GENERALIZED VOJTA-RÉMOND INEQUALITY

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ABSTRACT. Following and generalizing unpublished work of Ange, we prove a generalized version of Rémond’s generalized Vojta inequality. This generalization can be applied to arbitrary products of irreducible positive-dimensional projective varieties, defined over the field of algebraic numbers, instead of powers of one fixed such variety. The proof runs closely along the lines of Rémond’s proof.

1. Introduction

Let $m \geq 2$ be an integer and let $X_1, \ldots, X_m$ be a family of irreducible positive-dimensional projective varieties, defined over $\overline{\mathbb{Q}}$. We wish to extend Rémond’s results of [10] to the case of an algebraic point $x = (x_1, \ldots, x_m)$ in the product $X_1 \times \cdots \times X_m$. The following article is a further generalization of a generalization of these results by Thomas Ange. It draws heavily on a written account of this generalization by Ange [1].

In [4], we apply our generalized Vojta inequality to a relative version of the Mordell-Lang problem in an abelian scheme $A \to S$, where $S$ is an irreducible variety and everything is defined over $\overline{\mathbb{Q}}$. In the problem, one fixes an abelian variety $A_0$, defined over $\overline{\mathbb{Q}}$, a finite rank subgroup $\Gamma \subset A_0(\overline{\mathbb{Q}})$ and an irreducible closed subvariety $V \subset A$ and studies the points $p \in V$ of the form $\phi(\gamma)$ for an isogeny $\phi : A_0 \to A_{\pi(p)}$, $A_{\pi(p)}$ denoting the fiber of the abelian scheme over $\pi(p)$, and $\gamma \in \Gamma$.

In this application, it is crucial that we allow the $X_i$ to lie in different fibers of the abelian scheme. If the abelian scheme $A$ is constant, an analogue of the intended height bound has been obtained by von Buhren in [12]. In his case, the generalized Vojta inequality from [10], where $X_1 = X_2 = \cdots = X_m = X$, was sufficient, however for our intended application it is necessary to allow the $X_i$ to be different.

Let us recall the hypotheses which come into play. We use (almost) the same notation as in [10] and we refer to that article for the history of Vojta’s inequality.

For an $m$-tuple $a = (a_1, \ldots, a_m)$ of positive integers, we write

$$\mathcal{N}_a = \bigotimes_{i=1}^{m} p_i^* \mathcal{L}_i \otimes^{a_i},$$

where $\mathcal{L}_i$ is a fixed very ample line bundle on $X_i$ and $p_i : X_1 \times \cdots \times X_m \to X_i$ is the natural projection. We fix a non-empty open subset $U^0 \subset X_1 \times \cdots \times X_m$ and relate $a$ to an irreducible projective variety $\mathcal{X}$, provided with an
open immersion $U^0 \subset \mathcal{X}$ and a proper morphism $\pi : \mathcal{X} \to X_1 \times \ldots \times X_m$ such that $\pi|_{U^0} = \text{id}_{U^0}$, as well as to a nef line bundle $M$ on $\mathcal{X}$ which satisfies some further conditions, specified below.

We assume that there exists a very ample line bundle $P$ on $\mathcal{X}$, an injection $P \hookrightarrow N^{' \otimes t_1}$ which induces an isomorphism on $U^0$ and a system of homogeneous coordinates $\Xi$ for $P$ which are (by means of the aforementioned injection) monomials of multidegree $t_1a$ in the homogeneous coordinates $W^{(i)} \subset \Gamma(X_i, L_i)$, fixed in advance (we denote $\pi^*N_a$ also by $N_a$ and identify $p_i^*W^{(i)}$ and $\pi^*p_i^*W^{(i)}$ with $W^{(i)}$). By (a system of) homogeneous coordinates for a very ample line bundle we mean the set of pull-backs of the homogeneous coordinates on some $\mathbb{P}^{N'}$ under a closed embedding into $\mathbb{P}^{N'}$ that is associated to that line bundle.

We also assume that there exists an injection $(P \otimes \mathcal{M}^{-1}) \hookrightarrow N_a^{' \otimes t_2}$ which induces an isomorphism on $U^0$ and that $P \otimes \mathcal{M}^{-1}$ is generated by a family $Z$ of $M$ global sections on $\mathcal{X}$ which are polynomials $P_1, \ldots, P_M$ of multidegree $t_2a$ in the $W^{(i)}$ such that the height of the family of coefficients of all these polynomials, seen as a point in projective space, is at most $\sum_i a_i \delta_i$. The height of any polynomial is defined by considering the family of its coefficients as a point in an appropriate projective space. On projective space, the height is defined as in Definition 1.5.4 of [2] by use of the maximum norm at the archimedean places.

The integer parameters $t_1, t_2, M$ and the real parameters $\delta_1, \ldots, \delta_m$ (all at least 1) are fixed independently of the triple $(a, \mathcal{X}, \mathcal{M})$. This triple permits to define the following two notions of height for an algebraic point $x \in U^0(\overline{\mathbb{Q}})$:

$h_M(x) = h(\Xi(x)) - h(Z(x))$,

$h_{N_a}(x) = a_1h(W^{(1)}(x)) + \ldots + a_mh(W^{(m)}(x))$.

Our goal is to prove an inequality among these two numbers under certain assumptions about the intersection numbers of $\mathcal{M}$. Let therefore $\theta \geq 1$ and $\omega \geq -1$ be two integer parameters and set (with $\omega' = 3 + \omega$)

$\Lambda = \theta(2t_1u_0)^{u_0} \left( \max_{1 \leq i \leq m} N_i + 1 \right) \prod_{i=1}^m \deg(X_i)$,

$\psi(u) = \prod_{j=u+1}^{u_0} (\omega' j + 1)$,

$c_1 = c_2 = \Lambda \psi(0)$,

$c_3^{(i)} = \Lambda^{2\psi(0)}(Mt_2)^{u_0}(h(X_i) + \delta_i) \quad (i = 1, \ldots, m)$,

where $u_0 = \dim(X_1) + \ldots + \dim(X_m)$, $N_i + 1 = \#W^{(i)}$ and the degrees and heights are computed with respect to the embeddings given by the $W^{(i)}$. We use here the (normalized) height of a closed subvariety of projective space as defined in [3] (via Arakelov theory) or [6] (via Chow forms). The two definitions yield the same height by Théorème 3 of [11].

The following theorem therefore generalizes Théorème 1.2 of [10].

**Theorem 1.1.** Let $x \in U^0(\overline{\mathbb{Q}})$ be an algebraic point and $(a, \mathcal{X}, \mathcal{M})$ a triple as defined above. Suppose that, for every subproduct of the form $Y = Y_1 \times \ldots \times$
where we have the following estimate
\[
(\mathcal{M}^{\text{dim}(Y)} \cdot \mathcal{Y}) \geq \theta^{-1} \prod_{i=1}^{m} (\deg(Y_i))^{-\omega_i \text{dim}(Y_i)},
\]
where \(\mathcal{Y}\) denotes the closure of \(\pi^{-1}(Y \cap U^0)\) in \(\mathcal{X}\). Then we have
\[
h_{\mathcal{X}_x}(x) \leq c_1 h_{\mathcal{M}}(x)
\]
if furthermore \(c_2 a_{i+1} \leq a_i\) for every \(i < m\) and \(c_3^{(i)} \leq h(W^{(i)}(x_i))\) for every \(i \leq m\).

The fact that for each \(i\) there is a different constant \(c_3^{(i)}\) is the main difference with Ange’s work, where there is just one \(\delta\) instead of \(\delta_1, \ldots, \delta_m\) (in our set-up, \(\delta\) can be taken as \(\max_{1 \leq i \leq m} \delta_i\)) and there is just one constant \(c_3\) defined as
\[
\Lambda^{2\psi(0)}(Mt_2)^{u_0} \max \left\{ \max_{1 \leq i \leq m} h(X_i), \delta \right\}.
\]
The condition that \(x_i\) has large height then reads \(c_3 \leq h(W^{(i)}(x_i))\). Ange’s inequality is a direct generalization of Rémond’s inequality (up to the slightly different definitions of \(\Lambda\) and \(c_1\)).

If we set \(\delta = \max_{1 \leq i \leq m} \delta_i\), then the inequality \(c_3^{(i)} \leq h(W^{(i)}(x_i))\) follows from \(2c_3 \leq h(W^{(i)}(x_i))\), so Theorem 1.1 really is a generalization of Rémond’s work (up to the factor 2 and the slightly different definitions of \(\Lambda\) and \(c_1\)). In the application in [4], the fact that \(c_3^{(i)}\) depends only on \(h(X_i)\) and \(\delta_i\) and not on \(h(X_j)\) or \(\delta_j\) (\(j \neq i\)) is crucial. Ange’s version of the inequality is therefore not sufficient for the application.

Naturally, we follow the proof in [10] very closely with some minor changes: Firstly, the term \(12\psi(u)\) that appears in the last equation of [10] should be replaced by \(4\psi(u)\); that is why we do not use Lemme 5.4 of [10] and define \(\Lambda\) slightly differently. Secondly, Corollaire 5.1 of [10] does not apply if \(x_j^{(i)} = 0\), which means that Corollaire 3.2 of [10] has to be made more precise. Thirdly, in the last inequality in the proof of Proposition 4.2 of [10], a term bounding the contribution of the archimedean places when the \(P_i\) are raised to the \(d\)-th power is missing. Fourthly, the factor 8 in the upper bound \(8(N + 1)D_i \log(N + 1)D_i\) for \(\log 2f_2(u_i, D_i)\) given in the proof of Proposition 5.3 of [10] has to be increased. Fifthly, we had to impose that \(\mathcal{M}\) is nef in order to be able to translate the lower bound on its top self-intersection number into a lower bound for the dimension of a space of global sections.

2. REDUCTION TO A MINIMAL SUBPRODUCT

We first consider a subproduct \(Y = Y_1 \times \cdots \times Y_m\) of minimal total dimension \(u = u_1 + \cdots + u_m\), satisfying the following conditions:

(i) \(x_i \in Y_i(\bar{\mathbb{Q}})\) for all \(1 \leq i \leq m\);
(ii) \(d_i \leq \deg(X_i)\Lambda^{\psi(u)-1}\) for all \(1 \leq i \leq m\);
(iii) \(\prod_{i=1}^{m} d_i \leq (\prod_{i=1}^{m} \deg(X_i)) \Lambda^{\psi(u)-1}\);
(iv) \(\sum_{i=1}^{m} a_i(h_i + \delta_i) \leq 2^{-1} \Lambda^{2\psi(u)}(Mt_2)^{u_0-u} \sum_{i=1}^{m} (a_i(h(X_i) + \delta_i))\),
where $u_i = \dim(Y_i)$, $d_i = \deg(Y_i)$ and $h_i = h(Y_i)$ (in the projective embedding defined by $W^{(i)}$). Such a subproduct certainly exists, since $X_1 \times \ldots \times X_m$ satisfies these conditions. Furthermore, we have $u > 0$ since otherwise $Y = \{x\}$ and therefore

$$\sum_{i=1}^{m} a_i c_3^{(i)} \leq h_{N_k}(x) \leq \sum_{i=1}^{m} a_i h_i \leq \frac{1}{2} \sum_{i=1}^{m} a_i c_3^{(i)}.$$ 

We use the definition of an adapted projective embedding on p. 466 of [10]. By Proposition 2.2 of [10], we may define an embedding adapted to the closed subvariety $Y_i$ of $X_i$ by putting

$$V_j^{(i)} = \sum_{k=0}^{N_i} M_{jk} W_j^{(i)}$$

with $M^{(i)} \in \text{GL}_{N_i+1}(\mathbb{Q})$, where the coefficients of the matrix $M^{(i)}$ are integers and bounded by max $\left(1, \frac{d_i}{2}\right)$ in absolute value, at least if $Y_i \neq \mathbb{P}^{N_i}$. If $Y_i = \mathbb{P}^{N_i}$, then the notion of an adapted projective embedding is not defined in [10], but we may set $V_j^{(i)} = W_j^{(i)}$ ($j = 0, \ldots, N_i$) and check that all the assertions about adapted embeddings made in this article also hold true in this case.

We now prove the equivalent of Proposition 3.1 in [10], introducing

$$\Lambda_h = \sum_{i=1}^{m} a_i (h_i + \delta_i + d_i (u_i + 1) \log 2d_i (N_i + 1)),$$

which we will prove to verify

$$\Lambda_h < \Lambda^{2\psi(u)} (M_t)^{u_0-u} \sum_{i=1}^{m} (a_i (h(X_i) + \delta_i)) = \Lambda^{2\psi(u) - 2\psi(0)} (M_t)^{-u} \sum_{i=1}^{m} a_i c_3^{(i)}.$$ \hfill (1)

In order to show this inequality (given condition (iv) from above), it suffices to show that

$$\sum_{i=1}^{m} a_i d_i (u_i + 1) \log 2d_i (N_i + 1) < 2^{-1} \Lambda^{2\psi(u)} (M_t)^{u_0-u} \sum_{i=1}^{m} (a_i (h(X_i) + \delta_i))$$

or even $\sum_{i=1}^{m} d_i (u_i + 1) \log 2d_i (N_i + 1) < 2^{-1} \Lambda^{2\psi(u)} (M_t)^{u_0-u}$. But since

$$\log 2d_i (N_i + 1) < 2d_i (N_i + 1),$$

it follows from (ii) that the left-hand side is at most

$$(u + m) \Lambda^{2\psi(u)-2} \max_{1 \leq i \leq m} \{2 \deg(X_i)^2 (N_i + 1)\} < 2^{-1} \Lambda^{2\psi(u)}$$

and now the claim is obvious.

**Proposition 2.1.** There does not exist any pair $(l, U)$ such that $1 \leq l \leq m$ and $U$ ($V^{(l)}$) is a homogeneous polynomial in the first adapted coordinates $V_0^{(l)}, \ldots, V_u^{(l)}$ satisfying

(a) $U (V^{(l)})(x_l) = 0$;
(b) $U$ is not the zero polynomial;
(c) $\deg(U) \leq \Lambda^{\omega u \psi(u)}$;
(d) \( a_l h(U) \leq \Lambda^{2\psi(u-1)-2\psi(u)} \left( \frac{M_{t_2}}{4d_l} \right) \Lambda_h \).

**Proof.** We assume the contrary and define \( Y'_l \) as an irreducible component containing \( x_l \) of the closed subvariety of \( Y_l \) defined by the equation \( U(Y^{(l)}) = 0 \) and we verify that the subproduct \( Y' \) obtained by replacing \( Y_l \) by \( Y'_l \) in \( Y \) contradicts the minimality of the latter. We have \( Y' = Y'_1 \times \cdots \times Y'_n \) with \( Y'_i = Y_i \) for all \( i \neq l \). By (a), condition (i) holds for \( Y' \). By (b) and the definition of an adapted embedding, \( Y' \) is a proper subvariety of \( Y \).

The polynomial \( U(Y^{(l)}) \) corresponds by means of \( M^{(l)} \) to a polynomial \( U'(W^{(l)}) \), where \( \deg(U') = \deg(U) \) and

\[
h(U') \leq h(U) + \deg(U) \log(N_l + 1) \max \left( 1, \frac{d_l}{2} \right) + \log \left( \frac{\deg(U) + u_l}{\deg(U)} \right).\]

As

\[
\left( \frac{\deg(U) + u_l}{\deg(U)} \right) = \prod_{i=1}^{\deg(U)} \left( 1 + \frac{u_l}{i} \right) \leq (1 + u_l)^{\deg(U)},
\]

it follows that

\[
h(U') \leq h(U) + \deg(U) \log d_l(N_l + 1)(u_l + 1) \quad (2)
\]

The (arithmetic as well as geometric) theorems of Bézout yield

\[
\deg(Y'_l) \leq \deg(U')d_l
\]

and

\[
h(Y'_l) \leq \deg(U')h_l + d_l \left( h(U') + \sqrt{N_l} \right).
\]

For the arithmetic Bézout theorem, we use Théorème 3.4 and Corollaire 3.6 of [8], where the modified height \( h_m \) used there can be bounded thanks to Lemme 5.2 of [9]. Together with (c), the first line implies that \( \deg(Y'_l) \leq d_l \Lambda^{\psi(u-1)-\psi(u)} \), since by definition \( \psi(u-1) = (\omega' u + 1)\psi(u) \). This shows that \( Y' \) satisfies conditions (ii) and (iii).

From the second line together with \((2)\), (c) and (d), we deduce that

\[
\sum_{i=1}^{m} a_i(h(Y'_l) + \delta_i) \leq d_m a_l h(U) + d_m a_l \Lambda^{\omega'\psi(u)} \log d_l(N_l + 1)(u_l + 1)
\]

\[
+ d_l a_l \sqrt{N_l} + \Lambda^{\omega'\psi(u)} \sum_{i=1}^{m} a_i(h_i + \delta_i) \leq d_m a_l h(U) + 3\Lambda^{\omega'\psi(u)} \Lambda_h
\]

\[
\leq \Lambda^{2\psi(u-1)-2\psi(u)} \left( \frac{M_{t_2}}{4} \right) \Lambda_h + 3\Lambda^{\omega'\psi(u)} \Lambda_h.
\]

Finally, we have \( 3\Lambda^{\omega'\psi(u)} \leq \Lambda^{2\omega'\psi(u)} \left( \frac{M_{t_2}}{4} \right) = \Lambda^{2\psi(u-1)-2\psi(u)} \left( \frac{M_{t_2}}{4} \right) \). It then follows from (1) that \( Y' \) satisfies condition (iv) as well and we get the desired contradiction. \( \square \)

We proceed to deduce from this an equivalent of Corollaire 3.2 in [10] (with a modification of the last assertion). Let us mention that by Lemme 2.3 of [10], there exist polynomial relations

\[
P_j^{(i)} \left( V_0^{(i)}, \ldots, V_{u_i}^{(i)}, Y_j^{(i)} \right) = Q_j^{(i)} \left( V_0^{(i)}, \ldots, V_{u_i}^{(i)}, W_j^{(i)} \right) = 0 \quad \text{in} \quad \Gamma \left( Y_i, L_i^{\otimes d_l} \right)
\]
for all $1 \leq i \leq m$ and all $0 \leq j \leq N_i$. The polynomials $P_j^{(i)}(T)$ and $Q_j^{(i)}(T)$ are homogeneous of degrees $d_i$, monic in their last variable $T_{u_{i+1}}$ and equal to a power of an irreducible polynomial (we denote the corresponding exponent for $Q_j^{(i)}$ by $b_{i,j}$). Furthermore, we know from the same lemma that the height of the family $B_i$ of all the coefficients of the $P_j^{(i)}$ and the $Q_j^{(i)}$ for fixed $i$ (seen as a point in projective space) can be estimated from above as

$$h(B_i) \leq b_i + d_i(u_i + 1) \log d_i(N_i + 1).$$

(3)

**Corollary 2.2.** For every index $1 \leq i \leq m$, we have that

1. the morphism $\rho_i : Y_i \to \mathbb{P}^u$, defined by the first adapted coordinates $V_0^{(i)}, \ldots, V_{u_i}^{(i)}$, is finite, surjective and étale at $x_i \in Y_i(\overline{\mathbb{Q}})$;
2. $V_0^{(i)}(x_i) \neq 0$;
3. for every index $0 \leq j \leq N_i$ such that $W_j^{(i)} \neq 0$ in $\Gamma(Y_i, \mathcal{L}_i)$, we have

$$W_j^{(i)} \frac{\partial b_{i,j} Q_j^{(i)}}{\partial T_{u_{i+1}}} \left( \frac{V_0^{(i)}, \ldots, V_{u_i}^{(i)}}{V_0^{(i)}, \ldots, V_{u_i}^{(i)}} \right) (x_i) \neq 0.$$

Proof. That the morphism $\rho_i$ is finite and surjective follows from the definition of adapted embeddings (see [10], Section 2.1). If one of the three assertions were not true, we could construct a pair $(i, U(V^{(i)}))$ that would contradict Proposition [2.1] with $\deg(U) \leq 2d_i^2$ and $h(U) \leq 6N_i d_i^2 + 2d_i h(B_i)$.

We refer to Corollaire 3.2 of [10] for the proof – in the case that $W_j^{(i)} \neq 0$, $W_j^{(i)}(x_i) = 0$ it suffices to take $U(V^{(i)}) = Q_j^{(i)}(V_0^{(i)}, \ldots, V_{u_i}^{(i)}, 0)$. Note that $P_{u_i+1}^{(i)}$ is not only a power of an irreducible polynomial, but in fact irreducible, since its degree is equal to the degree of $Y_i$, which is also equal to the degree of any irreducible factor of $P_{u_i+1}^{(i)}$. Hence, its discriminant does not vanish identically. That the morphism $\rho_i$ is étale at $x_i$ is proved in the same way as in the proof of Lemme 4.3 in [7].

3. Constructing a section of small height

Following Section 4 of [10], we set

$$\epsilon = \frac{1}{2u \theta (t_{1,m})^\mu} \prod_{i=1}^m d_i^{1-\omega}$$

and define a family of sections $Z_d^{(i)} \subset \Gamma(\mathcal{X}, \mathcal{M}_{-d} \otimes \mathcal{P}_{\otimes d} \otimes \mathcal{N}_{a_{de}})$ of cardinality $M' = M(N_i + 1) \cdots (N_m + 1)$ for every $d \in \epsilon^{-1} \mathbb{N} \subset \mathbb{N}$ by

$$Z_d^{(i)} = \left\{ \zeta \otimes \left( W_j^{(i)} \right)^{\otimes d_{e,i}} \otimes \cdots \otimes \left( W_j^{(m)} \right)^{\otimes d_{e,m}} : \zeta \in Z, W_j^{(i)} \in W^{(i)} \right\}.$$

The proof of Proposition 4.1 of [10] then goes through without any major modifications (given that $\mathcal{M}$ is nef, see below). It yields a natural number $d_0$ that we choose sufficiently large so that for each natural number $d \geq d_0$ there exists a basis of $\Gamma(\mathcal{Y}, \mathcal{P}_{\otimes d})$ that consists of monomials of degree $d$ in the elements of $\mathcal{M}$. We obtain the following equivalent of Proposition 4.2 in [10].
Proposition 3.1. For \( d \in \mathbb{N} \setminus \{0\} \), we write \( Q_d = M^{\otimes d} \otimes N_a^{\otimes -d} \) and fix a basis of \( \Gamma(\mathcal{Y}, P^{\otimes d}) \) that consists of monomials in the sections \( \Xi \) of degree \( d \).

Then there exists a section \( 0 \neq s \in \Gamma(\mathcal{Y}, Q_d) \) such that the height of \( s \), defined as the height of the family of coefficients of the sections \( s \otimes Z'_d \) with respect to the fixed basis, seen as a point in projective space, satisfies

\[
h(s) \leq \frac{2M'd}{u\epsilon}(t_1 + 2t_2 + \epsilon)\Lambda_h + o(d).
\]

Proof. The dimension estimate

\[
\dim \Gamma(\mathcal{Y}, Q_d) \geq \frac{d^n}{4^n u!} \prod_{i=1}^m d_i^{-\omega a_i u_i} + O(d^{u-1})
\]
given in Proposition 4.1 of [10] is still valid, since the intersection numbers are formally the same. Here, we need however that \( M \) is nef in order to translate the lower bound for its top self-intersection number into a lower bound for the dimension of a space of global sections through the asymptotic Riemann-Roch theorem (see [3], Theorem VI.2.15).

In the Faltings complex on \( \mathcal{Y} \) defined by the family \( Z'_d \) of cardinality \( M' \)

\[
0 \to Q_d \to \left(P^{\otimes d}\right)^{\otimes M'} \to \left(N_a^{\otimes d(t_1 + t_2 + \epsilon)}\right)^{\otimes (M')^2}
\]

the image of \( \Gamma(\mathcal{Y}, Q_d) \) in \( F = \Gamma(\mathcal{Y}, P^{\otimes d})^{M'} \) coincides with the kernel of a family of linear forms in the coordinates with respect to the fixed basis. This family can be chosen such that the coefficients of the linear forms lie in a number field that is independent of \( d \) and the height of the set of all coefficients, seen as a point in projective space, is at most \( d(t_1 + 2t_2 + \epsilon)\Lambda_h + o(d) \): in order to show this, we follow the proof of Proposition 4.2 of [10] by applying Lemme 2.5 in [10] with \( n_i = N_i \) and use [3] to bound \( h(B_i) \). Note that when estimating \( h(P_1^{d_1}, \ldots, P_M^{d_M}) \) as in the proof of Proposition 4.2, one obtains by well-known height estimates an upper bound of

\[
d \sum_{i=1}^m a_i \delta_i + dt_2 \sum_{i=1}^m a_i \log(N_i + 1)
\]

(the second summand, coming from the archimedean places, is missing in [10]).

Furthermore, the injection \( P^{\otimes d} \hookrightarrow N_a^{\otimes dt_1} \) yields that

\[
\dim F \leq M' \prod_{i=1}^m d_i^{u_1}(dt_1 a_i)^{u_i} + o(d^u)
\]

and so \( \log \dim F = o(d) \). Hence, the Dirichlet exponent of the system can be estimated as

\[
\frac{\dim F}{\dim \Gamma(\mathcal{Y}, Q_d)} \leq \frac{2M'}{u\epsilon} + o(1)
\]

and the proposition follows from the Siegel lemma (Lemme 2.6 in [10]). \( \square \)
4. The index is small

We now replace \( \mathcal{Y} \) by a sufficiently small open subset of \( \mathcal{Y} \) that contains \( x \). According to Corollary \ref{cor:vanishing}

we can in particular assume that each section \( V_0^{(i)} \) vanishes nowhere on this subset and suppose that the sheaf of differentials \( \Omega_{\mathcal{Y}/\mathcal{Q}} \) is generated by the differentials of the \( V_j^{(i)}/V_0^{(i)} \) \( (i = 1, \ldots, m, 1 \leq j \leq u_i) \). We can furthermore suppose that \( \mathcal{P}, \mathcal{M} \) and \( \mathcal{N}_\mathfrak{a} \) all can be trivialized over this subset.

We fix an isomorphism \( \mathcal{Q}_d \simeq \mathcal{O}_{\mathcal{Y}} \) and consider the index \( \sigma \) (as defined in Section 5.2 of \cite{[10]} of the section \( s_d \in \Gamma(\mathcal{Y}, \mathcal{Q}_d) \) that was constructed in the preceding proposition with respect to the weight \( dt_1 a \) in \( x \).

**Lemma 4.1.** With notations as above, we have

\[
\sigma \leq (4t_1 \max_i d_i(N_i + 1))^{-1} \epsilon
\]

for \( d \in \epsilon^{-1} \mathbb{N} \cap d_0 \mathbb{N} \) sufficiently large.

**Proof.** We assume that the inequality is false and derive a contradiction.

We can estimate

\[
\sigma \prod_{i=1}^{m} d_i^{-1} \geq (4t_1 \max_i d_i(N_i + 1))^{-1} \epsilon \prod_{i=1}^{m} d_i^{-1} \\
\geq (8u_1 t_1^{u_1} m^u \max(N_i + 1))^{-1} \prod_{i=1}^{m} d_i^{-1}
\]

It then follows from (iii) that

\[
\sigma \prod_{i=1}^{m} d_i^{-1} \geq (8u_1 t_1^{u_1} m^u \max(N_i + 1))^{-1} \prod_{i=1}^{m} \left( \deg X_i \right)^{-\omega_1} \Lambda^{-\omega_1(\psi(u))^{-1}}
\]

and hence \( \sigma \prod_{i=1}^{m} d_i^{-1} \geq \sigma_0 = m \Lambda^{-\omega_1(\psi(u))^{-1}} \).

Then, we can construct a non-zero multihomogeneous polynomial \( G(V) \) of multidegree \( dt_1(d_1 \cdots d_m)a \) in the adapted coordinates \( V_j^{(i)}, 0 \leq j \leq u_i \), of height bounded by

\[
h(G) \leq (d_1 \cdots d_m) \left( h(s_d) + dt_1 \sum_{i=1}^{m} a_i(h(B_i) + \log(2(u_i + 1))) \right) + o(d)
\]

and of index at least \( \sigma \) in \( \rho(x) \) with respect to the weight \( dt_1 a \), where \( \rho = (\rho_1 \circ p_1 |_{\mathcal{Y}}, \ldots, \rho_m \circ p_m |_{\mathcal{Y}}) \).

For this, we choose \( \zeta' \in Z'_d \) which does not vanish at \( x \). We write

\[
s_d \otimes \zeta' = \alpha \left( \left( V_0^{(1)} \right)^{\otimes dt_1 a_1} \otimes \cdots \otimes \left( V_0^{(m)} \right)^{\otimes dt_1 a_m} \right),
\]

where \( \alpha \) is a polynomial in the \( W_j^{(i)}/V_0^{(i)} \) with coefficients in \( \bar{\mathcal{Q}} \). Consider the norm \( N(\alpha) \) of \( \alpha \) with respect to the field extension \( \bar{\mathcal{Q}}(\mathcal{Y})/L \), where \( L \) is the subfield of \( \bar{\mathcal{Q}}(\mathcal{Y}) \) generated by the \( V_j^{(i)}/V_0^{(i)} \) \( (j = 1, \ldots, u_i, i = 1, \ldots, m) \). We can take

\[
G = \left( V_0^{(1)} \right)^{dt_1(d_1 \cdots d_m)a_1} \cdots \left( V_0^{(m)} \right)^{dt_1(d_1 \cdots d_m)a_m} N(\alpha).
\]

On the one hand, this is a quotient of multihomogeneous elements of \( R = \bar{\mathcal{Q}}[V_j^{(i)}; 1 \leq i \leq m, 0 \leq j \leq u_i] \). As such it has a multidegree, which
is exactly $dt_1(d_1 \cdots d_m)a$. On the other hand, it is the norm of the multihomogenization of $\alpha$, which is a multihomogeneous polynomial in the $W_j^{(i)}$, with respect to the field extension $$\tilde{L}/\bar{Q}(V_j^{(i)}; 1 \leq i \leq m, 0 \leq j \leq u_i),$$ where $\tilde{L}$ is the fraction field of the multihomogeneous coordinate ring of $Y_1 \times \cdots \times Y_m \hookrightarrow \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$. As such $G$ is integral over $R$ and therefore lies in $R$. So $G$ is in fact a multihomogeneous polynomial of the desired multidegree.

Note that $\beta = N(\alpha)\alpha^{-1} = \prod_{\tau \neq \text{id}} \tau(\alpha)$ lies in $\bar{Q}(Y)$ and is integral over $\mathcal{O}_{\mathbb{P}^{u_1} \times \cdots \times \mathbb{P}^{u_m}, \rho(x)}$ because $\alpha$ is. Here, $\tau$ runs over the embeddings of $\bar{Q}(Y)$ into a normal closure of the extension $\bar{Q}(Y)/L$. Hence, $\beta$ is integral over $\mathcal{O}_{Y,x}$. As $\rho$ is étale at $x$, this local ring is normal and hence contains $\beta$. So the index of $N(\alpha)$ in $x$ (or equivalently, the index of $G$ in $\rho(x)$) is greater or equal than the index of $\alpha$ in $x$. For the bound for $h(G)$, see Lemme 5.5 of [10].

We can then apply Théorème 5.6 (Faltings’ product theorem) of [10] with the value of $\sigma_0$ above and obtain in this way a contradiction with Proposition 2.1. The hypotheses of the theorem are satisfied, since

$$\frac{a_i}{a_{i+1}} \geq c_2 \geq \left(\frac{m}{\sigma_0}\right)^u \geq (2u^2)^u$$

and $G$ has index at least $\sigma$ with respect to the weight $dt_1a$ in $\rho(x)$, hence has index at least $\sigma \prod_{i=1}^m d_i^{-1} \geq \sigma_0$ with respect to the weight $dt_1(d_1 \cdots d_m)a$ in $\rho(x)$.

We obtain a pair $(l, U)$ with $U(V^{(i)})(x_l) = 0$, $U$ non-zero, $\deg(U) \leq \left(\frac{m}{\sigma_0}\right)^u = \Lambda^{\omega' \psi(u)}$ and

$$a_i h(U) \leq u_l \left(\frac{m}{\sigma_0}\right)^u \left(\frac{h(G)}{dt_1d_1 \cdots d_m} + \sum_{i=1}^m a_i(u_i \log(u_i + 1) + \log 2)\right) + a_l \left(\frac{m}{\sigma_0}\right)^u \left(u_l + 1\right) + a_l \log \left(\frac{\deg(U) + u_l}{u_l}\right) + o(1).$$

Here, we used that the height of projective $n$-space is bounded from above by $n \log(n + 1)$.

After some simplification and by using that $u_l \geq 1$ (which is a consequence of the product theorem) and $\frac{m}{\sigma_0} = \Lambda^{\omega' \psi(u)}$, we deduce that

$$a_i h(U) \leq u_l \Lambda^{\omega' \psi(u)} \left(\frac{h(s_d)}{dt_1} + 2\Lambda_h + 2a_l \log \left(\frac{m}{\sigma_0}\right)^u (u_l + 1)\right) + o(1)$$

$$\leq u_l \Lambda^{\omega' \psi(u)} \left(\frac{2M'}{u \epsilon} (1 + 2t_2 + 3\epsilon)\Lambda_h + 2a_l \log \left(\frac{m}{\sigma_0}\right)^u\right) + o(1).$$

For the last inequality, we used that $2a_l \log(2(u_l + 1)) \leq 2\Lambda_h$ and $\frac{2M'}{u} \geq 2$.

We can now estimate

$$2a_l u_l \log \left(\frac{m}{\sigma_0}\right)^u \leq 2\Lambda_h \omega' u \psi(u) \log \Lambda \leq \Lambda_h \Lambda^{(\omega' \psi(u) - 1)\psi(u)},$$
By definition, there exists such a $V$ of the $D_0$ for every operator $polynomials of degree \( d \) phism, precisely since $D_0$. In order to define the right-hand side, one has to fix an isomorphism $Y$ the proof of Theorem 1.1 by considering the following height

We write

$$D_{t1} \quad \text{monomials of multidegree } \xi,$$

Furthermore, the sections $\Xi$ themselves are monomials of multidegree $t$. Hence, the right choice of isomorphism shows that $Y$.

5. Finishing the proof

We now have established that the section $s_d \in \Gamma(Y, Q_d)$ given by Proposition 3.1 has index (in $x$ and with respect to the weight $dt_{1}a$) bounded as

$$\sigma \leq (4t_1 \max_i (N_i + 1))^{-1} \epsilon.$$

We write $D$ for a differential operator associated to that index and finish the proof of Theorem 1.1 by considering the following height

$$-h_{Q_d}(x) = dh(Z(x)) - dh(\Xi(x)) + \epsilon \sum_{i=1}^{m} a_i h(W^{(i)}(x))$$

$$= h(Z'_d(x)) - dh(\Xi(x)).$$

By definition, there exists such a $D$ with $D(s_d)(x) \neq 0$ and we have $D'(s_d)(x) = 0$ for every operator $D'$ of index $\sigma' < \sigma$, hence by the product formula

$$h(Z'_d(x)) = h \left( (D(s_d) \otimes Z'_d)(x) \right) = h \left( (D(s_d) \otimes \zeta')(x) \right).$$

In order to define the right-hand side, one has to fix an isomorphism $P^{\otimes d} \simeq \mathcal{O}_Y$. The right-hand side is however independent of the choice of isomorphism, precisely since $D$ is an operator associated to the index of $s_d$.

Let us recall that the sections $s_d \otimes \zeta' \in \Gamma(Y, P^{\otimes d})$ are homogeneous polynomials of degree $d$ in the sections $\Xi$ and that $\log \dim \Gamma(Y, P^{\otimes d}) = o(d)$.

Furthermore, the sections $\Xi$ themselves are monomials of multidegree $t_1a$ in the coordinates $W^{(i)}$. Hence, the right choice of isomorphism shows that

$$h(Z'_d(x)) \leq h(s_d) + h((D(\xi')(x))_{\zeta'}) + o(d),$$

where $\xi'$ runs over the monomials of degree $d$ in the sections $\Xi$ (seen as monomials of multidegree $dt_{1}a$ in the $W^{(i)}$) divided by appropriate products of the $V_0^{(i)}$ ($i = 1, \ldots, m$).
We can estimate the height of the $D(\xi^v(x))$ by using Leibniz’ formula as well as Corollaire 5.1 and Lemma 5.2 of [10] (corrected). For $1 \leq i \leq m$ and $l = (l_1, \ldots, l_{u_i}) \in (\mathbb{N} \cup \{0\})^{u_i}$, we define the operator

$$
\partial^{i,l} = \prod_{j=1}^{u_i} \frac{1}{l_j!} \left( \partial_{\nu_j}^{l_j} / \nu_{0}^{l_j} \right) : \mathcal{O}_{Y,x} \to \mathcal{O}_{Y,x}.
$$

If $w = (w_1, \ldots, w_k) \in (\mathbb{N} \cup \{0\})^k$ is a multi-index, we write $|w| = w_1 + \cdots + w_k$.

**Lemma 5.1.** Let $1 \leq i \leq m$ be an integer and let $K$ be a number field that contains the coordinates $\left( W_j^{(i)} / V_0^{(i)} \right)(x)$, the families $B_i$ and the products

$$
c_i = \prod_j \left( \frac{W_j^{(i)} / V_0^{(i)}}{b_{i,j}^\omega} \right) \frac{\partial^{b_{i,j}} Q_j^{(i)}}{\partial T_{u_i}^{b_{i,j}}} \left( 1, \frac{V_1^{(i)}}{V_0^{(i)}}, \ldots, \frac{V_{u_i}^{(i)}}{V_0^{(i)}}, \frac{W_j^{(i)}}{V_0^{(i)}} \right) (x_i) \right) ^{\frac{1}{b_{i,j}}},
$$

where $j$ runs over the indices satisfying $W_j^{(i)} \neq 0$ in $\Gamma(Y_i, L_i)$. Then for every place $v$ of $K$ and every multi-index $l \in (\mathbb{N} \cup \{0\})^{u_i}$, we have

$$
\left| \partial^{i,l} \left( W_j^{(i)} / V_0^{(i)} \right)(x) \right|_v \leq \left| \left( W_j^{(i)} / V_0^{(i)} \right)(x) \right|_v \left( |c_i|^{-2} C_{i,v} \right)^{|l|}
$$

with

$$
C_{i,v} = 2^{-\epsilon_v} \left( (d_i(N_i + 1))^{6d_i \epsilon_v} \max_{b \in B_i} |b|_v \max_{0 \leq k \leq N_i} \left| \left( W_k^{(i)} / V_0^{(i)} \right)(x) \right|_v \right)^{2(N_i+1)}
$$

and $\epsilon_v = 1$ if $v$ is infinite, 0 if $v$ is finite.

**Proof.** Recall that by Corollary 2.2 the number $c_i \in \overline{\mathbb{Q}} \setminus \{0\}$ is well defined (up to the choice of the root which can be made arbitrarily).

If $W_j^{(i)} = 0$ in $\Gamma(Y_i, L_i)$, the derivative $\partial^{i,l} \left( W_j^{(i)} / V_0^{(i)} \right)$ is zero and the inequality holds. Otherwise, we may apply Corollaire 5.1 of [10] and follow the proof of Lemme 5.2 of [10] with $N = N_i$, using at the end that

$$
2f_2(u_i, d_i) = 2(2u_i+4)^{1+\frac{d_i}{2}d_i} \left( d_i + u_i \right)^{2(N_i+2)} \left( d_i + 1 \right)^{2(2(N_i+1)+2d_i)}
$$

is bounded from above by

$$
2^{2+3d_i+2(N_i+1)d_i} (N_i + 2)^{1+\frac{d_i}{2}d_i} (N_i + 1)^{2(2N_i+2)d_i} d_i^{2(N_i+1)d_i} ((N_i + 1) d_i)^{2(N_i + 1)d_i}
$$

and

$$
\leq (N_i + 1)^{N_i+1+\frac{d_i}{2}d_i+\log_2(1+\frac{3}{2}d_i)+5(N_i+1)} d_i^{N_i+1+\frac{d_i}{2}d_i} ((N_i + 1) d_i)^{2(N_i+1)d_i}
$$

$$
\leq (N_i + 1)^{2(N_i+1)+2(N_i+1)d_i+5(N_i+1)d_i} d_i^{N_i+1+\frac{d_i}{2}d_i} ((N_i + 1) d_i)^{2(N_i+1)d_i}
$$

$$
\leq (d_i(N_i + 1))^{12d_i(N_i+1)}.
$$

For $D = \prod_{i=1}^{m} \partial^{i;\nu_i}$, we obtain the following bound (cf. the proof of Proposition 5.3 in [10])

$$
|D(\xi^v(x))|_v \leq |\xi^v(x)|_v \prod_{i=1}^{m} 2^{(|\nu_i|+d_1u_{i;\nu_i})\epsilon_v} (|c_i|^{-2} C_{i,v})^{\nu_i}.
$$
and hence thanks to the product formula for the $c_i$

$$h((D(ξ^u)(x))ξ^v) \leq dh(ξ(x)) + \sum_{i=1}^{m} 2(N_i + 1)|κ_i|(h(B_i) + d_i h(W^{(i)}(x)))$$

$$+ \sum_{i=1}^{m} (dt_i u_i a_i \log 2 + 12d_i(N_i + 1)|κ_i| \log d_i(N_i + 1)).$$

We proceed with bounding

$$|κ_i| \leq dt_i u_i σ \leq (4(N_i + 1)d_i)^{-1}de a_i,$$

which implies that $\sum_{i=1}^{m} 2d_i(N_i + 1)|κ_i|h(W^{(i)}(x)) \leq \frac{de}{2}h_{N_u}(x)$. Together with [3], the bound also implies that

$$\sum_{i=1}^{m} 2(N_i + 1)|κ_i|(h(B_i) + 6d_i \log d_i(N_i + 1))$$

$$\leq de \sum_{i=1}^{m} a_i \left( \frac{h_i + d_i(u_i + 1) \log d_i(N_i + 1)}{2d_i} + 3 \log d_i(N_i + 1) \right) \leq 4dεΛ_h.$$

Finally we know that $\sum_{i=1}^{m} u_i a_i \log 2 \leq Λ_h$ and putting all these estimates together we get

$$eh_{N_u}(x) - h_{M}(x) = \frac{h_{Q_d}(x)}{d} \leq \frac{h(s_d)}{d} + \frac{ε}{2}h_{N_u}(x) + (t_1 + 4ε)Λ_h + o(1).$$

Thanks to Proposition [3.1] and [II], it follows that

$$\frac{ε}{2}h_{N_u}(x) - h_{M}(x) \leq \frac{2M'}{uc}(t_1 + 2t_2 + ε)Λ_h + (t_1 + 4ε)Λ_h + o(1)$$

$$\leq \left( \frac{2M'}{uc} \right) (2t_1 + 2t_2 + 5ε)Λ_h + o(1)$$

$$\leq \left( \frac{M't_1}{ε^2} \right) 8(2 + 2t_2 + 5ε)Λ^{2\psi(u) - 2\psi(0)}(Mt_2)^{-u} \left( \frac{ε}{4} \sum_{i=1}^{m} a_i c_{3}^{(i)} \right),$$

where the strict inequality in [II] allowed us to sweep the $o(1)$ under the rug (for $d$ large enough).

We have $8(2 + 2t_2 + 5ε) \leq 42t_2 \leq Λ^2t_2$ and it follows from (iii) that

$$(M't_1)ε^{-2} \leq Mt_1 \max_i (N_i + 1)^m (2uθ)^2(t_1 m)^{2u} \left( \prod_{i=1}^{m} \deg(X_i) \right)^{2(1+ω)} Λ^{2(1+ω)(ψ(u) - 1)}$$

$$\leq MA^{\max(3, m) + 2(1+ω)ψ(0)} \leq MA^{2ω'ψ(u) - 2} = MA^{2ψ(u - 1) - 2ψ(u) - 2},$$

where we used that $\max\{3, m\} \leq 4uψ(u) - 2$.

Hence, we can deduce that

$$\frac{ε}{2}h_{N_u}(x) - h_{M}(x) \leq (Mt_2)^{-u} Λ^{2(ψ(u - 1) - 2ψ(0))} \frac{ε}{4} \sum_{i=1}^{m} a_i c_{3}^{(i)} \leq \frac{ε}{4} h_{N_u}(x),$$
from which it follows that \( h_{\mathcal{M}_a}(x) \leq 4\epsilon^{-1} h_{\mathcal{M}}(x) \). The theorem follows, since by (iii)
\[
4\epsilon^{-1} \leq 8u\theta(t_1m)^u \left( \prod_{i=1}^{m} \text{deg}(X_i) \right)^{1+\omega} \Lambda^{(1+\omega)(\psi(u)-1)} \leq \Lambda^{(1+\omega)\psi(u)+2}
\]
and \( \Lambda^{(1+\omega)\psi(u)+2} \leq \Lambda^\omega u^\omega(u) \leq c_1 \).

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