A PRIORI ESTIMATES, UNIQUENESS AND NON-DEGENERACY OF POSITIVE SOLUTIONS OF THE CHOQUARD EQUATION

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Abstract. We consider the positive solutions for the nonlocal Choquard equation
\[ -\Delta u + u - (|\cdot|^{-\alpha} |u|^{p})|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^d. \]
Compared with ground states, positive solutions form a larger class of solutions and lack variational information. Within the range of parameters of Ma-Zhao's result [25] on symmetry, we prove a priori estimates for positive solutions, generalizing the classical method of De Figueiredo-Lions-Nussbaum [10] to the unbounded domain and the nonlocal nonlinearity in our model. As an application, we show uniqueness and non-degeneracy results for the positive solution of the Choquard equation when \( d \in \{3, 4, 5\} \), \( p \geq 2 \) and \((\alpha, p)\) close to \((d - 2, 2)\).

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1. INTRODUCTION

1.1. Introduction. In this paper, we consider the equation
\[ -\Delta u + u - (|\cdot|^{-\alpha} |u|^{p})|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^d \tag{Choquard} \]
with \( d \geq 1 \), \( \alpha \in (0, d) \), \( p \in (1, \infty) \) and \( u \) a real-valued measurable function.

This equation (Choquard) is usually referred to as Choquard or Choquard-Pekar equation. The case \( d = 3, \alpha = d - 2, p = 2 \) appears in various physical contexts, including quantum mechanics for polaron at rest [30] and one-component plasma [22]. It is also known as the Schrödinger-Newton equation by coupling the Schrödinger equation of quantum physics with nonrelativistic Newtonian gravity [3]. Besides, every solution \( u \) of (Choquard) relates to a solitary wave solution
\[ i\partial_t \psi = -\Delta \psi - (|\cdot|^{-\alpha} |\psi|^{p})|\psi|^{p-2}\psi, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.1} \]
When $p = 2$, (1.1) is called Hartree equation, appearing in the study of Boson stars and other physical phenomena [31]. Please refer to [28] and the references therein for more mathematical and physics background of the Choquard equation (Choquard).

Solutions of (Choquard) are formally critical points of the action functional
\[
\mathcal{A}(u) := \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2) - \frac{1}{2p} \int_{\mathbb{R}^d} (|\cdot|^{-\alpha} * |u|^p)|u|^p.
\] (1.2)

One of the most interesting solution is the groundstate, defined as the minimizer of $\mathcal{A}$ on the Nehari manifold
\[
\mathcal{A}(u) = \inf \left\{ \mathcal{A}(v) : v \in H^1(\mathbb{R}^d) \backslash \{0\}, \langle \mathcal{A}'(u), u \rangle = 0 \right\}.
\]

There are many studies of groundstates in the variational and elliptic viewpoint [22, 23, 27, 28] and for the corresponding solitary wave in generalized Hartree equation (1.1) [7, 26] as well.

In this paper, we focus on a larger class of solutions for (Choquard): the positive solutions. We say $u$ is a solution of (Choquard) in the sense that $u \in H^1(\mathbb{R}^d) \cap L^{\frac{2dp}{d-\alpha}}(\mathbb{R}^d)$ and for any test function $\varphi \in C_c(\mathbb{R}^d)$,
\[
\int_{\mathbb{R}^d} (\nabla u \cdot \nabla \varphi + u \varphi - (|\cdot|^{-\alpha} * |u|^p)|u|^{p-2}u \varphi) \, dy = 0.
\]

Note that groundstates must be positive from the variational structure and regularity properties (see [27, Proposition 5.1]). The lack of variational information makes the study of positive solutions rather harder.

Before coming to our results, we recall some basic results of positive solutions for (Choquard).

**Theorem 1.1** (see [27]). For $d \geq 1$, $\alpha \in (0, d)$, $p \in (1, \infty)$ and
\[
\frac{d}{2d-\alpha} > \frac{1}{p} > \frac{d-2}{2d-\alpha},
\] (1.3)
then the following results hold.

1. *(Existence)* There exists at least one groundstate for (Choquard). In particular, there exists at least one positive solution for (Choquard).
2. *(Regularity)* If $u$ is a positive solution for (Choquard), then $u \in W^{2,r}(\mathbb{R}^d) \cap C_\text{loc}^{\infty}(\mathbb{R}^d)$ for $r \in (1, \infty)$.
3. *(Decay)* If $u$ is a positive solution for (Choquard) and moreover $p \geq 2$, then there exists $\gamma > 0$ such that $u(x) \leq C(u)e^{-\gamma|x|}$.

To make use of all these properties, we will restrict our discussion to (Choquard) with parameters $(d, \alpha, p)$ in the range
\[
d \geq 1, \alpha \in (0, d), \ p \in (1, \infty), \ \frac{1}{2} \geq \frac{1}{p} > \frac{d-2}{2d-\alpha},
\] (1.4)

Notice that $p \geq 2$ also guarantee $\mathcal{A}$ to be twice Fréchet-differentiable on $H^1(\mathbb{R}^d)$ [28, Proposition 3.1], which is essential in our discussion on uniqueness and non-degeneracy in §1.3.

Next we recall a more involved result on the symmetry of positive solutions. For groundstates, the minimizing property enables a standard rearrangement argument, inferring the radially decreasing property around a fixed point\footnote{In this paper, a non-negative function $u$ is radially decreasing means $u(x) = u(|x|)$ and $\partial_r u(r) \leq 0$. We say $u$ is radially decreasing around a fixed point $x_0$ to indicate $u(\cdot - x_0)$ is radially decreasing.} for parameters of full range (1.3) [27, Proposition 5.2]. As for
positive solutions, we can merely rely on the information given by the elliptic equation. Ma-Zhao [25] managed to apply a moving plane method in the integral form to prove this symmetry in a narrower range. To clarify, we first define the following assumption on parameters:

**Assumption 1.2.** For \((d, \alpha, p)\) satisfying (1.3) and \(p \geq 2\), we assume there exist constants

\[
\begin{align*}
& r, r_1, r_2, r_3 \in \left[2, \frac{2d}{d-2}\right], \\
& t, t_1 \in \left\{t : \frac{1}{t} \in \left[\frac{p(d-2)}{2d} - \frac{d-\alpha}{d}, \frac{\alpha}{d}\right) \cap (0, 1)\right\}, \\
& \frac{1}{s} \in \left[\frac{2}{d}, 1\right] \cap \left[\frac{1}{r}, \frac{2}{r - \frac{d}{d}}\right]
\end{align*}
\]

such that

\[
\begin{align*}
& r_1 \geq p - 2, r_2 \geq p - 1, r_3 \geq p - 1, \\
& \frac{1}{t_1} + \frac{p-2}{r_1} + \frac{1}{r} = \frac{1}{s}, \\
& \frac{1}{t} + \frac{p-1}{r_2} = \frac{1}{s}, \\
& \frac{1}{t} + \frac{d-\alpha}{d} = \frac{1}{r_3} + \frac{1}{r}.
\end{align*}
\]

**Remark 1.3.** This assumption still includes a wide range of interesting cases.

- One subcase (see [25, Remark 3]) is that
  \[
  2 < \alpha < d, \quad \frac{1}{2} \geq \frac{1}{p} > \frac{d-2}{2d-\alpha}.
  \]

In particular, if we define the critical scaling index \(s_c := \frac{d}{2} - \frac{d+2-\alpha}{2(p-1)}\) for the generalized Hartree equation (1.1)\(^2\), then (1.5) includes the whole interrange case \(0 < s_c < 1\) when \(p = 2\).

- Another subcase is the perturbation of \((\alpha, p) = (d - 2, 2)\):
  \[
  |\alpha - (d - 2)| \leq \frac{1}{100}, \quad 0 \leq p - 2 \leq \frac{1}{100}, \quad d \in \{3, 4, 5\}.
  \]

The case \(d = 5\) is contained in (1.5). For \(d = 3, 4\) we can directly verify

\[
\begin{align*}
 r &= r_1 = \frac{8}{3}, r_2 = \frac{8}{3}(p-1), r_3 = \frac{12(p-1)}{3-4(\alpha-1)}, t = \frac{24}{7}, t_1 = \frac{24}{7-9(p-2)}, s = \frac{3}{2}, \quad \text{when } d = 3; \\
 r &= r_1 = \frac{8}{3}, r_2 = \frac{8(p-1)}{3}, r_3 = \frac{8(p-1)}{3-2(\alpha-2)}, t = 4, t_1 = \frac{8}{2-3(p-2)}, s = \frac{8}{5}, \quad \text{when } d = 4.
\end{align*}
\]

Now we can state their result as

**Theorem 1.4** ([25, Theorem 2]). Any positive solution of (Choquard) with \((d, \alpha, p)\) satisfying (1.3), \(p \geq 2\) and Assumption 1.2 must be radially decreasing around some fixed point.

This gives us motivation to study radially decreasing positive solutions of (Choquard).

### 1.2. A priori estimates.

Recall that Theorem 1.1 indicates that every positive solution of (Choquard) is bounded in \(W^{2,r}(\mathbb{R}^d)\) and decays exponentially. Our first main result indicates that if we further require the positive solution to be radially decreasing, it has an a priori upper bound, depending only on \((d, \alpha, p)\) in a uniform way.

\(^2\)That is, \(H^{s_c}(\mathbb{R}^d)\) norm of the solution to (1.1) is invariant under the scaling symmetry of (1.1).
**Theorem 1.5.** For a radially decreasing positive solution $u$ of (Choquard) with $(d, \alpha, p)$ satisfying (1.4), for $r \in (1, \infty)$, there exists constants $C(d, \alpha, p; r)$ and $C'(d, \alpha, p)$ such that

$$
\|u\|_{W^{2,r}(\mathbb{R}^d)} \leq C(d, \alpha, p; r),
$$

(1.7)

$$
|\nabla u(s)| + u(s) \leq C'(d, \alpha, p)e^{-\frac{s}{2}}, \quad s \geq 0.
$$

(1.8)

Moreover, these constants depend continuously on $(\alpha, p)$, indicating that this is a uniform bound for $(\alpha, p)$ satisfying (1.4) taking values in a compact subset.

Combine Theorem 1.4 and Theorem 1.5, we can remove the radially decreasing condition under Assumption 1.2.

**Theorem 1.6.** For any positive solution $u$ of the Choquard equation (Choquard) with $(d, \alpha, p)$ satisfying (1.4) and Assumption 1.2, we have the same uniform a priori bounds (1.7) and (1.8).

There is a long history of studying a priori bounds for positive solutions of elliptic equations [4, 10, 11, 17, 32, 34]. Such a priori bounds provide lots of information about existence of positive solution and the structure of the positive solutions set, see [6, 24, 29] and §1.3. In more recent work of proving uniqueness for nonlinear groundstates of fractional Laplacians [14, 15], one crucial step is to derive an a priori bound for the global bifurcation branch.

Among these works, three methods have been used to derive a priori bounds. In [14], Frank-Lenzmann controls the solutions in the branch through its ”evolution” in the parameter space. It requires a non-degeneracy result to construct the bifurcation branch, which is highly non-trivial. The second approach is a blowup method from Gidas-Spruck [17]. It is a contradictory argument, reducing the problem of a priori bounds to Liouville result of some simple equation by rescaling. This method is powerful and may also work for our problem to get an $L^\infty$ bound. However, this is not enough to control $L^p$ norm in unbounded space, and such contradictory argument provides relatively little information on the shape of the solution. The third method due to De Figueiredo-Lions-Nussbaum [10] derives the bound in a direct way. They exploit the positive eigenfunction to get some a priori bound and get the desired bound with functional identities and the subcriticality nature of the equation. We adapt this method to our problem.

To our knowledge, however, the second and third method have only been applied for problems on bounded domains with local nonlinearities (although the blowup method can tackle systems of equations, see for example [34]). Perhaps the main innovation of this paper is to deal with the unboundedness of domain and nonlocality of nonlinearity. Nevertheless, one good point in our setting is the symmetry of $\mathbb{R}^d$, so we can add radially decreasing property to our positive solutions thanks to Theorem 1.4.

The unboundedness of domain causes trouble in two ways. On the one hand, we lack of eigenfunctions of Laplacian to get initial a priori information. This can be substituted by a nonlinear positive eigenfunction, namely the groundstate of nonlinear elliptic equation. It gives weaker information but enough for us. On the other hand, a more challenging problem is to control the solution in unbounded domains. Naturally, the exponential decay when $u$ small is a good enough bound in some exterior region. The local argument in [10] basically ensures good bound for any fixed interior region, but the question is whether the non-exponential-decay interior region can be arbitrarily large. The crux is to exclude flatness in the connection region. We apply the a priori information and an ODE argument for this. As a model case, we study the following semilinear elliptic equation with a local nonlinearity

$$
-\Delta u + u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^d
$$

(Model)

where $p \in (1, \infty)$ for $d = 1, 2$ and $p \in (1, \frac{d+2}{d-2})$ for $d \geq 3$. We have the following conclusion
Theorem 1.7. The positive $H^1$ solution $u$ of (Model) has a priori bounds

$$
\|u\|_{H^1} \leq C(p, d),
$$

(1.9)

where $C(p, d)$ depends continuously on $p$.

Although this result can be obtained from the existence [5] and uniqueness [20] of the positive solution, this theorem may still of its own interest and possible to generalize to more complicated nonlinearity.

Regarding the second difficulty of (Choquard), the nonlocal nonlinearity $|\cdot|^{-\alpha} * |u|^p|u|^{p-2}u$, we will exploit the good structure of this nonlocality to bound some norm of $u$ (see Proposition 2.1). Also, we view it as another radial function and study its evolution (Proposition 2.2) along with that of $u$. Finally, after generalizing the argument of subcriticality in [10] to nonlocal case, we can complete the proof of Theorem 1.5 in a similar framework as Theorem 1.7.

1.3. Non-degeneracy and uniqueness. Thereafter, we consider the uniqueness and non-degeneracy (explained later) of the positive solution of (Choquard). These results are essential for discussing the dynamics of the corresponding solitary wave solution $\psi(t, x) = e^{it}u(x)$ of the focusing time-dependent generalized Hartree equation. The uniqueness clarifies what makes up the minimal obstruction of global-wellposedness and scattering, and the non-degeneracy provides suitable spectral condition for perturbative analysis. See [18,19,36] and see [21] for more applications.

However, relatively little is known on the uniqueness and non-degeneracy for positive solutions or groundstates of (Choquard). For the isolated case $(d, \alpha, p) = (3, 1, 2)$, Lieb [22] used the special structure of Newtonian potential $|x|^{-(d-2)}$ to prove the uniqueness of the radial positive solution (hence also of the groundstate) and non-degeneracy was verified by [21,37,38]. These results can be easily generalized to the positive solution of $d \in \{4, 5\}$, $(\alpha, p) = (d - 2, 2)$ (see [2, Appendix A] for uniqueness and [9] for non-degeneracy). The radial condition was removed by Ma-Zhao’s result [25]. Besides, we have uniqueness and non-degeneracy of the groundstate for $d \geq 3$, $p \in (2, \frac{2d}{d-2})$ and $\alpha$ close to 0 or $d$ [33] and for $d \in \{3, 4, 5\}$, $\alpha = d - 2$ and $p \in [2, 2 + \delta]$ for some $\delta > 0$ [40]. Both are proved by perturbative arguments. We also mention another nonlocal elliptic problem, the nonlinear fractional Laplacian equation, where the uniqueness and non-degeneracy of groundstates were resolved partially in [13] and later completely in [14,15]. All these results for continuous exponents utilize more or less variational information of groundstates and thus not work for positive solutions.

In this paper, with our a priori bound Theorem 1.5, we prove the uniqueness and nondegeneracy of the positive solution for (Choquard) with parameters $d \in \{3, 4, 5\}$ and $(\alpha, p) \in [d - 2 - \delta, d - 2 + \delta] \times [2, 2 + \delta]$ for some $\delta > 0$. It can be viewed as generalization of [40] to all positive solutions, two-dimensional perturbation of parameters and also to multiple dimensions, or generalization of [25] to a neighborhood of exponents.

Define the corresponding linearized operator $L_+$ associated with a function $Q$ and for parameters $(d, \alpha, p)$ to be

$$
L_{+, Q, d, \alpha, p} \xi := -\Delta \xi + \xi - (p - 1)(|\cdot|^{-\alpha} * Q^p)Q^{p-2}\xi - p(|\cdot|^{-\alpha} * (Q^{p-1}\xi))Q^{p-1}
$$

(1.10)
as a nonlocal operator on $L^2(\mathbb{R}^d)$. Then we can state our results on non-degeneracy and uniqueness.

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3 Usually, we take $Q$ to be $Q_{d, \alpha, p}$, a positive solution for (Choquard) with these parameters. So we may omit these parameters and denote $L_{+, Q_{d, \alpha, p}}$ as $L_{+, Q_{d, \alpha, p}}$ for simplicity. We may even leave out $d$ if $(\alpha, p)$ are emphasized and no ambiguity occurs.
Theorem 1.8 (Non-degeneracy). For \( d \in \{3, 4, 5\} \), there exists \( \delta > 0 \) such that for \((\alpha, p) \in [d - 2 - \delta, d - 2 + \delta] \times [2, 2 + \delta], \) any positive solution \( Q_{\alpha, p} \) of (Choquard) is non-degenerate, namely
\[
\text{Ker}L_{+, Q_{\alpha, p}} = \text{span}\{\partial_\ell Q_{\alpha, p}\}_{\ell=1}^d
\]  
(1.11)

Theorem 1.9 (Uniqueness). For \( d \in \{3, 4, 5\} \), there exists \( \delta > 0 \) such that for \((\alpha, p) \in [d - 2 - \delta, d - 2 + \delta] \times [2, 2 + \delta], \) the positive solution of (Choquard) is unique up to translations.

We remark that \( \text{Ker}L_{+, Q_{\alpha, p}} \supset \text{span}\{\partial_\ell Q_{\alpha, p}\}_{\ell=1}^d \) holds for any solution \( Q_{\alpha, p} \) of (Choquard) by differentiating the equation. So (1.11) indicates that there exist no other vanishing modes for \( L_{+, Q} \), which explains the meaning of non-degeneracy. For more illustration of non-degeneracy and the operator \( L_{+, Q} \), please see [21, 40].

As for the proof, our starting point is the uniqueness and non-degeneracy for \( d \in \{3, 4, 5\} \), \((\alpha, p) = (d - 2, 2)\) (see [2, Appendix A] and [9]). Theorem 1.4 and Remark 1.3 reduce these theorems our questions to the discussion of radially decreasing positive solutions (with \( \delta \leq \frac{1}{10} \)), and then we use a perturbative strategy. The first ingredient is a compactness result Proposition 5.1, confirming that every positive solution approximates \( Q_{d-2,2} \) (the unique solution for \((\alpha, p) = (d - 2, 2)\)) when the parameter \((\alpha, p)\) approximates. It is here that we exploit the a priori bounds Theorem 1.5 to discuss such asymptotic behavior for positive solutions rather than groundstates. Then since 0 is an isolated eigenvalue for \( L_{+, Q_{d-2,2}} \), the non-degeneracy Theorem 1.8 comes from perturbation of Fredholm operator. The uniqueness Theorem 1.9 easily follows this compactness result and a local uniqueness theorem (Proposition 5.2) from implicit function theorem. We remark that the linearized operator is actually not \( C^1 \) near \( Q_{d-2,2} \) but just at \( Q_{d-2,2} \) (see Lemma C.3), which requires us to be more careful when applying the implicit function theorem to prove Proposition 5.2.

### 1.4. Structure of the paper and notations.

The structure of this paper is as follows. In §2, we show some estimates of nonlinearity and functional identity for the Choquard equation (Choquard) as a preparation. Then we prove the a priori bounds for the model case (Theorem 1.7) and Choquard equation (1.5) in §3 and §4 respectively. The non-degeneracy and uniqueness (Theorem 1.8 and 1.9) follow in §5. We put some complicated computations and the refined implicit function theorem in the appendix, which are used in the last two sections.

Our notations are standard. We employ \( L^p(\mathbb{R}^d), W^{k,r}(\mathbb{R}^d), H^k(\mathbb{R}^d) \) and \( C^k(\mathbb{R}^d) \) for those Sobolev spaces of real-valued functions. For Banach space \( X \) and \( Y \), we use \( L(X, Y) \) to denote the space of bounded linear operators from \( X \) to \( Y \). In particular, \( L(X) := L(X, X) \). We use \( B_R(x) \) with \( x \in \mathbb{R}^d \) to denote the Euclidean open ball centered at \( x \) with radius \( R > 0 \). Two related notations are \( B^c_R(x) := \mathbb{R}^d \setminus B_R(x) \) and \( B_R := B_R(0) \).

We also write \( X \lesssim Y \) or \( Y \gtrsim X \) to indicate \( X \leq CY \) for some constant \( C > 0 \), and write \( X \sim Y \) for \( X \lesssim Y \lesssim X \). If \( C \) depends upon some additional parameters, we will indicate this with subscripts. For example, \( X \lesssim_{\phi} Y \) means \( X \leq C(\phi)Y \). We use \( X_n = o_n(Y_n) \) to denote that for any \( \epsilon > 0 \), there exists \( N > 0 \) such that \( |X_n| \leq \epsilon Y_n \) for any \( n \geq N \). The subscript can also other parameters with prescribed limiting process, for example \( p \to 2 \).

### 2. Preliminaries for the Choquard equation

In this section, we first show some propositions for the nonlocal term \(|\cdot|^{-\alpha} * f\) (also called Riesz potential) when \( f \) is non-negative and radially decreasing. Evidently, \(|\cdot|^{-\alpha} * f\) is also a non-negative and radial function.

We recall a pointwise equivalence result, which is essential in our proof of a priori bounds.
Proposition 2.1 ([12, corollary 2.3]). Let $f$ be a non-negative, radially decreasing function in $\mathbb{R}^d$ and $\alpha \in (0, d)$. We have

\[
( | \cdot |^{-\alpha} \ast f )( r ) \sim_{d, \alpha} r^{-\alpha} \int_0^r f(s)s^{d-1}ds + \int_r^\infty f(s)s^{d-1-\alpha}ds.
\]

(2.1)

Moreover, the constant is uniformly bounded for $\alpha$ in a compact subset of $(0, d)$.

We recall its proof for completeness.

Proof. Firstly we claim that for any $r \geq 0$ and $x \in \mathbb{R}^d$ with $|x| = r$,

\[
( | \cdot |^{-\alpha} \ast f )( r ) \sim_{d, \alpha} r^{-\alpha} \int_{B_r} f(y)dy + \int_{B_r^c} f(y)|y|^{-\alpha}dy + \int_{\mathbb{R}^d \setminus B_r} \frac{f(y)}{|x-y|^{-\alpha}}dy.
\]

(2.2)

Those three terms on the right hand side correspond respectively to integration on $B_r \setminus B_r^c(x)$, $B_r^c \setminus B_r^c(x)$ and $B_r^c(x)$. The equivalence requires non-negativity.

Next we will prove

\[
\int_{B_r^c(x)} \frac{f(y)}{|x-y|^{-\alpha}} \leq_{d, \alpha} r^{-\alpha} \int_0^r f(s)s^{d-1}ds,
\]

(2.3)

which combined with (2.2) implies (2.1). The radially decreasing property implies

\[
f \left( \frac{r}{2} \right) \leq \frac{4}{r} \int_{B_r^c} f(s) \left( \frac{4s}{r} \right)^{d-1}ds \leq \frac{4^d}{r^d} \int_0^r f(s)s^{d-1}ds
\]

and

\[
\int_{B_r^c(x)} \frac{f(y)}{|x-y|^{-\alpha}} \leq_d f \left( \frac{r}{2} \right) \left( \frac{r}{2} \right)^{d-\alpha}.
\]

They yield (2.3). \hfill \Box

Then we discuss the evolution of $| \cdot |^{-\alpha} \ast f$.

Proposition 2.2. Let $f$ be a non-negative, radially decreasing $C^1$ function in $\mathbb{R}^d$ and $\alpha \in (0, d)$. Then for any $\epsilon_0 \leq \frac{1}{d}$, we have

\[
- \partial_r ( | \cdot |^{-\alpha} \ast f ) ( r ) \gtrsim_{d, \alpha, \epsilon_0} f \left( \frac{2}{3}r \right) - f \left( \frac{1}{2d}r \right) r^{d-1-\alpha}.
\]

(2.4)

The constant here is uniformly bounded for $(\alpha, \epsilon_0)$ taking values in compact subset of $(0, d) \times (0, \frac{1}{d})$. In particular, $| \cdot |^{-\alpha} \ast f$ is strictly radially decreasing if $f$ vanishes at infinity.

As a preparation, we define $\chi_\Omega$ to be the characteristic function of $\Omega \subset \mathbb{R}^d$

\[
\Psi_{R_1,R_2}(x) := (\chi_{B_{R_1}} \ast \chi_{B_{R_2}})(x) = |B_{R_1}(0) \cap B_{R_2}(x)|.
\]

(2.5)

Some properties of $\Psi$ are discussed in the lemma below.

Lemma 2.3 (Properties of $\Psi$). For $R_1 \geq R_2 > 0$, $\Psi_{R_1,R_2}$ satisfies the followings:

1. $\Psi_{R_1,R_2}$ is non-negative and radially decreasing. $\Psi_{R_1,R_2} = \Psi_{R_2,R_1}$. $\Psi_{R_1,R_2}(x) = R_2^d$ in $B_{R_1-R_2}$ and $\Psi_{R_1,R_2}(x) = 0$ in $B_{R_1+R_2}^c$.

2. Scaling property:

\[
\Psi_{R_1,R_2}(r) = R_2^d \Psi_{\frac{R_1}{R_2}, 1} \left( \frac{r}{R_2} \right)
\]
(3) Derivative: As a radial function, $\Psi_{R_1,R_2}$ is absolutely continuous on $[0, \infty)$ and
\[ -\frac{d}{dr} \Psi_{R_1,R_2}(r) = |\partial B_{R_1}(0) \cap B_{R_2}(r)|, \quad r \in (R_1 - R_2, R_1 + R_2). \tag{2.6} \]

(4) Lower bound of the derivative: for $\epsilon_0 \in (0, 1)$,
\[ -\frac{d}{dr} \Psi_{R_1,R_2}(r) \gtrsim_{d, \epsilon_0} R_2^{d-1}, \quad \text{for } |r - R_1| \leq (1 - \epsilon_0)R_2. \tag{2.7} \]

Proof. (1)-(3) directly follows the definition (2.5). From (2) and (3), the estimate (4) boils down to
\[ \inf_{R \geq 1} \inf_{t \in [-1+\epsilon_0, 1-\epsilon_0]} |\partial B_R(0) \cap B_1(R + t)| > 0 \tag{2.8} \]
for any $\epsilon_0 \in (0, 1)$. For $d = 1$, LHS of (2.8) = 1 > 0. Next we consider the case $d \geq 2$.

Assuming the center of $B_1(R + t)$ to be $(R + t, 0, \ldots, 0) \in \mathbb{R}^d$, the symmetry around $x_1$-axis implies that $\partial B_R(0) \cap \partial B_1(R + t)$ lies in a hyperplane $\{x_1 = a(R, t)\}$ with $a(R, t) \in (0, R)$. And $\partial B_R(0) \cap B_1(R + t) = \partial B_R(0) \cap \{x_1 \geq a(R, t)\}$.

Using spherical coordinates, we see for $d \geq 3$ (and similar for $d = 2$)
\[ |\partial B_R(0) \cap \{x_1 \geq a(R, t)\}| = R^{d-1} \int_0^{\arccos \frac{a}{R}} \frac{d \theta_1}{\pi} \prod_{k=1}^{d-2} \sin^k \theta_1 \sim_d R^{d-1} \left( \frac{R - a}{R} \right)^{\frac{d-1}{2}} \sim_d \left( R^2 - a^2 \right)^{\frac{d-1}{2}}, \]
where we made use of
\[ \frac{R - a}{R} = \cos 0 - \cos \arccos \frac{a}{R} = \int_0^{\arccos \frac{a}{R}} \sin \theta d\theta \sim \left( \arccos \frac{a}{R} \right)^2. \]

Thus it suffices to give a uniform lower bound on $\left( R^2 - a^2(R, t) \right)^{\frac{d-1}{2}}$, which is the radius of the $(d - 1)$-dimensional ball $\partial B_R(0) \cap \{x_1 = a(R, t)\}$. Actually, we are just using the measure of this sectional ball to bound the measure of the corresponding spherical cap. Elementary trigonometric calculations and estimates indicate that
\[ R^2 - a(R, t)^2 = \frac{R}{R + t} \left( 1 - \frac{1 - t^2}{4R(R + t)} \right) \left( 1 - t^2 \right) \geq \frac{1}{2} \left( 1 - \frac{1 - t}{4R} \right) \left( 1 - t^2 \right) \geq \frac{\epsilon_0}{4} \]
for any $R \geq 1$ and $|t| \leq 1 - \epsilon_0$. \hfill \Box

Proof of Proposition 2.2. For the given $f$, denote $r_\lambda := \inf \{ r \geq 0 : f(r) \leq \lambda \}$. Using the layer cake representation, we have
\[ (| \cdot |^{-\alpha} * f)(x) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \chi_{\{z: |z - r| > t\}}(y) \chi_{\{z: f(z) > \lambda\}}(x - y) dy ds dt \]
\[ = \int_0^\infty \int_0^\infty \Psi_{r, t^{-\frac{1}{\alpha}}}(x) ds dt \]

Thus from (2.7), we have
\[ \frac{d}{dr} (| \cdot |^{-\alpha} * f)(r) \gtrsim_{d, \epsilon_0} \int_0^{\infty} \int_{r - \epsilon_0}^{\infty} \chi_{\{r - r < |r| \leq (1 - \epsilon_0)t^{-\frac{1}{\alpha}}\}}(r) t^{-\frac{d-1}{\alpha}} ds \]
\[ = \int_0^\infty f(r) \int_{r - \epsilon_0}^{(r - \epsilon_0)^{-\alpha}} t^{-\frac{d-1}{\alpha}} ds \]
\[ = \int_0^\infty f(r) \left( (r - \epsilon_0)^{-\alpha} \right) t^{-\frac{d-1}{\alpha}} ds \]
The last equality comes from this observation: the inner integral is non-zero only if $|r - r_s| \leq (1 - \epsilon_0)r_s$, which means $\frac{r}{2-\epsilon_0} \leq r_s \leq \frac{r}{\epsilon_0}$ and thus $f\left(\frac{r}{2-\epsilon_0}\right) \leq s \leq f\left(\frac{r}{\epsilon_0}\right)$ from the definition of $r_s$.

By changing variables $\tilde{t} := r^\alpha t$, the inner integral turns into

$$
\int_{r_s^{-\alpha}}^{\left(\frac{|r-r_s|}{1-\epsilon_0}\right)^{-\alpha}} t^{-\frac{d-1}{\alpha}} dt = r^{d-1-\alpha} \int_{\left(\frac{r_s}{r}\right)^{-\alpha}}^{\left(\frac{r_{s,0}}{r}\right)^{-\alpha}} \tilde{t}^{-\frac{d-1}{\alpha}} d\tilde{t} =: r^{d-1-\alpha} g_{\epsilon_0,\alpha} \left(\frac{r_s}{r}\right) \tag{2.9}
$$

Now we claim

$$
g_{\epsilon_0,\alpha}(t) \gtrsim_{d,\alpha,\epsilon_0} 1 \quad t \in \left[\frac{2}{3}, \frac{1}{2\epsilon_0}\right]. \tag{2.10}
$$

Since $\frac{r_s}{r} \in \left[\frac{2}{3}, \frac{1}{2\epsilon_0}\right]$ is equivalent to $s \in \left[f\left(\frac{2r}{3}\right), f\left(\frac{r}{2\epsilon_0}\right)\right]$ and $\epsilon_0 \leq \frac{1}{2}$ indicates $\frac{1-\epsilon_0}{2-\epsilon_0} < \frac{2}{3}$,

$$
-d \frac{d}{dr} \left(\cdot \left(-\frac{r}{\epsilon_0}, f\right)\right) \gtrsim_{d,\epsilon_0} \int_{\frac{f(2r)}{r}}^{f(\frac{r_s}{r})} r^{d-1-\alpha} g_{\epsilon_0,\alpha} \left(\frac{r_s}{r}\right) ds \gtrsim_{d,\alpha,\epsilon_0} \left[f\left(\frac{2r}{3}\right) - f\left(\frac{1}{2\epsilon_0}r\right)\right] r^{d-1-\alpha}.
$$

Finally we check the claim (2.10). Computing the integral in (2.9), we also get another representation of $g_{\epsilon_0,\alpha}$

$$
g_{\epsilon_0,\alpha}(t) = \left\{ \begin{array}{ll}
\frac{d-1-\alpha}{d-\frac{1}{\alpha}} \left[t^{d-1-\alpha} - \left(\frac{1-t}{1-\epsilon_0}\right)^{d-1-\alpha}\right], & \alpha \in (0, d) \backslash \{d - 1\}, \\
-(d-1) \ln \left(\frac{1-t}{1-\epsilon_0}\right), & \alpha = d - 1.
\end{array} \right.
$$

Evidently, $g_{\epsilon_0,\alpha}(t)$ is $C^1$ on $[\frac{1-\epsilon_0}{2}, 1) \cup (1, \frac{1}{2\epsilon_0}]$. Also at $t = 1$, we always have $\lim_{t \to 1-0} g_{\epsilon_0,\alpha}(t) = \lim_{t \to 1+0} g_{\epsilon_0,\alpha}(t)$. Thus an elementary computation on $g_{\epsilon_0,\alpha}(t)$ verifies that for all $\alpha \in (0, d)$ and $\epsilon_0 \in (0, 1/4]$, $g_{0,\alpha}$ first monotonically increases and then monotonically decreases with respect to $t \in \left[\frac{1-\epsilon_0}{2}, \frac{1}{\epsilon_0}\right)$. In particular,

$$
\inf_{t \in \left[\frac{2}{3}, \frac{1}{2\epsilon_0}\right]} g_{\epsilon_0,\alpha}(t) = \min \left\{ g_{\epsilon_0,\alpha} \left(\frac{2}{3}\right), g_{\epsilon_0,\alpha} \left(\frac{1}{2\epsilon_0}\right) \right\} \gtrsim_{d,\alpha,\epsilon_0} 1.
$$

The last inequality comes easily from the integral representation (2.9). And that complete the proof of (2.10) and this proposition.

Finally, we recall some useful functional identities for solutions of Choquard equations.

**Proposition 2.4** (Functional Identities, [27, Proposition 3.1]). Let $u \in H^1(\mathbb{R}^d)$ be a solution of (Choquard) with parameters $(d, \alpha, p)$ satisfying (1.4). Then

$$
\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = \int_{\mathbb{R}^d} (|\cdot|^{-\alpha} * |u|^p) |u|^p dx \tag{2.11}
$$

$$
\frac{d-2}{2} \|\nabla u\|^2_{L^2} + \frac{d}{2} \|u\|^2_{L^2} = \frac{2d-\alpha}{2p} \int_{\mathbb{R}^d} (|\cdot|^{-\alpha} * |u|^p) |u|^p dx \tag{2.12}
$$

In particular,

$$
\|\nabla u\|^2_{L^2} = \frac{pd - (2d-\alpha)}{2p} \int_{\mathbb{R}^d} (|\cdot|^{-\alpha} * |u|^p) |u|^p dx \tag{2.13}
$$

$$
\|u\|^2_{L^2} = \frac{(2d-\alpha) - p(d-2)}{2p} \int_{\mathbb{R}^d} (|\cdot|^{-\alpha} * |u|^p) |u|^p dx \tag{2.14}
$$
The last two inequalities follow directly from the first two, which are derived by taking inner product of (Choquard) with \( u \) and \( x \cdot \nabla u \) respectively. Theorem 1.1(2) ensures enough smoothness to do integration by parts. We mention that (2.12) is usually called Pohozaev identity. During the proof of Theorem 1.5, we will also derive and utilize a localized version of them ((4.26) and (4.27)).

An immediate corollary is the following lower bound of \( H^1 \) norm.

**Corollary 2.5.** Let \( u \in H^1(\mathbb{R}^d) \) be a solution of (Choquard) with parameters \((d, \alpha, p)\) satisfying (1.4). In addition, suppose \( u \) is not zero. Then

\[
\|u\|_{H^1(\mathbb{R}^d)} \gtrsim_{d, \alpha, p} 1
\]

where the constant depends continuously on \((\alpha, p)\).

**Proof.** By Hölder inequality, Hardy-Littlewood-Sobolev inequality and Sobolev embedding, whose constants depend on the index continuously, we see

\[
\int_{\mathbb{R}^d} \left(|\cdot|^{-\alpha} \ast |u|^p\right) |u|^p \, dx \leq \left\| |\cdot|^{-\alpha} \ast |u|^p \right\|_{L^{\frac{2d}{d-\alpha}}} \|u\|_{L^{\frac{2dp}{2d-\alpha}}} \lesssim_{d, \alpha, p} \|u\|_{H^1}^{2p},
\]

where we used (1.4) to check that \( H^1 \hookrightarrow L^{\frac{2dp}{2d-\alpha}} \). Also from (2.11) and \( p > 1 \), we have

\[
\|u\|_{H^1}^2 \lesssim_{d, \alpha, p} \|u\|_{H^1}^{2p},
\]

which confirms the lower bound if \( u \) is not identically zero. \( \square \)

### 3. A priori estimates for the model case

In this subsection, we prove the a priori bound Theorem 1.7 for (Model) with \( p \in (1, \infty) \) if \( d = 1, 2 \), \( p \in (1, \frac{4d+2}{2d-2}) \) if \( d \geq 3 \).

By moving plane method [16], one can show that any \( H^1 \) positive solution of (Model) will be radially decreasing around some fixed point. So again we only need to prove that bound for positive, radially decreasing solution \( u \). Also we know \( u \) is Schwartz from a standard iterative argument to improve the regularity (see for example [35, Proposition B.7]), which enables us to freely taking derivatives and integrating by parts.

**Step 1. A priori nonlinear eigenfunction estimate**

We fix \( d \) and arbitrarily take a \( p_0 = p_0(d) \) within the range above. Then we take a Schwartz, positive, radially decreasing solution \( u \) of (Model). That is,

\[
-\Delta Q + Q - Q^{p_0} = 0 \quad \text{in} \quad \mathbb{R}^d.
\]

Using the equation of \( Q \) and \( u \), we have

\[
\int u^p Q = \int (-\Delta + 1)uQ = \int u(-\Delta + 1)Q = \int uQ^{p_0}
\]

Fix \( C_0 := 2\|Q\|_{L^\infty}^{p_0-1} \), we have

\[
C_0 \int uQ = \int_{\{x: u(x) \leq C_0^{\frac{1}{p_0-1}}\}} C_0 uQ + \int_{\{x: u(x) > C_0^{\frac{1}{p_0-1}}\}} C_0 uQ
\]

\[
\leq \int u^p Q + C_0^{\frac{p}{p-1}} \int Q = \int uQ^{p_0} + C_0^{\frac{p}{p-1}} \int Q
\]

\[
\leq \frac{C_0}{2} \int uQ + C_0^{\frac{p}{p-1}} \int Q.
\]

\(^4\)For example, we may take a minimizer of the corresponding Weinstein functional (see [39] or [35, Appendix B]).
Thus
\[ \int uQ \leq 2C_0^{\frac{1}{p-1}} \|Q\|_{L^1} \lesssim_{d,p} 1. \]  (3.2)

With (3.1), we also obtain
\[ \int u^pQ \leq C_0^{\frac{p}{p-1}} \|Q\|_{L^1} \lesssim_{d,p} 1. \]  (3.3)

**Step 2. Exponential decay far away**

Since \(u\) is radially decreasing and positive, we denote \(r_\lambda\) with \(\lambda \in (0, u(0))\) to be
\[ r_\lambda := \inf\{r \geq 0 : u(r) \leq \lambda\}. \]  (3.4)

Consider
\[ \delta_0 = (2p)^{-\frac{1}{p-1}} \sim_p 1, \]  (3.5)
and
\[ R_0 := \max\{r_{\delta_0}, d^\frac{1}{2}\}. \]  (3.6)

Then for any \(r \geq R \geq R_0\), we will characterize the exponential decay of \(u(r)\) and \(-\partial_r u(r)\) related to \(R\).

For \(r \geq r_{\delta_0}\), \(u(r)^{p-1} \leq (2p)^{-1}\) and hence
\[ \left(-\Delta + \frac{1}{4}\right) u = \left(-\frac{3}{4} + u^{p-1}\right) u \leq 0 \]

We take \(v(r) := \frac{r^{-\beta(d)}e^{-\frac{r}{2}}}{R^{-\beta(d)}e^{-\frac{R}{2}}} u(R)\), where \(\beta(d) = 0\) for \(d = 1, 2\) and \(\beta(d) = \frac{d-1}{2}\) for \(d \geq 3\). It satisfies \((-\Delta + \frac{1}{4}) v \geq 0\) and \(v(R) = u(R)\). The classical comparison theorem for \(u\) and \(v\) on \(B_R^c\) implies an upper bound of \(u\)
\[ u(r) \leq v(r) = \frac{r^{-\beta(d)}e^{-\frac{r}{2}}}{R^{-\beta(d)}e^{-\frac{R}{2}}} u(R), \quad r \geq R \geq R_0. \]  (3.7)

Similarly, the positivity of \(u\) implies
\[ (-\Delta + 1)u = u^p \geq 0. \]

And we compare \(u\) on \(B_R^c\) with a multiple of \(r^{-\gamma(d)}e^{-r}\), where \(\gamma(d) = \frac{d-1}{2}\) for \(d \leq 3\) and \(\gamma(d) = d-2\) for \(d \geq 4\) to ensure \((-\Delta + 1)(r^{-\gamma(d)}e^{-r}) \leq 0\). It results in a lower bound
\[ u(r) \geq \frac{r^{-\gamma(d)}e^{-r}}{R^{-\gamma(d)}e^{-\frac{R}{2}}} u(R), \quad r \geq R \geq R_0. \]  (3.8)

Taking \(\partial_r\) on the original elliptic equation (Model) and using the radial symmetry of \(u\), we get
\[ \left(-\Delta + 1 + \frac{d-1}{r^2} - pu^{p-1}\right) (-\partial_r u) = 0. \]  (3.9)

From the definition of \(R_0\), we have \(1 + \frac{d-1}{r^2} - pu^{p-1} \in (\frac{1}{2}, 2)\) for \(r \geq R_0\). So non-negativity of \(-\partial_r u\) and similar comparison argument infers the following bounds,
\[ -\partial_r u(r) \leq \frac{r^{-\gamma(d)}e^{-\frac{r}{2}}}{R^{-\gamma(d)}e^{-\frac{R}{2}}} (-\partial_r u(R)), \quad r \geq R \geq R_0, \]  (3.10)
\[ -\partial_r u(r) \geq \frac{r^{-\gamma(d)}e^{-2r}}{R^{-\gamma(d)}e^{-2R}} (-\partial_r u(R)), \quad r \geq R \geq R_0. \]  (3.11)

Integrate (3.11) from \(R\) to \(+\infty\) and use Lemma A.1,
\[ u(R) = \int_R^\infty (-\partial_r u)(r)\, dr \gtrsim_d (-\partial_r u)(R). \]  (3.12)
Step 3. Controlling $r_{\delta_0}$.

In this step, our goal is to prove an a priori bound for $r_{\delta_0}$. Namely, there exists some $R(p, d) > 0$ (continuously depending on $p$) such that

$$r_{\delta_0} \leq R(p, d).$$  

(3.13)

This part is essential for dealing with the unbounded domain.

Without loss of generality, we assume $r_{\delta_0} \geq d^{\frac{1}{2}}$ so that $R_0 = r_{\delta_0}$ as (3.6). The exponential decay bounds (3.7), (3.8), (3.10), (3.11) and (3.12) in Step 2 hold for $r \geq r_{\delta_0}$. In particular,

$$-\partial_r u(r_{\delta_0}) \lesssim d u(r_{\delta_0}) = \delta_0 \sim_p 1.$$  

From the upper bounds (3.7), (3.10), Lemma A.1 and radially decreasing of $u$, we have

$$\| u \|^2_{L^2(B_{r_{\delta_0}})} \gtrsim_{d, p} r_{\delta_0}^d, \quad (3.14)$$  

$$\| u \|^2_{L^2(B_{r_{\delta_0}^c})} \lesssim_{d, p} r_{\delta_0}^{d-1}, \quad (3.15)$$  

$$\| \nabla u \|^2_{L^2(B_{r_{\delta_0}^c})} \lesssim_{d, p} r_{\delta_0}^{d-1}.\quad (3.16)$$

Similar as Proposition 2.4 for the Choquard equation, we get functional identities via inner product (Model) respectively with $u$

$$\| \nabla u \|^2_{L^2(\mathbb{R}^d)} + \| u \|^2_{L^2(\mathbb{R}^d)} - \| u \|_{L^{p+1}(\mathbb{R}^d)} = 0,$$  

and $x \cdot \nabla u$ (Pohozaev identity)

$$\frac{d - 2}{2} \| \nabla u \|^2_{L^2(\mathbb{R}^d)} + \frac{d}{2} \| u \|^2_{L^2(\mathbb{R}^d)} - \frac{d}{p + 1} \| u \|_{L^{p+1}(\mathbb{R}^d)} = 0.$$  

Eliminating the last term, we find

$$\| \nabla u \|^2_{L^2(\mathbb{R}^d)} = \frac{d(p - 1)}{(d + 2) - p(d - 2)} \| u \|^2_{L^2(\mathbb{R}^d)} \sim_{d, p} \| u \|^2_{L^2(\mathbb{R}^d)} \quad (3.17)$$

Now we can find an $R_1 = R_1(d, p) \gg 1$ depending on constants in (3.14)-(3.17), such that if $r_{\delta_0} \geq R_1$, then

$$10 \| \nabla u \|^2_{L^2(B_{r_{\delta_0}})} \leq \| \nabla u \|^2_{L^2(\mathbb{R}^d)} \sim_{d, p} \| u \|^2_{L^2(\mathbb{R}^d)} \leq \frac{11}{10} \| u \|^2_{L^2(B_{r_{\delta_0}^c})}.$$  

Thus the main contribution to $H^1$ norm comes from $B_{r_{\delta_0}}$.

$$\| \nabla u \|^2_{L^2(B_{r_{\delta_0}})} \gtrsim \frac{9}{10} \| \nabla u \|^2_{L^2(\mathbb{R}^d)} \gtrsim_{d, p} r_{\delta_0}^d \quad (3.18)$$

However, we claim that the following two estimates in the connecting region $B_{r_{\delta_0}} \setminus B_1$ and the center region $B_1$ hold.

$$\| \nabla u \|^2_{L^2(B_{r_{\delta_0}} \setminus B_1)} \lesssim_{d, p} r_{\delta_0}^{d-1}, \quad (3.19)$$  

$$\| \nabla u \|^2_{L^2(B_1)} \lesssim_{d, p} 1. \quad (3.20)$$

These three estimates imply an a priori bound $R_2(d, p)$ for $r_{\delta_0}$ depending on $(d, p)$, and take $R(d, p) := \max\{d^{1/2}, R_1(d, p), R_2(d, p)\}$, we get the bound (3.13) and finish this step. Next we prove these two estimates (3.19) and (3.20). For simplicity, we denote

$$y(r) := -\partial_r u(r).$$

Step 3(a). Upper bound for $H^1$ norm in the connecting region $B_{r_{\delta_0}} \setminus B_1$
From the a priori bound (3.2) in Step 1 and radially decreasing property of \( u \), we have
\[
    u(1) \lesssim_d \int_{B_1} u \lesssim_d Q(1) \int_{B_1} u \lesssim \int_{B_1} uQ \lesssim_{d,p} 1,
\]
which implies
\[
    \int_1^\infty y(r)dr = y(1) \lesssim_{d,p} 1.
\]
We will see that this implies
\[
    y(r) \lesssim_{d,p} 1, \quad \forall r \in [1, r_{\delta_0}].
\]
from the ODE evolution. Indeed, (Model) and (3.21) indicate that there exists \( C_1(d,p) \) such that
\[
    \left| y'(r) + \frac{d-1}{r} y(r) \right| = |u(r) - u^p(r)| \leq C_1(d,p), \quad r \in [1, \infty).
\]
Since \((r^{d-1}y)' = r^{d-1}(y' + \frac{d-1}{r} y)\), we integrate this inequality from \( \tilde{r} \) to \( r > \tilde{r} \geq 1 \) to get
\[
    \left| r^{d-1}y(r) - \tilde{r}^{d-1}y(\tilde{r}) \right| \leq \int_{\tilde{r}}^{r} s^{d-1}C_1(d,p)ds = \frac{C_1(d,p)}{d} (r^d - \tilde{r}^d) \leq C_1(d,p)(r - \tilde{r})r^{d-1}
\]
and thus
\[
    y(r) \geq \left( \frac{\tilde{r}}{r} \right)^{d-1} y(\tilde{r}) - C_1(d,p)(r - \tilde{r}), \quad r > \tilde{r} \geq 1.
\]
So if for some \( \tilde{r} \geq 1 \) we have \( y(\tilde{r}) \geq 2^dC_1(d,p) \), then for \( r \in [\tilde{r}, \tilde{r} + 1] \),
\[
    y(r) \geq 2^{-(d-1)}y(\tilde{r}) - C_1(d,p) \geq 2^{-d}y(\tilde{r}),
\]
and
\[
    \int_1^\infty y(r)dr \geq \int_{\tilde{r}}^{\tilde{r}+1} y(r)dr \geq 2^{-d}y(\tilde{r}).
\]
(3.22) and (3.25) confirm (3.23).

(3.19) follows immediately the \( L^1([1, r_{\delta_0}]) \) bound (3.22) and \( L^\infty([1, r_{\delta_0}]) \) bound (3.23):
\[
    \|\nabla u\|_{L^2(B_{r_{\delta_0}} \setminus B_1)} \lesssim_d \int_{1}^{r_{\delta_0}} y(r)2^{d-1}dr \leq r_{\delta_0}^{d-1} \int_{1}^{r_{\delta_0}} y(r)dr \left( \sup_{r \in [1, r_{\delta_0}]} y(r) \right) \lesssim_{d,p} r_{\delta_0}^{d-1}.
\]

Step 3(b). Upper bound for \( \dot{H}^1 \) norm in the center region \( B_1 \)

Let \( a := u(1) \) and \( v(r) := u(r) - a \), then \( v \) is a radially decreasing, positive solution for the following elliptic problem on \( B_1 \):
\[
    \begin{cases}
        \Delta v + g_p(v; a) = 0 & \text{in } B_1, \\
        v = 0 & \text{on } \partial B_1,
    \end{cases}
\]
where \( g_p(v; a) := -(v + a) + (v + a)^p \). According to (3.21) and (3.23), \( a = u(1) \), \(-\partial_r u(1)\) are both bounded from above. So we can directly apply the method of [10], using subcriticality to verify (3.20).

Denote
\[
    G_p(v; a) := \int_0^v g_p(s; a)ds = -\frac{(v + a)^2 - a^2}{2} + \frac{(v + a)^{p+1} - a^{p+1}}{p + 1}.
\]
We still multiply this elliptic equation with $v$ and $x \cdot \nabla v$ respectively and integrate within $B_1$ to get
\[
\|\nabla v\|_{L^2(B_1)}^2 - \int_{B_1} vg_p(v;a)dx = 0
\] (3.27)
\[-(d-2)\|\nabla v\|_{L^2(B_1)}^2 + 2d \int_{B_1} G_p(v;a)dx = 2 \int_{\partial B_1} |\nabla v(x)|^2dS
\] (3.28)
Eliminating $\|\nabla v\|_{L^2(B_1)}^2$, we get
\[
2d \int_{B_1} G_p(v;a)dx - (d-2) \int_{B_1} vg_p(v;a)dx = 2 \int_{\partial B_1} |\nabla v(x)|^2dS \lesssim_{d,p} 1
\] (3.29)
Note that for $d \geq 3$, $p < \frac{d+2}{d-2}$. Comparing the highest order term of $v$ for $G_p(v;a)$ and $vg_p(v;a)$, it’s easy to see that there exists $t_{d,p} \gg 1$ (continuously depending on $p$) such that for every $0 \leq a = u(1) \lesssim_{d,p} 1$ as (3.21) and $t \geq t_{d,p}$,
\[
\left(\frac{d}{d-2} + \frac{p+1}{2}\right) G_p(t;a) - tg_p(t;a) \geq 0,
\]
\[
G_p(t;a) \geq \frac{1}{2(p+1)} t^{p+1}.
\] (3.30)
Thus (3.29) implies
\[
\int_{B_1 \cap \{x: v(x) \geq t_{d,p}\}} \frac{d+2}{d-2} - p t^{p+1}(x)dx + \int_{B_1 \cap \{x: v(x) < t_{d,p}\}} 2dG_p(v;a) - (d-2)vg_p(v;a)dx \lesssim_{d,p} 1
\]
Then immediately we have
\[
\|v\|_{L^{p+1}(B_1)}^{p+1} \lesssim_{d,p} 1
\] (3.31)
for $d \geq 3$. If $d = 1, 2$, then (3.28) directly implies
\[
\int_{B_1} G_p(v;a)dx \lesssim_{d,p} 1.
\]
So we similarly take a $t_{d,p}$ such that (3.30) holds for all $a \lesssim_{d,p} 1$ as (3.21) and $t \geq t_{d,p}$. This implies (3.31) for $d = 1, 2$.

Now from (3.21), (3.27) and the form of $g_p(v;a)$, we get (3.20)
\[
\|\nabla u\|_{L^2(B_1)}^2 = \|\nabla v\|_{L^2(B_1)}^2 = \int_{B_1} vg_p(v;a)dx \lesssim_{d,p} 1 + \|v\|_{L^{p+1}(B_1)}^{p+1} \lesssim_{d,p} 1.
\]
This concludes Step 3.

**Step 4. Concluding the proof.**

Using the a priori bound for $r_{\delta_0}$ (3.13) in Step 3, we estimate $\|u\|_{L^2(B_{R(d,p)}^c)}$ and $\|u\|_{L^2(B_{R(d,p)}^c)}$ respectively.

On the exterior region $B_{R(d,p)}^c$, we have exponential decay in Step 2 since $R_0 = \max\{r_{\delta_0}, d^4\} \leq R(d,p)$. Thus by taking $R = R(d,p)$ and integrate (3.7) with Lemma A.1, we see
\[
\|u\|_{L^2(B_{R(d,p)}^c)} \lesssim_{d,p} R(d,p)^{d-1} \lesssim_{d,p} 1.
\] (3.32)

For the interior region $B_{R(d,p)}$, we try to use the argument of Step 3(b). First we bound the boundary values $u(R(d,p)) \lesssim_{d,p} 1$ and $-\partial_r u(R(d,p)) \lesssim_{d,p} 1$ from (3.2) and (3.12) respectively.
Thereafter we can define $v := u - u(R(d, p))$ and use exactly the same argument as Step 3(b) to get
\[ \|v\|_{L^{p+1}(B_{R(d, p)})} \lesssim_{d,p} 1. \]

Finally,
\[ \|u\|_{L^2(B_{R(d, p)})} \lesssim d \|v\|_{L^2(B_{R(d, p)})} + u(R(d, p))R(d, p)^{\frac{d}{q}} \]
\[ \lesssim d \|v\|_{L^{p+1}(B_{R(d, p)})} R(d, p)^{\left(\frac{1}{2} - \frac{1}{p+1}\right)} + u(R(d, p))R(d, p)^{\frac{d}{q}} \lesssim_{d,p} 1. \tag{3.33} \]

Combine (3.32) and (3.33), we get the $L^2$ bound and therefore the $H^1$ bound via (3.17).

4. A priori estimates for the Choquard equation

In this section, we prove Theorem 1.5, the a priori estimates for radially decreasing positive solutions of (Choquard). The whole proof follows a similar framework, but differs in many ways due to the nonlinearity’s nonlocal dependence on $u$. We define
\[ H_{d, \alpha, p}[u](x) := (|\cdot|^{-\alpha} + |u|^p)(x)|u|^{p-2}(x). \tag{4.1} \]
For simplicity, we refer to it as $H[u](x)$ or even $H(x)$ if no ambiguity occurs. From Proposition 2.2, $H[u]$, as a function of $x$, is positive and radially strictly decreasing. So we also define $s_\lambda$ by
\[ H[u](s_\lambda) = \lambda \tag{4.2} \]
when $\lambda \in (0, H[u](0))$ and $s_\lambda = 0$ when $\lambda \geq F[u](0)$.

**Step 1. A priori nonlinear eigenfunction estimate**

We pick the same Schwartz, positive, radially decreasing $Q$ satisfying
\[ -\Delta Q + Q - Q^{p_0(d)} = 0 \]
as in Step 1. of §3. Then we have
\[ \int H[u]uQ = \int (\Delta + 1)uQ = \int u(\Delta + 1)Q = \int uQ^{p_0} \tag{4.3} \]
Similarly, we define $C_0 := 2Q(0)^{p_0-1}$, and
\[ C_0 \int uQ \leq \int_{H \leq C_0} C_0 uQ + \int_{H > C_0} C_0 uQ \]
\[ \leq \int_{B_{C_0}} C_0 uQ + \int_{H > C_0} H uQ = \int_{B_{C_0}} C_0 uQ + \int uQ^{p_0} \]
\[ \leq \int_{B_{C_0}} C_0 uQ + \frac{C_0}{2} \int uQ. \]
So we have
\[ \int_{B_{C_0}} uQ \leq \int_{B_{C_0}} uQ \tag{4.4} \]
Using the monotonicity of $u$ this implies
\[ u(sC_0) \int_{B_{sC_0}} Q \leq \int_{B_{C_0}} uQ \leq \int_{B_{C_0}} uQ \leq u(sC_0) \int_{B_{sC_0}} Q \]
This indicates an a priori bound for $sC_0$ depending merely on $d$: there exists $R_1(d)$ such that
\[ sC_0 \leq R_1. \tag{4.5} \]
Step 2. Exponential decay far away.
This step resembles the one in §3 very much. We take
\[
\delta_0 := \min \left\{ \frac{1}{2}, \frac{1}{2(p-1)} \right\},
\]
and
\[
R_0 := \max \{ s_{\delta_0}, d_2^{\frac{1}{2}} \}.
\]
Then for \( r \geq R \geq R_0 \), using the equation (Choquard)
\[
(-\Delta + 1)u = H[u], u \in (0, \frac{1}{2}u)
\]
we get decay estimates from below and above by the classical elliptic comparison theorem
\[
u(r) \geq \frac{r^{-\gamma(d)} e^{-r}}{R^{-\gamma(d)} e^{-r}} u(R), \quad r \geq R \geq R_0.
\]
\[
u(r) \leq \frac{r^{-\beta(d)} e^{-\frac{r}{2}}}{R^{-\beta(d)} e^{-\frac{R}{2}}} u(R), \quad r \geq R \geq R_0,
\]
where
\[
\beta(d) := \begin{cases} 0 & \text{if } d = 1, 2, \\ \frac{d-1}{2} & \text{if } d \geq 3, \end{cases}
\]
and
\[
\gamma(d) := \begin{cases} \frac{d-1}{2} & \text{if } d = 1, 2, \\ d-2 & \text{if } d \geq 3.
\end{cases}
\]
Taking \( \partial_r \) on (Choquard), we find
\[
\left(-\Delta + 1 + \frac{d-1}{r^2} - (p-1)H \right) (-\partial_r u) = -\partial_r (| \cdot |^{-\alpha} * |u|^p) u^{p-1} \geq 0.
\]
The non-negativity of the right hand side follows Proposition 2.2. So we get a lower bound of \( -\partial_r u \)
\[
-\partial_r u(r) \geq \frac{r^{-\gamma(d)} e^{-2r}}{R^{-\gamma(d)} e^{-2r}} (-\partial_r u)(R), \quad r \geq R \geq R_0,
\]
Again, integrate this lower bound (4.11) from \( R \) to \( +\infty \) with Lemma A.1, we get
\[
u(R) = \int_{R}^{\infty} (-\partial_r u)(r) dr \gtrsim_{d} (-\partial_r u)(R).
\]
This also implies an exponential decay of \( -\partial_r u(r) \)
\[
-\partial_r u(r) \gtrsim_{d} \frac{r^{-\beta(d)} e^{-\frac{r}{2}}}{R^{-\beta(d)} e^{-\frac{R}{2}}} u(R), \quad r \geq R \geq R_0,
\]
Step 3. Controlling \( s_{\delta_0} \).
We want to obtain an an a priori bound for \( s_{\delta_0} \). Namely, there exists some \( R(d, \alpha, p) \sim_{d, \alpha, p} 1 \) (continuously depending on \( (\alpha, p) \)) such that
\[
s_{\delta_0} \leq R(d, \alpha, p).
\]
This time, we discuss the evolution of \( u \) and \( H[u] \) more carefully and utilize the structure of \( H[u] \).
First we derive pointwise control of \( u \) on a large interval via information from \( H[u] \). Then on such interval, the value of \( u \) implies a non-trivial change in \( H[u] \), which finally indicates a large variation in the evolution of \( u \). This contradicts with the previous pointwise bound when \( s_{\delta_0} \) is too large.

Step 3(a). Pointwise bound of \( u \) on a large interval.
Firstly, we may assume \( s_{\delta_0} \geq \max\{R_1(d), d^{1/2}\} \) from (4.5). Next, apply (2.1), we see

\[
\delta_0 = H[u](s_{\delta_0}) \sim_{d,\alpha} u(s_{\delta_0})^{p-2} \left( s_{\delta_0}^{-\alpha} \int_0^{s_{\delta_0}} u^p(s) s^{d-1} ds + \int_{s_{\delta_0}}^{\infty} u^p(s) s^{d-1-\alpha} ds \right).
\]

Also note that from the monotonicity of \( u \), exponential decay (4.9) and Lemma A.1 those two terms on the right hand side have different order of \( s_{\delta_0} \)

\[
s_{\delta_0}^{-\alpha} \int_0^{s_{\delta_0}} u^p(s) s^{d-1} ds \gtrsim_d u^p(s_{\delta_0}) s_{\delta_0}^{-\alpha}, \tag{4.15}
\]

\[
\int_{s_{\delta_0}}^{\infty} u^p(s) s^{d-1-\alpha} ds \gtrsim_{d,\alpha,p} u^p(s_{\delta_0}) s_{\delta_0}^{d-\alpha - 1}. \tag{4.16}
\]

So we have an \( R_2 = R_2(d,\alpha,p) \) and \( C_1 = C_1(d,\alpha,p) \), such that when \( s_{\delta_0} \geq R_2 \), (4.15) dominates and thus

\[
u(s_{\delta_0})^{p-2} s_{\delta_0}^{-\alpha} \int_0^{s_{\delta_0}} u^p(s) s^{d-1} ds \geq C_1. \tag{4.17}
\]

In this step, we assume \( s_{\delta_0} \geq \max\{R_2(d,\alpha,p), R_1(d), d^{1/2}\} \) later on.

Recall from Step 1 that \( s_{C_0} \leq R_1 \). We apply (2.1) to see for every \( r \geq R_1 \geq s_{C_0} \),

\[
C_0(d) \geq H[u](r) \gtrsim_{d,\alpha,p} u(r)^{p-2} r^{-\alpha} \int_0^{r} u^p(s) s^{d-1} ds \gtrsim_d u(r)^{2p-2} r^{d-\alpha}.
\]

Hence there exists \( C_2 = C_2(d,\alpha,p) \) and \( C_3 = C_3(d,\alpha,p) \), such that

\[
u(r) \leq C_2 r^{-\frac{d-\alpha}{2p-2}}, \quad \forall r \geq R_1, \tag{4.18}
\]

\[
u(R_1)^{p-2} R_1^{-\alpha} \int_0^{R_1} u^p(s) s^{d-1} ds \leq C_3.
\]

Take \( M \in [1, \frac{s_{\delta_0}}{R_1}] \) to be specified later, we have

\[
u^{p-2}(s_{\delta_0}) \int_0^{\frac{s_{\delta_0}}{M}} u^p(s) s^{d-1} ds \leq u(R_1)^{p-2} \int_0^{R_1} u^p(s) s^{d-1} ds + u \left( \frac{s_{\delta_0}}{M} \right)^{p-2} \int_{\frac{s_{\delta_0}}{M}}^{s_{\delta_0}} u^p(s) s^{d-1} ds
\]

\[
\leq C_3 + \left[ C_2 \left( \frac{s_{\delta_0}}{M} \right)^{-\frac{d-\alpha}{2p-2}} \right]^{p-2} \int_{\frac{s_{\delta_0}}{M}}^{s_{\delta_0}} \left(C_2 s^{-\frac{d-\alpha}{2p-2}} \right)^p s^{d-1} ds
\]

\[
\leq C_3 + C_2^{2p-2} \left( d - \frac{(d-\alpha)p}{2p-2} \right)^{-1} \left( \frac{s_{\delta_0}}{M} \right)^{\alpha}, \tag{4.19}
\]

where we used \( p \geq 2 > \frac{2d-\alpha}{d} \), so that \( -\frac{d-\alpha}{2p-2} p + d = \frac{p(d+\alpha)-2d}{2p-2} > 0 \). Now take \( M(d,\alpha,p) := \left( 4C_1C_2^{2p-2} \left( d - \frac{(d-\alpha)p}{2p-2} \right)^{-1} \right)^{\frac{1}{\alpha}} \) and \( R_3(d,\alpha,p) := \max\{R_1M, (4C_1C_2)^{\frac{1}{\alpha}} \} \). If \( s_{\delta_0} \geq \max\{R_1, R_2, R_3, d^{1/2}\} \), then (4.17) and (4.19) imply

\[
\int_{\frac{s_{\delta_0}}{M}}^{s_{\delta_0}} u^{p-2}(s) s^{d-1} ds \geq u^{p-2}(s_{\delta_0}) \int_{\frac{s_{\delta_0}}{M}}^{s_{\delta_0}} u^p(s) s^{d-1} ds \geq \frac{1}{2} C_1 s_{\delta_0}^{\alpha}. \tag{4.20}
\]

Combining this integral lower bound with the pointwise upper bound (4.18), we get a pointwise lower bound on a large subset of \([M^{-1}s_{\delta_0}, s_{\delta_0}]\): there exists an \( \mu(d,\alpha,p) \in (0,1) \) and \( C_4(d,\alpha,p) > 0 \), such that

\[
\left\{ r \in [M^{-1}s_{\delta_0}, s_{\delta_0}] : u(r) \geq C_4 r^{-\frac{d-\alpha}{2p-2}} \right\} \geq \mu s_{\delta_0}.
\]
To be more specific, there exists an $\delta_1(d, \alpha) > 0$ such that

$$H(r) - H\left(r + \frac{T_1 - T_0}{8}\right) \geq \delta_1, \quad r \in [T_0, T_0 + 7T_1].$$

(4.24)

Step 3(c). Large evolution of $u$.

Now we divide into two cases with respect to the position of $s_1$, namely $F(s_1) = 1$, and derive an a priori bound of $s_{\delta_0}$ in each case.

**Case 1.** $s_1 \leq \frac{T_0 + T_1}{8}$.

(4.24) implies that for $r \geq \frac{3T_0 + 5T_1}{8}$,

$$H(r) \leq H\left(r - \frac{T_1 - T_0}{8}\right) - \delta_1 \leq H(s_1) - \delta_1 = 1 - \delta_1,$$

The evolution estimate (4.22) then indicates a lower bound for $y(r)$ with $r_1 = r \in \left[\frac{3T_0 + 5T_1}{8}, \frac{T_0 + 3T_1}{4}\right]$ and $r_2 = T_1$:

$$y(r) \geq r^{-(d-1)} \int_r^{T_1} (1 - H(s))u(s)s^{d-1}ds \geq \frac{-(d-1)^{1}}{s_{\delta_0}^{\frac{d-\alpha}{2(p-1)}} + d-1} = \frac{1}{s_{\delta_0}^{\frac{d-\alpha}{2(p-1)}}}. $$
So we integrate it to get
\[ u \left( \frac{3T_0 + 5T_1}{8} \right) \geq \int_{\frac{T_0+3T_1}{2}}^{\frac{T_0+3T_1}{4}} y(r)dr \geq d,\alpha,p \ s_{\delta_0}^{-\frac{2-d-\alpha}{2(p-1)}}. \]

This implies a bound \( s_{\delta_0} \leq R'_4(d, \alpha, p) \) from (4.18).

**Case 2.** \( s_1 \geq \frac{T_0+T_1}{2} \).

Conversely, we focus on the interval \([T_0, \frac{T_0+T_1}{2}]\) in this case. For \( r \in \left[ T_0, \frac{5T_0+3T_1}{8} \right] \),
\[ H(r) \geq H \left( r + \frac{T_1 - T_0}{2} \right) + \delta_1 \geq H(s_1) + \delta_1 = 1 + \delta_1, \]

And for \( r \in \left[ \frac{3T_0+T_1}{4}, \frac{5T_0+3T_1}{8} \right] \), we take \( r_1 = T_0 \), \( r_2 = r \) and apply (4.22)
\[ y(r) \geq r^{-(d-1)} \int_{T_0}^{r} (H(s) - 1)u(s)d^d-1ds \geq d,\alpha,p \ s_{\delta_0}^{-\frac{(d-1)+1}{2(p-1)}+d-1} = \frac{1-\frac{d-\alpha}{2(p-1)}}{s_{\delta_0}}. \]

Similarly, we integrate that on \([\frac{3T_0+T_1}{4}, \frac{5T_0+3T_1}{8}]\) to get a lower bound of \( u \left( \frac{3T_0+5T_1}{8} \right) \). Combining with (4.18), this provides us with a bound \( s_{\delta_0} \leq R'_4(d, \alpha, p) \).

So now we can conclude this step by taking \( R(d, \alpha, p) := \max \{ d^2, R_1, R_2, R_3, R_4, R'_4 \} \) in (4.14).

**Step 4. A priori \( L^2 \) bound.**

In this step, we derive an a priori bound of \( L^2(\mathbb{R}^d) \) norm
\[ \|u\|_{L^2(\mathbb{R}^d)} \lesssim_{d,\alpha,p} 1 \]
with its constant depending continuously on \( (\alpha, p) \). We control \( \|u\|_{L^2(B_{\delta_0}^R)} \) and \( \|u\|_{L^2(B_{2R})} \) respectively with \( R \) from (4.14).

The control on exterior region follows directly from the exponential decay (4.9) and the pointwise control (4.18). Moreover, from (4.12), we also get \(-\partial_t u(2R) \lesssim_{d} u(2R) \lesssim_{d,\alpha,p} 1 \).

Regarding the interior region \( B_{2R} \), we use the local argument similar to Step 3(b) in §3 (originated in [10]), but more involved due to the essentially nonlocal nature of the nonlinearity \( H[u] \).

**Remark 4.1.** The structure of \( H[u] \) (specifically, (2.1)) also implies a simple way to obtain the desired bound when \( p = 2 \). Indeed,
\[ \delta_0 \geq H[u](2R) = (| \cdot |^{-\alpha} * u^2)(2R) \geq_{d,\alpha} R^{-\alpha} \int_{B_{2R}} u^2 dx \sim_{d,\alpha,p} \|u\|_{L^2(B_{2R})}^2. \]

However, this argument does not work for \( p > 2 \) due to the lack of lower bound of \( u(2R) \). Our following argument solves this problem and presents a uniform control for \( p \geq 2 \) as well.

Denote \( a := u(2R) \). We first claim the following two functional identities (the local version of (2.11) and (2.12))
\[ \int_{B_{2R}} (|\nabla u|^2 + u^2 - H[u]u^2) \, dx = I \]
\[ \int_{B_{2R}} \left( -\frac{d-2}{2} |\nabla u|^2 - \frac{d}{2} u^2 + \frac{2d-\alpha}{2p} H[u]u^2 \right) \, dx = II_1 + II_2 + II_3 \]
where

\[
I = a \int_{\partial B_{2R}} \partial_{\nu} u d\sigma,
\]

\[
II_1 = 2R \int_{\partial B_{2R}} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p} H[u] u^2 - \frac{1}{2} u^2 \right) d\sigma
\]

\[
II_2 = -\frac{\alpha}{2p} \int_{B_{2R}} \int_{B_{2R}^c} |x - y|^{-\alpha} u^p(y) u^p(x) dy dx
\]

\[
II_3 = \frac{\alpha}{p} \int_{B_{2R}} \int_{B_{2R}^c} |x - y|^{-\alpha} u^p(y) u^p(x) \frac{(x - y) \cdot x}{|x - y|^2} dy dx
\]

Indeed, (4.26) and (4.27) come from multiplying (Choquard) with \( u \) and \( x \cdot \nabla u \) respectively, integrating on \( B_{2R} \) and using integration by parts. The only tricky point is to apply the symmetry

\[
\int_{B_{2R}} \int_{B_{2R}} |x - y|^{-\alpha} u^p(y) u^p(x) \frac{(x - y) \cdot x}{|x - y|^2} dy dx = \int_{B_{2R}} \int_{B_{2R}} |x - y|^{-\alpha} u^p(y) u^p(x) \frac{(y - x) \cdot y}{|x - y|^2} dy dx
\]

\[
= \frac{1}{2} \int_{B_{2R}} \int_{B_{2R}} |x - y|^{-\alpha} u^p(y) u^p(x) dy dx.
\]

to derive (4.27).

Next, we claim that right hand sides of (4.26) and (4.27) are bounded for our \( u \).

**Claim.**

\[
|I| + |II_1| + |II_2| + |II_3| \lesssim_{d,\alpha,p} 1.
\]

We postpone its proof and see how this implies the \( L^2(B_{2R}) \) bound. Eliminating \( |\nabla u|^2 \) in (4.26) and (4.27), we get from the claim that

\[
\left| \int_{B_{2R}} \left[ \frac{2d - \alpha - p(d - 2)}{2p} H[u] - 1 \right] u^2 dx \right| \lesssim_{d,\alpha,p} 1
\]

(4.29)

Note that \( 2d - \alpha - p(d - 2) > 0 \) follows from (1.4). Take \( A := \frac{4p}{2d - \alpha - p(d - 2)} \). If \( s_A \geq 2R \), (4.29) yields the desired \( L^2 \) bound on \( B_{2R} \). So we consider \( s_A \in [0, 2R) \), in which case (4.29) indicates

\[
\int_{B_{s_A}} u^2 dx \lesssim_{d,\alpha,p} 1
\]

(4.30)

We further discuss \( u(s_A) \) to control \( L^2 \) norm on \( B_{2R} \setminus B_{s_A} \):

- If \( u(s_A) \leq 1 \), then

\[
\int_{B_{2R} \setminus B_{s_A}} u^2 dx \leq \int_{B_{2R} \setminus B_{s_A}} dx \lesssim_{d,\alpha,p} 1
\]

- If \( u(s_A) \geq 1 \), then the argument of Remark 4.1 works.

\[
A = (|\cdot|^{-\alpha} * u^p)(s_A) u^{p-2}(s_A) \geq (|\cdot|^{-\alpha} * u^p)(s_A)
\]

\[
\gtrsim_{d,\alpha} s_A^{-\alpha} \int_0^{s_A} u^p s^{d-1} ds + \int_{s_A}^{2R} u^p s^{d-1-\alpha} ds
\]

\[
\geq (2R)^{-\alpha} \int_{B_R} u^p dx \gtrsim_{d,\alpha,p} \|u\|_{L^p(B_{2R})}^p \gtrsim_{d,\alpha,p} \|u\|_{L^2(B_{2R})}^p.
\]

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These discussions and (4.30) imply (4.25).

To end this step, we now prove the claim. $I$ and $II_1$ are bounded from the pointwise bound of $u(2R)$ (by (4.18)), $-\partial_t u(2R)$ (by (4.13)) and $H[u](2R) \leq H[u](s_\delta) \leq \delta_0$. For $II_2$, we first check

\[
\sup_{x \in B_{2R}} \int_{B_{2R}^c} u^2(y)|x - y|^{-\alpha} dy \leq \sup_{x \in B_{2R}} \int_{B_{2R}^c} u^2(y)|x - y|^{-\alpha} dy + \int_{B_{4R} \setminus B_{2R}} u^2(y)|x - y|^{-\alpha} dy \leq_d \int_0^\infty u^2(r) \left( \frac{r}{2} \right)^{-\alpha} r^{d-1} dr + \int_{B_{6R}^c} |z|^{-\alpha} dz \lesssim_{d,\alpha,p} 1, \tag{4.31}
\]

where the last inequality follows the exponential decay (4.9) and $\alpha < d$. This yields

\[
|II_2| \leq \frac{\alpha}{2p} u^{p-2}(2R) \int_{B_{2R}} u^p(x) dx \left[ \sup_{x \in B_{2R}} \int_{B_{2R}^c} u^2(y)|x - y|^{-\alpha} dy \right] \lesssim_{d,\alpha,p} 1
\]

where we also apply (2.1) to obtain $u^{p-2}(2R) \int_{B_{2R}} u^p(x) dx \lesssim_{d,\alpha} R^\alpha H[u](2R) \leq R^\alpha \delta_0$.

Finally, we will control $II_3$. We integrate by parts to avoid the weight $|x - y|^{-\alpha - 1}$ which is not integrable when $\alpha \geq d - 1$

\[
pII_3 = \sum_{i=1}^d \int_{B_{2R}} x_i u^p(x) \int_{B_{2R}^c} \partial_y_i |x - y|^{-\alpha} u^p(y) dy dx
\]

\[
= \sum_{i=1}^d \int_{B_{2R}} x_i u^p(x) \int_{B_{2R}^c} \left[ \partial_y_i \left( |x - y|^{-\alpha} u^p(y) \right) - |x - y|^{-\alpha} \partial_y_i (u^p(y)) \right] dy dx
\]

\[
= - \sum_{i=1}^d \int_{B_{2R}} x_i u^p(x) \int_{\partial B_{2R}} |x - y|^{-\alpha} u^p(y) e_i \cdot d\sigma dx
\]

\[
- p \sum_{i=1}^d \int_{B_{2R}} x_i u^p(x) \int_{B_{2R}^c} |x - y|^{-\alpha} u^{p-1}(y) \partial_i u(y) dy dx := II_{31} + II_{32}
\]

where $e_i$ is the $i$-th vector of the standard basis.

Similar as (4.31), with the exponential decay of $-\partial_t u$ from (4.13), we have

\[
\sup_{x \in B_{2R}} \int_{B_{2R}^c} |x - y|^{-\alpha} u(y)|\nabla u(y)| dy \lesssim_{d,\alpha,p} 1. \tag{4.32}
\]

Then

\[
|II_{32}| \lesssim_{d,\alpha,p} 1
\]

follows in the same way as controlling $II_2$. And $II_{31}$ comes as

\[
|II_{31}| \leq 2dR \left[ \int_{B_{4R}} u^p(x) \int_{\partial B_{2R}} |x - y|^{-\alpha} u^p(2R) d\sigma dx + \int_{B_{2R} \setminus B_{4R}} u^p(x) \int_{\partial B_{2R}^c} |x - y|^{-\alpha} u^p(2R) d\sigma dx \right]
\]

\[
\leq 2dR \left[ \int_{B_{4R}} u^p(x) dx R^{-\alpha} u^p(2R) \int_{\partial B_{2R}} d\sigma + u^p(R) u^p(2R) \int_{B_{4R}} |z|^{-\alpha} dz \int_{\partial B_{2R}} d\sigma \right]
\]

\[
\lesssim_{d,\alpha,p} H[u](R) u^2(2R) + u^p(R) u^p(2R) \lesssim_{d,\alpha,p} 1.
\]

This finishes the proof of that claim and this step.

**Step 5. Concluding the proof.**

In this final step, we use the $L^2$ a priori bound (4.25) to verify (1.7) and (1.8).
The $L^2$ bound implies an $H^1$ bound immediately from Proposition 2.4. Then for given $r \in (1, \infty)$, we apply the proof of [27, Proposition 4.1], the classic bootstrap method for semilinear elliptic problems plus the Hardy-Littlewood-Sobolev inequality to improve the regularity of solutions. The $H^1$ bound therefore implies a $W^{2,r}$ bound (1.7) after finite times of bootstrap iterations. The uniformity of $(\alpha, p, r)$ in (1.7) comes from the uniformity of constants in the Hardy-Littlewood-Sobolev inequality and the elliptic estimates in every iteration.

As for (1.8), we use Sobolev embedding to get a $C^1(\mathbb{R}^d)$ bound from (1.7). Then we get pointwise bound of $u(R)$ from (4.18) and (4.14), and thus obtain exponential decay of $u(r)$, $|\nabla u(r)| = -\partial_r u(r)$ when $r \geq R(\alpha, p, R)$ from (4.9) and (4.13). These two facts yield (1.8) and finish the whole proof of Theorem 1.5.

5. Non-degeneracy and Uniqueness

In this section, we prove the non-degeneracy and uniqueness of positive solution for $(\alpha, p) \in [d - 2 - \delta, d - 2 + \delta] \times [2, 2 + \delta]$ with $d \in \{3, 4, 5\}$, $\delta \ll 1$. The starting point will be non-degeneracy and uniqueness for the Newtonian case $(\alpha, p) = (d - 2, 2)$ with $d \in \{3, 4, 5\}$. We refer the reader to [9] for non-degeneracy and [2, Appendix A] for uniqueness, also to [21, 22] for the original proof for $d = 3$.

5.1. Compactness analysis. As a preparation, we first establish the following compactness result for radial positive solutions. It makes use of the a priori bound Theorem 1.5.

**Proposition 5.1.** Let $d \in \{3, 4, 5\}$, there exists $\delta > 0$ such that the following holds. For any sequence $\{(\alpha_n, p_n)\} \subset [d - 2 - \delta, d - 2 + \delta] \times [2, 2 + \delta]$ with $(\alpha_n, p_n) \to (d - 2, 2)$ when $n \to \infty$, and $Q_n$ be an $H^1$ radial positive solution for (Choquard) with parameters $(d, \alpha_n, p_n)$. Then

$$Q_n \to Q_0 \quad \text{in } H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

where $Q_0$ is the unique radial positive solution for $(d, \alpha, p) = (d, d - 2, 2)$.

**Proof.** It suffices to show that for any sequence $\{Q_n\}$ as above, we can find subsequence $Q_{n_k} \to Q_0$ in $H^1$ and $L^\infty$. Before starting, we take $\delta$ small enough such that $(\alpha_n, p_n)$ satisfies (1.4).

**Step 1. Bounds and convergences.**

Theorem 1.5 and the range of $(\alpha_n, p_n)$ indicate a uniform upper bound

$$\|Q_n\|_{H^1} + \|Q_n\|_{L^\infty} \lesssim_{d, \delta} 1.$$  \hfill (5.1)

Note that $[\frac{11}{3}, 3] \subset (2, \frac{2d}{d-2})$ for $d \in \{3, 4, 5\}$. Thus by the compact embedding $H^1_{rad}(\mathbb{R}^d) \hookrightarrow L^q_{rad}(\mathbb{R}^d)$ for $q \in (2, \frac{2d}{d-2})$, there exist $Q \in H^1_{rad}(\mathbb{R}^d)$ and a subsequence (still denoted by $\{Q_n\}$) such that

$$Q_n \rightharpoonup Q \quad \text{in } H^1; \quad Q_n \to Q \quad \text{in } L^\frac{2d}{d} \cap L^3.$$  \hfill (5.2)

A further subsequence (still denoted by $\{Q_n\}$) implies almost everywhere convergence

$$Q_n \to Q \quad \text{a.e.}$$  \hfill (5.3)

With these bounds and convergences, we claim the following convergence.

**Claim.** For any uniformly bounded $L^\frac{2d}{d}$ functions $\{\phi_n\}$ and $\phi$ with $\phi_n \to \phi$ in $L^\frac{2d}{d}$, we have

$$\int_{\mathbb{R}^d} (|\cdot|^{-\alpha_n} * Q_n^p) Q_n^p \phi_n dx \to \int_{\mathbb{R}^d} (|\cdot|^{-(d-2)} * Q^2) Q \phi dx$$  \hfill (5.4)
Step 2. End of the proof with the claim.

Postponing the proof of (5.4) to Step 3, we finish the proof of this proposition with that convergence. Firstly, for any $\phi \in H^1(\mathbb{R}^d)$, we take $\phi_n = \phi$ for all $n \in \mathbb{N}$ and use (5.2) and (5.4), then

$$\int_{\mathbb{R}^d} \left[ \nabla Q \cdot \nabla \phi + Q \phi - \left( | \cdot |^{-(d-2)} + Q^2 \right) Q \phi \right] dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} \left[ \nabla Q_n \cdot \nabla \phi + Q_n \phi - \left( | \cdot |^{-\alpha_n} * Q_n^p \right) Q_n^{p-1} \phi \right] dx = 0.$$ 

Hence $Q$ is an $H^1$ radial solution of (Choquard) with $(d, \alpha, p) = (d, d - 2, 2)$. Taking $\phi_n := Q_n \to Q =: \phi$ in $L^{\frac{2d}{d-2}}$, which follows (5.2) and that $\frac{4d}{3} < \frac{4d}{d-2} < 3$ for $d \in \{3, 4, 5\}$, we have

$$\int_{\mathbb{R}^d} \left( | \cdot |^{-\alpha_n} * Q_n^p \right) Q_n^{p-1} dx \to \int_{\mathbb{R}^d} \left( | \cdot |^{-(d-2)} + Q^2 \right) Q^2 dx.$$ 

(5.5)

Also by Corollary 2.5 and (2.11), for $(\alpha_n, p_n) \in [d - 2 - \delta, d - 2 + \delta] \times [2, 2 + \delta]$, we have a uniform lower bound

$$\int_{\mathbb{R}^d} \left( | \cdot |^{-\alpha_n} * Q_n^p \right) Q_n^p dx = \|Q_n\|_{H^1}^2 \gtrsim_{d, \delta} 1.$$

So the limit in (5.5) is nonzero and $Q$ is nonzero. Notice that (5.3) ensures that $Q$ is also nonnegative and, moreover, positive from the strong maximal principle. So far, we have verified $Q$ to be a positive and radial $H^1$ solution for (Choquard) with $(d, \alpha, p) = (d, d - 2, 2)$. Thus $Q = Q_0$ by uniqueness.

Together with (2.13), (2.14) and $(\alpha_n, p_n) \to (d - 2, 2)$, we see

$$\|Q_n\|_{L^2}^2 \to \|Q\|_{L^2}^2, \quad \|\nabla Q_n\|_{L^2}^2 \to \|\nabla Q\|_{L^2}^2.$$

Thus (5.2) can be improved to be strong $H^1$ convergence.

As for $L^\infty$ convergence, we use similar strategy as in Step 5 of §4 to improve the regularity. Take the difference of equations of $Q_n$ and $Q_0$, we have

$$(-\Delta + 1)(Q_n - Q_0) = (| \cdot |^{-\alpha_n} * Q_n^p)Q_n^{p-1} - (| \cdot |^{-(d-2)} * Q_0^2)Q_0.$$ 

(5.6)

Using (5.26) in Lemma C.5 with $u_0 = Q_0$, $u = Q_n$, $(\alpha, p) = (\alpha_n, p_n)$, we get from $\|Q_n - Q_0\|_{H^1} = o_n(1)$ that

$$\|(\cdot |^{-\alpha_n} * Q_n^p)Q_n^{p-1} - (| \cdot |^{-(d-2)} * Q_0^2)Q_0\|_{L^{\frac{2d}{d-2}}} = o_n(1).$$

Then since $W^{2, \frac{2d}{d-2}}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ when $d < 6$, (5.6) implies

$$\|Q_n - Q_0\|_{L^\infty} \lesssim_d \|Q_n - Q_0\|_{W^{2, \frac{2d}{d-2}}} \lesssim_d \|(| \cdot |^{-\alpha_n} * Q_n^p)Q_n^{p-1} - (| \cdot |^{-(d-2)} * Q_0^2)Q_0\|_{L^{\frac{2d}{d-2}}} = o_n(1)$$

which is our desired $L^\infty$ convergence.

Step 3. Proof of the claim.

Finally, we verify the claim. Decompose

$$\int_{\mathbb{R}^d} (| \cdot |^{-\alpha_n} * Q_n^p)Q_n^{p-1}\phi_n dx - \int_{\mathbb{R}^d} (| \cdot |^{-(d-2)} * Q_n^2)Q\phi dx := I + II + III$$

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where
\[ I := \int_{\mathbb{R}^d} (| \cdot |^{-\alpha_n} * Q_n^p) (Q_n^{p_n-1} \phi_n - Q \phi) dx, \]
\[ II := \int_{\mathbb{R}^d} (| \cdot |^{-\alpha_n} (Q_n^{p_n} - Q^2)) Q \phi dx, \]
\[ III := \int_{\mathbb{R}^d} \left( (| \cdot |^{-\alpha_n} - | \cdot |^{-(d-2)}) * Q^2 \right) Q \phi dx. \]

Then we may restrict \( \delta \) to be smaller to estimate each term on the right hand side.

For \( I \),
\[ |I| \leq \left| \int_{\mathbb{R}^d} (| \cdot |^{-\alpha_n} * Q_n^p) (Q_n^{p_n-1} - Q) \phi_n dx \right| + \left| \int_{\mathbb{R}^d} (| \cdot |^{-\alpha_n} * Q_n^p) Q (\phi_n - \phi) dx \right| \]
\[ \leq \| | \cdot |^{-\alpha_n} * Q_n^p \|_{L^{\frac{2d}{d+2}}} \left[ \|Q_n^{p_n-1} - Q\|_{L^{\frac{4d}{d+2}}} \|\phi_n\|_{L^{\frac{4d}{d+2}}} + \|Q\|_{L^{\frac{4d}{d+2}}} \|\phi_n - \phi\|_{L^{\frac{4d}{d+2}}} \right]. \]

Using Hardy-Littlewood-Sobolev inequality,
\[ \| | \cdot |^{-\alpha_n} * Q_n^p \|_{L^{\frac{2d}{d+2}}} \lesssim_{d, \delta} \|Q_n^{p_n}\|_{L^{\frac{4d}{d+2}}} \lesssim_{d, \delta} \|Q_n\|_{H^\delta} \lesssim_{d, \delta} 1 \quad (5.7) \]
where \( \frac{2dp_n}{3d - 2 - 2\alpha_n} = \frac{4d}{d+2} + \alpha_n(1) \) so that we can take \( \delta \) small enough such that
\[ \frac{2dp}{3d - 2 - 2\alpha} \in \left[ \frac{11}{5}, 3 \right] \quad \forall (\alpha, p) \in [d - \delta, d - 2 + \delta] \times [2, 2 + \delta]. \quad (5.8) \]

Also, we introduce \( T \gg 1 \) to estimate
\[ \|Q_n^{p_n-1} - Q\|_{L^{\frac{4d}{d+2}}} \leq \|Q_n^{p_n-1} - Q_n\|_{L^{\frac{4d}{d+2}}} + \|Q_n - Q\|_{L^{\frac{4d}{d+2}}} \]
\[ \leq \|Q_n^{p_n-2} - 1\|_{L^{\frac{12d}{6d-1}}(B_T)} \|Q_n\|_{L^3(B_T)} + \left( \|Q_n\|_{L^{\frac{4d}{4d(p_n - 1)}}(B_T^{p_n})} + \|Q_n\|_{L^{\frac{4d}{4d+2}}(B_T^{p_n})} \right) + \|Q_n - Q\|_{L^{\frac{4d}{d+2}}} \]

From the \( L^\infty \) bound (5.1) and almost everywhere convergence (5.3), the dominant convergence implies that \( \|Q_n^{p_n-2} - 1\|_{L^{\frac{12d}{6d-1}}(B_T)} = o_n(1) \) for fixed \( T \). We require \( \delta \) sufficiently small such that
\[ \frac{4d(p - 1)}{d + 2} \in \left[ \frac{11}{5}, 3 \right] \quad \forall p \in [2, 2 + \delta], \quad (5.9) \]
so (5.2) ensures that \( \sup_n \left( \|Q_n\|_{L^{\frac{4d(p_n - 1)}{4d+2}}(B_T^{p_n})} + \|Q_n\|_{L^{\frac{4d}{d+2}}(B_T^{p_n})} \right) = o_{T \to \infty}(1). \) Thus
\[ \|Q_n^{p_n-1} - Q\|_{L^{\frac{4d}{d+2}}} = o_n(1) \quad (5.10) \]

Combining (5.7), (5.10) and assumption on \( \phi_n \), we arrive at
\[ |I| = o_n(1). \]

For \( II \), by Hölder inequality and Hardy-Littlewood-Sobolev inequality,
\[ |II| \lesssim_{d, \delta} \|Q_n - Q^2\|_{L^{\frac{4d}{d+2}}} \|Q\|_{L^{\frac{4d}{d+2}}} \|\phi\|_{L^{\frac{4d}{d+2}}} \]
\[ \leq \|Q_n^{p_n-1} - Q\|_{L^{\frac{4d}{d+2}}} \|Q\|_{L^{\frac{4d}{d+2}}} \|\phi\|_{L^{\frac{4d}{d+2}}} \]
We further require \( \delta \ll 1 \) such that
\[ \frac{4d}{5d - 6 - 4\alpha_n} \in \left[ \frac{11}{5}, 3 \right] \quad \forall \alpha \in [d - 2 - \delta, d - 2 + \delta]. \quad (5.11) \]
(5.1) and (5.3) indicate \(|Q|_{L^\infty} \lesssim d, \delta 1\), so we can prove \(|Q^{n-1} - Q|_{L^{2+\delta}} = o_{n \to \infty}(1)\) as before.

These estimates imply
\[ |II| = o_n(1). \]

Regarding III, we use dominant convergence. Obviously \(\left( (| \cdot |^{-\alpha_n} - | \cdot |^{-\alpha}) * Q^2 \right) Q\phi \to 0\) almost everywhere when \(n \to \infty\). Also, since
\[
\left( (| \cdot |^{-\alpha_n} - | \cdot |^{-(d-2)}) * Q^2 \right) Q\phi \leq \left( (| \cdot |^{-d+2+\delta} + | \cdot |^{-(d+2-\delta)}) * Q^2 \right) Q\phi,
\]
and
\[
\int_{\mathbb{R}^d} \left( | \cdot |^{-d+2+\delta} * Q^2 \right) Q\phi dx \lesssim_{d, \delta} \|Q\|^2_{L^{d+2+\delta}} \|Q\|^4_{L^{d+2}} \|\phi\|_{L^{2}} \lesssim_{d, \delta} 1
\]
if
\[
\frac{11}{5} \leq \frac{4d}{d+2+2\delta} \leq 3, \tag{5.12}
\]
then \((| \cdot |^{-d+2+\delta} + | \cdot |^{-(d+2-\delta)}) * Q^2 \) is a feasible dominant function to derive
\[ |III| = o_n(1). \]

To conclude, if we take \(\delta\) small enough such that (5.8), (5.9), (5.11) and (5.12) hold, then the claim is true and thus the proposition is proven. \(\square\)

5.2. Non-degeneracy. In this subsection, we prove Theorem 1.8. From Theorem 1.4 and the translation-invariance of \(L_+\), we only need to discuss radial positive solutions with \((\alpha, p)\) close to \((d - 2, 2)\). We will denote a radial positive solution of (Choquard) with parameters \((d, \alpha, p)\) by \(Q_{d, \alpha, p}\) if omitting \(d\) causes no trouble.

To begin with, we decompose \(L_{+, u, d, \alpha, p}\) as (1.10) to be
\[
L_{+, u, d, \alpha, p} = -\Delta + 1 - (p - 1)V_{u, d, \alpha, p} - pA_{u, d, \alpha, p} \tag{5.13}
\]
where (omitting \(d\) for convenience)
\[ V_{u, \alpha, p} := | \cdot |^{-\alpha} |u|^p |u|^{p-2}, \tag{5.14} \]
\[ A_{u, \alpha, p} := | \cdot |^{-\alpha} (|u|^p - 2u\xi)|u|^{p-2}u. \tag{5.15} \]

In Appendix C, we discuss properties of these operators for varying \((u, \alpha, p)\): boundedness, compactness and continuous dependence on \((u, \alpha, p)\). They lay the foundation for the perturbative argument in proving Theorem 1.8 and Theorem 1.9.

Consider the kernel of \(L_{+, \alpha, p}\). From the non-degeneracy of Choquard equation at \((\alpha, p) = (d - 2, 2)\) for \(d \in \{3, 4, 5\}\), we have
\[
\text{Ker} L_{+, Q_{d-2, 2}} = \text{span} \{ \partial_i Q_{d-2, 2} \}_{i=1}^d \tag{5.16}
\]
Also by differentiating (Choquard), we have
\[
\text{Ker} L_{+, Q_{d, \alpha, p}} \supseteq \text{span} \{ \partial_i Q_{d, \alpha, p} \}_{i=1}^d \tag{5.17}
\]
for any positive solution \(Q_{\alpha, p}\) of (Choquard) with \((\alpha, p)\) satisfying (1.4). We will use an argument of spectral perturbation to show the other side of (5.17)
\[
\dim \text{Ker} L_{+, Q_{\alpha, p}} \leq \dim \text{Ker} L_{+, Q_{d-2, 2}} = d \tag{5.18}
\]
when \((\alpha, p)\) close to \((d - 2, 2)\).

\[\text{We remark that our argument here does not require uniqueness of positive solutions for } (\alpha, p) \neq (d - 2, 2).\]

\[\text{In particular, for } Q_{\alpha, p}, \text{ we may further simplify the notation to be } L_{+, Q_{\alpha, p}}, V_{Q_{\alpha, p}}, \text{ and } A_{Q_{\alpha, p}}.\]
From Lemma C.4 and $Q_{d-2,2} \in W^{2,r}(\mathbb{R}^d)$ for $r \in (1, \infty)$, $(-\Delta + 1)^{-1}V_{Q_{d-2,2}}$ and $(-\Delta + 1)^{-1}A_{Q_{d-2,2}}$ are compact operators on $L^2(\mathbb{R}^d)$ and thus $L_{+,Q_{0,p}}$ is a compact perturbation of $-\Delta + 1$.

$\sigma_{ess}(L_{+,Q_{0,p}}) = \sigma_{ess}(-\Delta + 1) = [1, \infty)$. In particular, 0 is an isolated eigenvalue of $L_{+,Q_{d-2,2}}$. So we can define the Riesz projection

$$P_{0,Q_{d-2,2}} := \frac{1}{2\pi i} \oint_{\partial D_r} (L_{+,Q_{d-2,2}} - z)^{-1}dz,$$

where $D_r := \{z \in \mathbb{C} : |z| < r\}$ and $r$ sufficiently small such that $\text{Im} P_{0,Q_{d-2,2}} = \text{Ker} L_{+,Q_{d-2,2}}$.

Notice that

$$L_{+,Q_{0,p}} - L_{+,Q_{d-2,2}} = -(p-1)(V_{Q_{0,p}} - V_{Q_{d-2,2}}) - p(A_{Q_{0,p}} - A_{Q_{d-2,2}}) - (p-2)(V_{Q_{d-2,2}} + A_{Q_{d-2,2}}).$$

Lemma C.4, Lemma C.5 and the $H^1 \cap L^\infty$ approximation from Proposition 5.1 imply

$$\|L_{+,Q_{0,p}} - L_{+,Q_{d-2,2}}\|_{L^2 \to L^2} = o_{(\alpha,p)}(d-2,2)(1).$$

Thus for the $r$ taken as above and $\delta_1 \leq \delta$ small enough, we have

$$\|(L_{+,Q_{0,p}} - z)^{-1}\|_{L^2 \to L^2} \leq 2\|(L_{+,Q_{d-2,2}} - z)^{-1}\|_{L^2 \to L^2}$$

when $(\alpha,p) \in [d-2 - \delta_1, d-2 + \delta_1] \times [2, 2 + \delta_1]$ and thereafter

$$\|(L_{+,Q_{0,p}} - z)^{-1} - (L_{+,Q_{d-2,2}} - z)^{-1}\|_{L^2 \to L^2} = o_{(\alpha,p)}(d-2,2)(1).$$

So for such $(\alpha,p)$,

$$P_{0,Q_{0,p}} := \frac{1}{2\pi i} \oint_{\partial D_r} (L_{+,Q_{d-2,2}} - z)^{-1}dz$$

is a bounded operator on $L^2$ and

$$\|P_{0,Q_{0,p}} - P_{0,Q_{d-2,2}}\|_{L^2 \to L^2} = o_{(\alpha,p)}(d-2,2)(1).$$

Note that $P_{0,Q_{d-2,2}}$ is a Fredholm operator as a finite-rank projection. Via the theory of perturbation of Fredholm operators, there exists a $\delta_2 \leq \delta_1$ such that (5.18) holds for $(\alpha,p) \in [d-2 - \delta_2, d-2 + \delta_2] \times [2, 2 + \delta_2]$. This and (5.17) concludes the proof of Theorem 1.8.

5.3. Uniqueness. In this subsection, we prove Theorem 1.9. We start with defining

$$X_d := L^2_{rad}(\mathbb{R}^d) \cap L^{10}_d(\mathbb{R}^d), \quad d \in \{3, 4, 5\}.$$  

(5.21)

Note that $H^2(\mathbb{R}^d) \hookrightarrow X_d$ for such $d$.

Now we can state and prove a local uniqueness result.

**Proposition 5.2.** For $d \in \{3, 4, 5\}$. Let $Q_{d-2,2}$ be the unique radial positive solution for $(\alpha,p) = (d-2,2)$ of (Choquard). Then there exist $\delta_1 > 0$ and a $C^0$ map $\tilde{Q}_0 : [d-2-\delta_1, d-2+\delta_1] \times [2, 2+\delta_1] \to X_d$ such that the following holds, where we denote $\tilde{Q}_{\alpha,p} := \tilde{Q}(\alpha,p)$.

1. $\tilde{Q}_{\alpha,p}$ is an $H^1$ radial solution of (Choquard) with parameters $(d,\alpha,p)$.

2. There exists $\epsilon > 0$ such that $\tilde{Q}_{\alpha,p}$ is the unique $H^1$ radial solution of (Choquard) with parameters $(d,\alpha,p)$ in the neighborhood $\{u \in X_d : \|u - Q_{d-2,2}\|_{X_d} \leq \epsilon\}$. In particular, $\tilde{Q}_{d-2,2} = Q_{d-2,2}$.

**Proof.** For $d \in \{3, 4, 5\}$ and $(d,\alpha,p)$ satifying (1.4), it’s easy to see $u$ is an $H^1$ solution of (Choquard) if and only if $u \in X_d$ is a solution of

$$u - (-\Delta + 1)^{-1}[(|\cdot|^{-\alpha} * |u|^p)|u|^{p-2}u] = 0.$$  

Define $F : X_d \times [d-2-\delta_1, d-2+\delta_1] \times [2, 2+\delta_1] \to X_d$ by

$$F(u,\alpha,p) = u - (-\Delta + 1)^{-1}[(|\cdot|^{-\alpha} * |u|^p)|u|^{p-2}u].$$  

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where $\delta_1$ is small enough and to be determined. First we require $\delta_1$ to be smaller than the $\delta$ in Lemma C.3, then $F$ is well-defined, continuous and differentiable w.r.t. $u$, and

$$\partial_u F(u, \alpha, p) = \text{Id} - (-\Delta + 1)^{-1} [(p - 1)V_{u, \alpha, p} + pA_{u, \alpha, p}] = (-\Delta + 1)^{-1} L_{+, \alpha, p}$$

(5.23)
is continuous at $(Q_{d-2,2}, d - 2, 2)$. If $\partial_u F(Q_{d-2,2}, d - 2, 2)$ is invertible in $L(X_d)$, then these facts enable us to apply the implicit function theorem Proposition B.1. The assertions (1) and (2) follow directly from its conclusion and the above equivalence.

To conclude the proof, we check the invertibility of $\partial_u F$ at $(Q_{d-2,2}, d - 2, 2)$. Again from Lemma C.4 and regularity of $Q_{d-2,2}$, $(-\Delta + 1)^{-1} V_{Q_{d-2,2}}$ and $(-\Delta + 1)^{-1} A_{Q_{d-2,2}}$ are compact operators on $L^2(\mathbb{R}^d)$ and $L^{10}(\mathbb{R}^d)$ respectively, and thus also compact on $X_d$. This indicates that $\partial_u F(Q_{d-2,2}, d - 2, 2)$ is a Fredholm operator on $X_d$ by (5.23). On the other hand, non-degeneracy of $L_{+, Q_{d-2,2}}$ indicates that

$$\text{Ker} L_{+, Q_{d-2,2}}|_{L^2_{rad}(\mathbb{R}^d)} = \{0\},$$

which implies that $\partial_u F(Q_{d-2,2}, d - 2, 2)$ is injective on $X_d$. Properties of Fredholm operators show that $\partial_u F(Q_{d-2,2}, d - 2, 2)$ is also bijective and therefore has a bounded inverse. That finishes the proof.

Now we are in place to prove Theorem 1.9.

Proof of Theorem 1.9. From Theorem 1.4, we only need to show the uniqueness of the $H^1$ radial positive solution. Given $d \in \{3, 4, 5\}$, take $\delta_1$ and $\epsilon$ as in Proposition 5.2. Proposition 5.1 indicates that there exists $\delta_2 > 0$ such that any $H^1$ radial positive solution $Q_{\alpha, p}$ for (Choquard) with parameters $(\alpha, p) \in [d - 2 - \delta_2, d + \delta_2] \times [2, 2 + \delta_2]$ satisfying

$$\|Q_{\alpha, p} - Q_{d-2,2}\|_{X} \leq \epsilon.$$  

(5.24)

Now taking $\delta = \min\{\delta_1, \delta_2\}$, for every $(\alpha, p) \in [d - 2 - \delta, d + \delta] \times [2, 2 + \delta]$, Theorem 1.1 (1) indicates that there exists an $H^1$ radial positive solution $Q_{\alpha, p}$ satisfying (5.24), which, by Proposition 5.2, must equal to $Q_{\alpha, p}$ and be unique in this neighborhood. Since Proposition 5.1 also guarantees the non-existence of $H^1$ radial positive solutions outside this neighborhood, $Q_{\alpha, p}$ is exactly the unique $H^1$ radial positive solution for (Choquard).

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Appendix A. A computational lemma

We derive the following lemma estimating the integration of exponential function multiplied by a polynomial.

Lemma A.1. For $\alpha \in \mathbb{R}$, $\beta \geq \frac{1}{2}$ and $R \geq 1$, we define

$$I(R; \alpha, \beta) := \int_{R}^{\infty} r^{-\alpha} e^{-\beta r} dr.$$  

(A.1)

Then we have

$$I(R; \alpha, \beta) \sim_{\alpha, \beta} R^{-\alpha} e^{-\beta R},$$  

(A.2)

and the constant can be taken uniformly for $(\alpha, \beta)$ in a compact subset of $\mathbb{R} \times [\frac{1}{2}, \infty)$.
Thus we only need to prove (A.2) for $\beta = 1$ and $R \geq \frac{1}{2}$.

The upper bound for $\alpha \geq 0$ comes easily as

$$I(R; \alpha, 1) \leq R^{-\alpha} \int_{R}^{\infty} e^{-r} dr = R^{-\alpha} e^{-R}.$$

Similarly, the lower bound for $\alpha \leq 0$ holds.

As for the lower bound for $\alpha > 0$, it’s a classical result for $\alpha = n \in \mathbb{N}_+$ (refer to [1, 5.1.19])

$$I(R; n, 1) \geq e^{-R} R^{-n+1} \frac{1}{R+n} \geq \frac{1}{2n+1} R^{-n} e^{-R}, \quad R \geq \frac{1}{2}, \; n = 1, 2, 3 \ldots$$

And for the interrange case $\alpha \in (n-1, n)$ with $n \in \mathbb{N}_+$,

$$I(R; \alpha, 1) \geq \int_{R}^{\infty} R^{-\alpha} r^{-n} e^{-r} dr \geq \frac{1}{2\alpha + 3} R^{-\alpha} e^{-R}.$$

Finally, we check the upper bound for $\alpha < 0$ case. Using integration by parts

$$I(R; \alpha, 1) = R^{-\alpha} e^{-R} - \alpha I(R; \alpha + 1, 1), \quad \alpha \neq 0,$$

$I(R; \alpha, 1)$ with $-\alpha \in (n-1, n]$ can be bounded within $n$ times of iterations

$$I(R; \alpha, 1) = R^{-\alpha} e^{-R} + \sum_{k=1}^{n-1} \left( \prod_{j=1}^{k} (-\alpha -1 +j) \right) R^{-\alpha-k} e^{-R}$$

$$+ \left( \prod_{j=1}^{n} (-\alpha -1 +j) \right) I(R; \alpha + n, 1) \lesssim_{\alpha} R^{-\alpha} e^{-R}.$$

The final inequality utilizes the upper bound for $\alpha \geq 0$ case and $R \geq \frac{1}{2}$. \qed

**Appendix B. A refined implicit function theorem**

We carefully check the proof of the classical implicit function theorem in Banach space (see for example [8, Theorem 1.2.1]) to slightly relax the $C^1$ condition to $C^1$ at one point. It will be used in proving Proposition 5.2. In this subsection, we use $B_r(x), \bar{B}_r(x)$ to denote open and closed balls respectively in general Banach spaces.

**Proposition B.1.** Let $X,Y,Z$ be Banach spaces, $U \subset X \times Y$ be an open set. Suppose that $f \in C(\bar{U}, Z)$ is differentiable w.r.t. $y$. For a point $(x_0, y_0) \in \bar{U}$, if $f_y : \bar{U} \to L(Y,Z)$ is continuous at $(x_0, y_0)$ and

$$f(x_0, y_0) = 0,$$

$$f_y^{-1}(x_0, y_0) \in L(Z,Y),$$

then there exist $r, r_1 > 0$ and a $C^0$ map $u : \bar{B}_r(x_0) \to \bar{B}_{r_1}(y_0)$, such that

$$\begin{cases} \bar{B}_r(x_0) \times \bar{B}_{r_1}(y_0) \subset \bar{U}, \\ u(x_0) = y_0, \\ f(x, u(x)) = 0 \quad \forall \; x \in \bar{B}_r(x_0), \end{cases}$$

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Furthermore, if \( f(x, y) = 0 \) for some \((x, y) \in \bar{B}_r(x_0) \times \bar{B}_{r_1}(y_0)\), then \( y = u(x) \).

**Proof.** Consider

\[
g(x, y) := f_y^{-1}(x_0, y_0) \circ f(x + x_0, y + y_0).
\]

We look for the solution \( y = u(x) \in \bar{B}_{r_1}(0) \) of \( g(x, y) = 0 \) for \( x \in \bar{B}_r(0) \), with \( r, r_1 \) small enough and determined later. Define

\[
R(x, y) := y - g(x, y),
\]

and then we will check that \( R(x, \cdot) \) for \( x \in \bar{B}_r(0) \) is a contraction mapping on \( \bar{B}_{r_1}(0) \).

Firstly, for any \( x \in \bar{B}_r(0) \) and \( y_1, y_2 \in \bar{B}_{r_1}(0) \), we have

\[
\|R(x, y_1) - R(x, y_2)\|_Y = \|y_1 - y_2 - [g(x, y_1) - g(x, y_2)]\|_Y
\]

\[
\leq \int_0^1 \|y - g(x, y)\|_Y dt \cdot \|y_1 - y_2\|_Y.
\]

From the continuity of \( f_y \) at \((x_0, y_0)\), there exist \( r, r_1 \ll 1 \) such that for \((x, y) \in \bar{B}_r(x_0) \times \bar{B}_{r_1}(y_0) \subset U\),

\[
\|f_y(x, y) - f_y(x_0, y_0)\|_{L(Y,Z)} \leq \frac{1}{2} \|f_y^{-1}(x_0, y_0)\|_{L(Z,Y)}^{-1}.
\]

This leads to

\[
\|R(x, y_1) - R(x, y_2)\|_Y \leq \frac{1}{2} \|y_1 - y_2\|_Y. \tag{B.1}
\]

Secondly, we verify \( R(x, \cdot) : \bar{B}_{r_1}(0) \to \bar{B}_{r_1}(0) \). Indeed, note that

\[
\|R(x, y)\|_Y \leq \|R(x, 0)\|_Y + \|R(x, y) - R(x, 0)\|_Y
\]

\[
\leq \|f_y^{-1}(x_0, y_0)\|_{L(Z,Y)} \|f(x + x_0, y_0)\|_Z + \frac{1}{2} \|y\|_Y.
\]

Using continuity of \( f \), we shrink \( r \) further to guarantee

\[
\sup_{x \in \bar{B}_r(0)} \|f(x + x_0, y_0)\|_Z \leq \frac{r_1}{2} \|f_y^{-1}(x_0, y_0)\|_{L(Z,Y)}^{-1},
\]

which leads to \( R(x, y) \leq r_1 \) for \((x, y) \in \bar{B}_r(0) \times \bar{B}_{r_1}(0)\).

Thus \( R \) is a contraction and for all \( x \in \bar{B}_r(0) \), there exists a unique \( y \in \bar{B}_{r_1}(0) \) satisfying \( g(x, y) = 0 \), namely \( f(x + x_0, y + y_0) = 0 \). We denote by \( y(x) \) the solution \( y \), and \( u(x) := y(x-x_0)+y_0 \).

Finally we prove the continuity of \( u : \bar{B}_r(x_0) \to \bar{B}_{r_1}(y_0) \), which is equivalent to that of \( v : \bar{B}_r(0) \to \bar{B}_{r_1}(0) \). Using (B.1), for \( x, x' \in \bar{B}_r(0) \)

\[
\|v(x) - v(x')\|_Y = \|R(x, u(x)) - R(x', u(x'))\|_Y
\]

\[
\leq \frac{1}{2} \|v(x) - v(x')\|_Y + \|R(x, u(x)) - R(x', u(x))\|_Y.
\]

So we obtain

\[
\|v(x) - v(x')\|_Y \leq 2\|R(x, u(x)) - R(x', u(x))\|_Y.
\]

The continuity of \( R \) implies that of \( v \). \( \square \)
APPENDIX C. Properties of $V, A$ and Regularity of $F$

In this section, we discuss boundedness, compactness and continuous dependence of linear operators $V_{u,d,\alpha,p}$ and $A_{u,d,\alpha,p}$. They are related to the linearized operator $L_{+Q_{\alpha,p}}^{}$ for non-degeneracy §5.2, the $H_{d}$-valued function $F$ in §5.3 and an improvement of regularity argument in §5.1. We recall their definitions from §5.2:

$$V_{u,d,\alpha,p} := (| \cdot |^{-\alpha} + |u|^{p})|u|^{p-2},$$
$$A_{u,d,\alpha,p} := (| \cdot |^{-\alpha} + (|u|^{p-2}u|\xi|))|u|^{p-2}u.$$

For simplicity of notation, we denote the region of the $(\alpha, p)$ we consider to be

$$\Omega_{d,\delta} := [d-2-\delta, d-2+\delta] \times [2, 2+\delta] \quad (C.1)$$

with $\delta \ll 1$. Obviously $(\alpha, p) \in \Omega_{d,\delta}$ satisfies (1.4).

One estimate (and its perturbation) will be frequently used

$$||(| \cdot |^{-d-2} \ast f)g||_{L^2} \lesssim ||f||_{L_d^{\frac{2d}{d-2}}} ||g||_{L_d^4}. \quad (C.2)$$

C.1. For $u \in H_{d}$. First we consider $u \in H_{d} := L_{rad}^{2}(\mathbb{R}^{d}) \cap L_{rad}^{10}(\mathbb{R}^{d})$ for $d \in \{3, 4, 5\}$.

**Lemma C.1.** For $d \in \{3, 4, 5\}$, $(\alpha, p) \in \Omega_{d,\delta}$ and $\delta \ll 1$. If $u \in H_{d}$, $V_{u,\alpha,p}$ and $A_{u,\alpha,p}$ are bounded from $H_{d}$ to $L_{rad}^{2}$.

**Proof.** Using (C.2),

$$||V_{u,\alpha,p}f||_{L^2} \leq ||| \cdot |^{-\alpha} + |u|^{p}|u|^{p-2}f||_{L_d^d} \lesssim_{d,\alpha,p} ||u||_{L_{d}^{\frac{2d}{d-2}}} ||u||_{L_{d}^{d(p-1)}} \lesssim ||u||_{H_{d}} ||f||_{H_{d}},$$
$$||A_{u,\alpha,p}f||_{L^2} \leq ||| \cdot |^{-\alpha} |u|^{p-2}uf||_{L_d^d} \lesssim_{d,\alpha,p} ||u||_{L_{d}^{d(p-1)}} \lesssim ||u||_{H_{d}} ||f||_{H_{d}},$$

where we need $\delta$ not large so that for any

$$\frac{2dp}{3d-2\alpha-2}, d(p-1) \in [2, 10], \quad \forall \ d \in \{3, 4, 5\}, \ (\alpha, p) \in \Omega_{d,\delta} \quad (C.3)$$

**Lemma C.2.** For $d \in \{3, 4, 5\}$, $\delta \ll 1$, the maps $V : \mathbb{X} \times \Omega_{d,\delta} \to L(H_{d}, L^{2})$ and $A : \mathbb{X} \times \Omega_{d,\delta} \to L(H_{d}, L^{2})$ are well-defined. The following statements are true.

1. $A$ is continuous on $\mathbb{X} \times \Omega_{d,\delta}$.
2. $V$ is continuous on $\mathbb{X} \times \{(\alpha, p) \in \Omega_{d,\delta} : p > 2\}$.
3. $V$ is continuous at $(Q_{\alpha,2}, \alpha, 2)$ and discontinuous at $(Q_{\alpha,2}, R, 2)$ for any $|\alpha - (d-2)| \leq \delta$ and $R > 0$. Here $Q_{\alpha,2}$ is a positive and radially decreasing solution for (Choquard) with parameters $(d, \alpha, 2)$, and $\varphi_{R}$ is a smooth cutoff compactly supported in $B_{2R}$ and equals 1 on $B_{R}$.

**Proof.** We first let $\delta$ be required by Lemma C.1, namely (C.3). We may put on further requirements during the proof.

1. Given $(u, \alpha, p) \in \mathbb{X}_{d} \times \Omega_{d,\delta}$, We show that for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that for $(u_1, \alpha_1, p_1) \in \mathbb{X}_{d} \times \Omega_{d,\delta}$ and $\|u - u_1\|_{\mathbb{X}_{d}} + |\alpha - \alpha_1| + |p - p_1| < \delta_1$,

$$\|A_{u,\alpha,p}f - A_{u_1,\alpha_1,p_1}f||_{L^2} \leq \epsilon \|f||_{\mathbb{X}_{d}}.$$
We will take $\delta$ small enough to ensure
\begin{align}
\| A_{u,\alpha,p} f - A_{u,\alpha,p} f \|_{L^2} &\leq \frac{\epsilon}{3} \| f \|_{X_d}, \\
\| A_{u,\alpha,p} f - A_{u,\alpha_1,p} f \|_{L^2} &\leq \frac{\epsilon}{3} \| f \|_{X_d}, \\
\| A_{u,\alpha_1,p} f - A_{u,\alpha_1,p} f \|_{L^2} &\leq \frac{\epsilon}{3} \| f \|_{X_d},
\end{align}

and they conclude the proof of (1).

Firstly
\begin{align}
\| A_{u,\alpha,p} f - A_{u,\alpha,p} f \|_{L^2} \\
\leq & \left[ \| \cdot \|_{-\alpha} \left( \| u \|_{P''-2} f - |u|^{p''-2} u f \right) \right] |u|^{p''-2} u \|_{L^2} \\
&+ \left[ \| \cdot \|_{-\alpha} \left( |u|^{p''-2} u f \right) \right] |u|^{p''-2} u - \| u \|^{p''-2} u \|_{L^2}.
\end{align}

These two terms are estimated in a similar way, so we only do (C.7) as an example. If $p_1 = p$, then there is nothing we need to do. Assume $p_1 > p$ (the case $p < p_1$ can be treated similarly). We take $M > 1$ and partition the range of $|u|$ to get
\begin{align}
\| u \|^{p''-2}(r) - |u|^{p''-2}(r) &\leq \begin{cases} 
2 |u|^{p''-2}(r) & r \in \{ |u| > M \}, \\
C(M)|p_1 - p| \| u \|^{p''-2}(r) & r \in \{ 1/M \leq |u| \leq M \}, \\
2 |u|^{p''-2}(r) & r \in \{ |u| < 1/M \}.
\end{cases}
\end{align}

We still use the exponent as Lemma C.1
\begin{align}
\| \| \cdot \|_{-\alpha} \left( \| u \|^{p''-2} f - |u|^{p''-2} u f \right) \| u \|^{p''-2} u \|_{L^2} \\
\lesssim \| u \|^{p''-2} u - \| u \|^{p''-2} u \|_{L^2} \| f \|_{L^{2d\alpha}} \| u \|^{p''-2} u \|_{L^{2d\alpha}} \\
\leq 2 \| u \|^{p''-2} u \|_{L^{2d\alpha}} \| f \|_{L^{2d\alpha}} + C(M)|p_1 - p| \| u \|^{p''-2} u \|_{L^{2d\alpha}} \\
\leq 2 \| u \|^{p''-2} \| f \|_{X_d} + C(M)|p_1 - p| \| u \|^{p''-2} \| f \|_{X_d},
\end{align}

where we also require $\delta_1$ small such that $\frac{2(d\alpha)(\alpha_1-1)}{2d\alpha(d\alpha-2)(\alpha_1-1)} \in [2,10]$. Now first take $M \gg 1$ depending on $u$ such that $\| u \|_{X_d} \| u \|_{X_d} \ll \epsilon(\| u \|_{X_d}^{1+\delta}+1)^{-1}$, and then require $\delta \ll 1$ to make $C(M)|p_1 - p| \ll \epsilon(\| u \|_{X_d}^{p+1-p-2})$. We see (C.7) can be bounded by $\frac{\epsilon}{6} \| f \|_{X_d}$. (C.8) comes in a similar way. So (C.4) is confirmed.

Next, consider (C.5), the variation of $\alpha$. Take a $N \gg 1$, then
\begin{align}
\| |x|^{-\alpha} - |x|^{-\alpha_1} \| \lesssim \begin{cases} 
\| \cdot \|_{-\alpha}^{-\delta_0} + \| \cdot \|_{-\alpha}^{\alpha+\delta_0} & |x| < 1/N \text{ or } |x| > N, \\
\| \cdot \|_{-\alpha} \| \cdot \|_{-\alpha} \| \cdot \|_{-\alpha} & \| x \| < 1/N \text{ or } \| x \| > N.
\end{cases}
\end{align}

so we easily see
\begin{align}
\| |x|^{-\alpha} - |x|^{-\alpha_1} \| \ll \| \cdot \|_{-\alpha}^{-\delta_0} + \| \cdot \|_{-\alpha}^{\alpha+\delta_0} = o_{|\alpha-\alpha_1|-0}(1).
\end{align}

Hence by Young’s inequality and similar estimates as (C.2),
\begin{align}
\| A_{u,\alpha_1,p} f - A_{u,\alpha_1,p} f \|_{L^2} = & \| \| \cdot \|_{-\alpha} \| \cdot \|_{-\alpha_1} \| \cdot \|_{-\alpha_1} \| |u|^{p''-2} u f \| |u|^{p''-2} u \|_{L^2} \\
\lesssim & \| u \|^{p''-2} \| f \|_{X_d} \cdot o_{|\alpha-\alpha_1|-0}(1)
\end{align}

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where we require $\delta$ small such that $\frac{2d}{d+2+\delta} \geq 1$ and $\frac{2dp}{d+2+\delta} \in [2,10]$ for any $p \in [2,2+\delta]$. Then (C.5) follows this and $\delta_1$ small enough.

Finally, for the variation of $u$, we have

$$
\|A_{u,\alpha_1,p_1} f - A_{u,\alpha_1,p_1} f\|_{L^2} \\
\leq ||[\cdot - \alpha_1 \ast (|u|^{p_1-2} u f - |u_1|^{p_1-2} u_1 f)]|u_1|^{p_1-2} u_1\|_{L^2} \\
+ ||[\cdot - \alpha_1 \ast (|u|^{p_1-2} u f)](|u|^{p_1-2} u - |u_1|^{p_1-2} u_1)\|_{L^2} \\
\lesssim_{d,\alpha,p} \|f\|_{L_{d-2-\delta+2}^{2d-2\alpha-2}} \left(\|u\|_{L_{d-2-\delta+2}}^{p_1-2} + \|u_1\|_{L_{d-2-\delta+2}}^{p_1-2}\right) \|u - u_1\|_{L_{d-2-\delta+2}} \|u_1\|_{L_{d(p-1)}}^{p_1-1} \\
+ \|f\|_{L_{2d-2\alpha-2}^{2dp}} \|u_1\|_{L_{d(p-1)}}^{p_1-1} \left(\|u\|_{L_{d(p-1)}}^{p_1-2} + \|u_1\|_{L_{d(p-1)}}^{p_1-2}\right) \|u - u_1\|_{L_{d(p-1)}}
$$

The last inequality comes from pointwise estimate

$$
|\|x|^{p_1-2} x - |y|^{p_1-2} y| \lesssim_{p_1} (|x|^{p-2} + |y|^{p-2})|x - y|
$$

and nonlinear estimate as Lemma C.1. Taking $\delta_1$ small enough and we obtain the last control (C.6).

(2) Similarly, we will show that for $(u,\alpha,p) \in \mathbb{X}_d \times \Omega_{d,\delta}$ and every $\epsilon > 0$, there exists $\delta_1 > 0$ such that for $(u_1,\alpha_1,p_1) \in \mathbb{X}_d \times \Omega_{d,\delta}$ and $\|u - u_1\|_{\mathbb{X}_d} + |\alpha - \alpha_1| + |p - p_1| < \delta_1$, we have

$$
\|V_{u,\alpha,p_1} f - V_{u,\alpha,p_1} f\|_{L^2} \leq \frac{\epsilon}{3} \|f\|_{\mathbb{X}_d}, \quad (C.10)
$$

$$
\|V_{u,\alpha,p_1} f - V_{u_1,\alpha,p_1} f\|_{L^2} \leq \frac{\epsilon}{3} \|f\|_{\mathbb{X}_d}, \quad (C.11)
$$

$$
\|V_{u_1,\alpha,p_1} f - V_{u_1,\alpha,p_1} f\|_{L^2} \leq \frac{\epsilon}{3} \|f\|_{\mathbb{X}_d}, \quad (C.12)
$$

It is easy to check that (C.11) and (C.12) follows almost the same estimates as (C.5) and (C.6) respectively, which also works when $p_1 = 2$.

For (C.10),

$$
\|V_{u,\alpha,p_1} f - V_{u,\alpha,p_1} f\|_{L^2} \\
\leq \||[\cdot - \alpha \ast (|u|^p - |u|^{p_1})]|u|^{p_1-2} f\|_{L^2} + \||[\cdot - \alpha \ast |u|^p]|(|u|^{p-2} - |u|^{p_1-2})f\|_{L^2}. \quad (C.13)
$$

Like (1), we assume $0 < p_1 - p < 1$ and partition the range of $|u|$ with respect to $M > 1$ and $1/M$. Then using (C.9), the first term follows in a similar way as (1)

$$
\| [\cdot - \alpha \ast (|u|^p - |u|^{p_1})]|u|^{p_1-2} f\|_{L^2} \\
\lesssim_{d,\alpha,p} ||u|^p - |u|^{p_1}||_{L_{d-2-\delta+2}^{2d-2\alpha-2}} \|f\|_{L_{d(p-1)}} \|u|^{p_1-2}\|_{L_{p_1-2}^{d(p-1)}} \\
\leq \left[2||u||_{\mathbb{X}_d(|u|<1/M)}^p + 2||u||_{\mathbb{X}_d(|u|>M)}^p + C(p,M)|p_1 - p||u||_{\mathbb{X}_d}^p\right] \|u|^{p_1-2}\|f\|_{\mathbb{X}_d}.
$$
We remark that this estimate also works when \( p = 2 \) or \( p_1 = 2 \), since \( \|u\|^{p-2}L^d((2d-\alpha)(p_1-2)) = \|u\|^{p-2}_{L^{\infty}} = 1 \). For the second term,

\[
\| [\cdot | -\alpha | u]^p \| (|u|^{p-2} - |u|^{p_1-2})f \| L^2 \\
\leq_{d,\alpha, p} \|u\|^{p-2} \|f\|_{L^d((p-1)}) \|u\|^{p_1-2} - |u|^{p_2-2} \| L^{\frac{d(p-1)}{p_1-2}} ((\|u\| \leq M)) \\
+ \|f\|_{L^d((p-1)} \|u\|^{p_1-2} - |u|^{p_2-2} \| L^{\frac{d(p-1)}{p_1-2}} ((\|u\| > M)) \\
\leq \|u\|^{p-2}_{X_d} \|f\|_{X_d} \left[ 2\|u\|^{p-2} \|f\|_{X_d(\|u\| < 1/M) \right] + 2\|u\|^{p_1-2}_{X_d(\|u\| > M)} + C(p, M)\|p| - p\|u\|^{p-2}_{X_d} \right].
\]

Then \( p, p_1 > 2 \) is necessary to get the smallness from \( \|u\|_{X_d(\|u\| < 1/M) \|u\|_{X_d(\|u\| > M) = o_M(\infty) \).
So we can take \( M \gg 1 \) and then \( \delta = 1 \) (so that \( p_1 - 2 \) and \( p - 2 \) have a lower positive bound) to guarantee (C.10).

(3) To discuss the continuity at \( (Q_{\alpha,2}, \alpha, 2) \) or \( (Q_{\alpha,2}\varphi_R, \alpha, 2) \), we still consider another \( (u_1, \alpha_1, p_1) \in X_d \times \Omega_d \) and the three parts as (C.10)-(C.12). As in (2), (C.11) and (C.12) hold all these cases. What distinguishes the cases is (C.10).

To be more specific, the trouble is the second term \( \| [\cdot | -\alpha | u]^2 \| (1 - |u|^{p_1-2})f \| L^2 \) in the further partition (C.13). When \( u = Q_{\alpha,2} \), we can use its strict positivity, \( L^\infty \)-bounded and radially decreasing to derive smallness. Let \( Q_{\alpha,2} < 1/M \) := \( B^c_R(M) \), then the crux is that \( R(M) \rightarrow \infty \) as \( M \rightarrow \infty \). Taking \( M > \|Q_{\alpha,2}\|_{L^\infty} \), we have

\[
\| [\cdot | -\alpha | u]^2 \| (1 - Q_{\alpha,2}^{p_1-2})f \| L^2 \\
\leq \| [\cdot | -\alpha | u]^2 \| (1 - Q_{\alpha,2}^{p_1-2})f \| L^2(\{Q_{\alpha,2} < 1/M \}) \}
\]

Using Proposition 2.1,

\[
\| [\cdot | -\alpha | u]^2 \|_{L^\infty} \leq_{d,\alpha} \int_0^\infty Q_{\alpha,2}(r)^{d-1-\alpha} dr \leq_{d,\alpha} \|Q_{\alpha,2}\|_{L^\infty}^2 L^\infty \infty < \infty
\]

So \( M \gg 1 \) and then \( \delta = 1 \) will ensure the smallness of \( \| [\cdot | -\alpha | u]^2 \| (1 - Q_{\alpha,2}^{p_1-2})f \| L^2 \) and thereafter (C.10) is verified. That is the continuity at \( (Q_{\alpha,2}, \alpha, 2) \) for \( \alpha \in (0, d) \).

Regarding the discontinuity for \( u = Q_{\alpha,2}\varphi_R, R > 0 \), we claim that

\[
\|V_{Q_{\alpha,2}\varphi_R, \alpha} - V_{Q_{\alpha,2}\varphi_R, \alpha} \| L(\|u\|_{L^2}) \geq \|V_{Q_{\alpha,2}\varphi_R, \alpha} \| \chi_B^2 R \| L(\|u\|_{L^2}) > 0 \]

for any \( p_1 > 2 \). Indeed, for any \( f \) supported on \( B^c_R \),

\[
\|V_{Q_{\alpha,2}\varphi_R, \alpha} f - V_{Q_{\alpha,2}\varphi_R, \alpha} \|_{L^2} = \| [\cdot | -\alpha | (Q_{\alpha,2}\varphi_R)^2 | (1 - (Q_{\alpha,2}\varphi_R)^{p_1-2}) f \|_{L^2} \\
\geq \| [\cdot | -\alpha | (Q_{\alpha,2}\varphi_R)^2 \|_{L^2}.
\]

\[\square\]
Now we apply the results above to derive regularity of $F$ in §5.1. In particular, the discontinuity will also appear and thus hinder the application of the common version of the implicit function theorem. We first recall the definition of $F$:

$$F(u, \alpha, p) = u - (-\Delta + 1)^{-1}(|\cdot|^{-\alpha} * |u|^p)|u|^{p-2}u.$$  

**Lemma C.3.** There exists $\delta > 0$ such that $F : X_d \times \Omega_{d,\delta} \to X_d$ is well-defined, continuous w.r.t. $(u, \alpha, p)$ and differentiable w.r.t. $u$; $F_u : X_d \times \Omega_{d,\delta} \to L(X_d)$ is continuous at $(Q_{d-2,2}, d-2, 2)$. Besides, $\partial_u F$ is discontinuous on \{ $u \in X_d : \|u - Q_{d-2,2}\|_{X_d} \leq \epsilon$ and $\Omega_{d,\epsilon}$ for any $\epsilon > 0$. 

**Proof.** We will frequently use the following nonlinear estimate given by (C.2):

$$\| (\cdot|^{-\alpha} * (|f_1|^{p-2}f_2f_3)) |f_4|^{p-2}f_5 \|_{L^2} \lesssim_{d,\alpha,p} \| f_1 \|_{X_d}^{p-2}\| f_2 \|_{X_d} \| f_3 \|_{X_d} \| f_4 \|_{X_d} \| f_5 \|_{X_d}. \quad \text{(C.15)}$$

(1) Firstly, we check that $F$ is well-defined. Using Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow X_d$ and (C.15), we easily see

$$\| F(u, \alpha, p) \|_{X_d} \lesssim_d \| u \|_{X_d} + \| V_{u,\alpha, p} u \|_{L^2} \lesssim_{d,\alpha,p} \| u \|_{X_d} + \| u \|_{X_d}^{2p-1}.\]$$

(2) Next we prove the continuity of $F$. Note that

$$\| F(u, \alpha, p) - F(u_1, \alpha_1, p_1) \|_{X_d} \lesssim_d \| V_{u,\alpha, p} u - V_{u_1,\alpha_1, p_1} u_1 \|_{L^2}. $$

We only need to show the following estimate

$$\| V_{u,\alpha, p} u - V_{u,\alpha, p} u_1 \|_{L^2} = o_{p-\alpha_1 \to 0}(1) \quad \text{(C.16)}$$

and apply (C.11) and (C.12). Due to the existence of $u$, the left hand side is exactly the same as $A_{u,\alpha, p} u - A_{u,\alpha, p} u_1$, and thus this estimate follows (C.4).

(3) Then we turn to the Fréchet differentiability of $F$ w.r.t. $u$. We claim that

$$\partial_u F(u, \alpha, p) = \text{Id} - (-\Delta + 1)^{-1} \left[ (p-1)V_{u,\alpha, p} + pA_{u,\alpha, p} \right]. \quad \text{(C.17)}$$

We need to prove that for the $\partial_u F$ defined above, for any $h \in X_d$,

$$\| F(u + h, \alpha, p) - F(u, \alpha, p) - \partial_u F(u, \alpha, p) h \|_{X_d} = o_{\|h\|_{X_d} \to 0}(\|h\|_{X_d}). \quad \text{(C.18)}$$

Again, from $H^2(\mathbb{R}^d) \hookrightarrow X_d$ and a direct computation

$$\| F(u + h, \alpha, p) - F(u, \alpha, p) - \partial_u F(u, \alpha, p) h \|_{X_d} \lesssim_{d,\alpha,p} \| [\cdot|^{-\alpha} * (|u + h|^p - |u|^p - p|u|^{p-2}uh)] |u + h|^{p-2}(u + h) \|_{L^2}$$

$$+ \| (\cdot|^{-\alpha} * |u|^{p-2}u)h (|u + h|^{p-2}(u + h) - |u|^{p-2}u) \|_{L^2}$$

$$+ \| (\cdot|^{-\alpha} * |u|^p) [u + h|^{p-2}(u + h) - |u|^{p-2}u - (p-1)|u|^{p-2}h] \|_{L^2},$$

the estimate (C.18) follows elementary pointwise estimates

$$\| u + h|^{p-2}(u + h) - |u|^{p-2}u - (p-1)|u|^{p-2}h \|_{L^2} \lesssim_{p} \left\{ \begin{array}{ll} (|u|^{p-3} + |h|^{p-3})|h|^2 & \text{if } p > 3 \\ |h|^{p-1} & \text{if } p \in [2,3] \end{array} \right. $$

and (C.15). Note that $\partial_u F(u, \alpha, p) \in L(X_d)$ comes directly from Sobolev embedding and Lemma C.1.
(4) Next we show the continuity of $\partial_u F$ at $(Q_{d-2,2}, d-2, 2)$. Indeed
\[\|\partial_u F(Q_{d-2,2}, d-2, 2) - \partial_u F(u, \alpha, p)\|_{L(X_d)} \leq (p-2)\|Q_{d-2,2}\|_{L(X_d, L^2)} + (p-2)\|A_{Q_{d-2,2}}\|_{L(X_d, L^2)} + (p-1)\|Q_{d-2,2} - V_{u, \alpha, p}\|_{L(X_d, L^2)} + p\|A_{Q_{d-2,2}} - A_{u, \alpha, p}\|_{L(X_d, L^2)}.\] (C.19)

So Lemma C.2 (1)(3) and Lemma C.1 imply the continuity.

(5) Finally, the discontinuity again follows from the discontinuity argument in Lemma (C.2) (3). For every $\epsilon > 0$, there exists $R \gg 1$ such that $\|Q_{d-2,2}\varphi_R - Q_{d-2,2}\|_{L(X_d)} < \epsilon$ and $\partial_u F$ is discontinuous at $(Q_{d-2,2}\varphi_R, d-2, 2)$, where $\varphi_R$ was defined in Lemma C.2. Indeed, for any $p > 2$,
\[\|\partial_u F(Q_{d-2,2}\varphi_R, d-2, 2) - \partial_u F(Q_{d-2,2}\varphi_R, d-2, 2)\|_{L(X_d)} \geq ||(-\Delta + 1)^{-1}(Q_{d-2,2}\varphi_R, d-2, 2) - V_{Q_{d-2,2}\varphi_R, d-2, 2})\|_{L(X_d)} - C(d)(p-2)\|Q_{d-2,2}\varphi_R, d-2, 2)\|_{L(X_d, L^2)} + (p-2)\|A_{Q_{d-2,2}\varphi_R, d-2, 2)} - A_{Q_{d-2,2}\varphi_R, d-2, 2}p\|_{L(X_d, L^2)}\right).\]

Considering the support of $(Q_{d-2,2}\varphi_R)^{p-2}$, we see the first term is lower bounded by $||(\Delta + 1)^{-1}(Q_{d-2,2}\varphi_R, d-2, 2)\|_{L(X_d)} - C(d)\|Q_{d-2,2}\varphi_R, d-2, 2)\|_{L(X_d, L^2)}$ and the negative part is $o_p(1)$ due to the boundedness and continuity from Lemma C.1 and Lemma C.2 (1). So this estimate indicates the discontinuity. □

C.2. For $u = Q_{\alpha, p}$. Now we discuss the boundedness, compactness and continuity of $V_{u, \alpha, p}$ and $A_{u, \alpha, p}$ for $u$ with better bound than $X_d$. In the main text, $u$ will be taken as $Q_{\alpha, p}$. Besides, here we will not restrict to spaces of radial functions.

**Lemma C.4.** For $d \in \{3, 4, 5\}$, $(\alpha, p) \in \Omega_d$ and $q \in (1, \infty)$. For $u \in L^2 \cap L^\infty$, we have

1. $V_{u, \alpha, p} : L^q \to L^q$ is bounded.
2. If in addition $u \in C^0$, then $V_{u, \alpha, p} : W^{1,q} \to L^q$ is compact.
3. $A_{u, \alpha, p} : L^q \to L^q$ is bounded.
4. If in addition $u \in W^{1,r}$ for $r \in (1, \infty)$, then $A_{u, \alpha, p} : L^q \to L^q$ is compact.

**Proof.** (1) Using H"older inequality, we estimate the $L^\infty$ norm of $V_{u, \alpha, p}$ as a function
\[\|V_{u, \alpha, p}\|_{L^\infty} \leq ||| \cdot |^{\alpha} |u|^p\|_{L^\infty} ||u|^{p-2}\|_{L^\infty} \leq ||| \cdot |^{\alpha} |L^1|L^\infty |||u|^{p-2}\|_{L^{1\cap L^\infty}} ||u|^{p-2}\|_{L^{1\cap L^\infty}} \] 1

This immediately implies that $V_{u, \alpha, p} : L^q \to L^q$ is bounded for any $q \in (1, \infty)$.

(2) Moreover, when $u \in C^0$, we claim that the function $V_{u, \alpha, p}$ vanishes at infinity and is uniformly continuous: for every $\epsilon > 0$, there exists $\delta > 0$ such that
\[|V_{u, \alpha, p}(x) - V_{u, \alpha, p}(y)| \leq \epsilon, \quad \forall |x - y| \leq \delta. \] (C.20)

Indeed, the vanishing comes from $u \in L^\infty$ and $| \cdot |^{\alpha} |u|^p$ decays at infinity since $| \cdot |^{\alpha} \in L^1 + L^\infty$ and $|u|^p \in L^1 \cap L^\infty$. The uniform continuity of $V_{u, \alpha, p}$ then follows its local continuity which comes from $u \in C^0$.

Now we prove the compactness via Fréchet-Kolmogorov compactness theorem. Let $\{f_n\} \subset W^{1,q}$ be a bounded sequence. Then $\{V_{u, \alpha, p}f_n\}$ is uniformly bounded in $L^q$ from the argument above.

We need to show $\{V_{u, \alpha, p}f_n\}$ is equicontinuous and uniformly localized in $L^q$. Indeed,
• Equicontinuous: For $h \in \mathbb{R}^d$, (C.20) indicates
\[
\|(V_{u,\alpha,p}f_n)(\cdot + h) - V_{u,\alpha,p}f_n\|_{L^q}
\leq \|(V_{u,\alpha,p}f_n)(\cdot + h) - V_{u,\alpha,p}f_n\|_{L^q} + \|V_{u,\alpha,p}(f_n(\cdot + h) - f_n)\|_{L^q}
\leq \|f_n\|_{L^q} \cdot o_{h \to 0}(1) + \|V_{u,\alpha,p}\|_{L^\infty} \|f_n\|_{W^{1,q}} |h| = o_{h \to 0}(1).
\]

Uniform localization: since $V_{u,\alpha,p}$ vanishes at infinity,
\[
\|V_{u,\alpha,p}f_n\|_{L^q(B_R^c)} \leq \|V_{u,\alpha,p}\|_{L^\infty(B_R^c)} \|f_n\|_{L^q} = o_{R \to \infty}(1).
\]
So Fréchet-Kolmogorov compactness theorem verifies the compactness.

3. We divide two cases for boundedness. When $q \in (1, \frac{2d}{d-\alpha})$, boundedness follows
\[
\|A_{u,\alpha,p}f\|_{L^q} \leq \left\| \frac{1}{|x|^{\alpha + 1}} \right\|_{L^q} \| |x|^{\beta(d,\alpha)} \left( |x| + h \right) |x|^{-\alpha - \beta(d,\alpha)} + |x|^{-\alpha - \beta(d,\alpha)} \right\| \|f\|_{L^q}.
\]
And when $q \geq \frac{2d}{d-\alpha} > 2$, we estimate as
\[
\|A_{u,\alpha,p}f\|_{L^q} \leq \left\| \frac{1}{|x|^{\alpha + 1}} \right\|_{L^q} \| |x|^{\beta(d,\alpha)} \left( |x| + h \right) |x|^{-\alpha - \beta(d,\alpha)} + |x|^{-\alpha - \beta(d,\alpha)} \right\| \|f\|_{L^q}.
\]

4. Again, take a bounded sequence $\{f_n\}_n \subset L^q$. We verify the equicontinuity and uniform localization of $\{A_{u,\alpha,p}f_n\}_n$ to confirm compactness of $A_{u,\alpha,p}$.

• Equicontinuous: We first prove a pointwise estimate: for any $x, h \in \mathbb{R}^d$, $\alpha \in (0, d)$,
\[
|x + h|^{-\alpha} - |x|^{-\alpha} \leq \beta(d, \alpha) \left( |x + h|^{-\alpha - \beta(d,\alpha)} + |x|^{-\alpha - \beta(d,\alpha)} \right),
\]
where
\[
\beta(d, \alpha) = \min\left\{1, \frac{d-\alpha}{2} \right\}.
\]
Indeed, when $|x| \geq 2|h|$, we have
\[
|x + h|^{-\alpha} - |x|^{-\alpha} \leq \left| \int_0^1 \frac{1}{-\alpha} |x + th|^{-\alpha - 2} (x + th) \cdot hdt \right|
\leq -\alpha^{-1} \left( \frac{1}{2} |x| \right) -\alpha^{-1} |h| \leq \beta(d, \alpha) |x|^{-\alpha - \beta} |h|^\beta.
\]
And when $|x| < 2|h|$, we have max$\{|x|, |x + h|\} \leq 3|h|$, so
\[
|x + h|^{-\alpha} - |x|^{-\alpha} \leq |x + h|^{-\alpha} + |x^{-\alpha} \leq \left( |x + h|^{-\alpha - \beta} + |x|^{-\alpha - \beta} \right) 3|h|^\beta.
\]
For any $h \in \mathbb{R}^d$,
\[
\|(A_{u,\alpha,p}f_n)(\cdot + h) - A_{u,\alpha,p}f_n\|_{L^q}
\leq \left\| \left( |\cdot|^{-\alpha} \right) \left( |u|^{-2} |u|^{p-2} u f_n \right)(\cdot) (\cdot + h) \right\|_{L^q}
\leq \left\| \left( |\cdot|^{-\alpha} \right) \left( |u|^{-2} |u|^{p-2} u f_n \right) \|_{L^q} \right\|_{L^q}.
\]
the term (C.25) is estimated as (C.15) and (C.22) to be \( o_{h \to 0}(1) \). And from (C.23),
\[
\left| (\cdot | -\alpha \ast (|u|^{p-2}u f_n))(x) - (\cdot | -\alpha \ast (|u|^{p-2}u f_n))(x) \right| \\
\lesssim_{d,\alpha} |h|^\beta \left[ \left| (\cdot | -\alpha \ast (|u|^{p-2}u f_n))(x) \right| + \left| (\cdot | -\alpha \ast (|u|^{p-2}u f_n))(x) \right| \right].
\]
Note that \( \alpha + \beta \in (0, d) \) and \( \beta > 0 \), we can similarly bound (C.24) to be \( o_{h \to 0}(1) \).

• Uniform localization: Estimate as (C.21) and (C.22),
\[
\|A_{u,\alpha,p}f_n\|_{L^q(B_R^\circ)} = \|((\cdot | -\alpha \ast (|u|^{p-2}u f_n))u \chi_{B_R^\circ})\|_{L^q}^{p-1} \\
\lesssim_{d,\alpha} \left\{ \begin{array}{ll}
\|f_n\|_{L^q}\|u\|^{p-1}_{L^q} & q \in (1, \frac{2d}{d-\alpha}), \\
\|f_n\|_{L^q}\|u\|^{p-1}_{L^{\frac{2d}{d-\alpha}}(B_R^\circ)} & q \in [\frac{2d}{d-\alpha}, \infty)
\end{array} \right.
\]
\[
= o_{R \to \infty}(1).
\]

The following miscellaneous estimates will be used in proving Proposition 5.1 and Theorem 1.8 in §5.

**Lemma C.5.** For \( d \in \{3, 4, 5\} \), there exists \( \delta > 0 \) such that the following statements hold. For \( (\alpha, p) \in \Omega_{d,\delta} \), \( u_0, u \in L^2 \cap L^\infty \) and \( u_0 \) positive and radially decreasing, we have
\[
\|V_{u_0,d-2}u_0 - V_{u_0,\alpha,p}u\|_{L^\frac{2d}{d-\alpha}} \lesssim_{d,\delta,u_0} o_{\alpha \dashv (d-2)+|p-2| \to 0}(1) + \|u_0 - u\|_{L^2 \cap L^\frac{2d}{d-\alpha}} \quad (C.26)
\]
\[
\|V_{u_0,d-2}u_0 - V_{u_0,\alpha,p}\|_{L^\infty} \lesssim_{d,\delta,u_0} o_{\alpha \dashv (d-2)+|p-2|+\|u_0-u_0\|_{L^2 \cap L^\infty} \to 0}(1) \quad (C.27)
\]
\[
\|A_{u_0,d-2}u_0 - A_{u_0,\alpha,p}\|_{L^2 \cap L^2} \lesssim_{d,\delta,u_0} o_{\alpha \dashv (d-2)+|p-2| \to 0}(1) + \|u_0 - u\|_{L^2 \cap L^\infty}. \quad (C.28)
\]

**Proof.** The proof of these inequalities are like those in Lemma C.2, with main difference comes from the choice of norms since we have better control of \( u_0 \) and \( u \) this time. So we only sketch the proof.

(1) Proof of (C.26). It suffices to show the following two estimates:
\[
\|V_{u_0,d-2}u_0 - V_{u_0,\alpha,p}u\|_{L^\frac{2d}{d-\alpha}} \lesssim_{d,\delta,u_0} o_{\alpha \dashv (d-2)+|p-2| \to 0}(1) \quad (C.29)
\]
\[
\|V_{u_0,d-2}u_0 - V_{u_0,\alpha,p}\|_{L^\infty} \lesssim_{d,\delta,u_0} o_{\alpha \dashv (d-2)+|p-2| \to 0}(1) \quad (C.30)
\]
\[
\|A_{u_0,d-2}u_0 - A_{u_0,\alpha,p}\|_{L^2 \cap L^2} \lesssim_{d,\delta,u_0} o_{\alpha \dashv (d-2)+|p-2| \to 0}(1) + \|u_0 - u\|_{L^2 \cap L^\infty}. \quad (C.31)
\]

Inequality (C.29) is exactly included in the proof of (10) in Lemma C.2 (3) (where we utilize the positive and radially decreasing of \( u \)). For (C.30), we adjust the exponents in proving (C.11):
\[
\|V_{u_0,d-2}u_0 - V_{u_0,\alpha,p}\|_{L^\infty} \leq \|u_0\|_{L^2 \cap L^\infty} \left( |(\cdot | -|d-2| \ast |\cdot | -|\alpha|) \ast |u_0| \|u_0\|_{L^\infty} \right)
\]
\[
\leq \|u_0\|_{L^\infty} \cdot |(\cdot | -|d-2| \ast |\cdot | -|\alpha|) \ast |u_0| \|u_0\|_{L^\infty} \]
\[
\leq \|u_0\|_{L^\infty} \cdot |(\cdot | -|d-2| \ast |\cdot | -|\alpha|) \ast |u_0| \|u_0\|_{L^\infty} \leq \|u_0\|_{L^2 \cap L^\infty} \cdot o_{\alpha \dashv (d-2)+|p-2| \to 0}(1)
\]

And for (C.31), it can be deduced easily from Hölder and Hardy-Littlewood-Sobolev estimates like (C.6).

(2) Proof of (C.26). Given (C.29) and (C.30), we only need to estimate
\[
\|V_{u_0,\alpha,p} - V_{u_0,\alpha,p}\|_{L^\infty} \leq \|((\cdot | -\alpha \ast (|u_0|^p - |u|^p)) |u_0|^{p-2} \|_{L^\infty} + \|((\cdot | -\alpha \ast |u|^p) (|u_0|^p - |u|^{p-2}) \|_{L^\infty}.
\]

The only tricky part is the second term, where we take an \( M > \|u_0\|_{L^\infty} \) and let \( R(M) \) be such that \( B_{R(M)} = \{ u \geq M^{-1} \} \). Then when \( \|u_0 - u\|_{L^\infty} \) is bounded, we have
\[
\|((\cdot | -\alpha \ast |u|^p) (|u_0|^p - |u|^{p-2}) \|_{L^\infty}(B_{R(M)}) = o_{M \to \infty}(1)
\]
due to the vanishing of $| \cdot |^{-\alpha} * |u|^p$ at infinity. And suppose $\| u_0 - u \|_{L^\infty} \leq M^{-2}$, we can estimate the $L^\infty(B_{R(M)})$ part using

$$\| u_0 \|_{L^p} - |u|_{L^p} = |u_0|_{L^p} - \left( 1 - \left| \frac{|u_0| - |u|}{|u_0|} \right| \right)^{p-2} \leq |u_0|_{L^\infty} \left| \frac{|u_0| - |u|}{|u_0|} \right| \| u_0 - u \|_{L^\infty} \frac{p-2}{M-1}. $$

(3) Proof of (C.28). This is almost the same as (C.4)-(C.6) using perturbation of (C.2). □

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