A SUFFICIENT CONDITION FOR COMPACTNESS OF HANKEL OPERATORS

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ABSTRACT. Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$. We show that if $\varphi \in C^1(\overline{\Omega})$ is holomorphic along analytic varieties in $b\Omega$, then $H_\varphi^q$, the Hankel operator with symbol $\varphi$, is compact. We have shown the converse earlier ([ÇŞ20]), so that we obtain a characterization of compactness of these operators in terms of the behavior of the symbol relative to analytic structure in the boundary. A corollary is that Toeplitz operators with these nonvanishing symbols are Fredholm (of index zero).

1. INTRODUCTION

This paper is a follow up to our recent [ÇŞ20]. There we proved the following necessary condition for compactness of a Hankel operator on a bounded convex domain $\Omega$ in $\mathbb{C}^n$ ([ÇŞ20, Theorem 1]). Let $n \geq 2$, $0 \leq q \leq (n-1)$ and suppose $\varphi \in C(\overline{\Omega})$. If the Hankel operator $H_\varphi^q$ on $(0,q)$–forms is compact, then the symbol $\varphi$ is holomorphic along complex varieties of dimension $(q+1)$ (and higher) in the boundary (such varieties are actually affine and given by the intersection of $\overline{\Omega}$ with affine subspaces, see the discussion in [ÇŞ20]). We also proved a rather partial converse: if the boundary contains only ‘finitely many’ varieties, then the converse holds ([ÇŞ20, Theorem 2]). That is, in this case, if the symbol is holomorphic along the varieties, then the Hankel operator is compact. We conjectured that this restriction was an artifact of the proof. This is indeed the case: the main purpose of this paper is to prove the full converse (via a different method), with one small caveat: we need the symbol to be in $C^1(\overline{\Omega})$.

We first recall some notation and terminology from [ÇŞ20]. The first section of that paper also contains an introduction to the subject and a discussion of previous work to which we refer the reader. For a bounded domain $\Omega$, we denote by $K^2_{(0,q)}(\Omega)$ the space of square integrable $\overline{\partial}$–closed $(0,q)$–forms on $\Omega$; its subspace of forms with holomorphic coefficients is denoted by $A^2_{(0,q)}(\Omega)$. The operator $P_{q}$ denotes the Bergman projection on $(0,q)$–forms, i.e. the orthogonal projection $P_{q} : L^2_{(0,q)}(\Omega) \to K^2_{(0,q)}(\Omega)$. For a symbol $\varphi \in L^\infty(\Omega)$, the associated Hankel operator $H_\varphi^q : K^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ is defined as

$$H_\varphi^q f = \varphi f - P_{q}(\varphi f) \quad (1)$$

for $f \in K^2_{(0,q)}(\Omega)$. 

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**Theorem 1.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{C}^n \), \( \varphi \in C^1(\overline{\Omega}) \), and \( 0 \leq q \leq n - 1 \). Assume that \( \varphi \) is holomorphic along the varieties in the boundary of dimension \( q + 1 \) (and higher). Then \( H^q_\varphi \) is compact on \( K^2_{(0,q)}(\Omega) \).

Combining Theorem 1 with [ÇSS20, Theorem 1], we have a complete characterization of compactness of \( H^q_\varphi \) for symbols in \( C^1(\overline{\Omega}) \).

**Corollary 1.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{C}^n \), \( \varphi \in C^1(\overline{\Omega}) \), and \( 0 \leq q \leq n - 1 \). Then \( H^q_\varphi \) is compact if and only if \( \varphi \) is holomorphic along complex varieties of dimension \( (q + 1) \) (and higher) in the boundary.

**Remark 1.** Theorem 1 in [ÇSS20] only assumes that the Hankel operator is compact on \( A^2_{(0,q)}(\Omega) \). Thus, on a bounded convex domain, if the symbol \( \varphi \) is in \( C^1(\overline{\Omega}) \), then \( H^q_\varphi \) is compact on \( K^2_{(0,q)}(\Omega) \) if and only if it is compact on \( A^2_{(0,q)}(\Omega) \). The analogous fact holds for the \( \bar{\partial} \)–Neumann operator, see [FS98]. For related questions, see [Has14, Theorem 12.12].

**Remark 2.** The converse of Theorem 1, [ÇSS20, Theorem 1], holds for \( \varphi \) only assumed in \( C(\overline{\Omega}) \). We do not know whether the conclusion of Theorem 1 still holds under this weaker assumption on \( \varphi \). One natural approach would be to approximate \( \varphi \in C(\overline{\Omega}) \) by suitable symbols in \( C^1(\overline{\Omega}) \). Our proof below indicates that it suffices to approximate \( \varphi \in C(\overline{\Omega}) \), holomorphic along the varieties in the boundary, uniformly on \( \overline{\Omega} \) to within \( \varepsilon \) by \( \varphi_\varepsilon \in C^1(\overline{\Omega}) \) with \( |\bar{\partial}_V \varphi_\varepsilon| \leq \eta(\varepsilon) \), where \( \lim_{\varepsilon \to 0^+} \eta(\varepsilon) = 0 \) (rather than with \( \bar{\partial}_V \varphi_\varepsilon = 0 \)). Here, \( \bar{\partial}_V \) denotes \( \bar{\partial}_V \) along \( V \subset b\Omega \).

One of the reasons compactness of Hankel operators is of interest in operator theory is the connection to the Fredholm theory of Toeplitz operators. For \( \varphi \in L^\infty(\Omega) \), the Toeplitz operator \( T^q_\varphi \) is the bounded operator on \( K^2_{(0,q)}(\Omega) \) defined by \( T^q_\varphi f = P_q(\varphi f) \) (see e.g. [FK72], [Ven72, Section IV.4]). This connection provides the following consequence of Theorem 1, which gives the Fredholm property for certain Toeplitz operators in the absence of compactness in the \( \bar{\partial} \)–Neumann problem.

**Corollary 2.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{C}^n \), \( \varphi \in C^1(\overline{\Omega}) \), and \( 0 \leq q \leq n - 1 \). Assume that \( \varphi \) is holomorphic along the varieties in the boundary of dimension \( q + 1 \) (and higher), and that \( \varphi \) does not vanish on \( b\Omega \) \( (q = 0) \), or \( \varphi \) does not vanish on \( \overline{\Omega} \) \( (1 \leq q \leq (n - 1)) \). Then the Toeplitz operator \( T^q_\varphi \) is Fredholm of index zero.

We provide the details of the argument in Section 3, but note here that as usual, index zero results form the fact that the boundary of a convex domain (for \( q = 0 \) and the domain itself (for \( 1 \leq q \leq (n - 1) \)) are simply connected. The distinction between \( q = 0 \) and \( q > 0 \) arises because the restriction to a compact subset of \( \Omega \) is a compact operator on \( K^2_{(0,q)}(\Omega) \) when \( q = 0 \) (i.e. on holomorphic functions), while it is not when \( q > 0 \). Alternatively, one could consider Toeplitz operators on \( A^2_{(0,q)}(\Omega) \), or on \( \ker(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \subset K^2_{(0,q)}(\Omega) \), endowed with the graph norm, to avoid this issue.
Our proof of Theorem 1, given in Section 2, owes much to [Zim21]. Zimmer’s use of Frankel’s work ([Fra91]), and in particular his realization that the canonical potential for the Bergman metric on a convex domain has self bounded (complex) gradient, are important ingredients in this proof. The other two main ingredients come from [Str06, Str10], specifically from the proof given there of McNeal’s result ([McN02]) that property \( P_q \) implies compactness of the \( \partial \)-Neumann operator on \( (0,q) \)-forms (proof of Theorem 2.1 in [Str06], proof of Theorem 4.29 in [Str10]). The work in [Zim21] also suggests possible generalizations of Theorem 1 (as well as of the main result in [CSS20]), but we do not pursue this direction here.

2. PROOF OF THEOREM 1

We first explain the strategy of the proof. Compactness of \( H^q_{\phi} \) will be established by showing that we have (a family of) compactness estimates: for all \( \varepsilon > 0 \) there exists a compact operator \( K_\varepsilon \) from \( K^2_{(0,q)}(\Omega) \) to some Hilbert space \( Z_\varepsilon \), such that

\[
\| H^q_{\phi} \|^2 \leq \varepsilon \| f \|^2 + \| K_\varepsilon f \|^2
\]

(see [Str10, Lemma 4.3, part (ii)]). Observe that if \( \phi \in C^1(\Omega) \) and \( f \in K^2_{(0,q)}(\Omega) \), then Kohn’s formula \( P_q = Id - \partial^* N_{q+1} \partial \) gives

\[
\| H^q_{\phi} f \|^2 = \langle \partial^* N_{q+1} (\partial \phi \wedge f), \partial^* N_{q+1} (\partial \phi \wedge f) \rangle = \langle N_{q+1} (\partial \phi \wedge f), \partial \phi \wedge f \rangle;
\]

here, \( N_{q+1} \) denotes the \( \partial \)-Neumann operator on \( (0,q+1) \)-forms, and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2_{(0,q)}(\Omega) \) and \( L^2_{(0,q+1)}(\Omega) \), respectively. The goal is now to establish the estimates (2) for the right hand side of (3).

Let us assume for this paragraph that \( q = 0 \), to simplify the discussion. If \( N_1 \) were compact, i.e. if there were no varieties in the boundary, compactness of \( H^0_{\phi} \) would be immediate from (3). Compactness of \( N_1 \) in the absence of varieties is established by showing that there exist families of plurisubharmonic functions, uniformly bounded or with gradient uniformly self bounded (see below), that have Hessians as large as we please (property \( P \) or property \( \tilde{P} \), respectively). The large Hessians are used to produce compactness estimates. An analytic disc \( V \) in the boundary is an obstruction to the existence of these families. On the other hand, in the right hand side of (3), only the component of \( N_1(f \partial \phi) \) along \( \partial \phi \) enters. If \( \phi \) is holomorphic along \( V \), this component is transverse to \( V \). In order to estimate this component, we therefore only need families of functions with big Hessians in directions transverse to \( V \). Such families should exist, the presence of \( V \) notwithstanding. This is indeed the case, and the desired families can be obtained form \( \log B(z,z) \), where \( B \) is the Bergman kernel.

We begin with a lemma that makes this discussion rigorous. Denote by \( (g^i_{jk})_{j,k=1,...,n}(z) \) the Bergman metric at the point \( z \in \Omega \), i.e. \( g^i_{jk} = \partial^2 \log B(z,z)/\partial z_j \partial \bar{z}_k \). A good reference for properties of the Bergman metric is [JP13, Chapter VI].
Lemma 1. Let $\Omega$ be a bounded convex domain, $0 \leq q \leq (n-1)$, and $\varphi \in C^1(\overline{\Omega})$. Assume that $\varphi$ is holomorphic along varieties in $b\Omega$ of dimension $(q+1)$ (or higher). Then, for every $\varepsilon > 0$, there exists a relatively compact subdomain $\Omega_\varepsilon$ of $\Omega$ with the following property. For each $z \in \Omega \setminus \Omega_\varepsilon$ and each set $t_1(z), \ldots, t_{q+1}(z)$ of orthonormal eigenvectors of $(g_{jk})(z)$, if the sum of the corresponding eigenvalues is $\leq \varepsilon^{-1}$, then $\left| (\partial \varphi(z), t_s(z)) \right| \leq \varepsilon, s = 1, \ldots, (q+1)$.

We have identified $\overline{\partial} \varphi(z)$ with a vector in the obvious way, and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{C}^n$.

Proof. First of all, let $0 \leq q \leq (n-2)$. We argue indirectly. Assume that the conclusion of the lemma is false. Then there exist a sequence $\{z_m\} \subset \Omega, z_0 \in b\Omega$ and orthonormal eigenvectors $t_1(z_m), \ldots, t_{q+1}(z_m)$ for the matrix $(g_{jk})(z_m)$ with associated eigenvalues $\sigma_1(z_m), \ldots, \sigma_{q+1}(z_m)$ with the following three properties:

(i) $z_m \to z_0$ as $m \to \infty$,

(ii) $\sigma_1(z_m) + \cdots + \sigma_{q+1}(z_m) \leq \varepsilon^{-1}$,

(iii) for every $m$ there is $s_m, 1 \leq s_m \leq (q+1)$, with $| (\partial \varphi(z_m), t_{s_m}(z_m) | > \varepsilon$.

After passing to a suitable subsequence, we may assume that $s_m = s_0$ for all $m (1 \leq s_0 \leq (q+1))$ and that the frames $t_1(z_m), \ldots, t_{q+1}(z_m)$ converge to an orthonormal frame $t_1(z_0), \ldots, t_{q+1}(z_0)$ at $z_0$ which spans a $(q+1)$–dimensional affine subspace of $\mathbb{C}^n$.

We next show that this affine subspace intersects $\overline{\Omega}$ in a nontrivial $(q+1)$–dimensional (affine) variety contained in $b\Omega$. In order to do this, we need a notion from [Fra91] that is clearly relevant for dealing with affine varieties in the boundary. For $z \in \Omega$ and $v \in \mathbb{C}^n$, set

$$
\delta_\Omega(z, v) := \min \{|w - z| : w \in b\Omega \cap (z + \mathbb{C}v)\}.
$$

On convex domains, the quantity $|v|/\delta_\Omega(z, v)$ (referred to in [Fra91] as the complex $1/d$–metric) satisfies a crucial comparison to the Bergman metric ([Fra91], [Zim21, Theorem 4.3] and see [NPZ11] for this result on $\mathbb{C}$–convex domains): for all $n \in \mathbb{N}$, there is a constant $A_n$ such that if $\Omega$ is a bounded convex domain in $\mathbb{C}^n$, then

$$
\frac{1}{A_n} \frac{|v|}{\delta_\Omega(z, v)} \leq \left((g_{jk})(z)(v, v)\right)^{1/2} \leq A_n \frac{|v|}{\delta_\Omega(z, v)}.
$$

The left inequality gives

$$
\delta_\Omega(z_m, t_s(z_m)) \geq \varepsilon^{1/2} A_n, \quad s = 1, \ldots, (q+1), \quad m = 1, 2, \ldots
$$

(since each $\sigma_s(z_m) \leq 1/\varepsilon$). Therefore for $1 \leq s \leq (q+1)$, the affine discs $V_s := \{z_0 + \zeta t_s(z_0) : |\zeta| \leq (\varepsilon^{1/2}/A_n)\}$ are contained in $\overline{\Omega}$. As a convex domain, $\Omega$ admits a plurisubharmonic defining function (obtained from its Minkowski functional or gauge function, which is convex ([Roc70, page 35], [CS01, Lemma 6.2.4]) and therefore plurisubharmonic). The value of such a defining function at the center of $V_s$ (i.e. at $z_0$) is zero, while it is less than or equal to zero
throughout the disc. The strong maximum principle for subharmonic functions implies that the value is zero throughout, that is $V_s \subset b\Omega$. Essentially the same argument shows that the convex hull of $\bigcup_{s=1}^{j+1} V_s$ is also contained in the boundary. By assumption, $\varphi$ is holomorphic along this variety. In particular, $(\partial \varphi(z_0), t_{s_0}(z_0)) = 0$. This contradicts the fact that $|\langle \partial \varphi(z_m), t_{s_0}(z_m) \rangle| > \varepsilon$.

If $q = (n - 1)$, (5) shows that $\sigma_1(z_m) + \cdots + \sigma_n(z_m) \to \infty$ when $z_m \to b\Omega$, since $\delta_{t\Omega}(z_m, v) \to 0$ for $v$ transverse to $b\Omega$. So this sum will always be greater than $\varepsilon^{-1}$ for $z$ outside a relatively compact subdomain $\Omega_\varepsilon$, and the statement of the lemma holds. □

We are now ready for the proof of Theorem 1.

**Proof of Theorem 1.** Denote by $\lambda(z)$ the Bergman potential; $\lambda(z) := \log B(z, z)$. The property that is crucial for us is that $\lambda$ has self bounded gradient on $\Omega$:

\[
\left| \sum_{j=1}^{n} \frac{\partial \lambda}{\partial z_j}(z)w_j \right|^2 \leq C \sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k}(z)w_j \overline{w_k}; \quad z \in \Omega, \ w \in C^n.
\]

That is, $\partial \lambda$, measured in the metric induced by $\frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k}$, is bounded ([Zim21, Proposition 4.6]).

In order to carry out the proof idea sketched above, we need a family of functions $\lambda_\varepsilon$ with uniformly (in $\varepsilon$) self bounded gradients, smooth in a neighborhood of the boundary, and with $\frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k}$ satisfying the properties from Lemma 1, for the given $\varepsilon$. Without loss of generality, assume that $0 \in \Omega$. Let $\Omega_\varepsilon$ be from Lemma 1 and $0 < \varepsilon' \leq \varepsilon$ small enough so that $\tilde{\Omega}_\varepsilon := (1 + \varepsilon')\Omega_\varepsilon \subset \Omega$. We set

\[
\lambda_\varepsilon(z) := \lambda \left( \frac{z}{1 + \varepsilon'} \right).
\]

Then $\lambda_\varepsilon \in C^\infty((1 + \varepsilon')\Omega)$ and

\[
\partial \lambda_\varepsilon(z) = \frac{1}{1 + \varepsilon'} \partial \lambda \left( \frac{z}{1 + \varepsilon'} \right); \quad \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k}(z) = \frac{1}{(1 + \varepsilon')^2} \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k} \left( \frac{z}{1 + \varepsilon'} \right).
\]

Because $\lambda$ has self bounded gradient, one easily checks that the $\lambda_\varepsilon$ have gradients that are self bounded uniformly in $\varepsilon$.

Let $z$ be in $\Omega \setminus \tilde{\Omega}_\varepsilon$ and assume that the matrix $\begin{pmatrix} \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k}(z) \end{pmatrix}$ has a set of $(q + 1)$ eigenvalues $\sigma_{\varepsilon,1}(z), \ldots, \sigma_{\varepsilon,q+1}(z)$ with corresponding eigenvectors $t_{\varepsilon,1}(z), \ldots, t_{\varepsilon,q+1}(z)$ such that $\sigma_{\varepsilon,1}(z) + \cdots + \sigma_{\varepsilon,q+1}(z) \leq \varepsilon^{-1}$. We claim that then

\[
|\langle \overline{\partial \varphi(z)}, t_{\varepsilon,s}(z) \rangle| \leq \omega(\varepsilon); \quad 1 \leq s \leq (q + 1),
\]

where $\omega(\varepsilon)$ denotes a quantity that is independent of $z$ and tends to zero as $\varepsilon$ tends to zero. To see this, note that the eigenvalues of $\begin{pmatrix} \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k}(z) \end{pmatrix}$ are $1/(1 + \varepsilon')^2$ times those of $\begin{pmatrix} \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k}(z/(1 + \varepsilon')) \end{pmatrix}$ (see (9)), and the associated eigenvectors are the same. Because $\varphi \in C^1(\overline{\Omega})$, we also have that $|\overline{\partial \varphi(z)} - \overline{\partial \varphi(z/(1 + \varepsilon'))}|$ tends to zero uniformly for $z \in \overline{\Omega}$ as $\varepsilon$ tends to zero.
two observations with a triangle inequality argument yields (10); the resulting quantity \(\omega(\varepsilon)\) involves, among other things, the modulus of continuity of \(\overline{\partial}\varphi\) on \(\Gamma\).

We are now in a position to establish (2). Fix \(\varepsilon > 0\) and pick \(\Omega_\varepsilon\) and \(\lambda_\varepsilon\) from Lemma 1 and (8), respectively. To simplify notation, set

\[
v := N_{q+1}(\overline{\partial}\varphi \wedge f).
\]

Then \(v \in \text{ker}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*)\) and \(\|v\| + \|\overline{\partial}^* v\| \lesssim \|f\|\). As in the proof of Theorem 4.29 in \cite{Str10} or Theorem 2.1 in \cite{Str06} (that property \((\tilde{P}_q)\) implies compactness of \(N_q\)), we need an estimate that brings the Hessian of \(\lambda_\varepsilon\) into play. If \((\lambda_\varepsilon)\) is a family of functions with uniformly self bounded gradient, and \(u \in \text{ker}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \subseteq L^2(0,q)\), then

\[
\int_{\Omega} \sum' \sum_{j,k} \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k} (e^{-\lambda_\varepsilon/2} u)_{jk} (e^{-\lambda_\varepsilon/2} u)_{kk} dV \lesssim \|\overline{\partial}^* (e^{-\lambda_\varepsilon/2} u)\|^2,
\]

with a constant that does not depend on \(\varepsilon\). This estimate results from the Morrey–Kohn–Hörmander formula with weight \(\lambda_\varepsilon\) (see, for instance, \cite[Proposition 2.4]{Str10}). On the left hand side, we have distributed the weight factor \(e^{-\lambda_\varepsilon}\) into the form \(e^{-\lambda_\varepsilon/2} u\). There is no \(\overline{\partial}\)–term on the right hand side, as \(\overline{\partial}u = 0\), and replacing the (weighted) \(\overline{\partial}^*\)–term in \cite[(2.24)]{Str10} by the right hand side of (12) makes an error of the order \(|\partial \lambda_\varepsilon(e^{-\lambda_\varepsilon/2} u)|\). This error can be absorbed into the left hand side by the self bounded gradient condition. This assumes that the constant in the self bounded gradient estimate is small enough. This property can always be achieved by scaling \(\lambda_\varepsilon\) to \(c\lambda_\varepsilon\) if necessary (the two sides of (7) scale with \(c^2\) and \(c\), respectively). This scaling does not affect the argument, and we continue with \(\lambda_\varepsilon\). Details are in \cite{Str10}, estimate (4.80). There is a compactly supported term on the right hand side in (4.80) in \cite{Str10}, but this term is the result of the function \(\lambda_M\) there satisfying the self bounded gradient assumption only in a neighborhood of the boundary, and so does not occur here.

The factor \(e^{-\lambda_\varepsilon/2}\) with the form \(u\) in (12) appears problematic at first, as there is no such factor in the right hand side of (3) (the quantity we want to estimate). However, the following observation from \cite{Str06, Str10} allows to handle this difficulty. If \(v\) is \(\overline{\partial}\)–closed and we define

\[
u_\varepsilon := P_{q+1,\lambda_\varepsilon/2} (e^{\lambda_\varepsilon/2} v) \in \text{ker}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*),
\]

then one has

\[
P_{q+1} (e^{-\lambda_\varepsilon/2} u_\varepsilon) = v; \text{ and then also } \overline{\partial}^* (e^{-\lambda_\varepsilon/2} u_\varepsilon) = \overline{\partial}^* v.
\]

Here, \(P_{q+1}\) and \(P_{q+1,\lambda_\varepsilon/2}\) are the Bergman projection and the weighted Bergman projection on \((0,q+1)\)-forms on \(\Omega\), respectively. One can check (14) by pairing with \(\overline{\partial}\)–closed forms, see \cite[(4.83) – (4.85)]{Str10}. The point of (14) in the present context is that because \(\overline{\partial}\varphi \wedge f\) is \(\overline{\partial}\)–closed,
the quantity we need to estimate now becomes
\begin{equation}
\langle v, \bar{\partial} \varphi \wedge f \rangle = \langle e^{-\lambda_{\varepsilon}/2} u_{\varepsilon}, \bar{\partial} \varphi \wedge f \rangle,
\end{equation}
and we have the exponential factor. Moreover, by (14), the right hand side of (12) equals $\bar{\partial}^* v = \bar{\partial}^* N_{q+1}(\bar{\partial} \varphi \wedge f)$, and so is dominated by $\|f\|$ (independently of $\varepsilon$). Therefore, we will get good estimates on $e^{-\lambda_{\varepsilon}/2} u_{\varepsilon}$, hence on $\langle v, \bar{\partial} \varphi \wedge f \rangle$, in directions where $\lambda_{\varepsilon}$ has big Hessian. In the other directions, we will estimate $\langle v, \bar{\partial} \varphi \wedge f \rangle$ via (10).

The argument proceeds as follows. Let $z \in \Omega \setminus \widetilde{\Omega}_{\varepsilon}$ (fixed for now). We want to estimate
\begin{equation}
\left( e^{-\lambda_{\varepsilon}/2} u_{\varepsilon}, \bar{\partial} \varphi \wedge f \right)(z) = \sum'_{|j|=q+1} \left( e^{-\lambda_{\varepsilon}/2} u_{\varepsilon} \right)_j (\bar{\partial} \varphi \wedge f)(z).
\end{equation}
To do this, we work relative to a basis $\{t_{\varepsilon,1}(z), \ldots, t_{\varepsilon,n}(z)\}$ of eigenvectors of $(\frac{\partial^2 \lambda_{\varepsilon}}{\partial z_j \partial \overline{z}_k})(z)$ with associated eigenvalues $\sigma_{\varepsilon,1}(z), \ldots, \sigma_{\varepsilon,n}(z)$. Fix a multi index $J = (j_1, \ldots, j_{q+1})$. Then
\begin{equation}
\sum'_{|K|=q, j} \sum_{j} \frac{\partial^2 \lambda_{\varepsilon}}{\partial z_j \partial \overline{z}_k} (z) (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_j K (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{K K}
= \sum'_{|K|=q, j} \sum_{j} \sigma_{\varepsilon,j}(z) (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{j K}^2 \geq (\sigma_{\varepsilon,j_1}(z) + \cdots + \sigma_{\varepsilon,j_{q+1}}(z)) (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{j}^2;
\end{equation}
compare equation (4.30) in [Str10]. The reason for the last inequality is that $(u_{\varepsilon})_j$ appears via $(q + 1)$ pairs $(j, K)$, namely for $j = j_1, \ldots, j_{q+1}$; the other terms not involving $J$ are nonnegative. If $\sigma_{\varepsilon,j_1}(z) + \cdots + \sigma_{\varepsilon,j_{q+1}}(z) \geq 1/\varepsilon$, we obtain
\begin{equation}
|\left( e^{-\lambda_{\varepsilon}/2} u_{\varepsilon} \right)_j (z)| \leq \varepsilon \sum'_{|K|=q, j} \sum_{j} \frac{\partial^2 \lambda_{\varepsilon}}{\partial z_j \partial \overline{z}_k} (z) (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{j K} (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{K K}.
\end{equation}
If $\sigma_{\varepsilon,j_1}(z) + \cdots + \sigma_{\varepsilon,j_{q+1}}(z) < 1/\varepsilon$, then $|\left( \bar{\partial} \varphi(z), t_{\varepsilon,s}(z) \right)| \leq \omega(1/\varepsilon)$ for $1 \leq s \leq q + 1$ (see (10); $z \in \Omega \setminus \widetilde{\Omega}_{\varepsilon}$). Therefore
\begin{equation}
|\left( \bar{\partial} \varphi \wedge f \right)(z)| = |\left( \bar{\partial} \varphi \wedge f \right)(t_{\varepsilon,j_1}, t_{\varepsilon,j_2}, \ldots, t_{\varepsilon,j_{q+1}})|
\leq \sum_{s=1}^{q+1} \left| \left( \bar{\partial} \varphi, t_{\varepsilon,j_s} \right)(z) \right| \left| \left( \bar{\partial} \varphi, t_{\varepsilon,j_s} \right)(z) \right| \leq \omega(\varepsilon) |f(z)|.
\end{equation}
To control $e^{-\lambda_{\varepsilon}/2} u_{\varepsilon}$ in this case, we use that $(\frac{\partial^2 \lambda_{\varepsilon}}{\partial z_j \partial \overline{z}_k})$ is uniformly bounded from below; this can be seen for example from (5), because $\Omega$ is bounded. Therefore
\begin{equation}
|e^{-\lambda_{\varepsilon}/2} u_{\varepsilon}(z)|^2 = \frac{1}{q + 1} \sum'_{|K|=q} \sum_{j} |e^{-\lambda_{\varepsilon}/2} (u_{\varepsilon})_{j K}(z)|^2
\leq \frac{1}{q + 1} \sum'_{|K|=q} \sum_{j} \frac{\partial^2 \lambda_{\varepsilon}}{\partial z_j \partial \overline{z}_k} (z) (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{j K} (e^{-\lambda_{\varepsilon}/2} u_{\varepsilon})_{K K} ;
\end{equation}
in the last step \( \lesssim \) is independent of \( f \) and \( \varepsilon \). Thus we obtain from both cases that when \( z \in \Omega \setminus \tilde{\Omega}_\varepsilon \),
\[
\left| \left( e^{-\lambda_\varepsilon/2} u_\varepsilon \bar{\partial} \varphi \wedge f \right)(z) \right|^2 \lesssim (\varepsilon + \omega^2(\varepsilon)) \left( \sum' \sum_{|k|=q} \frac{\partial^2 \lambda_\varepsilon}{\partial \bar{z}_j \partial \bar{z}_k} (z) (e^{-\lambda_\varepsilon/2} u_\varepsilon)_{j,k} (e^{-\lambda_\varepsilon/2} u_\varepsilon)_{j,k} \right) |f(z)|^2.
\]

In the estimate above \( \lesssim \) is independent of \( \varepsilon \) and \( f \). Setting \( \tilde{\omega}(\varepsilon) := \sqrt{\varepsilon + \omega^2(\varepsilon)} \), we have in view of (12)
\[
\left| \langle e^{-\lambda_\varepsilon/2} u_\varepsilon \bar{\partial} \varphi \wedge f \rangle_{\Omega \setminus \tilde{\Omega}_\varepsilon} \right| \lesssim \tilde{\omega}(\varepsilon) \left( \int_{\Omega} \sum' \sum_{|k|=q} \frac{\partial^2 \lambda_\varepsilon}{\partial \bar{z}_j \partial \bar{z}_k} (e^{-\lambda_\varepsilon/2} u_\varepsilon)_{j,k} (e^{-\lambda_\varepsilon/2} u_\varepsilon)_{j,k} dV \right)^{1/2} \left( \int_{\Omega} |f|^2 dV \right)^{1/2}
\]
(21)
\[
\lesssim \tilde{\omega}(\varepsilon) \left( \| \tilde{\partial} (e^{-\lambda_\varepsilon/2} u_\varepsilon) \|^2_{\Omega} + \| f \|^2_{\Omega} \right)
\lesssim \tilde{\omega}(\varepsilon) \left( \| \tilde{\partial} v \|^2_{\Omega} + \| f \|^2_{\Omega} \right) \lesssim \tilde{\omega}(\varepsilon) \| f \|^2_{\Omega}.
\]

We have used here that the integrands are nonnegative, so that the integrals over \( \Omega \) dominate those over \( \Omega \setminus \tilde{\Omega}_\varepsilon \); the second inequality is \( |ab| \leq a^2 + b^2 \).

It remains to control \( \langle e^{-\lambda_\varepsilon/2} u_\varepsilon \bar{\partial} \varphi \wedge f \rangle_{\tilde{\Omega}_\varepsilon} \). To this end, consider the following three linear operators:
\[
K_{1,\varepsilon} : K^2_{(0,q)}(\Omega) \to \text{dom}(\tilde{\partial}) \cap \text{dom}(\tilde{\partial})^* \subset L^2_{(0,q)}(\Omega),
\]
\[
K_{2,\varepsilon} : \text{dom}(\tilde{\partial}) \cap \text{dom}(\tilde{\partial}^*) \subset L^2_{(0,q)}(\Omega) \to \text{dom}(\tilde{\partial}) \cap \text{dom}(\tilde{\partial}^*) \subset L^2_{(0,q)}(\Omega),
\]
\[
K_{3,\varepsilon} : \text{dom}(\tilde{\partial}) \cap \text{dom}(\tilde{\partial}^*) \subset L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\tilde{\Omega}_\varepsilon),
\]
defined as follows: \( K_{1,\varepsilon} f = N_q (\bar{\partial} \varphi \wedge f) \), \( K_{2,\varepsilon} v = e^{-\lambda_\varepsilon/2} (P_{q,\lambda_\varepsilon/2})(e^{-\lambda_\varepsilon/2} v) \), and \( K_{3,\varepsilon} w = w|_{\tilde{\Omega}_\varepsilon} \). Then \( K_{1,\varepsilon} \) and \( K_{2,\varepsilon} \) are continuous, and \( K_{3,\varepsilon} \) is compact (by interior elliptic regularity, see for example Proposition 5.1.1 in [CS01], and the fact \( W^1(\tilde{\Omega}_\varepsilon) \hookrightarrow L^2(\tilde{\Omega}_\varepsilon) \) is compact). Consequently, the operator \( K_\varepsilon = K_{3,\varepsilon} \circ K_{2,\varepsilon} \circ K_{1,\varepsilon} \) is compact. We have
\[
\left| \langle e^{-\lambda_\varepsilon/2} u_\varepsilon \bar{\partial} \varphi \wedge f \rangle_{\tilde{\Omega}_\varepsilon} \right| = \left| \langle K_\varepsilon f, \bar{\partial} \varphi \wedge f \rangle_{\tilde{\Omega}_\varepsilon} \right| \lesssim \| K_\varepsilon f \|_{\tilde{\Omega}_\varepsilon} \| f \|_{\Omega} \lesssim \| e^{-1} K_\varepsilon f \|^2_{\tilde{\Omega}_\varepsilon} + \varepsilon^2 \| f \|^2_{\Omega};
\]
with the constants in the two inequalities being independent of \( \varepsilon \) (and \( f \)). The second inequality is the usual small constant–large constant estimate.

To complete the proof of Theorem 1, it now suffices to combine (3), (11), (15), (21), and (22) to obtain
\[
\| H^0_{\phi,\varepsilon} f \|^2 \leq C \left( (\tilde{\omega}(\varepsilon) + \varepsilon^2) \| f \|^2 + \| e^{-1} K_\varepsilon f \|^2 \right),
\]
where \( C \) is independent of \( \varepsilon \). Because \( e^{-1} K_\varepsilon \) is compact, and \( \lim_{\varepsilon \to 0^+} (\tilde{\omega}(\varepsilon) + \varepsilon^2) = 0 \), we can rescale to obtain the required family of estimates (2). \( \square \)
3. Proof of Corollary 2

The properties of Fredholm operators we use can be found for example in [Con85, Chapter XI]. We first note that we may assume that \( \varphi \neq 0 \) on \( \Omega \) also when \( q = 0 \). Indeed, if \( \varphi \neq 0 \) on \( b\Omega \), then there is \( \tilde{\varphi} \in C^1(\overline{\Omega}) \) with \( \tilde{\varphi} \neq 0 \) on \( \Omega \) and \( \varphi - \tilde{\varphi} \) compactly supported. Then \( T^0_\varphi = T^0_{\tilde{\varphi}} + T^0_{\varphi - \tilde{\varphi}} \). Since \( f \mapsto (\varphi - \tilde{\varphi})f \) is compact when \( q = 0 \) the operator \( T^0_{\varphi - \tilde{\varphi}} \) is compact. Hence \( T^0_\varphi \) and \( T^0_{\tilde{\varphi}} \) are simultaneously Fredholm or non-Fredholm, with equal indices in the first case.

So assume now that \( \varphi \in C^1(\overline{\Omega}) \) is nonvanishing on \( \overline{\Omega} \), and \( 0 \leq q \leq (n - 1) \). Then \( (1/\varphi) \in C^1(\Omega) \) and \( (1/\varphi) \) is also holomorphic along varieties in the boundary. We have

\[
T^q_\varphi T^q_{(1/\varphi)} = T^q_{(1/\varphi)}(1/\varphi) - (H^q_{(1/\varphi)})^* H^q_{(1/\varphi)},
\]

and

\[
T^q_{(1/\varphi)} T^q_\varphi = T^q_{(1/\varphi)}(1/\varphi) - (H^q_{1/\varphi})^* H^q_{\varphi},
\]

where * denotes the adjoint. These equations are the standard relations between (semi) commutators of Toeplitz operators and Hankel operators; they follow by direct computation. Of course, \( T^q_{\varphi(1/\varphi)} = T^q_{(1/\varphi)\varphi} = I \), the identity. By Theorem 1, both \( H^q_{(1/\varphi)} \) and \( H^q_{\varphi} \) are compact. So (24) and (25) say that \( T^q_\varphi \) is invertible modulo compact operators (with inverse in the Calkin algebra given by \( T^q_{(1/\varphi)} \)), that is, \( T^q_\varphi \) is Fredholm.

It remains to see that \( \text{ind}(T^q_\varphi) = 0 \). Because \( \overline{\Omega} \) is simply connected, there is \( \hat{\varphi} \in C(\overline{\Omega}) \) such that \( \varphi = e^{\hat{\varphi}} \) \((z \mapsto e^z \) is a covering map of \( C \) onto \( C \setminus \{0\} \)). Moreover, \( \hat{\varphi} \in C^1(\overline{\Omega}) \), and it is holomorphic along varieties in \( b\Omega \). This is clear for a local branch of \( \log \varphi \), and such a branch differs from \( \hat{\varphi} \) by a constant. We can now apply a standard homotopy argument. Namely, for all \( t \in [0, 1] \), \( T^q_{e^{t\hat{\varphi}}} \) is Fredholm (by Theorem 1). Also, \( t \mapsto T^q_{e^{t\hat{\varphi}}} \) is continuous from \([0, 1] \) to \( B(K^2_{(0,q)}(\Omega)) \), the bounded operators on \( K^2_{(0,q)}(\Omega) \) \((\varphi \mapsto T^q_\varphi \) is continuous form \( C(\Omega) \) to \( B(K^2_{(0,q)}(\Omega)) \). The index is continuous from the Fredholm operators with the topology inherited from \( B(K^2_{(0,q)}(\Omega)) \) to \( \mathbb{Z} \) (with the discrete topology). Thus \( t \mapsto \text{ind}(T^q_{e^{t\hat{\varphi}}}) \) is continuous from \([0, 1] \) to \( \mathbb{Z} \), and therefore is constant. We obtain \( 0 = \text{ind}(T^q_{e^{t\hat{\varphi}}})|_{t=0} = \text{ind}(T^q_{e^{t\hat{\varphi}}})|_{t=1} = \text{ind}(T^q_\varphi) \), and the proof of Corollary 2 is complete.

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