GRADED SKEW SPECHT MODULES AND CUSPIDAL MODULES FOR
KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF AFFINE TYPE A

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Abstract. Kleshchev, Mathas and Ram (2012) gave a presentation for graded Specht modules over Khovanov-Lauda-Rouquier algebras of finite and affine type A. In this paper we show that this construction can be applied more generally to skew shapes to give a presentation of graded skew Specht modules, which appear as subquotients of restrictions of Specht modules. As an application, we show that cuspidal modules in affine type A are skew Specht modules associated to certain hook shapes.

1. Introduction

Let \( O \) be a commutative ring with identity, and let \( S_d \) be the symmetric group on \( d \) letters. To every partition \( \lambda \) of \( d \), or equivalently, every Young diagram with \( d \) nodes, there is an associated \( O S_d \)-module \( S_\lambda \) called a Specht module, which has \( O \)-basis in correspondence with standard \( \lambda \)-tableaux. Specht modules arise as cell modules for the cellular structure on the group algebra of \( S_d \), going back to [15].

Over the complex numbers, the group algebra of \( S_d \) is semisimple, and it is well known that \( t S_\lambda C |_{\lambda} \) is a complete set of irreducible representations. For \( k \leq d \), we consider \( S_k \) a subgroup of \( S_d \) with respect to the first \( k \) letters, and denote the copy of \( S_k \) embedded in \( S_d \) with respect to the last \( k \) letters as \( S_k' \). For \( \lambda \vdash d \) and \( \mu \vdash k \),

\[
S^{\lambda/\mu}_C := \text{Hom}_{S_k'}(S^{\mu}_C, \text{Res}_{S_k} S^\lambda_C).
\]

is a \( C S_{d-k} \)-module. In fact, \( S^{\lambda/\mu}_C = 0 \) unless the Young diagram for \( \mu \) is contained in that of \( \lambda \), so going forward we assume that is the case. The set of nodes \( \lambda/\mu \) in the complement is called a skew diagram, and \( S^{\lambda/\mu}_C \) is called the skew Specht module. As a \( C \)-vector space, \( S^{\lambda/\mu}_C \) has basis in correspondence with standard \( \lambda/\mu \)-tableaux, and there is an analogue of Young's orthogonal form for skew Specht modules, see e.g. [8, §2].

More generally, to an \( l \)-multipartition \( \lambda \), one may associate a Specht module \( S^\lambda \) over a cyclotomic Hecke algebra of level \( l \), of which the group algebra of \( S_d \) is a special (level one) case. Brundan and Kleshchev [2] showed that over an arbitrary field such algebras are isomorphic to a certain cyclotomic quotient \( R^\lambda_\alpha \) of the Khovanov-Lauda-Rouquier (KLR) algebra \( R_d = \bigoplus_{h(k) = d} R_k \). Importantly, KLR algebras and their cyclotomic quotients carry a grading, thereby allowing one to consider the graded representation theory of cyclotomic Hecke algebras via this isomorphism. In [3], Brundan, Kleshchev and Wang showed Specht modules are gradable. In [9], Kleshchev, Mathas and Ram gave a presentation for \( S^\lambda \) over \( R_\alpha \), in terms of a 'highest weight' generator \( v^\lambda \) and relations which include a homogeneous version of the classical Garnir relations for Specht modules.

In this paper we define graded skew Specht modules over \( R_\alpha \). The approach of (1.1) fails in general, since arbitrary Specht modules \( S^\lambda \) are not generally semisimple nor indecomposable. However, it turns out that if we extend, in the most obvious way, the presentation of [9] to skew diagrams \( \lambda/\mu \), we achieve the desired result; a graded \( R_\alpha \)-module \( S^{\lambda/\mu}_\alpha \) with homogeneous basis in correspondence with standard \( \lambda/\mu \)-tableaux. As an analogue of (1.1), we prove that for \( \lambda \) of content \( \beta + \alpha \), the \( R_\beta \otimes R_\alpha \)-module \( \text{Res}_{\beta,\alpha} S^\lambda \) has an explicit (graded) filtration with subquotients of the form \( S^{\mu} \boxtimes S^{\lambda/\mu} \).
Our motivation for defining skew Specht modules arose from the study of cuspidal modules over KLR algebras of affine type $\mathfrak{A}$. The theory of cuspidal systems for affine KLR algebras, described in [7, 10, 14], building on the ideas of [11, 13] for finite types, provides a classification of irreducible modules over $R_\alpha$. Cuspidal modules (along with the so-called imaginary modules) are the building blocks of this theory; to every positive real root $\alpha \in \Phi_+$, one associates an irreducible $R_\alpha$-module $L_\alpha$ characterized by specific properties (see [5]). We show that under a balanced convex preorder (part of the data of the cuspidal system), $L_\alpha$ is isomorphic to $S^{\lambda/\mu}$ up to some shift, where $\lambda/\mu$ is a skew hook diagram described by an inductive process and dependent on the chosen preorder. This gives a presentation for cuspidal modules, along with a graded character which can be read off from the skew hook diagram. This result can be seen as an affine analogue of a result by Kleshchev and Ram [11, §8.4], which showed that in finite type $\mathfrak{A}$, the cuspidal modules are Specht modules associated to certain hook partitions.

In Section 2, we collect combinatorial facts and notation surrounding Young diagrams, skew diagrams, and their tableaux. In Section 3, we recall the definition of the KLR algebra $R_\alpha$, and some facts about its representation theory. In Sections 4 and 5, we define permutation modules and skew Specht modules. Readers familiar with the construction in [9] and the ‘spanning’ half of the Specht module basis theorem, and to whom the phrase ‘extend to skew diagrams in the obvious way’ makes sense, may reasonably skip ahead to Section 6, as up to this point most of the arguments from [9] carry over to the skew diagram case with little significant alteration. In the key Section 6, we prove that skew Specht modules arise as subquotients of restrictions of Specht modules, and complete the ‘linear independence’ half of the basis theorem for skew Specht modules. In Section 7, we prove some useful results on characters of certain skew Specht modules. In Section 8, we briefly describe the theory of cuspidal systems, along with some related notions we’ll need in Section 9, where we demonstrate the connection between cuspidal modules and skew hook Specht modules.

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2. Preliminaries

2.1. Lie theoretic notation. We use notation similar to [9, 7]. Let $e \in \{0, 2, 3, 4, \ldots\}$ and $I = \mathbb{Z}/e\mathbb{Z}$. Let $\Gamma$ be the quiver with vertex set $I$ and a directed edge $i \to j$ if $j = i - 1 \pmod{e}$. Thus $\Gamma$ is a quiver of type $A_e$, if $e = 0$ or $A_{e-1}$ if $e > 0$. The corresponding Cartan matrix $C = (a_{i,j})_{i,j \in I}$ is defined by

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j; \\ 0 & \text{if } j \neq i, i \pm 1; \\ -1 & \text{if } i \to j \text{ or } i \leftarrow j; \\ -2 & \text{if } i \equiv j. \end{cases}$$

Let $(h, \Pi, \Pi^\vee)$ be a realization of $(a_{i,j})_{i,j \in I}$, with root system $\Phi$, positive roots $\Phi_+$, simple roots $\{\alpha_i \mid i \in I\}$, fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and normalized invariant form $(\cdot, \cdot)$ such that $(\alpha_i, \alpha_j) = a_{i,j}$ and $(\Lambda_i, \alpha_j) = \delta_{i,j}$. If $e > 0$, the null-root is $\delta = \alpha_0 + \cdots + \alpha_{e-1}$. Let $P_+$ be the set of dominant integral weights, and $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ the positive root lattice. For $\alpha \in Q_+$, let the height $ht(\alpha)$ of $\alpha$ be the sum of the coefficients when $\alpha$ is expanded in terms of the simple roots.

Fix a positive integer $l$, referred to as the level, and an ordered $l$-tuple $\kappa = (k_1, \ldots, k_l) \in I^l$ called a multicharge. Define the corresponding dominant weight $\Lambda = \Lambda(\kappa) := \sum_{j=1}^l \Lambda_{k_j} \in P_+$. 

2.2. Words. Sequences of elements of \( I \) will be called **words**, and the set of all words is denoted \( \langle I \rangle \). If \( i = i_1 \cdots i_d \in \langle I \rangle \), then \( |i| := \alpha_{i_1} + \cdots + \alpha_{i_d} \in Q_+ \). For \( \alpha \in Q_+ \), denote
\[ \langle I \rangle_\alpha := \{ i \in \langle I \rangle \mid |i| = \alpha \}. \]
If \( \alpha \) is of height \( d \), then \( \mathcal{S}_d \) with simple transpositions \( s_1, \ldots, s_{d-1} \) has a left action on \( \langle I \rangle_\alpha \) via place permutations.

2.3. Young diagrams. An \( l \)-multipartition \( \lambda \) of \( d \) is an \( l \)-tuple of partitions \( (\lambda^{(1)}, \ldots, \lambda^{(l)}) \) such that \( \sum_{i=1}^l |\lambda^{(i)}| = d \). For \( 1 \leq i \leq l \), let \( n(\lambda, i) \) be the number of nonzero parts of \( \lambda^{(i)} \). When \( l = 1 \), we will usually write \( \lambda = \lambda^{(1)} \). The **Young diagram** of the partition \( \lambda \) is
\[ \{(a, b, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \ldots, l\} \mid 1 \leq b \leq \lambda_a^{(m)}\}. \]
The elements of this set are the **nodes** of \( \lambda \). We will usually identify the partition with its Young diagram. To each node \( A = (a, b, m) \) we associate its **residue**
\[ \text{res} \, A = \text{res}^c \, A = k_m + (b - a) \pmod{c}. \]
An \( i \)-node is a node of residue \( i \). Define the **residue content** of \( \lambda \) to be \( \text{cont}(\lambda) := \sum_{A \in \lambda} \text{res} \, A \in Q_+ \). Denote
\[ \mathcal{P}_\alpha := \{ \lambda \in \mathcal{P} \mid \text{cont}(\lambda) = \alpha \}, \quad (\alpha \in Q_+). \]
and set \( \mathcal{P}_d^\alpha := \bigcup_{\text{ht}(\alpha) = d} \mathcal{P}_\alpha \).

A node \( A \in \lambda \) is **removable** if \( \lambda \setminus \{A\} \) is a Young diagram, and a node \( B \notin \lambda \) is **addable** if \( \lambda \cup \{B\} \) is a Young diagram. We use the notation \( \lambda_A := \lambda \setminus \{A\} \) and \( \lambda^B := \lambda \cup \{B\} \) for the corresponding partitions.

Let \( \lambda, \mu \in \mathcal{P}_d^\alpha \). We say \( \lambda \) **dominates** \( \mu \), and write \( \lambda \succ \mu \), if

\[ \sum_{a=1}^{m-1} |\lambda^{(a)}| + \sum_{b=1}^{m} \lambda^{(m)}_b \geq \sum_{a=1}^{m-1} |\mu^{(a)}| + \sum_{b=1}^{m} \mu^{(m)}_b \]

for all \( 1 \leq m \leq l \) and \( c \geq 1 \).

Let \( \lambda = (\lambda^{(1)}_1, \ldots, \lambda^{(1)}_l) \) signify the **conjugate partition** to \( \lambda \), where \( \lambda^{(i)}_j \) is obtained by swapping the rows and columns of \( \lambda^{(i)} \).

2.4. Tableaux. Let \( \lambda \in \mathcal{P}_d^\alpha \). A **\( \lambda \)-tableau** \( T \) is obtained by inserting \( 1, \ldots, d \) into the nodes of \( \lambda \) with no repeats. If the node \( A = (a, b, m) \in \lambda \) is occupied by \( r \) in \( T \), then we write \( T(a, b, m) = r \) and set \( \text{res}_T(r) = \text{res} \, A \). The **residue sequence** of \( T \) is
\[ i(T) = i_1(T) \cdots i_d(T) = \text{res}_T(1) \cdots \text{res}_T(d) \in \langle I \rangle. \]

A \( \lambda \)-tableau is **row-strict** (resp. **column-strict**) if its entries increase from left to right (resp. top to bottom) along the rows (resp. columns) of \( T \). We say \( T \) is **standard** if it is row- and column-strict. Let \( \text{Tab}(\lambda) \) (resp. \( \text{St}(\lambda) \)) be the set of all (resp. standard) \( \lambda \)-tableaux.

Let \( T \) be a \( \lambda \)-tableau and suppose that \( 1 \leq r \neq s \leq d \) and \( r = T(a_1, b_1, m_1) \) and \( s = T(a_2, b_2, m_2) \). We write \( r \nearrow_T s \) if \( m_1 = m_2, a_1 > a_2 \) and \( b_1 < b_2 \); informally, if \( r \) and \( s \) are in the same component of \( \lambda \) and \( s \) is strictly to the northeast of \( r \). The symbols \( \searrow_T, \nwarrow_T, \downarrow_T \) have similarly obvious meanings.

Let \( \lambda \in \mathcal{P}_d^\alpha, i \in I, A \) be a removable \( i \)-node, and \( B \) be an addable \( i \)-node of \( \lambda \). We set
\[ d_A(\lambda) := \# \{ \text{addable } i \text{-nodes strictly below } A \} - \# \{ \text{removable } i \text{-nodes strictly below } A \} \]
\[ d_B(\lambda) := \# \{ \text{addable } i \text{-nodes strictly above } B \} - \# \{ \text{removable } i \text{-nodes strictly above } B \} \]

The **degree** of \( T \) is defined in [3 Section 3.5] as follows. If \( d = 0 \), then \( T = \emptyset \) and \( \text{deg} \, T := 0 \). Otherwise, let \( A \) be the node occupied by \( d \) in \( T \). Let \( T_{<d} \in \text{St}(\lambda_A) \) be the tableau obtained by removing this node, and set
\[ \text{deg} \, T := d_A(\lambda) + \text{deg} \, T_{<d}. \]
Similarly, define the dual notion of codegree of $T$ by $\text{codeg } \emptyset = 0$ and
\[
\text{codeg } T := d^A(\lambda) + \text{codeg } T_{\leq d}.
\]

The group $\mathcal{S}_d$ acts on the set of $\lambda$-tableaux from the left by acting on entries. Let $T^\lambda$ be the $\lambda$-tableau in which the numbers $1, 2, \ldots, d$ appear in order from left to right along the successive rows, starting from the top. Let $T_{\lambda}$ be the $\lambda$-tableau in which the numbers $1, 2, \ldots, d$ appear in order from top to bottom along the successive columns, working from the leftmost column to the rightmost column within a component and moving from the $l$th component up to the first component. Then $T^\lambda_{\lambda} = T^\lambda$, where the conjugate of a tableau is defined in the obvious way.

For each $\lambda$-tableau $T$, define permutations $w^T$ and $w_T \in \mathcal{S}_d$ by the equations:
\[
w^T T^\lambda = T = w_T T^\lambda.
\]

2.5. Bruhat order. Let $\ell$ be the length function on $\mathcal{S}_d$ with respect to the Coxeter generators $s_1, \ldots, s_{d-1}$. Let $\leq$ be the Bruhat order on $\mathcal{S}_d$, so that $1 \leq w$ for all $w \in \mathcal{S}_d$. Define a partial order $\preceq$ on $\text{St}(\lambda)$ as follows:
\[
S \preceq T \iff w^S \preceq w^T.
\]

2.6. Skew diagrams and tableaux. Let $\lambda, \mu \in \mathcal{P}^\kappa$, with $\mu \subset \lambda$ as Young diagrams. Then we call $\lambda/\mu := \lambda\backslash\mu$ a skew diagram. A (level one) skew diagram is called a skew hook if it is connected and does not have two nodes on the same diagonal. We may consider a Young diagram as a skew diagram with empty inner tableau. With $\mu$ fixed, let $\mathcal{J}^\kappa_\mu$ be the set of skew diagrams $\lambda/\mu$ such that $|\lambda/\mu| = d$. Let $\mathcal{J}^\kappa_\mu = \bigcup_{\lambda \in \mathcal{P}^\kappa_\mu} \mathcal{J}^\kappa_\mu$. Residue and content for skew diagrams are defined as before; for example $\text{cont}(\lambda/\mu) := \sum_{A \in \lambda/\mu} \alpha_{\text{res } A} \in Q_+$. Denote
\[
\mathcal{J}^\kappa_{\mu, \alpha} = \{ \lambda/\mu \in \mathcal{J}^\kappa_\mu \mid \text{cont}(\lambda/\mu) = \alpha \}.
\]

For $\lambda/\mu, \nu/\mu \in \mathcal{J}^\kappa_\mu$, we say that $\lambda/\mu$ dominates $\nu/\mu$, or $\lambda/\mu \succeq \nu/\mu$, if $\lambda \succeq \nu$.

For $\lambda/\mu \in \mathcal{J}^\kappa_{\mu, \alpha}$, a $\lambda/\mu$-tableau $\tau$ is defined by inserting the integers $1, \ldots, d$ into the nodes of $\lambda/\mu$ with no repeats. Let $\text{Tab}(\lambda/\mu)$ be the set of $\lambda/\mu$-tableaux. We define the residue sequence of $i(\tau)$ in the same manner as for Young tableaux. Let $t^{\lambda/\mu}$ be the $\lambda/\mu$-tableau in which the numbers $1, \ldots, d$ appear in order from left to right, starting from the top. For every $\lambda/\mu$-tableau $\tau$, define a $\lambda$-tableau $\mathcal{Y}(\tau)$ by setting $\mathcal{Y}(\tau)(a, b, m) = \ell(\tau(a, b, m))$ for $(a, b, m) \in \mu$ and $\mathcal{Y}(\tau)(a, b, m) = \tau(a, b, m) + |\mu|$ for $(a, b, m) \in \lambda/\mu$.

For example, if $l = 1$, $\lambda = (4, 4, 1)$, and $\mu = (2, 1, 1)$, then
\[
t^{4/1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}, \quad \text{and} \quad \mathcal{Y}(t^{4/1}) = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 8 & 9 \\ 4 \end{bmatrix}.
\]

Let $\text{St}(\lambda/\mu)$ be the set of standard (i.e. row- and column-strict) $\lambda/\mu$-tableaux. For $\tau \in \text{St}(\lambda/\mu)$, we define
\[
\deg \tau := \deg \mathcal{Y}(\tau) - \deg \tau^\mu.
\]

The symmetric group $\mathcal{S}_d$ acts on $\text{Tab}(\lambda/\mu)$ in the obvious fashion. For $\tau \in \text{Tab}(\lambda/\mu)$, define $w^\tau$ by $w^\tau t^{\lambda/\mu} = \tau$. Define a partial order on $\text{Tab}(\lambda/\mu)$ as follows:
\[
s \preceq \tau \quad \text{if and only if} \quad w^s \preceq w^\tau.
\]

Lemma 2.1. Let $s, \tau \in \text{Tab}(\lambda/\mu)$. Then $s \preceq \tau$ if and only if $\mathcal{Y}(s) \preceq \mathcal{Y}(\tau)$.

Proof. Let $\widehat{w}^\tau$ be the image of $w^\tau$ under the ‘right side’ embedding $\mathcal{S}_d \hookrightarrow \mathcal{S}_{|\mu|} \times \mathcal{S}_d \hookrightarrow \mathcal{S}_{|\lambda|}$. Then $w^\tau(\tau) = \widehat{w}^\tau \mathcal{Y}(\tau^{\lambda/\mu})$, with $\ell(w(\tau)) = \ell(w^\tau) + \ell(\mathcal{Y}(\tau^{\lambda/\mu}))$, and similarly for $w^\tau(\mathcal{Y}(s))$. Since $w^s \preceq w^\tau$ if and only if $\widehat{w}^s \preceq \widehat{w}^\tau$, the result follows. \qed
Remark 2.2. In order to translate between the orders in the various papers cited, we provide the following dictionary. Our partial order on partitions and tableaux agrees with that of [3]. In [9] the order on tableaux (which we'll call \( \preceq_U \)) amounts to \( S \preceq_U T \iff w^S \preceq w^T \). As is shown in [9 Lemma 2.18(ii)], when \( S, T \in \text{St}(\mu) \), we have \( S \preceq_U T \iff S \supseteq T \). In [12], the reverse Bruhat order \( (1 \geq w) \) is used on elements of \( \mathfrak{S}_d \), and the order on tableaux (which we'll call \( \preceq_M \)) is defined (on row-strict tableaux) by the shape condition in Lemma 2.5. Thus Lemma 2.5 will give \( S \preceq_M T \iff S \supseteq T \) when \( S, T \) are row-strict.

The following lemmas are proved in [3] and [12] in the context of Young diagrams, but the proofs carry over to skew shapes without significant alteration. The first lemma is obvious.

Lemma 2.3. Let \( t \in \text{St}(\lambda/\mu) \). Then \( s, t \in \text{St}(\lambda/\mu) \) if and only if \( r \not\leq_t r + 1 \), or \( r + 1 \not\leq_t r \).

For nodes \( A, B \) in \( \lambda/\mu \), we say that \( A \) is earlier than \( B \) if \( t^{\lambda/\mu}(A) < t^{\lambda/\mu}(B) \); i.e. \( A \) is above or directly to the left of \( B \), or in an earlier component.

Lemma 2.4. Let \( s, t \in \text{Tab}(\lambda/\mu) \). Then \( s \preceq t \) if and only if \( s = (a_1b_1) \cdots (a_rb_r) t \) for some transpositions \( (a_1b_1), \ldots, (a_rb_r) \) such that for each \( 1 \leq n \leq r \) we have \( a_n < b_n \) and \( b_n \) is in an earlier node in \( (a_{n+1}b_{n+1}) \cdots (a_rb_r)t \) than \( a_n \).

Proof. This follows from applying Lemma 2.1 and [9] Lemma 3.4 to \( \Upsilon(s) \) and \( \Upsilon(t) \). \( \square \)

Given \( \lambda/\mu \in \mathcal{S}_\mu^{\leq} \) and a row-strict \( t \in \text{Tab}(\lambda/\mu) \), for all \( 1 \leq a \leq d \) define \( t_{<a} \) to be the tableau obtained by erasing all nodes occupied by entries greater than \( a \).

Lemma 2.5. Let \( s, t \) be row-strict \( \lambda/\mu \)-tableaux. Then \( s \preceq t \) if and only if \( \text{sh}(\Upsilon(s)_{<a}) \supseteq \text{sh}(\Upsilon(t)_{<a}) \) for each \( a = |\mu| + 1, \ldots, |\mu| + d \).

Proof. This follows from Lemma 2.1 and [12] Theorem 3.8. \( \square \)

Lemma 2.6. Let \( \lambda/\mu \in \mathcal{S}_\mu^{\leq} \) and \( s, t \in \text{St}(\lambda/\mu) \), and \( r \in \{1, \ldots, d-1\} \) such that \( r \not\leq_t r + 1 \) or \( r \rightarrow_t r + 1 \). Then \( s \prec t \) implies \( s \preceq t \).

Proof. By [9] Lemma 3.7, \( \Upsilon(s) \subsetneq \Upsilon(s, t) = \Upsilon(s_{r+1}) \Upsilon(t) \) if and only if \( \Upsilon(s) \supsetneq \Upsilon(t) \), and the result follows by Lemma 2.4. \( \square \)

3. KLR Algebras

3.1. Definition. Let \( O \) be a commutative ring with identity, let \( \alpha \in Q_+ \), and assume \( e > 0 \) (resp. \( e = 0 \)). The **KLR algebra** \( R_\alpha = R_{\alpha}(O) \) of type \( A^{(1)}_{e-1} \) (resp. \( A_{e} \)) is an associative graded unital \( O \)-algebra generated by

\[
\{1_i \mid i \in \langle \alpha \rangle \} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots, \psi_{d-1}\}
\]

and subject only to the following relations:

\[
\begin{align*}
1_i1_j &= \delta_{i,j}1_i; & \sum_{i \in \langle \alpha \rangle} 1_i &= 1; & y_r1_i &= 1_iy_r; & y_ry_t &= y_ty_r; \\
\psi_r1_i &= 1_{s,r}\psi_r; & (y_t\psi_r - \psi_r y_{s_r(t)})1_i &= \delta_{i, s_r+1}(\delta_{i, r+1} - \delta_{i, r})1_i; \\
\psi_r^21_i &= \begin{cases} 
0 & i_r = i_{r+1}, \\
1_i & i_{r+1} \neq i_r, i_r \pm 1, \\
(y_r - y_{r+1})1_i & i_r \rightarrow i_{r+1}, \\
(y_r + y_{r+1})1_i & i_r \leftarrow i_{r+1}, \\
(y_r + y_{r+1})(y_r - y_{r+1})1_i & i_r \leftrightarrow i_{r+1}; 
\end{cases} \\
(\psi_r \psi_{r+1} - \psi_{r+1} \psi_r)1_i &= \begin{cases} 
1_i & i_{r+2}=i_r \leftrightarrow i_{r+1}, \\
-1_i & i_{r+2}=i_r \rightarrow i_{r+1}, \\
(-y_r + 2y_{r+1} - y_{r+2})1_i & i_{r+2}=i_r \leftrightarrow i_{r+1}, \\
0 & \text{otherwise}. 
\end{cases}
\end{align*}
\]
The grading on $R_\alpha$ is defined by setting

$$\deg(1_i) = 0, \quad \deg(y_i) = 2, \quad \deg(\psi_i 1_i) = -a_{i, i, i+1}.$$ 

**Remark 3.1.** In [6, Khovanov and Lauda present a convenient diagrammatic approach to computations in $R_\alpha$ which will be used often in arguments in this paper.

3.2. **Properties.** Suppose that $\alpha \in Q_+$ and $\operatorname{ht}(\alpha) = d$. For all $w \in \mathcal{S}_d$, fix a preferred reduced expression $w = s_{r_1} \cdots s_{r_m}$. Define $\psi_w = \psi_{r_1} \cdots \psi_{r_m} \in R_\alpha$ for all $w \in \mathcal{S}_d$. In general $\psi_w$ depends on the choice of reduced expression. When $w$ is fully commutative however, i.e., when one can go from any reduced expression for $w$ to any other using only the braid relations of the form $s_r s_t = s_t s_r$ for $|r-t| > 1$, the element $\psi_w$ is independent of the choice of reduced expression.

For $\lambda/\mu \in \mathcal{F}_{\alpha, 0}'$ and $t \in \operatorname{Tab}(\lambda/\mu)$, define

$$\psi^t := \psi_{w^t}.$$

**Lemma 3.2.** Let $t \in \operatorname{St}(\lambda/\mu)$. If $w^t = s_{r_1} \cdots s_{r_m}$ is a reduced decomposition in $\mathcal{S}_d$, then

$$\deg t - \deg t^{\lambda/\mu} = \deg(\psi_{r_1} \cdots \psi_{r_m} 1_{i_\lambda}).$$

**Proof.** Write $c = |\mu|$. Let $s_{t_1} \cdots s_{t_m}$ be a reduced decomposition for $w^{c(t)}(t^{\lambda/\mu})$. Then $w^t = s_{r_1+c} \cdots s_{r_m+c}$ is reduced and $w^{c(t)} = w^{c(t^{\lambda/\mu})} = s_{r_1+c} \cdots s_{r_m+c} s_{s_1} \cdots s_{t_m}$ is reduced. Then by [28 Corollary 3.13] we have

$$\deg Y(t) - \deg T^\lambda = \deg(\psi_{r_1+c} \cdots \psi_{r_m+c} 1_{i_\lambda}) + \deg(\psi_{t_1} \cdots \psi_{t_m} 1_{i_\lambda})$$

and

$$\deg Y(t^{\lambda/\mu}) - \deg T^\lambda = \deg(\psi_{t_1} \cdots \psi_{t_m} 1_{i_\lambda}),$$

which implies the result. \qed

**Proposition 3.3.** Let $f(y) = f(y_1, \ldots, y_d) \in \mathcal{O}[y_1, \ldots, y_d]$ be a polynomial in the generators $y_i$ of $R_\alpha$. Let $1 \leq r_1, \ldots, r_m \leq d - 1$. Then

(i) $f(y) \psi_{r_1} \cdots \psi_{r_m} 1_i$ is an $\mathcal{O}$-linear combination of elements of the form $\psi_{r_1}^{c_1} \cdots \psi_{r_m}^{c_m} g(y) 1_i$, where $g(y) \in \mathcal{O}[y_1, \ldots, y_d]$, each $\epsilon_i \in \{0, 1\}$, and $s_{r_1}^{c_1} \cdots s_{r_m}^{c_m}$ is a reduced expression.

(ii) If $w = s_{r_1} \cdots s_{r_m}$ is reduced, and $s_{t_1} \cdots s_{t_m}$ is another reduced expression for $w$, then

$$\psi_{r_1} \cdots \psi_{r_m} 1_i = \psi_{t_1} \cdots \psi_{t_m} 1_i + \sum_{u \in w} d_u \psi_u g_u(y) 1_i,$$

where each $d_u \in \mathcal{O}$, $g_u(y) \in \mathcal{O}[y_1, \ldots, y_d]$, and each $u$ in the sum is such that $\ell(u) \leq m - 3$. Alternatively,

$$\psi_{r_1} \cdots \psi_{r_m} 1_i = \psi_{t_1} \cdots \psi_{t_m} 1_i + (\ast),$$

where $(\ast)$ is an $\mathcal{O}$-linear combination of elements of the form $\psi_{r_1}^{c_1} \cdots \psi_{r_m}^{c_m} g(y)$, where $g(y) \in \mathcal{O}[y_1, \ldots, y_d]$, each $\epsilon_i \in \{0, 1\}$, $\epsilon_i = 0$ for at least three distinct $i$’s, and $s_{r_1}^{c_1} \cdots s_{r_m}^{c_m}$ is a reduced expression.

**Proof.** This is proved in [3 Lemma 2.4] and [3 Proposition 2.5], for the case of cyclotomic KLR algebras, but the cyclotomic relation is not used in the proof. \qed

**Theorem 3.4.** [6 Theorem 2.5], [16 Theorem 3.7] Let $\alpha \in Q_+$. Then

$$\{\psi_{w y_1 m_1} \cdots y_d m_d 1_i \mid w \in \mathcal{S}_d, m_1, \ldots, m_d \in \mathbb{Z}_{\geq 0}, i \in \langle I \rangle_\alpha\}$$

is an $\mathcal{O}$-basis for $R_\alpha$. 
3.3. **Representation theory of** $R_\alpha$. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Denote by $R_\alpha$-mod the abelian category of finite dimensional graded left $R_\alpha$-modules, with degree-preserving module homomorphisms. Let $[R_\alpha$-mod] denote the corresponding Grothendieck group. $[R_\alpha$-mod] is a $\mathcal{A}$-module via $q^m [M] := [M(m)]$, where $M(m)$ denotes the module obtained by shifting the grading up by $m$. The **graded dimension** of a module $M$ is

$$\dim_q M := \sum_{n \in \mathbb{Z}} (\dim V_n) q^n \in \mathcal{A}.$$  

Every irreducible graded $R_\alpha$-module is finite dimensional [6 Proposition 2.12], and there are finitely many irreducible $R_\alpha$-modules up to grading shift [6 §5.2]. For every irreducible module $L$ there is a unique choice of grading shift so that $L^\oplus \cong L$, and in particular, the grading of $L$ is symmetric about zero. When speaking of irreducible $R_\alpha$-modules we often assume by fiat that the shift has been chosen in this way.

For $i \in \langle I \rangle_\alpha$ and $M \in R_\alpha$-mod, the $i$-word space of $M$ is $M_i := 1_i M$. We say $i$ is a word of $M$ if $M_i \neq 0$. We have $M = \bigoplus_{i \in \langle I \rangle_\alpha} M_i$. The **character** of $M$ is

$$\text{ch}_q M := \sum_{i \in \langle I \rangle_\alpha} (\dim_q M_i) i \in \mathcal{A} \langle I \rangle_\alpha.$$  

The character map $\text{ch}_q : R_\alpha$-mod $\to \mathcal{A} \langle I \rangle_\alpha$ factors through to give an **injective** $\mathcal{A}$-linear map $\text{char}_q : [R_\alpha$-mod] $\to \mathcal{A} \langle I \rangle_\alpha$ by [6 Theorem 3.17].

3.4. **Induction and restriction.** Given $\alpha, \beta \in Q_+$, we set $R_{\alpha, \beta} := R_\alpha \otimes R_\beta$. Let $M \boxtimes N$ be the outer tensor product of the $R_\alpha$-module $M$ and the $R_\beta$-module $N$. There is an injective homogeneous (non-unital) algebra homomorphism $R_{\alpha, \beta} \hookrightarrow R_{\alpha + \beta}$ mapping $1_i \otimes 1_j$ to $1_{ij}$, where $ij$ is the concatenation of the two sequences. The image of the identity element of $R_{\alpha, \beta}$ under this map is $1_{\alpha, \beta} := \sum_{i \in \langle I \rangle_\alpha, j \in \langle I \rangle_\beta} 1_{ij}$.

Let $\text{Ind}_{\alpha, \beta}$ and $\text{Res}_{\alpha, \beta}$ be the corresponding induction and restriction functors:

$$\text{Ind}_{\alpha, \beta} := R_{\alpha + \beta} 1_{\alpha, \beta} \otimes R_{\alpha, \beta} ? : R_{\alpha, \beta}$-$mod $\to $ R_{\alpha + \beta}$-$mod,$

$$\text{Res}_{\alpha, \beta} := 1_{\alpha, \beta} R_{\alpha + \beta} \otimes R_{\alpha + \beta} ? : R_{\alpha + \beta}$-$mod $\to $ R_{\alpha, \beta}$-$mod.$

The functor $\text{Ind}_{\alpha, \beta}$ is left adjoint to $\text{Res}_{\alpha, \beta}$. Both functors are exact and send finite dimensional modules to finite dimensional modules. These functors have obvious generalizations to $n \geq 2$ factors:

$$\text{Ind}_{\gamma_1, \ldots, \gamma_n} : R_{\gamma_1, \ldots, \gamma_n}$-$mod $\to $ R_{\gamma_1 + \ldots + \gamma_n}$-$mod,$

$$\text{Res}_{\gamma_1, \ldots, \gamma_n} : R_{\gamma_1 + \ldots + \gamma_n}$-$mod $\to $ R_{\gamma_1, \ldots, \gamma_n}$-$mod.$

If $M_\alpha \in R_{\gamma_\alpha}$-mod, for $\alpha = 1, \ldots, n$, we define

$$M_1 \circ \cdots \circ M_n := \text{Ind}_{\gamma_1, \ldots, \gamma_n} M_1 \boxtimes \cdots \boxtimes M_n.$$

4. **Permutation modules**

In this section we recall the permutation modules defined in [9], generalizing to the case of skew diagrams.

4.1. **The permutation module** $M(\bar{s})$. For $i \in I$, $N \in \mathbb{Z}_{\geq 1}$, let $s = s(i, N) \in \langle I \rangle$ be the word $j_1 \cdots j_N$ with $j_r = i + r - 1 \pmod{e}$, i.e., the segment of length $N$ starting at $i$. If $\alpha = |s| \in Q_+$, define the graded $R_\alpha$-module $M(s) := \mathcal{O} \cdot m(s)$ to be the free $\mathcal{O}$-module of rank one on the generator $m(s)$ of degree 0, with action

$$1_i m(s) = \delta_{i, s} m(s), \quad \psi_r m(s) = 0, \quad y_r m(s) = 0$$

for all admissible $i$, $r$, and $t$.

If $\bar{s} = (s(1), \ldots, s(n))$ is an ordered tuple of such segments, with $\beta_i = |s(i)|$, define the **permutation module**

$$M(\bar{s}) := M(s(1)) \circ \cdots \circ M(s(n)).$$
over $R_\beta$, where $\beta = \beta_1 + \cdots + \beta_n$. Then $M(\tilde{s})$ is generated by the vector
\[ m(\tilde{s}) := 1 \otimes m(s(1)) \otimes \cdots \otimes m(s(n)) \]
Let $\lambda_i := \text{ht}(\beta_i)$. Considering $(\lambda_1, \ldots, \lambda_n)$ as a composition of $d$, let $G_\lambda$ be the parabolic subgroup $G_{\lambda_1} \times \cdots \times G_{\lambda_n}$ of $G_d$, where $d = \sum_{i=1}^{n} \lambda_i$. Consider the concatenation $j(\tilde{s}) := s(1) \cdots s(n) \in \langle I \rangle_{\beta}$. Let $K(\tilde{s})$ be the left ideal of $R_\beta$ generated by
\[ \{1_i - \delta_{i,j}(\tilde{s}), \psi_i \mid i \in \langle I \rangle_{\beta}, 1 \leq r \leq d, 1 \leq t \leq d - 1 \text{ such that } s_t \in G_\lambda \} \]
Then we have $M(\tilde{s}) \cong R_\alpha/K(\tilde{s})$. As a consequence of Theorem 3.3, we have the following

**Theorem 4.1.** Let $D_\alpha$ be the set of the shortest length left coset representatives of $G_\lambda$ in $G_d$. Then $\{\psi_w m(\tilde{s}) \mid w \in D_\alpha\}$ is an $O$-basis of $M(\tilde{s})$. Moreover each basis element is homogeneous of degree equal to the degree of the element $\psi_w 1_j(\tilde{s}) \in R_\beta$, and $\psi_w m(\tilde{s}) \in 1_{w^{-1}j}(\tilde{s}) M(\tilde{s})$.

**4.2. The row permutation module** $M^{\lambda/\mu}$. Let $\lambda/\mu \in \mathcal{S}_d^{\pm,\alpha}$ with nonempty rows $R_1, \ldots, R_g$ counted from top to bottom. If a row $R_i$ has length $N$ and the leftmost node of $R_i$ has residue $i$, we associate to the segment $r(a) := s(i,N)$ to $R_i$. Let $\tilde{r} = (r(1), \ldots, r(g))$, and put
\[ M^{\lambda/\mu} = M^{\lambda/\mu}(O) := M(\tilde{r})(\deg t^{\lambda/\mu}). \]
The module $M^{\lambda/\mu}$ is generated by the vector $m^{\lambda/\mu} := m(\tilde{r})$ of degree $\deg t^{\lambda/\mu}$. For any $\lambda/\mu$-tableau $t$ we define
\[ m^t := \psi^t m^{\lambda/\mu}. \]
As a special case of Theorem 4.1, we have

**Theorem 4.2.** Suppose that $\alpha \in Q_+$ and $\lambda/\mu \in \mathcal{S}_d^{\pm,\alpha}$. Then
\[ \{m^t \mid t \text{ is a row-strict } \lambda/\mu\text{-tableau}\} \]
is an $O$-basis of $M^{\lambda/\mu}$.

The following lemma will be useful in manipulating elements of $M^{\lambda/\mu}$.

**Lemma 4.3.** Let $t \in \text{Tab}(\lambda/\mu)$. Let $s_{r_1} \cdots s_{r_m}$ be a not necessarily reduced subword (resp. proper subword) of $t$ for a reduced expression for $w^t$. Then $\psi_{r_1} \cdots \psi_{r_m} m^{\lambda/\mu}$ is an $O$-linear combination of terms of the form $m^s$, where $s$ is a row-strict tableau and $s \preceq t$ (resp. $s \preceq u \preceq t$).

**Proof.** We go by induction on the Bruhat order on tableaux. The base case $t = t^{\lambda/\mu}$ is trivial, as $t$ is row-strict. For the induction step, take $t \in \text{Tab}(\lambda/\mu)$ and assume the claim holds for all $u \preceq t$. First assume that the subword $s_{r_1} \cdots s_{r_m} =: w^a \in G_d$ is reduced and proper. Then, using Lemma 3.3, we have
\[ \psi_{r_1} \cdots \psi_{r_m} m^{\lambda/\mu} = \psi^a m^{\lambda/\mu} + \sum_{u \preceq w^a} c_u \psi_u m^{\lambda/\mu} \]
for some constants $c_u \in O$. But $vt^{\lambda/\mu} \preceq u \preceq t$, for all $v$ in the sum on the right, so by the induction assumption $\psi^a m^{\lambda/\mu} = m^a$ and $\psi_v m^{\lambda/\mu} = m^{vt^{\lambda/\mu}}$ are $O$-linear combinations of terms of the form $m^s$, where $s$ is a row-strict tableau and $s \preceq u \preceq t$.

Now assume that the subword $s_{r_1} \cdots s_{r_m} =: w^a \in G_d$ is unreduced (and proper). Then by Lemma 3.3 we have that $\psi_{r_1} \cdots \psi_{r_m} m^{\lambda/\mu}$ is an $O$-linear combination of terms of the form $\psi_{s_{r_{t_1}}} \cdots \psi_{s_{r_{t_m}}} m^{\lambda/\mu}$, where $s_{r_{t_1}} \cdots s_{r_{t_m}}$ is a reduced proper subword of $s_{r_1} \cdots s_{r_m}$, and thus a reduced proper subword of $w^t$, reducing to the previous case.

Finally, assume $s_{r_1} \cdots s_{r_m} = w^a$. If $t$ is row-strict, we are done, so assume otherwise. Write $w^t = wv$, where $w \in D_\alpha$ and $id \neq v \in G_\lambda$. Then using Lemma 3.3,
\[ \psi_{r_1} \cdots \psi_{r_m} m^{\lambda/\mu} = \psi_w \psi_v m^{\lambda/\mu} + \sum_{u \preceq v^a} c_u \psi_u m^{\lambda/\mu}. \]
5. Skew Specht modules

In this section we define the graded skew Specht module $S_{\lambda/\mu}^A$.

5.1. Garnir skew tableaux. Let $A = (a, b, m)$ be a node of $\lambda/\mu \in \mathcal{S}$. We say $A$ is a Garnir node if $(a + 1, b, m)$ is also a node of $\lambda/\mu$. The $A$-Garnir belt $B^A$ is the set of nodes

$$B^A = \{(a, c, m) \in \lambda/\mu \mid c \geq b\} \cup \{(a + 1, c, m) \in \lambda/\mu \mid c \leq b\}.$$

For example, if $l = 1, A = (2, 6, 1), \lambda = (11, 10, 7, 2, 2),$ and $\mu = (7, 4, 3, 1)$, then $B^A$ is highlighted below:

```
\begin{center}
\begin{tabular}{cccc}
\hline
\hline
\hline
\end{tabular}
\end{center}
```

The $A$-Garnir tableau is the $\lambda/\mu$-tableau $g^A$ that is equal to $t^{\lambda/\mu}$ outside the Garnir belt, and with numbers $t^{\lambda/\mu}(a, b, m)$ through $t^{\lambda/\mu}(a + 1, b, m)$ inserted into the Garnir belt, in order from bottom left to top right. Continuing the example, we have:

$$g^A = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 9 & 10 & 11 & 12 & 13 \\
6 & 7 & 8 & 14 \\
15 & 16 & 17
\end{array}$$

Lemma 5.1. Suppose that $\lambda/\mu \in \mathcal{S}$, $A$ is a Garnir node of $\lambda/\mu$, and $t \in \text{Tab}(\lambda/\mu)$. If $t \equiv g^A$, then $t$ agrees with $t^{\lambda/\mu}$ outside the $A$-Garnir belt.

Proof. Since $w^A$ fixes the entries outside the Garnir belt, $w^A \equiv w^A$ must do the same. □

Lemma 5.2. Suppose that $\lambda/\mu \in \mathcal{S}$ and $t$ is a row-strict nonstandard $\lambda/\mu$-tableau. Then there exists a Garnir tableau $g^A$, for some $A \in \lambda/\mu$, and $w \in \mathcal{S}$ such that $wg^A = t$ and $\ell(w^A) = \ell(w^A) + \ell(w)$.

Proof. This is proved in [12] Lemma 3.13] in the context of Young tableaux, but the proof is equally valid in the case of skew tableaux. We present it here for convenience.

Let $t \in \text{Tab}(\lambda/\mu)$ be row-strict, and $A = (a, b, m)$ a node such that $t(a, b, m) > t(a + 1, b, m)$.

We first claim that either $t = g^A$, or $t' := st < t$ for some $s = (n, n + 1) \in \mathcal{S}$, with $t'(a, b, m) > t'(a + 1, b, m)$, and $t'$ row-strict. Let $X$ be the set of nodes earlier than $(a, b, m)$, and let $Y$ be the set of nodes later than $(a + 1, b, m)$. Let $(c, d)$ be the earliest node in $X$ such that $t(c, d, m') \neq t^{\lambda/\mu}(c, d, m')$, if one exists. Then $n := t(c, d, m') > t^{\lambda/\mu}(c, d, m')$, and $n - 1$ is in a later node, so $s = (n - 1, n)$ satisfies the claim. If no such node exists in $X$, let $(c, d, m')$ be the latest node in $Y$ such that $t(c, d, m') \neq t^{\lambda/\mu}(c, d, m')$, if one exists. Then $n := t(c, d, m') < t^{\lambda/\mu}(c, d, m')$, and $n + 1$ is in an earlier node, so $s = (n, n + 1)$ satisfies the claim. If no such node exists in $Y$, then $t$ agrees with $t^{\lambda/\mu}$ outside $B^A$. But then since $t$ is row-strict and nonstandard, it follows that $t = g^A$, and the claim holds.

The claim shows that we may induct on the dominance order, taking the base case to be $g^A$, in order to prove the lemma. For the base case we just take $w = \text{id}$. For the induction step, let $t$ be row-strict, with $t(a, b, m) > t(a + 1, b, m)$. By the above, there exists some...
s = (n, n + 1) ∈ S_d such that t' := st < t, with t'(a, b, m) > t'(a + 1, b, m) and t' is row-strict. By the induction assumption, we have some w' ∈ S_d such that w'C = t', with ℓ(w') = ℓ(wC) + ℓ(w'). Then, setting w := sw', we have wg = t, and

\[ ℓ(w) = ℓ(w') + 1 = ℓ(wC) + ℓ(w') + 1 = ℓ(wC) + ℓ(w), \]

completing the proof. □

5.2. Bricks. Take λ/μ ∈ S_c and Garnir node A = (a, b, m) ∈ λ/μ. A brick is a set of nodes

\[ \{(c, d, m'), (c, d + 1, m'), \ldots, (c, d + e - 1, m')\} \subseteq B^A \]

such that res(c, d, m') = res(A). Then B^A is a disjoint union of bricks, together with less than e nodes at the beginning of row a + 1 and less than e nodes at the end of row a. Let k^A be the number of bricks in B^A, and let f^A to be the number of bricks in row a of B^A. Label the bricks B_1^A, B_2^A, \ldots, B_k^A in order, beginning at the bottom left.

Going back to our example with l = 1, A = (2, 6, 1), λ = (11, 10, 7, 2, 2), and μ = (7, 4, 3, 1), if e = 2, the bricks in B^A are shown below:

Now assume that k^A > 0 and let n^A be the smallest entry in g^A contained in a brick. In the example above, n^A = 7. Define

\[ w_r^A = \prod_{a=n^A+r-1}^{n^A+re-1} (a, a + e) \in S_d \quad (1 \leq r \leq k^A) \]

Informally, w_r^A swaps the bricks B_r^A and B_{r+1}^A. The elements w_1^A, \ldots, w_{k-1}^A are Coxeter generators of the symmetric group

G^A := \langle w_1^A, \ldots, w_{k-1}^A \rangle \cong S_k

We call S_d the brick permutation group. By convention, G^A is the trivial group if k^A = 0.

Let Gar^A be the set of row-strict λ/μ-tableaux which are are obtained by acting on G^A by brick permutations. Thus all tableaux in Gar^A save g^A are standard. By Lemma 2.5, g^A is the unique maximal element of Gar^A, and there exists a unique minimal element t^A, which has the bricks B_1^A, \ldots, B_{f^A}^A in order from left to right in row a, and the remaining bricks B_{f^A+1}^A, \ldots, B_{k^A}^A in order from left to right in row a + 1. By definition, if t ∈ Gar^A, then \( i(t) = i(g^A) \). Define \( i^A \) as this common residue sequence.

Let D^A be the set of minimal length left coset representatives of S_f × S_{k-f} in G^A. By definition G^A is a subgroup of S_d, so its elements act on λ/μ-tableaux. Thus we have

Gar^A = \{ wt^A | w ∈ D^A \}.

Continuing the example above, the elements of Gar^A are g^A = w_1^A w_2^A t^A (shown above), and the following two tableaux:

\[ t^A = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 7 & 8 & 9 \\ 6 & 11 & 12 & 14 \\ 15 & 16 & 17 \end{array}, \quad w_2^A t^A = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 7 & 8 & 11 \\ 6 & 9 & 10 & 14 \\ 15 & 16 & 17 \end{array} \]
Taking \( \kappa = (1) \), the residues of nodes in \( \lambda/\mu \) are as follows:

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
\end{array}
\]

In diagrammatic notation,

\[
\psi^{t_A} 1_{\lambda/\mu} = \ldots
\]

\[
\psi^{w_1^{i_A}} 1_{\lambda/\mu} = \ldots
\]

\[
\psi^{g_A} 1_{\lambda/\mu} = \ldots
\]

The colors in the diagram are intended to highlight the action on bricks in \( B^A \), and beyond this have no mathematical meaning.

**Lemma 5.3.** Suppose that \( \lambda/\mu \in \mathcal{F}^\kappa \) and \( A \in \lambda/\mu \) is a Garnir node. Then

\[
\text{Gar}^A \{ g_A \} = \{ t \in \text{St}(\lambda/\mu) \mid t \equiv g_A \text{ and } i(t) = i^A \}.
\]

Moreover, \( \text{deg} t = \text{deg} s_i g_A - a_{i_r,i_{r+1}} \) for all \( t \in \text{Gar}^A \{ g_A \} \), where \( r = g_A(A) - 1 \).

**Proof.** The first statement is clear from the preceding discussion and Lemma 5.1.

For the second statement, we instead prove that \( \text{codeg}(s_i g_A) = \text{codeg}(t) = -a_{i_r,i_{r+1}} \), which is equivalent by [3, Lemma 3.12]. By the definition of codegree, and the fact that \( Y(t) \) and \( Y(s_i g_A) \) agree outside of the bricks of the Garnir belt, it is enough to consider the case where \( \lambda/\mu \) is a two-row Young diagram, with \( \lambda = \lambda/\mu = ((k^A e - 1, (k^A - f^A)e)) \), and \( A = (1, (k^A - f^A)e, 1) \). Pictorially,

\[
\lambda = \ldots
\]

Each brick contributes 0 to \( \text{codeg}(t) \), and every brick contributes 0 to \( \text{codeg}(s_i g_A) \), except for \( B_{k^A-f^A} \) (the rightmost brick in the bottom row), which contributes 2 if \( e = 2 \) and 1 if \( e > 2 \). Thus \( \text{codeg}(t) = 0 \) and \( \text{codeg}(s_i g_A) = -a_{i_r,i_{r+1}} \), and the result follows. \( \square \)

### 5.3. The skew Specht module \( S^{\lambda/\mu} \)

Fix a Garnir node \( A \in \lambda/\mu \). Define

\[
\sigma^A := \psi_{w_A^1} 1_{i_A} \quad \text{and} \quad \tau^A := (\sigma^A + 1) 1_{i_A}.
\]

Write \( u \in \mathcal{G}^A \) as a reduced product \( w_{r_1}^A \cdots w_{r_n}^A \) of simple generators in \( \mathcal{G}^A \). If \( u \in \mathcal{G}^A \), then \( u \) is fully commutative, and thus we have well-defined elements

\[
\{ \tau^A_u := \tau_{r_1}^A \cdots \tau_{r_n}^A \mid u \in \mathcal{G}^A \}.
\]

Then for any \( s \in \text{Gar}^A \), we may can write \( w^u = u^s w^{t_A} \) so that \( \ell(w^u) = \ell(u^s) + \ell(w^{t_A}) \) and \( u^s \in \mathcal{G}^A \), and the elements \( \psi_{w^u}, \psi^{t_A} \) and \( \psi^u = \psi_{w^u} \psi^{t_A} \) are all independent of the choice of
Definition 5.4. Let $\lambda/\mu \in \mathcal{P}_{\mu,\alpha}$, and $A \in \lambda/\mu$ be a Garnir node. The Garnir element is

$$g^A := \sum_{u \in \mathcal{D}^A} \tau_u^A \psi^A \in R_{\alpha}.$$  

In the module $M^{\lambda/\mu}$ we have

$$g^A m^{\lambda/\mu} = \sum_{u \in \mathcal{D}^A} \tau_u^A m^A.$$  

By Lemma 5.3 all summands on the right side of (5.2) are of the same degree.

Definition 5.5. Let $\lambda/\mu \in \mathcal{P}_{\mu,\alpha}$, and $A \in \lambda/\mu$ be a Garnir node. The Garnir element is

$$m^A := m^A = \psi^A m^{\lambda/\mu} \in M^{\lambda/\mu}.$$  

Definition 5.6. Let $\alpha \in Q_+$, $d = \text{ht}(\alpha)$, and $\lambda/\mu \in \mathcal{P}_{\mu,\alpha}$. Define the graded Specht module $S^{\lambda/\mu} = S^{\lambda/\mu}(O)$ to be the graded $R_{\alpha}$-module generated by the vector $z^{\lambda/\mu}$ in degree $\deg t^{\lambda/\mu}$ and subject only to the following relations:

(i) $1_j z^{\lambda/\mu} = \delta_{j,\lambda/\mu} z^{\lambda/\mu}$ for all $j \in \langle I \rangle_{\alpha}$;
(ii) $y_r z^{\lambda/\mu} = 0$ for all $r = 1, \ldots, d$;
(iii) $\psi_r z^{\lambda/\mu} = 0$ for all $r = 1, \ldots, d - 1$ such that $r \rightarrow_{\lambda/\mu} r + 1$;
(iv) $g^A z^{\lambda/\mu} = 0$ for all Garnir nodes $A \in \lambda/\mu$.

In other words, $S^{\lambda/\mu} = (R_{\alpha}/J_{\alpha}^{\lambda/\mu})(\deg(t^{\lambda/\mu}))$, where $J_{\alpha}^{\lambda/\mu}$ is the homogeneous left ideal of $R_{\alpha}$ generated by the elements

(i*) $1_j - \delta_{j,\lambda/\mu}$ for all $j \in \langle I \rangle_{\alpha}$;
(ii*) $y_r$ for all $r = 1, \ldots, d$;
(iii*) $\psi_r$ for all $r = 1, \ldots, d - 1$ such that $r \rightarrow_{\lambda/\mu} r + 1$;
(iv*) $g^A$ for all Garnir nodes $A \in \lambda/\mu$.

The elements (i*)-(iii*) generate a left ideal $K^{\lambda/\mu}$ such that $R_{\alpha}/K^{\lambda/\mu} \cong M^{\lambda/\mu}$. So we have a natural surjection $M^{\lambda/\mu} \rightarrow S^{\lambda/\mu}$ with kernel $J^{\lambda/\mu}$ generated by the Garnir relations $g^A m^{\lambda/\mu} = 0$. This surjection maps $m^{\lambda/\mu}$ to $z^{\lambda/\mu}$ and $J^{\lambda/\mu} = J_{\alpha}^{\lambda/\mu} m^{\lambda/\mu}$.

Remark 5.7. Let $g^A$ be the only tableau in $\text{Gar}^A$, e.g., when $e = 0$ or $e > d$, the Garnir relation simply says that $\psi^A z^{\lambda/\mu} = 0$.

5.4. A spanning set for $S^{\lambda/\mu}$. Let $\alpha \in Q_+$ with $\text{ht}(\alpha) = d$, and $\lambda/\mu \in \mathcal{P}_{\mu,\alpha}$. Recall that $S^{\lambda/\mu} \cong M^{\lambda/\mu}/J^{\lambda/\mu}$ and $z^{\lambda/\mu} = m^{\lambda/\mu} + J^{\lambda/\mu}$. Set

$$z^A := m^A + J^{\lambda/\mu} \in S^{\lambda/\mu}$$

for any Garnir node $A \in \lambda/\mu$. Associate to every $t \in \text{Tab}(\lambda/\mu)$ the homogeneous element

$$v^t := \psi^A z^{\lambda/\mu} = m^t + J^{\lambda/\mu}$$

Define the brick permutation space $T^{\lambda/\mu} \subseteq M^{\lambda/\mu}$ to be the $O$-span of all elements of the form $\sigma_{e^1} \cdots \sigma_{e^d} m^A$.

The following two lemmas are proved in [9, Corollary 5.12, Lemma 5.13] in the context of Young diagrams, but the proofs are valid in our more general context. The results rely on [9, Theorem 5.11], which needs no alteration for application to the skew diagram case.

Lemma 5.8. Suppose that $\lambda/\mu \in \mathcal{P}_{\mu,\alpha}$, $A$ is a Garnir node of $\lambda/\mu$, and $s = ut^A \in \text{Gar}^A$ for some $u \in \mathcal{D}^A$. Then

$$m^s = \tau_u^A m^A + \sum_{u \in \mathcal{D}^A, u \neq A} c_u \tau_u^A m^A$$

for some constants $c_u \in O$. In particular, $\{v^t \mid t \in \text{Gar}^A\}$ is an $O$-basis of $T^{\lambda/\mu} \cdot A$. 
Lemma 5.8. Suppose that $\lambda/\mu \in \mathcal{P}^\kappa_{\mu,\alpha}$ and $A$ is a Garnir node of $\lambda/\mu$. Then
\[ v^A = \sum_{t \in \text{Gar}^A, t \not\equiv g^A} c_t v^t \]
for some constants $c_t \in \mathcal{O}$.

Lemma 5.9. If $\lambda/\mu \in \mathcal{P}^\kappa_{\mu,\alpha}$, and $t \in \text{Tab}(\lambda/\mu)$ is row-strict, then $v^t$ is an $\mathcal{O}$-linear combination of terms of the form $v^s$, where $s$ is standard and $s \preceq t$.

Proof. We argue by induction. The base case $v^{\lambda/\mu}$ is clear as $t^{\lambda/\mu}$ is standard. For the induction step, assume $t$ is row-strict and the result holds for all $s \prec t$. If $t$ is standard, we’re done. Otherwise by Lemma 5.2, there is a Garnir tableau $g$ and $w \in \mathfrak{S}_d$ such that $t = wg$ and $\ell(w^t) = \ell(w) + \ell(w^g)$. Then by Proposition 3.3,
\[ v^t = \psi^t z^{\lambda/\mu} = \psi^t z^{\lambda/\mu} + \sum_{u \leq w^t} c_u \psi^u z^{\lambda/\mu}, \]
for some $c_u \in \mathcal{O}$. By Lemma 4.3 the terms in the sum on the right can be written as an $\mathcal{O}$-linear combination of terms of the form $v^u$, where $u$ is row-strict and $u \preceq w^{t^{\lambda/\mu}} \prec t$. Each $v^u$ can in turn be written as an $\mathcal{O}$-linear combination of terms of the form $v^s$, where $s$ is standard, and $s \preceq u \prec t$, by the induction assumption.

It remains to consider $\psi^t z^{\lambda/\mu}$. By Lemma 5.8 we see that
\[ \psi^t z^{\lambda/\mu} = \psi^t z^{\lambda/\mu} = \sum_{u \in \text{Gar}^t, u \not\equiv g^t} c_u \psi^u z^{\lambda/\mu}, \]
for some constants $c_u \in \mathcal{O}$. Take any $u$ appearing in the sum. If $w = s_1 \cdots s_r$ and $w^u = s_{r+1} \cdots s_r$ are the preferred reduced words for $w$ and $w^u$, then $s_1, \ldots, s_r$ is a subword of a reduced expression for $wu^g = w^t$, since $w^u \prec w^g$. But then by Lemma 4.3, $\psi^t z^{\lambda/\mu}$ is an $\mathcal{O}$-linear combination of terms of the form $v^s$, where $s$ is row-strict and $s \preceq g^t \prec t$, and each $v^s$ is an $\mathcal{O}$-linear combination of terms of the form $v^x$, for $x \preceq s \prec t$, by the induction assumption.

Lemmas 4.8 and 5.9 give us

Corollary 5.10. Let $\lambda/\mu \in \mathcal{P}^\kappa_{\mu,\alpha}$, and $t \in \text{Tab}(\lambda/\mu)$. Let $s_1 \cdots s_r$ be a (not necessarily reduced) subword (resp. proper subword) of a reduced expression for $w$. Then $\psi^{s_1} \cdots \psi^{s_r} z^{\lambda/\mu}$ is an $\mathcal{O}$-linear combination of terms of the form $v^s$, where $s$ is a standard tableau and $s \preceq t$ (resp. $s \preceq t$).

Proposition 5.11. The elements of the set
\[ \{v^t \mid t \in \text{St}(\lambda/\mu)\} \]
span $S^{\lambda/\mu}$ over $\mathcal{O}$. Moreover, we have $\deg(v^t) = \deg(t)$ for all $t \in \text{St}(\lambda/\mu)$.

Proof. Using Lemma 5.2,
\[ \deg(v^t) = \deg(\psi^t 1_{\lambda/\mu} z^{\lambda/\mu}) = \deg(\psi^t 1_{\lambda/\mu}) + \deg(z^{\lambda/\mu}) = \deg(t) - \deg(t^{\lambda/\mu}) + \deg(z^{\lambda/\mu}), \]
which proves the second statement, as $\deg(z^{\lambda/\mu}) = \deg(t^{\lambda/\mu})$ by definition. For the first statement, we apply Corollary 5.10 to see that for any $s \in \text{Tab}(\lambda/\mu)$, $v^s$ can be written as an $\mathcal{O}$-linear combination of elements in (5.3).

6. Connecting skew Specht modules with restrictions of Specht modules

In this section we show that for $\lambda \in \mathcal{P}^\kappa_{\alpha,\beta}$, the $R_{\alpha,\beta}$-module $\text{Res}_{\alpha,\beta} S^\lambda$ has a filtration with subquotients isomorphic to $S^\mu \boxtimes S^{\lambda/\mu}$, where $\mu \in \mathcal{P}^\kappa_{\alpha}$ and $\lambda/\mu \in \mathcal{P}^\kappa_{\mu,\beta}$. As a consequence, we get that (5.3) is an $\mathcal{O}$-basis for $S^{\lambda/\mu}$. For the case of Young diagrams, this was shown in [20 Corollary 6.24]:

Theorem 6.1. Let $\lambda \in \mathcal{P}^\kappa_{\alpha}$. Then $S^\lambda$ has $\mathcal{O}$-basis $\{v^T \mid T \in \text{St}(\lambda)\}$. 

6.1. Submodules of $\text{Res}_{\alpha,\beta} S^\lambda$. Let $\alpha, \beta \in \mathbb{Q}^+$ and $\text{ht}(\alpha) = a, \text{ht}(\beta) = b$. Let $\lambda \in \mathcal{P}_{\alpha + \beta}$. By Theorem 6.1, $S^\lambda_{\alpha, \beta} := \text{Res}_{\alpha, \beta}(S^\lambda)$ has $\mathcal{O}$-basis $\{ v^T | T \in B \}$, where

$$B = \{ T \in \text{St}(\lambda) | \text{cont}(\text{sh}(T_{\leq a})) = \alpha \},$$

since $1_{\alpha, \beta} v^T = v^T$ if and only if $i(T) = i_1 \cdots i_{a+b}$ has $\alpha_1 + \cdots + \alpha_i = \alpha$, and is zero otherwise. Define

$$B_\mu = \{ T \in B | \text{sh}(T_{\leq a}) \supseteq \mu \} \quad \text{and} \quad C_\mu = \mathcal{O}\{ v^T \in S^\lambda | T \in B_\mu \}.$$

Lemma 6.2. If $\mathcal{U}, T \in B$, $\mathcal{U} \approx T$, and $T \in B_\mu$, then $\mathcal{U} \in B_\mu$.

Proof. If $\mathcal{U} \nsupseteq T$, then by Lemma 2.5, $\text{sh}(U_{\leq a}) \nsupseteq \text{sh}(T_{\leq a}) \nsupseteq \mu$. \hfill $\square$

Lemma 6.3. $C_\mu$ is an $R_{\alpha, \beta}$-submodule of $S^\lambda_{\alpha, \beta}$.

Proof. We show that $C_\mu$ is invariant under the action of generators of $R_{\alpha, \beta}$. For idempotents $1_{ij}$ this is clear. Let $T \in B_\mu$.

1. For $1 \leq j \leq a+b$, $y_j v^T$ is an $\mathcal{O}$-linear combination of $v^T \in B$ for $\mathcal{U} \approx T$, by [3, Lemma 4.8]. By Lemma 6.2, each $v^T$ is in $C_\mu$.

2. For $j \in \{ 1, \ldots, a-1, a+1, \ldots, a+b \}$, where $j \rightarrow j+1$ or $j \leftarrow j+1$, then $y_j v^T$ is a linear combination of $v^T \in B$ for $\mathcal{U} \approx T$, by [3, Lemma 4.9], and the result follows by Lemma 6.2.

3. For $j \in \{ 1, \ldots, a-1, a+1, \ldots, a+b-1 \}$, where $j + 1 \not\approx j$, then $y_j v^T$ is a linear combination of $v^T \in B$ for $\mathcal{U} \approx T$, by [2, Lemma 2.14], and the result follows by Lemma 6.2.

4. Assume $j \in \{ 1, \ldots, a-1, a+1, \ldots, a+b-1 \}$, and $j \not\approx j+1$. Then $s_j T \approx T$, and $s_j v^T = w^{s_j} v^T$, with $\ell(w^{s_j}) = \ell(v^T) + 1$. Then by Lemma 3.3, $y_j v^T = v^T + \sum_{0 \leq s \leq j} c_s v^T$ for some constants $c_s \in \mathcal{O}$. But $(s_j T)_{\leq a} = T_{\leq a}$, so $s_j T \in B_\mu$ and the result follows by Lemma 6.2.

This exhausts the possibilities for $T$ and completes the proof. \hfill $\square$

Now define $B_{\neq \mu} = \bigcup_{\nu \supseteq \mu} B_{\nu} = \{ T \in B | \text{sh}(T_{\leq a}) \supsetneq \mu \}$. Then $C_{\neq \mu} := \sum_{\nu \supsetneq \mu} C_{\nu} = \mathcal{O}\{ v^T \in S^\lambda | \text{sh}(T_{\leq a}) \supsetneq \mu \}$ is an $R_{\alpha, \beta}$-submodule of $S^\lambda_{\alpha, \beta}$. Define $N_\mu = C_\mu / C_{\neq \mu}$, and write

$$x^T = v^T + C_{\neq \mu} \in S^\lambda_{\alpha, \beta} / C_{\neq \mu},$$

for $T \in B$. To cut down on notational clutter in what follows, write $\xi$ for $\lambda / \mu$, $\xi(\nu)$ for the components $\lambda(\nu) / \mu(\nu)$ of $\lambda / \mu$, and $\xi(j)$ for the $j$th row of nodes in $\xi$ for $j \not\approx j$. Then for $T \in \text{Tab}(\mu)$, $\tau \in \text{Tab}(\xi)$, define $T \tau \in \text{Tab}(\lambda)$ such that $(T \tau)_{\leq a} = T$ and $T \tau(A) = \lambda(\tau)(A)$ for nodes $A \in \xi$. From the definition it is clear that $N_\mu$ has homogeneous $\mathcal{O}$-basis

$$\{ x^T | T \in \text{St}(\xi), \text{sh}(\text{cont}(T_{\leq a})) = \mu \} = \{ x^T | T \in \text{St}(\mu), \tau \in \text{St}(\xi) \}.$$ 

Write $T^\mu \xi := T^\mu \xi = Y(\tau \xi)$, and write $x^\mu \xi$ for $x^T \xi$.

6.2. Constructing a morphism $S^\mu \boxtimes S^{\lambda(\mu)} \rightarrow N_\mu$. Define a (graded) morphism $f$ from the free module $R_{\alpha, \beta}(\deg T^\mu + \deg \xi)$ to $N_\mu$ by $f : 1_{ij} \rightarrow x^\mu \xi$.

Proposition 6.4. The kernel of $f$ contains the left ideal $K^\mu_{\alpha} \otimes R_\beta + R_{\alpha} \otimes K^\xi_{\beta}$.

Proof. We show that the relevant generators of $K^\mu_{\alpha} \otimes R_\beta$, given by (i*)–(iii*) in Definition 5.5 are sent to zero by $f$. The proof for $R_{\alpha} \otimes K^\xi_{\beta}$ is similar.

(i*) First we consider idempotents.

$$f[(1_j - \delta_j, i^\nu) \otimes 1_b] = (1_j - \delta_j, i^\nu) x^\mu \xi = \sum_{k \in I^b} 1_{jk} x^\mu \xi - \delta_j, i^\nu x^\mu \xi = \sum_{k \in I^b} \delta_{jk}, i^\nu x^\mu \xi - \delta_j, i^\nu x^\mu \xi = \delta_{j}, i^\nu x^\mu \xi - \delta_j, i^\nu x^\mu \xi = 0.$$
(ii*) For $1 \leq r \leq a$, we have by [3] Lemma 4.8 that $f(y_r) = y_r \cdot x^{\mu \xi}$ is an $O$-linear combination of $x^U$, where $U \in B$ and $U \triangleright T^{\mu \xi}$. But $T^{\mu \xi}$ is minimal such that $sh(T_{\leq a}) = \mu$, so each $U \in B_{\geq a}$, and thus $f(y_r) = 0.$

(iii*) Note that $r \rightarrow r + 1$ implies $r \rightarrow w_r \cdot r + 1$, so by [3] Lemma 4.9 it follows that for $1 \leq r \leq a - 1$, $f(y_r) = y_r \cdot x^{\mu \xi}$ is an $O$-linear combination of $x^U$, where $U \in B$ and $U \triangleleft T^{\mu \xi}$. But then as in (2) this implies that $f(y_r) = 0.$

\[\square\]

The goal in the rest of this section is to show that in fact, the kernel of $f$ contains the the left ideal $J_\mu \circ R_\beta + R_\alpha \circ j^{\xi}$, i.e., $g^{\lambda \mu} \circ 1_\beta$ (resp. $1_\alpha \circ g^{A \xi}$) are sent to zero by $f$, for Garnir nodes $A_\mu \in \mu$ (resp. $A_\xi \in \xi$). As the proofs for $\mu$ and $\xi$ are similar (see Remark 6.10), we focus on the former and leave the latter for the reader to verify. We will occasionally need to make use of the following lemma, proved in [4] Lemma 2.16:

**Lemma 6.5.** Suppose $\lambda \in \mathcal{P}^\alpha$, $\tau \in \text{St}(\lambda)$, $j_1, \ldots, j_r \in \{1, \ldots, d - 1\}$, and that when $\psi_{j_1} \cdots \psi_{j_r} z^\lambda$ is expressed as a linear combination of standard basis elements, $v^\tau$ appears with non-zero coefficient. Then the expression $s_{j_1} \cdots s_{j_r}$ has a reduced expression for $w^\tau$ as a subexpression.

Note that $w^{\mu \xi} = w^{\mu \xi}$ is in $\mathcal{S}_{a,b} \setminus \mathcal{S}_{a+b}/\mathcal{S}_{\mu_1(1) \xi_1(1)} \times \mathcal{S}_{\mu_2(1) \xi_2(1)} \times \cdots \times \mathcal{S}_{\mu_n(\lambda, l) \xi_n(\lambda, l)}$, and as such is fully commutative. Writing $n := n(\lambda, l)$, in diagrammatic form we have

\[
w^{\mu \xi} = \begin{array}{c}
\mu_1(1) \\
\xi_1(1) \\
\mu_2(1) \\
\xi_2(1) \\
\vdots \\
\mu_{n-1}(1) \\
\xi_{n-1}(1) \\
\mu_n(1) \\
\xi_n(1)
\end{array}
\]

Here we are letting $\mu_i(1)$ in the diagram stand for $(a_1, \ldots, a_k)$, where $a_1, \ldots, a_k$ are the entries (in order) in $T^\lambda$ of the nodes contained in the ith row of $\mu_i(1)$, and similarly for $\xi_i(1)$.

Let $1 \leq i \leq l$, $1 \leq j \leq n(\lambda, l)$. It will be useful to write $w^{\mu \xi} = w_{i,j}^{L \xi_i}, w_{i,j}^{R \xi_i}, w_{i,j}^{D \xi_i}$, the decomposition into fully commutative elements of $\mathcal{S}_{a+b}$ given as follows:

\[
w^{\mu \xi} = \begin{array}{c}
w_{i,j}^{L \xi_i} \\
w_{i,j}^{R \xi_i} \\
w_{i,j}^{D \xi_i}
\end{array}
\]
Define \( \psi_{i,j}^X := \psi_{i,j}^X \) for \( X \in \{L, R, D\} \), and set
\[
c_i,j = \sum_{1 \leq h \leq j-1} \mu_k^{(h)} + \sum_{1 \leq k \leq j} \mu_k^{(i)},
\]
\[
d_i,j = \sum_{1 \leq h \leq j-1} \xi_k^{(h)} + \sum_{1 \leq k \leq j-1} \xi_k^{(i)}.
\]

If \( \Psi := \psi_{r_1} \cdots \psi_{r_s} \) for some \( r_1, \ldots, r_s \), then we will write \( \Psi[c] := \psi_{r_1+c} \cdots \psi_{r_s+c} \) for admissible \( c \in \mathbb{Z} \). The following lemma will aid us in translating between Garnir relations defining \( S^\lambda \) and those defining \( S^\mu \).

**Lemma 6.6.** Assume \( r_1, \ldots, r_s \) are such that \( c_i,j+1 \leq r_1, \ldots, r_s \leq a-1 \), and \( \Psi = \psi_{r_1} \cdots \psi_{r_s} \).

Then
\[
\Psi_x^{\mu \xi} = \psi_{r_1}^{D} \psi_{r_1}^{L} \Psi[d_{i,j}] \psi_{r_1}^{R} x^\lambda.
\]

**Proof.** We go by induction on \( s \), the base case \( s = 0 \) being trivial. By assumption we have
\[
\Psi_x^{\mu \xi} = \psi_{r_1} \cdots \psi_{r_s} x^{\mu \xi} = \psi_{r_1}^{D} \psi_{r_1}^{L} \psi_{r_1}^{R} x^{\mu \xi}.
\]

Write \( i^{(h)}_k \) for the residue sequence associated with the nodes in \( \mu^{(h)}_k \) in \( T^\lambda \), and similarly for \( i_k^{(r)} \). In terms of Khovanov-Lauda diagrams, with the vector \( x^\lambda \) pictured as being at the top of the diagram, we must show that

![Diagram](image-url)
Let \( j = w_{i,j}^L w_{i,j+1}^R \cdots w_{i,r_j-1}^R i^\lambda \). Since \( s_{i,j} w_{i,j}^D = w_{i,j}^D s_{i,j} \) and \( \ell(s_{i,j} w_{i,j}^D) = \ell(w_{i,j}^D) + \ell(s_{i,j}) \), it follows from Lemma \ref{lem:subdivision} that

\[
\psi_{i,j} \psi_{i,j}^D 1_j = \psi_{i,j}^D \psi_{i,j+1}^D 1_j + \sum_{u \in w_{i,j}^D} c_u \psi_u \psi_{i,j+1}^D f_u(y) 1_j
\]

for some constants \( c_u \in \mathcal{O} \), polynomials \( f_u(y_1, \ldots, y_{d-1}) \), and \( \epsilon_u \in \{0,1\} \). Thus it remains to show that

\[
\psi_u \psi_u^\epsilon f_u(y) \psi_{i,j+1}^D 1_j + \sum_{u \in w_{i,j}^D} c_u \psi_u \psi_{i,j+1}^D f_u(y) 1_j = 0 \in S^\lambda/C_{\mathcal{O}} \mu
\]

for all \( u \) in the sum in (6.1). Let \( s_{i,j}^R \cdots s_{i,j}^{R_{N_L}} \) be the preferred reduced expression for \( w_{i,j}^R \) and similarly for \( w_{i,j}^L \). Pushing the \( y \)'s to the right to act (as zero) on \( x^\lambda \), this is by Lemma \ref{lem:subdivision} and \( \mathcal{O} \)-linear combination of terms of the form

\[
\psi_u \psi_u^\epsilon f_u(y) \psi_{i,j+1}^D 1_j + \sum_{u \in w_{i,j}^D} c_u \psi_u \psi_{i,j+1}^D f_u(y) 1_j = 0 \in S^\lambda/C_{\mathcal{O}} \mu
\]

for some \( \epsilon_i \in \{0,1\} \). Write \( \Theta \) for the sequence of \( \psi \)'s in (6.2). Assume \( \psi^0 \) appears with nonzero coefficient when \( \Theta \psi^\lambda \) is expanded in terms of basis elements. Then it follows from Lemma \ref{lem:subdivision} that one can write \( w^U \) diagrammatically by removing crossings from the diagram

and in particular, removing at least one crossing from \( w_{i,j}^D \), the third row of the diagram, since \( u \prec w_{i,j}^D \). But in any case, this implies that there is a pink strand that ends to the left of a blue strand, i.e., some \( t \leq a \) such that \( (w^U)^{-1}(t) \) is in \( \xi_{i,j}^{(h)} \) for some \( h, k \). Then \( sh(U_{\leq a}) \neq \mu \). But since \( N_{\mu} \) is an \( R_{\alpha,\beta} \)-submodule, we must have \( U \in B_{\mu} \). This implies that \( U \in B_{\mathcal{O}} \mu \), and hence \( x^U = 0 \in S^\lambda/C_{\mathcal{O}} \mu \). \qed

Let \( A_\mu \) be a Garnir node in \( \mu \). This is also a Garnir node of \( \lambda \), and when we consider it as such, we will label it with \( A_\lambda \). Let \( B^{A_k} \) be the Garnir belt associated with \( A_\lambda \), and let \( B^{A_{\mu}} \) be the Garnir belt of nodes in \( \mu \). Assume \( A_\lambda \) is in row \( j \) of the \( i \)th component of \( \lambda \). We subdivide the sets of nodes of \( \mu_j^{(i)}, \mu_{j+1}^{(i)} \) and \( \xi_j^{(i)} \) in the following fashion:

1. We subdivide \( \mu_j^{(i)} \) into three sets:
   a. Let \( \mu_{j,1}^{A} \) be the nodes of \( \mu_j^{(i)} \) not contained in \( B^A \mu \).
   b. Let \( \mu_{j,2}^{A} \) be the nodes of \( \mu_j^{(i)} \) contained in bricks in \( B^A \mu \).
   c. Let \( \mu_{j,3}^{A} \) be the nodes of \( \mu_j^{(i)} \) contained in \( B^A \mu \), but not contained in any brick.

2. We subdivide \( \xi_j^{(i)} \) into three sets:
   a. Let \( \xi_{j,1}^{A} \) be the nodes of \( \xi_j^{(i)} \) contained in a brick in \( B^A \lambda \) which contains nodes of \( \mu \).
   b. Let \( \xi_{j,2}^{A} \) be the nodes of \( \xi_j^{(i)} \) contained in a brick in \( B^A \lambda \) which is entirely contained in \( \xi \).
(c) Let $\xi_{A,3}^{(i)}$ be the nodes of $\xi_{j}^{(i)}$ contained in $B_{A}^{\lambda}$, but not contained in any brick.

(3) We subdivide $\mu_{j+1}^{(i)}$ into three sets:

(a) Let $\mu_{A,1}^{(i)}$ be the nodes of $\mu_{j+1}^{(i)}$ contained in $B_{A}^{\lambda}$ but not contained in any brick.

(b) Let $\mu_{A,2}^{(i)}$ be the nodes of $\mu_{j+1}^{(i)}$ contained in bricks in $B_{A}^{\mu}$.

(c) Let $\mu_{A,3}^{(i)}$ be the nodes of $\mu_{j+1}^{(i)}$ not contained in $B_{A}^{\mu}$.

For example, if $l = 1$, $\mu = (14, 10)$, $\lambda = (23, 18)$, $A = (1, 8, 1)$ and $e = 3$, then the subdivisions are as follows:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\mu_{A,1}^{(i)} & \mu_{A,2}^{(i)} & \mu_{A,3}^{(i)} & \xi_{A,1}^{(i)} & \xi_{A,2}^{(i)} & \xi_{A,3}^{(i)} \\
\hline
\end{array}
\]

Now write $w_{R_{i,j}} = w_{R_{i,j}}^{L} w_{R_{i,j}}^{R}$, where $w_{R_{i,j}}^{L}$, $w_{R_{i,j}}^{R}$ are given as follows:

\[
w_{R_{i,j}}^{L} = \left\{ \begin{array}{c}
\mu_{i+1}^{(i)} \xi_{1}^{(i)} \mu_{i+2}^{(i)} \ldots \mu_{j}^{(i)} \xi_{A,1}^{(i)} \mu_{A,1}^{(i)} \mu_{A,2}^{(i)} \mu_{A,3}^{(i)} \mu_{j+1}^{(i)} \mu_{j+2}^{(i)} \ldots \mu_{n}^{(i)} \xi_{n}^{(i)} \\
\end{array} \right. 
\]

Let $G_{A}^{\lambda} = \omega T_{A}^{\lambda}$ and $G_{A}^{\mu} = \zeta T_{A}^{\mu}$, where $\omega \in \mathcal{D}_{A}^{\lambda}$ and $\zeta \in \mathcal{D}_{A}^{\mu}$. Then $\omega = \omega_{2} \omega_{1}$, where $\omega_{1}, \omega_{2} \in \mathcal{G}_{A}^{\lambda}$ are given as follows:

\[
w_{G_{A}^{\lambda}} = \left\{ \begin{array}{c}
\mu_{1}^{(i)} \xi_{1}^{(i)} \ldots \mu_{j-1}^{(i)} \xi_{j-1}^{(i)} \mu_{A,1}^{(i)} \mu_{A,2}^{(i)} \mu_{A,3}^{(i)} \xi_{A,1}^{(i)} \mu_{A,1}^{(i)} \mu_{A,2}^{(i)} \mu_{A,3}^{(i)} \ldots \mu_{n}^{(i)} \xi_{j-1}^{(i)} \\
\end{array} \right. 
\]

Then

\[
(6.3) \quad \zeta w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}} = w_{R_{i,j}}^{A_{\mu}} w_{R_{i,j}}^{A_{\lambda}}.
\]
This is best seen diagrammatically. On the right side of (6.3) we have

\[ w = (w_{r_1}^{A_1})^{s_1} \cdots (w_{r_{n_1}}^{A_1})^{s_{n_1}} (w_{r_1}^{A_3})^{s_1} \cdots (w_{r_{n_2}}^{A_3})^{s_{n_2}} \in \mathcal{A}_k, \]

where each \( s_k \in \{0, 1\} \). In other words, \( w \) is achieved by deleting simple transpositions in \( \mathcal{A}_k \) from \( \omega \).

**Lemma 6.7.** If \( s_k = 0 \) for some \( 1 \leq k \leq n_2 \), then

\[ \psi_{i,j}^D \psi_{i,j}^L \psi_{i,j}^R (\sigma_{r_1}^{A_1})^{r_1} \cdots (\sigma_{r_{n_1}}^{A_1})^{r_{n_1}} (\sigma_{r_1}^{A_3})^{r_1} \cdots (\sigma_{r_{n_2}}^{A_3})^{r_{n_2}} \psi^A \lambda \lambda \psi^A = 0. \]

**Proof.** By Lemma 3.3

\[ \psi_{i,j}^D \psi_{i,j}^L \psi_{i,j}^R (\sigma_{r_1}^{A_1})^{r_1} \cdots (\sigma_{r_{n_1}}^{A_1})^{r_{n_1}} (\sigma_{r_1}^{A_3})^{r_1} \cdots (\sigma_{r_{n_2}}^{A_3})^{r_{n_2}} \psi^A \lambda \lambda \psi^A \]

is an \( O \)-linear combination of elements of the form \( w^T \), where a reduced expression for \( w^T \) appears as a subexpression in the (not necessarily reduced) concatenation of reduced expressions associated with

\[ w_{i,j}^D w_{i,j}^L w_{i,j}^R (w_{r_1}^{A_1})^{s_1} \cdots (w_{r_{n_1}}^{A_1})^{s_{n_1}} (w_{r_1}^{A_3})^{s_1} \cdots (w_{r_{n_2}}^{A_3})^{s_{n_2}} w^A. \]

In other words, one can write \( w^T \) by removing crossings in (6.4), and in particular (since \( s_k = 0 \) for some \( k \)), removing at least one of the pink/blue crossings in the second row.
any case then, there is some pink strand that $w^T$ sends to the left side, i.e., some $c \leq a$ such that $(w^T)^{-1}(c) \in \xi_k^{(h)}$ for some $h,k$. Then $\text{sh}(T_{c,a}) \neq \mu$. But since $w^T$ is obtained by removing crossings in $w^{(A^\mu)}$, we have $T \preceq G^\mu \cdot \xi$. If $s_r$ is the transposition such that $s_r G^\mu \cdot \xi \in \text{St}(\lambda)$, then Lemma 2.6 implies that $T \preceq s_r G^\mu \cdot \xi \in B_\mu$, which in turn implies by Lemma 6.2 that $T \in B_\mu$. But then $T \in B_{\preceq \mu}$, and thus $x^T = 0 \in S^\lambda / C_{\preceq \mu}$. \hfill \Box

Every $w \in \mathcal{D}^A_\lambda$ can be written as a reduced expression of the form (6.3) for some $c_k^\mu \in \{0, 1\}$. If $c_k^\mu = 0$ for some $1 \leq k \leq n_2$, or equivalently, if there is some node $(a, b, m)$ in $\xi$ such that $wT^\lambda(a, b, m) \neq G^\lambda(a, b, m)$, then the above lemma implies that

$$
\psi_{i,j}^\mathcal{D} \psi_{i,j}^R \psi_{i,j}^{A_\lambda} \psi_{i,j}^{T_\lambda} x^\lambda = 0,
$$

and

$$
\psi_{i,j}^\mathcal{D} \psi_{i,j}^R \psi_{i,j}^{A_\lambda} \psi_{i,j}^{T_\lambda} x^\lambda = \psi_{i,j}^\mathcal{D} \psi_{i,j}^R \psi_{i,j}^{A_\lambda} \psi_{i,j}^{T_\lambda} x^\lambda
$$

otherwise. Let $f^{A_\lambda}$ and $f^{A_\mu}$ denote the number of bricks in the top row of $B^{A_\lambda}$ and $B^{A_\mu}$ respectively. Note that $w = (w_{r_j}^{A_\lambda})^{c_1} \cdots (w_{r_{n_1}}^{A_\lambda})^{c_2} \omega_1$ is a reduced expression for an element in $\mathcal{D}^A_{\lambda}$ if and only if $(w_{r_j}^{A_\lambda})^{c_1} \cdots (w_{r_{n_1}}^{A_\lambda})^{c_2}$ is a reduced expression for an element in $\mathcal{D}^{f^{A_\mu}, k^{A_\lambda}} - f^{A_\lambda}$. Since $k^{A_\mu} = k^{A_\lambda} - (f^{A_\lambda} - f^{A_\mu})$, this allows us to associate $\mathcal{D}^{A_\mu}$ with $\mathcal{D}^{A_\lambda}$ in the following way. Let $\tilde{\mathcal{D}}^{A_\lambda}$ be the set of all $w \in \mathcal{D}^{A_\lambda}$ such that $c_k^\mu \neq 0$ for all $k$. Then there is a bijection between $\mathcal{D}^{A_\mu}$ and $\tilde{\mathcal{D}}^{A_\lambda}$ given by $u \mapsto u[d_{i,j}]^{\omega_2}$.

**Lemma 6.8.** For all $u \in \mathcal{D}^{A_\mu}$,

$$
\tau_{\mu} \psi_{i,j}^{T^{A_\mu}} \psi_{i,j}^{\mu \xi} x^\lambda = \psi_{i,j}^\mathcal{D} \psi_{i,j}^R \psi_{i,j}^{A_\lambda} \psi_{i,j}^{T_\lambda} [d_{i,j}] \psi_{i,j}^{T^{A_\lambda}} x^\lambda.
$$

**Proof.** This is easily seen in terms of Khovanov-Lauda diagrams, with $x^\lambda$ pictured as being at the top of the diagram. The left side:
is by Lemma 6.6 equal to

\[ \psi_{i,j}^{R} \]

which, after an isotopy of strands, becomes the right side in the lemma statement:

\[ \psi_{i,j}^{D} \]

completing the proof. \[\square\]

**Lemma 6.9.** Let \( A_\mu \) be a Garnir node of \( \mu \). Then \( f(g^A_\mu \otimes 1_\beta) = 0 \).
Proof. We make use of Lemma 6.8 and the bijection between $\mathcal{D}^\mu$ and $\mathcal{D}^\lambda$:

$$f(g^\mu \otimes 1_\beta) = g^\mu \cdot x^\xi = \left(\sum_{w \in \mathcal{D}^\mu} \tau_w x^\lambda \right) x^\lambda$$

$$= \sum_{w \in \mathcal{D}^\mu} \psi_i^w \psi_{i,j}^w \psi_{i,j}^{R^w} \tau_w \left[ d_{i,j} \right] \psi \psi_\omega x^\lambda x^\lambda$$

$$= \psi_i^w \psi_{i,j}^w \psi_{i,j}^{R^w} \sum_{w \in \mathcal{D}^\mu} \tau_w \left[ d_{i,j} \right] \psi \psi_\omega x^\lambda x^\lambda$$

$$= \psi_i^w \psi_{i,j}^w \psi_{i,j}^{R^w} \sum_{w \in \mathcal{D}^\lambda} \tau_w x^\lambda x^\lambda$$

$$= \psi_i^w \psi_{i,j}^w \psi_{i,j}^{R^w} g^\lambda x^\lambda$$

$$= 0.$$

\[ \square \]

Remark 6.10. Although we have focused on Garnir nodes in $\mu$, there are obvious analogues (whose proofs are entirely analogous) of Lemmas 6.6, 6.7, and 6.8 which imply the analogue of Proposition 6.9

$$f(1_\alpha \otimes g^\xi) = 0$$

for Garnir nodes in $\xi$.

Proposition 6.11. The map $f : R_\alpha,\beta(\deg T^\mu + \deg T^\xi) \to N_\mu$ induces a graded isomorphism $f : S^\mu \otimes S^\xi \to N_\mu$.

Proof. We have that $f$ factors through to a map $R_\alpha,\beta/(J_\alpha^\mu \otimes R_\beta + R_\alpha \otimes J_\beta^\xi)(\deg T^\mu + \deg T^\xi) \to N_\mu$ by Lemmas 6.4, 6.9 and Remark 6.10. However, we also have

$$R_\alpha,\beta/(J_\alpha^\mu \otimes R_\beta + R_\alpha \otimes J_\beta^\xi)(\deg T^\mu + \deg T^\xi) \cong R_\alpha/J_\alpha^\mu(\deg T^\mu) \otimes R_\beta/J_\beta^\xi(\deg T^\xi) = S^\mu \otimes S^\xi.$$

Moreover, for all $T \in \text{St}(\mu)$, $\tau \in \text{St}(\xi)$,

$$f(v^T \otimes v^\tau) = f(\psi^T v^\mu \psi^\tau) = \psi^T v^\mu \psi^\tau[a] x^\mu x^\lambda = \psi^T v^\mu \psi^\tau[a] x^\lambda = x^\tau + \sum_{\tau \in \text{St}(\xi)} d_{\tau} x^\tau$$

for some constants $d_{\tau}$, by Lemma 6.3. Since $\{v^T \otimes v^\tau \mid T \in \text{St}(\mu), \tau \in \text{St}(\xi)\}$ is a spanning set for $S^\mu \otimes S^\xi$ and $\{x^\tau \mid T \in \text{St}(\mu), \tau \in \text{St}(\xi)\}$ is a basis for $N_\mu$, it follows that $f$ is an isomorphism. \[ \square \]

6.3. A basis for $S^{\lambda/\mu}$ and a filtration for $\text{Res}_{\alpha,\beta}S^\lambda$. Proposition 6.11 in hand, we may now prove two theorems which complete the analogy with the definition 1.1 in the semisimple case, and justify our use of the term skew Specht module for $S^{\lambda/\mu}$.

Theorem 6.12. Let $\lambda/\mu \in \mathcal{S}^\kappa$. Then $S^{\lambda/\mu}$ has a homogeneous $O$-basis

$$\{v^\tau \mid \tau \in \text{St}(\lambda/\mu)\}.$$

Proof. By Proposition 6.3 the set 6.6 spans $S^{\lambda/\mu}$ over $O$, and the set is linearly independent by Proposition 6.11. \[ \square \]

Theorem 6.13. Let $\lambda \in \mathcal{S}^\kappa$. Let $\{\mu_1, \ldots, \mu_k\} = \{\mu \in \mathcal{S}^\kappa \mid \mu \subset \lambda\}$ and assume the labels are such that $\mu_i \triangleright \mu_j \iff i < j$. Write

$$V_i := \sum_{j=1}^{i} C_{\mu_j} = O \{v^\tau \in S^\lambda \mid \tau \in \text{St}(\lambda), \text{sh}(T_{\omega \alpha}) = \mu_j \text{ for some } j < i\}$$
for all $i$. Then

$$0 = V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_k = \text{Res}_{\alpha,\beta} S^\lambda$$

is a graded filtration of $\text{Res}_{\alpha,\beta} S^\lambda$ by $R_{\alpha,\beta}$-submodules, with subquotients

$$V_i / V_{i-1} \cong S^{\mu_i} \otimes S^\lambda / S^{\mu_i}.$$ 

**Proof.** The fact that $V_k = \text{Res}_{\alpha,\beta} S^\lambda$ follows from the fact that $B = \bigcup_{j=1}^k B_{\mu_j}$ and $\{v^T \mid T \in B\}$ is a basis for $\text{Res}_{\alpha,\beta} S^\lambda$. Since $C_{\mu_i} \geq C_{\mu_j}$ if $\mu_j \cong \mu_i$, we have

$$V_i = \sum_{j=1}^i C_{\mu_j} = C_{\mu_i} \oplus \sum_{j<i} C_{\mu_j}$$

and

$$V_{i-1} = \sum_{j=1}^{i-1} C_{\mu_j} = C_{\mu_i} \oplus \sum_{j<i} C_{\mu_j} = C_{\mu_i} \oplus \sum_{j<i} C_{\mu_j},$$

which implies that $V_i / V_{i-1} \cong C_{\mu_i} / C_{\mu_j} = N_{\mu_i} \cong S^{\mu_i} \otimes S^\lambda / S^{\mu_i}$. $\Box$

### 6.4. Induction product of skew Specht modules
The following theorem was proved in [9, Theorem 8.2] in the context of Young diagrams, but the proof is applicable with no significant alteration to the more general case of skew diagrams.

**Theorem 6.14.** Suppose that $\lambda/\mu \in \mathcal{F}_\alpha$. Then

$$S^{\lambda/\mu} \cong S^{\lambda(1)/\mu(1)} \circ \cdots \circ S^{\lambda(l)/\mu(l)} \langle d_{\lambda/\mu} \rangle,$$

as graded $R_{\alpha}$-modules, where

$$d_{\lambda/\mu} = \deg(t^{\lambda/\mu}) - \deg(t^{\lambda(1)/\mu(1)}) - \cdots - \deg(t^{\lambda(l)/\mu(l)}).$$

### 7. Joinable diagrams

In this section we present a useful, albeit rather technical, result regarding the graded characters of skew Specht modules whose associated component diagrams jibe with each other in a specific sense. This result, together with Theorem 6.14, will make it possible for us to identify cuspidal modules in [9] while operating solely at the level of characters.

**Definition 7.1.** Let $l = 2$, $\kappa = (k_1, k_2)$, and $\lambda = (\lambda(1), \lambda(2)) \in \mathcal{F}_\alpha$. Write $x_1 := n(\lambda, i)$, and $y_1 := \lambda_1^{(1)}$. If $(x_1, 1, 1)$ (the bottom left node in $\lambda(1)$) and $(1, y_2, 2)$ (the top right node in $\lambda(2)$) are such that $\text{res}(x_1, 1, 1) = \text{res}(1, y_2, 2) + 1$, we call $\lambda$ joinable.

In this section we will assume that $\lambda$ is joinable. Define the one-part multicharges $\kappa^* := (k_2 + x_1)$ and $\kappa_* := (k_2 + x_1 - 1)$. We now define $\lambda^*/\mu^* \in \mathcal{F}_\alpha$ and $\lambda_* / \mu_* \in \mathcal{F}_\alpha$ by setting:

$$\lambda^* := (\lambda(1) + y_2 - 1, \ldots, \lambda_{x_1}^{(1)} + y_2 - 1, \lambda_1^{(2)}, \ldots, \lambda_{x_2}^{(2)}),$$

and

$$\lambda_* := (\lambda(1) + y_2, \ldots, \lambda_{x_1}^{(1)} + y_2, \lambda_1^{(2)}, \ldots, \lambda_{x_2}^{(2)}), \quad \mu_* := (y_2^{x_1 - 1}).$$

In other words, $\lambda^*/\mu^*$ is achieved by shifting the Young diagram associated with $\lambda(1)$ until its bottom-left node lies directly above the top-right node of $\lambda(2)$, and then viewing this as a one-part skew diagram. Similarly, $\lambda_* / \mu_*$ is achieved by shifting the Young diagram associated with $\lambda(1)$ until its bottom-left node lies directly to the right of the top-right node of $\lambda(2)$.

There is an obvious bijection $\tau^*$ (resp. $\tau_*$) between nodes of $\lambda$ and $\lambda^*/\mu^*$ (resp. $\lambda_* / \mu_*$), given by
Lemma 7.3. Let \( \lambda^{(1)} \equiv (a, b, 1) \) and \( \lambda^{(2)} \equiv (a, b, 2) \) be joinable.

\[
\begin{align*}
\lambda^{(1)} & \equiv (a, b, 1) \xrightarrow{\tau^*} (a, b + y_2 - 1) \in \lambda^*/\mu^* \\
\lambda^{(2)} & \equiv (a, b, 2) \xrightarrow{\tau^*} (a + x_1, b) \in \lambda^*/\mu^*
\end{align*}
\]

and, respectively,

\[
\begin{align*}
\lambda^{(1)} & \equiv (a, b, 1) \xrightarrow{\tau_*} (a, b + y_2) \in \lambda_*/\mu_* \\
\lambda^{(2)} & \equiv (a, b, 2) \xrightarrow{\tau_*} (a + x_1 - 1, b) \in \lambda_*/\mu_*
\end{align*}
\]

Note that the charges \( \kappa^* \) and \( \kappa_* \) are chosen so that residues of nodes are preserved under this bijection. Let \( T \in \text{St}(\lambda) \). Viewing the tableau as a function \( \{1, \ldots, d\} \to \lambda \), then composing with \( \tau^* \) (resp. \( \tau_* \)) gives a \( \lambda^*/\mu^* \)-tableau (resp. \( \lambda_*/\mu_* \)-tableau). Define

\[
T^* := \tau^* \circ T \quad \text{and} \quad T_* := \tau_* \circ T.
\]

Then we have bijections

\[
\{ T \in \text{St}(\lambda) \mid T(x_1, 1, 1) < T(1, y_2, 2) \} \xrightarrow{\tau^*} \text{St}(\lambda^*/\mu^*)
\]

and

\[
\{ T \in \text{St}(\lambda) \mid T(x_1, 1, 1) > T(1, y_2, 2) \} \xrightarrow{\tau_*} \text{St}(\lambda_*/\mu_*)
\]

Example 7.2. Let \( e = 3, \kappa = (0, 1), \lambda^{(1)} = (3, 2, 2) \) and \( \lambda^{(2)} = (2, 2) \). Then \( \lambda \) is joinable since \( \text{res}(3, 1, 1) = 1 = 0 + 1 = \text{res}(1, 2, 1) = 1 \). Then, with respect to the row- and column-leading tableaux, we have:

\[
T^\lambda = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12
\end{array}
\]

\[
(T^\lambda)^* = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12
\end{array}
\]

\[
T_\lambda = \begin{array}{ccc}
5 & 8 & 11 \\
6 & 9 & 10 \\
1 & 3 & 7 \\
2 & 4 & 10
\end{array}
\]

\[
(T_\lambda)^* = \begin{array}{ccc}
5 & 8 & 11 \\
6 & 9 & 10 \\
1 & 3 & 7 \\
2 & 4 & 10
\end{array}
\]

Lemma 7.3. Let \( \lambda \in \mathcal{B}_\kappa \) be joinable, \( \text{res}(1, y_2, 2) = i \), and let \( T \in \text{St}(\lambda) \). Then

\[
\deg T^* = \deg T - \left( \Lambda_i, \text{cont}(\lambda^{(1)}) \right)
\]

if \( T(x_1, 1, 1) < T(1, y_2, 2) \), and

\[
\deg T_* = \deg T - \left( \Lambda_{i+1}, \text{cont}(\lambda^{(1)}) \right)
\]

if \( T(x_1, 1, 1) > T(1, y_2, 2) \).

Proof. We prove the first statement. The second is similar. Let \( U = T_{\leq t} \) for some \( t \). We’ll show that the claim holds for \( U \):

\[
(7.1) \quad \deg(U^*) = \deg(U) - \left( \Lambda_i, \text{cont}(\text{sh}(U)^{(1)}) \right)
\]

going by induction on the size of \( \text{sh}(U) \).

For the base case we have \( \text{sh}(U) = (\emptyset, \emptyset) \), so that \( \deg(U) = 0 = \deg(U^*) = 0 \).

Now we attack the induction step. By the inductive definition of degree for tableaux, we just need to show that for every removable node \( A \) in \( U \),

\[
(7.2) \quad d_{\tau^*(A)}(\text{sh}(U^*)) - d_A(\text{sh}(U)) = \begin{cases} 
-1 & A \in \lambda^{(1)} \text{ and } \text{res}(A) = i, \\
0 & \text{otherwise}.
\end{cases}
\]
By the construction of $\mathbb{U}^*$, it is clear that for $1 \leq r \leq x_1 - 1$ and $j \in I$, the $r$-th row of $\text{sh}(\mathbb{U})^{(1)}$ has an addable (resp. removable) $j$-node if and only if the corresponding $r$-th row in $\text{sh}(Y(\mathbb{U}^*))$ has an addable (resp. removable) $j$-node. Similarly, for $2 \leq r \leq x_2$, the $r$-th row of $\text{sh}(\mathbb{U})^{(2)}$ has an addable (resp. removable) $j$-node if and only if the corresponding $(x_1 + r)$-th row in $\text{sh}(Y(\mathbb{U}^*))$ has an addable (resp. removable) $j$-node. Thus it remains to compare addable/removable nodes of rows $x_1, x_1 + 1$ in $\text{sh}(\mathbb{U})^{(1)}$ and row 1 in $\text{sh}(\mathbb{U})^{(2)}$ with the rows $x_1, x_1 + 1$ in $\text{sh}(Y(\mathbb{U}^*))$.

For simplicity, we label

- $B := (x_1 - 1, 1, 1)$, the bottom-left node in $\lambda^{(1)}$. Write $B^* := \tau^*(B)$. Both $B$ and $B^*$ have residue $i + 1$.
- $C := (1, y_2 - 1, 2)$, the node to the left of the top-right node in $\lambda^{(2)}$. Write $C^* := \tau^*(C)$. Both $C$ and $C^*$ have residue $i - 1$.
- $D := (1, y_2, 2)$, the top-right node in $\lambda^{(2)}$. Write $D^* := \tau^*(D)$. Both $D$ and $D^*$ have residue $i$.

For instance, in $\mathbb{U}$ these labels correspond to the nodes shown below:

| Row $x_1$ of $\lambda^{(1)}$ --- | Row $x_1$ of $\lambda^*/\mu^*$ --- |
| --- | --- |
| $B$ | $B^*$ |
| Row $x_1 + 1$ of $\lambda^{(1)}$ --- | Row $x_1 + 1$ of $\lambda^*/\mu^*$ --- |
| $C$ | $D$ |

There are five cases to consider.

1. $\{B, C, D\} \cap \text{sh}(\mathbb{U}) = \emptyset$.
   - Row $x_1$ of $\text{sh}(\mathbb{U})^{(1)}$ has addable node $B$ iff $B^*$ is addable in $\text{sh}(Y(\mathbb{U}^*))$. Row $x_1$ of $\text{sh}(\mathbb{U})^{(1)}$ has no removable nodes.
   - Row $x_1 + 1$ of $\text{sh}(\mathbb{U})^{(1)}$ has no addable/removable nodes.
   - Row 1 of $\text{sh}(\mathbb{U})^{(2)}$ has an addable (resp. removable) $j$-node iff row $x_1 + 1$ of $\text{sh}(Y(\mathbb{U}^*))$ has an addable (resp. removable) $j$-node.
   - Row $x_1$ of $\text{sh}(Y(\mathbb{U}^*))$ has a removable $i$-node (the bottom-right node of $\mu_*$ to be precise).

From this follows.

2. $\{B, C, D\} \cap \text{sh}(\mathbb{U}) = \{B\}$.
   - Row $x_1$ of $\text{sh}(\mathbb{U})^{(1)}$ has an addable (resp. removable) $j$-node iff row $x_1$ of $\text{sh}(Y(\mathbb{U}^*))$ has an addable (resp. removable) $j$-node.
   - Row $x_1 + 1$ of $\text{sh}(\mathbb{U})^{(1)}$ has an addable $i$-node, and no removable nodes.
   - Row 1 of $\text{sh}(\mathbb{U})^{(2)}$ has an addable (resp. removable) $j$-node if row $x_1 + 1$ of $\text{sh}(Y(\mathbb{U}^*))$ has an addable (resp. removable) $j$-node.

From this follows.

3. $\{B, C, D\} \cap \text{sh}(\mathbb{U}) = \{C\}$.
   - Row $x_1$ of $\text{sh}(\mathbb{U})^{(1)}$ has addable node $B$ iff $B^*$ is addable in $\text{sh}(Y(\mathbb{U}^*))$. Row $x_1$ of $\text{sh}(\mathbb{U})^{(1)}$ has no removable nodes.
   - Row $x_1 + 1$ of $\text{sh}(\mathbb{U})^{(1)}$ has no addable/removable nodes.
   - Row 1 of $\text{sh}(\mathbb{U})^{(2)}$ has an addable $i$-node $D$. Row 1 of $\text{sh}(\mathbb{U})^{(2)}$ has removable node $C$ iff $C^*$ is removable in row $x_1 + 1$ of $\text{sh}(Y(\mathbb{U}^*))$.
   - Row $x_1$ of $\text{sh}(Y(\mathbb{U}^*))$ has no removable nodes.
   - Row $x_1 + 1$ of $\text{sh}(Y(\mathbb{U}^*))$ has no addable nodes.

From this follows.

4. $\{B, C, D\} \cap \text{sh}(\mathbb{U}) = \{B, C\}$. 
• Row $x_1$ of $\text{sh}(U)^{(1)}$ has an addable (resp. removable) $j$-node iff row $x_1$ of $	ext{sh}(Y(U^*))$ has an addable (resp. removable) $j$-node.

• Row $x_1 + 1$ of $\text{sh}(U)^{(1)}$ has an addable $i$-node and no removable node.

• Row 1 of $\text{sh}(U)^{(2)}$ has an addable (resp. removable) $j$-node iff row $x_1 + 1$ of $\text{sh}(Y(U^*))$ has an addable (resp. removable) $j$-node.

From this (7.2) follows,

(5) $\{B, C, D\} \cap \text{sh}(U) = \{B, C, D\}$.

• Row $x_1$ of $\text{sh}(U)^{(1)}$ has an addable (resp. removable) $j$-node to the right of $B$ iff row $x_1$ of $\text{sh}(Y(U^*))$ has an addable (resp. removable) $j$-node to the right of $B^*$. The $(i + 1)$-node $B$ is not removable in row $x_1$ of $\text{sh}(U)^{(1)}$ iff row $x_1 + 1$ of $\text{sh}(Y(U^*))$ has an addable $(i + 1)$-node to the right of $D^*$.

• Row $x_1 + 1$ of $\text{sh}(U)^{(1)}$ has an addable $i$-node and no removable node.

• Row 1 of $\text{sh}(U)^{(2)}$ has an addable $(i + 1)$-node. Row 1 of $\text{sh}(U)^{(2)}$ has removable node $D$ iff $D^*$ is removable in row $x_1 + 1$ of $\text{sh}(Y(U^*))$.

From this (7.2) follows.

Thus in all cases, (7.2) is satisfied, and the lemma follows by induction. \hfill \qed

**Definition 7.4.** We say that an arbitrary skew diagram $\lambda/\mu$ is **minimal** if $\mu_1^{(i)} < \lambda_1^{(i)}$ and $\mu_n^{(i)}(\lambda, \mu) = 0$ for all $i$. Less formally, a skew diagram is minimal if, in each component, it has nodes in the top row and in the leftmost column.

**Definition 7.5.** Let $l = 2$. We say that $\lambda/\mu \in S^* \kappa$ is **joinable** if it is minimal and $\lambda$ is joinable.

Assuming $\lambda/\mu$ is joinable, define $\kappa^*, \kappa, x_i, y_i$ as before, with respect to $\lambda$. In the same vein as before we construct a skew tableau $\lambda^*/\mu^*$ by shifting the skew diagram $\lambda^{(1)}/\mu^{(1)}$ until the lower left node lies above the upper right node of $\lambda^{(2)}/\mu^{(2)}$, and we construct a skew tableau $\lambda^*/\mu^*$ by shifting the skew diagram $\lambda^{(1)}/\mu^{(1)}$ until the lower left node lies directly to the right of the upper right node of $\lambda^{(2)}/\mu^{(2)}$. Specifically, define $\lambda^*/\mu^* \in S^* \kappa^*$ and $\lambda^*/\mu^* \in S^* \kappa^*$ by setting:

$$
\lambda^* := (\lambda_1^{(1)} + y_2 - 1, \ldots, \lambda_{x_1}^{(1)} + y_2 - 1, \lambda_2^{(2)}, \ldots, \lambda_{x_2}^{(2)}),
$$

$$
\mu^* := (\mu_1^{(1)} + y_2 - 1, \ldots, \mu_{x_1}^{(1)} + y_2 - 1, \mu_2^{(2)}, \ldots, \mu_{x_2}^{(2)}),
$$

and

$$
\lambda_* := (\lambda_1^{(1)} + y_2, \ldots, \lambda_{x_1}^{(1)} + y_2, \lambda_2^{(2)}, \ldots, \lambda_{x_2}^{(2)}),
$$

$$
\mu_* := (\mu_1^{(1)} + y_2, \ldots, \mu_{x_1}^{(1)} + y_2, \mu_2^{(2)}, \ldots, \mu_{x_2}^{(2)}).
$$

With $\tau^*_*$ and $\tau^*$ defined as before with respect to $\lambda$, we define

$$
\text{Tab}(\lambda^*/\mu^*) \ni \tau^* := \tau \circ \tau \quad \text{and} \quad \text{Tab}(\lambda^*/\mu^*) \ni \tau_* := \tau \circ \tau
$$

for $u \in \text{Tab}(\lambda/\mu)$. We have bijections

$$
\{ t \in \text{St}(\lambda/\mu) \mid t(x_1, 1, 1) < t(1, y_2, 2) \} \xrightarrow{\tau^*} \text{St}(\lambda^*/\mu^*)
$$

and

$$
\{ t \in \text{St}(\lambda/\mu) \mid t(x_1, 1, 1) > t(1, y_2, 2) \} \xrightarrow{\tau_*} \text{St}(\lambda^*/\mu^*)
$$

**Proposition 7.6.** Let $\lambda/\mu \in S^* \kappa$ be joinable, with the top right node in $\lambda^{(2)}$ having residue $i$. Let $t \in \text{St}(\lambda/\mu)$. Then

$$
\deg \tau^* = \deg t - \left( A_i, \text{cont}(\lambda^{(1)}/\mu^{(1)}) \right)
$$
if $t(x_1,1,1) < t(y_2,2,1)$, and
\[
\deg t_\ast = \deg t - \left( \Lambda_{i+1}, \cont(\lambda^{(1)}) / \mu^{(1)} \right)
\]
if $t(x_1,1,1) > t(y_2,2,1)$.

**Proof.** We prove the first statement. The second is similar. Let $\nu = \lambda^\ast \setminus \sh((t^\lambda)^\ast)$. Then by definition,

(7.3) \[
\deg t^\ast = \deg Y(t^\ast) - \deg T^{(\mu^\ast)}
\]
(7.4) \[
\deg t = \deg Y(t) - \deg T^{\mu}
\]
(7.5) \[
\deg Y(t)^\ast = \deg Y(Y(t)^\ast) - \deg T^{\nu}
\]

Lemma 7.3 gives us

(7.6) \[
\deg Y(t)^\ast = \deg Y(t) - \left( \Lambda_i, \cont(\lambda^{(1)}) \right)
\]

Note that $Y(Y(t)^\ast)$ and $Y(t)^\ast$ agree outside of $\mu^\ast$, so

\[
\deg Y(t)^\ast + \deg T^{\nu} - \deg Y(t)^\ast = \deg Y(Y(t)^\ast) - \deg Y(t)^\ast
\]
\[
= \deg Y(Y(t)^\ast)_{\leq |\mu^\ast|} - \deg Y(t)^\ast_{\leq |\mu^\ast|}
\]
\[
= \deg Y(\left( T^{\mu} \right)^{\ast}) - \deg T(\mu^\ast)
\]
\[
= \deg (T^{\mu})^\ast + \deg T^{\nu} - \deg T^{\mu^\ast}
\]
\[
\deg T^{\mu} - \left( \Lambda_i, \cont(\mu^{(1)}) \right) + \deg T^{\nu} - \deg T^{\mu^\ast},
\]

using (7.1) in the last step. Combining equations (7.3)–(7.7) yields the result. \qed

**Lemma 7.7.** Let $\lambda / \mu \in \mathcal{S}^n$ be a joinable skew diagram, and assume the top right node in $\lambda^{(2)}$ has residue $i$. With $\lambda^\ast / \mu^\ast \in \mathcal{S}^n$ and $\lambda_i / \mu_i \in \mathcal{S}^2$ defined as above, we have

\[
\ch_q(S_{\lambda / \mu}) = q^{d^\ast} \ch_q \left( S_{\lambda^\ast / \mu^\ast} \right) + q^{d_\ast} \ch_q \left( S_{\lambda_i / \mu_i} \right) = q^{d_{\lambda / \mu}} \ch_q \left( S_{\lambda^{(1)} / \mu^{(1)}} \circ S_{\lambda^{(2)} / \mu^{(2)}} \right),
\]

where

\[
d^\ast = \left( \Lambda_i, \cont(\lambda^{(1)}) / \mu^{(1)} \right)
\]
\[
d_\ast = \left( \Lambda_{i+1}, \cont(\lambda^{(1)}) / \mu^{(1)} \right)
\]
\[
d_{\lambda / \mu} = \deg t^{\lambda / \mu} - \deg t^{\lambda^{(1)} / \mu^{(1)}} - \deg t^{\lambda^{(2)} / \mu^{(2)}}.
\]

**Proof.** The first equality follows from Corollary 6.12 and Proposition 7.6 via the bijections that $\tau^\ast$ and $\tau_\ast$ induce on basis elements. The second equality is Theorem 6.14. \qed

8. CUSPIDAL SYSTEMS

Our primary motivation in developing the theory of skew Specht modules was to describe an important class of $R_\alpha$-modules called *cuspidal modules*. In this section we very briefly describe the notion of cuspidal systems for KLR algebras of type $k^{(1)}$. See [7], [14] for a thorough treatment.

8.1. Convex preorders. For the rest of this paper, we consider the case $e > 0$ and $\mathcal{O} = F$, an arbitrary ground field. Recall from 2.3.1 the set of positive roots $\Phi_+$. It is known that $\Phi_+ = \Phi^+_r \cup \Phi^+_i$, where $\Phi^+_r$ are real roots, and $\Phi^+_i = \{ n\delta \mid n \in \mathbb{Z}_{>0} \}$ are the imaginary roots. Take a convex preorder on $\Phi_+$, i.e., a preorder $\leq$ such that for all $\alpha, \beta \in \Phi_+$:

1. $\alpha \leq \beta$ or $\beta \leq \alpha$;
2. if $\alpha \leq \beta$ and $\alpha + \beta \in \Phi_+$, then $\alpha \leq \alpha + \beta \leq \beta$;
3. $\alpha \leq \beta$ and $\beta \leq \alpha$ if and only if $\alpha$ and $\beta$ are proportional.
Then \( \Phi^e = \Phi^e_{\ast} \cup \Phi^e_{\ast}' \), where \( \Phi^e_{\ast} := \{ \alpha \in \Phi^e_{\ast} \mid \alpha > \delta \} \) and \( \Phi^e_{\ast}' := \{ \alpha \in \Phi^e_{\ast} \mid \alpha < \delta \} \). Let \( C' \)
be the Cartan matrix of finite type corresponding to the subset of vertices \( I' = \Gamma \setminus \{0\} \), and let \( \Phi' \)
be the corresponding root system with positive roots \( \Phi'_{\ast} \). In what follows we make the additional
assumption that the convex preorder is balanced:

\[
\Phi^e_{\ast} = \{ m \delta + \alpha \mid \alpha \in \Phi'_{\ast}, m \in \mathbb{Z}_{\geq 0} \} = \{ m \delta + \alpha_1 + \cdots + \alpha_j \mid m \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq j \leq e-1 \}
\]

\[
\Phi^e_{\ast}' = \{ m \delta - \alpha \mid \alpha \in \Phi'_{\ast}, m \in \mathbb{Z}_{\geq 1} \} = \{ m \delta - \alpha_i + \cdots + \alpha_j \mid m \in \mathbb{Z}_{\geq 1}, 1 \leq i \leq j \leq e-1 \}.
\]

Balanced convex preorders exist, see [1]. One can label the real roots so that

\[
\Phi^e_{\ast} = \{ \rho_1 > \rho_2 > \cdots \} \quad \text{and} \quad \Phi^e_{\ast}' = \{ \cdots > \rho_{-2} > \rho_{-1} \}.
\]

8.2. Cuspidal systems. A cuspidal system associated to the convex preorder \( \leq \) is the following data:

1. (Cusp) An irreducible \( R_{\rho} \)-module \( L_{\rho} \) assigned to every \( \rho \in \Phi^e_{\ast} \), with the property: if \( \beta, \gamma \in Q_{+} \setminus \{0\} \) are such that \( \rho = \beta + \gamma \) and \( \text{Res}_{\beta, \gamma} L_{\rho} \neq 0 \), then \( \beta \) is a sum of positive roots
less than \( \rho \) and \( \gamma \) is a sum of positive roots greater than \( \rho \).

2. (Imag) An irreducible \( R_{m \delta} \)-module \( L(\nu) \) assigned to every \((e-1)\)-multipartition \( m \)
for every \( m \in \mathbb{Z}_{\geq 0} \), with the property: if \( \beta, \gamma \in Q_{+} \setminus \{0\} \) are such that \( m \delta = \beta + \gamma \) and \( \text{Res}_{\beta, \gamma} L(\nu) \neq 0 \), then \( \beta \) is a sum of real roots less than \( \delta \) and \( \gamma \) is a sum of real roots
greater than \( \delta \). Moreover \( L(\nu) \neq L(\zeta) \) unless \( \nu = \zeta \).

We call modules of the first type cuspidal and modules of the second type imaginary.

Let \( \alpha \in Q_+ \). Define the set \( \Pi(\alpha) \) of root partitions of \( \alpha \) to be the set of all pairs \((M, \nu)\),
where \( M = (m_1, m_2, \ldots ; m_0, \ldots, m_{-2}, m_{-1}) \) is a sequence of nonnegative integers,
and \( \nu \) is an \((e-1)\)-multipartition of \( n_0 \) such that \( m_0 \delta + n_0 \rho_0 = \alpha \). There is a bilexicographic
partial order \( \leq \) on \( \Pi(\alpha) \), see [7].

Given a root partition \((M, \nu) \in \Pi(\alpha)\), define the proper standard module

\[
\sum(M, \nu) := L_{\rho_1}^{m_1} \circ L_{\rho_2}^{m_2} \circ \cdots \circ L(\nu) \circ \cdots L_{\rho_{-2}}^{m_{-2}} \circ L_{\rho_{-1}}^{m_{-1}} \langle \text{shift}(M, \nu) \rangle,
\]

where \( \text{shift}(M, \nu) = \sum_{n \neq 0}(\rho_n, m_n)(n_m - 1)/4 \).

Much of the importance of cuspidal systems lies in the following classification theorem:

**Theorem 8.1.** [2] Main Theorem] For any convex preorder there exists a cuspidal system. Moreover:

(i) For every root partition \((M, \nu)\), the proper standard module \( \sum(M, \nu) \) has irreducible
head, denoted \( L(M, \nu) \).

(ii) \( \{ L(M, \nu) \mid (M, \nu) \in \Pi(\alpha) \} \) is a complete and irredundant system of irreducible \( R_{\alpha} \)-
modules up to isomorphism.

(iii) \( \sum(M, \nu) : L(M, \nu) \rangle_{\zeta} = 1, \) and \( \sum(M, \nu) : L(M, \zeta) \rangle_{\zeta} \neq 0 \) implies \((N, \nu) \leq (M, \zeta)\).

(iv) \( L(M, \nu)^\Diamond \cong L(M, \nu) \).

A canonical definition of the imaginary modules \( L(\nu) \) is provided in [10].

8.3. Minuscule imaginary representations. The ‘smallest’ imaginary modules, those in
\( R_{m} \)-mod, are of particular importance. By the above, they are in bijection with \((e-1)\)-
multipartitions of \( 1 \). We label them \( L_{m; i} \), for \( i \in I \setminus \{0\} \), and call them minuscule imaginary
modules.

**Proposition 8.2.** For each \( i \in I \setminus \{0\} \), \( L_{m; i} \) can be characterized up to isomorphism as the
unique irreducible \( R_{m} \)-module such that \( i_1 = 0 \) and \( i_e = i \) for all words \( i \) of \( L_{m; i} \).

**Proof.** This is [2] Lemma 5.1, Corollary 5.3. \(\Box\)
8.4. **Minimal pairs.** Let \( \rho \in \Phi^r_+ \). A pair of positive roots \((\beta, \gamma)\) is called a minimal pair for \( \rho \) if

(i) \( \beta + \gamma = \rho \) and \( \beta \succeq \rho \);

(ii) for any other pair \((\beta', \gamma')\) satisfying (i) we have \( \beta' \succeq \beta \) or \( \gamma' \prec \gamma \).

**Lemma 8.3.** Let \( \rho \in \Phi^r_+ \) and \((\beta, \gamma)\) be a minimal pair for \( \rho \). If \( L \) is a composition factor of \( \Delta(\beta, \gamma) = L_\beta \circ L_\gamma \), then \( L \cong L(\beta, \gamma) \) or \( L \cong L_\rho \), up to shift.

**Proof.** This follows immediately from the minimality of \((\beta, \gamma) \in \Pi(\rho) \setminus \{\rho\} \) and Theorem 8.1(iii).

One can be more precise in the case that \((\beta, \gamma) \in \Phi^r_+ \). Define

\[
\rho_{\beta, \gamma} := \max\{n \in \mathbb{Z}_{\geq 0} \mid \beta - n\gamma \in \Phi_+\}.
\]

**Lemma 8.4.** \([\text{Remark 6.5}]\). Let \( \rho \in \Phi^r_+ \), and let \((\beta, \gamma)\) be a real minimal pair for \( \rho \). Then

\[
[L_\beta \circ L_\gamma] = [L(\beta, \gamma)] + q^{\rho_{\beta, \gamma}}[L_\rho],
\]

and

\[
[L_\gamma \circ L_\beta] = q^{-\rho_{\beta, \gamma}}[L(\beta, \gamma)] + q^{-\rho_{\beta, \gamma}}[L_\rho].
\]

Lemmas 8.3 and 8.4 are useful in inductively constructing cuspidal modules.

8.5. **Extremal words.** Let \( i \in I \). Define \( \theta^*_i : \langle I \rangle \to \langle I \rangle \) by

\[
\theta^*_i(j) = \begin{cases} j_1 \cdots j_{d-1} & \text{if } j_d = i; \\ 0 & \text{otherwise.} \end{cases}
\]

Extend \( \theta^*_i \) linearly to a map \( \theta^*_i : \mathcal{A}\langle I \rangle \to \mathcal{A}\langle I \rangle \). Let \( x \in \mathcal{A}\langle I \rangle \), and define

\[
\varepsilon_i(x) := \max\{k \geq 0 \mid (\theta^*_i)^k(x) \neq 0\}.
\]

**Definition 8.5.** A word \( i_1^{a_1} \cdots i_b^{a_b} \in \langle I \rangle \), with \( a_1, \ldots, a_b \in \mathbb{Z}_{\geq 0} \), is called extremal for \( x \) if

\[
a_b = \varepsilon_{i_b}(x), a_{b-1} = \varepsilon_{i_{b-1}}((\theta^*_{i_b})^{a_b}(x)), \ldots, a_1 = \varepsilon_{i_1}((\theta^*_{i_2})^{a_2} \cdots (\theta^*_{i_b})^{a_b}(x)).
\]

A word \( i \in \langle I \rangle \) is called extremal for \( M \in R_\alpha\text{-mod} \) if it is an extremal word for the quantum integer \( n \) if

\[
\text{We have the quantum integer } [n] := (q^n - q^{-n})/(q - q^{-1}) \in \mathcal{A} \text{ for } n \in \mathbb{Z} \text{, and the quantum factorial } [n]! := [1][2] \cdots [n]. \text{ The following lemma is useful in establishing multiplicity-one results for } R_\alpha\text{-modules.}
\]

**Lemma 8.6.** \([\text{Lemma 2.28}]\). Let \( L \) be an irreducible \( R_\alpha\)-module, and \( i = i_1^{a_1} \cdots i_b^{a_b} \in \langle I \rangle_\alpha \) be an extremal word for \( L \). Then \( \dim_q L_i = [a_1]^! \cdots [a_b]^! \).

9. **Cuspidal modules are skew hook Specht modules**

Take a balanced convex preorder \( \preceq \) on \( \Phi_+ \), as described in Section 8.1. In this section we prove that the cuspidal modules \( L_\rho \), for \( \rho \in \Phi^r_+ \) are skew Specht modules associated to certain skew hook shapes, and provide an inductive process for identifying them as such.
9.1. Cuspidal modules for a balanced convex preorder. Throughout this section we work with Young diagrams and skew diagrams of level \( l = 1 \). Let \( \kappa = (i) \). For \( i \in I \), Let \( \nu_i = (1) \in \mathcal{P}_\alpha^\kappa \). The following is clear:

Lemma 9.1. For \( i \in I \), \( L_{\kappa} \cong S^{i^i} \).

Let \( \kappa = (0) \), and \( \eta_i \in \mathcal{P}_\delta^\kappa \) be the hook partition of content \( \delta \) with a node of residue \( i \) in the bottom row, depicted below with residues shown:

\[
\begin{array}{cccc}
0 & 1 & \cdots & i-2 & i-1 \\
\vdots \\
\end{array}
\]

\[
\begin{array}{cccc}
i-1 \\
\vdots \\
1 \\
\end{array}
\]

Let \( X_0 = 0 \) and define \( X_{i-1} := F\{v \in S^n | \text{res}_T(e) = i - 1 \} \subseteq S^n \) for \( 1 < i \leq e - 1 \). The following lemma describes minuscule imaginary modules as quotients of \( S^n \).

Lemma 9.2.

1. \( X_{i-1} \) is a submodule of \( S^n \).
2. \( X_{i-1} \cong L_{\delta,i-1}(1) \) if \( i > 1 \).
3. \( S^n/X_{i-1} \cong L_{\delta,i} \).

Proof. For \( i > 1 \), it is easy to see that

\[
\{ T \in \text{St}(\eta_i) | \text{res}_T(e) = i - 1 \} = \{ T \in \text{St}(\eta_i) | \deg T = 1 \}
\]

and

\[
\{ T \in \text{St}(\eta_i) | \text{res}_T(e) = i \} = \{ T \in \text{St}(\eta_i) | \deg T = 0 \},
\]

give a partition of \( \text{St}(\eta_i) \), and hence \( X_{i-1} \) is the span of degree 1 elements in \( S^n \). As there are no repeated entries in words of \( S^n \), it follows that every negatively-graded element of \( R_{\delta} \) acts as zero on \( S^n \), and hence \( X_{i-1} \) is a submodule. Moreover all words of \( X_{i-1} \) are of the form \((0, \ldots, i - 1)\), and all word spaces are 1-dimensional and in degree 1. Thus it follows from Proposition \ref{prop:module} that \( X_{i-1} \cong L_{\delta,i-1}(1) \). Then all words of \( S^n/X_{i-1} \) are of the form \((0, \ldots, i)\) and all word spaces are 1-dimensional and in degree 0, so again it follows from Proposition \ref{prop:module} that \( S^n/X_{i-1} \cong L_{\delta,i} \).

For \( 1 \leq i \leq e - 1 \), \( m \in \mathbb{Z}_{\geq 0} \), let \( \lambda^{m,i}/\mu^{m,i} \) be the skew hook diagram in \( \mathcal{P}_{m\delta,\alpha_i}^\kappa \), where \( l = 1 \), \( \kappa = ((1-m)i \ (\text{mod} \ e)) \),

\[
\lambda^{m,i} = (mi + 1, ((m - 1)i + 1)^{e-i}, \ldots, (i + 1)^{e-i}, 1^{e-i})
\]

and

\[
\mu^{m,i} = (((m - 1)i)^{e-i}, \ldots, (2i)^{e-i}, i^{e-i}).
\]
In other words, $\lambda^{m,i}/\mu^{m,i}$ is the minimal (in the sense of Definition 7.4) skew hook diagram with residues shown:

$$\begin{array}{cccc}
0 & 1 & \ldots & -1 \\
\vdots \\
-1 & 1 & 0 & \ldots \\
\end{array}$$

$$\begin{array}{cccc}
0 & 1 & \ldots & -1 \\
\vdots \\
-1 & 1 & 0 & \ldots \\
\end{array}$$

where the 0-node appears on the inner corners $m$ times, and the $i$-node appears on the outer corners $m+1$ times.

**Lemma 9.3.** For $1 \leq i \leq e-1$, $m \in \mathbb{Z}_{\geq 0}$, $L(m\delta + \alpha_i) \cong S^{\lambda^{m,i}/\mu^{m,i}}$.

**Proof.** We prove this by induction on $m$. As $\lambda^{0,i}/\mu^{0,i} = t_i$, the claim follows by Lemma 9.1. Now assume that $L(m\delta + \alpha_i) \cong S^{\lambda^{m,i}/\mu^{m,i}}$. It is easy to see that $(m\delta + \alpha_i, \delta)$ is a minimal pair for $(m+1)\delta + \alpha_i$ (see [7 §6.1]). By Lemma 9.2, the factors of $S^n_p$ are $L_{\delta,i}$ and $L_{\delta,i-1}$. Thus by Lemma 9.3, the only possible factors (up to shift) of $S^{\lambda^{m,i}/\mu^{m,i}} \circ S^n_p$ are

$$L((m+1)\delta + \alpha_i) \quad \text{and} \quad L(m\delta + \alpha_i, \delta^{(j)})$$

for some number of times, with shifts.

Note that $\varphi/\mu := (\lambda^{m,i}, \eta_i)/(\mu^{m,i}, \emptyset)$ is joinable, with $\alpha_s/\mu_s$ (as defined in [7]) equal to $\lambda^{m+1,i}/\mu^{m+1,i}$, so by Lemma 7.7 we have

$$\operatorname{ch}_q(S^{\lambda^{m,i}/\mu^{m,i}} \circ S^n_p) = q^a \operatorname{ch}_q(S^{\lambda^{m+1,i}/\mu^{m+1,i}}) + q^b \operatorname{ch}_q(S^{\lambda^{n}/\mu^{n}})$$

for some $a, b \in \mathbb{Z}$. By injectivity of the character map [5, Theorem 3.17], it follows that the only factors of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$, are those in [12.2], up to some shift. If $t \in \text{St}(\lambda^{m+1,i}/\mu^{m+1,i})$, with $i(t) = i_1 \cdots i_k$, note that $\alpha_{i_{k-1}+1} + \cdots + \alpha_{i_k} \neq \delta$, i.e., there is no sequence of removable nodes in $[6,1]$ whose residues add up to $\delta$, as is easily seen. Thus $\text{Res}_{m\delta + \alpha_i, \delta} S^{\lambda^{m+1,i}/\mu^{m+1,i}} = 0$. But by adjointness and Theorem 8.1, $S_{m\delta + \alpha_i \delta} S(m\delta + \alpha_i, \delta^{(j)}) \neq 0$ for all $j \in \Gamma \setminus \{0\}$.

Hence $L(m\delta + \alpha_i, \delta^{(j)})$ is not a factor of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$ for any $j$, and the only possible factor is $L((m+1)\delta + \alpha_i)$ some number of times, with shifts.

Consider the extremal word

$$i = 0^{m+1}1^{m+1} \cdots (i-1)^{m+1}(e-1)^{m+1}(i+1)^{m+1}i^{m+2}$$

of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$. There are $((m+1)!)^{e-1}(m+2)!$ distinct $t \in \text{St}(\lambda^{m+1,i}/\mu^{m+1,i})$ such that $i(t) = i$, so this is the (ungraded) dimension of the $i$-word space of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$. By Lemma 8.6, the dimension of a module with extremal word $i$ must be exactly $((m+1)!)^{e-1}(m+1)!$, which implies that $L((m+1)\delta + \alpha_i)$ can only appear once in $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$, with some shift.

Let $t^{\text{top}} \in \text{St}(\lambda^{m+1,i}/\mu^{m+1,i})$ be the tableau achieved by entering $1, \ldots, m$ in the 0-nodes of $\lambda^{m+1,i}/\mu^{m+1,i}$ from top to bottom, then $m+1, \ldots, 2m$ in the 1-nodes from top to bottom,
and so forth, until the \((i-1)\)-nodes are filled, then fill the nodes with residue \(e-1, e-2, \ldots, i\) in the same fashion, working from top to bottom. Then \(t^{\text{top}}\) has residue sequence \(i\), and
\[
\deg t^{\text{top}} = \frac{(e-1)m(m+1)}{2} + \frac{(m+1)(m+2)}{2}.
\]
Let \(t^{\text{bot}}\) be constructed in the same fashion, except with nodes filled from bottom to top. Then
\[
\deg t^{\text{bot}} = -\frac{(e-1)m(m+1)}{2} - \frac{(m+1)(m+2)}{2}.
\]
As these degrees are the greatest and least in the expression \([[m + 1]]^{e-1}[m + 2]]\) it follows that \(S^{\lambda_{m+1,i}/\mu_{m+1,i}}\) is symmetric with respect to grading, and hence \(S^{\lambda_{m+1,i}/\mu_{m+1,i}} \cong L((m + 1)\delta + \alpha_i)\) with no shift.

For \(1 \leq i \leq e-1, \ m \in \mathbb{Z}_{\geq 1}\), let \(l = 1, \ k = ((1-m)i \ (\text{mod} \ e))\), and let \(\lambda_{m,i}/\mu_{m,i}\) be the skew hook diagram in \(\mathcal{R}^\kappa\), where
\[
\lambda_{m,i} = (m,((m-1)i+1)^{e-i},\ldots,(i+1)^{e-i},1^{e-i-1})
\]
\[
\mu_{m,i} = (((m-1)i)^{e-i},\ldots,(2i)^{e-i},i^{e-i}).
\]
In other words, \(\lambda_{m,i}/\mu_{m,i}\) is the minimal skew hook diagram with residues shown:

\[
\begin{array}{cccc}
0 & 1 & \cdots & \underbrace{-2} & \cdots \\
\vdots & & & & \\
\vdots & & & & \\
\end{array}
\]

(9.3)

where the 0-node appears on the inner corners \(m\) times, and the \(i\)-node appears on the outer corners \(m - 1\) times.

**Lemma 9.4.** For \(1 \leq i \leq e-1, \ m \in \mathbb{Z}_{\geq 1}\), \(L(m\delta - \alpha_i) \cong S^{\lambda_{m,i}/\mu_{m,i}}\langle 1-m \rangle\).

**Proof.** We go by induction on \(m\), and the proof proceeds in the same manner as Lemma 9.3. The base case is slightly different however. \(S^{\lambda_{1,i}/\mu_{1,i}}\) is the following hook partition, with residues shown:

\[
\begin{array}{cccc}
0 & 1 & \cdots & \underbrace{-2} & \cdots \\
\vdots & & & & \\
\vdots & & & & \\
\end{array}
\]

By [7, Lemma 5.2], \(L(\delta - \alpha_i)\) factors through the cyclotomic quotient to become the unique irreducible \(R^{\Lambda_0}_{\delta - \alpha_i}\)-module. Consideration of the words of \(S^{\lambda_{1,i}/\mu_{1,i}}\) shows that it factors
through the cyclotomic quotient as well. Moreover, all of its word spaces are 1-dimensional and in degree 0, so it follows that $S^{\lambda_{1,i}/\mu_{1,i}} \cong L(\delta - \alpha_i)$.

The induction step proceeds as in Lemma 9.3 with $(\delta, m\delta - \alpha_i)$ used as a minimal pair for $(m + 1)\delta - \alpha_i$. Considering the induction product $S^{\mu_i} \circ S^{\lambda_{1,i}/\mu_{1,i}}$, and using Lemma 7.7 one sees that the only possible factor of $S^{\lambda_{1,i}/\mu_{1,i}}$ is $L((m + 1)\delta - \alpha_i)$, some number of times, with shifts. Consideration of the extremal word

$$i = 0^{m+1}1^{m+1} \cdots (i - 1)^{m+1}(e - 1)^{m+1} \cdots (i + 1)^{m+1}1^m$$

shows that $L((m + 1)\delta - \alpha_i)$ appears but once as a factor of $S^{\lambda_{1,i}/\mu_{1,i}}$, with some shift. $L((m + 1)\delta - \alpha_i)$ must have $i$-word space of graded dimension $c(m + 1))^{e-1}[m]^1$. As before, we define two standard $\lambda_{1,i}/\mu_{1,i}$-tableaux; $t^{\text{top}}$, where the nodes are filled in from top to bottom according to their order in $i$, and $t^{\text{bot}}$, where the nodes are filled similarly from bottom to top. Then

$$\text{deg } t^{\text{top}} = \left[\frac{(e - 1)m(m + 1)}{2} + \frac{(m - 1)m}{2}\right] - m$$

$$\text{deg } t^{\text{bot}} = \left[-\frac{(e - 1)m(m + 1)}{2} - \frac{(m - 1)m}{2}\right] - m.$$

On the right we have the greatest and least degrees in the expression $(m + 1))^{e-1}[m]^1$, shifted by $-m$, hence $L((m + 1)\delta - \alpha_i) \cong S^{\lambda_{1,i}/\mu_{1,i}}(1 - (m + 1))$, completing the proof. \(\Box\)

9.2. Identifying cuspidal modules as skew hook Specht modules. We now present an inductive process for identifying cuspidal modules as skew hook Specht modules with a certain shift.

**Proposition 9.5.** Let $\alpha$ be a real positive root, and assume that for all real positive roots $\beta$ with $\text{ht}(\beta) < \text{ht}(\alpha)$, we have $L_\beta \cong S^{\lambda_\alpha/\mu_\alpha}(c_\beta)$ for some skew hook diagram $\lambda_\alpha/\mu_\alpha \in \mathcal{S}_\beta^\kappa$, where $\kappa = (k)$ for some $k \in I$ and $c_\beta \in \mathbb{Z}$. Then the following process gives a skew hook diagram $\lambda_\alpha/\mu_\alpha$ and $c_\alpha \in \mathbb{Z}$, such that $L_\alpha \cong S^{\lambda_\alpha/\mu_\alpha}(c_\alpha)$.

1. If $\alpha = m\delta + \alpha_i$ for some $m \in \mathbb{Z}_{\geq 0}$ and $i \in I \setminus \{0\}$, then $\lambda_\alpha/\mu_\alpha = \lambda_\alpha_\alpha/\mu_\alpha$ and $c_\alpha = 0$.
2. If $\alpha = m\delta - \alpha_i$ for some $m \in \mathbb{Z}_{\geq 1}$ and $i \in I \setminus \{0\}$, then $\lambda_\alpha/\mu_\alpha = \lambda_{\alpha_\alpha}/\mu_{\alpha_\alpha}$ and $c_\alpha = 1 - m$.
3. Else there is a real minimal pair $(\beta, \gamma)$ for $\alpha$.
   a. If $\lambda/\mu := (\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$ is joinable, then $\lambda_\alpha/\mu_\alpha = \lambda_\alpha/\mu_\alpha$ and
   $$c_\alpha = c_\beta + c_\gamma - p_{\beta, \gamma} + (\beta, \gamma) + d_{\alpha} - d_{\lambda/\mu};$$
   b. Else $\lambda/\mu := (\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$ is joinable, $\lambda_\alpha/\mu_\alpha = \lambda_\alpha/\mu_\alpha$ and
   $$c_\alpha = c_\beta + c_\gamma + p_{\beta, \gamma} + d^* - d_{\lambda/\mu},$$

where $d_{\alpha}, d^*$ are as in Lemma 7.7, $d_{\lambda/\mu}$ as in Lemma 6.14 and $p_{\beta, \gamma}$ as in (8.7).

**Proof.** (1) and (2) are Lemmas 9.3 and 9.4, so assume we are in case (3). There exists a real minimal pair $(\beta, \gamma)$ for $\alpha$ by [2] Lemma 6.9]. By assumption $L(\beta) \cong S^{\lambda_\beta/\mu_\beta}(c_\beta)$, and $L(\gamma) \cong S^{\lambda_\gamma/\mu_\gamma}(c_\gamma)$. We have $\beta = m\delta + (-1)^e(\alpha_i + \cdots + \alpha_j)$ for some $s \in \{0, 1\}$, $1 \leq i \leq j \leq e - 1$ and $\gamma = m\delta + (-1)^e(\alpha_{i'} + \cdots + \alpha_{j'})$ for some $s' \in \{0, 1\}$, $1 \leq i' \leq j' \leq e - 1$. Since $\beta + \gamma$ is a real root, one of the following must be true:

$s = s', j + 1 = i'$ or $s = s', j' + 1 = i$ or $s = -s', j = j'$ or $s = -s', i = i'$.

Note that since $\lambda_\beta/\mu_\beta$ (resp. $\lambda_\gamma/\mu_\gamma$) is a skew hook diagram, $s = 0$ (resp. $s' = 0$) implies that the lower left node of $\lambda_\beta/\mu_\beta$ (resp. $\lambda_\gamma/\mu_\gamma$) has residue $i$ (resp. $i'$), and the top right node has residue $j$ (resp. $j'$). If $s = 1$ (resp. $s' = 1$), then the lower left node of $\lambda_\beta/\mu_\beta$ (resp. $\lambda_\gamma/\mu_\gamma$) has residue $j + 1$ (resp. $j' + 1$), and the top right node has residue $i - 1$ (resp. $i' - 1$). In any case then, we see that one of $(\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$ or $(\lambda_\gamma, \lambda_\beta)/(\mu_\gamma, \mu_\beta)$ must be joinable.
Assume the former, and set $\lambda/\mu := (\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$. Then, using Lemma 8.4,

$$[S^{\lambda_\beta/\mu_\beta} \circ S^{\lambda_\gamma/\mu_\gamma}] = q^{-c_\beta-c_\gamma} [L_\beta \circ L_\gamma] = q^{-c_\beta-c_\gamma} [L(\beta, \gamma)] + q^{-c_\beta-c_\gamma+p_\beta,\gamma-(\beta, \gamma)} [L_\alpha].$$

Using Lemma 7.7 and the fact that $ch_q$ is injective on $[R_\alpha\text{-mod}]$, we also have

$$[S^{\lambda_\beta/\mu_\beta} \circ S^{\lambda_\gamma/\mu_\gamma}] = q^{d_\beta-d_\lambda} [S^{\lambda_\beta/\mu_\beta}] + q^{d_\gamma-d_\lambda} [S^{\lambda_\gamma/\mu_\gamma}].$$

Thus, $L_\alpha$ must be (a shift of) $S^{\lambda_\beta/\mu_\beta}$ or $S^{\lambda_\gamma/\mu_\gamma}$. But $1_{\beta,\gamma} \cdot \lambda_\beta/\mu_\beta \neq 0$, so $Res_{\beta,\gamma} S^{\lambda_\beta/\mu_\beta} \neq 0$, and thus the cuspidality property of $L(\alpha)$ implies it must be the latter, proving the validity of step (3)(a).

If instead, $\lambda/\mu := (\lambda_\beta, \lambda_\gamma, \mu_\beta, \mu_\gamma)$ is joinable, we use the second statement in Lemma 8.4 and a similar argument to prove the validity of step (3)(b). □

**Corollary 9.6.** For a balanced convex order, all real cuspidal modules of $R_\alpha$ are skew hook Specht modules up to some shift.

**Proof.** Apply Proposition 9.5 inductively, with base case given by Lemma 9.1. □

**Remark 9.7.** In [11 §8.4], Kleshchev and Ram showed that in finite type $A$, the cuspidal modules (associated to a convex lexicographic order) are Specht modules associated to hook partitions. Thus one can view Corollary 9.6 as an affine analogue of this fact.

#### 9.3. An explicit formulation of cuspidal modules.

For a certain balanced convex order we can bypass the inductive process and explicitly describe the skew hook diagram (and shift) corresponding to a real positive root. Take the following balanced $e$-row order (in the sense of [5]) on $\Phi_+$:

1. $m\delta + \alpha > m'\delta > m''\delta - \alpha$, for all $m \in \mathbb{Z}_{\geq 0}$, $m', m'' \in \mathbb{Z}_{\geq 1}$, $\alpha \in \Phi'_+$.
2. $m\delta + \alpha_i + \cdots + \alpha_j > m'\delta + \alpha_{i'} + \cdots + \alpha_{j'}$ if $i < j'$; or $i = i', m < m'$; or $i = i', m = m', j < j'$.
3. $m\delta - \alpha_i - \cdots - \alpha_j > m'\delta - \alpha_{i'} - \cdots - \alpha_{j'}$ if $i > j'$; or $i = i', m > m'$; or $i = i', m = m', j < j'$.

Under this order, it is easy to see that for any $\alpha > \delta$ not of the form $m\delta + \alpha_i$, the positive root $\beta > \alpha$ immediately preceding $\alpha$ in the order constitutes the lefthand side of a real minimal pair $(\beta, \alpha - \beta)$ for $\alpha$. Similarly, for $\alpha < \delta$ not of the form $m\delta - \alpha_i$, the positive root $\alpha > \beta$ immediately succeeding $\alpha$ in the order constitutes the righthand side of a real minimal pair $(\alpha - \beta, \beta)$ for $\alpha$. Then, applying the inductive process in Proposition 9.5 we arrive at:
(1) For $1 \leq i \leq j \leq e - 1$ and $m \in \mathbb{Z}_{\geq 0}$, $L(m\delta + \alpha_i + \cdots + \alpha_j) \cong S^{\lambda/\mu}$, where $\lambda/\mu$ is the minimal skew hook diagram with residues shown:

\[
\begin{array}{c}
\vdots \\
| \alpha_i | & \alpha_{i+1} \\
| \alpha_j | & \alpha_{j+1} \\
| \vdots | & | \alpha_{e-1} | \\
\end{array}
\]

where the 0-node appears on the inner corners $m$ times, and the $i$-node appears on the outer corners $m + 1$ times.

(2) For $1 \leq i \leq j \leq e - 1$ and $m \in \mathbb{Z}_{\geq 1}$, $L(m\delta - \alpha_i - \cdots - \alpha_j) \cong S^{\lambda/\mu}(1-m)$, where $\lambda/\mu$ is the minimal skew hook diagram with residues shown:

\[
\begin{array}{c}
\vdots \\
| \alpha_i | & \alpha_{i+1} \\
| \alpha_j | & \alpha_{j+1} \\
| \vdots | & | \alpha_{e-1} | \\
\end{array}
\]

where the 0-node appears on the inner corners $m$ times, and the $i$-node appears on the outer corners $m - 1$ times.

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