Continuous quantum phase transitions beyond Landau’s paradigm in a large-\(N\) spin model

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We study a large-\(N\) generalization of \(J_1-J_2\) Heisenberg model on square lattice – an \(Sp(2N)\) spin model. The possible quantum spin liquid phases of the \(Sp(2N)\) model are studied using the \(SU(2)\) projective construction. We find several spin liquid states at least in large \(N\) limit, which include \(SU(2)\) \(\pi\)-flux state, \(SU(2)\) chiral spin state and \(Z_2\) spin liquid states. All those spin liquid states have non-trivial quantum orders. We show how projective symmetry group, which characterizes quantum order, protects the stability of even gapless spin liquids. We also study the continuous quantum phase transition from the \(SU(2)\) \(\pi\)-flux state to the \(SU(2)\) chiral spin liquid state, and from the \(SU(2)\) \(\pi\)-flux state to the \(Z_2\) spin liquid state. We show that those phase transitions are beyond the paradigm of Landau symmetry-breaking theory. The first phase transition, although a \(Z_2\) symmetry breaking transition, contains critical exponents that are different from those obtained from the Ginzburg-Landau theory of Ising universality class. The second transition does not even involve a change of symmetry and has no symmetry breaking order parameter.

I. INTRODUCTION

The \(SU(2)\) projective construction (or, more generally, slave-boson theory [1, 2]) for 2D spin liquids in spin-1/2 systems has been studied over ten years [3, 4]. Upon doping, the \(SU(2)\) projective construction provides a quite complete theory for underdoped high \(T_c\) superconductors [5, 6]. The theory explains the strange Fermi surfaces [5, 6], the strong \((\pi, \pi)\) spin fluctuations [7, 8], and the temperature dependence of superfluid density [9, 10] for underdoped samples.

At half filling, the \(SU(2)\) projective construction can also be used to describe various spin liquids. In fact, the \(SU(2)\) projective construction (or more generally, the slave boson theory) have predict many different spin liquids, such as the algebraic spin liquid [8, 11, 12], chiral spin liquid [13, 14], \(Z_2\) spin liquids [15–18]. It was also shown that the \(SU(2)\) projective construction is capable of describing hundreds of different spin liquids that have the same symmetry but different quantum orders [12].

The above predictions of spin liquids and the classification of spin liquids were based on mean-field calculations. At the mean-field level, it is not very hard to design a spin Hamiltonian that realizes a spin liquid that has one of a few hundreds quantum orders. It is also not hard to find the mean-field ground state for a given spin Hamiltonian. The real issue is whether we should trust the mean-field results. It was argued [12, 16] that, if the obtained mean-field ground state is unstable (i.e. if the mean-field fluctuations cause diverging interactions at low energies), then the mean-field result cannot be trusted and the mean-field state does not correspond to any real physical spin state. It was also argued that, if the mean-field ground state is stable (if the mean-field fluctuations cause vanishing interactions at low energies), then the mean-field result can be trusted and the mean-field state does correspond to a real physical spin liquid state.

However, the above statement about stable mean-field states is too optimistic. A ‘stable mean-field state’ does not have diverging fluctuations at low energies. So it does not have to be unstable. On the other hand, it does not have to be stable either. This is because short-distance fluctuations, if strong enough, can also cause phase transitions and instabilities. Therefore, in order for a mean-field result to be reliable, the mean-field state must be stable (i.e. no infrared divergence) and the short-distance fluctuations must be weak. As we do not have any small parameters in the \(SU(2)\) slave-boson theory for spin-1/2 systems, the short-distance fluctuations are not weak, even for stable mean-field states. Because of this, it is not clear if the mean-field results, even for the stable states, can be applied to the spin-1/2 model or not.

In this paper, we will generalize the spin-1/2 model to a large-\(N\) model. The large-\(N\) model can be solved approximately using the \(SU(2)\) projective construction. We will show that the \(SU(2)\) projective construction for the large-\(N\) model have weak short-distance fluctuations. Thus, the stable \(SU(2)\) mean-field states for the large-\(N\) model do correspond to real physical spin liquid states. The \(SU(2)\) mean-field results for the stable states, such as fractionalization, emergent gauge structures and emergent Fermi statistics, can be applied to the large-\(N\) model.

We concentrate on a model with nearest-neighbor \(J_1\) coupling and next-nearest-neighbor \(J_2\) coupling. We find several mean-field phases that include \(SU(2)\) \(\pi\)-flux state, \(SU(2)\) chiral spin state and \(Z_2\) spin liquid state.

The \(SU(2)\) \(\pi\)-flux state is described by a low energy
effective theory that includes gapless Dirac fermions coupled to SU(2) gauge fields. Due to the non-vanishing interaction between the fermions and the gauge bosons down to zero energy, there is no free fermionic or bosonic low energy quasiparticles in the SU(2) π-flux state. Despite this, we show that the SU(2) π-flux state is a stable spin liquid state. It is a realization of algebraic spin liquids. [8, 12]

The SU(2) chiral spin state and Z₂ spin liquid state are both gapped and, thus, naturally stable. Both state carry non-trivial topological orders. Within our J₁-J₂ model, the quantum transition (i.e. the zero-temperature transition) between the SU(2) π-flux state and the SU(2) chiral spin state turns out to be a continuous transition. The transition breaks the time-reversal symmetry and has a well defined Z₂ order parameter. However, we show that the critical properties of the transition are not described by the 3D Ising universality class of the Ginzburg-Landau theory. This is because the transition not only break the Z₂ symmetry, it also changes the quantum/topological order. The continuous quantum transition between the SU(2) π-flux state and the SU(2) chiral spin state is a new class of quantum transition.

It has been shown that continuous quantum transitions are possible between two states with the same symmetry (but different topological orders). [19–23] The continuous quantum transitions are also possible between two states with the incompatible [37] symmetries. In this paper, we show that even symmetry-breaking transitions with well defined order parameters, sometimes are not described by Landau’s symmetry breaking theory. So it appear that most quantum continuous transitions are not described by Ginzburg-Landau theory, regardless if they have symmetry breaking and order parameter or not.

We also study the transitions from the SU(2) π-flux state to a Z₂-linear spin liquid state, and from the SU(2) π-flux state to a U(1)-linear spin liquid state. The transition between the SU(2) π-flux state to the Z₂-linear state is found to be a continuous quantum transition that does not break any symmetry. The transition between the SU(2) π-flux state to the U(1)-linear state, on the other hand, is a continuous quantum transition that breaks lattice rotaion and translation symmetry. What is surprising is that the two seemingly very different transitions are described by the same critical point with the same set of critical exponents.

II. FORMULATION OF SU(2) PROJECTIVE CONSTRUCTION

First we would like to briefly review the SU(2) projective construction. We will mainly follow the notation of Ref. [12, 25]. Let us start with spin-1/2 Heisenberg Model, and introduce the fermion representation of spins:

\[ H = \sum_{(i,j)} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \]  

\[ \mathbf{S}_i = \frac{1}{2} f_{i\alpha}^{\dagger} \sigma_{\alpha\beta} f_{i\beta} \]  

If we regard \( H \) as an operator acting on the fermion Hilbert space, then we have enlarged the Hilbert space. Therefore we have to add extra constraints to reduce the enlarged Hilbert space to the original one for the spin system. The constraints are the one-particle per site constraints:

\[ f_{i\alpha}^{\dagger} f_{i\alpha} = 1 \]  

\[ f_{i\alpha} f_{i\beta} \epsilon_{\alpha\beta} = 0, \quad f_{i\alpha}^{\dagger} f_{i\beta}^{\dagger} \epsilon_{\alpha\beta} = 0 \]  

The two extra constraints in Eq. (4) are results of the first one in Eq. (3). However in path integral formalism, if we enforce all constraints simultaneously by introducing some Lagrangian multipliers, a lattice gauge theory with SU(2) gauge group can be derived. Let us introduce some notations:

\[ \hat{\eta}_{ij} = f_{i\alpha}^{\dagger} \epsilon_{\alpha\beta} f_{j\beta} \]  

\[ \hat{\chi}_{ij} = f_{i\alpha}^{\dagger} \epsilon_{\alpha\beta} f_{j\beta} \]  

These are the only singlet bilinear forms of the pairing between site \( i \) and \( j \). After some rearrangements, one has:

\[ \mathbf{S}_i \cdot \mathbf{S}_j = -\frac{1}{4} \hat{\eta}_{ij}^{\dagger} \hat{\eta}_{ij} - \frac{1}{4} \hat{\chi}_{ij}^{\dagger} \hat{\chi}_{ij} + \frac{1}{4}. \]  

Ignoring the irrelevant constant, the path integral Lagrangian of the Heisenberg model Eq. (1) turn out to be:

\[ L' = \sum_i f_{i\alpha}^{\dagger} i\partial_t f_{i\alpha} \]  

\[ -\sum_i \left( -\frac{1}{2} a_{0,i}^{\dagger} \hat{\eta}_{i\dagger} \hat{\eta}_{ii} + \frac{1}{2} a_{0,i}^{\dagger} \hat{\chi}_{ii}^{\dagger} \hat{\chi}_{ii} - \frac{1}{2} a_{0,i}^{\dagger} \right) \]  

\[ \frac{1}{4} \sum_{(i,j)} J_{ij} \left( \hat{\eta}_{ij}^{\dagger} \hat{\eta}_{ij} + \hat{\chi}_{ij}^{\dagger} \hat{\chi}_{ij} \right) \]  

The constraint of one particle per site has been encoded by the Lagrangian multipliers in the second line:

\[ \begin{pmatrix} a_{0,i}^{\dagger} \\ a_{0,i} \\ a_{0,i}^{\dagger} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} (a_{0,i}^{1} - ia_{0,i}^{2}) \\ \frac{1}{2} (a_{0,i}^{1} + ia_{0,i}^{2}) \end{pmatrix} \]  

If we do a particle-hole transformation of the spin-down fermions \( f_{i\downarrow} \) together with Hubbard-Stratonovich transformation, the Lagrangian can be written in a form with the explicit SU(2) gauge invariance:

\[ L = \sum_i \left[ \psi_i^{\dagger} (i\partial_t - a_{0,i}^{\dagger} t^\tau) \psi_i \right] \]  

\[ -\sum_{(i,j)} \frac{1}{4} J_{ij} \left[ \frac{1}{2} \text{Tr} U_{ij} U_{ij}^{\dagger} + \left( \psi_i^{\dagger} U_{ij} \psi_j + h.c. \right) \right] \]  

(7)
where

\[ \psi_i = \begin{pmatrix} f_i^\uparrow \\ f_i^\downarrow \end{pmatrix} \]  
\[ U_{ij} = \begin{pmatrix} \chi_{ij} & \eta_{ij} \\ \eta_{ij} & -\chi_{ij} \end{pmatrix} \]  

The path integral that describes the spin-1/2 system is given by

\[ Z = \int D(\psi)D(a_{0,i}^d)D(U_{ij})e^{i\int dt L} \]

The \( SU(2) \) gauge transformation is given by

\[ \psi_i \rightarrow \psi_i' = W_i \psi_i, \]
\[ a_{0,i}^d \tau^l \rightarrow a_{0,i}^d \tau^l W_i^\dagger + (i\partial_t W_i) W_i^\dagger \]
\[ U_{ij} \rightarrow W_{ij} = W_i U_{ij} W_j^\dagger \]

where all \( W_i \in SU(2) \).

There are some other useful relations. For each site, the conjugate of fundamental representation of \( SU(2) \) is equivalent to the fundamental representation itself. One therefore can introduce another \( SU(2) \) doublet (neglecting the site label):

\[ \hat{\psi} = i\sigma_2 \psi^* = \begin{pmatrix} f_i \\ -f_i^\dagger \end{pmatrix} \]  

Thus there are only three \( SU(2) \) gauge invariant bilinear forms for \( \psi \) fields on the same site

\[ S^+ = \frac{1}{2} \psi_\alpha^\dagger \hat{\psi}_\alpha = f_i^\dagger f_i \]  
\[ S^- = \frac{1}{2} \psi_\alpha^\dagger \hat{\psi}_\alpha = f_i f_i^\dagger \]  
\[ S^z = \frac{1}{2} (\psi_\alpha^\dagger \hat{\psi}_\alpha - 1) = \frac{1}{2} (1 - \psi_\alpha^\dagger \hat{\psi}_\alpha) \]
\[ = \frac{1}{2} (f_i^\dagger f_i - f_i f_i^\dagger) \]  

They turned out to be the generators of the spin rotation symmetry.

In the zeroth order approximation, or at the mean-field level, one assumes that the boson fields \( a_{0,i}^d \) and \( U_{ij} \) get condensed, or more specifically, can be replaced by some time-independent c-numbers. In this approximation, the system is described by the following mean-field Hamiltonian

\[ H_{\text{mean}} = \sum_i \psi_i^\dagger a_{0,i}^d \tau^l \psi_i \]
\[ + \sum_{(ij)} \frac{1}{4} J_{ij} \left[ \frac{1}{2} \text{Tr} U_{ij} U_{ij}^\dagger + (\psi_i^\dagger U_{ij} \psi_j + \text{h.c.}) \right] \]  

\[ \text{(14)} \]

**III. A LARGE-\( N \) LIMIT OF SU(2) PROJECTIVE CONSTRUCTION**

The above mean-field approximation is a good one only when there are some reasons to suppress the fluctuations of the boson fields \( a_{0,i}^d \) and \( U_{ij} \). One way to suppress these fluctuations is to go to a large-\( N \) limit. Actually the mean-field result is exact when \( N = \infty \). In this section, we try to answer the following questions:

- What is the lattice spin model that the large-\( N \) limit corresponds to?
- What is ground state of this lattice spin model?

Here we present our answers of these problems first. Later we will see the detailed derivation of these answers:

- The lattice spin model is a \( Sp(2N) \) spin model described by Eq. 18. The \( Sp(2N) \) spin model is a generalization of the usual \( SU(2) \) spin model (which corresponds to \( N = 1 \) case).
- The ground states of the model are \( Sp(2N) \) singlets. For small \( J_2 \) (the next-nearest-neighbor coupling), the system is in an algebraic spin liquid state.[8, 12] The excitations are the fractionalized particles (spinons) coupled to \( SU(2) \) gauge fields. The gauge field is deconfined. For larger \( J_2/J_1 \), there is a continuous quantum phase transition and the system goes into the chiral spin state which breaks the time reversal symmetry.[14]

It was debated for long time the existence of featureless Mott insulator if the unit cell does not have even number of electrons. In that case, it seems that in order to have a Mott insulator state, the ground state must break translation symmetry to enlarge the unit cell such that the number of electrons per unit cell becomes even. Here we present a counter-example. The ground state of \( Sp(2N) \) when \( N \) is large can be featureless Mott insulator with odd numbers of electrons per unit cell.

We want to introduce a large-\( N \) limit and maintain the \( SU(2) \) gauge structure. The simplest way to do this is to introduce \( N \) flavors of fermions (later in this paper we also denote \( N \) as \( N_f \) to emphasize it represents the number of flavors of fermions):

\[ \psi_i^a = \begin{pmatrix} f_i^a \\ f_i^a \end{pmatrix}, \quad a = 1, 2, \cdots, N. \]

Then the Lagrangian of the \( N \)-flavor model is:

\[ L = \sum_i \psi_i^a \left( i\partial_t - a_{0,i}^d \tau^l \right) \psi_i^a \]
\[ + \sum_{(ij)} \frac{1}{4} J_{ij} \left[ \frac{N}{2} \text{Tr} U_{ij} U_{ij}^\dagger + (\psi_i^a U_{ij} \psi_j^a + \text{h.c.}) \right] \]

where the repeated index \( a \) is summed. The large-\( N \) model is clearly invariant under the \( SU(2) \) gauge transformation. Comparing with the original model Eq. (7),
after integrating out the fermions, the effect of the \( N \)-flavor is to put a factor of \( N \) in front of the boson fields Lagrangian, which controls the fluctuation of them when \( N \) is large.

Now we want to find out the corresponding spin model for this large-\( N \) theory. It should be some generalization of the spin-1/2 Heisenberg model. To do so, we need to integrate out the boson fields. If we go back to the \( f \) picture, we have:

\[
L = \sum_i f^a_{iα} \partial_\tau f^a_{iα} - \sum_i \left[ \frac{1}{2} \tilde{a}^{aα}_i \tilde{a}^{aα}_i + \frac{1}{4} a^a_{iα} a^{aα}_i + \frac{1}{2} a^3_{iα} (\tilde{χ}^{αα}_i - N) \right] + \sum_{i,j} \frac{J_{ij}}{4N} (\tilde{χ}^{αα}_{ij} \tilde{χ}^{ββ}_{ij} + \tilde{χ}^{αα}_{ij} \tilde{χ}^{ββ}_{ij}),
\]

where

\[
\tilde{χ}^{αβ}_{ij} = f^a_{iα} \tilde{a}^{bβ}_{jβ} = ψ^a_{iα} ψ^b_{jβ} - \bar{ψ}^a_{iα} ψ^b_{jβ},
\]

\[
\tilde{χ}^{αβ}_{ij} = f^a_{iα} \delta^β_γ \delta^β_γ = f^a_{iα} \tilde{a}^{bβ}_i ψ^b_{iβ} + \bar{ψ}^a_{iα} ψ^b_{iβ}.
\]

Here one immediately see the constraints become \( χ^{αα}_{ii} = N \) and \( χ^{αα}_{ii} = 0 \), or

\[
f^a_{iα} f^a_{iα} = N \quad f^a_{iα} \delta^β_γ f^a_{jγ} = h.c. = 0
\]

for each site. The Hamiltonian under these constraints are:

\[
H = - \sum_{i,j} \frac{J_{ij}}{4N} \left( \tilde{χ}^{αα}_{ij} \tilde{χ}^{ββ}_{ij} + \tilde{χ}^{αα}_{ij} \tilde{χ}^{ββ}_{ij} \right)
\]

\[
= \sum_{i,j} \frac{J_{ij}}{4N} \left( -f^a_{iα} f^b_{jβ} f^a_{jα} f^β_{iα} \right)
\]

\[
+ f^a_{iα} f^b_{jβ} f^a_{jβ} f^β_{iα} + N \right) \]

(17)

We want to understand the symmetry of this model and try to rewrite it in a form of spin coupling. Here, the symmetry that we are studying is not the local gauge invariance, but some global physical symmetry in analogy with the spin rotation symmetry for the \( N = 1 \) model.

Motivated by Eq. (11-13), we construct all the gauge invariant bilinear forms of \( ψ \) for each site (neglecting the site label). Let

\[
\tilde{ψ}^a = iσ_2 \tilde{ψ}^a = \left( f^a_+ - f^a_- \right)
\]

The \( SU(2) \) gauge invariant bilinears are

\[
S^{ab+} = \frac{1}{2} \tilde{ψ}^a_+ \tilde{ψ}^b_+ = \frac{1}{2} \left( f^a_+ f^b_+ + f^b_+ f^a_+ \right)
\]

\[
S^{ab-} = \frac{1}{2} \tilde{ψ}^a_+ \tilde{ψ}^b_- = \frac{1}{2} \left( f^a_+ f^b_- + f^b_- f^a_+ \right)
\]

\[
S^{ab3} = \frac{1}{2} \left( \tilde{ψ}^a_+ \tilde{ψ}^b_- - δ^{ab} \right) = \frac{1}{2} \left( δ^{ab} - \tilde{ψ}^a_+ \tilde{ψ}^b_- \right)
\]

\[
= \frac{1}{2} \left( f^a_+ f^b_- - f^a_- f^b_+ \right)
\]

What is the group generated by these \( S \) operators? First let us count how many of them there are. For \( S^{ab+} \) or \( S^{ab-} \), the label is symmetric for \( a \) and \( b \), so there are \( N(N+1) \) operators of each type. For \( S^{ab3} \), the labels are not symmetric, so there are simply \( N^2 \) of them. Totally we have \( N(N+1) + N^2 = 2N^2 + N \) of them. One can further examine their commutation relations:

\[
[S^{ab-}, S^{cd+}] = 0, \quad [S^{ab+}, S^{cd-}] = 0
\]

\[
[S^{ab3}, S^{cd3}] = \frac{1}{2} \left( δ^{bc} S^{bd3} - δ^{ad} S^{bd3} \right)
\]

\[
[S^{ab3}, S^{cd+}] = \frac{1}{2} \left( δ^{bc} S^{ad+} + δ^{ad} S^{bc+} \right)
\]

\[
[S^{ab3}, S^{cd-}] = -\frac{1}{2} \left( δ^{ad} S^{bc-} + δ^{bc} S^{ad-} \right)
\]

\[
[S^{ab+}, S^{cd-}] = \frac{1}{2} \left( δ^{ac} S^{bd3} + δ^{bd} S^{ac3} \right)
\]

These are the relations for \( SP(2N) \) algebra, so all the \( S \) operators are the \( 2N^2 + N \) generators which generate an \( SP(2N) \) group. When \( N = 1 \), \( SP(2) \) is isomorphic to \( SO(3) \). After some rearrangements, the Hamiltonian Eq. (17) can be rewritten as:

\[
H = \sum_{i,j} \frac{J_{ij}}{4N} \left[ \frac{1}{2} S^{ab+}_i S^{ab-}_j + \frac{1}{2} S^{ab3}_i S^{ab3}_j \right]
\]

(18)

If we define:

\[
S^{ab1}_i = S^{ab1}_i = \frac{1}{2} \left( S^{ab+}_i + S^{ab-}_i \right)
\]

\[
S^{ab2}_i = S^{ab2}_i = \frac{1}{2i} \left( S^{ab+}_i - S^{ab-}_i \right),
\]

then the vector

\[
S^{ab}_i = \left( S^{ab1}_i, S^{ab2}_i, S^{ab3}_i \right)
\]

can simplify the Hamiltonian to:

\[
H_{SP(2N)} = \sum_{i,j} \frac{J_{ij}}{N} S^{ab}_i \cdot S^{ba}_j
\]

(19)

Here we have to mention that the three components of \( S^{ab}_i \) are actually not on the same footing, since the first two are symmetric with respect to the flavor labels but the third one is not. It is a simple task to check that all the generators commute with the Hamiltonian. And one can even check that the \( SP(2N) \) is indeed the full symmetry group of it.

Now we want to know what the physical Hilbert space is. It should be the subspace of the enlarged Hilbert space in which the constraints Eq. (15) are satisfied. When \( N = 1 \), the physical Hilbert space is the spin-1/2 Heisenberg’s, where we have only two states on each site:

\[
| ↑ \rangle, \quad | ↓ \rangle
\]
When $N = 2$, we have 5 states on each site:

$$|1\uparrow 2\uparrow\rangle, \ |1\uparrow 2\downarrow\rangle, \ |1\downarrow 2\uparrow\rangle, \ |1\downarrow 2\downarrow\rangle,$$

$$\frac{1}{\sqrt{2}} (|1\uparrow 2\downarrow 0\rangle - |1\downarrow 2\uparrow 0\rangle)$$

Here, for example, $|1\uparrow 2\uparrow\rangle$ represents a fermion of flavor 1 and spin up and another fermion of flavor 2 and spin down; and $|1\uparrow 2\downarrow\rangle$ represents a fermion of flavor 1 and spin up, another fermion of flavor 1 and spin down, and no fermion of flavor 2. One can check the dimension of the Hilbert space on each site for larger $N$'s:

- $N=3$ dimension = 14,
- $N=4$ dimension = 43,
- $N=5$ dimension = 142,
- $N=6$ dimension = 429, …

This Hilbert space turns out to be an irreducible representation of the $SP(2N)$ symmetry group. If we label an irreducible representation by its highest weight state for a particular Cartan basis. The Cartan basis for $SP(2N)$ can be chosen to be the $z$-component spins of each flavor:

$$S^{a3}, \text{ where } a = 1, 2, \cdots, N,$$

with $N$ generators. Then the highest weight state in our Hilbert space is simply:

$$|1\uparrow 2\uparrow; 3\uparrow \cdots N\uparrow\rangle$$

IV. PHASE DIAGRAM OF THE $Sp(2N)$ MODEL ON 2D SQUARE LATTICE

In this section we would like to calculate the zero temperature phase diagram for the $Sp(2N)$ model Eq. (19) on 2D square lattice. We assume that only the nearest-neighbor couplings and the next-nearest-neighbor couplings are non-zero

$$J_{i,i+x} = J_{i,i+y} = J_1,$$

$$J_{i,i+x+y} = J_{i,i+x-y} = J_2.$$

We also assume $J_1 + J_2 = 1$.

There are two mean-field approaches to the $Sp(2N)$ model. In the first mean-field approach, we use the ground state $|\Phi^{(m^b)}\rangle$ of a trial Hamiltonian

$$H_{trial} = \sum_{ij} \frac{J_{ij}}{N} m_{ij}^{ab} \cdot S_{ij}^{ba}$$

as the trivial wave function and obtain the mean-field ground state of the $Sp(2N)$ model Eq. (19) by minimizing

$$\langle \Phi^{(m^b)}| H_{Sp(2N)} | \Phi^{(m^b)} \rangle$$

as we vary the variational parameters $m_{ij}^{ab}$. The obtained ground state corresponds to a $Sp(2N)$ spin polarized state. One can show that the ground state energy obtained in this mean-field approach is always of order 1 per site in the large $N$ limit.

In the second mean-field approach (the projective construction)[1, 25] we start with a fermion trial Hamiltonian

$$H_{trial} = \sum_{ij} \psi^a_i \psi^a_j$$

where $u_{ij} = u_{ij}^\dagger \tau^\mu$ are two by two complex matrices that satisfy

$$u_{ij}^1 = u_{ji}^\dagger, \quad u_{ij}^0 = \text{imaginary}, \quad u_{ij}^l = |l=1,2,3\rangle\text{real}$$

Let $|\Phi^{(\text{mean})}\rangle$ be the ground state of the above trial Hamiltonian. $u_{ij}^{\alpha\beta} = a_{\alpha\beta}^{ij}$ are chosen such that the constraints

$$\psi_i^{\alpha\dagger} \tau^\mu \psi_j^\beta = 0$$

are satisfied on average:

$$\langle \Phi^{(\text{mean})}| \psi_i^{\alpha\dagger} \tau^\mu \psi_j^\beta | \Phi^{(\text{mean})} \rangle = 0$$

We then project $|\Phi^{(\text{mean})}\rangle$ to the physical Hilbert space and obtain $P|\Phi^{(\text{mean})}\rangle$. The physical Hilbert space is formed by states $|\text{phys}\rangle$ that satisfy the constraints Eq. (15) or

$$\psi_i^{\alpha\dagger} \tau^\mu \psi_j^\beta |\text{phys}\rangle = 0$$

The projected wave function $P|\Phi^{(\text{mean})}\rangle$ is our trial wave function with $u_{ij}$ as variational parameters. Minimizing

$$\langle \Phi^{(\text{mean})}| P H_{Sp(2N)} P |\Phi^{(\text{mean})}\rangle$$

by varying $u_{ij}$, we obtain the approximated ground state.

Since in the large-$N$ limit, the fluctuations of the Lagrangian multiplier $a_{\alpha\beta}^{ij}(t)$ are weak, we expect that removing the projection $P$ only causes an error of order $1/N$. Also, other mean-field fluctuations are weak in the large-$N$ limit, so we expect that the minimized mean-field energy

$$E = \langle \Phi^{(\text{mean})}| H_{Sp(2N)} | \Phi^{(\text{mean})} \rangle$$

$$= - \sum_{ij} \frac{J_{ij}}{4N} (\eta_{ij}^{a\alpha} \eta_{ij}^{b\beta} + \chi_{ij}^{a\alpha} \chi_{ij}^{b\beta}) + O(1)$$

to be the true ground state energy in the leading order in the large-$N$ expansion. Here

$$\chi_{ij}^{a\beta} = \langle \Phi^{(\text{mean})}| \chi_{ij}^{a\beta} | \Phi^{(\text{mean})} \rangle, \quad \eta_{ij}^{a\beta} = \langle \Phi^{(\text{mean})}| \eta_{ij}^{a\beta} | \Phi^{(\text{mean})} \rangle$$

note that the above minimized energy is of order $-N$ per site. Thus the states obtained in the first mean-field approximation Eq. 20 cannot be the ground state.

Within the $SU(2)$ projective construction and using the translation invariant ansatz $u_{ij}$ with only nearest- and next-nearest-neighbor couplings, we find many local minima of $- \sum_{ij} (\eta_{ij}^{a\alpha} \eta_{ij}^{b\beta} + \chi_{ij}^{a\alpha} \chi_{ij}^{b\beta})$ as we vary $u_{ij}$. We plot those minima in Fig. 1 as functions of $J_2$ (note $J_1 + J_2 = 1$).
As we change $J_2$, the energy of the state changes smoothly along each curve. So there is no quantum phase transition as we move from one point of a curve to another point on the same curve. The ansatzs on the same curve belong to the same phase. However, if two curves cross each other, the crossing point represents a quantum phase transition. This is because the ground state energy is not analytic at the crossing point. If the slopes of the curves at the crossing point are different, the quantum phase transition is first order. If the slopes at the crossing point are the same, the quantum phase transition is second order.

From Fig. 1, we see second-order (or continuous) phase transitions (at mean-field level) between the following pairs of phases: (A,D), (A,G), (B,G), (C,E), and (B,H). We used to believe that all the second-order phase transitions are caused by symmetry breaking. So a natural question is what symmetries are broken for the above five second-order phase transitions?

It turns out that, except phase (D) and phase (E), all other phases have the same symmetry. In other words, the projected ground state wave functions $P(\Phi^{\text{mean}}_{u_{ij}})$ for the ansatz $u_{ij}$ associated with those phases have identical symmetry. Thus the three continuous transitions (B,G), (B,H) and (A,G) do not change any symmetries. It was pointed out that those phases, despite having the same symmetry, contain different quantum orders [12]. The projective symmetry group (PSG), defined as the invariant group of the ansatz $u_{ij}$, is introduced to describe this new class of orders [12].

The ansatzs on the same curve have the same PSG and correspond to the same quantum phase. On the other hand, the ansatzs on the different curves have different PSG’s. We see that a quantum phase transition is characterized by a change in PSG. Those quantum phase transitions represent a new class of phase transitions beyond the Landau’s symmetry breaking theory. Other examples of phase transitions beyond the Landau’s theory can be found in Ref. [19–23, 26].

In the following we will discuss quantum orders (or PSG’s) for mean-field phases in Fig. 1. The PSG’s for those quantum orders are labeled by labels which look like Z2A0013 [12].

The phase (A) [27] is the $\pi$-flux state, or the SU2Bn0 state

$$u_{i,i+x} = i\chi, \quad u_{i,i+y} = -i^z\chi, \quad a^0_0 = 0.$$  \hspace{1cm} (22)

The low energy excitations are described by massless Dirac fermions with a linear dispersion and gapless SU(2) gauge fluctuations. Therefore we also call such a state SU(2)-linear state.

The phase (B) [12] is a state with two independent uniform RVB states [1] on the diagonal links. The gapless fermions have finite Fermi surfaces. The fermions interact with SU(2) × SU(2) gauge fluctuations. Such a state is called SU(2) × SU(2)-gapless state (SU(2) × SU(2) indicates the low energy gauge group and “gapless” indicates finite Fermi surface). Its ansatz is given by

$$u_{i,i+x+y} = \chi\tau^3, \quad u_{i,i+x-y} = \chi\tau^3, \quad a^0_0 = 0.$$  \hspace{1cm} (23)

The phase (C) [12] is a state with two independent $\pi$-flux states on the diagonal links. It has SU(2) × SU(2) gauge fluctuations at low energies and will be called an SU(2) × SU(2)-linear state. Its ansatz is given by

$$u_{i,i+x+y} = \chi(\tau^3 + \tau^1), \quad u_{i,i+x-y} = \chi(\tau^3 - \tau^1), \quad a^0_0 = 0.$$  \hspace{1cm} (24)

The low energy excitations are SU(2) × SU(2) gauge fluctuations and massless Dirac fermions.

The phase (D) is the chiral spin state [14]

$$\chi_{i,i+x} = i\chi_1, \quad \chi_{i,i+y} = i\chi_1(-)^{i_x}, \quad a^0_0 = 0, \quad \chi_{i,i+x+y} = -i\chi_2(-)^{i_x}, \quad \chi_{i,i+x-y} = i\chi_2(-)^{i_x}.$$  \hspace{1cm} (25)

Both fermionic excitations and SU(2) gauge excitations are gapped. The gap of the SU(2) gauge excitations is due to an SU(2) Chern-Simons term.

The phase (E) [12] is described by an ansatz

$$u_{i,i+x+y} = \chi_1\tau^1 + \chi_2\tau^2, \quad u_{i,i+x-y} = \chi_1\tau^1 - \chi_2\tau^2, \quad a^0_0 = 0.$$  \hspace{1cm} (26)

which breaks the 90° rotation symmetry. It is a U(1)-linear state, i.e., the low lying excitations are massless U(1) gauge fluctuations interacting with massless Dirac fermions.
The phase (F) [12] is described by the following U1Cn00x ansatz

\[ u_{i,i+x} = \eta r^1 \quad u_{i,i+y} = \eta r^1 \]
\[ u_{i,i+x+y} = \chi r^3 \quad u_{i,i+x-y} = \chi r^3 \]
\[ a_0^3 = \lambda, \quad a_1^{1,2} = 0 \]  \hspace{1cm} (27)

The U1Cn00x state can be a U(1)-linear state where fermions are gapless with a linear dispersion relation (if \( a_0^3 \) is small) or a U(1)-gapped state where the fermions are gapped (if \( a_0^3 \) is large). The state for phase (F) turns out to be a U(1)-gapped state. The only low energy excitations are massless U(1) gauge bosons.

The phase (G) [12] is described by the Z2A0013 ansatz

\[ u_{i,i+x} = \chi r^1 - \eta r^2, \quad u_{i,i+y} = \chi r^1 + \eta r^2, \]
\[ u_{i,i+x+y} = -\gamma r^1, \quad u_{i,i-x+y} = +\gamma r^1, \]
\[ u_{i,i+2x} = u_{i,i+2y} = 0, \quad a_0^{1,2,3} = 0. \]  \hspace{1cm} (28)

The SU(2) gauge structure is broken down to a Z2 gauge structure. Hence there is no gapless gauge fluctuations. The only low energy excitations are massless Dirac fermions. Such a state is called a Z2-linear state.

The phase (H) [18] is described by the Z2A0013 ansatz

\[ a_0^3 \neq 0, \quad a_0^{0,1,2} = 0, \]
\[ u_{i,i+x} = \chi r^1 + \eta r^2, \quad u_{i,i+y} = \chi r^1 - \eta r^2, \]
\[ u_{i,i+x+y} = +\gamma r^3, \quad u_{i,i-x+y} = +\gamma r^3. \]  \hspace{1cm} (29)

It is also a Z2-linear state.

The phase (I) is the uniform RVB state [1], i.e., the SU(2)-gapless state SU2An0

\[ u_{i,i+x} = \chi, \quad u_{i,i+y} = \chi, \quad a_0^0 = 0. \]  \hspace{1cm} (30)

It has gapless SU(2) gauge fluctuations and gapless fermionic excitations that form a finite Fermi surface.

From Fig. 1, we see continuous phase transitions (at mean-field level) between the following pairs of phases: (A,D), (A,G), (B,G), (C,E), and (B,H). For the three continuous transitions (B,G), (B,H) and (A,G) that do not change any symmetries, we observe that the SU(2) gauge structure in the phase (A) breaks down to Z2 in the continuous transition from the phase (A) to the phase (G). The SU(2) \times SU(2) gauge structure in the phase (B) breaks down to Z2 in the two transitions (B,G) and (B,H).

V. A STABLE ALGEBRAIC SPIN LIQUID – SU(2)-LINEAR SPIN LIQUID

In this section, we will study the SU(2)-linear state given by Eq. (22) which describes the phase A in phase diagram Fig. 1. In the mean-field theory, the interactions between the excitations are ignored. In this section, we include those interactions and study how those interactions affect the low energy properties of the mean-field

 SU(2)-linear state. We will show that, after including those interactions, the gapless excitations in the mean-field SU(2)-linear state remain gapless, which leads to a stable algebraic spin liquid.

A. The low energy effective theory of the SU(2)-linear state

To obtain the low energy effective theory of the SU(2)-linear state in the continuum limit, we choose the unit cell as in Fig. 2. It contains 4 sites; each site has spin up and down, so totally 8 fermions. Let us write down the mean field Hamiltonian:

\[ H_{\text{mean}} = \sum_i \tau_i \left[ \psi_i \psi_i^\dagger + \psi_i^\dagger \psi_i + \psi_i^\dagger \psi_i + \psi_i^\dagger \psi_i + \psi_i^\dagger \psi_i + \psi_i^\dagger \psi_i + \psi_i^\dagger \psi_i + \psi_i^\dagger \psi_i \right] + h.c. \]
\[ = \sum_k \tau_k \left[ \psi_k \psi_k^\dagger + \psi_k^\dagger \psi_k + \psi_k^\dagger \psi_k + \psi_k^\dagger \psi_k + \psi_k^\dagger \psi_k + \psi_k^\dagger \psi_k + \psi_k^\dagger \psi_k + \psi_k^\dagger \psi_k \right] + h.c. \]
\[ = \sum_k \left( \psi_k^\dagger \psi_k, \psi_k^\dagger \psi_k, \psi_k^\dagger \psi_k, \psi_k^\dagger \psi_k \right) M \left( \psi_k \psi_k^\dagger \psi_k \psi_k^\dagger \right) \]  \hspace{1cm} (31)

where

\[ M = \chi \left( \begin{array}{cccc} 0 & i - i e^{-i k_x} & i & -i e^{-i k_y} \\ -i + i e^{i k_x} & 0 & 0 & -i + i e^{-i k_y} \\ -i + i e^{i k_y} & 0 & 0 & -i + i e^{-i k_x} \\ 0 & i - i e^{i k_y} & -i + i e^{i k_x} & 0 \end{array} \right) \]  \hspace{1cm} (32)

Here we have assumed the lattice constant to be 1/2, so we have \(-\pi < k_x, k_y < \pi\). Note that here \( \psi_k, i \) is actually SU(2)-doublet, corresponding to the spin up and down components in the f-formalism; so still totally 8 fermions.
After some rearrangements:

\[
M = 2\chi\begin{pmatrix}
0 & -e^{-ik_x} \sin k_x & 0 & -e^{-ik_y} \sin k_y \\
-e^{ik_x} \sin k_x & 0 & 0 & e^{-ik_y} \sin k_y \\
-e^{ik_y} \sin k_y & 0 & 0 & -e^{-ik_x} \sin k_x \\
0 & e^{ik_x} \sin k_x & -e^{ik_y} \sin k_y & 0
\end{pmatrix}
\]  

(33)

In the continuous limit, the energy spectrum for fermion is characterized by a single fermi point at \((0, 0)\). When \(k \approx 0\):

\[
M = 2\chi\begin{pmatrix}
0 & -k_x & -k_y & 0 \\
-k_x & 0 & 0 & k_y \\
-k_y & 0 & 0 & -k_x \\
0 & k_y & -k_x & 0
\end{pmatrix}
\]  

(34)

We can do an extra rotation to make it the usual form of Dirac fermion:

\[
\psi \rightarrow \bar{\psi} = R^\dagger \psi
\]  

(35)

\[
R = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\]  

(36)

Then

\[
H_{\text{mean}} = \bar{\psi}^\dagger \gamma_0 [ik_x \gamma_1 + ik_y \gamma_2] \psi
\]  

(37)

where the \(\gamma\) matrices in Euclidean space are:

\[
\gamma_0 = \begin{pmatrix} \sigma_3 & -\sigma_3 \\ -\sigma_3 & \sigma_3 \end{pmatrix}
\]

\[
\gamma_1 = \begin{pmatrix} \sigma_1 & -\sigma_1 \\ -\sigma_1 & \sigma_1 \end{pmatrix}
\]

\[
\gamma_2 = \begin{pmatrix} \sigma_2 & -\sigma_2 \\ -\sigma_2 & \sigma_2 \end{pmatrix}
\]  

(38)

Here we can also introduce the other two \(\gamma\) matrices:

\[
\gamma_3 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\]

\[
\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = i \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}
\]  

(39)

where \(I\) is the 2 by 2 identity matrix. Notice that both \(\gamma_3\) and \(\gamma_5\) anticommute with all space-time components of \(\gamma\) matrices: \(\gamma_0, \gamma_1\) and \(\gamma_2\). We should also include gauge field fluctuations above the mean-field theory. The full lagrangian is:

\[
L = \bar{\psi} \left( \partial_\mu - i a_\mu^{ij=t} \gamma_5 \right) \gamma_\mu \psi + \frac{1}{2g^2} \text{Tr} \left[ f_{\mu\nu}^l f_{\mu\nu}^l \right] + \cdots
\]  

(40)

where \(\bar{\psi} = \psi^\dagger \gamma_0\). So the low energy effective theory of the \(SU(2)\)-linear state is a QCD3 with a \(SU(2)\) gauge field and two massless 4-component Dirac fermions (or \(2N_f\) 4-component Dirac fermions which form \(N_f\) \(SU(2)\) gauge doublets in the large \(N_f\) limit, notice \(N_f = 1\) is the physical Heisenberg model’s case).

The \(\cdots\) in Eq. (40) represents other terms which may be generated by the interactions as we integrate out high energy fluctuations. Understanding those terms is the key to understand the low energy behavior of the model. Those terms must be consistent with the underlying lattice symmetry. So in the following, we will study the symmetry properties of the effective theory Eq. (40). We will show that none of terms allowed by the symmetry are relevant at low energies. None of those terms can cause infrared instability. As a result the mean-field \(SU(2)\)-linear state leads to a stable algebraic spin liquid.

### B. Space translation and rotation symmetry

Now let us think about the corresponding lattice symmetry in continuous limit. Firstly let us discuss translation by one lattice site along \(x\)-direction \(T_x\), in terms of the lattice fields:

\[
T_x : \begin{cases}
\psi_{i+1} \rightarrow \psi_i = \psi_i - x \\
u_{ij} = u_{ij} - x.\end{cases}
\]  

(41)

It seems that the translation symmetry is broken since \(u_{ij}\) is not invariant:

\[
\{ u_{i,i+x} = i\chi, \\
u_{i,i+y} = i(-)^i \chi \} \rightarrow \{ \psi_i, W_i \psi_i \}
\]  

(42)

But as shown in Fig.3 one can do an extra local \(SU(2)\) gauge transformation \(W_T\) to transform \(u_{ij}\) back. Let

\[
W_i = (-)^i\nu
\]  

(43)

Then

\[
W_T : \begin{cases}
\psi_i \rightarrow \psi_i' = W_i \psi_i = (-)^i\psi_i - x \\
u_{ij} \rightarrow \nu_{ij}' = W_i \nu_{ij} W_j^\dagger
\end{cases}
\]  

(44)

where

\[
\{ u_{i,i+x}' = u_{i,i+x} = i\chi = u_{i,i+x}, \\
u_{i,i+y}' = -u_{i,i+y} = i(-)^i \chi = u_{i,i+y}
\]  

(45)

Here we point out that the combination of \(W_T\) and \(T_x\) is a transformation leaving \(U_{ij}\) invariant. We call such a transformation an element of PSG. PSG, by definition, is the collection of all transformations leaving the ansatz \(U_{ij}, a_{ij}^{\delta}\) invariant. Here in \(SU(2)\)-linear state, \(a_{ij}^{\delta}\) is zero, so it is also invariant.

After choosing the unit cell as in Fig.2, in terms of the four component \(\psi\) fermion, the combination of \(W_T\) and \(T_x\) transforms:

\[
\begin{pmatrix}
\psi_0 \\
\psi_{i1} \\
\psi_{i2} \\
\psi_{i3}
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\psi_0 \\
\psi_{i1} \\
\psi_{i2} \\
\psi_{i3}
\end{pmatrix}
\]  

(46)

Here we assumed in the continuous limit, \(\psi_i\) and \(\psi_{i+x}\) are on the same position. We can do the extra rotation
FIG. 3: Figures (a)–(d) illustrate how the $SU(2)$-linear state respects translation symmetry $T_x$: (a) original ansatz (b) after $T_x$ (c) do $W_{T_x}$ (d) go back to original ansatz.

to transform into $\tilde{\psi}$, the usual Dirac fermion:

$$\tilde{\psi} \rightarrow \tilde{\psi}' = R^{\dagger} \psi' = R^{\dagger} \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array} \right) R \tilde{\psi}$$

$$= \left( \begin{array}{cccc} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \tilde{\psi} = -\gamma_3 \tilde{\psi}$$ (47)

Eventually we know that the PSG element $W_{T_x} \circ T_x$, which is the lattice symmetry, if translated into continuous limit, is the internal symmetry $-\gamma_3$. Note that the minus sign actually corresponds to a global $SU(2)$ transformation: $W_i = -1$. So we have the correspondence:

$$W_{T_x} \circ T_x \leftrightarrow \gamma_3$$ (48)

Similarly one can study the translation by one lattice site along $y$-direction: we find:

$$W_{T_y} \circ T_y \leftrightarrow \gamma_5$$ (49)

For the reflection $P_x : x \rightarrow -x$:

$$W_{P_x} \circ P_x \leftrightarrow \gamma_1$$ (50)

For the reflection $P_y : y \rightarrow -y$:

$$W_{P_y} \circ P_y \leftrightarrow \gamma_2$$ (51)

For the reflection $P_{xy} : x \rightarrow y, y \rightarrow x$:

$$W_{P_{xy}} \circ P_{xy} \leftrightarrow \frac{1}{2}(\gamma_1 - \gamma_2)(\gamma_3 + \gamma_5)$$ (52)

Note that $T_x, T_y, P_x, P_y, P_{xy}$ already give the full space-time symmetry. For example, rotation by 90 degree $R_{90} : x \rightarrow y, y \rightarrow -x$ is a combination of $P_x$ and $P_{xy}$:

$$R_{90} = P_{xy} \circ P_x$$ (53)

C. Time-reversal symmetry

Now let us study the time-reversal symmetry $T$. In terms of spin operator:

$$T : S \rightarrow -S$$ (54)

One should be cautious that $T$ is not a usual linear operator, instead it is an anti-linear operator:

$$Tf = -iT$$ (55)

What does this operator correspond to in terms of lattice fermion and bond variable $U_{ij}$? We know that

$$S = \frac{1}{2} f^\dagger \sigma f$$ (56)

So the corresponding transformation on $f$ fermion is:

$$T : f \rightarrow i\sigma_2 f, \quad T : f^\dagger \rightarrow f^\dagger (i\sigma_2)^\dagger$$ (57)

$$TST^{-1} = Tf^\dagger T^{-1} \sigma T^{-1} fT^{-1} = f^\dagger (i\sigma_2)^\dagger (\sigma_1, -\sigma_2, \sigma_3) (i\sigma_2) f$$

$$= f^\dagger (-\sigma_1, -\sigma_2, -\sigma_3) f$$

$$= -S$$ (58)

Here we used the anti-linear property of $T$ operator. Therefore the $T$ transformation on $\psi$ fermion is:

$$T : \psi = \left( \begin{array}{c} f_1 \\ f_2^\dagger \end{array} \right) \rightarrow \left( \begin{array}{c} f_2^\dagger \\ -f_1 \end{array} \right) = i\tau_2 \psi^*$$ (59)

Our convention of notation is that in terms of $f$ fermion, we use $\sigma$ to denote Pauli matrices; while in terms of $\psi$ fermion, we use $\tau$ to denote them.

What is the time-reversal transformation on $U_{ij}$? Here we notice that $U_{ij}$ has two meanings: in Eq.(7) it means the operator defined as in Eq.(6); while in Eq.(14) it means the average value of the operator on mean-field ground state. Let us take a look at how the operator transform under time-reversal:

$$T\tilde{\chi}_{ij}T^{-1} = T f^\dagger_{i\alpha} f_{j\beta} T^{-1} = T f^\dagger_{i\alpha} T^{-1} f_{j\alpha} T^{-1}$$

$$= f^\dagger_{i\alpha} (i\sigma_2)^\dagger (i\sigma_2) f_{j\beta} = f^\dagger_{i\alpha} f_{j\beta} = \tilde{\chi}_{ij}$$ (60)

$$T\tilde{\gamma}_{ij}T^{-1} = T f^\dagger_{i\alpha} \epsilon_{\alpha\beta} f^\dagger_{j\beta} T^{-1} = T f^\dagger_{i\alpha} T^{-1} \epsilon_{\alpha\beta} T f^\dagger_{j\beta} T^{-1}$$

$$= f^\dagger_{i\alpha} (i\sigma_2)_{\gamma\alpha\beta} (i\sigma_2)_{\delta\beta\gamma} f_{j\delta} = f^\dagger_{i\gamma} \epsilon_{\gamma\delta} f_{j\delta} = \tilde{\gamma}_{ij}$$ (61)
Therefore we know that the $\hat{U}_{ij}$, as an operator, is invariant under $T$. Here it is helpful to write down $\hat{U}_{ij}$ in terms of $\psi$ operators:

$$\hat{U}_{ij} = \left( \hat{\chi}_{ij} \hat{\eta}_{ij} \right) = -\psi_j^* \psi_i^\dagger - (i\tau_2 \psi_j^*) (i\tau_2 \psi_i^*)^\dagger$$ \hspace{0.5cm} (62)

Notice Eq.(59), under $T$ transformation, the first term transforms into the second term, and the second transforms into the first. So the whole $\hat{U}_{ij}$ is invariant. One can also check that together with transformation Eq.(59), the Lagrangian Eq.(7) is invariant under $T$. This is expected since the original Heisenberg Hamiltonian Eq.(1) is $T$ invariant.

But things are different if $\hat{U}_{ij}$ condense, more specifically, if it has some non-zero average value: $U_{ij} = \langle \Psi | \hat{U}_{ij} | \Psi \rangle$. Because $T$ is an anti-linear operator, it transforms $U_{ij}$ into $U_{ij}^*$. We know that for two arbitrary states $|\alpha\rangle$, $|\beta\rangle$, the anti-linear property of $T$ gives

$$\langle \alpha | \beta \rangle = \langle T\beta | T\alpha \rangle$$ \hspace{0.5cm} (63)

Therefore for an arbitrary linear operator $\hat{O}$,

$$\langle \alpha | \hat{O} | \beta \rangle = \langle \hat{O}^\dagger T\beta | T\alpha \rangle = \langle T\beta | T\hat{O}^\dagger T^{-1} | T\alpha \rangle$$ \hspace{0.5cm} (64)

let $\hat{O} = \hat{\chi}_{ij}^\dagger$, and $|\alpha\rangle = |\beta\rangle = |\Psi\rangle$, one immediately see that under $T$ transformation, the average value of $\hat{\chi}_{ij}$ operator transforms as:

$$\chi_{ij} = \langle \Psi | \hat{\chi}_{ij} | \Psi \rangle \rightarrow \chi_{ij}^* = \langle T\Psi | \hat{\chi}_{ij} | T\Psi \rangle = \langle \chi_{ij}^\dagger | \Psi \rangle = \chi_{ij}^\dagger$$ \hspace{0.5cm} (65)

Here we know that the mean-field variable $U_{ij}$, as the average value of $\hat{U}_{ij}$ operator, transforms into $U_{ij}^*$ under $T$. This is consistent with our understanding of anti-linear $T$ operator, since it transforms any $c$-number into its complex conjugate.

$$T : U_{ij} \rightarrow T U_{ij} T^{-1} = U_{ij}^*$$ \hspace{0.5cm} (66)

In general, $U_{ij}$ can be written as:

$$U_{ij} = u_{ij}^0 \tau_0 + u_{ij}^1 \tau_1 + u_{ij}^2 \tau_2 + u_{ij}^3 \tau_3$$ \hspace{0.5cm} (67)

where $\tau_0$ is identity matrix. For spin rotation invariant system, one can show that $u_{ij}^0$ is pure imaginary, while $u_{ij}^i$ with $i = 1, 2, 3$ are pure real.

How does mean-field Hamiltonian transform under $T$? Here we need combine transformations on $\psi$ and $U_{ij}$:

$$\psi_i^\dagger U_{ij} \psi_j \rightarrow \psi_i^\dagger (i\tau_2) U_{ij}^* (i\tau_2) \psi_j^*$$

(using Eq.(67))

$$= \psi_i^\dagger \psi_j^*$$

$$= \psi_i^\dagger U_{ij}^\dagger \psi_i$$

$$\langle U_{ij} = U_{ji}^* \rangle = \psi_j^\dagger U_{ij} \psi_i$$ \hspace{0.5cm} (68)

Comparing with the term in original mean-field Hamiltonian $\psi_i^\dagger U_{ij} \psi_j$, one concludes that the transformation of mean-field hamiltonian under $T$ can be simply expressed as $U_{ij} \rightarrow U_{ij}^*$, with no fermion transformation. If $U_{ij}^*$ and original $U_{ij}$ can be related by a $SU(2)$ gauge transformation, the system has time-reversal symmetry; otherwise the $T$ was broken.

For our $SU(2)$-linear state, $U_{ij}^*$ and $U_{ij}$ are indeed related by a gauge transformation $W_T$.

$$W_T : W_t = -(-)^{t_s + t_y}$$ \hspace{0.5cm} (69)

After choosing unit cell, in terms of four-component fermion $\psi$,

$$W_T : \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$ \hspace{0.5cm} (70)

In terms of usual Dirac fermion $\tilde{\psi}$:

$$W_T : \tilde{\psi} \rightarrow \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \tilde{\psi} = \gamma_0 \tilde{\psi}$$ \hspace{0.5cm} (71)

The combined transformation $W_T \circ T$ leaves $U_{ij}$ invariant. So $W_T \circ T$ is another element if PSG. And we know in terms of Dirac fermion $\psi$,

$$W_T \circ T : \tilde{\psi} \rightarrow \tilde{\psi}' = (i\tau_2) \gamma_0 \tilde{\psi}$$ \hspace{0.5cm} (72)

This is very similar to the time-reversal transformation in usual Dirac field theory.

### D. Spin rotation: “charge conjugation”

There is another important symmetry, the global spin rotation. Let us think about rotation around $y$-axis by 180 degree:

$$R_{spin} : (S_x, S_y, S_z) \rightarrow (-S_x, S_y, -S_z)$$ \hspace{0.5cm} (73)

In terms of $f$ spinon,

$$R_{spin} : f \rightarrow f' = (i\sigma_2) f = \begin{pmatrix} f_\uparrow \\ -f_\uparrow \end{pmatrix}$$ \hspace{0.5cm} (74)

In terms of $\psi$ fermion,

$$R_{spin} : \psi' = \begin{pmatrix} f_\uparrow \\ f_\uparrow \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} f_\uparrow \\ -f_\uparrow \end{pmatrix} = (i\tau_2) \psi^*$$ \hspace{0.5cm} (75)

It seems Eq.(75) is identical to Eq.(59), which means $T$ and $R_{spin}$ are identical. This is obviously wrong, the difference here lies in the fact that $T$ is anti-linear but $R_{spin}$ is linear.
In terms of 4-component Dirac fermion $\tilde{\psi}$:

$$R_{\text{spin}}: \tilde{\psi} = R^\dagger \tilde{\psi} \rightarrow R^\dagger (i\pi_2) \tilde{\psi}^* = (i\pi_2) R^\dagger \tilde{\psi}^*$$

$$= (i\pi_2)(-\gamma_1\gamma_5) \tilde{\psi}^*$$  \hspace{1cm} (76)

The transformation rule above is in real space. In momentum space,

$$R_{\text{spin}}: \tilde{\psi}_k = \frac{1}{\sqrt{N}} \sum_j e^{-ik_j} \tilde{\psi}_j$$

$$\rightarrow (-i\pi_2)(\gamma_1\gamma_5) \frac{1}{\sqrt{N}} \sum_j e^{-ik_j} \tilde{\psi}^*_j$$

$$= (-i\pi_2)(\gamma_1\gamma_5) \tilde{\psi}^*_{-k}$$  \hspace{1cm} (77)

i.e., the spin rotation transformation $R_{\text{spin}}$ also flip the sign of momentum!

Now think about how $\hat{U}_{ij}$ transforms under $R_{\text{spin}}$. The form of Eq.(62) is explicitly $R_{\text{spin}}$ invariant, so $\hat{U}_{ij}$ is $R_{\text{spin}}$ invariant. This is expected since we have spin rotation invariant $\tilde{\chi}$ and $\tilde{\eta}$. This is actually the point of this slave-boson mean-field approach. We choose the mean-field variables to be spin rotation singlets, such that even it acquires non-zero average value, it doesn’t break spin-rotation symmetry, which describes spin liquid states. Moreover, since $R_{\text{spin}}$ is a linear operator, we know that the average value of $\hat{U}_{ij}$ is also $R_{\text{spin}}$ invariant.

$$R_{\text{spin}}: U_{ij} \rightarrow R_{\text{spin}} U_{ij} R_{\text{spin}}^{-1} = U_{ij}$$  \hspace{1cm} (78)

This is consistent with the usual property of linear operator: it commutes with c-numbers.

What is the transformation on mean-field Hamiltonian? We should put transformations on fermionEq.(75) and $U_{ij}$ Eq.(78) together:

$$R_{\text{spin}}: \psi^T_i U_{ij} \psi_j \rightarrow \psi^T_i (i\pi_2)^T U_{ij} (i\pi_2) \psi^*_j$$

using Eq.(67) $\psi^T_i = (U_{ij})^\dagger \psi^*_j$ $\rightarrow \psi^T_i U_{ij} \psi_i = \psi^T_i U_{ij} \psi_i$

This is just another term in the original mean-field Hamiltonian. We conclude that $H_{\text{mean}}$ is invariant under $R_{\text{spin}}$. This again is expected since our spin-liquid states should not break $R_{\text{spin}}$. So $R_{\text{spin}}$ is also an element of PSG.

Here we point out that the form of $R_{\text{spin}}$ transformation Eq.(76) is very similar to the charge-conjugation transformation $C$ in usual Dirac field theory. In fact we can denote $R_{\text{spin}}$ as $C$ in our later discussion. We summarize the transformations on Dirac fermion in Table I.

**E. Transformations of fermion bilinears**

Let us focus on $SU(2)$-linear state. We know that the mean-field Hamiltonian is characterized by a $SU(2)$ doublet of massless Dirac fermions. But can the mass gap be generated after including quantum fluctuations? If yes then the mean-field theory cannot describe the real physical system at all since the low energy behavior is completely different. In this section we will show that this is not the case.

We know that for a certain mean-field state, it has a certain PSG, which is the symmetry of the mean-field Hamiltonian. After we include fluctuation, this symmetry will still be respected. Suppose we go through an renormalization group process to find the low energy theory, any counter-term explicitly breaking PSG will not be generated when taking care of fluctuations.

This can be viewed in the following way: let $L(\psi_i, U_{ij}, a_{ij}^0)$ in Eq.(7) be the Lagrangian describing the dynamics of fermion and gauge fields. In the original theory, we have a huge "symmetry" group leaving $L$ invariant, for example, the translation along $x$-axis by one lattice site $T_x$:

$$L(\psi_i, \hat{U}_{ij}) = L(T_x \psi_i T_x^{-1}, T_x \hat{U}_{ij} T_x^{-1})$$

$$= L(\psi_{i-x}, \hat{U}_{i-x,j-x})$$  \hspace{1cm} (79)

or an arbitrary local $SU(2)$ gauge "symmetry" transformation $W$ (here the meaning of quotation mark is that gauge "symmetry" is not a physical symmetry, instead it is just a many-to-one bad labelling):  \hspace{1cm} (80)

$$L(\psi_i, \hat{U}_{ij}) = L(W_i \psi_i, W_i \hat{U}_{ij} W_j^\dagger)$$

But after $\hat{U}_{ij}$ condense, things are different. The above huge "symmetry" group will be "spontaneously" broken (the unbroken state must have $\hat{U}_{ij} = 0$). PSG is the remaining unbroken "symmetry" group after this symmetry breaking. PSG is defined as the collection of all transformations leaving the Lagrangian $L(\psi_i, \hat{U}_{ij}, a_{ij}^0)$ invariant and also leaving average value $U_{ij}$ invariant. Let $P$ be an element of PSG, then

$$L(\psi_i, \hat{U}_{ij}) = L(P \psi_i P^{-1}, P \hat{U}_{ij} P^{-1})$$

$$U_{ij} = PU_{ij} P^{-1}$$  \hspace{1cm} (81) \hspace{1cm} (82)

We can consider fluctuation around the average value $U_{ij}$:

$$\hat{U}_{ij} = U_{ij} + \delta \hat{U}_{ij}$$  \hspace{1cm} (83)
Plugging into Eq.(81)

\[ L(\psi_i, U_{ij} + \delta U_{ij}) = L(P\psi_i P^{-1}, P(U_{ij} + \delta U_{ij})P^{-1}) \]
\[ = L(P\psi_i P^{-1}, U_{ij} + P\delta U_{ij}P^{-1}) \] (84)

So the Lagrangian for the fluctuations \( L(\psi_i, \delta U_{ij}) \) must be invariant under PSG transformations.

We want to find out the transformations of fermion bilinears under PSG. Because if they all transform non-trivially under PSG, they will be all forbidden. That is why the fermions remain to be massless after including fluctuations.

Here we will consider only the fermion bilinears of form \( \psi^\dagger \psi \). The bilinears of forms \( \bar{\psi} \psi \) and \( \psi^{\dagger} \bar{\psi} \) are not invariant under the spin \( S_z \) rotation and are not allowed in the effective Lagrangian. Since we are using 4-component fermion, there should be \( 4 \times 4 = 16 \) different fermion bilinears of form \( \psi^\dagger \psi \). From Table I, it is quite easy to find the transformations of all fermion bilinears under PSG. It turns out that among these 16 bilinears, as shown in Table II, there are four 1-dimensional representations and six 2-dimensional representations of PSG. All the fermion bilinears transform non-trivially under the PSG. So perturbatively, the fermions remain massless after inclusion of fluctuations.

Now let us consider the fermion bi-linear terms that also contain a single spatial derivative. Those terms represent marginal perturbations when \( N_f = \infty \). From the table II, we see that the only term that is allowed by the PSG is \( \bar{\psi} \partial_x \gamma_1 \psi + \bar{\psi} \partial_y \gamma_2 \psi \). All other terms are forbidden. The reason is as follows: The 1-dimensional representations together with a spatial derivative cannot be Lorentz singlet, so are ruled out. Among 2-dimensional representations together with a spatial derivative, only \( \bar{\psi} \partial_x \gamma_1 \psi + \bar{\psi} \partial_y \gamma_2 \psi \) is invariant under translation \( T_x^{PSG} \) (in fact, it is invariant under full PSG); all others are ruled out. But the term \( \bar{\psi} \partial_x \gamma_1 \psi + \bar{\psi} \partial_y \gamma_2 \psi \), which is already present in Eq.(40), only changes the velocity of the fermions. In a RG study, this counter-term means a wavefunction renormalization. The low-energy effective theory Eq.(40) remains valid.

The next question is, will there be 4-fermion interaction terms? The answer is yes. For example, we choose a certain 1-dimension representation in Table II, say \( \bar{\psi} \gamma_5 \psi \), then couple this term to itself to make a 4-fermion interaction. It is obvious that this 4-fermion term is PSG invariant, which is allowed in the Lagrangian. Will this kind of 4-fermion term change the low energy behavior drastically? The answer is no. This is because we are in \( 2+1 \) space-time dimension, and by power counting 4-fermion terms are of dimension 4, so they are irrelevant couplings. Same argument can be done for fermion bilinear terms with second order derivatives; those terms are also irrelevant.

In summary, we have discussed the possible fermion self-interactions. Our conclusion is that in perturbative sense, these fermion self-interactions will not change the low energy behavior from the mean-field result. We may say that the \( SU(2) \)-linear mean-field result is stable under fermion self-interactions.

F. Emergent \( Sp(4) \) Physical Symmetry

In this section we will discuss the emergent symmetry for \( SU(2) \)-linear phase whose low energy effective theory is Eq.(40). We already know that there are two fermion-chiral symmetry generators \( \gamma_3 \) and \( \gamma_5 \), and they are anti-commuting. (Here please note that we are talking about fermion-chiral symmetry, which is different from the spin-chirality in the later discussion about chiral spin liquid.) Therefore the symmetry of \( SU(2) \)-linear phase contains at least a global \( SU(2) \) fermion-chiral Lie group whose generators are \( \gamma_3, \gamma_5 \) and \( i\gamma_3\gamma_5 \).

We also know that the theory should be global \( SU(2) \) spin-rotation invariant, since we are talking about a spin liquid phase here. Thus there should be at least another \( SU(2) \) spin-rotation symmetry group. However after the particle-hole transformation we made in Eq.(8), this spin-rotation symmetry was hidden in our formalism.

The full physical symmetry group of \( SU(2) \)-linear phase should contain both the \( SU(2) \) fermion-chiral and \( SU(2) \) spin-rotation as its subgroups. The naive guess for the full group is \( SU(2) \times SU(2) \) but it turns out to be wrong. We will show in this section that the correct full symmetry group is \( Sp(4) \). We noticed that the same \( Sp(4) \) symmetry was found earlier by Tanaka and Hu[28] by viewing the \( \pi \)-flux state as a fermionic mean field state, i.e., ignoring the effect of \( SU(2) \) gauge field. Here we clarified the gauge field effect and obtained the same global flavor symmetry. Then we can classify all the fermion bilinears according to their transformation rules under \( Sp(4) \) group.

Later, we will show that after including the \( SU(2) \) gauge fluctuations, the \( SU(2) \)-linear state remains gapless and the correlations between various operators remain algebraic. But the exponents of algebraic correlations may be modified by the \( SU(2) \) gauge interaction. Classifying fermion bilinears according to their transformation under the \( Sp(4) \) group is very important in understanding the scaling properties of those operators. The operators that belong to the same irreducible \( Sp(4) \) representation will have the same scaling dimension.

First we consider the spin rotation group; basically it will mix \( \psi \) and \( \psi^* \). To make the spin rotation transformation explicit, it is convenient to reintroduce the \( \bar{\psi} \) formalism in Eq.(10):

\[ \bar{\psi} = i\sigma_2 \psi^* = \begin{pmatrix} f^i_i \\ -f_i^j \end{pmatrix} \] (85)

Let us look at lattice fermion \( \psi \) at certain site. If we put \( \psi \) (2-component, corresponding to spin up and down) and


\[
\tilde{\psi} \text{ (2-component) together to form a 4-component vector:}
\]

\[
\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}
\]  

(86)

then it is straightforward to write down the spin rotation transformation. For example, the rotation around z-axis:

\[
\psi \to \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \psi \quad \tilde{\psi} \to \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tilde{\psi}
\]

(87)

Therefore

\[
\Psi \to 1 \otimes e^{i\theta_3/2}\Psi
\]

(88)

where the identity matrix labels the internal space of \(\psi\) (spin up and down), while \(e^{i\theta_3/2}\) acts on the space mixing \(\psi\) and \(\tilde{\psi}\).

What about rotation along y-axis? Suppose we do an infinitesimal transformation

\[
\begin{align*}
 f_1 &\to f_1 + \frac{\theta}{2} f_4 \\
 f_4 &\to f_4 - \frac{\theta}{2} f_1
\end{align*}
\]

(89)

it implies:

\[
\begin{align*}
 \psi_1 &\to \psi_1 + \frac{\theta}{2} \psi_2^* \\
 \psi_2 &\to \psi_2 - \frac{\theta}{2} \psi_1^*
\end{align*}
\]

(90)

thus

\[
\psi \to \psi + \frac{\theta}{2} \tilde{\psi}
\]

\[
\tilde{\psi} \to \tilde{\psi} - \frac{\theta}{2} \psi
\]

(91)

in terms of \(\Psi\):

\[
\Psi \to 1 \otimes e^{i\theta_1/2}\Psi
\]

(92)

Similarly the rotation along x-axis is:

\[
\Psi \to 1 \otimes e^{i\theta_2/2}\Psi
\]

(93)

To summarize, we know that the spin rotation is acting on the space mixing \(\psi\) and \(\tilde{\psi}\).

Let us go to continuous limit, and consider the 4-component Dirac fermion \(SU(2)\) doublet \(\psi\) in Eq.(36).

Again we write it together with \(\tilde{\psi} = (-\gamma_1\gamma_5)i\sigma_2\tilde{\psi}^*\), where the \((-\gamma_1\gamma_5)\) is inserted to make the spin-rotation have a simple form:

\[
\bar{\Psi} = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}
\]

(94)

Note that actually \(\bar{\Psi}\) has 16 components and 16 = 4\(\times 2\times 2\) where 4 is the number of Dirac components, the first 2 is for \(SU(2)\) gauge doublet and the second 2 is for the space mixing \(\tilde{\psi}\) and \(\tilde{\psi}\). From the above discussion, the

|                  | \(T_x^{PSG}\) | \(T_y^{PSG}\) | \(P_x^{PSG}\) | \(P_y^{PSG}\) | \(P_z^{PSG}\) | \(C^{PSG}\) |
|------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(\tilde{\psi}\) | -1             | -1             | -1             | -1             | +1             | -1             |
| \(\tilde{\psi}\) | +1             | +1             | +1             | +1             | +1             | -1             |
| \(i\tilde{\psi}\) | +1             | +1             | -1             | -1             | -1             | +1             |
| \(i\tilde{\psi}\) | -1             | +1             | +1             | -1             | -1             | +1             |

TABLE II: Transformations of 16 fermion bilinears. All transform non-trivially.
space mixing $\tilde{\psi}$ and $\tilde{\psi}$ is actually spin rotation space. A generic transformation $G$ on fermion field can be written as a transformation on $\tilde{\Psi}$:

$$G = G_{\text{Dirac}} \otimes G_{\text{gauge}} \otimes G_{\text{spin}}$$

(95)

where the transformations with subscripts act on each corresponding space.

The three spin rotation generators are, from above discussion:

$$1 \otimes 1 \otimes \sigma_1, \quad 1 \otimes 1 \otimes \sigma_2, \quad 1 \otimes 1 \otimes \sigma_3$$

(96)

The fermion-chiral generator $\gamma_3$ is acting on $\tilde{\psi}$. One can easily check that while acting on $\tilde{\Psi}$, since $\tilde{\psi} = (-\gamma_3 \gamma_5) \sigma_2 \tilde{\psi}^*$, the generator has the form: $\gamma_3 \otimes 1 \otimes \sigma_3$. Similarly one can find the other two generators of fermion-chiral transformation. In summary, they are:

$$\gamma_3 \otimes 1 \otimes \sigma_3, \quad \gamma_5 \otimes 1 \otimes \sigma_3, \quad i\gamma_3 \gamma_5 \otimes 1 \otimes 1$$

(97)

Now if we do commutations between Eq.(96) and Eq.(97), the full set of symmetry generators can be found:

$$1 \otimes 1 \otimes \sigma, \quad \gamma_3 \otimes 1 \otimes \sigma, \quad \gamma_5 \otimes 1 \otimes \sigma, \quad i\gamma_3 \gamma_5 \otimes 1 \otimes 1$$

(98)

Totally $3 + 3 + 3 + 1 = 10$ elements, which satisfy $Sp(4)$ algebraic relation.

Here one thing we need to mention is that the three gauge transformation generators:

$$1 \otimes \tau \otimes 1$$

(99)

will also keep the Lagrangian Eq.(40) invariant. But they are gauge transformations and should not be taken as physical symmetries.

We just showed the emergent $Sp(4)$ global symmetry. Can the emergent continuous symmetry group larger than $Sp(4)$? The answer is no, as one can see in the following. We have totally 8 components of fermions, and they form four $SU(2)$ gauge doublets. For global symmetry we should only consider transformations invariant in the gauge sector, which means we should consider the transformation between the 4 doublets only (i.e., in flavor space but not in gauge space), including the mixing between $\tilde{\psi}$ and $\tilde{\psi}^\dagger$. In Majorana fermion representation, it is obvious that the allowed flavor transformations form $SO(8)$ group. The Lorentz transformations $i\gamma_0 \gamma_1, i\gamma_0 \gamma_2, i\gamma_1 \gamma_2$ generate $SO(3)$ group in Euclidean space, and we also know that $Sp(4) = SO(5)$. The flavor symmetry $SO(5)$ and Lorentz symmetry $SO(3)$ actually commute. This $SO(5)$ is the largest continuous subgroup of $SO(8)$ which can commute with $SO(3)$ and has no common element with $SO(3)$ except for identity. Therefore $Sp(4)$ is the largest continuous global symmetry.

If we introduce $N_f$ flavors of fermions, it turns out that the emergent symmetry group is $Sp(4N_f)$. We should also include the Lorentz symmetry. Here by Lorentz Group we mean the continuous group $SO(2,1)$, generated by $\gamma_1 \gamma_2, \gamma_0 \gamma_1, \gamma_0 \gamma_2$. Note that the physical lattice rotation is not identical to the rotation element in this emergent Lorentz group. For example, according to Table 1, the rotation on lattice $R_{90} = P_{xy} \circ P_x$ is given by

$$R_{90} = \frac{1}{2}(\gamma_1 - \gamma_2)(\gamma_3 + \gamma_5)\gamma_1$$

$$= \frac{1}{2}(1 + \gamma_1 \gamma_2)(\gamma_3 + \gamma_5)$$

$$= e^{\frac{\pi}{2} \gamma_1 \gamma_2} \frac{1}{\sqrt{2}}(\gamma_3 + \gamma_5)$$

(100)

in the continuum limit. We can see that the physical rotation on lattice is a combination of the Dirac rotation and an element in the $Sp(4)$: $\frac{1}{\sqrt{2}}(\gamma_3 + \gamma_5)$. This element actually exchanges $\gamma_3$ and $\gamma_5$.

We should also include certain discrete symmetries such as time-reversal $T$, spatial reflections $P_x, P_y, P_{xy}$, total parity $-1$ and charge conjugation. But we know that charge conjugation is related to the spin rotation, which is included in the $Sp(4)$; $P_{xy}$ is related to Dirac rotation, $P_x$ and element $\frac{1}{\sqrt{2}}(\gamma_3 + \gamma_5)$ in $Sp(4)$; and $-1$ is included in $Sp(4)$ as well, namely $e^{\pi \gamma_3}$. Therefore the full symmetry group of the low energy effective theory for the $SU(2)$-linear state is $Sp(4N_f) \times$ Lorentz Group $\times T \times P_x \times P_y$. Such a symmetry group is certainly much larger than the symmetry group of the lattice model. (The effective theory for the $SU(2)$-linear state does contain terms that violate the $Sp(4N_f) \times$ Lorentz Group. But all those terms are irrelevant and have vanishing effects at low energies.)

One can classify the fermion bilinears according to their transformation rules under $Sp(4)$ and Lorentz group. It is convenient to rewrite the 16 bilinears in terms of $\tilde{\psi}$, then in terms of $\tilde{\Psi}$:

$$\tilde{\psi} \tilde{\psi} = -\tilde{\psi} \tilde{\psi}$$

$$\tilde{\psi} \gamma_0 \tilde{\psi} = -\tilde{\psi} \gamma_0 \tilde{\psi}$$

$$\tilde{\psi} \gamma_3 \gamma_5 \tilde{\psi} = \tilde{\psi} \gamma_3 \gamma_5 \tilde{\psi}$$

$$\tilde{\psi} \gamma_1 \gamma_2 \tilde{\psi} = \tilde{\psi} \gamma_1 \gamma_2 \tilde{\psi}$$

$$\tilde{\psi} \gamma_1 \tilde{\psi} = -\tilde{\psi} \gamma_1 \tilde{\psi}$$

$$\tilde{\psi} \gamma_2 \tilde{\psi} = -\tilde{\psi} \gamma_2 \tilde{\psi}$$

$$\tilde{\psi} \gamma_0 \gamma_1 \tilde{\psi} = \tilde{\psi} \gamma_0 \gamma_1 \tilde{\psi}$$
Here if there is no minus sign, it transforms as singlet under spin rotation. If there is a minus sign after the equal sign, it means the fermion bilinear has a $\sigma_3$ in spin space, which in turn means the fermion bilinear transforms as triplet under spin rotation. Triplet should have 3 components, but in our 16 bilinears we only included one of them (the one along $z$-axis). And the other two are fermion bilinears of form $\bar{\psi}$ and $\bar{\psi}^\dagger$. 

One can express all the fermion bilinears in terms of $\bar{\Psi}$. In summary, one can organize them as in Table III. Notice that there are 10 conserved currents of Sp(4), so they all have zero anomalous dimension.

In Table III, we enumerate all the fermion bilinears. But what do they correspond to in our original spin model? For example, let us look at a particular fermion bilinear $\bar{\Psi} \psi_{\gamma_0 \gamma_3} \otimes 1 \otimes \sigma \bar{\Psi}$. We will show that this term corresponds to the spin triplet bond order: $(-)^i S_i \times S_{i+x}$. Let us write this fermion bilinear interaction in terms of the lattice fermion operators in a unit cell $\psi_i$ ($i = 0, 1, 2, 3$), as shown in Fig.2:

$$H_1 = \bar{\Psi} \psi_{\gamma_0 \gamma_3} \otimes 1 \otimes \sigma \bar{\Psi} = \begin{pmatrix} \psi_0^\dagger & \psi_1^\dagger & \psi_2^\dagger & \psi_3^\dagger \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

(102)

Now we can write down the Hamiltonian $H_{SU(2)}$ on lattice. For example, the interaction between site-0 and site-1, in terms of $f$-operator, is

$$i\chi f^\dagger_0 f_{1\alpha} + \theta f^\dagger_0 \sigma_3 f_{1\beta} + h.c.$$  

(103)

which basically tells us that $\langle f^\dagger_0 f_{1\uparrow} \rangle = i\chi + \theta$, and $\langle f^\dagger_0 f_{1\downarrow} \rangle = i\chi - \theta$.

On the other hand, we can write down the spin operator $S_0 \times S_1$ in terms of $f$ operators. Focusing on the $z$-component:

$$\langle (S_0 \times S_1)_z \rangle = -2i\langle (f^\dagger_0 f_{1\uparrow} f^\dagger_0 f_{1\downarrow} f^\dagger_{0\downarrow} f_{1\downarrow} f^\dagger_{0\uparrow} f_{1\uparrow}) \rangle$$

$$= -2i \left( \langle f^\dagger_0 f_{1\uparrow} \rangle \langle f^\dagger_{0\downarrow} f_{1\downarrow} \rangle - \langle f^\dagger_{0\downarrow} f_{1\downarrow} \rangle \langle f^\dagger_{0\uparrow} f_{1\uparrow} \rangle \right)$$

$$= -8\chi \theta \neq 0$$

(104)

Similarly one can show that $\langle (S_2 \times S_3)_z \rangle = 8\chi \theta$, and our correspondence is established.

In Table IV we list the correspondence between the field theory operators and original spin operators. From this table we know that fermomagnetic order, triplet VBS order and staggered spin chirality order all have zero anomalous dimension and their correlation function all scale as $1/\rho$, even after the inclusion of the $SU(2)$ gauge interaction. We also know that the Neel order and VBS order have the same anomalous dimension which turns out to be non-zero after inclusion of the $SU(2)$ gauge interaction. We noticed that the same Sp(4) emergent group for $\pi$-flux state which rotates Neel order into VBS order was found[28] when ignoring the $SU(2)$ gauge field effect.

Here we should mention the work done by Hermele et.al.[29], where the $U(1)$-linear spin liquid was discussed and the emergent symmetry is $SU(4)$, and similar classification of totally 64 fermion bilinears was done. One can recover their result from our formalism. From our formulation of $SU(2)$-linear phase, the $U(1)$-linear phase can be regarded as a Higgs phase where $SU(2)$ gauge field is broken down to $U(1)$. Let us assume the remaining $U(1)$-gauge symmetry is along $\tau_3$ direction. The only things one should add to recover their result are: first the gauge invariant transformations not only include those in Eq.(98); we should also include

$$\gamma_3 \otimes \tau_3 \otimes 1, \gamma_5 \otimes \tau_3 \otimes 1, i\gamma_3 \gamma_5 \otimes \tau_3 \otimes \sigma.$$  

(105)

So totally 15 elements, and they form a $SU(4)$ algebra. Secondly the bilinear with a $\tau_3$ in the gauge sector is also gauge invariant, as shown in Table V.

**G. The effect of $SU(2)$ gauge interaction on $SU(2)$-linear spin liquid**

We know that in the continuous limit, the full lagrangian should be Eq.(40). The question is, will the $SU(2)$ gauge interaction change the low energy behavior of mean-field theory drastically? The answer is complicated. There are two main concerns: spontaneous chiral symmetry breaking (SCSB) and confinement. SCSB means that fermion mass is generated dynamically. And confinement means that no excitation with gauge charge can show up in the spectrum, and gauge interaction is linearly confining. If any of these happens, the low energy behavior of the system is changed drastically and we say that the mean-field state is unstable under gauge fluctuation. In this case, the mean-field $SU(2)$-linear state
TABLE III: Under $Sp(4)$ and Lorentz group, all fermion bilinears can be classified into 3 groups. A group of Dirac scalar and $Sp(4)$ 5-dimension representation, a group of Dirac scalar and $Sp(4)$ singlet, and a group with Dirac vectors in it. For a given group of bilinears, they are connected by $Sp(4)$ transformation in each row, and connected by Lorentz group in each column. Totally there are 36 bilinears. All elements in a given group have the same scaling dimension. In the group of Dirac vector, we actually have conserved current corresponding to each column, totally 10 conserved currents. Those are the conserved $Sp(4)$ currents, and they all have zero anomalous dimension after inclusion of the $SU(2)$ gauge interaction.

| Dirac Scalar (5 elements) | $\Psi \otimes 1 \otimes \sigma \bar{\Psi}$, $\Psi \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$ |
|--------------------------|--------------------------------------------------------------------------------------------------|
| Dirac Scalar (1 element) | $\Psi \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_7 \gamma_3 \otimes 1 \otimes \sigma \bar{\Psi}$, $\Psi \gamma_7 \gamma_5 \otimes 1 \otimes \sigma \bar{\Psi}$ |
| Dirac Vector (30 elements) | $\Psi \gamma_7 \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_7 \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_7 \gamma_{12} \otimes 1 \otimes 1 \bar{\Psi}$ |

TABLE IV: The correspondence between fermion field operators and original spin operators. In the group of Dirac vector, only 4) 5-dimension representation, a group of Dirac scalar and $Sp(4)$ singlet, and a group with Dirac vectors in it. For a given group of bilinears, they are connected by $Sp(4)$ transformation in each row, and connected by Lorentz group in each column. Totally there are 36 bilinears. All elements in a given group have the same scaling dimension. In the group of Dirac vector, we actually have conserved current corresponding to each column, totally 10 conserved currents. Those are the conserved $Sp(4)$ currents, and they all have zero anomalous dimension after inclusion of the $SU(2)$ gauge interaction.

| Dirac Scalar | $\Psi \otimes 1 \otimes \sigma \bar{\Psi}$, $\Psi \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$ | Neel Order: $(-)^i S_i$ |
|--------------|--------------------------------------------------------------------------------------------------|-------------------|
| $\Psi \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$ | $(-)^{i+y} S_i \cdot S_{i+x}$ | $\Psi \gamma_7 \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_7 \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$ |
| $\Psi \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$ | $(-)^{i+y} S_i \cdot S_{i+y}$ |

Dirac Scalar | $\Psi \gamma_3 \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$ | Uniform Spin Chirality: $S_i \cdot (S_{i+x} \times S_{i+x+y})$ |

Dirac Vector | $\Psi \gamma_7 \gamma_3 \otimes 1 \otimes 1 \bar{\Psi}$, $\Psi \gamma_7 \gamma_5 \otimes 1 \otimes 1 \bar{\Psi}$ | Ferromagnetic Order: $S_i$ |

Does not lead to a stable algebraic spin liquid. (Actually, we do not know the low energy properties of the model beyond the mean-field theory.)

This problem is a famous and difficult problem in QCD, since both effects are non-perturbative. And these two effects are related: if there is a mass gap generated for the fermion, then below the mass gap there is effectively no fermion to screen the gauge interaction and we only have pure gauge field. We know that pure gauge interaction is confining. So logically SCBS will induce confinement. The other way, whether confinement will induce SCBS, is not clear yet.

Usually it is believed that for a $SU(N)$ gauge theory coupled to $N_f$ flavor of massless fermions, there is a critical $N_f^c[30-32]$. If $N_f$ is smaller than $N_f^c$, the system have both SCBS and confinement. However if $N_f$ is larger than $N_f^c$, there is a conformal invariant IR fixed point.

In particular, in 2+1 dimension, QCD always has a stronger interaction at low energies. But if $N_f$ is large enough, the fermion screening effect is dominant and the renormalization group flow will terminate at an IR fixed point $g_s^2 \sim \frac{N_f}{N}$. The low energy behavior is governed by that fixed point, which is deconfined and has no SCBS. When $N_f \rightarrow \infty$, $g_s \rightarrow 0$, we are in the perturbative region.

To have a quantitative study, we need work within large $N_f$ limit[33]. Before we study the $SU(2)$ gauge fluctuation, it is worthwhile to mention the $U(1)$ case[8]. The main result there was that in the large $N_f$ case, the model remains gapless and the spin-spin correlation function is a power law with an anomalous dimension $\gamma$. This anomalous dimension $\gamma$ is calculable in large $N_f$ expansion, and up to the $1/N_f$ (leading) order, $\gamma$ is found to be $-\frac{32}{3\pi^2 N_f}$.

Now we look at the $SU(2)$ case. In particular, the low energy effective theory is Eq.(40). Technically there are two ways to do large-$N_f$ limit. The first way is a complete renormalization group analysis. To have an controlled calculation, one first does an $\epsilon$-expansion, then studies the renormalization group flow, finds out the IR fixed point, and the scaling dimension of operators at that fixed point, finally sets $N_f$ to be large. This way is generally accepted and the result is thought to be reliable. However the calculation is complicated.

Here we do the large-$N_f$ calculation in a different...
Dirac Scalar (15 elements) 
\[
\begin{align*}
\Psi^1 & \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^1 \gamma_5 \otimes 1 \otimes 1_\Psi, \ \Psi_i^1 \gamma_5 \otimes 1 \otimes \Psi \\
\Psi & \otimes 1 \otimes \gamma_5 \otimes 1_\Psi, \ \Psi_i \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \otimes 1 \otimes \gamma_5 \otimes \sigma_\Psi \\
\Psi_i & \gamma_5 \otimes 1 \otimes \gamma_5 \otimes 1_\Psi.
\end{align*}
\]

Dirac Scalar (1 element) 
\[
\Psi_i^1 \gamma_5 \otimes 1 \otimes 1_\Psi.
\]

Dirac Vector (45 elements) 
\[
\begin{align*}
\Psi_i^1 & \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^1 \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^1 \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^1 \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^1 \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi \\
\Psi_i & \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi \\
\Psi_i^2 & \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^2 \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^2 \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^2 \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi \\
\Psi_i & \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi \\
\Psi_i^3 & \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^3 \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^3 \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i^3 \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi \\
\Psi_i & \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1 \otimes \sigma_\Psi \\
\Psi & \otimes 1 \otimes \gamma_5 \otimes 1_\Psi, \ \Psi_i \gamma_5 \otimes 1 \otimes 1 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \otimes 1 \otimes \gamma_5 \otimes \sigma_\Psi, \ \Psi_i \gamma_5 \otimes 1 \otimes \gamma_5 \otimes \sigma_\Psi
\end{align*}
\]

Dirac Vector (3 elements) 
\[
\Psi_i \gamma_5 \otimes 1 \otimes 1_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes 1_\Psi, \ \Psi_i \gamma_5 \gamma_2 \otimes 1 \otimes \gamma_5 \otimes 1_\Psi.
\]

TABLE V: Under SU(4) and Lorentz group, the total 64 bilinears can be classified into 4 groups: A group of Dirac scalars and SU(4) adjoint representation with 15 elements, a group of Dirac scalar and SU(4) singlet with one element, a group of Dirac vectors and SU(4) 12-dimension representation with 3 × 15 = 45 elements, and group of Dirac vectors and SU(4) singlets with 3 elements.

way[33]. Taking large-\(N_f\) limit:
\[
N_f \rightarrow \infty, \quad N_f g^2 \rightarrow \text{const.} \quad (106)
\]

The fermion contribution to any physical quantity can be expanded in \(1/N_f\) systematically. This is just a way to organize the summation of Feynman diagrams. For example, the leading order term usually corresponds to summation of fermion one-loop diagrams. The IR fixed point can be found by cancellation of leading correction to scaling. We will discuss this in detail soon. Here we want to discuss whether this approach and the first approach are equivalent. There is no general proof that these two approaches are equivalent, but in [33] quite a lot of examples are presented and it was found that these two approaches are equivalent, for example, in the case of Dirac fermions coupling to \(U(1)\) gauge field, the scaling dimension and fixed point found in the two approaches are consistent. Gracey et al.[34, 35] calculated large-\(N_f\) expansion of anomalous dimensions of many quantities in QED\(_3\) and QCD\(_3\), including fermion mass, which is gauge invariant. Then they compared the results with results from usual MS renormalization+dimensional regularization, and found they are consistent.

First let us take a look at the gauge interaction. From Eq.(106), \(g^2\) is of order \(1/N_f\). In Figure 5, all leading order contributions to gauge propagator are summed together. The double line represents the leading order dressed gauge propagator.

One can calculate the gauge-gauge two point function in Fig.5. The result is:
\[
- \frac{1}{2g^2} A^a_{\mu} (k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}) A^a_{\nu} \\
- N_f \text{Tr} \text{Tr} \tau^a \tau^b A^a_{\mu} (k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}) \left( \frac{1}{64k} - \frac{1}{48\pi \Lambda} \right) A^b_{\nu} \quad (107)
\]

Here \(1\) is the identity matrix in Dirac spinor space. In our \(SU(2)\)-linear example, \(\text{Tr} 1 = 4\). And \(\tau^a\) are usual \(SU(2)\) Pauli matrices. Note that the term \(1/64k\) is independent of regularization scheme, but \(-1/(48\pi \Lambda)\) is dependent. In different regularization schemes, the coefficients in front of \(1/\Lambda\) are different. The result here \(-1/(48\pi \Lambda)\) is from the Pauli-Villar regularization.

One immediately sees that at low energy \((k \rightarrow 0)\), the term \(1/64k\) dominates. The low energy two point function of gauge field is:
\[
N_f \text{Tr} \frac{1}{2g^2} A^a_{\mu} (k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}) \frac{1}{64k} A^a_{\nu} \quad (108)
\]

The IR fixed point is found by cancellation of the leading correction to scaling. Here it means that the \(-1/2g^2\) term and the \(-1/(48\pi \Lambda)\) term should cancel:
\[
\frac{1}{2g^2} = \frac{N_f \text{Tr} 1}{2 \pi} \frac{1}{48\pi \Lambda} \\
\Rightarrow g^2 = \frac{12\pi \Lambda}{N_f} \quad (109)
\]

The dimensionless coupling \(g^*/\Lambda\) describes the strength of gauge interaction at the energy scale \(\Lambda\). We see that dimensionless coupling \(g^*/\Lambda \rightarrow 0\) as the cut-off energy scale \(\Lambda \rightarrow 0\) and \(N_f \rightarrow \infty\), as we expected in the renormalization group flow diagram Fig. 4. Remember that if the number of flavors of fermions is small, confinement
will happen, renormalization group flow will go to some fixed point of $g_2^2/\Lambda \sim 1$ as $\Lambda \to 0$ and perturbation theory breaks down. The message is that the fermion screening effect in our large $N_f$ model drives the gauge interaction to weak limit at low energies.

From Eq.(107), we know that the gauge field remains massless, and in deconfined phase. So the $SU(2)$-linear state is also stable under gauge fluctuation. We can also see that the scaling dimension of gauge field $A$ is $d_A = 1$ at leading order. With $1/N_f$ correction, we expect to have $d_A = 1 + O(1/N_f)$. Now we can understand why the fixed point $g_2^2$ is IR stable. Suppose we are slightly away from the fixed point:

$$L = \sum_{i=1}^{N_f} \bar{\psi}_i (\partial_\mu - i \gamma_\mu A^\mu ) \gamma_5 \psi_i + \frac{1}{2g_2^2} \text{Tr} \left[ f^{\mu \nu}_{\mu \nu} f^{\mu \nu}_{\mu \nu} \right]$$

$$+ \delta g \text{Tr} \left[ f^{\mu \nu}_{\mu \nu} f^{\mu \nu}_{\mu \nu} \right]$$

(110)

By power counting, the scaling dimension of $\delta g$ is $4 + O(1/N_f)$, so it is irrelevant. Therefore at low energy $\delta g$ flows to zero and $g_2^2$ is IR stable.

In the above discussion, we have shown that the $SU(2)$-linear mean-field state is stable under fermion self-interaction and gauge fluctuation in large $N_f$ limit. So it is a stable phase. Here by “stable” we mean that the low energy behavior does not change drastically from mean-field result. When $N_f = \infty$ we are back to the mean-field result, but if $N_f$ large but finite, the low energy behavior is changed from the mean-field result, but not drastically. Below we will see what this change is.

VI. SPIN-SPIN CORRELATION FUNCTION IN $SU(2)$-LINEAR PHASE

In this section we investigate the spin-spin correlation function. In a frustrated spin liquid, there is always a strong trend to antiferromagnetic long range order. To describe this trend, we can calculate the staggered spin-spin correlation function in fermion $\psi$ formalism:

$$(-1)^x S^z(x) S^z(0) = \frac{1}{64} \left( \langle \bar{\psi} \psi(x) \bar{\psi} \psi(0) \rangle - \langle \bar{\psi} \psi \rangle^2 \right)$$

(111)

Here $\bar{\psi}$ is actually an 8-component fermion operator, with both spinor indices and $SU(2)$ gauge indices.

One can see that in field theory language, staggered spin-spin correlation function is nothing but the mass operator correlation function. At zeroth order, we have the free fermion Feynman diagram (Fig. 6). The staggered spin-spin correlation function in momentum space at zeroth order is

$$< S^z_\alpha(q) S^z_\beta(-q)_0 > = -\frac{1}{64} \int \frac{dp^3}{(2\pi)^3} \text{Tr} \left[ G_{\alpha \beta}(p) G_{\beta \alpha}(q - p) \right]$$

$$= -\frac{\sqrt{g_0^2 + g_2^2}}{128}$$

(112)

FIG. 6: Spin-spin correlation function at zeroth order. Cross means spin operator insertion.

FIG. 7: Contribution to spin-spin correlation function at order of $1/N_f$.

The first order in $1/N_f$ expansion involves diagrams in Fig.7. Note that the double line represents the dressed gauge field propagator:

$$D^{\mu \nu}_{\mu \nu} = \frac{1}{N_f} k \left( \delta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab}$$

(113)

These diagrams will give the spin-spin correlation function an anomalous dimension. According to Eq.(112), the spin-spin correlation function in momentum space scales like $\sim |k|$. With $1/N_f$ correction, the correlation function scales like $\sim |k|^{1 + 2\gamma}$. Here $\gamma$ is called the anomalous dimension. The contributions from the three diagrams in Fig.7 give (See Appendix A):

$$\gamma = -\frac{16}{\pi^2 N_f}$$

(114)

We see that in the $SU(2)$-linear state, spin-spin correlation function remains gapless and algebraic, but the scaling dimensions of physical operators, for example spin operator, are unusual due to the gapless gauge interaction. Actually there is no quasi-particle in this phase.

VII. PHASE TRANSITION BETWEEN SPIN LIQUIDS

A. Phase transition between $SU(2)$-linear phase and $SU(2)$-chiral phase

1. The effective field theory for phase transition

In this section we study the phase transition between $SU(2)$-linear phase (A in phase diagram Fig.1) and $SU(2)$-chiral phase (D in Fig.1).

We have argued that the $SU(2)$-linear state is a stable phase. What about the $SU(2)$-chiral state in Eq.(25)? One can check that the fermion spectrum is gapped at mean-field level. As for gauge field, because time-reversal symmetry is broken in the $SU(2)$-chiral state, a Chern-Simons term is generated after integrating out...
the fermions. And that will also give gauge field a mass gap. So the SU(2)-chiral state is a fully gapped state. We know quantum fluctuation cannot kill a gapped system perturbatively, so SU(2)-chiral state is also a stable phase.

Now we have two stable phases. According to phase diagram Fig. 1, there is a phase transition between the two phases around $J_2 = 0.35$. It also seems that the energies of the two phases connect smoothly on the same diagram, which indicates the phase transition may be continuous. We will study this phase transition in this section.

Fig. 8 shows the plot of the ansatzs of the two phases on lattice. Our notation here is

\[ SU(2)\text{-linear phase: } u_{i,i+x} = i\chi, \quad u_{i,i+y} = i\chi(-)^{i_x}, \]
\[ SU(2)\text{-linear phase: } u_{i,i+x-y} = i\sigma(-)^{i_x}. \]  

In Fig. 8, we also show the phases one fermion gains after hopping around a plaquette. In SU(2)-linear case, this phase for the square plaquette is $\pi$; while in SU(2)-chiral case, this phase for the triangle plaquette is $\frac{\pi}{2}$. Those phases indicate the fluxes through the plaquettes. After a time-reversal transformation, the flux will change sign. For SU(2)-linear case, we have $-\pi$ flux, and $-\pi$ differs from $\pi$ by $2\pi$, thus equivalent. This indicates the SU(2)-linear phase respects time-reversal symmetry. However for the SU(2)-chiral phase, we have $\frac{-\pi}{2}$, which is physically different from $\frac{\pi}{2}$. So the SU(2)-chiral phase breaks time-reversal symmetry and the parity symmetry. Other than that, one can show that SU(2)-chiral phase has full translation and rotation symmetry:

Physical symmetry of SU(2)-chiral phase
\[ \{T_x, T_y, R_{90^\circ}, C\} \] (116)

The two phases involved in the phase transition, SU(2)-linear and SU(2)-chiral states, have different PSGs, because even their physical symmetries are different. The phase transition breaks time-reversal symmetry.

The low energy physics at mean-field level of the systems can be derived by taking the lattice model into continuous limit:

SU(2)-linear phase:
\[ L_{\text{mean}} = \bar{\psi} \partial_\mu \gamma_\mu \psi \] (117)

SU(2)-chiral phase:
\[ L_{\text{mean}} = \bar{\psi} \partial_\mu \gamma_\mu \psi + \sigma \bar{\psi} [i\gamma_3 \gamma_5] \psi \] (118)

To save notation, we dropped the tilde above the fermion operator $\psi$.

Here we see that the $\sigma$ boson field is driving the phase transition. In SU(2)-linear phase, $\langle \sigma \rangle = 0$; in SU(2)-chiral phase, $\langle \sigma \rangle \neq 0$. Therefore to understand the phase transition, we need to know the dynamics of the $\sigma$ field. Let us include quantum fluctuations of all fields, the low energy effective theory is:

\[ L = \sum_{i=1}^{N_f} \bar{\psi}_i \left( \partial_\mu - i a_\mu^I \right) \gamma_\mu \psi_i + \frac{1}{2g^2} \text{Tr} \left[ f_{\mu\nu}^I f_{\mu\nu}^I \right] \]
\[ + \sigma \bar{\psi} [i\gamma_3 \gamma_5] \psi + \frac{1}{2g^2} (\partial_\mu \sigma)^2 + V(\sigma) \] (119)

Here to have a controlled calculation, we again introduced $N_f$ flavors of fermion. The first line is the Dirac fermion coupling to SU(2) gauge field. The second line is the coupling between the fermion and $\sigma$ boson, and the dynamics of $\sigma$ boson field. The potential $V(\sigma)$ is not known yet. Nevertheless we know in SU(2)-linear phase, the dynamics of $\sigma$ boson gives $\langle \sigma \rangle = 0$; while in SU(2)-chiral phase, $\langle \sigma \rangle \neq 0$, and the time-reversal symmetry is broken. This is similar to the usual formalism of the phase transition of symmetry breaking, except for the fact that we have gauge field involved here.

2. The correct effective theory from PSG consideration

Eq. (119) is the effective Lagrangian for both phases. What is the symmetry that the lagrangian should respect? It should respect the full symmetry of the lattice model. Before the symmetry breaking the SU(2)-linear phase has a symmetry described by the SU(2)-linear PSG. Thus, the effective theory for the phase transition should respect the symmetry described by the SU(2)-linear PSG.

We have shown that, under the SU(2)-linear PSG, the fermion bilinears transform in the way described by Table II. For example, the transformation $P_x^{PSG}$:
\[ P_x^{PSG} : i\bar{\psi}[\gamma_3 \gamma_5] \psi \rightarrow \bar{\psi}[\gamma_3 \gamma_5] \psi \] (120)
and
\[ P_x^{PSG} : \sigma \rightarrow -\sigma \] (121)

In fact, $\sigma$ is the average of $\bar{\psi}[\gamma_3 \gamma_5] \psi$ and transforms in the same way as $i\bar{\psi}[\gamma_3 \gamma_5] \psi$ under the SU(2)-linear PSG.
(see Table II), thus the term $\sigma \bar{\psi}i\gamma_3\gamma_5\psi$ is invariant under the full $SU(2)$-linear PSG. We also see that the other three possible couplings $\sigma \bar{\psi}\psi$, $\sigma \bar{\psi}\gamma_0\psi$, and $\sigma \bar{\psi}\gamma_1\gamma_2\psi$ are not invariant under the $SU(2)$-linear PSG and hence are not allowed the effective theory.

Similarly, the potential $V(\sigma)$ should also respect $\sigma \rightarrow -\sigma$ symmetry and take a form

$$V(\sigma) = \frac{m^2}{2}\sigma^2 + \lambda\sigma^4$$  \hspace{1cm} (122)

up to quartic order. There is no cubic term since it breaks the $F^P_x$.

Here we see that PSG tells us the correct form of low energy effective theory:

$$L = \sum_{i=1}^{N_f} \bar{\psi}_i \left( \partial_\mu - i a'_\mu \tau^3 \right) \gamma_\mu \psi_i + \frac{1}{2g^2} \text{Tr} \left[ f_{\mu\nu} f_{\mu\nu} \right]
+ \sigma \bar{\psi}i\gamma_3\gamma_5\psi + \frac{1}{2\mu^2} (\partial_\mu \sigma)^2 + \frac{m^2}{2}\sigma^2 + \lambda\sigma^4$$  \hspace{1cm} (123)

At the mean-field level, we already have the picture for this phase transition, as shown in Fig. 9. We see that when $m^2 > 0$, $\langle \sigma \rangle = 0$, we are in the $SU(2)$-linear phase; when $m^2 < 0$, $\langle \sigma \rangle \neq 0$, we are in the $SU(2)$-chiral phase. $m^2 = 0$ is the phase transition point.

At this level the phase transition is second-order, $\langle \sigma \rangle$ changes continuously from zero to nonzero. The next question is, will this transition be second-order after including quantum fluctuations?

3. The $T$ breaking phase transition does not belong to the Ising class

To answer the above question, we need to count the number of relevant coupling constants at the phase transition fixed point in the renormalization group sense. If there is only one relevant coupling $m^2$, that means the phase transition is indeed second-order, and $m^2 = 0$ is the critical point.

We can estimate the scaling dimension $d_\sigma$ of $\sigma$ field. In tree level, power counting gives us $d_\sigma = \frac{2}{N_f}$. But in large-$N_f$ limit, the fermion dressing changes $d_\sigma$ strongly. The leading order $\sigma$ propagator in $\frac{1}{N_f}$ expansion is shown in Fig.10. the $\sigma$ boson two point function at leading order

$$\begin{array}{c}
\text{FIG. 9: The behavior of potential } V(\sigma) \text{ before(left) and after(right) the phase transition from } SU(2)\text{-linear phase to } SU(2)\text{-chiral phase.}
\end{array}$$

which indicates that the scaling dimension $d_\sigma = 1 + O(\frac{1}{N_f})$. Therefore the scaling dimension of $(\partial\sigma)^2$ and $\sigma^4$ are both $4 + O(\frac{1}{N_f})$, which is larger than space-time dimension 3 and thus irrelevant. As for the gauge coupling $g$, the argument in the end of section VG is still valid. So $\delta g$ is also irrelevant. Now we can safely say that the only relevant coupling is $\frac{m^2}{2}\sigma^2$, whose scaling dimension is $2 + O(\frac{1}{N_f})$.

We can calculate the scaling behavior at the critical point where $\sigma$ boson is also massless. For example, we again compute the staggered spin-spin correlation function. At the critical point, apart from the contribution from massless gauge field in Fig.7, we have the contribution from massless $\sigma$ boson in Fig.11 as well.

So at the critical point, the staggered spin-spin correlation function not only receives an anomalous dimension from gauge field $\gamma = -\frac{16}{\pi^2N_f}$ (Eq.(114)), but also receives an anomalous dimension from the gapless $\sigma$ boson $\gamma'$. Detailed calculation shows that at order of $\frac{1}{N_f}$, $\gamma' = \frac{4}{3\pi^2N_f}$ (See Appendix A).

The total anomalous dimension $\gamma_{\text{total}}$ is the sum of $\gamma$ and $\gamma'$:

$$\gamma_{\text{total}} = \gamma + \gamma' = -\frac{44}{3\pi^2N_f}$$  \hspace{1cm} (125)

Fig.12 shows the change of scaling dimension of staggered spin-spin correlation function during the phase transition. Note that in terms of symmetry breaking, this phase transition is quite normal: it simply breaks the time-reversal, which is a $Z_2$-like symmetry. In our usual understanding of phase transition, the Landau-Ginzburg paradigm, the symmetry determines the universality of the phase transition. Therefore our usual understanding for this phase transition should be a $Z_2$-like or Ising-like phase transition. However, our study shows it is not the
case. Although the phase transition is characterized by the breaking of a $Z_2$-like symmetry, it is obviously not Ising-like. For example, an Ising like transition is gapless only at the critical point, whereas in our transition the $SU(2)$-linear phase is also gapless. The scaling behavior is very different from the Ising-universality. Actually with different value of $N_f$, the scaling exponent can have infinite number of values, which indicates infinite number of universalities.

The above discussion is at zero temperature, but in experiments, people can only measure the system at finite temperature. What people can see in experiments actually should be crossover behavior between disordered phase and quantum critical region, as shown in Fig.13.

\begin{align*}
\alpha &= 4 - \frac{\pi}{2} N \\
\langle (-1)^S(x)S(0) \rangle &= \frac{1}{N^\gamma} \\
\end{align*}

\[ (126) \]

FIG. 12: Change of scaling dimension of staggered spin-spin correlation function during phase transition.

\begin{align*}
\eta \tau_1 &= +i \chi \\
\eta \tau_2 &= i (-1)^y \chi \\
\chi &= +i \chi \\
\end{align*}

FIG. 13: At finite temperature $T$, system shows crossover behavior between $SU(2)$-linear phase and quantum critical region (dashed line). But the $SU(2)$-chiral phase and quantum critical region are still separated by a phase transition since there is a physical symmetry breaking (solid line).

\begin{align*}
\eta \tau_1 &= +i \chi \\
\eta \tau_2 &= i (-1)^y \chi \\
\chi &= +i \chi \\
\end{align*}

\[ (126) \]

FIG. 14: $SU(2)$-linear phase $Z_2$-linear phase

\begin{align*}
\eta \tau_1 &= +i \chi \\
\eta \tau_2 &= i (-1)^y \chi \\
\chi &= +i \chi \\
\end{align*}

\[ (126) \]

FIG. 15: The 4 fermi points of $Z_2$-linear state.

B. Phase transition between $SU(2)$-linear phase and $Z_2$-linear phase.

1. $Z_2$-linear phase.

In this section we study the phase transition between $SU(2)$-linear phase (A in phase diagram Fig.1) and $Z_2$-linear phase (G in Fig.1). The following is the ansatz of the $Z_2$-linear state (Fig.14).

\begin{align*}
\bar{u}_i, i + x &= i \chi, \\
\bar{u}_i, i + y &= i (-1)^y \chi \\
\bar{u}_i, i + x + y &= i \eta \tau_1, \\
\bar{u}_i, i + x + y &= i \eta \tau_2. \\
\end{align*}

The fermion energy spectrum of $Z_2$-linear state at mean-field level is characterized by 4 fermi points as shown in Fig.15.

The low energy effective theory is the massless fermion coupling to $Z_2$ gauge field. At mean-field level where we do not include gauge fluctuation yet, after taking continuous limit, we have

\[ \bar{L}_{\text{mean}} = \bar{\psi} \partial_i \gamma_\mu \psi + \bar{\psi} [i \eta \gamma_1 \gamma_5 + \eta^2 i \gamma_2 \gamma_3] \psi \]  

Once again we need to argue that the $Z_2$-linear state is a stable phase. Here by stable we still mean that the low energy behavior is not changed after including quantum fluctuations. Similar to what we did in Section V, we can discuss the stability of $Z_2$-linear state.

Through a PSG analysis we can show that the fermion bilinear term is not allowed and the fermions remain massless after including quantum fluctuation[12]. Therefore the PSG protects the masslessness of fermion in the
$Z_2$-linear state. In addition, $Z_2$ gauge fluctuation is always gapped. For pure $Z_2$ gauge theory on lattice in $2+1$ dimension, it can be in deconfined phase or confined phase[36]. Here we have massless fermion coupling to $Z_2$ gauge field, fermion dressing effect should drive the system even more likely to be deconfined. We assume that the gauge field is deconfined, then the gapped $Z_2$ gauge fluctuation should be irrelevant. Therefore $Z_2$-linear state is also a stable phase.

In Eq.(127), $\eta^1$ and $\eta^2$ describe the fluxes through the red and blue triangle plaquettes in Fig.14. Since the two fluxes are not colinear, the $SU(2)$ gauge group breaks down to $Z_2$ gauge group. In Eq.(127), one can say that the $\eta$ field is driving the phase transition. When $\langle \eta \rangle = 0$, we go back to $SU(2)$-linear phase, and when $\langle \eta \rangle \neq 0$, we are in $Z_2$-linear phase.

We notice that $\eta$ field is not gauge invariant. After a local $SU(2)$ gauge transformation, the direction of $\tau^1$ and $\tau^2$ in Eq.(127) will be rotated. Thus we should use a vector field $\vec{n}$ to describe the fluctuations of $\eta$. Furthermore, since the $\eta$ in front of $\gamma_1\gamma_5$ and the $\eta$ in front of $\gamma_2\gamma_3$ should be able to fluctuate independently, we should have two vector fields, say $\vec{n}_1$ and $\vec{n}_2$, to describe them. So the gauge invariant way to write Eq.(127) would be:

$$L_{\text{mean}} = \bar{\psi} \partial_{\mu} \gamma_{\mu} \psi + \bar{\psi} \left[ (\vec{n}_1 \cdot \vec{\tau}) i\gamma_1\gamma_5 + (\vec{n}_2 \cdot \vec{\tau}) i\gamma_2\gamma_3 \right] \psi$$

(128)

where $\vec{n}_1$ and $\vec{n}_2$ transform as the adjoint representation of $SU(2)$ gauge group. Now we include dynamics of $\vec{n}$ fields. To have controlled calculation, we introduce $N_f$ flavors of fermions, too. The following is the low energy effective theory of the system:

$$L = \sum_{i=1}^{N_f} \bar{\psi}_i \partial_{\mu} \gamma_{\mu} \psi_i + \frac{1}{2\kappa^2} \left( (D_{\mu} \vec{n}_1)^2 + (D_{\mu} \vec{n}_2)^2 \right)$$

$$+ \sum_{i=1}^{N_f} \bar{\psi}_i \left[ (\vec{n}_1 \cdot \vec{\tau}) i\gamma_1\gamma_5 + (\vec{n}_2 \cdot \vec{\tau}) i\gamma_2\gamma_3 \right] \psi_i + V(\vec{n}_1, \vec{n}_2)$$

(129)

where $D_{\mu}$ is the covariant derivative of $SU(2)$ gauge theory, and the form of potential $V(\vec{n}_1, \vec{n}_2)$ is unknown yet.

The phase transition is described by a Higgs mechanism, $\vec{n}_1$ and $\vec{n}_2$ are Higgs bosons. When $\langle \vec{n}_1 \rangle = \langle \vec{n}_2 \rangle = 0$, we are in the $SU(2)$-linear phase; when $\langle \vec{n}_1 \rangle \neq 0$, $\langle \vec{n}_2 \rangle \neq 0$ and $\langle \vec{n}_1 \rangle \perp \langle \vec{n}_2 \rangle$, we are in the $Z_2$-linear phase.

2. The low energy effective theory from PSG consideration

What is the symmetry that the low energy effective Lagrangian Eq.(129) should respect? Again it should be the full symmetry described by the PSG of the $SU(2)$-linear state. We simply need to review Table II again to see how $\vec{n}$ fields transform under PSG. For example:

$$T_x^{\text{PSG}}: \bar{\psi}_1 \gamma_1 \gamma_5 |\psi_1| \rightarrow -\bar{\psi}_1 \gamma_1 \gamma_5 |\psi_1|$$

(130)

$$T_y^{\text{PSG}}: \bar{\psi}_1 \gamma_1 |\psi_1| \rightarrow \bar{\psi}_1 \gamma_1 |\psi_1|$$

(131)

$$P_x^{\text{PSG}}: \bar{\psi}_1 \gamma_2 \gamma_3 |\psi_1| \rightarrow -\bar{\psi}_1 \gamma_2 \gamma_3 |\psi_1|$$

(132)

To have the term $\bar{\psi}_1 (\vec{n}_1 \cdot \vec{\tau}) i\gamma_1\gamma_5 + (\vec{n}_2 \cdot \vec{\tau}) i\gamma_2\gamma_3 |\psi_1|$ in Eq.(129) invariant under PSG, we should have:

$$T_x^{\text{PSG}}: \vec{n}_1 \rightarrow -\vec{n}_1$$

(133)

$$T_y^{\text{PSG}}: \vec{n}_2 \rightarrow \vec{n}_2$$

(134)

$$P_x^{\text{PSG}}: \vec{n}_1 \rightarrow -\vec{n}_2$$

(135)

In summary, the following three transformations of $\vec{n}$ should be the symmetry of the potential $V(\vec{n}_1, \vec{n}_2)$:

$$\vec{n}_1 \rightarrow -\vec{n}_1 \quad \vec{n}_2 \rightarrow -\vec{n}_2 \quad \vec{n}_1 \leftrightarrow \vec{n}_2$$

(136)

which strongly constrains the form of potential $V(\vec{n}_1, \vec{n}_2)$. The only allowed gauge invariant form of $V(\vec{n}_1, \vec{n}_2)$ up to quartic order is:

$$V(\vec{n}_1, \vec{n}_2) = \frac{1}{2} m^2 (\vec{n}_1^2 + \vec{n}_2^2) + a(\vec{n}_1^4 + \vec{n}_2^4) + b(\vec{n}_1^2 \vec{n}_2^2) + c(\vec{n}_1 \cdot \vec{n}_2)^2$$

(137)

Terms like $\vec{n}_1 \cdot \vec{n}_2$ and $(\vec{n}_1 \cdot \vec{n}_2)(\vec{n}_1)^2$ are forbidden since they break PSG.

3. A phase transition with no breaking symmetry.

One can show that the two phases involved in the phase transition, the $SU(2)$-linear phase and the $Z_2$-linear phase, have different PSGs. In [12], PSG of the $SU(2)$-linear phase was labelled as $SU(2)Bn$, whereas PSG of $Z_2$-linear phase was labelled as $Z2Azz$. But after projection, the physical symmetries of the two phases are identical. They both have the full symmetry of translation, rotation and time-reversal:

Physical symmetry of $SU(2)$-linear and $Z_2$-linear states

$$= \{T_x, T_y, P_x, P_y, P_{xy}, T, C\}$$

(138)

We are investigating a phase transition with no breaking of physical symmetry. Here the introduction of quantum order, or PSG is inevitable. Otherwise we do not know what is changed during the phase transition.

At mean-field level, we already have the picture for the phase transition. With different values of coupling
FIG. 16: Various Higgs condensed phases with different values of $a$, $b$, and $c$. In phase I, $|\langle \vec{n}_1 \rangle| = |\langle \vec{n}_2 \rangle|$ and $\langle \vec{n}_1 \rangle \perp \langle \vec{n}_2 \rangle$, it is the $Z_2$-linear phase. In phase II, $\langle \vec{n}_1 \rangle = 0$ and $\langle \vec{n}_2 \rangle \neq 0$ or vice versa. It is a $U(1)$-linear phase which breaks translation and rotation symmetry. In phase III, $|\langle \vec{n}_1 \rangle| = |\langle \vec{n}_2 \rangle|$ and $\langle \vec{n}_1 \rangle \parallel \langle \vec{n}_2 \rangle$. It is another $U(1)$-linear phase which breaks translation and rotation symmetry.

constant in potential $V(\vec{n}_1, \vec{n}_2)$, the Higgs bosons $\vec{n}_1, \vec{n}_2$ may or may not condense. If they do not condense, we are in the $SU(2)$-linear phase. If they condense in such a fashion that $\langle \vec{n}_1 \rangle \parallel \langle \vec{n}_2 \rangle$, we are in the $Z_2$-linear phase. Detailed study of the potential shows that if $m^2 > 0$, Higgs bosons do not condense. If $m^2 < 0$, Higgs bosons do condense, and the way of condensation is determined by the value of parameters $a, b$, and $c$ as shown in Fig. 16. There are three different Higgs condensed phases, labelled by I, II, and III.

Our $Z_2$-linear phase is phase I. On the mean-field level, the phase transition from the $SU(2)$-linear phase to the $Z_2$-linear phase can be described by changing $m^2$ from positive to negative, and $m^2 = 0$ is the phase transition point.

The next question is whether this mean-field picture survives after inclusion of quantum fluctuations. If there is only one relevant coupling constant $m^2$, our mean-field picture remains valid, otherwise it will fail. We now estimate the scaling dimension of $\vec{n}$ field. Again the fermion dressing is strong in the large-$N_f$ limit. Similar to our argument for $\sigma$ boson in section VII A 3, we know that the scaling dimension of $\vec{n}$ field is $d_\vec{n} = 1 + O(1/N_f)$. Therefore by power counting, the terms $(\partial_\mu \vec{n})^2$ and $\vec{n}^4$(those $a, b, c$ terms) are both of scaling dimension $4 + O(1/N_f)$, which are irrelevant. However since the Lagrangian Eq.(129) is not Lorentz invariant, calculating the $1/N_f$ correction to $d_\vec{n}$ would be complicated.

We have just concluded that there is only one relevant coupling $m^2$, so the phase transition is second-order and $m^2 = 0$ is the critical point. Although $a, b, c$ couplings are irrelevant at the critical point, they are important to determine which Higgs condensed phase the system would end up. Therefore they are dangerous irrelevant couplings. This can be seen from Fig.17. Although the critical theories for the phase transition from the $SU(2)$-linear phase to Higgs condensed phases are the same, the system may change into different Higgs condensed phases depending the values of couplings $a, b, c$. Different Higgs phases separate from each other by first order transition boundaries.

The transition from the $SU(2)$-linear state to the $Z_2$-linear state is a phase transition without breaking of any symmetry. What are the changes in physically measurable quantities during the phase transition? Let us think about the staggered spin-spin correlation function again. On both sides of the phase transition, the fermions are massless so the spin-spin correlation functions are of power law. But the values of power are different. As shown in Fig.18, in the $Z_2$-linear phase, since $Z_2$ gauge field is gapped, spin-spin correlation does not receive anomalous dimension. At the critical point, due to the existence of the massless Higgs fields, the correlation function will have another scaling exponent.

4. Phase transition from spin liquids to ordered phases

Spin liquids phases can also experience phase transitions into ordered phases. Our discussion of phase transition between $SU(2)$-linear phase and chiral spin liquid is an example where the time-reversal is broken, but no
space translation or spin rotation symmetries are broken. Here we focus on the phase transition from spin liquids to phases breaking space translation or spin rotation symmetries. For example, it can go into ferromagnetic phase, anti-ferromagnetic phase or VBS phase.

Suppose our starting point is the $SU(2)$-linear phase (or $\pi$-flux phase). Table IV is very useful. For example, a phase transition from $SU(2)$-linear phase to Neel phase can be easily understood as the opening of a mass gap $\psi \psi$. Then because at energy scale below the mass gap there is no fermion, we are left with a pure $SU(2)$ gauge field which is confined in 2+1 dimension. This phase transition then have two ingredients in it: fermion-chiral symmetry breaking and confinement. Because of the confinement effect the spinons are always bounded together and do not appear in physical excitations.

Similarly one can study the phase transition from $SU(2)$-linear phase to VBS order and ferromagnetic order. The features for these phase transitions are all similar: opening of a mass gap and the confinement.

VIII. CONCLUSION

In this article, we studied the stability of various spin liquids, including gapless spin liquid (also called algebraic spin liquid). Spin liquids are defined as the disordered phases of a spin system. They have the full space-time translation and spin rotation symmetries. Different spin liquids may have the same physical symmetry. To understand the physics of these phases, first one needs to understand why they are different despite they have the same symmetry. That means we have to classify quantum states in greater detail than those achieved through usual symmetry group. This is the main motivation to introduce the idea of quantum order. The PSG is just one attempt to characterize quantum order mathematically.

We find that PSG is very important in understanding the stability of algebraic spin liquid. Without PSG, it is hard to understand why fermions can remain massless after including fluctuations around the mean-field state. After considering certain PSG transformations originated from lattice symmetry, we find that such PSG transformations turns into chiral symmetry in continuous limit, which guarantees the masslessness of fermions. Using this idea, we show the existence of an algebraic spin liquid – $SU(2)$-linear state – whose low energy effective theory is a QCD$_3$ with $SU(2)$ gauge group. The spin-spin correlation function is also calculated. We find that the $SU(2)$-linear state has a large emergent symmetry – $Sp(4) \times \text{Lorentz} \times T \times P_x \times P_y$ (or $Sp(4N) \times \text{Lorentz} \times T \times P_x \times P_y$ for the large $N$ model) – at low energies. The lattice model does contain terms that violate the $Sp(4) \times \text{Lorentz}$ Group symmetry. But all those terms are shown to be irrelevant with the help of the PSG analysis.

We also discussed the continuous quantum phase transition between spin liquids. Again, the PSG plays a key role here. The first transition we studied is a quantum phase transition that breaks time reversal and parity symmetries, which is the transition between the $SU(2)$-linear state and the $SU(2)$-chiral state. Such a $Z_2$ symmetry breaking transition has a well defined order parameter. However, as one can see from the calculated critical exponents, the critical point at the transition does not belong to the Ising universality class. It is interesting to see that even some symmetry breaking continuous phase transitions are beyond Landau’s symmetry breaking paradigm in the sense that critical properties are different for those obtained from Ginzburg-Landau theory.

The second transition that we studied is a continuous quantum phase transition between the $SU(2)$-linear state and the $Z_2$-linear state. The two states have the same symmetry. Hence we show that continuous phase transitions even exist between two phases with the same symmetry[19–23].

The third transition that we studied is a continuous quantum phase transition between the $SU(2)$-linear state and a $U(1)$-linear state. Such a transition breaks the lattice translation and lattice rotation symmetry. Amazingly, we found that the third transition and the second transition are described by the same critical theory with the same set of critical exponents. So it is possible for transitions between very different states to have the same critical point.

All above phenomena are beyond Landau-Ginzburg paradigm for phase and phase transition. Those discoveries suggest that we need rethink Landau-Ginzburg symmetry breaking approach to phase and phase transition. We know that stable phases and critical points can all be viewed as fixed point in the renormalization group sense. If a fixed point has no relevant operator that is allowed by the symmetry (or PSG), the fixed point will represent a stable phase. If a fixed point has only one relevant operator that is allowed by the symmetry (or PSG), the fixed point will represent a critical point between two phases. In this paper, we found that one cannot use symmetry to characterize all the possible fixed points. New kind of order beyond the symmetry description exists. We showed that how to use the PSG analysis to capture the new physics beyond Landau-Ginzburg symmetry breaking paradigm.

The phase transitions studied here are characterized by a change of quantum order in addition to a possible change of symmetry. This is why those phases and phase transitions are beyond Landau-Ginzburg paradigm of breaking symmetry. We emphasize that quantum order, or PSG, is necessary to understand the correct low energy effective theory and the critical phenomena. Also, an experimental discovery of a new critical point (with unusual critical exponents) implies a discovery of new quantum orders. Thus it is very important to measure critical exponents even for seemingly ordinary symmetry breaking transitions.

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APPENDIX A: DETAILED CALCULATION OF ANOMALOUS DIMENSION $\gamma$

The scaling dimension of staggered spin-spin correlation function $\langle (-)^a S(x) S(y) \rangle$ is calculated by the large-$N_f$ expansion of quantum field theory. In our formalism, one can show that the staggered spin-spin correlation function is just the fermion mass operator $\langle \psi(x) \bar{\psi}(y) \rangle$ correlation function in the effective theory Eq.(119). By power counting, the scaling behavior should change into $g^2 \delta_{ab} \frac{1}{k^2}$, but quantum fluctuations change it into $\langle \psi(x) \bar{\psi}(y) \rangle$, where $\gamma_{\psi\bar{\psi}}$ is called the anomalous dimension of fermion mass operator. It turns out that the easiest way of calculating $\gamma_{\psi\bar{\psi}}$ is not to calculate $\langle \psi(x) \bar{\psi}(y) \rangle$ directly, but to calculate the correlation function of fermion field $\psi$: $\langle \psi(x) \bar{\psi}(y) \rangle$, and the three-point correlation function $\langle \bar{\psi}(x) \psi(y) \bar{\psi}(0) \rangle$.

Let us firstly calculate the staggered spin-spin correlation function in $SU(2)$-linear phase, where the low energy effective theory is Eq.(40). We need to understand the gauge interaction. In the large-$N_f$ limit, the gauge field is strongly screened by fermions, and under renormalization group the coupling $g$ will flow to an IR stable conformal invariant fixed point $g^2 \sim \frac{1}{N_f}$. Here $\Lambda$ is the UV cut-off of our theory. To the leading order of $\frac{1}{N_f}$, the dressed gauge propagator is as Fig.19. Let us work within Euclidean space and Landau gauge, where the bare gauge propagator is:

$$G_{\mu\nu,\text{bare}}(x, y) = \langle A_{\mu}^a(x) A_{\nu}^b(y) \rangle = \int \frac{dk^3}{(2\pi)^3} \frac{\delta_{\mu\nu}}{k^2} \frac{g^2 k_{a} k_{b}}{k^2}$$  \hspace{1cm} (A1)

The bare fermion propagator is:

$$\langle \psi_i(x) \bar{\psi}_j(y) \rangle = \int \frac{dp^3}{(2\pi)^3} e^{ip(x-y)} \frac{-ip\delta_{ij}}{p^2} \hspace{1cm} (A2)$$

where $i, j$ label the gauge components. The dressed gauge propagator can be calculated as:

$$G_{\mu\nu,\text{dressed}}(k) = \frac{g^2 \delta_{\mu\nu}}{k^2(1 + \Pi)} \left( \delta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right)$$  \hspace{1cm} (A3)

where if we do Pauli-Villar regularization,

$$(k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}) \Pi = N_f g^2 T_F \int \frac{dq^3}{(2\pi)^3} \frac{\text{Tr} \left[ \gamma_{\mu} \gamma_{\nu} (\bar{\psi} \gamma_{-} \psi) \right]}{q^2(q - k)^2}$$

$$= (k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}) N_f g^2 T_F \left( \frac{1}{8k} \frac{1}{6\pi \Lambda} \right)$$  \hspace{1cm} (A4)

in which

$$\text{Tr} \left[ \tau^a \tau^b \right] = T_F \delta^{ab}$$

At the fixed point, where $g^2 = \frac{\delta^a A}{N_f T_F}$, the dressed gauge propagator is:

$$G_{\mu\nu,\text{dressed}}(k) = \frac{8 \delta^{ab}}{N_f T_F k} \left( \delta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2} \right)$$  \hspace{1cm} (A6)

we now study the fermion correlation function with first order correction in $\frac{1}{N_f}$ expansion, as shown in Fig.20. The dressed fermion propagator is:

$$S_{ij}(k) = \frac{-i \delta_{ij}}{k^2} (1 + \Sigma)$$  \hspace{1cm} (A7)

where

$$\rho \Sigma = i \int \frac{dq^3}{(2\pi)^3} \frac{\gamma_{\mu} (-i)(\bar{\psi} \gamma_{+} \psi) C_F 8}{(k + q)^2 N_f T_F q} \frac{\delta_{\mu\nu}}{q^2}$$

$$= - \rho \frac{8 C_F}{3 \pi^2 N_f T_F} \log \frac{\Lambda}{m}$$  \hspace{1cm} (A8)

and

$$\tau^a \tau^a = C_F I$$

Thus we know that the anomalous dimension of $\psi$ is:

$$\gamma_{\psi} = - \frac{1}{2} \frac{8 C_F}{3 \pi^2 N_f T_F}$$  \hspace{1cm} (A10)

Then we look at the dressed three-point correlation function $\langle \bar{\psi}(x) \psi(y) \bar{\psi}(0) \rangle$ at the order of $\frac{1}{N_f}$, as shown in Fig.21. Suppose we fix the momentum of $\bar{\psi} \psi$ to be $2k$, while $\psi$ and $\bar{\psi}$ each carry momentum $k$, then the tree level three point correlation function will be $G_3(2k, k, k) = \frac{-ik - i(-k)}{k^2} = \frac{1}{k^2}$. From the contributions of diagrams in
FIG. 21: Gauge dressed three point correlation function at order of $\frac{1}{N_f}$.

FIG. 22: The contribution of $\sigma$-boson to fermion propagator at order of $\frac{1}{N_f}$, where the double dashed line is the dressed $\sigma$-boson propagator at leading order.

Fig.21, we will have the dressed three point correlation function:

$$G_3(2k, k, k) = \frac{1}{k^2} \left( 1 + (A + B + C) \log \frac{k}{\Lambda} \right)$$  \hspace{1cm} (A11)

where $A, B, C$ are the contributions from each corresponding diagram. Actually we know that $A + B + C = \gamma \bar{\psi} \psi + 2 \gamma \psi$. So by calculating $A + B + C$, we will know the anomalous dimension of fermion mass operator $\gamma \bar{\psi} \psi$.

It is easy to see that $A, B$ are just from the dressed fermion propagator, which has been calculated above: $A = B = 2 \gamma \psi$. New calculation needs to be done for vertex correction in $C$.

$$C \log \left( \frac{k}{\Lambda} \right)$$

where the second term comes from the contribution of massless $\sigma$-boson. The change of scaling behavior during this phase transition is shown in Fig.12.

Now we can compute $\gamma \bar{\psi} \psi$:

$$\gamma \bar{\psi} \psi = A + B + C - 2 \gamma \psi = C + 2 \gamma \psi$$

$$= - \frac{32C_F}{3\pi^2 N_f T_F} = - \frac{16}{\pi^2 N_f}$$  \hspace{1cm} (A13)

We can also calculate the spin-spin correlation function at the critical point in a similar fashion. The only difference is that the $\sigma$ boson becomes massless at critical point and contributes to the anomalous dimension of correlation functions. The contribution of $\sigma$ boson to fermion propagator and three-point correlation function are shown in Fig.22 and Fig.23. After similar calculation, we find that at the critical point,

$$\gamma \bar{\psi} \psi = - \frac{16}{\pi^2 N_f} + \frac{4}{3\pi^2 N_f}$$  \hspace{1cm} (A14)

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If the symmetry group of one phase is not a subgroup of the other phase and vice versa, then the two phases are said to have incompatible symmetries. According to Landau’s symmetry breaking theory, two phases with incompatible symmetries cannot have continuous phase transition between them.