On the protective layer boundary determining error in the thermal conductivity inverse problem

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Abstract. The article studies the problem of determining the error introduced by the inaccuracy of determining the thickness of a protective heat-resistant coating for composite materials. The mathematical problem is the heat conduction equation on an inhomogeneous half-line. The temperature on the outer side of the half-line \((x = 0)\) is considered unknown on an infinite time interval. To find it, the temperature is measured at the media section at the point \(x = x_0\). An analytical study of the direct problem is carried out in the work. It made it possible to formulate the inverse problem mathematically rigorously and to define functional spaces in which it is convenient to solve the inverse problem. The main difficulty to be solved in the article is to obtain an estimate of the approximate solution error. The projection regularization method is used to estimate the modulus of conditional correctness, with the help of which order-exact estimates are obtained.

1. Introduction

There are structures in modern technology, the nodes of which are subject to thermal effects. So the properties of the materials can change markedly as a result. However, direct temperature measurement is not possible as a rule. Therefore, it is desirable to be able to solve the thermal diffusivity inverse problem.

You can also apply a protective coating. The coating will protect the device for some time, but it will make it difficult to solve the inverse problem [1], which will have to be posed for the composite material already.

The coating can be destroyed by heat and other factors. The thickness of the protective layer and the thermal effect of the heat source on the material change accordingly.

We propose to consider the heating of the composite medium in this work. We assume at the same time that the protective layer destruction process does not lead to the release of heat, which goes to its destruction – the layer simply changes its thickness, without introducing any other changes in the heat conduction equation. This layer disappears from consideration having become ruined, moreover. Our task is to find the permissible error in determining the thickness of the protective layer knowing the properties of the material and the properties of the medium.

We have considered the exact solution of the direct problem, formulated the inverse problem, and constructed an estimate for the order of accuracy in this paper.
2. Direct mathematical problem

The direct mathematical problem has the form [2]

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= a_1^2 \frac{\partial^2 u_1(x,t)}{\partial x^2}, \quad x \in (0; x_0), \quad t \in (0; +\infty), \\
\frac{\partial u_2(x,t)}{\partial t} &= a_2^2 \frac{\partial^2 u_2(x,t)}{\partial x^2}, \quad x \in (x_0; +\infty), \quad t \in (0; +\infty), \\
u_1(0,t) &= q(t), \quad t \in [0; +\infty), \quad u_2(+\infty, t) = 0, \quad t \in [0; +\infty), \\
u_1(x_0, t) &= u_2(x_0, t), \quad t \in [0; +\infty), \\
u_1(x, 0) &= 0, \quad x \in [0; x_0], \\
u_2(x, 0) &= 0, \quad x \in [x_0; +\infty). \\
\end{align*}
\]

(1)

The formal solution of the problem (1) can be written by the formulas

\[
\begin{align*}
u_1(x, t) &= \int_0^t \frac{x}{\sqrt{4\pi a_1^2(t-\tau)^3}} \exp \left( -\frac{x^2}{4a_1^2(t-\tau)} \right) q(\tau) d\tau, \\
u_2(x, t) &= \int_0^t \frac{x + l_1}{\sqrt{4\pi a_2^2(t-\tau)^3}} \exp \left( -\frac{(x + l_1)^2}{4a_2^2(t-\tau)} \right) q(\tau) d\tau,
\end{align*}
\]

where \( l_1 = \frac{a_2 - a_1}{a_1} x_0. \)

**Lemma 2.1.** Let \( q(t) \in C^1[0; +\infty) \) and \( \exists M_0, \sup_{t \in [0; +\infty)} (|q(t)|, |q'(t)|) < M_0, \) and the integrals \( J(x, t), J_1(x, t), J_2(x, t) \) are given by formulas:

\[
\begin{align*}
J(x, t) &= \int_0^t \frac{xq(\tau)d\tau}{\sqrt{4\pi a_1^2(t-\tau)^3}} \exp \left( -\frac{x^2}{4a_1^2(t-\tau)} \right), \\
J_1(x, t) &= \frac{\partial}{\partial x} \int_0^t \frac{xq(\tau)d\tau}{\sqrt{4\pi a_1^2(t-\tau)^3}} \exp \left( -\frac{x^2}{4a_1^2(t-\tau)} \right), \\
J_2(x, t) &= \frac{\partial^2}{\partial x^2} \int_0^t \frac{xq(\tau)d\tau}{\sqrt{4\pi a_1^2(t-\tau)^3}} \exp \left( -\frac{x^2}{4a_1^2(t-\tau)} \right).
\end{align*}
\]

Then these integrals are uniformly bounded \( \forall x \in [0; x_0], \forall t \geq 0. \)

**Proof.** Let us prove the first assertion of the lemma. We transform \( J(x, t) \) as follows:

\[
J(x, t) = -\int_0^t \frac{q(\tau)}{\sqrt{\pi}} \exp \left( -\frac{x^2}{4a_1^2(t-\tau)} \right) d\left( \frac{x}{2a_1 \sqrt{t-\tau}} \right).
\]

We introduce the variable \( \xi = \frac{x}{2a_1 \sqrt{t-\tau}} \) and obtain the upper bound for the integral

\[
\frac{1}{\sqrt{\pi}} \int_0^t q(\tau) \exp \left( -\frac{x^2}{4a_1^2(t-\tau)} \right) d\left( \frac{x}{2a_1 \sqrt{t-\tau}} \right) \leq \frac{M_0}{\sqrt{\pi}} \int_0^{+\infty} e^{-\xi^2} d\xi \leq M_0.
\]

The uniform convergence of the integrals \( J_1(x, t) \) and \( J_2(x, t) \) can be proved in a similar way.
3. Existence and uniqueness of a solution to the problem (1)

**Theorem 3.1.** Let \( q(t) \in C^1[0;+\infty) \) and \( \exists M_0, \) so that \( \sup_{t \geq 0} |q(t)|, |q'(t)| < M_0. \) Then the solution to the problem (1) exists, is unique and can be given by the formula (2).

**Proof.** The existence of a solution is proved in the lemma 2.1.

To prove uniqueness, suppose that the solution is not unique, and another solution to the problem (1) exists: functions \( w_1(x, t), x \in [0; x_0], t \in [0; +\infty), \) \( w_2(x, t), x \in [x_0; +\infty), t \in [0; +\infty). \) Let us construct the difference of solutions \( z_1(x, t) = w_1(x, t) - w_2(x, t), x \in [0; x_0], t \in [0; +\infty), \) \( z_2(x, t) = w_2(x, t) - w_2(x, t), x \in [x_0; +\infty), t \in [0; +\infty). \) The boundary value problem is obtained for the functions \( z_1(x, t), z_2(x, t): \)

\[
\begin{align*}
\frac{\partial z_1(x, t)}{\partial t} &= a_1^2 \frac{\partial^2 z_1(x, t)}{\partial x^2}, x \in (0; x_0), t \in (0; +\infty), \\
\frac{\partial z_2(x, t)}{\partial t} &= a_2^2 \frac{\partial^2 z_2(x, t)}{\partial x^2}, x \in (x_0; +\infty), t \in (0; +\infty), \\
z_1(0, t) = 0, t \in [0; +\infty), z_2(+\infty, t) = 0, t \in [0; +\infty), \\
z_1(x_0, t) = z_2(x_0, t), t \in [0; +\infty), a_1 \frac{\partial z_1(x_0, t)}{\partial x} = a_2 \frac{\partial z_2(x_0, t)}{\partial x}, t \in [0; +\infty), \\
z_1(x, 0) = 0, x \in [0; x_0), z_2(x, 0) = 0, x \in [x_0; +\infty).
\end{align*}
\]

(3)

The solution to this equation can be constructed, it is bounded to the region \( x \in [0; +\infty), t \in [0; +\infty), \) and, therefore, there is its Laplace transform in \( t : \tilde{z}(x, p) = \int_0^{+\infty} e^{-pt} z(x, t)dt. \) As the result, we get the problem:

\[
\begin{align*}
p\tilde{z}_1(x, p) &= a_1^2 \frac{\partial^2 \tilde{z}_1(x, p)}{\partial x^2}, x \in (0; x_0), p \in \mathbb{Z}, \\
p\tilde{z}_2(x, p) &= a_2^2 \frac{\partial^2 \tilde{z}_2(x, p)}{\partial x^2}, x \in (x_0; +\infty), p \in \mathbb{Z}, \\
\tilde{z}_1(0, p) = 0, p \in \mathbb{Z}, \tilde{z}_2(+\infty, p) = 0, p \in \mathbb{Z}, \\
\tilde{z}_1(x_0, p) = \tilde{z}_2(x_0, p), p \in \mathbb{Z}, a_1 \frac{\partial \tilde{z}_1(x_0, p)}{\partial x} = a_2 \frac{\partial \tilde{z}_2(x_0, p)}{\partial x}, p \in \mathbb{Z}.
\end{align*}
\]

It is easy to see that the solution to this equation is unique and equal to zero, which means the uniqueness of the solution to the problem (1).

4. Statement of the inverse problem

Let us now proceed to the solution of the inverse problem. Suppose that the exact value of the function \( f_0(t) \) is not known, and instead of it there is some approximate function \( f_\delta(t) \in C[0; \infty), f_\delta(0) = 0 \) and the error level \( \delta > 0, \) such that

\[
\sup_{t \in [0; \infty)} |f_\delta(t) - f_0(t)| \leq \delta.
\]

(4)

Therefore, consider the problem (8) with another condition along with the problem (8) with the condition \( u_2(x_0, t) = f(t), \)

\[
u_2(x_0, t) = f_\delta(t), \quad t \geq 0.
\]

(5)

The solution to the problem (5) is denoted by \( u_2(x_0, t, \delta). \)

We first solve the problem for \( u_2(x, t), x \in [x_0; +\infty), t \in [0; +\infty), \) find \( \frac{\partial u_2(x_0, t)}{\partial x}. \) Then we obtain a boundary value problem for \( u_1(x, t), x \in [0; x_0], t \in [0; +\infty), \) using the matching
conditions:
\[
\begin{cases}
\frac{\partial u_2(x,t)}{\partial t} = a_2^2 \frac{\partial^2 u_2(x,t)}{\partial x^2}, x \in (x_0; +\infty), t \in (0; +\infty), \\
u_2(x_0,t) = f(t), t \in [0; +\infty), u_2(+\infty, t) = 0, t \in [0; +\infty), \\
u_2(x,0) = 0, x \in [x_0; +\infty).
\end{cases}
\]
(6)

The following theorem is true:

**Theorem 4.1.** The solution to the problem (6) exists, is unique and can be written as:

\[
u_2(x,t) = \int_0^t \frac{(x-x_0)f(\tau)}{\sqrt{4a_2^2\pi(t-\tau)^3}} \exp\left(-\frac{(x-x_0)^2}{4a_2^2(t-\tau)}\right) d\tau.
\]
(7)

To find the derivative \(\frac{\partial u_2(x_0,t)}{\partial x}\), we use the original information \(f_\delta(t), \delta\).

From the system of equations
\[
\begin{cases}
\frac{\partial u_2(x,t)}{\partial t} = a_2^2 \frac{\partial^2 u_2(x,t)}{\partial x^2}, x \in (x_0; +\infty), t \in (0; 0), \\
u_2(x,0) = 0, x \in [x_0; +\infty), u_2(+\infty, t) = 0, t \geq 0,
\end{cases}
\]
(8)

we find the function \(u_2(x,t)\) for \(x \leq x, t \geq 0\):

\[
u_2(x,t) = \int_0^t \frac{x-x_0}{\sqrt{4\pi a_2^2(t-\tau)^3}} f(\tau) \exp\left(-\frac{(x-x_0)^2}{4a_2^2(t-\tau)}\right) d\tau.
\]

Lemma 2.1 implies that the function \(u_2(x,t) \in C^{1,2}([0, \infty) \times (x_0, +\infty))\). A solution \(u_2(x,t)\) to the problem (8) exists for \(f(t) = f_0(t)\).

**Lemma 4.2.** Let \(\phi(x,t) = u_2(x,t) - u_2(x,t, \delta)\). Then for any \(x \in (x_0; +\infty), t \geq 0\) \(|\phi(t,x)| \leq \delta\).

**Proof.** A solution to the problem is the \(\phi(t,x)\) function
\[
\begin{cases}
\frac{\partial \phi(x,t)}{\partial t} = a_2^2 \frac{\partial^2 \phi(x,t)}{\partial x^2}, x \in (x_0,1), t \in (0, \infty), \\
\phi(x,t) = 0, x \in [x_0; +\infty), f_\delta(t) - f_0(t), t \geq 0, \phi(t,1) = 0, t \geq 0.
\end{cases}
\]
(9)

Let \(\Omega = [x_0; +\infty) \times [0, \infty)\) is the set on which the problem (9) is defined, and \(\partial \Omega = \Gamma_1 \cup \Gamma_2\), where \(\Gamma_1 = \{(x,t) : x_0 \leq x < +\infty\}, \Gamma_2 = \{(x,t) : 0 \leq t < \infty\}\). Then \(\sup_{(x,t) \in \Omega} |\phi(x,t)| \leq \sup_{(x,t) \in \partial \Omega} |\phi(x,t)|\).

Since \(\sup_{(x,t) \in \Gamma_1} |\phi(x,t)| = 0\) and \(\sup_{(x,t) \in \Gamma_2} |\phi(x,t)| \leq \delta\), then by the maximum principle \(\sup_{(x,t) \in \Omega} |\phi(x,t)| \leq \delta\).

The derivative \(\frac{\partial u_2(x_0,t)}{\partial x}\) is denoted by \(g_0(t)\),

\[
\frac{\partial u_2(x_0,t)}{\partial x} = \int_0^t \frac{1}{\sqrt{4\pi a_2^2(t-\tau)^3}} f_0(\tau) \exp\left(-\frac{(x-x_0)^2}{4a_2^2(t-\tau)}\right) d\tau -
\int_0^t \frac{(x-x_0)^2}{\sqrt{64\pi a_2^2(t-\tau)^5}} f_0(\tau) \exp\left(-\frac{(x-x_0)^2}{4a_2^2(t-\tau)}\right) d\tau \equiv g_0(t).
\]
It is required to define the function \( g_\delta(t) \) and the number \( \sigma(\delta) \) such that \( \sup_{t \in [0, \infty)} |g_\delta(t) - g_0(t)| < \sigma(\delta) \) using the initial data \( (u_2(x_0, t, \delta), \delta) \).

Since the problem of calculating the derivative in the space \( C[a, b] \) is incorrect \([3], [4]\), we construct a regularizing family of operators \( \{R_\nu : \nu > 0\} \), which for any \( \nu > 0 \) maps the space \( C[0, \infty) \) to \( C[0, \infty) \) and are defined by the formula \([5]\): \( R_\nu f(t) = \frac{u_2(x_0 + \nu, t) - f_0(t)}{\nu}, f(t) \in C[0, \infty), \ f(0) = 0, \ \nu > 0, \ t \in [0, \infty). \)

At the points \( (x_0, t) \) the derivative \( \frac{\partial u_2(x_0, t)}{\partial x} \), existence of which follows from Theorem 3.1, is replaced by the increment relation

\[
R_\nu u_2(t, x_0, \delta) = \frac{u_2(t, x_0 + \nu, \delta) - u_2(t, x_0, \delta)}{\nu}, \nu = \nu(\delta) = \sqrt{\frac{2\delta}{d}}. \tag{10}
\]

**Lemma 4.3.** Let \( R_\nu f_0(t) \) be defined by formula (10), and \( d \geq \left\| \frac{\partial^2 u_2(x, t)}{\partial x^2} \right\|_{C([0, \infty) \times [x_0; +\infty))} \).

Then \( \sup_{t \geq 0} |g_0(t) - R_\nu f_0(t)| \leq d \cdot \nu. \)

**Proof.** \([6]\)

Let’s set the inverse problem. We know the function \( u_2(x_0, t) \) and the derivative \( \frac{\partial u_2(x_0, t)}{\partial x}, t \geq 0 \). It is necessary to find a solution to problem:

\[
\begin{align*}
\frac{\partial u_1(x, t)}{\partial t} &= a_1^2 \frac{\partial^2 u_1(x, t)}{\partial x^2}, x \in (0; x_0), t \in (0; +\infty), \\
u_1(0, t) &= q(t), t \in [0; +\infty), u_1(x_0, t) = f(t), t \in [0; +\infty), \\
u_1(x_0, t) &= g(t), t \in [0; +\infty), u(x, 0) = 0, x \in [0; +\infty). \tag{11}
\end{align*}
\]

**5. Estimates for layer error**

Now consider the following situation: let the heat-shielding layer collapsed during the heating of the body to a thickness \( \zeta \). Let this value \( \zeta \) be small in comparison with \( x_0 : \frac{\zeta}{x_0} \ll 1 \). Let us find out what destruction thickness of the thermal protection layer will introduce an error that does not go beyond the error limits \( \delta \).

Let us assume that destruction (in essence, peeling of a part of the protection) occurred at the very beginning of the thermal process and in reality, at \( x = x_0 \) not the temperature \( f(t) = u_2(x_0, t) \) is measured, but the temperature \( f_1(t) = u_2(x_0 - \zeta, t) \) — temperature corresponding to a thermal layer that differs from the initial one by the thickness \( \zeta \). Then the temperature, corresponding to a thermal layer that differs from the initial one by thickness \( \delta q(t) \), caused by these factors, is:

\[
\delta q(t) = \int_0^t \frac{f_1(\tau)(x_0 - \zeta)d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \exp \left( -\frac{(x_0 - \zeta)^2}{4a_1^2(t - \tau)} \right) - \int_0^t \frac{x_0 f(\tau)d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \exp \left( -\frac{x_0^2}{4a_1^2(t - \tau)} \right). \tag{12}
\]
We assume that
\[ \max_{t \in [0; +\infty)} |f(t) - f_1(t)| < \delta. \]
Then (12):
\[
\|\delta q(t)\|_C \leq \zeta \|f(t)\|_C \cdot \int_0^t \frac{x_0 d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \exp\left(-\frac{(x_0 - \zeta)^2}{4a_1^2(t - \tau)}\right) + \\
+ \|f(t)\|_C \int_0^t \frac{x_0 d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \left\{ \exp\left(-\frac{x_0^2}{4a_1^2(t - \tau)}\right) - \exp\left(-\frac{(x_0 - \zeta)^2}{4a_1^2(t - \tau)}\right) \right\},
\]
where \( \|f(t)\|_C \equiv \sup_{t \geq 0} |f(t)|. \) The estimate is valid:
\[
\left| \int_0^t \frac{x_0 d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \exp\left(-\frac{(x_0 - \zeta)^2}{4a_1^2(t - \tau)}\right) \right| \leq 1,
\]
and equations are valid:
\[
\int_0^t \frac{x_0 d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \exp\left(-\frac{x_0^2}{4a_1^2(t - \tau)}\right) = \frac{1}{2} - \Phi\left(\frac{x_0}{2a_1 \sqrt{t}}\right),
\]
\[
\int_0^t \frac{(x_0 - \zeta) d\tau}{\sqrt{4\pi a_1^2(t - \tau)^3}} \exp\left(-\frac{(x_0 - \zeta)^2}{4a_1^2(t - \tau)}\right) = \frac{1}{2} - \Phi\left(\frac{x_0 - \zeta}{2a_1 \sqrt{t}}\right),
\]
where \( \Phi(s) \) is the Laplace function.

Due to the (14), (15), (16) the estimate (13):
\[
\|\delta q(t)\|_C \leq \zeta \|f(t)\|_C + \|f(t)\|_C \left| \Phi\left(\frac{x_0}{2a_1 \sqrt{t}}\right) - \Phi\left(\frac{x_0 - \zeta}{2a_1 \sqrt{t}}\right) \right|.
\]
Let us estimate the difference of the Laplace functions. We get:
\[
\left| \Phi\left(\frac{x_0}{2a_1 \sqrt{t}}\right) - \Phi\left(\frac{x_0 - \zeta}{2a_1 \sqrt{t}}\right) \right| \leq \frac{\zeta}{2a_1 \sqrt{t}} \exp\left(-\frac{x_0^2}{4a_1^2 t}\right).
\]
The exponential expression is bounded and reaches its largest value at \( t = \frac{x_0^2}{2a_1^2} \). Therefore, the upper bound is true:
\[
\frac{\zeta}{2a_1 \sqrt{t}} \exp\left(-\frac{x_0^2}{4a_1^2 t}\right) \leq \frac{\zeta}{x_0 \sqrt{2\pi e}},
\]
It follows from the formulas (19), (18) and (17):
\[
\delta = \frac{\zeta}{x_0 \|f(t)\|_C} \left(1 + \frac{1}{\sqrt{2\pi e}}\right).
\]
6. Reduction of the inverse problem (11) to the calculating of the unbounded operator values

Let’s introduce the correctness class $M_{\delta}$

$$M_{\delta} = \left\{ q(t) \cdot e^{-t} : q(t) \in C^2[0; +\infty), \int_0^{+\infty} |q(t) \cdot e^{-t}|^2 dt + \int_0^{+\infty} |q'(t) \cdot e^{-t}|^2 dt \leq b^2 \right\},$$

$\delta$ – known positive number. Let us suppose that for $f(t) = f_0(t) \in C^1[0, \infty)$ and $g(t) = g_0(t) \in C^1[0, \infty)$ there exists the solution $q_0(t)$ to the inverse problem (11), but the functions $f(t)$ and $g(t)$ are unknown to us. Some approximate functions $f_\delta(t) \in C^1[0; \infty), \ g_\delta(t) \in C^1[0; \infty)$ are known instead of them and error levels $\delta > 0$ and $\sigma(\delta)$, such that

$$\sup_{t \in (0;\infty)} |f_\delta(t) - f(t)| \leq 2\delta, \quad \sup_{t \in (0;\infty)} |g_\delta(t) - g(t)| \leq 2\sqrt{2 \cdot d \cdot \delta}. \quad (21)$$

Moreover, note that $2\delta$ in the expression $\sup_{t \in (0;\infty)} |f_\delta(t) - f(t)| \leq 2\delta$ is due to the error measurements of $f_\delta(t)$, and the error introduced by the destruction of the protective layer.

It is required to determine an approximate solution $q_\delta$ to the problem (11) using $f_\delta(t), g_\delta(t), \delta$ and $M_{\delta}$.

Note that the Fourier transform with respect to $t$ plays an important role in obtaining an estimate of the error in the inverse problem approximate solution. Let’s make a substitution to use the Fourier transform

$$z(x,t) = e^{-t}w_1(x,t), \quad 0 \leq x \leq x_0, \ t > 0. \quad (22)$$

We get the problem:

$$\left\{ \begin{array}{ll}
\frac{\partial z(x,t)}{\partial t} = a_1 \frac{\partial^2 z(x,t)}{\partial x^2} - z(x,t), & 0 < x < x_0, \ t \geq 0, \\
z(x,0) = 0, & 0 \leq x \leq x_0, \\
z(x_0,t) = s(t), & s(t) = f(t) \cdot e^{-t}, \ t \geq 0, \\
a_1 \frac{\partial z(x_0,t)}{\partial x} = p(t), & p(t) = g(t) \cdot e^{-t}, \ t \geq 0.
\end{array} \right. \quad (23)$$

where $q(t) \cdot e^{-t} \in M_{\delta}$. Note that this technique is discussed in detail in [6], where the necessary theorems are proved.

Let us introduce the operator $F$, that maps $L_2[0; \infty) \cap L_1[0; \infty)$ into $L_2(\infty; \infty) \cap C_0(\infty; \infty)$ and is defined by formula

$$\hat{q}(\tau) = F[q(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(t) e^{-i\tau t} dt, \ q(t) \in L_2[0; \infty) \cap L_1[0; \infty), \ q(t) = \left\{ \begin{array}{ll}
q(t), & t > 0, \\
0, & t \leq 0.
\end{array} \right. \quad (24)$$

Define the function $\hat{m}(\tau)$, using the functions $\hat{s}(\tau)$ and $\hat{p}(\tau)$,

$$\hat{m}(\tau) = \hat{s}(0, \tau), \ -\infty < \tau < \infty, \quad (25)$$

for which is true [6]:

$$\hat{m}(\tau) = T^1(\tau) \cdot \hat{s}(\tau) + T^2(\tau) \cdot \hat{p}(\tau), \quad (26)$$

where

$$T^1(\tau) = \cosh \left( \frac{x_0 \sqrt{1 + i\tau}}{a_1} \right), \ T^2(\tau) = -\frac{a_2}{\sqrt{1 + i\tau}} \sinh \left( \frac{x_0 \sqrt{1 + i\tau}}{a_1} \right).$$
Lemma 7.1. Let $\hat{\mathcal{M}}_\delta$ be the set defined by the formula

$$\hat{\mathcal{M}}_\delta = \left\{ \hat{m}(\tau) : \hat{m}(\tau) \in L_2[0, \infty), \int_{-\infty}^{\infty} (1 + \tau^2) |\hat{m}(\tau)|^2 d\tau \leq 2b^2 \right\}.$$ 

Since $m_0(t) \in \mathcal{M}_\delta$, implies $\hat{m}_0(\tau) \in \hat{\mathcal{M}}_\delta$.

7. The solution to the problem (26)–(28)

We use the projection regularization method to solve the problem (26)–(28). This method is based on the regularizing family of operators $\{T_\alpha : \alpha > 0\}$, defined by the formula

$$T_\alpha \{\hat{s}(\tau), \hat{p}(\tau)\} = \left\{ T\{\hat{s}(\tau), \hat{p}(\tau)\}, |\tau| \leq \alpha, 0, |\tau| > \alpha. \right\}$$

The regularized solution $\hat{q}_\delta^\alpha(\tau)$ to the problem (26) is defined by the formula

$$\hat{q}_\delta^\alpha(\tau) = T_\alpha \{\hat{q}_\delta(\tau), \hat{p}_\delta(\tau)\}; \ |\tau| \geq 0, \alpha > 0, \delta > 0. \ (29)$$

To choose the regularization parameter $\alpha = \alpha(\delta, b)$ in the formula (29), consider the estimate

$$\|\hat{q}_\delta^\alpha(\tau) - \hat{q}_0(\tau)\| \leq \|\hat{q}_\delta^\alpha(\tau) - \hat{q}_\delta^\alpha(\tau)\| + \|\hat{q}_\delta^\alpha(\tau) - \hat{q}_0(\tau)\|, \ (30)$$

where $\hat{q}_\delta^\alpha(\tau) = T_\alpha \{\hat{q}_\delta(\tau), \hat{p}_\delta(\tau)\}$.

Let’s consider an estimate of the $\|T_\alpha\|$.

Lemma 7.1. There exists $\alpha_1 > 0$ such that $\forall \alpha, \alpha \geq \alpha_1$

$$\|T_\alpha(\tau)\| \leq 4 \cdot e^{\sqrt{\alpha/2} \cdot \frac{x_0}{a_1}}$$

is true.

Proof. By definition, $\|T_\alpha\| = \sup_{0 \leq |\tau| \leq \alpha} |T(\tau)|$. Let’s consider $|T(\tau)| \leq |T_1(\tau)| + |T_2(\tau)|$. We get from $\cosh z = \sqrt{\cosh^2 x - \cosh^2 y}, z = x + iy$ and

$$\sqrt{1 + i\tau} = \sqrt{\frac{1 + \tau^2 + 1}{2}} + i \sqrt{\frac{1 + \tau^2 - 1}{2}}.$$

$$|\cosh \left( \frac{x_0 \sqrt{1 + \tau}}{a_1} \right) | \leq \cosh \left( \sqrt{\frac{1 + \tau^2 + 1}{2}} \cdot \frac{x_0}{a_1} \right) \leq \cosh \left( \sqrt{\frac{1}{2} \cdot \frac{x_0}{a_1} } \right).$$

In this way, $|T_1(\tau)| \leq 2 \exp \left\{ \frac{\sqrt{\tau} \cdot x_0}{\sqrt{2} \cdot a_1} \right\}$.

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Considering \(|\sinh z_2| = \sqrt{\cosh^2 x_2 - \cos^2 y_2}\) and \(z_2 = x_2 + iy_2\), we get that \(\exists c_5 > 0\)
\[|T^2(\tau)| = \left| \frac{a_2}{\sqrt{1 + i\tau}} \sinh \left( \frac{x_0 \sqrt{1 + i\tau}}{a_1} \right) \right| \leq c_5 \exp \left( \frac{\sqrt{\tau} \cdot x_0}{\sqrt{2}} \right).\]

The above implies \(\exists \tau_1 > 0\) such that \(\forall \tau, |\tau| \geq \tau_1\) the relation \(|T^2(\tau)| \leq |T^1(\tau)| \leq 2\exp \left( \frac{\sqrt{\tau/2} \cdot x_0}{a_1} \right)\) is valid.

Let \(\alpha_1 = \tau_1\), then \(\forall \alpha, \alpha \geq \alpha_1\) is true \(\|T_\alpha(\tau)\| \leq 4 \cdot e^{\sqrt{\alpha/2} \cdot x_0/\alpha_1}\).

This proves the lemma.

Let’s get an estimate for \(\|\tilde{m}_0^\alpha(\tau) - \hat{m}_0(\tau)\|\) in (30). Let
\[
\Delta^2_\alpha = \sup \left\{ \int_{-\infty}^{-\alpha} |\tilde{m}_0(\tau)|^2 d\tau + \int_{\alpha}^{\infty} |\hat{m}_0(\tau)|^2 d\tau : \hat{m}_0(\tau) \in \hat{M}_b \right\}.
\]

We obtain from (28) that under the condition \(\hat{m}_0(\tau) \in \hat{M}_b\)
\[
\int_{-\infty}^{-\alpha} (1 + \tau^2)|\tilde{m}_0(\tau)|^2 d\tau + \int_{\alpha}^{\infty} (1 + \tau^2)|\hat{m}_0(\tau)|^2 d\tau \leq 2\beta^2.
\]

From (31) and (32) it follows \(\Delta^2_\alpha \leq 2\beta^2/(1 + \alpha^2)\).

Thus, from (30), (32), the previous relation and Lemma 7.1 we obtain \(\|\tilde{m}_0^\alpha(\tau) - \hat{m}_0(\tau)\| \leq 2\sqrt{2} \cdot \frac{\beta}{\sqrt{1 + \alpha^2}}; \|\tilde{m}_0^\alpha(\tau) - \hat{m}_0(\tau)\| \leq 8 \cdot (\delta + \sqrt{d \cdot \delta}) \cdot e^{\frac{\pi \alpha}{4}} \sqrt{\alpha/2}.

From the above, we obtain \(\exists \alpha_0 > 0\) such that \(\|\tilde{m}_0^\alpha(\tau) - \hat{m}_0(\tau)\| \leq 2\sqrt{2} \cdot \frac{\beta}{\sqrt{1 + \alpha^2}} + 8 \cdot c_6 \cdot \sqrt{\delta} \cdot e^{\frac{\pi \alpha}{4}} \sqrt{\alpha/2}.

Regularization parameter \(\alpha = \alpha(\delta)\) in formula (29) is to be selected from the condition
\[
\sqrt{1 + \alpha^2} \cdot \|T_\alpha\| \cdot c_6 \cdot \sqrt{\delta} = 2\sqrt{2} \cdot \frac{\beta}{\sqrt{\delta}}.
\]

We get considering the above
\[
\|\tilde{m}_0(\tau)\| \leq 4\sqrt{2} \cdot \frac{\beta}{\sqrt{1 + \alpha^2}}(\delta).
\]

Since \(\sqrt{1 + \alpha^2} \exp \left( \frac{x_0}{a_1} \sqrt{\alpha^2/2} \right)\) increases strictly in \(\alpha\) and changes from 1 to \(\infty\), then there is a unique solution \(\alpha(\delta, \beta)\) to equation (33) and \(\alpha(\delta, \beta) \geq \alpha_1\).

We approximate the equation (33) by a pair of equations, since it has no solution in elementary functions:
\[
\exp \left( \frac{x_0}{a_1} \sqrt{\alpha_1^2/2} \right) = \frac{\beta}{\sqrt{\delta}} \quad \exp \left( \frac{2x_0}{a_1} \sqrt{\alpha_2^2/2} \right) = \frac{\beta}{4 \cdot \sqrt{\delta}}.
\]

The solutions to the equations (35) are denoted by \(\alpha_1(\delta, \beta)\) and \(\alpha_2(\delta, \beta)\) respectively. The following relations are true for sufficiently small values of \(\delta\)
\[
\alpha_2(\delta, \beta) \leq \alpha(\delta, \beta) \leq \alpha_1(\delta, \beta).
\]

We obtain from (35) \(\alpha_1(\delta, d) = \frac{2\alpha_1^2}{x_0^2} \ln^2 \left( \frac{\beta}{\sqrt{\delta}} \right)\) and \(\alpha_2(\delta, d) = \frac{\alpha_2^2}{2x_0^2} \ln^2 \left( \frac{\beta}{4 \cdot \sqrt{\delta}} \right)\). And from (36):
\[
\alpha(\delta, \beta) \sim \ln^2 \delta, \delta \to 0.
\]
The solution to the problem (26)–(28) is defined by the formula
\[ \hat{m}_\delta(\tau) = \hat{m}_0(\tau). \]
Then it follows from the relation (34) that
\[ \|\hat{m}_\delta(\tau) - \hat{m}_0(\tau)\| \leq \frac{4\delta}{\sqrt{1 + \pi^2_1(\delta, b)}}. \] (38)

Let denote by \( \varpi(\tau, b) \) the problem conditional correctness modulus (26)–(28) (the modulus of continuity of the operator \( T \) on the set \( \hat{M}_b \))
\[ \varpi(\tau, b) = \sup\{\|T\hat{f}\| : T\hat{f} \in \hat{M}_b, \|\hat{f}\| \leq \tau\}, \quad \tau, b > 0. \] (39)

**Theorem 7.2.** The problem (26)–(28) is conditionally well-posed for the conditional well-posedness module of this problem on the set \( \hat{M}_b \) for sufficiently small values of \( \delta \). The estimate is
\[ \varpi(\tau, b) \leq \frac{8\delta}{\sqrt{1 + \alpha^2_1(\tau, b)}}. \]

The proof follows from the formulas (38) and (39).

Thus, the solution \( m_\delta(t) \) to the inverse problem (23) and (27) is defined by the formula
\[ m_\delta(t) = \begin{cases} \text{Re}F^{-1}[\hat{m}_\delta(\tau)], & t \geq 0, \\ 0, & t < 0, \end{cases} \] (40)
where \( F^{-1} \) is the inverse of \( F \).

Considering the above, for \( m_\delta(t) \) the estimate
\[ \|m_\delta(t) - m_0(t)\| \leq \frac{4\delta}{\sqrt{1 + \pi^2_1(\delta, b)}}. \] (41)

(37) and (41) implies \( \exists d_1 > 0 \) such that for any sufficiently small \( \delta \) the estimate \( \|m_\delta(t) - m_0(t)\| \leq d_1 \cdot b \cdot \log^{-2} \delta \).

**8. Conclusion**

The problem we have considered allows us to estimate the level of error, including the error in the layer thickness. If the error in determining the thickness of the heat-shielding layer \( \zeta \) coincides with the measurement error \( \delta \), the following relation is true
\[ \frac{\delta}{\sup_{t \geq 0} |f(t)|} = \frac{\zeta}{x_0} \left(1 + \frac{1}{\sqrt{2\pi e}}\right). \]

The level of error is determined by the formula:
\[ \|m_\delta(t) - m_0(t)\| \leq d_1 \cdot b \cdot \ln^{-2}(\delta). \]

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