SPACE QUASICONFORMAL MAPPINGS AND NEUMANN EIGENVALUES IN FRACTAL DOMAINS

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Abstract. We study the variation of the Neumann eigenvalues of the $p$-Laplace operator under quasiconformal perturbations of space domains. This study allows to obtain lower estimates of the Neumann eigenvalues in fractal type domains. The suggested approach is based on the geometric theory of composition operators in connections with the quasiconformal mapping theory.

1. Introduction

Let $\Omega$ be a bounded domain in the $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. We consider the Neumann eigenvalue problem for the $p$-Laplace operator, $p > 1$,

\[
\begin{align*}
-\text{div}(|\nabla u|^{p-2}\nabla u) &= \mu_p |u|^{p-2}u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

This classical formulation is correct for bounded Lipschitz domains. The weak statement of this spectral problem is as follows: a function $u$ solves the previous problem, if and only if, $u \in W^{1,p}(\Omega)$ and

\[
\int_{\Omega} (|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)) \, dx = \mu_p \int_{\Omega} |u(x)|^{p-2}u(x)v(x) \, dx
\]

for all $v \in W^{1,p}(\Omega)$ and is correct for any bounded domains $\Omega$ in $\mathbb{R}^n$.

The spectral stability estimates for elliptic operators were discussed, for example, in [4, 5, 6, 7, 8, 21, 23]. The main result of the article gives estimates of the variation of the first non-trivial Neumann eigenvalue $\mu_p$ under quasiconformal perturbations of domains:

**Theorem A.** Let a bounded domain $\Omega$ in $\mathbb{R}^n$ be a $(p, p)$-Sobolev-Poincaré domain, $1 < p < \infty$. Assume that there exists a Lipschitz $K$-quasiconformal homeomorphism $\varphi: \Omega \to \tilde{\Omega}$ of a domain $\Omega$ onto a bounded domain $\tilde{\Omega}$ such that

\[
Q_p(\Omega) := \text{ess sup}_{x \in \Omega} |D\varphi(x)|^\frac{p-n}{p} < \infty.
\]
Then
\[
\frac{1}{\mu_p(\Omega)} \leq KQ_p^p(\Omega)\|D\varphi|^n | L_\infty(\Omega)\| \cdot \frac{1}{\mu_p(\Omega)}.
\]

The lower estimates of the first non-trivial Neumann eigenvalues \(\mu_p(\Omega)\) in basic domains \(\Omega\) which can be union of convex domains will be considered in Section 3. Note, that if \(\mu_p(\Omega)\) is calculated, then Theorem A gives lower estimates of \(\mu_p(\Omega)\) in a large class of (non)convex domains in the terms of quasiconformal geometry of \(\Omega\). The lower estimates of first non-trivial Neumann eigenvalues for convex domains in terms of Euclidean diameters and isoperimetric inequalities were intensively studied in the last decades (see, for example, [1, 10, 11, 25]).

As an example we consider the non-convex star-shaped domain \(\Omega_\delta = \Omega_1 \cup \Omega_2\), \(\delta > 0\) given, \(\alpha = \delta(\sqrt{3} - 1)/2\), where
\[
\Omega_1 = \{(x', x_n) \in \mathbb{R}^n : \max\{|x'| - \delta, -\alpha\} < x_n < \alpha\}
\]
and
\[
\Omega_2 = \{(x', x_n) \in \mathbb{R}^n : -\alpha < x_n < \min\{\delta - |x'|, \alpha\}\}.
\]

Let \(n = 3\). Then, \(\Omega = \Omega_1 \cup \Omega_2\) is a \((\delta(\sqrt{3} - 1)/2, \delta\sqrt{2})\)-John domain and there exists a \(K\)-quasiconformal mapping \(\varphi : \mathbb{R}^3 \to \mathbb{R}^3\) such that \(\varphi(B^3(0, 1)) = \Omega_\delta\). By Theorem A (Example C) for \(p > 3\)
\[
\mu_p(\Omega_\delta) \geq \frac{\sqrt{2(4 - \sqrt{6} + \sqrt{2})}}{\delta^2(4 + \sqrt{6} + \sqrt{2})^{1/4}(\sqrt{4 + \sqrt{6} - \sqrt{2}} + \sqrt{4 - \sqrt{6} - \sqrt{2}})^2} \mu_p(B_3(0, 1)).
\]

The proof of Theorem A is based on estimates of constants in the Sobolev-Poincaré inequalities.

Let \(1 \leq r, p \leq \infty\). A bounded domain \(\Omega\) in \(\mathbb{R}^n\) is called a \((r, p)\)-Sobolev-Poincaré domain, if for any function \(f \in L_p^r(\Omega)\), the \((r, p)\)-Sobolev-Poincaré inequality
\[
\inf_{c \in \mathbb{R}} \|f - c | L_r(\Omega)\| \leq B_{r,p}(\Omega)\|\nabla f | L_p(\Omega)\|
\]
holds.

It is well known that the constant \(B_{r,p}(\Omega)\) depends on the geometry of \(\Omega\), see, for example, [24]. We prove

**Theorem B.** Let a bounded domain \(\Omega\) in \(\mathbb{R}^n\) be a \((r, q)\)-Sobolev-Poincaré domain, \(1 < q \leq r < \infty\). Suppose that there exists a \(K\)-quasiconformal homeomorphism \(\varphi : \Omega \to \Omega\) of a domain \(\Omega\) onto a bounded domain \(\Omega\), so that \(\varphi\) belongs to the Sobolev space \(L_{\alpha}^1(\Omega)\) for some \(\alpha > n\). Suppose additionally that
\[
Q_{p,q}(\Omega) := \left(\int_{\Omega} |D\varphi|^\frac{(p-n)\alpha}{\alpha - n} \, dx\right)^\frac{\alpha}{\alpha - n} < \infty.
\]
for some \(p \in [q, r)\). Then for \(1 \leq s = \frac{\alpha - n}{\alpha} r\) in the domain \(\tilde{\Omega}\) the \((s, p)\)-Sobolev-Poincaré inequality holds and
\[
B_{s,p}(\tilde{\Omega}) \leq K \frac{q}{r} \min_{1 \leq q < p} \left(Q_{p,q}(\Omega)\|D\varphi | L_\alpha(\Omega)\|^\frac{q}{p}\right) \cdot B_{r,q}(\Omega),
\]
where $B_{r,q}(\Omega)$ is the best constant in the $(r,q)$-Sobolev-Poincaré inequality in the domain $\Omega$.

The suggested method of investigation is based on the geometric theory of composition operators \cite{27, 29} and its applications to the Sobolev type embedding theorems \cite{13, 14}.

In the recent works \cite{2, 3, 15, 16, 17} the spectral stability problem and lower estimates of Neumann eigenvalues in planar domains were considered. In \cite{18} spectral estimates in space domains using the theory of weak $p$-quasiconformal mappings were obtained.

2. Quasiconformal Composition Operators and Neumann Eigenvalues

2.1. Notation. For any domain $\Omega$ in $\mathbb{R}^n$ and any $1 \leq p < \infty$ we consider the Lebesgue space of measurable functions with the finite norm

$$\|f|_{L^p(\Omega)}\| := \left(\int_{\Omega} |f(x)|^p \, dx\right)^{1/p} < \infty.$$  

The space $L^\infty(\Omega)$ is the space of essentially bounded Lebesgue measurable functions with the finite norm

$$\|f|_{L^\infty(\Omega)}\| := \inf\{b : |f(x)| \leq b \text{ for almost every } x \in \Omega\} < \infty.$$  

We define the Sobolev space $W^{1,p}(\Omega)$, $1 \leq p < \infty$, as a Banach space of weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f|_{W^{1,p}(\Omega)}\| := \left(\int_{\Omega} |f(x)|^p \, dx\right)^{1/p} + \left(\int_{\Omega} |\nabla f(x)|^p \, dx\right)^{1/p}.$$  

We define also the homogeneous seminormed space $L^1_p(\Omega)$ of weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f|_{L^1_p(\Omega)}\| := \left(\int_{\Omega} |\nabla f(x)|^p \, dx\right)^{1/p}.$$  

We recall that any element of $L^1_p(\Omega)$ is in $L^{p,\text{loc}}(\Omega)$, that is, the space of functions which are locally integrable to the power $p$ in $\Omega$, \cite{24}.

A mapping $\varphi : \Omega \to \mathbb{R}^n$ is weakly differentiable on $\Omega$, if its coordinate functions have weak derivatives on $\Omega$. Hence the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x, \varphi)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ of the matrix $D\varphi(x)$ is the norm of the corresponding linear operator. We will use the same notation for this matrix and the corresponding linear operator.

We recall that any mapping $\varphi : \Omega \to \tilde{\Omega}$ is called $K$-quasiconformal if $\varphi \in W^{1,n}_{\text{loc}}(\Omega)$ and there exists a constant $K < \infty$ such that

$$|D\varphi(x)|^n \leq K|J(x, \varphi)| \text{ for almost all } x \in \Omega.$$  

A mapping $\varphi : \Omega \to \mathbb{R}^n$ possesses the Luzin $N$-property if an image of any set of measure zero has measure zero. Note that any Lipschitz mapping possesses the Luzin $N$-property.
2.2. Composition Operators on Lebesgue Spaces. The following theorem about composition operators on Lebesgue spaces is well known, we refer to [29].

**Theorem 2.1.** Let a homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) between two domains \( \Omega \) and \( \tilde{\Omega} \) be weakly differentiable. Then the composition operator

\[
\varphi^* : L_s(\tilde{\Omega}) \to L_s(\Omega), \quad 1 \leq s < \infty,
\]

is bounded, if and only if, \( \varphi^{-1} \) possesses the Luzin \( N \)-property and

\[
\left( \int_\Omega \left| J(y, \varphi^{-1}) \right|^{\frac{r}{s}} \, dy \right)^{\frac{s}{r}} = K < \infty, \quad \text{for } 1 \leq s < r < \infty,
\]

\[
\left| J(y, \varphi^{-1}) \right|^{\frac{1}{s}} = K < \infty, \quad \text{for } 1 \leq s = r < \infty.
\]

The norm of the composition operator is \( \| \varphi^* \| = K \).

2.3. Composition Operators on Sobolev Spaces. By the standard definition functions of \( L^1_p(\Omega) \) are defined only up to a set of measure zero, but they can be redefined quasi-everywhere i.e. up to a set of \( p \)-capacity zero. Indeed, every function \( u \in L^1_p(\Omega) \) has a unique quasicontinuous representation \( \tilde{u} \in L^1_p(\Omega) \). A function \( \tilde{u} \) is termed quasicontinuous if for any \( \varepsilon > 0 \) there is an open set \( U_\varepsilon \) such that the \( p \)-capacity of \( U_\varepsilon \) is less then \( \varepsilon \) and on the set \( \Omega \setminus U_\varepsilon \) the function \( \tilde{u} \) is continuous (see, for example [20, 24]).

Let \( \Omega \) and \( \tilde{\Omega} \) be domains in \( \mathbb{R}^n \). We say that a homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq \infty,
\]

by the composition rule \( \varphi^*(f) = f \circ \varphi \), if for any function \( f \in L^1_p(\tilde{\Omega}) \), the composition \( \varphi^*(f) \in L^1_q(\Omega) \) is defined quasi-everywhere in \( \Omega \) and there exists a constant \( K_{p,q}(\varphi; \Omega) < \infty \) such that

\[
\| \varphi^*(f) \|_{L^1_q(\Omega)} \leq K_{p,q}(\varphi; \Omega) \| f \|_{L^1_p(\tilde{\Omega})}.
\]

The main result of [27] gives the analytic description of composition operators on Sobolev spaces (we also refer to [29]) and asserts that

**Theorem 2.2.** [27] A homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) between two domains \( \Omega \) and \( \tilde{\Omega} \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q < p < \infty,
\]

if and only if, \( \varphi \in W^{1,1}_{\text{loc}}(\Omega) \), \( \varphi \) has finite distortion, and

\[
K_{p,q}(\varphi; \Omega) = \left( \int_\Omega \left( \frac{|D\varphi(x)|^p}{|J(x, \varphi)|} \right)^{\frac{q}{p}} \, dx \right)^{\frac{p-1}{pq}} < \infty.
\]

We prove the following property of quasiconformal homeomorphisms:

**Lemma 2.3.** Let \( \varphi : \Omega \to \tilde{\Omega} \) be a \( K \)-quasiconformal homeomorphism. Then \( \varphi \) generates by the composition rule \( \varphi^*(f) = f \circ \varphi \) a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q < p < \infty,
\]
if and only if,
\[ Q_{p,q}(\Omega) = \left( \int_{\Omega} |D\varphi|^\frac{p-n+q}{p-q} \, dx \right)^{\frac{p-q}{p-n}} < \infty. \]

**Proof.** Let \( f \in L^1_p(\tilde{\Omega}) \) be a smooth function. Then, because quasiconformal homeomorphisms possess the Luzin \( N \) and \( N^{-1} \) properties, the composition \( g = \varphi^*(f) \) is well defined almost everywhere in \( \Omega \) and belongs to \( L^1_{1,\text{loc}}(\Omega) \). Hence, using Theorem 2.2 we obtain
\[
\|g|_{L^q_1(\Omega)}\| \leq \left( \int_{\Omega} \left( \frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-n}} \, dx \right)^{\frac{p-q}{p-n}} \|f|_{L^1_p(\tilde{\Omega})}\|
\]
\[
= \left( \int_{\Omega} \left( \frac{|D\varphi(x)|^n|D\varphi(x)|^{p-n}}{|J(x,\varphi)|} \right)^{\frac{q}{p-n}} \, dx \right)^{\frac{p-q}{p-n}} \|f|_{L^1_p(\tilde{\Omega})}\|
\]
\[
\leq K^{\frac{q}{p-n}} Q_{p,q}(\Omega) \|f|_{L^1_p(\tilde{\Omega})}\|.
\]

By approximating an arbitrary function \( f \in L^1_p(\tilde{\Omega}) \) by smooth functions we obtain the required inequality.

Now, let the composition operator \( \varphi^*: L^1_p(\tilde{\Omega}) \rightarrow L^1_q(\Omega) \), \( 1 \leq q < p < \infty \), be bounded. Then, using the Hadamard inequality:
\[ |J(x,\varphi)| \leq |D\varphi(x)|^n \text{ for almost all } x \in \Omega, \]
and Theorem 2.2, we have
\[
Q_{p,q}(\Omega) = \left( \int_{\Omega} |D\varphi|^\frac{p-n+q}{p-q} \, dx \right)^{\frac{p-q}{p-n}} \leq \left( \int_{\Omega} \left( \frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-n}} \, dx \right)^{\frac{p-q}{p-n}} < +\infty.
\]

□

**Corollary 2.4.** Let \( \varphi: \Omega \rightarrow \tilde{\Omega} \) be a \( K \)-quasiconformal homeomorphism such that
\[
Q_{p,q}(\Omega) = \left( \int_{\Omega} |D\varphi|^\frac{p-n+q}{p-q} \, dx \right)^{\frac{p-q}{p-n}} < \infty.
\]

Then
\[
\|\varphi^* f|_{L^1_q(\Omega)}\| \leq K^{\frac{q}{p-n}} Q_{p,q}(\Omega) \|f|_{L^1_p(\tilde{\Omega})}\| \text{ for any } f \in L^1_p(\tilde{\Omega}).
\]

2.4. **Sobolev-Poincaré inequalities.** Let \( 1 < r, p \leq \infty \). We recall that a bounded domain \( \Omega \) in \( \mathbb{R}^n \) is called a \((r,p)\)-Sobolev-Poincaré domain, if for any function \( f \in L^1_p(\Omega) \), the \((r,p)\)-Sobolev-Poincaré inequality
\[
\inf_{c \in \mathbb{R}} \|f - c|_{L^r(\Omega)}\| \leq B_{r,p}(\Omega) \|\nabla f|_{L^p(\Omega)}\|
\]
holds.

**Theorem B.** Let a bounded domain \( \Omega \) in \( \mathbb{R}^n \) be a \((r,q)\)-Sobolev-Poincaré domain, \( 1 < q \leq r < \infty \). Suppose that there exists a \( K \)-quasiconformal homeomorphism
\( \varphi : \Omega \to \tilde{\Omega} \) of a domain \( \Omega \) onto a bounded domain \( \tilde{\Omega} \), such that \( \varphi \) belongs to the Sobolev space \( L^\alpha_\alpha(\Omega) \) for some \( \alpha > n \). Suppose additionally that
\[
Q_{p,q}(\Omega) = \left( \int_{\Omega} |D\varphi|^{\frac{m-n}{m}} dx \right)^{\frac{m-n}{mp}} < \infty.
\]
for some \( p \geq q \). Then for \( 1 \leq s = \frac{\alpha-n}{\alpha} r \) in the domain \( \tilde{\Omega} \) the \((s, p)\)-Sobolev-Poincaré inequality holds and
\[
B_{s,p}(\tilde{\Omega}) \leq K_{\frac{1}{r}} \min_{1 \leq q < p} \left( Q_{p,q}(\Omega) \| D\varphi | L^\alpha_\alpha(\Omega) \| \right) \cdot B_{r,q}(\Omega),
\]
where \( B_{r,q}(\Omega) \) is the best constant in the \((r, q)\)-Sobolev-Poincaré inequality in the domain \( \Omega \).

\textbf{Proof.} By the assumptions there exists a quasiconformal homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \). Then, using the change of variable formula we obtain:
\[
\inf_{c \in \mathbb{R}} \left( \int_{\tilde{\Omega}} |f(y) - c|^s dy \right)^{\frac{1}{s}} = \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(\varphi(x)) - c| |J(x, \varphi)| dx \right)^{\frac{1}{s}}.
\]
Now we choose \( r = \alpha s / (\alpha - n) \). Then, using the Hölder inequality we have:
\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(\varphi(x)) - c|^s |J(x, \varphi)| dx \right)^{\frac{1}{s}} \\
\leq \left( \int_{\Omega} |J(x, \varphi)|^{\frac{s}{s-s}} dx \right)^{\frac{s-s}{s}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r dx \right)^{\frac{1}{r}} \\
\leq \left( \int_{\Omega} |D\varphi(x)|^{\frac{m-n}{m}} dx \right)^{\frac{m-n}{mp}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r dx \right)^{\frac{1}{r}} \\
= \left( \int_{\Omega} |D\varphi(x)|^{\alpha} dx \right)^{\frac{\alpha}{\alpha r}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r dx \right)^{\frac{1}{r}}.
\]
Hence, applying the Sobolev-Poincaré inequality in the \((r, q)\)-Sobolev-Poincaré domain \( \Omega \)
\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r dx \right)^{\frac{1}{r}} \leq B_{r,q}(\Omega) \left( \int_{\Omega} |\nabla g(x)|^q dx \right)^{\frac{1}{q}}
\]
we have
\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^s dy \right)^{\frac{1}{s}} \leq \| D\varphi \| L^\alpha_\alpha(\Omega) \| g \| L^1_q(\Omega).
\]
By Lemma 2.3 we have
\[
\| g \| L^1_q(\Omega) \leq K_{\frac{1}{r}} Q_{p,q}(\Omega) \| f \| L^1_p(\tilde{\Omega}).
\]
Therefore
\[
\inf_{c \in \mathbb{R}} \left( \int_{\mathcal{O}} |f(y) - c|^s \, dy \right)^{1/s} \leq K^{1/s} Q_{p,q}(\mathcal{O}) \|D\varphi| L_\alpha(\mathcal{O})\| \cdot B_{r,q}(\mathcal{O}) \left( \int_{\mathcal{O}} |\nabla f|^p \, dy \right)^{1/p}.
\]

\[ \square \]

As an application we consider the Neumann spectral problem for the \( p \)-Laplace operator
\[
\begin{cases}
- \text{div}(|\nabla u|^{p-2} \nabla u) = \mu_p |u|^{p-2} u & \text{in } \mathcal{O}, \\
\frac{\partial u}{\partial n} \bigg|_{\partial \mathcal{O}} = 0.
\end{cases}
\]

By the generalized version of Rellich-Kondrachov compactness theorem (see, for example, [24], [19], [9]) and the \((r, p)\)-Sobolev-Poincaré inequality for \( r > p \) the embedding operator
\[
i : W^{1,p}(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})
\]
is compact in domains which satisfy conditions of Theorem B.

Hence, the first nontrivial Neumann eigenvalue \( \mu_p(\mathcal{O}) \) can be characterized as
\[
\mu_p(\mathcal{O}) = \min \left\{ \int \frac{|\nabla u(x)|^p \, dx}{\int |u(x)|^p \, dx} : u \in W^{1,p}(\mathcal{O}) \setminus \{0\}, \int |u|^{p-2} u \, dx = 0 \right\}.
\]

Moreover, \( \mu_p(\mathcal{O})^{-\frac{1}{p}} \) is the best constant \( B_{p,p}(\mathcal{O}) \) (for example we refer to [1]) in the following Poincaré inequality
\[
\inf_{c \in \mathbb{R}} \| f - c | L^p(\mathcal{O}) \| \leq B_{p,p}(\mathcal{O}) \| \nabla f | L^p(\mathcal{O}) \|, \quad f \in W^{1,p}(\mathcal{O}).
\]

So from Theorem B in the case \( s = p \) we obtain

**Theorem 2.5.** Let a bounded domain \( \mathcal{O} \) be in \( \mathbb{R}^n \) be a \((r, q)\)-Sobolev-Poincaré domain, \( 1 < q \leq r < \infty \). Assume that there exists a \( K \)-quasiconformal homeomorphism \( \varphi : \mathcal{O} \to \tilde{\mathcal{O}} \) of a domain \( \mathcal{O} \) onto a bounded domain \( \tilde{\mathcal{O}} \), so that \( \varphi \) belongs to the space \( L^1_\alpha(\mathcal{O}) \) for \( \alpha = nr/(r-p) \), \( r > p \). Suppose that
\[
Q_{p,q}(\mathcal{O}) = \left( \int_{\mathcal{O}} |D\varphi|^{\frac{(p-n)\alpha}{p-q}} \, dx \right)^{\frac{p-q}{p-n}} < \infty.
\]

for some \( p > q \). Then
\[
\frac{1}{\mu_p(\mathcal{O})} \leq K \min_{1 \leq q < p} \left( Q_{p,q}^p(\mathcal{O}) \|D\varphi| L_\alpha(\mathcal{O})\| \right)^n \cdot B_{r,q}(\mathcal{O}),
\]

where \( B_{r,q}(\mathcal{O}) \) is the best constant in the \((r, q)\)-Sobolev-Poincaré inequality in the domain \( \mathcal{O} \).

In the limit case, when a quasiconformal mapping \( \varphi : \mathcal{O} \to \tilde{\mathcal{O}} \) is Lipschitz homeomorphism, we have:
Theorem A. Let a bounded domain $\Omega$ in $\mathbb{R}^n$ be a $(p, p)$-Sobolev-Poincaré domain, $1 < p < \infty$, and there exists a Lipschitz $K$-quasiconformal homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ of a domain $\tilde{\Omega}$ onto a bounded domain $\tilde{\Omega}$ such that

$$Q_p(\Omega) = \text{ess sup}_{x \in \Omega} |D\varphi(x)| \frac{\|D\varphi\|_{L^p(\tilde{\Omega})}}{\mu_p(\Omega)} < \infty.$$ 

Then

$$\frac{1}{\mu_p(\Omega)} \leq KQ_p^p(\Omega) \|D\varphi\|_{L^p(\tilde{\Omega})} \cdot \frac{1}{\mu_p(\Omega)}.$$ 

Let us give an illustration of Theorem A.

Example C. Consider the domain $\Omega_\delta = \Omega_1 \cup \Omega_2$, $\delta > 0$ given, $\alpha = \delta(\sqrt{3}-1)/2$, where

$$\Omega_1 = \{(x', x_n) \in R^n : \max\{|x'| - \delta, -\alpha\} < x_n < \alpha\}$$

and

$$\Omega_2 = \{(x', x_n) \in R^n : -\alpha < x_n < \min\{\delta - |x'|, \alpha\}\}.$$ 

Let $n = 3$. Then, $\Omega_\delta = \Omega_1 \cup \Omega_2$ is a $(\delta(\sqrt{3}-1)/2, \delta\sqrt{2})$-John domain and there exists a $K$-quasiconformal mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(B^3(0, 1)) = \Omega_\delta$.

The domain $\Omega_\delta$ is starshaped with respect to the origin. By [12] with the angle $\alpha = \pi/12$ we obtain

$$K^2 \leq \frac{2}{4 - \sqrt{6}} \frac{\sqrt{4 + \sqrt{6} + \sqrt{2}}}{4 + \sqrt{6} - \sqrt{2}}$$

and

$$|D\varphi(x)|^3 \leq \delta^3 \left(\frac{\sqrt{4 + \sqrt{6} + \sqrt{2}}}{\sqrt{6} - \sqrt{2}}\right)^3.$$

By Theorem A for $p > 3$

$$\frac{1}{\mu_p(\Omega_\delta)} \leq \frac{\sqrt{2(4 + \sqrt{6} + \sqrt{2})}^{1/4}}{\sqrt{4 - \sqrt{6} - \sqrt{2}}} \left(\frac{\sqrt{4 + \sqrt{6} - \sqrt{2}} + \sqrt{4 - \sqrt{6} + \sqrt{2}}}{\sqrt{6} - \sqrt{2}}\right)^{3/4} \frac{\mu_p(B^3(0, 1))^{1/p}}{2 \delta}.$$ 

If $p = 2$ then the first non-trivial Neumann eigenvalue in the unit ball is

$$\mu_2(B^n(0, 1)) = p_{n/2},$$

where $p_{n/2}$ denotes the first positive zero of the function $(t^{1-n/2} J_{n/2}(t))'.$ In particular, if $n = 2$, we have $p_1 = j'_{1, 1} \approx 1.84118$ where $j'_{1, 1}$ denotes the first positive zero of the derivative of the Bessel function $J_1$. And $p_{3/2}$ denotes the first positive zero of the function $(t^{1/2} J_{3/2}(t))'.$

If $p > 2$, then by [10]

$$\mu_p(B^n(0, 1)) \geq \left(\frac{\pi_p}{2}\right)^p$$

where

$$\pi_p = 2 \int_0^{(p-1)\frac{2}{p}} \frac{dt}{(1 - t^p/(p-1))^{1/p}} = 2\pi \frac{(p-1)\frac{2}{p}}{p \sin(\pi/p)},$$
3. Poincaré inequalities for Whitney complexes

The aim of this section is to study the upper estimates of the Poincaré constants $B_{p,p}(W)$ for the $(p,p)$-Poincaré-Sobolev inequalities:

$$\inf_{c \in \mathbb{R}} \| f - c | L_p(W) \| \leq B_{p,p}(W) \| \nabla f | L_p(W) \|$$

for functions $f$ of the space $L^1_p(W)$ defined in the fractal type domains what we call the Whitney complex $W$ in $\mathbb{R}^n$.

**Lemma 3.1.** [22] Let $1 \leq p < \infty$. Let $A$ be a measurable subset of a domain $\Omega$ in $\mathbb{R}^n$ such that $|A| > 0$ and let $f$ be in $L^p(\Omega)$. Then for each $c \in \mathbb{R}$

$$\| f - f_A | L_p(\Omega) \| \leq 2 \left( \frac{|\Omega|}{|A|} \right)^{1/p} \| f - c | L_p(W) \| .$$

**Lemma 3.2.** Suppose that $Q_1, Q_2, Q_3$ are bounded convex domains. If $|Q_1 \cap Q_2| > 0$ and $|Q_2 \cap Q_3| > 0$ and $Q_1$ and $Q_3$ are disjoint we call the set $A = Q_1 \cup Q_2 \cup Q_3$ a Whitney triple.

Now we use Lemma 3.2 for Whitney triples.

**Lemma 3.3.** Let $A$ in $\mathbb{R}^n$ be a Whitney triple. Then,

$$\int_A | f(x) - f_A |^p dx \leq B_{p,p}(A) \int_A | \nabla f(x) |^p dx ,$$

where

$$B_{p,p}(A) \leq 2^{4p-1} \left( \frac{|Q_1 \cup R_2|}{|R_2|} \right) \left( \frac{|Q_1|}{|Q_1 \cap R_2|} \right) B_{p,p}(Q_1)$$

$$+ 2^{4p-1} \left( \frac{|Q_1 \cup R_2|}{|Q_1 \cap R_2|} + \frac{|Q_3 \cup R_2|}{|Q_3 \cap R_2|} \right) B_{p,p}(R_2)$$

$$+ 2^{4p-1} \left( \frac{|Q_3 \cup R_2|}{|R_2|} \right) \left( \frac{|Q_3|}{|Q_3 \cap R_2|} \right) B_{p,p}(Q_3) .$$
Proof. Using Lemma 3.1

\[ \| f - f_A \|_{L^p(A)} \leq 2^p \| f - f_{R_2} \|_{L^p(A)} \]

\[ \leq 2^p \left( \int_{Q_1 \cup R_2} |f(x) - f_{R_2}|^p dx + \int_{R_2 \cup Q_3} |f(x) - f_{R_2}|^p dx \right) \]

\[ \leq 2^{2p} \frac{|Q_1 \cup R_2|}{|R_2|} \int_{Q_1 \cup R_2} |f(x) - f_{Q_1 \cup R_2}|^p dx \]

\[ + 2^{2p} \frac{|R_2 \cup Q_3|}{|R_2|} \int_{R_2 \cup Q_3} |f(x) - f_{R_2 \cup Q_3}|^p dx. \]

By Lemma 3.2

\[ \| f - f_A \|_{L^p(A)} \leq 2^{2p} \frac{|Q_1 \cup R_2|}{|Q_1 \cap R_2|} \left( |Q_1| \| B_{p,p}^p(Q_1) \| \int_{Q_1} |\nabla f(y)|^p dy + |R_2| \| B_{p,p}^p(R_2) \| \int_{R_2} |\nabla f(y)|^p dy \right) \]

\[ + 2^{2p} \frac{|R_2 \cup Q_3|}{|R_2 \cap Q_3|} \left( |R_2| \| B_{p,p}^p(R_2) \| \int_{R_2} |\nabla f(y)|^p dy + |Q_3| \| B_{p,p}^p(Q_3) \| \int_{Q_3} |\nabla f(y)|^p dy \right). \]

\[ \square \]

Definition 3.4. If \( A_j \) are Whitney triples and \( |A_j \cap A_{j+1}| > 0 \) we call the set \( W = \bigcup_{j=1}^{\infty} A_j \) a Whitney complex.

Theorem 3.5. Let \( W = \bigcup_{i=1}^{\infty} A_j \), be a Whitney complex. Then

\[ \int_W |f(x) - f_{A_i}|^p dx \leq 2^{p-1} \sum_{j=1}^{\infty} B_{p,p}^p(A_j) \int_{A_j} |\nabla f(x)|^p dx \]

\[ + 2^{2p} \sum_{j=1}^{\infty} \int_{A_j} \left( \sum_{\mu=1}^{j-1} \frac{B_{p,p}^p(A_\mu)}{|A_\mu \cap A_{\mu+1}|} \int_{A_\mu} |\nabla f(x)|^p dx \right) dy. \]

Proof. We have to estimate the integral

\[ \int_W |f(x) - f_{A_i}|^p dx \leq 2^{p-1} \sum_{j=1}^{\infty} \int_{A_j} |f(x) - f_{A_j}|^p dx + 2^{p-1} \sum_{j=1}^{\infty} \int_{A_j} |f_{A_j} - f_{A_i}|^p dx. \]

The integral

\[ \int_{A_j} |f(x) - f_{A_i}|^p dx \]

was handled in Lemma 3.3. We estimate the integral

\[ \int_{A_i} |f_{A_i} - f_{A_i}|^p dx. \]
Let us write $f_{A_j} = f_j$. By the triangle inequality

$$|f_i - f_1| \leq \sum_{k=1}^{i-1} |f_k - f_{k+1}|$$

where

$$|f_k - f_{k+1}|^p \leq \frac{2^{p-1}}{|A_k \cap A_{k+1}|} \left( \int_{A_k} |f(x) - f_k|^p \, dx + \int_{A_{k+1}} |f(x) - f_{k+1}|^p \, dx \right).$$

Hence,

$$\int_{A_i} |f_{A_i} - f_{A_1}|^p \, dx \leq \int_{A_i} \left( \sum_{k=1}^{i-1} |f_k - f_{k+1}|^p \right) \, dx$$

$$\leq \int_{A_i} \left( \sum_{k=1}^{i-1} \frac{2^{1-1/p}}{|A_k \cap A_{k+1}|^{1/p}} \left( \int_{A_k} |f(x) - f_k|^p \, dx + \int_{A_{k+1}} |f(x) - f_{k+1}|^p \, dx \right)^{1/p} \right)^p \, dx.$$

Thus, by Lemma 3.3

$$\int_{A_i} |f_{A_i} - f_{A_1}|^p \, dx \leq \int_{A_i} \left( \sum_{k=1}^{i} \left( \frac{2^p}{|A_k \cap A_{k+1}|} \int_{A_k} |f(x) - f_k|^p \, dx \right)^{1/p} \right)^p \, dy$$

$$\leq \int_{A_i} \left( \sum_{k=1}^{i} \frac{2^p B_{p,p}(A_k)}{|A_k \cap A_{k+1}|} \int_{A_k} |\nabla f(x)|^p \, dx \right)^p \, dy$$

Hence,

$$\int_W |f(x) - f_{A_1}|^p \, dx \leq 2^{p-1} \sum_{j=1}^{\infty} B_{p,p}(A_j) \int_{A_j} |\nabla f(x)|^p \, dx$$

$$+ 2^{2p} \sum_{j=1}^{\infty} \int_{A_j} \sum_{k=1}^{j-1} \frac{B_{p,p}(A_k)}{|A_k \cap A_{k+1}|} \int_{A_k} |\nabla f(x)|^p \, dx \, dy.$$

Theorem 3.6. Let

$$W = \bigcup_{i=1}^{\infty} A_j,$$

be a Whitney complex. Then

$$\int_W |f(x) - f_{A_1}|^p \, dx \leq 2^{p-1} \sum_{j=1}^{\infty} B_{p,p}(A_j) \int_{A_j} |\nabla f(x)|^p \, dx$$

$$+ 2^{2p} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{B_{p,p}(A_i)}{|A_i \cap A_{i+1}|} \int_{A_i} |\nabla f(x)|^p \, dx \, dy.$$

Proof. There is a reformulation of the second term on the right hand side in Theorem 3.5.

The following extension theorem is needed for fractal type examples.
Theorem 3.7. Let $W$ be a fractal tree in $\mathbb{R}^n$, $n \geq 2$, and let $\Delta_0$ be the starting domain for the tree $W$. Let

$$W = \bigcup_k \Delta_k$$

where $\Delta_k$ are the elements of the tree. When the element $\Delta_k$ is fixed, let

$$(\Delta, \Delta_0, \Delta_k) = \left\{ \text{all the elements } \Delta \text{ of the tree to where we go from } \Delta_0 \text{ through } \Delta_k \right\}.$$ 

Let $1 \leq p < \infty$. Let $B_{p,p}(\Delta_k)$ be the Poincaré constant of the element of $\Delta_k$ of the tree. Then,

$$\int_W |f(x) - f_{\Delta_0}|^p \, dx \leq 2^{p-1} \sum_{\Delta_k} B_{p,p}(\Delta_k) \int_{\Delta_k} |\nabla f(x)|^p \, dx +$$

$$2^{p-1} \sum_{\Delta_k} \sum_{\Delta \in (\Delta_0, \Delta_k, \Delta)} \# \{ \text{steps from } \Delta_0 \text{ to } \Delta \}^{p-1} |\Delta|^{p-1}|\Delta_k| \int_{\Delta_k} |\nabla f(x)|^p \, dx.$$ 

Proof. A modification of the proof of Theorem 4.4 in [22].

This theorem allows estimate the Poincaré constants in fractal domains:

Example 3.8. Let $W$ be a fractal tree which is a modification of the snowflake definition given in [26].

1. The starting point is a triangle $\Delta_0$ with the sidelength $a$.

2. We form three new triangles $\Delta_1$ with the sidelength $a/3$, it is the 1st step.

3. We form for the previous three triangles $\Delta_1$ each two new triangles $\Delta_2$ with the sidelength $a/3^2$, so altogether $3 \times 2$ new triangles $\Delta_2$, it is the 2nd step.

4. We form for the previous $3 \times 2^{j-1}$ triangles $\Delta_j$, to each $\Delta_j$ two new triangles with the sidelength $a/3^{j+1}$ so altogether $3 \times 2^j$ new triangles $\Delta_{j+1}$, it is the $j + 1$ step.

So, when we have $\Delta_j$ from the step $j$, its sidelength is $a/3^j$ and its area is

$$\frac{\sqrt{3}a^2}{4 \cdot 3^j}$$

and

$$\# \left\{ \Delta_j : |\Delta_j| = \frac{\sqrt{3}a^2}{2^j \cdot 3^j} \right\} = 3 \cdot 2^{j-1}.$$ 

We denote by $\Delta_j^*$ the triangle which is obtained from $\Delta_j$ by extending $\Delta_j$ to inside $\Delta_{j-1}$ so that

$$c_1 |\Delta_j| \leq |\Delta_{j-1} \cap \Delta_j^*| \leq c_2 |\Delta_j|.$$
We have to estimate the last term in Theorem 3.7:
\[
\sum_{j=1}^{\infty} \# \left\{ \Delta_j : |\Delta_j| = \frac{\sqrt{3}a_j^2}{2^j \cdot 32^j} \right\} \sum_{\Delta_i \in (\Delta_0, \Delta_j, \Delta_i)} j^{p-1}2^{-j}|\Delta_j^i| \frac{B_{p,p}(\Delta_j^i)}{|\Delta_j^i|} \int_{\Delta_j^i} |\nabla u(x)|^p \, dx
\]
\[
\leq \sum_{j=1}^{\infty} 3 \cdot 2j^{-1} \sum_{i=j}^{\infty} j^{p-1}2^{-j} \frac{2^i}{3^{2i}} \frac{3^p}{\pi_p} \frac{a^p}{3^{j(p-2)}} \int_{\Delta_j^i} |\nabla u(x)|^p \, dx
\]
\[
\leq \sum_{j=1}^{\infty} 3 \cdot 2j^{-1} 2^{-j} \sum_{i=j}^{\infty} j^{p-1}2^{-j} \frac{2^i}{3^{2i}} \frac{3^p}{\pi_p} \frac{a^p}{3^{j(p-2)}} \int_{\Delta_j^i} |\nabla u(x)|^p \, dx.
\]
Since,
\[
\sum_{i=j}^{\infty} j^{p-1} \left( \frac{2}{3^2} \right)^i \leq \left( \frac{2}{3^2} \right)^{(1-\epsilon)}
\]
where \( \epsilon > 0 \) is an arbitrary small number,
\[
\sum_{j=1}^{\infty} 3 \cdot 2j^{-1} 2^{-j} \sum_{i=j}^{\infty} j^{p-1}2^{-j} \frac{2^i}{3^{2i}} \frac{3^p}{\pi_p} \frac{a^p}{3^{j(p-2)}}
\]
\[
\leq \frac{2^{p-1}}{\pi_p} a^p \sum_{j=1}^{\infty} \left( \frac{2}{3^2} \right)^{j(1-\epsilon)} \leq \frac{1}{3^{(p-2)}} = \frac{2^{p-1}}{\pi_p} a^p \sum_{j=1}^{\infty} \frac{2^j}{3^{(p-2)}}
\]
where the sum converges, since \( \epsilon > 0 \) can be taken arbitrarily small.

Results of Section 2 allow obtain variation of Poincaré constants under quasi-conformal perturbations fractal type domains.

**Theorem 3.9.** Let \( \tilde{W} \) be an image of the Whitney complex \( W \) under a Lipschitz \( K \)-quasiconformal homeomorphism \( \varphi : W \to \tilde{W} \) such that
\[
Q_p(W) = \text{ess sup}_{x \in W} |D\varphi(x)|^{\frac{p-n}{p}} < \infty.
\]

Then
\[
\frac{1}{\mu_p(W)} \leq KQ_p^p(W) \| |D\varphi|^n \| L_{\infty}^p(W) \| \cdot \frac{1}{\mu_p(W)}.
\]

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