On convergence to the global optima

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Abstract

We show that there is no general algorithm which computes a sequence of points converging to the global optima of any continuous function.

1 Introduction and Preliminaries

We consider the problem of finding the global minima of a non-convex continuous function $f : C \to \mathbb{R}$, where $C \subset \mathbb{R}^d$ is a closed, compact subset. Global minima is the point $x^* \in C$ which satisfies the following property: $f(x^*) \leq f(x)$ for all $x \in C$. The function $f$ attains this minimum at least once by extreme value theorem. Our goal is to find one such point. This problem is well-studied with many books written on the subject, see for example [1].

We note that our setting is different from the computability of similar problems studied for example in [2]. In these works they study computable real numbers and computable real functions. But we consider finite-precision numbers and the general case of any continuous function.

Since computers have limited memory, oracles with real-numbers are not a very useful model. We consider only finite-precision reals. Since they are bounded, by multiplying an appropriate constant we can consider them as natural numbers. Our main result is that there is no algorithm that computes a sequence of points (finite-precision) converging to the global minima of any continuous function.

1.1 Finite Precision Reals

We now briefly explain what we mean by this. Consider any real number $x \in \mathbb{R}$. Let $n_0$ be the largest integer such that $n_0 \leq x$. Having chosen $n_0, n_1, \ldots, n_{k-1}$ choose largest positive integer $n_k$ such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} \leq x.$$

This is the decimal expansion of the number. Now by finite precision we specify a number $k$ (the precision). And any real $x \in \mathbb{R}$, the numbers $n_0, n_1, \ldots, n_k$ is its finite precision representation. Note that $n_i, 0 \leq i \leq k$ can be zero. We can consider this finite-precision real number as a natural number by representing it as $n_0 \cdot 10^k + n_1 \cdot 10^{k-1} + \ldots + n_k$. For $x \in \mathbb{R}^d$, we define precision length to be the sum of all finite-precision lengths of its coordinates. Note that though we give binary representations to the Turing machine, for simplicity we assume precisions denote the decimal precisions.

1.2 The Problem

We assume there is an oracle for our continuous function $f$. This oracle gives the value $f(x)$ upto any finite-precision for an given value $x$. The Turing-machine has access to this function.
oracle. We give also give a value \( \delta > 0 \) as input to the Turing machine. It needs to write the any point \( x_\alpha \) or finite precision length such that \( |f(x_\alpha) - f(x^*)| < \delta \) i.e., it should find \( \delta \)-approximation of the global optima. We show that this problem is not computable.

Let us assume we have a three-tape Turing machine, one is used for calculation, one is for checking whether \( h \mid \delta \) is continuous. Let \( n \) be such that \( |x - y| < \epsilon \) implies \( |f(x) - f(y)| < \delta \). Such an \( \epsilon > 0 \) exists for all \( \delta > 0 \) because the function \( f \) is continuous. Let \( n \) be the precision length required to represent numbers with gap \( \epsilon/10 \) between consecutive numbers. For the global minima \( x^* \) then there exists a point \( x_n^* \) with precision length \( n \) such that \( |x_n^* - x^*| < \epsilon \).

**Definition 1.** Turing machine has a three infinite tapes divided into cells, a reading head which scans one cell of the tape at a time, and a finite set of internal states \( Q = \{q_0, q_1, \ldots, q_n\}, n \geq 1 \). Each cell is either blank or has symbol 1 written on it. In a single step the machine may simultaneously (1) change the from one state to another; (2) change the scanned symbol \( s \) to another symbol \( s' \in S = \{1, B\}; (3) move the reading head one cell to the right \((R)\) or left \((L)\).

This operation of machine is controlled by a partial map \( \Gamma : Q \times S^3 \rightarrow Q \times (S \times \{R, L\})^3 \). The map \( \Gamma \) viewed as a finite set of quintuples is called a Turing program. The interpretation is that if \((q, s_1, s_2, s_3, q', s'_1, X_1, s'_2, X_2, s'_3, X_3) \in \Gamma\), in state \( q \), scanning symbols \( s_1, s_2, s_3 \) changes state to \( q' \) and in the tape \( i \) input symbol to \( s'_i \) and moves to scan one square to the right if \( X_i = R \) (or left if \( X_i = L \)) in the tape \( i \).

## 2 Main Theorem

Given the function \( f \), let the set of global minima be denoted by \( G_f \). Now consider \( \delta \)-approximation to the global minima.

**Lemma 1.** For all \( \delta > 0 \) there exists a point \( x_n^* \) of finite precision length \( n \) such that \( |f(x^*) - f(x_n^*)| < \delta \).

**Proof.** Let \( \epsilon > 0 \) be such that \( |x - y| < \epsilon \) implies \( |f(x) - f(y)| < \delta \). Such an \( \epsilon > 0 \) exists for all \( \delta > 0 \) because the function \( f \) is continuous. Let \( n \) be the precision length required to represent numbers with gap \( \epsilon/10 \) between consecutive numbers. For the global minima \( x^* \) then there exists a point \( x_n^* \) with precision length \( n \) such that \( |x_n^* - x^*| < \epsilon \).

**Definition 2.** Let \( G_{\delta, k} \) be the set of points with given finite precision length \( k \geq 0 \) where the function value is \( \delta > 0 \) close to the global minima. And \( G_{\delta} \) be the union of all such sets.

We consider only finite-precision numbers. As the domain is a compact set, this can be regarded as a subset of natural numbers by multiplying by an appropriate constant. And in particular the set \( G_{\delta, k} \) is countable. Since we would like an algorithm to converge to a single point, for simplicity we assume the global optima is unique i.e., \( G_f \) is a singleton. We can state the main theorem.

**Theorem 1.** There is no sequence of computable points with finite-precision length converging to the global minima.

**Proof.** Let assume we have a sequence of \( \{\delta_k\} \) which goes to zero. Let \( x_k \) be any point of some finite precision length \( n_k \) such that \( |f(x^*) - f(x_k)| < \delta_k \). Such a point exists by Lemma 1 i.e., the set \( G_{\delta, n_k} \) is non-empty for \( \delta > 0 \). The length \( n_k \) can increase with \( k \).

We consider an equivalent problem. We define \( h_{x_k}^\delta(x) := \max\{0, f(x_k) - f(x) - \delta\} \). This function is identically zero if and only if \( |x_k - x^*| < \epsilon \). Note that \( x_k \) and \( x \) are represented with some finite precision. Our problem of finding the approximation to global minimum is same as checking whether \( h_{x_k}^\delta(\cdot) \) is identically zero. Since our objective function \( f \) is continuous, \( h_{x_k}^\delta(\cdot) \) is also continuous.

The set of all points with finite precision that are \( \delta \) close to the global minima \( G_{\delta, k} \) is also the set of all points \( x_k \) where the function \( h_{x_k}^\delta(\cdot) \) is identically zero. To check if \( x_k \) belongs
to $G'_{\delta,n_k}$ to is same as checking if whether $h^\delta_{x_k}(x) \equiv 0$ (function is identically zero). But this cannot be checked for a particular $x_k$ unless it is checked for all $x$ of finite precision length. As there are infinitely many such points, there is no Turing machine which can compute (halt) if a function is zero at infinitely many points. Hence checking if $\{x_k\}$ belongs to $G'_{\delta,n_k}$ is not computable or in other words we have shown the theorem.

**Corollary 1.** The problem of approximating the global minima of a continuous function by an arbitrary $\delta > 0$ is not computable.

**Corollary 2.** The problem of checking whether local minima $z$ is global is not computable as this involves checking whether $h^\delta_z(\cdot)$ is identically zero.

### 3 Conclusion

We have given a simple proof that there is no algorithm which computes a sequence (of finite-precision) points converging to global optima of any continuous function $f$.

### References

[1] R. Horst and H. Tuy., Global Optimization: Deterministic Approaches, Springer-Verlag (1996).

[2] R.I. Soare., Turing Computability: Theory and Applications, Springer-Verlag (2016).

[3] M. Pour-El and J. Richards., Computability in analysis and physics, Springer, Heidelberg (1989).