Discrete and Ultradiscrete Mixed Soliton Solutions

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We propose a new type of soliton equation, which is obtained from
the generalized discrete BKP equation. The obtained equation admits
two types of soliton solutions. The signs of amplitude and velocity of
the soliton solution are opposite to the other. We also propose the ultra-
discrete analogues of them. The ultradiscrete equation also admits the
similar properties. In particular it behaves the original Box-Ball system
in a special case.

Key words: Discrete BKP equation; Ultradiscrete soliton equation; Box-Ball System.

1 Introduction

Soliton equations are known as nonlinear equations which have exact solutions. Among these equations, the soliton equations are classified in some hierarchies. For example, the KP equation, BKP equation[1]. In the KP equation, the KdV equation or the Toda equation have been researched especially. This is because they have good properties and to easy to handle. Similarly, the Sawada-Kotera equation is also known as a good example in the BKP equation.

Soliton equations can be divided into continuous, discrete and ultradiscrete equations by its discreteness of dependent and independent variables. In discrete soliton equations, there are also the discrete KP, BKP equations similar to the continuous equations. Recently, one of the authors proposed the generalized discrete BKP equation and its soliton solution[2]. The equation is expressed by

\[
\begin{align*}
& z_1\tau(p+1,q,r)\tau(p,q+1,r+1) + z_2\tau(p,q+1,r)\tau(p+1,q,r+1) \\
& + z_3\tau(p,q,r+1)\tau(p+1,q+1,r) - z_4\tau(p,q,r)\tau(p+1,q+1,r+1) = 0,
\end{align*}
\]  

1
where \( z_1, z_2, z_3, z_4 \) are arbitrary parameters satisfying \( z_1 + z_2 + z_3 - z_4 = 0 \). We note (1) reduces to the discrete KP equation when \( z_4 = 0 \). Moreover, when we set
\[
\begin{align*}
z_1 &= (a + b)(a + c)(b - c), \\
z_2 &= (b + c)(b + a)(c - a), \\
z_3 &= (c + a)(c + b)(a - b), \\
z_4 &= -(a - b)(b - c)(c - a),
\end{align*}
\]
then (1) reduces to the discrete BKP equation. This fact means the generalized discrete BKP equation includes the discrete KP, BKP equations. Conversely, as mentioned in [2], (2) have solutions for \( a, b, c \) when \( z_1 z_2 z_3 z_4 \neq 0 \). In other words, the generalized discrete BKP equation (1) coincides with the discrete BKP equation for most values of \( z_i \). However, the advantage of adopting the expression of (1) than that of the discrete BKP equation is the coefficients can be taken freely. This freedom gives us a simple expression of soliton solutions. Furthermore it enables us to ultradiscretize equations and solutions easily.

In this paper we present a new type of soliton equation from (1) through the reduction. The obtained equation possesses exact solution which contains two different types of soliton solutions. The signs of amplitudes and the velocities are opposite each other. We also give its ultradiscrete analogues [5]. The obtained ultradiscrete solution also contains two different types of solutions. One behaves as the original Box-Ball system and the other does negative soliton solution.

This paper composed below. In Section 2, we derive a certain discrete soliton equation from (1). Also we give the ultradiscrete equation. In Section 3, we give the discrete soliton solutions and show its properties. In Section 4, we ultradiscretize the discrete soliton solutions. In Section 5, we take the continuous limit and show the relation between our equation and the KdV-Sawada-Kotera equation. We give concluding remarks in Section 6.

## 2 Discrete and Ultradiscrete Equations

In this Section, we derive new discrete and ultradiscrete equations from (1) by imposing some conditions. First we introduce new independent variable transformations
\[
p = n - m - 2l, \quad q = -l, \quad r = m + l.
\]

Then (1) is rewritten by
\[
\begin{align*}
z_1 \hat{\tau}(l, m + 1, n) &\hat{\tau}(l + 1, m - 1, n + 1) + z_2 \hat{\tau}(l, m, n - 1) \hat{\tau}(l + 1, m, n + 2) \\
+ \ z_3 \hat{\tau}(l, m, n) &\hat{\tau}(l + 1, m, n + 1) - z_4 \hat{\tau}(l + 1, m - 1, n) \hat{\tau}(l, m + 1, n + 1) = 0,
\end{align*}
\]
where \( \hat{\tau} \) depends on \( l, m, n \). In particular, we set
\[
\begin{align*}
z_1 &= d_1, \\
z_2 &= d_2, \\
z_3 &= 1 - d_2, \\
z_4 &= 1 + d_1,
\end{align*}
\]
with \( 0 < d_2 < d_1 < 1 \). By imposing \( \hat{\tau} \) does not depend on \( l \), that is, \( \hat{\tau}(l + 1, m, n) = \hat{\tau}(l, m, n) \) holds, then we finally obtain
\[
(1 + d_1) f_n^{m-1} f_{n+1}^m = d_1 f_n^{m+1} f_{n+1}^{m-1} + d_2 f_n^{m-1} f_{n+2}^m + (1 - d_2) f_n^m f_{n+1}^m,
\]
with
where $f_m^n$ denotes $\tau(l, m, n)$. We note (3) is reduced to the discrete KdV equation in bilinear form when $d_2 = 0$. In addition we note (4) can be also obtained from the discrete BKP equation by putting suitable parameters $a, b, c \in \mathbb{C}$ into (2), however the expressions of them are complicated to deal with.

If we introduce dependent variable transformations

$$u_n^m = \frac{f_m^n + f_{m+1}^n}{f_m^n f_{n+1}^m}, \quad v_n^m = \frac{f_m^{n+1} f_{m+1}^{n-1}}{(f_m^n)^2}, \quad x_n^m = \frac{f_m^{n+1} f_m^{n-1}}{(f_m^n)^2}, \quad \text{(7)}$$

for (3), we obtain

$$\frac{u_n^m}{u_{n-1}^m} = u_n^m, \quad (8a)$$
$$\frac{(1 + d_1)v_{n+1}^m}{v_n^m} = (1 - d_2)v_n^m u_n^m - d_1 u_n^m (u_n^m - 1)^2 + d_2 x_n^m x_n^{m+1} u_n^m - 1, \quad (8b)$$
$$\frac{x_n^{m+1}}{x_n^{m-1}} = \frac{u_n^{m+1}}{u_n^{m-1}}. \quad (8c)$$

We regard $m$ and $n$ as time and space variables. Then the time evolutions of $u_n^m, v_n^m, x_n^m$ are described by (8a), (8b) and (8c) under the boundary condition $\lim_{n \to -\infty} v_n^m = \text{const}.$

Next, we shall derive the ultradiscrete analogues [5]. By introducing transformations $d_i = e^{-\delta_i}, f_n^m = e^{f_n^m/\varepsilon},$ and taking the limit $\varepsilon \to +0$ for (3), we obtain the ultradiscrete equation.

$$F_{n+1}^{m+1} + F_{n+1}^{m-1} = \max(F_{n+1}^m + F_n^m, F_{n+1}^m + F_{n-1}^m - \delta_1, F_n^m + F_{n-1}^m - \delta_2), \quad \text{(9)}$$

where $0 < \delta_1 < \delta_2.$ Here we use the key formula $\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B).$ We note (3) reduces to the ultradiscrete KdV equation in bilinear form when $\delta_2$ is sufficiently large [6]. We can also obtain the ultradiscrete analogues of (3) with the transformations $u_n^m = e^{U_n^m/\varepsilon}, v_n^m = e^{V_n^m/\varepsilon}, x_n^m = e^{X_n^m/\varepsilon}$ by the similar procedure.

$$U_n^m = U_{n-1}^m + V_n^m - V_{n+1}^m, \quad \text{(10a)}$$
$$V_{n+1}^m = \max(0, V_n^m + U_n^m - \delta_1, X_n^m + X_{n+1}^m - \delta_2) + U_{n-1}^m, \quad \text{(10b)}$$
$$X_n^m = X_{n-1}^m + U_{n-1}^m - V_n^m. \quad \text{(10c)}$$

Notice that the equations (10a) and (10b) take the form of the conservation law. This procedure determines $X_n^m$ from the values of $X_{n-1}^m$ and $U_{n-1}^m$ from (10c). Then, if we assume boundary condition $\lim_{n \to -\infty} V_n^m$ we can determine the values of $V_{n+1}^m$ from the values of $U_{n-1}^m, V_n^m, X_n^m$ using (10b). Finally $U_n^m$ is determined from the values of $U_{n-1}^m$ and $V_n^m$ from (10a). Repeating this procedure to the new values $U_n^m, V_n^m, X_n^m,$ we can obtain the time evolution.

Eq. (10) can be considered as a kind of Box-Ball systems. Let us consider $U_n^m$ and $V_n^m$ as a number of balls in $n$th box at time $m$ and a carrier of balls respectively. Also let us consider $\delta_1$ as the capacity of the boxes. Assume the boundary conditions

$$\lim_{n \to -\infty} U_n^m = \lim_{n \to -\infty} V_n^m = \lim_{n \to -\infty} X_n^m = 0.$$
Then the time evolution rule of (10) is described as follows. From time $m - 1$ to $m$, the carrier moves from the $-\infty$ site to the $\infty$ site. It passes each box from the left to the right. At $n$th box, the carrier loads the balls as many as possible from the $n$th box and at the same time, unloads the balls from the carrier as many as the vacant spaces of the $n$th box, which corresponds to $\delta_1 - U_{m-1}^m$. Moreover, if the maximum value of RHS in (10b) is given by $X_{m}^n + 1 - X_{m}^n - \delta_2$, then the carrier loads more balls even if the number of balls becomes negative. In this case we consider the box has ‘negative balls’ as a debt.

As a special case, if the condition $-\delta_2 + X_{m}^n + X_{m}^{n+1} \leq 0$ always holds, then (10b) becomes

$$V_{m+1}^n = \max(0, V_{m}^n + U_{m-1}^m - \delta_1) + U_{m-1}^n.$$ (11)

Using the conservation law (10a) to the left hand side of this equation, we obtain the following.

$$U_{m}^n = \min(V_{m}^n, \delta_1 - U_{m-1}^n).$$ (12)

We can also rewrite the dependent variable $V_{m}^n$ as

$$V_{m}^n = \sum_{k=-\infty}^{n} (V_k - V_{k-1}) = \sum_{k=-\infty}^{n} (U_{k-1}^m - U_k^m),$$ (13)

using the conservation law (10a) together with the boundary condition $\lim_{n \to -\infty} V_{m}^n = 0$. Substituting this representation to the equation (12), the equations (11) and the conservation law (10a) turns out to be

$$U_{m}^n = \min\left(\sum_{k=-\infty}^{n} (U_{k-1}^m - U_k^m), \delta_1 - U_{m}^n\right).$$ (14)

This is nothing but the Box-Ball system with the box capacity $\delta_1$.

Exact solutions and examples for (10) are given in Section 4.

3 Discrete Soliton Solution

The soliton solution of (1) is expressed by

$$\tau(p,q,r) = \sum_{0 \leq k \leq N} \sum_{I_k \subset [N]} \left( \prod_{i \in I_k} c_i \varphi(t) \varphi(s) \prod_{i < j} b_{ij} \right),$$ (15)

where $N$ is a positive integer, $[N]$ denotes $\{1, 2, \ldots, N\}$ and $\sum_{I_k \subset [N]}$ the summation over all $k$-element subsets $I_k = \{i_1, i_2, \ldots, i_k\}$ chosen from $[N]$. When $k = 0$, we define $\sum_{I_0 \subset [N]}$ is 1. Function $\varphi(t)$ and $b_{ij}$ are defined by

$$\varphi(t) = (a_1(t))^p (a_2(t))^q (a_3(t))^r,$$

$$a_1(t) = \frac{z_2 z_4 + z_3 t}{z_1 - t}, \quad a_2(t) = \frac{z_1 z_4 - z_3 t}{z_2 + t}, \quad a_3(t) = t,$$ (16)
Proposition 3.1 Set \( t'_i = -\frac{z_1 z_2 z_4}{s z_i}, \) \( s'_i = -\frac{z_1 z_2 z_4}{s t_i} \). Then 

\[
\varphi(t_i) = \varphi(t'_i) \frac{\varphi(s'_i)}{\varphi(s_i)}, \\
b_{ij} = b_{ij'} = b_{i'j'} = b'_{ij'}
\] (18)

hold for any \( i, j = 1, 2, \ldots, N \) (\( i \neq j \)). Here \( b_{ij} \) denotes 

\[
b_{ij} = \frac{cf(t'_i, t_j) cf(s'_i, s_j)}{cf(s'_i, t_j) cf(t'_i, s_j)}.
\] (19)

Proof. The relations (18) can be obtained by confirming 

\[
\frac{a_k(t)}{a_k(s)} = \frac{a_k(t')}{a_k(s')}, \\
cf(t_i, t_j) = -\frac{1}{z_1 z_2 z_3 z_4 cf(t_i, s'_j)} = cf(s'_i, s'_j)
\] (20)

for \( k = 1, 2, 3 \) from the definitions.

From Proposition 3.1 we may assume either \( t_i \) or \( s_i \) is always positive for any \( i \) without loss of generality. Actually, the solution with \( t_i, s_i \) and the one with \( t'_i, s'_i \) correspond. Moreover, considering the term \((a_3(t_i)/a_3(s_i))^{r} = (t_i/s_i)^{r} \) in \( \varphi(t_i)/\varphi(s_i) \), we have the condition both \( t_i \) and \( s_i \) should be positive so that \( \varphi(t_i)/\varphi(s_i) > 0 \). Similarly, by considering the positivity of \( a_1(t_i)/a_1(s_i) \), \( a_2(t_i)/a_2(s_i) \), we obtain the following proposition.

Proposition 3.2 If parameters \( t, s \) satisfy one of the conditions,

1. \( 0 < t, s < z \),
2. \( z < t, s < \bar{z} \),
3. \( \bar{z} < t, s \),

where \( z \) and \( \bar{z} \) denote \( \min(z_1, \frac{z_2 z_4}{z_3}) \), \( \max(z_1, \frac{z_2 z_4}{z_3}) \) respectively, then \( \varphi(t)/\varphi(s) \) take positive values for any \( p, q, r \in \mathbb{Z} \).

We also have the following.

Proposition 3.3 Suppose \( t_i, s_i \) are positive for any \( i = 1, 2, \ldots, N \). If parameters \( t_i, s_i \) satisfy one of the conditions,

1. Both \( t_i \) and \( s_i \) take values between \( t_j \) and \( s_j \),
2. Both \( t_j \) and \( s_j \) take values between \( t_i \) and \( s_i \),
3. \( \max(t_i, s_i) < \min(t_j, s_j) \),

\[
b_{ij} = \frac{cf(t_i, t_j) cf(s_i, s_j)}{cf(s_i, t_j) cf(t_i, s_j)}, \\
cf(t, s) = \frac{t - s}{tsz_3 + z_1 z_2 z_4}.
\] (17)
4. \( \max(t_j, s_j) < \min(t_i, s_i) \),

for any combination \( i, j \) (\( i < j \)), then \( b_{ij} \) takes a positive value.

**Proof.** Figure 1 shows an example of the parameters satisfying the conditions. The conditions in the proposition mean each line connected between \( t_i \) and \( s_i \) does not intersect with others. The conditions derive \( (t_i - t_j)(s_i - s_j)(t_i - s_j)(s_i - t_j) > 0 \) holds for any \( i, j \) and it gives \( b_{ij} > 0 \) from the definition.

These propositions hence can be summarized by the following theorem.

**Theorem 3.1** Suppose \( c_i > 0 \) and parameters \( t_i \) and \( s_i \) satisfy the conditions given in Propositions 3.2 and 3.3 for \( i = 1, 2, \ldots, N \), then the solution \( \tau(p, q, r) \) take positive values for any \( p, q, r \in \mathbb{Z} \).

Now let us derive the soliton solution of (6) under the assumptions given in Theorem 3.1. Hereafter we set \( z_k \) as (5). By applying the transformations (3), the solution (15) is transformed to

\[
\hat{\tau}(l, m, n) = \sum_{0 \leq k \leq N} \sum_{k \subset [N]} \left( \prod_{i \in I_k} c_i \frac{\phi(t_i)}{\phi(s_i)} \prod_{i<j \in I_k} b_{ij} \right), \tag{21}
\]

where

\[
\phi(t) = \left( \frac{a_3(t)}{(a_1(t))^2 a_2(t)} \right)^l \left( \frac{a_3(t)}{a_1(t)} \right)^m (a_1(t))^n. \tag{22}
\]

If \( t_i, s_i \) satisfy the relation

\[
a_3(t_i) \over (a_1(t_i))^2 a_2(t_i) = a_3(s_i) \over (a_1(s_i))^2 a_2(s_i) \tag{23}
\]

for \( i = 1, 2, \ldots, N \), then the constraint condition \( \hat{\tau}(l+1, m, n) = \hat{\tau}(l, m, n) \) holds, hence (21) satisfies (6). Considering the fluctuation of \( \psi(t) := a_3(t)/a_1^2(t)/a_2(t) \) from elementary calculus, one can find the following (See Figure 2).

- There exist three points \( \alpha_i \) (\( i = 1, 2, 3 \)) such that \( \frac{d}{dt} \psi(\alpha_i) = 0 \) on \( 0 < t < \infty \).
  Each \( \alpha_i \) satisfies \( \alpha_1 \in (\frac{\alpha_1}{z_3}, \infty) \), \( \alpha_2 = z_3 \), \( \alpha_3 \in (0, z_1) \) respectively.

- \( \psi(t) \) is monotone increasing on \( (z_1 z_4/z_3, \alpha_1), (z_1, z_1 z_4/z_3) \) and \( (0, \alpha_3) \).
\[ \psi(t) - z^2 \alpha_3 t \psi(t) \]

FIG. 2. Plot of \( \psi(t) \). The right figure shows near the origin.

- \( \psi(t) \) is monotone decreasing on \((\alpha_1, \infty)\) and \((\alpha_3, z_1)\).
- \( \psi(t) \geq 0 \) on \((0, z_1 z_4 / z_3)\). \( \psi(t) < 0 \) on \((z_1 z_4 / z_3, \infty)\).

Therefore if \( t \) belongs to either \( J_1 = \left( z_1 z_4 / z_3, \infty \right) \) or \( J_2 = (0, z_1) \), except for \( \alpha_1, \alpha_3 \), then there exists a unique value \( s \) in the same interval such that \( s \neq t \) and satisfying (23). In particular, these parameters \( t_1, t_2, \ldots, t_N, s_1, s_2, \ldots, s_N \) hold the conditions in Theorem 3.1. These results are summarized as the following theorem.

**Theorem 3.2** The solution of (6) is expressed by

\[
f_m^n = \sum_{0 \leq k \leq N} \sum_{I_k \subset [N]} \left( \prod_{i \in I_k} c_i \phi(t_i) \prod_{i,j \in I_k, i < j} b_{ij} \right),
\]

where

\[
\phi(t) = \left( \frac{a_3(t)}{a_1(t)} \right)^m \left( a_1(t) \right)^n.
\]

Here the parameters \( t_i, s_j \) belong to the same intervals \( J_1 \) or \( J_2 \), and they satisfy the dispersion relation (23) for \( i = 1, 2, \ldots, N \). Moreover \( f_m^n \) take positive values for any \( m, n \) under \( c_i > 0 \).

Figure 3 shows the one soliton solution with \( t, s \in J_1 \) and Fig. 4 does with \( t, s \in J_2 \). Here \( m \) and \( n \) denote time and space variables. In Fig 3 we set \( t = 3, (d_1, d_2) = (1/2, 1/3) \) and \( s \) so that satisfying (23). The solitary wave moves to the right side. In Fig 4 we set \( t = 3/10, (d_1, d_2) = (1/2, 1/3) \) and \( s \) satisfies (23). The solitary wave moves to the left side slowly. In each case, the velocity of solitary wave is given by \( 1 - \log(a_3(t)/a_3(s))/\log(a_1(t)/a_1(s)) \). Figure 5 shows the interaction of two different soliton solutions. We set \( (t_1, t_2) = (3, 3/10), (d_1, d_2) = (1/2, 1/3), s_1 \) and \( s_2 \) satisfy (23) respectively.

It is noted distinct parameters \( t, s \) in \( J_2 \) do not exist when \( d_2 = 0 \) since \( \psi(t) \) is monotone decreasing on \( J_2 \). It means the discrete KdV equation does not admit the solution with \( t, s \) in \( J_2 \).
FIG. 3. 1-soliton solution \( u_n^m \) with \( t, s \in J_1 \).

FIG. 4. 1-soliton solution \( u_n^m \) with \( t, s \in J_2 \).

FIG. 5. Mixed (1 + 1)-soliton solution \( u_n^m \) with \( t_1, s_1 \in J_1 \) and \( t_2, s_2 \in J_2 \).
4 Ultradiscrete Soliton Solutions

4.1 One Soliton Solution

In order to derive ultradiscrete solutions for (9), we introduce new parameters for each $J_1, J_2$:

$$
t = t^{(1)} + \frac{z_1 z_4}{z_3} \quad \text{for } t \in J_1,
$$

$$
t = \frac{t^{(2)} z_1}{1 + t^{(2)}} \quad \text{for } t \in J_2.
$$

These replacements enable us to take $t^{(i)}$ for any positive values. Then for $t \in J_1$, by using $z_1 + z_2 + z_3 - z_4 = 0$, we have

$$
a_1(t) = \frac{z_2 z_4 + z_3 t}{z_1 - t} = -\frac{z_3(z_4(z_1 + z_2) + z_3 t^{(1)})}{z_1(z_1 + z_2) + z_3 t^{(1)}}.
$$

We may neglect the signature since it can be cancelled with $a_1(s)$. Thus, from (5) and $d_1 = e^{-\delta_1/\varepsilon}$, $d_2 = e^{-\delta_2/\varepsilon}$, $t^{(i)} = e^{-T^{(i)}/\varepsilon}$, we obtain the ultradiscrete analogue of $|a_1(t)|$ as

$$
|a_1(t)| \to \max(-\delta_1, T^{(1)}) - \max(-2\delta_1, T^{(1)}) =: A_1(T^{(1)}).
$$

Similarly, we also obtain the ultradiscrete analogues of $|a_2(t)|, |a_3(t)|$ as follows.

$$
|a_2(t)| \to A_2(T^{(1)}) = T^{(1)} - \max(T^{(1)}, -\delta_1),
$$

$$
|a_3(t)| \to A_3(T^{(1)}) = \max(T^{(1)}, -\delta_1).
$$

Hence we obtain the ultradiscrete analogue of the one soliton solution with $t, s \in J_1$.

$$
F_{m}^{n} = \max(0, m(A_3(T^{(1)}) - A_3(S^{(1)}) - A_1(T^{(1)}) + A_1(S^{(1)}))) + n(A_1(T^{(1)}) - A_1(S^{(1)}) + C),
$$

where $T^{(1)}, S^{(1)}$ satisfy the dispersion relation,

$$
A_3(T^{(1)}) - 2A_1(T^{(1)}) - A_2(T^{(1)}) = A_3(S^{(1)}) - 2A_1(S^{(1)}) - A_2(S^{(1)}),
$$

which is obtained by ultradiscretizing (23). The above is simplified as

$$
|T^{(1)} + 2\delta_1| = |S^{(1)} + 2\delta_1|.
$$

This leads $S^{(1)} = -T^{(1)} - 4\delta_1$ due to $T^{(1)} \neq S^{(1)}$. If we denote $\Omega, K$ as the coefficients of $m, n$:

$$
\Omega = A_3(T^{(1)}) - A_3(S^{(1)}) - A_1(T^{(1)}) + A_1(S^{(1)}),
$$

$$
K = A_1(T^{(1)}) - A_1(S^{(1)}),
$$

$$
9
$$
then $\Omega$ can take any value since $\Omega = T^{(1)} + 2\delta_1$ and we find $K = \frac{1}{2}(|\Omega - \delta_1| - |\Omega + \delta_1|)$ holds. Therefore, (30) is rewritten by

$$F_n^m = \max(0, \Omega m + Kn + C).$$  (34)

This is the one soliton solution for (29). It is noted this solution corresponds to the one of the ultradiscrete Toda equation[8].

For $t \in J_2$, we obtain

$$|a_1(t)| \rightarrow A_1(T^{(2)}) = \max(T^{(2)} - \delta_1, -\delta_2) + \delta_1,$$
$$|a_2(t)| \rightarrow A_2(T^{(2)}) = \max(T^{(2)} - 2\delta_1, -\delta_1) - \max(T^{(2)} - \delta_1, -\delta_2),$$
$$|a_3(t)| \rightarrow A_3(T^{(2)}) = T^{(2)} - \delta_1 - \max(0, T^{(2)}).$$  (35)

The dispersion relation

$$A_3(T^{(2)}) - 2A_1(T^{(2)}) - A_2(T^{(2)}) = A_3(S^{(2)}) - 2A_1(S^{(2)}) - A_2(S^{(2)})$$  (36)

holds by setting

$$S^{(2)} = -T^{(2)} + \delta_1 - \delta_2 - \max(T^{(2)} - \delta_1, 0) - \frac{1}{2}\max(-T^{(2)} - \delta_2, 0).$$  (37)

If we denote $Q, P$ as the coefficients of $m, n$,

$$Q = A_3(T^{(2)}) - A_3(S^{(2)}) - A_1(T^{(2)}) + A_1(S^{(2)}),$$
$$P = A_1(T^{(2)}) - A_1(S^{(2)}),$$

then $P$ can take any value and $Q = \max(0, P - \delta_2) + \min(0, P + \delta_2)$ holds. Therefore, we obtain another one soliton solution for (29).

$$F_n^m = \max(0, Qn + Pm + C).$$  (39)

It is noted this solution corresponds to the one of the ultradiscrete Toda equation[8].

### 4.2 Mixed Soliton Solution

We consider the ultradiscrete analogue of $b_{ij}$. We define

$$B(T^{(\alpha)}_i, T^{(\beta)}_j) = Cf(T^{(\alpha)}_i, T^{(\beta)}_j) + Cf(S^{(\alpha)}_i, S^{(\beta)}_j) - Cf(T^{(\alpha)}_i, S^{(\beta)}_j) - Cf(S^{(\alpha)}_i, T^{(\beta)}_j)$$  (40)

for $\alpha, \beta = 1, 2$, where $Cf(T^{(\alpha)}_i, T^{(\beta)}_j)$ is the ultradiscrete analogue of $|cf(t_i, t_j)|$, in which $t_i$ and $t_j$ are replaced as $t_i^{(\alpha)}, t_j^{(\beta)}$. For example, $\alpha = 1, \beta = 2$, we have

$$cf(t^{(1)}, t^{(2)}) = \frac{1}{3} \frac{z_3t^{(1)} + z_3t^{(2)} + z_1t^{(1)}z_1 + z_2t^{(2)}z_2}{z_3 + z_1t^{(1)}z_1 + z_2t^{(2)}z_2 + (1 + t^{(2)})z_1z_2}$$

$$\rightarrow \max(T^{(1)}, T^{(1)} + T^{(2)}, T^{(2)} - 2\delta_1, -\delta_1)$$

$$- \max(T^{(1)} + T^{(2)}, T^{(2)} - \delta_1, -\delta_2) + \delta_1$$

$$= :Cf(T^{(1)}, T^{(2)}).$$  (41)
Notice \( C f(S^{(1)}, S^{(2)}) \) can be expressed by \( T^{(1)} \) and \( T^{(2)} \) through the dispersion relations. We can calculate

\[
B(T_i^{(1)}, T_j^{(1)}) = 2 \max(\min(\Omega_i, -\Omega_j), \min(-\Omega_i, \Omega_j)),
\]

\[
B(T_i^{(2)}, T_j^{(2)}) = \max(\min(-2P_i - Q_i, 2P_j + Q_j), \min(2P_i + Q_i, -2P_j - Q_j)),
\]

and

\[
B(T_i^{(1)}, T_j^{(2)}) = \begin{cases} 
- \max(\min(\Omega_i + 2K_i, 2Q_j), \min(\Omega_i + 2K_i - P_j + \delta_2, 0)) & (\Omega_i > 0, P_j > 0) \\
\max(\min(\Omega_i + 2K_i, -2Q_j), \min(\Omega_i + 2K_i + P_j + \delta_2, 0)) & (\Omega_i > 0, P_j < 0) \\
\max(\min(-\Omega_i - 2K_i, 2Q_j), \min(-\Omega_i - 2K_i - P_j + \delta_2, 0)) & (\Omega_i < 0, P_j > 0) \\
- \max(\min(-\Omega_i - 2K_i, -2Q_j), \min(-\Omega_i - 2K_i + P_j + \delta_2, 0)) & (\Omega_i < 0, P_j < 0)
\end{cases}
\]

(43)

where \( \Omega_i, K_i, P_i, Q_i \) are defined by (33) and (38). Due to \(|b_{ij}| = |b_{ji}|, \)

\( B(T_i^{(2)}, T_j^{(1)}) \) can be also obtained. Thus the following holds.

**Theorem 4.1** The solution of (44) is expressed by

\[
F_n^m = \max_{0 \leq k \leq N + M} \max_{I_k \subset [N + M]} \left( \sum_{i \in I_k} \Phi_i(m, n) + \sum_{i, j \in I_k} B_{ij} \right),
\]

(44)

where \( N, M \) are non negative integers. When \( k = 0, \) we define \( \max_{I_0 \subset [N + M]} \) is 0. Function \( \Phi_i(m, n) \) is defined by

\[
\Phi_i(m, n) = \begin{cases} 
\Omega_i m + K_i n + C_i & (i = 1, 2, \ldots, N) \\
Q_i m + P_i n + C_i & (i = N + 1, N + 2, \ldots, N + M)
\end{cases}
\]

(45)

\[
K_i = \frac{1}{2} \left( |\Omega_i - \delta_1| - |\Omega_i + \delta_1| \right), \quad Q_i = \max(0, P_i - \delta_2) + \min(0, P_i + \delta_2),
\]

(46)

with arbitrary parameters \( P_i, \Omega_i \) and \( C_i \). The interaction factor \( B_{ij} \) is defined by

\[
B_{ij} = \begin{cases} 
2 \max(\min(\Omega_i, -\Omega_j), \min(-\Omega_i, \Omega_j)) & (i < j \leq N) \\
\max(\min(-2P_i - Q_i, 2P_j + Q_j), \min(2P_i + Q_i, -2P_j - Q_j)) & (N < i < j), \\
B'_{ij} & (i \leq N < j)
\end{cases}
\]

(47)

where

\[
B'_{ij} = \begin{cases} 
- \max(\min(\Omega_i + 2K_i, 2Q_j), \min(\Omega_i + 2K_i - P_j + \delta_2, 0)) & (\Omega_i > 0, P_j > 0) \\
\max(\min(\Omega_i + 2K_i, -2Q_j), \min(\Omega_i + 2K_i + P_j + \delta_2, 0)) & (\Omega_i > 0, P_j < 0) \\
\max(\min(-\Omega_i - 2K_i, 2Q_j), \min(-\Omega_i - 2K_i - P_j + \delta_2, 0)) & (\Omega_i < 0, P_j > 0) \\
- \max(\min(-\Omega_i - 2K_i, -2Q_j), \min(-\Omega_i - 2K_i + P_j + \delta_2, 0)) & (\Omega_i < 0, P_j < 0)
\end{cases}
\]

(48)
We note the solution in case $M = 0$ corresponds to the soliton solution of the ultradiscrete KdV equation. In other words, this solution $U^n_m$ behaves as the original Box-Ball system through the transformations

$$
U^n_m = F_{n+1}^m + F_n^{m+1} - F_n^{m} - F_{n+1}^m,
$$
$$
V^n_m = F_{n+1}^{m+1} + F_n^{m-1} - 2F_n^m,
$$
$$
X^n_m = F_{n+1}^m + F_{n-1}^m - 2F_n^m,
$$

(49)

which are the ultradiscrete analogues of \( \sigma \). In fact, we can prove $-\delta_2 + X_n^m + X_{n+1}^m < 0$ holds for any $m, n$ when $M = 0$\(^9\). Figure 6 shows the behaviour of the ultradiscrete soliton solution $U^n_m$. The left figure in Fig 6 shows $(3 + 0)$ soliton solution with $(\Omega_1, \Omega_2, \Omega_3) = (1, 3, 4)$ and $(\delta_1, \delta_2) = (1, 2)$. As mentioned in Section 2, its evolution obeys the original Box-Ball system with the box capacity $\delta_1$. The middle figure in Fig 6 shows $(0 + 3)$ soliton solution with $(P_1, P_2, P_3) = (3, 4, 5)$. The notation $\underline{1}$ denotes $-1$. We can observe $U^n_m$ take negative values and move to the left side as discrete solutions. The right figure in Fig 6 shows $(2 + 1)$-mixed soliton solution with $(\Omega_1, \Omega_2, P_1) = (2, 4, 4)$. We can observe they pass through each other without losing their identities.

5 Continuous limit

In this Section, we investigate the continuous limit of the equation \( \sigma \). Let us start with rewriting the equation \( \sigma \) using the Hirota’s D-derivative.

$$
\begin{align*}
-(1 + d_1) \cosh \left( -D_m - \frac{1}{2}D_n \right) + d_1 \cosh \left( D_m - \frac{1}{2}D_n \right) \\
+ d_2 \cosh \left( \frac{3}{2}D_n \right) + (1 - d_2) \cosh \left( \frac{1}{2}D_n \right)
\end{align*}

f \cdot f = 0
$$

(50)

Here, $D_m, D_n$ are the Hirota’s D-derivative corresponding to the variables $m, n$ respectively.
To take the continuous limit we introduce the new variables \( x_1, x_5 \) and relate them with the original variables as

\[
D_m = 2\epsilon D_1 + \frac{2}{3} a_2 \epsilon^3 D_1 + \frac{2}{5} a_3 \epsilon^5 D_5, \\
D_n = 2k \epsilon D_1 + \frac{2}{3} b_2 \epsilon^3 D_1 + \frac{2}{5} b_3 \epsilon^5 D_5.
\]

Here, \( D_1, D_5 \) are the Hirota’s D-derivative of the variables \( x_1, x_5 \), and \( k, a_2, a_3, b_2, b_3 \) are arbitrary complex parameters. \( \epsilon \) is the real parameter which represents the lattice spacing of the discrete variables. Furthermore, we replace the parameters \( d_1, d_2 \) as

\[
d_1 = -\frac{1}{2} + \frac{-k + b_2 \epsilon^2}{4k^2}, \quad d_2 = \frac{a_2 \epsilon^2 + 1}{4k^2}.
\]

Then equation (50) becomes as follows

\[
\left\{ (k(2k - 1) + b_2 \epsilon^2) \cosh \left( -D_m - \frac{1}{2} D_n \right) + (k(2k + 1) - b_2 \epsilon^2) \cosh \left( D_m - \frac{1}{2} D_n \right) \right. \\
+ (-1 - a_2 \epsilon^2) \cosh \left( \frac{3}{2} D_n \right) + (1 - 4k^2 + a_2 \epsilon^2) \cosh \left( \frac{1}{2} D_n \right) \right\} f \cdot f = 0
\]

Taking the small \( \epsilon \) limit we obtain the following equation as the coefficient of the sixth order term of the parameter \( \epsilon \).

\[
\left\{ 108k(a_3 k - b_3)D_1 D_5 - 8k^2(k^2 - 1)(4k^2 - 1)D_1^6 \\
- 40k(k^2 - 1)(a_2 k + 2b_2)D_4^1 + 40(a_2 k - b_2)^2 D_5^2 \right\} f \cdot f = 0 \quad (51)
\]

This equation contains the terms of \( D_1 D_5 f \cdot f, D_1^6 f \cdot f, D_4^1 f \cdot f \) and \( D_5^2 f \cdot f \). As a special case, let us assume the following form of the parameters \( a_i, b_i \).

\[
a_2 = c \left( \frac{-2\sqrt{(k^2 - 1)(4k^2 - 1)} - 4k^2 + 1}{3} \right), \\
b_2 = ck \left( \frac{\sqrt{(k^2 - 1)(4k^2 - 1)} - 4k^2 + 1}{3} \right), \\
a_3 = \frac{2}{3} + \frac{10}{3} k^2 - \frac{8}{3} k^4, \\
b_3 = 0
\]

Substituting these expressions into the equations (51), we obtain the following equation.

\[
- \frac{3k^2(k^2 - 1)(4k^2 - 1)}{45} (9D_1 D_5 + D_1^6 - 5c D_4^1 - 5c^2 D_5^2) f \cdot f = 0
\]

This is nothing but the bilinear form of the KdV-Sawada-Kotera equation appearing in the paper [10].
6 Concluding Remarks

We have derived a soliton equation and its solution from the generalized discrete BKP equation. The equation admits two types of solitary wave solutions. One is the solution which moves to the positive direction and has the positive amplitude while the other does the negative direction and amplitude. The type of the solution depends on whether the parameter \( t \) belongs to \( J_1 \) or \( J_2 \). Interestingly while these solutions have the same form and dispersion relation, their behaviours are quite different. We also have derived its ultradiscrete analogues. The dispersion relations in the ultradiscrete soliton solutions have the same expressions of the ultradiscrete KdV and ultradiscrete Toda equation. Furthermore we have shown the relation between our equation and the KdV-Sawada-Kotera equation through the continuous limit.

Acknowledgement

The authors are grateful to Prof. Ralpf Willox for valuable advices.

Appendix A. Transformation

We shall derive (15) from

\[
\tau(p, q, r) = \sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_N=1}^{2} \prod_{1 \leq i < j \leq N} c_f(t_i^{(k_i)}, t_j^{(k_j)}) \prod_{i=1}^{N} c_i^{(k_i)} \varphi(t_i^{(k_i)}),
\]  
(A.1)

which is a soliton solution of (1) given in [2]. Here \( t_i^{(k_i)} \) and \( c_i^{(k_i)} \) are arbitrary parameters, \( \varphi(t) \) is defined by

\[
\varphi(t) = (a_1(t))^p(a_2(t))^q(a_3(t))^r.
\]  
(A.2)

Using the gauge transformation, that is, by multiplying (A.1) with

\[
\frac{1}{\prod_{1 \leq i < j \leq N} c_f(t_i^{(2)}, t_j^{(2)}) \prod_{i=1}^{N} c_i^{(2)} \varphi(t_i^{(2)})},
\]

we have another form of soliton solution,

\[
\tau(p, q, r) = \sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_N=1}^{2} \prod_{1 \leq i < j \leq N} c_f(t_i^{(k_i)}, t_j^{(k_j)}) \prod_{i=1}^{N} c_i^{(k_i)} \varphi(t_i^{(k_i)}).
\]  
(A.3)

The terms with

\[
k_i = \begin{cases} 
1 & (i \in \mathcal{I} := \{i_1, i_2, \ldots, i_l\}) \\
2 & (i \in \mathcal{I}^c := [N] - \{i_1, i_2, \ldots, i_l\}) 
\end{cases}
\]
in (A.3) are expressed by
\[
\prod_{i \in I} c_i^{(1)} \varphi(t_i^{(1)}) \left( \prod_{1 \leq i < j \leq N} \frac{c_f(t_i^{(1)}, t_j^{(1)})}{c_f(t_j^{(1)}, t_i^{(1)})} \right) \prod_{i \in I, j \in I^c} c_f(t_i^{(1)}, t_j^{(1)}) \prod_{i \in I, j \in I^c} c_f(t_i^{(1)}, t_j^{(2)}) \prod_{i \in I, j \in I^c} c_f(t_i^{(2)}, t_j^{(1)}) \prod_{i \in I, j \in I^c} c_f(t_i^{(2)}, t_j^{(2)}) \right).
\]

If we replace free parameters \( c_i \) as
\[
\frac{c_i^{(1)}}{c_i^{(2)}} \rightarrow c_i \prod_{j=1, i \neq j}^N \frac{c_f(t_i^{(2)}, t_j^{(2)})}{c_f(t_i^{(2)}, t_j^{(1)})},
\]
then (A.4) is reduced to
\[
\prod_{i \in I} c_i \varphi(t_i^{(1)}) \prod_{i \in I} \varphi(t_i^{(2)}) \prod_{i < j} b_{ij}
\]
(A.5)

since \( c_f(t, s) = -c_f(s, t) \) and
\[
\prod_{i \in I, j \neq i} b_{ij} = \prod_{i < j} b_{ij} \prod_{i < j} b_{ji} \prod_{i \in I, j \in I^c} b_{ij} \prod_{i \in I, j \in I^c} b_{ji}
\]
hold. Therefore (A.3) corresponds to (15).

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