A new numerical treatment of soil heterogeneity for modeling unsaturated flow

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Abstract

We develop a new approach to solve the Richardson-Richards equation for modeling unsaturated flow through heterogeneous porous media. The main idea of the proposed techniques is the use of the Kirchhoff transformation, the Brooks and Corey model for the capillary pressure function and a power-law relation in saturation for the relative permeability function. The new approach allows us to avoid the technical issues encountered in the Kirchhoff transformation due to soil heterogeneity. This transformation is applied to reduce the non-linearity of the model which is solved using a numerical scheme based on a local radial basis function method (RBF). To validate the developed approach for predicting the dynamics of unsaturated flow in porous media, numerical experiments are performed in one, two, and three-dimensional soils. The numerical results demonstrate the efficiency and accuracy of the proposed techniques for modeling infiltration through heterogeneous soils.

Keywords: Richardson-Richards equation, Heterogeneous soils, Brooks-Corey model, Kirchhoff transformation, Meshfree methods, Radial basis function

1. Introduction

Understanding infiltration through soils is of great importance in the fields of agriculture, hydrology, and water resources and environmental management. The modeling of infiltration processes is time-consuming and there is a need in the development of efficient techniques for these processes in the case of heterogeneous soils. The Richardson-Richards equation [1] describes the dynamic of unsaturated flow through porous medium which is due to the actions of gravity and capillarity. Richardson-Richards’ equation is highly nonlinear because of the largely nonlinear dependencies of both unsaturated hydraulic conductivity and capillary pressure on saturation [2, 3, 4]. The van Genuchten [4] and Brooks-Corey models [3] are often used for the capillary pressure function of unsaturated soils. In terms of numerical analysis, the Gardner model [2] is important since large class of analytical

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solutions are available [5, 6, 7, 8]. However, this model has some limitations in practical applications for describing unsaturated flow in soils [9].

The design of efficient numerical techniques for solving the Richardson-Richards equation is very challenging due to the highly non-linearity of the equation and the technical issues encountered in the numerical treatment of soil heterogeneity. Various classes of approaches have been developed for modeling infiltration in soils such as finite difference methods [10, 11, 12, 13], finite element methods [14, 15, 16, 17] and finite volume methods [18, 19, 20, 21]. While many methods have been developed for modeling unsaturated flow in soils, there is still a need for more efficient techniques to deal with the soil heterogeneity and the nonlinearity of the medium hydraulic properties as functions of saturation [3, 4]. Most available approaches used iterative methods, such as Newton and Picard schemes [11, 22, 23, 24, 25, 26, 27], to linearize the system to be solved. While these iterative algorithms produce accurate results, they are expensive in terms of computational cost and they may have convergence issues for some flow conditions because of the highly non-linearity of the Richardson-Richards equation and soil heterogeneity [24, 28, 29, 27].

Among the numerical techniques proposed to solve the Richardson-Richards equation is the Kirchhoff transformation approach [10, 30, 15, 28, 31, 32]. Several studies have shown the efficiency of this approach because it reduces the non-linearity of the Richardson-Richards equation [33, 28, 34, 35]. However, this approach is developed for homogeneous soils or particular soil heterogeneity and capillary pressure functions [10, 36, 28, 31, 32]. Most of available techniques using the Kirchhoff transformation are limited to Gardner model for the capillary pressure function [37, 38, 39, 40, 28, 32, 41]. For instance, Suk and Park [32] recently developed a new numerical method based on the Kirchhoff transformation and the Gardner model to solve the Richardson-Richards equation for layered soils. In their approach, the authors used a truncated Taylor series expansion to the Kirchhoff head at the material interface.

We note that the Gardner model is limited for practical applications compared to van Genuchten and Brooks-Corey models which are suitable for the entire range of pressure head [32]. Incorporating both high non-linear models for the capillary pressure [3, 4] and soil heterogeneity is still a challenge in solving the Richardson-Richards equation using the Kirchhoff transformation [32].

Here, we develop a new technique using the Kirchhoff transformation to solve the Richardson-Richards equation for heterogeneous soils based on the Brooks and Corey model [3] for the capillary pressure function. In our approach, a power-law relation in saturation for the relative permeability function is used to avoid the technical issues encountered in the Kirchhoff transformation due to the non-linearity of capillary pressure function and the heterogeneity of soils [32]. At the discrete level, in order to avoid discontinuity issues for the Kirchhoff variable between different soil materials, the spatial operators in our system are approximated based on the approach used by Celia et al. [11]. To solve our system, the localized radial basis function (RBF) method [42] is used. This meshfree method doesn’t require connection between the collocation points of the computational domain, which minimizes the computing cost and makes the proposed techniques less expensive and effective for solving high-dimensional infiltration problems [42, 43].
The paper is organized as follows. In Section 2, we introduce the Richardson-Richards model and the developed approach based on the Kirchhoff transformation, the Brooks and Corey model for the capillary pressure function, and a power-law relation in saturation for the relative permeability function. The proposed numerical model based on the localized RBF method is described in Section 3. In Section 4, numerical simulations are conducted to validate the developed approach for modeling infiltration through soils. Finally, some concluding remarks are provided in Section 5.

2. Material and models

2.1. Richardson-Richards’ model

We consider the traditional Richardson-Richards equation describing infiltration through soils [1]:

\[
\frac{\partial \theta(h)}{\partial t} - \nabla \cdot (K_s(x)k_r(h)\nabla (h + z)) = s(x, t), \quad x \in \Omega,
\]

where \( \theta [L^3/L^3] \) is the water content, \( h [L] \) is the pressure head, \( K_s [L/T] \) is the saturated hydraulic conductivity which depends on the medium’s spatial heterogeneity, \( k_r [-] \) is the water relative permeability, \( s(x, t) \) is a source or sink term which may include evaporation and plant-root extraction, \( \Omega \) denotes an open subset of \( \mathbb{R}^3 \), \( x [L] \) is the spatial coordinate and \( z [L] \) is the upward vertical coordinate.

We consider the water-saturation \( S = (\theta - \theta_r)/(\theta_s - \theta_r) \) to write Richardson-Richards’ equation without source/sink term:

\[
\phi \frac{\partial S}{\partial t} - \nabla \cdot (K_s(x)k_r(S)\nabla h) - \frac{\partial (K_s k_r)}{\partial z} = 0, \quad x \in \Omega,
\]

where \( \theta_s [L^3/L^3] \) is the saturated water content, \( \theta_r [L^3/L^3] \) is the residual water content and the parameter \( \phi [-] \) is given by \( \phi = (\theta_s - \theta_r) \).

Equation (2.2) is highly non-linear due to the nonlinear dependencies of the capillary pressure and relative permeability functions on saturation. Empirical constitutive relationships have been developed for these functions using experiments [2, 3, 4] and are used in previous studies [11, 44, 45, 43, 46].

2.2. Capillary pressure and relative permeability functions

Here, we used the Brooks and Corey’s model [3] for the capillary pressure function. The saturation is given by:

\[
S(h) = \begin{cases} 
\left( \frac{h}{h_d} \right)^{-\lambda}, & \text{if } h \leq h_d, \\
1, & \text{if } h > h_d,
\end{cases}
\]

where \( \lambda [-] \) is the Brooks-Corey parameter, \( h_d = -h_{\text{cap}} [L] \) and the characteristic capillary rise \( h_{\text{cap}} \) is determined by the Leverett scaling formula [47]. The capillary pressure function can be expressed as follows:

\[
h(S) = h_d J(S),
\]
where the Leverett $J$-function is given by $J(S) = S^{-1/\lambda}$. We propose to use the power-law relation in saturation for the relative permeability $k_r(S) = S^\beta$ which can be expressed using capillary pressure as follows:

$$k_r(h) = \begin{cases} 
\left( \frac{h}{h_d} \right)^{-\lambda \beta}, & \text{if } h \leq h_d \\
1, & \text{if } h > h_d,
\end{cases} \quad (2.5)$$

where $\beta > 1$.

Let’s introduce a reference constant $\bar{h} [L]$ for the capillary pressure in Equation (2.2) in order to use a dimensionless form of the capillary pressure function. For $h \leq h_d$, we obtain the following expression where we set $\omega = \bar{h}/h_d$:

$$K_s k_r \nabla h = K_s \omega^{-\lambda \beta} \left( \frac{h}{\bar{h}} \right)^{-\lambda \beta} \nabla h, \quad (2.6)$$

and $\bar{h}$ can be taken as:

$$\bar{h} = \frac{1}{V} \int \int \int_V h_d(x,y,z) dx dy dz, \quad (2.7)$$

where $V$ is the volume of the whole domain. Since $\omega$ depends only on space ($\omega = \omega(x)$), we obtain:

$$\frac{\partial S}{\partial t} = \omega^{-\lambda} \frac{\partial}{\partial t} \left[ \left( \frac{h}{\bar{h}} \right)^{-\lambda} \right], \quad (2.8)$$

and

$$\frac{\partial (K_s k_r)}{\partial z} = \frac{\partial}{\partial z} \left[ K_s \omega^{-\lambda \beta} \left( \frac{h}{\bar{h}} \right)^{-\lambda \beta} \right]. \quad (2.9)$$

We substitute Equations (2.6), (2.8) and (2.9) into Equation (2.2), we obtain the new form of the Richardson-Richards equation associated with the Brooks and Corey model for $h \leq h_d$:

$$\phi \omega^{-\lambda} \frac{\partial}{\partial t} \left[ \left( \frac{h}{\bar{h}} \right)^{-\lambda} \right] - \nabla \cdot \left[ K_s \omega^{-\lambda \beta} \left( \frac{h}{\bar{h}} \right)^{-\lambda \beta} \nabla h \right] - \frac{\partial}{\partial z} \left[ K_s \omega^{-\lambda \beta} \left( \frac{h}{\bar{h}} \right)^{-\lambda \beta} \right] = 0, \quad (2.10)$$

where we separate the terms which are dependent on the capillary pressure function and those which depend only on space due to soil heterogeneity.

For $h > h_d$, we obtain:

$$-\nabla (K_s \nabla h) - \frac{\partial K_s}{\partial z} = 0. \quad (2.11)$$
2.3. Kirchhoff transformation

In this study, we propose to use the transformation of Kirchhoff which allows us to reduce the nonlinearity of the model equation. The Kirchhoff integral transformation is defined as:

$$\varphi(h) = \bar{h} \int_{+\infty}^{h/\bar{h}} \sigma^{-\lambda \beta} d\sigma,$$

where we used the variable \(\sigma = h/\bar{h}\). Based on Equation (2.5), the transformation (2.12) can be rewritten as follows:

$$\varphi(h) = \begin{cases} \frac{\bar{h}}{1 - \lambda \beta} \left( \frac{h}{\bar{h}} \right)^{(1 - \lambda \beta)} & \text{if } h \leq h_d, \\ \frac{\bar{h}}{1 - \lambda \beta} \left( \frac{h}{\bar{h}} \right)^{(1 - \lambda \beta)} + \left( \frac{h_d}{h} \right)^{-\lambda \beta} (h - h_d) & \text{if } h > h_d, \end{cases}$$

(2.13)

where, we assume that \(\lambda \beta > 1\). From Equation (2.13), we can determine the pressure head \(h\) as a function of \(\varphi\):

$$h = \begin{cases} \bar{h} \left( \frac{(1 - \lambda \beta)}{h} \varphi \right)^{1/(1 - \lambda \beta)} & \text{if } \varphi \leq \frac{\bar{h}}{1 - \lambda \beta} \left( \frac{h_d}{h} \right)^{(1 - \lambda \beta)}, \\ (h_d/h)^{\lambda \beta} \varphi + h_d - \frac{h_d}{1 - \lambda \beta} & \text{if } \varphi > \frac{\bar{h}}{1 - \lambda \beta} \left( \frac{h_d}{h} \right)^{(1 - \lambda \beta)}. \end{cases}$$

(2.14)

In our approach, in the transformation of the model equation, the variation of the intrinsic permeability is assumed dominant in the effect of heterogeneity variability [47, 46]. We have for \(h \leq h_d\):

$$\nabla \varphi(h) = \left( \frac{h}{\bar{h}} \right)^{-\lambda \beta} \nabla h,$$

(2.15)

and

$$\nabla \cdot \left[ K_{s \omega}^{-\lambda \beta} \left( \frac{h}{h} \right)^{-\lambda \beta} \nabla h \right] = \nabla \cdot \left( K_{s \omega}^{-\lambda \beta} \nabla \varphi \right),$$

(2.16)

and the third term of Equation (2.10) becomes:

$$\frac{\partial}{\partial z} \left[ K_{s \omega}^{-\lambda \beta} \left( \frac{h}{h} \right)^{-\lambda \beta} \varphi \right] = \frac{\partial}{\partial z} \left[ K_{s \omega}^{-\lambda \beta} \left( \frac{1 - \lambda \beta}{h} \right) \left( \frac{h}{h} \right)^{-1} \varphi \right].$$

(2.17)

The time derivative in the first term of Equation (2.10) can be rewritten under the same assumption \(h \leq h_d\) by:

$$\frac{\partial}{\partial t} \left[ \left( \frac{h}{h} \right)^{-\lambda} \right] = \frac{-\lambda}{h} \left( \frac{h}{h} \right)^{-\lambda - 1} \frac{\partial h}{\partial t},$$

(2.18)
and
\[ \frac{\partial \varphi}{\partial t} = \left( \frac{h}{\bar{h}} \right)^{-\lambda \beta} \frac{\partial h}{\partial t}, \]
which implies:
\[ \frac{\partial}{\partial t} \left[ \left( \frac{h}{\bar{h}} \right)^{-\lambda} \right] = -\lambda \left( \frac{h}{\bar{h}} \right)^{\lambda \beta - \lambda - 1} \frac{\partial \varphi}{\partial t}. \]  
(2.20)

Similarly, for \( h > h_d \), Equation (2.11) can be written in terms of \( \varphi \): 
\[ -\nabla.(K_s \omega^{-\lambda \beta} \nabla \varphi) - \frac{\partial K_s}{\partial z} = 0. \]  
(2.21)

For simplicity, we will use the following parameters:
\[ \chi = K_s \omega^{-\lambda \beta}, \]  
(2.22)

\[ E = \begin{cases} \phi \left( \frac{h}{\bar{h}} \right)^{\lambda \beta - \lambda - 1}, & \text{if } h \leq h_d, \\ 0, & \text{if } h > h_d, \end{cases} \]  
(2.23)

\[ F = \begin{cases} (1 - \lambda \beta) \left( \frac{h}{\bar{h}} \right)^{-1}, & \text{if } h \leq h_d, \\ 0, & \text{if } h > h_d, \end{cases} \]  
(2.24)

\[ G = \begin{cases} 0, & \text{if } h \leq h_d, \\ K_s, & \text{if } h > h_d. \end{cases} \]  
(2.25)

Finally, we obtain the new form of the model equation using the Kirchhoff \( \varphi \):
\[ E \frac{\partial \varphi}{\partial t} - \nabla.(\chi \nabla \varphi) - \frac{\partial}{\partial z} \left( \chi F \varphi \right) - \frac{\partial G}{\partial z} = 0, \]  
(2.26)

where the non-linearity of the original Richardson-Richards model has been reduced since only the terms \( E \) and \( F \) are nonlinear and \( \chi \) depends only on spatial coordinates \( \mathbf{x} \) due to the heterogeneity of soils.

3. Numerical model

3.1. Approximation methods

In this section, we describe the approximation methods used to solve Equation (2.26). Let \( \Delta t > 0 \) a time step and \( \nu^p = p\Delta t \) with \( p \geq 0 \) denotes the time level. Temporal discretization of Equation (2.26) using the backward Euler method may be written as:
\[ \frac{E^{p+1} \varphi^{p+1}}{\Delta t} - \varphi^p - \nabla.(\chi \nabla \varphi^{p+1}) - \frac{\partial}{\partial z} \left( \chi F^{p+1} \varphi^{p+1} \right) - \frac{\partial G}{\partial z} = 0, \]  
(3.1)
respectively. We then obtain:

\[
E^{p+1,m} \frac{\varphi^{p+1,m+1} - \varphi^p}{\Delta t} - \nabla.(\chi \nabla \varphi^{p+1,m+1}) - \frac{\partial}{\partial z} \left( \chi F^{p+1,m} \varphi^{p+1,m+1} \right) - \frac{\partial G}{\partial z} = 0,
\]  

(3.2)

with \(m\) identifies iteration level. The solution is assumed to be known both at time level \(p\) and at iteration level \(m\). Let \(\{x_i = (x_i, y_i, z_i)\}_{i=1}^{N_i} \subset \Omega\) be \(N_i\) uniform distinct points and \(\{x_i\}_{i=N_i+1}^{N} \subset \partial \Omega\) be \(N_b\) distinct nodes, where \(N_i\) denotes the number of interior points and \(N_b\) denotes the number of points on the boundary (\(N = N_i + N_b\)).

For each point \(\{x_i\}_{i=1}^{N_i}\), we discretize \(\nabla.(\chi \nabla \varphi^{p+1,m+1})\) as follows:

\[
\nabla.(\chi \nabla \varphi^{p+1,m+1}) = \frac{\partial}{\partial x} \left( \chi \frac{\partial \varphi^{p+1,m+1}}{\partial x} \right) + \frac{\partial}{\partial y} \left( \chi \frac{\partial \varphi^{p+1,m+1}}{\partial y} \right) + \frac{\partial}{\partial z} \left( \chi \frac{\partial \varphi^{p+1,m+1}}{\partial z} \right).
\]  

(3.3)

For simplicity, we take the following expressions:

\[
\mathcal{L}_d^m \varphi_i = \frac{\partial}{\partial x^d} \left( \chi_i \frac{\partial \varphi_i}{\partial x^d} \right),
\]  

(3.4)

\[
\mathcal{L}_4^m \varphi_i = \frac{\partial}{\partial z} \left( \chi_i F_i \varphi_i \right),
\]  

(3.5)

where \(d = \{1, 2, 3\}\) and \((x^{(1)}, x^{(2)}, x^{(3)}) = (x, y, z)\). By extending the 1D spatial approximation used in [11] to 3D case, we obtain:

\[
\mathcal{L}_d^m \varphi_i = \frac{1}{(\Delta x^d)^2} \left( \chi_{i+1/2} \varphi^{(d)}_{iR} - \varphi_i^{(d)} - \chi_{i-1/2} \varphi^{(d)}_{iL} \right),
\]  

(3.6)

where the expressions \(\chi_{i+1/2}^{(d)}\) and \(\chi_{i-1/2}^{(d)}\) are given by:

\[
\begin{cases}
\chi_{i+1/2}^{(d)} = \frac{1}{2} \left( \chi_i^{(d)} + \chi_{iR}^{(d)} \right), \\
\chi_{i-1/2}^{(d)} = \frac{1}{2} \left( \chi_i^{(d)} + \chi_{iL}^{(d)} \right).
\end{cases}
\]  

(3.7)

Along the \(x^{(d)}\)-axis, \(\varphi^{(d)}_{iR}\) and \(\chi_{iR}^{(d)}\) are the corresponding values at the right of \(\varphi_i\) and \(\chi_i\) respectively. Similarly, \(\varphi^{(d)}_{iL}\) and \(\chi_{iL}^{(d)}\) are the corresponding values at the left of \(\varphi_i\) and \(\chi_i\) respectively. We then obtain:

\[
\nabla.(\chi \nabla \varphi^{p+1,m+1}) = \mathcal{L}_1^m \varphi^{p+1,m+1} + \mathcal{L}_2^m \varphi^{p+1,m+1} + \mathcal{L}_3^m \varphi^{p+1,m+1}.
\]  

(3.8)

On the other hand, we approximate the following spatial operators as follows [11]:

\[
\mathcal{L}_4^m \varphi_i = \frac{1}{(\Delta z)^2} \left( \chi_{i+1/2}^{(3)} F_{i+1/2}^{(3)} \varphi_{i+1/2}^{(3)} - \chi_{i-1/2}^{(3)} F_{i-1/2}^{(3)} \varphi_{i-1/2}^{(3)} \right),
\]  

(3.9)
\[ \frac{\partial G_i}{\partial z} = \frac{1}{(\Delta z)} \left( G_{i+1/2}^{(3)} - G_{i-1/2}^{(3)} \right). \tag{3.10} \]

We then obtain:

\[ \frac{\partial}{\partial z} (\chi_i F_{p+1,m}^{i+1} \varphi_i^{p+1,m+1}) = \mathcal{L}_m^{i+1} \varphi_i^{p+1,m+1}. \tag{3.11} \]

For simplicity, we use the following expressions:

\[ L_m \varphi_{p+1,m+1}^{i+1} = \frac{E_{p+1,m}^{i+1}}{\Delta t} \varphi_{p+1,m+1}^{i+1} - (L_1^m + L_2^m + L_3^m) \varphi_{p+1,m+1}^i - L_4^m \varphi_i^{p+1,m+1}, \tag{3.12} \]

\[ f_{p+1,m} = \frac{E_{p+1,m}^{i+1}}{\Delta t} \varphi_{p+1,m+1}^{i+1} + \frac{1}{\Delta z} (G_{i+1/2}^{(3)} - G_{i-1/2}^{(3)}). \tag{3.13} \]

The operator \( \mathcal{L}^m \) is linear for each iteration level \( m \). In the case of homogeneous soils, \( \mathcal{L}^m \) is given by:

\[ \mathcal{L}_m \varphi_{p+1,m+1} = \frac{E_{p+1,m}^{i+1}}{\Delta t} \varphi_{p+1,m+1}^{i+1} - \chi \nabla^2 \varphi_{p+1,m+1} - D_{p+1,m} \frac{\partial \varphi_{p+1,m+1}}{\partial z}, \tag{3.14} \]

where \( h \), in this case, is equal to \( h_d \) and:

\[ D = \begin{cases} \frac{-\lambda \beta}{h} \left( \frac{h}{h_d} \right)^{-1}, & \text{if } h \leq h_d, \\ 0, & \text{if } h > h_d. \end{cases} \tag{3.15} \]

In addition to initial and boundary conditions, Equation (3.2) may be rewritten as follows:

\[
\begin{cases}
\mathcal{L}_m \varphi_{p+1,m+1}(x) = f_{p+1,m}(x), & x \in \Omega, \\
\mathcal{B}_m \varphi_{p+1,m+1}(x) = \varphi_T(x), & x \in \partial \Omega, \\
\varphi^{0,m+1}(x) = \varphi_0(x), & x \in \Omega.
\end{cases} \tag{3.16}
\]

\( \varphi_0 \) and \( \varphi_T \) are functions associated with the initial and boundary conditions. For each time level \( p \), the linear system (3.16) is solved at each iteration level of Picard \( m \) until the following inequality is satisfied at all collocation points:

\[ \delta^m = |\varphi_{p+1,m+1} - \varphi_{p+1,m}| \leq Tol, \tag{3.17} \]

where \( Tol \) is the error tolerance.

### 3.2. Approach using local radial basis functions

In this study, we use the local RBF meshfree method \[42\] which has advantages in terms of operational memory and calculation procedures where only inversions of small size matrices are required.

For any point \( x_s \in \Omega \), the k-d tree algorithm is used \[48\] to create a localized influence...
domain $\Omega[s] = \{x_k[s]\}_{k=1}^{n_s} \subset \bar{\Omega}$. It contains $n_s$ nearest nodal points from $x_s$.

In the local RBF approach, the transformed Kirchhoff variable $\varphi^p,m+1[s]$ is approximated in each localized influence domain $\Omega[s]$ as follows:

$$
\varphi^p,m+1(x_s) = \sum_{i=1}^{n_s} \alpha_i^{p+1,m+1} \psi(\|x_s - x_i[s]\|),
$$

(3.18)

where $\{\alpha_i^{p+1,m+1}\}_{i=1}^{n_s}$ are constants to be determined and $\psi$ is a RBF for which, in our approach, we chose the exponential function given by $\psi(r) = \exp(-(cr)^2)$, where $r = \|x_s - x_i[s]\|$ denotes the distance between $x_s$ and $x_i[s]$ and $c > 0$ is the shape parameter.

According to Equation (3.18), we obtain:

$$
\varphi^p,m+1 = \psi[s] \alpha^{p+1,m+1}[s],
$$

(3.19)

where we used the matrix $\psi[s] = \{\psi(\|x_i[s] - x_j[s]\|)\}_{1 \leq i, j \leq n_s}$ and the vectors:

$$
\varphi^{p+1,m+1}[s] = \begin{bmatrix} \varphi^{p+1,m+1}(x_1[s]), \varphi^{p+1,m+1}(x_2[s]), \ldots, \varphi^{p+1,m+1}(x_{n_s}[s]) \end{bmatrix}^T,
$$

$$
\alpha^{p+1,m+1}[s] = \begin{bmatrix} \alpha_1^{p+1,m+1}, \alpha_2^{p+1,m+1}, \ldots, \alpha_{n_s}^{p+1,m+1} \end{bmatrix}^T.
$$

From Equation (3.19), we obtain:

$$
\alpha^{p+1,m+1}[s] = (\psi[s])^{-1}\varphi^{p+1,m+1}[s].
$$

(3.20)

Applying the linear operator $L^m$ to Equation (3.18) at each $x_s \in \Omega[s]$, we have:

$$
L^m \varphi^{p+1,m+1}[s](x_s) = \sum_{i=1}^{n_s} \alpha_i^{p+1,m+1} L^m \psi(\|x_s - x_i[s]\|) = \sum_{i=1}^{n_s} \alpha_i^{p+1,m+1} \Psi^m(\|x_s - x_i[s]\|)
$$

(3.21)

$$
= \Gamma^m[s] \alpha^{p+1,m+1}[s] = \Gamma^m[s] (\psi[s])^{-1} \varphi^{p+1,m+1}[s] = \Upsilon^m[s] \varphi^{p+1,m+1}[s],
$$

where $\Psi^m = L^m \psi$, $\Gamma^m[s] = \{\Psi(\|x_s - x_1[s]\|), \ldots, \Psi(\|x_s - x_{n_s}[s]\|)\}$ and $\Upsilon^m[s] = \Gamma^m[s] (\psi[s])^{-1}$.

In order to reformulate Equation (3.21) in terms of the global vector $\varphi^{p+1,m+1}$ instead of $\varphi^{p+1,m+1}[s]$, $\Upsilon^m$ is considered as the expansion of $\Upsilon^m[s]$ by inserting zeros in the proper position. It follows that:

$$
L^m \varphi^{p+1,m+1}[s](x_s) = \Upsilon^m \varphi^{p+1,m+1},
$$

(3.22)

where $\varphi^{p+1,m+1} = \begin{bmatrix} \varphi^{p+1,m+1}(x_1), \varphi^{p+1,m+1}(x_2), \ldots, \varphi^{p+1,m+1}(x_N) \end{bmatrix}^T$.

Similarly, for $x_s \in \partial\Omega$, we apply the linear operator $B$:

$$
B \varphi^{p+1,m+1}[s](x_s) = \sum_{i=1}^{n_s} \alpha_i^{p+1,m+1} B \psi(\|x_s - x_i[s]\|) = (B \psi[s]) \alpha^{p+1,m+1}[s]
$$

(3.23)

$$
= (B \psi[s])^{-1} \varphi^{p+1,m+1}[s] \varphi^{p+1,m+1}[s] = v[s] \varphi^{p+1,m+1},
$$

where $v[s] = (B \psi[s])^{-1}$ and $v$ is the global expansion of $v[s]$ by adding zeros in the proper location.
From Equations (3.22) and (3.23), we get the system below:

\[
\mathcal{L}^m \varphi^{p+1,m+1}(x_s) = \Upsilon^m(x_s) \varphi^{p+1,m+1} = f^{p+1,m}(x_s),
\]
\[
\mathcal{B} \varphi^{p+1,m+1}(x_s) = u(x_s) \varphi^{p+1,m+1} = \varphi_\Gamma(x_s).
\] (3.24)

We obtain the following sparse linear system:

\[
\begin{pmatrix}
\Upsilon^m(x_1) \\ \Upsilon^m(x_2) \\ \vdots \\ \Upsilon^m(x_{N_1}) \\ v(x_{N_1+1}) \\ \vdots \\ v(x_N)
\end{pmatrix}
\begin{pmatrix}
\varphi^{p+1,m+1}(x_1) \\ \varphi^{p+1,m+1}(x_2) \\ \vdots \\ \varphi^{p+1,m+1}(x_{N_1}) \\ \varphi^{p+1,m+1}(x_{N_1+1}) \\ \vdots \\ \varphi^{p+1,m+1}(x_N)
\end{pmatrix}
= \begin{pmatrix}
f^{p+1,m}(x_1) \\ f^{p+1,m}(x_2) \\ \vdots \\ f^{p+1,m}(x_{N_1}) \\ \varphi_\Gamma(x_{N_1+1}) \\ \vdots \\ \varphi_\Gamma(x_N)
\end{pmatrix}.
\] (3.25)

The localized RBF approach leads to system of sparse equations (3.25) which allows us to reduce the size of the dense matrices and avoid ill-conditioned problems arising from the global approach [49]. The approximate solutions \( \varphi^{p+1,m+1} = \{ \varphi^{p+1,m+1}(x_i) \}_{i=1}^N \) can be obtained by solving the system (3.25).

3.3. Initial and boundary conditions

At \( t = 0 \), we assume that \( h(x, 0) = h_0(x) \) which implies in terms of the Kirchhoff variable that \( \varphi(x, 0) = \varphi_0(x) \), where

\[
\varphi_0(x) = \frac{\tilde{h}}{(1 - \lambda \beta)} \left( \frac{h_0}{\bar{h}} \right)^{(1-\lambda \beta)}.
\]

The boundary conditions are expressed in terms of the Kirchhoff variable. For Dirichlet conditions, we consider \( h = g_b \) for \( z = 0 \) and \( z = L \) which implies that:

\[
\varphi_b = \frac{\tilde{h}}{(1 - \lambda \beta)} \left( \frac{g_b}{h} \right)^{(1-\lambda \beta)},
\]

where \( g_b \) is given by:

\[
g_b = \begin{cases}
    h_0, & z = 0, \\
    0, & z = L.
\end{cases}
\]

For Neumann conditions, we consider \(-K \frac{\partial h}{\partial x} = 0 \) for \( x = 0 \) and \( x = l_1 \) which implies that \(-\chi \frac{\partial \varphi}{\partial x} = 0 \). In the same way, we assume that \(-\chi \frac{\partial \varphi}{\partial y} = 0 \) at the lateral boundaries \( y = 0 \) and \( y = L \).
and \( y = l_2 \). Therefore, the linear operator corresponding to the boundary conditions is given by:

\[
B\varphi = \begin{cases} 
\varphi_b, & z = \{0, L\}, \\
-\chi \frac{\partial \varphi}{\partial x}, & x = \{0, l_1\}, \\
-\chi \frac{\partial \varphi}{\partial y}, & y = \{0, l_2\}.
\end{cases}
\]  

(3.26)

4. Numerical experiments

To validate the proposed approach for modeling unsaturated flow through heterogeneous soils, we present numerical solutions of Richardson-Richards equation in one-, two- and three-dimensional systems. The computational domain \( \Omega = [0, l_1] \times [0, l_2] \times [0, L] \) is used to perform 3D numerical simulations and we consider the domains \( \Omega = [0, l_1] \times [0, L] \) and \( \Omega = [0, L] \), respectively, for 2D and 1D numerical simulations.

In Section (4.1), numerical tests are performed using the developed model to simulate flow in unsaturated homogeneous soils. Section (4.2) presents numerical simulations of infiltration through heterogeneous soils. To investigate the accuracy of the developed technique, the results of numerical tests are used to compute the \( \text{RMSE} \) and \( L^1_{er} \) errors based on the following formulas:

\[
RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [\theta(x_i) - \theta_{ref}(x_i)]^2}, \quad (4.1)
\]

\[
L^1_{er} = \sum_{i=1}^{N} \frac{(\theta(x_i) - \theta_{ref}(x_i))^2}{\sum_{i=1}^{N} \theta_{ref}(x_i)^2}, \quad (4.2)
\]

where \( \theta(x_i) \) represents the approximate solution for the water content and \( \theta_{ref}(x_i) \) represents a reference solution and \( N \) is the number of collocation points.

4.1. Infiltration in homogeneous soils

4.1.1. Infiltration in 1D homogeneous soils

We consider different soil samples with a depth \( L = 1 \) m. The hydraulic parameters of these soils are shown in Table 1. The initial condition \( h_0 \) is chosen based on the value of \( \theta_0 \).

In this numerical test, we used \( N_z = 1001 \) uniform nodes with \( n_s = 3 \) number of neighboring points and a time step \( \Delta t = 0.0001 \). The water content profiles are shown in Figure 1 for the considered types of soils. Table 2 illustrates the \( RMSE \), \( L^1_{er} \) errors between the numerical solutions and the reference solutions which are obtained using 1D-Hydrus [50]. We obtain accurate results and the predictions are in good agreement with the reference solutions simulated using 1D-Hydrus.

We compute the evolution of the total mass of water \( I \) for each numerical solution:

\[
I(t) = \int_0^L \theta(z, t)dz. \quad (4.3)
\]
Table 1: Parameters of soils.

| Soil Type       | $\theta_r$ ($m^3/m^3$) | $\theta_s$ ($m^3/m^3$) | $\theta_0$ ($m^3/m^3$) | $K$ ($m/day$) | $h_d$ ($m$) | $\lambda$ | $\lambda\beta$ |
|-----------------|-------------------------|-------------------------|-------------------------|---------------|-------------|-----------|---------------|
| 1 Clay          | 0.09                    | 0.475                   | 0.226                   | 0.0144        | -0.3731     | 0.131     | 2.393         |
| 2 Clay loam     | 0.075                   | 0.366                   | 0.130                   | 0.040         | -0.2590     | 0.194     | 2.582         |
| 3 Sand          | 0.04                    | 0.354                   | 0.0819                  | 5.04          | -0.01471    | 1.051     | 5.153         |
| 4 Silty clay    | 0.056                   | 0.479                   | 0.212                   | 0.0216        | -0.3425     | 0.127     | 2.381         |

Figure 1: The water content profiles of the numerical and the reference solutions.

Figure 2 shows the evolution of the total mass of water for the numerical solutions obtained using the proposed techniques and 1D-solution obtained using Hydrus. The results show the effectiveness of the proposed method in terms of conservation of mass.
Table 2: The computed errors between the numerical and reference solutions.

| Soils             | $T$  | $RMSE$  | $L^1_{cr}$ |
|-------------------|------|---------|------------|
| 1                 | 12 h | $1.2 \times 10^{-5}$ | $8.31 \times 10^{-4}$ |
|                   | 3 days | $7.7 \times 10^{-5}$ | $3 \times 10^{-3}$ |
| 2                 | 9 h  | $6.4 \times 10^{-5}$ | $3.6 \times 10^{-3}$ |
|                   | 1.5 day | $8.6 \times 10^{-5}$ | $5.5 \times 10^{-3}$ |
| 3                 | 5.04 min | $4.9 \times 10^{-5}$ | $1.6 \times 10^{-3}$ |
|                   | 25.92 min | $9.6 \times 10^{-5}$ | $7.4 \times 10^{-3}$ |
| 4                 | 12 h | $1.4 \times 10^{-3}$ | $1.2 \times 10^{-3}$ |
|                   | 2 days | $3.5 \times 10^{-3}$ | $1.7 \times 10^{-3}$ |

4.1.2. Infiltration in 2D homogeneous soils

In this section, numerical simulations are performed for 2D infiltration problem using the local RBF method. We consider the same physical parameters of soils taken previously, and $l_1 = 0.3 \text{ m}$. The clay and sandy clay soils are chosen for this numerical test. We set $\Delta t = 0.0001$, $N_x = 200$, $N_z = 1001$, $c = 1$ and $n_s = 5$. Figure 3 shows the numerical simulations of the saturation for the selected soils. The cross sectional average in the vertical direction of the total mass of water of 2D numerical solutions and the 1D-Hydrus reference solutions ($l_x = 1$) are shown in Figure 4. We obtain a good correspondence between the
Clay Sandy clay

Figure 3: Maps of water saturation of the clay and sandy clay soils.

solutions which confirm the accuracy of the developed method for modeling two-dimensional unsaturated flow in soils.

4.1.3. Infiltration in 3D homogeneous soils

Here, we perform numerical simulations for 3D infiltration problem, where we consider a block of soil having the dimensions $l_1 = l_2 = 0.3$ m and $L = 1$ m. We consider the same physical parameters as the previous tests and we set $c = 0.6$, $n_s = 7$, $N_x = N_y = 90$, $N_z = 300$ and $\Delta t = 0.0001$. Figures 5 and 6 show the 3D evolution of saturation (left) for the silty clay and clay loam soils respectively. The results on the right side are the $x$-slices of saturation ($x = 0$, $x = l_1/4$, $x = l_1/2$, $x = 3l_1/4$, $x = l_1$). The cross sectional average in the vertical direction of the total mass of water of 3D numerical solutions and the 1D-Hydrus reference solutions ($l_x = l_y = 1$) are shown in Figure 7. The results confirm the accuracy
4.2. Infiltration in heterogeneous soil

In the following sections, we perform numerical tests to study the robustness of the developed numerical method in modeling one-, two- and three-dimensional heterogeneous medium.

4.2.1. Infiltration in 1D-layered soils

In this numerical test, we perform simulations of infiltration using a column of soil \( L = 25.5 \text{ cm} \) with three layers. The layered soil consists of a thin surface crust \((0.5 \text{ cm})\), a tilled layer \((10 \text{ cm})\) and a subsoil layer \((15 \text{ cm})\). The hydraulic properties \[19\] for the layers soil are shown in Table 3. Numerical simulations are performed for two cases using

\[ h_0 = -100 \text{ cm} \text{ and } -1000 \text{ cm} \]. We set \( c = 0.6, n_s = 3, N_z = 1001 \text{ and } \Delta t = 0.005 \). Figure 8 displays the time evolution of the water content (left) and pressure head (right). The numerical solutions are in good agreement with the 1D-Hydrus simulations. Table 4 presents the \( RMSE \) and \( L^2_z \) errors between the numerical solutions and the results obtained using 1D-Hydrus for the soil water content. The results confirm the effectiveness of the developed method in terms of accuracy in modeling unsaturated flow in heterogeneous soils.

| Layer         | Elevation (cm) | \( \theta_s \) | \( K_s \) (cm/h) | \( h_d \) (cm) | \( \lambda \) | \( \lambda \beta \) |
|---------------|----------------|----------------|------------------|----------------|--------------|----------------|
| Surface crust | \( 25 \leq z \leq 25.5 \) | 0.562          | 0.0616           | -4.55          | 0.1470       | 2.4410        |
| Tilled layer  | \( 15 \leq z \leq 25 \) | 0.562          | 1.396            | -4.55          | 0.0751       | 2.2253        |
| Sub-soil      | \( 0 \leq z \leq 15 \) | 0.440          | 0.312            | -9.50          | 0.0751       | 2.2253        |
4.2.2. Infiltration in 2D-layered soils

This two-dimensional numerical test is performed for unsaturated flow through heterogeneous porous medium. We consider the physical parameters of soils given in Table 3 and
Figure 6: The 3D evolution of saturation of the clay loam soil.

Figure 7: Time-evolution of the total mass of water of the selected soils.
l_1 = 5 cm. Figure 9 shows the time-evolution of saturation for h_0 = -1000 cm obtained using the proposed method. The results are obtained using N_x = 100, N_z = 1001, Δt = 0.005, n_s = 5 and c = 0.6. We present in Figure 10 the cross sectional average in the vertical direction of the total mass of water of 2D numerical solutions and the 1D-Hydrus reference solutions (l_x = 1) where we consider two cases using h_0 = -100 cm and -1000 cm. The results show a good correspondence between the numerical and reference solutions which confirms the accuracy of the proposed method.

4.2.3. Infiltration in 3D-layered soils

Here, we investigate the capability of the developed numerical model in predicting infiltration through heterogeneous three-dimensional porous medium. We consider the same hydraulic properties of soils as the previous test. We perform numerical simulations using c = 0.6, n_s = 7, N_x = N_y = 100, N_z = 501 and Δt = 0.001. In Figure 11, we display the
Figure 9: The time evolution of saturation of the considered soil.

Figure 10: Time-evolution of the total mass of water for $h_0 = -100\,\text{cm}$ and $-1000\,\text{cm}$.

3D evolution of saturation (left) for the considered soils for $h_0 = -1000\,\text{cm}$. The results on the right side present the $x$-slices of saturation ($x = 0$, $x = l_1/4$, $x = l_1/2$, $x = 3l_1/4$, $x = l_1$). The cross sectional average in the vertical direction of the total mass of water of 3D numerical solutions and the 1D-Hydrus reference solutions ($l_x = l_y = 1$) are shown in Figure 12. The comparison between the results of the total mass shows the accuracy of the developed numerical model for infiltration in three-dimensional layered soils.

5. Conclusion

In this study, an efficient approach is developed for modeling unsaturated flow through heterogeneous porous media. The proposed techniques are based on the Kirchhoff transfor-
Figure 11: The 3D evolution of saturation of the layered soils.

Figure 12: Time-evolution of the total mass of water for $h_0 = -1000 \text{ cm}$.

formation, the Brooks and Corey model for the capillary pressure function and a power-law relation for the relative permeability function. The proposed approach allows us to avoid
technical issues associated with the use of the Kirchhoff transformation in heterogeneous soils and to reduce the nonlinearity of the model equation. The resulting system is solved based on the localized radial basis function method which is very effective for solving high-dimensional problems since it doesn’t require mesh generation and has a computational advantage of using reduced memory. The performance and robustness of the developed numerical model are demonstrated based on comparisons between numerical and reference solutions. Numerical experiments are performed to simulate the infiltration in one, two, and three-dimensional soils. The numerical results demonstrate the accuracy of the proposed techniques for modeling infiltration through heterogeneous porous media.

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