On integrability of the Killing equation

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Abstract
Killing tensor fields have been thought of as describing the hidden symmetry of space(-time) since they are in one-to-one correspondence with polynomial first integrals of geodesic equations. Since many problems in classical mechanics can be formulated as geodesic problems in curved space and spacetime, solving the defining equation for Killing tensor fields (the Killing equation) is a powerful way to integrate equations of motion. Thus it has been desirable to formulate the integrability conditions of the Killing equation, which serve to determine the number of linearly independent solutions and also to restrict the possible forms of solutions tightly. In this paper, we show the prolongation for the Killing equation in a manner that uses Young symmetrizers. Using the prolonged equations, we provide the integrability conditions explicitly.

Keywords: Killing tensor, integrable system, integrability condition, prolongation, Young symmetry

1. Introduction

Many problems in classical mechanics can be formulated as geodesic problems in curved space(-time). For instance, Euler’s equations for a rigid body, such as the Euler top, are described as the geodesic equations on the Lie groups with the corresponding metrics (see, e.g. [1]). In general relativity, the motion of a particle in a gravitational field is described as geodesics in a curved spacetime. Less well known is that, even in the presence of an external force, the motion of a particle can be framed as the geodesic problem in an effective curved space(-time) in the same or higher dimensions (see [2] for a review).

In the Hamiltonian formulation, the geodesic equations in a curved space(-time) \( M \) with a metric \( g_{\alpha\beta} \) are given by Hamilton’s equations \( \dot{x}^\alpha = \partial H / \partial p_\alpha, \quad \dot{p}_\alpha = - \partial H / \partial x^\alpha \) with the Hamiltonian \( H = (1/2) g^{\alpha\beta} p_\alpha p_\beta \). According to the Liouville–Arnold theorem, if there exist
first integrals that are in involution as many as the number of dimensions, Hamilton’s equations are integrable in the Liouville sense. However, it is not always easy to find first integrals for a given Hamiltonian. This has motivated many authors to develop various methods to investigate the integrability structure of the geodesic equations.

In this paper we focus on Killing tensor fields (KTs) in a curved space(-time), which have been thought of as describing the hidden symmetry of space(-time) since they are in one-to-one correspondence with polynomial first integrals of the geodesic equations. A KT of order $p$, denoted by $K_{a_1\cdots a_p}$, is a symmetric tensor field $K_{a_1\cdots a_p} = K_{(a_1\cdots a_p)}$ that satisfies the Killing equation

$$\nabla_{(b} K_{a_1\cdots a_p)} = 0,$$

where $\nabla$ is the Levi-Civita connection, and the round brackets denote symmetrization over the enclosed indices. Square brackets over indices will be used for antisymmetrization. A metric is a trivial KT, which is always a solution of the Killing equation. Hence it has been asked whether the Killing equation has nontrivial solutions for a given metric. KTIs of order 1 are known as Killing vector fields (KVs), which have been actively studied as spacetime symmetry (isometry). KTIs of order 2 have also been considerably studied in connection with separation of variables in Hamilton–Jacobi equations [3–5]. In general relativity, a nontrivial KT of order 2 has been discovered in the Kerr spacetime [6, 7]. Since then, nontrivial KTIs have been investigated in various black hole spacetimes in four and higher dimensions [8–15]. In recent years higher-order KTIs have attracted much interest [16–18].

The main purpose of this paper is to write out the integrability conditions of the Killing equation, which has the distinguished role of computing the number of linearly independent solutions and also restricting the possible forms of solutions tightly. To this end, one can employ the so-called prolongation procedure (see [19–21]). For instance, it is well known that the Killing equation for KVs,

$$\nabla (\nabla^a \xi_b) = 0,$$

leads to the first-order linear partial differential equations for $\xi_a$ and $\omega_{ab}$ (see, e.g. [22]),

$$\nabla \nabla^a \xi_b = \omega_{ab},$$

$$\nabla \omega_{bc} = R^{d}_{cda} \xi_d,$$

where $\omega_{ab} = \nabla (\nabla^a \xi_b)$ and $R^{d}_{cda}$ is the Riemann curvature tensor. The procedure used here in deriving equations (3) and (4) from equation (2) is known as prolongation. Hereafter, such equations obtained by prolongation are referred to as the prolonged equations.

The prolonged equations (3) and (4) can be viewed as equations for parallel sections of the vector bundle $E^{(1)} \equiv T^*M \oplus \Lambda^2 T^*M$, where $\xi_a$ and $\omega_{ab}$ are sections of $T^*M$ and $\Lambda^2 T^*M$, respectively. Since it is shown that Killing vector fields are in one-to-one correspondence with parallel sections of $E^{(1)}$, the dimension of the space of KVs is bound by the rank of $E^{(1)}$, i.e. $n(n + 1)/2$ in $n$ dimensions. Similarly, it is shown that KTIs of order $p$ are in one-to-one correspondence with the parallel sections of a certain vector bundle $E^{(p)}$ [23, 24]. This leads to the Barbance–Delong–Takeuchi–Thompson (BDTT) formula [25–28]

$$\dim K^p(M) \leq \frac{1}{n} \left(\begin{array}{c} n + p \\ p + 1 \end{array}\right) \left(\begin{array}{c} n + p - 1 \\ p \end{array}\right) = \text{rank } E^{(p)},$$

where $K^p(M)$ denotes the space of KTIs of order $p$ in an $n$-dimensional space(-time) $M$. The equality is attained if and only if $M$ is of constant curvature.
In principle, if the prolonged equations are provided, it is possible to obtain the integrability conditions. Indeed, they can be found for order 1 [29] and order 2 [29–32]. The integrability conditions for \( p \geq 3 \) were discussed in [33]. Similar techniques have been applied to other hidden symmetries [34–39].

While such a structure of the prolongation for the Killing equation has been realized, it is not so easy to write out the integrability conditions for \( p \geq 2 \). Even writing the prolonged equations out is rather hard since the expressions become more and more complicated as \( p \) becomes larger. Thus the integrability conditions for \( p \geq 3 \) have never been provided explicitly. The expressions provided in [33] are still complicated for elucidating the underlying structure.

To challenge this task, our strategy is to take advantage of the structure of prolongation. We use Young symmetrizers. A Young symmetrizer is the operator that acts on a tensor field of order \( p \) and projects it onto an irreducible representation of \( GL(n) \). Since Young symmetrizers have many useful properties, we can derive the prolonged equations of the Killing equation for a general order and obtain the integrability conditions explicitly for \( p = 1, 2 \) and \( 3 \). To our knowledge, the integrability condition for \( p = 3 \) is new in the sense that the expressions are written explicitly, and they will be useful to investigate KTs of order 3 for various metrics. For \( p > 3 \), although we do not write out the integrability conditions, we make a conjecture on them from several observations for \( p = 1, 2 \) and \( 3 \).

This paper is organized as follows: In section 2, we formulate the prolonged equations for the Killing equation in a manner that uses Young symmetrizers. At the same time, we will briefly explain the definitions and properties of Young symmetrizers there. For readers who are not familiar with them, refer to appendix A and references therein. The integrability conditions of the Killing equation are investigated in section 3. We explicitly provide the integrability conditions for order \( p = 1, 2 \) and \( 3 \). We also make a conjecture on the integrability condition for a general order and show a method for computing the dimension of the space of KTs. In section 4 we conclude with a brief discussion. Four appendices are devoted to technical details. Appendix A describes the properties and some formulas of Young symmetrizers. Appendix B presents a prolongation procedure for the Killing equation up to order 2. A derivation of the integrability condition (37) is given in appendix C. In appendix D, we discuss the Killing–Yano equation by using our analysis.

## 2. Prolongation of the Killing equation

We shall formulate the prolonged equations for the Killing equation of order \( p \) in a manner that uses Young symmetrizers. For this purpose, we commence by giving a brief definition of Young symmetrizers (see appendix A for details).

A Young symmetrizer \( Y_\Theta \) is the projection operator corresponding to a Young tableau \( \Theta \), which makes row-by-row symmetrization and column-by-column antisymmetrization sequentially. Specifically, it reads

\[
Y_\Theta \equiv \alpha_\Theta \prod_{C_j \in \text{col}(\Theta)} \hat{A}_{C_j} \prod_{R_i \in \text{row}(\Theta)} \hat{S}_{R_i},
\]

where \( \theta \) is the shape of \( \Theta \), i.e. the Young diagram eliminates the numbers from the tableau \( \Theta \), \( \hat{S}_{R_i} \) (\( \hat{A}_{C_j} \)) denotes the (anti-)symmetrization of the slots corresponding to the entries in the \( i \)-th row (column) of the tableau \( \Theta \), and \( \alpha_\Theta \) is the normalization factor determined by the idempotency \( Y_\Theta^2 = Y_\Theta \).
Practically, $\alpha_\Theta$ is calculated as follows: since $\hat{S}_b$ and $\hat{A}_c$ also have the normalization factors determined by the idempotency, $\alpha_\Theta$ is given by the product of the normalization factors of all $\hat{S}_b$ and $\hat{A}_c$ over the product of the hook lengths of all the boxes of $\Theta$, say $\|\Theta\|$. For instance, let $\Theta$ be the Young tableau $\begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
\end{array}$. Then, $Y_{\Theta} = \alpha_{\Theta,\hat{S}_b,\hat{A}_c} = \frac{(2!)^4}{\|\Theta\|} = 4/3$. In particular, $Y_{\Theta}$ projects a tensor field of order 4 onto the part of the representation $\begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
\end{array}$, which has the same representation as the Riemann curvature tensor.

2.1. Prolonged equations

In this subsection, we first provide the prolonged equations for the Killing equation without any proof. A sketch of the proof is shown later in this subsection.

To provide the prolonged equations, we introduce the prolongation variables, $K_{b_1\ldots b_q a_1\ldots a_1}^{(q)} = Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \sum_{i=2}^{q} Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \nabla_{b_1\ldots b_q} K_{a_1\ldots a_1}, \quad (1 \leq q \leq p) \quad (7)

where $\nabla_{a_1\ldots c} \equiv \nabla_{a_1} \nabla_{b_1} \cdots \nabla_{c_1} K_{a_1\ldots a_1}$ is a KT of order $p$. We remark that one needs $(p + 1)$ prolongation variables to carry out the prolongation for KT's, while that for Killing–Yano tensor fields involves only two prolongation variables for any order (see [34, 36] or appendix D). This fact complicates the prolongation for the Killing equation (1).

We are now ready to provide the prolonged equations. The prolongation for the Killing equation of order $p$ can be achieved as follows:

$\nabla_c K_{a_1 \cdots a_1} = Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} K_{a_1 \cdots a_1}, \quad (8)$

$\nabla_c K_{b_1 \cdots b_q a_1 \cdots a_1}^{(q)} = Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \left( Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} + \sum_{i=2}^{q} Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \nabla_{b_1} K_{a_1 \cdots a_1} \right), \quad (1 \leq q \leq p - 1) \quad (9)$

$\nabla_c K_{b_1 \cdots b_q a_1 \cdots a_1}^{(p)} = Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \left( Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} + \sum_{i=2}^{p} Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \nabla_{b_1} K_{a_1 \cdots a_1} \right), \quad (10)$

where the slashed index $b_i$ is deleted from the Young tableau.

It is worth noting that the derivative terms look like being left on the right-hand side. However, by virtue of the properties of Young symmetrizers, those terms can be replaced with non-derivative terms whose coefficients consist of the Riemann curvature tensor and its derivatives. The proof is given by induction with respect to $q$ as follows: For a fixed $q$ ($1 \leq q \leq p$), the first term in the parenthesis of equations (9) and (10) reads

$\sum_{i=2}^{q} Y_{\Theta} \begin{array}{cccc}
abla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\nabla & \nabla & \nabla & \nabla \\
\end{array} \nabla_{c_1} (A_{a_1 b_1} A_{a_2 b_2} \cdots A_{a_q b_q} K_{a_1 \cdots a_1}) \right). \quad (11)$

Performing the symmetrization over the indices $b_1, \ldots, b_q$ in the above expression, we obtain the $q!$ terms. For each term, we then exchange $b_1$ with the index immediately to the left repeatedly as

$\nabla_{c b_1 \cdots [b_2 b_q]} = \nabla_{c b_1 \cdots b_2} + \nabla_{c b_1 \cdots [b_2 b_q]} = \nabla_{c b_1 \cdots b_2} + \nabla_{c b_1 \cdots [b_2 b_q]} = \cdots$,
until $b_1$ comes next to $c$. After that, we act $\hat{A}_{ab|cd}$ on the resulting terms so as to replace the outer two derivatives $\nabla_{cb}$ with the Riemann curvature tensor, confirming that equation (11) can be cast in the prolongation variables of lower orders than $q$ with the coefficients of the Riemann curvature tensor and its derivatives. Similarly, the summands in equations (9) and (10) can read

\[
\nabla_{cb} \cdots \nabla_{cb} K_{ab \cdots a_1} = 2 \nabla_{cb} \cdots [b_{cb}] K_{ab \cdots a_1},
\]

\[
\nabla_{cb} \cdots \nabla_{cb} K_{ab \cdots a_1} = (2 \nabla_{cb} \cdots [b_{cb}] b_1 K_{ab \cdots a_1} + 2 \nabla_{cb} \cdots b_1 b_2 b_1 K_{ab \cdots a_1}),
\]

and so on. We deduce that all the summands can also be cast in the prolongation variables of lower orders than $q$ with the coefficients of the Riemann curvature tensor and its derivatives. We therefore conclude that equations (8)–(10) are sufficient to state that the prolongation has been completed.

We show the explicit forms of the prolonged equations for $p = 1, 2$ and 3. The derivation of these results can be found in appendix B; the prolonged equations for KVs are given by

\[
\nabla_{cb} K_{ba} = K_{ba}^{(1)},
\]

\[
\nabla_{cb} K_{ba}^{(1)} = Y Y Y Y R_{cba} d K_d,
\]

which completely agree with equations (3) and (4); for order 2, the prolonged equations are given by

\[
\nabla_{cb} K_{ba}^{(2)} = Y Y \left[ K_{dcba}^{(2)} - \frac{1}{5} R_{dac} m K_{mb} - 2 R_{dab} m K_{mc} + \frac{1}{2} R_{acb} m K_{md} \right],
\]

\[
\nabla_{cb} K_{ba}^{(2)} = Y \left[ -\frac{1}{2} (\nabla_d R_{mbc} m) K_{me} - \frac{1}{3} (\nabla_d R_{mc} m) K_{md} - 8 (\nabla_d R_{mde} m) K_{mc} - 12 R_{eac} m K_{mbd}^{(1)} - 4 R_{eab} m K_{mec}^{(1)} - \frac{1}{2} R_{ebd} m K_{mde}^{(1)} + \frac{1}{2} R_{eab} m K_{mec}^{(1)} \right].
\]

Compared with the results of [29–32], our results have simpler forms. Taking one more step, we can write out the prolonged equations for order 3 explicitly.

\[
\nabla_{cb} K_{ba}^{(3)} = Y Y \left[ K_{dcba}^{(3)} - 3 R_{ead} m K_{hce} - 5 R_{eab} m K_{mde} - R_{dab} m K_{mce} \right],
\]

\[
\nabla_{cb} K_{ba}^{(3)} = Y Y Y Y \left[ K_{jdcba}^{(3)} + 2 (\nabla_d R_{ead} m) K_{mef} + 2 (\nabla_d R_{eac} m) K_{mdf} + 2 (\nabla_d R_{fac} m) K_{mdc} + 6 (\nabla_d R_{fad} m) K_{mce} - 2 (\nabla_d R_{efa} m) K_{mde} - 32 R_{bhe} m K_{mced}^{(1)} - 10 R_{bhe} m K_{mced}^{(1)} - 20 R_{bhe} m K_{mced}^{(1)} + \frac{10}{7} R_{bea} m K_{mfcde}^{(1)} + \frac{10}{7} R_{bea} m K_{mfcde}^{(1)} \right].
\]
\[ \nabla_c K_{fedba}^{(3)} = Y_{\begin{array}{c} a \\ b \end{array}} \left[ -24 R_{efjc}^{ \ m K_{mbdeca}^{(2)}} + 6 R_{fjcl}^{ \ m K_{mbeca}^{(2)}} + 4 R_{jfc}^{ \ m K_{mbecau}^{(2)}} - 12 R_{jfc}^{ \ m K_{mbcdeau}^{(2)}} \right. \\
- 20 (\nabla_d R_{ea}^{ \ m K_{mbcf}}^{(1)}) + 12 (\nabla_d R_{age}^{ \ m K_{mbcf}}^{(1)}) - 2 (\nabla_f R_{gde}^{ \ m K_{mbcf}}^{(1)}) \\
- \frac{1}{3} (\nabla_d R_{arf}^{ \ m K_{mbcf}}^{(1)}) - 16 (\nabla_d R_{aeb}^{ \ m K_{mbcf}}^{(1)}) - 3 (\nabla_f R_{gde}^{ \ m K_{mbcf}}^{(1)}) \\
+ \frac{9}{2} (\nabla_{fe} R_{dab}^{ \ m K_{mbcf}}^{(1)}) - \frac{9}{2} (\nabla_{ed} R_{cfg}^{ \ m K_{mbcf}}^{(1)}) - 3 (\nabla_{fe} R_{cfg}^{ \ m K_{mbcf}}^{(1)}) \\
+ 6 R_{fe}^{ \ m R_{efc}^{ n K_{mbaf} a} a} + 5 R_{gfc}^{ \ m R_{dgf}^{ n K_{mbaf} a} a} + 6 R_{fj}^{ \ m R_{fj}^{ n K_{mbaf} a} a} - 9 R_{dfe}^{ \ m R_{efc}^{ n K_{mbaf} a} a} \\
- 2 R_{fj}^{ \ m R_{mbaf} n K_{nag} a} + \frac{11}{2} R_{fe}^{ \ m R_{nbgd} n K_{nag} a}, \right]. \] (20)

Let us provide a sketch of the proof for the results (8)–(10). Since \( K_{ap...a1} \) is totally symmetric, we have
\[ \nabla_c K_{ap...a1} = Y_{\begin{array}{c} a \\ c \end{array}} \ id_{p+1} \ nabla_c K_{ap...a1}, \]
where \( id_{p+1} \) is the identity operator. Using the completeness of the Young symmetrizers with Littlewood’s correction (A.5) yields
\[ Y_{\begin{array}{c} a \\ c \end{array}} \ id_{p+1} \ nabla_c K_{ap...a1} = Y_{\begin{array}{c} a \\ c \end{array}} \left( L_{\begin{array}{c} a \\ b \end{array}} + \cdots \right) \nabla_c K_{ap...a1}. \]
The round brackets contain a lot of the Young symmetrizers. However, most of these symme-

hydrators vanish due to Pierì’s formula (A.15) and the Killing equation (1), leaving only
\[ L_{\begin{array}{c} a \\ b \end{array}} \]
Thus we obtain
\[ \nabla_c K_{ap...a1} = Y_{\begin{array}{c} a \\ c \end{array}} L_{\begin{array}{c} a \\ b \end{array}} \ nabla_c K_{ap...a1}. \] (21)
The tableau \( \begin{array}{c} a \\ c \end{array} \) is row-ordered and then \( \begin{array}{c} a \\ b \end{array} \) is equal to \( Y_{\begin{array}{c} a \\ b \end{array}} \), confirming equation (8). Similarly, differentiating the \( q \)th prolongation variable (7) for \( 1 \leq q \leq p \) gives
\[ \nabla_c K_{bq...a1}^{(q)} = Y_{\begin{array}{c} a \\ b \end{array}} \ nabla_c K_{bq...a1}, \]
\[ = Y_{\begin{array}{c} a \\ b \end{array}} \left[ L_{\begin{array}{c} a \\ c \end{array}} + L_{\begin{array}{c} a \\ d \end{array}} + L_{\begin{array}{c} a \\ e \end{array}} + \cdots + \sum_{i=2}^{q} L_{\begin{array}{c} a \\ i \end{array}} \right] \]
\[ \times \ nabla_c K_{bq...a1}. \] (22)
As we see from equations (A.10) and (A.11), all Littlewood’s corrections in the above expression vanish and thus \( L_{\begin{array}{c} a \\ b \end{array}} \) equals \( Y_{\begin{array}{c} a \\ b \end{array}} \). We therefore obtain equation (9), concluding the proof. Note that the expression (22) is also valid for \( q = p \) if \( L_{\begin{array}{c} a \\ b \end{array}} \) is omitted.

2.2. Geometric interpretation

Once the prolonged equations (8)–(10) have been formulated, one may forget the definitions of the prolongation variables (7) because one can reconstruct equations (1) and (7) from the prolonged equations (8)–(10) under the assumption
\[ K_{bq...a1}^{(q)} = Y_{\begin{array}{c} a \\ b \end{array}} K_{bq...a1}^{(q)}, \] (23)
which means that the prolonged equations (8)–(10) with the assumption (23) are equivalent to
the Killing equation (1). A proof of this assertion is given as follows: Suppose the prolonged
equations (8)–(10) with the assumption (23) hold. First, multiplying both sides of equation (8)
by \( \nabla_{(c} K_{ap}^{(1)} \) from the left gives
\[
\nabla_{(c} K_{ap}^{(1)} = Y c Y a \cdots Y a = 0,
\]
confirming the Killing equation (1). We have used the orthogonality of Young symmetrizers
(A.2) here. Next, multiplying both sides of equation (9) by \( Y a \cdots Y a \) from the left yields
\[
\nabla_{(c} K_{ap}^{(q)} = Y c Y a \cdots Y a \cdots Y a \cdots Y a \cdots Y a \cdots Y a = 0,
\]
which leads to
\[
K_{ap}^{(q+1)} = Y c Y a \cdots Y a \cdots K_{ap}^{(q+1)} = K_{ap}^{(q+1)},
\]
where we have used the identity
\[
Y a \cdots Y a = Y a \cdots Y a
\]
which follows from Schur’s lemma (A.13) and Raicu’s formula (A.14).

Geometrically, the set of variables (23) can be viewed as a section of the vector bundle \( E^{(p)} \)
over \( M \)
\[
E^{(p)} = \text{p boxes} \oplus \text{(p+1) boxes} \oplus \cdots \oplus \text{2p boxes}
\]
where the fibers are irreducible representations of \( \text{GL}(n) \) corresponding to the Young diagrams. Moreover, the prolonged equations (8)–(10) can be viewed as the parallel equation for
a section of \( E^{(p)} \),
\[
D_a K = 0,
\]
where \( D_a \equiv \nabla_{a} - \Omega_{a} \) is the connection on \( E^{(p)} \) and \( K \) is a section of \( E^{(p)} \). \( \Omega_a \in \text{End}(E^{(p)}) \)
depends on the Riemann curvature tensor and its derivatives up to \((p-1)\)th order which can be read off from the right-hand side of the prolonged equations (8)–(10). Hence it turns out that there is a one-to-one correspondence between KTs of order \( p \) and the parallel sections.

3. Integrability conditions

This section is devoted to investigating the integrability condition of the \( q \)th prolongation variable
of a KT of order \( p \), which arises as a consistency condition:
\[
0 = 2\Omega_{a_1 \cdots a_p} K_{a_1 \cdots a_p a_1} \equiv \nabla_{a_1} K_{a_1 \cdots a_p a_1} - \nabla_{a_1} K_{a_1 \cdots a_p a_1} - 2\nabla_{[a_1} K_{a_1 \cdots a_p a_1]},
\]
where the first two terms in equation (30) are evaluated by the \( q \)th prolonged equation (9); the last term in equation (30) is described by the defining equation of the Riemann curvature tensor.
3.1. Main results

Calculating the integrability condition (30) up to \( p = 3 \), we obtain the following results:

\[ p = 1 \]

\[
I_{abcd}^{(1,0)} = 0, \tag{31}
\]

\[
I_{abcd}^{(1,1)} = Y_{abcd} \left[ (\nabla_d R_{cba}^m) K_m - 2R_{cba}^m K^{(1)}_m \right], \tag{32}
\]

\[ p = 2 \]

\[
I_{abcde}^{(2,0)} = I_{abcde}^{(2,1)} = 0, \tag{33}
\]

\[
I_{abcdef}^{(2,2)} = Y_{abcdef} \left[ 3(\nabla_f R_{ecb}^m) K_{ma} + 2R_{edc}^m R_{mfb}^n K_{na} - 5R_{edc}^m R_{mfb}^n K_{na} \
+ 3(\nabla_f R_{edc}^m) K_{mab}^{(1)} - 9(\nabla_f R_{edc}^m) K_{mbc}^{(1)} - 8R_{edc}^m K_{mcb}^{(2)} \right], \tag{34}
\]

\[ p = 3 \]

\[
I_{abcdef}^{(3,0)} = I_{abcdef}^{(3,1)} = I_{abcdefg}^{(3,2)} = 0, \tag{35}
\]

\[
I_{abcdefh}^{(3,3)} = Y_{abcdefh} \left[ 6(\nabla_f R_{gde}^m) K_{nba} - 27(\nabla_h R_{gfe}^m) R_{mdc}^n K_{nba} - 34R_{gfh}^m (\nabla_e R_{mdc}^n) K_{nba} \
- 15(\nabla_d R_{gte}^m) R_{mcb}^n K_{nba} + 15(\nabla_h R_{gfe}^m) R_{dcb}^n K_{nma} - 20R_{gfh}^m (\nabla_e R_{mdc}^n) K_{nba} \
+ 20R_{gfh}^m (\nabla_e R_{dcb}^n) K_{nma} - 24(\nabla_h R_{gfe}^m) R_{mbc}^n K_{nma} + 12(\nabla_h R_{gfe}^m) R_{mbc}^n K_{nma} \
+ 50R_{gfh}^m R_{mec}^n K_{nba}^{(1)} + 40R_{gfh}^m R_{mec}^n K_{nba}^{(1)} - 24R_{gfh}^m R_{mbu}^n K_{nba}^{(1)} \
- 20R_{gfh}^m R_{edu}^n K_{nba}^{(1)} - 35(\nabla_e R_{ffe}^m) K_{mab}^{(2)} - 5(\nabla_e R_{ffe}^m) K_{mab}^{(2)} \
- 20R_{gfh}^m K_{nbd}^{(3)} \right]. \tag{36}
\]

From equations (31)–(36), it is observed that in the case \( q < p \) the integrability condition of the \( q \)th prolongation variable is automatically satisfied. More precisely, we can confirm that all of these conditions vanish identically up to the first and second Bianchi identities, \( R_{[abc]} = \nabla_e [R_{bc}] de = 0 \). In contrast, the integrability condition at \( q = p \) provides nontrivial relations among all the prolongation variables. This is consistent with the result in [40].

It is intriguing to note that the integrability condition of the \( p \)th prolongation variable belongs to the Young diagram of shape \((p + 1, p + 1)\). If we act the curvature operator on the \( p \)th prolongation variable, we obtain a \((2p + 2)\)th order tensor belonging to the representation \((1, 1) \otimes (p, p)\). It can be decomposed into the irreducible representations \((p, p, 1, 1), (p + 1, p, 1)\) and \((p + 1, p + 1)\) that are respectively described by the Young diagrams

\[
\begin{array}{c}
| & | \\
| & | \\
| & | \\
| & | \\
| & | \\
\end{array}
\]

However, we observe from equations (32), (34) and (36) that the integrability condition of the \( p \)th prolongation variable makes a non-trivial contribution only for the representation \((p + 1, p + 1)\). For instance the integrability condition \( I_{abcd}^{(1,1)} \) could have the representations

\[
\begin{array}{c}
| & | \\
| & | \\
| & | \\
| & | \\
| & | \\
\end{array}
\]

8
However, the result (32) claims that the first two representations do not appear for some reason. Thus we are led to make the following conjecture:

**Conjecture.** The integrability condition of the \( p \)th prolongation variable belongs to the representation described by the rectangular Young diagram \((p + 1, p + 1)\).

In general, it is not easy to write out the integrability condition of the \( p \)th prolongation variable. This difficulty becomes more prominent as the order of KTs increases. However, if this conjecture holds true for \( p \geq 4 \), then there is no need to calculate the terms that belong to the representations \( \cdots \) and \( \cdots \), thereby allowing us to obtain the formula

\[
I^{(p,p)}_{a_1 \cdots a_p b_1 \cdots b_p c d} = \sum_{a_1} \left[ 2 \nabla_d [c b_p] \cdots b_1 K_{a_p \cdots a_1} + 3 \nabla_d [b_p] \cdots b_1 K_{a_p \cdots a_1} \\
+ 4 \nabla_d [b_p] \cdots b_1 K_{a_p \cdots a_1} + \cdots + (p + 1) \nabla_d [b_p] \cdots b_1 K_{a_p \cdots a_1} - \nabla_d [c b_p] \cdots b_1 K_{a_p \cdots a_1} \right]
\]

\[+ \text{(the terms that belong to the representations } \cdots \text{ and } \cdots \). \] (37)

We have confirmed that for the cases \( p \leq 3 \), the last term exactly vanishes up to the first and second Bianchi identities, \( R_d [a b c] = 0 \). The proof of the formula (37) is given by appendix C.

As is the case with the prolonged equations (8)–(10), there are still a lot of derivative terms left in the right-hand side of equation (37). Then again, we can rewrite all these terms in equation (37) to non-derivative terms by using the prolonged equations. For \( p \geq 4 \), this is a challenging and daunting task that is beyond our scope here and will be considered in the future.

### 3.2. Application

As an application of the integrability conditions, we show a method for computing the number of linearly independent solutions to the Killing equation.

Let us recall the parallel equation (29). We introduce the curvature of the connection \( D_a \) as \( R^b_{abc} \equiv [D_a, D_b] K \). We call this the Killing curvature. All the integrability conditions of a KT of order \( p \) can be collectively expressed as

\[
R^b_{abc} = 0. \] (38)

By repeatedly differentiating the condition (38), we obtain the set of linear algebraic equations

\[
R^b_{abc} = 0, \quad (D_a R^b_{abc}) K = 0, \quad (D_a D_b R^b_{abc}) K = 0, \quad \ldots. \] (39)

After working out \( r \) differentiations, we are led to the system

\[
R^b_{abc} = 0, \] (40)

where the coefficient matrix \( R^b_{abc} \) depends on the Killing curvature and its derivatives. For example,
\[ R^D_0 = (R^D_{ab}) , \quad R^D_1 = \begin{pmatrix} R^D_{ab} \\ D_a R^D_{bc} \end{pmatrix} , \quad R^D_2 = \begin{pmatrix} R^D_{ab} \\ D_a D_b R^D_{cd} \end{pmatrix} . \]  

(41)

It is known that by applying the Frobenius theorem to the condition (40), the following theorem holds true (see, e.g. [41]).

**Theorem.** If we find the smallest natural number \( r_0 \) such that

\[ \text{rank} \ R^D_{r_0} = \text{rank} \ R^D_{r_0 + 1}, \]  

(42)

then it follows that \( \text{rank} \ R^D_{r_0} = \text{rank} \ R^D_{r_0 + r} \) for any natural number \( r \) and the dimension of the space of the KT reads

\[ \dim K^p = \text{rank} \ E^{(p)} - \text{rank} \ R^D_{r_0}, \]  

(43)

where \( \text{rank} \ E^{(p)} \) is given by the BDTT formula (5).

The condition (42) means that the components of the \((r_0 + 1)\)th order derivatives of the Killing curvature, \( D_{a_1} \cdots D_{a_{r_0+1}} R^D_{bc} \), can be expressed as the linear combinations of the components of the lower order derivatives than \( r_0 + 1 \). Hence, the components of the one-higher order derivatives, \( D_{a_1} \cdots D_{a_{r_0+2}} R^D_{bc} \), can also be expressed as the linear combinations of the lower order derivatives than \( r_0 + 1 \). By induction, we can conclude that the theorem holds true.

It should be remarked that computing the rank of the matrix \( R^D_j \) boils down to solving a system of the linear algebraic equations

\[ I^{(p)}_{a_1 \cdots a_p b_1 \cdots b_p c d} = 0, \quad \cdots, \quad \nabla e_1 \cdots e_p I^{(p)}_{a_1 \cdots a_p b_1 \cdots b_p c d} \bigg|_{DK=0} = 0, \]  

(44)

where \( I^{(p)}_{a_1 \cdots a_p b_1 \cdots b_p c d} \) is the integrability condition of the \( p \)th prolongation variable of a KT of order \( p \) defined by equation (30). If \( r_0 \) exists, equation (43) allows us to have the value of \( \dim K^p \). Otherwise differentiating the integrability condition (38) reveals a large number of additional conditions. We can stop the differentiation and conclude that no KT of order \( p \) exists if \( \text{rank} \ R^D_{r_0} \) is equal to \( \text{rank} \ E^{(p)} \). Based on this fact, we can determine the dimension of the space of KT.

To demonstrate the efficacy of our method, let us take the Kerr metric in Boyer–Lindquist coordinates:

\[ ds^2 = - \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4aMr \sin^2 \theta}{\Sigma} dr d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2, \]  

(45)

with

\[ \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \]  

(46)

and determine the number of the solutions to the Killing equation up to \( p = 2 \). As a higher order KT includes reducible ones, e.g. \( \xi(a, \phi) \) is a trivial KT if \( \xi^a \) and \( \xi^\phi \) are KVs, at first we must solve the integrability condition for \( p = 1 \).
For the \( p = 1 \) case, a section of the bundle \( E^{(1)} \) can be written as
\[
K = \begin{pmatrix} K_{ca} \\ K_{ba}^{(1)} \end{pmatrix}, \quad \text{with} \quad K_{ba}^{(1)} \in \mathbb{F}.
\] (47)

By solving the linear systems \( R_D^1 K = 0 \) and \( R_D^2 K = 0 \), that is
\[
I_{abcd}^{(1,1)} = 0, \quad \nabla_{K} I_{abcd}^{(1,1)} \big|_{DK=0} = 0,
\] (48) and
\[
I_{abcd}^{(1,1)} = 0, \quad \nabla_{K} I_{abcd}^{(1,1)} \big|_{DK=0} = 0, \quad \nabla_{ef} I_{abcd}^{(1,1)} \big|_{DK=0} = 0,
\] (49)
we find that \( \text{rank} R_D^1 = \text{rank} R_D^2 = 8 \). In other words, the adjoined equation in equation (49), \( \nabla_{ef} I_{abcd}^{(1,1)} \big|_{DK=0} = 0 \), does not change the rank. Since the maximal number of the KVs is \( \text{rank} E^{(1)} = 10 \), we can conclude that \( \dim K^1 = 2 \). This is consistent with our knowledge: the two vector fields \( \xi^a = (\partial_t)^a \) and \( \zeta^a = (\partial_\phi)^a \) are the only KVs in the Kerr metric (45).

Similarly, a section of the bundle \( E^{(2)} \) is given by
\[
K = \begin{pmatrix} K_{ba} \\ K_{cb}^{(1)} \\ K_{dcba}^{(2)} \end{pmatrix}, \quad \text{with} \quad K_{cb}^{(1)} \in \mathbb{F}, \quad K_{dcba}^{(2)} \in \mathbb{F}.
\] (50)

After solving the linear systems \( R_D^1 K = 0 \) and \( R_D^2 K = 0 \), we find that \( \text{rank} R_D^1 = \text{rank} R_D^2 = 45 \). It also follows from equation (5) that \( \text{rank} E^{(2)} = 50 \). This amounts to \( \dim K^2 = 5 \). We know that four of them
\[
g_{ab}, \quad \xi_\alpha (\xi_\beta), \quad \xi_\alpha (\zeta_\beta), \quad \zeta_\alpha (\zeta_\beta),
\] (51)
are reducible KTs while only one is not
\[
K_{ab} = a^2 \left[ \frac{\Delta \cos^2 \theta + r^2 \sin^2 \theta}{\Delta} \right] (dt)^2_{ab} - a^2 \Delta \cos^2 \theta \left[ (d\phi)^2_{ab} + r^2 \Sigma (d\phi)^2_{ab} \right] \\
+ \frac{\sin^2 \theta}{\Sigma} \left[ r^2 (a^2 + r^2) + a^4 \Delta \cos^2 \theta \sin^2 \theta \right] (d\phi)^2_{ab} \\
- 2a \sin^2 \theta \left[ r^2 (a^2 + r^2) + a^2 \Delta \cos^2 \theta \right] (dt)_{(a} (d\phi)_{b)}. \] (52)

Using the same method, we investigate the first and second order KTs in several regular black hole metrics, as shown in table 1. It would be of great interest to make a systematic investigation of higher-order KTs in various spacetimes. We leave it as a future work.
4. Summary and discussion

In this paper we have formulated the prolonged equations and the integrability conditions of the Killing equation (1) in a manner that uses Young symmetrizers. In particular, we have provided the explicit forms of the prolonged equations for a general order and the integrability conditions for \( p = 1, 2 \) and \( 3 \). We have also made a conjecture on the integrability conditions for \( p \geq 4 \) and shown their application for computing the dimension of the space of KTs.

While integrability conditions are simply the consequence of the requirement that mixed partial derivatives must commute, the explicit forms of them have brought us essential insights into general relativity, such as the Gauss–Codazzi equations in the Hamiltonian formulation of general relativity and the Raychaudhuri equations in the derivation of the singularity theorems. Similarly, the integrability conditions of the Killing equation for \( p = 1 \) lead to an immediate corollary (see, e.g. [42]). After some algebra, we can show that

\[
\nabla_{a_1 \cdots a_r} f^{(1)}_{b c d e} \bigg|_{DK=0} = 0 \iff \mathcal{L}_K \nabla_{a_1 \cdots a_r} R_{b c d e} \big|_{DK=0} = 0.
\]

(53)

where \( \mathcal{L}_K \) is the Lie derivative along a KV \( K \). This implies that if \( Q \) is the scalar constructed out of the Riemann curvature tensor and its derivatives, then \( \mathcal{L}_K Q \) must be zero. So if the set of the 1-form \( dQ^{(1)}, \ldots, dQ^{(n)} \) are linearly independent, the \( n \)-form

\[
dQ^{(1)} \wedge \cdots \wedge dQ^{(n)},
\]

(54)

must also be zero for \( n \)-dimensional metrics and is called a curvature obstruction. If any of the possible obstructions is not vanishing, such a metric admits no KV. We therefore expect that further analysis of the explicit forms of the integrability condition for \( p > 1 \) will lead to similar obstructions for KTs.

A natural question to ask is whether we can formulate the existence condition or hopefully the value of \( r_0 \) in equation (42). Answering this question may be linked to the conjecture we made in section 3. In fact, if the conjecture holds true, we obtain a criteria

\[
C = \frac{\text{rank } f^{(p,p)}}{\text{rank } E^{(p)}} = \frac{n(n-1)}{(p+2)(p+1)},
\]

(55)

where \( n \) and \( p \) are the dimension of space(-time) \( M \) and the order of KTs, respectively. Here, \( \text{rank } E^{(p)} \) agrees with the upper limit of the BDTT formula (5); \( \text{rank } f^{(p,p)} \) denotes the number of linearly independent components of the integrability condition (37). If \( C \leq 1 \), we definitely need the derivatives of \( f^{(p,p)} \) to determine the dimension of the space of KTs and thus \( r_0 > 0 \). The equality is attained when \( n = p + 2 \). Namely, our conjecture serves to formulate the lower bound on the value of \( r_0 \).

As shown in appendix D, our analysis based on Young symmetrizers has effective applications to other types of overdetermined PDE systems. The immediate examples include the Killing–Yano equation, the \( (p, q) \)-type Killing spinor equations and massless higher-spin field equations in four-dimensional spacetimes. To analyze the conformal Killing–Yano equations, we need not irreducible representations of \( \text{GL}(n) \) but those of \( \text{SO}(n) \). So, it will be necessary to incorporate the trace operation into our analysis. Such modifications have not been pursued in this paper but will be considered in the future.

It is worth noting the significance of our results in application to computer programs. In recent years various software implemented with a computer algebraic system, such as Mathematica and Maple, have been developed. Each piece of software prepares many packages available for solving individual problems in mathematics and physics, and we then find packages for solving the Killing equation for Killing vector fields as well as Killing and
Killing–Yano tensor fields. However most of these packages do not solve the Killing equations efficiently, as they merely use a built-in PDE solver without the integrability conditions. For fairness it should be mentioned that we found one package (e.g. KillingVectors in Maple), which does use the integrability conditions, albeit only for Killing vector fields and not for Killing and Killing–Yano tensor fields. Hence, in order to make such packages more efficient, especially for Killing tensor fields, our results in section 3 are significant to provide the formulas of the integrability conditions to be implemented.

Finally, we close this paper with a comment on the properties of Young symmetrizers. In this paper, we have used many properties on Young symmetrizers. For example, Raicu’s formula (A.14) has been used repeatedly to a simplification of the product of two Young symmetrizers. We remark that this formula can be applied only to the case when one of the two Young tableaux is contained in the other one. Thus we needed other formulas (A.16) and (A.17) on the product of two Young symmetrizers when we calculated the integrability conditions. To obtain the conditions for \( p > 3 \), other new formulas would be required.

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Appendix A. Young symmetrizer

The aim of this appendix is to introduce the reader to some properties and formulas of Young symmetrizers that play central roles in our calculations in appendix B. See [43] for more on Young tableaux and the representation theory of symmetric groups.

A.1. Basic properties

We briefly present the basic properties of Young symmetrizers defined by equation (6). In our notation the Latin letters \((a, b, c, \ldots)\) are identified as a naturally ordered set \((1, 2, 3, \ldots)\). Therefore, for instance, the standard Young tableau \(Y_1 Y_2\) is equated with \(Y_1 Y_2\) which is more suitable for tensor calculus. We also order the subscripted Latin letters \((a_1, a_2, \ldots, b_1, b_2, \ldots)\) as \(a_1 < a_2 < \cdots < b_1 < b_2 < \cdots\).

In what follows, we shall denote the set of all standard Young tableaux with \(k\) boxes by \(\mathcal{B}_k\), e.g.

\[ \mathcal{B}_2 = \left\{ \begin{array}{c} 1 \ 2 \\ 1 \ 3 \end{array} \right\}, \quad \mathcal{B}_3 = \left\{ \begin{array}{c} 1 \ 2 \ 3 \\ 1 \ 2 \ 4 \\ 1 \ 3 \ 5 \\ 2 \ 3 \ 6 \end{array} \right\}. \]

Young symmetrizers are endowed with the following three properties.

Idempotence

\[ Y^2 = Y, \quad \forall Y \in \mathcal{B}. \]  \hspace{1cm} (A.1)
Orthogonality

\[ Y_\Theta Y_\Phi = \delta_{\Theta \Phi}, \quad \forall \Theta, \Phi \in \mathfrak{Y}, \quad (k = 1, 2, 3, 4). \quad (A.2) \]

Completeness

\[ \sum_{\Theta \in \mathfrak{Y}} Y_\Theta = \text{id}_k, \quad (k = 1, 2, 3, 4). \quad (A.3) \]

It is worth mentioning that the orthogonality (A.2) holds for general \( k \) if the shapes of the tableaux \( \Theta \) and \( \Phi \) are different.

For tensor calculus, the completeness relation (A.3) plays an important role. For instance, a decomposition of a tensor field of order 2 into an irreducible representation of the general linear group GL(2) can be done as follows.

\[ T_{ab} = (Y_{ab} + Y_{ba})T_{ab} = T_{(ab)} + T_{[ab]} . \]

For a tensor field of order 3, a similar calculation reads

\[ T_{abc} = (Y_{abc} + Y_{acb} + Y_{bac} + Y_{bca})T_{abc} = T_{(abc)} + \frac{2}{3} \hat{A}_{ab} T_{(ab)c} + \frac{2}{3} \hat{A}_{ac} T_{(a|b)c} + T_{[abc]} . \]

But at \( k = 5 \) the subsequent calculation reaches a deadlock since the completeness relation (A.3) no longer holds for \( k \geq 5 \). The standard example of this is

\[ Y_{abc}Y_{def} = 0 \quad \text{but} \quad Y_{abc}Y_{def} \neq 0 . \]

A.2. Littlewood’s correction

In practical use, the failure of the orthogonality and completeness of the Young symmetrizers for \( k \geq 4 \) is crucial. This failure is complemented by Littlewood’s correction [44]. We shall present it here. For other prescriptions, see [45, 46].

Before going into the details, we introduce the following two definitions:

**Definition A.1 (Row-word of a Young tableau).** Let \( \Theta \in \mathfrak{Y} \) be a Young tableau. The row-word of \( \Theta \), say \( \text{row}(\Theta) \), is defined as the row vector whose entries are those of \( \Theta \) read row-wise from top to bottom.

For instance, suppose \( \Theta = \) \begin{equation} \begin{array}{c} a \hline b \\ c \hline d \end{array} \end{equation} \end{equation} Then the row-word of \( \Theta \) reads \( \text{row}(\Theta) = (a, b, c, d) \).

**Definition A.2 (Row-order relation).** Let \( \Theta \) and \( \Phi \) be two Young tableaux of the same shape. Denoting \( \text{row}(\Theta_i) \), as the \( i \)th component of \( \text{row}(\Theta) \), it is said that \( \Theta \) precedes \( \Phi \) and write \( \Theta \prec \Phi \) if \( \text{row}(\Theta_i) < \text{row}(\Phi_i) \), for the leftmost \( i \) where \( \text{row}(\Theta_i) \) and \( \text{row}(\Phi_i) \) differ.

Using the row-order relation, we can order the Young tableaux of the same shape, e.g.

\begin{align*}
\begin{array}{c|c|c|c|c}
\begin{array}{c} \hline a \\ \hline b \\ \hline c \\ \hline d
\end{array} & \prec & \begin{array}{c} \hline a \\ \hline b \\ \hline c \\ \hline e
\end{array} & \prec & \begin{array}{c} \hline a \\ \hline d \\ \hline b \\ \hline c
\end{array} & \prec & \begin{array}{c} \hline a \\ \hline d \\ \hline c \\ \hline e
\end{array}
\end{array}
\end{align*}

The following result is an easy consequence of the row-order relation: let \( \{\Theta_1, \Theta_2, \Theta_3, \ldots\} \) be the set of all Young tableaux in \( \mathfrak{Y} \) with a particular shape. Suppose this set be ordered as \( \Theta_i \prec \Theta_j \) whenever \( i < j \), one can see by inspection that the one-sided orthogonality...
\[ Y_{\Theta_i} Y_{\Theta_j} = 0, \quad (A.4) \]

holds.

We are now able to state Littlewood’s correction. The Young symmetrizer with Littlewood’s correction, say \( L_{\Theta_i} \), corresponding the tableau \( \Theta_i \in \{ \Theta_1, \Theta_2, \Theta_3, \ldots \} \) is iteratively defined by

\[
L_{\Theta_i} \equiv Y_{\Theta_i} \left( 1 - \sum_{j=1}^{i-1} L_{\Theta_j} \right),
\]

or the factorized form

\[
L_{\Theta_i} = Y_{\Theta_i} \prod_{j=1}^{i-1} \left( 1 - Y_{\Theta_{i-j}} \right).
\]

As proven in [44], the Young symmetrizers with correction (A.5) recover the orthogonality,

\[
L_{\Theta} L_{\Phi} = \delta_{\Theta \Phi} L_{\Phi}, \quad \forall \Theta, \Phi \in Y_k, \quad (A.7)
\]

and the completeness,

\[
\sum_{\Theta \in Y_k} L_{\Theta} = \text{id}_k, \quad (A.8)
\]

for general \( k \).

We note that all the corrections in (A.5) vanish for \( k \leq 4 \), so that \( L_{\Theta_i} \) are equivalent to \( Y_{\Theta_i} \).

Even for \( k > 4 \), many corrections vanish. For example, for \( k = 5 \), only two symmetrizers

\[
L_{a_{12} b_{12}} = Y_{a_{12} b_{12}} \left( 1 - Y_{a_{12} b_{12}} \right) \quad \text{and} \quad L_{a_{1} b_{1}} = Y_{a_{1} b_{1}} \left( 1 - Y_{a_{1} b_{1}} \right),
\]

differ from their original counterparts. Thus it is useful for practical use to make it clear what kind of the Young symmetrizes with Littlewood’s correction are equivalent to the original counterparts. Since the tableau \( \begin{array}{|c|c|}
1 & 2 \\
3 & 4 \\
\end{array} \) is row-ordered, it follows from the definition

\[
L_{\begin{array}{|c|c|}
1 & 2 \\
3 & 4 \\
\end{array}} = Y_{\begin{array}{|c|c|}
1 & 2 \\
3 & 4 \\
\end{array}}.
\]

It is also shown that

\[
L_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}} = Y_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}} \left( 1 - Y_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}} \right) = Y_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}}.
\]

\[
L_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}} = \begin{array}{|c|c|c|}
7 & 8 & 9 \\
10 & 11 & 12 \\
\end{array} \left( 1 - Y_{\begin{array}{|c|c|c|}
7 & 8 & 9 \\
10 & 11 & 12 \\
\end{array}} \right) \left( 1 - Y_{\begin{array}{|c|c|c|}
7 & 8 & 9 \\
10 & 11 & 12 \\
\end{array}} \right) = Y_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}}.
\]

and so on, where we have only used the relations \( \hat{S}_{a_1 b_1} \hat{A}_{a_1 b_1} = 0 \) and \( \hat{S}_{a_1 b_1} \hat{A}_{a_2 b_1} = 0 \). In general, the corrected symmetrizer

\[
L_{\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}} \quad \text{with} \quad p \geq q \geq 1, \quad q \geq i \geq 2, \quad (A.12)
\]

is coincident with its original counterpart by a trivial relation \( \hat{S}_{a_{i-1} b_1} \hat{A}_{a_{i-1} b_1} = 0 \).
A.3. Useful formulas

We collect some useful formulas to perform calculations in appendix B.

We first glance at the result referred to as Schur’s lemma in the context of the representation theory of symmetric groups. Let \( \Theta \) and \( \Phi \) be Young tableaux with \( k \) boxes. If \( Y_\Theta \) and \( Y_\Phi \) are orthogonal, that is \( Y_\Theta \ Y_\Phi = 0 \), then

\[
Y_{\Theta \sigma} \ Y_\Phi = 0, \tag{A.13}
\]

holds true for an arbitrary permutation \( \sigma \).

Second, we state Raicu’s formula as follows\(^3\). Let \( \Theta \in \mathcal{Y}_k \) and \( \Phi \in \mathcal{Y}_{k+1} \). Suppose that the unique entry in \( \Phi \) outside \( \Theta \) is located in the right edge of the tableau \( \Phi \), then

\[
Y_\Theta \ Y_\Phi = Y_\Phi, \tag{A.14}
\]

holds.

Next, we state an important result from the representation theory of symmetric groups. Let \( \theta \) and \( \phi \) be two Young diagrams with \( k \) and \( k+1 \) boxes respectively. It is said that \( \phi \) includes \( \theta \) and write \( \theta \subset \phi \) if \( \theta \) is a subdiagram of \( \phi \). Let \( \Theta \) and \( \Phi \) be Young tableaux of shapes \( \theta \) and \( \phi \) respectively, then

\[
Y_\Theta \ Y_\Phi = Y_\Phi \ Y_\Theta = 0, \tag{A.15}
\]

holds if \( \phi \) does not include \( \theta \). This result is called Pieri’s formula.

It should be noted that the first Bianchi identity, \( R^d_{[abc]} = 0 \), can be recaptured by Pieri’s formula. We know that \( R_{abcd} \) belongs to \( \mathcal{Y}_{a c b d} \) and hence the first Bianchi identity can be written in terms of Young symmetrizers as

\[
Y_a \ b \ c \ \ Y_a \ c \ b \ d = 0, \tag{A.16}
\]

which is clearly a type of Pieri’s formula. Therefore, we can say that Pieri’s formula is a generalization of the Bianchi identity.

At last, we state our findings as pertains to the Young diagram of shape \((p + 1, p + 1)\):

\[
Y a b c d [1 + (-1)^{p-1} \prod_{i=1}^{p} (a_i, b_i)] = 0, \tag{A.16}
\]

\[
Y a b c d \left[ 1 + \sum_{i=1}^{p-1} (a_i, b_i) \right] Y a b c d = 0, \tag{A.17}
\]

where \((a, b)\) denotes the permutation that swaps indices \(a\) and \(b\).

These formulas can be applicable to a simplification of the product of two Young symmetrizers or of the operands of the rectangular Young symmetrizer. We look at three examples before closing this appendix.

(i) Consider the product \( Y a b c d \ Y a b c d \). It is simplified to

\[
Y a b c d \ Y a b c d = Y a b c d \sum_{\Theta \in \mathcal{Y}_k} Y_\Theta = Y a b c d \ Y a b c d = Y a b c d \ Y a b c d = Y a b c d, \tag{A.18}
\]

\(^3\)To be precise, this result is an example of Raicu’s theorem. A complete wording of Raicu’s theorem can be found in [47].
where the second and third equalities follow from Schur’s lemma (A.13) and Raicu’s formula (A.14) respectively.

(ii) Another example is the product $\sum_{\Theta \in \mathcal{Y}} Y_{\Theta}$. A similar simplification can be done as

$$
\sum_{\Theta \in \mathcal{Y}} Y_{\Theta} = \sum_{\Theta \in \mathcal{Y}} \left( Y_{\Theta} + Y_{\Theta} + Y_{\Theta} \right)
= \sum_{\Theta \in \mathcal{Y}} \left( Y_{\Theta} + Y_{\Theta} + Y_{\Theta} + Y_{\Theta} \right),
$$

where in the first equality Pieri’s formula (A.15) has been used.

(iii) A simplification of the operands of the rectangular Young symmetrizer goes as follows.

$$
\left[ (\nabla_a R_{bcd}^m) K_m + (\nabla_c R_{dab}^m) K_m \right] = \left[ 2(\nabla_a R_{bcd}^m) K_m \right],
$$

where the formula (A.16) has been used. Another example is

$$
\left[ R_{ade}^m (\nabla_b R_{cfh}^n) K_{mgh} - R_{ade}^m (\nabla_c R_{hbc}^n) K_{mah} \right] = \left[ R_{ade}^m (\nabla_f R_{hbe}^n) K_{mgh} \right],
$$

where we have applied the formula (A.17).

Appendix B. Prolongation procedure

In this appendix we shall demonstrate a prolongation procedure for KTs of order 1 and 2. Our calculation below can be extended to the higher order cases that we have skipped here due to space considerations. Throughout this appendix, we repeatedly use the properties and formulas of Young symmetrizers shown in appendix A.

B.1. Killing vector fields

We start our investigation in the case of Killing vector $K_a$. Calculating the derivative of $K_a$ can be done as

$$
\nabla_b K_a = (Y_{\Theta} + Y_{\Theta}) \nabla_b K_a = \nabla_b K_a \equiv K_{ba}^{(1)},
$$

where we have inserted the completeness relation (A.3) in the first equality and used the Killing equation in the second.

Calculating the derivative of $K_{ba}^{(1)}$ goes as

$$
\nabla_c K_{ba}^{(1)} = Y_{\Theta} \nabla_c K_a = Y_{\Theta} \left( Y_{\Theta} + Y_{\Theta} + Y_{\Theta} \right) \nabla_c K_a = Y_{\Theta} Y_{\Theta} \nabla_c K_a
= Y_{\Theta} Y_{\Theta} \left( 2 \nabla_{[cb]} K_a + \nabla_{bc} K_a \right)
= Y_{\Theta} Y_{\Theta} R_{cba} \theta K_d.
$$

In the fourth equality, no term has survived besides the third because of the Killing equation and the first Bianchi identity. The above calculations confirm the results (12) and (13).
B.2. Killing tensor fields of order 2

We shift our focus to the Killing tensor field $K^{ba}$. Calculating the derivative of $K_{ba}$ gives

$$\nabla_c K_{ba} = Y^{dcba} \nabla_c K_{ba} = Y^{dcba} \left( Y^{a} + Y^{d} + Y^{c} + Y^{b} \right) \nabla_c K_{ba}$$

$$= Y^{dcba} Y^{a} \nabla_c K_{ba} = Y^{dcba} K^{(1)}_{cba} .$$

where the third equality follows from the Killing equation and a trivial relation \( \hat{S}_{ab} \hat{A}_{ab} = 0 \).

Calculating the derivative of $K^{(1)}_{cba}$ goes as

$$\nabla_d K^{(1)}_{cba} = Y^{dcba} \nabla_d K_{ba} = Y^{dcba} (Y^{d} + Y^{c} + Y^{b}) \nabla_d K_{ba}$$

$$= Y^{dcba} \left( Y^{b} + 2 Y^{d} \right) \nabla_d K_{ba} + K^{(2)}_{dcb} = Y^{dcba} \left( Y^{b} + 2 Y^{d} \right) R_{dcb}^\mu K_{bca} + K^{(2)}_{dcb} .$$

where in the second equality the result (A.19) and the Killing equation have been used. Expanding the two Young symmetrizers in parentheses in equation (B.1), we obtain

$$\nabla_d K^{(1)}_{cba} = Y^{dcba} \left[ K^{(2)}_{dcb} - \frac{\epsilon}{2} R_{dic} \pi K_{mb} - 2R_{dcb} \pi K_{mc} + \frac{1}{2} R_{acb} \pi K_{md} \right] .$$

At last, calculating the derivative of $K^{(2)}_{dcb}$ can be done as

$$\nabla_e K^{(2)}_{dcb} = Y^{dcba} \nabla_e K_{ba} = Y^{dcba} (L^{e} + L^{d} + L^{c} + L^{b}) \nabla_e K_{ba}$$

$$= Y^{dcba} (Y^{e} + Y^{d} + Y^{c} + Y^{b}) (1 - Y^{a} + Y^{d} - Y^{c}) \nabla_e K_{ba}$$

$$= Y^{dcba} (Y^{e} + Y^{d} + Y^{c} + Y^{b}) \nabla_e K_{ba} .$$

Now the number of boxes of the tableaux exceeds 4, so we have need to take Littlewood’s correction (A.5) into account. However, all these corrections are dropped by trivial relations \( \hat{S}_{ad} \hat{A}_{ad} = 0 \) and \( \hat{S}_{be} \hat{A}_{be} = 0 \). Hence, our calculation can be pursued as

$$\nabla_e K^{(2)}_{dcb} = Y^{dcba} \left( Y^{e} + Y^{d} + Y^{c} + Y^{b} \right) \nabla_e K_{ba}$$

$$= Y^{dcba} \left\{ Y^{e} \nabla_e K_{ba} + Y^{d} \left( 2 \nabla_{e[d]} K_{ba} + 2 \nabla_{[d]e} K_{ba} \right) + Y^{c} \left( 2 \nabla_{[e]d} K_{ba} + 2 \nabla_{d[e]} K_{ba} \right) \right\}$$

$$= Y^{dcba} \left\{ Y^{e} \nabla_e K_{ba} + Y^{d} \left( 2 \nabla_{e[d]} K_{ba} + 2 \nabla_{[d]e} K_{ba} \right) \right\}$$

$$+ Y^{dcba} \left( R_{dcb}^\mu K^{[1]}_{bca} + 2 R_{dcb}^\mu K^{(1)}_{mca} - 4 R_{dcb}^\mu K^{(1)}_{mca} + 2 \nabla_{d R_{eb}^\mu} K^{(1)}_{bca} \right) .$$

(B.2)
Regarding the first term, a straightforward calculation gives
\[ Y_{abc} \nabla_{edc} K_{ba} = \frac{1}{8} \left( 4 R_{ecd} \frac{m}{m} K_{mab}^{(1)} - 9 R_{ecd} \frac{m}{m} K_{mba}^{(1)} - 9 R_{ecd} \frac{m}{m} K_{mbe}^{(1)} + 5 R_{ecd} \frac{m}{m} K_{mba}^{(1)} + 5 R_{ecd} \frac{m}{m} K_{mba}^{(1)} \right) + 2 (\nabla_c R_{edc \frac{m}{m}}) K_{mb} + 2 (\nabla_c R_{edc \frac{m}{m}}) K_{ne} - 2 (\nabla_c R_{edc \frac{m}{m}}) K_{nd} \]  \hspace{1cm} (B.3)
Combining the results of (B.2) and (B.3) leads to the conclusion that
\[ \nabla_c K_{dcb \frac{a}{a}} = Y_{abc} \left( -\frac{4}{3} (\nabla_a R_{edc \frac{m}{m}}) K_{mb} - \frac{2}{3} (\nabla_a R_{edc \frac{m}{m}}) K_{nc} - \frac{2}{3} (\nabla_a R_{edc \frac{m}{m}}) K_{nd} \right) - 12 R_{ead} \frac{m}{m} K_{mcb}^{(1)} - 4 R_{ead} \frac{m}{m} K_{mcd}^{(1)} - \frac{2}{3} R_{ead} \frac{m}{m} K_{mcd}^{(1)} + \frac{2}{3} R_{ead} \frac{m}{m} K_{mcd}^{(1)} , \]
where we have expanded the three Young symmetrizers in parentheses in equation (B.2) explicitly.

Appendix C. Derivation of the integrability conditions

In this appendix we shall verify the integrability conditions (37). We begin with the integrability conditions of the \( p \)th prolonged equation. Evaluating the expression (30) at \( q = p \), one finds that
\[
i^{(p, p)}_{a_{1}...a_{p}b_{1}...b_{p}c_{d}} = Y_{abc} \left[ Y_{abc} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} + Y_{abc} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} \right. \]
\[ + \sum_{i=2}^{p} Y_{abc} \left( \nabla_{dcb...b_{i}K_{a_{p}...a_{1}}} - \nabla_{dcb...b_{i}K_{a_{p}...a_{1}}} \right] . \]  \hspace{1cm} (C.1)

The last term in equation (C.1) can be treated as
\[
Y_{abc} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} = Y_{abc} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} \sum_{\Theta \in \mathcal{P}_{p-1}} L_{\Theta} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} \]
\[ = Y_{abc} \left( Y_{abc} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} + Y_{abc} \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} \right) \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} , \]
where Pieri’s formula (A.15) is used and Littlewood’s correction (A.5) is dropped by a relation \( \hat{S}_{a_{\gamma}} A_{a_{\gamma}} = 0 \). Hence, equation (C.1) can be rewritten as
\[
i^{(p, p)}_{a_{1}...a_{p}b_{1}...b_{p}c_{d}} = Y_{abc} \left[ \left( Y_{abc} + Y_{abc} \right) \left( \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} - \nabla_{dcb...b_{1}K_{a_{p}...a_{1}}} \right) \right. \]
\[ + \sum_{i=2}^{p} Y_{abc} \left( \nabla_{dcb...b_{i}K_{a_{p}...a_{1}}} \right] . \]  \hspace{1cm} (C.2)
Suppose now that the conjecture in section 3 holds true. We then ignore the first symmetrizer \( Y_{abc} \) since it does not induce the representations belonging to \( (p + 1, p + 1) \). The products of Young symmetrizers in equation (C.2) can be simplified to
The first result (C.3) follows immediately from Raicu’s formula (A.14). The second result (C.4) can be confirmed by a direct calculation. By using the relations (C.3) and (C.4), the equation (C.2) can be rewritten as

\[ I (p, p) = 1 \prod_{j=1}^{p-i} (b_{p+1-j}, b_{p-j}) \cdot \]

Expanding \( \text{id}_{2p+2} \) and the antisymmetrizations of the operands yields the result (37).

**Appendix D. Killing–Yano equation**

Our analysis based on Young symmetrizers has effective applications to other types of overdetermined PDE systems. In this appendix we discuss the Killing–Yano equation

\[ \nabla (b F_{a_1 \ldots a_p}) = 0, \] (D.1)

where \( F_{a_1 \ldots a_p} = F_{[a_1 \ldots a_p]} \) is a Killing–Yano tensor field (KY). If we have a KY, then we can obtain a KT of order 2 as

\[ K_{ab} \equiv F_{ac_1 \ldots c_{p-1}} F^b_{c_1 \ldots c_{p-1}}, \] (D.2)

but the converse is not generally true. While the prolonged equations of the Killing–Yano equation and their integrability conditions have been known in [36–38], we revisit the results by using Young symmetrizers.

Let \( F_{ab} \) be a KY and consider its derivatives. Since \( \nabla \right F_{ba} \) is a type \((0, 3)\) tensor field, its decomposition to the irreducible representations reads

\[ \nabla \right F_{ba} = Y \text{id}_3 \nabla \right F_{ba} = Y \left( Y + Y + Y \right) \nabla \right F_{ba} = Y \nabla \right F_{ba} \equiv F_{1ba} \right, \] (D.3)

where we have used Pieri’s formula (A.15) and the Killing–Yano equation (D.1). We next consider \( \nabla \left F_{1ba} \right \) as the above result is not yet closed. Its decomposition to the irreducible representations reads

\[ \nabla \left F_{1ba} \right = Y \text{id}_4 \nabla \left F_{1ba} \right = Y \left( Y + Y + Y + Y \right) \nabla \left F_{1ba} \right = Y \nabla \left F_{1ba} \right \equiv F_{1ba} \left, \] (D.4)

where we have used Riemann–Christoffel symbols.
A similar calculation, taking into account Littlewood’s corrections, yields the conclusion that the Killing–Yano equation (D.1) is equivalent to the prolonged equations

\[ \nabla_b F_{a p \cdots a_1} = F^{(1)}_{b a p \cdots a_1}, \] (D.5)

\[ \nabla_c F_{a p \cdots a_1} = p Y_{a b} Y_{c d} R_{d b p}^{m} F_{m a p \cdots a_1}, \] (D.6)

where

\[ F^{(1)}_{b a p \cdots a_1} \equiv Y_{a b} \nabla_b F_{a p \cdots a_1}. \] (D.7)

After a calculation analogous to that in appendix C, we obtain the integrability condition for equation (D.5)

\[ Y_{a b} \left[ R_{c a}^{m} F_{m b p} \cdots a_1 \right] = 0, \quad \text{for} \quad p > 1. \] (D.8)

It can be confirmed that the integrability condition for equation (D.6) is involved in the derivative of equation (D.8). Therefore, equation (D.8) and its derivatives are enough to discuss the integrability condition of the Killing–Yano equation. Once again, we face a situation similar to the one just discussed in section 3. Namely, there is only a representation in equation (D.8), even though the possible representations of equation (D.8) are three

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