PRODUCT OF MASSES ON BOUNDEDNESS, BLOW-UP AND CONVERGENCE IN A TWO-SPECIES AND TWO-STIMULI CHEMOTAXIS SYSTEM WITH/WITHOUT LOOP

KE LIN AND TIAN XIANG\(^*\)

ABSTRACT. In this work, we study dynamic properties of classical solutions to a homogenous Neumann initial-boundary value problem (IBVP) for a two-species and two-stimuli chemotaxis model with/without chemical signalling loop in a 2D bounded and smooth domain. We detect the product of two species masses as a feature to determine boundedness, blow-up and convergence of classical solutions for the corresponding IBVP. More specifically, we first show generally a smallness on the product of both species masses, thus allowing one species mass to be suitably large, is sufficient to guarantee global boundedness, higher order gradient estimates and \(W^{j,\infty}(j \geq 1)\)-convergence with rates of convergence to constant equilibria; and then, in a special case, we detect a straight line of masses on which blow-up occurs for large product of masses. Our findings provide new understandings about the underlying model, and thus, improve and extend greatly the existing knowledge relevant to this model.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In this work, we further study dynamic properties of classical solutions to the Neumann initial-boundary value problem for the following two-species and two-stimuli chemotaxis model with/without chemical signalling loop:

\[
\begin{align*}
    u_t &= \nabla \cdot (\nabla u - \chi_1 u \nabla v) & \text{in } \Omega \times (0, \infty), \\
    \tau_1 v_t &= \Delta v - v + w & \text{in } \Omega \times (0, \infty), \\
    w_t &= \nabla \cdot (\nabla w - \chi_2 w \nabla z - \chi_3 w \nabla v) & \text{in } \Omega \times (0, \infty), \\
    \tau_2 z_t &= \Delta z - z + u & \text{in } \Omega \times (0, \infty), \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
    (u, \tau_1 v, w, \tau_2 z) &= (u_0, \tau_1 v_0, w_0, \tau_2 z_0) & \text{in } \Omega \times \{0\}.
\end{align*}
\]

Here, \(\Omega \subset \mathbb{R}^2\) is a bounded and smooth domain and \(\frac{\partial}{\partial \nu}\) denotes the outer normal derivative on the boundary \(\partial \Omega\). \(u = u(x,t)\) and \(w = w(x,t)\) respectively denote the unknown density of macrophages and tumor cells, while \(v = v(x,t)\) and \(z = z(x,t)\) represent the concentration of chemical signals secreted by \(w\) and \(u\), respectively. The modelling parameters \(\chi_i > 0, \tau_i \geq 0\) \((i = 1, 2)\) and \(\chi_3 \in \mathbb{R}\) are given constants.

2010 Mathematics Subject Classification. Primary: 35K59, 35B25, 35B44, 35K51; Secondary: 92C17, 92D25.

Key words and phrases. Chemotaxis with/without loop, product of mass, boundedness, blow-up, gradient estimates, asymptotic stability.

\(^*\) Corresponding author.
Model \((1.1)\) involves four unknown variables \(u, v, w, z\) and describes a two-species and two-stimuli chemotaxis model with/without chemical signalling loop, depending on \(\chi_3 = 0\) or not: macrophages \(u\) secrete a chemical signal \(z\), called gradient epidermal growth factor, which has an attractive impact on tumor cells \(w\) and further stimulates them to secrete the other chemical signal \(v\), called the colony stimulating factor 1, which attracts macrophages \(u\) to aggregate and binds to receptors of the macrophages \(u\), continuing the activation of them in return \((1.1)\). This model contains two widely-studied sub-models: upon setting \(u = z = 0\) or \(\chi_1 = \chi_2 = 0\), the well-known one-species and one-stimuli minimal Keller-Segel model follows:

\[
\begin{align*}
\tau_1 v_t &= \Delta v - v + w \quad \text{in } \Omega \times (0, \infty), \\
w_t &= \nabla \cdot (\nabla w - \chi_3 w v) \quad \text{in } \Omega \times (0, \infty).
\end{align*}
\]

(1.2)

This minimal KS model is well-known to exhibit critical mass blow-up striking future in 2D (small mass \(m_2\chi_3 < \pi^*\), defined by Lemma \(2.4\) below, yields boundedness \(8, 26\), otherwise, blow-up may occur \(9, 10, 14, 25\)) and generic blow-up in \(\geq 3D\), see the review papers \(1, 11, 37, 38\) for more. The second sub-model is the two-species and two stimuli chemotaxis model obtained by setting \(\chi_3 = 0\):

\[
\begin{align*}
\tau_1 v_t &= \Delta v - v + w \quad \text{in } \Omega \times (0, \infty), \\
w_t &= \nabla \cdot (\nabla w - \chi_2 w z) \quad \text{in } \Omega \times (0, \infty), \\
\tau_2 z_t &= \Delta z - z + w \quad \text{in } \Omega \times (0, \infty).
\end{align*}
\]

(1.3)

When \(\tau_1 = \tau_2 = 0\), Tao and Winkler \(34\) systematically studied the boundedness vs blow-up, wherein \(\chi_1\) and \(\chi_2\) are allowed to be real: for either \(\chi_1 < 0\) or \(\chi_2 < 0\), boundedness for large initial data is guaranteed in \(\leq 3D\); in the challenging while more interesting case when both \(\chi_1 > 0\) and \(\chi_2 > 0\), boundedness vs blow-up is determined by the total mass of both species: writing

\[
m_1 = \int_{\Omega} u_0, \quad m_2 = \int_{\Omega} w_0,
\]

(1.4)

then boundedness is ensured for \(\max\{m_1, m_2\} < C_0\) with some \(C_0 > 0\), whereas, for \(\chi_1 = \chi_2 = 1\), finite time blow-up in 2D may occur for \(\min\{m_1, m_2\} > 4\pi\). These results were improved by Yu et. al. in \(12\) by showing that \(C_0 = 4\pi\) and a blow-up criterion that

\[
\frac{1}{m_2\chi_1} + \frac{1}{m_1\chi_2} < \frac{1}{2\pi}.
\]

(1.5)

Very recently, we observed in \(24\) that the chemotactic signaling loop between two cell types bridges certain relationship between \(u\) and \(w\), and therefore, the dynamics of one species shall be essentially determined by the other. To verify that, we considered a 2D much simplified version of \((1.1)\) in the unit ball \(\Omega = B_1(0) \subset \mathbb{R}^2\) with the second and fourth equation respectively replaced by

\[
0 = \Delta v - \bar{w}_0 + w, \quad 0 = \Delta z - \bar{u}_0 + u, \quad \bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0.
\]

(1.6)

In this setup, the problem essentially becomes two 1D scalar parabolic equations, which renders parabolic comparison principles applicable. Then substantial progresses on the simultaneous boundedness and finite-time blow-up are provided and, in particular, the previous boundedness for both small masses was improved to
be \( \min \{ m_1 \chi_2, m_2 \chi_1 \} < 4 \pi \), requiring only one mass be small. While, those arguments, especially [24 Lemma 3.1], seem to be hardly adapted to \((1.3)\) even in radial settings. Even through suitable largeness of both masses are known to produce blow-ups \((1.5)\) (c.f. also \((1.3)\)), however, in non-radial settings, as a starting motivation of this project, we are wondering

(Q1) whether suitable largeness of one mass is still able to ensure boundedness and further convergence?

When \( \tau_1 > 0, \tau_2 > 0 \), in this fully parabolic case, much less seems to be known except that Li and Wang [20] provided boundedness for \((1.3)\) under an implicit smallness condition on both \( m_1 \) and \( m_2 \). On the other hand, to our best knowledge, so far, no blow-up has been detected yet and there seems even no available result on large time behavior of bounded solutions to either \((1.3)\) or \((1.1)\), except that Li and Wang [20] provided boundedness for \((1.3)\) under an implicit smallness condition on both \( m \) except that \( \tau \).

(Q2) whether suitable largeness of both masses induces blow-up?

As a continuation of mainly works [20, 24, 34, 42], our purpose is to provide further understandings about global dynamics of the two-species and two-stimuli chemotaxis model \((1.1)\) with/without signal loop motivated by the non-obvious questions (Q1) and (Q2) for the cases of \( \tau_1 = \tau_2 = 0 \) and \( \tau_1, \tau_2 > 0 \). Roughly, going far beyond (Q1) and (Q2), our findings first show that only a smallness of product \( m_1 m_2 \chi_1 \chi_2 \) is needed to ensure global boundedness, higher order gradient estimates and \( W^{1, \infty}(j \geq 1) \)-convergence with rates of convergence; and, moreover, we are wondering how much the blow-up criterion \((1.5)\) in the elliptic case can be carried over to the fully parabolic case by asking

(Q2) whether suitable largeness of both masses induces blow-up?

Theorem 1.1. Let \( \chi_i > 0, \tau_i \geq 0 \) \((i = 1, 2)\), \( \chi_3 \in \mathbb{R}, \Omega \subset \mathbb{R}^2 \) be a bounded and smooth domain, and let the initial data \((u_0, \tau_1 v_0, u_0, \tau_2 z_0)\) be nontrivial and nonnegative and take respectively from \( C^0(\Omega) \times W^{1, \infty}(\Omega) \times C^0(\Omega) \times W^{1, \infty}(\Omega) \).

(B1) [Uniform Boundedness / Assume]

\[
m_1 m_2 \chi_1 \chi_2 < \begin{cases} (\pi^* - m_2 \chi_3) \pi^*, & \text{if } \tau_1 = \tau_2 = 0, \\ \sqrt{1 - 4 m_2 \chi_3 C_{GN}}, & \text{if } \tau_1, \tau_2 > 0. \end{cases} \tag{1.9}
\]

Then the IBVP \((1.1)\) admits a unique global-in-time classical solution which is uniformly bounded in time according to

\[
\|u(t)\|_{L^\infty} + \|\tau(t)\|_{W^{1, \infty}} + \|v(t)\|_{L^\infty} + \|z(t)\|_{W^{1, \infty}} \leq C_1, \quad t \geq 0. \tag{1.10}
\]
\[\text{(B2) [Gradient Estimates]} \text{ The uniform } L^1\text{-boundedness of } (u \ln u, w \ln w) \text{ indeed implies the higher order gradient estimate away from } t = 0, \text{ say } t \geq 1:\]
\[\| (u(t), u(t)) \|_{W^2,1} + \| (u(t), w(t)) \|_{W^3,1} + \| (v(t), z(t)) \|_{W^{3,1}} \]
\[\leq C_2, \quad \forall t \geq 1. \quad (1.11)\]
\[\text{(B3) [Exponential Convergence]} \text{ When } \tau_1 = \tau_2 = 0, \text{ assume}\]
\[k^2m_1m_2\chi_1\chi_2 + km_2\Omega|\chi_3^+| < 4|\Omega|^2, \quad \chi_3^+ = \max\{\chi_3, 0\}; \quad (1.12)\]
\[\text{when } \tau_1 > 0 \text{ and } \tau_2 > 0, \text{ assume}\]
\[\left\{ \begin{array}{l}
2 - \frac{\chi_3^+}{3} < \frac{k^2m_1m_2\chi_1\chi_2}{|\Omega|} < \sqrt{2}, \\
k^2m_1m_2\chi_1\chi_2 < \frac{2\chi_3^+}{3}|\Omega|^2 \min\left\{ 1, \frac{4}{2 + \frac{k^2m_1m_2\chi_1\chi_2}{|\Omega|}} \right\}. \end{array} \right. \quad (1.13)\]
\[\text{Then the unique global solution of } (1.1) \text{ decays exponentially according to:}\]
\[\left\{ \begin{array}{l}
\| (u(t) - \bar{u}_0, w(t) - \bar{u}_0) \|_{W^{1,\infty}} \leq C_3 e^{-\frac{\chi_3^+}{4}t}, \quad \forall t \geq 1, \\
\| (v(t) - \bar{u}_0, z(t) - \bar{u}_0) \|_{W^{3,1}} \leq C_4 e^{-\frac{\chi_3^+}{44}t}, \quad \forall t \geq 1, \quad \text{if } \tau_1 = \tau_2 = 0, \\
\| (v(t) - \bar{u}_0, z(t) - \bar{u}_0) \|_{W^{3,\infty}} \leq C_5 e^{-\frac{\chi_3^+}{44}t}, \quad \forall t \geq 1, \quad \text{if } \tau_1, \tau_2 > 0. \end{array} \right. \quad (1.14)\]
\[\text{(B4) [Finite time Blow-up]} \text{ Assume that } \tau_1 = \tau_2 \text{ and } \chi_3 = 0. \text{ Then on the straight line } m_1\chi_2 = m_2\chi_1, \text{ there exists a family of initial data } (u_0, \tau_1v_0, w_0, \tau_2z_0) \text{ with}\]
\[m_1m_2\chi_1\chi_2 > (\pi^*)^2, \quad (1.15)\]
\[\text{such that for some finite } T > 0 \text{ the corresponding unique solution of the IBVP } (1.1) \text{ exists classically on } \Omega \times (0, T) \text{ but blows up at } t = T \text{ in the sense that}\]
\[\limsup_{t \uparrow T} (\| (u \ln u) (t) \|_{L^1} + \| (w \ln w) (t) \|_{L^1}) = \infty. \quad (1.16)\]

Here and below, \( m_i \) \((i = 1, 2) \) are defined in (1.4), \( \pi^* \) is an explicit positive number defined in Lemma [2.4], \( C_{GN} \) and \( k \) are defined in (1.7) and (1.8), respectively, \( I_{(\tau_1, 0)} = 1 \) if \( \tau_1 = 0 \), otherwise, it is zero; \( \zeta(k) = \min\{\frac{1}{\tau_1}, \frac{1}{\tau_2}, \frac{\tau(k)}{2} \} \)
and \( \sigma(k) = \mu(k) \) if \( \tau_1 = \tau_2 = 0 \) and \( \sigma(k) = \delta(k) \) if \( \tau_1, \tau_2 > 0 \) with \( \mu(k) \)
defined by (1.12) and \( \delta(k) \) by (1.38), both of them are functions of \( m_1, \chi_1, \tau_1 \) and \( k \); \( C_i = C_i(u_0, \tau_1v_0, w_0, \tau_2z_0, \chi_1, |\Omega|) \) are positive constants, \( \bar{u}_0 \), the average of \( u_0 \), is defined in (1.6), similarly for \( \bar{w}_0 \), vector notation is understood as componentwise, and, finally, the commonly abbreviated notations are used: for instance, for a generic function \( f \),
\[\|f(t)\|_{L^p} = \|f(\cdot, t)\|_{L^p} = \|f(\cdot, t)\|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x, t)|^p \, dx \right)^{\frac{1}{p}}.\]

Remark 1.2 (Product of masses on boundedness, blow-up and convergence).
\[\text{(P1) Our higher order gradient estimate (B2) and the } W^{3,\infty}(j \geq 1)\text{-convergence in (B3) seem to appear the first time in chemotaxis-related systems.}\]
\[\text{(P2) The feature of our main results is that we detect a smallness on the product } m_1m_2\chi_1\chi_2 \text{ to ensure boundedness, higher order gradient estimates and exponential convergence with rates of convergence thanks to (B2) and (B3).}\]
\[\text{These together with Remark 1.7 show that our results also extend and improve existing boundedness and convergence of solutions to the one-species and one stimuli Keller–Segel model [182], c.f. [8, 37].}\]
(P3) When $\chi_3 = 0$, our boundedness (B1) under the (explicit) smallness (1.9) improves greatly the existing boundedness under smallness of both $m_1$ and $m_2$ [20, 34, 42]. When $\chi_3 \neq 0$, boundedness results (B1)-(B3) extend our previous one under radial and elliptic simplification (cf. (1.6)) in [24]. Moreover, the sign effect of $\chi_3$ is exhibited, showing damping effect of repulsion on boundedness and convergence, especially, when $\tau_1 = \tau_2 = 0$.

(P4) No matter $\tau_1 = \tau_2 = 0$ or $\tau_1, \tau_2 > 0$, (B4) detects a blow-up line on $m_2\chi_1 = m_1\chi_2$. We should point out that it (along with (B1)) is not a blow-up criterion, while, it gives a rough lower bound for $C_{GN}$, i.e., $C_{GN}^4 \geq \frac{1}{2\pi}$.

Also, in convex domains, $k$ has a lower bound: $k \geq \frac{\sqrt{d}}{\sqrt{\pi}}$, cf. Lemma 2.3. In non-symmetric domains, i.e., $\pi^* = 4\pi$, it is easy to see that this blow-up line is inside the range enclosed by the blow-up criterion (1.5), which is inside the range enclosed by (1.15). Hence, (B1) is not optimal even in the case of $\tau_1 = \tau_2 = 0$. Together with (B1), we see that the critical curve that distinguishes boundedness and blow-up for (1.3) must contain $(m_2\chi_1, m_1\chi_2) = (\pi^*, \pi^*)$ as a boundary point. We conjecture that a general version of the blow-up criterion (1.6) for (1.3) continues to hold in the case of $\tau_1, \tau_2 > 0$, namely,

$$\frac{1}{m_2\chi_1} + \frac{1}{m_1\chi_2} < \frac{2}{\pi^*}.$$ 

We leave this open problem as a future investigation.

The point of our project is that we detect the product $m_1m_2\chi_1\chi_2$ as a feature to determine boundedness, blow-up and convergence for (1.1). First, the smallness of $m_1m_2\chi_1\chi_2$ in (1.9) or (1.12) or (1.13) allows us to choose suitably large mass of one species and small for the other to ensure global boundedness, higher order gradient estimates and exponential convergence. This is in sharp contrast to those of [20, 34, 42], wherein smallness of both masses are needed to have boundedness. Second, we find a line of masses on which blow-up occurs for large product of masses. While, we have to mention that, even in the case $\tau_1 = \tau_2 = 0$, critical mass phenomenon for (1.3) has not been detected yet. Critical mass phenomenon does exist in the minimal classical KS model with one-species and one-stimuli [14, 25, 26, 27], while, for more complex or multi-species chemotaxis systems, boundedness and blow-up exist [6, 13, 20, 24, 31, 34, 42], but critical mass blow-up occurs rarely [5, 15]. In a future exploration, we shall aim to determine a critical curve which separates boundedness and blow-up for (1.1) or simplified version like (1.3).

In the remaining of this section, we outline the structure of this paper, which contains four main sections.

In the present section, we provide an introduction to our two-species and two-stimuli chemotaxis model with/without chemical signalling loop that encompasses two important widely-studied sub-models, and then we formulate our main motivations and state our main findings in Theorems (1.1) on product of masses on boundedness, higher order gradient estimates, blow-up and exponential convergence.

In Section 2, we first state the local existence and extensibility of smooth solutions to the IBVP (1.1), and then, we obtain a standard $W^{1,q}$-estimate for an inhomogeneous heat/elliptic equation, cf. Lemma 2.2. and then, upon an observation of best constant for the Poincaré inequality [28], we find an explicit lower bound
for $k$ defined in (1.8) in convex domains, cf. Lemma 2.3, and, finally, for convenience, we state a version of Trudinger-Moser inequality [26] and the widely-used 2D Gagliardo-Nirenberg inequality [7], which will be used later on.

To make the flow of our ideas more clear, we divide Section 3 into 4 subsections to prove our stated boundedness, gradient estimates and finite time blow-up in (B1), (B2) and (B4). Our analysis begins with a general identity associated with (1.1) which becomes a conditional Lyapunov functional in the case of $\tau_1 = \tau_2 = 0$ and small product of masses, cf. Lemmas 3.1 and 3.2, and thus yields the key starting uniform $L^1$-boundedness of $(u \ln u, w \ln w)$. In the fully parabolic case, the same boundedness is derived based on estimating the differential of a well-selected combined energy together with subtle analysis, cf. Lemma 3.4. Then, using quite known testing procedure, we raise the obtained $L^1$-boundedness of $(u \ln u, w \ln w)$ eventually to the one stated in (1.10), cf. Lemma 3.7. Right after that, we are devoted to showing the uniform $L^1$-boundedness of $(u \ln u, w \ln w)$ indeed implies higher order gradient estimates as in (B2). To achieve that goal, we progressively use energy method together with the 2D G-N interpolation inequality, $W^{2,p}$-elliptic and $W^{1,q}$-parabolic estimate to derive uniform estimates for the following route map of mainly $u$ (similar for $w$):

$$\|\nabla u\|_{L^2} \to \|\nabla u\|_{L^4} \to \|\Delta u\|_{L^2} + \|(v, z)\|_{W^{2,\infty}} \to \|\Delta u\|_{L^4} + \|(v, z)\|_{W^{3,\infty}}.$$ 

Finally, in Subsection 3.4, on the straight line $m_2 \chi_1 = m_1 \chi_2$, we construct initial data satisfying (1.15) so that the corresponding solution of (1.1) blows up in finite time according to (1.16), cf. Lemma 3.10. In this case, upon an observation that our two-species and two-stimuli model (1.1) is a two-copy of the one-species and one-stimuli minimal KS model (cf. Lemma 3.9), we make use of the well-known blowup knowledge about the minimal model ([9, 10, 11, 14, 25, 26, 27]) to construct the existence of finite time blow-up for (1.1) as in (B4), cf. Lemma 3.10.

In Section 4, inspired from [8], we first transform our model (1.1) conveniently into an equivalent one in (4.1), and then we construct two well-chosen testing functionals involving $(U \ln U, W \ln W)$ in (4.1) and (4.13), which become genuine Lyapunov functionals and decay exponentially with precise rates under (1.12) or (1.13), cf. Lemmas 4.1 and 4.3. Here, $(U, W) = (\bar{u}, \bar{w})$, cf. (4.2). As consequences, we obtain the crucial starting $L^1$-exponential convergence of $(U \ln U, W \ln W)$ with precise convergence rates. Then with the aid of the Csizszár-Kullbach-Pinsker inequality (cf. [3]), we indeed obtain $L^p$ ($p \geq 1$)-exponential convergence of $(U - 1, W - 1)$, cf. Lemma 4.6. With these information at hand, one can use (commonly used, cf. [22, 23, 37]) the standard $W^{2,p}$-estimate in the case of $\tau_1 = \tau_2 = 0$ or the $L^p$-$L^q$-smoothing estimate for the Neumann heat semigroup $e^{t\Delta}$ in the case of $\tau_1, \tau_2 > 0$ to derive the exponential decay of bounded solutions in up to $L^\infty$-norm. Here, thanks to our uniform higher order gradient estimates as in (B2), instead, we readily utilize the G-N interpolation inequality to improve the $L^p$-convergence to $W^{j,\infty}$ ($j \geq 1$)-convergence of $(U, V, W, Z)$ with rate of convergence. Upon simple translations, we achieve the $W^{j,\infty}$-convergence for our original model (1.1) indeed more than what has been stated in (B4), cf. Lemma 4.7.

2. Preliminaries and basic results on our model

We first state the well-established local well-posedness and extensibility of solutions to the IBVP (1.1) and elementary $L^1$-properties of local solutions.
Lemma 2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $\chi_i, \tau_i$ ($i = 1, 2$) be negative constants, and finally, let the nontrivial initial data $(u_0, v_0, w_0, \tau_2 z_0) \in (C(\bar{\Omega}))^2$ and $(\tau_1 v_0, \tau_2 z_0) \in (W^{1,\infty}(\Omega))^2$. Then there exist a maximal existence time $T_m \in (0, \infty]$ and a uniquely determined pair of positive functions $(u, v, w, z) \in (C(\bar{\Omega} \times [0, T_m])) \cap C^{2,1}(\bar{\Omega} \times (0, T_m)))$ that solve the IBVP \((1.1)\) classically on $\bar{\Omega} \times (0, T_m)$ and fulfill the following $L^1$-properties: for $t \in (0, T_m)$,

$$
\begin{align*}
\|u(t)\|_{L^1} &= \|u_0\|_{L^1}, \quad \|w(t)\|_{L^1} = \|w_0\|_{L^1}, \\
\|v(t)\|_{L^1} &= \|v_0\|_{L^1} + \begin{cases} 
0, & \text{if } \tau_1 = 0, \\
(\|v_0\|_{L^1} - \|u_0\|_{L^1}) e^{-\tau_1}, & \text{if } \tau_1 > 0,
\end{cases} \\
\|z(t)\|_{L^1} &= \|u_0\|_{L^1} + \begin{cases} 
0, & \text{if } \tau_2 = 0, \\
(\|z_0\|_{L^1} - \|u_0\|_{L^1}) e^{-\tau_2}, & \text{if } \tau_2 > 0.
\end{cases}
\end{align*}
$$

(2.1)

Moreover, the local solution $(u, v, w, z)$ fulfills the following extensibility criterion:

$$
T_m < \infty \Rightarrow \lim_{t \to T_m} \sup_{t < T_m} (\| (u(t), w(t)) \|_{L^\infty} + \| (\tau_1 v(t), \tau_2 z(t)) \|_{W^{1,\infty}}) = \infty. \tag{2.2}
$$

Proof. The local well-posedness and extensibility of solutions to the IBVP \((1.1)\) and thus \((2.1)\) have been well-established via Banach contraction principle and parabolic regularity; see e.g. \([1, 12, 34, 30, 32, 43]\) for closely-related chemotaxis systems. The conservations of $u$ and $w$ follow upon integration by parts on the first and third equation in \((1.1)\). By a simple integration of the $v$-equation and using the homogeneous Neumann boundary conditions, one has

$$
\frac{\tau_1}{d}{\int}_{\Omega} v + \int_{\Omega} v = \int_{\Omega} w = \int_{\Omega} w_0,
$$

which implies the $L^1$-norm of $v$ in \((2.1)\). Likewise, the $L^1$-norm of $z$ follows. \hfill \Box

Lemma 2.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded and smooth domain and let

$$
q \in \left[1, \frac{2p}{2p-1}\right], \quad \text{if } 1 \leq p \leq 2,
$$

$$
q \in [1, \infty], \quad \text{if } p > 2. \tag{2.3}
$$

Then there exist $C_1 = C_1(p, q, \tau_1 v_0, \Omega) > 0$ and $C_2 = C_2(p, q, \tau_2 z_0, \Omega) > 0$ such that the unique local-in-time classical solution $(u, v, w, z)$ of \((1.1)\) satisfies

$$
\begin{align*}
\|v(t)\|_{W^{1, q}} &\leq C_1 \left(1 + \sup_{s \in (0, t)} \|w(s)\|_{L^p}\right), \quad \forall t \in (0, T_m), \\
\|z(t)\|_{W^{1, q}} &\leq C_2 \left(1 + \sup_{s \in (0, t)} \|u(s)\|_{L^p}\right), \quad \forall t \in (0, T_m).
\end{align*}
$$

(2.4)

In particular, for any $q \in [1, 2]$, there exists $C_3 = C_3(q, \tau_1 v_0, \tau_2 z_0, \Omega) > 0$ such that

$$
\begin{align*}
\|v(t)\|_{L^{\frac{2p}{2p-1}}} + \|z(t)\|_{L^{\frac{2p}{2p-1}}} + \|v(t)\|_{W^{1, q}} + \|z(t)\|_{W^{1, q}} &\leq C_3, \quad \forall t \in (0, T_m).
\end{align*}
$$

(2.5)

Proof. In the case of $\tau_1, \tau_2 > 0$, using the widely known smoothing $L^p-L^q$ estimates of the Neumann heat semigroup \(\{e^{t\Delta}\}_{t \geq 0}\) in $\Omega$, see, e.g. \([27, 4]\) and applying those estimates to the second and fourth equation in \((1.1)\), we can readily deduce \((2.4)\), cf. \([12, 13, 50]\). In the case of $\tau_1 = \tau_2 = 0$, the standard well-known $W^{2, p}$-elliptic theory (see e.g. \([19]\)) easily lead to \((2.4)\). Because of the conservations of $u$ and $w$ in \((2.1)\), we first take $p = 1$ in \((2.3)\), and then from \((2.5)\) and Sobolev embedding, we arrive at the desired estimate \((2.5)\). \hfill \Box
Based on Sobolev and Hölder inequalities, the following Poincaré-type inequality follows. In convex domains, upon an observation of the optimal constant for Poincaré inequality \( [28] \), we find an explicit lower bound for the involving constant.

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded and smooth domain and let \( k \) be defined in (1.8). Then for any a.e nonnegative function \( \varphi \in W^{1,2}_\Omega \) with \( \bar{\varphi} = 1 \), one has

\[
\| \varphi - 1 \|_{L^2}^2 \leq k \| \nabla \varphi \|_{L^2}^2. \tag{2.6}
\]

If, furthermore, \( \Omega \) is convex, then \( k \geq \frac{4d^2}{\pi^2} \) with \( d \) being the diameter of \( \Omega \).

**Proof.** The validity of (2.6) is proven in [8, Lemma 2.3]. We here re-show the simple proof with emphasis on the explicit lower bound of \( k \) in convex domains. In such cases, it is known from [28] that the optimal constant for the Poincaré inequality

\[
\| \varphi - 1 \|_{L^2}^2 \leq \mu_1 \| \nabla \varphi \|_{L^2}^2, \tag{2.7}
\]

is given by

\[
\mu_1 = \frac{d^2}{\pi^2}.
\]

The 2D Sobolev embedding \( W^{1,1}(\Omega) \hookrightarrow L^2(\Omega) \) implies there exists \( \mu_2 > 0 \) such that

\[
\| \varphi - 1 \|_{L^2}^2 \leq \mu_2 \| \nabla \varphi \|_{L^1}^2. \tag{2.8}
\]

This shows that \( k \) defined in (1.8) makes sense, and, the optimal constant of \( \mu_2 \) is

\[
\mu_2 = \frac{k}{4|\Omega|}. \tag{2.9}
\]

Thus, by Hölder inequality and the optimality of \( \mu_1 \) in (2.7), we have

\[
\mu_2 |\Omega| \geq \mu_1 = \frac{d^2}{\pi^2}.
\]

By (2.8) with (2.9), Hölder inequality and the fact that \( \| \varphi \|_{L^1} = |\Omega| \), we deduce

\[
\| \varphi - 1 \|_{L^2}^2 \leq \mu_2 \| \nabla \varphi \|_{L^1}^2 = 4 \mu_2 \| \varphi^\frac{1}{2} \cdot \nabla \varphi^{\frac{1}{2}} \|_{L^2}^2 \\
\leq 4\mu_2 |\Omega| \cdot \| \nabla \varphi^{\frac{1}{2}} \|_{L^2}^2 = k \| \nabla \varphi^{\frac{1}{2}} \|_{L^2}^2,
\]

which readily shows (2.6) with \( k \geq \frac{4d^2}{\pi^2} \), as desired. \( \square \)

For convenience of reference, we state the following version of Trudinger-Moser inequality and the widely-known 2D Gagliardo-Nirenberg interpolation inequality.

**Lemma 2.4 (Trudinger-Moser inequality [26]).** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded and smooth domain. Define

\[
\pi^* = \begin{cases} 
8\pi, & \text{if } \Omega = B_R(0), \\
4\pi, & \text{otherwise}.
\end{cases}
\]

Then, for any \( \epsilon > 0 \), there exists \( C_\epsilon = C(\epsilon, \Omega) > 0 \) such that

\[
\int_\Omega \exp |f| \leq C_\epsilon \exp \left( \frac{1}{2\pi^*} + \epsilon \| \nabla f \|_{L^2(\Omega)}^2 + \frac{2}{|\Omega|} \| f \|_{L^1(\Omega)} \right), \quad \forall f \in H^1(\Omega).
\]
Lemma 2.5 (2D Gagliardo-Nirenberg interpolation inequality \[7, 10, 21\]). Let \( \Omega \subset \mathbb{R}^2 \) be a bounded and smooth domain. Let \( i \) and \( j \) be any integers satisfying \( 0 \leq i < j \), and let \( 0 < p, q, r \leq \infty \), and \( \frac{1}{j} \leq \theta \leq 1 \) such that
\[
\frac{1}{p} - \frac{i}{2} = \theta \left( \frac{1}{q} - \frac{j}{2} \right) + (1 - \theta) \frac{1}{r}.
\]
Then there exists \( C > 0 \) depending only on \( \Omega, p, q, r, i \) and \( j \) such that
\[
\|D^i f\|_{L^p} \leq C \left( \|D^j f\|_{L^q}^\theta \|f\|_{L^r}^{1-\theta} + \|f\|_{L^r} \right), \quad \forall f \in W^{j,q}_0 \tag{2.10}
\]
with the following exception: if \( 1 < q < \infty \) and \( j - i - 2 \frac{q}{r} \) is a nonnegative integer, then \( (2.10) \) holds only for \( \theta \) satisfying \( \frac{1}{j} \leq \theta < 1 \).

3. PRODUCT OF MASSES ON BOUNDEDNESS VS BLOW-UP

In the section, we shall prove boundedness and gradient estimates as in (B1) and (B2) of classical solutions to the IBVP (1.1) under small product of masses. We divide this section into four subsections to make the flow of our ideas more smooth.

3.1. From \( L^1 \) to \( L^2 \). We start with the following energy identity, which plays a crucial role for our purpose, especially when \( \tau_1 + \tau_2 = 0 \) or \( \chi_2 = 0 \).

Lemma 3.1. The local-in-time classical solution \((u, v, w, z)\) of (1.1) fulfills
\[
\mathcal{F}'(t) = - (\tau_1 + \tau_2) \chi_1 \chi_2 \int_{\Omega} v t z - \tau_1 \chi_1 \chi_3 \int_{\Omega} v^2 - \chi_2 \int_{\Omega} u |\nabla (\ln u - \chi_1 v)|^2
\]
\[
- \chi_1 \int_{\Omega} w |\nabla (\ln w - \chi_2 z - \chi_3 v)|^2, \quad t \in (0, T_m),
\]
where \( \mathcal{F}(t) \) is defined by
\[
\mathcal{F}(t) = \chi_2 \int_{\Omega} u \ln u + \chi_1 \int_{\Omega} w \ln w - \chi_1 \chi_2 \int_{\Omega} (uv + wz) - \chi_1 \chi_3 \int_{\Omega} w v
\]
\[
+ \chi_1 \chi_2 \int_{\Omega} (wz + \nabla v \cdot \nabla z) + \frac{\chi_1 \chi_3}{2} \int_{\Omega} (v^2 + |\nabla v|^2), \quad t \in (0, T_m). \tag{3.2}
\]

Proof. Multiplying the first equation in (1.1) by \( \ln u - \chi_1 v \) and integrating by parts over \( \Omega \), we obtain upon noticing \( \int_{\Omega} u_t = 0 \) that
\[
- \int_{\Omega} u |\nabla (\ln u - \chi_1 v)|^2 = \int_{\Omega} u_t (\ln u - \chi_1 v)
\]
\[
= \frac{d}{dt} \int_{\Omega} (u \ln u - \chi_1 uv) + \chi_1 \int_{\Omega} w v_t. \tag{3.3}
\]
Similarly, multiplying the third equation in (1.1) by \( \ln w - \chi_2 z - \chi_3 v \), we see that
\[
- \int_{\Omega} w |\nabla (\ln w - \chi_2 z - \chi_3 v)|^2
\]
\[
= \frac{d}{dt} \int_{\Omega} (w \ln w - \chi_2 wz - \chi_3 wv) + \chi_2 \int_{\Omega} wz_t + \chi_3 \int_{\Omega} w v_t. \tag{3.4}
\]
Next, from the facts that \( u = \tau_2 z_t - \Delta z + z \) and \( w = \tau_1 v_t - \Delta v + v \) due to (3.1), we deduce from integration by parts that

\[
\int_{\Omega} uv_t + \int_{\Omega} vz_t = \int_{\Omega} (\tau_2 z_t - \Delta z + z) v_t + \int_{\Omega} (\tau_1 v_t - \Delta v + v) z_t
\]

\[
= (\tau_1 + \tau_2) \int_{\Omega} u_t z_t + \frac{d}{dt} \int_{\Omega} \nabla v \cdot \nabla z + \frac{d}{dt} \int_{\Omega} v z
\]

(3.5)

and that

\[
\int_{\Omega} uv_t = \int_{\Omega} (\tau_1 v_t - \Delta v + v) v_t = \tau_1 \int_{\Omega} v_t^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} (v^2 + |\nabla v|^2).
\]

(3.6)

Finally, multiplying (3.3) by \( \chi_2 \) and (3.4) by \( \chi_1 \) and then using (3.5) and (3.6), we can readily end up with (3.1) with \( \mathcal{F} \) given by (3.2).

**Lemma 3.2.** When \( \tau_1 = \tau_2 = 0 \), with \( \pi^* \) defined in Lemma 2.4, assume that

\[
m_1 m_2 \chi_1 \chi_2 < (\pi^* - m_2 \chi_3) \pi^*.
\]

(3.7)

Then there exists \( C = C(u_0, w_0, \Omega) > 0 \) such that

\[
\|u(t)\|_{L^1} + \|w(t)\|_{L^1} + \|v(t)\|_{H^1} + \|z(t)\|_{H^1} \leq C, \quad \forall t \in (0, T_m).
\]

(3.8)

**Proof.** For our later purpose, thanks to (3.7), we first pick positive constants \( a, b \) and \( \epsilon \) according to

\[
a = \frac{\pi^*}{m_1}, \quad b = \frac{\pi^*}{m_2}, \quad \epsilon = \frac{(\pi^* - m_2 \chi_3) \pi^* - m_1 m_2 \chi_1 \chi_2}{6 (\pi^*)^3},
\]

(3.9)

and then, by direct but tedious computations, we find that

\[
\begin{cases}
  A := [(1 - 2 \pi^* \epsilon) \frac{\pi^*}{2m_2} - \frac{\chi_2}{2}] \chi_1 > 0, \\
  B := (1 - 2 \pi^* \epsilon) \frac{\pi^*}{2m_2} > 0,
\end{cases}
\]

(3.10)

Since \( (\tau_1, \tau_2) = (0, 0) \), we first infer from (3.1) that

\[
\begin{cases}
  f_\Omega |\nabla z|^2 + f_\Omega z^2 = f_\Omega u z, \\
  f_\Omega |\nabla v|^2 + f_\Omega v^2 = f_\Omega u v,
\end{cases}
\]

(3.11)

then, by the definition of \( \mathcal{F} \) in (3.2) and the choices of \( a, b \) in (3.9), we deduce that

\[
\mathcal{F}(t) = \chi_2 \int_{\Omega} u \ln u + \chi_1 \int_{\Omega} w \ln w - \chi_1 \chi_2 \int_{\Omega} wz - \frac{\chi_1 \chi_3}{2} \int_{\Omega} (v^2 + |\nabla v|^2)
\]

\[
= \chi_2 \int_{\Omega} (u \ln u - au z) + \chi_1 \int_{\Omega} (w \ln w - bwv)
\]

\[
+ a \chi_2 \int_{\Omega} uz + b \chi_1 \int_{\Omega} vw - \chi_1 \chi_2 \int_{\Omega} wz - \frac{\chi_1 \chi_3}{2} \int_{\Omega} (v^2 + |\nabla v|^2)
\]

(3.12)

\[
= -\chi_2 \int_{\Omega} u \ln \frac{e^{\epsilon z}}{u} - \chi_1 \int_{\Omega} w \ln \frac{e^{bw}}{w} + \left( b - \frac{\chi_3}{2} \right) \chi_1 \int_{\Omega} (v^2 + |\nabla v|^2)
\]

\[
- \chi_1 \chi_2 \int_{\Omega} (vz + \nabla v \cdot \nabla z) + a \chi_2 \int_{\Omega} (z^2 + |\nabla z|^2).
\]

Observe that \( -\ln p \) is a convex function in \( p \): \( \int_{\Omega} \frac{u}{m_1} = 1 \) and \( \int_{\Omega} \frac{w}{m_2} = 1 \) due to mass conservation of \( u \) and \( w \). Then Jensen’s inequality tells us that

\[
-\ln \left( \frac{1}{m_1} \int_{\Omega} e^{az} \right) = -\ln \int_{\Omega} \frac{e^{az}}{m_1} u \leq -\frac{1}{m_1} \int_{\Omega} u \ln e^{az} u
\]
and, similarly,
\[- \ln \left( \frac{1}{m_2} \int_{\Omega} e^{b_0} \right) \leq - \frac{1}{m_2} \int_{\Omega} w \ln \frac{e^{b_0}}{w}.\]

Now, for the specifications of $a, b, \epsilon$ in (3.9), we apply the Trudinger-Morser inequality in Lemma 2.4 along with the boundedness $\|z\|_{L^1} = \|u_0\|_{L^1}$ and $\|v\|_{L^1} = \|w_0\|_{L^1}$ due to $(\tau_1, \tau_2) = (0, 0)$ to find a $C = C(\Omega) > 0$ such that

\[- \chi_2 \int_{\Omega} u \ln \frac{e^{a z}}{w} - \chi_1 \int_{\Omega} w \ln \frac{e^{b_0}}{w} \geq -m_1 \chi_2 \ln \left( \frac{1}{m_1} \int_{\Omega} e^{a z} \right) - m_2 \chi_1 \ln \left( \frac{1}{m_2} \int_{\Omega} e^{b_0} \right)\]

\[- \chi_2 \int_{\Omega} u \ln \frac{e^{a z}}{w} - \chi_1 \int_{\Omega} w \ln \frac{e^{b_0}}{w} \geq -m_1 \chi_2 \left[ \ln \frac{C}{m_1} + \frac{1}{2\pi^*} + \epsilon \right] + m_2 \chi_1 \ln \left( \frac{1}{m_2} \int_{\Omega} e^{b_0} \right)\]

\[- \chi_2 \int_{\Omega} u \ln \frac{e^{a z}}{w} - \chi_1 \int_{\Omega} w \ln \frac{e^{b_0}}{w} \geq -m_1 \chi_2 \left[ \ln \frac{C}{m_1} + \frac{1}{2\pi^*} + \epsilon \right] + m_2 \chi_1 \ln \left( \frac{1}{m_2} \int_{\Omega} e^{b_0} \right)\]

\[- \chi_2 \int_{\Omega} u \ln \frac{e^{a z}}{w} - \chi_1 \int_{\Omega} w \ln \frac{e^{b_0}}{w} \geq -m_1 \chi_2 \left[ \ln \frac{C}{m_1} + \frac{1}{2\pi^*} + \epsilon \right] + m_2 \chi_1 \ln \left( \frac{1}{m_2} \int_{\Omega} e^{b_0} \right)\]

where $D$ is a finite number and is defined by

\[D = m_1 \chi_2 \left[ \ln \frac{C}{m_1} + \frac{2a}{|\Omega|} \|u_0\|_{L^1} \right] + m_2 \chi_1 \left[ \ln \frac{C}{m_2} + \frac{2b}{|\Omega|} \|w_0\|_{L^1} \right].\]

Substituting (3.13) into (3.12) and using (3.10), we conclude that

\[\mathcal{F}(t) - \int_{\Omega} \left[ \frac{\pi^*}{m_2} \chi_1 v^2 - \chi_1 \chi_2 vz + \frac{\pi^* \chi_2}{m_1} z^2 \right] + D\]

\[\geq \int_{\Omega} \left( A \|v\|^2 - \chi_1 \chi_2 v^2 + B \|v\|^2 \right)\]

\[\geq \int_{\Omega} \left( A \|v\|^2 - \chi_1 \chi_2 v^2 + B \|v\|^2 \right)\]

\[\geq \int_{\Omega} \left( A \|v\|^2 - \chi_1 \chi_2 v^2 + B \|v\|^2 \right)\]

\[\geq \int_{\Omega} \left( A \|v\|^2 - \chi_1 \chi_2 v^2 + B \|v\|^2 \right)\]

By the 2D G-N inequality in (2.10), for any $\eta > 0$, there exists $C_\eta > 0$ such that

\[\|\phi\|_{L^2}^2 \leq \eta \|\nabla \phi\|_{L^2}^2 + C_\eta \|\phi\|_{L^1}^2, \; \forall \phi \in H^1.\]

Combining this with (3.14), (3.11) and the decreasing monotonicity of $\mathcal{F}$ implied by (5.1) with $(\tau_1, \tau_2) = (0, 0)$, we infer there exists a constant $E > 0$ such that

\[\|v\|_{H^1}^2 + \|z\|_{H^1}^2 \leq E \mathcal{F}(t) + E \leq E \mathcal{F}(0) + E,\]
which, along with (3.12) and (3.11), further enables us to deduce that
\[
\chi_2 \int_{\Omega} |u \ln u| + \chi_1 \int_{\Omega} |w \ln w| \\
= \chi_2 \left( \int_{\Omega} u \ln u - 2 \int_{\{w \leq 1\}} u \ln u \right) + \chi_1 \left( \int_{\Omega} w \ln w - 2 \int_{\{w \leq 1\}} w \ln w \right) \\
\leq \chi_2 \int_{\Omega} u \ln u + \chi_1 \int_{\Omega} w \ln w + 2(\chi_1 + \chi_2)e^{-1}|\Omega| \\
= \mathcal{F}(t) + \chi_1 \chi_2 \int_{\Omega} (e^z + \nabla v \cdot \nabla z) + \frac{\chi_1 \chi_2}{2} |v|^2_{H^1} + 2(\chi_1 + \chi_2)e^{-1}|\Omega| \\
\leq \mathcal{F}(0) + (\chi_2 + |\chi_3|) \chi_1 \left( ||v||^2_{H^1} + ||\zeta||^2_{H^1} \right) + 2(\chi_1 + \chi_2)e^{-1}|\Omega|.
\]
(3.16)

Consequently, our desired estimate (3.8) follows readily from (3.15) and (3.16). \( \Box \)

**Remark 3.3.** By simpler arguments, when \(\chi_1 \chi_2 = 0\) and \(m_2 \chi_3 < \pi^*\), no matter whether \((\tau_1, \tau_2) = (0, 0)\) or not, one can easily show that (3.8) is still valid.

In the fully parabolic case, we shall derive an analog of Lemma 3.2 under an implicit smallness condition on the product of masses of \(u\) and \(w\).

**Lemma 3.4.** In the fully parabolic case, i.e., \(\tau_1 > 0, \tau_2 > 0\), assume that
\[
4m_1 m_2 \chi_1 \chi_2 C_{GN}^8 < \sqrt{1 - 4m_2 \chi_3 C_{GN}^4}.
\]
(3.17)

Then there exists \(C = C(u_0, v_0, w_0, z_0, \tau_1, \chi_i, \Omega) > 0\) such that
\[
||u \ln u(t)||_{L^1} + ||w \ln w(t)||_{L^1} + ||v(t)||_{H^1} + ||\zeta(t)||_{H^1} \leq C, \ \forall t \in (0, T_m).
\]
(3.18)

**Proof.** Multiplying the first and the third equation in (3.11) by \(\ln u\) and \(\ln w\), and then, multiplying the second and the fourth equation by \(-\Delta v\) and \(-\Delta \zeta\), respectively, and finally integrating over \(\Omega\) by parts, we compute, for \(t \in (0, T_m)\), that
\[
\begin{aligned}
\frac{d}{dt} \int_{\Omega} u \ln u + 4 \int_{\Omega} |\nabla u|^2 = -\chi_1 \int_{\Omega} u \Delta v, \\
\frac{d}{dt} \int_{\Omega} w \ln w + 4 \int_{\Omega} |\nabla w|^2 = -\chi_2 \int_{\Omega} w \Delta \zeta - \chi_3 \int_{\Omega} w \Delta v, \\
\frac{\tau_1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 = -\int_{\Omega} w \Delta v, \\
\frac{\tau_2}{2} \frac{d}{dt} \int_{\Omega} |\nabla \zeta|^2 + \int_{\Omega} |\nabla \zeta|^2 + \int_{\Omega} |\nabla \zeta|^2 = -\int_{\Omega} u \Delta \zeta.
\end{aligned}
\]
(3.19)

Given any positive constants \(a, b\) and \(c\), to be specified below as in (3.20), through an elementary linear combination of (3.11), we arrive at
\[
\begin{aligned}
\frac{d}{dt} \int_{\Omega} \left( u \ln u + a w \ln w + \frac{b \tau_1}{2} |\nabla v|^2 + \frac{c \tau_2}{2} |\nabla \zeta|^2 \right) + 4 \int_{\Omega} |\nabla u|^2 \\
+ 4a \int_{\Omega} |\nabla w|^2 + b \int_{\Omega} |\Delta v|^2 + b \int_{\Omega} |\nabla v|^2 + c \int_{\Omega} |\Delta \zeta|^2 + c \int_{\Omega} |\nabla \zeta|^2 \\
= -\int_{\Omega} \left( \chi_1 u + a \chi_2 w + bw \right) \Delta v - \int_{\Omega} \left( a \chi_2 w + au \right) \Delta \zeta.
\end{aligned}
\]
(3.20)
Using basic Cauchy-Schwarz inequality, we estimate the right-hand side as

\[- \int_\Omega (\chi_1 u + a\chi_3 w + bw) \Delta v - \int_\Omega (a\chi_2 w + cu) \Delta z\]

\[\leq b \int_\Omega |\Delta v|^2 + \frac{1}{4b} \int_\Omega (\chi_1 u + a\chi_3 w + bw)^2\]

\[+ c \int_\Omega |\Delta z|^2 + \frac{1}{4c} \int_\Omega (a\chi_2 w + cu)^2\]

\[\leq b \int_\Omega |\Delta v|^2 + \frac{1}{2b} \int_\Omega [\chi_1^2 u^2 + (a\chi_3 + b)^2 w^2]\]

\[+ c \int_\Omega |\Delta z|^2 + \frac{1}{2c} \int_\Omega (a^2 \chi_2^2 w^2 + c^2 u^2)\]

\[= b \int_\Omega |\Delta v|^2 + c \int_\Omega |\Delta z|^2 + \left(\frac{\lambda^2}{2b} + c\right) \int_\Omega u^2\]

\[+ \left(\frac{(a\chi_3 + b)^2}{2b} + \frac{a^2 \chi_2^2}{2c}\right) \int_\Omega w^2, \quad \forall t \in (0, T_m).\]  

By the 2D Gagliardo-Nirenberg interpolation inequality in Lemma 2.5 and the elementary fact that \((X + Y)^4 \leq 2^4(X^4 + Y^4)\) for all \(X, Y \geq 0\), we infer there exists a constant \(C_{GN} = C_{GN}(\Omega) > 0\) such that

\[\int_\Omega \phi^2 = \|\phi\|_{L^2}^4 \leq C_{GN}^4 \left(\|\nabla \phi \|_{L^2}^2 \|\phi\|_{L^2}^2 + \|\phi\|_{L^2}^4\right)\]  

\[\leq 8C_{GN}^4 \|\phi\|_{L^2} \|\nabla \phi \|_{L^2} + 8C_{GN}^4 \|\phi\|_{L^2}, \quad \forall \phi \in W^{1,2}.\]

Recalling that \(\|u\|_{L^1} = \|u_0\|_{L^1} = m_1\) and \(\|w\|_{L^1} = \|w_0\|_{L^1} = m_2\) by (2.1), we employ (3.22) twice to finally estimate (3.21) as

\[- \int_\Omega (\chi_1 u + a\chi_3 w + bw) \Delta v - \int_\Omega (a\chi_2 w + cu) \Delta z\]

\[\leq b \int_\Omega |\Delta v|^2 + c \int_\Omega |\Delta z|^2 + 4m_1 \left(\frac{\lambda^2}{b} + c\right) C_{GN}^4 \int_\Omega |\nabla u|^2\]

\[+ 4m_2 \left(\frac{(a\chi_3 + b)^2}{b} + \frac{a^2 \chi_2^2}{c}\right) C_{GN}^4 \int_\Omega |\nabla w|^2 + C_1, \quad \forall t \in (0, T_m).\]

where \(C_1\) is a finite number given by

\[C_1 = 4m_1^2 \left(\frac{\lambda^2}{b} + c\right) C_{GN}^4 + 4m_2^2 \left(\frac{(a\chi_3 + b)^2}{b} + \frac{a^2 \chi_2^2}{c}\right) C_{GN}^4.\]

Finally, substituting (3.22) into (3.20), we end up with a key ODE as follows:

\[\frac{d}{dt} \int_\Omega \left(\frac{u}{b} \ln u + aw \ln w + \frac{br_1}{2} |\nabla v|^2 + \frac{cr_2}{2} |\nabla z|^2\right)\]

\[+ 4A \int_\Omega |\nabla u|^2 + 4B \int_\Omega |\nabla w|^2 + b \int_\Omega |\nabla v|^2 + c \int_\Omega |\nabla z|^2\]

\[\leq C_1, \quad \forall t \in (0, T_m).\]

where the constants \(A\) and \(B\) are given by

\[\begin{cases}
A = 1 - m_1 \left(\frac{\lambda^2}{b} + c\right) C_{GN}^4 := p^{-1} \left(p - \frac{\lambda^2}{b} - c\right), \\
B = a - m_2 \left(\frac{(a\chi_3 + b)^2}{b} + \frac{a^2 \chi_2^2}{c}\right) C_{GN}^4 := q^{-1} \left(aq - \frac{(a\chi_3 + b)^2}{b} - \frac{a^2 \chi_2^2}{c}\right).
\end{cases}\]
To gain something out of (3.24), we wish that both $A$ and $B$ be positive, which is possible only when $16\chi_1^2 \chi_3^2 < r^2 q (q - 4 \chi_3)$, equivalent to our assumption (3.17). In such case, we can specify, for instance, positive $a, b, c$ as

\[
\begin{cases}
    b = \frac{\gamma q (q - 4 \chi_3)}{8 \chi_2} > 0, \\
    a = \frac{(bq - \chi_1^2)(q - 2 \chi_3)b}{2[bq - \chi_1^2 + (bq - \chi_1^2) \chi_3]} > 0, \\
    c = \frac{(bq - \chi_1^2) + a^2 b \chi_2^2}{2[bq - (a \chi_3 + b) \chi_3]} > 0
\end{cases}
\tag{3.26}
\]

so that $A$ and $B$ defined in (3.25) are positive. Next, notice, for any $\epsilon > 0$, one has that $s \ln s \leq \epsilon s^2 + C$, with finite $C_\epsilon = \sup\{s \ln s - \epsilon s^2 : s > 0\}$. Therefore, one can readily deduce from (3.22), for some $C_2, C_3 > 0$, that

\[
\int_{\Omega} u \ln u \leq A \int_{\Omega} |\nabla u|^2 + C_2, \quad a \int_{\Omega} w \ln w \leq B \int_{\Omega} |\nabla w|^2 + C_3.
\]

Combining this with (3.24), we finally find a positive $C_4 > 0$ such that

\[
\frac{d}{dt} \int_{\Omega} \left( u \ln u + aw \ln w + \frac{br_1}{2} |\nabla v|^2 + \frac{cr_2}{2} |\nabla z|^2 \right) + \min \left\{ 1, \frac{2}{\tau_1}, \frac{2}{\tau_2} \right\} \int_{\Omega} \left( u \ln u + aw \ln w + \frac{br_1}{2} |\nabla v|^2 + \frac{cr_2}{2} |\nabla z|^2 \right)
\]

\[
\leq C_4, \quad \forall t \in (0, T_m).
\]

Solving this simple Gronwall inequality and using the simple trick used in (3.15), we find a positive $C_5 > 0$ such that

\[
\|u \ln u\|_{L^1} + \|w \ln w\|_{L^1} + \|\nabla v\|_{L^2} + \|\nabla w\|_{L^2} \leq C_5, \quad \forall t \in (0, T_m),
\]

which along with (2.5) with $q = 1$ yields our desired estimate (3.18). \hfill $\Box$

 Armed with the key uniform boundedness of $(u \ln u, w \ln w)$ as obtained in Lemmas 3.2 and 3.3, it is quite standard for us to show higher $L^p$-boundedness ($p > 1$) and, eventually, $L^\infty$-boundedness as in 2.D setting, see similar situations in [41, 29, 30]. We here would like to supply a short argument for (1.1) for the sake of completeness and for clarity of deriving higher order gradients in Subsection 3.3.

**Lemma 3.5.** When $\tau_1 = \tau_2 = 0$, assume that (3.7) holds. Then there exists a constant $C = C(u_0, w_0, \Omega) > 0$ such that

\[
\|u(t)\|_{L^1} + \|w(t)\|_{L^1} + \|v(t)\|_{L^2} + \|z(t)\|_{H^2} \leq C, \quad \forall t \in (0, T_m);
\]

and, for any $q \in (1, \infty)$, there exists $C_q = C(q, u_0, w_0, \Omega) > 0$ such that

\[
\|v(t)\|_{W^{1,q}} + \|z(t)\|_{W^{1,q}} \leq C_q, \quad \forall t \in (0, T_m).
\]

**Proof.** Applying the elliptic estimate in [10] Lemma 2.7 to the second and fourth equation with $\tau_1 = \tau_2 = 0$ in (1.1), we see, for any $\epsilon > 0$ and $p > 1$, there exists a positive constant $C_\epsilon > 0$ such that

\[
\int_{\Omega} (v^p, w^p) \leq \epsilon \int_{\Omega} (w^p, w^p) + C_\epsilon.
\]

(3.29)
Using the equations in \[\text{(1.1)}\] with \(\tau_1 = \tau_2 = 0\), performing integration by parts and using Young’s inequality and \[\text{(3.29)}\], we compute that

\[
3 \frac{d}{dt} \int_\Omega (u^2 + w^2) + 6 \int_\Omega |\nabla u|^2 + 6 \int_\Omega |\nabla w|^2
= 3 \chi_1 \int_\Omega u^2 (w - v) + 3 \chi_2 \int_\Omega w^2 (u - z) + 3 \chi_3 \int_\Omega w^2 (w - v)
\leq (4 \chi_1 + \chi_2) \int_\Omega u^3 + (\chi_1 + |\chi_3|) \int_\Omega v^3
+ (\chi_1 + 4 \chi_2 + 5 |\chi_3|) \int_\Omega w^3 + \chi_2 \int_\Omega z^3
\leq (4 \chi_1 + 4 \chi_2 + 6 |\chi_3|) \int_\Omega (u^3 + w^3) + C_1, \quad t \in (0, T_m).
\]

Due to the uniform \(L^1\)-boundedness of \((u \ln u, w \ln w)\) in \[\text{(3.8)}\], the 2D G-N inequality involving logarithmic functions from \[\text{[33, Lemma A.5]}\] implies that

\[
\int_\Omega (u^3, w^3) \leq C \int_\Omega (|\nabla u|^2, |\nabla w|^2) + C_\eta, \quad \forall \eta > 0.
\]

Of course, the above inequality or the usual 2D G-N inequality simply shows

\[
\int_\Omega (u^2, w^2) \leq C \int_\Omega (|\nabla u|^2, |\nabla w|^2) + C_\sigma, \quad \forall \sigma > 0.
\]

Based on \[\text{(3.32)}, \text{(3.31)}\] and \[\text{(3.30)}\], one can readily derive an ODI of the form

\[
\frac{d}{dt} \int_\Omega (u^2 + w^2) + \int_\Omega (u^2 + w^2) \leq C(u_0, w_0, \Omega), \quad \forall t \in (0, T_m),
\]

yielding directly the uniform boundedness of \(\|u\|_{L^2} + \|w\|_{L^2}\) and the \(H^2\)-boundedness of \((v, z)\) by the \(H^2\)-elliptic estimate (cf. \[\text{[19]}\]) in \[\text{(3.7)}\]. The \(W^{1,2}\)-estimate of \((v, z)\) in \[\text{(3.28)}\] follows from Lemma \[\text{[22]}\] with \(p = 2\).

**Lemma 3.6.** In the fully parabolic case, i.e., \(\tau_1 > 0, \tau_2 > 0\), assume that \[\text{(3.17)}\] holds. Then there exists a constant \(C = C(u_0, w_0, \Omega) > 0\) such that

\[
\|u(t)\|_{L^2} + \|w(t)\|_{L^2} + \|v(t)\|_{H^2} + \|z(t)\|_{H^2} \leq C, \quad t \in (\min\{1, \frac{T_m}{2}\}, T_m);
\]

and, for any \(q \in (1, \infty)\), there exists \(C_q = C(q, u_0, w_0, \Omega) > 0\) such that

\[
\|v(t)\|_{W^{1,q}} + \|z(t)\|_{W^{1,q}} \leq C_q, \quad \forall t \in (0, T_m).
\]

**Proof.** Using the equations and homogeneous Neumann boundary conditions in the IBVP \[\text{(1.1)}\], we find, upon integration by parts, for \(t \in (0, T_m)\), that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 + 2 \int_\Omega |\nabla u|^2 &= -\chi_1 \int_\Omega u^2 \Delta v, \\
\frac{1}{2} \frac{d}{dt} \int_\Omega w^2 + 2 \int_\Omega |\nabla w|^2 &= -\chi_2 \int_\Omega w^2 \Delta z - \chi_3 \int_\Omega w^2 \Delta v, \\
\tau_1 \frac{d}{dt} \int_\Omega |\Delta v|^2 + 2 \int_\Omega |\Delta v|^2 + 2 \int_\Omega |\nabla v|^2 &= -2 \int_\Omega \nabla w \nabla \Delta v, \\
\tau_2 \frac{d}{dt} \int_\Omega |\Delta z|^2 + 2 \int_\Omega |\Delta z|^2 + 2 \int_\Omega |\nabla z|^2 &= -2 \int_\Omega \nabla u \nabla \Delta z.
\end{align*}
\]
Adding those identities in (3.35) together, we obtain, for any \( \epsilon > 0 \), that
\[
\frac{d}{dt} \int_{\Omega} \left( u^2 + w^2 + \tau_1 |\Delta v|^2 + \tau_2 |\Delta z|^2 \right) + 2 \int_{\Omega} |\nabla u|^2 + 2 \int_{\Omega} |\nabla w|^2 \\
+ 2 \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\nabla \Delta v|^2 + 2 \int_{\Omega} |\Delta z|^2 + 2 \int_{\Omega} |\nabla \Delta z|^2 \\
= -\chi_1 \int_{\Omega} u^2 \Delta v + \chi_2 \int_{\Omega} w^2 \Delta z - \chi_3 \int_{\Omega} w^2 \Delta v \\
- 2 \int_{\Omega} \nabla w \Delta v - 2 \int_{\Omega} \nabla u \Delta z \\
\leq (\chi_1 + |\chi_3|) \epsilon \int_{\Omega} |\Delta v|^3 + \frac{2 \chi_1}{3 \sqrt{3} \epsilon} \int_{\Omega} u + 2 \frac{\chi_2 + |\chi_3|}{3 \sqrt{3} \epsilon} \int_{\Omega} w^3 \\
+ \chi_2 \epsilon \int_{\Omega} |\Delta z|^3 + \int_{\Omega} |\nabla w|^2 + \int_{\Omega} |\nabla \Delta v|^2 + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla \Delta z|^2,
\] (3.36)
where we have applied the Young’s inequality with epsilon a couple of times:
\[
abla \cdot \mathbf{u} \leq \epsilon + \frac{b}{(\epsilon p)^{\frac{p}{q}}}, \quad p > 0, q > 0, \frac{1}{p} + \frac{1}{q} = 1, \quad \forall a, b \geq 0.
\] (3.37)
Then it is straightforward to see from (3.36) that
\[
\frac{d}{dt} \int_{\Omega} (u^2 + w^2 + \tau_1 |\Delta v|^2 + \tau_2 |\Delta z|^2) + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla w|^2 \\
+ 2 \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\nabla \Delta v|^2 + 2 \int_{\Omega} |\Delta z|^2 + 2 \int_{\Omega} |\nabla \Delta z|^2 \\
\leq (\chi_1 + \chi_2 + |\chi_3|) \epsilon \int_{\Omega} (|\Delta v|^3 + |\Delta z|^3) \\
+ \frac{2 (\chi_1 + \chi_2 + |\chi_3|)}{3 \sqrt{3} \epsilon} \int_{\Omega} (u^3 + w^3), \quad \forall \epsilon > 0.
\] (3.38)
Applying the 2D G-N interpolation inequality in Lemma 2.5, Sobolev interpolation inequality and the boundedness of \( ||v||_{H^1} + ||z||_{H^1} \) ensured by (3.18), we infer (see details, for instance, in (3.9)), for some \( C_1 > 0 \), that
\[
\int_{\Omega} (|\Delta v|^3, |\Delta z|^3) \leq C_1 \int_{\Omega} (|\nabla \Delta v|^2, |\nabla \Delta z|^2) + C_1.
\] (3.39)
Based on (3.39), (3.31) and (3.32), upon suitably choosing \( \epsilon, \eta, \sigma \), from (3.38), we can easily deduce, for \( t \in (\min(1, \frac{1}{\sigma}), T_m) \), a final ODI of the form that
\[
\frac{d}{dt} \int_{\Omega} (u^2 + w^2 + \tau_1 |\Delta v|^2 + \tau_2 |\Delta z|^2) \\
+ \min \left\{ \frac{2}{\tau_1}, \frac{2}{\tau_2} \right\} \int_{\Omega} (u^2 + w^2 + \tau_1 |\Delta v|^2 + \tau_2 |\Delta z|^2) \leq C_2.
\]
This along with the standard elliptic \( H^2 \)-estimate and Lemma 3.7 with \( p = 2 \) yields (3.33) and (3.34), as wished.

3.2. From \( L^2 \) to \( L^\infty \): In this subsection, we shall prove the global boundedness claimed in (1.11) and thus global existence of solutions to (1.1).

**Lemma 3.1.** Under the conditions of Lemma 3.3 or 3.4 the classical solution \((u, v, w, z)\) of (1.1) is global in time and is uniformly bounded according to (1.10).
Proof. Multiplying the \( u \)-equation in (1.1) by \( 3u^2 \), integrating over \( \Omega \) by parts and applying the \((L^2, L^\infty)\)-boundedness of \((u, \nabla v)\) in Lemma A.1 or 3.3 Young’s inequality \((3.37)\) and the 2D G-N inequality, we conclude, for \( t \in (0, T_m) \), that
\[
\frac{d}{dt} \int_{\Omega} u^3 + \int_{\Omega} u^3 + 3 \int_{\Omega} u |\nabla u|^2 \leq 3 \lambda_1^2 \int_{\Omega} u^3 |\nabla v|^2 + \int_{\Omega} u^3 \\
\leq 3 \lambda_1^2 \int_{\Omega} u^3 + \frac{3 \lambda_1^2}{4} \int_{\Omega} |\nabla v|^8 + \int_{\Omega} u^3 \\
\leq 4 \lambda_1^2 \|u\|_{L^\infty}^2 \|\nabla v\|_{L^8}^8 + C_1 \\
\leq 4 \lambda_1^2 C_2 \left( \|\nabla u\|_{L^2}^2 \|u\|_{L^\infty}^{\frac{8}{3}} + \|u\|_{L^\infty}^{\frac{8}{3}} \right) + C_1 \\
\leq C_3 \|\nabla u\|_{L^2}^2 + C_3 \\
\leq \int_{\Omega} |\nabla u|^2 + C_4,
\]
from which the uniform \( L^3 \)-boundedness of \( u \) follows. Applying the same argument to \( w \)-equation and noticing the \((L^2, L^\infty, L^\infty)\)-boundedness of \((w, \nabla v, \nabla z)\), one can readily show the uniform \( L^3 \)-boundedness of \( w \). Consequently, a simple application of Lemma 2.2 gives rise to
\[
\|u\|_{L^3} + \|w\|_{L^3} + \|v\|_{W^{1,\infty}} + \|z\|_{W^{1,\infty}} \leq C_5, \; \forall t \in (0, T_m).
\]
(3.40)
To derive the \( L^\infty \)-boundedness of \( u \), based on (3.40), we employ the variation-of-constants formula for the \( u \)-equation in (1.1) and the well-known smoothing \( L^p \)-estimates for the Neumann heat semigroup \( \{e^{tA}\}_{t \geq 0} \) (\ref{3.40}) to conclude that
\[
\|u(t)\|_{L^\infty} \leq \|e^{tA} u_0\|_{L^\infty} + \lambda_1 \int_0^t \left\| e^{(t-s)A} \nabla \cdot ((u \nabla v)(s)) \right\|_{L^\infty} ds \\
\leq \|u_0\|_{L^\infty} + C_6 \lambda_1 \int_0^t \left( 1 + (t-s)^{-\frac{1}{4}} \right) e^{-\lambda_1 (t-s)} \|u \nabla v\|_{L^5} ds \\
\leq \|u_0\|_{L^\infty} + C_6 \lambda_1 \int_0^t \left( 1 + (t-s)^{-\frac{1}{4}} \right) e^{-\lambda_1 (t-s)} \|u\|_{L^3} \|\nabla v\|_{L^\infty} ds \\
\leq \|u_0\|_{L^\infty} + C_7 \lambda_1 \int_0^t \left( 1 + \sigma^{-\frac{1}{4}} \right) e^{-\lambda_1 \sigma} d\sigma \\
\leq \|u_0\|_{L^\infty} + C_8 \lambda_1, \; \forall t \in (0, T_m).
\]
Here, \( \lambda_1 (> 0) \) is the first nonzero eigenvalue of \(-\Delta\) under homogeneous Neumann boundary condition. Performing the same argument to the variation-of-constants formula for the \( w \)-equation and using (3.40), we get the uniform \( L^\infty \)-boundedness of \( w \). To sum up, we have shown that
\[
\|u(t)\|_{L^\infty} + \|w(t)\|_{L^\infty} + \|v(t)\|_{W^{1,\infty}} + \|z(t)\|_{W^{1,\infty}} \leq C_9, \; \forall t \in (0, T_m).
\]
(3.41)
By the extensibility criterion \( (2.2) \) in Lemma 2.1 we first infer that \( T_m = \infty \), and then, the desired uniform boundedness \((1.10)\) is simply \( (3.35) \); that is, the classical solution \((u, v, w, z)\) of (1.1) is global in time and is uniformly bounded. \( \square \)

3.3. Higher order gradient estimates. For our stabilization purpose below, given the uniform \( L^1 \)-boundedness of \((u \ln u, w \ln w)\), we proceed to show further higher order gradient estimates away from the initial time \( t = 0 \) as stated in (1.11), which is of interest for its own sake, on the other hand.
Lemma 3.8. Under the uniform $L^1$-boundedness of $(u \ln u, w \ln w)$, there exists $C = C(u_0, \tau_1 v_0, w_0, \tau_2 w_0, \chi, |\Omega|) > 0$ such that (1.11) holds.

Proof. In light of the uniform $L^1$-boundedness of $(u \ln u, w \ln w)$, one can use the same arguments as Lemma 3.5 or 3.6 to show the uniform boundedness of $\|v\|_{H^2}$ and $\|z(t)\|_{H^2}$ for $t \in \{\min\{1, \frac{T}{m}\}, T\}$, and repeating the argument in previous subsections, one can easily obtain first the uniform estimate (3.41) with $T_m = \infty$, and then $(u, v, w, z) \in (C^2, 1)(\bar{\Omega} \times [1, \infty))$. Then, to get higher order gradient estimates, we begin to test the $u$-equation in (1.1) by $-2\Delta u$ and use (3.41) to get

$$\frac{d}{dt} \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^2 + 2 \int_\Omega |\Delta u|^2 = 2\chi_1 \int_\Omega (\nabla u \nabla v + u \Delta v) \Delta u + \int_\Omega |\nabla u|^2$$

$$\leq \int_\Omega |\Delta u|^2 + C_1 \int_\Omega |\nabla u|^2 + C_2.$$  (3.42)

The 2D Gagliardo-Nirenberg inequality and the $H^2$-elliptic estimate together imply

$$C_1 \|\nabla u\|_{L^2}^2 \leq C_3 \left(\|D^2 u\|_{L^2}^2 \|u\|_{L^2}^2 + \|u\|_{L^2}^2 \right)^2 \leq C_4 \left(\|D^2 u\|_{L^2}^2 + 1 \right)^2 \leq \|\Delta u\|_{L^2} + C_5. $$  (3.43)

Substituting (3.43) into (3.42), we end up with

$$\frac{d}{dt} \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^2 \leq C_6,$$

yielding directly the uniform boundedness of $\|\nabla u\|_{L^2}$. The same type argument applied to the $u$-equation in (1.11) gives the uniform boundedness of $\|\nabla w\|_{L^2}$.

Now, we again use the $u$-equation in (1.1) to calculate that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^4 + \int_\Omega |\nabla \nabla u|^2 |^2 + 2 \int_\Omega |\nabla u|^2 |D^2 u|^2$$

$$= -2\chi_1 \int_\Omega |\nabla u|^2 \nabla u \cdot \nabla (\nabla u \nabla v + u \Delta v) + \int_{\partial \Omega} |\nabla u|^2 \frac{\partial}{\partial v} |\nabla u|^2.$$  (3.44)

By direct computations, we discover that

$$\nabla (\nabla u \nabla v) = u_{x_1} \nabla v_{x_1} + v_{x_1} \nabla u_{x_1} + u_{x_2} \nabla v_{x_2} + v_{x_2} \nabla u_{x_2}.$$  (3.45)
With these, we then employ (3.44) and (3.41) to estimate, for any $\epsilon_i > 0$, that

\[-2\chi_1 \int_\Omega |\nabla u|^2 u_{x_1} \nabla u \nabla v_{x_1} \leq \epsilon_1 \int_\Omega |\nabla u|^6 + C_{\epsilon_1} \int_\Omega |\nabla v_{x_1}|^6,
\]

\[-2\chi_1 \int_\Omega |\nabla u|^2 v_{x_1} \nabla u \nabla u_{x_1} \leq \epsilon_2 \int_\Omega |\nabla u|^2 |D^2 u|^2 + C_{\epsilon_2} \int_\Omega |\nabla u|^4,
\]

\[-2\chi_1 \int_\Omega |\nabla u|^2 u_{x_2} \nabla u \nabla v_{x_2} \leq \epsilon_3 \int_\Omega |\nabla u|^6 + C_{\epsilon_3} \int_\Omega |\nabla v_{x_2}|^6,
\]

\[-2\chi_1 \int_\Omega |\nabla u|^2 v_{x_2} \nabla u \nabla u_{x_2} \leq \epsilon_4 \int_\Omega |\nabla u|^2 |D^2 u|^2 + C_{\epsilon_4} \int_\Omega |\nabla u|^4,
\]

\[-2\chi_1 \int_\Omega |\nabla u|^4 \Delta v \leq \epsilon_5 \int_\Omega |\nabla u|^6 + C_{\epsilon_5} \int_\Omega |\Delta v|^3,
\]

\[-2\chi_1 \int_\Omega u |\nabla u|^2 \nabla u \Delta v \leq \epsilon_6 \int_\Omega |\nabla u|^6 + C_{\epsilon_6} \int_\Omega |\Delta v|^3.
\]

\[C_{\epsilon_1} \int_\Omega |\nabla v_{x_1}|^3 + C_{\epsilon_3} \int_\Omega |\nabla v_{x_2}|^3 \leq C_{\epsilon_1,\epsilon_3} \int_\Omega |D^2 v|^3 + C_{\epsilon_1,\epsilon_3} \int_\Omega |\Delta v|^3 + \tilde{C}_{\epsilon_1,\epsilon_3} \int_\Omega |\Delta v|^3 + \tilde{C}_{\epsilon_1,\epsilon_3} \leq \int_\Omega |\nabla \Delta v|^2 + \tilde{C}_{\epsilon_1,\epsilon_3},\]

where we have applied the uniform boundedness of $\|v\|_{L^2}$, the $W^{2,3}$-elliptic estimate (cf. [14]) and the 2D G-N interpolation inequality in the last estimate.

In view of the uniform boundedness of $\|\nabla u\|_{L^2}$ and the fact fact $\frac{\partial u}{\partial \nu} = 0$ on $\partial \Omega$, by the 2D G-N inequality and boundary trace embedding, the following two estimates are quite known (cf. [14] (3.31) and (3.32)) for example:

\[\int_\Omega |\nabla u|^6 \leq C_7 \int_\Omega |\nabla |\nabla u|^2|^2 + C_7,
\]

\[\int_{\partial \Omega} |\nabla u|^2 \frac{\partial}{\partial \nu} |\nabla u|^2 \leq \epsilon_7 \int_\Omega |\nabla |\nabla u|^2|^2 + C_{\epsilon_7}.
\]

Substituting these estimates into (3.44) and choosing sufficiently small $\epsilon_i$, we infer

\[\frac{d}{dt} \int_\Omega |\nabla u|^4 + \int_\Omega |\nabla u|^4 \leq C_8 \int_\Omega |\nabla \Delta v|^2 + C_8. \tag{3.46}
\]

Finally, we combine (3.46) with (3.35) to derive an ODI as follows:

\[\frac{d}{dt} \int_\Omega (|\nabla u|^4 + C_{\gamma_1} |\Delta v|^2) + \int_\Omega (|\nabla u|^4 + 2C_9 |\Delta v|^2) \leq C_9,
\]

which trivially yields the uniform boundedness of $\|\nabla u\|_{L^4}$. Doing the same argument to the $w$-equation in (1.1) shows the uniform boundedness of $\|\nabla w\|_{L^4}$. 
Now, we once again use the $u$-equation in (1.1) and use (3.30), (3.31), (3.35) and the elliptic $H^2$-estimate to bound
\[
\frac{d}{dt} \int_{\Omega} |\Delta u|^2 + \int_{\Omega} |\Delta u|^2 + 2 \int_{\Omega} |\nabla \Delta u|^2 \\
= 2 \chi_1 \int_{\Omega} \nabla (\nabla u \nabla v + u \Delta v) \nabla \Delta u + \int_{\Omega} |\Delta u|^2 \\
\leq \int_{\Omega} |\nabla \Delta u|^2 + C_{10} \int_{\Omega} |D^2 u|^2 + C_{10} \int_{\Omega} |\nabla u|^4 + \int_{\Omega} |\Delta u|^2 \\
+ C_{10} \int_{\Omega} |D^2 v|^2 + C_{10} \int_{\Omega} |\Delta v|^4 + C_{10} \int_{\Omega} |\nabla \Delta v|^2 \\
\leq \int_{\Omega} |\nabla \Delta u|^2 + C_{11} \int_{\Omega} |\Delta u|^2 + C_{11} \int_{\Omega} |\Delta u|^2 + C_{11} \int_{\Omega} |\nabla \Delta v|^2 + C_{11} \\
\leq 2 \int_{\Omega} |\nabla \Delta u|^2 + C_{12} \int_{\Omega} |\nabla \Delta v|^2 + C_{12},
\]
where we have used the uniform $(H^1, H^2)$-boundness of $(u, v)$ in the last two lines. Combining (3.47) with (3.35), we then derive a key ODI as follows:
\[
\frac{d}{dt} \int_{\Omega} (|\Delta u|^2 + C_{12} \tau_1 |\Delta v|^2) + \int_{\Omega} (|\Delta u|^2 + 2 C_{12} |\Delta v|^2) \leq C_{13},
\]
showing the uniform boundedness of $\|\Delta u\|_{L^2}$. The same argument applied to the $w$-equation in (1.1) shows the uniform boundedness of $\|\Delta w\|_{L^2}$. In light of our gained uniform $H^1$-boundedness of $(u, w)$, the $W^{2,2}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ for all $q \in (1, \infty)$, we end up with
\[
\begin{align*}
\|u(t)\|_{W^{2,2}} + \|w(t)\|_{W^{2,2}} & \leq C_{14}, \quad t \geq 1, \\
q < \infty, \quad \|u(t)\|_{W^{1,q}} + \|w(t)\|_{W^{1,q}} & \leq C_{q}, \quad t \geq 1.
\end{align*}
\]
Now, differentiating the $v$ and $z$ equations in (1.1) twice with respect to $x_i$ and then $x_j$ for $i, j = 1, 2$, we discover that
\[
\begin{align*}
\tau_1 (v_{x,x_i}, t) = & \Delta (v_{x,x_i}) - v_{x,x_i} + w_{x,x_i} & & \text{in } \Omega \times (1, \infty), \\
\tau_2 (z_{x,x_i}, t) = & \Delta (z_{x,x_i}) - z_{x,x_i} + w_{x,x_i} & & \text{in } \Omega \times (1, \infty), \\
\frac{\partial v_{x,x_i}}{\partial \nu} = & \frac{\partial z_{x,x_i}}{\partial \nu} = 0 & & \text{on } \partial \Omega \times (1, \infty).
\end{align*}
\]
By the facts that $(u_{x,x_i}(t), w_{x,x_i}(t)) \in (C(\bar{\Omega}))^2$, the standard Schauder regularity says that $(v_{x,x_i}(t), z_{x,x_i}(t)) \in (C^2(\Omega))^2$. Therefore, applying $W^{2,2}$-estimate or $W^{1,q}$-estimate in Lemma (2.2) to (3.49), we obtain, for $i, j = 1, 2$, that
\[
\begin{align*}
\|v_{x,x_i}(t)\|_{W^{2,2}} + \|z_{x,x_i}(t)\|_{W^{2,2}} & \leq C_{15}, \quad t \geq 1, & & \text{if } \tau_1 = \tau_2 = 0, \\
q > 1, \quad \|v_{x,x_i}(t)\|_{W^{1,q}} + \|z_{x,x_i}(t)\|_{W^{1,q}} & \leq C_{q}, \quad t \geq 1, & & \text{if } \tau_1, \tau_2 > 0.
\end{align*}
\]
This upon Sobolev embedding yields the existence of $C_{16} > 0$ such that
\[
\|v(t)\|_{W^{2,\infty}} + \|z(t)\|_{W^{2,\infty}} \leq C_{16}, \quad t \geq 1.
\]
Progressively, we again utilize the \( u \)-equation in (1.1) and utilize (3.48), (3.50) and (3.51) to bound
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta u|^4 + \int_{\Omega} |\Delta u|^4 + 6 \int_{\Omega} |\Delta u|^2 |\nabla \Delta u|^2 \\
= 6\chi_1 \int_{\Omega} (\Delta u)^2 \nabla (\nabla u \nabla v + u \Delta v) \nabla (\Delta u) + \int_{\Omega} |\Delta u|^4 \\
\leq \int_{\Omega} |\Delta u|^2 |\nabla \Delta u|^2 + \int_{\Omega} |\Delta u|^4 + C_{17} \int_{\Omega} |\Delta u|^2 |\nabla u|^2 \\
+ C_{17} \int_{\Omega} |\Delta u|^2 |D^2 u|^2 + C_{17} \int_{\Omega} |\Delta u|^2 |\nabla \nu|^2 \\
\leq \int_{\Omega} |\Delta u|^2 |\nabla \Delta u|^2 + C_{18} \int_{\Omega} |\nabla u|^4 + C_{18} \int_{\Omega} |\Delta u|^4 \\
+ C_{18} \int_{\Omega} |D^2 u|^4 + C_{18} \int_{\Omega} |\nabla \nu|^4 \\
\leq \int_{\Omega} |\Delta u|^2 |\nabla \Delta u|^2 + C_{19} \int_{\Omega} |\Delta u|^4 + C_{19} \int_{\Omega} |D^2 u|^4 + C_{19}.
\] (3.52)

We next use the uniform boundedness of \( \|u\|_{H^2} \), the \( W^{2,4}\)-elliptic estimate and the 2D G-N interpolation inequality in (2.10) to infer, for any \( \epsilon > 0 \), that
\[
\|D^2\|_{L^4}^4 \leq C_{20} \left( \|\Delta u\|_{L^4}^4 + \|u\|_{L^4}^4 \right)^2 \\
\leq C_{21} \left( \|\Delta u\|_{L^4}^4 + 1 \right) \\
\leq C_{22} \left( \|\nabla (\Delta u)^2\|_{L^2} \|\Delta u\|_{L^2}^2 + \|\Delta u\|_{L^4}^2 \|\Delta u\|_{L^2}^2 \right) \leq \epsilon \int_{\Omega} |\Delta u|^2 |\nabla \Delta u|^2 + C_{\epsilon}.
\]

This along with (3.52) enables us to see that
\[
\frac{d}{dt} \int_{\Omega} |\Delta u|^4 + \int_{\Omega} |\Delta u|^4 \leq C_{24},
\]
showing the uniform boundedness of \( \|\Delta u\|_{L^4} \). The same argument applied to the \( v \)-equation in (1.1) entails the uniform boundedness of \( \|\Delta u\|_{L^4} \). Due to our established uniform \( H^2 \)-boundedness of \((u, w)\), the \( W^{2,4}\)-elliptic estimate and the 2D Sobolev embedding \( W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \), we finally conclude that
\[
\begin{cases}
\|u(t)\|_{W^{2,4}} + \|w(t)\|_{W^{2,4}} \leq C_{25}, \quad t \geq 1, \\
\|u(t)\|_{W^{1,\infty}} + \|w(t)\|_{W^{1,\infty}} \leq C_{26}, \quad t \geq 1.
\end{cases}
\] (3.53)

Then applying \( W^{2,4}\)-estimate or \( W^{1,q}\)-estimate Lemma (2.2) to (3.49), we see that
\[
\begin{cases}
\|v_{x, y}(t)\|_{W^{2,4}} + \|z_{x, y}(t)\|_{W^{2,4}} \leq C_{27}, \quad t \geq 1, \quad \text{if } \tau_1 = \tau_2 = 0, \\
\|v_{x, y}(t)\|_{W^{1,\infty}} + \|z_{x, y}(t)\|_{W^{1,\infty}} \leq C_{28}, \quad t \geq 1, \quad \text{if } \tau_1, \tau_2 > 0.
\end{cases}
\] (3.54)

Then our desired higher order estimate (1.11) follows from (3.53) and (3.54). \( \square \)

3.4. Finite time blow-up. In this subsection, we show (B4) by detecting a line of masses on which the solution of (1.1) blows up in finite time under (1.14).
Then, if \((w_0, \tau_2 z_0) = (\chi_{12} u_0, \chi_{12} \tau_1 v_0)\), the unique solution of (1.1) on its maximal existence time is given by \((u, v, w, z) = (n, c, \frac{\chi_1}{\chi_2} n, \frac{\chi_1}{\chi_2} c)\).

**Proof.** By direct computations, one sees first that \((n, c, \frac{\chi_1}{\chi_2} n, \frac{\chi_1}{\chi_2} c)\) solves (1.1), and then it is the unique solution of (1.1) by uniqueness. \(\square\)

Based on this observation, we use the well-known blow-up knowledge about (3.55) to detect a blow-up line for our two-species and two-stimuli model (1.1).

**Lemma 3.10.** Let \(\tau_1 = \tau_2, \chi_3 = 0\) and \((w_0, \tau_2 z_0) = (\chi_{12} u_0, \chi_{12} \tau_1 v_0)\). Assume that (1.15) is satisfied and \(\int_{\Omega} u_0(x)^{2} x - x_0)^2\) is sufficiently small for \(x_0 \in \Omega\). Then the solution of the IBVP (1.1) blows up in a finite time \(T > 0\) according to (1.16).

**Proof.** It follows from \(m_2 = \int_{\Omega} u_0 = \int_{\Omega} \chi_{12} u_0 = \chi_{12} m_1\) that \(m_1 m_2 \chi_1 \chi_2 = (m_1 \chi_2)^2\). Thus, the large product condition (1.15) directly gives \(m_1 \chi_2 > \pi^*\). By the well-known blow-up results about (3.55) (cf. [9, 10, 11, 14, 25, 26, 27]), we know that the solution \((n, c)\) of (3.55) blows up in a finite time \(T > 0\) in the sense that

\[
\limsup_{t \to T} (\|n(t)\|_{L^\infty} + \|c(t)\|_{L^\infty}) = \infty.
\]

Then Lemma 3.9 simply says that \((u, v, w, z) = (n, c, \frac{\chi_1}{\chi_2} n, \frac{\chi_1}{\chi_2} c)\) is a classical solution of (1.11) on \(\Omega \times [0, T]\) which blows up at \(t = T\) even in the sense of (1.16). Otherwise, the uniform \(L^1\)-boundedness of \((\ln u, \ln w)\) implies global boundedness (and thus no blow-up) by previous subsections, cf. Lemma 3.8. \(\square\)

4. Convergence for small product of masses

So far, we have proved the global boundedness of solutions to the IBVP (1.1) under certain smallness assumption on the product of masses and blow-up for certain large product. In this section, we turn our attention to study the large time behavior of bounded solutions under (B3). To this end, we find that it is more convenient to work on its equivalent system:

\[
\begin{aligned}
U_t &= \nabla \cdot (\nabla U - U \nabla V) & & \text{in } \Omega \times (0, \infty), \\
\tau_1 V_t &= \Delta V - V + \eta_1 (W - 1) & & \text{in } \Omega \times (0, \infty), \\
W_t &= \nabla \cdot (\nabla W - W \nabla Z - \chi W \nabla V) & & \text{in } \Omega \times (0, \infty), \\
\tau_2 Z_t &= \Delta Z - Z + \eta_2 (U - 1) & & \text{in } \Omega \times (0, \infty), \\
\frac{\partial U}{\partial \nu} &= \frac{\partial V}{\partial \nu} - \frac{\partial W}{\partial \nu} = \frac{\partial Z}{\partial \nu} = 0 & & \text{on } \partial \Omega \times (0, \infty), \\
(U, \tau_1 V, W, \tau_2 Z) &= (U_0, \tau_1 V_0, W_0, \tau_2 Z_0) & & \text{in } \Omega \times \{0\}.
\end{aligned}
\]
Here, the newly introduced variables satisfy the following transformations:

\[
\begin{align*}
U &= \frac{u}{u_0}, \quad U_0 = \frac{u_0}{u_0}, \quad V = \chi_1(v - \bar{v}), \quad V_0 = \chi_1(v_0 - \bar{v}_0), \\
W &= \frac{w}{w_0}, \quad W_0 = \frac{w_0}{w_0}, \quad Z = \chi_2(z - \bar{z}), \quad Z_0 = \chi_2(z_0 - \bar{z}_0),
\end{align*}
\]

(4.2)

Then it follows simply from (4.1) and (4.2) that

\[
\dot{U} = 1 = W, \quad \dot{V} = 0 = \dot{Z}.
\]

(4.3)

Then an easy use of Lemma 2.3 provides a \( k = k(\Omega) > 0 \) such that

\[
\begin{align*}
\|U - 1\|_{L^2}^2 &\leq k\|\nabla U^\frac{1}{2}\|_{L^2}^2, \\
\|W - 1\|_{L^2}^2 &\leq k\|\nabla W^\frac{1}{2}\|_{L^2}^2.
\end{align*}
\]

(4.4)

Let us begin with the simpler case when \( \tau_1 = \tau_2 = 0 \). In this case, our convergence will rely on building a conditional Lyapunov functional of the form:

\[
G(t) = \frac{\eta_2}{\eta_1} \int_\Omega U \ln U + \int_\Omega W \ln W.
\]

(4.5)

To that purpose, we multiply the first equation by \( \ln U \) and the third equation by \( \ln W \) in (4.1) and then integrate over \( \Omega \) by parts to infer

\[
\frac{d}{dt} \int_\Omega U \ln U + 4 \int_\Omega |\nabla U^\frac{1}{2}|^2 = \eta_1 \int_\Omega (U - 1)(W - 1) - \int_\Omega (U - 1)V,
\]

(4.6)

and, similarly,

\[
\frac{d}{dt} \int_\Omega W \ln W + 4 \int_\Omega |\nabla W^\frac{1}{2}|^2 = \eta_2 \int_\Omega (U - 1)(W - 1) - \int_\Omega (W - 1)Z + \eta_1 \chi \int_\Omega (W - 1)^2 - \chi \int_\Omega (W - 1)V.
\]

(4.7)

Thanks to \( \tau_1 = \tau_2 = 0 \), we see from the second and fourth equation in (4.1) that

\[
\begin{align*}
2 \int_\Omega |\nabla V|^2 + 2 \int_\Omega V^2 &= 2 \eta_1 \int_\Omega (W - 1)V \leq \int_\Omega V^2 + \eta_1^2 \int_\Omega (W - 1)^2, \\
2 \int_\Omega |\nabla Z|^2 + 2 \int_\Omega Z^2 &= 2 \eta_2 \int_\Omega (U - 1)Z \leq \int_\Omega Z^2 + \eta_2^2 \int_\Omega (U - 1)^2.
\end{align*}
\]

(4.8)

With those computations, we next derive the derivative of \( G \) and its decay property.

**Lemma 4.1.** When \( \tau_1 = \tau_2 = 0 \), the derivative of \( G \) defined in (4.5) satisfies

\[
\begin{align*}
G'(t) + \frac{4\eta_2}{\eta_1} \int_\Omega |\nabla U^\frac{1}{2}|^2 + 4 \int_\Omega |\nabla W^\frac{1}{2}|^2 &= 2\eta_2 \int_\Omega (U - 1)(W - 1) - \frac{\eta_2}{\eta_1} \int_\Omega (U - 1)V \\
&\quad - \int_\Omega (W - 1)Z + \eta_1 \chi \int_\Omega (W - 1)^2 - \chi \int_\Omega (W - 1)V, \quad t \in [0, T_m).
\end{align*}
\]

(4.9)

Moreover, if

\[
k^2 \eta_1 \eta_2 + k \eta_1 \chi^+ < 4, \quad \chi^+ = \max\{\chi, \ 0\},
\]

(4.10)

then \( G \) decays exponentially according to

\[
0 \leq G(t) \leq G(0)e^{-\mu t}, \quad t \in [0, T_m),
\]

(4.11)
where the positive and explicit decay rate $\mu$ is given by
\[
\mu = \frac{(4 - k^2 \eta_1 \eta_2 - k \eta_1 \chi^+)}{2k^2} \min \left\{ k, \frac{4}{k^2 \eta_1 \eta_2 + 2(4 - k^2 \eta_1 \eta_2 - k \eta_1 \chi^+)} \right\}.
\] (4.12)

A direct consequence of (4.11) and (4.15) is
\[
\eta_2 \int_{\Omega} U \ln U + \eta_1 \int_{\Omega} W \ln W \leq \eta_1 \mathcal{G}(0)e^{-\mu t}, \quad t \in [0, T_m).
\] (4.13)

Proof. A simple linear combination from (4.6), (4.7) and (4.8) entails (4.9).

Next, since $s \ln s \geq -1$ for any $s > 0$, upon integration, we obtain from the facts that $\bar{U} = \bar{W} = 1$ that
\[
\int_{\Omega} U \ln U \geq \int_{\Omega} (U - 1) = 0, \quad \int_{\Omega} W \ln W \geq \int_{\Omega} (W - 1) = 0.
\] (4.14)
which together with the definition of $\mathcal{G}$ in (4.3) immediately shows $\mathcal{G} \geq 0$.

Now, employing repeatedly Young’s inequality with epsilon and (4.8), for $\epsilon_i > 0$ to be fixed as (4.10), we bound the right-hand of (4.10) term by term as
\[
\begin{align*}
2\eta_2 \int_{\Omega} (U - 1)(W - 1) & \leq \epsilon_1 \int_{\Omega} (U - 1)^2 + \frac{\eta_2^2}{\epsilon_1} \int_{\Omega} V^2, \\
- \frac{\eta_1}{\eta_2} \int_{\Omega} (U - 1)V & \leq \epsilon_2 \int_{\Omega} (U - 1)^2 + \frac{\eta_2^2}{4 \epsilon_1} \int_{\Omega} V^2, \\
- \eta_1 \int_{\Omega} (W - 1)Z & \leq \frac{\eta_2}{\eta_1} \int_{\Omega} Z^2 + \frac{\eta_2^2}{4 \epsilon_3} \int_{\Omega} V^2, \\
- \chi \int_{\Omega} (W - 1)V & \leq \frac{\eta_2}{\eta_1} \int_{\Omega} (W - 1)^2 + \frac{\chi^2}{2 \epsilon_1} \int_{\Omega} V^2 - \frac{\chi^2}{\eta_1} \int_{\Omega} V^2
\end{align*}
\]
Combining these inequalities with (4.4), we bound (4.9) as follows:
\[
\mathcal{G}'(t) + \frac{4\eta_2}{\eta_1} \int_{\Omega} |\nabla U|^2 + 4 \int_{\Omega} |\nabla W|^2
\leq \left( \epsilon_1 + \epsilon_2 + \epsilon_3 \right) \int_{\Omega} (U - 1)^2 + \left( \frac{\eta_2}{\epsilon_1} + \frac{\eta_2^2}{4 \epsilon_2} + \frac{\eta_2^2}{4 \epsilon_3} + \eta_1 \chi^+ \right) \int_{\Omega} (W - 1)^2
\] (4.15)
\[
\leq \left( \epsilon_1 + \epsilon_2 + \epsilon_3 \right) k \int_{\Omega} |\nabla U|^2 + \left( \frac{\eta_2}{\epsilon_1} + \frac{\eta_2^2}{4 \epsilon_2} + \frac{\eta_2^2}{4 \epsilon_3} + \eta_1 \chi^+ \right) k \int_{\Omega} |\nabla W|^2.
\] Thanks to (4.11), we now fix, for instance,
\[
\epsilon_1 = \frac{2\eta_2}{k \eta_1}, \quad \epsilon_2 = \frac{\eta_2}{k \eta_1}, \quad \epsilon_3 = \left[ \frac{\eta_1}{\eta_2} + \frac{4 - k^2 \eta_1 \eta_2 - k \eta_1 \chi^+}{k \eta_2^2} \right]^{-1}
\] (4.16)
so that
\[
\begin{align*}
A & := \frac{4 \eta_2}{\eta_1} - \left( \epsilon_1 + \epsilon_2 + \epsilon_3 \right) k \\
& = \frac{2 \eta_2 (4 - k^2 \eta_1 \eta_2 - k \eta_1 \chi^+)}{k \eta_1^2 + 2(4 - k^2 \eta_1 \eta_2 - k \eta_1 \chi^+)} > 0, \\
\tilde{A} & := 4 - \left( \frac{\eta_2}{\epsilon_1} + \frac{\eta_2^2}{4 \epsilon_2} + \frac{\eta_2^2}{4 \epsilon_3} + \eta_1 \chi^+ \right) k \\
& = \frac{1}{2} \left( 4 - k^2 \eta_1 \eta_2 - k \eta_1 \chi^+ \right) > 0.
\end{align*}
\] (4.17)
Upon directly integrating and using (4.4) and the facts that \( \parallel \)\( U \parallel \) or equivalently \( \parallel (0 \leq U \leq a) \) uniformly bounded on \( (k = 8, 22, 23) \) in other situations:

For this purpose, we shall construct a conditional Lyapunov functional of the form

Substituting (4.18) into (4.15) and recalling (4.5) and (4.17), we finally end up with a key ODI for \( G \) as follows:

\[
\mathcal{G}'(t) \leq -\min \left\{ \frac{\lambda}{k}, \frac{\alpha}{k} \right\} \mathcal{G}(t), \quad t \in [0, T_m),
\]

which along with (4.17) gives rise to (4.11) with \( \mu \) given by (4.12).

**Remark 4.2.** One can easily check the proof of Lemma 4.1 works simply for the limiting case of \( \eta = 0 \). Thus, by setting \( U = Z \equiv 0 \) formally, the \( L^1 \)-convergence \( W \ln W \) for the minimal KS model holds under \( k\eta_1 \chi^+ < 4 \).

If \( k^2 \eta_2 \leq 4 \), it follows easily from the proof of this lemma that \( \|U \ln U\|_{L^1} + \|W \ln W\|_{L^1} \) is uniformly bounded on \( (0, T_m) \), and then our Section 3 implies that the solution to (1.1) or equivalently (1.3) exists globally in time and is bounded on \( \Omega \times (0, \infty) \).

Next, we explore the convergence property for the case that \( \tau_1 > 0 \) and \( \tau_2 > 0 \). For this purpose, we shall construct a conditional Lyapunov functional of the form (cf. [8, 22, 23] in other situations):

\[
\mathcal{H}(t) = \frac{\alpha}{k} \int_\Omega U \ln U + \frac{\tau_1 \alpha}{2} \int_\Omega |\nabla V|^2 + \frac{\tau_1 \alpha}{2} \left( 1 + 2\beta + \frac{\eta_1}{k\eta_2} \right) \int_\Omega V^2
\]

\[
+ \frac{1}{k} \int_\Omega W \ln W + \frac{\tau_2}{2} \int_\Omega |\nabla Z|^2 + \frac{\tau_2}{2} \left( 1 + 2\beta + \frac{\eta_2}{k\eta_2} \right) \int_\Omega Z^2,
\]

where \( k \) is given in (4.4) and nonnegative \( \alpha, \beta, \gamma_i (i = 1, 2) \) will be detailed in (4.20).

**Lemma 4.3.** The time derivative of \( \mathcal{H} \) defined in (4.19) fulfills

\[
\mathcal{H}'(t) + \frac{4\alpha}{k} \int_\Omega |\nabla U|^2 + \frac{4}{k} \int_\Omega |\nabla W|^2 \leq \frac{\tau_1 \alpha}{k\eta_1} \int_\Omega (V^2 + |\nabla V|^2)
\]

\[
+ \frac{\tau_2}{2} \int_\Omega \left( Z^2 + |\nabla Z|^2 \right) + \frac{\tau_2}{2} \int_\Omega Z^2
\]

\[
= \left( \frac{\eta_1 + \eta_2}{k} \right) \int_\Omega (U - 1) (W - 1) - \frac{\tau_1 \alpha}{k} \int_\Omega (U - 1) V_t
\]

\[
- \frac{\alpha}{k} \int_\Omega (U - 1) V - \frac{\tau_2}{k} \int_\Omega (W - 1) Z_t - \frac{1}{k} \int_\Omega (W - 1) Z
\]

\[
+ \frac{\eta_1}{k} \int_\Omega (W - 1)^2 + \tau_2 \eta_2 \int_\Omega (U - 1) Z_t + \frac{\tau_2}{k} \int_\Omega (U - 1) Z
\]

\[
+ \left( \eta_1 \alpha - \frac{\chi}{k} \right) \tau_1 \int_\Omega (W - 1) V_t + \left( \gamma_1 \alpha - \chi \right) \int_\Omega (W - 1) V
\]

\[
+ 2\beta \alpha \tau_1 \int_\Omega V V_t + 2\beta \tau_2 \int_\Omega ZZ_t, \quad t \in [0, T_m).
\]
Proof. By (4.1) and (4.19), we see that the differentiation of $H$ solves
\begin{equation}
H'(t) = \frac{\alpha}{k} \int_{\Omega} U_t \ln U + \tau_1 \alpha \int_{\Omega} \nabla V_t \cdot \nabla V + \tau_1 \alpha \left( 1 + 2\beta + \frac{\gamma_1}{k \eta_1} \right) \int_{\Omega} V_t V \\
+ \frac{1}{k} \int_{\Omega} W_t \ln W + \tau_2 \int_{\Omega} \nabla Z_t \cdot \nabla Z + \tau_2 \left( 1 + 2\beta + \frac{\gamma_2}{k \eta_2} \right) \int_{\Omega} Z_t Z.
\end{equation}
(4.21)

For the terms $\frac{\alpha}{k} \int_{\Omega} U_t \ln U$ and $\frac{1}{k} \int_{\Omega} W_t \ln W$, testing the first equation of (4.1) by $\frac{1}{k} \ln U$ and the third equation by $\frac{1}{k} \ln W$, we conclude that
\begin{align*}
\frac{\alpha}{k} \int_{\Omega} U_t \ln U + &\frac{4 \alpha}{k} \int_{\Omega} |\nabla U|^2 \\
= &-\frac{\alpha}{k} \int_{\Omega} (U - 1) \cdot \Delta V, \quad (4.22)
\end{align*}

and, similarly,
\begin{align*}
\frac{d}{dt} \frac{\eta_2}{k} \int_{\Omega} W \ln W + &\frac{4}{k} \int_{\Omega} |\nabla W|^2 \\
= &\frac{\eta_2}{k} \int_{\Omega} (U - 1)(W - 1) - \frac{\tau_2}{k} \int_{\Omega} (W - 1) Z_t - \frac{1}{k} \int_{\Omega} (W - 1) Z \\
&+ \frac{\eta_2}{k} \int_{\Omega} (W - 1)^2 - \frac{\tau_2}{k} \int_{\Omega} (W - 1) V_t - \frac{1}{k} \int_{\Omega} (W - 1) V. \quad (4.23)
\end{align*}

As to the second and third terms in (4.21), we use the second equation in (4.1) to calculate that
\begin{equation}
\tau_1 \alpha \int_{\Omega} \nabla V_t \cdot \nabla V = \tau_1 \eta_1 \alpha \int_{\Omega} (W - 1) V_t - \tau_1^2 \alpha \int_{\Omega} V_t^2 - \tau_1 \alpha \int_{\Omega} V_t V, \quad (4.24)
\end{equation}

and that
\begin{align*}
\tau_1 \alpha \left( 1 + 2\beta + \frac{\gamma_1}{k \eta_1} \right) \int_{\Omega} V_t V \\
= &\left( 1 + 2\beta \right) \tau_1 \alpha \int_{\Omega} V_t V + \frac{\tau_1 \gamma_1 \alpha}{k \eta_1} \int_{\Omega} V_t V \\
= &\left( 1 + 2\beta \right) \tau_1 \alpha \int_{\Omega} V_t V + \frac{\gamma_1 \alpha}{k \eta_1} \int_{\Omega} \left[ \Delta V - V + \eta_1 (W - 1) \right] V, \quad (4.25)
\end{align*}

In the same reasoning, we find that
\begin{equation}
\tau_2 \int_{\Omega} \nabla Z_t \cdot \nabla Z = \tau_2 \eta_2 \int_{\Omega} (U - 1) Z_t - \tau_2^2 \int_{\Omega} Z_t^2 - \tau_2 \int_{\Omega} Z_t Z, \quad (4.26)
\end{equation}

and that
\begin{align*}
\tau_2 \left( 1 + 2\beta + \frac{\gamma_2}{k \eta_2} \right) \int_{\Omega} Z_t Z \\
= &\left( 1 + 2\beta \right) \tau_2 \int_{\Omega} Z_t Z - \frac{\tau_2 \eta_2}{k \eta_2} \int_{\Omega} \left[ Z^2 + |\nabla Z|^2 \right] + \frac{\gamma_2}{k} \int_{\Omega} (U - 1) Z. \quad (4.27)
\end{align*}

Substituting (4.22), (4.23), (4.24), (4.25), (4.26), (4.27) into (4.21), upon rearrangement, we finally accomplish our stated dissipation identity (4.20). \qed
Based on (4.28), the function $H$ will decay exponentially under a smallness assumption on the product of initial masses, as provided below.

**Lemma 4.4.** Let $\tau_1, \tau_2 > 0$, $\chi \in \mathbb{R}$ and $\eta_1, \eta_2$ being from (4.2) satisfy
\[
2 - \sqrt{\frac{2}{3}} < k\eta_1 \chi < \sqrt{2}, \quad k^2 \eta_1 \eta_2 < \frac{2\sqrt{2}}{3} \min \left\{1, \frac{3}{2} + k\eta_1 \chi \right\}. \tag{4.28}
\]

Then, for the specifications
\[
\begin{cases}
\alpha = \frac{1}{2} \left( \max \left\{\frac{\chi^2}{2}, \frac{\eta_1}{\sqrt{2}\eta_1} \right\} + \frac{3 - 2k\eta_1 \chi}{3k^2 \eta_1^2} \right), \\
\beta = \frac{1 - k^2 \eta_1^2}{k^2 \eta_1^2}, \quad \gamma_1 = \frac{k\eta_1 \chi + 1}{k\eta_1 \alpha}, \quad \gamma_2 = \frac{\alpha}{k\eta_2},
\end{cases} \tag{4.29}
\]
the function $H$ defined in (4.19) decays exponentially according to
\[
0 \leq H(t) \leq H(0) e^{-\delta t}, \quad t \in [0, T_m). \tag{4.30}
\]

Here, the rate $\delta = \delta(\eta_1, \eta_2, \tau_1, \tau_2, \chi, k)$ can be made precise as in (4.31) below.

A direct consequence from (4.30) and (4.19) follows: for some $C > 0$,
\[
||U \ln U||_{L^1} + ||W \ln W||_{L^1} + ||V||_{H^1}^2 + ||Z||_{H^1}^2 \leq C e^{-\delta t}, \quad \forall t \geq 0. \tag{4.31}
\]

**Proof.** Thanks to (4.28), we first find that
\[
\begin{cases}
\max \left\{\frac{\chi^2}{2}, \frac{\eta_1}{\sqrt{2}\eta_1} \right\} < \alpha < \frac{3 + 2k\eta_1 \chi}{3k^2 \eta_1^2}, \\
\eta_1^2 \alpha^2 - 3k^2 \eta_1^2 \alpha^2 + 2\eta_1^2 > 0, \\
0 < \alpha < \frac{1}{k^2 \eta_1^2},
\end{cases} \tag{4.32}
\]
which entails that $\alpha, \beta, \gamma_1, \gamma_2$ defined in (4.20) are positive, and so the function $H$ is nonnegative by (4.11) and (4.19); moreover, along with (4.28), (4.32) implies that
\[
\begin{cases}
(1 + \beta)k^2 \eta_2^2 < 2\alpha, \\
\alpha^2 \gamma_1^2 - 2 \left( \chi + \frac{1}{k\eta_1} \right) \alpha \gamma_1 + (1 - 2\beta)\alpha + \chi^2 < 0, \\
2(1 + \beta)k^2 \eta_2^2 \alpha^2 - 4\alpha + \chi^2 < 0, \\
k^2 \gamma_2 \gamma_2^2 - 2\alpha \gamma_2 + (1 - 2\beta)k\alpha \eta_2 < 0.
\end{cases}
\]

In light of this, (4.29) and (4.32), we further compute that
\[
\begin{align*}
A_1 &:= - \left( (1 + \beta)\eta_2^2 - \frac{2\eta_2}{k\eta_2} \right) = \frac{1}{k^2 \eta_2^2} (\alpha - \frac{\eta_2}{\sqrt{2}\eta_2}) (\alpha + \frac{\eta_2}{\sqrt{2}\eta_2}) > 0, \\
A_2 &:= - \frac{1}{4} \left( (1 - 2\beta)\alpha + (\gamma_1 \alpha - \chi)^2 - \frac{2\eta_1 \alpha}{k\eta_2} \right) = \frac{1}{4k^2 \eta_2^2} \left( \frac{3 + 2k\eta_1 \chi}{3k^2 \eta_1^2} - \alpha \right) > 0, \\
A_3 &:= - \frac{1}{2} \left( 2(1 + \beta)\alpha \eta_2^2 - \frac{4}{k\eta_2} + \frac{\chi^2}{k \eta_2} \right) = \frac{1}{k^2 \eta_2^2} \left( \alpha - \frac{\chi^2}{2} \right) > 0, \\
A_4 &:= - \frac{1}{2} \left( (1 - 2\beta) + \frac{\gamma_2^2}{\alpha} - \frac{2\eta_2}{k\eta_2} \right) = \frac{(\eta_1^2 \alpha^2 - 3k^2 \eta_1^2 \eta_2^2 + 2\eta_1^2)}{2k^2 \eta_2^2 \eta_1^2} > 0.
\end{align*}
\tag{4.33}
\]
With these preparations, we next apply Young’s inequality multiple times to bound the terms on the right-hand side of (4.20) as follows:

\[
\begin{align*}
\frac{\eta_1 \alpha}{k} \int_\Omega (U - 1)(W - 1) &\leq \frac{\alpha}{2k^2} \int_\Omega (U - 1)^2 + \frac{\eta_1^2 \alpha}{2} \int_\Omega (W - 1)^2, \\
\frac{\eta_2}{k} \int_\Omega (U - 1)(W - 1) &\leq \frac{\eta_2^2}{2} \int_\Omega (U - 1)^2 + \frac{1}{2k^2} \int_\Omega (W - 1)^2, \\
- \frac{\tau_1 \alpha}{k} \int_\Omega (U - 1)V_t &\leq \frac{\tau_1^2 \alpha}{2} \int_\Omega V_t^2 + \frac{\alpha}{2k^2} \int_\Omega (U - 1)^2, \\
- \frac{\alpha}{k} \int_\Omega (U - 1)V &\leq \frac{\alpha}{2k^2} \int_\Omega (U - 1)^2 + \frac{\alpha}{2} \int_\Omega V^2, \\
- \frac{\tau_2}{k} \int_\Omega (W - 1)Z_t &\leq \frac{\tau_2^2}{2} \int_\Omega Z_t^2 + \frac{1}{2k^2} \int_\Omega (W - 1)^2, \\
- \frac{1}{k} \int_\Omega (W - 1)Z &\leq \frac{1}{2k^2} \int_\Omega (W - 1)^2 + \frac{1}{2} \int_\Omega Z^2,
\end{align*}
\]

and further

\[
\begin{align*}
\frac{\tau_2}{k} \int_\Omega (U - 1)Z_t &\leq \frac{\tau_2^2}{2} \int_\Omega Z_t^2 + \frac{\eta_2^2}{2} \int_\Omega (U - 1)^2.
\end{align*}
\]

Moreover, from the second and fourth equation in (4.1), we deduce that

\[
\begin{align*}
\begin{cases}
2\tau_1 \int_\Omega VV_t + 2 \int_\Omega |\nabla V|^2 + \int_\Omega V^2 &\leq \eta_1^2 \int_\Omega (W - 1)^2, \\
2\tau_2 \int_\Omega ZZ_t + 2 \int_\Omega |\nabla Z|^2 + \int_\Omega Z^2 &\leq \eta_2^2 \int_\Omega (U - 1)^2.
\end{cases}
\end{align*}
\]

Collecting (4.34), (4.35), (4.36) and (4.20), we obtain that

\[
\begin{align*}
\mathcal{H}'(t) + \frac{4\alpha}{k} \int_\Omega |\nabla U|^2 + \frac{4}{k} \int_\Omega |\nabla W|^2 &+ \left( \frac{\tau_1}{k\eta_1} + 2\beta \right) \alpha \int_\Omega |\nabla V|^2 + \left( \frac{\tau_2}{k\eta_2} + 2\beta \right) \int_\Omega |\nabla Z|^2 \\
&\leq \frac{2\alpha^2}{k^2} + (1 + \beta)\eta_1^2 \int_\Omega (U - 1)^2 + \frac{1}{2} \left( 1 - 2\beta \right) \alpha + \frac{1}{2} \int_\Omega (W - 1)^2 \\
&+ \frac{1}{2} \left[ 1 + 2\beta \eta_1^2 + 4 \frac{\eta_1 \alpha}{k} + \frac{1}{2} \left( \eta_2^2 - \chi \right)^2 + \frac{2\eta_1 \chi}{k} \right] \int_\Omega (U - 1)^2 \\
&+ \frac{1}{2} \left( 1 - 2\beta + \frac{\tau_2^2}{\alpha} - \frac{2\tau_2}{k\eta_2} \right) \int_\Omega Z^2,
\end{align*}
\]
we see that the solution to (4.1) or equivalently (1.1) exists globally in time and is repeating the arguments on boundedness in our Section 3, especially Lemma 3.8, which together with (4.4) and (4.33) enables us to derive a key ODI for gradient estimate away from which trivially gives rise to the desired exponential decay estimate (4.30).

\[ H'(t) + \left( \frac{\gamma_1}{k \eta_1} + 2 \beta \right) \alpha \int_{\Omega} \nabla V^2 + \left( \frac{\gamma_2}{k \eta_2} + 2 \beta \right) \int_{\Omega} |\nabla Z|^2 \]

\[ \leq \left[ (1 + \beta) \eta^2 - \frac{2 \alpha}{k^2} \right] \int_{\Omega} (U - 1)^2 \]

\[ + \frac{1}{2} \left[ (1 - 2 \beta) \alpha + (\gamma_1 \alpha - \chi)^2 - \frac{2 \gamma_1 \alpha}{k \eta_1} \right] \int_{\Omega} V^2 \]

\[ + \frac{1}{2} \left[ 2(1 + \beta) \alpha \eta^2 - \frac{4}{k^2} + \frac{\chi^2}{k^2 \alpha} \right] \int_{\Omega} (W - 1)^2 \]

\[ + \frac{1}{2} \left[ (1 - 2 \beta + \frac{\gamma_2^2}{\alpha} - \frac{2 \gamma_2}{k \eta_2} \right] \int_{\Omega} Z^2 \]

\[ = -A_1 \int_{\Omega} (U - 1)^2 - A_2 \int_{\Omega} V^2 - A_3 \int_{\Omega} (W - 1)^2 - A_4 \int_{\Omega} Z^2. \]

Now, writing

\[ \delta = \min \left\{ \frac{A_1 k}{\alpha}, \frac{2 A_2}{\alpha \tau_1 (1 + 2 \beta + \frac{\alpha}{k \eta_1})}, A_3 k, \frac{2 A_4}{\tau_2 (1 + 2 \beta + \frac{\alpha}{k \eta_2})}, 2 \left( \frac{\gamma_1}{k \eta_1 \tau_1} + \frac{2 \beta}{\tau_1} \right), 2 \left( \frac{\gamma_2}{k \eta_2 \tau_2} + \frac{2 \beta}{\tau_2} \right) \right\}, \]

and recalling that \( U = W = 1 \), from (4.38), (4.39) and (4.40), we finally derive a simple ODI for \( H \) of the form:

\[ H'(t) \leq -\delta H(t), \quad t \in [0, T_m), \]

which trivially gives rise to the desired exponential decay estimate (4.30). \( \square \)

**Remark 4.5.** By setting \( \chi = 0, (\eta_1, \tau_1) = (\eta_2, \tau_2), (W_0, \tau_2 Z_0) = (U_0, \tau_1 V_0) \), we see by uniqueness that (4.1) reduces to two copies of the minimal KS model:

\[ \begin{align*}
U_1 &= \nabla \cdot (\nabla U - U \nabla V) \quad \text{in } \Omega \times (0, \infty), \\
\tau_1 V_1 &= \Delta V - V + \eta_1 (U - 1) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial U}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
(U_1, \tau_1 V_1) &= (U_0, \tau_1 V_0) \quad \text{in } \Omega \times \{0\}.
\end{align*} \]

These together with our subsequent \( W^{3, \infty} \) \((j \geq 1)\)-convergence offer an exponential decay for (4.39) with convergence rates, extending and detailing those of [3].

Given the crucial starting \( L^1 \)-convergence of \((U \ln U, W \ln W)\) provided in Lemmas 4.1 and 4.4 which simply yields the uniform \( L^1 \)-boundedness of \((U \ln U, W \ln W)\), repeating the arguments on boundedness in our Section 3, especially Lemma 3.8, we see that the solution to (4.1) or equivalently (1.1) exists globally in time and is bounded in \( L^\infty (\Omega \times (0, \infty); \text{ moreover, we have the following uniform higher order gradient estimate away from } t = 0, \text{ for some } C > 0, \]

\[ \| (U(t), W(t)) \|_{W^{2,4}} + \| (U(t), W(t)) \|_{W^{1, \infty}} \]

\[ + \| (I_{\{\tau_1 = 0\}} V(t), I_{\{\tau_2 = 0\}} Z(t)) \|_{W^{4,4}} + \| (V(t), Z(t)) \|_{W^{3, \infty}} \leq C, \quad \forall t \geq 1. \]
Lemma 4.6. For any $p \geq 1$, there exists $C_p = C(p, \eta_1, \eta_2, \tau_1, \tau_2, \chi, k) > 0$ such that
\[
\|U(t) - 1\|_{L^p} + \|W(t) - 1\|_{L^p} \leq C_p e^{-\frac{\delta}{\omega} t}, \quad t > 0. \tag{4.41}
\]
There exist constants $D_i > 0$ depending on $(\eta_1, \eta_2, \chi, k)$ such that
\[
\begin{align*}
\{ & \|V(t)\|_{L^1}, \|Z(t)\|_{L^1} \leq D_1 e^{-\frac{\delta}{\omega} t}, \quad t > 0, & \text{if } \tau_1 = \tau_2 = 0, \\
\leq & D_2 \left( e^{-\frac{\delta}{\omega} \min \left\{ \frac{1}{\eta_1}, \frac{1}{\tau_1} \right\} t}, e^{-\frac{\delta}{\omega} \min \left\{ \frac{1}{\eta_1}, \frac{1}{\tau_1} \right\} t} \right), \quad t > 0, & \text{if } \tau_1, \tau_2 > 0.
\end{align*}
\tag{4.42}
\]
We remind here again $\sigma = \mu$ if $\tau_1 = \tau_2 = 0$ and $\sigma = \delta$ if $\tau_1, \tau_2 > 0$.

Proof. In view of the Csiszár-Kullback-Pinsker inequality (cf. [3]) and the facts that $\bar{U} = 1 = \bar{W}$ and (4.13) or (4.31), we infer, for some $C_1, C_2 > 0$, that
\[
\begin{align*}
\{ & \|U - \bar{U}\|_{L^1} \leq 2 \int_0^t U \ln \frac{U}{\bar{U}} = 2 \int_0^t U \ln U \leq C_1 e^{-\sigma t}, \quad t > 0, \\
\leq & \|W - \bar{W}\|_{L^1} \leq 2 \int_0^t W \ln \frac{W}{\bar{W}} = 2 \int_0^t W \ln W \leq C_2 e^{-\sigma t}, \quad t > 0.
\end{align*}
\]
Hence, for any $p \geq 1$, the $L^\infty$-boundedness of $(U, W)$ provides some $C_3, C_4 > 0$ depending on $p, \eta_1, \tau_1, \chi$ and $\Omega$ such that
\[
\begin{align*}
\|U - 1\|_{L^p} \leq & \|U - 1\|_{L^\infty} \|U\|_{L^1} \leq C_3 e^{-\frac{\delta}{\omega} t}, \quad t > 0, \\
\|W - 1\|_{L^p} \leq & \|W - 1\|_{L^\infty} \|W\|_{L^1} \leq C_4 e^{-\frac{\delta}{\omega} t}, \quad t > 0.
\end{align*}
\tag{4.43}
\]
Now, by the $V$-equation in (4.1), we have
\[
\tau_1 \int_\Omega V^2 + 2 \int_\Omega (\nabla V)^2 + \int_\Omega V^2 \leq \eta_1 \int_\Omega (W - 1)^2. \tag{4.44}
\]
Thus, when $\tau_1 = 0$, we deduce from (4.44) and (4.43) that
\[
\|V(t)\|_{L^1} \leq \|\Omega\|^{\frac{1}{2}} \|V(t)\|_{L^2} \leq \eta_1 \|\Omega\|^{\frac{1}{2}} \|W - 1\|_{L^2} \leq C_5 e^{-\frac{\delta}{\omega} t}, \quad t > 0; \tag{4.45}
\]
and, when $\tau_1 > 0$, we get (4.44) and (4.43) that
\[
\int_\Omega V^2 \leq C_6 e^{-\frac{\delta}{\omega} t},
\]
which enables us to derive that
\[
\|V(t)\|_{L^1} \leq \|\Omega\|^{\frac{1}{2}} \|V(t)\|_{L^2} \leq C_7 e^{-\frac{\delta}{\omega} \min \left\{ \frac{1}{\eta_1}, \frac{1}{\tau_1} \right\} t}, \quad t > 0. \tag{4.46}
\]
One can easily use the same argument to the $W$-equation in (4.1) to infer that
\[
\begin{align*}
\{ & \|V(t)\|_{L^1} \leq \|\Omega\|^{\frac{1}{2}} \|V(t)\|_{L^2} \leq C_8 e^{-\frac{\delta}{\omega} t}, \quad t > 0, & \text{if } \tau_2 = 0, \\
\leq & C_9 e^{-\frac{\delta}{\omega} \min \left\{ \frac{1}{\eta_1}, \frac{1}{\tau_1} \right\} t}, \quad t > 0, & \text{if } \tau_2 > 0.
\end{align*}
\tag{4.47}
\]
Now, our decay estimate (4.42) follows trivially from (4.45), (4.46) and (4.47). \qed

At this position, based on the exponential decay estimate (4.41) and (4.42), one can use (commonly used, cf. [22, 23, 37]) the standard $W^{2,p}$-estimate in the case of $\tau_1 = \tau_2 = 0$ or the $L^p$-$L^q$-smoothing estimate for the Neumann heat semigroup $e^{t\Delta}$ in the case of $\tau_1, \tau_2 > 0$ to derive the exponential decay of bounded solutions in up to $L^\infty$-norm. Here, thanks to our uniform higher order gradient estimates as in (4.40), instead, we can easily apply the $G$-N interpolation inequality to improve the $L^p$-convergence to $W^{j,\infty}(j \geq 1)$-convergence of $(U, V, W, Z)$. 

Lemma 4.7. Under Lemma 4.4 or 4.6, there exist constants $L_i > 0$ depending on $(\eta_1, \eta_2, \eta_3, \chi, k)$ such that

$$\begin{align*}
\| (U(t) - 1, \nabla U(t), W(t) - 1, \nabla W(t)) \|_{L^\infty} & \leq L_1 \left( e^{-\frac{\eta_1}{2} t}, e^{-\frac{\eta_2}{2} t}, e^{-\frac{\eta_3}{2} t}, e^{-\frac{\eta_4}{2} t} \right), \quad \forall t \geq 1; \\
\| (V, \nabla V, D^2 V, D^3 V) \|_{L^\infty} & \leq L_2 \left( e^{-\frac{\eta_1}{2} t}, e^{-\frac{\eta_2}{2} t}, e^{-\frac{\eta_3}{2} t}, e^{-\frac{\eta_4}{2} t} \right), \quad \forall t \geq 1;
\end{align*}$$

and, in the case of $\tau_1 = \tau_2 = 0$,

$$\begin{align*}
\| (V, \nabla V, D^2 V) \|_{L^\infty} & \leq L_3 \left( e^{-\frac{\eta_1}{2} t}, e^{-\frac{\eta_2}{2} t}, e^{-\frac{\eta_3}{2} t}, e^{-\frac{\eta_4}{2} t} \right), \quad \forall t \geq 1; \\
\| (Z, \nabla Z, D^2 Z) \|_{L^\infty} & \leq L_4 \left( e^{-\frac{\eta_1}{2} t}, e^{-\frac{\eta_2}{2} t}, e^{-\frac{\eta_3}{2} t}, e^{-\frac{\eta_4}{2} t} \right), \quad \forall t \geq 1.
\end{align*}$$

Here, $\zeta_i = \min \left\{ \frac{\eta_i}{\tau_i}, \frac{\eta_i}{\tau_j} \right\}$, $i = 1, 2$ and $\sigma$ is defined in Lemma 4.6.

Proof. Based on (4.40), (4.41) and the 2D G-N inequality in (2.10), we deduce that

$$\begin{align*}
\| U - 1 \|_{L^\infty} & \leq C_1 \| \nabla U \|_{L^\infty}^{\frac{3}{2}} \| U - 1 \|_{L^1}^{\frac{1}{2}} + C_2 \| U - 1 \|_{L^1}, \\
& \leq C_2 \| U - 1 \|_{L^1}^{\frac{1}{2}} \leq C_3 e^{-\frac{\eta_1}{2} t}, \quad t \geq 1.
\end{align*}$$

and

$$\begin{align*}
\| \nabla (U - 1) \|_{L^\infty} & \leq C_4 \| D^2 U \|_{L^\infty}^{\frac{3}{2}} \| U - 1 \|_{L^1}^{\frac{1}{2}} + C_4 \| U - 1 \|_{L^1}, \\
& \leq C_4 \| U - 1 \|_{L^1}^{\frac{1}{2}} \leq C_6 e^{-\frac{\eta_1}{2} t}, \quad t \geq 1.
\end{align*}$$

The same reasonings applied to the $W$-component give us that

$$\begin{align*}
\| W - 1 \|_{L^\infty} & \leq C_7 e^{-\frac{\eta_1}{2} t}, \quad t \geq 1, \\
\| \nabla (W - 1) \|_{L^\infty} & \leq C_8 e^{-\frac{\eta_1}{2} t}, \quad t \geq 1.
\end{align*}$$

The $W^{1,\infty}$-decay of $(U, W)$ in (4.48) follows from (4.51), (4.52) and (4.53). When $\tau_1 = \tau_2 = 0$, we use (4.40), (4.41) and the 2D G-N inequality to infer that

$$\begin{align*}
\| V \|_{L^\infty} & \leq C_9 \| \nabla V \|_{L^\infty}^{\frac{3}{2}} \| V \|_{L^1}^{\frac{1}{2}} + C_9 \| V \|_{L^1} \leq C_{10} e^{-\frac{\eta_1}{2} t}, \quad t \geq 1, \\
\| \nabla V \|_{L^\infty} & \leq C_{11} \| D^2 V \|_{L^\infty}^{\frac{3}{2}} \| V \|_{L^1}^{\frac{1}{2}} + C_{11} \| V \|_{L^1} \leq C_{12} e^{-\frac{\eta_1}{2} t}, \quad t \geq 1, \\
\| D^2 V \|_{L^\infty} & \leq C_{13} \| D^3 V \|_{L^\infty}^{\frac{3}{2}} \| V \|_{L^1}^{\frac{1}{2}} + C_{13} \| V \|_{L^1} \leq C_{14} e^{-\frac{\eta_1}{2} t}, \quad t \geq 1, \\
\| D^3 V \|_{L^\infty} & \leq C_{15} \| D^4 V \|_{L^\infty}^{\frac{10}{3}} \| V \|_{L^1}^{\frac{1}{3}} + C_{15} \| V \|_{L^1} \leq C_{16} e^{-\frac{\eta_1}{2} t}, \quad t \geq 1.
\end{align*}$$

In a similar way via replacing $V$ by $Z$ in (4.54), we obtain that

$$\begin{align*}
\| (Z, \nabla Z, D^2 Z, D^3 Z) \|_{L^\infty} & \leq C_{17} \left( e^{-\frac{\eta_1}{2} t}, e^{-\frac{\eta_2}{2} t}, e^{-\frac{\eta_3}{2} t}, e^{-\frac{\eta_4}{2} t} \right), \quad t \geq 1.
\end{align*}$$

Then the $W^{3,\infty}$-decay of $(V, Z)$ follows from (4.54) and (4.55), by recalling $\sigma = \mu$.

When $\tau_1, \tau_2 > 0$, in a similar way to (4.48), we readily apply (4.40), (4.41) and the 2D G-N inequality in (2.10) to derive the $W^{2,\infty}$-decay of $(V, Z)$ in (4.50). \qed
Proof of the \( W^{1,\infty} \)-exponential convergence in (B3). Lemma 4.7 actually proves more detailed exponential convergence about each order derivative of solution components than what has been stated in (1.14) of (B3). Here, we present a short proof of (B3). Indeed, using the transformations in (4.2) that links (4.1) with (1.1) and translating Lemmas 4.4 and 4.7 back to our original model (1.1), we obtain the \( W^{1,\infty} \)-exponential convergence for \((u, v)\) as in (1.14) of (B3). As for \((v, z)\), noticing the facts from (2.15) that

\[
\| (\vec{v}(t) - \vec{u}_0, \vec{z}(t) - \vec{u}_0) \|_{W^{1,\infty}} = \begin{cases} 
(0, 0), & \text{if } \tau_1 = \tau_2 = 0, \\
(\| v_0 - \bar{u}_0 \| e^{-\frac{t}{\tau_1}}, \| z_0 - \bar{u}_0 \| e^{-\frac{t}{\tau_2}}), & \text{if } \tau_1, \tau_2 > 0,
\end{cases}
\]

and then, in the case of \( \tau_1, \tau_2 > 0 \), using (4.2) and (4.50), we estimate

\[
\| \chi_1(\vec{v}(t) - \vec{u}_0), \chi_2(\vec{z}(t) - \vec{u}_0) \|_{W^{2,\infty}} 
\leq \| \chi_1(\vec{v}(t) - \vec{u}_0), \chi_2(\vec{z}(t) - \vec{u}_0) \|_{W^{2,\infty}}
\leq \| (V(t), Z(t)) \|_{W^{2,\infty}} + C_{18} \left( e^{-\frac{t}{\tau_1}}, e^{-\frac{t}{\tau_2}} \right)
\leq C_{19} \left( e^{-\frac{\tau_1}{\tau_1}t}, e^{-\frac{\tau_2}{\tau_2}t} \right), \quad \forall t \geq 1.
\]

In the simple case of \( \tau_1 = \tau_2 = 0 \), we have from (4.39) that, for \( t \geq 1 \),

\[
\| \chi_1(\vec{v}(t) - \vec{u}_0), \chi_2(\vec{z}(t) - \vec{u}_0) \|_{W^{3,\infty}} = \| (V(t), Z(t)) \|_{W^{3,\infty}} \leq C_{20} e^{-\frac{\tau_1}{\tau_1}t}.
\]

Then our claimed \( W^{j,\infty} \) \((j = 2, 3)\)-exponential convergence for \((v, z)\) in (B3) in the Introduction follows directly from (4.56) and (4.57). \( \square \)

Acknowledgments K. Lin is supported by the NSF of China (No. 11801461), and T. Xiang is supported by the NSF of China (No. 11601516 and 11871226) and the Research Funds of Renmin University of China (No. 2018030199).

References

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[2] T. Black, Global existence and asymptotic stability in a competitive two-species chemotaxis system with two signals, Discrete Contin. Dyn. Syst. Ser. B, 22 (2017), 1253–1272.
[3] J. Carrillo, A. –, N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[4] C. Conca, E. Espejo and K. Vilches, Remarks on the blow up and global existence for a two species chemotactic Keller-Segel system in R2, European J. Appl. Math., 22 (2011), 553–580.
[5] E. Espejo, A. Stevens and J. Velázquez, Simultaneous finite time blow-up in a two-species model for chemotaxis, Analysis (Munich), 29 (2009), 317–338.
[6] A. Friedman, Partial Differential Equations, Holt, Rinehart Winston, New York, 1969.
[7] H. Gajewski and K. Zacharias, Global behaviour of a reaction-diffusion system modelling chemotaxis, Math. Nachr., 195 (1998), 77–114.
[8] M. Herrero and J. Velázquez, A blow-up mechanism for a chemotaxis model, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 24 (1997), 633–683.
[9] D. Horstmann and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math., 12 (2001), 159–177.
[11] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, I, Jahresber. Deutsch. Math. Verien., 105 (2003), 103–165.
[12] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005), 52–107.
[13] D. Horstmann, Generalizing the Keller-Segel model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species, J. Nonlinear Sci., 21 (2011), 231–270.
[14] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc., 329 (1992), 819-824.
[15] H. Jin, and Z. Wang, Boundedness, blowup and critical mass phenomenon in competing chemotaxis, J. Differential Equations, 260 (2016), 162–196.
[16] H. Jin and T. Xiang, Repulsion effects on boundedness in a quasilinear attraction-repulsion chemotaxis model in higher dimensions, Discrete Contin. Dyn. Syst. Ser. B, 23 (2018), 3071–3085.
[17] H. Knútsdóttir, E. Pálsson and L. Edelstein-Keshet, Mathematical model of macrophage-facilitated breast cancer cells invasion, J. Theor. Biol., 357 (2014), 184-199.
[18] R. Kowalczyk and Z. Szymańska, On the global existence of solutions to an aggregation model, J. Math. Anal. Appl., 343 (2008), 379–398.
[19] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural’eva, Linear and Quasi-linear Equations of Parabolic Type, Amer. Math. Soc. Transl. 23, AMS, Providence, RI, 1968.
[20] X. Li and Y. Wang, Boundedness in a two-species chemotaxis parabolic system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B, 22 (2017), 2717–2729.
[21] Y. Li and J. Lankeit, Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion, Nonlinearity, 29 (2016), 1564–1595.
[22] K. Lin and C. Mu, Global existence and convergence to steady states for an attraction-repulsion chemotaxis system, Nonlinear Anal. Real World Appl., 31 (2016), 630–642.
[23] K. Lin, C. Mu and D. Zhou, Stabilization in a higher-dimensional attraction-repulsion chemotaxis system if repulsion dominates over attraction, Math. Models Methods Appl. Sci., 28 (2018), 1105–1134.
[24] K. Lin and T. Xiang, On global solutions and blow-up for a short-ranged chemical signaling loop, J. Nonlinear Sci., 29 (2019), 551–591.
[25] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., 5 (1995), 581–601.
[26] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac., 40 (1997), 411–433.
[27] T. Nagai, Blow-up of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in twodimensional domains, J. Inequal. Appl., 6 (2001), 37–55.
[28] L. Payne and H. Weinberger, An optimal Poincaré inequality for convex domains, Arch. Rational Mech. Anal., 5 (1960), 286–292.
[29] H. Qiu and S. Guo, Global existence and stability in a two-species chemotaxis system, Discrete Contin. Dyn. Syst. Ser. B, 24 (2019), 1569–1587.
[30] C. Stinner, J. Tello and M. Winkler, Competitive exclusion in a two-species chemotaxis model, J. Math. Biol., 68 (2014), 1607–1626.
[31] Y. Tao and Z. Wang, Competing effects of attraction vs. repulsion in chemotaxis, Math. Models Methods Appl. Sci. 23 (2013), 1–36.
[32] Y. Tao and M. Winkler, A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source, SIAM J. Math. Anal., 43 (2011), 685–704.
[33] Y. Tao and M. Winkler, Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, J. Differential Equations, 257 (2014), 784–815.
[34] Y. Tao and M. Winkler, Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), 3165–3183.
[35] J. Tello and M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, Nonlinearity, 25 (2012), 1413–1425.
[36] X. Tu, C. Mu, P. Zheng and K. Lin, Global dynamics in a two-species chemotaxis-competition system with two signals, Discrete Contin. Dyn. Syst., 38 (2018), 3617–3636.
[37] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), 2889–2905.
[38] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl., 100 (2013), 748–767.

[39] T. Xiang, Boundedness and global existence in the higher-dimensional parabolic-parabolic chemotaxis system with/without growth source, J. Differential Equations, 258 (2015), 4275–4323.

[40] T. Xiang, Global dynamics for a diffusive predator-prey model with prey-taxis and classical Lotka-Volterra kinetics, Nonlinear Anal. Real World Appl., 39 (2018), 278–299.

[41] T. Xiang, Sub-logistic source can prevent blow-up in the 2D minimal Keller-Segel chemotaxis system, J. Math. Phys., 59 (2018), 081502, 11 pp.

[42] H. Yu, W. Wang and S. Zheng, Criteria on global boundedness versus finite time blow-up to a two-species chemotaxis system with two chemicals, Nonlinearity, 31 (2018), 502–514.

[43] Q. Zhang, X. Liu and X. Yang, Global existence and asymptotic behavior of solutions to a two-species chemotaxis system with two chemicals, J. Math. Phys. 58 (2017), 111504, 9 pp.

[44] Q. Zhang, Competitive exclusion for a two-species chemotaxis system with two chemicals, Appl. Math. Lett., 83 (2018), 27–32.

[45] P. Zheng and C. Mu, Global boundedness in a two-competing-species chemotaxis system with two chemicals, Acta Appl. Math., 148 (2017), 157–177.

School of Economics and Mathematics, Southwestern University of Economics and Finance, Chengdu, 610074 SICHU China

E-mail address: linke@swufe.edu.cn

Institute for Mathematical Sciences, Renmin University of China, Beijing, 100872, China

E-mail address: txiang@ruc.edu.cn