Zooming in on infinitesimal $1 - .9\ldots$ in a post-triumvirate era

Karin Usadi Katz and Mikhail G. Katz
Bar Ilan University, Israel

Abstract
The view of infinity as a metaphor, a basic premise of modern cognitive theory of embodied knowledge, suggests in particular that there may be alternative ways in which one could formalize mathematical ideas about infinity. We discuss the key ideas about infinitesimals via a conceptual analysis of the meaning of the ellipsis “…” in the real formula $.999\ldots = 1$. Infinitesimal-enriched number systems accommodate quantities in the half-open interval $[0, 1)$ whose extended decimal expansion starts with an unlimited number of repeated digits 9. Do such quantities pose a challenge to the unital evaluation of the symbol “.999…”? We present some non-standard thoughts on the ambiguity of the ellipsis, in the context of the cognitive concept of generic limit of B. Cornu and D. Tall. We analyze the vigorous debates among mathematicians concerning the idea of infinitesimals.

Keywords: decimal representation, generic limit, hypernatural number, infinitesimal, limit, unital evaluation.

1. Introduction
Prior to the creation of operative infinitesimal-enriched continua in the 20th century, many mathematicians thought of infinitesimals in terms of “naive befogging” and “vague mystical ideas”. Thus, R. Courant (1937, p. 101) presents a rather dim perspective on infinitesimals. On that single page, Courant manages to describe infinitesimals as
- incompatible with the clarity of ideas;
- entirely meaningless;
- vague mystical ideas;
- fog which hang round the foundations;
- hazy idea.

Historically speaking, infinitesimals have been difficult to conceptualize, while a metaphor conveying the idea of proximity, such as approaching or getting closer, has
often been used to make sense of their elusive properties. In this spirit, Roh notes that

the words used in expressing the dynamic imagery of limits, such as ‘approaching’ or ‘getting close to’, do not precisely convey the mathematical meaning of the concept of limit . . . These expressions instead convey an everyday sense of such words, without the precision of the mathematically rigorous meaning of limits (Roh, 2008, p. 218).

We would have been more comfortable with Roh’s conclusion had Roh written, in place of the phrase mathematically rigorous meaning of limit, the more accurate phrase Weierstrassian definition of limit, given the existence of an alternative, dynamic, definition in an infinitesimal-enriched number system. R. Ely (2010) has argued that student nonstandard conceptions, routinely perceived and dismissed as erroneous conceptions, can actually serve as valuable tools in learning calculus.

We propose an approach to infinitesimals via an examination of the meaning of the real formula .999 . . . = 1, henceforth referred to as the unital evaluation of the symbol “.999 . . .”.

In Section 2, we examine the cognitive concept of generic limit of Cornu and Tall, in relation to a hyperreal approach to limits, and exhibit a hyperreal quantity in [0, 1) with an unlimited number of repeated 9s. In Section 3, we represent such a quantity graphically by means of an infinite-magnification microscope, already exploited for pedagogical purposes by Keisler (1986) and Tall (1980), and use it to calculate \( f'(1) \) where \( f(x) = x^2 \), in the framework of Tall’s locally straight approach. The historical Section 4 contains an examination of the views of R. Courant, I. Lakatos, E. Bishop, and A. Heyting as pertaining to infinitesimals. In Section 5, we develop an applied-mathematical model of a hypercalculator so as to explain a familiar phenomenon of a calculator returning a string of 9s in place of an integer. In Section 6 we propose that research be undertaken to see if infinitesimals are a better support for students to come to grips with the notion of limits, and draw some conclusions regarding the foundations of analysis and the ramifications of the use of an infinitesimal-enriched continuum in teaching.

2. Generic limit

D. Tall (2009) notes that “the infinite decimal 0.999 . . . is intended to signal the limit of the sequence 0.9, 0.99, 0.999, . . . which is 1, but in practice it is often imagined as a limiting process which never quite reaches 1.” Tall (1991, 2009)

---

1See footnote for a discussion of such an approach in the context of an infinitesimal-enriched continuum.

2Such as the conception of an infinitesimal as represented by “.000 . . .1” (with infinitely many zeros preceding the digit 1) being the difference between 1 and “.999 . . .9” (with infinitely many digits 9); cf. Lightstone’s notation in footnote.

3See also Sad, Teixeira, Baldino (2001, p. 286).
describes a concept in cognitive theory he calls a *generic limit concept* in the following terms:

[I]f a quantity repeatedly gets smaller and smaller and smaller without ever being zero, then the limiting object is naturally conceptualised as an extremely small quantity that is not zero (Cornu, 1991). Infinitesimal quantities are natural products of the human imagination derived through combining potentially infinite repetition and the recognition of its repeating properties.

Nonstandard student conceptions about infinitesimals were recently analyzed by R. Ely (2007, 2010). J. Monaghan (2001, p. 248), based on field studies, concluded as follows: “do infinite numbers of any form exist for young people without formal mathematical training in the properties of infinite numbers? The answer is a qualified ‘yes’.” Numbers with infinitely many digits are not beyond the intellectual capacity of children.

An examination of the meaning of the real formula \(0.999\ldots = 1\) in the context of an infinitesimal-enriched number system, suggests a mathematical realisation of the cognitive concept of *generic limit*, in terms of a choice of a hypernatural number, thought of as an encapsulation in the context of a metaphor of an “ever larger natural number”, as follows. In the familiar finite domain, we evaluate the formula

\[
1 + r + r^2 + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}
\]

at \(r = \frac{1}{10}\), we obtain

\[
1 + \frac{1}{10} + \frac{1}{100} + \ldots + \frac{1}{10^{n-1}} = \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}},
\]

or alternatively

\[
\frac{1.11\ldots1}{n} \frac{1}{n} = \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}}.
\]

Multiplying by \(\frac{9}{10}\), we obtain

\[
\underbrace{.999\ldots9}_n = \frac{9}{10} \left( \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \right)
\]

\[
= 1 - \frac{1}{10^n}
\]

for every \(n \in \mathbb{N}\). As \(n\) increases without bound, the formula

\[
\underbrace{.999\ldots9}_n = 1 - \frac{1}{10^n}
\]

(1)
becomes

\[ .999 \ldots = 1; \]

at any rate, so goes the traditional mathematical account of the matter of “.999 \ldots”.

Cognitively speaking, the “underbrace” formula (1) suggests exploiting a generic (in Tall’s sense) “infinite natural number”, which we will denote \([\mathbb{N}]\), encapsulating, as it were, the ever increasing sequence

\[ \langle \ldots, n - 1, n, n + 1, \ldots \rangle, \]

so as to obtain

\[ \underbrace{.999 \ldots}_{[\mathbb{N}]} = 1 - \frac{1}{10^{[\mathbb{N}]}}. \] (2)

The latter formula is suggestive of an infinitesimal difference in the minute amount of \( \frac{1}{10^{[\mathbb{N}]}} \), with the quantity \( \underbrace{.999 \ldots}_{[\mathbb{N}]} \) falling just short of 1. The procedure presented so far can be described as inhabiting the proceptual\[ ^4 \] symbolism world, namely the second of Tall’s three worlds (see Tall, 2010).

A remarkable passage by Leibniz is a testimony to the enduring appeal of the metaphor of infinity, even in its, paradoxically, terminated form. In a letter to Johann Bernoulli dating from June 1698, Leibniz speculated concerning “lines \ldots which are terminated at either end, but which nevertheless are to our ordinary lines, as an infinite to a finite” (see Jesseph, 1998). And again, he speculates as to the possibility of

a point in space which can not be reached in an assignable time by uniform motion. And it will similarly be required to conceive a time terminated on both sides, which nevertheless is infinite, and even that there can be given a certain kind of eternity \ldots which is terminated.

To be sure, Leibniz rejected such metaphysics, and ultimately conceived of both infinitesimals and infinite quantities as ideal numbers, falling short of a metaphysical reality of familiar appreciable quantities. To formalize such quantities mathematically, the transition to Tall’s third world of axiomatic formalism can be accomplished through a “generic” analysis of the classical construction of the reals as equivalence classes of Cauchy sequences of rational numbers, that parallels our “generic” analysis of the real formula \( .999 \ldots = 1 \). Here one first relaxes the equivalence relation, so that certain null sequences (i.e. sequences tending to zero) will become inequivalent (and represent distinct infinitesimals). Furthermore, one extends the relation to arbitrary (not merely Cauchy) sequences of rational numbers. Thus, the sequence

\[ \langle u_n, n = 1, 2, 3, \ldots \rangle = \langle \mathbb{N} \rangle, \]

\[ ^4 \text{The term procept was coined by Gray and Tall (1991).} \]
listing all the natural numbers in increasing order, will be (at the level of the new equivalence class) encapsulated as an infinite hypernatural number, denoted \([N]\).

The resulting alternative evaluation (2) of the symbol “.999...” thus employs the string of natural numbers \(\langle 1, 2, 3, \ldots \rangle\), and could be accordingly termed its natural string evaluation, a “generic” competitor to unital evaluation. Ultimately, the traditional insistence on the unital evaluation of the string with an unlimited number of 9s, is necessitated by the absence of infinitesimals in the traditional number system.

3. Infinitesimals under magnifying glass

Tall’s “locally straight” approach employing computer graphics and the metaphor of an infinite microscope, can also be used effectively in the context of an analysis of the real equality .999... = 1, so to give students a geometric feel for the process which is occurring just short of 1. Recall that the symbol “\(\infty\)” is employed in standard real analysis to define a formal completion of the real line \(\mathbb{R}\), namely

\[
\mathbb{R} \cup \{\infty\}.
\]

Sometimes a formal point “\(-\infty\)” is added, as well. Such a formal device is helpful in simplifying the statements of certain theorems (which would otherwise have a number of subcases).\(^5\)

\(^5\)Such a construction is known as the ultrapower construction and is due to Luxemburg in 1962, see (Luxemburg 1964) and (Goldblatt, 1998). To obtain the full hyperreal field, one starts with sequences of real numbers.

\(^6\)Each real number is accompanied by a cluster (alternative terms are prevalent in the literature, such as halo, but the term cluster has the advantage of being self-explanatory) of hyperreals infinitely close to it. The standard part function collapses the entire cluster back to the standard real contained in it. The cluster of the real number 0 consists precisely of all the infinitesimals. Every infinite hyperreal decomposes as a triple sum \(H + r + \epsilon\), where \(H\) is a hyperinteger (see below), \(r\) is a real number in \([0, 1)\), and \(\epsilon\) is infinitesimal. Varying \(\epsilon\) over all infinitesimals, one obtains the cluster of \(H + r\). A hyperreal number \(H\) equal to its own integer part: \(H = [H]\) is called a hyperinteger. Here the integer part function is the natural extension of the real one. Limited (finite) hyperintegers are precisely the standard ones, whereas the unlimited (infinite) hyperintegers are sometimes called non-standard integers. The limit \(L\) of a convergent sequence \(\langle u_n \rangle\) is the standard part \(st\) of the value of the sequence at an infinite hypernatural: \(L = st(u_H)\), for instance at \(H = [N]\). In connection with .999..., I. Stewart notes that

\[
\text{[t]he standard analysis answer is to take ‘...’ as indicating passage to a limit. But in non-standard analysis there are many different interpretations (Stewart, 2009, p. 176).}
\]

Some additional details to be found in (Katz & Katz, 2010). Historically, there have been two main approaches to infinitesimals. The approach of Leibniz postulates the existence of infinitesimals of arbitrary order, while B. Nieuwentijdt favored nilpotent (nilsquare) infinitesimals (see J. Bell 2009). Bell notes that a Leibniz infinitesimal is implemented in the hyperreal continuum of Robinson, whereas a Nieuwentijdt infinitesimal is implemented in the smooth infinitesimal analysis of F.W. Lawvere, based on intuitionistic logic.

\(^7\)The symbol is used in a different sense in projective geometry, where adding a point at infinity \(\{\infty\}\) to \(\mathbb{R}\) results a circle: \(\mathbb{R} \cup \{\infty\} \approx S^1\).
ZOOMING IN ON INFINITESIMAL 1 − .9.. IN A POST-TRIUMVIRATE ERA

Figure 1. Three infinitely close hyperreals under a microscope

We have refrained from exploiting the symbol “∞” to denote an infinite hypernatural number, as in

\[ \underbrace{.999\ldots}_{\infty} \]

so as to avoid the risk of creating a false impression of the uniqueness, as in (3), of such an infinite point (see Figure 1). We represent the quantity

\[ \underbrace{.999\ldots}_{H} \]

visually by means of an infinite-resolution microscope already exploited for pedagogical purposes by Keisler (1986). Similarly, we can follow Tall’s “locally straight” approach, and exploit our “generic” analysis of “.999…” in order to calculate the slope of the tangent line to the curve \( y = x^2 \) at \( x = 1 \).
Thus, we first compute the ratio

\[
\frac{\Delta y}{\Delta x} = \frac{(.9..)^2 - 1^2}{.9.. - 1} = \frac{(.9.. - 1)(.9.. + 1)}{.9.. - 1} = .9.. + 1,
\]

where we have deleted the underbrace \(\underbrace{\text{H}}\) and also shortened the symbol \(.999\ldots\) to \(.9..\), so as to lighten the notation for the purposes of this section only. Next, we note that the “standard part” function, denoted “st”, associates to each finite quantity, the unique real number infinitely close to it. The slope is then computed as the standard part

\[
\frac{dy}{dx} = \text{st} \left( \frac{\Delta y}{\Delta x} \right) = \text{st}(.9.. + 1) = \text{st}(.9..) + 1 = 1 + 1 = 2,
\]

as illustrated in Figure 2.

In the context of geometric visualisation of infinitesimals, perhaps a helpful parallel is provided by the famous animated film Flatland, based on the classic book (Abbott, 2008). Here the two-dimensional creatures are unable to conceive of what we think of as the sphere in 3-space, due to their dimension limitation. Similarly, one can conceive of the difficulty in the understanding of the unital evaluation of “.999\ldots”, as due to the limitation of the standard real vision.

The flatlanders see the sphere, as it cuts across their existential plane, as the creation of a point, continuously growing through a family of concentric circles, and

\footnote{Note that \(\Delta x = .9.. - 1 = -.0\ldots01\) in Lightstone’s notation (see Lightstone, 1972), where the digit “1” appears at infinite decimal rank \(H\).}
then decreasing in radius and petering out as the circle shrinks to a point and disappears. Meanwhile, E. A. Abbott sees, of course, a sphere as it moves across a plane in 3-space.

Over half a millenium earlier, in 1310, Yitzchak HaYisraeli (1310) may have seen more than what Mr. Abbott saw. Before the disappearance of the shrinking circle, he perceives “the smallest of them, a cap very, very small so you would think there is no cap smaller than it, with the point of the pole fixed, as it were, in its midst as its center” (HaYisraeli, 1310, p. 10). Note that he mentions the pole as the center of the very, very small cap, as distinct from the cap itself. This indicates the the cap has not yet degenerated to a point in the author’s mind. Yet it is very, very small and there is no cap smaller than it, indicating that he is referring to an infinitesimal cap.

Perhaps Yitzchak HaYisraeli can be thought of as a precursor of the generic limit, all this a century prior to Nicholas of Cusa. Nicholas of Cusa (15th century) considered the circle as a polygon with an infinite number of sides, inspiring Kepler to formulate his bridge of continuity (see Tall, 2010). Yitzchak HaYisraeli’s book Yesod Olam was written over a decade before the birth of Nicole Oresme, another pioneer of infinitesimals. HaYisraeli (1310, p. 12 and fol.) also presents a detailed statement of the spherical law of sines and both cases of the spherical law of cosines for right-angled triangles, and mentions, as one of his illustrious predecessors at Toledo, the astronomer Ibrahim al-Zarqali (see HaYisraeli, 1310).

That the notion of an infinitesimal appeals to intuition is acknowledged even by its critics, see R. Courant’s comment in Section 4. The infinitesimal would not go away inspite of what is, by now, over a century of \((\varepsilon, \delta)\)-ideology, see Bishop’s comment in Section 4.

4. Second thoughts about infinitesimals

In the introduction, we quoted Courant’s unflattering appraisal (predating Robinson’s work) of infinitesimals. Yet, Courant acknowledges the appeal of the infinitesimal: “For a great many simple-minded people it undoubtedly has a certain charm, the charm of mystery which is always associated with the word ‘infinite’.” Courant was unable to peer through the hazy mystical fog the way Robinson would. It should be kept firmly in mind that Courant’s criticism predated Robinson’s theory, unlike more recent criticisms. Courant’s criticism was not without merit at the time of its writing, namely, a quarter century prior to Robinson’s work. I. Lakatos wrote in 1966 as follows:

Robinson’s work . . . offers a rational reconstruction of the discredited infinitesimal theory which satisfies modern requirements of rigour and which is no weaker than Weierstrass’s theory. This reconstruction makes

\(^9\)See (Goldwurm, 2001, p. 104) for biographical information on HaYisraeli.
infinitesimal theory an almost respectable ancestor of a fully fledged, powerful modern theory, lifts it from the status of pre-scientific gibberish, and renews interest in its partly forgotten, partly falsified history (Lakatos, 1978, p. 44). \footnote{The falsification problem is analyzed in Ehrlich (2006).}

Not everyone was persuaded by Lakatos’ appreciation of the significance of Robinson’s work. A decade later, in 1975, Courant’s duty of avoiding every such hazy idea was taken up under a constructivist banner by E. Bishop, affectionately described as the “reluctant guru” of constructive mathematics by his advisor, P. Halmos (1985, p. 162). In his essay (Bishop, 1975) cast in the form of an imaginary dialog between Brouwer and Hilbert, Bishop anchors his foundational stance in a species of mathematical constructivism. Thus, Bishop’s opposition to Robinson’s infinitesimals, expressed in a vitriolic\footnote{Historians of mathematics have noted the vitriolic tenor of Bishop’s criticism, see e.g. (Dauben, 1996), p. 139.} review (Bishop, 1977) of Keisler’s textbook,\footnote{Namely, review of an earlier edition of (Keisler, 1986).} was to be expected (and in fact was anticipated by editor Halmos).

Two years earlier, Bishop had already expressed his views about non-standard analysis and its use in teaching in a brief paragraph toward the end of his essay “Crisis in contemporary mathematics”, see Bishop (1975, p. 513-514). Having discussed Hilbert’s formalist program, he writes:

A more recent attempt at mathematics by formal finesse\footnote{The description of Hilbert’s program as “formal finesse” has been objected to by many authors. Avigad and Reck (2001) provide a detailed discussion of the significance, and meaning, of Hilbert’s program.} is non-standard analysis. I gather that it has met with some degree of success, whether at the expense of giving significantly less meaningful proofs I do not know. My interest in non-standard analysis is that attempts are being made to introduce it into calculus courses.

Bishop concludes: “It is difficult to believe that debasement of meaning could be carried so far.” Bishop’s view of the introduction of non-standard analysis in the classroom as no less than a debasement of meaning, was duly noted by Dauben (1992).

Bishop’s sentiments toward non-standard calculus stand in sharp contrast with those of his fellow intuitionist A. Heyting (1973, p. 136) who felt that

[Robinson] connected [an] extremely abstract part of model theory with a theory apparently so far apart as the elementary calculus. In doing so [he] threw new light on the history of the calculus by giving a clear sense to Leibniz’s notion of infinitesimals.\footnote{Note that Schubring (2005) attributes the first systematic use of infinitesimals as a foundational concept, to Johann Bernoulli.}
Bishop clarified what exactly it was he had in mind when he spoke of *debasement of meaning* in an earlier text (Bishop, 1985) distributed in 1973 and eventually published in 1985. Here Bishop writes: “Brouwer’s criticisms of classical mathematics were concerned with what I shall refer to as ‘the debasement of meaning’ ” (1985, p. 1). In Bishop’s own words, the *debasement of meaning* expression, employed in his *Crisis* text to refer to non-standard calculus, is in fact a criticism of *classical mathematics* as a whole, exposing a thinly disguised foundational agenda in his criticism of non-standard calculus as well.

To illustrate how Bishop anchors his foundational stance in a species of mathematical constructivism, note that he writes:

To my mind, it is a major defect of our profession that we refuse to distinguish . . . between integers that are computable and those that are not . . . the distinction between computable and non-computable, or constructive and non-constructive is the source of the most famous disputes in the philosophy of mathematics... (Bishop, 1975, pp. 507-508).

On page 511, Bishop defines a principle (LPO) as the statement that “it is possible to search ‘a sequence of integers to see whether they all vanish’ ”, and goes on to characterize the dependence on the LPO as a procedure both Brouwer and Bishop himself reject. S. Feferman explains the principle as follows:

Bishop criticized both non-constructive classical mathematics and intuitionism. He called non-constructive mathematics “a scandal”, particularly because of its “deficiency in numerical meaning”. What he simply meant was that if you say something exists you ought to be able to produce it, and if you say there is a function which does something on the natural numbers then you ought to be able to produce a machine which calculates it out at each number (Feferman, 2000).

Elsewhere, Feferman identifies LPO as a special case of the law of excluded middle.

It recently came to light (Manning, 2009) that Bishop never uttered his criticism of non-standard calculus in his oral presentation at the 1974 workshop (which helps explain the absence of any critical reaction to such *debasement* on the part of the audience in the discussion session, included at the end of the published version of his talk), but rather inserted it at the galley proof stage of publication. Bishop fails to acknowledge in his review in the *Bulletin* that his criticism is motivated by his foundational preoccupation with the law of excluded middle, and with what he calls *numerical meaning* in his *Crisis* essay. In short, Bishop is criticizing apples for not being oranges: the critic (Bishop) and the criticized (non-standard analysis) do not share a common foundational framework. Note that a similar point was mentioned by M. Davis (1977, p. 1008). Note that the foundational framework of non-standard analysis, namely the Zermelo-Frankel set theory with the axiom of choice (ZFC), is the framework of the vast majority of the readers of the *Bulletin*, at variance with
Bishop’s preferred framework, in which his *debasement of meaning* criticism would apply equally well to all of mainstream mathematics. This point was alluded to in Feferman’s comment cited above.

What Bishop sees as *debasement of meaning* in classical mathematics is the alleged absence of *numerical meaning* already alluded to above. Classical logic relies on the *law of excluded middle* (LEM), the key ingredient in proofs by contradiction. Bishop tends to conflate his narrow notion of *numerical meaning* (narrowly defined as the avoidance of LEM) with *meaning* in a wider epistemological sense.

Meanwhile, infinitesimal calculus as developed by the founders of the discipline possesses numerical meaning of the post-LEM variety, as expressed in explicit formulas for the derivatives and integrals of standard functions. Infinitesimal calculus remains in the post-LEM category even after Robinson’s work, due to the nature of the construction of the hyperreal number system.

Dauben (1996, p. 135) presents a detailed analysis, in the areas both of pedagogy and of meaning, of Bishop’s criticisms, and describes them as ultimately “unfounded”. Dauben (1996, p. 133) identifies an intriguing point of convergence between Bishop and Robinson, namely that the history of the calculus has been, in Bishop’s words, “systematically distorted to support the status quo” (see (Bishop, 1975, p. 508)). Dauben describes as “one of the most important achievements of Robinson’s work in non-standard analysis”, “his conclusive demonstration of the poverty of [the] kind of historicism” that focuses exclusively on an alleged triumph of “Weierstrassian epsilontics over infinitesimals in making the calculus “rigorous” in the course of the 19th century”. Non-standard calculus in the classroom was analyzed in the Chicago study by K. Sullivan (1976). Sullivan showed that students following the non-standard calculus course were better able to interpret the sense of the mathematical formalism of calculus than a control group following a standard syllabus. Such a conclusion was also noted by M. Artigue (1994, p. 172). A synthesis of such experiments was made by Bernard Hodgson (1994) in 1992, and presented at the ICME-7 at Quebec. G. Schubring (2005, p. 153) points out that an alternative approach to calculus developed by a German mathematician “has been . . . unable to win as much celebrity and as many adherents as [non-standard analysis]”. Leibniz historian H. Bos (1974, p. 13) acknowledged that Robinson’s hyperreals provide a preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities.

F. Medvedev (1998) further points out that nonstandard analysis “makes it possible to answer a delicate question bound up with earlier approaches to the history of classical

---

15See footnote 14 for a historical clarification.

16An alternative infinitesimal-enriched intuitionistic continuum has been developed by Lawvere, see footnote 6.

17The issue of the falsification of the history of the calculus was discussed by Lakatos, see main text at footnote 10.
analysis. If infinitely small and infinitely large magnitudes are [to be] regarded as inconsistent notions, how could they [have] serve[d] as a basis for the construction of so [magnificent] an edifice of one of the most important mathematical disciplines?”

5. Hypercalculator returns .999...

Our goal here is to illustrate how a non-standard number “.999…” can be exploited in an explanation of a familiar phenomenon from routine calculator use. This topic is too advanced to be presented at a highschool level, as it involves Newton’s algorithm. The latter is an iterative procedure for finding a sequence \(x_0, x_1, \ldots\) rapidly converging to a zero of a smooth function, under suitable hypotheses on the derivatives, and thus requires a familiarity with calculus. On the other hand, it could be presented as an “enrichment” topic to a class of college students, already familiar with standard calculus, in the context of a first exposure to the hyperreals.

Everyone who has ever held an electronic calculator is familiar with the curious phenomenon of it sometimes returning the value

\[ .999999 \]

in place of the expected 1.000000. For instance, a calculator programmed to apply Newton’s method to find the zero of a function, may return the .999999 value as the unique zero of the function

\[ \log x. \]

Developing a model to account for such a phenomenon is complicated by the variety of the degree of precision displayed, as well as the greater precision typically available internally than that displayed on the LCD. To simplify matters, we will consider an idealized model, called a hypercalculator, of a theoretical calculator that applies Newton’s method precisely \(H\) times, where \(H\) is a fixed infinite hypernatural, e.g. \([N]\), as discussed in Section 2.

It can be established that \(if f is a concave strictly increasing differentiable function with domain an open interval (1 − \(\epsilon\), 1 + \(\epsilon\)) and vanishing at its midpoint, then the hypercalculator applied to \(f\) will return a hyperreal decimal .999… with an initial segment consisting of an unlimited number of repeated 9’s.

Indeed, it is well-known that Newton’s algorithm converges under the above hypotheses. We will reproduce the main calculation of the standard proof, emphasizing the novelty that the strict inequality can be retained in the end, as in \(\text{[5]}\).

Assume for simplicity that \(f(x_0) < 0\). We have

\[ x_1 = x_0 + \frac{|f(x_0)|}{f'(x_0)}. \]
By the mean value theorem, there is a point $c$ such that $x_0 < c < 1$ where $f'(c) = \frac{|f(x_0)|}{1-x_0}$, or

$$\frac{|f(x_0)|}{f'(c)} = 1 - x_0.$$ 

Since $f$ is concave, its derivative $f'$ is decreasing, hence

$$x_1 = x_0 + \frac{|f(x_0)|}{f'(x_0)} < x_0 + 1 - x_0 = 1.$$

Thus $x_1 < 1$. Inductively, the point $x_{n+1} = x_n + \frac{|f(x_n)|}{f'(x_n)}$ satisfies $x_n < 1$ for all $n$. By the transfer principle of non-standard analysis, the hyperreal $x_H$ satisfies a strict inequality

$$x_H < 1,$$

as well. Hence the hypercalculator returns a value strictly smaller than 1 yet infinitely close to 1, and therefore starts with an unlimited number of 9s.

### 6. Conclusions

The utility of infinitesimals is known to transcend the “.999...” issue. In the fall of 2008, the second-named author taught a course in calculus using Keisler’s textbook *Elementary Calculus* (Keisler, 1986) to a group of 25 freshmen. The TA had to be trained as well, as the material was new to the TA. The students have never been so excited about learning calculus, according to repeated reports from the TA. Two of the students happened to be highschool teachers (they were somewhat exceptional in an otherwise teenage class). They said they were so excited about the new approach that they had already started using infinitesimals in teaching basic calculus to their 12th graders. After the class was over, the TA paid a special visit to the professor’s office, so as to place a request that the following year, the course should be taught using the same approach. Furthermore, the TA volunteered to speak to the chairman personally, so as to impress upon him the advantages of the approach exploiting an infinitesimal-enriched number system. *Restoring* an infinitesimal-enriched number system to the classroom will enhance the presentation of a million calculus teachers around the globe.

The difficulty of the Weierstrassian limit concept is no secret either to professional mathematicians, or to professional educators. One approach to the difficulty has been to view it as intrinsic in the subject matter. Thus, C. Boyer (1949, p. 298) sees the crowning of 2500 years of “investigations leading to the calculus” from Pythagoras onward, in the “satisfactory definitions of number and the infinite” established by “the great triumvirate: Weierstrass, Dedekind, and Cantor”. To his credit, Boyer notes on page 299 that “[t]here is a strong temptation on the part of

---

18Infinitesimals were in classroom use as late as the 1930s, if sometimes surreptitiously, see (Roquette 2008) and (Luzin 1931).
professional mathematicians and scientists to seek always to ascribe great discoveries and inventions to single individuals.” On page 287, we learn that “the limit concept does not involve the idea of approaching, but only a static state of affairs.” On page 298, it turns out that “any criticism of the use of the infinite in defining irrational number or in the limit concept is answered by Cantor’s work, which clarifies the situation.” Boyer asserts that such criticisms were answered by Cantor. Or perhaps not entirely? Boyer equips his phrase with a footnote 92, which reads as follows: “It should, perhaps, be observed at this point that the theory of infinite aggregates has resulted also in a number of puzzling and as yet unresolved antimonies.” Nonetheless, “the real number of Dedekind is in a sense a creation of the human mind, independent of intuitions of space and time” (Boyer, 1949, p. 292).

A cognitive scientist would agree emphatically that the real number is a creation of the human mind; he would disagree that such a creation (as all embodied knowledge) could be independent of intuitions of space and time. However, on page 305 we are told that “[t]he bases of the calculus were then defined formally in terms only of number and infinite aggregates, with no corroboration through an appeal to the world of experience either possible or necessary.”

One wonders how many calculus instructors may have adopted Boyer’s view concerning a lack of necessity of appeals to the world of experience, in formally defining the bases of the calculus. On page 308, Boyer describes mathematics as “the symbolic logic of possible relations”, and is content to conclude his book on page 309 with a final flourish, describing mathematics as “a syllogistic elaboration of arbitrary premises”.

Boyer has thus invested the great triumvirate with the highest ministry in a disembodied world of mathematics, stripped of both intuition and motion, but firmly afloat upon the twin whales of a Cantorian set-theoretic paradise of infinite aggregates, on the one hand, and unassailable symbolic logic, on the other. Let us therefore hear out the logicians themselves—the best of them.

Abraham Robinson, speaking at Bristol in 1973, confides as follows: “I do not believe in the primacy of set theory over all other branches of mathematics” (Robinson, 1975, p. 48); (Robinson, 1979, p. 563). But what about the infinite aggregates? “[M]athematical theories which, allegedly, deal with infinite totalities do not have any detailed . . . reference” (Robinson, 1975, p. 42; 1979, p. 557). Two decades before Lakoff & Núñez (2000), Robinson describes mathematical infinities as lacking a detailed reference, i.e., as metaphors. But is it conceivable that the empirical scientists should dare throw off the yoke of the great triumvirate? On this point, Robinson’s deference to the empirical scientist is as palpable as it is visionary:

At this point, we should also consider the possibility of future developments in the empirical sciences which will affect the areas with which we are concerned. I can think of one such development, which will surely occur in the fullness of time, although I have no means of judging when. This is the possibility of analyzing in detail the neurophysiological processes in
the brain which correspond to its mathematical activity (Robinson, 1975, p. 48-49); (Robinson, 1979, p. 563-564).

An increasing sentiment in foundational and educational circles is that there are alternative ways in which to formalize mathematical ideas (in this case, ideas about infinity). It is not uncommon to hear opinions to the effect that one needs to break the dogma that foundations in mathematics are a ‘solid building’, or that it is not possible to have different explanations to justify formal results. Other educators feel that mathematical ideas can be, and have been, controversial.

Acknowledgments

We are grateful to David Ebin and David Tall for a careful reading of an earlier version of the manuscript, and for making numerous helpful suggestions. We thank the editor of the article and the anonymous referees for an exceptionally thorough analysis of the shortcomings of the version originally submitted, resulting, through numerous intermediate versions, in a more focused text.

References

Abbott, E. A. (2008) The annotated Flatland. A romance of many dimensions. With an introduction and notes by Ian Stewart. Basic Books, New York. Original work published 2002.

Artigue, M. (1994) Analysis, in Advanced Mathematical Thinking (ed. David Tall), Springer-Verlag, p. 172 (“The non-standard analysis revival and its weak impact on education”).

Avigad, J.; Reck, E. (2001) “Clarifying the nature of the infinite”: the development of metamathematics and proof theory. December 11, 2001. Carnegie Mellon Technical Report CMU-PHIL-120.

Bell, J. L. (2009) Continuity and infinitesimals. Stanford Encyclopedia of philosophy. Revised 20 july ’09.

Bishop, E. (1975) The crisis in contemporary mathematics, Historia Math. 2, no. 4, 507-517.

Bishop, E. (1977) Review: H. Jerome Keisler, Elementary calculus, Bull. Amer. Math. Soc. 83, 205-208.

Bishop, E. (1985) Schizophrenia in contemporary mathematics. In Errett Bishop: reflections on him and his research (San Diego, Calif., 1983), 1–32, Contemp. Math. 39, Amer. Math. Soc., Providence, RI.

Bos, H. J. M. (1974) Differentials, higher-order differentials and the derivative in the Leibnizian calculus. Arch. History Exact Sci. 14, 1–90.

In a letter to M. Vygodskii, the mathematician Luzin questioned whether the Weierstrassian approach to the foundations of analysis “corresponds to what is in the depths of our consciousness . . . I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the Integral calculus and the incomparably artificial and complex work of the ‘justification’ and their ‘proofs’ ” (Luzin, 1931). The publication of the text Fundamentals of Infinitesimal Calculus, by Vygodskii, in 1931, provoked sharp criticisms. Luzin wrote his (two) letters to counterbalance such criticisms, and took the opportunity to elaborate his own views concerning infinitesimals.
ZOOMING IN ON INFINITESIMAL 1 − .9... IN A POST-TRIUMVIRATE ERA

Boyer, C. (1949) *The concepts of the calculus*. Hafner Publishing Company.

Cornu, B. (1991) *Limits*, pp. 153-166, in *Advanced mathematical thinking*. Edited by David Tall. Mathematics Education Library, 11. Kluwer Academic Publishers Group, Dordrecht.

Courant, R. (1937) *Differential and integral calculus*. Vol. I. Translated from the German by E. J. McShane. Reprint of the second edition. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

Dauben, J. (1992) “Appendix (1992): revolutions revisited”. pp. 72–82 in *Revolutions in mathematics*. Edited and with an introduction by Donald Gillies. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.

Dauben, J. (1996) *Arguments, logic and proof: mathematics, logic and the infinite. History of mathematics and education: ideas and experiences* (Essen, 1992), 113–148, *Stud. Wiss. Soz. Bildungsgesch. Math.*, 11, Vandenhoeck & Ruprecht, Göttingen.

Davis, M. (1977) Review: J. Donald Monk, *Mathematical logic*, *Bull. Amer. Math. Soc.* 83, 1007-1011.

Ely, R. (2007) Student obstacles and historical obstacles to foundational concepts of calculus. PhD thesis, The University of Wisconsin - Madison.

Ely R. (2010) Nonstandard student conceptions about infinitesimals. *Journal for Research in Mathematics Education* 41, no. 2, 117-146.

Ehrlich, P. (2006) The rise of non-Archimedean mathematics and the roots of a misconception. I. The emergence of non-Archimedean systems of magnitudes. *Arch. Hist. Exact Sci.* 60, no. 1, 1–121.

Feferman, S. (2000) Relationships between constructive, predicative and classical systems of analysis. Proof theory (Roskilde, 1997), 221-236, Synthese Lib., 292, Kluwer Acad. Publ., Dordrecht.

Goldblatt, R. (1998) *Lectures on the hyperreals. An introduction to nonstandard analysis*. Graduate Texts in Mathematics, 188. Springer-Verlag, New York.

Goldwurm, R’ H. (2001) *The Rishonim. Biographical sketches of the prominent early rabbinic sages and leaders from the tenth-fifteenth centuries*. Second edition. Artscroll History Series. Mesorah Publications.

Gray, E.; Tall, D. (1991) Duality, Ambiguity and Flexibility in Successful Mathematical Thinking. *Proceedings of PME 15*, Assisi, 2, 72-79.

Halmos, P. (1985) *I want to be a mathematician. An automathography*. Springer-Verlag, New York.

HaYisraeli, Yitzchak (1310) *Yesod Olam*. In “*Poel hashem*”, Bnei Braq.

Heijting, A. (1973) Address to Professor A. Robinson. At the occasion of the Brouwer memorial lecture given by Prof. A. Robinson on the 26th April 1973. *Nieuw Arch. Wisk.* (3) 21, 134–137.

Hodgson, B.R. (1994) *Le calcul infinitésimal*. In D.F. Robitaille, D.H. Wheeler et C. Kieran, dir., Choix de conférence du 7e Congrès international sur l’enseignement des mathématiques (ICME-7). Presses de l’Université Laval, pp. 157-170.

Jesseph, D. (1998) Leibniz on the foundations of the calculus: the question of the reality of infinitesimal magnitudes. Leibniz and the sciences. *Perspect. Sci.* 6, no. 1-2, 6–40.

Katz, K.; Katz, M. (2010) When is .999... less than 1? *The Montana Mathematics Enthusiast*, 7, no. 1, 3–30.
Keisler, H. Jerome (1986) *Elementary Calculus: An Infinitesimal Approach*. Second Edition. Prindle, Weber & Schmidt, Boston.

Lakoff, G.; Núñez, R. (2000) *Where mathematics comes from. How the embodied mind brings mathematics into being*. Basic Books, New York.

Lakatos, I. (1978) Cauchy and the continuum: the significance of nonstandard analysis for the history and philosophy of mathematics. Math. Intelligencer 1, no. 3, 151–161 (originally published in 1966).

Lightstone, A. H. (1972) Infinitesimals. *Amer. Math. Monthly* 79, 242–251.

Luxemburg, W. (1964) *Nonstandard analysis. Lectures on A. Robinson's Theory of infinitesimals and infinitely large numbers*. Pasadena: Mathematics Department, California Institute of Technology, second corrected edition.

Luzin, N. N. (1931) Two letters by N. N. Luzin to M. Ya. Vygodskiı. With an introduction by S. S. Demidov. Translated from the 1997 Russian original by A. Shenitzer. *Amer. Math. Monthly* 107 (2000), no. 1, 64–82.

Manning, K. (2009) private communication, july 2009.

Medvedev, F. A. (1998) Nonstandard analysis and the history of classical analysis. Translated by Abe Shenitzer. *Amer. Math. Monthly* 105, no. 7, 659–664.

Monaghan, J. (2001) Young peoples’ ideas of infinity. *Educational Studies in Mathematics* 48, no. 2-3, 239-257.

Robinson, A. (1966) *Non-standard analysis*. North-Holland Publishing Co., Amsterdam.

Robinson, A. (1975) Concerning progress in the philosophy of mathematics. Logic Colloquium ’73 (Bristol, 1973), pp. 41–52. Studies in Logic and the Foundations of Mathematics, Vol. 80, North-Holland, Amsterdam.

Robinson, A. (1979) Selected papers of Abraham Robinson. Vol. II. Nonstandard analysis and philosophy. Edited and with introductions by W. A. J. Luxemburg and S. Krner. Yale University Press, New Haven, Conn.

Robinson, A. (1996) *Non-standard analysis*. Reprint of the second (1974) edition. With a foreword by Wilhelmus A. J. Luxemburg. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ.

Roh, K.H. (2008) Students’ images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics* 69, 217-233.

Roquette, P. (2008) Numbers and Models, standard and nonstandard. “Algebra days”, may 2008, Antalya.

Sad, L. A.; Teixeira, M. V.; Baldino, R. B. (2001) Cauchy and the problem of point-wise convergence. *Arch. Internat. Hist. Sci.* 51, no. 147, 277–308.

Schubring, G. (2005) *Conflicts between generalization, rigor, and intuition. Number concepts underlying the development of analysis in 17–19th Century France and Germany*. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag, New York.

Stewart, I. (2009) *Professor Stewart’s Hoard of Mathematical Treasures*. Profile Books.

Sullivan, K. (1976) The Teaching of Elementary Calculus Using the Nonstandard Analysis Approach, *Amer. Math. Monthly* 83, 370-375.

Tall, D. (1980) Looking at graphs through infinitesimal microscopes, windows and telescopes, *Mathematical Gazette* 64, 22-49.
Tall, D. (1982) Elementary axioms and pictures for infinitesimal calculus, *Bulletin of the IMA*, 18, 43–48.

Tall, D. (1991) The psychology of advanced mathematical thinking, in *Advanced mathematical thinking*. Edited by David Tall. Mathematics Education Library, 11. Kluwer Academic Publishers Group, Dordrecht.

Tall, D. (2009) Dynamic mathematics and the blending of knowledge structures in the calculus, pp. 1-11 in Transforming Mathematics Education through the use of Dynamic Mathematics, *ZDM* (June 2009).

Tall, D. (2010) *How humans learn to think mathematically* (to appear).

Tall, D; Schwarzenberger, R. (1978) Conflicts in the Learning of Real Numbers and Limits. *Mathematics Teaching* 82, 44-49.