Existence of weak solutions to a generalized nonlinear multi-layered fluid-structure interaction problem with the Navier-slip boundary conditions

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Abstract

We consider a fluid-structure interaction problem with Navier-slip boundary conditions in which the fluid is considered as a Non-Newtonian fluid and the structure is described by a nonlinear multi-layered model. The fluid domain is driven by a nonlinear elastic shell and thus is not fixed. Therefore, to analyze the problem, we map the fluid problem into a fixed domain by applying an arbitrary Lagrange Euler mapping. Unlike the classical method by which we consider the problem as its entirety, we utilize the time-discretization and split the problem into a fluid subproblem and a structure subproblem by an operator splitting scheme. Since the structure subproblem is nonlinear, Lax-Milgram lemma does not hold. So we prove the existence and uniqueness by means of the traditional semigroup theory. Noticing that the Non-Newtonian fluid possesses a \( p - \)Laplacian structure, we show the existence and uniqueness of solutions to the fluid subproblem by considering the Browder-Minty theorem. With the uniform energy estimates, we deduce the weak and weak* convergence respectively. By a generalized Aubin-Lions-Simon Lemma, we obtain the strong convergence.

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1 Introduction

This paper deals with a generalized multi-layered fluid-structure interaction problem with Navier-slip boundary conditions. More precisely, the problem consists of a generalized fluid, a nonlinear thin structure and a thick structure.

1.1 Model description

We consider a half cylindrical fluid domain \( \Omega_F(t) \) composed by a moving boundary \( \Gamma^\eta(t) \) and three rigid boundaries \( \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_h =: \Sigma \), i.e., \( \partial \Omega_F(t) = \Gamma^\eta(t) \cup \Sigma \) (see Figure 1). The displacement of thin structure is depicted by \( \eta : [0, T] \times \Gamma \rightarrow \mathbb{R}^2 \). Assume that the length of

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fluid domain is \( L \) and the reference radius is \( r = 1 \). Then we have the parameterized fluid domain as

\[
\Omega^\eta_F(t) = \{(z, r) \in \mathbb{R}^2 : z \in (0, L), r \in (0, 1 + \eta \cdot \nu)\},
\]

where \( \nu = (0, 1) \) and the interface boundary as

\[
\Gamma^\eta(t) = \{(z, r) \in \mathbb{R}^2 : z \in (0, L), r = 1 + \eta \cdot \nu\}.
\]

![Figure 1: Geometry of fluid-structure interaction problem](image)

Subsequently, we model fluid motion by the two dimensional incompressible Navier-Stokes equations in \( \Omega^\eta_F(t) \):

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= \nabla \cdot \sigma \quad \text{in } (0, T) \times \Omega^\eta_F(t), \\
\nabla \cdot u &= 0
\end{align*}
\]

where \( u \) is the fluid velocity, \( \sigma = -\pi I + 2S(D(u)) \) denotes the stress tensor, \( \pi \) is the fluid pressure. \( S(D(u)) = \mu_F(D(u))D(u) \) represents the viscous effects with viscosity \( \mu_F(D(u)) \), which is a nonlinear term. \( D(u) = \frac{1}{2}(\nabla u + \nabla^T u) \) is the symmetrized gradient. In this manuscript, we consider the Non-Newtonian fluid whose viscosity is the so called ‘power law’ proposed by Carreau in his Ph.D. Thesis (see also e.g., \[1, 11\]):

\[
\mu_F(D(u)) = (1 + |D(u)|^2)^{\frac{p-2}{2}}, \quad 2 < p < \infty.
\]

The associated initial data of problem (1) is

\[
u(0, \cdot) = u_0.
\]

On the rigid part of the fluid domain boundaries, we have

\[
\begin{align*}
u \cdot \nu_F &= 0 \quad \text{on } (0, T) \times \Gamma_b, \\
\partial_{\nu} u_{\nu} &= 0
\end{align*}
\]

\[
\begin{align*}
\pi + \frac{1}{2} |u|^2 &= P_{\text{in/out}}(t) \quad \text{on } (0, T) \times \Gamma_{\text{in/out}}, \\
u \cdot \tau_F &= 0
\end{align*}
\]

where \( \nu_F \) and \( \tau_F \) are outer normal and tangential vectors of fluid domain, respectively, and \( P_{\text{in/out}}(t) \) denotes \( P_{\text{in}}(t) \) or \( P_{\text{out}}(t) \), provides “inlet” or “outlet” boundary data.
On the elastic part of fluid domain boundary, interaction between $u$ and $\eta$ occurs. Let \( \Gamma = \Gamma^0(0) = (0, L) \) be the Lagrangian domain. Then the thin structure elastodynamic problem is given by

\[
\begin{align*}
\partial_{tt}\eta + \mathcal{L}_e \eta + f(\eta) &= h, & \text{on } (0, T) \times \Gamma, \\
\eta &= 0, & \text{on } (0, T) \times \partial\Gamma,
\end{align*}
\]

where \( f \) is a nonlinear term that will be assigned later. \( \mathcal{L}_e \) is a continuous, self-adjoint, coercive, linear operator defined on \( [H^2_0(\Gamma)]^2 \) such that

\[
\langle \mathcal{L}_e \eta, \eta \rangle \geq \delta_0 \| \eta \|_{[H^2_0(\Gamma)]^2}^2, \quad \forall \ \eta \in [H^2_0(\Gamma)]^2
\]

with \( \delta_0 \) a positive constant.

Moreover, follows are the coupled conditions:

- The kinematic condition:

\[
\begin{align*}
\boldsymbol{u} \cdot \nu_F = \partial_t \eta \cdot \nu_F & \quad \text{(Continuity of normal velocity on } \Gamma^0(t)\text{)}, \\
(\partial_t \eta - \boldsymbol{u}) \cdot \tau_F = \alpha \sigma \nu_F \cdot \tau_F & \quad \text{(Slip effect on } \Gamma^0(t)\text{)}.
\end{align*}
\]

- The dynamic coupling condition:

\[
h = -J_F^0 \sigma \nu_F - S \nu_S, \quad \text{on } (0, T) \times \Gamma.
\]

where \( \alpha > 0 \) is the ratio constant of the slip effect. \( J_F^0 \) denotes the Jacobian of transformation from Eulerian to Lagrangian formulations. \( S \) is the stress acted on thin structure from thick structure and \( \nu_S \) is the outer normal vector of thick structure. We note here that on the interface, \( \nu_F = -\nu_S \).

The other side of thin structure is the thick structure with thickness \( H \). We define the thick structure domain as

\[
\Omega_S = (0, L) \times (1, 1 + H),
\]

with boundary \( \partial\Omega_S = \Gamma \cup \Gamma_{\text{in/out}}^S \cup \Gamma_{\text{top}} \). Using Lagrangian formulation, the motion of thin layer defined on \( \Omega_S \) is described by a linear elastic equation:

\[
\partial_{tt} \boldsymbol{d} = \nabla \cdot \boldsymbol{S}, \quad \text{in } (0, T) \times \Omega_S,
\]

with boundary conditions:

\[
\begin{align*}
\boldsymbol{d} &= \eta, & \text{on } (0, T) \times \Gamma, \\
\boldsymbol{d} &= 0, & \text{on } (0, T) \times \Gamma_{\text{in/out}}, \\
S \nu_S &= 0, & \text{on } (0, T) \times \Gamma_{\text{top}},
\end{align*}
\]

where \( \boldsymbol{S} \) is the first Piola-Kirchhoff stress tensor given by \( \boldsymbol{S} = 2\mu_S \boldsymbol{D}(\boldsymbol{d}) + \lambda (\nabla \cdot \boldsymbol{d}) \mathbb{I} \) and \( \mu_S > 0 \) is the viscosity of thick structure.

In addition, problem (1)–(11) satisfies the initial conditions

\[
\begin{align*}
\boldsymbol{u}(0, \cdot) &= \boldsymbol{u}_0, \quad \eta(0, \cdot) = \eta_0, \quad \partial_t \eta(0, \cdot) = \nu_0, \quad \boldsymbol{d}(0, \cdot) = \boldsymbol{d}_0, \quad \partial_t \boldsymbol{d}(0, \cdot) = \boldsymbol{V}_0,
\end{align*}
\]

and necessary compatibility conditions \[32\].
• The initial fluid velocity must satisfy:
\[
\begin{align*}
\mathbf{u}_0 &\in L^2(\Omega_F^0)^2, \quad \nabla \cdot \mathbf{u}_0 = 0, \quad \text{in } \Omega_F^0, \\
\mathbf{u}_0 \cdot \nu &= 0, \quad \text{on } \Gamma_b, \\
\mathbf{u}_0 \cdot \nu_0 &= \mathbf{v}_0 \cdot \nu_0, \quad \text{on } \Gamma^0,
\end{align*}
\]
where \( \Omega_F^0 = \Omega_F^0(0), \Gamma^0 = \Gamma^0(0), \nu_0 = \nu(0, \cdot) \).

• The initial domain must be such that there exists a diffeomorphism \( \varphi^0 \in C^1(\Omega)_F \) such that
\[
\varphi^0(\Omega_F) = \Omega_F^0, \quad \det \nabla \varphi^0 > 0, \quad (\varphi - I)|_\Gamma = \eta_0,
\]
and the initial displacement \( \eta_0 \) is such that
\[
\| \eta_0 \|_{H^{11/6}} \leq c, \quad \text{where } c \text{ is small.}
\]

1.2 Motivation

In recent years, mathematical problems of fluid-structure interaction have been studied continuously. These problems arise from several applications in different fields, such as biomechanics, blood flow dynamics, aeroelasticity, hydroelasticity and so on. There are many research works that are investigating the problems from different aspects. For fluid-elastic interaction problem with strong solutions results, Beirão da Veiga [2] consider 2D fluid and 1D linear elastic model with periodic boundary conditions. They proved the local strong solutions by the linearization of fluid equation and fixed point theorem. Coutand and Shkoller showed the existence of a unique regular local solution of a 3D fluid-3D structure (linear [9] or quasi-linear [10] elasticity) immersed in the fluid. Then Cheng and Shkoller [8] extended the strong solution results to nonlinear elastic Koiter shell. Lequeurre [24, 25] expanded the results of Beirão da Veiga. They removed the small condition on initial data, obtained the local strong solution results and showed the globally existence of strong solutions with small initial data. Later, Grandmont and Hillairet [13] considered a 2d fluid equation coupled with elastic beam equation and obtain the globally strong solution by means of a regularized system. Subsequently, Grandmont, Hillairet and Lequeurre [14] generalized the beam equation in [13] with different parameters combination. They improved the existence result of strong solutions with the same regularity of initial data by a regularization method. More results concerning strong solution can be found in [18, 19, 20, 21, 36, 37].

For fluid-elastic interaction problem with weak solutions, there are also many interesting results. Chambolle, Desjardins, Esteban and Grandmont [6] investigated an interaction problem between an incompressible fluid and a viscoelastic plate in 3D. After that, Grandmont [12] studied a 2D incompressible fluid interacted with a 1D elastic plate. They construct a perturbed viscoelastic term in plate equation and analyzed the limiting problem. In [22] Lengeler and Růžička took a nature method to discuss the compactness issue in a 3D fluid-Koiter shell interaction problem. All the results above were obtained by constructing a regularized system and making energy estimate so that to pass variables to the limit with compactness principle. At the same time, Muha and Čanić [5, 28, 29, 30, 32] studied the existence of weak solutions to a series of different fluid-structure interaction problems involving incompressible viscous fluid. They came up with a numerical-like way to prove this type of results inspired by the numerical scheme in [15]. Their methods includes taking Arbitrary Lagrangian Euler mapping (ALE) to fix moving fluid domain, splitting the problem by Lie operator splitting, constructing approximation solutions with the idea of time-discrete iterative solution and proving the existence of fluid subproblem and structure subproblem respectively, so as to show the existence of the weak solution according to the compactness principle. More specifically, they
discussed the interaction between 2D case in [29], while in [28, 31] 3D cylindrical case, and linear, nonlinear Koiter shell equations were studied respectively. In [30], they consider a more realistic model associated with human arteries vessel which contains multi-layers. The model was abstracted into a 2D fluid interacting with a multi-layered structure including a 1D thin and a 2D thick elastic structure. In [32], Muha and Čanić considered different boundaries and took Navier-slip boundary into account in the system, which will allow both longitudinal and tangential components of displacement. The system was simplified to a two-dimensional case for subsequent analysis. Later, Trifunović and Wang combined this method with a hybrid approximation scheme to deal with a 3D incompressible fluid coupled with a nonlinear plate equation. They used the Galerkin method for the structure subproblem and passed both spatial variable $k$ and time discrete variable $\Delta t$ to the limit and obtained the existence of the weak solutions. The latest work was [5] done by Čanić, Galić and Muha. They addressed a 3D nonlinear fluid-mesh-shell interaction problem with moving boundary, in which they added a net of 1D hyperbolic equations to model the elastodynamics of an elastic mesh of curved rods. The results improved the simple problem [4] defined in a fixed fluid domain.

In this paper, we consider a generalized multi-layered fluid-structure interaction problem with Navier-slip boundary condition, in which 2D Non-Newtonian fluid is bounded on one side by a nonlinear thin structure and a linear elastic thick structure, while the interaction of fluid and thin structure is driven by Navier-slip effects. In all the studies mentioned above, the viscosity of fluid was treated as a constant. However, in nature, ideal fluid is almost nonexistent, which means the viscosity of the fluid decreases with the increase of the shear strain rate (pseudoplastic, shear thinning), while in other case it behaves just the opposite (Dilatant, shear thickening). Therefore, we want to study the fluid-structure interaction problem for Non-Newtonian fluid and multi-layered structure in order to model blood flow in human artery. For the work of Non-Newtonian fluid-structure interaction problems, we notice that Lengeler [23] generalized the viscous Newtonian fluid in [22] to a Non-Newtonian fluid interacting with a linear elastic Koiter shell. They introduced a shear-dependent viscosity, which obeys “power law”, and resolved the issue of additional stress (Non-Newtonian) limit. Finally, they used the regularized system to obtain the relative compact for $p > \frac{3}{2}$. In [17], Hundertmark-Zaušková, Lukáčová-Medviďová and Nečasová set $p > 2$ in power law, which means the fluid is shear-thickening, and investigated the existence of weak solutions by fixed point procedure. The techniques dealing with Non-Newtonian limit also can be found in [11, 26, 40].

1.3 Methodology and features

In present work, we analyze (1)–(15) by the method proposed by Muha and Čanić. More specifically, there are following steps:

- Applying the ALE mapping to the problem and obtaining the weak formulation in fixed reference fluid domain $\Omega_F$, see Section 2.3
- Taking Lie splitting method to decompose system (1)–(15) into a fluid subproblem and a structure subproblem, showing the existence and uniqueness for both subproblems in each time subinterval and deriving the uniform estimates, see Section 3.1–3.5
- Concluding the weak and weak* convergences from the uniform boundedness, see Section 3.6–3.7
- Combining the compactness Lemma to compact embeddings to derive the strong convergences of velocities, displacement and geometry parameters, see Section 3.8
• Passing to the limit as $N \to \infty$, see Section 4.

Though our framework may be similar to the works done by Muha and Čanić, we have the following unique features in solving our problem:

(i) Muha and Čanić used Lax-Milgram Theorem to show the existence and uniqueness of subproblem in [28, 29, 30, 32]. However, in our paper, Lax-Milgram Theorem does not hold due to the nonlinear term $f(\eta)$. Unlike [39], in which Galerkin method was used, we prove the existence and uniqueness of solutions to system [46] by means of the traditional semigroup theory [35].

(ii) Since the Non-Newtonian constitutive relation is nonlinear, we can not apply the standard Lax-Milgram Lemma. Muha and Čanić made use of the Schaefer’s fixed point theorem in [31] to deal with the nonlinear Koiter membrane. We notice the $p$-structure of fluid subproblem and show that using the Browder-Minty theorem [7, Theorem 9.14–1] for $p > 2$ works well in our problem.

(iii) When we summarize the weak and weak* convergence, we get obtain the $L^p$ weak convergence for symmetrized gradient $D^{\eta_N}(u_N)$ and $L^q$ weak convergence ($\frac{1}{p} + \frac{1}{q} = 1$) for $S(D^{\eta_N}(u_N))$. Their limit cannot be deduced directly due to the variant subscript $\eta_N$ (related to fluid domain) and nonlinearity from $S$. We modified the proof of Proposition 7.6 in [30] by introducing the localized Minty’s Trick to obtain the limit.

2 Preliminaries and main result

Since the problem (1)–(15) is defined in a moving fluid domain which is part of unknowns, we can not define its weak solutions directly. To overcome this difficult, we introduce an Arbitrary Lagrangian Eulerian (ALE) mapping, which is common in numerical simulations of fluid-structure interaction problems. This maps our problem to a fixed domain so that we can carry out our analysis. There are many applications of ALE mapping in fluid-structure problems, see e.g., [5, 28, 29, 30, 31, 32, 33].

Before we perform the ALE mapping in Section 2.3, we provide some useful facts and assumptions in Section 2.1 and the energy differential inequality associated with (1)–(15) in Section 2.2. Section 2.4 and 2.5 are devoted to transformation and space settings, respectively. Finally, we present our main result (Theorem 2.1) in Section 2.6.

2.1 Some useful facts and assumptions

Lemma 2.1. It can be easily checked that for $p > 2$, $S$ satisfies [11]

1. Coercivity:
   
   \[ S(D) : D \geq \kappa_1 D^p - \kappa_2; \quad (16) \]

2. Growth:
   
   \[ |S(D)| \leq \kappa_3 \left(|D|^{p-1} + 1\right); \quad (17) \]

3. Monotonicity:
   
   \[ (S(D_1) - S(D_2)) : (D_1 - D_2) > 0, \text{ if } D_1 \neq D_2. \quad (18) \]

Here, $D_1 = D(u_1)$, $D_2 = D(u_2)$ and the notations $\kappa_i$, $i = 1, 2, 3$ are constants, depending at most on $p$, such that $\kappa_i > 0$, $i = 1, 3$ and $\kappa_2 \in \mathbb{R}$. We remark that these three results will used in the proof of existence of the system.
Assumptions:

(f1) \( f \) is locally Lipschitz from \( H^{2-\epsilon} \) into \( H^2 \), i.e.
\[
\| f(\eta_1) - f(\eta_2) \|_{H^2(\Gamma)} \leq C_R \| \eta_1 - \eta_2 \|_{H^{2-\epsilon}(\Gamma)},
\]
for some \( \epsilon > 0 \), a constant \( C_R > 0 \) and for every \( \| \eta_i \|_{H^{2-\epsilon}(\Gamma)} \leq R \) (\( i = 1, 2 \)).

(f2) \( f \) has a potential in \( H^2(\Gamma) \), i.e., there is a Fréchet differentiable functional \( \Pi(\eta) \) in \( H^2(\Gamma) \) such that \( \Pi'(\eta) = \langle f(\eta), \partial\eta \rangle \).

Remark 2.1. Since the bound of approximate solutions \( \eta_N \) is obtained in \( H^2(\Gamma) \), the assumption (f1) is used to pass to the limit of nonlinear term \( f(\eta_N) \). We note here that this assumption is a little bit stronger for the nonlinear term \( f(\eta) \), while in [39], Trifunović and Wang make two weaker Lipschitz assumptions for \( f(\eta) \) from \( H^2(\Gamma) \) into \( H^{-2}(\Gamma) \) for some \( \epsilon > 0 \) and \( H^2(\Gamma) \) into \( H^{-a}(\Gamma) \) for some \( 0 \leq a < 2 \), i.e.,
\[
\| f(\eta_1) - f(\eta_2) \|_{H^{-2}} \leq C_R \| \eta_1 - \eta_2 \|_{H^{2-\epsilon}}(\Gamma) \quad (19a)
\]
\[
\| f(\eta_1) - f(\eta_2) \|_{H^{-a}} \leq C_R \| \eta_1 - \eta_2 \|_{H^2}(\Gamma) \quad (19b)
\]

(f1) is used to pass the convergence of \( f(\eta) \) when the bound of approximate solutions \( \eta_N \) is obtained in \( H^2(\Gamma) \). It is a weaker Lipschitz condition than (f1). (19b) depicts the order of nonlinearity precisely and is useful in determining the minimal time, which is related to the approximate solution convergence. This requirement is from the Galerkin approximation method in [39] to handle the nonlinear structure subproblem. However, instead of using this "hybrid approximation scheme" as in [39], we choose the classical semigroup method in our study for the structure subproblem. Thus, we make a stronger assumption and need only one Lipschitz condition.

2.2 Energy differential inequality

For simplicity, we denote the bilinear form associated with the elastic energy of the thick structure by
\[
a_S(d, \psi) = \int_{\Omega_S} 2\mu_S D(d) : D(\psi) + \lambda(\nabla \cdot d)(\nabla \cdot \psi).
\]

Remark 2.2. For any two vectors \( d_1 \) and \( d_2 \), we have
\[
a_S(d_1, d_1 - d_2) = \frac{1}{2} (a_S(d_1, d_1) + a_S(d_1 - d_2, d_1 - d_2) - a_S(d_2, d_2))
\]
where we have used \( A : (A - B) = \frac{1}{2} (A : A + (A - B) : (A - B) - B : B) \) and \( a(a - b) = \frac{1}{2}(a^2 + (a - b)^2 - b^2) \). Here \( A, B \) are vectors and \( a, b \) are scalar quantities.

Next, we derive the energy for (1)–(15). We first multiplying (1), (4), (8) by \( u, \partial_t \eta, \partial_t d \) and integrate by parts over \( \Omega_S \), \( \Gamma, \Omega_S \), respectively. Then we add the results and combine with boundary conditions to obtain
\[
\frac{d}{dt} \| u \|_{L^2(\Omega_S)}^2 + \| \partial_t \eta \|_{L^2(\Gamma)}^2 + \| \partial_t d \|_{L^2(\Omega_S)}^2 + a_S(d, d) + 2\Pi(\eta) \leq C,
\]
where energy \( E(t) \) and dissipation \( D(t) \) are denoted by
\[
E(t) := \frac{1}{2} \left( \| u(t) \|_{L^2(\Omega_S)}^2 + \| \partial_t \eta(t) \|_{L^2(\Gamma)}^2 + \| \partial_t d(t) \|_{L^2(\Omega_S)}^2 + a_S(d, d) + 2\Pi(\eta) \right),
\]
\[
D(t) := \| u \|_{W^{1,p}(\Omega_S)}^p + \frac{1}{\alpha} \left( \| \partial_t \eta(t) - u(t) \|_{L^2(\Gamma)}^2 \right),
\]
\( C \) is determined by the boundary data.
2.3 The ALE mapping

Due to the Navier-slip effects, we consider both radial (vertical) and longitudinal displacements in this study and these create some additional issue when we pass to the limit (see e.g., Section 4). Thus, we follow the procedure in [32, Section 3.2] to define our ALE mapping. First, we denote the corresponding deformation of the elastic boundary by \( \varphi^\eta \), i.e.,

\[
\varphi^\eta(t, z) = \text{id} + \eta(t, z), \quad (t, z) \in [0, T] \times \Gamma.
\]

Then we introduce a family of ALE mappings \( A_\eta \) parameterized by \( \eta \):

\[
A_\eta(t) : \Omega \rightarrow \Omega^\eta(t),
\]

\[
(z, r) \mapsto (x, y) = A_\eta(t)(z, r),
\]

and it is defined for each \( \eta \) as a harmonic extension of deformation \( \varphi^\eta \). This means that \( A_\eta(t) \) is the solution of the following boundary value problem on the reference domain \( \Omega_F \):

\[
\Delta A_\eta(t, \cdot) = 0 \quad \text{in} \ \Omega_F,
\]

\[
A_\eta(t)|_\Gamma = \varphi^\eta(t, \cdot),
\]

\[
A_\eta(t)|_{\Sigma} = \text{id},
\]

where \( \Sigma = \partial \Omega_F \setminus \Gamma \) denotes the rigid part of the boundary. The Jacobian of ALE mapping \( A^\eta \) is defined by

\[
J^\eta = \det \nabla A^\eta(t),
\]

and the ALE velocity is

\[
w^\eta = \frac{d}{dt} A_\eta.
\]

From [32], we know that if the compatibility conditions (13), (14), (15) hold, then it can be deduced that there exists a \( T' > 0 \) such that

\[
J^\eta \geq C > 0, \quad \text{on} \quad (0, T') \times \Omega_F.
\]

In addition, there exists a \( T'' > 0 \) such that for every \( t \in [0, T''] \), the ALE mapping \( A_\eta(t) \) is an injection.

We note here that both conditions \( J^\eta > 0 \) and \( A_\eta \) is injective are to make sure that the fluid domain is not degenerate for some time. Let \( T = \min \{T', T''\} \), then this new time determines the maximal existence time interval for the weak solution. In this sense, our weak solution exists for a maximal time \( T \), at which either \( J^\eta = 0 \) or \( A_\eta \) is no longer injective (see e.g., [32, Fig. 3]).

2.4 Transformation settings

In order to define our associated weak solutions in the fixed domain \( \Omega_F \), we map the functions on the moving domain \( \Omega^\eta(t) \) onto the reference domain \( \Omega_F \) by the ALE mapping given above in Section 2.3. For the function depending on \( \eta \), we make use of a superscript \( \eta \) to denote it. Specifically, for a function \( g \) defined on \( \Omega^\eta(t) \), whether it is a scalar or a vector, we denote it by

\[
g^\eta(t, z, r) = g(t, A_\eta(t)(z, r)).
\]

Correspondingly, the gradient and divergence are defined by

\[
\nabla^\eta g^\eta := (\nabla g)^\eta = \frac{\nabla g^\eta}{\nabla A_\eta}, \quad \nabla^{\eta, n} = \text{tr} (\nabla^\eta g^\eta)
\]

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and the symmetrized gradient is denoted by

\[ D^n(g^n) := \frac{1}{2} (\nabla^n g^n + (\nabla^n g^n)^\top) \]

\[ = \frac{1}{2} \left( (\nabla g)^n + ((\nabla g)^n)^\top \right) \]

\[ = \frac{1}{2} \left( \frac{\nabla g^n}{\nabla A^n} + \frac{(\nabla g^n)^\top}{\nabla A^n} \right) \]

\[ = \frac{D(g^n)}{\nabla A^n}. \]  \( (23) \)

Moreover, we define the ALE derivative on the fixed reference domain \( \Omega_F \):

\[ \partial_t g|_{\Omega_F} = \partial_t g + (w^n \cdot \nabla) g. \]  \( (24) \)

Consequently, we can rewrite the Navier-Stokes equations in the ALE formulation as:

\[ \partial_t u|_{\Omega_F} + ((u - w^n) \cdot \nabla) u = \nabla \cdot \sigma, \quad \text{in } \Omega_F^0(t), \]  \( (25) \)

where \( \partial_t u|_{\Omega_F} \) and \( w^n \) are composed with \( (A^n(t))^{-1} \) and we find that transformed divergence-free condition is

\[ \nabla^n \cdot u^n = 0. \]

Under the above transformation with ALE mapping, we give the space settings related to weak solutions of problem \( (1)-(15) \) in the next section.

### 2.5 Space settings

We denote Lebesgue-Sobolev spaces \([W^{k,p}]^2, [H^p]^2\) and \([L^p]^2\) by \( W^{k,p}, H^p \) and \( L^p \) respectively. Now, we describe the functional spaces associated with the weak solutions of problem \( (1)-(15) \).

Motivated by the energy inequality \( (20) \) and the boundary conditions, we denote the four spaces of fluid velocity, “improved” fluid velocity, thin structure displacement and thick structure displacement, respectively by

\[ V^n_F = \left\{ u^n \in \left[ C^1(\Omega) \right]^2 : \nabla^n \cdot u^n = 0, u^n \cdot \tau^n = 0, \text{ on } \Gamma_{\text{in/out}}, \right\}. \]  \( (26) \)

\[ V^n_F = V^n_F \cap W^{1,p}(\Omega_F). \]  \( (27) \)

\[ V_W = H^2_0(\Gamma). \]  \( (28) \)

\[ V_S = \left\{ d \in H^1(\Omega_S) : d \cdot \tau_S = 0, \text{ on } \Gamma, d = 0, \text{ on } \Gamma_{\text{in/out}} \right\}. \]  \( (29) \)

Subsequently, the associated evolution spaces for fluid, thin structure and thick structure can be written as:

\[ W^n_F = L^\infty(0,T; L^2(\Omega_F)) \cap L^p(0,T; V^n_F), \]  \( (30) \)

\[ W_W = W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^2(0,T; V_W), \]  \( (31) \)

\[ W_S = W^{1,\infty}(0,T; L^2(\Omega_S)) \cap L^2(0,T; V_S). \]  \( (32) \)

Including the slip boundary condition, we have the following solution space:

\[ W^n = \{ (u, \eta, d) \in W^n_F \times W_W \times W_S : u \cdot \nu^n_F = \partial_t \eta \cdot \nu^n_F, d \cdot \nu_S = \eta \}. \]  \( (33) \)

The corresponding test space is denoted by

\[ Q^n = \left\{ (q, \phi, \psi) \in C^1_c([0,T), V^n_F \times V_W \times V_S) : q^n \cdot \nu^n_F = \phi \cdot \nu^n_F, \phi = \psi, \text{ on } \Gamma \right\}. \]  \( (34) \)
2.6 Weak solutions and main result

Since the remainder analysis are based the problem defined on the reference domain $\Omega_F$, we need to transform the problem \((1) - (15)\) by ALE mapping. To establish the definition of the weak solutions, we first consider the transformed Navier-Stokes equation \((25)\). Multiplying \((25)\) by $q$ and integrating over $\Omega_F^\eta(t)$, we have

$$\int_0^T \int_{\Omega_F^\eta(t)} \partial_t u|_{\Omega_F} \cdot q + \int_0^T \int_{\Omega_F^\eta(t)} ((u - w^\eta) \cdot \nabla) u \cdot q = \int_0^T \int_{\Omega_F^\eta(t)} \nabla \cdot \sigma \cdot q, \quad (35)$$

where we have dropped the superscript $\eta$ in $u^\eta$ for easier reading.

Integrating by parts of the second term on the left-hand side of \((35)\) we get

$$\int_0^T \int_{\Omega_F^\eta(t)} ((u - w^\eta) \cdot \nabla) u \cdot q = \frac{1}{2} \int_0^T \int_{\Omega_F^\eta(t)} ((u - w^\eta) \cdot \nabla) u \cdot q - \frac{1}{2} \int_0^T \int_{\Omega_F^\eta(t)} ((u - w^\eta) \cdot \nabla) q \cdot u$$

$$+ \int_0^T \int_{\Omega_F^\eta(t)} (\nabla \cdot w^\eta) u \cdot q + \int_0^T \int_{\Gamma^\eta(t)} (u - w^\eta) \cdot \nu^\eta_F (u \cdot q),$$

For the term on the right-hand side of \((35)\), it follows from the divergence theorem that

$$- \int_0^T \int_{\Omega_F^\eta(t)} \nabla \cdot \sigma \cdot q = 2 \int_0^T \int_{\Omega_F^\eta(t)} \mathbb{S}(D(u)) : D(q) - \int_0^T \int_{\partial \Omega_F^\eta(t)} \sigma \nu^\eta_F \cdot q.$$

Due to the slip effect, which leads to the non-zero tangential and normal component of velocity at interface boundary, we expand the last term above as

$$\int_0^T \int_{\partial \Omega_F^\eta(t)} \sigma \nu^\eta_F \cdot q = \int_{\Gamma^\eta(t)} \left( (\sigma \nu^\eta_F \cdot \nu^\eta_F) q \cdot \nu^\eta_F + (\sigma \nu^\eta_F \cdot \tau^\eta_F) q \cdot \tau^\eta_F \right)$$

$$+ \int_{\Gamma_{in/out}} \pi q \nu^\eta_F$$

$$= \int_{\Gamma^\eta(t)} \left( (\sigma \nu^\eta_F \cdot \nu^\eta_F) \phi \cdot \nu^\eta_F + \frac{1}{\alpha} (\partial_t \eta - u) \cdot \tau^\eta_F (q \cdot \tau^\eta_F) \right)$$

$$+ \int_{\Gamma_{in/out}} \pi q \nu^\eta_F. \quad (36)$$

We note here that the first term on the right-hand side cancels with the same term in thin structure equation. By means of integration by parts, we obtain

$$\int_0^T \int_{\Omega_F^\eta(t)} \partial_t u|_{\Omega_F} \cdot q = \int_0^T \int_{\Omega_F} J^\eta \partial_t u \cdot q^\eta$$

$$= - \int_0^T \int_{\Omega_F} \partial_t J^\eta u \cdot q^\eta - \int_0^T \int_{\Omega_F} J^\eta u \cdot \partial_t q^\eta$$

Since we have (see e.g., [16, pp. 77])

$$\partial_t J^\eta = J^\eta \nabla \cdot w^\eta,$$
where

\[ \int_0^T \int_{\Omega_F(t)} \partial_t u_{|\Omega_F} \cdot q = - \int_0^T \int_{\Omega_F} \left( J^n \left( \nabla u^n \cdot w^n \right) \left( u \cdot q^n \right) - J^n u \cdot \partial_t q^n \right) \]

\[ - \int_{\Omega_F} J_0 u_0 \cdot q^n(0, \cdot). \]

Now, we multiply the elastic equation of \( \eta \) and \( d \) by \( \phi \) and \( \psi \) and integrate by parts over \( (0, T) \times \Gamma \) and \( (0, T) \times \Omega_S \), respectively. Then, we add the results with \( 35 \) together to obtain the weak formulation.

**Definition 2.1 (Weak solution).** Assume that assumptions \([f1]\) and \([f2]\) hold. Then \((u, \eta, d) \in W^n\) is a weak solution of \([1]–[15]\) if for every \((q, \phi, \psi) \in Q^n\), the following equality holds:

\[
\frac{1}{2} \int_0^T \int_{\Omega_F} J^n \left( \left( (u - w^n) \cdot \nabla \right) u \cdot q - \left( (u - w^n) \cdot \nabla \right) q \cdot u - \left( \nabla u^n \cdot \nabla w^n \right) \cdot q \cdot u \right) \\
- \int_0^T \int_{\Omega_F} J^n u \cdot \partial_t q + 2 \int_0^T \int_{\Omega_F} J^n S(D^n(u)) : D^n(q) + \int_{\Omega} \langle f(\eta), \phi \rangle \\
+ \frac{1}{\alpha} \int_0^T \int_{\Gamma} \left( \partial_t \eta_{\tau \beta} - \partial_t \eta_{\tau \beta} \right) q_{\tau \beta} J^n d z d t - \int_0^T \int_{\Gamma} \partial_t \eta \partial_t \phi + \int_{\Gamma} \langle L_c \eta, \phi \rangle \\
+ \frac{1}{\alpha} \int_0^T \int_{\Gamma} \left( \partial_t \eta_{\tau \beta} - \eta_{\tau \beta} \right) \phi_{\tau \beta} J^n d z d t - \int_0^T \int_{\Omega_S} \partial_t d \cdot \partial_t \psi + \int_{\Omega_S} a_S(d, \psi) \\
= \int_0^T \langle R, q \rangle + \int_{\Omega_F} J_0 u_0 \cdot q(0) + \int_{\Gamma} v_0 \phi(0) + \int_{\Omega_S} v_0 \cdot \psi(0).
\]

where \( \langle R, q \rangle = \int_{\Gamma_{in/out}} P_{in/out} q \cdot \nu_F \).

In the following, we state the main result.

**Theorem 2.1 (Main result).** Let \( R \in L^2(0, \infty; H^1(\Omega_{max}')) \). Suppose that the initial datum \( u_0 \in L^2(\Omega_F), \eta_0 \in H^1_0(\Gamma), \nu_0 \in L^2(\Gamma), \eta_0 \in H^1(\Omega_S) \) and \( V_0 \in L^2(\Omega_S) \) satisfy the compatibility conditions \([13]\), \([14]\) and \([15]\), then under the assumptions \([f1]\) and \([f2]\), there exist a \( T > 0 \) and a weak solution \((u, \eta, d)\) to \([1]–[15]\) on \((0, T)\) in the sense of Definition 2.1 such that the following energy estimate holds:

\[
E(t) + \int_0^t D(s) d s \leq E_0 + C \| R \|_{L^2(0, \infty; H^1(\Omega_{max}'))}^2,
\]

where \( E(t) \) and \( D(t) \) are

\[
E(t) = \frac{1}{2} \left( \| u \|^2_{L^2(\Omega^n_F(t))} + \| \partial_t \eta \|^2_{H^1_0(\Gamma)} + \| \eta \|^2_{H^1_0(\Gamma)} + \| \partial_t d \|^2_{L^2(\Omega_S)} + a_S(d, d) + 2\Pi(\eta) \right),
\]

\[
D(t) = \| u \|^2_{W^1,p(\Omega^n_F(t))} + \frac{1}{\alpha} \left( \| \partial_t \eta_{\tau \beta} - \eta_{\tau \beta} \|^2_{L^2(\Omega^n(t))} \right),
\]

and \( E_0 = E(0) \).

## 3 Approximate solutions

### 3.1 Operator splitting scheme

In this section, backward Euler scheme is used to define a sequence of approximate solutions of the fluid-structure interaction problem. For every fixed \( T > 0 \) and \( N \geq 1 \), we devide
the interval \([0, T]\) into \(N\) subintervals of length \(\Delta t = \frac{T}{N}\) with \(0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\). In the subintervals of \([0, T]\) we separate the problem into two parts by Lie operator splitting method as follows.

First, we rewrite the problem (1)–(15) as

\[
\begin{cases}
\frac{dX}{dt} = AX, & t \in (0, T), \\
X|_{t=0} = X^0.
\end{cases}
\]  

(39)

where \(X = (u, \eta, v, d, V)^T\) and \(X^0 = (u_0, \eta_0, v_0, d_0, V_0)^T\).

Then, we decompose \(A = A_1 + A_2\), where \(A_1\) and \(A_2\) are non-trivial, and for \(n = 0, 1, \ldots, N - 1, i = 1, 2\), we obtain

\[
\begin{cases}
\frac{dX^{n+\frac{i}{2}}}{dt} = A_i X^{n+\frac{i}{2}}, & t \in (t_n, t_{n+1}) \\
X^{n+\frac{i}{2}}|_{t=t_n} = X^{n+\frac{i-1}{2}}.
\end{cases}
\]

which can be solved for the approximate vector solutions

\[X^{n+\frac{i}{2}} = (u^{n+\frac{i}{2}}, \eta^{n+\frac{i}{2}}, v^{n+\frac{i}{2}}, d^{n+\frac{i}{2}}, V^{n+\frac{i}{2}})^T,\]

with \(X^0 = (u_0, \eta_0, v_0, d_0, V_0)^T\).

### 3.2 Energy and dissipation under time discretization

According to the decomposition of problem (1)–(15), we define the semi-discrete kinematic energy, elastic energy and dissipation by

\[
E_{kin,N}^{n+\frac{i}{2}} = \frac{1}{2} \left( \int_{\Omega_F} J^n \left| u^{n+\frac{i}{2}} \right|^2 + \left\| v^{n+\frac{i}{2}} \right\|^2_{L^2(\Gamma)} + \left\| V^{n+\frac{i}{2}} \right\|^2_{L^2(\Omega_S)} \right),
\]

\[
E_{el,N}^{n+1} = \frac{1}{2} \left( \int_{\Omega_F} L_n \left( \eta^{n+\frac{i}{2}}, \eta^{n+\frac{i}{2}} \right) + 2\mu_S \left\| D\left( d^{n+\frac{i}{2}} \right) \right\|^2_{L^2(\Omega_S)} + \left\| \nabla \cdot d^{n+\frac{i}{2}} \right\|^2_{L^2(\Omega_S)} + 2\Pi(\eta^{n+\frac{i}{2}}) \right),
\]

\[
E_N^{n+\frac{i}{2}} = E_{kin,N}^{n+\frac{i}{2}} + E_{el,N}^{n+1},
\]

\[
D_N^{n+1} = \kappa_1 \Delta t \int_{\Omega_F} J^n \left| D^n(u^{n+1}) \right|^p + \frac{\Delta t}{\alpha} \left\| (v^{n+1} - u^{n+1})_\tau \right\|^2_{L^2(\Gamma)}.
\]

In the following, we write the subproblems under the time discretization.

### 3.3 The structure subproblem

In the structure subproblem, we notice that \(u\) does not change, then we denote

\[u^{n+\frac{1}{2}} = u^n.\]

Let

\[\mathcal{H} = \mathcal{V}_W \times \mathcal{V}_W \times \mathcal{V}_S \times \mathcal{V}_S\]
and

\[ \mathcal{H} := \{ (\phi, \psi)^T \in \mathcal{V}_W \times \mathcal{V}_S : \psi|_\Gamma = \phi \}. \]

Fixing \( \Delta t \) and defining the solution of structure subproblem by \( (\eta^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, d^{n+\frac{1}{2}}, V^{n+\frac{1}{2}}) \in \mathcal{H} \), we have the weak form of structure subproblem:

For \( (\eta^n, d^n)^T \in \mathcal{H} \), find \( (\eta^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, d^{n+\frac{1}{2}}, V^{n+\frac{1}{2}})^T \in \mathcal{H} \) such that

\[
\begin{align*}
\frac{d^{n+\frac{1}{2}}}{\Delta t} &= \eta^{n+\frac{1}{2}}, \\
\frac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t} &= v^{n+\frac{1}{2}}, \\
\frac{d^{n+\frac{1}{2}} - d^n}{\Delta t} &= V^{n+\frac{1}{2}}, \\
\int_{\Gamma} v^{n+\frac{1}{2}} - v^n \cdot \phi + \int_{\Omega_S} V^{n+\frac{1}{2}} - V^n \cdot \psi + \langle \mathcal{L}_e \eta^{n+\frac{1}{2}}, \phi \rangle &+ a_S(d^{n+\frac{1}{2}}, \psi) + \int_{\Gamma} f(\eta^{n+\frac{1}{2}}) \cdot \phi = 0,
\end{align*}
\]

for all \( (\phi, \psi)^T \in \mathcal{H} \).

In (44), equations in the first row are kinematic coupling conditions and the second row is the weak form. We solve \( (\eta^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, d^{n+\frac{1}{2}}, V^{n+\frac{1}{2}})^T \in \mathcal{H} \) with \( (\eta^n, d^n)^T \) under the invariance of \( u^{n+\frac{1}{2}} = u^n \).

**Lemma 3.1.** For a fixed \( \Delta t > 0 \), there exists a unique weak solution \( (\eta^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, d^{n+\frac{1}{2}}, V^{n+\frac{1}{2}})^T \in \mathcal{H} \) to subproblem (44) with \( (\eta^n, d^n)^T \in \mathcal{H} \).

As stated in Section 1.3, Lax-Milgram Theorem does not hold any more. So we rewrite the structure subproblem as the form of continuous ordinary differential system in a subinterval \((t_n, t_{n+1})\) and apply the Theorem 6.1.2 in [35] to complete the proof as follows.

First, define two linear self-adjoint operator as follows:

\[ \langle \mathcal{L}_1 \eta, v \rangle = \langle S(d) \nu_S, v \rangle_{\Gamma}, \]

and

\[ \langle \mathcal{L}_2 d, d' \rangle = a_S(d, d') - \langle S(d) \nu_s, \partial_t d' \rangle_{\Gamma}. \]

Denoting \( \mathcal{U}(t) \) by

\[ \mathcal{U}(t) = (\eta, v, d, V)^T, \]

we note that for \( U, \mathcal{U} = (\bar{\eta}, \bar{v}, \bar{d}, \bar{V})^T \in \mathcal{H}, \)

\[
\left< \mathcal{U}, \mathcal{U} \right>_{\mathcal{H}} = \left< \mathcal{L}_e \eta, \bar{\eta} \right> + \left< v, \bar{v} \right> + a_S(d, \bar{d}) + \left< V, \bar{V} \right>.
\]

Then the equivalent continuous structure sub-problem in \((t_n, t_{n+1})\) is:

For \( \mathcal{U}(t_n) = (\eta^n, v^n, d^n, V^n)^T \in \mathcal{H} \), find \( \mathcal{U} = \mathcal{U}(t_{n+\frac{1}{2}}) \), such that

\[
\frac{d}{dt} \mathcal{U} + A \mathcal{U} = \mathcal{F}(\mathcal{U})
\]

with

\[
A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \mathcal{L}_e + \mathcal{L}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \mathcal{L}_2 & 0 \end{pmatrix}, \quad \mathcal{F}(\mathcal{U}) = \begin{pmatrix} 0 \\ -f(\eta) \\ 0 \\ 0 \end{pmatrix},
\]

and

\[ \mathcal{D}(A) = \{ \mathcal{U} \in \mathcal{H} | \mathcal{L}_e \eta + \mathcal{L}_1 \eta \in \mathcal{V}_W, \mathcal{L}_2 d \in \mathcal{V}_S \}. \]

In the sequel, we will prove Lemma 3.1.
Proof of Lemma 3.1. Step 1. \( \langle AU, U \rangle_\mathcal{H} \geq 0 \).

From the definition of \( \mathcal{A} \), we have

\[
\mathcal{A}U = (-v, \mathcal{L}_c \eta + \mathcal{L}_1 \eta, -V, \mathcal{L}_2 d)^T,
\]

then, we get

\[
\langle AU, U \rangle_\mathcal{H} = \langle -\mathcal{L}_c v, \eta \rangle + \langle \mathcal{L}_c \eta, v \rangle + \langle \mathcal{L}_1 \eta, v \rangle + a_S(-V, d) + \langle \mathcal{L}_2 d, V \rangle
\]

\[
= \langle S \nu_S, v \rangle_\Gamma - a_S(V, d) + a_S(d, V) + \langle S \nu_S, v \rangle_\Gamma
\]

\[
= 0 \geq 0,
\]

where we have used the boundary condition \( \partial_t d = V = v \) on \( \Gamma \).

Step 2. \( R(\mathbb{I} + \mathcal{A}) = \mathcal{H} \).

First, we prove that \( \mathbb{I} + \mathcal{A} \) is surjective, i.e., for every \( G = (g_1, g_2, g_3, g_4)^T \in \mathcal{H} \), there exists \( V = (v_1, v_2, v_3, v_4)^T \in \mathcal{D}(\mathcal{A}) \) such that

\[
(\mathbb{I} + \mathcal{A})V = G,
\]

that is,

\[
\begin{cases}
  v_1 - v_2 = g_1, \\
  v_2 + \mathcal{L}_c v_1 + \mathcal{L}_1 v_1 = g_2, \\
  v_3 - v_4 = g_3, \\
  v_4 + \mathcal{L}_2 v_3 = g_4.
\end{cases}
\]  

Adding the first two equations and last two equations in (48) respectively, we obtain

\[
\begin{cases}
  v_1 + \mathcal{L}_c v_1 + \mathcal{L}_1 v_1 = g_1 + g_2, \\
  v_3 + \mathcal{L}_2 v_3 = g_3 + g_4,
\end{cases}
\]

Multiplying (49) by \( (\tilde{v}_1, \tilde{v}_3)^T \in \tilde{\mathcal{H}} \) and integrating equations over \( \Gamma, \Omega_S \) respectively, we find that

\[
\begin{cases}
  \int_\Gamma v_1 \cdot \tilde{v}_1 + \int_\Gamma \mathcal{L}_c v_1 \cdot \tilde{v}_1 + \int_\Gamma \mathcal{L}_1 v_1 \cdot \tilde{v}_1 = \int_\Gamma g_1 \cdot \tilde{v}_1 + \int_\Gamma g_2 \cdot \tilde{v}_1, \\
  \int_{\Omega_S} v_3 \cdot \tilde{v}_3 + \int_{\Omega_S} \mathcal{L}_2 v_3 \cdot \tilde{v}_3 = \int_{\Omega_S} g_3 \cdot \tilde{v}_3 + \int_{\Omega_S} g_4 \cdot \tilde{v}_3.
\end{cases}
\]

Since \( \langle \mathcal{L}_1 v_1, \tilde{v}_1 \rangle + \langle \mathcal{L}_2 v_3, \tilde{v}_3 \rangle = a_S(v_3, \tilde{v}_3) \) with \( v_1 = v_3 \) and \( \tilde{v}_1 = \tilde{v}_3 \) on \( \Gamma \), we have the following variational formulation:

\[
B ((v_1, v_3)^T, (\tilde{v}_1, \tilde{v}_3)^T) = \tilde{B} ((\tilde{v}_1, \tilde{v}_3)^T), \quad \forall (\tilde{v}_1, \tilde{v}_3)^T \in \tilde{\mathcal{H}}.
\]

where

\[
B ((v_1, v_3)^T, (\tilde{v}_1, \tilde{v}_3)^T) = \int_\Gamma v_1 \cdot \tilde{v}_1 + \int_\Gamma \mathcal{L}_c v_1 \cdot \tilde{v}_1 + \int_{\Omega_S} v_3 \cdot \tilde{v}_3 + a_S(v_3, \tilde{v}_3),
\]

and

\[
\tilde{B} ((\tilde{v}_1, \tilde{v}_3)^T) = \int_\Gamma g_1 \cdot \tilde{v}_1 + \int_\Gamma g_2 \cdot \tilde{v}_1 + \int_{\Omega_S} g_3 \cdot \tilde{v}_3 + \int_{\Omega_S} g_4 \cdot \tilde{v}_3.
\]

Now we introduce the norm of the Hilbert space \( \tilde{\mathcal{H}} \) as follows:

\[
\| (v_1, v_3) \|_{\tilde{\mathcal{H}}}^2 = \| v_1 \|_{L^2(\Gamma)}^2 + \| v_3 \|_{H^2(\Gamma)}^2 + \| v_3 \|_{L^2(\Omega_S)}^2 + a_S(v_3, v_3).
\]
Then it can be deduced that bilinear $B(\cdot,\cdot)$ and functional $\tilde{B}(\cdot)$ are bounded. Furthermore, it follows that there exists a positive constant $\delta_1$ such that

$$B((v_1,v_3)^T,(v_1,v_3)^T) = \int_{\Gamma} v_1 \cdot v_1 + \int_{\Gamma} L_e v_1 \cdot v_1 + \int_{\Omega_S} v_3 \cdot v_3 + a_S(v_3,v_3) \geq \delta_1 \|(v_1,v_3)\|_{\tilde{H}}^2,$$

which means $B(\cdot,\cdot)$ is coercive.

By applying the Lax-Milgram Lemma [35], system (50) has a unique solution $(v_1,v_3)^T \in \tilde{H}$. In (48) and (48)_{3}, we see that $v_2 \in V_W$, $v_4 \in V_S$.

Then it follows from (48)_{2} and (48)_{4} that

$$L_e v_1 + L_1 v_1 = g_2 - v_2 \in V_W, \quad L_2 v_3 = g_4 - v_4 \in V_S. \quad (51) \quad (52)$$

As a consequence, there exists a unique solution $U \in D(A)$ such that (47) is satisfied, which means that $I + A$ is surjective.

Step 1 and step 2 tell us that $A$ is a maximal monotone operator. Then by the Lumer-Phillips theorem [35, Theorem 1.4.3], $A$ generates a semigroup of contractions in $H$.

**Step 3.** $F$ is locally Lipschitz in $H$.

In this step, we show that $f$ is locally Lipschitz from $H^2$ into $H^2$ and this is true due to the assumption [H].

By [35 Theorem 6.1.2], we have the existence and uniqueness of the solutions.

For readability, we will replace the superscript $\eta^n$ of fluid variables with $n$ in the sequel.

### 3.4 The fluid subproblem

Suppose that we have solved problem (44). Since $\eta$ and $d$ do not change in the fluid subproblem, setting $\eta^{n+1} = \eta^{n+\frac{1}{2}}$ and the fixed displacement by $\tilde{\eta}_N^{n+1} = \eta_N^{n+1}(n+1)\Delta t$ in fluid subproblem, then we have the discrete ALE velocity

$$w^{n+1} = A^{n+1} - A^n, \quad w^{n+1} |_{\Gamma} = \frac{\eta^{n+1} - \eta^n}{\Delta t},$$

the “unchanged” thin structure velocity

$$\partial_t \eta^{n+1} = \frac{\eta^{n+1} - \eta^n}{\Delta t},$$

and the Jacobian of the ALE mapping

$$J^{n+1} = \det \nabla \tilde{A}^{n+1}, \quad \tilde{A}^{n+1} : \Omega_F \rightarrow \tilde{\Omega}_F^{n+1}.$$
Banach space and let $A$ be a real separable reflexive Banach space and let $A : V \to V'$ be a coercive and hemicontinuous monotone operator. Then $A$ is surjective, i.e., given any $f \in V'$, there exists $u$ such that

$$u \in V \quad \text{and} \quad A(u) = f.$$ 

If $A$ is strictly monotone, then $A$ is also injective.

**Proof of Lemma 3.2.** Hemicontinuity. From the definition of $\mathcal{H}$, we know that $\mathcal{A} : \mathcal{H} \to \mathcal{H}'$ is a bounded operator due to the Hölder’s inequality and (17). Since all parts except the Non-Newtonian term are linear, the boundedness implies the continuity. It remains to be shown that the nonlinear part is hemicontinuous. For that we define the operator $\mathcal{N} : \Omega_F^n \to (\Omega_F^n)'$ such that

$$\langle \mathcal{N}(u^{n+1}), q \rangle = \int_{\Omega_F} (J^{n+1} - J^n) u^{n+1} \cdot q + \frac{1}{2} \int_{\Omega_F} (J^{n+1} + J^n) u^{n+1} \cdot q,$$

$$+ \frac{\Delta t}{2} \int_{\Omega_F} J^{n+1} \left( \left( (u^n - w^{n+1}) \cdot \nabla \eta^{n+1}_N \right) u^{n+1} \cdot q \right),$$

$$+ 2 \Delta t \int_{\Omega_F} J^{n+1} S(D\tilde{\eta}^{n+1}_N(u^{n+1})) : D\tilde{\eta}^{n+1}_N(q),$$

$$+ \frac{\Delta t}{\alpha} \int_{\Gamma} (u^n_{\tau^{n+1}} - v^n_{\tau^{n+1}}) q_{\tau^{n+1}} J^{n+1} + \int_{\Gamma} v^{n+1} \cdot \phi,$$

$$+ \frac{\Delta t}{\alpha} \int_{\Gamma} (v^n_{\tau^{n+1}} - u^n_{\tau^{n+1}}) \hat{\phi}_{\tau^{n+1}} J^{n+1} + 1,$$

and

$$\left\langle \mathcal{F}(u^{n+1}, v^{n+1}, q, \phi), (q, \phi)^T \right\rangle = \int_{\Omega_F} J^n u^{n+1} \cdot q + \int_{\Gamma} v^{n+1} \cdot \phi + \Delta t \left( R^{n+1}, q \right),$$

for $(q, \phi)^T \in \mathcal{H}$ and $R^{n+1} = \frac{1}{\Delta t} \int_{n \Delta t}^{(n+1) \Delta t} R$. The corresponding weak form is

$$\langle \mathcal{A}(u^{n+1}, v^{n+1}), (q, \phi)^T \rangle = \left\langle \mathcal{F}(u^{n+1}, v^{n+1}, q, \phi), (q, \phi)^T \right\rangle.$$

**Remark 3.1.** Notice that the solution $u^{n+1}$ and the test function $q$ are all defined in $\Omega_F^n$ in which the functions are all in domain $\Omega_F^n$. This is because in a fixed time subinterval $(n \Delta t, (n+1) \Delta t)$, the fluid domain $\Omega_F^n(t)$ is parameterized by $\eta^n$, not by $\eta^{n+1}$.

**Lemma 3.2.** For a fixed $\Delta t > 0$, there exists a unique weak solution $(u^{n+1}, v^{n+1})^T \in \mathcal{H}$ to subproblem [53] with $(u^{n+1}, v^{n+1})^T \in \mathcal{H}$.

In our manuscript, inspired by the $p$-structure of $\mu_F(D(u))$, we solve the fluid subproblem by the Browder-Minty theorem [7, Theorem 9.14–1].

First, we state the Browder-Minty theorem.

**Proposition 3.1** (Browder-Minty theorem [3] [7] [27]). Let $V$ be a real separable reflexive Banach space and let $A : V \to V'$ be a coercive and hemicontinuous monotone operator. Then $A$ is surjective, i.e., given any $f \in V'$, there exists $u$ such that

$$u \in V \quad \text{and} \quad A(u) = f.$$
is continuous with respect to $s$. Therefore, $\mathcal{N} : \mathcal{V}_F^n \to (\mathcal{V}_F^n)'$ is hemicontinuous.\footnote{Proof of Theorem 9.14-2} and $\mathcal{A}$ is hemicontinuous.

**Coercivity.** Taking $q = u^{n+1}$ and $\psi = v^{n+1}$, we find that there is a $\delta_2$, such that

$$
\langle \mathcal{A}(u^{n+1}, v^{n+1}), (u^{n+1}, v^{n+1}) \rangle = \int_{\Omega_F} J^n |u^{n+1}|^2 + \frac{1}{2} \int_{\Omega_F} (J^{n+1} - J^n) |u^{n+1}|^2
$$

$$
+ 2\Delta t \int_{\Omega_F} J^{n+1} S(D_{n+1}^{\mathcal{N}}(u^{n+1}))(D_{n+1}^{\mathcal{N}}(u^{n+1}))
$$

$$
+ \frac{\Delta t}{\alpha} \int_{\Gamma} \left| u_{\tau_F^{\mathcal{N}+1}} - v_{\tau_F^{\mathcal{N}+1}} \right|^2 J_{n+1} + \int_{\Gamma} |v^{n+1}|^2
$$

$$
\geq \delta_2 \left( \int_{\Omega_F} J^n |u^{n+1}|^2 + \int_{\Omega_F} J^{n+1} |D_{n+1}^{\mathcal{N}}(u^{n+1})|^2 + \|v^{n+1}\|^2_{L^2(\Gamma)} \right),
$$

where we have used the property \footnote{16} to deal with the $p$-structure of $\mu_F$. Then the coercivity of $\mathcal{A}$ is verified.

**Strict monotone.** For two different variables pairs $(u_1^{n+1}, v_1^{n+1})^T$ and $(u_2^{n+1}, v_2^{n+1})^T$, it can be shown from \footnote{18} that

$$
\langle \mathcal{A}(u_1^{n+1}, v_1^{n+1}) - \mathcal{A}(u_2^{n+1}, v_2^{n+1}), (u_1^{n+1} - u_2^{n+1}, v_1^{n+1} - v_2^{n+1})^T \rangle
$$

$$
= \int_{\Omega_F} J^n |u_1^{n+1} - u_2^{n+1}|^2 + \frac{1}{2} \int_{\Omega_F} (J^{n+1} - J^n) |u_1^{n+1} - u_2^{n+1}|^2
$$

$$
+ 2\Delta t \int_{\Omega_F} J^{n+1} (S(D_1) - S(D_2)) : (D_1 - D_2)
$$

$$
+ \int_{\Gamma} |v_1^{n+1} - v_2^{n+1}|^2 + \frac{\Delta t}{\alpha} \int_{\Gamma} \left| u_{\tau_F^{\mathcal{N}+1}} - v_{\tau_F^{\mathcal{N}+1}} \right|^2 J_{n+1}
$$

$$
> 0,
$$

where $D_1 = D_{n+1}^{\mathcal{N}}(u_1^{n+1})$ and $D_2 = D_{n+1}^{\mathcal{N}}(u_2^{n+1})$. Thus $\mathcal{A}$ is strictly monotone from the definition.

Therefore, the proof is complete by means of Proposition \footnote{3.1}.

3.5 Uniform energy estimates

**Lemma 3.3.** Solution of subproblem \footnote{44} satisfies the following semi-discrete energy inequality

$$
E_{\mathcal{N}}^{n+\frac{1}{2}} + \frac{1}{2} \left( \left\| v^{n+\frac{1}{2}} - v^n \right\|^2_{L^2(\Gamma)} + \left\| V^{n+\frac{1}{2}} - V^n \right\|^2_{L^2(\Omega_S)} \right)
$$

$$
+ \frac{1}{2} \left\| \eta^{n+\frac{1}{2}} - \eta^n \right\|^2_{H_0^1(\Gamma)} + \frac{1}{2} a_S(d_{n+\frac{1}{2}} - d^n, d_{n+\frac{1}{2}} - d^n) \leq E_{\mathcal{N}}^n
$$

(55)

**Proof.** Let $\phi = \eta^{n+\frac{1}{2}} = \frac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t}$ and $\psi = V^{n+\frac{1}{2}} = \frac{d^{n+\frac{1}{2}} - d^n}{\Delta t}$ in \footnote{44}. More precisely, taking $\phi = v^{n+\frac{1}{2}}$ in the first term and $\psi = V^{n+\frac{1}{2}}$ in the second term in \footnote{44}, $\phi = \frac{\eta^{n+\frac{1}{2}} - \eta^n}{\Delta t}$ and $\psi = \frac{d^{n+\frac{1}{2}} - d^n}{\Delta t}$ in the other terms, by means of Remark \footnote{2.2} we have

$$
\left\| v^{n+\frac{1}{2}} \right\|^2_{L^2(\Gamma)} + \left\| v^{n+\frac{1}{2}} - v^n \right\|^2_{L^2(\Gamma)} + \left\| V^{n+\frac{1}{2}} - V^n \right\|^2_{L^2(\Omega_S)} + \left\| \eta^{n+\frac{1}{2}} - \eta^n \right\|^2_{H_0^1(\Gamma)} + 2\Pi(\eta^{n+\frac{1}{2}}) + a_S(d^{n+\frac{1}{2}}, d^{n+\frac{1}{2}})
$$

$$
+ a_S(d^{n+\frac{1}{2}} - d^n, d^{n+\frac{1}{2}} - d^n) \leq \left\| v^n \right\|^2_{L^2(\Gamma)} + \left\| V^n \right\|^2_{L^2(\Omega_S)} + \langle \mathcal{L}_\psi \eta^n, \eta^n \rangle + 2\Pi(\eta^n),
$$

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where
\[
\int_{\Gamma} f(\eta^{n+\frac{1}{2}}) \cdot v^{n+\frac{1}{2}} = \int_{\Gamma} f(\eta^{n+\frac{1}{2}}) \cdot \partial_t \eta^{n+\frac{1}{2}} = \frac{d}{dt} \Pi(\eta^{n+\frac{1}{2}}) = \frac{\Pi(\eta^{n+\frac{1}{2}}) - \Pi(\eta^n)}{\Delta t}.
\]

By adding \( \int_{\Omega_F} J^n |u^n|^2 \) on both sides, it follows from \( \eta^n = \eta^{n+\frac{1}{2}} \) and \( d^n = d^{n+\frac{1}{2}} \) in fluid subproblem that (55) holds.

**Lemma 3.4.** Solution of subproblem (53) satisfies the following semi-discrete energy inequality
\[
E_{kin,n}^{n+1} + \frac{1}{2} \int_{\Omega_F} J^n \left| u^{n+1} - u^{n+\frac{1}{2}} \right|^2 + \frac{1}{2} \| v^{n+1} - v^{n+\frac{1}{2}} \|^2_{L^2(\Gamma)} + D_{N}^{n+1} \leq E_{kin,n}^{n+\frac{1}{2}} + C \Delta t \left( \| R^{n+1} \|_{H^1(\Omega_F)^t} + 1 \right),
\]
where \( C \) depends on \( \Omega_F, \kappa_1, \kappa_2 \).

**Proof.** Taking \( q = u^{n+1}, \phi = v^{n+1} \) in (54), combining with (16) and using Remark 2.2, we find that
\[
\frac{1}{2} \int_{\Omega_F} J^n \left( |u^{n+1}|^2 + |u^{n+1} - u^{n+\frac{1}{2}}|^2 - |u^n|^2 \right) + \frac{1}{2} \int_{\Omega_F} (J^{n+1} - J^n) |u^{n+1}|^2 + 2 \kappa_1 \Delta t \int_{\Omega_F} J^n \left( |D_{N}^{n+1}(u^{n+1})|^p - \kappa_2 \right) + \frac{\Delta t}{\alpha} \int_{\Gamma} |u_{n+1} - v_{n+1}|^2 J_{F}^{n+1} + \frac{1}{2} \left( \| v^{n+1} \|^2_{L^2(\Gamma)} + \| v^{n+1} - v^n \|^2_{L^2(\Gamma)} - \| v^n \|^2_{L^2(\Gamma)} \right) \leq \Delta t \langle R^{n+1}, u^{n+1} \rangle.
\]

By using the trace inequality and Sobolev inequality, we obtain that for \( p > 2 \),
\[
\langle R^{n+1}, u^{n+1} \rangle \leq c \| R^{n+1} \|_{(H^1(\Omega_F)^t)} \| u^{n+1} \|_{H^1(\Omega_F(t))} \leq \frac{c}{4\varepsilon} \| R^{n+1} \|_{(H^1(\Omega_F)^t)} + c\varepsilon \| u^{n+1} \|_{H^1(\Omega_F(t))} \leq \frac{c}{4\varepsilon} \| R^{n+1} \|_{(H^1(\Omega_F)^t)} + c^* \varepsilon \| u^{n+1} \|_{H^1,\varepsilon(\Omega_F(t))}.
\]

Proceeding as in [30, 32], we choose \( \varepsilon > 0 \) small enough such that \( c^* \varepsilon \leq \kappa_1 \) to complete the proof. \( \square \)

**Lemma 3.5** (Uniform energy estimates). Let \( \Delta t > 0 \) and \( N = T/\Delta t \), then we have the following estimates:

1. \( E_{N}^{n+\frac{1}{2}} \leq C, E_{N}^{n+1} \leq C, \) for all \( n = 0, \ldots, N - 1 \).

2. \( \sum_{j=1}^{N} D_{N}^{j} \leq C \).

3. \( \sum_{n=0}^{N-1} \left( \int_{\Omega_F} J_{N}^{n} \left| u_{N}^{n+1} - u_{N}^{n} \right|^2 + \left| v_{N}^{n+1} - v_{N}^{n+\frac{1}{2}} \right|^2_{L^2(\Gamma)} + \left| v_{N}^{n+\frac{1}{2}} - v_{N}^{n} \right|^2_{L^2(\Gamma)} + \| V_{N}^{n+1} - V_{N}^{n} \|^2_{L^2(\Omega_S)} \right) \leq C, \)

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4. \[ \sum_{n=0}^{N-1} \left( \| \eta_{n+1}^N - \eta_n^N \|_{H^2_0(\Gamma)}^2 + a_S(d_{n+1}^N - d_n^N, d_{n+1}^N - d_n^N) \right) \leq C, \]

where \( C = E_0 + \tilde{C} \left( \| R \|_{L^2(0,T; H^1(\Omega_F))'}^2 + 1 \right) \) and \( \tilde{C} \) depends on \( \Omega_F \) and other parameters in the problem.

**Proof.** Adding (55) and (56), and taking sum for \( n \) from 0 to \( N - 1 \), we find that

\[
E_N^N + \sum_{n=0}^{N-1} D_{n+1}^N + \sum_{n=0}^{N-1} \left( \int_{\Omega_F} J_n^N \left| u_{n+1}^N - u_n^N \right|^2 + \left| v_{n+1}^N - v_n^N \right|^2 \right)_{L^2(\Gamma)} + \left| v_{n+1}^N - v_n^N \right|_{L^2(\Omega_S)}^2 + \left( \left| \eta_{n+1}^N - \eta_n^N \right|_{H^2_0(\Gamma)}^2 + a_S(d_{n+1}^N - d_n^N, d_{n+1}^N - d_n^N) \right) \
\leq E_0 + C \sum_{n=0}^{N-1} \Delta t \left\| R_{n+1}^N \right\|_{(H^1(\Omega_F))'}^2.
\]

By definition of \( R_{n+1}^N \), the last term in (57) becomes

\[
\sum_{n=0}^{N-1} \Delta t \left\| R_{n+1}^N \right\|_{(H^1(\Omega_F))'}^2 = \Delta t \sum_{n=0}^{N-1} \left\| \int_{n\Delta t}^{(n+1)\Delta t} R \right\|_{(H^1(\Omega_F))'}^2.
\]

Therefore, we have the estimates 2, 3, and 4. It can be deduced form summing from 0 to \( n \) instead of from 0 to \( N - 1 \) that \( E_{n+1}^N \leq C \). Next, (55) implies \( E_{n+1}^N \leq E_n^N \leq C \).

### 3.6 Uniform boundedness

Let \( \eta_N, \tilde{v}_N, d_N, V_N \) be the solutions of (44) given in Lemma 3.1 and \( u_N, v_N \) be the solutions of (53) given in Lemma 3.2. As discussed in Section 2.3, we need to determine the time interval of existence of solutions. With compatibility conditions (13)–(15), we have the following proposition.

**Proposition 3.2 (32).** There is a \( T \) small enough and a positive constant \( C \) such that \( A_N^N \) is an injection, and

\[ J_N^N = \det \nabla A_N^N > 0, \quad \Delta t > 0, n = 1, \ldots, N. \]

**Proof.** It has been proved in [32 Proposition 5], so we omit it here.

From Proposition 3.2, we have the maximal existence time \( T \) before which the domain will not degenerate. Then Lemma 3.3 implies the following boundedness properties.

**Lemma 3.6.** For a fixed \( \Delta t = \frac{T}{N} > 0 \) and \( p > 2 \), the following holds:

1. The sequence \( \{ \eta_N \}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0,T; H^2_0(\Gamma)) \).
2. The sequence \( \{ \tilde{v}_N \}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0,T; L^2(\Gamma)) \).
3. The sequence \( \{ d_N \}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0,T; H^1(\Omega_S)) \).
4. The sequence \( \{ V_N \}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0,T; L^2(\Omega_S)) \).
5. The sequence \( \{u_N\}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0,T;L^2(\Omega_F)) \).

6. The sequence \( \{v_N\}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0,T;L^2(\Gamma)) \).

Let us denote \( \tilde{\eta}_N = T_N \eta_N(t,\cdot) := \eta_N(t-\Delta t,\cdot), \Delta t = T/N \). We can deduce the following lemma.

**Lemma 3.7.** For a pair of conjugate indices \( p \) and \( q \) which satisfies \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

1. The sequence \( \{D\tilde{\eta}_N(u_N)\}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^p((0,T) \times \Omega_F) \),
2. The sequence \( \{\mathcal{S}(D\tilde{\eta}_N(u_N))\}_{N \in \mathbb{N}} \) is uniformly bounded in \( L^q((0,T) \times \Omega_F)^2 \).

**Proof.** The estimate 2 in Lemma 3.5 implies that

\[
\sum_{n=0}^{N-1} \kappa_1 \int_{\Omega_F} J_N^n |D\tilde{\eta}_N(u_{n+1})|^p \Delta t \leq C.
\]

Since the Jacobian \( J_N^n \) of the ALE mapping \( A_N \) is uniformly bounded from below by a positive constant (see Proposition 3.2), we have the boundedness of \( D\tilde{\eta}_N(u_N) \). Combining this boundedness with (17), we obtain

\[
\|\mathcal{S}(D\tilde{\eta}_N(u_N))\|_{L^q((0,T) \times \Omega_F)^2} \leq C'.
\]

This completes the proof. \( \square \)

### 3.7 Weak and weak* convergence

**Lemma 3.8** (Weak* convergence). For a fixed \( \Delta t = \frac{T}{N} > 0 \), there exist subsequences \( \{\eta_N\}_{N \in \mathbb{N}}, \{\tilde{v}_N\}_{N \in \mathbb{N}}, \{d_N\}_{N \in \mathbb{N}}, \{V_N\}_{N \in \mathbb{N}}, \{u_N\}_{N \in \mathbb{N}} \) and \( \{v_N\}_{N \in \mathbb{N}} \) and functions \( \eta \in L^\infty(0,T;H^1_0(\Gamma)) \), \( v, \tilde{v} \in L^\infty(0,T;L^2(\Gamma)) \), \( u \in L^\infty(0,T;L^2(\Omega_F)) \), \( d \in L^\infty(0,T;H^1(\Omega_S)) \) and \( V \in L^\infty(0,T;L^2(\Omega_S)) \) such that

- \( \eta_N \to \eta \) weakly* in \( L^\infty(0,T;H^1_0(\Gamma)) \),
- \( \tilde{v}_N \to \tilde{v} \) weakly* in \( L^\infty(0,T;L^2(\Gamma)) \),
- \( d_N \to d \) weakly* in \( L^\infty(0,T;H^1(\Omega_S)) \),
- \( V_N \to V \) weakly* in \( L^\infty(0,T;L^2(\Omega_S)) \),
- \( u_N \to u \) weakly* in \( L^\infty(0,T;L^2(\Omega_F)) \),
- \( v_N \to v \) weakly* in \( L^\infty(0,T;L^2(\Gamma)) \).

By means of reflexivity and Lemma 3.7, we have the following weak convergence results.

**Lemma 3.9** (Weak convergence). For a fixed \( \Delta t = \frac{T}{N} > 0 \), \( p > 2 \) and \( q \) which satisfies \( \frac{1}{p} + \frac{1}{q} = 1 \), there exist subsequences \( \{D\tilde{\eta}_N(u_N)\}_{N \in \mathbb{N}} \) and \( \{\mathcal{S}(D\tilde{\eta}_N(u_N))\}_{N \in \mathbb{N}} \) and functions \( M \in L^p((0,T) \times \Omega_F), G \in L^q((0,T) \times \Omega_F)^2 \) such that

- \( D\tilde{\eta}_N(u_N) \to M \) weakly in \( L^p((0,T) \times \Omega_F) \),
- \( \mathcal{S}(D\tilde{\eta}_N(u_N)) \to G \) weakly in \( L^q((0,T) \times \Omega_F)^2 \).

**Remark 3.2.** Here, \( M \) and \( G \) are unknown since the gradient are not equal in \( \nabla\tilde{\eta}_N u_N \) and \( \nabla u \). In the limiting process, we can show that \( M = D\eta(u) \) and \( G = \mathcal{S}(D\eta(u)) \) (see Lemma 4.2).
3.8 Strong convergence

In this section, we prove the strong convergence of weak solution. This convergence is useful when we finally pass to the limit.

3.8.1 Strong convergence for velocities

First, we establish the strong convergence of \( u_N \) and \( v_N \). In [33], Muha and Čanić proved a generalized Aubin-Lions-Simon Lemma to deal with the specific problems for which the spatial domain depends on time. More precisely, they made use of the classical Simon’s theorem in [38, Theorem 1] together with the uniform estimates of the problem and provided the \( L^2(0,T;H) \) compactness for the moving domains. This theorem is effective in processing this type of problem and Muha and Čanić gave three examples in [33], whose strong convergences were proved by a more sophisticated procedure before. After [33], the generalized Aubin-Lions-Simon Lemma was applied in several works, see [5, 34, 39].

With the help of [33, Theorem 3.1], we have the following compactness theorem:

**Theorem 3.1.** The sequence \( \{(u_N, v_N, V_N)\}_{N \in \mathbb{N}} \), introduced in Lemma 3.8, is relatively compact in \( L^2(0,T;H) \), where \( H = L^2(\Omega_{\text{max}}) \times H^{-s}(\Gamma) \times H^{-s}(\Omega_S) \).

**Proof.** Since the model of fluid-thin structure interaction in our work can be found in [32] and [33, Section 4.3], the proof of Theorem 3.1 can be easily established by [33, Theorem 3.1] with some modifications. We need to take \( L^p(0,T;W^{1,p}) \hookrightarrow L^2(0,T;H^1) \) into account and make use of the estimates in Lemma 3.5 to verify all the conditions of [33, Theorem 3.1], more specifically, Properties (A), (B) and (C). Here, we give a sketch of the proof.

**Property (A).** To show (A1) and (A2) of Properties (A), we define the corresponding spaces as follows.

\[
V = H^s(\Omega_{\text{max}}) \times L^2(\Gamma) \times L^2(\Omega_S),
\]

and

\[
H = L^2(\Omega_{\text{max}}) \times H^{-s}(\Gamma) \times H^{-s}(\Omega_S), \quad 0 < s < \frac{1}{2}.
\]

Then we have \( V \subset\subset H \). In addition, we choose the moving velocity spaces \( V^n_N \) and moving test spaces \( Q^n_N \) such that

\[
V^n_N = \left\{ (u, v, V) \in V^n_F \times H^2(\Gamma) \times L^2(\Omega_S) : (u|_{\Gamma_n} - v) \cdot \nu_{F,N}^{-1} = 0 \right\},
\]

\[
Q^n_N = \left\{ (q, \phi, \psi) \in (V^n_F \cap H^4(\Omega_{F,N})) \times V_W \times V_S : q|_{\Gamma_n} = \phi, \phi = \psi|_{\Gamma} \right\}.
\]

and \( (V^n_N, Q^n_N) \hookrightarrow V \times V \). It follows from the trace theorem \( \|v^n_N\|^2_{H^1(\Gamma)} \leq \|u^n_N\|^2_{H^1(\Omega_F)} \) that

\[
\sum_{n=1}^{N} \|(u^n_N, v^n_N, V^n_N)\|_{V^n_N}^2 \Delta t
= \sum_{n=1}^{N} \left( \|u^n_N\|^2_{H^1(\Omega_F)} + \|v^n_N\|^2_{H^2(\Gamma)} + \|V^n_N\|^2_{L^2(\Omega_S)} \right) \Delta t
\leq C \sum_{n=1}^{N} \left( \|u^n_N\|^2_{H^1(\Omega_F)} + \|V^n_S\|^2_{L^2(\Omega_S)} \right) \Delta t
\]

(59)
\( \leq C. \)

From the embedding \( L^2 \hookrightarrow H^{-s}, 0 < s < \frac{1}{2} \) and the uniform boundedness in Lemma 3.8 we deduce that

\[
\| (u_N, v_N, V_N) \|_{L^\infty(0,T;H^s)}^2 = \| u_N \|_{L^\infty(0,T;L^2(\Omega_{\max}))}^2 + \| v_N \|_{L^\infty(0,T;H^{-s}(\Gamma))}^2 + \| V_N \|_{L^\infty(0,T;H^{-s}(\Omega_S))}^2 \leq \| u_N \|_{L^\infty(0,T;L^2(\Omega_{\max}))}^2 + \| v_N \|_{L^\infty(0,T;L^2(\Gamma))}^2 + \| V_N \|_{L^\infty(0,T;L^2(\Omega_S))}^2 \leq C.
\]

Then (59) and (60) verify (A1) and (A2) respectively. Since it has been shown in [33] Theorem 3.2] that (A3) can be obtained by Property (B), we omit it here.

**Property (B).** In the following, we prove the Property (B). Direct calculation yields

\[
\left\| P^{n+1}_N \frac{(u^{n+1}_N, v^{n+1}_N, V^{n+1}_N) - (u^n_N, v^n_N, V^n_N)}{\Delta t} \right\|_{(Q^{n+1}_N)^*} = \sup_{\|(q, \phi, \psi)\|_{(Q^{n+1}_N)}} \left| \int_{\Gamma^{n+1}_F} \frac{u^{n+1}_N - u^n_N}{\Delta t} \cdot q + \int_{\Omega^{n+1}_S} \frac{v^{n+1}_N - v^n_N}{\Delta t} \cdot \phi + \int_{\Omega^{n+1}_S} \frac{V^{n+1}_N - V^n_N}{\Delta t} \cdot \psi \right|
\]

\[
\leq \sup_{\|(q, \phi, \psi)\|_{(Q^{n+1}_N)}} \left| \int_{\Omega^{n+1}_F} \frac{\tilde{u}^{n}_N - u^n_N}{\Delta t} \cdot q + \int_{\Omega^{n+1}_S} \frac{\tilde{v}^{n}_N - v^n_N}{\Delta t} \cdot \phi + \int_{\Omega^{n+1}_S} \frac{\tilde{V}^{n}_N - V^n_N}{\Delta t} \cdot \psi \right|
\]

By means of the weak formulation, we obtain

\[
\left| \int_{\Omega^{n+1}_F} \frac{u^{n+1}_N - u^n_N}{\Delta t} \cdot q + \int_{\Gamma^{n+1}_S} \frac{v^{n+1}_N - v^n_N}{\Delta t} \cdot \phi + \int_{\Gamma^{n+1}_S} \frac{V^{n+1}_N - V^n_N}{\Delta t} \cdot \psi \right|
\]

\[
\leq \left( \| \tilde{u}^{n}_N \| + \| \tilde{w}^{n+1} \| \right) \| \nabla u^{n+1}_N \| \| q \|_L^\infty + (\| \tilde{w}^{n+1}_N \| + \| w^{n+1} \|) \| \nabla q \| \| u^{n+1}_N \|_p + \| u^{n+1}_N \|_p + \| v^{n+1}_N \|_p + \| V^{n+1}_N \|_p + \| u^{n+1}_N \|_p \| \nabla q \|_p \| u^{n+1}_N \|_p
\]

\[
+ \| \phi \|_p \| u^{n+1}_N \|_p \| \nabla \phi \|_p + \| \phi \|_p \| \nabla \phi \|_p + \| \phi \|_p \| \nabla \phi \|_p + \| \phi \|_p \| \nabla \phi \|_p + \| \phi \|_p \| \nabla \phi \|_p
\]

\[
+ \| \Delta \eta \|_p \| \Delta \phi \|_p + \| \Delta \eta \|_p \| \Delta \phi \|_p + \| \Delta f(0) \|_p + \| \Delta f(0) \|_p + \| \Delta f(0) \|_p + \| \Delta f(0) \|_p + \| \Delta f(0) \|_p + \| \Delta f(0) \|_p
\]

\[
\leq C \left( \| (u^{n+1}_N, v^{n+1}_N, V^{n+1}_N) \|_{V^{n+1}_N} + 1 \right) \| (q, \phi, \psi) \|_{(Q^{n+1}_N)},
\]

where we have used the Assumption [1] that

\[
\| f(\eta) \|_H^2 = \| f(\eta) - f(0) \|_H^2 
\]

\[
\leq \| f(\eta) - f(0) \|_H^2 + \| f(0) \|_H^2 
\]

\[
\leq C \| \eta \|_H^2 + \| f(0) \|_H^2.
\]

Following the same procedure in [5] and [33], we get

\[
\left| \int_{\Omega^{n+1}_F} \frac{\tilde{u}^{n}_N - u^n_N}{\Delta t} \cdot q \right| \leq C \| (u^{n}_N, v^{n}_N, V^{n}_N) \|_{V^{n}_N} \| (q, \phi, \psi) \|_{(Q^{n}_N)}.
\]

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Consequently,
\[
\left\| p_{n+1}^{n+1} \left( u_{n+1}^{n+1}, v_{n+1}^{n+1}, V_{n+1}^{n+1} \right) - (u^n, v^n, V^n) \Delta t \right\|_{(Q_{N,1}^{n+1})'} \\
\leq C \left( \|(u^n, v^n, V^n)\|_{V_N^2} + 1 \right),
\]
which proves the Property (B).

**Property (C).** We notice that the rest properties needed to be shown follow from the same procedure in [5]. Therefore, we provide the definition of several operators and spaces. First, we denote the operator \( J_{N,l,n}^i : Q_{N}^{n,i} \rightarrow Q_{N}^{n+2} \) by
\[
J_{N,l,n}^i (q, \phi, \psi) = \left( q|_{\Omega_F^{n,i}}, q|_{\Gamma^{n,i}}, \psi \right),
\]
and space \( Q_{N}^{n,i} \) by
\[
Q_{N}^{n,i} = \left\{ (q, \phi, \psi) \in \left( V_{F}^{n,i} \cap H^4 (\Omega_{F}^{n,i}) \right) \times V_{W} \times V_{S} : q|_{\Gamma^{n}} \cdot \nu_{F}^{n} = \phi \cdot \nu_{F}^{n}, \phi = \psi|_{\Gamma} \right\}.
\]
Moreover, to establish Property (C2), we need the operator \( I_{N,l,n}^i : V_{N}^{n+2} \rightarrow V_{N}^{n,i} \) as
\[
I_{N,l,n}^i (u_{N}^{n+2}, v_{N}^{n+2}, V_{N}^{n+2}) = \left( u_{N}^{n+2}|_{\Omega_{F}^{n,i}}, (u_{N}^{n+2}|_{\Gamma^{n}} \cdot \nu_{F}^{n}) \cdot \nu_{F}^{n} + (v_{N}^{n+2} \cdot \tau_{F}^{n}) \cdot \tau_{F}^{n}, V_{N}^{n+2} \right),
\]
with space
\[
V_{N}^{n,i} = \left\{ (u, v, V) \in H^4 (\Omega_{F}^{n,i}) \times L^2 (\Gamma) \times L^2 (\Omega_{S}) : \nabla \cdot u = 0, (u|_{\Gamma^{n}} - v) \cdot \nu_{F}^{n} = 0 \right\}.
\]
Finally, we complete the proof following [5] and [33].

**Remark 3.3.** In our work, the fluid is motioned by a generalized Non-Newtonian constitutive which results in a \( L^p \) regularity. However, in order to pass to the limits in the next section, we need the \( L^2 \) strong convergence instead of the \( L^p \) strong convergence. This is because the convergence is used in the first and second convergence of equation [69] and it needs a \( L^2 \) regularity.

The compactness stated in Theorem 3.1 implies the following strong convergence results.

**Corollary 3.1.** As \( N \rightarrow \infty \), the following strong convergence results hold:
1. \( u_N \rightarrow u \) in \( L^2 (0, T ; L^2 (\Omega_F)) \);
2. \( v_N \rightarrow v \) in \( L^2 (0, T ; L^2 (\Gamma)) \),
3. \( v_N \rightarrow v \) in \( L^2 (0, T ; L^2 (\Gamma)) \).

### 3.8.2 Strong convergence for displacement and geometry parameters

In the following, we give the strong convergence of the thin structure displacement. To achieve this goal, we notice that \( n_N \) is uniformly bounded in \( W^{1,\infty} (0, T ; L^2 (\Gamma)) \cap L^\infty (0, T ; H_0^2 (\Gamma)) \), then from the continuous embedding
\[
W^{1,\infty} (0, T ; L^2 (\Gamma)) \cap L^\infty (0, T ; H_0^2 (\Gamma)) \hookrightarrow C^{0,1-\beta} ([0, T] ; H^{2\beta} (\Gamma)), \quad 0 < \beta < 1,
\]
and we obtain
\[
C^{0,1-\beta} ([0, T] ; H^{2\beta} (\Gamma)) \hookrightarrow C^{0,1-\beta} ([0, T] ; H_0^{2\beta} (\Gamma)).
\]

Finally, we complete the proof following [5] and [33].
we have uniform boundedness of $\eta_N$ in $C^{0,1-\beta}([0,T]; H^{2\beta}(\Gamma))$. Due to the compact embedding of $H^{2\beta} \hookrightarrow H^{2\beta-\epsilon}$ for every fixed $\epsilon > 0$ and the fact that functions in $C^{0,1-\beta}([0,T]; H^{2\beta}(\Gamma))$ are uniformly continuous in time on finite interval, we find, by applying the Arzela-Ascoli Theorem, that as $N \to \infty$

$$\eta_N \to \eta \text{ in } C([0,T]; H^{2s}(\Gamma)), \quad 0 < s < 1,$$

and

$$T_N \eta_N \to \eta \text{ in } C([0,T]; H^{2s}(\Gamma)), \quad 0 < s < 1.$$

Then by the similar procedure in [29, Lemma 3], we have the following strong convergence results for structure displacement.

**Theorem 3.2.** We have the following strong convergence results as $N \to \infty$:

1. $\eta_N \to \eta$ in $L^\infty(0,T; H^{2s}(\Gamma))$, $0 < s < 1$;
2. $T_\Delta \eta_N \to \eta$ in $L^\infty(0,T; H^{2s}(\Gamma))$, $0 < s < 1$.

Consequently, considering our 2D fluid problem and 1D structure problem, we have $H^2(\Gamma) \hookrightarrow C^1(\Gamma)$ for $s > \frac{3}{2}$, Theorem 3.2 implies the following result.

**Corollary 3.2** (Convergence for displacement). The following uniform convergence results hold as $N \to \infty$:

1. $\eta_N \to \eta$ in $L^\infty(0,T; C^1(\Gamma))$;
2. $T_\Delta \eta_N \to \eta$ in $L^\infty(0,T; C^1(\Gamma))$.

To pass to the limit in the weak formulation, we still need the convergence of geometry parameters due to the effect of Navier-slip. In this sense, both normal and tangential structure displacements are considered to be non-zero. This may bring additional difficulties when take $N \to \infty$. By means of above statements and the explicit formulas of the normals $\nu_{F,N}$, $\tau_{F,N}$ and quantities associated with $A_N$, we can deduce the corresponding strong convergence result as follows:

**Corollary 3.3** (Convergence for geometry quantities, see also [32]). For $\nu_{F,N}$, $\tau_{F,N}$ and quantities associated with $A_N$ as defined earlier, we have the following convergence as $N \to \infty$:

1. $\nu_{F,N} \to \nu_{F}^\eta$ in $L^\infty(0,T; C(\Gamma))$;
2. $\tau_{F,N} \to \tau_{F}^\eta$ in $L^\infty(0,T; C(\Gamma))$;
3. $w_N \to w^\eta$ in $L^2(0,T; H^1(\Omega_F))$;
4. $J_{F,N} \to J_{F}^\eta$ in $L^\infty(0,T; C(\Gamma))$;
5. $J_N \to J^\eta$ in $L^\infty(0,T; C(\Omega_F))$;
6. $T_N J_N \to J^\eta$ in $L^\infty(0,T; C(\Omega_F))$;
7. $\frac{1}{\sqrt{A_N}} \to \frac{1}{\sqrt{A^\eta}}$ in $L^\infty(0,T; C(\Omega_F))$.

4 The limiting problem

In the first part of this section, we construct the suitable test functions that converge to the test functions in weak formulation. Then, we pass to the limit of approximate problem by means of the weak and strong convergence results we obtained before.
4.1 Construction of suitable test functions

Since the test functions in [34] for the limiting problem depend on \( \eta \), we are in the position to construct the test functions for limiting problem and for the approximate problem due to the fact that test functions rely on the parameter \( N \).

To eliminate the dependence of test functions on \( N \), we follow the same ideas proposed in [6, 30]. Our goal is to restrict the space of all test functions \( Q^\eta(0,T) \) to a dense subset, which is denoted by \( X^\eta(0,T) \). Then we construct a sequence of \( q_N \) of test functions such that for every \( q \in X^\eta(0,T) \), \( q_N \to q \) in suitable norms. This idea has been used in different fluid-structure interaction problems, see e.g., [5, 29, 30, 32, 39].

First, we denote define the uniform domain which contains all the approximate domains as

\[
\Omega_{\text{max}} = \bigcup_{\Delta t > 0, n \in N} \Omega_{\tilde{F}_N}^n.
\]

Next, we introduce

\[
X_{\text{max}} = \{ r \in C_c^1([0,T); C^p(\overline{\Omega_{\text{max}}})) : \nabla \cdot r = 0, r \cdot \tau = 0, \text{ on } \Gamma_{\text{in/out}}, r \cdot \nu_F = 0, \text{ on } \Gamma_b \}
\]

and

\[
X^\eta(0,T) = \left\{ (q, \phi, \psi) : \begin{array}{l}
q(t,\cdot) = r(t,\cdot)|_{\Omega_{\tilde{F}_N}^n(t)} \circ A_{\eta}^n(t), r \in X_{\text{max}}, \\
\psi|_\Gamma = \phi, (\nu|_{\Gamma_F} - \phi) \cdot \nu_F^n = 0, \phi \in H^2_0(\Gamma), \psi \in H^1_0(\Omega_S) \end{array} \right\}.
\]

It can be easily checked that \( X^\eta(0,T) \) is dense in \( Q^\eta(0,T) \).

Then we define the approximate test functions \((q_N, \phi_N)\) in \((n-1)\Delta t, n\Delta t]\) as

\[
q_N(t,\cdot) = q_N^n := r_n(n\Delta t,\cdot)|_{\Omega_{\tilde{F}_N}^n(t)} \circ A_{\eta}^n(t),
\]

\[
\phi_N(t) = \phi_N^n := \phi(n\Delta t).
\]

It is clear that \((q_N(t,\cdot), \phi_N(t,\cdot), \psi(t,\cdot))\) \(\in W^n\) for \( t \in ((n-1)\Delta t, n\Delta t] \). Fixing \((q, \phi, \psi) \in X^\eta(0,T)\) with \( q(t,\cdot) = r(t,\cdot)|_{\Omega_{\tilde{F}_N}^n(t)} \circ A_{\eta}^n(t), r \in X_{\text{max}}, \phi \in H^2_0(\Gamma), \psi \in H^1_0(\Omega_S) \), we obtain the following lemma using the idea from [29, 30, 32].

**Lemma 4.1** ([32]). For every \((q, \phi, \psi) \in X^\eta(0,T),\) we have

1. \((q_N, \phi_N) \to (q, \phi)\) in \( L^\infty(0,T; C^1(\Omega_F)) \times L^\infty(0,T; C^1(\Gamma)) \),
2. \( d\eta_N \to \partial_t q \) in \( L^2(0,T; L^2(\Omega_F)) \).

**Lemma 4.2** (Convergence of gradients). \( M = D^n u \) and \( G = S(D^n(u)) \), where \( u \) and \( \eta \) are the weak* limits given by Lemma 3.8. \( M \) and \( G \) are the weak limits given by Lemma 3.9.

**Proof.** As in [30, 5], it is helpful to map the approximate fluid velocities and the limiting fluid velocity onto the physical domains. For that purpose, we introduce the following functions

\[
\chi^N g(t,x) = \begin{cases} 
\{g, & x \in \Omega_{\tilde{F}_N}^n(t), \\
0, & x \notin \Omega_{\tilde{F}_N}^n(t)
\end{cases}, \quad \chi\tilde{g}(t,x) = \begin{cases} 
\{g, & x \in \tilde{\Omega}_F^n(t), \\
0, & x \notin \tilde{\Omega}_F^n(t)
\end{cases},
\]

where \( A_{\eta} \) is the ALE mapping defined in Section 2.3 and \( \eta \) is the weak* limit in Lemma 3.8. We follow the ideas in [30, Proposition 7.6] and [5, Section 10.2], and provide a sketch of proof in the following three steps:
Step 1. The strong convergence
\[ \chi^N u^N \to \chi \hat{u}, \text{ strongly in } L^2((0, T) \times \Omega_{\text{max}}). \]
can be easily checked from [30, Proposition 7.6] so we omit it here.

Step 2. We need to prove
\[ \chi^N D(u^N) \to \chi D(\hat{u}), \text{ weakly in } L^p((0, T) \times \Omega_{\text{max}}), \]
\[ \chi^N S(D(u^N)) \to \chi S(D(\hat{u})), \text{ weakly in } L^q((0, T) \times \Omega_{\text{max}}), \]
where \( D(u^N) = D\tilde{\eta}_N(u_N). \) From Lemma 3.7 there exist \( \tilde{M} \) and \( \tilde{G}, \) such that \( \chi^N D(u^N) \to \tilde{M} \) weakly in \( L^p((0, T) \times \Omega_{\text{max}}) \) and \( \chi^N S(D(u^N)) \to \tilde{G} \) weakly in \( L^q((0, T) \times \Omega_{\text{max}}), \) i.e.,
\[
\int_0^T \int_{\Omega_{\text{max}}} \tilde{M} \cdot y = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}} \chi^N D(u^N) \cdot y, \quad y \in C_c^\infty((0, T) \times \Omega_{\text{max}}),
\]
\[
\int_0^T \int_{\Omega_{\text{max}}} \tilde{G} \cdot y = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}} \chi^N S(D(u^N)) \cdot y, \quad y \in C_c^\infty((0, T) \times \Omega_{\text{max}}),
\]
In order to obtain \( \tilde{M} = \chi D(\hat{u}) \) and \( \tilde{G} = \chi S(D(\hat{u})), \) we divide \( \Omega_{\text{max}} \) into \( \Omega_{\text{max}}^p(t) \) and \( \Omega_{\text{max}} \setminus \Omega_{\text{max}}^p(t). \) Taking the test function \( y \) such that \( y \) is supported in \( (0, T) \times \Omega_{\text{max}} \setminus \Omega_{\text{max}}^p(t) \) and combining the uniform convergence of \( \tilde{\eta}_N = \mathcal{T}_N \eta_N, \) we find that \( \tilde{M} = \tilde{G} = 0 \) in \( (0, T) \times (\Omega_{\text{max}} \setminus \Omega_{\text{max}}^p(t)). \)

Next, taking a test function \( z \) such that \( \text{supp } z \subset ((0, T) \times \Omega_{\text{max}}^p(t)) \) and combining the uniform convergence of \( \eta_N = \mathcal{T}_N \eta_N, \) we get
\[
\int_0^T \int_{\Omega_{\text{max}}} \tilde{M} \cdot z = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}} \chi^N D(u^N) \cdot z = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}^p(t)} D(u^N) \cdot z,
\]
\[
\int_0^T \int_{\Omega_{\text{max}}} \tilde{G} \cdot z = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}} \chi^N S(D(u^N)) \cdot z = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}^p(t)} S(D(u^N)) \cdot z.
\]
Since the strong convergence \( \chi^N u^N \to \chi \hat{u} \) holds as shown in Step 1, we find that on the set \( \text{supp } z, \) both \( u^N \to \hat{u} \) and \( D(u^N) \to D(\hat{u}) \) in the sense of distributions. Consequently, we have
\[
\int_0^T \int_{\Omega_{\text{max}}} \tilde{M} \cdot z = \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}^p(t)} D(u^N) \cdot z = \int_0^T \int_{\Omega_{\text{max}}^p(t)} D(\hat{u}) \cdot z, \quad (61)
\]
for all the test functions \( z \) supported in \( (0, T) \times \Omega_{\text{max}}^p(t). \) Due to the uniqueness of the limit, we obtain
\[ \tilde{M} = D(\hat{u}) \text{ a.e. in } (0, T) \times \Omega_{\text{max}}^p(t). \]

However, since the viscosity is nonlinear, we need a different approach to show \( S(D(u^N)) \to S(D(\hat{u}))) \) in the sense of distributions. We notice the \( p \)-Laplacian structure of \( S \) and proceed to use the “Minty’s trick” to obtain the value of \( \tilde{G}. \) Before doing that, we need a localized Minty’s Trick to make sure the domain is unchanged since our domain here is \( (0, T) \times \Omega_{\text{max}}. \)

The following localization of the Minty’s Trick is due to [10, Appendix A].

Proposition 4.1 (Localized Minty’s Trick [10, Appendix A]). Let \( u_m \in L^p(0, T; W^{1,p}(\Omega)) \) and \( \zeta_m \in L^\infty(Q) \) with \( Q = (0, T) \times \Omega. \) If for \( a_0 = \text{const} > 0, \)
\[
0 \leq \zeta_m \leq a_0 \text{ a.e. in } Q, \quad m \in \mathbb{N},
\]
Then

\[ \tilde{\zeta} = S(D(u)) \zeta \text{ a.e. in } Q. \]  

Let \( \Omega = \Omega_{\text{max}}, u_m = u^N, u = \tilde{u} \) and \( \tilde{S} = \tilde{G} \) in Proposition 4.1. We define \( \zeta_N(t, \cdot) \), \( \zeta(t, \cdot) \) as

\[
\zeta_N(t, x) = \begin{cases} 
1, & x \in \Omega_{F}^N(t), \\
0, & x \in \Omega_{\text{max}} \setminus \Omega_{F}^N(t),
\end{cases}
\]

\[
\zeta(t, x) = \begin{cases} 
1, & x \in \Omega_{F}^N(t), \\
0, & x \in \Omega_{\text{max}} \setminus \Omega_{F}^N(t).
\end{cases}
\]

It is easy to check \( \zeta_N \to \zeta \) a.e. in \( Q \) as \( N \to \infty \), which means that \( \tilde{G} \) and \( (65) \) are satisfied.

Also, we have \( D(u_N) \to \tilde{D}(\tilde{u}) \) in \( L^p((0, T) \times \Omega_{\text{max}}) \) and \( S(D(u_N)) \rightharpoonup \tilde{G} \) in \( L^q((0, T) \times \Omega_{\text{max}}) \) as obtained earlier. To verify \( \text{(66)} \) in Proposition 4.1, we carry out a direct calculation to find

\[
\left| \int_Q S(D(u^N)) : D(u^N) \zeta_N - \int_0^T \int_Q \tilde{G} : \tilde{D}(\tilde{u}) \zeta \right| \\
\leq \left| \int_Q \left( S(D(u^N)) - \tilde{G} \right) : D(u^N) \zeta_N \right| + \left| \int_Q \tilde{G} : (D(u^N) - D(\tilde{u})) \zeta_N \right| \\
+ \left| \int_Q \tilde{G} : \tilde{D}(\tilde{u}) (\zeta_N - \zeta) \right|.
\]

By the convergences of \( S(D(u^N)), D(u^N) \) and \( \zeta_N \), we have \( \text{(66)} \). Then from Proposition 4.1, we achieve \( \tilde{G} \zeta = S(D(\tilde{u})) \zeta \) a.e. in \( Q \), which means

\[
\tilde{G} = S(D(\tilde{u})) \text{ a.e. in } (0, T) \times \Omega_{F}^N(t).
\]

**Step 3.** Finally, we are in the position to show that

\[
\int_0^T \int_{\Omega_F} \tilde{M} : \tilde{q} = \int_0^T \int_{\Omega_F} \tilde{D}^\gamma(u) : \tilde{q},
\]

\[
\int_0^T \int_{\Omega_F} \tilde{G} : \tilde{q} = \int_0^T \int_{\Omega_F} S(D^\gamma(u)) : \tilde{q},
\]

for every test function \((\tilde{q}, 0, 0) \in \mathcal{X}^\gamma(0, T)\). It follows from the results of Step 2, the uniform boundedness and convergence of gradients \( D^\gamma_N(u_N) \) and \( S(D^\gamma_N(u_N)) \) provided by Lemma 3.9, the strong convergence of \( \frac{1}{\sqrt{A_N^N}} \) given in Corollary 3.3 and the strong convergence of the test functions \( q_N \to \tilde{q} \) obtained in Lemma 4.1 as well as \((23)\) that

\[
\int_0^T \int_{\Omega_F} \tilde{M} : \tilde{q} = \lim_{N \to \infty} \int_0^T \int_{\Omega_F} \tilde{D}^\gamma_N(u_N) : q_N \\
= \lim_{N \to \infty} \int_0^T \int_{\Omega_{\text{max}}} \frac{1}{\sqrt{A_N^N}} \chi_N D(u^N) : q \\
= \int_0^T \int_{\Omega_F} \frac{1}{\sqrt{A_N^N}} D(\tilde{u}) : q.
\]
\[ \int_0^T \int_{\Omega_F} D^n(u) : \tilde{q} \, dV \, dt = \int_0^T \int_{\Omega_F} D^\eta(u) : \eta \, dV \, dt, \]

\[ \int_0^T \int_{\Omega_F} G : \tilde{q} = \lim_{N \to \infty} \int_0^T \int_{\Omega_F} \mathcal{S}(D^\eta_N(u_N)) : q_N \]

\[ = \lim_{N \to \infty} \int_0^T \int_{\Omega_F} \frac{1}{\nabla A_N} \chi^N \mathcal{S}(D(u_N)) : q \]

\[ = \int_0^T \int_{\Omega_F} \frac{1}{\nabla A_N} \mathcal{S}(D(\tilde{u})) : q \]

\[ = \int_0^T \int_{\Omega_F} \mathcal{S}(D^\eta(u)) : \tilde{q}. \]

where we have used \( D(u^N) = D^\eta_N(u_N) \) and \( D(\tilde{u}) = D^\eta(u) \) from Section 2.3. This completes the proof. \( \square \)

By the analogous argument above, it is easy to deduce the following corollary.

**Corollary 4.1** ([30]). For every \((q, \phi, \psi) \in \mathcal{X}^n(0, T)\), we have \( D^\eta_N(q_N) \to D(q) \) in \( L^p(0, T; L^p(\Omega_F)) \).

### 4.2 Pass to the limit

To get the weak formulation of the coupled problem, setting \((\phi_N, \psi)\) as the test functions in (44) and integrating it over \((n+1)\Delta t\), taking \((q_N, \phi_N)\) as the test functions in (54), multiplying \( \frac{1}{\Delta t} \), again integrating it over \((n+1)\Delta t\), and adding the two equations together, we have

\[ \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_F} J_N^+ \frac{u_N^{n+1} - u_N^n}{\Delta t} \cdot q_N + \frac{1}{2} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_F} J_N^+ \frac{u_N^{n+1} - u_N^n}{\Delta t} u_N^{n+1} \cdot q_N + \frac{1}{2} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_F} J_N^+ \mathcal{S}(D(u_N^{n+1})) : D(q_N) + \frac{1}{\alpha} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Gamma} (u_N^{n+1} - u_N^n) \cdot \phi_N \cdot \psi + \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_S} v_N^{n+1} \cdot \mathcal{S}(D(u_N^{n+1})) : D(q_N) + \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Gamma} (\mathcal{L} u_N^{n+1}, \phi_N) + \int_{n\Delta t}^{(n+1)\Delta t} \alpha_S(D(u_N^{n+1}), \psi_N) + \int_{n\Delta t}^{(n+1)\Delta t} \langle f(u_N^{n+1}), \phi_N \rangle = \int_{n\Delta t}^{(n+1)\Delta t} \langle R^{n+1}, q_N \rangle. \]

Summing (68) from \( n = 0, 1, \ldots, N \), we obtain the weak formulation of approximate problem over \((0, T)\) as follows:

\[ \int_0^T \int_{\Omega_F} \mathcal{T}_N J_N \partial_t u_N^* : q_N + \frac{1}{2} \int_0^T \int_{\Omega_F} J_N - \mathcal{T}_N J_N \frac{u_N \cdot q_N}{\Delta t}. \]
\[\begin{align*}
+ \frac{1}{2} & \int_0^T \int_{\Omega_F} J_N \left( ((T_N w - w_N) \cdot \nabla \eta_N) u_N \cdot q_N 
- ((T_N u - w_N) \cdot \nabla \eta_N) q_N \cdot u_N \right) \\
+ 2 & \int_0^T \int_{\Omega_F} J_N S(\mathcal{D}^{\eta N}(u_N)) : \mathcal{D}^{\eta N}(q_N) \\
+ \frac{1}{\alpha} & \int_0^T \int_{\Gamma} (u_N, \tau_{F,N} - v_N, \tau_{F,N}) (q_N, \tau_{F,N} - \phi_{N}, \tau_{F,N}) J_{F,N} \\
+ & \int_0^T \int_{\Omega_S} \partial_t v_N^* \cdot \phi_N + \int_0^T \int_{\Omega_S} \partial_t V_{N}^* \cdot \psi \\
+ & \int_0^T \langle \mathcal{L}_e \eta_N, \phi_N \rangle + \int_0^T a_S(d_N, \psi) + \int_0^T \langle f(\eta_N), \phi_N \rangle \\
= & \int_0^T \langle R_N, q_N \rangle,
\end{align*}\]

where \(u^*_N, v^*_N\) and \(V_N^*\) are the piece-wise linear approximations of \(u_N, v_N\) and \(V_N\), that is for \(t \in [n \Delta t, (n + 1) \Delta t]\),

\[\begin{align*}
   u^*_N(t) &= u^*_N(t) = u^*_N(t) + \frac{u^{n+1}_N - u^n_N}{\Delta t} (t - \Delta t), \\
   v^*_N(t) &= v^*_N(t) = \frac{v^{n+1}_N - v^n_N}{\Delta t} (t - \Delta t), \\
   V^*_N(t) &= V^*_N(t) = V^{n+1}_N - \frac{V^n_N}{\Delta t} (t - \Delta t).
\end{align*}\]

In the sequel, we will take the limit \(N \to \infty\), which means \(\Delta t = \frac{T}{N} \to 0\). Here, we denote \((9)\) as \(\sum_{i=1}^{11} I_i = I_{12}\) and pass to the limit for each term.

1. \(I_1\) and \(I_2\): Using integration by parts and the convergence results earlier, we obtain

\[\begin{align*}
   \int_0^T \int_{\Omega_F} T_N J_N \partial_t v^*_N \cdot q_N + \int_0^T \int_{\Omega_F} \frac{J_N - T_N J_N}{\Delta t} u_N \cdot q_N \\
   \to - \int_0^T \int_{\Omega_F} f_N u \cdot \partial_t q - \int_0^T \int_{\Omega_F} f_N (\nabla q) u \cdot q - \int_{\Omega_F} f_N u_0 q(0).
\end{align*}\]

More details can be found in \([32, \text{Proposition 8}]\).

2. \(I_3\): We need to show the convergence of each term in \(I_3\). First, the convergence of \(T_N J_N, u_N, w_N, q_N, \nabla \eta_N u_N\) and \(\nabla \eta_N q_N\) can be derived directly from the previous Lemmas. From the estimate 3 in Lemma \(3.5\), we have

\[\|T_N u_N - u_N\|_{L^2(0,T;L^2(\Omega))} \leq C \sum_{n=1}^{N} \|u^{n+1}_N - u^n_N\|_{L^2(\Omega)} \Delta t \leq C \Delta t,\]

which means \(T_N u_N \to u\). Therefore, we obtain the convergence of \(I_3\).

3. \(I_4\): From the convergence of \(J_N, S(\mathcal{D}^{\eta N}(u_N))\) and \(\mathcal{D}^{\eta N}(q_N)\) in some appropriate function spaces, we have

\[\begin{align*}
   \int_0^T \int_{\Omega_F} J_N S(\mathcal{D}^{\eta N}(u_N)) : \mathcal{D}^{\eta N}(q_N) - \int_0^T \int_{\Omega_F} f^{\eta N} S(\mathcal{D}^{\eta N}(u)) : \mathcal{D}^{\eta N}(q) \to \int_0^T \int_{\Omega_F} f^{\eta N} S(\mathcal{D}^{\eta N}(u)) : \mathcal{D}^{\eta N}(q).
\end{align*}\]
\[
\begin{align*}
&= \int_0^T \int_{\Omega_F} J_N (S(D^N(u_N)) - S(D(u))) : D^N(q_N) \\
&\quad + \int_0^T \int_{\Omega_F} (J_N - J^\eta) S(D(u)) : D^N(q_N) \\
&\quad + \int_0^T \int_{\Omega_F} J^\eta S(D(u)) : (D^N(q_N) - D(q)) \\
&\to 0, \text{ as } N \to \infty.
\end{align*}
\]

4. \(I_5\) and \(I_6\): It is easy to get the convergence of \(I_5\) and \(I_6\) by using the convergence of \(u_N\), \(v_N\) and the geometric quantities in Corollary 3.3.

5. \(I_7\) and \(I_8\): Using the convergence of \(v^*_N\) and \(V^*_N\), and integrating by parts yield

\[
\begin{align*}
&\int_0^T \int_{\Gamma} \partial_t v^*_N \cdot \phi_N \to - \int_0^T \int_{\Gamma} v \cdot \partial_t \phi - \int_{\Gamma} v_0 \cdot \phi(0), \\
&\int_0^T \int_{\Omega_S} \partial_t V^*_N \cdot \psi_N \to - \int_0^T \int_{\Omega_S} V \cdot \partial_t \psi - \int_{\Omega_S} V_0 \cdot \psi(0).
\end{align*}
\]

6. \(I_9\) and \(I_{10}\): From the convergence of \(\eta_N\), \(d_N\) and \(\phi_N\) in some proper spaces, we can deduce the convergence of \(I_9\) and \(I_{10}\) directly.

7. \(I_{11}\): Since \(f\) is locally Lipschitz from \(H^{2-\epsilon}\) to \(H^{-2}\), it follows from the convergence of \(\eta_N\) and \(\phi_N\) that

\[
\begin{align*}
&\int_0^T \langle f(\eta_N), \phi_N \rangle - \int_0^T \langle f(\eta), \phi \rangle \\
&= \int_0^T \langle f(\eta_N) - f(\eta), \phi_N \rangle + \int_0^T \langle f(\eta), \phi_N - \phi \rangle \\
&\leq \int_0^T \| f(\eta_N) - f(\eta) \|_{H^{2-\epsilon}(\Gamma)} \| \phi_N \|_{H^{-2}(\Gamma)} + \int_0^T \| f(\eta) \|_{L^2(\Gamma)} \| \phi_N - \phi \|_{L^2(\Gamma)} \\
&\leq C \int_0^T \| \eta_N - \eta \|_{H^{2-\epsilon}(\Gamma)} \| \phi_N \|_{L^2(\Gamma)} + \int_0^T \| f(\eta) \|_{L^2(\Gamma)} \| \phi_N - \phi \|_{L^2(\Gamma)} \\
&\to 0, \text{ as } N \to \infty.
\end{align*}
\]

8. \(I_{12}\): By the definition of \(R_N\), the convergence of \(q_N\) and \(u\) leads to the convergence of \(I_{12}\).

Therefore, we have shown that the limiting functions \(u\), \(\eta\) and \(d\) as \(N \to \infty\) satisfy the weak form of original problem in the sense of (37) in Definition 2.1 for all test functions \(q\), \(\phi\) and \(\psi\), which are dense in the test space \(Q^{\eta}(0,T)\). This means that the approximate solutions we constructed converge to a weak solution of problem (1)–(15).

Now, we prove Theorem 2.1.

**Proof of Theorem 2.1.** From the above analysis, we have the existence of the weak solution.

To derive (38), we take the limit of estimates 1 and 2 in Lemma 3.5. Thanks to the semi-continuity properties of norms, we recover the energy estimate in (38).

This completes the proof.
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