Non-zero momentum level reduction in path integrals for dynamical systems with symmetry given on a product manifold consisting of the total space of the principal fiber bundle and a vector space

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Abstract

The case of non-zero momentum level reduction in Wiener path integrals for a mechanical system with symmetry describing the motion of two scalar particles with interaction on a Riemannian product manifold with the given action a compact semisimple Lie group is considered. The original product manifold consists of the vector space and a smooth compact finite-dimensional Riemannian manifold, which, due to the action of the group, can be regarded as the total space of the principal fiber bundle. The integral relation between the path integrals representing the fundamental solutions of the backward Kolmogorov equation defined on the total space of the principal fiber bundle (the original Riemannian product manifold) and the corresponding backward Kolmogorov equation on the space of the sections of the associated covector bundle is obtained.

KeyWords: Marsden-Weinstein reduction, Kaluza-Klein theories, Path integral, Stochastic analysis.
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1 Introduction

This work can be considered as a supplement to our previous work \cite{1}, where
the reduction procedure in path integrals for a special dynamical system
was considered. The interest in studying this system is due to the fact that
it can serve as a model for describing the interaction of the gauge fields
with the scalar fields. In our previous work, we considered the case of the
zero momentum level of the Marsden-Weinstein reduction \cite{2} in the original
mechanical system. Here we will study the general case of the reduction onto
the non-zero momentum level. In the path integrals, the reduction procedure
is performed

The path integral reduction is based on two transformation of the path
integrals. The first transformation is the factorization of the path integral
measure which can be done by using the non-linear filtering equation from the
stochastic process theory. The second transformation of the path integrals
is due to the Girsanov transformation of the stochastic processes. In the
present work we will deal with the Girsanov transformation since the first
part of the path integral transformation was already studied in \cite{1}.

2 Definition

Let us briefly recall the main definitions used in our previous works.

We are interested in the backward Kolmogorov equation which is given
on a smooth compact Riemannian manifold $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$:

$$
\begin{cases}
\left( \frac{\partial}{\partial t_a} + \frac{1}{2} \mu^2 \kappa \left[ \triangle_{\mathcal{P}}(p_a) + \triangle_{\mathcal{V}}(v_a) \right] + \frac{1}{\mu^2 \kappa m} V(p_a, v_a) \right) \psi_{t_a}(p_a, t_a) = 0, \\
\psi_{t_b}(p_b, v_b, t_b) = \phi_0(p_b, v_b), \quad (t_b > t_a),
\end{cases}
$$

(1)

where $\mu^2 = \hbar/m$, $\kappa$ is a real positive parameter, $V(p, f)$ is the group-invariant
potential term: $V(pg, g^{-1}v) = V(p, v)$, $g \in \mathcal{G}$, $\triangle_{\mathcal{P}}(p_a)$ is the Laplace–
Beltrami operator on a manifold $\mathcal{P}$ and $\triangle_{\mathcal{V}}(v)$ is the Laplacian on the vector
space $\mathcal{V}$. Locally, in the coordinates $(Q^A, f^a)$ of the point $(p, v)$\footnote{In our formulas we assume that there is sum over the repeated indices. The indices
denoted by the capital letters ranged from 1 to $n_\mathcal{P} = \dim \mathcal{P}$, and the small Latin letters,
except $i, j, k, l$, – from 1 to $n_\mathcal{V} = \dim \mathcal{V}$.} the Laplacian is given by

$$
\triangle_{\mathcal{P}}(Q) = G^{-1/2}(Q) \frac{\partial}{\partial Q^A} G^{AB}(Q) G^{1/2}(Q) \frac{\partial}{\partial Q^B},
$$

(2)
where $G = \text{det}(G_{AB})$ and $G_{AB}$ are the components of the initial Riemannian metric given in the coordinate basis $\{\frac{\partial}{\partial \eta}\}$.

$\Delta_{\psi}$ is given by

$$\Delta_{\psi}(f) = G^{ab} \frac{\partial}{\partial f^a} \frac{\partial}{\partial f^b}.$$ 

By the assumption used in the paper, the matrix $G_{ab}$ consists of fixed constant elements. It is admitted that $G_{ab}$ may have off-diagonal elements.

In case of fulfillment of smooth requirements imposed on the coefficients and the initial function of equation (1), the solution of equation, as it follows from [3], can be represented as follows:

$$\psi_b(p_a, v_a, t_a) = E\left[\phi_0(\eta_1(t_b), \eta_2(t_b)) \exp\left\{ \frac{1}{\mu^2 \kappa m} \int_{t_a}^{t_b} V(\eta_1(u), \eta_2(u)) du \right\}\right]$$

$$= \int_{\Omega_{-}} d\mu^p(\eta) \phi_0(\eta(t_b)) \exp\{\ldots\}, \tag{3}$$

where $\eta(t)$ is a global stochastic process on a manifold $\tilde{P} = \mathcal{P} \times \mathcal{V}$, formed from the processes $\eta_1(t)$ and $\eta_2(t)$; $\mu^p$ is the path integral measure on the path space $\Omega_{-} = \{\omega(t) = \omega^1(t) \times \omega^2(t) : \omega^{1,2}(t_a) = 0, \eta_1(t) = p_a + \omega^1(t), \eta_2(t) = v_a + \omega^2(t)\}$ given on manifold $\tilde{P}$.

In a local chart $(U_p \times U_v, \varphi)$ of the manifold $\tilde{P}$, the process $\eta(t)$ is given by the solution of two stochastic differential equations:

$$d\eta^A(t) = \frac{1}{2} \mu^2 \kappa G^{-\frac{1}{2}} \frac{\partial}{\partial Q^B}(G^{1/2} G^{AB}) dt + \mu \sqrt{\kappa} \chi^A_M(\eta_1(t)) dw^M(t), \tag{4}$$

and

$$d\eta^b(t) = \mu \sqrt{\kappa} \chi^b_\alpha dw^\alpha(t). \tag{5}$$

$(\chi^A_M$ and $\chi^b_\alpha$ are defined by the local equalities $\sum_{K=1}^{n_P} \chi^A_M \chi^B_K = G^{AB}$ and $\sum_{\alpha=1}^{n_v} \chi^b_\alpha \chi^b_\alpha = G^{bc}$, $dw^M(t)$ and $dw^\alpha(t)$ are the independent Wiener processes. Here and what follows we denote the Euclidean indices by over-barred indices).

The global semigroup determined by equation (3) is defined in [3] by the limit (under the refinement of the time interval) of the superposition of the local semigroups. In our case it is given by

$$\psi_b(p_a, v_a, t_a) = U(t_b, t_a) \phi_0(p_a, v_a) = \lim_{q \to 0} \tilde{U}_\eta(t_a, t_1) \ldots \tilde{U}_\eta(t_{n-1}, t_b) \phi_0(p_a, v_a), \tag{6}$$

where each of the local semigroup $\tilde{U}_\eta$ is as follows:

$$\tilde{U}_\eta(s, t) \phi(p, v) = E_{s, p, v} \phi(\eta_1(t), \eta_2(t)) \quad s \leq t \quad \eta_1(s) = p, \eta_2(s) = v.$$
These local semigroups are also given by the path integrals with the integration measures determined by the local representatives \( \varphi^\mathcal{P}(\eta_t) \equiv \{ \eta^1_t(t), \eta^2_t(t) \} \) of the global stochastic process \( \eta(t) \).

On the Riemannian manifold \( \tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V} \), we are given a smooth isometric free and proper action of a compact semisimple Lie group \( \mathcal{G} \): \( (\tilde{p}, \tilde{v}) = (p, v)g = (pg, g^{-1}v) \). In a local coordinates \( (Q^A, f^a) \), this action is given as follows:

\[
\tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^b = \tilde{D}^b_a(g)f^a,
\]

where \( \tilde{D}^b_a(g) \equiv D^b_a(g^{-1}) \), and by \( D^b_a(g) \) we denote the matrix of the finite-dimensional representation of the group \( \mathcal{G} \) acting on the vector space \( \mathcal{V} \).

The Riemannian metric of the manifold \( \tilde{\mathcal{P}} \) can be written as follows:

\[
ds^2 = G_{AB}(Q)dQ^A dQ^B + G_{ab} df^a df^b. \tag{7}
\]

The Killing vector fields for the metric (7) given on the manifold \( \tilde{\mathcal{P}} \) are the vector fields on \( \mathcal{P} \) and \( \mathcal{V} \). In local coordinates \( (Q^A, f^a) \), they are represented as \( K^A_\alpha(Q) \partial/\partial Q^A \) with \( K^A_\alpha(Q) = \partial Q^A/\partial a^\alpha |_{a=e} \) and \( K^b_\alpha(f) \partial/\partial f^b \) with

\[
K^b_\alpha(f) = \partial \tilde{f}^b/\partial a^\alpha |_{a=e} = \partial \tilde{D}^b_c(a)/\partial a^\alpha |_{a=e} f^c \equiv (\bar{J}_\alpha)_c f^c.
\]

The generators \( \bar{J}_\alpha \) of the representation \( \tilde{D}^b_c(a) \) satisfy the commutation relation \( [\bar{J}_\alpha, \bar{J}_\beta] = c^\gamma_{\alpha\beta} \bar{J}_\gamma \), where the structure constants \( c^\gamma_{\alpha\beta} = -c^\gamma_{\beta\alpha} \).

Based on (7), we conclude that the action of a group \( \tilde{\mathcal{G}} \) on the manifold \( \tilde{\mathcal{P}} \) leads to the principal fiber bundle \( \pi' : \mathcal{P} \times \mathcal{V} \rightarrow \mathcal{P} \times_{\mathcal{G}} \mathcal{V} \). This allow us to regard the initial manifold \( \tilde{\mathcal{P}} \) as a total space of this principal fiber bundle \( \mathcal{P}(\mathcal{M}, \mathcal{G}) \). By \( \mathcal{M} \) we denoted the orbit space manifold \( \mathcal{P} \times_{\mathcal{G}} \mathcal{V} \) – the base space of the bundle \( \pi' \).

From this it follows that we can express the local coordinates \( (Q^A, f^a) \) of the point \((p, v) \in \tilde{\mathcal{P}} \) in terms of the coordinates of the principal fiber bundle. It is done with the help of the adapted coordinates. They are the group coordinates \( a^\alpha(Q) \) and the invariant coordinates \( x^i(Q) \) determined from the equation

\[
Q^{*A}(x^i) = F^A(Q, a^{-1}(Q)),
\]

which satisfy \( \chi^\alpha(Q^{*A}(x^i)) = 0 \), where \( \chi^\alpha(Q) \) is a “gauge”- the function by which the local surface \( \Sigma \) in \( \mathcal{P} \) is defined. This surface is needed to determine the coordinates in the principal fiber bundle.

The group coordinates \( a^\alpha(Q) \) of a point \( p \in \mathcal{P} \) are defined by the solution of the following equation:

\[
\chi^\alpha(F^A(Q, a^{-1}(Q))) = 0.
\]

\(^2\pi' : (p, v) \rightarrow [p, v] \), where \([p, v]\) is the equivalence class formed by the relation \((p, v) \sim (pg, g^{-1}v)\).
This group element carries the point \( p \) to the submanifold \( \Sigma \), so that
\[
Q^* = F^A(Q, a^{-1}(Q)).
\]

With the principal fiber bundle coordinates, the point \((p, v) \in \tilde{P}\), whose coordinates were \((Q^A, f^b)\), obtains the adapted coordinates \((x^i(Q), \tilde{f}^a, a^\alpha(Q))\). The replacement of the coordinates \((Q^A, f^b)\) of a point \((p, v)\) for a new coordinates is performed as follows:
\[
Q^A = F^A(Q^*(x^i), a^\alpha), \quad f^b = \tilde{D}^b_c(a) \tilde{f}^c,
\]
where \(a^\alpha = a^\alpha(Q)\) is obtained as before.

In a new coordinates the Riemannian metric can be represented in the following form:
\[
G_{AB} = \begin{pmatrix}
\tilde{h}_{ij} + \mathcal{A}_{ij}^{\mu} \mathcal{A}_{\mu}^{\nu} d_{\mu \nu} & 0 & \mathcal{A}_{ij}^{\mu} d_{\mu \nu} \tilde{a}_{\nu}^\alpha(a) \\
0 & G_{ab} & \mathcal{A}_{ab}^{\mu} d_{\mu \nu} \tilde{a}_{\nu}^\alpha(a) \\
\mathcal{A}_{ij}^{\mu} d_{\mu \nu} \tilde{a}_{\nu}^\alpha(a) & \mathcal{A}_{ab}^{\mu} d_{\mu \nu} \tilde{a}_{\nu}^\alpha(a) & d_{\mu \nu} \tilde{a}_{\nu}^\alpha(a) \tilde{a}_{\nu}^\alpha(a)
\end{pmatrix},
\]
where \(\tilde{h}_{ij}(x, \tilde{f}) = Q^i A H Q^J B, \quad \tilde{G}^H_{AB} = G_{AB} - G_{AC} K^{C} d_{\mu \nu} K_{\nu} D G_{DB}.\) Further, we will also denote expressions that include \(d_{\mu \nu},\) with a tilde mark above the character associated with that expression.

The metric is written in terms of the components \((\mathcal{A}_{i}^{\alpha}, \mathcal{A}_{b}^{\alpha})\) of the mechanical connection that exists in the principal fiber bundle \(P(M, G)\). These components are determined as follows:
\[
\mathcal{A}_{i}^{\alpha}(x, \tilde{f}) = d^{\alpha \beta} K_{\beta}^{C} G_{DC} Q^*_i D, \quad \mathcal{A}_{b}^{\alpha}(x, \tilde{f}) = d^{\alpha \beta} K_{\beta}^{C} G_{b p}.
\]

The inverse matrix \(G^{AB}\) is given by
\[
G^{AB} = \begin{pmatrix}
\tilde{h}_{ij} & \mathcal{A}_{ij}^{\mu} K_{\mu}^{\alpha} h_{mj} & -h_{ij} \mathcal{A}_{ij}^{\mu} \tilde{a}_{\mu}^\beta \\
\mathcal{A}_{i}^{\mu} K_{\mu}^{b} h_{ni} & G^{AB} N_{A}^{\alpha} N_{B}^{\beta} + G_{ab} & -G^{EC} \Lambda_{E}^{\mu} A_{C}^{\beta} K_{\mu}^{b} \tilde{a}_{\beta}^{\alpha} \\
-h_{ki} \mathcal{A}_{k}^{\mu} \tilde{a}_{\mu}^\beta & -G^{EC} \Lambda_{E}^{\mu} \Lambda_{C}^{\beta} K_{\mu}^{\alpha} \tilde{a}_{\beta}^{\alpha} & G^{BC} \Lambda_{B}^{\beta} \Lambda_{C}^{\beta} \tilde{a}_{\alpha}^\alpha \tilde{a}_{\beta}^\beta
\end{pmatrix}.
\]

Here \(\Lambda_{E}^{\beta} = (\Phi^{-1})^{\beta}_{\mu} \chi_{E}^{\mu}, h_{ij}\) is an inverse matrix to the matrix \(h_{ij} = Q_{i} A H Q_{j}^{*} B\) with
\[
G_{AB}^{H} = G_{AB} - G_{AC} K_{D}^{\gamma} \gamma_{\alpha}^{\beta} K_{\beta}^{C} G_{CB}.
\]

By \(\mathcal{A}_{b}^{\alpha}\) we denote the mechanical connection formed from the orbit metric \(\gamma_{\mu \nu}.\)
\[
\mathcal{A}_{b}^{\alpha} = \gamma^{\mu \nu} K_{\nu}^{A} G_{AB} Q_{m}^{* B}.
\]
By $\chi^a_B$ we denote $\chi^a_B = \partial \chi^a(\varphi)/\partial Q^B|_{Q=\varphi(x)}$, $(\Phi)^a_B = K^A_B \chi^a_A$ is the Faddeev-Popov matrix, $N^a_B = -K^a_B (\Phi)^a_B \chi^a_B \equiv -K^a_B \Lambda^a_B$ is one of the components of a particular projector on a tangent space to the orbit space.

The determinant of matrix (9) is equal to

$$
\det G_{AB} = \left( \det d_{\alpha\beta} \right) \left( \det \bar{u}_\mu^\nu (a) \right)^2 \det \begin{pmatrix} \hat{h}_{ij} & \tilde{G}^H_{Aa} Q^*_A \\
\hat{G}^H_{Aa} Q^*_j & \tilde{G}^H_{ba} \end{pmatrix},
$$

(11)

where $\tilde{G}^H_{Aa} = -G_{AB} K^B_\mu d^{\mu\nu} K^\nu_{ba}$, $\tilde{G}^H_{ba} = G_{ba} - G_{bc} K^c_\mu d^{\mu\nu} K^p_\nu G_{pa}$.

Note that the last determinant on the right hand side of (11) is the determinant of the metric defined on the orbit space $\tilde{M} = P \times G$ of the principal fiber bundle $P(M, G)$. In what follows we will denote this determinant by $H$.

Also note that in the matrix $G^{AB}$, the upper left quadrant of the matrix (11) is the matrix that represents the inverse metric to the metric in the orbit space of our principal fiber bundle.

The introduction of new coordinates leads to a transformation of the original stochastic process, the measures that they generate, and, consequently, the local evolutionary semigroups. By gluing together these local semigroups on their common intersection domains, we obtain, after passing to the corresponding limit, a new global semigroup.

The next local semigroup transformation performed in [1] was a transformation that leads to factorization of path integral measures in local semigroups, which allows obtaining the corresponding global semigroup with factorized measure.

Thus, as a result of factorization of the measure in the original path integral (11) it became possible to derive the integral relation between the Green functions given on the total space $\tilde{P}$ of the principal fiber bundle $P(M, G)$ and on the orbit space $\tilde{M}$ of this bundle. The obtained integral relation looks as follows:

$$
G^\lambda_{mn}(x_b, \tilde{f}_b, t_b; x_a, \tilde{f}_a, t_a) = \int_G G_{\tilde{P}}(p_\theta, v_\theta, t_b; p_a, t_a) D^\lambda_{nm}(\theta) d\mu(\theta),
$$

$$(x, \tilde{f}) = \pi'(p, v).$$

(12)

The path integral which represent the Green function $G^\lambda_{mn}$ is written
symbolically as
\[ G^\lambda_{mn}(\pi'(p_b), \pi'(v_b), t_b; \pi'(p_a), \pi'(v_a), t_a) = \]
\[ = \int_{\xi(t_a)=\pi(p_a), \xi(t_b)=\pi(p_b)} d\mu^x \exp\left\{ \frac{1}{\mu^2 \kappa m} \int_{t_a}^{t_b} \tilde{V}(\xi_1(u), \xi_2(u)) du \right\} \]
\[ \times \exp \int_{t_a}^{t_b} \left\{ \mu^2 K \left[ \frac{1}{2} d^{\mu\nu}(x(u), \tilde{f}(u))(J_\alpha)^{\lambda}_{pn}(J_\nu)^{\lambda}_{pm} \right. \right. \]
\[ - \frac{1}{2} \frac{\partial}{\sqrt{d} \partial x^k} \left( \sqrt{d} H h^{km} \omega^\mu_m(x(u)) \right) (J_\nu)^{\lambda}_{pm} \]
\[ - \frac{1}{2} \left( G^{EC} \Lambda^\mu E \Lambda^\nu C \right) \frac{1}{\sqrt{d} \partial f^b} \left( \sqrt{d} H K^b_\mu \right) (J_\nu)^{\lambda}_{pm} du \]
\[ - \mu \sqrt{K} \left[ \omega^\nu_a (x(u)) \tilde{X}_m^k(u)(J_\nu)^{\lambda}_{pm} d\tilde{w}^m(u) + \omega^\nu_a \tilde{X}_b^a(u)(J_\nu)^{\lambda}_{pm} d\tilde{w}^b(u) \right] \right\} (13) \]

The differential generator of the matrix semigroups with the kernel \( G^\lambda_{mn} \) (without the potential term) is given by
\[ \frac{1}{2} \mu^2 K \left\{ \Delta_{\tilde{M}} + h^{ni} \left( \frac{1}{\sqrt{d}} \partial(\sqrt{d}) + \omega^\nu_n \frac{1}{\sqrt{d}} \partial(\sqrt{d} K^b_\nu) \right) \frac{\partial}{\partial x^i} \right. \]
\[ + \left( h^{ni} \omega^\nu_m K^a_n \frac{1}{\sqrt{d}} \partial(\sqrt{d}) + (G^{ab} + G^{AB} N^a_A N^b_B) \frac{1}{\sqrt{d}} \partial(\sqrt{d}) \right) \frac{\partial}{\partial f^a} \right] (I)^{\lambda}_{pq} \]
\[ - 2h^{ni} \omega^\beta_{\mu a} (J_\beta)^{\lambda}_{pq} \frac{\partial}{\partial x^i} - 2h^{nk} \omega^\beta_{\mu a} \omega^\mu_n K^a_n (J_\beta)^{\lambda}_{pq} \frac{\partial}{\partial f^a} \]
\[ - 2(\gamma^\alpha_{\mu b} K^a_\beta + G^{ab}) \omega^\beta_{\mu a} (J_\beta)^{\lambda}_{pq} \frac{\partial}{\partial f^b} \]
\[ - \left[ \frac{1}{\sqrt{d} \partial f^a} \frac{\partial}{\partial x^i} \left( \sqrt{d} H h^{ni} \omega^\beta_{\mu a} \right) + (G^{EC} \Lambda^\beta E \Lambda^\nu C) \frac{1}{\sqrt{d} \partial f^b} \right] (J_\beta)^{\lambda}_{pq} \]
\[ + (\gamma^\alpha_{\mu b} + h^{ij} \omega^\beta_{\mu a} \omega^\beta_{\nu j} (J_\alpha)^{\lambda}_{pq} (J_\nu)^{\lambda}_{pq} \right\} (14) \]

(Here \( (I)^{\lambda}_{pq} \) is a unity matrix.)

This operator acts in the space of the sections \( \Gamma(\tilde{M}, V^*) \) of the associated co-vector bundle \( E^* = \tilde{P} \times_{\mathcal{G}} V^* \) (\( \tilde{P} = P \times V \)) with the scalar product
\[ (\psi_n, \psi_m) = \int_{\tilde{M}} \langle \psi_n, \psi_m \rangle_{V^*} \sqrt{d(x, \tilde{f})} dv_{\tilde{M}}(x, \tilde{f}), \]
where \( dv_{\tilde{M}}(x, \tilde{f}) = \sqrt{H(x, \tilde{f})} dx^1 ... dx^n d\tilde{f}^1 ... \tilde{f}^{n \nu} \) is the Riemannian volume element on the manifold \( \tilde{M} \).
The measure $\mu^\xi$ in (13) is generated by the stochastic process $\xi(t)$ given by the solution of the stochastic differential equation on the manifold $\tilde{\mathcal{M}}$. The local form of this equation is represented by an equation that combines the equations for the local processes $x^i(t)$ and $\tilde{f}^a(t)$:

$$d\xi_{loc}^i(t) = \frac{1}{2} \mu^2 \kappa \left( \begin{array}{c} \tilde{b}^i \\ \tilde{b}^a \end{array} \right) dt + \mu \sqrt{\kappa} \left( \begin{array}{cc} \tilde{X}^i_m & 0 \\ \tilde{X}^a_m & \tilde{X}^a_b \end{array} \right) \left( \begin{array}{c} d\tilde{\omega}^m \\ d\tilde{\omega}^b \end{array} \right)$$

(15)

with

$$\left( \begin{array}{c} \tilde{b}^i \\ \tilde{b}^a \end{array} \right) = \left( \begin{array}{c} \tilde{b}^i \\ \tilde{b}^a \end{array} \right) + \left( \begin{array}{c} h^{ij} \\ h^{ij} \end{array} \right) \left( \begin{array}{cc} \alpha^\mu_m K^b_m h^m_i & \mu^2 \kappa_i^a \\ \alpha^\mu_m K^b_m h^m_i & \mu^2 \kappa_a^b \end{array} \right) \left( \begin{array}{c} 1 \sqrt{\frac{1}{\partial \tilde{f}^a} \partial \tilde{f}^b} d\tilde{w}^m \\ 1 \sqrt{\frac{1}{\partial \tilde{f}^a} \partial \tilde{f}^b} d\tilde{w}^b \end{array} \right),$$

in which

$$\left( \begin{array}{c} h^{ij} \\ h^{ij} \end{array} \right) \left( \begin{array}{cc} \alpha^\mu_m K^b_m h^m_i & \mu^2 \kappa_i^a \\ \alpha^\mu_m K^b_m h^m_i & \mu^2 \kappa_a^b \end{array} \right) = \left( \begin{array}{c} \tilde{h}^{ij} \\ \tilde{h}^{ij} \end{array} \right) \left( \begin{array}{cc} \tilde{h}^i_0 & \tilde{h}^a_j \\ \tilde{h}^i_0 & \tilde{h}^a_j \end{array} \right),$$

$$\tilde{b}^i = \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^i} \left( \sqrt{H} h^{ij} \right) + \alpha^\mu_i K^a_{(\gamma)} h^a_i \frac{1}{\sqrt{H}} \frac{\partial}{\partial \tilde{f}^b} \left( \sqrt{H} K_m^b \right) \text{ and}$$

$$\tilde{b}^a = \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^j} \left( \sqrt{H} h^{mj} \alpha^\mu_m \right) K_m^a + \left( G^{ab} + G^{AB} N^a_A N^b_B \right) \frac{1}{\sqrt{H}} \frac{\partial}{\partial \tilde{f}^b} \left( \sqrt{H} \right)$$

3 Girsanov transformation of the path integral measure

The drift coefficients $b^i$ and $b^a$ of the equation (15) include the terms with the partial derivatives of the determinant of the metric $d_{\alpha\beta}(x, \tilde{f})$ given on the orbits of the principal fiber bundle $P(\tilde{\mathcal{M}}, \mathcal{G})$.

These terms are not intrinsic for the orbit space $\tilde{\mathcal{M}}$ and hence are not necessary for description of the evolution on this space. Therefore, we must perform such a transformation of the path integral in which the measure $\mu^\xi$ is replaced by a new measure $\mu^\tilde{\xi}$ associated with a new stochastic process $\tilde{\xi}(t)$ with the following local stochastic differential equations:

$$d\tilde{\xi}_{loc}^i(t) = \frac{1}{2} \mu^2 \kappa \left( \begin{array}{c} \tilde{b}^i \\ \tilde{b}^a \end{array} \right) dt + \mu \sqrt{\kappa} \left( \begin{array}{cc} \tilde{X}^i_m & 0 \\ \tilde{X}^a_m & \tilde{X}^a_b \end{array} \right) \left( \begin{array}{c} d\tilde{\omega}^m \\ d\tilde{\omega}^b \end{array} \right),$$

(16)
Such transformation is known as the Girsanov transformation of the stochastic processes. This transformation is based on the assumption that the partial differential equation whose solution is represented by the path integral have a unique solution. In our case, this allows us to find a rule according to which the multiplicative stochastic integral under the path integral sign in (13) is transformed.

The Girsanov transformation means that the following equality of the path integrals must hold:

\[
\int d\mu \hat{\exp}_\xi(...)_{pq}^\lambda \varphi_p(\xi) = \int d\mu \hat{\exp}_\xi(...)_{pq}^\lambda \varphi_p(\xi),
\]

where the multiplicative stochastic integral \(\hat{\exp}_\xi(...)_{pq}^\lambda\) is as in (13). As for the multiplicative stochastic integral \(\hat{\exp}_\xi(...)_{pq}\), we assume that it has the following form:

\[
\hat{\exp}_\xi(...)_{pq}^\lambda = \left(\exp_\xi \int \left(\tilde{L} dw + \tilde{M}' dt\right)_{pq}^\lambda\right),
\]

where

\[
\tilde{L} dw = - (\mu \sqrt{\kappa}) \left((\hat{\Gamma}_3n)_{pq}^\lambda \tilde{X}_n^p d\tilde{w}^\tilde{n} + (\hat{\Gamma}_4c)_{pq}^\lambda \tilde{X}_c^p d\tilde{w}^\tilde{c}\right),
\]

and

\[
\tilde{M}' = (\mu^2 \kappa) \tilde{M}'_{pq} dt
\]

The unknown coefficients \(\hat{\Gamma}_3n, \hat{\Gamma}_4c\) and \(\tilde{M}'\) can be determined by comparing the differential generators for the evolution matrix semigroups that represented by the path integrals in (17).

The differential generator for the matrix semigroup defined by the left-hand side of (17) is (14). The differential generator for the matrix semigroup, given on the right-hand side of (17), can be obtained by taking the mathematical expectation of the Ito’s differential of \(\varphi_p(\tilde{\xi}) \tilde{Z}_{pq}^\lambda(\tilde{\xi})\):

\[
d(\varphi_p \tilde{Z}_{pq}^\lambda) = d\varphi_p Z_{pq}^\lambda + \varphi_p dZ_{pq}^\lambda + d\varphi_p d\tilde{Z}_{pq}^\lambda,
\]

where \(\tilde{Z}_{pq}^\lambda\) is notation for \(\hat{\exp}_\xi(...)_{pq}^\lambda\).

\[\text{3Here the summation over repeated indices is implied.}\]
Taking the mathematical expectation of the terms represented by $d\varphi_p Z^\lambda_{pq}$, we get the following expression:

$$\left(\frac{\partial \varphi_p}{\partial x^i} \frac{\partial}{\partial x^i} + \frac{\partial^2 \varphi_p}{\partial f^a \partial f^a} \tilde{X}_m^i \tilde{X}_m^j + \frac{1}{2} \frac{\partial^2 \varphi_p}{\partial f^a \partial f^b} (X_m^a X_n^b + \tilde{X}_a^i \tilde{X}_b^i) + \frac{\partial^2 \varphi_p}{\partial x^i \partial f^a} (X_m^a \tilde{X}_n^b + \tilde{X}_a^i \tilde{X}_b^i)\right) (1^\lambda)_{pq}.$$ 

This expression can also be represented as follows:

$$\left(\frac{\partial \varphi_p}{\partial x^i} \frac{\partial}{\partial x^i} + \frac{\partial^2 \varphi_p}{\partial f^a \partial f^a} \tilde{h}^i j + \frac{1}{2} \frac{\partial^2 \varphi_p}{\partial f^a \partial f^b} \tilde{h}^{ab} + \frac{\partial^2 \varphi_p}{\partial x^i \partial f^a} \tilde{h}^{ia}\right) (1^\lambda)_{pq}. \quad (18)$$

The mathematical expectation of the terms given by $d\varphi_p d\tilde{Z}^\lambda_{pq}$ are

$$-\left(\frac{\partial \varphi_p}{\partial x^i} \tilde{X}_m^i (\tilde{\Gamma}_{3n})^\lambda_{pq} \tilde{X}_m^n - \left(\frac{\partial \varphi_p}{\partial f^a}\right) (\tilde{\Gamma}_{3n})^\lambda_{pq} \tilde{X}_m^a + (\tilde{\Gamma}_{4c})_{pq} \tilde{X}_m^a\right),$$

or written in another form

$$-\left(\frac{\partial \varphi_p}{\partial x^i}\right) \tilde{h}^{im} (\tilde{\Gamma}_{3n})^\lambda_{pq} - \left(\frac{\partial \varphi_p}{\partial f^a}\right) (\tilde{\Gamma}_{3n})^\lambda_{pq} \tilde{h}^{na} + (\tilde{\Gamma}_{4c})_{pq} \tilde{h}^{ac} \right), \quad (19)$$

where $\tilde{h}^{im} = h^{im}$, $\tilde{h}^{an} = \omega^\mu_m K^a_n h^{mn}$, $\tilde{h}^{ac} = (\gamma^{ab} K^a_m K^b_n + G^{ac})$.

Comparing those terms of the differential operators that stand at the first derivatives of $\varphi_p$ in (14) with the analogous terms in (18) and (19), one can obtain that

$$(\tilde{\Gamma}_{3n})^\lambda_{pq} = \omega_{\beta}^\gamma (J_{\beta})^\lambda_{pq} - \frac{1}{2} \frac{\partial (\sqrt{d})}{\partial x^m} (1^\lambda)_{pq} - \frac{1}{2} \frac{\partial (\sqrt{d})}{\partial f^b} (1^\lambda)_{pq}$$

and

$$(\tilde{\Gamma}_{4c})^\lambda_{pq} = -\frac{1}{2} \frac{\partial (\sqrt{d})}{\partial f^c} (1^\lambda)_{pq} + \omega_{c}^\beta (J_{\beta})^\lambda_{pq}.$$ 

The linear in $\varphi_p$ terms in the differential generator (14), and obtained from $\varphi_p dZ^\lambda_{pq}$ terms in the differential generator for the matrix semigroup related with the process $\tilde{\xi}$, are used to define the drift term $\tilde{M}^\lambda_{pq}$ in the multiplicative stochastic integral.

"$\varphi_p dZ^\lambda_{pq}$" terms are given by

$$\langle \mu^2 \rangle \left\{ \tilde{M}^\lambda_{pq} + \frac{1}{2} \left( (\tilde{\Gamma}_{3n})^\lambda_{pq} (\tilde{\Gamma}_{3k})^\lambda_{qq} \tilde{h}^{nk} + (\tilde{\Gamma}_{4a})^\lambda_{pq} (\tilde{\Gamma}_{4c})^\lambda_{qq} \tilde{h}^{ac} \right) \right\} \varphi_p. \quad (20)$$
(\hat{\Gamma}_{3n})^{\lambda}_{pq}(\hat{\Gamma}_{3k})^{\lambda}_{q'q} has the following explicit representation:

\[
\omega^{\beta}_{n}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} - \frac{1}{2} d_{n}\omega^{\alpha}_{k}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} - \frac{1}{2} d_{b}K^{b}_{\nu}\omega^{\nu}_{n}(\lambda J_{pq}^{\lambda})
\]

\[
-\frac{1}{2}d_{k}\omega^{\beta}_{n}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \frac{1}{4} d_{n}d_{k}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \frac{1}{4} d_{b}d_{k}K^{b}_{\nu}\omega^{\nu}_{n}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}
\]

\[
-\frac{1}{2}d_{b}K^{b}_{\nu}\omega^{\alpha}_{k}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \frac{1}{4} d_{n}d_{b}K^{b}_{\nu}\omega^{\alpha}_{k}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \frac{1}{4} d_{b}d_{c}K^{b}_{\nu}\omega^{\nu}_{n}K^{c}_{\mu}\omega^{\alpha}_{k}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}.
\]

An explicit expression of \((\hat{\Gamma}_{4a})^{\lambda}_{pq}(\hat{\Gamma}_{4e})^{\lambda}_{q'q}\) is

\[
\omega^{\beta}_{c}\omega^{\alpha}_{a}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} - \frac{1}{2} d_{c}\omega^{\alpha}_{a}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} - \frac{1}{2} d_{a}\omega^{\beta}_{c}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \frac{1}{4} d_{a}d_{c}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}.
\]

In the above formulas, it was used a new notation by which \(d_{n} = \frac{1}{\sqrt{d}}\frac{\partial(\sqrt{\hat{A}})}{\partial x_{n}}\) and \(d_{b} = \frac{1}{\sqrt{d}}\frac{\partial(\sqrt{\hat{A}})}{\partial x_{b}}\).

Thus, \(M^{\lambda}_{pq}\) is defined from the equation in which the left-hand side is given by \(20\), and the right-hand side of the equation consists of the corresponding terms obtained as a result of applying the differential operator \(14\) to \(\varphi_{p}\).

First, it can be shown that in this equation there is a cancellation of the terms that include as a factor only one term \(d_{i}\) (or \(d_{a}\)) and \((\lambda J_{pq}^{\lambda})^{\lambda}_{pq}\).

For these terms we will have

\[
d_{b}K^{b}_{\nu}h^{nk}_{\epsilon}\omega^{\nu}_{n}\omega^{\alpha}_{k}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}(-\frac{1}{4} - \frac{1}{4}) + d_{a}\omega^{\beta}_{c}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}(\gamma^{\alpha\beta}K^{c}_{\mu}K^{\gamma}_{\nu} + G^{ac})(-\frac{1}{4} - \frac{1}{4}) =
\]

\[
-\frac{1}{2} d_{b}K^{b}_{\nu}(G^{EC}\Lambda^{\mu}_{C})(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}.
\]

Since \(G^{EC}\Lambda^{\mu}_{C} = \gamma^{\mu\nu} + h^{nk}_{\epsilon}\omega^{\nu}_{n}\omega^{\alpha}_{k}\), we get

\[
-d_{a}d^{b\epsilon}_{\mu}K^{\epsilon}_{\mu}G_{ec}(\gamma^{\mu\nu}K^{\alpha}_{\mu}K^{\nu}_{\nu} + G^{ac}) = -d_{b}K^{b}_{\nu}\gamma^{\beta\nu}.
\]

But this equality lead to the identity \(d_{a}K^{\alpha}_{\mu}(d^{b\sigma}\gamma^{\epsilon}_{\sigma\nu}\gamma^{\mu\nu} + d^{b\mu} - \gamma^{b\nu}) \equiv 0\).

(We recall that \(d_{a\beta} = \gamma_{a\beta} + \gamma'_{a\beta}\) and \(\gamma'_{a\beta} = K^{a}_{\alpha}G_{ab}K^{b}_{\beta}\).)

From the equation for \(\hat{M}^{\lambda}_{pq}\) it follows that its solution can be represented as the sum of three group of terms:

\[
\hat{M}^{\lambda}_{pq} = (\mu^{2}\kappa)(\{\ldots\}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \{\ldots\}^{\beta}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq} + \{\ldots\}^{\alpha\beta}(\lambda J_{pq}^{\lambda})^{\lambda}_{pq}^{(J_{pq}^{\lambda})^{\lambda}_{pq}}).
\]
The diagonal part of the solution, \(\{\ldots\}(\ell^1)_{pq}\), is
\[
\begin{align*}
&\frac{-1}{2} \left( \frac{1}{4} h^{nk} d_n d_k + \frac{1}{4} d_n d_k K^b_{\nu} \omega^\nu_n h^{nk} + \frac{1}{4} (d_n d_b K^b_{\alpha} \omega^\alpha_n h^{nk})_{(\gamma)} \right) \\
&\quad + \frac{1}{4} d_n d_b h^{nk} K^c_{\nu} K^e_{\mu} \omega^\nu_n \omega^\mu_c + \frac{1}{4} (\gamma^\mu \nu K^a_{\mu} K^c_{\nu} + G^{ac}) d_n d_b \right) (\ell^1)_{pq}.
\end{align*}
\]
By denoting \(\sigma = \ln d\) so that \(d_n = \frac{1}{2} \sigma_n\), we rewrite above terms as a quadratic form consisting of the partial derivatives of \(\sigma\):
\[
\begin{align*}
&\frac{-1}{8} \left( \frac{1}{4} h^{nk} \sigma_n \sigma_k + \frac{1}{2} \sigma_b \sigma_k \tilde{h}^{bk} + \frac{1}{4} \sigma_d \sigma_c \tilde{h}^{ac} \right) (\ell^1)_{pq} = -\frac{1}{32} \partial \sigma, \partial \sigma = \mathcal{M} (\ell^1)_{pq}.
\end{align*}
\]
The second group of terms in the solution for \(\mathcal{M}^\lambda_{pq}\) is given by
\[
\begin{align*}
\{\ldots\}^\beta (\ell^1)_{pq} = & \ - \frac{1}{2} \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^k} \left( \sqrt{H} h^{km} \mathcal{A}_m^{\beta} (x(u)) \right) (\ell^1)_{pq} \\
&\quad - \frac{1}{2} (G^{EC} \Lambda^\beta E^\nu \Lambda^\mu_C) \frac{1}{\sqrt{f^k}} \frac{\partial}{\partial f^b} \left( \sqrt{H} K^b_{\mu} \right) (\ell^1)_{pq}.
\end{align*}
\]
The last group can be obtained by transforming the following equality
\[
\begin{align*}
\{\ldots\}^{\alpha\beta} + \frac{1}{2} (\gamma^\alpha \beta K^\alpha_n K^\beta_n + G^{ac}) = \frac{1}{2} \gamma^{\alpha\beta}
\end{align*}
\]
into \(\{\ldots\}^{\alpha\beta} + \frac{1}{2} (-\delta^{\alpha\beta}) = 0\). This is done using an explicit representation for the connection \(\mathcal{A}^{\beta}_{\gamma} = d^\nu K^b_{\nu} \mathcal{A}_{bc}\), and by substituting \(\gamma^\nu_{\beta} = d^\mu_{\beta} - \gamma^\nu_{\beta}\).

As a result of the performed calculation the diffusion terms in the multiplicative stochastic integral become as follows:
\[
- (\Gamma_{3n})_{pq}^\lambda \tilde{X}^n_m d\tilde{w}^m = - \mathcal{A}^{\beta}_{\gamma} (\ell^1)_{pq}^\gamma \tilde{X}^n_m d\tilde{w}^m + \frac{1}{4} (\sigma_n \tilde{X}^n_m d\tilde{w}^m + \sigma_b \tilde{X}^b_m d\tilde{w}^m) (\ell^1)_{pq},
\]
where \(\tilde{X}^b_m = \mathcal{A}^{\nu}_{\gamma} K^b_{\nu} \tilde{X}^n_m\), and
\[
- (\Gamma_{4n})_{pq}^\lambda \tilde{X}^n_b d\tilde{w}^b = - \mathcal{A}^{\beta}_{\gamma} (\ell^1)_{pq}^\gamma \tilde{X}^n_b d\tilde{w}^b + \frac{1}{4} (\sigma_c \tilde{X}^c_b d\tilde{w}^b) (\ell^1)_{pq}.
\]
After substitution of the obtained diffusion and drift terms into \(\tilde{X} - \) dependent multiplicative stochastic integral from (17), it will contain, together with the off-diagonal terms, the terms of the diagonal matrices. The exponential represented by the diagonal matrix can be factor out of the multiplicative
\[
\begin{align*}
\mathcal{M}^\lambda_{pq} = & \ - \frac{1}{2} \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^k} \left( \sqrt{H} h^{km} \mathcal{A}_m^{\beta} (x(u)) \right) (\ell^1)_{pq} \\
&\quad - \frac{1}{2} (G^{EC} \Lambda^\beta E^\nu \Lambda^\mu_C) \frac{1}{\sqrt{f^k}} \frac{\partial}{\partial f^b} \left( \sqrt{H} K^b_{\mu} \right) (\ell^1)_{pq}.
\end{align*}
\]
stochastic integral and can be treated independently. Therefore, the multiplicative stochastic integral on the right-hand side \([17]\) can be represented as follows:

\[
\exp \frac{1}{4} \int_{t_a}^{t_b} \left\{ \mu \sqrt{\kappa} \left[ \sigma_n \dot{X}_n^\mu d\tilde{w}^\mu + \sigma_b(\dot{X}_b^\mu d\tilde{w}^\mu + \dot{X}_a^\mu d\tilde{w}^\mu) \right] - \frac{1}{8} \mu^2 \kappa < \partial \sigma, \partial \sigma > \tilde{\mathcal{M}} du \right\} \left( \Lambda^\lambda_{pq} \right)
\]

\[
\times \exp \int_{t_a}^{t_b} \left\{ \mu^2 \kappa \left[ \frac{1}{2} d^\mu \sigma(x(u), \tilde{f}(u))(J_\alpha)^\lambda_{\mu n}(J_\nu)^\lambda_{\nu m} - \frac{1}{2} \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^k} \left( \sqrt{H} h_{km} \mathcal{A}_{(\gamma)}^\nu (x(u)) \right) (J_\nu)^\lambda_{\nu m} \right] d\nu - \mu \sqrt{\kappa} \left[ \mathcal{A}_{(\gamma)}^\nu (x(u)) \dot{X}_n^\mu (u)(J_\nu)^\lambda_{\nu m} d\tilde{w}^\mu (u) + \mathcal{A}_{(\gamma)}^\nu (u)(J_\nu)^\lambda_{\nu m} d\tilde{w}^\mu (u) \right] \right\} (21)
\]

In the same way as in case of the reduction onto the zero momentum level \([1]\) the exponential with the stochastic integral in the first factor of the multiplicative stochastic integral \((21)\) can be transformed by the Ito’s identity. As a result, we arrive at the following representation of the first factor in \((21)\):

\[
\left( \frac{\exp(\sigma(x(t_b), \tilde{f}(t_b)))}{\exp(\sigma(x(t_a), \tilde{f}(t_a)))} \right)^{1/4} \exp\left\{ - \frac{1}{8} \mu^2 \kappa \int_{t_a}^{t_b} (\mathcal{A}_{(\gamma)} \sigma + \frac{1}{4} < \partial \sigma, \partial \sigma > \tilde{\mathcal{M}}) du \right\} (22)
\]

that is, to the path integral reduction Jacobian.

In \([5]\), it was obtained the geometrical representation of this Jacobian. The integrand \(\tilde{J}\) of this Jacobiana was expressed as follows:

\[
\tilde{J} = R_{\tilde{\mathcal{P}}} - R_{\tilde{\mathcal{M}}} - R_{\tilde{\mathcal{G}}} - \frac{1}{4} d_{\mu \nu} \mathcal{F}_{\lambda' \lambda''} \mathcal{F}_{\nu \lambda' \lambda''} - ||j||^2,
\]

where \(R_{\tilde{\mathcal{P}}}\) is the scalar curvature of the original manifold, \(R_{\tilde{\mathcal{M}}}\) is the scalar curvature of the orbit space manifold \(\tilde{\mathcal{M}}\), \(R_{\tilde{\mathcal{G}}}\) is the scalar curvature of the orbit, \(\mathcal{F}_{\lambda' \lambda''}\) is the curvature of the mechanical connection, \(||j||^2\) is the trace of the square of the fundamental form \(j_{\alpha \beta}\) of the orbit taken on \(\tilde{\mathcal{M}}\).

Thus, the Girsanov transformation allows us to rewrite the integral relation \(\left(12\right)\) as follows:

\[
d_{b}^{-1/4} d_{a}^{-1/4} G_{pq}(x_b, \tilde{f}_b, t_b; x_a, \tilde{f}_a, t_a) = \int_{\tilde{\mathcal{G}}} G_{\tilde{\mathcal{P}}}(p_{\theta}, v_{\theta}, t_b; p_a, v_a, t_a) D_{\tilde{\mathcal{G}}}(\theta) d\mu(\theta),
\]

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where \( d_b = d(x_b, \tilde{f}_b) \), \( d_a = d(x_a, \tilde{f}_a) \) and the Green’s function \( \tilde{G}^\lambda_{pq} \) is given by the following path integral

\[
\tilde{G}^\lambda_{mn}(\pi'(p_b), \pi'(v_b), t_b; \pi'(p_a), \pi'(v_a), t_a) = \\
\int_{\xi(t_a) = \pi'(p_a, v_a)}^{t_b} d\mu^\xi \exp \left\{ \int_{t_a}^{t_b} \left( \frac{\tilde{V}(\xi(u))}{\mu^2 \kappa m} - \frac{1}{8} \mu^2 \kappa J \right) du \right\}
\]

\[
\times \exp \int_{t_a}^{t_b} \left\{ \mu^2 \kappa \left[ \frac{1}{2} \delta^\nu_{\mu}(x(u), \tilde{f}(u))(J_\nu)_{pn}^\lambda (J_\nu)_{rn}^\lambda \\
- \frac{1}{2} \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^k} \left( \sqrt{H} h^{km} \mathcal{A}^\nu_m(x(u)) \right) (J_\nu)_{pn}^\lambda \\
- \frac{1}{2} (G^{EC} \mathcal{A}^\nu_m \mathcal{A}^\mu_n) \frac{1}{\sqrt{H}} \frac{1}{\sqrt{f^b}} \left( \sqrt{H} K^b_\mu \right) (J_\nu)_{pn}^\lambda \right] du \\
- \mu \kappa \sqrt{\mathcal{A}^\nu_m(x(u)) \mathcal{X}^\mu_m(u) (J_\nu)_{pn}^\lambda d\tilde{\omega}^m(u) + \mathcal{A}^\nu_m \mathcal{X}^\mu_m(u) (J_\nu)_{pn}^\lambda d\tilde{\omega}^b(u) \right}\} (23)
\]

The differential generator of the matrix semigroups with the kernel \( \tilde{G}^\lambda_{mn} \) is given by

\[
\frac{1}{2} \mu^2 \kappa \left\{ \left[ \Delta_{\tilde{M}} + \frac{2 \tilde{V}}{(\mu^2 \kappa)^2 m} - \frac{1}{4} \tilde{J} \right] (I^\lambda)_{pq} \\
- 2 h^{ni} \mathcal{A}^\beta_{pq} (J_\beta)_{pq}^\lambda \frac{\partial}{\partial x^i} - 2 h^{nk} \mathcal{A}^\beta_{pq} K^a_\mu (J_\beta)_{pq}^\lambda \frac{\partial}{\partial f^a} \\
- 2 (\gamma^{\alpha \beta} K^a_\mu K_{\beta} + G^{ab}) \mathcal{A}^\beta_{pq} (J_\beta)_{pq}^\lambda \frac{\partial}{\partial f^b} \\
- \left[ \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^i} \left( \sqrt{H} h^{ni} \mathcal{A}^\beta_{pq} \right) + (G^{EC} \mathcal{A}^\beta_{pq} \mathcal{A}^\mu_n) \frac{1}{\sqrt{H}} \frac{\partial(\sqrt{H} K^b_\mu)}{\partial f^b} \right] (J_\beta)_{pq}^\lambda \\
+ (\gamma^{\alpha \beta} + h^{ij} \mathcal{A}^{ij}_{pq} \mathcal{A}^\beta_{pq}^i) (J_\alpha)_{pq}^\lambda (J_\beta)_{pq}^\lambda \right\} (24)
\]

This operator acts in the space of the sections \( \Gamma(\tilde{M}, V^*) \) of the associated covector bundle \( E^* = \tilde{P} \times_g V^*_\Lambda \) (\( \tilde{P} = P \times V \)) with the following scalar product

\[
(\varphi_p, \varphi_q) = \int_{\tilde{M}} \langle \varphi_p, \varphi_q \rangle_{V^*_\Lambda} dv_{\tilde{M}}(x, \tilde{f}),
\]

where \( dv_{\tilde{M}}(x, \tilde{f}) = \sqrt{H(x, \tilde{f})} dx^1 \ldots dx^n dm d\tilde{f}^1 \ldots d\tilde{f}^{n_y} \).

It is not difficult to show that the differential operator \( (24) \) can also be expressed in terms of the horizontal Laplacian. Usually, for a covector bundle,
this operator is defined as follows:

\[
(\triangle^*_\lambda)_{pq} = \sum_{\mathcal{M}} \left( \nabla_{X^A_{\mathcal{M}}} \nabla_{X^B_{\mathcal{M}}} - \nabla_{X^A_{\mathcal{M}}} \nabla_{X^B_{\mathcal{M}}} \right)_{pq}^{\lambda},
\]

in which \(X^A_M\) is a “square root” of the metric: \(\sum_{\mathcal{M}} X^A_M X^B_M = G^{AB}\) and the connection is \((\Gamma^\lambda)_{Bpq} = \mathcal{A}_B^\alpha (J^\alpha)_{pq}^{\lambda}\).

The covariant derivative \(\nabla^*_\lambda\) is defined as

\[
\nabla_{\pi^\lambda}^*_p = (I^\lambda_{pq} \frac{\partial}{\partial Q^D} - \mathcal{A}_D (J^\lambda_{pq}) \varphi^q,\]

and

\[
\nabla_{\pi^\lambda}^*_p = \lambda^4 \Gamma^\lambda_{AB} e^C.
\]

It should be noted that in our case we are dealing with the product manifold, therefore, each capital index in the previous formulas means an abbreviation for two small indices.

Taking this into account, one can find that the differential operator has the following representation:

\[
\frac{1}{2} \mu^2 \kappa \left[ (\triangle^*_\lambda)_{pq} + d^\mu (J^\mu)_{pq}^\lambda (J^\nu)_{q}^\lambda \right] + \left( \frac{1}{\mu^2 \kappa m} \dot{V} - \frac{1}{8} \mu^2 \kappa J \right) (I^\lambda)_{pq}.
\]

Also note, that the Green function \(G^\lambda_{mn}\) satisfies the forward Kolmogorov equation with the operator

\[
\hat{H}_\kappa = \frac{\hbar \kappa}{2m} \left[ (\triangle^\lambda)_{pq} + d^\mu (J^\mu)_{pq}^\lambda (J^\nu)_{q}^\lambda \right] = \frac{\hbar \kappa}{2m} \left[ \hat{J} \right] I^\lambda_{pq} + \dot{V} I^\lambda_{pq},
\]

in which the horizontal Laplacian \((\triangle^\lambda)_{pq}\) is

\[
(\triangle^\lambda)_{pq} = \sum_{\mathcal{M}} \left( \nabla_{X^A_{\mathcal{M}}} \nabla_{X^B_{\mathcal{M}}} - \nabla_{X^A_{\mathcal{M}}} \nabla_{X^B_{\mathcal{M}}} \right)_{pq}^{\lambda},
\]

At \(\kappa = i\) the forward Kolmogorov equation becomes the Schrödinger equation with the Hamilton operator \(\hat{H}_\xi = -\frac{\hbar}{\kappa} \hat{H}_\kappa |_{\kappa = i}\). The operator \(\hat{H}_\xi\) acts in the Hilbert space of the sections of the associated vector bundle \(\mathcal{E} = \hat{P} \times \mathcal{G} V_\lambda\).
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