Exponential Convergence to the Maxwell Distribution Of Solutions of Spatially Inhomogenous Boltzmann Equations

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Abstract

We consider the rate of convergence of solutions of spatially inhomogenous Boltzmann equations, with hard sphere potentials, to some equilibriums, called Maxwellians. Maxwellians are spatially homogenous static Maxwell velocity distributions with different temperatures and mean velocities. We study solutions in weighted space $L^1(\mathbb{R}^3 \times \mathbb{T}^3)$. We prove a conjecture of C. Villani in [28]: assume the solution is sufficiently localized and sufficiently smooth, then the solution, in $L^1$-space, converges to a Maxwellian, exponentially fast in time.

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1 Formulation of the problem

In this paper we consider the Boltzmann’s equation

$$\partial_t g + v \cdot \nabla_x g = Q(g, g)$$

(1.1)

with initial condition

$$g(v, x, 0) = g_0(v, x) \geq 0, \quad v \in \mathbb{R}^3, \quad x \in \mathbb{R}^3/(2\pi\mathbb{Z})^3$$

satisfying $\int_{\mathbb{R}^3 \times \mathbb{T}^3} g_0(v, x) \, d^3v d^3x = 1$. The nonlinearity $Q(g, g)$ is chosen to correspond to a hard-sphere potential:

$$Q(g, g)(v, x) := \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega|[g(u', x)g(v', x) - g(u, x)g(v, x)] \, d^3u \, d^2\omega, \quad (1.2)$$
where \( u', v' \in \mathbb{R}^3 \) are given by 
\[
\begin{align*}
\dot{u}' &= u - [(u - v) \cdot \omega] \omega, \\
\dot{v}' &= v + [(u - v) \cdot \omega] \omega.
\end{align*}
\]

The equation has the following properties, for any time \( t \geq 0 \), provided that the solution exists,

(A) \( g(v, x, t) \geq 0 \) if \( g_0(v, x) \geq 0 \);

(B) \[
\int_{\mathbb{R}^3 \times T^3} g(v, x, t) \, d^3v \, d^3x = \int_{\mathbb{R}^3 \times T^3} g_0(v, x) \, d^3v \, d^3x = 1; \tag{1.3}
\]

(C) \[
\int_{\mathbb{R}^3 \times T^3} v_k g(v, x, t) \, d^3v \, d^3x = \int_{\mathbb{R}^3 \times T^3} v_k g_0(v, x) \, d^3v \, d^3x, \quad k = 1, 2, 3; \tag{1.4}
\]

(D) \[
\int_{\mathbb{R}^3 \times T^3} |v|^2 g(v, x, t) \, d^3v \, d^3x = \int_{\mathbb{R}^3 \times T^3} |v|^2 g_0(v, x) \, d^3v \, d^3x. \tag{1.5}
\]

The equation has a family of Maxwellian solutions \( M_{T, \mu} \) defined as 
\[
M_{T, \mu}(v) := \frac{1}{(2\pi)^3} \frac{1}{(2\pi T)^{\frac{3}{2}}} e^{-|v-\mu|^2/2T}, \tag{1.6}
\]

where \( T \) is the temperature, and \( \mu \in \mathbb{R}^3 \) is the mean velocity of the gas.

The purpose of this paper is to prove asymptotic stability of Maxwellians. The main objective is to prove a conjecture of C. Villani, namely the solution will converge to a Maxwellian exponentially fast, under the assumption of the smoothness and boundedness (uniform in time \( t \)) of the solution. For the complete statement, see Main Theorem 2.1.

In the literature, one finds many results on the asymptotic stability of Maxwellians for the Boltzmann equation. One circle of results concerns the spatially homogeneous case, where \( g(v, x, t) \) is independent of the position \( x \). This direction of research has been pioneered by H.Grad in [16]. Further results can be found in [5, 11, 15, 24]. Another circle of results concerns the Boltzmann equation on an exponentially weighted \( L^2 \) space; see, e.g. [31, 20, 21, 17, 18, 6]. The advantage of working in such spaces is that spectral theory on Hilbert space can be used. There are also results in [30, 29, 22, 7, 23].

In this context, the existence of weak global solutions has been established in [13]. In [12], the asymptotic stability of Maxwellians, for general initial conditions, has been studied under the
assumption that global smooth solutions exist. In the spatially homogeneous case, such results appear, e.g. in [1, 33, 24, 8].

There is an earlier proof of Villani’s conjecture due to Maria Gualdani, Stephane Mischler and Clement Mouhot in [19]. In the present paper an alternative proof is presented. For a non-constructive proof, see [2].

In this paper, the main difficulty is to study the properties of a certain linear operator $L$ defined in Equation (3.3), below. An important step in our analysis consists in proving an appropriate decay estimate for the linear evolution given by $e^{-tL}(1 - P)$, where $P$ is the Riesz projection onto the eigenspace of $L$ corresponding to the eigenvalue 0. The difficult is that, as in [14], the spectrum of the operator $L$ occupies the entire right half of the complex plane, except for a strip of strictly positive width around the imaginary axis that only contains the eigenvalue 0; see Figure 6.1, below. Rewriting $e^{-tL}(1 - P)$ in terms of the resolvent, $(L - z)^{-1}$, of $L$,

$$e^{-tL}(1 - P) = -\frac{1}{2\pi i} \oint_{\Gamma} e^{-tz}(L - z)^{-1} dz,$$  (1.7)

(see, e.g., [26]), where the integration contour $\Gamma$ encircles the spectrum of $L$, except for the eigenvalue 0, we encounter the problem of proving strong convergence of the integral on the right hand side of (1.7) on $L^1$. This problem is solved in Section 6.

Our paper is organized as follows. The main Theorem will be stated in Section 2. The operator obtained by linearization around Maxwellian will be derived and studied in Section 3. Based on the spectrum of the linear operator, the solution will be decomposed into several components. The estimates on these components will be a reformulation of the Main Theorem. This will take place in Section 4. The main theorem will be proved in Section 5. In the rest of the paper, namely from Section 6, we prove the decay estimate for the propagator.

In the present paper the meaning of $a \lesssim b$ is that, for some fixed constant $C$,

$$a \leq Cb.$$

(1.8)

2 Main Theorem

We start with formulating C. Villani conjecture, see [28] [12].

The conjecture is formulated under assumptions that $g$, the solution to Boltzmann equation (1.1), satisfies several conditions, including the following two:

1. For some sufficiently large constant $\phi > 0$,

$$\sup_{t \geq 0} \|\langle v \rangle^\phi g(\cdot, t)\|_{L^1(\mathbb{R}^3 \times T^3)} \lesssim 1.$$  (2.1)
(2) For some sufficiently large natural number $L$,

$$\sup_{t \geq 0} \sum_{|k| \leq L} \| \partial_x^k g(\cdot, t) \|_{L^2(\mathbb{R}^3 \times T^3)} \lesssim 1. \quad (2.2)$$

By assuming these and some more assumptions, L. Desvilletes and C. Villani proved in [12] that the solution converges to a Maxwellian faster than $t^{-N}$ in space $L^1$, for any $N \geq 0$. C. Villani conjectured the convergence rate is exponential, see [28].

It is worth pointing out that there are examples satisfying all the assumptions, by the results of Guo in [20, 21].

In what follows we state the main result of the present paper, which is an affirmative answer to the conjecture. The initial conditions we choose need to be sufficiently close to a Maxwellian, and this is satisfied by solution at a large time, proved by C. Villani, see [28].

Before stating the main result, we choose $T, \mu$ for initial conditions $g_0$. Recall that $g_0$ is the initial conditions for Boltzmann equation (1.1), and $M_{T, \mu}, T \in \mathbb{R}^+, \mu \in \mathbb{R}^3$, are Maxwellian solutions. It is not difficult to see that there exist unique $T$ and $\mu$ such that

$$\int_{\mathbb{R}^3 \times T^3} v_k g_0(v, x, t) \, dv \, dx = \int_{\mathbb{R}^3 \times T^3} v_k M_{T, \mu} \, dv \, dx, \quad k = 1, 2, 3, \quad (2.3)$$

$$\int_{\mathbb{R}^3 \times T^3} |v|^2 g_0(v, x, t) \, dv \, dx = \int_{\mathbb{R}^3 \times T^3} |v|^2 M_{T, \mu} \, dv \, dx.$$

The main result is

**Theorem 2.1.** Assume the solution $g$ of Boltzmann equation satisfies the estimates in [1] and [2] above, and assume that the initial conditions $g(\cdot, 0)$ is sufficiently close to a Maxwellian $M_{T_0, \mu_0}$ for some $T_0, \mu_0$, in the sense that for some $\delta = \delta(T_0) > 0$

$$\| g(\cdot, 0) - M_{T_0, \mu_0} \|_{L^1(\mathbb{R}^3 \times T^3)} \leq \delta. \quad (2.4)$$

Then for the $T, \mu$ chosen in (2.3), there exist constants $C_0, C_1 > 0$, such that for any time $t \geq 0$

$$\| g(\cdot, t) - M_{T, \mu} \|_{L^1(\mathbb{R}^3 \times T^3)} \leq C_1 e^{-C_0 t}. \quad (2.5)$$

This theorem will be proven in Section 5.

### 3 The linearization around the Maxwellian

We start with defining a linear operator, obtained by linearizing around the Maxwellian solutions.
Recall that $M_{T,\mu}$ are solutions to the equation

$$-v \cdot \nabla xg + Q(g, g) = 0.$$  \hfill (3.1)

We plug $g = M_{T,\mu} + f$ into the nonlinear operator $-v \cdot \nabla xg + Q(g, g)$ to find

$$-v \cdot \nabla xg + Q(g, g) = -L_{T,\mu}f + Q(f, f).$$  \hfill (3.2)

Here the linear operator $L$ is defined by

$$L_{T,\mu} := v \cdot \nabla x + \nu_{T,\mu}(v) + K_{T,\mu}.$$  \hfill (3.3)

where $\nu_{T,\mu}$ is the multiplication operator defined by

$$\nu_{T,\mu}(v) := \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega|M_{T,\mu}(u) \, d^3u d^2\omega, $$  \hfill (3.4)

and $K_{T,\mu}$ is an integral operator, defined by, for any function $f$,

$$K_{T,\mu}(f) := M_{T,\mu}(v) \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega|M_{T,\mu}(u) \, d^3u d^2\omega
$$

$$- \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega|M_{T,\mu}(u) f(v') \, d^3u d^2\omega
$$

$$- \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega|M_{T,\mu}(v) f(u') \, d^3u d^2\omega
$$

$$=: K_1 - K_2 - K_3$$  \hfill (3.5)

where the operators $K_l$, $l = 1, 2, 3$, are naturally defined.

Next we study the eigenvectors and eigenvalues of the operator $L_{T,\mu}$. By the fact that

$$-v \cdot \nabla x cM_{T,\mu} + Q(cM_{T,\mu}, cM_{T,\mu}) = 0.$$  \hfill (3.6)

for any $c \in \mathbb{R}$, $T > 0$, $\mu \in \mathbb{R}^3$, we obtain, after taking $c$, $T$ and $\mu$ derivatives on the equation above, that $L_{T,\mu}$ has five eigenvectors with eigenvalues zero

$$M_{T,\mu}, \partial_T M_{T,\mu}, \partial_{\mu_k} M_{T,\mu}, \, k = 1, 2, 3.$$  \hfill (3.7)

A key fact is that these are the only eigenvectors for $L_{T,\mu}$ with eigenvalue 0 in certain weighted $L^2$ space, see \cite{10, 20, 21, 25}.
Define its Riesz projection, onto the eigenvector space, by $P_{T, \mu}$. It takes the form

$$P_{T, \mu} h := \frac{1}{8\pi^2} e^{-\frac{|v-\mu|^2}{2T}} \int_{\mathbb{R}^3 \times \mathbb{T}^3} h(u, x) \, d^3u \, d^3x$$

(3.8)

$$+ \frac{1}{u_1^2} e^{-\frac{|u|^2}{2T}} \sum_{k=1}^3 e^{-\frac{|v_k-\mu|^2}{4T}} (v_k-\mu_k) \int_{\mathbb{R}^3 \times \mathbb{T}^3} (u_k - \mu_k) h(u, x) \, d^3u \, d^3x$$

$$+ \frac{1}{\int_{\mathbb{R}^3} (|u|^2 - 3T)^2 e^{-\frac{|u|^2}{2T}}} \frac{1}{d^3u \, 8\pi^2} e^{-\frac{|v-\mu|^2}{2T}} (|v-\mu|^2 - 3T) \int_{\mathbb{R}^3 \times \mathbb{T}^3} (|u-\mu|^2 - 3T) h(u, x) \, d^3u \, d^3x.$$

To prepare for our analysis, we state some estimates on the nonlinearity $Q$ and the operators $\nu_{T, \mu}, K_{T, \mu}$. Define a constant $\Lambda_T$ as

$$\Lambda_T := \inf_v \nu_{T, \mu}(v).$$

(3.9)

The results are:

**Lemma 3.1.** $\Lambda_T$ is positive, i.e.

$$\Lambda_T > 0.$$  

(3.10)

There exists a positive constant $C_T$ such that

$$\nu_{T, \mu}(v) \geq C_T (1 + |v - \mu|).$$

(3.11)

For any $m \geq 0$, there exists a positive constant $\Upsilon_{m, T}$ such that, for any functions $f, g \in L^1(\mathbb{R}^3)$,

$$\| (v - \mu)^m K_{T, \mu} f \|_{L^1(\mathbb{R}^3)} \leq \Upsilon_{m, T} \| (v - \mu)^{m+1} f \|_{L^1(\mathbb{R}^3)},$$

and

$$\| (v)^m Q(f, g) \|_{L^1(\mathbb{R}^3)} \leq C_m \left[ \| f \|_{L^1(\mathbb{R}^3)} \| (v)^{m+1} g \|_{L^1(\mathbb{R}^3)} + \| (v)^{m+1} f \|_{L^1(\mathbb{R}^3)} \| g \|_{L^1(\mathbb{R}^3)} \right].$$

(3.12)

This lemma is proven in Appendix A.

### 4 Reformulation of Main Theorem 2.1

To facilitate later analysis we reformulate equation (1.1) into a more convenient form.

For the $T, \mu$ chosen in (2.3), we define a function $f : \mathbb{R}^3 \times \mathbb{T}^3 \times \mathbb{R}^+ \to \mathbb{R}$ by

$$f(v, x, t) := g(v, x, t) - M_{T, \mu}(v).$$

(4.1)
By the conservation laws in (1.3)-(1.5), we have

$$\int_{\mathbb{R}^3 \times T^3} v_k f(v, x, t) \, dv^3 \, dx^3 = 0, \quad k = 1, 2, 3,$$

(4.2)

$$\int_{\mathbb{R}^3 \times T^3} |v|^2 f(v, x, t) \, dv^3 \, dx^3 = 0,$$

(4.3)

and

$$\int_{\mathbb{R}^3 \times T^3} f(v, x, t) \, dv^3 \, dx^3 = 0,$$

(4.4)

by the fact $\int_{\mathbb{R}^3 \times T^3} g(v, x, t) \, dv^3 \, dx^3 = \int_{\mathbb{R}^3 \times T^3} M_{T, \mu}(v) \, dv^3 \, dx^3 = 1.$

These orthogonality conditions for $f$ and the definition of $P^{T, \mu}$ in (3.8) imply

$$P^{T, \mu} f(\cdot, t) = 0.$$  

(4.5)

In what follows we derive effective governing equation for $f$.

Plug the decomposition of $g$ in (4.1) into Boltzmann equation (1.1) to derive

$$\partial_t f = -L_{T, \mu} f + Q(f, f).$$

(4.6)

Here the linear operator $L_{T, \mu}$ is defined in (3.3), and the nonlinear term $Q(f, f)$ is defined in (1.2).

For the initial conditions $f_0(v, x) := g_0(v, x) - M_{T, \mu}(v)$, we use the conditions imposed on $g_0$ in (2.4) and the assumed condition $\|\langle v \rangle^4 g_0\|_{L^1} < \infty$, see (2.1), to find that

$$\|f_0\|_{L^1} \leq \delta^2 \ll 1.$$  

(4.7)

To cast the equation for $\partial_t f$ into a convenient form, we apply the operator $1 - P^{T, \mu}$ on both sides of (4.6), and use that $P^{T, \mu} f = 0$, and that $P^{T, \mu}$ commutes with $L_{T, \mu}$, to obtain an effective equation for $f$,

$$\partial_t f = -L_{T, \mu} f + (1 - P^{T, \mu})Q(f, f).$$

(4.8)

Apply Duhamel’s principle on (4.3) to obtain

$$f = e^{-t L_{T, \mu}} f_0 + \int_0^t e^{-(t-s) L_{T, \mu}} (1 - P^{T, \mu})Q(f, f)(s) \, ds.$$  

(4.9)

The proof that $f$ decays exponentially fast in weighted $L^1$ norm, relies critically on the decay estimates of the propagator $e^{-t L_{T, \mu}} (1 - P^{T, \mu})$ acting on $L^1$. The result is
Theorem 4.1. If $m > 0$ is sufficiently large, then there exist constants $C_0$, $C_1$, $\Pi > 0$, such that for any function $h$, we have

\[
\| \langle v - \mu \rangle^m e^{-tL^T,\mu} (1 - P^T,\mu) h \|_{L^1(\mathbb{R}^3 \times T^3)} \leq \| \langle v - \mu \rangle^{m+\Pi} h \|_{L^1(\mathbb{R}^3 \times T^3)}.
\]  

(4.10)

The theorem will be proved in Section 6.

We continue to study the equation (4.9). Apply the propagator estimate in Theorem 4.1 and use that $(1 - P^T,\mu)f_0 = f_0$ to find that,

\[
\| \langle v - \mu \rangle^m f(\cdot, t) \|_{L^1} \lesssim e^{-C_0 t} \| \langle v - \mu \rangle^{m+\Pi} f_0 \|_{L^1} + \int_0^t e^{-C_0 (t-s)} \| \langle v - \mu \rangle^{m+\Pi} Q(f, f)(s) \|_{L^1} ds.
\]  

(4.11)

We will prove in Subsection 4.1 below, using the assumptions on the solution in Theorem 2.1, that

\[
\| \langle v - \mu \rangle^m f(s) \|_{L^1(\mathbb{R}^3 \times T^3)} \lesssim \| \langle v - \mu \rangle^m f(s) \|_{L^1(\mathbb{R}^3 \times T^3)}^\frac{5}{4} \leq e^{-\frac{5}{4} C_0 s} \mathcal{M}^{\frac{5}{4}}(t)
\]  

(4.12)

where $\mathcal{M}$ is a controlling function defined as

\[
\mathcal{M}(t) := \max_{0 \leq s \leq t} e^{C_0 s} \| \langle v - \mu \rangle^m f(s) \|_{L^1(\mathbb{R}^3 \times T^3)}. 
\]  

(4.13)

It is not hard to see that $\| \langle v - \mu \rangle^m f(s) \|_{L^1(\mathbb{R}^3 \times T^3)}$, from (4.9), is continuous in $t$, this implies that the function $\mathcal{M}$ is also continuous.

Suppose (4.12) holds, then by (4.11)

\[
\| \langle v - \mu \rangle^m f(\cdot, t) \|_{L^1} \lesssim e^{-C_0 t} \| \langle v - \mu \rangle^{m+\Pi} f_0 \|_{L^1} + \mathcal{M}^{\frac{5}{4}}(t).
\]  

(4.14)

Observe that $\mathcal{M}$ is an increasing function by definition, hence

\[
\mathcal{M}(t) \lesssim \| \langle v - \mu \rangle^{m+\Pi} f_0 \|_{L^1} + \mathcal{M}^{\frac{5}{4}}(t).
\]  

(4.15)

It turns out that (4.15) directly implies Main Theorem 2.1, see Section 5.

4.1 Proof of (4.12)

Without loss of generality, we only prove (4.12) for $\mu = 0$.

We divide the proof into two steps. In the first step we prove

\[
\| \langle v \rangle^{m+\Pi} Q(f, f) \|_{L^1} \lesssim \sum_{|\beta| \leq 4} \| \langle v \rangle^{m+\Pi+1} \partial_\beta^2 f \|_{L^1} \| f \|_{L^1}.
\]  

(4.16)
Then in the second step we prove, for $|\beta| \leq 4$

$$\|\langle v \rangle^{m+\Pi+1} \partial_x^\beta f \|_{L^1} \lesssim \|\langle v \rangle^m f \|_{L^1}^{1/4} \left[ 1 + \|\langle v \rangle^{3m+4\Pi+24} f \|_{L^1} + \| (1 - \partial_x^2)^{20} f \|_{L^2} \right].$$  \hspace{1cm} (4.17)

Suppose that (4.16) and (4.17) hold, then we apply the assumptions in Main Theorem 2.1 to obtain

$$\| (1 - \partial_x^2)^{20} f \|_{L^2} \leq \|\langle v \rangle^{2m+2\Pi+12} f \|_{L^1} \lesssim 1$$ \hspace{1cm} (4.18)

To see this, we use the assumptions (1) and (2) in Main Theorem 2.1 to obtain

$$\| (1 - \partial_x^2)^{20} f \|_{L^2} \leq \| (1 - \partial_x^2)^{20} g \|_{L^2} \leq (1 - \partial_x^2)^{20} M \|_{L^2} \lesssim 1, \quad \|\langle v \rangle^{2m+2\Pi+12} f \|_{L^1} \lesssim 1, \quad \|\langle v \rangle^{2m+2\Pi+12} M \|_{L^1} \lesssim 1,$$ \hspace{1cm} (4.19)

recall that $g = M + f$ in (4.1).

Plug (4.18) into (4.17) to find

$$\sum_{|\beta| \leq 4} \|\langle v \rangle^{m+\Pi+1} \partial_x^\beta f \|_{L^1} \lesssim \|\langle v \rangle^m f \|_{L^1}^{1/4},$$ \hspace{1cm} (4.20)

This together with (4.16) implies the desired estimate for $\|\langle v \rangle^{m+\Pi} Q(f,f) \|_{L^1}$, or (4.12). To complete the proof we need to prove (4.16) and (4.17).

We start with proving (4.16).

Use (3.13) to find

$$\|\langle v \rangle^{m+\Pi+1} Q(f,f) \|_{L^1} \leq \|\langle v \rangle^{m+\Pi+1} f \|_{L^1} \| f \|_{L^1} \| f \|_{L^1} \| f \|_{L^1} \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)},$$

$$\leq \| f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \max_{x \in \mathbb{T}^3} \|\langle v \rangle^{m+\Pi+1} f(\cdot, x) \|_{L^1(\mathbb{R}^3)}.$$

We claim that

$$\max_{x \in \mathbb{T}^3} \|\langle v \rangle^{m+\Pi+1} f(\cdot, x) \|_{L^1(\mathbb{R}^3)} \lesssim \sum_{|\beta| \leq 4} \|\langle v \rangle^{m+\Pi+1} \partial_x^\beta f \|_{L^1},$$

which together with the estimates above implies the desired estimate.

To verify the claim, we Fourier expand $f$ into the form

$$f(v,x) = \sum_{n \in \mathbb{Z}^3} e^{i n \cdot x} f_n(v)$$
and compute directly to have
\[
\max_{x \in \mathbb{T}^3} \| \langle v \rangle^{m+\Pi+1} f(\cdot, x) \|_{L^1(\mathbb{R}^3)} \leq \sum_{n \in \mathbb{Z}^3} \| \langle v \rangle^{m+\Pi+1} f_n \|_{L^1(\mathbb{R}^3)}
\]
\[
= \sum_{n \in \mathbb{Z}^3} \frac{1}{(|n|^2 + 1)^2} (1 + |n|^2)^2 \| \langle v \rangle^{m+\Pi+1} f_n \|_{L^1(\mathbb{R}^3)}. \tag{4.21}
\]

Observe that
\[
(|n|^2 + 1)^2 \| \langle v \rangle^{m+\Pi+1} f_n \|_{L^1(\mathbb{R}^3)} = \frac{1}{(2\pi)^3} \| (e^{i n \cdot x}, (1 - \partial_x^2)^2 \langle v \rangle^{m+\Pi+1} f)_{T^3} \|_{L^1(\mathbb{R}^3)}
\]
\[
\leq \frac{1}{(2\pi)^3} \| (1 - \partial_x^2)^2 \langle v \rangle^{m+\Pi+1} f \|_{L^1(\mathbb{R}^3 \times T^3)}.
\]

Put this back into (4.21) and use the fact that
\[
\sum_{n \in \mathbb{Z}^3} \frac{1}{(|n|^2 + 1)^2} < \infty
\]
to obtain the desired (4.16).

Next we prove (4.17), which is to control
\[
\| \langle v \rangle^m \partial_\beta x f \|_{L^1}, \quad |\beta| \leq 4.
\]

The key step is to prove, using some Hölder’s-type inequality to obtain, for any constant \( \epsilon > 0 \),
\[
\| \langle v \rangle^{m+\Pi+1} \partial_x^3 f \|_{L^1} \lesssim \| \frac{1}{\epsilon} \langle v \rangle^{2m+2\Pi+12} + \epsilon (v)^{-10} (1 - \partial_x^2)^{20} \|_{L^1}. \tag{4.22}
\]

This estimate takes an equivalent form, by defining \( g := \left[ \frac{1}{\epsilon} \langle v \rangle^{2m+2\Pi+12} + \epsilon (v)^{-10} (1 - \partial_x^2)^{20} \right] f \),
\[
\epsilon \| \langle v \rangle^{m+\Pi+11} \partial_\beta^3 \left[ \langle v \rangle^{2m+2\Pi+22} + \epsilon^2 (1 - \partial_x^2)^{20} \right]^{-1} g \|_{L^1} \lesssim \| g \|_{L^1}. \tag{4.23}
\]

For the latter, Fourier-expand \( g \)
\[
g(v, x) = \sum_{n \in \mathbb{Z}^3} e^{in \cdot x} g_n(v),
\]
and compute directly to obtain the desired result, recall that \( |\beta| \leq 4 \)
\[
\epsilon \| \langle v \rangle^{m+\Pi+11} \partial_\beta^3 \left[ \langle v \rangle^{2m+2\Pi+22} + \epsilon^2 (1 - \partial_x^2)^{20} \right]^{-1} g \|_{L^1(\mathbb{R}^3 \times T^3)}
\]
\[
\leq \sum_n \epsilon (1 + |n|^2)^4 \| \langle v \rangle^{m+\Pi+11} \|_{L^1(\mathbb{R}^3 \times T^3)}
\]
\[
\leq \sum_n \frac{1}{(1 + |n|^2)^6} \| g_n \|_{L^1(\mathbb{R}^3 \times T^3)} \lesssim \| g \|_{L^1(\mathbb{R}^3 \times T^3)} \tag{4.24}
\]
where in the second step we used the Hölder’s inequality, and in the last step we used that
\[ \| g_n \|_{L^1(\mathbb{R}^3)} = \frac{1}{(2\pi)^3} \| \langle e^{inx} \rangle g \|_{T^3} \leq \frac{1}{(2\pi)^3} \| g \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \]
and
\[ \sum_{n \in \mathbb{Z}^3} \frac{1}{(1 + |n|^2)^6} \leq 1. \]

After proving (4.22), we apply Hölder’s inequality on both terms to obtain
\[ \| \langle v \rangle^{m+\Pi+1} \partial_x^2 f \|_{L^1} \lesssim \frac{1}{\epsilon^1} \| \langle v \rangle^{2m+2\Pi+12} f \|_{L^1} + \epsilon \| \langle v \rangle^{-10} (1 - \partial_x^2)^{20} f \|_{L^1} \]
\[ \lesssim \frac{1}{\epsilon^1} \| \langle v \rangle^m f \|_{L^1} + \epsilon^2 \| \langle v \rangle^{3m+4\Pi+24} f \|_{L^1} + \epsilon \| (1 - \partial_x^2)^{20} f \|_{L^2}, \] (4.25)
where, we used the facts \( \langle v \rangle^{-10} \in L^2(\mathbb{R}^3) \), and \( L^2(\mathbb{T}^3) \subset L^1(\mathbb{T}^3) \) to obtain
\[ \| \langle v \rangle^{-10} (1 - \partial_x^2)^{20} f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \leq \| (1 - \partial_x^2)^{20} f \|_{L^2(\mathbb{R}^3)} \|_{L^1(\mathbb{T}^3)} \lesssim \| (1 - \partial_x^2)^{20} f \|_{L^2(\mathbb{R}^3 \times \mathbb{T}^3)}. \]

Next set
\[ \epsilon = \left[ \| \langle v \rangle^m f \|_{L^1} \right]^{\frac{1}{6}} \] (4.26)
in (4.25) to obtain the desired result (4.17).

5 Proof of the Main Theorem 2.1

By the choice of initial conditions, we have that,
\[ \mathcal{M}(0) \ll 1 \] and \( \| \langle v \rangle^{m+\Pi} f_0 \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \leq \| f_0 \|_{L^1}^{\frac{1}{6}} \| \langle v \rangle^{2m+2\Pi} f_0 \|_{L^1}^{\frac{1}{6}} \ll 1, \]
where we used the condition \( \| f_0 \|_{L^1}^{\frac{1}{6}} \ll 1 \) in (4.17), and the assumption \( \| \langle v \rangle^{2m+2\Pi} f_0 \|_{L^1}^{\frac{1}{6}} \ll 1 \) in Theorem 2.1.

This together with (4.15), and that \( \mathcal{M} \) is a continuous function, implies that for \( t \in [0, \infty) \),
\[ \mathcal{M}(t) \leq 2C \| \langle v \rangle^{m+\Pi} f_0 \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)}. \] (5.1)

This, together with the definition of \( \mathcal{M} \) in (4.13), proves Theorem 2.1. \( \square \)
6 Propagator Estimates: Proof of Theorem 4.1

To simplify the notations, we fix the constant $T$ and vector $\mu$ to be

\[ T = \frac{1}{2}, \quad \mu = 0 \]  \hspace{1cm} (6.1)

and for the operators $L^{1/2,0}, \nu^{1/2,0}, K^{1/2,0}$ and $P^{1/2,0}$ and for the Maxwellian $M^{1/2,0}$ we denote

\[ L := L^{1/2,0}, \quad \nu := \nu^{1/2,0}, \quad K := K^{1/2,0}, \quad P := P^{1/2,0}, \quad M := M^{1/2,0}. \]  \hspace{1cm} (6.2)

It is easy to see that our arguments, in what follows, can be easily adapted to general cases.

The proof are based on previous results in [10, 21, 20, 25], where it was proved that the operator $L$, mapping the space $M^{1/2 L^2} := \{ f : \mathbb{R}^3 \times \mathbb{T}^3 \to \mathbb{C} \mid \| M^{-1/2} f \|_{L^2} < \infty \}$ into itself, has an eigenvalue 0 with eigenvectors listed in (3.7), and it has a gap with the other parts of the spectrum. By these we establish the crucial identity (7.12) below.

Besides these, in proving Theorem 4.1, we adopt the same strategy as in [14], to circumvent the difficulty that the spectrum of $L$ is “too big”.

We start with outlining the general strategy of the proof.

There are two typical approaches to proving decay estimates for propagators. The first one is to apply the spectral theorem, (see e.g. [26]), to obtain

\[ e^{-t L} (1 - P) = \frac{1}{2\pi i} \oint \Gamma e^{-t \lambda} (\lambda - L)^{-1} d\lambda \]

where the contour $\Gamma$ is a curve encircling the spectrum of $L(1 - P)$. The obstacle is that the spectrum of $L(1 - P)$ occupies the entire right half of the complex plane, except for a strip in a neighborhood of the imaginary axis, as illustrated in Figure 6.1 below. This makes it difficult to prove strong convergence on $L^1$ of the integral on the right hand side.

The second approach is to use perturbation theory, which amounts to expanding $e^{-tL}$ in powers of the operator $K$, (see (6.15)):

\[ e^{-tL} = e^{-t(\nu \cdot v \cdot \nabla_x)} + \int_0^t e^{-(t-s)(\nu \cdot v \cdot \nabla_x)} K e^{-s(\nu \cdot v \cdot \nabla_x)} ds + \ldots. \]

It will be shown in Proposition 6.1 that each term in this expansion can be estimated quite well, but the fact that $K$ is unbounded forces us to estimate them in different spaces.
We will combine these two approaches to prove Theorem 4.1.

We expand the propagator $e^{-tL}(1 - P)$ using Duhamel’s principle:

$$
e^{-tL}(1 - P) = \sum_{k=0}^{12} (1 - P)A_k(t) + (1 - P)\tilde{A}(t),$$  \hspace{1cm} (6.3)

where the operators $A_k$ are defined recursively, with

$$A_0 = A_0(t) := e^{-t(\nu + v \cdot \nabla x)},$$  \hspace{1cm} (6.4)

and $A_k$, $k = 1, 2, \cdots, 12$, given by

$$A_k(t) := \int_0^t e^{-(t-s)(\nu + v \cdot \nabla x)}KA_{k-1}(s) \, ds.$$  \hspace{1cm} (6.5)

Finally $\tilde{A}$ is defined by

$$\tilde{A}(t) = \int_0^t e^{-(t-s)L}KA_{12}(s) \, ds.$$  \hspace{1cm} (6.6)

The exact form of $A_k$, $k = 0, 1, \cdots, 12$, implies the following estimates.

Recall that $\Lambda := \inf_\nu \nu(v) > 0.$
Proposition 6.1. For any $C_0 \in (0, \Lambda)$, there exists a positive constant $C_1$ such that, for any function $f : \mathbb{R}^3 \times \mathbb{T}^3 \to \mathbb{C}$,

$$\| \langle v \rangle^m A_k(t) f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \leq C_1 e^{-C_0 t} \| \langle v \rangle^{m+k} f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)}.$$  \hfill (6.7)

This proposition is proven in Subsection 6.1.

Next we estimate $\tilde{A}$, which is given by

$$\tilde{A} = \int_0^t e^{-(t-s_1)L} K \int_0^{s_1} e^{-(s_1-s_2)(\nu+v \cdot \nabla_x)} K \cdots \int_0^{s_{12}} e^{-(s_{12}-s_{13})(\nu+v \cdot \nabla_x)} K e^{-s_{13}(\nu+v \cdot \nabla_x)} ds_{13} \cdots ds_1.$$  \hfill (6.8)

We start with transforming $\tilde{A}$ into a more convenient form.

One of the important properties of the operators $L$ is that, for any function $g : \mathbb{R}^3 \to \mathbb{C}$ (i.e., independent of $x$) and $n \in \mathbb{Z}^3$, we have that

$$P e^{i n \cdot x} g = 0 \text{ if } n \neq 0,$$

$$L e^{i n \cdot x} g = e^{i n \cdot x} L_n g,$$

$$e^{i n \cdot x} g = e^{i n \cdot (\nu + n \cdot v)} g,$$  \hfill (6.9)

where the operator $L_n$ is unbounded and defined as

$$L_n := \nu + i n \cdot v + K.$$

(Recall that $P$ has been defined in (3.8).)

To make (6.8) applicable, we Fourier-expand the function $g : \mathbb{R}^3 \times \mathbb{T}^3 \to \mathbb{C}$ in the variable $x$, i.e.,

$$g(v, x) = \sum_{n \in \mathbb{Z}^3} e^{i n \cdot x} g_n(v).$$  \hfill (6.10)

Then use (6.8) and compute directly to obtain

$$\| (1 - P) \tilde{A} g \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \leq \sum_{n \in \mathbb{Z}^3} \| \tilde{A}_n g_n \|_{L^1(\mathbb{R}^3)},$$

where $\tilde{A}_n$ is defined as follows: If $n \neq (0,0,0)$ then

$$\tilde{A}_n := \int_0^t e^{-(t-s_1)L_n} K \int_0^{s_1} e^{-(s_1-s_2)(\nu + i v \cdot n)} K \cdots \int_0^{s_{12}} e^{-(s_{12}-s_{13})(\nu + i v \cdot n)} K e^{-s_{13}(\nu + i v \cdot n)} ds_{13} \cdots ds_1$$

and for $n = (0,0,0)$ we define

$$\tilde{A}_0 := \int_0^t (1 - P) e^{-(t-s_1)L_0} K \int_0^{s_1} e^{-(s_1-s_2)\nu} K \cdots \int_0^{s_{12}} e^{-(s_{12}-s_{13})\nu} K e^{-s_{13}\nu} ds_{13} \cdots ds_1.$$
Next, we study $\tilde{A}_n$, which is defined in terms of the operators $e^{-tL_n}$, $e^{-t[v+i\cdot v]}$ and $Ke^{-t[v+i\cdot v]}K$. It is easy to estimate $e^{-t[v+i\cdot v]}$: The fact that the function $\nu$ has a positive global minimum $\Lambda$ (see (3.9)) implies that
\[ \|e^{-t[v+i\cdot v]}\|_{L^1 \rightarrow L^1} \leq e^{-\Lambda t}. \] (6.11)

Next we consider operator $e^{-tL_n}$.

The result is:

**Lemma 6.2.** There exist constants $C_0, C_1 > 0$, such that if $n \neq (0,0,0)$, then
\[ \|e^{-tL_n}\|_{(\nu)^{-m}L^1(\mathbb{R}^3) \rightarrow (\nu)^{-m}L^1(\mathbb{R}^3)} \leq C_1 (1 + |n|)e^{-C_0 t}. \] (6.12)

and for $n = (0,0,0)$
\[ \|e^{-tL_0(1-P)}\|_{(\nu)^{-m}L^1(\mathbb{R}^3) \rightarrow (\nu)^{-m}L^1(\mathbb{R}^3)} \leq C_1 e^{-C_0 t}. \] (6.13)

This lemma will be proven in Section 7.

The most important step is to estimate $K(n)t$:

It is well known that the operator $K$, defined in (3.5), has an integral kernel $K(v,u)$: for any function $f: \mathbb{R}^3 \rightarrow \mathbb{C}$,
\[ K(f) = K_1(f) - K_2(f) - K_3(f) \] (6.14)

with integral kernels taking the form
\[ K_1(f) = \pi e^{-|v|^2} \int_{\mathbb{R}^3} |u - v| f(u) \, d^3u, \] (6.15)
\[ K_2(f) + K_3(f) = 2\pi \int_{\mathbb{R}^3} |u - v|^{-1} e^{-\frac{|(u-v)\cdot v|^2}{|u-v|^2}} f(u) \, d^3u. \]

Here we take the explicit form of $K$ from [15], (see also [16, 11]).

Then the integral kernel, $K_t(n)(v,u)$, of $K_t(n)$ is given by
\[ K_t(n)(v,u) = \int_{\mathbb{R}^3} K(v,z)e^{-t[v(z)+i\cdot z]}K(z,u) \, dz \]
for some properly defined function $K(v,u)$. The presence of the factor $e^{-i\cdot n \cdot z}$ plays a critically important role. It makes the operator $K_t(n)$ smaller, as $|n|$ becomes larger. Recall that $\Lambda := \inf_{\nu} \nu(v) > 0.
Lemma 6.3. There exists a positive constant \( C_1 \) such that, for any \( n \in \mathbb{Z}^3 \) and \( t \geq 0 \),
\[
\| K_t^{(n)} f \|_{L^1(\mathbb{R}^3)} \leq \frac{C_1}{1 + |n|} e^{-\lambda t} \| \langle \nu \rangle^{-m} f \|_{L^1(\mathbb{R}^3)}.
\] (6.16)

This lemma will be proven in Subsection 6.2.

The results in Proposition 6.1, Lemma 6.2 and Lemma 6.3 suffice to prove Theorem 4.1.

Proof of Theorem 4.1. In Equation (6.3) we have decomposed \( e^{-tL} (1 - P) \) into several terms. The operators \( A_k, k = 0, 1, 2, \ldots, 12 \), are estimated in Proposition 6.1.

In what follows, we study \( \tilde{A} \). By (6.10) we only need to control \( \tilde{A}_n, n \in \mathbb{Z}^3 \). For \( n = (0, 0, 0) \) it is easy to see that
\[
\| \langle \nu \rangle^m \tilde{A}_0 g_n \|_{L^1(\mathbb{R}^3)} \lesssim e^{-\lambda t} \| \langle \nu \rangle^{m+20} g_n \|_{L^1(\mathbb{R}^3)}
\] (6.17)

by collecting the different estimates in (6.11) and Lemma 6.2 and using the estimates on \( K \) in Lemma 6.1.

For \( n \neq 0 \), we observe that the integrands in the definitions of \( \tilde{A}_n \) are products of terms \( e^{-(t-s_1)L_n}, Ke^{-(s_k-s_{k+1})(\nu + in \cdot v)}K \) and \( e^{-(s_k-s_{k+1})(\nu + in \cdot v)} \), where \( k \in \{1, 2, \ldots, 13\} \) (we use the convention that \( s_{14} = 0 \)). Applying the bounds in (6.11), Lemma 6.2 and Lemma 6.3 we see that there is a constant \( C_0 > 0 \) such that
\[
\| \langle \nu \rangle^m \tilde{A}_n g_n \|_{L^1(\mathbb{R}^3)} \lesssim e^{-\lambda t} (1 + |n|) \| \langle \nu \rangle^{m+20} g_n \|_{L^1(\mathbb{R}^3)} \times \int_0^t \int_0^{s_1} \cdots \int_0^{s_{12}} [1 + |n|(s_{12} - s_{13})]^{-1} [1 + |n|(s_{10} - s_{11})]^{-1} \cdots [1 + |n|(s_2 - s_3)]^{-1} ds_{13} ds_{12} \cdots ds_1.
\]

Compute directly to find, for any positive constant \( C_0 \leq \lambda \), there exists a constant \( C_1 > 0 \) such that
\[
\| \langle \nu \rangle^m \tilde{A}_n g_n \|_{L^1(\mathbb{R}^3)} \leq C_1 e^{-C_0 t} \frac{1}{(1 + |n|)^4} \| \langle \nu \rangle^{m+20} g_n \|_{L^1(\mathbb{R}^3)}.
\]

Plugging this and (6.17) into (6.10), we find that
\[
\| \langle \nu \rangle^m (1 - P) \tilde{A} g \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \lesssim C_1 e^{-C_0 t} \sum_{n \in \mathbb{Z}^3} \frac{1}{(1 + |n|)^4} \| \langle \nu \rangle^{m+20} g_n \|_{L^1(\mathbb{R}^3)}.
\] (6.18)

The fact \( g_n = \frac{1}{(2\pi)^3} e^{i n \cdot x}, g_x \) enables to obtain
\[
\| \langle \nu \rangle^{m+20} g_n \|_{L^1(\mathbb{R}^3)} \leq (2\pi)^3 \| \langle \nu \rangle^{m+20} g \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)}.
\]

This, together with the fact that \( \sum_{n \in \mathbb{Z}^3} \frac{1}{(1 + |n|)^4} < \infty \), implies that
\[
\| \langle \nu \rangle^m (1 - P) \tilde{A} g \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \lesssim C_1 e^{-C_0 t} \| \langle \nu \rangle^{m+20} g \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)}.
\] (6.19)
Obviously Equation (6.3), Inequality (6.19) and Proposition 6.1 imply Theorem 4.1.

\[\square\]

6.1 Proof of Proposition 6.1

Recall the definition of the constant \( \Lambda = \frac{1}{2} > 0 \) in (3.9). The definition of \( A_0 \) (see (6.14)) implies that

\[
\|\langle v \rangle^m A_0(t) f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \leq e^{-\Lambda t} \|\langle v \rangle^m f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)}.
\] (6.20)

For \( A_1 \), we use the estimate for the unbounded operator \( K \) given in Lemma 3.1. Compute directly to obtain

\[
\|\langle v \rangle^m A_1(f) \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \leq \int_0^t e^{-\Lambda(t-s)} \|\langle v \rangle^m K e^{-s(\nu + v \cdot \nabla x)} f \|_{L^1(\mathbb{R}^3 \times \mathbb{T}^3)} \, ds
\]

\[
\lesssim \int_0^t e^{-\Lambda(t-s)} e^{-\Lambda s} \, ds \|\langle v \rangle^{m+1} f \|_{L^1} = e^{-\Lambda t} \|\langle v \rangle^{m+1} f \|_{L^1}.
\]

Similar arguments yield the desired estimates for \( A_k, k = 2, 3, \ldots, 12 \).

Thus, the proof of Proposition 6.1 is complete.

\[\square\]

6.2 Proof of Inequality (6.16)

Proof. We denote the integral kernel of the operator \( K \) by \( K(v, u) \) and infer its explicit form from (6.15). It is then easy to see that the integral kernel of the operator \( Ke^{-t(\nu + \mathbf{n} \cdot v)} \) is given by

\[
K_t^{(n)}(v, u) := \int_{\mathbb{R}^3} K(v, z) e^{-t(\nu(z) + \mathbf{n} \cdot z)} K(z, u) \, d^3 z.
\]

We use the oscillatory nature of \( e^{-it \mathbf{n} \cdot z} \) to derive some “smallness estimates” when \( |\mathbf{n}| \) is sufficiently large, by integrating by parts in the variable \( z \). Without loss of generality we assume that

\[
|n_1| \geq \frac{1}{3} |\mathbf{n}|.
\]
Integrate by parts in the variable $z_1$ to obtain

$$K_t^{(n)}(v, u) = \int_{\mathbb{R}^3} K(v, z)K(z, u)\frac{1}{-t[\partial_{z_1} \nu(z) + im_1]} \partial_{z_1} e^{-t[\nu(z) + imz]} \, d^3z$$

$$= \int_{\mathbb{R}^3} \partial_{z_1} [K(v, z)K(z, u)] \frac{1}{t[\partial_{z_1} \nu(z) + im_1]} e^{-t[\nu(z) + imz]} \, d^3z \quad (6.21)$$

The different terms in $\partial_{z_1} [K(v, z)K(z, u)] \frac{1}{t[\partial_{z_1} \nu(z) + im_1]}$ are dealt with as follows.

1. We claim that, for $l = 0, 1$, and for any $\Psi \geq 0$, there exists a constant $c(\Psi) > 0$ such that

$$\int_{\mathbb{R}^3} \langle v \rangle^\Psi |\partial_{z_1} K(v, z)| \, d^3v \leq c(\Psi) \langle z \rangle^{\Psi+2}, \quad \int_{\mathbb{R}^3} \langle z \rangle^\Psi |\partial_{z_1} K(z, u)| \, d^3z \leq C(\Psi) \langle u \rangle^{\Psi+2}. \quad (6.22)$$

2. By direct computation,

$$|\partial_{z_1} e^{-t[\nu(z) + imz]}| \lesssim \frac{1}{|n|^l} \quad \text{for } l = 0, 1. \quad (6.23)$$

These bounds and the fact that $e^{-tw} \lesssim e^{-\Lambda t}$ (see (3.9)) imply that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \langle v \rangle^\Psi |K_t^{(n)}(v, u)g(u)| \, d^3u \lesssim \frac{e^{-\Lambda t}}{|n|^l} \parallel \langle v \rangle^{\Psi+3} g \parallel_{L^1}. \quad (6.16)$$

To remove the non-integrable singularity in the upper bound at $t = 0$, we use a straightforward estimate derived from the definition of $K_t^{(n)}$ to obtain

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \langle v \rangle^\Psi |K_t^{(n)}(v, u)g(u)| \, d^3u \leq C(\Psi) e^{-\Lambda t} \parallel \langle v \rangle^{\Psi+3} g \parallel_{L^1}. \quad (6.16)$$

Combination of these two estimates yields (6.16).

We are left with proving (6.22). In the next we focus on proving (6.22) when $l = 1$, the case $l = 0$ is easier, hence omitted. By direct computation we find that

$$|\partial_{z_1} K(v, z)| \lesssim |\partial_{z_1} |z - v|^{-1} e^{-\frac{|(z - v) \cdot v|^2}{|v|^2}} | + |\partial_{z_1} |z - v| e^{-|v|^2}|$$

and, similarly, that

$$|\partial_{z_1} K(z, u)| \lesssim |\partial_{z_1} |z - u|^{-1} e^{-\frac{|(z - u) \cdot u|^2}{|u|^2}} | + |\partial_{z_1} |z - u| e^{-|u|^2}|.$$
Among the various terms we only study the most difficult one, namely \( \partial_{z_1} \tilde{K}(v, z) \), where \( \tilde{K}(v, z) \) is defined by

\[
\tilde{K}(v, z) := |z - v|^{-1} e^{-\frac{(z - v) \cdot v}{|z - v|^2}}.
\]

By direct computation

\[
|\partial_{z_1} \tilde{K}(v, z)| \lesssim \frac{1 + |v|}{|v - z|^2} e^{-\frac{1}{2} \frac{(z - v) \cdot v}{|z - v|^2}}.
\]

To complete our estimate we divide the set \((v, z) \in \mathbb{R}^3 \times \mathbb{R}^3\) into two subsets defined by \(|v| \leq 10|z|\) and \(|v| > 10|z|\), respectively. In the first subset we have that

\[
|\partial_{z_1} \tilde{K}(v, z)| \lesssim \frac{1}{|v - z|^2} (|v| + 1) \leq \frac{10(|z| + 1)}{|v - z|^2},
\]

and hence

\[
\int_{|v| \leq 8|z|} \langle v \rangle^\Psi |\partial_{z_1} \tilde{K}(v, z)| \, d^3v \leq 10(1 + |z|)^{\Psi + 1} \int_{|v| \leq 10|z|} \frac{1}{|v - z|^2} \, d^3v \lesssim (1 + |z|)^{\Psi + 2}.
\] (6.24)

In the second subset we have that \(z - v \approx -v\), which implies that \(\frac{(z - v) \cdot v}{|z - v|^2} \geq \frac{1}{2} |v|\). Thus,

\[
|\partial_{z_1} \tilde{K}(v, z)| \leq \frac{1 + |v|}{|v|^2} e^{-\frac{1}{8} |v|^2}.
\]

This obviously implies that

\[
\int_{|v| \geq 10|z|} \langle v \rangle^\Psi |\partial_{z_1} \tilde{K}(v, z)| \, d^3v \lesssim \int_{|v| \geq 10|z|} \langle v \rangle^\Psi \frac{1 + |v|}{|v|^2} e^{-\frac{1}{8} |v|^2} \, d^3v \lesssim 1.
\] (6.25)

By such estimates the proof of (6.22) can be easily completed. \(\square\)

7 Proof of Lemma 6.2

Proof. Before we study the linear unbounded operator

\[
L_n := \nu(v) + iv \cdot n + K, \quad n \in \mathbb{Z}^3,
\] (7.1)

mapping \(\langle v \rangle^{-m} L^1(\mathbb{R}^3)\) into the same space, we start with studying \(L_n\), mapping \(M_{\frac{1}{2}}^L L^2(\mathbb{R}^3)\) into itself. Here the definitions of the spaces \(\langle v \rangle^{-m} L^1(\mathbb{R}^3)\) and \(M_{\frac{1}{2}}^L L^2(\mathbb{R}^3)\) are

\[
\langle v \rangle^{-m} L^1(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \to \mathbb{C} | \| \langle v \rangle^m f \|_{L^1} < \infty \}\}
\] (7.2)
\[ M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{C} | \| M_{\frac{1}{2}}^2 f \|_{L^2} < \infty \}. \] (7.3)

Here recall that \( M = M_{\frac{1}{2}, 0} \) is the Maxwellian solution, see [6.2].

Denote the spectrum of the unbounded linear operator \( L_n \), mapping \( M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \) into itself, by \( \sigma(L_n) \). Then since \( K \) is a compact operator in the chosen space, we have that

\[ \sigma(L_n) = \sigma_d(L_n) \cup \sigma_{ess}(L_n). \] (7.4)

Recall that \( L_n \) is related to \( L := \nu(v) + v \cdot \nabla x + K \) by the fact that

\[ L e^{i n \cdot x} f = e^{i n \cdot x} L_n f. \]

Hence if \( f \) is an eigenvector for \( L_n \) in the space \( M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \), then \( e^{i n \cdot x} f \) is an eigenvector for \( L \) in the \( M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3 \times \mathbb{T}^3) \) space, with the same eigenvalue.

By this we have the following results.

**Lemma 7.1.** If \( f : \mathbb{R}^3 \rightarrow \mathbb{C} \) is an eigenvector for \( L_n \) in the space \( M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \), then \( e^{i n \cdot x} f \) is an eigenvector for \( L \) in the \( M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3 \times \mathbb{T}^3) \) space, with the same eigenvalue.

The set of eigenvalues of \( L_n : M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \rightarrow M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \), is a subset of that of \( L : M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3 \times \mathbb{T}^3) \rightarrow M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3 \times \mathbb{T}^3) \).

Moreover if \( f \in M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3 \times \mathbb{T}^3) \) is an eigenvector of \( L \) with eigenvalue \( \lambda \), and \( \langle f, e^{i n \cdot x} \rangle_{\mathbb{T}^3} = \int_{\mathbb{T}^3} f(v, x) e^{i n \cdot x} dx \neq 0 \), then \( \langle f, e^{i n \cdot x} \rangle_{\mathbb{T}^3} \in M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \) is an eigenvector of \( L_n \) with eigenvalue \( \lambda \).

To locate the essential spectrum in the space \( M_{\frac{1}{2}}^2 L^2(\mathbb{R}^3) \), we use the fact that \( K \) is compact to find

\[ \sigma_{ess}(L_n) = \{ \nu(v) + iv \cdot n | v \in \mathbb{R}^3 \}. \] (7.5)

By known results, see [9, 10, 25, 20, 21], and Lemma 7.1 we have that for \( n \neq (0, 0, 0) \)

\[ \sigma_d(L_n) = A_n, \text{ and } \sigma_{ess}(L_n) = \{ \nu(v) + iv \cdot n | v \in \mathbb{R}^3 \}; \] (7.6)

and for \( n = (0, 0, 0) \)

\[ \sigma_d(L_n) = \{ 0 \} \cup A_n, \text{ and } \sigma_{ess}(L_n) = \{ \nu(v) | v \in \mathbb{R}^3 \}. \] (7.7)

Here the sets \( A_n \) keep a uniform distance from the imaginary axis, specifically, there exists a positive constant \( \Lambda \) satisfying

\[ \Lambda \in (0, \inf_{v \in \mathbb{R}^3} \nu(v)). \] (7.8)
such that

\[ Re\lambda \geq \Lambda > 0 \text{ if } \lambda \in \bigcup_{n \in \mathbb{Z}^3} A_n. \]  \hspace{1cm} (7.9)

Figure 7.1: The spectrum of \( L_n \), the curve \( \Gamma_n \), and the region \( \Omega_n \)

In what follows we study \( L_n, \ n \neq (0, 0, 0) \). For \( n = (0, 0, 0) \), the analysis is similar except that 0 is an eigenvalue.

Based on the informations about the spectrum of \( L_n \) in (7.6) and (7.7), we have the following results.

For any \( n \in \mathbb{Z}^3\setminus\{0, 0, 0\} \), we define a curve \( \Gamma_n \) (see Figure 7.1) to encircle the spectrum of \( L_n \),

\[ \Gamma_n := \Gamma_1(n) \cup \Gamma_2(n) \cup \Gamma_3(n) \tag{7.10} \]

with

\[ \Gamma_1(n) := \{ \Theta + i\beta \mid \beta \in [-\Psi(|n| + 1), \Psi(|n| + 1)] \}; \]
\[ \Gamma_2(n) := \{ \Theta + i(|n| + 1)\Psi + \beta + i\Psi\beta(|n| + 1), \beta \geq 0 \}; \]
\[ \Gamma_3(n) := \{ \Theta - i(|n| + 1)\Psi + \beta - i\Psi\beta(|n| + 1), \beta \geq 0 \}. \]

Here \( \Psi \) is a large positive constant to be chosen later, see (7.11), Lemma 7.2 and (8.4) below; \( \Theta > 0 \) can be any constant in \((0, \frac{1}{2}\Lambda)\), with \( \Lambda \) being the same one in (7.9).
Moreover, we define $\Omega_n$ to be the complement of the region encircled by the curve $\Gamma_n$; see Figure 7.1.

For the multiplication operator $\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta$, if the constant $\Psi$ in the definition of the curves $\Gamma_{k,n}$, $k = 0, 1, 2$, in (7.10), are sufficiently large, then there exists a constant $C$ such that for any $\zeta \in \Gamma_n$

$$|\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta|^{-1} \leq C(1 + |\mathbf{v}| + |\mathbf{n} \cdot \mathbf{v}|)^{-1}. \tag{7.11}$$

It is straightforward, but a little tedious to verify this. Details are omitted.

Use the spectral theorem in [26] to find that, if $\Psi$ is large enough, then for $n \neq (0,0,0)$ and for any $g \in M^1_2L^2_1$, we have that

$$e^{-tL_n}g = \frac{1}{2\pi i} \oint_{\Gamma_n} e^{-t\zeta} [\zeta - L_n]^{-1} d\zeta \ g, \tag{7.12}$$

recall that $L_n - \zeta = \nu + i\mathbf{n} \cdot \mathbf{v} - \zeta + K$. To see $[\zeta - L_n]^{-1} = [\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta]^{-1}[1 + K(\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta)]^{-1}$ is uniformly well defined, we use the key fact that the operator $K : \ M^1_2L^2 \to M^1_2L^2$ is compact, and discuss two different cases:

(a) If $|\zeta| \gg 1$ or $|\mathbf{n}| \gg 1$, then this together with (7.11) implies that $|\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta| \gg 1$ everywhere except for a small set, this makes the operator $K(\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta)^{-1}$ small, hence $[\zeta - L_n]^{-1}$ is uniformly well defined. It is easy, but tedious, to prove that $|\nu + i\mathbf{n} \cdot \mathbf{v} - \zeta| \gg 1$ everywhere except for a small set. Moreover the techniques will be used to prove Propositions 8.2 and 8.3 below, which are more involved. Hence we choose to skip the details here.

(b) If $|\zeta|, |\mathbf{n}| = \mathcal{O}(1)$, then $\zeta \in \Gamma_1(\mathbf{n})$ if $\Phi$ in (7.10) is sufficiently large, here the uniformity is implied by Lemma 7.1 and the spectrum of $L$.

Motivated by Cook’s method, see [27], we consider the identity in the space $\langle \mathbf{v} \rangle^{-m}L^1_1(\mathbb{R}^3)$, defined as

$$\langle \mathbf{v} \rangle^{-m}L^1_1 := \{ f : \mathbb{R}^3 \to \mathbb{C} | \langle \mathbf{v} \rangle^m f \rangle_{L^1} < \infty \}. \tag{7.13}$$

By the wellposedness, to be proved in Section 8 below, we have that for any time $t \geq 0$, $e^{-tL_n}g \in \langle \mathbf{v} \rangle^{-m}L^1_1$ if $g \in \langle \mathbf{v} \rangle^{-m}L^1_1$.

For the term on the right hand side of (7.12), the following lemma provides an important estimate.

**Lemma 7.2.** There exists a constant $\Phi$ such that if $m \geq \Phi$, and if the positive constant $\Psi$ in (7.10) is sufficiently large, then there exists a constant $C = C(m)$ independent of $\mathbf{n}$ and $\zeta \in \Gamma_n$ such that, for any point $\zeta \in \Gamma_n$ and $\mathbf{n} \in \mathbb{Z}^3$, we have

$$\|(L_n - \zeta)^{-1}\|_{\langle \mathbf{v} \rangle^{-m}L^1 \to \langle \mathbf{v} \rangle^{-m}L^1} \leq C.$$
This lemma will be proven in section 8.

Applying Lemma 7.2 to (7.12), we obtain that, for \( g \in \langle v \rangle - m L^1 \cap M^{1/2} L^2 \),

\[
\| e^{-tL_n} g \|_{\langle v \rangle - m L^1} \lesssim \int_{\zeta \in \Gamma_1(n) \cup \Gamma_2(n) \cup \Gamma_3(n)} e^{-t \Re \zeta |d\zeta|} \| g \|_{\langle v \rangle - m L^1}.
\]

By the definition of \( \Gamma_1(n) \), it is easy to see that

\[
\int_{\zeta \in \Gamma_1} e^{-\Theta|d\zeta|} \lesssim e^{-\Theta(|n| + 1)}.
\]

Similarly, the definitions of \( \Gamma_2(n) \) and \( \Gamma_3(n) \) imply that for any \( t \geq 1 \),

\[
\int_{\zeta \in \Gamma_2(n) \cup \Gamma_3(n)} e^{-t \Re \zeta |d\zeta|} \lesssim (1 + |n|) \int_\Theta e^{-t\sigma} d\sigma \lesssim e^{-\Theta t (1 + |n|)}.
\]

Collecting the estimates above and using the fact that \( \langle v \rangle - m L^1 \cap M^{1/2} L^2 \) is dense in \( \langle v \rangle - m L^1 \), we prove (6.12), for \( t \geq 1 \).

The proof will be complete if we can show that the propagator \( e^{-tL_n} \) is bounded on \( L^1(\mathbb{R}^3) \) when \( t \in [0, 1] \). To prove this, we establish the local wellposedness of the linear equation

\[
\begin{align*}
\partial_t g &= [-\nu - i n \cdot v - K] g, \\
g(v, 0) &= g_0(v),
\end{align*}
\]

(7.14)

in Appendix B below, which shows that, there exists a constant \( C \), independent of \( n \), s.t. (7.14) has a unique solution in the time interval \( [0, 1] \) and it satisfies the estimate

\[
\| \langle v \rangle^m g(\cdot, t) \|_{L^1} \leq C \| \langle v \rangle^m g_0 \|_{L^1}.
\]

This completes the proof of Lemma 6.2.

8 Proof of Lemma 7.2

As stated in Lemma 7.2, we need \( m \) sufficiently large to make certain constants sufficiently small. In the rest of the paper, we keep track all the constants related to \( m \). The meaning \( a \lesssim b \) is that

\[
a \leq C b
\]

(8.1)

with \( C \) being a fixed constant, independent of \( m \).
We start by simplifying the arguments in Lemma 7.2. Using the definitions of the operators $L_n$, $n \in \mathbb{Z}^3$, in (6.8), $K$ in (6.15), and $\nu$ in (3.4) we find that

$$L_n = \nu + K + in \cdot v.$$ 

In order to prove the uniform invertibility of $L_n - \zeta$, $\zeta \in \Gamma_n$, we claim that it suffices to prove this property for $1 - K_{\zeta,n}$ with $K_{\zeta,n}$ defined by

$$K_{\zeta,n} := K(\nu + in \cdot v - \zeta)^{-1}. \quad (8.2)$$

To see that, rewrite $L_n - \zeta$ as

$$L_n - \zeta = [1 + K_{\zeta,n}](\nu + in \cdot v - \zeta). \quad (8.3)$$

For the multiplication operator $\nu + in \cdot v - \zeta$, if the constant $\Psi$ in the definition of the curves $\Gamma_{k,n}$, $k = 0, 1, 2$, in (7.10), are sufficiently large, then there exists a constant $C$ such that for any $\zeta \in \Gamma_n$

$$|\nu + in \cdot v - \zeta|^{-1} \leq C(1 + |v| + |n \cdot v|)^{-1}. \quad (8.4)$$

It is straightforward, but a little tedious to verify this. Details are omitted.

In what follows we study the linear operator $1 + K_{\zeta,n}$, the key result is the following: recall the definition of space

$$\langle v \rangle^{-m}L^1 := \{ f : \mathbb{R}^3 \to \mathbb{C} \mid \| \langle v \rangle^m f \|_{L^1} < \infty \}. \quad (8.5)$$

**Lemma 8.1.** Suppose that $m > 0$ is sufficiently large. Then for any point $\zeta \in \Gamma_n$ and $n \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$, we have that $1 + K_{\zeta,n} : \langle v \rangle^{-m}L^1 \to \langle v \rangle^{-m}L^1$ is invertible; its inverse satisfies the estimate

$$\| (1 + K_{\zeta,n})^{-1} \|_{\langle v \rangle^{-m}L^1 \to \langle v \rangle^{-m}L^1} \leq C(m),$$

where the constant $C(m)$ is independent of $n$ and $\zeta$.

This will be proven after presenting the key ideas.

The results above complete the proof of Lemma 7.2 assuming that Lemma 8.1 holds.

Next we present the key ideas in proving Lemma 8.1.

In proving 8.1, we divide the set $\{(n, \zeta) \mid n \in \mathbb{Z}^3, \zeta \in \Gamma_n\}$ into three subsets: namely for some large constants $N$ and $X$,

1. $|n| > N$,
2. $|n| \leq N$, and $|\zeta| > X,$

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(3) $|n| \leq N$, and $|\zeta| \leq X$.

To fix $N$ and $X$, we need the results in Propositions 8.2, 8.3 and 8.4 below.

Recall the constant $\Upsilon_m := \Upsilon_m, \frac{1}{2}$ defined in (3.12), and that $\Upsilon_m \to \infty$ as $m \to \infty$.

Proposition 8.2. There exists a constant $Y > 0$, such that if $m \geq Y$, then for any $\zeta \in \Gamma_n$, satisfying $|\zeta| \geq (1 + |n|^2)[\Upsilon_m + m]^4$, we have

$$\|\langle v \rangle^m (1 + K_{\zeta, n})^{-1} \langle v \rangle^{-m} \|_{L^1 \to L^1} \leq 2.$$ (8.6)

The proposition will be proved in subsection 8.1.

The basic ideas in the proof are easy. By the fact $K_{\zeta, n} = (K_1 - K_2 - K_3) (\nu + in \cdot v - \zeta)^{-1}$, we have that in the region $|v| \leq |\zeta|^\frac{1}{2}$, we have a smallness estimate

$$| (\nu + in \cdot v - \zeta)^{-1}| \leq \langle v \rangle^{-1} |\zeta|^{-\frac{1}{2}}.$$

For the region $|v| > |\zeta|^\frac{1}{2}$ it is relatively easy to prove that $K_{\zeta, n}$ is small.

Next, we state the second result. Let $\Upsilon_m$ be the same constant as in Proposition 8.3.

Proposition 8.3. There exists a constant $Y > 0$, such that if $m \geq Y$, and if $\Upsilon_m^2 |n|^{-\frac{1}{2}} + C(m)|n|^{-\frac{1}{2}}$ is sufficiently small for some fixed constant $C(m)$, then for any $\zeta \in \Gamma_n$,

$$\|\langle v \rangle^m (1 + K_{\zeta, n})^{-1} \langle v \rangle^{-m} \|_{L^1 \to L^1} \leq \Upsilon_m.$$ (8.7)

The proposition will be proved in subsection 8.2.

The basic ideas in proving Proposition 8.3 are easy. Recall that by definition

$$K_{\zeta, n} = (K_1 - K_2 - K_3) (\nu + in \cdot v - \zeta)^{-1}.$$

When $|n|$ is large, the purely imaginary part of $\nu + in \cdot v - \zeta$, which is $n \cdot v - Im \zeta$, is large except for a “small” set, for example where $v \perp n$ and $v = 0$. This will render $K_{\zeta, n}$ small except for a small set.

We also need the following key result: recall $\Lambda$ from (7.9),

Proposition 8.4. There exists a constant $\Phi$ such that if $m \geq \Phi$, then for any $\zeta$, satisfying the condition $Re \zeta \leq \frac{1}{2} \Lambda$, we have that, for some constants $C_{n, \zeta} > 0$,

$$\|\langle v \rangle^m (1 + K_{\zeta, n})^{-1} g \|_{L^1} \leq C_{n, \zeta} \|\langle v \rangle^m g \|_{L^1}.$$ (8.8)
The proof will be in Section 8.4 by the techniques and construction learned from [24, 19], see also [1, 33].

Based on Propositions 8.2, 8.3 and 8.4, we choose $N$ and $X$, to define the three regimes listed before Proposition 8.2.

Let $m \geq Y$, with $Y$ large enough such that Propositions 8.2 and 8.3 apply.

Then choose $N \in \mathbb{N}$ large enough to make $\Upsilon_m^2|N|^{-\frac{1}{2}} + C(m)|N|^{-\frac{1}{m}}$ sufficiently small, then by Proposition 8.3 for any $n$ satisfying $n \geq N$,

$$\|\langle v \rangle^m [1 + K_{\zeta,n}]^{-1} \langle v \rangle^{-m}\|_{L^1 \to L^1} \lesssim \Upsilon_m. \quad (8.9)$$

After choosing $N$, we choose $X$ as

$$X := (1 + |N|^2)(\Upsilon_m + m)^4, \quad (8.10)$$

so that for any $|n| \leq N$ and $\zeta \in \Gamma_n$ satisfying $|\zeta| \geq X$, Proposition 8.2 applies.

Now we are ready to prove Lemma 8.1.

**Proof.** For the first and second regimes, we use Propositions 8.2 and 8.3.

For the third regime, by the definition of $\Gamma_n$, we have $\zeta \in \Gamma_1$ if $\Psi$ is large enough. This together with (8.8) and the facts $|n| \leq N$ and $|\zeta| \leq X$ implies that $C_{n,\zeta}$ in Lemma 8.1 has a proper upper bound in this regime.

Collecting the estimates above, we complete the proof.

In the rest of this section, we prove Propositions 8.2 and 8.3. Upon completion of the work, we realize that, by reading known works such as [19], many of the estimates to be used in proving Propositions 8.2 and 8.3 can be obtained more efficiently by applying Povzner’s inequality, see also [4, 3, 34, 19, 24].

Before the proof we define three small constants.

Recall the definitions of operators $K_l$, $l = 1, 2, 3$, in (3.5). Define a new quantity $\delta_{m,0}$ by

$$\delta_{m,0} := \sum_{l=1}^{3} \|\chi_{>m}(v)^m K_l(v)^{-m-1} \chi_{>m}\|_{L^1 \to L^1}. \quad (8.11)$$

Here the cutoff function $\chi_{>m}$ is defined as

$$\chi_{>m}(v) = \begin{cases} 1 & \text{if } |v| > m \\ 0 & \text{otherwise} \end{cases} \quad (8.12)$$

The result is
Lemma 8.5. The quantity $\delta_{m,0}$ satisfies the following estimate

$$\delta_{m,0} \to 0 \text{ as } m \to +\infty. \quad (8.13)$$

The proof will be in subsection 8.3.

Define a constant $\delta_{m,1}$ as

$$\delta_{m,1} := \max_{a \geq 0} (1 + a^2)^{-\frac{m}{2}} \frac{a}{\sqrt{1 + a^2}}. \quad (8.14)$$

By the definition we have

$$\delta_{m,1} \to 0 \text{ as } m \to \infty. \quad (8.15)$$

We define another constant $\delta_{m,2}$ as

$$\delta_{m,2} := 2^m \int_{z \geq m^{\frac{3}{4}}} z^{2m} e^{-\frac{z^2}{4}} dz = 3 \cdot 2^{m-2} \int_{z \geq m} z^{\frac{3m}{2}} e^{-\frac{z^2}{4}} \, dz. \quad (8.16)$$

Then the fact $e^{-\frac{1}{4}z^2}$ decays faster, in $|z|$, than exponentially makes $\frac{\delta_{m+1,2}}{\delta_{m,2}} \ll 1$ if $m$ is large, hence

$$\delta_{m,2} \to 0 \text{ as } m \to \infty. \quad (8.17)$$

The proof of these two facts above are straightforward, hence omitted.

8.1 Proof of Proposition 8.2

We start by casting the expression into a convenient form, by transforming the operator $1 + K_{\zeta, n}$ into a $2 \times 2$ operator-valued matrix. Let $\chi_{\leq 2m}$ be a Heaviside function

$$\chi_{\leq 2m}(v) = \begin{cases} 1 & \text{if } |v| \leq 2m \\ 0 & \text{otherwise} \end{cases} \quad (8.18)$$

and naturally

$$\chi_{> 2m} := 1 - \chi_{\leq 2m}.$$ 

Decompose the $L^1(\mathbb{R}^3)$ space into a vector space, bijectively,

$$L^1(\mathbb{R}^3) \to \begin{bmatrix} \chi_{\leq 2m} L^1(\mathbb{R}^3) \\ \chi_{> 2m} L^1(\mathbb{R}^3) \end{bmatrix}, \text{ with } f \to \begin{bmatrix} \chi_{\leq 2m} f \\ \chi_{> 2m} f \end{bmatrix}. \quad (8.19)$$
with the norm \( \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{L^1} = \| f \|_{L^1} + \| g \|_{L^1} = \| f + g \|_{L^1} \), where in the last step we used a property uniquely holds for \( L^1 \) space (among \( L^p \) space, \( p \in [1, \infty) \)): namely if \( f \) and \( g \) have disjoint supports, then
\[
\| f \|_{L^1} + \| g \|_{L^1} = \| f + g \|_{L^1}.
\]
Consequently for any function \( f \),
\[
\| \langle v \rangle^m (1 + K_{\zeta,n}) f \|_{L^1} = \| \langle v \rangle^m (1 + D) \begin{bmatrix} \chi_{\leq 2m} \\ \chi_{>2m} 
\end{bmatrix} \|_{L^1}. 
\]
Here \( D \) is an operator-valued \( 2 \times 2 \) matrix defined as
\[
D := \begin{bmatrix}
\chi_{\leq 2m} K_{\zeta,n} \chi_{\leq 2m} & \chi_{\leq 2m} K_{\zeta,n} \chi_{>2m} \\
\chi_{>2m} K_{\zeta,n} \chi_{\leq 2m} & \chi_{>2m} K_{\zeta,n} \chi_{>2m}
\end{bmatrix}. 
\]
Next we prove that all entries in \( D \) are small. Recall that the small constants \( \delta_{m,l} \), \( l = 0, 1, 2 \), are defined in (8.11), (8.14) and (8.16).

**Lemma 8.6.** If
\[
|\zeta| \geq (1 + |n|^2)[\Upsilon_m + m]^4,
\]
then we have
\[
\| \chi_{\leq 2m} \langle v \rangle^m K_{\zeta,n} \langle v \rangle^{-m} \chi_{\leq 2m} \|_{L^1} \to L^1, \quad \| \chi_{>2m} \langle v \rangle^m K_{\zeta,n} \langle v \rangle^{-m} \chi_{\leq 2m} \|_{L^1} \to L^1 \leq |\zeta|^{-\frac{1}{2}}. \tag{8.23}
\]
Moreover
\[
\| \chi_{>2m} K_{\zeta,n} \chi_{>2m} \|_{L^1} \to L^1 \leq \delta_{m,0} \tag{8.24}
\]
and
\[
\| \chi_{\leq 2m} K_{\zeta,n} \chi_{>2m} \|_{L^1} \to L^1 \leq \delta_{m,0} + 2^{-m} \tag{8.25}
\]
The lemma will be proved in subsubsection 8.1.1.

The fact that \( D \) is small obviously implies that \( 1 + D \) is uniformly invertible, provided that \( \zeta \) and \( m \) are sufficiently large. This is the desired Proposition 8.2.

Next we prove Lemma 8.6 to complete the proof.
8.1.1 Proof of Lemma 8.6

Proof. We start with proving (8.23). Instead of proving the two estimates separately, it suffices to prove a stronger one:

\[ \| \langle v \rangle^m K_{\zeta,n} \langle v \rangle^{-m} \chi_{\leq 2m} \|_{L^1 \to L^1} \leq |\zeta|^{-\frac{1}{2}}. \] (8.26)

Recall that by definition

\[ K_{\zeta,n} := (K_1 - K_2 - K_3)(\nu(v) + in \cdot v - \zeta)^{-1}. \]

The key idea here is to exploit that \(|\zeta|\) is large.

Observe that, restricted to the set \(|v| \leq 2m\),

\[ |\nu(v) + in \cdot v - \zeta| \geq |\zeta| - |n||v| - |\nu(v)| \geq \frac{1}{2} |\zeta| \langle v \rangle. \] (8.27)

This together with the estimate for \(K_l, l = 1, 2, 3\), in (3.12) implies the desired result

\[ \| \langle v \rangle^m K_{\zeta,n} \langle v \rangle^{-m} \chi_{\leq 2m} \|_{L^1 \to L^1} \leq 2|\zeta|^{-1} \sum_{l=1}^{3} \| \langle v \rangle^m K_l(v)^{-m-1} \chi_{\leq 2m} \|_{L^1 \to L^1} \leq 2 \Upsilon_m |\zeta|^{-1} \leq |\zeta|^{-\frac{1}{2}}. \] (8.28)

Now we turn to (8.24), i.e. estimating \(\chi > 2m K_{\zeta,n} \chi > 2m\).

Use \(|\nu(v) + in \cdot v - \zeta|^{-1} \lesssim \langle v \rangle^{-1}\) from (8.4), and apply the estimate \(\delta_{m,0}\) in Lemma 8.5 to have the desired estimate

\[ \| \langle v \rangle^m \chi_{>2m} K_{\zeta,n} \langle v \rangle^{-m} \chi_{>2m} \|_{L^1 \to L^1} \lesssim 3 \sum_{l=1}^{3} \| \langle v \rangle^m \chi_{>2m} K_l(v)^{-m-1} \chi_{>2m} \|_{L^1 \to L^1} \leq \delta_{m,0}. \] (8.29)

Next we prove (8.25). Compute directly to find the desired result, for any function \(f\),

\[ \| \langle v \rangle^m \chi_{\leq 2m} K_{\zeta,n} \langle v \rangle^{-m} \chi_{>2m} f \|_{L^1} \]

\[ \lesssim \sum_{l=1}^{3} \| \langle v \rangle^m \chi_{\leq 2m} K_l(v)^{-m-1} \chi_{>2m} f \|_{L^1} \]

\[ \leq \sum_{l=1}^{3} \| \langle v \rangle^m \chi_{\leq 2m} K_l(v)^{-m-1} \chi_{>2m} f \|_{L^1} + \sum_{l=1}^{3} \| \langle v \rangle^m \chi_{>2m} K_l(v)^{-m-1} \chi_{>2m} f \|_{L^1}. \]

\[ \lesssim 2^{-m} \sum_{l=1}^{3} \| K_l(v)^{-1} \chi_{>2m} f \|_{L^1} + \delta_{m,0} \| \chi_{>2m} f \|_{L^1} \lesssim [2^{-m} + \delta_{m,0}] \| \chi_{>2m} f \|_{L^1}. \] (8.30)
where in the second last step we used the obvious estimate \( \langle v \rangle^m \lesssim 2^{-m} \) if \( |u| \geq 2m \) and \( |v| \leq m \), and the bound \( \leq \delta_{m,0}\|\chi_{>2m} f\|_{L^1} \) is from Lemma 8.5.

\[ \blacksquare \]

### 8.2 Proof of Proposition 8.3

We start with presenting the ideas. By the definition \( K_{\zeta, n} \) is defined as

\[ K_{\zeta, n} = K(\nu + i n \cdot v - \zeta)^{-1}. \]

The fact that \( |n| \) is large makes the purely imaginary part of \( \nu + i n \cdot v - \zeta \), which is \( n \cdot v - \zeta_2 \) with \( \zeta_2 := Im \zeta \), favorably large, except for a “small” set of \( v \). We divide this small set into two regimes:

1. \( |v| \leq |n|^{-\frac{1}{4}} \),
2. \( \frac{1}{|v| |n|} |n \cdot v - \zeta_2| \leq |n|^{-\frac{1}{4}} \).

To consider, separately, this “small but adverse” set, we define a Heaviside function \( \chi : \mathbb{R}^3 \to \{0, 1\} \)

\[ \chi(v) = \begin{cases} 1 & \text{if } |v| \leq |n|^{-\frac{1}{4}}, \\ 1 & \text{if } \frac{1}{|v| |n|} |n \cdot v + h| \leq |n|^{-\frac{1}{4}}, \\ 0 & \text{otherwise}. \end{cases} \]  

(8.31)

Then as in (8.20) we transform the linear operator \( K_{\zeta, n} \) into a \( 2 \times 2 \) operator valued matrix \( F \), defined as

\[ F := \begin{bmatrix} \chi K_{\zeta, n} & \chi K_{\zeta, n}(1 - \chi) \\ (1 - \chi) K_{\zeta, n} \chi & (1 - \chi) K_{\zeta, n}(1 - \chi) \end{bmatrix}, \]  

(8.32)

and for any function \( g \in L^1 \),

\[ \|\langle v \rangle^m (1 + K_{\zeta, n}) g\|_{L^1} = \|\langle v \rangle^m (1 + F) \left[ \begin{array}{c} \chi g \\ (1 - \chi) g \end{array} \right]\|_{L^1} \]  

(8.33)

Consequently, to prove the invertibility of \( 1 + K_{\zeta, n} \), it suffices to prove that for the matrix operator \( 1 + F \).

The entries in \( F \) satisfy the following estimates:

**Lemma 8.7.** There exists \( N \) such that if \( |n| \geq N \), then three entries of \( F \) are small

\[ \|\langle v \rangle^m \chi K_{\zeta, n} \chi f\|_{L^1} \lesssim \left[ C(m) |n|^{-\frac{1}{2}} + \delta_{m,0} + \delta_{m,2} \right] \|\langle v \rangle^m \chi f\|_{L^1}, \]  

(8.34)
with $C(m)$ being some constant depending only on $m$, and
\[
\|\langle v \rangle^m K_{\zeta, n}(1 - \chi) f\|_{L^1} \lesssim \Upsilon_m |n|^{-\frac{1}{4}} \|\langle v \rangle^m (1 - \chi) f\|_{L^1}. \tag{8.35}
\]
One (and only one) of the off-diagonal entries is possibly large,
\[
\|\langle v \rangle^m (1 - \chi) K_{\zeta, n} \chi f\|_{L^1} \lesssim \Upsilon_m \|\langle v \rangle^m \chi f\|_{L^1}. \tag{8.36}
\]

The lemma will be proved in subsection 8.2.1.

We are ready to prove Proposition 8.3.

**Proof.** The difficulty here is that an off-diagonal component, namely $(1 - \chi) K_{\zeta, n} \chi$, of the matrix operator $F$ possibly has a large norm.

The basic idea in proving the invertibility of $1 + F$ is motivated by inverting a $2 \times 2$ scalar matrix $\text{Id} + \tilde{F}$: suppose that $\tilde{F}$ takes the form $\tilde{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ with $|f_{11}|, |f_{12}|, |f_{22}| \ll 1$ and $|f_{21}| \gg 1$. If one has that $|f_{12} f_{21}| \ll 1$, then $\text{Id} + \tilde{F}$ is invertible and by direct computation,
\[
\| (\text{Id} + \tilde{F})^{-1} \| \lesssim 1 + |f_{21}|. \tag{8.37}
\]

For the present $2 \times 2$ operator-valued matrix $F = [f_{ij}]$, we have
\[
\| f_{11} \|, \| f_{22} \|, \| f_{12} \| \ll 1,
\]
and the only large entry $f_{21}$ satisfies
\[
\| f_{12} \| \| f_{21} \| \ll 1.
\]
By this we construct the inverse of the matrix operator $1 + F$ by first diagonalizing the matrix, and then finding the bound on the inverse as in (8.37). The process is easy but tedious. We omit the details here.

\[\square\]

### 8.2.1 Proof of Lemma 8.7

**Proof.** For (8.35), use the definition of $K$ in (3.5) to obtain
\[
K_{\zeta, n} f = (K_1 - K_2 - K_3)(\nu + i n \cdot v - \zeta)^{-1} f. \tag{8.38}
\]
To simplify our consideration, it suffices to prove a slightly general estimate, for $l = 1, 2, 3$,
\[
\|\langle v \rangle^m K_l (\nu + i n \cdot v - \zeta)^{-1} \langle v \rangle^{-m} (1 - \chi) f\|_{L^1} \lesssim \Upsilon_m |n|^{-\frac{1}{4}} \|(1 - \chi) f\|_{L^1}. \tag{8.39}
\]

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The key observation is that on the support of $1 - \chi$, where
\[ |v| \geq |n|^{-\frac{1}{4}} \quad \text{and} \quad \frac{1}{|v||n|} |n \cdot v + \zeta_2| \geq |n|^{-\frac{1}{4}}, \tag{8.40} \]
we have
\[ |\nu + in \cdot v - \zeta| \geq |n|^\frac{1}{2} |v| \geq |n|^\frac{1}{2} (|v|^2 + 1)^{\frac{1}{2}}. \]
Compute directly to obtain the desired result
\[
\| (v)^m K_l(\nu + in \cdot v - \zeta)^{-1}(v)^{-m}(1 - \chi)f \|_{L^1} \leq |n|^{-\frac{1}{4}} \| (v)^m K_l(v)^{-1-m}(1 - \chi)f \|_{L^1} \\
\leq \Upsilon_m |n|^{-\frac{1}{4}} \| (1 - \chi)f \|_{L^1}.
\]
Now we prove (8.36) by a direct computation, using (3.12),
\[
\| (v)^m (1 - \chi)K_\zeta n \chi f \|_{L^1} \lesssim \sum_{l=1}^3 \| (v)^m K_l(v)^{-1} \chi f \|_{L^1} \lesssim \Upsilon_m \| (v)^m \chi f \|_{L^1}. \tag{8.41}
\]
Next we prove (8.34).
Decompose $\chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi$ further as
\[
\chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi = \chi_{\leq 2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi + \chi_{>2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi
\]
\[
= \chi_{\leq 2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi + \chi_{>2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi_{\leq m} \chi
\]
\[
+ \chi_{>2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi_{>m} \chi \tag{8.42}
\]
with $\chi_{\leq 2m}, \chi_{>2m} = 1 - \chi_{\leq 2m}, \chi_{>m}$, and $\chi_{\leq m} = 1 - \chi_{>m}$ being Heaviside functions defined in a similar way as that in (8.18).

In estimating $\chi_{\leq 2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi$, $l = 1, 2, 3$, we use the fact that $|(\nu + in \cdot v - \zeta)^{-1}| \lesssim 1$ to find that, for any function $f$,
\[
\| (v)^m \chi_{\leq 2m} \chi K_l(\nu + in \cdot v - \zeta)^{-1} \chi f \|_{L^1} \lesssim \| \chi_{\leq 2m} \chi K_l \chi f \|_{L^1} \tag{8.43}
\]
We claim that, for some constant $C(m) > 0$,
\[
\| (v)^m \chi_{\leq 2m} \chi K_l \chi f \|_{L^1} \leq C(m) |n|^{-\frac{1}{4}} \| \chi f \|_{L^1}. \tag{8.44}
\]
We start with considering the terms involving $K_2$ and $K_3$. Use the integral kernel for $K_2 + K_3$ in (6.15) to find
\[
\sum_{l=2,3} \| (v)^m \chi_{\leq 2m} \chi K_l \chi f \|_{L^1} \leq \| (v)^m \chi_{\leq 2m} \chi [K_2 + K_3] f \|_{L^1} \lesssim \max_a \int_{\mathcal{R}} (v)^m |v - u|^{-1} dv^3 \| \chi f \|_{L^1},
\]
\[
\tag{8.45}
\]
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with $\mathcal{R} \subset \mathbb{R}^3$ defined as
\[
\mathcal{R} := \{ |v| \leq 2m, \quad \frac{1}{|v||n|} |n \cdot v - \zeta_2| \leq |n|^{-\frac{1}{4}}, \quad |v| \geq |n|^{-\frac{1}{4}} \}\].
\tag{8.46}

Observe that the large value of $n$, the smaller region $\mathcal{R}$ becomes.

Compute directly to find that, for some constant $C(m) > 0$,
\[
\max_u \int_{\mathcal{R}} \langle v \rangle^m |v - u|^{-1} dv^3 \lesssim (1 + 2m)^m \left[ \int_{|v - u| \leq |n|^{-\frac{1}{10}}} |v - u|^{-1} d^3v + |n|^{-\frac{1}{10}} \int_{\mathcal{R}} d^3v \right]
= (1 + 2m)^m \left[ \int_{|v| \leq |n|^{-\frac{1}{10}}} |v|^{-1} d^3v + |n|^{-\frac{1}{10}} \int_{\mathcal{R}} d^3v \right]
\leq C(m) |n|^{-\frac{1}{10}}
\tag{8.47}
\]
This, together with (8.45), implies the desired (8.44) for $K_l$, $l = 2, 3$.

The corresponding estimate for $K_1$ is easier, by the presence of the factor $e^{-|v|^2}$ in its integral kernel. We skip the details here.

For the third term in (8.42), we apply Lemma 8.5 to find the desired estimate
\[
\| \chi_{>2m} \langle v \rangle^m K_1(v + in \cdot v - \zeta)^{-1} \langle v \rangle^{-m} \chi \chi_{>m} \|_{L^1 \to L^1} \leq \| \chi_{>2m} \langle v \rangle^m K_1(v)^{-m-1} \chi_{>m} \|_{L^1 \to L^1} \leq \delta_{m,0}.
\tag{8.48}
\]

Turning to the second term in (8.42), we compute directly to find, for any function $f$,
\[
\| \chi_{>2m} \langle v \rangle^m K_1(v + in \cdot v - \zeta)^{-1} \langle v \rangle^{-m} \chi \chi_{\leq m} f \|_{L^1} \leq \| \chi_{>2m} \langle v \rangle^m K_1(v)^{-m-1} \chi_{\leq m} f \|_{L^1}.
\tag{8.49}
\]
For $K_1$, the presence of the factor $e^{-|v|^2}$ in its integral kernel makes the integral small,
\[
\| \chi_{>2m} \langle v \rangle^m K_1(v)^{-m-1} \chi_{\leq m} f \|_{L^1} \lesssim \int_{|v| \geq 2m} \langle v \rangle^{m+1} e^{-|v|^2} d^3v \| \chi_{\leq m} f \|_{L^1} \leq \delta_{m,2} \| \chi_{\leq m} f \|_{L^1}
\tag{8.50}
\]
with the small constant $\delta_{m,2}$ defined in (8.17).

For $K_2 + K_3$, we use its integral integral to find
\[
\sum_{l=2,3} \| \chi_{>2m} \langle v \rangle^m K_l(v)^{-m-1} \chi_{\leq m} f \|_{L^1} = \| \chi_{>2m} \langle v \rangle^m \sum_{l=2,3} K_l(v)^{-m-1} \chi_{\leq m} f \|_{L^1}
= \| \chi_{>2m} \langle v \rangle^m \int K(u, v) \langle v \rangle^{-m-1} \chi_{\leq m} f(u) \ d^3u \|_{L^1}
\tag{8.51}
\]

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where \( K(u, v) := 2\pi |u - v|^{-1} e^{-\frac{|(u-v)\cdot v|^2}{|u-v|^2}} \), see (6.15). The facts that \(|v| \geq 2m\) and \(|u| \leq m\) imply that the direction of the unit vector \( \frac{u-v}{|u-v|} \) is controlled by \( v \), hence
\[
\frac{|(u-v)\cdot v|^2}{|u-v|^2} \geq \frac{1}{4}|v|^2.
\]
This together with estimate \(|u-v|^{-1} \leq 2|v|^{-1}\) implies that
\[
K(u, v) \lesssim |v|^{-1} e^{-\frac{1}{4}|v|^2}.
\]
(8.52)

By these results, we compute directly to obtain the desired estimate,
\[
\sum_{l=2,3} \| \chi_{|v|>m} (\langle v \rangle)^m K_l (\langle v \rangle)^{-m-1} \chi_{|v|>m} f \|_{L^1} \lesssim \int_{|v| \geq 2m} \langle v \rangle^m |v|^{-1} e^{-\frac{1}{4}|v|^2} dv \| \chi_{|v|>m} f \|_{L^1} \leq \delta_m \| \chi_{|v|>m} f \|_{L^1}.
\]
(8.53)

Thus we complete the estimate for the second term on the right hand side of (8.42).

Collect the estimates above to complete the proof of (8.34).

\[\Box\]

8.3 Proof of Lemma 8.5

Proof. It is easy to prove that
\[
\frac{\| \chi_{|v|>m} (\langle v \rangle)^m K_l (\langle v \rangle)^{-m-1} \chi_{|v|>m} f \|_{L^1}}{\| f \|_{L^1}} \to 0 \quad \text{as} \quad m \to \infty
\]
by the presence of the factor \( e^{-|v|^2} \) in the integral kernel of \( K_l \) and that \( \lim_{m \to \infty} \int_{|v|>m} e^{-|v|^2} dv = 0 \).

The methods in estimating \( \| \chi_{|v|>m} (\langle v \rangle)^m K_l (\langle v \rangle)^{-m-1} \chi_{|v|>m} f \|_{L^1} \), \( l = 2, 3 \), are similar, hence we choose to study \( l = 2 \).

We start with casting the expression into a convenient form.

For \( \omega \in S^2 \) in the definition of \( K_2 \), we look for an unitary rotation \( U_\omega \) to make
\[
U_\omega^* \omega = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]
(8.54)

For any \( \omega \in S^2 \), there exist unique \( \theta \in [0, 2\pi) \) and \( \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}) \) such that
\[
\omega = \begin{bmatrix} \cos \theta \cos \alpha \\ \sin \theta \cos \alpha \\ \sin \alpha \end{bmatrix},
\]
(8.55)
then we choose $U^*$ in (8.54) as

$$
U^* = \begin{bmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-sin \alpha & 0 & \cos \alpha
\end{bmatrix}.
$$

Insert the rotation into appropriate places of $K_2$ and $f$, and compute directly to obtain

$$
\|\chi \cap (v)^m K_2(v)^{-1} \chi \cap (v) f\|_{L^1} \\
\leq \int_{\mathbb{R}^2} \int_{R(m)} e^{-|v_1|^2 - |u_2|^2 - |u_3|^2} \frac{\langle v \rangle^m |u_1 - v_1|}{(1 + u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} |f|((u_1, v_2, v_3) U_\omega) du dv \ dv \ dv \\
\lesssim \|f\|_{L^1} \sup_{v_1, v_2, v_3} \int_{\|v\| > m} e^{-|v_1|^2} \frac{\langle v \rangle^m |u_1 - v_1|}{(1 + u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} dv_1
$$

(8.56)

where $R(m)$ is the region defined as

$$
R(m) := \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | \|v\| > m, \sqrt{u_1^2 + v_2^2 + v_3^2} > m\},
$$

and the integral $\int_{R(m)}$ in the second step is over 6 variable $(u, v) \in \mathbb{R}^6$.

Consequently, to prove the desired result Lemma 8.5 it suffices to prove that

$$
\sup_{v_1, v_2, v_3} \int_{\|v\| > m} e^{-|v_1|^2} \frac{\langle v \rangle^m |u_1 - v_1|}{(1 + u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} dv_1 \rightarrow 0 \text{ as } m \rightarrow \infty.
$$

(8.57)

To see this we divide the integral region into two parts: $|v_1| \leq m^{\frac{2}{3}}$ and $|v_1| > m^{\frac{3}{4}}$.

For the first case $|v_1| > m^{\frac{2}{3}}$, we compute directly to find

$$
\int_{|v_1| \geq m^{\frac{2}{3}}} e^{-|v_1|^2} \frac{\langle v \rangle^m |u_1 - v_1|}{(1 + u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} dv_1 \leq 2^m \int_{|v_1| \geq m^{\frac{2}{3}}} e^{-|v_1|^2} |v_1|^m dv_1 \\
= 3 \cdot 2^{m-2} \int_{|u| \geq m} e^{-\frac{2}{3} u_1^2} |u|^\frac{m-1}{3} du \leq \delta_{m, 2}
$$

(8.58)

where, by (8.17) $\lim_{m \rightarrow \infty} \delta_{m, 2} = 0$, and in the last step we changed variable $u = |v_1|^{\frac{2}{3}}$.

In the bounded region, where $|v_1| \leq m^{\frac{2}{3}}$, $\|v\| \geq m$ and $m \gg 1$, we observe that

$$
e^{-\frac{4}{3}v_1^2 \langle v \rangle^m} \leq 2e^{-\frac{4}{3}v_1^2 (v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}}} \leq 2(v_2^2 + v_3^2)^{\frac{m}{2}}
$$

(8.59)
where in the last step, after some elementary manipulations, we took the power \( \left( \frac{v}{m} \right)^2 \) to find the following equivalent form, which can easily verified by using \( |v| \geq m \),

\[
1 + \frac{v_1^2}{v_2^2 + v_3^2} \leq e^{\frac{v_1^2}{m}} = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{v_1^2}{m} \right)^l.
\]

Apply (8.59) to find

\[
e^{-\frac{1}{2}v_1^2} \frac{(v)^m|u_1 - v_1|}{(1 + u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} \leq e^{-\frac{1}{2}v_1^2} \frac{(v)^m|u_1 - v_1|}{(u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} \leq \frac{2(v_2^2 + v_3^2)^{\frac{m+1}{2}}}{(u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} + \frac{2(v_2^2 + v_3^2)^{\frac{m+1}{2}}}{(u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}}. \tag{8.60}
\]

The second part is small by

\[
\frac{|v_1|}{(u_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}}} \leq m^{-\frac{1}{4}}.
\]

The first takes the form

\[
(1 + a^2)^{-\frac{m}{2}} \frac{a}{\sqrt{1 + a^2}} \leq \max_{b \geq 0} (1 + b^2)^{-\frac{m}{2}} \frac{b}{\sqrt{1 + b^2}} := \delta_{m,1}
\]

with \( a := \frac{|u_1|}{\sqrt{|v_2|^2 + |v_3|^2}} \), which goes to zero when \( m \) goes to \( \infty \).

Consequently

\[
\sup_{u_1, v_2, v_3} \int_{|v| > m, |v_1| \leq m^{\frac{3}{4}}} e^{-|v_1|^2} \frac{(v)^m|u_1 - v_1|}{(1 + u_1^2 + v_2^2 + v_3^2)^{\frac{m+1}{2}}} \, dv_1 \lesssim \int e^{-\frac{1}{2}|v_1|^2} \, dv_1[\delta_{m,1} + \delta_{m,2} + m^{-\frac{1}{4}}] \lesssim \delta_{m,1} + \delta_{m,2} + m^{-\frac{1}{4}}. \tag{8.61}
\]

This is of the desired form since as \( m \to \infty \), \( \delta_{m,1} + \delta_{m,2} + m^{-\frac{1}{4}} \to 0 \).

This together with (8.58) implies the desired (8.57), and hence completes the proof.

\[\square\]

### 8.4 Proof of Proposition 8.4

Proof. In what follows we use the techniques and constructions developed in [24, 19]. The collision integral of Boltzmann equation (1.1) is the \( \omega \) representation of the corresponding equation in [24], see [32]. Hence the results in [24,19] apply. It is worth pointing out that in [24], the space is chosen
to be $e^{-av^s}L^1$ for some $a, s > 0$ where we consider the space $\langle v \rangle^{-m}L^1$. However the results are applicable in (8.65) below by the definition of cutoff function in (8.63) below.

The first observation is that, the fact $Re\lambda \leq \frac{1}{2}\inf \nu(v)$, see (8.62), implies that there exists some $C > 0$ independent of $\lambda$ such that

$$|(\nu(v) + iv \cdot n - \lambda)^{-1}| \leq C(1 + |v|)^{-1}. \quad (8.62)$$

Hence the operator $\langle v \rangle^m(\nu + iv \cdot n - \lambda)^{-1}K(\nu)^{-m} : L^1 \rightarrow L^1$ is uniformly bounded.

In the next we use the construction of (right) inverse operator for $\langle v \rangle^m(L_n - \lambda)\langle v \rangle^{-m}$ in Proposition 4.1 of [24].

We take the same cutoff functions from [19], see also [24]: define a smooth cutoff function $\Theta_\delta = \Theta_\delta(v, u, \omega)$ satisfying the following conditions

$$\begin{align*}
\Theta_\delta &= 1 \text{ if } |v| \leq \delta^{-1}, \quad 2\delta < |v - u| \leq \delta^{-1}, \quad |\cos \theta| \leq 1 - 2\delta, \\
&= 0 \text{ if } |v| \leq 2\delta^{-1}, \quad \delta < |v - u| \leq 2\delta^{-1}, \quad |\cos \theta| \leq 1 - \delta,
\end{align*} \quad (8.63)$$

Here $|\cos \theta| = |\omega \cdot \frac{u - v}{|u - v|}|$.

Recall the definitions of the operators $K_l, l = 1, 2, 3$ from (3.3), we define $K_\delta$ as

$$K_\delta(f) := M(v) \int_{\mathbb{R}^3 \times S^2} \Theta_\delta(v, u, \theta) |(u - v) \cdot \omega| f(u) \, d^3u d^2\omega - \int_{\mathbb{R}^3 \times S^2} \Theta_\delta(v, u, \theta) |(u - v) \cdot \omega| M(u') f(v') \, d^3u d^2\omega - \int_{\mathbb{R}^3 \times S^2} \Theta_\delta(v, u, \theta) |(u - v) \cdot \omega| M(v') f(u') \, d^3u d^2\omega.$$
Hence if the operator

\[ [1 - \langle v \rangle^m M^{\frac{1}{2}} (L_n - \lambda) M^{\frac{1}{2}}]^{-1} \langle v \rangle^m M^{\frac{1}{2}} [\langle v \rangle^m A \delta \langle v \rangle^{-m}] [\langle v \rangle^m B \delta \langle v \rangle^{-m}]^{-1} : L^1(\mathbb{R}^3) \to L^1(\mathbb{R}^3) \]

is well defined, then \( \langle v \rangle^m (L_n - \lambda) \langle v \rangle^{-m} \) is invertible, which directly implies the desired result.

To verify the operator is well defined, we use the following facts:

(1) the mapping \( \langle v \rangle^m M^{\frac{1}{2}} : L^2 \to L^1 \) is obviously well defined since \( M \) decays rapidly fast,

(2) the mapping \( [M^{-\frac{1}{2}} (L_n - \lambda) M^{\frac{1}{2}}]^{-1} : L^2 \to L^2 \) is well defined since \( -\lambda \) is not an eigenvalue of \( L_n \), see [20, 21]. Recall that \( n \neq (0, 0, 0) \).

Recall that, in certain sense, the eigenvector space of \( L_n \) is a subset of that of \( L \), see Lemma 7.1.

(3) the mapping \( M^{-\frac{1}{2}} A \delta \langle v \rangle^{-m} \) can be considered in the setting

\[ M^{-\frac{1}{2}} A \delta \langle v \rangle^{-m} : L^1 \to L^2 \]  \hspace{1cm} (8.65)

by using that \( K_\delta h \) is “compactly supported” by the definition of cutoff function, see also (2.12) and (2.14) in [24], where the space was chosen to be \( e^{-av^s} L^1 \) for some \( a, s > 0 \). But the result still applies by the definition of the cutoff functions in (8.63). See also [19].

(4) To show that \( \langle v \rangle^m B \delta \langle v \rangle^{-m} : L^1 \to L^1 \) is invertible for large \( m \), we apply Lemma 4.4 of [19] where the Povzner lemma was applied, see also [4], together with (8.62), to find that there exists some \( c > 0 \) such that

\[ \| \langle v \rangle^m (\nu + iv \cdot n - \lambda)^{-1} (K - K_\delta) f \|_1 \leq \left[ \frac{c}{m - 1} + \epsilon_m(\delta) \right] \| \langle v \rangle^m f \|_1 \]  \hspace{1cm} (8.66)

where \( \epsilon_m(\delta) \to 0 \) as \( \delta \to 0 \) for each fixed \( m \).

Hence if we choose \( m \) to be large enough, and then choose \( \delta \) to be small enough, then

\[ \frac{c}{m - 1} + \epsilon_m(\delta) \leq \frac{1}{4}, \]  \hspace{1cm} (8.67)

and hence the operator \( 1 + \langle v \rangle^m (\nu + iv \cdot n - \lambda)^{-1} (K - K_\delta) \langle v \rangle^{-m} : L^1 \to L^1 \) is uniformly invertible.

The proof is complete. \( \square \)
A Proof of Lemma 3.1

It is easy to derive (3.10) and (3.11) by the definitions of $\nu_{T,\mu}$. It is an easy application of results in [4]. We therefore omit the details.

For (3.13), we start with proving that, for any functions $f$, $g : \mathbb{R}^3 \to \mathbb{C}$ we have

$$\|\langle v \rangle^m Q(f,g)\|_{L^1(\mathbb{R}^3)} \leq C_m [\|f\|_{L^1(\mathbb{R}^3)} \|\langle v \rangle^{m+1} g\|_{L^1(\mathbb{R}^3)} + \|\langle v \rangle^{m+1} f\|_{L^1(\mathbb{R}^3)} \|g\|_{L^1(\mathbb{R}^3)}].$$

(A.1)

A key observation in proving the estimate is that for any fixed $\omega \in \mathbb{S}^2$, the mapping from $(u, v) \in \mathbb{R}^6$ to $(u', v') \in \mathbb{R}^6$ is a linear symplectic transformation, hence

$$d^3 u d^3 v = d^3 u' d^3 v'$$

(A.2)

where, $u'$ and $v'$ are defined (1.1). This together with the observation that

$$\langle v \rangle^m \leq c(m)[\langle u' \rangle^m + \langle v' \rangle^m], \text{ and } |(u - v) \cdot \omega| \leq |u'| + |v'|$$

(A.3)

obviously implies (A.1), and hence the desired (3.13).

As one can infer from the definition $K$ in (3.5), (3.12) is a special case of (3.13) by setting $f$ or $g$ to be $M_{T,\mu}$.

□

B The local wellposedness of the linear equation

In the present appendix we study the local wellposedness of the linear problem

$$\partial_t g = (-\nu - i n \cdot v - K) g,$$

$$g(v,0) = g_0(v).$$

(B.1)

(B.2)

Recall that we study the solution in the space

$$\langle v \rangle^{-m} L^1(\mathbb{R}^3) := \{ f \ | \ \|\langle v \rangle^m f\|_{L^1} < \infty \},$$

with $m$ sufficiently large.

The main result is:

**Lemma B.1.** Suppose that $m > 0$ is large enough, then the equation (B.1) has a unique solution for any given $g_0 \in \langle v \rangle^{-m} L^1(\mathbb{R}^3)$. Moreover, for any $t \geq 0$, there exists a positive function $X$, independent of $n$, such that

$$\|\langle v \rangle^m g(\cdot, t)\|_{L^1} \leq X(t) \|\langle v \rangle^m g_0\|_{L^1}.$$

(B.3)
Proof. We start with casting the equation into a convenient form. Apply Duhamel’s principle to obtain
\[ g = e^{(-\nu - i n \cdot v)t} g_0 - \int_0^t e^{(-\nu - i n \cdot v)(t-s)} K g(s) \, ds. \]  
(B.4)

We start with simplifying the problem.

(1) Since the equation is linear, it suffices to prove the existence of solutions, in a small time interval.

(2) All the estimates made on the terms on the right hand side of (B.4) will be based on (B.6) and (B.7) below, which do not depend on n. Thus the estimates are “uniform in n”.

In the next, we define a Banach space to make the fixed point theorem applicable in proving the existence and uniqueness of the solution to (B.4).

We define the norm, for any function \( g : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{C} \), for any \( \tau \geq 0 \),
\[ \| g \|_\tau := \max_{s \in [0, \tau]} \left[ \| \langle v \rangle^m g(s) \|_{L^1} + \Phi^2 \int_0^s \| \chi_{\leq m} \langle v \rangle^{m+1} g(s_1) \|_{L^1} \, ds_1 + \Phi \int_0^s \| \chi_{> m} \langle v \rangle^{m+1} g(s_1) \|_{L^1} \, ds_1 \right]. \]  
(B.5)

where \( \Phi \gg 1 \) is to be chosen, later. The ideas in choosing the norm above are motivated directly by those used in [21, 20], see also the application of Lumer-Philipps Theorem in [19].

In the chosen Banach space, the following two results make the fixed point theorem applicable, hence establish the desired result Lemma B.1:

(A) for any \( \tau > 0 \), and \( g_0 \) satisfying \( \| \langle v \rangle^m g_0 \|_{L^1} < \infty \), we have
\[ \| e^{-\nu t} |g_0| \|_\tau < \infty, \]  
(B.6)

(B) if \( \tau > 0 \) is sufficiently small, then the linear mapping \( \int_0^t e^{(-\nu - i n \cdot v)(t-s)} K g(s) \, ds \) is contractive,
\[ \| \int_0^t e^{(-\nu - i n \cdot v)(t-s)} K g(s) \, ds \|_\tau \leq \frac{3}{4} \| g \|_\tau. \]  
(B.7)

To complete the proof, we need to prove the two key estimates (B.6) and (B.7). We start with proving (B.7). To simplify the notation, we define a linear operator \( H(g) \) by
\[ H(g)(t) := \sum_{l=1}^3 \int_0^t e^{-\nu(t-s)} K_l g(s) \, ds. \]  
(B.8)
We start with considering $\|\langle v \rangle^m H(g)\|_{L^1}$. Compute directly to obtain, recall $\Upsilon_m = \Upsilon_m^{\frac{1}{2}}$ from (B.12),

$$\|\langle v \rangle^m H(g)\|_{L^1} \lesssim \Upsilon_m \int_0^t \|\langle v \rangle^{m+1} g(s)\|_{L^1} \, ds = \Upsilon_m \int_0^t \|\chi_{\leq m} \langle v \rangle^{m+1} g(s)\|_{L^1} \, ds + \Upsilon_m \int_0^t \|\chi_{> m} \langle v \rangle^{m+1} g(s)\|_{L^1} \, ds \leq \frac{\Upsilon_m}{\Phi} \|v\|_t. \quad (B.9)$$

For $\Phi^2 \int_0^t \|\chi_{\leq m} \langle v \rangle^{m+1} H(g)\|_{L^1} \, ds$, compute directly to obtain

$$\Phi^2 \int_0^t \|\chi_{\leq m} \langle v \rangle^{m+1} H(g)\|_{L^1} \, ds \leq \Phi^2 \int_0^t \int_0^s \|\chi_{\leq m} e^{-(s-s_1)\nu} \langle v \rangle^{m+1} K_l g(s_1)\|_{L^1} \, ds_1 \, ds$$

$$\leq \Phi^2 \sum_{k=1}^3 \|\chi_{\leq m} \int_0^t \int_0^s e^{-(s-s_1)\nu} \langle v \rangle^{m+1} K_l |g(s_1)| \, ds_1 \, ds\|_{L^1}. \quad (B.10)$$

Integrate by parts in $s$, using $e^{-s\nu} = -\nu^{-1}\partial_s e^{-s\nu}$, to obtain

$$\chi_{\leq m} \int_0^t \int_0^s e^{-(s-s_1)\nu} \langle v \rangle^{m+1} K_l |g(s_1)| \, ds_1 \, ds = \chi_{\leq m} \int_0^t (1 - e^{-(t-s)\nu}) \nu^{-1} \langle v \rangle^{m+1} K_l |g(s)| \, ds.$$

(B.11)

The smallness is from $\chi_{\leq m}(1 - e^{-(t-s)\nu})$ if $t$, and hence $s \leq t$, are sufficiently small, then we have

$$\epsilon(tm) \to 0 \text{ as } tm \to 0$$

with $\epsilon(tm)$ defined as

$$\epsilon(tm) := \|\chi_{\leq m}(1 - e^{-t\nu})\|_{L^\infty} \leq \|\chi_{\leq m}(1 - e^{-(t-s)\nu})\|_{L^\infty}, \, s \leq t. \quad (B.12)$$

Plug this into (B.10), and use $\nu^{-1} \langle v \rangle \lesssim 1$, to obtain

$$\Phi^2 \int_0^t \|\chi_{\leq m} \langle v \rangle^{m+1} H(g)\|_{L^1} \, ds$$

$$\lesssim \Phi^2 \epsilon(tm) \Upsilon_m \left[ \int_0^t \|\chi_{\leq m} \langle v \rangle^{m+1} g(s)\|_{L^1} \, ds + \int_0^t \|\chi_{> m} \langle v \rangle^{m+1} g(s)\|_{L^1} \, ds \right]$$

$$\lesssim (\Phi + 1) \epsilon(tm) \Upsilon_m \|g\|_t. \quad (B.13)$$

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For \( \Phi \int_0^t \| \chi_{>m}(v)^{m+1} H(g) \|_{L^1} \, ds \), integrate by parts as in (B.11) to find

\[
\int_0^t \| \chi_{>m}(v)^{m+1} H(g) \|_{L^1} \, ds \lesssim \sum_{l=1}^3 \left[ \int_0^t \| \chi_{>m}(v)^{m} K_l \chi_{>m} g(s) \|_{L^1} \, ds + \int_0^t \| \chi_{>m}(v)^{m} K_l \chi_{\leq m} g(s) \|_{L^1} \, ds \right]
\]

\[
\lesssim \Phi \delta_{m,0} \int_0^t \| \chi_{>m}(v)^{m+1} g(s) \|_{L^1} \, ds + \Phi \Upsilon_m \int_0^t \| \chi_{\leq m}(v)^{m+1} g(s) \|_{L^1} \, ds
\]

\[
\lesssim [\delta_{m,0} + \frac{\Upsilon_m}{\Phi}] \| g \|_t. \tag{B.14}
\]

where \( \delta_{m,0} \) is defined before Lemma 8.5 and satisfies \( \delta_{m,0} \to 0 \) as \( m \to \infty \).

The estimates in (B.9), (B.13) and (B.14) imply that, for some constant \( c \),

\[
\| H(g) \|_t \leq c \left[ \frac{\Upsilon_m}{\Phi} + \delta_{m,0} + (\Phi + 1) \epsilon(tm) \Upsilon_m \right] \| g \|_t. \tag{B.15}
\]

Now we choose \( m, \Phi \) and \( t \) to make

\[
c \left[ \frac{\Upsilon_m}{\Phi} + \delta_{m,0} + (\Phi + 1) \epsilon(tm) \Upsilon_m \right] \leq \frac{3}{4}. \tag{B.16}
\]

This, together with (B.15), implies the desired (B.7).

Next we choose \( m, \Phi \) and \( t \) to make (B.16) valid. First choose \( m \) to be sufficiently large, so that

\[
c \delta_{m,0} \leq \frac{1}{4},
\]

secondly choose \( \Phi \) so that

\[
c \frac{\Upsilon_m}{\Phi} \leq \frac{1}{4},
\]

and lastly choose \( t \) small enough so that

\[
c (\Phi + 1) \epsilon(tm) \Upsilon_m \leq \frac{1}{4}.
\]

The proof of (B.7) is complete.

The proof of (B.6) is considerably easier, hence omitted.

\[\square\]

References

[1] L. Arkeryd. Stability in \( L^1 \) for the spatially homogeneous Boltzmann equation. Arch. Rational Mech. Anal., 103(2):151–167, 1988.
[2] L. Arkeryd, R. Esposito, and M. Pulvirenti. The Boltzmann equation for weakly inhomogeneous data. *Comm. Math. Phys.*, 111(3):393–407, 1987.

[3] A. V. Bobylev. Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems. *J. Statist. Phys.*, 88(5-6):1183–1214, 1997.

[4] A. V. Bobylev, I. M. Gamba, and V. A. Panferov. Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions. *J. Statist. Phys.*, 116(5-6):1651–1682, 2004.

[5] R. Bodmer. Zur Boltzmanngleichung. *Comm. Math. Phys.*, 30:303–334, 1973.

[6] F. Bonetto, M. Loss, and R. Vaidyanathan. The Kac model coupled to a thermostat. *J. Stat. Phys.*, 156(4):647–667, 2014.

[7] M. Briant. Stability of global equilibrium for the multi-species boltzmann equation in l? settings. *arXiv:1603.01497*, 2016.

[8] M. Briant and Y. Guo. Asymptotic stability of the Boltzmann equation with Maxwell boundary conditions. *J. Differential Equations*, 261(12):7000–7079, 2016.

[9] C. Cercignani. *The Boltzmann equation and its applications*, volume 67 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.

[10] C. Cercignani, R. Illner, and M. Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[11] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[12] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159(2):245–316, 2005.

[13] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.

[14] J. Fröhlich and Z. Gang. Exponential convergence to the Maxwell distribution for some class of Boltzmann equations. *Comm. Math. Phys.*, 314(2):525–554, 2012.

[15] R. T. Glassey. *The Cauchy problem in kinetic theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[16] H. Grad. Asymptotic theory of the Boltzmann equation, II. In *Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l’UNESCO, Paris, 1962)*, Vol. I, pages 26–59. Academic Press, New York, 1963.
[17] P. T. Gressman and R. M. Strain. Global classical solutions of the Boltzmann equation with long-range interactions. *Proc. Natl. Acad. Sci. USA*, 107(13):5744–5749, 2010.

[18] P. T. Gressman and R. M. Strain. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc.*, 24(3):771–847, 2011.

[19] M. Gualdani, S. Mischler, and C. Mouhot. Factorization for non-symmetric operators and exponential h-theorem. *arxiv.org/abs/1006.5523*, 2010.

[20] Y. Guo. The Vlasov-Poisson-Boltzmann system near Maxwellians. *Comm. Pure Appl. Math.*, 55(9):1104–1135, 2002.

[21] Y. Guo. The Vlasov-Maxwell-Boltzmann system near Maxwellians. *Invent. Math.*, 153(3):593–630, 2003.

[22] Y. Guo. Decay and continuity of the Boltzmann equation in bounded domains. *Arch. Ration. Mech. Anal.*, 197(3):713–809, 2010.

[23] C. Kim and D. Lee. The boltzmann equation with specular boundary condition in convex domains. *arXiv:1604.04342*, 2016.

[24] C. Mouhot. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. *Comm. Math. Phys.*, 261(3):629–672, 2006.

[25] C. Mouhot. Quantitative linearized study of the Boltzmann collision operator and applications. *Commun. Math. Sci.*, (suppl. 1):73–86, 2007.

[26] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York, 1972.

[27] M. Reed and B. Simon. *Methods of modern mathematical physics. III*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979. Scattering theory.

[28] F. Rezakhanlou and C. Villani. *Entropy methods for the Boltzmann equation*, volume 1916 of *Lecture Notes in Mathematics*. Springer, Berlin, 2008. Lectures from a Special Semester on Hydrodynamic Limits held at the Université de Paris VI, Paris, 2001, Edited by François Golse and Stefano Olla.

[29] R. M. Strain and Y. Guo. Stability of the relativistic Maxwellian in a collisional plasma. *Comm. Math. Phys.*, 251(2):263–320, 2004.

[30] R. M. Strain and Y. Guo. Almost exponential decay near Maxwellian. *Comm. Partial Differential Equations*, 31(1-3):417–429, 2006.
[31] S. Ukai. On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad.*, 50:179–184, 1974.

[32] C. Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 71–305. North-Holland, Amsterdam, 2002.

[33] B. Wennberg. Stability and exponential convergence in $L^p$ for the spatially homogeneous Boltzmann equation. *Nonlinear Anal.*, 20(8):935–964, 1993.

[34] B. Wennberg. The Povzner inequality and moments in the Boltzmann equation. In *Proceedings of the VIII International Conference on Waves and Stability in Continuous Media, Part II (Palermo, 1995)*, number 45, part II, pages 673–681, 1996.