LOCAL POINTS ON $p$-ADICALLY UNIFORMIZED SHIMURA VARIETIES

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Abstract. Using the $p$-adic uniformization of Shimura varieties we determine, for some of them, over which local fields they have rational points. Using this we show in some new curve cases that the Jacobians are even in the sense of [PS].

1. Introduction and notation

The real points on certain Shimura varieties were determined by Shimura in [Sh]. In [JL1] the case of $p$-adic points was treated for Shimura curves associated to maximal orders in indefinite rational quaternion division algebras. The case of good reduction turned out to reduce to the trace formula via Hensel’s lemma. The case of bad reduction was handled through the result of Čerednik and Drinfel’d, which assured that these curves admit a $p$-adic uniformization.

A general result of Varshavsky, Rapoport and Zink [Va1, Va2] gives the known cases when Shimura varieties admit a $p$-adic uniformization. Of special interest in this context is the case of curves, because of the recent results of [PS] and [JL3]. In this work we answer the question of existence of local points for some of these varieties. Using this we show in some new curve cases that the Jacobians are even in the sense of [PS].

We now set some notation. Let $F$ be a totally real number field, $g = [F : \mathbb{Q}]$. Denote by $\Sigma_\infty = \{\infty_1, \ldots, \infty_g\}$ the set of real embeddings of $F$ and by $\Sigma_f$ the set of finite places of $F$. Set $\Sigma = \Sigma_\infty \cup \Sigma_f$. Let $A = A_F$, $A_f$, $A_\infty$, $A_f^\infty$, $A_\infty^x$, $A_f^x$, and $A_\infty^x$ denote the adèles of $F$, the finite adèles of $F$, the finite adèles without the $v$ components for places $\nu$ in a finite set of places $\Xi \subseteq \Sigma_f$, the integral finite adèles without the $v$ components for places $\nu$ in a finite set of places $\Xi \subseteq \Sigma_f$, the idèles of $F$, the finite idèles of $F$, and the integral finite idèles of $F$, respectively. For an algebraic group $G/F$ we abbreviate $G = G(F)$, $G_\nu = G(F_\nu)$, $G_\infty = \prod_{i=1}^g G(F_\infty)$, $G_f = G(A_f)$, and $G_f^\infty = G(A_f^\infty)$. We view $G$ as contained in $G(A)$ and also in each $G_{f_1}^{\infty_1} \cdots ^{\infty_r}$ and in each $G_v$ via the natural projection.

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2. \( P \)-adic uniformization of Shimura varieties

We will consider two types of Shimura varieties (cf. \([\text{De}2], \ [\text{Va}2]\)):

**Case 1.** Let \( B^\text{int} / F \) be a quaternion division algebra, Let \( G^\text{int} / F \) be the algebraic group associated to its multiplicative group, and let \( \varphi^\text{int} : G^\text{int} \to \mathbb{G}_m \) be the \( F \)-morphism induced from the reduced norm from \( B^\text{int} \) to \( F \). Assume that \( B^\text{int} \otimes_{F, \infty} \mathbb{R} \) is indefinite for \( 1 \leq i \leq r \) and definite for \( r + 1 \leq i \leq g \), with \( r \geq 1 \). Fix identifications of \( B_\infty \), with \( \text{Mat}_{2 \times 2}(\mathbb{R}) \) for \( 1 \leq i \leq r \) and with the Hamilton quaternions \( \mathbb{H} \) for \( r + 1 \leq i \leq g \). Then \( B \otimes_{\mathbb{Q}} \mathbb{R} \cong \text{Mat}_{2 \times 2}(\mathbb{R})^r \times \mathbb{H}^{g-r} \). Let \( h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m(\mathbb{R}) \to \text{Res}_{\mathbb{F}/\mathbb{Q}} \mathbb{G}^\text{int}(\mathbb{R}) \) be the Hodge type

\[
h(z) = (m(z), m(z), \ldots, m(z), 1, 1, \ldots, 1) \in \text{GL}_2(\mathbb{R})^r \times (\mathbb{H}^x)^{g-r} = \text{Res}_{\mathbb{F}/\mathbb{Q}} \mathbb{G}^\text{int}(\mathbb{R}),
\]

where for \( z = x + \sqrt{-1}y \in \mathbb{C}^x = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m(\mathbb{R}) \) we set \( m(z) = \begin{bmatrix} z & y \\ -y & z \end{bmatrix}^{-1} \).

**Case 2.** Here let \( K \) be a CM extension of \( F \), let \( (x \mapsto \overline{x}) : K \to K \) be the conjugation over \( F \), and let \( d \geq 2 \) be an integer. Let \( D^\text{int} \) be a division algebra, central and of dimension \( d^2 \) over \( K \), with an involution of the second kind \( \alpha^\text{int} \). Let \( G^\text{int} = \text{GU}(D^\text{int}, \alpha^\text{int}) \) be the associated group of unitary similitudes and let \( \varphi^\text{int} : G^\text{int} \to \mathbb{G}_m \) be the \( F \)-morphism induced from the factor of similitudes. Also let \( N^\text{int} : G^\text{int} \to \text{Res}_{K/\mathbb{F}} \mathbb{G}_m \) be the morphism induced from the norm map from \( D^\text{int} \) to \( K \). Choose extensions \( \infty_1, \ldots, \infty_g \) to embeddings, denoted by the same letter, \( \infty_i : K \to \mathbb{C} \), and suppose that

\[
G^\infty \cong \begin{cases} \text{GU}_{d-1,1}(\mathbb{R}) & \text{for } 1 \leq i \leq r, \\ \text{GU}_d(\mathbb{R}) & \text{for } r + 1 \leq i \leq g. \end{cases}
\]

Define \( h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m(\mathbb{R}) \to \text{Res}_{\mathbb{F}/\mathbb{Q}} G^\text{int}(\mathbb{R}) \cong \text{GU}_{d-1}(\mathbb{R})^r \times \text{GU}_d(\mathbb{R})^{g-r} \) by

\[
h(z) = (m(z), \ldots, m(z), 1, \ldots, 1) \in \text{GU}_{d-1}(\mathbb{R})^r \times \text{GU}_d(\mathbb{R})^{g-r},
\]

where \( m(z) = \text{diag}(1, \ldots, 1, z/\overline{z})^{-1} \).

To handle cases 1 and 2 simultaneously, we formally set \( K = F \) and \( d = 2 \) in Case 1.

In both cases, let \( K_{\text{Gal}} \) be the Galois closure of \( \infty_1(K) \) in \( \mathbb{C} \), and let \( H \) be the subgroup of \( \text{Gal}(K_{\text{Gal}}/\mathbb{Q}) \) preserving the set \( \{ \infty_1, \ldots, \infty_g \} \). The reflex field is defined by \( E = K_{\text{Gal}}^H \).

Let \( X_\infty \) be the conjugacy class of \( h \). Our identifications allow us to view \( X_\infty \) as \( (\mathbb{C} \setminus \mathbb{R})^r \) in the case \( d = 2 \), and as \( (B^{d-1})^r \) in the case \( d \geq 3 \), where \( B^{d-1} \) is the unit ball in \( \mathbb{C}^{d-1} \). For a compact open subgroup \( S \subset G_f \), the corresponding Shimura variety is given complex analytically by

\[
\widetilde{X}_S^\text{int} = (X_\infty \times (S \setminus G_f^\text{int})) / G^\text{int},
\]

where \( G^\text{int} \) acts on the product \( X_\infty \times (S \setminus G_f^\text{int}) \) by the rule \( (x, g) \gamma := (\gamma^{-1}(x), g \gamma) \). This variety is denoted by \( S M_{\infty}(H, X_\infty) \) in \([\text{De}1]\).

The complex manifold \( X_S^\text{int} \) has a canonical algebraization \( \widetilde{X}_S^\text{int} \) to a complex projective variety. In fact each \( X_S^\text{int} \) admits a canonical model over the reflex field \( E \) and \( X_S^\text{int} \)’s form a projective system. The main result of \([\text{Va}2]\) is that \( X^\text{int} = \lim_S X_S^\text{int} \) admits a \( p \)-adic
uniformization, under some conditions which we will now specify. Let \( v \) be a finite place of \( E \) of residue characteristic \( p \) and choose a prime \( \tilde{v} \) of \( K_{\text{Gal}} \) above \( v \). Fix an embedding of the completion \( \tilde{K}_{\text{Gal},\tilde{v}} \) of \( K_{\text{Gal}} \) into \( \mathbb{C} \). The embeddings \( \infty_1, \ldots, \infty_r \) then induce places \( w_1, \ldots, w_r \) of \( K \) above \( p \). Moreover \( \{ w_1, \ldots, w_r \} \) depends only on \( v \), because \( \text{Gal}(K_{\text{Gal}}/E) \) preserves \( \{ \infty_1, \ldots, \infty_r \} \), and \( E_v \) is the compositum of \( K_{w_1}, \ldots, K_{w_r} \) in \( K_{\text{Gal},\tilde{v}} \) as in \([Va2\text{, Lemma 2.6}]\). Let \( v_i \) be the restriction of \( w_i \) to \( F \). Then \( X_v^\text{int} = X^\text{int} \otimes_F E_v \) has a \( p \)-adic uniformization \( ([Va2\text{, Theorems 5.3 and 2.13}] \) provided we have

1. The \( v_i \)'s are distinct and split in \( K/F \).
2. The Brauer invariant \( \text{inv}_{v_i} D^\text{int} \) is \( 1/d \) for each \( 1 \leq i \leq r \).

We shall now describe a special case of the result more precisely. This requires more notation.

In Case 1, let \( B \) be a quaternion algebra over \( F \) split at \( v_1, \ldots, v_r \); ramified at all infinite places; and with a fixed isomorphism

\[
B^\text{int} \otimes A_{\mathfrak{f}}^{v_1, \ldots, v_r} \cong B \otimes A_{\mathfrak{f}}^{v_1, \ldots, v_r}.
\]

Let \( G \) be the algebraic group over \( F \) corresponding to \( B^\times \), and let \( \nu : G \to \mathbb{G}_m \) be the \( F \)-morphism induced from the reduced norm from \( B \) to \( F \). Set \( \overline{G} = \gamma = G_{t}^{v_1, \ldots, v_r} \).

In Case 2, let \( D \) be a central simple \( K \)-algebra with an involution \( \alpha \) of the second kind. We suppose that \( D \) is split at all places over \( v_1, \ldots, v_r \) (which are \( w_1, \overline{w_1}, \ldots, w_r, \overline{w_r} \)), and that we are given an identification of \( (D, \alpha) \otimes_F F_v \) with \( (D^\text{int}, \alpha^\text{int}) \otimes_F F_v \) at the other finite places \( v \) of \( F \). Moreover we assume that \( \alpha \) is definite at all the infinite places of \( K \). Define an algebraic group \( G \) over \( F \) by \( G = GU(D, \alpha) \), and let \( \nu : G \to \mathbb{G}_m \) be the \( F \)-morphism induced from the factor of similitudes. Also let \( \text{Nm} : G \to \text{Res}_{K/F} \mathbb{G}_m \) be the morphism induced from the norm map from \( D \) to \( K \). Set \( \overline{G} = G_{t}^{v_1, \ldots, v_r} \). The decompositions

\[
D \otimes_F F_v \cong \text{Mat}_d(K_{w_i}) \oplus \text{Mat}_d(K_{\overline{w_i}}) \quad \text{and} \quad D^\text{int} \otimes_F F_v \cong D_{w_i}^\text{int} \oplus D_{\overline{w_i}}^\text{int},
\]

together with the \( F \)-rational maps \( \nu : G \to \mathbb{G}_m \) and \( \nu^{\text{int}} : G^{\text{int}} \to \mathbb{G}_m \) induce decompositions

\[
(1) \quad G_{v_i} \cong \text{GL}_d(K_{w_i}) \times F_{v_i}^\times \quad \text{and} \quad G_{v_i}^{\text{int}} \cong (D_{w_i}^{\text{int}})^\times \times F_{v_i}^\times.
\]

More precisely, the conjugation of \( K \) over \( F \) induces an isomorphism \( K_{w_i} \cong K_{\overline{w_i}} \) and \( \alpha \) (respectively \( \alpha^{\text{int}} \)) induces a compatible isomorphism of algebras \( D_{w_i} \cong D_{\overline{w_i}}^{\text{opp}} \) (respectively \( D_{w_i}^{\text{int}} \cong (D_{w_i}^{\text{int}})^{\text{opp}} \)). Viewing all the above isomorphisms as identifications, the decompositions \((1)\) above become

\[
(2) \quad G_{v_i} = \{(g, \lambda g) | g \in \text{GL}_d(K_{w_i}), \lambda \in F_{v_i}^\times \} \quad \text{and} \quad G_{v_i}^{\text{int}} = \{(g, \lambda g) | g \in (D_{w_i}^{\text{int}})^\times, \lambda \in F_{v_i}^\times \}.
\]

We can then view \( \gamma = \prod_i F_{v_i}^\times \times \gamma \) as a closed subgroup of both \( G_t \) and \( G_t^{\text{int}} \).
In both cases, fix for each $i$ a central division algebra $\tilde{D}_{w_i}$ over $K_{w_i}$ of invariant $1/d$ (= $1/2$ in Case 1). We then have that
\[
G_f \cong \prod_{i=1}^r \text{GL}_d(K_{w_i}) \times G' \quad \text{and} \quad G_{f \text{int}}^\infty \cong \prod_{i=1}^r \tilde{D}_{w_i}^\times \times G'.
\]

The ring of integers $O_{\tilde{D}_{w_i}}$ is normalized by $\tilde{D}_{w_i}^\times$, and $\tilde{D}_{w_i}^\times / O_{\tilde{D}_{w_i}}^\times$ is identified with $\mathbb{Z}$ via the valuation of the norm. Hence the group $G_{f \text{int}}$ acts on $X_{\text{int},p} := (\prod_{i=1}^r O_{\tilde{D}_{w_i}}^\times) \setminus X_{\text{int}}$ through its quotient
\[
G_{f \text{int}}^\infty \bigslant{\prod_{i=1}^r O_{\tilde{D}_{w_i}}^\times} \cong G' \times \mathbb{Z}^r.
\]

For a finite extension $L$ of $\mathbb{Q}_p$, let $\Omega_L = \Omega_L^d$ be Drinfeld’s symmetric space. Recall that the analytic space ($[\text{Ber}]$) $\Omega_L$ is obtained by removing all rational hyperplanes from $\mathbb{P}_d^{d-1}$. It is preserved by the action of $\text{PGL}_d(L)$. Consider an analytic space $\Omega_L^{\text{nr}} = \Omega_L \hat{\otimes}_L L^{\text{nr}}$ over $L$, where $L^{\text{nr}}$ is the completed maximal unramified extension of $L$. We let $g \in \text{GL}_d(L)$ act on $\Omega_L^{\text{nr}}$ via the natural (left) action on $\Omega_L$ and the action of $\text{Frob}_L^{\text{val(det} g)}$ on $L^{\text{nr}}$. We also let $n \in \mathbb{Z}$ act on $\Omega_L^{\text{nr}}$ through the action of $\text{Frob}_L^{-n}$ on $L^{\text{nr}}$. This gives an $L$-rational action of $\text{GL}_d(L) \times \mathbb{Z}$ on $\Omega_L^{\text{nr}}$. In our situation, set
\[
\Omega = \prod_{i=1}^r \left( \Omega_{K_{w_i}}^{\text{nr}} \hat{\otimes}_{K_{w_i}} E_v \right).
\]

This is an $E_v$-analytic space with an $E_v$-rational action of the group $\prod_{i=1}^r \text{GL}_d(K_{w_i}) \times \mathbb{Z}^r$.

Let $G$ act on the $i$th factor of $\Omega$ via the embedding $G(F) \hookrightarrow \text{GL}_d(K_{w_i})$, and on $G'$ through the natural embedding (in Case 2, let $G(F)$ act on each factor $F_{w_i}^\times$ of $G'$ via the similitude factor $\nu$).

For a compact open subgroup $S \subset G'$, the $E_v$-analytic space $(\Omega \times (S \backslash G'))/G(F)$ algebraizes canonically to a scheme $X_S$ over $E_v$. The inverse limit of $X_S$ over such $S$’s is a scheme $X$ over $E_v$ with an $E_v$-rational action by $G' \times \mathbb{Z}^r$.

A special case of the main result of $[\text{Va2}]$ is the following

**Theorem 2.1.** Under conditions 1 and 2 above, there exists a $G' \times \mathbb{Z}^r$-equivariant, $E_v$-rational isomorphism
\[
X_{\text{int},p} \otimes_E E_v \cong X.
\]

**3. THE CONNECTED COMPONENT**

A description of the set of connected components of a Shimura variety, with the Galois action, is known in general (see $[\text{De1} 2.1.3.1]$). We need an explicit form of this description in our case(s). For this it is simpler to use the $p$-adic uniformization, essentially repeating the standard argument there.
Define a torus over $F$ by $M = G/G_{\text{der}}$. In Case 1, $\nu$ induces an isomorphism $M \simeq \mathbb{G}_m/F$; in Case 2, $\text{Nm} \times \nu$ induces an isomorphism of $M$ with the torus over $F$ defined by

$$\{(x, \lambda) \in \text{Res}_{K/F} \mathbb{G}_m \times \mathbb{G}_m | x \lambda = \lambda^d\},$$

(Notice that $M$ is also isomorphic to $G^{\text{int}}/G_{\text{der}}^{\text{int}}$ through $\nu^{\text{int}}$ and $\text{Nm}^{\text{int}}$.)

In Case 1, $M_{\nu_i} \simeq F_{v_i}^\times$; in Case 2, we have $K \otimes_F F_{v_i} \cong K_{w_i} \oplus K_{\infty}$, and therefore we can and will make the identification

$$M_{\nu_i} = K_{w_i}^\times \times F_{v_i}^\times.$$

Using the decomposition (12), the quotient map $G_{\nu_i} \to M_{\nu_i}$ is then induced by $\text{Nm} \times \nu$.

Since the derived group $G_{\text{der}}$ is a form of $\text{SL}_2$ in the first case and $\text{SU}_d$ in the second case, it is simply connected. Using Hasse principle, combined with vanishing of cohomology of $p$-adic groups, and a local archimedean calculation, we get that the image of $G$ in $M$ is

$$M^+ = \{x \in M | \nu(x) \in F^\times \text{ is positive at all the infinite places of } F\}.$$  

For a compact open subgroup $S \subset G'$, let $T$ be the image of $S$ in $M_\text{f}$, and set $\tilde{T} = \left(\prod_{i=1}^{r} \mathcal{O}_{K_{w_i}}^\times\right) \times T$. Then $\tilde{T}$ is an open subgroup of $M_\text{f}$. Let $M_t^+$ be the connected component of $M(F \otimes_{\mathbb{Q}} \mathbb{R})$. Set $\text{Cl}_t^+ = (M_{\nu_i}^+ \tilde{T}) \setminus M(\mathbb{A})/M$. Let $f_i = f(E_v/K_{w_i})$ be the degree of the extension of residue fields. Let $\pi_{w_i}$ be a uniformizer of $K_{w_i}$. Using the decomposition (13), we can view $\pi_{w_i}$ as an element of $M_{\nu_i}$. The image $\varphi_+$ of $\prod_{i=1}^{r} \pi_{w_i}^\prime$ in $\text{Cl}_t^+$ is then independent of the choices of the $\pi_{w_i}$s.

For a local field $L$, denote by $L^{(r)}$ the extension of degree $r$ of $L$ in $L_{\text{nr}}$. We will now prove the following:

**Theorem 3.1.** The map $\nu$ in Case 1 and $\text{Nm} \times \nu$ in Case 2 induce an isomorphism $\pi_0(S \setminus X_{\text{int}, p}^{\nu}) \cong \text{Cl}_t^+$. Let $k_+$ be the order of $\varphi_+$ in $\text{Cl}_t^+$. Then the field of definition of each geometric connected component of $S \setminus X_{\text{int}, p}^{\nu}$ is $E_v^{(k_+)}$.

**Proof.** Fix an algebraic closure $\overline{E}_v$ of $E_v$. Since each $\Omega_{K_{w_i} \otimes K_{w_i}} E_v$ is geometrically connected, the set of geometrically connected components of $\Omega_{K_{w_i} \otimes K_{w_i}} E_v$ is in bijection with the set $\text{Emb}_i$ of continuous $K_{w_i}$-homomorphisms of $K_{w_i}^{\text{nr}}$ into the completion of $\overline{E}_v$. Each $\text{Emb}_i$ carries a natural $\text{Gal}(K_{w_i}^{\text{nr}}/K_{w_i}) \times \text{Gal}(\overline{E}_v/E_v)$ action. The two Galois actions are obviously related: the $\text{Gal}(\overline{E}_v/E_v)$-action factors through $\text{Gal}(K_{w_i}^{\text{nr}} E_v/E_v) = \text{Gal}(E_v^{\text{nr}}/E_v)$, and

$$\text{Frob}_{E_v} \in \text{Gal}(E_v^{\text{nr}}/E_v) \text{ acts like } \text{Frob}_{K_{w_i}}^{-f_i} \in \text{Gal}(K_{w_i}^{\text{nr}}/K_{w_i}).$$

The $\text{Gal}(K_{w_i}^{\text{nr}}/K_{w_i})$-action induces an action of $G \subset \text{GL}(2, K_{w_i})$ through

$$\text{GL}(2, K_{w_i}) \ni g \mapsto \text{Frob}_{K_{w_i}}^{\text{val}_{w_i}^{\nu}(\det g)}.$$
The set of connected components $\pi_0(S \backslash X^{\text{int,p}})$ is therefore canonically the separated quotient $(\prod_{i=1}^r \text{Emb}_i \times (S \backslash \mathcal{G}')/G$. By the strong approximation theorem, $G^\text{der} \subset G_{\text{der}}S$, and $G_{\text{der}}$ acts trivially on $\prod_{i=1}^r \text{Emb}_i$. Therefore
\[
\pi_0(S \backslash X^{\text{int,p}}) = \left(\prod_{i=1}^r \text{Emb}_i \times (S \backslash \mathcal{G}')\right)/G \cong \left(\prod_{i=1}^r \text{Emb}_i \times (T \backslash M')\right)/M^+,
\]
where $M'$ is the image of $\mathcal{G}'$ in $M_f$. Explicitly, $M' = M^{\text{nr}}_i \times \prod_{i=1}^r F_{v_i}^{\times}$ in Case 1 and $M' = M^{\text{nr}}_i \times \prod_{i=1}^r F_{v_i}^{\times}$ in Case 2. Now fix some embedding of each $K_{w_i}^{\text{nr}}$ in $E_v$. Then $\text{Emb}_i$ gets identified with $\text{Gal}(K_{w_i}^{\text{nr}}/K_{w_i}) \cong \hat{\mathbb{Z}}$, into which $K_{w_i}^{\times}/O_{w_i}$ embeds as a dense subset preserved by the $G$-action (through the $w_i$-valuation of the norm). Using (3), we get an embedding of $M_v/O_{K_{w_i}}^{\times} \cong K_{w_i}^{\times}/O_{K_{w_i}}^{\times} \times F_{v_i}^{\times}$ into $\text{Emb}_i \times F_{v_i}^{\times}$ as a dense subset, compatible with the $M^+$-action (through the $K_{w_i}$-component). Hence
\[
\pi_0(S \backslash X^{\text{int,p}}) \cong \left(\prod_{i=1}^r (K_{w_i}^{\times}/O_{w_i}^{\times}) \times (T \backslash M')\right)/M^+ = \hat{T} \backslash M_f/M^+ \cong \hat{T} M_{\infty}^{+} \backslash M(A)/M,
\]
using the weak approximation at the infinite places to get the last isomorphism. Of course this agrees with Deligne’s general description cited above, but now we see that $\text{Gal}((E_v^+/E_v)$ acts on $\pi_0(S \backslash X^{\text{int,p}})$ through its image in $\prod_{i=1}^r \text{Gal}(E_v^{\text{nr}}/E_v)$, and this image already factors through $\text{Gal}(E_v^{\text{nr}}/E_v)$. By (3) and (4), $\text{Frob}_v$ acts on $\text{Cl}_f^+$ as the idèle whose $v$th component is $\pi_{v_i}^f$, if $v = v_i$ and 1 otherwise. We also see that a power $\text{Frob}_v^l$ acts trivially on one component (equivalently, on all components) if and only if $\varphi_+^l = 1$ in $\text{Cl}_f^+$. This concludes the proof of the Theorem.

**Remark 3.2.** When $r = 1$ we have $E_v = K_{w_1}$, so that $\varphi_+ = \pi_{w_1}$.

4. **LOCAL POINTS OF TWISTED MUMFORD QUOTIENTS**

In this section only we change our notation, and let $F$ be a finite extension field of $\mathbb{Q}_p$. Let $d \geq 2$ be an integer, and let $\Gamma \subset \text{GL}_d(F)$ be a subgroup. Set $Z\Gamma := \Gamma \cap Z(\text{GL}_d(F))$, and for every subgroup $\Delta \subset \Gamma$ containing $Z\Gamma$, we will write $P\Delta$ instead of $\Delta/Z\Gamma$. Assume that:

A) the closure of $\Gamma$ has a finite covolume in $\text{GL}_d(F)$;

B) $P\Gamma$ is a (cocompact) lattice in $\text{PGL}_d(F)$.

C) (for convenience) $\det \Gamma \subset Z\Gamma$.

Let $\text{val}(\det \Gamma) = k_+\mathbb{Z}$ and $\text{val}(Z\Gamma) = k\mathbb{Z}$. Then by C), $k|k_+|d_\Gamma$. Let
\[
\Gamma' := \{ \gamma \in \Gamma | d_\Gamma \text{ divides } \text{val}(\det(\gamma)) \}.
\]

Then $\Gamma'$ is a normal subgroup of $\Gamma$ containing $Z\Gamma$, and $\Gamma/\Gamma'$ is cyclic of order $d_\Gamma/k_+$. For every subgroup $\Delta \subset \Gamma$ we denote by $\Delta'$ the intersection of $\Delta$ with $\Gamma'$.

Let $X := \Gamma \backslash (\Omega^d_F \otimes_F F^{nr})$, and let $X' := \Gamma' \backslash \Omega^d_F$. Then $X$ and $X'$ are projective and geometrically connected varieties over $F^{(k_+)}$ and $F$ respectively, and $X = (\Gamma/\Gamma') \backslash (X' \otimes_F F^{(dk)})$. Thus $X$ is a (Frobenius) twist of the Mumford uniformized variety $X'$.
Let $L$ be any finite extension of $F^{(k_+)}$, and let $e = e(L/F^{(k_+)})$ and $f = f(L/F^{(k_+)})$ be the ramification and the inertia degrees of $L$ over $F^{(k_+)}$ respectively. Notice that $[L : F] = efk_+.$

**Theorem 4.1.**  
a) If $X(L) \neq \emptyset$, then there exists a subgroup $\Delta \subset \Gamma$ containing $Z\Gamma$ such that:

(i) the group $P\Delta$ is finite;
(ii) $\text{val}(\text{det}(\Delta)) = dkZ + fk_+Z$;
(iii) $d|efk_+$, where $m$ is the order of $P\Delta$.

b) The converse of a) holds if we assume in addition either that $d$ is prime or that $d|efk_+$. In particular, the converse of a) holds if $P\Gamma'$ is torsion-free.

**Proof.** Let $d'$ be the least integer $\geq 1$ for which $L^{(d')} \cap F^{(k_+)}$. Then $d'$ is the smallest positive integer such that $dk$ divides $fk_+d'$, hence $d' = d/\gcd(d, fk_+/k)$. Observe that conditions (ii) and (iii) of the theorem can be restated as (ii') $\text{val}(\text{det}(\Delta)) = (dk/d')Z$ and (iii') $d'|efk_+$ respectively. Then $F^{(dk)} \otimes_{F^{(k_+)}} L \cong \oplus \text{Hom}_{F^{(k_+)}}(F^{(k_+)}, L^{(d')}) L^{(d')}$. This yields the following description of $X_L$:

$$X_L = X \otimes_{F^{(k_+)}} L \cong (\Gamma/\Gamma') \backslash \left( \bigcup \text{Hom}_{F^{(k_+)}}(F^{(k_+)}, L^{(d')}) X' \otimes_F L^{(d')} \right) \cong (\Gamma_1/\Gamma') \backslash (X' \otimes_F L^{(d')}).$$

where $\Gamma_1$ is the stabilizer in $\Gamma$ of any (and hence of each) connected component of $X' \otimes_F (F^{(dk)} \otimes_{F^{(k_+)}} L)$. Thus

$$\Gamma_1 = \{ \gamma \in \Gamma | dk/d' \text{ divides } \text{val}(\text{det}(\gamma)) \}.$$

The quotient $\Gamma_1/\Gamma'$ is cyclic of order $d'$, generated by the projection of any element $\gamma_0 \in \Gamma_1$ such that $\text{val}(\text{det}(\gamma_0)) = dk/d'$. Fix such a $\gamma_0$. Then

$$X(L) = \{ P' \in X'(L^{(d')}) | \gamma_0(P') = P' \}.$$

These considerations prove the following

**Lemma 4.1.** $X(L) \neq \emptyset$ if and only if there exists $P \in \Omega_F^d(\overline{T})$ such that for every $\sigma \in \text{Gal}(\overline{T}/L)$ there exists an element $\phi(\sigma) \in \Gamma_1$ satisfying $\sigma(P) = \phi(\sigma)(P)$ and $\phi(\sigma) \in \gamma_0 \Gamma'$ if $\sigma_{|L^{(d')}} = Fr_{\Gamma'}$. (Here $\Gamma_1$ acts on $\Omega_F^d(\overline{T})$ through its projection to $\text{PGL}_d(F)$.)

We return to the proof of the Theorem:

a) Assume that $X(L) \neq \emptyset$, and let $P$ be as in the lemma. Set $\Delta := \{ \gamma \in \Gamma | \gamma(P) = \tau(P) \text{ for some } \tau = \tau(\gamma) \in \text{Gal}(\overline{T}/L) \}$. Then $\Delta$ is a subgroup of $\Gamma_1$, containing $Z\Gamma$, so it will suffice to show that $\Delta$ satisfies the conditions (i)-(iii) of the Theorem.

(i) Since the natural $\text{GL}_d(F)$-equivariant map from $\Omega_F^d$ to the Bruhat-Tits building $\mathcal{B}_F^d$ of $\text{PGL}_d(F)$ is constant on the Galois orbits, $\Delta$ stabilizes a certain point of $\mathcal{B}_F^d$. By assumption B) on $\Gamma$, the group $P\Delta \subset \text{PGL}_d(F)$ is compact and discrete. Hence it is finite.

(ii) By Lemma 4.1, the existence of $P$ forces $\Delta$ to contain an element from $\gamma_0 \Gamma'$. Since $\Delta \supset Z\Gamma$, the statement follows.

(iii) Let $\delta_0$ be an element from $\gamma_0 \Gamma' \cap \Delta \subset \Gamma_1$. Multiplying $\delta_0$ by an element of $Z\Gamma$ we may and will assume that $\text{val}_F(\text{det}(\delta_0)) = \text{val}_F(\text{det}(\gamma_0)) = kd/d'$. Let $n$ be the order
of the image of $\delta_0$ in $P\Delta$, and let $\delta_0^n = a \in F^x \subset L^x$. Then $\det(\delta_0)^n = a^d$, hence $\val_L(a) = n \val_L(\det(\delta_0))/d = ekn/d'$.

Let $L'$ be the field of rationality of $P$ (corresponding by Galois theory to the stabilizer $H \subset \Gal(T/L)$ of $P$). Then for every $\sigma \in \Gal(T/L)$ and every $\tau \in H$ we have

$$\tau(\sigma(P)) = \tau(\phi(\sigma)(P)) = \phi(\sigma)(\tau(P)) = \phi(\sigma)(P) = \sigma(P),$$

hence $L'$ is a Galois extension of $L$. Also we have a natural surjective homomorphism, $\pi : \Delta \to \Gal(L'/L)$ such that $\pi(\delta)(P) = \delta(P)$ for each $\delta \in \Delta$.

Let $v \in (L')^d$ be a representative of $P \in \Omega^d_L(L') \subset \mathbb{P}^{d-1}(L')$. Then by our assumption, there exist $\tau \in \Gal(T/L)$ and $\lambda \in L'$ such that $\delta_0(v) = \lambda \tau(v)$. Since the action of $\delta_0 \in \GL_d(F)$ commutes with that of $\tau$, we get $av = \delta_0^n(v) = \lambda \tau(\lambda)\tau^2(\lambda)\ldots\tau^{n-1}(\lambda)(v)$. Hence $a = \lambda \tau(\lambda)\tau^2(\lambda)\ldots\tau^{n-1}(\lambda)$. Taking valuations, we get $\val_L(\lambda) = \val_L(a)/n = ek/d'$. Since $\val_{P'}(\lambda)$ is an integer, $d'|ek(\tau')/L'$, so it will suffice to show that $e(L'/L)$ divides $m$.

But $e(L'/L)$ obviously equals the ramification degree of $L'L^{(d')} \cong L^{(d')}$, hence it divides the degree $[L'L^{(d')} : L^{(d')}]$. This degree is equal to the number of conjugates of $P$ over $L^{(d')}$, but conjugates of $P$ over $L^{(d')}$ form a homogeneous space for the action of the group $\Delta'/\Gamma$, hence the number of conjugates divides the order of $\Delta'/\Gamma$, which is $m$. This completes the proof of part a).

b) We will show the existence of local points by a case-by-case analysis.

I) Assume first that $d|efk_+$ or, equivalently, that $d'|ek$. Let $\delta_0$, $n$ and $a$ be as in the proof of iii) in a). It will suffice us to show that there exists a point $P \in \Omega^d_{\mathbb{F}}(L^{(n)})$ such that $\delta_0(P) = \Fr_{L'}(P)$.

The assumption $d'|ek$ implies that $n \mid \val_L(a)$. Therefore there exist $\lambda \in L^{(n)}$ such that $N_{L^{(n)}/L}(\lambda) = a$. Hence $\delta' := \lambda^{-1}\delta_0 \in \GL_d(L^{(n)})$ satisfies $\Fr_{L}^{n-1}(\delta')\Fr_{L}^{n-2}(\delta')\ldots\delta' = 1$. By Hilbert Theorem 90 for $\GL_d$, there exists $B \in \GL_d(L^{(n)})$ such that $\delta' = \Fr_{L'}(B) \cdot B^{-1}$. Hence for every vector $v \in L^d$, the image $P = P_v$ of $Bv \in (L^{(n)})^d$ in $\mathbb{P}^{d-1}(L^{(n)})$ satisfies $\delta_0(P) = \Fr_{L'}(P)$. Therefore it remains to show that there exists $v \in L^d$ such that the corresponding $P_v$ belongs to $\Omega^d_{\mathbb{F}}$.

Let $W$ be the $L^{(n)}$-vector subspace of $\Mat_d(L^{(n)})$ spanned by the Galois conjugates of $B$ over $L$. Then $W$ is invariant under the action of the Galois group $\Gal(L^{(n)}/L)$, hence is defined over $L$ by descent theory. Therefore $W$ contains an element $B' \in \GL_d(L)$.

By our assumption, $d|efk_+$, hence $[L : F] = efk_+ \geq d$. In particular, $L$ contains $d$ elements which are linearly independent over $F$. Equivalently, there exists $v' \in L^d$ not lying in any $F$-rational hyperplane. We claim that for $v := (B')^{-1}(v')$ the corresponding $P_v$ lies in $\Omega^d_{\mathbb{F}}$. In fact, let $w \in F^m$ be a row vector such that $wBv = 0$. Since $w$ and $v$ are $L$-rational, we get $wCVv = 0$ for every matrix $C \in W$. In particular, we have $wv' = wB'v = 0$. By our choice of $v'$, the vector $w$ is therefore the trivial vector, so that $P_v \in \Omega^d_{\mathbb{F}}$.

II) Assume now that $d$ is prime but it does not divide $efk_+$. Then $d' = d$, $k_+ = k$ and $[\Delta : \Delta'] = d$. Let $\delta_0$, $n$ and $a$ be as above.
We claim that for every field extension \( \tilde{L} / F \) whose ramification degree \( e(\tilde{L} / F) \) is prime to \( d \) (in particular for \( \tilde{L} = L \)), the subalgebra \( \tilde{L}_0 = \tilde{L}[\delta_0] \subset \operatorname{Mat}_d(\tilde{L}) \) is a totally ramified field extension of \( \tilde{L} \) of degree \( d \). Indeed, since the minimal polynomial of \( \delta_0 \) divides \( x^n - a \), the algebra \( \tilde{L}_0 \) is either a field or a direct sum of fields. Let \( \tilde{L}_0' \) be one of the direct factors of \( \tilde{L}_0 \), and let \( \delta'_0 \) be the image of \( \delta_0 \) in \( \tilde{L}_0' \). Then \( (\delta'_0)^n = a \), therefore \( \operatorname{val}_{\tilde{L}_0'}(\delta'_0) = \operatorname{val}_{\tilde{L}_0}(a)/n = e(\tilde{L}_0'/F)ek/n = e(\tilde{L}_0'/F)ek/d \). Since \( \operatorname{val}_{\tilde{L}_0}(\delta'_0) \) is an integer, our assumption implies that \( d|e(\tilde{L}_0'/\tilde{L})| [\tilde{L}_0' : \tilde{L}] \leq [L_0 : \tilde{L}] = d \). From this the statement follows.

It follows from our assumption that \( d^2 \) divides the order of \( P\Delta \). We distinguish two cases: i) \( d^2 | n \) and ii) \( d^2 \) does not divide \( n \).

Case i) We will prove that there is a cyclic extension \( M / L \) of degree \( n \), whose ramification degree is \( d \) such that \( a \in \text{Nm}_{M/L} M^\times \). Write \( n \) as a product \( d' n' \), where \( n' \) is prime to \( d \). As before, \( \operatorname{val}_L(a) = ekn/d = d^{-1}ekn' \), hence \( n' | \operatorname{val}_L(a) \). It follows that \( a \) belongs to \( \text{Nm}_{L(n')/L}(L(n')^\times) \). If we find a cyclic extension \( M'/L \) of degree \( d' \) with ramification degree \( d \) such that \( a \in \text{Nm}_{M'/L}(M')^\times \), then the composite field \( M := M'L(n') \) satisfies the required property. Indeed, as \( M' \) and \( L(n') \) are linearly disjoint over \( L \), we have \( \text{Nm}_{M'/L}(M'^\times) = \text{Nm}_{M'/L}(M')^\times \cap \text{Nm}_{L(n')/L}(L(n')^\times) \).

By Local Class Field Theory to construct \( M' \) it is equivalent to construct an open subgroup \( H \subset L^\times \), containing \( a \) and contained in \( \{ l \in L^\times | d^{-1} \text{ divides } \operatorname{val}_L(l) \} \) such that \( L^\times / H \cong \mathbb{Z}/d' \mathbb{Z} \).

First we show that \( a \) is not a \( d^{th} \) power in \( L \). In fact, suppose that \( a = b^d \) for some \( b \in L \). Let \( \eta \) be a primitive \( d^{th} \) root of unity inside \( \mathfrak{T} \). Then the ramification degree of \( \tilde{L} := L(\eta) \) over \( L \) (hence over \( F \)) is prime to \( d \), so by the claim above the algebra \( \tilde{L}_0 = \tilde{L}[\delta_0] \subset \operatorname{Mat}_d(\tilde{L}) \) is a field. The equality \( (\delta_0^n/d - b)(\delta_0^n/d - \eta b) \cdots (\delta_0^n/d - \eta^{d-1}b) = \delta_0^n - a = 0 \) then implies that some \( \delta_0^n/d - \eta^b \) equal to 0. Hence \( \delta_0^n/d \) is central in \( \operatorname{Mat}_d(\tilde{L}) \), contradicting the fact that \( \delta_0 \) has an order \( n \) modulo center.

The group \( A := L^\times/(L^\times)^{d} \) is a finite abelian \( d \)-group. Let \( \tilde{a} \) and \( \tilde{\pi} \) be the images in \( A \) of \( a \) and of some uniformizer of \( L \) respectively, and set \( A' := \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{d} \subset A \). We claim that there exists a subgroup \( H' \subset A' \) such that \( A \) decomposes as a direct sum \( \langle \tilde{\pi} \rangle \oplus \langle \tilde{a} \rangle \oplus H' \), where by \( \langle \tilde{\pi} \rangle \) (resp. \( \langle \tilde{a} \rangle \)) we denote the cyclic subgroup generated by \( \tilde{\pi} \) (resp. \( \tilde{a} \)). Indeed, using the fact that \( d^{-1}| \operatorname{val}_L(a) \) and that \( a \) is not a \( d^{th} \) power in \( L \), we first see that that cyclic subgroups \( \langle \tilde{\pi} \rangle \) and \( \langle \tilde{a} \rangle \) have a trivial intersection. Put \( a' := a\pi^{-\operatorname{val}_L(a)} \in \mathcal{O}_L^\times \). It remains to show that \( \langle \tilde{a'} \rangle \) is a direct summand in \( A' \), or equivalently that \( a' \) is not a \( d^{th} \) power. As \( d| \operatorname{val}_L(a) \) and \( a \) is not a \( d^{th} \) power in \( L \), the statement follows. Now we can take \( H \subset L^\times \) be the inverse image of \( \langle \tilde{a} \rangle \oplus H' \) in \( L^\times \).

Let \( \sigma \) be a generator of \( \operatorname{Gal}(M/L) \) such that \( \sigma_{L(n'\bar{d})} = F_{\mathfrak{T}L} \). As in the previous case, it will suffice to find a point \( P \in \Omega^d_F(M) \) such that \( \delta_0(P) = \sigma(P) \). Let \( \lambda \) be an element of \( M \) such that \( \text{Nm}_{M/L}(\lambda) = a \). As before, there exists \( B \in \text{GL}_d(M) \) such that \( \sigma(B) = \lambda^{-1}\delta_0B \). Then for every \( v \in \mathbb{P}^d \) the image \( P = P_v \in \mathbb{P}^{d-1}(M) \) of \( Bv \in M^d \) satisfies \( \delta_0(P) = \sigma(P) \), so it remains to show that some \( Bv \) does not lie in an \( F \)-rational hyperplane. In fact,
suppose that \( wBv = 0 \) for some row vector \( w \in F^d \). Since \( \sigma(Bv) = \lambda^{-1}\delta_0 Bv \), it follows that \( w\delta_0^i Bv = 0 \) for all \( i \). Since the minimal polynomial of \( \delta_0 \) has degree \( d \), we get that for a generic \( v \) the vectors \( \delta_0^i Bv = B(B^{-1}\delta_0 B)^i v \) span all of \( M^d \). For such a \( v \) we therefore get \( w = 0 \), as claimed.

Case ii) Let \( \delta_0 \) be the image of \( \delta_0^{n/d} \) in \( P\Delta \), and let \( \Delta_d \) be a \( d \)-Sylow subgroup of \( P\Delta \) containing \( \delta_0 \). By our assumptions, \( \Delta_d \cap P\Delta' \) is a non-trivial \( d \)-subgroup, hence it contains a central element \( \delta_1 \) of \( \Delta_d \) of order \( d \). Since \( n/d \) is prime to \( d \), we get that \( \delta_0 \notin P\Delta' \). Therefore \( \delta_0 \) and \( \delta_1 \) generate a subgroup, isomorphic to \( (\mathbb{Z}/d\mathbb{Z})^2 \).

Let \( \delta_1 \in \Delta \) be a representative of \( \bar{\delta}_1 \), then \( (\delta_1)^d \in F^\times \). Replacing \( \delta_0 \) by \( \delta_0^{n/d} \), we get \( \delta_0\delta_1 = b\delta_1\delta_0 \) for some \( b \in F^\times \). Taking the determinants we see that \( b^d = 1 \). We claim that \( b \neq 1 \). In fact, if \( b \) would equal to \( 1 \), then \( \delta_1 \) would belong to the centralizer of \( \delta_0 \) in \( \text{Mat}_d(F) \), which is \( F(\delta_0) \). Since \( F(\delta_0) \) is a totally ramified extension of \( F \), the only elements of \( F(\delta_0)^\times \) whose \( d \)-th power is in \( F^\times \) are of the form \( c(\delta_0)^i \) for some \( c \in F^\times \) and some \( i \). But then we get \( \delta_1 = (\delta_0)^i \), contradicting to our assumptions. Hence \( b \) is a primitive \( d \)-th root of unity.

The characteristic polynomial of \( \delta_0 \) is irreducible over \( F \), therefore \( \delta_0 \) has \( d \) distinct eigenvectors \( V_1, ..., V_d \) (conjugate over \( F \)). They correspond to \( d \) conjugate fixed points \( P_1, ..., P_d \in \Omega_F^d \) of \( \delta_0 \). To finish the proof of the theorem it will suffice to show that the cyclic group generated by \( \delta_1 \) acts simply-transitively on the \( P_i \)'s.

Let \( \lambda_1, ..., \lambda_d \) be the eigenvalues of \( \delta_0 \), corresponding the \( V_i \)'s, that is, \( \delta_0(V_i) = \lambda_i V_i \) for all \( i = 1, ..., d \). Then for each \( i = 1, ..., d \) and each \( j = 1, ..., d - 1 \) we have \( \delta_0(\delta_1)^j(V_i) = b^j(\lambda_i \delta_0)(\delta_1)^j(V_i) = b^j\lambda_i(\delta_1)^j(V_i) \). So that \( (\delta_1)^j(P_i) \neq P_i \), as was claimed.

**Remarks 4.2.**

a) By similar analysis, we can show that converse of a) holds in some more cases, but we don’t know whether it holds in general.

b) We suspect that there should be a simpler “topological” proof of Theorem 4.1., which uses the fact that the Bruhat-Tits building of \( \text{PGL}_d(F) \) can be naturally embedded into (Berkovich’s) \( \Omega_F^d \).

c) The curve case of Theorem 4.1 was studied in [JL1], and generalizing these results to the higher dimensional case is one of our aims here — see also Section 4. It is in fact possible to obtain our results here through a generalization of the method of loc. cit., namely an analysis of the special fiber and the resolution of its singularities.

**Corollary 4.2.** If \( P \Gamma \) is torsion-free, then \( X(L) \neq \emptyset \) if and only if \( dk|f \).

**Proof.** Since \( \text{val}_F(\det(Z\Gamma)) = dk \mathbb{Z} \), the corollary follows immediately from the theorem. \( \square \)

5. Local Points on Shimura Curves

We shall now specialize to the case of curves, which is Case 1 of Section 3 with \( r = 1 \). We restrict the level at the prime \( \mathcal{P} \) of \( F \) corresponding to \( v_1 \) by taking a compact open subgroup \( S \) of \( \mathcal{G} = \mathcal{G}' \) and considering a geometrically connected component \( X \) of \( S \setminus X^{\text{int}, \mathcal{P}} \). By Theorem 3.1, \( X \) is defined over \( F_p^{(k_+)} \). Put \( \Omega_{F_p}^{2, \text{nr}} = \Omega_{F_p}^{2} \otimes_{F_p} F_p^{\text{nr}} \). By
Theorem 2.1. \(X\) is \(F_p^{(k+)}\)-isomorphic to \(\Gamma(gSg^{-1})\backslash \Omega_{F_p}^{2,\text{nr}}\) for some \(g \in G\), where we write \(\Gamma(T)\) instead of \(T \cap G(F)\). We also restrict the level away from \(P\) by assuming that (\(^*\)) the norm of \(S\) and its intersection with the center agree as subgroups of \((\mathbb{A}_f^P)^\times\).

Our restriction of the level at \(P\) means that the level (of \(G^\text{int}\)) at \(P\) is the maximal compact subgroup compact modulo the center. (Some cases of larger level subgroups at \(M\), positive elements in \(F/\mathfrak{m}\), some cases of larger level subgroups at \(M\), positive elements in \(F/\mathfrak{s}\).)

Proof. \(Nm \Gamma(S) \subseteq T \cap F_+^\times\), with \(F_+^\times\) be the totally positive elements of \(F\).

**Lemma 5.1.** a) \(Nm \Gamma(S) = T \cap F_+^\times\), with \(F_+^\times\) be the totally positive elements of \(F\).

b) \(k \mathbb{Z} = \text{val}_P \Gamma(S)\) and \(k_+ \mathbb{Z} = \text{val}_P Nm \Gamma(S)\).

Proof. \(Nm \Gamma(S) \subseteq T \cap F_+^\times\) holds since \(B\) is definite. Conversely, take \(t \in T \cap F_+^\times\) (viewed in \((\mathbb{A}_f^P)^\times\)). By the Hasse principle \(t = Nm b\) for some \(b \in B^\times\), and also \(t = Nm s\) for some \(s \in S\). Then \(Nm bs^{-1} = 1 \in G_p^\times\). By the Eichler-Kneser strong approximation theorem, \(bs^{-1} = b_1s_1\) for some \(b_1 \in B\) and \(s_1 \in S\), both of norm 1. Then \(b_1^{-1}s = s_1\) is in \(\Gamma(S)\) and has norm \(t\), proving Part a). Part b) is equivalent to the exactness of the two sequences

\[
\begin{align*}
Z \Gamma(S) & \to F_p^\times / \mathcal{O}_{F_p}^\times \to Cl_T^+ \quad \text{and} \quad \Gamma(S) \to F_p^\times / \mathcal{O}_{F_p}^\times \to Cl_T^+.
\end{align*}
\]

Here \(\alpha\) takes the \(P\)-component, \(\alpha'\) takes the \(P\)-component of the norm, and \(\beta, \beta'\) are the natural maps. The exactness is routine, and left to the reader.

We can now apply the results of Section 3. As \(d = 2\), we have two possibilities: either \(k_+ = k\), in which case \(X\) is the quadratic unramified twist of \(X'\) by \(w_\nu\) (\(=\) the image of \(\gamma_0\) in \(\text{Aut} X'\)) or \(k_+ = 2k\) in which case \(X \simeq X'\). In the latter case we will say that \(X\) is \(\text{Mumford-uniformized}\), even though strictly speaking \(X, X'\) are obtained from a variety which is Mumford-uniformized over \(F\) by extending scalars to \(F^{(k+)}\).

For a finite set \(W\) of finite primes \(\neq P\) of \(F\), we denote by \(K(W)\) the principal congruence subgroup of (squarefree) level \(\prod_{\mathfrak{q} \in W} \mathfrak{q}\) in \(G_p^\times\). We will now show that for a "sufficiently small" \(S\) we have \(k_+ = k\):

**Proposition 5.2.** Let \(W_1\) be a finite set of primes of \(F\). Then there exists a finite set \(W_2\) of finite primes of \(F\), disjoint from \(W_1 \cup \{ P \}\), such that \(k_+ = k\) for any open subgroup \(S \subseteq K(W_2)\).

Proof. By the Unit Theorem, we have \(\mathcal{O}_{F,P}^\times \simeq (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^g\). For \(0 \leq i \leq g\), choose \(\epsilon_i \in \mathcal{O}_{F,P}^\times\) whose classes modulo squares are a basis for the \(\mathbb{F}_2\)-vector space \(\mathcal{O}_{F,P}^\times / (\mathcal{O}_{F,P}^\times)^2\). A routine application of Chebotarev’s Theorem shows that there are primes \(Q_i, 0 \leq i \leq g\),
not in $W_1 \cup \{ P \}$, such that $\epsilon_i$ has a non-square residue modulo $Q_j$ if and only if $i = j$. Put $W_2 = \{ Q_i \}$. Then for any open subgroup $S \subset K(W_2)$ we have $Z \Gamma(S) = Z(S) \cap F^x \subset (O_{F,P})^2$, so a fortiori $Z \Gamma(S) = \text{Nm} \Gamma(S)$. The Proposition follows.

Remarks 5.3. 1. It follows from the proof that for $S \subset K(W_2)$ as above, the $P$-adic valuation of each $x \in Z \Gamma(S)$ is even, hence $k_+ = k$ is even.

2. The Proposition can be strengthened in several ways. If $\epsilon_1, \ldots, \epsilon_h$ are in the subgroup $O_{F,P}^+$ of the totally positive elements of $O_{F,P}$ and generate it modulo squares, it suffices to take $W_2 = \{ Q_i \}$, $h < i \leq g$. It is also possible to choose $W_2$ in a set of positive density (with the obvious definition of positive density here), not merely to a void $W_1 \cup \{ P \}$. In addition, if the primes of $W_1 \cup \{ P \}$ are all prime to 2, one can replace $K(W_2)$ by an open subgroup of $G_P$ whose level divides some power of 2. We shall not need these facts in the sequel.

3. The Proposition and its proof clearly generalize from $d = 2$ to any $d$.

We will need the following

Lemma 5.4. Let $\gamma \in \text{GL}_2(F_P)$ has finite order modulo the center and suppose that $\text{val}_P \det \gamma$ is odd. Then $\gamma$ reverses a unique edge $e_\gamma$ of the Bruhat-Tits tree $B^2_P$.

Proof. The subgroup of $\text{PGL}_2(F_P)$ generated by $\gamma$ is contained in a maximal compact subgroup, hence it fixes a vertex or an edge. Since $\gamma$ moves each vertex to an odd distance, it cannot fix any vertex, hence must reverse a unique edge.

Theorem 4.1 now says the following:

Theorem 5.5. a) If $k_+ = 2k$, or if $f$ is even, then $X(L) \neq \emptyset$.

b) Suppose $k_+ = k$ and $f$ odd. Then $X(L) \neq \emptyset$ if and only if there exists an element $\gamma \in \Gamma(S)$, of finite order modulo the center satisfying $\text{val}_P \det \gamma = k$, such that either $ek$ is even, or $ek$ is odd and the stabilizer $H$ of $e_\gamma$ in $P \Gamma(S)$ as an oriented edge has even order.

Proof. Since $d = 2$ is a prime, part a) follows immediately from Theorem 4.1 by taking $\Delta = Z \Gamma$. To deduce part b), suppose first $X(L) \neq \emptyset$. Then the subgroup $P \Delta$ of Theorem 4.1 contains an element $\gamma$ as needed, and either $ek$ is even or $P \Delta'$ has even order. But if $ek$ is odd, then $P \Delta'$ is precisely the stabilizer of $e_\gamma$ as an oriented edge. Conversely, if $ek$ is even let $\Delta$ be the subgroup of $P \Gamma$ generated by $\gamma$; and if $ek$ is odd let $\Delta$ be the stabilizer in $\Gamma$ of the unoriented edge $e_\gamma$, defined in Lemma 5.4. Then it is immediate that $\Delta$ has the properties required in Theorem 4.1, so $X(L) \neq \emptyset$.

If $S$ is small enough, then $P \Gamma(S)$ is torsion free. If $S$ moreover satisfies the condition of Proposition 7.2, we shall say we are in the asymptotic case. It then follows from the Theorem that

$$\text{(ASYMP)} \quad k_+ = k, \text{ and } X(L) \neq \emptyset \text{ if and only if } f \text{ is even}. $$

If we only assume that $P \Gamma'(S)$ is torsion free, the Theorem has the following
Corollary 5.6. If $P\Gamma(S)$ is torsion free, then $X(L) = \emptyset$ if and only if $k_+ = k$, $f$ is odd, and either $ek$ is also odd or $P\Gamma(S)$ is torsion free.

Proof. By the theorem, we may assume that $k_+ = k$, that $f$ is odd and that $P\Gamma(S)$ has a torsion. It remains to show that $X(L) = \emptyset$ if and only if $ek$ is odd. Assume first that $ek$ is odd. As $P\Gamma(S)$ is torsion free, no non-trivial element of $P\Gamma(S)$ stabilizes an oriented edge, so we get $X(L) = \emptyset$. Assume now that $ek$ is even. Let $\gamma' \in \Gamma$ be any torsion moduli center element. Since $P\Gamma(S)$ is torsion free, $val_{\mathcal{P}} \det \gamma' \equiv k \pmod{2k}$. Therefore modifying $\gamma$ by an element from $\mathbb{Z}\Gamma(S)$, we get an element $\gamma$ with $val_{\mathcal{P}} \det \gamma = k$, as claimed.

At the other extreme, we can pin down the situation rather precisely when $X(L) \neq \emptyset$ and $fek_+$ is odd (so that $k = k_+$). This will require two lemmas, which we state in slightly more general form than necessary.

For an $n^{th}$ root $\zeta$ of $1$, let $Q(\zeta)^+$ be the maximal totally real subfield of the field $Q(\zeta)$ of $n^{th}$ roots of $1$. Then we have the following

Lemma 5.7. Let $\zeta$ be a primitive $n^{th}$ root of $1$, with $n > 2$. Let $Q^+$ be a prime ideal of $Q(\zeta)^+$ of residue characteristic $q$. Then $val_{Q^+}(1+\zeta)(1+\overline{\zeta}) = 1$ if $n = 2q^r$ for $r \geq 1$. Otherwise this valuation is $0$.

Proof. Let $A$ be the positive integer $Nm_{Q(\zeta)/Q}(1+\zeta)$. We claim that $A = \ell$ if $n = 2\ell^r$ for a prime $\ell$, and $A = 1$ otherwise. Indeed, write $n = 2^tt$ with $t$ odd and $r \geq 0$. If $r = 0$ then $A = 1$ because $Nm_{Q(\zeta)/Q}(1-\zeta) = Nm_{Q(\zeta)/Q}(1-\zeta^2)$. For any $u$, let $\Phi_u$ denote the $u^{th}$ cyclotomic polynomial. If $r = 1$ then $A = \prod_u(1-\eta)$, where $\eta$ goes over all primitive $t^{th}$ roots of $1$, so $A = \Phi_t(1)$. Then if $t$ is a power of a prime $\ell$, we have $\Phi_t(x) = (x^t - 1)/(x^{t/\ell} - 1) = x^{(\ell-1)t/\ell} + \cdots + 1 = A = \ell$; else $t = t't''$ with $t'$, $t''$ odd and relatively prime. Then $\Phi_t(x)$ divides $Q(x) = (x^{t'-1}t'' + \cdots + 1)/(x^{t''-1} + \cdots + 1)$, so $A|Q(1) = 1$. Similarly, if $r \geq 2$, then $A = \Phi_n(1)$. Then if $t = 1$, we have $\Phi_n(x) = (x^n/2 + 1)$, so $A = 2$, whereas if $t > 1$, then $\Phi_n(x)$ divides $Q(x) = \Phi_2'(x^t)/\Phi_2(x) = (x^{n/2} + 1)/(x^{2^{r-1}} + 1)$, so $A|Q(1) = 1$. This proves our claim in all cases. The Lemma is now immediate if $n \neq 2q^r$, $r > 0$. Else, recall that $1 + \zeta$ is a uniformizer for the (totally ramified) prime $\mathfrak{Q}$ of $Q(\zeta)$ above $q$ and $Q^+$. Then $val_{Q^+}(1+\zeta)(1+\overline{\zeta}) = val_{\mathfrak{Q}}(1+\zeta) = 1$, proving the Lemma.

Assume that $k_+ = k$, and let $\gamma$ be as in Theorem 5.5. Then $F(\gamma)$ is a CM extension of $F$, since it splits the definite algebra $B$ and is $\not\equiv F$. Then $Nm \gamma$ is a (totally positive) generator of $\mathcal{P}^k$. Writing $Nm$ for $Nm_{K/F}$ for a quadratic extension $K$ of $F$ is unambiguous, because $Nm_{K/F}$ and the reduced norm of $B$ agree for any $F$-embedding of $K$ into $B$. Likewise we let $\overline{\gamma}$ denote both the main involution of $B$ and the (complex) conjugate of any element $x \in K \subset B$.

Lemma 5.8. Let $F(\gamma)$ be a (quadratic) CM extension of $F$, such that some power $\gamma^m$ is in $F$, with $m$ (not minimal). Suppose that $Nm \gamma$ generates a power $\mathcal{P}^k$, $k > 0$, of a prime ideal $\mathcal{P}$ of $F$, of residue characteristic $p$. Set $\zeta = \gamma/\overline{\gamma}$. Then $\gamma$ is an integer of $F(\gamma)$, $\zeta$ is a primitive $m^{th}$ root of $1$, and if $m$ is even, then $\gamma^m$ is totally negative. Moreover,
(i) If $m = 2$ then $\gamma^2 = -u$, where $u \in F$ is a positive generator of $\mathcal{P}^k$.
(ii) If $m \neq 2$ then $F(\gamma) = F(\zeta)$, and $\gamma = s(1 + \zeta)$ for some $s \in F^\times$.

Proof. $\gamma$ is an algebraic integer since its power is. The order of $\zeta$ is the smallest positive integer $n$ such that $\gamma^n = \overline{\gamma}^n$, or equivalently that $\gamma^n \in F$. Thus $n = m$, as claimed. If $m$ is even, then $\zeta^{m/2} = -1$. Hence $\gamma^m = -\gamma^{m/2}\overline{\gamma}^{m/2} = -\text{Nm}(\gamma^{m/2})$ is totally negative as asserted, and we also get (i). For (ii), define $s = \gamma\overline{\gamma}/(\gamma + \overline{\gamma})$, which makes sense since $\overline{\gamma} \neq -\gamma$. Then $\gamma = s(1 + \zeta)$ as asserted. \hfill $\square$

Remark 5.9. $s$ need not belong to $\mathcal{O}_{F,P}^\times$: take $F = \mathbb{Q}(\sqrt{5})$, $\mathcal{P} = (3, \sqrt{5})$, and $\gamma = \sqrt{5}(1 + \sqrt{-1})/2$. Then $m = 4$, and $\gamma = \sqrt{3}\zeta_8$, but $\sqrt{6}/2$ is not in $\mathcal{O}_{F,P}^\times$.

Let $t(F)$ be the maximal integer such that the maximal totally real subfield $\mathbb{Q}(\zeta_{t(F)})$ of $\mathbb{Q}(\zeta_{t(F)})$ is a subfield of $F$. Then $t(F) \geq 2$. Let $\tau$ be the unique prime of $\mathbb{Q}(\zeta_{t(F)})$ lying (and totally ramified) above 2. We now have the following

Proposition 5.10. In the notation of Theorem 5.3, suppose that $f_k$ is odd. Then $X(L) \neq \emptyset$ if and only if either of Conditions (a) or (b) below holds:

(a) $\mathcal{P}$ lies above $\tau$ with odd ramification index $e(\mathcal{P}/\tau)$, and there exists an element $\gamma'$ of $\Gamma(S)$, of order $2^e(\mathcal{P})$ modulo the center, such that $\text{val}_\mathcal{P}\text{Nm}(\gamma') = k$.
(b) There exist elements $\alpha, \gamma'$ in $\Gamma(S)$, of order 2 modulo the center, such that $\text{val}_\mathcal{P}\text{Nm}(\gamma') = k$, $\text{val}_\mathcal{P}\text{Nm}(\alpha) = 0$, and $\alpha\gamma = -\gamma\alpha$.

Proof. Suppose first that (a) or (b) hold. Lemma 5.1 and the fact that $f$ is odd enable us to modify $\gamma'$ by a central element in $Z\Gamma(S)$ and hence to replace it by $\gamma$ with the same order modulo the center and such that $\text{val}_\mathcal{P}\text{Nm}(\gamma) = f_k$. Since $f_k$ is odd, $\gamma$ reverses an edge. The stabilizer $H$ of this edge contains $\overline{\gamma}^2$ in case (a), and $\overline{\gamma}$ in case (b). Hence its order $|H|$ is even in either case. By Theorem 5.3, $X(L) \neq \emptyset$.

Conversely, if $X(L) \neq \emptyset$ let $\gamma$ be as in Theorem 5.3. Then an odd power of $\gamma$ is of order $2^t$, $t \geq 1$, modulo the center. Modifying this power by a central element of $\Gamma(S)$ we get an element $\gamma_1$, of order $2^t$ modulo the center, such that $\text{val}_\mathcal{P}\text{Nm}(\gamma_1) = f_k$. We now replace $\gamma$ by $\gamma_1$. The new $\gamma$ has the same fixed edge $d'$ as before, and satisfies the properties of Theorem 5.3 and its order modulo the center is $m = 2^t$.

Suppose first $t = 1$. Then $B_{\mathcal{P}}^2$ gives an element $\beta \in H$, of order 2 modulo the center. Let $A$ be the subgroup of $P\Gamma(S)$ generated by the images $\overline{\beta}$ and $\overline{\gamma}$ of $\beta$ and $\gamma$. Since $A$ acts discretely on $B_{\mathcal{P}}(S)$ fixing the unoriented edge $\{d', \overline{d'}\}$, it is a finite group. Since $A$ is generated by 2 involutions it is a dihedral group, and the cyclic subgroup $A_0$ generated by $\overline{\gamma}$ is of index 2. Also $\ell = |A_0|$ is even, because $A$ has another subgroup $A_1 \neq A_0$ of index 2, namely the elements of $A$ fixing $d'$. Hence the group generated by $\overline{\gamma}$ and $\overline{\gamma}$, where $\alpha = (\beta\gamma)^{t/2}$, is a non-cyclic group of order 4. Replacing $\alpha$ by $\alpha\gamma$ if necessary, we may assume that $\alpha\gamma = d'$. In $\Gamma(S) \subset B_{\mathcal{P}}(S)$ this implies that $\alpha^2$ and $\gamma^2$ are in $F^\times$, that $\alpha$ is in $\Gamma'(S)$, and that $\gamma\alpha = u\alpha\gamma$ for some $u \in F^\times$. Taking norms (to $F$) we see that $\text{Nm}_{B_{\mathcal{P}}(F)}(u) = u^2 = 1$, so that $u = \pm 1$. However $u = 1$ is impossible, since in this case $\alpha$ and $\gamma$ would generate a commutative subgroup of $GL_2(\mathbb{C})$ whose image modulo the center is finite and non-cyclic, which cannot happen. This gives case (b) of the Proposition.
Suppose next \( t \geq 2 \). By Lemma \[ \text{(5.8)} \), \( \gamma = s(1 + \zeta_{2^q}) \) and \( F(\gamma) = F(\zeta_{2^q})^2 \), so that \( F \) must contain \( F_0 = \mathbb{Q}(\zeta_{2^q})^+ \). Then \( Nm_{k} \gamma = s^2 \theta \) where \( \theta = (1 + \zeta_{2^q})(1 + \overline{\zeta_{2^q}}) \) generates the unique prime \( \tau_0 \) of \( F_0 \), lying over 2. By Lemma \[ \text{(5.7)} \), we get
\[
jk_+ = \text{val}_\mathcal{P} s^2 \theta \equiv e(\mathcal{P}/\tau_0) \text{val}_{\tau_0} \theta = e(\mathcal{P}/\tau_0) \quad (\text{mod } 2).
\]
This shows that \( e(\mathcal{P}/\tau_0) \) is odd.

Finally we check that \( t = t(F) \). From its definition \( t \leq t(F) \); but if \( t(F) \geq t + 1 \) then \( e(\mathcal{P}/\tau_0) \) would be divisible by the ramification index of \( \tau_0 \) in \( \mathbb{Q}(\zeta_{2^q+1})^+ \), which is 2, contradicting the fact that \( e(\mathcal{P}/\tau_0) \) is odd. Hence \( t = t(F) \), concluding the proof of the Proposition.

\[ \text{Remark 5.11.} \] In case (b) above, the isomorphism type of the quaternion algebras \( B \) and \( B^{\text{int}} \) is almost forced. Indeed, for \( a_1, a_2 \in F^\times \) let \( B(a_1, a_2) \) denote the quaternion algebra over \( F \) with basis \( 1, i, j, k \) satisfying \( i^2 = a_1, j^2 = a_2, \) and \( k = ij = -ji \). Then \( B(a_1, a_2) \) is ramified at a place \( v \) of \( F \) if and only if the Hilbert symbol \( (a_1, a_2)_v \) is \(-1\). In case (b) we have \( B \simeq B(-\text{Nm } \gamma, -\text{Nm } \alpha) \), so that \( B \), in addition to being totally definite, is unramified away from primes above 2 (we know it is unramified at \( \mathcal{P} \)). For example, if \( F = \mathbb{Q} \) we get that \( B \simeq \mathbb{H} = B(-1, -1) \), the Hamilton quaternions over \( \mathbb{Q} \); if \( F \) is (real) quadratic, then \( B \) is isomorphic to the base change of \( \mathbb{H} \) to \( F \), or is the totally definite algebra of discriminant 1 (or both). It is clear, however, that matters can get complicated when \( F \) has many primes above 2.

6. Some special cases

Let \( X \) be as before a geometrically connected component of \( S \backslash X^{\text{int}, p} \). We assume throughout this section that we are in the curve case. Then for certain classes of examples we will determine the deficient local fields for \( X \) in the sense of [JL], namely those local fields \( L \) containing the field of definition of \( X \) for which \( X(L) = \emptyset \). In what follows we shall make the following assumption
\[ \text{(Max)} \quad S \text{ is a maximal compact subgroup of } G = G'. \]
All such \( S \)'s are conjugate in \( G \). In fact they are the units of \( \mathcal{M} \otimes \mathcal{O}_L \mathcal{P} \) for any maximal \( \mathcal{O}_F \)-order \( \mathcal{M}^{\text{int}} \) of \( B^{\text{int}} \). Likewise the maximal \( S \)'s are also the units of \( \mathcal{M} \otimes \mathcal{O}_L \mathcal{P} \) for any maximal \( \mathcal{O}_F[1/\mathcal{P}] \)-order \( \mathcal{M} \) of \( B \). Such \( S \)'s certainly satisfy Assumption (*).

The conjugacy classes of the \( \mathcal{M}^{\text{int}} \)'s in \( B^{\text{int}} \) and of the \( \mathcal{M} \)'s in \( B \) are classified by appropriate quotients of \( \text{Cl}(F)/2\text{Cl}(F) \): (See e. g. [CF], Section 3) for the case of the \( \mathcal{M}^{\text{int}} \)'s. To avoid complications we shall therefore add the following assumption:
\[ \text{(Odd)} \quad \text{The narrow class number } h^+(F) \text{ is odd.} \]
We list some consequences of Assumption (Odd) in the following

\[ \text{Lemma 6.1.} \quad \text{(a) For every } 1 \leq i \leq g(= [F : \mathbb{Q}]) \text{ there exists a unit } \epsilon_i \text{ of } F \text{ satisfying } \infty_i(\epsilon_i) > 0 \text{ and } \infty_j(\epsilon_i) < 0 \text{ for any } j \neq i, 1 \leq j \leq g. \]
\[ \text{(b) Totally positive units in } F \text{ are squares.} \]
\[ \text{(c) All the } \mathcal{M}^{\text{int}} \text{'s are conjugate in } B^{\text{int}} \text{ and all the } \mathcal{M} \text{'s are conjugate in } B. \]
(d) All the geometrically connected components of $S \backslash X^\text{int,p}$ are isomorphic.
(e) The quotient of the normalizer of $M^\text{int}$ in $B^\text{int}_+$ by $Z(B^\text{int})^\times (M^\text{int})^\times$ is a finite group $(\mathbb{Z}/2\mathbb{Z})^r$, where $r$ is the number of primes dividing $\text{Disc} B^\text{int}$, and by $B^\text{int}_+$ we denote elements of $B^\text{int}$, whose norm is totally positive.

Proof. Though the proof is straightforward, we will sketch it for the convenience of the reader. Our assumption (Odd) is equivalent to the decomposition

$$A^\times = (A^\times)^2 \mathbb{O}^\times_f F^\times.$$  

Then $-\epsilon_i$ is an element of $F^\times$ appearing in the decomposition of the idele $-1 \in F^\times_{\infty_i} \subset A^\times_{\mathbb{F}_2}$, implying (a). (b) now follows from (a) together with the fact that $\{-\epsilon_1, \ldots, -\epsilon_g\}$ form a basis of the $\mathbb{F}_2$-vector space $\mathbb{O}^\times_f/(\mathbb{O}^\times_f)^2$.

Decomposition (8) implies decomposition $A^\times_f = (A^\times_f)^2 \mathbb{O}^\times_f F^\times$. Using strong approximation theorem together with (an analog of) (4), we get that

$$G^\text{int}(A_{F,\ell}) = Z(G^\text{int})(A_{\ell})G^\text{int}(\mathbb{O}_{\ell})G^\text{int}(F)_+,$$

where $G^\text{int}(F)_+ = B^\text{int}_+$, and similarly for $G$. This decomposition immediately implies part (d) and reduces the remaining parts to the corresponding local statements, which are clear.

Let $X$ be a connected component of $S \backslash X^\text{int,p}$. The field of definition $F'$ of $X$ is a Hilbert class field of $F$. Hence $F'$ is an abelian extension of $F$ of odd degree. In particular, it is totally real. $X$ has good reduction at all the (finite) primes of $F$ which do not divide $\text{Disc} B^\text{int}$. As in [JL1], the question of existence of local points can be settled in such cases via the trace formula and Hensel’s lemma. In [Sh] Shimura had settled the case of real points. His general results specialize in our case to the following

**Proposition 6.2.** Let $\infty$ be a place of $F'$ above $\infty_i$. Then $X(F_{\infty}) \neq \emptyset$ if and only if $F(\sqrt{\epsilon_i})$ does not split at any prime where $B^\text{int}$ ramifies, where $\epsilon_i$ is as in Lemma 6.1. (Equivalently, if no finite prime of $F$ dividing $\text{Disc} B^\text{int}$ splits in $F(\sqrt{\epsilon_i})$).

Proof. For the convenience of the reader and for comparison with the case of a finite prime $v|\text{Disc} B$, we shall briefly sketch the proof.

The theorem on conjugation of Shimura varieties allows us to assume that $B^\text{int}$ splits at $\infty_i$. Putting $H = \mathbb{C} \setminus \mathbb{R}$ we then get, using Lemma 6.1 (d), that $(X \otimes_{F'} \mathbb{C}_\infty)^\text{an} \simeq \Gamma(S) \backslash H$. Then

$$X(F'_{\infty}) \neq \emptyset \iff (\exists x \in H, \exists \gamma \in \Gamma(S) : \gamma(x) = x).$$

In this case, $\gamma^2(x) = x$, hence some power $\gamma^k$ of $\gamma$ is central. Also $\det \gamma < 0$, so $\gamma$ is conjugate in $\text{GL}_2(F_{\infty}) = \text{GL}_2(F_{\infty_i})$ to a matrix $\begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix}$ with $\alpha, \beta > 0$. Then

$$(\alpha/\beta)^k = 1$$

since $\gamma^k$ is central, so that $\alpha = \beta$. It follows that $\epsilon := \gamma^2 = \alpha^2$ is in $\Gamma(S) \cap F = \mathbb{O}_F^\times$. By Lemma 6.1 (b), we have $F(\sqrt{\epsilon}) = F(\sqrt{\epsilon})$, so that this field splits $B^\text{int}$ and hence cannot be split (over $F$) at any prime dividing $\text{Disc} B^\text{int}$.
Conversely, assume that \( F(\sqrt{\varepsilon_i}) \) is not split at any prime dividing \( \text{Disc} B^{\text{int}} \). Then \( F(\sqrt{\varepsilon_i}) \) embeds into \( B^{\text{int}} \), and the image \( \gamma \) of \( \sqrt{\varepsilon_i} \) must belong to some maximal order of \( B^{\text{int}} \). By Lemma 6.1 (c), we may assume \( \gamma \in \Gamma(S) \). Moreover in \( \text{GL}_2(F_{\infty}) \) \( \gamma \) is conjugate to a matrix \( \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \), since \( \gamma^2 = \varepsilon_i \) is a scalar negative at \( \infty_i \). Hence there exists \( x \in \mathcal{H} \) such that \( \gamma(x) = \overline{x} \), therefore \( X(F_{\infty}) \neq \emptyset \). \( \square \)

Lastly, Proposition 5.10 gives the following result concerning the deficiency of primes \( \mathcal{P} \mid \text{Disc} B^{\text{int}} \):

**Proposition 6.3.** Let \( X \) be as before, and suppose that Assumptions (Odd) and (Max) hold and that \( f_{ek} \) is odd. Let \( \mathcal{P} \) be a finite prime dividing \( \text{Disc} B^{\text{int}} \), and let \( B \) be the totally definite quaternion algebra over \( F \) of discriminant \( \text{Disc} B^{\text{int}} = \text{Disc} B/\mathcal{P} \). Let \( \pi \) be a totally positive generator of the principal ideal \( \mathcal{P}^{k_+} \). Then in the notation of Proposition 5.10 we have that \( X(L) \neq \emptyset \) if and only if either of Conditions (a) or (b) below holds:

(a) \( F(\zeta_{2^t(F)}) \) splits \( B \), and \( \mathcal{P} \) lies above \( \tau \) with odd ramification index. (Recall that \( \tau \) is the prime above 2 in \( \mathbb{Q}(\zeta_{2^t(F)})^+ \), where \( t(F) \) is the maximal integer for which \( F \) contains \( \mathbb{Q}(\zeta_{2^t(F)})^+ \).)

(b) \( B \) is isomorphic to \( B(-1,-\pi) \).

**Proof.** We will check that our conditions (a) and (b) are equivalent to the corresponding conditions of Proposition 5.10. The equivalence of conditions (b) follows from Remark 5.11 and Lemma 6.1 (b). Since \( t(F) \geq 2 \), Lemma 5.8 (ii) implies that our condition (a) follows from that of Proposition 5.10.

Finally assume our assumption (a) and choose an embedding of \( F(\zeta_{2^t(F)}) \) into \( B \). Observe that \( \gamma := 1 + \zeta_{2^t(F)} \) is an algebraic integer such that \( \text{Nm} \gamma \in \mathcal{O}_{F,\mathcal{P}}^\times \). Lemma 5.1 (c) implies that some conjugate \( \gamma'' \) of \( \gamma \) belongs to \( \Gamma(S) \). By Lemma 5.8, \( \gamma'' \) is of order \( 2^{t(F)} \) modulo the center, and by Lemma 5.7. 

\[
\text{val}_{\mathcal{P}} \text{Nm}(\gamma'') = e(\mathcal{P}/\tau) \text{val}_{\tau}(1 + \zeta_{2^t(F)})(1 + \overline{\zeta}_{2^t(F)}) = e(\mathcal{P}/\tau)
\]

is odd. Hence using Lemma 5.1 we can modify \( \gamma'' \) by an element of \( Z\Gamma(S) \) to get an element \( \gamma' \) satisfying \( \text{val}_{\mathcal{P}} \text{Nm}(\gamma') = k \). \( \square \)

7. **Evenness of Jacobians of Shimura curves**

In [PS] Poonen and Stoll defined a dichotomy of principally polarized abelian varieties over a number field \( E \) into even and odd cases. When the abelian variety is a jacobian of a curve \( C/E \) they gave a criterion for the evenness in terms of \( C \). Keeping assumptions (Max) and (Odd), let \( X \) be a component of an appropriate Shimura curve. The methods of [11.2] carry as are to our context. We will call a prime of \( F \) relevant if it is either infinite of divides \( \text{Disc} B^{\text{int}} \). As in [PS, Theorem 23] it follows from Corollary 10 there that the jacobian of \( X \) is even unless the genus of \( X \) is even and the number of relevant deficient primes for \( X \) is odd. We shall prove the following
Theorem 7.1. With $F$ quadratic, suppose that assumptions (Max) and (Odd) hold. For technical reasons assume also that $F$ is not isomorphic to $\mathbb{Q}(\sqrt{2})$ and that $\text{Disc} B^\text{int}$ is not a prime of residual characteristic 2. Then the jacobian of $X/F'$ is even.

Proof. We shall need the following

Proposition 7.2. Let $F = \mathbb{Q}(\sqrt{m})$ for a prime $m \equiv 1 \mod 4$. Suppose $\text{Disc} B^\text{int}$ is an odd prime $\mathcal{P}$ of $F$. Let $k_+$ be the order of $\mathcal{P}$ in $\text{Cl}^+(F)$, and let $\pi$ be a totally positive generator of $\mathcal{P}^{k_+}$. Then

(i) The number of infinite deficient primes of $X$ is even if and only if $(-1, -\pi)_{\mathcal{P}_p} = 1$.
(ii) The number of finite relevant deficient primes of $X$ is even if and only if $(-1, -\pi)_{\mathcal{P}_Q} = 1$ for $\mathcal{Q} = \mathcal{P}$ and for all $\mathcal{Q}$ of residue characteristic 2.

Proof. As $F'$ is an abelian extension of $F$ of odd degree, the number of primes $\propto_1, \propto_2$, and $\mathcal{P}'$ of $F'$ above each of $\propto_1, \propto_2$, and $\mathcal{P}$ is odd and they are deficient simultaneously. For the infinite primes, the fields $F(\sqrt{\epsilon_i})$ are unramified at $\mathcal{P}$ since $\mathcal{P}$ is odd. By Proposition 6.3, the number of deficient infinite primes is even if and only if $\mathcal{P}$ is split in both or inert in both these fields. This happens if and only if $(-1, -\pi)_{\mathcal{P}_p} = 1$, which is equivalent to $(-1, -\pi)_{\mathcal{P}_Q} = 1$, as $k_+$ and $\mathcal{P}$ are odd.

For (ii) notice as before that only case (b) of Proposition 6.3 can happen since $\mathcal{P}$ is odd. Thus the number of finite relevant deficient primes of $X$ is even if and only if the quaternion algebra $B(-1, -\pi)$ splits at all the finite places of $F$. The assertion now follows from Remark 5.11. □

Corollary 7.3. (a) If $m \equiv 5 \mod 8$ then the number of relevant deficient primes for $X$ is even.
(b) If $m \equiv 1 \mod 8$ and $\mathcal{P} = p\mathcal{O}_F$ with $p$ a rational prime (inert in $F$, i.e. $\left(\frac{m}{p}\right) = -1$) which is $\equiv 1 \mod 4$, then the number of relevant deficient primes for $X$ is odd.

Proof. (a) The condition on $m$ means that the rational prime 2 is inert in $F$. By the product formula, $(-1, -\pi)_{\mathcal{P}_2} = (-1, -\pi)_{\mathcal{P}_2}$, so the assertion follows.
(b) Since $p$ is inert in $F$ then without loss of generality $\pi = p^{k_+}$. Also $-1$ is a square in $F_p = F_{\mathcal{P}}$, so $(-1, -\pi)_{\mathcal{P}_p} = 1$. By the proposition, the number of infinite deficient primes is even. Now let $\lambda$ be a prime of $F$ above 2. As 2 splits in $F$ and $k_+$ is odd, we get

$(-1, -\pi)_{\mathcal{P}_\lambda} = (-1, -p)_{\mathcal{Q}_2} = (-1, -p)_{\mathbb{R}}(-1, -p)_{\mathcal{Q}_p} = -1$,

and the claim follows from the Proposition. □

We now prove Theorem 7.4. The discriminant of $B^\text{int}$ is a product of an odd number $r$ of finite primes of $F$. By Lemma 6.1 (e), the group $W \simeq (\mathbb{Z}/2\mathbb{Z})^r$ acts naturally on $X$. The stabilizer of a point $x \in \mathcal{H}$ in $W$ must be cyclic, hence trivial or of order 2. Let $n$ be the number of points of $X$ fixed by a nontrivial element of $W$. Then the genera $g(X)$ and $g(X/W)$ of $X$ and of $X/W$ are related by

$$2 - 2g(X) = |W|(2 - 2g(X/W)) - n |W|/2.$$
Hence if $r \geq 3$ it follows that $4|2 - 2g(X)$ so that $g(X)$ is odd and the jacobian $\text{Jac}(X)$ is certainly even. Now assume $r = 1$ so that $\text{Disc} B^{\text{int}} = \mathcal{P}$ for a prime $\mathcal{P}$ of $F$ of residual characteristic $p$. By genus theory ([BS], Ch. 3.8, Cor.]), Assumption (Odd) forces $F = \mathbb{Q}(\sqrt{m})$ for a prime $m$ which is either 2 or $\equiv 1 \pmod{4}$. The case $m = 2$ is excluded by our assumption. If the genus of $X$ is odd, there is nothing to prove. Else we must be in either case 1.(a) or 1.(b)(i) of [Sad, Theorem 1.1].

In the first of these cases $m \equiv 5 \pmod{12}$ so that 3 is inert in $F$ and $\mathcal{P} = 3\mathcal{O}_F$. Then $(-1, -3)_\mathcal{P} = 1$, so that the number of deficient infinite primes is even. We need therefore to show that the number of relevant deficient finite primes is even as well; equivalently, by Proposition 7.2, that $(-1, -3)_Q = 1$ for all finite primes $Q$ of residue characteristic 2. Set $(-1, -3)_2 = \prod_Q (-1, -3)_Q$, where the product is over $Q$ of residue characteristic 2. By the product formula $(-1, -3)_2 = 1$, so that if 2 is inert we are done. If 2 is split (the only case left), then for a prime $Q$ above 2 we have $F_Q = Q_2$, so that

$$(-1, -3)_Q = (-1, -3)_2 = (-1)^{\frac{(-1-1)(-3-1)}{2}} = 1,$$

proving again what we need.

In the other case, $m \equiv 5 \pmod{8}$, so corollary 7(a) shows that the number of relevant deficient primes is even, concluding the proof of the Theorem. □

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