Amplified emission and lasing in photonic time crystals

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Photic time crystals (PTCs), materials with a dielectric permittivity that is modulated periodically in time, offer new concepts in light manipulation. We study theoretically the emission of light from a radiation source placed inside a PTC and find that radiation corresponding to the momentum bandgap is exponentially amplified, whether initiated by a macroscopic source, an atom, or vacuum fluctuations, drawing the amplification energy from the modulation. The radiation linewidth becomes narrower with time, eventually becoming monochromatic in the middle of the bandgap, which enables us to propose the concept of nonresonant tunable PTC laser. Finally, we find that the spontaneous decay rate of an atom embedded in a PTC vanishes at the band edge because of the low density of photonic states.

Here, we explore the radiation emitted by a radiation source embedded in a PTC. We formulate the quantum theory describing the emission of light by atoms in an excited state and the classical theory of radiating dipoles embedded in PTCs, and show that radiation is always exponentially amplified when associated with the momentum gap and its linewidth becomes narrower with time. This effect allows us to propose nonresonant tunable PTC lasers which draw their energy from the modulation.

Our model consists of a PTC with a source of radiation inside (Fig. 1A). First, we consider an empty PTC medium (no radiation source) and derive the eigenmodes, and then add an arbitrary radiation source. Starting with Maxwell equations with $\varepsilon = \varepsilon(t)$, $\mu = 1$, we can write the wave equation for the magnetic field as follows:

$$\{ \partial_t [\varepsilon(t) \partial_t] + c^2 k^2 \} H_k = 0 \quad (1)$$

where we use a Fourier transform in space because the system is homogeneous and $k$ is a good quantum number. Physically, this means that the eigenmodes are shaped as plane waves, defined by their wave number $k$.

For each $k$, this equation has two Floquet eigenmodes:

$$H_k^{1,2}(t) = H_{0k}(t)e^{\pm i\omega_k^{1,2}t} \quad (2)$$

where $\omega_k$ are Floquet quasifrequencies and $H_{0k}(t)$ is a periodic function in time, constructed from harmonics of the modulation period $T$. We assume that $\varepsilon$ is real (i.e., the medium is lossless), so if $H(t)$ is an eigenmode, i.e., solution of (Eq. 1), then so is $H^*(t)$, which means that $\omega_k^2 = -\omega_0^2 = \omega_0$. Solving for the dispersion relation, we find that the dispersion curve forms a band structure (Fig. 1C). In the bands, the frequency $\omega_k$ is real and the two modes are oscillating at the same frequency, whereas in the gaps, $\omega_0$ has an imaginary part, with one mode exponentially growing with time and the other exponentially decaying. To explore the response of the PTC to the excitation, we add to Eq. (1) a radiation source associated with a temporally dependent current density $j(r)$:

$$\{ \partial_t [\varepsilon(t) \partial_t] + c^2 k^2 \} H_k(t) = 4\pi i \varepsilon_0 \mu_0 n_0^2 \int j(r) \cdot E(r) dr \quad (3)$$

where $j(r)$ is a Fourier $k$ component of current $j(r,t)$. For a point dipole, we assume $j(r,t) = \delta_0(\mathbf{r}) e^{i\omega_0 t}$, where $\omega_0$ is a Heaviside step function denoting that the current is turned on at $t = 0$. Physically, the field $H_k(t)$ is the response of the medium to this current. We can express it in a general form through Green’s function as follows:

$$H_k(t) = 4\pi i \int_{-\infty}^{t} G_k(t,t’)k \times j_k(t’) dt’ \quad (4)$$

Photonic time crystals (PTCs) are dielectric media with a refractive index that experiences large, ultrafast periodic variations in time (1–5). Generally, a wave propagating in a medium undergoing an abrupt change in the refractive index experiences time reflection and time refraction. The time reflection is especially interesting because causality imposes that the wave reflected from the temporal interface propagates backward in space rather than in time (6). Periodic modulation of the refractive index makes these time reflections and time refractions interfere giving rise to bands and bandgaps in the momentum (1, 3, 4). The dispersion relation of PTCs seems analogous to spatial photonic crystals (SPCs), in which the refractive index is periodic in space. However, despite the similarity, there are fundamental differences: SPCs are stationary in time so energy conservation governs most processes, whereas in PTCs, energy is not conserved and causality dictates the dynamics in the system. Conversely, waves propagating in SPCs exchange momentum with the spatial lattice, whereas in spatially homogeneous PTCs, momentum is conserved.

The most important feature of PTCs is the existence of a bandgap in momentum, because the modes associated with this gap have two solutions in which the mode amplitude grows or decays exponentially with time, and both solutions are physical. The exponential growth of the gap modes is nonresonant; it occurs for all wave vectors associated with the momentum gaps, which offers an avenue for amplification of radiation by drawing energy from the modulation. PTCs bear some relation to optical parametric amplifiers, but the latter are resonant phenomena: the frequency of the pump is equal to the sum of the frequencies of the signal and idler and phase matching guarantees conservation of momentum, so only a specific wave is amplified. In contrast, PTCs display a significant momentum gap in which every wave is amplified. For a detailed comparison of PTCs and optical parametric amplifiers, see (7).

Apart from a momentum band structure, the abrupt temporal modulation of the permittivity also opens up new possibilities such as a frequency conversion (2), photon pair creation (8–12), topological temporal edge states (5), antireflection temporal coatings (13), extreme energy transformations (14), interaction with free electrons (15), and amplified localization in temporally disordered media (16). Experimentally, time refraction has already been observed in phototons (17), whereas time reflection has thus far only been observed with water waves (18), acoustic waves (19), and elastic waves (20). This is because of the highly demanding requirements for observing time reflections: The refractive index change should act as a “wall,” analogous to a spatial interface causing Fresnel reflection. For light in the near infrared, the modulation should be at few femtosecond rates with an absolute permittivity change of $\Delta \varepsilon > 0.1$, which is difficult to realize in experimental conditions. However, recent progress with epsilon-near-zero materials (21–24) brings these ideas close to experimental realization (25).

The existence of momentum bands and gaps in a PTC raise fundamental questions about the emission of light by a radiation source embedded in a PTC. An analogous study has led to the discovery of the inhibition of spontaneous emission in the bandgap of SPCs (26), which has had major consequences, such as thresholdless lasing (27, 28).
Fig. 1. Emission by a point dipole embedded in a PTC. (A) Sketch of the PTC, with permittivity varying as \(\varepsilon(t) = \varepsilon_0 + \Delta/2\cos(\Omega t)\), \(\Omega = 2\pi/\tau\) with a dipole antenna inside. The dipole radiation is exponentially amplified with time. (B) Exponential growth of electromagnetic energy associated with the dipole emission for different dipole frequencies \(\omega_0\) and modulation amplitudes \(\Delta\). (C) Complex dispersion relation (band structure) of the PTC for \(\varepsilon_0 = 2, \Delta = 1\). The values of \(\omega_0\) at the bandgap around \(k_g\) are complex, indicating exponentially growing and decaying eigenmodes. (D) Power spectrum of dipole emission versus wave number as it evolves with time. Initially, the point dipole with frequency \(\omega_0\) excites all eigenmodes. \(k_0\) is a wave number of the mode resonantly excited by the dipole with frequency \(\omega_0\): \(\omega_0(k_0) = \omega_0\). The emission linewidth initially occupies all bands and gaps, located at \(k_g\), but eventually, after a short time, the radiation in the gap becomes dominant and narrower with time, reflecting the stronger emission at midgap. In each moment in time, the spectrum is normalized by the total radiation power. The horizontal axes in (C) and (D) coincide.

where

\[
(\partial_t^2 - c^2 k^2)G_k(t,t') = \delta(t-t')
\]

and then express this Green’s function through the eigenmodes from Eq. (2):

\[
G_k(t,t') = \begin{cases} 
0, & t < t' \\
H^2_k(t')H^2_k(t) - H^2_k(t')H^2_k(t) \\
\epsilon(t)(H^2_k(t')/c^2)H^2_k(t') - H^2_k(t')/c^2H^2_k(t'), & t > t'
\end{cases}
\]

Green’s function, \(G_k(t,t')\), represents the response of the medium at time \(t\) to a single homogeneous “flash” at time \(t'\). The detailed derivation of Eq. (6) is provided in (7). A closer look at Eqs. 4 to 6 reveals that, in the medium bandgap where \(\text{Im}(\omega_0) \neq 0\), the medium responds with exponentially growing emission even to the slightest flash of radiation emitted from the current source. This seemingly counterintuitive feature is a consequence of the lack of energy conservation in the medium. In fact, the energy deposited into the exponentially growing gap modes comes not from the source but rather from the external modulation of the medium. The exponentially growing dipole emission is shown in Fig. 1B for various dipole frequencies and permittivity profiles. The growth rate barely depends on the frequency of the dipole but strongly depends on the amplitude of the permittivity modulation. The larger the modulation, the sooner the growth takes place and the steeper it is. The energy spectrum \(k\) of the dipole emission and its evolution with time are depicted in Fig. 1D. The numerical simulation of the fields in Fig. 1, B and D, is described in detail in section 4 of (7). Initially, the point dipole with frequency \(\omega_0\) excites all the eigenmodes with proper wave number \(k_0\) such that \(\omega_0(k_0) = \omega_0\). This is because these waves lie on the dispersion curve and thus are perfectly phase matched. However, within a few oscillation cycles, the gap modes start to dominate even if \(k_0\) does not belong to the gap. These modes are not phase matched with the dipole frequency, but they nevertheless grow exponentially in time, which overshadows any phase matching.

We can understand the exponentially growing response in a PTC through Fig. 2, which shows the difference between excited gap modes in SPC and in PTC, where the excitation in the SPC is by a point source in real space and the excitation in the PTC is by a flash in time. The solution of Eq. (5) should be expressed through two eigenmodes on either side of the excitation point and stitched with two stitching conditions. The physical constraints in both cases reveal which contributions are unphysical and should be removed. In the case of the SPC (Fig. 2A), the solution must obey energy conservation, so only evanescent waves are allowed on either side of the excitation point in space. Therefore, the response to the excitation at a frequency in the gap of a SPC are evanescent waves. Conversely, in the PTC (Fig. 2B), two of the four modes are propagating back in time and therefore cannot be excited because they are restricted by causality. Thus, Green’s function must be expressed with two forward-propagating waves in time, one of which is exponentially decaying and the other exponentially growing, which is allowed because there is no energy conservation in PTCs.

This analysis explains the exponentially growing dipole emission in a PTC. The dipole excites the gap modes, which, once excited, grow exponentially regardless of the dipole, even when mismatched. The key issue here is that a point dipole excites modes with all \(k, j_k \neq 0\), including the exponentially growing gap modes. Thus, any point source in a PTC results in a exponentially growing emission, even when the excitation is a single flash in time. The emission from this flash will grow exponentially, drawing energy from the modulation.

Next, we quantize our model. First, we write the electromagnetic field Hamiltonian in a PTC different from that in SPC.
oscillates near some constant value, whereas photons, \(N_k\), fulfill the band structure is curved. (Image 37x715 to 557x728) 

In the classical case: It is time dependent in the classical case. The periodic variation of \(n(t)\) allows describing the dynamics of the excited atom, interacting with the radiation field described below, but the exponential growth of the number of photons in the momentum gap and the classical/semiclassical intuition allow us to make some safe statements on the dynamics in this unusual quantum system.

To derive the emission from excited atoms in PTC, we add the atomic and the interaction parts to the Hamiltonian (Eq. 7):

\[
H = H_f + H_a + H_{\text{int}}
\]

where \(H_f = \hbar \sum_k a_k^\dagger a_k\) is the creation (annihilation) operator for mode with the wave vector \(k\). This Hamiltonian is derived in (7) following the quantization procedure described in (29). It follows our intuition gained in the classical case: It is time dependent through \(n(t)\) and it conserves momentum, \(H_f, \sum_k K a_k^\dagger a_k = 0\), but it does not conserve the number of photons. The Hamiltonian (Eq. 7) allows describing the dynamics of the free field for each photon pair \(\{k, -k\}\) separately. The resulting dynamics agrees with the classical case: For modes with \(k\) associated with the band of the PTC, the expectation value of the number of photons, \(N_k(t) = \langle \psi(t)|a_k^\dagger a_k|\psi(t)\rangle\), oscillates near some constant value, whereas if \(k\) belongs to the PTC bandgap, \(N_k\) grows exponentially with time at the same rate as in the classical case. The periodic variation of \(n(t)\) allows introducing the Floquet eigenmodes \(|\psi_k(t)\rangle = e^{-i\omega_0 t}|\psi_k\rangle\) of the Hamiltonian (Eq. 7), with \(\omega_0\) being the Floquet frequency. Let us first list the main features of the quantum Floquet eigenmodes, the detailed analysis of which is provided in (7). In the bands, \(\omega_0\) coincides with the Floquet frequency calculated in the classical analysis, and the Floquet eigenstates experience weak oscillations in the number of photons. Conversely, in the bandgap, the eigenstates of the Hamiltonian cannot exist: By correspondence with the classical case, \(N_k\) in the gap eigenmodes should grow exponentially, which is impossible with Hermitian Hamiltonians such as the one in Eq. (7). The absence of eigenstates in the gap brings complexity in studying the dynamics of the excited atom, interacting with the radiation field described below, but the exponential growth of the number of photons in the momentum gap and the classical/semiclassical intuition allow us to make some safe statements on the dynamics in this unusual quantum system.

To describe the emission from excited atoms in PTC, we add the atomic and the interaction parts to the Hamiltonian (Eq. 7):

\[
H = H_f + H_a + H_{\text{int}}
\]

where \(H_{\text{int}} = \hbar \omega_0 \sigma_z\) is the coupling constant between the initial and the final Floquet eigenstates through \(H_{\text{int}}\) and \(k_m: \omega_0(k_m) = \omega_0 + m\Omega\) is the wave number of the mode corresponding to \(m\)th harmonic of the atomic transition (30). Analyzing the dynamics of the emission rate \(\gamma(k)\) within the band, we observe that there are two competing contributions: the closer to the band edge the larger the \(|V_{fi}^m|^2\), because for modes in the vicinity of the gap the oscillations are larger, whereas at the band edge, the density of states, \(\rho \propto \frac{k^2}{\bar{v}^2} \propto \frac{1}{k_m} \propto k_m\), is smaller. Fig. 3A shows that at the band edge, the rate of spontaneous emission vanishes because the density of states goes down to zero. The low density of states is apparent from the vertical slope of the dispersion near the band edge (Fig. 3B). The implication is intriguing: Even though the Floquet modes have larger oscillations closer to the band edge, which naturally increases the strength of the light-matter interaction, the emission rate at the edge goes to zero because there are no states to radiate into. Thus, an “atom” or a nano-antenna with directional emission at the band edge would stay in the excited state forever, unable to relax to the ground state through spontaneous emission.

In one-dimensional PTCs, the presence of a gap in the momentum alters the light-matter interactions in a profound way, bringing to question foundational issues such as the meaning of spontaneous and induced emission in such media and the lifetime of an atom in excited...
states. The exponential growth of energy in the modes associated with the PTC gap and the nonmonotonous growth rate raise the exciting idea of PTC lasers extracting their energy from the modulation. The simplest setting for such a laser is to construct a resonator by placing mirrors on either side of the dielectric medium with its permittivity modulated in time. The cavity length should be much larger than the wavelength of the waves of interest, such that momentum conservation applies despite the finite size of the resonator. CAVITIES with shorter lengths can also exhibit momentum gaps but require additional treatment of the spatial modes. Because the amplification of the waves associated with the gap modes attains a maximum at midgap, any saturation mechanism will eventually result in stable monochromatic emission. Thus, controllable periodic change of the permittivity can give rise to coherent radiation from an almost arbitrary source and, under some conditions, the emission can be shaped into pulses by designing the modulation.

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**Amplification in photonic time crystals**
Regular photonic crystals are structures in which the refractive index is spatially periodic and can suppress the spontaneous emission of light from an emitter embedded in the structure. In photonic time crystals, the refractive index is periodically modulated in time on ultrafast time scales. Lyubarov *et al.* explored theoretically what happens when an emitter is placed in such a time crystal (see the Perspective by Faccio and Wright). In contrast to the regular photonic crystals, the authors found that time crystals should amplify emission, leading to lasing. —ISO

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