Field of U-invariants of adjoint representation of the group \( \text{GL}(n, K) \)

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It is known that the field of invariants of an arbitrary unitriangular group is rational (see [1]). In this paper for adjoint representation of the group \( \text{GL}(n, K) \) we present the algebraically system of generators of the field of \( U \)-invariants.

Note that the algorithm for calculating generators of the field of invariants of an arbitrary rational representation of an unipotent group was presented in [2, Chapter 1]. The invariants was constructed using induction on the length of Jordan-Hölder series. In our case the length is equal to \( n^2 \); that is why it is difficult to apply the algorithm.

Let us consider the adjoint representation \( \text{Ad}_g A = g A g^{-1} \) of the group \( \text{GL}(n, K) \), where \( K \) is a field of zero characteristic, on the algebra of matrices \( \text{Mat}(n, K) \). The adjoint representation determines the representation \( \rho_g \) on \( K[\text{M}] \) (resp. \( K(\text{M}) \)) by formula \( \rho_g f(A) = f(g^{-1}Ag) \).

Let \( U \) be the subgroup of upper triangular matrices in \( \text{GL}(n, K) \) with units on the diagonal. A polynomial (rational function) \( f \) on \( \text{M} \) is called a \( U \)-invariant, if \( \rho_u f = f \) for every \( u \in U \). The set of \( U \)-invariant rational functions \( K(\text{M})^U \) is a subfield of \( K(\text{M}) \).

Let \( \{x_{i,j}\} \) be a system of standard coordinate functions on \( \text{M} \). Construct a matrix \( \text{X} = (x_{ij}) \). Let \( \text{X}^* = (x_{ij}^*) \) be its adjugate matrix, \( \text{X} \cdot \text{X}^* = \text{X}^* \cdot \text{X} = \det \text{X} \cdot E \). Denote by \( J_k \) the left lower corner minor of order \( k \) of the matrix \( \text{X} \).

We associate with each \( J_k \) the system of \( k \) determinants \( J_{k,i} \), where \( 0 \leq i \leq k - 1 \). The determinant \( J_{k,0} \) coincides with the minor \( J_k \). The first \( k - i \) rows in the determinant \( J_{k,i} \), coincide with the last \( k - i \) rows in the minor \( J_k \); the last \( i \) rows in \( J_{k,i} \) coincide with the similar rows in the minor \( J_k(\text{X}^*) \) of the adjugate matrix \( \text{X}^* = (x_{ij}^*) \).

**EXAMPLE 1.** Case \( n = 2 \), \( \text{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \text{X}^* = \begin{pmatrix} x_{11}^* & x_{12}^* \\ x_{21}^* & x_{22}^* \end{pmatrix} \)

\( J_{1,0} = x_{2,1}, \ J_{2,0} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \ J_{2,1} = \begin{vmatrix} x_{21} & x_{22} \\ x_{21}^* & x_{22}^* \end{vmatrix} \).

*The work is supported by the RFBR-grants 12-01-00070-a, 12-01-00137-a, 13-01-97000-Volga region-a*
EXAMPLE 2. Case \( n = 3 \),

\[
X = \begin{pmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix}, \quad X^* = \begin{pmatrix}
x_{11}^* & x_{12}^* & x_{13}^* \\
x_{21}^* & x_{22}^* & x_{23}^* \\
x_{31}^* & x_{32}^* & x_{33}^*
\end{pmatrix},
\]

\[
J_{1,0} = x_{31}, \quad J_{2,0} = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \quad J_{3,0} = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix},
\]

\[
J_{2,1} = \begin{vmatrix} x_{31} & x_{32} \\ x_{31}^* & x_{32}^* \end{vmatrix}, \quad J_{3,1} = \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{31}^* & x_{32}^* & x_{33}^* \end{vmatrix}, \quad J_{3,2} = \begin{vmatrix} x_{31} & x_{32} & x_{33} \\ x_{31}^* & x_{32}^* & x_{33}^* \end{vmatrix}.
\]

**Proposition 1.** Each polynomial \( J_{k,i} \), where \( 1 \leq k \leq n, \quad 0 \leq i \leq k - 1 \) is an \( U \)-invariant.

**Proof.** First, the formula \( \rho_u X = u^{-1} Xu \) implies that each left lower corner minor \( J_k \), where \( 1 \leq k \leq n \), is an \( U \)-invariant. Indeed, each \( J_k \) is invariant with respect to right and left multiplication by elements \( u \in U \). For each \( 1 \leq i \leq n \) we construct the \( n \times n \) matrix

\[
\mathbb{Y}_i = \begin{pmatrix} X_{n-i} & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
X_i^* & \cdots & \cdots
\end{pmatrix}.
\]

Here the block \( X_{n-i} \) has size \((n - i) \times n\); it is obtained from the matrix \( X \) deleting first \( i \) rows. The block \( X_i^* \) has size \( i \times n \); it is obtained from the matrix \( X^* \) by deleting first \( n - i \) rows.

Since \( \rho_u X = u^{-1} Xu \) and \( \rho_u X^* = u^{-1} X^* u \), we have

\[
\rho_u \mathbb{Y}_i = \begin{pmatrix} u_{n-i}^{-1} X_{n-i} u \\
\cdots & \cdots & \cdots \\
u_i^{-1} X_i^* u
\end{pmatrix} = u_* \mathbb{Y}_i u, \quad (1)
\]

where \( u_* = \begin{pmatrix} u_{n-i}^{-1} & 0 \\
0 & u_i^{-1}
\end{pmatrix} \), \( u_{n-i} \) (resp. \( u_i \)) is a right lower block of order \( n - i \) (resp. \( i \)) for \( u \in U \).

Note that for every \( 1 \leq i \leq k - 1 \) the determinant \( J_{k,i} \) is the left lower
corner minor of order \( k \) of the matrix \( Y_i \). The formula (1) implies that \( J_{k,i} \) is an \( U \)-invariant. \( \square \)

Consider a Zariski open subset \( \Omega \subset M \), that consists of all matrices obeying \( J_k \neq 0 \) for any \( 1 \leq k \leq n \). Denote by \( L \) the subset of matrices of form

\[
B = \begin{pmatrix}
0 & 0 & 0 & \ldots & b_{n,0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & b_{3,0} & \ldots & b_{n,n-3} \\
0 & b_{2,0} & b_{3,1} & \ldots & b_{n,n-2} \\
b_{1,0} & b_{2,1} & b_{3,2} & \ldots & b_{n,n-1}
\end{pmatrix}
\]

with entries from the field \( K \), \( b_{j,0} \neq 0 \) for \( 1 \leq j \leq n \). It is well known that \( L \subset \Omega \). Well known that the subset \( \Omega \) is invariant with respect to \( \text{Ad}_U \) and each \( \text{Ad}_U \)-orbit of \( \Omega \) intersects \( L \) in a unique point.

Since \( \Omega \) is dense in \( M \), we see that any \( f \in K(M)^U \) is uniquely determined by its restriction on \( L \). The map \( \pi \), that takes a rational function \( f \in K(M)^U \) its restriction on \( L \), is an embedding of the field \( K(M)^U \) into the field \( K(L) \).

Construct the matrix

\[
S = \begin{pmatrix}
0 & 0 & 0 & \ldots & s_{n,0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & s_{3,0} & \ldots & s_{n,n-3} \\
0 & s_{2,0} & s_{3,1} & \ldots & s_{n,n-2} \\
s_{1,0} & s_{2,1} & s_{3,2} & \ldots & s_{n,n-1}
\end{pmatrix},
\]

where \( s_{k,i} \) is a restriction of the corresponding element of the matrix \( X \) on \( L \) (more precisely, \( x_{a,b} \) with \( a = i - k + n + 1 \), \( b = k \)). It is obvious that \( s_{k,i}(B) = b_{k,i} \). The system of coordinate functions \( \{ s_{k,i} \} \) is algebraically independent and generate the field \( K(L) \).

Order \( \{ s_{i,j} \} \) in the following way

\[
s_{1,0} < \ldots < s_{n,0} < s_{2,1} < s_{3,2} < s_{3,1} < \ldots < s_{n,n-1} < \ldots < s_{n,1}. \tag{2}
\]

**Proposition 2.** For any pair \( (k,i) \), where \( 1 \leq k \leq n \), \( 0 \leq i \leq k - 1 \) there exist the rational functions \( \phi_{k,i} \neq 0 \) and \( \psi_{k,i} \) of coordinate functions \( \{ s_{a,b} \} \) that are smaller than \( s_{k,i} \) in the sense (2) such that

\[
\pi(J_{k,i}) = \phi_{k,i} s_{k,i} + \psi_{k,i}. \tag{3}
\]

**Proof.** We use the induction method on the coordinate functions given by (2). The statement is true for \( i = 0 \). Assume that the statement is proved for all coordinate functions \( < s_{k,i} \), where \( i \geq 1 \); let us prove the statement for \( s_{k,i} \).
Denote by $S_{k,i}$ the block of order $(k-i)$ of the matrix $S$, that is an intersection of last $k-i$ rows and columns $\{i+1, \ldots, k\}$. Let $C_i$ be the left lower corner block of order $i$ of the adjugate matrix $S^*$. By direct calculations we obtain

\[
\pi(J_{k,i}) = \begin{vmatrix} * & S_{k,i} \\ C_i & 0 \end{vmatrix}.
\] (4)

The coordinate function $s_{k,i}$ is in the right upper corner of block $S_{k,i}$. All other coordinate functions of $S_{k,i}$ are smaller than $s_{k,i}$. The minor $|S_{k,i}|$ does not equal to zero. The minor $|C_i|$ is a monomial of $s_{1,0} \cdots s_{n,0}$. Calculating (4) we obtain (3). \quad \Box

**EXAMPLE 1.** Case $n=2$, \quad $S = \begin{pmatrix} 0 & s_{2,0} \\ s_{1,0} & s_{21} \end{pmatrix}$, \quad $S^* = \begin{pmatrix} * & -s_{2,0} \\ -s_{1,0} & 0 \end{pmatrix}$,

\[
\pi(J_{1,0}) = s_{1,0}, \quad \pi(J_{2,0}) = -s_{1,0}s_{2,0},
\]

\[
\pi(J_{2,1}) = \begin{vmatrix} s_{1,0} & s_{2,1} \\ -s_{1,0} & 0 \end{vmatrix} = s_{1,0}s_{2,1}.
\]

**EXAMPLE 2.** Case $n=3$,

\[
S = \begin{pmatrix} 0 & 0 & s_{3,0} \\ 0 & s_{2,0} & s_{3,1} \\ s_{1,0} & s_{2,1} & s_{3,2} \end{pmatrix}, \quad S^* = \begin{pmatrix} * & * & s^*_{3,0} \\ * & s^*_{2,0} & 0 \\ s^*_{1,0} & 0 & 0 \end{pmatrix},
\]

\[
\pi(J_{1,0}) = s_{1,0}, \quad \pi(J_{2,0}) = -s_{1,0}s_{2,0}, \quad \pi(J_{3,0}) = -s_{1,0}s_{2,0}s_{3,0},
\]

\[
\pi(J_{2,1}) = -s^*_{1,0}s_{2,1}, \quad \pi(J_{3,2}) = \begin{vmatrix} s_{1,0} & s_{2,1} & s_{3,2} \\ * & s^*_{2,0} & 0 \\ s^*_{1,0} & 0 & 0 \end{vmatrix} = s^*_{1,0}s^*_{2,0}s_{3,2},
\]

\[
\pi(J_{3,1}) = \begin{vmatrix} 0 & s_{2,0} & s_{3,1} \\ s_{1,0} & s_{2,1} & s_{3,2} \\ s^*_{1,0} & 0 & 0 \end{vmatrix} = s^*_{1,0} \begin{vmatrix} s_{2,0}s_{3,1} \\ s_{2,1}s_{3,2} \end{vmatrix}.
\]

**Theorem.** The field of $U$-invariants of adjoint representation of the group $\text{GL}(n, K)$ is the field of rational functions of $\{J_{k,i} : 1 \leq k \leq n, \ 0 \leq i \leq k-1\}$.

**Proof.** As we say above, the restriction map $\pi$ is an embedding of the field $K(M)^U$ into the field $K(\mathcal{L})$. It follows from proposition 2 that the system
\{\pi(J_{k,i}) : 1 \leq k \leq n, \ 0 \leq i \leq k - 1\} \text{ is algebraically independent and generate the field } K(L). \text{ This implies the statement of theorem. } \Box

The authors are grateful to M.Brion, C. De Concini, D.F.Timashev, D.A.Shmelkin, E.B.Vinberg for useful discussions.

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