All-photonic architecture for scalable quantum computing with Greenberger-Horne-Zeilinger states

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Linear optical quantum computing is beset by the lack of deterministic entangling operations besides photon loss. Motivated by advancements at the experimental front in deterministic generation of various kinds of multiphoton entangled states, we propose an architecture for linear-optical quantum computing that harnesses the availability of three-photon Greenberger-Horne-Zeilinger (GHZ) states. Our architecture and its subvariants use polarized photons in GHZ states, polarization beam splitters, delay lines, optical switches and on-off detectors. We concatenate topological quantum error correction code with three-qubit repetition codes and estimate that our architecture can tolerate remarkably high photon-loss rate of 11.5%; this makes a drastic change that is at least an order higher than those of known proposals. Further, considering three-photon GHZ states as resources, we estimate the resource overheads to perform gate operations with an accuracy of $10^{-6}$ ($10^{-15}$) to be $2.07 \times 10^6$ ($5.03 \times 10^7$). Compared to other all-photonic schemes, our architecture is also resource-efficient. In addition, the resource overhead can be even further improved if larger GHZ states are available. Given its striking enhancement in the photon loss threshold and the recent progress in generating multiphoton entanglement, our scheme will make scalable photonic quantum computing a step closer to reality.

I. INTRODUCTION

Out of the many physical platforms available for quantum computing, optical platforms facilitate quicker gate operations compared to the decoherence time, fast readouts and efficient qubit transfer [1, 2]. These features make them one of the strongest contenders for realizing scalable quantum computing. Linear optical quantum computing [1, 3, 4] uses only beam splitters, phase shifters and photon detectors to process the quantum information encoded in light. Besides this apparent simplicity, the ability to operate at room temperature makes this approach attractive. Measurement-based quantum-computing [5] is particularly suitable for optical platforms [6] in terms of practical implementation. In this approach, a particular class of multi-qubit entangled states, known as cluster states, are first generated, and single-qubit-measurements are then performed on them to realize a universal set of gate operations. In optical platforms, linear optical elements are sufficient for implementing the entangling operations for the generation of cluster states as well as single-qubit-measurements.

However, one major shortcoming that besets linear optical quantum computing is the fact that a direct Bell-state measurement (BSM), which is an entangling operation used to form a larger entangled state from smaller ones, is not deterministic [2, 7]. Adding to the shortcoming, photon loss is ubiquitous in all optical platforms and specifically in integrated optics [8], which not only causes optical qubit states to leak out of the computational basis but also introduces dephasing or depolarizing noise into qubits, gate operations and readouts (measurements). Thus, the bare-bone measurement-based quantum-computing scheme in Ref. [5] is not tolerant against the aforementioned practical issues, and additional enhancements are necessary to ensure its successful execution in real experiments. To overcome indeterminism of BSM and quantum noise, we need fault-tolerant architectures that employ quantum error correction (QEC) [9, 10]. With QEC, it is possible to achieve scalable linear optical quantum computing using lossy optical components provided the photon-loss level is below a certain threshold. This threshold value is dependent on the QEC code and noise model considered in the fault-tolerant architecture [11–13].

Various kinds of linear-optical platforms have been proposed for quantum computing depending on the nature of variables used for encoding quantum information. Discrete-variable (DV) platforms manipulate level-structure properties of photons like polarization to encode quantum states. A BSM on DV qubits using linear optics is probabilistic with a success rate of 50% without additional resources [7, 14]. Fault-tolerant architectures for linear optical quantum computing in Refs. [6, 13, 15–18] overcome indeterminism of entangling operations by the repeat-until-success strategy. However, if we wish to carry out, say, m simultaneous and identical entangling operations successfully, the average resource overhead is $O(1/p_m^\Delta)$, which will result in an increase in overhead by a factor of $O(\Delta^m)$ that is exponential in m when the success rate $p_m$ of each entangling operation decreases by a factor of $\Delta$ [15]. In our protocol, which we detail in Sec. III, the strategy with low success rates of entangling operations is used only at a certain stage unlike other mentioned architectures. This makes our protocol competitive in terms of resource overheads. Furthermore, because of such probabilistic entangling operations, these schemes would require optical switches to pass the successfully entangled states to the next step, which is known to contribute significant photon loss [19]. Alternatively, continuous-variable (CV) platforms that employ coherent states described by a continuous parameter (amplitude) [20, 21] offers BSMs that can be nearly deterministic [22]. Here, the success rate of a BSM grows with the coherent-state amplitude. The corresponding architectures for linear optical quantum computing require lower resource overheads, but are sensitive to photon loss and can only tolerate smaller thresholds [22]. Also, there has been significant developments in using optical Gottesman-Kitaev-Preskill
states for fault-tolerant quantum computing \cite{25, 26}. These schemes need very large squeezing strengths (> 10 dB) to implement gates with high precision.

Recently, there have been efforts to combine the DV and CV approaches for quantum computing \cite{27, 31}. It was demonstrated that by using optical hybrid qubits, which are entangled states in the DV and CV domains, near-deterministic entangling operations can be implemented \cite{27, 28}. Moreover, many shortcomings faced individually by CV and DV qubit-based schemes are overcome. Importantly, quantum computing with hybrid qubits reduces resource overheads and also improves the photon-loss tolerance \cite{27, 29}. By increasing the amplitude of the CV part, incurred resources can also be reduced. However, if the coherent amplitude of hybrid qubits is too large, the dephasing noise level in the presence of photon loss will also be commensurately too high \cite{32}. Reference \cite{29} also supports the logic that quantum computing on a special cluster state known as Raussendorf–Harrington–Goyal (RHG) lattice \cite{33, 34} built with only DV qubits could tolerate higher photon loss, albeit at higher resource costs. Larger cluster states like the RHG lattice is built by performing BSMS on smaller entangled states.

Besides probabilistic BSMS and photon loss, practical implementation of linear optical quantum computing is greatly hindered by indeterministic generation of multiphoton entangled states, such as the GHZ and cluster states. There are theoretical proposals for on-demand generations of such multiphoton entangled states \cite{35, 36}. For instance, various multiphoton entangled states can be generated by coupling a multi-level ancillary system to a two-level photon-emitting source and performing sequential unitary operation on both the ancillary system and source \cite{35}. Here, the type of entangled states generated depends on the chosen unitary operation. In the experimental work \cite{37}, both the ancillary system and photon source are transmons, and are coupled via a flux-tunable superconducting quantum interference device. Furthermore, by interacting the single-photon sources, such as quantum dots, with spin-1/2 states, the sources can be made to emit multi-photon entangled states \cite{36}. Recent experimental realizations \cite{38, 39} alternatively make use of entanglement between the electron spin and polarization of photons emitted from optical excitations. An interesting point is that experimentally-realized three-photon and four-photon GHZ states respectively possess fidelities 0.9 and 0.82 \cite{37}. We can also observe that as the number of photons in the GHZ state increases, the fidelity drops.

Motivated by the state-of-the-art techniques in deterministic generation of multiphoton entangled states, in this work we propose a multiphoton-qubit-based topological quantum computing protocol (MTQC) that uses multiphoton GHZ polarization states from deterministic sources to build RHG lattice. Furthermore, we demonstrate that our protocol provides an exceptionally high tolerance against photon-loss that reaches 11.5%. Considering GHZ states as the raw ingredients, we also show that the protocol is resource-efficient than all known \cite{13, 17, 22, 27, 40, 42} non-hybrid qubit-based schemes. We further encode each qubit of the RHG lattice with a multiqubit repetition code \cite{41}—a concatenation of two QEC codes—to improve the photon-loss tolerance. Another salient and practically favorable feature of our protocol is that it requires only on-off detectors, unlike Ref. \cite{18} that demands detectors that can resolve photon numbers and are thus more difficult to implement with competitive accuracy. Also photon number resolving nano-wire \cite{43} and transition-edge \cite{44} based detectors cannot operate at room temperatures. However, on-off detectors can operate at room temperatures \cite{45}, which allows our MTQC protocol pave the way for scalable all-photon quantum computing.

The rest of the article is organized as follows. In Sec. \textbf{II}, we describe our MTQC in detail. Next, in Sec. \textbf{III}, we discuss the noise model we consider throughout the work. Section \textbf{IV} shall touch on the employment of concatenated QEC methods and their effects on further raising the threshold photon-loss rate that can be tolerated with our all-photon architecture, where as the numerical simulation procedure of QEC is separately outlined in Appendix \textbf{A}. In Sec. \textbf{V}, we present our results on the photon-loss thresholds, and the details concerning resource estimation, specifically the average number of resource states. We point out that like other DV protocols, the repeat-until-success strategy with low-success-rate entangling operations is employed only at this stage to generate these resource states. After which, we perform a collective BSM \cite{41} on multiphoton resource states, which is a near-deterministic entangling operation.

The collective BSM (described in Sec. \textbf{IIA}), is a crucial ingredient of our protocol. This is because the near-deterministic entangling operation requires only on-off detectors to boost its success rate of entangling arbitrarily close to 100%. This is unlike some other BSMS \cite{46, 47} that demand photon-number resolving detectors that can distinguish up to four photons to boost the success rate to 75%, and the ability to resolve higher photon numbers is essential for further enhancements. Another salient feature of the collective BSM is that it needs no optical switching to entangle two optical qubits like the other DV protocols when using repeat-until-success strategy. These features make MTQC practically more attractive. It is also important to note that once we start using collective BSMS in our protocol, the repeat-until-success strategy, even if employed, shall not drastically increase resource overheads as the success rate of a collec-
The foremost step is to create multiphoton three-qubit entangled states, to build an RHG lattice that is of a sufficiently high fidelity for optical quantum computing. But this approach has the disadvantage of reducing the effective size of the RHG lattice which will contribute to a large resource overhead. Also, in this situation the RHG lattice is less robust against dephasing errors. There is also an attempt to build the lattices with CV-based qubits in Ref. [23]. But this demands average photon numbers of CV qubits over 100 to build an RHG lattice that is of a sufficiently high fidelity for fault-tolerant computation. Such high average photon numbers are not achievable and lattices built under practical values [50] (average photon number of 2) are far from fault-tolerant. Recently, there has been progress in the generation of two-dimensional CV cluster-states without BSM [51, 52]. Another proposal [18] aims to build RHG lattices using BSMs with a boosted success rate of 75% (or higher) by adding single photons or Bell pairs and employing photon number resolving detectors. Here, the 14.5% failure-rate mark is overcome by the repeat-until-success strategy to create entanglement between qubits similar to Ref. [13]. Involvement of tree clusters [53, 54] render the scheme fault-tolerant against BSM failure, but this comes at the cost of feed-forward measurements. Depending on the failure or success of BSM remaining qubits of tree cluster are measured in appropriate basis [18]. These feed-forward operations form the bottleneck in linear optical quantum computing.

The state $|\mathcal{C}_L\rangle$ of an RHG lattice is a unique 3D cluster state [43] where qubits are entangled to their nearest neighbors represented by the edges of the RHG lattice. In general, a cluster state $|\mathcal{C}\rangle$ over a collection of qubits $\mathcal{C}$ is a state stabilized by the operators $X_a \otimes Z_{b \in \mathcal{C}(a)} Z_b$, where $a, b \in \mathcal{C}$, $Z_i$ and $X_i$ are the Pauli operators on the $i$th qubit, and $\text{nh}(a)$ denotes the adjacent neighborhood of qubit $a \in \mathcal{C}$:

$$|\mathcal{C}\rangle = \prod_{a \in \mathcal{C}} CZ_{a,b} \bigotimes_{a' \in \mathcal{C}} |+\rangle_{a'},$$  \hspace{1cm} (1)

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ are eigenkets of $X$, while $|0\rangle, |1\rangle$ are those of $Z$. The controlled-Z unitary gate $CZ_{a,b}$, which is an entangling operation, applies $Z$ on the target qubit $b$ if the source qubit $a$ is in the state $|1\rangle$. The successful action of $CZ_{a,b}$ on the lattice qubits on sites $a$ and $b$ is represented by an edge in the lattice. The state $|\mathcal{C}_L\rangle$ is currently the best available choice, to the best of our knowledge, to make linear-optical platform fault-tolerant; it can tolerate qubit loss [55, 56], probabilistic entangling operations [15, 48], dephasing and depolarizing noises [33, 34, 57], all of which are peculiar to the platform. Furthermore, QEC and gate operations on $|\mathcal{C}_L\rangle$ is topological in nature and thus offers the highest fault tolerance in the platforms where interactions are restricted to those of the nearest neighbors [58].

In MTQC, the logical basis for an $l$-photon qubit is

$$|0_i\rangle \equiv |H\rangle^{\otimes l}, \quad |1_i\rangle \equiv |V\rangle^{\otimes l}.$$  \hspace{1cm} (2)

where $|H\rangle, |V\rangle$ are the discrete orthonormal polarizations and are eigenkets of the Z Pauli operator. An $r$-photon GHZ ket has the form $|\text{GHZ}_r\rangle \propto |H\rangle^{\otimes r} + |V\rangle^{\otimes r}$ (up to normalization). Once there is a continuous and reliable supply of $|\text{GHZ}_r\rangle$ from deterministic sources, the following prescription forms the stages of our protocol to create $|\mathcal{C}_L\rangle$ and perform topological fault-tolerant quantum computing on it:

1. The foremost step is to create multiphoton three-qubit resource states using $|\text{GHZ}_r\rangle$ and BSMs.
2. Near-deterministic collective BSM are performed on these resource states to form star cluster states.
3. The star clusters then undergo collective BSM to form layers of $|\mathcal{C}_L\rangle$.
4. Finally, the qubits are measured layer by layer in a suitable basis to effect both QEC and Clifford-gate operations on the logical states of $|\mathcal{C}_L\rangle$. Initialization of $|\mathcal{C}_L\rangle$ to certain logical states and magic-state distillation are also possible by measurements which complete the universal set of gates for quantum computing.

Certain aspects of MTQC do bear some resemblance with the protocol in Ref. [28], which uses hybrid qubits as basic ingredients. However, this resemblance is only superficial. Since we use GHZ states as basic ingredients in MTQC, the very process of generating three-qubit resource states is different. The kind of entangling operations and apparatus employed are also very different. Here, we need on-off detectors whereas the protocol in Ref. [28] requires photon-number-parity detectors. Additionally, MTQC exploits the concatenation of error-correcting codes to improve the tolerance against photon loss. In what follows, all stages of the MTQC scheme are discussed in detail.

### A. Collective Bell-state measurement on multiphoton qubits

The Bell states of multiphoton qubits, each with $n$ modes, are $|\phi^\pm_n\rangle = (|0_n,0_n\rangle \pm |1_n,1_n\rangle)/\sqrt{2}$, $|\psi^\pm_n\rangle = (|0_n,1_n\rangle \pm |1_n,0_n\rangle)/\sqrt{2}$. Interestingly, Bell states of multiphoton qubit can be decomposed as Bell states of individual photon modes, that is $|\phi^\pm\rangle = (|HH\rangle \pm |VV\rangle)/\sqrt{2}$, $|\psi^\pm\rangle = (|HH\rangle \pm |VV\rangle)/\sqrt{2}$ as follows [41]:

$$|\phi^+_n\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_j \mathcal{P}_n \left[ |\phi^+\rangle \otimes |\psi^-\rangle \otimes |\psi^-\rangle \right]$$

$$|\phi^-_n\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_j \mathcal{P}_n \left[ |\phi^+\rangle \otimes |\psi^-\rangle \otimes |\psi^-\rangle \right]$$

$$|\psi^+_n\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_j \mathcal{P}_n \left[ |\psi^+\rangle \otimes |\phi^-\rangle \otimes |\psi^-\rangle \right]$$

$$|\psi^-_n\rangle = \frac{1}{\sqrt{2^{n-1}}} \sum_j \mathcal{P}_n \left[ |\psi^+\rangle \otimes |\phi^-\rangle \otimes |\psi^-\rangle \right]$$  \hspace{1cm} (3)
Here, for two states (or operators) $A$ and $B$ that are $i$- and $(n-i)$-partite respectively ($i \leq n$) and invariant under permutations of their supports, we define $\mathcal{P}_n[A, B] := \sum_{\mathcal{I} \in \mathcal{S}_n} A_I \otimes B_{Z_n \setminus I}$, where $Z_n := \{1, \cdots, n\}$, $\mathcal{S}_n,i := \{ I \subseteq \mathcal{Z}_n : |I| = i \}$, and $A_I$ indicates the state (or operator) $A$ supported on $I$. As an example, if $A$ and $B$ are single-partite operators, $\mathcal{P}_n[A \otimes Z, B \otimes Z] = A \otimes A \otimes B \otimes B + A \otimes B \otimes A \otimes B + A \otimes B \otimes B \otimes A + B \otimes A \otimes A \otimes B$.

From the above expression it is clear that if the states $|\phi^+\rangle$ and $|\psi^-\rangle$ can be distinguished at mode level, all the four Bell states can be identified at the logical level with success rate $1 - 2^{-n}$. Thus, as we add more photon-modes to qubits, the success rate of BSMs on them improves and becomes near-deterministic. The states $|\phi^+\rangle$ and $|\psi^-\rangle$ can be unambiguously distinguished by the BSM setup $B_S$ shown in Fig. 1. If $B_S$ registers even (odd) number of $|\phi^+\rangle$'s, the corresponding state at the logical level is $|\phi^+_{\mathcal{n}}\rangle (|\psi^+_{\mathcal{n}}\rangle)$. On the other hand if even (odd) number of $|\psi^-\rangle$'s are registered, the corresponding state at the logical level is $|\phi^-_{\mathcal{n}}\rangle (|\psi^-_{\mathcal{n}}\rangle)$.

This scheme is robust against failure of bare BSM due to photon loss. Employing this scheme of near-deterministic collective BSM has another advantage over protocols which use multiple attempts of BSMs to entangle two optical smaller cluster states. In the latter case, the bare BSMs must be applied in a sequence and thus there is associated waiting time for each of them during which photons may be lost. Also, when one of the BSMs succeed, the left over photons must be trimmed from the star cluster state. Thus, the process also demands extensive usage of delay lines and switching networks. But in our case all the BSMs are applied simultaneously removing the necessity for delay lines and switching networks, and the associated noise.

### B. Resource states

Our protocol for MTQC begins with the creation of the following two kinds of multiphoton resource states,

$$\begin{cases} 
|\mathcal{C}_{3,n,n,n}\rangle = \frac{1}{2} \left( |0_n 0_n 0_n\rangle + |0_n 0_n 1_n\rangle + |1_n 1_n 0_n\rangle - |1_n 1_n 1_n\rangle \right), \\
|\mathcal{C}_{3,n,m}\rangle = \frac{1}{\sqrt{2}} \left( |0_n 0_m 0_n\rangle + |1_n 1_m 1_n\rangle \right), 
\end{cases} \tag{4}$$

which are three-qubit entangled states at the logical level. Throughout the article, we suppose that $|\mathcal{C}_{3,n,n,n}\rangle$ has $n$ polarization photons in all qubits where as $|\mathcal{C}_{3,n,m}\rangle$ has $n$, $m$, and $n$ polarization photons in the first, second and third qubits, respectively. One can verify that $|\mathcal{C}_{3}\rangle$ is the result of a Hadamard operation on the first qubit of the three-qubit linear cluster state $|\mathcal{C}_{z1,2,z2,3}\rangle = |+n\rangle_1|+n\rangle_2|+n\rangle_3$. On the other hand, $|\mathcal{C}_{3,3} \rangle$ is obtained by a Hadamard operation on the first and third qubits of the three-qubit linear cluster state and is a logical GHZ state.

The resource states are created by entangling $r$-photon GHZ states from deterministic sources using $B_S$, as shown in Fig. 1(a). This way of creating resource states using $B_S$’s is possible when $r \geq 3$. Due to the usage of $B_S$’s, the process is probabilistic and the required states are generated only when all the $B_S$’s succeed. If one of the $B_S$’s fails, the state is discarded and the process is restarted; that is, the repeat-until-success strategy is employed. This is also reflected in the resource overhead calculations carried out in Sec. VI. A successful $B_S$ can distinguish $|\phi^+\rangle$ and $|\phi^-\rangle$. If the outcome is $|\phi^+\rangle$, a feed-forward $Z$ operation would be necessary on a photon in resultant GHZ state. However, there is no need for physical implementation of this operation as such a feed-forward procedure can be realized by updating the Pauli frame. Accordingly, measurement results on the photons must be in-
BSMs on the value of $k$ lines are employed to slow the passage of delay lines and switching networks. As an example, consider which is beyond the scope of this article.

Correction scheme to handle the failures, the discussion of additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of additional resources.

Interpreted in corroboration with the Pauli frame. On the other hand, failed BSMs make the involved GHZ states mixed, thus they cannot be restored by feed-forward operations without additional resources. One solution is to use a suitable error correction scheme to handle the failures, the discussion of which is beyond the scope of this article.

In this work, we consider $r = 3$, the smallest possible value, considering its experimental availability using current technology [37]. In principle, $|\text{GHZ}_r\rangle$’s with $r > 3$ can be used at this stage, which would not only reduce the required number of $B_S$, but also improve resource efficiency (average number of incurred $|\text{GHZ}_r\rangle$’s) for generating $|\mathcal{E}_3\rangle_{n,n,n}$ and $|\mathcal{E}_r\rangle_{n,m,n}$. However, a larger $r$ would imply a poorer average fidelity of the GHZ states generated experimentally [37].

The state $|\mathcal{E}_3\rangle_{n,n,n}$ is created by entangling two $|\text{GHZ}_{n+1}\rangle$’s and $H_{n+2}|\text{GHZ}_{n+2}\rangle$, where $H_j$ is the Hadamard operator acting on the $j$th photon, using $B_S$’s as shown in the final step of Fig. 2(a). The Hadamard operation is achieved by passing the polarization photons through a $\pi/2$-rotator. Similarly, procedure to generate $|\mathcal{E}_r\rangle_{n,m,n}$ is shown in the Fig. 2(b).

The states $|\text{GHZ}_{n+1}\rangle$ can be efficiently created by acting...
BS's on \(|\text{GHZ}_3\rangle\)'s in \([\log_2(n - 1)]\) steps as illustrated in Fig. 3. Here, \([\cdot]\) represents the ceiling of a number. For example, when the step number \(k = 3\), up to \(|\text{GHZ}_{10}\rangle\) (where \(n + 1 = 10\)) can be created (other possible states are listed in Tab. II). The resource states \(|c_{3}^{n,m,n}\rangle\) and \(|c_{3}^{n,m,n}\rangle\) are created with success rates \((2^\left\lfloor\log_2 n\right\rfloor + 2)^{-1}\) and \((2^\left\lfloor\log_2(n - 1)\right\rfloor + 2)^{-1}\), respectively. At this stage of the protocol, delay lines and optical switches are employed. Delay lines are essential to delay the passage of photons that are not undergoing measurement until the action of the current BS is complete. Therefore, quicker BS's permit shorter delay lines. Optical switches are needed to control the flow of photons through these delay lines, choose the successful outputs of the BS's, and send the larger GHZ states to the next step. Hence, many generation steps would entail a large number of optical switches and longer delay lines. Therefore, our resource-state generation protocol aims to minimize the usage of delay lines and optical switches that significantly contribute to photon loss. On the other hand, when deterministic sources capable of producing high-fidelity \(|\text{GHZ}_r\rangle\)'s with \(r \geq 4\) are used, resource states can be generated with a smaller number of time steps. The total number of steps to complete the generation of \(|c_{3}^{n,m,n}\rangle\) \((|c_{3}^{n,m,n}\rangle\) is \([\log_2 n] + 1\) \(([\log_2(n - 1)] + 1\), when \(m + 1 < n\).

Let us consider the creation of \(|c_{3}^{8,8,8}\rangle\) and \(|c_{3}^{8,2,8}\rangle\) as an example. To create this state, we need to entangle \(|\text{GHZ}_0\rangle\), \(|\text{GHZ}_{10}\rangle\) and \(|\text{GHZ}_0\rangle\). All the required GHZ states are generated at \((k = 3)\)th step using \(|\text{GHZ}_3\rangle\)'s. Having successful BS on the 9th photon of \(|\text{GHZ}_0\rangle\) and 1st photon of \(|\text{GHZ}_{10}\rangle\) and on the 10th photon of \(|\text{GHZ}_{10}\rangle\) and 1st photon of \(|\text{GHZ}_0\rangle\), we create \(|c_{3}^{8,8,8}\rangle\) in \((k = 4)\)th step. Similarly, when no \(H_{10}\) is performed and replacing \(|\text{GHZ}_{10}\rangle\) with \(|\text{GHZ}_4\rangle\), we end up having \(|c_{3}^{8,2,8}\rangle\).

### C. Star cluster states

After explaining the procedure to create resource states in detail, we shall now discuss the formation of the star cluster state \(|c_{s}\rangle\). Here, we use near-deterministic \(n\)-Bell state measurement \((n\text{-BSM})\) [41] which is a cascade of \(n\) BS's, as shown in Fig. 3(a), to entangle the resource states. An \(n\)-BSM fails when all the constituent BS's fail. Therefore, the success rate of \(n\)-BSM is \(1 - 1/2^n\) and arbitrarily approaches 1 with increasing \(n\). The working principle of \(n\)-BSM is explained in Sec. II A. Two \(|c_{3}^{n,m,n}\rangle\)'s are entangled using two \(n\)-BSMs to form a \(|c_{s}\rangle\) as shown in the Fig. 3(b). A \(|c_{s}\rangle\) is formed only when both the \(n\)-BSMs are successful in the process. In other failed cases, the desired \(|c_{s}\rangle\) is not formed and the resulting states are distorted as shown in Fig. 3(c) and Fig. 3(d). A successfully generated \(|c_{s}\rangle\) shall have \(m\) photons in the central qubit and \(n\) photons in the surrounding qubits. Having a larger \(n\) is desirable as the success rate of the \(n\)-BSM improves when \(|c_{s}\rangle\)'s are entangled to form layers of \(|c_{s}\rangle\). While having \(m > 1\) suppresses the probability of qubit loss on \(|c_{s}\rangle\) (as qubit is redundantly encoded with multiple photons) during photon loss, it also invites stronger dephasing on the qubit as described in Sec. III.

The \(n\)-BSM operations employed in our protocol scheme are surely prone to failures. At this point we introduce two subvariants of MTQC. In the first subvariant, referred to as MTQC-1, we do not employ any optical switches and use whatever star cluster states upon failure of \(n\)-BSMs. In the second subvariant, namely MTQC-2, we use optical switches circuit to choose intact \(|c_{s}\rangle\)'s and discard the distorted ones. This will make the photons to pass through another delay line and switch. Thus, MTQC-1 adds one extra time step in creating \(|c_{s}\rangle\) while MTQC-2 adds two. To be explicit, until now the process of creating \(|c_{s}\rangle\) has taken place in \(k + 1\) and \(k + 2\) time steps with MTQC-1 and MTQC-2, respectively. We shall study and compare the performances of both these subvariants in terms of photon-loss tolerance and resource efficiency.
for more details on logical errors. The logical errors distorted we refer the Reader to Refs. [15, 48]. However, in MTQC-1, tail in Sec. MTQC-1 would have more missing qubits (explained in de-
will be missing and such situations are handled by QEC on BSM fails, the corresponding edge between the central qubit s
is overcome by QEC where qubits associated with diagonal edges (refer also to Fig. 2(c) of Ref. [28]) along
diagonal edges distort the local RHG structure, which in turn makes usual error syndrome extrac-
structure, which in turn makes usual error syndrome extrac-
In MTQC-1 the formation of |\psi_L\rangle and |\psi_i\rangle takes place simultaneously; that is, both happen in the \((\log_2 n) + 2\)th time step. But in MTQC-2 the formation of |\psi_L\rangle happens in the next time step after the intact |\psi_i\rangle are formed. Therefore, |\psi_L\rangle is formed in \((\log_2 n) + 3\)th time step. Another point to be noted is that the qubits of |\psi_i\rangle that take part in entangling to future layer must wait for an extra time step. This completes the protocol for generating |\psi_i\rangle.

E. Universal quantum computing on RHC lattice

Following the RHG-lattice generation, where a faulty |\psi_L\rangle with missing edges and phase-flip errors is created, topological fault-tolerant quantum computing is carried out by making sequential single-qubit measurements in X and Z bases as dictated by the quantum gates being implemented. Measurements in the Z basis remove qubits from |\psi_L\rangle, creating defects which also creates logical states of lattice. These defects are braided to achieve two-qubit logical operations topologically \([33, 34]\). The X-basis measurement outcomes provide error syndromes and also effect Clifford gates on the logical states. The universal set of operations for quantum computing is complete with the inclusion of magic-state distillation for which measurements on the chosen qubits are carried out in the \((X \pm Y)/\sqrt{2}\) basis \([33, 34]\). Practically, Z-basis measurements on qubits are possible by measuring the polarization of any photon belonging to the qubit in the z direction. On the other hand, an X-basis measurement outcome of a lattice-qubit is given by the parity of X-basis measurements of the constituent photons.

Logical errors on |\psi_L\rangle occur when a chain of Z operators connects two defects or encircles a defect. Code distance, \(d\) (lattice size) is the minimum number of Z operations on the lattice qubits such that two defects are connected. In the following we explain how random Z operators (errors) on optical qubits happen due to photon-loss. These errors coupled with wrong inference in decoding during QEC go undetected and shall lead to logical errors \([33, 34]\). Also refer to Appendix [A] for more details on logical errors. The logical errors are faulty gate operations and can be minimized by choosing sufficiently large \(d\). If \(p_L\) denotes the logical error rate, our aim is to reduce it to a target error rate, \(p_{L_{\text{err}}}\) set by the end user.

III. NOISE MODEL

Apart from photon loss being a major source of errors \([2]\), the lattice |\psi_L\rangle built with linear optics suffer from missing edges due to probabilistic entangling operations. In this section, we study the effect of photon loss on multiphoton qubits, success rate of entangling operation and consequently on the MTQC protocol. Suppose that the overall photon-loss rate suffered by each photon due to imperfect optical components is \(\eta\) and the initial state of an \(n\)-photon qubit is defined by \(|\Psi^0\rangle = \alpha|H\rangle^{\otimes n} + \beta|V\rangle^{\otimes n}\). When \(\eta\) is nonzero, the state of the
multiphoton qubit is \([41]\)

\[
\tilde{\rho}_l = |\Psi_0^l\rangle\langle\Psi_0^l| + \frac{1}{2} \sum_{q=1}^{l} \eta^q (1 - \eta)^{l-q} \times \mathcal{P}_l (|\Psi_0^{l-q}\rangle\langle\Psi_0^{l-q}| + |\Psi_{l-q}\rangle\langle\Psi_{l-q}|, \mid \text{VAC}_q\rangle\langle \text{VAC}_q\mid).
\]

(5)

where \(|\Psi_{l-q}\rangle = \alpha |\Psi\rangle \otimes |\Psi\rangle - \beta |\Psi\rangle \otimes |\Psi\rangle \) and \(|\text{VAC}_q\rangle\) is a vacuum state. The probability of an \(l\)-photon qubit losing a photon or more is \(1 - (1 - \eta)^l\). When photons are lost, with probability 0.5 the state possesses the component \(|\Psi_l\rangle\langle\Psi_l|\), that is, undergoes dephasing.

If \(0 \leq \eta \leq 1\) is the overall photon-loss rate due to imperfect GHZ source (\(\eta_{\text{swc}}\)), delay lines (\(\eta_{\text{dly}}\)), optical switching network (\(\eta_{\text{osw}}\)) and detectors (\(\eta_{\text{det}}\)), the rate of dephasing on the multilayer qubits due to photon loss is

\[
P_Z = \frac{1 - (1 - \eta)^l}{2}.
\]

(6)

If photon losses in different components of quantum computing are statistically independent, we have the relation

\[
\eta = 1 - (1 - \eta_{\text{swc}})(1 - \eta_{\text{dly}})(1 - \eta_{\text{osw}})(1 - \eta_{\text{det}})
\]

(7)

that associate the overall photon-loss rate to the relevant individual component loss rates. For the delay lines \((1 - \eta_{\text{dly}}) = e^{-c \tau_0 \kappa / L_0}\), where \(L_0 = 22\) km is attenuation length of optical fiber for the standard telecom wavelength of 1550 nm \([59]\). \(c = 2 \times 10^8\) km/s is the speed of light in, say, acrylic (PMMA) optical fibers, \(\tau_0 = 150 \times 10^{-9}\) s \([2]\) is the time duration to complete one BSM, which can be made smaller using electrooptical modulators, and \(\kappa\) is the total time steps (in units of \(\tau_0\)) a photon of lattice-qubit; that is, central qubit of \(|\psi\rangle\) spends before being measured during fault tolerant quantum computing. Also, each photon has to pass through a network of \(\kappa\) switches. Therefore, \(\eta_{\text{swc}} = 1 - (1 - \eta^\kappa)^\kappa\), where \(\eta^k\) is photon-loss rate through one optical switch. It is important to note from Eq. (6) that photon-loss introduces larger rate of dephasing on qubits with larger number of photons.

The \(B_S\) operates on the lossy states and when input photons are lost the failure rate increases to \(1 - (1 - \eta)^2/2\). An \(n\)-BSM would fail only when all the constituent B's fail. Therefore, the failure rate of an \(n\)-BSM due to photon loss is

\[
P_f = \left[1 - \frac{(1 - \eta)^2}{2}\right]^n \approx \left(\frac{1}{2} + \eta\right)^n.
\]

(8)

From the above expression, we observe that when \(\eta\) is \(O(10^{-2})\) and \(n\) is large, \(P_f\) is not very sensitive to photon loss.

We point out that like other DV optical schemes \([17]\), photon loss does not necessarily imply lattice-qubit loss. The probability that photon loss leads to lattice-qubit loss, \(\eta_m\), is much smaller than \(P_f\) for \(\eta \sim 10^{-2}\). \(n \geq 5\) and \(m \geq 2\) considered in this work. Therefore, \(n\)-BSM failure has a dominant effect over qubit loss when calculating logical error rate on \(|\psi\rangle\) and the latter can be neglected. However, having large \(m\) is not favorable as it invokes stronger dephasing as inferred from Eq. (6). To mitigate this issue we set \(m = 2\) throughout the work, which also sufficient to neglect the effect of qubit loss.

The MTQC is operated at the same photon-loss rate \(\eta\) on all qubits and the number of photons in lattice qubits is \(m = 2\). In this case \(\kappa < k\) and it is crucial to note that the lattice qubits always spend \(\kappa = 3\) (4) time steps in MTQC-1 (MTQC-2) before being measured. Now one can appreciate that our protocol to generate \(|\psi\rangle_n, n, r\) makes sure that the photons of lattice qubits passes through least number of optical components. Therefore, for a fixed \(\eta\) component-wise photon-loss tolerance of MTQC is enhanced. Also, note that before performing \(n\)-BSMs to form \(|\psi\rangle\), the surrounding qubits of a \(|\psi\rangle\) has to pass through a larger number of lossy components as \(n > m\) and therefore have a larger photon-loss rate. Moreover, qubits that are part of the \(n\)-BSM between the current and future layers spend an extra time step and thus suffer from stronger losses. However, this time delay depends on the physical architecture that runs measurement-based quantum computation along with other technological details; in principle, different layers can be generated simultaneously. Therefore, we hereby neglect the influence of time delays on \(\eta\), which is reasonable if the time delay is sufficiently shorter than \(-L_0/c\ln \eta\).

For the star cluster states, qubit dephasing owing to photon loss happens locally on the central and other qubits. In addition, the surrounding qubits are consumed during an \(n\)-BSM and noise owing to photon loss can be dealt with by suitably encoding these qubits. Investigation of this procedure is a subject matter of our future work that is beyond the scope of this article. In this work, we consider only the dephasing noise due to photon loss on the central qubits. One can also consider other kinds of amplitude-damping noise \([60, 63]\) to evaluate the performance our MTQC protocols.

**IV. ENCODING LATTICE QUBITS WITH THE REPETITION CODE**

As described in the previous section, photon-loss leads to dephasing on the qubits. We observe from Eq. (6) that if the degree of dephasing can be reduced, MTQC can have a larger photon-loss threshold \(\eta_{\text{th}}\). To reduce the effect of dephasing, one intuitive approach would be to encode the qubits of \(|\psi\rangle\) with a multiqubit repetition code. This can be achieved by encoding the \(m\)-photon qubits of resource ket \(|\psi\rangle^\otimes_{N,m,n}\) with \(N\)-qubit repetition code; one would replace \(|0_m\rangle\) with \(|0_m\rangle \oplus |1_m\rangle \oplus |N\rangle\), and \(|1_m\rangle\) with \(|0_m\rangle - |1_m\rangle \oplus |N\rangle\) (up to normalization), where \(N\) is the repetition number. This gives the following encoded resource ket

\[
|\psi\rangle^\text{enc} = |0_m\rangle (|0_m\rangle + |1_m\rangle) \oplus |N\rangle (|0_m\rangle - |1_m\rangle)
\]

(9)

Note that the extreme qubits, each holding \(n\) photons, are not encoded as they shall be consumed by \(n\)-BSMs anyway.

As an example for demonstrating that we can increase \(\eta_{\text{th}}\), let us consider \(N = 3\). The generation of the encoded \(|\psi\rangle^\text{enc}\) kets using |GHZ3\rangle’s and \(B_S\) is explained in Appendix \(E\). The QEC
which corresponds to
\[ \text{N} \]
which involves the error function \( \text{erf}(\cdot) \). The infinite upper limit of the integral is justified by the extremely narrow width of the Gaussian integrand for large \( \text{N} \). Since for a large argument \( z \),
\[ \text{erf}(z) \approx 1 - \frac{e^{-z^2}}{\sqrt{\pi z}} , \]
we get \( ye^{z^2/8} \approx \sqrt{2/\pi/p_{\text{Z,enc}}} \), where
\[ y = \frac{1 - 2p_z}{\sqrt{p_z(1 - p_z)/N}} . \]
Squaring this relation allows us to write
\[ y^2 \approx 4W \left( 1/ \left( 2\pi p_{\text{Z,enc}}^2 \right) \right) . \]
which expresses the solution as a Lambert function \( W(\cdot) \). This leads to the physical solution
\[ p_z \approx \frac{1}{2} - \frac{1}{2\sqrt{1 + 4t}^\text{th}} , \quad t = \frac{\text{N}}{4W \left( 1/ \left( 2\pi p_{\text{Z,enc}}^2 \right) \right)} . \]

We may now immediately identify \((1 - \eta_{\text{th}}^{\text{enc}})^m \approx (1 + 4t)^{-1/2} \) with \( m \) being the number of photons of each qubit in the repetition code, which finally yields
\[ \eta_{\text{th}}^{\text{enc}} \approx 1 - \left[ 1 + \frac{\text{N}}{W \left( 1/ \left( 2\pi p_{\text{Z,enc}}^2 \right) \right)} \right]^{-1/(2m)} . \]

As \( \text{N} \) increases, we find that \( \eta_{\text{th}}^{\text{enc}} \approx 1 - O(1/\text{N}^{1/(2m)}) \rightarrow 1 \) for any designated value of \( p_{\text{Z,enc}} < p_z \) and \( m \). On the other hand, the function \( W \left( 1/ \left( 2\pi p_{\text{Z,enc}}^2 \right) \right) \) itself is a slowly increasing function of \( p_{\text{Z,enc}} \) as \( p_{\text{Z,enc}} \) decreases, originating from the large-argument expansion
\[ W(x) \approx \ln x - \ln \ln x + \frac{\ln \ln x - 2\ln x}{2\ln x^2} , \]
so that within the typical range \( 0.001 \leq p_{\text{Z,enc}} \leq 0.1 \) of interest, the order of magnitude for \( W \left( 1/ \left( 2\pi p_{\text{Z,enc}}^2 \right) \right) \) does not change. For a repetition code of fixed \( \text{N} \) and a given \( p_{\text{Z,enc}} \), increasing \( m \) reduces \( \eta_{\text{th}}^{\text{enc}} \).

V. RESULTS ON PHOTON-LOSS THRESHOLD

When carrying out QEC on \( |\text{C}_L \rangle \), the logical error rate \( p_L \) is determined against \( p_z \) for various code distances, \( d \) via simulations. This procedure is repeated for various values of \( \text{n} \), which determine the respective \( p_L \) of \( n \)-BSMs. The intersection point of the curves corresponding to various \( d \)'s is the threshold dephasing rate \( p_{\text{Z,th}} \) as marked in Fig. 5. The photon-loss threshold, \( \eta_{\text{th}} \) is determined using Eq. (6) by replacing \( p_z \) with \( p_{\text{Z,th}} \).

For example, from Fig. 5 which corresponds to \( \text{n} = 8 \), we have \( p_{\text{Z,th}} \approx 2.9 \times 10^{-2} (3.2 \times 10^{-2}) \) for MTQC-1 (MTQC-2). Considering that MTQC is operated below the threshold value (see Tab. 1) at \( \eta = 0.01 \), according to Eq. (8) we then have the associated value of \( p_L = 4.58 \times 10^{-3} \). Replacing \( p_z \) by \( p_{\text{Z,th}} \) in Eq. (6) we find that the total tolerable photon loss rate, \( \eta_{\text{th}} \) is \( 2.9 \times 10^{-2} (3.2 \times 10^{-2}) \) for MTQC-1 (MTQC-2). However, the photon loss rate tolerable by individual components can be deduced as follows. The total time steps spent by lattice-qubits before being measured is \( \kappa = 3 \) (4) for MTQC-1 (MTQC-2). Therefore, we have \( \eta_{\text{th}} = 4.1 \times 10^{-3} (5.4 \times 10^{-3}) \). Further, by inserting the value of \( \eta_{\text{th}} \) in Eq. (7) (\( m = 2 \)), we have \( \eta_{\text{th}}^{\text{stow}} = \eta_{\text{th}}^{\text{swt}} = \eta_{\text{th}}^{\text{det}} = 8.6 \times 10^{-3} (8.8 \times 10^{-3}) \) for MTQC-1 (MTQC-2). This implies that equal amount of photon-loss can be tolerated in GHZ source, optical switches and measurement. In this case each switch in the network can tolerate photon-loss rate of \( \eta^* = 2.8 \times 10^{-3} (1.7 \times 10^{-3}) \).
FIG. 5. Logical error rate $p_L$ plotted against the dephasing rate $p_Z$ for MTQC-1 and MTQC-2 of $n \in [4,9]$, where $m = 2$, accompanied by 99% confidence intervals (shaded regions) that are typically much narrower than the average values. For each $n$ value, $p_L$ corresponding to code distances (RHG lattice size) $d = 3, 5, 7, 9, 11, 13$ are plotted. The intersection point of the $p_L$ curves for various $d$ values corresponds to the threshold dephasing rate $p_{Z,th}$. We observe that as $n$ increases, the failure rate of $n$-BSMs, $p_f$, decreases, leading to a larger $p_{Z,th}$. It is important to note that when $n = 9$, $p_{Z,th}$ is close to that in the $p_f = 0$ case, so that considering $n > 9$ results in no visible advantage. The threshold value $\eta_{th}$ is shown for every figure panel.

While the $\eta_{th}$ in MTQC-2 is higher both subvariants offer comparable component-wise photon-loss thresholds. However, MTQC-2 imposes more stringent restriction on allowed photon-loss rate in switches. We lastly note that the time delays between consecutive layers can be neglected if they are sufficiently shorter than $-(L_0/c)\ln\eta_{th} \approx 4 \times 10^{-4}$ s for both MTQC-1 and MTQC-2, as discussed in Sec. III

The $\eta_{src}$, $\eta_{swc}$ and $\eta_{det}$ are complementary in nature as the overall tolerable photon-loss rate of the MTQC is fixed. Therefore, if the GHZ source and detectors are operated at lower loss levels, the MTQC protocol can tolerate a much higher photon-loss rate in the optical switches. Similarly, we
have repeated calculation for \( n = 5 \) through 9 and the values of \( \eta_a \) are tabulated in Tab. 1. We know from the Ref. 48 that when \( p_l = 0.145 \) in MTQC-2, the \( |C_L\rangle \) cannot tolerate any dephasing, and hence any photon loss. This threshold \( p_l \) is overcome when \( n \geq 3 \). However, the situation is different for MTQC-1 and is explained in the following.

In MTQC-1, distorted \( |\psi_L\rangle \)’s are allowed and this gives rise to \( |C_L\rangle \) with diagonal edges (refer also to Fig. 2(c) of Ref. [28]). This is overcome by removing the qubits at the ends of a diagonal edge during QEC. Therefore, the probability that a qubit survives on \( |C_L\rangle \) is \( 1 - p_l \). Note also that each qubit in \( |C_L\rangle \) is susceptible to losses from diagonal edges connected to \( \text{four} \) \( |\psi_L\rangle \)’s: two in the layer containing the qubit, and another two coming from different layers. Additionally, failure of a \( n \)-BSM that connects two \( |\psi_L\rangle \)’s would leave an edge between qubits missing. This situation is handled by removing one of the qubits associated with such a missing edge [48]. The survival probability of a lattice qubit in this case is \( 1 - p_l/2 \), where \( n \)-BSMs are involved. It follows that the total probability of loosing a qubit on \( |C_L\rangle \) is \( 1 - (1 - p_l)^3(1 - p_l/2)^4 \) and this number should not exceed 0.249 [55, 64] if \( |C_L\rangle \) should be useful for quantum computing. Therefore, the threshold value for \( p_l \) is 0.047, which is determined by solving the equation \( 1 - (1 - p_l)^3(1 - p_l/2)^4 = 0.249 \). So, MTQC-1 is possible only when \( n \geq 5 \). Note that lattices other than the RHG type have different values for threshold failure rates of entangling operation [65, 66]. Accordingly, the probability of missing qubits in MTQC-2 is \( 1 - (1 - p_l)^4 \). For a given value of \( p_l \), the resulting lattice in MTQC-1 has more missing qubits and is therefore of a relatively poorer quality.

The tolerable photon-loss thresholds for MTQC naturally increase with the usage of repetition codes in the manner discussed in Sec. IV. Let us consider an example of encoded lattice-qubit with \( n = 8 \), \( m = 2 \) and \( N = 3 \). In this encoded situation, the value for the overall photon-loss threshold with a 3-qubit repetition code, obtained by using Eq. (10), is \( \eta_{\text{rep}}^{\text{enc}} = 10.7\% \) (11.1\%). \( \kappa = 6 \) (7) for MTQC-1 (MTQC-2), following which we have \( \eta_{\text{rep}}^{\text{enc}} = 8.1 \times 10^{-3} (9.5 \times 10^{-3}) \). The component wise tolerance would be \( 3.2 \times 10^{-2} (3.4 \times 10^{-2}) \). In other words, encoding the lattice qubits with a three-qubit repetition code increases the \( \eta_{\text{rep}}^{\text{enc}} \) by nearly four times. Also, each switch can now tolerate a higher \( \eta_{l}^{\text{enc}} = 5.4 \times 10^{-3} (4.9 \times 10^{-3}) \). The results for other values of \( n \) are available in Tab. II. Additionally, a sufficiently large set of stabilizer measurements on encoded qubits can be used to reconstruct the underlying noisy quantum process acting on these qubits [67, 68].

VI. RESULTS ON RESOURCE OVERHEAD

In this work, we consider GHZ states as the raw ingredients for constructing \( |C_L\rangle \), on which fault-tolerant gate operations are performed via single-qubit measurements. Therefore, the resource overhead, \( N \), is the average number of GHZ’s consumed to build \( |C_L\rangle \) of required size. Other components used in MTQC, such as delay lines, detectors, optical switches and beam splitters scale proportionately to \( N \). Alternatively, Ref. [18] considers detectors to be resources. Logical gate operations are possible by passing defects through \( |C_L\rangle \) [33, 34]. As logical errors occur when a chain of \( Z \) errors connects two defects or encircles a defect, error-free operations would demand these defects be separated by a distance \( d \) and also have a perimeter of \( d \). When noise is below threshold, by increasing the value of \( d \), the logical error rate \( p_l \) can be reduced arbitrarily. If \( |C_L\rangle \) has sides of length \( l = 5d/4 \), it can accommodate a defect of perimeter \( d \) and other defects placed a distance \( d \) apart from each other (refer to Fig. 8 of Ref. [29]).

The time for simulation of QEC on \( |C_L\rangle \) increases drastically with \( d \), rendering the estimation of (a very small) \( p_l \) for arbitrarily large \( d \) unfeasible. So, the value of \( d \) at which an extremely small target \( p_l \) \( (p_l^\text{ targ}) \) is achieved can be estimated by extrapolating known values of \( p_l \). We can determine the target \( d \) required to achieve \( p_l^\text{ targ} = 10^{-6} \) and \( 10^{-15} \) using the following expression [56]

\[
p_L^\text{targ} = b \left( \frac{a}{b} \right) - (d - d_b)/2 ,
\]

where \( a \) and \( b \) are the values of \( p_l \) corresponding to the second largest \( d_a \) and the largest code distance \( d_b \), respectively considered in our simulations. For example, as seen from the Fig. 5 when \( p_Z = 0.01 \) we have \( d_a = 11 \) and \( d_b = 13 \).

As a practice, for \( n > 5 \), we operate MTQC at \( \eta = 0.01 \), which is below the photon-loss threshold value. This corresponds to \( p_L \approx 0.01 \) at \( m = 2 \) for the unencoded case and \( p_L, \text{enc} \approx 3 \times 10^{-4} \) for the encoded one. However, unencoded MTQC-1 with \( n = 5 \) yields a threshold rate of \( \eta_{\text{th}} \approx 6 \times 10^{-3} \) and is thus operated at \( \eta = 3 \times 10^{-3} \). Therefore, the \( N_{\text{p-\text{enc}}} \) is also determined at the operation point. In Eq. (19), \( a \) and \( b \) also correspond to the operation point. The reason for choosing the operation point away from the threshold is as follows: It is known empirically that \( p_l \approx (p/Z)p_{Z,\text{th}}(d+1)/2 \) when the minimum weight perfect matching decoder is used [69]. If we operate closer to the threshold, a larger \( d \) is essential to reach some pre-chosen \( p_l^\text{targ} \). Thus, the operation point is chosen away from the threshold. Also, sufficiently away from the threshold point the ratio \( a/b \) is reasonably constant and the estimation with Eq. (19) is reliable [54]. Once \( d \) for achieving \( p_L^\text{targ} \) is determined, \( N \) can be estimated by counting the average number of GHZ states required to build \( |C_L\rangle \) of side \( l = 5d/4 \). Only the central qubit of a \( |C_L\rangle \) stays in the lattice and rest of them are consumed by \( n \)-BSMs. A \( |C_L\rangle \) of sides \( l \) would have \( 6l^3 \) qubits. Therefore, we need \( 6l^3 \) \( |\psi_L\rangle \)’s per fault-tolerant gate operation.

As explained in Sec. IIC to create a \( |\psi_L\rangle \) we need two \( |\psi_L\rangle \)’s and one \( |\chi_L\rangle \). Based on the noise model in Sec. III in order to minimize dephasing effects on the central qubit, we set \( m = 2 \). According to Fig. 2, the creation of a \( |\chi_L\rangle \) necessitates the entanglement of two \( |\text{GHZ}_{n+1}\rangle \) and a \( |\text{GHZ}_{n+2}\rangle \) using two \( B_8 \)’s. As the failure rate of one \( B_8 \) is \( (1 + 2\eta)/2 \) given a photon-loss rate \( \eta \) (obtained by setting \( n = 1 \) in Eq. (8)), one needs \( 8(1 - 2\eta)^{-2} \) copies of \( |\text{GHZ}_{n+1}\rangle \) and \( 4(1 - 2\eta)^{-2} \) copies of \( |\text{GHZ}_{n+2}\rangle \), on average. Similarly, to create a \( |\chi_L\rangle \) one needs on average of \( 8(1 - 2\eta)^{-2} \)
TABLE I. Table of values for the dephasing-noise threshold $p_{Z,th}$, photon-loss threshold $\eta_{th}$ and resource overhead $N^{(i)}_{p_{th}}$ to achieve $p_{th}^{\text{arg}} = 10^{-6}$ and $10^{-15}$ concerning various instances $n$ of our scalable MTQC-1 and MTQC-2 protocols. The improvement in photon-loss threshold by encoding all lattice qubits with the $(N = 3)$-qubit repetition QEC code, $\eta_{th}^\text{enc}$ and their associated resource overhead, $N^\text{enc}_{p_{th}}$, are also listed for comparisons. The table starts from $n = 5$ since MTQC-1 is not possible when $n \leq 4$ as detailed in Sec. [3]. The benefits of encoding is apparent both in terms of $\eta_{th}$ and resource overheads. The asymptotically achievable value of $\eta_{th}$ when $p_{th} = 0$ is 0.033 (see Fig. 5). Therefore, increasing $n$ beyond 9 gives no visible advantage. However, $\eta_{th}^\text{enc}$ can be arbitrarily improved by increasing the repetition number $N$ in the encoding of lattice-qubits. As is expected, increasing the value of $n$ generally improves $\eta_{th}$ and $\eta_{th}^\text{enc}$. Interestingly, $N^\text{enc}_{p_{th}}$ reduces with $n$ until $n = 8$, beyond which the excessive amount of (GHZ)$\text{'s}$ required for resource-states generation quickly nullifies any subsequently insignificant improvement in $\eta_{th}$. In view of this, we conclude that MTQC-2 with $n = 8$ is the optimal case of un-encoded MTQC. Note that the resource overheads are calculated when MTQC operates at $\eta = 0.01$ which corresponds to $p_{Z} = 0.01$ and $p_{Z,\text{enc}} = 0.0003$. A similar practice is adopted in [34], where “operational overheads” are computed “at 1/3 of the fault-tolerance threshold.”

\[
\langle \text{GHZ}_{n+1} \rangle \text{ and } (4 - 2\eta)^{-2} \langle \text{GHZ}_{m+2} \rangle. \quad \text{Therefore,} \\
N^\text{enc} = \frac{4(6N_{n+1} + 2N_{n+2} + N_{m+2})}{(1 - 2\eta)^2} \\
\text{(20)}
\]

\[
\langle \text{GHZ}\rangle \text{'s consumed to create a } \langle \text{C}_{n} \rangle \text{ in MTQC-1, on average, where } N_{n} \text{ is the average number of [GHZ]'}\text{'s consumed to create a } \langle \text{GHZ}_{n} \rangle, \text{ example values of which can be read from Tab. [I] in Appendix [B]. On the other hand, one needs} \\
N^\text{enc}_{n} = \frac{4(6N_{n+1} + 2N_{n+2} + N_{m+2})}{(1 - 2\eta)^2(1 - p_{th})^2} \\
\text{(21)}
\]

\[
\langle \text{GHZ}\rangle \text{'s, on average, in MTQC-2. For example, consider the case when } n = 8 \text{ and } m = 2, \text{ and MTQC is operated at } \eta = 0.01. \text{ Looking at Tab. [I] and inserting the values of } N_{10}, N_{9} \text{ and } N_{4} \text{ in to Eq. (20) one can estimate that } 4(6 \times 55.16 + 68.00 + 4.08)(1 - 2 \times 0.01)^{-2} \approx 1962 \text{ [GHZ]'}\text{'s, on average, are consumed in MTQC-1 to create a } \langle \text{C}_{n} \rangle. \text{ Similarly, using Eq. (21) we estimate that approximately 1980 [GHZ]'}\text{'s are needed in MTQC-2.} \\
\text{Once the average number of [GHZ]'}\text{'s required to create a } \langle \text{C}_{n} \rangle \text{ is known, it is straightforward to calculate the resource overhead } N^\text{enc}_{p_{th}} \text{ to achieve some } p_{th}^{\text{arg}}. \text{ In the case of MTQC-1, we have} \\
N^\text{enc}_{p_{th}} = \frac{375(6N_{n+1} + 2N_{n+2} + N_{m+2})}{8(1 - 2\eta)^3}d^3, \\
\text{(22)}
\]

and for MTQC-2, it is

\[
N^\text{enc}_{p_{th}} = \frac{375(6N_{n+1} + 2N_{n+2} + N_{m+2})}{8(1 - 2\eta)^3(1 - p_{th})^2}d^3. \\
\text{(23)}
\]

Let us consider the same example when $n = 8$, $m = 2$ and similar value for $\eta$ to estimate resource overheads. Further, from the simulation results we estimate that one needs $d = 15$ (13) to attain $p_{th} \sim 10^{-6}$ in MTQC-1 (MTQC-2). Using the values of $d$ in Eqs. (22) and (23) we estimate that $N_{10} \sim 8.19 \times 10^7$ and $N_{10}^{\text{enc}} \sim 4.78 \times 10^7$. On the other hand, for $p_{th} \sim 10^{-15}$ one needs $d \approx 47$ (37) and thus $N_{10} \sim 2.33 \times 10^9$ and $N_{10}^{\text{enc}} \approx 1.14 \times 10^9$. Resource overhead and $d$ for other values of $n$ are presented in Tab [I].

In encoding the qubits of $\langle C \rangle$ with the three-qubit repetition QEC code $\langle C_{3} \rangle_{n,m,n}$ is replaced by $\langle C_{3} \rangle_{\text{enc}}$ which is created by replacing $\langle \text{GHZ}_{m+2} \rangle$ in Fig. 2 by

\[
\langle \text{enc} \rangle = |H\rangle (|H\rangle^{\otimes m} + |V\rangle^{\otimes m})^{\otimes 3} |H\rangle \\
+ |V\rangle (|H\rangle^{\otimes m} - |V\rangle^{\otimes m})^{\otimes 3} |V\rangle. \\
\text{(24)}
\]

The process to create $\langle \text{enc} \rangle$ is detailed in Appendix [C]. To estimate the resource overhead due to encoding, we first estimate the average number of [GHZ]’s consumed to generate $\langle \text{enc} \rangle$. For this a [GHZ] and a [GHZ] are needed in the first step (refer to Appendix [C]) and are entangled using $B_{8}$. When $\eta = 0.01$, $N_{\text{enc}} \approx 104.96$ [GHZ]’s are incurred, on average, to form a $\langle \text{enc} \rangle$ (deduced in Appendix [C]. Hereafter, the procedure to estimate resource overhead remains the same as
non-encoded case. Therefore, the resource overheads in the
encoded case are
\[
\frac{N_{\text{enc}}^{\text{th}}}{p_L^{\text{th}}} \bigg|_{m=2} = \frac{375(6N_{\eta+1} + 2N_{\eta+2} + N_{\text{enc}})}{8(1-2\eta)^2} d^3,
\]
for MTQC-1 and
\[
\frac{N_{\text{enc}}^{\text{th}}}{p_L^{\text{th}}} \bigg|_{m=2} = \frac{375(6N_{\eta+1} + 2N_{\eta+2} + N_{\text{enc}})}{8(1-2\eta)^2(1-p_L)^3} d^3
\]
for MTQC-2.

VII. COMPARISON

Now, we shall compare the performance of our MTQC to
other schemes for fault-tolerant linear optical quantum com-
puting. In Fig. 6 we present (a) photon-loss thresholds and
(b) resource overheads of known linear optical quantum com-
puting schemes [13, 17, 22, 27, 29, 40–42] with MTQC-1 and
MTQC-2. Clearly, MTQC shows exceptionally high loss tol-
erance compared to all known schemes, and is also highly
competitive in terms of resource efficiency. In the follow-
ing we shall briefly describe each scheme to which MTQC
is compared.

Reference [13] is one of the first works to determine the
region of \( \theta_b \) along with a dephasing error rate and an es-
timation of resource overheads for fault-tolerant linear optical
quantum computing. The scheme uses optical cluster
states built using polarization Bell pairs. This scheme coupled
7-qubit Steane QEC codes [9] with telecorrection (where
teleportation is used for error-syndrome extraction) for fault-
tolerance. Unlike schemes that use topological codes, the
concatenation of Calderbank–Shor–Steane (CSS) codes
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the scheme is too high for practical considerations. A subsequent scheme in Ref. [40] that encodes multiple polarization photons into a logical qubit in a parity state provides a smaller $\eta_h \approx 2 \times 10^{-3}$, but an improved resource efficiency compared to Ref. [13]. This scheme also uses 7-qubit Steane QEC codes with teleportation and multiple levels of concatenations. When $\eta = 4 \times 10^{-3}$ and depolarizing rate is $4 \times 10^{-4}$, the average number of Bell pairs consumed is $N_{10-6} \approx 6.8 \times 10^{14}$ and $N_{10-15} \approx 3.5 \times 10^{19}$ [29] with four and six levels of concatenation, respectively.

Later, Ref. [43] used error-detecting quantum state transfer where the underlying codes were capable of detecting errors in a way similar to the scheme in Ref. [70]. Here, QEC is done using streams of entangled polarization photons, a topological photonic quantum computing scheme, which involves generating $\eta$ that is achievable by creating $N$ catenations. When $\eta = 1 \times 10^{-4}$ and depolarizing rate is $1 \times 10^{-5}$, with five (seven) levels of concatenation, the average number of Bell pairs used is $N_{10-6} \approx O(10^{15})$ [13]. There is yet another multi-polarization-photon quantum computing scheme [44] that again utilizes teleportation based on 7-qubit Steane QEC codes and thus needs need the same levels of concatenation. Calculations in Ref. [28] show that $N_{10-6} \approx O(10^{15}) [N_{10-15} \approx O(10^{16})]$ Bell pairs are consumed when $\eta \approx O(10^{-6})$ and $p_Z \approx O(10^{-4})$.

Using streams of entangled polarization photons, a topological photonic quantum computing scheme, which involves creating $|\psi_L\rangle$, was proposed in Ref. [17]. This has $\eta_h \approx 5.3 \times 10^{-4}$ when the depolarizing error rate is zero. However, this scheme gives $\eta_h = 1.14 \times 10^{-3}$ for the hypothetical case of $\eta = 0$, which is higher by an order of magnitude compared to other non-topological fault-tolerant architectures. Calculations in Ref. [28] show that $N_{10-6} > 2 \times 10^9 (N_{10-15} > 4.2 \times 10^{10})$ for non-zero $\eta$.

The coherent-states $\{|\alpha\rangle, |\alpha\rangle\} = \{|\alpha\rangle, -|\alpha\rangle\}$ can be used as the logical basis for CV qubits [24, 25, 27, 21]. Reference [22] uses these qubits to develop a fault-tolerant quantum computing scheme. This also employs 7-qubit Steane QEC codes with teleportation and multiple levels of concatenations for tolerance against photon-loss and depolarizing errors. Here, superpositions of coherent states, $|\alpha\rangle \pm |-\alpha\rangle$ (up to normalization) [72, 73], are considered as resources. For this scheme $\eta_h \approx 2.3 \times 10^{-4}$ and $N_{10-6} \approx 2.1 \times 10^{11}$ $(N_{10-15} \approx 6.9 \times 10^{15})$ when $\eta = 8 \times 10^{-5}$ and $p_Z = 2 \times 10^{-4}$. The resource overhead is reduced by many orders of magnitude compared to Ref. [13], but this comes at the cost of a very low $\eta_h$. Reference [23] improved this situation by replacing coherent superposition states with hybrid states. The new scheme offers a better value of $\eta_h \approx 4.6 \times 10^{-4}$. Here $N_{10-6} \approx 8.2 \times 10^9 (N_{10-15} \approx 2.3 \times 10^{12})$ hybrid qubits are required when $\eta = O(10^{-6})$ and $p_Z = O(10^{-4})$.

By far, Ref. [28] shows the best $\eta_h$-to-resource-overhead ratio that is achievable by creating $|\psi_L\rangle$’s with hybrid qubits. In the scheme of Ref. [28], $\eta_h \approx 3.3 \times 10^{-3}$ and $N_{10-6} \approx 8.5 \times 10^5 (N_{10-15} \approx 1.7 \times 10^7)$ hybrid qubits are consumed when $\eta = 1.5 \times 10^{-3}$ and $p_Z = 3 \times 10^{-3}$. Subsequently, Ref. [29] demonstrated that $\eta_h$ can be further improved by spending more hybrid qubits. This scheme could achieve an improved $\eta_h \approx 5.7 \times 10^{-3}$ with $N_{10-6} \approx 2.9 \times 10^7 (N_{10-15} \approx 4.9 \times 10^8)$ when $\eta = 2.6 \times 10^{-3}$ and $p_Z = 2.3 \times 10^{-3}$.

After the previous overview of various existing linear optical schemes for fault-tolerant quantum computing and mentioning the associated numerical values of $\eta_h$ and $N_{\text{err}}$, we are now set for a comparison with MTQC. We shall compare with schemes that involve the creation of RHG lattices—the schemes in Refs. [17, 28, 29]. In comparison, MTQC outperforms the scheme in Ref. [17] both in terms of photon-loss tolerance and resource efficiency. Although the MTQC performs better than schemes in Refs. [28, 29] in terms of $\eta_h$, it falls short in terms of $N_{\text{err}}$ compared to the scheme in Ref. [28]. On the other hand, the MTQC outperforms all the known non-topological schemes Refs. [13, 17, 22, 27, 40, 42] both in terms of $\eta_h$ and resource efficiency.

We note that the error models in Refs. [13, 17, 40, 42] employed photon loss and depolarizing errors independently. On the other hand, the other schemes [22, 27, 41] considered more realistic models where photon loss and dephasing are related. In addition, the photon-loss thresholds in Refs. [13, 17, 40, 42] are obtained under the condition of zero depolarizing error, which is unrealistic. For these four schemes, when a non-zero depolarizing error is considered, the threshold values should then be lower than the ones presented as empty bars in Fig. 6(a), since additional depolarizing errors deteriorate $\eta_h$.

A very recent scheme [74] that also encodes lattice qubits with QEC codes claims to be able to tolerate a photon-loss rate of 10.4% occurring in entangling operations. This requires a particular type of 24-photon entangled states of which information concerning their (unreported) generation resource overheads could be of interest. On the contrary, for MTQC in this work, we begin only with 3-photon GHZ states that can be deterministically generated using current technology [57].

The scheme in Ref. [18] considers detectors as resources, about $O(10^9)$ of them, to create a tree-cluster state, a collection of which ultimately forms a lattice (similar to the fate of $|\psi_L\rangle$’s). This can tolerate, component-wise, a photon-loss rate of approximately $1 \times 10^{-3}$ (when the beam splitters are assumed to be lossless and the success rate of a BSM is 0.5). In our work, when $n = 9$, we need 2935 (GHZ$_3$)’s (encoded case) are consumed and this number also reflects the order of magnitude of detectors needed. Additionally, our protocol has component-wise $\eta_h \approx 3.7 \times 10^{-2}$, which is a significant improvement. The extravagant resource overhead originates from the usage of single photons as basic ingredients for forming (GHZ$_3$)’s (which are in turn used to generate tree-cluster states) with a low success rate of $1/32$ [54]. However, recent work has demonstrated that the success rate can be greater than $1/32$ [75].

VIII. ALTERNATIVE PLATFORMS FOR SCALABILITY ENHANCEMENT

While fault-tolerant MTQC can significantly improve resource overheads and error thresholds relative to other schemes, it requires resource-state generation and moderately-large collective BSMS that could pose a chal-
lellenge with current polarization-based optical platforms. In particular, repeat-until-success strategies based on polarization-encoded qubits, as discussed in this work and many of the cited references rely on repeated generation of resource states that involve a huge number of entangled multiphoton qubits.

The use of time-bin qubits (photons encoded into time-delayed pulses of well-separated arrival times that do not overlap one another [76]), which are alternative quantum-information encoding schemes, have been considered in the generation of multiphoton entangled states [77, 78]. In particular, it has been shown that such an encoding allows for a deterministic generation of GHZ states [79]. In this reference, specifically, |ψy⟩n,m,n of 2n + m = 3 and 4 were generated with the respective fidelity of 0.90 and 0.82. Improvement in the fidelity of larger time-bin GHZ states in order to generate |ψy⟩n,m,n of m ≥ 2 and n ≥ 8, which are optimal ranges for MTQC as shown in Tab. 1 is an important research direction.

In the grander scheme of things, it might be of interest to consider the generation of both |ψy⟩n,m,n and |ψy⟩n,m,m resource states using the full potential of temporal-mode (TM) optical qubits with both time and frequency content [80–84], where each mode possesses an infinite-dimensional Hilbert space that is encoded onto one physical qubit. By exploiting such a large number of degrees of freedom, it is, in principle, possible to encode multiquantum information onto a single physical photon.

Let us briefly highlight how such a logical TM encoding works. The central operation is the (unitary) quantum pulse gate (QPG) [80, 85–87]:

\[ Q_k^{(θ)} = 1 - |A_k⟩⟨A_k| - |C⟩⟨C| + \cos θ (|A_k⟩⟨A_k| + |C⟩⟨C|) + \sin θ (|C⟩⟨A_k| - |A_k⟩⟨C|), \]

(27)

which allows one to convert a mode-matched TM basis ket |A_k⟩⟨A_k| = δ_k, k’ into the superposition |A_k⟩⟨A_k| + |C⟩⟨C| sin θ, where |C⟩ is a TM in a different frequency band than |A_k⟩ such that |A_k⟩C = 0. On the other hand, a mode-mismatched action, \[ Q_k^{(θ)}|A_k⟩_{k’} = |A_k⟩_{k’}, \]

leaves the TM basis ket intact.

To present an alternative approach to multiquibit cluster-state generation, just as an example, we shall revisit the so-called type-I fusion scheme [88] that was first introduced to entangle two spatial photons using a PBS, followed by a photodetection after a 45° polarization rotation. If we now suppose that |A_0⟩_a logically represents |0⟩_1,m1,n_a for qubit a and |A_1⟩_b logically represents |1⟩_1,m1,n_b for qubit b, then, it has already been shown in [88] that a QPG-adapted type-I fusion together with deterministic spatial-mode combination (implicitly carried out throughout this analysis) permits the generation of \[ |ψy⟩_y ≡ (|A_0⟩_a + |A_1⟩_a)/\sqrt{2}; \]

\[ 50:50 \text{ beam splitter on } C \text{ modes} \]

\[ \text{single-photon heralding} \rightarrow |ψy⟩_y. \]

(28)

Here, single-photon heralding is performed consistently on one of the two output detectors in order to fix all relative phase factors in the final output pure state. One may similarly continue the above type-I fusion procedure until four logical TM basis kets are superposed, resulting in the formation of |ψy⟩_y ≡ (|A_0⟩_a + |A_1⟩_a + |A_2⟩_a - |A_3⟩_a)/2:

\[ |ψy⟩_y \rightarrow (|A_0⟩_a + |C⟩_a)/(|A_1⟩_a + |C⟩_a) \]

(29)

and finally,

\[ 50:50 \text{ beam splitter on } C' \text{ modes} \]

\[ \text{single-photon heralding} \rightarrow |ψy⟩_y. \]

(30)

Ideally, near-perfect TM manipulation such as the above exemplifying scheme could reduce the average number of photon-pair operations (PPOs) needed to generate resource states, since only two photons are handled at any round of the TM type-I fusion process via a 50:50 beam splitter on the C or C’ modes followed by detector-specific single-photon heralding. In practical scenarios, reducing the number of PPOs is equivalent to minimizing the number of detectors needed in a scheme as both scale commensurately with each other. As a basic comparison under ideal non-lossy conditions (n = 1), and hence a 50% chance of a BSM failure, we recall from Sec. VII and Fig. 2 that two GHZ_{m+1} states and one GHZ_{m+2} state are entangled via two BSMs to create a copy of |ψy⟩_y,m,n with polarization encoding. If we suppose that m = 2 and n = 8 are the target indices, then from Appendix B and the fact that the average number of PPOs needed to create two states and entangle them, given that these states were previously generated with the respective average number of PPOs l_1 and l_2, is 2(l_1 + l_2 + 1) in view of the 0.5 BSM failure rate, we require an average of 2 PPOs (via BSMs) to create a GHZ_4 state and 34 PPOs to create a GHZ_8 state. These numbers are obtained from the assumption that no PPOs are required to generate the basic GHZ_3 states (see Fig. 7 for a simple exposition). Therefore, an average of 218 PPOs are necessary to create a copy of |ψy⟩_y,8,8. On the other hand, according to [28], TM-adapted type-I fusions only require 4 PPOs on average to create any |ψy⟩_y,n,m,n since the success probability of beamsplit single-photon heralding on a specified detector is 0.25. Similarly, to create a copy of |ψy⟩_y,8,8 using polarization encoding, whereas 64 PPOs with TM adaptors [based on (28)–(30)] are sufficient to create any |ψy⟩_y,n,m,n.
We stress that this drastic improvement is made only with 3-photon GHZ states, passive linear-optical elements, and on-off detectors. We also demonstrated that by employing codes of larger repetition number for concatenation, the photon-loss threshold can be further improved. Further, resource overheads in terms of the average number of three-photon GHZ states incurred per gate operation corresponding to various values of $n$ are tabulated in Tab. 1. In MTQC-2, when $n = 9$ the resource overhead to reach the target logical error rate of $10^{-6}$ ($10^{-15}$) is $2.07 \times 10^6$ ($5.03 \times 10^7$). This is the most resource efficient case of MTQC. Interestingly, we observed that code concatenation not only improves photon-loss threshold but also favourably reduces the resource overheads of the MTQC. Comparing our results with known linear optical quantum computing schemes [13, 17, 22, 27, 29, 40–42], MTQC offers clearly the highest tolerance against photon loss. MTQC is also highly resource-efficient compared with known linear optical schemes and is comparable only with Ref. [28]. In principle, our protocol can be carried out even if we start with single photons as the basic ingredient. For this, the three-photon GHZ states can be generated using linear optics with a success rate of $1/32$ [54], or higher [89]. In this case, the resource overhead (average number of single photons) for MTQC would increase approximately by two orders of magnitude.

In the current work we have not used photon-number resolving detectors at any stage of the protocol that is essential to boost the success rate of direct BSM. The reason for this choice is that the on-off detectors are practically more efficient and can operate at room temperatures. If one chooses to employ photon-number resolving detectors and operate at cryogenic temperatures, the resource efficiency of MTQC can be further improved.

Our protocol can also be extended to the creation of lattices with different geometry [90, 91]. However, it remains to be examined if they can be made tolerant against entangling operation failures. MTQC also demonstrates the crucial need for the experimental development of high fidelity deterministic multiphoton entangled state generators in order for the advancement of the field of the scalable linear-optics-based quantum information processing. Given its significant enhancement in the photon-loss threshold and the recent progress in generating multiphoton entanglement, our scheme will make scalable photonic quantum computing a step closer to reality.

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FIG. 8. (a) Primal unit cell of an RHG lattice. Blue dots and lines indicate qubits and edges, respectively. (b) RHG lattice used for the simulation. The lattice is in a cuboid-shaped space with the size of \( d - 1 \), \( d - 1 \), and \( 4d + 1 \) along the \( x \), \( y \), and \( t \) (simulating time) axes, respectively, in the unit of a cell. The first primal cells along the \( t \) axis are shown as black solid lines. The boundaries are primal about the \( x \) and \( t \) axes, while dual about the \( y \) axis. An error chain connecting the two opposite \( x \) or \( y \) boundaries (orange dotted line) incurs a logical error.

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Appendix A: Simulation of QEC

Here we present the method to obtain the logical error rate \( p_L \) numerically for a given subvariant of MTQC (MTQC-1 or MTQC-2), failure rate of \( n \)-BSM \( (p_L) \), dephasing rate \( (p_Z) \), and code distance \( (d) \). We consider a three-dimensional space with the \( x \), \( y \), and \( simulating time \) \( (t) \) axes. We simulate an RHG lattice in a cuboid with the size of \( (d - 1 \), \( d - 1 \), \( T \) ) for \( T := 4d + 1 \) about the three axes in the unit of a cell, as shown in Fig. 8. The boundaries are primal about the \( x \) and \( t \) axes (that is, they are in contact with primal cells), while dual about the \( y \) axis (that is, they cut primal cells in a half).

An isolated dephasing error is detected by two check operators adjacent to the qubit. Generally, an error chain of dephasing errors is detected by two check operators located at its ends [33, 34, 57, 92]. However, error chains connecting two opposite boundaries are not detectable, since there are no check operators at their ends. If the number of such error chains regarding the \( x(y) \)-boundaries is odd, a primal (dual) logical error occurs. We take account of only primal logical errors in this simulation.

The code distance is determined by the widths about the \( x \) and \( y \) axes, not the \( t \) axis, thus \( T \) can be an arbitrary number. For fair comparison with different code distances or other computation schemes, we calculate the logical error rate per unit simulating time. \( T \) should be large enough to get a reliable value, thus we set it to \( 4d + 1 \).

We use the Monte Carlo method for the simulation; we repeat a sampling cycle many times enough to obtain a desired confidence interval of the logical error rate per unit simulating time. Each cycle is structured as follows: We first prepare a cluster state described above. Due to the failures of \( n \)-BSM with the probability of \( p_L \), qubits are randomly removed by the method described in Sec. [V]. Check operators containing the removed qubits are merged with adjacent check operators repeatedly until not containing any one of them [48]. If a qubit on a boundary is removed, the involved check operator is removed and the boundary is deformed to include the other qubits in the check operator. If the two opposite \( x \)-boundaries meet due to the deformation, we conclude that a logical loss occurs, namely, that the desired computation fails. In this case, we stop the cycle and start the next one immediately. Otherwise, dephasing errors are randomly assigned to the left qubits with the probability of \( p_Z \). For simplicity, qubits on the \( t \)-boundaries are assumed to be perfect; namely, they are neither removed nor have errors. This ensures that the \( t \)-boundaries cause no logical losses or errors. Such an unrealistic assumption has negligible effects if \( T \) is large enough.

Next, the outcomes of check operators are calculated, then decoded to deduce errors with Edmonds’ minimum-weight perfect matching algorithm (MWPM) [93–95] via Blossom V software [96]. Error chains connecting the two opposite \( x \)-boundaries are identified by comparing the assigned and decoded errors. We then count the number of distinct simulating times corresponding to the ends of the error chains at the boundary of \( x = 0 \), called erroneous simulating times.

After repeating enough cycles, we calculate the logical error rate per unit simulating time \( p_L \) by the ratio of the number of erroneous simulating times to the total simulating times. The error threshold \( p_{th} \) is obtained from the calculated \( p_L \) results for different values of \( d \) and \( p_Z \). \( p_L \) decreases as \( d \) increases if \( p_Z < p_{th} \) and vice versa otherwise.

| \( k \) | Possible GHZ states | Creation process | Average number of [GHZ] |
|------|---------------------|-----------------|------------------------|
| 1    | \([\text{GHZ}_4]\) | \([\text{GHZ}_3] + B_s \) \([\text{GHZ}_3]\) | 4 (4.08) |
| 2    | \([\text{GHZ}_5]\) | \([\text{GHZ}_4] + B_s \) \([\text{GHZ}_3]\) | 10 (10.37) |
| 3    | \([\text{GHZ}_7]\) | \([\text{GHZ}_5] + B_s \) \([\text{GHZ}_4]\) | 28 (29.50) |
| 4    | \([\text{GHZ}_{11}]\) | \([\text{GHZ}_7] + B_s \) \([\text{GHZ}_6]\) | 88 (94.19) |
|      | \([\text{GHZ}_{10}]\) | \([B_s B_s B_s] \) \([\text{GHZ}_{10}]\) | 256 (277.55) |

TABLE II. Possible GHZ states at each step \( k \) and the corresponding generation processes are tabulated. \( +B_s \) stands for the BSM \( B_s \). Average number of \([\text{GHZ}_3]\)‘s consumed in generation of each GHZ state too is tabulated. The numbers in the brackets in the last column corresponds to the average number of \([\text{GHZ}_3]\)‘s consumed in the presence of photon loss of rate \( \eta = 0.01 \). The photon loss also reduces the success rate of BSM to \((1 - 2\eta) / 2 \) which in turn increases the average number of \([\text{GHZ}_3]\)‘s consumed.
Let the total number of GHZ sources to be entangled using BSMs be \( N \). We now present in the following iterative scheme:

1. Let \( M \) denote the total number of ingredient GHZ states to be entangled using BSMs. When \( k = 1 \), for example, \( M = m - 2 \).
2. If \( M \) is odd, define the number of GHZ pairs \( n_p = (M - 1)/2 \) with \( n_{\text{left}} = 1 \) GHZ ket leftover. Otherwise, define \( n_p = M/2 \) and \( n_{\text{left}} = 1 \).
3. Proceed with \( n_p \) distinct pairwise BSM of GHZ states. If \( M \) is odd, \( n_{\text{left}} = 1 \) GHZ state will not be entangled, which shall be pairwise entangled with another GHZ state in the next step.
4. If \( M = 1 \), terminate the generation protocol. Otherwise, update \( n_p = n_p + n_{\text{left}} \) and raise \( k \) by one.

Using this numerical recipe, the (unbracketed) values in Tab. III can be generated. Upon numerical-pattern inspection, we find the explicit analytical formula,

\[
N_m = 3(m - 2) \cdot 2^{\lfloor \log_2(m - 2) \rfloor} - 2 \cdot 4^{\lfloor \log_2(m - 2) \rfloor},
\]

for the number of GHZ states needed to generate a GHZ state.

### Appendix C: Generation of concatenated resource state

In this section we describe how to create resource state concatenated with three-qubit repetition QEC code that is, \([|\text{enc}\rangle]_\text{3'}\) enc = \(|0_m\rangle (|0_m\rangle + |1_m\rangle)^{\otimes 2} |0_1\rangle + |1_1\rangle (|0_m\rangle - |1_m\rangle)^{\otimes 2} |1_1\rangle \). This state can be generated in the \( [k = \log_2(m - 1)] \) th step when using \( \text{BS}_3 \). To begin with, apply Hadamard operation on \( |\text{GHZ}_{m+1}\rangle \) so that we have \( H_{m+1} |\text{GHZ}_{m+1}\rangle = (|H\rangle^{\otimes m} + |V\rangle^{\otimes m}) |H\rangle + (|H\rangle^{\otimes m} - |V\rangle^{\otimes m}) |V\rangle \). Applying \( \text{BS}_3 \) between \( H_{m+1} |\text{GHZ}_{m+1}\rangle \) and \( |\text{GHZ}_3\rangle \), and rearranging the modes we get \( |H\rangle^{\otimes m} + |V\rangle^{\otimes m} |H\rangle^{\otimes 3} + |V\rangle (|H\rangle^{\otimes m} - |V\rangle^{\otimes m}) |V\rangle^{\otimes 3} \) in \( (k + 1) \) th step. Further, by entangling two \( H_{m+1} |\text{GHZ}_{m+1}\rangle \)'s with the above state the encoded central qubit,

\[
|\text{enc}\rangle = |H\rangle (|H\rangle^{\otimes m} + |V\rangle^{\otimes m})^{\otimes 3} |H\rangle
+ |V\rangle (|H\rangle^{\otimes m} - |V\rangle^{\otimes m})^{\otimes 3} |V\rangle,
\]

in \( k \) \& \( (k + 1) \) th step. Finally, two \( |\text{GHZ}_{m+1}\rangle \)'s are entangled on both sides of \( |\text{enc}\rangle \) to get the desired state, \(|\text{enc}\rangle_{\text{3'}}\). This final step is nothing but replacing \(|\text{GHZ}_{m+2}\rangle \) with \(|\text{enc}\rangle \) in the \( k \) th step in Fig. 2. In the step \( k = 2 \), \((1 + 10)/0.5 = 22 \) \(|\text{GHZ}_3\rangle \)'s are consumed on average. Further, \((22 + 2)/(0.5 \times 0.5) = 96 \) \(|\text{GHZ}_3\rangle \)'s are consumed on average in creation of \(|\text{enc}\rangle \). However, taking in to account the photon loss of rate \( \eta = 0.01 \), the average number of \(|\text{GHZ}_3\rangle \)'s consumed in \((k = 2)\) th step is \((1 + 10.37)/[(1 - 2\eta)/2] = 23.20 \) and finally it is \( (23.20 + 2)/[(1 - 2\eta)/2]^2 \approx 104.96 \). Figure 9 supplements these statements with a flowchart.

When we have \( m = 2 \), The first process in creation of \(|\text{enc}\rangle \) takes place in \( k = 2 \) time steps. Therefore, creation of \(|\text{enc}\rangle \) takes a total of 4 steps. In the unencoded case, the central qubit always waited for 2 time steps during creation of \(|\text{enc}\rangle \) and totally \( k = 4 (5) \) for \(|\text{enc}\rangle \) in MTQC-1 (MTQC-2). In the encoded case the central qubit has to wait for totally \( k = 6 (7) \) time steps before being measured for quantum computing.

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**FIG. 9.** Schematic representation of the process of generation of \(|\text{enc}\rangle \) using \(|\text{GHZ}_3\rangle \)'s and \( \text{BS}_3 \) in \( k = 4 \) steps. \( \text{H}_3 \) is the Hadamard operation on the third photon of \(|\text{GHZ}_3\rangle \) and is performed before feeding it to \( \text{BS}_3 \).
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