Intrinsic sound of anti-de Sitter manifolds

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Abstract As is well-known for compact Riemann surfaces, eigenvalues of the Laplacian are distributed discretely and most of eigenvalues vary viewed as functions on the Teichmüller space. We discuss a new feature in the Lorentzian geometry, or more generally, in pseudo-Riemannian geometry. One of the distinguished features is that $L^2$-eigenvalues of the Laplacian may be distributed densely in $\mathbb{R}$ in pseudo-Riemannian geometry. For three-dimensional anti-de Sitter manifolds, we also explain another feature proved in joint with F. Kassel [Adv. Math. 2016] that there exist countably many $L^2$-eigenvalues of the Laplacian that are stable under any small deformation of anti-de Sitter structure. Partially supported by Grant-in-Aid for Scientific Research (A) (25247006), Japan Society for the Promotion of Science.

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1 Introduction

Our “common sense” for music instruments says:

“shorter strings produce a higher pitch than longer strings”,

“thinner strings produce a higher pitch than thicker strings”.

Let us try to “hear the sound of pseudo-Riemannian locally symmetric spaces”. Contrary to our “common sense” in the Riemannian world, we find a phenomenon that compact three-dimensional anti-de Sitter manifolds have “intrinsic sound”
which is stable under any small deformation. This is formulated in the framework of spectral analysis of anti-de Sitter manifolds, or more generally, of pseudo-Riemannian locally symmetric spaces $X_\Gamma$. In this article, we give a flavor of this new topic by comparing it with the flat case and the Riemannian case.

To explain briefly the subject, let $X$ be a pseudo-Riemannian manifold, and $\Gamma$ a discrete isometry group acting properly discontinuously and freely on $X$. Then the quotient space $X_\Gamma := \Gamma \backslash X$ carries a pseudo-Riemannian manifold structure such that the covering map $X \to X_\Gamma$ is isometric. We are particularly interested in the case where $X_\Gamma$ is a pseudo-Riemannian locally symmetric space, see Section 3.2.

Problems we have in mind are symbolized in the following diagram:

| Geometry | Geometric existence problem | deformed v.s. rigidity |
|----------|----------------------------|-----------------------|
| Does cocompact $\Gamma$ exist? | Higher Teichmüller theory v.s. rigidity theorem |
| Analysis | Does $L^2$-spectrum exist? | Whether $L^2$-eigenvalues vary or not |
| (Problem A) | (Section 4.2) | (Problem B) |

2 A program

In [5, 6, 12] we initiated the study of “spectral analysis on pseudo-Riemannian locally symmetric spaces” with focus on the following two problems:

**Problem A** Construct eigenfunctions of the Laplacian $\Delta X_\Gamma$ on $X_\Gamma$. Does there exist a nonzero $L^2$-eigenfunction?

**Problem B** Understand the behaviour of $L^2$-eigenvalues of the Laplacian $\Delta X_\Gamma$ on $X_\Gamma$ under small deformation of $\Gamma$ inside $G$.

Even when $X_\Gamma$ is compact, the existence of countably many $L^2$-eigenvalues is already nontrivial because the Laplacian $\Delta X_\Gamma$ is not elliptic in our setting. We shall discuss in Section 2.2 for further difficulties concerning Problems A and B when $X_\Gamma$ is non-Riemannian.

We may extend these problems by considering joint eigenfunctions for “invariant differential operators” on $X_\Gamma$ rather than the single operator $\Delta X_\Gamma$. Here by “invariant differential operators on $X_\Gamma$” we mean differential operators that are induced from $G$-invariant ones on $X = G/H$. In Section 7 we discuss Problems A and B in this general formulation based on the recent joint work [6, 7] with F. Kassel.
2.1 Known results

Spectral analysis on a pseudo-Riemannian locally symmetric space $X_\Gamma = \Gamma \backslash X = \Gamma \backslash G/H$ is already deep and difficult in the following special cases:

1) (noncommutative harmonic analysis on $G/H$) $\Gamma = \{e\}$.
   In this case, the group $G$ acts unitarily on the Hilbert space $L^2(X_\Gamma) = L^2(X)$ by translation $f(\cdot) \mapsto f(g^{-1} \cdot)$, and the irreducible decomposition of $L^2(X)$ (Plancherel-type formula) is essentially equivalent to the spectral analysis of $G$-invariant differential operators when $X$ is a semisimple symmetric space.
   Noncommutative harmonic analysis on semisimple symmetric spaces $X$ has been developed extensively by the work of Helgason, Flensted-Jensen, Matsuki–Oshima–Sekiguchi, Delorme, van den Ban–Schlichtkrull among others as a generalization of Harish-Chandra’s earlier work on the regular representation $L^2(G)$ for group manifolds.

2) (automorphic forms) $H$ is compact and $\Gamma$ is arithmetic.
   If $H$ is a maximal compact subgroup of $G$, then $X_\Gamma = \Gamma \backslash G/H$ is a Riemannian locally symmetric space and the Laplacian $\Delta_{X_\Gamma}$ is an elliptic differential operator.
   Then there exist infinitely many $L^2$-eigenvalues of $\Delta_{X_\Gamma}$ if $X_\Gamma$ is compact by the general theory for compact Riemannian manifolds (see Fact 1). If furthermore $\Gamma$ is irreducible, then Weil’s local rigidity theorem [18] states that nontrivial deformations exist only when $X$ is the hyperbolic plane $SL(2, \mathbb{R})/SO(2)$, in which case compact quotients $X_\Gamma$ have a classically-known deformation space modulo conjugation, i.e., their Teichmüller space. Viewed as a function on the Teichmüller space, $L^2$-eigenvalues vary analytically [1, 20], see Fact 11.
   Spectral analysis on $X_\Gamma$ is closely related to the theory of automorphic forms in the Archimedian place if $\Gamma$ is an arithmetic subgroup.

3) (abelian case) $G = \mathbb{R}^{p+q}$ with $H = \{0\}$ and $\Gamma = \mathbb{Z}^{p+q}$.
   We equip $X = G/H$ with the standard flat pseudo-Riemannian structure of signature $(p, q)$ (see Example 1). In this case, $G$ is abelian, but $X = G/H$ is non-Riemannian. This is seemingly easy, however, spectral analysis on the $(p+q)$-torus $\mathbb{R}^{p+q}/\mathbb{Z}^{p+q}$ is much involved, as we shall observe a connection with Oppenheim’s conjecture (see Section 5.2).

2.2 Difficulties in the new settings

If we try to attack a problem of spectral analysis on $\Gamma \backslash G/H$ in the more general case where $H$ is noncompact and $\Gamma$ is infinite, then new difficulties may arise from several points of view:

1) Geometry. The $G$-invariant pseudo-Riemannian structure on $X = G/H$ is not Riemannian anymore, and discrete groups of isometries of $X$ do not always act properly discontinuously on such $X$. 

(2) Analysis. The Laplacian $\Delta_X$ on $X_\Gamma$ is not an elliptic differential operator. Furthermore, it is not clear if $\Delta_X$ has a self-adjoint extension on $L^2(X_\Gamma)$.

(3) Representation theory. If $\Gamma$ acts properly discontinuously on $X = G/H$ with $H$ noncompact, then the volume of $\Gamma \backslash G$ is infinite, and the regular representation $L^2(\Gamma \backslash G)$ may have infinite multiplicities. In turn, the group $G$ may not have a good control of functions on $\Gamma \backslash G$. Moreover $L^2(X_\Gamma)$ is not a subspace of $L^2(\Gamma \backslash G)$ because $H$ is noncompact. All these observations suggest that an application of the representation theory of $L^2(\Gamma \backslash G)$ to spectral analysis on $X_\Gamma$ is rather limited when $H$ is noncompact.

Point (1) creates some underlying difficulty to Problem B: we need to consider locally symmetric spaces $X_\Gamma$ for which proper discontinuity of the action of $\Gamma$ on $X$ is preserved under small deformations of $\Gamma$ in $G$. This is nontrivial. This question was first studied by the author [9, 11]. See [4] for further study. An interesting aspect of the case of noncompact $H$ is that there are more examples where nontrivial deformations of compact quotients exist than for compact $H$ (cf. Weil’s local rigidity theorem [18]). Perspectives from Point (1) will be discussed in Section 4.

Point (2) makes Problem A nontrivial. It is not clear if the following well-known properties in the Riemannian case holds in our setting in the pseudo-Riemannian case.

**Fact 1** Suppose $M$ is a compact Riemannian manifold.

1. The Laplacian $\Delta_M$ extends to a self-adjoint operator on $L^2(M)$.
2. There exist infinitely many $L^2$-eigenvalues of $\Delta_M$.
3. An eigenfunction of $\Delta_M$ is infinitely differentiable.
4. Each eigenspace of $\Delta_M$ is finite-dimensional.
5. The set of $L^2$-eigenvalues is discrete in $\mathbb{R}$.

**Remark 1.** We shall see that the third to fifth properties of Fact 1 may fail in the pseudo-Riemannian case, e.g., Example 6 for (3) and (4), and $M = \mathbb{R}^{2,1}/\mathbb{Z}^3$ (Theorem 7) for (5).

In spite of these difficulties, we wish to reveal a mystery of spectral analysis of pseudo-Riemannian locally homogeneous spaces $X_\Gamma = \Gamma \backslash G/H$. We shall discuss self-adjoint extension of the Laplacian in the pseudo-Riemannian setting in Theorem 13 and the existence of countable many $L^2$-eigenvalues in Theorems 8 and 12.

### 3 Pseudo-Riemannian manifold

#### 3.1 Laplacian on pseudo-Riemannian manifolds

A *pseudo-Riemannian manifold* $M$ is a smooth manifold endowed with a smooth, nondegenerate, symmetric bilinear tensor $g$ of signature $(p, q)$ for some $p, q \in \mathbb{N}$. 
(M, g) is a Riemannian manifold if q = 0, and is a Lorentzian manifold if q = 1. The metric tensor g induces a Radon measure dµ on X, and the divergence div. Then the Laplacian

\[ \Delta_M := \text{div grad}, \]

is a differential operator of second order which is a symmetric operator on the Hilbert space \( L^2(X, d\mu) \).

**Example 1.** Let \((M, g)\) be the standard flat pseudo-Riemannian manifold:

\[ R^{p,q} := (R^{p+q}, dx_1^2 + \ldots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2). \]

Then the Laplacian takes the form

\[ \Delta_{R^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}. \]

In general, \( \Delta_M \) is an elliptic differential operator if \((M, g)\) is Riemannian, and is a hyperbolic operator if \((M, g)\) is Lorentzian.

### 3.2 Homogeneous pseudo-Riemannian manifolds

A typical example of pseudo-Riemannian manifolds \( X \) with “large” isometry groups is semisimple symmetric spaces, for which the infinitesimal classification was accomplished by M. Berger in 1950s. In this case, \( X \) is given as a homogeneous space \( G/H \) where \( G \) is a semisimple Lie group and \( H \) is an open subgroup of the fixed point group \( G^\sigma = \{ g \in G : \sigma g = g \} \) for some involutive automorphism \( \sigma \) of \( G \). In particular, \( G \supset H \) are a pair of reductive Lie groups.

More generally, we say \( G/H \) is a *reductive homogeneous space* if \( G \supset H \) are a pair of real reductive algebraic groups. Then we have the following:

**Proposition 1.** Any reductive homogeneous space \( X = G/H \) carries a pseudo-Riemannian structure such that \( G \) acts on \( X \) by isometries.

**Proof.** By a theorem of Mostow, we can take a Cartan involution \( \theta \) of \( G \) such that \( \theta H = H \). Then \( K := G^\theta \) is a maximal compact subgroup of \( G \), and \( H \cap K \) is that of \( H \). Let \( g = \mathfrak{k} + \mathfrak{p} \) be the corresponding Cartan decomposition of the Lie algebra \( g \) of \( G \). Take an \( \text{Ad}(G) \)-invariant nondegenerate symmetric bilinear form \( \langle , \rangle \) on \( g \) such that \( \langle , \rangle|_{\mathfrak{k} \times \mathfrak{k}} \) is negative definite, \( \langle , \rangle|_{\mathfrak{p} \times \mathfrak{p}} \) is positive definite, and \( \mathfrak{k} \) and \( \mathfrak{p} \) are orthogonal to each other. (If \( G \) is semisimple, then we may take \( \langle , \rangle \) to be the Killing form of \( g \).)

Since \( \theta H = H \), the Lie algebra \( \mathfrak{h} \) of \( H \) is decomposed into a direct sum \( \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p}) \), and therefore the bilinear form \( \langle , \rangle \) is non-degenerate when restricted to \( \mathfrak{h} \). Then \( \langle , \rangle \) induces an \( \text{Ad}(H) \)-invariant nondegenerate symmetric bilinear form \( \langle , \rangle|_{\mathfrak{g}/\mathfrak{h}} \) on the quotient space \( \mathfrak{g}/\mathfrak{h} \), with which we identify the tangent space \( T_o(G/H) \)
at the origin $o = eH \in G/H$. Since the bilinear form $(\cdot, \cdot)_{g/H}$ is $\text{Ad}(H)$-invariant, the left translation of this form is well-defined and gives a pseudo-Riemannian structure $g$ on $G/H$ of signature $(\dim p/h \cap p, \dim k/h \cap k)$. By the construction, the group $G$ acts on the pseudo-Riemannian manifold $(G/H, g)$ by isometries. □

3.3 Pseudo-Riemannian manifolds with constant curvature, Anti-de Sitter manifolds

Let $Q_{p,q}(x) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ be a quadratic form on $\mathbb{R}^{p+q}$ of signature $(p, q)$, and we denote by $O(p, q)$ the indefinite orthogonal group preserving the form $Q_{p,q}$. We define two hypersurfaces $M_{p,q}^\pm$ in $\mathbb{R}^{p+q}$ by

$$M_{p,q}^\pm := \{ x \in \mathbb{R}^{p+q} : Q_{p,q}(x) = \pm 1 \}.$$ 

By switching $p$ and $q$, we have an obvious diffeomorphism

$$M_{p,q}^\pm \simeq M_{q,p}^\mp.$$ 

The flat pseudo-Riemannian structure $\mathbb{R}^{p,q}$ (Example 1) induces a pseudo-Riemannian structure on the hypersurface $M_{p,q}^\pm$ of signature $(p - 1, q)$ with constant curvature 1, and that on $M_{p,q}^\pm$ of signature $(p, q - 1)$ with constant curvature $-1$.

The natural action of the group $O(p, q)$ on $\mathbb{R}^{p,q}$ induces an isometric and transitive action on the hypersurfaces $M_{p,q}^\pm$, and thus they are expressed as homogeneous spaces:

$$M_{p,q}^\pm \simeq O(p,q)/O(p-1,q), \quad M_{p,q}^\pm \simeq O(p,q)/O(p,q-1),$$

giving examples of pseudo-Riemannian homogeneous spaces as in Proposition 1.

The anti-de Sitter space $\text{AdS}^n = M_{n-1,2}^-$ is a model space for $n$-dimensional Lorentzian manifolds of constant negative sectional curvature, or anti-de Sitter $n$-manifolds. This is a Lorentzian analogue of the real hyperbolic space $H^n$. For the convenience of the reader, we list model spaces of Riemannian and Lorentzian manifolds with constant positive, zero, and negative curvatures.

Riemannian manifolds with constant curvature:

$$S^n = M_{n+1,0}^+ \simeq O(n+1)/O(n) : \text{standard sphere},$$

$$\mathbb{R}^n : \text{Euclidean space},$$

$$H^n = M_{n-1}^- \simeq O(1,n)/O(n) : \text{hyperbolic space},$$

Lorentzian manifolds with constant curvature:
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\[ \text{dS}^n = M_{+}^{n,1} \simeq O(n,1)/O(n-1,1) : \text{de Sitter space,} \]
\[ \mathbb{R}^{n-1,1} : \text{Minkowski space,} \]
\[ \text{AdS}^n = M_{-}^{n-1,2} \simeq O(2,n-1)/O(1,n-1) : \text{anti-de Sitter space,} \]

4 Discontinuous groups for pseudo-Riemannian manifolds

4.1 Existence problem of compact Clifford–Klein forms

Let \( H \) be a closed subgroup of a Lie group \( G \), and \( \Gamma \) a discrete subgroup of \( G \). If \( H \) is compact, then the double coset space \( \Gamma \backslash G/H \) becomes a \( C^\infty \)-manifold for any torsion-free discrete subgroup \( \Gamma \) of \( G \). However, we have to be careful for noncompact \( H \), because not all discrete subgroups acts properly discontinuously on \( G/H \), and \( \Gamma \backslash G/H \) may not be Hausdorff in the quotient topology. We illustrate this feature by two general results:

**Fact 2**

(1) (Moore’s ergodicity theorem [13]) Let \( G \) be a simple Lie group, and \( \Gamma \) a lattice. Then \( \Gamma \) acts ergodically on \( G/H \) for any noncompact closed subgroup \( H \). In particular, \( \Gamma \backslash G/H \) is non-Hausdorff.

(2) (Calabi–Markus phenomenon ([2, 8])) Let \( G \) be a reductive Lie group, and \( \Gamma \) an infinite discrete subgroup. Then \( \Gamma \backslash G/H \) is non-Hausdorff for any reductive subgroup \( H \) with \( \text{rank}_\mathbb{R} G = \text{rank}_\mathbb{R} H \).

In fact, determining which groups act properly discontinuously on reductive homogeneous spaces \( G/H \) is a delicate problem, which was first considered in full generality by the author; we refer to [13, Section 3.2] for a survey.

Suppose now a discrete subgroup \( \Gamma \) acts properly discontinuously and freely on \( X = G/H \). Then the quotient space

\[ X_\Gamma := \Gamma \backslash X \simeq \Gamma \backslash G/H \]

carries a \( C^\infty \)-manifold structure such that the quotient map \( p : X \to X_\Gamma \) is a covering, through which \( X_\Gamma \) inherits any \( G \)-invariant local geometric structure on \( X \). We say \( \Gamma \) is a **discontinuous group** for \( X \) and \( X_\Gamma \) is a **Clifford–Klein form** of \( X = G/H \).

**Example 2**

(1) If \( X = G/H \) is a reductive homogeneous space, then any Clifford–Klein form \( X_\Gamma \) carries a pseudo-Riemannian structure by Proposition [1].

(2) If \( X = G/H \) is a semisimple symmetric space, then any Clifford–Klein form \( X_\Gamma = \Gamma \backslash G/H \) is a pseudo-Riemannian locally symmetric space, namely, the (local) geodesic symmetry at every \( p \in X_\Gamma \) with respect to the Levi-Civita connection is locally isometric.

By **space forms**, we mean pseudo-Riemannian manifolds of constant sectional curvature. They are examples of pseudo-Riemannian locally symmetric spaces. For simplicity, we shall assume that they are geodesically complete.
Example 3. Clifford–Klein forms of $M^{p+1,q}_+ = O(p+1,q)/O(p,q)$ (respectively, $M^{p,q+1}_- = O(p,q+1)/O(p,q)$) are pseudo-Riemannian space forms of signature $(p,q)$ with positive (respectively, negative) curvature. Conversely, any (geodesically complete) pseudo-Riemannian space form of signature $(p,q)$ is of this form as far as $p \neq 1$ for positive curvature or $q \neq 1$ for negative curvature.

A general question for reductive homogeneous spaces $G/H$ is:

**Question 1.** Does compact Clifford–Klein forms of $G/H$ exist?

or equivalently,

**Question 2.** Does there exist a discrete subgroup $\Gamma$ of $G$ acting cocompactly and properly discontinuously on $G/H$?

This question has an affirmative answer if $H$ is compact by a theorem of Borel. In the general setting where $H$ is noncompact, the question relates with a “global theory” of pseudo-Riemannian geometry: how local pseudo-Riemannian homogeneous structure affects the global nature of manifolds? A classic example is space form problem which asks the global properties (e.g. compactness, volume, fundamental groups, etc.) of a pseudo-Riemannian manifold of constant curvature (local property). The study of discontinuous groups for $M^{p+1,q}_+$ and $M^{p,q+1}_-$ shows the following results in pseudo-Riemannian space forms of signature $(p,q)$:

**Fact 3** Space forms of positive curvature are

1. always closed if $q = 0$, i.e., sphere geometry in the Riemannian case;
2. never closed if $p \geq q > 0$, in particular, if $q = 1$ (de Sitter geometry in the Lorentzian case [2]).

The phenomenon in the second statement is called the Calabi–Markus phenomenon (see Fact [2](2) in the general setting).

**Fact 4** Compact space forms of negative curvature exist

1. for all dimensions if $q = 0$, i.e., hyperbolic geometry in the Riemannian case;
2. for odd dimensions if $q = 1$, i.e., anti-de Sitter geometry in the Lorentzian case;
3. for $(p,q) = (4m,3)$ ($m \in \mathbb{N}$) or $(8,7)$.

See [13] Section 4] for the survey of the space form problem in pseudo-Riemannian geometry and also of Question 1 for more general $G/H$.

A large and important class of Clifford–Klein forms $X_\Gamma$ of a reductive homogeneous space $X = G/H$ is constructed as follows (see [8]).

**Definition 1.** A quotient $X_\Gamma = \Gamma\backslash X$ of $X$ by a discrete subgroup $\Gamma$ of $G$ is called standard if $\Gamma$ is contained in some reductive subgroup $L$ of $G$ acting properly on $X$.

If a subgroup $L$ acts properly on $G/H$, then any discrete subgroup of $\Gamma$ acts properly discontinuously on $G/H$. A handy criterion for the triple $(G,H,L)$ of reductive groups such that $L$ acts properly on $G/H$ is proved in [8], as we shall recall below.
Let $G = K \exp \mathfrak{a}$ be a Cartan decomposition, where $a$ is a maximal abelian subspace of $p$ and $\mathfrak{a}$ is the dominant Weyl chamber with respect to a fixed positive system $\Sigma^+(g,a)$. This defines a map $\mu : G \to \mathfrak{a}^+$ (Cartan projection) by

$$\mu(k_1 e^X k_2) = X \quad \text{for } k_1, k_2 \in K \text{ and } X \in a.$$ 

It is continuous, proper and surjective. If $H$ is a reductive subgroup, then there exists $g \in G$ such that $\mu(gHg^{-1})$ is given by the intersection of $\mathfrak{a}^+$ with a subspace of dimension $\text{rank } R_H$. By an abuse of notation, we use the same $H$ instead of $gHg^{-1}$.

With this convention, we have:

**Properness Criterion 5 ([8])**

$L$ acts properly on $G/H$ if and only if $\mu(L) \cap \mu(H) = \{0\}$.

By taking a lattice $\Gamma$ of such $L$, we found a family of pseudo-Riemannian locally symmetric spaces $X_\Gamma$ in [8, 13]. The list of symmetric spaces admitting standard Clifford–Klein forms of finite volume (or compact forms) include $M_{p,q+1} = O(p,q+1)/O(p,q)$ with $(p,q)$ satisfying the conditions in Fact 4. Further, by applying Properness Criterion 5 Okuda [16] gave examples of pseudo-Riemannian locally symmetric spaces $\Gamma \backslash G/H$ of infinite volume where $\Gamma$ is isomorphic to the fundamental group $\pi_1(\Sigma_g)$ of a compact Riemann surface $\Sigma_g$ with $g \geq 2$.

For the construction of stable spectrum on $X_\Gamma$ (see Theorem 10 and Theorem 12 (2) below), we introduced in [6, Section 1.6] the following concept:

**Definition 2.** A discrete subgroup $\Gamma$ of $G$ acts strongly properly discontinuously (or sharply) on $X = G/H$ if there exists $C, C' > 0$ such that for all $\gamma \in \Gamma$,

$$d(\mu(\gamma), \mu(H)) \geq C\| \mu(\gamma) \| - C'.$$

Here $d(\cdot, \cdot)$ is a distance in a given by a Euclidean norm $\| \cdot \|$ which is invariant under the Weyl group of the restricted root system $\Sigma(g,a)$. We say the positive number $C$ is the first sharpness constant for $\Gamma$.

If a reductive subgroup $L$ acts properly on a reductive homogeneous space $G/H$, then the action of a discrete subgroup $\Gamma$ of $L$ is strongly properly discontinuous ([6 Example 4.10]).

### 4.2 Deformation of Clifford–Klein forms

Let $G$ be a Lie group and $\Gamma$ a finitely generated group. We denote by $\text{Hom}(\Gamma, G)$ the set of all homomorphisms of $\Gamma$ to $G$ topologized by pointwise convergence. By taking a finite set $\{\gamma_1, \cdots, \gamma_k\}$ of generators of $\Gamma$, we can identify $\text{Hom}(\Gamma, G)$ as a subset of the direct product $G \times \cdots \times G$ by the inclusion:

$$\text{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(\gamma_1), \cdots, \varphi(\gamma_k)).$$

(1)
If \( \Gamma \) is finitely presentable, then \( \text{Hom}(\Gamma, G) \) is realized as a real analytic variety via (I).

Suppose \( G \) acts continuously on a manifold \( X \). We shall take \( X = G/H \) with noncompact closed subgroup \( H \) later. Then not all discrete subgroups act properly discontinuously on \( X \) in this general setting. The main difference of the following definition of the author [9] in the general case from that of Weil [18] is a requirement of proper discontinuity.

\[
R(\Gamma, G; X) := \{ \varphi \in \text{Hom}(\Gamma, G) : \varphi \text{ is injective}, \\
\quad \text{and } \varphi(\Gamma) \text{ acts properly discontinuously and freely on } G/H \}.
\]

Suppose now \( X = G/H \) for a closed subgroup \( H \). Then the double coset space \( \varphi(\Gamma) \backslash G/H \) forms a family of manifolds that are locally modelled on \( G/H \) with parameter \( \varphi \in R(\Gamma, G; X) \). To be more precise on “parameter”, we note that the conjugation by an element of \( G \) induces an automorphism of \( \text{Hom}(\Gamma, G) \) which leaves \( R(\Gamma, G; X) \) invariant. Taking these unessential deformations into account, we define the deformation space (generalized Teichmüller space) as the quotient set

\[
\mathcal{F}(\Gamma, G; X) := R(\Gamma, G; X)/G.
\]

**Example 4.**

1. Let \( \Gamma \) be the surface group \( \pi_1(\Sigma_g) \) of genus \( g \geq 2 \), \( G = \text{PSL}(2, \mathbb{R}) \), \( X = H^2 \) (two-dimensional hyperbolic space). Then \( \mathcal{F}(\Gamma, G; X) \) is the classical Teichmüller space, which is of dimension \( 6g - 6 \).
2. \( G = \mathbb{R}^n \), \( X = \mathbb{R}^n \), \( \Gamma = \mathbb{Z}^n \). Then \( \mathcal{F}(\Gamma, G; X) \simeq \text{GL}(n, \mathbb{R}) \) (see (4) below).
3. \( G = \text{SO}(2, 2) \), \( X = \text{AdS}^3 \), and \( \Gamma = \pi_1(\Sigma_g) \). Then \( \mathcal{F}(\Gamma, G; X) \) is of dimension \( 12g - 12 \) (see [6, Section 9.2] and references therein).

**Remark 2.** There is a natural isometry between \( X_{\varphi(\Gamma)} \) and \( X_{\varphi(\varphi(\Gamma)^{-1})} \). Hence, the set \( \text{Spec}_d(X_{\varphi(\Gamma)}) \) of \( L^2 \)-eigenvalues is independent of the conjugation of \( \varphi \in R(\Gamma, G; X) \) by an element of \( G \). By an abuse of notation we shall write \( \text{Spec}_d(X_{\varphi(\Gamma)}) \) for \( \varphi \in \mathcal{F}(\Gamma, G; X) \) when we deal with Problem \([3]\) of Section \([2]\).

### 5 Spectrum on \( \mathbb{R}^{p,q}/\mathbb{Z}^{p+q} \) and Oppenheim conjecture

This section gives an elementary but inspiring observation of spectrum on flat pseudo-Riemannian manifolds.

#### 5.1 Spectrum of \( \mathbb{R}^{p,q}/\varphi(\mathbb{Z}^{p+q}) \)

Let \( G = \mathbb{R}^n \) and \( \Gamma = \mathbb{Z}^n \). Then the group homomorphism \( \varphi : \Gamma \to G \) is uniquely determined by the image \( \varphi(e_j) \) (\( 1 \leq j \leq n \)) where \( e_1, \ldots, e_n \in \mathbb{Z}^n \) are the standard
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basis, and thus we have a bijection

\[ \text{Hom}(\Gamma, G) \xrightarrow{\sim} M(n, \mathbb{R}), \quad \varphi \mapsto g \]  

(3)

by \( \varphi_g(m) := gm \) for \( m \in \mathbb{Z}^n \), or equivalently, by \( g = (\varphi_g(e_1), \ldots, \varphi_g(e_n)) \).

Let \( \sigma \in \text{Aut}(G) \) be defined by \( \sigma(x) := -x \). Then \( H := G^\sigma = \{0\} \) and \( X := G/H \cong \mathbb{R}^n \) is a symmetric space. The discrete group \( \Gamma \) acts properly discontinuously on \( X \) via \( \varphi_g \), if and only if \( g \in GL(n, \mathbb{R}) \). Moreover, since \( G \) is abelian, \( G \) acts trivially on \( \text{Hom}(\Gamma, G) \) by conjugation, and therefore the deformation space \( \mathcal{F}(\Gamma, G; X) \) identifies with \( R(\Gamma, G; X) \). Hence we have a natural bijection between the two subsets of \( \{3\} \):

\[ \mathcal{F}(\Gamma, G; X) \xrightarrow{\sim} GL(n, \mathbb{R}). \]  

(4)

Fix \( p, q \in \mathbb{N} \) such that \( p + q = n \), and we endow \( X \cong \mathbb{R}^n \) with the standard flat indefinite metric \( \mathbb{R}^{p,q} \) (see Example \( \{1\} \)). Let us determine \( \text{Spec}_d(X_{\varphi_g}(\Gamma)) \cong \text{Spec}_d(\mathbb{R}^{p,q} / \varphi_g(\mathbb{Z}^n)) \) for \( g \in GL(n, \mathbb{R}) \cong \mathcal{F}(\Gamma, G; X) \).

For this, we define a function on \( X = \mathbb{R}^n \) by

\[ f_m(x) := \exp(2\pi \sqrt{-1} m g^{-1} x) \quad (x \in \mathbb{R}^n) \]

for each \( m \in \mathbb{Z}^n \) where \( x \) and \( m \) are regarded as column vectors. Clearly, \( f_m \) is \( \varphi_g(\Gamma) \)-periodic and defines a real analytic function on \( X_{\varphi_g}(\Gamma) \). Furthermore, \( f_m \) is an eigenfunction of the Laplacian \( \Delta_{\mathbb{R}^{p,q}} \):

\[ \Delta_{\mathbb{R}^{p,q}} f_m = -4\pi^2 Q_{g^{-1} p q g}^{-1} f_m, \]

where, for a symmetric matrix \( S \in M(n, \mathbb{R}) \), \( Q_S \) denotes the quadratic form on \( \mathbb{R}^n \) given by

\[ Q_S(y) := y^T S y \quad \text{for} \quad y \in \mathbb{R}^n. \]

Since \( \{f_m : m \in \mathbb{Z}^n\} \) spans a dense subspace of \( L^2(X_{\varphi_g}(\Gamma)) \), we have shown:

**Proposition 2.** For any \( g \in GL(n, \mathbb{R}) \cong \mathcal{F}(\Gamma, G; X) \),

\[ \text{Spec}_d(X_{\varphi_g}(\Gamma)) = \{ -4\pi^2 Q_{g^{-1} p q g}^{-1}(m) : m \in \mathbb{Z}^n \}. \]

Here are some observation in the \( n = 1, 2 \) cases.

**Example 5.** Let \( n = 1 \) and \( (p, q) = (1, 0) \). Then \( \text{Spec}_d(X_{\varphi_g}(\Gamma)) = \{ -4\pi^2 m^2 / g^2 : m \in \mathbb{Z} \} \) for \( g \in \mathbb{R} \cong GL(1, \mathbb{R}) \) by Proposition \( \{2\} \). Thus the smaller the period \( |g| \) is, the larger the absolute value of the eigenvalue \( -4\pi^2 m^2 / g^2 \) becomes for each fixed \( m \in \mathbb{Z} \setminus \{0\} \). This is thought of as a mathematical model of a music instrument for which shorter strings produce a higher pitch than longer strings (see Introduction).

**Example 6.** Let \( n = 2 \) and \( (p, q) = (1, 1) \). Take \( g = I_2 \), so that \( \varphi_g(\Gamma) = \mathbb{Z}^2 \) is the standard lattice. Then the \( L^2 \)-eigenspace of the Laplacian \( \Delta_{\mathbb{R}^{1,1}/\mathbb{Z}^2} \) for zero eigenvalue contains \( W := \{ \psi(x - y) : \psi \in L^2(\mathbb{R}/\mathbb{Z}) \} \). Since \( W \) is infinite-dimensional and \( W \not\subset C^0(\mathbb{R}^2 / \mathbb{Z}^2) \), the third and fourth statements of Fact \( \{1\} \) fail in this pseudo-Riemannian setting.
By the explicit description of $\text{Spec}_d(X_{\varphi}(\Gamma))$ for all $\varphi \in \mathcal{T}(\Gamma, G; X)$ in Proposition\textsuperscript{2} we can also tell the behaviour of $\text{Spec}_d(X_{\varphi}(\Gamma))$ under deformation of $\Gamma$ by $\varphi$. Obviously, any constant function on $X_{\varphi}(\Gamma)$ is an eigenfunction of the Laplacian $\Delta_{X_{\varphi}(\Gamma)} = \Delta_{\mathbb{R}^{p+q}}/\varphi(\mathbb{Z}^{p+q})$ with eigenvalue zero. We see that this is the unique stable $L^2$-eigenvalue in the flat compact manifold:

**Corollary 1 (non-existence of stable eigenvalues).** Let $n = p + q$ with $p, q \in \mathbb{N}$. For any open subset $V$ of $\mathcal{T}(\Gamma, G; X)$, 

$$ \bigcap_{\varphi \in V} \text{Spec}_d(X_{\varphi}(\Gamma)) = \{0\}. $$

### 5.2 Oppenheim’s conjecture and stability of spectrum

In 1929, Oppenheim [17] raised a question about the distribution of an indefinite quadratic forms at integral points. The following theorem, referred to as Oppenheim’s conjecture, was proved by Margulis (see [14] and references therein).

**Fact 6 (Oppenheim’s conjecture)** Suppose $n \geq 3$ and $Q$ is a real nondegenerate indefinite quadratic form in $n$ variables. Then either $Q$ is proportional to a form with integer coefficients (and thus $Q(\mathbb{Z}^n)$ is discrete in $\mathbb{R}$), or $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

Combining this with Proposition\textsuperscript{2} we get the following.

**Theorem 7.** Let $p + q = n$, $p \geq 2$, $q \geq 1$, $G = \mathbb{R}^n$, $X = \mathbb{R}^{p,q}$ and $\Gamma = \mathbb{Z}^n$. We define an open dense subset $U$ of $\mathcal{T}(\Gamma, G; X) \simeq GL(n, \mathbb{R})$ by 

$$ U := \{ g \in GL(n, \mathbb{R}) : g^{-1}f_{p,q}g^{-1} \text{ is not proportional to an element of } M(n, \mathbb{Z}) \}. $$

Then the set $\text{Spec}_d(X_{\varphi}(\Gamma))$ of $L^2$-eigenvalues of the Laplacian is dense in $\mathbb{R}$ if and only if $\varphi \in U$.

Thus the fifth statement of Fact\textsuperscript{1} for compact Riemannian manifolds do fail in the pseudo-Riemannian case.

### 6 Main results—sound of anti-de Sitter manifolds

#### 6.1 Intrinsic sound of anti-de Sitter manifolds

In general, it is not clear whether the Laplacian $\Delta_M$ admits infinitely many $L^2$-eigenvalues for compact pseudo-Riemannian manifolds. For anti-de Sitter 3-manifolds, we proved in [6, Theorem 1.1]:
Theorem 8. For any compact anti-de Sitter 3-manifold $M$, there exist infinitely many $L^2$-eigenvalues of the Laplacian $\Delta_M$.

In the abelian case, it is easy to see that compactness of $X_\Gamma$ is necessary for the existence of $L^2$-eigenvalues:

Proposition 3. Let $G = \mathbb{R}^{p+q}, X = \mathbb{R}^{p,q}, \Gamma = \mathbb{Z}^k$, and $\varphi \in R(\Gamma, G; X)$. Then $\text{Spec}_d(X_{\varphi(\Gamma)}) \neq \emptyset$ if and only if $X_{\varphi(\Gamma)}$ is compact, or equivalently, $k = p + q$.

However, anti-de Sitter 3-manifolds $M$ admit infinitely many $L^2$-eigenvalues even when $M$ is of infinite-volume (see [6, Theorem 9.9]):

Theorem 9. For any finitely generated discrete subgroup $\Gamma$ of $G = SO(2, 2)$ acting properly discontinuously and freely on $X = \text{AdS}^3$,

$$\text{Spec}_d(X_\Gamma) \supset \{l(l - 2) : l \in \mathbb{N}, l \geq 10C^{-3}\}$$

where $C \equiv C(\Gamma)$ is the first sharpness constant of $\Gamma$.

The above $L^2$-eigenvalues are stable in the following sense:

Theorem 10 (stable $L^2$-eigenvalues). Suppose that $\Gamma \subset G = SO(2, 2)$ and $M = \Gamma \setminus \text{AdS}^3$ is a compact standard anti-de Sitter 3-manifold. Then there exists a neighbourhood $U \subset \text{Hom}(\Gamma, G)$ of the natural inclusion with the following two properties:

$$U \subset R(\Gamma, G; \text{AdS}^3),$$

$$\#(\bigcap_{\varphi \in U} \text{Spec}_d(X_{\varphi})) = \infty.$$ (5)

The first geometric property (5) asserts that a small deformation of $\Gamma$ keeps proper discontinuity, which was conjectured by Goldman [3] in the AdS$^3$ setting, and proved affirmatively in [11]. Theorem 10 was proved in [6, Corollary 9.10] in a stronger form (e.g., without assuming “standard” condition).

Figuratively speaking, Theorem 10 says that compact anti-de Sitter manifolds have “intrinsic sound” which is stable under any small deformation of the anti-de Sitter structure. This is a new phenomenon which should be in sharp contrast to the abelian case (Corollary 11) and the Riemannian case below:

Fact 11 (see [20, Theorem 5.14]). For a compact hyperbolic surface, no eigenvalue of the Laplacian above $\frac{1}{4}$ is constant on the Teichmüller space.

We end this section by raising the following question in connection with the flat case (Theorem 7):

Question 3. Suppose $M$ is a compact anti-de Sitter 3-manifold. Find a geometric condition on $M$ such that $\text{Spec}_d(M)$ is discrete.


7 Perspectives and sketch of proof

The results in the previous section for anti-de Sitter 3-manifolds can be extended to more general pseudo-Riemannian locally symmetric spaces of higher dimension:

**Theorem 12 ([6, Theorem 1.5]).** Let \( X_\Gamma \) be a standard Clifford–Klein form of a semisimple symmetric space \( X = G/H \) satisfying the rank condition
\[
\text{rank} G/H = \text{rank} K/H \cap K.
\]

Then the following holds.

(1) There exists an explicit infinite subset \( I \) of joint \( L^2 \)-eigenvalues for all the differential operators on \( X_\Gamma \) that are induced from \( G \)-invariant differential operators on \( X \).

(2) (stable spectrum) If \( \Gamma \) is contained in a simple Lie group \( L \) of real rank one acting properly on \( X = G/H \), then there is a neighbourhood \( V \subset \text{Hom}(\Gamma, G) \) of the natural inclusion such that for any \( \phi \in V \), the action \( \phi(\Gamma) \) on \( X \) is properly discontinuous and the set of joint \( L^2 \)-eigenvalues on \( X_{\phi(\Gamma)} \) contains the infinite set \( I \).

**Remark 3.** We do not require \( X_\Gamma \) to be of finite volume in Theorem [12].

**Remark 4.** It is plausible that for a general locally symmetric space \( \Gamma \backslash G/H \) with \( G \) reductive, no nonzero \( L^2 \)-eigenvalue is stable under nontrivial small deformation unless the rank condition (7) is satisfied. For instance, suppose \( \Gamma = \pi_1(\Sigma_g) \) with \( g \geq 2 \) and \( R(\Gamma, G; X) \neq \emptyset \). (Such semisimple symmetric space \( X = G/H \) was recently classified in [16].) Then we expect the rank condition (7) is equivalent to the existence of an open subset \( U \) in \( R(\Gamma, G; X) \) such that
\[
\#(\bigcap_{\phi \in U} \text{Spec}_{\phi}(X_{\phi(\Gamma)})) = \infty.
\]

It should be noted that not all \( L^2 \)-eigenvalues of compact anti-de Sitter manifolds are stable under small deformation of anti-de Sitter structure. In fact, we proved in [7] that there exist also countably many negative \( L^2 \)-eigenvalues that are NOT stable under deformation, whereas the countably many stable \( L^2 \)-eigenvalues that we constructed in Theorem [9] are all positive. More generally, we prove in [7] the following theorem that include both stable and unstable \( L^2 \)-eigenvalues:

**Theorem 13.** Let \( G \) be a reductive homogeneous space and \( L \) a reductive subgroup of \( G \) such that \( H \cap L \) is compact. Assume that the complexification \( X_C \) is \( L_C \)-spherical. Then for any torsion-free discrete subgroup \( \Gamma \) of \( L \), we have:

(1) the Laplacian \( \Delta_{X_\Gamma} \) extends to a self-adjoint operator on \( L^2(X_\Gamma) \);

(2) \#\text{Spec}_{\phi}(X_{\Gamma}) = \infty \) if \( X_\Gamma \) is compact.
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By “$L_{\mathbb{C}}$-spherical” we mean that a Borel subgroup $L_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. In this case, a reductive subgroup $L$ acts transitively on $X$ by \cite[Lemma 5.1]{10}.

Here are some examples of the setting of Theorem 13, taken from \cite[Corollary 3.3.7]{13}.

### Table 1 [1]

|   | $G$                   | $H$                   | $L$                  |
|---|-----------------------|-----------------------|----------------------|
| (i) | $SO(2n,2)$           | $SO(2n,1)$           | $U(n,1)$             |
| (ii) | $SO(2n,2)$           | $U(n,1)$              | $SO(2n,1)$           |
| (iii) | $SU(2n,2)$           | $U(2n,1)$              | $Sp(n,1)$             |
| (iv) | $SU(2n,2)$           | $Sp(n,1)$             | $U(2n,1)$             |
| (v) | $SO(4n,4)$           | $SO(4n,3)$           | $Sp(1) \times Sp(n,1)$ |
| (vi) | $SO(8,8)$           | $SO(8,7)$            | $Spin(8,1)$           |
| (vii) | $SO(8,\mathbb{C})$ | $SO(7,\mathbb{C})$ | $Spin(7,1)$           |
| (viii) | $SO(4,4)$           | $Spin(4,3)$         | $SO(4,1) \times SO(3)$ |
| (ix) | $SO(4,3)$            | $G_2(\mathbb{R})$ | $SO(4,1) \times SO(2)$ |

Examples for Theorem 13 include Table 1 (ii) for all $n \in \mathbb{N}$, whereas we need $n \in 2\mathbb{N}$ in Theorem 12 for the rank condition (7).

The idea of the proof for Theorem 12 is to take an average of a (nonperiodic) eigenfunction on $X$ with rapid decay at infinity over $\Gamma$-orbits as a generalization of Poincaré series. Geometric ingredients of the convergence (respectively, nonzeroness) of the generalized Poincaré series include “counting $\Gamma$-orbits” stated in Lemma 1 below (respectively, the Kazhdan–Margulis theorem, cf. \cite[Proposition 8.14]{6}). Let $B(o, R)$ be a “pseudo-ball” of radius $R > 0$ centered at the origin $o = eH \in X = G/H$, and we set

$$N(x, R) := \# \{ \gamma \in \Gamma : \gamma \cdot x \in B(o, R) \}.$$  

**Lemma 1 (\cite[Corollary 4.7]{6}).**

1. If $\Gamma$ acts properly discontinuously on $X$, then $N(x, R) < \infty$ for all $x \in X$ and $R > 0$.
2. If $\Gamma$ acts strongly properly discontinuously on $X$, then there exists $A_x > 0$ such that

$$N(x, R) \leq A_x \exp \left( \frac{R}{C} \right) \quad \text{for all } R > 0,$$

where $C$ is the first sharpness constant of $\Gamma$.

The key idea of Theorem 13 is to bring branching laws to spectral analysis \cite[10]{12}, namely, we consider the restriction of irreducible representations of $G$ that are realized in the space of functions on the homogeneous space $X = G/H$ and analyze the $G$-representations when restricted to the subgroup $L$. Details will be given in \cite{7}.
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