GLOBAL FUJITA-KATO SOLUTION OF 3-D INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES SYSTEM

PING ZHANG

ABSTRACT. In this paper, we shall prove the global existence of weak solutions to 3D inhomogeneous incompressible Navier-Stokes system (INS) with initial density in the bounded function space and having a positive lower bound and with initial velocity being sufficiently small in the critical Besov space, $\dot{B}_{2,1}^{1}$. This result corresponds to the Fujita-Kato solutions of the classical Navier-Stokes system. The same idea can be used to prove the global existence of weak solutions in the critical functional framework to (INS) with one component of the initial velocity being large and can also be applied to provide a lower bound for the lifespan of smooth enough solutions of (INS).

Keywords: Inhomogeneous Navier-Stokes system, Besov space, weak solutions.
AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

In this paper, we consider the global existence of weak solutions to the following three dimensional incompressible inhomogeneous Navier-Stokes equations with initial density in the bounded function space and having a positive lower bound, and with initial velocity being sufficiently small in the critical Besov space, $\dot{B}_{2,1}^{1}$:

\[
\begin{align*}
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\rho (\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \pi &= 0, \\
\text{div} u &= 0, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0),
\end{align*}
\]

where $\rho, u$ stand for the density and velocity of the fluid respectively, $\pi$ is a scalar pressure function which guarantees the divergence free condition of the velocity field. Such a system describes a fluid which is obtained by mixing several immiscible fluids that are incompressible and that have different densities.

Let us first state three major basic features of system (1.1) in general Euclidean space $\mathbb{R}^d$, $d \geq 2$. Firstly, the incompressibility condition on the convection velocity field in the density transport equation ensures that

\[
\| \rho(t) \|_{L^\infty} = \| \rho_0 \|_{L^\infty} \quad \text{and} \quad \text{meas}\{ x \in \mathbb{R}^d \mid \alpha \leq \rho(t, x) \leq \beta \} \quad \text{is independent of } t \geq 0,
\]

for any pair of non-negative real numbers $(\alpha, \beta)$. Secondly, this system has the following energy law

\[
\frac{1}{2} \int_{\mathbb{R}^d} \rho(t, x) |u(t, x)|^2 dx + \int_0^t \| \nabla u(t', \cdot) \|_{L^2(\mathbb{R}^d)}^2 dt' = \frac{1}{2} \int_{\mathbb{R}^d} \rho_0(x) |u_0(x)|^2 dx.
\]

Date: June 12, 2018.
The third basic feature is the scaling invariance property: if \((\rho, u, \pi)\) is a solution of (1.1) on \([0, T] \times \mathbb{R}^d\), then the rescaled triplet \((\rho, u, \pi)_\lambda\) defined by

\[
(\rho, u, \pi)_\lambda(t, x) = (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 \pi(\lambda^2 t, \lambda x)), \quad \lambda \in \mathbb{R}
\]

is also a solution of (1.1) on \([0, T/\lambda^2] \times \mathbb{R}^d\). This leads to the notion of critical regularity.

Based on the energy estimate (1.3), Simon [27] (see [24] for general result on the variable viscosity case) constructed global weak solutions of the system (1.1) with finite energy. Ladyzhenskaya and Solonnikov [20] proved the local well-posedness to the system (1.1) with homogeneous Dirichlet boundary condition and with smooth initial data that has no vacuum. Motivated by (1.4), Danchin [8] established the well-posedness of (1.1) in the so-called critical functional framework for small perturbations of some positive constant density. The essential idea in [8] is to use functional spaces (or norms) that have the same scaling invariance as (1.4). In this framework, it has been stated in [1, 8] that for the initial data \((\rho_0, u_0, \pi_0)\) satisfying

\[
(\rho_0 - 1) \in \dot{B}^d_{p,1}(\mathbb{R}^d), \quad u_0 \in \dot{B}^d_{p,1}^{-1}(\mathbb{R}^d) \text{ with } \text{div } u_0 = 0
\]

and that for a small enough constant \(c\)

\[
\|\rho_0 - 1\|_{\dot{B}^d_{p,1}} + \|u_0\|_{\dot{B}^d_{p,1}^{-1}} \leq c,
\]

we have for any \(p \in [1, 2d]\)

- existence of global solution \((\rho, u, \nabla p)\) with \(\rho - 1 \in C([0, \infty]; \dot{B}^d_{p,1}(\mathbb{R}^d)), \ u \in C([0, \infty]; \dot{B}^d_{p,1}^{-1}(\mathbb{R}^d))\), and \(\partial_t u, \nabla^2 u, \nabla p \in L^1(\mathbb{R}^+, \dot{B}^d_{p,1}(\mathbb{R}^d))\);  
- uniqueness in the above space if in addition \(p \leq d\).

The above existence result was extended to general Besov spaces in [2, 9, 25] even without the size restriction for the initial density [3, 4]. The uniqueness of such solutions for \(p \in ]d, 2d[\) was obtained by Danchin and Mucha in [9].

In all these aforementioned works, the density has to be at least in the Besov space \(B^{d/p}_{p,\infty}(\mathbb{R}^d)\), which excludes the density function with discontinuities across some hypersurface. Indeed, the Besov regularity of the characteristic function of a smooth domain is only \(B^{1/p}_{p,\infty}(\mathbb{R}^d)\). Therefore, those results do not apply to a mixture flow composed of two separate fluids with different densities.

In particular, Lions proposed the following open question in [24]: suppose the initial density \(\rho_0 = 1_D\) for some smooth domain \(D\). Theorem 2.1 of [24] provides at least one global weak solution \((\rho, u, \pi)\) of (1.1) such that for all \(t \geq 0\), \(\rho(t) = 1_{D(t)}\) for some set \(D(t)\) with \(\text{vol}(D(t)) = \text{vol}(D)\). Then whether or not the regularity of \(D\) is preserved by time evolution? To avoid the difficulty caused by vacuum, Liao and the author [22] investigated the case when the system (1.1) is supplemented with the initial density, \(\rho_0(x) = \eta_1 1_{\Omega_0} + \eta_2 1_{\Omega_0^c}\), for a pair of positive constants \((\eta_1, \eta_2)\) with \(|\eta_1 - \eta_2|\) being sufficiently small, and where \(\Omega_0\) is a bounded, simply connected 2D domain with \(W^{k+2,p}\)-boundary regularity for \(k \in \mathbb{N}\). This smallness assumption for the difference between \(\eta_1\) and \(\eta_2\) was removed by the authors in [23]. Danchin and Zhang [13], Gancedo and Garcia-Juarez [18] proved the propagation of \(C^{k+\gamma}\) regularity of the density patch to (1.1). Lately Danchin and Mucha [11] can allow vacuum.

In the general case when \(\rho_0 \in L^\infty\) with a positive lower bound and initial velocity \(u_0 \in H^1\), Kazhikov [19] proved the local existence of weak solution to the system (1.1). While with \(u_0 \in H^2\), Danchin and Mucha [10] proved that the system (1.1) has a unique local in time solution. Paicu, the author and Zhang [26] improved the well-posedness results in [10] with
Here and in all that follows, we always denote $B^s$ for any $s > 0$, and with initial velocity in $H^1(\mathbb{R}^3)$. Chen, Zhang and Zhao [5] further improved the regularity of the initial velocity in 3D to be in $H^s(\mathbb{R}^3)$ for any $s > \frac{1}{2}$. Nevertheless, in either [5] or [26], the authors can not prove the propagation of the regularities for the initial velocity field, namely, they can not prove the velocity $u$ belongs to $C([0,T]; H^s)$. Furthermore, the norms of the initial velocity in [5, 26] is not critical in the sense that the norms are not scaling invariant under the transformation (1.4).

On the other hand, when $\rho_0 \equiv 1$, the system (1.1) reduces to the classical incompressible Navier-Stokes system (NS). Let us recall the following celebrated result by Fujita and Kato [17] on (NS):

**Theorem 1.1.** Given solenoidal vector field $u_0 \in \dot{H}^{\frac{1}{2}}$ with $\|u\|_{\dot{H}^{\frac{1}{2}}} \leq \varepsilon_0$ for $\varepsilon_0$ sufficiently small, then (NS) has a unique global solution $u \in C([0, \infty[; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \cap L^2(\mathbb{R}^3; \dot{H}^{\frac{3}{2}}))$.

Before proceeding, let us recall the definition of weak solutions to (1.1) from [16, 26]:

**Definition 1.1.** We call $(\rho, u, \nabla \pi)$ a global weak solution of (1.1) if

- for any test function $\phi \in C^\infty_c([0, \infty[ \times \mathbb{R}^3)$, there holds

$$
\int_0^\infty \int_{\mathbb{R}^3} \rho(\partial_t \phi + u \cdot \nabla \phi) \, dx \, dt + \int_{\mathbb{R}^3} \phi(0, x) \rho_0(x) \, dx = 0,
$$

$$
\int_0^\infty \int_{\mathbb{R}^3} \text{div} u \phi \, dx \, dt = 0,
$$

- for any vector valued function $\Phi = (\Phi^1, \Phi^2, \Phi^3) \in C^\infty_c([0, \infty[ \times \mathbb{R}^3)$, one has

$$
\int_0^\infty \int_{\mathbb{R}^3} \left\{ u \cdot \partial_t \Phi - (u \cdot \nabla u) \cdot \Phi - \frac{1}{\rho} \left( \Delta u - \nabla \pi \right) \Phi \right\} \, dx \, dt + \int_{\mathbb{R}^3} u_0 \cdot \Phi(0, x) \, dx = 0.
$$

The goal of the following theorem is to prove similar version of Theorem 1.1 for the inhomogeneous incompressible Navier-Stokes system (1.1), the proof of which will be based on the basic features of (1.1), namely, (1.2), (1.3) and (1.4).

**Theorem 1.2.** Let $(\rho_0, u_0)$ satisfy

$$
0 < c_0 \leq \rho_0(x) \leq C_0 < +\infty \quad \text{and} \quad u_0 \in B^{\frac{1}{2}}.
$$

Then there exists a constant $\varepsilon_0 > 0$ depending only on $c_0, C_0$ such that if

$$
\|u_0\|_{B^{\frac{1}{2}}} \leq \varepsilon_0,
$$

the system (1.1) has a global weak solution $(\rho, u)$ with $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ and $u \in C([0, \infty[, B^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+; B^{\frac{3}{2}})$ which satisfies

$$
c_0 \leq \rho(t, x) \leq C_0 \quad \text{for} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3,
$$

and

$$
\|u\|_{\dot{L}^{\infty}_t(B^{\frac{1}{2}})} + \|u\|_{\dot{L}^{2}_t(B^{\frac{3}{2}})} + \|\nabla u\|_{\dot{L}^{\infty}_t(B^{\frac{3}{2}})} + \|\nabla(tu, \pi)\|_{\dot{L}^{\frac{1}{2}}_t(B^{\frac{1}{2}})}
$$

$$
+ \|\nabla tu\|_{\dot{L}^{\frac{1}{2}}_t(B^{\frac{3}{2}})} + \|tu\|_{\dot{L}^{\infty}_t(B^{\frac{3}{2}})} + \|tD_t u\|_{\dot{L}^{\frac{1}{2}}_t(B^{\frac{3}{2}})} \leq C\|u_0\|_{B^{\frac{1}{2}}}.
$$

Here and in all that follows, we always denote $D_t \overset{\text{def}}{=} \partial_t + u \cdot \nabla$ to be the material derivative. For simplification, we always denote $B^s \overset{\text{def}}{=} B^{s}_{2,1}$ in this paper. The definitions of Besov spaces, $B^{s}_{p,r}$, and Chemin-Lerner type space, $\dot{L}^q_T(B^s)$, will be recalled in the Appendix A.
Theorem 1.3. Let $u_0 \in L^\infty$ and $u_0 = (u^{h}_0, u^{3}_0) \in B^{\frac{1}{2}}$. Then there exists a positive constant $\epsilon_0$ so that if

$$
\eta \overset{\text{def}}{=} (\|a_0\|_{L^\infty} + \|u_0^h\|_{B^{1/2}}) \exp(C\|u_0^3\|^{2}_{B^{1/2}}) \leq \epsilon_0,
$$

Remark 1.1. (1) We improve the regularity of the initial velocity in $H^s$ for any $s > \frac{1}{2}$ in [5] to be the critical space $B^{\frac{1}{2}}$. Moreover, we can propagate the regularity of the initial velocity field, namely, here $u \in C([0, \infty], B^{\frac{1}{2}})$. Whereas the velocity field in [5] belongs to some time-weighted integer Sobolev spaces. More precisely, for any $t \in [0, +\infty[$ and $\sigma(t) \overset{\text{def}}{=} \min(1, t)$, they have

$$\|\sigma(t)^{\frac{1}{2}} \nabla u\|_{L^p_\infty(L^2)} + \|\sigma(t)^{\frac{1}{2}} (\nabla^2 u, u_t, \nabla \pi)\|_{L^2(L^2)} \leq C\|u_0\|_{H^s}^2,$$

$$\|\sigma(t)^{1-\frac{s}{2}} (\nabla^2 u, u_t, \nabla \pi)\|_{L^p_\infty(L^2)} + \|\sigma(t)^{1-\frac{s}{2}} \nabla u_t\|_{L^2(L^2)} \leq C\|u_0\|_{H^s}^2 \exp(C(\|u_0\|_{L^2}^2 + \|u_0\|_{H^s}^4)).$$

(2) The time weight in (1.10) is optimal even for heat semigroup $e^{t\Delta} u_0$. Indeed, it follows from Lemma A.2 and Bernstein inequality that

$$\|\Delta_j (t^{\frac{s}{2}} \nabla e^{t\Delta} u_0)\|_{L^2} \leq t^{\frac{1}{2}} 2^j e^{-ct2^j} \|\Delta_j u_0\|_{L^2} \lesssim \|\Delta_j u_0\|_{L^2},$$

which implies that $\|t^{\frac{s}{2}} \nabla e^{t\Delta} u_0\|_{L^\infty(B^{\frac{1}{2}})} \lesssim \|u_0\|_{B^{s}}$ for any $s \in \mathbb{R}$.

(3) As in [5, 26], with a little bit more regularity assumption on the initial velocity field, we can also prove the uniqueness of such weak solutions of (1.1) constructed in Theorem 1.2.

Again in a critical functional framework, Huang, Paicu and the author [16] proved the global existence of weak solutions to (1.1) provided that the initial data satisfy the nonlinear smallness condition:

$$\left(\|a_0^{-1} - 1\|_{L^\infty} + \|u_0^h\|_{B^{-1+\frac{3}{p}}_{p,r}} \exp(C_r\|u_0^3\|_{B^{-1+\frac{3}{p}}_{p,r}}^{2r}) \right) \leq \epsilon_0$$

for some positive constants $\epsilon_0, C_r$ and $1 < p < 3$, $1 < r < \infty$, where $u_0^h = (u_0^h, u_0^3)$ and $u_0 = (u_0^h, u_0^3)$. With a little bit more regularity assumption on the initial velocity, they [16] also proved the uniqueness of such solutions. Danchin and the author extended this result to the half-space setting in [12]. Nonetheless as in [5, 26], the authors there can not prove the propagation of the fractional derivative for the initial velocity field. Moreover, The result in [16] does not work for the index $r = 1$ due to technical reason (the application of maximal regularity estimate for heat semi-group forbids the case for $r = 1$.) The purpose of the next theorem is to solve the aforementioned questions.

Toward this, let us denote $a \overset{\text{def}}{=} \frac{1}{p} - 1$. Then we can reformulate (1.1) as

$$
\begin{align*}
\begin{cases}
\partial_t a + u \cdot \nabla a = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t u + u \cdot \nabla u - (1 + a)(\Delta u - \nabla \pi) = 0, \\
\text{div} u = 0, \\
(a, u)|_{t=0} = (a_0, u_0),
\end{cases}
\end{align*}
$$

The second result of this paper states as follows:

Theorem 1.3. Let $a_0 \in L^\infty$ and $u_0 = (u^h_0, u^3_0) \in B^{\frac{1}{2}}$. Then there exists a positive constant $\epsilon_0$ so that if

$$
\eta \overset{\text{def}}{=} (\|a_0\|_{L^\infty} + \|u_0^h\|_{B^{1/2}}) \exp(C\|u_0^3\|_{B^{1/2}}^2) \leq \epsilon_0,
$$
(1.11) has a global weak solution \((a, u)\) in the sense of Definition 1.1, which satisfies \(u = v + e^{t\Delta}u_0\) and
\[
\begin{align*}
\|v\|_{\dot{L}^{\infty}(B^{3/2})} + \|v\|_{\dot{L}^{2}(B^{3/2})} + \|\sqrt{t}v\|_{\dot{L}^{\infty}(B^{\frac{3}{2}})} + \|\sqrt{t}v_t\|_{\dot{L}^{2}(B^{3/2})} + \|tv_t\|_{\dot{L}^{\infty}(B^{3/2})} + \|tD_tv\|_{\dot{L}^{2}(B^{3/2})} & \leq C\eta.
\end{align*}
\]

The idea of the proof to Theorem 1.2 can also be used to prove the following Theorem concerning the lifespan of smooth enough solution to (1.1), which in particular generalize the corresponding result for classical Navier-Stokes system (see Proposition 1.1 of [7]) to the inhomogeneous context.

**Theorem 1.4.** Let \(\rho_0\) satisfy (1.7), \(u_0 \in \dot{H}^{\frac{1}{2}+2\gamma}\) for \(\gamma \in [0, 1/4]\). Then (1.1) has a unique solution \((\rho, u)\) on \([0, T]\) so that \(u \in C([0, T]; \dot{H}^{\frac{1}{2}+2\gamma}) \cap L^2([0, T]; \dot{H}^{\frac{1}{2}+2\gamma})\). We denote \(T^*(u_0)\) to be the maximal time of existence of such solution. Then there exists a constant \(c_\gamma\) so that
\[
T^*(u_0) \geq c_\gamma\|u_0\|_{\dot{H}^{\frac{1}{2}+2\gamma}}^{-\frac{1}{2}} \text{ defined } T_\gamma.
\]
Moreover, for \(t \leq T_\gamma\), there holds
\[
\begin{align*}
\|u\|_{\dot{L}^{\infty}(\dot{H}^{\frac{1}{2}+2\gamma})} + \|u\|_{\dot{L}^{2}(\dot{H}^{\frac{1}{2}+2\gamma})} + \|\sqrt{t}u\|_{\dot{L}^{\infty}(\dot{H}^{\frac{1}{2}+2\gamma})} + \|\sqrt{t}u_t\|_{\dot{L}^{2}(\dot{H}^{\frac{1}{2}+2\gamma})} + \|tv_t\|_{\dot{L}^{\infty}(\dot{H}^{\frac{1}{2}+2\gamma})} + \|tD_tv\|_{\dot{L}^{2}(\dot{H}^{\frac{1}{2}+2\gamma})} & \leq C\|u_0\|_{\dot{H}^{\frac{1}{2}+2\gamma}}.
\end{align*}
\]

**Remark 1.2.** When \(\rho_0 = 1\) and \(\gamma = \frac{1}{4}\), (1.14) corresponds to the celebrated Leray estimate on the lifespan of strong solutions to the classical Navier-Stokes system in [21].

Let us end this introduction by some notations that will be used in all that follows.

For operators \(A, B\), we denote \([A; B] = AB - BA\) to be commutator of \(A\) and \(B\). For \(a \lesssim b\), we mean that there is a uniform constant \(C\), which may be different on different lines, such that \(a \leq C b\). We denote by \(\int f g dx\) the \(L^2(\mathbb{R}^3)\) inner product of \(f\) and \(g\). For \(X\) a Banach space and \(I\) an interval of \(\mathbb{R}\), we denote by \(C(I; X)\) the set of continuous functions on \(I\) with values in \(X\). For \(q \in [1, +\infty]\), the notation \(L^q(I; X)\) stands for the set of measurable functions on \(I\) with values in \(X\), such that \(t \mapsto \|f(t)\|_X\) belongs to \(L^q(I)\). Finally we always denote \((d_j)_{j \in \mathbb{Z}}\) (resp. \((c_j)_{j \in \mathbb{Z}}\)) to be a generic element of \(\ell^1(\mathbb{Z})\) (resp. \(\ell^2(\mathbb{Z})\)) so that \(\sum_{j \in \mathbb{Z}} d_j = 1\) (resp. \(\sum_{j \in \mathbb{Z}} c_j^2 = 1\)).

## 2. Ideas of the Proof and Structure of the Paper

First of all, let us recall that the classical idea to prove the local wellposedness of (1.11) in the critical functional framework in [1, 8] is first to apply the operator \(\Delta_j\) (see (A.1)) to the momentum equation of (1.11) and then taking \(L^2\) inner product of the result equation with \(\Delta_j u\), which gives
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_j u(t)\|_{L^2}^2 + \|\nabla \Delta_j u\|_{L^2}^2 = (\Delta_j(u \cdot \nabla u)\Delta_j u) + (\Delta_j[a(\Delta u - \nabla \pi)]\Delta_j u).
\]

The next idea is to apply the commutative estimate for
\[
[\Delta_j; a](\Delta u - \nabla \pi)
\]
and the smallness condition for \(a\) in the critical Besov space, \(\dot{B}^{\frac{3}{2}}_{2, \infty}\), to deal with the term \((\Delta_j[a(\Delta u - \nabla \pi)]\Delta_j u)\) as a perturbation term of the left-hand side. To go through this
process, one needs \( a \in L^\infty \cap B_2^{\frac{3}{2}} \). Nevertheless, here we only assume that \( a_0 \) belongs to the bounded function space. Hence the aforementioned program does not work for the proof of Theorems 1.2 to 1.4 here and the results in [5, 16, 26].

The main idea to the proof of Theorems 1.2 to 1.4 is motivated by the following lemma and its proof in [6]:

**Lemma 2.1** (Lemma 2.64 of [6]). Let \( s \) be a positive real number and \((p, r)\) be in \([1, \infty]^2\). A constant \( C_s \) exits such that if \((u_j)_{j \in \mathbb{Z}}\) is a sequence of smooth functions where \( \sum_{j \in \mathbb{Z}} u_j \) converges to \( u \) in \( S_h^t \) and

\[
N_s((u_j)_{j \in \mathbb{Z}}) = \| \left( \sup_{|\alpha| \leq |\alpha|_{0, \infty} + 1} 2^{j(s-|\alpha|)} \| \partial^\alpha u_j \|_{L^p} \right)_{j \in \mathbb{Z}} \|_{l^r(\mathbb{Z})} < \infty,
\]

then \( u \) is in \( B_{p,r}^s \) and \( \| u \|_{B_{p,r}^s} \leq C_s N_s((u_j)_{j \in \mathbb{Z}}) \).

More precisely, we shall explain how to combine the energy method with the proof of Lemma 2.1 in [6] to prove the following proposition:

**Proposition 2.1.** Let \((\rho, u)\) be a smooth enough solution of (1.1) on \([0, T^*]\). Then under the assumption of (1.7) and (1.8), we have (1.9) and

\[
\| u \|_{L^p_t(B_{2}^s)} + \| \nabla u \|_{L^r_t(B_{2}^s)} \leq C \| u_0 \|_{B_{2}^s} \quad \text{for } t < T^*.
\]

**Proof.** We first deduce from the classical theory on transport equation and (1.7) that there holds (1.9) for \( t < T^* \).

We now consider the coupled system of \((u_j, \nabla \pi_j)\) as follows:

\[
\begin{aligned}
\rho (\partial_t u_j + u \cdot \nabla u_j) - \Delta u_j + \nabla \pi_j &= 0, \\
\text{div} u_j &= 0, \\
u_j|_{t=0} &= \Delta_j u_0.
\end{aligned}
\]

Then we deduce from the uniqueness of local smooth solution to (1.1) that

\[
(2.3) \quad u = \sum_{j \in \mathbb{Z}} u_j, \quad \text{and} \quad \nabla \pi = \sum_{j \in \mathbb{Z}} \nabla \pi_j.
\]

By taking \( L^2 \) inner product of the momentum equation of (2.2) with \( u_j \) and using the transport equation of (1.1), we write

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_j|^2 dx + \int |\nabla u_j|^2 dx = 0.
\]

Integrating the above equation over \([0, t] \) leads to

\[
\frac{1}{2} \| \sqrt{\rho} u_j(t) \|_{L^2}^2 + \| \nabla u \|_{L^2_t(L^2)}^2 = \frac{1}{2} \| \sqrt{\rho} u_0 \|_{L^2}^2,
\]

and thus there holds

\[
(2.4) \quad \| u_j \|_{L^\infty_t(L^2)} + \| \nabla u \|_{L^2_t(L^2)} \leq C \| \Delta_j u_0 \|_{L^2} \leq C d_j 2^{-\frac{j}{2}} \| u_0 \|_{B_{2}^s}.
\]

Whereas taking \( L^2 \) inner product of the momentum equation of (2.2) with \( \partial_t u_j \) gives

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_j(t) \|_{L^2}^2 + \| \sqrt{\rho} \partial_t u_j \|_{L^2}^2 = - \int \rho u \cdot \nabla u_j \partial_t u_j \ dx
\]

\[
\leq C \| u \|_{L^2} \| \nabla u_j \|_{L^2} \| \sqrt{\rho} \partial_t u_j \|_{L^2}
\leq C \| u \|_{B_{2}^s} \| \nabla u_j \|_{L^2} \| \sqrt{\rho} \partial_t u_j \|_{L^2}.
\]
On the other hand, the momentum equation of (2.2) can be reformulated as

\begin{equation}
\begin{cases}
-\Delta u_j + \nabla \pi_j = -\rho \partial_t u_j - \rho u \cdot \nabla u_j, \\
\text{div} u_j = 0,
\end{cases}
\end{equation}

from which and the law of product (A.2), we infer

\begin{equation}
\|\nabla^2 u_j\|_{L^2} + \|\nabla \pi_j\|_{L^2} \leq C(\|\sqrt{\rho} \partial_t u_j\|_{L^2} + \|u \cdot \nabla u_j\|_{L^2}) \\
C(\|\sqrt{\rho} \partial_t u_j\|_{L^2} + \|u\|_{\dot{H}^\frac{1}{2}} \|\nabla^2 u_j\|_{L^2}).
\end{equation}

Let us denote

\begin{equation}
T_1^* \overset{\text{def}}{=} \sup\{\ t < T^* : \|u\|_{L^\infty_t(\dot{H}^\frac{1}{2})} \leq c_1 \}
\end{equation}

Then for \( c_1 \) being so small that \( Cc_1 \leq \frac{1}{2} \) and for \( t \leq T_1^* \), we have

\begin{equation}
\|\nabla^2 u_j\|_{L^2} + \|\nabla \pi_j\|_{L^2} \leq C(\|\sqrt{\rho} \partial_t u_j\|_{L^2} + \|\sqrt{\rho} \partial_t u_j\|_{L^2})
\end{equation}

Then for \( c_1 \) in (2.8) being so small that \( Cc_1 \leq \frac{1}{2} \), we achieve

\begin{equation}
\frac{d}{dt}\|\nabla u_j(t)\|_{L^2}^2 + \|\sqrt{\rho} \partial_t u_j\|_{L^2}^2 \leq 0 \quad \text{for} \ t \leq T_1^*,
\end{equation}

from which and (2.9), we deduce that

\begin{equation}
\|\nabla u_j\|_{L^\infty_t(L^2)} + \|\sqrt{\rho} \partial_t u_j, \nabla^2 u_j, \nabla \pi_j\|_{L^2_t(L^2)} \\
\leq C(\|\nabla u_j(0)\|_{L^2} + \|\nabla \Delta_j u_0\|_{L^2} \leq C d_j 2^{\frac{j}{2}} \|u_0\|_{B_1^\frac{1}{2}}).
\end{equation}

In view of (2.3), (2.4) and (2.10), we get, by applying Bernstein inequality, that

\begin{equation}
\|\Delta_j u\|_{L^\infty_t(L^2)} + \|\nabla \Delta_j u\|_{L^2_t(L^2)} \leq \sum_{j' \geq j} (\|\Delta_j u_{j'}\|_{L^\infty_t(L^2)} + \|\nabla \Delta_j u_{j'}\|_{L^2_t(L^2)}) \\
2^{-j} \sum_{j' \leq j} (\|\nabla \Delta_j u_{j'}\|_{L^\infty_t(L^2)} + \|\nabla \Delta_j u_{j'}\|_{L^2_t(L^2)}) \\
\leq d_j 2^{-\frac{j}{2}} \|u_0\|_{B_1^\frac{1}{2}}.
\end{equation}

This implies that (2.1) holds for \( t \leq T_1^* \). Then taking \( \varepsilon_0 \) in (1.8) so small that \( C\|u_0\|_{B_2^\frac{1}{4}} \leq C\varepsilon_0 \leq \frac{1}{2} \), for \( c_1 \) given by (2.8), we get, by using a continuous argument, that \( T_1^* \) determined by (2.8) equals any number smaller than \( T^* \). This in turn shows (2.1). \( \square \)

**Remark 2.1.** We remark that similar idea was first used by Hmidi and Keraani [14] for two dimensional incompressible Euler system, which also works (without change) for the transport diffusion equation. In the inhomogeneous context, similar idea was used by Liao and the author [23] in order to propagate fractional Besov regularities for the velocity field of the two dimensional incompressible inhomogeneous Navier-Stokes system.
Remark 2.2. By virtue of (2.1), we deduce from the classical theory of inhomogeneous Navier-Stokes system that $T^* = \infty$.

Along the same line to the proof of Proposition 2.1, we can also show that

**Proposition 2.2.** Under the assumptions of Proposition 2.1, we have
\[
\|\sqrt{t} \nabla u\|_{\dot{L}^2_t(B^{1/2}_x)} + \|t^{1/2} u_t\|_{\dot{L}^2_t(B^{1/2}_x)} \leq C\|u_0\|_{B^{1/2}_x} \quad \text{for any } t > 0.
\]

and

**Proposition 2.3.** Under the assumptions of Proposition 2.1, we have
\[
\|t \nabla u_t\|_{\dot{L}^2_t(B^{1/2}_x)} + \|t^{1/2} u_{tt}\|_{\dot{L}^2_t(B^{1/2}_x)} \leq C\|u_0\|_{B^{1/2}_x} \quad \text{for any } t > 0.
\]

The complete proof of the above two propositions will be presented in Section 3.

Now we are in a position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By mollifying the initial data $(\rho_0, u_0)$, we deduce from the classical theory of inhomogeneous incompressible Navier-Stokes system that (1.1) has a unique local solution $(\rho, u^\varepsilon)$ on $[0, T^*_\varepsilon]$. Moreover, we can show that (2.1), (2.11) and (2.12) hold for $(\rho^\varepsilon, u^\varepsilon)$. In particular, Proposition 2.1 ensures $T^*_\varepsilon = \infty$ provided that $\varepsilon_0$ is small enough in (1.8). Then exactly along the same line to proof of Theorem 1.2 in [26], we can complete the existence part of Theorem 1.2 by using the uniform estimates (2.1), (2.11), (2.12) and (3.25) for $(\rho^\varepsilon, u^\varepsilon)$ and a standard compactness argument, which we omit details here. In order to prove (1.10), it remains to show that
\[
\|t u_t\|_{L^\infty_t(B^{1/2}_x)} \leq C\|u_0\|_{B^{1/2}_x}.
\]

Indeed it follows from the law of product in Besov spaces and (2.11), (2.12) that
\[
\|t u_t\|_{L^\infty_t(B^{1/2}_x)} \leq \|t D_t u\|_{L^\infty_t(B^{1/2}_x)} + C\|\sqrt{t} u\|_{L^\infty_t(B^{1/2}_x)} \|\sqrt{t} \nabla u\|_{L^\infty_t(B^{1/2}_x)}
\]
\[
\leq C\|u_0\|_{B^{1/2}_x}.
\]

This completes the proof of Theorem 1.2. $\square$

In order to prove Theorem 1.3, we denote $u = v + e^{t\Delta} u_0$. Then by virtue of (1.11), $(a, v, \nabla \pi)$ verifies
\[
\begin{aligned}
\partial_t a + (v + e^{t\Delta} u_0) \cdot \nabla a &= 0, \\
\partial_t v + (v + e^{t\Delta} u_0) \cdot \nabla v + v \cdot \nabla e^{t\Delta} u_0 - (1 + a)(\Delta v - \nabla \pi) &= -e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} u_0 + a \Delta e^{t\Delta} u_0, \\
\text{div } v &= 0, \\
v|_{t=0} &= 0.
\end{aligned}
\]

Now let $(a, v, \nabla \pi)$ be a smooth enough solution of (2.15) on $[0, T^*]$, we construct $(v_j, \nabla \pi_j)$ via
\[
\begin{aligned}
\rho \partial_t v_j + \rho (v + e^{t\Delta} u_0) \cdot \nabla v_j + \rho v_j \cdot \nabla e^{t\Delta} u_0 - \Delta v_j + \nabla \pi_j &= \rho F_j + \rho a e^{t\Delta} \Delta_j u_0 \\
F_j &= \begin{pmatrix}
-e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} \Delta_j u_0^h \\
-e^{t\Delta} \Delta_j u_0^h \cdot \nabla e^{t\Delta} u_0^3 + e^{t\Delta} u_0^3 e^{t\Delta} \Delta_j \text{div} h u_0^h
\end{pmatrix}, \\
\text{div } v_j &= 0, \\
v_j|_{t=0} &= 0.
\end{aligned}
\]

Here and in Section 3, we always denote $v^h \overset{\text{def}}{=} (v_1, v_2), v = (v^h, v^3)$ and $\rho \overset{\text{def}}{=} \frac{1}{1+a}$.
Due to uniqueness result for smooth enough solution of (2.15) and \( \text{div} \, u_0 = 0 \), we have

\[
(2.17) \quad v = \sum_{j \in \mathbb{Z}} v_j \quad \text{and} \quad \nabla \pi = \sum_{j \in \mathbb{Z}} \nabla \pi_j.
\]

Then the proof of Theorem 1.3 consists of the following three propositions, the proof of which will be presented in Section 4:

**Proposition 2.4.** Let \((\rho, v)\) be a smooth enough solution of (2.15) on \([0, T^*]\). Then under the assumption of (1.12), we have

\[
(2.18) \quad \|v\|_{\bar{L}^\infty(B_{\frac{1}{2}}^2)} + \|\nabla v\|_{\bar{L}^2(\mathbb{R}^3)} \leq C\eta \quad \text{for} \quad t < T^*,
\]

where \(\eta\) is given by (1.12).

**Proposition 2.5.** Under the assumption of Proposition 2.4, we have

\[
(2.19) \quad \|\sqrt{t} \nabla v\|_{\bar{L}^\infty(B_{\frac{1}{2}}^2)} + \|\sqrt{t} v_t\|_{\bar{L}^2(\mathbb{R}^3)} \leq C\eta \quad \text{for any} \quad t > 0.
\]

**Proposition 2.6.** Under the assumptions of Proposition 2.4, for any \(t > 0\), we have

\[
(2.20) \quad \|t D_t v\|_{\bar{L}^\infty(B_{\frac{1}{2}}^2)} + \|t \nabla D_t v\|_{\bar{L}^2(\mathbb{R}^3)} \leq C\eta.
\]

**Proof of Theorem 1.3.** With the a priori estimates (2.18), (2.19) and (2.20), we can follow the same line as the proof of Theorem 1 of [16] to complete the existence part of Theorem 1.3. Moreover, there holds (1.13).

**Remark 2.3.** We emphasize that we crucially used the divergence free condition of \(u_0\) in the construction of \((v_j, \nabla \pi_j)\) in (2.16). We comment that the proof of Theorem 1.3 here is more concise than that of Theorem 1 in [16], where the authors first write the integral formulation for the velocity field \(u\) of (1.11), then perform the estimate of \(u^h\) and finally the estimate of \(u^3\) through the application of the maximal regularity estimate for heat-semi-group. I am not sure if we can go through the proof of Theorem 1.3 by performing energy estimate for \(u_j^h\) and then for the energy estimate of \(u_j^3\) for solutions \((u_j, \nabla \pi_j)\) of (2.2).

Finally, let us turn to the proof of Theorem 1.4, which relies on the following propositions:

**Proposition 2.7.** Let \((\rho, u)\) be a smooth enough solution of (1.1) on \([0, T^*]\). Then for \(\gamma \in [0, 1/4]\), there is a positive constant \(c_\gamma\) so that

\[
(2.21) \quad T^* > T_\gamma \overset{\text{def}}{=} c_\gamma \|u_0\|_{\dot{H}_{\frac{3}{4} + 2\gamma}^{-\frac{1}{4}}}.
\]

and for \(t \leq T_\gamma\), there holds

\[
(2.22) \quad \|u\|_{\bar{L}^\infty_t(\dot{H}_{\frac{3}{4} + 2\gamma})} + \|\nabla u\|_{\bar{L}^2_t(\dot{H}_{\frac{3}{4} + 2\gamma})} \leq C\|u_0\|_{\dot{H}_{\frac{3}{4} + 2\gamma}}.
\]

**Proposition 2.8.** Under the assumptions of Proposition 2.7, we have

\[
(2.23) \quad \|\sqrt{t} \nabla u\|_{\bar{L}^\infty_t(\dot{H}_{\frac{3}{4} + 2\gamma})} + \|\sqrt{t} u_t\|_{\bar{L}^2_t(\dot{H}_{\frac{3}{4} + 2\gamma})} \leq C\|u_0\|_{\dot{H}_{\frac{3}{4} + 2\gamma}} \quad \text{for} \quad t \leq T_\gamma.
\]

**Proposition 2.9.** Under the assumptions of Proposition 2.7, we have

\[
(2.24) \quad \|t D_t u\|_{\bar{L}^\infty_t(\dot{H}_{\frac{3}{4} + 2\gamma})} + \|t \nabla D_t u\|_{\bar{L}^2_t(\dot{H}_{\frac{3}{4} + 2\gamma})} \leq C\|u_0\|_{\dot{H}_{\frac{3}{4} + 2\gamma}} \quad \text{for} \quad t \leq T_\gamma.
\]

By summing up Proposition 2.7 to Proposition 2.9, we conclude the proof of Theorem 1.4. The detailed proof of Propositions 2.7 to 2.9 will be presented in Section 5.

For the convenience of the readers, we shall collect some basic facts on Littlewood-Paley theory in the Appendix A.
3. The Proof of Theorem 1.2

The purpose of this section is to present the proof of Propositions 2.2 to 2.3. We first deduce the following corollary from the proof of Proposition 2.1 in Section 2.

**Corollary 3.1.** Let \((u_j, \nabla \pi_j)\) be determined by (2.2), we have

\[
\|\sqrt{t} \nabla u_j\|_{L_t^\infty(L^2)} + \|\sqrt{t} \partial_t u_j, \nabla^2 u_j, \nabla \pi_j\|_{L_t^2(L^2)} \leq C d_j 2^{-\frac{1}{2}} \|u_0\|_{B^{\frac{1}{2}}} \quad \text{for any } t > 0.
\]

**Proof.** Indeed thanks to (2.5), we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u_j(t)\|^2_{L^2} + \|\sqrt{t} \partial_t u_j\|^2_{L^2} \leq C \|u\|_{L^\infty} \|\nabla u_j\|_{L^2} \|\sqrt{t} \partial_t u_j\|_{L^2}
\]

\[
\leq C \|u\|^2_{B^{\frac{1}{2}}} \|\nabla u_j\|^2_{L^2} + \frac{1}{2} \|\sqrt{t} \partial_t u_j\|^2_{L^2}.
\]

Multiplying the above inequality by \(t\) gives rise to

\[
\frac{d}{dt} \left( t \|\nabla u_j(t)\|^2_{L^2} \right) + t \|\sqrt{t} \partial_t u_j\|^2_{L^2} \leq \|\nabla u_j\|^2_{L^2} + C \|u\|^2_{B^{\frac{1}{2}}} t \|\nabla u_j\|^2_{L^2}.
\]

Applying Gronwall’s inequality and then using (2.1) and (2.4) that

\[
\|\sqrt{t} \nabla u_j\|_{L_t^\infty(L^2)} + \|\sqrt{t} \partial_t u_j\|_{L_t^2(L^2)} \leq \|\nabla u_j\|^2_{L_t^2(L^2)} \exp\left( C \|u\|^2_{L_t^2(B^{\frac{1}{2}})} \right)
\]

\[
\leq C d_j 2^{-\frac{1}{2}} \|u_0\|^2_{B^{\frac{1}{2}}},
\]

which together with (2.9) implies (3.1). \(\square\)

**Proof of Proposition 2.2.** Applying \(\partial_t\) to the momentum equation of (2.2) yields

\[
\rho \partial_t^2 u_j + \rho u \cdot \nabla \partial_t u_j - \Delta \partial_t u_j + \nabla \partial_t \pi_j = -\rho_t D_t u_j - \rho u_t \cdot \nabla u_j.
\]

Here and in all that follows, we always denote \(D_t \overset{\text{def}}{=} \partial_t + u \cdot \nabla\) to be the material derivative.

By taking \(L^2\) inner product of the above equation with \(\partial_t u_j\) and making use of the transport equation of (1.1), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\partial_t u_j|^2 dx + \|\nabla \partial_t u_j\|^2_{L_t^2(L^2)} = -\int_{\mathbb{R}^3} \rho_t D_t u_j |\partial_t u_j| dx - \int_{\mathbb{R}^3} \rho u_t \cdot \nabla u_j |\partial_t u_j| dx.
\]

Let us now handle term by term above.

- **Estimate for \(\int_{\mathbb{R}^3} \rho_t D_t u_j |\partial_t u_j| dx\).**

  By virtue of the transport equation of (1.1), we get, by using integration by parts, that

  \[
  -\int_{\mathbb{R}^3} \rho_t |\partial_t u_j| \partial_t u_j dx = \int_{\mathbb{R}^3} u \cdot \nabla \rho |\partial_t u_j|^2 dx
  \]

  \[
  = -2 \int_{\mathbb{R}^3} \rho u \cdot \nabla \partial_t u_j |\partial_t u_j| dx,
  \]

  which leads to

  \[
  \left| \int_{\mathbb{R}^3} \rho_t |\partial_t u_j| \partial_t u_j dx \right| \leq C \|u\|_{L^\infty} \|\sqrt{t} \partial_t u_j\|_{L^2} \|\nabla \partial_t u_j\|_{L^2}
  \]

  \[
  \leq C \|u\|^2_{B^{\frac{1}{2}}} \|\sqrt{t} \partial_t u_j\|^2_{L^2} + \frac{1}{6} \|\nabla \partial_t u_j\|^2_{L^2}.
  \]
Once again we get, by using integration by parts, that

\[- \int_{\mathbb{R}^3} \rho u_\cdot \nabla u_j \partial_t u_j \, dx = - \int_{\mathbb{R}^3} (pu \cdot \nabla u) \cdot \nabla u_j \partial_t u_j \, dx - \int_{\mathbb{R}^3} \rho (u \otimes u) : \nabla^2 u_j \partial_t u_j \, dx - \int_{\mathbb{R}^3} u \cdot \nabla u_j \rho u \cdot \nabla \partial_t u_j \, dx.\]

It follows from (2.9) that

\[
|\int_{\mathbb{R}^3} (pu \cdot \nabla u) \cdot \nabla u_j \partial_t u_j \, dx| \leq C \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|\nabla u_j\|_{L^6} \|\sqrt{\rho} \partial_t u_j\|_{L^2} \\
\leq C \|u\|^2_{B^{3/2}} \|\nabla^2 u_j\|_{L^2} \|\sqrt{\rho} \partial_t u_j\|_{L^2} \\
\leq C \|u\|^2_{B^{3/2}} \|\sqrt{\rho} \partial_t u_j\|^2_{L^2}.
\]

The same estimate holds for \( \int_{\mathbb{R}^3} \rho (u \otimes u) : \nabla^2 u_j \partial_t u_j \, dx \). Moreover, along the same line, we have

\[
|\int_{\mathbb{R}^3} u \cdot \nabla u_j \rho u \cdot \nabla \partial_t u_j \, dx| \leq C \|u\|_{L^\infty} \|\nabla u_j\|_{L^6} \|\nabla \partial_t u_j\|_{L^2} \\
= C \|u\|_{B^{3/2}} \|u\|_{B^{3/2}} \|\nabla^2 u_j\|_{L^2} \|\nabla \partial_t u_j\|_{L^2} \\
\leq C \|u\|^2_{B^{3/2}} \|\nabla^2 u_j\|^2_{L^2} + \frac{1}{6} \|\nabla \partial_t u_j\|^2_{L^2}.
\]

As a result, it comes out

\[
|\int_{\mathbb{R}^3} \rho u \cdot \nabla u_j \partial_t u_j \, dx| \leq C \left(1 + \|u\|^2_{B^{3/2}}\right) \|u\|^2_{B^{3/2}} \|\sqrt{\rho} \partial_t u_j\|^2_{L^2} + \frac{1}{6} \|\nabla \partial_t u_j\|^2_{L^2}.
\]

*Estimate for \( \int_{\mathbb{R}^3} pu \cdot \nabla u_j \partial_t u_j \, dx \)

\[
|\int_{\mathbb{R}^3} pu \cdot \nabla u_j \partial_t u_j \, dx| \leq C \|u_t\|_{L^2} \|\nabla u_j\|_{L^3} \|\partial_t u_j\|_{L^6} \\
\leq C \|u_t\|_{L^2} \|\nabla u_j\|^2_{L^2} \|\nabla^2 u_j\|_{L^2} \|\nabla \partial_t u_j\|_{L^2} \\
\leq C \|u_t\|^2_{L^2} \|\nabla u_j\|_{L^2} \|\nabla^2 u_j\|_{L^2} \|\sqrt{\rho} \partial_t u_j\|_{L^2} + \frac{1}{6} \|\nabla \partial_t u_j\|^2_{L^2} \\
\leq C \|u_t\|^2_{L^2} \|\nabla u_j\|_{L^2} \|\sqrt{\rho} \partial_t u_j\|_{L^2} + \frac{1}{6} \|\nabla \partial_t u_j\|^2_{L^2},
\]

where in the last step, we used (2.9).

Inserting the above estimates into (3.3) gives rise to

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \rho |\partial_t u_j|^2 \, dx + \|\nabla \partial_t u_j\|^2_{L^2(L^2)} \leq C \left(1 + \|u\|^2_{B^{3/2}}\right) \|u\|^2_{B^{3/2}} \|\sqrt{\rho} \partial_t u_j\|^2_{L^2} \\
+ \|u_t\|^2_{L^2} \|\nabla u_j\|_{L^2} \|\sqrt{\rho} \partial_t u_j\|_{L^2}.
\]

Multiplying the above inequality by \( t \) and using (2.1) yields

\[
\frac{d}{dt} \|\sqrt{t} \rho \partial_t u_j\|^2_{L^2} + \|\sqrt{t} \nabla \partial_t u_j\|^2_{L^2(L^2)} \leq \|\sqrt{\rho} \partial_t u_j\|^2_{L^2} + C \|t^{3/2} u_t\|^2_{L^2} \|\nabla u_j\|_{L^2} \\
+ C \left(\|u\|^2_{B^{3/2}} + \|t^{1/2} u_t\|^2_{L^2}\right) \|\sqrt{\rho} \partial_t u_j\|^2_{L^2}.
\]
By applying Gronwall’s inequality and using (1.9), we achieve
\begin{equation}
\| \sqrt{t} \partial_t u_j \|_{L^\infty \times L^2} + \| \sqrt{t} \nabla \partial_t u_j \|_{L^2(L^2)}^2 \leq C \exp \left( \int_0^t (\|u\|_{B^{\frac{1}{2}}}^2 + \|t^{\frac{1}{2}} u_t\|_{L^2}^2) \, dt' \right)
\times \left( \| \sqrt{t} \partial_t u_j \|_{L^2(L^2)}^2 + \| t^{\frac{1}{4}} u_t \|_{L^2(L^2)}^2 \| \nabla u_j \|_{L^\infty(L^2)}^2 \right).
\end{equation}

On the other hand, we deduce from (2.10) and (3.1) that
\begin{align*}
\| t^{\frac{1}{4}} \partial_t u_j \|_{L^2(L^2)} & \leq \| \partial_t u_j \|_{L^2(L^2)}^\frac{1}{2} \| \sqrt{t} \partial_t u_j \|_{L^2(L^2)}^\frac{1}{2} \\
& \leq C d_j \| u_0 \|_{B^{\frac{1}{2}}},
\end{align*}
so that it follows from (2.3)
\begin{equation}
\| t^{\frac{1}{4}} \partial_t u \|_{L^2(L^2)} \leq \sum_{j \in \mathbb{Z}} \| t^{\frac{1}{4}} \partial_t u_j \|_{L^2(L^2)} \leq C \| u_0 \|_{B^{\frac{1}{2}}},
\end{equation}

Therefore, by virtue of (2.1), (2.10) and (3.6), we obtain
\begin{align}
\| \sqrt{t} \partial_t u_j \|_{L^\infty(L^2)} + \| \sqrt{t} \nabla \partial_t u_j \|_{L^2(L^2)} \leq & C d_2 2^{\frac{j}{4}} \exp \left( C \| u_0 \|_{B^{\frac{1}{2}}}^2 \right) \| u_0 \|_{B^{\frac{1}{2}}} \\
\leq & C d_2 2^{\frac{j}{4}} \| u_0 \|_{B^{\frac{1}{2}}},
\end{align}
from which and (2.9), we achieve
\begin{equation}
\| \sqrt{t} \nabla^2 u \|_{L^\infty(L^2)} \leq C d_2 2^{\frac{j}{2}} \| u_0 \|_{B^{\frac{1}{2}}}.
\end{equation}

With (3.1), (3.7) and (3.8), we get, by a similar derivation of (2.1), that (2.11) holds for any $t > 0$.

**Corollary 3.2.** Under the assumptions of Proposition 2.1, for any $t > 0$, we have
\begin{equation}
\| t D_t u_j \|_{L^\infty(L^2)} + \| t \nabla D_t u_j \|_{L^2(L^2)} \leq C d_j 2^{-\frac{j}{2}} \| u_0 \|_{B^{\frac{1}{2}}},
\end{equation}

and
\begin{equation}
\| \sqrt{t} \nabla^2 u \|_{L^2(L^2)} \leq C \| u_0 \|_{B^{\frac{1}{2}}}.
\end{equation}

**Proof.** By multiplying (3.4) by $t^2$, we obtain
\[
\frac{d}{dt} \| t \sqrt{\nu} \partial_t u_j \|_{L^2}^2 + \| \sqrt{t} \nabla \partial_t u_j \|_{L^2(L^2)}^2 \leq 2 \| \sqrt{\nu} \partial_t u_j \|_{L^2(L^2)}^2 + C \| t^{\frac{1}{2}} u_t \|_{L^2}(L^2) \| \sqrt{t} \nabla u_j \|_{L^2(L^2)}^2 \\
+ C (\| u \|_{B^{\frac{1}{2}}}^2 + \| \sqrt{t} u_t \|_{L^2(L^2)}^2) \| \sqrt{\nu} \partial_t u_j \|_{L^2(L^2)}^2.
\]

Applying Gronwall’s inequality gives
\begin{equation}
\| t \partial_t u_j \|_{L^\infty(L^2)}^2 + \| t \nabla \partial_t u_j \|_{L^2(L^2)}^2 \leq \exp \left( C \int_0^t (\|u\|_{B^{\frac{1}{2}}}^2 + \|t^{\frac{1}{2}} u_t\|_{L^2}^2) \, dt' \right)
\times (\| t \partial_t u_j \|_{L^2(L^2)}^2 + C \| t^{\frac{1}{4}} u_t \|_{L^2(L^2)}^2 \| \sqrt{t} \nabla u_j \|_{L^\infty(L^2)}^2),
\end{equation}
which together with (3.1) and (3.6) implies
\begin{equation}
\| t \partial_t u_j \|_{L^\infty(L^2)}^2 + \| t \nabla \partial_t u_j \|_{L^2(L^2)}^2 \leq C d_2 2^{-j} \| u_0 \|_{B^{\frac{1}{2}}}^2.
\end{equation}
Then by using (2.11) and (3.1) once again, we get
\[ ||tD_t u_j||_{L^\infty_t(L^2)} \leq ||t\partial_t u_j||_{L^\infty_t(L^2)} + ||\sqrt{t}\nu||_{L^\infty_t(B^{\frac{1}{2}})} ||\sqrt{t}\nabla u_j||_{L^\infty_t(L^2)} \]
\[ \leq Cd_j 2^{-\frac{j}{4}} ||u_0||_{B^{\frac{1}{2}}}. \]
Whereas by applying the law of product in Sobolev space, (A.2), and using (2.9) gives rise to
\[ ||t\nabla D_t u_j||_{L^2_t(L^2)} \leq ||t\nabla \partial_t u_j||_{L^2_t(L^2)} + ||u||_{L^2_t(B^{\frac{1}{2}})} ||t\nabla^2 u_j||_{L^\infty_t(L^2)} \]
\[ \leq ||t\nabla \partial_t u_j||_{L^2_t(L^2)} + C ||u||_{L^2_t(B^{\frac{1}{2}})} ||\partial_t u_j||_{L^\infty_t(L^2)} \]
\[ \leq Cd_j 2^{-\frac{j}{4}} ||u_0||_{B^{\frac{1}{2}}}, \]
which implies (3.9).

On the other hand, note from (1.1) that
\[ \begin{cases} -\Delta u + \nabla \pi = -\rho(\partial_t u + u \cdot \nabla u), \\ \div u = 0, \end{cases} \]
from which and regularity theory of Stokes operator, we deduce from (2.1) and (2.11) that
\[ ||\sqrt{t}\nabla^2 u||_{L^2_t(L^3)} \leq C (||\sqrt{t}\partial_t u||_{L^2_t(L^3)} + ||u||_{L^2_t(L^\infty)} ||\sqrt{t}\nabla u||_{L^\infty_t(L^3)}) \]
\[ \leq C (||\sqrt{t}\partial_t u||_{L^2_t(B^{\frac{1}{2}})} + ||u||_{L^2_t(B^{\frac{1}{2}})} ||\sqrt{t}\nabla u||_{L^\infty_t(B^{\frac{1}{2}})}) \]
\[ \leq C ||u_0||_{B^{\frac{1}{2}}}, \]
which leads to (3.10). \(\square\)

**Proof of Proposition 2.3.** It is easy to observe that
\[ [D_t; \nabla] f = -\nabla u \cdot \nabla f, \quad \text{and} \]
\[ [D_t; \Delta] f = -\Delta u \cdot \nabla f - 2 \sum_{i=1}^{3} \partial_i u \cdot \nabla \partial_i f. \]
Then we get, by applying the operator \(D_t\) to the momentum equation of (2.2), that
\[ \rho D_t^2 u_j - \Delta D_t u_j + \nabla D_t \pi_j = -\Delta u \cdot \nabla u_j - 2 \sum_{i=1}^{3} \partial_i u \cdot \nabla \partial_i u_j + \nabla u \cdot \nabla \pi_j \overset{\text{def}}{=} f_j. \]
Moreover, due to \(\div u_j = 0\), we have
\[ \div D_t u_j = \sum_{i=1}^{3} \partial_i u \cdot \nabla u_j. \]
Then we get, by taking space divergence operator to (3.13), that
\[ \Delta D_t \pi_j = \sum_{i=1}^{3} \Delta (\partial_i u \cdot \nabla u_j) - \div (\rho D_t^2 u_j) + \div f_j, \]
from which, we infer
\[ ||\nabla D_t \pi_j||_{L^2} \leq C \left( ||\nabla \text{Tr}(\nabla u \nabla u_j)||_{L^2} + ||\rho D_t^2 u_j||_{L^2} + ||f_j||_{L^2} \right), \]
which together with (3.13) ensures that \(||\nabla^2 D_t u_j||_{L^2}\) shares the same estimate.
By using integration by parts, one has
\[
(3.17)
\]
\[
Then we have
\[
(3.15)
\]
so that there holds
\[
(3.16)
\]
which together with (2.1), (3.7) and (3.8) ensures that
\[
(3.19)
\]
\[
The same estimate holds for \( \| \sqrt{t} \nabla u_j \|_{L^2} \). Before proceeding, we notice from (2.6) that
\[
\left\| \sqrt{t}(\nabla^2 u_j, \nabla \pi_j) \right\|_{L^2_t(L^6)} \leq C \left( \| \sqrt{t} \nabla \partial_t u_j \|_{L^2_t(L^3)} + \| \sqrt{t} u \cdot \nabla u_j \|_{L^2_t(L^6)} \right) \\
\leq C \left( \| \sqrt{t} \nabla \partial_t u_j \|_{L^2_t(L^3)} + \| u \|_{L^\infty_t(L^\infty)} \| \sqrt{t} \nabla^2 u_j \|_{L^2_t(L^6)} \right),
\]
which together with (2.1), (3.7) and (3.8) ensures that
\[
(3.16)
\]
Then we have
\[
\left\| t \nabla \text{Tr}(\nabla u \nabla u_j) \right\|_{L^2_t(L^3)} \leq \| \sqrt{t} \nabla^2 u \|_{L^2_t(L^3)} \| \sqrt{t} \nabla u_j \|_{L^\infty_t(L^6)} + \| \sqrt{t} \nabla u \|_{L^\infty_t(L^3)} \| \sqrt{t} \nabla^2 u_j \|_{L^2_t(L^6)} \\
\leq C \left( \| \sqrt{t} \nabla^2 u \|_{L^2_t(L^3)} \| \sqrt{t} \nabla^2 u_j \|_{L^\infty_t(L^3)} \right) \\
+ \| \sqrt{t} \nabla u \|_{L^\infty_t(B^{3/2})} \| \sqrt{t} \nabla^2 u_j \|_{L^2_t(L^6)},
\]
from which, (3.8), (10) and (13.16), we infer
\[
(3.17)
\]
The same estimate holds for \( \| t \Delta u \cdot \nabla u_j \|_{L^2_t(L^2)} \) and \( \sum_{i=1}^3 \| t \partial_i u \cdot \nabla \partial_i u_j \|_{L^2_t(L^2)} \). Furthermore, we deduce from (3.16) that
\[
\left\| t \nabla u \cdot \nabla \pi_j \right\|_{L^2_t(L^2)} \leq \| \sqrt{t} \nabla u \|_{L^\infty_t(L^3)} \| \sqrt{t} \nabla \pi_j \|_{L^2_t(L^6)} \\
\leq C_d \| t \nabla u \|_{L^\infty_t(B^{3/2})} \| u_0 \|_{B^{3/2}},
\]
This along with (2.11) ensures that
\[
(3.18)
\]
and thus
\[
(3.19)
\]
\[
\text{Estimate for } \int_0^t \int_{\mathbb{R}^3} \nabla D_t u_j \| \nabla; D_t \| D_t u_j dx dt'.
\]
It is easy to observe that
\[
\left| \int_{\mathbb{R}^3} \nabla D_t u_j [\nabla; D_t] D_t u_j \, dx \right| \leq \|\nabla u\|_{L^3} \|\nabla D_t u_j\|^2_{L^3} \\
\leq C \|\nabla u\|_{L^3} \|\nabla D_t u_j\|_{L^2} \|\nabla^2 D_t u_j\|_{L^2}.
\]
Yet we deduce from (3.14), (3.17) and (3.18) that
\[
\|t \nabla^2 D_t u_j\|_{L_t^p(L^p)} + \|t \nabla D_t \pi_j\|_{L_t^p(L^p)} \\
\leq C \left( \|t \nabla \text{Tr}(\nabla u u_j)\|_{L_t^p(L^p)} + \|t \rho D_t^2 u_j\|_{L_t^p(L^p)} + \|t f_j\|_{L_t^p(L^p)} \right) \\
\leq C \left( d_j 2^j \|u_0\|_{B^{\frac{3}{2}}} + \|t \sqrt{\rho} D_t^2 u_j\|_{L_t^p(L^p)} \right).
\]
Then we obtain
\[
\left| \int_0^t (t')^2 \int_{\mathbb{R}^3} \nabla D_t u_j [\nabla; D_t] D_t u_j \, dx \, dt' \right| \leq \frac{1}{6} \|t \sqrt{\rho} D_t^2 u_j\|^2_{L_t^p(L^p)} \\
+ C \left( d_j 2^j \|u_0\|^2_{B^{\frac{3}{2}}} + \int_0^t \|\nabla u\|^2_{B^{\frac{3}{2}}} \|t \nabla D_t u_j\|^2_{L_t^p(L^p)} \right).
\]

**Estimate for** \( \int_0^t (t')^2 \int_{\mathbb{R}^3} \nabla D_t \pi_j [D_t^2 u_j] \, dx \, dt' \).

We first get, by using integration by parts, that
\[
\int_{\mathbb{R}^3} \nabla D_t \pi_j [D_t^2 u_j] \, dx = - \int_{\mathbb{R}^3} D_t \pi_j [\partial_t \text{div} D_t u_j] \, dx + \int_{\mathbb{R}^3} \nabla D_t \pi_j [u \cdot \nabla D_t u_j] \, dx \\
= - \int_{\mathbb{R}^3} D_t \pi_j [\partial_t (\nabla u \nabla u_j)] \, dx + \int_{\mathbb{R}^3} \nabla D_t \pi_j [u \cdot \nabla D_t u_j] \, dx \\
= \int_{\mathbb{R}^3} \nabla D_t \pi_j [\partial_t u \cdot \nabla u_j] \, dx + \int_{\mathbb{R}^3} \nabla D_t \pi_j [\partial_t u_j \cdot \nabla u] \, dx \\
+ \int_{\mathbb{R}^3} \nabla D_t \pi_j [u \cdot \nabla D_t u_j] \, dx.
\]

Next we handle term by term above. We first get, by applying product laws in Besov spaces, (A.2), that
\[
\left| \int_0^t (t')^2 \int_{\mathbb{R}^3} \nabla D_t \pi_j [\partial_t u \cdot \nabla u_j] \, dx \, dt' \right| \leq \int_0^t (t')^2 \|\nabla D_t \pi_j\|_{L^p} \|\partial_t u \cdot \nabla u_j\|_{L^p} \, dt' \\
\leq \|t \nabla D_t \pi_j\|_{L_t^p(L^p)} \|\sqrt{\rho} \partial_t u_j\|_{L_t^p(B^{\frac{3}{2}})} \|\sqrt{\rho} \nabla u_j\|_{L_t^p(B^{\frac{3}{2}})}.
\]

which together (2.11), (3.8) and (3.20) ensures that
\[
\left| \int_0^t (t')^2 \int_{\mathbb{R}^3} \nabla D_t \pi_j [\partial_t u \cdot \nabla u_j] \, dx \, dt' \right| \leq \frac{1}{18} \|t \sqrt{\rho} D_t^2 u_j\|^2_{L_t^p(L^p)} + C d_j 2^j \|u_0\|^2_{B^{\frac{3}{2}}}.
\]
Along the same line, we get, by applying (2.11), (3.7) and (3.20), that
\[
\left| \int_0^t (t')^2 \int_{\mathbb{R}^3} \nabla D_t \pi_j [\partial_t u_j \cdot \nabla u] \, dx \, dt' \right| \leq \|t \nabla D_t \pi_j\|_{L_t^p(L^p)} \|\sqrt{\rho} \partial_t u_j\|_{L_t^p(L^p)} \|\sqrt{\rho} \nabla u\|_{L_t^p(B^{\frac{3}{2}})} \\
\leq \frac{1}{18} \|t \sqrt{\rho} D_t^2 u_j\|^2_{L_t^p(L^p)} + C d_j 2^j \|u_0\|^2_{B^{\frac{3}{2}}}.
\]
Remark 3.2. We remark that the advantage of applying the material derivative, $D_t$, instead of $\partial_t$, to the momentum equation of (1.1) is that $D_t \rho = 0$. Indeed the energy estimate for $D_t u$ was first performed by Hoff in [15] for the isentropic compressible Navier-Stokes system. For
the inhomogeneous incompressible case, similar estimate was first obtained by Liao and the author in [23].

4. THE PROOF OF THEOREM 1.3

The goal of this section is to present the proof of Propositions 2.4 to 2.6.

Lemma 4.1. Let \((a, v, \nabla \pi)\) be a smooth enough solution of (2.15) on \([0, T^*]\). Let \((v_j, \nabla \pi_j)\) be determined by (2.16). Then for \(t < T^*\), one has

\[
\|v_j\|_{L^\infty_t(L^2)}^2 + \|\nabla v_j\|_{L^2_t(L^2)}^2 \leq C d_j^2 2^{-j} \left( \|u_0^h\|_{B^{\frac{3}{4}}_{2,2}}^2 + \|a_0\|_{L^\infty}^2 \right) \exp \left( C \|u_0\|_{B^{\frac{3}{4}}_{2,2}}^2 \right).
\]

Proof. We first get, by taking \(L^2\) inner product of the momentum equation of (2.16) with \(v_j\) and using the transport equation of (2.15), that

\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} v_j(t)\|_{L^2}^2 + \|\nabla v_j\|_{L^2}^2 = - \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t\Delta} u_0 |v_j| dx
\]

\[
- \int_{\mathbb{R}^3} \rho e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} \Delta_j u_0^h |v_j| dx + \int_{\mathbb{R}^3} \rho a \Delta e^{t\Delta} \Delta_j u_0 |v_j| dx
\]

\[
- \int_{\mathbb{R}^3} \rho e^{t\Delta} \Delta_j u_0^h \cdot \nabla e^{t\Delta} u_0^3 - e^{t\Delta} u_0^3 e^{t\Delta} \Delta_j \text{div} u_0^h |v_j|^3 dx.
\]

Next let us estimate term by term above.

- **Estimate for** \(\int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t\Delta} u_0 |v_j| dx\).

\[
\left| \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t\Delta} u_0 |v_j| dx \right| \leq \|e^{t\Delta} u_0\|_{L^3} \|v_j\|_{L^3}^2
\]

\[
\leq C \|e^{t\Delta} u_0\|_{B^{\frac{3}{4}}_{2,2}} \|v_j\|_{L^2} \|\nabla v_j\|_{L^2}
\]

\[
\leq C \|e^{t\Delta} u_0\|_{B^{\frac{3}{4}}_{2,2}}^2 \|v_j\|_{L^2}^2 + \frac{1}{2} \|\nabla v_j\|_{L^2}^2.
\]

- **Estimate for** \(\int_{\mathbb{R}^3} \rho e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} \Delta_j u_0^h |v_j|^3 dx\).

\[
\left| \int_{\mathbb{R}^3} \rho e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} \Delta_j u_0^h |v_j|^3 dx \right| \leq C \|e^{t\Delta} u_0\|_{L^\infty} \|e^{t\Delta} \Delta_j u_0^h\|_{L^2} \|v_j\|_{L^2}
\]

\[
\leq C \|e^{t\Delta} u_0\|_{L^\infty} \|\Delta_j u_0^h\|_{L^2} \|v_j\|_{L^2} \|v_j\|_{L^2}^2 + \frac{1}{2} \|\nabla e^{t\Delta} \Delta_j u_0^h\|_{L^2}^2.
\]

- **Estimate for** \(\int_{\mathbb{R}^3} \rho a \Delta e^{t\Delta} \Delta_j u_0 |v_j| dx\).

\[
\left| \int_{\mathbb{R}^3} \rho a \Delta e^{t\Delta} \Delta_j u_0 |v_j| dx \right| \leq C \|a\|_{L^\infty} \|\Delta e^{t\Delta} \Delta_j u_0\|_{L^2} \|v_j\|_{L^2}
\]

\[
\leq C \|a_0\|_{L^\infty} \|\Delta e^{t\Delta} \Delta_j u_0\|_{L^2} \|v_j\|_{L^2}.
\]

- **Estimate for** \(\int_{\mathbb{R}^3} \rho \left( e^{t\Delta} \Delta_j u_0^h \cdot \nabla \text{div} u_0^h - \Delta u_0^3 e^{t\Delta} \Delta_j \text{div} u_0^h \right) |v_j|^3 dx\).
It follows from the law of product (A.2) that
\[
\left| \int_{\mathbb{R}^3} \rho \left( -e^{t\Delta} \Delta_j u_0^h \cdot \nabla_h e^{t\Delta} u_0^3 - e^{t\Delta} u_0^3 e^{t\Delta} \Delta_j \operatorname{div}_h u_0^h \right) |v_j|^3 \, dx \right|
\leq \left( \| \nabla_h e^{t\Delta} u_0^3 \|_{B^\frac{1}{2}} + \| e^{t\Delta} u_0^3 \|_{L^\infty} \right) \| \nabla e^{t\Delta} \Delta_j u_0^h \|_{L^2} |v_j|_{L^2}
\leq C \| e^{t\Delta} u_0^3 \|^2_{B^\frac{1}{2}} |v_j|^2_{L^2} + \frac{1}{2} \| \nabla e^{t\Delta} \Delta_j u_0^h \|_{L^2}.
\]
Inserting the above estimates into (4.2) and then applying Gronwall’s and Young’s inequalities gives rise to
\[
\|v_j\|^2_{L^2(L^2)} + \| \nabla v_j \|^2_{L^2(L^2)} \leq C \exp(C \| e^{t\Delta} u_0 \|^2_{L^2(B^\frac{1}{2})})
\times \left( \| \nabla e^{t\Delta} \Delta_j u_0^h \|^2_{L^2(L^2)} + \| a_0 \|_{L^\infty} \| \Delta e^{t\Delta} \Delta_j u_0 \|_{L^2(L^2)} |v_j|_{L^2} \right)
\leq \frac{1}{2} \|v_j\|^2_{L^2(L^2)} + C \exp(C \| e^{t\Delta} u_0 \|^2_{B^\frac{1}{2}})
\times \left( \| \nabla e^{t\Delta} \Delta_j u_0^h \|^2_{L^2(L^2)} + \| a_0 \|_{L^\infty} \| \Delta e^{t\Delta} \Delta_j u_0 \|_{L^2(L^2)} \right),
\]
which together with Lemma A.2 implies that
\[
\|v_j\|^2_{L^2(L^2)} + \| \nabla v_j \|^2_{L^2(L^2)} \leq C d_j^2 2^{-j} \left( \| u_0 \|^2_{B^\frac{1}{2}} + \| a_0 \|^2_{L^\infty} \right) \exp(C \| u_0 \|^2_{B^\frac{1}{2}})
\leq C d_j^2 2^{-j} \left( \| u_0 \|^2_{B^\frac{1}{2}} + \| a_0 \|^2_{L^\infty} \right) \exp(C \| u_0 \|^2_{B^\frac{1}{2}}).
\]
This proves (4.1). □

**Lemma 4.2.** Under the assumptions of Lemma 4.1, we denote
\[
T^*_2 \overset{\text{def}}{=} \{ t < T^* : \| v \|_{L^2_t(L^\infty)} \leq 1 \}.
\]
Then for \( t < T^*_2 \), one has
\[
\| \nabla v_j \|^2_{L^2(L^2)} + \| \partial_t v_j \|^2_{L^2(L^2)} + \| \nabla (\partial_t v_j, \nabla^2 v_j, \nabla \pi_j) \|^2_{L^2(L^2)} \leq C d_j^2 2^{-j} \left( \| u_0 \|^2_{B^\frac{1}{2}} + \| a_0 \|^2_{L^\infty} \right) \exp(C \| u_0 \|^2_{B^\frac{1}{2}}).
\]
**Proof.** By taking \( L^2 \) inner product of the momentum equation of (2.16) with \( \partial_t v_j \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla v_j(t) \|^2_{L^2} + \| \sqrt{\rho} \partial_t v_j \|^2_{L^2}
= - \int_{\mathbb{R}^3} \rho (v + e^{t\Delta} u_0) \cdot \nabla v_j \partial_t v_j \, dx - \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t\Delta} u_0 \partial_t v_j \, dx
+ \int_{\mathbb{R}^3} \rho \Delta e^{t\Delta} \Delta_j u_0 \partial_t v_j \, dx - \int_{\mathbb{R}^3} \rho e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} \Delta_j u_0^h \partial_t v_j \, dx
- \int_{\mathbb{R}^3} \rho (e^{t\Delta} \Delta_j u_0^h \cdot \nabla_h e^{t\Delta} u_0^3 - e^{t\Delta} u_0^3 e^{t\Delta} \Delta_j \operatorname{div}_h u_0^h) \partial_t v_j \, dx.
\]
We now deal with term by term above.
\begin{itemize}
\item **Estimate for** \( \int_{\mathbb{R}^3} \rho (v + e^{t\Delta} u_0) \cdot \nabla v_j \partial_t v_j \, dx. \)
\[
\left| \int_{\mathbb{R}^3} \rho (v + e^{t\Delta} u_0) \cdot \nabla v_j \partial_t v_j \, dx \right| \leq C (\| v \|_{L^\infty} + \| e^{t\Delta} u_0 \|_{L^\infty}) \| \nabla v_j \|_{L^2} \| \rho \partial_t v_j \|_{L^2}
\leq C (\| v \|^2_{L^\infty} + \| e^{t\Delta} u_0 \|^2_{B^\frac{1}{2}}) \| \nabla v_j \|^2_{L^2} + \frac{1}{10} \| \sqrt{\rho} \partial_t v_j \|^2_{L^2}.
\]
\item **Estimate for** \( \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t\Delta} u_0 \partial_t v_j \, dx. \)
\end{itemize}
It follows from the law of product, (A.2), that
\[
\left| \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t \Delta} u_0 | \partial_t v_j \, dx \right| \leq \| \nabla v_j \|_{L^2} \| \nabla e^{t \Delta} u_0 \|_{B^{3/2}_2} \| \sqrt{\rho} \partial_t v_j \|_{L^2} \\
\leq C \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 \| \nabla v_j \|_{L^2}^2 + \frac{1}{10} \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2.
\]

- **Estimate for** \( \int_{\mathbb{R}^3} \rho a \Delta e^{t \Delta} \Delta_j u_0 | \partial_t v_j \, dx \). \[
\left| \int_{\mathbb{R}^3} \rho a \Delta e^{t \Delta} \Delta_j u_0 | \partial_t v_j \, dx \right| \leq C \| a \|_{L^\infty} \| \Delta e^{t \Delta} \Delta_j u_0 \|_{L^2} \| \sqrt{\rho} \partial_t v_j \|_{L^2} \\
\leq C \| a_0 \|_{L^\infty} \| \Delta e^{t \Delta} \Delta_j u_0 \|^2_{L^2} + \frac{1}{10} \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2.
\]

- **Estimate for** \( \int_{\mathbb{R}^3} \rho e^{t \Delta} u_0 \cdot \nabla e^{t \Delta} \Delta_j u_0^h | \partial_t v_j^h \, dx \). \[
\left| \int_{\mathbb{R}^3} \rho e^{t \Delta} u_0 \cdot \nabla e^{t \Delta} \Delta_j u_0^h | \partial_t v_j^h \, dx \right| \leq C \| e^{t \Delta} u_0 \|_{L^\infty} \| \nabla e^{t \Delta} \Delta_j u_0^h \|_{L^2} \| \sqrt{\rho} \partial_t v_j \|_{L^2} \\
\leq C \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 \| \nabla e^{t \Delta} \Delta_j u_0^h \|^2_{L^2} + \frac{1}{10} \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2.
\]

- **Estimate for** \( \int_{\mathbb{R}^3} \rho (e^{t \Delta} \Delta_j u_0^h \cdot \nabla h e^{t \Delta} u_0^3 - e^{t \Delta} u_0^3 \Delta_j \partial_t v_j^h) | \partial_t v_j^h \, dx \). \[
\text{It follows from the law of product, (A.2), that}
\]
\[
\left| \int_{\mathbb{R}^3} \rho (e^{t \Delta} \Delta_j u_0^h \cdot \nabla h e^{t \Delta} u_0^3 - e^{t \Delta} u_0^3 \Delta_j \partial_t v_j^h) | \partial_t v_j^h \, dx \right| \\
\leq (\| \nabla h e^{t \Delta} u_0^3 \|_{B^{3/2}_2} + \| e^{t \Delta} u_0^3 \|_{L^\infty}) \| \nabla e^{t \Delta} \Delta_j u_0^h \|_{L^2} \| \sqrt{\rho} \partial_t v_j \|_{L^2} \\
\leq C \| e^{t \Delta} u_0^3 \|_{B^{3/2}_2}^2 \| \nabla e^{t \Delta} \Delta_j u_0^h \|^2_{L^2} + \frac{1}{10} \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2.
\]

Inserting the above estimates into (4.5) leads to
\[
(4.6) \quad \frac{d}{dt} \| \nabla v_j(t) \|_{L^2}^2 + \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2 \leq C \left( \| v \|_{L^\infty}^2 + \| \Delta e^{t \Delta} \Delta_j u_0 \|_{L^2}^2 + \| e^{t \Delta} u_0 \|_{L^2} \| \nabla e^{t \Delta} \Delta_j u_0 \|_{L^2}^2 \right).
\]

Applying Gronwall’s inequality yields
\[
\| \nabla v_j \|_{L^2(L^2)}^2 + \| \sqrt{\rho} \partial_t v_j \|_{L^2(L^2)}^2 \leq C \exp \left( \| v \|_{L^2(L^\infty)}^2 + \| \Delta e^{t \Delta} \Delta_j u_0 \|_{L^2(B^{3/2}_2)}^2 \right) \\
\times \left( \| a_0 \|_{L^\infty} \| \Delta e^{t \Delta} \Delta_j u_0 \|_{L^2(B^{3/2}_2)}^2 + \| e^{t \Delta} u_0 \|_{L^2(B^{3/2}_2)}^2 \| \nabla e^{t \Delta} \Delta_j u_0 \|_{L^2(L^\infty)}^2 \right).
\]

Note from Lemma A.2 that
\[
\| e^{t \Delta} u_0 \|_{L^2(B^{3/2}_2)} \leq C \| u_0 \|_{B^{3/2}_2}, \quad \| \Delta e^{t \Delta} \Delta_j u_0 \|_{L^2(L^2)} \leq C d_j 2^{2\frac{3}{2}} \| u_0 \|_{B^{3/2}_2} \quad \text{and}
\]
\[
\| \nabla e^{t \Delta} \Delta_j u_0 \|_{L^\infty(L^2)} \leq C d_j 2^{\frac{3}{2}} \| u_0 \|_{B^{3/2}_2}.
\]

Then for \( t \leq T'_* \), we find
\[
(4.7) \quad \| \nabla v_j \|_{L^\infty(L^2)}^2 + \| \sqrt{\rho} \partial_t v_j \|_{L^2(L^2)}^2 \leq C d_j^2 2^{2\frac{3}{2}} \exp \left( C \| u_0 \|_{B^{3/2}_2}^2 \left( \| a_0 \|_{L^\infty}^2 + \| u_0^h \|_{B^{3/2}_2}^2 \right) \| u_0 \|_{B^{3/2}_2}^2 \right) \\
\leq C d_j^2 2^{2\frac{3}{2}} \exp \left( C \| u_0 \|_{B^{3/2}_2}^2 \left( \| a_0 \|_{L^\infty}^2 + \| u_0^h \|_{B^{3/2}_2}^2 \right) \right).
On the other hand, we deduce from (2.16) that
\[
\|\nabla^2 v_j, \nabla \pi_j \|_{L^2} \leq C\left(\|v\|_{L^\infty} + \|e^{t \Delta} u_0\|_{B^\frac{3}{2} +}}\right)\|\nabla v_j\|_{L^2} + \|\sqrt{\rho} \partial_t v_j\|_{L^2} \\
+ \|e^{t \Delta} u_0\|_{L^\infty}\|\nabla e^{t \Delta} \Delta_j u_0\|_{L^2} + \|a_0\|_{L^\infty}\|\Delta e^{t \Delta} \Delta_j u_0\|_{L^2}),
\]
Taking $L^2$ norm with respect to time yields
\[
\|\nabla^2 v_j, \nabla \pi_j \|_{L^2_t(L^2)} \leq C\left(\|v\|_{L^2_t(L^\infty)} + \|e^{t \Delta} u_0\|_{L^2_t(B^\frac{3}{2} +}}\right)\|\nabla v_j\|_{L^2_t(L^\infty)} \\
+ \|e^{t \Delta} u_0\|_{L^2_t(L^\infty)}\|\nabla e^{t \Delta} \Delta_j u_0\|_{L^2_t(L^\infty)} \\
+ \|\sqrt{\rho} \partial_t v_j\|_{L^2_t(L^2)} + \|a_0\|_{L^\infty}\|\Delta e^{t \Delta} \Delta_j u_0\|_{L^2_t(L^2)},
\]
which together with (4.7) ensures that
\[
\|\nabla^2 v_j, \nabla \pi_j \|_{L^2_t(L^2)} \leq Cd_j2^j\left(\|a_0\|_{L^\infty} + \|u_0^h\|_{B^\frac{3}{2} +}}\right) \exp(C\|u_0\|^2_{B^\frac{3}{2} +}).
\]
Along with (4.7), we conclude the proof of (4.4).

\[\Box\]

**Proof of Proposition 2.4.** With (4.1) and (4.4), we get, by a similar derivation of (2.1), that
\[
\|v\|_{L^\infty_t(B^\frac{3}{2} +}} + \|\nabla v\|_{L^1_t(B^\frac{3}{2} +)} \leq (\|a_0\|_{L^\infty} + \|u_0^h\|_{B^\frac{3}{2} +}) \exp(C\|u_0\|^2_{B^\frac{3}{2} +}) \text{ for } t \leq T^*_2,
\]
which together with (1.12) ensures that (2.18) holds for $t \leq T^*_2$.

Then for $t \leq T^*_2$, we have
\[
\|v\|_{L^2_t(L^\infty)} \leq C\|v\|_{L^2_t(B^\frac{3}{2} +)} \leq C\varepsilon_0,
\]
for $\varepsilon_0$ given by (1.12). In particular, if we take $\varepsilon_0$ in (1.12) so small that $C\varepsilon_0 \leq \frac{1}{2}$, we deduce by a continuous argument that $T^*_2$ determined by (4.3) can be any number smaller than $T^*$. This proves (2.18). \[\Box\]

**Remark 4.1.** Once again by virtue of (2.18), it follows from classical theory of inhomogeneous incompressible Navier-Stokes system that $T^* = \infty$.

**Corollary 4.1.** Under the assumptions of Proposition 2.4, for any $t > 0$ and $\eta$ given by (1.12), we have
\[
\|\sqrt{t}\nabla v_j\|^2_{L^\infty_t(L^2)} + \|\sqrt{t}(\partial_t v_j, \nabla^2 v_j, \nabla \pi_j)\|^2_{L^2_t(L^2)} \leq C\eta d^2 2^{-j}.
\]

**Proof.** We first get, by multiplying $t$ to (4.6) and then applying Gronwall’s inequality, that
\[
\|\sqrt{t}\nabla v_j\|^2_{L^\infty_t(L^2)} + \|\sqrt{t} \partial_t v_j\|^2_{L^2_t(L^2)} \leq C\exp(C\|v\|^2_{L^2_t(L^\infty)} + C\|e^{t \Delta} u_0\|^2_{L^2_t(B^\frac{3}{2} +))} \\
\times \left(\|\nabla v_j\|^2_{L^2_t(L^2)} + \|a_0\|^2_{L^\infty} \|\sqrt{t} e^{t \Delta} \Delta_j u_0\|^2_{L^2_t(L^2)} \\
+ \|e^{t \Delta} u_0\|^2_{L^2_t(B^\frac{3}{2} +))} \|\sqrt{t} \nabla e^{t \Delta} \Delta_j u_0\|^2_{L^2_t(L^2)} \right).
\]
Yet it follows Lemma A.2 that
\[
\|\sqrt{t} e^{t \Delta} \Delta_j u_0\|^2_{L^2_t(L^2)} \leq C 2^j \int_0^t t e^{-s 2^j} \|\Delta_j u_0\|^2_{L^2} ds \\
\leq C 2^j \int_0^t e^{-\varepsilon s 2^j} ds \|\Delta_j u_0\|^2_{L^2} \\
\leq C \|\Delta_j u_0\|^2_{L^2} \leq C 2^j 2^{-j} \|u_0\|^2_{B^\frac{3}{2} +}.
\]
and

\[ \| \sqrt{t} \nabla \Delta_j u_0^h(t) \|_{L^2}^2 \leq C t^{2j} e^{-ct^{2j}} \| \Delta_j u_0^h \|_{L^2}^2 \]

\[ \leq C \| \Delta_j u_0^h \|_{L^2}^2 \leq C d_j^2 2^{-j} \| u_0^h \|_{B^2_2}^2, \]

As a result, we deduce from Proposition 2.4 that

\[ \| \sqrt{t} \nabla v_j \|_{L^2}(L^2) + \| \sqrt{t} \partial_t v_j \|_{L^2}(L^2) \leq C \eta d_j^2 2^{-j}. \]

This together with (4.8) ensures that

\[ \| \sqrt{t}(\nabla^2 v_j, \nabla \pi_j) \|_{L^2}(L^2) \leq C \eta d_j^2 2^{-j}. \]

This proves (4.9). □

**Lemma 4.3.** Under the assumptions of proposition 2.4, for any \( t > 0 \), we have

\[
\frac{d}{dt} \| \sqrt{\rho_0} \partial_t v_j \|_{L^2}^2 + \| \nabla \partial_t v_j \|_{L^2}^2 \leq \| \nabla \partial_t e^{t \Delta} \Delta_j u_0^h \|_{L^2}^2 + C \| \sqrt{\rho_0} \partial_t v_j \|_{L^2}^2 \times (\| u \|_{B^{3/2}_2}^2 + \| t^{1/4} u_t \|_{L^2}^2 + \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 + \| \nabla e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 + \| \sqrt{t} e^{t \Delta} u_0 \|_{B^{3/2}_2}^2) \\
+ C \left[ (t^{-1} + t^{-\frac{1}{2}} \| u \|_{L^\infty}) \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 \right] \| t^{\frac{1}{4}} u_t \|_{L^2}^2 + \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 \right) \| \nabla v_j \|_{L^2}^2
+ C t^{-\frac{1}{2}} \| t^{1/4} u_t \|_{L^2}^2 \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 \| \nabla e^{t \Delta} \Delta_j u_0^h \|_{L^2} + \| a_0 \|_{L^\infty} \| \nabla^2 e^{t \Delta} \Delta_j u_0^h \|_{L^2} \| \nabla v_j \|_{L^2}
+ \| a_0 \|_{L^\infty} \| \nabla \partial_t e^{t \Delta} \Delta_j u_0^h \|_{L^2} \leq \| \nabla e^{t \Delta} \Delta_j u_0^h \|_{L^2}^2
+ \left( \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 + \| u \|_{L^\infty}^2 \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 + t^{-\frac{1}{2}} \| \partial_t e^{t \Delta} u_0 \|_{B^{3/2}_2}^2 \right) \| \nabla e^{t \Delta} \Delta_j u_0^h \|_{L^2}^2
+ \left( (t^{-1} + \| e^{t \Delta} u_0 \|_{B^{3/2}_2}^2) \| \nabla^2 e^{t \Delta} \Delta_j u_0^h \|_{L^2}^2 + \| a_0 \|_{L^\infty} \| \Delta \partial_t e^{t \Delta} \Delta_j u_0^h \|_{L^2} \| \sqrt{\rho_0} \partial_t v_j \|_{L^2}.
\]

**Proof.** In the rest of this section, we shall always denote \( u \overset{\text{def}}{=} v + e^{t \Delta} u_0 \) and \( D_t = \partial_t + u \cdot \nabla \). Then we get, by applying \( \partial_t \) to the \( v_j \) equation of (2.16), that

\[
\rho \partial_t^3 v_j + \rho u \cdot \nabla \partial_t v_j + \partial_t (\rho v_j \cdot \nabla e^{t \Delta} u_0) - \Delta \partial_t v_j + \nabla \partial_t \pi_j \]

\[ = - \rho_1 D_t v_j - \rho \partial_t u \cdot \nabla \partial_t v_j + \partial_t (\rho F_j) + \partial_t (\rho a \Delta e^{t \Delta} \Delta_j u_0). \]

Taking \( L^2 \) inner product of the above equation with \( \partial_t v_j \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} \partial_t v_j(t) \|_{L^2}^2 + \| \nabla \partial_t v_j \|_{L^2}^2 = - \int_{\mathbb{R}^3} (\rho_1 D_t v_j + \rho \partial_t u \cdot \nabla \partial_t v_j) \partial_t v_j \ dx
\]

\[ - \int_{\mathbb{R}^3} \partial_t (\rho v_j \cdot \nabla e^{t \Delta} u_0) \partial_t v_j \ dx + \int_{\mathbb{R}^3} \partial_t (\rho F_j) \partial_t v_j \ dx
\]

\[ + \int_{\mathbb{R}^3} \partial_t (\rho a \Delta e^{t \Delta} \Delta_j u_0) \partial_t v_j \ dx. \]

**Estimate for** \( \int_{\mathbb{R}^3} (\rho_1 D_t v_j + \rho \partial_t u \cdot \nabla \partial_t v_j) \partial_t v_j \ dx. **Observing that**

\[
\left| \int_{\mathbb{R}^3} u \cdot \nabla v_j | \rho u \cdot \nabla \partial_t v_j \ dx \right| \leq C \| u \|_{L^\infty} \| \nabla v_j \|_{L^2} \| \nabla \partial_t v_j \|_{L^2}
\]

\[ \leq C \| u \|_{L^\infty}^4 \| \nabla v_j \|_{L^2}^2 + \frac{1}{24} \| \nabla \partial_t v_j \|_{L^2}^2. \]
Then along the same line to the proof of (3.4), we have
\[
\left| \int_{\mathbb{R}^3} \left( \rho_t D_t v_j + \rho \partial_t u \cdot \nabla \partial_t v_j \right) |\partial_t v_j| \, dx \right| \leq C \left( \|u\|^2_{B^2} + \|\sqrt{\rho} \partial_t v_j\|^2_{L^2} + \|u\|^2_{L^\infty} \|\nabla v_j\|^2_{L^2} 
\right. \\
\left. + \|u_t\|^2_{L^2} \|\nabla v_j\|_{L^2} \left( \|\nabla^2 v_j\|_{L^2} + \|\sqrt{\rho} \partial_t v_j\|_{L^2} \right) \right) + \frac{1}{8} \|\nabla \partial_t v_j\|^2_{L^2},
\]
which together with (4.8) ensures that
\[
\left| \int_{\mathbb{R}^3} \left( \rho_t D_t v_j + \rho \partial_t u \cdot \nabla \partial_t v_j \right) |\partial_t v_j| \, dx \right| \leq C \left( \|u\|^2_{B^2} + \|t^{\frac{1}{4}} u_t\|^2_{L^2} \|\sqrt{\rho} \partial_t v_j\|^2_{L^2} 
\right. \\
\left. + \left[\|t^{\frac{1}{4}} u_t\|^2_{L^2} \left( t^{-1} + t^{-\frac{1}{4}} \|v\|_{L^\infty} + \|e^{t^2} u_0\|_{B^2} \right) \right] + \|u\|^2_{L^\infty} \|\nabla v_j\|^2_{L^2} 
\right. \\
\left. + \left( \|e^{t^2} u_0\|_{B^2} \|\nabla e^{t^2} \Delta v_0\|^2_{L^2} + \|a_0\|_{L^\infty} \|\Delta e^{t^2} \Delta v_0\|^2_{L^2} \right) \right) \\
\times t^{-\frac{1}{2}} \|t^{\frac{1}{4}} u_t\|^2_{L^2} \|\nabla v_j\|_{L^2} + \frac{1}{8} \|\nabla \partial_t v_j\|^2_{L^2}.
\]

- Estimate for \( \int_{\mathbb{R}^3} \partial_t (\rho v_j \cdot \nabla e^{t^2} u_0) |\partial_t v_j| \, dx \).

It is easy to observe that
\[
\int_{\mathbb{R}^3} \partial_t (\rho v_j \cdot \nabla e^{t^2} u_0) |\partial_t v_j| \, dx = \int_{\mathbb{R}^3} \partial_t \rho v_j \cdot \nabla e^{t^2} u_0 |\partial_t v_j| \, dx \\
+ \int_{\mathbb{R}^3} \rho \partial_t v_j \cdot \nabla e^{t^2} u_0 |\partial_t v_j| \, dx + \int_{\mathbb{R}^3} \rho v_j \cdot \nabla \partial_t e^{t^2} u_0 |\partial_t v_j| \, dx.
\]

Thanks to the transport equation of (1.1), we get, by using integration by parts, that
\[
\int_{\mathbb{R}^3} \partial_t \rho v_j \cdot \nabla e^{t^2} u_0 |\partial_t v_j| \, dx = \int_{\mathbb{R}^3} \rho (u \cdot \nabla v_j) \cdot \nabla e^{t^2} u_0 |\partial_t v_j| \, dx \\
+ \int_{\mathbb{R}^3} \rho v_j \otimes u : \nabla^2 e^{t^2} u_0 |\partial_t v_j| \, dx + \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t^2} u_0 |u \cdot \nabla \partial_t v_j| \, dx.
\]

It is easy to observe that
\[
\left| \int_{\mathbb{R}^3} \rho (u \cdot \nabla v_j) \cdot \nabla e^{t^2} u_0 |\partial_t v_j| \, dx \right| \leq C \|u\|_{L^\infty} \|\nabla v_j\|_{L^2} \|\nabla e^{t^2} u_0\|_{L^\infty} \|\sqrt{\rho} \partial_t v_j\|_{L^2} \\
\leq C \left( \|u\|^2_{L^\infty} \|\sqrt{\rho} \partial_t v_j\|^2_{L^2} + \|\nabla e^{t^2} u_0\|^2_{B^2} \|\nabla v_j\|^2_{L^2} \right).
\]

Similarly, one has
\[
\left| \int_{\mathbb{R}^3} \rho v_j \otimes u : \nabla^2 e^{t^2} u_0 |\partial_t v_j| \, dx \right| \leq C \|v_j\|_{L^6} \|u\|_{L^\infty} \|\nabla^2 e^{t^2} u_0\|_{L^3} \|\sqrt{\rho} \partial_t v_j\|_{L^2} \\
\leq C \left( \|u\|^2_{L^\infty} \|\sqrt{\rho} \partial_t v_j\|^2_{L^2} + \|\nabla e^{t^2} u_0\|^2_{B^2} \|\nabla v_j\|^2_{L^2} \right),
\]
and
\[
\left| \int_{\mathbb{R}^3} \rho v_j \cdot \nabla e^{t^2} u_0 |u \cdot \nabla \partial_t v_j| \, dx \right| \leq C \|v_j\|_{L^6} \|\nabla e^{t^2} u_0\|_{L^3} \|u\|_{L^\infty} \|\nabla \partial_t v_j\|_{L^2} \\
\leq C \|u\|^2_{L^\infty} \|e^{t^2} u_0\|^2_{B^2} \|\nabla v_j\|^2_{L^2} + \frac{1}{8} \|\nabla \partial_t v_j\|^2_{L^2}.
\]

Whereas we notice that
\[
\left| \int_{\mathbb{R}^3} \rho \partial_t v_j \cdot \nabla e^{t^2} u_0 |\partial_t v_j| \, dx \right| \leq \|\nabla e^{t^2} u_0\|_{L^\infty} \|\sqrt{\rho} \partial_t v_j\|^2_{L^2},
\]
and it follows from the law of product, (A.2), that
\[
\int_{\mathbb{R}^3} \rho v_j \cdot \nabla \partial_t e^t \Delta u_0 | \partial_t v_j \, dx \leq C \| \nabla v_j \|^2_{L^2} \| \nabla \partial_t e^t \Delta u_0 \|_{B^{1/2}_{p,2}} \| \sqrt{\rho} \partial_t v_j \|_{L^2}
\]
\[
\leq C t^{-\frac{1}{2}} \| \partial_t e^t \Delta u_0 \|_{B^{1/2}_{p,2}} \| \nabla v_j \|^2_{L^2} + t^{\frac{1}{2}} \| \partial_t e^t \Delta u_0 \|_{B^{1/2}_{p,2}} \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2.
\]
As a result, it comes out
\[
\int_{\mathbb{R}^3} \partial_t \left( \rho v_j \cdot \nabla e^t \Delta u_0 \right) | \partial_t v_j \, dx \leq \frac{1}{8} \| \nabla \partial_t v_j \|_{L^2}^2
\]
\[
+ C \left( \| u \|^2_{L^\infty} + \| \nabla e^t \Delta u_0 \|_{B^{1/2}_{p,2}} + t^{\frac{1}{2}} \| \partial_t e^t \Delta u_0 \|_{B^{1/2}_{p,2}} \right) \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2
\]
\[
+ \left( \| e^t \Delta u_0 \|^2_{B^{1/2}_{p,2}} + \| u \|^2_{L^\infty} \| \nabla e^t \Delta u_0 \|^2_{B^{1/2}_{p,2}} + t^{-\frac{1}{2}} \| \partial_t e^t \Delta u_0 \|_{B^{1/2}_{p,2}} \right) \| \nabla v_j \|^2_{L^2}.
\]

- **Estimate for** \( \int_{\mathbb{R}^3} \partial_t \left( \rho a e^t \Delta \Delta_j u_0 \right) | \partial_t v_j \, dx \). Again thanks to the transport equation of (1.11) and notice that \( \rho = \frac{1}{t + a} \), we get, by using integration by parts, that
\[
\int_{\mathbb{R}^3} \partial_t \left( \rho a e^t \Delta \Delta_j u_0 \right) | \partial_t v_j \, dx = \int_{\mathbb{R}^3} \frac{a}{1 + a} u \cdot \nabla \Delta e^t \Delta_j u_0 | \partial_t v_j \, dx
\]
\[
+ \int_{\mathbb{R}^3} \frac{a}{1 + a} \Delta e^t \Delta_j u_0 | u \cdot \nabla \partial_t v_j \, dx + \int_{\mathbb{R}^3} \frac{a}{1 + a} \partial_t e^t \Delta \Delta_j u_0 | \partial_t v_j \, dx,
\]
from which, we infer
\[
\int_{\mathbb{R}^3} \partial_t \left( \rho a e^t \Delta \Delta_j u_0 \right) | \partial_t v_j \, dx \leq C \| a_0 \|_{L^\infty} \left( \| u \|_{L^\infty} \left( \| \sqrt{\rho} \partial_t v_j \|_{L^2} \| \nabla e^t \Delta \Delta_j u_0 \|_{L^2} \right.ight.
\]
\[
\left. \left. + \| \nabla \partial_t v_j \|_{L^2} \| \Delta \partial e^t \Delta \Delta_j u_0 \|_{L^2} \right) + \| \partial_t e^t \Delta \Delta_j u_0 \|_{L^2} \| \sqrt{\rho} \partial_t v_j \|_{L^2} \right).
\]
Then applying Young’s inequality gives
\[
\int_{\mathbb{R}^3} \partial_t \left( \rho a e^t \Delta \Delta_j u_0 \right) | \partial_t v_j \, dx \leq \frac{1}{8} \| \nabla \partial_t v_j \|_{L^2}^2 + C \left( \| u \|^2_{L^\infty} \| \sqrt{\rho} \partial_t v_j \|_{L^2}^2
\]
\[
+ \| a_0 \|_{L^\infty} \| \Delta \partial e^t \Delta \Delta_j u_0 \|_{L^2} \| \sqrt{\rho} \partial_t v_j \|_{L^2}
\]
\[
+ \left( \| a_0 \|^2_{L^\infty} \left( \| \nabla e^t \Delta \Delta_j u_0 \|^2_{L^2} + \| u \|^2_{L^\infty} \| \Delta e^t \Delta \Delta_j u_0 \|^2_{L^2} \right) \right).
\]

- **Estimate for** \( \int_{\mathbb{R}^3} \partial_t \rho F_j | \partial_t v_j \, dx \). In view of the transport equation of (1.1), we get, by using integration by parts, that
\[
\int_{\mathbb{R}^3} \partial_t \rho F_j | \partial_t v_j \, dx = \int_{\mathbb{R}^3} \rho u \cdot \nabla F_j | \partial_t v_j \, dx + \int_{\mathbb{R}^3} F_j | \rho u \cdot \nabla \partial_t v_j \, dx,
\]
from which, we infer
\[
\int_{\mathbb{R}^3} \partial_t \rho F_j | \partial_t v_j \, dx \leq C \| u \|_{L^\infty} \left( \| \sqrt{\rho} \partial_t v_j \|_{L^2} \| \nabla F_j \|_{L^2} + \| F_j \|_{L^2} \| \nabla \partial_t v_j \|_{L^2} \right)
\]
\[
\leq C \| u \|^2_{L^\infty} \| \sqrt{\rho} \partial_t v_j \|^2_{L^2} + \| \nabla F_j \|^2_{L^2} + \| u \|^2_{L^\infty} \| F_j \|^2_{L^2} + \frac{1}{8} \| \nabla \partial_t v_j \|^2_{L^2}.
\]
Yet it follows from the law of product in Besove spaces that
\[
\| F_j \|_{L^2} \leq C \| e^t \Delta u_0 \|_{B^{1/2}_{p,2}} \| \nabla e^t \Delta \Delta_j u_0 \|_{L^2},
\]
and
\[ \| \nabla F_j \|_{L^2} \leq C ( \| e^{s \Delta} u_0 \|_{L^2} + \| e^{s \Delta} \Delta_j u_0^h \|_{L^2} + \| e^{s \Delta} u_0 \|_{L^2} + \| e^{s \Delta} \Delta_j u_0^h \|_{L^2} ) . \]

Hence we obtain
\[ \left| \int_{\mathbb{R}^3} \partial_t \rho F_j \partial_v v_j \, dx \right| \leq C \left( \| u \|_{L^\infty} \sqrt{\rho} \partial_v v_j \|_{L^2} + \| e^{s \Delta} u_0 \|_{L^2} + \| e^{s \Delta} \Delta_j u_0^h \|_{L^2} \right) \]
\[ + \| e^{s \Delta} u_0 \|_{L^2} ^2 \| \nabla e^{s \Delta} \Delta_j u_0^h \|_{L^2} + \| u \|_{L^\infty} \| e^{s \Delta} u_0 \|_{L^2} \| \nabla e^{s \Delta} \Delta_j u_0^h \|_{L^2} \right) + \frac{1}{8} \| \nabla \partial_v v_j \|_{L^2} ^2 . \]

\textbf{Estimate for } \int_{\mathbb{R}^3} \rho \partial_t F_j \partial_v v_j \, dx.

It follows from the law of product in Sobolev spaces (A.2) that
\[ \| \partial_t F_j \|_{L^2} \leq \| \partial_t e^{s \Delta} u_0 \|_{L^2} \| \nabla^2 e^{s \Delta} \Delta_j u_0^h \|_{L^2} + \| e^{s \Delta} u_0 \|_{L^2} \| \partial_t e^{s \Delta} \Delta_j u_0^h \|_{L^2} \]
\[ + \| \partial_t e^{s \Delta} \Delta_j u_0^h \|_{L^2} \| e^{s \Delta} u_0 \|_{L^\infty} + \| \nabla e^{s \Delta} \Delta_j u_0^h \|_{L^2} \| \partial_t e^{s \Delta} u_0 \|_{L^2} . \]

This implies that
\[ \int_{\mathbb{R}^3} \rho \partial_t F_j \partial_v v_j \, dx \leq C \left( \| t \partial_t e^{s \Delta} u_0 \|_{L^2} ^2 + \| e^{s \Delta} u_0 \|_{L^2} ^2 + \| t \partial_t e^{s \Delta} u_0 \|_{L^2} ^2 \right) \]
\[ + \| t \partial_t e^{s \Delta} u_0 \|_{L^2} ^2 \| \nabla \partial_t v_j \|_{L^2} ^2 + t \| e^{s \Delta} \Delta_j u_0^h \|_{L^2} ^2 \]
\[ + \| \nabla \partial_t e^{s \Delta} \Delta_j u_0^h \|_{L^2} ^2 + t \| e^{s \Delta} u_0 \|_{L^2} ^2 \| \partial_t e^{s \Delta} u_0 \|_{L^2} . \]

Inserting the above estimates into (4.12) leads to (4.10). This completes the proof of the Lemma.

Before proceeding, we also need the following lemma:

**Lemma 4.4.** For any $s \geq 0$, one has
\[ \| t^s e^{s \Delta} u_0 \|_{L^1_t(B^{2s+\frac{3}{2}}_2)} + \| t^s e^{s \Delta} u_0 \|_{L^2_t(B^{2s+\frac{3}{2}}_2)} \leq C \| u_0 \|_{B^{\frac{3}{2}}} . \]

\textbf{Proof.} In view of Definition A.1 and Lemma A.2, we have
\[ \| t^s e^{s \Delta} u_0 \|_{L^1_t(B^{2s+\frac{3}{2}}_2)} \leq C \sum_{j \in \mathbb{Z}} \int_0^t (t')^s 2^{j(2s+\frac{3}{2})} 2^{-ct'2^j} \| \Delta_j u_0 \|_{L^2} \, dt' \]
\[ \leq C \sum_{j \in \mathbb{Z}} 2^{j} \int_0^t 2^{-ct'2^j} \| \Delta_j u_0 \|_{L^2} \, dt' \leq C \sum_{j \in \mathbb{Z}} 2^{j} \| \Delta_j u_0 \|_{L^2} \leq C \| u_0 \|_{B^{\frac{3}{2}}} . \]

While we get, by applying Minkowsky’s inequality, that
\[ \| t^s e^{s \Delta} u_0 \|_{L^2_t(B^{2s+\frac{3}{2}}_2)} \leq \sum_{j \in \mathbb{Z}} 2^{j} \left( \int_0^t (t')^s 2^{j} \| \Delta_j e^{s \Delta} u_0 \|_{L^2} \, dt' \right)^{\frac{1}{2}} \]
\[ \leq \sum_{j \in \mathbb{Z}} 2^{j} \left( \int_0^t (t')^s \| e^{-ct2^j} \|_{L^2} \, dt' \right)^{\frac{1}{2}} \| \Delta_j u_0 \|_{L^2} \]
\[ \leq \sum_{j \in \mathbb{Z}} 2^{j} \| \Delta_j u_0 \|_{L^2} \leq C \| u_0 \|_{B^{\frac{3}{2}}} . \]

This completes the proof of the lemma. \qed
Proof of Proposition 2.5. By multiplying (4.10) by \( t \) and then applying Gronwall’s inequality, we obtain
\[
\| \sqrt{t} \partial_t v_j \|_{L_t^2(L^2)}^2 + \| \nabla \partial_t v_j \|_{L_t^2(L^2)}^2 
\leq C \exp \left( C \left( \| u \|_{L_t^2(B^2 \dot{\infty})}^2 + \| t^{\frac{1}{2}} u_t \|_{L_t^2(B^2 \dot{\infty})}^2 \right) \right)
\times \left( \| t^{\frac{1}{2}} e^{t \Delta} u \|_{L_t^2(B^2 \dot{\infty})}^2 + \| t^{\frac{1}{2}} \nabla e^{t \Delta} \Delta_j u_0 \|_{L_t^2(B^2 \dot{\infty})}^2 + \| t^{\frac{1}{2}} e^{t \Delta} u_0 \|_{L_t^2(B^2 \dot{\infty})}^2 \right)
+ \left( 1 + \left( \| t^{\frac{1}{2}} e^{t \Delta} u_0 \|_{L_t^2(B^2 \dot{\infty})}^2 + \| t^{\frac{1}{2}} u \|_{L_t^2(L^\infty)}^2 \right) \right) \| \nabla v_j \|_{L_t^2(L^2)}^2
\]
Let us denote
\[
T_3^* \overset{\text{def}}{=} \{ \ t > 0 \ : \ \| t^{\frac{1}{2}} v_t \|_{L_t^2(B^2 \dot{\infty})} \leq 1 \ \}.
\]
In view of (4.4) and (4.9), we get, by a similar derivation of (3.6), that
\[
\| t^{\frac{1}{2}} v_t \|_{L_t^2(L^2)} \leq C \eta,
\]
which together with Lemma 4.4 ensures that
\[
\| t^{\frac{1}{2}} u_t \|_{L_t^2(L^2)} \leq \| t^{\frac{1}{2}} \partial_t e^{t \Delta} u_0 \|_{L_t^2(L^2)} + \| t^{\frac{1}{2}} v_t \|_{L_t^2(L^2)} \leq \| t^{\frac{1}{2}} e^{t \Delta} u_0 \|_{L_t^2(B^2 \dot{\infty})} + \| t^{\frac{1}{2}} u \|_{L_t^2(L^\infty)} \leq C ( \eta + \| u \|_{L_t^2(B^2 \dot{\infty})} )
\]
Then we deduce from Lemmas A.2, 4.2 and 4.4 that
\[
\| t^{\frac{1}{2}} \partial_t v_j \|_{L_t^2(L^\infty)}^2 + \| \sqrt{t} \nabla \partial_t v_j \|_{L_t^2(L^2)}^2 \leq C \eta \delta^2 2^{j}
\]
for \( t \leq T_3^* \) and \( \eta \) given by (1.12).

By virtue of (4.8), (4.16) and Lemma A.2, we infer
\[
\| \sqrt{t} \nabla^2 v_j \|_{L_t^2(L^2)} \leq C \left( \| \sqrt{t} v \|_{L_t^2(L^\infty)} + \| \sqrt{t} e^{t \Delta} u_0 \|_{L_t^2(B^2 \dot{\infty})} \right) \| \nabla v_j \|_{L_t^2(L^\infty)}
+ \| \sqrt{t} \partial_t v_j \|_{L_t^2(L^\infty)} + \| \sqrt{t} e^{t \Delta} u_0 \|_{L_t^2(B^2 \dot{\infty})} \| \nabla e^{t \Delta} \Delta_j u_0 \|_{L_t^2(L^\infty)}
\]
\[
\| \nabla^2 \Delta^2 \Delta_j u_0 \|_{L_t^2(L^\infty)} \leq C \eta \delta^2 2^{j}
\]
for \( t \leq T_3^* \) and \( \eta \) given by (1.12).
Combining (4.9) with (4.16) and (4.17), we achieve (2.19) for \( t \leq T_3^* \). Then under the assumption of (1.12), we have

\[
\|t^{2}v\|_{L_t^\infty(B_2^\infty)} \leq \frac{1}{2} \quad \text{for } t \leq T_3^*
\]
as long as \( \varepsilon_0 \) in (1.12) is small enough. This contradict with the definition of \( T_3^* \) determined by (4.16). This in turn shows that \( T_3^* = \infty \), and we complete the proof of Proposition 2.5. \( \square \)

Exactly along the same line to the proof of (4.16) and (4.17), we have the following corollary:

**Corollary 4.2.** Under the assumption of proposition 2.5, for any \( t > 0 \), we have

\[
(4.18) \quad \|t(\partial_t v_j, \nabla^2 v_j)\|_{L_t^\infty(L^2)}^2 + \|t\nabla\partial_t v_j\|_{L_t^2(L^2)}^2 \leq C\eta d^2 t^{-2-j}
\]
for \( \eta \) given by (1.12)

Now let us turn to the proof of Proposition 2.6.

**Proof of Proposition 2.6.** The proof of this proposition basically follows from that of Proposition 2.3. By applying the operator \( D_t = \partial_t + u \cdot \nabla \) to the \( v_j \) equation of (2.16), we get, by a similar derivation of (3.13), that

\[
(4.19) \quad \rho D_t^2 v_j - \Delta D_t v_j + \nabla D_t \pi_j = -\Delta u \cdot \nabla v_j - 2 \sum_{i=1}^{3} \partial_t u \cdot \nabla \partial_t v_j + \nabla u \cdot \nabla v_j - \rho D_t (v_j \cdot \nabla e^{t\Delta} u_0) + \rho D_t F_j + \rho a D_t \Delta e^{t\Delta} \Delta_j u_0 \stackrel{\text{def}}{=} G_j.
\]

Then along the same line to the proof of (3.15), we write

\[
(4.20) \quad \frac{1}{2} \frac{d}{dt} \|\nabla D_t v_j(t)\|_{L^2}^2 + \|t\nabla D_t v_j\|_{L^2}^2 \leq \|\nabla D_t v_j\|_{L^2}^2 + t^2 \int_{\mathbb{R}^3} G_j D_t^2 v_j \, dx - t^2 \int_{\mathbb{R}^3} \nabla D_t v_j \cdot \nabla D_t v_j \, dx.
\]

We first deal with the estimate of \( \|tG_j\|_{L_t^2(L^2)} \). Notice from (2.16) that

\[
\begin{cases}
-\Delta v_j + \nabla \pi_j = -\rho(\partial_t v_j + u \cdot \nabla v_j + v_j \cdot \nabla e^{t\Delta} u_0 - F_j - a e^{t\Delta} \Delta_j u_0), \\
\text{div } v_j = 0,
\end{cases}
\]

from which, we deduce from the classical theory on Stokes operator that

\[
\|\nabla(\nabla^2 v_j, \nabla \pi_j)\|_{L_t^2(L^6)} \leq C\left( \|\nabla \partial_t v_j\|_{L_t^2(L^2)} + \|u\|_{L_t^2(L^\infty)} \|\nabla \nabla^2 v_j\|_{L_t^\infty(L^2)} + \|\nabla v_j\|_{L_t^\infty(L^2)} \|\nabla e^{t\Delta} u_0\|_{L_t^1(B_2^\infty)} + \|\nabla F_j\|_{L_t^2(L^6)} + \|a_0\|_{L^\infty} \|\nabla \nabla^2 \Delta_j u_0\|_{L_t^1(L^2)} \right).
\]

Whereas it follows from the law of product, (A.2), that

\[
\|\nabla F_j\|_{L_t^1(L^6)} \leq C \|\nabla F_j\|_{L_t^1(H^1)} \leq C \left( \|e^{t\Delta} u_0\|_{L_t^1(B_2^\infty)} \|\nabla \nabla^2 e^{t\Delta} \Delta_j u_0\|_{L_t^\infty(L^2)} + \|\nabla \nabla^2 e^{t\Delta} u_0\|_{L_t^1(B_2^\infty)} \|\nabla \nabla^2 \Delta_j u_0\|_{L_t^1(L^2)} \right) \leq C d^2 \frac{2}{3} \|u_0\|_{B_3^\infty}^{1/2} \|u_0^h\|_{B_3^\infty}^{1/2}.
\]
As a result, we deduce from Lemma A.2 and (4.16), (4.17) that
\begin{equation}
\|t \Delta u \cdot \nabla v_j + 2 \sum_{i=1}^{3} \partial_i u \cdot \nabla \partial_i v_j - \nabla u \cdot \nabla \pi_j \|_{L^2_t(L^2)} \leq C \sqrt{\eta} d_j 2^{\frac{j}{2}}.
\end{equation}
for \( \eta \) given by (1.12). Then we get, by a similar derivation (3.18), that
\begin{equation}
\|t \Delta u \cdot \nabla v_j + 2 \sum_{i=1}^{3} \partial_i u \cdot \nabla \partial_i v_j - \nabla u \cdot \nabla \pi_j \|_{L^2_t(L^2)} \leq C \sqrt{\eta} d_j 2^{\frac{j}{2}}.
\end{equation}
Note that
\begin{equation}
\|t D_t (v_j \cdot \nabla e^{t\Delta} u_0)\|_{L^2_t(L^2)} \leq \|t D_t v_j\|_{L^\infty_t(L^2)} \|\sqrt{t} e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})}
+ \|\nabla v_j\|_{L^\infty_t(L^2)} \|t D_t \nabla e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})}.
\end{equation}
Due to Lemma 4.4, we have
\begin{equation}
\|t D_t \nabla e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})} \leq C (\|t \partial_t e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})} + \|\sqrt{t} u\|_{L^\infty_t(B^{\frac{3}{2}})} \|\sqrt{t} e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})})
\leq \|u_0\|_{B^{\frac{3}{2}}} (1 + \|u_0\|_{B^{\frac{3}{2}}}),
\end{equation}
Hence by virtue of Lemmas 4.2, we obtain
\begin{equation}
\|t D_t (v_j \cdot \nabla e^{t\Delta} u_0)\|_{L^2_t(L^2)} \leq C \sqrt{\eta} d_j 2^{\frac{j}{2}}.
\end{equation}
Similarly, we deduce from Lemma A.2 that
\begin{equation}
\|t p a D_t \Delta e^{t\Delta} \Delta_j u_0\|_{L^2_t(L^2)} \leq C \|a_0\|_{L^\infty} \left( \|t \Delta \partial_t e^{t\Delta} \Delta_j u_0\|_{L^2_t(L^2)}
+ \|u\|_{L^2_t(L^\infty)} \|t \nabla e^{t\Delta} \Delta_j u_0\|_{L^2_t(L^\infty)} \right)
\leq C d_j 2^{\frac{j}{2}} \|a_0\|_{L^\infty} \|u_0\|_{B^{\frac{3}{2}}} (1 + \|u_0\|_{B^{\frac{3}{2}}}).
\end{equation}
Finally, we have
\begin{equation}
\|t D_t F_j\|_{L^2_t(L^2)} \leq \|t D_t e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})} \|\nabla e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)}
+ \|e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})} \|t D_t \nabla e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)}
+ \|\sqrt{t} \nabla e^{t\Delta} u_0^h\|_{L^2_t(B^{\frac{3}{2}})} \|\sqrt{t} D_t e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)}
+ \|t D_t \nabla \sqrt{t} e^{t\Delta} u_0^h\|_{L^2_t(B^{\frac{3}{2}})} \|\nabla e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)}.
\end{equation}
Whereas it follows from Lemma 4.4 that
\begin{equation}
\|t D_t e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})} \leq \|t \partial_t e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})} + \|\sqrt{t} u\|_{L^\infty_t(B^{\frac{3}{2}})} \|\sqrt{t} e^{t\Delta} u_0\|_{L^2_t(B^{\frac{3}{2}})}
\leq C \|u_0\|_{B^{\frac{3}{2}}} (1 + \|u_0\|_{B^{\frac{3}{2}}}).
\end{equation}
And it follows from Lemma A.2 that
\begin{equation}
\|t D_t \nabla e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)} \leq \|t \partial_t e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)} + \|\sqrt{t} u\|_{L^\infty_t(B^{\frac{3}{2}})} \|\sqrt{t} \nabla e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)}
\leq C d_j 2^{\frac{j}{2}} (1 + \|u_0\|_{B^{\frac{3}{2}}}) \|u_0^h\|_{B^{\frac{3}{2}}},
\end{equation}
and
\begin{equation}
\|\sqrt{t} D_t e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)} \leq \|\sqrt{t} \partial_t e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)} + \|\sqrt{t} u\|_{L^\infty_t(B^{\frac{3}{2}})} \|\nabla e^{t\Delta} \Delta_j u_0^h\|_{L^\infty_t(L^2)}
\leq C d_j 2^{\frac{j}{2}} (1 + \|u_0\|_{B^{\frac{3}{2}}}) \|u_0^h\|_{B^{\frac{3}{2}}}. 
\end{equation}
Hence we obtain
\[ \| tD_t F_j \|_{L_t^2(L^2)} \leq C d_j 2^j (1 + \| u_0 \|_{H_{1/2}^j}) \| u_0 \|_{B_{1/2}^j} \| u_0^i \|_{B_{1/2}^j}. \]

Therefore, by summing up the above estimates, we achieve
\[ \| G_j \|_{L_t^2(L^2)} \leq C \sqrt{\eta} d_j 2^j. \]

With the estimate (4.23), we can follow the proof of Proposition 2.3 to prove that
\[ \| t \nabla D_t v_j \|_{L_t^2(L^2)}^2 + \| (D_t^2 v_j, \nabla D_t \pi_j) \|_{L_t^2(L^2)}^2 \leq C \eta d_j^2 2^j. \]

Thanks to (4.18) and (4.24), we deduce (2.20) via a similar derivation of (2.1). This completes the proof of the proposition. \( \square \)

5. The proof of Theorem 1.4

In this section, we shall modify the proof of Theorem 1.2 to prove Theorem 1.4. Let us first present the proof of Proposition 2.7.

**Proof of Proposition 2.7.** Let \((\rho, u)\) be a smooth enough solution of (1.1) on \([0, T^*].\) We construct \((u_j, \nabla \pi_j)\) via (2.2). Then there holds (2.3). With \(u_0 \in \dot{H}_{1/2}^{1+2\gamma},\) we deduce from (2.4) that
\[ \| u_j \|_{L_t^\infty(L^2)}^2 + \| \nabla u_j \|_{L_t^2(L^2)}^2 \leq C \| \Delta_j u_0 \|_{L_t^2}^2 \leq C c_j^2 2^{-j(1+2\gamma)} \| u_0 \|_{\dot{H}_{1/2}^{1+2\gamma}}^2. \]

Here and in the rest of this section, we always denote \((c_j)_{j \in \mathbb{Z}}\) to be a generic element of \(\ell^2(\mathbb{Z})\) so that \(\sum_{j \in \mathbb{Z}} c_j^2 = 1.\)

Whereas thanks to (2.5) and (2.7), we infer that there exists a positive constant \(c\) so that
\[ \frac{d}{dt} \| \nabla u_j(t) \|_{L_t^2}^2 + 2c \| (\partial_t u_j, \nabla^2 u_j, \nabla \pi_j) \|_{L_t^2}^2 \leq C \| u \cdot \nabla u_j \|_{L_t^2}^2 \text{ for } t < T^*. \]

Yet due to \(\gamma \in [0, 1/4],\) it follows from the law of product in homogeneous Sobolev spaces, (A.2), that
\[ \| u \cdot \nabla u_j \|_{L_t^2} \leq C \| u \|_{\dot{H}_{1/2}^{1+2\gamma}} \| \nabla u_j \|_{\dot{H}_{1/2}^{-1-2\gamma}} \]
\[ \leq C \| u \|_{\dot{H}_{1/2}^{1+2\gamma}} \| \nabla u_j \|_{L_t^2}^2 \| \nabla^2 u_j \|_{L_t^2}^{1-2\gamma}. \]

Then applying Young’s inequality gives
\[ C \| u \cdot \nabla u_j \|_{L_t^2} \leq C \| u \|_{\dot{H}_{1/2}^{1+2\gamma}} \| \nabla u_j \|_{L_t^2}^2 + c \| \nabla^2 u_j \|_{L_t^2}^2. \]

Thus we achieve
\[ \frac{d}{dt} \| \nabla u_j(t) \|_{L_t^2}^2 + c \| (\partial_t u_j, \nabla^2 u_j, \nabla \pi_j) \|_{L_t^2}^2 \leq C \| u \|_{\dot{H}_{1/2}^{1+2\gamma}} \| \nabla u_j \|_{L_t^2}^2 \text{ for } t < T^*. \]

Applying Gronwall’s inequality leads to
\[ \| \nabla u_j \|_{L_t^\infty(L^2)}^2 + c \| (\partial_t u_j, \nabla^2 u_j, \nabla \pi_j) \|_{L_t^2(L^2)}^2 \]
\[ \leq C \| \nabla \Delta_j u_0 \|_{L_t^2}^2 \exp \left( C \gamma \int_0^t \| u \|_{\dot{H}_{1/2}^{1+2\gamma}} \, dt' \right) \]
\[ \leq C c_j^2 2^{j(1+2\gamma)} \| u_0 \|_{\dot{H}_{1/2}^{1+2\gamma}}^2 \exp \left( C \gamma \int_0^t \| u \|_{\dot{H}_{1/2}^{1+2\gamma}} \, dt' \right) \text{ for } t < T^*. \]
In view of (5.1) and (5.4), for $t < t^*$, we deduce by a similar derivation of (2.1) that
\[
\begin{aligned}
\|\Delta_j u\|_{L_1^2(L^2)} + \|\nabla \Delta_j u\|_{L_1^2(L^2)} &\leq \sum_{j' > j} \left( \|u_{j'}\|_{L_1^\infty(L^2)} + \|\nabla u_{j'}\|_{L_1^2(L^2)} \right) \\
+ 2^{-j} \sum_{j' \leq j} \left( \|\nabla u_{j'}\|_{L_1^\infty(L^2)} + \|\nabla^2 u_{j'}\|_{L_1^2(L^2)} \right)
\end{aligned}
\]
(5.5)
which implies for $t < t^*$
\[
\begin{aligned}
\|u\|_{L_t^\infty(H^{\frac{3}{2} + 2\gamma})}^2 + \|\nabla u\|_{L_t^2(H^{\frac{3}{2} + 2\gamma})}^2 &\leq C\|u_0\|_{H^{\frac{3}{2} + 2\gamma}}^2 \exp \left( \frac{C}{\gamma} \int_0^t \|u\|_{H^{\frac{3}{2} + 2\gamma}}^{\frac{1}{2}} \, dt' \right),
\end{aligned}
\]
(5.6)
Let us denote
\[
T_4^* \overset{\text{def}}{=} \left\{ t < T^* \mid \|u\|_{L_t^\infty(H^{\frac{3}{2} + 2\gamma})} \leq 2C\|u_0\|_{H^{\frac{3}{2} + 2\gamma}} \right\}.
\]
Then for $t \leq \min \left( T_4^*, \frac{\gamma \ln \left( \frac{T^*}{2} \right)}{2^{\frac{3}{2}} C^{1 + \frac{1}{\gamma}} \|u_0\|_{H^{\frac{3}{2} + 2\gamma}}^{\frac{1}{2}} \right) $, we deduce from (5.6) that
\[
\begin{aligned}
\|u\|_{L_t^\infty(H^{\frac{3}{2} + 2\gamma})}^2 + \|\nabla u\|_{L_t^2(H^{\frac{3}{2} + 2\gamma})}^2 &\leq 3C\|u_0\|_{H^{\frac{3}{2} + 2\gamma}}^2.
\end{aligned}
\]
(5.8)
This in turn shows that
\[
T^* \geq \frac{\gamma \ln \left( \frac{T^*}{2} \right)}{2^{\frac{3}{2}} C^{1 + \frac{1}{\gamma}} \|u_0\|_{H^{\frac{3}{2} + 2\gamma}}^{\frac{1}{2}} \overset{\text{def}}{=} T_\gamma,
\]
and for $t \leq T_\gamma$, there holds (2.22). This completes the proof of Proposition 2.7. \qed

**Remark 5.1.** It follows from the definition of $T_\gamma$ that
\[
\exp \left( \frac{C}{\gamma} \int_0^{T_\gamma} \|u(t)\|_{H^{\frac{3}{2} + 2\gamma}}^{\frac{1}{2}} \, dt \right) \leq 4.
\]
(5.9)

**Proof of Proposition 2.8.** Due to the energy conservation law, we have
\[
\begin{aligned}
\frac{1}{2} \|\sqrt{\rho} u\|_{L_t^\infty(L^2)}^2 + \|\nabla u\|_{L_t^2(L^2)}^2 &\leq \frac{1}{2} \|\sqrt{\rho_0} u_0\|_{L^2}^2 \quad \text{for} \quad t < T^*,
\end{aligned}
\]
from which and (2.22), we obtain
\[
\|u\|_{L_t^2(H^{\frac{3}{2}})} \leq C\|\nabla u\|_{L_t^\infty(H^{\frac{3}{2} + 2\gamma})} \leq C\|u_0\|_{H^{\frac{3}{2} + 2\gamma}} \quad \text{for} \quad t \leq T_\gamma.
\]
(5.10)
On the other hand, by multiplying (5.3) by $t$ and then applying Gronwall’s inequality, we get
\[
\begin{aligned}
\|\sqrt{t} \nabla u_j\|_{L_t^\infty(L^2)}^2 &+ \|\nabla (\partial_t u_j, \nabla^2 u_j, \nabla \pi_j)\|_{L_t^2(L^2)}^2 \\
&\leq C\|\nabla u_j\|_{L_t^2(L^2)}^2 \exp \left( \frac{C}{\gamma} \int_0^t \|u\|_{H^{\frac{3}{2} + 2\gamma}}^{\frac{1}{2}} \, dt' \right).
\end{aligned}
\]
(5.11)
which together with (5.1) and (5.9) implies that for $t \leq T_\gamma$
\[
\|\sqrt{t} \nabla u_j\|_{L_t^\infty(L^2)}^2 + \|\nabla (\partial_t u_j, \nabla^2 u_j, \nabla \pi_j)\|_{L_t^2(L^2)}^2 \leq C c_j 2^{-j(\frac{3}{2} + 2\gamma)} \|u_0\|_{H^{\frac{3}{2} + 2\gamma}}^2.
\]
With (5.10), we get, by a similar derivation of (5.11), that
\[\|\partial_t u_j\|_{L^2_t(L^2)} \lesssim d_j 2^{j\frac{4}{3}} \|u_0\|_{B^\frac{1}{2}},\]
which implies
\[\|t^{\frac{4}{3}} u_j\|_{L^2_t(L^2)} \leq C \|u_0\|_{B^\frac{1}{2}} \leq C \|u_0\|_{H^{\frac{1}{2}+2\gamma}}.\]

Then by virtue of (3.5), (5.10) and (5.12), we achieve
\[\|\sqrt{t} \partial_t u_j\|_{L^2_t(L^2)} + \|\sqrt{t} \nabla \partial_t u_j\|_{L^2_t(L^2)} \leq C \exp(\int_0^t (\|u\|_{B^\frac{1}{2}}^2 + \|t^{\frac{4}{3}} u_t\|_{L^2}^2) dt') \times (\|\sqrt{t} \partial_t u_j\|_{L^2_t(L^2)} + \|t^{\frac{4}{3}} u_t\|_{L^2_t(L^2)} \|\nabla u_j\|_{L^2_t(L^2)}) \lesssim C \gamma 2^{j(\frac{4}{3} - 2\gamma)} \|u_0\|_{H^{\frac{1}{2}+2\gamma}},\]

Whereas we deduce from (2.7) and (5.2) that
\[\|\nabla^2 u_j\|_{L^2} \leq C (\|\partial_t u_j\|_{L^2} + \|u \cdot \nabla u_j\|_{L^2}) \leq C (\|\partial_t u_j\|_{L^2} + \|u\|_{H^{\frac{1}{2}+2\gamma}} \|\nabla u_j\|_{L^2} \|\nabla^2 u_j\|_{L^2}^{1-2\gamma}) \leq C (\|\partial_t u_j\|_{L^2} + \|u\|_{H^{\frac{1}{2}+2\gamma}} \|\nabla u_j\|_{L^2} + \frac{1}{2} \|\nabla^2 u_j\|_{L^2},\]

which together with (5.4) and (5.13) ensures that
\[\|\sqrt{t} \nabla^2 u_j\|_{L^2_t(L^2)} \leq C (\|\sqrt{t} \partial_t u_j\|_{L^2_t(L^2)} + \sqrt{T} \|u\|_{L^2_t(H^{\frac{1}{2}+2\gamma})} \|\nabla u_j\|_{L^2_t(L^2)}) \lesssim C \gamma 2^{j(\frac{4}{3} - 2\gamma)} \|u_0\|_{H^{\frac{1}{2}+2\gamma}}.\]

Thanks to (5.11), (5.13) and (5.14), we conclude the proof of (2.23) by following the same line as (5.5).

**Remark 5.2.** Thanks to (5.10) and (5.12), we get, by a similar proof of Corollary 3.2, that
\[\|t D_t u_j\|_{L^2_t(L^2)} + \|t \nabla D_t u_j\|_{L^2_t(L^2)} \leq C \gamma 2^{j(\frac{4}{3} + 2\gamma)} \|u_0\|_{H^{\frac{1}{2}+2\gamma}},\]
and there holds (3.10) for \( t \leq T_\gamma \).
Furthermore, by virtue of (5.13) and (5.14), we get, by a similar derivation of (3.16), that
\[\|t \nabla^2 u_j, \nabla \pi_j\|_{L^2_t(L^6)} \leq C \gamma 2^{j(\frac{4}{3} + 2\gamma)} \|u_0\|_{H^{\frac{1}{2}+2\gamma}} \text{ for } t \leq T_\gamma \]

With Remark 5.2, we can follow the proof of Proposition 2.3 to conclude the proof of Proposition 2.9, which we omit the details here.

**Appendix A. Tool box on Littlewood-Paley theory**

For the convenience of the readers, we recall some basic facts on Littlewood-Paley theory from [6]. Let
\[\Delta_j a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|) \hat{a}), \text{ and } S_j a \overset{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{a}),\]
where \( \mathcal{F}a \) and \( \hat{a} \) denote the Fourier transform of the distribution \( a \), \( \chi(\tau) \) and \( \varphi(\tau) \) are smooth functions such that
\[\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \text{ and } \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1,\]
Supp $\chi \subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\}$ and $\forall \tau \in \mathbb{R}$, $\chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1$.

**Definition A.1.** Let $(p, r)$ be in $[1, \infty]^2$ and $s$ in $\mathbb{R}$. Let us consider $u$ in $\mathcal{S}'_h(\mathbb{R}^3)$, which means that $u$ is in $\mathcal{S}'(\mathbb{R}^3)$ and satisfies $\lim_{j \to -\infty} ||S_j u||_{L^\infty} = 0$. We set $$||u||_{\hat{B}^{s}_{p, r}} \overset{\text{def}}{=} \left(\sum_{j \in \mathbb{Z}} 2^{js} ||\Delta_j u||_{L^p} \right)_{j \in \mathbb{Z}}.$$

- if $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $\hat{B}^{s}_{p, r}(\mathbb{R}^3) \overset{\text{def}}{=} \{ u \in \mathcal{S}'_h(\mathbb{R}^3) / ||u||_{\hat{B}^{s}_{p, r}} < \infty \}$.
- if $k \in \mathbb{N}$ and if $\frac{3}{p} + k \leq \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then we define $\hat{B}^{s}_{p, r}(\mathbb{R}^3)$ as the subset of $u \mathcal{S}'_h(\mathbb{R}^3)$ such that $\partial^3 u$ belongs to $\hat{B}^{s-k}_{p, r}(\mathbb{R}^3)$ whenever $|\beta| = k$.

We remark that $\hat{B}^{s}_{2, 2}$ coincides with the classical homogeneous Sobolev spaces $\dot{H}^s$.

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin-Lerner type spaces $\dot{L}^q_T(\hat{B}^{s}_{p, r}(\mathbb{R}^3))$.

**Definition A.2.** Let $s \leq \frac{3}{p}$ (respectively $s \in \mathbb{R}$), $(q, p, r) \in [1, +\infty]^3$ and $T \in ]0, +\infty]$. We define $\dot{L}^q_T(\hat{B}^{s}_{p, r}(\mathbb{R}^3))$ as the completion of $C([0, T]; \mathcal{S}'_h(\mathbb{R}^3))$ by the norm $$||f||_{\dot{L}^q_T(\hat{B}^{s}_{p, r})} \overset{\text{def}}{=} \left(\sum_{j \in \mathbb{Z}} 2^{js} \left(\int_0^T \|\Delta_j f(t)||_{L^p}^q \, dt\right)^{\frac{r}{q}} \right)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For short, we just denote this space by $\dot{L}^q_T(\hat{B}^{s}_{p, r})$.

We also frequently use the following Lemmas from [6]:

**Lemma A.1** (Corollary 5.5 of [6]). Let $(s_1, s_2) \in [-3/2, 3/2]^2$, a constant $C$ exists such that if $s_1 + s_2$ is positive, then we have

\begin{equation}
A.2 \quad ||fg||_{\hat{B}^{s_1+s_2}_{2, 2}} \leq C ||f||_{\dot{H}^{s_1}} ||g||_{\dot{H}^{s_2}}.
\end{equation}

**Lemma A.2** (Lemma 2.4 of [6]). Let $C$ be an annulus. Positive constants $c$ and $C$ exist such that for any $p \in [1, \infty]$ and any couple $(\lambda, \tau)$ of positive real numbers, we have

$$\text{Supp} \hat{u} \subset \lambda C \Rightarrow ||e^{\lambda \Delta} f||_{L^p} \leq Ce^{-c\lambda^2} ||f||_{L^p}.$$

**Acknowledgments.** The author would like to thank Professor Jean-Yves Chemin for introducing him the reference [7]. P. Zhang is partially supported by NSF of China under Grants 11371347 and 11688101, and innovation grant from National Center for Mathematics and Interdisciplinary Sciences of The Chinese Academy of Sciences.

**References**

[1] H. Abidi, Équation de Navier-Stokes avec densité et viscosité variables dans l’espace critique, Rev. Mat. Iberoam., **23**(2) (2007), 537–586.

[2] H. Abidi and M. Paicu, Existence globale pour un fluide inhomogène, Ann. Inst. Fourier (Grenoble), **57** (2007), 883–917.

[3] H. Abidi, G. Gui and P. Zhang, On the wellposedness of 3-D inhomogeneous Navier-Stokes equations in the critical spaces, Arch. Ration. Mech. Anal., **204** (2012), 189-230.

[4] H. Abidi, G. Gui and P. Zhang, Wellposedness of 3-D inhomogeneous Navier-Stokes equations with highly oscillating initial velocity field, J. Math. Pures Appl., **100** (2013), 166–203.
[5] D. Chen, Z. Zhang and W. Zhao, Fujita-Kato theorem for the 3-D inhomogeneous Navier-Stokes equations, *J. Differential Equations*, 261 (2016), 738-761.

[6] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, 343, Springer-Verlag Berlin Heidelberg, 2011.

[7] J.-Y. Chemin and I. Gallagher, A non-linear estimate on the life span of solutions of the three dimensional Navier-Stokes equations. arXiv:1801.07439.

[8] R. Danchin, Density-dependent incompressible viscous fluids in critical spaces, *Proc. Roy. Soc. Edinburgh Sect. A*, 133 (2003), 1311–1334.

[9] R. Danchin and P. B. Mucha, A Lagrangian approach for the incompressible Navier-Stokes equations with variable density, *Comm. Pure. Appl. Math.*, 65 (2012), 1458-1480.

[10] R. Danchin and P. B. Mucha, Incompressible flows with piecewise constant density, *Arch. Ration. Mech. Anal.*, 207 (2013), 991-1023.

[11] R. Danchin and P. B. Mucha, The incompressible Navier-Stokes equations in vacuum. arXiv:1705.06061.

[12] R. Danchin and P. Zhang, Wellposedness to the initial boundary value problem of inhomogeneous Navier-Stokes equations with bounded density, *J. Funct. Anal.*, 267 (2014), 2371-2436.

[13] R. Danchin and X. Zhang, On the persistence of Hölder regular patches of density for the inhomogeneous Navier-Stokes equations, *J. Éc. Polytech. Math.*, 4 (2017), 781-811.

[14] T. Hmidi and S. Keraani, Incompressible viscous flows in borderline Besov spaces, *Arch. Ration. Mech. Anal.*, 189 (2008), 263-300.

[15] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations*, 120 (1995), 215-254.

[16] J. Huang, M. Paicu and P. Zhang, Global wellposedness to incompressible inhomogeneous fluid system with bounded density and non-Lipschitz velocity, *Arch. Ration. Mech. Anal.*, 209 (2013), 631-682.

[17] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, *Arch. Ration. Mech. Anal.*, 16 (1964), 269-315.

[18] F. Gancedo and E. Garcia-Juarez, Global regularity of 2D density patch for inhomogeneous Navier-Stokes. arXiv:1612.08665.

[19] A. Kazhikhov, Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid, *Dokl. Akad. Nauk SSSR*, 216 (1974), 1008-1010.

[20] O. A. Ladyženskaja and V. A. Solonnikov, The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. (Russian) Boundary value problems of mathematical physics, and related questions of the theory of functions, 8, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 52 (1975), 52–109, 218–219.

[21] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, *Acta Math.*, 63 (1934), 193-248.

[22] X. Liao and P. Zhang, On the global regularity of two-dimensional density patch for inhomogeneous incompressible viscous flow, *Arch. Ration. Mech. Anal.*, 220 (2016), 937-981.

[23] X. Liao and P. Zhang, Global regularity of 2-D density patches for viscous inhomogeneous incompressible flow with general density: low regularity case, *Comm. Pure Appl. Math.* in press.

[24] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*, Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.

[25] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system, *J. Funct. Anal.*, 262 (2012), 3556-3584.

[26] M. Paicu, P. Zhang and Z. Zhang, Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density, *Comm. Partial Differential Equations*, 38 (2013), 1208-1234.

[27] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, *SIAM J. Math. Anal.*, 21 (1990), 1093-1117.