A note on harmonic continuation of characteristic function

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Abstract We propose a necessary and sufficient condition for a real-valued function on the real line to be a characteristic function of a probability measures. The statement is given in terms of harmonic functions and completely monotonic functions.

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1 Introduction

If $\sigma$ is a probability measure on the real line $\mathbb{R}$, then

$$\Theta(x) = \int_{-\infty}^{\infty} e^{ixt} d\sigma(t)$$

is called the characteristic function of the probability measure $\sigma$. We will call $\Theta$ the characteristic function for short. Note that a characteristic function is real-valued if and only if it is even on $\mathbb{R}$.

The motivation of the present paper stems from Egorov’s paper [2, p. 567], where the following criterion for real-valued infinitely differentiable characteristic functions is given:

**Theorem 1**. Let $\varphi(t)$ be an infinitely differentiable absolutely integrable even function on $\mathbb{R}$ with real values. Let $\varphi(t)$ satisfy the condition $\varphi^{(m)}(t)t^{-2} = O(1)$ as $t \to \infty$, $m = 1, 2, \ldots$, and let the Fourier transform $\hat{\varphi}$ belong to $L^1(\mathbb{R})$. Then $\varphi(t)$ is a characteristic function if and only if

1) for all $0 < \delta < \infty$ and all $p = 0, 1, 2, \ldots$

$$(-1)^p \int_0^{\infty} \frac{1}{\delta + t^2} \varphi^{(2p)}(t) \, dt \geq 0,$$ 

$$(-1)^{p+1} \int_0^{\infty} \frac{t}{\delta + t^2} \varphi^{(2p+1)}(t) \, dt \geq 0;$$

2) $\varphi(0) = 1$.

We claim that some conditions of Theorem 1 are unnecessary. Furthermore, this criterion cannot be applied to the simplest infinitely differentiable characteristic functions $\varphi(t) \equiv 1$ and $\varphi(t) =
cos at, where $a \in \mathbb{R}$ (since they are not absolutely integrable on $\mathbb{R}$). Therefore, we will follow here a different approach to a similar criterion associated with the theory of harmonic functions. This approach allows us to simplify and strengthen the statement of Theorem 1; compare Corollary 3 below.

Let us start by introducing some notation and basic facts that will be used throughout this paper. Let $CB(\mathbb{R})$ denote the Banach space of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the usual uniform norm. If $f \in CB(\mathbb{R})$, then, as is well known (see, for example [5, Chapter II]), the Dirichlet problem for the upper half plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ has only unique solution in $CB(\overline{\mathbb{R}^2_+})$, where $\overline{\mathbb{R}^2_+}$ is the closure of $\mathbb{R}^2_+$. More precisely, given $f \in CB(\mathbb{R})$, there exists a function $u_f$ on $\mathbb{R}^2_+$ such that $u_f$ is bounded and continuous in $\mathbb{R}^2_+$, and

$$
\lim_{y \to 0} u_f(x,y) = f(x)
$$

(1.1)

for each $x \in \mathbb{R}$, and $u_f$ is harmonic in $\mathbb{R}^2_+$. This means that $u_f$ is infinitely differentiable in $\mathbb{R}^2_+$ and there satisfies

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_f = 0.
$$

We call $u_f$ the harmonic continuation of $f$ into $\mathbb{R}^2_+$. The solution of the Dirichlet problem for $\mathbb{R}^2_+$ can be obtained by using the Poisson kernel

$$
P(x,y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.
$$

(1.2)

Namely, the harmonic continuation $u_f$ can be obtained as a convolution with respect to $x$ of $f$ and $P(x,y)$

$$
u_f(x,y) = \left( f * P \right)(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} f(t) \, dt.
$$

(1.3)

where $(x,y) \in \mathbb{R}^2_+$.

We recall that a function $\omega : [0, \infty) \to \mathbb{R}$ is called completely monotonic if it is infinitely differentiable on $[0, \infty)$ and

$$
(-1)^n \omega^{(n)}(x) \geq 0
$$

(1.4)

for $x \in (0, \infty)$ and all $n = 0, 1, 2, \ldots$. The following elementary functions are immediate examples of completely monotonic functions, which is verified directly:

$$
e^{-ax}, \quad \frac{1}{(\alpha x + \beta)^p}, \quad \text{and} \quad \ln \left( b + \frac{c}{x} \right),
$$

where $a \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, and $p \geq 0$ with $\alpha$ and $\beta$ not both zero and $b \geq 1$, $c > 0$.

**Theorem 2.** Suppose that $f \in CB(\mathbb{R})$ is an even function and $f(0) = 1$. Then $f$ is a characteristic function if and only if the restriction of $u_f$ to the imaginary axis, i.e., the function $y \to u_f(0, y)$, $y \in [0, \infty)$, is completely monotonic.
Corollary 3. Let \( f \in CB(\mathbb{R}) \). Suppose

(i) \( f \) is infinitely differentiable on \( \mathbb{R} \) and \( f^{(k)} \in CB(\mathbb{R}) \) for all \( k = 1, 2 \ldots \);
(ii) \( f \) is even and \( f(0) = 1 \).

Then \( f \) is characteristic function if and only if

\[
\int_{-\infty}^{\infty} \Im \left( \frac{1}{t - iy} \right) f(t) \, dt \geq 0, \tag{1.5}
\]

\[
(-1)^{n+1} \int_{-\infty}^{\infty} \Re \left( \frac{1}{(t - iy)^2} \right) f^{(2n)}(t) \, dt \geq 0 \tag{1.6}
\]

and

\[
(-1)^n \int_{-\infty}^{\infty} \Im \left( \frac{1}{(t - iy)^2} \right) f^{(2n+1)}(t) \, dt \geq 0 \tag{1.7}
\]

for \( y > 0 \) and all \( n = 0, 1, 2, \ldots \).

2 PRELIMINARIES AND PROOFS

A complex-valued function \( \varphi \) on \( \mathbb{R} \) is said to be positive definite if

\[
\sum_{i,j=1}^{n} \varphi(x_i - x_j) c_i \overline{c_j} \geq 0
\]

for every choice of \( x_1, \ldots, x_n \in \mathbb{R} \), for every choice of complex numbers \( c_1, \ldots, c_n \in \mathbb{C} \), and all \( n \in \mathbb{N} \). The Bochner theorem (see [3, p. 150]) characterizes continuous positive definite functions: a continuous function \( \varphi : \mathbb{R} \to \mathbb{C} \) is positive definite if and only if there exists a finite non-negative measure \( \mu \) on \( \mathbb{R} \) such that

\[
\varphi(x) = \int_{-\infty}^{\infty} e^{ixt} \, d\mu(t).
\]

Note that \( \varphi \) is a characteristic function if and only if \( \varphi \) is continuous positive definite and \( \varphi(0) = 1 \). Any characteristic function \( \varphi \) satisfies:

(i) \( \varphi \) is uniformly continuous on \( \mathbb{R} \);
(ii) \( \varphi \) is bounded on \( \mathbb{R} \), i.e., \( |\varphi(x)| \leq \varphi(0) = 1 \) for all \( x \in \mathbb{R} \);
(iii) The real part of \( \theta(x) \) is also an even characteristic function.

In the case of completely monotonic functions, the Bernstein-Widder theorem (see [7, p. 161]) asserts that a function \( f : [0, \infty) \to \mathbb{R} \) is completely monotonic if and only if it is the Laplace transform of a finite non-negative measure \( \eta \) supported on \( [0, \infty) \), i.e.,

\[
\varphi(x) = \int_0^{\infty} e^{-xt} \, d\eta(t) \tag{2.1}
\]

for \( x \in [0, \infty) \).

Let \( f \in CB(\mathbb{R}) \). It is well known that harmonic continuation (1.3) has harmonic conjugate. Recall that if \( U \) is a harmonic function in \( \mathbb{R}^2_+ \), then another function \( V \) harmonic in \( \mathbb{R}^2_+ \) is called
harmonic conjugate of $U$, provided $U + iV$ is analytic in $C_+ = \{ z = x + iy \in \mathbb{C} : y > 0 \}$. The harmonic conjugate $V$ is unique, up to adding a constant. Let $v_f$ denote a harmonic conjugate of $u_f$. We can choose (see [4, p.p. 108-109])

$$v_f(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x-t)^2 + y^2} f(t) \, dt,$$

where $x \in \mathbb{R}$ and $y > 0$. Then

$$E_f(z) = E_f(x,y) = u_f(x,y) + iv_f(x,y)$$

(2.3)
is analytic in $C_+$.

We note that usually the following simpler integral

$$\tilde{v}_f(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x-t)^2 + y^2} f(t) \, dt$$

(2.4)
chosen as the harmonic conjugate of $u_f$. In that case the analytic function $u_f + i\tilde{v}_f$ coincides with the usual Cauchy type integral

$$(u_f + i\tilde{v}_f)(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} f(t) \, dt$$
for $z \in C_+$. However, the integral in (2.4) converges as long as

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|} \, dt < \infty.$$
For example, this condition is satisfied if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, but not in the case of an arbitrary $f \in CB(\mathbb{R})$.

**Proof of Theorem** Suppose that $f$ is a real-valued characteristic function. By the Bochner theorem, there exists a probability measure $\sigma$ on $\mathbb{R}$ such that

$$f(t) = \int_{-\infty}^{\infty} e^{ixt} \, d\sigma(x).$$
(2.5)
For each $y > 0$ the Poisson kernel (1.2) is a Lebesgue integrable function of $x$ on $\mathbb{R}$. Hence, by the Fubini theorem, it follows from (1.3) and (2.5) that

$$u_f(0,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} P(t,y) e^{ixt} \, dt \right) \, d\sigma(x).$$

Since

$$\int_{-\infty}^{\infty} P(t,y) e^{ixt} \, dt = e^{-|x|y},$$
it follows that

$$u_f(0,y) = \int_{-\infty}^{\infty} e^{-|x|y} \, d\sigma(x).$$
(2.6)
For each $y > 0$ and any $n = 0, 1, 1, 2, \ldots$, there exists $0 < a(y,n) < \infty$ such that

$$\sup_{x \in \mathbb{R}} \left| \frac{\partial^n}{\partial y^n} e^{-|x|y} \right| \leq a(y,n).$$
Therefore, if we recall that $\sigma$ is a finite measure, then we have that the partial derivatives of $u_f(0,y)$ in $y$ can be obtained by differentiation under the integral sign in (2.6) (see, for example, [1, p. 283]). Now the positiveness $\sigma$ implies that

$$(-1)^n \left( \frac{\partial^n}{\partial y^n} u_f \right)(0, y) = \int_{-\infty}^{\infty} |x|^n e^{-|x|y} d\sigma(x) \geq 0$$

for all $y > 0$ and all $n = 0, 1, 2, \ldots$. Note that the harmonic function $u_f(x, y)$ is continuous in $\mathbb{R}^2_+$. Therefore, by the Bernstein-Widder theorem, we obtain that $u_f(0, y)$ is completely monotonic on $[0, \infty)$.

Suppose $u_f(0, y)$ is completely monotonic for $y \in [0, \infty)$. By the Bernstein-Widder theorem, there exists a finite non-negative measure $\eta$ supported on $[0, \infty)$ such that

$$u_f(0, y) = \int_0^{\infty} e^{-yt} d\eta(t), \quad (2.7)$$

$y \in [0, \infty)$. According to (1.1), we get

$$\lim_{y \to 0} u_f(0, y) = f(0) = 1. \quad (2.8)$$

This means that $\eta$ is a probability measure. Set

$$K_\eta(z) = \int_0^{\infty} e^{izt} d\eta(t) \quad (2.9)$$

for $z \in \mathbb{C}_+ = \{ z = x + iy \in \mathbb{C} : y \geq 0 \}$. We can consider the measure $\eta$ as tempered distribution on $\mathbb{R}$. Then $K_\eta$ is the distributional Laplace transform of $\eta$ (see [6, p. 127]). Therefore, since $\eta$ is supported on $[0, \infty)$, we have that $K_\eta$ is analytic function in $\mathbb{R} + i(0, \infty) = \mathbb{C}_+$ and the following differentiation formula in $\mathbb{C}_+$ holds

$$\frac{d^n}{dz^n} K_\eta(z) = (i)^n \int_0^{\infty} e^{izt} \left( t^n e^{-yt} \right) d\eta(t),$$

[6, p.p. 127-128].

We claim that $K_\eta$ and (2.3) coincide in $\mathbb{C}_+$. Indeed, since $f$ is even on $\mathbb{R}$, it follows from (2.2) that $v_f(0, y) = 0$ for each $y > 0$. Hence

$$E_f(iy) = u_f(0, y) \quad (2.10)$$

for $y > 0$. On the other hand, using (2.7) and (2.9), we have

$$K_\eta(iy) = u_f(0, y), \quad y > 0.$$ 

If we combine this with (2.10) and use the uniqueness theorem for analytic functions, we obtain that $E_f = K_\eta$ in $\mathbb{C}_+$.

By (2.9), we have that for any fixed $y_0 \in (0, \infty)$ the function $x \to K_\eta(x + iy_0)$ is positive definite for $x \in \mathbb{R}$. Therefore the function $x \to \Re[K_\eta(x + iy_0)]$ also is positive definite for $x \in \mathbb{R}$. Since $E_f = K_\eta$ in $\mathbb{C}_+$, it follows from (2.3) that

$$u_f(x, y_0) = \Re[E_f(x, y_0)] = \Re[K_\eta(x + iy_0)]$$

for $x, y_0$.
for $x \in \mathbb{R}$. Thus each function $x \rightarrow u_f(x, y_0)$, where $y_0 \in (0, \infty)$, is positive definite on $\mathbb{R}$. Since the pointwise limit of positive definite functions also is positive definite function, (1.1) implies that $f$ is positive definite on $\mathbb{R}$. Finally, since $f$ is continuous on $\mathbb{R}$, it follows from (2.8) and the Bochner theorem that $f$ is a characteristic function. This proves Theorem 2.

**Proof of Corollary 3** By Theorem 2 and (1.4), a necessary and sufficient condition for $f$ to be a characteristic function is that

$$(-1)^k \left[ \frac{\partial^k}{\partial y^k} u_f(0, y) \right] \geq 0$$

(2.11)

for $k = 0, 1, 2, \ldots$, and $y > 0$. In the case $k = 0$, the condition (2.11) is

$$u_f(0,y) = \frac{1}{\pi} \int_0^\infty \frac{y}{t^2 + y^2} f(t) \, dt = \frac{1}{\pi} \int_{-\infty}^\infty \mathfrak{R} \left[ \frac{1}{t - iy} \right] f(t) \, dt \geq 0.$$  

(2.12)

Let us calculate the partial derivatives in (2.11). To this end, we recall that if $F$ is an analytic function, then the Cauchy-Riemann equations imply

$$\frac{d}{dz} F(z) = \frac{\partial}{\partial y} (\mathfrak{R} F(z)) - i \frac{\partial}{\partial y} (\mathfrak{I} F(z)).$$

Hence, in the case if $F$ is the function (2.3), then we get

$$\frac{d^{2n}}{dz^{2n}} E_f(z) = (-1)^n \left[ \frac{\partial^{2n}}{\partial y^{2n}} u_f(x, y) + i \frac{\partial^{2n}}{\partial y^{2n}} v_f(x, y) \right]$$

(2.13)

for $n = 1, 2, \ldots$, and

$$\frac{d^{2n+1}}{dz^{2n+1}} E_f(z) = (-1)^n \left[ \frac{\partial^{2n+1}}{\partial y^{2n+1}} v_f(x, y) - i \frac{\partial^{2n+1}}{\partial y^{2n+1}} u_f(x, y) \right]$$

(2.14)

for $n = 0, 1, 2, \ldots$.

On the other hand, by (1.3), (2.2) and (2.3), we have

$$E_f(z) = \frac{i}{\pi} \int_{-\infty}^\infty \left( \frac{1}{z - t} + \frac{t}{t^2 + 1} \right) f(t) \, dt.$$  

Hence

$$\frac{d^k}{dz^k} E_f(z) = (-1)^k k! \frac{i}{\pi} \int_{-\infty}^\infty \frac{f(t)}{(z - t)^{k+1}} \, dt = (-1)^k \frac{i}{\pi} \int_{-\infty}^\infty f^{(k-1)}(t) \, dt,$$

where $k = 1, 2, \ldots$. Therefore, (2.13) and (2.14) imply

$$\frac{\partial^{2n}}{\partial y^{2n}} u_f(0,y) = (-1)^n \mathfrak{R} \left[ \frac{d^{2n}}{dz^{2n}} E_f(z) \right] (z = iy) = (-1)^n \mathfrak{R} \left[ \frac{i}{\pi} \int_{-\infty}^\infty \frac{f^{(2n-1)}(t)}{(iy - t)^2} \, dt \right]$$

$$= \frac{(-1)^{n+1}}{\pi} \int_{-\infty}^\infty \mathfrak{I} \left[ \frac{1}{(t - iy)^2} \right] f^{(2n-1)}(t) \, dt$$

(2.15)
for $n = 1, 2, \ldots$, and
\[
\frac{\partial^{2n+1}}{\partial y^{2n+1}} u_f(0, y) = (-1)^{n+1} \Im \left[ d^{2n+1} \frac{\partial^{2n+1}}{\partial z^{2n+1}} E_f(z) \right] (z = iy)
\]
\[
= (-1)^{n+1} \Im \left[ \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{f^{(2n)}(t)}{(iy-t)^2} \, dt \right] = \frac{(-1)^n}{\pi} \int_{-\infty}^{\infty} \Re \left[ \frac{1}{(t-iy)^2} \right] f^{(2n)}(t) \, dt \quad (2.16)
\]
where $n = 0, 1, 2, \ldots$.
Finally, (2.12), (2.15), and (2.16) show that (2.11) is equivalent to the conditions (1.5)–(1.7).

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