg model \([41-43]\) and spherical models \([44-46]\) and it is defined as \([13, 23]\):

\[
B(E; n_1, n_2, n_3) = \frac{1}{\pi} \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} \frac{\cos(n_1 \mu) \cos(n_2 \nu) \cos(n_3 \omega)}{E - \cos \mu \cos \nu \cos \omega} \, d\mu \, d\nu \, d\omega. \tag{1}
\]

Where \(E \geq 1\), \(n_1, n_2,\) and \(n_3\) are either all even or all odd integers.

The LGF for the BCC lattice at the site \((0,0,0)\) which represents the origin of the lattice for \(E = 1\) (i.e., \(B(1;0,0,0) = b_o\)) was of so interests in physics and it was carried out first by Van Peijpe \([47]\) and later on by Watson \([48]\). They showed that:

\[
b_o = B(1;0,0,0) = \frac{4}{\pi^2} [K(1/\sqrt{2})]^2 = \frac{\Gamma^4(1/2)}{4\pi^3} = 1.3932039297. \tag{2}
\]

Where \(K\) is the complete elliptic integral of the first kind, and \(\Gamma\) is the Gamma function.

In a recent work the LGF for the infinite BCC lattice has been expressed rationally as \([23]\):

\[
B(1; n_1, n_2, n_3) = \sigma_1 b_o + \frac{\sigma_2}{\pi^2 b_o} + \sigma_3. \tag{3}
\]

Where \(\sigma_1, \sigma_2\) and \(\sigma_3\) are rational numbers.
3. Application: Evaluation of the resistance \(R(n_1, n_2, n_3)\) in an Infinite BCC Network

The aim of this section is to express the equivalent resistance between the origin \((0,0,0)\) and the lattice site \((n_1, n_2, n_3)\) in the infinite BCC network which is consisting from identical resistors rationally in terms of \(b_o\) and \(\pi\).

First of all, it has been showed that for a 3D infinite network consisting of identical resistors each of resistance \(R\), the equivalent resistance between the origin and any other lattice site is \([30]\):

\[
R(r) = 2[G(0) - G(r)].
\]  

(4)

Where \(r\) is the position vector of the lattices point, and for a \(d\)-dimensional lattice it has the following form:

\[
r = n_1a_1 + n_2a_2 + \ldots + n_da_d.
\]  

(5)

With \(n_1, n_2, \ldots, n_d\) are integers, and \(a_1, a_2, \ldots, a_d\) are independent primitive translation vectors.

Also, the equivalent resistance between the origin and any other lattice site is can be expressed in an integral form as \([30]\):

\[
\begin{align*}
R(n_1, n_2, \ldots, n_d) &= R\int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \frac{1 - \exp(in_1x_1 + in_2x_2 + \ldots + in_dx_d)}{2\sum_{i=1}^{d}(1 - \cos x_i)} \\
&= R\int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \exp(in_1x_1 + in_2x_2 + \ldots + in_dx_d) \\
&= 2\sum_{i=1}^{d}(1 - \cos x_i).
\end{align*}
\]  

(6)

On the other hand, the LGF for a 3D hypercube read as \([30]\):

\[
G(n_1, n_2, \ldots, n_d) = \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \frac{\exp(in_1x_1 + in_2x_2 + \ldots + in_dx_d)}{2\sum_{i=1}^{d}(1 - \cos x_i)}.
\]  

(7)

For cubic lattices \(d = 3\). Then substituting \(d = 3\) into Eqs. (6) and (7) and comparing them with Eq. (4) one get:

\[
R(n_1, n_2, n_3) = R[b_o - B(1; n_1, n_2, n_3)].
\]  

(8)

Now make use of Eq. (3) and Eq. (8) one yields
\[
\frac{R(n_1,n_2,n_3)}{R} = r_1 b_o + \frac{r_2}{\pi^2 b_o} + r_3. \quad (9)
\]

Where \( r_1 = 1 - \sigma_1, \) \( r_2 = -\sigma_2 \) and \( r_3 = -\sigma_3 \) are rational numbers. These rational numbers, for the sites from \((0,0,0)\) to \((8,8,8)\), can be gathered from [13, appendix A]. In Table 1 below we present these rational numbers.

Based on the recurrence formula presented in [13, Eq. (5.8)] we have calculated additional rational values for the sites from \((9,1,1)\) to \((10,0,0)\) and arranged them in Table 1 below.

Since the LGF is an even function (i.e., \( B(1; n_1, n_2, n_3) = B(1; -n_1, -n_2, -n_3) \)) and due to the fact that the infinite BCC network is pure and symmetric, then as a result \( R(n_1, n_2, n_3) = R(-n_1, -n_2, -n_3) \).

Finally, it is interesting to study the asymptotic behavior of the equivalent resistance for large separation between the origin \((0,0,0)\) and any other lattice site \((n_1, n_2, n_3)\).

The asymptotic form of \( B(1;0,0,n_3) \), as \( n_3 \to \infty \), is given as [13]:

\[
B(1;0,0,2n_3) = \frac{1}{n_3^2} \left(1 - \frac{1}{8n_3^2} + \frac{1}{128n_3^4} - \frac{173}{1024n_3^6}\right). \quad (10)
\]

While, for large value of \( |n| = \sqrt{n_1^2 + n_2^2 + n_3^2} \) it has been shown that [13] \( B(1; n_1, n_2, n_3) \) has the following asymptotic formula

\[
B(1; n_1, n_2, n_3) \approx 2\pi|n|^{-1} \left[1 - \frac{9}{8} |n|^2 + \frac{5}{8} |n|^4 (n_1^4 + n_2^4 + n_3^4) + \frac{15}{4} |n|^6 (n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2)\right]. \quad (11)
\]

Inserting Eq. (10) and Eq. (11) into Eq. (8), one gets the following two equations:

\[
\frac{R_o(0,0,2n_3)}{R} \to b_o - \frac{1}{n_3^2} \left(1 - \frac{1}{8n_3^2} + \frac{1}{128n_3^4} - \frac{173}{1024n_3^6}\right). \quad (12)
\]

\[
\frac{R_o(n_1,n_2,n_3)}{R} \to b_o - \frac{2}{\pi} |n|^{-1} \left[1 - \frac{9}{8} |n|^2 + \frac{5}{8} |n|^4 (n_1^4 + n_2^4 + n_3^4) - \right.
\]

\[
\left. \frac{15}{4} |n|^6 (n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2)\right].
\]
\[ \frac{15}{4} |n|^{-6} (n_1^2 n_2^2 + n_2^2 n_3^2 + n_1^2 n_3^2) \]. \quad (13)

The last asymptotic formula agrees with Eq. (12) for \( n_1 = 0, n_2 = 0 \) and for \( n_3 = 2n_3 \). In addition, the above two asymptotic formulas can be used to check the results obtained in Table 1 below. For example,

\[
\frac{R_o(8,0,0)}{R} \cong 1.31425;
\frac{R_o(8,8,6)}{R} \cong 1.34413;
\frac{R_o(8,8,8)}{R} \cong 1.34778;
\frac{R_o(9,9,9)}{R} \cong 1.35273;
\frac{R_o(10,0,0)}{R} \cong 1.32986. \quad (15)
\]

From the above two asymptotic formulas, one can see that as \( n_3 \to \infty \), or as \( |n| \to \infty \) then the resistance goes to a finite value (i.e., goes to \( b_o \)).

4. Results and Discussion

In this work we have expressed the equivalent resistance between the origin \((0,0,0)\) and any other lattice site \((n_1, n_2, n_3)\) in an infinite BCC network consisting of identical resistors each of resistance \( R \) rationally in terms of the two known values \( b_o \) and \( \pi \). The rational number \( r_1, r_2 \) and \( r_3 \) presented in Eq. (14) were calculated using some recurrence formulas. In Fig. 1 and Fig. 2 the equivalent resistance is plotted against the lattice site.

Figure 1 shows the resistance in an infinite BCC lattice against the site \((n_1, n_2, n_3)\) along the [100] direction. From this figure it is clear that the resistance is symmetric.

Figure 2 shows the resistance in an infinite BCC lattice against the site \((n_1, n_2, n_3)\) along the [111] direction. From this figure it is clear that the resistance is symmetric.

The above figures indicate that as the separation between the origin and the lattice site \((n_1, n_2, n_3)\) increases, then the equivalent resistance approaches a finite value (i.e., \( b_o = 1.3932039297 \)) as explained above.
Finally, it is worth mention that for the case of Face Centered Cubic (FCC) network the equivalent resistance as the separation between the origin and any other lattice site approaches a finite value \( f_o = 0.4482203944 \) \([37]\) where \( f_o \) is the LGF at the origin for the FCC lattice, while for the case of an infinite SC network it goes to the finite value \( g_o = 0.505462 \) \([30, 33]\), where \( g_o \) is the LGF at the origin for the SC lattice. Whereas for the infinite square lattice it goes to infinity \([30, 34]\)
Table Captions

Table 1: Values for selected rational numbers $r_1$, $r_2$, $r_3$ and $R(n_1,n_2,n_3)$ for sites (0,0,0) to (10,0,0)

| $(n_1,n_2,n_3)$ | $r_1$ | $r_2$ | $r_3$ | $R(n_1,n_2,n_3)/R$ |
|-----------------|-------|-------|-------|-------------------|
| 000             | 0     | 0     | 0     | 0                 |
| 111             | 0     | 0     | 1     | 1.0000            |
| 002             | 1     | -4    | 0     | 1.1023            |
| 022             | 0     | 16    | 0     | 1.16360           |
| 222             | -3    | -36   | 8     | 1.20228           |
| 113             | 2     | -8    | -1    | 1.20461           |
| 133             | -4    | 80    | 1     | 1.24521           |
| 333             | -18   | -504  | 63    | 1.26877           |
| 004             | 8/9   | 0     | 0     | 1.23840           |
| 024             | 25/9  | -36   | 0     | 1.25190           |
| 224             | 104/9 | 16    | -16   | 1.26285           |
| 044             | -112/9| 256   | 0     | 1.28003           |
| 244             | -407/9| 444   | 32    | 1.28626           |
| 444             | -360/9| -5376 | 448   | 1.30059           |
| 115             | -2/9  | 8     | 1     | 1.27220           |
| 135             | 120/9 | -224  | -1    | 1.28558           |
| 335             | 810/9 | 920   | -191  | 1.29564           |
| 155             | -652/9| 1392  | 1     | 1.30374           |
| 355             | -4266/9| 1192 | 575   | 1.30990           |
| 555             | 7650/9| -48840| 2369  | 1.31934           |
| 006             | 1     | -36/25| 0     | 1.28848           |
| 026             | -16/9 | 1296/25| 0    | 1.29327           |
| 226             | -155/9| 444/25| 24    | 1.29753           |
| 046             | 409/9 | -21316/25| 0 | 1.30487           |
| 246             | 1112/9| -42224/25| -48 | 1.30795           |
| 446             | 5481/9| 379804/25| -1952 | 1.31569         |
| 066             | -288  | 138384/25| 0  | 1.31802           |
| 266             | -5147/9| 249596/25| 72 | 1.31998           |
| 466             | -165600/36| -563056/25| 8048 | 1.32510         |
| 666             | 169317/9| -9083844/25| 216 | 1.33175           |
| 117             | 20/9  | -272/25| -1    | 1.30476           |
| 137             | -808/36| 10856/25| 1   | 1.31055           |
| 337             | -7616/36| -29888/25| 383 | 1.31541           |
| 157             | 9480/36| -125320/25| -1 | 1.31962           |
| 357             | 12620/9| -275440/25| -1151 | 1.32317         |
| 557             | 20584/9| 4757600/25| -17025 | 1.32903       |
| 177             | -57824/36| 769376/25| 1   | 1.32912           |
| 377             | -207128/36| 1964312/25| 2303 | 1.33153           |
| 577             | -1396840/36| -2848376/5| 95489 | 1.33566           |
| 777             | 9331056/36| -42996912/25| -236033 | 1.34058       |
| 008             | 1664/1764| 0     | 0     | 1.31422           |
| 028             | 8164/1764| -1764/25| 0   | 1.31641           |
| 228             | 48256/1764| -1648/25| -32  | 1.31845           |
| 048             | -183136/1764| 50176/25| 0   | 1.32213           |
| 448             | -3604288/1764| -737024/25| 4992 | 1.32815           |
|   | 068 | 2029540/1764 | -550564/25 | 0 | 1.32949 |
|---|-----|-------------|------------|---|--------|
| 268 | 3543104/1764 | -928496/25 | -96 | 1.33070 |
| 468 | 28134948/1764 | -664004/25 | -20288 | 1.33398 |
| 668 | -38421504/1764 | 49955984/25 | -114976 | 1.33849 |
| 088 | -12686080/1764 | 3444736/25 | 0 | 1.33687 |
| 288 | -19609372/1764 | 5280412/25 | 128 | 1.33772 |
| 488 | -11581796/1764 | 13932544/25 | 50944 | 1.34006 |
| 688 | -436242204/1764 | -216804644/25 | 975232 | 1.34340 |
| 888 | 5048578944/1764 | 165654528/25 | -4469248 | 1.34721 |
| 119 | -148/441 | 272/25 | 1 | 1.32669 |
| 139 | 17230/441 | -17912/25 | -1 | 1.32666 |
| 339 | 19368/49 | 30816/25 | -639 | 1.3293 |
| 159 | -306598/441 | 66616/5 | 1 | 1.33168 |
| 359 | -1425500/441 | 177776/5 | 1919 | 1.33381 |
| 559 | -6683968/441 | 475264/5 | 55681 | 1.33751 |
| 179 | 41833/63 | -3179392/25 | -1 | 1.33754 |
| 379 | 1200158/63 | -7803464/25 | -3839 | 1.33914 |
| 579 | 10571410/63 | 4255912/5 | -295681 | 1.34199 |
| 779 | -678572 | 435693424/25 | -322047 | 1.34551 |
| 199 | -17834440/441 | 19368352/25 | 1 | 1.3433 |
| 399 | -4533062/49 | 42106968/25 | 6399 | 1.34446 |
| 599 | -335991178/441 | 10935752/5 | 902401 | 1.34657 |
| 799 | -2907724/7 | -2645823088/25 | 8275455 | 1.34938 |
| 999 | 1270018116/49 | 7998622128/25 | -59378175 | 1.35244 |
| 0010 | 1 | -196/225 | 0 | 1.32985 |
Figure Captions

**Fig.1:** Resistance between the origin \((0,0,0)\) and the site \((n,0,0)\) along [100] direction for BCC network.

**Fig.2:** Resistance between the origin \((0,0,0)\) and the site \((n,n,n)\) along [111] direction for BCC network.
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