PT-symmetric models in curved manifolds

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Abstract

We consider the Laplace–Beltrami operator in tubular neighborhoods of curves on two-dimensional Riemannian manifolds, subject to non-Hermitian parity and time preserving boundary conditions. We are interested in the interplay between the geometry and spectrum. After introducing a suitable Hilbert space framework in the general situation, which enables us to realize the Laplace–Beltrami operator as an m-sectorial operator, we focus on solvable models defined on manifolds of constant curvature. In some situations, notably for non-Hermitian Robin-type boundary conditions, we are able to prove either the reality of the spectrum or the existence of complex conjugate pairs of eigenvalues, and establish similarity of the non-Hermitian m-sectorial operators to normal or self-adjoint operators. The study is illustrated by numerical computations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Many systems in nature can be under the first approximation described by linear second-order differential equations, such as the wave, heat or Schrödinger equation. The common denominator is the Helmholtz equation describing the stationary regime and leading to the spectral study of the Laplace operator. It is important to understand the influence of the geometry on the spectrum of the Laplacian, subject to various types of boundary conditions, and vice versa, to characterize geometric and boundary interface properties from given spectral data.
In this paper, we are interested in the interplay between the curvature of the ambient space and the spectrum of the Laplacian subjected to a special class of non-Hermitian boundary conditions. We choose probably the simplest non-trivial model, i.e. the spectral problem corresponding to the equation

$$-\Delta \psi = \lambda \psi \quad \text{in } \Omega,$$

where $\lambda$ is a spectral parameter, $\Omega$ is an $a$-tubular neighborhood of a closed curve $\Gamma$ (usually a geodesic) in a two-dimensional Riemannian manifold $\mathcal{A}$ (not necessarily embedded in $\mathbb{R}^3$), i.e.

$$\Omega := \{ x \in \mathcal{A} \mid \text{dist}(x, \Gamma) < a \},$$

and $-\Delta$ is the associated Laplace–Beltrami operator. The boundary conditions we consider are general ‘parity and time preserving’ boundary conditions introduced in section 2.3.2 below; a special example is given by the non-Hermitian Robin-type boundary conditions

$$\frac{\partial \psi}{\partial n} + i \alpha \psi = 0 \quad \text{on } \partial \Omega,$$

where $n$ is the unit normal vector field along $\Gamma$ extended to $\Omega$ by parallel transport along the geodesics orthogonal to $\Gamma$, and $\alpha$ is a real-valued function.

The Schrödinger equation in tubular neighborhoods of submanifolds of curved Riemannian manifolds has been extensively studied in the context of quantum waveguides and molecular dynamics (cf [57] for a recent mathematical paper and many references therein). Here the confinement to a vicinity of the submanifold is usually modeled by constraining potentials [41, 57] or Dirichlet boundary conditions [14, 34, 35].

Note that, in contrast, the non-Hermitian nature of boundary conditions (1.3) enables one to model a leak/supply of energy from/into the subsystem $\Omega$, since the probability current does not vanish on the boundary $\partial \Omega$ unless $\alpha = 0$. In fact, non-Hermitian boundary conditions of the type (1.3) have been considered in [29–31] to model open (dissipative) quantum systems. One also arrives at (1.3) when transforming a scattering problem to a (nonlinear) spectral one [17, example 9.2.4]. Quite recently it has been observed in [51] that the boundary conditions (1.3) appear in a supersymmetric counterpart of the one-dimensional quantum well. Finally, we observe that Robin boundary conditions are known as ‘impedance boundary conditions’ in classical electromagnetism, where they are conventionally used to approximate very thin layers [6, 11, 19].

Our primary motivation to consider the spectral problems (1.1) and (1.3) comes from the so-called $\mathcal{PT}$-symmetric quantum mechanics originated in [8], where the authors discussed a class of Schrödinger operators $H$ in $L^2(\mathbb{R})$ whose spectrum is real in spite of the fact that their potentials are complex. They argued that the rather surprising reality of the spectrum follows from the $\mathcal{PT}$-symmetry property:

$$[H, \mathcal{P} \mathcal{T}] = 0.$$  

Here the ‘parity’ $\mathcal{P}$ and ‘time reversal’ $\mathcal{T}$ operators are defined by $(\mathcal{P} \psi)(x) := \psi(-x)$ and $\mathcal{T} \psi := \overline{\psi}$. It is important to emphasize that $\mathcal{T}$ is an antilinear operator and that (1.4) is neither a sufficient nor necessary condition to ensure the reality of the spectrum of $H$.

Nevertheless, it was later observed in [9, 42–44] that if the spectrum of a $\mathcal{PT}$-symmetric operator $H$ in a Hilbert space $\mathcal{H}$ is indeed real (and some further hypotheses are satisfied), condition (1.4) actually implies that $H$ is ‘quasi-Hermitian’ [52], i.e. there exists a bounded invertible positive operator $\Theta$ with bounded inverse, called ‘metric’, such that

$$H^* = \Theta^{-1} H \Theta.$$  

(1.5)
In other words, $H$ is similar to a self-adjoint operator for which a conventional quantum-mechanical interpretation makes sense. We refer to recent reviews [7, 45] and proceedings [5, 21, 28] for further information and references about the concept of the $\mathcal{PT}$-symmetry.

In addition to the potential quantum-mechanical interpretation, we would like to mention the relevance of $\mathcal{PT}$-symmetric operators in view of their recent study in the context of superconductivity [48, 49], electromagnetism [33, 50] and fluid dynamics [12, 13, 16, 58].

A suitable mathematical framework to analyze $\mathcal{PT}$-symmetric Hamiltonians is either the theory of self-adjoint operators in Krein spaces [27, 39] or the $J$-self-adjointness [10]. The latter means that there exists an antilinear involution $J$ such that

$$H^* = J H J. \quad (1.6)$$

The concept (1.6) is not restricted to functional Hilbert spaces and it turns out that the majority of $\mathcal{PT}$-symmetric Hamiltonians existing in the literature are indeed $J$-self-adjoint. In general, however, properties (1.4)–(1.6) are all unrelated [53, 55].

Summing up, given a non-Hermitian operator $H$ satisfying (1.4), two fundamental questions arise. First,

(1) is the spectrum of $H$ real?

Second, if the answer to the previous question is positive,

(2) does there exist a metric $\Theta$ satisfying (1.5)?

It turns out that the questions constitute a difficult problem in the theory of non-self-adjoint operators.

For this reason, one of the present authors and his coauthors proposed in [37] (see also [36]) an elementary one-dimensional $\mathcal{PT}$-symmetric Hamiltonian, for which the spectrum and metric are explicitly computable. The simplicity of the Hamiltonian consists of the fact that it acts as the Hamiltonian of a free particle in a box, and the non-Hermitian interaction is introduced via the Robin-type boundary conditions (1.3) only. The model was later generalized to a two-dimensional waveguide in [10], where the variable coupling in the boundary conditions is responsible for the existence of real (or complex conjugate pairs of) eigenvalues outside the essential spectrum (see also [38]).

In this paper, we continue the generalization of the models of [10, 37] to curved Riemannian manifolds. This leads to a new, large class of $\mathcal{PT}$-symmetric Hamiltonians. Our main goal is to study the effect of curvature on the spectrum, namely the existence/absence of non-real eigenvalues and the metric.

The organization of this paper is as follows. In section 2, we introduce our model in a full generality, in the sense that the ambient geometry and boundary interactions of the spectral problem (1.1) are described by quite arbitrary (non-constant and non-symmetric) functions. Our main strategy to deal with the curved geometry is based on the use of Fermi coordinates. In section 3, we use the framework of sesquilinear forms to define the Laplace–Beltrami operator appearing in (1.1) as a (closed) $m$-sectorial operator in the Hilbert space $L^2(\Omega)$. We also explicitly determine the operator domain if the assumptions about the geometry and boundary-coupling functions are naturally strengthened. Moreover, we find conditions about the geometry under which the operator becomes $\mathcal{PT}$-symmetric (and $T$-self-adjoint). In order to study the effects of curvature on the spectrum, in section 4 we focus on solvable models. Assuming that the curvature and boundary-coupling functions are constant, the eigenvalue problem can be reduced to the investigation of (infinitely many) one-dimensional differential operators with $\mathcal{PT}$-symmetric boundary conditions. Here the previous results [36, 37] and the general theory of boundary conditions for differential operators [46, 47] are appropriate and helpful. In particular, since the $\mathcal{PT}$-symmetric boundary conditions are (except one case
excluded here by assumption) strongly regular ones, it is possible to show that the studied one-dimensional operators are ‘generically’ similar to self-adjoint or normal operators. However, it remains to decide whether this is true for their infinite sum, i.e. for the original two-dimensional Laplace–Beltrami operator. To answer this affirmatively, it turns out that the $J$-self-adjoint formulation of $\mathcal{PT}$-symmetry (cf the text around (1.6)) is fundamental, with $J = T$ playing the role of antilinear involution. The properties of the solvable models are illustrated by a numerical analysis of their spectra. The paper is concluded in section 5 where possible directions of future research are mentioned.

2. Definition of the model

We use the quantum-mechanical framework to describe our model.

2.1. The configuration space

We assume that the ambient space of a quantum particle is a connected complete two-dimensional Riemannian manifold $\mathcal{A}$ of class $C^2$ (not necessarily embedded in the Euclidean space $\mathbb{R}^3$). Furthermore, we suppose that the Gauss curvature $K$ of $\mathcal{A}$ is continuous, which holds under the additional assumption that $\mathcal{A}$ is either of class $C^3$ or embedded in $\mathbb{R}^3$.

On the manifold, we consider a $C^2$-smooth unit-speed embedded curve $\Gamma: [-l, l] \to \mathcal{A}$, with $l > 0$. Since $\Gamma$ is parameterized by the arc length, the derivative $T := \dot{\Gamma}$ is the unit tangent vector of $\Gamma$. Let $N$ be the unit normal vector of $\Gamma$ which is uniquely determined as the $C^1$-smooth mapping from $[-l, l]$ to the tangent bundle of $\mathcal{A}$ by requiring that $N(s)$ is orthogonal to $T(s)$ and that $\{T(s), N(s)\}$ is positively oriented for all $s \in [-l, l]$ (cf [56, section 7.B]). We denote by $\kappa$ the corresponding curvature of $\Gamma$ defined by the Frenet formula $\nabla_T T = \kappa N$, where $\nabla$ stands for the covariant derivative in $\mathcal{A}$. We note that the sign of $\kappa$ is uniquely determined up to the reparametrization $s \mapsto -s$ of the curve $\Gamma$ and that $\kappa$ coincides with the geodesic curvature of $\Gamma$ if $\mathcal{A}$ is embedded in $\mathbb{R}^3$.

The feature of our model is that the particle is assumed to be ‘confined’ to an $a$-tubular neighborhood $\Omega$ of $\Gamma$, with $a > 0$. $\Omega$ can be visualized as the set of points $q$ in $\mathcal{A}$ for which there exists a geodesic of length less than $a$ from $q$ meeting $\Gamma$ orthogonally. More precisely, we introduce a mapping $\mathcal{L}$ from the rectangle

$$\Omega_0 := (-l, l) \times (-a, a) \equiv J_1 \times J_2$$

(considered as a subset of the tangent bundle of $\mathcal{A}$) to the manifold $\mathcal{A}$ by setting

$$\mathcal{L}(x_1, x_2) := \exp_{T(x_1)}(N(x_1) x_2),$$

where $\exp_q$ is the exponential map of $\mathcal{A}$ at $q \in \mathcal{A}$, and define

$$\Omega := \mathcal{L}(\Omega_0).$$

Note that $x_1 \mapsto \mathcal{L}(x_1, x_2)$ traces the curves parallel to $\Gamma$ at a fixed distance $|x_2|$, while the curve $x_2 \mapsto \mathcal{L}(x_1, x_2)$ is a geodesic orthogonal to $\Gamma$ for any fixed $x_1$. See figure 1.

2.2. The Fermi coordinates

Throughout the paper we make the hypothesis that

$$\mathcal{L} : \Omega_0 \to \Omega$$

is a diffeomorphism. Since $\Gamma$ is compact, (2.4) can always be achieved for sufficiently small $a$ (cf [24, section 3.1]). Consequently, $\mathcal{L}$ induces a Riemannian metric $G$ on $\Omega_0$, and we can identify the tubular
neighborhood $\Omega \subset \mathcal{A}$ with the Riemannian manifold $(\Omega_0, G)$. In other words, $\Omega$ can be conveniently parameterized via the (Fermi or geodesic parallel) `coordinates' $(x_1, x_2)$ determined by (2.2). We refer to [24, section 2] and [25] for the notion and properties of Fermi coordinates. In particular, it follows by the generalized Gauss lemma that the metric acquires the diagonal form:

$$G = \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $f$ is continuous and has continuous partial derivatives $\partial_1 f$, $\partial_2 f$ satisfying the Jacobi equation

$$\partial_1^2 f + K f = 0 \quad \text{with} \quad \begin{cases} f(\cdot, 0) = 1, \\ \partial_1 f(\cdot, 0) = -\kappa. \end{cases}$$

Here $K$ is considered as a function of the Fermi coordinates $(x_1, x_2)$.

2.3. The Hamiltonian

We identify the Hamiltonian $H$ of the quantum particle in $\Omega$ with the Laplace–Beltrami operator $-\Delta_1 G$ in the Riemannian manifold $(\Omega_0, G)$, subject to a special class of non-self-adjoint boundary conditions.

2.3.1. The action of the Hamiltonian. Denoting by $G^{ij}$ the coefficients of the inverse metric $G^{-1}$ and $|G| := \det(G)$, we have

$$-\Delta_1 G = -|G|^{-1/2} \partial_j |G|^{1/2} G^{ij} \partial_i = -f^{-1} \partial_1 f f^{-1} \partial_1 = -f^{-1} \partial_2 f \partial_2.$$  

Here the first equality (in which the Einstein summation convention is assumed) is a general formula for the Laplace–Beltrami operator $-\Delta_G$ expressed in local coordinates in a Riemannian manifold equipped with a metric $G$. The second equality uses the special form (2.5), for which $|G| = f^2$ and $G^{-1} = \text{diag}(f^{-2}, 1)$. Henceforth, we assume that the Jacobian of (2.4) is uniformly positive and bounded, i.e.

$$f, f^{-1} \in L^\infty(\Omega_0),$$

so that $-\Delta_1 G$ is a uniformly elliptic operator. Again, (2.8) can be achieved for sufficiently small $a$, cf. (2.6).

Remark 2.1. Assumption (2.4) is not really essential. Indeed, abandoning the geometrical interpretation of $\Omega$ as a tubular neighborhood embedded in $\mathcal{A}$, $(\Omega_0, G)$ with (2.5) can be considered as an abstract Riemannian manifold for which (2.8) is the only important hypothesis. The results of this paper extend automatically to this more general situation.
2.3.2. The boundary conditions. We denote $\partial_i \Omega_0 = \Gamma_i^- \cup \Gamma_i^+$ the boundary in $x_i$ direction, $i \in \{1, 2\}$, see figure 1:
\begin{equation}
\Gamma_i^+ := \{ \pm l \} \times J_2, \quad \Gamma_i^- := J_1 \times \{ \pm a \}.
\end{equation}
Boundary conditions imposed respectively on $\partial_1 \Omega_0$ and $\partial_2 \Omega_0$ are of different nature. Having in mind the situation when $\Gamma$ is a closed curve, standard periodic boundary conditions are imposed on $\partial_1 \Omega_0$, i.e.
\begin{equation}
\psi(-l, x_2) = \psi(l, x_2), \quad \partial_1 \psi(-l, x_2) = \partial_1 \psi(l, x_2),
\end{equation}
for a.e. $x_2 \in J_2$, where $\psi$ denotes any function from the domain of $H$. We assume also the symmetry condition on the geometry
\begin{equation}
f(-l, x_2) = f(l, x_2),
\end{equation}
in order to have indeed periodic system in the $x_1$ direction.

On the other hand, non-self-adjoint $\mathcal{PT}$-symmetric boundary conditions are imposed on $\partial_2 \Omega_0$. A general form of $\mathcal{PT}$-symmetric boundary conditions was presented in [2]; further study and more general approach to extensions can be found in [3, 4]. Denoting $\Psi_1 := (\psi \partial_2 \psi)$,
\begin{equation}
\end{equation}
there are two types of conditions, separated and connected.

(I) Separated:
\begin{equation}
\left( \pm \beta(x_1) + i \alpha(x_1) \atop 0 \right) \begin{pmatrix} \psi(x_1, \pm a) \\ 1 \end{pmatrix} = 0
\end{equation}
for a.e. $x_1 \in J_1$, with $\alpha, \beta$ being real-valued functions.

(II) Connected:
\begin{equation}
\Psi(x_1, a) = B(x_1) \Psi(x_1, -a),
\end{equation}
for a.e. $x_1 \in J_1$, where the matrix $B$ has the form
\begin{equation}
B(x_1) := \begin{pmatrix} \sqrt{1 + b(x_1) c(x_1) e^{i \phi(x_1)}} & b(x_1) \\ c(x_1) & \sqrt{1 + b(x_1) c(x_1) e^{-i \phi(x_1)}} \end{pmatrix},
\end{equation}
with $b, c, \phi$ being real-valued functions satisfying $b > 0, c \geq -1/b, \phi \in [-\pi, \pi]$.

We specify assumptions on smoothness, boundedness and periodicity of the functions entering the boundary conditions later. The index $i \in \{I, II\}$ will be used throughout the paper to distinguish between the two types of boundary conditions.

The boundary conditions (2.13I) are $\mathcal{PT}$-symmetric in following sense: if a function $\psi$ satisfies (2.13I), then the function $\mathcal{PT} \psi$ satisfies (2.13I) as well. Here and in the following the symmetry operators $\mathcal{P}$ and $\mathcal{T}$ are defined as follows:
\begin{equation}
(\mathcal{P} \psi)(x_1, x_2) := \psi(x_1, -x_2), \quad \mathcal{T} \psi := \overline{\psi}.
\end{equation}

It is important to stress that the $\mathcal{PT}$-symmetric boundary conditions (2.13I) do not automatically imply that the operator $H$ is $\mathcal{PT}$-symmetric, unless additional assumption on the geometry of $\Omega_0$ is imposed. The assumption, ensuring the $\mathcal{PT}$-symmetry of $H$ (cf proposition 3.1 below), reads
\begin{equation}
\forall (x_1, x_2) \in \Omega_0 : \quad f(x_1, x_2) = f(x_1, -x_2).
\end{equation}
In view of (2.6), a necessary condition to satisfy the second requirement in (2.15) is that the curve $\Gamma$ is a geodesic, i.e. $\kappa = 0$. 

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2.3.3. The functional spaces. The space in which we give a precise meaning of $H$ is the Hilbert space $L^2(\Omega_0, G)$, i.e. the class of all measurable functions $\varphi, \psi$ on $\Omega_0$ for which the norm $\| \cdot \|_G$ induced by the inner product $(\varphi, \psi)_G := \int_{\Omega_0} \overline{\varphi(x)} \psi(x) |G(x)|^{1/2} \, dx$ (2.16) is finite. Assuming (2.8), the norm $\| \cdot \|_G$ in $L^2(\Omega_0, G)$ is equivalent to the usual one $\| \cdot \|$ in $L^2(\Omega_0)$. Moreover, the ‘energy space’ $W^{1,2}(\Omega_0, G)$ can be as a vector space identified with the usual Sobolev space $W^{1,2}(\Omega_0)$. However, this equivalence does not hold for $W^{2,2}$-spaces, unless one assumes extra regularity condition on $f$:

\[
\forall x_2 \in J_2 : \ f(\cdot, x_2), f^{-1}(\cdot, x_2) \in W^{1,\infty}(J_1). \tag{2.18}
\]

Under this assumption, which is actually equivalent to the Lipschitz continuity of $f$, $f^{-1}$ in the first argument (cf [20, chapter 5.8.2.b., theorem 4]), one can indeed identify the $W^{2,2}$-Sobolev space on the Riemannian manifold $(\Omega_0, G)$ (precisely defined, e.g. in [26, section 2.2]) with the usual Sobolev space $W^{2,2}(\Omega_0)$.

2.3.4. The schism: two definitions of the Hamiltonian. Although the above equivalence of the $W^{2,2}$-spaces under condition (2.18) is not explicitly used in this paper, it is in fact hidden in our proof that the particle Hamiltonian on $L^2(\Omega_0, G)$ naturally identified with

\[
H_\iota \psi := -\Delta_G \psi,
\]

is well defined (cf theorem 3.1). As mentioned in section 2.3.2, we use the notation $H_\iota$, with $\iota \in \{I, II\}$, to distinguish between separated (2.13I) and connected (2.13II) boundary conditions.

To avoid the additional assumption (2.18), one can always interpret (2.7) in the weak sense of quadratic forms, which gives rise to an alternative Hamiltonian $\tilde{H}_\iota$ (cf corollary 3.1). This is the content of the following section, where we also show that $H_\iota = \tilde{H}_\iota$ provided that (2.11) and (2.18), and some analogous hypotheses about the boundary-coupling functions hold.

3. General properties

The main goal of this section is to show that the Hamiltonian $H_\iota$ introduced in (2.19) is a well-defined operator, in particular that it is closed. This will be done by proving that $H_\iota = \tilde{H}_\iota$, where $\tilde{H}_\iota$ is the alternative operator defined through a closed quadratic form. Finally, we establish some general spectral properties of the Hamiltonians.

3.1. The Hamiltonian defined via quadratic form

Taking the sesquilinear form $(\varphi, H_\iota \psi)_G$ with $\varphi, \psi \in \text{Dom}(H_\iota)$ and integrating by parts, one arrives to a sesquilinear form, which is well defined for a wider class of functions $\varphi, \psi$, not necessarily possessing second (weak) derivatives. The function $f$ is assumed to satisfy (2.8) and (2.11); however, the extra regularity condition (2.18) is not required.
More precisely, exclusively under assumption (2.8) for a moment, we define the sesquilinear form
\[ h_i(\varphi, \psi) := h^1(\varphi, \psi) + h^2(\varphi, \psi), \]
where, for any \( \varphi, \psi \in \text{Dom}(h_i) \),
\[ h^1(\varphi, \psi) := (f^{-1} \partial_1 \varphi, f^{-1} \partial_1 \psi)_G + (\partial_2 \varphi, \partial_2 \psi)_G, \]
\[ h^2(\varphi, \psi) := (\varphi, (\beta + \i \omega)\psi)_G^+ + (\varphi, (\beta - \i \omega)\psi)_G^-, \]
\[ h^3(\varphi, \psi) := (\varphi, B_{12}^{-1} \psi)_G^+ + (\varphi, B_{12}^{-1} \psi)_G^- - (\varphi, B_{22} B_{12}^{-1} \psi)_G^+ - (\varphi, B_{11} B_{12}^{-1} \psi)_G^- \]
Here \( B_j \) denotes the elements of the matrix \( B \) defined in (2.13), the operator \( \mathcal{P} \) is introduced in (2.14) and
\[ (\varphi, \psi)^G_+ := \int_{-\ell}^\ell \varphi(x_1, \pm a) \psi(x_1, \pm a) f(x_1, \pm a) \, dx_1. \]

All the boundary terms should be understood in sense of traces [1].

**Lemma 3.1.** Let \( f \) satisfy (2.8). The forms \( h_i, h^1 \) are densely defined. \( h^1 \) is a symmetric, positive, closed form (associated with the self-adjoint Laplace–Beltrami operator in \( L^2(\Omega_0, G) \) with periodic boundary conditions on \( \partial_1 \Omega_0 \) and Neumann boundary conditions on \( \partial_2 \Omega_0 \)).

**Proof.** The density of the domains is obvious, properties of \( h^1 \) are well known, see the detailed discussion of a similar problem in [15, section 7.2]. \( \square \)

Although the forms \( h_i \) are not symmetric, we show that \( h_i^2 \) can be understood as small perturbations of \( h^1 \).

**Lemma 3.2.** Let \( b, 1/b, c, \alpha, \beta \in L^\infty(J_1) \) and let \( f \) satisfy (2.8). Then \( h_i^2 \) are relatively bounded with respect to \( h^1 \) with
\[ |h_i^2[\psi]| \leq \epsilon h^1[\psi] + \epsilon^{-1} C \|\psi\|_G^2, \] (3.1)
for all \( \psi \in W^{1,2}_{\text{per}}(\Omega_0) \) and any positive number \( \epsilon \). The constant \( C \) depends on the function \( f \), dimensions \( a, l \) and boundary-coupling functions \( \alpha, \beta \) or \( b, c, \phi \).

**Proof.** The proof is based on the estimate
\[ \int_{-\ell}^\ell |\psi(x_1, \pm a)|^2 \, dx_1 \leq \epsilon \|\nabla \psi\|^2 + \epsilon^{-1} \tilde{C} \|\psi\|^2, \] (3.2)
where \( \epsilon \) is an arbitrary positive constant and \( \tilde{C} \) is a positive constant depending only on \( a \) and \( l \). We give the proof for \( h_i^2 \) only because the other case is analogous. The assumptions on \( \alpha, \beta \) and property (2.8) allow us to estimate the functions \( |\alpha|, |\beta| \) and \( f \) by their \( L^\infty \)-norms. Consequence of application of (3.2) therefore yields
\[ |h_i^2[\psi]| \leq \epsilon \| f \|_{L^\infty(\Omega_0)} \|\nabla \psi\|^2 + \epsilon^{-1} 2\tilde{C}(\|\alpha\|_{L^\infty(J_1)} + \|\beta\|_{L^\infty(J_1)}) \| f \|_{L^\infty(\Omega_0)} \|\psi\|^2. \]

In order to replace the term \( \|\nabla \psi\|^2 \) by \( h^1[\psi] \), the regularity assumption on geometry (2.8) is used. Once we consider the equivalence of the norms \( \| \cdot \| \) and \( \| \cdot \|_G \) and the arbitrariness of \( \epsilon \), we obtain the estimate (3.1). \( \square \)

**Corollary 3.1.** Let \( b, 1/b, c, \alpha, \beta \in L^\infty(J_1) \) and let \( f \) satisfy (2.8). Then there exist the unique \( m \)-sectorial operators \( H_i \) in \( L^2(\Omega_0, G) \) such that
\[ h_i(\varphi, \psi) = (\varphi, H_i \psi)_G \] (3.3)
for all $\psi \in \text{Dom}(\tilde{H}_\iota)$ and $\phi \in \text{Dom}(h_\iota)$, where
\[
\text{Dom}(\tilde{H}_\iota) := \{ \psi \in W^{1,2}_{\text{per}}(\Omega_0) \mid \exists F \in L^2(\Omega_0, G), \quad \forall \phi \in W^{1,2}_{\text{per}}(\Omega_0), \quad h_\iota(\phi, \psi) = (\phi, F)_G \}.
\] (3.4)

**Proof.** With regard to lemmata 3.1, 3.2, and the perturbation result [32, theorem VI.3.4], the statement follows by the first representation theorem [32, theorem VI.2.1]. \qed

### 3.2. The equivalence of the two definitions

Under stronger assumptions on smoothness of functions appearing in boundary conditions (2.13) and on the function $f$ entering the metric tensor $G$, we show that operators $\tilde{H}_\iota$ associated with the forms $h_\iota$ are equal to the Hamiltonians $H_\iota$ defined in (2.19). To prove this, we need the following lemma. Let us introduce a space of Lipschitz continuous functions over $[-l, l]$ satisfying periodic boundary conditions:
\[
W^{1,\infty}_{\text{per}}(J_\iota) := \{ \psi \in W^{1,\infty}(J_\iota) \mid \psi(-l) = \psi(l) \}.
\]

**Lemma 3.3.** Let $\alpha, \beta, b, 1/b, c, \phi \in W^{1,\infty}_{\text{per}}(J_\iota)$ and let $f$ satisfy (2.8), (2.11), and (2.18). Then for every $F \in L^2(\Omega_0, G)$, a solution $\psi$ to the problem
\[
\forall \phi \in W^{1,2}_{\text{per}}(\Omega_0), \quad h_\iota(\phi, \psi) = (\phi, F)_G
\] (3.5)

belongs to $\text{Dom}(H_\iota)$ introduced in (2.19b).

**Proof.** We prove the separated boundary conditions case only, the connected case is analogous. For each $\psi \in W^{1,2}_{\text{per}}(\Omega_0)$ we introduce a difference quotient
\[
\delta \psi(x_1, x_2) := \frac{\psi_\delta(x_1, x_2) - \psi(x_1, x_2)}{\delta},
\] (3.6)
where $\psi_\delta(x_1, x_2) := \psi(x_1 + \delta, x_2)$ and $\delta$ is a small real number. The shifted value $\psi_\delta(x_1, x_2)$ is well defined for every $x_1 \in J_\iota$ and $\delta \in \mathbb{R}$ by extending $\psi$ periodically to $\mathbb{R}$. We use periodic extensions of other functions in $x_1$ direction throughout the whole proof without further specific comments. The estimate
\[
\|\delta \psi\| \leq \|\psi\|_{W^{1,2}(\Omega_0)}
\] (3.7)
is valid for $\delta$ small enough [20, section 5.8.2., theorem 3].

We express the difference of identities (3.5) for $\psi$ and $\psi_\delta$, whence we get for every $\psi \in W^{1,2}_{\text{per}}(\Omega_0)$
\[
(\partial_1 \psi, (\delta f^{-1}) \partial_1 \psi) + (\partial_1 \psi, f^{-1} \delta_1 (\delta \psi_\delta)) + (\partial_2 \psi, (\delta f) \partial_2 \psi)
+ (\partial_2 \psi, f_\delta \partial_2 (\delta \psi_\delta)) + (\phi, \delta(f(\beta + i\alpha))\psi_\delta)_{\Omega_1} + (\phi, f(\beta + i\alpha)(\delta \psi))_{\Omega_1}
+ (\phi, \delta(f(\beta - i\alpha))\psi_\delta)_{\Omega_1} + (\phi, f(\beta - i\alpha)(\delta \psi))_{\Omega_1}
= (\phi, (\delta f) F_\delta) + (\phi, f(\delta F)),
\] (3.8)
where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega_0)$ and
\[
(\phi, \psi)_{\Omega_1} := \int_{-l}^{l} \psi(x_1, \pm a) \psi(x_1, \pm a) \, dx_1.
\] (3.9)
We insert $\phi = \delta \psi$ into equation (3.8) and apply the ‘integration-by-parts’ formula [20, section 5.8.2] for difference quotients, i.e. $(\phi, \delta F) = -((-\delta \phi, F)$, in order to avoid the difference quotient of the arbitrary (e.g. possibly non-continuous) function $F \in L^2(\Omega_0, G)$. 


Using the embedding of $W^{1,2}(\Omega_0)$ in $L^2(\partial\Omega_0)$, the regularity assumptions on $\alpha$, $\beta$ and $f$, the Schwarz and Cauchy inequalities, and the estimate (3.7), we obtain

$$\|\delta\psi\|_{W^{1,2}(\Omega_0)} \leq C,$$

(3.10)

where $C$ is a constant independent of $\delta$. By standard arguments [20, D.4], this estimate yields that $\partial_1\psi \in W^{1,2}(\Omega_0)$. At the same time, standard elliptic regularity theory [23, theorem 8.8] implies that the solution $\psi$ to (3.5) belongs to $W^{2,2}_{\text{loc}}(\Omega_0)$. Thus, $\psi$ satisfies the equation

$$-\Delta_1 G\psi = F$$

(3.11)
a.e. in $\Omega_0$. If we express $\partial_2^2\psi$ from (3.11), we obtain that $\partial_2^2\psi \in L^2(\Omega_0)$.

It remains to check boundary conditions of $\text{Dom}(H_i)$. Once the $W^{2,2}$-regularity of the solution $\psi$ is established, this can be done using integration by parts in identity (3.5) and considering the arbitrariness of $\varphi$, see [10, lemma 3.2] for the more detailed discussion in an analogous situation. □

Let us write $H_I(\alpha, \beta)$ and $H_{II}(b, c, \phi)$ if we want to stress the dependence of the Hamiltonians on functions $\alpha$, $\beta$ and $b, c, \phi$ entering the boundary conditions.

**Theorem 3.1.** Let $\alpha, \beta, b, 1/b, c, \phi \in W^{1,\infty}(J_1)$ and let $f$ satisfy (2.8), (2.11) and (2.18). Then

1. $\hat{H}_i = H_i$,
2. $H_i$ are $m$-sectorial operators,
3. the adjoint operators $H^*_i$ can be found as

$$H^*_I(\alpha, \beta) = H_I(-\alpha, \beta), \quad H^*_{II}(b, c, \phi) = H_{II}(b, c, -\phi),$$

and
4. the resolvents of $H_i$ are compact.

**Proof.**

(Ad 1) It is easy to verify, using integration by parts, that if $\psi \in \text{Dom}(H_i)$ then $\psi \in \text{Dom}(\hat{H}_i)$; in fact, the function $F$ from (3.4) satisfies $F = -\Delta_1 G\psi$ in the distributional sense. Thus, $H_i \subset \hat{H}_i$. The more non-trivial inclusion $\hat{H}_i \subset H_i$ follows from lemma 3.3. Once the equality of the operators is established, the other properties readily follow from the corresponding properties for $\hat{H}_i$.

(Ad 2) $\hat{H}_i$ is $m$-sectorial by corollary 3.1.

(Ad 3) By [32, theorem VI.2.5], the adjoint operator $\hat{H}^*_i$ is associated with the adjoint form

$$h^*_i(\psi, \varphi) := \hat{h}_i(\psi, \varphi),$$

which establishes the required identities for $\hat{H}_i$.

(Ad 4) The compactness of the resolvents for $\hat{H}_i$ is provided by the perturbation result [32, theorem VI.3.4] and lemmata 3.1, 3.2. □

### 3.3. Spectral consequences

Since the Hamiltonians $H_i$ are $m$-sectorial by theorem 3.1, the spectrum (as a subset of the numerical range) is contained in a sector of the complex plane, i.e. there exists a vertex $\gamma \in \mathbb{R}$ and a semiangle $\theta \in [0, \pi/2)$ such that

$$\sigma(H_i) \subset \{\zeta \in \mathbb{C} | |\arg(\zeta - \gamma)| \leq \theta\}.$$ 

Furthermore, since the resolvents of $H_i$ are compact, the spectra of $H_i$ are purely discrete, as it is reasonable to expect for the Laplacian defined on a bounded manifold.

Under the additional assumptions on the geometry of model (2.15), one can show that $H_i$ are $\mathcal{PT}$-symmetric.
Proposition 3.1. Let \( \alpha, \beta, b, 1/b, c, \phi \in W^{1,\infty}_{\text{per}}(J_1) \) and let \( f \) satisfy (2.8), (2.15) and (2.18). Then Hamiltonians \( H_i \) are

1. \( \mathcal{PT} \)-symmetric, i.e. \( \mathcal{PT} H_i \subset H_i \mathcal{PT} \),
2. \( \mathcal{P} \)-pseudo-Hermitian, i.e. \( H_i = \mathcal{P} H_i^* \mathcal{P} \),
3. \( T \)-self-adjoint, i.e. \( H_i = T H_i^* T \),

where the operators \( \mathcal{P} \) and \( T \) are defined in (2.14).

Proof. Note that the \( \mathcal{PT} \)-symmetry relation means that whenever \( \psi \in \text{Dom}(H_i) \), \( \mathcal{PT} \psi \) also belongs to \( \text{Dom}(H_i) \) and \( \mathcal{PT} H_i \psi = H_i \mathcal{PT} \psi \). This can be verified directly using the definition of \( H_i \) via (2.19). The proofs of the remaining statements are based on the explicit knowledge of the adjoint operators, theorem 3.1.3.

Corollary 3.2. Under the hypotheses of proposition 3.1, the spectra of \( H_i \) are invariant under complex conjugation, i.e.

\[ \forall \lambda \in \mathbb{C}, \quad \lambda \in \sigma(H_i) \iff \bar{\lambda} \in \sigma(H_i). \]

Proof. Recall that the spectrum of \( H_i \) is purely discrete due to theorem 3.1.4. With regard to \( \mathcal{PT} \)-symmetry, it is easy to check that if \( \psi \) is the eigenfunction corresponding to the eigenvalue \( \lambda \), then \( \mathcal{PT} \psi \) is the eigenfunction corresponding to the eigenvalue \( \bar{\lambda} \).

4. Solvable models: constantly curved manifolds

In order to examine basic effects of curvature on the spectrum of the Hamiltonians we investigate solvable models now. We restrict ourselves to the spectral problem in constantly curved manifolds and subjected to constant interactions on the boundary, i.e. the functions \( K, \alpha, \beta, b, c, \phi \) are assumed to be constant. Moreover, we assume that \( \Gamma \) is a geodesic, i.e. \( \kappa = 0 \), to have (2.15).

4.1. Preliminaries

To emphasize the dependence of the Hamiltonians \( H_i \) on the curvature \( K \), we use the notation \( H_i(K) \) in this section. One can easily derive the scaling properties of eigenvalues for constant \( K \neq 0 \):

\[ \lambda_i(K,a,l) = |K| \lambda_i(\pm 1, \sqrt{|K|}a, \sqrt{|K|}l). \]

Hence, the decisive factor for qualitative properties of the spectrum is the sign of \( K \), while the specific value of curvature is not essential. Hereafter, we restrict ourselves to

\[ K \in \{-1, 0, 1\}. \]  

(4.1)

Possible realizations of the ambient manifolds \( A \) corresponding to these three cases are pseudosphere, cylinder and sphere, respectively, see figure 2.

Remark 4.1. The pseudosphere should be considered as a useful realization of \( A \) with \( K = -1 \) only locally, since no complete surface of constant negative curvature can be globally embedded in \( \mathbb{R}^3 \) (this is reflected by the singular equator in figure 2(a)). However, since \( \Omega \) is a precompact subset of \( A \), the incompleteness of the pseudosphere is not a real obstacle here.
Moreover, hereafter we put $l = \pi$, so that the length of the strip is $2\pi$. This provides an instructive visualization of $\Omega$ as a tubular neighborhood of a geodesic circle on the cylinder and the sphere, see figure 2.

For $\kappa = 0$ and constant curvatures (4.1), the Jacobi equation (2.6) admits the explicit solutions:

$$f_{(K)}(x_1, x_2) = \begin{cases} 
\cosh x_2 & \text{if } K = -1, \\
1 & \text{if } K = 0, \\
\cos x_2 & \text{if } K = 1.
\end{cases}$$

(4.2)

It follows that assumption (2.8) is satisfied for any positive $a$ if $K = -1, 0$, while one has to restrict to $a < \pi/2$ if $K = 1$. The latter is also sufficient to satisfy (2.4) for the sphere. There is no restriction on $a$ to have (2.4) if $\Gamma$ is the geodesic circle on the cylinder. In any case (including the pseudosphere), (2.4) can always be satisfied for sufficiently small $a$. The other hypotheses, i.e. (2.11), (2.15) and (2.18), clearly hold regardless of the curvature sign.

**Remark 4.2.** In view of remark 2.1, $a < \pi/2$ for $K = 1$ is the only essential restriction in the constant-curvature case (4.2).

Explicit structures of the Hamiltonians $H_{i(K)}$ introduced in (2.19) readily follow from (2.7) by using (4.2):

$$H_{i(K)} = \begin{cases} 
-\frac{1}{\cosh^2 x_2} \partial_1^2 - \frac{1}{\cosh x_2} \partial_2 \cosh x_2 \partial_2 & \text{if } K = -1, \\
\partial_1^2 - \partial_2^2 & \text{if } K = 0, \\
-\frac{1}{\cos^2 x_2} \partial_1^2 - \frac{1}{\cos x_2} \partial_2 \cos x_2 \partial_2 & \text{if } K = 1,
\end{cases}$$

(4.3)

on $\Dom(H_{i(K)})$. 

Figure 2. Realizations of the constantly curved manifolds.
4.2. Partial wave decomposition

Since both the coefficients of $H_{\kappa}(K)$ and the boundary conditions are independent of the first variable $x_1$, we can decompose the Hamiltonians into a direct sum of transverse one-dimensional operators. The decomposition is based on the following lemma.

**Lemma 4.1.**

\[ \forall \psi \in L^2(\Omega_0, G), \quad \psi(x_1, x_2) = \sum_{m \in \mathbb{Z}} \psi_m(x_2) \phi_m(x_1) \text{ in } L^2(\Omega_0, G), \quad (4.4) \]

where

\[ \phi_m(x_1) := \frac{1}{\sqrt{2\pi}} e^{imx_1}, \quad \psi_m(x_2) := (\phi_m, \psi(\cdot, x_2))_{L^2(J_1)}. \quad (4.5) \]

**Proof.** We may restrict the proof to $L^2(\Omega_0)$ only because the norms $\| \cdot \|$ and $\| \cdot \|_G$ are equivalent due to (2.8). Let us also stress that $G$ is independent of $x_1$ and \{\phi_m\}_{m \in \mathbb{Z}} forms an orthonormal basis of $L^2(J_1)$. Hence,

\[ \left\| \sum_{m \in \mathbb{Z}} \psi_m(x_2) \phi_m \right\|_{L^2(J_1)} = \| \psi(\cdot, x_2) \|_{L^2(J_1)} \in L^2(J_2). \quad (4.6) \]

The decomposition in $L^2(\Omega_0)$ can then be justified by using the dominated convergence theorem. \[ \square \]

Writing $\psi(x_1, x_2) = \sum_{m \in \mathbb{Z}} \phi_m(x_1) \psi_m(x_2)$ in the expression $H_{\kappa}(K) \psi$ and formally interchanging the summation and differentiation, we (formally) arrive at the decomposition

\[ H_{\kappa}(K) = \bigoplus_{m \in \mathbb{Z}} H^m_{\kappa}(K) B^m, \quad (4.7) \]

with

\[ H^m_{\kappa}(K) := \begin{cases} - \frac{1}{\cosh x_2} \frac{\partial^2 \cosh x_2 \partial_2 + m^2}{\cosh^2 x_2} & \text{if } K = -1, \\ -\partial_2^2 + m^2 & \text{if } K = 0, \\ - \frac{1}{\cos x_2} \frac{\partial_2 \cos x_2 \partial_2 + m^2}{\cos^2 x_2} & \text{if } K = 1, \end{cases} \]

where $B^m$ are bounded rank-one operators defined by

\[ (B^m \psi)(x_1, x_2) := (\phi_m, \psi(\cdot, x_2))_{L^2(J_1)} \phi_m(x_1). \quad (4.8) \]

The operators $H^m_{\kappa}(K)$ act in $L^2(J_2, d\nu(K))$ spaces with the measure

\[ d\nu(K)(x_2) := \begin{cases} \cosh x_2 \, dx_2 & \text{if } K = -1, \\ dx_2 & \text{if } K = 0, \\ \cos x_2 \, dx_2 & \text{if } K = 1. \end{cases} \quad (4.9) \]

The domains of $H^m_{\kappa}(K)$ are given by

\[ \text{Dom}(H^m_{\kappa}(K)) := \{ \psi \in W^{2,2}(J_2) | \psi \text{ satisfies (2.13)\iota} \}, \quad (4.10) \]

with obvious modification of the $\mathcal{P}\mathcal{T}$-symmetric boundary conditions (2.13\iota) to the one-dimensional situation.

To justify decomposition (4.7) in a resolvent sense, we need the following technical lemma specifying the numerical range of $H^m_{\kappa}(K)$.
Lemma 4.2. Let $\mathcal{Z}^m_{i(K)}$ denote the numerical range of $H^m_{i(K)}$. Then there exist real constants $c_0, c_1$ independent of $m \neq 0$ such that
\[
\mathcal{Z}^m_{i(K)} \subset \{ z \in \mathbb{C} \mid \text{Re } z \geq c_0 + m^2, \ |\text{Im } z| \leq c_1 \sqrt{\text{Re } z + |c_0| - m^2} \}.
\]

Proof. We give the proof for $H^m_{i+1}$ only, the other cases are analogous. We abbreviate $(\cdot, \cdot)_+ := (\cdot, \cdot)_{L^2(J_2; \nu_{i+1})}$ and define
\[
v^m(x_2) := \frac{m^2}{\cos^2 x_2}, \quad h[\psi] := (\psi, H^m_{i+1}\psi)_+\]
for every $\psi \in \text{Dom}(H^m_{i+1})$. Integration by parts yields the following expressions for real and imaginary parts of $h[\psi]$:
\[
\text{Re } h[\psi] = \|\psi\|_2^2 + (\psi, v^m\psi)_+ + \beta \cos a(|\psi(a)|^2 + |\psi(-a)|^2),
\]
\[
\text{Im } h[\psi] = a \cos a(|\psi(a)|^2 - |\psi(-a)|^2)
\]
for every $\psi \in \text{Dom}(H^m_{i+1})$. The estimates of $\text{Re } h[\psi]$ and $\text{Im } h[\psi]$ can easily be obtained taking into account the equivalence of the norm $\|\cdot\|_{L^2(J_2)}$ with $\|\cdot\|$, and using the one-dimensional version of the estimate \((3.2)\).

Now we are in a position to establish the main result of this subsection.

Proposition 4.1. $D := \bigcap_{m \in \mathbb{Z}} \varrho(H^m_{i(K)})$ is non-empty and $D \subset \varrho(H^m_{i(K)})$. For every $z \in D$
\[
(H^m_{i(K)} - z)^{-1} = \bigoplus_{m \in \mathbb{Z}} (H^m_{i(K)} - z)^{-1} B^m,
\]
where $(H^m_{i(K)} - z)^{-1}$ abbreviates $1 \otimes (H^m_{i(K)} - z)^{-1}$ acting on $L^2(J_1) \otimes L^2(J_2, \nu_{i(K)})$ and $B^m$ are defined in (4.8).

Proof. We give a proof for $H^m_{i+1}$ only, the remaining cases are analogous. Take $z \in D$, for every $\psi \in L^2(\Omega, G)$ and $m \in \mathbb{Z}$, we define
\[
U_m(x_2) := (H^m_{i+1} - z)^{-1}\psi_m(x_2),
\]
where $\psi_m$ was introduced in (4.5). It is clear that $U_m \in L^2(J_2, \nu_{i+1})$. With regard to lemma 4.2, take $m_0 \in \mathbb{N}$ such that for every $|m| > m_0, z \notin \mathcal{Z}^m_{i(K)}$. Using \cite[theorem V.2.3]{32} together with lemma 4.2, we obtain for $|m| > m_0$
\[
\|U_m\|_{L^2(J_2)} \leq C_1 \|\psi_m\|_{L^2(J_2)} / m^2 + 1
\]
where $C_1$ is a constant independent of $m$, nonetheless depending on $\zeta, |\alpha|, |\beta|$, and $a$. Let us remark that since $z \in D$, $\|U_m\|_{L^2(J_2)}$ are bounded for finitely many $|m| \leq m_0$. From the identity
\[
\|U''_m\|_2^2 + (\alpha a + \beta) \cos a |U_m(a)|^2 + \beta a |U_m(-a)|^2 + (\nu_m, U_m)_+ - \|U_m\|_2^2 = (\psi_m, U_m)_+,
\]
with $\nu_m$ and $(\cdot, \cdot)_+$ defined in lemma 4.2, we obtain the estimate for the norm of $U'_m$ for $|m| > m_0$:
\[
\|U'_m\|_{L^2(J_2)} \leq C_1 \|\psi_m\|_{L^2(J_2)} / \sqrt{m^2 + 1}
\]
Again, for finitely many $|m| \leq m_0, \|U_m\|_{L^2(J_2)}$ are clearly bounded. With regard to (4.6), (4.14) and (4.15), every function $R_m(x_1, x_2) := \phi_m(x_1)U_m(x_2)$ belongs to $W^{1,2}_{per}(\Omega_0)$. 

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Our goal is to show that $R := \sum_{m \in \mathbb{Z}} R_m$ is in $W^{1,2}(\Omega_0)$ as well.

The finite number of bounded terms with $|m| \leq m_0$ is included automatically in the following estimates and equalities without any other specific comments. Identity (4.6) and inequality (4.14) together with Fubini’s theorem imply

$$\left\| \sum_{m \in \mathbb{Z}} R_m \right\| \leq C_2 \| \Psi \|.$$ 

A similar estimate can be obtained for $\partial_1 R_m$ provided that we use the inequality (4.15). For $\partial_1 R_m$, we have

$$\left\| \sum_{m = -N}^N \partial_1 R_m \right\|^2 = \sum_{m = -N}^N m^2 \| u_m \|_{L^1(J_2)}^2 \leq C_2^2 \sum_{m = -N}^N \frac{m^2}{m^2 + 1} \| \psi_m \|_{L^2(J_2)}^2,$$

where we used inequality (4.14). The fraction in the sum on the right-hand side is bounded; therefore, using the Parseval equality, the limit $\sum_{m \in \mathbb{Z}} \partial_1 R_m$ remains in $L^2(\Omega_0)$. We conclude that $R$ belongs to $W^{1,2}(\Omega_0)$ and

$$\| R \|_{W^{1,2}(\Omega_0)} \leq C_3 \| \Psi \|_{L^2(\Omega_0)}.$$ 

It remains to verify that $R$ belongs to $W^{1,2}_{\text{per}}(\Omega_0)$. We introduce the partial sum $R_N := \sum_{m = -N}^N R_m$. The fact that $R_N \in W^{1,2}_{\text{per}}(\Omega_0)$ for every $N \in \mathbb{N}$ and the (trace) embedding of $W^{1,2}(\Omega_0)$ in $L^2(\partial \Omega_0)$ yields

$$\| (\varphi, R(-l, \cdot) - R(-l, \cdot), \varphi) \|_{L^1(J_2)} = \| (\varphi, R(-l, \cdot) - R_N(-l, \cdot), \varphi) \|_{L^1(J_2)} \leq 2C_4 \| \varphi \| \| R - R_N \|_{W^{1,2}(\Omega_0)}$$

for every $\varphi \in L^2(J_2, d\nu_{(l_1)})$; $C_4$ is a constant depending only on $\Omega_0$. Note that the left-hand side does not depend on $N$. Hence, the periodicity of $R$ is justified by taking the limit $N \to +\infty$ and considering the arbitrariness of $\varphi$.

Now, knowing that $R$ belongs to $W^{1,2}_{\text{per}}(\Omega_0)$, one can easily check that

$$\forall \varphi \in W^{1,2}_{\text{per}}(\Omega_0), \quad h_1(\varphi, R) - z(\varphi, R)_{L^1(\Omega_0, G)} = (\varphi, \Psi)_{L^1(\Omega_0, G)}.$$ 

This implies that $R \in \text{Dom}(H_{l+(1)}^{(1)})$, see lemma 3.3, and $(H_{l+(1)} - z) R = \Psi$. \hfill $\square$

Proposition 4.1 has the important consequence for the spectrum of $H_{l(K)}$. 

**Corollary 4.1.**

$$\sigma(H_{l(K)}) = \bigcup_{m \in \mathbb{Z}} \sigma(H_{l(K)}^m).$$

**Proof.** Resolvents on both sides of (4.12) are compact. The inclusion $\sigma(H_{l(K)}) \subset \bigcup_{m \in \mathbb{Z}} \sigma(H_{l(K)}^m)$ is proved (formulated for resolvent sets) in proposition 4.1. The other inclusion is trivial since the existence of an eigenfunction $\xi_{m_0}$ of $H_{l(K)}^m$ corresponding to an eigenvalue $\lambda_0$ implies that $\xi_{m_0}(x_2) \phi_{m_0}(x_1)$ is an eigenfunction of $H_{l(K)}$ corresponding to the same eigenvalue. \hfill $\square$

**Remark 4.3.** Note that the statement of corollary 4.1 relating the spectra of a direct sum of operators to their individual spectra does not hold in general (cf [17, theorem 8.1.12]). In our case, however, we have been able to prove the result due to the compactness of resolvents and additional information about the behavior of the numerical ranges of $H_{l(K)}$ (cf lemma 4.2).
4.3. Similarity to self-adjoint or normal operators

We proceed with an analysis of $H_m^\iota(K)$. For sake of simplicity, we drop the subscript 2 of the $x_2$ variable in the following. We remark that $\mathcal{PT}$-symmetry and $\mathcal{P}$-pseudo-Hermiticity of $H_m^\iota(K)$ is preserved with $\mathcal{P}$ and $\mathcal{T}$ being naturally restricted to $L^2(J_2, d\nu_{\iota(K)})$.

The operators $H_m^\iota(K)$ are neither self-adjoint nor normal, nevertheless we can show the following general result.

**Theorem 4.1.** For every $m \in \mathbb{Z}$ and $K \in \{-1, 0, 1\}$:

1. The families of operators $H_m^\iota(I(K))(\alpha, \beta)$, $H_m^\iota(II(K)) (b, c, \phi)$ are holomorphic with respect to parameters $\alpha$, $\beta$ and $b$, $c$, $\phi$ entering the boundary conditions.

2. The spectrum of $H_m^\iota(K)$ is discrete consisting of simple eigenvalues (i.e. the algebraic multiplicity being one), except of finitely many eigenvalues of algebraic multiplicity two and geometric multiplicity one that can appear for particular values of $\alpha$, $\beta$ and $b$, $c$, $\phi$.

3. If all the eigenvalues are simple, then
   a. the eigenvectors of $H_m^\iota(K)$ form a Riesz basis in $L^2(J_2, d\nu_{\iota(K)})$;
   b. $H_m^\iota(K)$ is similar to a normal operator, i.e. for every $m$ there exists a bounded operator $\varrho$ with bounded inverse such that $\varrho H_m^\iota(K) \varrho^{-1}$ is normal;
   c. if moreover all eigenvalues are real, then $H_m^\iota(K)$ is similar to a self-adjoint operator, i.e. $\varrho H_m^\iota(K) \varrho^{-1}$ is self-adjoint.

4. Let us denote by $\{\psi_{i,m}\}_{i \in \mathbb{N}}$ the eigenfunctions of $H_m^\iota(K)$. The set of eigenfunctions $\mathcal{B} := \{\phi_m \psi_{i,m}\}_{m \in \mathbb{Z}, i \in \mathbb{N}}$, where $\phi_m$ were introduced in (4.5), forms a Riesz basis of $L^2(\Omega_0, G)$.

**Remark 4.4.** We remark that while each $H_m^\iota(K)$ is similar to a normal (or self-adjoint) operator, the similarity transformation $\varrho$ depends on $m$ and there is a priori no uniform (in $m$) bound on $\varrho$ and $\varrho^{-1}$.

**Proof.**

(Ad 1) In view of [32, section VII, example 1.15], the Hamiltonians $H_m^\iota(I(K))(\alpha, \beta)$, considered as a family of operators depending on parameters $\alpha$, $\beta$ entering boundary conditions, are holomorphic. The same is true for $H_m^\iota(II(K))(b, c, \phi)$.

(Ad 2) The separated boundary conditions belong to the class of strongly regular boundary conditions [46, 47]. The connected $\mathcal{PT}$-symmetric boundary conditions are strongly regular as well because $\theta_1 = -b$, $\theta_{-1} = b$ (in Naimark’s notation) and $b$ is non-zero by the assumption in (2.13). Moreover, all the eigenvalues are simple [40] up to finitely many degeneracies that can appear: eigenvalues with algebraic multiplicity 2 and geometric multiplicity 1.

(Ad 3) With regard to the strong regularity of boundary conditions, the eigenfunctions of the Hamiltonian $H_m^\iota(K)$ form a Riesz basis [40], except the situations when the degeneracies appear. The existence of Riesz basis implies the similarity to a normal operator, and as a special case, the similarity to a self-adjoint operator if the spectrum of $H_m^\iota(K)$ is real.

In more details, let $[\psi_n]_{n \in \mathbb{N}}$ be the Riesz basis of eigenvectors of $H_m^\iota(K)$, i.e. $H_m^\iota(K) \psi_n = \lambda_n \psi_n$. By definition, there exists a bounded operator $\rho$ with bounded inverse such that $\{\rho \psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis that we denote by $[e_n]_{n \in \mathbb{N}}$. Then

$$
\rho H_m^\iota(K) \rho^{-1} = \sum_{n \in \mathbb{N}} \lambda_n e_n (e_n, \cdot)_{L^2(\Omega_0, G)}
$$

is a normal (self-adjoint if every $\lambda_n \in \mathbb{R}$) operator.
At first, we investigate the Hamiltonians

4.4. Separated boundary conditions

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Proposition 4.2. The spectrum of \( H_{\Omega_1}^m(\alpha, \beta) \) is real for all \( m \in \mathbb{Z} \). The eigenvalues \( \lambda_{j,m} \) and eigenfunctions \( \psi_{j,m} \) can be written in the following form, \( m \in \mathbb{Z} \):

\[
\lambda_{j,m} = \begin{cases} 
\alpha^2 + m^2 & \text{if } j = 0, \\
k_j^2 + m^2 & \text{if } j \geq 1,
\end{cases}
\]
\[ \psi_{j,m}(x) = \begin{cases} C_0 \exp(-i \alpha x) & \text{if } j = 0, \\ C_j \left( \cos(k_j x) + \frac{k_j \sin(k_j a) - i \alpha \cos(k_j a)}{k_j \cos(k_j a) + i \alpha \sin(k_j a)} \sin(k_j x) \right) & \text{if } j \geq 1, \end{cases} \]

where \( k_j := \frac{n \pi}{2a} \). If \( \alpha^2 \neq k_j^2 \), i.e. there is no level-crossing for the same \( m \), then the operator is similar to a self-adjoint operator or, equivalently, it is quasi-Hermitian.

Remark 4.6. Closed formulae for the metric operator \( \Theta_1 \) for \( H_{0}^{m}(\alpha, 0) \) are presented in [36, 37]. The similarity transformation \( \varrho \) can be found as \( \varrho = \sqrt{\Theta_1} \) or as any other decomposition of the positive operator \( \Theta_1 = \varrho^* \varrho \).

The \( \alpha \)-dependence of eigenvalues \( \lambda \) for \( m = 0, 1, 2 \) is plotted in figure 3.

The case of \( \beta \neq 0 \) is more complicated, and as was remarked in [37], the spectrum of \( H_{0}^{m}(\alpha, \beta) \) can be complex. More precise results follow from a further analysis, not presented in [37].

Proposition 4.3.

(1) If \( \beta > 0 \), then the spectrum of \( H_{0}^{m}(\alpha, \beta) \) is purely real for all \( m \in \mathbb{Z} \) and \( \alpha \in \mathbb{R} \).

(2) If \( \beta < 0 \), then the spectrum of \( H_{0}^{m}(\alpha, \beta) \) is either purely real or there is one pair of complex conjugated eigenvalues with real part located in the neighborhood of \( \alpha^2 + \beta^2 \). For fixed negative \( \beta \), the points \( \alpha_n \) where a pair of complex conjugated eigenvalues appears (by increasing of \( \alpha \)) are determined by \( \alpha_n^2 + \beta^2 = k_n^2 \), where \( k_n^2 := ((2n + 1) \pi/4a)^2 \) for some \( n \in \mathbb{N} \).

The eigenvalues \( \lambda = k^2 \) of \( H_{0}^{0}(\alpha, \beta) \) are determined (\( k = 0 \) is admissible only if \( \alpha = \beta = 0 \)) by the equation

\[ (k^2 - \alpha^2 - \beta^2) \sin(2ka) - 2\beta k \cos(2ka) = 0. \] (4.19)

The corresponding eigenfunctions read

\[ \psi(x) = C \left( \cos(kx) + \frac{k \sin(ka) - (i\alpha + \beta) \cos(ka)}{k \cos(ka) + (i\alpha + \beta) \sin(ka)} \sin(kx) \right). \] (4.20)

The eigenvalues of \( H_{0}^{m}(\alpha, \beta) \) are obtained by adding \( m^2 \) to the eigenvalues of \( H_{0}^{0}(\alpha, \beta) \).
Proof.} We proceed in a similar way as in the alternative proof [37, section 6.1] of the reality of the spectrum of $H_{0(0)}^1(\alpha, 0)$. The original eigenvalue problem (4.17) with $m = 0$ can be transformed, using $\phi(x) := e^{ax}\psi(x)$, into
\[
\begin{cases}
-\phi'' + 2a\phi' + a^2\phi = \lambda\phi & \text{in } (-a, a), \\
\phi'(\pm a) \pm \beta\phi(\pm a) = 0.
\end{cases}
\] (4.21)

We multiply equation (4.21) by $\bar{\phi}''$ and integrate over $(-a, a)$. Next we multiply the complex conjugated version of equation (4.21) by $\phi''$ and again we integrate over $(-a, a)$. By subtracting the results and integrating by parts with use of the boundary conditions in (4.21), we obtain the identity
\[-\alpha \beta^2(\phi(a))^2 - |\phi(-a)|^2 = \text{Im } \lambda \left( \| \psi \|_{L^2(J)}^2 + \beta(\phi(a))^2 + |\phi(-a)|^2 \right).\] (4.22)

If we perform the same procedure, however, with multiplication by $\bar{\phi}$, after some integration by parts we receive the relation
\[
\alpha(\phi(a))^2 - |\phi(-a)|^2 = \text{Im } \lambda \| \phi \|_{L^2(J)}^2.
\] (4.23)

Combining (4.22) with (4.23) leads to the identity
\[0 = \text{Im } \lambda \left( \| \psi \|_{L^2(J)}^2 + \beta(\phi(a))^2 + |\phi(-a)|^2 \right) + \beta^2\| \phi \|_{L^2(J)}^2.\] (4.24)

If $\beta$ is positive, then the whole term in the brackets is strictly positive and thus imaginary part of $\lambda$ must be zero. This proves the first item of the proposition.

If $\beta$ is negative, then complex eigenvalues can appear. If we divide the equation (4.19) by $k^2$ and leave only $\sin(2ka)$ term on the left-hand side, then it is clear that eigenvalues approach $(n\pi/2a)^2$ for $k$ real and large enough. After simple algebraic manipulation (4.19) becomes
\[
\tan(2ka) = \frac{2\beta k}{k^2 - a^2 - \beta^2}.
\] (4.25)

and the eigenvalues correspond to the intersections of the graphs of functions on left and right-hand side of (4.25). We denote $l(k)$ the function on the left-hand side, $r(k)$ the one on the right-hand side, and $k_0 := \sqrt{a^2 + \beta^2}$. The behavior of $r(k)$ for $k \in \mathbb{R}$ is summarized in table 1.

Graphs of the functions $l(k)$ and $r(k)$ are plotted in figure 4. It is clear from the holomorphic dependence of eigenvalues on $\alpha, \beta$ (a consequence of theorem 4.1) that eigenvalues are close to $(n\pi/2a)^2$, corresponding to zeros of $l(k) = \tan(2ka)$, except those in the neighborhood of $a^2 + \beta^2$. Since $\alpha = \beta = 0$ case corresponds to Neumann boundary conditions, for small $\alpha$ and $\beta$, all eigenvalues must be close to $(n\pi/2a)^2$. Hence, if we fix $\beta$ and increase $\alpha$, then two intersections of graphs of $l(k)$ and $r(k)$ are ‘lost’ precisely at the point where $a^2 + \beta^2 = k_n^2$ for some $n \in \mathbb{N}$, i.e. the asymptote of $r(k)$ corresponds to the asymptote of the tangent $l(k)$. This implies the creation of complex conjugate pair of eigenvalues. If we increase $\alpha$ more, two intersections appear again which means the annihilation of complex conjugate pair, i.e. the restoration of two real eigenvalues. The two intersections are lost at the next critical value $\alpha_{n+1}$. Very rough estimates give the location of restoration of real eigenvalues in the interval $(n\pi/2a, (2n + 1)\pi/4a)$.

In view of the presented arguments, only one complex conjugated pair can appear in the spectrum for fixed $\alpha$ and $\beta$ in the neighborhood of $a^2 + \beta^2$, and the other eigenvalues approach $(n\pi/2a)^2$ as the distance from $a^2 + \beta^2$ increases. Moreover, for fixed $\beta$, the enlarging of $\alpha$ results into the shift of eigenvalues from almost Neumann ones $(n\pi/2a)^2$, $n \in \mathbb{N}$, to Dirichlet ones $((n + 1)\pi/2a)^2$, $n \in \mathbb{N}$, for $\alpha$ large.

Finally, the equation for eigenvalues and eigenfunctions are found in a standard way. The general solution $A\cos(kx) + B\sin(kx)$ of $-\psi'' = k^2\psi$ is inserted into boundary
Figure 4. Graphs of $l(k)$ (full line) and $r(k)$ (dashed line), $\alpha = \pi/4$, $\beta = -0.5$.

Table 1. The behavior of $r(k)$.

| $k$            | Sign $r(k) > 0$, $r'(k) > 0$, $r''(k) > 0$ | Asymptotics |
|----------------|---------------------------------------------|-------------|
| $(-\infty, -k_0)$ | $\lim_{k \to -k_0} r(k) = +\infty$        |             |
| $(-k_0, 0)$     | $\lim_{k \to -k_0} r(k) = 0$              |             |
| $(0, k_0)$      | $\lim_{k \to -k_0} r(k) = -\infty$        |             |
| $(k_0, \infty)$ | $\lim_{k \to -k_0} r(k) = +\infty$        |             |

conditions (4.17) and the condition for existence of non-trivial solutions $A, B$ is the eigenvalue equation (4.19).

Figures 5 and 6 represent the $\alpha$-dependence of the first four eigenvalues as obtained by a numerical analysis of (4.19). The numerical results confirm the above described behavior. Let us remark that if $\beta$ is positive, then the graph of $r(k)$ is reflected by the $x$-axis and the effect of losing intersections is not possible; hence, the spectrum remains real.

4.4.2. Positive curvature. The eigenvalue problem for the Hamiltonian $H^{m}_{l(\psi)}$ reads

\[
\begin{align*}
-\psi''(x) + \tan x \psi'(x) + \frac{m^2}{\cos^2 x} \psi(x) &= k^2 \psi(x) \quad \text{in} \quad (-\alpha, \alpha), \\
\psi'(\pm\alpha) + i\alpha \psi(\pm\alpha) &= 0. 
\end{align*}
\] (4.26)

Solutions of (4.26) can be written down in terms of associated Legendre functions $P^{(m)}_\nu, Q^{(m)}_\nu$:

\[
\psi(x) = C_1 \psi_1(x) + C_2 \psi_2(x) \equiv C_1 P^{(m)}_\nu(\sin x) + C_2 Q^{(m)}_\nu(\sin x),
\] (4.27)

where

\[
\nu := \frac{1}{2}(\sqrt{1 + 4\lambda} - 1),
\] (4.28)

\[
C_2(\alpha \psi_2(-\alpha) - i\psi'_2(-\alpha)) = C_1(\alpha \psi_1(-\alpha) + i\psi'_1(-\alpha)).
\] (4.29)
Inserting the general solution (4.27) into boundary conditions in (4.26) and consequent search for non-trivial constants \(C_1, C_2\) yields the eigenvalue equation

\[
\begin{vmatrix}
\psi'_1(a) + i\alpha \psi_1(a) & \psi'_2(a) + i\alpha \psi_2(a) \\
\psi'_1(-a) + i\alpha \psi_1(-a) & \psi'_2(a) + i\alpha \psi_2(a)
\end{vmatrix}
= 0. 
\tag{4.30}
\]

In order to analyze the spectrum in more detail, we transform the Hamiltonian \(H_{l(\pm 1)}^m\) into a unitarily equivalent operator of a more convenient form. The proof of the lemma is a straightforward calculation.

**Lemma 4.3.** The unitary mapping. \(U_{(\pm 1)} : L^2(J_2, dx) \to L^2(J_2, d\nu_{(\pm 1)})\)

\[
(U_{(\pm 1)}\psi)(x) := (\cos x)^{-\frac{1}{2}} \psi(x)
\tag{4.31}
\]

transforms \(H_{l(\pm 1)}^m(\alpha, 0)\) into

\[
U_{(\pm 1)}^{-1} H_{l(\pm 1)}^m(\alpha, 0) U_{(\pm 1)} = H_{l(0)}^0 \left( \alpha, \frac{1}{2} \tan a \right) + V_{(\pm 1)}^m.
\tag{4.32}
\]
where
\[ V_m^{(e)}(x) := \frac{8m^2 - 3 - \cos 2x}{8\cos^2 x}. \] (4.33)

Equipped with the equivalent form of the Hamiltonian, we prove the following result.

**Proposition 4.4.** For every \( m \in \mathbb{Z} \) there exists a real number \( \Lambda_{m+1}^m \) such that all eigenvalues \( \lambda \) with \( \Re \lambda \geq \Lambda_{m+1}^m \) are real and simple (i.e. the algebraic multiplicity being one). The eigenvalues with \( \Re \lambda < \Lambda_{m+1}^m \) can be complex, ordered in complex conjugated pairs.

Eigenvalues are determined by equation (4.30) and eigenfunctions can be written in the form (4.27) with (4.29).

**Proof.** We follow standard arguments of perturbation theory (see e.g. [18] for spectral operators). Let us consider the transformed Hamiltonian (4.32) and forget about the potential for a moment, i.e. we understand the potential \( V_m^{(e)}(x) \) as a perturbation of \( H_0^{(0)}(\alpha, \frac{1}{2} \tan a) \). Since \( \tan a \) is positive under the assumption \( a < \pi / 2 \), the reality of the spectrum is guaranteed by proposition 4.3.1. The potential represents a bounded perturbation and it can shift eigenvalues only by \( C \| V_m^{(e)}(x) \| \). Here the constant \( C \) comes from the estimate of the norm of the resolvent \( \| R_{H_0}^{(0)}(\lambda) \| \leq C \| \lambda \| \), which is valid for \( H_0^{(0)}(\alpha, \frac{1}{2} \tan a) \) due to the similarity to a normal operator (cf theorem 4.1). The separation distance \( |\lambda_{m+1} - \lambda_m| \) of eigenvalues (ordered with respect to the real part) of the unperturbed operator \( H_0^{(0)}(\alpha, \frac{1}{2} \tan a) \) grows to infinity and two eigenvalues must collide at first to create a complex conjugate pair. Hence, the perturbed operator cannot have more than finitely many complex eigenvalues. Recall that due to \( PT \)-symmetry (corollary 3.2) the complex eigenvalues come in complex conjugated pairs. \( \square \)

**Remark 4.7.** In other words, we detected the effects of a positive curvature. It acts as the adding of real bounded potential \( V_m^{(e)}(x) \) and real ‘\( \beta \)-like’ term in the boundary conditions to the zero curvature Hamiltonian \( H_0^{(0)} \). The positive \( \frac{1}{2} \tan a \) term is decisive for the behavior of the spectrum; the bounded potential \( V_m^{(e)}(x) \) can affect substantially only the lowest eigenvalues. Nonetheless, we conjecture that the spectrum remains real for every \( m \in \mathbb{Z} \).

A numerical analysis of the equation (4.30) for \( \lambda = k^2 \) is presented in figure 7. Obvious similarity with figure 5 supports the perturbative results.

### 4.4.3. Negative curvature

The eigenvalue problem for the Hamiltonian \( H_m^{(-1)} \), reads
\[
\begin{cases}
-\psi''(x) - \tanh x \psi'(x) + \frac{m^2}{\cosh^2 x} \psi(x) = k^2 \psi(x) \quad \text{in} \quad (-a, a),
\psi'(\pm a) + i \alpha \psi(\pm a) = 0.
\end{cases}
\] (4.34)

The solutions of (4.34) can again be expressed via associated Legendre functions \( P_{-m}^{(\mu)}(\tanh x) \), \( Q_{-m}^{(\mu)}(\tanh x) \), but they have a more complicated form than (4.27):
\[
\psi(x) = C_1 \psi_1(x) + C_2 \psi(x) \equiv C_1 P_{-m}^{(\mu)}(\tanh x) \sqrt{\cosh x} + C_2 Q_{-m}^{(\mu)}(\tanh x) \sqrt{\cosh x},
\] (4.35)
where
\[
\mu := im - \frac{1}{2}, \quad \nu := \frac{1}{2} \sqrt{1 - 4 \lambda}.
\] (4.36)
Relations between $C_1, C_2$ can be obtained from equation (4.29), however, with $\psi_1, \psi_2$ corresponding to the negative curvature solutions (4.35); the same is true for the eigenvalue equation (4.30).

To explain the behavior of the spectrum in a deeper way, we use the same strategy as in the positive curvature case. The eigenvalue problem (4.34) can be transformed by an analogous unitary transformation leading to a modified zero curvature eigenvalue problem.

**Lemma 4.4.** The unitary mapping $U_{(-1)} : L^2(J_2, dx) \rightarrow L^2(J_2, d\nu_{(-1)})$

$$(U_{(-1)} \psi)(x) := (\cosh x)^{-\frac{1}{2}} \psi(x)$$

transforms $H_{l_{(-1)}}^m(\alpha, 0)$ into

$$U_{(-1)}^{-1} H_{l_{(-1)}}^m(\alpha, 0) U_{(-1)} = H_{l_{(0)}}^{0}(\alpha, -\frac{1}{2} \tanh a) + V_{(-1)}^m,$$

where

$$V_{(-1)}^m(x) := \frac{8m^2 + 3 + \cosh 2x}{8 \cosh^2 x}.$$  

**Proposition 4.5.** For every $m \in \mathbb{Z}$ there exists a real number $\Lambda_{m}^{(-1)}$ such that all eigenvalues $\lambda$ with $\text{Re} \lambda \geq \Lambda_{m}^{(-1)}$ are either real and simple (i.e. the algebraic multiplicity being one), or there is one complex conjugated pair of eigenvalues with a real part located in the neighborhood of $\alpha^2 + \beta^2$. The eigenvalues with $\text{Re} \lambda < \Lambda_{m}^{(-1)}$ can be complex, ordered in complex conjugated pairs.

Eigenvalues are determined by equation (4.30) with $\psi_1, \psi_2$ from (4.35). Eigenfunctions can be written in the form (4.35) with constants $C_1, C_2$ satisfying (4.29) with $\psi_1, \psi_2$ from (4.35).

**Proof.** The proof is the same as in the positive curvature case, cf the proof of proposition 4.4. The unperturbed Hamiltonian $H_{l_{(0)}}^{0}(\alpha, -\frac{1}{2} \tanh a)$ corresponds to the case analyzed in proposition 4.3.2. \(\square\)
Figure 8. \( \alpha \)-dependence of eigenvalues, negative curvature, \( \alpha = \pi/4 \). Red (full), green (dashed) and blue (dot-dashed) curves correspond to \( m = 0, 1, 2 \) respectively. See animation at stacks.iop.org/JPhysA/43/485204/mmedia, for an animated visualization of the \( \alpha \)-dependence of the eigenvalues.

Remark 4.8. The curvature effect is now represented by the bounded real potential \( V_m^{(\pm 1)} \) and the extra negative term \( -\frac{1}{2} \tanh \alpha \) in the boundary conditions. The lowest eigenvalues (in absolute values) can be complex; nonetheless, we showed that the creation of a complex pair of eigenvalues is always followed by its annihilation, i.e. the restoration of real eigenvalues, when parameter \( \alpha \) is increased.

A result of the numerical analysis of the eigenvalue problem is presented in figure 8. See animation at stacks.iop.org/JPhysA/43/485204/mmedia. The resemblance to zero curvature case with negative \( \beta \) in boundary conditions is obvious.

4.5. Connected boundary conditions

The connected boundary conditions are, by their nature, more complicated than the separated ones and moreover, they are given by three real parameters \( b, c, \phi \). Like for the separated boundary conditions, we can use the unitary transformations \( U_{(\pm 1)} \) introduced in (4.31) and (4.37) to transform the problems to the zero curvature case, however, with modified boundary conditions and with additional bounded real potentials \( V_m^{(\pm 1)} \), defined in (4.33) and (4.39). The modification of boundary conditions is presented in appropriate subsections below.

The spectrum is not analytically described even for the zero curvature model and it is beyond the scope of this article to proceed with this analysis. The main aim of this section is to show the effect of curvature, i.e. the transformation of curved models to the zero curvature case. Furthermore, we present some results of a numerical analysis for the ‘lowest’ eigenvalues: \( \phi \)-dependence for selected values of \( b, c \). We remark that the case \( b = c = 0, \phi = \pm \pi/2 \) corresponds to irregular boundary conditions and the spectrum of such operators is completely different from the cases presented here (cf [54]).

4.5.1. Zero curvature. We impose connected boundary conditions (2.13II) on the solutions of eigenvalue problem for \( H^0_{\Pi 0}(b, c, \phi) \) and we obtain the following equation for eigenvalues \( \lambda = k^2 \):

\[
- 2k + 2k \cos(2ak)\sqrt{1 + bc \cos \phi + (bk^2 - c) \sin(2ak)} = 0
\]

(4.40)
and eigenfunctions

\[ \psi(x) = C_1 \cos(kx) + C_2 \sin(kx), \]  

(4.41)

where the constants are further restricted by

\[ C_2((-1 + \sqrt{1 + bc} e^{i\phi}) \cos(ak) + bk \sin(ak)) = C_1((1 + \sqrt{1 + bc} e^{i\phi}) \sin(ak) - bk \cos(ak)). \]  

(4.42)

**Proposition 4.6.** Eigenvalues \( \lambda = k^2 \) of \( H_{II}^0(b, c, \phi) \) are determined by equation (4.40), eigenfunctions read (4.41) with (4.42). Eigenvalues for \( m \neq 0 \) can be obtained by the shift \( \lambda \mapsto \lambda + m^2 \) while the corresponding eigenfunctions remain the same.

Figure 9 illustrates the behavior of eigenvalues for a certain choice of the parameters.

4.5.2. **Positive curvature.** The solutions of the eigenvalue problem for \( H_{II}^m(b, c, \phi) \) with connected boundary conditions (2.13II) are the same as (4.27) except the constants \( C_1, C_2 \) now satisfy

\[ C_2(\sqrt{1 + bc} e^{i\phi} \psi_2(-a) - \psi_2(a) + b \psi'_2(-a)) = C_1(\sqrt{1 + bc} e^{i\phi} \psi_1(-a) - \psi_1(a) + b \psi'_1(-a)). \]  

(4.43)

The equation for eigenvalues reads

\[
\begin{vmatrix}
-\sqrt{1 + bc} e^{i\phi} \psi_1(-a) + \psi_1(a) - b \psi'_1(-a) & \sqrt{1 + bc} e^{i\phi} \psi_2(-a) + \psi_2(a) - b \psi'_2(-a) \\
-c \psi_1(-a) - \sqrt{1 + bc} e^{-i\phi} \psi'_1(-a) + \psi'_1(a) & -c \psi_2(-a) - \sqrt{1 + bc} e^{-i\phi} \psi'_2(-a) + \psi'_2(a)
\end{vmatrix} = 0.
\]

(4.44)

Figure 10 illustrates the behavior of eigenvalues for a certain choice of the parameters.

We employ the unitary transformation \( U_{(+)} \) introduced in lemma 4.3 to map \( H_{II}^m(b, c, \phi) \) to a zero curvature Hamiltonian.
Proposition 4.7. The unitary mapping $U_{(+)}$ defined in (4.31) transforms the Hamiltonian $H_{II(+)}(b, c, \phi)$ into

$$U_{(+)}^{-1} H_{II(+)}(b, c, \phi) U_{(+)} = \tilde{H}_{II(0)} + V_m^{(+)},$$

(4.45)

where $V_m^{(+)}$ is defined in (4.33) and $\tilde{H}_{II(0)} := -\frac{d^2}{dx^2}$ with the domain consisting of $\psi \in W^{2,2}(J_2)$ satisfying

$$\Psi(a) = B_{(+)} \Psi(-a), \quad \Psi(x) := \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} \quad \text{and} \quad B_{(+)} := \begin{pmatrix} \sqrt{1 + bc} e^{i\phi} - \frac{b}{2} \tan a \\ c - \sqrt{1 + bc} \tan a \cos \phi + \frac{b}{2} \tan^2 a \\ \sqrt{1 + bc} e^{-i\phi} - \frac{b}{2} \tan a \end{pmatrix}. \quad (4.46)$$

Eigenvalues $\lambda = k^2$ of $H_{II(+)}(b, c, \phi)$ are determined by equation (4.44), eigenfunctions read (4.27) with constants $C_1, C_2$ given by (4.43).

Remark 4.9. The boundary conditions (4.46) are $\mathcal{P}\mathcal{T}$-symmetric, but they are no more $\mathcal{P}$-pseudo-Hermitian. This result shows that although we reduced the problem to the zero curvature case (in the sense of previous sections), the investigation of spectrum must be done with more general boundary conditions than $\mathcal{P}\mathcal{T}$-symmetric and $\mathcal{P}$-pseudo-Hermitian at the same time.

4.5.3. Negative curvature. The solutions of the eigenvalue problem for $H_{II(-1)}^m(b, c, \phi)$ with connected boundary conditions (2.13II) are the same as in the separated conditions case (4.35), but the relation between constants $C_1, C_2$ is given by (4.43) with $\psi_1, \psi_2$ corresponding to the negative curvature solutions (4.35); the same is also valid for the eigenvalue equation (4.44).

Figure 11 illustrates the behavior of eigenvalues for a certain choice of parameters.

Proposition 4.8. The unitary mapping $U_{(-1)}$ defined in (4.37) transforms the Hamiltonian $H_{II(-1)}^m(b, c, \phi)$ into

$$U_{(-1)}^{-1} H_{II(-1)}^m(b, c, \phi) U_{(-1)} = \tilde{H}_{II(0)} + V_m^{(-1)},$$

(4.47)
where $V_m^{(-1)}(x)$ is defined in (4.39) and $\tilde{H}_I^{0}(0) := -\frac{d^2}{dx^2}$ with the domain consisting of $\psi \in W^{2,2}(J_2)$ satisfying

$$\Psi(a) = B_{(-1)}\Psi(-a), \quad \Psi(x) := \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}$$

and

$$B_{(-1)} := \begin{pmatrix} \sqrt{1 + bc} e^{i\phi} + \frac{1}{2} b \tanh a & b \\ c + \sqrt{1 + bc} \tanh a \cos \phi + \frac{1}{2} b \tanh^2 a & \sqrt{1 + bc} e^{-i\phi} + \frac{1}{2} b \tanh a \end{pmatrix}.$$  

Eigenvalues $\lambda = k^2$ of $H_m^{0}(b, c, \phi)$ are determined by equation (4.44) with $\psi_1, \psi_2$ from (4.35). The eigenfunctions read (4.35), where constants $C_1, C_2$ are given by (4.43) with $\psi_1, \psi_2$ from (4.35).

**Remark 4.10.** The boundary conditions (4.48) are ${\mathcal{P}}$-$\mathcal{T}$-symmetric, however not $\mathcal{P}$-pseudo-Hermitian, as for the positive curvature case. Therefore, it is necessary to investigate more general boundary conditions in zero curvature eigenvalue problem.

### 5. Concluding remarks

The goal of this paper is to introduce a new class of $\mathcal{P}\mathcal{T}$-symmetric Hamiltonians defined in curved manifolds and describe the effects of curvature on the spectrum. Although we were able to find these effects for both separated and connected boundary conditions, the absence of results on the reality of the spectrum for the latter (even in the case of zero curvature) did not allow us to present the conclusions in an entirely descriptive and explicit way. Let us therefore summarize the main features of the model for the separated Robin-type boundary conditions (1.3) only.

In table 2 we schematically (and very roughly) describe qualitative properties of the spectrum we observed in the constant-curvature cases. The entry describing the positive curvature case includes our conjecture (supported by numerical analysis) that all eigenvalues are real.

The most instructive results in the paper are probably lemmata 4.3 and 4.4, which enable one to understand the effect of curvature in terms of an additional effective potential and boundary-coupling interaction. For the $s$-wave modes (i.e. $m = 0$ in decomposition (4.7))
Table 2. A heuristic summary of our analytical and numerical analysis.

| Curvature    | Spectrum | Eigenvalues                       |
|--------------|----------|----------------------------------|
| Zero         | \( \mathbb{R} \) | Only some \( \alpha \)-dependent, crossings |
| Positive     | \( \mathbb{R} \) | All \( \alpha \)-dependent, no crossings |
| Negative     | \( \mathbb{C} \) | All \( \alpha \)-dependent, crossings, creation and annihilation of complex pairs |

and infinitesimally thin strips (i.e. \( a \ll l \)), it follows from the lemmata that the positive and negative curvature acts as an attractive and repulsive interaction, respectively. This is in agreement with a spectral analysis of similar models in the self-adjoint case of Dirichlet boundary conditions [34, 35]. However, the additional boundary interaction is not negligible for positive widths \( a \), and its effect is actually completely opposite (cf remarks 4.7, 4.8); the positive and negative curvature give rise to an attractive and repulsive Robin-type boundary condition, respectively. The interplay between these two effects is further complicated by the presence of the repulsive centrifugal term for \( |m| \geq 1 \), and the numerical analysis confirms that the overall picture of the spectrum can be quite complex.

It follows from previous comments and remarks that several open problems remain, e.g. the proof of the reality of all eigenvalues in the positive curvature model. Nonetheless, we would also like to mention some other interesting directions of potential future research: the spectral effect of curvature in non-constant curvature and non-constant boundary-coupling functions setting, the existence of Riesz basis for such setting or models defined on unbounded domains (waveguides) in curved spaces. The last case can be viewed as a natural continuation of [10] where a planar \( \mathcal{PT} \)-symmetric waveguide was studied.

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