LARGE TIME BEHAVIOR OF THE ON-DIAGONAL HEAT KERNEL FOR MINIMAL SUBMANIFOLDS WITH POLYNOMIAL VOLUME GROWTH

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Abstract. In this paper we provide a lower bound for the long time on-diagonal heat kernel of minimal submanifolds in a Cartan-hadamard ambient manifold assuming that the submanifold is of polynomial volume growth. In particular cases, that lower bound is related with the number of ends of the submanifold.

1. Introduction

Let $M^m$ be a $m$-dimensional minimally immersed submanifold into a simply connected ambient manifold $N^n$ with sectional curvatures $K_N$ bounded from above by $K_N \leq 0$. S. Markvorsen proved in [Mar86] -following [CLY84]- that the heat kernel $H$ of $M^m$ is bounded from above by the heat kernel $H_{m,0}$ of the Euclidean space $\mathbb{R}^m$, namely:

$$H(t, x, y) \leq H_{m,0}(t, r_x(y)) = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{(r_x(y))^2}{4t}},$$

where $r_x(y)$ is the distance in $N$ from $x$ to $y$. In particular for the on-diagonal heat kernel $H(t, x, x)$ of $M^m$ one can state that

$$H(t, x, x) \leq \left(\frac{4\pi t}{m}\right)^{\frac{m}{2}}.$$

This paper deals with lower bounds to the on-diagonal heat kernel assuming certain restriction on the volume growth. In order to define that appropriate behavior on the growth of the extrinsic volume, recall that given a minimal submanifold $M^m$ properly immersed in a Cartan-Hadamard manifold $N$ with sectional curvatures $K_N$ bounded from above by $K_N \leq 0$ and denoting by $\omega_m$ the volume of a radius one geodesic ball in $\mathbb{R}^m$ and by $B_N^R(p)$ the geodesic ball in $N$ of radius $R$ centered at $p$, by the monotonicity formula (see for instance [MP12, theorem 2.6.9] and [Pal99]) for any point $p \in M^m$ the function

$$Q(R) = \frac{\text{Vol}(M^m \cap B_N^R(p))}{\omega_m R^m},$$

is a non decreasing function. Throughout this paper a complete minimal submanifold properly immersed in a Cartan-hadamard ambient manifold is called a minimal submanifold of polynomial volume growth if there exists a constant $\mathcal{E}$ depending on $M^m$ such that:

$$\lim_{R \to \infty} Q(R) \leq \mathcal{E} < \infty.$$

Under such volume growth behavior we can state the behavior of the long time asymptotic for the on-diagonal heat kernel by the main theorem of this paper. The main theorem makes use of the following constant $C_m$ depending only on the dimension $m$ of the submanifold:

$$C_m := \frac{\Gamma\left(\frac{m}{2}, \frac{2}{m} \left(\frac{\pi}{2} \Gamma\left(\frac{m}{2}\right)\right)^{\frac{2}{m}}\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

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Figure 1. The catenoid, the Costa surface and the Scherk singly periodic surface are examples of minimal surfaces immersed in $\mathbb{R}^3$ with polynomial volume growth which is equivalent to quadratic area growth when the submanifold is a surface.

where $\Gamma(z)$ and $\Gamma(z_1, z_2)$ in the above expression denote the gamma function and the incomplete gamma function respectively, i.e,

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} \, dt.$$  
$$\Gamma(z_1, z_2) := \int_{z_2}^{\infty} t^{z_1-1} e^{-t} \, dt.$$  

For minimal submanifolds with an extrinsic volume growth controlled by the above constant $C_m$ we can state the main result of this paper:

**Main Theorem.** Let $M^m$ be a complete $m$-dimensional submanifold properly immersed in a simply connected ambient manifold $N$ with sectional curvatures $K_N$ bounded from above by $K_N \leq 0$. Suppose that $M^m$ is of polynomial volume growth, and that

$$\mathcal{E} < \frac{1}{C_m},$$

Then, the heat kernel $\mathcal{H}$ of $M^m$ satisfies

$$\limsup_{t \to \infty} (4\pi t)^{-\frac{m}{2}} \mathcal{H}(t, x, x) \leq 1.$$  

It is not hard to find examples of complete minimal submanifolds properly and minimally immersed in a Cartan-Hadamard ambient manifold with polynomial volume growth. Indeed, for a complete minimal surface embedded in $\mathbb{R}^3$, by a well known result (see [Oss86, JM83] and introduction in [GP13]), if the surface has finite total curvature then the surface has polynomial volume growth (quadratic area growth) and the constant $\mathcal{E}$ given in equation (1.4) is equal to the number of ends of the surface. This is the case of the catenoid or the Costa surface (with $\mathcal{E} = 2$ for the catenoid and $\mathcal{E} = 3$ for the Costa surface). It is also known that there exist other surfaces with quadratic area growth but without finite total curvature and even without finite topological type. An example of that kind of surface is the Scherk singly periodic surface (see introduction in [MW07]) which has $\mathcal{E} = 2$.

Since

$$C_2 \sim 0.14, \quad \frac{1}{C_2} \sim 7.39,$$

we can apply the main theorem to the catenoid, the Costa and the Scherk surface, obtaining

$$\frac{(1 - 0.28)^2}{2} \leq \limsup_{t \to \infty} (4\pi t) \mathcal{H}(t, x, x) \leq 1,$$

for the catenoid and the Scherk singly periodic surface, and

$$\frac{(1 - 0.41)^2}{3} \leq \limsup_{t \to \infty} (4\pi t) \mathcal{H}(t, x, x) \leq 1,$$
for the Costa surface.

As we have shown, there are several examples where the volume growth is related with the number of ends of the submanifold. In fact, the following theorem allows us to achieve inequality (1.4) under certain decay of the norm of the second fundamental form and to give a topological meaning to \( \lim_{R \to \infty} Q(R) \)

**Theorem 1.1** (see theorem 2.2 of [Qin95] and [GP12]). Let \( M^m \) be an \( m \)-dimensional complete immersed minimal submanifold in \( \mathbb{R}^n \) which satisfies

\[
\lim_{R \to \infty} \sup_{x \in M^m \cap r(x) \geq R} r(x) \| A \| (x) = 0,
\]

where \( A \) denotes the second fundamental form. Then, the number of ends \( E(M^m) \) of \( M^m \) is given by:

\[
\lim_{R \to \infty} Q(R) = E(M^m)
\]

provided either of the following two conditions is satisfied:

1. \( m = 2, n = 3 \) and each end of \( M^m \) is embedded.
2. \( m \geq 3 \).

Hence, we can state the following corollary showing the relation between the number of ends and the lower bound for the heat kernel under the assumptions of the above theorem (see introduction of [GSC09] for a complete overview on the two sides estimates for the heat kernel on manifolds with ends):

**Corollary 1.2.** Let \( M^m \) be an \( m \)-dimensional complete immersed minimal submanifold in \( \mathbb{R}^n \) which satisfies

\[
\lim_{R \to \infty} \sup_{x \in M^m \cap r(x) \geq R} r(x) \| A \| (x) = 0,
\]

and

1. if \( m = 2, n = 3 \) and each end of \( M^m \) is embedded. Or,
2. \( m \geq 3 \).

Then, if the number of ends \( E(M^m) \) of \( M^m \) is bounded from above by

\[
E(M^m) < \frac{1}{C_m},
\]

the heat kernel \( \mathcal{H} \) of \( M^m \) satisfies

\[
\frac{(1 - E(M^m)C_m)^2}{E(M^m)} \leq \limsup_{t \to \infty} (4\pi t)^{-\frac{m}{2}} \mathcal{H}(t, x, x) \leq 1.
\]

If \( M^2 \) is a minimal surface in \( \mathbb{R}^3 \), by the Gauss formula the second fundamental form is related with the Gaussian curvature \( K_G \) of \( M^2 \) by

\[
K_G = -\frac{1}{2} |A|^2,
\]

in view of [MPR13] theorem 1.2] it seems that in the particular case of complete embedded minimal surfaces in \( \mathbb{R}^3 \) if there exists a constant \( C \) such that \( |K_G| R^2 \leq C \), then:

\[
|K_G| R^2 \leq C \quad \Rightarrow \quad \int_{M^2} |K_G| < \infty \quad \Rightarrow \quad \lim_{R \to \infty} \sup_{x \in M^m \cap r(x) \geq R} r(x) |A| (x) = 0.
\]

Hence, the condition given in equation (1.10) in the above corollary can be replaced in the particular case of complete embedded minimal surfaces in \( \mathbb{R}^3 \) by

\[
|K_G| R^2 \leq C.
\]
Recall also that a particular case when equality (1.10) holds is (see [Qin95]) when

$$\int_{M^n} |A|^m \, dV < \infty$$

i.e., when the submanifold has finite scalar curvature (see also [And84]).

Let us finally remark that

**Remark a.** Given a manifold $M^n$ with non-negative Ricci curvature $\text{Rc} > 0$, Bishop-Gromov volume comparison theorem asserts that for any $o \in M^n$ the relative volume quotient

$$\frac{\text{Vol}(B_R^{M^n}(o))}{\omega_n R^n}$$

is non-increasing in the radius $R$ (being $B_R^{M^n}(o)$ the geodesic ball of radius $R$ centered at $o$). The relative volume quotient converges to a non-negative number $\Theta$:

$$\lim_{R \to \infty} \frac{\text{Vol}(B_R^{M^n}(o))}{\omega_n R^n} = \Theta \geq 0.$$ 

If $\Theta > 0$, one says that the manifold $M^n$ has maximal volume growth.

P. Li proved in [Li86] (see also [Xu13]) that if $M^n$ has $\text{Rc} > 0$ and maximal volume growth, then

$$\lim_{t \to \infty} \text{Vol} \left( B_{\sqrt{t}}^{M^n}(y) \right) \mathcal{H}(t, x, y) = \omega_n (4\pi)^{-\frac{n}{2}}.$$ 

Therefore

$$\lim_{t \to \infty} (4\pi t)^{\frac{n}{2}} \mathcal{H}(t, x, y) = \frac{1}{\Theta}.$$ 

In some sense, our main theorem can be understood (partially) as a reverse of the Li’s theorem because at least on dimension 2, by the Gauss formula (equation (1.13)), a submanifold properly and minimally immersed in a Cartan-Hadamard ambient manifold has non-positive sectional curvature (instead of $\text{Rc} > 0$) and because, by the monotonicity formula, the extrinsic quotient given in equation (1.3) is non-decreasing (instead of non-increasing like the relative volume quotient).

Despite of the weakness of the inequalities (1.7) in comparison to equality (1.15) observe, however, that a non-negatively Ricci-curved manifold with maximal volume growth must have finite fundamental group (see [Li86]) but that is not true for minimal submanifolds of a Cartan-Hadamard with polynomial volume growth (see for instance the singly periodic Scherk surface (figure 1)).

The most well known examples of heat kernels of minimal submanifolds $M^n$ in the Euclidean space $\mathbb{R}^n$ are when $M^n$ is a totally geodesic submanifold $\mathbb{R}^m$ in $\mathbb{R}^n$. Observe that in that case $\mathcal{E} = 1$ if $C_m = 0$ the inequality (1.7) would be an exact equality. Therefore, it is a natural question to ask the following open question

**Open question.** *Is it possible to improve the main theorem changing $C_m$ by 0?*

The structure of the paper is as follows

In §2 we recall the definition and several properties of the heat kernel on a Riemannian manifold and provide proposition 2.1 which states that every complete minimal submanifold with polynomial volume growth is stochastically complete. With those previous requirements we can, in §3, to prove the main theorem.

2. Preliminaries

Let $M$ be a Riemannian manifold with (possibly empty) smooth boundary $\partial M$, and denote by $\Delta$ the Laplacian on $M$. The heat kernel on $M$ is a function $\mathcal{H}(t, x, y)$ on $(0, \infty) \times M \times M$ which is the minimal positive fundamental solution to the heat equation

$$\frac{\partial v}{\partial t} = \Delta v.$$
In other words, the Cauchy problem with Dirichlet boundary conditions

\begin{equation}
\begin{cases}
\frac{\partial v}{\partial t} = \Delta v, \\
v|_{t=0} = v_0(x)
\end{cases}
\end{equation}

has a solution

\begin{equation}
v(x,t) = \int_M \mathcal{H}(t,x,y)v_0(y)d\mu_y,
\end{equation}

provided that \(v_0\) is a bounded continuous positive function. Moreover the heat kernel has the following properties:

1. Symmetry in \(x,y\) that is \(\mathcal{H}(t,x,y) = \mathcal{H}(t,y,x)\).
2. The semigroup identity: for any \(s \in (0,t)\)

\begin{equation}
\mathcal{H}(t,x,y) = \int_M \mathcal{H}(s,x,z)\mathcal{H}(t-s,z,y)dV(z).
\end{equation}

3. For all \(t > 0\) and \(x \in M\),

\begin{equation}
\int_M \mathcal{H}(t,x,y)dV(y) \leq 1.
\end{equation}

If \(M\) is the Euclidean space \(\mathbb{R}^n\) then, due to the homogeneity and isotropy of the Euclidean space, the heat kernel \(\mathcal{H}^{n,0}(t,x,y)\) depends only on \(t\) and \(\rho(x,y) = \text{dist}(x,y)\), and is given by the classical formula

\begin{equation}
\mathcal{H}^{n,0}(t,\rho(x,y)) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\rho^2(x,y)}{4t}}.
\end{equation}

A manifold \(M\) satisfying for all \(x \in M\) and all \(t > 0\)

\begin{equation}
\int_M \mathcal{H}(t,x,y)dV(y) = 1,
\end{equation}

is said to be stochastically complete.

In the following proposition is proved that a complete submanifold of polynomial volume growth is stochastically complete

**Proposition 2.1.** Let \(M^m\) be a \(m\)-dimensional complete minimal submanifold properly immersed in a Cartan-Hadamard ambient manifold. Suppose that \(M^m\) is of polynomial volume growth, then \(M^m\) is stochastically complete

**Proof.** Since \(M^m\) has polynomial volume growth by equation (1.4), for any \(o \in M\) and any \(R \in \mathbb{R}_+\) we have

\begin{equation}
\text{Vol}(M^m \cap B^N_R(o)) \leq \mathcal{E}\omega_m R^m.
\end{equation}

But since the geodesic ball \(B^N_R(o)\) of radius \(R\) in \(M^m\) is a subset of the extrinsic ball \(M^m \cap B^N_R(o)\), one obtains that

\begin{equation}
\int_0^\infty \frac{rdr}{\log(\text{Vol}(B^N_R(o)))} \geq \int_0^\infty \frac{rdr}{\log(\text{Vol}(M^m \cap B^N_R(o)))} \geq \int_0^\infty \frac{rdr}{\log(\mathcal{E}\omega_m R^m)} = \infty.
\end{equation}

Hence, by [Gri99 theorem 9.1] \(M^m\) is stochastically complete. \(\square\)

Finally in order to conclude this preliminary section let us recall here the coarea formula
Theorem 2.2 (Coarea formula, see [Sak98, Cha84]). Let \( f \) be a proper \( C^\infty \) function defined on a Riemannian manifold \((M^n, g)\). Now we set
\[
\Omega_t := \{ p \in M; f(p) < t \}, \quad V_t := \text{Vol}(\Omega_t),
\]
\[
\Gamma_t := \{ p \in M; f(p) = t \}, \quad A_t := \text{Vol}_{n-1}(\Gamma_t).
\]
Then for an integrable function \( u \) on \( M^n \) the following hold:

1. Let \( g_t \) be the induced metric on \( \Gamma_t \) from \( g \). Then
\[
\int_{M^n} u|\nabla f| dv_g = \int_{-\infty}^{\infty} dt \int_{\Gamma_t} u dv_{g_t}.
\]

2. \( t \to V_t \) is of class \( C^\infty \) at a regular value \( t \) of \( f \) such that \( V_t < +\infty \), and
\[
\frac{d}{dt} V_t = \int_{\Gamma_t} \frac{1}{|\nabla f|} dv_{g_t}.
\]

3. Proof of the main theorem

First of all, let us denote by \( D_R(x) \) the extrinsic ball of radius \( R \) centered at \( x \), i.e.,
\[
D_R(x) := M^n \cap B_R^e(x),
\]
therefore \( Q(R) \) is given by
\[
Q(R) = \frac{\text{Vol}(D_R(x))}{\omega_m R^m}.
\]
Note that \( D_R(x) \) is the sublevel set of the extrinsic distance function \( r_x \):
\[
D_R(x) = \{ p \in M^n; r_x(p) < R \}.
\]

Making use of the upper bounds for the heat kernel (equation (1.2) and the semigroup property of the heat kernel (equation (2.4))
\[
1 \geq (4\pi t)^{-\frac{n}{2}} \mathcal{H}(t, x, x) = (4\pi t)^{-\frac{n}{2}} \int_{M^n} \mathcal{H}(t/2, x, y)^2 d\mathcal{V}(y)
\]
\[
\geq (4\pi t)^{-\frac{n}{2}} \int_{D_R(x)} \mathcal{H}(t/2, x, y)^2 d\mathcal{V}(y),
\]
for any extrinsic ball \( D_R(x) \). Applying now the Cauchy–Schwarz inequality
\[
1 \geq (4\pi t)^{-\frac{n}{2}} \mathcal{H}(t, x, x) \geq (4\pi t)^{-\frac{n}{2}} \left( \frac{\int_{D_R(x)} \mathcal{H}(t/2, x, y)^2 d\mathcal{V}(y)}{\text{Vol}(D_R(x))} \right)^2,
\]
Since by proposition (2.1) \( M^n \) is stochastically complete
\[
1 \geq (4\pi t)^{-\frac{n}{2}} \mathcal{H}(t, x, x) \geq (4\pi t)^{-\frac{n}{2}} \left( 1 - \int_{M^n \setminus D_R(x)} \mathcal{H}(t/2, x, y)^2 d\mathcal{V}(y) \right)^2 \frac{\text{Vol}(D_R(x))}{\text{Vol}(M^n)}
\]
Applying the polynomial volume growth property
\[
1 \geq (4\pi t)^{-\frac{n}{2}} \mathcal{H}(t, x, x) \geq (4\pi t)^{-\frac{n}{2}} \left( 1 - \int_{M^n \setminus D_R(x)} \mathcal{H}(t/2, x, y)^2 d\mathcal{V}(y) \right)^2 \frac{\omega_m R^m}{\omega_m R^m}
\]
for all \( R > 0 \). If we choose
\[
R = R_t := \frac{(4\pi t)^{\frac{1}{2}}}{\omega_m^{\frac{1}{2}}} t^\frac{1}{2} = 2 \left[ \frac{m}{2} \Gamma \left( \frac{m}{2} \right) \right]^{\frac{1}{2}} t^\frac{1}{2},
\]
we obtain
\[
1 \geq (4\pi t)^{-\frac{n}{2}} \mathcal{H}(t, x, x) \geq \left( 1 - \int_{M^n \setminus D_R(x)} \mathcal{H}(t/2, x, y)^2 d\mathcal{V}(y) \right)^2 \frac{\omega_m R^m}{\omega_m R^m}.
\]
We need now the following proposition

**Proposition 3.1.** Suppose that
\[ \lim_{R \to \infty} Q(R) = \mathcal{E} \]
then
\[ \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}(t/2, x, y) dV(y) \leq \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}^{m,0}(t/2, r_x(y)) dV(y) \]
(3.8)
being \( \delta \) a smooth function with \( \delta \to 0 \) when \( t \to \infty \).

**Proof.** By inequality (1.1)
\[ \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}(t/2, x, y) dV(y) \leq \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}^{m,0}(t/2, r_x(y)) dV(y) \]
by coarea formula (theorem 2.2)
\[ \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}(t/2, x, y) dV(y) \leq \int_{R_t}^{\infty} \int_{\partial D_s(x)} \mathcal{H}^{m,0}(t/2, r_x(y)) \frac{dV_s(y) ds}{|\nabla r_x|} \]
(3.10)
\[ \leq \int_{R_t}^{\infty} \mathcal{H}^{m,0}(t/2, s) (Vol(D_s(x)))' ds. \]

The derivative \( \frac{d}{dR} Vol(D_R(o)) = (Vol(D_R))' \) satisfies
\[ (Vol(D_R))' = m \omega_m Q(R) R^{m-1} + \omega_m R^m Q(R) (log(Q(R)))' \]  
(3.11)
Therefore,
\[ \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}(t/2, x, y) dV(y) \leq \]
\[ \frac{\omega_m}{(2 \pi t)^{\frac{m}{2}}} \int_{R_t}^{\infty} e^{-\frac{m}{2} \frac{m}{2} s^{m-1}} + s^{m} Q(s) (log(Q(s)))' ds \leq \]
(3.12)
\[ \frac{\omega_m}{(2 \pi t)^{\frac{m}{2}}} \int_{R_t}^{\infty} e^{-\frac{m}{2} \frac{m}{2} s^{m-1}} + s^{m} (log(Q(s)))' ds \leq \]
\[ \frac{\omega_m}{(2 \pi t)^{\frac{m}{2}}} \left[ \int_{R_t}^{\infty} me^{-\frac{m}{2} \frac{m}{2} s^{m-1}} ds + \left( \sup_{s \in [0, \infty)} e^{-\frac{m}{2} \frac{m}{2} s^{m}} \right) \log \left( \frac{\mathcal{E}}{Q(R_t)} \right) \right] = \]
\[ \frac{\omega_m}{(2 \pi t)^{\frac{m}{2}}} \left[ m^2 e^{-\frac{m}{2} \frac{m}{2} s^{m-1}} + R_t^{2m} + e^{-\frac{m}{2} \frac{m}{2} s^{m}} \log \left( \frac{\mathcal{E}}{Q(R_t)} \right) \right]. \]

Taking into account the definition of \( R_t \) (equation (3.6)) and that \( \omega_m = \frac{2\pi^{\frac{m}{2}}}{m! (\frac{m}{2})^{\frac{m}{2}}} \),
\[ \int_{M^n \setminus D_{R_t}(x)} \mathcal{H}(t/2, x, y) dV(y) \leq \mathcal{E} [C_m + \delta(t)] \]
(3.13)
where
\[ \delta(t) := e^{-\frac{m}{2} \frac{m}{2} \frac{m}{2} t^{\frac{m}{2}}} \log \left( \frac{\mathcal{E}}{Q \left( 2 \left( \frac{m}{2} \Gamma \left( \frac{m}{2} \right) \right)^{\frac{m}{2}} t^{\frac{m}{2}} \right) \right). \]

Making use that \( Q(s) = \mathcal{E} \) when \( s \to \infty \) the proposition is proven. \( \square \)

Hence for \( t \) large enough we can apply the above proposition in equation (3.7)
\[ 1 \geq (4 \pi t)^{\frac{m}{2}} \mathcal{H}(t, x, x) \geq \left[ \frac{1 - \mathcal{E} (C_m + \delta(t))}{\mathcal{E}} \right]^2 \]
(3.14)
Therefore, taking limits the theorem follows.
REFERENCES

[And84] Michael T. Anderson, *The compactification of a minimal submanifold in euclidean space by the gauss map*, unpublished preprint, 1984.

[Cha84] Isaac Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press Inc., Orlando, FL, 1984, Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk, MR 768584 (86g:58140)

[CLY84] Shiu Yuen Cheng, Peter Li, and Shing-Tung Yau, *Heat equations on minimal submanifolds and their applications*, Amer. J. Math. 106 (1984), no. 5, 1033–1065. MR 761578 (85m:58171)

[GP12] Vicent Gimeno and Vicente Palmer, *Volume growth, number of ends, and the topology of a complete submanifold*, Journal of Geometric Analysis (2012), 1–22 (English).

[GP13] ———, *Extrinsic isoperimetry and compactification of minimal surfaces in Euclidean and hyperbolic spaces*, Israel J. Math. 194 (2013), no. 2, 539–553. MR 3047082

[Gri99] Alexander Grigor’yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 2, 135–249. MR 1659871 (99k:58195)

[GSC09] Alexander Grigor’yan and Laurent Saloff-Coste, *Heat kernel on manifolds with ends*, Ann. Inst. Fourier (Grenoble) 59 (2009), no. 5, 1917–1997. MR 2573194 (2011d:35499)

[JM83] Luquésio P. Jorge and William H. Meeks, III, *The topology of complete minimal surfaces of finite total Gaussian curvature*,Topology 22 (1983), no. 2, 203–221. MR 683761 (87i:58160)

[Li86] Peter Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*, Ann. of Math. (2) 124 (1986), no. 1, 1–21. MR 847950 (87k:58259)

[Mar86] Steen Markvorsen, *On the heat kernel comparison theorems for minimal submanifolds*, Proc. Amer. Math. Soc. 97 (1986), no. 3, 479–482. MR 840633 (87i:58160)

[MP12] William H. Meeks, III and Joaquín Pérez, *A survey on classical minimal surface theory*, University Lecture Series, vol. 60, American Mathematical Society, Providence, RI, 2012. MR 3012474

[MPR13] William H. Meeks, III, Joaquín Pérez, and Antonio Ros, *Local removable singularity theorems for minimal laminations*, 2013, arXiv:1308.6439

[MW07] William H. Meeks, III and Michael Wolf, *Minimal surfaces with the area growth of two planes: the case of infinite symmetry*, J. Amer. Math. Soc. 20 (2007), no. 2, 441–465. MR 2276776 (2007m:53008)

[Oss86] Robert Osserman, *A survey of minimal surfaces*, second ed., Dover Publications Inc., New York, 1986. MR 852409 (87j:53012)

[Pal99] Vicente Palmer, *Isoperimetric inequalities for extrinsic balls in minimal submanifolds and their applications*, J. London Math. Soc. (2) 60 (1999), no. 2, 607–616. MR 1724821 (2000j:53050)

[Qin95] Chen Qing, *On the volume growth and the topology of complete minimal submanifolds of a euclidean space*, J. Math. Sci. Univ. Tokyo 2 (1995), 657–669.

[Sak96] Takashi Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, Providence, RI, 1996, Translated from the 1992 Japanese original by the author. MR 1350760 (97f:53011)

[Xu13] Guoyi Xu, *Large time behavior of the heat kernel*, 2013, arXiv:1310.2382

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