THE UNIQUENESS OF THE SOLUTION TO INVERSE INTERPOLATION PROBLEMS

L. V. Veselova and O. E. Tikhonov

Abstract. We prove that an arbitrary Banach couple is uniquely determined by the collection of intermediate spaces that are interpolation spaces for the operators of rank one. From this we deduce a confirmation to the conjecture by Yu. A. Brudny˘ı and N. Ya. Kruglyak that a Banach couple is uniquely determined by the collection of all its interpolation spaces. Some relevant problems are also analyzed.

Introduction

The principal result of this paper is an exhaustive answer to a question which originates from the classical paper by N. Aronszajn and E. Gagliardo [1] (see also [3, §1, Subsection 9]): To what extent is a Banach couple determined by its interpolation spaces? Namely, we prove that an arbitrary Banach couple is uniquely determined by the collection of all its interpolation spaces. Note that Yu. A. Brudny˘ı and N. Ya. Kruglyak conjectured that this holds true and posed the corresponding problem [4, Conjecture 2.2.32 and Problem 2.7.4(a)].

Also, we examine a similar problem for exact interpolation. The two problems are closely connected: the exhaustive solution to the second is based on that to the first. However, as what we mean, these are rather distinct. For example, in the case when the spaces of a Banach couple coincide as linear ones, the first problem is trivial in contrast to that solving the second has required certain efforts (see [5]).

We examine the first problem in Section 2 and the second one in Section 3. In each of the cases, we first analyze the corresponding problem for operators of rank one, and then we obtain the main result of the corresponding section as a corollary. The main results are already published in [7] and [8], but in contrast to those papers we underline the role of rank one operators here.

1. Notation and Preliminaries

In this section, we introduce notation and recall some definitions and facts from the theory of interpolation of linear operators in Banach spaces (see, e.g., [2], [3], or [4]). Also, we present the proofs of some preliminary results.

This note is a revised and translated form of [6].

1991 Mathematics Subject Classification. Primary 46B70; Secondary 46M35.

Key words and phrases. Banach couple, interpolation space, exact interpolation space.

Supported by the Russian Foundation for Basic Research, grants no. 95–01–00025 and no. 98–01–00103.

Typeset by A44S-TEX
For a Banach space $E$, we will denote by $\| \cdot \|_E$ the norm in $E$, by $B_E$ and by $\overline{B}_E^E$ the closed and the open unit ball in $E$ respectively, and by $\overline{D}_E$ the closure in $E$ of a subset $D \subset E$. We will denote by $\| \cdot \|_{E^*}$ the norm in the conjugate space $E^*$.

Let $E$ and $F$ be two Banach spaces with $E \hookrightarrow F$. (Here and subsequently, the notation $E \hookrightarrow F$ for two Banach spaces $E$ and $F$ means that $E$ is embedded linearly and continuously into $F$.) The embedding constant, $\alpha(F, E)$, is defined by

$$\alpha(F, E) = \sup \{ \| x \|_F / \| x \|_E : x \in E \setminus \{0\} \}.$$  

If $\varphi \in F^*$ then, clearly, $\varphi|_E \in E^*$. We write $\| \varphi \|_E$ to denote $\| \varphi|_E \|_{E^*}$ and put

$$\beta(E^*, F^*) = \sup \{ \| \varphi \|_E / \| \varphi \|_{E^*} : \varphi \in F^* \setminus \{0\} \}.$$  

The restriction operator $\varphi \mapsto \varphi|_E$ from $F^*$ into $E^*$ is adjoint to the embedding operator $E \hookrightarrow F$, therefore $\alpha(F, E) = \beta(E^*, F^*)$.

**Lemma 1.1.** Let $E$ and $F$ be Banach spaces with $E \hookrightarrow F$. Suppose that $E$ and $F$ do not coincide as linear spaces. Then there exists a sequence $\{ \varphi_n \} \subset F^*$ such that $\| \varphi_n \|_{E^*} = 1$ and $\| \varphi_n \|_{F^*} \to 0$.

**Proof.** It is well-known (e.g., [2, Lemma I.1.1]) that the unit ball $B_E$ is nowhere dense in $F$. Therefore $nB_E^E$ does not contain $B_F$ for any natural number $n$. Hence there exists $x_n \in B_F$ such that $x_n \notin nB_E^E$. It follows from the Hahn—Banach theorem that there exists $\varphi_n \in F^*$ such that $\| \varphi_n \|_{F^*} = 1$ and

$$\sup\{ |\varphi_n(x)| : x \in nB_E^E \} < \varphi_n(x_n).$$

Hence we have

$$\| \varphi_n \|_{E^*} = \sup\{ |\varphi_n(x)| : x \in B_E \} \leq \varphi_n(x_n) / n \leq 1 / n.$$  

For two Banach spaces $E$ and $F$, the notation $E \simeq F$ will stand for the case when $E$ and $F$ coincide as linear spaces and their norms are equivalent, the notation $E \equiv F$ will stand for the case when the norms are proportional, and $E = F$ will stand for the case when the norms coincide.

Two Banach spaces $X$ and $Y$ are said to form a Banach couple $(X, Y)$ if they both are linearly and continuously embedded into a certain Hausdorff topological vector space. Note that if $X$ and $Y$ form a Banach couple and one of them is embedded into another as a set, then the embedding operator is linear and continuous [2, Lemma I.3.3], and consequently if $X$ and $Y$ coincide as sets, then $X \simeq Y$. A Banach couple $(X, Y)$ is called embedded if one of the spaces is embedded into another.

To each Banach couple $(X, Y)$ can be canonically associated two Banach spaces, the intersection $X \cap Y$ and the sum $X + Y$, with norms defined by

$$\| z \|_{X \cap Y} = \max\{ \| z \|_X, \| z \|_Y \} \quad (z \in X \cap Y)$$

and

$$\| z \|_{X + Y} = \inf\{ \| x \|_X + \| y \|_Y : x \in X, y \in Y, z = x + y \} \quad (z \in X + Y).$$
A Banach space \( Z \) is said to be *intermediate* for a Banach couple \((X, Y)\) if \( X \cap Y \hookrightarrow Z \hookrightarrow X + Y \). The set of all intermediate spaces for a Banach couple \((X, Y)\) will be denoted by \( I(X, Y) \).

We will use without reference the following assertions which hold true for any Banach couple \((X, Y)\):

\[ X \cap Y \text{ is dense in } Y \text{ if and only if } Y \text{ is dense in } X + Y; \]
\[ Y \cap X \text{ is dense in } X \text{ if and only if } X \text{ is dense in } X + Y; \]
\[ X \cap Y \text{ is dense in } X \text{ if and only if } X \cap Y \text{ is dense in } X + Y. \]

For two Banach couples \((X, Y)\) and \((V, W)\), we will write \((X, Y) \sim (V, W)\) if \( I(X, Y) = I(V, W) \). Obviously, \((X, Y) \sim (V, W)\) if and only if \( X \cap Y \simeq V \cap W \) and \( X + Y \simeq V + W \).

**Lemma 1.2** (see [1, the proof of Corollary 10.XIV]). *Suppose that for two Banach couples, \((X, Y)\) and \((V, W)\), the following conditions are fulfilled:*

1) \((X, Y) \sim (V, W), \)
2) \( X \hookrightarrow V, \)
3) \( Y \hookrightarrow W. \)

*Then \( X \simeq V \text{ and } Y \simeq W. \)*

**Lemma 1.3** (e.g., [2, Lemma I.3.4]). *Let, for a Banach couple \((X, Y)\), a subset \( D \subseteq X \cap Y \) be dense in \( X \) and an inequality \( \|x\|_Y \leq c\|x\|_X \) hold on \( D \). Then \( X \) is embedded into \( Y \) with an embedding constant less than or equal to \( c. \)*

Next, consider certain families of linear operators in a Banach couple and relevant classes of the intermediate spaces.

A linear operator \( T \) in \( X + Y \) is called a *bounded linear operator on a Banach couple \((X, Y)\)* if \( T \) maps \( X \) into \( X \) and \( Y \) into \( Y \) boundedly. Let \( L((X, Y)) \) stand for the set of all such operators. \( L((X, Y)) \) is a Banach space with respect to the norm

\[ \|T\|_{L((X, Y))} = \max\{|T|_{X \rightarrow X}, |T|_{Y \rightarrow Y}\}. \]

Let \( R1((X, Y)) \) stand for the subset of the operators of rank one.

A Banach space \( Z \in I(X, Y) \) is called an *interpolation space* relative to a Banach couple \((X, Y)\) if \( T(Z) \subset Z \) for every \( T \in L((X, Y)) \). The collection of all interpolation spaces for a Banach couple \((X, Y)\) will be denoted by \( \text{Int}(X, Y) \). It is well-known that if \( Z \in I(X, Y) \), \( T \in L((X, Y)) \), and \( T(Z) \subset Z \), then \( T|_{Z} \) is a bounded operator in \( Z \). Moreover, if \( Z \in \text{Int}(X, Y) \), then there exists a constant \( C > 0 \) such that the following “interpolation inequality” holds true for any \( T \in L((X, Y)) \):

\[ \|T\|_{Z \rightarrow Z} \leq C\|T\|_{L((X, Y))}. \]

If the inequality holds with \( C = 1 \), then \( Z \) is said to be an *exact interpolation space.* The collection of all exact interpolation spaces for a Banach couple \((X, Y)\) will be denoted by \( \text{Int}_{1}(X, Y) \).

When one restricts himself to considering operators of rank one, he has to distinguish between the concept of invariant spaces and that of interpolation spaces. We will say that a space \( Z \in I(X, Y) \) is *invariant* for the operators of rank one on a Banach couple \((X, Y)\) if \( T(Z) \subset Z \) for every \( T \in R1((X, Y)) \). The collection
of all such spaces will be denoted by \( R1-\text{Inv}(X,Y) \). If there exists, in addition, a constant \( C > 0 \) such that
\[
\|T\|_{Z \rightarrow Z} \leq C\|T\|_{L((X,Y))}
\]
for any \( T \in R1((X,Y)) \), then we will say that \( Z \) is an interpolation space for the operators of rank one. The collection of all such spaces will be denoted by \( R1-\text{Int}(X,Y) \). If the last inequality holds with \( C = 1 \), then we will say that \( Z \) is an exact interpolation space for the operators of rank one, and we will denote the collection of all such spaces by \( R1-\text{Inv}_1(X,Y) \).

It is immediate from the above definitions that
\[
\text{Int}_1(X,Y) \subset \text{Int}(X,Y) \subset I(X,Y),
\]
\[
R1-\text{Int}_1(X,Y) \subset R1-\text{Int}(X,Y) \subset R1-\text{Inv}(X,Y) \subset I(X,Y),
\]
and
\[
\text{Int}(X,Y) \subset R1-\text{Int}(X,Y).
\]
Note also that \( X \cap Y, X + Y \in \text{Int}_1(X,Y) \).

**Proposition 1.4.** Let \((X,Y)\) and \((V,W)\) be two Banach couples. Then
\[
\mathcal{J}(V,W) \subset \mathcal{J}(X,Y) \iff V,W \in \mathcal{J}(X,Y),
\]
where \( \mathcal{J} \) may be substituted by any of the symbols \( \text{I}, \text{R1-Inv}, \text{R1-Int}, \text{Int}, \text{R1-Int}_1, \) or \( \text{Int}_1 \).

**Proof.** Let us examine the case \( \mathcal{J} = \text{R1-Inv} \). The implication \( \implies \) is trivial. Let us prove the \( \iff \). Suppose that \( V,W \in \text{R1-Inv}(X,Y) \). Let \( Z \in \text{R1-Inv}(V,W) \) and \( T \in R1((X,Y)) \). Then \( T \) maps \( V \) into \( V \) and \( W \) into \( W \) boundedly. It follows that \( TV + TW \in R1((V,W)) \), hence \( T(Z) \subset Z \). Thus \( Z \in \text{R1-Inv}(X,Y) \).

Consider another case, \( \mathcal{J} = \text{Int}_1 \) for instance. Let us prove the \( \iff \). Suppose that \( V,W \in \text{Int}_1(X,Y) \). Let \( Z \in \text{Int}_1(V,W) \) and \( T \in L((X,Y)) \). We have \( \|T\|_{V \rightarrow V} \leq \|T\|_{L((X,Y))} \) and \( \|T\|_{W \rightarrow W} \leq \|T\|_{L((X,Y))} \). It follows that \( \|T\|_{Z \rightarrow Z} \leq \|T\|_{L((V,W))} \leq \|T\|_{L((X,Y))} \). Thus \( Z \in \text{Int}_1(X,Y) \).

The rest of the cases are treated similarly.

Let \((X,Y)\) and \((V,W)\) be two Banach couples. We write \((X,Y) \simeq (V,W)\) if either \( X \simeq V \) and \( Y \simeq W \) or \( X \simeq W \) and \( Y \simeq V \). We write \((X,Y) \cong (V,W)\) if either \( X \cong V \) and \( Y \cong W \) or \( X \cong W \) and \( Y \cong V \).

Let \( \mathcal{J} \) stand for one of the symbols: \( \text{Int}, \text{R1-Inv}, \text{R1-Int}, \) or \( \text{I} \). We say that a Banach couple \((X,Y)\) is uniquely determined by a collection \( \mathcal{J}(X,Y) \) if the implication
\[
\mathcal{J}(X,Y) = \mathcal{J}(V,W) \implies (X,Y) \simeq (V,W)
\]
holds true for any Banach couple \((V,W)\). Let \( \mathcal{U}(\mathcal{J}) \) stand for the class of all such couples \((X,Y)\).

Now, let \( \mathcal{J} \) stand for either \( \text{Int}_1 \) or \( \text{R1-Int}_1 \). We say that a Banach couple \((X,Y)\) is uniquely determined by a collection \( \mathcal{J}(X,Y) \) if the implication
\[
\mathcal{J}(X,Y) = \mathcal{J}(V,W) \implies (X,Y) \cong (V,W)
\]
holds true for any Banach couple \((V,W)\). Similarly to the above, we will denote by \( \mathcal{U}(\mathcal{J}) \) the class of all such couples \((X,Y)\).

It is clear that the inverse implications in (1) and (2) are always hold true.
Proposition 1.5.
  a) $\mathfrak{I}(I) \subset \mathfrak{I}(R_1-{\text{Inv}}) \subset \mathfrak{I}(R_1-{\text{Int}}) \subset \mathfrak{I}(\text{Int})$.
  b) $\mathfrak{I}(R_1-{\text{Int}}_1) \subset \mathfrak{I}(\text{Int}_1)$.

Proof. Let us prove that $\mathfrak{I}(R_1-{\text{Inv}}) \subset \mathfrak{I}(R_1-{\text{Int}})$. Let $(X, Y) \in \mathfrak{I}(R_1-{\text{Inv}})$ and let, for a Banach couple $(V, W)$, it hold

$$R_1-{\text{Int}}(V, W) = R_1-{\text{Int}}(X, Y).$$

By making use of Proposition 1.4, we get

$$R_1-{\text{Inv}}(V, W) = R_1-{\text{Inv}}(X, Y).$$

Hence $(V, W) \simeq (X, Y)$. Thus $(X, Y) \in \mathfrak{I}(R_1-{\text{Int}})$.

The other inclusions can be analyzed similarly.

Remark. A Banach couple $(X, Y)$ with $X \simeq Y$ gives a trivial example of a couple from $\mathfrak{I}(I)$.

2. The Uniqueness of the Solution to Inverse Problem of Interpolation of Linear Operators

Throughout this section, we will denote, for short, $X \cap Y$ and $X + Y$ by $\Delta$ and $\Sigma$, respectively. Also, we will denote by $\overline{Z}$ the closure in $\Sigma$ of a space $Z \in I(X, Y)$.

First, let us study when a Banach couple $(X, Y)$ is uniquely determined by the collection $R_1-{\text{Inv}}(X, Y)$.

Proposition 2.1. Let $(X, Y)$ be a Banach couple and $Z \in I(X, Y)$. Then $Z \in R_1-{\text{Inv}}(X, Y)$ if and only if at least one of the following four conditions is satisfied:

  (i) $\Delta \hookrightarrow Z \hookrightarrow \overline{\Delta}$,
  (ii) $X \hookrightarrow Z \hookrightarrow \overline{X}$,
  (iii) $Y \hookrightarrow Z \hookrightarrow \overline{Y}$,
  (iv) $Z \simeq \Sigma$.

Proof. By [1, Theorem 7.IV], if $Z \in \text{Int}(X, Y)$ then at least one of the four conditions is satisfied. However, it was actually proved there that one of (i)–(iv) was satisfied whenever $Z \in R_1-{\text{Inv}}(X, Y)$. Therefore, it remains to show that each of the conditions entails that $Z \in R_1-{\text{Inv}}(X, Y)$.

Let $T$ be an operator of rank one in $\Sigma$, which maps boundedly $X$ into $X$ and $Y$ into $Y$. Necessary, $T$ is of the form $T(\cdot) = \varphi(\cdot)x$ with $\varphi \in \Sigma^*$, $x \in \Sigma$. The following two cases are possible: $x \in \Delta$ and $x \notin \Delta$. In the first case, $T$ maps boundedly $Z$ into $Z$ for every $Z \in I(X, Y)$. Now, let $x \notin \Delta$. We will show that if one of the conditions (i)–(iv) is fulfilled then $T$ maps $Z$ into itself.

First, let (i) be satisfied. Since $T$ has to map $\Delta$ into itself, $\varphi|_{\Delta} = 0$, therefore $\varphi|_{\overline{\Delta}} = 0$, and hence $T$ maps $Z$ into itself.

Next, let (ii) be satisfied. If $x \in X$ then, clearly, $T$ maps $Z$ into itself. If $x \notin X$ then $\varphi|_{X} = 0$, therefore $\varphi|_{Z} = 0$, and hence $T$ maps $Z$ into itself.

The case (iii) is similar to (ii), and (iv) is trivial.

Remark. The proposition shows that a Banach couple is not determined in general by the collection of all its intermediate spaces which are invariant for the operators of rank one. In fact, if $(X, Y)$ is a regular Banach couple, then $R_1-{\text{Inv}}(X, Y) = R_1-{\text{Inv}}(\Delta, \Sigma) = I(X, Y)$, and if the couple is non-embedded, then $(X, Y) \not\simeq (\Delta, \Sigma)$. 
Lemma 2.2. Suppose that for two Banach couples, \((X, Y)\) and \((V, W)\), the following four conditions are fulfilled:

1) \(X \hookrightarrow Y\),
2) \(X\) is not dense in \(Y\),
3) \((X, Y) \sim (V, W)\),
4) \(\text{R1-Inv}(V, W) \subset \text{R1-Inv}(X, Y)\).

Then \((X, Y) \simeq (V, W)\).

Proof. To prove the lemma, it suffices to examine all the positions of \(V, W \in \text{R1-Inv}(X, Y)\) relative to \((X, Y)\), which are admissible by Proposition 2.1.

Lemma 2.3. Suppose that the following three relations hold for a Banach couple \((X, Y)\) and a non-embedded Banach couple \((V, W)\):

1) \(\Delta\) is dense in neither \(X\) nor \(Y\),
2) \((X, Y) \sim (V, W)\),
3) \(\text{R1-Inv}(V, W) \subset \text{R1-Inv}(X, Y)\).

Then \((X, Y) \simeq (V, W)\).

Proof. First, note that neither \(X\) nor \(Y\) is dense in \(\Sigma\). If one examines all admissible positions of \(V, W \in \text{R1-Inv}(X, Y)\) relative to \((X, Y)\) and rejects the cases which entail the embeddedness of \((V, W)\), then he easily sees that it suffices to consider the four cases only:

(i) \(\Delta \hookrightarrow V \hookrightarrow \overline{X}, \Delta \hookrightarrow W \hookrightarrow \overline{X}\);
(ii) \(X \hookrightarrow V \hookrightarrow \overline{X}, X \hookrightarrow W \hookrightarrow \overline{X}\);
(iii) \(\Delta \hookrightarrow V \hookrightarrow \overline{X}, \Delta \hookrightarrow W \hookrightarrow \overline{X}\);
(iv) \(X \hookrightarrow V \hookrightarrow \overline{X}, Y \hookrightarrow W \hookrightarrow \overline{Y}\).

Each of the cases (i), (ii), and (iii) leads to \(V + W \hookrightarrow \overline{X}\), but this contradicts the fact that \(V + W \simeq \Sigma\) and \(\overline{X} \not\in \Sigma\). By Lemma 1.2, (iv) implies \((X, Y) \simeq (V, W)\), which completes the proof.

It is easy to deduce the following theorem from Lemma 2.2 and Lemma 2.3.

Theorem 2.4. Suppose that a non-regular Banach couple satisfies one of the two conditions:

a) \((X, Y)\) is embedded,

b) \(\Delta\) is dense in neither \(X\) nor \(Y\).

Then \((X, Y) \in \mathfrak{U}(\text{R1-Inv})\).

We now turn to proving the fact that an arbitrary Banach couple \((X, Y)\) is uniquely determined by the collection \(\text{R1-Int}(X, Y)\).

Lemma 2.5. Suppose that for two Banach couples, \((X, Y)\) and \((V, W)\), the following conditions are satisfied:

1) \((X, Y) \sim (V, W)\),
2) \(\Delta\) is dense in \(Y\),
3) \(X \hookrightarrow V\),
4) \(V \not\simeq \Sigma\),
5) \(W \in \text{R1-Int}(X, Y)\).

Then \(X \simeq V\) and \(Y \simeq W\).

Proof. By Lemma 1.1, we can choose a sequence \(\{\varphi_n\} \subset \Sigma^*\) such that \(\|\varphi_n\|_{\Sigma} = 1\) and \(\|\varphi_n\|^Y \to 0\). Then from the condition 3) we obtain \(\|\varphi_n\|^X \to 0\). By making use of the well-known relation \(B_{\Sigma}^X = \text{conv}(B_{X}^{\varphi}, B_{Y}^{\varphi})\), we conclude that \(\|\varphi_n\|^Y \to 1\).
Since $\Sigma \simeq V + W$, there exists a constant $c > 0$ such that $\|\varphi_n\|^{V+W} \geq c$ for all $n$. Since $\|\varphi_n\|^V \to 0$, it follows that there exist a constant $c_1 > 0$ and a natural $n_0$ such that $\|\varphi_n\|^W \geq c_1$ as $n \geq n_0$.

By 5), there exists a constant $C > 0$ such that

$$\|T\|_{W \to W} \leq C\|T\|_{L((X,Y))}$$

for any $T \in R1((X,Y))$. Let us take an arbitrary $x \in \Delta$ and consider the linear operators $T_n$ of rank one given by the formula $T_n(\cdot) = \varphi_n(\cdot)x$. Then we have

$$\|T_n\|_{X \to X} = \|\varphi_n\|^X\|x\|_X \to 0;$$
$$\|T_n\|_{Y \to Y} = \|\varphi_n\|^Y\|x\|_Y \to \|x\|_Y;$$
$$\|T_n\|_{W \to W} = \|\varphi_n\|^W\|x\|_W \geq c_1\|x\|_W \text{ as } n \geq n_0.$$ 

It follows that if we take $n$ sufficiently large then we have

$$c_1\|x\|_W \leq \|T_n\|_{W \to W} \leq C\max\{\|T_n\|_{X \to X}, \|T_n\|_{Y \to Y}\} \leq 2C\|x\|_Y.$$ 

Hence $\|x\|_W \leq (2C/c_1)\|x\|_Y$ for all $x \in \Delta$. Since $\Delta$ is dense in $Y$, we have $Y \subset W$ by Lemma 1.3. Now, application of Lemma 1.2 completes the proof.

**Theorem 2.6.** Suppose that for two Banach couples, $(X,Y)$ and $(V,W)$, the following conditions are satisfied:

1) $(X,Y) \sim (V,W)$,
2) each of the couples $(X,Y)$ and $(V,W)$ is non-embedded,
3) $R1-Int(V,W) \subset R1-Int(X,Y)$.

Then $(X,Y) \simeq (V,W)$.

**Proof.** To prove the theorem, we analyze the three cases separately.

a) Suppose, first, that $\Delta$ is dense in neither $X$ nor $Y$. In this case, the assertion of the theorem follows from Lemma 2.3, since $R1-Int(V,W) \subset R1-Int(X,Y)$ implies $R1-Inv(V,W) \subset R1-Inv(X,Y)$ (see Proposition 1.4).

b) Next, suppose that $\Delta$ is dense in $Y$, but not in $X$. Recall that in this case $X$ is dense in $\Sigma$, but $Y$ is not. We examine all the positions of $V,W \in R1-Inv(X,Y)$ relative to $(X,Y)$, which are admissible by Proposition 2.1, and reject the positions which entail that $(V,W)$ is embedded. We see that it suffices to treaty the six cases:

- (b1) $\Delta \to V \to \overline{X}, \Delta \to W \to \overline{X}$;
- (b2) $Y \to V \to \overline{Y}, \Delta \to W \to \overline{X}$;
- (b3) $X \to V \to \overline{X}, \Delta \to W \to \overline{X}$;
- (b4) $Y \to V \to \overline{Y}, Y \to W \to \overline{Y}$;
- (b5) $X \to V \to \overline{X}, X \to W \to \overline{X}$;
- (b6) $X \to V \to \overline{X}, Y \to W \to \overline{Y}$.

Each of the cases (b1), (b2), and (b4) entails that $V + W \to \overline{Y}$, but this contradicts the fact that $V + W \simeq \Sigma$ and $\overline{\Sigma} \not\simeq \Sigma$. The case (b5) yields $X \to V \cap W$, which contradicts the fact that $V \cap W \simeq \Delta$ and $\overline{\Delta} \not\simeq X$. By Lemma 1.2, (b6) gives $X \simeq V$ and $Y \simeq W$. It remains to treaty (b3). In this case, all the assumptions of Lemma 2.5 turn out to be satisfied. Therefore, $(X,Y) \simeq (V,W)$.

c) Suppose, finally, that $\Delta$ is dense in both $X$ and $Y$. We will analyze two possibilities, c1) and c2).
c1) Let \( Y + V \simeq \Sigma \). We proceed similarly to the proof of Lemma 2.5. We choose a sequence \( \{ \varphi_n \} \subset \Sigma^* \) such that \( \| \varphi_n \|^{\Sigma} = 1 \) and \( \| \varphi_n \|^{V} \to 0 \). Then \( \| \varphi_n \|^\Sigma \to 1 \) and there exist a constant \( c > 0 \) and a natural \( n_0 \) such that \( \| \varphi_n \|^\Sigma \geq c \) as \( n \geq n_0 \). For \( x \in \Delta \), we estimate the norms of the operators \( T_n \) defined by \( T_n(\cdot) = \varphi_n(\cdot)x \), in each of the spaces \( V, X, \) and \( Y \). Then from \( V \in \text{Int}(X,Y) \) we infer that there exists a constant \( c_1 \) such that \( \|x\|_V \leq c_1\|x\|_X \) for all \( x \in \Delta \). But \( \Delta \) is dense in \( X \), and consequently \( X \subset V \). Now, as in the case (b3), all the assumptions of Lemma 2.5 turn out to be satisfied. Therefore, \( (X,Y) \simeq (V,W) \).

c2) Let \( Y + V \nsimeq \Sigma \). Again, we proceed similarly to the proof of Lemma 2.5. Here we take a sequence \( \{ \varphi_n \} \subset \Sigma^* \) such that \( \| \varphi_n \|^{\Sigma} = 1 \) and \( \| \varphi_n \|^{Y+V} \to 0 \). Then \( \| \varphi_n \|^Y \to 0 \) and \( \| \varphi_n \|^V \to 0 \). It follows that \( \| \varphi_n \|^X \to 1 \) and that there exist a constant \( c > 0 \) and a natural \( n_0 \) such that \( \| \varphi_n \|^W \geq c \) as \( n \geq n_0 \). For \( x \in \Delta \), we estimate the norms of the operators \( T_n(\cdot) = \varphi_n(\cdot)x \) in \( W, X, \) and \( Y \) and conclude that there exists a constant \( c_1 \) such that \( \|x\|_W \leq c_1\|x\|_X \) for all \( x \in \Delta \). From this we obtain \( X \subset W \), and application of Lemma 2.5 (with interchanged \( V \) and \( W \)) completes the proof.

Remark. N. Aronszajn and E. Gagliardo said in [1, Remark 10.XV] that they did not know of any example of two Banach couples \( (X,Y) \) and \( (V,W) \) with \( (X,Y) \nsimeq (V,W) \) and satisfying the following conditions:

1) \( (X,Y) \sim (V,W) \),
2) \( (X,Y) \nsimeq (\Delta, \Sigma) \),
3) \( (V,W) \nsimeq (\Delta, \Sigma) \),
4) \( \text{Int}(V,W) \subset \text{Int}(X,Y) \).

Taking into account Proposition 1.4, one sees that if two Banach couples satisfy all of 1)-4) then they satisfy the assumptions of Theorem 2.6. Therefore, the realization of the conditions 1)-4) does yield \( (X,Y) \simeq (V,W) \).

**Theorem 2.7.** Let \( (X,Y) \) be a regular non-embedded Banach couple. Then neither \( X \) nor \( Y \) belongs to \( \text{R1-Int}(\Delta, \Sigma) \).

**Proof.** Suppose on the contrary that \( Y \in \text{R1-Int}(\Delta, \Sigma) \), for instance.

Again, we proceed similarly to the proof of Lemma 2.5. We choose a sequence \( \{ \varphi_n \} \subset \Sigma^* \) such that \( \| \varphi_n \|^\Sigma = 1 \) and \( \| \varphi_n \|^X \to 0 \). Then \( \| \varphi_n \|^Y \to 1 \) and \( \| \varphi_n \|^\Delta \to 0 \). For \( x \in \Delta \), we estimate the norms of the operators \( T_n(\cdot) = \varphi_n(\cdot)x \) in each of the spaces \( Y, \Delta, \) and \( \Sigma \). Taking into account the fact that \( Y \in \text{R1-Int}(\Delta, \Sigma) \), we infer that there exists a constant \( c \) such that \( \|x\|_Y \leq c\|x\|_\Sigma \) for all \( x \in \Delta \). But \( \Delta \) is dense in \( \Sigma \), and consequently \( \Sigma \subset Y \). This contradicts the assumption of that \( (X,Y) \) is not embedded.

By combining Theorem 2.4, Theorem 2.6, Theorem 2.7, and Proposition 1.5, we obtain the following theorem and corollary, which concludes the section.

**Theorem 2.8.** Let \( (X,Y) \) and \( (V,W) \) be two Banach couples with

\[
\text{R1-Int}(X,Y) = \text{R1-Int}(V,W).
\]

Then \( (X,Y) \simeq (V,W) \).

**Corollary 2.9** [7]. Let \( (X,Y) \) and \( (V,W) \) be two Banach couples with

\[
\text{Int}(X,Y) = \text{Int}(V,W).
\]

Then \( (X,Y) \simeq (V,W) \).
3. The Uniqueness of the Solution to Inverse Problem of Exact Interpolation

As proportional norms on a linear space define the same operator norm, we will replace norms by proportional ones when this is convenient. This will involve no loss of generality. In particular, embedding constants will be frequently assumed to equal 1.

**Lemma 3.1.** Suppose that two Banach couples, \((X,Y)\) and \((V,W)\), satisfy the following conditions:
1) \(X \simeq V, Y \simeq W\);
2) \(X \not\cong X + Y\);
3) \(W \in \text{R1-Int}(X,Y), \ Y \in \text{R1-Int}(V,W)\).
Then \(Y \cong W\).

**Proof.** We examine two cases separately:

a) \(X\) is not dense in \(X + Y\),
b) \(X\) is dense in \(X + Y\).

a) Since \(X\) is not dense in \(X + Y\), we can find \(\varphi_0 \in (X+Y)^*\) such that \(\|\varphi_0\|_{X+Y} = 1\) and \(\varphi_0|_X = 0\). It is easy to deduce from \(B^X_{X+Y} = \text{conv}(B^X_X, B^Y_Y)\) that \(\|\varphi_0\|_Y = 1\). Replacing the norm in \(W\) by a proportional one if necessary, we may and do assume that \(\|y_0\|_Y = \|y_0\|_W = 1\) for some \(y_0 \in Y (\simeq W)\). Consider the linear operator of rank one given by the formula \(T_0(\cdot) = \varphi_0(\cdot)y_0\). Then we have

\[
\|T_0\|_{X \to X} = \|T_0\|_{V \to V} = 0,
\]
\[
\|T_0\|_{Y \to Y} = \|\varphi_0\|_Y \|y_0\|_Y = 1,
\]
\[
\|T_0\|_{W \to W} = \|\varphi_0\|_W \|y_0\|_W = \|\varphi_0\|_W.
\]
By 3), it follows that \(\|\varphi_0\|_W = 1\).

For an arbitrary \(y \in Y \cong W\), consider the operator given by \(S_0(\cdot) = \varphi_0(\cdot)y\). We have

\[
\|S_0\|_{X \to X} = \|S_0\|_{V \to V} = 0,
\]
\[
\|S_0\|_{Y \to Y} = \|y\|_Y,
\]
\[
\|S_0\|_{W \to W} = \|y\|_W.
\]
By 3), it follows that \(\|y\|_Y = \|y\|_W\), i.e., \(Y \cong W\).

b) Arguments similar to those in the proof of Lemma 2.5 show that we can choose a sequence \(\{\varphi_n\} \subset (X + Y)^*\) such that \(\|\varphi_n\|_X \to 0\) and \(\|\varphi_n\|_Y = 1\). Then, from 1), we obtain \(\|\varphi_n\|_V \to 0\) and \(c_1 \leq \|\varphi_n\|_W \leq c_2\) for some \(c_1, c_2 > 0\).

Since \(X\) is dense in \(X + Y\), \(X \cap Y\) is dense in \(Y\). In particular, \(X \cap Y \neq \{0\}\). Without loss of generality we assume that \(\|x_0\|_Y = \|x_0\|_W = 1\) for some \(x_0 \in X \cap Y\) (\(\simeq V \cap W\)). Then the norms of the rank one operators \(T_n(\cdot) = \varphi_n(\cdot)x_0\) satisfy

\[
\|T_n\|_{X \to X} = \|\varphi_n\|_X \|x_0\|_X \to 0,
\]
\[
\|T_n\|_{Y \to Y} = \|\varphi_n\|_Y \|x_0\|_Y = 1,
\]
\[
\|T_n\|_{V \to V} = \|\varphi_n\|_V \|x_0\|_V \to 0,
\]
\[
\|T_n\|_{W \to W} = \|\varphi_n\|_W \|x_0\|_W = \|\varphi_n\|_W.
\]
By 3), it follows that $\|\varphi_n\|^W = 1$ for all $n$ is large enough.

For an arbitrary $x \in X \cap Y$ and sufficiently large $n$'s, the norms of operators $S_n(\cdot) = \varphi_n(\cdot)x$ satisfy

$$
\|S_n\|_{X \to X} = \|\varphi_n\|^X \|x\|_X \to 0,
$$

$$
\|S_n\|_{Y \to Y} = \|x\|_Y,
$$

$$
\|S_n\|_{V \to V} = \|\varphi_n\|^V \|x\|_V \to 0,
$$

$$
\|S_n\|_{W \to W} = \|x\|_W.
$$

By 3), it follows that $\|x\|_Y = \|x\|_W$. Since $X \cap Y$ is dense in $Y (\simeq W)$, we have $Y = W$.

**Proposition 3.2.** Suppose that an embedded Banach couple $(X, Y)$ and another Banach couple $(V, W)$ satisfy the following conditions:

1) $(X, Y) \simeq (V, W)$;

2) $R1$-$\text{Int}_2(X, Y) = R1$-$\text{Int}_1(V, W)$.

Then $(X, Y) \cong (V, W)$.

The proof of the proposition is based on a sequence of lemmas which we present below. Throughout the lemmas and the proof of the proposition we assume that $X \hookrightarrow Y$ with $\alpha(Y, X) = 1$. Moreover, we fix a sequence $\{u_n\} \subset X$ with $\|u_n\|_Y = 1$ and $\|u_n\|_X^{-1} \to \alpha(Y, X) = 1$ as $n \to \infty$. We also fix a sequence $\{\theta_n\} \subset Y^*$ with $\|\theta_n\|^Y = 1$ and $\theta_n(u_n) = \|u_n\|_Y = 1$. By Lemma 3.3 below, $\|\theta_n\|^X \to \beta(X^*, Y^*) = 1$.

**Lemma 3.3.** Let $A$ be a bounded linear operator from a Banach space $E$ into a Banach space $F$. Suppose that we take $x_n \in E \setminus \{0\}$ and $\varphi_n \in F^* \setminus \{0\}$ such that $\|A x_n\|_F / \|x_n\|_E \to \|A\|$ ($n \to \infty$) and $\varphi_n(A x_n) = \|\varphi_n\|_F \|A x_n\|_F$. Then $\|A^* \varphi_n\|_F / \|\varphi_n\|_F \to \|A^*\|$ ($n \to \infty$).

**Proof.** The proof is straightforward.

**Lemma 3.4.** Let $Z \in R1$-$\text{Int}_1(X, Y)$. Then the following relations hold:

(i) $\|u_n\|_Y / \|u_n\|_Z$ (=$\|u_n\|_Z^{-1}$) $\to \alpha(Y, Z)$,

(ii) $\|\theta_n\|_Z / \|\theta_n\|_Y$ (=$\|\theta_n\|_Y^{-1}$) $\to \beta(Z^*, Y^*)$,

(iii) $\|u_n\|_Z / \|u_n\|_X \to \alpha(Z, X)$,

(iv) $\|\theta_n\|_X / \|\theta_n\|_Z \to \beta(X^*, Z^*)$.

**Proof.** For an arbitrary $\varphi \in Y^* \setminus \{0\}$, consider the linear operators $T_n$ of rank one defined by the formula $T_n(\cdot) = \varphi(\cdot)u_n$. Then we have

$$
\|T_n\|_{Y \to Y} = \|\varphi\|^Y \|u_n\|_Y = \|\varphi\|^Y,
$$

$$
\|T_n\|_{X \to X} = \|\varphi\|^X \|u_n\|_X \leq \|\varphi\|^Y \|u_n\|_X,
$$

$$
\|T_n\|_{Z \to Z} = \|\varphi\|^Z \|u_n\|_Z.
$$

Since $Z \in R1$-$\text{Int}_1(X, Y)$, it follows that

$$
\|\varphi\|^Z \|u_n\|_Z \leq \|\varphi\|^Y \|u_n\|_X.$$
Thus \(\|u_n\|_X/\|u_n\|_Z \geq \|\varphi\|^Z/\|\varphi\|^Y\) for every \(\varphi \in Y^* \setminus \{0\}\), therefore

\[
\|u_n\|_X/\|u_n\|_Z \geq \beta(Z^*, Y^*) = \alpha(Y, Z).
\]

Hence we obtain

\[
\alpha(Y, Z) \geq \frac{\|u_n\|^Y}{\|u_n\|^Z} = \frac{1}{\|u_n\|} \frac{\|u_n\|^X}{\|u_n\|^Z} \geq \|u_n\|^{-1} \alpha(Y, Z).
\]

Since \(\|u_n\|^{-1}_X \to 1\), it follows that \(\|u_n\|^{-1}_Z \to \alpha(Y, Z)\), and (i) is proved.

The relation (ii) follows from (i) and Lemma 3.3.

The proof of (iv) is a slight modification of that of (i). Namely, instead of \(T_n\) we consider for an arbitrary \(x \in X \setminus \{0\}\) the operators \(S_n\) defined by \(S_n(\cdot) = \theta_n(\cdot)x\).

To prove (iii), we note that

\[
\alpha(Z, X)\alpha(Y, Z) = \beta(X^*, Z^*)\beta(Z^*, Y^*)
\]

\[
= \lim \frac{\|\theta_n\|^X}{\|\theta_n\|^Z} \lim \frac{\|\theta_n\|^Z}{\|\theta_n\|^Y} = \lim \frac{\|\theta_n\|^X}{\|\theta_n\|^Y} = 1.
\]

Hence

\[
\frac{\|u_n\|_Z}{\|u_n\|_X} = \frac{\|u_n\|^Y/\|u_n\|^X}{\|u_n\|^Y/\|u_n\|^Z} \to \frac{1}{\alpha(Y, Z)} = \alpha(Z, X).
\]

**Lemma 3.5.** Let \(R1\text{-}\text{Int}_1(X, Y) = R1\text{-}\text{Int}_1(Z, Y)\). Then \(X \cong Z\).

**Proof.** We can and do assume that \(\alpha(Y, Z) = 1\). Clearly \(X \simeq Z\). Applying part (iii) and then part (i) of Lemma 3.4, we obtain for an arbitrary \(x \in X \setminus \{0\}\):

\[
\frac{\|x\|_Z}{\|x\|_X} \leq \alpha(Z, X) = \lim \frac{\|u_n\|_Z}{\|u_n\|_X} = \lim \frac{\|u_n\|^Y/\|u_n\|^X}{\|u_n\|^Y/\|u_n\|^Z} = 1
\]

Similarly \(\|x\|_X/\|x\|_Z \leq 1\). Thus \(X = Z\).

**Proof of Proposition 3.2.** If \(X\) and \(Y\) do not coincide as linear spaces then it suffices to apply Lemma 3.1 and Lemma 3.5.

Now, let \(X \simeq Y\). Then \(X \simeq Y \simeq V \simeq W\) and also \(X^* \simeq Y^* \simeq V^* \simeq W^*\).

From (i) in Lemma 3.4, we have \(\|u_n\|^{-1}_V \to \alpha(Y, V)\) and \(\|u_n\|^{-1}_W \to \alpha(Y, W)\). By interchanging the roles of \(X\) and \(Y\) in Lemma 3.4, we see that there exists a sequence \(\{\eta_n\} \subset X^*\) such that

\[
\|\eta_n\|^V = 1, \quad \frac{1}{\|\eta_n\|} \to \beta(Y^*, X^*),
\]

\[
\frac{1}{\|\eta_n\|^V} \to \beta(Y^*, V^*), \quad \text{and} \quad \frac{1}{\|\eta_n\|} \to \beta(Y^*, W^*).
\]

Consider the operators \(T_n\) of rank one defined by the formula \(T_n(\cdot) = \eta_n(\cdot)u_n\). It is easy to calculate their norms in each of the spaces \(X, Y, V,\) and \(W\). Then, from \(\text{Int}_1^N(X, Y) = \text{Int}_1^N(V, W)\), we obtain

\[
\max\{1, \|u_n\|_X \|\eta_n\|^X\} = \max\{\|u_n\|_W \|\eta_n\|^W, \|u_n\|_V \|\eta_n\|^V\}.
\]
Note that
\[ \lim \max \{1, \|u_n\|_X \|\eta_n\|_X^X\} = 1, \]
for otherwise we can find \( \varepsilon > 0 \) and an infinite family of indices \( \{n_k\} \) such that
\[ \|u_{n_k}\|_X \|\eta_{n_k}\|_X^X > 1 + \varepsilon. \]
Hence
\[ \alpha(Y, X)\alpha(X, Y) = \alpha(Y, X)\beta(Y^*, X^*) = \lim(\|u_n\|_X \|\eta_n\|_X^X)^{-1} < 1, \]
which is impossible.

Clearly \( V \) or \( W \) (say, \( W \)) satisfies the following: There exists an increasing infinite sequence of indices \( \{n_k\} \) such that
\[ \max \{1, \|u_{n_k}\|_X \|\eta_{n_k}\|_X^X\} = \|u_{n_k}\|_W \|\eta_{n_k}\|_W^W. \]
We can now write
\[ 1 = \lim_{k \to \infty} \|u_{n_k}\|_W \|\eta_{n_k}\|_W^W = 1/(\alpha(Y, W)\beta(Y^*, W^*)) = 1/(\alpha(Y, W)\alpha(W, Y)). \]
Thus \( \alpha(Y, W)\alpha(W, Y) = 1 \), and hence \( Y \cong W \). An application of Lemma 3.5 completes the proof.

**Proposition 3.6.** Suppose that two Banach couples, \((X, Y)\) and \((V, W)\), satisfy the conditions:
1) \((X, Y) \simeq (V, W)\),
2) \(\text{R1-Int}_1(X, Y) = \text{R1-Int}_1(V, W)\).
Then \((X, Y) \cong (V, W)\).

**Proof.** The case of an embedded couple \((X, Y)\) has been analyzed in Proposition 3.2. In the case where \((X, Y)\) is not embedded, we have \(X \not\cong X + Y\) and \(Y \not\cong X + Y\), therefore it suffices to apply Lemma 3.1 twice.

Finally, we present the complete analogues of Theorem 2.8 and Corollary 2.9.

**Theorem 3.7.** Let \((X, Y)\) and \((V, W)\) be two Banach couples with
\[ \text{R1-Int}_1(X, Y) = \text{R1-Int}_1(V, W). \]
Then \((X, Y) \cong (V, W)\).

**Proof.** Proposition 1.4 shows that if \(\text{R1-Int}_1(V, W) = \text{R1-Int}_1(X, Y)\) then \(\text{R1-Int}(V, W) = \text{R1-Int}(X, Y)\). By Theorem 2.8, the latter yields \((V, W) \simeq (X, Y)\), and Proposition 3.6 gives \((V, W) \cong (X, Y)\).

**Corollary 3.8**[8]. Let \((X, Y)\) and \((V, W)\) be two Banach couples with
\[ \text{Int}_1(X, Y) = \text{Int}_1(V, W). \]
Then \((X, Y) \cong (V, W)\).

**Proof.** It suffices to apply Proposition 1.5 b).

**Concluding Remarks and Unsolved Problems.**
1. The reformulated Theorem 2.8, Corollary 2.9, Theorem 3.7, and Corollary 3.8 can be summarized as follows: Each of the \(U(\text{Int}_1), U(\text{Int}), U(\text{Int}_1^N), \) and \(U(\text{Int}^N)\) exhausts all Banach couples.
2. In the remark that concludes Section 1, we have adduced the example of a Banach couple from \(U(I)\). Obviously, any non-embedded Banach couple does not belong to \(U(I)\). We believe that to characterize the Banach couples from \(U(I)\) is an interesting and nontrivial problem.
3. Another problem that remains unsolved is to describe \(U(\text{R1-Inv})\) (cf. Theorem 2.4 and Remark after Proposition 2.1).
ACKNOWLEDGMENT

We are indebted to N. M. Zobin who attracted our attention to inverse interpolation problems. We wish to express gratitude to Yu. A. Brudnyi, N. Ya. Kruglyak, and E. I. Pustylnik for valuable discussions and our warmest appreciation to M. Cwikel for his interest in this research.

REFERENCES

1. N. Aronszajn and E. Gagliardo, Interpolation spaces and interpolation methods, Ann. Mat. Pura ed Appl. 68 (1965), 51–117.
2. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, Interpolation of linear operators, “Nauka”, Moscow, 1978 (Russian); English transl., Amer. Math. Soc., Providence, 1982.
3. Yu. A. Brudnyi, S. G. Krein, and E. M. Semenov, Interpolation of linear operators, in Itogi Nauki i Tekhniki. Matem. Analiz 24, Moscow, 1986, pp. 3–163 (Russian); English transl., J. of Soviet Math. 42 (1988), 2009–2113.
4. Yu. A. Brudnyi and N. Ya. Kruglyak, Interpolation functors and interpolation spaces I, North-Holland, 1991.
5. L. V. Veselova and O. E. Tikhonov, On the uniqueness of the solution of the inverse exact interpolation problem, Funktsional'nyi Analiz i ego Prilozheniya 26 (1992), no. 2, 67–68 (Russian); English transl., Funct. Anal. Appl. 26 (1992), 129–131.
6. L. V. Veselova and O. E. Tikhonov, The uniqueness of the solution to inverse interpolation problems, Research Institute of Mathematics and Mechanics, Preprint no. 95–2, Kazan Mathematics Foundation, Kazan, 1995. (Russian)
7. O. E. Tikhonov and L. V. Veselova, A Banach couple is determined by the collection of its interpolation spaces, Proc. Amer. Math. Soc. 126 (1998), 1049–1054.
8. O. E. Tikhonov and L. V. Veselova, The uniqueness of the solution to the inverse problem of exact interpolation, Israel Math. Conf. Proc. 13 (1999), 208–214 (to appear).