PARTIAL SEPARABILITY/ENTANGLEMENT VIOLATES DISTRIBUTIVE RULES

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Abstract. We found three qubit Greenberger-Horne-Zeilinger diagonal states which tells us that the partial separability of three qubit states violates the distributive rules with respect to the two operations of convex sum and intersection. The gaps between the convex sets involving the distributive rules are of nonzero volume.

1. Introduction

Pure states in classical probability theory are uncorrelated, which is not the case in quantum probability theory, where this nonclassical form of correlation is called entanglement [17]. Beyond the conceptual questions it raises [24, 25], entanglement plays a key role in the physics of strongly correlated many-body systems [2], and also finds direct applications in quantum information theory [20]. Mixed states in classical probability theory arise as statistical mixtures (convex combinations) of pure, hence uncorrelated states. Again, this is not the case in quantum probability theory, and states which are mixtures of uncorrelated states are called separable, while the others are entangled [31].

In the case of multipartite systems, the partitions of the total system into subsystems give rise to various notions of partial separability. In the tripartite case with the elementary subsystems \( A, B \) and \( C \), we have three nontrivial partitions \( A-BC \), \( B-CA \) and \( C-AB \), and the corresponding partial separability properties are called \( A-BC \)-separability, \( B-CA \)-separability and \( C-AB \)-separability, respectively. We call these basic biseparabilities.

It is natural to consider the intersections (also called partial separability classes) and convex hulls of the three convex sets consisting of the above three kinds of basic biseparable states. For example, the intersection of them [3], the intersections of two of them [6, 7] and the convex hull of them [11, 23] have been considered. More recently, the convex hulls of two of them have also been considered together with intersections and complements of convex sets arising in the way [30, 15, 16], leading to the description of the hierarchy of the intersections [27]. See also [26, 28, 29] for further developments. Recall that the intersection and convex hull of the convex sets of three basic biseparable states give rise to \textit{fully biseparable} and \textit{biseparable} states, respectively. Tripartite states which are not biseparable are called genuinely multipartite entangled.

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In this context, we consider the lattice generated by three convex sets of three qubit basic biseparable states with respect to the above mentioned two operations, intersection and convex hull. In this way, we deal with convex hulls of intersections, as well as intersections of convex hulls of convex sets arising from basic biseparability. One may go further to investigate the whole structures of partial separability and partial entanglement. Due to technical reasons, we will consider the three convex cones $\alpha$, $\beta$ and $\gamma$ of all un-normalized $A$-$BC$, $B$-$CA$ and $C$-$AB$ biseparable three qubit states, respectively. We note that the convex hull $\sigma \vee \tau$ of two convex cones $\sigma$ and $\tau$ coincides with the sum $\sigma + \tau$ of them. The intersection of two convex cones $\sigma$ and $\tau$ will be denoted by $\sigma \wedge \tau$ following the lattice notations. In general, the two operations $\wedge$ and $\vee$ among convex sets obey associative rule and commutative rule. Furthermore, they also satisfy the following relations

$$\sigma \vee \sigma = \sigma, \quad \sigma \wedge \sigma = \sigma,$$

$$(\sigma \vee \tau) \wedge \sigma = \sigma, \quad (\sigma \wedge \tau) \vee \sigma = \sigma,$$

and so they give rise to a lattice.

We denote by $L$ the lattice generated by three convex cones $\alpha$, $\beta$ and $\gamma$. Therefore, $L$ is the smallest lattice containing the convex cones $\alpha$, $\beta$ and $\gamma$ in the 64-dimensional real vector space of all self-adjoint three qubit matrices. The purpose of this note is to show that the lattice $L$ does not satisfy the distributive rules. More precisely, we show that both inequalities

1. $$(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \leq \alpha \wedge (\beta \vee \gamma),$$
2. $$\beta \vee (\gamma \wedge \alpha) \leq (\beta \vee \gamma) \wedge (\beta \vee \alpha)$$

are strict. Furthermore, the gaps between the two sets are of nonzero volume, in both cases. We also show that the lattice $L$ does not satisfy the modularity which is weaker than distributivity.

We note that a state $\rho$ in the gap $\alpha \wedge (\beta \vee \gamma) \setminus (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ by the strict inequality in (1) has the following properties:

- $\rho$ is $A$-$BC$ biseparable, and it is a mixture of $B$-$CA$ and $C$-$AB$ biseparable states,
- but, it is not a mixture of a simultaneously $A$-$BC$ and $B$-$CA$ biseparable state and a simultaneously $A$-$BC$ and $C$-$AB$ biseparable state.

We will find such states among GHZ diagonal states. On the other hand, a state $\rho$ arising by the strict inequality in (2) has the following properties:

- $\rho$ is a mixture of $B$-$CA$ and $C$-$AB$ biseparable states, and it is also a mixture of a $B$-$CA$ and $A$-$BC$ biseparable states,
- but it is not a mixture of $B$-$CA$ biseparable state and a simultaneously $C$-$AB$ and $A$-$BC$ biseparable states.
2. $X$-shaped states

We will find required examples among so called $X$-shaped states whose entries are zeros except for diagonal and anti-diagonal entries by definition. A self-adjoint $X$-shaped three qubit matrix is of the form

$$X(a, b, z) = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & z_1 \\
a_2 & a_3 & z_2 & z_3 & \ddots \\
a_3 & z_2 & a_4 & \ddots & \ddots \\
a_4 & \ddots & \ddots & \ddots & z_4 \\
z_1 & z_2 & \ddots & \ddots & b_1 \\
z_4 & b_3 & \ddots & b_2 & \ddots \\
z_3 & \ddots & \ddots & b_2 & \ddots \\
b_4 & \ddots & \ddots & \ddots & \ddots \\
b_3 & \ddots & \ddots & \ddots & \ddots \\
b_2 & \ddots & \ddots & \ddots & \ddots \\
b_1 & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix},$$

with $a, b \in \mathbb{R}^4$ and $z \in \mathbb{C}^4$. Here, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is identified with the vector space $\mathbb{C}^8$ with respect to the lexicographic order of indices. Note that $X(a, b, z)$ is a state if and only if $a_i, b_i \geq 0$ and $\sqrt{a_i b_i} \geq |z_i|$ for each $i = 1, 2, 3, 4$.

We recall that every GHZ diagonal states [10] are in this form, and an $X$-state $X(a, b, z)$ is GHZ diagonal if and only if $a = b$ and $z \in \mathbb{R}^4$. In this case, we use the notation

$$X \left( \frac{a}{z} \right) = X(a, a, z).$$

By a pair $\{i, j\}$, we will mean an unordered set with two distinct elements for simplicity throughout this paper. For a given three qubit $X$-shaped state $\rho = X(a, b, z)$, we consider the inequalities

| Inequality | Condition |
|------------|-----------|
| $S_1[i, j]$ | $\min \{\sqrt{a_i b_i}, \sqrt{a_j b_j}\} \geq \max \{|z_i|, |z_j|\}$, |
| $S_2[i, j]$ | $\min \left\{\sqrt{a_i b_i} + \sqrt{a_j b_j}, \sqrt{a_i b_i} + \sqrt{a_j b_j}\right\} \geq \max \{|z_i| + |z_j|, |z_k| + |z_l|\}$, |
| $S_3$ | $\sum_{j \neq i} \sqrt{a_j b_j} \geq |z_i|, \quad i = 1, 2, 3, 4$ |

for a pair $\{i, j\}$, where $\{k, \ell\}$ is chosen so that $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. By [13] Proposition 5.2, we have the following:

- $\rho \in \alpha$ if and only if $S_1[1, 4]$ and $S_1[2, 3]$ hold,
- $\rho \in \beta$ if and only if $S_1[1, 3]$ and $S_1[2, 4]$ hold,
- $\rho \in \gamma$ if and only if $S_1[1, 2]$ and $S_1[3, 4]$ hold.

We also have the following

- $\rho \in \beta \land \gamma$ if and only if $S_2[1, 4]$ (equivalently $S_2[2, 3]$) holds,
- $\rho \in \gamma \land \alpha$ if and only if $S_2[1, 3]$ (equivalently $S_2[2, 4]$) holds,
- $\rho \in \alpha \lor \beta$ if and only if $S_2[1, 2]$ (equivalently $S_2[3, 4]$) holds.

by [16] Theorem 5.5. The inequality $S_3$ will not be used in this paper, but it is the characteristic inequality for the convex cone $\alpha \lor \beta \lor \gamma$ [9, 13, 16, 21].

We will consider the above inequality $S_2$ for arbitrary two pairs $\{i, j\}$ and $\{k, \ell\}$ as follows:
For this purpose, we will use the duality among convex cones in a real vector space. We are now working in the real vector spaces of all three qubit self-adjoint matrices, where the bi-linear pairing is defined by \( \langle x, y \rangle = \text{Tr}(yx^t) \), as usual. See [16].

Every closed convex cone \( C \) satisfies the relation \( C = (C^o)^o \), which tells us that \( x \in C \) if and only if \( \langle x, y \rangle \geq 0 \) for every \( y \in C^o \).

The dual cones of the cones in (3) have also been characterized in [16]. For a given \( \chi \)-shaped self-adjoint matrix \( W = \chi(s, t, u) \) with \( s_i, t_i \geq 0 \) and \( u \in \mathbb{C}^4 \), we have considered the inequalities \( W_1, W_2, W_3 \) given by

\[
\begin{align*}
W_1[i, j] &= \sqrt{s_i t_i} + \sqrt{s_j t_j} \geq |u_i| + |u_j|, \\
W_2[i, j] &= \sum_{k \neq j} s_{k l} t_{k l} \geq |u_i|, \\
W_3 &= \sum_{i=1}^4 \sqrt{s_i t_i} \geq \sum_{i=1}^4 |u_i|,
\end{align*}
\]

for a pair \( \{i, j\} \). Then we have the following by [16] Proposition 3.3:

- \( W \in \alpha^o \) if and only if \( W_1[1, 4] \) and \( W_1[2, 3] \) hold,
- \( W \in \beta^o \) if and only if \( W_1[1, 3] \) and \( W_1[2, 4] \) hold,
- \( W \in \gamma^o \) if and only if \( W_1[1, 2] \) and \( W_1[3, 4] \) hold.

On the other hand, we also have the following by [16] Theorem 5.2:

- \( W \in \beta^o \lor \gamma^o \) if and only if \( W_2[1, 4] \), \( W_2[2, 3] \) and \( W_3 \) hold,
- \( W \in \gamma^o \lor \alpha^o \) if and only if \( W_2[1, 3] \), \( W_2[2, 4] \) and \( W_3 \) hold,
- \( W \in \alpha^o \lor \beta^o \) if and only if \( W_2[1, 2] \), \( W_2[3, 4] \) and \( W_3 \) hold.

For given \( a, b \in \mathbb{R}^4 \) with nonzero entries, and \( z \in \mathbb{C}^4 \) with \( \arg z_i = \theta_i \), we consider the following self-adjoint matrices:

\[
W_{i,j[k, \ell]} = \chi \left( \frac{b_i}{a_i} E_i + \frac{b_j}{a_j} E_j, \frac{a_i}{b_i} E_i + \frac{a_j}{b_j} E_j, -e^{-i\theta_k} E_k - e^{-i\theta_\ell} E_\ell \right)
\]

for pairs \( \{i, j\} \) and \( \{k, \ell\} \), where \( \{E_i\} \) is the usual orthonormal basis of \( \mathbb{C}^4 \). Then the inequality

\[
\langle W_{i,j[k, \ell]}, \chi(a, b, z) \rangle \geq 0
\]
Proof. We will prove (i). For the ‘only if’ part, we first consider the case $g$ gives rise to the inequality by the inequalities for a given three qubit state $\rho = X(a, b, z)$, we have the following:

(i) $\rho \in \alpha \lor (\beta \land \gamma)$ if and only if $S_4[i, j|k, \ell]$ holds whenever $\{i, j\}, \{k, \ell\}$ are two of $\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\};$

(ii) $\rho \in \beta \lor (\gamma \land \alpha)$ if and only if $S_4[i, j|k, \ell]$ holds whenever $\{i, j\}, \{k, \ell\}$ are two of $\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\};$

(iii) $\rho \in \gamma \lor (\alpha \land \beta)$ if and only if $S_4[i, j|k, \ell]$ holds whenever $\{i, j\}, \{k, \ell\}$ are two of $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}.$

Proof. We will prove (i). For the ‘only if’ part, we first consider the case $a_i, b_i > 0$ for $i = 1, 2, 3, 4.$ Then we see that $W_{[i,j|k,\ell]}$ belongs to $\alpha^\circ \lor (\beta^\circ \land \gamma^\circ)$ whenever $\{i, j\}, \{k, \ell\}$ are two of $\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\},$ by checking the required inequalities $W_1[1, 4], W_1[2, 3], W_1[2, 4], W_1[3, 4]$ and $W_3.$ Now, the inequality $\langle W_{[i,j|k,\ell]}, \rho \rangle \geq 0$ gives rise to $S_4[i, j|k, \ell]$ for such $\{i, j\}, \{k, \ell\}.$ If $a$ or $b$ has a zero entry, then we consider $\rho + \epsilon X(1, 1, 0)$ with arbitrary $\epsilon > 0$ and take $\epsilon \to 0$ for the conclusion, where $1 = (1, 1, 1, 1)$ and $0 = (0, 0, 0, 0).$

For the converse, it suffices to show the following:

• if $\rho = X(a, b, z)$ satisfies $S_4[i, j|k, \ell]$ whenever $\{i, j\}, \{k, l\}$ are two of $\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\},$ and $W = X(s, t, u)$ satisfies $W_1[1, 4], W_1[2, 3], W_2[1, 4], W_2[2, 3]$ and $W_3,$ then $\langle W, \rho \rangle \geq 0,$

by Proposition 2.2 and Corollary 2.3 of [16]. If $\rho \in \alpha$ then there is nothing to prove, and so we may assume that $\rho \notin \alpha.$ Without loss of generality, we may also assume that $|z_4| > \sqrt{a_1b_1.}$ Putting $p = \max\{|z_2|, |z_3|\},$ we have

\[
\min\{\sqrt{a_2b_2}, \sqrt{a_3b_3}\} \geq |z_4| + (p - \sqrt{a_1b_1}),
\]

by the inequalities $S_4[i, j|k, \ell]$ for $\{i, j\} = \{1, 2\}, \{1, 3\}$ and $\{k, \ell\} = \{2, 4\}, \{3, 4\}.$ Therefore, we have

\[
\sum_{i=2}^{4} \sqrt{s_i}t_i \sqrt{a_i} - |u_4||z_4| \geq (\sqrt{s_2t_2} + \sqrt{s_3t_3}) \min\{\sqrt{a_2b_2}, \sqrt{a_3b_3}\} + \sqrt{s_4}t_4|z_4| - |u_4||z_4|
\]

\[
\geq \left(\sum_{i=2}^{4} \sqrt{s_i}t_i - |u_4|\right) |z_4| + (\sqrt{s_2t_2} + \sqrt{s_3t_3}) (p - \sqrt{a_1b_1})
\]

\[
\geq \left(\sum_{i=2}^{4} \sqrt{s_i}t_i - |u_4|\right) \sqrt{a_1b_1} + (\sqrt{s_2t_2} + \sqrt{s_3t_3}) (p - \sqrt{a_1b_1})
\]

\[
= (\sqrt{s_4}t_4 - |u_4|) \sqrt{a_1b_1} + (\sqrt{s_2t_2} + \sqrt{s_3t_3})p,
\]
by the inequality $W_2[4, 1]$ and the assumption $|z_4| > \sqrt{a_1b_1}$. We also have
\[ \sqrt{s_i1_1} a_1b_1 - \sum_{i=1}^{3} |u_i||z_i| \geq \sqrt{s_i1_1} a_1b_1 - |u_1|\sqrt{a_1b_1} - |u_2||z_2| - |u_3||z_3| \]
\[ \geq \sqrt{s_i1_1} a_1b_1 - |u_1|\sqrt{a_1b_1} - (|u_2| + |u_3|)p. \]
Summing up the above two inequalities, we have
\[ \sum_{i=1}^{4} (\sqrt{s_i1_1} a_i - |u_i||z_i|) \]
\[ \geq (\sqrt{s_1t_1} + \sqrt{s_4t_4} - |u_1| - |u_4|)\sqrt{a_1b_1} + (\sqrt{s_2t_2} + \sqrt{s_3t_3} - |u_2| - |u_3|)p, \]
which is nonnegative by the inequalities $W_1[1, 4]$ and $W_1[2, 3]$. Therefore, we have
\[ \frac{1}{2} \langle X(s, t, u), X(a, b, z) \rangle = \frac{1}{2} \sum_{i=1}^{4} (s_i a_i + t_i b_i + 2Re (u_i z_i)) \]
\[ \geq \sum_{i=1}^{4} (\sqrt{s_i1_1} a_i - |u_i||z_i|) \geq 0, \]
which completes the proof. □

3. Examples

In order to get analytic examples distinguishing the convex cones in the inequalities (1) and (2), we consider GHZ diagonal states
\[ \varnothing_{0, 0} = \frac{1}{8} X \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad \varnothing_{1, 0} = \frac{1}{8} X \left( \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array} \right), \quad \varnothing_{0, 1} = \frac{1}{12} X \left( \begin{array}{cccc} 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{array} \right), \]
and define
\[ \varnothing_{s, t} = (1 - s - t)\varnothing_{0, 0} + s\varnothing_{1, 0} + t\varnothing_{0, 1} \]
\[ = \frac{1}{8} X \left( \begin{array}{cccc} 1 + \frac{4}{3} t & 1 + s - \frac{1}{3} t & 1 - s - \frac{1}{3} t & 1 + \frac{1}{3} t \\ s + \frac{4}{3} t & \frac{2}{3} t & \frac{2}{3} t & s \end{array} \right), \]
for real numbers $s$ and $t$. We consider the convex set $\mathbb{D}$ of all three qubit states, which is a 63 affine dimensional convex body. We slice $\mathbb{D}$ by the 2-dimensional plane $\Pi$ determined by $\varnothing_{0, 0}$, $\varnothing_{1, 0}$ and $\varnothing_{0, 1}$ to get the pictures for various convex sets. We see that $\varnothing_{s, t}$ is a state if and only if
\[ |s + \frac{4}{3} t| \leq 1 + \frac{4}{3} t, \quad |\frac{2}{3} t| \leq 1 + s - \frac{1}{3} t, \quad |\frac{2}{3} t| \leq 1 - s - \frac{1}{3} t, \quad |s| \leq 1 + \frac{1}{3} t \]
if and only if $(s, t)$ belongs to the region
\[ R = \{(s, t) : s + t \leq 1, \quad -s + t \leq 1, \quad -s - \frac{5}{3} t \leq 1, \quad s - \frac{1}{3} t \leq 1\} \]
which is a quadrilateral on the $st$-plane with the four vertices $(1, 0)$, $(\frac{2}{3}, -1)$, $(-1, 0)$ and $(0, 1)$. Therefore, the 2-dimensional convex body $\mathbb{D} \cap \Pi$ is also a quadrilateral with the vertices
\[ (4) \quad \varnothing_{0, 1}, \quad \varnothing_{1, 0}, \quad \varnothing_{\frac{2}{3}, -1} = \frac{1}{12} X \left( \begin{array}{cccc} 1 & 3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{array} \right), \quad \varnothing_{-1, 0} = \frac{1}{8} X \left( \begin{array}{cccc} 1 & 0 & 2 & 1 \\ -1 & 0 & 0 & -1 \end{array} \right). \]
See Figure 1.
It is easily checked by $S_1[1, 4]$ and $S_1[2, 3]$ that the four states in $\mathcal{H}$ belong to the convex set $\alpha$, and so the convex set $\alpha$ on the plane $\Pi$ is represented by the quadrilateral $R$ itself. Using $S_1[1, 3]$, $S_1[2, 4]$ and $S_1[1, 2]$, $S_1[3, 4]$, it is also easy to characterize $(s, t)$ such that $\varrho_{s,t} \in \beta$ if and only if it is a state and satisfies both inequalities $2s + \frac{5}{3}t \leq 1$ and $-2s + \frac{1}{3}t \leq 1$. Therefore, the region for $\beta$ on the plane $\Pi$ is a pentagon with vertices

$$(-\frac{2}{5}, \frac{3}{5}), \quad (-\frac{2}{5}, \frac{9}{11}), \quad (\frac{1}{2}, -\frac{3}{5}), \quad (\frac{3}{5}, -1), \quad (-\frac{6}{11}, -\frac{3}{11}).$$

We also see that the region for $\gamma$ is determined by $\frac{2}{3}t \leq 1$, $-2s - t \leq 1$ and $2s + \frac{1}{3}t \leq 1$. This is also a pentagon with vertices

$$(-\frac{2}{5}, \frac{3}{5}), \quad (\frac{2}{5}, \frac{3}{5}), \quad (\frac{2}{7}, -1), \quad (-\frac{2}{7}, -\frac{3}{7}), \quad (-\frac{2}{5}, \frac{1}{3}).$$

It is clear that the region for $\alpha \lor \beta$ or $\gamma \lor \alpha$ on the plane $\Pi$ occupies all of the quadrilateral $R$. One may also easily check by $S_2[1, 4]$ that the four states in $\mathcal{H}$ belong to $\beta \lor \gamma$, and so the region for $\beta \lor \gamma$ coincides with the quadrilateral $R$. More precisely, the convex sets

$$(\alpha \lor \beta) \cap \Pi = (\beta \lor \gamma) \cap \Pi = (\gamma \lor \alpha) \cap \Pi = \alpha \cap \Pi$$
are represented by the quadrilateral $R$. The whole quadrilateral $R$ in Figure 1 thus represents the regions for the following convex sets

$$
\alpha, \quad \alpha \lor \beta, \quad \beta \lor \gamma, \quad \gamma \lor \alpha, \quad \alpha \lor \beta \lor \gamma, \quad \alpha \land (\beta \lor \gamma), \quad \alpha \land (\beta \land \gamma)
$$
on the $st$-plane. It should be noted that they are strictly bigger than the convex hull generated by $\beta \cap \Pi$ and $\gamma \cap \Pi$. For example, the state $\varrho_{1,0} \in \Pi$ in Figure 1 belongs to the convex hull $\beta \lor \gamma$, but it is not a mixture of states in $\beta \cap \Pi$ and $\gamma \cap \Pi$. In fact, if $\varrho_{1,0} = \varrho_{1} + \varrho_{2}$ with $\varrho_{1} \in \beta$ and $\varrho_{1} \in \gamma$ then one can easily see that the $X$-parts of $\varrho_{1}$ and $\varrho_{2}$ should be of the form

$$
\frac{1}{8} X \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & * & 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{8} X \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & * & 0 & 0 \end{pmatrix},
$$

respectively. Therefore, they never belong to the plane $\Pi$.

Now, we use Theorem 2.1 to find the region for the convex set $\beta \lor (\gamma \land \alpha)$. For pairs $\{i, j\} = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$, the form $8 \sqrt{a_{i} b_{i}} + 8 \sqrt{a_{j} b_{j}}$ for the state $\varrho_{s,t}$ has the values

$$
2 + s, \quad 2 - s, \quad 2 + \frac{2}{3} t, \quad 2 - \frac{2}{3} t, \quad 2 + s, \quad 2 - s,
$$

respectively. On the other hands, $8|z_{i}| + 8|z_{j}|$ becomes

$$
|s + \frac{3}{2} t| + \frac{2}{3}|t|, \quad |s + \frac{3}{2} t| + \frac{2}{3}|t|, \quad |s + \frac{1}{3} t| + |s|, \quad \frac{4}{3}|t|, \quad |s + \frac{2}{3} t|, \quad |s + \frac{2}{3} t|.
$$

Therefore, we see that a state $\varrho_{s,t}$ belongs to $\beta \lor (\gamma \land \alpha)$ if and only if it belongs to $\gamma \lor (\alpha \land \beta)$ if and only if the inequality

$$
\min \{2 + s, \quad 2 - s, \quad 2 + \frac{2}{3} t, \quad 2 - \frac{2}{3} t\}
$$

$$
\geq \max \{|s + \frac{3}{2} t| + \frac{2}{3}|t|, \quad |s + \frac{1}{3} t| + |s|, \quad \frac{4}{3}|t|, \quad |s + \frac{2}{3} t|, \quad |s + \frac{2}{3} t|\}
$$

holds. One may check a point $(s, t) \in R$ satisfies this inequality if and only if

$$
\frac{1}{2} s + \frac{3}{4} t \leq 1, \quad \frac{3}{2} s + \frac{3}{4} t \leq 1, \quad -\frac{3}{2} s - \frac{3}{4} t \leq 1,
$$

This region is represented by the bigger hexagon $H_{1}$ with the vertices

$$
(-\frac{3}{17}, \frac{6}{17}), \quad (\frac{3}{7}, \frac{3}{7}), \quad (\frac{6}{7}, -\frac{3}{7}), \quad (\frac{2}{7}, -1), \quad (-\frac{6}{17}, -\frac{3}{17}), \quad (-\frac{10}{17}, \frac{3}{17})
$$
in Figure 1. Therefore, the difference $R \setminus H_{1}$ consisting of three triangles gives us examples in

$$
(5) \quad (\beta \lor \gamma) \land (\beta \lor \alpha) \setminus \beta \lor (\gamma \land \alpha),
$$

which shows that the strict inequality holds in (2).

In order to consider the strict inequality (1), we first note the inequality

$$
(6) \quad (\alpha \land \beta) \lor (\alpha \land \gamma) \leq \alpha \land (\beta \lor (\gamma \land \alpha)) \leq \alpha \land (\beta \lor \gamma),
$$

which holds in general. By the first inequality, the region for $(\alpha \land \beta) \lor (\alpha \land \gamma)$ is also contained in $H_{1}$. In fact, it fills up all of $H_{1}$. Indeed, it is clear that five vertices of $H_{1}$
belong to \((\alpha \land \beta) \lor (\alpha \land \gamma)\) except for \((-\frac{10}{13}, \frac{3}{11})\) by \(S_1[i,j]\). We also see that

\[
52 \varrho_{-\frac{10}{13}, \frac{3}{11}} = X\begin{pmatrix} 7 & 1 & 11 & 7 \\ -3 & 1 & 1 & -5 \end{pmatrix} = X\begin{pmatrix} 3 & 1 & 3 & 3 \\ -3 & 1 & 1 & -1 \end{pmatrix} + X\begin{pmatrix} 4 & 0 & 8 & 4 \\ 0 & 0 & 0 & -4 \end{pmatrix}
\]

belongs to \((\alpha \land \beta) \lor (\alpha \land \gamma)\). Therefore, this coincides with \(\alpha \land (\beta \lor (\gamma \land \alpha))\) for states \(\varrho_{s,t}\), that is, the first inequality in (6) becomes an identity on the plane \(\Pi\). Therefore, the difference \(R \setminus H_1\) again gives rise to examples in

\[
(7) \quad \alpha \land (\beta \lor \gamma) \setminus (\alpha \land \beta) \lor (\alpha \land \gamma),
\]

for the strict inequality in (1).

Now, we proceed to show that the gaps (5) and (7) arising in the distributive inequalities have nonzero volume. We first note that the convex set \(\mathbb{S}\) of all fully separable three qubit states generates the same affine manifold as the convex set \(\mathbb{D}\) of all three qubit states. See the discussion at the end of Section 7 in [12]. Therefore, all the convex sets in (5) and (7) generate the same affine manifold. We also recall that a point \(x_0\) of a convex set \(C\) is called an interior point of \(C\) with respect to the affine space generated by \(C\). We note that the state \(\varrho_{0,0}\) is a common interior point of the convex sets appearing in (5) and (7) as well as \(\mathbb{S}\) and \(\mathbb{D}\).

If we consider the line segment \(x_t = (1-t)x_0 + tx_1\) between an interior point \(x_0\) of a convex set \(C\) and an arbitrary point \(x_1 \in C\) then \(x_t\) is also an interior point of \(C\) for \(0 < t < 1\). See [19, Lemma 2.3]. Therefore, every interior point of \([\alpha \land (\beta \lor \gamma)] \cap \Pi\) is actually an interior point of \(\alpha \land (\beta \lor \gamma)\). For example, \(\varrho_{4,0}\) is an interior point of \(\alpha \land (\beta \lor \gamma)\) which is a boundary point of \((\alpha \land \beta) \lor (\alpha \land \gamma)\). From this, we may conclude that the difference (7) has the nonempty interior by [12, Proposition 7.4]. The exactly same argument also shows that the difference (5) also has the nonempty interior.

We also consider the smaller hexagon \(H_2\) with vertices

\[
(-\frac{2}{3}, \frac{2}{3}), \quad (0, \frac{2}{3}), \quad (\frac{1}{2}, 0), \quad (\frac{2}{3}, -1), \quad (-\frac{2}{7}, -\frac{3}{7}), \quad (-\frac{1}{2}, 0)
\]

in Figure 1, which is the region for \(\varrho_{s,t}\) in \(\beta \land \gamma\). This represents also \(\alpha \land \beta \land \gamma\). Note that an \(X\)-shaped state belongs to \(\alpha \land \beta \land \gamma\) if and only if it is of positive partial transpose by [13, Theorem 5.3]. Therefore, the region \(H_2\) represents the region for PPT states for \(\varrho_{s,t}\).

Recall that a lattice is called modular if \(x \leq z\) implies \(x \lor (y \land z) = (x \lor y) \land z\). This is the case if and only if the modular identity

\[
(x \land z) \lor (y \land z) = ((x \land z) \lor y) \land z
\]

holds for every \(x, y\) and \(z\). Every distributive lattice is modular. See [4, 22] for elementary properties of modular lattices. We exhibit examples of states showing that the first inequality in (6) is also strict, to conclude that the lattice \(\mathcal{L}\) is not modular.
To do this, we consider $g_1 = \frac{1}{12} X \begin{pmatrix} 2 & 1 & 1 & 2 \\ 2 & 0 & 1 & 0 \end{pmatrix}$, and put

$$g_t = (1-t)g_{0,0} + t g_1 = \frac{1}{24} X \begin{pmatrix} 3 + t & 3 - t & 3 - t & 3 + t \\ 4t & 0 & 2t & 0 \end{pmatrix}.$$  

We also consider $W = X \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix}$, which satisfies the inequality $W_2[i,j]$ for $\{i,j\} = \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}$ and $W_3$. Therefore, we see that $W$ belongs to the convex cone $(\alpha^2 \lor \beta^2) \land (\alpha^2 \lor \gamma^2)$, which is the dual of the convex cone $(\alpha \land \beta) \lor (\alpha \land \gamma)$. Now, we see that $\langle W, g_t \rangle = \frac{1}{24} (6 - 8t) \geq 0$ if and only if $t < 1$.  

Therefore, we conclude that $g_t$ does not belong to $(\alpha \land \beta) \lor (\alpha \land \gamma)$ for $\frac{2}{3} < t \leq 1$. On the other hand, one can easily check that $g_t \in \alpha \land (\beta \lor (\gamma \land \alpha))$ by Theorem 2.1. Therefore, we see that $g_t$ also belongs to the same cone. In fact, $g_t$ is an interior point of the cone $\alpha \land (\beta \lor (\gamma \land \alpha))$ for $0 \leq t < 1$, because $g_{0,0}$ is an interior point. Hence, we also see that the gap for the first inequality in (8) has also nonzero volume.

Finally, we also characterize the full separability for the states we are considering. To do this, we summarize the results in [11, 14, 18]. See also [5, 12]. For a given GHZ diagonal state $\varrho = X(a, a, c)$ with $a, c \in \mathbb{R}^4$, we consider the following:

$$\lambda_5 = 2(c_1 + c_2 + c_3 + c_4), \quad \lambda_6 = 2(-c_1 - c_2 + c_3 + c_4),$$

$$\lambda_7 = 2(-c_1 + c_2 - c_3 + c_4), \quad \lambda_8 = 2(-c_1 + c_2 + c_3 - c_4),$$

$$t_1 = c_1(-c_1^2 + c_2^2 + c_3^2 + c_4^2) - 2c_2c_3c_4, \quad t_2 = c_2(+c_1^2 - c_2^2 + c_3^2 + c_4^2) - 2c_1c_3c_4,$$

$$t_3 = c_3(+c_1^2 + c_2^2 - c_3^2 + c_4^2) - 2c_1c_2c_4, \quad t_4 = c_4(+c_1^2 + c_2^2 + c_3^2 - c_4^2) - 2c_1c_2c_3,$$

When all the following inequalities

(8) \[ \lambda_5 \lambda_6 \lambda_7 \lambda_8 > 0, \quad t_1 t_4 \lambda_6 \lambda_7 < 0, \quad t_2 t_3 \lambda_5 \lambda_8 > 0 \]

hold, the state $\varrho = X(a, a, c)$ is fully separable if and only if the inequality

(9) \[ \min \{a_1, a_2, a_3, a_4\} \geq \sqrt{\frac{(\lambda_5 \lambda_6 + \lambda_7 \lambda_8)(\lambda_5 \lambda_7 + \lambda_6 \lambda_8)(\lambda_5 \lambda_8 + \lambda_6 \lambda_7)}{8 \sqrt{\lambda_5 \lambda_6 \lambda_7 \lambda_8}}} \]

are satisfied. In the other cases, $\varrho$ is fully separable if and only if it is of PPT. For the state $\varrho_{s,t}$, the conditions [8] are given by

$$s(3s + 4t) < 0, \quad (9s^2 + 18st + 4t^2)(9s^2 + 6st - 4t^2) > 0.$$  

We note that a point $(s, t)$ on the line $t = as$ satisfies this condition if and only if

$$-\frac{3}{4}(3 + \sqrt{5}) \leq a < -\frac{3}{4}.$$  

On the other hands, the square of right side of (9) is given by

$$\frac{t^2(9s^2 + 12st - 4t^2)}{432s(3s + 4t)}$$

respectively. See Figure 2.
Figure 2. Two line segments through the origin are given by the conditions \( (8) \), and two curves surrounding the region of full separability are given by \( (9) \).

4. Further questions

In this paper, we have considered the lattice \( \mathcal{L} \) generated by three basic convex sets \( \alpha, \beta \) and \( \gamma \) consisting of all \( A-BC \) biseparable, \( B-CA \) biseparable and \( C-AB \) biseparable three qubit states, respectively, with respect to the operations of convex hull \( \lor \) and intersection \( \land \). In this way, we may consider convex sets of partially separable states obtaining by arbitrary convex hulls and intersections of \( \alpha, \beta \) and \( \gamma \), and the whole structure of partial separability may be revealed by mathematical properties of the lattice \( \mathcal{L} \). For general theory for lattices, we refer to the monographs [4], [8] and [22].

We gave the negative answer to the first natural question asking if this lattice is distributive. The lattice \( \mathcal{L} \) is not even modular. Another interesting question is to ask if the lattice \( \mathcal{L} \) has infinitely many elements. We conjecture this is the case. This means that there are infinitely many kinds of partial separability and partial entanglement. It is known that a free lattice with three generators must have infinitely many elements. In this regard, it would be interesting to know if the lattice \( \mathcal{L} \) is free or not.

The next question is whether the lattice \( \mathcal{L} \) is complemented or not. A lattice is called complemented if every element \( x \) has a complement \( y \) which satisfies \( x \land y = 0 \) and \( x \lor y = 1 \), where 0 and 1 denote the least and greatest elements, respectively. The least and the greatest elements of \( \mathcal{L} \) are given by \( \alpha \land \beta \land \gamma \) and \( \alpha \lor \beta \lor \gamma \), respectively.
They represent the set of all fully biseparable and biseparable states, respectively. Especially, we would like to ask if $\alpha$ has a complement, that is, we ask if there exist $\sigma \in \mathcal{L}$ such that $\alpha \wedge \sigma = \alpha \wedge \beta \wedge \gamma$ and $\alpha \vee \sigma = \alpha \vee \beta \vee \gamma$. Recall that the set of all closed subspaces of a Hilbert space makes a lattice, the subspace lattice, with respect to the closed linear hull and intersection. This plays an important role in quantum logic and theory of operator algebras. We note that the subspace lattice is non-distributive, but complemented.

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