SMOOTH SCHEME MORPHISMS: A FRESH VIEW

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ABSTRACT. Relations between some kinds of formal and standard smoothness, for morphisms of schemes, are clarified in surprisingly simple and direct ways, bypassing much of the customarily employed machinery. Even the deep local-to-global property of formal smoothness has a fairly elementary proof, under mild additional hypotheses.

1. INTRODUCTION

Using methods as elementary as possible, accessible to those having only a basic understanding of scheme theory, we present efficient new proofs of the equivalence of some of the main kinds of smoothness of scheme morphisms. While smoothness is clearly important in algebraic geometry, there is no general agreement about which among several seemingly different definitions should be chosen as the basic one that best expresses the concept. In certain situations, one approach to smoothness can be distinctly easier to work with than others, so it is useful to make transitions whenever convenient. Our aim is to make such processes thoroughly transparent.

We innovate by proving key results using machinery so limited that concepts such as dimension, tangent spaces, fibres, regularity and flatness do not appear. Discussion of those, as well as the important topic of étale morphisms, has been relegated to another article on consequences and further characterizations of smoothness. Only trivial facts will be cited, except in many non-essential comments where various concepts and results are presupposed in order to make comparisons with other work. Kähler differentials are treated as forming modules rather than sheaves, and will play an important role in proofs. Unlike much written in this area, there will be no assumptions or reductions involving Noetherianity. Although they are useful for clarifying some aspects of smoothness, Grothendieck topologies other than the Zariski topology will not be treated here.

Stimulus for writing this article came from examining standard sources selected from a vast literature, notably Görtz and Wedhorn [3] and Vakil [12] as well as the encyclopedic Stacks Project book [11] and EGA [1]. The first two provide motivation and additional details, important for elucidating the significance of smoothness. Except where noted, our definitions coincide with those of [11]. Some of the most relevant will be repeated below. For reasons of stability, references to [11] are given in the form [Tag ...], and can be consulted on-line. In [11], as in [1] and many other...

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works (one interesting survey being [8]), further variations on notions of smooth-
ness appear, often phrased in terms of certain kinds of intersections or using I-adic
topology. Homological methods yield deeper results, some presented in [7]. Our
scope is far more limited.

Formal smoothness and standard smoothness will be regarded as properties
of morphisms between schemes, to be studied from Zariski-local and stalk-local
points of view as properties of (homo)morphisms of commutative rings. The first
of these notions of smoothness, due to Grothendieck, has an intrinsic (presentation-
independent) definition well suited for formal diagrammatic demonstrations. The
other notion, seemingly more concrete, asserts the existence of presentations by
generators and relations of a certain form. Precise details are given in the next
section.

A different approach, adopted for example in [11], takes smoothness to be at
heart a property of the sheaf of differentials. To properly formalize the idea, the
naive cotangent complex of a morphism [Tag 00S0] is introduced. This object is
shown in [Tag 031J] to be in some sense trivial precisely when the morphism is
formally smooth. An example in [Tag 0635], with a morphism not locally of finite
type, shows that the full cotangent complex need not behave as well. However,
smooth morphisms are by definition required to be locally finitely presented. In
that case, a refined approach to differentials allows formal smoothness to be ex-
ploited in a previously unknown way, yielding a proof of the first theorem below.

In what follows, the non-standard term ‘around \(x\)’ pertains to local properties,
those that hold on some restriction \(U \to V\) to affine open subschemes with \(x \in U\),
and on further such restrictions. We also write ‘at \(x\)’ for properties that are are
even more local, defined from a single map \(\mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}\) to the stalk at \(x\) from
the stalk at \(\varphi(x)\). Such maps are local homomorphisms \(R_{p'} \to S_p\) of local rings. In
the terminology just introduced, and with the definitions of smoothness as given in
the next section, the first fundamental result is:

**Theorem 1.** Let \(\varphi : X \to Y\) be a scheme morphism, locally of finite presentation.
For each point \(x\) of \(X\), the following are equivalent:
(a) \(\varphi\) is formally smooth around \(x\);
(b) \(\varphi\) is standard smooth around \(x\);
(c) \(\varphi\) is formally smooth at \(x\);
(d) \(\varphi\) is standard smooth at \(x\).

The \(x \in X\) satisfying these conditions form the point set of an open subscheme
of \(X\), the smooth locus of \(\varphi\). When this is \(X\), \(\varphi\) is said to be smooth. As a
byproduct of the proof techniques, the idea of studying smoothness from given
generators and relations can be justified. It will be clear after the proof of 1 that
it is straightforward to test at stalks. We even obtain a formula, too cumbersome
to be of practical use, for the smooth locus. Some open subsets of Spec \((S)\) needed
just below will be principal open sets \(D(f)\), \(f \in S\), while others are complements
of closed sets \(V(I)\), \(I\) an ideal of \(S\).
Theorem 2. If $S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$, the smooth locus of $\varphi : \text{Spec}(S) \to \text{Spec}(R)$ is a union of $(n+c)^n$ open sets $D(\Delta) \cap V(I)^c$. Each $\Delta$, a Jacobian determinant, and $I$, an ideal of the form $(I_0 : I_1)$, is defined by an explicit formula involving the given $f_i \in R[x_1, \ldots, x_n]$ and their formal partial derivatives $\frac{\partial f_i}{\partial x_j}$.

The final section concerns a distinctly deeper fundamental result: an arbitrary scheme morphism $\varphi : X \to Y$ is formally smooth precisely when it is locally formally smooth, by which we mean this holds around all points of $X$, with ‘around’ as defined above. Under an assumption that includes the case where $\varphi$ is locally of finite type, we show that elementary methods, carefully deployed, yield a proof technically simpler than previously known ones.

2. Definitions and notation

All rings considered will be commutative algebras (with 1, and allowing the case $1 = 0$) over a ring called $R$. Maps between rings, including derivations, are taken to be $R$-linear. As in [Tag 00TH], a ring morphism $R \to S$ is formally smooth if, in the category of $R$-algebras, every morphism $S \to A/I$, where $I$ is an ideal of $A$ with $I^2 = 0$, lifts to (or factors through) a morphism $S \to A$. In other words, the defining property is that all these $A \to A/I$ induce surjective functions $\text{Hom}_R(S, A) \to \text{Hom}_R(S, A/I)$.

A simple observation, not needed below, motivates the definition. On dropping the assumption $I^2 = 0$, when $S$ is formally smooth over $R$ there is instead a lift $S \to A/I^2$, then in turn a lift $S \to A/I^4$, and so on. One can easily interpose $A/I^3$ and other terms. Thus, after countably many choices, the given $S \to A/I$ factors through $S \to \hat{A}$, where $\hat{A}$ is the inverse limit $\lim_{\leftarrow} A/I^n$, the $I$-adic completion of $A$.

Formal smoothness of a scheme morphism $\varphi : X \to Y$ has an analogous definition [Tag 02GZ], best stated relative to the category of schemes over $Y$ [Tag 01JX]. We say $\varphi : X \to Y$ (or even just $X$) is formally smooth if every morphism $\text{Spec}(A/I) \to \text{Spec}(A)$ over $Y$, where $I^2 = 0$, induces a surjective function $\text{Hom}_Y(\text{Spec}(A), X) \to \text{Hom}_Y(\text{Spec}(A/I), X)$.

Among various results that are trivial consequences of the definition, we note that of [Tag 02H3]: every restriction of $\varphi : X \to Y$ to a morphism $U \to V$ between open subschemes of $X$ and $Y$ is formally smooth if $\varphi$ is. It is also clear that any such restriction $U \to V$ is formally smooth precisely when $U \to Y$ is. Further properties, not needed here, show the robustness of this notion. The really striking fact, mentioned earlier, is that the formal smoothness of a morphism turns out to be determined by local properties alone (it is local in the sense of [Tag 01SS]). One can find further details of interest, in the context of algebraic spaces, in [Tag 049R].

Given an explicit finite presentation of $S$ over $R$, say $S \cong P/(f_1, \ldots, f_c)$, where $P$ denotes the polynomial ring $P = R[x_1, \ldots, x_n]$, the image in $S$ of any $f$ in $P$ is also called $f$ when the meaning is clear. We prefer to use the name $I_0$ for the ideal $(f_1, \ldots, f_c)$ of $S$, reserving $I$ for other uses. The cases $c = 0$ or $n = 0$ are allowed, rewording where necessary.
Departing slightly from the presentation-dependent definition in [Tag 00T6], we say a ring morphism $R \to S$, or $S$ as an $R$-algebra, is \textit{standard smooth} over $R$ if $S$ can be presented as above, where $c \leq n$ and the $c \times c$ matrix $J$ with $(i,j)$-entry $\frac{\partial f_i}{\partial x_j}$ maps to an invertible matrix over $S$, or equivalently $\text{det}(J)$ is invertible in $S$. Also say $R \to S$ is \textit{essentially standard smooth} if $S$ is isomorphic to a localization of some standard smooth $R$-algebra. This is of most interest when $S$ arises from localizing at a prime ideal. At the other extreme, a principal localization $S = T_g$ ($g \in T$) of a standard smooth $R$-algebra $T$ remains standard smooth, from the now commonplace Rabinowitsch trick: introduce a new variable $x$ and relator $x,g - 1$.

A morphism $\varphi : X \to Y$ of schemes is \textit{standard smooth around a point} $x \in X$ if there are affine open subsets $U$, $V$ with $x \in U$ such that the restriction of $\varphi$ to $U \to V$ is induced from a standard smooth ring homomorphism. By definition $\{x \in X \mid \varphi\text{ is standard smooth around } x\}$ is open in $X$. On this, the smooth locus, $\varphi$ is said to be \textit{locally standard smooth}, a property with many desirable consequences. Maps between stalks cannot be expected to be finitely presented. Thus, when $\varphi$ is said to be \textit{standard smooth at} $x$, for some $x \in X$, we mean that the induced map $\mathcal{O}_Y,\varphi(x) \to \mathcal{O}_{X,x}$ is essentially standard smooth. Here $\mathcal{O}_{Y,\varphi(x)}$ plays the role of $R$.

\textbf{Example 1.} Suppose 2 is a unit of $R$ and $S = R[x,y]/(x^2 + y^2 - 1)$. While $R \to S$ is not standard smooth (which could be verified later, from the module of differentials), the morphism $\text{Spec}(S) \to \text{Spec}(R)$ is locally standard smooth. This follows at once, using the open cover $\{D(x), D(y)\}$ of $\text{Spec}(S)$.

### 3. Derivations and differentials

The usual differential on $P = R[x_1, \ldots, x_n]$ is $df = \sum_j \frac{\partial f}{\partial x_j} \, dx_j$, where to be definite the $dx_j$ (in order) are taken to be the elements of the canonical basis of the module $P^n$. With $S \cong P/I_0$ and $I_0 = (f_1, \ldots, f_c)$ as before, there is a derivation $d = d_S : S \to M(S)$, where $M(S)$, the $S$-module of differentials, is the quotient of $S^n$ by the submodule generated by $\{df \mid f \in I_0\}$. It suffices to impose the relations $df_i = 0$ ($1 \leq i \leq c$), since all $f_i$ act as 0 on $M(S)$, so $d(\sum_i g_i f_i) = \sum_i (g_i \cdot df_i + f_i \cdot dg_i) = 0$. However, finiteness of the number of generators and relations is clearly non-essential here, and indeed throughout this whole section.

Since $M(S)$ is a concrete representative of the Kähler module [Tag 07BK], satisfying (with $d_S$) the universal property [Tag 00R0] of being the freest possible $S$-module of $R$-differentials of $S$, we adopt the standard notation $\Omega_{S/R}$ in place of $M(S)$. A different representative, visibly presentation-independent, is furnished in [Tag 00RN]. A more refined notion of universality will appear in the next lemma.

The most subtle idea of the whole development is to compare certain derivations, as in [Tag 02HP] and [Tag 031I] (which includes a converse). These leave a few important details for readers to check. We follow roughly similar lines, placing more emphasis on universal concepts, proving exactly what is necessary for our purposes.

To set notation, given an ideal $I$ of an $R$-algebra $S$, write $\overline{S} = S/I$, $S' = S/I^2$, with $s \mapsto s' \mapsto \overline{s}$ under the maps $S \to S' \to \overline{S}$. Note that $\overline{S}$-modules can be regarded as $S$-modules annihilated by $I$, or as $S'$-modules annihilated by $I/I^2$. 

Similar remarks apply to derivations $d$. The universal property in the next result should be clear from the way it will be used. As always, $R$-linearity is implicit.

**Lemma 1.** The usual $d_S : S \to \Omega = \Omega_{S/R}$, universal for derivations from $S$ to $S$-modules, induces a $d'_S : S' \to \Omega' = \Omega/I\Omega$ that is universal for derivations from $S'$ to $\overline{\mathfrak{S}}$-modules, where $S' = S/I^2$ and $\overline{\mathfrak{S}} = S/I$.

**Proof.** Let $M$ be an arbitrary $\overline{\mathfrak{S}}$-module, so $I.M = 0$ for $M$ as an $S$-module. Every $d : S \to M$ must satisfy $d(I^2) = 0$, so induces some $d' : S' \to M$. This correspondence $d \mapsto d'$ is bijective, since any $d' : S' \to M$ lifts via $S \to S'$ to some $d : S \to M$ that in turn induces $d'$. The bijection has a naturality property: given an $\overline{\mathfrak{S}}$-module map $\alpha : M \to M'$, for every $d : S \to M$ we have $(\alpha \circ d)' = \alpha \circ d'$.

The derivation called $d'_S : S' \to \Omega'$ is the one induced from the composition of the universal $d_S : S \to \Omega$ and $\Omega \to \Omega'$. To verify the claimed universal property of $d'_S$, start with any $d' : S' \to M$, where $I.M = 0$. This is induced from some $d : S \to M$, which factors through $d_S$ as $S \to \Omega \to \Omega' \to M$ for some unique $\overline{\mathfrak{S}}$-linear map $\alpha : \Omega \to M$. By naturality, uniqueness also holds for the factorization $d' : S' \to \Omega' \to M$ through $d'_S$. \hfill $\Box$

**Proposition 1.** With notation as above, assume that $R \to \overline{\mathfrak{S}}$ is formally smooth. Then $d'_S : S' \to \Omega'$ restricts to an isomorphism of $\overline{\mathfrak{S}}$-modules between $I/I^2$ and a direct summand of $\Omega'$, giving a split exact sequence $0 \to I/I^2 \to \Omega \to \Omega' \to 0$. In addition, $\Omega'_{\overline{\mathfrak{S}}/R}$ is a projective $\overline{\mathfrak{S}}$-module.

**Proof.** Since $\overline{\mathfrak{S}}$ is formally smooth (over $R$) and isomorphic to $S'/(I/I^2)$, where $I/I^2$ has square zero, some injective ring map $\beta : \overline{\mathfrak{S}} \to S'$ is a right inverse of $S' \to \overline{\mathfrak{S}}$. Thus $S'$ is a split extension $(I/I^2) + \beta(\overline{\mathfrak{S}})$, a direct sum as $R$-modules. Projection yields an $R$-linear function $D : S' \to I/I^2$ that satisfies $s' = D(s') + \beta(\overline{\mathfrak{S}})$ ($s' \in S'$). A calculation of $s'.t'$ in $S'$, using $D(s').D(t') = 0$, shows that $D$ is a derivation:

$$D(s'.t') = s'.D(t') + t'.D(s') \quad (s', t' \in S').$$

By 1, $D$ factors as $d'_S : S' \to \Omega'$ followed by some $\overline{\mathfrak{S}}$-linear $\gamma : \Omega' \to I/I^2$. However, $D$, being a projection map, restricts to the identity function on $I/I^2$. Thus $d'_S$ maps $I/I^2$ isomorphically to an $\overline{\mathfrak{S}}$-submodule of $\Omega'$, with $\ker(\gamma)$ as a complementary direct summand. We have $\ker(\gamma) \cong \overline{\mathfrak{S}}/d'_S(I/I^2) \cong \Omega'_{\overline{\mathfrak{S}}/R}$.

To conclude, $\overline{\mathfrak{S}}$ can also be identified with a quotient $\overline{\mathfrak{P}} = P/I_0$, where $P$ is a polynomial ring over $R$ or (for later use) a localization of some such ring. Then $\Omega'_{\overline{\mathfrak{S}}/R} \cong \Omega_{\overline{\mathfrak{P}}/R}$. In the new split exact sequence, $\Omega' = \Omega_{P/I_0}/\Omega_{P/I_0}$, which is a free $\overline{\mathfrak{P}}$-module, using the following observation on localizations. \hfill $\Box$

Differentials behave well under localization [Tag 00RT(2)]. In brief, if $S \cong P/I_0$ is localized at a prime ideal $\mathfrak{p}$, with corresponding $\mathfrak{p}' \in \text{Spec}(R)$, $\mathfrak{p} \in \text{Spec}(P)$, there is a well-defined $d : P_{\mathfrak{p}} \to P_{\mathfrak{p}} \otimes_P \Omega_{P/I_0}$, $f/g \mapsto (1/g)df - (f/g^2)dg$, which induces $d : S_{\mathfrak{p}} \to S_{\mathfrak{p}} \otimes_S \Omega_{S/I_0}$, clearly universal for $S_{\mathfrak{p}}$. The last module can be identified with $\Omega_{S_{\mathfrak{p}}/R_{\mathfrak{p}'}}$, as $R$ acts through $R_{\mathfrak{p}'}$. Similar results hold for any localization of $S$. 
4. Local relations between formal and standard smoothness

Arguments in this section rely crucially on finiteness of presentation. We start with the well-known fact that formal smoothness holds for presentations satisfying a version of the Jacobian criterion. The usual proof, given below, uses ideas from deformation theory closely related to iterative methods for approximating solutions of systems of equations. As often occurs, a desired result is obtained on restricting to an open subscheme, here \( \text{Spec} (S_\Delta) \) within \( \text{Spec} (S) \). Renumberings of \( x_1, \ldots, x_n \) may alter \( \Delta \), producing various subschemes, some possibly empty.

**Proposition 2.** Suppose \( S = P/I_0 \), where \( P = R[x_1, \ldots, x_n] \), \( I_0 = (f_1, \ldots, f_c) \) and \( c \leq n \). Let \( J \) be the \( c \times c \) matrix with \((i, j)\)-entry \( \frac{\partial f_i}{\partial x_j} \in P \), with \( \Delta \) the image of \( \det(J) \) in \( S \). Then the principal localization \( S_\Delta \) is a formally smooth \( R \)-algebra.

**Proof.** One can obtain a presentation for \( S_\Delta \) from that of \( S \) by adding a generator \( x_{n+1} \) and relator \( x_{n+1} \Delta - 1 \). This produces a new matrix whose determinant is the square of the original one. Thus, changing notation, it can and will be assumed that \( \Delta \) is already a unit (invertible in \( S \)).

Given any map from \( S = P/I_0 \) to \( A = A/Z \), where \( Z \) is an ideal of square zero, for each generator \( x_j \) of \( P \) choose \( a_j \in A \) so that \( \overline{\partial f_j} \mapsto \overline{a_j} \) (\( 1 \leq j \leq n \)). This defines a homomorphism \( P \to A, x_j \mapsto a_j \). It will be deformed to one annihilating \( I_0 \), with \( x_j \mapsto a_j + z_j \), for certain \( z_j \in \mathbb{Z} \). The given map \( S \to A \) will then lift to \( S \to A \).

Since \( \mathbb{Z}^2 = 0 \), each condition that \( f_i \) maps to 0 is a linear (or affine) equation in the \( z_j \). Explicitly, in vector notation, \( f_i(\overline{a} + \overline{z}) = f_i(\overline{a}) + \sum_j \frac{\partial f_i}{\partial x_j}(\overline{a}).z_j = 0 \). All \( z_j \) with \( j > c \) can be freely chosen, say set to 0. The coefficient matrix is then an image of \( J \), defined above. The system can be solved because its determinant has the same image in \( A \) as \( \Delta = \det(J) \), hence is a unit of \( A \) since \( Z \) is nilpotent. \( \square \)

To prepare for use in arguments, we refine previous notation. Given ideals \( I_0 \subseteq I_1 \) of \( P = R[x_1, \ldots, x_n] \), define \( S = P/I_0, \overline{S} = P/I_1, I = I_1/I_0 \). Then \( I \) is an ideal of \( S \) for which \( S/I \cong \overline{S} \). To fix notation, write \( I_0 = (f_1, \ldots, f_{c_0}) \) (so \( c_0 = c \)) and \( I_1 = (f_1, \ldots, f_{c_1}) \). In practice, generators for \( I_1 \) will be given and \( I_0 \) will be defined by selecting a certain number \( c_0 \) of these. There are Jacobian matrices \( J_0 \) and \( J_1 \), of sizes \( c_0 \times n \), and \( c_1 \times n \), each with \((i, j)\)-entry the image of \( \frac{\partial f_i}{\partial x_j} \) in \( S \). Fix \( \overline{p} \in \text{Spec} (\overline{S}) \), lifting it to \( p \in \text{Spec} (S) \) and \( \overline{p} \in \text{Spec} (P) \). Let \( \overline{J_0}, \overline{J_1} \) denote the images of \( J_0, J_1 \) with entries in \( k(\overline{p}) \), the field onto which \( \overline{S} \) maps.

**Proposition 3.** With notation as just above, let \( c_0 \) be the rank of \( \overline{J_1} \) over \( k(\overline{p}) \). After some reordering of the \( f_i \), the rows of \( \overline{J_0} \), corresponding to generators \( f_1, \ldots, f_{c_0} \) of \( I_0 \), form a basis for the rowspace of \( \overline{J_1} \). The universal derivation \( d_S : S \to \Omega \), where \( S = P/I_0 \), maps the ideal \( I = I_1/I_0 \) into \( p\Omega \).

**Proof.** The rows of \( \overline{J_1} \) lie in \( k(\overline{p})^n \), so contain a maximal independent set of size \( c_0 \leq n \). We work with polynomials, and derivations induced from \( d : P \to P^n \), \( f \mapsto (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \), where \( P = R[x_1, \ldots, x_n] \). Let \( i \) range over \( [1, c_0] \). By the choice of generators \( f_i \) of \( I_0 \), for each \( f \in I_1 \) there are \( g_i \in P \) for which \( df - \sum g_i df_i \in \overline{p}^n \subset P^n \). Recall that \( S = P/I_0 \), so the \( df_i \) are relators of the universal \( S \)-module \( \Omega = \Omega_{S/R} \), a quotient of \( P^n \) (via \( S^n \)). Thus \( d_S : S \to \Omega \) maps \( I_1/I_0 \) into \( p\Omega \). \( \square \)
The following basic result will also be needed.

**Lemma 2.** Let $M$ be a direct sum $M_1 \oplus M_2$ of modules over $S_p$, where $p \in \Spec(S)$. If the submodule $M_1$ is finitely generated and contained in $pM$, then $M_1 = 0$.

**Proof.** Note that $\overline{M} = M/pM$ is a direct sum $\overline{M_1} \oplus \overline{M_2}$, as $pM$ is also a direct sum. Then $pM_1 = M_1$ since $\overline{M_1} = 0$. By Nakayama’s Lemma [Tag 00DV], $M_1 = 0$. \(\square\)

**Proof of 1.** All parts reduce to assertions about ring morphisms.

(b) $\Rightarrow$ (a): This was done in 2.

(a) $\Rightarrow$ (c): Easy arguments with diagrams involving $A \to A/I$, where $I^2 = 0$, show that formal smoothness of $R \to S$ is preserved under any localization of $S$, say $S_p$, then also for the induced $R_p \to S_p$, where $p \mapsto p'$ in $\Spec(S) \to \Spec(R)$.

(b) $\Rightarrow$ (d): This is trivial, from the definitions.

(d) $\Rightarrow$ (c): If the local ring $S_p$ is the localization of a standard smooth $R$-algebra $S_1$, by previous steps $R \to S_1$ is formally smooth and, by localizing, so is $R_{p'} \to S_p$.

(c) $\Rightarrow$ (b): To retain consistency with earlier notation, we begin with an $R$-algebra called $S$, finitely presented as $P/I$, and a formally smooth stalk map $R_{p'} \to S_{p'}$. It suffices to find some $\overline{p} \in S_p$ for which $\overline{S_{p'}}$ is standard smooth over $R$.

After choosing an ideal $I_0 \subset I_1$ of $P$ as in 3, write as before $S = P/I_0$, $I = I_1/I_0$, $S' = S/I^2$, and note that $\overline{S} \cong S/I$. By 3, $d_S : S \to \Omega$ maps $I$ into $p\Omega$ and so, from 1, $d_S : S' \to \Omega$ maps $I/I^2$ into $p\Omega$.

To localize, lift the given $\overline{p}$ to $\overline{p} \in \Spec(P)$ and work with analogous definitions from $P_\overline{p}$ in place of $P$. Let for example $(I_1)$ denote $I_1P_\overline{p}$ and $(I)$ be $(I_1)/(I_0)$. To simplify notation, $T$ in place of $S$ signifies localization. Thus the given $\overline{S_{p'}}$ becomes $\overline{T}$. We rename $\Omega$ as $\Omega_{T/R_{p'}}$ (isomorphic to $S_P \otimes_{S} \Omega_{S/R}$), and let $\overline{\Omega} = \Omega/(I)\Omega$.

The result involving $d_S'$ now yields: $d_S' : T' \to \overline{\Omega}$ maps $(I)/(I)^2$ into $p\overline{\Omega}$. By hypothesis $\overline{T} \cong T/(I)$ is formally smooth over $R_{p'}$, so 1 implies that $(I)/(I)^2$ maps isomorphically onto a direct summand of $\overline{\Omega}$. By 2, $(I) = (I_1)/(I_0)$. As $(I) = (I_1)/(I_0)$ is a finitely-generated proper ideal of the local ring $T = P_{\overline{p}}/(I_0)$, a form of Nakayama’s Lemma implies that $(I)$ is the zero ideal, so $(I_1) = (I_0)$ in $P_{\overline{p}}$. Returning to ideals of $P$, finite generation of $I_1$ now implies that $gI_1 \subset I_0$ for some $g \in P \setminus \overline{p}$. Thus $\overline{S_{p'}} \cong P_g/I_0P_g$, a standard smooth $R$-algebra by 2. \(\square\)

Recall that 3 shows how to choose a certain number $c_0$ of polynomials from any given list of generators of $I_1$. Assuming a formally smooth stalk map, it was shown just above that all remaining relators in the list become redundant, after passing to a suitable principal localization. As a rough restatement, within each stalk a failure of redundancy in the additional relators is equivalent to the failure of smoothness for the map at that stalk.
5. A FORMULA FOR THE SMOOTH LOCUS

Going beyond a test for the smoothness of individual stalk maps, we present a formula for the smooth locus of an open affine subscheme over $\mathbb{R}$, now written as $\text{Spec}(S)$, $S \cong P/I_1$, to conform with earlier conventions. Here $P = R[x_1, \ldots, x_n]$ and $I_1 = (f_1, \ldots, f_c)$, so $c = c_1$. The idea is to study square submatrices $J$ of the $c \times n$ Jacobian matrix defined from the given presentation of $S$.

Proof of 2. Given any $\mathfrak{p} \in \text{Spec}(S)$, let $\mathfrak{p}$ be the lift to $\text{Spec}(P)$. Following 3, there is a largest $c_0 \geq 0$ for which one can choose $c_0$ of the $f_i$, generating an ideal $I_0$, then $c_0$ of the variables $x_j$, used to form a submatrix $J$ of the Jacobian matrix such that $\det(J) \notin \mathfrak{p}$. If $\mathfrak{p}$ is in the smooth locus of $\overline{S}$, we know from the end of the proof of 1 that $gI_1 \subset I_0$ for some $g \notin \mathfrak{p}$, or equivalently $\mathfrak{p} \notin V((I_0 : I_1))$, where $(I_0 : I_1) = \{g \in P \mid gI_1 \subset I_0\}$. Conversely, if $\mathfrak{p}$ satisfies this last condition then $\mathfrak{p}$ is in the smooth locus, by 2 and preservation of smoothness (of either kind) under localization at prime ideals.

Reversing the previous point of view, one now starts by choosing a natural number $c_0 \leq \min(c, n)$ and selecting $c_0$ members from each of the lists $f_1, \ldots, f_c$ and $x_1, \ldots, x_n$. From these one defines an ideal $I_0 \subset I_1$ and a $c_0 \times c_0$ submatrix $J$ of the Jacobian matrix. Let $\Delta$ be the image in $S$ of $\det(J)$. As just above, each $\mathfrak{p}$ in $D(\Delta) \cap V((I_0 : I_1))^c$ lies in the smooth locus of $\overline{S}$. Conversely, varying the choices, such open subsets of $\text{Spec}(S)$ cover the whole smooth locus, by arguments in the previous paragraph.

The number of open sets used is the number of square submatrices (including the empty matrix) of a general $c \times n$ matrix. This is symmetric in $c$ and $n$, so to count we may suppose $c \leq n$. The number is $\sum_{i=0}^{c} \binom{n}{i} \binom{c}{c-i} = \sum_{i=0}^{c} \binom{n}{i} \binom{c-c+i}{c} = \binom{n+c}{c}$. □

6. LOCAL TO GLOBAL FOR FORMALLY SMOOTH MORPHISMS

The nontrivial direction of the following fundamental theorem is a remarkable local-to-global property of formal smoothness. It is basically a cohomological result from deformation theory, but we will show how it follows from a careful analysis using elementary methods. Instead of a local hypothesis of the form ‘$R \to S$ of finite type’, it suffices to require only ‘essentially of finite type’ [Tag 00QM]: $S$ is a localization of an $R$-algebra of finite type. Examples of such morphisms, not locally finite, are easily found using transcendental extensions of fields. With ‘around’ in the sense defined in the introduction, the result we prove is:

Theorem 3. For every scheme morphism $\varphi : X \to Y$ that is locally essentially of finite type, the following are equivalent:
(a) $\varphi$ is formally smooth;
(b) $\varphi$ is formally smooth around all points of $X$.

The equivalence is in fact true for arbitrary $\varphi$, but the given hypothesis on $\varphi$ is sufficiently general to include all situations that normally occur, and significantly simplifies the proof. In this context it is usual to cite a result on the triviality of Čech cohomology for quasicoherent sheaves on affine schemes, but we do not do so.
Some historical background is given for those interested. The very last (fourth) issue of [1], Vol. 4 states the equivalence in full generality as Proposition (17.1.6), but the proof cites something that requires $\varphi$ to be locally of finite presentation. Grothendieck [4, Remarque 9.5.8] (also see [9]) discussed that gap, found soon after publication. He stated that there is no problem for locally finite morphisms, adding that the general result would follow from a proof of a conjecture on the structure of locally projective modules. A few years later, in 1971, the deep analysis carried out by Raynaud and Gruson [10], building on earlier work of Kaplansky [6], implied a full solution, once some technical details had been corrected in [5].

We ignore all this, and later related work, as the case we treat can be handled using only rudimentary machinery. This provides an instructive contrast to more conventional methods. At the end, adjustments are made via partitions of unity analogous to ones used in complex analysis to prove certain theorems of Dolbeault and de Rham—see for example [2, p. 311].

Just as in [Tab 01UP], a one-line calculation (omitted) yields:

**Lemma 3.** Let $\bar{A} = A/I$, where $I^2 = 0$. Given a ring morphism $\sigma : R \to A$, regard $I$ as an $R$-module via the induced map $\sigma : R \to \bar{A}$ composed with the natural action of $\bar{A}$ on $I$. Then the morphisms $R \to A$ that induce $\sigma$ are the functions of the form $\sigma + \delta$ for which $\delta : R \to I$ is an $R$-derivation.

**Proof of 3.** One direction is trivial, from observations made in Section 2. Given (b), one sees from the hypotheses that for each $x \in X$ there are affine sets $U \subset X$ and $V \subset Y$ with $x \in U$ such that $f$ restricts to a formally smooth morphism $U \to V$ that is essentially of finite type. Using notation $\bar{A} = A/I$, where $I^2 = 0$, and $T' = \text{Spec}(\bar{A})$, with underlying set identified throughout with that of $T = \text{Spec}(A)$, it must be shown that any morphism $\psi : T' \to X$ factors through the usual $T' \to T$. The morphisms here are already over $Y$, via compositions with $\varphi : X \to Y$.

As a first step, in which notation will be defined and various choices made, each $t \in T'$ lies in a principal affine open set contained in some $\psi^{-1}(U)$ with $U$ as above. The affine scheme $T'$ is covered by finitely many of these sets, henceforth called $T'_1, \ldots T'_l$, of the form $\text{Spec}(\bar{A}_f)$. We write the localization $A_{\bar{f}}$ as $A_i$ and use a similar convention for the $\bar{A}$-module $I$, but not more generally. Variables $i, j, k$ range over $[1,l]$, the initial focus being on a single $i$. Also choose as above affine opens $U_i \subset X$, $V_i \subset Y$ so that $\psi$ and $\varphi$ restrict to maps $T'_i \to U_i \to V_i$. Each corresponding ring homomorphism $R_i \to S_i \to \bar{A}_i$ is regarded as forming a morphism $S_i \to A_i$ of $R_i$-algebras, with $S_i$ essentially finite over $R_i$. By the formal smoothness assumption, one can choose a lift $\sigma_i : S_i \to A_i$.

To study morphisms on subschemes of $T = \text{Spec}(A)$, notation will be extended in an obvious way and natural identifications made, so for example $T_{ij}$ denotes the scheme $\text{Spec}(A_{ij})$ on the set $T_i \cap T_j$. The case $j = i$ is not excluded but will be of no interest. The morphism $T_{ij} \to U_i$, obtained from composing $\sigma_i$ with $A_i \to A_{ij}$, factors through $U_{ij} = U_i \cap U_j$, a scheme which need not be affine but is open in $U_i$ and in $U_j$. Thus both $\sigma_i$ and $\sigma_j$ will induce scheme morphisms $T_{ij} \to U_{ij}$. Composing with the inclusion $U_{ij} \to U_i$ produces two lifts $S_i \to A_{ij}$ of the same morphism $S_i \to \bar{A}_i \to \bar{A}_{ij}$. By 3, their difference (in the order $i, j$) is an $R_i$-linear derivation $\delta_{ij} : S_i \to I_{ij}$. Recall that $\delta_{ij}$ then factors as $S_i \to \Omega_{S_i/R_i} \to I_{ij}$. 
By assumption $S_i$ is a localization of the $R_i$-subalgebra generated by some finite set $\{s_\gamma \mid \gamma \in I_i\}$ of its elements. Each $\delta_{ij}(s_\gamma)$ is of the form $z_{i,j}/f_{ij}^{N(\gamma,j)}$, for some chosen $z_{i,j} \in I_i$ that minimizes $N(\gamma,j) \in \mathbb{N}$, with $i$ implicitly determined by the index $\gamma$. We can then choose some upper bound $N \in \mathbb{N}$ for all the $N(\gamma,j)$, letting $i$ also vary.

From the end of the proof of 1, $\Omega_{S_i/R_i}$ can be identified with a direct summand of $\Omega_{P_i'/R_i'}$, where $P_i'$ is a suitable localization of the polynomial algebra with free generators $x_\gamma$ ($\gamma \in I_i$), such that $x_\gamma \mapsto s_\gamma$ defines a surjective morphism $P_i' \rightarrow S_i$. Passing to differentials, the $dx_\gamma$ form a $P_i'$-basis of $\Omega_{P_i'/R_i'}$. The projection $dx_\gamma \mapsto ds_\gamma$ fixes $\Omega_{S_i/R_i}$, and the above factorization of $\delta_{ij}$ gives a map with $ds_\gamma \mapsto z_{i,j}/f_{ij}^{N(\gamma,j)}$. A multiple of the composition lifts to a map $\Omega_{P_i'/R_i'} \rightarrow I_i$, $dx_\gamma \mapsto f_{ij}^{N-\min(N(\gamma,j))}z_{i,j}$. Restriction to $\Omega_{S_i/R_i}$ then yields an $R_i$-linear derivation $\delta^{(N)}_{ij} : S_i \rightarrow \Omega_{S_i/R_i}$, which, composed with $I_i \rightarrow I_{ij}$, is $f_{ij}^N\delta_{ij} : S_i \rightarrow I_{ij}$. As $N$ may later be increased further, note that for $M \in \mathbb{N}$ we have $\delta^{(M+N)}_{ij} = f_{ij}^M\delta^{(N)}_{ij}$.

In much the same way that $\delta_{ij}$ was defined, one can find a derivation $S_i \rightarrow I_{ij}$ that is the difference of two homomorphisms $S_i \rightarrow A_{ij}$ and, when calculated on the stalks of $U_{ij}$, which are algebras over both $R_i$ and $R_j$, has the formula $\delta^{(N)}_{ik} - \delta^{(N)}_{jk} - f_k^N\delta_{ij}$. From the definitions, the image of each element of $S_i$ under this derivation lies in the kernel of $I_{ij} \rightarrow I_{ijk}$, so is annihilated in $I_{ij}$ by some power of $f_k$. The derivation is determined by its effect on the finite set $\{s_\gamma \mid \gamma \in I_i\}$, and is multiplied by $f_k^M$ if $N$ is increased by $M$. Thus, for a sufficiently large $N$, all these derivations $S_i \rightarrow I_{ij}$, with $i,j,k$ now varying over $1..l$, are identically zero. One then has a formula, valid on stalks of $U_{ij}$:

$$\delta^{(N)}_{ik} - \delta^{(N)}_{jk} = f_k^N\delta_{ij}.$$  

The sets $D(f_k)$ cover $\text{Spec}(A)$, so there are $h_k \in A$ with $\sum_k h_k f_k^N = 1$. For each $i$, define the ring morphism $\sigma_i' : S_i \rightarrow A_i$, a new lift of the original map $S_i \rightarrow A_i$ by $\sigma_i' = \sigma_i - \sum_k h_k \delta_{ik}^{(N)}$. To compare corresponding scheme morphisms $T_i \rightarrow U_i$ and $T_j \rightarrow U_j$ on the overlap $T_i \rightarrow U_{ij}$, one can take differences of the ring homomorphisms on stalks (or use suitable affine subschemes of $U_{ij}$). Using previous formulas, calculations taking values in stalks of $T_{ij}$, then later just in $A_{ij}$, give:

$$\sigma_i' - \sigma_j' = (\sigma_i - \sigma_j) - \sum_k h_k f_k^N\delta_{ij} = \delta_{ij} - \delta_{ij} = 0.$$  

Thus the new morphisms $T_i \rightarrow U_i \rightarrow X$ patch together, forming a map $T = \text{Spec}(A) \rightarrow X$ through which the initially given $\psi : T' = \text{Spec}(A) \rightarrow X$ factors. $\Box$

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