MIXED NORM ESTIMATES FOR THE CESÁRO MEANS ASSOCIATED WITH DUNKL–HERMITE EXPANSIONS

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Abstract. Our main goal in this article is to study mixed norm estimates for the Cesàro means associated with Dunkl–Hermite expansions on $\mathbb{R}^d$. These expansions arise when one considers the Dunkl–Hermite operator (or Dunkl harmonic oscillator) $H_\kappa := -\Delta_\kappa + |x|^2$, where $\Delta_\kappa$ stands for the Dunkl–Laplacian. It is shown that the desired mixed norm estimates are equivalent to vector-valued inequalities for a sequence of Cesàro means for Laguerre expansions with shifted parameter. In order to obtain the latter, we develop an argument to extend these operators for complex values of the parameters involved and apply a version of three lines lemma.

1. Introduction and main results

The Dunkl operators were introduced by C. F. Dunkl in [9], where he built a framework for a theory of special functions and integral transforms in several variables related to reflection groups. Such operators are relevant in physics, namely for the analysis of quantum many body systems of Calogero–Moser–Sutherland type (see [8, 14]). From the mathematical analysis point of view, the importance of Dunkl operators lies on the fact that they generalize the theory of symmetric spaces of Euclidean type. There is a vast literature related to Dunkl transform and Dunkl Laplacian, see for instance [1, 5, 7, 12, 18, 23, 24].

In [17] M. Rösler studied the Dunkl–harmonic oscillator (which we will also call Dunkl–Hermite operator)

$$H_\kappa := -\Delta_\kappa + |x|^2,$$

where $\Delta_\kappa$ stands for the Dunkl–Laplacian (2.2), and introduced the Dunkl–Hermite functions $\Phi^{(d)}_{\mu,\kappa}$ as eigenfunctions of $H_\kappa$. When $\kappa$, which is called a multiplicity function (see (2.1)), is the null function, the situation is reduced to the standard Hermite operator $H$ and $\Phi^{(d)}_{\mu}$ become the usual Hermite functions $\Phi^{(d)}_{\mu}$, see [17] page 521. It is known that the set $\{\Phi^{(d)}_{\mu,\kappa}\}_{\mu \in \mathbb{N}^d}$ forms an orthonormal basis for $L^2(\mathbb{R}^d, h_\kappa^2 \, dx)$ where $h_\kappa^2$ is a suitable weight function defined in terms of the corresponding reflection group and $\kappa$, see precise definitions in Section 2. Thus we have the orthogonal expansion

$$f = \sum_{\mu \in \mathbb{N}^d} (f, \Phi^{(d)}_{\mu,\kappa}) \Phi^{(d)}_{\mu,\kappa},$$

which converges to $f$ in $L^2(\mathbb{R}^d, h_\kappa^2 \, dx)$. In short we can rewrite this as

$$f = \sum_{j=0}^{\infty} P^{(d)}_{j,\kappa} f, \quad \text{where} \quad P^{(d)}_{j,\kappa} f = \sum_{|\mu|=j} (f, \Phi^{(d)}_{\mu,\kappa}) \Phi^{(d)}_{\mu,\kappa}.$$

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It is known that the Dunkl–Hermite functions are of Schwartz class and hence the projections $P_{j,\kappa}^{(d)} f$ in (1.2) make sense for any $f \in L^p(\mathbb{R}^d, h_\kappa^2 \, dx)$. However, when $p \neq 2$, we do not have any convergence results for $\sum_{j=0}^{\infty} P_{j,\kappa}^{(d)} f$. In the case of $\kappa \equiv 0$, where the above series reduces to standard Hermite expansions, it is well known that the Hermite expansions fail to converge in $L^p(\mathbb{R}^d, dx)$ for $p \neq 2$, unless $d = 1$. Even when $d = 1$, the series converges in $L^p(\mathbb{R}, dx)$ (or, equivalently, the corresponding partial sum operators are uniformly bounded) if and only if $\frac{4}{3} < p < 4$ according to a theorem by R. Askey and S. Wainger [3].

In the absence of convergence results for the partial sums, we are led to consider other summability methods such as Cesàro means or Bochner–Riesz means. For $N \in \mathbb{N}$ and $\delta > 0$ we define the Cesàro means of order $\delta$ associated with the Dunkl–Hermite expansions by

$$\sigma_{N,\kappa}^{\delta,d} f(x) := \frac{1}{A_N^\delta} \sum_{j=0}^{N} A_N^{\delta} P_{N-j,\kappa}^{(d)} f(x),$$

where

$$A_N^\delta = \binom{j+\delta}{j} = \frac{\Gamma(j+\delta+1)}{\Gamma(j+1)\Gamma(\delta+1)}$$

are the binomial coefficients. When $\kappa \equiv 0$ and $d \geq 2$ it is known that $\sigma_{N,0}^{\delta,d} f$ converges to $f$ in $L^p(\mathbb{R}^d, dx), 1 \leq p < \infty$, whenever $\delta > \frac{d-1}{2}$, see [21, 22]. Actually, more precise results are known, giving critical indices of summability for any given $p$, $1 \leq p < \infty$.

In the one dimensional case there is only one reflection group, viz. $\mathbb{Z}_2$, and the generalized Hermite expansion in this case has been studied by Ó. Ciaurri and J. L. Varona [4]. Therein, the authors studied weighted norm inequalities for the Cesàro means. However, for the higher dimensional case we are not aware of any work dealing with Cesàro or Riesz means associated with Dunkl–Hermite expansions, which is the main concern of the present work.

The techniques used to study Cesàro means $\sigma_{N,0}^{\delta,d} f$ for the standard Hermite expansions are not available in the case of Dunkl–Hermite expansions. This is mainly due to the lack of explicit formulas for $\Phi_{\mu_0}^{(d)}$ and their asymptotic properties. However, we can obtain several formulas and identities in the Dunkl–Hermite setting, analogous to those in the standard Hermite setting. First, Mehler’s formula has an analogue in the Dunkl–Hermite setting, see (2.7). We also deduce an analogue of Funk–Hecke formula for spherical harmonics in the Dunkl setting (which we will call spherical $h$-harmonics) given in Corollary 2.2. Then, if we expand a function $f$ into spherical $h$-harmonics, this Funk–Hecke formula allows us to compute the spherical $h$-harmonic coefficients. Since spherical $h$-harmonics form an orthonormal basis for $L^2(S^{d-1}, h_\kappa^2 \, d\sigma)$ (see Subsection 2.3 for details), it is natural to look for mixed norm estimates for the Cesàro means $\sigma_{N,\kappa}^{\delta,d} f$. Let $L^p,2(\mathbb{R}^d, h_\kappa^2 \, dx)$ stand for the space of all functions on $\mathbb{R}^d$ for which

$$(1.4) \quad \|f\|_{(p,2)} := \left( \int_0^\infty \left( \int_{S^{d-1}} |f(rx')|^2 h_\kappa^2(x') \, d\sigma(x') \right)^\frac{p}{2} r^{d+2\gamma-1} \, dr \right)^\frac{1}{p}$$

are finite. Here $\gamma$ is given in terms of the multiplicity function $\kappa$, see Section 2 for the precise definition. We can now state our main result.

**Theorem 1.1.** Let $d \geq 2$ and $\delta > \frac{d+2\gamma-1}{2}$ where $\gamma$ is as in (2.3). Then, for any $1 < p < \infty$, we have the uniform estimates

$$\|\sigma_{N,\kappa}^{\delta,d} f\|_{(p,2)} \leq C \|f\|_{(p,2)}.$$

Consequently, $\sigma_{N,\kappa}^{\delta,d} f$ converges to $f$ in $L^p,2(\mathbb{R}^d, h_\kappa^2 \, dx)$ as $N \to \infty$.

The outline of the proof is the following. We prove a Funk–Hecke formula for the spherical $h$-harmonic coefficients of $\sigma_{N,\kappa}^{\delta,d} f$ that allows us to express the Cesàro kernels $\sigma_{N,\kappa}^{\delta,d}(x,y)$ for the
Dunkl–Hermite expansions in terms of the Cesàro kernels for the standard Hermite expansions \( \sigma_{N}^{\delta,d}(x,y) \). This is contained in Proposition 5.1. From here, we are able to express \( C_{N,\kappa}^{\delta,d}f \) in terms of an spherical \( h \)-harmonic expansion of a sequence of certain integral linear operators \( T_{N,m}^{\delta,\gamma} \) whose kernels are related to \( \sigma_{N}^{\delta,d}(x,y) \), see Corollary 5.2. In turn, we show in Proposition 5.4 that these new kernels are nothing but Cesàro kernels for Laguerre expansions of different types. Thus, it follows that the mixed norm inequality for \( \sigma_{N,\kappa}^{\delta,d} \) in Theorem 1.1 is equivalent to a vector valued inequality for a sequence of Cesàro means associated with Laguerre expansions with shifted parameters.

**Theorem 1.2** (Vector valued inequalities for \( T_{N,m}^{\delta,\gamma} \)). Let \( 0 \leq \gamma \leq \frac{1}{2}, \delta > \frac{d+2\gamma-1}{2} \) and \( T_{N,m}^{\delta,\gamma} \) be the operators defined in (5.6). Then, for any \( 1 < p < \infty \), there is a constant \( C \) independent of \( N \) such that

\[
\left\| \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |T_{N,m}^{\delta,\gamma}f_{m,j}(r)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{+},r^{d+2\gamma-1}dr)} \leq C \left\| \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{+},r^{d+2\gamma-1}dr)},
\]

where \( d(m) \) is as in (2.4), for any sequence of functions \( f_{m,j} \in L^p(\mathbb{R}^{+},r^{d+2\gamma-1}dr) \) for which the right hand side is finite.

In order to prove Theorem 1.2, we extend the Cesàro means for complex values of \( \delta \) and also complexify the type of the Laguerre functions involved. Then again, we can express the kernels of such extended operators in terms of Cesàro kernels of standard Hermite expansions and a clever use of three lines lemma, we get the result.

The paper is organized as follows. In Section 2 we recall basics about the general Dunkl context, Dunkl harmonic oscillator, Dunkl–Hermite functions and Funk–Hecke identity and Mehler’s formula in the Dunkl setting. Section 3 is devoted to the study of Cesàro means for the standard Hermite expansions and a vector-valued inequality for an operator related to these Cesàro means is proved. In Section 4, we prove some technical results, namely, integral formulas for Bessel functions which involve ultraspherical polynomials with real and complex parameters. Finally, in Section 5 we express the Cesàro kernels for the Dunkl–Hermite expansions in terms of the same for the standard Hermite expansions, and we bring out the connection with Laguerre expansions, which allow us to introduce an analytic family of operators. With this, we prove the main results in the final Subsection 5.4.

**Notation.** Throughout the paper, we will use the following notation. For \( x,y \in \mathbb{R}^{d} \), we shall take \( x = rx' \) and \( y = sy' \), \( r,s \in \mathbb{R}^{+} \) and \( x',y' \in \mathbb{S}^{d-1} \), where \( \mathbb{S}^{d-1} \) is the unit sphere in \( \mathbb{R}^{d}, \ d \geq 2 \). Analogous notation will be valid for \( x,y \in \mathbb{R}^{d+1} \). For \( 1 \leq p \leq \infty \), \( p' \) will denote its adjoint, \( 1/p + 1/p' = 1 \). Finally, we shall write \( C \) to denote positive constants independent of significant quantities the meaning of which can change from one occurrence to another.

2. THE DUNKL HARMONIC OSCILLATOR

For the sake of completeness, in this section we collect several facts concerning the general Dunkl setting and the Dunkl harmonic oscillator. For a more detailed exposition on these topics, we refer the reader to [3,10,17].

2.1. The general Dunkl setting. We use the notation \( \langle \cdot, \cdot \rangle \) for the standard inner product on \( \mathbb{R}^{d} \). For \( \nu \in \mathbb{R}^{d} \setminus \{0\} \), we denote by \( \sigma_{\nu} \) the orthogonal reflection in the hyperplane perpendicular to \( \nu \), i.e.,

\[
\sigma_{\nu}(x) = x - 2 \frac{\langle \nu, x \rangle}{|\nu|^2} \nu.
\]
A finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ is a root system if $\sigma_{\nu}(R) = R$, for all $\nu \in R$. Each root system can be written as a disjoint union $R = R_+ \cup (-R_+)$, where $R_+$ and $-R_+$ are separated by a hyperplane through the origin. Such $R_+$ is called the set of all positive roots in $R$. The group $G$ generated by the reflections $\{\sigma_{\nu} : \nu \in R\}$ is called the reflection group or Coxeter group associated with $R$. A function
\begin{equation}
\kappa : R \to [0, \infty)
\end{equation}
which is invariant under the action of $G$ on the root system $R$ is called a multiplicity function. Let $T_j, j = 1, 2, \ldots, d$, be the difference -differential operators defined by
\[ T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\nu \in R_+} \kappa(\nu)\nu_j \frac{f(x) - f(\sigma_{\nu} x)}{\langle \nu, x \rangle}. \]
These operators, known as Dunkl operators, form a family of commuting operators. The Dunkl Laplacian $\Delta_\kappa$ is then defined to be the operator
\begin{equation}
\Delta_\kappa = \sum_{j=1}^{d} T_j^2
\end{equation}
which can be explicitly calculated, see [10, Theorem 4.4.9]. It is known that the operators $T_j$ have a joint eigenfunction $E_\kappa(x, y)$ satisfying
\[ T_j E_\kappa(x, y) = y_j E_\kappa(x, y), \quad j = 1, \ldots, d. \]
The function $(x, y) \mapsto E_\kappa(x, y)$ is called the Dunkl kernel or the generalized exponential kernel on $\mathbb{R}^d \times \mathbb{R}^d$, which is the generalization of the exponential function $e^{(x,y)}$. Associated with the root system $R$ and the multiplicity function $\kappa$, the weight function $h_\kappa^2(x)$ is defined by
\[ h_\kappa^2(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{2\kappa(\nu)}. \]
The nonnegative real number
\begin{equation}
\gamma = \sum_{\nu \in R_+} \kappa(\nu)
\end{equation}
defined in terms of the multiplicity function $\kappa(\nu)$ plays an important role in Dunkl theory. Note that $h_\kappa^2(x)$ is homogeneous of degree $2\gamma$, which motivates the definition of mixed norm spaces as in [14].

Now we move to some basic facts about spherical $h$-harmonics (or simply $h$-harmonics); a good reference for $h$-harmonics is [10, Chapter 5]. These are the restrictions of solid $h$-harmonics to $\mathbf{S}^{d-1}$ where by solid $h$-harmonics we mean homogeneous polynomials $P(x)$ satisfying $\Delta_\kappa P(x) = 0$. The $h$-harmonics are analogues of spherical harmonics and defined using $\Delta_\kappa$ in place of $\Delta$. Let $\mathcal{H}^d_m$ be the space of all $h$-harmonics of degree $m$. Then the space $L^2(\mathbf{S}^{d-1}, h_\kappa^2 d\sigma)$ is the orthogonal direct sum of the finite dimensional spaces $\mathcal{H}^d_m$ over $m = 0, 1, 2, \ldots$. Thus there is an orthonormal basis
\[ \{Y_{m,j}^h : j = 1, 2, \ldots, d(m), m = 0, 1, 2, \ldots\}, \]
where
\begin{equation}
\dim(\mathcal{H}^h_m) = \sum_{m=0}^{\infty} \frac{d(m)}{m!}
\end{equation}
for $L^2(\mathbf{S}^{d-1}, h_\kappa^2 d\sigma)$ so that for each $m$, $\{Y_{m,j}^h : j = 1, 2, \ldots, d(m)\}$ is an orthonormal basis of $h$-harmonics of degree $j$ for $\mathcal{H}^d_m$. The $h$-harmonic expansion of a function $f$ is given by
\[ f(rx') = \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} f_{m,j}(r) Y_{m,j}^h(x'), \]
where the $h$-harmonic coefficients are
\begin{equation}
(2.5) \quad f_{m,j}(r) = \int_{\mathbb{S}^{d-1}} f(rx') Y_{m,j}^h(x') h^2_{\kappa}(x') d\sigma(x').
\end{equation}

In the Dunkl setting we also have a Funk–Hecke formula for $h$-harmonics. The classical Funk–Hecke formula for spherical harmonics states the following. For any continuous function $f$ on $[-1,1]$ and a spherical harmonic $Y_{m,j}$ of degree $m$ we have
\[
\int_{\mathbb{S}^{d-1}} f(\langle x', y' \rangle) Y_{m,j}(y') d\sigma(y') = \Lambda_m(f) Y_{m,j}(x')
\]
where $\Lambda_m(f)$ is a constant defined by
\[
\Lambda_m(f) = \frac{\omega_d \Gamma(d/2)}{\sqrt{\pi} \Gamma((d - 1)/2)} \int_{-1}^1 f(u) P_m^d(u)(1 - u^2)^{-\frac{d-1}{2}} du.
\]

Here $P_m^d$ stands for the normalized ultraspherical polynomials of type $\lambda > -\frac{1}{2}$ and degree $m$ and $\omega_d := \int_{\mathbb{S}^{d-1}} d\sigma(\omega)$. To state the Funk–Hecke formula for $h$-harmonics, we need to recall the intertwining operator in the Dunkl setting. It is known that there is an operator $T_j \kappa$ satisfying $T_j \kappa V_{\kappa} = V_{\kappa} \frac{\partial}{\partial x_j}$. However, the explicit form of $V_{\kappa}$ is not known, except in a couple of simple cases, but it is a useful operator. In particular, the Dunkl kernel is given by $E_{\kappa}(x,y) = V_{\kappa} e^{\langle x, y \rangle}(x)$. The Funk–Hecke formula for $h$-harmonics is as follows (see [6, Theorem 7.2.7] or [10, Theorem 5.3.4]).

**Theorem 2.1.** Let $f$ be a continuous function defined on $[-1,1]$ and $\lambda = \frac{d}{2} + \gamma - 1$. Then for every $Y_{m,j}^h \in \mathcal{H}_m^d$,
\[
\int_{\mathbb{S}^{d-1}} V_{\kappa} f(\langle x', \gamma \rangle)(y') Y_{m,j}^h(y') h^2_{\kappa}(y') d\sigma(y') = \Lambda_m^\kappa(f) Y_{m,j}^h(x')
\]
where $\Lambda_m^\kappa(f)$ is a constant defined by
\[
\Lambda_m^\kappa(f) = \frac{\omega_d^\kappa \Gamma(\lambda + 1)}{\sqrt{\pi} \Gamma(\lambda + 1/2)} \int_{-1}^1 f(u) P_m^\lambda(u)(1 - u^2)^{\lambda - \frac{d-1}{2}} du
\]
with
\begin{equation}
(2.6) \quad \omega_d^\kappa := \int_{\mathbb{S}^{d-1}} h^2_{\kappa}(\omega) d\sigma(\omega).
\end{equation}

By applying Theorem 2.1 to the function $f(t) = e^{rst}$, $r, s \geq 0$, and using the fact $V_{\kappa} f(\langle x', y' \rangle) = E_{\kappa}(rx', sy')$, we immediately obtain the following.

**Corollary 2.2** (Funk–Hecke for Dunkl kernel). Let $\lambda = \frac{d}{2} + \gamma - 1$. Then for every $Y_{m,j}^h \in \mathcal{H}_m^d$,
\[
\int_{\mathbb{S}^{d-1}} E_{\kappa}(rx', sy') Y_{m,j}^h(y') h^2_{\kappa}(y') d\sigma(y') = \frac{\omega_d^\kappa \Gamma(\frac{d}{2} + \gamma)}{\sqrt{\pi} \Gamma(\frac{d}{2} + \gamma + \frac{1}{2})} \left( \int_{-1}^1 e^{rsu} P_m^\lambda(u)(1 - u)^{\lambda - \frac{d-1}{2}} du \right) Y_{m,j}^h(x'),
\]
where $\omega_d^\kappa$ is as in (2.6).

**2.2. Mehler’s formula for Dunkl–Hermite functions and Hecke–Bochner identity.** For each $\mu \in \mathbb{N}^d$, we consider the generalized Hermite functions (or Dunkl–Hermite functions) $\Phi^{(d)}_{\mu, \kappa}$, which are eigenfunctions of the operator $H_{\kappa}$ with eigenvalues $(2|\mu| + d + 2\gamma)$, where $\gamma$ is as in (2.3), and $|\mu| = \sum_{\ell=1}^d \mu_\ell$. Generalized Hermite polynomials and generalized Hermite functions associated with Coxeter groups were studied in [17], where the precise definitions can be found. For
our purposes, the most important result is the generating function identity or the *Mehler’s formula for the Dunkl–Hermite functions*. For $|w| < 1$, one has (see [17, Theorem 3.12])

$$(2.7) \quad \sum_{\mu \in \mathbb{N}^d} \phi_{\mu, \kappa}^{(d)}(x) \phi_{\mu, \kappa}^{(d)}(y) w^{\mu} = \frac{2}{\omega^d \Gamma \left( \frac{d}{2} + \gamma \right)} (1 - w^2)^{-\frac{d}{2} - \gamma} e^{-\frac{1}{2} \left( \frac{1 + w^2}{1 - w^2} \right) |x|^2 + |y|^2} E_\kappa \left( \frac{2w}{1 - w^2}, y \right).$$

Let $P^{(d)}_{j, \kappa}$ be the orthogonal projection associated with the eigenspace corresponding to the eigenvalue $(2j + d + 2\gamma)$ of $H_\kappa$ described in (1.2). Then $P^{(d)}_{j, \kappa}$ is given by the kernel operator

$$(2.8) \quad P^{(d)}_{j, \kappa} f(x) = \int_{\mathbb{R}^d} \phi_{j, \kappa}^{(d)}(x, y) f(y) h_\kappa^2(y) dy$$

where

$$(2.9) \quad \phi_{j, \kappa}^{(d)}(x) = \sum_{|\mu| = j} \phi_{\mu, \kappa}^{(d)}(x) \phi_{\mu, \kappa}^{(d)}(y).$$

For these projections $P^{(d)}_{j, \kappa}$ we can also prove a Hecke–Bochner identity as in the case of standard Hermite projections $P^{(d)}_j$ (see [21, Theorem 3.4.1]). The Hecke–Bochner identity says that the projection $P^{(d)}_{j, \kappa}$ of functions of the form $f(x) = f_0(|x|) Y_m^h \left( \frac{x}{|x|} \right)$, where $f_0$ is a radial function and $Y_m^h \left( \frac{x}{|x|} \right)$ is an $h$-harmonic of degree $m$, is also of the same form. Indeed, for $\alpha > -1$, let $\psi_\alpha^h$ be the normalized Laguerre functions given by

$$\psi_\alpha^h(r) = \left( \frac{2\Gamma(k + 1)}{\Gamma(k + \alpha + 1)} \right)^{\frac{1}{2}} L_k^\alpha(r^2) e^{-\frac{1}{2} r^2}, \quad k = 0, 1, \ldots,$$

where $L_k^\alpha$ are the Laguerre polynomials of order $\alpha$, see [15, page 76]. We prove the following theorem which, although not used for our purposes in the present paper, is of independent interest.

**Theorem 2.3** (Hecke–Bochner for Dunkl–Hermite projections). Let $f(x) = f_0(|x|) Y_m^h \left( \frac{x}{|x|} \right)$, where $Y_m^h$ is a spherical $h$-harmonic of degree $m$. Then one has $P^{(d)}_{2j + m, \kappa} f(x) = F(|x|) Y_m^h \left( \frac{x}{|x|} \right)$ where

$$F(r) = \int_0^\infty (rs)^m \psi_j^{\lambda+m}(r) \psi_j^{\lambda+m}(s) f_0(s) s^{2\lambda+1} ds$$

with $\lambda = \frac{d}{2} + \gamma - 1$. For other values of $\ell$, $P^{(d)}_{\ell, \kappa} f = 0$.

**Proof.** The proof is similar to the proof of [21, Theorem 3.4.1]. Here we need to use the Mehler’s formula for the generalized Hermite functions $\phi_{\mu, \kappa}^{(d)}(x)$ (2.7) and Funk–Hecke formula for the Dunkl kernel $E_\kappa(x, y)$ in Corollary 2.2. The details of the proof are left to the reader. \qed

### 3. Cesàro means for the standard Hermite expansions

One of the key points to prove Theorem 1.1 is to express Cesàro kernels for the Dunkl–Hermite expansions in terms of Cesàro kernels for the standard Hermite expansions and then extend these operators for complex values of the parameters involved. In this section we recall some basic results concerning the $L^p$ boundedness of Cesàro means for the standard Hermite expansions and prove others concerning the extended operators.

As explained in the introduction, when $\kappa \equiv 0$ the Dunkl–Hermite functions reduce to the standard Hermite functions $\Phi^{(d)}_{\mu, \kappa}$ on $\mathbb{R}^d$, which are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$ with eigenvalues $(2|\mu| + d)$. Cesàro means of order $\delta \geq 0$ associated with the Hermite expansions are then defined by

$$\sigma^\delta_N f = \frac{1}{A_N^\delta} \sum_{j=0}^N A_{N-j}^\delta P^{(d)}_j f.$$
The operators $\sigma_N^{\delta,d} f$ can be described as integrals operators with a kernel $\sigma_N^{\delta,d}(x,y)$, which is explicitly given by

$$\sigma_N^{\delta,d}(x,y) = \frac{1}{A_N^d} \sum_{j=0}^N A_{N-j}^d \Phi_j^{(d)}(x,y),$$

where $\Phi_j^{(d)}(x,y)$ is the kernel of the $j$th projection associated with the Hermite operator (see [21 page 6]). Mehler’s formula for $\Phi_j^{(d)}(x,y)$ reads as

$$\sum_{j=0}^\infty \Phi_j^{(d)}(x,y) w^j = \pi^{-\frac{d}{2}} (1 - w^2)^{-\frac{d}{2}} e^{-\frac{1}{2} (\frac{1+w^2}{1-w^2})(|x|^2+|y|^2)+\frac{2w}{1-w^2}x\cdot y}. \tag{3.1}$$

From Mehler’s formula it follows that $\Phi_j^{(d)}(rx',sy')$ and consequently $\sigma_N^{\delta,d}(rx',sy')$ is a function of $r$, $s$ and $u = x' \cdot y'$. Hence, sometimes we will write $\Phi_j^{(d)}(r,s;u)$ instead of $\Phi_j^{(d)}(rx',sy')$ and $\sigma_N^{\delta,d}(r,s;u)$ instead of $\sigma_N^{\delta,d}(rx',sy')$.

On the other hand, the Bochner–Riesz means associated with the Hermite expansions are defined by

$$S_R^{\delta,d} f = \sum_{j=0}^{\infty} \left(1 - \frac{2j + d}{R}\right)^\delta P_j^{(d)} f,$$

where $R > 0$ and $(1-s^2)_+ = \max\{1-s^2,0\}$. In the literature, the boundedness of both Cesàro and Bochner–Riesz means have been studied. Their behaviour are similar in the sense that is possible to express the Cesàro means $\sigma_N^{\delta,d} f$ in terms of $S_R^{\delta,d} f$ and vice-versa. Indeed, we have the following theorem due to J. J. Gergen [11], adapted to our context.

**Theorem 3.1** (Gergen). Let $m$ be the integral part of $\delta$. Then there exist two functions $U$ and $V$, $U(x) = O(x^{-2})$ as $x \to \infty$, $U(x) = O(x^{m-\delta+1})$ as $x \to 0$, and $V(x) = O(x^{-2})$ as $x \to \infty$, $V(x) = O(x^\delta)$ as $x \to 0$, such that

$$\sigma_R^{\delta,d}(x,y) = R^{-\delta} \sum_{k \leq R} V(R-k) A_k^d \sigma_k^{\delta,d}(x,y)$$

and

$$\sigma_N^{\delta,d}(x,y) = \frac{1}{A_N^\delta} \int_0^{N+1} U(N+1-t) t^{\delta} S_t^{\delta,d}(x,y) dt, \quad \text{for } N = 0,1,\ldots$$

In view of Gergen’s theorem, we can readily prove a version of [21 Theorem 3.3.3] (that states pointwise estimates for Bochner–Riesz) for Cesàro means.

**Theorem 3.2.** Let $d \geq 2$ and $\delta > \frac{d-1}{2}$. Then for any $q \geq 2$ and $f \in L^q(\mathbb{R}^d)$ we have the pointwise inequality

$$\sup_N |\sigma_N^{\delta,d} f(x)| \leq CM_q f(x)$$

where $M_q f(x) = (M|f|^q)^{\frac{1}{q}}(x)$, $M$ being the Hardy–Littlewood maximal function.

In order to prove Theorem 3.1 we need to consider Cesàro means $\sigma_N^{\delta,d}$ when $\delta$ is complex. Let us write

$$\delta(\zeta) = \frac{d-1}{2} + \zeta \quad \text{for } \zeta \in \mathbb{C} \tag{3.2}$$

and consider $\sigma_N^{\delta(\zeta),d} f$ with $\text{Re}(\delta(\zeta)) \geq 0$. Let us recall also the definition of a function of admissible growth; we say that a function $F(y)$, $y \in \mathbb{R}$, is of admissible growth if there exist constants $a < \pi$ and $b > 0$ such that $|F(y)| \leq e^{be^{a|y|}}$. As a corollary of Theorem 3.2 we get the following result.
Theorem 3.3. Let \( d \geq 2 \). Then for any \( q \geq 2 \) and \( f \in L^q(\mathbb{R}^d) \) we have the pointwise inequality

\[
\sup_N \left| \sigma_N^{\delta(\beta + \varepsilon),d} f(x) \right| \leq C_{\varepsilon}(\beta)M_q f(x)
\]

for a fixed \( \varepsilon > 0 \) and \( \beta \in \mathbb{R} \). Here, the function \( C_{\varepsilon}(\beta) \) is of admissible growth. Moreover, the operator \( \sup_N \left| \sigma_N^{\delta(\beta + \varepsilon),d} f \right| \) is bounded on \( L^p(\mathbb{R}^d) \) for any \( p > 2 \).

Proof. From the definition of \( \sigma_N^{\delta,d} \) it follows that

\[
(3.3) \quad \sigma_N^{\delta(\beta + \varepsilon),d} f(x) = \frac{1}{A_N^{\delta(\beta + \varepsilon)}} \sum_{j=0}^{N} A_{N-j}^{\delta(\varepsilon/2)} A_j^{\delta(\varepsilon/2)} |D_j f(x)|.
\]

Indeed, the right hand side in the identity above is

\[
\frac{1}{A_N^{\delta(\beta + \varepsilon)}} \sum_{j=0}^{N} A_{N-j}^{\delta(\varepsilon/2)} |D_j f(x)| = \frac{1}{A_N^{\delta(\beta + \varepsilon)}} \sum_{j=0}^{N} P_j^{(\varepsilon/2)} f(x) \sum_{j=0}^{N} A_{N-j}^{\delta(\varepsilon/2)} A_j^{\delta(\varepsilon/2)},
\]

so (3.3) will be proved once we check that

\[
A_{N-k}^{\delta(\beta + \varepsilon)} = \sum_{j=k}^{N} A_{N-j}^{\delta(\varepsilon/2)} A_j^{\delta(\varepsilon/2)},
\]

or equivalently

\[
(3.4) \quad A_{N-k}^{\delta(\beta + \varepsilon)} = \sum_{j=0}^{N} A_{N-j}^{\delta(\varepsilon/2)} A_j^{\delta(\varepsilon/2)}.
\]

Recall the following basic facts concerning generating functions. Since \( (1-w)^{-r-1} = \sum_{k=0}^{\infty} \binom{r+k}{k} w^k \) and \( (1-w)^{-s-1} = \sum_{k=0}^{\infty} \binom{s+k}{k} w^k \), if we multiply these together, we get \( (1-w)^{-r+s+1} = (1-w)^{-r-1}(1-w)^{-s-1} \). Equating coefficients gives us

\[
(3.5) \quad \left( \binom{r+s+1+n}{n} \right) = \sum_{k=0}^{n} \binom{r+n-k}{n-k} \binom{s+k}{k}.
\]

With this, we see that (3.4) is true.

Since \( \delta(\varepsilon/2) > \frac{d-1}{2} \), it follows by Theorem 3.2 that \( \sup_j \left| \sigma_j^{\delta(\varepsilon/2),d} f(x) \right| \leq C M_q f(x) \) and hence the theorem will be proved once we show that

\[
\sum_{j=0}^{N} \left| A_{N-j}^{\delta(\beta + \varepsilon)} A_j^{\delta(\varepsilon/2)} \right| \leq C_{\varepsilon}(\beta) \left| A_N^{\delta(\beta + \varepsilon)} \right|,
\]

but this is proved in Lemma 3.4. Finally, the last statement of Theorem 3.3 follows from the fact that the maximal function \( M_q \) is bounded on \( L^p(\mathbb{R}^d) \) for any \( p > q \geq 2 \). \( \square \)

Lemma 3.4. For \( \zeta \in \mathbb{C} \), let \( \delta(\zeta) \) be defined as in (3.2). For any \( \varepsilon > 0 \) and \( \beta \in \mathbb{R} \) we have the estimate

\[
\frac{1}{|A_N^{\delta(\beta + \varepsilon)}|} \sum_{j=0}^{N} \left| A_{N-j}^{\delta(\beta + \varepsilon)} A_j^{\delta(\varepsilon/2)} \right| \leq C_{\varepsilon}(1 + |\beta|) \cosh(\pi \beta)
\]

where \( C_{\varepsilon} \) is independent of \( N \).
Proof. In order to prove the lemma we make use of the fact that

\[(3.6) \quad |\Gamma(\alpha + i\beta)| \leq \Gamma(\alpha),\]

for \(\alpha, \beta \in \mathbb{R}, \alpha > 0\), which follows from the very definition of Gamma function. We also have the Beta function

\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 (1-t)^{x-1}t^{y-1} \, dt
\]

which leads to the estimate

\[(3.7) \quad \left| \frac{\Gamma(\alpha)\Gamma(i\beta + 1)}{\Gamma(\alpha + i\beta + 1)} \right| \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)}\]

for any \(\alpha > 0\). We rewrite

\[
\sum_{j=0}^{N} A_{j}^{N-j} A_{j}^{\delta(\varepsilon/2)} = \frac{\Gamma(\varepsilon/2)}{\Gamma(\varepsilon/2 + i\beta)} \sum_{j=0}^{N} \frac{\Gamma(N - j + \varepsilon/2 + i\beta)}{\Gamma(N - j + 1)\Gamma(\varepsilon/2)} A_{j}^{\delta(\varepsilon/2)}.
\]

By \((3.6)\) and \((3.5)\) we have

\[
\left| \sum_{j=0}^{N} A_{j}^{\delta(\varepsilon/2)} \right| \leq \frac{\Gamma(\varepsilon/2)}{\Gamma(\varepsilon/2 + i\beta)} \sum_{j=0}^{N} \frac{\Gamma(N - j + \varepsilon/2)}{\Gamma(N - j + 1)\Gamma(\varepsilon/2)} A_{j}^{\delta(\varepsilon/2)}
\]

Thus, we are left with estimating

\[
\left| \frac{\Gamma(\varepsilon/2) A_{N}^{\delta(\varepsilon)}}{\Gamma(\varepsilon/2 + i\beta) A_{N}^{\delta(i\beta + \varepsilon/2)}} \right| = \left| \frac{\Gamma(\varepsilon/2)}{\Gamma(\varepsilon/2 + i\beta)} \frac{\Gamma(\delta(\varepsilon) + i\beta + 1)}{\Gamma(\delta(\varepsilon) + 1)} \frac{\Gamma(N + \delta(\varepsilon) + 1)}{\Gamma(N + \delta(\varepsilon) + i\beta + 1)} \right|.
\]

The middle term in the right hand side is clearly bounded by \((3.6)\). The first term can be written as

\[
\frac{(\varepsilon/2 + i\beta)\Gamma(\varepsilon/2)\Gamma(i\beta + 1)}{\Gamma(i\beta + 1)\Gamma(\varepsilon/2 + i\beta + 1)}
\]

which leads, by \((3.7)\), to the estimate

\[
\left| \frac{\Gamma(\varepsilon/2)}{\Gamma(\varepsilon/2 + i\beta)} \right| \leq \frac{\Gamma(\varepsilon/2)}{\Gamma(\varepsilon/2 + 1)} \frac{(1 + |\beta|)}{|\Gamma(i\beta + 1)|}.
\]

Similarly, the third term gives the estimate \(\left| \Gamma(i\beta + 1) \right|^{-1}\). Therefore, the expression in Lemma \(3.4\) is bounded by

\[
\frac{\Gamma(\varepsilon/2)}{\Gamma(\varepsilon/2 + 1)} (1 + |\beta|) |\Gamma(i\beta + 1)|^{-2}.
\]

Once again the expression with the Beta function

\[
\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} + i\beta)}{\Gamma(1 + i\beta)} = \int_0^1 (1-t)^{-\frac{1}{2}i\beta - \frac{1}{2}} \, dt
\]

leads to the estimate \(\left| \Gamma(i\beta + 1) \right|^{-1} \leq \sqrt{\pi} |\Gamma(\frac{1}{2} + i\beta)|^{-1}\). Therefore,

\[
\left| \Gamma(i\beta + 1) \right|^{-2} \leq \pi \left| \Gamma(\frac{1}{2} + i\beta) \right|^{-2} = \pi \left( \Gamma(\frac{1}{2} + i\beta) \Gamma(\frac{1}{2} - i\beta) \right)^{-1}
\]

In view of the identity \(\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\) we obtain

\[
\left| \Gamma(i\beta + 1) \right|^{-2} \leq \sin \pi \left( \frac{1}{2} + i\beta \right) = \cosh \beta.
\]

This completes the proof of the lemma. \(\square\)
Further, we introduce one more operator $S_N^{δ_d}$ related to $σ_N^{δ_d}$ as follows. For any $f \in L^p(\mathbb{R}^+, dr)$ we define

$$S_N^{δ_d} f(r) = r^{d-1 \over p} \int_0^∞ s^{d-1 \over p} \left( \int_{-1}^1 |σ_N^{δ_d}(r, s; u)|(1 - u^2)^{d-3 \over 2} du \right) f(s) ds. \tag{3.8}$$

We will require a vector-valued inequality for the maximal function associated with $S_N^{δ_d}$. In order to get this, recall the following result about vector-valued extensions for general bounded operators by J. L. Krivine (see [13] or [16, Thm. 1.f.14]).

**Theorem 3.5 (Krivine).** Let $X$ and $Y$ be two Banach lattices and let $T : X \to Y$ be a bounded linear operator. Then, for every choice of $\{x_i\}_{i=1}^n$ in $X$, we have

$$\left\| \left( \sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\| \leq K_G \|T\| \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2},$$

where $K_G$ is the universal Grothendieck constant.

**Theorem 3.6.** Let $d \geq 2$ and for $ζ \in \mathbb{C}$, let $δ(ζ)$ be defined as in (3.2). Then for any $p > 2$ we have the vector-valued inequality

$$\left( \int_0^∞ \left( \sum_{m=0}^∞ \sum_{j=1}^{d(m)} \left( \sup_N |S_N^{δ(ζ+ε),d} f_{m,j}(r)| \right)^2 \right)^{1 \over 2} dr \right)^{1 \over p} \leq C(β) \left( \int_0^∞ \left( \sum_{m=0}^∞ \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{1 \over 2} dr \right)^{1 \over p}$$

for any sequence of functions $f_{m,j} \in L^p(\mathbb{R}^+, dr)$ for which the right hand side is finite. Moreover, $C(β)$ is of admissible growth.

**Proof.** Observe that we can consider $\sup_N |S_N^{δ(ζ+ε),d} f|$ as a linear operator mapping $L^p(\mathbb{R}^+)$ into $L^p(\mathbb{R}^+, l^∞(\mathbb{N}))$. Then, appealing to Theorem 3.5 it is enough to show that this maximal operator is bounded on $L^p(\mathbb{R}^+, dr)$.

Consider the radial function $F$ defined by $F(x) = |x|^{d-1 \over p} f(|x|)$ which belongs to $L^p(\mathbb{R}^d)$ as $f \in L^p(\mathbb{R}^+, dr)$. In terms of $F$ we can consider $S_N^{δ(ζ+ε),d} f$ as a radial function on $\mathbb{R}^d$ given by

$$|x|^{d-1 \over p} \int_{\mathbb{R}^d} |σ_N^{δ(ζ+ε),d}(x, y)| |F(y)| dy.$$

Thus the boundedness of $\sup_N |S_N^{δ(ζ+ε),d} f|$ on $L^p(\mathbb{R}^+, dr)$ follows from the fact that the maximal operator

$$\sup_N \left( \int_{\mathbb{R}^d} \left| σ_N^{δ(ζ+ε),d}(x, y) \right| |F(y)| dy \right)$$

is bounded on $L^p(\mathbb{R}^d)$, which is true by Theorem 3.5. The theorem is proved. \hfill \square

4. Some results on ultraspherical polynomials and Bessel functions

We will prove some technical results involving ultraspherical polynomials and modified Bessel functions with complex parameters. Let us recall several facts concerning both special functions.

Rodrigues’ formula for the normalized ultraspherical polynomials takes the form

$$\lambda = 1/2 P_m^\lambda(u) = \frac{(-1)^m}{2^m(\lambda + 1/2)_m} \frac{d^m}{du^m} \left((1 - u^2)^m\lambda - 1/2\right), \quad m = 0, 1, \ldots, \tag{4.1}$$
where \((\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}\) is the Pochhammer symbol. The normalized ultraspherical polynomials are also given by the explicit formula
\[
P^\lambda_m(\cos \theta) = C^\lambda_m(\cos \theta) = \frac{(\lambda)_k (\lambda)_{m-k}}{k!(m-k)!} \cos (m - 2k) \theta
\]  
for \(\theta \in [0, \pi]\). It is immediate from (4.2) that
\[
|P^\lambda_m(u)| \leq 1.
\]
Since the functions \(\lambda \mapsto (\lambda)_k\) are holomorphic for each fixed \(u \in [-1, 1]\), the ultraspherical polynomials \(P^\lambda_m(u)\) can be extended as a holomorphic function of \(\lambda\) on the domain \(\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > -\frac{1}{2}\}\).

Let \(I_\lambda\) be the modified Bessel function of the first kind and order \(\lambda\), defined as
\[
I_\lambda(z) = \sum_{m=0}^{\infty} \frac{1}{m! (m + \lambda + 1)} \left(\frac{z}{2}\right)^{2m+\lambda}.
\]
Note that the functions \(I_\lambda\) can also be defined for complex values of \(\lambda\). We will use Schlöfli’s integral representation of Poisson type for modified Bessel function, see [15] (5.10.22)
\[
I_\lambda(z) = \frac{(z/2)^\lambda}{\sqrt{\pi} \Gamma(\lambda + 1/2)} \int_{-1}^{1} e^{zu}(1-u^2)^{\lambda-1/2} du, \quad |\arg z| < \pi, \quad \lambda > -\frac{1}{2}.
\]

The following lemma will be useful in expressing the Dunkl–Cesàro mean kernel in terms of the Cesàro kernel for the Laguerre expansions in Subsection 5.2.

**Lemma 4.1.** Let \(z \in \mathbb{C}\) and \(\lambda > -\frac{1}{2}\). Then the following holds
\[
\int_{-1}^{1} e^{zu} P^\lambda_m(u)(1-u^2)^{\lambda-1/2} du = \sqrt{\pi} \Gamma(\lambda + 1/2)(z/2)^{-\lambda} I_{\lambda+m}(z), \quad m = 0, 1, \ldots
\]

**Proof.** In view of Rodrigues’ formula (4.1), we get
\[
\int_{-1}^{1} e^{zu} P^\lambda_m(u)(1-u^2)^{\lambda-1/2} du = \frac{(-1)^m}{2^m(\lambda + 1/2)_m} \int_{-1}^{1} e^{zu} \frac{d^m}{du^m}(1-u^2)^{\lambda+1/2} du
\]
Integrating by parts and using (4.4) we see that
\[
\frac{(-1)^m}{2^m(\lambda + 1/2)_m} \int_{-1}^{1} e^{zu} \frac{d^m}{du^m}(1-u^2)^{\lambda+1/2} du = \frac{(z/2)^m}{(\lambda + 1/2)_m} \int_{-1}^{1} e^{zu} (1-u^2)^{m+\lambda-1/2} du
\]
\[
= \sqrt{\pi} \Gamma(\lambda + 1/2)(z/2)^{-\lambda} I_{\lambda+m}(z)
\]
since \(\Gamma(\lambda + 1/2) = \frac{\Gamma(\lambda+m+1/2)}{(\lambda+1/2)_m}\). This proves the lemma. \(\square\)

In order to estimate the Cesàro kernels for Dunkl–Hermite expansions, we need to consider a variant of Lemma 4.1 where the parameter \(\lambda\) has to be taken complex. In the following lemma we express \(I_{\lambda+m}\) in terms of certain variants of ultraspherical polynomials defined for complex \(\lambda\). Moreover, we also obtain some good estimates for these variants.

**Lemma 4.2.** Let \(\varepsilon > 0\) and \(\lambda = \alpha + i\beta\) with \(\alpha > \frac{1}{2}\) and \(\beta \in \mathbb{R}\). For \(u \in [-1, 1]\), define
\[
Q^\lambda_\varepsilon_m(u) := \frac{1}{\Gamma(i\beta + \varepsilon)} \int_{0}^{1} \chi_{[-u, u]}(s) P^\alpha_m \left(\frac{u}{s}\right) (s^2 - u^2)^{\alpha-1/2} (1-s)^{i\beta+\varepsilon-1} s^{m+1} ds
\]
for \(m = 0, 1, 2, \ldots\). Then, for \(z \in \mathbb{C}\), we have the identity
\[
\int_{-1}^{1} e^{zu} Q^\lambda_\varepsilon_m(u) du = \frac{\sqrt{\pi} \Gamma(\alpha + 1/2)}{2} \cdot \frac{I_{\lambda+m+\varepsilon}(z)}{(z/2)^{\lambda+\varepsilon}}.
\]
Moreover, we have the uniform estimates

$$|Q_m^{\lambda, \varepsilon}(u)| \leq C \frac{(1 - u^2)^{\alpha - \frac{1}{2}}}{|\Gamma(i\beta + 1)|},$$

for all $-1 \leq u \leq 1$. Consequently,

$$Q_m^\lambda(u) := \lim_{\varepsilon \to 0} Q_m^{\lambda, \varepsilon}(u)$$

exists, satisfies (4.2) and

$$(4.7) \quad \int_{-1}^{1} e^{zu} Q_m^\lambda(u) \, du = \frac{\sqrt{\pi} \Gamma(\alpha + 1/2)}{2} \cdot I_{\lambda + m}(z).$$

Proof. For $\text{Re}(\nu) > 0$ and $\text{Re}(\mu) > 0$, we have (see [2, Theorem 4.11.1])

$$\frac{J_{\nu + \mu}(z)}{z^{\nu + \mu}} = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_0^1 \frac{J_{\nu}(sz)}{(sz)^\nu} (1 - s^{2})^{\mu-1} s^{2\nu+1} \, ds,$$

where $J_{\nu}$ is the Bessel function of order $\nu$. Using the relation $\frac{J_{\nu}(iz)}{(iz)^\nu} = \frac{I_{\nu}(z)}{(z/2)^\nu}$ and taking $\nu = \alpha + m$ and $\mu = \varepsilon + i\beta$, i.e. $\nu + \mu = \lambda + m + \varepsilon$, the above can be written as

$$I_{\lambda + m + \varepsilon}(z) = \frac{2}{\Gamma(i\beta + \varepsilon)} \int_0^1 \frac{I_{\alpha + m}(sz)}{(sz/2)^{\alpha + m}} (1 - s^{2})^{i\beta + \varepsilon - 1} s^{2\alpha + 2m + 1} \, ds$$

$$= \frac{2}{\Gamma(i\beta + \varepsilon)} \int_0^1 \frac{I_{\alpha + m}(sz)}{(sz/2)^{\alpha}} (1 - s^{2})^{i\beta + \varepsilon - 1} s^{2\alpha + m + 1} \, ds.$$

Therefore, in view of this and (4.5), we get

$$\frac{I_{\lambda + m + \varepsilon}(z)}{(z/2)^{\lambda + \varepsilon}} = \frac{2}{\Gamma(i\beta + \varepsilon)\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^1 \int_{-1}^{1} e^{zu} P_m^{\alpha}(u) (1 - u^{2})^{\alpha - \frac{1}{2}} du (1 - s^{2})^{i\beta + \varepsilon - 1} s^{2\alpha + m + 1} \, ds du.$$

By making a change of variables and then a change of the order of integration, the right hand side of the above can be written as

$$\frac{2}{\Gamma(i\beta + \varepsilon)\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_{-1}^{1} e^{zu} \int_{[u, 1]} \chi_{[|u|, 1]}(s) P_m^{\alpha} \left( \frac{u}{s} \right) (s^{2} - u^{2})^{\alpha - \frac{1}{2}} (1 - s^{2})^{i\beta + \varepsilon - 1} s^{2\alpha + m + 1} \, ds du.$$

Thus we have the desired identity

$$\frac{I_{\lambda + m + \varepsilon}(z)}{(z/2)^{\lambda + \varepsilon}} = \frac{2}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_{-1}^{1} e^{zu} Q_m^{\lambda, \varepsilon}(u) \, du.$$

We now proceed to show that $\lim_{\varepsilon \to 0} Q_m^{\lambda, \varepsilon}(u)$ exists and satisfies the required estimate. In order to do so, let us write

$$f_m(s) = P_m^{\alpha}(s) (1 - s^{2})^{\alpha - \frac{1}{2}}, \quad g_m(s) = s^{2\alpha + m - 1}$$

so that we can write

$$Q_m^{\lambda, \varepsilon}(u) = \frac{i\beta + \varepsilon}{\Gamma(i\beta + \varepsilon + 1)} \int_{[u]} f_m \left( \frac{u}{s} \right) (1 - s^{2})^{i\beta + \varepsilon - 1} g_m(s) \, ds.$$

Integrating by parts and noting that the boundary terms vanish (since $\alpha > \frac{1}{2}$) we have

$$Q_m^{\lambda, \varepsilon}(u) = \frac{1}{2\Gamma(i\beta + \varepsilon + 1)} \int_{[u]} \frac{d}{ds} \left( f_m \left( \frac{u}{s} \right) g_m(s) \right) (1 - s^{2})^{i\beta + \varepsilon} \, ds.$$
Observe that $|Q_m^{\lambda,\varepsilon}(u)| \leq \frac{1}{2\pi \Gamma(i\beta + 1)} \int_{|u|}^1 \frac{d}{ds} \left(f_m \left(\frac{u}{s}\right)g_m(s)\right) (1 - s^2)^{i\beta} \, ds$. Besides, it is easy to check that we can now pass to the limit as $\varepsilon \to 0$ and define

$$Q_m^\lambda(u) := \lim_{\varepsilon \to 0} Q_m^{\lambda,\varepsilon}(u) = \frac{1}{2\pi \Gamma(i\beta + 1)} \int_{|u|}^1 \frac{d}{ds} \left(f_m \left(\frac{u}{s}\right)g_m(s)\right) (1 - s^2)^{i\beta} \, ds.$$

On one hand, the boundedness $\|P_m^\delta(u)\|$ of $P_m^\delta(u)$ leads to the estimate

$$\int_{|u|}^1 \left|f_m \left(\frac{u}{s}\right)\right| g_m(s) \, ds \leq C \left(g_m(1) - g_m(|u|)\right) \leq 2C.$$

On the other hand, from Rodrigues’ formula (4.1) it is easy to see that $|f_m'(u)| \leq C(1 - u^2)^{\alpha - \frac{3}{2}}$ and therefore,

$$\int_{|u|}^1 \left|f_m' \left(\frac{u}{s}\right)\right| \left|\frac{|u|}{s^2}\right| g_m(s) \, ds \leq C \int_{|u|}^1 (s^2 - u^2)^{\alpha - \frac{3}{2}} \frac{|u|}{s} s^{m+1} \, ds$$

which gives the estimate (as $|u| \leq s \leq 1$)

$$|Q_m^{\lambda,\varepsilon}(u)| \leq \frac{C}{\Gamma(i\beta + 1)} (1 - u^2)^{\alpha - \frac{3}{2}}.$$

As $Q_m^\lambda(u)$ is defined as the limit of $Q_m^{\lambda,\varepsilon}(u)$ it follows that $Q_m^\lambda(u)$ also satisfies the same estimate. Passing to the limit in (4.6) we see that

$$\int_{-1}^1 e^{zn} Q_m^\lambda(u) \, du = \frac{\sqrt{\pi} \Gamma(\alpha + 1/2)}{2} \cdot \frac{I_{\lambda+m}(z)}{(z/2)^\lambda}.$$  

This completes the proof. 

\(\square\)

5. Cesàro means for the Dunkl–Hermite expansions: proof of the main results

In this section we establish various relations that exist between Cesàro kernels for Dunkl–Hermite expansions, Cesàro kernels for Hermite expansions and Cesàro kernels for Laguerre expansions with shifted parameters. These relations can also be extended to complex values of the parameters involved. These facts will lead us to a proof of Theorem 1.2 and, as a consequence, of Theorem 1.1

5.1. Cesàro kernels for Hermite and Dunkl–Hermite expansions. In this subsection we obtain an expression for the Cesàro kernel for the Dunkl–Hermite expansions in terms of the Cesàro kernel for the standard Hermite expansions. For $\delta \geq 0$, it is clear from (1.3) and (2.8) that the kernel of the Cesàro means $\sigma_{N,k}^{\delta,d}$ is given by

$$\sigma_{N,k}^{\delta,d}(x, y) := \frac{1}{A_N^\delta} \sum_{j=0}^N A_{N-j}^{\delta} \Phi^{(d)}_{j,k}(x, y),$$

where $\Phi^{(d)}_{j,k}(x, y)$ are those in (2.9). Observe that, when $\kappa = 0$, we have that $\sigma_{N,0}^{\delta,d} = \sigma_N^{\delta,d}$. In view of the Mehler’s formula (2.7) it follows that $\Phi^{(d)}_{j,k}(r x', s y')$ is a function of $r, s$ and $u = x' \cdot y'$. The same is true for the Cesàro kernels and so we will sometimes write $\Phi^{(d)}_{j,k}(r; s; u)$ instead of $\Phi^{(d)}_{j,k}(r x', s y')$ and $\sigma_{N,k}^{\delta,d}(r, s; u)$ instead of $\sigma_{N,k}^{\delta,d}(r x', s y')$. The following proposition contains expressions for the $h$-harmonic coefficients of Dunkl–Cesàro kernels in terms of $d$-dimensional and $(d + 1)$-dimensional Cesàro kernels.
**Proposition 5.1** (Funk–Hecke for Cesàro–Dunkl–Hermite). For $\gamma$ defined as in (2.3), let

\begin{equation}
(5.1) \quad c_{d,\gamma} = \frac{2\pi^{(d-1)/2}}{\Gamma\left(d\frac{d-1}{2} + \gamma\right)}
\end{equation}

and $\lambda = \frac{d}{2} + \gamma - 1$. Then, for any spherical $h$-harmonic $Y_{m,\ell}^h$ of degree $m$, we have the identity

\begin{equation}
(5.2) \quad \int_{S^{d-1}} \sigma_{N,\kappa}^d(rx', sy')Y_{m,\ell}^h(y')h_k^2(y')\,d\sigma(y')
= c_{d,\gamma} \sum_{j=0}^{[N/2]} A_j^{d-1} A_{N-2j}^d \left( \int_{-1}^{1} \sigma_{N-2j}^d(r, s; u)P_m^\lambda(u)(1 - u^2)^{\lambda - 1/2}\,du \right) Y_{m,\ell}^h(x').
\end{equation}

Moreover, we also have

\begin{equation}
(5.3) \quad \int_{S^{d-1}} \sigma_{N,\kappa}^d(rx', sy')Y_{m,\ell}^h(y')h_k^2(y')\,d\sigma(y')
= c_{d+1,\gamma} \sum_{j=0}^{[N/2]} A_j^{d+1} A_{N-2j}^d \left( \int_{-1}^{1} \sigma_{N-2j}^{d+1}(r, s; u)P_m^\lambda(u)(1 - u^2)^{\lambda - 1/2}\,du \right) Y_{m,\ell}^h(x').
\end{equation}

**Proof.** By integrating Mehler’s formula for the Dunkl–Hermite functions (2.7) against $Y_{m,\ell}^h(y')$ and using Funk–Hecke formula in Corollary 2.2 we get

\begin{equation}
(5.4) \quad \sum_{k=0}^{\infty} w^k \int_{S^{d-1}} \Phi_{k,\kappa}^d(rx', sy')Y_{m,\ell}^h(y')h_k^2(y')\,d\sigma(y')
= 2(1 - w^2)^{-d/2 + \gamma} e^{-\frac{1}{2} \left( \frac{1 + w^2}{1 - w^2} \right)(r^2 + s^2)} \left( \int_{-1}^{1} e^{\frac{2\pi r}{1 - w^2}u}P_m^\lambda(u)(1 - u)^{\lambda - 1/2}\,du \right) Y_{m,\ell}^h(x').
\end{equation}

Now, comparing the right hand side of the above with the Mehler’s formula for the Hermite expansions (3.1), we conclude that

\begin{equation}
(5.5) \quad \sum_{k=0}^{\infty} w^k \int_{S^{d-1}} \Phi_{k,\kappa}^d(rx', sy')Y_{m,\ell}^h(y')h_k^2(y')\,d\sigma(y')
= \frac{2\pi^{(d-1)/2}(1 - w^2)^{-\gamma}}{\Gamma\left(d\frac{d-1}{2} + \gamma\right)} \sum_{k=0}^{\infty} \left( \int_{-1}^{1} \Phi_{k}^d(r, s; u)P_m^\lambda(u)(1 - u)^{\lambda - \frac{1}{2}}\,du \right) w^k Y_{m,\ell}^h(x').
\end{equation}

By multiplying the identity (5.5) by $(1 - w)^{-\delta - 1}$ and using the expansion $(1 - w)^{-\delta - 1} = \sum_{k=0}^{\infty} A_k^d w^k$, we obtain

\begin{align*}
\sum_{k=0}^{\infty} w^k \sum_{j=0}^{k} A_{k-j}^d \left( \int_{S^{d-1}} \Phi_{j,\kappa}^d(rx', sy')Y_{m,\ell}^h(y')h_k^2(y')\,d\sigma(y') \right) \\
= c_{d,\gamma}(1 - w)^{\gamma} \sum_{k=0}^{\infty} w^k \sum_{j=0}^{k} A_{k-j}^d \left( \int_{-1}^{1} \Phi_{j}^d(r, s; u)P_m^\lambda(u)(1 - u)^{\lambda - \frac{1}{2}}\,du \right) Y_{m,\ell}^h(x') \\
= c_{d,\gamma}(1 - w)^{\gamma} \sum_{k=0}^{\infty} w^k A_k^d \left( \int_{-1}^{1} \sigma_{k}^d(r, s; u)P_m^\lambda(u)(1 - u^2)^{\lambda - 1/2}\,du \right) Y_{m,\ell}^h(x').
\end{align*}
Comparing the coefficients of \( w^k \) on both sides in the above we obtain (5.2), namely

\[
\int_{\mathbb{S}^{d-1}} \sigma_{N,\kappa}^{\delta,d}(r \sigma', s \sigma') Y_{m,\ell}^h(\sigma') h^2_\kappa(\sigma') \, d\sigma(\sigma') = \frac{c_{d,\gamma}}{N^2} \sum_{k=0}^{[N/2]} A^j_k A^{N-2j} \left( \int_{-1}^{1} \sigma_{N-2}(r, s; u) P^\lambda_m(u)(1 - u^2)^{\lambda-1/2} \, du \right) Y_{m,\ell}^h(x').
\]

Observe that the right hand side of (5.4) remains the same if we replace \( d \) by \( d+1 \) and \( \gamma \) by \( \gamma - \frac{1}{2} \). Consequently, in view of Mehler’s formula for the Hermite expansions (3.1), the left hand side of (5.4) is also equal to

\[
\frac{2\pi^{d/2}(1 - u^2)^{-\gamma + \frac{1}{2}}}{\Gamma(\frac{d}{2} + \gamma)} \sum_{k=0}^{\infty} \left( \int_{-1}^{1} \Phi_k^{(d+1)}(r, s; u) P^\lambda_m(u)(1 - u^2)^{\lambda-1/2} \, du \right) w^k Y_{m,\ell}^h(y).
\]

Therefore, by repeating an analogous reasoning, we also have

\[
\int_{\mathbb{S}^{d-1}} \sigma_{N,\kappa}^{\delta,d}(r \sigma', s \sigma') Y_{m,\ell}^h(\sigma') h^2_\kappa(\sigma') \, d\sigma(\sigma') = \frac{c_{d+1,\gamma} - \frac{1}{2}}{N^2} \sum_{j=0}^{[N/2]} A^j_k A^{N-2j} \left( \int_{-1}^{1} \sigma_{N-2}(r, s; u) P^\lambda_m(u)(1 - u^2)^{\lambda-1/2} \, du \right) Y_{m,\ell}^h(x').
\]

This completes the proof of the proposition. \( \square \)

Let \( T_{N,m} \) be the linear operators defined by

\[
T_{N,m} f_{m,j}(r) := \int_{0}^{\infty} K_{N,m}^{\delta,\gamma}(r, s) f_{m,j}(s) s^{d+2\gamma-1} \, ds
\]

where \( f_{m,j}(r) \) are the \( h \)-harmonic coefficients of \( f \) defined in (2.5) and the kernels \( K_{N,m}^{\delta,\gamma} \) are given by

\[
K_{N,m}^{\delta,\gamma}(r, s) := \frac{c_{d,\gamma}}{N^2} \sum_{j=0}^{[N/2]} A^j_k A^{N-2j} \left( \int_{-1}^{1} \sigma_{N-2}(r, s; u) P^\lambda_m(u)(1 - u^2)^{\lambda-1/2} \, du \right),
\]

where \( c_{d,\gamma} \) as in (5.1). By expanding \( \sigma_{N,\kappa}^{\delta,d}(\sigma') \) in terms of \( h \)-harmonics and using Proposition 5.1 we can easily deduce the following.

**Corollary 5.2.** For any Schwartz class function \( f \) we have

\[
\sigma_{N,\kappa}^{\delta,d}(x') = \sum_{m=0}^{N} \sum_{j=1}^{d(m)} T_{N,m}^{\delta,\gamma}(r) Y_{m,j}(x').
\]
Remark 5.3. In view of identity (5.3) in Proposition 5.1, the kernels $K_{N,m}^{\delta,\gamma}$ can also be written in the form

$$K_{N,m}^{\delta,\gamma}(r,s) = \frac{c_{d+1,\gamma-1}}{A_{N}^{d}} \sum_{j=0}^{[N/2]} A_{j}^{\gamma - \frac{1}{2}} A_{N-2j}^{d} \int_{-1}^{1} A_{N-2j}^{d+1}(r,s;u)P_{m}^{\lambda}(u)(1-u^{2})^{\lambda-1/2} \, du.$$  

Therefore, we can express the kernels $K_{N,m}^{\delta,\gamma}(r,s)$ and consequently the operators $T_{N,m}^{\delta,\gamma} f$ in terms of both $d$-dimensional and $(d+1)$-dimensional Hermite Cesàro kernels. Indeed, Corollary 5.2 is also valid for both expressions.

5.2. The Laguerre connection. We are going to express the kernel $K_{N,m}^{\delta,\gamma}(r,s)$ in (5.7) in terms of Laguerre functions. This is contained in Proposition 5.4 below. This result will not be enough for our purposes; an improved version with complex parameters is required, see Proposition 5.5. Nevertheless, we first show the non-complex version in detail since it is interesting to understand that, philosophically, there is an underlying phenomenon of transplantation.

Let us recall some basic facts about Laguerre functions. We have the following generating function identity for Laguerre functions (see [21, (1.1.47)])

$$\sum_{k=0}^{\infty} \psi_{k}^{\alpha}(r)\psi_{k}^{\alpha}(s)w^{2k} = 2(1-w^{2})^{-1}(rs)^{-\alpha}e^{-\frac{1}{2}(1+k^{2}r^{2})/2} I_{\alpha}(2rs).$$

From this identity, we easily deduce that

$$\sum_{k=0}^{\infty} (rs)^{m}\psi_{k}^{\alpha+m}(r)\psi_{k}^{\alpha+m}(s)w^{2k+m} = 2(1-w^{2})^{-1}e^{-\frac{1}{2}(1+k^{2}r^{2})/2} I_{\alpha+m}(2rs).$$

Proposition 5.4. For $d \geq 2$, let $\lambda = \frac{d}{2} + \gamma - 1$. Then the kernel $K_{N,m}^{\delta,\gamma}$ in (5.7) (or in (5.8)) can be written as

$$K_{N,m}^{\delta,\gamma}(r,s) = \frac{1}{A_{N}^{d}} \sum_{j=0}^{[(N-m)/2]} A_{N-m-2j}^{\delta}(rs)^{m}\psi_{j}^{\lambda+m}(r)\psi_{j}^{\lambda+m}(s)$$

provided $N \geq m$. For other values of $N$, $K_{N,m}^{\delta,\gamma}(r,s) = 0$.

Proof. We integrate Mehler’s formula for $\Phi_{N}(x,y)$ in (5.11) against $P_{m}^{\lambda}(u)(1-u^{2})^{\lambda-1/2}$ over $(-1,1)$, and apply Lemma 4.1 with $z = \frac{2w}{1-w^{2}}rs$, so that we get

$$\sum_{N=0}^{\infty} w^{N} \int_{-1}^{1} \Phi_{N}(r,s;u)P_{m}^{\lambda}(u)(1-u^{2})^{\lambda-1/2} \, du$$

$$= \pi^{-d/2}(1-w^{2})^{-d/2}e^{-\frac{1}{2}(1+k^{2}r^{2})/2} \int_{-1}^{1} P_{m}^{\lambda}(u)(1-u^{2})^{\lambda-1/2}(u)e^{zu} \, du$$

$$= \pi^{-(d-1)/2}\Gamma(\lambda + 1/2)(1-w^{2})^{-d/2}e^{-\frac{1}{2}(1+k^{2}r^{2})/2} I_{\lambda+m}(z).$$

Comparing this with the generating function identity for the Laguerre functions (5.9) we obtain

$$\sum_{N=0}^{\infty} w^{N} \int_{-1}^{1} \Phi_{N}(r,s;u)P_{m}^{\lambda}(u)(1-u^{2})^{\lambda-1/2} \, du$$

$$= \frac{\Gamma(\lambda + 1/2)}{2\pi^{(d-1)/2}}(1-w^{2})^{\gamma}(wrs)^{m} \sum_{N=0}^{\infty} \psi_{N}^{\lambda+m}(r)\psi_{N}^{\lambda+m}(s)w^{2N}.$$
On the one hand, multiplying left hand side in (5.10) by \((1 - w)^{-\delta - 1}\), we have
(5.11)
\[
(1 - w)^{-\delta - 1} \left( \sum_{N=0}^{\infty} w^N \Phi_N^{(d)} (r, s; u) \right) = \sum_{N=0}^{\infty} \left( \sum_{j=0}^{N} A_{N-j}^{\delta, d}(r, s; u) \right) w^N = \sum_{N=0}^{\infty} A_N^{\delta, d}(r, s; u) w^N.
\]

On the other hand, we also have
(5.12)
\[
(1 - w)^{-\delta - 1} \frac{\Gamma(\lambda + 1/2)}{2\pi^{(d-1)/2}} (1 - w^2)^\gamma (wrs)^m \left( \sum_{N=0}^{\infty} \psi_N^{\lambda+m}(r) \psi_N^{\lambda+m}(s) w^{2N} \right)
= \frac{\Gamma(\lambda + 1/2)}{2\pi^{(d-1)/2}} (1 - w^2)^\gamma (wrs)^m \sum_{N=0}^{[N/2]} \left( \sum_{j=0}^{[N/2]} A_{N-2j}^{\delta, d}(r, s; u) \right) w^N.
\]

Therefore, in view of (5.11) and (5.12), from (5.10), we get
\[
\sum_{N=0}^{\infty} A_N^{\delta} \left( \int_{-1}^{1} \sigma_N^{\delta, d}(r, s; u) P_m^\lambda(u) (1 - u^2)^{\lambda-1/2} du \right) w^N
= \frac{\Gamma(\lambda + 1/2)}{2\pi^{(d-1)/2}} (1 - w^2)^\gamma (wrs)^m \sum_{N=0}^{[N/2]} \left( \sum_{j=0}^{[N/2]} A_{N-2j}^{\delta, d}(r, s; u) \right) w^N.
\]

Now, we multiply both sides by \((1 - w^2)^{-\gamma}\) and rearrange to get
\[
\sum_{N=0}^{\infty} \left( \sum_{j=0}^{[N/2]} A_{N-2j}^{\delta, d}(r, s; u) \int_{-1}^{1} \sigma_N^{\delta, d}(r, s; u) P_m^\lambda(u) (1 - u^2)^{\lambda-1/2} du \right) w^N
= \frac{\Gamma(\lambda + 1/2)}{2\pi^{(d-1)/2}} \sum_{N=0}^{[N/2]} \left( \sum_{j=0}^{[N/2]} A_{N-2j}^{\delta, d}(r, s; u) \right) w^{N+m}
= \frac{\Gamma(\lambda + 1/2)}{2\pi^{(d-1)/2}} \sum_{N=0}^{[N/2]} \left( \sum_{j=0}^{[N/2]} A_{N-2j}^{\delta, d}(r, s; u) \right) w^N.
\]

Equating the coefficients of \(w^N\) on both sides, by (5.7) we see that
\[
K_{N,m}^{\delta, \gamma}(r, s) = \frac{1}{A_N^{\delta}} \sum_{j=0}^{[N/2]} A_{N-2j}^{\delta, d}(r, s; u) \psi_j^{\lambda+m}(r) \psi_j^{\lambda+m}(s)
\]
for \(N \geq m\). It is clear that \(K_{N,m}^{\delta, \gamma}(r, s) = 0\) for \(N < m\). This completes the proof of the proposition.

Repeating the same procedure, but starting with Mehler’s formula (3.1) for \(\Phi_N^{(d+1)}(x, y)\), \(x, y \in \mathbb{R}^{d+1}\), we obtain the proposition for the kernel \(K_{N,m}^{\delta, \gamma}(r, s)\) expressed as in (5.8).

5.3. Extension to complex parameters. In order to prove Theorem 1.2 we need to extend our operators for complex parameters and get proper expressions for them.

First, observe that the kernels \(K_{N,m}^{\delta, \gamma}\), which are expressible in terms of Laguerre functions \(\psi_k^{\lambda+m}\) by Proposition 5.4, make sense even if \(\delta\) and \(\gamma\) are complex. For \(\zeta \in \mathbb{C}\) recall the definition of \(\delta(\zeta)\) in (3.2) and let us also define
(5.13)
\[
\lambda(\zeta) = \frac{d}{2} + \zeta - 1.
\]
Fix $\varepsilon > 0$ and consider the sequence of operators

$$T_{N,m}^\varepsilon(\zeta)f(r) = \int_0^\infty K_{N,m}^\varepsilon(r,s)f(s)\,ds$$

defined for $f \in L^p(\mathbb{R}^+, dr)$ with kernel

$$K_{N,m}^\varepsilon(r,s) = \frac{r^{2\zeta}(r)}{s^{\zeta}(s)} K_{N,m}^\delta(\zeta+\varepsilon, \zeta)(r,s)$$

where $p > 2$. More explicitly, by Proposition 5.4,

$$K_{N,m}^\varepsilon(r,s) = \left(\frac{r s}{A^\delta(\zeta+\varepsilon)}\right)^{\frac{\varepsilon}{2}} \frac{\sum_{k=0}^{(N-m)/2} \sum_{j=0}^{(N-m-2j)/2} A_{N-m-2j}^1 \psi_j^{\lambda(\zeta)+m}(r) \psi_j^{\lambda(\zeta)+m}(s)}{}.$$  

Then, we shall prove that the kernels $K_{N,m}^\varepsilon(r,s)$ can be expressed in terms of the $d$-dimensional and $(d+1)$-dimensional kernels of Cesàro means, for certain values of $\zeta$. This result is contained in Proposition 5.5 that is a complex variant of Proposition 5.4.

**Proposition 5.5.** Let $C_d = \frac{4\pi(d-1)^2}{1(d-1)^2}$. For any $\beta \in \mathbb{R}$ we have

$$K_{N,m}^{\beta/2, \varepsilon}(r,s) = \frac{C_d}{A_N} \sum_{k=0}^{\varepsilon/2} \frac{\varepsilon}{N-k} K_{N,m}^{\beta/2, \varepsilon}(r,s)$$

and

$$K_{N,m}^{1/2, \beta, \varepsilon}(r,s) = \frac{C_d+1}{A_N} \sum_{k=0}^{\varepsilon/2} \frac{\varepsilon}{N-k} K_{N,m}^{1/2, \beta, \varepsilon}(r,s)$$

where the kernels $\tilde{K}_{N,m}^{\beta/2, \varepsilon}$ and $\tilde{K}_{N,m}^{1/2, \beta, \varepsilon}$ are defined by

$$\tilde{K}_{N,m}^{\beta/2, \varepsilon}(r,s) = \sum_{j=0}^{[k/2]} A_j^{\beta/2-1} \frac{s^{\beta/2}}{r^\delta(\beta+\varepsilon)} \int_{-1}^1 \sigma_{k-2j}^{\delta(\beta+\varepsilon)}(r,s;u)Q_m^{\lambda(\beta)}(u)\,du$$

and

$$\tilde{K}_{N,m}^{1/2, \beta, \varepsilon}(r,s) = \sum_{j=0}^{[k/2]} A_j^{1/2-1} \frac{s^{1/2}}{r^\delta(1+\beta)} \int_{-1}^1 \sigma_{k-2j}^{\delta(1+\beta)}(r,s;u)Q_m^{\lambda(1+\beta)}(u)\,du.$$  

**Proof.** Let $Q_m^{K(\zeta)}(u)$ be as in Lemma 4.2, where $\lambda(\zeta)$ is taken as in (5.13). We integrate Mehler’s formula for $\Phi_N^{(d)}(x,y)$ in (3.1) against $Q_m^{K(\zeta)}(u)$, and use (4.7) with $z = \frac{1-w^2}{1-w^2}rs$, so that

$$\sum_{N=0}^\infty w^N \int_{-1}^1 \Phi_N^{(d)}(r,s;u)Q_m^{\lambda(\beta)}(u)\,du$$

$$= \frac{\Gamma((d-1)/2)}{2\pi(d-1)^2} \frac{1}{(1-w^2)^{d/2}} e^{-\frac{1}{2} \frac{4+u^2}{4+u^2}} \int_{-1}^1 Q_m^{K(\zeta)}(u) e^{zu}\,du$$

$$= \frac{\Gamma((d-1)/2)}{2\pi(d-1)^2} \frac{1}{(1-w^2)^{d/2}} e^{-\frac{1}{2} \frac{4+u^2}{4+u^2}} I_{K(\zeta)+m}(z) \left(\frac{z}{2}\right)^{\lambda(\beta)}.$$  

From this and (5.9) we have

$$\sum_{N=0}^\infty w^N \int_{-1}^1 \Phi_N^{(d)}(r,s;u)Q_m^{\lambda(\beta)}(u)\,du$$
On the other hand, from right hand side of (5.17) we get

\[
(1 - w^2)^{-\frac{\varepsilon}{2}} - i^\beta (1 - w)^{-\delta(i^\beta + \varepsilon) - 1},
\]

which we rewrite as (1 - w^2)^{-\frac{\varepsilon}{2}} - i^\beta (1 - w)^{-\delta(i^\beta + \varepsilon) - 1}(1 + w)^{\frac{\varepsilon}{2}},

where \( \delta(\zeta) \) is as in (3.2). Then, left hand side of (5.17) delivers

\[
(1 - w^2)^{-\frac{\varepsilon}{2}} - i^\beta (1 - w)^{-\delta(i^\beta + \varepsilon) - 1}(1 + w)^{\frac{\varepsilon}{2}} \sum_{N=0}^{\infty} w^N \left( \sum_{j=0}^{[N/2]} \mathcal{A}_j^\varepsilon + i^\beta - \frac{\varepsilon}{2} \int_{-1}^{1} \frac{\Phi_d(r, s; u)Q_m^\lambda(i^\beta)(u) du}{\sigma_{N-2j}} \right)\]

On the other hand, from right hand side of (5.17) we get

\[
C_d^{-1}(1 - w^2)^{-\frac{\varepsilon}{2}} - i^\beta (1 - w)^{-\delta(i^\beta + \varepsilon) - 1}(1 + w)^{\frac{\varepsilon}{2}} \sum_{N=0}^{\infty} (rs)^m \psi_N^\lambda(i^\beta) + m(r) \psi_N^\lambda(i^\beta) + m(s) w^{2N+m}
\]

\[
= C_d^{-1}(1 - w)^{-\delta(i^\beta + \varepsilon) - 1} \sum_{N=0}^{\infty} (rs)^m \psi_N^\lambda(i^\beta) + m(r) \psi_N^\lambda(i^\beta) + m(s) w^{2N+m}
\]

\[
= C_d^{-1} \sum_{N=m}^{\infty} \left( \sum_{j=0}^{[N-m/2]} \mathcal{A}_j^\delta(i^\beta + \varepsilon) \mathcal{A}_{N-m-2j}^\delta(i^\beta + \varepsilon) \psi_j^\lambda(i^\beta) + m(r) \psi_j^\lambda(i^\beta) + m(s) \right) w^N
\]

where the last equality is true in view of Proposition 5.4. Altogether, we have

\[
\sum_{N=0}^{\infty} w^N \left( \sum_{j=0}^{[N/2]} \mathcal{A}_j^\varepsilon + i^\beta - \frac{\varepsilon}{2} \int_{-1}^{1} \frac{\Phi_d(r, s; u)Q_m^\lambda(i^\beta)(u) du}{\sigma_{N-2j}} \right) = \frac{(1 + w)^{-\frac{\varepsilon}{2}}}{C_d} \sum_{N=m}^{\infty} w^N \mathcal{A}_N^\delta(i^\beta + \varepsilon) \mathcal{K}^\delta(i^\beta + \varepsilon, i^\beta)(r, s).
\]

Multiplying both sides above by \( r^{-2\delta(i^\beta)} s^{-2\delta(i^\beta)} (1 + w)^{\varepsilon/2} \) and using \( (1 + w)^{\varepsilon/2} = \sum_{j=0}^{\infty} \left( \frac{\varepsilon}{2} \right)^j ) \), we see that

\[
\sum_{N=0}^{\infty} w^N \left( \sum_{k=0}^{N} \left( \frac{\varepsilon}{2} \right)^{-k} \sum_{j=0}^{[k/2]} \mathcal{A}_j^\varepsilon + i^\beta - \frac{\varepsilon}{2} \int_{-1}^{1} \frac{\Phi_d(r, s; u)Q_m^\lambda(i^\beta)(u) du}{\sigma_{k-2j}} \right) = \frac{1}{C_d} \sum_{N=m}^{\infty} w^N \mathcal{A}_N^\delta(i^\beta + \varepsilon) \mathcal{K}^\delta(i^\beta + \varepsilon, i^\beta)(r, s).
\]

Equating the coefficients of \( w^N \) on both sides, we get

\[
\mathcal{K}^\delta(i^\beta + \varepsilon)(r, s) = \frac{C_d}{\mathcal{A}_N^\delta(i^\beta + \varepsilon)} \sum_{k=0}^{N} \left( \frac{\varepsilon}{2} \right)^{-k} \mathcal{K}^\delta(i^\beta + \varepsilon, i^\beta)(r, s).
\]

Similarly, by using the Mehler’s formula for \( \Phi_d^{(d+1)}(x, y), x, y \in \mathbb{R}^{d+1} \), we have
We consider the function
\[ F_N \] supported on \( \mathbb{R} \) that we need the following variant of three lines lemma proved in [20, Ch. V, Lemma 4.2].

In view of Corollary 5.2 we have that

**Proof of Theorem 1.1.**

Mixed norm estimates for the Cesàro means: proofs of Theorems 1.1 and 1.2.

5.4. Mixed norm estimates for the Cesàro means: proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** In view of Corollary 5.2 we have that

\[
\|\sigma_{N,r}^{d,d} f\|_{L^p(\mathbb{R}^d)} = \left( \int_0^\infty \left( \int_{S^{d-1}} \left| \sigma_{N,r}^{d,d} f(r,x') \right|^2 \, d\sigma(x') \right)^\frac{2}{p} \, dr \right)^{\frac{q}{2}} = \left( \int_0^\infty \left( \int_{S^{d-1}} \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} |T_{N,m}^\delta \gamma f_{m,j}(r)|^2 \, d\sigma(x') \right)^\frac{2}{p} \, dr \right)^{\frac{q}{2}}
\]

Then, with Theorem 1.2 we can conclude.

Therefore, it remains to prove Theorem 1.2. Given sequences of functions \((f_{m,j})\) and \((g_{m,j})\) such that

\[
\int_0^\infty \left( \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{q}{2}} \, dr = \int_0^\infty \left( \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} |g_{m,j}(r)|^2 \right)^{\frac{q}{2}} \, dr = 1
\]

we define \(\tilde{f}_{m,j}(r) = f_{m,j}(r) r^{\delta(\gamma) - \frac{N}{d}}\) and \(\tilde{g}_{m,j}(r) = g_{m,j}(r) r^{\delta(\gamma) - \frac{N}{d}}\), where \(\delta(\gamma)\) is as in (3.2). Then it follows that

\[
\int_0^\infty \left( \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} |\tilde{f}_{m,j}(r)|^2 \right)^{\frac{q}{2}} \, dr = \int_0^\infty \left( \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} |\tilde{g}_{m,j}(r)|^2 \right)^{\frac{q}{2}} \, dr = 1.
\]

We consider the function \(F_N(\zeta)\) on the strip \(0 \leq \text{Re}(\zeta) \leq \frac{1}{2}\) defined by

\[
F_N(\zeta) := \int_0^\infty \left( \sum_{m=0}^{d(m)} \sum_{j=1}^{d(m)} T_{N,m}^\zeta f_{m,j}(r) \tilde{g}_{m,j}(r) \right) \, dr,
\]

where \(T_{N,m}^\zeta(\zeta)\) are the family of operators defined in (5.13). It is clear that when \(f_{m,j}\) are compactly supported on \(\mathbb{R}^d\), \(F_N(\zeta)\) is a holomorphic function in the interior of the strip \(0 \leq \text{Re}(\zeta) \leq \frac{1}{2}\) and continuous up to the boundary. We will prove that \(F_N(\text{Re}(\zeta))\) is bounded in the strip. In order to do that, we need the following variant of three lines lemma proved in [20, Ch. V, Lemma 4.2].
Lemma 5.6 (Stein-Weiss). Suppose $F$ is a function defined and continuous on the unit strip $S = \{ z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1 \}$ that is analytic in the interior of $S$ satisfying
\[
\sup_{x+iy \in S} e^{-a|y|} \log |F(x + iy)| < \infty
\]
for some $a < \pi$. Then
\[
(5.18) \quad \log |F(x)| \leq \frac{1}{2} \sin \pi x \int_{-\infty}^{\infty} \left( \frac{\log |F(iy)|}{\cosh \pi y - \cos \pi x} + \frac{\log |F(1 + iy)|}{\cosh \pi y + \cos \pi x} \right) dy
\]
whenever $0 < x < 1$.

Proposition 5.7. Let $\zeta := \gamma + i\beta$, with $0 \leq \gamma \leq 1/2$ and $\beta \in \mathbb{R}$. We have
\[
|F_N(i\beta)| \leq C_0(\beta) \quad \text{and} \quad |F_N(1/2 + i\beta)| \leq C_1(\beta)
\]
where $C_0(\beta)$ and $C_1(\beta)$ are independent of $N$ and of admissible growth. Moreover,
\[
|F_N(\gamma)| \leq C, \quad \text{for} \quad 0 \leq \gamma \leq 1/2.
\]

Proof. In view of (5.15) in Proposition 5.5, the operator $T^e_{N,m}(i\beta)$ is given by
\[
T^e_{N,m}(i\beta) = \sum_{k=0}^{N} \left( \frac{\varepsilon/2}{N-k} \right) \tilde{T}^e_{N,m}(i\beta)
\]
where $\tilde{T}^e_{N,m}(i\beta)$ is the operator whose kernel is $\tilde{K}_{k,m}(r,s)$, given in Proposition 5.5. The operators $T^e_{N,m}(i\beta)$ can be bounded in terms of the operators $S^e_{N,m}(i\beta)$ in (3.8). Indeed, in view of Proposition 5.5 and the fact that $\sum_{j=0}^{\infty} |(\varepsilon/2_j)| < \infty$, we have
\[
|T^e_{N,m}(i\beta) f_{m,j}(r)| \leq C_0(\beta) \sup_{N} |S^e_{N,m}(i\beta, \delta (\varepsilon/2_j)) f_{m,j}(r)|
\]
provided we have the estimate
\[
\frac{C_d}{A^e_{N}} \sum_{j=0}^{[k/2]} |A^e_j A^e_{k-2j}| \leq C_0(\beta)
\]
for all $0 \leq k \leq N$, $N = 1, 2, 3, \ldots$. This estimate will follow once we have
\[
\frac{C_d}{A^e_{k}} \sum_{j=0}^{[k/2]} |A^e_j A^e_{k-2j}| \leq C(\beta)
\]
for any $k$. This can be proved similarly as in the case of Lemma 3.4; we leave the details to the reader.

With the above observations, we obtain, for $p > 2$,
\[
|F_N(i\beta)| \leq \int_{0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |T^e_{N,m}(i\beta) f_{m,j}(r)|^2 \right)^{1/2} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |g_{m,j}(r)|^2 \right)^{1/2} dr
\]
\[
\leq C_0(\beta) \int_{0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} (\sup_{N} |S^e_{N,m}(i\beta, \delta (\varepsilon/2_j)) f_{m,j}(r)|)^2 \right)^{1/2} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |g_{m,j}(r)|^2 \right)^{1/2} dr
\]
\[
\leq C_0(\beta) \left( \int_{0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} (\sup_{N} |S^e_{N,m}(i\beta, \delta (\varepsilon/2_j)) f_{m,j}(r)|)^2 \right)^{p/2} dr \right)^{1/p}
\]
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Proof of Theorem 1.2. We first observe that when $\sigma$ the standard Hermite expansion and hence Theorem 1.2 is true. Indeed, we have
\[
\left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} \left| g_{m,j}(r) \right|^2 \right)^{p/2} \leq C_0(\beta) \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} \left| f_{m,j}(r) \right|^2 \right)^{p/2} = C_0(\beta),
\]
where in the last inequality we applied Theorem 3.6. Similarly, by (5.16) in Proposition 5.5 we get the estimate
\[
|T_{N,m}^\epsilon(1/2 + i\beta) f_{m,j}(r)| \leq C_1(\beta) \sup_N \left| S_N^{\delta(i+\epsilon/2),2} f_{m,j}(r) \right|.
\]
Then, an analogous reasoning with proper modifications, leads us to $|F_N(1/2 + i\beta)| \leq C_1(\beta)$.

Observe that with the estimates just proven, we apply Lemma 5.6 to our function $F_N$, so that right hand side of (5.18) is finite, therefore ensuring the boundedness of $F_N(\gamma)$, for $0 < \gamma < 1/2$.

Proof of Theorem 1.2. We first observe that when $\gamma = 0$ the Dunkl–Hermite expansion reduces to the standard Hermite expansion and hence Theorem 1.2 is true. Indeed, we have $L^p(\mathbb{R}^d)$ boundedness of $\sigma_N^{\delta,d}$ for any $\delta > \frac{d-1}{4}$, $1 \leq p < \infty$ (see [22]). On the other hand, it is easy to check that the Cesàro kernels associated to standard Hermite expansions are rotation invariant. Then, from a theorem of Rubio de Francia (see [19], Theorem 3.1 and the corresponding Remark (a) therein), we immediately get the desired vector-valued extension. When $\gamma = \frac{1}{2}$, the same reasoning holds by considering Cesàro means $\sigma_N^{\delta,d+1}$. Hence we can restrict ourselves to the case $0 < \gamma < \frac{1}{2}$.

Given $\delta > \frac{d+2\gamma-1}{2}$ we can choose $\varepsilon > 0$ so that $\delta = \frac{d+2\gamma-1}{2} = \delta(\gamma + \varepsilon)$, the last defined as in (3.2). It suffices to prove that, for $p > 2$,
\[
\left\| \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} \left| T_{N,m}^{(\epsilon+\delta),\gamma} f_{m,j}(r) \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, r^{1/d-\gamma-1}dr)} \leq C \left\| \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} \left| f_{m,j}(r) \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^+, r^{1/d-\gamma-1}dr)}.
\]

We can write
\[
\int_0^\infty \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} T_{N,m}^{(\epsilon+\delta),\gamma} f_{m,j}(r) \overline{g_{m,j}(r)} \right) r^{d+2\gamma-1} dr
\]
\[
= \int_0^\infty \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} K_{N,m}^{\delta(\epsilon+\delta),\gamma}(r,s) f_{m,j}(s) s^{d+2\gamma-1} ds \overline{g_{m,j}(r)} \right) r^{d+2\gamma-1} dr
\]
\[
= \int_0^\infty \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} \int_0^\infty \frac{2\alpha(\gamma)}{\alpha(\gamma)} r^{2\alpha(\gamma)} s^{2\gamma} \overline{K_{N,m}^{\delta(\epsilon+\delta),\gamma}}(r,s) f_{m,j}(s) s^{2\alpha(\gamma)} ds \overline{g_{m,j}(r)} \right) r^{d+2\gamma-1} dr
\]
\[
= \int_0^\infty \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} \int_0^\infty K_{N,m}^{\gamma,\epsilon}(r,s) \overline{f_{m,j}(s) g_{m,j}(r)} ds \right) dr
\]
\[
= \int_0^\infty \left( \sum_{m=0}^{\infty} d(m) \sum_{j=1}^{r} T_{N,m}^{\gamma,\epsilon}(r) \overline{f_{m,j}(r) g_{m,j}(r)} \right) dr = F_N(\gamma),
\]
and the last one is bounded, by Proposition 5.7. By an argument of duality, the estimate is valid for all $1 < p < \infty$.

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References

[1] B. Amri and M. Sifi, *Riesz transforms for Dunkl transform*, Ann. Math. Blaise Pascal 19 (2012), no. 1, 247–262.
[2] G. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge, 1999.
[3] R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. 87 (1965), 695–708.
[4] Ó. Ciaurri and J. L. Varona, *Two-weight norm inequalities for the Cesàro means of generalized Hermite expansions*, J. Comput. Appl. Math. 178 (2005), 99–110.
[5] F. Dai and H. Wang, *A transference theorem for the Dunkl transform and its applications*, J. Funct. Anal. 258 (2010), 4052–4074.
[6] F. Dai and Y. Xu, *Approximation theory and harmonic analysis on spheres and balls*, Springer Monographs in Mathematics, Springer, 2013.
[7] M. F. E. De Jeu, *The Dunkl transform*, Invent. Math. 113 (1993), 147–162.
[8] J. F. van Diejen and L. Vinet, *Calogero–Sutherland–Moser Models*, CRM Series in Mathematical Physics, Springer-Verlag, New York, 2000.
[9] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. 311 (1989), no. 1, 167–183.
[10] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*. Encyclopedia of Mathematics and its Applications, 81, Cambridge University Press, Cambridge, 2001.
[11] J. J. Gergen, *Summability of double Fourier series*, Duke Math. J. 3 (1937), 133–148.
[12] S. Hassani and M. Sifi, *Spectral multipliers for the Dunkl Laplacian*, Commun. Stoch. Anal. 6 (2012), 547–563.
[13] J. L. Krivine, *Théorèmes de factorisation dans les espaces réticulés*, Séminaire Maurey–Schwartz, 1973–1974: Espaces $L^p$, applications radonifiantes et géométrie des espaces de Banach, Exp. 22–23, École Polytech., Paris, 1974.
[14] L. Lapointe and L. Vinet, *Exact operator solution of the Calogero–Sutherland model*, Comm. Math. Phys. 178 (1996), 425–452.
[15] N. N. Lebedev, *Special functions and its applications*, Dover, New York, 1972.
[16] J. Lindenstrauss and L. Tzafriri, *Classical Banach function spaces. II. Function spaces*, 97, Springer-Verlag, Berlin, 1979.
[17] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192 (1998), 519–542.
[18] M. Rösler, *An uncertainty principle for the Dunkl transform*, Bull. Austral. Math. Soc. 59 (1999), 353–360.
[19] J. L. Rubio de Francia, *Transference principles for radial multipliers*, Duke Math. J. 58 (1989), 1–19.
[20] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series 32, Princeton University Press, Princeton, NJ, 1971.
[21] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes 42. Princeton University Press, Princeton, NJ, 1993.
[22] S. Thangavelu, *Summability of Hermite expansions. II*, Trans. Amer. Math. Soc. 314 (1989), no. 1, 143–170.
[23] S. Thangavelu and Y. Xu, *Convolution operator and maximal function for Dunkl transform*, J. Anal. Math. 97 (2005), 25–55.
[24] S. Thangavelu and Y. Xu, *Riesz transform and Riesz potentials for Dunkl transform*, J. Comput. Appl. Math. 199 (2007), 181–195.

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