GEOMETRY OF ASYMPTOTIC BIAS REDUCTION OF PLUG-IN ESTIMATORS WITH ADJUSTED LIKELIHOOD

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A geometric framework to improve a plug-in estimator in terms of asymptotic bias is developed. It is based on an adjustment of a likelihood, that is, multiplying a non-random function of the parameter, called the adjustment factor, to the likelihood. The condition for the second-order asymptotic unbiasedness (no bias up to \( O(n^{-1}) \) for a sample of size \( n \)) is derived. Bias of a plug-in estimator emerges as departure from a kind of harmonicity of the function of the plug-in estimator, and the adjustment of the likelihood is equivalent to modify the model manifold such that the departure from the harmonicity is canceled out. The adjustment is achieved by solving a partial differential equation. In some cases the adjustment factor is given as an explicit integral. Especially, if a plug-in estimator is a function of the geodesic distance, an explicit representation in terms of the geodesic distance is available, thanks to differential geometric techniques for solving partial differential equations. As an example of the adjustment factor, the Jeffreys prior is specifically discussed. Some illustrative examples are provided.

1. Introduction. Derivation of the celebrated James-Stein estimator [13, 20] for mean of the multivariate normal model with known covariance is a well-known example that an estimator is improved by a geometric consideration. The James-Stein estimator demonstrates that the sample mean is inadmissible in terms of the mean squared error if the dimension is larger than two. For a sample of \( d(\geq 3) \)-dimensional vectors

\[
x_i \overset{i.i.d.}{\sim} N_d(z, I), \quad i \in \{1, \ldots, n\},
\]

where \( z \) is the mean vector in \( \mathbb{R}^d \) and \( I \) is the identity matrix, the James-Stein estimator of the mean vector \( z \) is given as an improvement of the sample mean, \( \bar{x} := n^{-1} \sum_{i=1}^{n} x_i \):

\[
\hat{z}_j = \bar{x}_j + \frac{1}{n} \partial_j \log h(\bar{x}), \quad \partial_j := \frac{\partial}{\partial x_j}, \quad j \in \{1, \ldots, d\},
\]

where \( h \) is a superharmonic function, that is, the Hessian of \( h \) satisfies \( \Delta h \leq 0 \). Here, \( \Delta := \sum_{i=1}^{d} \partial_i^2 \) is the Laplacian in the \( d \)-dimensional Euclidean space. If a superharmonic function \( h \) satisfies the Laplace equation, \( \Delta h = 0 \), \( h \) is called harmonic. A basic harmonic function is the fundamental solution (also called Green’s function) of \( \Delta h = 0 \) [10],

\[
h(x, x_0) = \frac{\Gamma\left\{(d-2)/2\right\}}{4\pi^{(d-2)/2}} |x - x_0|^{2-d}
\]

defined on \((x, x_0) \in \{(x, x_0) : \mathbb{R}^d \times \mathbb{R}^d, x \neq x_0\}\) and satisfying

\[
\Delta \int_{\mathbb{R}^d} h(x, x_0)f(x)dy = f(x_0)
\]
for any function \( f \in C_0^\infty(\mathbb{R}^d) \), where \( C_0^\infty(\mathbb{R}^d) \) is the set of infinitely differentiable functions defined on \( \mathbb{R}^d \) with a compact support. Substituting (2) with \( x_0 = 0 \) into (1), we obtain the James-Stein estimator

\[
\hat{x} = \left(1 - \frac{d-2}{n|x|^2}\right)\bar{x}.
\]

The factor \( \{1 - (d-2)/(n|x|^2)\} \) is called the shrinkage factor, because the sample mean \( \bar{x} \) is shrunk toward the ground mean, 0.

As an analogue of the James-Stein estimation, Efron and Morris [5, 6] considered estimation of the means \( z_i \in \mathbb{R} \), \( i \in \{1, \ldots, n\} \) of a normal mixed model with unknown variance \( \sigma^2 > 0 \):

\[
(3) \quad x_i|z_i \sim \text{N}(z_i, 1), \quad z_i \sim \text{N}(0, \sigma^2), \quad i \in \{1, \ldots, n\}.
\]

The best linear unbiased predictor of the means \( z_i \), which minimizes the mean squared error \( \mathbb{E}\{(\hat{z}_i - z_i)^2\} \) among all linear unbiased predictors is

\[
(4) \quad \hat{z}_i(\sigma^2) = \{1 - b(\sigma^2)\}x_i, \quad b(\sigma^2) := \frac{1}{1 + \sigma^2}, \quad i \in \{1, \ldots, n\}.
\]

We now change to call \( b(\sigma^2) \) the shrinkage factor because the predictors \( \hat{z}_i \) are shrunk toward the ground mean, 0, as the variance \( \sigma^2 \) decreases. If we regard \( \text{N}(0, \sigma^2) \) as the prior distribution for the means \( z_i \), the estimator (4) gives an empirical Bayes estimator of the means, if \( \sigma^2 \) maximizes the marginal likelihood

\[
(5) \quad e^{l(\sigma^2;x_1, \ldots, x_n)} = \left\{2\pi(1 + \sigma^2)\right\}^{-n/2} \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2(1 + \sigma^2)}\right),
\]

which is obtained by integrating out the means \( z_i \) in the model (3). Here, we simply call the estimator \( \hat{\sigma}_0^2 \) which maximize the likelihood (5) the maximum likelihood estimator of \( \sigma^2 \).

It is \( \hat{\sigma}_0^2 = \sum_{i=1}^n x_i^2/n - 1 \), where \( n(1 + \sigma_0^2)/(1 + \sigma^2) \sim \chi^2(n) \), because \( x_i \sim \text{N}(0, 1 + \sigma^2) \). Although \( \hat{\sigma}_0^2 \) is unbiased, the plug-in estimator of the shrinkage factor is biased:

\[
(6) \quad \mathbb{E}\{b(\hat{\sigma}_0^2) - b(\sigma^2)\} = \frac{1}{2} \frac{d^2 b}{d(\sigma^2)^2} \mathbb{E}\{(\hat{\sigma}_0^2 - \sigma^2)^2\} + o(n^{-1}) = \frac{1}{1 + \sigma^2} \text{Var}(\hat{\sigma}_0^2) + o(n^{-1}) = \frac{2}{(1 + \sigma^2)n} + o(n^{-1}),
\]

where the expectation is with respect to the product probability measure \( e^{l(\sigma^2;x_1, \ldots, x_n)} \prod_{i=1}^n dx_i \) given by (5). Suppose an adjustment of the log-likelihood by adding a non-random function of parameter as

\[
(7) \quad l(\sigma^2; x_1, \ldots, x_n) + \tilde{l}(\sigma^2), \quad \tilde{l}(\sigma^2) := \log(1 + \sigma^2).
\]

The adjustment of a log-likelihood with adding a non-random function, called an adjustment factor, was proposed by Lahiri and Li [16] in frequentist approach to linear mixed models. The maximum likelihood estimator of the adjusted likelihood, which we will call the adjusted maximum likelihood estimator, is \( \hat{\sigma}^2 = \sum_{i=1}^n x_i^2/(n - 2) - 1 \). Now, \( \hat{\sigma}^2 \) is biased:

\[
\mathbb{E}(\hat{\sigma}^2) = \frac{n}{n - 2}(1 + \sigma^2) - 1 = \sigma^2 + \frac{2(1 + \sigma^2)}{n} + o(n^{-1}),
\]

but the plug-in estimator of the shrinkage factor is unbiased up to \( O(n^{-1}) \) for large \( n \):

\[
(8) \quad \mathbb{E}\{b(\hat{\sigma}^2) - b(\sigma^2)\} = \frac{db}{d(\sigma^2)} \mathbb{E}(\hat{\sigma}^2 - \sigma^2) + \frac{1}{2} \frac{d^2 b}{d(\sigma^2)^2} \mathbb{E}\{(\hat{\sigma}^2 - \sigma^2)^2\} + o(n^{-1}) = o(n^{-1}).
\]
Note that the expectation is still with respect to the probability measure given by (5); the adjusted likelihood (7) no longer gives a probability measure.

As is mentioned above, the bias reduction of plug-in estimators based on adjusted likelihoods introduced here was proposed in [16] for estimation of the dispersion parameter in linear mixed models. After that, the adjusted maximum likelihood estimation has been investigated as in Li and Lahiri [18], Yoshimori and Lahiri [21, 22], and Hirose and Lahiri [11, 12]. For the case that the mean of \( z_i \) is unknown, Hirose and Lahiri [11] obtained a plug-in estimator of the shrinkage factor without bias up to \( O(n^{-1}) \) with using a specific adjusted maximum likelihood estimator of \( \sigma^2 \).

What we have seen on the bias reduction of the plug-in estimation of the shrinkage factor in the normal mixed model (3) has a simple geometrical interpretation. In the expression (6), the bias of the plug-in estimator emerges with the Hessian of the shrinkage factor, \( \frac{d^2 b}{d(\sigma^2)^2} \). Then, in the expression (8), the bias is canceled out with using the gradient of the shrinkage factor, \( \frac{db}{d(\sigma^2)} \). To cancel out the bias, the ratio of the first centered moment and the second centered moment of the estimator \( \hat{\sigma}^2 \) should be tuned. This tuning is implemented as the maximization of the adjusted likelihood. The bias reduction of the plug-in estimator is achieved by a proper choice of the adjustment factor, and a geometric view is useful for the purpose. Motivated by this perspective, we have developed a novel geometric framework to improve a plug-in estimator in terms of asymptotic unbiasedness, which is presented in this paper.

Although this paper has rather frequentist flavor, we mention here some connections with improvement of Bayesian predictive distributions. Komaki [15] showed that the Bayesian predictive distribution based on a prior, which is the product of the fundamental solution of the Laplace equation and the Jeffreys prior, asymptotically dominates the Bayesian predictive distribution based on the Jeffreys prior in terms of the Kullback-Leibler loss. In the same line of thought, Eguchi and Yanagimoto [7] discussed asymptotic improvement of maximum likelihood estimators in terms of the Kullback-Leibler loss. They considered adding a correction term to the maximum likelihood estimator. The correction term was given as the gradient of some function as in the James-Stein estimator (1), where \( h \) could be a prior density function. Our approach to improve an estimator is different from that in [7]; we do not correct an estimator directly, rather, we improve an estimator through adjusting the likelihood. The improved estimator is derived as the adjusted maximum likelihood estimator. Aitchson demonstrated that a Bayesian prediction improves a plug-in prediction [1]. The improvement can be attributed to the fact that the Bayesian predictive density can exit from the model manifold, but the plug-in density cannot. Komaki [14] constructed such an optimal predictive density by shifting model manifold in the direction orthogonal to the model manifold. The adjustment of the likelihood is another way to modify the model manifold such that a plug-in estimator has better performance in terms of bias than that with the original model manifold. Rather than reducing the Kullback-Leibler divergence, the modification of the model manifold shifts the minimizer of the Kullback-Leibler divergence, and the minimizer can give better plug-in estimator than the original one.

The previous works on improvement of an estimator in terms of measures of discrepancy, such as the squared loss or the Kullback-Leibler loss, essentially rely on the fundamental solution of the Laplace equation. More precisely, the previous works do not demand solving partial differential equations from scratch. On the other hand, the choice of the adjustment factor discussed in this paper is equivalent to solving a partial differential equation. We sometimes have to solve a partial differential equation involving a variant of Laplace-Beltrami operator, which we call the skewed-Laplacian. For a plug-in estimator of a function of the geodesic distance, some explicit results are available thanks to differential geometric techniques for solving partial differential equations, which was developed by Hadamard [8] and Riesz [19].
This paper is organized as follows. In Section 2, it is shown that the adjusted maximum likelihood estimation is the minimization of the Kullback-Leibler divergence between the empirical distribution and the denormalized measure determined by an adjustment factor. This interpretation does not always hold, but provides a geometric idea how the adjusted maximum likelihood estimation works. Our main result is presented in Section 3. A way to state it is as follows: the bias of a plug-in estimator emerges as the departure from the skewed-harmonicity of the function of the plug-in estimator, and to erase the bias, the foliation determined by the adjustment factor should be chosen such that the departure from the harmonicity is canceled out at each point of the model manifold. This condition is explicitly integrated in one-dimensional cases. As a by-product, we establish a dual geometric property in plug-in estimation for an exponential family: the bias of the plug-in estimator of the expectation of the sufficient statistics with the maximum likelihood estimator of the natural parameter is $o(n^{-1})$. Then, we discuss the bias of what estimator can be reduced by a given adjustment factor by exemplifying with the Jeffreys prior. For a given function, choice of an adjustment factor is not at all unique. In Section 4, we discuss the case that the foliation determined by the adjustment factor is the same as the foliation determined by the function of the plug-in estimator. In this case, the adjustment factor is given as an explicit integral, and further explicit expressions in terms of the normal coordinate system are available. Section 5 is devoted to presentation of some simple but typical examples to illustrate how adjustment factors are constructed. The examples include the multivariate normal model with known covariance, where the model manifold is the Euclidean space, the location-scale model, where the model manifold is the hyperbolic space, and the nested error regression model in small area estimation, where the model manifold is dually flat.

This paper is intended to be a starting point of discussion of geometry of asymptotic bias reduction of plug-in estimators with adjusted likelihoods. Many issues remain to be investigated. For example, as the examples in Section 5 suggest, the construction of an adjustment factor could be discussed in homogeneous spaces, because the associated partial differential equations are tractable [9]. Characterization of the skewed-Laplacian should be given. Practical methods to construct an adjustment factor and further applications should be provided. These issues will be discussed in subsequent papers.

2. Adjustment as denormalization of model manifold. Let us recall some basic facts around the Kullback-Leibler divergence and denormalized measures, which give us a geometric interpretation of the adjusted maximum likelihood estimation.

The Kullback-Leibler divergence between two non-negative finite measures $p$ and $q$ on a sample space $\mathcal{X}$ are defined as

\begin{equation}
D(p\|q) := \int_{\mathcal{X}} q(x)dx - \int_{\mathcal{X}} p(x)dx + \int_{\mathcal{X}} p(x) \log p(x)dx - \int_{\mathcal{X}} p(x) \log q(x)dx.
\end{equation}

Jansen’s inequality leads $D(p\|q) \geq 0$, and it is saturated if and only if $p = q$. The denormalization of a model manifold is employed in Chapters 2 and 3 of [3] to describe the geometry of dual connections. If we consider inferences of an unknown probability measure $p$ with an unnormalized parametric model $q(x; \xi)$ with parameters $\xi$, the Kullback-Leibler divergence (9) between $p$ and $q(x; \xi)$ becomes

\begin{equation}
D\{p\|q(x; \xi)\} = \int_{\mathcal{X}} q(x; \xi)dx - 1 + \int_{\mathcal{X}} p(x) \log p(x)dx - \int_{\mathcal{X}} p(x) \log q(x; \xi)dx.
\end{equation}

If we use a normalized model, the minimization of the Kullback-Leibler divergence is equivalent to the minimization of the cross entropy

\begin{equation}
C\{p\|q(x; \xi)\} = -\int_{\mathcal{X}} p(x) \log q(x; \xi)dx.
\end{equation}
Note that only the last term in (10) is relevant in the minimization. Based on an iid sample \((x_1, \ldots, x_n) \sim p\), we may estimate \(\hat{\xi}_0\) as a minimizer of the cross entropy between the empirical distribution \(p\) and the model \(q(x; \xi)\):

\[
(11) \quad \hat{\xi}_0(x_1, \ldots, x_n) = \arg \min_{\xi} \left\{ -\frac{1}{n} \sum_{i=1}^{n} \log q(x_i; \xi) \right\}.
\]

On the other hand, if we use an unnormalized model, the first term in (10) is relevant in the minimization of the Kullback-Leibler divergence. Let us consider a denormalized model \(q(x; \xi) = q_0(x; \xi) z(\xi), \quad z(\xi) := \int_X q(x; \xi) \, dx\) of a normalized model \(q_0(x; \xi)\). The minimization of the Kullback-Leibler divergence between the empirical distribution of \(p\) and the model \(q(x; \xi)\) is now achieved by

\[
(12) \quad \hat{\xi}(x_1, \ldots, x_n) = \arg \min_{\xi} \left\{ -\frac{1}{n} \left( \sum_{i=1}^{n} \log q_0(x_i; \xi) + \tilde{l}(\xi) \right) \right\},
\]

where

\[
(13) \quad \tilde{l}(\xi) := n \{ 1 - z(\xi) + \log z(\xi) \} \leq 0
\]

is the adjustment factor discussed in Introduction. The equality is saturated if and only if \(q(x; \xi)\) is normalized. The adjusted maximum likelihood estimator \(\hat{\xi}\) is an M-estimator satisfying the random equation

\[
\frac{\partial}{\partial \xi_j} \sum_{i=1}^{n} \left\{ \log q_0(x_i; \hat{\xi}(x)) + \frac{1}{n} \tilde{l}(\hat{\xi}(x)) \right\} = 0, \quad j \in \{1, \ldots, d\},
\]

and the adjusted log-likelihood is

\[
l(\xi; x_1, \ldots, x_n) + \tilde{l}(\xi),
\]

where \(l(\xi; x_1, \ldots, x_n) = \sum_{i=1}^{n} \log q_0(x_i; \xi)\).

As we have seen, the adjusted maximum likelihood estimator introduced in Introduction can be interpreted as the minimization of the Kullback-Leibler divergence between the empirical distribution and the denormalized measure determined by an adjustment factor. Throughout this paper, we will assume an adjustment factor \(\tilde{l}(\xi)\) to be \(O(1)\). Therefore, as (13) implies, we consider a small denormalization of \(O(n^{-1})\). Unfortunately, this interpretation does not always hold. If we regard an adjustment of a likelihood as a denormalization of a model manifold, as in (13), the adjustment factor \(\tilde{l}(\xi)\) should be negative. If \(\tilde{l}(\xi)\) is positive, as in the normal mixed model in Introduction, an adjustment of a likelihood should not be regarded as a denormalization of a model manifold. In such a case, the adjusted maximum likelihood estimation implemented as the minimization (12) should be simply regarded as a modification of the cross-entropy minimization (11).

There is a reason why a denormalization of a model manifold leads to negative adjustment factor: a denormalization increases the Kullback-Leibler divergence between the empirical distribution and a point in the parametric model, because the empirical distribution is a realization of a normalized measure. Nevertheless, in the minimization of the Kullback-Leibler divergence between the empirical distribution and the denormalized model, the minimizer will shift and it can give better estimation than the original normalized model in terms of bias. We will use this room to improve an estimator in the adjusted maximum likelihood estimation.
3. The skewed-Laplacian and asymptotic bias reduction. In this section we present our main result and the direct consequences. To present the results, the definition of skewed-Laplacian and related concepts in differential geometry of statistical model manifolds are inevitable. We begin this section with preparing them. For more backgrounds on differential geometry of statistical model manifolds, sometimes called information geometry, see e.g. books [2, 3] and a concise survey in [17]. In the following, we adopt the summation convention that whenever an index appears in an expression as upper and lower, we sum over that index.

For a sample space \( \mathcal{X} \), a parametric model \( q(\cdot; \xi) \) with parameters \( \xi \in \Xi \) is a family of probability measures \( \mathcal{P} \) induced by a map \( U \to \Xi \subset \mathbb{R}^d \), where \( U \) is an open subset of a family of probability measures and the parameter space \( \Xi \) is an open subset of \( \mathbb{R}^d \). We assume \( q(\cdot; \xi) \) to be a \( C^\infty \)-function of \( \xi \). Then, we get a family of probability measures \( \mathcal{P} \) equipped with a differentiable structure, which admits various local coordinate systems. Under suitable regularity assumptions, \( \mathcal{P} \) is equipped with a Riemannian metric, called the Fisher metric tensor

\[
\gamma_{ij}(\xi) = \mathbb{E}[(\partial_i l)(\partial_j l)], \quad \partial_i := \frac{\partial}{\partial \xi^i},
\]

where \( l(\xi; x) = \log q(x; \xi) \) and the expectation is with respect to the probability measure \( q(x; \xi) \). The inner product of vectors \( f \) and \( h \) are denoted by \( \langle f, h \rangle := g^{ij} f_i h_j \), and the determinant of the Fisher metric tensor is denoted by \( g \). The \( \alpha \)-connections are defined by

\[
\Gamma^{(\alpha)}_{ij,k} := \mathbb{E}[(\partial_i (\partial_j l))(\partial_k l)] + \frac{1 - \alpha}{2} S_{ijk}, \quad \alpha \in \mathbb{R},
\]

where the symmetric tensor of order three,

\[
S_{ijk} := \mathbb{E}[(\partial_i l)(\partial_j l)(\partial_k l)]
\]

is called the skewness tensor (also called the Amari-Chentsov tensor). The contracted vector, the skewness vector, \( S_i \equiv g_i^k S_{ijk} \) will sometimes appear. The 0-connection is called the Riemannian connection (also called the Levi-Civita connection). The \( \alpha \)-connection and the \((-\alpha)\)-connection are dual to each other with respect to the Fisher metric tensor (14), namely, \( \partial_k g_{ij} = \Gamma^{(\alpha)}_{k,i,j} + \Gamma^{(-\alpha)}_{k,i,j} \). The covariant derivative of the Fisher metric tensor is

\[
\nabla^{(\alpha)}_i g_{jk} := \partial_i g_{jk} - \Gamma^{(\alpha)}_{ik,j} - \Gamma^{(\alpha)}_{ij,k} = \alpha S_{ijk}.
\]

The \( \alpha \)-connection (15) is torsion free, and is the unique connection satisfies (16) with respect to the skewness tensor [17]. Because the covariant derivative of the metric in 0-connection is 0, the 0-connection is given by derivatives of the metric. Now, a model manifold is a Riemannian manifold with the skewness tensor, denoted by a triple \( (\mathcal{M}, g, S) \), where \( \mathcal{M} \) is an \( d \)-dimensional \( C^\infty \)-manifold, \( g \) is the Fisher metric tensor, and \( S \) is the skewness tensor.

Let us define \( \alpha \)-Laplacian operating on a scalar \( f \):

\[
\Delta^{(\alpha)} f := \nabla^{(\alpha)}_i \nabla^{(\alpha)}_i f = g^{ij} \partial_i \partial_j f - g^{ij} g^{kr} \Gamma^{(\alpha)}_{kr,i} \partial_j f.
\]

If \( \alpha = 0 \), 0-Laplacian is the Laplace-Beltrami operator with 0-connection satisfying

\[
\Delta^{(0)} f = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f).
\]

If a function \( f \) satisfies \( \Delta^{(\alpha)} f = 0 \), we will say the function \( f \) is \( \alpha \)-harmonic. The \( \alpha \)-Laplacian is an example of an extension of the Laplacian proposed in [7]. Using the \( \alpha \)-Laplacian, let us define the skewed-Laplacian, denoted by \( \Delta^S \). If a function \( f \) satisfies \( \Delta^S f = 0 \), we will say the function \( f \) is skewed-harmonic.
DEFINITION 3.1. The skewed-Laplacian operating on a scalar $f$ is defined as
\begin{equation}
\Delta^S f := \Delta f + g^{ij} g^{kr} (\Gamma^{(1)}_{jk,r} - S_{jk,r}) \partial_i f = g^{ij} \partial_i \partial_j f + g^{ij} g^{kr} (2\Gamma^{(1)}_{jk,r} - \partial_r g_{jk}) \partial_i f
\end{equation}
for $\alpha \in \mathbb{R}$.

REMARK 3.2. As is seen in the right hand side of (17), the skewed Laplacian does not depend on $\alpha$. However, specific choices of coordinate system are helpful for each problem. Especially, for an $\alpha(\neq 0)$-flat manifold with an $\alpha$-affine coordinate system, with using (16) and $g^{ij} \partial_k g_{ij} = \partial_k \log g$, we have a simple expression of the skewed-Laplacian:
\begin{equation}
\Delta^S f = g^{ij} \partial_i \partial_j f - \frac{1}{\alpha} g^{ij} (\partial_i \log g) \partial_j f.
\end{equation}

We prepare the following lemma. The proof is given in Appendix.

LEMMA 3.3. Under the regularity conditions B1-B6 in Appendix, for an adjustment factor $\tilde{l}(\xi) \in C^1_b(\mathbb{R}^d)$ of $O(1)$, where $C^1_b(\mathbb{R}^d)$ is the set of differentiable bounded functions defined on $\mathbb{R}^d$, we have
(i) $\mathbb{E}[\hat{\xi} - \xi]^i = g^{ij} \left\{ \partial_j \tilde{l} + g^{kr} (\Gamma^{(1)}_{jk,r} - \frac{1}{2} \partial_r g_{jk}) \right\} + o(n^{-1})$, and
(ii) $\mathbb{E}[(\hat{\xi} - \xi)^i(\hat{\xi} - \xi)^j] = g^{ij} + o(n^{-1})$,
for large $n$.

Then, the main result of this paper is stated in the following lemma, whose special forms for specific models have appeared in [21, 11]. The inner product of $\partial_i f$ and $\partial_j g$ will be denoted by $\langle \partial_i f, \partial_j g \rangle$.

LEMMA 3.4. Under the regularity conditions of Lemma 3.3, for a function $f(\xi) \in C^2_b(\mathbb{R}^d)$ and an adjustment factor $\tilde{l}(\xi) \in C^1_b(\mathbb{R}^d)$ of $O(1)$ satisfying
\begin{equation}
\Delta^S f + 2 \langle \partial_i \tilde{l}, \partial_i f \rangle = o(n^{-1}),
\end{equation}
the bias of the plug-in estimator $f(\hat{\xi})$, where $\hat{\xi}$ is the adjusted maximum likelihood estimator of $\xi$ with the adjustment factor $\tilde{l}(\xi)$, is $o(n^{-1})$.

PROOF. With Lemma 3.3, the bias is evaluated as
\begin{align*}
\mathbb{E}\{f(\hat{\xi}) - f(\xi)\} &= \mathbb{E}\{\hat{\xi} - \xi\} \partial_i f + \mathbb{E}\{(\hat{\xi} - \xi)^i(\hat{\xi} - \xi)^j\} \frac{1}{2} \partial_i \partial_j f + o(n^{-1}) \\
&= g^{ij} \left\{ \partial_j \tilde{l} + g^{kr} (\Gamma^{(1)}_{jk,r} - \frac{1}{2} \partial_r g_{jk}) \right\} \partial_i f + \frac{1}{2} \partial_i \partial_j f + o(n^{-1}).
\end{align*}
The assertion follows from the definitions of the skewed-Laplacian (17).

REMARK 3.5. An estimator whose asymptotic bias is $o(n^{-1})$ is called a second-order unbiased estimator. Lemma 3.4 says that the plug-in estimator $f(\hat{\xi})$ of a function $f(\xi)$ satisfying the condition (19) is a second-order unbiased estimator.

Note that (19) is a scalar condition. It does not depend on a choice of a coordinate system, so we may choose any coordinate system for our convenience. As we will see, specific choices of coordinate system are helpful for considering each problems; various coordinate systems and $\alpha$-connections will be employed.
The condition (19) is satisfied with
\[
\Delta^S f + 2 \langle \partial \tilde{l}, \partial f \rangle = 0,
\]
which is interpreted as follows: the bias of a plug-in estimator emerges as departure from the skewed-harmonicity, \( \Delta^S f \neq 0 \), and the adjustment of the likelihood is equivalent to modify the model manifold with the adjustment factor \( \tilde{l} \) such that the departure from the harmonicity is canceled out. The estimator \( f \) determines a foliation of the model manifold \( M \), say \( \{ N_f : f \in f(\mathbb{R}) \} \), because \( \partial_i f, i \in \{1, \ldots, d\} \), is the normal vector to a hypersurface of codimension one, \( N_{f_0} = \{ \xi : f(\xi) = f_0 \}, f_0 \in f(\mathbb{R}) \). In the same manner, the adjustment factor \( \tilde{l} \) determines a foliation of \( M \), say \( \{ \tilde{N}_f : \tilde{l} \in \tilde{l}(\mathbb{R}) \} \).

For the one-dimensional model manifold \( M \) (\( d = 1 \)), the condition (19) can be integrated explicitly. We omit the proof because it is a straightforward computation.

**Theorem 3.6.** Consider a one-dimensional model manifold. Under the regularity conditions of Lemma 3.3, for a function \( f \in C^2_b(\mathbb{R}) \) with \( f'(\xi) > 0 \), the bias of the plug-in estimator \( \hat{f}(\xi) \), where \( \hat{f} \) is the adjusted maximum likelihood estimator of \( \xi \), is erased up to \( O(n^{-1}) \) by an adjustment factor \( \tilde{l}(\xi) \in C^1_b(\mathbb{R}) \) satisfying
\[
e^{\tilde{l}(\xi)} \propto \frac{1}{\sqrt{f'(\xi)}} e^{\frac{1}{2} \int \xi^T T_1(\xi) d\xi}.
\]
Moreover, if \( \alpha 
eq 0 \), for an \( \alpha \)-flat manifold with an \( \alpha \)-affine coordinate system \( \xi \),
\[
e^{\tilde{l}(\xi)} \propto \frac{\{g(\xi)\}^{1/(2\alpha)}}{\sqrt{f'(\xi)}}.
\]

**Remark 3.7.** The condition \( f'(\xi) > 0 \) appears in Theorem 3.6 (and also in Lemma 4.1 and Theorem 4.2) is not essential as long as \( f'(\xi) \neq 0 \). If we have a function \( f'(\xi) < 0 \), we can switch to consideration of the function \( -f(\xi) \).

**Remark 3.8.** For a 1-flat manifold with a 1-affine coordinate system, the Jeffreys prior, \( e^{\tilde{l}(\xi)} \propto \sqrt{g(\xi)} \), erases the bias of the plug-in estimator of a linear function \( f(\xi) \) up to \( O(n^{-1}) \). A multi-dimensional analogue is Corollary 3.12.

**Example (A normal mixed model).** The log-likelihood (5),
\[
l(\xi) = \xi \sum_{i=1}^{n} y_i^2 + \frac{n}{2} \log(\xi) + \text{const.,}
\]
determines the 1-flat manifold with the 1-affine coordinate system \( \xi = 1/(2(1 + \sigma^2)) \). The determinant of the Fisher metric tensor is \( g = n/(2\xi^2) \). For the plug-in estimator of the shrinkage factor, substituting \( f(\xi) = -b(\sigma^2) = 2\xi \) into (21), we obtain the adjustment factor \( e^{\tilde{l}(\xi)} \propto \xi^{-1} \propto (1 + \sigma^2) \). This is what we have seen in Introduction. Moreover, the adjustment factor is the Jeffreys prior.

We collect some consequences of Lemma 3.4 for multi-dimensional models. In the following discussion, the assumptions of Lemma 3.4 will be assumed.

**Corollary 3.9.** The bias of the plug-in estimator \( f(\hat{\xi}) \), where \( \hat{\xi} \) is the (non-adjusted) maximum likelihood estimator of \( \xi \), for a skewed-harmonic function \( f(\xi) \), that is, a function \( f \) satisfying
\[
\Delta^S f = 0,
\]
is \( o(n^{-1}) \).
Moreover, for a 1-flat model manifold, there is an interesting dual property.

**Corollary 3.10.** For a 1-flat model manifold with a 1-affine coordinate system, the bias of the plug-in estimator \( \eta(\xi) \), where \( \xi \) is the (non-adjusted) maximum likelihood estimator of 1-affine coordinate \( \xi \), and \( \eta(\xi) \) is the \((-1)\)-affine coordinate as a function of \( \xi \), is \( o(n^{-1}) \). The assertion holds if we interchange 1-flat and 1-affine by \((-1)\)-flat and \((-1)\)-affine, respectively.

**Proof.** For a 1-flat model manifold, the \((-1)\)-affine coordinate system, \( \eta_i, i \in \{1, \ldots, d\} \) satisfies \( g_{ij} = \partial_i \eta_j \) (see Chapter 3.5 of [3]). Therefore, if we choose \( f(\xi) = \eta_k(\xi) \) for \( k \in \{1, \ldots, d\} \), it is skew-harmonic, because \( (18) \) yields

\[
\Delta^S \eta_k = g^{ij} \partial_i \partial_j \eta_k - g^{ij} (\partial_i \log g) \partial_j \eta_k = g^{ij} \partial_i g_{jk} - g^{ij} \partial_i g_{jk} = 0.
\]

Then, the assertion follows from Corollary 3.9. The second assertion follows in the same way, but \( g^{ij}(\eta) = \partial^2 \eta / \partial \eta_i \partial \eta_j \) and \( \partial^2 \log g(\eta) = -g_{jk}(\eta) \{ \partial g^{jk}(\eta) / \partial \eta_i \} \) are used.

**Remark 3.11.** This corollary yields a property of a plug-in estimation of an exponential family: the bias of a plug-in estimator of the expectation of the sufficient statistics with the maximum likelihood estimator of the natural parameter is \( o(n^{-1}) \).

We may ask the bias of what estimator can be reduced by a given adjustment factor. In fact, Hirose and Lahiri [11, 12] discussed adjustment factors as Bayesian priors. If we regard an adjustment factor as a prior, the adjusted maximum likelihood estimate is the maximum a posteriori estimate. Here, as an example, we employ the Jeffreys prior as the adjustment factor, because the Jeffreys prior, \( \sqrt{g(\xi)} \), has a special meaning in differential geometry, that is, \( \sqrt{g(\xi)} d^1 \xi \wedge \cdots \wedge d^d \xi \) is the volume form of an orientable Riemannian manifold. The condition \( (19) \) becomes

\[
\Delta^S f + g^{ij} (\partial_i \log g) \partial_j f = o(n^{-1}).
\]

In particular, for a 1-affine model manifold with a 1-affine coordinate system, by \( (18) \) the condition \( (23) \) becomes \( \Delta^{(1)} f = g^{ij} \partial_i \partial_j f = o(n^{-1}) \) and thus we have the following corollary.

**Corollary 3.12.** For a 1-flat model manifold with a 1-affine coordinate system, the bias of the plug-in estimator of a 1-harmonic function \( f(\xi) \), where \( \xi \) is the adjusted maximum likelihood estimator of \( \xi \), is erased by the Jeffreys prior up to \( O(n^{-1}) \).

**Remark 3.13.** The assertion does not hold if 1-flat and 1-affine are replaced with \((-1)\)-flat and \((-1)\)-affine, respectively. Corollary 3.12 shows an affinity of the Jeffreys prior and exponential families.

For a generic (not necessarily \( \alpha \)-flat) model manifold, by using \( \partial_i \log g = 2 g^{jk} T_{ij,k}^{(0)} \), the condition \( (23) \) in a 0-coordinate system reduces to the following form:

\[
\Delta^{(0)} f + g^{ij} \left\{ \frac{3}{2} (\partial_i \log g) - S_i \right\} \partial_j f = o(n^{-1}).
\]

This is a homogeneous elliptic partial differential equation of \( f \). Hadamard demonstrated in his book [8] that the fundamental solution of (mainly hyperbolic) partial differential equations can be constructed as a convergent series of square of the geodesic distance associated with the Laplace-Beltrami operator. Riesz further developed the theory [19]. Their discussion is based on the Hamiltonian formulation of the variational problem of obtaining
the 0-geodesic as the curve of shortest length connecting two points. Note that the length-minimizing property distinguishes the 0-geodesics from other $\alpha$-geodesics. Our equation (24) can be integrated as a function of the geodesic distance, as is shown in Theorem 3.15.

We prepare the following lemma.

**Lemma 3.14 ([19]).** The geodesic distance $r = r(\xi, \xi_0)$ of $\xi$ from $\xi_0$ along with the 0-geodesic extends satisfies

(i) $\frac{1}{2} \partial_i r^2 = g_{ij} \frac{d \xi^i}{d r}$, 
(ii) $\Delta^{(0)} f = f'(r^2) \Delta^{(0)}(r^2) + 4r^2 f''(r^2)$, and 
(iii) $\Delta^{(0)}(r^2) = 2d + r \frac{d}{d r} \{ \log \bar{g}(\zeta) \}$, where $\zeta$ is the normal coordinate system whose origin is $\xi_0$.

(i), (ii), and (iii) are equations (37), (55), and (57), respectively, in Chapitre VII of [19].

**Theorem 3.15.** Under the regularity conditions of Lemma 3.3, the Jeffreys prior erases the bias of the plug-in estimator $f(r(\hat{\xi}))$, where $\hat{\xi}$ is the adjusted maximum likelihood estimator of $\xi$,

$$ f(r) \propto \int_0^r \frac{1}{r^{d-1}} \left\{ \frac{\bar{g}(\zeta)}{r} \right\}^2 \exp \left\{ \int \frac{\bar{S}_i(\zeta)}{r} d \bar{\xi}^i \right\} d \bar{r}, $$

up to $O(n^{-1})$. Here, $r := \text{dis}(\xi, \xi_0)$ is the geodesic distance of $\xi$ from $\xi_0$, and $\bar{g}(\zeta)$ and $\bar{S}_i(\zeta)$ are the determinant and the skewness vector in the normal coordinate systems $\zeta$ whose origin is $\xi_0$, respectively.

**Proof.** It is convenient to work with $f(r^2)$. We observe

$$ g^{ij} g^{kr} \frac{\partial g_{kr}}{\partial \xi^i} \frac{\partial f}{\partial \xi^j} = g^{ij} g^{kr} \frac{\partial \bar{g}_{kr}}{\partial \bar{\xi}^i} \frac{\partial f}{\partial \bar{\xi}^j} = \bar{g}^{ij} \frac{\partial \log \bar{g}(\zeta)}{\partial \bar{\xi}^j} \frac{\partial f}{\partial \bar{\xi}^j}, $$

where the first equality follows by the coordinate transformation of the scalar, and the last equality follows from a property of the geodesic distance: $g^{ij}(\partial r/\partial \xi^i)(\partial r/\partial \xi^j) = 1$ for any coordinate system. Substituting (i) and (ii) of Lemma 3.14 and (25) into (24) with the right hand side is 0, we obtain

$$ f''(r^2) + \left[ \frac{d}{2r^2} + \frac{1}{r} \frac{d}{d r} \{ \log \bar{g}(r) \} - \frac{1}{2r} \bar{S}_i \frac{\partial r}{\partial \xi^i} \right] f'(r^2) = 0. $$

The integral is

$$ \log f'(r^2) + \frac{d}{2} \log r^2 + 2 \log \bar{g}(r) - \int_S \bar{S}_i \frac{\partial \bar{r}}{\partial \bar{\xi}^i} d \bar{r} = \text{const.} $$

Using (iii) of Lemma 3.14, we have

$$ \bar{S}_i \frac{\partial r}{\partial \bar{\xi}^i} = \frac{S_i d \xi^i}{r} \frac{d}{d r} = \bar{S}_i \frac{d \bar{\xi}^i}{r \frac{d}{d r}}. $$

Integration of (26) yields the assertion.
For a given function, choice of an adjustment factor is not at all unique. In the next section, we will consider the case that an adjustment factor is chosen such that the foliation of the model manifold determined by the adjustment factor is the same as the foliation of the model manifold determined by the function of the plug-in estimator. In such a case, the adjustment factor is given as an explicit integral. Nevertheless, other choice of adjustment factor would be useful. See Section 5 for such examples.

4. Adjustment within the foliation determined by the estimator. Let us seek an adjustment factor such that the foliation of the model manifold $\mathcal{M}$ determined by the adjustment factor $\{\tilde{N}_\ell : \ell \in \tilde{I}(\mathbb{R}^d)\}$ is the same as the foliation of $\mathcal{M}$ determined by the function $f$ of the plug-in estimator, $\{\tilde{N}_f : f \in \tilde{f}(\mathbb{R}^d)\}$. In such a case, an adjustment factor is given as an explicit integral, because the choice of the adjustment factor essentially reduces to a one-dimensional integration. To construct such an foliation of $\mathcal{M}$, let us assume there is a $C^2$ map $\gamma : \mathbb{R}^d \to \mathbb{R}$, and consider parameterization $\tilde{l} = \tilde{l}(\gamma(\xi))$ and $f = f(\gamma(\xi))$, $\xi \in \mathbb{R}^d$ such that $\tilde{l}(\gamma(\mathbb{R}^d)) = \tilde{I}(\mathbb{R}^d)$ and $f\{\gamma(\mathbb{R}^d)\} = \tilde{f}(\mathbb{R}^d)$. Then, we may represent the foliations of $\mathcal{M}$ determined by the adjustment factor or the function $f$ of the plug-in estimator as $\{\tilde{N}_\gamma : \gamma \in \gamma(\mathbb{R}^d)\}$. Here, $\gamma$ is a curve in $\mathcal{M}$ and the foliation $\{\tilde{N}_\gamma : \gamma \in \gamma(\mathbb{R}^d)\}$ consist of hypersurfaces of codimension one perpendicular to the curve.

Under this assumption, the condition (19) can be integrated explicitly. We omit the proof because it is a straightforward computation.

Lemma 4.1. Under the regularity conditions of Lemma 3.3, for a function $f \in C^2_0(\mathbb{R})$ with $f'(\gamma) > 0$, the bias of the plug-in estimator $f(\gamma(\hat{\xi}))$, where $\hat{\xi}$ is the adjusted maximum likelihood estimator of $\xi$, is erased up to $O(n^{-1})$ by an adjustment factor $\tilde{l}(\gamma)$ satisfying

$$e^{\tilde{l}(\gamma)} \propto \frac{1}{\sqrt{f'(\gamma)}} \exp \left( -\frac{1}{2} \int_\gamma \frac{\Delta^\gamma}{\partial \gamma_x \partial \gamma_y} d\gamma ight).$$

More explicit result is available in a 0-coordinate system if the curve $\gamma$ is a 0-geodesic.

Theorem 4.2. Under the regularity conditions of Lemma 3.3, for a function $f \in C^2_0(\mathbb{R})$ with $f'(r) > 0$, the bias of the plug-in estimator $f(r(\hat{\xi}))$, where $\hat{\xi}$ is the adjusted maximum likelihood estimator of $\xi$, is erased up to $O(n^{-1})$ by an adjustment factor $\tilde{l}(r)$ satisfying

$$e^{\tilde{l}(r)} \propto \frac{1}{\sqrt{f''(r^2) r^{d_g} g(\xi)}} \exp \left( \frac{1}{2} \int S_i(\xi) d\xi^i \right),$$

where $r := \text{dis}(\xi, \xi_0)$ is the geodesic distance of $\xi$ from $\xi_0$, and $g(\xi)$ and $S_i(\xi)$ are the determinant of and the skewness tensor in the normal coordinate systems $\xi$ whose origin is $\xi_0$, respectively.

Proof. In the 0-coordinate system, the condition (19) with the right hand side is 0 becomes

$$\Delta^{(0)} f + 2 \langle \partial l, \partial f \rangle = 0, \quad l = \tilde{l} - \frac{1}{2} \int \tilde{S}_i d\xi^i + \frac{1}{4} \log \tilde{g}.$$ 

Then, the condition (19) becomes

$$\tilde{l}'(r^2) = -\frac{1}{2} \left\{ \frac{f''(r^2)}{f'(r^2)} + \frac{\Delta(0)(r^2)}{4r^2} \right\}.$$
Using (iii) of Lemma 3.14, the integration gives
\[ \int \frac{\Delta^{(0)}(r^2)}{r^2} dr^2 = 2d \log r^2 + \int \frac{1}{r} d r^2 (\log \bar{g}) dr^2 = 2d \log r^2 + 2 \log \bar{g} + \text{const.} \]
and we obtain
\[ \bar{I}(r^2) = -\frac{1}{2} \log \left\{ f'(r^2) r^d \sqrt{\bar{g}} \right\} + \text{const.} \]

\[ \Box \]

5. Examples. In this section several simple examples are presented to illustrate how adjustment factors introduced in previous sections are constructed.

5.1. The multivariate normal model with known covariance. Let us consider the \( d \)-variate normal model \( N_d(\xi, \Sigma) \), where \( \xi = (\xi^1, ..., \xi^d) \in \mathbb{R}^d \) is an unknown mean vector and \( \Sigma \) is a known variance-covariance matrix. We consider cases with \( d \geq 2 \). For simplicity of expressions, we set \( \Sigma = I \), where \( I \) is the identity matrix. The model manifold is the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) endowed with the Fisher metric tensor \( g_{ij} = \delta_{ij} \) and the zero skewness tensor. The condition (19) of Lemma 3.4 is satisfied if
\[ \Delta^{(0)} f + 2 \langle \partial \bar{I}, \partial f \rangle = 0, \quad \Delta^{(0)} = \sum_{i=1}^{d} \partial_i^2. \]
Since \( \alpha \)-Laplacians do not depend on \( \alpha \), we use the 0-Laplacian. Let us introduce the polar coordinate system:
\[ \begin{align*}
\xi^1 &= r \cos \theta_1, \\
\xi^i &= r \prod_{j=1}^{i-1} \sin \theta_j \cos \theta_i, \quad i \in \{2, 3, ..., d-2\}, \\
\xi^{d-1} &= r \cos \phi \prod_{j=1}^{d-2} \sin \theta_j, \\
\xi^d &= r \sin \phi \prod_{j=1}^{d-2} \sin \theta_j,
\end{align*} \]
for \( d \geq 3 \), where \( r \in [0, +\infty) \), \( \theta_i \in [0, \pi) \), \( 1 \leq i \leq d-2 \), and \( \phi \in [0, 2\pi) \). If \( d = 3 \), the second equation should be ignored. If \( d = 2 \), \( (\xi^1, \xi^2) = (r \cos \phi, r \sin \phi) \). Let \( \omega = (\theta, \phi) \in S^{d-1} \), where \( S^{d-1} \) is the \((d-1)\)-dimensional unit hypersphere. The 0-geodesics are straight lines depart from the origin, and the radial coordinate
\[ r(\xi) = |\xi| = \sqrt{\sum_{i=1}^{d} \xi_i^2} = \text{dis}(\xi, 0) \]
is the geodesic distance of \( \xi \) from 0. The 0-Laplacian is written as
\[ \Delta^{(0)} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_s, \]
where \( \Delta_s \) is the Laplacian on \( S^{d-1} \). If a group \( G \) acts transitively on a manifold, the manifold is called a homogeneous space. A homogeneous space \( X \) has a coset representation, say, \( X = G/H \), where \( H \) is the stabilizer subgroup. If we consider a homogeneous space with isometry \( G/H \), the zonal spherical function \( \varphi \) is an eigenfunction of all invariant differential operators on \( X \) satisfying \( \varphi(hx) = \varphi(x) \), \( \forall x \in X \), \( \forall h \in H \) with \( \varphi(eH) = 1 \). The (oriented) hypersphere \( S^{d-1} \) is a homogeneous space with isometry \( SO(d)/SO(d-1) \), the
Laplacian $\Delta_s$ is the invariant differential operator on $S^{d-1}$ and the zonal spherical function is the Gegenbauer polynomial.

Let us consider a plug-in estimator of a function of the geodesic distance $r(\xi)$. In other words, we are interested in SO($d$)-invariant estimation. The estimator $r(\xi)$ is an estimator of deviation of the mean from the origin. The family of hyperspheres $(N_r : r \in \mathbb{R}_{>0})$, where

$$N_{r_0} = \{(r, \omega) : r = r_0, \omega \in S^{d-1}\}, \quad r_0 \in \mathbb{R}_{>0},$$

constitutes a foliation of $\mathbb{R}^d$ with the origin, and diagonally intersects with the 0-geodesics.

If we seek an adjustment factor $\tilde{l}$ in functions of $r$, the foliation determined by the adjustment factor is the same as the foliation $(N_r : r \in \mathbb{R}_{>0})$. Since $\Delta_s f(r) = 0$, the condition (28) for $\tilde{l}$ is immediately integrated and we have

$$\tilde{l}(r) = -\frac{1}{2} \log(r^{d-1} \frac{df}{dr}) + \text{const}, \quad (29)$$

which is consistent with Theorem 4.2 (note that $f'(r^2) = f'(r)/(2r)$). We may take an adjustment factor $\tilde{l}(\xi)$ such that the foliation determined by the adjustment factor is different from the foliation $(N_r : r \in \mathbb{R}_{>0})$. For example, we may take

$$\xi^i \partial_i \tilde{l} = -\frac{1}{2} \{ (d - 1) + r \frac{d}{dr} (\log \frac{df}{dr}) \}.$$

If we are interested in a function of the form $f(r) = r^\beta$, $\beta \in \mathbb{R}$, a possible choice is

$$\tilde{l}(\xi) = -\frac{1}{\beta}(d + \beta - 2) \log \xi^1. \quad \text{The foliation determined by this choice of the adjustment factor consists of hyperplanes } \xi_1 = \text{const.} \text{ and is different from the foliation } (N_r : r \in \mathbb{R}_{>0}).$$

If $f(r)$ is 0-harmonic, the bias of the plug-in estimator $f(r(\xi))$ is $o(n^{-1})$, where $\xi$ is the (non-adjusted) maximum likelihood estimator of $\xi$. The fundamental solution of $\Delta^{(0)} f = 0$, $r > 0$ is $f(r) \propto r^{-(d-2)}$ for $d \geq 3$ and $f(r) \propto \log r$ for $d = 2$. Another 0-harmonic example is

$$|\xi - \xi_0|^{-(d-2)} = \sum_{i \geq 0} \frac{r^i}{|\xi_0|^{i+d-2}} C_i^{(d-2)/2} (\langle \omega, \omega_0 \rangle), \quad d \geq 3 \quad (30)$$

for $\xi = r\omega$ and fixed $\xi_0 = |\xi_0|\omega_0$, where $C_i^{(\cdot)}(\cdot)$ is the Gegenbauer polynomial. Here, $|\xi - \xi_0| = \text{dis}(\xi, \xi_0)$ is the geodesic distance of $\xi$ from $\xi_0$. Theorem 3.15 implies that the Jeffreys prior erases the bias of the plug-in estimator of a function $f(r) \propto r^{-(d-2)}$ for $d > 2$, and $f(r) \propto \log r$ for $d = 2$ up to $O(n^{-1})$. This is consistent with the above result, because the Jeffreys prior is 1, namely, we do not need an adjustment factor.

Let us consider a plug-in estimator of a function which may depend on $\omega$. If we seek the adjustment factor $\tilde{l}$ in functions of $r$, the condition (28) can be solved as follows. We consider the cases with $d \geq 3$. Substituting $\langle \partial_l, \partial f \rangle = l' \partial f / \partial r$ into (28), we obtain a partial differential equation

$$r^2 \frac{\partial^2 f}{\partial r^2} + \{(d - 1)r + 2r^2p\} \frac{\partial f}{\partial r} = -\Delta_s f.$$

Since $r$ and $\omega$ are separated in the left hand side and the right hand side of the equation, a solution has the form $f_\lambda(\xi) = R_\lambda(r) \Phi_\lambda(\omega)$ for some constant $\lambda \in \mathbb{C}$, where $R_\lambda$ and $\Phi_\lambda$ satisfy

$$r^2 R''_\lambda + \{(d - 1)r + 2r^2p\} R'_\lambda - \lambda R_\lambda = 0 \quad (31)$$

and

$$-\Delta_s \Phi_\lambda = \lambda \Phi_\lambda, \quad (32)$$
respectively. For the partial differential equation (32), the spectrum is known as an increasing sequence \( \lambda_i = i(d - 2 + i), \ i = 0, 1, 2, ..., \) and the corresponding eigenfunctions are the Gegenbauer polynomials \( C_i^{(d-2)/2}(\omega, \omega_0) \). Hence, a general solution \( f(\xi) \) can be represented as
\[
f(\xi) = \sum_{i \geq 0} R_{\lambda_i}(r) C_i^{(d-2)/2}(\omega, \omega_0).
\]
However, note that \( \tilde{l} \) may depend on \( i \). There are two possible types of solutions: the one is that \( \tilde{l} \) does not depend on \( i \), and the other is that \( R_{\lambda_i} \) are zero except for a single \( i \). Integrating (31) yields a condition for \( \tilde{l} \):
\[
\tilde{l}_i(r) = -\frac{1}{2} \log R_{\lambda_i} - \frac{d - 1}{2} \log r + \frac{i(d - 2 + i)}{2} \int^r \frac{R_{\lambda_i}}{r^2} dr + \text{const.}, \quad i \in \mathbb{Z}_{\geq 0}
\]
If \( R_{\lambda_i} \propto r^i \) or \( \propto r^{-i-d+2} \), then \( \tilde{l} = \text{const.} \). The expression (30) is the case of \( R_{\lambda_i} \propto r^i \). If we choose \( R_{\lambda_i} = \delta_{i0} \), then (33) reduces to (29).

5.2. Location-scale models. Let \( p(z), \ z \in \mathbb{R} \) is a probability density symmetric about the origin. The location-scale family of \( p(z) \) is given by
\[
\frac{1}{\sigma} p\{(x - \mu)/\sigma\} dx = p(z) dz, \quad z = (x - \mu)/\sigma,
\]
for the location parameter \( \mu \in \mathbb{R} \) and the scale parameter \( \sigma > 0 \). For the log-likelihood \( l(\mu, \sigma) = l_0(z) - \log \sigma \), where \( l_0(z) = \log p(z) \), we have
\[
\frac{\partial l}{\partial \mu} = \frac{l_0'}{\sigma}, \quad \frac{\partial l}{\partial \sigma} = -\frac{z l_0'}{\sigma} - \frac{1}{\sigma}, \quad \frac{\partial^2 l}{\partial \mu^2} = \frac{l_0''}{\sigma^2},
\]
\[
\frac{\partial^2 l}{\partial \sigma^2} = \frac{2 z l_0''}{\sigma^2} + \frac{z l_0'}{\sigma} + \frac{1}{\sigma^2}, \quad \frac{\partial^2 l}{\partial \sigma \partial \mu} = \frac{l_0'}{\sigma^2} + \frac{z l_0''}{\sigma^2}, \quad l_0' := \frac{dl_0(z)}{dz}.
\]
It can be seen that \( \mathbb{E}[\partial_\mu l] = 0, \mathbb{E}[\partial_\sigma l] = 0, \mathbb{E}[\partial_\mu \partial_\sigma l] = 0, \) and
\[
\mathbb{E} \left(-\frac{\partial^2 l}{\partial \mu^2}\right) = -\frac{1}{\sigma^2} \mathbb{E}[l_0''], \quad \mathbb{E} \left(-\frac{\partial^2 l}{\partial \sigma^2}\right) = \left\{1 - \mathbb{E}(z^2 l_0'')\right\} \frac{1}{\sigma^2},
\]
where the expectations are taken with respect to the location-scale family, \( \sigma^{-1} p\{(x - \mu)/\sigma\} dx \). For simplicity, we consider the density \( p(z) \) satisfying a condition \( \mathbb{E}(l_0'') = \mathbb{E}(z^2 l_0'') - 1 = -R^2 \). Then, the Fisher metric tensor becomes \( g_{ij} = (R/\sigma)^2 \delta_{ij} \) (if \( \mathbb{E}(l_0'') \neq \mathbb{E}(z^2 l_0'') - 1 \), then \( g_{\mu\mu} \neq g_{\sigma\sigma} \)). The skewness tensors is
\[
S_{\mu\mu} = S_{\mu\sigma\sigma} = 0, \quad S_{\mu\sigma} = \frac{c_1}{\sigma^3}, \quad S_{\sigma\sigma} = \frac{c_2}{\sigma^3},
\]
so \( S_{\mu} = 0 \) and \( S_{\sigma} = c/\sigma^4 \) with \( c = c_1 + c_2 \), where \( c_1 = -\mathbb{E}(z l_0'' + l_0') \), \( c_2 = -\mathbb{E}(z l_0' + 1)^2 \). The \( \alpha \)-connection is
\[
\Gamma_{\sigma \sigma \mu}^{(\alpha)} = \Gamma_{\mu \sigma \sigma}^{(\alpha)} = \Gamma_{\mu \mu \sigma}^{(\alpha)} = 0, \quad \Gamma_{\sigma \sigma \sigma}^{(\alpha)} = -\frac{R^2}{\sigma^3} - \frac{\alpha}{2} \frac{c_2}{\sigma^3},
\]
\[
\Gamma_{\mu \sigma \mu}^{(\alpha)} = -\frac{R^2}{\sigma^3} - \frac{\alpha}{2} \frac{c_1}{\sigma^3}, \quad \Gamma_{\mu \mu \sigma}^{(\alpha)} = \frac{R^2}{\sigma^3} - \frac{\alpha}{2} \frac{c_1}{\sigma^3}.
\]
The model manifold is the hyperbolic space of sectional curvature \(-R^{-2}\), denoted as \( H(-R^{-2}) \), endowed with the Fisher metric tensor and the skewness tensor derived above.
Further specification of the density \( p(z) \) determines the skewness tensor; if we choose \( p(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} \), we have \( c_1 = 4 \), \( c_2 = 8 \), and \( R^2 = 2 \).

Let us consider a plug-in estimator of a function of \( r = \text{dis}((\mu, \sigma), (0, 1)) \), which is is the geodesic distance of \((\mu, \sigma)\) from \((0, 1)\) along with the 0-geodesic. The estimator \( r(\hat{\mu}, \tilde{\nu}) \) is an estimator of deviation of the model from the standardized one: \( p(z) \). The 0-geodesic equations are

\[
\frac{\dot{\mu}}{\sigma} - 2 \frac{\ddot{\mu}}{\sigma} = 0, \quad \dot{\sigma} + \frac{\ddot{\mu}^2 - \ddot{\sigma}^2}{\sigma} = 0,
\]

where \( \ddot{\mu} := d^2\mu/dr^2 \), \( \dot{\mu} := d\mu/dr \), \( \ddot{\sigma} := d^2\sigma/dr^2 \), and \( \dot{\sigma} := d\sigma/dr \). A solution is

\[
\mu(r) = d_1 \tanh \frac{r - d_2}{R} + d_3, \quad \sigma(r) = \frac{d_1}{\cosh \frac{r - d_2}{R}},
\]

or \( \mu = d_3, \sigma = e^{(r-d_2)/R} \) for constants \( d_1, d_2 \) and \( d_3 \). Then, the geodesic distance of \((\mu(r), \sigma(r))\) from \((\mu(0), \sigma(0)) = (0, 1)\) is

\[
\text{dis}((0, 1), (\mu(r), \sigma(r))) = \int_0^r \frac{R}{\sigma} \sqrt{\dot{\mu}^2 + \dot{\sigma}^2} d\bar{r} = r,
\]

as expected. A 0-geodesic is a semi-circle on the upper-half plane:

\[(\mu - d_3)^2 + \sigma^2 = d_1^2, \quad \sigma > 0,
\]

and thus the coordinate system \((\mu, \sigma)\) is called Poincaré’s half-plane model. For example, if we consider the 0-geodesic with \( d_2 = d_3 = 0 \), it is a portion of a semi-circle whose center is at the origin, and the foliation \( \mathcal{N}_r : r \in \mathbb{R}_{>0} \) consists of closed curves equidistant from \((0, 1)\). If we seek an adjustment factor \( \tilde{l} \) in functions of \( r \), the foliation determined by the adjustment factor is the same as the foliation \( \mathcal{N}_r : r \in \mathbb{R}_{>0} \). Theorem 4.2 gives the adjustment factor which erases the bias of the plug-in estimator \( f(r(\hat{\mu}, \tilde{\nu})) \) up to \( O(n^{-1}) \):

\[
e^{\tilde{l}(r)} \propto \frac{\sigma \tilde{\sigma} r}{r \sqrt{\tilde{l}'(r^2)} \tilde{g}(\zeta)}.
\]

The derivation of the determinant \( \tilde{g}(\zeta) \) in the normal coordinate system \( \zeta(r) \) is as follows. The solutions of the geodesic equations integrated from \((0, 1)\) are

\[
\mu(r) = \mu(0)r + \mu(0)\sigma(0)r^2/2, \quad \sigma(r) = 1 + \sigma(0)r + \{(\dot{\sigma}(0))^2 - (\ddot{\mu}(0))^2\}r^2/2,
\]

where \( \mu(0) = 0 \) and \( \dot{\sigma}(0) = 1/R \) if the geodesic is parallel to \( \mu \)-axis and

\[
\dot{\mu}(0) = \frac{1}{d_1 R}, \quad \dot{\sigma}(0) = \frac{d_3}{d_1 R}
\]

otherwise. The normal coordinate system \( \zeta = (\tilde{\mu}, \tilde{\sigma}) \), whose origin is \((0, 1)\), is given as

\[
\tilde{\mu}(r) := \mu(0)r + \mu(0)\sigma(0)r^2 = \mu(r) - \tilde{\mu}(r)\sigma(r), \quad \tilde{\sigma}(r) := \sigma(0)r + \{(\dot{\mu}(0))^2 - (\ddot{\sigma}(0))^2\}r^2/2 = \sigma(r) - 1 + \{(\tilde{\mu}(0))^2 - (\tilde{\sigma}(0))^2\}r^2/2.
\]

Then, the metric in the normal coordinate is

\[
\bar{g}_{\mu\mu}(\zeta) = \frac{\partial \mu}{\partial \mu} \frac{\partial \mu}{\partial \mu} + \frac{\partial \mu}{\partial \mu} \frac{\partial \sigma}{\partial \mu} g_{\sigma \sigma} = (1 + \sigma)^2 g_{\mu \mu} + \mu^2 g_{\sigma \sigma},
\]

\[
\bar{g}_{\sigma \sigma}(\zeta) = \frac{\partial \sigma}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma} g_{\sigma \sigma} = \mu^2 g_{\mu \mu} + (1 + \sigma)^2 g_{\sigma \sigma}, \quad \bar{g}_{\mu \sigma}(\zeta) = 0.
\]
Hence, the square of the determinant in the normal coordinate system is
\[
\sqrt{\bar{g}(\xi)} = \left\{ (1 + \hat{\sigma})^2 + \hat{\mu}^2 \right\} \frac{R^2}{\sigma^2} = R^2 \left\{ (1 + \hat{\sigma})^2 + \hat{\mu}^2 \right\} \left\{ 1 + \sigma + \frac{2}{\sigma} - \frac{\hat{\mu}^2}{\sigma} \right\}^{-2}
\]
\[
= R^2 (1 + O(r^2)),
\]
and the adjustment factor \( \tilde{l} \) we are looking for is
\[
e^{\tilde{l}(r)} \propto \frac{1}{\sqrt{f(r^2)}} \left( \frac{1}{r} + \frac{c\hat{\sigma}(0)}{2R^2} + O(r) \right),
\]
which reduces to the result for the Euclidean space (29) in the limit of \( R \to \infty \).

To discuss 0-harmonicity, working in the geodesic polar coordinate system of Poincaré’s disc model is more convenient. In the geodesic polar coordinate system, the radial coordinate is the 0-geodesic, as in the polar coordinate system of the Euclidean space. The origin of the geodesic polar coordinate system is the point \((0,1)\) in Poincaré’s upper half plane model. The metric of the Poincaré’s disc model is the set \( \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} \) with the Riemannian metric
\[
ds^2 = R^2 \left( 1 - \frac{u^2 + v^2}{4} \right)^{-2} (du^2 + dv^2).
\]
It can be seen that the geodesic distance of \((u,v) = (t \cos \theta, t \sin \theta), 0 < t < 2\) from the origin along with the 0-geodesic is \( r = 2R \tanh^{-1}(t/2) \). In the geodesic polar coordinate system \((r,\phi)\), the metric (35) becomes
\[
ds^2 = dr^2 + R^2 \sinh^2(\frac{r}{R}) d\phi^2.
\]
The 0-Laplacian operating on a scalar \( f \) is given as
\[
\Delta^{(0)} = \partial_r^2 f + \frac{1}{R \tanh(\frac{r}{R})} \partial_r f + \frac{1}{R^2 \sinh^2(\frac{r}{R})} \partial_{\phi}^2 f.
\]
The fundamental solution of \( \Delta^{(0)} f(r) = 0 \) is
\[
f(r) \propto \int^{r}_0 \frac{ds}{\sinh(s/R)} = \log \tanh \frac{\frac{r}{2R}}{2} = \log \frac{r}{2R} - \frac{r^2}{12R^2} + O(r^4).
\]
Substituting (36) into (34), the adjustment factor which erases the bias of the plug-in estimator of a function (36) up to \( O(n^{-1}) \) is
\[
e^{\tilde{l}(r)} \propto 1 + \frac{c\hat{\sigma}(0)}{2R^2} r + O(r^2).
\]
In contrast to the result in the Euclidean space, the adjustment factor which erases the bias of the plug-in estimator of the fundamental solution is not proportional to the Jeffreys prior
\[
\sqrt{\bar{g}(\xi)} \propto 1 - 2\hat{\sigma}(0)r + O(r^2).
\]
Another interesting estimator is a plug-in estimator of a function \( f(\gamma(\mu,\sigma)) \), where \( \gamma(\mu,\sigma) = \sigma/\mu \) is the coefficient of variation. If we seek an adjustment factor \( \tilde{l} \) in functions of \( \gamma \), the foliation determined by the adjustment factor is the same as the foliation \( (N_{\gamma} : \gamma \in \mathbb{R}) \). The condition (27) of Lemma 4.1 yields the adjustment factor which erases bias of the plug-in estimator up to \( O(n^{-1}) \):
\[
e^{\tilde{l}(\gamma)} \propto \frac{1}{\sqrt{f(\gamma)}} \gamma^{1+c/(2R^2)} (1 + \gamma^2)^{-1-c/(4R^2)}
\]
Remark 5.1. In we choose \( p(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} \), the \( \alpha \)-geodesic equations are
\[
\tilde{\mu} - 2 \frac{1 + \alpha}{\sigma} \tilde{\mu} \tilde{\sigma} = 0, \quad \tilde{\sigma} + \frac{1 - \alpha}{\sigma} \tilde{\mu}^2 - \frac{1 + 2\alpha}{\sigma} \tilde{\sigma}^2 = 0
\]
and an \( \alpha \)-geodesic is a quadratic curve:
\[
(1 - \alpha)(\mu - d_3)^2 + \sigma^2 = d_1^2, \quad \sigma > 0.
\]
Therefore, the line \( \sigma = \gamma \mu \) becomes an \( \alpha \)-geodesic if and only if \( \alpha = 1 + \gamma^2 \) [17]. In general, a \((1 + \gamma^2)\)-geodesic is a hyperbola. The foliation \( (N'_{\gamma} : \gamma \in \mathbb{R}) \) consists of \((1 + \gamma^2)\)-geodesics extend from the origin.

Corollary 3.9 states that the bias of the plug-in estimator \( f(\hat{\mu}, \hat{\sigma}) \) for a skewed-harmonic function \( f \) is \( o(n^{-1}) \), where \( (\hat{\mu}, \hat{\nu}) \) are the (non-adjusted) maximum likelihood estimators of \( (\mu, \nu) \). Let us find a generic form of a skewed-harmonic function for the location-scale model. A skewed-harmonic function \( f(\xi) \) should satisfy the partial differential equation (22), or
\[
\partial^2_\mu f = -\partial^2_\sigma f + \frac{2 + c/R^2}{\sigma} \partial_\sigma f.
\]
Since \( \mu \) and \( \sigma \) are separated in the left hand side and the right hand side of this equation, a solution has the form of \( f(\lambda, \sigma) = M_\lambda(\mu)S_\lambda(\sigma) \) for some constant \( \lambda \in \mathbb{C} \), where \( M_\lambda \) and \( S_\lambda \) satisfy
\[
\partial^2_\mu M_\lambda = \lambda^2 M_\lambda, \quad \text{(37)}
\]
and
\[
\partial^2_\sigma S - \frac{2 + c/R^2}{\sigma} \partial_\sigma S + \lambda^2 S = 0. \quad \text{(38)}
\]
Then, solutions of (37) and (38) are expressed as
\[
M_\lambda(\mu) = d_1(\lambda) \cosh(\lambda \mu) + d_2(\lambda) \sinh(\lambda \mu)
\]
and
\[
S_\lambda(\sigma) = \sigma^{3/2+c/(2R^2)} \{d_3(\lambda)J_{3/2+c/(2R^2)}(\lambda \sigma) + d_4(\lambda)Y_{3/2+c/(2R^2)}(\lambda \sigma)\},
\]
respectively, for constants \( d_1, d_2, d_3 \) and \( d_4 \). Here, \( J_{3/2+c/(2R^2)}(\cdot) \) and \( Y_{3/2+c/(2R^2)}(\cdot) \) are Bessel functions of the first kind and the second kind, respectively. The eigenvalues \( \lambda \) are not discretized.

It is worth pointing out that the separation of variables demonstrated above in the coordinate system \( (\mu, \sigma) \) of the Poincaré’s upper half plane model does not work in the geodesic polar coordinate system \( (r, \phi) \). To obtain skewness tensor in the geodesic polar coordinate, the parameter transformation between \( (\mu, \sigma) \) and \( (r, \phi) \) is needed. Cayley’s transform
\[
\frac{(u + \sqrt{-1}v) + 2\sqrt{-1}(\sqrt{-1}(u + \sqrt{-1}v) + 2)} = \mu + i\sigma
\]
yields
\[
\mu = \frac{2 \tanh(r/2R) \cos \phi}{1 - 2 \tanh(r/2R) \sin \phi + \tanh^2(r/2R)}, \quad \sigma = \frac{1 - \tanh^2(r/2R)}{1 - 2 \tanh(r/2R) \sin \phi + \tanh^2(r/2R)},
\]
and we obtain
\[
\begin{align*}
S^r &= \frac{c}{R^3} \sin(\phi - \mu \cos \phi), \\
S^\phi &= \frac{c\mu}{R^4 \sinh^2(r/R)}.
\end{align*}
\]
In the geodesic polar coordinate system, the partial differential equation (22) becomes

\[
\partial_r^2 f + \frac{1}{R \tanh(r/R)} \partial_r f - \frac{2 + c/R^2}{R} (\sin \phi - \mu \cos \phi) \partial_r f \\
+ \frac{1}{R^2 \sinh^2(r/R)} \partial_\phi^2 f - \frac{2 + c/R^2}{R^2 \sinh^2(r/R)} \mu \partial_\phi f = 0.
\]

It is obvious that we cannot separate the variables \((r, \phi)\).

5.3. Nested error regression model. Let us consider a specific linear mixed model, which is an extension of the normal mixed model (3):

\[
x_{ij} | z_i \sim N(z_i, d), \quad j \in \{1, \ldots, n_i\}, \quad z_i \overset{iid}{\sim} N(0, a), \quad i \in \{1, \ldots, m\},
\]

where the variances \(a > 0\) and \(d > 0\) are unknown. This model is called nested error regression model without covariates (in a general model \(z^i\) depends on covariates). In small area estimation, this model is called a unit-level model, where \(i\) is the index of ‘units’ and \(n_i\) is the size of the \(i\)-th unit. Let the total size be \(n := \sum_{i=1}^m n_i\). If we set \(d/n_i = d\) with known \(d_i, y_i = \sum_{j=1}^{n_i} y_{ij}/n_i\) follows the Fay-Herriot model without covariates, which is a well-known model in small area estimation under the simple random sampling design. If we set \(n_1 = \cdots = n_m = d = 1\), this model reduces to the normal mixed model discussed in Introduction. The best linear unbiased predictors of the means \(z_i\) are

\[
\hat{z}_i = \{1 - b(i)(a, d)\} \bar{x}_i, \quad b(i)(a, d) := \frac{d/n_i}{d/n_i + a}, \quad i \in \{1, \ldots, m\},
\]

The shrinkage factors of the unit level model are \(b(i)(a, d)\). Note that the shrinkage factors are chosen for each unit. The index \((i), i \in \{1, \ldots, m\}\) should not be confused with the index of the coordinate of the model manifold. Let us discuss the plug-in estimator of the shrinkage factors \(b(i)(a, d)\). See [12] for the backgrounds.

The log-likelihood is

\[
l(a, d) = \frac{1}{2d} \left( \sum_{i=1}^m \frac{a n_i \bar{y}_i^2}{d/n_i + a} - \sum_{i=1}^m \sum_{j=1}^{n_i} y_{ij}^2 \right) - \frac{1}{2} \sum_{i=1}^m \log \left\{ (d/n_i + a)^{n_i-1} \right\} + \text{const}.
\]

The Fisher metric tensor is [12]

\[
(g_{ij}) = \frac{1}{2} \left( \sum_{i=1}^m \frac{1}{n_i(d/n_i + a)^2} \sum_{i=1}^m \frac{1}{n_i(d/n_i + a)^2} + \frac{n-m}{d^2} \right)
\]

with the determinant

\[
g = \frac{1}{4} \sum_{i<j} \frac{1}{(d/n_i + a)^2(d/n_j + a)^2} \left( \frac{1}{n_i} - \frac{1}{n_j} \right)^2 + \frac{n-m}{d^2} \sum_{i=1}^m \frac{1}{(d/n_i + a)^2} > 0.
\]

We exclude the case of \(n_i = 1, \forall i\), because the determinant vanishes. In fact, for the case the model is not identifiable. After some calculation, we obtain the skewness tensor

\[
S_{aaa} = \sum_{i=1}^{n_i} \frac{1}{(d/n_i + a)^3}, \quad S_{aad} = \sum_{i=1}^{n_i} \frac{1}{n_i(d/n_i + a)^3},
\]

\[
S_{add} = \sum_{i=1}^{n_i} \frac{1}{n_i^2(d/n_i + a)^3}, \quad S_{ddd} = \sum_{i=1}^{n_i} \frac{1}{n_i^2(d/n_i + a)^3} - \frac{m-n}{d^3}.
\]
The model manifold is the open orthant \( \{(a, d) \in \mathbb{R}^2 : a > 0, d > 0\} \) endowed with the Fisher metric tensor and the skewness tensor derived above. Moreover, we can confirm that the \((-1)-\)connection vanishes. Therefore, the model manifold is \((-1)-\)flat and thus dually-flat with the \((-1)-\)affine coordinate system \((a, d)\).

The expression (18) is now becomes

\[
\Delta^{S} b^{(i)} = \Delta^{(-1)} b^{(i)} + g^{ik}(\partial_j \log g)\partial_k b^{(i)}, \quad i \in \{1, ..., m\},
\]

with

\[
\Delta^{(-1)} b^{(i)} = \frac{1}{g} \left\{ \sum_{j \neq i} \frac{n_j}{(d/n_j + a)(d/n_i + a)^3} \left( \frac{1}{n_i} - \frac{1}{n_j} \right)^2 + \frac{n - m}{n_i(d/n_i + a)^3} \right\} > 0.
\]

Analogy with the normal mixed model (3) gives us an anticipation that the adjustment factor \(\bar{l}^{(i)}(a, d) = \log(d/n_i + a)\) erases the bias of the plug-in estimator of \(b^{(i)}\) up to \(O(n^{-1})\). However, this anticipation is false. In fact, we can confirm that

\[
\Delta^{(-1)} b^{(i)} + 2\langle \partial \bar{l}^{(i)}, \partial b^{(i)} \rangle = 0.
\]

and the condition (20) fails. Fortunately, we can obtain the correct answer. By comparing (40) with (39), we see that the correct answer is given by sifting the adjustment factor:

\[
\bar{l}^{(i)}(a, d) = \log(d/n_i + a) - \log \sqrt{g}.
\]

Hirose and Lahiri [12] obtained the adjustment factor for the case of \(d\) is known, but did not solve the unknown case because of the non-uniqueness of the adjustment. By Corollary 3.9, the bias of the plug-in estimator \(b^{(i)}(\hat{a}, \hat{d})\), where \((\hat{a}, \hat{d})\) are the adjusted maximum likelihood estimators of \((a, d)\) with the adjustment factor \(\bar{l}^{(i)}(a, d)\), is \(o(n^{-1})\). The estimator \(b^{(i)}\) and the adjustment factor \(\bar{l}^{(i)}\) we obtained determine different foliations of the model manifold. In fact, for \(b^{(i)} = b_0\) determines the \((-1)-\)geodesic \(d = a(b_0 n_i)/(1 - b_0)\), while \(\bar{l}^{(i)} = l_0\) does not determine a \((-1)-\)geodesic.

Let us consider taking an adjustment factor \(\bar{l}\) such that the foliation determined by the adjustment factor is the same as the foliation \((N_b : b \in (0, 1))\). The shrinkage factor is expressed as \(b^{(i)}(\gamma) = (1 + n_i \gamma)^{-1}\), where \(\gamma = a/d > 0\) is the ratio of the variances. Let us consider a function \(f^{(i)}(\gamma) = -b^{(i)}(\gamma)\) such that \(f^{(i)}(\gamma) > 0\). Since

\[
\frac{\Delta^{S} \gamma}{\langle \partial \gamma, \partial \gamma \rangle} = \frac{m}{n \gamma} - \frac{1}{n} \sum_{j=1}^{m} \frac{1}{\gamma(1 + n_j \gamma)^2} - (\log g(\gamma))',
\]

Lemma 4.1 yields the adjustment factor:

\[
e^{\bar{l}(\gamma)} \propto \frac{(1 + n_i \gamma)}{\sqrt{g(\gamma)}} \left\{ \prod_{j} \frac{e^{(1+n_j \gamma)^{-1}}}{1 + n_j \gamma} \right\}^{1/n} \frac{(a + d/n_i - d)^{-1}}{\sqrt{g(\gamma)}} \left\{ \prod_{j} \frac{e^{(1+n_j a/d)^{-1}}}{1 + n_j a/d} \right\}^{1/n},
\]

which is far more complicated than (41). This result demonstrates that our anticipation may give a simpler adjustment factor than that sought in the foliation determined by the function of the plug-in estimator. Since both of the adjustment factor has bias of \(o(n^{-1})\), simpler adjustment factor is more preferable in practice. Note that no randomness is involved in adjustment factors.
APPENDIX: PROOF OF LEMMA 3.3

Consider a likelihood function $l(\xi; x_1, \ldots, x_n) = \sum_{i=1}^{n} \log q(x; \xi)$ of a sample $(x_1, \ldots, x_n) \in \mathcal{X}^n$ with a parametric model $q(x; \xi)$, $\xi \in \Xi$. The parameter space $\Xi$ is a open subset of $\mathbb{R}^d$ for a fixed $d \in \mathbb{Z}_{>0}$ for large $n$. The true value of the parameter, $\xi_0$, assumed to be interior of $\Xi$. Expectations are taken with respect to the product probability measure $P_{\xi_0}(dx) = e^{l(\xi_0; x)} \prod_{i=1}^{n} dx_i$ and a derivative is denoted by $\partial_i := \partial / \partial \xi_i$.

We prepare regularity conditions:

A1: The map $\mathcal{X} \ni x \mapsto l(\xi; x)$ is measurable for each $\xi \in \Xi$;
A2: The map $\Xi \ni \xi \mapsto l(\xi; x)$ is three times differentiable for each $\xi \in \Xi$;
A3: $\sum_{k=1}^{n} \partial_i l \log q(x_k; \theta_0)$ is square integrable with respect to $P_{\xi_0}(dx)$;
A4: The largest eigenvalue of $-C^{-1}GC^{-1}$, $\lambda_{\text{max}}$, for a matrix $C = (c_{ij}) = \text{diag}(c_1, \ldots, c_d)$ with $c_i > 0$ such that $c_{\ast} := \min_i c_i \to \infty$ as $n \to \infty$ satisfies $\lim \sup_{n \to \infty} \lambda_{\text{max}} \in (-\infty, 0)$;
A5: For an $r \in \mathbb{Z}_{>0}$, the $r$-th moments of the following are bounded:

$$\frac{1}{c_i} |\partial_i l(\xi_0; x)|, \quad \frac{1}{\sqrt{c_i c_j}} |\partial_i \partial_j l(\xi_0; x) + g_{ij}(\xi_0)|, \quad \frac{c_{\ast}}{c_i c_j c_k} M_{ijk}(\xi_0),$$

where $M_{ijk}(\xi_0) := \sup_{\hat{\xi} \in B_{\delta}(\xi_0)} |\partial_i \partial_j \partial_k l(\hat{\xi}; x)|$ with a ball

$$B_{\delta}(\xi_0) := \{\hat{\xi} : |\hat{\xi} - \xi_0| \leq \delta c_{\ast}/c_i, i \in \{1, \ldots, d\}\}.$$

Under the regularity conditions A1-A5, Das et al.[4] proved, as their Theorem 2.1, the following theorem for asymptotic representation of $\hat{\xi} - \xi_0$ to study mean squared error of empirical predictor in linear mixed models. Yoshimori and Lahiri [21] also used their result for estimation of a dispersion parameter.

**THEOREM A.1.** Under the regularity conditions A1-A5,

(i) A $\hat{\xi} \in \Xi$ exists such that for any $\rho \in (0, 1)$ there is a set of events $\mathcal{E}$ satisfying for large $n$ and on $\mathcal{E}$, $\partial_i l(\hat{\xi}; x) = 0$, $|c_{ij}(\hat{\xi} - \xi_0)^2| < c_{\ast}^{1-\rho}$ and

$$\hat{\xi} = \xi_0 + g_{ij}(\xi_0) \partial_j l(\xi_0; x) + R, \quad |R| \leq c_{\ast}^{-2\rho} u_{\ast}$$

with $\mathbb{E}(u_{\ast})$ bounded;

(ii) $\mathbb{P}(\mathcal{E}^c) \leq c_0 c_{\ast}^{-\tau r}$ for some constant $c_0$ and $\tau := (1/4) \wedge (1 - \rho)$.

This theorem states that the solution of $\partial_i l(\xi; x) = 0$, $\hat{\xi}$, exists, and lies in the parameter space $\Xi$ with probability tending 1.

Let the adjusted log-likelihood of a sample $(x_1, \ldots, x_n) \in \mathcal{X}^n$ denoted by

$$l_{ad}(\xi; x_1, \ldots, x_n) := l(\xi; x_1, \ldots, x_n) + \tilde{l}(\xi), \quad \xi \in \Xi,$$

where the adjustment factor $\tilde{l}(\xi)$ is $O(1)$. The adjusted maximum likelihood estimator is denoted by $\hat{\xi}_n$. Expectations are still taken with respect to the product probability measure $P_{\xi_0}(dx) = e^{l(\xi_0; x)} \prod_{i=1}^{n} dx_i$. Note that $\mathbb{E}\{\sum_{k=1}^{n} \partial_i l_{ad}(\xi_0; x_k) \partial_j l_{ad}(\xi_0; x_k)\} = g_{ij}(\xi_0) + O(1)$. Theorem A.1 is used in the following proof of Lemma 3.3. We set $c_1 = \cdots = c_n = c_{\ast} = \sqrt{n}$, because a single asymptotics is sufficient for our purpose.

We prepare the regularity conditions with some modifications:

B1: The map $\mathcal{X} \ni x \mapsto l_{ad}(\xi; x)$ is measurable for each $\xi \in \Xi$;
B2: The map $\Xi \ni \xi \mapsto l_{ad}(\xi; x)$ is four times differentiable for each $\xi \in \Xi$;
B3: $\sum_{k=1}^{n} \partial_i \log q(x_k; \theta_0)$ is square integrable with respect to $P_{\xi_0}(dx)$;
B4: $G = g_{ij}, g_{ij}(\xi_0) := \mathbb{E}\{\sum_{k=1}^{n} \partial_i \log q(x_k; \xi_0) \partial_j \log q(x_k; \xi_0)\}$ is a regular matrix;
B5: For an \( r \in \mathbb{Z}_{\geq 9} \), the \( r \)-th moments of the following are bounded:
\[
\frac{1}{\sqrt{n}} |\partial_l l_ad(\hat{\xi}_n; x)|, \quad \frac{1}{n} |\partial_l \partial_j \partial_k l_ad(\hat{\xi}_n; x)|, \quad \frac{1}{\sqrt{n}} |\partial_l \partial_j l_ad(\hat{\xi}_n; x) + g_{ij}(\xi_0)|, \\
\frac{1}{n} |\partial_l \partial_j \partial_k l_ad(\hat{\xi}_n)|, \quad \frac{1}{n} |\partial_l \partial_j \partial_k l_ad(\hat{\xi}_n; x) - \mathbb{E}\{\partial_l \partial_j \partial_k l_ad(\hat{\xi}_n; x)\}|.
\]

B6: \( \max_{i \in \{1, \ldots, d\}} |\hat{\xi}_n^i| < d_0 n^s \) for some constant \( d_0 \) and \( 0 < s < r/16 - 1/2 \).

**Proof of Lemma 3.3.** With setting \( \rho \in (2/3, 3/4) \), Theorem A.1 concludes that (a): A \( \hat{\xi}_n \in \Xi \) exists such that there is a set of events \( \mathcal{E} \) satisfying for large \( n \) and on \( \mathcal{E} \), \( \partial_l l_ad(\hat{\xi}_n; x) = 0, \ |(\hat{\xi}_n - \xi_0)^j| < n^{-\rho/2} \) and
\[
\hat{\xi}_n = \xi_0 + g^{ij}(\xi_0)\partial_j l_ad(\hat{\xi}_n; x) + R, \quad |R| \leq n^{-\rho} u_+ \quad (42)
\]
with \( \mathbb{E}(u_+^s) \) bounded; and (b): \( \mathbb{P}(\mathcal{E}^c) \leq c_0 n^{-r/8} \) for some constant \( c_0 \). For simplicity of expressions, in the following, \( g_{ij}(\xi_0) \) will be denoted by \( g_{ij} \). In addition, \( \mathbb{E}^{\mathcal{E}}(\cdot) \) and \( \mathbb{E}^{\mathcal{E}^c}(\cdot) \) will denote \( \mathbb{E}(\cdot|1_{\mathcal{E}}) \) and \( \mathbb{E}(\cdot|1_{\mathcal{E}^c}) \), respectively. By Taylor’s theorem,
\[
\partial_l l_ad(\hat{\xi}_n; x) = \partial_l l_ad(\xi_0; x) + \frac{1}{2} (\hat{\xi}_n - \xi_0)^j (\hat{\xi}_n - \xi_0)^k \partial_l \partial_j \partial_k l_ad(\xi_0; x) + R_1
\]
\[
= - (\hat{\xi}_n - \xi_0)^j g_{ij}(\xi_0) + (\hat{\xi}_n - \xi_0)^j \{\partial_l \partial_j l_ad(\xi_0; x) + g_{ij}(\xi_0)\}
\]
\[
+ \frac{1}{2} (\hat{\xi}_n - \xi_0)^j (\hat{\xi}_n - \xi_0)^k \partial_l \partial_j \partial_k l_ad(\xi_0; x) + R_1, \quad (43)
\]
where
\[
R_1 = \frac{1}{3!} (\hat{\xi}_n - \xi_0)^j (\hat{\xi}_n - \xi_0)^k (\hat{\xi}_n - \xi_0)^l \partial_l \partial_j \partial_k \partial_r l_ad(\hat{\xi}_n; x), \quad \hat{\xi}_n \in (\hat{\xi}_n, \xi_0).
\]
Since \( \partial_l l_ad(\hat{\xi}_n; x) = 0 \), we have
\[
(\hat{\xi}_n - \xi_0)^j = g^{ij}\{\partial_j l_ad(\xi_0; x) + (\hat{\xi}_n - \xi_0)^k (\partial_j \partial_k l_ad(\xi_0; x) + g_{jk}) + \frac{1}{2} (\hat{\xi}_n - \xi_0)^k (\hat{\xi}_n - \xi_0)^l \partial_l \partial_j \partial_k \partial_r l_ad(\xi_0; x) + R_1\}
\]
\[
= g^{ij} \partial_j l_ad(\xi_0; x) + R_2. \quad (44)
\]
Here, \( g^{ij} = O(n^{-1}) \) and (42) gives \( |R_2| \leq n^{-\rho} u_+ \) with \( \mathbb{E}(u_+^s) \) bounded. Then, substituting (44) into (43), we have
\[
\mathbb{E}^{\mathcal{E}}(\{(\hat{\xi}_n - \xi_0)^j\}) = g^{ij} \mathbb{E}^{\mathcal{E}}(\partial_j l_ad(\xi_0; x)) + g^{ij} \mathbb{E}^{\mathcal{E}}(R_1)
\]
\[
+ g^{ij} \mathbb{E}^{\mathcal{E}}(\{g^{kr} \partial_r l_ad(\xi_0; x) + R_2(\partial_j \partial_k l_ad(\xi_0; x) + g_{jk})\})
\]
\[
+ g^{ij} \frac{1}{2} \mathbb{E}^{\mathcal{E}}(\{g^{kr} \hat{\xi}_n l_ad(\xi_0; x) + R_2(g^{rt} \partial_t l_ad(\xi_0; x) + R_2) \partial_j \partial_k \partial_r l_ad(\xi_0; x)\})
\]
Cauchy-Schwarz’s inequality, the condition B5, and \( |R_2| \leq n^{-\rho} u_+ \) leads to
\[
g^{ij} \mathbb{E}^{\mathcal{E}}(\{\partial_j l_ad(\xi_0; x) + g_{jk}\}) = g^{ij} [\mathbb{E}^{\mathcal{E}}(R_2^2)^{1/2} [\mathbb{E}^{\mathcal{E}}(\{(\partial_j \partial_k l_ad(\xi_0; x) + g_{jk})^2\})^{1/2}]
\]
\[
\leq (\mathbb{E}(u_+^2))^{1/2} o(n^{-1}),
\]
In the same way, we observe that
\[
g^{ij} g^{ks} \mathbb{E}^{\mathcal{E}}(\partial_s l_ad(\xi_0; x) R_2 \partial_j \partial_k \partial_r l_ad(\xi_0; x)), \quad g^{ij} \mathbb{E}^{\mathcal{E}}(R_2^2 \partial_j \partial_k \partial_r l_ad(\xi_0; x))
are $o(n^{-1})$. The condition B5 and $|\xi - \xi_0|^i < n^{-\rho/2}$ leads to
\[
g^{ij} \mathbb{E}_E(R_1) < g^{ij} \frac{1}{3^i} n^{-3\rho} \mathbb{E}_E \{ \partial_r \partial_j \partial_k \partial_r \partial_l \partial_x l_1(\tilde{\xi}; x) \} = o(n^{-1}),
\]
where $\rho > 2/3$ is demanded. Therefore, we have
\[
\mathbb{E}_E \{ (\xi_n - \xi_0)^i \} = 2g^{ij} \mathbb{E}_E \{ \partial_j l_{ad}(\xi_0; x) \} + g^{ij} g^{kr} \mathbb{E}_E \{ \partial_r l_{ad}(\xi_0; x) \partial_j \partial_k \partial_l \partial_x l_1(\tilde{\xi}; x) \} + o(n^{-1})
\]
\[
= 2g^{ij} \mathbb{E}_E \{ \partial_j l_{ad}(\xi_0; x) \} + g^{ij} g^{kr} \mathbb{E}_E \{ \partial_r l_{ad}(\xi_0; x) \partial_j \partial_k \partial_l \partial_x l_1(\tilde{\xi}; x) \} + o(n^{-1})
\]
\[
= 2g^{ij} \mathbb{E}_E \{ \partial_j l_{ad}(\xi_0; x) \} + g^{ij} g^{kr} \mathbb{E}_E \{ \partial_r l_{ad}(\xi_0; x) \partial_j \partial_k \partial_l \partial_x l_1(\tilde{\xi}; x) \} + o(n^{-1})
\]
where the second equality holds from the fact follows from B5:
\[
\text{Cov}^E(\partial_r l_{ad}(\xi_0; x) \partial_j \partial_k \partial_x l_1(\xi_0; x)) = o(n^2).
\]
Since $\mathbb{E}_E(\cdot) \leq \mathbb{E}(\cdot)$, the last expression is bounded from the above by
\[
\mathbb{E}_E \{ (\xi_n - \xi_0)^i \} \leq \mathbb{E}_E \{ (\xi_n - \xi_0)^i \} + o(n^{-1}).
\]
and thus $\mathbb{E}(\xi_n - \xi_0)^i = \mathbb{E}(\xi_n - \xi_0)^i + o(n^{-1})$. Therefore, we establish
\[
\mathbb{E}(\xi_n - \xi_0)^i = g^{ij} \partial_j \tilde{l}_1(\xi_0)
\]
\[
+ \frac{g^{ij} g^{kr}}{2} \mathbb{E}_E \{ \partial_j l(\xi_0; x) \} + \frac{1}{2} \mathbb{E}_E \{ \partial_r \partial_k \partial_j l(\xi_0; x) \} + o(n^{-1})
\]
\[
= g^{ij} \left[ \partial_j \tilde{l}_1(\xi_0) + g^{kr} \left( \Gamma_{jk,r}^{(1)} - \frac{1}{2} \partial_r g_{jk} \right) \right] + o(n^{-1}).
\]
for large $n$, where $\mathbb{E}(\partial_j l(\xi_0; x)) = 0$ is used in the first equality, and the last equality holds because $\mathbb{E}(\partial_r l(\xi_0; x) \partial_j \partial_k l(\xi_0; x)) = \Gamma_{jk,r}^{(1)}$ and $\mathbb{E}(\partial_j \partial_k \partial_j l(\xi_0; x)) = \partial_r g_{jk}$. Hence, the assertion (i) is established. For the assertion (ii), by using (42), we have
\[
\mathbb{E}(\xi_n - \xi_0)^i (\xi_n - \xi_0)^j = g^{ij} g^{kr} E(\partial_j l_{ad}(\xi; x) \partial_k l_{ad}(\xi; x)) + o(n^{-1}) = g^{ij} + o(n^{-1})
\]
in a similar way as in the proof of the assertion (i). \hfill \Box

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REFERENCES

[1] AITCHISON, J. (1975). Goodness of Prediction Fit. Biometrika 62 547–554
[2] AMARI, S. (1985). Differential-Geometrical Methods in Statistics. Lecture Notes in Statistics 28. Springer, New York.
[3] AMARI, S., NAGAOKA, H. (2000). Methods of Information Geometry. American Mathematical Society.
[4] DAS, K., JIANG, J., RAO, J. N. K. (2004). Mean squared error of empirical predictor. Ann. Statist. 32 818–840.
[5] EFRON, B., MORRIS, C. (1972). Empirical Bayes on vector observations: an extension of Stein’s method. Biometrika 59 335–347.
[6] EFRON, B., MORRIS, C. (1976). Multivariate empirical Bayes and estimation of covariance matrices. Ann. Statist. 4 22–32.
[7] EGUCHI, S., YANAGIMOTO, T. (2008). Asymptotical improvement of maximum likelihood estimators on Kullback-Leibler loss. J. Statist. Plann. Inference 138 3502–3511.
[8] HADAMARD, J. (1932). Le problème de Cauchy: et les Équations aux Dérivées Partielles Linéaires Hyperboliques. Herman, Paris.
[9] HELGASON, S. (1984). Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions. Academic Press, Orlando.
[10] HELMS, L.L. (1969). Introduction to potential theory. John Wiley and Sons.
[11] HIROSE, Y. M., LAHIRI, P. (2018). Estimating variance of random effects to solve multiple problems simultaneously, Ann. Statist. 46 1721–41.
[12] HIROSE, Y. M., LAHIRI, P. (2020). Multi-goal prior selection: a way to reconcile Bayesian and classical approaches for random effects models, J. Amer. Statist. Assoc. to appear. https://doi.org/10.1080/01621459.2020.1737532
[13] JAMES, W., STEIN, C. (1961). Estimation with quadratic loss. Proc. Fourth. Berkeley Symp. Math. Statist. Probab. 1 361–380. Univ. California Press.
[14] KOMAKI, F. (1996). On asymptotic properties of predictive distributions. Biometrika 83 299–313.
[15] KOMAKI, F. (2006). Shrinkage priors for Bayesian prediction. Ann. Statist. 34 808–819.
[16] LAHIRI, P., LI, H. (2009). Generalized maximum likelihood method in linear mixed models with an application in small area estimation. In Proceedings of the Federal Committee on Statistical Methodology Research Conference.
[17] LAURITZEN, S. L. (1987). Statistical manifolds. in Differential Geometry in Statistical Inference, IMS lecture notes-monograph series. 163–216.
[18] LI, H., LAHIRI, P. (2010). An adjusted maximum likelihood method for solving small area estimation problems. J. Multivariate Anal. 101 882–892.
[19] RIESZ, M. (1949). L’intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math. 81 1–223.
[20] STEIN, C. (1981). Estimation of the mean of a multivariate normal distribution. Ann. Statist. 9 1135–1151.
[21] YOSHIMORI, M., LAHIRI, P. (2014). A second-order efficient empirical Bayes confidence interval. Ann. Statist. 42 1233–1261.
[22] YOSHIMORI, M., LAHIRI, P. (2014). A new adjusted maximum likelihood method for the Fay-Herriot small area model. J. Multivariate Anal. 124 281–294.