Critical regularity for elliptic equations from Littlewood-Paley theory

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ABSTRACT. Using simple facts from harmonic analysis, namely Bernstein inequality and Plancherel isometry, we prove that the pseudodifferential equation $\Delta^\alpha u + Vu = 0$ improves the Sobolev regularity of solutions provided the potential $V$ is integrable with the critical power $n/2\alpha > 1$.

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1 Introduction

In this paper we prove the following local regularity result for (complex pseudodifferential) elliptic equations. Let $B_R$ denote a ball of radius $R$ and center $0$ in $\mathbb{R}^n$, $n = 2, 3, \ldots$.

Theorem 1.1 Let $u \in H^s(B_1)$ solve

\begin{equation}
\Delta^\alpha u + Vu = 0 \quad \text{in} \quad B_1
\end{equation}

with $V \in L^{n/2\alpha}(B_1)$, $0 < 2\alpha < n$, and $0 < 2s < n$. Assume also that

\begin{equation}
2\alpha - \frac{n}{2} < s < 2\alpha.
\end{equation}

Then

\begin{equation}
\tag{1.3}
u \in H^{s+\varepsilon}(B_{1/2})
\end{equation}

for some $\varepsilon > 0$, $\varepsilon = \varepsilon(n, \alpha, s)$.

By $H^s$ we denote the Sobolev Hilbert space of order $s \in \mathbb{R}^1$. We write $u \in H^s(B_R)$ if $u$ is a distribution on $B_R$ such that

$$u = \tilde{u}|_{B_R} \quad \text{for some} \quad \tilde{u} \in H^s(\mathbb{R}^n).$$

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For any $\alpha > 0$ the operator $\Delta^\alpha$ is a nonlocal pseudodifferential operator defined on $H^s(\mathbb{R}^n)$,

$$\Delta^\alpha: H^s(\mathbb{R}^n) \to H^{s-2\alpha}(\mathbb{R}^n),$$

for any $s \in \mathbb{R}^n$.

Thus the theorem states that, if a distribution $u \in H^s(B_1)$ has an extension $\tilde{u}$ with $\Delta^\alpha \tilde{u} \in H^{s-2\alpha}(\mathbb{R}^n)$ satisfying (1.1) in $B_1$, then we can find a distribution from $H^{s+\varepsilon}(\mathbb{R}^n)$, $\varepsilon > 0$, coinciding with $u$ in $B_{1/2}$. The particular case of integer $s = \alpha$ arises in the calculus of variations. In this case our theorem says that equation (1.1) improves the regularity of $H^\alpha$-solutions if $0 < 2\alpha < n$.

We will prove the theorem using only two simple facts from Littlewood-Paley theory, namely, the Planesherel isometry and the Bernstein inequality. Regularity for (1.1) for the end-point relations between $\alpha$, $\varepsilon$, and $s$ cannot apparently be established with such simple tools. The only non-obvious assumption on the parameters in Theorem 1.1 is the lower bound for $\varepsilon$. Together with the Sobolev embedding it guarantees that the product $V u$ is defined as a distribution.

Theorem 1.1 can be derived from the results of Y.Y.Li [3], see Theorem 1.3 there. Equation (1.1) is treated in [5] as an integral equation in the physical space and the frequency space is not used there at all. Technique developed in the present paper does not depend on the structure of the fundamental solution of $\Delta^\alpha$. In particular, it allows to establish the local regularity for more general pseudodifferential equations on smooth manifolds [7].

The function $V$ is integrable with the critical power in the theorem meaning the following: if $V \in L^p(B_1)$ with $p < n/2\alpha$ then in general (1.3) does not hold for any $\varepsilon > 0$ as the family of examples below shows. If $V \in L^p(B_1)$ with $p > n/2\alpha$ then the improved regularity (1.3) is easy to prove. Indeed, in this case Sobolev and Holder inequalities imply at once that

$$\Delta^\alpha u = f \quad \text{in} \quad B_1 \quad \text{with} \quad f \in L^p(B_1), \quad \frac{1}{p} < \frac{2\alpha}{n} + \frac{1}{2} - \frac{s}{n}. \quad (1.4)$$

Now (1.3) is the straightforward consequence of the Calderon-Zygmund estimate and Sobolev inequality.

The main purpose and application of Theorem 1.1 is deriving the full regularity for quasilinear (complex pseudodifferential) elliptic equations with the critical growth nonlinearity. For such application any $\varepsilon > 0$ in (1.3) works equally well. It is for this reason that we do not care about the sharp value of $\varepsilon(n, \alpha, s)$ in Theorem 1.1. For example, consider a weak solution $u \in H^\alpha$, $0 < 2\alpha < n$, of the equation

$$\Delta^\alpha u + g(x, u) = 0 \quad \text{in} \quad B_1.$$ 

Assume that $g$ is a smooth, possibly complex valued function of the critical growth:

$$|g(x, t)| \leq C \left(1 + |t|^{(n+2\alpha)/(n-2\alpha)}\right) \quad \text{for all} \quad x \in B_1, t \in \mathbb{C}.$$

We can write

$$g(x, u) = \frac{g(x, u)}{1 + |u|} + \left(\frac{g(x, u)}{1 + |u|} \frac{|u|}{u}\right)u = f + Vu$$

with $f$ as in (1.4) and $V \in L^{n/2\alpha}$. Now the application of Theorem 1.1 combined with Calderon-Zygmund and Sobolev inequalities improve the integrability of $u$. Then Schauder estimates imply that $u \in C^{\infty}(B_{1/2})$.

This way of proving the regularity for the critical semilinear equations was suggested for $\alpha = 1$ by Brezis and Kato [1], see also Appendix B in [13]. These authors improved integrability of $u$ using Moser’s iteration technique. Earlier Trudinger [10] (also in the case $\alpha = 1$) had already used Moser’s iterations.
to prove the full regularity for the nonlinear problem directly. The case of an integer \( \alpha > 1 \) has attracted recent attention in \[2\], \[17\] due to its applications in conformal geometry. In a related paper \[6\] Y. Y. Li proved the full regularity for the equation

\[(1.5) \quad \Delta^{\alpha} u + u^{(n+2\alpha)/(n-2\alpha)} = 0, \quad u > 0, \quad 0 < 2\alpha < n.\]

The main goal in \[6\] was to establish Liouville-type theorems for \( (1.5) \) in \( \mathbb{R}^n \) using the moving spheres method. Earlier Liouville theorems were proved for \( (1.5) \) in \[8\], \[3\], \[5\] using the moving plane method. The Littlewood-Paley approach was used in \[4\] and in \[14\] to give proofs of regularity of Hölder-continuous harmonic maps and harmonic maps from surfaces into spheres respectively.

Elliptic equations with supercritical nonlinearity do not improve the regularity of solutions. For example, for \( \alpha = 1 \) and any \( p > (n+2)/(n-2) \), \( n \geq 3 \), the function

\[ u(x) = \frac{A}{|x|^a}, \quad a = 2/(p-1), \quad A = (a(n+a-2))^{1/(p-1)}, \]

satisfies

\[ \Delta u + u^p = 0 \quad \text{in} \quad B_1, \quad u \in H^1(B_1). \]

However, \( u \) is not smooth in \( B_{1/2} \). Pohozaev in \[10\] investigated local regularity for supercritical semilinear problems, and established some sharp low regularity results.

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2 Proof of Theorem 1.1

Let \( \{\hat{\varphi}_j\}_{j=-\infty}^{+\infty} \) be the standard smooth partition of unity in the Littlewood-Paley theory \[9\], \[15\], \[11\], \[12\]. Thus \( \hat{\varphi}_j = \hat{\varphi}(\cdot/2^j) \) is supported in, say, the ring

\[ \{\xi : 2^j/5 \leq |\xi| \leq 2^{j+5}/3 \} \subset (B_{2^{j+1}} \setminus B_{2^{j-1}}). \]

Let \( P_j \) denote the Littlewood-Paley projection,

\[ (P_j f)^\wedge = \hat{\varphi}_j \hat{f}, \quad f \in \mathcal{S}' \]

We also set

\[ P_{a < b} = \sum_{j=a+1}^{b-1} P_j. \]

Distributions with the localised Fourier transform enjoy the important Bernstein inequality: for \( f \in \mathcal{S}' \) and \( 1 \leq p \leq q \leq \infty \)

\[ \|f\|_q \lesssim 2^{nj((1/p)-(1/q))}\|f\|_p \quad \text{provided} \quad \text{supp} \, \hat{f} \subset B_{2^j}. \]

For \( s \in \mathbb{R}^+ \) the Sobolev space \( H^s(\mathbb{R}^n) \) consists of distributions with the finite norm

\[ \|f\|_{H^s} = \|P_{\leq 0} f\|_2 + \left( \sum_{j=1}^{\infty} 2^{2js} \|P_j f\|_2^2 \right)^{1/2}. \]

The Plancherel isometry implies that for \( s = 1, 2, \ldots \) the space \( H^s(\mathbb{R}^n) \) consists of distributions with all derivatives up to the order \( s \) lying in \( L^2(\mathbb{R}^n) \).
PROOF. (OF THEOREM 1.1) 1. First, we localise the problem. Take a cutoff function \( \eta_\rho \),

\[
\eta_\rho = 1 \quad \text{in} \quad B_\rho, \quad \eta_\rho = 0 \quad \text{outside} \quad B_{2\rho}.
\]

The commutator of the multiplication by \( \eta_\rho \) and \( \Delta^\alpha \) is a pseudodifferential operator of order \( 2\alpha - 1 \).

For integer \( \alpha \) this is just the Leibnitz formula for the derivative of the product. Hence, for some \( F \in H^{s-2\alpha+1}(\mathbb{R}^n) \) we obtain

\[
\Delta^\alpha(\eta_\rho u) = \Delta^\alpha(\eta_\rho \hat{u}) = \eta_\rho \Delta^\alpha \hat{u} + F
\]

(2.1)

To economize on notations denote \( u\eta_\rho \) by \( u \) and \( V\eta_\rho \) by \( V \). Then in (2.1) we have \( u \in H^s(\mathbb{R}^n) \), \( \text{supp}(u) \subset B_{2\rho} \). Moreover, the \( L^{s\alpha/2} \)-norm of \( V \) is small when \( \rho \) is small. In the proof we, by making this norm small enough, will establish that

\[
u = \eta_\rho u \in H^{s+\epsilon}(\mathbb{R}^n).
\]

Statement (1.3) then follows by covering \( B_{1/2} \) with small balls. Therefore the goal is to choose a suitable \( \rho \) so that for some constant \( C > 0 \), \( C = C(u, V, \rho, n, \alpha, s) \),

\[
\|P_k u\|_2 \leq \frac{C}{2^{(s+\epsilon)k}} \quad \text{for all} \quad k \geq 1.
\]

(2.2)

Clearly it is enough to prove (2.2) only for large \( k \).

2. The product in the right hand side of (2.1) is an integrable function as a result of (1.2). Hence, applying the Littlewood-Paley projection, we derive that

\[
2^{2\alpha k} \|P_k u\|_2 \lesssim \|P_k(V u)\|_2 + \|P_k F\|_2 \\
\lesssim \|P_k(V u)\|_2 + C F 2^{(2\alpha - s-1)k}.
\]

(2.3)

Thus to prove (2.2) we need to estimate \( P_k(V u) \). We take into account the localisation of the Littlewood-Paley projections in the frequency space. It implies that for \( f, g \in S' \) the distribution \( P_k(P_f P_g) \) vanishes identically if

\[
\left( B_{2^{k+1}} \setminus B_{2^{k+1}} \right) \cap \left( B_{2^{k+1}} \setminus B_{2^{k-1}} \right) = \emptyset.
\]

Consequently for a fixed \( k \in \mathbb{Z} \)

\[
P_k(V u) = \sum_{i,j \in \mathbb{Z}} P_k(P_i V P_j u)
\]

\[
= \left\{ \sum_{i,j \in LL} + \sum_{i,j \in LH} + \sum_{i,j \in HL} + \sum_{i,j \in HH} \right\} P_k(P_i V P_j u)
\]

(2.4)

where \( LL \), \( LH \), \( HL \), and \( HH \) are the low-low, low-high, high-low, and high-high frequencies interaction zones on the integer lattice:

\[
LL = \{i,j \in \mathbb{Z}: k - 5 \leq i,j \leq k + 7, \text{min}\{i,j\} \leq k + 5\},
\]

\[
LH = \{i,j \in \mathbb{Z}: i < k - 5, k - 3 \leq j \leq k + 3\},
\]

\[
HL = \{i,j \in \mathbb{Z}: k - 3 \leq i \leq k + 3, j < k - 5\},
\]

\[
HH = \{i,j \in \mathbb{Z}: i,j > k + 5, |i-j| \leq 3\}.
\]
We are going to estimate the four terms in (2.4) separately. For brevity set

\[
\delta = \|V\|_{n/2^{\alpha}}.
\]

As mentioned above, we can make \(\delta\) as small as we wish by choosing a small enough \(\rho\) in (2.1). We will always assume that \(k\) is big enough, say \(k \geq 10\).

3. By properties of \(P_k\) and the Bernstein inequality

\[
\|I\|_2 \lesssim \sum_{i,j \in LL} \|P_i V P_j u\|_2 \\
\lesssim \sum_{i,j \in LL} \|P_i V\|_\infty \|P_j u\|_2 \\
\lesssim 2^{nk(2\alpha/n)} \delta \sum_{j=k-5}^{k+7} \|P_j u\|_2.
\]

Term \(II\) is estimated exactly the same way. It is convenient to record the final estimate in the following form

(2.5)

\[
\|I\|_2 + \|II\|_2 \lesssim \delta 2^{(2\alpha-s)k} \sum_{j=k-5}^{k+7} 2^{sj} \|P_j u\|_2
\]

4. To estimate \(III\) we distinguish two cases. First, assume that

(2.6)

\[
n \leq 4\alpha,
\]

and hence \(n/2^{\alpha} \leq 2\). Apply the Holder inequality to derive

\[
\|III\|_2 \lesssim \|P_{k-3} \leq k+3 V P_{\leq 0} u\|_2 + \sum_{j=1}^{k-5} \|P_{k-3} \leq k+3 V P_j u\|_2 \\
\lesssim \|P_{k-3} \leq k+3 V\|_2 \|P_{\leq 0} u\|_\infty \\
+ \sum_{j=1}^{k-5} \|P_{k-3} \leq k+3 V\|_2 \|P_j u\|_\infty \\
= X + Y.
\]

From the Bernstein inequalities we deduce that

\[
X \lesssim 2^{nk((2\alpha/n)-(1/2))} \|V\|_{n/2^{\alpha}} \|P_{\leq 0} u\|_2 \\
\lesssim 2^{2ak-(n/2)k} \delta \|P_{\leq 0} u\|_2,
\]

and similarly

\[
Y \lesssim \sum_{j=1}^{k-5} 2^{nk((2\alpha/n)-(1/2))} \delta 2^{nj/2} \|P_j u\|_2 \\
\lesssim \delta \sum_{j=1}^{k-5} 2^{2ak-(n/2)k} 2^{(n/2)j-sj} 2^{sj} \|P_j u\|_2.
\]

Consequently, in the case of (2.6) we can write the final estimate for \(III\) as

(2.7)

\[
\|III\|_2 \lesssim \delta 2^{2\alpha-(n/2)k} \|P_{\leq 0} u\|_2 \\
+ \delta 2^{(2\alpha-s)k} \sum_{j=1}^{k-5} (2^{sj} \|P_j u\|_2) 2^{((n/2)-s)(j-k)}.
\]
Next assume that
\[(2.8) \quad n > 4\alpha.\]

Hence
\[
\frac{2\alpha}{n} + \frac{n - 4\alpha}{2n} = \frac{1}{2}, \quad \text{and} \quad \frac{n}{2\alpha}, \frac{2n}{n - 4\alpha} > 2.
\]

By the Holder inequality
\[
\|III\|_2 \lesssim \|P_{k-3 \leq k+3} V\|_{n/2\alpha} \|P_{\leq 0} u\|_{2n/(n-4\alpha)}
\]
\[
+ \sum_{j=1}^{k-5} \|P_{k-3 \leq k+3} V\|_{n/2\alpha} \|P_j u\|_{2n/(n-4\alpha)}
\]
\[
= Z + W.
\]

The Bernstein inequalities imply that
\[
Z \lesssim \delta \|P_{\leq 0} u\|_2,
\]
and
\[
W \lesssim \delta \sum_{j=1}^{k-5} 2^{n j ((1/2)-(1/2)+(2\alpha/n))} \|P_j u\|_2
\]
\[
\lesssim \delta \sum_{j=1}^{k-5} 2^{(2\alpha - s)j} 2^{s j} \|P_j u\|_2.
\]

Consequently, in the case of \((2.8)\), the final estimate for \(III\) can be written as
\[
\|III\|_2 \lesssim \delta \|P_{\leq 0} u\|_2
\]
\[
+ \delta 2^{(2\alpha - s)k} \sum_{j=1}^{k-5} 2^{(2\alpha - s)(j-k)} \left(2^sj\|P_j u\|_2\right).
\]

5. To estimate \(IV\) we also need to consider two cases. First assume that \((2.6)\) holds. By the Holder inequality
\[
\|P_k (P_i V P_j u)\|_2 \lesssim 2^{nk/2} \|P_k (P_i V P_j u)\|_1
\]
\[
\lesssim 2^{nk/2} \|P_i V\|_{n/2\alpha} \|P_j u\|_{n/(n-2\alpha)}.
\]

According to \((2.6)\) we have
\[
\frac{n}{n - 2\alpha} \geq 2.
\]

Therefore we can continue with the help of Bernstein inequality and derive that
\[
\|P_k (P_i V P_j u)\|_2 \lesssim 2^{nk/2} \delta 2^{nj((1/2)-(1/2)+(2\alpha/n))} \|P_j u\|_2.
\]

After the summation over \(i\) and \(j\) lying in the \(HH\) zone we discover that
\[
\|IV\|_2 \lesssim \delta 2^{(2\alpha - s)k} \sum_{j=k}^{\infty} 2^{(n/2-2\alpha+s)(k-j)} \left(2^sj\|P_j u\|_2\right)
\]
\[
\text{provided (2.6) holds.}
\]

Next assume that \((2.8)\) holds. Then define \(q\), \(1 \leq q \leq 2\) by writing
\[
\frac{1}{q} = \frac{1}{2} + \frac{2\alpha}{n}.
\]
Bernstein and Holder inequalities imply that
\[ \| P_k(\Pi^j P_i u) \|_2 \lesssim 2^{nk(2\alpha/n)} \| P_k(\Pi^j P_j u) \|_q \]
\[ \lesssim 2^{nk(2\alpha/n)} \| \Pi^j P_i u \|_q \]
\[ \lesssim 2^{2\alpha k} \| \Pi^j u \|_2. \]

Summing this estimate over \( i \) and \( j \) in the \( HH \) region, we conclude that in the case of \( (2.8) \)
\[ (2.11) \quad \| IV \|_2 \lesssim \delta 2^{(2\alpha-s)k} \sum_{j=k}^\infty 2^{s(k-j)} (2^{sj} \| \Pi^j u \|_2). \]

6. Now we can prove the desired estimate \( (2.2) \). If \( (2.6) \) holds, then substitute \( (2.5), (2.7), \) and \( (2.10) \) into \( (2.4) \). If \( (2.8) \) holds then use \( (2.5), (2.9), \) and \( (2.11) \). To express the result define
\[ \theta = \begin{cases} \min\{1, (n/2) - s, (n/2) + s - 2\alpha\} & \text{for } n \leq 4\alpha \\ \min\{1, s, 2\alpha - s\} & \text{for } 4\alpha < n. \end{cases} \]

According to assumptions of the theorem, \( \theta > 0 \). Then we derive from \( (2.12) \) that for \( k \geq 10 \)
\[ 2^{sk} \| P_k u \|_2 \leq C_1(u, \rho) 2^{-\theta k} \]
\[ + C_2(n, \alpha, s) \delta \sum_{j=0}^\infty (2^{sj} \| \Pi^j u \|_2) 2^{-|j-k|}. \]

For convenience set
\[ a_k = 2^{sk} \| P_k u \|_2, \quad k = 0, 1, \ldots. \]

We intend to use elementary iteration Lemma 2.1 below to bound the sequence \( \{a_k\} \). First take \( \varepsilon = \theta/2 \). Next, find \( \rho > 0 \) such that in \( (2.12) \) we have
\[ \overline{\delta} \overset{\text{def}}{=} C_2 \delta < (1 - 2^{-\varepsilon/100})/2. \]

Then utilising \( (2.12) \) we can choose \( J = J(u, \rho) \) such that
\[ a_k \leq \frac{1}{2^k} + \delta \sum_{j=0}^\infty \frac{a_j}{2^{|j-k|}} \quad \text{for } k \geq J \]

with \( \overline{\delta} \) satisfying \( (2.13) \). Now, utilising the definition of \( H^s \)-norm find \( K = K(u, \rho) \) such that
\[ a_k \leq 1 \quad \text{for } k \geq K. \]

Finally set
\[ S = J + K. \]

All assumptions of Lemma 2.1 now hold and we derive \( (2.2) \). \( \blacksquare \)

The following lemma is a statement about number sequences. The proof of the lemma is a careful but straightforward iteration of its assumptions. Actually we establish a stronger statement: the proof shows that \( (2.13) \) holds even if in \( (2.11) \) we replace \( 2\varepsilon \) by any \( \varepsilon' > \varepsilon \).

**Lemma 2.1** Let \( \varepsilon > 0 \), let \( \delta \) satisfy
\[ (2.13) \quad 0 < \delta < (1 - 2^{-\varepsilon})/2, \]
and let the sequence \( \{a_k\} \) satisfy

\[
0 \leq a_k \leq 1 \quad \text{for} \quad k \geq S,
\]

\[
a_k \leq \frac{1}{2^{c_k}} + \delta \sum_{j \geq 0} \frac{a_j}{2^{c|k-j|}} \quad \text{for} \quad k \geq S,
\]

with some \( S \geq 0 \). Then

\[
a_k \leq \frac{M}{2^{c_k}}, \quad k = 0, 1, \ldots,
\]

with a constant \( M \geq 0 \), \( M = M(\varepsilon, \delta, S, \|\{a_k\}\|_{l^\infty}) \).

**Proof.** 1. From the bounds on \( a_k \) we derive at once that

\[
a_k \leq \frac{A}{2^{c_k}} + \delta \sum_{j \geq S} \frac{a_j}{2^{c|k-j|}} \quad \text{for all} \quad k \geq S
\]

with a constant \( A > 0 \), \( A = A(\varepsilon, \delta, S, \|\{a_k\}\|_{l^\infty}) \). Define

\[
C_\varepsilon = \frac{2}{(1 - 2^{-\varepsilon})}.
\]

Then, replacing \( a_j \) in (2.16) by 1 , we also have

\[
a_k \leq \frac{A}{2^{c_k}} + \frac{2}{1 - 2^{-\varepsilon}}
\]

\[
\leq \frac{A}{2^{c_k}} + \delta C_\varepsilon \quad \text{for all} \quad k \geq S.
\]

2. We claim that for any \( k \geq S \) and any \( N \geq 0 \) the estimate

\[
a_k \leq \frac{A}{2^{c_k}} \left( 1 + \delta C_\varepsilon + \cdots + (\delta C_\varepsilon)^N \right) + (\delta C_\varepsilon)^{N+1}
\]

holds. Indeed, for \( N = 0 \) and all \( k \geq S \) this is just (2.17). Assume now that (2.18) holds for some \( N \) and all \( k \geq S \). Then substitute (2.18) into (2.16) to discover that for any \( k \geq S \)

\[
a_k \leq \frac{A}{2^{c_k}} + \delta A \left( 1 + \delta C_\varepsilon + \cdots + (\delta C_\varepsilon)^N \right) \sum_{j \geq S} \frac{1}{2^{c_j}2^{c|k-j|}}
\]

\[
+ \delta (\delta C_\varepsilon)^{N+1} \sum_{j \geq S} \frac{1}{2^{c_j}2^{c|k-j|}}
\]

\[
\leq \frac{A}{2^{c_k}} \left( 1 + \delta C_\varepsilon + \cdots + (\delta C_\varepsilon)^{N+1} \right) + (\delta C_\varepsilon)^{N+2},
\]

because for \( k \geq S \)

\[
\sum_{j \geq S} \frac{1}{2^{c_j}2^{c|k-j|}} = \sum_{j=S}^{k} \frac{2^{c_j}}{2^{c_k}} + \sum_{j=k+1}^{\infty} \frac{2^{c_k}}{2^{c_j}}
\]

\[
\leq \frac{1}{2^{c_k}} \left( \frac{1}{1 - 2^{-\varepsilon}} + \frac{1}{2^{c_k}} \left( \frac{1}{1 - 2^{-\varepsilon}} \right) \right)
\]

\[
\leq \frac{C_\varepsilon}{2^{c_k}}.
\]

Hence (2.18) is proved.
3. Finally, sending $N$ to infinity in (2.18), we deduce according to (2.13) that $\delta C_\varepsilon < 1$ and

$$a_k \leq \left( \frac{A}{1 - \delta C_\varepsilon} \right) \frac{1}{2^{|k|}}$$

for all $k \geq S$.

Thus (2.16) holds. ■

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