Do Killing–Yano tensors form a Lie algebra?

David Kastor, Sourya Ray and Jennie Traschen

Department of Physics, University of Massachusetts, Amherst, MA 01003, USA

Received 9 May 2007, in final form 1 June 2007
Published 4 July 2007
Online at stacks.iop.org/CQG/24/3759

Abstract

Killing–Yano tensors are natural generalizations of Killing vectors. We investigate whether Killing–Yano tensors form a graded Lie algebra with respect to the Schouten–Nijenhuis bracket. We find that this proposition does not hold in general, but that it does hold for constant curvature spacetimes. We also show that Minkowski and (anti)-deSitter spacetimes have the maximal number of Killing–Yano tensors of each rank and that the algebras of these tensors under the SN bracket are relatively simple extensions of the Poincaré and (A)dS symmetry algebras.

PACS numbers: 04.20.–q, 02.40.–k

1. Introduction

The connection between symmetries and conservation laws in physics has both fundamental significance and great practical utility. The symmetries of curved spacetimes are generated by Killing vectors. Killing vectors, in turn, yield conserved quantities, both for geodesic motion within a given spacetime and for the spacetime as a whole. For geodesics, it is well known that if \( K^a \) is a Killing vector and \( u^a \) is the tangent vector to the geodesic (in affine parameterization), then the scalar \( u^a K_a \) is constant along the geodesic. For the spacetime as a whole, the conserved Komar charge is given by \( \int d^3 x \sqrt{-g} K^0 \), where the integral is taken over the boundary of a spacelike slice at infinity. In addition, ADM-like conserved charges may be defined for any class of spacetimes that are asymptotic at spatial infinity to a background spacetime admitting one or more Killing vectors [1].

Killing–Yano tensors are generalizations of Killing vectors that are also associated with conserved quantities in a number of ways. Let us call totally anti-symmetric contravariant tensor fields multivectors. A Killing–Yano tensor of rank \( n \) is a multivector \( K^{a_1} \cdots a_n \) that satisfies the property [2]

\[
\nabla_{(a_1} K_{a_2) a_3 \cdots a_{n+1}} = 0. 
\]

(1)

This clearly reduces to Killing’s equation for \( n = 1 \). It is useful to note (see, e.g., [3]) that condition (1) is equivalent to the statement \( \nabla_{a_1} K_{a_2 \cdots a_{n+1}} = \nabla_{[a_1} K_{a_2 \cdots a_{n+1}]} \). Killing–Yano tensors of all ranks yield conserved quantities for geodesic motion. It is straightforward to show that the multivector \( u^{a_1} K_{a_2 \cdots a_{n+1}} \) is parallel transported along a geodesic with tangent
vector $u^a$. Such conservation laws, for example, underlie the integrability of geodesics in the $d = 4$ Kerr spacetime [4, 5] and its higher dimensional generalizations [6–8].

It was further shown in [9] that transverse asymptotically flat spacetimes have conserved Y-ADM charges that are associated with the Killing–Yano tensors of flat spacetime. Transverse asymptotically flat spacetimes [10], such as those describing $p$-branes, become flat near infinity only in a subset of the full set of spatial directions, i.e. those directions transverse to the brane. The construction of these Y-ADM charges, which are given by integrals over the boundary of a codimension $n$ slice at transverse spatial infinity, follows closely the derivation of ADM conserved charges given in [1]. In a $p$-brane spacetime, the Y-ADM charges associated with rank $n = p + 1$ Killing–Yano tensors are charge densities, e.g. mass per unit of world-volume area. Positivity properties of the Y-ADM mass density were studied in [11].

Given that Killing–Yano tensors give rise to conserved quantities in general relativity, it is reasonable to ask whether Killing–Yano tensors are also associated with symmetries in some appropriately generalized sense. In the case of Killing vectors, the fact that they generate a group of continuous symmetries is reflected in their satisfying a closed Lie algebra. One concrete way to investigate the possible association of Killing–Yano tensors with symmetries is then to ask, as we will in this paper, whether Killing–Yano tensors satisfy a closed Lie algebra amongst themselves.

Part of the motivation for the present work is to lay ground work for developing a clearer understanding of the Y-ADM charges mentioned above. For example, in the Hamiltonian formalism, the ADM charges of asymptotically flat spacetimes satisfy a Poisson bracket algebra that is isomorphic to the Lie algebra satisfied by the corresponding Killing vectors of Minkowski spacetime [12], i.e. the Poincaré algebra. We would like to know if a similar structure holds for Y-ADM charges. Because Y-ADM charges are given by boundary integrals on codimension $n$ submanifolds, we expect that the proper setting for such a bracket algebra will be a generalized Hamiltonian formalism in which data are similarly specified on a codimension $n$ submanifold and propagated via the Hamiltonian equations of motion in the remaining $n$ directions. Such a formalism for general relativity has been developed in [13]. We also need to know whether the Killing–Yano tensors of flat spacetime satisfy a Lie algebra that generalizes the Poincaré algebra of Killing vectors. We will see below that, although Killing–Yano tensors do not generally form Lie algebras, in fact they do in Minkowski or other maximally symmetric spacetimes. Moreover, we will see that these Lie algebras are natural extensions of the Poincaré and (A)dS symmetry algebras.

Returning to Killing vectors, we know that the whole set of vector fields on a manifold $M$ forms a Lie algebra. The Lie bracket of two vector fields $A^a$ and $B^a$ is defined to be the Lie derivative of $B^a$ with respect to $A^a$,

$$[A, B]^a = (\mathcal{L}_A B)^a,$$

$$= A^b \partial_b B^a - B^b \partial_b A^a. \quad (3)$$

This Lie algebra depends only on the manifold structure of $M$. The Killing vectors with respect to a particular spacetime metric $g_{ab}$ on $M$ turn out to form a subalgebra of this larger algebra of vector fields. One way to see this is to consider Killing’s equation in the form

$$(\mathcal{L}_A g)_{ab} = 0. \quad (4)$$

The Lie derivative acting on tensor fields of arbitrary type satisfies the relation

$$\mathcal{L}_{[A, B]} = \mathcal{L}_A \mathcal{L}_B - \mathcal{L}_B \mathcal{L}_A. \quad (5)$$

Therefore, if two vector fields $A^a$ and $B^a$ satisfy Killing’s equation, so does the vector that results from taking their Lie bracket. We would like to know whether a similar construction holds for Killing–Yano tensors. Fortunately, it is known that the set of all multi-vector fields
on a manifold $M$ forms a (graded) Lie algebra with respect to the Schouten–Nijenhuis (SN) bracket [14–16]. The SN bracket serves as a starting point for our investigations below.

The plan of the paper is as follows. In section 2, we introduce the SN bracket and present some of its useful properties. In section 3, we study whether Killing–Yano tensors form a subalgebra of the full algebra of multi-vector fields with respect to the SN bracket. We demonstrate that this property holds for constant curvature spacetimes. However, our calculations in section 3 are inconclusive as to whether it is true in general spacetimes admitting Killing–Yano tensors. In section 4, we look at two examples of spacetimes, having non-constant curvature, that admit rank 2 Killing–Yano tensors, $D = 4$ Kerr and $D = 4$ Taub-NUT. In each of these cases, we find that the SN bracket of Killing–Yano tensors fails to satisfy the Killing–Yano equation (1). These serve as explicit counter-examples to the general conjecture. In section 5, we present results on the Lie algebra of Killing–Yano tensors in maximally symmetric spaces. We show that flat spacetime as well as (A)dS spacetimes admits the maximal number of Killing–Yano tensors of each rank and discuss the structures of the corresponding Lie algebras. We offer some concluding remarks in section 6.

2. The Schouten–Nijenhuis bracket

There exists a generalization to multivectors of the Lie bracket of vector fields that is known as the Schouten–Nijenhuis (SN) bracket [14–16]. The SN bracket of a rank $p$ multivector $A$ and a rank $q$ multivector $B$ is a rank $(p + q - 1)$ multivector, given in component form by

$$\left[ A, B \right] = \sum_{a=1}^{p+q-1} a \partial_{a}^B \left[ A_{a_1 a_2 \ldots a_{p+q-1}} B^{a_1 a_2 \ldots a_{p+q-1}} \right] + (p - 1) \sum_{b=1}^{q} b \partial_{b}^B \left[ A_{a_1 a_2 \ldots a_{p+q-1}} B^{b a_1 a_2 \ldots a_{p+q-1}} \right].$$

The spacetime dimension $D$ serves as the maximal rank for multivectors. Therefore, the SN bracket vanishes if $p + q - 1 > D$. The SN bracket reduces to the Lie bracket of vector fields (3) for $p = q = 1$, and more generally for $p = 1$ and $q$ arbitrary to the Lie derivative acting on a rank $q$ multivector. Like the Lie bracket, the SN bracket is a map from tensors to tensors that depends only on the manifold structure. The partial derivatives in the SN bracket may be replaced by covariant derivatives, as all the Christoffel symbol terms cancel out.

The SN bracket is well known in the mathematics literature, playing a central role, for example, in the theory of Poisson manifolds. The properties of the SN bracket are discussed in detail in this context in [17]. It is demonstrated there that the SN bracket satisfies a number of important relations. First is the exchange property

$$\left[ A, B \right] = (-1)^{pq} \left[ B, A \right],$$

which shows the $\mathbb{Z}_2$ grading of the SN bracket. It can also be shown that the SN bracket satisfies the graded product rule

$$\left[ A, B \wedge C \right] = \left[ A, B \right] \wedge C + (-1)^{(p+1)q} B \wedge \left[ A, C \right],$$

and the graded Jacobi identity

$$(-1)^{(p+1)r} [A, [B, C]] + (-1)^{(p+1)q} [B, [C, A]] + (-1)^{(q+1)r} [C, [A, B]] = 0,$

where $C$ is a multivector field of rank $r$. The wedge product between multivectors, denoted by the symbol $\wedge$ in equation (8), is defined in analogy with the wedge product of differential forms. These relations imply that the set of multivector fields on a manifold satisfy a $\mathbb{Z}_2$-graded Lie algebra.

For later use, we note that we can show that the bracket of two rank 2 multivectors $A$ and $B$ is symmetric, and yields the rank 3 multivector

$$1 \text{ The proposition that the SN bracket preserves the Killing–Yano property has been previously investigated in [3]. It is claimed there, without a proof being presented, that the propositions are valid. The counter-examples presented in section 4 demonstrate that this is not the case.}$$
\[ [A, B]^{abc} = 2(A^{d[\alpha} \nabla_\beta B^{\beta \gamma]} + B^{d[\alpha} \nabla_\beta A^{\beta \gamma]}). \] (10)

One can easily check in this example that the Christoffel symbols on the right-hand side cancel out.

3. A Lie algebra of Killing–Yano tensors?

The next question to ask is whether the Killing–Yano tensors on a Riemannian manifold, i.e. multi-vector fields satisfying equation (1) with respect to the given metric, form a subalgebra of the full algebra of multivector fields. Ideally at this point, we would want to proceed as we did with Killing vectors in the introduction, where we made use of Killing’s equation in the form given in equation (4). However, unlike the case of vector fields, the bracket of higher rank multivector fields cannot be extended to give an action of multivectors on tensors of general type analogous to the Lie derivative (see [17] for a discussion of this point). Thus, we have no way of rewriting the Killing–Yano condition (1) in a form analogous to equation (4).

Instead we follow a more direct approach. As a warm-up, we consider the Lie bracket of two Killing vectors and show that the resulting vector satisfies Killing’s equation. We then consider the SN bracket of a Killing vector with a rank 2 Killing–Yano tensor and show that the resulting rank 2 multivector is again a Killing–Yano tensor. Finally, we consider the SN bracket of two rank 2 Killing–Yano tensors and ask whether, or not, the resulting rank 3 multivector satisfies the Killing–Yano condition (1). Our result, in this case, is inconclusive. We go on to show that for constant curvature spacetimes, the answer is yes. However, in section 4 we give two explicit counterexamples which demonstrate that the property does not hold in general.

Before proceeding with our calculations, it is worth noting that there is also an SN bracket for totally symmetric contravariant tensors, which gives the set of all such fields on a manifold a natural Lie algebra structure. The symmetric SN bracket may be used to show that the set of symmetric Killing tensors forms a Lie subalgebra of this larger algebra of all symmetric tensor fields. The proof is very similar to the one given for Killing vectors in the introduction. Killing tensors are defined by the condition
\[ \nabla^a (a_1 K_{a_2 \cdots a_n}) = 0. \] (11)

The inverse metric \( g^{ab} \), in particular, is clearly a symmetric Killing tensor. One can show that equation (11) is equivalent to the condition that the SN bracket of \( K^{a_1 \cdots a_n} \) with the inverse metric vanishes [18]. It then follows from the Jacobi identity for the symmetric SN bracket that the bracket of two symmetric Killing tensors is again a symmetric Killing tensor\(^2\). Killing vectors may equally well be considered to be either totally symmetric or totally anti-symmetric. It seems interesting that with regard to the considerations above, Killing vectors seem to have more in common with symmetric Killing tensors.

Returning to Killing–Yano tensors, we proceed in a direct manner to check whether, or not, the SN bracket preserves the Killing–Yano property. It is instructive to go through the calculation first for Killing vectors. The necessary ingredient in this calculation is the property that acting with two derivatives on a Killing vector \( A^a \) gives
\[ \nabla_a \nabla_b A_c = -R_{bca}^d A_d. \] (12)

Let \( A^a \) and \( B^a \) be two Killing vectors. In order to check that their bracket \( C^a = A^b \nabla_b B^a - B^b \nabla_b A^a \) also satisfies Killing’s equation, we compute

\(^2\) The SN bracket plays a natural role in the Hamiltonian formulation of geodesic motion. If \( K^{a_1 \cdots a_n} \) is a symmetric Killing tensor, then the scalar quantity \( K^{a_1 \cdots a_n} u_{a_1} \cdots u_{a_n} \) is conserved along a geodesic with tangent vector \( u^a \). The condition that the Poisson bracket of two such conserved quantities vanishes is simply that the SN bracket of the two Killing tensors vanishes. See [19] for a discussion of this topic.
\[ \nabla_a C_{bc} = (\nabla_a A_c) \nabla^c B_b - (\nabla_a B_c) \nabla^c A_b + A_c \nabla_a \nabla^c B_b - B_c \nabla_a \nabla^c A_b \] 

\[ = - (\nabla_c A_a) \nabla^c B_b + (\nabla_c A_b) \nabla^c B_a - R_{abcd} A^c B^d \] 

\[ = \nabla_c [a C_b], \] 

where Killing’s equation for \( A^a \) and \( B^a \) is used in processing the first derivative terms and equation (12) is used together with basic properties of the Riemann tensor in processing the second derivative terms.

We now proceed to show that the SN bracket of a Killing vector \( A^a \) with a Killing–Yano tensor \( B^{ab} \) is again a Killing–Yano tensor. Let \( C^{abc} = [A, B]^{abc} \), which in this case is simply the Lie derivative of \( B^{ab} \) with respect to the Killing vector \( A^a \). The calculation requires the analogue of equation (12) for a rank 2 Killing–Yano tensor. This is given by

\[ \nabla_a \nabla_b B_{cd} = \frac{3}{2} R_{[bc|a|} B_{de]} \] 

(16)

The starting point is then the expression \( C^{abc} = A^c \nabla_a B^{bc} - B^{cb} \nabla_a A^c - B^{ac} \nabla_a \nabla^c A^b \). Making use of the Killing–Yano condition for \( A^{ab} \) and \( B^{ab} \), equation (16) and properties of the Riemann tensor, one can show that

\[ \nabla_a C_{bc} = -3 (\nabla_d A[a] \nabla^d B_{bc]) + \frac{3}{2} A^d R_{de[ab]} B_{ce} \] 

(17)

We see that \( \nabla_a C_{bc} \) \( = \nabla_c [a C_b] \) and that therefore \( C_{abc} \) is again a Killing–Yano tensor.

Finally, let us now assume that \( A^{ab} \) and \( B^{ab} \) are two rank 2 Killing–Yano tensors and ask whether, or not, \( C^{abc} = [A, B]^{abc} \) is also a Killing–Yano tensor. From equation (10), we have that

\[ C^{abc} = 2 (A^d [a \nabla_d B^{bc}] + B^{d[a} \nabla_d A^{bc}]). \] 

(18)

In this case, we were not able to show generally that \( \nabla_a C_{bcd} = \nabla[d C_{bcd}] \). There are many equivalent expressions that may be given for \( \nabla_a C_{bcd} \), one of which is

\[ \nabla_a C_{bcd} = -4 (\nabla_e A_{[ab]} \nabla^e B_{cd]) + 2 (A_{[e[b} \nabla_{|a|} \nabla^e B_{cd]} + B_{e[b} \nabla_{|a|} \nabla^e A_{cd]}). \] 

(19)

Here we see that the terms quadratic in first derivatives, which have been processed using the Killing–Yano condition \( A^{ab} \) and \( B^{ab} \), do display the necessary anti-symmetry consistent with \( C^{abc} \) having the Killing–Yano property. The second derivative terms may be rewritten in many equivalent forms using equation (16) and the identities \( \nabla_d [a \nabla_b B_{cd}] = \nabla_b [a \nabla_d B_{cd}] = 0 \). However, none of these forms appears to be totally anti-symmetric in the free indices. We confirm the conclusion that \( C^{abc} \) is not generally a Killing–Yano tensor in the following section, by presenting two explicit counterexamples.

Our conclusion, however, does not imply that the SN bracket of Killing–Yano tensors is never a Killing–Yano tensor. It is interesting to consider the special case of constant curvature spacetimes. Constant curvature spacetimes locally have the maximal number of Killing–Yano tensors of each rank. Specializing even further to flat spacetime, the second derivative terms in (19) vanish by virtue of equation (16). Hence, given the anti-symmetry of the remaining term on the right-hand side of (19), we see that the Killing–Yano tensors of flat spacetime do form a graded Lie algebra with respect to the SN bracket. More generally, for a constant curvature spacetime the Riemann tensor is given by \( R_{abcd} = \alpha (g_{ac} g_{bd} - g_{bc} g_{ad}) \) and we find that a rank 2 Killing–Yano tensor satisfies

\[ \nabla_a \nabla_b A_{cd} = -3 \alpha g_{[a[b A_{cd]}}. \] 

(20)

Equation (19) then becomes

\[ \nabla_a C_{bcd} = -4 (\nabla_e A_{[ab]} \nabla^e B_{cd}) - 4 \alpha A_{[ab} B_{cd]} = \nabla[d C_{bcd}] \] 

(21)

and we see that the SN bracket of rank 2 Killing–Yano tensors in a constant curvature spacetime is again a Killing–Yano tensor.
It is straightforward to show that the SN bracket always preserves the Killing–Yano property in spacetimes with constant curvature. Acting with two derivatives on a Killing–Yano tensor of rank \( n \) gives
\[
\nabla_a \nabla_b A_{b_2 \ldots b_{n+1}} = (-1)^{n+1} \left( \frac{n+1}{2} \right) R^{d}_{a[b_1 \ldots b_{n+2}]d}.
\]
(22)

For constant curvature spacetimes, we then have
\[
\nabla_a \nabla_b A_{b_2 \ldots b_{n+1}} = -(n+1) \alpha g_{a[b_1 \ldots b_{n+2}]}.
\]
(23)

Let \( A \) and \( B \) be Killing–Yano tensors of ranks \( m \) and \( n \) respectively; then \( C = [A, B] \) is a multivector of rank \( m + n - 1 \). One finds that
\[
\nabla_{a_1} \ldots \nabla_{a_m} C_{a_1 \ldots a_m} = -(m + n) \left\{ (\nabla_c A_{[a_1 \ldots a_m]} \nabla^c B_{a_{m+1} \ldots a_{n+m}} + \alpha A_{[a_1 \ldots a_m} B_{a_{m+1} \ldots a_{n+m}]} \right\},
\]
(24)

which reduces to equation (21) for \( m = n = 2 \) and shows generally that the Killing–Yano tensors of a constant curvature spacetime form a graded Lie algebra under the SN bracket.

4. Two counterexamples

In the preceding section, we were unable to verify, or definitively falsify, the proposition that the SN bracket of two Killing–Yano tensors is necessarily another Killing–Yano tensor, for the general case of metrics with non-constant curvature. In this section, we show that the proposition is indeed false by providing two explicit counterexamples.

**Counterexample 1**: \( D = 4 \) Kerr. The first counterexample is in the four-dimensional Kerr spacetime, which has a rank 2 Killing–Yano tensor \( A_{ab} \), which was originally found by Penrose [5] and Floyd [4]. The tensor \( A_{ab} \) plays a central role in the integrability of geodesic motion and the separability of various wave equations (see [20] for a discussion and further references). Recall from equation (7) that the SN bracket is \( Z_2 \)-graded, and in particular that the bracket of two rank 2 multivectors is symmetric under interchange. Therefore, one can consider the tensor \( B_{abc} = [A, A]_{abc} \) and ask whether, or not, it is a Killing–Yano tensor.

The Kerr metric in Boyer–Lindquist coordinates is given by
\[
ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta \, d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2) \, dr)^2 - \frac{\rho^2}{\Delta} \, dr^2 + \rho^2 \, d\theta^2,
\]
(25)

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \) and \( \Delta = r^2 + a^2 - 2mr \). The non-zero components of the Killing–Yano tensor \( A_{ab} \) in these coordinates are (from [20])
\[
A'^t = -\frac{a \cos \theta (r^2 + a^2)}{\rho^2}, \quad A'^\phi = -\frac{a^2 \cos \theta}{\rho^2},
\]
\[
A^\theta = \frac{ar \sin \theta}{\rho^2}, \quad A^{\phi \phi} = \frac{r}{\rho^2 \sin \theta}.
\]
(26)

The tensor \( B_{abc} = [A, A]_{abc} \) may then be computed from equation (10), giving the non-zero components
\[
B'^r \phi = \frac{4ar}{3\rho^2}, \quad B^{\theta \phi} = \frac{4a \cos \theta}{3\rho^2 \sin \theta}.
\]
(27)

We now want to check whether, or not, the tensor \( B_{abc} \) is itself a Killing–Yano tensor. It requires very little work to check that the condition \( \nabla_a B_{abc} = 0 \) is satisfied. This is consistent with the Killing–Yano property (1). However, checking in more detail whether equation (1) is satisfied, it turns out that many components of the tensor \( \nabla_{a_b} B_{b_2 \ldots b_{n+m} a_{n+m}} \) are non-vanishing. For
example, one finds that
\[ \nabla_r B_{\theta\phi} + \nabla_\theta B_{r\phi} = \frac{2}{3} am \sin 2\theta \left( \frac{a^2 \cos^2 \theta - 3r^2}{\rho^2} \right). \]  
(28)

We see, therefore, that the SN bracket of the Killing–Yano tensor \( A_{ab} \) of \( D = 4 \) Kerr with itself does not satisfy the Killing–Yano property. The only exception to this is the case \( m = 0 \) (note that the tensor \( B_{abc} \) vanishes for \( a = 0 \)). In this case, all the components of \( \nabla_a B_{bcd} \) vanish and hence \( B_{abc} \) is a Killing–Yano tensor. This is consistent with our results of section 3. For \( m = 0 \), the Kerr metric is flat, and we have shown in section 3 that the SN bracket of Killing–Yano tensors is always a Killing–Yano tensor in flat spacetime. The counterexample presented above, however, serves to falsify the general proposition that the SN bracket of Killing–Yano tensors is always a Killing–Yano tensor.

Counterexample 2: Euclidean Taub-NUT. We have now seen the SN brackets of Killing–Yano tensors are not generally Killing–Yano tensors. However, we have also seen in section 3 that this property does hold in the special case of constant curvature. It would be interesting to know whether there are other spacetimes for which the property holds as well. The result for constant curvature spacetimes suggests that we look at other spacetimes with Killing–Yano tensors for which the Riemann tensor enjoys some other special property. The Euclidean Taub-NUT metric has a self-dual Riemann tensor. It admits four rank 2 Killing–Yano tensors (see [21] for a detailed discussion). We have checked whether the SN brackets of these tensors satisfy the Killing–Yano condition and found that they do not.

The metric of the Euclidean Taub-NUT spacetime is given by
\[ ds^2 = V(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + \frac{16m^2}{V(r)}(d\chi + \cos \theta d\phi)^2, \quad V(r) = 1 + \frac{4m}{r}. \]  
(29)

The four rank 2 Killing–Yano tensors are given in covariant form by
\[ f_Y = 8m(d\chi + \cos \theta d\phi) \wedge dr + 4r(r + 2m) \left( 1 + \frac{r}{4m} \right) \sin \theta d\theta \wedge d\phi \]  
(30)
\[ f_i = 8m(d\chi + \cos \theta d\phi) \wedge dx_j - \epsilon_{ijk} dx_i \wedge dx_j, \]  
(31)
where \( i, j = 1, 2, 3 \) and the Cartesian coordinates \( x_i \) are related to the spherical coordinates \( (r, \theta, \phi) \) in the standard way. The 2-forms \( f_i \) are covariantly constant and therefore have trivially vanishing SN brackets. The tensors \( B^{abc} = [f_Y, f_Y]^{abc} \) and \( B[^{(i)}abc] = [f_i, f_Y]^{abc} \) are non-zero. However, we find that the tensors \( \nabla_a B_{bcd} \) and \( \nabla_a B[^{(i)}bcd] \) are not totally anti-symmetric. Therefore \( B^{abc} \) and \( B[^{(i)}abc] \) are not Killing–Yano tensors.

5. Maximally symmetric spacetimes

It was shown in section 3 that the Killing–Yano tensors of a constant curvature spacetime do form a Lie algebra with respect to the SN bracket. In this section, we will study the Killing–Yano tensors of \( d \)-dimensional Minkowski and (anti-)de Sitter spacetimes. These spacetimes are well known to have the maximal number of Killing vectors. We will see that they also have the maximal number of Killing–Yano tensors\(^3\). Calculation of the full Lie algebra of Killing–Yano tensors proves to be quite tedious. It will be clear, however, from the explicit forms of the Killing–Yano tensors that these algebras are, in principle, simple extensions of the Poincaré and (A)dS algebras.

\(^3\) Y-ADM charges for asymptotically AdS spacetimes have been studied in [22, 23].
Let us denote the maximal number of Killing–Yano tensors of rank \( n \) in a given spacetime dimension \( d \) by \( N(d, n) \). This number is determined \([9]\) via a simple generalization of the argument for Killing vectors. By virtue of equation (12), a Killing vector \( A^a \) is determined everywhere on a manifold by its value at one point together with the values of its first derivatives \( \nabla_a \xi_b \) at that point (see, e.g., appendix C of \([24]\)). Since \( \nabla_a \xi_b \) is anti-symmetric, it follows that there are at most \( N(d, 1) = d + d(d - 1)/2 \) linearly independent Killing vectors.

The counting works similarly for Killing–Yano tensors of rank \( n \), where a generalization of equation (16) determines all second and higher order derivatives in terms of the values of the tensor and its first derivatives. Since Killing–Yano tensor \( A_{a_1 \cdots a_n} \) and its first derivative \( \nabla_a A_{a_1 \cdots a_n} \) are both anti-symmetric, it follows that

\[
N(d, n) = \frac{d!}{n!(d - n)!}, \quad N_R(d, n) = \frac{d!}{(n + 1)!(d - (n + 1))!},
\]

which together add up to the maximal number. The rank \( n \) translations and boost/rotations can be labelled \( T_{(a_1 \cdots a_n)} \) and \( R_{(a_1 \cdots a_n \xi_1 \cdots \xi_{d - n + 1})} \). They are anti-symmetric in their label indices. Their components forms are

\[
T^{b_1 \cdots b_n}_{(a_1 \cdots a_n)} = n! \delta^{b_1}_{a_1} \cdots \delta^{b_n}_{a_n}, \tag{34}
\]

\[
R^{b_1 \cdots b_n}_{(a_1 \cdots a_n \xi_1 \cdots \xi_{d - n + 1})} = (n + 1)! \eta^c_{(a_1} \delta^{b_1}_{a_2} \cdots \delta^{b_n}_{a_{n+1})}. \tag{35}
\]

It is straightforward to check that these tensors satisfy the Killing–Yano condition (1). It is likewise straightforward, in principle, to calculate the SN brackets of the collection of whole complex of Killing–Yano tensors of Minkowski spacetime. However, the many anti-symmetrizations make this a tedious task. It is simple, however, to see the general form that the full algebra of Killing–Yano tensors take. Translational Killing–Yano tensors are independent of the spacetime coordinates, while boost/rotations are linear in the spacetime coordinates. The SN bracket has one derivative. Therefore, the full algebra of Killing–Yano tensors will have a structure very similar to that of the Poincaré algebra. The bracket of two translations will vanish. The bracket of a translation with a boost/rotation will be a translation, while the bracket of two boost/rotations will be another boost/rotation. Hence, the boosts and rotations form a Lorentz-like subalgebra.

We now turn to (A)dS\(_d\), which we realize as hyperboloids in \((d + 1)\)-dimensional flat spacetime with coordinates \( X^i \) with \( A = 0, \ldots, d \) and signature \((d, 1)\) for dS\(_d\) and \((d - 1, 2)\) for AdS\(_d\). The hyperboloids are then given by

\[
-(X^0)^2 + (X^1)^2 + \cdots + (X^{d-1})^2 + \kappa (X^d)^2 = r^2, \quad \kappa = \pm 1
\]

where \( \kappa = +1 \) for dS\(_d\) and \( \kappa = -1 \) for AdS\(_d\). If we now introduce coordinates \( x^a \) with \( a = 0, \ldots, d - 1 \) on the (A)dS\(_d\) hyperboloids and use the radius \( R \) as an additional coordinate, then the flat \((d + 1)\)-dimensional metric may be written as

\[
ds_{d+1}^2 = \kappa dR^2 + R^2 k_{ab} \, dx^a \, dx^b, \tag{37}
\]
where $g_{ab} = R^2 k_{ab}$ is the (A)dS$_d$ metric and $k_{ab} = k_{ab}(x^c)$. Let $\nabla_a$ denote the (A)dS$_d$ covariant derivative operator and $\xi^a$ the projection of a flat-space Killing vector onto the (A)dS$_d$ hyperboloid. One can then show that

$$\nabla_a \xi_b + \nabla_b \xi_a = -\frac{2k}{R} g_{ab} \xi^R,$$

where $\xi^R$ is the component of the Killing vector field normal to the hyperboloid. Killing vectors of the flat embedding space that are tangent to the hyperboloid are then seen to be Killing vectors of (A)dS$_d$. The rotational Killing vectors of the flat embedding spacetime satisfy this property. There are $N_R(d+1,1)$ of these vectors, and since $N_R(d+1,1) = N(d,1)$ we see that (A)dS$_d$ are maximally symmetric spacetimes. We can also see from equation (38) that the projections of the translational Killing vectors of the flat embedding space onto the (A)dS$_d$ hyperboloid yield conformal Killing vectors, since for these $\xi^R$ will be non-zero.

The situation for higher rank Killing–Yano tensors is quite similar. Let $\xi^{a_1 \cdots a_n}$ be the projection of a rank $n$ Killing–Yano tensor of the flat embedding spacetime onto its components tangent to the (A)dS$_d$ hyperboloid. The tensor $\xi^{a_1 \cdots a_n}$ then satisfies the equation

$$2\nabla_{(a_1} \xi_{a_2 a_3 \cdots a_{n-1})} = -\frac{k}{R} \left\{ g_{a_1 a_2} \xi_{a_3 \cdots a_{n-1} a_R} + \cdots + g_{a_1 a_{n-1}} \xi_{a_2 a_3 \cdots a_R} \right\},$$

$$+ g_{a_2 a_1} \xi_{a_3 \cdots a_{n-1} a_R} + \cdots + g_{a_{n-1} a_{n-2}} \xi_{a_1 a_2 \cdots a_R} \right\}. (39)$$

Killing–Yano tensors of the flat embedding spacetime that are tangent to the hyperboloid, i.e. for which $\xi_{a_1 \cdots a_{n-1}} = 0$, are then also Killing–Yano tensors of (A)dS$_d$. It is straightforward to check that the rotational Killing–Yano tensors $R_{(a_1 \cdots a_{n-1})}$ have this property. Moreover, they provide precisely the maximal number of (A)dS$_d$ Killing–Yano tensors of each rank. This can be seen by noting the equality

$$N(d,n) = N_R(d+1,n) = \frac{(d+1)!}{n!(d+1-n)!}. (40)$$

Hence (A)dS$_d$ has the maximal number of Killing–Yano tensors of each rank. We also see from this construction that the Lie algebra of Killing–Yano tensors for (A)dS$_d$ coincides with the subalgebra of boost/rotational Killing–Yano tensors in the corresponding flat $(d+1)$-dimensional embedding space.

The translational Killing–Yano tensors of the embedding spacetime have non-vanishing radial components. Their projections onto the (A)dS$_d$ hyperboloid yield conformal Killing–Yano tensors. These are defined in [25] to be anti-symmetric tensor fields satisfying the condition

$$\nabla_b \xi_{a_1 \cdots a_p} + \nabla_{a_1} \xi_{b a_2 \cdots a_p} = 2g_{b a_1} \chi_{a_2 \cdots a_p} = \sum_{i=2}^{p} (-1)^i \left( g_{b a_1} \chi_{a_2 \cdots a_i a_{i+1} \cdots a_p} + g_{a_1 a_i} \chi_{b a_2 \cdots a_i a_{i+1} \cdots a_p} \right)$$

for some anti-symmetric tensor $\chi_{a_1 \cdots a_p}$, of one-degree lower rank.

6. Conclusions

We have shown that, although the proposition that the SN bracket preserves the Killing–Yano property is false in general, it does hold at least locally for constant curvature spacetimes. Through a counting argument in section 5, we have also found the maximum possible number of Killing–Yano tensors of a given rank in a given spacetime dimension. We have then seen via explicit construction that Minkowski and (A)dS spacetimes have the maximal possible number of Killing–Yano tensors of each rank. The algebra of these tensors under the SN bracket extends the structure of the Poincaré and (A)dS Lie algebras respectively.

These results suggest a number of interesting questions for additional work. A few of these are the following. Are there other classes of spacetimes in which the Killing–Yano...
tensors form an algebra under the SN bracket? Do the Y-ADM charges of asymptotically flat and (A)dS spacetimes, as discussed in the introduction, form algebras with respect to some form of generalized Poisson brackets? Finally, we can ask whether the representation theories of the graded Lie algebras of Killing–Yano tensors for Minkowski and (A)dS spacetimes have any physical significance.

Acknowledgments

We thank Ivan Mirkovic for helpful conversations. This work was supported by NSF grant PHY-0555304.

References

[1] Abbott L F and Deser S 1982 Stability of gravity with a cosmological constant Nucl. Phys. B 195 76
[2] Yano K 1952 Some remarks on tensor fields and curvature Ann. Math. 55 328
[3] Dolan P 1984 A generalization of the Lie derivative Classical General Relativity (Proceedings of the 1983 London Conference on Classical (Non-Quantum) General Relativity) ed W B Bonnor, J N Islam and M A H MacCallum (Cambridge: Cambridge University Press)
[4] Floyd R 1973 The dynamics of Kerr fields PhD Thesis (University of London)
[5] Penrose R 1973 Naked singularities Ann. N. Y. Acad. Sci. 224 125
[6] Krtous P, Kubiznak D, Page D N and Frolov V P 2007 Killing–Yano tensors, rank-2 Killing tensors, and conserved quantities in higher dimensions J. High Energy Phys. JHEP02(2007)004 (Preprint hep-th/0612029)
[7] Page D N, Kubiznak D, Vasudevan M and Krtous P 2007 Integrability of geodesic motion in general Kerr-NUT-AdS spacetimes Phys. Rev. Lett. 98 061102 (Preprint hep-th/0611083)
[8] Kubiznak D and Frolov V P 2007 Hidden symmetry of higher dimensional Kerr-NUT-AdS spacetimes Class. Quantum Grav. 24 F1 (Preprint gr-qc/0610144)
[9] Kastor D and Traschen J 2004 Conserved gravitational charges from Yano tensors J. High Energy Phys. JHEP08(2004)045 (Preprint hep-th/0406052)
[10] Townsend P K and Zamaklar M 2001 The first law of black brane mechanics Class. Quantum Grav. 18 5269 (Preprint hep-th/0107228)
[11] Kastor D, Shiromizu T, Tomizawa S and Traschen J 2005 Positivity bounds for the Y-ADM mass density Phys. Rev. D 71 104015 (Preprint hep-th/0410289)
[12] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity Ann. Phys. 88 286
[13] Grant J D E and Moss I G 1997 Hamiltonians for reduced gravity Phys. Rev. D 56 6284 (Preprint gr-qc/9703078)
[14] Schouten J A 1940 Uber Differentialkomitanten zweier kontravarianter Grössen. (German) Ned. Akad. Wet. Proc. Ser. A 17 449–52
[15] Schouten J A 1954 On the Differential Operators of First Order in Tensor Calculus (Convegno Internazionale di Geometria Differenziale, Italia, 1953) (Roma: Edizioni Cremonese) pp 1–7
[16] Nijenhuis A 1955 Jacobi-type identities for bilinear differential concomitants of certain tensor fields: I Ned. Akad. Wet. Proc. Ser. A 58 390–7
[17] Nijenhuis A 1955 Jacobi-type identities for bilinear differential concomitants of certain tensor fields: II Ned. Akad. Wet. Proc. Ser. A 59 398–403
[18] Vaisman I 1994 Lectures on the Geometry of Poisson Manifolds (Progress in Mathematics vol 118) (Basel: Birkhauser)
[19] Geroch R 1970 Multipole moments: I. Flat space J. Math. Phys. 11 1955
[20] Thompson G 1986 Killing tensors in spaces of constant curvature J. Math. Phys. 27 2693
[21] Gibbons G W, Rietdijk R H and van Holten J W 1993 SUSY in the sky Nucl. Phys. B 404 42 (Preprint hep-th/9303112)
[22] Gibbons G W and Ruback P J 1988 The hidden symmetries of multicenter metrics Commun. Math. Phys. 115 267
[23] Cebeci H, Saroglou O and Tekin B 2006 Gravitational charges of transverse asymptotically AdS spacetimes Phys. Rev. D 74 124021 (Preprint hep-th/0611011)
[24] Iwashita Y, Kastor D, Shiromizu T and Traschen J 2005 Y-ADM charges in asymptotically AdS spacetimes, unpublished
[25] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[26] Kashiwada T 1968 On conformal Killing tensor Natur. Sci. Rep. Ochanomizu Univ. 19 67