On Surfaces of finite Chen-type

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Abstract
We investigate some relations concerning the first and the second Beltrami operators corresponding to the fundamental forms \( I, II, III \) of a surface in the Euclidean space \( E^3 \) and we study surfaces which are of finite type in the sense of B.-Y. Chen with respect to the fundamental forms \( II \) and \( III \).

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Introduction. Let \( M \) be a (connected) surface in a Euclidean 3-space \( E^3 \) referred to any system of coordinates \( u^1, u^2 \), and let \( n: M \to S^2 \subset E^3 \) be its Gauss map. Denote by \( \Delta_I \) the Laplace operator corresponding to the first fundamental form

\[
I = g_{ij} \, du^i \, du^j.
\]

Then the position vector \( x = x(u^1, u^2) \) and the mean curvature \( H \) of \( M \) in \( E^3 \) satisfy the relation

\[
\Delta_I x = -2H \, n,
\]
(with sign convention such that $\Delta^I = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$). From (1) we know the following two facts [7]

- $M$ is minimal if and only if all coordinate functions of $x$ are eigenfunctions of $\Delta^I$ with eigenvalue 0.
- $M$ lies in an ordinary sphere $S^2$ if and only if all coordinate functions of $x$ are eigenfunctions of $\Delta^I$ with a fixed nonzero eigenvalue.

In 1983 B.-Y. Chen introduced the notion of Euclidean immersions of finite type [1]. In terms of B.-Y. Chen theory a surface $M$ is said to be of finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian $\Delta^I$. Then the above facts can be stated as follows

- $M$ is minimal if and only if $M$ is of null 1-type.
- $M$ lies in an ordinary sphere $S^2$ if and only if $M$ is of 1-type.

B.-Y. Chen posed in [2], [3] the problem of classifying the finite type surfaces in $E^3$. Many authors have been concerned with this problem. For a more detailed report and a recent survey we refer the reader to [3] and [4]. Up to this moment the only known finite Chen-type surfaces in $E^3$ are portions of spheres, circular cylinders and minimal surfaces.

In this paper we consider the second Beltrami operators $\Delta^{II}$ and $\Delta^{III}$ corresponding to the second and to the third fundamental form respectively of a surface $M$ in $E^3$. Our purpose is to find surfaces of finite Chen-type with respect to the second and to the third fundamental form. In paragraph 1 formulas for $\Delta^{II}x$, $\Delta^{III}x$, $\Delta^{II}n$ and $\Delta^{III}n$ are established. In paragraphs 2-4 we state and prove the main results.

Throughout this paper we make full use of the tensor calculus. The reader is referred to [5] for symbols and formulas.

1. We consider a surface $M$ which does not contain parabolic points and we denote by $b_{ij}$ the components of the second and by $e_{ij}$ the components of the third fundamental form of $M$. Furthermore, when we deal with the second fundamental form, we suppose that $M$ consists only of elliptic points. Let $\varphi$ and $\psi$ be two sufficient differentiable functions on $M$. Then the first differ-
ential parameter of Beltrami with respect to the fundamental form \( J = I, II, III \) of \( M \) is defined by [5]

\[
\nabla^J(\varphi, \psi)_i = a^i_j \varphi_j \psi, \\

\]

where \( \varphi_j = \frac{\partial \varphi}{\partial u^j} \) and \( (a^i_j) \) denotes the inverse tensor of \( (g_{ij}), (b_{ij}) \) and \( (e_{ij}) \) for \( J = I, II \) and \( III \) respectively.

We first prove the following relations:

\[
\nabla^I(\varphi, x) + \nabla^{II}(\varphi, n) = 0, \\

\]

\[
\nabla^{II}(\varphi, x) + \nabla^{III}(\varphi, n) = 0. \\

\]

For the proof of (1.2) we use (1.1) and the Weingarten equations [5, p.128]

\[
\nabla^{II}(\varphi, n) = b_{ij} \varphi_j n_{i} = -b_{ij} \varphi_j b_{jk} g^{km} x_{m} = -g^{im} \varphi_i x_{m} = -\nabla^{I}(\varphi, x), \\

\]

being (1.2).

We have similarly

\[
\nabla^{III}(\varphi, n) = e_{ij} \varphi_j n_{i} = -e_{ij} \varphi_j e_{jk} b^{km} x_{m} = -b^{im} \varphi_i x_{m} = -\nabla^{II}(\varphi, x), \\

\]

which is (1.3).

The second differential parameter of Beltrami with respect to the fundamental form \( J = I, II, III \) of \( M \) is defined by [5]

\[
\Delta^J \varphi = -a^i_j \nabla^J_i \varphi_j, \\

\]

where \( \varphi \) is a sufficient differentiable function, \( \nabla^J_i \) is the covariant derivative in the \( u^i \) direction with respect to the fundamental form \( J \) and \( (a^i_j) \) stands as in definition (1.1) for the inverse tensor of \( (g_{ij}), (b_{ij}) \) and \( (e_{ij}) \) for \( J = I, II \) and \( III \) respectively.

We first compute \( \Delta^J x \) for \( J = II \) and \( III \).
Recalling the equations [5, p.128]

\[ \nabla_j^{\mu} x_\mu = -\frac{1}{2} b^{km} \nabla_k^{\mu} b_{ij} x_m + b_{ij} n, \]

and inserting these into

\[ \Delta^{\mu} x = -b^{ij} \nabla_j^{\mu} x_\mu \]

we have

(1.5) \[ \Delta^{\mu} x = \frac{1}{2} b^{ij} b^{km} \nabla_k^{\mu} b_{ij} x_m - b^{ij} b_{ij} n. \]

By using the Mainardi-Codazzi equations [5, p.21]

(1.6) \[ \nabla_k^{\mu} b_{ij} - \nabla_i^{\mu} b_{jk} = 0, \]

relation (1.5) becomes

(1.7) \[ \Delta^{\mu} x = \frac{1}{2} b^{ij} b^{mk} \nabla_i^{\mu} b_{jk} x_m - 2 n. \]

We consider the Christoffel symbols of the second kind with respect to the first and second fundamental form respectively

\[ \Gamma^{ij}_k: = \frac{1}{2} g^{km} (-g_{ij/m} + g_{im/j} + g_{jm/i}), \quad \Pi^{ij}_k: = \frac{1}{2} b^{kr} (-b_{ij/m} + b_{im/j} + b_{jm/i}), \]

and we put

(1.8) \[ T^{ij}_k: = \Gamma^{ij}_k - \Pi^{ij}_k. \]

It is known that [5, p.125]

(1.9) \[ T^{ij}_k = -\frac{1}{2} b^{km} \nabla_m^{\mu} b_{ij}, \]

and

(1.10) \[ \Gamma^{ij}_j: = \frac{g_{ij}}{2g}, \quad \Pi^{ij}_j: = \frac{b_{ij}}{2b}, \]

where \( g: = \det(g_{ij}) \) and \( b: = \det(b_{ij}) \).
For the Gauss curvature $K$ of $M$ we know that

$$K = \frac{b}{g}.$$ 

From this, by using (1.8), (1.9) and (1.10), we find

$$\frac{K_{ij}}{K} = \frac{b_{ik}}{b} \frac{g_{jk}}{g} = 2(\Pi_{ij}^j - \Gamma_{ij}^j) = -2T_{ij} = b^{ij} \nabla_i b_{kj},$$

and therefore

$$\frac{1}{2K} \nabla^{iii}(K,n) = \frac{1}{2K} \epsilon^{ks} K_i^n s_j = \frac{1}{2} \epsilon^{ks} b^{ij} \nabla_i b_{kj} n_j.$$ 

Hence, because of (1.4), we find

$$\frac{1}{2K} \nabla^{iii}(K,n) = -\frac{1}{2} \epsilon^{ks} b^{ij} \epsilon_{sr} b^{sm} \nabla^l b_{kj} x^n = -\frac{1}{2} b^{ij} b^{km} \nabla^l b_{kj} x^m.$$ 

By simple substitution in (1.7) we obtain

(1.12) $$\Delta^{ii}x = -\frac{1}{2K} \nabla^{iii}(K,n) - 2n.$$ 

We use now the equations [5, p.128]

$$\nabla^m_j x_i = -b^{km} \nabla^n_i b_{ij} x_k + b^n_i n$$

and

$$\Delta^{iii}x = -\epsilon^{ij} \nabla^m_i x_j$$

to get

(1.13) $$\Delta^{iii}x = \epsilon^{ij} b^{km} \nabla^n_i b_{ij} x_k - \epsilon^{ij} b^n_i n.$$ 

We consider the Christoffel symbols of the second kind with respect to the third fundamental form

$$\Lambda^{k}_{ij} = \frac{1}{2} \epsilon^{km} (-\epsilon_{ij/m} + \epsilon_{im/j} + \epsilon_{jm/i}),$$

and we put

$$\tilde{T}^{k}_{ij} = \Lambda^{k}_{ij} - \Pi^{k}_{ij}.$$
It is known that [5, p.22]

\( \tilde{T}^k_{ij} = -\frac{1}{2} b^{km} \nabla^\text{III}_m b_{ij} \)

and

\( T^k_{ij} + \tilde{T}^k_{ij} = 0. \)

On the other hand, using Ricci’ s Lemma

\( \nabla^\text{III}_j e^{jk} = 0 \)

and the formula

\( \frac{2H}{K} = e^{jk} b_{jk}, \)

we have

\( \nabla^\text{III}_m (e^{jk} b_{jk}) = e^{jk} \nabla^\text{III}_m b_{jk}. \)

By combining of (1.9), (1.14), (1.15) and (1.18) we obtain

\( e^{ij} b^{km} \nabla^l_m b_{ij} = -2e^{ij} T^k_{ij} = 2e^{ij} \tilde{T}^k_{ij} = -e^{ij} b^{km} \nabla^\text{III}_m b_{ij} = -b^{km} \left( \frac{2H}{K} \right)_{/m} \)

and so

\( e^{ij} b^{km} \nabla^l_m b_{ij} x_k = -b^{km} \left( \frac{2H}{K} \right)_{/m} x_k = -\nabla^\text{III}(\frac{2H}{K} x). \)

From (1.13), (1.17) and (1.19) we find

\( \Delta^\text{III} x = -\nabla^\text{III}(\frac{2H}{K} x) - \frac{2H}{K} n. \)

Finally, using (1.3) we arrive at

\( \Delta^\text{III} x = \nabla^\text{III}(\frac{2H}{K} n) - \frac{2H}{K} n. \)

We focus now our interest to the computation of \( \Delta^J n \) for \( J = \text{II} \) and \( \text{III} \). Firstly we mention the well-known formula

\( \Delta^J n = 2\nabla^J(H,x) + 2(2H^2 - K) n. \)
Next we take into consideration the equations [5, p.128]

\[ \nabla^m_{j \ n} = - \frac{1}{2} b^k_{m \ j} \nabla^m_{n \ k} - e_{ij \ n}, \]

so that

\[ \Delta^{ii} n = -b^i_{ji} \nabla^m_{i \ n}, \]

takes the form

\[ \Delta^{ii} n = \frac{1}{2} b^i_{ji} b^k_{m \ j} \nabla^m_{n \ k} + b^i_{ji} e_{ij \ n}. \]

On account of

\[ 2H = b_{ik} g^{ik} = e_{ik} b^{ik}, \]

and (1.14) we obtain

\[ \Delta^{ii} n = -b^i_{ji} T^{j \ k}_{ij \ n} + 2H n. \]

On use of (1.6), (1.9) and (1.15) we have

\[ \Delta^{ii} n = b^k_{m \ j} T_{mj}^{j \ k \ n} + 2H n. \]

On the other hand using (1.11) we have

\[ \frac{1}{2K} \nabla^{ii}(K, n) = \frac{1}{2K} b^k_{m \ n} K_{m \ k} n = -b^k_{m \ j} T_{mj}^{j \ k \ n}. \]

Inserting this in (1.21) we get in view of (1.2)

\[ \Delta^{ii} n = \frac{1}{2K} \nabla^{ii}(K, x) + 2H n. \]

Finally from [5, p.128]

\[ \nabla^m_{k \ i} n_i = -e_{ik \ n}, \]

we have

\[ \Delta^{iii} n = -e^{ik} \nabla^m_{k \ i} n_i = e^{ik} e_{ik \ n}, \]

so that we conclude
Remark. The first-named author proved in [6] relations (1.20) and (1.24) using Cartan’s method of the moving frame.

2. Outgoing from [1] we say that a surface $M$ is of finite type with respect to the fundamental form $J$, or briefly of finite $J$-type, $J = II, III$, if the position vector $x$ of $M$ can be written as a finite sum of nonconstant eigenvectors of the operator $\Delta^J$, that is if

$$x = c + \sum_{i=1}^{m} x_i, \quad \Delta^J x_i = \lambda_i x_i, \quad i = 1, \ldots, m,$$

where $c$ is a constant vector and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of $\Delta^J$; when there are exactly $k$ nonconstant eigenvectors $x_1, \ldots, x_k$ appearing in (2.1) which all belong to different eigenvalues $\lambda_1, \ldots, \lambda_k$, then $M$ is said to be of $J$-type $k$, and when $\lambda_i = 0$ for some $i = 1, \ldots, k$, then $M$ is said to be of null $J$-type $k$.

Theorem 2.1. A surface $M$ in $E^3$ is of $II$-type 1 if and only if $M$ is part of a sphere.

Proof. Let $M$ be a part of a sphere of radius $r$ centered at the origin. Then

$$\Delta^II x = \frac{2}{r} x.$$

By using (1.12) and (2.2) we find

$$\Delta' x = \frac{2}{r} x.$$

Therefore $M$ is of $II$-type 1 with corresponding eigenvalue $\lambda = \frac{2}{r}$.

Conversely, let $M$ be of $II$-type 1. Then we have

$$\Delta^II x = \lambda x, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0.$$

From (1.1), (1.12) and (2.3), we get

$$2\lambda K x = -e^{ik} K_{ik} n_k - 4K n,$$

whence we find for the inner product of $x$ and $n$
Differentiating covariantly (2.4) in the $u^l$ direction with respect to the third fundamental form $III$ and by using (1.23), (2.4) and Ricci’s Lemma (1.16) we find

\[ 2\lambda K_{j} x + 2\lambda K x_{j} = -e^{ik} \nabla J_i K_{j} n_{ik} - 3K_{j} n - 4K n_{j}. \]

Taking the inner product of both sides of the last equation with $n$, we find in view of (2.5) $K_{j} = 0, j = 1, 2$. Thus $K = \text{const}$. Then (2.4) yields $\lambda x = -2 n$, from which we have $|x| = \left| \frac{2}{\lambda} \right|$ and $M$ is part of a sphere. □

**Theorem 2.2.** The Gauss map of a surface $M$ in $E^3$ is of II-type 1 if and only if $M$ is part of a sphere.

**Proof.** Let $M$ be a part of a sphere of radius $r$ centered at the origin. From (1.22) and (2.2) we get

\[ \Delta^I n = \frac{2}{r} n, \]

and the Gauss map of $M$ is of II-type 1 with eigenvalue $\lambda = \frac{2}{r}$.

Conversely, let the Gauss map of $M$ be of II-type 1. Then we have

\[ \Delta^I n = \lambda n, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0. \]

From this relation and (1.22) we obtain

\[ \frac{1}{2K} \nabla^l (K, x) + (2H - \lambda) n = 0, \]

whence we deduce that $H = \text{const.} \neq 0$ and $K = \text{const.} \neq 0$, because $\nabla^l (K, x)$ is tangent to $M$, while $n$ normal to $M$. So $M$ is part of a sphere. □
3. We turn now to the study of surfaces, which are of finite type with respect to the third fundamental form considering first the minimal surfaces.

For the position vector of a minimal surface $M$ we find from (1.20)

$$\Delta^{III}x = 0x,$$

thus $M$ is of null $III$-type 1.

Conversely, let $M$ be of null $III$-type 1. From (1.20) and (3.1) it follows

$$\nabla^{III}(\frac{2H}{K}, n) - \frac{2H}{K} n = 0.$$

But $\nabla^{III}(\frac{2H}{K}, n)$ is tangent to $M$ while $n$ is normal to $M$. Therefore $\frac{2H}{K}$ vanishes and $M$ is minimal. So we have [6]

**Theorem 3.1.** A surface $M$ in $E^3$ is of null $III$-type 1 if and only if $M$ is minimal.

From (1.24) we get [6]

**Corollary 3.2.** The Gauss map of every surface $M$ in $E^3$ is of $III$-type 1. The corresponding eigenvalue is $\lambda = 2$.

Next we prove

**Theorem 3.3.** A surface $M$ in $E^3$ is of $III$-type 1 if and only if $M$ is part of a sphere.

**Proof.** Let $M$ be a part of a sphere of radius $r$ centered at the origin. By using (1.20) and (2.2) we find

$$\Delta^{III}x = 2x$$

which means that $M$ is of $III$-type 1.

Conversely, let $M$ be a surface of $III$-type 1. Then

$$\Delta^{III}x = \lambda x, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0.$$
From (1.1), (1.20) and (3.2) we get

\[ \lambda x = e^{ik} \left( \frac{2H}{K} \right)_{ij} n_{ik} - \frac{2H}{K} n. \]

We differentiate now covariantly in the \( u^i \) direction with respect to the third fundamental form \( III \). Then we use (1.23) and Ricci’s Lemma (1.16) to deduce

\[ \lambda x_{ij} = e^{ik} \left[ \nabla^{III}_{ij} \left( \frac{2H}{K} \right) \right] n_{ik} - 2 \left( \frac{2H}{K} \right)_{ij} n - \frac{2H}{K} n_{ij}, \]

whence, taking the inner product of both sides with \( n \), we infer that \( \left( \frac{2H}{K} \right)_{jj} = 0 \) for \( j = 1, 2 \). Thus \( \frac{H}{K} = \mu = \text{const.} \). Consequently (3.3) yields \( \lambda x = -2\mu n \), which implies \( |x| = \left| \frac{2\mu}{\lambda} \right| \) and \( M \) is obviously part of a sphere. □

4. Let \( M \) be a surface of \( III \)-type \( k \) with position vector \( x = x(u^1, u^2) \) and \( M^* \) be a parallel surface of \( M \) in (directed) distance \( \rho = \text{const.} \neq 0 \), so that

\[ 1 - 2\rho H + \rho^2 K \neq 0. \]

Then \( M^* \) possesses the position vector

\[ x^* = x + \rho n. \]

Denoting by \( H^* \) the mean and by \( K^* \) the Gauss curvature of \( M^* \) we mention the following relation [5]

\[ \frac{H^*}{K^*} = \frac{H}{K} - \rho. \]

Furthermore the surfaces \( M, M^* \) have common unit normal vector and spherical image. Thus \( III = III^* \) and

\[ \Delta^{III^*} = \Delta^{III}. \]
For the surface \( M \) we have
\[
x = x_1 + x_2 + \ldots + x_k, \quad \Delta^{III} x_i = \lambda_i x_i, \quad i = 1, 2, \ldots, k.
\]

On use of (1.24), (4.1) and (4.3) we observe that the position vector of the surface \( M^* \) can be written as
\[
x^* = x_1 + x_2 + \ldots + x_k + x_{k+1}, \quad x_{k+1} = \rho n,
\]
where
\[
\Delta^{III*} x_i = \lambda_i x_i \quad \text{for} \quad i = 1, 2, \ldots, k,
\]
and
\[
\Delta^{III*} x_{k+1} = 2 x_{k+1}.
\]
This result implies the following [6]

**Theorem 4.1.** Every parallel surface \( M^* \) of a surface \( M \) of finite III-type \( k \) is of III*-type \( k \) or \( k+1 \).

This theorem includes obviously the following special case, which we express as

**Corollary 4.2.** When \( M \) is of null III-type \( k \), then every parallel surface \( M^* \) of \( M \) is of null III*-type \( k \) or \( k+1 \).

We prove now the following

**Theorem 4.3.** Every parallel surface \( M^* \) of a minimal \( M \) is of null III*-type 2.

**Proof.** Let \( M: x = x(u^1, u^2) \) be the minimal surface and \( M^* \) the parallel in distance \( \rho \), \( \rho \in \mathbb{R} \setminus \{0\} \). According to Corollary 4.2 \( M^* \) is of null III*-type 1 or 2.

Let \( M^* \) be of null III*-type 1. Then by Theorem 3.1 \( M^* \) is minimal. Thus there would be in the bundle of the parallel surfaces of \( M \) two minimal surfaces, i.e. \( M \) and \( M^* \). This is possible only in case that \( M \) is part of a plane. But then \( K = 0 \), which we have excluded.

So \( M^* \) is of null III*-type 2. \( \Box \)

Noticing from (4.2) that for the parallel \( M^* \) of the minimal surface \( M \) yields
\[
\frac{H^*}{K^*} = -\rho = \text{const.} \neq 0,
\]

we show the following

**Theorem 4.4.** Let \( M \) be a surface satisfying \( \frac{H}{K} = \mu = \text{const.} \neq 0 \), which is not part of a sphere. Then \( M \) is of null III-type 2.

**Proof.** We consider the parallel surface \( M^* \) of \( M \) in distance \( \rho = \mu = \frac{H}{K} \).

From (4.2) we have
\[
\frac{H^*}{K^*} = 0,
\]

consequently \( M^* \) is minimal. But \( M \) is the parallel of \( M^* \) in distance \(-\mu\).

Therefore \( M \), being parallel of a minimal surface, is of null III-type 2. \( \square \)

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