Modelling the water droplet motion on a leaf surface

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Abstract. The main object of the research given in this paper is modelling the water droplet movements on the leaf surface which is an important factor in realising how pesticide, nutrient or water engrossed through the surface. Therefore, a physical model based on mathematics proposed here for producing a realistic trajectory of droplet traversing a leaf surface. A vital feature of our methodology is to build the leaf surface using a recently developed method, by the author, so-called a hybrid CloughTocher cubic polynomial interpolation (CT-CPI) method. The leaf surface consisted of a mesh of triangles over which the hybrid CT-CPI method is build from 3D real life data gathered using a laser scanner. The droplet motion in our model affected by friction, resistance and gravity forces. The model verified using Matlab programming; the outcomes are promising and seem to capture reality well.

Keywords: Mathematical modelling, Virtual leaf, Finite elements methods, Surface fitting, CloughTocher method, cubic polynomial interpolation method.

1. Introduction
The application of fertilizers, pesticides and herbicides is significant in agriculture. They offer bettered plants growing conditions to which they applied. Lately, researches have been commenced on the plants to model the movement and adhesion of these solutions [1]-[3], [6], [7]-[9]. Realistic plant models industrialised to model accurately the conduct of these solutions.

When a droplet of water impacts on a surface, it may rebound off or spread out lengthwise the surface, depending on the surface inclination and surface nature, the drop size, drop speed, and some properties including surface tension and the viscosity. Rioboo et al. [26] state that their experiments propose the results comprise, corona splash, prompt splash, deposition, partial rebound, receding break up and complete rebound. Moreover, Yarin [27] in his reviewed article described these qualitatively. Further, Kim et al. [28] and Thoroddsen et al. [29], studied spreading drops which characterized by instabilities leading to viscous fingering. [30] used meta-balls in a gravitational area to simulate droplet shapes. Tong et al. [31] used meta-balls to model droplet motion by suggesting a volume-preserving technique. Lanfen et al. [32] proposed a physical simulation for couple or more large droplets morphing on a plane. Kaneda et al. [33] presented an approach to generate a droplet animation on a glass plate. A significant fact is that the complete fluid mechanics of these outcomes is pretty sophisticated and demands great level mathematical modelling, comprising stability and asymptotic analysis and an costly
experimental setup. The author [9] previously developed a model for a water droplet motion on the surface of Anthurium leaf. However, the forces that affected the motion were only the gravity and friction forces. In this research, we extended the model to include the resistance force also and we studied the motion on a Flame leaf surface.

The aims of this research paper are first, reconstructing virtual leaf surface such as Flame leaf surface from 3D scattered data, see Fig. 1. Secondly, modelling the behaviour of a droplet motion on the constructed surface to provide a realistic simulation of the droplet motion. In this framework, the virtual leaf surfaces necessitated to be smooth to enable the mathematical formulas that controlled the droplet motion. Our droplet model is effectively one-dimensional model where the droplet motion explained as a curved arcs polygonal path. Moreover, a stopping criterion developed for the droplet motion. The droplet movements verified using Matlab and exhibits that it capture reality pretty well on the surface.

To simulate droplet movements on a virtual leaf surface, firstly the surface need to be constructed. Loch [5] employed two methods based on finite element method to construct the leaf surface. In our previous work [10]-[15] and [17]-[21] we proposed a original surface fitting approach based on combining the CT method [4, 18] with radial basis functions (RBF) [25] to construct the leaf surface. Moreover, a new hybrid CT-CPI approach is proposed by the author lately for the same purpose [34].

The research presented in this paper consists of four sections. In section 2, a model for Flame leaf surface reconstruction using the hybrid CT-CPI method is presented. Moreover, leaf reference plane and leaf surface triangulation is given in this section which are essential for modelling leaf surface. In section 3, we proposed a model for simulating a water droplet motion on the constructed Flame leaf surface. Finally, conclusion of our work is given in section 4.

2. Leaf Surface Model
To be able to simulate droplet motion on a virtual leaf surface, we need to employ surface fitting method to create the surface. Therefore, recently the author proposed a new hybrid CT-CPI method for this purpose [34] which is applied in this context to construct the Flame leaf surface,
Figure 2. The laser scanner photo taken from Loch [5].

see Fig.6 and Fig.7. The hybrid CT-CPI approaches based on a huge number of 3D data points collected by a laser scanner, 2, from a real Flame leaf surface. The Flame leaf data consists of two groups. The first group contains 5705 surface points, while the second group contains 60 boundary points, see Fig.4. However, a preprocessing stage are compulsory to apply the hybrid CT-CPI to the leaf data, including the new reference plane determination for the data and then surface triangulation [13, 17].

Table 1. RMS and Maximum error computed using the hybrid CT-CIP method for the Flame leaf model.

| CT-CIP Method                |       |
|-----------------------------|-------|
| Relative RMS                | $6.0e^{-2}$. |
| Maximum Error               | $8.1e^{-3}$. |
| Number of points tested     | 5705. |
| Number of triangulation points | 587. |
| Number of triangles         | 1160. |
| Number of boundary points   | 60. |

Table 1 shows the RMS and maximum errors for the flame leaf model using the hybrid CT-CPI method, see Fig.6 and Fig.7. Note that the displayed leaf triangulations comprised of 1160 triangles (see Fig.5), giving a total of 5,118 data points to evaluate the surface representation accuracy. Also note that the EasyMesh triangulations consisted 587 vertices comprising 60 boundary points, see Fig.2.2.
2.1. Leaf Reference Plane

In order to fit a smooth surface through the data points a coordinate system needs to be positioned so that the surface may be represented by a function from $\mathbb{R}^2$ to $\mathbb{R}$.

The laser scanner may returns not suitable coordinate for interpolation because of the multivalued surfaces. So to overcome this problem, we need to use a reference plane (linear least square plane) to the data. Afterward, the reference plane ($xy$ plane) is obtained by rotating the coordinate system, see Fig.4. This process lead us to use the first two coordinates for a triangulation, while the third coordinate is the $z$-coordinate that represents the surface height, for more details see the authors work [10]-[15] and [17]-[21]. Now we summarize this process in the following paragraphs:

Given $P_i = (x_i, y_i, z_i)$, $i = 1, ..., n$, data points. The plane $p(x, y) = a_1x + a_2y + a_3$, is fitted in the least squares sense by minimizing the quantity

$$E(p) = \sum_{i=1}^{n}(z_i - p(x_i, y_i))^2$$

to obtain the plane of best fit through the given leaf data points. $E$ has a minimum when the partial derivatives $\frac{\partial E}{\partial a_j} = 0$, $j = 1, 2, 3$. These conditions lead to a system of three linear equations in $a_j$, known as the normal equations $(A^TA)a = A^Tz$, where:

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Next, we rotate all data points so that the reference plane becomes parallel to the xy-plane. To achieve this, we at first rotate the normal vector of the reference plane about the x-axis and then about the y-axis, see Fig.4-(b) and (d), using the rotation matrices $R_x$ and $R_y$ given respectively by:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{pmatrix} \quad \text{and} \quad R_y = \begin{pmatrix} \cos \delta & 0 & \sin \delta \\ 0 & 1 & 0 \\ -\sin \delta & 0 & \cos \delta \end{pmatrix}.$$  \hspace{1cm} (1)
After applying the two rotations, the normal vector to the least squares plane coincides with the z-axis.

2.2. Triangulation Methods

After the raw data points of the Flame leaf projected to the reference plane, it is now possible to apply the hybrid CT-CPI method to the leaf data. At first, however, we need to triangulate the surface. The Flame leaf surface represented by a large data points \((N = 5705)\), we select only a subset of 587 points, see Fig., of the complete data set to generate the triangulation, see Fig.5, in order to reduce the overall computational expense.

The triangulation of flame surface is then constructed using the \textit{EasyMesh} which is a software written by Niceno [22] in the C language, see Fig.5. \textit{EasyMesh} produces 2D Delaunay and constrained Delaunay triangulations of the domains. An input file must be provided that comprises the nodes (boundary points) of the leaf and the preferred triangle element edge length for the mesh. \textit{EasyMesh} then returns three output files which are the node file, the element file and the edge file. The node file contains the same nodes (boundary points), jointly with further nodes added by \textit{EasyMesh}, along with the set of points distributed inside the leaf (interior points) that represent the mesh triangle vertices. The number of triangles, or nodes, in the mesh can be controlled by increasing or decreasing the length of the triangle edge. Typically, if we

Figure 4. Flam leaf points before projection into reference plane (a) and (c) and after projection (b) and (d) for two different orientations. The blue dots exhibit the leaf surface points, while the red dots exhibit the leaf boundary points.
Figure 5. 3D Triangulation of the Flam leaf surface created by EasyMesh

increase the length of the triangle sides the number of nodes will decrease and if we decrease this length the number of nodes will increase. For more information see the authors work [10]-[15] and [17]-[21].

The following algorithm was implemented in Matlab to compute the piecewise cubic Clough Tocher surface at a point $x$:

**Algorithm hybrid CT-CPI Surface Evaluation** $(V, T, x, z)$

- *Input*: vertices $V$, a point $x \in \mathbb{R}^2$.
- *Output*: $z$, Estimated function value at $x$. $T$, The triangulation.

1. Generate the triangulation $T$ of the given vertices $V$ using the EasyMesh.
2. Find the triangle $t \in T$ that contains $x$, using the Matlab command `tsearch(V,T,x)`. If there is no such triangle containing the point, it can be concluded that the point lies outside the triangulation and it is ignored.
3. approximate the gradients at the vertices and midpoints of $t$ using the constructed CPI technique.
4. Transform the nodal values for triangle $t$ onto the standard triangle and then calculate the surface functions for the standard triangle.
5. Find the subtriangle of $t$ that contains $x$ and use the CT basis functions to obtain the desired leaf surface value at $x$.

3. Water Droplet Model
In this article we proposed a single water droplet model to simulate the motion of the droplet over the Flame leaf surface. Moreover, the proposed model offers many benefits such as, it is not difficult to control the droplet location and motion at any given time. However, the model in the future will be extended to several droplet model. The triangulation form the crucial component of the droplet model. Using the triangulations has many advantages for instant, the droplet animation and droplet location are easy to compute over each triangle. The computational
overheads increases considerably by employing too many triangles, thus to acquire less triangles a subset of the surface points is employed which act the main surface features, see Fig.5.

The authors [9] previously developed a model for the droplet movements on Anthurium leaf surface such that the droplet motion affected by the gravity and friction forces only. In this research we will adopt the same idea and extend the model to include the resistance force also to investigate the droplet movements on the Flame leaf surface. In the current model, the droplet movement affects by two forces, which include the internal force ($F_i$) and external force ($F_e$). The internal force consists of friction force ($F_{fr}$) and resistance force $F_{R}$ while the external force consists of gravity $F_g$. Moreover, $F_g$ projected in direction of the droplet movement $D_d$, as follows:

$$D_d = F_g - (F_g . N)N \tag{2}$$
where $N$ is the surface unit normal vector. The $F_{fr}$ is treated as a force of retarding with a fixed negative factor $K$ consequent of the roughness of the surface:

$$F_{fr} = -KV(t),$$  \hspace{1cm} (3)$$

where $V(t)$ is the droplet velocity at time $t$. Note that if $F_e$ exceeds $F_i$ then the droplet begins to move down on the surface. The resistance force $F_R$ is opposite to the $D_d$ and its arises from the interfacial tension that occurs between the surface and the droplet [23],[24] and its given by:

$$F_R = -RD_d,$$  \hspace{1cm} (4)$$

where $0 \leq R = (10)^{-4} \leq 1$ is the affinity degree and its used to model the $F_R$. Moreover, over each triangle $R$ presumed to be constant and set experimentally.

As mentioned earlier, the Flame surface consists of a triangular mesh over which appear the droplet moves. The height of the droplet is calculated on each triangle, furthermore, if a set tolerance ($\xi = 10^{-5}$) is greater than the droplet height (see Fig.8-(b) and (c), Fig.9-(c) and(d), Fig.10-(c) and (d)) then the droplet stop, else the droplet animation continues to the next element (triangle), (see Fig.8-(a),(b) and (d), Fig.9-(a), Fig.10-(a) and (b)).

### 3.1. The droplet motion over the Flame surface

Newton's second law $F = Ma$ is used to identify the droplet movements at any given time $T$. Consequently, the droplet has position $p$, acceleration $a$, velocity $V$ and mass $M$ where the mass is assumed to be constant in our model. Thus, the droplet model is constructed as follow:

$$MD_d - KV(T) - RD_d = MV'(T),$$  \hspace{1cm} (5)$$

Stokes's law for a sphere of radius $r = 0.001(m)$ is used estimated the friction coefficient $K \sim O(10)^{-8}(kgm/s^2)$.

To begin our simulation, first the initial velocity $V_0$, time $T_0$, position $p_0$, the frame time $F_i$, the transit time $T_i$ and initial triangle should be specify. Afterward, Eq. 2 is employed to identify $D_d$ and allow the droplet to transfer to the subsequent triangle. The new velocity and position of the droplet over the updated triangle is calculated using Eqs. 6 and 7 respectively. Finally, the droplet transfer to the subsequent triangle if the time frame is greater than the transit time and a defined tolerance $\epsilon = 10^{-5}$ is less than the droplet height.

$$V(T) = -\frac{M}{K}D_d^n\left(1 + \frac{R}{M}\right) + \left[V_0(T_n) + \frac{M}{K}D_d^n\left(1 + \frac{R}{M}\right)\right]\frac{K}{M}T_n;$$  \hspace{1cm} (6)$$

$$p(T) = p_0(T_n) - \frac{M}{K}D_d^n\left(1 + \frac{R}{M}\right)T_n + \frac{M}{K}\left[V_0(T_n) + \frac{M}{K}D_d^n\left(1 + \frac{R}{M}\right)\right]\left(e^{\frac{K}{M}T_n} - 1\right),$$  \hspace{1cm} (7)$$

where $T_n$ is the entrance time of the droplet the $n^{th}$ triangle. The leaving time denoted by $T_l = T_n + T_i$ as well as the exit velocity and position respectively $v(T_e)$ and $p(T_e)$ are calculated. Furthermore, we employed Newton algorithm to calculate $T_l$ by intersect the triangle three edges with the droplet trajectory (Eq. 7).

Our simulation of droplet movements over the surface is performed using MATLAB software, see Fig.8, Fig.9 and Fig.10. As expose in these figures, our model generates genuine droplet motion over the Flame surface and the model catches the droplet motion fairly well. Moreover, the droplet motion obey the surface curves and drop off the surface contingent by the model parameters, see Fig.8-(a),(b) and (d), Fig.9-(a) and Fig.10-(a) and (b). Furthermore, the droplet stops rolling if the height of the droplet is less than a set tolerance, see Fig.8-(b) and (c), Fig.9-(c) and (d) and Fig.10-(c) and (d).
Figure 8. The figure shows the droplet movements across the surface from different starting positions. Figures (b) and (c) show that the droplet stopped moving at some stage according to the droplet stopping criteria.

4. Conclusion
A physical model based on mathematics for simulating droplet movement on leaf surface is presented. Moreover, the model allows the researchers to realise of how a droplet transfers over a surface of leaf and how a slight changes in the model parameters produces different droplet movements. Also, the model generates genuine droplet motion over the Flame surface and catches the droplet motion fairly well. Our future research is to extend the model by including more forces that affect the droplet motion and dominant factors to generate more physical simulation of several droplets path.
Figure 9. Figure (a) shows the droplet movement across the surface. Figures (b), (c) (d) show that the droplet stopped moving at some stage according to the droplet stopping criteria.

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Figure 10. The figure (a) and (b) show that the droplet movements for two different starting points where the droplets move across the leaf vein. Figures (c) and (d) show that the droplet stopped moving at some stage according to the droplet stopping criteria.

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