Finding and using exact solutions of the Einstein equations

M.A.H. MacCallum

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, LONDON E1 4NS, U.K.
Email: m.a.h.maccallum@qmul.ac.uk

Abstract. The evolution of the methods used to find solutions of Einstein’s field equations during the last 100 years is described. Early papers used assumptions on the coordinate forms of the metrics. Since the 1950s more invariant methods have been deployed in most new papers. The uses to which the solutions found have been put are discussed, and it is shown that they have played an important role in the development of many aspects, both mathematical and physical, of general relativity.

Keywords: General relativity; exact solutions; global properties; black holes; gravitational waves

PACS: 04.20-q; 04.20.Dw; 04.20.Jb; 04.25-g; 04.30.-w; 04.40.Nr

1. INTRODUCTION

In an hour’s talk such as this it is impossible to cover all that is known on the subject, especially since in some respects detail is of the essence in dealing with exact solutions. The three major recent reviews of rather general character [1, 2, 3] have a total of over 1200 pages and that is before moving to more specialized reviews such as [4, 5]. (The short summary of what is new in [1] as compared with the first edition, given below as an appendix, illustrates some ways in which the field has changed in the last 25 years.) Thus I shall be selective, dwelling at some length on particular solutions rather than trying to cram in as many as possible in the time. For example, I give particular attention to the Schwarzschild solution, the first solution known that did not have constant curvature. I apologize to the authors of the many excellent pieces of work I do not touch on.

Before learning the techniques for finding solutions of the Einstein equations and discovering their properties, one should ask “why this is a worthwhile endeavour?”. Some colleagues do seem to regard it as something of a backwater in the theory. The reason it is still important stems from the nonlinearity of general relativity, one of its essential features. To understand the meaning of the theory, there are really three approaches. One can seek to prove global results, such as are described in [6] and were the subject of the Isaac Newton Institute programme in progress at the time of writing (see http://www.newton.cam.ac.uk/webseminars/pg+ws/2005/gmr/). One can try to use approximation, either in the form of iterated perturbation methods or numerical solutions. Finally, one can use exact solutions: as Mason and Woodhouse said, “they combine tractability with nonlinearity, so they make it possible to explore nonlinear phenomena while working with explicit solutions” [7], and as we shall see below, they have had considerable impact on the theory.

We also need to ask: “what is a solution?”. We could assume a form for the metric,
calculate its Einstein tensor, and so obtain a form for the energy-momentum through
Einstein’s equations. The pointlessness of such ‘solutions’ was made clear by Synge [8]. More generally, choosing a more complicated form of energy-momentum with a
simple form of metric usually reduces the number of equations to be solved and makes
the task of finding solutions easier. The vacuum case, and cases with an equivalent set
of equations to actually solve, are more difficult.

Of course, exact solutions are very special cases. However, as Bicak [3] noted, Feyn-
man said “The physicist is always interested in the special case. He is talking about
something, he is not talking abstractly about anything. He wants to discuss the gravity
law in three dimensions: he never wants the arbitrary force case in $n$ dimensions. So a
certain amount of reducing is necessary...”. The special cases known exactly can be very
useful as examples in guiding the global or approximative approaches.

Indeed without such uses finding exact solutions would be more like stamp-collecting
than science. It requires ingenuity and the objects found may be beautiful, which is fine
if you think relativity is a branch of pure mathematics, but if you think it is physics, that
is not so good, despite Feynman’s point. It is not the formulae for the solutions that are
really of interest: “At present the main problem concerning solutions, in our opinion, is
not to construct more but rather to understand more completely the known solutions
with respect to their local geometry, symmetries, singularities, sources, extensions,
completeness, topology and stability” (my emphasis). This remark by Ehlers and Kundt
in 1962 [9] is as true now as when it was written, perhaps more so as a result of the large
number of further solutions found since then. Part of my purpose here is to illustrate the
crucial role such understanding of exact solutions has played in the development of our
understanding of general relativity itself\(^1\).

The illustrations of uses of specific solutions given below naturally focus on some
of the best-known solutions. One might imagine that the very large number of other
solutions known have little use. For many that may be true, at least so far, but many
others have helped the exploration of the physics.

The effort required to find solutions is not trivial. Compared with Newtonian gravity,
general relativity has one more independent variable, 9 more dependent ones (taking
the metric approach), and equations of degree 8 in these variables rather than one, the
result of these changes being that the general form of the field equations expands to 10
partial differential equations each with thousands of terms (in terms of the metric and
coordinates). This complexity is the reason only some special solutions can be found,
principally those with some special symmetry or algebraic property.

One might nevertheless hope for a general solution. The closest to this which I know
of (the formalism of [10]) is however largely intractable. So I feel hopes for a useful
general solution are slim and instead one has to make the simplifying assumptions
already mentioned. Here the choices have developed over the last 90 years.

One disadvantage of those choices is that different ones may lead us to the same
solution. I repeat a jeu d’esprit I first gave in a review some years ago [11], by giving a

---

\(^1\) These aspects are covered at greater length in Bicak’s review [3], which I found very useful when
selecting points to cover below.
list of the most commonly-rediscovered solutions:

1. Flat space
2. The Schwarzschild solution
3. The Kasner solutions
4. Plane waves
5. Conformally flat perfect fluids
6. The Taub-NUT family of solutions
7. Static spherically symmetric perfect fluids
8. Cylindrically symmetric stationary electrovac solutions
9. Plane symmetric fluid and electrovac solutions
10. Spherically symmetric shearfree fluids

The 7th and subsequent places in this list are actually rather open to debate, for example the Harris Zund class are strong contenders. I will say something later about how to recognize known solutions.

I should add that in this review I am going to stick firmly to the number of dimensions that I know I exist in, i.e. 4, and avoid trespassing on the 5- and higher-dimensional work described by others. The development of exact solutions in such theories seems to me to be for the most part still at the stage of using only very simple metric forms.

2. THE PRE-EXISTING SOLUTIONS

Since the title of this meeting is ‘A century of relativity’, not the 90 of General Relativity, I am obliged to begin at the beginning, meaning the first solution, the spacetime of special relativity, Minkowski space:

\[ ds^2 = dx^2 + dy^2 + dz^2 - dt^2. \] (1)

It is flat, empty and is usually taken to have \( \mathbb{R}^4 \) topology.

The role of this solution has been considerable, for example:

- It provides the prototype for asymptotic flatness (see Ehlers’ contribution to this volume)
- It is ideal for ‘cutting and pasting’, which was a technique used to great effect while the concepts of causal structure were being developed⁴
- It was the setting for the first work on acceleration horizons (the Unruh effect)
- In suitable coordinates, it is the Milne universe which is often used as the extreme Robertson-Walker model with \( k = -1 \). Apart from the use of this example in astrophysical predictions, this second choice of slicing of flat space, together with the three foliations of de Sitter space, helps to illustrate the fact that the ’open’, ’flat’ or ’closed’ nature of spacetimes can depend on the slicing.

---

⁴ As a student I had the pleasure of watching part of this work as member of Sciama’s research group in Cambridge.
• It is a special case of many other models, whence its frequent rediscovery
• It is the main background for quantum field theory (QFT), and plays a role even for QFT in curved spaces.

The other solutions one could regard as in some sense known before GR was discovered were the other spaces of constant curvature, the de Sitter and anti-de Sitter spaces:

\[ ds^2 = \frac{dx^2 + dy^2 + dz^2 - dt^2}{\left[ 1 + \frac{1}{4}K(x^2 + y^2 + z^2 - t^2) \right]^2}, \]

where \( K = \pm 1 \). These have played a role in, for example,
• Inflation (for de Sitter)
• The AdS/CFT correspondence (anti de Sitter)
• The “no hair” theorems (de Sitter)
• The development of our understanding of particle horizons and event horizons.

Bicak [3] notes that stability against general non-linear vacuum perturbations has been proved for these three solutions. This is a ‘robustness’ result: it raises the question of how much of what we do is robust in this sense.

3. FINDING SOLUTIONS: THE FIRST PHASE

3.1. Methods used

Up to the 1950s the main method of finding new solutions started by postulating a coordinate form of the metric. Authors generally assumed one of:
• spherical symmetry,
• cylindrical symmetry,
• staticity or stationarity and axisymmetry, or
• a plane wave form.

In the large bibliography amassed as background for writing [1], only a part of which appears in the book itself, I found that nearly all papers before 1950 belonged to one of these groups, by far the largest group being the spherically symmetric cases. It should be noted that the metric forms were usually just written down, whereas now we would derive them from group-theoretic or other invariant assumptions.

As successful examples of this method, one can cite the work of Schwarzschild, Droste, Levi-Civita, Kasner, Chazy, Curzon, Brinkmann, Baldwin and Jeffrey, Lewis, van Stockum, Papapetrou, and Gödel, refs. [12] – [25], and of course the Robertson-Walker metrics. Many of these solutions have been important in later discussions, for example in establishing the reality of gravitational waves or elucidating the nature of directional singularities, but I will focus only on the two best-known.
3.2. The Schwarzschild solution

The original form of the Schwarzschild solution [12] used a radial coordinate \( r \) which had its origin at the horizon: for clarity I denote this \( r_0 \) below. Schwarzschild also gave (contrary to some statements in the literature) the form most often quoted now:

\[
\begin{align*}
\text{ds}^2 &= r^2(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) + A^{-1}dr^2 - Adt^2,
\end{align*}
\]

where \( A = 1 - 2m/r \). Here I have renamed Schwarzschild’s \( R \) as \( r \). Schwarzschild regarded this as an auxiliary form because it did not fulfil the condition \(|\text{det}(g)| = 1\) which Einstein had imposed in the initial formulation of General Relativity, a condition of course later discarded.

This solution initiated a discussion (see e.g. [26, 27]) on the meaning of the surface \( r = 2m \) and the absence of a solution clearly analogous to a point mass in Newtonian theory. There is of course no alternative point mass solution to consider due to the uniqueness of (3) as the spherically symmetric vacuum solution (Birkhoff’s theorem).

There are still authors who argue that Schwarzschild’s original \( r_0 = 0 \) should be regarded as a singularity representing a point mass. The horizon clearly is not a regular point since the area of surrounding spheres has a limit \( 4\pi(2m)^2 \) (which is part of the reason \( r = 2m \) is now understood as a sphere, not a point). The argument can be made at various levels of sophistication, the simplest being that the metric components are singular (similar things could be said of the axis of spherical polars, but nobody argues this is singular!). To counter such arguments, it helps to note first that the horizon \( r = 2m \) \((r_0 = 0)\) is not in the coordinate patch for (3). (The waters here have been somewhat muddied by Hilbert’s arguments that these coordinates could be continued to the interior, in which he overlooked that at the horizon the three coordinates \((t, \theta, \varphi)\) do not parametrize a three-dimensional manifold, as becomes immediately apparent on passing to the Kruskal-Szekeres picture.) The whole Schwarzschild patch \( r > 2m \) is isometric to a region of the Kruskal-Szekeres solution (region I in the conformal diagram given as Fig. 1),

\[
\begin{align*}
\text{ds}^2 &= r^2(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) - 32m^3r^{-1}e^{-r/2m}du dv,
\end{align*}
\]

where \( r \) is defined implicitly by the equations giving \( u \) and \( v \) in terms of the previous \( t \) and \( r \):

\[
\begin{align*}
u &= -(r/2m - 1)^{1/2}e^{r/4m}e^{-t/4m},
\end{align*}
\]

\[
\begin{align*}
u &= (r/2m - 1)^{1/2}e^{r/4m}e^{t/4m}.
\end{align*}
\]

By inspection of this form of the solution we see that considering the bounding surface \( r = 2m \) at a given time as a point amounts to topologically identifying all the points on a sphere. One can produce an analogous effect by cutting and pasting flat space (in \( r_0 \)-like coordinates).

One thing to note in these arguments is the ambivalence with which we treat coordinates. Introductions to General Relativity always emphasize the covariance of the equations, but practical examples often implicitly communicate the importance of particular coordinates. The problems this causes are seen at their worst when refereeing weak papers on exact solutions, where authors often refuse to accept that their solution is not new on the grounds its coordinate form is different from the known ones.
FIGURE 1. Conformal diagram of the Kruskal-Szekeres form of the Schwarzschild solution. Each point shown represents a two-sphere parametrized by $\theta$ and $\varphi$. Coordinates $u$ and $v$ are constant along lines at $45^\circ$. The $45^\circ$ lines crossing in the centre of the figure are the horizon $r = 2m$. The $45^\circ$ lines at the sides represent null infinity $\mathcal{I}$. Region I is isometric to the Schwarzschild region $r > 2m$. Region II is a second exterior. Region III is the black hole interior and region IV a white hole interior. The dotted curves running to $\mathcal{I}^+$ represent surfaces of constant $t$, and the ones through $\mathcal{I}^+$ surfaces of constant $r$. Note that for all $t$, $r = 2m$ is the single point at the centre, i.e. a single sphere.

The full understanding of the Schwarzschild horizon was an important element in the general understanding of global properties of spacetimes which developed in the 1960s. Conformal pictures drawn on the same principles as Fig. 1 were also developed for the Schwarzschild solution’s generalizations given by

$$A = 1 - 2m/r + e^2/r^2 - \frac{1}{3}\Lambda r^2,$$

the Reissner-Nordström solution being given by $\Lambda = 0 \neq e$ and the Köttler solution by $\Lambda \neq 0 = e$. Incidentally, these provide two illustrations of a ‘metatheorem’ that all named results have the wrong name: the Köttler solution is commonly called Schwarzschild-anti-de Sitter, though as far as I know neither Schwarzschild nor de Sitter gave this solution, and Weyl gave the general class to which the Reissner-Nordström metric belongs before Nordström’s paper appeared. Other examples of this are the “Tolman-Bondi” solutions, due to Lemaître [28], whose paper Tolman [29] cited, and the “Bertotti-Robinson” solution, which Robinson himself noted was first found by Levi-Civita [14].

The Schwarzschild solution had a pivotal role in two other developments. Approximations to it were used in predictions of the classical tests of general relativity: those approximations were later generalized to the PPN formalism which provides the basis of analysis of solar system tests of relativity. It was also the setting in which Hawking made the link between the laws of black hole mechanics, quantum field theory and thermodynamics in his discovery of Hawking radiation.

The thermodynamic identification of black hole surface area with entropy is in general not fully understood in terms of microscopic states, unlike entropy in normal statistical mechanics (see [30]). However, for the ‘extreme’ charged case$^3$ (which is supersymmet-

---

$^3$ I am grateful to Ingemar Bengtsson and Gary Horowitz for correcting a misapprehension on my part about this work.
ric) or near-extreme charged black holes, a microscopic basis in string theory has been
given [31]. Yet again the Schwarzschild solution’s generalizations are playing a major
role in advancing our understanding.

To summarize, the Schwarzschild solution assisted the development of our under-
standing of the following:

• there is no “point mass” solution in relativity, since there is no point centre;
• The role of coordinates has to be properly understood;
• global concepts such as
  black holes,
  event horizons,
  apparent horizons,
  trapped surfaces and the singularity theorems,
  cosmic censorship and naked singularities;
• PPN expansions; and
• Hawking radiation, QFT in curved spaces and black hole entropy.

### 3.3. The Robertson-Walker metric form: FLRW solutions

These are the solutions with the metric

\[
d s^2 = -d t^2 + a^2(t)[dr^2 + \Sigma^2(r,k)(d\theta^2 + \sin^2 \theta d\phi^2)].
\]

where

\[
\Sigma(r,k) = \sin r, r \text{ or } \sinh r, \text{respectively, when } k = 1, 0 \text{ or } -1.
\]

It would be impossible to overemphasize the importance of these solutions in cos-

mology. They are fundamental to the inferences from observation of the presence on
the cosmological scale of dark matter and “dark energy” (above the densities required
by dynamics of galaxy clusters). These matters were discussed by other speakers at the
meeting so I will not give details here.

There are many specific solutions due to Einstein himself, de Sitter, Friedman, Ed-
dington, Lemaître, and so on (see Ch. 14 of [1]). The pivotal role of the ones due to
Friedman and Lemaître led to those authors’ names being coupled with the names of
Robertson and Walker in the short name FLRW. The role of Robertson and Walker was
that they independently elucidated the geometrical basis of the metric form in the 1930s.

One may also note the wide use of the Lemaître-Tolman-Bondi models, spherically
symmetric inhomogeneous dust models, to describe collapse and voids, primordial black
holes, and other inhomogeneities in cosmology (see [2]). For example, it has been shown
that the number distribution of galaxies can be modelled by a non-evolving population
in an LTB spacetime rather than an evolving population in FLRW.
4. FINDING MORE SOLUTIONS: THE SECOND PHASE

4.1. Two major themes

Two very big steps forward were made in the early 1950s, which provided the framework for many of the new solutions found up to now.

The first of these was Taub’s 1951 paper [32] which contained the introduction of group-theoretic and differential geometric methods used in an essential way. (There was also some interaction with parallel work of Gödel.) This paper also discovered the Taub portion of the important Taub-NUT solution, and hinted at tetrad methods, though these latter were not fully developed until the 1960s, by Ellis [33, 34], Estabrook and Wahlquist [35], Newman and Penrose [36] and others (a development which I helped to codify [37] as far as the orthonormal tetrad version was concerned).

The second was Petrov’s classification of the Weyl tensor ([38]). This, together with the solutions of the Kundt class, played an important part in the development of gravitational radiation theory and formed a big step on the road to invariant classification of solutions, discussed later.

One can briefly describe the Petrov classification as following from the fact that for any non-zero Weyl tensor $C_{abcd}$, as defined by

\[ R_{abcd} = C_{abcd} - \frac{1}{3} R \delta_{[a}^{[c} \delta_{b]}^{d]} + 2 \delta_{[a}^{[c} R_{b]}^{d]}, \]  

there are four null vectors $k^a$, the principal null directions, which satisfy the equation

\[ k_{[i} C_{a]bc} k_f k^c = 0. \]  

If these four vectors are distinct, one has the general case, known as Petrov type I. The remaining cases, where two or more coincide, are called algebraically special. If just one pair coincide, we have type II, if three coincide, type III, and if all four coincide, type N: the other possible degeneracy, two coinciding pairs, is called type D. The case of zero Weyl tensor, where the spacetime is conformally flat, is sometimes called type O.

These two developments led to the two main themes of the organization of the exact solutions book [1], i.e. classification by symmetry groups and algebraically special metrics.

Work on the latter area was much assisted by the development of the already-mentioned Newman-Penrose formalism [36], a calculation technique not based on coordinates but on a tetrad of null vectors, and therefore well adapted to the study of algebraically special spacetimes. These ideas led to the discovery of a number of important solutions, for example the following.

- The Taub-NUT solution (or rather its NUT part): [39].
- The Kerr solution, the rotating black hole: [40].
- The Robinson-Trautman class: [41].
- (Later) the Kerr-Schild ansatz and solution class: [42].

The excellent review of Ehlers and Kundt [9], perhaps the first modern general review of exact solutions, summarized the first decade of the resulting developments. I will now...
pause in describing methods for finding solutions and discuss the uses of three of these solutions or solution classes.

4.2. The Taub-NUT solution

The metric in this case can be given as

\[ ds^2 = -U^{-1} d\tau^2 + (2\ell^2)U (d\psi + \cos \theta d\phi)^2 + (\tau^2 + \ell^2) (d\theta^2 + \sin^2 \theta d\phi^2), \]

(10)

where \( U \), which is positive in the Taub region and negative in the NUT region, is given by

\[ U(\tau) = -1 + \frac{2m \tau + \ell^2}{\tau^2 + \ell^2}. \]

(11)

This played such a role that Misner described it as a “counterexample to almost anything” [43]. Papers by Misner [43] and Misner and Taub [44] established that the Taub and NUT regions can be joined, that the NUT region contains closed timelike lines and no sensible Cauchy surfaces, that there are two inequivalent maximal analytic extensions of the Taub region (or one non-Hausdorff manifold with both extensions), that Taub-NUT space is nonsingular in the sense of a curvature singularity, and that there are geodesics of finite affine parameter length.

To summarize, it has the following properties:

- the topology of group orbits changes at the horizon;
- there are closed timelike lines in the NUT region;
- the boundary of the Taub region has closed null geodesics;
- there is geodesic incompleteness at finite affine parameter without a curvature singularity; and
- there are inequivalent extensions, or a non-Hausdorff one.

The solutions also have applications and generalizations outside strict general relativity, e.g. in string theory or Euclidean quantum gravity.

The solution has thus had a great influence on studies of exact solutions and cosmological models which are spatially-homogeneous, and more generally on those which are hypersurface-homogeneous and self-similar (see e.g. the discussion in [45]), on cosmology in general, and on our understanding of global analysis and singularities in space-times.

4.3. pp-waves and plane waves

The plane waves were first found by Brinkmann [19] but their significance, showing among other things that gravitational waves were definitely not a coordinate effect, was not appreciated until the 1950s, in work of Bondi, Pirani and Robinson [46], Peres [47],
Hely [48] and others. These are the metrics

\[ ds^2 = 2d\zeta d\bar{\zeta} - 2dudv - 2Hu^2, \quad H = H(\zeta, \bar{\zeta}, u), \] (12)

with \( H = A(u)\zeta^2 + \bar{A}(u)\bar{\zeta}^2 + B(u)\zeta\bar{\zeta} \). They are members of the more general \( pp \)-wave class, which are given, for electrovacuum cases, by

\[ H = f(\zeta, u) + \bar{f}(\bar{\zeta}, u) + \kappa_0 F(\zeta, u)\bar{F}(\bar{\zeta}, u), \quad F_{ab} = 2k_{[a}F_{b]}, \] (13)

where the functions \( f \) and \( F \) are arbitrary functions analytic in \( \zeta \) and dependent on the retarded time coordinate \( u \). Here \( \kappa_0 \) is the constant in the Einstein equations.

Among their properties are:

- the wave front speed is the speed of light;
- they have a transverse character;
- there are focusing effects leading to caustics;
- there is no global Cauchy surface (i.e. no initial value problem).

These solutions are comprehensively discussed in the review of Ehlers and Kundt [9], so the treatment in the exact solutions book [1] adds virtually nothing. They have interesting singularity and horizon behaviour, and gave rise to a number of special cases of interest, such as sandwich waves and impulsive waves. One can also study collisions between them [49, 50, 51, 52] and the resulting metrics have been studied in the monograph of Griffiths [4]: see also Ch. 25 of [1].

They have more recently been used in understanding higher-dimensional theories and as examples in quantum field theory (they have the property of having no quantum corrections).

### 4.4. The Kerr solution

Together with the Robertson-Walker and Schwarzschild solutions, this is probably the best-known exact solution, because it represents the unique rotating vacuum black hole. It has a number of interesting mathematical properties, having indeed been found as the complement to the NUT investigation of Petrov type D vacuum solutions (the NUT solutions had initially been thought to be the only such solutions). The metric is

\begin{align*}
  ds^2 &= \left(1 - \frac{2mr}{r^2 + a^2\cos^2\vartheta}\right)^{-1} \left[ (r^2 - 2mr + a^2) \sin^2\vartheta d\varphi^2 
  &\quad + (r^2 - 2mr + a^2) \cos^2\vartheta \left( d\vartheta^2 + \frac{dr^2}{r^2 - 2mr + a^2} \right) 
  &\quad - \left(1 - \frac{2mr}{r^2 + a^2\cos^2\vartheta}\right) \left( dt + \frac{2mar\sin^2\vartheta d\varphi}{r^2 - 2mr + a^2\cos^2\vartheta} \right)^2 \right] 
\end{align*}

(14)

When the global structure of this metric was worked out, it emerged that it has a ring singularity (for \( m > a \)) through which further exterior regions can be connected: see e.g.
the summary in [53]. Moreover, the Hamilton-Jacobi and Klein-Gordon equations are separable in this metric, which is related to the fact that it has a non-trivial Killing tensor. It exhibits the phenomenon of an ergosphere, a region outside the black hole horizon but within which any particle has to corotate around the hole. There is a relation to work on characterization of stationary axisymmetric spacetimes by multipole moments: the Kerr solution has very specific relations between its moments which do not appear to be found in physical bodies of rotating fluid, posing a question about possible sources or the process of approach to the Kerr solution as the eventual black hole outcome of a collapse.

Work on its mathematical properties, however, is likely to be exceeded by the many papers on its astrophysical implications. For example, the ergosphere classically allows the Penrose process, in which a part of a body dividing within the ergosphere can emerge with more energy than the original body entered with. The wave version of this is superradiance, where the scattered wave has more energy than the incident wave, and this phenomenon was one of the stimuli to the laws of black hole mechanics later explained by Hawking. It is also related to the Blandford-Znajek mechanism [54], in which a magnetic field threading the black hole can extract rotational energy from the hole.

The most important of all the astrophysical uses of the Kerr solution is probably as a basis for accretion disk physics, thought, for example, to be responsible for the X-ray emission of X-ray binaries in the sky, and used in explanations of larger objects such as jets in active galactic nuclei. Observations of astronomical accretion disks are now suggesting the objects at their centres really are Kerr black holes, as the only way to explain both their short periods and other orbital data [55, 56]. Chandrasekhar remarked4 “In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein’s equations, discovered by the New Zealand mathematician Roy Kerr, provides an absolutely exact representation of untold numbers of massive black holes that populate the Universe...”

### 4.5. Finding more solutions: generating techniques

A third set of novel techniques appearing for the first time in papers in the 50s and early 60s (by Buchdahl, Ehlers, and Bonnor, for instance [57, 58, 59]) took rather longer to grow to maturity than the first two methods described in this section: this third topic is that of the generating techniques. Although they exist for metrics with one Killing vector, they are most used for stationary axisymmetric solutions, and the other classes with two commuting Killing vectors: cylindrical waves and colliding plane waves, boost-rotation symmetric spacetimes and cosmologies with two commuting spacelike Killing vectors. They work not only for vacuum, but for other forms of matter with characteristic propagation speed equal to the speed of light: massless scalar fields (or 'stiff fluid' in the case of a timelike gradient of the field), (massless) neutrinos, and electromagnetism.

---

4 This remark came to my attention in Bicak’s review [3].
This area exploded in the 1970s and 1980s in work of many people. The number of methods proliferated, and created a secondary industry of understanding the relations between them. At the same time the number of applications grew hugely. There is so much work that I shall not attempt to give references here but instead refer the reader to [1] (see also [4] and [5]). The best known formulation for the basic equations is that due to Ernst. For the stationary axisymmetric metrics

\[ ds^2 = e^{-2U}(\gamma_{MN}dx^Mdx^N + W^2d\phi^2) - e^{2U}(dt + Ad\phi)^2, \]  

(15)

where the metric functions \( U, \gamma_{MN}, W, A \) depend only on the coordinates \( x^M = (x^1, x^2) \) which label the points on the 2-surfaces \( S_2 \) orthogonal to the orbits, and the electromagnetic field is given by a complex potential \( \Phi \), it reads

\[
\begin{align*}
(\text{Re} \epsilon + \Phi\overline{\Phi})W^{-1}(\epsilon\phi_M)^M &= \epsilon_M(\epsilon^M + 2\Phi\Phi^M), \\
(\text{Re} \epsilon + \Phi\overline{\Phi})W^{-1}(\Phi\phi_M)^M &= \Phi_M(\epsilon^M + 2\Phi\Phi^M).
\end{align*}
\]

(16, 17)

To recover the metric one needs the equations

\[
\begin{align*}
A_M &= We^{-4U}\epsilon_{MN}\omega^N, \\
e^{2U} &= \text{Re} \epsilon + \Phi\overline{\Phi},
\end{align*}
\]

(18, 19)

where \( \epsilon_{MN} \) is the Levi-Civita tensor in the \( S_2 \).

Although I will not attempt a review here of the methods, the works of Geroch, Neugebauer and Kramer, Hoenselaers, Kinnersley and Xanthopoulos (HKX), Kinnersley and Chitre, Harrison, Belinski and Zakharov, Hauser and Ernst, Yamazaki, and Cosgrove were so influential their names should be mentioned.

These methods themselves can now be seen as embedded in an even more general context of symmetries of differential equations and integrable systems, involving concepts such as inverse scattering and Lax pairs, Bäcklund transformations, Riemann-Hilbert problems, prolongation and so on (see [7] or Ch. 10 of [1]). Thus the solutions have offshoots in mathematics and in other physical theories. They provide a unification of results on known solutions (for example, all stationary axisymmetric electrovacuum solutions in which a portion of the rotation axis is regular can be generated from flat space).

They also enable an infinite number of solutions to be obtained. At one stage there were a significant number of papers which exploited this possibility by exhibiting specific solutions, but it quickly became apparent that merely obtaining a new solution, when it can be done infinitely often (at least in principle), is pointless. Attention is now normally directed to ways to generate solutions with predetermined characteristics: for example one can ask what class of axis data gives a certain feature to the solution?

Of the solutions obtained or obtainable by these methods, a number have been of importance or interest, for example the Tomimatsu-Sato family, the double Kerr solution, the Neugebauer-Meinel dust disk, and a number of colliding wave and cosmological solutions. Brevity precludes detailed discussion and the reader is referred to the literature already cited.
5. FINDING AND USING MORE SOLUTIONS: RECENT DEVELOPMENTS

I now return to the fundamental question raised earlier: how can we compare solutions? In particular how can we test for local isometric equivalence? This is called the equivalence problem, and belongs to the general class of recognition problems. It has largely been answered, in theory and in practice, using ideas due to Cartan, Brans, Karlhede and others, with practical implementation and development by Åman, and later myself and my group, although in a formal sense its final step is undecideable. I will briefly describe the procedure now: a full description would be a lecture in itself. For a review see Ch. 9 of [1].

The key point is that scalar polynomial invariants, i.e. polynomials in the Riemann tensor and its derivatives in which all indices are contracted over, are insufficient. Instead we need curvature invariants of a more general sort, the ‘Cartan invariants’. These are, for example, given by the components of the Weyl tensor referred to a tetrad chosen using the principal null directions.

These enable local characterization of the metric. The basic idea is to use invariantly defined tetrads, like the one from the principal null directions, and take components of the Riemann tensor and its derivatives in this frame. Thus the method has links with the Petrov classification and tetrad methods which were introduced in the 1950s and 1960s. Counting functionally independent invariants gives the dimension of the symmetry group, thus linking to the group theoretic ideas of that period, and additional information available can give the group structure.

These characterization methods can be used to check if solutions found are really new, or to search among known solutions for examples with desired local properties. They could in principle be used to find solutions, as well as classify known ones, and first examples of this method have been developed by Bradley, Karlhede and Marklund.

Other uses of this approach, i.e. the direct use of invariants, have been in understanding the limits of families of solutions without needing trial and error for the appropriate coordinate transformations (for example, studying the limits of the Schwarzschild family as \( m \to \infty \)); proving (non) existence of matchings by characterizing the geometries of the proposed matching surfaces (e.g. in work of my student Daniel Cox [60]); and in providing a method to ‘unravel’ directional singularities (in work of my student John Taylor [61]). The classifying quantities might also be used to give a topology on the space of solutions, which could even be of interest in numerical relativity.

Summarizing, the main methods for finding solutions in current use are still those outlined in Section 4, but the understanding and classification of these solutions can now be done in an invariant manner which should enable better use of the solutions found and may also provide a fresh and more invariant way to find more.

APPENDIX

The second edition of the exact solutions book contains about 400 pages of new material, covering hundreds of new solutions and references. In its preparation the authors read about 4000 new papers (as well as the 3000 read for the first edition). So there is far more
material than I could talk about. Most of the new material is integrated into and expands existing chapters and sections: additional sections were added on the GHP formalism and other calculi, junction conditions and so on, and the chapter on solutions obtained by generating techniques was almost completely re-written.

The entirely new chapters or part-chapters were on:

- Homotheties
- Characterization by invariants
- Generating techniques themselves
- Dynamical systems methods
- Inhomogeneous solutions with two spacelike Killing vectors
- Colliding plane waves
- Special vector and tensor fields.

REFERENCES

1. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein’s field equations*, 2nd edition, Cambridge University Press, Cambridge, 2003.
2. A. Krasiński, *Inhomogeneous cosmological models*, Cambridge University Press, Cambridge, 1997.
3. J. Bicak, “The role of exact solutions of Einstein’s equations in the developments of general relativity and astrophysics: selected themes,” in *Einstein’s field equations and their physical implications*, Springer Verlag, Heidelberg, 2000, vol. 540 of *Lecture Notes in Physics*.
4. J. Griffiths, *Colliding plane waves in general relativity*, Oxford University Press, Oxford, 1991.
5. V. Belinski, and E. Verdaguer, *Gravitational solitons*, Cambridge University Press, Cambridge, 2001.
6. S. Hawking, and G. Ellis, *The large-scale structure of space-time*, Cambridge University Press, Cambridge, 1973.
7. L. Mason, and N. Woodhouse, *Integrability, Self-duality and twistor theory*, vol. 15 of *London Mathematical Society Monographs*, Oxford Science Publications, 1996.
8. J. Synge, *Relativity: the general theory*, North-Holland, Amsterdam, 1960.
9. J. Ehlers, and W. Kundt, “Exact solutions of the gravitational field equations,” in *Gravitation: an introduction to current research*, edited by L. Witten, Wiley, New York and London, 1962, pp. 49–101.
10. D. Sciama, P. Waylen, and R. Gilman, *Phys. Rev. A* 187, 1762–6 (1969).
11. M. MacCallum, “An overview of exact solutions of Einstein’s equations and their classification,” in *Highlights in gravitation and cosmology (Proceedings of the Goa conference 1987)*, edited by B. Iyer, A. Kembhavi, J. Narlikar, and C. Vishveshwara, Cambridge University Press, London and New York, 1989, pp. 3–14.
12. K. Schwarzschild, *Sitz. Preuss. Akad. Wiss.* pp. 189–196 (1916).
13. J. Droste, *Kon. Akad. Wetensch. Amsterdam*, Proc. Sec. Sci. 19, 197–215 (1916-17).
14. T. Levi-Civita, *Rend. R. Accad. Lincei, Cl. sci. fis., mat. nat* 26, 519–531 (1917).
15. H. Weyl, *Ann. Phys. (Germany)* 54, 117 (1917).
16. E. Kasner, *Amer. J. Math.* 43, 217 (1921).
17. J. Chazy, *Bull. Soc. Math. France* 52, 17 (1924).
18. H. Curzon, *Proc. London Math. Soc.* 23, 477 (1924).
19. H. Brinkmann, *Math. Ann.* 94, 119–145 (1925).
20. O. Baldwin, and G. Jeffery, *Proc. Roy. Soc. Lond. A* 111, 95 (1926).
21. T. Lewis, *Proc. Roy. Soc. Lond.* A 136, 176 (1932).
22. W. van Stockum, *Proc. Roy. Soc. Edinburgh A* 57, 135 (1937).
23. A. Papapetrou, *Proc. Roy. Irish Acad.* A 51, 191 (1947).
24. A. Papapetrou, *Ann. Phys. (Germany)* 12, 309 (1953).
25. K. Gödel, *Rev. Mod. Phys.* 21, 447 (1949).
26. J. Eisenstaedt, *Arch. Hist. Exact Sci.* 27, 157–198 (1982).
27. J. Eisenstaedt, *Arch. Hist. Exact Sci.* **37**, 275–357 (1987).
28. G. Lemaître, *Ann. Soc. Sci. Bruxelles A* **53**, 51 (1933), translation by M.A.H. MacCallum in *Gen. Rel. Grav.* **29**, 935-943 (1997).
29. R. Tolman, *Proc. Nat. Acad. Sci. (Wash.)* **20**, 169 (1934), reprinted in *Gen. Rel. Grav.* **29**, 935-943 (1997).
30. R. Wald, *The thermodynamics of black holes*, http://relativity.livingreviews.org/Articles/lrr-2001-6, Albert Einstein Institute (2001).
31. G. Ellis, On general relativistic fluids and cosmological models, Ph.D. thesis, Cambridge University (1964).
32. A. Taub, *Ann. Math.* **53**, 472 (1951), reprinted, with editorial introduction by M.A.H. MacCallum, in *Gen. Rel. Grav.* **36**, 2699-2719 (2004).
33. G. Ellis, *J. Math. Phys.* **8**, 1171 (1967).
34. F. Estabrook, and H. Wahlquist, *J. Math. Phys.* **5**, 1629 (1964).
35. E. Newman, and R. Penrose, *J. Math. Phys.* **3**, 566 (1962).
36. M. MacCallum, “Cosmological models from the geometric point of view,” in *Cargese Lectures in Physics*, vol. 6, edited by E. Schatzman, Gordon and Breach, New York, 1973, pp. 61–174.
37. A. Petrov, *Scientific Proceedings of Kazan State University (named after VI. Ulyanov-Lenin), Jubilee (1804-1954) Collection* **114**, 55–69 (1954), translation by J. Jezierski and M.A.H. MacCallum, with introduction by M.A.H. MacCallum, *Gen. Rel. Grav.* **32**, 1661-1685 (2000).
38. E. Newman, L. Tamburino, and T. Unti, *J. Math. Phys.* **4**, 915 (1963).
39. R. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963).
40. R. Genzel, R. Schoedel, T. Ott, A. Eckart, T. Alexander, F. Lacombe, D. Rouan, and B. Aschenbach, *Nature* **425**, 934–937 (2003).
41. J. Hély, *Comptes Rendus Acad. Sci. (Paris)* **249**, 1867–1868 (1959).
42. R. Penrose, *Rev. Mod. Phys.* **37**, 215 (1965).
43. C. Misner, “Taub-NUT space as a counterexample to almost anything,” in *Relativity theory and astrophysics, vol. 1: Relativity and cosmology*, edited by J. Ehlers, Lectures in applied mathematics, volume 8, American Mathematical Society, Providence, R.I., 1963, pp. 160–169.
44. C. Misner, and A. Taub, *Zh. Eks. Teor. Fiz.* **55**, 233 (1968).
45. R. Jantzen, “Higher Dimensional Cosmological Models: the View from Above,” in *Atti del convegno sulla relatività generale: problemi dell’energia e ondi gravitationali*, Barbèra, Firenze, 1965, p. 222.
46. A. Peres, *Phys. Rev. Lett.* **3**, 571–572 (1959).
47. J. Ehlers, *Konstruktionen und charakterisierungen vonlösungen der Einsteinschen gravitationsfeld-gleichungen*, Dissertation, Hamburg (1957).
48. W. Bonnor, *Z. Phys.* **161**, 439 (1961).
49. D. Cox, *Physical Review D* **68**, 124008 (2003).
50. J. Taylor, *Class. Quant. Grav.* **22**, 4961–4971 (2005).