Letter

Hydrodynamics of the atomic Bose–Einstein condensate beyond the mean-field approximation

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Abstract
Several hydrodynamic models of the atomic Bose–Einstein condensate (BEC) obtained beyond the mean-field approximation are discussed together from a single point of view. All these models are derived from the microscopic quantum description. The derivation is made within the many-particle quantum hydrodynamics method suggested by L Kuz’menkov. The derivation is demonstrated and discussed for the mean-field regime revealing the Gross–Pitaevskii equation as the simplest illustration. It appears in the first order by the interaction radius. Generalization of the hydrodynamic Euler equation obtained in the third order by the interaction radius is discussed. It includes the contribution of the isotropic short-range interaction (SRI) presented by the third space derivative of the square of concentration. The Euler equation also includes the contribution of the anisotropic part of the SRI proportional to the second order spherical function. A systematic account of the quantum fluctuations in terms of the many-particle quantum hydrodynamics method requires the extension of the set of hydrodynamic equations from the couple continuity and Euler equations to the set of four equations which also includes the pressure evolution equation and the evolution equation for the third rank tensor of the pressure flux. The pressure evolution equation contains no interaction contribution in the first order by the interaction radius. The source of the quantum fluctuations is in the interaction caused term existing in the third rank tensor evolution equation which is obtained in the first order by the interaction radius. The presented models are considered for the single-component BECs.

Keywords: quantum hydrodynamics, pressure evolution equation, extended hydrodynamics, quantum fluctuations, mean-field approximation

(Some figures may appear in colour only in the online journal)

1. Introduction
The majority of physical effects in the Bose–Einstein condensates (BECs) of ultracold atomic gases can be described by the Gross–Pitaevskii (GP) equation [1–5]. Which is the nonlinear Schrödinger equation with the cubic nonlinearity found in the mean-field approximation for the short-range interaction (SRI). Here, the interaction is reduced to the scattering of atoms with the zeroth transfer of momentum. Therefore, the interaction is characterized by the s-wave scattering length a. The GP equation can be represented in the form of hydrodynamic equations for the curl-free ‘fluid’.
In recent years, there has been a growth of the number of phenomena in neutral BECs which requires model obtained beyond the mean field approximation. Therefore, several generalized approaches are presented in literature.

The famous example of the analytical model obtained beyond the mean-field approximation is the model of quantum fluctuations in BECs [6]. It includes the presence of bosons in the excited states at the zero temperature [6]. The generalization of this model for the dipolar BEC (see [7] and [8]) is the crucial instrument in the modeling of the quantum-droplets formation [9–11], which are observed experimentally. The contribution of quantum fluctuations can be approximately presented in the generalized GP equation, which contains the fourth-degree nonlinearity along with the third degree nonlinearity. While the model of quantum fluctuations included in the GP equation for the dipolar BECs is the dominating model in the description of the redistribution of the cloud of the trapped bosons to the number of smaller clouds [11], the three-particle interaction is also studied as the mechanism for the formation of the self-bound states [12].

The characteristic feature of the interaction of neutral atoms is the short range of the interaction (except the dipolar BEC). This short range is comparable with the radius of atoms, but it is small in compare with the average interparticle distance or the characteristic length of the collective effects such as the wavelength of waves and the width of solitons. Therefore, we can expand the general functions describing the interaction on the small parameter which is the nontrivial interparticle distance, where the interaction is nonzero. At the larger distances the potential goes to zero, it is not relevant. The main contribution appears in the first order on the small parameter (the first order by the interaction radius). It leads to the GP equation.

However, we can consider the next orders of the expansion. The non-zero contribution appears in the third order by the interaction radius (TOIR) [13]. It is obtained under the assumption that all bosons are in the quantum state with the lowest energy. Therefore, this generalization of the GP model does not related to the quantum fluctuations described above. The terms existing in the TOIR are easily presented in terms of the hydrodynamic model, but this approximation does not allow to obtain any non-linear Schrödinger equation. The TOIR approximation is presented by the higher derivatives of the concentration in the Euler equation. Moreover, it introduces the second interaction constant. The interaction constant in the GP equation is the zeroth moment of the potential of interatomic interaction:

\[
g = \int dr U(r).
\]  

This is the single interaction constant for the single-component BECs. The TOIR approximation gives the second interaction constant which is proportional to the second moment of the potential

\[
g_2 \sim \int dr^2 U(r).
\]  

The notion of moment of the function is similar to the similar notion in the probability theory, but the similarity exists just in the structure of the integrals. Different approach to the non-local generalization of the GP equation is presented in [14, 15]. Same interaction constant (2) appears at the consideration of the p-wave scattering in quantum gases [16, 17], while the interaction constant (1) is associated with the s-wave scattering [1]. The higher-order generalization of the GP model is also considered in recent paper [18], where the model is developed in similarity with the Swift–Hohenberg theory of pattern formation.

The term in the Euler equation appearing in the TOIR approximation leads to novel types of solitons in BECs [19, 20]. Actually, it creates conditions for existence of the bright soliton in the repulsive BECs [19]. Similar phenomenon is found experimentally in [21].

If we include the anisotropy of the SRI it leaves no trace in the mean-field model obtained in the first order by the interaction radius (the GP approximation). However, the part of anisotropy described by the spherical function of the second order \(Y_{20}\) leads to the additional term in the TOIR approximation [20]. It also introduces the third interaction constant.

It is noticed that the collisions of solitons in BECs requires modeling of BEC beyond mean-field approximation [22]. To this end, corresponding models are developed [23–25]. They are based on the microscopic dynamics traced through the numerical methods.

It is possible to derive the quantum fluctuations purely from the quantum hydrodynamics based on the microscopic quantum dynamics of bosons. It requires to extend the set of hydrodynamic equations from two equations (the continuity and Euler equations) to the set of four equations (the continuity equation, the Euler equation, the pressure evolution equation, and the third rank tensor evolution equation). The pressure is the second rank tensor constructed as the average of the product of two velocities in the comoving frame. The third rank tensor is the flux of pressure in the comoving frame, hence it is the average of the product of three velocities. The nonzero velocities in the comoving frame is the trace of the particles in the excited states. The third rank tensor evolution equation contains the contribution of the SRI which has nonzero value in the first order by the interaction radius at the zero temperature. It is proportional to novel (fourth) interaction constant, which is proportional to the zeroth moment of the second derivative of the potential:

\[
g_4 \sim \int dr U''(r).
\]  

The third rank tensor evolution equation includes the source existing at the zero temperature caused by the SRI. Therefore, it is interpreted as the quantum fluctuations.

The quantum hydrodynamic method [13, 20, 26] leads to finite (but almost infinite) set of hydrodynamic equations or, in other words the conservation laws. But, in recent years, other methods of similar set of equations are developed. See for instance [27], where a hydrodynamic theory for infinitely
many conservation laws is derived in close relation to the processes of local relaxation to the Gibbs state. It is applied there to the study of heat flows in some integrable models.

We described some results obtained within the quantum hydrodynamics method. However, the ‘generalized hydrodynamics’ is also method of description of collective phenomena including quantum effects [28, 29]. Particularly, the quantum fluctuations in BECs are discussed within this method [28]. Mentioned hydrodynamic models are described in more details in the text below.

The major part of discussion presented above is focused on different models. Let us point out some phenomena in quantum gases which require application of beyond mean-field models for their description. First, let us repeat the existence of the self-bound states or quantum droplets related to the quantum fluctuations in BECs. This is a phenomenon in which the cloud of trapped atoms being in the BEC state splits on the number of smaller clouds. So, the large scale instability leads to the local collapses of BEC is stabilized at smaller scales. This phenomenon can occur in the mixture of two nondipolar BECs [30], where the quantum fluctuations related to the SRI play essential role. The self bound states can be related to the three-particle interaction described by the nonlinear Schrödinger equation with the fifth order nonlinearity [12]. The self-bound states are observed for the highly dipolar atoms [11], where two parts of quantum fluctuation related to the SRI and the dipole–dipole long range interaction contribute in the formation of droplets.

Second, the novel types of bright solitons in compare with the mean-field are possible in the repulsive BECs [21]. This phenomenon is predicted in terms of the beyond mean-field model, where the higher derivatives of the concentration are included in the Euler equation [19]. But it can also modeled within the three-particle interaction [21]. Such similarity between the three-particle interaction and the contribution of the higher derivatives of concentration is discussed in [20]. This similarity rises the question on the possibility of the formation of the self-bound droplets in the model with the higher derivatives of concentration. The third branch of the beyond mean-field phenomena is the proper description of collisions between mean-field solitons [22–25]. While extensive analysis of the solitons in the mean-field regime is given in [31].

The number of atoms in the experiments with quantum gases in relatively low. It is of order of 10⁴ atoms. Hence the thermodynamical limit traditional for majority of theories of collective phenomena is not possible. It leads to variations from the mean-field model. Particularly, the correlation effects are essential for few-boson systems and finite atomic ensembles [32–37].

This paper is organized as follows. In section 2 the mean-field approximation including the derivation of the GP equation by the quantum hydrodynamic method is described. In section 3 the generalized equations of quantum hydrodynamics obtained beyond mean-field approximation are shown. In section 4 a brief summary of the obtained results is presented.

### 2. Mean-field model of BEC

BECs of different atoms are obtained over last 25 years. Earlier experiments were focused on alkali atoms, including sodium-23 and rubidium-87. Recent experiments are made with BECs of lanthanide atoms, such as ¹⁶⁴Dy, where self-bound states are observed.

The paper is focused on the models of atomic BEC beyond mean-field approximation. However, we start of paper with the description of the mean-field approximation as the solid background for the generalizations.

The BEC is the collection of bosons being in the single quantum state with the minimal energy. Weakly interacting atomic BEC of neutral particles interacting via the SRI can be approximately described by the well-known GP equation [1]:

\[
\hat{H} = -\frac{\hbar^2}{2m} \nabla \Phi + V_{\text{ext}} \Phi + g |\Phi|^2 \Phi, \tag{4}
\]

where \(\hbar\) is the Planck constant, \(m\) is the mass of particle, \(i\) is the imaginary unit, \(\partial_t\) is the derivative on time, \(\nabla\) is the Laplace operator, \(V_{\text{ext}}\) is the potential of the external field, \(g\) is the interaction constant (1), which can be interpreted via the scattering length \(a\): \(g = 4\pi\hbar^2a/m\), \(\Psi\) is the effective macroscopic wave function. Equation (4) is the non-linear Schrödinger equation, where the nonlinearity is caused by the interaction of bosons. Here, the interaction is presented approximately via the major term. GP equation (4) can be represented in the form of hydrodynamic equation.

Well-known derivation of the GP equation from the Schrödinger equation is given via the psi-operators, which corresponds to the quantum mechanics in the representation of the second quantization [38]. In this derivation the psi-operators are reduced to the functions of coordinate and time and the interaction potential is replaced by the quasi-potential.

Another derivation of the GP equation directly from the many-particle Schrodinger equation in the coordinate representation is made via the quantum hydrodynamics method [13]. Let us present this derivation here.

#### 2.1. Derivation of the GP equation within the quantum hydrodynamic method

The derivation of the GP equation is splitted on two parts. First, derivation of general form of hydrodynamic equations is presented. Second, the GP equation is found at the approximate analysis of term describing the interaction.

#### 2.1.1. Derivation of the general structure of the continuity and Euler equations from the microscopic quantum motion

Microscopic quantum dynamics of systems of particles is described within the Schrödinger equation \(i\hbar \partial_t \psi(R,t) = \hat{H} \psi(R,t)\). Different quantum systems differ from each other by the explicit form of the Hamiltonian \(\hat{H}\) and by the symmetry of wave function \(\psi(R,t)\) relatively to permutations of arguments.
In this paper, we consider the single species of neutral spin-0 bosons. Therefore, the wave function is symmetric relatively permutations of any pair of arguments. Masses of all particles are equal to each other \( m \equiv m \). System of interacting bosons placed in the external field with the potential \( V_{\text{ext}}(r_i, t) \) is described by the following Hamiltonian in the coordinate representation:

\[
H = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m_i} + \sum_{i=1}^{N} V_{\text{ext}}(r_i, t) + \frac{1}{2} \sum_{i,j,i \neq j} U(|r_i - r_j|),
\]

where \( N \) is the number of particles, \( \hat{p}_i = -i\hbar \nabla_i \) is the momentum operator of the \( i \)th particle, and \( U(|r_i - r_j|) \) is the interaction potential. The number of particles is included in the wave function via the number of arguments \( R = \{r_1, \ldots, r_N\} \).

We present the derivation in accordance with the many-particle quantum hydrodynamics method suggested by L. Kuz’menkov [26] (see also [39]), which is adopted for the ultracold gases in [13, 40–42].

The concentration (number density) of particles is the simplest fundamental hydrodynamic function for the classical and quantum description. It can be defined in the vicinity of the point \( r \) of the physical space as the operator \( \hat{n} = \sum_{i=1}^{N} \delta (r - r_i) \) averaged with the many-particle microscopic wave function \( \psi(R, t) \):

\[
n(r, t) = \int dR \sum_i \delta(r - r_i) \psi^*(R, t) \psi(R, t), \tag{6}
\]

where \( dR = \prod_{i=1}^{N} dr_i \) is the element of volume in \( 3N \) dimensional configurational space.

Operator \( \hat{n} \) is obtained by quantization of the classic microscopic concentration of particles, which is the sum of delta-functions. The coordinates of classic particles are replaced by the operator of coordinate in accordance with the quantization rule. The coordinate operator of \( i \)-th particle has trivial form in the coordinate representation of quantum mechanics \( \hat{r}_i = r_i \). Let us point out that the delta-function is the traditional element of the quantum description. One of major examples is the eigen-function of the coordinate operator in the coordinate representation of quantum mechanics.

Hydrodynamics describes the time evolution of the physical systems. Therefore, it is necessary to derive equation for the concentration evolution. Number of particles is conserved in our description since we neglect the molecular formation which happens in the traps and leads to the destruction of the condensate cloud. Hence, it is expected that the concentration satisfies the continuity equation. However, it is essential to derive the continuity equation since the derivation gives us the explicit form of the second hydrodynamic function in terms of the many-particle wave function \( \psi(R, t) \).

We consider the time derivative of the concentration (6). The time derivative acts on the argument of wave functions which can be found from the Schrödinger equation with Hamiltonian (5). After straightforward calculations we obtain that the term under consideration is the divergence of the following vector function:

\[
\mathbf{j}^\alpha(r, t) = \int dR \sum_i \delta(r - r_i) \times \frac{1}{2m_i} [ (\hat{p}_i^\alpha | \psi^*(R, t) \psi(R, t) + \psi^*(R, t) (\hat{p}_i^\alpha \psi)(R, t) ],
\]

where index \( \alpha \) is the tensor index (all Greek indexes below have meaning of tensor indexes), which stands for Cartesian coordinates \( x, y, z \). Vector function \( \mathbf{j}^\alpha \) is the particle current proportional to the local momentum density of the bosons. We find that the concentration of bosons (6) satisfies the continuity equation:

\[
\partial_t n + \nabla \mathbf{j} = 0, \tag{8}
\]

with the current (7).

Microscopic definitions of the concentration and the current allows to derive equations for their evolution, but the hydrodynamic functions cannot be calculated from the definition containing the wave function since the microscopic many-particle Schrödinger equation cannot be solved. The continuity equation shows that the evolution of the concentration is governed by the current, but we need to calculate the evolution of the current as well. Equation for its evolution can be found using the Schrödinger equation with Hamiltonian (5). Let us present the result of calculation:

\[
\partial_t \mathbf{j} = \frac{1}{m} \partial_\beta \Pi^{\alpha \beta} (r, t) = -\frac{1}{m} n(r, t) \nabla^\alpha V_{\text{ext}}(r, t) - \frac{1}{m} \int dr' (\nabla \cdot U(r, r')) n_2(r, r', t). \tag{9}
\]

All hydrodynamic functions are defined as the fields in the three-dimensional physical space \( r \) changing with time \( t \). Therefore, we mostly drop the arguments of these functions. Correlation functions like \( n_2(r, r', t) \) have more complex structure of arguments, hence we keep them explicit.

The left-hand side of equation (9) contains the kinetic functions while the right-hand side contains the interaction. We keep this structure for all hydrodynamic equations below. The first term on the right-hand side describes the interaction of the bosons with the external field. The second term presents the interaction between bosons. Tensor \( \Pi^{\alpha \beta} \) is the quantum tensor of the momentum flux:

\[
\Pi^{\alpha \beta}(r, t) = \int dR \sum_i \delta(r - r_i) \frac{1}{4m_i} [ \psi^*(R, t) (\hat{p}_i^\alpha \hat{p}_i^\beta \psi)(R, t) + (\hat{p}_i^\alpha \psi)^*(R, t) (\hat{p}_i^\beta \psi)(R, t) + \text{c.c.} ], \tag{10}
\]

where \( \text{c.c.} \) is the complex conjugation. It is the symmetric tensor \( \Pi^{\alpha \beta} = \Pi^{\beta \alpha} \).
The interaction between the particles in equation (9) is expressed via the two-particle concentration $n_2(r, r', t)$ normalized over $N(N - 1)$:

$$n_2(r, r', t) = \int dR \sum_{i,j \neq j} \delta(r - r_i)\delta(r' - r_j)\psi^*(R, t)\psi(R, t). \quad (11)$$

Function $n_2(r, r', t)$ splits on the product of concentrations $n(r, t) \cdot n(r', t)$ if there is no two-particle correlations.

Equations (8) and (9) are the fundamental pair of hydrodynamic equations. They obtained for bosons with two-particle potential interaction, but same equations can be obtained for the fermions. Strength of interaction or the radius of the influence of the potential are not included in the derivation at this stage. These properties are included below.

2.2. Short-range of interaction

Our goal is the derivation of the GP equation for the weakly interacting bosons with the SRI between bosons. Hence, we need to include these features in the momentum balance equation (9). First we include the short range nature of the boson-boson interaction. Interaction in equation (9) is written via the two-particle concentration (11). To consider the small area of influence of the potential we need to rewrite the force field via the microscopic wave function $\psi(R, t)$. It helps to trace the symmetry relatively permutations of particles. Original form of the force field is:

$$F^\alpha = -\int dR \sum_{i,j \neq j} \delta(r - r_i)\left(\nabla^\alpha U(r_j)\right)\psi^*(R, t)\psi(R, t). \quad (12)$$

This expression does not depend on $i$ and $j$. Replacing indexes $i$ and $j$ we find different form of the force field:

$$F^\alpha = \int dR \sum_{i,j,i \neq j} \delta(r - r_j)\left(\nabla^\alpha U(r_j)\right)\psi^*(R, t)\psi(R, t). \quad (13)$$

To include the symmetries we consider the half sum of these expressions:

$$F^\alpha = -\frac{1}{2} \int dR \sum_{i,j \neq j} \left(\delta(r - r_i) - \delta(r - r_j)\right)$$

$$\times \left(\nabla^\alpha U(r_j)\right)\psi^*(R, t)\psi(R, t). \quad (14)$$

Small rance of the potential action $U(r_j)$ corresponds to nontrivial value of potential $U(r_j)$ at small $r_j$. The SRI potential goes to zero at the large interparticle distance $r_j$. We need to trace areas of small and large interparticle distances $r_j$ in all functions under the integral in the force field expression (14). Therefore, we introduce the following variables: the interparticle distance for two particles $r_{ij}$, and the center of mass for the pair of particles $R_{ij}$, which are defined as follows:

$$R_{ij} = \frac{1}{2}(r_i + r_j), \quad r_{ij} = r_i - r_j. \quad (15)$$

Therefore, we replace the coordinates of particles $r_i$ and $r_j$ via the relative distance $r_{ij}$ and the center of mass $R_{ij}$. It includes the corresponding substitution of the arguments of the wave function $\psi(R, t) = \psi(r_1, r_2, r_3, r_4, ..., r_N, t)$.

Nonzero value of the function under integral in (14) exists at the small relative distance $r_{ij}$. Therefore, we can expand the function under the integral on relative distance $r_{ij}$ in the Taylor series. The zeroth term of the expansion is equal to zero. Main contribution is given by the first term of expansion. The result can be written in the following form:

$$F^\alpha = -\partial_\beta \sigma^\alpha\beta, \quad (16)$$

where

$$\sigma^\alpha\beta(r, t) = -\frac{1}{2} \int dR \sum_{i,j \neq j} \delta(r - R_{ij})$$

$$\times \frac{r_{ij}^\alpha r_{ij}^\beta}{|r_{ij}|} \frac{\partial U(r_{ij})}{|\partial r_{ij}|} \psi^*(R, t)\psi(R, t), \quad (17)$$

is the quantum stress tensor. The explicit form of the quantum stress tensor (17) shows that it is symmetric with respect to permutation of indexes $\alpha, \beta$: $\sigma^\alpha\beta = \sigma^\beta\alpha$.

The momentum balance equation (9) can be rewritten via the quantum stress tensor:

$$\partial_t f^\alpha + \frac{1}{m} \partial_\beta \left(\Pi^\alpha\beta + \sigma^\alpha\beta\right) = -\frac{n}{m} \nabla^\alpha v_{ext}, \quad (18)$$

where the interparticle interaction is placed on the left-hand side.

Tensor $\Pi^\alpha\beta$ is the kinetic momentum flux. It can be considered in the following way. Consider some area of space surrounded with some surface. Motion of particles leads to the loss and acquisition of particles which cross the surface during their motion. Each particle carries the momentum. Hence, the loss (acquisition) of momentum located in the considered area of space. This mechanism of change of momentum is not related to the interaction. Same mechanism exists in the ideal gas. The quantum stress tensor $\sigma^\alpha\beta$ can be interpreted as the momentum flux caused by the interaction. The interaction of particles within the considered area of space does not change the full momentum in this area. However, the interaction of particles being inside with particles being outside gives the change of momentum in this area of space. We consider the SRI between particles. Hence, the particles located near the surface can interact and transfer the momentum across the surface. The superposition $\Pi^\alpha\beta + \sigma^\alpha\beta$ gives the full momentum flux.

Expression (17) is found for the arbitrary strength of interaction. However, tensor $\sigma^\alpha\beta$ in form (17) has no expression via the hydrodynamic functions. The application of the weak strength of interaction leads to the representation of the quantum stress tensor $\sigma^\alpha\beta$ via the hydrodynamic functions. The weakly interacting limit of bosons is presented in the next subsection. Here, we focus on another item.
2.3. The Madelung transformation and the velocity field

The momentum balance equation (9) or (18) has small resemblance to the traditional Euler equation. The Euler equation is written via the hydrodynamic velocity field \(v\). Consequently, we need to make transition of the form of hydrodynamic equation based on the velocity field.

The microscopic many-particle wave function \(\psi(R, t)\) is the complex function of the real arguments. It can be rewritten via two real functions:

\[
\psi(R, t) = a(R, t) \exp \left( \frac{iS(R, t)}{\hbar} \right),
\]

(19)

where \(a(R, t)\) is the amplitude of the wave function, and \(S(R, t)\) is the phase of the wave function.

Analysis of current (7) leads to the following structure:

\[
v_i(R, t) = \frac{1}{m_i} \nabla_i S(R, t),
\]

(20)

which can be interpreted as the velocity of \(i\)-th quantum particle. Velocity field \(v(r, t)\) is determined via the current and concentration:

\[
j = n v.
\]

(21)

There is nonzero difference between two introduced functions \(v_i(R, t)\) and \(v(R, t)\). It is the velocity of \(i\)-th quantum particle in the comoving frame \(u_i(R, t) = v_i(R, t) - v(R, t)\). It can be interpreted as the quantum analog of the velocity of thermal motion.

Next, we need to separate the thermal motion of the particles with velocities \(\mathbf{u}_i\) and the motion with the hydrodynamic velocity field \(\mathbf{v}(r, t)\) in the continuity equation and the momentum balance equation (18). We obtain to the following equations:

\[
\partial_t n + \nabla \cdot (n \mathbf{v}) = 0,
\]

(22)

and

\[
n \partial_t (\mathbf{v} \nabla - \mathbf{v} \partial) \mathbf{v} + \partial_t (p^{\alpha \beta} + \sigma^{\alpha \beta} + T^{\alpha \beta}) = -n \nabla \mathbf{v} \mathbf{v}_{cd}.
\]

(23)

The introduction of the velocity field in the Euler equation requires some straightforward calculations.

The kinetic momentum flux tensor \(\Pi^{\alpha \beta}\) splits on three part during the introduction of the velocity field:

\[
\Pi^{\alpha \beta} = m^\alpha v^\beta + p^{\alpha \beta} + T^{\alpha \beta}.
\]

(24)

The second term in the expression (24) is the kinetic pressure:

\[
p^{\alpha \beta}(r, t) = \int dR \sum_{i=1}^N \delta(r - r_i) a^2(R, t) m_i u_i^\alpha u_i^\beta.
\]

(25)

It is related to the distribution of particles over quantum states with different values of the momentum.

The last term in equation (24) is the quantum Bohm potential:

\[
T^{\alpha \beta}(r, t) = -\frac{\hbar^2}{2m} \int dR \sum_{i=1}^N \delta(r - r_i) a^2(R, t) \frac{\partial^2 \ln a}{\partial x_{\alpha} \partial x_{\beta}}.
\]

(26)

It is proportional to \(\hbar^2\) and has purely quantum origin. The quantum Bohm potential can be simplified for the system of noninteracting particles located in one quantum states:

\[
T^{\alpha \beta} = -\frac{\hbar^2}{4m} \left( \partial^\alpha \partial^\beta n - \frac{1}{n} (\partial^\alpha n)(\partial^\beta n) \right).
\]

(27)

Divergence of tensor (27) is usually presented in the form:

\[
\partial_\beta T^{\alpha \beta} = -\frac{\hbar^2}{2m} n \partial^\alpha \frac{\Delta \sqrt{n}}{\sqrt{n}}.
\]

(28)

There is no distribution of particles on different quantum states in the BECs (if there is no effects of the quantum fluctuations). Therefore, the pressure is equal to zero:

\[
p^{\alpha \beta} = 0.
\]

(29)

Let us point out that it is the kinetic pressure. Sometimes, at phenomenological analysis, the pressure is introduced at the superposition of the kinetic pressure \(p^{\alpha \beta}\) and the quantum stress tensor \(\sigma^{\alpha \beta}\). Hence, the full pressure \(P^{\alpha \beta} = p^{\alpha \beta} + \sigma^{\alpha \beta}\) is nonzero, but its nonzero part is caused by the interaction located in the quantum stress tensor \(\sigma^{\alpha \beta}\).

2.4. Derivation of the GP equation from the general hydrodynamic equation in the first order by the interaction radius

For the derivation of the GP equation we need to calculate the quantum stress tensor (17) in the weakly interacting regime. The integral over the particle distance in the expression (17) can be separated from other integrals. Hence, the quantum stress tensor (17) can be rewritten in the following form:

\[
\sigma^{\alpha \beta}(r, t) = -\frac{1}{2} Tr(\sigma^{\alpha \beta}(r, t')) \int dr' r' \frac{\partial U(r)}{\partial r}.
\]

(30)

where the trace of function of two arguments is used:

\[
Tr f(r, r') = f(r, r).
\]

(31)

The two-particle concentration (11) is rewritten in the following form:

\[
n_2(r, r', t) = N(N - 1) \int dR_{N-2} \langle n_1, n_2, \ldots | r, r', R_{N-2}, t \rangle \times \langle r, r', R_{N-2}, t | n_1, n_2, \ldots \rangle,
\]

(32)

where \(dR_{N-2} = \prod_{k=3}^N dr_k\) is the element of volume in \(N - 2\) dimensional configurational space.
Equation (32) contains the microscopic many-particle wave function $\psi(R, t)$ which is written in the representation of occupation numbers $n_\alpha$ (the number of particles in the quantum states described by the set of quantum numbers). The delta functions in definition (11) gives the projection of coordinates $r$ and $r_i$ (after replacement via relative distance and the expansion for the small relative distances) in the 3D physical space arithmetization with coordinate $r$. The wave function is written in the following form: $\psi(R, t) = \psi(r_1, r_2, R_{\alpha}, t) \rightarrow \psi(r_1, r_2, R_{\alpha}, t) = \delta(f - q)\delta(f' - q') + \delta(f - q')\delta(f' - q), \tag{35}$ for two particles in two different quantum states, and $\langle n_1, \ldots (n_f - 2), \ldots | n_1, \ldots (n_q - 2), \ldots \rangle = \delta(f - q), \tag{36}$ for two particles in the single quantum state.

Described calculation gives the expression of the two-particle concentration:

$$n_2(r, r', t) = n_n(r, t)n_n(r', t) + |\rho(r, r', t)|^2 + \sum_g n_g(n_g - 1)|\psi_g(r, t)|^2|\psi_g(r', t)|^2, \tag{37}$$

which is partially presented via the concentration written via the single particle wave function in accordance with definition (6) and expansion (33):

$$n(r, t) = \sum_g n_g\psi_g^*(r, t)\psi_g(r, t). \tag{38}$$

Moreover, equation (37) contains the density matrix written via the single particle wave functions:

$$\rho(r, r', t) = \sum_g n_g\psi_g^*(r, t)\psi_g(r', t). \tag{39}$$

The first and second terms in two-particle concentration (37) appear from the first term in expansion (34). They are related to the interaction of two particles being in different quantum states. It described the interaction of BEC with the normal fluid presented by the bosons in the excited states and the selfaction of the normal fluid. The last term in equation (37) appear from the last term in expansion (34). It describes the interaction of two particles being in the same quantum state. It describes the selfaction of the normal fluid if two bosons are in the excited states. The last term in equation (37) also describes the selfaction of BEC.

Next, we use equation (37) for calculation of the quantum stress tensor $\sigma^{\alpha\beta}$ given by equation (30). For this step we take trace of two-particle concentration (37) and find the expression for the quantum stress tensor:

$$\sigma^{\alpha\beta}(r, t) = \frac{1}{2} g \delta^{\alpha\beta} \left[ 2n^n(r, t) + \sum_g n_g(n_g - 1)|\psi_g(r, t)|^4 \right], \tag{40}$$

where we use that the trace of density matrix is equal to the concentration. The integral over the relative distance demonstrated in equation (30) is represented via the interaction constant $g_s^{\alpha\beta}$ which appears in the following form:

$$g_s = -\frac{4\pi}{3} \int dr (r) \frac{\partial U(r)}{\partial r}. \tag{41}$$

We obtain traditional form of the interaction constant (1) by integration by parts, where we use that $r^3U(r)$ tends to zero at $r$ tending to zero and to infinity.

Let us discuss equation (40) for the quantum stress tensor. It contains the square of concentration like the expressions
for the two-particle concentration presented above. However, the concentration square is written in formal way. This term refer to the interaction of particles being in different quantum states while the square of concentration includes the product of the same quantum states. Let us introduce $\tilde{n}^2$, where $\tilde{n}$ is the reduced concentration corresponding to the product of particles being in different quantum states. Hence, equation (40) appears as: $\sigma^{a\beta} = \frac{1}{2} g\delta^{a\beta}[2\tilde{n}^2 + \sum_{k} n_k (n_k - 1)|\varphi_{k}|^4]$. The missing term (corresponding to the product of the same quantum states) can be extracted from the second term $\sum_{k} n_k^2 |\varphi_{k}|^4$. However, we need twice of it. Therefore, the expression for the quantum stress tensor via full concentration is $\sigma^{a\beta} = \frac{1}{2} g\delta^{a\beta}[2n^2 - \sum_{k} n_k (n_k - 1)|\varphi_{k}|^4]$. We can neglect 1 in compare with $n_k$ for the large occupation numbers. The account of the product of the same quantum states does not change the expressions for $\sigma^{a\beta}$. Hence, we neglect them, but we keep the product of the same quantum state for the lowest energy state $\sigma^{a\beta} = \frac{1}{2} g\delta^{a\beta}[2n^2 - n_k^2 |\varphi_{k}|^4]$, where $g_0$ are the quantum numbers for the quantum state with the lowest energy. The last term is the square of the concentration of the BEC $n_{BEC} = n_0 |\varphi_0|^2$ in accordance with equation (38). Complete concentration $n$ is the sum of the concentration of BEC $n_{BEC}$ and the concentration of the normal fluid $n_n$: $n = n_{BEC} + n_n$. It gives the final expression for the quantum stress tensor for finite temperature:

$$\sigma^{a\beta}_{BEC,n} = \frac{1}{2} g\delta^{a\beta}\left(2n_{BEC}n_n + 2n_n^2 + n_{BEC}^2\right).$$

(42)

The quantum stress tensor simplifies at the zero temperature. For the BEC it appears as the single term:

$$\sigma^{a\beta}_{BEC} = \frac{1}{2} g\delta^{a\beta} n_{BEC}^2.$$  

(43)

It allows us to present the truncated version of the Euler equation (23) for the BEC:

$$mn(\partial_t + v \nabla)v - \frac{\hbar^2}{2m} n \nabla \sqrt{n} = -n \nabla V_{ext} - gn \nabla n.$$  

(44)

The concentration $n$ can be dropped in all terms in equation (44).

For the eddy-free motion (the curl-free condition) $v(r,t) = \nabla \phi(r,t)$ the second term in the Euler equation (44) simplifies to $(v \nabla)v = \frac{1}{2} \nabla v^2$. The Euler equation (44) is the final form of the Euler equation for the weakly-interacting BECs of neutral particles.

The GP equation can be found as the representation of hydrodynamic equations (22) and (44) via the non-linear Schrödinger equation. The condition of eddy-free (zero curl of the velocity field) motion allows to find the Cauchy-Lagrangian integral of the Euler equation (44):

$$\partial_t \phi + \frac{1}{2} v^2 + \frac{1}{m} gn - \frac{\hbar^2}{2m^2} \Delta \sqrt{n} + \frac{1}{m} V_{ext} = \text{const.}$$  

(45)

The non-linear Schrödinger equation is obtained for the effective macroscopic wave function which is constructed of the macroscopic hydrodynamic functions:

$$\Phi(r,t) = \sqrt{n(r,t)} \exp\left(\frac{\imath}{\hbar} m \phi(r,t)\right).$$  

(46)

We consider the time derivative of function (46) using the hydrodynamic equations: the continuity equation (22), and the Cauchy-Lagrangian integral of the Euler equation (45). After calculations we find the GP equation:

$$ih \partial_t \Phi = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext} + g |\Phi|^2\right) \Phi.$$  

(47)

It is presented in section 2 (see equation (4)), but we repeat it here as the result of presented derivation. The wave function $\Phi$ is normalized on the number of particles in the system:

$$\int d\mathbf{r} \Phi^* (\mathbf{r},t) \Phi (\mathbf{r},t) = N.$$  

(48)

3. Extended set of the quantum hydrodynamic equations for the BEC: model of BEC beyond the mean-field approximation

The method of derivation demonstrated in section 2 can be applied to the derivation of more general models. Let us present the most general extended quantum hydrodynamics model obtained today for the BECs. This model consists of four equations for tensors of different ranks: the scalar field concentration $n$, the vector field velocity $v$, the second rank tensor field of pressure $P^{a\beta}$, and the third rank tensor field of the pressure flux $Q^{a\beta\gamma}$.

The Euler equation for the velocity field evolution is derived with the account of interaction up to the TOIR. While the pressure evolution equation and the third rank tensor field evolution equation $Q^{a\beta\gamma}$ giving the quantum fluctuations contribution are found in the first order by the interaction radius.

The third rank tensor field $Q^{a\beta\gamma}$ has the following definition in terms of microscopic parameters:

$$Q^{a\beta\gamma}(r,t) = \int d\mathbf{R} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) a^2(R,t) a_\gamma^\dagger a_\beta^\dagger a_i^\dagger,$$  

(49)

where function $a(R,t)$ is the amplitude of the wave function introduced in equation (19). It appears as the flux of the pressure (25) in the comoving space.

Let us present the explicit form of the four described hydrodynamic equations. First, we repeat the continuity equation:

$$\partial_t n + \nabla \cdot (nv) = 0.$$  

(50)
The Euler equation:

\[ mm \partial_t \psi^\alpha + mm (v \cdot \nabla) \psi^\alpha - \frac{\hbar^2}{2m} n \nabla^2 \psi^\alpha = \frac{\hbar^2}{2m} n \Delta \psi^\alpha - mn^2 \nabla \Delta \psi^\alpha + \partial_{\beta\gamma} \psi^\alpha \]

\[ = \frac{g_2}{2} \partial^\alpha \partial_n^2 + \frac{g_3}{2} \partial_{\beta\gamma} \partial_{\alpha\delta} \partial_{n^2} n^2, \]  

(51)
differs from equation (44) by the presence of three terms.

The third and fourth term on the right-hand side are caused by the SRI included up to the TOIR approximation [13, 19, 20]. These terms contain the higher derivatives of the concentration similarly to the quantum Bohm potential given by the third term on the left-hand side. The isotropic part of the interaction in the TOIR approximation should be smaller than the GP term. However, it can be comparable with the quantum Bohm potential. Therefore, the interaction can considerably change the Bohm contribution of the ‘quantum’ term up to changing its sign. It can dramatically change the behavior of some phenomena like the dispersive (nondissipative) shock waves [44–46], since the existence of higher space derivatives is crucial for their existence.

The kinetic pressure tensor \( p_{\alpha\beta}^g \) in the Euler equation (51) is kept nonzero since it is the source of the quantum fluctuations [47]. The subindex \( g \) refers to the nature of this term. Importance of the higher rank tensors for the description of classic and quantum fluid is discussed in [48].

The Euler equation contains the following new notations:

\[ I^0_{\beta\gamma} = \delta^{\beta\gamma} \delta \alpha \gamma + \delta^{\beta\gamma} \delta \beta \delta \gamma + \delta^{\beta\gamma} \delta \delta \beta \gamma, \]  

(52)

and \( I^2_{\beta\gamma} \). Tensor \( I^2_{\beta\gamma} \) is the nonsymmetric tensor. It is partially symmetric. There is the symmetry relatively permutations of the last three indexes. However, there is no symmetry relatively permutations of the first index and other indexes. Tensor \( I^2_{\beta\gamma} \) has the following elements: \( I^{XXX}_{XXX} = I^{XXY}_{XXX} = 1, I^{XXX}_{XYY} = -2 \), \( I^{XXX}_{YXX} = -2/3 \), \( I^{XXX}_{YYX} = I^{XXX}_{XYY} = I^{XXX}_{YXX} = 1/3 \) and allowed permutations of indexes, other elements are equal to zero.

Euler equation (51) contains the following interaction constants:

\[ g_2 = \frac{1}{24} \int r^2 U(r) dr, \]  

(53)

and

\[ g_3 = \frac{1}{24\sqrt{3}} \int r^2 U_2(r) dr. \]  

(54)

Function \( U_2 \) in equation (54) corresponds to the anisotropic part of the SRI potential:

\[ U(r, \theta) = \sqrt{4\pi} \sum_{k=0}^{\infty} Y_{2}(0, \theta) U_2(r), \]  

(55)

where two terms give nonzero contribution in the Euler equation at the account of interaction up to the TOIR approximation: \( U(r, \theta) = U(r) + \sqrt{2} (1 - 3 \cos^2 \theta) U_2(r) \).

The quantum fluctuation reveal themselves via the nonzero value of the pressure. The pressure evolution is calculated in the first order by the interaction radius. It appears that there is no contribution of the interaction in this regime. However, the evolution of the second rank tensor of pressure is caused by the nonzero perturbations of the third rank tensor \( Q_{\alpha\beta}^{\gamma} \):

\[ \partial_t \psi_{\alpha\beta}^{\gamma} + \partial_\gamma (v^\alpha \partial_{\gamma} \psi_{\alpha\beta}^{\gamma}) + p_{\alpha\beta}^{\gamma} \partial_\gamma v^\beta + p_{\beta\gamma}^{\delta} \partial_\gamma \psi_{\delta\alpha}^{\gamma} + \partial_\gamma Q_{\alpha\beta}^{\gamma} = 0. \]  

(56)

No external field depending on the coordinates and time \( V_{\text{ext}}(r, t) \) gives contribution in the pressure evolution equation.

The nonzero value of perturbations of the third rank tensor \( Q_{\alpha\beta}^{\gamma} \) can be found from the third rank tensor evolution equation [47]:

\[ \partial_t Q_{\alpha\beta}^{\gamma} + \partial_\gamma (\psi^\delta Q_{\alpha\beta}^{\gamma}) + Q_{\alpha\beta}^{\gamma} \partial_\gamma v^\beta + \partial_\gamma Q_{\alpha\beta}^{\gamma} \partial_\gamma v^\alpha + Q_{\alpha\beta}^{\gamma} \partial_\gamma v^\beta + \partial_\gamma Q_{\alpha\beta}^{\gamma} \partial_\gamma v^\alpha + \partial_\gamma Q_{\alpha\beta}^{\gamma} \partial_\gamma v^\alpha + \partial_\gamma Q_{\alpha\beta}^{\gamma} \partial_\gamma v^\alpha = \frac{\hbar^2}{4m} \partial_\gamma^2 \partial_\gamma^2 \partial_\gamma^2 \partial_\gamma^2 V_{\text{ext}}, \]  

(57)

where

\[ g_4 = \frac{2}{3} \int dr U''(r). \]  

(58)

The right-hand side of equation (57) is caused by the interaction and the action of the external field. The interaction is calculated in the zero temperature regime in the first order by the interaction radius. It characterized by the independent interaction constant \( g_4 \). The method of calculation of the interaction term in equation (57) is similar to the method described above for the Euler equation (44). The parabolic traps are widely used in experiments on quantum gases \( V_{\text{ext}} = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2. \) However, the pressure flux evolution equation (57) contains the third derivative of the external potential. Hence, the parabolic potentials give no modification of the pressure flux in compare with the uniform medium.

The single component BECs are considered in this paper. However, same formalism can be generalized for the description of mixtures of different BECs, including the binary mixtures (BECs of different atoms or different isotopes of one type of atoms). BEC of each species is described by set of hydrodynamic equations, which also contain interspecies interaction terms.

Let us consider the reduction of equations (56) and (57) for the small amplitude perturbations in form of plane waves \( p_{\alpha\beta} = P \exp(-i \omega t + ikx) \delta^{\alpha\beta} \delta^{3k} \) and \( Q_{\alpha\beta} = P \exp(-i \omega t + ikx) \delta^{\alpha\beta} \delta^{3k} \delta^{3k} \delta^{3k} \) with zero equilibrium values of these two parameters.

The third rank tensor evolution equation gives \( Q_{\alpha\beta}^{XXX} = \frac{\hbar^2}{4m} \partial_\gamma^2 \partial_\gamma^2 \partial_\gamma^2 \partial_\gamma^2 n^2, \) where \( Q_{\alpha\beta}^{XXX} = L^{XXX}_{XXX} = 3 \). We obtain \( P_{\alpha\beta}^{XXX} = \frac{k}{(k/\omega)^2} Q_{\alpha\beta}^{XXX} \) from the pressure evolution equation (56). The Euler equation contains the following contribution of pressure \( \partial_\gamma p_{\alpha\beta}^{XXX} = ik p_{\alpha\beta}^{XXX} \delta^{3k} \). Combine all of these together and find \( \partial_\gamma p_{\alpha\beta}^{XXX} = -\frac{2ik}{4m(\omega /\omega)} \partial_\gamma p_{\alpha\beta}^{XXX} n^2 \). We also use the approximate relation between frequency \( \omega \) and the wave vector \( k: \omega^2 \approx \frac{\hbar \omega}{m} k^2 \).
Hence, we obtain the following reduction of the Euler equation (51):

\[ mn\partial_{\nu}^2 + mn(\nu \cdot \Delta)^2 = \frac{\hbar^2}{2m} n \Delta \sqrt{n} \]

\[ = -n\partial_n V_{ext} - gn\partial_n n + \frac{3g_\lambda\hbar^2}{4m^2 g} \partial_n n. \tag{59} \]

We drop the nonlocal terms for simplicity. The last term in equation (59) gives an approximate form of the contribution of the quantum fluctuations in the Euler equation.

The Lee-Huang-Yang term contains the nonlinear contribution of the scattering length. In the first Born approximation of the scattering theory the scattering length is the integral of the potential. It means that the Lee-Huang-Yang term is nonlinear on the potential. All hydrodynamic functions are defined as the binary function of the many-particle wave function (6), (7) and (10). The time evolution of the wave function is described by the Schrödinger equation. Hence, all evolution equations are linear on the potential of interaction. Some form of the approximate substitution of the pressure from the equations of the pressure evolution and the pressure flux evolution provides some nonlinear contributions of the potential in the Euler equation. Approximate form of the pressure perturbation based on the analysis of small amplitude perturbations is used to find equation (59), but this approach does not match the Lee-Huang-Yang result.

Similar extended hydrodynamic models obtained for degenerate repulsive fermions, where the interaction is included up to the TOIR approximation, and the set of hydrodynamic equations is extended up to the second rank tensor (the pressure evolution equation) [42, 43, 49].

Equation (59) is obtained as the reduction of the extended hydrodynamic model including the pressure evolution equation (56), and the pressure flux third rank tensor evolution equation (57). Application of two equations in addition to the traditional pair of the continuity and Euler equations results in the similarity of spectrum of collective excitation for the single species to the spectrum of two species considered in terms of the minimal coupling hydrodynamic model constructed of the continuity and Euler equations.

The quantum droplet formation is experimentally obtained for the high magnetic dipolar BECs [11], where it is explained via the existence of the superposition of the short-range and dipolar parts of the Lee-Huang-Yang (LHY) energy, which is included in the GP equation. The possibility of the self-bound states in the mixture of two nondipolar BECs is demonstrated in [30], where the mean-field attraction is compensated by the repulsive Lee-Huang-Yang term.

Let us present the GP equation with the LHY energy for better comparison with our model:

\[ \hbar \partial_t \Phi = -\frac{\hbar^2}{2m} \Delta \Phi + (V_{ext} + g |\Phi|^2 + g_{af} |\Phi|^4)\Phi, \tag{60} \]

where

\[ g_{af} = \frac{32}{3\sqrt{\pi}} g \sqrt{a^3}. \tag{61} \]

The last term in equation (60) appears from the interaction caused chemical potential:

\[ \mu_i = gn \left( 1 + \frac{32}{3\sqrt{\pi}} \sqrt{a^3} \right), \tag{62} \]

with the standard relation between the s-scattering amplitude and the interaction constant \( g = 4\pi\hbar^2/alm \). The LHY term is obtained for the uniform Bose gas [50, 51]. However, the local density approximation can be applied to generalize this result for the nonuniform mediums like the trapped BECs [52]. Particularly, the generalization of equation (62) for the dipolar BECs is found in [6] from the Bogoliubov-de-Gennes equations.

The Thomas-Fermi approximation refers to the regime, where higher derivatives (like the quantum Bohm potential term) are dropped. However, the quantum fluctuations described in terms of the extended GP equation with the fourth order nonlinearity appear in the Thomas-Fermi approximation. Presented hydrodynamic model of quantum fluctuations shows that the quantum fluctuations appear in equations via term proportional to the first derivative of the concentration. However, the further generalization of the obtained result beyond the Thomas-Fermi approximation is possible if the TOIR corrections are found in the pressure flux third rank tensor evolution equation (57).

4. Conclusion

This paper has a narrow area of discussion. It is focused on the hydrodynamic description of the BECs and generalizations of the GP equation demonstrating the beyond mean-field effects. It is focused on the models obtained within the ‘quantum hydrodynamics’ method. The described generalizations are based on more detailed account of interaction in the Euler equation. However, it also includes the interaction caused effects in equations for the higher rank tensors, where source of the quantum fluctuations is located.

The presented models demonstrate that the hydrodynamics as the method gives variety of approximations in a systematic way. For instance, the presented equations show that there are two sources for the additional quantum fluctuations. If we continue our calculations in the first order by the interaction radius we can derive the equation for evolution of the fifth rank tensor which provides nontrivial contribution proportional to the following interaction constant: \( g_5 \sim \int U^{(5)}(r)d^3r \), where (4) means the fourth derivative of the potential.

The second source of additional quantum fluctuations is the equation for evolution of the third rank tensor \( Q^{\alpha\beta\gamma} \), which can be generalized up to the TOIR approximation, like it is made for the Euler equation. Moreover, there is the open question: can we obtain some contribution of interaction of bosons in the pressure evolution equation if we consider interaction up to the TOIR at the zero temperature? The quantum fluctuations and the terms appearing in the TOIR approximation are essential for understanding of properties of BECs since they lead to novel phenomena (as described in section 1).
The studies presented in [22–25] formulate the open questions for the further development of the quantum hydrodynamic model. For which approximation would be able to properly describe the soliton collisions.

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