Photoacoustic imaging taking into account thermodynamic attenuation

Sebastián Acosta\textsuperscript{1,3} and Carlos Montalto\textsuperscript{2}

\textsuperscript{1} Department of Pediatrics—Cardiology, Baylor College of Medicine and Texas Children’s Hospital, Houston, TX, USA
\textsuperscript{2} Department of Mathematics, University of Washington, Seattle, WA, USA

E-mail: sacosta@bcm.edu and montcruz@uw.edu

Received 5 February 2016, revised 19 August 2016
Accepted for publication 24 August 2016
Published 19 September 2016

Abstract

In this paper we consider a mathematical model for photoacoustic imaging which takes into account attenuation due to thermodynamic dissipation. The propagation of acoustic (compressional) waves is governed by a scalar wave equation coupled to the heat equation for the excess temperature. We seek to recover the initial acoustic profile from knowledge of acoustic measurements at the boundary. We recognize that this inverse problem is a special case of boundary observability for a thermoelastic system. This leads to the use of control/observability tools to prove the unique and stable recovery of the initial acoustic profile in the weak thermoelastic coupling regime. This approach is constructive, yielding a solvable equation for the unknown acoustic profile. Moreover, the solution to this reconstruction equation can be approximated numerically using the conjugate gradient method. If certain geometrical conditions for the wave speed are satisfied, this approach is well-suited for variable media and for measurements on a subset of the boundary. We also present a numerical implementation of the proposed reconstruction algorithm.

Keywords: thermoacoustic and photoacoustic imaging, observability estimates, medical imaging, hybrid and multiwave methods, attenuation, dissipation, damping

(Some figures may appear in colour only in the online journal)
1. Introduction

Photoacoustic tomography (PAT) is an imaging technique that takes advantage of the high-contrast exhibited by optical absorption and the high-resolution carried by broadband acoustic waves in soft biological tissues. Details concerning this type of imaging modalities are found in [1–11]. Qualitative photoacoustic imaging consists of recovering an initial pressure profile from acoustic measurements acquired on the boundary of a region of interest. The successful transformation of boundary measurements into the sought interior pressure profile requires mathematical algorithms that have been studied by numerous researchers. Some of them are based on explicit formulas valid for waves propagating in free-space and homogeneous media [12–19]. Others seek to account for variable wave speed and/or the presence of boundaries [20–34]. See also the reviews [35–38] for additional references.

There have been recent efforts to incorporate acoustic attenuation in the modeling of PAT and thermoacoustic tomography (TAT). See [39–51] and references therein. Most of these results aim at modeling attenuation in the frequency domain to account for dissipation and dispersion. In this paper, however, we adopt a model where the propagation of acoustic waves is thermodynamically coupled to the diffusion of heat. The photoacoustic effect, on which PAT is physically based, consists of two transformations of energy. First, electromagnetic energy is absorbed and transformed into heat. Second, there is a conversion of heat into mechanical energy due to thermal expansivity of the tissues. Concerning this second step, due to the thermodynamic interaction between temperature and pressure, the reverse transformation of energy also takes place. Since heat diffuses, this process attenuates the energy of the thermoacoustic waves. See [52, chapter 8] for an introduction to thermoelasticity in biomechanics. We claim that this type of attenuation should be naturally considered in PAT because PAT itself is based on the thermo-elastic interaction.

We realize that, in mathematical terms, the PAT problem coincides with a problem of boundary observability—the ability to determine the solution of a partial differential equation from knowledge of overdetermined boundary data (Dirichlet and Neumann). This is one of the central concepts of control theory for partial differential equations [53–56]. We have already employed similar tools to address the PAT problem in an enclosure [30] and other related problems [57, 58]. The objective of this paper is to constructively employ the tools of observability for hyperbolic equations together with certain regularity properties of parabolic equations to solve the PAT problem in the presence of thermodynamic attenuation. For the thermoelastic system, there is a series of works on establishing exact, approximate and null controllability or observability estimates [55, 59–66]. In these works, however, either the boundary condition or the distribution of control/observation is not of the type we need to model the PAT problem. Therefore, we modify some ideas provided by these references to seek a solution for PAT in the weak coupling regime. Although it might be possible to use Carleman estimates from [55, 65, 66] to treat the strongly coupled system, we refrain from doing so because the thermoelastic coupling in PAT is known to be relatively weak. Hence, we claim that the results of this paper are sufficient for the nature of PAT in biological tissues.

2. Mathematical formulation and main results

In this paper we study the PAT problem in the presence of thermodynamic dissipation. This is modeled by the linear equations of elasticity coupled with thermal diffusivity [52]. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected domain with boundary $\partial \Omega$ where $n \geq 2$. The propagation of thermoelastic waves in isotropic media is governed by the following system
\[
\rho \partial_t^2 \mathbf{u} - \nabla (\lambda \text{div} \mathbf{u}) - \text{div} (\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T)) + \beta K \nabla \theta = 0, \quad \text{in} \ (0, \tau) \times \Omega \\
\partial_t \theta - \alpha \Delta \theta + \frac{\theta_{\text{ref}} \beta K}{\rho c_p} \text{div} \partial_t \mathbf{u} = 0, \quad \text{in} \ (0, \tau) \times \Omega
\]

for the displacement \( \mathbf{u} \) and where \( \theta \) denotes the deviation from the reference temperature \( \theta_{\text{ref}} \). Also, \( \rho \) is the mass density, \( \lambda \) and \( \mu \) are the Lamé coefficients, \( \alpha \) denotes the thermal diffusivity and \( c_p \) is the specific heat at constant pressure. The thermoelastic coupling is given by \( \beta K \) where \( \beta \) is the coefficient of volumetric thermal expansion and \( K \) is the bulk modulus. In soft biological tissue, \( \lambda \gg \mu \) which implies that \( K/(\lambda + 2\mu) \approx 1 \). For the moment let us assume that \( \lambda, \mu, K \) and \( \rho \) are constants in \( \Omega \). Later, we will drop this assumption.

Since photoacoustic imaging is primarily concerned with the compressional waves, we define the pressure \( p = -(\lambda + 2\mu) \text{div} \mathbf{u} \) and the square of the compressional wave speed \( c^2 = (\lambda + 2\mu)/\rho \) and proceed to obtain a scalar model for the thermoacoustic waves. Simultaneously, we seek to reveal the strength of the thermoelastic coupling by writing the governing equations in unitless form. Let \( L \) be a characteristic length scale of the domain \( \Omega \) (such as its diameter). Let the characteristic time scale be given by \( T = L/c_{\text{ref}} \) where \( c^2_{\text{ref}} = K/\rho \) is the square of a reference wave speed. We define the following unitless variables and parameters:

- Pressure: \( \hat{p} = p/K \) and temperature: \( \hat{\theta} = \theta/\theta_{\text{ref}} \).
- Length: \( \tilde{x} = x/L \), and time: \( \tilde{t} = t/T \).
- Parameters: \( \tilde{c}^2 = c^2 T^2/L^2 \), \( \tilde{\alpha} = \alpha T/L^2 \), \( \tilde{\sigma} = K / (\theta_{\text{ref}} \rho c_p) \) and \( \epsilon = \beta \theta_{\text{ref}} \).

The unitless coupling parameter \( \epsilon > 0 \) is introduced to analyze the case where \( \epsilon \) is sufficiently small. This is valid for a small coefficient of thermal expansion \( \beta \). The unitless product \( \tilde{G} = \epsilon \sigma \) is known as the Grüneisen coefficient. Table 1 displays rough estimates for the values of these physical parameters for soft biological tissues. We obtain that \( 0.05 \lesssim \epsilon \lesssim 0.1 \) and \( 0.5 \lesssim \sigma \lesssim 10 \).

For notational convenience, we assume that \( \sigma = 1 \). This presents no impediment to the theory as we could easily treat the case \( \sigma > 0 \). At this point, in order to alleviate the notation, we also drop the caret to denote the unitless quantities. The unitless scalar governing system then becomes

\[
\partial_t^2 \hat{p} - \epsilon c_0^2 \Delta \hat{p} - \epsilon c_0^2 \Delta \hat{\theta} = 0 \quad \text{in} \ (0, \tau) \times \Omega, \quad (1) \\
\partial_t \hat{\theta} - \alpha \Delta \hat{\theta} - \epsilon \partial_t \hat{p} = 0 \quad \text{in} \ (0, \tau) \times \Omega. \quad (2)
\]

| Physical parameter      | Symbol | Range     | Units       |
|-------------------------|--------|-----------|-------------|
| Bulk modulus            | \( K \) | 2000–2500 | \( 10^6 \) Pa |
| Density                 | \( \rho \) | 900–1100 | Kg m\(^{-3}\) |
| Ref. temperature        | \( \theta_{\text{ref}} \) | 290–310 | K |
| Coeff. thermal expansion| \( \beta \) | 200–300 | \( 10^{-6} \) K\(^{-1}\) |
| Specific heat           | \( c_p \) | 500–5000 | J/(Kg K) |
We take (1) and (2) as the starting point for the modeling of this problem. While the equivalence of the thermoelastic system and equations (1) and (2) only holds when $\lambda$, $\mu$, $K$ and $\rho$ are constants, we will consider a positive wave speed $c \in C^2(\overline{\Omega})$. This follows the common practice of considering a variable wave speed to model heterogeneous media, even when the wave equation is not in divergence form. Similarly, we assume a positive thermal diffusivity $\alpha \in C^2(\overline{\Omega})$. The system (1) and (2) is augmented by the following initial and boundary conditions

\[ p = p_0, \quad \partial_t p = p_1 \quad \text{and} \quad \theta = \theta_0 \quad \text{on} \quad \{t = 0\} \times \Omega, \quad (3) \]
\[ \partial_t p + \gamma \partial_p p = 0 \quad \text{and} \quad \partial_t \theta = 0 \quad \text{on} \quad (0, \tau) \times \partial \Omega. \quad (4) \]

Here, $\gamma : \partial \Omega \to [0, \infty)$ denotes the acoustic impedance coefficient at the boundary $\partial \Omega$. We assume that $\gamma \in C^2(\partial \Omega)$. Physically, $\gamma = 0$ models an acoustically hard surface (such as reflectors) and $\gamma \to \infty$ approximates an acoustically soft boundary. In general, we allow $\gamma$ to vary on the boundary $\partial \Omega$ to model the heterogeneous nature of an enclosing surface and the interface with sensors or air. The length of the observation window of time is given by $\tau < \infty$ which is defined below. In (4), the symbol $\partial_n$ denotes the outward normal derivative at the boundary $\partial \Omega$.

Concerning the initial conditions, it is common in the modeling of PAT to assume the following [67].

**Assumption 2.1 (Rapid deposition of heat).** The initial conditions (3) satisfy

\[ p_1 = 0 \quad \text{and} \quad \theta_0 = \epsilon \, p_0. \]

These two conditions are respectively known as *stress confinement* and *thermal confinement* [67]. They are valid when the pressure relaxation and thermal diffusion are negligible in the very short lapse of heat deposition from the optical source. These conditions can be achieved in biological tissues by using nanosecond optical pulses [1–6, 67–69]. The assumption $\theta_0 = \rho_0$ is mathematically crucial because it removes an important degree of freedom in the analysis. It would not be possible to recover, in stable manner, an independent initial condition for the thermal field. This is a well-known consequence of the smoothing effect of the heat equation. See details in [59–61, 64–66].

In order to consider partial measurements, we divide the boundary as the disjoint union $\partial \Omega = \Gamma \cup \{\partial \Omega \setminus \Gamma\}$ where $\Gamma$ is the portion where we make observations of the acoustic field. We also assume that $\{x \in \partial \Omega : \gamma(x) > 0\} \subset \Gamma$ so that the absorptive part of the boundary (where $\gamma > 0$) is contained within the observable part of the boundary. As reviewed in the next section, the forward problem (1)–(4) has a unique solution, and we can define the measurement map given by

\[ \mathcal{M}p_0 = p_{\Gamma(0, \tau) \times \Gamma}, \quad (5) \]

where $p$ is the solution of (1)–(4) with initial conditions satisfying assumption 2.1. The goal of the PAT problem is to find the initial profile $p_0$ from knowledge of $\mathcal{M}p_0$. This is a challenging problem with intricate dependencies between the domain $\Omega$, the partial boundary $\Gamma$, the wave speed $c$ and the time interval $(0, \tau)$. The admissible dependencies are made precise by a sophisticated assumption of geometric character. Following Bardos et al [54], we assume that our problem enjoys the *geometric control condition* for the Riemannian manifold $(\Omega, c^{-2} \text{d}x^2)$ with only the portion $\Gamma$ of the boundary $\partial \Omega$ being accessible for observation. We assume that $\Gamma$ is a smooth open domain relative to $\partial \Omega$ and that all the geodesics of
(Ω, e⁻²dx²) have finite contact order with the boundary ∂Ω. Under this condition, the geodesic rays of (Ω, e⁻²dx²) can be uniquely extended when they encounter the boundary ∂Ω. See mathematical details in [54]. The geometric assumption needed for our main result is the following.

**Assumption 2.2 (Geometric condition).** There exists τ₀ < ∞ such that any geodesic ray of the manifold (Ω, e⁻²dx²), originating from any point in Ω at t = 0, eventually reaches Γ at a non-diffractive point (after possible geometrical reflections on ∂Ω \ Γ) before time t = τ₀. Also assume that τ > τ₀.

A geodesic ray is non-diffractive if, in the absence of the boundary, the ray leaves Ω. See the precise mathematical definition in [54]. In physical terms, assumption 2.2 means that acoustic signals have an interaction with the boundary Γ strong enough for all acoustic signals to deliver a non-negligible amount of their energy to the boundary.

The main theoretical result of this paper is that even though the thermodynamic attenuation affects the acoustic waves, the pressure measurements acquired on the boundary t ≤ G₀, are sufficient to stably recover the initial state of the pressure field (provided that the thermoelastic coupling is sufficiently weak). We make this statement precise in the form of a theorem.

**Theorem 2.3 (Main result).** Suppose that assumptions 2.1 and 2.2 hold. There exists ε₀ > 0 so that if 0 ≤ ε < ε₀, the map p₀ ↦ Mp₀ is injective and satisfies a stability estimate of the form

\[ \|p₀\|_{H(Ω)} ≤ C\|Mp₀\|_{H(0, τ) × Γ)}, \]

for all p₀ ∈ H¹(Ω), and some positive constant C = C(ε₀) independent of ε.

By contrast to existing results for the observability/controllability of solutions to thermoelastic equations, we highlight that the conclusion of theorem 2.3 is true without having to observe the temperature field θ on the boundary Γ. However, this novel result depends critically on the thermal confinement assumption—that the initial temperature profile is proportional to the initial pressure profile. See assumption 2.1.

The proof of theorem 2.3 is presented in section 3. In addition to this theoretical result, we also propose a convergent iterative reconstruction algorithm which is described in section 4.

### 3. Proof of the main result

In order to properly analyze the inverse problem we must first state some mathematical properties of the initial boundary value problem (1)–(4). Our guiding references are [61, 70–75]. We denote by \(H^k(Ω)\) for \(k ∈ ℤ\) the Sobolev space of order \(k\) over \(L²(Ω)\). Notice, \(H⁰(Ω) = L²(Ω)\). See [71, sections 5.2–5.9] for an introduction to Sobolev spaces as well as the Bochner spaces \(H¹((0, τ); H⁰(Ω))\) and \(C¹((0, τ); H¹(Ω))\) which employ in this section. We use the following definition of energy for the thermoacoustic system,

\[ E(t) = \frac{1}{2} \int _Ω ((\nabla p(t, x))² + c⁻²(x) |\partial_t p(t, x)|² + |\nabla θ(t, x)|²) dV(x) \]

for any triplet \((p(t), θ(t), \partial_t p(t, τ)) ∈ H¹(Ω) × H⁰(Ω) × H¹(Ω)\). Notice that any pair \(p = \text{const}\) and \(θ = \text{const}\) is a solution of (1), (2) and (4) with zero energy. There are
Lemma 3.1. Given \((p_0, p_1, \theta_0) \in \mathcal{H} = \{(p_0, p_1, \theta_0) \in H^1(\Omega) \times H^0(\Omega) \times H^1(\Omega) : (6)\text{–}(7) \text{ are satisfied}\}\). Notice that \(\mathcal{H}\) is a closed subspace of \(H^1(\Omega) \times H^0(\Omega) \times H^1(\Omega)\). So it is complete under the norm of \(H^1(\Omega) \times H^0(\Omega) \times H^1(\Omega)\) as well as under the energy norm. We seek a weak solution \((p, \theta)\) of (1)–(4) in the space \(\mathcal{H}\). We say that the functions \(p, \theta \in H^1((0, \tau); H^{-k}(\Omega))\) for \(k = 0, 1, 2\) and \(\theta \in H^1((0, \tau); H^{-j}(\Omega))\) for \(j = 0, 1\) are a weak solution of (1)–(4) provided that

\[
\langle -2\partial_j^2 p(t), \psi \rangle _\Omega + \langle \nabla \theta(t), \nabla \psi \rangle _\Omega + \epsilon \langle \Delta \theta(t), \psi \rangle _\Omega + a J q J a J - \langle \gamma \partial p(t), \psi \rangle _\Omega = 0,
\]

\[
\langle \alpha^{-1} \partial_j \theta(t), \varphi \rangle _\Omega + \langle \nabla \theta(t), \nabla \varphi \rangle _\Omega - \epsilon \langle \partial_j p(t), \varphi \rangle _\Omega = 0,
\]

for all \(\psi, \varphi \in H^1(\Omega)\) and \(t \in (0, \tau)\), such that \((p(0), \partial p(0), \theta(0)) = (p_0, p_1, \theta_0) \in \mathcal{H}\).

The problem (1)–(4) is well-posed on the space \(\mathcal{H}\) and the energy is non-increasing. The proof follows from the standard analysis of partial differential equations and semigroup theory [61, 70–73]. The well-posedness can be established using energy estimates (see details in [71, chapter 7] and [74]) or by expressing the governing equations (1) and (2) as a system with first-order time derivatives to analyze the spectral properties of the corresponding infinitesimal generator for the strongly continuous semigroup. The details of this latter approach for the thermoacoustic system (1) and (2) are found in [61, section 2] and [56, chapter 2–3]. We state this in the form of a lemma.

Lemma 3.1. Given \((p_0, p_1, \theta_0) \in \mathcal{H}\), the unique weak solution of (1)–(4) satisfies \((p, \partial p, \theta) \in \mathcal{H}\) for \(t \geq 0\), and \(p \in C^\infty([0, \tau]; H^{-k}(\Omega))\) for \(k = 0, 1\) and \(\theta \in C([0, \tau]; H^1(\Omega))\). Moreover, \(E(t) \leq E(0)\) for all \(t \geq 0\) and all \(\epsilon \geq 0\). In fact (due to parabolic regularity) the energy \(E \in H^0(0, \tau)\) and

\[
\frac{dE}{dt}(t) = -\int _\Omega \alpha |\Delta \theta(t, x)|^2 dV(x) - \int _\partial \Omega \gamma(x) |\partial_j p(t, x)|^2 dS(x).
\]
where
\[ p_{0, \text{const}} = \left( \int_{\Omega} \gamma(x) dS(x) \right)^{-1} \left( \int_{\Omega} c^{-2}(x) p_0(x) dV(x) + \int_{\Omega} \gamma(x)p_0(x) dS(x) \right). \]
\[ \theta_{0, \text{const}} = \frac{1}{|\Omega|} \int_{\Omega} \left( \theta_0(x) - \epsilon p_0(x) \right) dV(x) + \epsilon q_{0, \text{const}}. \]

Therefore, the evolution of the triplet \((p_0, p_1, \theta_0) \in H^1(\Omega) \times H^0(\Omega) \times H^1(\Omega)\) can be decomposed into the evolution of the initial condition \((p_0 - p_{0, \text{const}}, p_1, \theta_0 - \theta_{0, \text{const}})\) in \(\mathcal{H}\) plus a time-independent solution given by \((p_{0, \text{const}}, \theta_{0, \text{const}})\) for \(t \geq 0\). Notice that the energy of this particular solution is zero because it is constant both in space and time. This leads to the following result using lemma 3.1 and the decomposition described above.

**Theorem 3.2 (Forward well-posedness).** Given \((p_0, p_1, \theta_0) \in H^1(\Omega) \times H^0(\Omega) \times H^1(\Omega)\), the unique weak solution of (1)–(4) satisfies \(p \in C([0, \tau]; H^{1-k}(\Omega))\) for \(k = 0, 1\) and \(\theta \in C([0, \tau]; H^1(\Omega))\). Moreover, \(E(t) \leq E(0)\) for all \(t \geq 0\) and all \(\epsilon \geq 0\).

Now we proceed to re-state and prove theorem 2.3.

**Theorem.** (Main result) Suppose that assumptions 2.1 and 2.2 hold. There exists \(\epsilon_0 > 0\) so that if \(0 \leq \epsilon < \epsilon_0\), the map \(p_0 \mapsto \mathcal{M}p_0\) is injective and satisfies a stability estimate of the form
\[ \|p_0\|_{H^1(\Omega)} \leq C\|\mathcal{M}p_0\|_{H^1(0, \tau) \times \Gamma}, \]
for all \(p_0 \in H^1(\Omega)\), and some positive constant \(C = C(\epsilon_0)\) independent of \(\epsilon\).

**Proof of theorem 2.3.** Consider the problem (1)–(4) for \(p_0 \in H^1(\Omega)\) and \(p_1 = 0\) and \(\theta_0 = \epsilon p_0\) (assumption 2.1). From theorem 3.2 we have that the solution to this problem satisfies \(p \in C([0, \tau]; H^{1-k}(\Omega))\) for \(k = 0, 1\) and \(\theta \in C([0, \tau]; H^1(\Omega))\). Now notice that \(\theta\) has initial condition in \(H^1(\Omega)\), homogeneous Neumann boundary condition, and a forcing term \(\epsilon \partial_t p \in C([0, \tau]; H^1(\Omega) \subset H^1(0, \tau; H^1(\Omega))\). Hence, from regularity theory for the parabolic equation (see [70, theorem 4.3, section 4, chapter 4, vol II] or [71, section 7.1.3]), we obtain that \(\theta \in H^2((0, \tau); H^1(\Omega)) \cap H^2((0, \tau); H^1(\Omega))\) and an estimate of the form
\[ \|\Delta \theta\|^2_{H^1(0, \tau) \times \Omega} \leq C(\|\partial_t p\|^2_{H^1(0, \tau) \times \Omega} + \|\nabla p_0\|^2_{H^1(\Omega)}), \]
for some constant \(C > 0\) independent of \(\epsilon \geq 0\). From the energy estimate in theorem 3.2, we also have that
\[ \|\partial_t p\|^2_{H^1(0, \tau) \times \Omega} \leq C(1 + \epsilon^2)\|\nabla p_0\|^2_{H^1(\Omega)}. \]
Now, under the geometric condition in assumption 2.2, the acoustic problem for \(p\) (governed by the wave equation (1) with source term \(\epsilon c^2 \Delta \theta\)) enjoys the following observability property (see details in [56, chapter 7], [53, chapter 6], [76, lemma 3.3] and [54, 55])
\[ \|p_0\|^2_{H^1(\Omega)} \leq C(\epsilon^2 \|\Delta \theta\|^2_{H^1(0, \tau) \times \Omega} + \|p\|^2_{H^1(0, \tau) \times \Gamma}), \]
where \(C > 0\) is also independent of \(\epsilon \geq 0\). Combining the above three inequalities, we obtain that
\[ \|p_0\|^2_{H^1(\Omega)} \leq C(\|p\|^2_{H^1(0, \tau) \times \Gamma} + \epsilon^4(2 + \epsilon^2)\|\nabla p_0\|^2_{H^1(\Omega)}). \]
Therefore, we select \( \epsilon_0 \) so that \( C_{\epsilon_0}^4 (2 + \epsilon_0^2) < 1 \). Then for any \( 0 \leq \epsilon < \epsilon_0 \), the second term on the right-hand side of the above inequality can be absorbed into the left-hand side to obtain the desired estimate. This concludes the proof. \( \square \)

4. Reconstruction algorithm

In this section we explicitly recover the initial acoustic profile \( p_0 \) in terms of the boundary measurements \( \mathcal{M}p_0 = p|_{(0,\tau) \times \Gamma} \). This is accomplished by using theorem 2.3 obtained in the previous section which leads to the invertibility of the normal operator \( (\mathcal{M}^\alpha \mathcal{M}) \).

In order to obtain an applicable expression for the operator \( \mathcal{M}^\alpha \), in this section we state the dual or adjoint problem associated with (1)–(4). This is equivalent to constructing the well-known Hilbert uniqueness method for control of partial differential equations. See [53, 77] for an overview of these ideas and their historical origin. Throughout, we assume that \( \epsilon > 0 \) is sufficiently small for theorem 2.3 to apply. This adjoint problem is to find a solution \((\psi, \xi)\) (defined by transposition as in [70, chapter 3, section 9] or [54, section 4]) for the following IBVP

\[
\begin{align*}
\partial_t^2 \psi - c^2 \Delta \psi - \epsilon c^2 \alpha^{-1} \partial_t \xi &= 0 \quad \text{in} \quad (0, \tau) \times \Omega, \\
\partial_t \xi + \alpha \Delta \xi - \epsilon \alpha \partial \psi &= 0 \quad \text{in} \quad (0, \tau) \times \Omega, \\
\psi &= 0, \quad \partial_\nu \psi = 0 \quad \text{and} \quad \xi = 0 \quad \text{on} \quad \{t = \tau\} \times \Omega, \\
\partial_\nu \psi - \gamma \partial_t \psi &= \eta \quad \text{and} \quad \partial_\nu \xi - \epsilon \partial_\nu \psi = 0 \quad \text{on} \quad (0, \tau) \times \partial \Omega.
\end{align*}
\]

for a given \( \eta \in H^{-1}((0, \tau) \times \Gamma) \) (extended as zero on \((0, \tau) \times \partial \Omega \setminus \Gamma\)). Notice that this problem is solved backwards in time with vanishing Cauchy data at time \( t = \tau \) and that the signs of the terms \( \partial_t \xi \) and \( \alpha \partial \psi \) are consistent with solving the heat equation backwards in time in a stable manner. In fact, the well-posedness of the dual system (11)–(14) is equivalent to the well-posedness of the primal system (1)–(4). See [70, chapter 3, section 9], [56, section 2.8] and [54, 77] for details. We obtain the following definition.

**Definition 4.1.** Let \( S \) be the mapping \( \eta \mapsto -\partial_t \psi|_{\nu=0} \), where \( \psi \) is the solution of (11)–(14) for the provided \( \eta \).

Integrating by parts the terms of equations (1) and (2) against \((\psi, \xi)\), where \((\psi, \xi)\) is the solution of (11)–(14), we easily obtain that

\[
\langle p_0, S\eta \rangle_{\Omega} = \langle \mathcal{M}p_0, \eta \rangle_{(0,\tau) \times \Gamma}, \quad \text{for all} \ \eta \in H^{-1}((0, \tau) \times \Gamma) \text{ and } p_0 \in H^1(\Omega).
\]

Hence, by definition we have that \( \mathcal{M}^\alpha = \mathcal{R} \mathcal{S} \mathcal{Q}^{-1} : H^1((0, \tau) \times \Gamma) \rightarrow H^1(\Omega) \) where \( \mathcal{R} : H^{-1}(\Omega) \rightarrow H^1(\Omega) \) and \( \mathcal{Q} : H^{-1}((0, \tau) \times \Gamma) \rightarrow H^1((0, \tau) \times \Gamma) \) are the Riesz representation unitary operators. Now, if we choose \( \eta = Q^{-1} \mathcal{M}p_0 \) and use the estimate from theorem 2.3, we obtain that

\[
\langle p_0, (\mathcal{M}^\alpha \mathcal{M})p_0 \rangle_{H^1(\Omega)} = \| \mathcal{M}p_0 \|_{H^1((0,\tau) \times \Gamma)}^2 \geq C \| p_0 \|_{H^1(\Omega)}^2,
\]

for all \( p_0 \in H^1(\Omega) \) and some constant \( C > 0 \). Therefore, the operator \( \mathcal{M}^\alpha : H^1((0, \tau) \times \Gamma) \rightarrow H^1(\Omega) \) is surjective and \( (\mathcal{M}^\alpha \mathcal{M}) : H^1(\Omega) \rightarrow H^1(\Omega) \) is coercive. With these results, we can establish the following controllability theorem.

**Theorem 4.2 (Acoustic control).** Let the geometric condition 2.2 hold. For sufficiently small \( \epsilon > 0 \), the operator \( S : H^{-1}((0, \tau) \times \Gamma) \rightarrow H^{-1}(\Omega) \) given in definition 4.1 is...
surjective. Therefore, for any $\phi \in H^{-1}(\Omega)$, there exists a boundary control $\eta \in H^{-1}((0, \tau) \times \Gamma)$ such that the solution $(\psi, \xi)$ of (11)–(14) satisfies

$$\partial_t \psi = -\phi,$$

at time $t = 0$.

Among all such boundary controls, there exists $\eta_{\min}$ which is uniquely determined by $\phi$ as the minimum norm control and satisfies the following stability condition

$$\|\eta_{\min}\|_{H^{-1}((0, \tau) \times \Gamma)} \leq C\|\phi\|_{H^{-1}(\Omega)}$$

for some constant $C > 0$. As a consequence, the mapping $\phi \mapsto \eta_{\min}$ defines a bounded control operator $C : H^{-1}(\Omega) \to H^{-1}((0, \tau) \times \Gamma)$, that satisfies $C = S^*(SS^*)^{-1}$. It also follows that $QCR^{-1} = M(M^*M)^{-1}$.

Let $(\psi, \xi)$ be the solution of (11)–(14) with $\eta = C\phi$ and $\phi \in H^{-1}(\Omega)$ arbitrary. Then by construction,

$$\langle p_0, \phi \rangle_{H^1(\Omega) \times H^{-1}(\Omega)} = \langle M p_0, C\phi \rangle_{H^1(0, \tau) \times \Gamma} \times H^{-1}((0, \tau) \times \Gamma),$$

for all $\phi \in H^{-1}(\Omega)$, which implies that the unknown initial condition $p_0$ is explicitly recovered as follows

$$p_0 = C^*M p_0,$$

(15)

where $C^* : H^1((0, \tau) \times \Gamma) \to H^1(\Omega)$ is the adjoint of the control operator $C : H^{-1}(\Omega) \to H^{-1}((0, \tau) \times \Gamma)$ defined in theorem 4.2. The reconstruction algorithm is based on the identity (15) and an iterative algorithm to approximate the action of $C^*$. This algorithm is based on the following points provided by theorem 4.2 (see [53, 77]):

1. The observability operator $C^* = (M^*M)^{-1}M^*$, where $(M^*M) : H^1(\Omega) \to H^1(\Omega)$ is coercive.
2. For $\zeta \in H^1(\Omega)$, the solution to $(M^*M)\phi = \zeta$ can be approximated using the conjugate gradient method.

Now we proceed to describe how the action of $C^*$ can be approximated using the conjugate gradient method. See [78, section 4.6] for a standard description of the conjugate gradient method in a Hilbert space setting. For sake of completeness, we describe the inversion of a generic equation $(M^*M)\phi = \zeta$. Let $\phi_0$ be an initial guess for the true solution $\phi_0$. Define $r_0 = \zeta - (M^*M)\phi_0$ as the initial residue and $s_0 = r_0$. For $k \geq 0$, define

$$\phi_{k+1} = \phi_k + \alpha_k s_k,$$

$$r_{k+1} = \zeta - (M^*M)\phi_{k+1}$$

$$s_{k+1} = r_{k+1} + \beta_k s_k,$$

where

$$\alpha_k = \frac{\|r_k\|^2_{H^1(\Omega)}}{\langle s_k, (M^*M)s_k \rangle_{H^1(\Omega)}},$$

$$\beta_k = \frac{\|r_{k+1}\|^2_{H^1(\Omega)}}{\|r_k\|^2_{H^1(\Omega)}}.$$

Since the operator $(M^*M) : H^1(\Omega) \to H^1(\Omega)$ is bounded and coercive, then there are positive constants $m$ and $M$ such that

$$m\|\phi\|^2_{H^1(\Omega)} \leq \langle \phi, (M^*M)\phi \rangle_{H^1(\Omega)} \leq M\|\phi\|^2_{H^1(\Omega)}.$$

The conjugate gradient iterates can be shown to converge as follows (see [78, section 4.6] and references therein)
\[ \| \phi_k - \phi_k \|_{H^1(\Omega)} \leq e^{-\sigma k} \| \phi_0 - \phi_0 \|_{H^1(\Omega)}, \quad \text{for } k \geq 0, \text{ where } \sigma = \ln \left( \frac{M + m}{M - m} \right) \]

Notice that at each iteration, one must apply the operator \((M^*M)\) which amounts to solve the problem (1)–(4) under assumption 2.1 followed by solving the adjoint problem (11)–(14). In practice, this can be approximated using numerical methods for PDEs. However, depending on the method of choice, there are intrinsic complications that may prevent a convergence estimate such as (16) from being satisfied in the limit as the discretization is refined. We shall not elaborate any further on these complications as they lie outside of the scope of this paper. For details on these numerical issues we refer to [53, 79–82] and references therein. In this paper, we adopted the two-grid approach described in [53, 81] using second order finite difference methods. For the two-grid approach, recall that the computation of residual \(r_{k+1} = \zeta - (M^*M)\phi_{k+1}\) is understood in the \(H^1(\Omega)\)-sense which means that \(r_{k+1}\) solves the equation \(\langle \nabla (\zeta - r_{k+1}), \nabla \psi \rangle = F(\psi)\) for all \(\psi \in H^1(\Omega)\) where \(F = (M^*M)\phi_{k+1}\) acts as a functional. This elliptic equation is solved using a grid that is coarser than the grid employed to propagate the wave fields. This computation on a coarser grid has a filtering effect which removes high-frequency oscillations from the residual. In turn, this procedure regulates the convergence of the algorithm as the grids are refined [53, 81].

5. Numerical results

Now we present some numerical results to illustrate the performance of the reconstruction algorithm described in section 4. We implemented a numerical solver for the governing system (1)–(4) and its adjoint (11)–(14) based on second order finite differences. To avoid spurious numerical instabilities, we adopted the two-grid approach described in [53, 81]. We worked in \(\mathbb{R}^2\) where the domain \(\Omega\) was taken as the unit-square. The initial profile \(p_0\) corresponds the Shepp–Logan phantom.

We present two examples. One with constant wave speed \(c(x) \equiv 1\), and the other with variable wave speed \(c = c(x)\) defined below. In both cases, we used the following parameters: impedance \(\gamma(x) = c^{-1}(x)\) over the boundary of \(\Omega\), thermal diffusivity \(\alpha = 0.01\), and coupling parameter \(\epsilon = 0.1\). The observability time was chosen to be \(\tau = 2\) which is enough for more than 99% of the energy contained in the initial profile to dissipate or leave the domain through the boundary when the wave speed is constant.

We shall compare the results from the proposed algorithm against the results from purely acoustic time-reversal. The latter is accomplished by producing measurements using the thermoacoustic forward solver \(M\), and then back-propagating the boundary measurements in a purely acoustic medium (\(\epsilon = 0\)), that is, by ignoring the thermodynamic attenuation. See details in [23, 30] for the purely acoustic time-reversal approach. The acoustic time-reversal is approximated using the same finite difference method. The initial guess for the conjugate gradient algorithm is the approximate solution obtained from the purely acoustic time-reversal algorithm. Although the proposed reconstruction algorithm has been described in the \(H^1(\Omega)\) setting, a similar study could be performed in the \(H^0(\Omega)\) setting where the inner-products in the conjugate gradient algorithm would need to be understood appropriately. In this section, we present results from the implementation both in the \(H^1(\Omega)\) and \(H^0(\Omega)\) formulations.
5.1. Constant wave speed

For the first example where \( c \equiv 1 \), figure 1 displays the exact initial profile and the reconstructions. The relative errors in the \( H^1(\Omega) \) and \( H^0(\Omega) \) formulations are reported in table 2 for

**Table 2.** Constant wave speed example.

| Iter | \( H^1(\Omega) \)-norm | \( H^0(\Omega) \)-norm |
|------|--------------------------|--------------------------|
| 0    | 52.6%                    | 31.1%                    |
| 1    | 19.8%                    | 12.8%                    |
| 2    | 10.6%                    | 5.7%                     |
| 3    | 6.3%                     | 4.4%                     |
| 4    | 4.5%                     | 3.8%                     |
| 5    | 3.8%                     | 3.1%                     |

Note. Relative error at each iteration of the conjugate gradient method described in section 4. Iter = 0 corresponds to the initial guess given by a purely acoustic time-reversal algorithm.

**Figure 1.** Exact initial acoustic profile (top-left), the reconstruction from purely acoustic time-reversal (top-right), and the reconstruction from the proposed algorithm described in section 4 using 1 iteration (bottom-left) and 5 iterations (bottom-right).
the first few iterations of the conjugate gradient algorithm. We notice that by ignoring the thermodynamic attenuation in the purely acoustic time-reversal reconstruction (Iter = 0), the edges in the Shepp–Logan phantom are blurred considerably. Some of the sharpness is recovered by accounting for the attenuation in the proposed algorithm even after a single iteration.

5.2. Variable wave speed

For the second example we have selected a variable wave speed defined as a layer of higher speed surrounding the smaller ellipses in the Shepp–Logan phantom. The actual profile is illustrated in the top-right panel of figure 2. The relative errors in the $H^1(\Omega)$ and $H^0(\Omega)$ formulations are reported in table 3 for the first few iterations of the conjugate gradient algorithm. Again, as shown in the lower panels of figure 2, we see great improvements over the purely acoustic time-reversal reconstruction. We highlight the ability in capturing the jump discontinuities and the reduction of the artifacts introduced by ignoring the attenuation.

![Figure 2. Exact initial acoustic profile (top-left), wave speed profile (top-right), the reconstruction from purely acoustic time-reversal (bottom-left) and the reconstruction from the proposed algorithm described in section 4 using 5 iterations (bottom-right).](image-url)
6. Conclusion

We have presented a PAT/TAT model based on thermoelasticity. The thermoelastic coupling accounts for how pressure changes can induce temperature changes in a body and vice versa. This coupling between temperature and deformation is a fundamental feature of PAT/TAT. The current literature dealing with PAT/TAT only considers one side of the thermoelastic interaction (the photoacoustic effect). By considering both effects simultaneously we account for a natural attenuation phenomenon.

We related the thermoelastic model of PAT/TAT with boundary observability for the thermacoustic system. We showed uniqueness and stability of recovering the initial pressure profile from boundary data provided that the thermoelastic coupling is weak. The recovery analysis of the initial wave profile is valid under a geometric assumption on the wave speed (see assumption 2.2). We also proposed a reconstruction algorithm based on the conjugate gradient method. We carried out proof-of-concept numerical simulations to illustrate the implementation of the reconstruction algorithm for synthetic data. The authors are in the process of applying the proposed algorithm to actual experimental data. As soon as meaningful results are obtained from these efforts, they will be reported in a forthcoming publication.

For soft biological tissues, the unitless coupling parameter $\epsilon$ of the thermoelastic model is approximately between 0.05 and 0.1 (as obtained from table 1). Theorem 2.3 requires $\epsilon$ to be sufficiently small. Given that in PAT/TAT the thermodynamic interaction is small, such a condition on $\epsilon$ is reasonable. Nonetheless, it might be possible to remove this condition by using Carleman estimates for the coupled thermoelastic system (e.g., [55, 65, 66]). The attenuation experienced by the shear waves has not been included in the present work. This may become relevant when the pressure waves interact with solid layers such as the skull [83–85]. In that case, it may be appropriate to incorporate the thermodynamic attenuation into the full elastic model of PAT/TAT [26].

Acknowledgments

The authors would like to thank Plamen Stefanov for recommendations on the first draft of the paper. We also want to thank Benjamín Palacios for fruitful discussions and for exploring the possibility of a Neumann series approach for this problem.

| Iter | $H^1(\Omega)$-norm | $H^0(\Omega)$-norm |
|------|---------------------|---------------------|
| 0    | 55.4%               | 34.3%               |
| 1    | 24.2%               | 16.6%               |
| 2    | 15.5%               | 8.0%                |
| 3    | 11.8%               | 4.7%                |
| 4    | 10.2%               | 4.0%                |
| 5    | 9.7%                | 3.7%                |

Note. Relative error at each iteration of the conjugate gradient method described in section 4. Iter = 0 corresponds to the initial guess given by a purely acoustic time-reversal algorithm.
References

[1] Wang L V and Wu H 2007 Biomedical Optics: Principles and Imaging (Hoboken, NJ: Wiley)
[2] Wang L V (ed) 2009 Photoacoustic Imaging and Spectroscopy (Optical Science and Engineering vol 144) (Boca Raton, FL: CRC Press)
[3] Cox B T, Lauffer J G and Beard P C 2009 The challenges for quantitative photoacoustic imaging Proc. SPIE 7177 717713
[4] Wang K and Anastasio M 2011 Photoacoustic and thermoacoustic tomography: image formation principles Handbook of Mathematical Methods in Imaging ed O Scherzer (New York: Springer) pp 781–815
[5] Wang L V and Hu S 2012 Photoacoustic tomography: in vivo imaging from organelles to organs Science 335 1458–62
[6] Wang X, Pang Y, Ku G, Xie X, Stoica G and Wang L V 2003 Noninvasive laser-induced photoacoustic tomography for structural and functional in vivo imaging of the brain Nat. Biotechnol. 21 803–6
[7] Zhang H F, Maslov K, Sivaramakrishnan M, Stoica G and Wang L V 2007 Imaging of hemoglobin oxygen saturation variations in single vessels in vivo using photoacoustic microscopy Appl. Phys. Lett. 90 053901
[8] Shah J, Park S, Aglyamov S, Larson T, Ma L, Sokolov K, Johnston K, Milner T and Emelianov S Y 2008 Photoacoustic imaging and temperature measurement for photothermal cancer therapy J. Biomed. Opt. 13 034024
[9] Pramanik M and Wang L V 2009 Thermoacoustic and photoacoustic sensing of temperature J. Biomed. Opt. 14 054024
[10] Cox B, Lauffer J G, Arridge S R and Beard P C 2012 Quantitative spectroscopic photoacoustic imaging: a review J. Biomed. Opt. 17 061202
[11] Finch D, Patch S and Rakesh 2004 Determining a function from its mean values over a family of spheres SIAM J. Math. Anal. 35 1213–40
[12] Kunyansky L 2007 Explicit inversion formulas for the spherical mean Radon transform Inverse Problems 23 373–83
[13] Finch D, Haltmeier M and Rakesh 2007 Inversion of spherical means and the wave equation in even dimensions SIAM J. Appl. Math. 68 392–412
[14] Kunyansky L 2011 Reconstruction of a function from its spherical (circular) means with the centers lying on the surface of certain polygons and polyhedra Inverse Problems 27 025012
[15] Wang K and Anastasio M 2012 A simple Fourier transform-based reconstruction formula for photoacoustic computed tomography with a circular or spherical measurement geometry Phys. Med. Biol. 57 N493
[16] Natterer F 2012 Photo-acoustic inversion in convex domains Inverse Problems Imaging 6 315–20
[17] Palamodov V 2012 A uniform reconstruction formula in integral geometry Inverse Problems 28 065014
[18] Haltmeier M 2014 Universal inversion formulas for recovering a function from spherical means SIAM J. Math. Anal. 46 214–32
[19] Anastasio M A, Zhang J, Modgill D and La Rivière P J 2007 Application of inverse source concepts to photoacoustic tomography Inverse Problems 23 S21
[20] Hristova Y, Kuchment P and Nguyen L 2008 Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media Inverse Problems 24 055006
[21] Hristova Y 2009 Time reversal in thermoacoustic tomography—an error estimate Inverse Problems 25 055008
[22] Stefanov P and Uhlmann G 2009 Thermoacoustic tomography with variable sound speed Inverse Problems 25 075011
[23] Stefanov P and Uhlmann G 2011 Thermoacoustic tomography arising in brain imaging Inverse Problems 27 045004
[24] Kunyansky L, Holman B and Cox B T 2013 Photoacoustic tomography in a rectangular reflecting cavity Inverse Problems 29 125010
[25] Titelfitz J 2012 Thermoacoustic tomography in elastic media Inverse Problems 28 055004
[27] Huang C, Wang K, Nie L, Wang L V and Anastasio M 2013 Full-wave iterative image reconstruction in photoacoustic tomography with acoustically inhomogeneous media IEEE Trans. Med. Imaging 32 1097–110
[28] Wang K, Xia J, Li C, Wang L V and Anastasio M 2014 Fast spatiotemporal image reconstruction based on low-rank matrix estimation for dynamic photoacoustic computed tomography J. Biomed. Opt. 19 056007
[29] Holman B and Kunyansky L 2015 Gradual time reversal in thermo- and photo-acoustic tomography within a resonant cavity Inverse Problems 31 035008
[30] Acosta S and Montalto C 2015 Multiwave imaging in an enclosure with variable wave speed Inverse Problems 31 065009
[31] Stefanov P and Yang Y 2015 Multiwave tomography in a closed domain: averaged sharp time reversal Inverse Problems 31 065007
[32] Nguyen L and Kunyansky L 2016 A dissipative time reversal technique for photoacoustic tomography in a cavity. SIAM J. Imaging Sci. 9 748–69
[33] Chervova O and Oksanen L 2016 Time reversal method with stabilizing boundary conditions for photoacoustic tomography (arXiv:1605.07817v1)
[34] Stefanov P and Yang Y 2016 Multiwave tomography with reflectors: Landweber’s iteration arXiv:1603.07045v3
[35] Agranovsky M, Kuchment P and Kunyansky L 2009 On reconstruction formulas and algorithms for the thermoacoustic and photoacoustic tomography Photoacoustic Imaging and Spectroscopy (Boca Raton, FL: CRC Press)
[36] Kuchment P and Kunyansky L 2008 Mathematics of thermoacoustic tomography Eur. J. Appl. Math. 19 191–224
[37] Kuchment P and Kunyansky L 2011 Mathematics of photoacoustic and thermoacoustic tomography Handbook of Mathematical Methods in Imaging ed O Scherzer (New York: Springer) pp 817–65
[38] Bao G 2011 Hybrid inverse problems and internal functionals Inverse Problems Appl.: Inside Out II 60 325–68
[39] La Riviere P, Zhang J and Anastasio M 2009 Image reconstruction in optoacoustic tomography accounting for frequency-dependent attenuation Photoacoustic Imaging and Spectroscopy ed L V Wang (Boca Raton, FL: CRC Press) pp 145–54
[40] Kowar R 2010 Integral equation models for thermoacoustic imaging of acoustic dissipative tissue Inverse Problems 26 095005
[41] Treeby B E and Cox B T 2010 Modeling power law absorption and dispersion for acoustic propagation using the fractional laplacian J. Acoust. Soc. Am. 127 2741–8
[42] Kowar R and Scherzer O 2012 Attenuation models in photoacoustics Mathematical Modeling in Biomedical Imaging II: Optical, Ultrasound, and Opto-Acoustic Tomographies (Lectures Notes in Mathematics vol 2035) ed H Ammari (Berlin: Springer) pp 85–130
[43] Ammari H, Breitin E, Jugnon V and Wahab A 2012 Photoacoustic imaging for attenuating acoustic media Mathematical Modeling in Biomedical Imaging II: Optical, Ultrasound, and Opto-acoustic Tomographies (Lectures Notes in Mathematics vol 2035) ed H Ammari (Berlin: Springer) pp 57–84
[44] Treeby B, Zhang E and Cox B 2010 Photoacoustic tomography in absorbing acoustic media using time reversal Inverse Problems 26 115003
[45] Röttner H and Burgholzer P 2011 Efficient modeling and compensation of ultrasound attenuation losses in photoacoustic imaging Inverse Problems 27 015003
[46] Cook J, Bouchard R and Emelianov S 2011 Tissue-mimicking phantoms for photoacoustic and ultrasonic imaging Biomed. Opt. Express 2 3193–206
[47] Huang C, Nie L, Schoonover R W, Wang L V and Anastasio M A 2012 Photoacoustic computed tomography correcting for heterogeneity and attenuation J. Biomed. Opt. 17 061211
[48] Kalimeris K and Scherzer O 2013 Photoacoustic imaging in attenuating acoustic media based on strongly causal models Math. Methods Appl. Sci. 36 2254–64
[49] Homan A 2013 Multi-wave imaging in attenuating media Inverse Problems Imaging 7 1235–50
[50] Kowar R 2014 Time reversal for photoacoustic tomography based on the wave equation of Nachman, Smith, and Waag Phys. Rev. E 89 023203
[51] Palacios B 2016 Reconstruction for multiwave imaging in attenuating media with arbitrary large damping coefficient (arXiv:1604.06068)
[52] Athanasiou K A and Natoli R M 2000 Introduction to Continuum Biomechanics (Synthesis Lectures on Biomedical Engineering vol 19) (San Rafael, California: Morgan & Claypool Publishers)

[53] Glowinski R, Lions J-L and He J 2008 Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach (Encyclopedia of Mathematics and its Applications vol 117) (Cambridge: Cambridge University Press)

[54] Bardos C, Lebeau G and Rauch J 1992 Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary SIAM J. Control Optim. 30 1024–65

[55] Croke C, Lasiecka I, Uhlmann G and Vogelius M (ed) 2004 Geometric Methods in Inverse Problems and PDE Control (IMA Volumes in Mathematics and its Applications vol 137) (New York: Springer)

[56] Tucsnak M and Weiss G 2009 Observation and Control for Operator Semigroups (Birkhauser Advanced Texts Basler Lehrbcher) (Basel: Birkhauser)

[57] Acosta S 2013 Time reversal for radiative transport with applications to inverse and control problems Inverse Problems 29 085014

[58] Acosta S 2015 A control approach to recover the wave speed (conformal factor) from one measurement Inverse Problems Imaging 9 301–15

[59] Zuazua E 1994 Controllability of the linear system of thermoelasticity: Dirichlet–Neumann boundary conditions Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena (INSM International Series of Numerical Mathematics vol 118) ed W Desch et al (Basel: Birkhäuser) pp 391–402

[60] de Teresa L and Zuazua E 1996 Controllability of the linear system of thermoelastic plates Adv. Differ. Equ. 1 369–402

[61] Lebeau G and Zuazua E 1998 Nullcontrollability of a system of linear thermoelasticity Arch. Ration. Mech. Anal. 141 297–329

[62] Liu W 1998 Partial exact controllability and exponential stability in higher-dimensional linear thermoelasticity ESAIM: Control Optimisation Calculus Variations 3 23–48

[63] Liu W 1998 Correction to partial exact controllability and exponential stability in higher-dimensional linear thermoelasticity ESAIM: Control Optimisation Calculus Variations 3 323–7

[64] Liu W-J and Williams G H 1998 Partial exact controllability for the linear thermo-viscoelastic model Electron. J. Differ. Equ. 1998 1–11 (electronic only)

[65] Eller M, Lasiecka I and Triggiani R 2000 Simultaneous exact/approximate boundary controllability of thermo-elastic plates with variable thermal coefficient and moment control J. Math. Anal. Appl. 251 452–78

[66] Albano P and Tataru D 2000 Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system Electron. J. Differ. Equ. 2000 1–15 (electronic only)

[67] Xia J, Yao J and Wang L 2014 Photoacoustic tomography: principles and advances Electromagn. Waves 147 1–22

[68] Kruger R, Reinecke D and Kruger G 1999 Thermooaoustic computed tomography—technical considerations Med. Phys. 26 1832–7

[69] Larina I, Larin K and Esenaliev R 2005 Real-time optoacoustic monitoring of temperature in tissues J. Phys. D: Appl. Phys. 38 2633

[70] Lions J-L and Magenes E 1972 Non-Homogeneous Boundary Value Problems and Applications (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen vol I–III) (Berlin: Springer)

[71] Evans L C 1998 Partial Differential Equations (Graduate Studies in Mathematics vol 19) (Providence, RI: American Mathematical Society)

[72] Renardy M and Rogers R 2004 An Introduction to Partial Differential Equations (Texts in Applied Mathematics vol 13) (New York: Springer)

[73] Engel K-J and Nagel R 2000 One-parameter Semigroups for Linear Evolution Equations (Graduate Texts in Mathematics vol 194) (New York: Springer)

[74] Dafermos C 1968 On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity Arch. Ration. Mech. Anal. 29 241–71

[75] Lebeau G and Zuazua E 1997 Sur la decroissance non uniforme de l’energie dans le systeme de la thermoelasticite lineaire C. R. Acad. Sci., Paris I 324 409–15

[76] Alabau-Boussouira F and Leuxaut M 2013 Indirect controllability of locally coupled wave-type systems and applications J. Math. Pures Appl. 99 544–76
[77] Lions J-L 1988 Exact controllability, stabilization and perturbations for distributed systems *SIAM Rev.* 30 1–68
[78] Atkinson K and Han W 2001 *Theoretical Numerical Analysis: A Functional Analysis Framework (Texts in Applied Mathematics vol 39)* (New York: Springer)
[79] Ervedoza S and Zuazua E 2012 The wave equation: control and numerics *Control of Partial Differential Equations (Lecture Notes in Mathematics vol 2048)* (Berlin: Springer) pp 245–339
[80] Ervedoza S 2009 Spectral conditions for admissibility and observability of wave systems: applications to finite element schemes *Numer. Math.* 113 377–415
[81] Asch M and Lebeau G 1998 Geometrical aspects of exact boundary controllability for the wave equation—a numerical study *ESAIM: Control Optimisation Calculus Variations* 3 163–212
[82] Ervedoza S and Zuazua E 2013 *Numerical Approximation of Exact Controls for Waves (Springer Briefs in Mathematics)* (New York: Springer)
[83] Schoonover R W and Anastasio M A 2011 Compensation of shear waves in photoacoustic tomography with layered acoustic media *J. Opt. Soc. Am. A* 28 2091–9
[84] Huang C, Nie L, Schoonover R W, Guo Z, Schirra C O, Anastasio M A and Wang L V 2012 Aberration correction for transcranial photoacoustic tomography of primates employing adjunct image data *J. Biomed. Opt.* 17 066016
[85] Schoonover R W, Wang L V and Anastasio M A 2012 Numerical investigation of the effects of shear waves in transcranial photoacoustic tomography with a planar geometry *J. Biomed. Opt.* 17 061215