Asymptotic analysis of a model of autoresonance

Leonid Kalyakin
Institute of Mathematics, Ufa Sci. Centre, Russian Acad. of Sci.
112, Chernyshevsky str., Ufa,
450077, Russia
E-mail: klenru@mail.ru

Abstract. Autoresonance is a phase locking phenomenon occurring in nonlinear oscillatory system, which is forced by oscillating perturbation. Many physical applications of the autoresonance are known in nonlinear physics. The essence of the phenomenon is that the nonlinear oscillator selfadjusts to the varying external conditions so that it remains in resonance with the driver for a long time. This long time resonance leads to a strong increase in the response amplitude under weak driving perturbation. An analytic treatment of a simple mathematical model is done here by means of asymptotic analysis using a small driving parameter. The main result is finding threshold for entering the autoresonance.

1. Introduction
We consider a simple mathematical model of forcing oscillations given by the nonlinear ordinary differential equation

\[ u'' + (1 + \gamma u^2)u = 2\alpha f(t) \cos(\varphi(t)), \quad t > 0; \quad 0 < \alpha \ll 1 \]  \hspace{1cm} (1)

where the right hand side represents a small external force. A small initial data is added here

\[ (u, u')|_{t=0} = \alpha^{1/3}(u_0, u_1), \quad u_0, u_1 = \text{const} \]  \hspace{1cm} (2)

so the system is starting in the neighborhood of the stable equilibrium. A nonlinearity may have a different signs: \( \gamma = \pm 1 \). The driving amplitude \( f \) is a slow varying function in contrast to the phase function so that \( f'/\varphi'(t) = o(1) \), \( \alpha \to 0 \). In the paper we derive the criterion of the autoresonance phenomenon. Namely, we find a condition under which the system’s energy \( \mathcal{E}(t, \alpha) \equiv [(u')^2 + u^2]/2 + \gamma u^4/4 - u2\alpha f \cos(\varphi) \) grows up to the order of \( O(1) \) as \( t \to \infty \) while the driver is being small: \( 0 < \alpha \ll 1 \), \( f(t) = O(1) \).

Great growth in the energy (and in the swing amplitude as well) usually takes place due to the resonance in oscillating system. The simplest model of the resonance phenomenon is given by the linear equation of harmonic oscillator under periodic force

\[ u'' + \omega^2 u = \cos(\nu t). \]

The energy of this system grows infinitely in time, if the driving frequency coincides with the free frequency \( \omega^2 = \nu^2 \). In the case of nonlinear system such type of phenomena is referred to as autoresonance [1, 2, 3, 4]. Unlike linear systems nonlinear systems must have two distinctive
features to be able to enter autoresonance. The first one is that the driving frequency must be varying, because the free frequency, depending on the energy, is varying in accordance with the growth of the oscillating amplitude. The second feature is that the driving amplitude must exceed a sharp threshold, depending on the chirp rate of the forcing, as it was noticed recently [1, 2].

An analytic treatment of the threshold phenomenon is done here by means of asymptotic analysis of the problem (1),(2), using a small parameter $\alpha$. We believe that an accurate study of the simple example provides understanding of more complicated problems. In particular, the fact of the cubic nonlinearity in (1) does not play any role. Similar results take place for each oscillating system with a smooth nonlinearity.

As was pointed out above the free frequency depends on the energy in nonlinear system. In order to reach large energy of the swing, using a resonance effect, it is necessary to vary the driving frequency $d\phi/dt$ in time, starting from initial value $d\phi/dt|_{t=0} = 1$. In this way one has to take into account that energy is small and varying slowly through initial stage until the amplitude, starting from zero, increases up to the order of unity. Hence there are two additional small parameters in the setting of the problem. The first one is the characteristic frequency mismatch, the second is the chirp rate, i.e. the rate of change of the mismatch. We shall identify both small parameters as $\varepsilon$ and $\varepsilon \lambda$ in the phase function:

$$\varphi = t + \varepsilon^{-\lambda} \Phi(\varepsilon^{1+\lambda} t), \quad (0 < \varepsilon \ll 1, \ \lambda = \text{const} \geq 0),$$

(3)

In this work the small parameter $\varepsilon$ will be related to the amplitude parameter $\alpha$ and two types of the data $f, \varphi(t, \alpha)$ will be considered. Two different cases are distinguished one from another by the rate of change of the slow varying functions $f, \varphi(t, \alpha)$ and they are sometimes called as “the rigid frequency chirping” and “the loose frequency chirping”. In each case a threshold for the driving amplitude is found and entering the autoresonance is proved in the paper.

We consider here just only the initial stage of amplitude rising and omit the following stage at which the evolution of the large amplitude of the order of unity occurs.

Our theory may be considered as an asymptotic analysis of a small amplitude solution. This approach gives an appropriate tool in finding the autoresonance phenomena in cases under consideration. Note, similar results for both problems of the nonlinear resonance and of the synchronization of oscillation [5, 6] are known. However, they are not directly related to the autoresonance phenomenon.

2. **Anzatz of asymptotic solution**

Our approach is a version of two scale methods, [7, 8, 9, 10]. The WKB type anzatz is taken as an asymptotic solution of the problem (1)–(4):

$$u = \alpha^{1/3} \left[ A(\varepsilon t, \varepsilon) \exp(i\varphi(t, \varepsilon)) + \text{c.c.} + \alpha^{2/3} u_0(t, \varepsilon) + \mathcal{O}(\alpha^{4/3}) \right], \quad \alpha \to 0.$$  

(4)

This is a small amplitude approximation because of the factor $\alpha^{1/3}$. The exponent 1/3 is chosen here in order to take into account both the nonlinearity and the driver in the slow modulation of the amplitude $A$. For similar reasons in order to take into account the frequency mismatch $\varphi' - 1$, a small parameter $\varepsilon$ is related to a driving parameter: $\varepsilon = \alpha^{2/3}$. In this way the nonlinear equation under the zero initial data is derived for the leading order amplitude

$$2iA' - 2\Phi' A + 3\gamma |A|^2 A = f, \quad A(\tau)|_{\tau=0} = A_0; \quad (\tau = \alpha^{2/3} t).$$

(5)

A modulation of the complex amplitude $A = A(\tau)$ in slow scale $\tau = \alpha^{2/3} t$ is described by this equation. Our main discovery is a class of data $\Phi', f$, under which a solution $A(\tau)$ is infinitely increasing as $\tau \to \infty$. This slow increasing is interpreted as the initial stage of the autoresonance.
3. Rigid driving mode

The phase function (3) at \( \lambda = 0 \) is considered in the section. Below this case will be referred to as rigid driving mode.

It is expedient to represent a complex amplitude \( A \) by means of a pair of real functions \( R, \psi \):

\[
A = R(\tau) \exp(i[\psi(\tau) - \Phi(\tau)]).
\]

One can obtain explicit solution of the equation (5) for some special data \( f, \Phi \). For example,

\[
R(\tau) = \frac{1}{2} \int_0^\tau f(\xi) d\xi, \quad \psi = \Phi - \pi/2, \quad \Phi(\tau) = \frac{3\gamma}{2} \int_0^\tau R^2(\zeta) d\zeta.
\]  

(6)

So in the case \(|f(\tau)| \geq f_0 > 0\) the amplitude increases linearly \(|R(\tau)| = \mathcal{O}(\tau)\), \( \tau \to \infty \) if the driving phase \( \Phi(\tau) \) is related with \( f(\tau) \) as implied by (6). As a result the leading order term of the asymptotic solution (4) has the order of unity \( u(t, \alpha) = \mathcal{O}(1) \) at time \( \tau = \mathcal{O}(\alpha^{-1/3}) \), or what is the same, \( t = \mathcal{O}(\alpha^{-1}) \).

In general case the existence theorem takes place

**Theorem 1** Let the functions \( f(\tau), \Phi(\tau) \) be continuous and the \( f(\tau) \) be uniformly bounded for all \( \tau \geq 0 \). Then there exist a unique solution of the problem (5) for \( \forall \tau > 0 \).

This result gives a basis for construction of an asymptotics of the amplitude module

\[
R = R_1 \tau + R_0 + \mathcal{O}(\tau^{-1/2}), \quad \tau \to \infty.
\]

Under assumption that the data have a smooth asymptotics at infinity:

\[
f(\tau) = f_0 + f_1 \tau^{-1} + \mathcal{O}(\tau^{-2}), \quad \Phi(\tau) = \Phi_3 \tau^3 + \Phi_2 \tau^2 + \Phi_1 \tau + \Phi_0 + \mathcal{O}(\tau^{-1}), \quad \tau \to \infty
\]

(7)

leading coefficients of an amplitude asymptotics are calculated: \( \gamma R_1^2 = 2 \Phi_3, \quad 3\gamma R_1 R_0 = 2\Phi_2 \). So the autoresonance phenomenon, when \( R_1 \neq 0 \), may occur just only under conditions:

\[
\gamma \Phi_3 > 0, \quad f_0^2 > 8|\Phi_3|.
\]

(8)

The second inequality may be considered as the threshold condition for entering the autoresonance.

The first correction \( u_1 \) in (4) is calculated from linearized equations. There is a secular term in \( u_1 \), which increases at infinity \( u_1 = \mathcal{O}(\tau^3), \quad \tau \to \infty \). Hence the first correction in expansion (4) remains lesser than the leading term \( \alpha |u_1| \ll \alpha^{1/3}|A| \) over a time interval, which is not too long \( 0 < t \ll \alpha^{-1} \).

The main result in the rigid driving mode case is as follows

**Theorem 2** Let the right hand side in the equation (1) as given by (3) under \( \lambda = 0 \), \( \varepsilon = \alpha^{2/3}, \quad (\tau = \alpha^{2/3} t) \) have the properties (7),(8). Then the system under appropriate initial data enters autoresonance. The amplitude of the leading order term in (4) found from equation (5) increases linearly \( |A(\tau)| = \sqrt{2|\Phi_3|} \tau + \mathcal{O}(1), \quad \tau \to \infty \). Asymptotics (4) is available over a time interval, which is not too long \( 0 < t \leq \alpha^{-1+\nu}, \quad \forall \nu > 0 \).

There are many solutions which have a growing asymptotics of that type [11]

4. Loose driving mode

The amplitude equation (5) is nonautonomous and in general case it cannot be solved in the explicit form. However if both the coefficient \( \Phi'(\delta \tau) \) and the right hand side \( f(\delta \tau) \) depend on the slow time \( \delta \tau \) with a small parameter \( \delta < \delta \ll 1 \) one can use an asymptotic approach known as adiabatic approximation. This case will be referred to as loose driving mode below and will be considered in this section.

We consider the problem (1),(2) with phase driving function \( \varphi \) given by (3) with a small parameter \( \varepsilon = \alpha^{2/3} \) and exponent \( \lambda > 0 \); so the adiabatic parameter in amplitude equation (9)
may be taken $\delta = \alpha^{2\lambda/3}$. Our main result is an asymptotics of the amplitude $A = A(\tau, \delta)$ as $\delta \to 0$, which is valid for large time $\tau \gg \delta^{-1}$. Using these asymptotics we find the condition under which the amplitude modulo $|A(\tau, \delta)|$ is infinitely increasing as $\delta \tau \to \infty$. This slow increasing is interpreted as the initial stage of the autoresonance.

Let us go over to the asymptotic constructions. First, the scaling transformation is performed in the equation (0)

$$A(\tau, \delta) = (f/3\gamma)^{1/3}B(\zeta, \delta), \quad \zeta = \eta/\delta, \quad \eta = (3\gamma)^{1/3}2 \int_0^{\delta \tau} f^{2/3}(\xi) d\xi. \quad (9)$$

Then the problem for the complex amplitude $B = P+iQ$ is represented by a system of differential equations for two functions, namely for real and imaginary parts

$$Q' - [P^2 + Q^2 - \Omega(\eta)]P = -1 + \delta F(\eta)Q, \quad Q(\zeta, \delta)|_{\zeta=0} = q_0, \quad (10)$$

$$P' + [P^2 + Q^2 - \Omega(\eta)]Q = \delta F(\eta)P, \quad P(\zeta, \delta)|_{\zeta=0} = p_0. \quad (11)$$

Coefficients

$$\Omega(\eta) = 2\Phi'(3\gamma)^{-1/3}f^{-2/3}(\delta \tau), \quad F(\eta) = -2f'(3\gamma)^{-1/3}f^{-5/3}(\delta \tau)$$

depend on the new slow time $\eta = \delta \zeta, \ 0 < \delta \ll 1$. Initial data $q_0, p_0$ are calculated from (2). These equations are considered either on the semiaxis $\zeta > 0$ or $\zeta < 0$ depending upon the signs of nonlinearity $\gamma = \pm 1$.

A small parameter $\delta$ occurs as the factor in equations (10),(11) and it defines the slow time in data as well. To solve such perturbed problem we apply an asymptotic method, which is ascribed to Bogolubov, Krylov, Kuzmak, Haberman, [8,11,12]. The basis of the approach is a two-parametric periodic solution of the unperturbed problem under frozen data $\Omega \equiv \text{const}$. In the case under consideration the unperturbed system (as $\delta = 0, \ \Omega \equiv \text{const}$) is autonomous and hamiltonian:

$$Q' - [P^2 + Q^2 - \Omega]P = -1,$$

$$P' + [P^2 + Q^2 - \Omega]Q = 0.$$

The system is integrable and an explicit integral representation of the solution can be obtained. But there is no need in an explicit formula to grasp the situation and to find the condition for entering the autoresonance. To this end the Hamiltonian as the first integral may be used

$$\frac{1}{4}(P^2 + Q^2 - \Omega)^2 - P = \text{const.}$$

One can see the family of periodic solutions from the picture of the phase space portrait, where nearly all trajectories are closed except for two embedded separatrices. To parametrize the solution it is reasonable to use both the plane area $\Pi$ covered by trajectory and the phase shift $s$ in the form $P, Q = P_0, Q_0(\zeta + s, \Pi, \Omega)$. Moreover the solution depends on the parameter $\Omega$ since $\Omega$ is present in the equations.

The behavior of separatrices depending upon the $\Omega$ has crucial role in detection of the autoresonance. Both these lines tend to the large circle $P^2 + Q^2 = \Omega$ as $\Omega \to \infty$, while the distance between them is small and has the order $O(\Omega^{-1/4})$ outside some neighbourhood of the saddle point. However, the area between separatrix loops increases and it is minorized by a large magnitude $m\Omega^{1/4}$, ($m = \text{const} > 0$), since the separatrix length has the order of $O(\Omega^{1/2})$. In the considered case with frozen data the solution $P_0, Q_0$ starting from zero is periodic and finite at any time $\zeta$. Its phase space trajectory depends on the parameter $\Omega$. It may be located either inside the inner separatrix or between them.
We return to the perturbed problem (10), (11) in which either \( \Omega(\eta) \neq \text{const} \) and \( \delta F(\eta) \neq 0 \) or only one of the inequalities \( \Omega(\eta) \neq \text{const}, \delta F(\eta) \neq 0 \) holds. One can construct an asymptotic solution which is valid over a long time interval by using both the adiabaticity of the coefficient \( \Omega \) and the smallness of the factor \( \delta \). Following [7, 9, 10] we introduce the two-scale representation of the solution. In this approach the leading order term is taken as the unperturbed solution \( \Omega \) and the smallness of the factor \( \delta \) solution which is valid over a long time interval by using both the adiabaticity of the coefficient and only one of the inequalities \( \Omega(\eta) \neq \text{const} \). In this approach it is necessary to find slow varying functions \( \Pi, S_1, S_0(\eta) \) in order to identify the leading order term of the asymptotic solution.

We are interested in the solution in which the amplitude \(|P| + |Q|\) is an increasing function of the slow time \( \eta \). Evidently, the structure of the fast variable \( \sigma \) is not needed at this stage. The main result in the section is based on finding the slow varying function \( \Pi(\eta) \). It is obtained from differential equation, [9, 10] which turns up trivial due to the appropriate parametrization

\[
\partial_\eta \Pi = F(\eta)\Pi, \quad \Pi(\eta)|_{\eta=0} = \Pi_0.
\]

Here \( \Pi_0 \) is the area covered by the initial trajectory (as \( \eta = 0 \)). In this way a remarkably simple formula is obtained:

\[
\Pi(\eta) = \Pi_0 |f(0)/f(\delta\tau)|.
\]

In the special case when \( f \equiv \text{const} \) the area is an adiabatic invariant \( \Pi \equiv \text{const} \).

Thus using (6), one can write down an adiabatic approximation of the solution of the amplitude problem (5) as follows

\[
A(\tau, \delta) = (f(\delta\tau)/3\gamma)^{1/3}[P_0(\sigma, \Pi(\eta), \Omega(\eta)) + iQ_0(\sigma, \Pi(\eta), \Omega(\eta))] + \tilde{A}(\tau, \delta).
\]

Here the remainder \( \tilde{A}(\tau, \delta) \to 0 \) as \( \delta \to 0 \). As to the remainder it is known [12] that it is evaluated through small parameter \( \tilde{A}(\tau, \delta) = O(\delta), \delta \to 0 \) and the estimate is uniform over long time interval \( 0 \leq \zeta \leq \eta_0\delta^{-1}, (\eta_0 = \text{const} > 0) \). But this strict result is not sufficient for us, because the autoresonance phenomenon is detected only at very far time \( \zeta \gg \delta^{-1} \) where the leading term of the asymptotics (12) may be increasing. There is no conflict since the small remainder \( \tilde{A}(\tau, \delta) = o(1), \delta \to 0 \) over a longer interval, on which the slow time can become large \( \eta = \delta\zeta \gg 1 \). However, one has to keep in mind that the remainder becomes worse with the growth of time. Hence the approximation given by (12) is valid so far as the remainder can be neglected with respect to leading term. Analysis of the first correction in the adiabatic asymptotic may suggest a limiting time interval but we do not treat corrections in this work. One can say a priori that an adiabatic approximation of the amplitude is valid until the small amplitude expansion (4) is suitable.

The main result of the section is derived from the formula (12) under the relation \( \delta = \alpha^\mu, (\mu = 2\lambda/3) \). For the first we consider zero initial data \( u_0 = u_1 = 0 \) which imply \( p_0 = q_0 = 0 \).

**Theorem 3** Let the right hand side in the equation (1) have both the amplitude and the phase \( f = f(\theta), \varphi = 1 + \alpha^{-\mu}\Phi(\theta), (\theta = \alpha^{\mu+2/3} t, \mu > 0) \) which satisfy \(|f(\theta)| \geq |f(0)|, |\Phi'(\theta)| \to \infty \) as \( \theta \to \infty \); let the function \( \Phi'/\gamma^{1/3} f^{2/3}(\theta) \to \infty \) be monotonous and increase to infinity of \( \theta \). Then entering the autoresonance depends on the initial value of the parameter \( \Omega(0) = 2\Phi'(0)/(3\gamma)^{1/3} f^{2/3}(0) \). Autoresonance does not occurs if \( \Omega(0) > 3^{2/3} \). Autoresonance occurs under \( \Omega(0) < 3^{2/3} \). In the last case the leading order term of an amplitude asymptotics is determined by the driving frequency as follows

\[
|A(\tau, \alpha^\mu)| = \left( \frac{2}{3} |\Phi'(\theta)| \right)^{1/2} + O(|\Phi'|^{-1/4}) + O(\alpha^\mu), \quad \alpha \to 0, \quad \theta \to \infty.
\]
In order to prove the theorem it is enough to observe a slow deformation of the phase plane trajectory $P_0, Q_0(\sigma, \Pi(\eta), \Omega(\eta))$ under variation of the parameter $\eta$. In the first case the trajectory is located inside loop of the inner separatrix and it encircles the equilibrium point near zero. In the second case the trajectory is located between separatrix loops and it encircles another equilibrium point near $P = \sqrt{\Omega(\eta)}$. Under given conditions the plane area $\Pi$ covered by trajectory is not increasing, while separatrix loops grows like the circle $P^2 + Q^2 = \Omega$ as $\Omega \to \infty$. Hence in the first case the trajectory remains in bounded part of the phase plane for all time. In the second case one has to take into account that the plane area between loops is increasing. Hence the trajectory remains between loops at any time and it is going to infinity as $\Omega \to \infty$.

In general case, when initial data (2) is not zero, autoresonance occurs under the initial point $q_0, p_0$ is located between separatrix loops.

5. Acknowledgments
This work may be considered as an attempt of a mathematician to understand physicist’s papers [1-4]. The author is grateful prof. A. Shagalov for suggestion of the subject. This research has been supported by the Russian Foundation of the Fundamental Research under Grants 03-01-00716, 04-02-97503, Grant of Scientific School 1446.2003.1 and by INTAS under Grant 03-51-4286

References
[1] Meerson B., Friedland L. 1990 *Physical Review A* 41 5233–5236
[2] Friedland L. 1997 *Physical Review E* 55 1929–1939
[3] Friedland L., Shagalov A. G. *Physical Review Letters* 1998. 81 4357–4360
[4] Friedland L. 2000 *Physical Review E* 61 3732–3735
[5] Andronov A. A., Vitt F. F. and Khaikin S. E. 1959 *Theory of Oscillations* (Moskow: FizMatGiz) (in Russian)
[6] Sagdeev R. Z., Usikov D. A and Zaslavsky G. M. 1988 *Nonlinear Physics: From Pendulum to Turbulent and Chaos* (New York: Harwood Academie)
[7] Bogolyubov N. N. and Mitropol’skii Yu.A., 1962 *Asymptotic Methods in the Theory of Nonlinear Oscillations* (New York: Gordon and Breach)
[8] Nayfeh A. H. 1981 *Perturbation Methods* (New York: Wiley–Interscience)
[9] Kuzmak G. E. 1959 *Prikl. Mat. Mekh.*. 23 515-526 (in Russian)
[10] Bourland F. J., Haberman R. 1988 *SIAM J. Appl.Math.*. 48 737-748
[11] Kalyakin L. A. 2003 *Doklady RAS* 388 305–308 (in Russian)
[12] Azhotkin V. D. and Babich V. M. 1985 *Prikl. Mat. Mekh.*. 49 377–383 (in Russian)