The 1/3-2/3 Conjecture for $N$-free ordered sets

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Abstract

A balanced pair in an ordered set $P = (V, \leq)$ is a pair $(x, y)$ of elements of $V$ such that the proportion of linear extensions of $P$ that put $x$ before $y$ is in the real interval $[1/3, 2/3]$. We prove that every finite $N$-free ordered set which is not totally ordered has a balanced pair.

Keywords: Ordered set; Linear extension; $N$-free; Balanced pair; 1/3-2/3 Conjecture.

1 Introduction

Throughout, $P = (V, \leq)$ denotes a finite ordered set, that is, a finite set $V$ and a binary relation $\leq$ on $V$ which is reflexive, antisymmetric and transitive. A linear extension of $P = (V, \leq)$ is a linear ordering $\preceq$ of $V$ which extends $\leq$, i.e. such that $x \preceq y$ whenever $x \leq y$.

Suppose an unknown linear extension $L$ of $P$ is to be determined using only comparisons between pairs of elements. At each step we ask a question of the form "is it true that $x \prec y$?". We will get the answer before we can ask another question. How many comparisons do we need to perform (in
the worst case) in order to determine \( L \) completely? This is known as the problem of *comparison sorting*.

Suppose that at each step we can find a pair \((x, y)\) of incomparable elements such that the proportion of linear extensions of \( P \) that put \( x \) before \( y \), denoted \( P(x < y) \), equals \( \frac{1}{2} \). Then we need at least \( \log_2(e(P)) \) comparisons where \( e(P) \) denotes the number of linear extensions of \( P \). This is not always possible as shown by the example (i) depicted in Figure 1. Indeed, in that example the only possible values for \( P(x < y) \) are 1/3 or 2/3.

Call a pair \((x, y)\) of elements of \( V \) a *balanced pair* in \( P = (V, \leq) \) if \( 1/3 \leq P(x < y) \leq 2/3 \). The 1/3-2/3 Conjecture states that every finite ordered set which is not totally ordered has a balanced pair. If true, the example (i) depicted in Figure 1 would show that the result is best possible. The 1/3-2/3 Conjecture first appeared in a paper of Kislitsyn [6]. It was also formulated independently by Fredman in about 1975 and again by Linial [7].

The 1/3-2/3 Conjecture is known to be true for ordered sets with a non-trivial automorphism [5], for ordered sets of width two [7], for semiorders [2], for bipartite ordered sets [10], for 5-thin posets [4], and for 6-thin posets [8]. See [3] for a survey.

In this paper we prove the 1/3–2/3 Conjecture for \( N \)-free ordered sets.

\[ \text{(i)} \]

\[ \text{(ii)} \]

\[ \text{(iii)} \]

**Figure 1:**

Let \( P = (V, \leq) \) be an ordered set. For \( x, y \in V \) we say that \( y \) is an *upper cover* of \( x \) or that \( x \) is a *lower cover* of \( y \) if \( x < y \) and there is no element \( z \in V \) such that \( x < z < y \). Also, we say that \( x \) and \( y \) are *comparable* if \( x \leq y \) or \( y \leq x \); otherwise we say that \( x \) and \( y \) are *incomparable*. A *chain* is a totally ordered set.

A 4-tuple \((a, b, c, d)\) of distinct elements of \( V \) is an \( N \) in \( P \) if \( b \) is an upper cover of \( a \) and \( c \), \( d \) is an upper cover of \( c \) and if these are the only comparabilities between the elements \( a, b, c, d \) (See Figure 1(ii)). The ordered
set $P$ is $N$-free if it does not contain an $N$ (the ordered set depicted in Figure \ref{fig:N-free}(iii) is $N$-free and the one depicted in Figure \ref{fig:N-free}(ii) is not).

Notice that every finite ordered set can be embedded into a finite $N$-free ordered set (see for example \cite{9}). It was proved in \cite{1} that the number of (unlabeled) $N$-free ordered sets is

$$2^n \log_2(n) + o(n \log_2(n)).$$

Our main result is this.

**Theorem 1.** Every finite $N$-free ordered set which is not totally ordered has a balanced pair.

The proof of Theorem \ref{thm:main} is similar to the proof of Theorem 2 of \cite{7} stating that the 1/3-2/3 Conjecture is true for finite ordered sets of width two (these being the ordered sets covered by two chains).

## 2 Proof of Theorem \ref{thm:main}

We start this section by stating some useful properties of $N$-free ordered sets.

**Lemma 2.** Let $P = (V, \leq)$ be an $N$-free ordered set. If $x, y \in V$ have a common upper cover, then $x$ and $y$ have the same upper covers. Dually, if $x, y \in V$ have a common lower cover, then $x$ and $y$ have the same lower covers.

Let $P = (V, \leq)$ be an ordered set. An element $m \in V$ is called minimal if for all $x \in V$ comparable to $m$ we have $x \geq m$. We denote by $\text{Min}(P)$ the set of all minimal elements of $P$. We recall that the decomposition of $P$ into levels is the sequence $P_0, \ldots, P_h, \ldots$ defined by induction by the formula

$$P_i := \text{Min}(P - \cup\{P_{l'} : l' < l\}).$$

In particular, $P_0 = \text{Min}(P)$.

**Lemma 3.** Let $P = (V, \leq)$ be an $N$-free ordered set and let $P_0, \ldots, P_h$ be the sequence of its levels. Then for every $x \in V$, there exists $i \leq h$ such that all upper covers of $x$ are in $P_i$. 

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Proof. If $x$ has at most one upper cover, then the conclusion of the lemma holds. So we may assume that $x$ has at least two distinct upper covers $x_1$ and $x_2$ belonging to two distinct levels. Let $j < k$ be such that $x_1 \in P_j$ and $x_2 \in P_k$. Then $x_2$ has a lower cover $x_3 \in P_{k-1}$. We claim that $(x_3, x_2, x, x_1)$ is an $N$ in $P$ contradicting our assumption that $P$ is $N$-free. Indeed, since $x_1$ and $x_2$ are upper covers of $x$ we infer that they must be incomparable. Moreover, $x_1$ and $x_3$ are incomparable because otherwise $x_1 < x_3 < x_2$ (notice that $x_3 < x_1$ is not possible since $j \leq k - 1$) which contradicts our assumption that $x_2$ is an upper cover of $x$. Similarly we have that $x$ and $x_3$ are incomparable proving our claim. The proof of the lemma is now complete. 

Let $P = (V, \leq)$ be an ordered set. For $x \in V$ define $D(x) := \{y \in V : y < x\}$ and $U(x) := \{y \in V : x < y\}$.

**Lemma 4.** Let $P$ be an $N$-free ordered set and let $P_0, \ldots, P_h$ be the sequence of its levels. Let $0 \leq i \leq h$ be such that $i$ is the largest with the property that $P_i$ contains two distinct elements with the same set of lower covers. Then for every $x \in P_i$ we have that $U(x) \cup \{x\}$ is a chain.

**Proof.** Let $x \in P_i$ be such that $U(x) \neq \emptyset$ and suppose that $U(x)$ is not a chain. There is then an element $y \in U(x) \cup \{x\}$ having at least two distinct upper covers, say $y_1, y_2$. From Lemma 3 we deduce that $y_1$ and $y_2$ are in the same level $P_j$ with $i < j$. Because $P$ is $N$-free it follows from Lemma 2 that $y_1$ and $y_2$ have the same set of lower covers. This contradicts our choice of $i$. 

We recall that an incomparable pair $(x, y)$ of elements is critical if $U(y) \subseteq U(x)$ and $D(x) \subseteq D(y)$. The following lemma is true for ordered sets that are not necessarily $N$-free.

**Lemma 5.** Suppose $(x, y)$ is a critical pair in $P$ and consider any linear extension of $P$ in which $y < x$. Then the linear order obtained by swapping the positions of $y$ and $x$ is also a linear extension of $P$. Moreover, $P(x < y) \geq \frac{1}{2}$.

**Proof.** Let $L$ be a linear extension that puts $y$ before $x$ and let $z$ be such that $y < z < x$ in $L$. Then $z$ is incomparable with both $x$ and $y$ since $(x, y)$ is a critical pair of $P$. Therefore, the linear order $L'$ obtained by swapping $x$ and $y$ is a linear extension of $P$. The map $L \mapsto L'$ from the set of linear
extensions that put $y$ before $x$ into the set of linear extensions that put $x$ before $y$ is clearly one-to-one. Hence, $P(y < x) \leq P(x < y)$ and therefore $P(x < y) \geq \frac{1}{2}$.

We now prove Theorem [1].

**Proof.** Let $P = (V, \leq)$ be an $N$-free ordered set not totally ordered and $P_0, \cdots, P_h$ be the sequence of its levels. If $P_0$ is a singleton, say $P_0 = \{p_0\}$, then $p_0$ will be the minimum element in any linear extension of the ordered set. Therefore, nothing will change if $p_0$ is deleted from the ordered set. So we may assume without loss of generality that $P_0$ has at least two distinct elements. Notice that any two such elements have the same set of lower covers: the empty set. Now let $0 \leq i \leq h$ be such that $i$ is the largest with the property that $P_i$ contains two distinct elements with the same set of lower covers and let $a, b \in P_i$ be such elements. If $U(b) = U(a) = \emptyset$, then $P(a < b) = \frac{1}{2}$ and we are done. Otherwise we may suppose without loss of generality that $U(b) \neq \emptyset$. From Lemma [4] we deduce that $U(b) \cup \{b\}$ is a chain, say $U(b) \cup \{b\}$ is the chain $b = b_1 < \cdots < b_n$. We prove the theorem by contradiction. We may assume without loss of generality that

$$P(a < b_1) < \frac{1}{3}.$$ 

Indeed, if $U(a) \neq \emptyset$, then the situation is symmetric with respect to $a$ and $b$ and therefore such an assumption is possible. Otherwise, $U(a) = \emptyset$ and hence $(b_1, a)$ is a critical pair (this is because $D(a) = D(b_1)$ by assumption) yielding $P(b_1 < a) > \frac{2}{3}$ (Lemma [5]) or equivalently $P(a < b_1) < \frac{1}{3}$.

Define now the following quantities

$$q_1 = P(a < b_1),$$

$$q_j = P(b_{j-1} < a < b_j)(2 \leq j \leq n),$$

$$q_{n+1} = P(b_n < a).$$

**Lemma.** The real numbers $q_j$ ($1 \leq j \leq n + 1$) satisfy:

(i) $0 \leq q_{n+1} \leq \cdots \leq q_1 \leq \frac{1}{3},$

(ii) $\sum_{j=1}^{n+1} q_j = 1.$
Proof. Since $q_1, \cdots, q_{n+1}$ is a probability distribution, all we have to show is that $q_{n+1} \leq \cdots \leq q_1$. To show this we exhibit a one-to-one mapping from the event that $b_j \prec a \prec b_{j+1}$ whose probability is $q_{j+1}$ into the event that $b_{j-1} \prec a \prec b_j$ whose probability is $q_j$ ($1 \leq j \leq n$). Notice that in a linear extension for which $b_j \prec a \prec b_{j+1}$ every element $z$ between $b_j$ and $a$ is incomparable to both $b_j$ and $a$. Indeed, such an element $z$ cannot be comparable to $b_j$ because otherwise $b_j < z$ in $P$ but the only element above $b_j$ is $b_{j+1}$ which is above $a$ in the linear extension. Now $z$ cannot be comparable to $a$ as well because otherwise $z < a$ in $P$ and hence $z < b = b_1 < b_j$ (by assumption we have that $D(a) = D(b)$). The mapping from those linear extensions in which $b_j \prec a \prec b_{j+1}$ to those in which $b_{j-1} \prec a \prec b_j$ is obtained by swapping the positions of $a$ and $b_j$. This mapping clearly is well defined and one-to-one.

Theorem 1 can be proved now: let $r$ be defined by

$$\sum_{j=1}^{r-1} q_j \leq \frac{1}{2} < \sum_{j=1}^{r} q_j$$

Since $\sum_{j=1}^{r-1} q_j = \mathbb{P}(a \prec b_{r-1}) \leq \frac{1}{2}$, it follows that $\sum_{j=1}^{r-1} q_j < \frac{1}{2}$. Similarly $\sum_{j=1}^{r} q_j = \mathbb{P}(a \prec b_r)$ must be $> \frac{2}{3}$. Therefore $q_r \geq \frac{1}{3}$, but this contradicts $\frac{1}{3} > q_1 \geq q_r$.

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