ON THE INTERACTION BETWEEN LEAVITT PATH ALGEBRAS AND PARTIAL SKEW GROUP RINGS

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Abstract. Let $E$ be a directed graph, $K$ be a field, and $F$ be the free group on the edges of $E$. In this work, we use the isomorphism between Leavitt path algebras and partial skew group rings to endow $L_K(E)$ with an $F$-gradation and study some algebraic properties of this gradation. More precisely, we show that graded cleanness, graded unit-regularity, and strong gradeness of $L_K(E)$ are all equivalent.

1. Introduction

The Leavitt path algebra associated with a directed graph $E$ and a field $K$ is a ring $L_K(E)$ which is an algebraic analogue of the graph Cuntz–Krieger C*-algebra. Leavitt path algebras were introduced in [7] and simultaneously in [3]. Since their introduction, Leavitt path algebras have gained great importance, and several of their algebraic properties have been studied in terms of properties of the underlying graph (see for instance [1, 4, 5, 6]). Each Leavitt path algebra may be viewed as a $\mathbb{Z}$-graded ring (see Section 4 in [17]) and a systematic study of properties of the $\mathbb{Z}$-gradation is considered in [14, 15, 17, 18, 20]. Moreover, in [16] the authors

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study $G$-graded Leavitt path algebras that contain strongly graded rings as direct summands, where $G$ is an arbitrary group.

On the other hand, partial actions of groups and partial skew group rings are introduced and studied in [9] as algebraic analogues of $C^*$-partial crossed products and have become a useful tool in several branches of mathematics, [8]. The connection between partial crossed products and Leavitt path algebras is made in [10], where Leavitt path algebras are realized as a partial skew group ring. From this realization, a new grading is obtained on Leavitt path algebras, namely a grading over the free group $\mathbb{F}$ on the edges of the graph. Moreover, new proofs of the Cuntz–Krieger uniqueness theorem and the simplicity criteria for Leavitt Path Algebras are obtained via partial crossed product theory as well as a characterization of Artinian Leavitt Path Algebras, [19].

The interaction between Leavitt path algebras and partial skew group rings is extended to include Leavitt path algebras associated with ultragraphs (which are generalizations of graphs) in [11]. In this context, which includes Leavitt path algebras of graphs, the skew group ring point of view is used to prove the reduction theorem, [12], and to describe graphs for which the grading (both by $\mathbb{Z}$ and $\mathbb{F}$) is strong or epsilon-strong, see [13]. Nevertheless, to the author’s knowledge, the development of the properties of the $\mathbb{F}$-gradation on a Leavitt path algebra is still in its infancy.

In this work, we study some structural properties of the $\mathbb{F}$-gradation on a Leavitt path algebra $L_K(E)$, such as graded cleanness and graded unit-regularity (see Proposition 10 and Proposition 11, respectively) and show that these conditions are equivalent to $L_K(E)$ being strongly graded, see Theorem 12. In the last section of this work, given any graph, we modify the construction in [10] (including some formal series) and present an alternative partial skew group ring that is isomorphic to the Leavitt path algebra associated with the graph.

2. Preliminaries

In this section, we present some notions and results on graded rings and partial actions which will be useful in the work.

2.1. Graded rings. Let $R$ be a ring and $G$ be a group with identity $e$. We say that $R$ is graded by $G$, or $G$-graded, if $R = \bigoplus_{g \in G} R_g$, where $R_g$ is an additive subgroup of $R$ for each $g \in G$, and $R_g R_h \subseteq R_{gh}$ for any $g, h \in G$ (where for $X, Y$ subsets of $R$ the product $XY$ is defined as the set of finite sums of elements $xy$, with $x \in X$ and $y \in Y$, respectively). If the equality $R_g R_h = R_{gh}$ holds, we say that $R$ is strongly graded. It is not difficult to see that for a graded ring $R$, the set $R_e$ is a subring of $R$ and $R_g R_{g^{-1}}$ is an ideal of $R_e$, for any $g \in G$. Moreover, the non-zero elements in $\bigcup_{g \in G} R_g$ are called homogeneous.

An element $x$ in a unital ring $R$ is clean if $x = u + a$ for some unit $u$ and some idempotent $a$. If every element of $R$ is clean we say that $R$ is clean. In the case of graded rings, we have, from [20], the following definition.

Definition 1. Let $R$ be a unital ring. A graded ring $R$ is graded clean if every homogeneous element $x \in R$ is a sum of a homogeneous unit and a homogeneous idempotent. We call such a sum a graded clean decomposition of $x$.

The following result, proved in [20] Lemma 2.2], gives us a characterization of graded clean rings. We include the proof here for completeness.
Lemma 2. \cite{20} Lemma 2.2] Let $R$ be a $G$-graded unital ring. Then $R$ is graded clean if, and only if, $R_e$ is clean and each nonzero element of $R_g$ is invertible for every $g \neq e$.

Proof. Suppose that $R$ is graded clean. We prove that $R_e$ is clean: let $x \in R_e$. Since $R$ is graded clean, there is a homogeneous invertible element $u \in R$ and a homogeneous idempotent $a$ such that $x = u + a$. So, we only need to prove that $u, a \in R_e$, which is true since $R_e$ is a direct summand of $R$. Now, we prove that any element of $R_g$ is invertible for all $g \neq e$. Let $x \in R_g$. Since $R$ is graded clean, we have that $x = u + a$ for a homogeneous invertible element $u$ and a homogeneous idempotent $a$. Since $a$ is a homogeneous idempotent, we have that $a \in R_e$, in which case $a = 0$ and $x = u$ is invertible.

Conversely, suppose that $R_e$ is clean and each element of $R_g$ is invertible for all $g \neq e$. Let $x \in R_g$. If $g \neq e$ then $x$ is invertible and $x = x + 0$. If $g = e$ then, since $R_e$ is clean, we can write $x = u + a$ with $u$ invertible, $a$ idempotent and $u, a \in R_e$.

We conclude that $R$ is graded clean. \hfill \qed

A unital ring $R$ is unit-regular if for any $x \in R$ there is an invertible $u \in R$ such that $x = xux$. The notion of unital regularity in the realm of graded rings is defined as follows (see \cite{20}).

Definition 3. Let $R$ be a unital graded ring. We say that $R$ is graded unit-regular if each homogeneous element $x \in R$ can be written as $x = xux$, where $u$ is an invertible and homogeneous element.

Lemma 4. If a graded ring $R$ is graded unit-regular, then every nonzero component $R_g$ contains an invertible element.

Proof. Let $0 \neq x \in R_g$. Then, there exists $u$ homogeneous and invertible such that $x = xux$, which implies that $u \in R_g^{-1}$ and its inverse $u^{-1}$ belongs to $R_g$. \hfill \qed

2.2. Partial actions. A partial action of the group $G$ on a ring $R$ is a pair $\alpha = (D_g, \alpha_g)_{g \in G}$, where for each $g \in G$, $D_g$ is an ideal of $R$, $\alpha_g : D_{g^{-1}} \to D_g$ is an isomorphism of rings, and for each $(g, h) \in G \times G$,

\hspace{1cm} (P1) $\alpha_e = \text{id}_R$;  
\hspace{1cm} (P2) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;  
\hspace{1cm} (P3) if $r \in D_{g^{-1}} \cap D_{gh}$, then $\alpha_g(\alpha_h(r)) = \alpha_{gh}(r)$.

Moreover, we say that $\alpha$ is global if $D_g = R$, for any $g \in G$.

Associated with the notion of partial action, we have the generalization of the concept of the skew group ring.

Definition 5. Given a partial action $\alpha = (D_g, \alpha_g)_{g \in G}$ on a ring $R$, the partial skew group ring $R \star_{\alpha} G$ is the direct sum $\bigoplus_{g \in G} D_g \delta_g$, in which the $\delta_g$’s are formal symbols. Addition is defined point-wise and multiplication is defined by the bi-additive extension of the relations

$$(r \delta_g)(r' \delta_h) = \alpha_g(\alpha_{g^{-1}}(r)r') \delta_{gh},$$

for $g, h \in G$, $r \in D_g$ and $r' \in D_h$.

In general, the ring $R \star_{\alpha} G$ is not associative, as shown in \cite{9} Example 3.5]. Nevertheless, it follows by \cite{21} Corollary 3.2] that $R \star_{\alpha} G$ is associative provided that $D_g$ is idempotent or non-degenerated, for any $g \in G$. 

Proposition 6. Let $\alpha = (D_g, \alpha_g)_{g \in G}$ be a partial action such that $D_g$ is idempotent for any $g \in G$. Then $\mathbb{R} \rtimes_{\alpha} G$ is strongly graded if, and only if, $\alpha$ is global.

Proof. Suppose that $\mathbb{R} \rtimes_{\alpha} G$ is strongly graded. Then, for each $g \in G$

$$D_g \delta_e = (D_g \delta_g) \cdot (D_{g^{-1}} \delta_{g^{-1}}) = \alpha_g(\alpha_{g^{-1}}(D_g)D_{g^{-1}}) \delta_e = \alpha_g(D_{g^{-1}}D_{g^{-1}}) \delta_e = D_g \delta_e$$

and hence $D_g = D_e = R$ for all $g \in G$ and $\alpha$ is global.

Reciprocally, if $\alpha$ is global, we have that $D_g = R$ for every $g \in G$ and hence, for $g, h \in G$, we have

$$(D_g \delta_g) \cdot (D_h \delta_h) = \alpha_g(\alpha_{g^{-1}}(D_g)D_h) \delta_{gh}$$

$$= \alpha_g(R^2) \delta_{gh} = \alpha_g(R) \delta_{gh}$$

$$= D_{gh} \delta_{gh}$$

and $\mathbb{R} \rtimes_{\alpha} G$ is strongly graded. \hfill $\Box$

3. Leavitt path algebras as partial crossed products

A directed graph $E = (E_0, E_1, r, s)$ consists of two countable sets $E_0$, $E_1$ and maps $r, s : E_1 \to E_0$ (called the range and source maps, respectively). The elements of $E_0$ are called vertices and the elements of $E_1$ are called edges. A vertex $v$ such that $s^{-1}(v) = \emptyset$ is called a sink, such that $r^{-1}(v) = \emptyset$ is called a source and such that $s^{-1}(v) = \emptyset = r^{-1}(v)$ is called isolated. A vertex $v \in E_0$ such that $|s^{-1}(v)| = \infty$ is called an infinite emitter. A path $\mu$ in $E$ is a sequence of edges $\mu = \mu_1 \ldots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i \in \{1, \ldots, n-1\}$. In such case, $s(\mu) := s(\mu_1)$ is the source of $\mu$, $r(\mu) := r(\mu_n)$ is the range of $\mu$ and $n$ is the length of $\mu$.

Definition 7. Leavitt path algebra $\mathbb{K}$. Let $E$ be any directed graph and let $\mathbb{K}$ be any field. The Leavitt path $\mathbb{K}$-algebra $L_{\mathbb{K}}(E)$ of $E$ with coefficients in $\mathbb{K}$ is the $\mathbb{K}$-algebra generated by a set $\{v \mid v \in E_0\}$ of pairwise orthogonal idempotents, together with a set of elements $\{f \mid f \in E_1\} \cup \{f^* \mid f \in E_1\}$, which satisfy the following relations:

1. $s(f)f = fr(f) = f$, for all $f \in E_1$;
2. $r(f)f^* = f^*s(f) = f^*$, for all $f \in E_1$;
3. $f^*f = \delta_{f, r}r(f)$, for all $f, f^* \in E_1$;
4. $v = \sum_{\{f \in E_1 \mid s(f) = v\}} ff^*$, for every $v \in E_0$ for which $s^{-1}(v)$ is non-empty and finite.

In [10], the authors have shown that the Leavitt path algebra $L_{\mathbb{K}}(E)$ is isomorphic as a $\mathbb{K}$-algebra to a partial skew group ring $D(X) \rtimes_{\alpha} \mathbb{F}$, where $D(X)$ is a certain commutative $\mathbb{K}$-algebra and $\mathbb{F}$ is the free group generated by $E_1$. For the reader's benefit, we briefly recall the construction of $D(X) \rtimes_{\alpha} \mathbb{F}$.

Let $W$ denote the set of all finite paths in $E$, and let $W^\infty$ denote the set of all infinite paths in $E$. Denote by $\mathbb{F}$ the free group generated by $E_1$. We define a partial action of $\mathbb{F}$ on the set

$$X = \{\xi \in W \mid r(\xi) \text{ is a sink}\} \cup \{v \in E_0 \mid v \text{ is a sink}\} \cup W^\infty.$$ 

For each $g \in \mathbb{F}$, let $X_g$ be defined as follows:

- $X_e := X$, where $e$ is the identity element of $\mathbb{F}$.
- $X_{g^{-1}} := \{\xi \in X \mid s(\xi) = r(b)\}$, for all $b \in W$.
- $X_a := \{\xi \in X \mid \xi_1\xi_2\ldots\xi_{|a|} = a\}$, for all $a \in W$. 

• $X_{ab^{-1}} := \{ \xi \in X \mid \xi_1 \xi_2 \cdots \xi_{|a|} = a \} = X_a$, for $ab^{-1} \in F$ with $a, b \in W$, $r(a) = r(b)$ and $ab^{-1}$ in its reduced form.
• $X_g := \emptyset$, for all other $g \in F$.

Let $\theta_\delta : X_e \to X_e$ be the identity map. For $b \in W$, if $r(b)$ is not a sink, $\theta_b : X_{b^{-1}} \to X_b$ is defined by $\theta_b(\xi) = b\xi$ and $\theta_{b^{-1}} : X_b \to X_{b^{-1}}$ by $\theta_{b^{-1}}(\eta) = \eta|_{|b|+1}\eta|_{|b|+2}\cdots$. On the other hand, if $r(b)$ is a sink, then $X_{b^{-1}} = \{r(b)\}$ and $X_b = \{b\}$, in which case we define $\theta_b(r(b)) = b$ and $\theta_{b^{-1}}(b) = r(b)$. Finally, for $a, b \in W$ with $r(a) = r(b)$ and $ab^{-1}$ in reduced form, $\theta_{ab^{-1}} : X_{ba^{-1}} \to X_{ab^{-1}}$ is defined by $\theta_{ab^{-1}}(\xi) = a\xi|_{|b|+1}\xi|_{|b|+2}\cdots$, with inverse $\theta_{ba^{-1}} : X_{ab^{-1}} \to X_{ba^{-1}}$ defined by $\theta_{ba^{-1}}(\eta) = b\eta|_{|a|+1}\eta|_{|a|+2}\cdots$.

The pair $(X_g, \theta_g)_{g \in F}$ is a partial action on the set level, which gives a partial action on a ring level $(F(X_g), \alpha_g)_{g \in F}$, where, for each $g \in F$, $F(X_g)$ denotes the algebra of all functions from $X_g$ to $K$, and $\alpha_g : F(X_{g^{-1}}) \to F(X_g)$ is defined by the rule $\alpha_g(f) = f \circ \theta_{g^{-1}}$. We define another partial action in the following way:

For each $g \in F$, and for each $v \in E^0$, define the characteristic maps $1_g := \chi_{X_g}$ and $1_v := \chi_{X_v}$, where $X_v = \{\xi \in X \mid s(\xi) = v\}$. Notice that $1_g$ is the identity element of $F(X_g)$. Finally, let

\[ D(X) = D_e = \text{span}_K \{1_g \mid g \in F \setminus \{0\}\} \cup \{1_v \mid v \in E^0\}, \]

(1) (where $\text{span}_K$ means the $K$-linear span) and, for each $g \in F \setminus \{0\}$, let $D_g \subseteq F(X_g)$ be defined as $1_g D_e$, that is,

\[ D_g = \text{span}_K \{1_g 1_h \mid h \in F\}. \]

By $[10]$ Lemma 2.4, $D(X)$ is a $K$-algebra and $D_g$, for $g \in F$, is an ideal of $D(X)$. Using that $\alpha_g(1_g^{-1} 1_h) = 1_g 1_{gh}$ (see $[10]$ Lemma 2.6), for each $g \in F$, one obtains that $\alpha_g$ is a bijection from $D_g^{-1}$ onto $D_g$. To simplify notation, this restriction map will also be denoted by $\alpha_g$. Clearly, $\alpha_g : D_g^{-1} \to D_g$ is an isomorphism of $K$-algebras and, furthermore, $(\alpha_g, D_g)_{g \in F}$ is a partial action. By $[10]$ Proposition 3.2] the map

\[ \varphi : L_K(E) \to D(X) \star_F F, \varphi(f) = 1_f \delta_f, \varphi(f^*) = 1_{f^{-1}} \delta_{f^{-1}} \] and $\varphi(v) = 1_v \delta_v$, for any $f \in E^1$ and $v \in E^0$, is an isomorphism of $K$-algebras.

Our goal is to show some results of the study of the $F$-gradation on Leavitt path algebras induced by the isomorphism defined above.

**Proposition 8.** The ideal $D_g$ defined above is idempotent for any $g \in F$.

**Proof.** For each $g \neq e$ we have that $D_g$ is idempotent, since it is generated by the central idempotent $1_g$. It remains to show that $D_e$ is idempotent. For this, it is enough to prove that $D_e D_e \supseteq D_e$, since the other inclusion is clear.

Let $x \in D_e$ and write

\[ x = \sum_{i=1}^{n} \alpha_{p_i q_i^{-1}} + \sum_{i=1}^{m} \gamma_{j} 1_{v_j}, \]

with $p_i, q_i \in F$ for all $i \in \{1, 2, \cdots, n\}$, $v_j \in E^0$ for all $j \in \{1, 2, \cdots, m\}$, and $\alpha_i, \gamma_j \in K$. Define $V$ as the set

\[ V = \{s(p_i q_i^{-1}) \mid 1 \leq i \leq n\} \cup \{v_j \mid 1 \leq i \leq m\}, \]
and write $y = \sum_{v \in V} 1_v$. Then,
\[
xy = \sum_{v \in V} \left( \sum_{i=1}^{n} \alpha_i p_i q_i^{-1} 1_v + \sum_{i=1}^{m} \gamma_j 1_v 1_v \right)
\]
\[
= \sum_{v \in V} \left( \sum_{i=1}^{n} \alpha_i (\delta_{s(p_i q_i^{-1})}, v) 1_v + \sum_{i=1}^{m} \gamma_j 1_v 1_v \right)
\]
\[
= \sum_{i=1}^{n} \alpha_i 1_{p_i q_i^{-1}} 1_{s(p_i q_i^{-1})} + \sum_{i=1}^{m} \gamma_j 1_{v_j}
\]
\[
= \sum_{i=1}^{n} \alpha_i 1_{p_i q_i^{-1}} + \sum_{i=1}^{m} \gamma_j 1_{v_j}
\]
\[
= x,
\]
and $x \in D_c D_e$ as desired. \qed

As a consequence of the above proposition, we obtain that the associated partial skew group ring $D(X) \rtimes_{\alpha} F$ is associative (see the comment after Definition 3). Below we describe when this partial skew ring is strongly graded (this result was proved, in the context of ultragraphs, in [13, Theorem 6.3], but our proof is shorter and simpler).

From now on, we consider only graphs with at least one edge, since we want the free group to be non-trivial.

**Proposition 9.** Let $E$ be a graph with at least one edge. The ring $D(X) \rtimes_{\alpha} F$ is strongly graded if, and only if, the graph $E$ is a loop.

**Proof.** Suppose that $D(X) \rtimes_{\alpha} F$ is strongly graded. It follows by Proposition 3 and Proposition 8 that $\alpha$ is global. If there exists $f, l \in E^1$ with $l \neq f$ then we have that $X_f \neq X_l$, which implies that $1_f \neq 1_l$ and hence $D_f \neq D_l$, which contradicts that the action is global. So, $f = l$ and $E^1 = \{f\}$. Let $v \in E^0$. Then, $1_v \in D_e = D_f$, and $1_f$ is the unit of $D_c$. Thus, $1_v = 1_f 1_v = 1_f 1_{s(f)} 1_v$. Since the functions $1_u$, with $u \in E^0$, are orthogonal, we get that $v = s(f)$. Analogously, we obtain that $v = r(f)$. Thus $s(f) = r(f)$ and the graph is a loop.

Reciprocally, suppose that $E$ is a loop with $E^1 = \{f\}$. Then $X = W^\infty = \{f^n\}$, with $f^\infty$ the infinite path that arises by concatenating the edge $f$, and $F = \{f\}$, so that every element $p \in F$ is of the form $p = f^n$, for some $n \in \mathbb{Z}$. Hence, $X = X_f = X_p$, and consequently $1_f = 1_p$ for each $p \in F$. Thus, $D_p = D_f$ for each $p \in F$ and the action is global, that is $D(X) \rtimes_{\alpha} F$ is strongly graded. \qed

From now on we assume that $L_K(E)$ has the $F$-gradation given by the isomorphism $\overline{\gamma}$. Next, we describe when $L_K(E)$ is $F$-graded clean and $F$-graded unit-regular.

**Proposition 10.** Let $E$ be a graph with at least one edge. The Leavitt path algebra $L_K(E)$ is $F$-graded clean if, and only if, $L_K(E)$ is strongly graded.

**Proof.** Suppose that $L_K(E) \cong D(X) \rtimes_{\alpha} F$ is $F$-graded clean. Then it is unital and thus has a finite number of vertices. By Proposition 9 we must verify that the graph is a loop. We first prove that the graph has only one vertex. Let $f \in E^1$ and suppose that there exists $v \neq s(f)$. Then, by Lemma 4 $0 \neq 1_f \delta_f \in D_f \delta_f$ is invertible with inverse $u$. Hence, $0 \neq v = v(fu) = (vf)u = 0$, which is a contradiction. Therefore,
s(f) is the only vertex in E and 1_{L_K(E)} = s(f). Now, we prove that f is the only edge of E. For this, we consider the set of idempotent elements

\[ S = \left\{ \sum_{i=1}^{n} f_i f_i^* \in L_K(E) \mid f_i \in E^1 \text{ and } n \in \mathbb{N} \right\} \]

and define a partial order in S by p \leq q if, and only if, pq = qp = p. Take an edge l \neq f. Since f^* f = r(f) = s(f) = 1_{L_K(E)} = fu, we have that f^* = f^*(fu) = (f^* f)u = u. Thus, ff^* = 1_{L_K(E)} and so ff^* \geq ff^* + ll^*. Clearly, ff^* \leq ff^* + ll^*. Hence ff^* = ff^* + ll^*, which implies that ll^* = 0 and thus l = lr(l) = ll^*l = 0l = 0, which is a contradiction. It follows that E is a loop and therefore L_K(E) is strongly graded.

Conversely, suppose that D(X) \rtimes_{\alpha} \mathbb{F} is strongly graded. By Proposition 9, the graph must be a loop with E^0 = \{v\} and E^1 = \{f\}, in which case X = \{f^\infty\} and 1, \delta_v is the unit of D(X) \rtimes_{\alpha} \mathbb{F}. Since 1_p = 1_v for all p \in \mathbb{F}, we have that (1_p \delta_p)(1_{p^{-1}} \delta_{p^{-1}}) = 1_v \delta_v, that is, each nonzero element of D_p \delta_p has an inverse and D_e \delta_e is clean. Therefore, by Lemma 2, D(X) \rtimes_{\alpha} \mathbb{F} is graded clean. □

**Proposition 11.** Suppose that E has at least one edge. The Leavitt path algebra L_K(E) is \mathbb{F}-graded unit-regular if, and only if, E is a loop.

**Proof.** Assume that the Leavitt path algebra L_K(E) is graded unit-regular, and suppose by contradiction that the graph is not a loop. This implies that either there are at least two distinct edges in E^1 or there is a single edge with a finite number of isolated vertices. We separate the proof accordingly to these two cases.

**Case 1:** Suppose that there are two distinct edges, say l, f \in E^1.

Then, by hypothesis, there exists an invertible element u \in D_{l^{-1} \delta_{l^{-1}}} such that (1_l \delta_l)u(1_l \delta_l) = 1_l \delta_l. Write

\[ u = \sum_{j=1}^{n} l_{l^{-1} \delta_{l^{-1}}} q_j, \delta_{l^{-1}} \text{ and } u^{-1} = \sum_{i=1}^{m} 1_l \delta_i, \]

with q_j, p_i \in \mathbb{F} for all i, j. We know that u^{-1}u(1_f \delta_f) = 1_f \delta_f, but since l \neq f we have that

\[ u(1_f \delta_f) = \left( \sum_{j=1}^{n} l_{l^{-1} \delta_{l^{-1}}} q_j \right)(1_f \delta_f) \]

\[ = \alpha_{l^{-1}} \left( \alpha_l \left( \sum_{j=1}^{n} l_{l^{-1} \delta_{l^{-1}}} q_j \right) 1_f \right) \delta_{l^{-1} f} \]

\[ = \sum_{j=1}^{n} \alpha_{l^{-1}} (l l_{l^{-1} \delta_{l^{-1}}} q_j) \delta_{l^{-1} f} \]

\[ = \sum_{j=1}^{n} \alpha_{l^{-1}} (q_j \ delta_{l^{-1} f}) = 0, \]

where we note that l l_{l^{-1} \delta_{l^{-1}}} q_j = 0 for all j \in \{1, \cdots, n\} since X_f \cap X_e = \emptyset.

The above implies that u^{-1}u(1_f \delta_f) = u^{-1}0 = 0 = 1_f \delta_f and hence X_f = \emptyset, a contradiction.
Case 2 Suppose that $E$ has a single edge $l$ together with a finite number of isolated vertices. We show that the graph $E$ cannot have sinks. Let $u$ be as in Case 1. Observe that

$$u^{-1}u = \left(\sum_{i=1}^{n} l_{i}1_{p_{i}} \delta_{l} \right) \left(\sum_{j=1}^{n} l_{j-1}1_{q_{j}} \delta_{l-1} \right).$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{i}(\alpha_{i-1}(1_{l}1_{p_{i}})1_{l-1}1_{q_{j}}) \delta_{e}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} l_{i}1_{p_{i}}1_{q_{j}} \delta_{e},$$

which gives

$$\sum_{i=1}^{m} \sum_{j=1}^{n} l_{i}1_{p_{i}}1_{q_{j}} \delta_{e} = \sum_{v \in E^{0}} 1_{v} \delta_{e} = 1\text{D}(X) \rtimes_{\alpha} \mathbb{F}.$$  

Hence,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} l_{i}1_{p_{i}}1_{q_{j}} = \sum_{v \in E^{0}} 1_{v}$$

and taking $w \in E^{0}$ a sink with $s(l) \neq w$ we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} l_{i}1_{p_{i}}1_{q_{j}}(w) = 0,$$

but $\sum_{v \in E^{0}} 1_{v}(w) = 1_{w}(w) = 1$, which gives a contradiction. From this we get that $E$ is a graph with one edge and one vertex (note that $s(e) = r(e)$ otherwise $r(e)$ is a sink), that is, $E$ is a loop.

Conversely, assume that the graph is a loop with $E^{1} = \{e\}$ and $E^{0} = \{v\}$. Note that $X = \{e^{\infty}\}$ and $X_{p} = X_{v} = X_{p^{-1}}$ for all $p \in \mathbb{F}$, which implies that $1_{p} = 1_{v} = 1_{p^{-1}}$ for all $p \in \mathbb{F}$. Furthermore, $1_{v} \delta_{e}$ is the identity of $D(X) \rtimes_{\alpha} \mathbb{F}$. With this in mind, notice that for each $p \in \mathbb{F}$, we have that $(1_{p} \delta_{p})(1_{p^{-1}} \delta_{p^{-1}})(1_{p} \delta_{p}) = 1_{p} \delta_{p}$ where $1_{p^{-1}} \delta_{p^{-1}}$ is invertible with inverse $1_{p} \delta_{p}$. Hence, $D(X) \rtimes_{\alpha} \mathbb{F}$ is graded unit-regular.

Combining Propositions 9, 10 and 11, we obtain our main theorem.

**Theorem 12.** Let $E$ be a graph with at least one edge. Consider the Leavitt path algebra $L_{K}(E)$ endowed with the $\mathbb{F}$-grading induced by the isomorphism (2). The following assertions are equivalent.

- $L_{K}(E)$ is strongly graded;
- $L_{K}(E)$ is graded clean;
- $L_{K}(E)$ is graded unit-regular;
- $E$ is a loop.

4. On the isomorphism between $D(X) \rtimes_{\alpha} \mathbb{F}$ and $L_{K}(E)$

We finish this work proposing a slight modification on the partial skew ring $D(X) \rtimes_{\alpha} \mathbb{F}$ constructed in [10], but still obtaining the algebra $L_{K}(E)$. In [10], the construction of the algebra $D(X)$ requires to endogenously add the characteristic functions $1_{v}$, for $v \in E^{0}$, to $D(X)$. In this section, we describe a construction where it is not
Definition 13. A formal series on a $\mathbb{K}$-algebra $A$ is an infinite sum that is considered independent of any notion of convergence and can be manipulated with the usual algebraic operations on series. More precisely, the formal series satisfy the following rules:

- $\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} (a_i + b_i)$;
- $\sum_{i=1}^{\infty} k \cdot a_i = \sum_{i=1}^{\infty} k \cdot a_i$ for all $k \in \mathbb{K}$;
- $\left(\sum_{i=1}^{\infty} a_i\right) \left(\sum_{i=1}^{\infty} b_i\right) = \left(\sum_{i=1}^{\infty} c_i\right)$, where $c_i = \sum_{i=m+n} a_m a_n$.

Following what is done in [10] (and recalled in Section 3), we consider the partial action at the set level given by $(X, \theta_p)_{p \in \mathbb{F}}$ and its induced partial action on the algebra level $(\mathbb{F}(X), \alpha_p)_{p \in \mathbb{F}}$.

Now, let $v \in E^0$. If $v$ is an infinite emitter, then the formal series $\sum_{i=1}^{\infty} 1_{(f_i)_v}$, where $\{(f_1)_v, (f_2)_v, \ldots\}$ is the set of all edges such that $s(f) = v$. Let

$$D(X) = \text{span} \left\{1_p \mid p \in \mathbb{F} \setminus \{0\} \cup \left\{\sum_{i=1}^{\infty} 1_{(f_i)_v} : v \text{ is an infinite emitter} \right\} \right\},$$

(where span means the $\mathbb{K}$-linear span) and for each $p \in \mathbb{F} \setminus \{0\}$, let $D_p \subseteq F(X_p)$ be defined as $1_p D(X)$; that is

$$D_p = \text{span}\{1_p q \mid q \in \mathbb{F}\}.$$

Remark 14. Notice that each element in $D(X)$ can be expressed as a formal series, where all summands are zero except for a finite number of them.

Again, we define $\alpha_p : D_{p^{-1}} \to D_p$ as the restriction of the isomorphisms $\alpha_p$ of the partial action (these isomorphisms do not change for $p \in \mathbb{F} \setminus \{e\}$ and $\alpha_e$ remains as the identity function) and so we get a partial action $\alpha = (D_p, \alpha_p)_{p \in \mathbb{F}}$ and the associated partial skew group ring $D(X) \rtimes_\alpha \mathbb{F}$

Remark 15. Recall that if $v = r(f)$, then $X_v = X_{f^{-1}}$ and so $1_v = 1_{f^{-1}}$.

Next, we will construct the isomorphism between $L_K(E)$ and $D(X) \rtimes_\alpha \mathbb{F}$. We need to replace the functions $1_v$ in the proof of [10 Proposition 3.2]. Let us analyze the possible cases:

- If $v$ is a sink: Since the graph has no isolated vertices, then there exists $f \in E^1$ such that $r(f) = v$. So, we can choose $f_v \in \{g \in E^1 \mid r(g) = v\}$ and thus replace the function $1_v$ by $1_{f^{-1}}$ (note that $1_v = 1_{f^{-1}}$ by Remark 15).
- If $v$ is a regular vertex: In this case, Item (4) in Definition 7 suggests that we replace $1_v$ by the sum $\sum_{i=1}^{n} 1_{(f_i)_v}$, where $\{(f_1)_v, \ldots, (f_n)_v\}$. 

necessary to include such maps. For the reader’s convenience, we recall the notion of formal series below.

**Definition 13.** A formal series on a $\mathbb{K}$-algebra $A$ is an infinite sum that is considered independent of any notion of convergence and can be manipulated with the usual algebraic operations on series. More precisely, the formal series satisfy the following rules:

- $\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} (a_i + b_i)$;
- $\sum_{i=1}^{\infty} k \cdot a_i = \sum_{i=1}^{\infty} k \cdot a_i$ for all $k \in \mathbb{K}$;
- $\left(\sum_{i=1}^{\infty} a_i\right) \left(\sum_{i=1}^{\infty} b_i\right) = \left(\sum_{i=1}^{\infty} c_i\right)$, where $c_i = \sum_{i=m+n} a_m a_n$.

Following what is done in [10] (and recalled in Section 3), we consider the partial action at the set level given by $(X, \theta_p)_{p \in \mathbb{F}}$ and its induced partial action on the algebra level $(\mathbb{F}(X), \alpha_p)_{p \in \mathbb{F}}$.

Now, let $v \in E^0$. If $v$ is an infinite emitter, then the formal series $\sum_{i=1}^{\infty} 1_{(f_i)_v}$, where $\{(f_1)_v, (f_2)_v, \ldots\}$ is the set of all edges such that $s(f) = v$. Let

$$D(X) = \text{span} \left\{1_p \mid p \in \mathbb{F} \setminus \{0\} \cup \left\{\sum_{i=1}^{\infty} 1_{(f_i)_v} : v \text{ is an infinite emitter} \right\} \right\},$$

(where span means the $\mathbb{K}$-linear span) and for each $p \in \mathbb{F} \setminus \{0\}$, let $D_p \subseteq F(X_p)$ be defined as $1_p D(X)$; that is

$$D_p = \text{span}\{1_p q \mid q \in \mathbb{F}\}.$$
• If $v$ is an infinite emitter: We replace $1_v$ by the formal series $\sum_{i=1}^{\infty} 1_{(f_i)_v}$, since $\{f_i \in E^1 \mid s(f_i) = v\} = X_v = \bigcup_{i=1}^{\infty} X_{(f_i)_v}$.

With the above, one can show that the sets $\{1_f \mid f \in E^1\}$, $\{1_{f^{-1}} \mid f \in E^1\}$ and $\{1_{f^{-1}} \delta_f \mid v \text{ is a sink}\} \cup \left(\sum_{i=1}^{n_v} 1_{(f_i)_v} \mid v \text{ is a regular vertex}\right) \cup \left(\sum_{i=1}^{\infty} 1_{(f_i)_v} \mid v \text{ is an infinite emitter}\right)$, satisfy the relations that define the Leavitt path algebra. Using the universal property of $L_k(E)$ (see [2, Remark 1.2.5]), we obtain the desired homomorphism from $L_k(E)$ to $D(X) \rtimes_F \mathbb{F}$. Following analogously to the proof of [10, Proposition 3.2], we conclude that this homomorphism is an isomorphism.

References

[1] G. Abrams, Leavitt path algebras: The first decade, Bull. Math. Sci. 5(1) (2015) 59–120.
[2] G. Abrams, P. Ara and M. S. Molina. Leavitt Path Algebras, Lecture Notes in Mathematics 2191, Springer-Verlag (2017)
[3] G. Abrams and G. Aranda Pino. The Leavitt path algebras of arbitrary graphs, Houston J. Math. 34 (2008), no. 2, 423–442.
[4] G. Abrams, G. Aranda Pino, and M Siles Molina, M.. Finite-dimensional Leavitt path algebras. J. Pure Appl. Algebra 209 (2007) (3):753–762.
[5] G. Abrams, P. N. Ánh, A. Louly and E. Pardo. The classification question for Leavitt path algebras, J. Algebra 320(5):1983–2026.
[6] P. Ara and E. Pardo. Stable rank of Leavitt path algebras. Proc. Amer. Math. Soc. (2008) 136(7):2375–238.
[7] P. Ara, M. A. Moreno and E. Pardo. Nonstable K-theory for graph algebras, Algebr. Represent. Theory 10 (2) (2007) 157–178.
[8] M. Dokuchaev. Recent developments around partial actions. SãO Paulo J. Math. Sci. 13, 195–247 (2019)
[9] M. Dokuchaev and R. Exel. Associativity of crossed products by partial actions, enveloping actions and partial representations. Trans. Amer. Math. Soc. 357, 1931–1952 (2005).
[10] D. Gonçalves, and D. Royer. Leavitt path algebras as partial skew group rings. Comm. Algebra, 42(8), 3578-3592 (2014).
[11] D. Gonçalves, and D. Royer, Simplicity and chain conditions for ultragraph Leavitt path algebras via partial skew group ring theory. J. Aust. Math. Soc., 109(3), 299-319, (2020).
[12] D. Gonçalves, and D. Royer. (2014). Representations and the reduction theorem for ultragraph Leavitt path algebras. J. Algebraic Combin., 53, 505-526, (2021).
[13] D. Gonçalves, and D. Royer. Properties of the gradings on ultragraph algebras via the underlying combinatorics. To appear at J. Algebraic Comb.[arXiv:2107.01227][math.RA] (2021).
[14] R. Hazrat. The graded structure of Leavitt path algebras. Israel J. Math. 195(2) (2013) 833–895.
[15] P. Lundstrom, and J. Öinert. Strongly graded Leavitt path algebras. J. Algebra Appl., 2250141, (2021).
[16] L. Martínez, H. Pinedo and Y. Soler. Ring theoretical properties of epsilon-strongly graded rings and Leavitt path algebras. Comm. Algebra, 50(8), 3201–3217, (2022).
[17] P. Nystedt and J. Öinert. Group gradations on Leavitt path algebras. J. Algebra Appl. 19(9):2050165, (2020).
[18] P. Nystedt and J. Öinert. Strongly graded Leavitt path algebras. J. Algebra Appl. 21(7), 2250141, 9 pp, (2022).
[19] P. Nystedt, J. Öinert and H. Pinedo. Artinian and noetherian partial skew groupoid rings, *J. Algebra* 503, 433–452, (2018).

[20] L. Vaš. Cancellation properties of graded and nonunital rings. Graded clean and graded exchange Leavitt path algebras. *J. Algebra Appl.*, 2350050, (2021).