S. S. Shahverdiyev\textsuperscript{a}, I. V. Tyutin\textsuperscript{b}, B. L. Voronov\textsuperscript{c}

On Local Variational Differential Operators in Field Theory

Department of Theoretical Physics, P. N. Lebedev Physical Institute, Leninsky prospect 53, 117924 Moscow, Russia

Abstract

We propose and develop a new calculus for local variational differential operators. The main difference of the new formalism with the canonical differential calculus is that the image of higher order operators on local functionals does not contain indefinite quantities like $\delta(0)$. We apply this formalism to BV formulation of general gauge field theory and to its $Sp(2)$-symmetric generalization. Its relation to a quasiclassical expansion is also discussed.
1 Introduction

In the functional formulation of local quantum field theory, it is rather common to encounter a problem that is known as "the problem of \( \delta(0) \)." The essence of the problem can be illustrated by the following simple example. Let \( S(\varphi) \) be a local functional of a field \( \varphi(x) \), say the action
\[
S(\varphi) = \int dx L(\varphi(x), \partial_\mu \varphi(x), \ldots), \quad \partial_\mu = \frac{\partial}{\partial x_\mu}.
\]

Then the second order variational derivative \( \delta^2 S(\varphi) / \delta \varphi(x_1) \delta \varphi(x_2) \) of this functional is a quasilocal distribution (for the notion of a quasilocal distribution see subsection 2.3), symbolically, \( \delta^2 S(\varphi) / \delta \varphi(x_1) \delta \varphi(x_2) \sim \delta(x_1 - x_2) \). Therefore, the second order variational differential operators of the type
\[
\Delta_2 = \int dx E_2(\varphi(x), \partial_\mu \varphi(x), \ldots) \left( \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(x)} \right)
\]
are not defined on local functionals: \( \Delta_2 S \sim \delta(0) \). These operators are not defined on the products \( S_1(\varphi)S_2(\varphi) \ldots \) of local operators, and generally on the functions \( f(S_1, S_2, \ldots) \) of the latter ones, if we assume the standard Leibnitz rule
\[
\Delta_2(S_1(\varphi)S_2(\varphi) \ldots) = (\Delta_2 S_1(\varphi))S_2(\varphi) \ldots + 2 \int dx E_2(...) \frac{\delta S_1}{\delta \varphi(x)} \frac{\delta S_2}{\delta \varphi(x)} \ldots + S_1(\Delta_2 S_2) \ldots + \ldots
\]
The analogous assertions are valid for the higher order differential derivatives \( \delta^n S(\varphi) / \delta \varphi(x_1) \ldots \delta \varphi(x_n) \), \( n > 2 \), and therefore for the local higher order differential operators
\[
\Delta_n = \int dx E_n(\varphi(x), \partial_\mu \varphi(x), \ldots)(\delta / \delta \varphi(x))^n.
\]

However, the emergence of such operators is unavoidable in local field theory. For instants, they arise when we consider the change of variables in the functional integrals that define a theory, in particular, when we seek an invariant measure for local general gauge field theory. Moreover, the \( \Delta_2 \)-type operator underlies the BV formulation of gauge theories [1]. The \( \delta(0) \)-terms that arise in local theories must be compensated by the corresponding "local measure" that is initially singular. It can happen, as "for example" in Yang – Mills or supersymmetric theories, that the coefficient at \( \delta(0) \) is equal to zero from the very beginning because of specific algebraic reasons, then the indefiniteness \( 0 \times \infty \) is resolved in favor of the zero.

In a general case, the standard way of avoiding the problem of \( \delta(0) \) consists in chattering something like this:"if using dimensional regularization [2], the corresponding singularity \( \sim \delta(0) \) is equal to zero." In other words, the formal rule \( \delta(0) = 0 \) is adopted at intermediate stages of the formulation. The successful experience of operating with this rule makes us to suggest the possibility of such a mathematically consistent formalism, where \( \delta(0) \) does not arise at all. In this paper we realize such
a possibility in the framework of a formalism that is proposed below. In this formalism, the absence of $\delta(0)$ is a consequence of the definitions. First of all, the proposed formalism is applied to the BV formulation of general gauge field theory. It turns out that all the contents of the theory remains unchanged. The proposed formalism is specific just for local field theory and has no direct analogs in the theory of functions of finite number of variables. This fact cautions once more against possible attempts to justify some, basic sometimes, assertions in local field theory by referring to the finite-dimensional analogs, and even more, to replace the proofs by such references [3]. On the other hand, as it will be seen below, the proposed formalism furnishes the consistent quasiclassical (with respect to the Plank constant $\hbar$) quantization.

The paper is organized as follows. In sec. 2, we define the appropriate classes of functionals and operators, formulate the rules of the action of the operators on the functionals and discuss their properties. In sec. 3, we consider the special class of operators of the type $\exp \nabla$ and the special class of functionals of the type $\exp S$, as well as the relation of the proposed formalism to the quasiclassical expansion. In sec.4, we consider the local changes of variables. In sec.5 and sec.6 respectively, we apply the proposed formalism to the Lagrangian BV formulation of gauge theories and to its $Sp(2)$-symmetric generalization.

2 Functionals, operators $\nabla$

2.1 Fields

We let $\Gamma = \{\Gamma^\alpha(x)\}$ denote the initial fields and use the condensed notations like $\Gamma^\alpha_k = \Gamma^\alpha_k(x_k), E_n^A(x_1, \ldots, x_n; \Gamma).$ The fields take its values in the Berezin algebra that includes Bose and Fermi fields on equal footing, their Grassmann parities are denoted by $\varepsilon(\Gamma^\alpha) = \varepsilon_A.$ The derivatives with respect to the fields are left,

$$\frac{\delta}{\delta \Gamma^\alpha_k(x_k)} = \delta_A_k, \quad \varepsilon(\delta_A_k) = \varepsilon_A_k.$$

In addition, we use the following abbreviations:

$$(\delta_A)^n = \delta_A_1 \cdots \delta_A_n, \quad \varepsilon((\delta_A)^n) = \sum_{k=1}^n \varepsilon_A_k,$$

$$(A)_m = A_1 \cdots A_m.$$  

The index, which appears twice in the same term, implies a summation (integration).

2.2 Functionals

The essential starting point is the restriction on the class of the functionals $\Phi$ of the fields $\Gamma.$ First of all, it includes all smooth local functionals $S_i(\Gamma), i = 1, 2, ..., $ with the certain Grassmann parity $\varepsilon(S_i) = \varepsilon(i),$ of the type

$$S_i(\Gamma) = \int dx \mathcal{L}_i(x, \Gamma(x), \partial_\mu \Gamma(x), ...),$$
where a local density $L$ is a smooth function of coordinates $x$, fields $\Gamma(x)$ and their finite order derivatives. The locality of the functional $S(\Gamma)$ implies that its variational derivatives $(\delta A)^n S(\Gamma) = \delta/\delta A_1 (x_1) \ldots \delta/\delta A_k (x_k) S(\Gamma)$ are quasilocal distributions, symbolically

$$(\delta A)^n S(\Gamma)) \sim \delta(x_1 - x_2) \ldots \delta(x_{n-1} - x_n).$$

The functionals $\Phi$ of a general type are smooth functions of local functionals:

$$\Phi(\Gamma) = f(S_1, S_2, ...) = f(S), \quad (2)$$

usually with the certain Grassmann parity $\varepsilon(f)$; we will call them the multilocal functionals. The derivatives with respect to $S_i$ are left,

$$\frac{\partial}{\partial S_i} = \partial_{i}, \quad \varepsilon(\partial_i) = \varepsilon(i).$$

In the sequel, we use the following natural abbreviations in our notations:

$$(S_i)^n = S_{i_1} \ldots S_{i_n}, \quad \varepsilon((S_i)^n) = \sum_{k=1}^{n} \varepsilon(i_k),$$

$$(\delta A S_i)^n = \delta A_1 S_{i_1} \ldots \delta A_n S_{i_n}, \quad \varepsilon((\delta A S_i)^n) = \sum_{k=1}^{n} \varepsilon A_k + \sum_{k=1}^{n} \varepsilon(i_k),$$

$$(\delta A \circ S_i)^n = (\delta A S_i)^n (-1)^{\sum_{j=1}^{n} \varepsilon(i_j)} \sum_{k=1}^{n} \varepsilon A_k,$$

$$f_{;i_1 \ldots i_n} = \partial_{i_1} \ldots \partial_{i_n} f,$$

$$(i)_n = i_1 \ldots i_n, \quad \overline{(i)}_n = i_n \ldots i_1,$$

such that for instance

$$f_{;(i)_n} = f_{;i_1 \ldots i_n}, \quad f_{;\overline{(i)}_n} = f_{;i_n \ldots i_1}.$$

### 2.3 Local variational differential operators $\nabla$

On the functional space considered, we define the multilocal operators, the epithet “multilocal” means that the image of the operator lies in the same space of multilocal functionals. We begin with the homogenous local differential operators $\nabla_n$ of an arbitrary order $n$. Let us consider the local differential expression

$$\nabla_n = \int dx_1 \ldots dx_n E_{n}^{A_1 \ldots A_n}(x_1, \ldots, x_n; \Gamma) \frac{\delta}{\delta \Gamma^{A_1}(x_1)} \ldots \frac{\delta}{\delta \Gamma^{A_n}(x_n)} = E_n^{(A)n}(\delta A)^n, \quad (3)$$

with the certain Grassmann parity $\varepsilon(\nabla_n)$, and with the coefficient functions

$$E_{n}^{A_1 \ldots A_n}(x_1, \ldots, x_n; \Gamma) = \sum_{m_1 \ldots m_n} \int dx E_{n[m_1 \ldots m_n]}^{A_1 \ldots A_n}(x, \Gamma(x), \partial_{\mu} \Gamma(x), ...) \prod_{k=1}^{n} (\partial_{\mu})^{m_k} \delta(x - x_k), \quad (4)$$
where \( E_{n|m_1...m_n}^{A_1...A_n}(x, \Gamma(x), \partial_\mu \Gamma(x), ...) \) are smooth functions of their arguments. The coefficients functions
\[
E_{n|m_1...m_n}^{A_1...A_n}(x_1, \ldots, x_n; \Gamma) = E_{n}^{(A)}
\]
are distributions with respect to the space arguments \( x_1, \ldots, x_n \) with the support at the coinciding points \( x_1 = x_2 = \ldots = x_n \). Following the tradition established in field theory, we call such distributions the quasilocal distributions. \( E_{n}^{(A)} \) depend locally on the field \( \Gamma \), so the variational derivatives
\[
\delta_{A_{n+1}...A_{n+m}} E_{n}^{(A)}
\]
are quasilocal distributions with respect to all space arguments \( x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \). \( E_{n}^{(A)} \) have the following symmetry properties with respect to the simultaneous permutations of indices and coordinates
\[
E_{n}^{A_1...A_kA_{k+1}...A_n}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n; \Gamma) =
\]

\[
= (-1)^{\varepsilon A_k \varepsilon A_{k+1}} E_{n}^{A_1...A_{k+1}A_k...A_n}(x_1, \ldots, x_{k+1}, x_k, \ldots, x_n; \Gamma),
\]

and the Grassmann parity
\[
\varepsilon(E_{n}^{(A)}) = \varepsilon(\nabla_n) + \sum_{1}^{n} \varepsilon A_k.
\]

\( \nabla_n \) defines the operator on the functions \( f(S) \) as follows:
\[
(\nabla_n f)(S) = E_{n}^{(A)}(\delta_A \circ S_i)^n f, \frac{\partial}{\partial_{(i)_n}} (S) \equiv S_{\nabla_n(i)_n} f, \frac{\partial}{\partial_{(i)_n}} (S),
\]

(5)

where \( S_{\nabla_n(i)_n} = S_{\nabla_n(i)_n}[S] = \frac{1}{n!} \nabla_n(S_i)^n = E_{n}^{(A)}(\delta_A \circ S_i)^n \) is the derivative from \( S_{i_1}, \ldots, S_{i_n} \) functional with the united (multi)index “\( \nabla_n(i)_n \)”, \( \varepsilon(S_{\nabla_n(i)_n}) = \varepsilon(\nabla_n) + \sum_{k=1}^{n} \varepsilon(i_k) \). It is evident that \( (\nabla_n f)(S) \) is a functional of the type considered. Definition (5) can obviously be extended to the case where \( S_i \) are not necessarily local. The general local differential operator \( \nabla \) is a linear combination of the operators \( \nabla_n \)
\[
\nabla = \sum_{n=0}^{c_n} \nabla_n,
\]
c\( n \) are constants (independent of \( x \) and \( \Gamma \)). \( \nabla_0 = S_0 \) is an arbitrary local functional: formula (5) enables us to consider the multiplication by a local functional as the zero order differential operator. Below, we extend this definition to infinite sums and, in such a case, operate with formal series postponing the question on the sense of series convergence.

The general form of the operators that are ”well” defined on the considered space of multilocal functionals is a smooth function \( \omega(\nabla) = \omega(\nabla_{n_1}, \nabla_{n_2}, \ldots) \) of the local operators \( \nabla_n \); we call them the multilocal operators. We assume the functions \( \omega \) are given by formal series that also defines the rule of the action of multilocal operators on multilocal functionals. It is evident that the image of a multilocal operator belongs to the space of multilocal functionals.

Let us now return to the definition of the operators \( \nabla_n \). The given definition of the operator \( \nabla_n \) differs from its canonical understanding, as the differential operator that satisfies Leibnitz rule and is then denoted by \( \Delta_n \) (below, we call such operators the canonical differential operators), by the additional requirement that it is forbidden
for two or more derivatives $\delta/\delta \Gamma^A(x)$ at the same point $x$ to act on one and the same local functional. This requirement is equivalent to the rule $\delta/\delta \Gamma^A_1(x)\delta/\delta \Gamma^A_2(x)S(\Gamma) = 0$, i.e. the prescription $\delta(0) = 0$, in particular,

$$\nabla_n(S_i)^m = 0, \quad n \geq m + 1.$$  

The relationship between the operators $\nabla_n$ and $\Delta_n$ can also be explained from the quasiclassical expansion point of view. It was noted long ago that the emergence of $\delta(0)$ in quantum field theory is accompanied by additional smallness in powers of the Plank constant $\hbar$, the parameter of the quasiclassical expansion (the loop expansion). Therefore, the operator $\nabla_n$ corresponding to differential expression (ref3) can be considered as a leading quasiclassical approximation to the canonical differential operator $\Delta_n$. The exact meaning of this statement is the following. Let us make the substitutions $S_i \rightarrow S_i/\hbar$, $f(S) \rightarrow f(S/\hbar)$ in functionals, $\delta/\delta \Gamma \rightarrow \hbar \delta/\delta \Gamma$ in derivatives with respect to the fields, and, at last, $\nabla_n \rightarrow \nabla_n(\hbar) = (1/\hbar)E_n^{(A)}(\hbar\delta_A)^n$ in differential expression (3).

In order that the equations to follow might have a sense, it is necessary to refuse from locality: of either the differential operators, by regularizing their coefficient functions $E_n^{(A)}(x_1, ..., x_n)$, or functionals, by assuming $S_i$ to be regularized nonlocal functionals (naturally, we can assume this and that). Then it is easy to see that we have for the action of the canonical differential operator

$$\Delta_n(\hbar)f(S/\hbar) = \left(\frac{1}{\hbar}(\nabla_n)^{(i)}(S_i)\right)f_{\xi(\hbar)}(S/\hbar) +$$

$$+ \sum_{k=1}^n \hbar^k \left(\frac{1}{\hbar}(\nabla_n)^{(i)}_{n-k}\right)f_{\xi(\hbar)}(S/\hbar),$$

where functionals $S_{\nabla_n(i)}_{n-k}$ are given by

$$S_{\nabla_n(i)}_{n-k} = E_n^{(A)}(B)_{n-k}(\delta_A)^k(\delta_B \circ S_i)^{n-k},$$

such that

$$\Delta_n(\hbar)f(S/\hbar) = \nabla_n(\hbar)f(S/\hbar) + O(\hbar, S/\hbar),$$

and $O(\hbar, S/\hbar)$ as a function of $S/\hbar$ has a nonzero order of smallness in $\hbar$. When removing the regularization, the functionals $S_{\nabla_n(i)}_{n-k}$, $k \geq 1$ become singularities $\sim \delta(0)$ and so does $O(\hbar, S/\hbar)$.

It is convenient to set in correspondence to every operator $\nabla_n$ its symbol $\tilde{\nabla}_n = \tilde{\nabla}_n(\Gamma, p)$, the functional of the fields $\Gamma = \{\Gamma^A(x)\}$ and their conjugate variables $p = \{p_A(x)\}$, $\varepsilon(p_A) = \varepsilon(\Gamma^A) = \varepsilon_A$, that is obtained from differential expression (3) by formally substituting the variable $p_A(x)$ for the variational derivative symbol $\delta/\delta \Gamma^A(x)$:

$$\tilde{\nabla}_n = \int dx_1 \ldots dx_n E_n^{A_1 \ldots A_n}(x_1, ..., x_n; \Gamma)p_{A_1}(x_1) \ldots p_{A_n}(x_n) = E_n^{(A)}(p_A)^n.$$
2.4 Operator algebra

The considered space of functionals is closed under the action of the multilocal operators $\omega(\nabla)$, so those form an associative algebra. The algebra of the local operators as an associative algebra is not closed in itself: the product of two operators $\nabla_n \nabla_m$ is not a local differential operator of the $\nabla$-type. Really, the result of the action of the (bilocal) operator $\nabla_n \nabla_m$ on a functional $f(S)$ contains second order variational derivatives of local functionals $S_k$ at different points, what is impossible for the action of a local operator of the $\nabla$-type.

However, as the Lie algebra, the algebra of the $\nabla$-operators proves to be closed: if $\nabla(1)$, $\nabla(2)$ are local, then their commutator

$$[\nabla(1), \nabla(2)] = \nabla(1) \nabla(2) - (-1)^{\varepsilon(\nabla(1))\varepsilon(\nabla(2))} \nabla(2) \nabla(1)$$

is also a local differential operator. In other words, we may say that the space of the local operators is closed under the $\text{ad}$-operation,

$$\text{ad}\nabla(1)(\nabla(2)) = [\nabla(1), \nabla(2)] = \nabla(3).$$

It is sufficient to verify this fact for the homogeneous operators $\nabla_n$. If

$$\nabla_m = E_m^{(A)m}(\delta_A)^m, \quad \nabla_n = E_n^{(A)n}(\delta_A)^n$$

are local, then

$$[\nabla_m, \nabla_n] = m E_m^{(A)m-1}\delta_C E_n^{(B)n}(-1)^{\varepsilon(\nabla_n)}\sum_{k=1}^{m-1} \varepsilon_{A_k}(\delta_A)^{m-1}(\delta_B)^n -$$

$$-(-1)^{\varepsilon(\nabla_m)\varepsilon(\nabla_n)}m E_n^{(A)n-1}\delta_C E_m^{(B)m}(-1)^{\varepsilon(\nabla_m)}\sum_{k=1}^{n-1} \varepsilon_{A_k}(\delta_A)^{n-1}(\delta_B)^m \equiv$$

$$\equiv E_{m+n-1}^{(m,n)(A)m+n-1}(\delta_A)^{m+n-1} = \nabla_{m+n-1}^{(m,n)}$$

is local too. Really, the coefficient functions $E_{m+n-1}^{(m,n)(A)m+n-1}$ defined by the last but one equality are quasilocal distributions in all coordinates $x_1, ..., x_{m+n-1}$ (because, in fact, they are convolutions of quasilocal distributions); they are defined uniquely in the class of distributions that are symmetric with respect to simultaneous permutations of coordinates and indices. This fact is simply formulated in terms of the $\tilde{\nabla}$-symbols. Let us consider the Lie algebra of functionals $F(\Gamma, p)$ generated by the canonical Poison bracket: if $F = F(\Gamma, p)$, $G = G(\Gamma, p)$, then

$$\{F, G\} = \frac{\delta_r F}{\delta p_A} \frac{\delta G}{\delta \Gamma_A} - (-1)^{\varepsilon(F)\varepsilon(G)} \frac{\delta_r G}{\delta p_A} \frac{\delta F}{\delta \Gamma_A},$$

here and below, the index $r$ denotes the right derivative. $\Gamma$ and $p$ are conjugate just in the sense of this bracket:

$$\{p_A(x_1), \Gamma^B(x_2)\} = \delta^B_A \delta(x_1 - x_2).$$

Then it is easy to verify that if $\tilde{\nabla}_m$, $\tilde{\nabla}_n$ are the symbols of $\nabla_m$ and $\nabla_n$ respectively, $\tilde{\nabla}_{m+n-1}^{(m,n)}$ is the symbol of their commutator, then

$$\tilde{\nabla}_{m+n-1}^{(m,n)} = \{\tilde{\nabla}_m, \tilde{\nabla}_n\}.$$

On any $\nabla$, this result extends by linearity.
### 3 Operators \( \exp \nabla \) and functionals \( \exp S \)

Later on, we need the multilocal operators of the form \( \exp \nabla \) treated as formal series

\[
e^{\nabla} = \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^n.
\]

The significance of these operators is determined by the fact that many transformations in field theory, in particular, a change of variables \( \Gamma \to \Gamma' \) in functionals, are formulated in their terms. Let us turn our attention to the properties of these operators following from the fact proved just now that the Lie algebra of the \( \nabla \)-operators is closed, or equivalently, that the set of these operators is invariant under the \( \text{ad} \)-operation: if \( \nabla^{(1)}, \nabla^{(2)} \) are local operators, then \( \text{ad} \nabla^{(1)}(\nabla^{(2)}) \) is also a local differential operator. Then the same is valid for

\[
\left( \text{ad} \nabla^{(1)} \right)^n(\nabla^{(2)}) = \left[ \nabla^{(1)}, \left[ \nabla^{(1)}, \ldots, \left[ \nabla^{(1)}, \nabla^{(2)} \right] \ldots \right] \right], \quad n = 0, 1, 2, \ldots,
\]

the proof is evident by induction, and, at last, for

\[
\text{Ad}(e^{\nabla^{(1)}})(\nabla^{(2)}) = e^{\nabla^{(1)}} \nabla^{(2)} e^{-\nabla^{(1)}} = e^{\text{ad} \nabla^{(1)}}(\nabla^{(2)}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad} \nabla^{(1)})^n(\nabla^{(2)}) = \nabla^{(3)}, \quad (7)
\]

\( \nabla^{(3)} \) is a local operator. In other words, the space of local differential operators is invariant under the \( \text{Ad} \)-operation too. In addition, it is true that

\[
e^{\nabla^{(1)}}e^{\nabla^{(2)}} = e^{\nabla^{(3)}}, \quad (8)
\]

where \( \nabla^{(3)} \) is a local differential operator. This fact is a direct consequence of the Baker-Hausdorff-Dynkin formula \([5]\):

\[
\nabla^{(3)} = \nabla^{(1)} + \nabla^{(2)} + \frac{1}{2}[\nabla^{(1)}, \nabla^{(2)}] + \ldots,
\]

the dots stand for a series in repeated commutators of \( \nabla^{(1)} \) and \( \nabla^{(2)} \).

In other words, the set of the operators \( \exp \nabla \) is invariant under associative multiplication (and defines the corresponding associative algebra in a natural way).

The functionals of the form \( f(S) = \exp S \), where \( S \) is a local functional of the form \([4]\), play a special role in quantum theory. Below, we restrict ourselves to the case of the Bose-functionals, \( \varepsilon(S) = 0 \). For such functionals, the action of a local operator \( \nabla = \sum_{n=0}^{\infty} c_n \nabla_n \) reduces to the multiplication by a local functional:

\[
\nabla e^S = S \nabla [S] e^S, \quad S \nabla [S] = \sum_{n=0}^{\infty} c_n \frac{1}{n!} \nabla_n S^n = \sum_{n=0}^{\infty} c_n E_n^{(A)}(\delta_A S)^n. \quad (9)
\]

The set of the functionals of this type proves to be invariant under the action of the operators \( \exp \nabla \)

\[
e^{\nabla} e^S = e^{S'}, \quad (10)
\]
where $S'$ is a local functional. To prove this, let us consider the one-parameter family of the operators $\exp(\alpha \nabla)$, where $0 \leq \alpha \leq 1$. Let us write

$$e^{\alpha \nabla} e^S = e^{S'(\alpha)}, \quad (11)$$

where the functional $S'(\alpha)$ depends smoothly on $\alpha$, $S'(0) = S$, $S'(1) = S'$. 

$$S'(\alpha) = S + \alpha S^{(1)} + \ldots = \sum_{m=0}^{\infty} \alpha^m S^{(m)}, \quad S^{(0)} = S. \quad (12)$$

For this functional, a differential equation naturally arises: differentiating Eq(11) with respect to $\alpha$, we obtain

$$\nabla e^{S'(\alpha)} = \frac{\partial S'(\alpha)}{\partial \alpha} e^{S'(\alpha)}. \quad (13)$$

But according to Eq(9) that holds true also for functionals $S$ not necessarily local,

$$\nabla e^{S'(\alpha)} = S_{\nabla}[S'(\alpha)] e^{S'(\alpha)}, \quad S_{\nabla}[S'(\alpha)] = \sum_{n=0}^{\infty} c_n E_n^{(A)} (\delta_A S'(\alpha))^n,$$

so we finally obtain the differential equation for $S_{\nabla}'(\alpha)$

$$\frac{\partial S'(\alpha)}{\partial \alpha} = S_{\nabla}[S'(\alpha)] \quad (13)$$

with the initial condition

$$S'(0) = S. \quad (14)$$

In the class of functionals smooth in $\alpha$, the solution of Eq(13, 14) is unique. Namely, substituting expansion (12) into Eq(13), we obtain the recurrent relations for $S^{(m)}$. In particular,

$$S^{(1)} = S_{\nabla}[S].$$

It is interesting to note that the operation $\exp \nabla$ is in agreement with the quasiclassical expansion in the following sense. Let us make the substitutions

$$\nabla \rightarrow \nabla(\hbar) = \sum_{n=0}^{\infty} c_n \nabla_n(\hbar) = \sum_{n=0}^{\infty} c_n \left( \frac{1}{\hbar} E_n^{(A)} (\hbar \delta_A)^n \right), \quad S \rightarrow \frac{1}{\hbar} S.$$ 

Consider a triple of operators $\nabla(i)$, $i = 1, 2, 3$. Then it turns out that if $\nabla(i)$ are related by Eq(5), the same holds true for $\nabla(i)(\hbar)$

$$[\nabla^{(1)}, \nabla^{(2)}] = \nabla^{(3)} \rightarrow [\nabla^{(1)}(\hbar), \nabla^{(2)}(\hbar)] = \nabla^{(3)}(\hbar).$$

It is sufficient to verify this for the homogenous operators $\nabla_n$. As a consequence, if $\nabla(i)$ are related by Eq(5) or Eq(9), the same holds true for $\nabla(i)(\hbar)$

$$e^{\nabla^{(1)}(\hbar)} e^{\nabla^{(2)}(\hbar)} e^{-\nabla^{(1)}(\hbar)} = \nabla^{(3)}(\hbar), \quad e^{\nabla^{(1)}(\hbar)} e^{\nabla^{(2)}(\hbar)} e^{-\nabla^{(1)}(\hbar)} = \nabla^{(3)}(\hbar),$$

$$e^{\nabla^{(1)}} e^{\nabla^{(2)}} = e^{\nabla^{(3)}}, \quad e^{\nabla^{(1)}(\hbar)} e^{\nabla^{(2)}(\hbar)} = e^{\nabla^{(3)}(\hbar)}.$$
It is easy to verify also that if an operator $\nabla$, functionals $S$ and $S_\nabla[S]$ are related by \(\text{Eq}(9)\), the same holds true for $\nabla(h), S/h, S_\nabla/h$ (with the same $S_\nabla$)

$$\nabla(h) e^{\frac{1}{h} S} = \frac{1}{h} S_\nabla[S] e^{\frac{1}{h} S}.$$ 

Finally, if an operator $\nabla$, functionals $S$ and $S'$ are related by \(\text{Eq}(10)\), then it is true that

$$e^{\nabla(h)} e^{\frac{1}{h} S} = e^{\frac{1}{h} S'},$$

$S'$ being independent of $h$. The proof repeats the proof of \(\text{Eq}(10)\): $h$ happily cancels in the equation for $S'(\alpha)$ that coincides with \(\text{Eq}(13), (14)\). The solution $S'(\alpha)$ does not depend on $h$.

### 4 Change of variables

Multilocal functionals $\Phi(\Gamma)$ and multilocal differential operators $\omega(\nabla)$ admit local change of variables

$$\Gamma^A = \Gamma^A(x) \rightarrow \Gamma'^A = \Gamma'^A(x, \Gamma(x), \partial_\mu \Gamma(x), \ldots),$$

where $\Gamma'^A(x, \Gamma(x), \partial_\mu \Gamma(x), \ldots)$ are smooth functions of fields $\Gamma(x)$ and their finite order derivatives. This means that $\delta \Gamma'^A(x)/\delta \Gamma^B(y)$ are quasilocal distributions

$$\frac{\delta \Gamma'^A(x)}{\delta \Gamma^B(y)} = P^A_B(x, \Gamma(x), \partial_\mu \Gamma(x), \ldots, \partial_\mu) \delta(x - y),$$

$P^A_B(x, \Gamma(x), \partial_\mu \Gamma(x), \ldots, \partial_\mu)$ is a polynomial in $\partial_\mu$ with coefficients that are local functions of the fields $\Gamma(x)$ and their derivatives. We assume that the change of variables conserves the Grassmann parity $\varepsilon(\Gamma'^A) = \varepsilon(\Gamma^A)$. We also assume that the change is invertible (at least perturbatively in powers of the field derivatives), the inverse change of variables

$$\Gamma'^A \rightarrow \Gamma = \Gamma^A(x, \Gamma'(x), \partial_\mu \Gamma'(x), \ldots)$$

being also local

$$\frac{\delta \Gamma^A(x)}{\delta \Gamma'^B(y)} Q^A_B(x, \Gamma(x), \partial_\mu \Gamma(x), \ldots, \partial_\mu) \delta(x - y).$$

Hence, the local changes of variables form a group (of diffeomorphisms), at least in some neighborhood of the point $\Gamma^A = 0$.

The local change of variables defines the local transformation of functionals in a natural way.

We assume that the functionals $\Phi(\Gamma)$ transform according to the scalar (anti)representation of the diffeomorphism group

$$\Phi'(\Gamma) = \Phi(\Gamma').$$

It is evident that if $S(\Gamma)$ is a local functional of the form (1), then $S'(\Gamma)$ is also local. For multilocal functionals of general form (2) we have

$$(f'(S))(\Gamma) = (f(S'))(\Gamma) = (f(S))(\Gamma').$$
Thus, the local transformations take the considered set of functionals into itself.
Correspondingly, the local changes of variables induce the transformations of local
differential operators $\nabla$
$$\nabla_{(\Gamma)} \rightarrow \nabla'_{(\Gamma')} = \nabla_{(\Gamma')},$$
(15)
This transformations are naturally defined as follows:
$$(\nabla f(S))(\Gamma) = \nabla_{(\Gamma)} f(S(\Gamma)) \rightarrow (\nabla' f'(S))(\Gamma') = (\nabla f(S))(\Gamma') = \nabla_{(\Gamma')}, f(S(\Gamma')).$$
For differential operator $\nabla_n$ (3), (4) this transformation law takes the form
$$\nabla'_n = E'_{n(A)}(\delta_A)^n,$$
(16)
where
$$E^{'A_1...A_n}(x_1, \ldots, x_n; \Gamma) = \int dy_1 \ldots dy_n (-1)^{\sum_{k=1}^{n-1} \epsilon_{A_k} \sum_{i=k+1}^{n} (\epsilon_{A_i} + \epsilon_{B_i})} \times$$
$$E_{n^{B_1...B_n}}(y_1, \ldots, y_n; \Gamma') \delta_{\Gamma'^{A_1}(y_1)} \cdots \delta_{\Gamma'^{A_n}(y_n)}.$$
The coefficient functions $E^{'A_1...A_n}(x_1, \ldots, x_n; \Gamma)$ transform according to the tensor (anti)representation.
It is evident that if $E^{'A_1...A_n}(x_1, \ldots, x_n; \Gamma)$ is a quasilocal distribution, then $E^{'A_1...A_n}(x_1, \ldots, x_n; \Gamma)$ is a quasilocal distribution too (as a convolution of quasilocal distributions). Therefore, the operator $\nabla'_n$ is local. For arbitrary operators $\nabla = \sum c_n \nabla_n$, the rule for change of variables extends by linearity. Finally, for multilocal operators $\omega(\nabla)$, the rule for change of variables has the form
$$\omega(\nabla) \rightarrow \omega(\nabla').$$
It is evident that the analogous properties hold true if we define the transformations of the functionals and the differential operators under the change of variables according to the scalar representation
$$\Phi'(\Gamma') = \Phi(\Gamma), \ nabla'(\Gamma') = nabla(\Gamma).$$
The local changes of variables are realized by operators $e^{\nabla_1}$. Namely, the following formulas take place:
$$\Gamma'^{A}(x) = e^{\nabla_1} \Gamma^{A}(x)$$
with some local differential operator $\nabla_1$,
$$S'(\Gamma) = S(\Gamma') = e^{\nabla_1} S(\Gamma),$$
$$e^{\nabla_1} f(S) = f(S').$$
If $\Phi(\Gamma)$ is considered as a multiplication operator in the space of functionals, then
$$e^{\nabla_1} \Phi(\Gamma) e^{-\nabla_1} = \Phi'(\Gamma) = \Phi(\Gamma'),$$
whereas
$$e^{\nabla_1} \frac{\delta}{\delta \Gamma^A(x)} e^{-\nabla_1} = \frac{\delta}{\delta \Gamma^A(x)} = \int dy \frac{\delta \Gamma^B(y)}{\delta \Gamma^A(x)} \frac{\delta}{\delta \Gamma^B(y)},$$
correspondingly
$$e^{\nabla_1} \nabla e^{-\nabla_1} = \nabla'.$$
5 Antibracket, master equation, G transformations

5.1 Antibracket, canonical transformations

Let $\nabla_2$ be a local differential operator of the second order

$$\nabla_2 = \frac{1}{2} \int dx_1 dx_2 \bar{E}_2^{A_1A_2}(x_1, x_2; \Gamma) \frac{\delta}{\delta \Gamma^{A_1}(x_1)} \frac{\delta}{\delta \Gamma^{A_2}(x_2)},$$

(17)

with Grassmann parity equal to unity, $\varepsilon(\nabla_2) = 1$, and nilpotent,

$$\nabla_2^2 = \frac{1}{2} [\nabla_2, \nabla_2] = 0.$$

(18)

This operator defines a bilinear operation on local functionals that is called antibracket. Namely, let $S_1$, $S_2$ be arbitrary local functionals, then the antibracket $(S_1, S_2)$ is defined as

$$(S_1, S_2) = (-1)^{\varepsilon(S_1)} \nabla_2 (S_1 S_2) = 2 (-1)^{\varepsilon(S_1)} S_{\nabla_2^2} = \int dx_1 dx_2 \bar{E}_2^{A_1A_2}(x_1, x_2; \Gamma) (-1)^{\varepsilon(A_2+1)\varepsilon(S_1)} \frac{\delta S_1}{\delta \Gamma^{A_1}(x_1)} \frac{\delta S_2}{\delta \Gamma^{A_2}(x_2)},$$

(19)

$$\varepsilon((S_1, S_2)) = \varepsilon(S_1) + \varepsilon(S_2) + 1,$$

the antibracket is evidently a local functional. We can now write down (see (3))

$$\nabla_2 f(S) = \frac{1}{2} (-1)^{\varepsilon(S_1)} (S_{i_1}, S_{i_2}) f_{i_1 i_2}.$$

(20)

It is convenient to represent the antibracket in the following form:

$$(S_1, S_2) = \int dx_1 dx_2 \frac{\delta S_1}{\delta \Gamma^{A_1}(x_1)} \bar{E}_2^{A_1A_2}(x_1, x_2; \Gamma) \frac{\delta S_2}{\delta \Gamma^{A_2}(x_2)},$$

where the metric tensor $\bar{E}_2^{A_1A_2}$ of the antibracket is

$$\bar{E}_2^{A_1A_2} = (-1)^{\varepsilon(A_1)\varepsilon(A_2+1)} E_2^{A_1A_2} = -(\varepsilon(A_1+1)\varepsilon(A_2+1)) \bar{E}_2^{A_2A_1}.$$

(21)

The antibracket evidently has the antisymmetry property

$$(S_1, S_2) = -(\varepsilon(S_1)+1)(\varepsilon(S_2)+1) (S_2, S_1).$$

As it follows from (19), (20), the nilpotency property (18) is equivalent to the Jacoby identity for the antibracket

$$\nabla_2^2 (S_1 S_2 S_3) \equiv 0 \rightarrow (-1)^{\varepsilon(S_1)+1}\varepsilon(S_2)+1) (S_1, (S_2, S_3)) + \text{cycle}(1, 2, 3) = 0.$$

(22)

In terms of the coefficient functions $\bar{E}_2^{A_1A_2}$ this condition means (it is sufficient to take $S_1 = \Gamma^A(x)$, $S_2 = \Gamma^B(y)$, $S_3 = \Gamma^C(z)$ in (22))

$$(-1)^{\varepsilon(A_1+1)\varepsilon(C+1)} \int du \frac{\delta}{\delta \Gamma_D(u)} \bar{E}_2^{BC}(y, z; \Gamma) + \text{cycle}(Ax, By, Cz) = 0,$$
or, in the condensed notations,

$$(-1)^{cA+1}(cC+1)E_2^{AD}\delta_D E_2^{BC} + \text{cycle}(A, B, C) = 0. \quad (23)$$

Of course, the converse is also true: if the tensor $\tilde{E}_2^{A_1A_2}$ satisfies antisymmetry condition (21) and Jacoby identity (23), then the operator $\nabla_2$ (17) with $E_2^{A_1A_2}$ defined by (21) is nilpotent. It is a standard to assume that under the change of variables $\Gamma \to \Gamma'$, the antibracket transforms, $(..) \to (..)'$ (the index “$'$” at the bracket is the symbol of a new antibracket), according to the following rule:

$$(S_1, S_2)(\Gamma) \to (S_1', S_2' \Gamma) = \delta_{rA} S_1' \delta_{B} S_2' \Gamma = (S_1, S_2)(\Gamma'),$$

wherefrom it follows that the metric tensor of the antibracket transforms according to the tensor (anti)representation:

$$\tilde{E}_2'^{A_1A_2}(x_1, \ldots, x_n; \Gamma) = \int dy_1 dy_2 \frac{\delta_{rA} \Gamma A_1(x_1)}{\delta \Gamma B_1(y_1)} \tilde{E}_2^{B_1B_2}(y_1, y_2; \Gamma') \delta_{B} \Gamma A_2(x_2).$$

This transformation properties of the antibracket are in agreement with the rules that are obtained from definition (19) of the antibracket: $(S_1', S_2' \Gamma) = (-1)^{c(S_1)} \nabla_2(S_1, S_2)$, $(S_1, S_2)' = (-1)^{c(S_1)} \nabla_2'(S_1, S_2)$, and transformation properties (13) of the operators $\nabla$.

In connection with the antibracket (as well as with the master equation, see below), the question arises on the field transformation $\Gamma \to \Gamma' = e^{\nabla_1} \Gamma$ that conserve the antibracket (local canonical changes of variables)

$$(S_1, S_2)'(\Gamma) = (S_1, S_2)(\Gamma).$$

In terms of operators, this means

$$e^{\nabla_1} \nabla_2 e^{-\nabla_1} = \nabla_2,$$

or

$$[e^{\nabla_1}, \nabla_2] = 0,$$

i.e. the transformation conserves the operator $\nabla_2$. In terms of the coefficient functions, this means

$$\tilde{E}_2'^{A_1A_2}(x_1, x_2; \Gamma) = \tilde{E}_2^{A_1A_2}(x_1, x_2; \Gamma).$$

The class of such transformations depends essentially on the properties of the coefficient functions $\tilde{E}_2^{A_1A_2}$ (for example, in the trivial case $\tilde{E}_2^{A_1A_2} = 0$, it includes all transformations). In the finite-dimension case, where the coefficient functions are finite-dimensional matrices $E_2^{ij}$, this class of transformations are determined by the rank of $E_2^{ij}$.

It is easy to see that property (24) is satisfied if

$$\nabla_1 = [\nabla_2, F] = \nabla_2 F + F \nabla_2, = -\int dx_1 dx_2 \frac{\delta_{rF}}{\delta \Gamma A_1(x_1)} \tilde{E}_2^{A_1A_2}(x_1, x_2; \Gamma) \frac{\delta}{\delta \Gamma A_2(x_2)},$$

where $F$ is a local fermion functional, $\varepsilon(F) = 1$. This follows from the relation

$$[\nabla_1, \nabla_2] = 0.$$
that is easy to verify. It is obvious that
\[ \nabla_1 S = (S, F). \]

Thus, the transformation
\[ \Gamma^A \rightarrow \Gamma'^A = e^{[\nabla_2, F]} \Gamma^A \]  
(25)
is canonical.

In the finite-dimensional case, we can show that if the matrix \( E^{ij}_2 \) is nondegenerate, then the transformations generated by \( \exp [\nabla_2, F] \) cover all transformations conserving the antibracket or, what is the same, \( \nabla_2 \).

5.2 Master equation, G transformations

The nilpotent local operator \( \nabla_2 \), \( \varepsilon(\nabla_2) = 1 \) defines the master equation for local functionals \( S(\Gamma) \)
\[ \nabla_2 e^{\frac{i}{\bar{\hbar}} S} = 0 \]
or
\[ (S, S) = \nabla_2 S^2 = 0. \]  
(26)

This equation is nontrivial only for Bose functionals. It is evident that this equation is invariant under the canonical transformations of functionals induced by local canonical transformations of fields \( \Gamma \) that conserve the antibracket. In particular, if \( S \) is a solution of the master equation, then
\[ S' = e^{[\nabla_2, F]} S, \quad \varepsilon(F) = 1, \]  
(27)
is a solution too. Transformation (27) can be represented as
\[ e^{\frac{i}{\bar{\hbar}} S} \rightarrow e^{\frac{i}{\bar{\hbar}} S'} = e^{[\nabla_2, F]} e^{\frac{i}{\bar{\hbar}} S}. \]  
(28)
The invariance of the master equation, as well as of the antibracket, under canonical transformations (28) results from the fact that for a nilpotent operator \( \nabla_2 \), the operator \( [\nabla_2, F] \) commutes with \( \nabla_2 \). On the other hand, any operator of the form \( [\nabla_2, \nabla] \), with an arbitrary \( \nabla \), commutes with \( \nabla_2 \). It follows that any transformation of the form
\[ e^{\frac{i}{\bar{\hbar}} S} \rightarrow e^{\frac{i}{\bar{\hbar}} S'} = e^{[\nabla_2(h), \nabla(h)]} e^{\frac{i}{\bar{\hbar}} S}, \]  
(29)
with an arbitrary local \( \nabla \), \( \varepsilon(\nabla) = 1 \), takes a local solution \( S \) of the master equation to a local solution \( S' \) (independent of \( \hbar \)). We call the transformations of the form (29) the gauge (G) transformations. Note that to represent G transformations as an operator acting directly on \( S \) is a rather complicated problem. It is worth noting also that the G transformation operators \( \exp [\nabla_2, \nabla] \) form an associative algebra.

However, it turns out that for the solutions of the master equation, any G transformation (29) reduces to the canonical transformation (28):
with some local Fermi functional \( F(\Gamma) \). The fact that the functional \( F \) does exist can be proved, for example, as follows. Instead of the operator \( \nabla \), we consider the operator \( \alpha \nabla \). Then the relation
\[
e^{\alpha [\nabla (\bar{h}), \nabla (\bar{h})]} e^{\frac{i}{\hbar} S} = e^{[\nabla (\bar{F}(\alpha)), \nabla (\bar{h})]} e^{\frac{i}{\hbar} S}
\]
will hold true if, for example, \( F(\alpha) \) is a solution of the equation
\[
\int_0^1 d\beta e^{\beta [\nabla (\bar{h})]} \frac{\partial F(\alpha)}{\partial \alpha} = S(\alpha), \quad F(0) = 0,
\]
and the local independent of \( \hbar \) Fermi functional \( S(\alpha) \) is defined by
\[
\hbar \nabla (\bar{h}) e^{\alpha [\nabla (\bar{h}), \nabla (\bar{h})]} e^{\frac{i}{\hbar} S} = S(\alpha) e^{\alpha [\nabla (\bar{h}), \nabla (\bar{h})]} e^{\frac{i}{\hbar} S}.
\]
The formal perturbative in \( \alpha \) local solution of Eq.(30) for \( F(\alpha) \) exists.

The answer to the question whether \( G \) transformations act on the solutions transitively is determined by the properties of the operator \( \nabla_2 \), i.e. of its coefficient functions \( E^{A_1 A_2} \), and by additional conditions on \( S \) (it is clear that \( S=0 \) is a solution anyway).

In the framework of the standard BV formulation of gauge theories, the metric \( \tilde{E}^{A_1 A_2} \) is nonsingular whereas the fields \( \Gamma \) are Darboux coordinates for the metric
\[
\Gamma^A(x) = \{ \Phi^a(x), \Phi^*_a(x) \}, \quad \varepsilon(\Phi^*_a) = \varepsilon(\Phi^a) + 1,
\]
\[
\nabla_2 = \int dx (-1)^{e(\Phi^a)} \frac{\delta}{\delta \Phi^a(x)} \frac{\delta}{\delta \Phi^*_a(x)}.
\]
The fields \( \Phi^a \) are divided into two groups: \( \Phi^a = \{ \varphi^i, C^\alpha \} \), where \( \varphi^i \) are the fields of an initial classical theory, \( C^\alpha \) are ghost fields. The fields \( \Phi^*_a \) (called antifields) are divided analogously. The ghost number \( g\hbar \) is ascribed to all variables: \( g\hbar (\varphi^i) = 0, \ g\hbar (C^\alpha) = 1, \ g\hbar (\Phi^*_a) = -g\hbar (\Phi^a) - 1 \). The solution \( S \) of master equation (26) is sought in the form of the power series expansion in \( C \):
\[
S = \mathcal{S} + \sum_{n=1} S^{(n)}, \quad S^{(n)} = O(C^n), \quad \varepsilon(S) = g\hbar(S) = 0,
\]
\( \mathcal{S} \) is a local action of the classical theory. The action \( \mathcal{S} \) is assumed to have a gauge symmetry, i.e. satisfies the gauge identities:
\[
R^i_\alpha(x, \varphi(x), \partial_\mu \varphi(x), \ldots, \partial_\mu) \frac{\delta \mathcal{S}}{\delta \varphi^i(x)} = 0,
\]
where \( R^i_\alpha(x, \varphi(x), \partial_\mu \varphi(x), \ldots, \partial_\mu) \) is a polynomial in \( \partial_\mu \) with local in \( x \) coefficients. In addition, \( \mathcal{S} \) satisfies some regularity conditions (see [6] and references therein), whose essence is that in an arbitrary set of the extremals \( \delta \mathcal{S}/\delta \varphi^i(x) \) and their space derivatives, it is possible to distinguish linearly (with local coefficients) dependent and linearly independent elements, and the only relations between them are linear combinations (with local coefficients) of the gauge identities and their space derivatives. Under these regularity conditions, it was proved [6] that the master equation does have the local solutions of form (32) and the arbitrariness in these solutions is described by
canonical transformations (27), i.e. the canonical transformations act on the space of the solutions of form (32) transitively.

Let us compare the expressions given above with the analogous formal expressions in the BV formalism [4]. The analog of the nilpotent operator $\nabla_2$ is the canonical second order differential operator $\Delta_2$ (we restrict ourselves to the case where the variables $\Gamma^A(x)$ are the Darboux coordinates for the metric $E_{12}$, i.e., the differential expressions for $\nabla_2$ and $\Delta_2$ have form (31)). Formally, the action in quantum field theory must satisfy the quantum master equation

$$\Delta_2 e^{i\hbar S} = 0$$

or

$$\frac{1}{2} (S, S) = i\hbar \Delta_2 S.$$  (33)

Equation (26) and formal equation (33) differ by the terms proportional to $\sim \hbar\delta(0)$, i.e. master equation (26) is a quasiclassical approximation to master equation (33), and the solutions of master equation (26) are the zero order in $\bar{\hbar}$ (quasiclassical) approximations to the solutions of quantum master equation (33).

A formal operator of the $G$ transformations is

$$e^{[\Delta_2, F]} = De^{[\nabla_2, F]} = e^{\frac{i}{\hbar}(-i\hbar \ln D)} e^{[\nabla_2, F]},$$  (34)

where

$$D = \frac{D(\Phi')}{D(\Phi)} = \left(\frac{D(\Gamma')}{D(\Gamma)}\right)^{1/2}, \quad \Gamma'^A = e^{[\nabla_2, F]} \Gamma^A.$$

A functional $S'$ related to a functional $S$ by

$$e^{\frac{i}{\hbar} S'} = e^{[\Delta_2, F]} e^{\frac{i}{\hbar} S},$$

is a solution of the quantum master equation if $S$ is. Thus, the operator $\exp \left([\nabla_2, F]\right)$ is a quasiclassical approximation to formal operator (34).

Another fact is also worth noting. As a standard, the following boundary conditions are imposed on a solution of master equation (33):

$$S|_{\bar{\hbar}=0, C=0} = S.$$  

In this case, although operators (34) take a solution of the master equation to a solution but do not act on the solutions transitively [4].

### 6 $Sp(2)$ master equation

In this section, we consider the $Sp(2)$ master equation [8] that describes the $Sp(2)$-symmetric generalization of the BV formalism. The full set of variables $\Gamma^\Sigma$ of the theory (in this section we change a little the notations for the indices at the variables $\Gamma$) is divided into the groups $\Phi^A, \Phi^*_a, \Phi_A, a, b, c = 1, 2, \bar{\hbar}$, every variable being a function of the coordinates $x$. In its turn, the variables $\Phi^A$ are divided into groups $\Phi^A = (\varphi^i, C^{ab}, B^a)$,
where \( \varphi \) are the classical theory variables, \( C^{ab} \) are the ghost and antighost fields, and \( B^a \) are the gauge introducing fields. The Grassmann parity is ascribed to all fields:

\[
\varepsilon(\Phi_A) = \varepsilon(\Phi_A^*) = \varepsilon_A, \quad \varepsilon(C^{ab}) = \varepsilon(c) = \varepsilon_A + 1, \quad \varepsilon(B^a) = \varepsilon_a, \quad \varepsilon(C^{ab}) = \varepsilon_a + 1,
\]

as well as the new ghost number \( \text{ ngh } \): \( \text{ ngh}(\varphi) = 0, \quad \text{ ngh}(C^{ab}) = 1, \quad \text{ ngh}(B^a) = 2, \quad \text{ ngh}(\Phi) = -2 \). In the Sp(2) formalism the effective action \( S(\Phi, \Phi^*, \bar{\Phi}) \) satisfies the Sp(2) master equation

\[
\frac{1}{2}(S, S)^a + \int dx \varepsilon^{ab} \frac{\delta}{\delta \Phi(x)} S = 0,
\]

and the boundary condition

\[
S|_{\Phi^*_B=\Phi=0} = S(\varphi),
\]

where \( S(\varphi) \) is the initial classical action with a gauge symmetry, \( R_a \delta S/\delta \varphi_i = 0 \). In (35) \((.,.)^a\) denotes the doublet of the antibrackets

\[
(S_1, S_2)^a \equiv \int dx \left( \frac{\delta S_1}{\delta \Phi(x)} \frac{\delta S_2}{\delta \Phi^*_A(x)} - \frac{\delta S_1}{\delta \Phi^*_A(x)} \frac{\delta S_2}{\delta \Phi(x)} \right).
\]

We introduce the doublet of the operators

\[
\nabla^a = \nabla^a_2 + \nabla^a_1, \quad \varepsilon(\nabla^a) = 1,
\]

\[
\nabla^a_2 = \int dx (-1)^{\varepsilon_A} \frac{\delta}{\delta \Phi^*_A(x)} \frac{\delta}{\delta \Phi^*_A(x)} = \varepsilon(\nabla^a_2) = 1,
\]

\[
\nabla^a_1 = i \int dx \varepsilon^{ab} \Phi_A(x) \frac{\delta}{\delta \Phi_A(x)} = \varepsilon(\nabla^a_1) = 1.
\]

These operators are nilpotent

\[
\nabla^a_2 \nabla^b_2 + \nabla^b_2 \nabla^a_2 = 0, \quad \nabla^a_1 \nabla^b_1 + \nabla^b_1 \nabla^a_1 = 0, \quad \nabla^a_2 \nabla^b_1 + \nabla^b_2 \nabla^a_1 + \nabla^c_1 \nabla^b_1 + \nabla^b_1 \nabla^a_2 = 0,
\]

\[
\nabla^a_1 \nabla^b_1 + \nabla^b_1 \nabla^a_1 = 0.
\]

The doublet of the antibrackets is related to the doublet of the operators \( \nabla^a_2 \) by

\[
(S_1, S_2)^a = (-1)^{\varepsilon(S)} \nabla^a_2(S_1 S_2).
\]

Using the operators \( \nabla^a \), we can write the Sp(2) master equation in the following equivalent form:

\[
\nabla^a(h) e^\frac{i}{\hbar} S = 0.
\]

Under the same assumptions that were described in subsec. 5.2, in [9] it was shown that the Sp(2) master equation has local solutions, as well as the arbitrariness in the general solution was found. We shall show that the formalism considered here is a natural apparatus for this purpose.
We introduce the class of operators, which we call the gauge (G) transformation operators,

\[ K = e^U, \quad U = \frac{1}{2} \varepsilon_{ab}[\nabla^b, [\nabla^a, \nabla]], \quad \varepsilon(U) = 0, \tag{37} \]

where \( \nabla \) is an arbitrary local operator, \( \varepsilon(\nabla) = 0 \). It is evident that the operator \( U \) has the property

\[ [\nabla^a, U] = 0, \tag{38} \]

and, as a consequence, the G transformation operator commutes with the operators \( \nabla^a \)

\[ [\nabla^a, K] = [\nabla^a, e^U] = 0. \tag{39} \]

It follows from (38) that the operators \( U \) form the Lie algebra

\[ [U^{(1)}, U^{(2)}] = \frac{1}{2} \varepsilon_{ab}[\nabla^b, [\nabla^a, \nabla^3]] = U^{(3)}, \quad \nabla^3 = [U^{(1)}, \nabla^{(2)}]. \]

Then, according to the Baker–Hausdorff–Dynkin formula, we have

\[ e^{U^{(1)}}e^{U^{(2)}} = e^{U^{(3)}}, \]

therefore, the G transformation operators form the associative algebra. It follows from the results of sec. 3 that if \( S \) is a local functional and independent of \( \hbar \), then the functional \( S' \) defined by

\[ e^{\frac{i}{\hbar}S'} = K(\hbar)e^{\frac{i}{\hbar}S}, \]

\[ K(\hbar) = e^{U(\hbar)}, \quad U(\hbar) = \frac{1}{2} \varepsilon_{ab}[\nabla^b(\hbar), [\nabla^a(\hbar), \nabla(\hbar)]], \]

is local and independent of \( \hbar \). In addition, if a functional \( S \) satisfies \( Sp(2) \) master equation (35), then because of (39), the functional \( S' \) satisfies the \( Sp(2) \) master equation too

\[ \nabla^a(\hbar)e^{\frac{i}{\hbar}S'} = 0. \]

For the functionals \( S \) satisfying the \( Sp(2) \) master equation the relation

\[ e^{\frac{i}{\hbar}S'(\alpha)} = e^{\frac{i}{\hbar}S_{ab}[\nabla^b(h), [\nabla^a(h), \nabla(h)]]}e^{\frac{i}{\hbar}S} = e^{\beta \frac{i}{\hbar}S_{ab}[\nabla^b(h), [\nabla^a(h), \nabla\alpha(h)]]}e^{\frac{i}{\hbar}S} \]

\[ \nabla_0(h) = \frac{1}{\hbar}Y, \]

holds true, where \( Y \) is some local functional that is independent of \( \hbar, \varepsilon(Y) = 0 \). Indeed, if instead of the operator \( \nabla \), we consider the operator \( \alpha \nabla^a \), then we can show that the relation

\[ e^{\frac{i}{\hbar}S'_{ab}(\alpha)} = e^{\frac{i}{\hbar}S_{ab}[\nabla^b(h), [\nabla^a(h), \nabla(h)]]}e^{\frac{i}{\hbar}S} = e^{\beta \frac{i}{\hbar}S_{ab}[\nabla^b(h), [\nabla^a(h), \nabla\alpha(h)]]}e^{\frac{i}{\hbar}S} \]

holds true, for instance, for \( Y(\alpha) \) satisfying the equation

\[ \int_0^1 d\beta e^{\frac{\beta}{\hbar}S_{ab}[\nabla^b(h), [\nabla^a(h), \frac{1}{\hbar}Y(\alpha)]]} \frac{\partial Y(\alpha)}{\partial \alpha} e^{-\beta \frac{i}{\hbar}S_{ab}[\nabla^b(h), [\nabla^a(h), \frac{1}{\hbar}Y(\alpha)]]} e^{\frac{i}{\hbar}S'_{ab}(\alpha)} = \]

\[ = \nabla(h)e^{\frac{i}{\hbar}S'_{ab}(\alpha)} = \frac{i}{\hbar}S_{ab}(\alpha)e^{\frac{i}{\hbar}S'_{ab}(\alpha)}(Y(0) = 0, \]

\[ (Y(\alpha)) = 0. \]
The perturbative in $\alpha$ solution of the latter equation exists. Thus, if we consider the action of the $G$ transformation operators (37) on the functionals $\exp (\frac{i}{\hbar}S)$, where $S$ obeys $Sp(2)$ master equation, we can restrict ourselves to the $G$ transformation operators with $\nabla = \nabla_0$.

Note that any change of the fields $\Phi^A$,

$$\Phi^A \rightarrow \Phi'^A = e^{T^B(\Phi)\delta_B \Phi^A},$$

can be extended to the $G$ transformation in the following sense:

$$e^{\frac{i}{\hbar} \varepsilon_{ab}[\nabla_a, [\nabla^a, \Phi^A]]} G(\Gamma) \big|_{\Phi^A = \Phi = 0} = e^{T^A(\Phi)\delta_A g(\Phi)} = g(\Phi').$$

$$g(\Phi) = G(\Gamma) \big|_{\Phi^* = \Phi = 0}.$$

Let us now proceed to describing the arbitrariness in the solution of the $Sp(2)$ master equation. Let $S = S + \sum_{n=1} S(n)$, $S(n) \sim C^k B^{n-k}$ be a solution of the $Sp(2)$ master equation with

$$S(1) = \int dx \left(C^{a\alpha} R^i_{\alpha\beta} \phi^* + \varepsilon^{ab} C^{*}_{ab} B^{\alpha} + B^a R^i_{\alpha\beta} \phi^* (-1)^{\epsilon_i + \epsilon_a} \right).$$

It was shown in [9] that if two solutions $S$ and $S_1$ coincide up to the $n$-th order of the series expansion in variables $C^{a\alpha}, B^\alpha$, then their difference $\delta S_{(n+1)}$ in the $(n+1)$-th order

$$S_1 - S = \delta S_{(n+1)} + O(C^k B^{n+2-k}), \quad \delta S_{(n+1)} \sim C^k B^{n+1-k},$$

can be represented in the form

$$\delta S_{(n+1)} = \frac{1}{2} \varepsilon_{ab} \omega^b \omega^a X_{(n+1)}, \quad X_{(n+1)} \sim C^k B^{n+1-k},$$

where

$$\omega^a = \int dx \left( -(-1)^{\epsilon_i} L_i \frac{\delta}{\delta \phi^*} - (-1)^{\epsilon_a} R^i_{\alpha\beta} \phi^* \frac{\delta}{\delta C^{*}_{ab}} + \right.$$

$$+((-1)^{\epsilon_i} R^i_{\alpha\beta} \phi^* + \varepsilon^{ab} C^{*}_{ab} \phi^*) \frac{\delta}{\delta B^{\alpha}} - (-1)^{\epsilon_a} \varepsilon^{ab} B^a \frac{\delta}{\delta C^{*}_{ab}} + \varepsilon^{ab} \phi^* \frac{\delta}{\delta \Phi^A} \right),$$

$$L_i(x) \equiv \delta S / \delta \phi^i(x),$$

$X_{(n+1)}$ is a local functional, $\varepsilon(X_{(n+1)}) = 0$. Let us introduce the operators $W^a$

$$W^a = (S, . . )^a + \int dx \varepsilon^{ab} \phi^* \frac{\delta}{\delta \Phi^A}.$$

The following relation is valid

$$\frac{1}{2} \varepsilon_{ab} \omega^b \omega^a X_{(n+1)} = \frac{1}{2} \varepsilon_{ab} W^b W^a X_{(n+1)} \big|_{n+1}. $$

Taking into account (35), (36), we can directly verify the validity of the equality

$$\frac{i}{\hbar} \left( \frac{1}{2} \varepsilon_{ab} W^b W^a X_{(n+1)} \right) e^{\frac{i}{\hbar} S} = -\frac{i}{\hbar} \varepsilon_{ab} \left[ \nabla_b(h), [\nabla^a(h), \frac{1}{\hbar} X_{(n+1)}] \right] e^{\frac{i}{\hbar} S}. $$
With (39) taken into account, we obtain that the action $S'_1$,

$$e^{\pm S'_1} = e^{\pm \varepsilon_{ab} [\bar{\nabla}^a(h),[\bar{\nabla}^b(h),\frac{1}{\hbar} X_{(n+1)}]]} e^{\pm S_1},$$

satisfies the $Sp(2)$ master equation and differs from $S$ starting from the $(n + 2)$-th order. As far as all solutions coincide in the zero order in $C$ and $B$, then using the induction method, we finally obtain the following result: the general solution $\tilde{S}$ of the $Sp(2)$ master equation can be constructed from a particular solution $S$ with the help of the G transformation operator

$$e^{\pm \tilde{S}} = K(h)e^{\pm S},$$

i.e., the G transformations act on the solutions of the $Sp(2)$ master equation transitively, and we can restrict ourselves to the G transformation operators of form (37) with $\nabla = \nabla_0$.

As in subsec. 5.2, we can verify that the operators $\bar{\nabla}^a$ and the G transformation operators (37) are the quasiclassical approximations to the formal nilpotent operators $\bar{\Delta}^a$ and to the G transformation operators of form (B7) with $U = \frac{1}{2} \varepsilon_{ab} [\bar{\Delta}^b, [\bar{\Delta}^a, \Delta]]$, respectively, the differential expressions for the canonical operators $\bar{\Delta}^a$ and $\Delta$ coinciding with the corresponding expressions for the operators $\bar{\nabla}^a$ and $\nabla$. In this case the quantum action must satisfy, formally, the quantum $Sp(2)$ master equation

$$\Delta^a(h)e^{\pm \tilde{S}} = 0,$$

or

$$\frac{1}{2} (S, S)^a + \int dx \varepsilon^{ab} \Phi^b_A(x) \frac{\delta}{\delta \Phi^a(x)} S = i\hbar \Delta^a e^{\pm \tilde{S}},$$

the differential expression for $\Delta^a$ coinciding with the differential expression for $\nabla^a$.

**Acknowledgments**

The work of S. S. S. is supported by Russian Foundation for Basic Researches under the Grant RFBR–99–02–17916 and by Human Capital and Nobility Program of the European Community under the Projects INTAS 96–0308, RFBR–INTAS–95–829. I. V. T. is partially supported by Russian Foundation for Basic Researches under the Grant RFBR–99–01–00980 and by Human Capital and Nobility Program of the European Community under the Project RFBR–96–0308.

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