CONSTRUCTION OF THE MODULI SPACE OF REDUCED GRÖBNER BASES

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Abstract. For a given monomial ideal $J \subset k[x_1, \ldots, x_n]$ and a given monomial order $\prec$, the moduli functor of all reduced Gröbner bases with respect to $\prec$ whose initial ideal is $J$ is determined. In some cases, such a functor is representable by an affine scheme of finite type over $k$, and a locally closed subfunctor of a Hilbert scheme. The moduli space is called the Gröbner basis scheme, the Gröbner strata and so on if it exists. This paper introduces an alternative procedure for explicitly constructing a defining ideal of the Gröbner basis scheme and its Zariski tangent spaces by studying combinatorics on the standard set associated to $J$. That is a generalization of Robbiano and Lederer’s technique. We also see that we can make an implementation of that. Moreover, as a generalization of Robbiano’s result, we show that if the Gröbner basis scheme for $\prec$ and $J$ defined over the rational numbers $\mathbb{Q}$ is nonsingular at the $\mathbb{Q}$-rational point corresponding to $J$, then the Gröbner basis scheme for $\prec$ and $J$ defined over any commutative ring $k$ is isomorphic to an affine space over $k$.

1. Introduction

Let $k$ be a commutative ring and $S = k[x] = k[x_1, \ldots, x_n]$ a polynomial ring. We fix a monomial ideal $J$ and a monomial order $\prec$. Then we can consider the moduli of all reduced Gröbner bases in $S$ whose initial ideal is $J$ with respect to $\prec$. In set-theoretically the moduli space is computable using by Buchberger’s criterion and computing reduced Gröbner basis for some weighted homogeneous ideal [RT10, LR11]. Moreover, there is a submoduli parameterizing homogeneous reduced Gröbner bases. That submoduli space can be realized as a representable locally closed subfunctor of a Hilbert functor if $J$ is a saturated strongly stable ideal [LR16].

The main purpose of this paper is to give an alternative technique for directly constructing the moduli space as a representable functor without division algorithm, and we will see that we can make an implementation computing a defining ideal. For that, we generalize the technique induced in [Rob09, Led11] that is obtained by combinatorics on the standard set $\Delta \subset \mathbb{N}^n$ attached to $J$. Thus we can deal with any monomial ideal and any monomial order over any commutative ring.

More precisely, we consider a moduli functor for a monomial ideal $J$ in $S$ and a monomial order $\prec$ on $\mathbb{N}^n$;

$$\mathcal{Hilb}_k^\Delta : (k\text{-Alg}) \to (\text{Set})$$

$$B \mapsto \{G \subset B[x] \mid G \text{ is a reduced Gröbner basis with } \langle \text{LM}(G) \rangle = J_\Delta \}.$$
Here $\Delta = \{ \beta \in \mathbb{N}^n \mid x^\beta \notin J \}$ is the standard set attached to $J$ and $J_\Delta = \langle x^\alpha \mid \alpha \in \mathbb{N}^n \setminus \Delta \rangle$ is the monomial ideal in $B[x] = B \otimes_k S$ determined by $\Delta$ (Definition 1.1, Example 1.1). We denote by $\text{LM}(G)$ the set of leading monomials of $G$ with respect to $\prec$. In this paper, we call this functor $\mathcal{H}(\text{Hilb}_k^{\prec\Delta})$ the Gröbner basis functor or, for short, the Gröbner functor [Rob09, LR16]. Note that other authors call this functor or the scheme representing this the Gröbner strata [Led11].

The first discussion in Section 3 to Section 8 of this paper is about representability of Gröbner functors. In fact, the Gröbner functor $\mathcal{H}(\text{Hilb}_k^{\prec\Delta})$ is representable by an affine scheme of finite type over $k$, called the Gröbner (basis) scheme $\text{Hilb}_k^{\prec\Delta}$, if we assume one of the following conditions:

- $\Delta$ is finite. Equivalently, $J$ is a zero-dimensional monomial ideal [Rob09, Led11].
- $\prec$ is reliable, namely, for any $\alpha \in \mathbb{N}^n$, the set $\{ \gamma \in \mathbb{N}^n \mid \gamma \prec \alpha \}$ is finite [RT10].

For instance, we can obtain a defining ideal of $\text{Hilb}_k^{\prec\Delta}$ in some affine space by using Buchberger’s criterion [RT10]: a reduced Gröbner basis $G$ in $S$ with $\langle \text{LM}(G) \rangle = J_\Delta$ is in the form

$$G = \left\{ g_\alpha = x^\alpha - \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta) \right\}, \quad (*)$$

where $\mathcal{C}(\Delta)$ is the set of all exponents of minimal generators of $J = J_\Delta$ (Lederer characterizes $\mathcal{C}(\Delta)$ as the set of corners of $\Delta$, see Definition 1.2). If we assume that $\Delta$ is finite or $\prec$ is reliable, replacing coefficients $a_{\alpha, \beta}$ to indeterminates $T_{\alpha, \beta}$ and taking remainders of the division of $S$-polynomials by $G$, then we obtain a finite set of polynomials $\mathcal{P}$ in $k[T_{\alpha, \beta} \mid \alpha \in \mathcal{C}(\Delta), \beta \in \Delta, \alpha \prec \beta]$ such that $\{ x^\alpha - \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta) \}$ is Gröbner basis if and only if the point $(a_{\alpha, \beta})$ lies in the zero set of $\mathcal{P}$. This method also works for any monomial order and any pair $(\mathcal{D}, \Delta)$ such that $\mathcal{D}$ is a finite subset of $\Delta$. We say a reduced Gröbner basis as in $(*)$ is of type $(\mathcal{D}, \Delta)$ if all non-leading monomials lie in $\{ x^\beta \mid \beta \in \mathcal{D} \}$, and we define the Gröbner functor of type $(\mathcal{D}, \Delta)$ denoted by $\mathcal{H}(\text{Hilb}_k^{\prec(\mathcal{D}, \Delta)})$ in the same way. So we can also find a finite set of polynomials with the same property for reduced Gröbner bases of type $(\mathcal{D}, \Delta)$.

However, since this method deals with only $k$-rational points $\mathcal{H}(\text{Hilb}_k^{\prec\Delta})(k)$, it seems to be that this construction does not lead representability of Gröbner functors in immediately (but in certainly $\mathcal{P}$ is a defining ideal of the Gröbner scheme). Furthermore, we can not see an explicit form of $\mathcal{P}$ before running algorithm.

In the case of $\Delta$ is finite, Robbiano and Lederer introduce another technique for constructing a defining ideal of $\text{Hilb}_k^{\prec\Delta}$ which describe an explicit form of generators and directly induces representability of the Gröbner functor [Rob09, Led11]. We define a larger functor $\mathcal{H}(\mathcal{D})$ containing $\mathcal{H}(\text{Hilb}_k^{\prec\Delta})$, called the border basis functor or the marked family, for that: the set

$$\mathcal{B}(\Delta) = \bigcup_{i=1}^{n} (\Delta + e_i) \setminus \Delta$$
is called the **border** of \( \Delta \), where \( e_i \) is the \( i \)th canonical vector in \( \mathbb{N}^n \) (Definition [12]). Also we recall a \((\Delta-)\)**border prebasis** [KR05] 6.4 Border bases], that is a finite set of polynomials in the form

\[
F = \left\{ g_\alpha = x^\alpha - \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \middle| \alpha \in \mathcal{B}(\Delta) \right\}. \quad (**)
\]

We say a border prebasis as in (**) is a **border basis** or marked basis if the composition

\[
kx \Delta \hookrightarrow S \rightarrow S/\langle F \rangle
\]

is an isomorphism of \( k \)-modules, where \( kx \Delta \) is a \( k \)-submodule of \( S \) generated by \( \{x^\beta \mid \beta \in \Delta\} \). A border prebasis \( F \) is a border basis if and only if its coefficients satisfy two types relations [Rob09, Led11]:

\[
a_{\alpha+\lambda, \beta} = \sum_{\gamma \in \Delta} a_{\alpha, \gamma} a_{\gamma+\lambda, \beta}
\]

if \( \alpha \in \mathcal{B}(\Delta) \), \( \lambda \) is a canonical vector such that \( \alpha + \lambda \in \mathcal{B}(\Delta) \) and \( \beta \in \Delta \).

\[
\sum_{\gamma \in \Delta} a_{\alpha, \gamma} a_{\gamma+\lambda, \beta} = \sum_{\gamma \in \Delta} a_{\alpha', \gamma} a_{\gamma+\lambda', \beta}
\]

if \( \alpha, \alpha' \in \mathcal{B}(\Delta) \), \( \lambda, \lambda' \) are canonical vectors such that \( \alpha + \lambda = \alpha' + \lambda' \notin \mathcal{B}(\Delta) \) and \( \beta \in \Delta \).

We note that these relations are perfectly independent of a choice of \( k \). We also say a finite set of polynomials in the form \((*)\) is a reduced \textit{Gröbner prebasis}. If \( \Delta \) is finite, then the border \( \mathcal{B}(\Delta) \) is also finite, hence in fact we can inductively create a border prebasis \( F = F_G \) from a reduced Gröbner prebasis \( G \) with a property that \( G \) is a reduced Gröbner basis if and only if \( F_G \) is a border basis [Led11] (24 after Corollary 1). This operation \( G \mapsto F_G \) is independent of a choice of \( k \). Therefore the Gröbner functor \( \mathcal{H}ilb_\mathcal{D}^{(\Delta)} \) is representable if \( \Delta \) is finite.

Our first main theorem introduces such an operation \( G \mapsto F_G \) for any pair \((\mathcal{D}, \Delta)\) in the case of \( \Delta \) is infinite. More precisely,

**Theorem 1.1.** Let \( \Delta \) be a standard set and \( \mathcal{D} \) a finite subset of \( \Delta \). Put a \( k \)-algebra \( R = k[T_{\alpha, \beta} \mid \alpha \in \mathcal{C}(\Delta), \beta \in \mathcal{D}, \alpha \succ \beta] \). Take a map \( \nu : \mathcal{B}(\Delta) \setminus \mathcal{C}(\Delta) \to E = \{e_1, \ldots, e_n\} \) such that \( \alpha - \nu(\alpha) \in \mathcal{B}(\Delta) \). Then there exists a polynomial family \( \{U_{\alpha, \beta} \mid \alpha \in \Delta \cup \mathcal{B}(\Delta), \beta \in \Delta\} \) in \( R \) such that

1. \( U_{\alpha, \beta} = \delta_{\alpha, \beta} \) if \( \alpha, \beta \in \Delta \).
2. \( U_{\alpha, \beta} = T_{\alpha, \beta} \) if \( \alpha \in \mathcal{C}(\Delta), \beta \in \mathcal{D} \) and \( \alpha \succ \beta \).
3. \( U_{\alpha, \beta} = 0 \) if \( \alpha \in \mathcal{B}(\Delta) \) and \( \beta \in \Delta \) such that \( \alpha \prec \beta \).
4. For any \( \alpha \in \mathcal{B}(\Delta) \), \( U_{\alpha, \beta} = 0 \) expect finitely many \( \beta \in \Delta \).
5. For any \( \alpha \in \mathcal{B}(\Delta) \setminus \mathcal{C}(\Delta) \) and \( \beta \in \Delta \),

\[
U_{\alpha, \beta} = \sum_{\gamma \in \Delta} U_{\alpha - \nu(\alpha), \gamma} U_{\gamma + \nu(\alpha), \beta}.
\]

Thus the Gröbner functor \( \mathcal{H}ilb_k^{\mathcal{D}(\Delta)} \) of type \((\mathcal{D}, \Delta)\) is always representable. As a corollary, we also obtain a scheme represents the **homogeneous Gröbner functor** \( \mathcal{H}ilb_k^{\mathcal{D}(h, \Delta)} \) that consists of homogeneous reduced
Gröbner bases in $Hilb^<_k\Delta$. The Gröbner functor $Hilb^<_k\Delta$ can be described as an inductive limit of representable functors

$$
Hilb^<_k\Delta = \lim_{\mathcal{D} \subset \Delta} Hilb^<(\mathcal{D},\Delta)
$$

whose all morphisms $Hilb^<(\mathcal{D},\Delta) \rightarrow Hilb^<(\mathcal{D}',\Delta)$ are closed immersion. Hence the Gröbner functor $Hilb^<_k\Delta$ is an ind-scheme for a general standard set $\Delta$ and a monomial order $\prec$.

Moreover, we little bit refine the set of generators of a defining ideal of the Gröbner scheme rather than \cite{Led11}. We show that the Gröbner scheme $Hilb^<_k(\mathcal{D},\Delta)$ is defined by a finitely generated ideal in some affine space over $k$ even if $k$ is not Noetherian and if $\Delta$ is not finite (Theorem \ref{thm:finite_generators}). Therefore the Gröbner scheme can be computable in this way. We discuss about computation of Gröbner scheme in Section 9 to Section 11.

The representability of the moduli functor reveals at least two information on the moduli space. One is Zariski tangent spaces on the moduli space, and other is the universal family of the moduli. We explicitly describe these in Section 11 and Section 12. In particular, in Section 11, we find a universal embedding of the Gröbner scheme. This is a generalization of Robbiano’s idea in the proof of \cite{Robbiano} Corollary 3.7.

**Theorem 1.2.** Let $N$ be the rank of the Zariski tangent space on the Gröbner scheme $H$ at $k$-rational point $J_\Delta$. Then there exists a closed immersion

$$
H \hookrightarrow \mathbb{A}^N_k
$$

such that all elements of the defining ideal have no linear terms and no constant terms. In particular, the followings are equivalent:

1. $H$ is isomorphic to $N$-dimensional affine space over $k$.
2. $H$ has the same dimension as $\mathbb{A}^N_k$. In other words, $H$ is non-singular at $J$.

This closed immersion is universal. Namely, for any ring morphism $k \rightarrow k'$, the diagram

$$
\begin{array}{ccc}
\mathbb{A}^N_{k'} & \longrightarrow & \mathbb{A}^N_k \\
\downarrow & & \downarrow \\
H_{k'} & \longrightarrow & H_k
\end{array}
$$

is cartesian. Here we denote by $H_k$ the Gröbner scheme defined over $k$.

### 2. Notation

- Let $k$ be a commutative ring and $S = k[x] = k[x_1, \ldots, x_n]$ a polynomial ring over $k$. For each subset $G$ of $S$, $\langle G \rangle$ is the ideal of $S$ generated by $G$.
- Let $\mathbb{N}$ be the set of all non-negative integers, we regard $\mathbb{N}^n$ as the set of all monomials in $S$ using a notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. 

• The degree of $\alpha \in \mathbb{N}^n$ is the number $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For a subset $A$ of $\mathbb{N}^n$, we put $A_{\geq r} = \{ \alpha \in A \mid |\alpha| \geq r \}$ and $A_{\leq r} = \{ \alpha \in A \mid |\alpha| \leq r \}$.

• For any subset $A$ of $\mathbb{N}^n$, we define a $k$-submodule $kx^A = \bigoplus_{\alpha \in A} kx^\alpha = \left\{ \sum_{\alpha \in A} c_{\alpha} x^\alpha \mid c_{\alpha} \in k \right\}$ in $S$.

3. Monomial order and Gröbner basis

First we recall basic definitions. Lemma 3.1 is most important idea in this paper. From that, we can use the technique given by [Rob09, Led11] for infinite standard sets (i.e. non-zero-dimensional monomial ideals).

**Definition 3.1.** A total order $\prec$ on $\mathbb{N}^n$ is a **monomial order** if it satisfies the followings:

• For any $\alpha \in \mathbb{N}^n \setminus \{0\}$, $\alpha \succ 0$.

• For any $\alpha, \beta, \gamma \in \mathbb{N}^n$, if $\alpha \prec \beta$, then $\alpha + \gamma \prec \beta + \gamma$.

**Example 3.1.** The **lexicographic order** $\prec_{\text{lex}}$ on $\mathbb{N}^n$ ($\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{R}^n$): for any vectors $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we say $\alpha \prec_{\text{lex}} \beta$ if $\alpha_i < \beta_i$ where $i = \min \{ j \mid \alpha_j \neq \beta_j \}$.

**Proposition 3.1.** ([Bay82, Chapter I, (1.6)]) A monomial order $\prec$ is a well-order on $\mathbb{N}^n$, i.e. every non-empty subset of $\mathbb{N}^n$ has a minimum element with respect to $\prec$.

Robbiano gave a classification of all monomial orders [Rob85]. The next proposition is a part of that.

**Proposition 3.2.** ([Eis95, Exercise 15.12], [Rob85, Theorem 4]) Let $\prec$ be a monomial order on $\mathbb{N}^n$. Then there exist a positive integer $m$ and vectors $u_1, \ldots, u_m \in \mathbb{Z}^n$ such that

$$u : (\mathbb{N}^n, \prec) \to (\mathbb{Z}^m, \prec_{\text{lex}}) ; u(\alpha) = (\alpha \cdot u_1, \ldots, \alpha \cdot u_m)$$

is an order preserving injection.

Using this injection, we can make a filtration of the set $\mathbb{N}^n$ with respect to $\prec$. We use this filtration for explicitly constructing coefficients of a border prebasis $\{ x^\alpha - \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{B}(\Delta) \}$ from given reduced Gröbner prebasis $\{ x^\alpha - \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta) \}$ (Theorem 8.1). Note that it is a hard point in the case of $\Delta$ is infinite because $\mathcal{B}(\Delta)$ is infinite and it is possible that a naive inductive method using Proposition 3.1 does not cover an infinite subset of $\mathbb{N}^n$. However, we can do it by using the induction along this filtration.

**Lemma 3.1.** Let $\prec$ be a monomial order on $\mathbb{N}^n$. Then there exist a positive integer $m$ and a family of subsets of $\mathbb{N}^n$

$$\{ L(s_1, \ldots, s_i) \mid 1 \leq i \leq m, \ s_1, \ldots, s_i \in \mathbb{N} \}$$

satisfying the followings:
(1) If \((s_1, \ldots, s_i) \prec_{\text{lex}} (t_1, \ldots, t_i)\) in \(\mathbb{N}^i\), then \(L(s_1, \ldots, s_i) \prec L(t_1, \ldots, t_i)\), where \(A \prec B\) means that \(\alpha \prec \beta\) for any \(\alpha \in A\) and \(\beta \in B\).

(2) \(\mathbb{N}^n = \bigcup_{s \in \mathbb{N}} L(s)\).

(3) \(L(s_1, \ldots, s_i) = \bigcup_{s \in \mathbb{N}} L(s_1, \ldots, s_i, s)\) for each \(1 \leq i \leq m - 1\).

(3) For each \((s_1, \ldots, s_m) \in \mathbb{N}^m\), \(L(s_1, \ldots, s_m)\) has at most one element.

Proof. We take the vectors \(u_1, \ldots, u_m \in \mathbb{Z}^n\) as in Proposition 3.2. We put

\[ L(s) = \{\alpha \in \mathbb{N}^n \mid \alpha \cdot u_1 = s\} \]

for each \(s \geq 0\). Clearly \(s < t\) implies \(L(s) \prec L(t)\), and we have

\[ \mathbb{N}^n = \bigcup_{s \in \mathbb{N}} L(s). \]

Using an induction, we may assume that there exist families

\[ \{L(s_1, \ldots, s_i) \mid s_1, \ldots, s_i \in \mathbb{N}\} \]

for each \(1 \leq i \leq j\) with the following conditions (a)--(d):

(a) \(\mathbb{N}^n = \bigcup_{s \in \mathbb{N}} L(s)\).

(b) \(L(s_1, \ldots, s_i) = \bigcup_{s \in \mathbb{N}} L(s_1, \ldots, s_i, s)\) for each \(1 \leq i \leq j - 1\).

(c) \(s < t\) implies \(L(s_1, \ldots, s_i, s) \prec L(s_1, \ldots, s_i, t)\) for each \(0 \leq i \leq j - 1\).

(d) \(u_1, u_2, \ldots, \) and \(u_i\) are constant on \(L(s_1, \ldots, s_i)\) as functions for each \(1 \leq i \leq j\).

We take a \(j\)-tuple \((s_1, \ldots, s_j) \in \mathbb{N}^j\). If \(L(s_1, \ldots, s_j) = \emptyset\), then we define \(L(s_1, \ldots, s_j, s) = \emptyset\) for each \(s \in \mathbb{N}\), otherwise we consider \(u_{j+1} : L(s_1, \ldots, s_j) \to \mathbb{Z}\). From the condition (d), for any \(\alpha, \beta \in L(s_1, \ldots, s_j)\), \(u_{j+1}(\alpha) < u_{j+1}(\beta)\) implies \(\alpha \prec \beta\). Therefore the image \(u_{j+1}(L(s_1, \ldots, s_j))\) is a well-ordered set with respect to the ordinary order on \(\mathbb{Z}\). Hence we can give a numbering as

\[ u_{j+1}(L(s_1, \ldots, s_j)) = \{z_0 < z_1 < z_2 < \cdots\}. \]

Then we define

\[ L(s_1, \ldots, s_j, s) = \{\alpha \in L(s_1, \ldots, s_j) \mid u_{j+1}(\alpha) = z_s\} \]

for each \(s \in \mathbb{N}\). Now we get a family \(\{L(s_1, \ldots, s_i) \mid s_1, \ldots, s_i \in \mathbb{N}\}\) with the conditions (a)--(d) for \(i = j + 1\), and for each \(1 \leq i \leq m\) inductively. The condition (d) and Proposition 3.2 imply that \(L(s_1, \ldots, s_m)\) has at most one element. Therefore this family

\[ \{L(s_1, \ldots, s_i) \mid 1 \leq i \leq m, s_1, \ldots, s_i \in \mathbb{N}\} \]

satisfies the conditions (1)--(4).

\[ \square \]

Definition 3.2. Let \(\prec\) be a monomial order on \(\mathbb{N}^n\). For a non-zero polynomial \(f \in S \setminus \{0\}\), we define the followings:

- \(\text{coef}(f, x^\alpha)\) is the coefficient of \(f\) at \(x^\alpha\).
- \(\text{supp}(f) = \{x^\alpha \mid \text{coef}(f, x^\alpha) \neq 0\}\) is called the support of \(f\).
- The leading monomial of \(f\), denoted by \(\text{LM}(f)\), is the maximum element of \(\text{supp}(f)\) with respect to \(\prec\).
- \(\text{LC}(f) = \text{coef}(f, \text{LM}(f))\) is called the leading coefficient of \(f\).
- \(\text{LT}(f) = \text{LC}(f) \text{LM}(f)\) is called the leading term of \(f\).
- The leading exponent of \(f\), denoted by \(\text{LE}(f)\), is the exponent of \(\text{LM}(f)\), i.e. \(\text{LM}(f) = x^{\text{LE}(f)}\).
Furthermore, for each subset $I$ of $S$, we put
$$\text{LM}(I) = \{\text{LM}(f) \mid f \in I \setminus \{0\}\}.$$Also we put $\text{LT}(I)$ and $\text{LE}(I)$ in the same way. The initial ideal of $I$ is the ideal generated by $\text{LM}(I)$.

The initial ideal $\langle \text{LM}(I) \rangle$ is generated by monomials, hence it is a monomial ideal.

**Proposition 3.3.** ([CLO97, pp.67-68, Lemma 2, and Lemma 3]) Let $k$ be a non-zero ring. Assume that $J$ is a monomial ideal.

- There exists a decomposition $J = \bigoplus_{\alpha \in \mathbb{N}^n} J \cap kx^\alpha$.
  
  In other words, for a polynomial $f = \sum_i a_i x^{\alpha_i}, f \in J$ if and only if $x^{\alpha_i} \in J$ for each $i$.
- If $J$ is generated by $\{x^\alpha \mid \alpha \in A\}$, then for any $\beta, \gamma \in \mathbb{N}^n$, $x^\beta \in J$ if and only if there exists $\alpha \in A$ such that $\beta - \alpha \in \mathbb{N}^n$.

**Definition 3.3.** Let $\prec$ be a monomial order on $\mathbb{N}^n$. A finite subset $G$ of $S$ is a Gröbner basis (of $I = \langle G \rangle$) if
$$\langle \text{LT}(I) \rangle = \langle \text{LT}(G) \rangle.$$We say that $I$ has a Gröbner basis if there exists a Gröbner basis in $S$ with $I = \langle G \rangle$.

**Proposition 3.4.** Let $G$ be a Gröbner basis in $S$ with respect to a monomial order $\prec$ and $I$ the ideal generated by $G$. Then $\langle \text{LM}(I) \rangle = \langle \text{LM}(G) \rangle$, thus $\text{LE}(I) = \text{LE}(G) + \mathbb{N}^n = \{\text{LE}(g) + \gamma \mid g \in G, \gamma \in \mathbb{N}^n\}$.

**Proof.** This is easy to show. Thus we omit. $\square$

4. Standard set

We recall a standard set in $\mathbb{N}^n$. We use the same notation as in [Led11].

**Definition 4.1.** A subset $\Delta$ in $\mathbb{N}^n$ is a standard set if $$(\mathbb{N}^n \setminus \Delta) + \mathbb{N}^n \subset (\mathbb{N}^n \setminus \Delta).$$Equivalently, for any $\alpha, \beta \in \mathbb{N}^n$, if $\alpha + \beta \in \Delta$, then $\alpha, \beta \in \Delta$.

**Example 4.1.** Let $\prec$ be a monomial order on $\mathbb{N}^n$. For any ideal $I$ of $S$,
$$\Delta = \Delta_I = \mathbb{N}^n \setminus \text{LE}(I)$$is a standard set. We call $\Delta_I$ the standard set attached to $I$. Conversely, let $\Delta$ be a standard set. If $k$ is a non-zero ring, then
$$J_\Delta = \langle x^\alpha \mid \alpha \in \mathbb{N}^n \setminus \Delta \rangle$$is a unique monomial ideal such that $\text{LE}(J_\Delta) = \mathbb{N}^n \setminus \Delta$ for any monomial order $\prec$ on $\mathbb{N}^n$. Therefore there exists a one-to-one correspondence between the set of all standard sets and the set of all monomial ideals.
Definition 4.2. ([Led11] Definition 2]) Let $\Delta$ be a standard set. We define followings:

- **The set of corners**
  \[
  \mathcal{C}(\Delta) = \{\alpha \in \mathbb{N}^n \setminus \Delta \mid \alpha - \lambda \notin \mathbb{N}^n \setminus \Delta \text{ for all } \lambda \in E\},
  \]
  where $E = \{e_1, \ldots, e_n\}$ is the set of all canonical bases of $\mathbb{N}^n$ (see the points C in Figure 1).

- **The border** of $\Delta$
  \[
  \mathcal{B}(\Delta) = \bigcup_{\lambda \in E} (\Delta + \lambda) \setminus \Delta
  \]
  (see the points o in Figure 1).

- A point $\varepsilon \in \Delta$ is an **edge point** of $\Delta$ (see the points E in Figure 1) if there exist $\lambda, \mu \in E$ such that $\varepsilon + \lambda, \varepsilon + \mu \in \mathcal{B}(\Delta)$ and $\varepsilon + \lambda + \mu \in \mathcal{C}(\Delta \cup \mathcal{B}(\Delta))$. We call such a triple $(\varepsilon; \lambda, \mu)$ an **edge triple** of $\Delta$.

In fact, the set of corners $\mathcal{C}(\Delta)$ determines a minimal generator of $J_\Delta$. Furthermore, if a subset $A$ of $\mathbb{N}^n$ satisfies $\mathbb{N}^n \setminus \Delta = A + \mathbb{N}^n$, then $\mathcal{C}(\Delta) \subset A$. 
**Proposition 4.1.** Let $\Delta$ be a standard set. Then the set of corners $C(\Delta)$ is a finite set.

**Proof.** This is a well known fact called Dickson’s Lemma. See [CLO97, p.69, Theorem 5]. □

### 5. Monic ideal and Reduced Gröbner basis

We fix a monomial order $\prec$ from now on.

**Definition 5.1.** Let $\Delta$ be a standard set in $\mathbb{N}^n$ and $G$ be a Gröbner basis in $S$ with $\text{LE}(\langle G \rangle) = \mathbb{N}^n \setminus \Delta$. We call $G$ a reduced Gröbner basis with respect to $\prec$ if $G$ satisfies following three conditions:

- For any $g \in G$, $g$ is monic, i.e. $\text{LC}(g) = 1$.
- $\text{LE}(G) = C(\Delta)$.
- For any $g \in G$, $g - \text{LM}(g) \in kx^\Delta$.

In other words, $G$ has the form

$$G = \left\{ g_\alpha = x^\alpha - \sum_{\beta \in \Delta} a_{\alpha,\beta} x^\beta \mid \alpha \in C(\Delta) \right\}$$

with $\text{LE}(g_\alpha) = \alpha$.

If $k$ is a field, then for any ideal $I$ of $S$, there exists a unique reduced Gröbner basis of $I$ [CLO97, p.90, Proposition 6]. However, if $k$ is a not filed, there can exist an ideal of $S$ such that it has no reduced Gröbner basis. We recall a characterization of an ideal having a reduced Gröbner basis [Wib07, Theorem 4].

**Definition 5.2.** ([Wib07, Definition 4], [Led11, Definition 1]) An ideal $I$ of $S$ is a monic ideal with respect to $\prec$ if $I$ satisfies following four equivalent conditions:

- $\langle \text{LT}(I) \rangle = \langle \text{LM}(I) \rangle$.
- $\langle \text{LT}(I) \rangle$ is a monomial ideal.
- For any $f \in I \setminus \{0\}$, there exists a monic polynomial $f' \in I \setminus \{0\}$ such that $\text{LM}(f) = \text{LM}(f')$.
- For any monomial $x^\alpha$, the ideal $\text{LC}(I, x^\alpha) = \{ \text{LC}(f) \mid f \in I \setminus \{0\}, \text{LM}(f) = x^\alpha \} \cup \{0\}$ is either the zero ideal or the unit ideal of $k$.

**Proposition 5.1.** (See also [Led11, Lemma 2.]) Let $\Delta$ be a standard set and $I$ an ideal with $\text{LE}(I) = \mathbb{N}^n \setminus \Delta$. Then $I$ is monic if and only if there exists a unique family of polynomials $\{ g_\alpha \in I \mid \alpha \in \mathbb{N}^n \setminus \Delta \}$ such that $g_\alpha$ satisfies following three conditions for each $\alpha \in \mathbb{N}^n \setminus \Delta$:

- $g_\alpha$ is monic.
- $\text{LE}(g_\alpha) = \alpha$.
- $g_\alpha - \text{LM}(g_\alpha) \in kx^\Delta$.

Namely, each $g_\alpha$ has the form

$$g_\alpha = x^\alpha - \sum_{\beta \in \Delta} a_{\alpha,\beta} x^\beta$$
with $\text{LE}(g_\alpha) = \alpha$. We call this family the monic basis of $I$.

**Proof.** If there exists such a family, since each $g_\alpha$ is monic, then $I$ is monic by definition. Conversely, we assume that $I$ is monic. The uniqueness of such a family comes from $I \cap kx^\Delta = \{0\}$. We show the existence. We use a reduction to the absurd. Let $A$ be the set of all vectors $\alpha \in \mathbb{N}^n \setminus \Delta$ such that there does not exist a polynomial $g_\alpha \in I$ with the three conditions for $\alpha$. If $A$ is not empty, we can take the minimum element $\alpha$ of $A$ with respect to $\prec$. Now $I$ is monic with $\text{LE}(I) = \mathbb{N}^n \setminus \Delta$, so there exists a monic polynomial $f \in I$ such that $\text{LE}(f) = \alpha$. Since $\alpha$ is minimum in $A$, for each $\gamma \in B = \{ \gamma \in \mathbb{N}^n \setminus \Delta \mid x^\gamma \in \text{supp}(f) \setminus \{x^\alpha\} \}$, there exists a polynomial $g_\gamma \in I$ with the three conditions. We put

$$g = f - \sum_{\gamma \in B} \text{coef}(f, x^\gamma) g_\gamma.$$ 

Then $g$ is included in $I$ and satisfies the three conditions for $\alpha$, but it is a contradiction to $\alpha \in A$. □

Clearly, if $\{g_\alpha \in I \mid \alpha \in \mathbb{N}^n \setminus \Delta\}$ is the monic basis of $I$, then the finite subset $\{g_\xi \mid \xi \in \mathcal{B}(\Delta)\}$ is a reduced Gröbner basis of $I$. Conversely, an ideal having a reduced Gröbner basis is monic.

**Proposition 5.2.** ([Wib07, Theorem 4]) An ideal $I$ of $S$ has a reduced Gröbner basis if and only if $I$ is monic.

**Corollary 5.1.** Any ideal $I$ of $S$ has at most one reduced Gröbner basis with respect to $\prec$.

**Proposition 5.3.** Let $I$ be a monic ideal with $\text{LE}(I) = \mathbb{N}^n \setminus \Delta$. Then the composition of $k$-module morphisms

$$kx^\Delta \xrightarrow{i} S \twoheadrightarrow S/I$$

is an isomorphism of $k$-modules between $kx^\Delta$ and $S/I$, where $i$ is the inclusion map and $S \twoheadrightarrow S/I$ is the natural surjection. Furthermore, the monic basis $\{g_\alpha \mid \alpha \in \mathbb{N}^n \setminus \Delta\}$ of $I$ is a $k$-basis of $I$.

**Proof.** Since $I \cap kx^\Delta = \{0\}$, this morphism is injective. For an arbitrary $f \in S$, we suppose

$$f = \sum_{\alpha \in \mathbb{N}^n \setminus \Delta} c_\alpha x^\alpha + \sum_{\beta \in \Delta} c_\beta x^\beta.$$ 

Then a polynomial $g = f - \sum_{\alpha \in \mathbb{N}^n \setminus \Delta} c_\alpha g_\alpha$ is included in $kx^\Delta$ and $f - g \in I$, therefore the morphism is surjective. Furthermore, if $f \in I$, then $g \in I \cap kx^\Delta = \{0\}$, therefore $\{g_\alpha \mid \alpha \in \mathbb{N}^n \setminus \Delta\}$ is a $k$-basis of $I$. □

In other words, the set of polynomials $\{g_\alpha \mid \alpha \in \mathcal{B}(\Delta)\}$ is the $\Delta$-border basis of $I$ (see [KR05] pp.419ff., 6.4 Border bases] or the introduction of this paper).
From Proposition 5.2 and Proposition 5.3, we have maps:
\[ \{ G \mid G \text{ is a reduced Gröbner basis in } S \text{ with } \text{LE}(G) = \mathcal{C}(\Delta) \} \]
\[ \cong \{ I \mid I \text{ is a monic ideal of } S \text{ with } \text{LE}(I) = \mathbb{N}^n \setminus \Delta \} \]
\[ \subset \{ G \mid G \text{ is a } \Delta\text{-border basis in } S \} \]
\[ \cong \{ I \mid I \text{ is an ideal of } S \text{ and } kx^\Delta \hookrightarrow S \to S/I \text{ is an isomorphism} \} \]
\[ \subset \text{Hom}_{k\text{-Mod}}(S, kx^\Delta). \]
These correspondences preserve coefficients of a monic basis. To find algebraic relations in coefficients of a reduced Gröbner basis, we focus on \( k \)-module morphisms \( \phi : S \to kx^\Delta \) whose Ker \( \phi \) is a monic ideal with \( \text{LE}(\text{Ker} \phi) = \mathbb{N}^n \setminus \Delta \) \[ \text{Rob09, Led11}. \]

**Proposition 5.4.** ([Led11 Proposition 2]) Let \( \phi : S \to kx^\Delta \) be a \( k \)-module morphism whose Ker \( \phi \) is an ideal of \( S \). For \( \alpha \in \mathbb{N}^n \), we put \( \phi(x^\alpha) = \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \). Then Ker \( \phi \) is a monic ideal with \( \text{LE}(\text{Ker} \phi) = \mathbb{N}^n \setminus \Delta \) if and only if \( \phi \) satisfies following two properties:

- For \( \gamma, \beta \in \Delta \), \( a_{\gamma, \beta} = \delta_{\gamma, \beta} \), where \( \delta_{\gamma, \beta} \) is the Kronecker delta. In other words, the composition \( kx^\Delta \hookrightarrow S \xrightarrow{\phi} kx^\Delta \) is the identity map on \( kx^\Delta \).
- For any \( \alpha \in \mathcal{C}(\Delta) \) and \( \beta \in \Delta \), if \( \alpha \prec \beta \), then \( a_{\alpha, \beta} = 0 \).

If \( \phi \) satisfies these properties, \( \phi \) also satisfies:

- For any \( \alpha \in \mathbb{N}^n \setminus \Delta \) and \( \beta \in \Delta \), if \( \alpha \prec \beta \), then \( a_{\alpha, \beta} = 0 \).

**Proposition 5.5.** ([KR05, p.434, Theorem 6.4.30], [Led11, 8. Representing the functors]) Let \( \phi : S \to kx^\Delta \) be a \( k \)-module morphism and \( a_{\alpha, \beta} \) be the coefficient of \( \phi(x^\alpha) \) at \( x^\beta \). We assume that the composition \( kx^\Delta \hookrightarrow S \xrightarrow{\phi} kx^\Delta \) is the identity map on \( kx^\Delta \). Then Ker \( \phi \) is an ideal of \( S \) if and only if \( a_{\alpha + \lambda, \beta} = \sum_{\gamma \in \Delta} a_{\alpha, \gamma} a_{\gamma + \lambda, \beta} \) for any \( \alpha \in \mathbb{N}^n \), \( \beta \in \Delta \) and \( \lambda \in E = \{ e_1, \ldots, e_n \} \).

Therefore if Ker \( \phi \) is a monic ideal, the coefficients of \( \alpha + \lambda \) are inductively determined from the coefficients at vectors less than \( \alpha + \lambda \) with respect to \( \prec \) (see also [Led11] (24) after Corollary 1)). To give a well expression of this relation, we define products on \( kx^\Delta \) induced by \( \phi \).

**Definition 5.3.** Let \( \phi : kx^{\Delta \cup \beta(\Delta)} \to kx^\Delta \) be a \( k \)-module morphism with \( \phi \circ \iota = \text{id} \). We suppose \( \phi(x^\alpha) = \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \). For \( f = \sum_{\beta \in \Delta} c_\beta x^\beta \in kx^\Delta \) and \( \lambda \in E \), the virtual product of \( f \) and \( \phi(x^\lambda) \) is
\[ f \ast \phi(x^\lambda) = \sum_{\beta, \gamma \in \Delta} c_\gamma a_{\gamma + \lambda, \beta} x^\beta \in kx^\Delta. \]
This virtual product is commutative if
\[ (f \ast \phi(x^\lambda)) \ast \phi(x^\mu) = (f \ast \phi(x^\mu)) \ast \phi(x^\lambda). \]
for any $f \in kx^\Delta$ and $\lambda, \mu \in E$.

**Proposition 5.6.** Let $\phi : kx^{\Delta \cup \mathcal{B}(\Delta)} \to kx^\Delta$ be a $k$-module morphism. If the virtual product induced by $\phi$ satisfies

$$\phi(x^{\alpha+\lambda}) = \phi(x^\alpha) \ast \phi(x^\lambda)$$

for any $\alpha \in \mathcal{B}(\Delta)$ and $\lambda \in E$ such that $\alpha + \lambda \in \mathcal{B}(\Delta)$, and if this virtual product is commutative, then we can uniquely extend $\phi$ to a $k$-module morphism $\phi : S \to kx^\Delta$ whose Ker $\phi$ is an ideal of $S$. Furthermore, we put $\phi(x^\alpha) = \sum_{\beta \in \Delta} a_{\alpha,\beta} x^\beta$ and $g_\alpha = x^\alpha - \sum_{\beta \in \Delta} a_{\alpha,\beta} x^\beta \in \text{Ker } \phi$. Then Ker $\phi$ has a $\Delta$-border basis $\{g_\alpha \mid \alpha \in \mathcal{B}(\Delta)\}$.

**Proof.** See [Led11, proof of Proposition 3., induction step].

**Remark 5.1.** The hypotheses of Proposition 5.6 are equivalent to the following two type relations:

$$a_{\alpha+\lambda,\beta} = \sum_{\gamma \in \Delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta}$$

for each $\alpha \in \mathcal{B}(\Delta), \lambda \in E$ such that $\alpha + \lambda \in \mathcal{B}(\Delta)$ and $\beta \in \Delta$.

$$\sum_{\gamma' \in \Delta} a_{\gamma+\lambda,\gamma'} a_{\gamma'+\mu,\beta} = \sum_{\gamma' \in \Delta} a_{\gamma+\mu,\gamma'} a_{\gamma'+\lambda,\beta}$$

for each $\beta, \gamma \in \Delta$ and $\lambda, \mu \in E$.

6. **Gröbner functor and Gröbner scheme**

We recall our main objects.

**Definition 6.1.** The border basis functor, or the marked family $\mathcal{H}ilb^\Delta_k : (k\text{-Alg}) \to (\text{Set})$ is the functor such that:

- For any $k$-algebra $B$,
  $$\mathcal{H}ilb^\Delta_k(B) = \{ G \subset B[x] \mid G \text{ is a } \Delta\text{-border basis} \}$$
  $$\cong \left\{ I \subset B[x] \mid \begin{array}{c} I \text{ is an ideal,} \\
  Bx^\Delta \hookrightarrow B[x] \to B[x]/I \text{ is an isomorphism} \end{array} \right\}.$$  

- For any $k$-algebra morphism $\sigma : A \to B$,
  $$\mathcal{H}ilb^\Delta_k(\sigma) : \mathcal{H}ilb^\Delta_k(A) \to \mathcal{H}ilb^\Delta_k(B)$$
  $$\phi : A[x] \to Ax^\Delta \mapsto \phi \otimes \sigma : A[x] \otimes_A B \to Ax^\Delta \otimes_A B.$$

**Definition 6.2.** We define a subfunctor $\mathcal{H}ilb^\Delta_k$ of $\mathcal{H}ilb^\Delta_k$, called the Gröbner basis functor or the Gröbner functor, as follows:

For any $k$-algebra $B$,

$$\mathcal{H}ilb^\Delta_k(B) = \left\{ G \subset B[x] \mid G \text{ is a reduced Gröbner basis with } \right\}$$

$$\cong \left\{ I \subset B[x] \mid I \text{ is a monic ideal with } \text{LE}(I) = \mathbb{N}^n \setminus \Delta \right\}.$$
We say \( G \in \mathcal{H}ilb^\Delta_k(B) \) is of type \((D, \Delta)\) if all non-leading monomials are in \( \{ x^\beta \mid \beta \in D \} \). We define a subfunctor \( \mathcal{H}ilb^\prec \) consisting of all reduced Gröbner basis of type \( (D, \Delta) \) called the Gröbner functor of type \( (D, \Delta) \).

Gröbner functors (and also border basis functors) naturally expand over the category of \( k \)-schemes as Zariski sheaves [Led11, LR16].

**Proposition 6.1.** Let \( X \) be a non-empty \( k \)-scheme and \( \mathcal{I} \) be an ideal sheaf of the \( \mathcal{O}_X \)-algebra \( \mathcal{O}_X[x] \). Then the following properties are equivalent:

- For any open affine subscheme \( V = \text{Spec} \, B \) of \( X \), the ideal \( \Gamma(V, \mathcal{I}|_V) = I \) of \( B[x] \) is monic with \( \text{LE}(I) = \mathbb{N}^n \setminus \Delta \).
- There exists an open affine covering \( \{ V_a = \text{Spec} \, B_a \}_{a \in A} \) of \( X \) such that for each \( a \in A \), the ideal \( \Gamma(V_a, \mathcal{I}|_{V_a}) = I_a \) of \( B_a[x] \) is monic with \( \text{LE}(I_a) = \mathbb{N}^n \setminus \Delta \).

**Proof.** We can check easily that the first property implies the second property. Conversely, we take the monic basis \( \{ g^{(a)}_\alpha \mid \alpha \in \mathcal{E}(\Delta) \} \) of \( I_a \) and its coefficients \( a^{(a)\beta}_{\alpha} \in B_a = \Gamma(V_a, \mathcal{O}_X) \). Gluing sections \( \{ a^{(a)\beta}_{\alpha} \}_{a \in A} \) along \( V = \bigcup_{a \in A} V \cap V_a \), we get sections \( a_{\alpha,\beta} \in \Gamma(V, \mathcal{O}_X) = B \). Those sections satisfy the relations in Proposition [5.4] and Proposition [5.5]. Therefore \( I \) has a reduced Gröbner basis with coefficients \( a_{\alpha,\beta} \). \( \square \)

A representable functor over the Zariski site \((k,-\text{Sch})\) is a Zariski sheaf. Hence it is possible that a Gröbner functor and a border basis functor are representable. More strongly, we will show that these functors are ind-schemes. We say a scheme \( X \) is a **Gröbner scheme** if it represents a Gröbner functor. In particular, if \( \mathcal{H}ilb^\Delta_k, \mathcal{H}ilb^\prec \) or \( \mathfrak{h}ilb^\prec \) is representable, we denote by \( \mathcal{H}ilb^\Delta_k, \mathcal{H}ilb^\prec \) or \( \mathfrak{h}ilb^\prec \) the scheme which represents that respectively.

In fact, \( \mathcal{H}ilb^\prec \) and \( \mathfrak{h}ilb^\prec \) are always representable. If \( \Delta \) is finite or \( \prec \) is reliable, then \( \mathcal{H}ilb^\Delta_k \) is also representable. We see examples of Gröbner functor and Gröbner scheme in the next section. We show representability of these Gröbner functors in Section 8.

7. **Examples of Gröbner functor**

Here is examples of Gröbner schemes computed by the procedure introduced in Section 11. We use Risa/Asir with default setting, so we take \( k = \mathbb{Q} \) in this section. Probably, almost examples are correct for other commutative rings \( k \).

**Example 7.1.** Let \( \prec \) be a graded lexicographic order on \( S = \mathbb{Q}[x, y, z] \) such that \( x \succ y \succ z \). Let \( \Delta \subset \mathbb{N}^3 \) be the standard set whose \( \mathcal{E}(\Delta) = \{ (1, 1, 0), (1, 0, 1) \} \). Namely, consider a monomial ideal \( J_\Delta = \langle xy, xz \rangle \) in \( S \). The Gröbner scheme \( \mathcal{H}ilb^\Delta_k \) is isomorphic to \( \mathbb{A}^3_{\mathbb{Q}} \). More explicitly, for any \( \mathbb{Q} \)-algebra \( B \), \( \mathcal{H}ilb^\Delta_k(B) \) consists of all polynomial pairs such that

\[
G = \left\{ xy - ay^2 - byz - cx - dy + bcz + (cd + ac^2) \right\},
\[
xx - bz^2 - ayz - ex + aey + (be - d - ac)z + (ed + ace) \right\},
\]

where \( a, b, c, d, e \in B \).
Example 7.2. Let $\prec$ be a graded lexicographic order on $S = \mathbb{Q}[x, y, z]$ such that $x > y > z$. Let $\Delta_1 \subset \mathbb{N}^3$ be the standard set whose $J_{\Delta_1} = \langle x^3, x^2y, xz, z^2 \rangle \subset \mathbb{Q}[x, y, z]$. See Figure 2. Then there exists a closed immersion $\text{Hilb}_{\prec \Delta_1} S/k \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$ whose defining ideal is generated by 2 polynomials.

Example 7.3. We little bit modify the standard set in the Example 7.2. Let $\Delta_0 \subset \mathbb{N}^3$ be the standard set whose $J_{\Delta_0} = \langle x^2, xz, z^2 \rangle \subset \mathbb{Q}[x, y, z]$. See Figure 3. Then the Gröbner scheme $\text{Hilb}_{\prec \Delta_0} S/k$ is isomorphic to the affine space $\mathbb{A}^9_{\mathbb{Q}}$.

Example 7.4. Let $\Delta_2 \subset \mathbb{N}^3$ be the standard set whose $J_{\Delta_2} = \langle x^2, xz, yz^2, z^3 \rangle \subset \mathbb{Q}[x, y, z]$. See Figure 4. Then there exists a closed immersion $\text{Hilb}_{\prec \Delta_2} S/k \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$ whose defining ideal is generated by 10 polynomials.

Example 7.5. Let $\prec$ be a lexicographic order on $S = \mathbb{Q}[x, y, z]$ such that $x > y > z$. Let $\Delta \subset \mathbb{N}^3$ be the standard set whose $J_{\Delta} = \langle x^2, xy, y^4, xz^2 \rangle$ (see Figure 5). The homogeneous Gröbner scheme $\text{Hilb}_{\prec(h, \Delta)} S/k$ is isomorphic to a scheme

$$\text{Spec} \mathbb{Q}[a, b, c, d, e, f, g, h, i]/\langle d(ad + 2bdg + 3cdg^2 - 4dg^3 - 2e - 4fg + h) \rangle.$$ 

In fact, $ad + 2bdg + 3cdg^2 - 4dg^3 - 2e - 4fg + h$ is irreducible. Hence $\text{Hilb}_{\prec(h, \Delta)} S/k$ has two 8-dimensional irreducible components and both components are isomorphic to hyper surfaces in $\mathbb{A}^9_{\mathbb{Q}}$.

![Figure 2.](image-url)
Figure 5.
8. Representability of Gröbner functor

We show that Gröbner functors $\text{Hilb}_k^{<(D,\Delta)}$ and $\text{Hilb}_k^{<(h,\Delta)}$ are representable. Furthermore, if $\Delta$ is finite or $\prec$ is reliable, then $\text{Hilb}_k^{\leq \Delta}$ is also representable.

We fix a standard set $\Delta$, a finite subset $D$ of $\Delta$ and a monomial order $\prec$. We put a $k$-algebra

$$R = k[T_{\alpha,\beta} \mid \alpha \in C(\Delta), \beta \in D, \alpha \succ \beta].$$

By definition, for each $\alpha \in B(\Delta) \setminus C(\Delta)$, there exists a canonical vector $\nu \in E$ such that $\alpha - \nu \in B(\Delta)$. Therefore we can take a map $\nu : B(\Delta) \setminus C(\Delta) \to E$ such that $\alpha - \nu(\alpha) \in B(\Delta)$ (of course we assume the axiom of choice).

**Theorem 8.1.** Take a map $\nu : B(\Delta) \setminus C(\Delta) \to E$ such that $\alpha - \nu(\alpha) \in B(\Delta)$. Then there exists a family of polynomials $\{U_{\alpha,\beta} \in R \mid \alpha \in \Delta \cup B(\Delta), \beta \in \Delta\}$ such that

1. $U_{\alpha,\beta} = \delta_{\alpha,\beta}$ if $\alpha, \beta \in \Delta$.
2. $U_{\alpha,\beta} = T_{\alpha,\beta}$ if $\alpha \in C(\Delta)$, $\beta \in D$ and $\alpha \succ \beta$.
3. $U_{\alpha,\beta} = 0$ if $\alpha \in B(\Delta)$ and $\beta \in \Delta$ such that $\alpha \prec \beta$.
4. For any $\alpha \in B(\Delta)$, $U_{\alpha,\beta} = 0$ expect finitely many $\beta \in \Delta$.
5. For any $\alpha \in B(\Delta) \setminus C(\Delta)$ and $\beta \in \Delta$, $U_{\alpha,\beta} = \sum_{\gamma \in \Delta} U_{\alpha - \nu(\alpha),\gamma} U_{\gamma + \nu(\alpha),\beta}$.

**Proof.** The condition (1)–(3) are not essential. First we discuss the condition (5). By the condition (2), we arrange the condition (5) in the next form

$$U_{\alpha,\beta} = \sum_{\gamma \in \Delta} U_{\alpha - \nu(\alpha),\gamma} U_{\gamma + \nu(\alpha),\beta}.$$

Now we use Lemma 3.1, then we can inductively find polynomials $T_{\alpha,\beta}$ for all vectors $\alpha \in B(\Delta)$ with the condition (5). The condition (4) is implied from (5). □

**Theorem 8.2.** Let $R$ be the same ring in Theorem 8.1. We take a family $\{T_{\alpha,\beta} \mid \alpha \in \Delta \cup B(\Delta), \beta \in \Delta\}$ satisfying conditions as in Theorem 8.1. Let $A_1$ be the ideal of $R$ generated by all relations

$$T_{\alpha + \lambda, \beta} - \sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma + \lambda, \beta}$$

for all vectors $\alpha \in C(\Delta)$, $\lambda \in E$ such that $\alpha + \lambda \in B(\Delta)$ and $\beta \in \Delta$. Let $A_2$ be the ideal of $R$ generated by all relations

$$\sum_{\gamma \in \Delta} T_{\varepsilon + \lambda, \gamma} T_{\gamma + \mu, \beta} - \sum_{\gamma \in \Delta} T_{\varepsilon + \mu, \gamma} T_{\gamma + \lambda, \beta}$$

for all edge triples $(\varepsilon; \lambda, \mu)$ of $\Delta$ and vectors $\beta \in \Delta$. We put an ideal $A = A_1 + A_2$.

Then the affine scheme

$$\text{Hilb}_k^{<(D,\Delta)} = \text{Spec} R/A$$

represents $\text{Hilb}_k^{<(D,\Delta)}$. 
Proof. For any $k$-algebra $B$, we define a natural transformation

$$\mathcal{H}ilb_k^{(D,\Delta)}(B) \to \text{Hom}_{k\text{-Alg}}(R/A, B)$$

$$\phi(x^\alpha) = \sum_{\beta \in \Delta} a_{\alpha,\beta} x^\beta \mapsto "T_{\alpha,\beta} \mapsto a_{\alpha,\beta}".$$ 

It is well-defined and injective for any $k$-algebra $B$ from Corollary 5.1 and Remark 5.1. We prove its surjectivity in two steps. In this proof, $\equiv$ means the congruence modulo $A$.

Step 1 First we show that the ideal $A$ includes the relation

$$H = H(\beta, \gamma, \lambda, \mu) = \sum_{\gamma' \in \Delta} T_{\gamma+\lambda, \gamma'} T_{\gamma'+\mu, \beta} - \sum_{\gamma' \in \Delta} T_{\gamma+\mu, \gamma'} T_{\gamma'+\lambda, \beta}$$

for any $\beta, \gamma \in \Delta$ and $\lambda, \mu \in E$. We use a reduction to the absurd.

We suppose that there exists a 4-tuple such that $H \notin A$. Let $(\beta, \gamma, \lambda, \mu)$ be a 4-tuple whose sum $\gamma + \lambda + \mu$ is minimum on such 4-tuples. To obtain the contradiction, we do a case analysis. We see that the minimality implies $H = H(\beta, \gamma, \lambda, \mu) \in A$ in all cases.

(i) If $\gamma + \lambda + \mu \in \Delta$, then $\gamma + \lambda, \gamma + \mu \in \Delta$, hence the both summations in $H$ are $T_{\gamma+\lambda+\mu, \beta}$. Therefore $H = 0$.

(ii) If $\gamma + \lambda + \mu \in B(\Delta)$, we separate this case into four parts.

(ii-a) If $\gamma + \lambda, \gamma + \mu \in \Delta$, then $H = 0$ as like as in the case (i).

(ii-b) If $\gamma + \lambda \in \Delta$ and $\gamma + \mu \in C(\Delta)$, then $H$ is a generator of $A_1$.

(ii-c) If $\gamma + \lambda, \gamma + \mu \in C(\Delta)$, since $(\gamma + \lambda) + \mu = (\gamma + \mu) + \lambda \in B(\Delta)$, we have

$$T_{\gamma+\lambda+\mu, \beta} \equiv \sum_{\gamma' \in \Delta} T_{\gamma+\lambda, \gamma'} T_{\gamma'+\mu, \beta}$$

and

$$T_{\gamma+\lambda+\mu, \beta} \equiv \sum_{\gamma' \in \Delta} T_{\gamma+\mu, \gamma'} T_{\gamma'+\lambda, \beta}.$$

Therefore $H \in A$.

(ii-d) If $\gamma + \lambda \in B(\Delta) \setminus C(\Delta)$, we put $\nu = \nu(\gamma + \lambda)$. Thus from the condition (5) in Theorem 8.1, the equality

$$T_{\gamma+\lambda, \gamma'} = \sum_{\delta \in \Delta} T_{\gamma+\lambda, \nu, \delta} T_{\delta+\nu, \gamma'}$$

holds. We arrange the first summation in $H$

$$\sum_{\gamma' \in \Delta} T_{\gamma+\lambda, \gamma'} T_{\gamma'+\mu, \beta} = \sum_{\gamma' \in \Delta} \sum_{\delta \in \Delta} T_{\gamma+\lambda, \nu, \delta} T_{\delta+\nu, \gamma'} T_{\gamma'+\mu, \beta}.$$ 

By the condition (2) in Theorem 8.1 we may suppose that the indexes in the right side run through the set of pairs $\gamma', \delta \in \Delta$ such that $\gamma + \lambda + \mu \succ \delta + \nu + \mu \succ \gamma' + \mu \succ \beta$. Thus from the
We show that \( \alpha \) may assume that \((\ast)\) for each commutative. As the finish of this proof, we check \( \psi \) is surjective. We take any \( \psi \in \Hom_{\Alg}(R/A, B) \). Since \( \gamma + \lambda - \nu \in \mathcal{B}(\Delta) \), we have \( \lambda \neq \nu \) and \( \gamma - \nu \in \Delta \). Hence using minimality over and over again (Note \( \gamma - \nu + \lambda + \mu < \gamma + \lambda + \mu \)), we obtain
\[
\sum_{\gamma' \in \Delta} T_{\gamma + \lambda, \gamma} T_{\gamma' + \mu, \beta} = \sum_{\gamma' \in \Delta} \sum_{\delta \in \Delta} T_{\gamma + \lambda - \nu, \delta} T_{\delta + \nu, \gamma} T_{\gamma' + \mu, \beta} \\
= \sum_{\gamma' \in \Delta} \sum_{\delta \in \Delta} T_{\gamma + \lambda - \nu, \delta} T_{\delta + \mu, \gamma} T_{\gamma' + \nu, \beta}.
\]
Since \( \gamma + \lambda - \nu \in \mathcal{B}(\Delta) \), we have \( \lambda \neq \nu \) and \( \gamma - \nu \in \Delta \). Hence using minimality over and over again (Note \( \gamma - \nu + \lambda + \mu < \gamma + \lambda + \mu \)), we obtain
\[
\sum_{\gamma' \in \Delta} T_{\gamma + \lambda, \gamma} T_{\gamma' + \mu, \beta} \\
= \sum_{\gamma' \in \Delta} \sum_{\delta \in \Delta} T_{\gamma + \lambda + \mu, \delta} T_{\delta + \nu, \gamma} T_{\gamma' + \mu, \beta} \\
= \sum_{\gamma' \in \Delta} \sum_{\delta \in \Delta} T_{\gamma + \lambda + \mu, \delta} T_{\delta + \nu, \gamma} T_{\gamma' + \lambda + \beta} \\
= \sum_{\gamma' \in \Delta} \sum_{\delta \in \Delta} T_{\gamma + \lambda + \mu, \delta} T_{\delta + \mu, \gamma} T_{\gamma' + \lambda + \beta} \\
= \sum_{\gamma' \in \Delta} T_{\gamma + \lambda, \gamma} T_{\gamma' + \lambda + \beta}.
\]

(iii) Finally, we consider the case of \( \gamma + \lambda + \mu \in \mathcal{B}(\Delta \cup \mathcal{B}(\Delta)) \). In this case \( \gamma + \lambda \) and \( \gamma + \mu \) are in \( \mathcal{B}(\Delta) \). Hence if \( \gamma + \lambda + \mu \in \mathcal{C}(\Delta \cup \mathcal{B}(\Delta)) \), the triple \( (\gamma; \lambda, \mu) \) is an edge triple of \( \Delta \), so \( H \in \mathcal{A}_2 \). As the last case, we assume that \( \gamma + \lambda + \mu \in \mathcal{B}(\Delta \cup \mathcal{B}(\Delta)) \setminus \mathcal{C}(\Delta \cup \mathcal{B}(\Delta)) \). Taking a canonical vector \( \eta \) such that \( \gamma + \lambda + \mu - \eta \in \mathcal{B}(\Delta \cup \mathcal{B}(\Delta)) \), we can show \( H \in \mathcal{A} \) as like as in the case (ii-d).

Therefore the relation \( H(\beta, \gamma, \lambda, \mu) \) is always in \( \mathcal{A} \).

**Step 2** We show that
\[
\mathcal{Hilb}_{k}^{-}(\mathcal{D}(\Delta))(B) \rightarrow \Hom_{\Alg}(R/A, B)
\]
is surjective. We take any \( \psi \in \Hom_{\Alg}(R/A, B) \). We set \( a_{\alpha, \beta} = \psi(T_{\alpha, \beta}) \) for each \( \alpha \in \Delta \cup \mathcal{B}(\Delta) \) and \( \beta \in \Delta \). Using the condition (4) in Theorem 5.6, we can take a \( B \)-module morphism \( \phi : B_{X \Delta \cup \mathcal{B}(\Delta)} \rightarrow B_{X \Delta} \) such that \( \phi(x^\alpha) = \sum_{\beta \in \Delta} a_{\alpha, \beta} x^\beta \). From Step 1, the virtual product defined from \( \phi \) is commutative. As the finish of this proof, we check
\[
\phi(x^{\alpha + \lambda}) = \phi(x^\alpha) \ast \phi(x^\lambda) \ast \cdots (\ast)
\]
for each \( \alpha \in \mathcal{B}(\Delta) \) and \( \lambda \in E \) such that \( \alpha + \lambda \in \mathcal{B}(\Delta) \) (Proposition 5.6). If \( \alpha \in \mathcal{C}(\Delta) \), then \((\ast)\) holds by taking the generators of \( \mathcal{A}_1 \), so we may assume that \((\ast)\) holds for any \( \alpha' \in \mathcal{B}(\Delta) \) such that \( \alpha' \prec \alpha \). Since \( \alpha + \lambda \in \mathcal{B}(\Delta) \), we have
\[
T_{\alpha + \lambda, \beta} = \sum_{\gamma \in \Delta} T_{\alpha + \lambda, \gamma} T_{\gamma + \nu, \beta},
\]
where \( \nu = \nu(\alpha + \lambda) \). Namely,
\[
\phi(x^{\alpha+\lambda}) = \phi(x^{\alpha+\lambda-\nu}) \ast \phi(x^\nu)
\]
holds. If \( \lambda = \nu \), then \((*)\) holds for \( \alpha \) and \( \lambda \). If \( \lambda \neq \nu \), then \( \alpha - \nu \in \Delta \cup B(\Delta) \), hence we have
\[
\phi(x^{\alpha+\lambda}) = \phi(x^{\alpha+\lambda-\nu}) \ast \phi(x^\nu) = (\phi(x^{\alpha-\nu}) \ast \phi(x^\lambda)) \ast \phi(x^\nu) = (\phi(x^{\alpha-\nu}) \ast \phi(x^\nu)) \ast \phi(x^\lambda).
\]

\[\square\]

**Remark 8.1.** For any standard set \( \Delta \), the set of corners \( C(\Delta) \) and the set of all edge triples are both finite sets. Therefore \( A \) is finitely generated even if \( k \) is not Noetherian. Hence more strongly, the Gröbner scheme \( \text{Hilb}_k <\Delta \) is a **finitely presented affine scheme over** \( k \).

**Corollary 8.1.** By taking a huge finite subset \( D \), the homogeneous Gröbner functor \( \text{Hilb}_k <(h,\Delta) > \) is represented by the closed subscheme of \( \text{Hilb}_k <(D,\Delta) > \) defined by the ideal
\[
\langle T_{\alpha,\beta} \mid \alpha \in C(\Delta), \beta \in D, |\alpha| \neq |\beta| \rangle.
\]
A monomial order \( \prec \) is **reliable** if \( \{ \beta \in \mathbb{N}^n \mid \beta \prec \alpha \} \) is a finite set for any \( \alpha \in \mathbb{N}^n \). The Gröbner functor \( \text{Hilb}_k <\Delta \) is representable if \( \prec \) is reliable.

Take the vectors \( u_1, \ldots, u_m \in \mathbb{Z}^n \) as in Proposition 3.2 for a monomial order \( \prec \). A monomial order \( \prec \) is reliable if and only if \( u_1 \in \mathbb{N}^n \) [RT10]. A monomial order \( \prec \) is **graded** if \( |\alpha| > |\beta| \) implies \( \alpha \succ \beta \). Clearly a graded monomial order is reliable. In particular, there exists the Gröbner scheme \( \text{Hilb}_k <\Delta \) for the graded lexicographic order \( \prec = \prec_{\text{grlex}} \), or the graded reverse lexicographic order \( \prec = \prec_{\text{grevlex}} \).

Each reduced Gröbner basis, or border basis, has finite non-leading terms. Hence we get
\[
\lim_{\mathcal{D} \subset \Delta} \text{Hilb}_k <(\mathcal{D},\Delta) > = \text{Hilb}_k <\Delta \>
\]
as functors. Then the Gröbner functor \( \text{Hilb}_k <\Delta \) (and in the same way the border basis functor \( \text{Hilb}_k ^B \)) is an ind-scheme for a general monomial order \( \prec \) and a standard set \( \Delta \). Here an **ind-scheme** means that an inductive limit of some inductive family of schemes whose all morphisms are closed immersions. We can easily give a proof of this fact by the definition of types, so we omit.

9. **Edge triples of standard set**

We need to determine the set of all edge triples for the defining ideal of the Gröbner scheme as in Theorem 8.2. However, if \( \Delta \) is infinite, we can not take all elements of \( \Delta \) in finite steps. On the other hand, a standard set \( \Delta \) is defined by a finite set of vectors \( C(\Delta) \). Here we introduce a procedure for determining the set of all edge triples of a standard set \( \Delta \) from the set of corners \( C(\Delta) \) in finite steps.

**Lemma 9.1.** Let \( A \) be a finite subset of \( \mathbb{N}^n \). Then the followings are equivalent:
• There exists a standard set \( \Delta \) in \( \mathbb{N}^n \) such that \( \mathcal{C}(\Delta) = A \).

• For each \( \alpha \in A \), \( \alpha \not\in A \setminus \{ \alpha \} + \mathbb{N}^n \).

\textit{Proof.} Let \( A \) be the set of corners of a standard set \( \Delta \). For \( \alpha \in A \), if there exists \( \alpha' \in \mathcal{C}(\Delta) \) and \( \gamma \in \mathbb{N}^n \) such that \( \alpha = \alpha' + \gamma \), then we obtain \( \gamma = 0 \) by the definition of corners. Therefore \( \alpha \not\in A \setminus \{ \alpha \} + \mathbb{N}^n \). Conversely, suppose the second condition. Let \( \Delta \) be the complement set of \( A + \mathbb{N}^n \) in \( \mathbb{N}^n \). Clearly \( \Delta \) is a standard set in \( \mathbb{N}^n \). Since \( \mathcal{C}(\Delta) \) is the minimum generators of \( \mathbb{N}^n \setminus \Delta \), \( \mathcal{C}(\Delta) \) is contained in \( A \). For any \( \alpha \in A \), if there exists a canonical vector \( \lambda \) such that \( \alpha - \lambda \in \mathbb{N}^n \setminus \Delta = A + \mathbb{N}^n \), then there exist \( \alpha' \in A \) and \( \gamma \in \mathbb{N}^n \) such that \( \alpha - \lambda = \alpha' + \gamma \). However, it is incompatible with our hypothesis since \( \gamma + \lambda \neq 0 \). Therefore \( \alpha - \lambda \not\in \mathbb{N}^n \setminus \Delta \) for any canonical vector \( \lambda \), thus \( \alpha \in \mathcal{C}(\Delta) \). \( \square \)

\textbf{Example 9.1.} We give a set of vectors \( \{(3,0,0), (2,1,0), (1,0,1), (0,0,2)\} \). By Lemma 9.1, this set is the set of corners of the standard set \( \Delta \subset \mathbb{N}^3 \) whose \( J_3 = \langle x^3, x^2 y, x z, z^2 \rangle \subset k[x, y, z] \). See Example 7.2.

\textbf{Proposition 9.1.} Let \( \Delta \) be a standard set in \( \mathbb{N}^n \). We denote by \( \alpha_i \) the \( i \)-th coordinate of a vector \( \alpha \). Put a number \( \theta_i = \max \{ \alpha_i \mid \alpha \in \mathcal{C}(\Delta) \} \) for each \( i = 1, \ldots, n \). Then for all \( \xi \in \mathcal{C}(\Delta \cup \mathcal{B}(\Delta)) \), \( |\xi| \leq \sum_i \theta_i + n \). In particular, \( |\varepsilon| \leq \sum_i \theta_i + n - 2 \) if \( \varepsilon \) is an edge point of \( \Delta \).

\textit{Proof.} We put a subset \( A = \{ \eta \in \mathbb{N}^n \setminus (\Delta \cup \mathcal{B}(\Delta)) \mid \eta_i \leq \theta_i + 1 \text{ for all } i \} \). By definition, we obtain \( \sum_i (\theta_i + 1) e_i \in A \), so \( A \) is not empty. We have \( \mathbb{N}^n \setminus (\Delta \cup \mathcal{B}(\Delta)) = A + \mathbb{N}^n \). Indeed, let \( \eta \) be an element of \( \mathbb{N}^n \setminus (\Delta \cup \mathcal{B}(\Delta)) \). If \( \eta \not\in A \), then there exists an index \( i \) such that \( \eta_i \geq \theta_i + 2 \). Hence \( \eta - e_i \in \mathbb{N}^n \setminus (\Delta \cup \mathcal{B}(\Delta)) \) since \( \eta - e_i - e_j \not\in \Delta \) for any \( j \). Thus, in inductively, there exists a canonical vector \( \lambda \) such that \( \eta - \gamma \in A \). Since the set of corners \( \mathcal{C}(\Delta \cup \mathcal{B}(\Delta)) \) is the minimal generators of \( \mathbb{N}^n \setminus (\Delta \cup \mathcal{B}(\Delta)) \), we get \( \mathcal{C}(\Delta \cup \mathcal{B}(\Delta)) \subset A \). \( \square \)

Now we get a procedure.

\textbf{Procedure. Input:} The set of corners \( \mathcal{C}(\Delta) \) of a standard set \( \Delta \).

\textbf{Output:} The set of all edge triples of \( \Delta \).

1. Put a number \( \theta_i = \max \{ \alpha_i \mid \alpha \in \mathcal{C}(\Delta) \} \) for each \( i = 1, \ldots, n \). Put \( L = \sum_i \theta_i + n \).
2. For each \( \varepsilon \in \mathbb{N}^n_{\leq L} \) and \( \lambda, \mu \in E \), determine if \( (\varepsilon; \lambda, \mu) \) is an edge triple of \( \Delta \) or not.

\textbf{Example 9.2.} Consider the same standard set as in Example 9.1. Namely, assume \( \mathcal{C}(\Delta) = \{(3,0,0), (2,1,0), (1,0,1), (0,0,2)\} \).

1. We get \( \theta_1 = 3, \theta_2 = 2 \) and \( \theta_3 = 2 \). Therefore \( L = 3 + 2 + 2 = 9 \).
2. Search edge triples in \( \mathbb{N}^3_{\leq 9} \). In fact, the set of all edge triples of \( \Delta \) is

\[ \{(0,0,1);e_3,e_3),(0,0,1);(1,1,0);e_3,((2,0,0);e_3,((2,0,0),(2,0,0);e_2,e_3)\} \]

See Fig 6 so the name “edge point” is not suitable in this case. However, it seems that almost edge points are on “edges”, thus I use this name.
10. Tangent space of Gröbner scheme

We again fix a standard set $\Delta$, an enough huge finite subset $D$ of $\Delta$ and a monomial order $\prec$. From now on, we denote the Gröbner scheme $\text{Hilb}_{k}(D,\Delta)$, $\text{Hilb}_{k}^{\Delta}$ or $\text{Hilb}_{k}^{(h,\Delta)}$ by $H$ in a lump. Also let $H$ be the Gröbner functor $\text{Hilb}_{k}(D,\Delta)$, $\text{Hilb}_{k}^{\Delta}$ or $\text{Hilb}_{k}^{(h,\Delta)}$. We determine Zariski tangent spaces on the Gröbner scheme $H$ as sets of sections in $H(D)$, where $D = k[\delta]/\langle \delta^{2} \rangle$ is the dual number ring over $k$ ([Har77, II. Schemes Exercise 2.8]). If $k$ is not a field, it is not usually Zariski tangent space in strictly. However, that is important for constructing a closed immersion of $H$ to an affine space over $k$ whose dimension is the embedding dimension of $H$. That closed immersion is induced in [Rob09, 3. Consequences and problems]. Moreover, that can be explicitly constructed by computing a reduced Gröbner basis of defining ideal of $H$ [RT10, Proposition 3.4], [LR11, Proposition 4.3], so it leads an simplification of generators of a defining ideal.

In this section, we see that such closed immersion is universal in some sense, and it can be determined by only some linear algebra without any computation of reduced Gröbner basis of the defining ideal of $H$.

Lemma 10.1. ([Bay82, Proposition 1.8]) For any finite subset $A$ of $\mathbb{N}^{n}$, there exists a positive weight function $u : \mathbb{N}^{n} \to \mathbb{N}$ such that for any $\alpha, \beta \in A$, $\alpha \prec \beta$ if and only if $u(\alpha) < u(\beta)$.

We take a positive weight function $u$ as in Lemma 10.1 for $C(\Delta) \cup D$. Then we can induce a weight function $W$ on $R = k[T_{\alpha,\beta} | \alpha \in C(\Delta), \beta \in D, \alpha \succ \beta]$ such that $W(T_{\alpha,\beta}) = u(\alpha) - u(\beta)$. By Theorem 8.2 and Corollary 8.1 the defining ideal $\mathcal{A}$ of $H$ in $R$ is a $W$-homogeneous ideal contained in $\langle T_{\alpha,\beta} | \alpha \in C(\Delta), \beta \in D, \alpha \succ \beta \rangle$. Therefore $H$ is a quasi-cone with respect to $W$.

Proposition 10.1. ([Led11 Proposition 5.1]) Let $W$ be the weight function in the above. The morphism

$$\mathbb{A}_{k}^{1} \times_{k} H \to H$$

induced by

$$T_{\alpha,\beta} \mapsto t^{u(\alpha) - u(\beta)} T_{\alpha,\beta}$$

satisfies the following properties:

- The restriction to $\{ (t - 1) \} \times H \cong H$ is the identity map.
- For all points $p \in H$, the image of $\mathbb{A}_{k}^{1} \times \{ p \}$ contains the monomial ideal $J_{\Delta}$.

In particular, if Spec $k$ is connected, then $H$ is connected.

Proof. It is clear by taking the images on $t = 0, 1$. □

Definition 10.1. Let $R = k[T] = k[t_{1},\ldots,t_{M}]$ be a polynomial ring over $k$. For each $F \in R$, $L(F)$ is the usually linear component of $F$. We put a $k$-module $L(\mathcal{A}) = k \{ L(F) | F \in \mathcal{A} \}$ for an ideal $\mathcal{A}$ of $R$.

If $k$ is a filed, the defining ideal of a quasi-cone has a very useful reduced Gröbner basis with respect to some suitable order. It implies an embedding $H \hookrightarrow \mathbb{A}_{k}^{N}$ to the $N$-dimensional affine space, where $N$ is the dimension of...
the Zariski tangent space on $H$ at the origin called the embedding dimension [RT10, Definition 3.3. and Proposition 3.4.].

**Definition 10.2.** Let $X$ be a $k$-scheme and $x : \text{Spec } k \to X$ a $k$-scheme morphism. In this paper, we define the **Zariski tangent space** on $X$ at $x$, denoted by $T_{x,X}$, as the set of all $k$-scheme morphisms $f : \text{Spec } D \to X$ whose diagram

$$
\begin{array}{ccc}
\text{Spec } D & \xrightarrow{f} & X \\
& \uparrow x & \\
\text{Spec } D/(\delta) & \cong & \text{Spec } k
\end{array}
$$

is commutative (see also [Har77, II. Schemes Exercise 2.8]).

We consider the case of $X$ is the Gröbner scheme $H = \text{Spec } R/\mathcal{A}$, where $R$ and $\mathcal{A}$ is the same as in Section 8. The canonical map $x : \text{Spec } R/(T) \to H$ corresponds to the monomial ideal $J = J_{\Delta}$ in $k[\mathbf{x}]$ and we have

$$
\text{Hom}_{(k,-\text{Sch})}(\text{Spec } D, H) \cong \mathcal{H}(D).
$$

The right side is the set of all reduced Gröbner bases $G$ in $D[\mathbf{x}]$ such that

$$
G = \left\{ x^\alpha - \sum_{\beta \in \Delta} (a_{\alpha,\beta} + b_{\alpha,\beta}\delta)x^\beta \mid \alpha \in \mathcal{C}(\Delta) \right\}.
$$

Taking the monic basis of $\langle G \rangle$, we get relations of coefficients:

$$
(a_{\alpha+\lambda,\beta} + b_{\alpha+\lambda,\beta}\delta) = \sum_{\gamma \in \Delta} (a_{\alpha,\gamma} + b_{\alpha,\gamma}\delta)(a_{\gamma+\lambda,\beta} + b_{\gamma+\lambda,\beta}\delta),
$$

Hence we have

$$
a_{\alpha+\lambda,\beta} = \sum_{\gamma \in \Delta} a_{\alpha,\gamma}a_{\gamma+\lambda,\beta}
$$

and

$$
b_{\alpha+\lambda,\beta} = \sum_{\gamma \in \Delta} a_{\alpha,\gamma}b_{\gamma+\lambda,\beta} + b_{\alpha,\gamma}a_{\gamma+\lambda,\beta}.
$$

If $G$ lies in the Zariski tangent space $T_{J,H}$ on $H$ at $J$, then $a_{\alpha,\beta} = 0$ and the relation can be arranged into

$$
b_{\alpha+\lambda,\beta} = b_{\alpha,\beta-\lambda},
$$

where $b_{\alpha,\beta-\lambda} = 0$ if $\beta - \lambda \not\in \mathbb{N}^n$.

Then the Zariski tangent space $T_{J,H}$ can be determined as follows: take a map $\nu : \mathcal{B}(\Delta) \setminus \mathcal{C}(\Delta) \to E$ such that $\alpha - \nu(\alpha) \in \mathcal{B}(\Delta)$. We put a free $k$-module $\Omega$ consisting of all arrays of coefficients $(b_{\alpha,\beta})$, where subscripts run through all $\alpha \in \mathcal{C}(\Delta)$ and $\beta \in \Delta$ such that $\alpha > \beta$ (moreover, if we consider the homogeneous Gröbner scheme, we also claim that $b_{\alpha,\beta} = 0$ if $|\alpha| \neq |\beta|$). For each $(b_{\alpha,\beta}) \in \Omega$, we inductively create new coefficients $b_{\xi,\beta}$ for $\xi \in \mathcal{B}(\Delta) \setminus \mathcal{C}(\Delta)$ and $\beta \in \Delta$ such that

$$
b_{\xi,\beta} = b_{\xi-\nu(\xi),\beta-\nu(\xi)}.
$$
Then the Zariski tangent space \( T_{J,H} \) is isomorphic to the subspace of \( \Omega \) consisting of all arrays \( (b_{\alpha,\beta}) \) satisfies the followings:

- \( b_{\alpha+\lambda,\beta} = b_{\alpha,\beta-\lambda} \) for \( \alpha \in \mathcal{C}(\Delta), \lambda \in \mathcal{E}(\Delta) \) and \( \beta \in \Delta \).
- \( b_{\varepsilon+\lambda,\beta-\mu} = b_{\varepsilon+\mu,\beta-\lambda} \) for all edge triples \( (\varepsilon; \lambda, \mu) \).

These represent the relations between linear components of generators as in Theorem 8.2. Indeed, we have

\[
L(T_{\alpha+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma+\lambda,\beta}) = L(T_{\alpha+\lambda,\beta} - T_{\alpha,\beta-\lambda}),
\]

\[
L(\sum_{\gamma \in \Delta} T_{\varepsilon+\lambda,\gamma} T_{\gamma+\mu,\beta} - \sum_{\gamma \in \Delta} T_{\varepsilon+\mu,\gamma} T_{\gamma+\lambda,\beta}) = L(T_{\varepsilon+\lambda,\beta-\mu} - T_{\varepsilon+\mu,\beta-\lambda}),
\]

and

\[
L(T_{\alpha+\lambda,\beta}) = L(T_{\alpha,\beta-\lambda}).
\]

Here we use a notation \( T_{\alpha,\beta} = 0 \) if \( \beta \in \mathbb{Z}^n \setminus \mathbb{N}^n \). Therefore linear components of generators as in Theorem 8.2 consist of at most two terms with coefficients 1 or \(-1\). Then we obtain the next proposition (see also [RT10, Proposition 3.4] and [LR11, Proposition 4.3]).

**Proposition 10.2.** Put \( R = k[T_{\alpha,\beta} \mid \alpha \in \mathcal{C}(\Delta), \beta \in \mathcal{D}, \alpha \succ \beta] \). Let \( A \) be the defining ideal of the Gröbner scheme \( H \) in \( \text{Spec } R \) as in Section 8. Then the Zariski tangent space \( T_{J,H} \) can be identified with the algebraic set defined by \( L(A) \) as free \( k \)-modules of finite rank. Furthermore, the **embedding dimension** \( \text{rank}_k T_{J,H} \) of \( H \) is independent from a choice of the base ring \( k \).

By the relations defining \( T_{J,H} \) in \( \Omega \), \( T_{J,H} \) is the kernel of a finite size matrix \( A \) whose each row has at most two non-zero entries 1 or \(-1\). Doing row reduction of \( A \), we get matrices \( P \) and \( B = PA \) such that \( B = PA \) is a normal form of \( A \). From the construction, \( P \) is regarded as an operator summing linear components of two generators of \( A \) finitely many times. Since the rows of \( A \) represents linear components of the generators of \( A \), \( P \) also can be regarded as an operator summing two of the generators finitely many times. Then we obtain a new set of generators \( F_1 \) of \( A \). Let \( T' \) be variables of \( R \) which correspond to columns of \( B \) having an entry 1. Clearly \( \#(T') = \text{rank } A = \dim R - \text{rank } T_{J,H} \). Moreover, watching behaviors of linear terms and using a fact that each generator of \( A \) is \( W \)-homogeneous, \( F_1 \) is in the following form (see also [Rob09, Corollary 3.7]):

\[
F_1 = \{ t' + g_{t'} \mid t' \in T' \} \cup \{ f_1, \ldots, f_s \} \quad (\star)
\]

with \( g_{t'} \in k[T \setminus \{ t' \}] \) and \( f_i \in (T) \). Finally, after the substitution of \( g_{t'} \) for \( t' \), we may suppose \( g_{t'} \in k[T \setminus T'] \) and \( f_i \in (T \setminus T')^2 \subset k[T \setminus T'] \). We call such \( T' \) an **eliminable variables**.

The consequence is the following.

**Theorem 10.1.** Let \( N = \text{rank}_k T_{J,H} \) be the embedding dimension of \( H \). Then there exists a closed immersion

\[
H \hookrightarrow \mathbb{A}^N_k
\]
such that all elements of the defining ideal have no linear terms and no constant terms. In particular, the followings are equivalent:

(1) $H$ is isomorphic to $N$-dimensional affine space over $k$.
(2) $H$ has the same dimension as $\mathbb{A}_k^N$. In other words, $H$ is non-singular at $J$.

Furthermore, if $k$ is an integral domain, then (1) and (2) are equivalent to

(3) In the $(\ast)$, $f_1, \ldots, f_s$ are zero in $k[T \setminus T']$.

This closed immersion is universal. Namely, for any ring morphism $k \to k'$, the diagram

$$
\begin{array}{ccc}
\mathbb{A}_k^N & \longrightarrow & \mathbb{A}_{k'}^N \\
\uparrow & & \uparrow \\
H_k & \longrightarrow & H_{k'}
\end{array}
$$

is cartesian. Here we denote by $H_k$ the Gröbner scheme defined over $k$.

**Proof.** By the construction, the diagram

$$
\begin{array}{ccc}
\text{Spec } k' & \longrightarrow & \text{Spec } k \\
\uparrow & & \uparrow \\
\mathbb{A}_{k'}^N & \longrightarrow & \mathbb{A}_k^N \\
\uparrow & & \uparrow \\
H_{k'} & \longrightarrow & H_k
\end{array}
$$

is commutative. Furthermore, the square on the upper and the outer square are cartesian. Then the square on the lower is cartesian. \qed

**Corollary 10.1.** Assume that $k$ is an integral domain. Let $K$ be the quotient field of $k$ and $\overline{K}$ the algebraic closure of $K$. Then the followings are equivalent:

(1) $H_{\overline{K}}$ is non-singular at $\overline{K}$-rational point $J_\Delta$.
(2) $H_K$ is non-singular at $K$-rational point $J_\Delta$.
(3) For any $k$-algebra $k'$, $H_{k'}$ is isomorphic to $N$-dimensional affine space over $k'$, where $N$ is the embedding dimension.

**Proof.** This comes from a fact that the diagram

$$
\begin{array}{ccc}
H_{k'} & \longrightarrow & H_k \\
\uparrow & & \uparrow \\
\text{Spec } k' & \longrightarrow & \text{Spec } k
\end{array}
$$

is cartesian for any ring morphism $k \to k'$. \qed

**Corollary 10.2.** The followings are equivalent:

(1) $H_{\overline{Q}}$ is non-singular at $\overline{Q}$-rational point $J_\Delta$.
(2) $H_Q$ is non-singular at $Q$-rational point $J_\Delta$.
(3) For any commutative ring $k$, the Gröbner scheme $H_k$ over $k$ is isomorphic to the $N$-dimensional affine space $\mathbb{A}_k^N$ over $k$, where $N = \dim_Q T_{J_HQ}$ is the embedding dimension.
Example 10.1. We assume the same situation as in Example 7.2. Then \( \mathcal{C}(\Delta) = \{(3,0,0),(2,1,0),(1,0,1),(0,0,2)\} \) and the set \( \{\langle \alpha, \beta \rangle \mid \alpha \in \mathcal{C}(\Delta), \beta \in \Delta, \alpha \succ \beta \} \) consists of 32 elements.

1. We obtain the following:

\[
\begin{align*}
&b_{(1,0,1),(0,0,0)} = 0, b_{(1,0,1),(0,1,0)} = 0, b_{(1,0,1),(0,2,0)} = 0, \\
&b_{(0,0,2),(0,0,0)} = 0, b_{(0,0,2),(0,1,0)} = 0, b_{(0,0,2),(1,0,0)} = 0, \\
&b_{(2,1,0),(0,0,0)} = 0, b_{(2,1,0),(0,1,0)} = 0, b_{(2,1,0),(0,2,0)} = 0, \\
&b_{(2,1,0),(0,3,0)} = 0, b_{(2,1,0),(1,0,0)} = 0, b_{(2,1,0),(0,0,0)} = 0, \\
&b_{(3,0,0),(0,0,1)} = 0, b_{(3,0,0),(0,1,0)} = 0, b_{(3,0,0),(0,1,1)} = 0, \\
&b_{(3,0,0),(0,2,0)} = 0, b_{(3,0,0),(0,2,1)} = 0, b_{(3,0,0),(0,3,0)} = 0, \\
&b_{(2,1,0),(0,1,0)} - b_{(3,0,0),(1,0,0)} = 0, b_{(2,1,0),(0,2,0)} - b_{(3,0,0),(1,1,0)} = 0, \\
&b_{(2,1,0),(0,3,0)} - b_{(3,0,0),(1,2,0)} = 0.
\end{align*}
\]

2. We do row reduction. A normal form is the following:

\[
\begin{align*}
&b_{(1,0,1),(0,0,0)} = 0, b_{(1,0,1),(0,1,0)} = 0, b_{(1,0,1),(0,2,0)} = 0, \\
&b_{(0,0,2),(0,0,0)} = 0, b_{(0,0,2),(0,1,0)} = 0, b_{(0,0,2),(1,0,0)} = 0, \\
&b_{(2,1,0),(0,0,0)} = 0, b_{(2,1,0),(0,1,0)} = 0, b_{(2,1,0),(0,2,0)} = 0, \\
&b_{(2,1,0),(0,3,0)} = 0, b_{(2,1,0),(1,0,0)} = 0, b_{(3,0,0),(0,0,0)} = 0, \\
&b_{(3,0,0),(0,0,1)} = 0, b_{(3,0,0),(0,1,0)} = 0, b_{(3,0,0),(0,1,1)} = 0, \\
&b_{(3,0,0),(0,2,0)} = 0, b_{(3,0,0),(0,2,1)} = 0, b_{(3,0,0),(0,3,0)} = 0, \\
&b_{(3,0,0),(1,0,0)} = 0, b_{(3,0,0),(1,1,0)} = 0, b_{(3,0,0),(1,2,0)} = 0.
\end{align*}
\]

In particular, the embedding dimension equals to 32 − 21 = 11.

3. Therefore the following set of variables is an eliminable variables:

\[
\begin{align*}
&T_{(1,0,1),(0,0,0)}, T_{(1,0,1),(0,1,0)}, T_{(1,0,1),(0,2,0)}, \\
&T_{(0,0,2),(0,0,0)}, T_{(0,0,2),(0,1,0)}, T_{(0,0,2),(1,0,0)}, \\
&T_{(2,1,0),(0,0,0)}, T_{(2,1,0),(0,1,0)}, T_{(2,1,0),(0,2,0)}, \\
&T_{(2,1,0),(0,3,0)}, T_{(2,1,0),(1,0,0)}, T_{(3,0,0),(0,0,0)}, \\
&T_{(3,0,0),(0,0,1)}, T_{(3,0,0),(0,1,0)}, T_{(3,0,0),(0,1,1)}, \\
&T_{(3,0,0),(0,2,0)}, T_{(3,0,0),(0,2,1)}, T_{(3,0,0),(0,3,0)}, \\
&T_{(3,0,0),(1,0,0)}, T_{(3,0,0),(1,1,0)}, T_{(3,0,0),(1,2,0)}.
\end{align*}
\]

11. Procedure for computing Gröbner schemes

We introduce a procedure for computing the Gröbner scheme \( \text{Hilb}_k^\Delta \) for a graded monomial order \( \prec \) and for a standard set \( \Delta \). The procedure is also effective to the homogeneous Gröbner scheme \( \text{Hilb}_k^{(h,\Delta)} \) for any monomial order \( \prec \) and for a standard set \( \Delta \).

We fix a graded monomial order \( \prec \). We obtained the following two type relations by Theorem 8.2

\[
T_{\alpha + \lambda, \beta} = \sum_{\gamma \in \Delta} T_{\alpha, \gamma} T_{\gamma + \lambda, \beta}
\]
for each $\alpha \in \mathcal{C}(\Delta)$, $\lambda \in E$ such that $\alpha + \lambda \in \mathcal{B}(\Delta)$ and $\beta \in \Delta$.

$$\sum_{\gamma \in \Delta} T_{\varepsilon+\lambda,\gamma} T_{\gamma+\mu,\beta} = \sum_{\gamma \in \Delta} T_{\varepsilon+\mu,\gamma} T_{\gamma+\lambda,\beta}$$

for each edge triple $(\varepsilon; \lambda, \mu)$ of $\Delta$.

In the first type, we may only take $\gamma$ which satisfies $|\alpha| \geq |\gamma|$. Also we may only take $\gamma$ which satisfies $|\varepsilon| + 1 \geq |\gamma|$ in the second. Therefore we get the following procedure.

**Procedure. Input:** The set of corners $\mathcal{C}(\Delta)$ of a standard set $\Delta$  

**Output:** The defining ideal $\mathcal{A}$ of the Gröbner scheme $\text{Hilb}^{\prec}_{\Delta} k$ in $\text{Spec} k[\alpha, \beta \mid \alpha \in \mathcal{C}(\Delta), \beta \in \Delta, \alpha \succ \beta]$ and closed immersion $\text{Hilb}^{\prec}_{\Delta} k \hookrightarrow A^N_k$, where $N$ is the embedding dimension.

1. Determine the set of all edge triples of $\Delta$ (Section 9).
2. Put $D = \max\{|\alpha| + 1 \mid \alpha \in \mathcal{C}(\Delta)\} \cup \{|\varepsilon| + 2 \mid \varepsilon \text{ is an edge point of } \Delta\}$.
3. Sort $\mathcal{B}(\Delta) \leq D$ with respect to $\prec$, and give a numbering such that $\mathcal{B}(\Delta) \leq D = \{ \alpha_1 \prec \alpha_2 \prec \cdots \}$.
4. Take a map $\nu : \mathcal{B}(\Delta) \leq D \to E$ such that $\alpha - \nu(\alpha) \in \mathcal{B}(\Delta)$. Thus make a family $\{ U_{\alpha, \beta} \mid \alpha \in \mathcal{B}(\Delta) \leq D, \beta \in \Delta \leq D \}$ as in Theorem 8.1.
5. Calculate a set of generators of a defining ideal $\mathcal{A}$ of $\text{Hilb}^{\prec}_{\Delta} k$ in $k[\alpha, \beta \mid \alpha \in \mathcal{C}(\Delta), \beta \in \Delta, \alpha \succ \beta]$ by Theorem 8.2.
6. Determine the Zariski tangent space $T_{J, H}$ on $H = \text{Hilb}^{\prec}_{\Delta} k$ at $J = J_{\Delta}$ as the kernel of a matrix $A$. Do row reduction of $A$, then take an eliminable variables $T'$ for $A$ and a new generators of $\mathcal{A}$ (Section 10).

12. **Universal family of Gröbner scheme**

Finally, we determine the universal family of the Gröbner scheme.

**Proposition 12.1.** Let $\mathcal{A}$ be the defining ideal of $\text{Hilb}^{\prec(D, \Delta)} k$ in the $R = k[\alpha, \beta \mid \alpha \in \mathcal{C}(\Delta), \beta \in D, \alpha \succ \beta]$. Then the universal family of $\text{Hilb}^{\prec(D, \Delta)} k$ is the affine scheme

$$U^{\prec(D, \Delta)}_{S/k} = \text{Spec}(R/A)[x]/(x^\alpha - \sum_{\beta \in D} T_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta))$$

and the natural morphism

$$U^{\prec(D, \Delta)}_{S/k} \to \text{Hilb}^{\prec(D, \Delta)} k$$

induced by

$$R/A \to (R/A)[x]/(x^\alpha - \sum_{\beta \in D} T_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta)).$$
In other words, for any $k$-algebra $B$ and reduced Gröbner basis $G$ of type $(\mathcal{D}, \Delta)$ in $B[x]$, the diagram

\[
\begin{array}{ccc}
\text{Spec } B[x]/\langle G \rangle & \xrightarrow{U_{\mathcal{G}/k}} & \text{Hilb}^<(\mathcal{D}, \Delta) \\
\downarrow & & \downarrow \\
\text{Spec } B & \xrightarrow{\phi} & \text{Hilb}^<(\mathcal{D}, \Delta)
\end{array}
\]

is cartesian, where $\phi$ is the morphism corresponding to $G$. We can also determine the universal family of other type Gröbner schemes in the same way.

**Proof.** The universal family of a moduli space $X$ equals to the universal object of $\text{Hom}(\mathcal{-}, X)$, and the universal object corresponds to the identity map $id_X \in \text{Hom}(X, X)$ (see [ML98 III. Universals and Limits]). The identity map $id \in \text{Hom}(\text{Hilb}_k^<(\mathcal{D}, \Delta), \text{Hilb}_k^<(\mathcal{D}, \Delta)) \cong \text{Hilb}_k^<(\mathcal{D}, \Delta)(R/A)$ corresponds to the ideal $\langle x^\alpha - \sum_{\beta \in \mathcal{D}} T_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{G}(\Delta) \rangle$ in $(R/A)[x]$, then the universal family of the Gröbner scheme $\text{Hilb}_k^<(\mathcal{D}, \Delta)$ is $\text{Spec}(R/A)[x]/\langle x^\alpha - \sum_{\beta \in \mathcal{D}} T_{\alpha, \beta} x^\beta \mid \alpha \in \mathcal{G}(\Delta) \rangle$.

\[ \square \]

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