Open-closed homotopy algebra in superstring field theory

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Abstract

We construct open-closed superstring interactions based on the open-closed homotopy algebra structure. It provides a classical open superstring field theory on general closed-superstring-field backgrounds described by classical solutions of the nonlinear equation of motion of the closed superstring field theory. We also give the corresponding WZW-like action through the map connecting the homotopy-based and WZW-like formulations.
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1 Introduction

It is known that several homotopy algebras are naturally realized as algebraic structures in string field theories and play a significant role. This was first recognized in closed bosonic string field theory [1, 2], where the $L_\infty$ structure determines the (classical) gauge-invariant action. Open bosonic string field theory was first formulated as a cubic theory using the (Witten) associative product [3] but can be extended to that with an $A_\infty$ structure more generally [4, 5]. This is also deformed to the theory on general closed string backgrounds [6–8] based on the open-closed homotopy algebra (OCHA) structure [9–11]. In the superstring field theories, the homotopy algebra structure is more important. Since it seems inevitable to avoid associativity anomaly [12], the $A_\infty$ structure becomes essential to determine the gauge-invariant action in the open superstring field theory [13–15]. The $L_\infty$ structure again plays the role of guiding principle to determine the action with appropriate picture numbers in the heterotic and type II superstring field theories [16–20].

On the other hand, the current understanding is that there is no essential difference between the theory of open string/closed string mixed system and the theory of purely closed string. It merely describes the perturbation on the different backgrounds, those with and without a D-brane [21, 22]. They should be derived from non-perturbatively formulated fundamental theory such as string field theory, but it is not a priori clear which one should be considered more fundamental. The closed string field theory is simpler, but the open-closed string field theory has a larger symmetry structure, the OCHA structure [1]. The purpose of this paper is to construct an open-closed string field theory realizing the OCHA structure. The action obtained explains the classical open string field theory on general closed-string backgrounds.

The paper is organized as follows. In section 2 we briefly review the open superstring field theory with general $A_\infty$ structure. After introducing some conventions and fundamental ingredients, we show how we construct the open superstring field theory based on the $A_\infty$ structure. The superstring products with appropriate picture numbers satisfying the $A_\infty$ relations can be obtained by recursively solving the differential equations. We similarly review the closed superstring field theory with the $L_\infty$ structure in section 3. We define the string products multiplying both open and closed string field in section 4 and show the relations they must satisfy to form the OCHA. We also give the differential equations that the products with OCHA structure should follow. They provide an action of the open superstring field theory on the general closed superstring backgrounds. In section 5 we obtain, as a byproduct, the corresponding WZW-like action through the map connecting the homotopy-based and WZW-like formulations, which is

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1In a formulation that introduces an auxiliary degree of freedom, the open-closed superstring field theory has already been constructed [23]. It does not, however, decrease the worthwhile to construct the theory based on the OCHA structure.
a generalization considered in [24]. Section 6 is devoted to the summary and discussion. In order to help the construction of concrete string interactions, some explicit procedure for solving the differential equations is given in Appendix A. Appendix B is added to make the paper self-contained. We introduce two composite string fields, the pure-gauge open string field and the associated open string field, which is nontrivial in the theory with general $A_\infty$ structure.

2 Open superstring field theory with $A_\infty$-structure

We summarize in this section how the open superstring field theory is constructed based on the $A_\infty$ algebra structure.

2.1 Open superstring field

The first-quantized Hilbert space of open superstring is composed of two sectors: $\mathcal{H}_o = \mathcal{H}_{NS} + \mathcal{H}_R$. Correspondingly, the open superstring field $\Psi$ has two components: $\Psi = \Psi_{NS} + \Psi_R$, both of which are Grassmann odd and have ghost number 1. The component $\Psi_{NS}$ ($\Psi_R$) has picture number $-1$ ($-1/2$) and represents space-time bosons (fermions). We impose on it a constraint

$$ P^o_{XY} \Psi = \Psi, \quad P^o_{XY} = G^o(G^o)^{-1}, \quad (2.1) $$

with

$$ G^o = \pi^0 + X^o \pi^1, \quad (G^o)^{-1} = \pi^0 + Y^o \pi^1, \quad (2.2) $$

where $\pi^0$ and $\pi^1$ are the projection operators onto the NS and R components, respectively: $\pi^0 \Psi = \Psi_{NS}$ and $\pi^1 \Psi = \Psi_R$. The picture changing operator (PCO) of open superstring $X^o$ and its inverse $Y^o$ are defined by

$$ X^o = -\delta(\beta_0)G + (\gamma_0 \delta(\beta_0) + \delta(\beta_0) \gamma_0)b_0, \quad Y^o = -\frac{G}{L_0} \delta(\gamma_0). \quad (2.3) $$

The PCO $X^o$ is BRST exact in the large Hilbert space:

$$ X^o = [Q, \Xi^o], \quad \Xi^o = \xi_0 + (\Theta(\beta_0) \eta \xi_0 - \xi_0)P_{-3/2} + (\xi_0 \eta \Theta(\beta_0) - \xi_0)P_{-1/2}, \quad (2.4) $$

where $P_{-3/2}$ ($P_{-1/2}$) is the projection operator onto the states with picture number $-3/2$ ($-1/2$). We call the Hilbert space restricted by the constraint in Eq. (2.1) the restricted Hilbert space and denote $\mathcal{H}_o^{res}$. Note that $G^o$ and $(G^o)^{-1}$ satisfy

$$ G^o(G^o)^{-1}G^o = G^o, \quad (G^o)^{-1}G^o(G^o)^{-1} = (G^o)^{-1}, \quad [Q, G^o] = 0, \quad (2.5) $$

and thus $P^o_{XY}$ is a projection operator that is compatible with the BRST cohomology: $QP^o_{XY} = P^o_{XY}QP^o_{XY}$. The open superstring field satisfying Eq. (2.1) is expanded in the ghost zero-modes.
as
\[
\Psi = (\phi_{NS} - c_0 \psi_{NS}) + \left( \phi_R - \frac{1}{2}(\gamma_0 + c_0 G) \psi_R \right) \in \mathcal{H}^\text{res}_o. \quad (2.6)
\]

Natural symplectic form \( \omega^o \) and \( \Omega^o \) in \( \mathcal{H}_o \) and \( \mathcal{H}^\text{res}_o \), respectively, are defined by using the BPZ inner product as
\[
\begin{align*}
\omega^o(\Psi_1, \Psi_2) &= (-1)^{\deg(\Psi_1)} \langle \Psi_1 | \Psi_2 \rangle, \quad (2.7) \\
\Omega^o(\Psi_1, \Psi_2) &= (-1)^{\deg(\Psi_1)} \langle \Psi_1 | (\mathcal{G}^o)^{-1} | \Psi_2 \rangle, \quad (2.8)
\end{align*}
\]
where \( \deg(\Psi) = 1 \) or \( 0 \) if \( \Psi \) is Grassmann even or odd, respectively. We also use a natural symplectic form \( \omega^o \) in the large Hilbert space \( \mathcal{H}^l_o \), which is similarly defined using the BPZ inner product in \( \mathcal{H}^l_o \), and related to \( \omega^o \) as \( \omega^o(\xi_0 \Psi_1, \Psi_2) = \omega^o(\Psi_1, \Psi_2) \) if \( \Psi_1, \Psi_2 \in \mathcal{H}^o_s \).

### 2.2 Interaction with \( A_\infty \)-structure

Open superstring interactions are described by the string products \( M_n \) mapping \( n \) open superstring fields to an open superstring field as
\[
M_n : (\mathcal{H}^\text{res}_o)^\otimes n \rightarrow \mathcal{H}^\text{res}_o, \quad (n \geq 1),
\]
\[
\Psi_1 \otimes \cdots \otimes \Psi_n \mapsto M_n(\Psi_1, \cdots, \Psi_n). \quad (2.9)
\]
We identify the one-string product as the open superstring BRST operator: \( M_1 = Q_o \). Note that the conditions
\[
\mathcal{P}_{XY}^o M_n(\Psi_1, \cdots, \Psi_n) = M_n(\Psi_1, \cdots, \Psi_n) \quad (2.10)
\]
hold by definition. The multi-linear maps \( M_n \) further satisfy the \( A_\infty \) relations
\[
\sum_{m=0}^{n} \sum_{k=0}^{n-m} (-1)^{\epsilon(1,k)} M_{n-m+1}(\Psi_1, \cdots, \Psi_k, M_{m+1}(\Psi_{k+1}, \cdots, \Psi_{k+m+1}), \Psi_{k+m+2}, \cdots, \Psi_n) = 0, \quad (2.11)
\]
where \( \epsilon(1,k) = \sum_{i=1}^{k} \deg(\Psi_i) \), and cyclicity with respect to the symplectic form \( \Omega^o \),
\[
\Omega^o(\Psi_1, M_n(\Psi_2, \cdots, \Psi_{n+1})) = -(-1)^{\deg(\Psi_1)} \Omega^o(M_n(\Psi_1, \cdots, \Psi_n), \Psi_{n+1}). \quad (2.12)
\]
The linear maps satisfying (2.11) and (2.12) form the cyclic \( A_\infty \) algebra \( (\mathcal{H}^\text{res}_o, \Omega^o, \{M_n\}) \).

Coalgebra representation allows us to describe these infinite number of relations of maps \( M_n \) concisely [25]. The set of maps \( \{M_n\} \) are represented by a degree-odd coderivation \( M = \sum_{n=1}^\infty M_n \) acting on the tensor algebra \( T \mathcal{H}_o = \sum_{n=0}^\infty (\mathcal{H}^\text{res}_o)^\otimes n \) as
\[
M = \sum_{n=1}^\infty M_n = \sum_{n=1}^\infty \sum_{k,l=0}^\infty (\mathbb{I}^\otimes k \otimes M_n \otimes \mathbb{I}^\otimes l) \pi^o_{k+n+l}, \quad (2.13)
\]

where $\pi^o_m$ is the projection operator onto $(\mathcal{H}_o^{\text{res}})^{\otimes m} \subset \mathcal{T}\mathcal{H}_o$. Then the $A_\infty$ relations in Eq. (2.11) is concisely written as

$$[M, M] = 0.$$  \hspace{1cm} (2.14)

For open superstring field theory, the string interaction $M_n$ must be defined for each combination of NS and R inputs so that the picture number must be conserved.

$$M_{n+1} = \sum_{p+r=n} M^{(p)}_{p+r+1} |2r\rangle,$$  \hspace{1cm} (2.15)

where $p$ is the picture number that the map itself has and $2r$ is the Ramond number (number of Ramond inputs − number of Ramond output).

The action with $A_\infty$ structure is given by

$$I_o = \int_0^1 dt \Omega^o \left( \Psi, \pi^o_1 M \left( \frac{1}{1-t\Psi} \right) \right),$$  \hspace{1cm} (2.16)

where we introduce a real parameter $t \in [0, 1]$ and the group-like element $\frac{1}{1-\Psi}$ defined by

$$\frac{1}{1-\Psi} = \mathbb{I}_{\mathcal{T}\mathcal{H}_o} + \sum_{n=1}^{\infty} \psi^{\otimes n}.$$  \hspace{1cm} (2.17)

Here, $\mathbb{I}_{\mathcal{T}\mathcal{H}_o}$ is the identity in $\mathcal{T}\mathcal{H}_o$ satisfying $\mathbb{I}_{\mathcal{T}\mathcal{H}_o} \otimes V = V = V \otimes \mathbb{I}_{\mathcal{T}\mathcal{H}_o}$ for $\forall V \in \mathcal{T}\mathcal{H}_o$. The arbitrary variation of $I_o$ is given by

$$\delta I_o = \Omega^o \left( \delta \Psi, \pi^o_1 M \left( \frac{1}{1-\Psi} \right) \right),$$  \hspace{1cm} (2.18)

where we used the cyclicity in Eq. (2.12). We can show that the action in Eq. (2.16) is invariant under the gauge transformation

$$\delta_{\Lambda} \Psi = \pi^o_1 M \left( \frac{1}{1-\Psi} \otimes \Lambda \otimes \frac{1}{1-\Psi} \right),$$  \hspace{1cm} (2.19)

using the $A_\infty$ relation in Eq. (2.11) and cyclicity in Eq. (2.12):

$$\delta_{\Lambda} I_o = \Omega^o \left( \pi^o_1 M \left( \frac{1}{1-\Psi} \otimes \Lambda \otimes \frac{1}{1-\Psi} \right), \pi^o_1 M \left( \frac{1}{1-\Psi} \right) \right)$$

$$= \Omega^o \left( \Lambda, \pi^o_1 M \left( \frac{1}{1-\Psi} \otimes \pi^o_1 M \left( \frac{1}{1-\Psi} \right) \otimes \frac{1}{1-\Psi} \right) \right)$$

$$= \Omega^o \left( \Lambda, \pi^o_1 MM \left( \frac{1}{1-\Psi} \right) \right) = 0.$$  \hspace{1cm} (2.20)

2In this paper, $[,]$ denotes the graded commutator.

3Whether the output is NS or R string is determined by the space-time fermion number conservation.
2.3 Explicit construction of interactions

The cyclic $A_\infty$ algebra $(\mathcal{H}_o^{res}, \Omega^o, M)$ for open superstring field theory is constructed in two steps. First, we consider a cyclic $A_\infty$ algebra $(\mathcal{H}_l^o, \omega_l^o, Q - \eta + A)$. A degree odd coderivation

$$A = \sum_{p,r=0}^{\infty} A^{(p)}_{p+r+1} |^{2r} (A^{(0)}_{1})^0 \equiv 0$$

is defined respecting the cyclic Ramond number (= number of Ramond inputs + number of Ramond output) to make it easier to realize cyclicity. This other $A_\infty$ algebra can be decomposed into two mutually commutative $A_\infty$ algebras $(\mathcal{H}_l, D)$ and $(\mathcal{H}_l, C)$ with

$$\pi_1 D = \pi_1 Q + \pi_1^0 A, \quad \pi_1 C = \pi_1 \eta - \pi_1^1 A$$

depending on the picture number deficit of the output. The $A_\infty$ relation $[A, A] = 0$ can also be decomposed as

$$[Q, A] + \frac{1}{2}[A, A]_1 = 0, \quad \text{(2.23a)}$$

$$[\eta, A] - \frac{1}{2}[A, A]_2 = 0, \quad \text{(2.23b)}$$

where the bracket with subscript $[\cdot, \cdot]^1$ or $2$ is defined by projecting the intermediate state onto the NS or R state after taking the (graded) commutator. The relation $[\cdot, \cdot] = [\cdot, \cdot]^1 + [\cdot, \cdot]^2$ holds since the intermediate state is either the NS state or R state. If such $A_\infty$ algebras are obtained, we can transform them by the cohomomorphism

$$\hat{F}^{-1} = \pi_1 I - \Xi^o \pi_1^1 A$$

(2.24)

to the cyclic $A_\infty$ algebra of interest $(\mathcal{H}_o^{res}, \Omega^o, M)$ and a (trivial) $A_\infty$ algebra $(\mathcal{H}_l^o, \eta)$:

$$\pi_1 \hat{F}^{-1} D \hat{F} = \pi_1 Q + \Xi^o \pi_1 A \hat{F} \equiv \pi_1 M, \quad \pi_1 \hat{F}^{-1} C \hat{F} = \pi_1 \eta.$$  

(2.25)

We consider a generating function

$$A(s,t) = \sum_{p,m,r=0}^{\infty} s^m t^p A^{(p)}_{m+p+r+1} |^{2r} \equiv \sum_{p=0}^{\infty} t^p A^{(p)}(s)$$

(2.26)

for constructing the $A_\infty$ algebra $(\mathcal{H}_l^o, \omega_l^o, Q - \eta + A)$ and extend the $A_\infty$ relations in Eqs. (2.23)

$$I(s,t) \equiv [Q, A(s,t)] + \frac{1}{2}[A(s,t), A(s,t)]_{01(s)} = 0, \quad \text{(2.27a)}$$

$$J(s,t) \equiv [\eta, A(s,t)] - \frac{1}{2}[A(s,t), A(s,t)]_{02(t)} = 0, \quad \text{(2.27b)}$$

Note that the cyclic Ramond number has the upper bound $p + 2 \geq r$. We consider $A^{(p)}_{p+r+1} |^{2r} \equiv 0$ against the outside of this region.
by introducing parameters \( s \) and \( t \) counting the picture number deficit and the picture number, respectively. Here, in Eqs. \((2.27)\), \( \mathcal{O}_1(s) = \pi^0 + s\pi^1 \), \( \mathcal{O}_2(t) = t\pi^1 \) and the bracket with subscript \([\cdot,\cdot]_\mathcal{O}\) is another simple notation for \([\cdot,\cdot]^{1,2}_\mathcal{O}\) and is defined by inserting the operator \( \mathcal{O} \) into the intermediate state after taking (graded) commutation relation,

\[
\pi_1[\mathcal{D}, \mathcal{D}']_\mathcal{O} = \sum_n \pi_1\left(D_n(\mathcal{O}\pi_1 \mathcal{D}' \wedge \mathbb{I}_{n-1}) - (-1)^{DD'+\mathcal{O}(D+D')}D'_n(\mathcal{O}\pi_1 \mathcal{D} \wedge \mathbb{I}_{n-1})\right). \tag{2.28}
\]

At \((s, t) = (0, 1)\), the generating function in Eq. \((2.26)\) and the relations in Eqs. \((2.27)\) reduce to \( A(0, 1) = \mathcal{A} \) and the \( A_\infty \) relations in Eqs. \((2.23)\), respectively.

Then, we can show that if \( \mathcal{A}(s, t) \) satisfies the differential equations

\[
\begin{align*}
\partial_t \mathcal{A}(s, t) &= [\mathcal{Q}, \mu(s, t)] + [\mathcal{A}(s, t), \mu(s, t)]_{\mathcal{O}_1(s)} \tag{2.29a} \\
\partial_s \mathcal{A}(s, t) &= [\eta, \mu(s, t)] - [\mathcal{A}(s, t), \mu(s, t)]_{\mathcal{O}_2(s)} \tag{2.29b}
\end{align*}
\]

with introducing the degree even coderivation

\[
\mu(s, t) = \sum_{p, m, r=0}^{\infty} s^m t^p \mu^{(p+1)}(s, t) \equiv \sum_{p=0}^{\infty} t^p \mu^{(p+1)}(s), \tag{2.30}
\]

the \( t \) derivative of the left hand sides of the relations in Eqs. \((2.27)\) become

\[
\begin{align*}
\partial_t I(s, t) &= [I(s, t), \mu(s, t)]_{\mathcal{O}_1(s)}, \tag{2.31} \\
\partial_t J(s, t) &= [J(s, t), \mu(s, t)]_{\mathcal{O}_1(s)} - [I(s, t), \mu(s, t)]_{\mathcal{O}_2(s)} - \partial_s I(s, t). \tag{2.32}
\end{align*}
\]

Thus, if

\[
\begin{align*}
I(s, 0) &= [\mathcal{Q}, \mathcal{A}(s, 0)] + \frac{1}{2} [\mathcal{A}(s, 0), \mathcal{A}(s, 0)]_{\mathcal{O}_1(s)} = 0, \tag{2.33a} \\
J(s, 0) &= [\eta, \mathcal{A}(s, 0)] = 0, \tag{2.33b}
\end{align*}
\]

then \( I(s, t) = J(s, t) = 0 \). However, the relations in Eqs. \((2.33)\) are nothing less than those satisfied by the geometric string products constructed similarly to those for the bosonic \( A_\infty \) algebra, \( \mathcal{Q} + M_B(s) \), without any insertion:

\[
M_B(s) \equiv \sum_{m, r=0}^{\infty} s^m (M_B)_{m+r+1} t^r. \tag{2.34}
\]

We can obtain the cyclic \( A_\infty \) algebra, \( (\mathcal{H}_q^r, \omega^r_q, \mathcal{Q} - \eta + \mathcal{A}) \) by recursively solving the differential equations in Eqs. \((2.29)\), or equivalently,

\[
(p + 1)\mathcal{A}^{(p+1)}(s) = [\mathcal{Q}, \mu^{(p+1)}(s)] + \sum_{q=0}^{p} [\mathcal{A}^{(p-q)}(s), \mu^{(q+1)}(s)]_{\mathcal{O}_1(s)}, \tag{2.35a}
\]

\[
[\eta, \mu^{(p+1)}(s)] = \partial_s \mathcal{A}^{(p)}(s) + \sum_{q=0}^{p-1} [\mathcal{A}^{(p-q-1)}(s), \mu^{(q+1)}(s)]^2. \tag{2.35b}
\]
The second equation at $p = 0$ can be solved for $\mu^{(1)}(s)$ as

$$\mu^{(1)}(s) = \xi_0^0 \circ \partial_s M_B(s) \quad (2.36)$$

under the initial condition $A(s, 0) = A^{(0)}(s) = M_B(s)$, where $\xi_0^0$ is the operation defined on general coderivation $A = \sum_{n=0}^{\infty} A_{n+2}$ by

$$\xi_0^0 \circ A = \sum_{n,k,l=0}^{\infty} (\mathbb{I}^k \otimes (\xi_0^0 \circ A_{n+2}) \otimes \mathbb{I}^l) \pi_{k+l+n+2}^0, \quad (2.37a)$$

$$\xi_0^0 \circ A_{n+2} = \frac{1}{n+3} \left( \xi_0^0 A_{n+2} - (-1)^{\deg(A)} \sum_{m=0}^{n+1} A_{n+2} \left( \mathbb{I}^{(n-m+1)} \otimes \xi_0^0 \otimes \mathbb{I}^m \right) \right). \quad (2.37b)$$

Substituting (2.36) into (2.35a) at $p = 0$, we obtain $A^{(1)}(s)$. Repeating the procedure, we can recursively obtain $\mu^{(p+1)}(s)$ and $A^{(n+1)}(s)$ from Eqs. (2.35b) and (2.35a), respectively. Then finally, the cohomomorphism in Eq. (2.24) gives the cyclic $A_\infty$ algebra $(\mathcal{H}_{o}^{res}, \Omega^o, M)$.  

3 Closed superstring field theory with $L_\infty$-structure

Similarly to the open superstring field theory, closed (type II) superstring field theory is constructed based on the $L_\infty$ algebra structure. We next summarize it in this section.

3.1 Closed superstring field

The first-quantized Hilbert space, $\mathcal{H}_c$, of type II (closed) superstring is composed of four sectors: $\mathcal{H}_c = \mathcal{H}_{NS-NS} + \mathcal{H}_{R-NS} + \mathcal{H}_{NS-R} + \mathcal{H}_{R-R}$. Correspondingly, the type II superstring field $\Phi$ has four components, $\Phi = \Phi_{NS-NS} + \Phi_{R-NS} + \Phi_{NS-R} + \Phi_{R-R}$, all of which are Grassmann even and have ghost number 2. The components $\Phi_{NS-NS}$ and $\Phi_{R-R}$ have picture numbers $(-1, -1)$ and $(-1/2, -1/2)$, respectively and represent space-time bosons. The components $\Phi_{R-NS}$ and $\Phi_{NS-R}$ have picture numbers $(-1/2, -1)$ and $(-1, -1/2)$ and represent space-time fermions. We impose it closed string constraints

$$b_0^{-1} \Phi = L_0 \Phi = 0, \quad (3.1)$$

and also an extra constraint

$$\mathcal{P}_{XY}^c \Phi = \Phi, \quad \mathcal{P}_{XY}^c = \mathcal{G}^c(\mathcal{G}^c)^{-1}, \quad (3.2)$$

Note that the part of $A^{(p+1)}(s)$ with a fixed number of inputs contains only a finite number of terms. We can calculate any $A_{p+r+1}^{(p)} |_{2r}$ with a finite procedure.
with

\[ G^c = \pi^{(0,0)} + X^c \pi^{(1,0)} + X^c \pi^{(0,1)} + X^c Y^c \pi^{(1,1)} , \]  
\[ (G^c)^{-1} = \pi^{(0,0)} + Y^c \pi^{(1,0)} + Y^c \pi^{(0,1)} + Y^c Y^c \pi^{(1,1)} , \]

(3.3)  
(3.4)

where \( \pi^{(0,0)} \), \( \pi^{(1,0)} \), \( \pi^{(0,1)} \), and \( \pi^{(1,1)} \), are the projection operators onto the NS-NS, R-NS, NS-R, and R-R components respectively: \( \pi^{(0,0)} = \Phi_{NS-NS} \), \( \pi^{(1,0)} = \Phi_{R-NS} \), \( \pi^{(0,1)} = \Phi_{NS-R} \), and \( \pi^{(1,1)} = \Phi_{R-R} \). The PCO \( X^c \) (\( \tilde{X}^c \)) and its inverse \( Y^c \) (\( \bar{Y}^c \)) are defined by

\[ X^c = -\delta(\beta_0) G + \frac{1}{2}(\gamma_0 \delta(\beta_0) + \delta(\beta_0) \gamma_0) b_0^+ , \quad Y^c = -2 \frac{C}{L_0} \delta(\gamma_0) , \]  
(3.5)  
\[ X^c = -\delta(\bar{\beta}_0) G + \frac{1}{2}(\bar{\gamma}_0 \delta(\bar{\beta}_0) + \delta(\bar{\beta}_0) \bar{\gamma}_0) \bar{b}_0^+ , \quad \bar{Y}^c = -2 \frac{\bar{C}}{L_0} \delta(\bar{\gamma}_0) . \]  
(3.6)

The PCOs \( X^c \) and \( \tilde{X}^c \) are BRST exact in the large Hilbert space:

\[ \Xi^c = \xi_0 + (\Theta(\beta_0) \eta \xi_0 - \xi_0) P_{-3/2} + (\xi_0 \eta \Theta(\beta_0) - \xi_0) P_{-1/2} , \]  
(3.7)  
\[ \tilde{\Xi}^c = \bar{\xi}_0 + (\Theta(\bar{\beta}_0) \bar{\eta} \bar{\xi}_0 - \bar{\xi}_0) \tilde{P}_{-3/2} + (\bar{\xi}_0 \bar{\eta} \Theta(\bar{\beta}_0) - \bar{\xi}_0) \tilde{P}_{-1/2} . \]  
(3.8)

where \( P_{-3/2} \) and \( P_{-1/2} \) (\( \tilde{P}_{-3/2} \) and \( \tilde{P}_{-1/2} \)) are the projectors onto the states with the left-moving (right-moving) picture numbers \(-3/2\) and \(-1/2\), respectively. We denote the restricted Hilbert space of type II superstring as \( \mathcal{H}^c_{res} \). Similarly to the relations in Eq. (2.5) for the open superstring, \( G^c \) and \( (G^c)^{-1} \) satisfy the relations

\[ G^c (G^c)^{-1} = G^c , \quad (G^c)^{-1} G^c (G^c)^{-1} = (G^c)^{-1} , \quad [Q, G^c] = 0 , \]  
(3.9)

and thus, \( G^c (G^c)^{-1} \) is a projector that is compatible with the BRST cohomology: \( Q\mathcal{P}^c_{XY} = \mathcal{P}^c_{XY} Q\mathcal{P}^c_{XY} \). The type II superstring field satisfying the constraint in Eq. (3.2) is expanded in the ghost zero-modes as

\[ \Phi = (\phi_{NS-NS} - c_0^+ \psi_{NS-NS}) + \left( \phi_{R-R} - \frac{1}{2} (\gamma_0 \bar{G} - \bar{\gamma}_0 G + 2c_0^+ G \bar{G}) \psi_{R-R} \right) \]  
\[ + \left( \phi_{R-NS} - \frac{1}{2} (\gamma_0 + 2c_0^+ G) \psi_{R-NS} \right) + \left( \phi_{NS-R} - \frac{1}{2} (\bar{\gamma}_0 + 2c_0^+ \bar{G}) \psi_{NS-R} \right) \in \mathcal{H}^c_{res} . \]  
(3.10)

Natural symplectic forms \( \omega_s^c \) and \( \Omega^c \) in \( \mathcal{H}_c \) and \( \mathcal{H}^c_{res} \), respectively, are defined by using the BPZ inner product as

\[ \omega_s^c (\Phi_1, \Phi_2) = (-1)^{\Phi_1} \langle \Phi_1 | c_0^+ | \Phi_2 \rangle , \]  
(3.11)  
\[ \Omega^c (\Phi_1, \Phi_2) = (-1)^{\Phi_1} \langle \Phi_1 | \tilde{G} (G^c)^{-1} | \Phi_2 \rangle . \]  
(3.12)

Natural symplectic form \( \omega_s^c \) in the large Hilbert space \( \mathcal{H}_f \) is similarly defined by using the BPZ inner product in \( \mathcal{H}_c \), and related to \( \omega_s^c \) as \( \omega_s^c (\xi_0 \bar{c}_0 \Phi_1, \Phi_2) = \omega_s^c (\Phi_1, \Phi_2) \) if \( \Phi_1, \Phi_2 \in \mathcal{H}_c^c \).
3.2 Interaction with $L_\infty$-structure

Type II superstring interactions are described by the string products $L_n$ that map $n$ closed superstring fields to a closed superstring field as

$$L_n : (\mathcal{H}^{res}_c)^n \rightarrow \mathcal{H}^{res}_c, \quad (n \geq 1),$$

$$\Phi_1 \wedge \cdots \wedge \Phi_n \mapsto L_n(\Phi_1, \cdots, \Phi_n),$$

where $\Phi_1 \wedge \cdots \wedge \Phi_n$ is the symmetrized tensor product defined by

$$\Phi_1 \wedge \cdots \wedge \Phi_n = \sum_\sigma \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(n)},$$

We identify the one-string product as the closed superstring BRST operator: $L_1 = Q_c$. By definition, these products must satisfy

$$b_0 L_n(\Phi_1, \cdots, \Phi_n) = L_0^{-1} L_n(\Phi_1, \cdots, \Phi_n) = 0,$$

$$\mathcal{P}_{XY} L_n(\Phi_1, \cdots, \Phi_n) = L_n(\Phi_1, \cdots, \Phi_n).$$

We further impose the $L_\infty$ relations

$$\sum_{\sigma} \sum_{m=1}^{n} (-1)^{\ell(\sigma)} \frac{1}{m!(n-m)!} L_{n-m+1}(L_m(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}) = 0,$$

and cyclicity

$$\Omega^c(\Phi_1, L_n(\Phi_2, \cdots, \Phi_{n+1})) = -(-1)^{|\Phi_1|} \Omega^c(L_n(\Phi_1, \cdots, \Phi_n), \Phi_{n+1}).$$

The linear maps satisfying Eqs. (3.17) and (3.18) form the cyclic $L_\infty$ algebra $(\mathcal{H}^{res}_c, \Omega^c, \{L_m\})$.

The linear maps in Eq. (3.13) are also represented by a degree-odd coderivation $L = \sum_{n=1}^{\infty} L_n$ acting on the symmetrized tensor algebra $S\mathcal{H}_c = \sum_{n=0}^{\infty} (\mathcal{H}^{res}_c)^\wedge n$ as

$$L = \sum_{n=1}^{\infty} L_n = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (L_n \wedge \mathbb{I}_m) \pi^c_{n+m}. $$

with $\mathbb{I}_m = \frac{1}{m^c} \mathbb{I}^\wedge m = \mathbb{I}^{\circ m}$, where $\pi^c_m$ is the projection operator onto $(\mathcal{H}^{res}_c)^\wedge m \subset S\mathcal{H}_c$.

Then, the $L_\infty$ relations in Eq. (3.17) is written as

$$[L, L] = 0.$$"
where we used the diagonal matrix representation $L_{n,m} = \delta_{n,m} L_n$. The superscript $p$ ($\bar{p}$) is the left-moving (right-moving) picture number that the map itself has, and the subscript $2r$ ($2\bar{r}$) is the left-moving (right-moving) Ramond number.

Introducing a real parameter $t \in [0, 1]$, the action with $L_\infty$ structure is given by

$$I_c = \int_0^1 dt \Omega^c (\Phi, \pi_1^c L(e^{\wedge \Phi})), \quad (3.22)$$

with the group-like element

$$e^{\wedge \Phi} = I_{\mathcal{SH}c} + \sum_{n=1}^\infty \frac{1}{n!} \Phi^{\wedge n}, \quad (3.23)$$

where $I_{\mathcal{SH}c}$ is the identity in $\mathcal{SH}c$ that satisfies $I_{\mathcal{SH}c} \wedge V = V$ for $\forall V \in \mathcal{SH}c$. The arbitrary variation of $I_c$ is given by

$$\delta I_c = \Omega^c (\delta \Phi, \pi_1^c L(e^{\wedge \Phi})). \quad (3.24)$$

We can show the action in Eq. (3.22) is invariant under the gauge transformation

$$\delta \Lambda \Phi = \pi_1^c L(e^{\wedge \Phi} \wedge \Lambda), \quad (3.25)$$

using the $L_\infty$ relation in Eq. (3.17) and cyclicity in Eq. (3.18):

$$\delta \Lambda I_c = \Omega^c (\Lambda, \pi_1^c L(e^{\wedge \Phi} \wedge \Lambda)), \quad (3.26)$$

3.3 Explicit construction of interactions

The cyclic $L_\infty$ algebra $(\mathcal{H}_{c}^{res}, \Omega^c, L)$ is constructed in two steps. We consider first an $L_\infty$ algebra $(\mathcal{H}_l, O)$ with

$$\pi_1 O = \pi_1 (Q - \eta - \bar{\eta} + B) - \left(1 - \frac{1}{2} (X + \bar{X})\right) \pi_1^{(1,1)} B, \quad (3.27)$$

introducing the degree odd coderivation

$$B = \sum_{p,r=0}^\infty B^{(p,\bar{p})}_{p+r+1,\bar{p}+\bar{r}+1} |^{(2r,2\bar{r})}. \quad (3.28)$$

This $L_\infty$ algebra is equivalent to three mutually commutative $L_\infty$ algebras $(\mathcal{H}_l, D)$, $(\mathcal{H}_l, C)$, and $(\mathcal{H}_l, \bar{C})$ with

$$\pi_1 D = \pi_1 Q + \pi_1^{(0,0)} B, \quad (3.29)$$

$$\pi_1 C = \pi_1 \eta - \left(\pi_1^{(1,0)} + \frac{1}{2} X \pi_1^{(1,1)}\right) B, \quad \pi_1 \bar{C} = \pi_1 \bar{\eta} - \left(\pi_1^{(0,1)} + \frac{1}{2} X \pi_1^{(1,1)}\right) B. \quad (3.30)$$
decomposed according to the picture number deficit. Then, the $L_\infty$ relations are written as

\begin{align}
[Q, B] + \frac{1}{2} [B, B]^{11} &= 0, \quad (3.31a) \\
[\eta, B] - \frac{1}{2} [B, B]^{21} - \frac{1}{4} [B, B]_{\bar{X}}^{22} &= 0, \quad (3.31b) \\
[\bar{\eta}, B] - \frac{1}{2} [B, B]^{12} - \frac{1}{4} [B, B]_{\bar{X}}^{22} &= 0. \quad (3.31c)
\end{align}

Here, the bracket $[\cdot, \cdot]^{11,21,12,22}$ is defined by projecting the intermediate state of the (graded) commutator to the NS-NS, R-NS, NS-R, or R-R state. We also define the bracket $[\cdot, \cdot]^{22}_{\bar{X} \text{ or } \bar{X}}$ by further inserting $X$ or $\bar{X}$ at the intermediate R-R state. If such $L_\infty$ algebras are found, we transform them by cohomomorphism

$$\pi_1 \hat{\mathcal{F}}^{-1} = \pi_1 \mathcal{I} - \left( \Xi_{\pi_1}^{(1,0)} + \Xi_{\pi_1}^{(0,1)} + \frac{1}{2} (\Xi \bar{X} + X \bar{\Xi}) \right) B$$

(3.32)

to the cyclic $L_\infty$ algebra ($\mathcal{H}_c^{\text{res}}, \Omega^c, L$) and two (trivial) $L_\infty$ algebras ($\mathcal{H}_l^c, \eta$) and ($\mathcal{H}_r^c, \bar{\eta}$) as

$$\pi_1 \hat{\mathcal{F}}^{-1} \mathcal{D} \hat{\mathcal{F}} = \pi_1 Q + \mathcal{G}^* \pi_1 B \hat{\mathcal{F}} \equiv \pi_1 L,$$

$$\pi_1 \hat{\mathcal{F}}^{-1} \mathcal{C} \hat{\mathcal{F}} = \pi_1 \eta, \quad \pi_1 \hat{\mathcal{F}}^{-1} \bar{\mathcal{C}} \hat{\mathcal{F}} = \pi_1 \bar{\eta}.$$  

Note that the $L_\infty$ algebra ($\mathcal{H}_l^c, \mathcal{O}$) is not cyclic with respect to $\omega^c_l$ unlike the open superstring case. However, we can show, in a similar way given in the Appendix C of Ref. [18], that the $L$ in Eq. (3.33) is cyclic with respect to $\Omega^c$ if $B$ is cyclic with respect to $\omega^c_l$.

In the next step, we consider a generating function

$$B(s, \bar{s}, t) = \sum_{m,p,r=0}^{\infty} \sum_{\bar{m},\bar{p},\bar{r}=0}^{\infty} \tilde{s}^m s^p \bar{t}^{\bar{r}} \bar{t}^{\bar{p}} B_{m+p+r+1,\bar{m}+\bar{p}+\bar{r}+1}^{(p,\bar{p})} (2r,2\bar{r})$$

(3.35)

and extend the $L_\infty$ relations in Eq. (3.31) to

\begin{align}
I(s, \bar{s}, t) &\equiv [Q, B(s, \bar{s}, t)] + \frac{1}{2} [B(s, \bar{s}, t), B(s, \bar{s}, t)]_{c_1(s, \bar{s}, t)} = 0, \quad (3.36a) \\
J(s, \bar{s}, t) &\equiv [\eta, B(s, \bar{s}, t)] - \frac{1}{2} [B(s, \bar{s}, t), B(s, \bar{s}, t)]_{c_2(t)} = 0, \quad (3.36b) \\
\bar{J}(s, \bar{s}, t) &\equiv [\bar{\eta}, B(s, \bar{s}, t)] - \frac{1}{2} [B(s, \bar{s}, t), B(s, \bar{s}, t)]_{\bar{c}_2(t)} = 0. \quad (3.36c)
\end{align}

for constructing the $L_\infty$ algebra ($\mathcal{H}_l^c, \mathcal{O}$). The parameters $s$, $\bar{s}$, and $t$ counting the left-moving picture number deficit, right-moving picture number deficit, and the total picture number, respectively. The bracket with subscript is defined by inserting

\begin{align}
c_1(s, \bar{s}, t) &= \pi^{(0,0)} + s \pi^{(1,0)} + \bar{s} \pi^{(0,1)} + (s \bar{s} + t (s \bar{X} + \bar{s} X)) \pi^{(1,1)}, \quad (3.37) \\
c_2(t) &= t \pi^{(1,0)} + \frac{t^2}{2} \bar{X} \pi^{(1,1)}, \quad \bar{c}_2(t) = t \pi^{(0,1)} + \frac{t^2}{2} X \pi^{(1,1)}, \quad (3.38)
\end{align}
at the intermediate state. At $s, \bar{s}, t = (0, 0, 1)$, the generating function in Eq. (3.35) and the relations in Eq. (3.36) reduce to $B(0, 0, 1) = B$ and the $L_\infty$ relations in Eqs. (3.31), respectively.

We can show that if $B(s, \bar{s}, t)$ satisfies the differential equations

\[
\partial_\bar{s} B(s, \bar{s}, t) = [Q, (\lambda + \bar{\lambda})(s, \bar{s}, t)] + \frac{1}{2} [B(s, \bar{s}, t), B(s, \bar{s}, t)]_{\partial(s, \bar{s})} \tag{3.39a}
\]

\[
\partial_s B(s, \bar{s}, t) = [\eta, \lambda(s, \bar{s}, t)] - [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} \tag{3.39b}
\]

\[
\partial_t B(s, \bar{s}, t) = [\bar{\eta}, \bar{\lambda}(s, \bar{s}, t)] - [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} \tag{3.39c}
\]

and $[\bar{\eta}, \lambda(s, \bar{s})] = [\eta, \bar{\lambda}(s, \bar{s})] = 0$ with $\partial(s, \bar{s}) = (\bar{s} \Xi + s \bar{\Xi})^{(1,1)}$ and the degree even coderivations

\[
\lambda(s, \bar{s}, t) = \sum_{m, p, r=0}^{\infty} \sum_{m, \bar{p}, \bar{r}=0}^{\infty} s^m \bar{s}^{\bar{m}} p^p \bar{p}^{\bar{p}} \lambda_{m+p+r+2, m+\bar{p}+\bar{r}+1}^{(2r, 2\bar{r})} \tag{3.40}
\]

\[
\bar{\lambda}(s, \bar{s}, t) = \sum_{m, p, r=0}^{\infty} \sum_{m, \bar{p}, \bar{r}=0}^{\infty} s^m \bar{s}^{\bar{m}} p^p \bar{p}^{\bar{p}} \bar{\lambda}_{m+p+r+1, m+\bar{p}+\bar{r}+2}^{(2r, 2\bar{r})} \tag{3.41}
\]

the $t$ derivative of the left hand sides of the relations in Eqs. (3.36) become

\[
\partial_t I(s, \bar{s}, t) = [I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} + [I(s, \bar{s}, t), B(s, \bar{s}, t)]_{\partial(s, \bar{s})} \tag{3.42a}
\]

\[
\partial_t J(s, \bar{s}, t) = [J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} + [J(s, \bar{s}, t), B(s, \bar{s}, t)]_{\partial(s, \bar{s})} - \partial_s I(s, \bar{s}, t) - [I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} \tag{3.42b}
\]

\[
\partial_t \bar{J}(s, \bar{s}, t) = [\bar{J}(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} + [\bar{J}(s, \bar{s}, t), B(s, \bar{s}, t)]_{\partial(s, \bar{s})} - \partial_s \bar{I}(s, \bar{s}, t) - [I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]_{\partial(s, \bar{s})} \tag{3.42c}
\]

They imply if

\[
I(s, \bar{s}, 0) = [Q, B(s, \bar{s}, 0)] + \frac{1}{2} [B(s, \bar{s}, 0), B(s, \bar{s}, 0)]_{\partial(s, \bar{s})} = 0 \tag{3.43a}
\]

\[
J(s, \bar{s}, 0) = [\eta, B(s, \bar{s}, 0)] = 0 \tag{3.43b}
\]

\[
\bar{J}(s, \bar{s}, 0) = [\bar{\eta}, B(s, \bar{s}, 0)] = 0 \tag{3.43c}
\]

then $I(s, \bar{s}, t) = J(s, \bar{s}, t) = \bar{J}(s, \bar{s}, t) = 0$. The relations in Eqs. (3.43) are those satisfied by the geometric string product without any insertion, which can be constructed similarly to that for the bosonic $L_\infty$ algebra, $Q + L_B(s, \bar{s})$:

\[
L_B(s, \bar{s}) \equiv \sum_{m, r=0}^{\infty} \sum_{m', \bar{r}=0}^{\infty} s^m \bar{s}^{\bar{m}} (L_B)_{m+r+1, m'+\bar{r}+1} |^{(2r, 2\bar{r})}. \tag{3.44}
\]

Similar to the open-superstring case in the previous section, we can obtain the $L_\infty$ algebra $(\mathcal{H}_s^t, \mathcal{O})$ by solving the differential equations in Eqs. (3.39) with the initial condition

\[
B(s, \bar{s}, 0) = L_B(s, \bar{s}) \tag{3.45}
\]
The concrete procedures are slightly complicated, so Appendix A shows the lower-order results. The cohomomorphism in Eq. (3.32) gives the cyclic $L_\infty$ algebra $(\mathcal{H}^{res}_c, \Omega^c, L)$ if we choose the solution $B$ to be cyclic with respect to $\omega^c_l$.

4 Open-closed superstring field theory with OCHA-structure

Now, we are ready to discuss OCHA, the main subject of this paper. In this section, we first see what OCHA is and how it is realized in the superstring field theory and then give a prescription to construct them explicitly.

4.1 Interaction with OCHA structure

We define classical interactions among the open and closed (type II) superstrings mixed system by the vertices described by the following two kinds of surfaces:

- A sphere with $n \geq 3$ closed-superstring punctures;
- A disk with $n \geq 0$ closed-superstring punctures on the bulk and $l + 1 \geq 1$ open-superstring punctures on the boundary with $n + l \geq 1$.

We can identify the former as the linear maps $\{L_n\}$ given in the previous section, which form the cyclic $L_\infty$ algebra $(\mathcal{H}^{res}_c, \Omega^c, \{L_n\})$. The latter includes both the open-superstring interactions ($n = 0$) and interactions between open and closed superstrings ($n > 0$) and is described by the string products $N_{n,l}$ that maps $n$ closed-superstring fields and $l$ open-superstring fields to an open-superstring field:

$$N_{n,l} : (\mathcal{H}^{res}_c)^n \otimes (\mathcal{H}^{res}_o)^l \longrightarrow \mathcal{H}^{res}_o, \quad (n, l \geq 0, n + l > 0),$$

with the identification $N_{0,l} = M_l$. By definition, the condition

$$\mathcal{P}_{XY} N_{n,l}(\Phi_1, \cdots, \Phi_n; \Psi_1, \cdots, \Psi_l) = N_{n,l}(\Phi_1, \cdots, \Phi_n; \Psi_1, \cdots, \Psi_l)$$

holds. The linear maps $\{L_n, N_{n,l}\}$ satisfying the OCHA relation

$$0 = \sum_{\sigma} \sum_{m=1}^{n} (-1)^{\epsilon(\sigma)} \frac{1}{m! (n - m)!} N_{n-m+1,l}(L_m(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}; \Psi_1, \cdots, \Psi_l)$$

$$+ \sum_{\sigma} \sum_{m=0}^{n} \sum_{j=0}^{l-l} \sum_{i=0}^{n} (-1)^{\mu_{m,i}(\sigma)} \frac{1}{m! (n - m)!} N_{m,l-j+1}(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}; \psi_1, \cdots, \psi_i, \psi_{i+1}, \cdots, \psi_{i+j}, \psi_{i+j+1}, \cdots, \psi_l),$$

(4.1)
and the cyclicity condition
\[
\Omega^p(\Psi_1, N_{n,l}(\Phi_1, \cdots, \Phi_n; \Psi_2, \cdots, \Psi_{l+1})) = -(-1)^{\deg(\Psi_1)}(\Phi_1 + \cdots + \Phi_l)^{+1} \Omega^o(N_{n,l}(\Phi_1, \cdots, \Phi_n; \Psi_1, \cdots, \Psi_l), \Psi_{l+1})
\] (4.4)
form the cyclic OCHA \((\mathcal{H}_c \oplus \mathcal{H}_o, \Omega^o, \{L_n, N_{n,l}\})\). Here, the sign factor \(\mu_{m,i}(\sigma)\) in Eq. (4.3) is given by
\[
\mu_{m,i}(\sigma) = \epsilon(\sigma) + \sum_{j=1}^m |\Phi_{\sigma(j)}| + \sum_{j=1}^i |\Psi_j|(1 + \sum_{k=m+1}^{n} |\Phi_{\sigma(k)}|).
\] (4.5)
Note that the OCHA relation in Eq. (4.3) includes the \(A_\infty\) relation in Eq. (2.11) as \(n = 0\), at which the cyclicity condition becomes the one for the open superstring in Eq. (2.12). In other words, the purely open-superstring interactions \(N_{0,l}\) form the cyclic \(A_\infty\) algebra \((\mathcal{H}_o, \Omega^o, \{N_{0,l}\})\).

The linear maps in Eq. (4.1) are also represented by a degree odd coderivation
\[
N = \sum_{n,l=0}^{\infty} N_{n,l}
\] (4.6)
acting on \(S\mathcal{H}_c^{res} \otimes T\mathcal{H}_o^{res}\) as
\[
N = \sum_{n,l=0}^{\infty} N_{n,l} = \sum_{n,l=0}^{\infty} \sum_{m,j,k=0}^{\infty} \left( I_m \otimes (I \otimes N_{n,l} \otimes I \otimes k) \right) \pi_{n+m,j+k+l},
\] (4.7)
where \(\pi_{n,l}\) is the projector onto the subspace \((\mathcal{H}_c^{res})^n \otimes (\mathcal{H}_o^{res})^l\). By extending \(L\) to the coderivation acting on \(S\mathcal{H}_c^{res} \otimes T\mathcal{H}_o^{res}\) as
\[
L = \sum_{n=1}^{\infty} L_n = \sum_{n=1}^{\infty} \sum_{m,l=0}^{\infty} \left( (L_n \wedge I_m) \otimes I \otimes l \right) \pi_{n+m,l},
\] (4.8)
we can consider the coderivation \(L + N\). The OCHA relation in Eq. (4.3) can then be written as
\[
[L, N] + \frac{1}{2}[N, N] = 0,
\] (4.9)
which we can rewrite as
\[
[L + N, L + N] = 0,
\] (4.10)
by combining with the \(L_\infty\) relation \([L, L] = 0\). For open-closed superstring field theory, the string interaction \(N_{n,l}\) must be defined for any combination of four sectors of closed superstring and two sectors of open superstring so that the sum of three kinds (open, left-moving, and right-moving) of picture numbers are conserved:
\[
N = \sum_{p,n,m,r=0}^{\infty} N_{n,l}^{(p)} |2r \delta_{p+r,2n+m-1},
\] (4.11)
where $2r$ is the total Ramond number defined by

\[
\text{total Ramond number} = \\
\# \text{ of } R - NS \text{ inputs} + \# \text{ of } NS - R \text{ input} + 2(\# \text{ of } R - R \text{ inputs}) + \# \text{ of open } R \text{ inputs} - \# \text{ of open } R \text{ output}.
\]

The action with OCHA structure is given by

\[
I_{oc} = \int_0^1 dt \, \Omega^o \left( \Psi, \pi_1 N \left( e^{\wedge \Phi} \otimes \frac{1}{1 - r \Psi} \right) \right),
\]

which describes the open superstring field theory on the closed-superstring background. The open-superstring field $\Psi$ is dynamical and the closed-superstring field $\Phi$ is the background field satisfying the equation of motion

\[
\pi_1 L(e^{\wedge \Phi}) = 0.
\]

The arbitrary variation of $I_{oc}$ is given by

\[
\delta I_{oc} = \Omega^o \left( \delta \Psi, \pi_1 N \left( e^{\wedge \Phi} \otimes \frac{1}{1 - \Psi} \right) \right).
\]

We can show that the action in Eq. (4.12) is invariant under the gauge transformation

\[
\delta_\Lambda \Psi = \pi_1 N \left( e^{\wedge \Phi} \otimes \left( \frac{1}{1 - \Psi} \otimes \Lambda \otimes \frac{1}{1 - \Psi} \right) \right),
\]

using the relation in Eq. (4.9):

\[
\delta_\Lambda I_{oc} = \Omega^o \left( \Lambda, \pi_1 N \left( e^{\wedge \Phi} \otimes \frac{1}{1 - \Psi} \otimes \pi_1 N \left( e^{\wedge \Phi} \otimes \frac{1}{1 - \Psi} \right) \otimes \frac{1}{1 - \Psi} \right) \right) = 0.
\]

The open-closed superstring interaction $N$ deforms by the background closed superstring field $\Phi$ gives a weak $A_\infty$ algebra ($\mathcal{H}_o^{res}, M(\Phi)$) with

\[
M(\Phi) = (N(e^\Phi \otimes I)) \pi^o.
\]

Here, $I$ is the identity map in $T\mathcal{H}_o^{res}$ and $\pi^o$ is the projector onto $T\mathcal{H}_o^{res}$.

---

6We omitted here the terms corresponding to a disk with closed strings in the bulk and no open strings on the boundary, which are included in the action proposed in Ref. [7]. These terms give a constant determined by a closed string background but do not relevant to symmetry structure of the theory [10].
4.2 Explicit construction of interactions

Let us construct a cyclic OCHA \((\mathcal{H}_c^{res} \oplus \mathcal{H}_o^{res}, \Omega^c \oplus \Omega^o, L + N)\). We assume that the cyclic sub-\(L_\infty\)-algebra \(L\) is already constructed in the way given in the previous subsection. Similarly to the previous cases, we can construct \(N\) satisfying the relation in Eq. \((1.9)\) and the cyclicity condition in Eq. \((4.4)\) in the following two steps. First consider a degree odd nilpotent coderivation

\[
\pi_1 \mathcal{O} = \pi_1 (Q - \eta + A + B + C) - \left(1 - \frac{1}{2}(X + \bar{X})\right)\pi_1^{(1,1)} B
\]

(4.18)
satisfying \([\mathcal{O}, \mathcal{O}] = 0\), or equivalently two mutually commutative coderivations

\[
\pi_1 \mathcal{D} = \pi_1 Q + \pi_1^{(0,0)} B + \pi_1^0 (A + C),
\]

(4.19)

\[
\pi_1 \mathcal{C} = \pi_1 \eta - \left(\pi_1^{(1,0)} + \pi_1^{(0,1)} + \frac{1}{2}(X + \bar{X})\pi_1^{(1,1)}\right) B - \pi_1^1 (A + C),
\]

(4.20)
satisfying \([\mathcal{D}, \mathcal{D}] = [\mathcal{C}, \mathcal{C}] = [\mathcal{D}, \mathcal{C}] = 0\), where \(Q\) acts as \(Q_c\) or \(Q_o\) on \(\mathcal{H}_c\) or \(\mathcal{H}_o\), respectively, and similarly \(\eta\) acts as \(\eta + \bar{\eta}\) or \(\eta\) on on \(\mathcal{H}_c\) or \(\mathcal{H}_o\), respectively. Degree odd coderivations \(A\) and \(B\) are those for constructing \(A_\infty\) and \(L_\infty\) algebras in Eqs. \((2.21)\) and \((3.28)\), and \(C\) is the one for constructing open-closed interaction defined by respecting the cyclic Ramond number:

\[
C = \sum_{p,n,l,r=0}^{\infty} \delta_{p+r,2n+l+1} C_{n+1,l}^{(p)} |^{2r}.
\]

(4.21)

The OCHA relations can be written as the \(L_\infty\) relations in Eq. \((3.31)\) and the relations

\[
[Q, C] + [A, C]^1 + \frac{1}{2} [C, C]^1 + [B, C]^11 = 0,
\]

(4.22a)

\[
[\eta, C] - [A, C]^2 - \frac{1}{2} [C, C]^2 - [B, C]^21 - [B, C]^{22}X + \bar{X} = 0.
\]

(4.22b)

If we find such \(A\), \(B\), and \(C\), the cohomomorphism

\[
\pi_1 \hat{\mathcal{F}}^{-1} = \pi_1 I - \left(\Xi^c \pi_1^{(1,0)} + \Xi^c \pi_1^{(0,1)} + \frac{1}{2}(\Xi^c X^c + \Xi^c \bar{X}^c)\pi_1^{(1,1)}\right) B - \Xi^o \pi_1^1 (A + C)
\]

(4.23)
transforms \(\mathcal{D}\) and \(\mathcal{C}\) to the ones we eventually construct as

\[
\pi_1 \hat{\mathcal{F}}^{-1} \mathcal{D} \hat{\mathcal{F}} = \pi_1 Q + G^c \pi_1 B \hat{\mathcal{F}} + G^0 \pi_1 (A + C) \hat{\mathcal{F}} = \pi_1 (L + N),
\]

(4.24)

\[
\pi_1 \hat{\mathcal{F}}^{-1} \mathcal{C} \hat{\mathcal{F}} = \eta.
\]

(4.25)

We can construct \(C\) similarly to \(A\) and \(B\), which we already find. By introducing parameters \(s\) and \(t\), we consider a generating functions in Eq. \((2.26)\), and Eq. \((3.35)\), and

\[
C(s, t) = \sum_{p,n,l,r=0}^{\infty} \delta_{m+p+r,2n+l+1} s^m t^p C_{n+1,l}^{(p)} \equiv \sum_{p=0}^{\infty} t^p C^{(p)}(s),
\]

(4.26)
and extend the relations in Eqs. (4.22) to

\[
I(s, t) \equiv [Q, C(s, t)] + [A(s, t), C(s, t)]_{o_1(s)} + \frac{1}{2}[C(s, t), C(s, t)]_{o_1(s)} + [B(s, s, t), C(s, t)]_{c_1(s, s, t)} = 0, \tag{4.27a}
\]

\[
J(s, t) \equiv [\eta, C(s, t)] - [A(s, t), C(s, t)]_{o_2(t)} - \frac{1}{2}[C(s, t), C(s, t)]_{o_2(t)} - [B(s, s, t), C(s, t)]_{c_2(s) + c_2(s)} = 0. \tag{4.27b}
\]

We can show that if \(C(s, t)\) satisfy

\[
\partial_t C(s, t) = [Q, \nu(s, t)]
\]

\[
+ [A(s, t), \nu(s, t)]_{o_1(s)} + [C(s, t), \mu(s, t)]_{o_1(s)} + [C(s, t), \nu(s, t)]_{o_1(s)}
\]

\[
+ [B(s, s, t), \nu(s, t)]_{c_1(s, s, t)} + [C(s, t), (\lambda + \bar{\lambda})(s, s, t)]_{c_1(s, s, t)}
\]

\[
+ [B(s, s, t), C(s, t)]_{b(s, s)}, \tag{4.28a}
\]

\[
\partial_s C(s, t) = [\eta, \nu(s, t)]
\]

\[
- [A(s, t), \nu(s, t)]_{o_2(t)} - [C(s, t), \mu(s, t)]_{o_2(t)} - [C(s, t), \nu(s, t)]_{o_2(t)}
\]

\[
- [B(s, s, t), \nu(s, t)]_{c_2(s) + c_2(s)} - [C(s, t), (\lambda + \bar{\lambda})(s, s, t)]_{c_2(s) + c_2(t)} \tag{4.28b}
\]

with degree even coderivation

\[
\nu(s, t) = \sum_{p, n, l, r, m=0}^\infty \delta_{p+m+r, 2n+l} s^m t^n \nu^{(p+1)}_{n+l, m} |2^r \equiv \sum_{p=0}^\infty t^p \nu^{(p+1)}(s), \tag{4.29}
\]

then, the \(t\) derivative of the left hand sides of (4.27) become

\[
\partial_t I(s, t) = [I(s, t), (\mu(s, t) + \nu(s, t))]_{o_1(s)} + [I(s, t), (\lambda + \bar{\lambda})(s, s, t)]_{c_1(s, s, t)}
\]

\[
+ [I(s, t), B(s, s, t)]_{b(s, s)}, \tag{4.30a}
\]

\[
\partial_t J(s, t) = [J(s, t), (\mu(s, t) + \nu(s, t))]_{o_1(s)} + [J(s, t), (\lambda + \bar{\lambda})(s, s, t)]_{c_1(s, s, t)}
\]

\[
+ [J(s, t), B(s, s, t)]_{b(s, s)} - \partial_t I(s, t) - [I(s, t), (\mu(s, t) + \nu(s, t))]_{o_2(t)}, \tag{4.30b}
\]

by using the differential equations (2.29) and

\[
\partial_t B(s, s, t) = [Q, (\lambda + \bar{\lambda})(s, s, t)]
\]

\[
+ [B(s, s, t), (\lambda + \bar{\lambda})(s, s, t)]_{c_1(s, s, t)} + \frac{1}{2}[B(s, s, t), B(s, s, t)]_{b(s, s)}, \tag{4.31}
\]

\[
\partial_s B(s, s, t) = [\eta + \bar{\eta}, (\lambda + \bar{\lambda})(s, s, t)] - [B(s, s, t), (\lambda + \bar{\lambda})(s, s, t)]_{c_2(s) + c_2(t)}, \tag{4.32}
\]

satisfied by \(A(s, t)\) and \(B(s, s, t)\), respectively. Therefore, if the relations at \(t = 0\)

\[
I(s, 0) = [Q, C(s, 0)] + [A(s, 0), C(s, 0)]_{o_1(s)} + \frac{1}{2}[C(s, 0), C(s, 0)]_{o_1(s)}
\]

\[
+ [B(s, s, 0), C(s, 0)]_{c_1(s, s, 0)} = 0, \tag{4.33a}
\]

\[
J(s, 0) = [\eta, C(s, 0)] = 0, \tag{4.33b}
\]
hold, then $I(s,t) = J(s,t) = 0$ for any $t$. We can easily find that the coderivation $C(s,0)$ satisfying Eq. (4.33) has no picture number and is given by setting

$$C(s,0) = C^{(0)}(s) = N_B(s),$$

(4.34)

with

$$N_B(s) = \sum_{m,n,l,r=0}^{\infty} s^m \delta_{m+r,2n+l+1}(N_B)_{n+1,t} |^{2r},$$

(4.35)

which can constructed similarly to those of the bosonic open-closed string field theory [7]. Therefore, we can obtain $C(s,t)$ satisfying Eq. (4.27) by recursively solving the differential equations in Eqs. (4.28) under the initial condition in Eq. (4.34) to be cyclic with respect to $\omega_o$.

In Appendix A, we give a concrete procedure to solve them for some lower orders. The cyclic OCHA $(\mathcal{H}^{res}, \Omega, L + N)$ is eventually constructed by transforming using cohomomorphism in Eq. (4.23).

5 Mapping to WZW-like action

The WZW-like formulation is the other complementary way to construct superstring field theories using the large Hilbert space [18–20, 26–31]. We can map the action we constructed in the previous section to the WZW-like action as in the open, heterotic, and type II superstring field theories [18–20, 27].

Let us first focus on the NS $\oplus$ NS-NS sector, which we simply call the NS sector in this section. The map between two formulations, the homotopy-based and WZW-like formulations, is given by the cohomomorphism $g = g_c \otimes g_o$ [14, 18, 20] with

$$g_c = \mathcal{P} \exp \left( \int_0^1 dt (\lambda + \bar{\lambda})^{NS}(0,t) \right), \quad g_o = \mathcal{P} \exp \left( \int_0^1 dt \mu^{NS}(0,t) \right),$$

(5.1)

where

$$(\lambda + \bar{\lambda})^{NS}(s,t) = \sum_{m,p=0}^{\infty} \sum_{\bar{m},\bar{p}=0}^{\infty} s^{m+\bar{m}} t^{p+\bar{p}} \left( \lambda^{(p+1,\bar{p})}_{m+p+2,\bar{m}+\bar{p}+2} |^{(0,0)} + \bar{\lambda}^{(p,\bar{p}+1)}_{m+p+1,\bar{m}+\bar{p}+2} |^{(0,0)} \right),$$

(5.2)

$$\mu^{NS}(s,t) = \sum_{m,p=0}^{\infty} s^{m} t^{p} \mu^{(p+1)}_{m+p+2} |^{0}. \quad (5.3)$$

This cohomomorphism maps the string fields $(\Phi_{NS-NS}, \Psi_{NS})$ to those in the WZW-like formulation $(V_o, V_c)$ as

$$\pi_c g_c(e^{\Phi_{NS-NS}}) = G_c(V_c), \quad \pi_o g_o \left( \frac{1}{1 - \Psi_{NS}} \right) = G_o(V_o),$$

(5.4)
where
\[ G_c(V) = \eta\bar{\eta}V + \frac{1}{2} \left( L_2^o(\eta\bar{\eta}V, \bar{\eta}V) + \eta L_2^o(\eta\bar{\eta}V, V) \right) + \cdots, \] (5.5)
is the pure-gauge string fields of type II superstring identically satisfying
\[ L^o_\eta(e^{G_c(V)}) = 0, \quad L^\bar{\eta}_\eta(e^{G_c(V)}) = 0, \] (5.6)
with \((L^o_\eta, L^\bar{\eta}_\eta) = (\hat{g}_o \eta \hat{g}_o^{-1}, \hat{g}_c \eta \hat{g}_c^{-1})\). The pure-gauge string field \(G_o(V_o)\) of the open superstring is similarly defined by a composite string field of \(V_o\) identically satisfying the equation
\[ L^o_\eta \left( \frac{1}{1 - G_o(V_o)} \right) = 0, \] (5.7)
with \(L^o_\eta = \hat{g}_o \eta \hat{g}_o^{-1}\). We give a prescription to obtain explicit form of \(G_o(V_o)\) in Appendix B.
The (dynamical) equation of motion of the open superstring is mapped as
\[ \pi_1 \tilde{N}_{NS} \left( e^{G_c(V_c)} \otimes \frac{1}{1 - G_o(V_o)} \right) = 0, \] (5.8)
with
\[ \tilde{N}_{NS} = \hat{g} N_{NS} \hat{g}^{-1} = Q_o + \hat{g}(N_{NS} - M_{NS}) \hat{g}^{-1}, \] (5.9)
where \(V_c\) is a background field satisfying the equation of motion of the closed-superstring \(Q_c G_c(V_c) = 0\). In order to give the WZW-like action deriving this equation of motion in Eq. (5.8), we define the associated string field as
\[ B_d(V_o) = \pi_1^o \hat{g}_d \xi_d \left( \frac{1}{1 - \Psi_{NS}} \right), \] (5.10)
where \(d = t, \delta\) or \(Q\) and \(\xi_d\) is the coderivation derived from \(\xi \partial_t, \xi \delta\) or \(-\xi \pi_1 M_{NS}\), respectively.
We can show that the relations
\[ dG_o(V_o) = (-1)^d D_\eta B_d(V_o), \] (5.11)
\[ D_\eta (\partial_t B_\delta(V_o) - \delta B_\partial(V_o)) = 0. \] (5.12)
hold\(^7\), where \(D_\eta\) is the nilpotent linear operator defined by
\[ D_\eta \varphi = \pi_1^o L^o_\eta \left( \frac{1}{1 - G_o(V_o)} \otimes \varphi \otimes \frac{1}{1 - G_o(V_o)} \right) \]
\[ = \pi_1^o L^o_\eta \left( e^{G_c(V_c)} \otimes \left( \frac{1}{1 - G_o(V_o)} \otimes \varphi \otimes \frac{1}{1 - G_o(V_o)} \right) \right), \] (5.13)
\(^7\)Note that \(\text{deg}(V_o) = 1\) and \(\text{deg}(B_d) = \text{deg}(d) + 1\).
acting on an open superstring field $\varphi \in \mathcal{H}_{NS}$. The coderivation $L^g$ acts as $L^g_c + L^g_o$ on $\mathcal{H}_{c}$ and as $L^g_o$ on $\mathcal{H}_{o}$. Then, the WZW-like action for the NS sector is given by

$$I_{WZW}^{NS} = \int_0^1 dt \omega^o \left( B_t(V_o), \pi_1 \hat{N} \left( e^{\Lambda_t G_c(V_c)} \otimes \frac{1}{1 - G_o(V_o)} \right) \right),$$

(5.14)

which is invariant under the gauge transformation

$$B_\delta(V_o) = \pi_1 \hat{N} \left( e^{\Lambda_t G_c(V_c)} \otimes \left( \frac{1}{1 - G_o(V_o)} \otimes \Lambda \otimes \frac{1}{1 - G_o(V_o)} \right) \right) + D_\eta \Omega.$$

(5.15)

It is straightforward to extend these results of the NS sector to all the sectors. Since $\hat{g}_c$ and $\hat{g}_o$ act as the identity operators outside the NS sector, we find that

$$\pi_1 \hat{g}_c \left( e^{\Lambda_t \Phi} = \pi_1 \hat{g}_c \left( e^{\Lambda_t \Phi_{NS-NS}} \right) + \Phi_{R-NS} + \Phi_{NS-R} + \Phi_{R-R},$$

(5.16)

$$\pi_1 \hat{g}_o \left( \frac{1}{1 - \Psi} = \pi_1 \hat{g}_o \left( \frac{1}{1 - \Psi_{NS}} \right) + \Psi_R,$$

(5.17)

and can identify the components $(\Phi_{R-NS}, \Phi_{NS-R}, \Phi_{R-R}; \Psi_R)$ to those in the WZW-like formulation $(\Psi_c, \bar{\Psi}_c, \Sigma_c; \Psi_o)$:

$$\Phi_{R-NS} = \Psi_c, \quad \Phi_{NS-R} = \bar{\Psi}_c, \quad \Phi_{R-R} = \Sigma_c, \quad \Psi_R = \Psi_o.$$

(5.18)

Thus, these components are also annihilated by $\eta_c$ and $\bar{\eta}_c$ (or $\eta_o$) and satisfy the constraint in Eq. (3.2) (or Eq. (2.1)). The WZW-like action of the open superstring field theory on the general closed-string backgrounds is eventually written as

$$I_{WZW} = \int_0^1 dt \omega^o \left( B_t(V_o(t)), (\mathcal{G}^o)^{-1} \pi_1 \hat{N} \left( e^{\Lambda_t (G_c(V_c)) + \Psi_c + \bar{\Psi}_c + \Sigma_c} \otimes \frac{1}{1 - G_o(V_o(t)) - \Psi_o(t)} \right) \right),$$

(5.19)

where $\hat{N} = \hat{g} \mathcal{N} \hat{g}^{-1}$ and

$$B_t(V_o(t)) = B_t(V_o(t)) + \xi_0 \partial_t \Psi_o(t).$$

(5.20)

The closed superstring backgrounds $(V_c, \Psi_c, \bar{\Psi}_c, \Sigma_c)$ satisfy

$$\pi_1 \hat{L} \left( e^{\Lambda_t (G_c(V_c)) + \Psi_c + \bar{\Psi}_c + \Sigma_c} \right) = 0$$

(5.21)

with $\hat{L} = \hat{g} L \hat{g}^{-1}$. Note that, since $\hat{g}$ acts as the identity except on the NS sector, $\hat{N}$ and $\hat{L}$ preserve the constraints in Eqs. (2.1) and (3.2), respectively. The WZW-like action in Eq. (5.19) is invariant under the gauge transformation

$$B_\delta(V_o) = \pi_1 \hat{N} \left( e^{\Lambda_t (G_c(V_c)) + \Psi_c + \bar{\Psi}_c + \Sigma_c} \otimes \left( \frac{1}{1 - G_o(V_o(t)) - \Psi_o(t)} \otimes (\Lambda + \xi \lambda) \otimes \frac{1}{1 - G_o(V_o(t)) - \Psi_o(t)} \right) \right),$$

(5.22)

which is also obtained through the map $\hat{g}$. Here, $\Lambda$ and $\lambda$ are the gauge parameters in the NS and R sectors, respectively, and $\lambda$ is annihilated by $\eta$ and satisfies the constraint in Eq. (2.1).
6 Summary and discussion

In this paper, we constructed interactions for the open-closed superstring field theory based on the OCHA structure. It provides the open-closed superstring field theory on general closed-superstring backgrounds. We also give a corresponding WZW-like action for open-closed superstring field theory through a field redefinition.

Recently, the open string field theory deformed with a gauge invariant open-closed coupling is studied [24, 32–35]. The effective open superstring field theory is governed by a weak $A_\infty$ structure which includes non-trivial tadpole term, destabilizing the initial perturbative vacuum. It requires to shift the vacuum to a new equilibrium point. The open-closed superstring field theory, given in this paper, provides a basis for such an analysis on more general closed-superstring backgrounds described by classical solutions of the nonlinear equation of motion of the closed superstring field theory.

In order to quantize the classical superstring field theory, we must extend the classical action to the quantum master action satisfying the quantum BV equation. Such an open-closed superstring field theory is recently given in Ref. [23] based on the formalism using the extra free field [36, 37]. It is interesting to give a quantum master action using the formulation based on the homotopy algebra, which requires to extend the OCHA structure to the quantum OCHA structure [38]. The quantum open-closed superstring field theory is also practically useful to study the string dynamics on the Ramond-Ramond backgrounds [39], the D-brane backgrounds [40–44], and so on, which are difficult in the first-quantized formulation using the RNS formalism. The (quantum) OCHA structure should shed new light on such nonperturbative studies.

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A Explicit procedure for solving (3.39) and (4.28)

Similar to the open superstring case in section 2, the differential equations (3.39) for constructing a $L_\infty$ algebra can be solved recursively with the initial condition in Eq. (3.45). The concrete procedure is, however, complicated due to the fact that the parameter $t$ counts only the total picture number without independently counting the left- and right-moving picture numbers.
We first rewrite the differential equations \([3.39]\) in the form
\[
\sum_{q=0}^{p+1} (p + 1) B^{(p-q+1,q)}(s, \bar{s}) = \sum_{q=0}^{p} [Q, (\lambda^{(p-q+1,q)} + \bar{\lambda}^{(p-q,q+1)})(s, \bar{s})] + \sum_{q=0}^{p} \sum_{l=0}^{q} \sum_{m=0}^{l} [B^{(p-q,q-l)}(s, \bar{s}), (\lambda^{(l-m+1,m)} + \bar{\lambda}^{(l-m,m+1)})(s, \bar{s})]c_0^q(s, \bar{s}) + \sum_{q=0}^{p} \sum_{l=0}^{q} \sum_{m=0}^{l} [B^{(p-q-1,q-l)}(s, \bar{s}), (\lambda^{(l-m+1,m)} + \bar{\lambda}^{(l-m,m+1)})(s, \bar{s})]c_1^q(s, \bar{s}) + \frac{1}{2} \sum_{q=0}^{p} \sum_{l=0}^{q} \sum_{m=0}^{l} [B^{(p-q,q-l)}(s, \bar{s}), B^{(l-m,m)}(s, \bar{s})]\eta(s, \bar{s}),
\]
\[
\sum_{q=0}^{p} [\eta, \lambda^{(p-q+1,q)}(s, \bar{s})] = \sum_{q=0}^{p} \partial_s B^{(p-q,q)}(s, \bar{s}) - \sum_{q=0}^{p} \sum_{l=0}^{q-1} \sum_{m=0}^{l} [B^{(p-q-1,q-l)}(s, \bar{s}), (\lambda^{(l-m+1,m)} + \bar{\lambda}^{(l-m,m+1)})(s, \bar{s})]^{21} - \sum_{q=0}^{p} \sum_{l=0}^{q-2} \sum_{m=0}^{l} [B^{(p-q-2,q-l)}(s, \bar{s}), (\lambda^{(l-m+1,m)} + \bar{\lambda}^{(l-m,m+1)})(s, \bar{s})]^{22}_X,
\]
\[
\sum_{q=0}^{p} [\bar{\eta}, \lambda^{(p-q+1,q)}(s, \bar{s})] = \sum_{q=0}^{p} \partial_{\bar{s}} B^{(p-q,q)}(s, \bar{s}) - \sum_{q=0}^{p} \sum_{l=0}^{q-1} \sum_{m=0}^{l} [B^{(p-q-1,q-l)}(s, \bar{s}), (\lambda^{(l-m+1,m)} + \bar{\lambda}^{(l-m,m+1)})(s, \bar{s})]^{12} - \sum_{q=0}^{p} \sum_{l=0}^{q-2} \sum_{m=0}^{l} [B^{(p-q-2,q-l)}(s, \bar{s}), (\lambda^{(l-m+1,m)} + \bar{\lambda}^{(l-m,m+1)})(s, \bar{s})]^{22}_X,
\]
where we expanded \(c_1(s, \bar{s}, t)\) in the power of \(t\) as \(c_1(s, \bar{s}, t) = c_0^1(s, \bar{s}) + tc_1^1(s, \bar{s})\) with \(c_0^1(s, \bar{s}) = \pi^{(0,0)} + s \pi^{(1,0)} + \bar{s} \pi^{(0,1)} + ss \pi^{(1,1)}\) and \(c_1^1(s, \bar{s}) = (s \bar{X} + \bar{s} X) \pi^{(1,1)}\). The first one \([A.1]\) determines several \(B^{(p,q)}(s, \bar{s})\) with the same total picture number simultaneously. We must split them by each left- and right-moving picture number. The explicit decomposition for the NS-NS sector.
was given in Ref. [16], but we have not yet extend it to the whole sectors in a closed form. Instead, we give an explicit decomposition for some lower picture numbers and show how the equations determine $B^{(n,p)}(s,\bar{s})$ for all the higher picture numbers. First, setting $p = 0$ in Eqs. (A.2) and (A.3), we have

$$[\eta, \lambda^{(1,0)}(s,\bar{s})] = \partial_s B^{(0,0)}(s,\bar{s}) , \quad [\eta, \bar{\lambda}^{(0,1)}(s,\bar{s})] = \partial_s B^{(0,0)}(s,\bar{s}).$$  

(A.4)

We can solve them as

$$\lambda^{(1,0)}(s,\bar{s}) = \xi_0 \circ \partial_s L_B(s,\bar{s}) , \quad \bar{\lambda}^{(0,1)}(s,\bar{s}) = \bar{\xi}_0 \circ \partial_s L_B(s,\bar{s}),$$

(A.5)

under the initial conditions $B^{(0,0)}(s,\bar{s}) = L_B(s,\bar{s})$. The operations $\xi_0^\circ$ and $\bar{\xi}_0^\circ$ are defined on general coderivation $B = \sum_{n=0}^\infty B_{n+2}$ by

$$\xi_0^\circ B = \sum_{n,k=0}^\infty \left( (\xi_0^\circ B_{n+2}) \wedge I_k \right) \pi^\circ_{n+k+2},$$

(A.6a)

$$\xi_0^\circ B_{n+2} = \frac{1}{n+3} \left( \xi_0^\circ B_{n+2} - (-1)^{\deg(B)} B_{n+2} (\xi_0^\circ \wedge I_{n+1}) \right),$$

(A.6b)

and those replacing $\xi_0^\circ$ with $\bar{\xi}_0^\circ$. Eq. (A.4) at $p = 0$ splits into two equations

$$B^{(1,0)}(s,\bar{s}) = [Q, \lambda^{(1,0)}(s,\bar{s})] + [L_B(s,\bar{s}), \lambda^{(1,0)}(s,\bar{s})]_{\xi_0^\circ} + \frac{s}{2} [L_B(s,\bar{s}), L_B(s,\bar{s})]_{\xi_0^\circ}^{22},$$

(A.7)

$$B^{(0,1)}(s,\bar{s}) = [Q, \bar{\lambda}^{(0,1)}(s,\bar{s})] + [L_B(s,\bar{s}), \bar{\lambda}^{(0,1)}(s,\bar{s})]_{\xi_0^\circ} + \frac{s}{2} [L_B(s,\bar{s}), L_B(s,\bar{s})]_{\xi_0^\circ}^{22}. $$

(A.8)

Substituting Eq. (A.5) in these expression, we obtain $B^{(1,0)}(s,\bar{s})$ and $B^{(0,1)}(s,\bar{s})$ independently. Next, setting $p = 1$ in Eqs. (A.2) and (A.3), we have

$$[\eta, \lambda^{(2,0)}(s,\bar{s})] = \partial_s B^{(1,0)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(1,0)}(s,\bar{s})]_{\xi_0^\circ}^{21},$$

(A.9)

$$[\eta, \lambda^{(1,1)}(s,\bar{s})] = \partial_s B^{(0,1)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(0,1)}(s,\bar{s})]_{\xi_0^\circ}^{21},$$

(A.10)

$$[\eta, \bar{\lambda}^{(1,1)}(s,\bar{s})] = \partial_s B^{(1,0)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(1,0)}(s,\bar{s})]_{\xi_0^\circ}^{12},$$

(A.11)

$$[\eta, \bar{\lambda}^{(0,2)}(s,\bar{s})] = \partial_s B^{(0,1)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(0,1)}(s,\bar{s})]_{\xi_0^\circ}^{12}. $$

(A.12)

Since all the quantities in the right hand sides are already known, we can solve these equations for $\lambda^{(2,0)}(s,\bar{s})$, $\lambda^{(1,1)}(s,\bar{s})$, $\bar{\lambda}^{(1,1)}(s,\bar{s})$, and $\bar{\lambda}^{(0,2)}(s,\bar{s})$ by acting $\xi_0^\circ$ or $\bar{\xi}_0^\circ$ as

$$\lambda^{(2,0)}(s,\bar{s}) = \xi_0^\circ \left( \partial_s B^{(1,0)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(1,0)}(s,\bar{s})]_{\xi_0^\circ}^{21} \right),$$

(A.13)

$$\lambda^{(1,1)}(s,\bar{s}) = \xi_0^\circ \left( \partial_s B^{(0,1)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(0,1)}(s,\bar{s})]_{\xi_0^\circ}^{21} \right),$$

(A.14)

$$\bar{\lambda}^{(1,1)}(s,\bar{s}) = \bar{\xi}_0^\circ \left( \partial_s B^{(1,0)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(1,0)}(s,\bar{s})]_{\xi_0^\circ}^{12} \right),$$

(A.15)

$$\bar{\lambda}^{(0,2)}(s,\bar{s}) = \bar{\xi}_0^\circ \left( \partial_s B^{(0,1)}(s,\bar{s}) + [L_B(s,\bar{s}), \lambda^{(0,1)}(s,\bar{s})]_{\xi_0^\circ}^{12} \right).$$

(A.16)
At $p = 1$, Eq. (A.1) can be split as

\[
2B^{(2,0)}(s, \bar{s}) = [Q, \lambda^{(2,0)}(s, \bar{s})] + [B^{(1,0)}(s, \bar{s}), \lambda^{(1,0)}(s, \bar{s})]_{\xi_1} + \bar{s}[B^{(0,0)}(s, \bar{s}), \lambda^{(0,0)}(s, \bar{s})]_{\lambda_1}^{22} + \bar{s}[B^{(0,0)}(s, \bar{s}), B^{(1,0)}(s, \bar{s})]_{\Xi_1}^{22},
\]

(A.17a)

\[
2B^{(1,1)}(s, \bar{s}) = [Q, (\lambda^{(1,1)} + \bar{\lambda}^{(1,1)})(s, \bar{s})] + [B^{(0,0)}(s, \bar{s}), (\lambda^{(1,1)} + \bar{\lambda}^{(1,1)})(s, \bar{s})]_{\xi_1} + \bar{s}[B^{(0,0)}(s, \bar{s}), \lambda^{(0,1)}(s, \bar{s})]_{\lambda_1}^{22} + \bar{s}[B^{(0,0)}(s, \bar{s}), B^{(0,1)}(s, \bar{s})]_{\Xi_1}^{22},
\]

(A.17b)

\[
2B^{(0,2)}(s, \bar{s}) = [Q, \bar{\lambda}^{(0,2)}(s, \bar{s})] + [B^{(0,1)}(s, \bar{s}), \lambda^{(0,1)}(s, \bar{s})]_{\xi_1} + \bar{s}[B^{(0,0)}(s, \bar{s}), \lambda^{(0,1)}(s, \bar{s})]_{\lambda_1}^{22} + \bar{s}[B^{(0,0)}(s, \bar{s}), B^{(0,1)}(s, \bar{s})]_{\Xi_1}^{22}.
\]

(A.17c)

All the quantities in the right hand sides have been obtained in the previous steps, and thus, Eqs. (A.17) determine $B^{(2,0)}(s, \bar{s})$, $B^{(2,0)}(s, \bar{s})$, and $B^{(2,0)}(s, \bar{s})$. Repeating the procedure, we can obtain $B^{(p,\bar{p})}(s, \bar{s})$ for arbitrary $p$ and $\bar{p}$ independently. Similar but slightly different analysis was give in Ref. [20].

The differential equations in Eqs. (4.28) for open-closed interactions is also solved recursively with the initial condition in Eq. (4.34). The differential equations in Eqs. (4.28) are rewritten as

\[
(p + 1)C^{(p+1)}(s) = [Q, \nu^{(p+1)}(s)] + \sum_{q=0}^{p} \left( \left( [A^{(p-q)}(s) + C^{(p-q)}(s)], \nu^{(q+1)}(s) \right)_{\sigma_1} + \left( C^{(p-q)}(s), \mu^{(q+1)}(s) \right)_{\xi_1} \right)
\]

\[
+ \sum_{q=0}^{p} \sum_{l=0}^{p-q} \left( [B^{(p-q-l+1)}(s, \bar{s}), \nu^{(l+1)}(s)]_{\xi_1} + \left. \left( C^{(p-q-l+1)}(s, \bar{s}), (\lambda^{(l+1)} + \bar{\lambda}^{(l+1)})(s, \bar{s}) \right)_{\xi_1} \right) \right)
\]

\[
+ \sum_{q=0}^{p} \sum_{l=0}^{p-q-1} \left( [B^{(p-q-l-1)}(s, \bar{s}), \nu^{(l+1)}(s)]_{\lambda_1}^{22} + \left. \left( C^{(p-q-l-1)}(s, \bar{s}), (\lambda^{(l+1)} + \bar{\lambda}^{(l+1)})(s, \bar{s}) \right)_{\lambda_1}^{22} \right) \right)
\]

\[
+ \sum_{q=0}^{p} \sum_{l=0}^{p-q} \left. \left( [B^{(p-q-l+1)}(s, \bar{s}), C^{(l)}(s)]_{\Xi_1}^{22} \right) \right).
\]

(A.18)
\[ [\eta, \nu^{(p+1)}(s)] = \partial_s C^{(p)}(s) \]
\[ + \sum_{q=0}^{p-1} \left( \left( (A^{(p-q-1)}(s) + C^{(p-q-1)}(s)), \nu^{(q+1)}(s) \right)^2 + C^{(p-q-1)}(s), \mu^{(q+1)}(s) \right)^2 \]
\[ + \sum_{q=0}^{p-1} \sum_{l=0}^{p-q-1} \left( [B^{(p-q-l-1,q)}(s), \nu^{(l+1)}(s)]^{21+12} + [C^{(p-q-l-1)}(s), (\lambda^{(l+1,q)} + \bar{\lambda}^{(q,l+1)})(s, s)]^{21+12} \right) \]
\[ + \frac{1}{2} \sum_{q=0}^{p-2} \sum_{l=0}^{p-q-2} \left( [B^{(p-q-l-2,q)}(s), \nu^{(l+1)}(s)] \right)^{22}_{\bar{\lambda} + \bar{\lambda}} \]
\[ + [C^{(p-q-l-2)}(s), (\lambda^{(l+1,q)} + \bar{\lambda}^{(q,l+1)})(s, s)]^{22}_{\bar{\lambda} + \bar{\lambda}} \), \quad (A.19) \]

where we denote \([A, B]^{21} + [A, B]^{12}\) as \([A, B]^{21+12}\) for notational simplicity. We assume that \(A^{(p)}(s), B^{(p,\rho)}(s, \bar{s}), \mu^{(p+1)}(s), \lambda^{(p+1,\rho)}(s, \bar{s})\), and \(\bar{\lambda}^{(p,\rho+1)}(s, \bar{s})\) are independently determined by solving the differential equations in Eqs. \((2.29)\) and \((3.39)\).

We start from Eq. \((A.19)\) at \(p = 0\) with the initial condition in Eq. \((4.34)\):
\[ [\eta, \nu^{(1)}(s)] = \partial_s N_B(s) \). \quad (A.20)\]

This is solved as
\[ \nu^{(1)}(s) = \chi_0^\circ \partial_s N_B(s) \), \quad (A.21)\]
so as to respect the cyclicity, where \(\chi_0^\circ\) is defined on general coderivation \(C = \sum_{n,l=0}^\infty C_{n+1,l}\) by
\[ \chi_0^\circ \circ C = \sum_{n,l=0}^\infty \sum_{m,j,k=0}^\infty (\mathbb{I}_m \otimes \mathbb{I}_{\otimes j} \otimes \chi_0^\circ \circ C_{n+1,l} \otimes \mathbb{I}_{\otimes k}) \otimes \pm_{m+n+1,j+k+l} \). \quad (A.22)\]
\[ \chi_0^\circ \circ C_{n+1,l} = \frac{1}{l+1} \left( \chi_0^\circ C_{n+1,l} - (-1)^{\text{deg}(C)} \sum_{m=0}^{l-1} C_{n+1,l} \otimes \mathbb{I}_{n+1} \otimes \mathbb{I}_{\otimes (l-m-1)} \otimes \chi_0^\circ \otimes \mathbb{I}_{\otimes m} \right) \). \quad (A.23)\]

Then, Eq. \((A.18)\) at \(p = 0\),
\[ C^{(1)}(s) = [Q, \nu^{(1)}(s)] \]
\[ + \left( (A^{(0)}(s) + C^{(0)}(s)), \nu^{(1)}(s) \right)_{\varphi_1(s)} + C^{(0)}(s), \mu^{(1)}(s) \right)_{\varphi_1(s)} \]
\[ + \left( B^{(0,0)}(s, s), \nu^{(1)}(s) \right)_{\xi_0^0(s)} + C^{(0)}(s), (\lambda^{(1,0)} + \bar{\lambda}^{(0,1)})(s, s) \right)_{\xi_0^0(s)} \]
\[ + s \left( B^{(0,0)}(s, s), C^{(0)}(s) \right)_{\Xi^+ + \Xi} \). \quad (A.24)\]
determines \( C^{(1)}(s) \). Next, we solve Eq. (A.19) at \( p = 1 \),

\[
[\eta, \nu^{(2)}(s)] = \partial_s C^{(1)}(s) \\
+ \left[ (A^{(0)}(s) + C^{(0)}(s)) \nu^{(1)}(s) + [C^{(0)}(s), \mu^{(0)}(s)] \right] \\
+ \left[ (B^{(0)}(s) + C^{(0)}(s)) \nu^{(2)}(s) |_{\nu_1} \right] + \left[ C^{(0)}(s), \mu^{(0)}(s) \right] \\
+ \left[ (B^{(1)}(s) + B^{(0)}(s)) \nu^{(1)}(s) |_{\nu_1} \right] + \left[ (B^{(1)}(s) + B^{(0)}(s)) \nu^{(2)}(s) |_{\nu_1} \right] \\
+ \left[ C^{(0)}(s), (\lambda^{(1)} + \bar{\lambda}^{(0,1)})(s) \right] \\
\text{as}
\]

\[
\nu^{(2)}(s) = \xi_0 \circ \left( \partial_s C^{(1)}(s) + \left[ (A^{(0)}(s) + C^{(0)}(s)) \nu^{(1)}(s) + [C^{(0)}(s), \mu^{(0)}(s)] \right] \\
+ \left[ (B^{(0)}(s) + C^{(0)}(s)) \nu^{(2)}(s) |_{\nu_1} \right] + \left[ C^{(0)}(s), (\lambda^{(1)} + \bar{\lambda}^{(0,1)})(s) \right] \right) .
\]

Equation (A.18) at \( p = 1 \) determines \( C^{(2)}(s) \) as

\[
2C^{(2)}(s) = [Q, \nu^{(2)}(s)] \\
+ \left[ (A^{(1)}(s) + C^{(1)}(s)) \nu^{(1)}(s) + [C^{(1)}(s), \mu^{(1)}(s)] \right] \\
+ \left[ (B^{(1)}(s) + B^{(0)}(s)) \nu^{(1)}(s) \right] \\
+ \left[ (B^{(1)}(s) + B^{(0)}(s)) \nu^{(2)}(s) \right] \\
+ \left[ C^{(1)}(s), (\lambda^{(1)} + \bar{\lambda}^{(0,1)})(s) \right] \\
\text{as}
\]

\[
\nu^{(2)}(s) = \xi_0 \circ \left( \partial_s C^{(1)}(s) + \left[ (A^{(0)}(s) + C^{(0)}(s)) \nu^{(1)}(s) + [C^{(0)}(s), \mu^{(0)}(s)] \right] \\
+ \left[ (B^{(0)}(s) + C^{(0)}(s)) \nu^{(2)}(s) |_{\nu_1} \right] + \left[ C^{(0)}(s), (\lambda^{(1)} + \bar{\lambda}^{(0,1)})(s) \right] \right) .
\]

Equation (A.18) at \( p = 1 \) determines \( C^{(2)}(s) \) as

\[
2C^{(2)}(s) = [Q, \nu^{(2)}(s)] \\
+ \left[ (A^{(1)}(s) + C^{(1)}(s)) \nu^{(1)}(s) + [C^{(1)}(s), \mu^{(1)}(s)] \right] \\
+ \left[ (B^{(1)}(s) + B^{(0)}(s)) \nu^{(1)}(s) \right] \\
+ \left[ (B^{(1)}(s) + B^{(0)}(s)) \nu^{(2)}(s) \right] \\
+ \left[ C^{(1)}(s), (\lambda^{(1)} + \bar{\lambda}^{(0,1)})(s) \right] \\
\text{as}
\]

\[
\nu^{(2)}(s) = \xi_0 \circ \left( \partial_s C^{(1)}(s) + \left[ (A^{(0)}(s) + C^{(0)}(s)) \nu^{(1)}(s) + [C^{(0)}(s), \mu^{(0)}(s)] \right] \\
+ \left[ (B^{(0)}(s) + C^{(0)}(s)) \nu^{(2)}(s) |_{\nu_1} \right] + \left[ C^{(0)}(s), (\lambda^{(1)} + \bar{\lambda}^{(0,1)})(s) \right] \right) .
\]

One can obtain any \( C^{(p)}(s) \) one wants by repeating the procedure.

Finally, it makes sense to mention that if we specify the type and number of inputs, the procedure ends in finite steps. We can explicitly determine any \( C^{(p)}_{n+1,0} \) \( 2^r \) you want in order from the one with the smallest number of inputs\(^8\). The one with the smallest number of inputs is the open-closed interaction:

\[
C^{(0)}_{1,0}(s) = C^{(0)}_{1,0} \left| 2 + s C^{(0)}_{1,0} \right| 0 ,
\]

\[
C^{(1)}_{1,0}(s) = C^{(1)}_{1,0} \left| 0 ,
\]

with \( \nu^{(1)}_{1,0}(s) = \nu^{(1)}_{1,0} | 0 \). They are determined by Eqs. (A.21), and (A.24) under the initial condition in Eq. (A.34) as

\[
C^{(0)}_{1,0} \left| 2 = (N_B)_{1,0} \left| 2 , C^{(0)}_{1,0} \left| 0 = (N_B)_{1,0} \left| 0 , C^{(1)}_{1,0} \left| 0 = X_0^o(N_B)_{1,0} \left| 0 ,
\]

\(^8\)The closed string input is counted as 2.
with \( \nu_{1,0}^{(1)} |^0 = \xi_0^{(0)} \circ (N_B)_{1,0} |^0 \). It is a little more non-trivial for \( C_{1,1}^{(p)}(s) \):

\[
C_{1,1}^{(0)}(s) = C_{1,1}^{(0)} |^4 + s C_{1,1}^{(0)} |^2 + s^2 C_{1,1}^{(0)} |^0, \quad (A.31)
\]

\[
C_{1,1}^{(1)}(s) = C_{1,1}^{(1)} |^2 + s C_{1,1}^{(1)} |^0, \quad (A.32)
\]

\[
C_{1,1}^{(2)}(s) = C_{1,1}^{(2)} |^0, \quad (A.33)
\]

with

\[
\nu_{1,1}^{(1)}(s) = \nu_{1,1}^{(1)} |^2 + s \nu_{1,1}^{(1)} |^0, \quad (A.34)
\]

\[
\nu_{1,1}^{(2)}(s) = \nu_{1,1}^{(2)} |^0. \quad (A.35)
\]

These are determined by Eqs. \((A.21), (A.24), (A.26), \) and \((A.27)\) under the initial condition in Eq. \((4.34)\) as

\[
C_{1,1}^{(0)} |^4 = (N_B)_{1,1} |^4, \quad C_{1,1}^{(0)} |^2 = (N_B)_{1,1} |^2, \quad C_{1,1}^{(0)} |^0 = (N_B)_{1,1} |^0, \quad (A.36)
\]

\[
C_{1,1}^{(1)} |^2 = [Q, \nu_{1,1}^{(1)} |^2] + [(M_B)_2 |^2, \nu_{1,1}^{(1)} |^0] + [(N_B)_{1,0} |^2, \mu_{2}^{(1)} |^0], \quad (A.37)
\]

\[
C_{1,1}^{(1)} |^0 = [Q, \nu_{1,1}^{(1)} |^0] + [(M_B)_2 |^0, \nu_{1,1}^{(1)} |^0] + [(N_B)_{1,0} |^0, \mu_{2}^{(1)} |^0], \quad (A.38)
\]

\[
C_{1,1}^{(2)} |^0 = \frac{1}{2} \left[ [Q, \nu_{1,1}^{(2)} |^0] + [A_{2}^{(1)} |^0, \nu_{1,1}^{(1)} |^0] + [C_{1,1}^{(1)} |^0, \mu_{2}^{(1)} |^0] \right], \quad (A.39)
\]

with

\[
\nu_{1,1}^{(1)} |^2 = \xi_0^{(0)} \circ (N_B)_{1,1} |^2, \quad \nu_{1,1}^{(1)} |^0 = 2 \xi_0^{(0)} \circ (N_B)_{1,1} |^0, \quad (A.40)
\]

\[
\nu_{1,1}^{(2)} |^0 = \xi_0^{(0)} \circ C_{1,1}^{(1)} |^0. \quad (A.41)
\]

Those for \( C_{2,0}^{(p)}(s) \) are similarly obtained by Eqs. \((A.19)\), and \((A.18)\) at \( p = 0, 1, 2 \) under the initial condition in Eq. \((4.34)\). We can continue these steps as long as we need.

**B  Composite string fields in open superstring field theory**

In this Appendix we show that the pure-gauge string field \( G_o(V) \) for the open superstring field theory with general \( A_\infty \) structure is obtained in a similar way given in the heterotic string field theory \([29]\). The pure-gauge string field \( G_o(V_o) \) is associated with a finite form of the “gauge transformation”

\[
\delta_{\delta V_o} \Psi = \pi_1 L_{\omega}^{\alpha} \left( \frac{1}{1 - \Psi} \otimes \delta V_o \otimes \frac{1}{1 - \Psi} \right)
\]

\[
= \eta \Lambda_{\alpha} + L_{\beta}^{\gamma}(\Psi, \delta V_o) + L_{\gamma}^{\beta}(\delta V_o, \Psi) + \cdots, \quad (B.1)
\]
with the infinitesimal parameter $\delta V_o$, and is obtained by integrating along a straight line connecting $0$ and $V_o$ that we parameterize as $\tau V_o$ with $0 \leq \tau \leq 1$. Considering that the difference between $G_o(\tau V_o + d\tau V_o)$ and $G_o(\tau V_o)$ is an infinitesimal gauge transformation, we obtain a differential equation

$$\partial_\tau G_o(\tau V_o) = \pi_1 L_o^n \left( \frac{1}{1 - g G_o(\tau V_o)} \otimes V_o \otimes \frac{1}{1 - g G_o(\tau V_o)} \right), \quad (B.2)$$

where we introduced a coupling constant $g$ for convenience. The pure-gauge string field $G_o(V_o)$ corresponds to $G_o(\tau V_o)$ at $\tau = 1$ and is obtained by solving this differential equation with the initial condition $G_o(0) = 0$. Expanding $G_o$ in the power of $g$ as $G_o = \sum_{n=0}^{\infty} g^n G_o^{(n)}$, we can sequentially solve the equation. The equation at $O(g^0)$ is given by $\partial_\tau G_o^{(0)}(\tau V_o, \tau V_o) = \eta V_o$ and is integrated as $G_o(\eta V_o) = \tau \eta V_o$. At $O(g)$, the equation becomes

$$\partial_\tau G_o^{(1)} = L_2^g(\eta V_o, V_o) + L_2^g(V_o, \tau \eta V_o) \quad (B.3)$$

and is solved as

$$G_o^{(1)} = \frac{\tau^2}{2} \left( L_2^g(\eta V_o, V_o) + L_2^g(V_o, \eta V_o) \right). \quad (B.4)$$

Similarly, we can find $G_o$ up to any order of $g$ we want:

$$G_o(V_o) = \eta V_o + \frac{1}{2} \left( L_2^g(\eta V_o, V_o) + L_2^g(V_o, \eta V_o) \right)$$

$$+ \frac{1}{3} \left( L_3^g(\eta V_o, \eta V_o, V_o) + L_3^g(\eta V_o, V_o, \eta V_o) + L_3^g(V_o, \eta V_o, \eta V_o) \right)$$

$$+ \frac{1}{3!} \left( L_2^g(L_2^g(V_o, \eta V_o), V_o) + L_2^g(V_o, \eta V_o, V_o) + L_2^g(V_o, L_2^g(\eta V_o, V_o)) + L_2^g(V_o, L_2^g(\eta V_o, V_o)) \right) + \cdots. \quad (B.5)$$

In order to find an explicit form of associated string field $B_d(V_o)$ ($d = \partial_t, \delta$ or $Q$), we consider

$$\mathcal{I}(\tau) = \pi_1 L_o^n \left( \frac{1}{1 - G_o(\tau V_o)} \otimes B_d(\tau; V_o, dV_o) \otimes \frac{1}{1 - G_o(\tau V_o)} \right) - (-1)^d G_o(\tau V_o), \quad (B.6)$$

and its $\tau$ derivative

$$\partial_\tau \mathcal{I}(\tau) = \pi_1 L_o^n \left( \frac{1}{1 - G(\tau V_o)} \otimes \left( \partial_\tau B_d(\tau; V_o, dV_o) - J(\tau) \right) \otimes \frac{1}{1 - G(\tau V_o)} \right)$$

$$- \pi_1 L_o^n \left( \frac{1}{1 - G(\tau V_o)} \otimes \mathcal{I}(\tau) \otimes \frac{1}{1 - G(\tau V_o)} \right) \otimes \frac{1}{1 - G(\tau V_o)}$$

$$+ (-1)^d \left( \frac{1}{1 - G(\tau V_o)} \otimes V_o \otimes \frac{1}{1 - G(\tau V_o)} \otimes \mathcal{I}(\tau) \otimes \frac{1}{1 - G(\tau V_o)} \right). \quad (B.7)$$
where
\[
J(\tau) = dV_o + \pi^2 L_o^2 \left( \frac{1}{1 - G(\tau V_o)} \otimes V_o \otimes \frac{1}{1 - G(\tau V_o)} \otimes B_d(\tau; V_o, dV_o) \otimes \frac{1}{1 - G(\tau V_o)} \right) \\
- \left( -1 \right)^d \frac{1}{1 - G(\tau V_o)} \otimes B_d(\tau; V_o, dV_o) \otimes \frac{1}{1 - G(\tau V_o)} \otimes V_o \otimes \frac{1}{1 - G(\tau V_o)} \right). \tag{B.8}
\]

If \(B_d(\tau; V_o, dV_o)\) satisfies the differential equation
\[
\partial_\tau B_d(\tau; V_o, dV_o) = J(\tau), \tag{B.9}
\]
with the initial condition \(B_d(0; V_o, dV_o) = 0\), then \(\partial_\tau I(\tau)\) is proportional to \(I(\tau)\) with \(I(0) = 0\), and thus \(I(\tau) = 0\) for \(\forall t\) due to Eq. (B.7). Since \(I(1) = 0\) is nothing but the relation in Eq. (5.11) characterizing the associated string field, we can obtain the associated field \(B_d(V_o, dV_o)\) by solving the differential equation in Eq. (B.9). Expanding \(B_d = \sum_{n=0}^\infty g^n B_d^{(n)}\) with scaling \(G_o \rightarrow gG_o\), we find that
\[
B_d(V_o, dV_o) = dV_o + \frac{1}{2} \left( L_2^0(V_o, dV_o) - L_2^0(dV_o, V_o) \right) \\
+ \frac{1}{3} \left( L_3^0(\eta V_o, V_o, dV_o) + L_3^0(V_o, \eta V_o, dV_o) + L_3^0(\eta V_o, V_o, dV_o) \right) \\
- L_3^0(\eta V_o, dV_o, V_o + \cdot \cdot \cdot ) + \frac{1}{3!} \left( L_4^0(V_o, L_2^0(V_o, dV_o)) - L_4^0(V_o, L_2^0(dV_o, V_o)) \right) \\
- L_2^0(L_2^0(V_o, dV_o), V_o) + L_2^0(L_2^0(dV_o, V_o), V_o) \right) + \cdots. \tag{B.10}
\]

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