Optimal Tracking Control for Unknown Linear Systems with Finite-Time Parameter Estimation

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Abstract—The optimal control input for linear systems can be solved from algebraic Riccati equation (ARE), from which it remains questionable to get the form of the exact solution. In engineering, the acceptable numerical solutions of ARE can be found by iteration or optimization. Recently, the gradient descent based numerical solutions has been proven effective to approximate the optimal ones. This paper introduces this method to tracking problem for heterogeneous linear systems. Differently, the parameters in the dynamics of the linear systems are all assumed to be unknown, which is intractable since the gradient as well as the allowable initialization needs the prior knowledge of system dynamics. To solve this problem, the method named dynamic regressor extension and mix (DREM) is improved to estimate the parameter matrices in finite time. Besides, a discounted factor is introduced to ensure the existence of optimal solutions for heterogeneous systems. Two simulation experiments are given to illustrate the effectiveness.

Index Terms—optimal tracking control, unknown linear systems, finite-time parameter estimation, dynamic regressor extension, gradient descent, linear quadratic regulation (LQR)

I. INTRODUCTION

The research on linear controlled systems has lasted for decades and can still receive great attention nowadays. In addition to studying the internal properties such as stability and controllable, it is also hoped in some scenarios to control the system state to follow a predefined or specific trajectory, namely tracking control. Control optimization for linear systems under complex environment is an important and developing research direction [1]. Similarly, the optimization of tracking control is also a topic worthy of studying. The ideas for solving such problems are very clear: construct optimization goals and constraints based on the information of the systems, such as dynamics or measurable state, then obtain the optimal solution as the controller candidate. For linear systems under quadratic performance index, including the linear leader-follower systems with the goal of state tracking, the key information needed for optimization is the prior knowledge about the parameter matrices of system dynamics [1]. Then the optimization problem is simplified to solve algebraic Riccati equations (ARE) related to the parameter matrices [2]. However, it is difficult to directly obtain the exact solution of ARE, things go even worse when the parameter matrices are unknown.

For unknown systems, data including measurable state, input and output can be collected to approximate some internal information of systems or to obtain the solutions of a specific target e.g. LQR performance. In terms of optimal tracking control, the error state of leader-follower systems together with designed control input can be used to analyze the effect of some candidate solutions, then choose the better solutions or adjust the solutions to perform better. For nonlinear completely unknown systems, an offline integral reinforcement learning algorithm was presented in [3] to approximate the solutions with convergence and stability guaranteed. The framework of actor-critic network was used for robust tracking control with unmatched uncertainties in [4]. Unfortunately, the implementation of this algorithm using neural networks [3–5] is impractical because it needs to find the appropriate input features and dimensions [6]. Although the universal approximation ability of neural networks [7] has been taken used in many researches on optimization for nonlinear systems [8–10], in research on optimization for linear cases, more practical algorithms are needed. For linear and completely unknown systems, an online model-free ADP method was introduced in [11] to obtain the approximate solution of ARE according to the state and input. The conservatism of this method lies in full rank assumption and stabilizing initialization, which was afterwards compared with concurrent learning and interval excitation conditions in [12]. Same assumptions were conducted in [13] that proposed an exponential output tracking algorithm with experience replay for unknown linear systems. Besides, as an exploitation of this algorithm, the unknown parameter matrices of exosystem dynamics could be also estimated [13]. This further exploitation inspires that: instead of directly obtaining the optimal control solution from the data, it should be better to first restore the system dynamics using data, and then approximate the optimal solutions. There are two reasons for this view. On the one hand, the data is typically the input and output of system dynamics, it is much more tractable to identify the dynamics using collective data rather than to obtain the solutions of Hamilton-Jacobi-Bellman (HJB) equation [3–2–4–10–14]. One the other hand, typically for linear systems, the more complex HJB equation is simplified to less complex ARE [1–2] which is decoupled.
from system state and input. It is much easier and optimality
guaranteed to obtain the numerical solutions for linear systems.

The identification for dynamics of linear systems can be
transformed into the extended regression problem through
filters \cite{13}, estimators \cite{16}, etc. In order to ensure conver-
gence, a common assumption is persistently excitation (PE)
condition \cite{17}, which is unverifiable and impractical. To relax
this, interval excitation was presented by \cite{18} and taken used
in many researches. By mixing a data-storage based item in
the updating law, namely concurrent learning, the adaptive
controller as well as uncertain dynamics can be tracked with
global stability under IE condition in \cite{19}. This idea has been
further presented in \cite{12, 13, 16} for approximate optimal
control, while in \cite{12} IE condition was compared with full
rank condition \cite{11}. The disadvantage of concurrent learning is
that, the data-storage mechanism should be intelligent enough
to capture sufficiently rich data \cite{12}. As another exploitation
of IE condition, in the field of parameter estimation, global
stability even finite-time estimation can be achieved without
either full rank or data-storage assumptions. In \cite{20}, a set of
filtered auxiliary variables was introduced to estimate the
parameters of manipulators in finite time under IE condition.
A more general technique is called dynamic regressor extension
and mixing (DREM), which was proposed in \cite{21} to estimate
parameters for linear time-varying systems without PE condition.
For linear time-invariant systems, DREM could improve the
temporary stability and realize finite-time convergence under
modified IE condition \cite{22}. A modified updating law of
DREM with short finite-time stability was studied in \cite{23}
in which robustness of this method was also analyzed. It
is of most importance to realize the finite-time convergence
for the objective of dynamics restoring and then optimal
solution approximating, typically for deterministic systems.
Specifically, only accurate model estimation can ensure the
accuracy of numerical optimal solutions and provide validation
strategy for initialization legitimacy, which is always neglected
in most researches e.g. \cite{3, 6, 9, 11}. However, DREM, DREM with
finite-time convergence has not yet been introduced in general
linear invariant systems except in \cite{22} where DREM for
multivariable systems was proposed without drift dynamics.

With the estimated parameters, the approximate optimal
controller can be obtained by solving ARE in an offline way
without additional measurement of state or input. It has been
independently developed using iterative method to obtain the
numerical solutions of ARE, e.g. the well known Kleinman’s
algorithm \cite{24} which needs an initial stabilizing gain. For
deterministic systems with known parameters, it is verifiable and
will converge to the optimal solutions. Alternatively, learning
methods such as policy iteration and value iteration have also
been developed \cite{25, 26}. The stability, convergence rate as
well as computational complexity of these methods has been
analyzed thoroughly, see \cite{27} for detailed review. Conversely,
the gradient based method has been rarely studied, yet it
may be more suitable to approximate the solutions under constraints e.g. equality-type constraints \cite{28}. As is already
analyzed, the optimization problem with ARE can be noncon-
 vex. Recently, a variable substitution method was proposed in
\cite{29} to reparameterize the problem into convex one, proving
the global exponential convergence via gradient method. In
addition, the solutions optimal output regulation using gradient
method was studied in \cite{28} according to $L-$smooth property
of the objective function. What is more, the gradient can also
be approximated through searching algorithm for model-free
setting with convergence guaranteed \cite{30}. Unfortunately, these
methods commonly need a stabilizing initial value, which is
unverifiable for unknown systems. In terms of tracking control
of heterogeneous systems, a discounted factor will be added
to the original ARE, see \cite{3} for details. It is also a task to
analyze applicability of solving modified ARE via gradient
method for leader-follower heterogeneous systems.

Motivated by the observations, this paper investigates the
optimal tracking control for unknown linear invariant sys-
tems using finite-time parameter estimation and gradient-based
searching method. The main contributions are listed below.

1) The optimal tracking control for unknown heteroge-
neous systems is studied. The parameter estimation
and numerical solutions for optimal controller have
exponential convergence rate under mild conditions. In
addition, the conditions on generation of signals and
initialization of control gain are online verifiable, which
are always neglected in most researches of unknown
systems.

2) A finite-time parameter estimation of general linear
time-invariant systems with multivariables via DREM
is proposed under modified interval excitation assump-
tion. Compared with \cite{23}, the drift dynamics as well as
coupled terms are considered. Moreover, the modified
filter-based derivative estimator is more capable than
that used in \cite{21, 22}, which can be used in DREM
while the derivative are unmeasurable.

3) Because of parameter mismatches of leader-follower
systems, a discounted factor is introduced to ensure
the stability of LQR optimization. A gradient-based
method is used to approximate the optimal solutions
of modified ARE with discounted factor. The criteria for
exponential convergence rate and optimality is given
with rigorous mathematical analysis.

Notations: In this paper, $\mathbb{R}$ denotes the set of real number,
$\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the sets of $n$ dimensional real vector
and $m \times n$ dimensional real matrix, respectively. \$\|\cdot\|$ is used
to denote the Frobenius norm for a vector or a matrix. For a
matrix $A \in \mathbb{R}^{n \times n}$, $\text{det} \{A \}$, $\text{tr} \{A \}$, $\text{adj} \{A \}$, $A^T$ and $A^{-1}$
represent the determinant, trace, adjoint matrix, transpose matrix
and the inverse matrix of $A$, respectively. $A > 0$ ($A \geq 0$)
means $A$ is symmetric and positive definite (semi-
positive definite). $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ are the the maximum
and minimum eigenvalues of $A$, respectively. $I$ and $0$ represent
identity matrix and zero matrix with appropriate dimensions.

II. Preliminaries

In this paper, the linear time-invariant systems take the following
form

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) + Cv(t), \quad x(0) = x_0 \quad (1) \\
\dot{v}(t) &= Dw(t), \quad v(0) = v_0 \quad (2)
\end{align}
where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^q$ are the system state, control input, exosystem state and exosystem input, respectively. Dynamics matrix $A \in \mathbb{R}^{n \times n}$ and gain matrices $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times q}$ and $D \in \mathbb{R}^{n \times q}$ are all unknown. For simplicity, states and inputs in autonomous systems (1) and (2) is abbreviated as $x$, $v$, $u$ and $w$ without losing generality.

The goal of tracking is to regulate the state of $x$ to track the state $v$ by designing appropriate control input $u$. The rigorous definition is as follows.

**Definition 1.** Define error state $e_s = x - v$ and the allowable error region $\Omega_e$. System $x$ is said to track system $v$ such that $\exists t_e > 0, \forall t > t_e$ there is $e_s(t) \in \Omega_e$.

It might be impossible to realize $e_s(t) = 0, \forall t > t_e$, when the dynamics of leader signal $v$ and follower system $x$ are different and control input $u$ can not compensate for this difference. For a controllable system, an appropriate control input can narrow the error bound, i.e. $\Omega_e$. Therefore, the solution for tracking problem for heterogeneous systems is a trade-off between the power consumption of control input and error bound, which can be described by LQR problem in which the performance index takes

$$J = \int_0^\infty r(e_s,u) \, d\tau$$

where $r(e_s,u) = e^T_s Q e_s + u^T R u$, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ and $Q, R > 0$. For linear systems, the performance $J$ takes the quadratic form of system state or error state of leader-follower systems. It is noted that the parameter mismatches may lead $J$ to take a more complex form rather than quadratic one. The intuitive explanation is that for two systems with different drift dynamics, the control input should perform real-time compensation to force the state consensus. In this case, $J$ increases even if $\|e_s\| \to 0$ due to $\|u\| \neq 0$. Therefore, a modified performance index with discounted factor $\gamma > 0$ inspired by reinforcement learning has been studied in many research works [21, 22, 23, 24]. Then, the optimization problem is described as

$$\min_u J' = \int_0^\infty e^{-\gamma t} r(e_s,u) \, d\tau, \quad (3)$$

subject to the dynamics (1) and (2). Then, define $J' = e^T_s P e_s$, the optimization problem can be transformed into the Hamilton-Jacobi-Bellman (HJB) equation:

$$e^T_s Q e_s + u^T R u + \frac{\partial V}{\partial e_s} \dot{e}_s - \gamma V = 0. \quad (4)$$

It was discussed in [23] that there exists an unique solution $u^*$ of (3) as long as $\gamma$ satisfies certain bounded condition for nonlinear system. However, since the parameter matrices are all unknown, the Riccati equation that is natural simplified for linear systems is intractable to obtain. Some researches regard such unknown linear systems as affined nonlinear systems, e.g. using data-driven RL method to approximate the solution according to the HJB (4) as [24, 25, 26]. It works but is difficult to guarantee convergence to optimal solutions. Besides, the initial control input is always assumed to be admissible, which is unverifiable under unknown circumstance. In all, these methods complicate the problem to some extend.

An alternative is to introduce observers to estimate the unknown parameters. At present, the research on linear regression has been more comprehensive. For a linear regression problem as

$$y_r = u_r^T \theta_r, \quad (5)$$

in which $y_r \in \mathbb{R}$ and $x_r \in \mathbb{R}^r$ is bound but unknown, one should design a regressor $\hat{\theta}_r$ to estimate $\theta_r$ using collective data. Traditionally, $\hat{\theta}_r$ can evolve along with gradient descent direction of (5) as

$$\dot{\hat{\theta}}_r = -\alpha_r x_r^T (x_r^T \hat{\theta}_r - y_r) = -\alpha_r x_r x_r^T \hat{\theta}_r,$$

where $\alpha_r > 0$ represents the learning rate, $\hat{\theta}_r = \hat{\theta}_r - \theta_r$. As $\theta_r$ is constant, one has

$$\dot{\hat{\theta}}_r = -\alpha_r x_r x_r^T \hat{\theta}_r.$$

To ensure convergence, the measurable signal $x_r x_r^T$ should be persistently excited, which is formally defined as follows.

**Definition 2.** A signal $\phi(t)$ is said to be persistently excited (PE) if there exists $t^\prime > 0$ and positive constant $\sigma$ such that

$$\int_{t}^{t^\prime} \phi(t) \phi^T(t) \, d\tau \geq \sigma I.$$

The PE condition is hard to achieve because it needs to activate the systems all the time which is unverifiable either. DREM relaxes this condition [21] and can be seen in [22], with IE condition defined in section [3] finite-time parameter estimation can be achieved.

After estimation, HJB equation (4) reduces to algebraic Riccati equation (ARE). Then, it is possible to approximate the optimal control input with guaranteed convergence rate. The most widely used method to transform the numerical solution is Kleinman’s algorithm [24] as follows.

**Lemma 1.** For $\dot{x} = Ax + Bu$, let $A_k = A - B_k$, $k = 0, 1, 2, \ldots$. If $A_0$ is stabilizable, then with the following iteration

$$A_k^T P_k + P_k A_k + Q + K_k R K_k = 0,$$

where $P_k$ is the unique positive definite matrix, there is

$$\lim_{k \to \infty} P_k = P^*, \quad \lim_{k \to \infty} K_k = K^*.$$

The Kleinman’s algorithm is simple and efficient with known parameter matrices in (1). An alternative way to approximate the optimal solutions of ARE is gradient-based method, considering $K$ as matrix variable [31]. A distinct advantage of gradient method is the feasibility to solve general ARE with constraints, e.g. equality constraint [28]. In the next two sections, DREM and gradient-based optimization are developed to deal with the optimal tracking control for heterogeneous linear systems.
III. Finite-time Parameter Estimation of System Dynamics

For the unknown heterogeneous systems with measurable state and input $x$, $v$, $w$ and $u$, it can be extended to multivariable regression problem to estimate these unknown matrices. The first objective is to obtain the parameters in $D$ of drift dynamics of $v$ in finite time, dependent only on the measurement of $v$ and $w$. Using linear filter $H_0 = \frac{1}{p+\lambda}$ with $p = \frac{d}{dt}$ to obtain the regression model, one has

$$
\dot{y} = -\lambda y + \dot{v}, \quad y(0) = 0 \tag{6}
$$

$$
\dot{v} = -\lambda v_1 + v, \quad v_1(0) = 0 \tag{7}
$$

$$
\ddot{z} = -\lambda z + w, \quad w_1(0) = 0 \tag{8}
$$

Defining $\Theta^* = D^T$ induces

$$
y = \Theta^* T z. \tag{9}
$$

Since the derivative $\dot{x}$ is not measurable, the filter state $y$ is solved as

$$
y(t) = e^{-\lambda t} \int_0^t e^{\lambda \tau} \dot{v}(\tau) d\tau, \tag{10}
$$

which can be further transformed into a measurable form using integration by parts as

$$
y(t) = v(t) - e^{-\lambda t} v(0) - \lambda v_1(t). \tag{11}
$$

**Remark 1.** These filters are used in [12, 13] as the regressor of system derivative, eliminating the assumption of known derivative information [21]. Compared with the filters used in [22], (11) can track the real derivative trajectory with faster convergence speed.

Introduce the one-input-$q$-output linear operators $H[.]$ as

$$
H[.] = col(H_1[.], H_2[.], \ldots, H_q[,]), \tag{12}
$$

where linear operator $H_i = \frac{1}{p+\lambda}$ is $\mathcal{L}_{\infty}$-stable with $\lambda_i > 0$, $\lambda_i \neq \lambda_j$, $\forall i, j \in \{1, \ldots, q\}$. Applying the operators on the transpose of (9), one has the matrix identity

$$
Y = Z \Theta^* \tag{13}
$$

with $Y = H[y^T] \in \mathbb{R}^{q \times n}$ and $Z = H[z^T] \in \mathbb{R}^{q \times q}$. With the help of adjoint matrix $adj\{Z\}$, define $\delta(t) = det\{Z\}$ and multiply $adj\{Z\}$ on the left of both sides of (13) derives

$$
\mathcal{Y} = \delta(t) \Theta^* \tag{14}
$$

in which $\mathcal{Y} = adj\{Z\} Y$. Furthermore, for $\delta$ is a scalar,

$$
\mathcal{Y}_{ij} = \delta(t) \Theta^*_{ij}, \quad \forall i, j \in \{1, \ldots, q\} \tag{15}
$$

which is the regression model of each element in $D$. A gradient-based approach for scalar variables is feasible to approximate each parameter in $\Theta^*$.

**Proposition 1.** Consider the linear regression function (14) and (15). If $\delta(t) \notin \mathcal{L}_2$, i.e. $\int_0^\infty \delta^2(t) dt = \infty$, then under the following update law,

$$
\dot{\Theta} = \alpha \delta(t) (\mathcal{Y} - \delta(t) \Theta), \tag{16}
$$

in which $\alpha > 0$ represents the learning rate, the estimation error $\dot{\Theta} = \Theta - \Theta^*$ will converge to zero. Moreover, the estimation error of each element $\dot{\Theta}_{ij}$ is monotonous satisfying $|\dot{\Theta}_{ij}(t)| \leq |\dot{\Theta}_{ij}(t)|$, $\forall t \geq t_2$.

**Proof.** Substituting (14) into (16) induces the dynamics of matrix-wise estimation error and element-wise estimation error as

$$
\dot{\Theta} = -\alpha \delta^2(t) \dot{\Theta}, \quad \dot{\Theta}_{ij} = -\alpha \delta^2(t) \dot{\Theta}_{ij}. \tag{17}
$$

Solving the differential equations yields

$$
\dot{\Theta}_{ij}(t) = -\alpha \delta^2(t) \Theta_{ij}(0). \tag{18}
$$

As $\delta^2(t) \geq 0$, $\dot{\Theta}_{ij}(t)$ is monotonic non-increasing. When $\delta(t) \notin \mathcal{L}_2$, all the elements in $\dot{\Theta}$ converge to zero, i.e. $\dot{\Theta}$ will converge to zero.

**Remark 2.** When discussing the conservativeness of PE, it should be noted that PE condition ensures strong regression ability, specifically, exponential stability of estimating the time-varying parameters. As proven in [22], it is of asymptotical stability for DREM to estimate time-varying parameters with $\delta^2(t) \notin \mathcal{L}_2$. In addition, it was discussed that $\delta^2(t) \notin \mathcal{L}$ is weaker that $z(t)$ satisfying PE condition. It can be verified by considering $\delta(t) = \frac{1}{\sqrt{\tau + 1}}$, evidently $\delta^2(t) \notin \mathcal{L}_2$ but not satisfies PE condition.

For linear time-invariant systems, the PE condition as well as $\delta^2(t) \notin \mathcal{L}_2$ is over conservative to estimate the unknown parameters. It is shown next that the parameter matrices in drift dynamics of multivariable systems can be estimated in finite time interval as long as $\delta(t)$ satisfies a modified IE condition which is formally defined below.

**Definition 3.** A signal $\phi(t)$ is said to be interval excitation (IE) if there exists time instance $t_c > 0$ and positive constant $\sigma$ such that

$$
\int_0^{t_c} \phi(\tau) \phi^T(\tau) d\tau \geq \sigma I. \tag{19}
$$

To further introduce the finite-time convergence of the parameter estimation, a modified IE condition can be defined as

$$
\int_0^{t_c} \phi(\tau) \phi^T(\tau) d\tau \geq -\frac{1}{\alpha} \ln (1 - \sigma), \tag{20}
$$

where $\sigma \in (0, 1)$ and $\alpha$ is the learning rate defined in (16).

**Proposition 2.** [23] Define the scalar signal $s_0$ with the following dynamics

$$
\dot{s}_0(t) = -\alpha \delta^2(t) s_0(t), \quad s_0(0) = 1. \tag{21}
$$

Consider the linear regression function (14) and (15). With the updating law (16), if $\delta(t)$ satisfies IE condition (20), the elaborate parameter matrix $\Theta_F$ with the following definition

$$
\Theta_F = \frac{1}{1 - s} (\Theta - s \Theta(0)) \tag{22}
$$

will converge to $\Theta^*$ within $(0, t_{c_e})$, where $s$ is the continuous signal defined by

$$
s = \begin{cases} 
1 - \sigma, & \text{if } s_0(t) > 1 - \sigma; \\
1, & \text{if } s_0(t) \leq 1 - \sigma.
\end{cases} \tag{23}
$$
Proof. It is noticed that $s \leq 1 - \sigma < 1$. When $t \in [0, t_c)$, one has 
$$\Theta_F = \frac{1}{\sigma} (\Theta - (1 - \sigma)\Theta(0)),$$
which implies no finite escape time in the calculation process of $\Theta_F$. Solving the differential equation \((21)\) yields 
$$s_0(t) = e^{-\alpha \int_0^t \sigma(\tau)^2 \, d\tau} s_0(0) = e^{-\alpha \int_0^t \sigma(\tau)^2 \, d\tau}. \quad (24)$$
According to \((18)\), there is 
$$\Theta = s_0(t) \Theta \implies \Theta - \Theta^* = s_0(t) (\Theta(0) - \Theta^*). \quad (25)$$
When IE condition \((20)\) is met and $t \geq t_c$, i.e. $s_0 \leq 1 - \sigma$, one has $s = s_0(t)$. The original parameter matrix can be obtained by 
$$\Theta^* = \frac{1}{1 - s} (\Theta - s\Theta(0)) = \Theta_F, \quad (26)$$
which implies that $\Theta_F$ converges to $\Theta^*$, $\forall t \geq t_c$.

Lemma 2. There exist some other methods to realize finite-time parameter estimation based on DREM as studied in \([23]\). The idea is simple but effective: modify the updating law of estimator \([16]\) into an adaptive law with low order feedback, like 
$$\dot{\tilde{\Theta}} = a\delta(t) (\gamma - \delta(t)) \Theta^* = \Theta_F \quad (27)$$
such that the tracking effect is strengthened when the tracking error becomes small, e.g. when $\tilde{\Theta}_{ij} \in (-1, 1)$. Therefore, finite-time convergence is ensured solving modified \((17)\). The robustness of this method was analyzed in \([21]\) under disturbances. However, these methods are more complicated than that proposed in Proposition \([2]\) which is more difficult to be introduced to general heterogeneous linear systems \([1]\).

So far, the parameter matrix in drift dynamics of linear systems can be estimated as illustrated. It is also feasible to estimate the coefficient matrices $A$, $B$ and $C$ using the same method. To obtain the information of derivative $\dot{x}(t)$, the filters are applied as 
$$\dot{\psi}_x = -\lambda \psi_x + x, \quad \psi_x(0) = 0, \quad (27)$$
$$\dot{\psi}_u = -\lambda \psi_u + u, \quad \psi_u(0) = 0, \quad (28)$$
$$\dot{\psi}_v = -\lambda \psi_v + v, \quad \psi_v(0) = 0. \quad (29)$$
Then, defining $\psi_{\psi} = -\lambda \psi_X + x$, one has 
$$\psi_{\psi} = A\psi_X + B\psi_u + C\psi_v = \Psi^T \psi_z, \quad (30)$$
where 
$$\Psi^* = [A^T, B^T, C^T]^T \psi_{\psi} = [\psi_{\psi}^T, \psi_{\psi}^u, \psi_{\psi}^v]^T \psi_{\psi}^T. \quad (31)$$
Introducing the one-input-$(n + m + q)$-output linear operators $H_{\psi}[]$ and following the same transformation, one has the matrix-wise regression function 
$$\mathcal{E} = \Delta(t) \Psi^*, \quad (32)$$
with $\mathcal{E} = adj \{F\} H_{\psi} \psi_{\psi}^T \in R^{n+m+q} \times n$, $F = H_{\psi} \psi_{\psi}^T \in \mathbb{R}^{(n+m+q) \times (n+m+q)}$ and $\Delta(t) = det\{F\}$, and the element-wise regression function 
$$\mathcal{E}_{ij} = \Delta(t) \Psi_{ij}, \quad \forall i, j \in 1, \ldots, (n + m + q). \quad (33)$$

Theorem 1. For linear deterministic systems \([1]\), the unknown parameter matrices $A$, $B$ and $C$, alternatively $\Psi^*$, can be estimated asymptotically if $\Delta \not\in L_2$ with the following law 
$$\dot{\Psi} = \alpha \Delta(t) (E - \Delta(t) \Psi), \quad (34)$$
where $\Psi$ as the estimation of the coefficient matrix $\Psi^*$. Furthermore, the finite-time convergence can be guaranteed if $\Delta(t)$ satisfies IE condition \((20)\) with \((21), (23)\) and 
$$\Psi_F = \frac{1}{1 - s} (\Psi - s\Psi(0)), \quad (35)$$
where $\Psi_F$ is the finite-time estimated matrix of $\Psi^*$.

The proof process is the same as that in Proposition \([1]\) and \([2]\) which is omitted for brevity.

Remark 3. The finite-time estimation \((35)\) can be computed only once when $t = t_c$, which means that $\Delta(t)$ satisfies modified IE condition \((20)\) within $t \in (0, t_c]$. The reverse calculation of \((35)\) can be more interesting. Suppose that $\sigma$ in modified IE condition \((20)\) is not available at the beginning of estimation. Finite-time estimation in $t \in (0, t_c]$ can be still achieved if $\Delta(t)$ is excited. When $t = t_c$, one can calculate $\sigma = 1 - s_0(t)$ inversely, and assign $s_0$ and $\Psi_F$ accordingly, which is a kind of potential high-gain injection. This holds true only if the signal $\Delta(t)$ satisfies IE condition and $t_c$ is not too small, or there exists finite escape time. As $s_0$ is computed online, the IE condition is online verifiable.

IV. Finite-time Parameter Approximation of Optimal Controller

Since the derivative of error state $e_s$ contains the state $v$, which is not feasible to obtain ARE directly. Therefore, the augmented systems are given by 
$$\dot{X} = AX + BU, \quad X_0 = \col (x_0 - v_0, v_0) \quad (36)$$
where 
$$X = \col (e_s, v), \quad A = \begin{bmatrix} A_{xx} & A_{xu} + C - D \end{bmatrix}, \quad B = \begin{bmatrix} B_{ux} \end{bmatrix} \quad (37)$$
and $U = u$. The tracking value function is defined as 
$$J(X, U) = \int_0^\infty e^{-\gamma\tau} r(X(\tau), U(\tau)) \, d\tau \quad (38)$$
where $r(X, U) = X^T \hat{Q} X + U^T R U$ with $\hat{Q} = \diag\{Q, 0\}$ and $\gamma > 0$ is the discounted factor for stability. For linear systems with linear feedback controller $U^* = -K^* X$, the value function takes the quadratic form as 
$$J(X, U) = f(K) = X_0^T P(K) X_0 \quad (39)$$
where $f(K)$ is the function of $K$ to be minimize and $P(K) > 0$ is the solution of the following algebraic Riccati equation (ARE) 
$$P(K) (A - BK) + (A - BK)^T P(K) \quad (40)$$
$$- \gamma P(K) + K^T R K + \hat{Q} = 0, \quad (41)$$
where $K$ should be in the set of stabilizing gains defined as 
$$\mathcal{S} = \{ K \in \mathbb{R}^{n \times 2n} \mid (A - BK - 0.5\gamma I) \text{ is Hurwitz} \}. \quad (42)$$
According to the stationarity condition of optimality, the optimal controller is
\[
K^* = R^{-1}B^TP^*.
\] (40)

Then, the ARE equation is transformed into
\[
P^*A + A^TP^* - \gamma P^* - P^*BR^{-1}B^TP^* + \hat{Q} = 0.
\] (41)

It is discussed in [14] that ARE (41) has a unique solution if \((A - 0.5\gamma I, B)\) is stabilizable. However, it is difficult to get the explicit form of \(P^*\). The gradient-based optimization method encounters the fact that the feasible solution domain \(S\) can be nonconvex with non-smooth boundary [28].

It is proven in [29, 31] that the LQR problem of linear systems can have convex representation through variable changes within an admissible domain. With the help of finite-time parameter estimation, it is possible to study the specific search space instead of entire parameter space to ensure the convergence and optimality of the gradient method. Specifically, the LQR problem can be solved using direct search for the optimal control gain matrix, as
\[
\min_K f(K)
\] (42)

where
\[
f(K) = \begin{cases} 
tr(P(K)\Pi), & K \in S \\
+\infty, & \text{otherwise}
\end{cases}
\]
\[
\Pi = X_0X_0^T \text{ and } P(K) = \int_0^\infty e^{AT}Qe^{AT}d\tau \text{ is the unique positive definite solution of Lyapunov equation (38) with } \Psi = A-BK - 0.5\gamma I, \Omega = K^TRK + Q. \text{ To optimize this problem, the gradient of } f(K) \text{ to } K \text{ is computed as}
\]
\[
\nabla_K f(K) = 2(RK - B^TP)Z
\] (43)

where operator \(\nabla_K = d/dK\) and \(Z\) is the solution of \(2(A-BK)Z = -\Pi [28]\). Performing gradient descent on \(K\) obtains the continuous-time searching policy
\[
\dot{K} = -\nabla_K f(K), \quad K(t_0) = K_0,
\] (44)

where \(t_0 \geq 0\) is the beginning time point of optimization which is unnecessary to be 0. \(K_0 \in \mathbb{R}^{m \times n}\) is the initial control gain.

As stated above, the search space \(S\) is nonconvex and descent method e.g. gradient descent may not find the optimal solution. With \(A, B\) estimated, within finite time interval \((0, t_c)\), it is feasible to set the initial control gain \(K_0 \in S\) and \(t_0 = t_c\). On the contrary, initial value \(K_0\) is unverifiable in \(S\) for unknown systems even if the parameters is approximated with exponential rate, owning to \(t_0 \to +\infty\). A new search space \(S_{K_0}\) can be set with the help of initial stabilizing control gain \(K_0\) as
\[
S_{K_0} = \{K \in S \mid f(K) \leq f(K_0)\}.
\] (45)

The following statement referred from [32] will be used to analyze the convergence and optimality of (44).

**Lemma 3.** [32] For a continuous, twice differentiable functional \(f(x(t))\) defined in Hilbert space whose derivative is defined by \(g(x(t))\), if there exists a minimum value \(f^* = \inf f(x^*) \neq -\infty\), the gradient descent iterative policy \(\dot{x} = -g(x(t))\) will lead to
\[
\lim_{t \to +\infty} g(x(t)) = 0, \quad \|x(t) - x^*\| \leq r e^{-rt}
\] (46)
in the region \(S' = \{x \mid \|x(t) - x(0)\| \leq \rho\}\) provided that the following conditions are met
\[
\|g(x(t_1)) - g(x(t_2))\| \leq R \|x(t_1) - x(t_2)\|, \quad \|g(x(t))\|^2 \geq 2r (f(x(t)) - f^*),
\] (47)

where \(r, R, \rho > 0\) and \(t = \sqrt{2R/(f(x(0)) - f^*)} \leq 1\). It means that \(g(x(t))\) is Lipschitz-continuous [37] and satisfies Polyak-Lojasiewicz (PL) condition [43].

It is shown in [30] that \(S\) is connected. Moreover, the new search space \(S_{K_0}\) is in \(S\) and the constraint \(tr(P(\Pi)) \leq tr(P(K_0)\Pi)\) is convex such that the new set \(S_{K_0}\) is connected as well. It is also shown in [28] that \(S_{K_0}\) is bounded and strongly convex in the neighborhood of \(K^*\). Because of the connectivity and boundedness of \(S_{K_0}\) and \(S, x^*\) is in \(S_{K_0}\) and should not be in \(S \setminus S_{K_0}\). Therefore, the defined search space is sufficiently satisfying the requirement in Lemma 5.

**Proposition 3.** [28] Assume that \(\Pi\) is positive definite. For \(K_0 \in S\) and \(K \in S_{K_0}\), the function \(\nabla_K f(K)\) is Lipschitz-continuous (namely \(\nabla_K f(K)\) is Lipschitz-continuous) with Lipschitz constant
\[
R = \frac{2f(K_0)}{\lambda_{\min}(Q)} (\lambda_{\max}(R) + \|B\| \zeta)
\] (49)

and satisfies
\[
\|\nabla_K f(K)\|^2 \geq f(K) - f(K^*) \geq \frac{f(K_0)}{\lambda_{\min}(\Pi)\lambda_{\min}(Q)} (\|A - BK\| +\|B\|)^2 \lambda_{\max}(S) \|\nabla_K f(K)\|^2
\] (50)

where \(\zeta = \sqrt{8f(K_0)} \left(\Gamma_{K_0} + \sqrt{(\Gamma_{K_0}^2 + \lambda_{\max}(R))^2}\right), \Gamma_{K_0} = \frac{f(K_0)}{\lambda_{\min}(\Pi)\lambda_{\min}(Q)}\) and \(S > 0\) is the solution of
\[
(A - BK)S + S(A - BK)^T - \gamma S + \Pi = 0.
\] (51)

**Proof.** The main analysis process can be found in [28]. One difference is that a discounted factor \(\gamma\) is introduced in the LQR modeling [36] for the augmented systems [35]. Following the same process, one can denote \(A_\gamma = A - \frac{1}{2}\gamma I\) and obtain the same results by replacing \(A\) with \(A_\gamma\). Another difference is that this paper only considers the Frobenius norm due to simplicity which can be extended to obtain a tighter bound if needed.

The PL condition in Lemma 5 is satisfied as discussed in the following statement by finding the upper bound of \(\lambda_{\max}(S)\) and \(\|K\|\).

**Corollary 1.** Assume that \(\Pi\) is positive definite. For \(K_0 \in S\) and \(K \in S_{K_0}\), the function \(f(x(t))\) satisfies PL condition as
\[
\|\nabla_K f(K)\|^2 \geq 2\mu (f(K) - f(K^*))
\] (52)

where
\[
\mu = \frac{\lambda_{\min}(R)\lambda_{\min}(\Pi)\lambda_{\min}(Q)}{8f(K_0) (\|A\| + 0.5\gamma n + \|B\|^2 f(K_0))}.
\]
Proof. According to (51), $f(x(t))$ can be written as
\[
f(K) = tr(P(K)\Pi) = tr(S(Q + K^T RK)).
\]
For positive semi-definite matrices $X$ and $Y$, there is
\[
\lambda_{\min}(X)tr(Y) \leq tr(XY) \leq \lambda_{\max}(X)tr(Y).
\]
Therefore, on the one hand,
\[
f(K) \geq \lambda_{\min}(Q + K^T RK) tr(S) \\
\geq \lambda_{\min}(Q + K^T RK) \lambda_{\max}(S),
\]
which yields
\[
\lambda_{\max}(S) \leq \frac{f(K)}{\lambda_{\min}(Q + K^T RK)}.
\]
On the other hand,
\[
f(K) \geq tr(SK^T RK) \geq \lambda_{\min}(S) \lambda_{\min}(R) \|K\|^2.
\]
According to (51) and Lyapunov equation (33),
\[
\lambda_{\min}(S) \geq 2\|A - 0.5\gamma I - BK\|_F \\
\geq 2\lambda_{\min}(R) \\
\geq 4\|A\| + \gamma_n + 2\|B\| \|K\|.
\]
Then, a scalar quadratic inequality of $\|K\|$ is obtained as
\[
\|K\|^2 - b\|K\| - c \leq 0
\]
where $b = 2\|B\| f(K)/(\lambda_{\min}(R))$ and $c = 2(\|A\| + 0.5\gamma_n) f(K)/(\lambda_{\min}(R))$. Solving this inequality to obtain the largest solution of $\|K\|$, one has
\[
\|K\| \leq \frac{b + \sqrt{b^2 + 4ac}}{2} \\
\leq \frac{2\|B\| f(K)}{\lambda_{\min}(R)} + \frac{2\|A\| + \gamma_n}{2\|B\|}.
\]
Taking the fact that $\sqrt{1 + a} \leq 1 + a$ if $a \geq 0$. Notice that $f(K_0) \geq f(K)$, $\forall K \in S_{K_0}$. Substituting (54) and (55) into (50) can result in the PL condition (52). The proof is complete.

It has been shown that in the search space $S_{K_0}$, the objective function $f(K)$ is $L$-smooth and satisfies PL inequality (49). The convergence and optimality of the gradient descent search algorithm (43) is ensured in the following theorem.

Theorem 2. Assume that $\Pi$ is positive definite. For $K_0 \in S$ and $K \in S_{K_0}$, there is $\lim_{t \to \infty} \nabla_K f(K) = 0$ and
\[
\|K - K^*\| \leq \rho e^{-\tau(t-\alpha)}, \quad \phi(t) \leq \phi(0)e^{-\rho(t-\alpha)} \quad (56)
\]
where $\rho = \frac{\sqrt{2\rho_0(0)}}{\mu}$ and $\phi(t) = f(K(t)) - f(K^*)$.

Proof. (Convergence) As the descent algorithm is based on gradient, taking the derivative of $f(K(t))$, one has
\[
\frac{d}{dt} f(K(t)) = \frac{d}{dt} K^T \nabla_K f(K) = -\|\nabla_K f(K)\|^2 \leq 0, \quad (57)
\]
which implies that $f(K)$ is monotonic non-increasing and $K \in S_{K_0}$. Define $\hat{K} \in S_{K_0}$ and $f(K) \geq f(K^*)$ be the minimum value of $f(K)$ while searching. Within any neighborhood $B(K) = \{K|K \in (K + \Delta), \|\Delta\| \leq \epsilon, \epsilon > 0\}$, for $\Delta K \in B_{\hat{K}}$ there is
\[
f(K + \Delta K) = f(K) + \Delta K^T \nabla_K f(K).
\]
Selecting the direction $\Delta K = -\nabla_K f(K)$ results in
\[
f(K + \Delta K) = f(K) - \|\nabla_K f(K)\|^2 \leq f(K).
\]
It is obvious that $\|\nabla_K f(K)\| = 0$. Since $f(K)$ is twice differentiable, $K \to \hat{K} = \nabla_K f(K) \to 0$.
(Optimality) Denote $\phi(t) = f(K(t)) - f(K^*)$ whose derivative is given by
\[
\frac{d}{dt} \phi(t) = -\|\nabla_K f(K(t))\|^2 \leq -2\mu \phi(t).
\]
Using comparison principle gives $\frac{d}{dt} \phi(t) \leq \phi(t_0)e^{-2\mu(t-t_0)}$ reflecting that $f(K)$ converges to $f(K^*)$. Meanwhile, for $t_1 > t_2 \geq t_0$, one has
\[
\|K(t_1) - K(t_2)\| \leq \int_{t_1}^{t_2} \|\nabla_K f(K(t))\| dt \\
\leq \sqrt{2\rho \int_{t_1}^{t_2} \phi(t) dt} \\
\leq \frac{2\rho \phi(t_0)}{\mu} \left(e^{-\mu(t_1-t_0)} - e^{-\mu(t_2-t_0)}\right),
\]
which implies that $\|K(t) - K^*\| \leq \sqrt{2\rho \phi(0)}e^{-\mu(t-t_0)}$ since there is only one optimal solution of $K^*$ when PL inequality is satisfied.

The optimal tracking control protocol and learning algorithm are listed in Algorithm (1).

Remark 4. Since $f(K)$ is strongly convex in the neighborhood of $K^*$, it is suggested to improve the gradient-based updating law (43) by other efficient or robust gradient-based methods such as adaptive gradient methods (44).

V. SIMULATIONS

Two simulation experiments are presented in this section to show the the effectiveness of parameter estimation and optimal solution searching, respectively.

Example 1. For an uncoupled LCL inverter-based generation system defined in (53), its dynamics is given as
\[
x(t) = \begin{bmatrix} -\frac{R_2}{L_1} & -\frac{1}{L_1} & 0 \\ \frac{1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{1}{L_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} u(t) \quad (58)
\]
in which $x(t) = [I_L, V_C, i_o]^T$ and $u(t) = V_I$. Without loss of generality, the parameter values are modified for simplicity. The values of electronic components are chosen as $R_1 = 0.2\Omega, R_2 = 0.5\Omega, L_1 = 25mH, L_2 = 55mH$ and $C = 0.3F$. When the values of some components are unknown but the state $x(t)$ and input $u(t)$ are measurable, one can use Step I in Algorithm (1) to estimate these values.

The initial state $x(0) = [0, 0, 0]^T$. Set the input voltage as $u(t) = 100sin(10t)V$, $t \in [0, 25s]$ to force system working and collect the data of $x(t)$ and $u(t)$, while computing $s_0(t)$ as (21). According to (50), the parameters of designed filters
Algorithm 1 Optimal Tracking Control Algorithm for Unknown Linear Systems

**Input:** dimensions of $A$, $B$, $C$ and $D$ are given, states of $x$, $v$ and $\omega$ are measurable and $u$ is designed to satisfy IE condition.

**Output:** Optimal Tracking Controller $u^*$.

**Step 1. Finite-time Parameter estimation.**

Set $\psi \in \mathbb{R}^{k \times n}$ where $k = 2n + m + q$ with initial value $\psi(0) = \psi_0$. Introduce one-input-$k$-output linear operator $H_\psi$ and get $F = H_\psi [\psi] \in \mathbb{R}^{k \times k}$.

while $t \in [0, t_c]$:

- Calculate adjoint matrix $\text{adj} \{ F \}$ and $\Delta = \det \{ F \}$.
- Determine the learning rate $\alpha > 0$ for descent updating policy \((33)\) and record the signal state $s$ according to \((23)\).
- if $t = t_c$
  - Determine the parameter matrices by $\psi_F$ with \((34)\).
- end if

**end while**

**Step 2. Optimal Gain Matrix Searching.**

Initialize $K(t_c) = K_0 \in S_0$ with estimated parameter matrices.

Decide the allowable error bound $\epsilon_b$

while $\| \nabla_k f(K) \| > \epsilon_b$

- Calculate $P(K)$ and $Z$ according to \((38)\) and \((43)\).
- Update $K$ with gradient descent law \((44)\).

**end while**

**Step 3. Optimal Controller Design.**

Record the value of $K$.

The optimal controller is $u^* = -K \text{col}(x - v, v)$.

is $\lambda = 0.1$, $\lambda_1 = 0.001$, $\lambda_2 = 0.01$, $\lambda_3 = 0.1$ and $\lambda_4 = 0.5$. Following the Algorithm 1 the estimation matrix $\Psi \in \mathbb{R}^{4 \times 3}$ with initial value 0 of all elements takes the learning law \((33)\) with $\alpha = 0.05$. It is shown in Fig. 1(a) that the parameters are not converge to the exact solution. Then, the evolutionary trajectory of $\Delta(t)$ is shown in Fig. 1(b) where $\Delta(t) \in L_2$ that is contrary to the assumptions in Proposition 1. Therefore, the convergence under \((33)\) may not be ensured. Note that PE condition is not satisfied either in this situation. Since IE condition is satisfied, it is feasible to introduce finite-time algorithm \((22)\) to solve the problem. According to Fig. 1(b) let the learning interval be $(0, 25s)$, it is observed that $\int_0^{25} \Delta^2(t)dt \geq 40$. It is verified that IE condition \((20)\) holds when setting $\sigma = 0.60$. It is shown in Fig. 1(c) that the estimation matrix $\Psi$ converges to the exact solution within 25s, which illustrates the effectiveness. The norm-based error is shown in Fig. 1(d). Last but not least, both algorithm achieves element-wise approximation, i.e. each element is approximated asymptotically, as analyzed in \((15)\).

**Example 2.** A controlled system is described by

\[
\dot{x} = \begin{bmatrix} 0 & 1.0 \\ -1.0 & 0 \end{bmatrix} x + \begin{bmatrix} 3.0 \\ 2.0 \end{bmatrix} u,
\]

\[
\dot{v} = \begin{bmatrix} 0.0 & 0.8 \\ -0.8 & -0.2 \end{bmatrix} v,
\]

where all of the parameter matrices are unknown. The control target is to design the controller $u$ with measurable state $x$ and $v$ to minimize the tracking performance \((56)\) with $\gamma = 0.5$, $Q = I$ and $R = 1$. With the ARE \((41)\) including the discounted factor, the numerical solution of $P^*$ and optimal control gains $K^*$ can be obtained via LQR toolbox as

\[
P^* = \begin{bmatrix} 0.4645 & -0.2166 & 0.0459 & 0.0170 \\ -0.2166 & 0.4543 & -0.0524 & 0.0052 \\ 0.0459 & -0.0524 & 0.8389 & 0.0607 \\ 0.0170 & 0.0052 & 0.0607 & 0.7348 \end{bmatrix},
\]

\[
K^* = \begin{bmatrix} 1.1701266 \\ 0.1659746 \\ 0.0753407 \\ 0.0886391 \end{bmatrix}.
\]

Noticing that this method only works when the parameter matrices of system dynamics are known exactly and the toolbox

![Fig. 1. Signals' and estimation parameters' trajectories in Example 1](image1)

(a) Parameter estimation using updating law \((33)\), where the blue dotted lines are the nonzero values of parameters in $A$ and the red dotted line represents nonzero value in $B$. (b) $\Delta(t) \in L_2$ satisfying IE condition. (c) Parameter convergence under \((33)\) within $40s$, where the black line represents the maximum allowable time point. (d) Parameter estimation errors under updating law \((33)\) and finite-time law \((22)\).

![Fig. 2. Signals’ and estimated parameters’ trajectories in Example 2](image2)

(a) Parameter estimation of $A$ and $B$ by $v$ under tuning law \((33)\). (b) Parameter estimation of $A$ and $B$ by $\psi_D$ under tuning law \((33)\). (c) Parameter estimation of $D$ by $\psi_D$ under tuning law \((33)\). (d) Parameter estimation of $D$ by $\psi_D^*$ under tuning law \((33)\).
is callable. When these conditions are hard to be met, the optimal control input can be approximated using Algorithm 1. For simplicity, the settings of filters, control inputs and learning rate are the same as Example 1. In this way, the parameter estimation of $K(t)$ initialized by zero matrices, which implies that the estimation is done within 15s.

Because the parameter matrices are estimated and known after 15s, the optimal solution can be obtained by searching based method, i.e. Step 2 in Algorithm 1. Let $K_0 = [0, 0, 0, 0]$. The eigenvalues of $(A - BK_0) - 0.25I$ are $(-0.25, i), (-0.25, -i), (0.35, 0.7937i)$ and $(-0.35, -0.7937i)$ verifying that $K_0 \in S_0$. Updating $K(t)$ with (44), the evolutionary trajectories of each element in $K(t)$ and error of $K(t) - K^*$ are shown in Fig. 3. Within 10s, the control gain $K(t)$ approximates the numerical solution of ARE (41) and $K(25) = [1.170126 \ 0.1659746 \ 0.0753407 \ 0.0886391]$. With this approximated solution, the optimal controller can be designed by $u(t) = -K(25)X(t)$ where $X(t) = \text{col}(x(t) - v(t), v(t))$. The tracking trajectories can be seen in Fig. 4 which implies that two heterogeneous systems achieve the goal of state. Moreover, it is seen from (3b) that the gradient-based method can approximate the optimal solutions with exponential convergence rate, which supports the result in Theorem 2.

VI. CONCLUSION

This paper has investigated the optimal tracking control problem for unknown heterogeneous linear systems. The parameter estimation has been proposed first using measurable state and input, proving the finite-time accurate estimation under modified IE condition. With the estimated matrices of system dynamics, the optimal controller can be solved from a new ARE of augmented systems with discounted factor. A gradient descent method has been utilized to approximate the optimal solutions. It has been proven that the gradient-based method with appropriate initialization would result in exponential convergence rate and optimality of the numerical solutions. Last but not least, all of the conditions are online verifiable.
On the one hand, the robustness of presented method needs further analysis because the time delay, unmeasurable disturbances and uncertain terms always exist in real systems. On the other hand, the validation of gradient method under more complex conditions such as sparse constraint remains an open topic.

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