All-loop ABJM amplitudes from projected, bipartite amplituhedron

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We define a new geometry obtained from the all-loop amplituhedron in $\mathcal{N} = 4$ SYM, by projecting its external and loop momenta to three dimensions. Focusing on the simplest four-point case, we provide strong evidence that the canonical form of this “projected amplituhedron” gives all-loop integrand of ABJM four-point amplitude! In addition to various all-loop cuts manifested by the geometry, we present explicitly new results for the integrand up to five loops, which are much simpler than $\mathcal{N} = 4$ SYM results. Such all-loop simplications become most transparent for projected negative geometries, where projected loop variables live in AdS$_3$ space and mutual negativity conditions lead to time orderings; amazingly we find that only bipartite graphs, where each node is either a source or sink in time ordering, survive the projection. Our results suggest an unexpected relation between four-point amplitudes in these two theories.

I. INTRODUCTION

The amplituhedron in planar $\mathcal{N} = 4$ SYM\textsuperscript{13} is one of the most surprising mathematical structures of scattering amplitudes we have seen, in that locality and unitarity are derived from underlying geometric picture (as opposed to being basic principles). All-loop integrands (and tree amplitudes) are given by canonical forms, which (only) have logarithmic singularities on boundaries, of the amplituhedron (positive geometry)\textsuperscript{4, 5}. As a beautiful mathematical object with remarkable physical properties, the amplituhedron has been extensively studied both at tree and loop level (c.f.\textsuperscript{2} 6\textsuperscript{11}, and in particular it can be used to make all-loop predictions about (cuts of) the integrand \textsuperscript{15} 16, which seem impossible to get using other method. On the other hand, it has been notoriously difficult to compute its canonical form at multi-loop level: even for the four-point ($n = 4$) $L$-loop amplituhedron, the geometry becomes very complicated as $L$ increases, and to our best knowledge an explicit computation for $L \geq 4$ has not been performed (though $n = 4$ integrand has been known up to $L = 10$\textsuperscript{17} 20). Besides, despite various interesting ideas extending such geometric pictures beyond planar $\mathcal{N} = 4$ SYM\textsuperscript{21} 31, we were not aware of an example of all-loop amplituhedron (even for $n = 4$) in a different theory.

In this letter, by “dimensional reducing” external and loop (region) momenta of the amplituhedron, we find that “projected” amplituhedron geometries have rich structures but the computation for canonical forms are greatly simplified, at least for the $n = 4$ case which is a $3L$-dimensional geometry in the space of $L$ loop variables. Remarkably, we find this simplified $n = 4$ geometry to be the long-sought-after all-loop amplituhedron for four-point amplitudes in Aharony-Bergman-Jafferis-Maldacena (ABJM) theory\textsuperscript{22}. Much less is known about ABJM integrands beyond $L = 2$\textsuperscript{33} 34 (the only data available is a conjecture for $n = 4$, $L = 3$ in\textsuperscript{35}), and even tree (momentum) amplituhedron has only been proposed recently\textsuperscript{36} 37. Here we will not only show that the forms of $n = 4$ projected amplituhedron manifest various highly non-trivial all-loop cuts of ABJM amplitudes, but also push the frontier significantly by presenting compact formulas for integrands up to $L = 5$.

In $\mathcal{N} = 4$ SYM, it is beneficial to decompose $n = 4$ amplituhedron into building blocks called negative geometries\textsuperscript{38}, and at each loop, the non-trivial geometries combine to give the integrand for an infrared-finite observable closely related to logarithm of amplitudes (or equivalently Wilson loops with a single insertion)\textsuperscript{39} 44. Analogous decomposition of projected amplituhedron reveals enormous simplications from $D = 4$ to $D = 3$, since we only have a tiny fraction of negative geometries, namely those correspond to bipartite graphs, contribute to the integrand, with very simple pole structures. We will see that the origin of such all-loop simplications lies at the fact that projected loop variables live in AdS$_3$ with natural time orderings, and the combination of negative geometries undergoes a huge reduction to those graphs with sources and sinks only.

II. FOUR-POINT ABJM AMPLITUDES FROM PROJECTED GEOMETRIES

Recall that the $n$-point amplituhedron is defined in the space of $n$ momentum twistors\textsuperscript{45}, $Z_a^i$ with $a = 1, 2, \cdots, n$ for external kinematics, as well as $L$ lines in twistor space, $(AB)^{ij}_a$ with $i = 1, \cdots, L$ for loop momenta; here $I, J, \cdots, 4$ are $SL(4)$ index, and $SL(4)$ invariant is defined as $\langle abcd \rangle = \epsilon_{ijkL}Z^i_aZ^j_bZ^k_cZ^L_d$ (and
similarly for \((AB)_i(ab)\) and \((AB)_i((AB)_j)\). In [46], external kinematics in \(D = 3\) was defined by dimensional reducing any external line, \(Z_a Z_a = 1\), in a completely analogous manner, here we also need dimension reduction of any line for loop variable \((AB)_i\), both of which are achieved by the so-called symplectic conditions:

\[
\Omega_{IJ} Z_a^I Z_a^J = \Omega_{IJ} A^I_i B^J_i = 0
\]  

(1)

for \(a = 1, 2, \ldots, n\) and \(i = 1, \ldots, L\), where \(\Omega = \begin{pmatrix} 0 & \epsilon_{2x2} \\ \epsilon_{2x2} & 0 \end{pmatrix}\) with totally antisymmetric matrix \(\epsilon_{2x2}\).

One could define the projected amplituhedron for any \(n, k, L\) by restricting the \(D = 4\) amplituhedron on the subspace defined by [41], if it has non-vanishing support there. In this letter we focus on the special case \(n = 4\) (with only \(k = 0\)), and it is clear that we have a 3L-dimensional geometry defined in projected \((AB)_i = 1, \ldots, L\) space. An important subtlety is that \((1234) < 0\) for real \(Z\)’s satisfying symplectic conditions, thus we need to flip the overall sign for the definition of \(D = 4\) amplituhedron [41]: we require \((AB)_{12}, (AB)_{23}, (AB)_{34}, (AB)_{14} < 0\) and \((AB)_{13}, (AB)_{24} > 0\), for any loop \((AB)_i\), as well as \((AB)_i(AB)_j < 0\), all on the support of \([41]\).

A convenient parametrization is \((A, B)_i = (Z_1 + z_1 Z_2 - w_1 Z_3, y_1 Z_2 + Z_3 + z_1 Z_4) [42]\), and symplectic condition on \((AB)_i\) becomes \(x_i z_i + y_i w_i - 1 = 0\); the \(n = 4\) geometry is defined by \((x_i, y_i) = (x_i - x_{i+1})\).

We denote this geometry as \(A_L\) with canonical form \(\Omega(A_L) := \Omega_L\), and our main claim is that \(\Omega_L\) gives the \(L\)-loop planar integrand for four-point ABJM amplitudes (after stripping off the overall tree amplitude).

A. Soft and vanishing cuts to all loops

Let us take a first look at \(L = 1\), where the geometry is defined as \(x, y, z, w > 0\) and \(xz + yw = 1\). In this special case, its canonical form is nothing but the projection of \(D = 4\) form on the 3d subspace:

\[
\Omega_1 = \frac{dx \, dy \, dz \, dw}{x \, y \, z \, w} \delta(xz + yw - 1)
\]

and we can rewrite it in covariant form and find:

\[
\Omega_1 = \frac{d^3 A B (1234)^{3/2} ((AB)_{12}) \delta(1234) \delta(AB)_{12}}{(AB)_{12} (AB)_{23} (AB)_{34} (AB)_{14}}
\]

(3)

where the measure is \(d^3 A B := \langle A B ^D A \rangle \delta(1234) \delta(AB)_{12}\), and the numerator, \((AB)_{12} (AB)_{23} (AB)_{34} (AB)_{14}\), turns out to be proportional to the famous \(\epsilon\) numerator of DCI box in \(D = 3\) [33]. Thus the projection of one-loop box in \(N = 4\) SYM gives that in ABJM (with \(\epsilon\) numerator), which confirms our claim at one loop.

Now we provide strong evidence that \(\Omega_L\) gives \(L\)-loop integrand for \(L > 1\). Let us first rewrite the inequalities [4] by solving for \(x\) variables \(x_i = (1 - y_i w_i)/z_i\), and we arrive at the following equivalent definition

\[
w_i, y_i, z_i > 0, w_i y_i < 1, \\
d_{ij} := (w_i z_j - w_j z_i)(y_j z_j - y_j z_i) - (z_j - z_i)^2 < 0,
\]

(4)

for \(i, j = 1, \ldots, L\). Before proceeding to explicit computations, we see that [4] allows us to study some all-loop cuts in a simple way. An important cut of four-point ABJM amplitudes is the so-called soft cut, where we take \((AB)_{12} = (AB)_{23} = (AB)_{34} = 0\), or equivalently \(y = z = w = 0\) for any given loop [47], and the result is \((L-1)\)-loop integrand. From geometry, with \(y_i = z_i = w_i = 0\) clearly \(d_{ij} < 0\) is trivially satisfied for any \(j \neq i\), thus the geometry reduces to \((L-1)\)-loop one:

\[
d_{ij} = 0, y_i = z_i = w_i = 0 A_L = A_{L-1}.\]

The soft cut is satisfied!

Moreover, certain cuts are known to vanish due to presence of vanishing odd-point amplitudes: by cutting \((AB)_{12} = (AB)_{13} (AB)_{14} = (AB)_{12} = 0\) (or change the last one to \((AB)_{34}\), we isolate a three-point (or five-point, respectively) amplitude, which must vanish. These are equivalent to setting \(w_i = d_{ij} = w_j = 0\) or \(w_i = d_{ij} = y_j = 0\); in either case, \(d_{ij} = z_{ij}^2 < 0\) is trivially satisfied on the support of the other two conditions, and the residue vanishes as expected. These and other vanishing cuts are nicely guaranteed by the geometry.

B. ABJM integrands to five loops from geometry

We present explicitly the canonical form up to five loops and leave the detailed derivation later. It is well known that the logarithm of amplitudes, denoted as \(\Omega_L\), takes a more compact form, and \(\Omega_L\) can be trivially recovered from it. To save space, we introduce shorthand notation \(\langle A B \rangle_i := \ell_i\), and it turns out the logarithm at \(L = 2\), \(\Omega_2 := \Omega(\ell_1, \ell_2) - \frac{1}{2} \Omega(\ell_1) \Omega(\ell_2)\), is simply

\[
\Omega_2 = -\frac{d^3 \ell_1 d^3 \ell_2 (1234)^2}{(\ell_1 \ell_2 \ell_3 \ell_4 \ell_1 \ell_2 \ell_3 \ell_4 \ell_1 \ell_2)} (\ell_1 \leftrightarrow \ell_2)
\]

(5)

This is nothing but a double-triangle integral where external region momenta correspond to (12), (34) for \(\ell_1\), and (23), (14) for \(\ell_2\), and vice versa. One can check easily that by adding back one-loop squared, we recover the well-known two-loop result [33].

One interesting feature of \(\Omega_2\) is that for each term, \(\ell_i\) contains only two poles, \((\ell_i \ell_2) \langle \ell_3 \rangle \langle \ell_4 \rangle\) or \((\ell_i \ell_3) \langle \ell_2 \rangle \langle \ell_4 \rangle\) (similarly for \(\ell_2\)). We denote these combinations and mutual conditions, which include all possible poles, as:

\[
s_i := \langle \ell_1 \ell_2 \rangle \langle \ell_3 \rangle, \quad t_i := \langle \ell_1 \ell_3 \rangle \langle \ell_2 \rangle, \quad D_{ij} := -\langle \ell_i \ell_j \rangle.
\]

We also denote the \(\epsilon\) numerator as \(\epsilon_1 := \langle \ell_1 \ell_3 \rangle \langle 12 \rangle \langle 13 \rangle\), and the constant (in loop
variables) \( c \equiv \langle 1234 \rangle \). In this notation, the integrands for \( L = 1, 2 \) (with \( \prod_{i=1}^{L} d^3 \ell_i \) stripped off) become

\[
\Omega_1 = \frac{c\epsilon}{s_{1} t_{1}}, \\
\Omega_2 = \frac{2c^2}{D_{12}^{2}} (\frac{1}{s_{1} t_{2}} + \frac{1}{t_{1} s_{2}}) = \frac{1}{2} + \frac{1}{2}.
\]

We represent \( \tilde{\Omega}_2 \) as a “chain” with \( s, t \) pole structures represented by black/white coloring, and we will discuss such “bipartite” structures shortly.

Now we are ready to move to \( L = 3 \). Remarkably we find that \( \Omega_3 \) only receives contributions with pole structures of 3 chains (each with two possible choices of \( s \) and \( t \)), as represented by 6 bipartite graphs. It reads

\[
\tilde{\Omega}_3 = \frac{4c^2 \epsilon}{s_1 t_2 s_3 D_{12} D_{23}} + (s \leftrightarrow t) + 2 \text{ perms}.
\]

where for each chain we have two terms with \( s, t \) swapped, and similar to \( L = 1 \) the \( \epsilon \)-numerator makes correct weight [48]. Each term is again a ladder integral with two triangles and a middle box (with \( \epsilon \) numerator). Very non-trivially, when converting back to \( \Omega_3 \), it agrees with the conjecture from general unitarity [45] and we have performed various new checks to be outlined in appendix.

After familiarized with the notation, we present the \( L = 4 \) result in a very compact form, and it turns out \( \Omega_4 \) only gets contributions from three topologies. We give all bipartite graphs in Fig. [1B] The first type consists of \( 12 \times 2 \) chain bipartite graphs

\[
C = \left( \frac{1}{2} + \frac{3}{2} \right) + 11 \text{ perms}
\]

\[
S = \left( \frac{1}{2} + \frac{3}{2} \right) + 3 \text{ perms}
\]

\[
B = \left( \frac{1}{2} + \frac{3}{2} \right) + 2 \text{ perms}
\]

where we define combinations similar to \( s, t \) but for two \( \ell \)'s, \( N^3_{13} := \langle \ell_1 12 \rangle \langle \ell_3 4 \rangle + \langle \ell_3 12 \rangle \langle \ell_1 4 \rangle \) and \( N^2_{24} := \langle \ell_2 14 \rangle \langle \ell_4 23 \rangle + \langle \ell_4 14 \rangle \langle \ell_2 23 \rangle \), as well as that with four, \( N^{\text{cyc}}_{i,j,k,l} := \langle \ell_i 12 \rangle \langle \ell_j 34 \rangle \langle \ell_k 14 \rangle \langle \ell_l 32 \rangle + \text{cyc}(1,2,3,4) \) (it is cyclically invariant in external legs 1, 2, 3, 4 as indicated by cyc(1, 2, 3, 4)); \( s \leftrightarrow t \) denotes the symmetrization in the pairs of dual points (12, 34) and (23, 14).

The final result for \( L = 4 \) reads

\[
\hat{\Omega}_4 = -C - S + B.
\]

We compute these forms using a method whose details are given in the appendix: after writing down denominators according to bipartite graphs, we use an ansatz for each numerator which consists of all possible terms with appropriate weight and \( \epsilon \) structures, and then we fix all parameters (with cross-checks) from various boundaries whose canonical forms can be computed directly.

Finally, we compute the five-loop form \( \Omega_5 \), which consists of 5 topologies: three tree graphs with 4 edges, a box with an external line (5 edges), and one with 2 nodes connected to 3 nodes (6 edges). We have

\[
\hat{\Omega}_5 = \left( \frac{1}{2} + \frac{3}{2} \right) - \left( \frac{1}{2} + \frac{3}{2} \right) + \left( \frac{1}{2} + \frac{3}{2} \right)
\]

where only topologies without labels or color are shown, and \( T_m \) denotes the total contribution from graphs with \( m \) edges. The contribution of all trees, \( T_4 \), takes similar forms as lower trees (e.g. \( C, S \) for \( L = 4 \)), as we present explicitly in [18] (which are special cases of a general tree formula in [15]). We follow the same method for computing the remaining contributions: \( 60 \times 2 \) bipartite graphs for \( T_5 \) and \( 10 \times 2 \) for \( T_6 \), which are analogous to \( B \) (especially \( T_5 \) takes a very similar form). The upshot is that the full five-loop result boils down to just these two new functions, which are given in [26] and [27].

We have checked our new results for \( L = 4, 5 \) thoroughly: in addition to various all-loop checks, we have computed unitarity cuts as detailed in appendix, and find that both \( \Omega_4 \) and \( \Omega_5 \) satisfy optical theorem: double-cuts are given by products of various lower-loop integrands.

III. PROJECTED NEGATIVE GEOMETRIES, ADS, AND BIPARTITE GRAPHS

In this section, we study \( A_L \) and associated negative geometries, and show that they are much simpler than \( D = 4 \) case while having very rich structures. A nice trick for studying the \( n = 4 \) amplituhedron [2] is to rewrite it as the sum of negative geometries represented by all possible graphs with \( L \) nodes (and \( E \) edges, with sign factors \((-1)^E\)). It suffices to consider all connected graphs, whose (signed) sum gives the geometry for the logarithm of amplitudes [55]. For example, \( \Omega_3 \) is given by the sum
of $\Omega_1^3/3!$ ($E = 0$), $-\tilde{\Omega}_2\Omega_1/2$ ($E = 1$), and the connected part $\tilde{\Omega}_3$ (with chain and triangle graphs):

$$\Omega_3 = \tilde{\Omega}_3 + \bullet \bullet \bullet - \bullet$$

\[
\tilde{\Omega}_3 = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array} - \begin{array}{c}
\triangle \\
\end{array}
\]

Similarly the connected part of $L = 4$, is given by the sum of graphs with 6 topologies (and so on for higher $L$),

$$\tilde{\Omega}_4 = \begin{array}{ccc}
\bullet & \bullet & \bullet & \bullet & \triangle \\
\end{array} + \begin{array}{c}
\square \\
\end{array} - \begin{array}{c}
\triangle \\
\end{array} - \begin{array}{c}
\square \\
\end{array} + \begin{array}{c}
\triangle \\
\end{array} - \begin{array}{c}
\square \\
\end{array} - \begin{array}{c}
\triangle \\
\end{array}$$

What is new in $D = 3$ is that we will see that most of these geometries do not contribute at all! We find that the projected negative geometries are directly related to time-ordered regions for points in AdS$_3$; this then lead to the surprising conclusion that only those associated with bipartite graphs survive, with very simple pole structures.

Now we observe that two loops are connected by an edge, i.e. with mutual negativity conditions $D_{i,j} < 0$, iff the two points in AdS$_3$ are time-like separated

\[
z_{i,j}^2 - y_{i,j}^2 < 0
\]

Positivity conditions for each loop read $y_i > 0, z_i > 0$, but $w_i > 0$, as well as $z_i^2 - y_i^2 > 0$: the last condition implies that it is space-like separated from the special point $(y_i', z_i') = (0, 0, 0) \equiv 0$.

The simple observation has important implications for projected negative geometries. When two points are time-like separated, there is a natural ordering in that $y_i', z_i', w_i', w_i''$ are ordered in the same way, which we will denote as $i < j$ or $j < i$ (with an arrow of time) from $i$ to $j$ or $j$ to $i$; this means that any connected negative geometry is the sum of time-ordered regions represented by acyclic orientations of the graph since any closed loop of time orderings leads to contradictions, e.g. $1 \prec 2 \prec 3 \prec 1$ is impossible. In this way, we write $\mathcal{A}_g$, as the sum of all directed acyclic graphs (DAGs) weighted by $(-E)^g$, and $\tilde{\Omega}_L$ is given by the canonical forms for those with connected DAGs. As the first non-trivial example, for $L = 3$ we have $6 + 4 \times 3 = 18$ DAGs: 6 for the triangle, and 4 for each of the 3 chain graphs.

### Table I. Counting of topologies and DAGs for connected graphs vs. bipartite ones.

| $L$ | top. of $G$ | top. of $g$ | DAGs of $G$ | $\tilde{\Omega}_L$ |
|-----|-------------|-------------|---------------|-----------------|
| 2   | 1           | 1           | 2             | 1               |
| 3   | 2           | 1           | 18            | 3               |
| 4   | 6           | 3           | 446           | 19              |
| 5   | 21          | 5           | 26430         | 195             |
| 6   | 112         | 17          | 3596762       | 3031            |
| 7   | 853         | 44          | 1111506858    | 67263           |

### a. Transitive reduction: from DAGs to bipartite graphs

Essentially due to its transitivity, the time-ordering implies enormous reduction for the ( combinations of) DAGs. If we have $3 < 2 < 1$ where $1, 2$ and $2, 3$ are time-like separated, then so are $1, 3$, thus the geometry with edges $(12), (23), (13)$ is equivalent to one with only $(12), (23)$, for this ordering. This means that we can delete edge $(13)$ in the presence of $(12), (23)$: a triangle with ordering $3 < 2 < 1$ equals a chain with the same ordering, which is the simplest example of the so-called transitive reduction for DAGs:

$$\tilde{\Omega}_3 = \left(\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array} + \begin{array}{c}
\triangle \\
\end{array}\right) + 2\text{ perms}$$

Note that in the combination of negative geometries, the triangle and chain differ by a sign thus they cancel each other; all 6 such pairs cancel, and we are left with $2 \times 3$ chains with $1, 3 < 2$ (no ordering for 1, 3) etc.

$$\tilde{\Omega}_3 = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array} + \begin{array}{c}
\triangle \\
\end{array} + 2\text{ perms}$$

Similarly transitive reduction simplifies the combination of DAG to higher loops. For example, in the sum of graphs with six topologies for $\tilde{\Omega}_4$, most of the DAGs cancel each other under transitive reductions, and only the box, the star and chain graphs survive, each with two types of orderings, as shown in Fig. 1B. These remaining DAGs are known as bipartite graphs where each node is either a source or sink, which we draw as black or white node, respectively (without the need of putting arrows). An important theorem, which we prove in appendix, says that the reduction to bipartite graphs works to all loops.

### Theorem
Under transitive reductions, we have

$$\sum_{\text{DAG } G} (-E)^{\mathcal{A}_G} = \sum_{\text{bipartite } \text{DAG } g} (-E)^{\mathcal{A}_g}$$

For any connected graph (or each connected component) of an undirected bipartite graph, there are exactly two DAGs (or assignments of black/white nodes).

We record the number of topologies for connected graphs vs. bipartite ones, as well as corresponding numbers of DAGs for $L = 2, 3, \ldots, 7$. As we have seen, only 3 and 5 functions (one for each topologies) are needed for $L = 4, 5$. The fraction of bipartite topologies and that of DAGs quickly tend to zero as $L$ increases.
at $L = 10$, these two fractions already drop to $10^{-5}$ and $10^{-9}$, respectively. Moreover, bipartite graphs are relatively the simplest among all graphs: the most complicated of them are the complete bipartite graphs with white half and black nodes (or differ by one for odd $L$), which have $|L^2/4|$ edges only (such as the box for $L = 4$ and those in $T_6$ for $L = 5$); any graph with more edges (all the way up to the complete one with $\binom{L}{2}$ edges) are gone. This implies that most of the complicated topologies of DCI integrals (such as “tennis court” for $L = 3$ and those beyond box for $L = 4$) are absent for ABJM. In particular, the “deepest cut” in ABJM is roughly half as deep as that in $\mathcal{N} = 4$ SYM [15].

B. Bipartite geometries: pole structures and canonical forms

We have seen that the decomposition of the projected amplituhedron into negative geometries is highly redundant, and a much more efficient way is to formulate $\mathcal{A}_L$ directly in terms of bipartite graphs (RHS of (17)), which we call bipartite geometries. To make our lives even easier, we show that such bipartite geometries have remarkably simple boundary structures, which greatly simplify the computation of their canonical forms!

For two loops which are time ordered, e.g. $i < j$, we see that $y'_i > 0$ and $w'_j > 0$ are no longer co-dimension 1 boundaries, since $y'_i > y'_j > 0$ (and similarly for $w'$), thus $j$ cannot have a pole with $y_j w_j = 0$. On the other hand, since $j$ is space-like separated from 0, i.e. $z_j^2 - y'_j w'_j > 0$, so is $i$ (otherwise it leads to contradiction with time ordering), thus $z_i^2 - y'_i w'_i \propto x_i z_i = 0$ is not a co-dimension 1 boundary for $i$ and cannot be a pole [53]. This leads to the following rule of poles for any bipartite graph: for a source/sink (black/white node), which is “earlier/later” than all points connected to it, it can only have $s \propto y w$ pole, or $t \propto x z$ pole, respectively,

$$
\begin{array}{cccc}
  i & i & j & j \\
  \cdot & \cdot & \cdot & \cdot \\
  \frac{1}{x_i} & \frac{1}{y_j} & \frac{1}{z_i} & \frac{1}{t_j}
\end{array}
$$

Note that this also implies that many DAGs (in particular, all complete graphs) vanish identically; we will not need this fact and postpone the discussion to appendix.

This rule fixes the denominator of the form for any bipartite geometry. Let us go through a few simple examples. For $L = 2$, we have two orderings $1 < 2$ and $2 < 1$: except for $D_{12} = 0$, we can only have $s_1$ and $t_2$ poles for the former, and only $t_1$ and $s_2$ poles for the latter. From weight, we can only have a constant numerator, which is fixed to be $2c^2$ by requiring unit residue, resulting in (9). Now for $L = 3$, we have 6 bipartite graphs, and $e.g.$ for the first one we only have $s_1, t_2, s_3$ poles (in addition to $D_{12}, D_{23}$); the only choice of the numerator must be $e_2$ for correct weight (times a constant $4c^2$), and we obtain (7) essentially without any computation!

For higher $L$, more works are needed to determine the form from its poles. We leave the details for such computations to appendix, but here we point out that for “tree” graphs, which are always bipartite, there is a general formula similar to that in [35]. Let us denote the set of sources and sinks as $B$ and $W$ (for $L$ even, we have $|B| = |W| = L/2$ and for $L$ odd, $|B| = (L+1)/2$, $|W| = (L−1)/2$ or vice versa), then we can write

$$
\Omega_{\text{tree}}(A_y) = \frac{2^{L−1}L^{L/2+1}N_g}{\prod_{i \in B(g)} s_i \prod_{j \in W(g)} t_j \prod_{e \in E(g)} D_e} \Omega_{L−1}
$$

where $2^{L−1}L^{L/2+1}$ is needed for unit residue, and it is easy to see that the weight of the denominator for loop $i$ is $v_i + 2$ with $v_i$ the valency, thus we need $N_g$ to have weight $v_i − 1$ for $i$. One can show that for $i \in B$ with odd $v_i$, all we need is a factor $t_i^{(v_i−1)/2}$, and for even $v_i$, $t_i^{v_i/2−1}e_i/e_i^{1/2}$; the same for $j \in W$ with $s_j$ instead. As a consistency check, it is easy to see that for even/odd $L$, there are even/odd number of $e$ factors.

We can also write the formula recursively: if we construct a $L$-node tree by attaching node $i$ (white) to $i$ (black), which has valency $v_i$ in the original $(L−1)$-node tree, all we need is an “inverse-soft factor”

$$
\Omega_L^{\text{tree}}(j \rightarrow i) = \Omega_{L−1}^{\text{tree}} \times T_{j \rightarrow i}
$$

where $T_{j \rightarrow i} = \frac{2v_j}{c_i c_j}$ for $v_i$ odd, and $\frac{2v_j/2}{c_i c_j}$ for $v_i$ even. Similarly with $t \rightarrow s$ if we have black $j$ attached to white $i$. Forms for all trees are determined in this way.

IV. CONCLUSIONS AND OUTLOOK

In this letter we have discovered a surprising connection between four-point amplitudes in $\mathcal{N} = 4$ SYM and ABJM: by dimensional reducing from $D = 4$ to $D = 3$, the amplituhedron of the former becomes that of the latter, which we have checked explicitly to five loops and for various all-loop cuts. The projected geometries exhibit remarkable structures and simplicity.

One pressing question is if projected amplituhedra for higher $n$ and $k$ give ABJM amplitudes for $k = n/2 - 2$ or perhaps null polygonal Wilson loops [54, 55]? On the other hand, our $n = 4$ integrand clearly contains higher-point ones via unitarity, e.g. their single-cuts give forward-limit of six-point amplitudes [56, 57] (see [58]). It would be fascinating to compute such higher-point forms at $L \geq 3$ and reveal possible geometries.

Last but not least, integrating the forms produces an interesting, finite observable in ABJM theory (analogous to that in $\mathcal{N} = 4$ SYM [35]). It is straightforward to do so for $L \leq 3$, and we expect that cusp anomalous dimension in the theory can be extracted from it. We also expect that resummation for some of these bipartite geometries would allow us to study their contributions non-perturbatively, which will be reported elsewhere.
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\[ \text{the undirected ones since we just multiply by 2 for directed ones.} \]

[53] Note that naively we could have spurious boundaries with $y'_j = y'_i$ or $w'_j = w'_i$, but they are forbidden by the mutual conditions $z'_{i,j} - y'_{i,j}w'_{i,j} < 0$, thus only physical poles can appear for each bipartite geometry.

APPENDIX: DETAILS ABOUT TRANSITIVE REDUCTION AND BIPARTITE GEOMETRIES

As we have seen in section III, the edge $i \rightarrow j$ or the mutual negativity condition $D_{i,j} < 0$ means that $i$ and $j$ are time-like separated in AdS$_3$, and there is a natural time-ordering of $i$ and $j$: recall our convention is that $i \rightarrow j$ if $i < j$ ($j$ is in the forward light-cone of $i$), which divides each geometry corresponding to an undirected diagram $G$ into the summation of geometries corresponding to all possible acyclic orientations of $G$.

A huge reduction happens here due to a simple fact that some directed edges (mutual negativity conditions) may be useless in a DAG. If there are $i \rightarrow j$ and $j \rightarrow k$, we can see that these two edges imply $i \rightarrow k$ from the light-cone picture, so we can delete this edge (if it exists) while the geometry leaves unchanged. Therefore, for a DAG $G$ we try to find a minimal subdigram $H$ such that all edges of $G$ can be implied by edges in $H$, then $A_G = A_H$. This minimal subdiagram $H$ is called a transitive reduction of $G$, a well-known concept in graph theory. For a finite DAG, there exists a unique transitive reduction, which can be easily found by the depth-first search. Transitive reductions lead to the following theorem.

**Theorem** Under transitive reductions, we have

\[ \sum_{\text{DAG } G} (-1)^E A_G = \sum_{\text{bipartite DAG } g} (-1)^E A_g, \]  

(17)

where all graphs are connected.

Each bipartite DAG on RHS has only sources or sinks, so we want to prove that the other diagrams all cancel in the summation on LHS.

We define $G \sim H$ if transitive reductions of $G$ and $H$ are the same, then $A_G = A_H$. This relation is an equivalence relation that divides all DAGs into disjoint equivalence classes. In fact, if two classes are not disjoint, any DAG in the intersection will have different transitive reductions, which contradicts the uniqueness of the transitive reduction.

From the minimal diagram $H$ in an equivalence class, suppose its edges can imply $m$ extra edges, then we can see that there are $2^m$ diagrams in this class, $\binom{m}{k}$ for adding $k$ edges into $H$. Therefore, if $m > 0$, the summation of all DAGs in this equivalence class is

\[ \sum_{\text{DAG } G \sim H} (-1)^E A_G = A_H \sum_{\text{DAG } G \sim H} (-1)^E = A_H (-1)^E H \sum_{k=0}^{m} (-1)^k \binom{m}{k} = A_H (-1)^E H (1 - 1)^m = 0. \]

The only left graphs are those with $m = 0$, which are exactly graphs with only sources and sinks! In fact, if there is a node which is not a source nor sink, there exists a subdigram $i \rightarrow j \rightarrow k$ where we can add $i \rightarrow k$, so $m > 0$.

Therefore, we have proven the theorem. We use $L = 4$ as an example. There are 10 topological inequivalent minimal
DAGs, they are

\[ m = 0 : \quad \text{[Diagram]} \]
\[ m = 1 : \quad \text{[Diagram]} \]
\[ m = 2 : \quad \text{[Diagram]} \]
\[ m = 3 : \quad \text{[Diagram]} \]

where we group them into different \( m \). For each DAG, there are \( 2^m \) DAGs in the corresponding equivalence class. For example, the equivalence class of the first DAG with \( m = 2 \) in the above table contains four DAGs

\[ \{ \text{[Diagram]} \} \]

and it’s easy to see that

\[ \text{[Diagram]} + \text{[Diagram]} = 0. \]

The counting of all equivalence class for general \( L \) are still unknown, but here we list the counting for \( L = 2, 12, 146, 3060, 101642, 5106612, \ldots \) from \( L = 2 \), which can be also found as A001927 at oeis.org and understood as the number of connected partially ordered sets with \( L \) labeled points. If we are not interested in relabels, there are \( 1, 3, 10, 44, 238, 1650, 14512, 163341, \ldots \) topologically inequivalent classes for \( L = 2, 3, \ldots \), which is A000608 at oeis.org.

**APPENDIX: COMPUTATIONS OF CANONICAL FORMS AND CONSISTENCY CHECKS**

In this appendix, we provide some details of computations for the canonical forms of bipartite geometries. As mentioned, our method is first writing down all the poles for a given bipartite graph, and then construct possible numerators which satisfy various symmetries, and finally fix all parameters using constraints from various lower boundaries. Here symmetries include dual conformal symmetry (weight), parity (total number of \( \epsilon \)'s) and the symmetry of the graph. This is very effective for bipartite geometries, since unnecessary poles (which do not correspond to co-dimension 1 boundaries) have been removed with only \( s \) pole for black vertices and \( t \) pole for white vertices left.

**A. Vanishing regions and tree formulas**

Before we study more non-trivial cases, let us first consider a special case, where some regions (DAGs) simply have vanishing forms. These are (non-bipartite) DAGs with valency-2 vertex which is neither source nor sink. From our rule of poles, if vertex \( i \) is neither source nor sink, the form cannot have \( s_i \) or \( t_i \) poles, but only mutual poles \( D_{ij} = 0 \) (for \( j \)'s connected to \( i \)). But since it has valency 2, there are only 2 such poles thus the canonical form for \( \ell_i \) must vanish. Said in a different way, we cannot write down any numerator such that \( \ell_i \) has correct weight since the denominator has weight 2, thus it must vanish.

This fact implies that the forms for many DAGs vanish (though they are cancelled any way in the combination). In particular, the complete graph with \( D_{ij} < 0, \forall i, j = 1, 2, \ldots, L \), e.g. triangle at \( L = 3 \) and tetrahedron at \( L = 4 \). There are \( L! \) DAGs for all possible orderings of the \( L \) points, and all of them are equivalent. Let us consider the region with ordering \( 1 < 2 \prec \cdots \prec L \); by transitive reduction, all edges \( (ij) \) with \( |j - i| > 1 \) can be deleted, resulting in a chain graph where \( L - 2 \) points, \( 2, 3, \cdots, L - 1 \), have valency 2. Any of them is neither source nor sink, thus any such DAG vanishes (with respect to any of these \( L - 2 \) loops). We should also notice that although at \( L = 3 \) the
non-vanishing DAGs are already bipartite graphs, this is no longer true starting \( L = 4 \). It is crucial to sum over all DAGs with \((-)^E\) in order to cancel all of them except for the remaining bipartite geometries.

Moving to non-vanishing graphs, we have proposed a nice formula for the forms of general tree graphs \([15]\), which one should be able to prove inductively by adding a vertex at a time. Let us spell out these results at \( L = 5 \), where we have 3 topologies \((2 \times (60 + 5 + 60)\) bipartite graphs) as follows:

\[
T_4 = \left(\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\end{array}
\end{array}\right) + 59 \text{ perms}\right) + \left(\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\end{array}
\end{array}\right) + 4 \text{ perms}\right) + \left(\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\end{array}
\end{array}\right) + 59 \text{ perms}\right)
\]

\[
= \left(\frac{16c^3\epsilon_2\epsilon_4}{s_1t_2s_3t_4D_{12}D_{23}D_{34}D_{45}}\right) + (s \leftrightarrow t) + 59 \text{ perms}\right) + \left(\frac{16c^3\epsilon_1s_3}{t_1s_2s_3s_4s_5D_{12}D_{13}D_{14}D_{15}}\right) + (s \leftrightarrow t) + 4 \text{ perms}\right)
\]

\[
= \left(\frac{16c^3\epsilon_3\epsilon_4}{s_1t_2s_3t_4D_{12}D_{13}D_{14}}\right) + (s \leftrightarrow t) + 59 \text{ perms}\right).
\]

(18)

Notice that the last graph also provides a nontrivial consistency check of our general formula \([15]\). As shown in \([18]\). On the one hand, we can get it from adding (red) link 3 – 4 to \( L = 4 \) star; on the other hand, we can get it from adding (red) link 1 – 5 to \( L = 4 \) chain, and the two result are identical:

\[
1 \quad 3 \quad 4 \\
\]

\[
= \frac{8t_1}{s_1t_2t_3t_5D_{12}D_{13}D_{15}} \times \frac{2c_3}{s_4D_{34}}
\]

(19)

\[
1 \quad 3 \quad 4 \\
\]

\[
= \frac{8c_3}{s_1t_2t_3s_4D_{12}D_{13}D_{14}} \times \frac{2t_1}{t_5D_{15}}
\]

(20)

**B. Determinations for non-trivial graphs at \( L = 4, 5 \)**

Now we move to more non-trivial graphs at \( L = 4, 5 \), and we use the box as an example. In this case, the poles in the denominator are \( s_1t_2s_3t_4D_{12}D_{23}D_{34}D_{45} \). Since the denominator has weight 4 for each loop, the numerator must have weight 1 in \( \ell_i \) (\( i = 1, 2, 3, 4 \)), and by taking into account the weight in 1, 2, 3, 4 and parity, we can write an ansatz which consists of the following six types of terms

\[
\begin{aligned}
\{ &\epsilon_i, \epsilon_i, \epsilon_i, (l_{i_1}p_{i_1}q_{i_1}) (l_{i_4}p_{i_4}q_{i_4}) (1234), \epsilon_i, \epsilon_i, (l_{i_3} \ell_{i_3}) (1234)^2 \\
&\epsilon_i, \epsilon_i, (l_{i_3} l_{i_4}) (l_{i_3} p_{i_3} q_{i_3}) (l_{i_4} p_{i_4} q_{i_4}) (1234)^3, (l_{i_1} l_{i_2}) (l_{i_1} l_{i_2}) (1234)^4 \}.
\end{aligned}
\]

(21)

Here we can have 0, 2, 4 \( \epsilon \)'s in total, and \( p, q = 1, 2, 3, 4 \) are external legs with correct weight; after imposing symmetry between \( \ell_1, \ell_3 \) and \( \ell_2, \ell_4 \) of the graph, we find the numbers of parameters in these six terms to be 1, 7, 3, 9, 14, 2, respectively.

We then use forms on boundaries to determine these parameters. Usually, such forms can be computed by direct triangulations since their mutual poles are simplified. In the \( L = 4 \) box, we have used the following conditions; the form with \( y_1 = y_3 = w_1 = 0 \) is

\[
4 \left( x_2 x_4 z_1^2 z_3^2 + z_2 z_4 + z_1 z_3 (-4 + x_4 z_2 + x_2 z_4) \right) ;
\]

(22)

the form with \( y_1 = y_3 = x_2 = 0 \) is

\[
4 w_1 w_3 z_1 z_3 (-1 + x_4 z_4) - 4 w_2 w_4 (z_1 (-4 + x_4 z_2) z_3 + z_2 z_4) ;
\]

(23)
the form with \( y_1 = w_3 = x_2 = x_4 = 0 \) is

\[
-4 \left( \frac{(4) + w_1 y_3)z_1 z_3 + z_2 z_4}{w_1 z_2 y_3 z_4 \left( D_{12} D_{23} D_{34} D_{14} \right)} \right)_{y_1 = w_3 = x_2 = x_4 = 0},
\]

(24)

and the form with \( y_1 = w_3 = x_2 = z_4 = 0 \) is

\[
-4 \left( \frac{(4) + w_1 y_3 + x_4 z_2}{w_1 z_2 y_3 x_4 \left( D_{12} D_{23} D_{34} D_{14} \right)} \right)_{y_1 = w_3 = x_2 = z_4 = 0}.
\]

(25)

Very nicely, these conditions suffice to fix all parameters for the box.

The same method can be applied to determine the forms of non-trivial diagrams in \( L = 5 \). After writing down all possible terms in the numerators, we find 60 parameters for any graph in \( T_5 \) and 172 parameters for that in \( T_6 \). In these cases we do need to impose more conditions compared to box at \( L = 4 \), but after imposing such boundary conditions we can indeed fix all parameters and arrive at the unique canonical form of \( T_5 \) and \( T_6 \):

\[
T_5 = 8c \frac{4 \epsilon_1 \epsilon_3 \epsilon_4 s_2 - \epsilon_1 \epsilon_2 \epsilon_3 N_{24}^l - c(-\epsilon_1 t_2 N_{34}^l - \epsilon_3 t_2 N_{14}^l + \epsilon_4 s_2 N_{13}^l + \epsilon_2 N_{1,2,3,4}^c)}{s_1 t_2 s_4 s_5 D_{12} D_{23} D_{34} D_{41} D_{52}} + (s \leftrightarrow t) + 59 \text{ perms},
\]

(26)

\[
T_6 = c \frac{4}{s_1 t_2 s_4 s_5 D_{12} D_{13} D_{14} D_{25} D_{35} D_{15}} \left( -8 \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 s_5 N_{15}^s + c \epsilon_2 \epsilon_4 s_4 P_a + c \epsilon_1 \epsilon_2 \epsilon_3 P_b + (\ell_1 \leftrightarrow \ell_5) + \epsilon_1 \epsilon_2 \epsilon_5 P_c + \text{cyc}(\ell_2, \ell_3, \ell_4) \right)
\]

+ \epsilon^2 \epsilon_1 P_d + (\ell_1 \leftrightarrow \ell_5) + \epsilon^2 \epsilon_2 P_e + \text{cyc}(\ell_2, \ell_3, \ell_4)) + (s \leftrightarrow t) + 9 \text{ perms};
\]

(27)

where in \( T_6 \), we have polynomials \( P_a, P_b, \ldots, P_c \) with certain weights in \( \ell_1, \ldots, \ell_5 \), and here we record their expressions:

\[
P_a := -20 s_1 s_5 + 16 t_1 t_5 + (N_{15}^s)^2, \quad P_b := 6 s_5 N_{14}^l, \quad P_c := N_{15}^s N_{34}^l - 4 N_{1,3,5,4}^c.
\]

(28)

\[
P_d := -s_5 (N_{12}^s N_{34}^l + \text{cyc}(\ell_2, \ell_3, \ell_4)) + 2 \epsilon_1 \epsilon_3 (15 \ell_{12})^3 (15 \ell_{34}) (15 \ell_{34}) (15 \ell_{34}) + \text{cyc}(1, 2, 3, 4)
\]

+ 2 t_5 (15 \ell_{14}) (15 \ell_{34}) (15 \ell_{23}), \quad P_e := 2 s_1 s_5 (N_{34}^l - N_{34}^s) - 4 t_1 t_5 N_{14}^l - s_5 (15 \ell_{12})^3 (15 \ell_{34}) (15 \ell_{34}) + (15 \ell_{12}), (15 \ell_{34})
\]

+ N_{15}^s (15 \ell_{14}) (15 \ell_{34}) (15 \ell_{23}) (15 \ell_{23}) + (15 \ell_{12}), (15 \ell_{34})
\]

(29)

\[
C. \text{ Consistency checks}
\]

Finally, we give more details regarding what we have checked about the claim that the canonical form of projected amplituhedron gives ABJM four-point integrands. What we have computed up to \( L = 5 \) automatically satisfy those all-loop cuts discussed above, which are essentially trivialized by geometries. Such cuts are very strong, \( e.g. \) soft cuts can determine coefficients of a large proportion of dual conformal integrals in unitary method. However, soft cuts can only probe disconnected graphs and cannot constrain those negative geometries contributing to \( \Omega_L \) \cite{38}, while all vanishing cuts become manifest in the pole structure. Therefore, we should consider more general cut conditions where all topologies of graphs (including connected and the disconnected ones) will contribute to. The double cuts, which follow from unitarity, seem to be perfect for that.

We remark that there are also many other simple cuts, such as ladder and next-to-ladder cuts, which can be derived easily from geometries. However, due to lack of data for \( L \geq 4 \) from physics side we do not use them as independent checks; we did confirm that such ladder-type cuts of our \( L = 3 \) result, which agrees with the conjecture of \cite{35}, are satisfied by cutting those contributing Feynman diagrams.

The double cut is a particular cut condition on a single loop:

\[
\langle AB12 \rangle = \langle AB34 \rangle = 0 \quad \text{or equivalently} \quad w_1 = y_1 = 0.
\]

(31)

From optical theorem, it is given in terms of products of lower-loop amplitudes with momentum shifted:

\[
\sum_{k=0}^{l-1} x_{1,3}^{2k} \langle p_1 p_4 \rangle \langle p_4 p_1 \rangle M_4^{k-\text{loop}} (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_a) M_4^{(l-k-1)-\text{loop}} (\vec{x}_1, \vec{x}_a, \vec{x}_3, \vec{x}_4)
\]

(32)
Here, $\vec{x}_i$ is the dual variable and $\vec{x}_{i+1} - \vec{x}_i \equiv \vec{p}_i$. The dual momentum variables of the left amplitude are $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_a$ and the dual variables of the right amplitude are $\vec{x}_1, \vec{x}_a, \vec{x}_3, \vec{x}_4$. The overall prefactor becomes unity when expressed in terms of $w, x, y, z$ variables.

To be more explicit, let us first consider the $L = 3$ case. After cutting out $w_1 = y_1 = 0$, the amplitude factorizes to $A_{\text{tree}}^1 \otimes A_{\text{2-loop}}^2, A_{\text{1-loop}}^1 \otimes A_{\text{1-loop}}^4$, and $A_{\text{2-loop}}^2 \otimes A_{\text{tree}}^4$. Here the momentum is shifted and the $\epsilon$-numerator is shifted to $c \epsilon_{3, R} \epsilon_{3, L}$ on the left side and to $c \epsilon_{3, L} \epsilon_{3, R}$ on the right side, which are denoted as $\epsilon_{i, L}$ and $\epsilon_{i, R}$ respectively. Also $t_i$ is shifted to $c \epsilon_{3, R} \epsilon_{3, L}$ on the left and $c \epsilon_{3, L} \epsilon_{3, R}$ on the right, denoted as $t_{i, L}$ and $t_{i, R}$ respectively.

We find that the residue of $L = 3$ integrand at $w_1 = y_1 = 0$ nicely matches with the sum of the three terms:

\[
\text{Double cut } w_1 = y_1 = 0 \text{ of the 3-loop integrand} = 1 \times \left( \frac{1}{2} \frac{c^2 \epsilon_{2, R} \epsilon_{3, R}}{s_2 t_{2, R} s_3 t_{3, R}} - \frac{2c^2 x_1}{s_2 t_{2, R} s_3 t_{3, R} D_{23}} \right) + \frac{c \epsilon_{2, L}}{s_3 t_{2, L}} \times \frac{c \epsilon_{2, R}}{s_2 t_{2, R}} + \frac{1}{2} \frac{c \epsilon_{2, L} \epsilon_{3, L}}{s_2 t_{2, L} s_3 t_{3, L}} - \frac{2c^2 z_1}{s_2 t_{3, L} D_{23}} \right) \times 1 + (2 \leftrightarrow 3) \quad (33)
\]

For higher $L$, we proceed in exactly the same way: we first confirm that $L = 4$ form (after including all disconnected graphs) has the correct double cut, which uses the shifted amplitude of one- to three- loops; then we use the four-loop result and shift it to express the double cut of the $L = 5$ form, which again has the correct double cut. This has provided very strong evidence the $L \leq 5$ results obtained from geometry are indeed correct!