Dissecting the string theory dual of QCD

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ABSTRACT: Input from QCD and string theory is used in order to elucidate basic features of the string theory dual of QCD. It is argued that the relevant string theory is a five-dimensional version of the type-0 superstring. The vacuum solution is asymptotically AdS$_5$, and the geometry near the boundary is stringy. The structure of YM perturbation theory however emerges near the boundary. In the IR, the theory is argued to be well-approximated by a two-derivative truncation that takes into account strong coupling effects. This explains the success of previously proposed five-dimensional Einstein-dilaton gravity with an appropriate potential to describe salient features of the strong YM dynamics.

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1. Introduction and outlook

QCD is a very successful theory of strong interactions. It is also a theory that is hard to calculate with, due to the strong coupling region in the IR. As any kind of observable physics passes via low-energy filters it has complicated efforts in the past three decades to test the theory and make predictions. Our theoretical understanding of QCD stems from several sources/techniques.
• Weak-coupling perturbation theory. This is valid in the UV, because of asymptotic freedom and has been a key element in the understanding of the strong force. Its impact is relying of the factorization of processes into hard and soft components.

• Euclidean Lattice techniques based on numerical estimates of the QCD path integral. To date this is the most direct non-perturbative technique that has provided the first clues to confinement, and numbers for the hadron spectrum that could be compared with data, among other things. This is a non-perturbative approach that is fully mature and its main technical limitation today is computing power. As it is inherently Euclidean it cannot however address ab initio a class of problems that describe time-dependent dynamics. This includes finite temperature dynamical phenomena, as well as scattering. Although some quantities can be obtained by analytic continuation extra input is needed in order for analytic continuation to be performed reliably.

• Special purpose phenomenological models and approximations. For specific problems, phenomenological models can give deep insights into physics that is not directly under analytical control in QCD. One notable example is Chiral Perturbation Theory. This is a low-energy effective field theory for the light meson sector based on ideas of chiral symmetry breaking. Other examples include resumations of perturbative effects based on some assumptions, truncation and solutions of Schwinger-Dyson equations, applications of the Large-$N_c$ expansion and associated matrix models, etc.

In the past decade there have been two developments that stirred the field of strong-interaction physics. The first is data from the RHIC collider that gave the first solid indications for the physics of the quark-gluon plasma [1]. The second is new intuition and results on the large-$N_c$ expansion of gauge theories that changed our perception of the description of strongly-coupled large-$N_c$ gauge theories, [2, 3, 4]. The prototype example has been the AdS/CFT correspondence as exemplified by the (well studied by now) duality of $\mathcal{N} = 4$ super Yang-Mills theory and IIB string theory on $AdS_5 \times S^5$. Further studies focused on providing examples that are closer to real world QCD, [5, 6]. It is fair to say that we now have a good holographic understanding of phenomena like confinement, chiral symmetry and its breaking as well as several related issues. The finite temperature dynamics of gauge theories, has a natural holographic counterpart in the thermodynamics of black-holes on the gravity side, and the thermal properties of various holographic constructions have been widely studied, [3, 4, 5, 6, 7, 8], exhibiting the holographic version of deconfinement and chiral restoration transitions.

The simplest top-down string theory model of QCD involves $D_4$ branes with supersymmetry breaking boundary conditions for fermions [3], as well as a flavor
sector that involves pairs of $D_8 - \overline{D}_8$ probe branes inserted in the bulk, \[1\]. The qualitative thermal properties of this model closely mimic what we expect in QCD, \[8\]. Although such theories reproduced the qualitative features of IR QCD dynamics, they contain Kaluza-Klein modes, not expected in QCD, with KK masses of the same order as the dynamical scale of the gauge theory. Above this scale the theories deviate from QCD. Therefore, although the qualitative features of the relevant phenomena are correct, a quantitative matching to real QCD is difficult.

Despite the hostile environment of non-critical theory, several attempts have been made to understand holographic physics in lower dimensions in order to avoid the KK contamination, based on two-derivative gravitational actions, \[12, 13\]. Indeed, large N QCD is expected to be described by a 5-dimensional theory. The alternative problem in non-critical theories is that curvatures are of string scale size and the truncation of the theory to the zero mode sector is subtle and may be misleading.

A different and more phenomenological bottom-up approach was developed and is now known as AdS/QCD. The original idea described in \[14\] was successfully applied to the meson sector in \[15\], and its thermodynamics was analyzed in \[9\]. The bulk gravitational background consists of a slice of AdS$_5$, and a constant dilaton. There is a UV and an IR cutoff. The confining IR physics is imposed by boundary conditions at the IR boundary. This approach, although crude, has been partly successful in studying meson physics, despite the fact that the dynamics driving chiral symmetry breaking must be imposed by hand via IR boundary conditions. Its shortcomings however include a glueball spectrum that does not fit very well the lattice data, the fact that magnetic quarks are confined instead of screened, and asymptotic Regge trajectories for glueballs and mesons that are quadratic instead of linear.

A phenomenological fix of the last problem was suggested by introducing a soft IR wall, \[16\]. Although this fixes the asymptotic spectrum of mesons and meson dynamics is in principle self-consistent, it does not allow a consistent treatment of the glue sector both at zero and finite temperature. In particular, neither dilaton nor metric equations of motion are solved. Therefore the “on-shell” action is not really on-shell. The entropy computed from the BH horizon does not match the entropy calculated using standard thermodynamics from the free energy computed from the action, etc.

A well-motivated way to obtain linear Regge trajectories for mesons was advocated in \[21\]. In particular it was pointed out that the natural order parameter for chiral symmetry breaking in the context of flavor branes is the open-string tachyon. By studying the tachyon dynamics, it was shown that confinement in a wide class of backgrounds is enough to guarantee chiral symmetry breaking, with linear meson trajectories. When flavor branes are at distances larger than the string scale, the analogue of the tachyon order parameter was investigated in \[22\]. This is relevant for flavor sectors that resemble more the Sakai-Sugimoto setup.
An improved holographic model that lies somewhere between bottom up and top down approaches has been proposed, [17, 18]. It is a five-dimensional Einstein dilaton system, with an appropriately chosen dilaton potential. The vacuum solution involves an asymptotically logarithmically AdS solution near the boundary. The bulk field $\lambda$, dual to the ’t Hooft coupling, is vanishing logarithmically near the boundary in order to match the expected QCD behavior. This implies that the potential must have a regular Taylor expansion as $\lambda \to 0$, and that $\lambda = 0$ is not an extremum of the potential. This is unlike almost all asymptotically AdS solutions discussed so far in the literature. In particular the canonically normalized scalar (the dilaton) is diverging at the boundary $r \to 0$ as $\phi \sim -\log(-\log r)$. The coefficients of the UV Taylor expansion of the potential are in one-to-one correspondence with the holographic $\beta$-function.

In the IR, the potential must have an appropriate behavior so that the theory is confined, has a mass gap and a discrete spectrum. This selects a narrow range of asymptotics that roughly obey

$$ V(\lambda) \sim \lambda^{2Q}, \quad \lambda \to \infty \quad \text{with} \quad \frac{2}{3} \leq Q < \frac{4}{3}. \quad (1.1) $$

The vacuum solution always ends in a naked singularity in the bulk. Demanding that this is a “good” singularity in the classification of Gubser [23] implies $Q < 4/3$. Simple interpolations between the UV and IR asymptotics reproduce very well the low-lying glueball spectrum as well as the perturbative running of the ’t Hooft coupling [18]. At finite temperature this model exhibits the behavior expected from QCD: There is a deconfining transition and the thermodynamics is very close to what one expects from lattice QCD, [19, 20].

In this paper we will go through several arguments originating both in string theory and QCD as we understand it, that will help us analyse in more detail the structure of the string theory dual of QCD. We will see that although there are ambiguities in several places, a picture emerges that seems consistent and gives some hope that we may one day tame the associated string theory. Even today, it may be used as qualitative litmus test of ambitious holographic models.

There are several directions that have not yet been explored. An important one concerns the behavior of one and two-point functions in the IR. This is an important issue as it stands at the heart of justifying the neglect of vevs of higher-dimension operators in a holographic context. Although techniques similar to what we use can be used in this direction, we will not attempt this here.

Another issue is the theoretical definition and practical viability of a hybrid model for QCD. In such a model, physics in the UV is described via perturbative QCD that is used to generate boundary conditions, at a rather low scale (in the few GeV region). Below this scale a holographic model should be used. In such a hybrid model, the UV region near the boundary (that as we argue here is stringy)
can be altogether avoided. The IR region (that as we argue here can be reasonably well-described by a two derivative action) can be handled with standard holographic techniques. An attempt in this direction can be found in [24].

An interesting issue is the cosmological evolution of strongly coupled matter, both made of glue and quarks. In the former case the setup is almost identical to the one studied in the context of Randall-Sundrum cosmology as was shown in [25]. Indeed the simple solution for conformal matter described in [25] has an alternative description in terms of lowering the UV cutoff brane inside AdS. The Randall-Sundrum tuning corresponds to the choice of coupling gravity to renormalized rather than bare sYM operators (the vacuum energy in particular). A similar study for a non-conformal theory like QCD involves a few extra ingredients, the most important of which is establishing the geodesic motion of the UV boundary in the bulk, and in particular the dilaton couplings to the boundary. This is interesting as it may give new tools to study the impact of the deconfinement phase transition in a cosmological setup.

2. General remarks on the string theory dual

The first question we may pose is: in how many dimensions is the string theory dual of QCD living? A way to answer this question proceeds via the intuition developed in the past 20 years from matrix and other large $N$ theory duals to string theory. Indeed, the intuition is as follows. The large N-gauge theory contains several adjoint fields living on a d-dimensional space $M_d$. Typically the eigenvalues of the adjoint matrices becomes new continuous dimensions. For example in the case of the “old matrix models” the single eigenvalue distribution increases the spacetime dimension by one.

Not all adjoint fields provide independent eigenvalue distributions and therefore holographic dimensions. Fields related by global symmetries must be reduced appropriately. Also if the symmetry is not there, but there is a symmetric theory related by RG flow to the previous one, than there is a reduction in the number of holographic dimensions. For example in $\mathcal{N} = 4$ superYM, there are 4 adjoint matrices from the vectors, 8 adjoint matrices from the fermions and 6 adjoint matrices from the scalars. Four-dimensional Lorentz invariance of the vacuum indicates there is a single independent eigenvalue distribution from the vectors. Similarly the SO(6) symmetry implies that that there are 5 independent eigenvalue distributions from the scalars (nicely exposed in the work of Berenstein and collaborators, [26]).

In the case of QCD the situation is simpler. We have four (a vector) adjoint fields. Since the theory and its vacuum on flat space are expected to be Lorentz invariant, only one eigenvalue distribution is independent. Therefore we expect one extra holographic dimension and therefore the string theory dual is expected to live in 5 dimensions.
QCD unlike successful gauge theories in the holographic domain, is asymptotically free. This is a property that has marred attempts in the past 10 years to work out a trustworthy dual string theory. Intuition coming from $\mathcal{N} = 4$ superYM indicates that the 't Hooft coupling is directly related to the spacetime curvature as $\lambda \sim \ell_{s}^{4}$. If this is taken at face value, it would imply that the putative QCD dual will have singular curvatures near the UV, where the QCD coupling vanishes.

There are several caveats to this line of reasoning, that we will mention here.

- The relation between the 't Hooft coupling and spacetime curvature as implied by $\mathcal{N} = 4$ superYM may not be universally applicable and in particular its extrapolation to weak coupling may not be trustworthy. We have many examples of such behavior, where non-linear behavior of effective actions (notably the DBI and CFT actions) smooths out the leading singular behavior. Controlable examples in closed string theory itself, that include WZW and coset models do indeed behave differently at strong curvature. A suggestive example is SU(2)$_{k}$ WZW model where the limit of strong curvature has either curvature of order the string scale ($k=1$) or contains no space at all ($k=0$).

It is plausible that a respectable alternative is that in the limit of the vanishing coupling, curvatures remain at the string scale. Therefore although the regime is stringy, it is not singular.

- The 't Hooft coupling in $\mathcal{N} = 4$ is constant. As such, one can rescale the metric and compute the curvature in different “frames”. Although different rescaled curvatures behave differently, going from one to the other is a simple process. On the other hand the choice of frame is a relevant question in the string theory dual of QCD, where $\lambda$ is expected to be a function of the holographic coordinate, reproducing the RG running of the YM coupling. Different frames with include derivatives of the $\lambda$ in the curvature and can radically alter its behavior.

- Another issue is the approximate conformal invariance that characterizes QCD in the extreme UV. We do expect that this will be geometrically encoded in the string theory dual. The generic lesson of holography in controlled examples is that conformal invariance and its approximations are encoded in the structure of the string geometry via AdS spaces and their deformations. AdS is a very symmetric spacetime, and carries a single boundary where the UV definition of the holographic theory resides via sources. It would be a rather unfortunate situation that the space is singular at the place where we are supposed to define the UV of the theory.

Because of all of the above, a reasonable expectation of the structure of the string background describing the QCD dual is a metric that is asymptotically $AdS_{5}$ near its
boundary region (UV). We expect that this will be modified as we flow towards the
IR signaling the breaking of conformal invariance. Moreover as we will argue more
in detail further on, the AdS$_5$ curvature near the boundary should be comparable to
the fundamental string length $\ell_s$.

2.1 The low-energy string spectrum: a gauge theory view

An important question that is crucial in setting up the vacuum problem in the string
theory dual is to estimate which of the bulk fields are important in determining the
vacuum structure. The lowest dimension fields are certainly the most important in
the UV. QCD has no (strongly) relevant operators. The first non-trivial operators
start at dimension 4 and are given by the quadratic trace of the field strengths

$$Tr[F_{\mu\nu}F_{\rho\sigma}]$$

To distinguish different operators we may use the U(d) decomposition

$$(\begin{array}{c} \text{symmetric} \\ \end{array}) = \begin{array}{c} \text{symmetric} + \text{antisymmetric} \\ \text{symmetric} - \text{antisymmetric} \\ \text{antisymmetric} \end{array}$$

Finally we must remove traces to construct the irreducible representations of O(d):

$$\begin{array}{c} \text{symmetric} \oplus \text{antisymmetric} \\ \text{antisymmetric} \end{array}$$

where the line under a Young tableau implies that all possible traces have been
removed and it therefore represents an irreducible representation of O(d). A $\bullet$ stands
for the singlet of O(d).

The two singlets are the scalar (YM Lagrangian, dual to the string theory dilaton)
and pseudoscalar (instanton) densities:

$$\phi \leftrightarrow Tr[F^2] \quad , \quad a \leftrightarrow Tr[F \wedge F]$$

Each carries a single d.o.f. The t’ Hooft coupling $\lambda$ is related to the dilaton as

$$\lambda \sim N_ce^{\phi}$$

at least in the UV. The instanton density should be dual to a pseudoscalar bulk
axion, $a$. The next operator is a traceless conserved symmetric tensor

$$T_{\mu\nu} = Tr \left[ F^2_{\mu\nu} - \frac{1}{4} g_{\mu\nu} F^2 \right] \quad , \quad (T)_{\mu} = 0 \quad , \quad \partial\mu T_{\mu\nu} + 0$$

It is dual to the 5-dimensional graviton, $g_{\mu\nu}$. Finally

$$T^{4}_{\mu\nu,\rho\sigma} = Tr \left[ F_{\mu\nu} F_{\rho\sigma} - \frac{1}{2} (g_{\mu\rho} F^2_{\nu\sigma} - g_{\nu\rho} F^2_{\mu\sigma} - g_{\mu\sigma} F^2_{\nu\rho} + g_{\nu\sigma} F^2_{\mu\rho}) + \frac{1}{6} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) F^2 \right]$$
It has 10 independent d.o.f and should be dual to a similar massive tensor in the bulk string theory.

Unlike $\mathcal{N} = 4$ superYM, near the UV, the dimensions of these operators are reliable, as they are given by their free-field theory values plus small corrections. Of all these operators with $\Delta = 4$, only the last one correspond to a field that is massive in the bulk string theory. Therefore it is expected to be less important at least in the UV.

Operators with $\Delta = 5$ are given by $\text{Tr} \left[ \nabla_\mu F_{\nu \rho} F_{\sigma \tau} \right]$ (with $\nabla$ being the gauge covariant derivative) while operators with $\Delta = 6$ are given by $\text{Tr} \left[ \nabla_\mu F_{\nu \rho} \nabla^{\mu'} F_{\sigma \tau} \right]$ and $\text{Tr} \left[ F_{\mu \nu} F_{\mu' \nu'} F_{\rho \sigma} \right]$. Of all the higher dimension operators one is expected to correspond to the NS antisymmetric tensor, namely

$$B_{\mu \nu} \sim \text{Tr} \left[ F_{[\mu a} F^{a b} F_{b \nu]} + \frac{1}{4} F_{a b} F^{a b} F_{\mu \nu} \right]$$

and should be massive with a UV mass $M = \frac{4}{\ell_{\text{AdS}}}$ in order to have the correct scaling dimension in the UV. This is happening because of a combination of two effects: the fact that $B$ appears in the five-form field strength together with the RR two-form $C_2$ and the fact that the RR-five form has a vev. This provides a mixing of $B_2$ and $C_2$ that gives both a mass. This is similar to what happens in $\mathcal{N} = 4$ superYM.

The conclusion of this section is that the massless bulk fields should be dual to the operators, $\text{Tr} \left[ F^2 \right]$ (dilaton), $T_{\mu \nu}$ (metric), $\text{Tr} \left[ F \wedge F \right]$ (axion).

### 2.2 Bosonic string or superstring?

We will now present an argument that suggests that the string theory dual to QCD should have a RR sector, which furthermore implies that it is a superstring theory. This does not necessarily imply (broken) spacetime supersymmetry but rather that the world-sheet gauge symmetry of the theory is some form of supersymmetry, which implies in particular the existence of a (bi-fermionic), RR sector. Note however that pure YM does not have gauge invariant fermionic operators. Therefore, the string theory, although a superstring theory, it should not contain spacetime fermions (NS-R, and R-NS sectors). Type 0 theories, have precisely this property, and have been candidate dual grounds for YM for some time [28].

It should be noted here that even after we add quarks, all gauge invariant operators in QCD (with the exception of baryon operators) are bosonic. Baryon operators on the other hand should correspond to appropriate “solitonic” D-branes in the string theory as we now understand in many similar examples [30]. Therefore the standard spectrum of the string theory dual to QCD should also contain no spacetime fermions.

A first candidate for a RR field should be the RR four-form, $C_4$, that in standard examples is known to provide the flux responsible for introducing the (large number of ) $D_3$ branes into the background geometry. Five dimensions is a very special
dimension however for $C_4$ as it does not contain any propagating degrees of freedom, and this makes therefore its presence a bit murkier. There is however another bulk field that (a) is propagating and (b) sources one of the couplings of the YM theory. This is the axion, dual to the YM instanton density. This field must be a RR field (as indeed happens in the $N = 4$ superYM example), in order to match known properties of the CP-odd sector of large-$N_c$ YM [27].

The action of large-$N_c$ QCD including a $\theta$ angle can be written in the form

$$S_{YM} = \int d^4x \, Tr \left[ \frac{1}{4g^2} F^2 + \frac{\theta}{8\pi^2} F \wedge F \right] = N_c \int d^4x \, Tr \left[ \frac{1}{4g^2 N_c} F^2 + \frac{\theta}{8\pi^2 N_c} F \wedge F \right]$$  

(2.9)

The proper scaling in the 't Hooft large-$N_c$ limit is to keep $\lambda = g^2 N_c$ and $\zeta = \frac{\theta}{N_c}$ finite and fixed. Consider now the $\theta$-dependent vacuum energy, to leading order in the $1/N_c$ expansion

$$E_{YM} = N_c^2 F[\lambda, \zeta]$$  

(2.10)

This must be invariant under the $\theta$-angle periodicity shift $\theta \rightarrow \theta + 2\pi$. This however is impossible if $F$ is a smooth function. This can be achieved only if $F$ is multibranched function (obtained by minimizing a collection of many nearly degenerate minima).

In view of this we can write the vacuum energy as [27],

$$E_{YM} = N_c^2 \min_{k \in \mathbb{Z}} f \left[ \lambda, \frac{\theta + 2\pi k}{N_c} \right]$$  

(2.11)

which shows that it is periodic but it is not a continuous function of $\theta$. The CP transformation $\theta \rightarrow -\theta$ implies that $f(\lambda, \theta) = f(\lambda, -\theta)$. CP is unbroken only if $\theta = 0, \pi$.

The integer $k$ labels different vacua that are related by integer shifts of the $\theta$-angle. The absolute minimum is expected at $\theta = 0$ at the $k = 0$ vacuum. Taking all this into account we can write at large $N_c$

$$E_{YM} = N_c^2 f[\lambda, 0] + \frac{1}{2} \frac{\partial f[\lambda, 0]}{\partial \zeta} \min_{k \in \mathbb{Z}} (\theta + 2\pi k)^2 + \mathcal{O}(N_c^{-2})$$  

(2.12)

The quantity $\chi = \frac{\partial f[\lambda, 0]}{\partial \zeta}$ is known as the topological susceptibility and it is known to be non-zero [31]. We therefore observe that the leading $\theta$ dependence of the vacuum energy is coming in at order $\mathcal{O}(1)$ while higher terms are further suppressed with $N_c$.

We will now show that this property on the string theory side is due to the special properties of RR fields. The axion is dual to the instanton density. This implies that its source gives the UV value of the $\theta$ angle

$$a(r) = \theta_{UV} + \mathcal{O}(r^4)$$  

(2.13)
Assuming it is a RR field we may write its tree-level effective action as

\[ S = M^3 \int d^5x \sqrt{g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 + \frac{e^{2\phi}}{2} (\partial a)^2 + C_4 e^{4\phi} (\partial a)^4 + \cdots \right] \quad (2.14) \]

where it should be noted the peculiar dependence on the dilaton of the axion terms in agreement with standard string-theory dilaton counting. We now translate to variables that have a smooth large-$N_c$ limit \( \lambda = N_c e^\phi \) to rewrite the action with explicit \( N_c \) dependence

\[ S = N_c^2 M^3 \int d^5x \sqrt{g} \frac{1}{\lambda^2} \left[ R + 4 \left( \frac{\partial \lambda}{\lambda} \right)^2 + \frac{\lambda^2}{2N_c^2} (\partial a)^2 + C_4 \frac{\lambda^4}{N_c^2} (\partial a)^4 + \cdots \right] \quad (2.15) \]

As in the on-shell action \((\partial a)^2 \sim \theta^2, (\partial a)^4 \sim \theta^4\) etc, we observe that we obtain the same large-$N_c$ scaling of the different \( \theta \)-dependent terms as in the field theory side, (2.12). More details on the vacuum solutions and action for the QCD axion can be found in [18].

The upshot of the previous analysis is that the axion in the string theory dual of QCD is a RR field, indicating that the string theory is a superstring theory in the type-0 class.

### 2.3 The minimal low-energy spectrum: a string-theory view

From our discussion so far we have argued that the string theory dual must have a NS-NS sector with the usual fields, \( g_{\mu\nu}, B_{\mu\nu}, \phi \), as well as a RR sector that contains at least the axion \( a \equiv C_0 \) and the four-form gauge potential \( C_4 \) necessary for generating the color flux, which will be proportional to the number of colors \( N_c \).

Two main issues require a discussion. The first is the RR sector. The minimal possibility is a spinor × spinor in five dimensions. Five-dimensional spinors are not chiral, therefore no gaps are expected in the expansion of the RR bispinor. A direct group-theoretic expansion gives

\[ \text{spinor}_5 \times \text{spinor}_5 = F_0 + F_1 + F_2 + F_3 + F_4 + F_5 \quad (2.16) \]

where \( F_p = dC_{p-1} \) is a p-form field strength. Moreover, the truly independent fields, are \( F_{0,1,2} \) as the rest are related by Poincaré duality: \( F_5 = *F_0, F_4 = *F_1, F_3 = *F_2 \) as implied by the properties of spinors.

- \( F_5 = *F_0 \). \( F_5 \sim N_c \) and the associated \( C_4 \) is not a propagating field in five dimensions. Therefore the 5-flux is important as a background, and as argued in [17, 18] is responsible for non-trivial dilaton dependence in the tree-level string effective action. It provides an IR effective potential for the dilaton (=QCD coupling), and other important dynamical effects in the flavor sector, in particular the correct flavor anomaly generating CS terms [21, 18]. Having no propagating degree of freedom, it is not dual to any operator of large-$N_c$ YM.
\( F_4 = *F_1 \). \( F_1 \) is the field strength of the axion \( a \) dual to the QCD instanton density. Its dual form is a three-form \( C_3 \).

\( F_3 = *F_2 \). \( F_2 \) is the field strength of a RR two-form, \( C_2 \). Its dual is a one-form \( C_1 \). Although these forms belong to the lower level of the RR spectrum, they should correspond to massive states of the string theory, and therefore to higher (than four) dimension operators in QCD. \( C_2 \) becomes massive due to its (CS related) mixing with \( B_2 \) from the NS-NS sector. Ignoring the axion we may write the leading order (string-frame) action for the three- and five-forms as

\[
S = -M^3 \int d^5x \sqrt{g} \left[ \frac{e^{-2\phi}}{2 \cdot 3!} H_3^2 + \frac{1}{2 \cdot 3!} F_3^2 + \frac{1}{2 \cdot 5!} F_5^2 \right]
\]

\( F_3 = dC_2 \), \( H_3 = dB_2 \), \( F_5 = dC_4 - C_2 \wedge H_3 \)

The equations of motion that stem from this action are

\[
\nabla\mu(e^{-2\phi} H_{3,\mu\rho}) + \frac{1}{4} F_{5,\rho\alpha\beta\gamma} F_3^{\alpha\beta\gamma} = 0 \quad \nabla\mu F_{3,\mu\nu} + \frac{1}{4} F_{5,\rho\alpha\beta\gamma} H_3^{\alpha\beta\gamma} = 0
\]

\[
\nabla\mu F_{5,\mu\rho\sigma\tau} = 0 \rightarrow F_{5,\mu\rho\sigma\tau} = \frac{\epsilon_{\mu\nu\rho\sigma\tau} w N_c}{\sqrt{g}} \frac{N_c}{\ell_s}
\]

where \( w \) is a dimensionless constant. Substituting the five-form in the three-form equations we obtain,

\[
\nabla\mu(e^{-2\phi} H_{3,\mu\rho}) + \frac{w N_c \epsilon_{\mu\rho\alpha\beta\gamma}}{4\ell_s} F_3^{\alpha\beta\gamma} = 0 \quad \nabla\mu F_{3,\mu\nu} + \frac{w N_c \epsilon_{\mu\rho\alpha\beta\gamma}}{4\ell_s} H_3^{\alpha\beta\gamma} = 0
\]

We may decouple the two equations by direct manipulation to obtain

\[
\nabla\mu \left[ \nabla\nu(e^{-2\phi} H_{3,\mu\rho} + \text{cyclic}) + \frac{3w^2 N_c^2}{16 \cdot 5! \ell_s^2} H_{3,\mu\rho\sigma} \right] = 0
\]

and a similar one for \( F_3 \). The \( N_c \) dependence is also appropriate as the equation (2.22) is \( N_c \)-independent if we introduce the appropriate \( N_c \)-independent variable \( \lambda = N_c e^\phi \) proportional to the YM ‘t Hooft coupling in the UV.

The upshot of this analysis is that both \( B_2 \) and \( C_2 \) combine to a massive two-tensor, that should be dual to the \( C - \text{odd} \) non-conserved operator \( Tr[F_{\mu\alpha}F^{ab}F_{b\nu}] + \frac{1}{4} F_{ab} F^{ab} F_{\mu\nu} \) with UV dimension 6.

Therefore, this minimal spectrum includes all fields we expect to be massless in five dimensions \((g_{\mu\nu}, \phi, a)\) as well as the flux-generating four-form. In simple type-0

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1 These equations are consistent only if the dilaton is constant, which is the case in the \( \mathcal{N} = 4 \) sYM case. Here non-linearities are important as we will see later one, but the basic mixing mechanism is similar.
vacua in ten dimensions, there is a doubling of RR fields. This is due to the fact
that there is effectively no chirality projection. However, in 5 dimensions, there is
no chirality and it is expected that the RR sector would have the form advocated
above.

There is another point that needs discussion: in type-0 vacua in ten and six
dimensions there is a closed-string “tachyon” scalar. It is indeed a tachyon in flat
ten-dimensional space [29], but it may be massless or even massive in curved non-
critical backgrounds [32]. There is certainly no place for a tachyonic or massless
scalar near the boundary of AdS in the dual string theory of YM as that would
imply the presence of another relevant or marginal operator in YM. However all
such operators have already accounted for. Therefore, if the string theory has such
a zero-th level scalar, it should be massive. Moreover, it is not at all obvious what
operator should be the dual of such a scalar.

Of the minimum set of fields that we mentioned above, only \( g_{\mu\nu}, \phi, a, F_5 \) can
have vevs (non-trivial profiles) in the vacuum in order to preserve O(1,3) Lorentz
invariance. In particular this precludes vectors and two-index antisymmetric tensors
from obtaining a vev.

\[ \text{2.4 The relevant charged defects} \]

Several strings and branes are expected to exist as solitonic objects in this string
theory, in analogy with critical string theory. We will enumerate them trying to
elucidate the nature of each defect.

- **The fundamental string.** It couples electrically to \( B_{\mu\nu} \). This is expected
to represent the YM flux tube. Its tension, \( T_F \sim \frac{1}{16\pi} \) is \( N_c \)-independent. It
should not be confused with the QCD string tension \( \sigma \) that is multiplying the
linear term in the inter-quark potential. This is proportional to \( T_F \) but the
string-frame scale factor also enters [18].

- **NS\(_0\) brane.** In five dimensions it couples magnetically to \( B_{\mu\nu} \). It is the ana-
logue of the NS\(_5\) brane of critical string theory. It is a “point-like” soliton with
tension that scales as \( T_{NS0} \sim O(N_c^2) \). It should be thought of as a magnetic
baryon vertex that binds together \( N_c \) magnetic quarks (each having a mass
that scales as \( O(N_c) \)).

- **\( D_{-1} \) branes.** They are the YM instantons. They couple electrically to the
axion. Their “tension” is \( O(N_c) \) in agreement with what we expect from the
instanton action.

\[ ^2 \text{Some intriguing observations on the couplings of such a scalar to probe D-branes were made in [32].} \]
• **$D_0$ branes.** They couple magnetically to $C_2$. They are the baryon vertices in five dimensions. Their WZ couplings are responsible for binding $N_c$ fundamental strings. Their tension is $\mathcal{O}(N_c)$. On flavor branes they couple to baryon number that is equivalent to instanton number of the flavor $U(N_f)$ gauge fields.

• **$D_1$ branes.** They are the magnetic strings, namely flux tubes between magnetic quarks. They couple electrically to $C_2$, and have a tension of order $\mathcal{O}(N_c)$.

• **$D_2$ branes.** They couple magnetically to the axion. They are domain walls in 4 dimensions that separate different oblique confinement vacua. As one moves across a $D_2$ brane, $\theta$ jumps by $2\pi$.

• **$D_3$ branes.** They couple electrically to the four-form. They generate the gauge group of the gauge theory.

• **$D_4$ branes.** They are space filling branes that generate flavor in the YM theory. Since the string theory is oriented, tadpole cancelation implies that an equal number $N_f$ of $D_4$ and $\overline{D_4}$ branes must be introduced in 5 dimensions. The strings stretched between $D_3$ and $D_4$ branes generate the left-handed quarks while the ones stretched between $D_3$ and $\overline{D_4}$ branes generate the right-handed quarks.

### 2.5 Why a spectrum truncation might work for the vacuum structure?

Large-$N_c$ YM has an infinite number of single-trace operators. The UV definition of the theory, involves only $T_{\mu\nu}$, and $Tr[F^2]$. If $\theta_{UV}$ is non-zero, then $Tr[F \wedge F]$ is also involved. In the holographic dual this translates into the statement that in the vacuum solution, only $\phi$, $g_{\mu\nu}$ and potentially $a$ have a source term in the UV boundary. This guarantees that their profile in the string-theory vacuum solution is necessarily non-zero. Although all the other (infinite tower) of bulk fields have no UV sources, this does not necessarily imply that their profiles in the vacuum solution vanish. They can have non-trivial vevs, that would trigger a non-trivial solution profile in the holographic direction.

In a theory with exact conformal invariance, non-zero one-point functions, can be redefined to zero by a subtraction. Because the theory is conformal, once they are set to zero at a given scale, they remain zero at all scales. This is not the case in theories where conformal symmetry is broken, as in QCD. Although one can subtract a one-point correlator at a given scale, this does not guarantee that the one-point function remains zero at all scales. Of course, bulk fields that break Lorentz invariance cannot acquire vevs, and therefore have trivial profiles in the vacuum. But this is not the case for example for bulk scalars. One therefore may ask, can we neglect the non-trivial profiles of all other single trace operators in the YM/QCD string vacuum solution?
The answer in the UV is rather simple: the higher the scaling dimension and spin of an operator, the larger its bulk mass, and the smaller its influence in the equations of motion of basic fields, $g_{\mu\nu}, \phi, a$. This is a well-known effect both in asymptotically AdS space-times and asymptotically free QFTs. This is the reason we can truncate the infinite coupled system of RG equations near a free field theory, and study a small number of flows corresponding to the most relevant operators. Therefore in the UV of YM, (free-field) scaling dimensions and spin determine the relative importance of operators.

The situation in the IR is more complicated and strictly speaking beyond control in QCD. Generally speaking, higher-dimension operators can and do sometimes become important in theories that are strongly coupled in the IR. A typical example involves (massive) KK fields in the bulk that their vevs can be important in resolving IR singularities in the bulk. Is this expected to happen in QCD? The direct, short and honest answer is: we do not know. There are however a few tantalizing arguments and indications in the past two-three decades that point towards the following answer: that for many, IR relevant and simple observables, vevs of higher dimension (in the UV) operators are not that important for the IR physics. A large class of such arguments are summarized by the surprising successes of the SVZ sums rules [33] that seem to hint at the previous statement. A holographic argument in the same direction, elaborated in the context of the string theory dual will also be presented later on in this article. In view of this we will entertain the possibility that for several observables it is enough to consider the vacuum structure as described by “light string spectrum”, $g_{\mu\nu}, \phi, a$ together with the the non-propagating four-form potential.

2.6 The vacuum solution ansatz

We will consider YM and QCD on Minkowski space, or Euclidean four-dimensional (flat) space. More exotic geometries maybe considered but we will not explore them here to keep the discussion simple. The theory is Lorentz invariant and we do not expect that Lorentz invariance will be broken in the quantum theory. Therefore the form that the metric and other fields will take in the vacuum is very constrained

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = b(r)^2 \left[ dr^2 + dx^M dx_M \right] , \quad \phi \rightarrow \phi(r) , \quad a \rightarrow a(r)$$

(2.23)

where $g_{\mu\nu}$ is the five-dimensional metric, $x^M$ are the four-dimensional Minkowski coordinates and $r$ is the radial coordinate. One can still perform radial reparametrizations, which have been used to bring the metric to the form above. There is also the five-form field strength $F_5$. 4d Lorentz invariance implies that

$$F_{\mu_1\mu_2\cdots\mu_5} = f(r) \epsilon_{\mu_1\mu_2\cdots\mu_5}$$

(2.24)

The fact that we are in five dimensions and the fact that Lorentz-invariance constraints the metric as in (2.23) already implies that capped geometries, like the one
in \cite{35} that have been popular in order to describe backgrounds that are confining in the IR are not possible here. The reason is that there is no extra holographic coordinate beyond the radial one to generate the cigar geometry familiar from Euclidean black holes. Lorentz invariance on the other hand prohibits the used of a Minkowski coordinate to that effect.

From now and for the rest of this paper we will neglect the axion. As discussed in section 2.2, its kinetic term is large-$N_c$ suppressed and it does not therefore contribute to the vacuum structure at leading order in $1/N_c$, except for CP-odd observables. The YM axion has been discussed in \cite{17, 18}.

3. The string effective action

In this section in preparation for our exploration of the “vacuum” of the string theory dual to YM we will investigate the general action at string tree level. As we will see, at least in the UV, the geometry is expected to be stringy, therefore we do not expect a few terms in the derivative expansion to be a good guide. In the absence of an exact string description we will use the language of arbitrary functionals of local curvature invariants as a guide. Although this intuition may sometimes fail we will trust it to obtain qualitative conclusions.

The tree-level string effective action that we will start (in the string frame) is

\[
S_{\text{tree}} = M^3 \int d^5x \sqrt{g} e^{-2\phi} \left[ 4(\partial \phi)^2 + F(R, \xi) \right], \quad \xi \equiv -e^{2\phi} F_5^2 \frac{5!}{5!}
\]  

In this action we have included the fields that will be non-trivial in the vacuum solution, namely $g_{\mu\nu}$, $\phi$, $F_5$. We have suppressed most possible distinct tensor structures under which the curvature and the five-form field strength enter in the above action. In particular, the effective action is expected to be a non-linear function of the Riemann tensor and its covariant derivatives. Although such terms are important ingredients in several string theory observables\footnote{In particular the holographic conformal anomaly \cite{36} and the shear viscosity \cite{37} can depart from their universal values only if higher derivative structures involving the Riemann tensor are present.}, for the arguments that will be made, a simplified action involving only the scalar curvature $R$ and the five-form square $F_5^2$ will suffice. Note that the five-form is always accompanied by a power of $e^\phi$ as is the case for RR forms.

The function $F$ in (3.1) will be taken arbitrary at this stage. Notice also that we did not add non-linear terms in the kinetic term of the dilaton, in the string frame. Following many works and explicit analysis of the string $\sigma$-model up to four-loops it was conjectured \cite{38} that there is only a linear term in $(\partial \phi)^2$. We will assume this to be true here.

\[
\text{--- 15 --}
\]
For weak curvatures we can expand

\[ F = \frac{2 \delta c}{3 \ell_s^2} + R + \frac{1}{2} \xi + \mathcal{O}(R^2, R\xi, \xi^2) \quad , \quad \delta c = 10 - 5 = 5 \quad (3.2) \]

to obtain the standard two derivative tree effective action including the dilaton potential (the first term) present due to the fact that we are in non-critical string theory.

As the four-form potential is non-propagating in five dimensions, it is appropriate to “integrate it out”. To do this we derive its equations of motion, solve them, we then substitute back into the equations of motion of the others fields (metric and dilaton) and we find the new action from which these equations stem by variation.

Varying (3.1) with respect to the four-form we obtain the equation

\[ \nabla^\mu (F_\xi F_{\mu\nu\rho\sigma}) = 0 \quad (3.3) \]

where \( F_\xi \equiv \frac{\partial F(R, \xi)}{\partial \xi} \). The solution of (3.3) is

\[ F_\xi F_{\mu\nu\rho\sigma} = \frac{N_c}{\ell_{AdS}^2} \frac{\epsilon_{\mu\nu\rho\sigma}}{\sqrt{g}} \quad (3.4) \]

where called the constant \( N_c \) and inserted the AdS scale \( \ell_{AdS} \) that will be introduced later in (4.9) to make \( N_c \) dimensionless\(^4\). All we know of course is that this constant is linear in the number of colors, but unlike the critical \( \mathcal{N} = 4 \) case we do not know the precise coefficient. To obtain this we need to know the \( D_3 \) solutions in the non-critical theory in question and their tensions. We will still call the flux constant \( N_c \) though from now on, keeping in mind that this is only proportional to the number of colors. What is important is that we are working in the limit where this flux is sent to infinity keeping

\[ \lambda \equiv N_c e^\phi \quad (3.5) \]

fixed. Indeed this combination is proportional to the 't Hooft coupling of YM, at least in the UV. Squaring (3.4) we obtain

\[ \xi F_\xi^2 = \frac{\lambda^2}{\ell_{AdS}^2} \quad (3.6) \]

This is an algebraic equation that involves, \( \xi, R, \lambda \), and we must solve it implicitly to obtain \( \xi \) as a function of \( R, \lambda \). This can be used to obtain the following differential formulae,

\[ (F_\xi + 2\xi F_{\xi\xi})d\xi + 2\xi F_{\xi\xi}dR = 2\xi F_\xi d\phi \quad , \quad \frac{d\xi}{d\phi} = \frac{2\xi F_\xi}{F_\xi + 2\xi F_{\xi\xi}} \quad (3.7) \]

\(^4\) Although the AdS length will emerge later, it is inserted here for economy, in order to make later formulae simpler.
useful for the variations of the action with respect to the other fields. The equations for the dilaton and graviton read

\[ \Box \phi - (\partial \phi)^2 + \frac{1}{4} (F - \xi F_\xi) = 0 \]  
\[ (3.8) \]

\[ F_R \, R_{\mu\nu} + e^{2\phi} (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) (e^{-2\phi} F_R) + 4 \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (4(\partial \phi)^2 + F) g_{\mu\nu} - F_\xi \frac{e^{2\phi} F^2_{\mu\nu}}{4!} = 0 \]
\[ (3.9) \]

where \( F_R \equiv \frac{\partial F(R, \xi)}{\partial R} \). Substituting from (3.6) into (3.9) we obtain the equation

\[ F_R \, R_{\mu\nu} + e^{2\phi} (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) (e^{-2\phi} F_R) + 4 \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (4(\partial \phi)^2 + F) g_{\mu\nu} + \xi F_\xi g_{\mu\nu} = 0 \]
\[ (3.10) \]

Equations (3.8) and (3.10) can be obtained from the following equivalent action

\[ S_{\text{tree}} = \int d^5 x \sqrt{g} e^{-2\phi} \left[ 4(\partial \phi)^2 + F(R, \xi) - 2\xi F_\xi (R, \xi) \right] , \]
\[ (3.11) \]

where in this action \( \xi \equiv \xi(R, \lambda) \) is an algebraic solution of (3.6). Note that this action is \( \mathcal{O}(N_c^2) \) if we use variables that are finite in the large-\( N_c \) limit as follows

\[ S_{\text{tree}} = M^3 N_c^2 \int d^5 x \sqrt{g} \frac{1}{\lambda^2} \left[ 4 \frac{\partial \lambda^2}{\lambda^2} + F(R, \xi) - 2\xi F_\xi (R, \xi) \right] , \]
\[ (3.12) \]

The conclusion of this analysis is that in the end of the day we must solve the following two equations subject to the algebraic condition (3.6)

\[ F_R \, R_{\mu\nu} + e^{2\phi} (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) (e^{-2\phi} F_R) + 4 \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (4(\partial \phi)^2 + F) g_{\mu\nu} + \xi F_\xi g_{\mu\nu} = 0 \]
\[ (3.13) \]

\[ 4\Box \phi - 4(\partial \phi)^2 + F - \xi F_\xi = 0 \]
\[ (3.14) \]

4. The UV regime

It was already argued in section 2 that the most reasonable description of the asymptotic UV geometry is as an AdS_5 near-boundary region. This description geometrizes the asymptotic conformal invariance, and brings us in line with what was understood in the best studied case of \( \mathcal{N} = 4 \) superYM.

The conformal invariance in QCD however comes together with a coupling constant that vanishes in the UV. Therefore we expect that as we approach the boundary \( \lambda \to 0 \). In the metric ansatz (2.23) our expectation translates into the fact that near the boundary

\[ b(r) \to \frac{\ell_{\text{AdS}}}{r} \]
\[ (4.1) \]

the AdS_5 warp-factor in Poincaré coordinates. Moreover in this regime the intuition is similar to the \( \mathcal{N} = 4 \) case: \( r \) is serving as the inverse of the energy scale.
QCD tells us that near the UV, the ’t Hooft coupling depends on the energy as
\[ \frac{1}{\lambda} = b_0 \log \frac{E}{\Lambda} + \mathcal{O}(\log \log \frac{E}{\Lambda}) \] (4.2)

Using \( r \) as the inverse of the energy we deduce that the solution for the ’t Hooft coupling \( \lambda \) near the AdS boundary must look like
\[ \frac{1}{\lambda} = -b_0 \log r\Lambda + \mathcal{O}(\log \log r\Lambda) \] (4.3)

and therefore \( \lambda \to 0 \) as \( \frac{1}{\log r\Lambda} \) as we approach the boundary \( r \to 0 \).

We conclude that asymptotically close to the boundary \( R \to R_* = -\frac{20}{\ell_{\text{AdS}}^2} \) and \( \lambda \to 0 \).

Consider now equation (3.6). Since the right-hand side vanishes near the boundary, the left hand side must vanish also. This can happen in two possible ways: as \( \lambda \to 0 \) either \( \xi \to 0 \) or \( \xi \to \xi_* \neq 0 \), with \( F_\xi(\xi_*) = 0 \). We will examine both options in turn and show that only the second one can consistently happen.

### 4.1 Vanishing \( \xi \) in the UV

If \( \xi \to 0 \), as \( \lambda \to 0 \) then \( \xi \) must vanish as\(^5\)
\[ \xi \simeq \frac{\lambda^2}{F_\xi^2(R_*, 0) \ell_{\text{AdS}}^2} + \mathcal{O}(\lambda^2) \] (4.4)

Assuming an AdS\(_5\) (constant curvature) solution \( R = R_* \), equations (3.11)-(3.14) imply to leading order \((\lambda = 0)\) that
\[ F(R_*, 0) = F_R(R_*, 0) = 0 \] (4.5)

This in turn implies that the AdS curvature \( R_* \) must be a double root of \( F(R, 0) \).

We now move to the next order and perturb the leading solution \( R = R_* + \delta R \) and non-zero but small \( \lambda \),
\[ \xi = \frac{\lambda^2}{F_\xi^2(R_*, 0) \ell_{\text{AdS}}^2} \left[ 1 - \frac{F_{\xi R}(R_*, 0)}{F_\xi^2(R_*, 0) \ell_{\text{AdS}}^2} \frac{2 \lambda^2}{F_\xi(R_*, 0)} \frac{F_{\xi R}(R_*, 0)}{F_\xi(R_*, 0)} \delta R + \cdots \right] \] (4.6)
\[ F \simeq \frac{\lambda^2}{F_\xi^2(R_*, 0) \ell_{\text{AdS}}^2} + \cdots, \quad \xi F_\xi = \frac{\lambda^2}{F_\xi(R_*, 0) \ell_{\text{AdS}}^2} + \cdots \] (4.7)

where we have used (4.3). To investigate this next order solution we use the desired asymptotic form of the dilaton
\[ \lambda \simeq -\frac{1}{b_0 \log(\Lambda r)} + \cdots \to \phi \simeq \text{constant} - \log[-b_0 \log(\Lambda r)] + \cdots \] (4.8)

\(^5\)This assumes that \( \xi F_\xi \) remains constant as \( \xi \to 0 \). Otherwise it can be shown that there is no solution.
and using the AdS metric
\[ ds^2 = \frac{\ell_{\text{AdS}}^2}{r^2} [dr^2 + dx^\mu dx_\mu] , \quad R_* = -\frac{20}{\ell_{\text{AdS}}^2} \] (4.9)
to leading order we obtain
\[ \Box \phi \simeq \frac{4}{\ell_{\text{AdS}}^2 \log(\Lambda r)} + \cdots , \quad (\partial \phi)^2 \simeq \frac{1}{\ell_{\text{AdS}}^2 \log(\Lambda r)^2} + \cdots \] (4.10)

Equations (4.7) and (4.9) are now incompatible with (3.14). Therefore, the starting assumption cannot be correct for the string theory we are seeking.  

4.1.1 Non-vanishing $\xi$ in the UV  
In this case the function $F_\xi$ must have a zero as a function of $\xi$ at a non-trivial value $\xi_*(R)$. We can therefore parametrize for convenience
\[ F \simeq c_0(R) + \frac{c_1(R)}{2}(\xi - \xi_*(R))^2 + \mathcal{O}[(\xi - \xi_*(R))^3] \] (4.11)
and obtain\footnote{A more exotic possibility is to cancel the offending term against an $\delta R^2$ term from the next order expansion of $F$. This however will give an effective expansion in powers of $\lambda^\frac{a}{2}$ and not integral powers of $\lambda$, a fact at odds with QCD perturbation theory.}
\[ \xi \equiv \xi_*(R) + \delta \xi \simeq \xi_*(R) \pm \frac{\lambda}{c_1(R) \ell_{\text{AdS}} \sqrt{\xi_*(R)}} + \mathcal{O}(\lambda^2) \] (4.12)
We keep both signs here but below we will see that only the minus sign is relevant.

Again the gravitational equation implies that for AdS to be the leading solution (at $\lambda = 0$) we must have
\[ c_0(R_*) = 0 , \quad \frac{\partial c_0}{\partial R} \bigg|_{R=R_*} = 0 \] (4.13)
Using the above we obtain that $F$ is zero to next order and the first non-trivial contribution is at quadratic order
\[ F(R, \xi) = \frac{\lambda^2}{2c_1(R_*) \ell_{\text{AdS}}^2 \xi_*(R_*)} + \frac{1}{2} \frac{\partial^2 c_0}{\partial R^2}(R_*)(R - R_*)^2 + \cdots \] (4.14)
and therefore subleading while
\[ \xi F_\xi = \pm \sqrt{\xi_*(R_*)} \frac{\lambda}{\ell_{\text{AdS}}} + \cdots \] (4.15)

\footnote{For similar reasons the more general possibility $F \simeq c_0(R) + \frac{c_1(R)}{2}(\xi - \xi_*(R))^{a+1} + \cdots$ with $a > 0$ would imply $\lambda \sim \log(\Lambda r)^{-a}$. Therefore only the case $a = 1$ is relevant for YM.}
Now it is possible to solve (3.14) to leading order and match
\[ b_0 = \frac{\ell_{\text{AdS}} \sqrt{\xi_s(R_*)}}{16} \] (4.16)
To have asymptotic freedom we must choose the minus sign in (4.12).
Continuing further to the trace of the gravitational equation (3.13) we obtain
\[ F_R R + 4F_R - 16\partial\phi\partial F_R - 8F_R\partial\phi + 10(\partial\phi)^2 - \frac{5}{2} F + 5\xi F_\xi = 0 \] (4.17)
Taking into account that to leading order \( F_R = 0 \), to next order \( F = 0 \) and (4.13) we obtain that (4.17) becomes to next order,
\[ (4\Box + R_*)\delta R = \frac{5 + \frac{\delta \xi(R_*)}{\xi_s(R_*)} R_*}{c''_0(R_*)} 2\sqrt{\xi_s(R_*)} \frac{\lambda}{\ell_{\text{AdS}}} \] (4.18)
where we have used the fact \( \Box \lambda = -\frac{4}{b_0 \ell_{\text{AdS}}^2 \log(\Lambda r)^2} \) which gives a subleading contribution. This equation gives the following leading modification to the AdS\(_5\) metric
\[ b = \frac{\ell}{r} \left[ 1 + \frac{w}{\log(\Lambda r)} + \cdots \right] , \quad \delta R = \frac{40w}{\ell^2 \log(\Lambda r)} + \cdots \] (4.19)
with
\[ w = -5 + \frac{\delta \xi(R_*)}{\xi_s(R_*)} R_* \frac{\xi_s(R_*)}{80 R_*} \] (4.20)
Let us pause and review what we have found. Near the boundary the leading solution is AdS\(_5\) and the dilaton is such that \( \lambda = 0 \).

- As all dilaton-dependent parts of the effective action (3.11) are subleading near the boundary, the leading AdS\(_5\) solution must be supported by curvature alone. This is the essence of the conditions (4.13). In particular the non-critical dilaton potential and the corrections coming from integrating out the four-form are subleading near the boundary due to asymptotic freedom.

The fact that the AdS solution is supported by curvature alone implies that modulo accidents, the AdS curvature scale \( \ell_{\text{AdS}} \) is of the same order of magnitude as the the string scale \( \ell_s \). Do we know string backgrounds with such a property? In a sense yes, although the backgrounds we know are somewhat simpler. In particular coset CFTs, [39] share some features with what should happen here. In coset CFTs there is a special frame in which the solution to the \( \sigma \) model conditions can be thought as exact, but in other schemes it obtains corrections. However, unlike what is expected to happen here, in coset models one can vary (in a discrete fashion) the curvatures by varying the levels of the current algebra. In a sense we must have a solution where the curvature cannot be varied.
• Unlike situations in critical string theory, the asymptotic AdS geometry here is not supported by RR flux. Therefore, to leading order near the boundary the $\sigma$-model is "conventional. It is to next order that the flux enters the solution and the coupling starts to run.

• So far we have seen the importance of non-linearity of the curvature part of the effective string action in order to find an leading AdS solution. Note however that in order for the function $F_\xi$ to have a root at a non-zero value of $\xi$, the function $F(\xi)$ must also be non-linear. Therefore, we also need the higher derivative corrections of the four-form to find the asymptotic solution of YM.

• Many authors have suggested that the key to understanding the UV region of the QCD string theory is a theory of an infinite number of massless higher spin fields. The idea behind is coming from $\mathcal{N} = 4$ holographic intuition. There, $\lambda \to 0$ implies that $\alpha' \to \infty$ and the whole string spectrum is becoming massless. We see here that things work differently. The string has finite stiffness at the boundary, although it is soft, with a fundamental string tension that is of the same order of magnitude as the background curvature. On the other hand, all three and higher-point connected correlators will vanish near the boundary for a simple reason. When properly normalized, they are all multiplied with positive powers of $\lambda$ that vanishes near the boundary.

• In this section we worked out explicitly the first non-trivial order of the string equations. We have "imposed" both an asymptotic AdS solution and a leading running of the coupling constant. For the rest, assuming genericity, there is a regular perturbation theory in inverse powers of logs, that is similar to the perturbative expansion in perturbative QCD. The various coefficients arise from the expansion coefficients of the non-linear string effective action around the vacuum solution. Without complete control of the non-critical string theory they cannot be calculated. Despite this, the structure of the near-boundary perturbation theory is clear. Moreover the simplifying assumptions we made about the effective action do not seem to modify the conclusions above.

5. The IR regime

We have seen that the general structure of the string effective action and some simple assumptions on the UV asymptotics indicate the presence of the standard YM perturbation theory in the UV. The situation in the IR is much murkier for a the simple reason, that no guiding principle like perturbation theory is known to exist. We do have specific expectations however from the strong coupling region of QCD. In particular we expect confinement and a discrete and gapped glueball spectrum. However these requirements are fairly indirect to guide us in the IR.
A first question we would like to ponder is what happens to the dilaton in the IR. The most natural expectation is that increases without bound, so that $\lambda \to \infty$. There have been minority claims of an IR fixed point in QCD but such claims are not in our opinion credible. There is also the alternative that the coupling asymptotes to a finite large value in the IR. Although this is not excluded, we will not entertain it here for two reasons. The first is that we did not find a good way to implement this possibility while keeping all the properties we expect from YM in the IR, in particular confinement. The second is that even if $\lambda \to \infty$ in the IR for most observables, there may be a maximum finite value for the 't Hooft coupling if all low lying wave-functions have support away from the $\lambda \to \infty$ limit.

Vacua with a runaway dilaton are known in string theory. The simplest is the linear dilaton vacuum that is simple enough to define, but has consistently puzzled researchers for two decades. Due to advances in the understanding of Liouville theory, the associated matrix models and more recent advances in the associated world-sheet CFTs we have today a fairly good idea of the physics in such a background. A crucial ingredient is that there should be a sufficient screening of the strong coupling singularity, that in Liouville theory is achieved via the “Liouville wall”. We will see later on that a linear dilaton background in the IR is marginally compatible with what we expect from YM at strong coupling. In particular we can show that we have confinement and a mass gap, but the spectrum is continuous above the gap. However, by slightly modifying the background we will obtain also a discrete spectrum as well as linear asymptotic trajectories. Moreover as will see, there is a sense in which the strong coupling singularity is screened: all local low-energy observables do not get contributions from arbitrarily close to the singularity. This is what we will call a “repulsive” singularity that is a more constrained concept than the “good” singularities of Gubser, [23].

Another intuition is emerging from the $\mathcal{N} = 4$ paradigm of AdS/CFT: at strong coupling we could expect a good effective description in terms of a two-derivative action. Although this is rather transparent in standard AdS/CFT, it is less clear here, because the 't Hooft coupling is not constant. We will however take it as a principle and we will see how far we can go. An important ingredient in order to implement this idea is that the curvature in the string frame must be very small in the IR. This ties well together with the linear dilaton paradigm as in that case the curvature in the string frame vanishes. We would therefore investigate the possibility that there is a vacuum solution in the IR with $\lambda \to \infty$, a small curvature in the string frame, and subleading contributions from higher derivative terms.

To investigate this we will expand now the string effective action in (3.12) in powers of the curvature, remembering that $\xi$ is an implicit function of $R$ and $\lambda$ from
\[ F(R, \xi) - 2\xi F_\xi(R, \xi) \equiv \sum_{n=0}^{\infty} Z_n(\lambda) R^n \]  

(5.1)

so that the action becomes

\[ S_{\text{tree}} = \int d^5x \sqrt{g} e^{-2\phi} \left[ 4(\partial \phi)^2 + Z_0(\lambda) + Z_1(\lambda) R + \sum_{n=2}^{\infty} Z_n(\lambda) R^n \right] \]  

(5.2)

We now make a conformal transformation

\[ g_{\mu\nu} \rightarrow f(\phi) g_{\mu\nu} \quad , \quad R \rightarrow \frac{1}{f} \left[ R - 4\Box \log f - 3(\partial \log f)^2 \right] \quad , \quad \Box \rightarrow \frac{1}{f} \left[ \Box + \frac{3}{2} \nabla^\mu (\log f) \nabla_\mu \right] \]  

(5.3)

with

\[ f = e^{4\phi} Z_1^{-2/3} \]  

(5.4)

to obtain

\[ S_{\text{tree}} = \int d^5x \sqrt{g} \left[ R - \frac{4}{3} \left[ 4 - 3\left( \frac{1 + Z_1}{Z_1} \right) \right] (\partial \phi)^2 + \frac{e^{4\phi} Z_0}{Z_1^{4/3}} + \sum_{n=2}^{\infty} \frac{Z_n(\lambda) Z_1^{4(2n-5)}}{e^{4\phi(n-1)}} R^n \right] \]  

(5.5)

We may now define a new scalar \( \Phi \) with a canonical kinetic term

\[ \frac{d\Phi}{d\phi} = \sqrt{4 - 3\left( \frac{1 + Z_1}{Z_1} \right)} \]  

(5.6)

Asking for positivity inside the square root, severely constraints on how \( Z_1 \) depends on \( \phi \). In particular we find that

\[ Z_1(\phi) = C_1 e^{4\phi} + \text{subleading} \quad , \quad \text{as} \quad \phi \rightarrow \infty. \]  

(5.7)

The action (5.5) can now be written as

\[ S_{\text{tree}} = \int d^5x \sqrt{g} \left[ R - \frac{4}{3} (\partial \phi)^2 + V_0(\Phi) + \sum_{n=2}^{\infty} V_n(\Phi) R^n \right] \]  

(5.8)

with

\[ V_0(\Phi) = \frac{e^{4\phi} Z_0}{Z_1^{4/3}} \quad , \quad V_n(\Phi) = \frac{Z_n(\lambda) Z_1^{4(2n-5)}}{e^{4\phi(n-1)}} \]  

(5.9)

We may now return to a “string frame” for \( \Phi \) as \( g_{\mu\nu} \rightarrow e^{-\frac{4}{3}\phi} g_{\mu\nu} \) to obtain

\[ S_{\text{ms}} = \int d^5x \sqrt{g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 + W_0(\Phi) + \sum_{n=2}^{\infty} W_n(\Phi) R^n \right] \]  

(5.10)
where

$$W_0(\Phi) = \frac{Z_0}{Z_1} \frac{1}{F^{\frac{2}{3}}} , \quad W_n(\Phi) = \frac{Z_n}{Z_1} F^{\frac{2(n-1)}{3}} , \quad F = e^{2(\Phi - \phi)} \quad (5.11)$$

$$\mathcal{R} = R + \frac{8}{3} \Box \log F - \frac{4}{3} (\partial \log F)^2 \quad (5.12)$$

From now on we will always call $\lambda = e^\Phi$ the ’t Hooft coupling and use it interchangeably with $\Phi$.

Before we proceed further we should discuss our expectations on the dependence of the coefficients $Z_n$ on the ’t Hooft coupling in the IR. We do expect the leading dependence to be the same for all $Z_n$. Moreover, we expect that their growth in the IR is bounded. This is already suggested by the positivity bound on $Z_1$ that we pointed out in (5.7).

In [18] an analysis of various dilaton potentials was performed, assuming a two-derivative action, in order to find which ones give properties that we expect from YM, namely, confinement, mass gap and discrete spectrum. At a two-derivative level, positive energy implies that all Lorentz invariant ansatze lead to an IR singularity (the only other alternative is an IR AdS space). Therefore part of the criteria we used included that this singularity is repulsive\(^8\): spectra can be computed without the need of extra boundary conditions at the singularity and string world-sheets should not come very close to the singularity.

It was found that potentials fall in several categories, that we parametrize their asymptotics as $V \sim \lambda^Q (\log \lambda)^P$. The presence of the log terms matter only when $Q = \frac{4}{3}$.

- Potentials with $Q < \frac{4}{3}$ do not confine.

- Potentials of the form $V \sim \lambda^{4/3} (\log \lambda)^{a - 1}$, $a > 0$, satisfy all criteria, and moreover they have the property that the string frame curvature vanishes in the IR. For $a = 2$ they generate asymptotically linear trajectories. $a = 1$ is the marginal case of the linear dilaton vacuum. This case does not have a discrete spectrum.

- Potentials with $Q = 4/3$ and $P > 1$, confine properly, but the string frame curvature blows up at the singularity

- Potentials with $\frac{4}{3} \leq Q \leq \frac{4 + \sqrt{2}}{3}$ confine properly but the string frame curvature blows up at the singularity.

- Potentials with $Q \geq \frac{4 + \sqrt{2}}{3}$ have a “bad” (non-repulsive) IR singularity.

\(^8\)This is a stronger condition than the one in Gubser’s classification, [23].
In [18] a detailed analysis of the glueball spectra of different confining potentials were performed. It was found that only the “soft potentials” \( V \sim \lambda^{4/3}(\log \lambda)^{\frac{a-1}{a}} \), \( a > 0 \) are capable of giving spectra that are reasonably close to the lattice glueball spectra. All other potentials produced glueball splittings (after adjusting/fitting parameters) that cannot be accommodated by the lattice data.

In view of this we may analyze the possibilities in the IR using the actions (5.8) and (5.10) and the principles that \( Z_n/Z_1 \to \text{constant} \) as \( \lambda \to \infty \) and that the bound (5.7) applies.

We find the following possibilities

1. When \( Z_1 \sim e^{\frac{4}{3}\phi} \) we find a potential that leads to non-confining behavior, and higher derivative corrections that are no controlable.

2. When \( Z_1 \sim e^{\phi b c} \) with \( b < \frac{4}{3} \) leads to non-controlable higher derivatives with the exception of \( b = -4 + 2\sqrt{7} \simeq 1.29 \). In the latter case the potential is \( V_0 \sim e^{\frac{4}{3}\phi} \Phi^\frac{4}{3}C \) and this is an good confining potential with vanishing string frame curvature provided \( c < \frac{3}{4} \). In this case all higher derivative corrections are subleading.

3. When \( Z_1 \sim \text{constant plus subleading} \) we end up with a non-controlable higher derivative behavior with the exception of \( Z_1 = 1 - \frac{C}{2\phi} + \cdots \). In that case the leading effective potential is \( V \sim e^{\frac{4}{3}\phi} \Phi^C \) which gives rise to confining behavior and vanishing string frame curvature. In this case the higher derivative corrections are again subleading.

4. Any other asymptotic behavior leads to non-controlable higher-derivative corrections.

It is not obvious which of the two favorable possibilities found above are realized in QCD. We cannot even exclude the cases that have non-controlable higher derivative corrections. It is noteworthy though that both favorable cases 2 and 3, give the same order of magnitude corrections to the IR running of \( \lambda \). Starting from the Einstein frame effective action and keeping the leading quadratic term (that is \( \Box \log F \)) we can directly estimate that its influence on the leading order solution scales as

\[
\frac{\delta \lambda}{\lambda} \sim \frac{1}{(\log \lambda)^2}
\]  

for all soft potentials \( V \sim \lambda^{\alpha} \lambda^{\frac{a-\alpha}{a}} \), \( a > 1 \).

With a bit of optimism and guided by the intuition above we will accept that the IR of QCD is governed by a soft potential, that once taking the existence of linear trajectories into account, it should have the following asymptotics \( V \sim \lambda^{\frac{4}{3}} \sqrt{\log \lambda} \). Moreover, in such a case the higher-derivative corrections are suppressed.
5.1 The ’t Hooft coupling in the IR

We have already seen that in the IR, the canonically normalized dilaton field $\Phi$ is non-linearly related to the original string theory dilaton $\phi$. In particular in case 2 of this section with $b = -4 + 2\sqrt{7}$ this relation is

$$\Phi = (3 - \sqrt{7})\phi + c \log \phi + \cdots \tag{5.14}$$

where $c$ a constant, while for case 3

$$\Phi = \phi + c \log \phi + \cdots \tag{5.15}$$

where $c$ is again a constant.

There is however another source of corrections to the identification of the ’t Hooft coupling, if it defined using the $D_3$ brane world-volume action. The general form of the kinetic term for the gauge fields on a $D_3$ brane is expected to be

$$L_{F^2} = e^{-\phi}Z(R, \xi) \text{Tr}[F^2] \tag{5.16}$$

where $Z(R, \xi)$ is an (unknown) function of curvatures and the five-form field strength. At weak background fields $Z \simeq -\frac{1}{4} + \cdots$.

In the UV regime, using the formulae of section 4.1.1 we obtain

$$L_{F^2} = N_c \text{Tr}[F^2] \frac{1}{\lambda} \left[ \frac{Z(R_s, \xi_s) - \frac{Z_{\xi}(R_s, \xi_s)}{\xi_{\xi}(R_s, \xi_s)\sqrt{\xi_s} \ell_{\text{AdS}}}}{\xi_{\xi}(R_s, \xi_s)\sqrt{\xi_s} \ell_{\text{AdS}}} + \mathcal{O}(\lambda^2) \right] \tag{5.17}$$

from which we can identify the QCD ’t Hooft coupling as

$$\lambda_{QCD} = \frac{\lambda}{Z(R_s, \xi_s)} + \frac{Z_{\xi}(R_s, \xi_s)}{Z(R_s, \xi_s)^2 \xi_{\xi}(R_s, \xi_s)\sqrt{\xi_s} \ell_{\text{AdS}}} \lambda^2 + \mathcal{O}(\lambda^3) \tag{5.18}$$

indicating that there are “perturbative” corrections to the identification to the ’t Hooft coupling. In the IR, the relation is more complicated and to leading order in the string curvature it becomes

$$\lambda_{QCD} = \frac{\lambda}{Z(0, \xi_s(0, \lambda))} \tag{5.19}$$

We do not know what this non-trivial function of the dilaton is. We should stress though that this is a particular definition (scheme) for the ’t Hooft coupling constant (that we call the $D_3$ scheme, and it may be very different (especially in the IR) from other schemes used in lattice calculations.)

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We thank K. Kajantie whose question prompted this discussion.

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\[9\]We thank K. Kajantie whose question prompted this discussion.
6. Schemes and scheme-dependence

As mentioned in the previous section, there are several choices in the theory that amount to a choice of scheme. Schemes in QFT correspond to coordinate choices in the space of couplings. One can parametrize the space of couplings in different fashions but that is not affecting physical observables that should “coordinate invariant”. A particular scheme choice is that of the renormalization group scale used in renormalizing the theory. Here again physical observables should be RG-invariant.

In the holographic context scheme dependence related to coupling redefinitions translates into field redefinitions for the bulk fields. As the bulk theory is on-shell, all on-shell observables (that are evaluated at the single boundary of spacetime) are independent of the field redefinitions showing the scheme-independence is expected. Invariance under radial reparametrizations of scalar bulk invariants is equivalent to RG invariance. Because of renormalization effects, the boundary is typically shifted and in this case field redefinitions must be combined with appropriate radial diffeomorphisms that amount to RG-transformations.

In particular, the holographic definition of the “Energy” should not affect physical quantities, provided it is reasonable: (a) it is monotonic and (b) it vanishes at the ultimate IR. Of course quantities like the β-function do depend on the definition of the energy (as well as the definition of the coupling). This is expected as a β-function is not an invariant (physical) but a vector in coupling constant space, as well as in energy space and therefore changes in different schemes.

7. A phenomenological model

In view of the discussion in the previous sections, we should ask the question: to what extent a tractable simplified model can be devised that is reasonably close to what we expect to happen in the low-energy theory.

The answer to this question in the IR is straightforward. We have argued that the leading IR asymptotics are described by a dilaton gravity system with a soft potential, and no higher-derivative corrections.  

\[ S_{HQCQ} = M_p^3 N_c^2 \int d^5 x \sqrt{g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V_0(\lambda) \right] , \quad V_0 \sim \lambda^{\frac{5}{2}} \sqrt{\log \lambda} \quad \text{as} \quad \lambda \to \infty \]  

(7.1)

The non-trivial issue is the UV. There we have seen that higher-derivative corrections, especially in the curvature, are essential for the appearance of the asymptotically AdS5 solution and a logarithmically-vanishing coupling constant. Implementing this at the present stage seems out of reach if we insist in a model that we

\footnote{Dilaton gravity with a potential was also argued to have similar thermodynamic properties to QCD in [49].}
can calculate observables. There is short-cut though that has the advantage that it can capture some of the salient features of the UV solution and dynamics while it remains manageable. As expected, it has also shortcomings that we will list below.

The shortcut is to assume a similar action as in (7.1) but with a potential that has different weak coupling asymptotics.

\[ V_0(\lambda) = \frac{12}{\ell_{\text{AdS}}^2} \left[ 1 + v_1 \lambda + v_2 \lambda^2 + v_3 \lambda^3 + \cdots \right] \quad \text{as} \quad \lambda \to 0 \quad (7.2) \]

This form is indicated by the fact that if \( v_1 \neq 0 \) (that implies that the ultimate boundary is not a minimum of the potential) and the dilaton boundary conditions are carefully chosen (as detailed in [18, 20]), the vacuum solution for the 't Hooft coupling and the metric has the same structure in terms of the expansion in inverse logs, as the one advocated here in section 4 and which agrees with QCD perturbation theory. In particular once we chose a definition of the energy scale\(^{11}\), then the dimensionless coefficients \( v_n \) are in one to one correspondence with the \( \beta \)-function coefficients.

Moreover the basic property that as we reach the boundary three- and higher-point connected correlators vanish is satisfied, since \( \lambda \to 0 \).

There are further consistency checks that such an approach in the UV captures quite a few important properties of the system. We list them below

- It captures the correct number of UV boundary conditions in accordance with YM
- It gives the correct perturbative running of the coupling once the analogue of \( \Lambda_{QCD} \) is set. This was verified in [18] by reproduced the experimentally measured value of the strong coupling constant at \( E = 1.2 \text{ GeV} \), using as an input the lowest glueball mass.
- It provides the correct UV boundary conditions for the physical fluctuations of the system, the glueballs, [11, 18].
- It is compatible with the existence of a finite temperature phase with the correct physical properties, [19, 20].
- It provides the correct free-field limit at high temperature and its first logarithmic correction.

There are also issues that are not as they should be. They include

\(^{11}\)In [17, 18] it was argued that a good (but not unique) choice is the scale factor of the metric in the Einstein frame. Moreover different choices that are perturbatively related affect the \( \beta \)-function coefficients beyond the two-loop level.
• The conformal anomaly. It is well known \cite{36} that any theory with a leading two-derivative bulk description must be such that the two anomaly coefficients are equal $a = c$ to leading order, $\mathcal{O}(N_c^2)$. On the other hand, QCD with or without quarks, has $a \neq c$ to leading order in $N_c^{12}$.

• The shear viscosity. It was shown in \cite{40} that two-derivative theories of gravity coupled to matter, have all the minimum constant value of the $\frac{\eta}{s}$ ratio. For the IR regime, as we argued earlier this is probably a very good approximation. However as we approach the UV, this fact is more and more at odds with what happens in perturbative QCD where $\frac{\eta}{s}$ diverges as the coupling vanishes.

• In the UV there are some spurious extra logs in some quantities. They reflect the fact that in the string theory dual of YM, the asymptotic AdS metric is achieved in the string frame and not the Einstein frame. This is a non-trivial issue and can be ascertained from various points of view. The analysis in section 3 is performed in the string frame. It would not go through as such if an AdS metric in the Einstein frame is advocated. On the other hand, for the phenomenological model discussed in this section, it is not consistent with the equations of motion to impose the AdS metric in the string frame. We therefore impose it on the Einstein frame. The difference is an extra multiplicative factor of $(\log(\Lambda r))^\frac{4}{3}$ since the dilaton is asymptotically logarithmic. The effects of this result in mild differences in some quantities. For example a calculation of the short distance asymptotics of the static quark potential produces an an extra log factor of the distance, multiplying the $1/r$ of the Coulomb law. This modification is mild and was shown to fit as well baryonium data as the standard Cornell potential, \cite{42}.

• Higher order correlators of the basic operators of the theory, or higher perturbative (i.e. log corrections) to low-order correlators may be quantitatively different although generically they remain qualitatively correct.

7.1 Parameters, fits and predictions

The action \eqref{S} contains as parameters two scales, $M_p$ and $\ell_{AdS}$ from the potential \eqref{V} as well as in principle many dimensionless parameters hidden in the potential. There is also the string scale $\ell_s$ that enters the fundamental string action. We do not know ab initio how such scales are related. However we may observe that $\ell_{AdS}$ is not really a physical parameter by a choice of energy scale.

Moreover there is dependence on boundary conditions for the vacuum solution. There are three boundary conditions needed for the gravity-dilaton system. One of

\footnote{It can be shown the for CFTs, with a minimum of $\mathcal{N} = 1$ supersymmetry only fundamental matter can contribute to $a - c$, \cite{13}. For theories with a weak coupling limit, conformal invariance implies that only theories with the $\mathcal{N} = 4$ spectrum have $a = c$.}
them is a gauge artifact, as it can be identified with a translation of origin in the radial coordinate. Another is fixed by demanding that correct perturbative asymptotics of the dilaton and the absence of a bad singularity in the bulk. The third corresponds to $\Lambda_{\text{QCD}}$. It is defined in a reparametrization invariant way as

$$\Lambda = \ell^{-1} \lim_{\lambda \to 0} \left\{ b(\lambda) \frac{\exp \left[ -\frac{1}{b_0 \lambda} \right]}{\lambda^{b_1/b_0}} \right\},$$  (7.3)

where $b(\lambda)$ is the scale factor of the Einstein metric in (2.23) written as a function of $\lambda$, and $b_0, b_1$ are the one and two-loop coefficients of the perturbative YM $\beta$-function. $\Lambda$ is fixed once we specify the value of the scale factor $b(\lambda)$ at a given $\lambda_0$.

$M_p$ does not affect spectra of particles but it affects interactions. It also affects the size of the free-energy at finite temperature. It can be fixed,

$$(M_p \ell_{\text{AdS}})^{-3} = 45 \pi^2,$$  (7.4)

by demanding that the free energy for $T \to \infty$ asymptotes to that of a free gas of gluons. The fundamental string scale can be fixed by calculating the inter-quark potential and comparing with the associated lattice results for the effective string tension.

In practice the rest of hidden parameters of the potential are truncated to a small set. In [18] a one-parameter fit could reproduce all known lattice glueball spectra with values inside the error bars, and all the characteristic features of the thermodynamics, [19]. Adding a second parameter the full equilibrium thermodynamics fits perfectly well, [44].

Once the parameters have been fixed, then one can calculate other quantities. They include the full spectrum [18], two and higher-point functions, the full equilibrium thermodynamic quantities [44], as well as transport data.

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References

[1] J. Adams et al. [STAR Collaboration], “Experimental and theoretical challenges in the search for the quark gluon plasma: The STAR collaboration’s critical assessment of the evidence from RHIC collisions,” Nucl. Phys. A 757 (2005) 102 [ArXiv:nucl-ex/0501009].

B. B. Back et al., “The PHOBOS perspective on discoveries at RHIC,” Nucl. Phys. A 757 (2005) 28 [ArXiv:nucl-ex/0410022].

I. Arsene et al. [BRAHMS Collaboration], “Quark gluon plasma and color glass condensate at RHIC? The perspective from the BRAHMS experiment,” Nucl. Phys. A 757 (2005) 1 [ArXiv:nucl-ex/0410020].

K. Adcox et al. [PHENIX Collaboration], “Formation of dense partonic matter in relativistic nucleus nucleus collisions at RHIC: Experimental evaluation by the PHENIX collaboration,” Nucl. Phys. A 757 (2005) 184 [ArXiv:nucl-ex/0410003].

[2] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [ArXiv:hep-th/9711200].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253 [ArXiv:hep-th/9802150].

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998) 105 [ArXiv:hep-th/9802109].

[5] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2 (1998) 505 [ArXiv:hep-th/9803131].

[6] J. M. Maldacena and C. Nunez, “Towards the large N limit of pure N = 1 super Yang Mills,” Phys. Rev. Lett. 86 (2001) 588 [ArXiv:hep-th/0008001].

I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and $\chi$SB-resolution of naked singularities,” JHEP 0008 (2000) 052 [ArXiv:hep-th/0007191].

[7] S. W. Hawking, C. J. Hunter and M. Taylor, “Rotation and the AdS/CFT correspondence,” Phys. Rev. D 59 (1999) 064005 [ArXiv:hep-th/9811056].
A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, “Charged AdS black-holes and catastrophic holography,” Phys. Rev. D 60 (1999) 064018 [ArXiv:hep-th/9902170].

M. Cvetic and S. S. Gubser, “Phases of R-charged black-holes, spinning branes and strongly coupled gauge theories,” JHEP 9904 (1999) 024 [ArXiv:hep-th/9902195].

T. Harmark and N. A. Obers, “Thermodynamics of spinning branes and their dual field theories,” JHEP 0001 (2000) 008 [ArXiv:hep-th/9910036].

D. Mateos, R. C. Myers and R. M. Thomson, “Holographic phase transitions with fundamental matter,” Phys. Rev. Lett. 97, 091601 (2006) [ArXiv:hep-th/0605046].

D. Mateos, R. C. Myers and R. M. Thomson, “Thermodynamics of the brane,” JHEP 0705, 067 (2007) [ArXiv:hep-th/0701132].

[8] O. Aharony, J. Sonnenschein and S. Yankielowicz, “A holographic model of deconfinement and chiral symmetry restoration,” Annals Phys. 322 (2007) 1420 [ArXiv:hep-th/0604161].

[9] C. P. Herzog, “A holographic prediction of the deconfinement temperature,” Phys. Rev. Lett. 98, 091601 (2007) [ArXiv:hep-th/0608151].

[10] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik and I. Kirsch, “Chiral symmetry breaking and pions in non-supersymmetric gauge / gravity duals,” Phys. Rev. D 69, 066007 (2004) [ArXiv:hep-th/0306018].

[11] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113 (2005) 843 [ArXiv:hep-th/0412141].

[12] S. Kuperstein and J. Sonnenschein, “Non-critical supergravity (d > 1) and holography,” JHEP 0407 (2004) 049 [ArXiv:hep-th/0403254].

S. Kuperstein and J. Sonnenschein, “Non-critical, near extremal AdS6 background as a holographic laboratory of four dimensional YM theory,” JHEP 0411 (2004) 026 [ArXiv:hep-th/0411009].

I. R. Klebanov and J. M. Maldacena, “Superconformal gauge theories and non-critical superstrings,” Int. J. Mod. Phys. A 19 (2004) 5003 [ArXiv:hep-th/0409133].

R. Casero, A. Paredes and J. Sonnenschein, “Fundamental matter, meson spectroscopy and non-critical string / gauge duality,” JHEP 0601 (2006) 127 [ArXiv:hep-th/0510110].

A. Paredes, “On unquenched N = 2 holographic flavor,” JHEP 0612 (2006) 032 [ArXiv:hep-th/0610270].

C. Csaki and M. Reece, “Toward a systematic holographic QCD: A braceless approach,” JHEP 0705 (2007) 062 [ArXiv:hep-ph/0608266].
[13] F. Bigazzi, R. Casero, A. L. Cotrone, E. Kiritsis and A. Paredes, “Non-critical holography and four-dimensional CFT’s with fundamentals,” JHEP 0510 (2005) 012 [ArXiv:hep-th/0505140].

[14] J. Polchinski and M. J. Strassler, “Hard scattering and gauge/string duality,” Phys. Rev. Lett. 88 (2002) 031601 [ArXiv:hep-th/0109174]; “Hard scattering and gauge/string duality,” Phys. Rev. Lett. 88 (2002) 031601 [ArXiv:hep-th/0109174].

[15] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, “QCD and a holographic model of hadrons,” Phys. Rev. Lett. 95, 261602 (2005) [arXiv:hep-ph/0501128]; L. Da Rold and A. Pomarol, “Chiral symmetry breaking from five dimensional spaces,” Nucl. Phys. B 721, 79 (2005) [ArXiv:hep-ph/0501218].

[16] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, “Linear confinement and AdS/QCD,” Phys. Rev. D 74 (2006) 015005 [ArXiv:hep-ph/0602229].

[17] U. Gursoy and E. Kiritsis, “Exploring improved holographic theories for QCD: Part I,” JHEP 0802 (2008) 032 [ArXiv:0707.1324 [hep-th]].

[18] U. Gursoy, E. Kiritsis and F. Nitti, “Exploring improved holographic theories for QCD: Part II,” JHEP 0802 (2008) 019 [ArXiv:0707.1349 [hep-th]].

[19] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Deconfinement and Gluon Plasma Dynamics in Improved Holographic QCD,” Phys. Rev. Lett. 101 (2008) 181601 [ArXiv:0804.0899 [hep-th]].

[20] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, “Holography and Thermodynamics of 5D Dilaton-gravity,” [ArXiv:0812.0792 [hep-th]].

[21] R. Casero, E. Kiritsis and A. Paredes, “Chiral symmetry breaking as open string tachyon condensation,” Nucl. Phys. B 787 (2007) 98 [ArXiv:hep-th/0702155].

[22] O. Aharony and D. Kutasov, “Holographic Duals of Long Open Strings,” Phys. Rev. D 78 (2008) 026005 [ArXiv:0803.3547 [hep-th]].

[23] S. S. Gubser, “Curvature singularities: The good, the bad, and the naked,” Adv. Theor. Math. Phys. 4, 679 (2002) [ArXiv:hep-th/0002160].

[24] N. Evans, A. Tedder and T. Waterson, “Improving the Infra-red of Holographic Descriptions of QCD,” JHEP 0701 (2007) 058 [ArXiv:hep-ph/0603249].

[25] E. Kiritsis, “Holography and brane-bulk energy exchange,” JCAP 0510 (2005) 014 [ArXiv:hep-th/0504219].

[26] D. Berenstein, R. Cotta and R. Leonardi, “Numerical tests of AdS/CFT at strong coupling,” Phys. Rev. D 78 (2008) 025008 [ArXiv:0801.2739 [hep-th]]. D. E. Berenstein, M. Hanada and S. A. Hartnoll, “Multi-matrix models and emergent geometry,” [ArXiv:0805.4658 [hep-th]].
[27] E. Witten, “Current Algebra Theorems For The U(1) Goldstone Boson,” Nucl. Phys. B 156, 269 (1979).

[28] A. M. Polyakov, “The wall of the cave,” Int. J. Mod. Phys. A 14 (1999) 645 [ArXiv:hep-th/9809057].

[29] I. R. Klebanov and A. A. Tseytlin, “D-branes and dual gauge theories in type 0 strings,” Nucl. Phys. B 546 (1999) 155 [ArXiv:hep-th/9811035].

[30] E. Witten, “Baryons and branes in anti de Sitter space,” JHEP 9807 (1998) 006 [ArXiv:hep-th/9805112].

[31] E. Vicari and H. Panagopoulos, “Theta dependence of SU(N) gauge theories in the presence of a topological term,” ArXiv:0803.1593 [hep-th].

[32] T. Harmark, V. Niarchos and N. A. Obers, “Stable non-supersymmetric vacua in the moduli space of non-critical superstrings,” Nucl. Phys. B 759 (2006) 20 [ArXiv:hep-th/0605192].

D. Israel and V. Niarchos, “Tree-level stability without spacetime fermions: Novel examples in string theory,” JHEP 0707 (2007) 065 [ArXiv:0705.2140 [hep-th]].

[33] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “QCD And Resonance Physics. Sum Rules,” Nucl. Phys. B 147 (1979) 385; “QCD And Resonance Physics: Applications,” Nucl. Phys. B 147 (1979) 448;

M. A. Shifman, “QCD sum rules: The Second decade,” ArXiv:hep-ph/9304253.

[34] S. R. Das and S. P. Trivedi, “Three brane action and the correspondence between $N = 4$ Yang Mills theory and anti de Sitter space,” Phys. Lett. B 445 (1998) 142 [ArXiv:hep-th/9804149].

S. Ferrara, M. A. Lledo and A. Zaffaroni, “Born-Infeld corrections to D3 brane action in $AdS(5) \times S(5)$ and $N = 4$, $d = 4$ primary superfields,” Phys. Rev. D 58 (1998) 105029 [ArXiv:hep-th/9805082].

[35] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2, 505 (1998) [ArXiv:hep-th/9803131].

[36] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP 9807 (1998) 023 [ArXiv:hep-th/9806087].

[37] G. Policastro, D. T. Son and A. O. Starinets, “The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma,” Phys. Rev. Lett. 87 (2001) 081601 [ArXiv:hep-th/0104066].

[38] R. R. Metsaev and A. A. Tseytlin, “Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor,” Nucl. Phys. B 293 (1987) 385.
A. A. Tseytlin, “Sigma Model Weyl Invariance Conditions And String Equations Of Motion,” Nucl. Phys. B 294 (1987) 383; “Sigma model approach to string theory,” Int. J. Mod. Phys. A 4 (1989) 1257.

[39] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, “String propagation in a black hole geometry,” Nucl. Phys. B 371 (1992) 269.
I. Bars and K. Sfetsos, “Conformally exact metric and dilaton in string theory on curved space-time,” Phys. Rev. D 46 (1992) 4510 [ArXiv:hep-th/9206006]
A. A. Tseytlin, “Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories,” Nucl. Phys. B 411 (1994) 509 [ArXiv:hep-th/9302083].

[40] A. Buchel and J. T. Liu, “Universality of the shear viscosity in supergravity,” Phys. Rev. Lett. 93 (2004) 090602 [ArXiv:hep-th/0311175].

[41] E. Kiritsis and F. Nitti, “On massless 4D Gravitons from 5D Asymptotically AdS Space-times,” Nucl. Phys. B 772 (2007) 67 [ArXiv:hep-th/0611344].

[42] D. f. Zeng, “Heavy quark potentials in some renormalization group revised AdS/QCD models,” Phys. Rev. D 78 (2008) 126006 [ArXiv:0805.2733 [hep-th]].

[43] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, “Viscosity Bound Violation in Higher Derivative Gravity,” Phys. Rev. D 77 (2008) 126006 [ArXiv:0712.0805 [hep-th]].

[44] U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti, Tuning Improved Holographic QCD at finite Temperature, to appear

[45] S. S. Gubser and A. Nellore, “Mimicking the QCD equation of state with a dual black hole,” Phys. Rev. D 78 (2008) 086007 [ArXiv:0804.0434 [hep-th]]
S. S. Gubser, A. Nellore, S. S. Pufu and F. D. Rocha, “Thermodynamics and bulk viscosity of approximate black hole duals to finite temperature quantum chromodynamics,” Phys. Rev. Lett. 101 (2008) 131601 [ArXiv:0804.1950 [hep-th]]
S. S. Gubser, S. S. Pufu and F. D. Rocha, “Bulk viscosity of strongly coupled plasmas with holographic duals,” JHEP 0808 (2008) 085 [ArXiv:0806.0407 [hep-th]].

[46] J. Heitger, H. Simma, R. Sommer and U. Wolff [ALPHA collaboration], “The Schrödinger functional coupling in quenched QCD at low energies,” Nucl. Phys. Proc. Suppl. 106 (2002) 859 [ArXiv:hep-lat/0110201].