Mesh stiffness modeling considering actual tooth profile geometry for a spur gear pair

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Abstract. Some tooth profile geometric features, such as root fillet area, flank modification and wear are of nonnegligible importance for gear mesh stiffness. However, due to complexity of analytical description, their influence on mesh stiffness was always ignored by existing research works. The present work derives analytical formulations for time-varying gear mesh stiffness by using parametric equations of flank profile. Tooth geometry formulas based upon a rack-type tool are derived following Litvin’s vector approach. The root fillet area and tooth profile deviations can therefore be fully considered for spur gear tooth stiffness evaluation. The influence of gear fillet determined by tip fillet radius of the rack-type tool is quantified parametrically. The proposed model is validated to be effective by comparing with a finite element model. Further, the model is applied to investigate the stiffness variations produced by tooth addendum modification, tooth profile nonuniform wear and modification.

Keywords: Mesh stiffness / root fillet curve / tooth profile modification / addendum modification / nonuniform wear

1 Introduction

Health monitoring and fault diagnosis of a gear transmission system are always conducted based on its dynamic characteristics, wherein vibration is widely recognized as the most important one. Elastic deformation on engaged teeth, determined by the coupled effect between the distributed load on contacting teeth and gear mesh stiffness, is one of the main reasons leading to vibration in a gear transmission system [1]. However, when applying a constant load, the load distribution between two contacting tooth pairs is directly related to mesh stiffness. Furthermore, the variations of teeth number in contact and mesh position during meshing result in a changing gear mesh stiffness, inducing extra complexity on mesh stiffness calculation. Therefore, accurate and effective calculation of meshing stiffness is of great importance to gear dynamics [2–4].

In ISO standard 6336-2 [5] and AGMA standard 2001-D04 [6], Navier’s equation was applied for cylindrical gear bending calculation based upon assuming the load is uniformly distributed along the line of contact. It is known that the load distribution is usually different at each contact point and it depends on the meshing stiffness. In order to obtain better evaluation of gear mesh stiffness and load sharing, both finite element method (FEM) and analytical method have been adopted. Numerous works have been published [7–11] by taking the benefit of the FEM. Lin et al. [10] developed a three-dimensional (3D) FEM to derive mesh stiffness and tooth load distribution under both static and dynamic loading during meshing process. Ma et al. [11] established a finite element model for a spur gear pair in meshing. Time-varying mesh stiffness (TVMS) under different crack depths was then obtained based on the proposed model. The FEM has been proved to be able to obtain pretty accurate evaluation of gear mesh stiffness. However, related operation utilizing the FEM is time-consuming and difficult to model [12,13].

On the other hand, analytical methods provide simple and effective ways to calculate tooth stiffness, which were adopted by many researchers. Pedrero et al. [14] proposed a non-uniform load distribution model along the line of contact for involute external gears by employing the minimum elastic potential energy criterion. Chaari, Fakhfakh and Haddar [13] derived an analytical formulation for mesh stiffness of spur gears based upon the widely used Weber equation. However, their research did not provide detailed method on specific tooth profile geometry modeling. Fernandez et al. [15] used an analytical approach, combined with the FEM, to formulate the deformation at each contact point on the gear tooth.
The meshing stiffness can therefore be obtained accordingly. Ma et al. [16] studied the influence of torque variation on the mesh stiffness by using a newly proposed analytical method and FEM. Yang and Lin [17] introduced a potential energy method to compute the stiffness for mating gear teeth by considering bending, axial compressive and Hertzian contact stiffnesses. Further, Tian et al. [18] thought that the shear stiffness should also be considered to enhance the method proposed by Yang and Lin [17]. The improved potential energy method was then utilized by Saxena et al. [19] to investigate the influence of time-varying friction coefficient on the total effective mesh stiffness for spur gear pairs, by Liang et al. [12] to evaluate the mesh stiffness of a planetary gear set. In most of existing related studies, the gear body was treated as rigid; very few research works have modeled the gear as elastic body to investigate the influence of gear-body deformation on mesh stiffness. Sainsot and Velex [20] assumed that stress variations at dedendum circle were linear and constant, and then developed an analytical formula for gear body-induced tooth deflection. Nevertheless, the research on actual tooth profile geometry modeling and its effect on gear mesh stiffness is limited. This study focuses on actual tooth profile geometry, i.e., root fillet area, profile modification and wear, and investigation of its effect on the TVMS.

When computing the mesh stiffness of a spur gear pair, the gear tooth was usually modeled as a cantilever beam that starts from the base circle without root fillet curve [4,17,18]. Actually, as shown in Figure 1a and b, the gear tooth starts from the root circle. When the base circle is bigger than root circle (such tooth is defined as “R-B tooth” in the present work), the models in the above work ignored the gear tooth part between the base and root circles. However, when base circle is smaller than root circle (defined as “B-R tooth” instead), the models considered the tooth part of involute curve extension between the two circles. Some other methods [12,21] discussed separately for both the types of gears whose base circles are bigger or smaller than root circle. In the work of Liang et al. [12], a more rigorous cantilever beam model for gear tooth was developed based on the assumption that a gear tooth starts from the root circle. When base circle is smaller or bigger than root circle, they used straight lines, presented by the red lines in Figure 1(a) and (b), respectively, to simplify the root fillet curves. However, the accurate root fillet curves were still not considered. In Ref. [13], the authors thought that the stiffness of a whole tooth from the addendum to dedendum circles can be evaluated by modeling the tooth as a nonuniform cantilever beam with an effective length. But they did not provide the explicit analytical equations of tooth profile. Tooth geometry formulas on the basis of a rack-type tool following Litvin’s vector approach [22] were mentioned in Refs. [15,23,24], but the influence of tool tip fillet radius change on meshing stiffness was not discussed. Feng et al. [24] built a uniform wear model to analyze mesh stiffness for gears with tooth wear. But it is known that wear is not as severe at the pitch point as it is on the addendum or dedendum of the tooth. Chen and Shao [25] proposed a method to calculate mesh stiffness of spur gear pairs with tooth profile modification. However, they did not provide the detailed model for mesh stiffness of a single tooth pair with profile deviations.

In this paper, an analytical approach with accurate consideration of tooth root fillet curves and profile deviations is developed to evaluate the gear mesh stiffness. The influence of the tip fillet radius of rack-type tools and addendum modification on gear mesh stiffness is investigated. Some profile deviations, such as tooth profile modification and wear, are further studied for a better understanding of its influence on gear mesh stiffness.

2 Machining tool and its parametric equations

An involute gear tooth profile consists of three parts, i.e., involute profile, root fillet curve and dedendum circle, as shown in Figure 2. A is the intersection point of the dedendum circle and root fillet curve. B is the starting point of the involute profile, as well as the end point of the root fillet. C is the starting point of the contact zone on the tooth flank. D is the tooth end point. Rack cutter or hob is widely used for the generation of spur involute gears in industry. The meshing between a hob and a gear being generated can be considered as a rack-gear meshing. The rotation of the hob is always viewed as the translational movement of the imaginary rack [22]. Figure 3 shows the tooth profile of a
When a conventional setting of the rack-cutter is employed, the middle-line of a rack-cutter is tangent to the gear pitch circle [26]. In this case, rack-cutter centrode employed, the middle-line of a rack-cutter is tangent to the gear center. Therefore, the length of $A_1B_1$ cannot be less than 0. The value of $\rho$ should satisfy the following inequality

$$\rho \leq \frac{\cos \alpha}{1 - \sin \alpha} \left(\frac{\pi m}{4} - h_f \tan \alpha\right)$$

(3)

To avoid interference between tooth root fillet area and gear tip during meshing, pressure angle of involute profile starting point (point $B$ in Fig. 2) should be less than that of the starting point of the contact zone on tooth profile (point $C$ in Fig. 2). Therefore, the value of $\rho$ should also satisfy

$$\rho \leq \frac{1}{1 - \sin \alpha} \times \left[h_f - \frac{m}{2} \sin \alpha \left(\sqrt{Z_0^2 \sin^2 \alpha + 4Z_0 h_a^* + 4h_a^{*2} - Z_0 \sin \alpha}\right)\right]$$

(4)

where $Z_0$ denotes the number of teeth on the mating gear of the gear being generated.

In order to prevent undercutting, $\rho$ should be greater than the minimum limit [22] expressed as

$$\rho \geq \frac{2h_f - Zm \sin^2 \alpha}{2(1 - \sin \alpha)}$$

(5)

where $Z$ represents the tooth number of the machined gear. The above contents suggest that the variation range of $\rho$ must greatly shrink in order to satisfy the above three in equations simultaneously.

### 2.2 Non-conventional settings of rack-cutter

Nonstandard gears are also generated by a standardized tool used for standard gears, but with a modified setting corresponding to the cutting gear [22]. When a modified setting of the rack-cutter is employed, its middle-line $M-M$ moves a distance of $me^*$ with respect to the centrode of the rack-cutter. The displacement of the rack-cutter brings a thickness change to gear tooth. The expressions for $h_f$ and $\Delta$ become

$$h_f = (h_a^* + c^* - e^*)m$$

(6)

$$\Delta = |O_1O_2| = \pi m/4 - [h_f - (1 - \sin \alpha)\rho + e^*m] \tan \alpha - \rho \cos \alpha$$

(7)

where $e^*$ is the modification coefficient. $e^* > 0$ if the displacement is performed outward from the machined gear center; $e^* < 0$ if the displacement is performed toward the gear center.
2.3 Equations of a rack-cutter flank profile

In tooth cutting process the rack-cutter reciprocates parallel to the machined gear rotation axis. The gear tooth is shaped into the envelope for the family of rack-cutter shapes which are expressed in coordinate system with rigid connection to the gear being machined [26]. In coordinate system \( S_1 \), parametric equations for the arc part \( A_1A_2 \) on the rack-cutter are expressed as follows

\[
\begin{align*}
  x_1 &= r \sin \alpha/C_0 \sqrt{r^2/C_0^2 - u^2} \\
  y_1 &= r \cos \alpha/C_0 \sqrt{r^2/C_0^2 - u^2} \quad u \in \left[0, r \cos \alpha/C_1\right]
\end{align*}
\]

The parametric equations for the straight-line portion \( A_2B_2 \) of the cutting tool in coordinate system \( S_1 \) are given by the following formulas

\[
\begin{align*}
  x_2 &= u \\
  y_2 &= \cot \alpha(u - r \cos \alpha) + h \\
  u &\in [r \cos \alpha, (mh_n + h) \tan \alpha + r \cos \alpha]
\end{align*}
\]

3 Derivation of gear mesh stiffness considering root fillet area

3.1 Tooth flank profile equations

The schematic for coordinate transformation of gear cutting process is shown in Figure 4. Coordinate system \( S_1 \) is rigidly connected to the tool tooth that performs a translational motion with regard to the fixed coordinate system \( S_r \). A detailed description about the two coordinates is demonstrated in Section 2. Coordinate system \( S_0(x, y) \) is introduced with setting the symmetrical line of the gear tooth as \( x \) axis. The direction of coordinate axes \( y_2 \) and \( y_f \) coincides with \( A' A \) (see Fig. 2). \( S_2(x_2, y_2) \) and \( S_f(x_f, y_f) \) are two coordinate systems rigidly connected to the machined gear with gear center as original point. \( \theta \) is the angle between coordinate axes \( y_2 \) and \( y_f \). The difference is that \( y_2 \) and \( y_f \) axes are parallel to \( y_1 \) and \( x \) axes, respectively. The initial position of the coordinate axis \( y_2 \) and point \( A_1 \) on the rack-cutter coincide with the axis \( y_1 \) and point \( A \) on the tooth profile, respectively (Fig. 4a). \( \phi \) is the angle of gear rotation from initial position.

The equations for spur gear tooth profiles involving involute profile and root fillet curve can be derived. The matrices \( M_{21}, M_{22} \) and \( M_{0f} \) are defined to describe the translations from \( S_1(x_1, y_1) \) to \( S_2(x_2, y_2) \), \( S_2(x_2, y_2) \) to \( S_f(x_f, y_f) \) and \( S_f(x, y) \) to \( S_0(x, y) \), respectively, which are expressed as follows

\[
\begin{align*}
  M_{21} &= \begin{bmatrix}
    \cos \varphi & \sin \varphi & r(\sin \varphi - \varphi \cos \varphi) \\
    -\sin \varphi & \cos \varphi & r(\cos \varphi + \varphi \sin \varphi) \\
    0 & 0 & 1
  \end{bmatrix} \\
  M_{22} &= \begin{bmatrix}
    \cos \theta & -\sin \theta & 0 \\
    \sin \theta & \cos \theta & 0 \\
    0 & 0 & 1
  \end{bmatrix} \\
  M_{0f} &= \begin{bmatrix}
    0 & 1 & -L \end{bmatrix}
\end{align*}
\]
where

\[ \theta = \pi/z - \Delta/r \]  
\[ L = O_2O = r \cos \theta \]  

Accordingly, the transition matrix \( \mathbf{M}_{01} \) relating \( S_1(x_1, y_1) \) to \( S_0(x, y) \) can be obtained

\[ \mathbf{M}_{01} = \mathbf{M}_0 \mathbf{M}_1 \mathbf{M}_{21} \]  

The curve equations of gear root fillet, generated by arc part \( A_1A_2 \) on the tool profile, can be derived in coordinate system \( S_0(x, y) \) through Eqs. (8) and (12).

\[
\begin{align*}
  x_{AB} &= (-u + r \varphi) \sin (\varphi - \theta) + (r + \rho \sin \alpha - h) - \sqrt{\rho^2 - u^2} \cos (\varphi - \theta) - L \\
  y_{AB} &= (-u + r \varphi) \cos (\varphi - \theta) - (r + \rho \sin \alpha - h) - \sqrt{\rho^2 - u^2} \sin (\varphi - \theta)
\end{align*}
\]

where \( \varphi(u) = (\rho \sin \alpha - h)u / (\rho^2 - u^2), u \in [0, \rho \cos \alpha] \)

The involute profile of gear tooth is yielded by part \( A_1B_1 \) of the tool profile. According to Eqs. (9) and (12), involute profile equations in coordinate system \( S_0(x, y) \) can be expressed by

\[
\begin{align*}
  x_{BD} &= (-u + r \varphi) \sin (\varphi - \theta) + [r + \cot \alpha(u - \rho \cos \alpha) - h] \cos (\varphi - \theta) - L \\
  y_{BD} &= (-u + r \varphi) \cos (\varphi - \theta) - [r + \cot \alpha(u - \rho \cos \alpha) - h] \sin (\varphi - \theta)
\end{align*}
\]

where \( \varphi(u) = [u \sec^2 \alpha - (\rho \cot \alpha \cos \alpha + h) \cot \alpha] / r, u \in [\rho \cos \alpha, (mh_u + h) \tan \alpha + \rho \cos \alpha] \), \( u \) can also be expressed as a function of \( \varphi \)

\[ u(\varphi) = u = \varphi r \sin^2 \alpha + (\rho \cos^2 \alpha + h \sin \alpha) \cos \alpha \]

3.2 Equations of tooth profile with deviations

For a gear pair with tooth profile deviations, such as tooth wear and profile modification, its tooth flank equations can be expressed as follows

\[
\begin{align*}
  x'_{BD} &= x_{BD} - \Delta h \sin \beta_u \\
  y'_{BD} &= y_{BD} - \Delta h \cos \beta_u
\end{align*}
\]

where \( \Delta h \) is the profile normal offset; \( \beta_u \) means the angle between the normal direction of tooth flank and the axis y.

3.3 Potential energy method for gear mesh stiffness

In previous work [8,12,17,18,21], the potential energy method is employed to evaluate the mesh stiffness of fixed-shaft gear pairs with the assumption that the gear system is perfect without friction and transmission error. The same assumptions is also adopted in this paper. The strain energy of a spur tooth, \( U \), can be represented as the sum of bending energy \( U_b \), shear energy \( U_s \), radial compression energy \( U_a \) and the Hertzian contact energy \( U_h \), which correspond to bending stiffness \( k_b \), shear stiffness \( k_s \), compression stiffness \( k_a \) and Hertzian contact stiffness \( k_h \), respectively. According to elastic mechanics [27], the relations between elastic potential energies and stiffnesses are expressed as

\[
\begin{align*}
  U_b &= \frac{F^2}{2k_b}, U_s &= \frac{F^2}{2k_s}, U_a &= \frac{F^2}{2k_a}, U_h &= \frac{F^2}{2k_h}
\end{align*}
\]

where \( F \) means the action force at contact point along the action line. As shown in Figure 5, \( F_a \) and \( F_b \) are two orthogonal components of \( F \).

\[ F_a = F \sin \beta, F_b = F \cos \beta \]

For an external gear, the gear teeth may be either B-R or R-B teeth. In Refs. [12,21], the two types of teeth were discussed separately. While in this paper, the mesh stiffness of both B-R and R-B teeth can be evaluated by using the same formulas. As shown in Figure 5, \( r_B \) denotes the distance from the gear center to the end point B of root fillet curve. Generally, \( r_B \geq r_h \) is satisfied no matter the root circle is smaller or bigger than the base circle. In the following work bending, shear and axial compressive stiffnesses for the two cases will be evaluated by the same formulas, rather than be treated separately as in Ref. [12].
can be obtained \[17,18\]

\[
U_b = \frac{F^2}{2k_b} = \int_0^1 \frac{[F_b(d_i - x) - F_a h_i]^2}{2E_x} dx
\]  

(19)

\[
U_s = \frac{F^2}{2k_s} = \int_0^1 \frac{1.2F_b^2}{2G_A} dx
\]  

(20)

\[
U_a = \frac{F^2}{2k_a} = \int_0^1 \frac{F_a^2}{2E_A} dx
\]  

(21)

where \(E\) and \(G\) represent elastic and shear moduli, respectively, \(h_i\) is the distance from the gear contact point to tooth central line, \(d_i\) is the distance between the contact point and the tooth root (the coordinate axis \(y\)). \(A_x\) and \(I_x\) denote respectively the area and the moment of inertia of the cross section at the contact point.

According to the equations of the root fillet and involute curves (i.e., Eqs. (13) and (14)), the above equations can be expressed as follows

\[
U_b = \int_0^{(\frac{\rho \cos \alpha}{\cos \alpha})} \frac{[F_b(d_i - x_{AB}(u)) - F_A h_i]^2}{\frac{4}{3} E_B g_{AB}(u)} \cdot \frac{dx_{AB}(u)}{du} \cdot du + \int_0^{(\frac{\rho \cos \alpha}{\cos \alpha})} \frac{1.2F_b^2}{\frac{4}{3} E_B g_{AB}(u)} \cdot \frac{dx_{BD}(u)}{du} \cdot du
\]  

(22)

\[
U_s = \int_0^{(\frac{\rho \cos \alpha}{\cos \alpha})} \frac{1.2F_b^2}{4 E_B g_{AB}(u)} \cdot \frac{dx_{AB}(u)}{du} \cdot du + \int_0^{(\frac{\rho \cos \alpha}{\cos \alpha})} \frac{1.2F_b^2}{4 E_B g_{BD}(u)} \cdot \frac{dx_{BD}(u)}{du} \cdot du
\]  

(23)

\[
U_a = \int_0^{(\frac{\rho \cos \alpha}{\cos \alpha})} \frac{F_a^2}{4 E_B g_{AB}(u)} \cdot \frac{dx_{AB}(u)}{du} \cdot du + \int_0^{(\frac{\rho \cos \alpha}{\cos \alpha})} \frac{F_a^2}{4 E_B g_{BD}(u)} \cdot \frac{dx_{BD}(u)}{du} \cdot du
\]  

(24)

where

\[
u(u - \gamma) = (\theta - \gamma) \rho \sin^2 \alpha + (\rho \cos^2 \alpha + \lambda \sin \alpha) \cos \alpha
\]  

(25)

\[
\alpha_i = \arccos \frac{r_i}{r_f}
\]  

(26)

\[
d_i = r_i \cos \gamma - r_f \cos \theta
\]  

(27)

\[
\gamma = \pi + 4e \tan \alpha
\]

(28)

\[
h_i = r_i \sin \gamma
\]  

(29)

\[
\beta = \alpha_i - \gamma
\]  

(30)

Therefore, the mesh stiffness can be expressed as a function of \(r_i\), which denotes the distance from gear center to a contact point on tooth flank profile. In order to facilitate the observation for TVMS, the stiffness can also be represented as a function of \(\phi\), the rotation angular displacement of the driving gear, by using the following equation.

\[
\phi = \frac{\sqrt{r_i^2 - r_f^2} - \sqrt{r_C^2 - r_f^2}}{p_h Z}
\]  

(31)

where \(p_h\) denotes the pinion base pitch, \(r_i\) means the distance between the contact point and the pinion center. \(\phi\) can be calculated when the meshing engagement of the tooth starts.

3.4 Hertzian stiffness

According to the research by Yang and Sun\[28\], the Hertzian contact stiffness of a pair of meshing teeth, \(k_{bh}\), is virtually constant along the whole action line independent of both the interpenetration depth and the contact position. \(k_{bh}\) can be expressed as

\[
k_{bh} = \frac{\pi b}{2} \left[ \frac{1 - v_1^2}{E_1} + \frac{1 - v_2^2}{E_2} \right]^{-1}
\]  

(32)

where \(b\) is the tooth width; \(E_1, E_2\) and \(v_1, v_2\) denote elastic moduli and Poisson’s ratios of the pinion and wheel, respectively.

3.5 Stiffness considering fillet-foundation deflection

In addition to deformation of gear tooth, fillet-foundation deflection, \(\delta_f\), also affects the gear mesh stiffness. The derived analytical formula for \(\delta_f\) was proposed by Sainsot and Velex\[20\] on the basis of the theory of Muskhelishvili\[29\].

\[
\delta_f = \frac{F}{k_f} = \frac{F \cos^2 \beta}{Eb} \left\{ L^* \left( \frac{u_f}{S_f} \right)^2 + M^* \left( \frac{u_f}{S_f} \right) + P^* (1 + Q^* \tan^2 \beta) \right\}
\]  

(33)

where \(k_f\) is the corresponding fillet-foundation deflection stiffness, \(b\) is the tooth width, \(\beta\) is the load angle. Parameters \(u_f\) and \(S_f\) are defined in Figure 6. The coefficients \(L^*, M^*, P^*\) and \(Q^*\) can be computed according to the following polynomial functions\[20\],

\[
X^*(\theta_f, \gamma_f) = \frac{A_i}{\theta_f} + B_i h_i^2 + C_i h_i \theta_f + D_i \theta_f + E_i h_i + F_i
\]  

(34)

\(X^*\) indicates the coefficients \(L^*, M^*, P^*\) and \(Q^*\); \(r_i, r_{int}\) and \(\theta_f\) are given in Figure 6; \(h_f = r_f / r_{int}\); the values of \(A_i, B_i, C_i, D_i, E_i\) and \(F_i\) are listed in Table 1.

3.6 Overall mesh stiffness

Contact ratio of spur gears is generally between 1 and 2, but for high contact ratio gear pairs it may be between 2 and 3. Thus, the number of tooth pair in contact varies between 1 and 2 or 2 and 3 alternatively during meshing. For a single tooth pair in contact, the total mesh stiffness can be
The stiffness of a single meshing tooth pair is then computed as

\[ k = \frac{F}{\delta} \]  

(38)

The major parameters of a single-stage spur gear system introduced in this model are listed in Table 2. The diameters of inner bore of pinion and wheel are 20 mm and 30 mm, respectively. The listed pinion and wheel belong to R-B tooth and B-R tooth, respectively. In this way, the following study can investigate the influence of both R-B and B-R teeth on mesh stiffness determination.

Figure 8 shows the evolution of the single tooth pair mesh stiffness, \( k_{i} \), computed by the proposed analytical and finite element models. The tool tip fillet radius \( \rho \) in Figure 8 is 0.38 m. The position where the tooth pair starts to engage is set as the initial position of the pinion. A good agreement between the results yielded by the FEM and the proposed model is obtained. The maximum difference between the mesh stiffnesses obtained by the two methods is 5.72%. Therefore, the effectiveness of the proposed analytical model for determining the mesh stiffness with considering root fillet area is verified.

4.2 Influence of actual root fillet area on gear mesh stiffness

The evolutions of single tooth pair mesh stiffness and TVMS yielded by models considering tooth root fillet area in different ways are compared, as illustrated in Figure 9a and b, respectively. Three models are involved. The first one is the simplified model with setting the studied gear tooth starting from the base circle [18]. This model ignored the gear tooth part between the root and base circles of an R-B tooth. Instead, for a B-R tooth, a tooth part of involute to tooth starting from the base circle [18]. This model ignored the gear tooth part between the root and base circles of an R-B tooth. Instead, for a B-R tooth, a tooth part of involute curve extension between the two circles is added. For the case in Table 2, the simplification on the tooth pair results in average increments of single tooth pair stiffness and TVMS about 14.5% and 14.1%, respectively, compared with those of the proposed model. This is because the simplification enhances the rigidity of the pinion tooth significantly, meanwhile, it brings a relatively small reduction of the rigidity of the wheel tooth. The second model [12] made some improvements on the first one with the introduction of simplified root fillets exhibited as straight line for R-B tooth and involute for B-R tooth. However, this model underestimated the stiffness for both the R-B and B-R teeth (decrease about 39% for both the
computed single tooth pair stiffness and TVMS). The deviations of the results obtained by the simplified methods comparing with those of the proposed analytical model demonstrate that it is of significant importance to consider gear tooth profile with actual root fillet area.

Table 1. Coefficients of the polynomial in Eq. (34).

|  | \(A_i\) | \(B_i\) | \(C_i\) | \(D_i\) | \(E_i\) | \(F_i\) |
|---|---|---|---|---|---|---|
| \(L^*(h_f, \theta_f)\) | \(-5.574 \times 10^{-5}\) | \(-1.9986 \times 10^{-3}\) | \(-2.3015 \times 10^{-4}\) | \(4.7702 \times 10^{-3}\) | 0.0271 | 6.8045 |
| \(M^*(h_f, \theta_f)\) | \(60.111 \times 10^{-5}\) | \(28.100 \times 10^{-3}\) | \(-83.431 \times 10^{-4}\) | \(-9.9256 \times 10^{-3}\) | 0.1624 | 0.9086 |
| \(P^*(h_f, \theta_f)\) | \(-50.952 \times 10^{-5}\) | \(185.50 \times 10^{-3}\) | \(0.0538 \times 10^{-4}\) | \(53.300 \times 10^{-3}\) | 0.2895 | 0.9236 |
| \(Q^*(h_f, \theta_f)\) | \(-6.2042 \times 10^{-5}\) | \(9.0889 \times 10^{-3}\) | \(-4.0964 \times 10^{-4}\) | \(7.8297 \times 10^{-3}\) | \(-0.1472\) | 0.6904 |

The asterisk represents the ratio of the parameter to module.

Table 2. Parameters of the gears.

|                     | Pinion | Wheel |
|---------------------|--------|-------|
| Teeth number        | \(Z_1 = 19\) | \(Z_2 = 48\) |
| Module (mm)         | \(m = 4\) | \(m = 4\) |
| Face width (mm)     | \(b = 16\) | \(b = 16\) |
| Pressure angle (°)  | \(\alpha = 20\) | \(\alpha = 20\) |
| Addendum coefficient| \(h_a^* = 1\) | \(h_a^* = 1\) |
| Tip clearance coefficient | \(c^* = 0.25\) | \(c^* = 0.25\) |
| Addendum modification| \(e_1^* = 0, e_2^* = 0, -0.1, 0.1, 0.3, 0.5\) | \(-0.3, -0.5\) |
| Tool tip radius coefficient | \(\rho_1^* = 0.38\) | \(\rho_2^* = 0.38\) |
| Modulus of elasticity (GPa) | \(E_1 = 206.8\) | \(E_2 = 206.8\) |
| Poisson’s ratio      | \(v_1 = 0.3\) | \(v_2 = 0.3\) |

Fig. 7. Finite element model of a meshing tooth pair.

Fig. 8. Comparison between single tooth pair mesh stiffness evolutions yielded by the FEM and the proposed analytical model.
4.3 Influence of tool tip fillet radius on gear mesh stiffness

Tool tip fillet radius $\rho$, determining the value of $S_f$ and $\theta_f$ in Figure 6, is an important factor for tooth fillet-foundation deflection stiffness $k_f$. According to constraint conditions defined in Section 2.1, $\rho = 0.21 - 0.47 \text{ m}$ and is applicable for the pinion–wheel set in Table 2. Figures 10a shows the evolutions of $k_f$ of gears machined by tools with various tip fillet radius, while Figure 10b shows the evolutions of relative value of $k_f$ that denotes the ratio of $k_f (\rho = 0.47 \text{ m})$ to $k_f (\rho = 0.21 \text{ m})$. The value of $k_f$ declines with the decrease of tool tip fillet radius for both the pinion and wheel. The reduction of $k_f$ for pinion is between 6.5% and 16.9%, while for wheel it is 2.6–12.7%. The influence of $\rho$ on the pinion is more pronounced than on the wheel. The nearer the mesh point is to the tooth top, the greater $k_f$ decreases. For a high contact ratio (HCR) gear pair of which $h_a^* = 1.33$ and $Z_1 = Z_2 = 48$, the feasible interval of $\rho$ is $[0, 0.3]$. The corresponding evolutions are shown in Figure 11a and b. In addition to the aforesaid conclusions, we can observe that $k_f$ of HCR gear pair is more susceptible to $\rho$.

The influence of tool tip fillet radius on the total mesh stiffness $k_t$ of the machined gears (Table 2) is studied. The computational results are presented in Figure 12a and b. There is a reduction of stiffness when a gear pair machined by a tool with a relatively small tip fillet. Specifically, for single tooth pair mesh stiffness, the maximum and mean differences between gears cut by tools with $\rho = 0.21 \text{ m}$ and $0.45 \text{ m}$ are 5.94% and 5.21%, respectively. For the other stiffness, corresponding values are 5.36% and 5.18%, respectively. For the studied gear set, the analysis result suggests that an increase in tool tip fillet radius will augment the stiffness of the gear pair.

The evolutions of $k_t$ are further compared between a HCR gear pair ($h_a^* = 1.33$, $Z_1 = Z_2 = 48$ and $\varepsilon = 2.252$) and a low contact ratio (LCR) gear pair ($h_a^* = 1$, $Z_1 = Z_2 = 48$ and $\varepsilon = 1.748$). Figure 13a and b show the evolutions of
single tooth pair stiffness and overall stiffness for HCR and LCR gear pair, respectively. In Figure 13c, the relative values of $k_t$ are shown. As the value of $\rho$ increases from 0 to 0.3m, the increment of overall mesh stiffness ranges from 6.8% to 6.9% and 5.6% to 6.2% for the HCR and LCR gear pair, respectively. We also find that $k_t$ of a HCR gear pair is more susceptible to $\rho$ than that of a LCR one.

4.4 Influence of addendum modification on mesh stiffness

Addendum modification, introduced by a rack-cutter with non-conventional settings, will change the gear tooth thickness and geometry of root fillet area. The variations of mesh stiffnesses with respect to different addendum
Modification coefficients, $e_1^*$ and $e_2^*$, are illustrated in Figure 14a–d, respectively. The tool tip fillet radius coefficient $\rho = 0.38$ for all cases. Long-addendum brings an increment of $k_f$, while short-addendum results in a reduction of that. For the pinion, the rangeability of $k_f$ resulted from the variation of addendum modification coefficient is greater than that of the wheel (Fig. 14a). The gear modification reduces contact ratio and changes the mesh stiffnesses (Fig. 14b). Modified gear pairs for both $e_1^* = 0.1$, $e_2^* = -0.1$ and $e_1^* = 0.3$, $e_2^* = -0.3$ have higher stiffnesses than a standard gear pair when they are at single-tooth-pair meshing stage (Figure 14c and d). However, greater modification coefficients, i.e., $e_1^* = 0.5$, $e_2^* = -0.5$, may also have an opposite influence on mesh stiffness when the gear pair varies from double-tooth-pair meshing stage to single-tooth-pair meshing stage.

4.5 Influence of tooth profile wear and modification on gear mesh stiffness

Early stage wear is inevitable for gear pairs. Although this kind of wear is always quite slight, it still leads to a decrease in gear mesh stiffness. Fortunately, wear cannot arise on root fillet area since contact does not take place in this region during meshing. Thus, the involute profile part BC (see Fig. 5) and the fillet of a wearing gear can still be obtained by using Eq. (13). For a pair of involute spur gears, the relative sliding velocity between the contact tooth flanks is proportional to the distance from contact point to pitch point. With the increase of sliding velocity, the tooth wear is likely to increase. Thus, the wear on tooth flank is usually nonuniform. Assuming that early wear along the tooth surface, shown in Figure 15a, is proportional to $\Delta s$. Set proportion coefficient as $I_h$. Then the contact zone (part CD) of involute profile should be acquired by following equations

$$
\begin{align*}
    x'_{CD} &= x_{CD} - \Delta h \sin \beta_u \\
    y'_{CD} &= y_{CD} - \Delta h \cos \beta_u
\end{align*}
$$

(39)

where $\Delta h = \Delta s I_h$, $\Delta s = |u - \rho \cos \alpha - h \tan \alpha | cot \alpha / \sin \alpha$, $\beta_u$ means the angle between the normal direction of tooth flank and the axis $y$, $\sin \beta_u = \sin \alpha \cos \gamma - \cos \alpha \sin \gamma$, $\cos \beta_u = \cos \alpha \cos \gamma + \sin \alpha \sin \gamma$, $u \in [u_s, u_e]$. 

---

**Fig. 14.** Evolutions of gear mesh stiffness for different addendum modification coefficients: (a) $k_f$ (b) single tooth pair stiffness (c) TVMS (d) TVMS.
\[
\sin \alpha_i = \frac{\sqrt{y_{CD}^2 + (x_{CD} + L)^2 + y_{CD}^2}}{\sqrt{x_{CD}^2 + (x_{CD} + L)^2 + y_{CD}^2}}, \quad \cos \alpha_i = \frac{r_b}{\sqrt{x_{CD}^2 + (x_{CD} + L)^2 + y_{CD}^2}},
\]
\[
\sin \gamma = \frac{x_{CD}}{\sqrt{x_{CD}^2 + (x_{CD} + L)^2 + y_{CD}^2}}, \quad \cos \gamma = \frac{x_{CD} + L}{\sqrt{x_{CD}^2 + (x_{CD} + L)^2 + y_{CD}^2}},
\]
\[
u_s = \rho \cos \alpha + h \tan \alpha - \frac{m}{\rho} \sin \alpha \tan \alpha \left(\sqrt{Z_0^2 \sin^2 \alpha + 4Z_0 h_a^* + 4h_a^* + Z_0 \sin \alpha}, \quad u_c = (m h_a^* + h) \tan \alpha + \rho \cos \alpha.\right.
\]

Tooth profile modification, shown in Figure 15b, is another important source producing profile deviation. Since root relief of one member has the same effect as tip relief of the mating member, tip reliefs on both the pinion and the wheel are adopted in this paper. All modifications start at the highest point of single tooth contact (HPSTC). The tip relief part of tooth profile can also be obtained by Eq. (40).

For a gear pair with tooth deviations, such as profile nonuniform wear and modification, the overall mesh stiffness in double-tooth engagement can be obtained by the following formulas [25]

\[
k_t = \frac{k_1 + k_2}{1 + k_2 E_{12}/F_n} E_1 - E_2 = E_{12} < 0
\]

\[
k_t = \frac{k_1 + k_2}{1 - k_1 E_{12}/F_n} E_1 - E_2 = E_{12} > 0
\]

where \( E_1 = \Delta h_{p1} + \Delta h_{w1}, \quad E_2 = \Delta h_{p2} + \Delta h_{w2}. \) Subscripts 1 and 2 mean the first and the second tooth pair meshing at the same time, respectively.

The mesh stiffness influenced by nonuniform early stage wear of gear pairs can be obtained by using the developed method. It is assumed that the wear on both flanks of gear tooth is the same. For mating gear pairs made of the same material, the pinion usually gets worn more severely in that its tooth contact frequency is higher comparing with that of the mating gear. Therefore, \( I_{h1} \) for the pinion should be larger than that for the wheel. In following cases, we apply \( I_{h1} \) for the wheel and \( I_{h1} = I_{h2}/Z_2 \) for the pinion. The reductions of single tooth pair mesh stiffness for gear pairs with nonuniform wears, in which \( I_s = 0.5–8 \mu m/mm \), are shown in Figure 16a. The maximum wear appears on tooth tip of the
wheel and the approach point (AP) of the pinion tooth. When $I_b$ ranges from 0.5 to 6 μm/mm, the maximum wear is 5.16–82.56 μm. The tool tip fillet radius coefficient is fixed at 0.38. An increase in wear value results in severer reductions of mesh stiffnesses. However, a nonuniform wear causes decreases of mesh stiffness in various degrees at different meshing points. The reduction is the largest while the pinion tooth is engaging out. Figure 16b shows the evolutions of the overall mesh stiffness of gear pairs. In spite of only a little change of the single tooth mesh stiffness, a significantly reduction of overall mesh stiffness in double-tooth engagement is generated form the profile deviations due to wear and it decreases with the applied load increasing.

For the cases simulated in this paper, the applied load $T = 200$ Nm, the amount of profile modification $C_a = 25.94$ μm, $C_a = 26.29$ μm for the pinion and the wheel, respectively. Evolutions of TVMS and load transmission errors of gear pairs with different profile modifications are shown in Figure 17a and b, respectively. Although the reductions of single tooth mesh stiffness due to tip relief are less than 0.1%, the total mesh stiffness in double tooth contact duration decrease obviously in Figure 17a. In this paper, evolutions of TVMS for both the linear and the parabolic modification tooth pair are almost equal to each other. However, the corresponding load transmission errors are quite different. The transmission error variations of modified gear pairs are obviously smaller than that of a non-modified gear pair, which helps to deduct vibration and noise.

5 Conclusion

In this article, an analytical model for evaluating mesh stiffness of gear pairs machined by a rack-type cutter considering actual tooth profile geometry was proposed. Related formulations are derived according to axial compressive, bending, shear, Hertzian contact and fillet-foundation deflection stiffnesses. A comparison between the stiffness results from the proposed model and FEM verified the effectiveness of the new model. The following conclusions are obtained:

- It is of significant importance to model the gear tooth starting from the dedendum circle with actual root fillet area for an accurate evaluation of gear mesh stiffness.
- The TVMS decreases with the increasing of tool tip fillet radius. TVMS of a HCR gear pair is more susceptible to tool tip fillet radius than that of a LCR one.
- Long-addendum brings an increment of $k_f$, while short-addendum results in a reduction of that. The addendum modification of gear pairs reduces contact ratio and changes gear mesh stiffness. Appropriately addendum modified gear pair may have higher mesh stiffnesses than a standard one.
- An increase in wear value leads to severer reduction of mesh stiffness. A nonuniform wear causes obviously decreases of mesh stiffness in double tooth contact duration.
- Tooth profile modifications decrease the gear mesh stiffness. Different forms of profile modification have the same influence on TVMS, but they will produce diverse load transmission errors.

Nomenclature

| Symbol | Description |
|--------|-------------|
| $A_x$ | Area of the section where the distance to the gear root is $d_a$, mm$^2$ |
| $b$ | Width of the gear tooth, mm |
| $d_i$ | Distance between the contact point and the tooth root, mm |
| $E$ | Elastic modulus, GPa |
| $e^*$ | Modification coefficient |
| $F$ | Action load, N |
| $G$ | Shear modulus, GPa |
| $h$ | Distance between tool fillet and tool middle-line, mm |
| $h_i$ | Distance between contact point and the tooth central line, mm |
\( h_a \) Addendum, mm
\( h_o \) Addendum coefficient
\( h_f \) Dedendum, mm
\( h_f^* \) Dedendum coefficient
\( I_x \) Area moment of inertia of the section where the distance to the gear root is \( d_o \), mm\(^4\)
\( k_a \) Radial compression stiffness, N/mm
\( k_b \) Bending stiffness, N/mm
\( k_f \) Stiffness considering fillet-foundation deflection, N/mm
\( k_h \) Hertzian contact stiffness, N/mm
\( k_s \) Shear stiffness, N/mm
\( k_t \) Overall mesh stiffness, N/mm
\( m \) Module, mm
\( p_b \) Base pitch of pinion
\( r \) Reference radius, mm
\( r_B \) Base radius, mm
\( r_f \) Distance between the gear center and the end point of root fillet curve, mm
\( r_r \) Root radius, mm
\( r_i \) Distance from gear center to contact point on tooth flank profile, mm
\( u \) Parameter of rack-cutter flank profile equations
\( v \) Poisson’s ratio
\( Z \) Number of teeth
\( Z_o \) Number of teeth on a mating gear of the gear being generated
\( \alpha \) Profile angle of the rack cutter, rad
\( \alpha_B \) Pressure angle at starting point of the contact zone (point \( B \)), rad
\( \alpha_i \) Pressure angle at contact point, rad
\( \beta \) Load angle (angle of the force component \( F_b \) and the force \( F \)), rad
\( \beta_u \) Angle between the normal of tooth flank and the axis y, rad
\( \gamma \) Angle between tooth symmetrical line and the connection between gear center and contact point, rad
\( \delta \) Deformation of the contact teeth due to the force \( F \), mm
\( \Delta \) Distance between tool tip fillet arc center and tool symmetric line, mm
\( \Delta h \) Uniform wear along the tooth surface, mm
\( \Delta s \) Distance between contact point and reference point, mm
\( \theta \) Half tooth angle on the root circle, rad
\( \rho \) Tool tip radius, mm
\( \rho^* \) Tool tip radius coefficient
\( \varphi \) Angle of coordinate system \( S_2 \) rotation, rad
\( \phi \) Rotation angular displacement of the driving gear, degree

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