ON HILBERT EXTENSIONS OF WEIERSTRASS’ THEOREM WITH WEIGHTS

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Abstract. In this paper we study the set of functions $G$-valued which can be approximated by $G$-valued continuous functions in the norm $L^\infty_G(I, w)$, where $I$ is a compact interval, $G$ is a real and separable Hilbert space and $w$ is certain $G$-valued weakly measurable weight. Thus, we obtain a new extension of celebrated Weierstrass approximation theorem.

Key words and phrases. Weierstrass’ theorem, $G$-valued weights, $G$-valued polynomials, $G$-valued continuous functions.

2001 Mathematics Subject Classification. Primary 41, 41A10. Secondary 43A32, 47A56.

To Maíta, in memoriam.

INTRODUCTION.

If $I$ is any compact interval, Weierstrass’ approximation theorem says that $C(I)$ is the largest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere. Weierstrass proved this theorem in 1885. Also, in that time he proved the density of trigonometric polynomials in the class of $2\pi$-periodic continuous real-valued functions. These results were -in a sense- a counterbalance to Weierstrass’ famous example of 1861 on the existence of a continuous nowhere differentiable function (see [1]).

Two subject of interest for Weierstrass were complex function theory and the possibility of representing functions by power series. The result obtained in his paper in 1885 should be viewed from that perspective, moreover, the title of the paper emphasizes such viewpoint (his paper was titled On the possibility of giving an analytic representation to an arbitrary function of real variable, see [10]). Weierstrass’ perception on analytic functions was ‘functions that could be represented by power series’. This paper of Weierstrass was reprinted in Weierstrass’
Mathematische Werke (collected works) with some notable additions, for example a short ‘introduction’. This reprint appeared in 1903 and contains the following statement:

The main result of this paper, restrict to the one variable case, can be summarized as follows:

Let \( f \in C(\mathbb{R}) \). Then there exists a sequence \( f_1, f_2, \ldots \) of entire functions for which

\[
f(x) = \sum_{i=1}^{\infty} f_i(x),
\]

for each \( x \in \mathbb{R} \). In addition the convergence of above sum is uniform on every finite interval.

Let us observe that there is not mention of fact that the \( f_i \) may be assumed to be polynomial. We state Weierstrass’ approximation theorem, not as given in his paper, but as it is currently stated and understood.

**Theorem 1.1.** *(K. Weierstrass).*

Given \( f : [a, b] \to \mathbb{R} \) continuous and an arbitrary \( \epsilon > 0 \) there exists an algebraic polynomial \( p \) such that

\[
|f(x) - p(x)| \leq \epsilon, \quad \forall \ x \in [a, b].
\]

Two papers of Runge published about the same time also provided a proof of this result. But unfortunately, the theorem was not titled Weierstrass-Runge theorem. The impact of Weierstrass’ approximation theorem on the mathematical world was immediate: there were later proofs of famous mathematicians such as Picard (1891), Volterra (1897), Lebesgue (1898), Mittag-Leffler (1900), Landau (1908), de la Vallée Poussin (1912). The proofs more commonly taken at level of undergraduate courses in Mathematics are those of Fejér (1900) and Bernstein (1912) (see, for example [2], [6], [16], [22] or [20]).

There have been many improvements, generalizations and ramifications of Weierstrass’ approximation theorem. For instance, if \( f \) is a \( \mathcal{V} \)-valued function, with \( \mathcal{V} \) a real (or complex) finite dimensional linear space, and if \( f \) is a function of several real variables. Each one of these cases have an easy formulation of results. While, the case in which \( f \) is a function of several complex variables requires a more profound study, with skillful adaptations of both hypothesis and conclusion. A detailed presentation of such results may be found in [30], [3], [7], [12] and [13]. Also, we must recall the Bernstein’s problem on approximation by polynomials on the whole real line (see [16], [17] and [18]), and the approximation problem for unbounded functions in \( I \) (see for example, [11]).
In recent years it has arisen a new focus on the generalizations of Weierstrass’ approximation theorem, which uses the weighted approximation. More precisely, if $I$ is a compact interval, the approximation problem is studied with the norm $L^\infty(I, w)$ defined by

$$\|f\|_{L^\infty(I, w)} := \text{ess sup}_{x \in I} |f(x)|w(x),$$

where $w$ is a weight, i.e., a non-negative measurable function and the convention $0 \cdot \infty = 0$ is used. Observe that (1.2) is not the usual definition of the $L^\infty$ norm in the context of measure theory, although it is the correct definition when we work with weights (see e.g. [4] and [8]).

Considering weighted norms has been proved to be interesting mainly because of two reasons: first, they allow to wider the set of approximable functions (since the functions in $L^\infty(I, w)$ can have singularities where the weight tends to zero); and, second, it is possible to find functions which approximate a given function $f$, whose qualitative behavior is similar to qualitative behavior of $f$ at those points where the weight tends to infinity. The reader may find in [24], [26], [27] and [28] recent and detailed studies about such subject.

Another special kind of approximation problems arises when we consider simultaneous approximation which includes derivatives of certain functions; this is the case of versions of Weierstrass’ theorem in weighted Sobolev spaces. With respect to recent developments about this subject we refer to [26] and [27]: In the first paper, under enough general conditions concerning the vector weights (the so called type 1) defined in a compact interval $I$, the authors characterize to the closure in the weighted Sobolev space $W^{(k, \infty)}(I, w)$ of the spaces of polynomials, $k$-differentiable functions, and infinite differentiable functions, respectively. In the second paper, the reader may find shaper results for the case $k = 1$.

In this paper we obtain a new result on Weierstrass’ approximation theorem with weights when considering approximation in Hilbert spaces. We consider a real and separable Hilbert space $G$, a compact interval $I \subset \mathbb{R}$, the space of all the $G$-valued essentially bounded functions $L^\infty_0(I)$, a weakly measurable function $w : I \rightarrow G$, the space of all $G$-valued continuous functions $C(I; G)$, the space of all the $G$-valued functions $L^\infty_0(I, w)$, which are bounded with respect to the norm defined by

$$\|f\|_{L^\infty_0(I, w)} := \text{ess sup}_{t \in I} \|f(t)(w(t))\|G.$$
often used throughout the text, we shall use standard notation or it will be properly introduced whenever needed. In Section 3 we present the main result about approximation in $L^\infty_G(I, w)$.

2. Preliminaries.

In what follows, $I$ stands for any compact interval in $\mathbb{R}$. By $l^2(\mathbb{R})$ we denote the real linear space of all sequences $\{x_n\}_{n \in \mathbb{Z}^+}$ with $\sum_{n=0}^{\infty} |x_n|^2 < \infty$, and $(\mathcal{G}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ stands for a real and separable Hilbert space with associated norm denoted by $\| \cdot \|_{\mathcal{G}}$.

It is well-known that every real and separable Hilbert space $\mathcal{G}$ is isomorphic either to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ or to $l^2(\mathbb{R})$. In each case, $\mathcal{G}$ has structure of commutative Banach algebra with the coordinatewise operations. In the first case, we have commutative Banach algebra with identity and the second case, this is commutative Banach algebra without identity. The reader is referred to [9], [14] or [33] for more details about these statements.

However, in many applications this isomorphism is not interesting: for instance, we may be dealing with Hilbert spaces of, say, analytic or differentiable functions, and in this interplay between the Hilbert space structure and the properties of individual functions, the second can be fruitful. Despite cases as the previous, to know such isomorphism is valuable, because it allows to determine as far as the properties of Hilbert spaces can be useful for us, and furthermore, we can just think in $l^2(\mathbb{R})$ or $\mathbb{R}^n$ when we want.

2.1. On weighted spaces. A detailed discussion about properties of weighted spaces may be found in [6], [10], [15], [19] or [23]. We recall here some important tools and definitions which will be used throughout this paper.

**Definition 2.1.** A scalar weight $w$ is a measurable function $w : \mathbb{R} \to [0, \infty]$. If $w$ is only defined in $A \subset \mathbb{R}$, we set $w := 0$ in $\mathbb{R} \setminus A$.

**Definition 2.2.** Given a measurable set $A \subset \mathbb{R}$ and a scalar weight $w$, we define the space $L^\infty(A, w)$ as the space of equivalence classes of measurable functions $f : A \to \mathbb{R}$ with respect to the norm

$$\|f\|_{L^\infty(A, w)} := \text{ess sup}_{x \in A} |f(x)| w(x).$$

This space inherits some properties from the classical Lebesgue space $L^\infty(A)$ and it allows us to study new functions, which could not be in the classical $L^\infty(A)$ (see, for example [6], [29]). Another properties of $L^\infty(A, w)$ have a strong relation with the nature of the weight $w$: in fact, if $A = I$ and $w$ has multiplicative inverse, (i.e. there exists a
weight \( w^{-1} : I \rightarrow \mathbb{R} \), such that \( w(t)w^{-1}(t) = 1, \ \forall t \in I \) then, it is easy to see that \( L^{\infty}(I, w) \) and \( L^{\infty}(I) \) are isomorphic, since the map \( \Psi_w : L^{\infty}(I, w) \rightarrow L^{\infty}(I) \) given by \( \Psi_w(f) = fw \) is a linear and bijective isometry, and therefore, \( \Psi_w \) is also homeomorphism, or equivalently, for all \( Y \subseteq L^{\infty}(I, w) \), we have \( \Psi_w(Y) = \overline{\Psi_w(Y)} \), where we take each closure with respect to the norms \( L^{\infty}(I, w) \) and \( L^{\infty}(I) \), respectively. Also, for all \( A \subseteq L^{\infty}(I) \), \( \Psi_w^{-1}(A) = \overline{\Psi_w^{-1}(A)} \) and \( \Psi_w^{-1} = \Psi_w^{-1} \). Then using Weierstrass’ theorem we have,

\[
(2.4) \quad \Psi_w^{-1}(\overline{I}) = \overline{\Psi_w^{-1}(I)} = \{ f \in L^{\infty}(I, w) : fw \in C(I) \}.
\]

Unfortunately, the last equality in (2.4) does not allow to obtain information on local behavior of the functions \( f \in L^{\infty}(I, w) \) which can be approximated. Furthermore, if \( f \in L^{\infty}(I, w) \), then in general \( fw \) is not continuous function, since its continuity also depends of the singularities of weight \( w \) (see [6], [16], [17], [28], [23], [27]).

The next definition presents the classification of the singularities of a scalar weight \( w \) done in [27] to show the results about density of continuous functions in the space \( L^{\infty}(supp(w), w) \).

**Definition 2.3.** Given a scalar weight \( w \) we say that \( a \in supp(w) \) is a singularity of \( w \) (or singular for \( w \)) if

\[
\operatorname{ess \lim \inf}_{x \to a} w(x) = 0.
\]

We say that a singularity \( a \) of \( w \) is of type 1 if \( \operatorname{ess \lim \sup}_{x \to a} w(x) = 0 \).

We say that a singularity \( a \) of \( w \) is of type 2 if \( 0 < \operatorname{ess \lim \sup}_{x \to a} w(x) < \infty \).

We say that a singularity \( a \) of \( w \) is of type 3 if \( \operatorname{ess \lim \sup}_{x \to a} w(x) = \infty \).

We denote by \( S \) and \( S_i \) (\( i = 1, 2, 3 \)) respectively, the set of singularities of \( w \) and the set of singularities of \( w \) of type \( i \).

We say that \( a \in S_i^+ \) (respectively \( a \in S_i^- \)) if \( a \) verifies the property in the definition of \( S_i \) when we take the limit as \( x \to a^+ \) (respectively \( x \to a^- \)). We define \( S^+ := S_1^+ \cup S_2^+ \cup S_3^+ \) and \( S^- := S_1^- \cup S_2^- \cup S_3^- \).

**Definition 2.4.** Given a scalar weight \( w \), we define the right regular and left regular points of \( w \), respectively, as

\[
R^+ := \{ a \in \text{supp}(w) : \operatorname{ess \lim \inf}_{x \to a^+} w(x) > 0 \},
\]

\[
R^- := \{ a \in \text{supp}(w) : \operatorname{ess \lim \inf}_{x \to a^-} w(x) > 0 \}.
\]

The following result was proved in [27] and it states a characterization for the functions in \( L^{\infty}(\text{supp}(w), w) \) which can be approximated by continuous functions in norm \( L^{\infty}(\text{supp}(w), w) \) for every \( w \).
Theorem 2.1. (Portilla et al. [27], Theorem 1.2). Let \( w \) be any scalar weight and

\[
H_0 := \left\{ f \in L^\infty(\text{supp}(w), w) : \begin{array}{l}
  \text{\( f \) is continuous to the right at every point of \( R^+ \),} \\
  \text{\( f \) is continuous to the left at every point of \( R^- \),} \\
  \text{for each } a \in S^+, \quad \text{ess lim}_{x \to a^+} |f(x) - f(a)| w(x) = 0, \\
  \text{for each } a \in S^-, \quad \text{ess lim}_{x \to a^-} |f(x) - f(a)| w(x) = 0
\end{array} \right\}.
\]

Then:
(a) The closure of \( C(\mathbb{R}) \cap L^\infty(w) \) in \( L^\infty(w) \) is \( H_0 \).
(b) If \( w \in L^\infty_{\text{loc}}(\mathbb{R}) \), then the closure of \( C^\infty(\mathbb{R}) \cap L^\infty(w) \) in \( L^\infty(w) \) is also \( H_0 \).
(c) If \( \text{supp}(w) \) is compact and \( w \in L^\infty(\mathbb{R}) \), then the closure of the space of polynomials is \( H_0 \) as well.

Theorem 2.1 is going to be an important tool which is going to allow us to obtain the key for the first result about Hilbert extensions of Weierstrass’ theorem with weights in the present paper.

2.2. \( \mathcal{G} \)-valued functions.

Definition 2.5. Let \( \mathcal{G} \) be a real and separable Hilbert space and we consider any sequence \( \{x_n\} \subset \mathcal{G} \). We say that the support of \( \{x_n\} \) is the set of \( n \) for which \( x_n \neq 0 \). We denote to support of \( \{x_n\} \) by \( \text{supp}(x_n) \).

Definition 2.6. Let \( \mathcal{G} \) be a real and separable Hilbert space, a weight \( w \) on \( \mathcal{G} \) is a weakly measurable function \( w : I \to \mathcal{G} \).

Let \( \mathcal{G} \) be a real and separable Hilbert space. A \( \mathcal{G} \)-valued polynomial on \( I \) is a function \( \phi : I \to \mathcal{G} \), such that

\[
\phi(t) = \sum_{n \in \mathbb{N}} \xi_n t^n,
\]

where \( (\xi_n)_{n \in \mathbb{N}} \subset \mathcal{G} \) has finite support.

Let \( \mathbb{P}(\mathcal{G}) \) be the space of all \( \mathcal{G} \)-valued polynomials on \( I \). It is well-known that \( \mathbb{P}(\mathcal{G}) \) is a subalgebra of the space \( C(I; \mathcal{G}) \) of all continuous \( \mathcal{G} \)-valued functions on \( I \).

For \( 1 \leq p \leq \infty \), \( L^p_\mathcal{G}(I) \) denotes the set of all weakly measurable functions \( f : I \to \mathcal{G} \) such that

\[
\int_I \|f(t)\|_\mathcal{G}^p dt < \infty, \quad \text{if } 1 \leq p < \infty,
\]

or

\[
\text{ess sup}_{t \in I} \|f(t)\|_\mathcal{G} < \infty, \quad \text{if } p = \infty.
\]
Then $L^2_G(I)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{L^2_G(I)} = \int_I \langle f(t), g(t) \rangle_G \, dt.$$ 

$P(G)$ is also dense in $L^p_G(I)$, for $1 \leq p < \infty$.

More details about these spaces may be found in [31].

**Definition 2.7.** Let $G$ be a real and separable Hilbert space, a weight $w$ on $G$ is a weakly measurable function $w : I \to G$.

**Definition 2.8.** Let $w$ be a weight on $G$, we define the space $L^\infty_G(I, w)$ as the space of equivalence classes of all the $G$-valued weakly measurable functions $f : I \to G$ with respect to the norm

$$\|f\|_{L^\infty_G(I, w)} := \text{ess sup}_{t \in I} \|(fw)(t)\|_G,$$

where $fw : I \to G$ is defined as follows: If $\dim G < \infty$, we have the functions $f$ and $w$ can be expressed by $f = (f_1, \ldots, f_{n_0})$ and $w = (w_1, \ldots, w_{n_0})$, respectively, where $f_j, w_j : I \to \mathbb{R}$, for $j = 1, \ldots, n_0$, with $n_0 = \dim G$. Then

$$(fw)(t) := (f_1(t)w_1(t), \ldots, f_{n_0}(t)w_{n_0}(t)), \text{ for } t \in I.$$ 

If $\dim G = \infty$, let $\{\tau_j\}_{j \in \mathbb{Z}^+}$ be a complete orthonormal system, then for $t \in I$ the functions $f$ and $w$ can be expressed as $f(t) = \sum_{j=0}^{\infty} \langle f(t), \tau_j \rangle_G \tau_j$ and $w(t) = \sum_{j=0}^{\infty} \langle w(t), \tau_j \rangle_G \tau_j$, respectively. So, we can define

$$(fw)(t) := \sum_{j=0}^{\infty} \langle f(t), \tau_j \rangle_G \langle w(t), \tau_j \rangle_G \tau_j, \text{ for } t \in I.$$ 

On this way, we can study our approximation problem using the properties of commutative Banach algebra of $l^2(\mathbb{R})$.

The next Proposition shows a result about algebraic properties and density of $P(G)$ in $C(I; G)$. The analogous result, when $G$ is a complex separable Hilbert space, appears in [31].

**Proposition 2.1.**

i) $P(G)$ is a subalgebra of the space of all $G$-valued continuous functions on $I$.

ii) The closure of $P(G)$ in $L^\infty_G(I)$ is $C(I; G)$.

**Proof.**

i) It is straightforward.

ii) It is enough to prove that $C(I; G) \subset \overline{P(G)}$, since $\overline{P(G)} \subset C(I; G) = C(I, G)$.

**Case 1:** $\dim G < \infty$. 
Let us assume that \( \dim \mathcal{G} = n_0 \). Given an orthonormal basis \( \{ \tau_1, \ldots, \tau_{n_0} \} \) of \( \mathcal{G} \), \( \epsilon > 0 \) and \( f \in C(I; \mathcal{G}) = C(I, \mathbb{R}^{n_0}) \), then \( f \sim (f_1, \ldots, f_{n_0}) \) with \( f_j \in C(I) \), \( j = 1, \ldots, n_0 \). The Weierstrass’ theorem guarantees that there exists \( p_k \in \mathbb{P} \) such that

\[
\| f_j - p_j \|_{L^\infty(I)} < \frac{\epsilon}{\sqrt{n_0}}, \quad j = 1, \ldots, n_0.
\]

If we consider the polynomial \( p \in \mathbb{P}(\mathcal{G}) \) such that \( p \sim (p_1, \ldots, p_{n_0}) \), then we have that

\[
\| f - p \|_{L^\infty(I)} = \text{ess sup}_{t \in I} \| (f - p)(t) \|_{\mathcal{G}}
\]

\[
= \text{ess sup}_{t \in I} \left[ \sum_{j=1}^{n_0} |\langle f(t) - p(t), \tau_j \rangle_{\mathcal{G}}|^2 \right]^{1/2}
\]

\[
\leq \text{ess sup}_{t \in I} \left\| (f_1 - p_1)(t), \ldots, (f_{n_0} - p_{n_0})(t) \right\|_{\mathbb{R}^{n_0}} < \epsilon.
\]

**Case 2:** \( \mathcal{G} \) is infinite-dimensional.

Let \( f \in C(I; \mathcal{G}) \) and \( \{ \tau_j \}_{j \in \mathbb{Z}_+} \) a complete orthonormal system, then for each \( t \in I \)

\[
f(t) = \sum_{j=0}^{\infty} \langle f(t), \tau_j \rangle_{\mathcal{G}} \tau_j,
\]

consequently, given \( \epsilon > 0 \) there exists \( m_0 \in \mathbb{Z}_+ \) such that

\[
\left\| f(t) - \sum_{j=0}^{n} \langle f(t), \tau_j \rangle_{\mathcal{G}} \tau_j \right\|_{\mathcal{G}} < \epsilon,
\]

whenever \( n \geq m_0 \).

Now, let us consider the functions \( f_j : I \to \mathbb{R} \) given by \( f_j(t) = \langle f(t), \tau_j \rangle_{\mathcal{G}} \). We have that \( f \sim \{ f_j \} \) with \( \sum_{j \in \mathbb{Z}_+} |f_j(t)|^2 < \infty \), for each \( t \in I \) and \( f_j \in C(I) \).

So, Weierstrass’ approximation theorem guarantees that there exists a sequence \( \{ p_j \}_{j \in \mathbb{Z}_+} \subset \mathbb{P} \) such that

\[
\| f_j - p_j \|_{L^\infty(I)} < \frac{\epsilon}{j + 1}, \quad j \in \mathbb{Z}_+.
\]

We define the \( \mathcal{G} \)-polynomials \( \tilde{p}_j \in \mathbb{P}(\mathcal{G}) \) by \( \tilde{p}_j(t) = p_j(t)\tau_j \), for each \( j \in \mathbb{Z}_+ \). Then for \( n \geq m_0 \) we have
ON HILBERT EXTENSIONS OF WEIERSTRASS’ THEOREM WITH WEIGHTS

\[ \left\| f(t) - \sum_{j=0}^{n} \tilde{p}_j(t) \right\|_G \leq \left\| f(t) - \sum_{j=0}^{n} f_j(t) \tau_j \right\|_G + \left\| \sum_{j=0}^{n} f_j(t) \tau_j - \sum_{j=0}^{n} \tilde{p}_j(t) \right\|_G \]

\[ \leq \epsilon + \left( \sum_{j=0}^{\infty} |f_j(t) - p_j(t)|^2 \right)^{1/2} \]

\[ < \epsilon + \left( \sum_{j=0}^{\infty} \left( \frac{\epsilon}{j+1} \right)^2 \right)^{1/2} \]

\[ \leq \epsilon \left( 1 + \left( \sum_{j=0}^{\infty} \frac{1}{(j+1)^2} \right)^{1/2} \right). \]

From these inequalities we can deduce that for \( n \) large enough there exists \( q_n(t) = \sum_{j=0}^{n} \tilde{p}_j(t) \) such that

\[ \| f - q_n \|_{L^G_\infty(I)} < C \epsilon. \]

This completes the proof. \( \square \)

3. The main results.

In this section, we only deal with weights \( w \) such that \( \text{supp}(w) = I \).

3.1. Approximation in \( L^\infty_\mathcal{G}(I, w) \).

Definition 3.1. Let \( \mathcal{G} \) be a real and separable Hilbert space and let \( w \) be a weight on \( \mathcal{G} \). We say that \( w \) is admissible* if one of the following conditions is satisfied

i) If \( \dim \mathcal{G} < \infty \) then each one of the components \( w_j, 1 \leq j \leq \dim \mathcal{G} \), is a scalar weight.

ii) If \( \dim \mathcal{G} = \infty \), \( \{ \tau_j \}_{j \in \mathbb{Z}_+} \) is a complete orthonormal system, and \( w(t) = \sum_{j=0}^{\infty} \langle w(t), \tau_j \rangle \mathcal{G} \tau_j \), then each one of the functions \( \langle w(t), \tau_j \rangle \mathcal{G} \) is a weight function.

Let us observe that if \( \dim \mathcal{G} = \infty \) and \( w \) is admissible*, then it induces a family of weighted \( L^2(\mathbb{R}) \) spaces, \( \{ L^2(\mathbb{R}; w) : t \in I \} \) given by

\[ L^2(\mathbb{R}; w) = \left\{ \{ x_j \}_{j \in \mathbb{Z}_+} : \sum_{j=0}^{\infty} \langle w(t), \tau_j \rangle \mathcal{G} |x_j|^2 < \infty \right\}. \]

And for each \( t \in I \), the function \( w_j(t) = \langle w(t), \tau_j \rangle \mathcal{G} \) also induces a linear isometry

\[ \Psi^t_{w_j} : L^2(\mathbb{R}; w_j) \to L^2(\mathbb{R}) \] given by

\[ \Psi^t_{w_j} \left( \{ x_j \}_{j \in \mathbb{Z}_+} \right) = \left\{ w_j(t) x_j \right\}_{j \in \mathbb{Z}_+} = \left\{ \langle w(t), \tau_j \rangle \mathcal{G} x_j \right\}_{j \in \mathbb{Z}_+}. \]

For a brief study on weighted \( L^2(\mathbb{R}) \) spaces the reader is referred to [9]. In order to characterize the \( \mathcal{G} \)-valued functions which can be
approximated in $L^\infty(I, w)$ by functions in $C(I; \mathcal{G}) \cap L^\infty(I, w)$, our argument requires an admissible* weight $w$. It is clear that in the one-dimensional case an admissible* weight is an arbitrary scalar weight on $I$, and therefore the Theorem 2.1 in [27] holds in this case.

**Theorem 3.1.** Let $\mathcal{G}$ be a real and separable Hilbert space and let $w$ be an admissible* weight on $\mathcal{G}$. Let us define

$$H := \left\{ f \in L^\infty(I, w) : f \sim (f_1, \ldots, f_{n_0}) \text{ and } f_j \in H_j, 1 \leq j \leq n_0 \text{ with } n_0 = \dim \mathcal{G}, \right\},$$

where

$$H_j := \left\{ f_j \in L^\infty(I, w) : f_j \text{ is continuous to the right at every point of } R^+, \right. \text{ for each } a \in S^+, \text{ ess lim}_{x \to a^+} |f_j(x) - f_j(a)| w_j(x) = 0, \right.$$

$$\left. \text{ for each } a \in S^-, \text{ ess lim}_{x \to a^-} |f_j(x) - f_j(a)| w_j(x) = 0 \right\}.$$}

Then the closure of $C(I; \mathcal{G}) \cap L^\infty(I, w)$ in $L^\infty(I, w)$ is $H$. Furthermore, if $w \in L^\infty(I)$ then the closure of the space of $\mathcal{G}$-valued polynomials is $H$ as well.

**Proof.** Let us assume that $\dim \mathcal{G} = n_0$. If $f \in C(I; \mathcal{G}) \cap L^\infty(I, w)$, then $f \sim (f_1, \ldots, f_{n_0})$, with $f_j : I \to R$, $1 \leq j \leq n_0$. Given $\epsilon > 0$, there exists $g \in C(I; \mathcal{G}) \cap L^\infty(I, w)$ such that $\|f - g\|_{L^\infty(I, w)} < \epsilon$. Let us consider $(g_1, \ldots, g_{n_0})$ such that $g_j \in C(I) \cap L^\infty(I, w_j)$ and $g \sim (g_1, \ldots, g_{n_0})$, then

$$|(f_j(t) - g_j(t))w_j(t)| \leq \text{ess sup}_{s \in I} \left[ \sum_{j=1}^{n_0} |(f_j(s) - g_j(s))w_j(s)|^2 \right]^{1/2} \quad \text{a.e.}$$

On other hand, $\left[ \sum_{j=1}^{n_0} |(f_j(s) - g_j(s))w_j(s)|^2 \right]^{1/2} = \|f - g\|_{L^\infty(I, w)}$, as consequence of $\mathcal{G}$ is isomorphic to $R^{n_0}$ and the Parseval identity (see [9] or [33]). Then,

$$\|f_j - g_j\|_{L^\infty(I, w_j)} \leq \|f - g\|_{L^\infty(I, w)} < \epsilon.$$}

Hence, $f_j \in C(I) \cap L^\infty(I, w_j)$ for $1 \leq j \leq n_0$, and the part (a) of Theorem 2.1 gives that $H$ contains $C(I; \mathcal{G}) \cap L^\infty(I, w)$.

In order to see that $H$ is contained in $C(I; \mathcal{G}) \cap L^\infty(I, w)$, let us fix $f \in H$ and $\epsilon > 0$, and let us consider each one of its component functions $f_j \in H_j, j = 1, \ldots, n_0$. By the part (a) of Theorem 2.1 there exists $g_j \in C(I) \cap L^\infty(I, w_j)$, $j = 1, \ldots, n_0$, such that

$$\|f_j - g_j\|_{L^\infty(I, w_j)} < \frac{\epsilon}{\sqrt{n_0}}.$$
We define the function $g \in C(I; G)$ such that $g \sim (g_1, \ldots, g_n)$, then

$$
\|f - g\|_{L^\infty(I, w)} = \text{ess sup}_{t \in I} \|((f - p)w)(t)\|_G
= \text{ess sup}_{t \in I} \left[ \sum_{j=1}^{n_0} \left| (f_j(t) - g_j(t))w_j(t) \right|^2 \right]^{1/2} < \epsilon.
$$

If $w \in L^\infty(I)$, the closure of the $G$-valued polynomials is $H$ as well, as a consequence of Proposition 2.1.

In a similar way, if $\dim G = \infty$, $\{\tau_j\}_{j \in \mathbb{Z}^+}$ is a complete orthonormal system and $f \in C(I; G) \cap L^\infty(I, w)$, then $f(t) = \sum_{j=0}^{\infty} \langle f(t), \tau_j \rangle \tau_j$. Given $\epsilon > 0$, there exists $g \in C(I; G) \cap L^\infty(I, w)$ such that $\|f - g\|_{L^\infty(I, w)} < \epsilon$.

Let us consider $\{g_j\}_{j \in \mathbb{Z}^+}$ such that $g_j \in C(I) \cap L^\infty(I, w_j)$ and $g \sim \{g_j\}_{j \in \mathbb{Z}^+}$, then

$$
|(f_j(t) - g_j(t))w_j(t)| \leq \text{ess sup}_{s \in I} \left[ \sum_{j=0}^{\infty} \left| (f_j(s) - g_j(s))w_j(s) \right|^2 \right]^{1/2} \text{ a.e.}
$$

On other hand, $\left[ \sum_{j=0}^{\infty} \left| (f_j(s) - g_j(s))w_j(s) \right|^2 \right]^{1/2} = \|f - g\|_{L^\infty(I, w)}$, as consequence of $G$ is isomorphic to $l^2(\mathbb{R})$ and the Parseval identity (see [9] or [33]). Then,

$$
\|f_j - g_j\|_{L^\infty(I, w_j)} \leq \|f - g\|_{L^\infty(I, w)} < \epsilon.
$$

Hence, $f_j \in C(I) \cap L^\infty(I, w_j)$ for $j \in \mathbb{Z}^+$, and the part (a) of Theorem 2.1 gives that $H$ contains $C(I; G) \cap L^\infty_{G}(I, w)^{L^\infty_{G}(I, w)}$.

In order to see that $H$ is contained in $C(I; G) \cap L^\infty_{G}(I, w)^{L^\infty_{G}(I, w)}$, let $f \in H$ and $\epsilon > 0$, and let us consider the component functions $f_j \in H_j$ of $f$, $0 \leq j < \infty$. Since $w_j(t) = \langle w(t), \tau_j \rangle$ is a weight, by the part (a) of Theorem 2.1 there exists $g_j \in C(I) \cap L^\infty(I, w_j)$, $0 \leq j < \infty$, such that

$$
\|f_j - g_j\|_{L^\infty(I, w_j)} < \frac{\epsilon}{j + 1}, \quad j \in \mathbb{Z}^+.
$$
We define the function $g : I \rightarrow G$ by $g(t) = \sum_{j=0}^{\infty} g_j(t) \tau_j$, then

\[
\|f - g\|_{L^\infty(I, w)} = \text{ess sup}_{t \in I} \|((f - g)w)(t)\|_G \\
= \text{ess sup}_{t \in I} \|\{(f_j(t) - g_j(t))w_j(t)\}\|_{l^2(\mathbb{R})} \\
= \text{ess sup}_{t \in I} \left[ \sum_{j=0}^{\infty} |f_j(t) - g_j(t)|^2 w_j^2(t) \right]^{1/2} \\
\leq \left[ \sum_{j=0}^{\infty} \left( \frac{\epsilon}{j+1} \right)^2 \right]^{1/2} = \epsilon \left[ \sum_{j=0}^{\infty} \frac{1}{(j+1)^2} \right]^{1/2}
\]

This result is similar when $G$ is a complex separable Hilbert space and it can also be extended to $L^\infty_{(I, w)}(G)$, where $L(G)$ is the space of operators on $G$.

**Acknowledgment.**

The author wishes to thank the referee and Professor José Manuel Rodríguez for their suggestions and comments which have improved the presentation of the paper.

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