A probabilistic model of dynamic allocation of resources by investment region

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Abstract. The article discusses the most general, probabilistic model of the dynamic distribution of investment labor resources, describes a method for the optimal distribution of labor resources by investment region and determining the optimal modes of their redistribution taking into account changing conditions. In several regions of investment, several types of labor resources are invested simultaneously.

1. Introduction
At the initial moment, a multitude of investment labor resources and investment regions were identified; the effectiveness of investing an investment resource in a particular investment region is also known. By the efficiency of investing an investment resource in a certain investment region, one can understand, for example, a value, a number equal to the value (in value terms) obtained as a result of this investment in a given investment region. Each investment resource at any time can be in one of a finite number of states and be invested in one of a finite number of regions. It is necessary to designate resources (types of capital) for investment in investment regions in an optimal way, that is, so that the total investment efficiency is maximum. The arisen static assignment problem can be solved in various ways (for example, by the Hungarian method, etc.). By the beginning of the next period of time, depending on the decision taken at the previous stage and, possibly, for other reasons, many regions of labor investment may change (in some regions there will be no more investment enterprises and there will not be any, in other regions, invested enterprises may appear), many labor resources, as well as the effectiveness of investments of a particular investment resource in a particular investment region. Consequently, a new situation arises in which it is necessary to solve a new problem of optimal appointment. Let there be a finite set of such situations and the probabilities of transition from one situation to another be determined.
2. Formal statement of the problem

There are many \( M \) types of investment labor. We number them by index \( m = 1, 2, \ldots, |M| \). There are many \( L \) regions of investment. We number them by index \( l = 1, 2, \ldots, |L| \). The investment process takes place over \( T \) time periods \( t = 0, 1, 2, \ldots, T \). At any time \((t, t + 1)\) each investment resource is in one of a finite number of states \( M \). There are a finite set of investment modes \( Q_i : q_i = 1, 2, \ldots, |Q| \), with each mode associated probabilities \( P_{ij}, \) transition of an investment resource from a state \( i \) to state \( j \), and the effectiveness of his investment in a particular investment region. Investment efficiency \( m \) investment resource in \( l \) investment region in its simplest form is defined as follows:

\[
E_{ml}(q_i) = \sum_{k \in K} d_{kml} x_{lk} C_k
\]

Thus, we can say that in \( |L| \) regions of investment, the process of investing a united system (hereinafter simply a system), combining \( |M| \) individual types of labor resources investing. Many states \( \bar{I}_t \) this system is defined as a direct product of the sets of states of the individual labor resources of the investment, that is

\[
\bar{I}_t = \prod_{m=1}^{|M|} I_t^m
\]

Many modes \( \bar{Q}_t \), in which the system can work, we define as a direct product of the sets of operating modes of individual types of labor resources, that is

\[
\bar{Q}_t = \prod_{m=1}^{|M|} Q_t^m
\]

Obviously, the number of states in which systems can be located is

\[
|\bar{I}_t| = |I_t|^{|M|}
\]

The number of modes in which the system can work is

\[
|\bar{Q}_t| = |Q_t|^{|M|}
\]

The probability of its transition from one state to another is associated with each mode of operation of the system.

Let the state \( \bar{i} \) the system is determined by the state vector of the invested labor resources \( \bar{i} = \{i_{m_1}^1, i_{m_2}^2, \ldots, i_{m|M|}^M\} \), and condition \( \bar{j} \) the system is determined by the vector \( \bar{j} = \{j_{m_1}^1, j_{m_2}^2, \ldots, j_{m|M|}^M\} \). For simplicity of presentation, we will assume that investment resources pass from one state to another independently of each other. Then the probabilities of the transition of the system from one state \( \bar{i} \) to state \( \bar{j} \) defined as follows:

\[
V_{\bar{i} \bar{j}}(\bar{q}_i) = \prod_{k=1}^{|M|} P_{i_{k|\bar{i}} j_{k|\bar{j}}} (q_j)
\]
Each mode of use of investment labor resources is associated with income generation $r_{it_{t+1}}(\bar{q}_i)$. Revenue over time $(t, t + 1)$ can be calculated in various ways (for example, by solving the static problem of the optimal assignment and calculating the total efficiency of labor use). Full income from the functioning of the system during $T$ periods of time is obtained by calculating the mathematical expectation and convolutions in one way or another (for example, assigning importance factors in different time periods to income values with their subsequent addition). To calculate the optimal income for the final $T$ you can use the recursive dynamic programming relationships. Let be $V^{T-t}(\tilde{i}_t)$ - maximum income from the functioning of the system during $T - t$ time periods from state $\tilde{i}_t$, where $\tilde{i}_t = 1, 2, \ldots, T$ under optimal policy. Maximum income from the functioning of the system for one period of time $(T = 1)$ determined by the formula

$$V^1(\tilde{i}_0) = \max_{\tilde{q} \in \tilde{Q}_0} \left\{ \sum_{i=1}^{I} P_{i\tilde{q}_i}(\tilde{q}) r_{i\tilde{q}_i}(\tilde{q}) \right\}.$$  

(6)

The maximum income from the functioning of the system for two periods of time is delivered by the expression

$$V^2(\tilde{i}_0) = \max_{\tilde{q} \in \tilde{Q}_0} \left\{ \sum_{i=1}^{I} P_{i\tilde{q}_i}(\tilde{q}) \left[ r_{i\tilde{q}_i}(\tilde{q}) + V^1(\tilde{i}_1) \right] \right\}.$$  

(7)

Where $V^1(\tilde{i}_1)$ - maximum income from the functioning of the system in a one-step process. To calculate the optimal income, we have the following functional relation

$$V^{T-t}(\tilde{i}_t) = \max_{\tilde{q} \in \tilde{Q}_{t+1}} \left\{ \sum_{i=1}^{I} P_{i\tilde{q}_i}(\tilde{q}) \left[ r_{i\tilde{q}_i}(\tilde{q}) + V^{T-t-1}(\tilde{i}_{t+1}) \right] \right\}.$$  

(8)

$V^0(\tilde{i}_t)$ it is calculated using well-known algorithms for solving the assignment problem, which deliver the initial or boundary conditions for performing calculations using the dynamic programming method. Can also put $V^0(\tilde{i}_T)$ equal to zero, which is natural. Applying relation (8) sequentially for $t = T - 1, T - 2, \ldots, 1, 0$, we calculate $V^1(\tilde{i}_{T-1}), V^2(\tilde{i}_{T-2}), V^3(\tilde{i}_{T-3}), \ldots, V^{T-1}(\tilde{i}_1), V^T(\tilde{i}_0)$, we can indicate the optimal distribution of labor resources by region of investment at any time. If the investment process takes place over a large number of time periods, then the Howard iterative method can be used to calculate the average income for one period. The Howard iterative solution consists of two steps:

1) we find a solution to system (8) with a fixed policy $\bar{q}$;

2) using received $\nu_T$, find such $\bar{q}$, which at any $\tilde{i}$ deliver

$$\max_{\tilde{q}} \left\{ \sum_{i=1}^{I} \bar{P}_{i\tilde{q}_i}(\tilde{q}) (r_{i\tilde{q}_i}(\tilde{q}) + \nu) - \nu_i \right\}.$$  

The process can be started from any stage (if from the second, then putting everything $\nu_T$ equal to zero). The process ends when obtaining the optimal solution.

3. Hungarian method for solving the static assignment problem

3.1. Description of the Hungarian Method

We now give a brief description of the Hungarian method of solving the static assignment problem, the algorithm of which can be used to calculate income in certain time periods when solving a dynamic problem. There is a matrix $|C_{ij}|_N$ investment efficiency $i$ resource type in $j$ region of investment.
Denote by \( x_{ij} \) number: 
\[
x_{ij} = \begin{cases} 
1, & \text{if the } i \text{ resource is invested in the } j \text{ region of investing} \\
0, & \text{otherwise.}
\end{cases}
\]
Here is to find \( x_{ij} \), delivering maximum expression
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij} x_{ij}
\]
under conditions
\[
\sum_{i=1}^{N} x_{ij} = 1, \quad j = 1,2,\ldots,N
\]
\[
\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1,2,\ldots,N
\]

**Definition 1.** Zero matrix elements \( d_{ij} \), \( z_1, z_2, \ldots, z_k \) will be called independent zeros if for any \( 1 \leq i \leq k \) row and column at the intersection of which the element lies \( z_i \), does not contain elements \( z_k \) for all \( i \neq k \).

**Definition 2.** Two rectangular matrices \( C_{ij} \) and \( d_{ij} \) call equivalent \( (C \approx D) \), if the ratio is satisfied: 
\[
C_{ij} = d_{ij} + d_i + \beta_j.
\]
Problems defined by equivalent matrices will be called equivalent problems. In the process of solving some rows (columns) will be marked with “+”. Their elements will be called selected elements.

### 3.2. Algorithm for solving a dynamic problem

The algorithm consists of a preparatory stage and no more than \( N - 2 \) iterations. The preparatory stage is as follows.

1. In each column we find the maximum element and subtracting from it all the elements of this column (we put the result in the corresponding positions). As a result, we get a matrix with non-negative elements that contains at least one zero element in each column.
2. From each element of each row we subtract the minimum element of this row (we put the result in the corresponding position). The result is a matrix with non-negative elements, containing at least one zero element in each row and in each column. Let her \( C_0 \).
3. In the first column, mark with an asterisk “*” arbitrary zero
4. We look through all the other columns and, if found, mark each zero in such a way that the row containing it does not contain already marked zeros.
5. Thus obtained independent zeros
6. The end of the preparatory phase.
7. Mark with a “+” sign columns containing zeros marked with an asterisk “*”.

**Separate iteration.**

6. If in \( C_k N \) zeros marked with an asterisk, the procedure ends (the optimal solution is determined by the positions of the marked zeros). If there are fewer zeros, then go to step 7.

**First step.**

7. If among the unselected elements there are no zero, then go to step 8.
8. check: does the string containing the zero found contain the zero marked with an asterisk?
   If so, go to step 10.
   If not, go to step 9.
9. We mark this zero with a “/” and go to step 12.
10. Mark this zero with the “/”, sign, the line containing it with the “+” sign. We destroy the “+” sign, which selects a column containing zero in this row, marked with a sign.
11. Check: are there still unselected zeros? 
If so, go to paragraph 8. If not, go to paragraph 16. 
The end of the first stage.

Second phase.

12. We build a chain from $0'$ to $0^*$ column and from $0^*$ to $0'$ by line. 
13. We mark with asterisk all zeros of the chain marked with a prime, and with a dash - zeros marked 
with an asterisk. 
14. Destroy the strokes of all zeros of the matrix. 
15. Destroy all the signs of the line selection. The result - there are one more independent zeros. Go to 
step 5. 
The end of the second stage. 
The end of a single iteration.

The third stage.

16. Choose among unselected elements the minimum. 
17. Subtract $h$ of all elements of unselected lines. 
18. We add $h$ to all elements of selected columns. Go to step 7. 
19. The end of the third stage.

4. Numerical example

To illustrate the described method for solving a dynamic problem, consider a simple example. Two 
types of labor resources are used in two investment regions. Each type of resource can be in one of two 
possible states and can be used in one of four possible states defined by the following table 1.

| Table 1. System Status. |
|-------------------------|
| State of the system     | 1 | 2 | 3 | 4 |
| The state of the first type of labor | 1 | 1 | 2 | 2 |
| The state of the second type of labor | 1 | 2 | 1 | 2 |

From (4) it follows that the invested resources can be used in one of four possible modes defined by the 
following table 2.

| Table 2. System Modes. |
|------------------------|
| System mode            | 1 | 2 | 3 | 4 |
| The mode of operation of the first type of labor | 1 | 1 | 2 | 2 |
| The mode of operation of the second type of labor | 1 | 2 | 1 | 2 |

The probability of the transition of labor resources from one state to another is determined by the 
following table 3.

| Table 3. The Probability of the transition of labor resources from one state to another. |
|---------------------------------|
| condition | Mode | Types of labor | Transition probability |
|-----------|------|----------------|------------------------|
| 1         | 1    | 1              | 0.6 0.4                |
| 1         | 1    | 2              | 0.3 0.7                |
| 1         | 2    | 1              | 0.4 0.6                |
| 1         | 2    | 2              | 0.7 0.3                |
| 2         | 1    | 1              | 0.7 0.3                |
| 2         | 1    | 2              | 0.4 0.6                |
| 2         | 2    | 1              | 0.3 0.7                |
| 2         | 2    | 2              | 0.6 0.4                |
We calculate the state of the system, the modes of operation of the system, the efficiency of labor resources investing in the regions, the optimal destination and the income received, as well as the probabilities of the system moving from one state to another, depending on the modes of use of the invested labor resources and the value of the resulting income (or damage). The modes of operation of the system, in which the efficiency of its operation for four periods of time will be maximum, are summarized in table 4.

### Table 4. Modes of maximum efficiency of the system for four periods of time.

| Number of steps remaining | States |
|---------------------------|--------|
|                           | 1  | 2  | 3  | 4  |
| 1                         | 2  | 2  | 2  | 2  |
| 2                         | 4  | 2  | 1  | 1  |
| 3                         | 4  | 2  | 1  | 1  |
| 4                         | 4  | 2  | 1  | 1  |

5. Conclusion

In many cases, the performance of a particular type of resource in a particular investment region can be determined differently than what is presented in the article. To determine the effectiveness of the invested labor resources, you can use any other method, which essentially does not change the described method for determining the optimal distribution of labor resources of investment.

Thus, a dynamic problem can be solved by the method of dynamic programming, and various methods for solving it can be applied to the static problem of optimal assignment that arises at each step. The considered method for solving a dynamic problem allows for the optimal distribution of various types of labor resources by investment region, taking into account the changing situation. The specified method can be implemented on a computer.

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