Half shuffle identities in arity 3 on Zinbiel and Tortkara algebras and a structure theorem for streamed information

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Abstract

We identify the free half shuffle (or Zinbiel) algebra of Schützenberger [Sch58] with an algebra of real-valued functions on paths, where the algebraic half shuffle product ≺ mirrors the analytic integration of a coordinate path against another. We then provide two, to our knowledge, new basic identities in arity 3 involving the commutator (area) and anti-commutator (shuffle) of the half shuffle product, and show that these are sufficient to recover the Zinbiel and Tortkara identities introduced by Dzhumadil’dae in [Dzh07]. We then use these identities to provide a direct proof of the main result in [DLPR20] stating that polynomials in iterated areas generate the half shuffle algebra.

In addition, we introduce minimal sets of iterated path integrals defined iteratively in terms of Hall sets. We then provide a proof based on elementary combinatorial arguments of the main result by Sussmann in [Sus86] stating that polynomial functions on these minimal sets of integrals freely generate the half shuffle algebra. We interpret this result as a structure theorem for streamed information, loosely analogous to the unique prime factorisation of integers, allowing to split any real valued function on streams into two parts: a first that extracts and packages the streamed information into recursively defined atomic objects, and a second that evaluates a polynomial in these objects without further reference to the original stream.

Finally, we construct a canonical, but to our knowledge, new decomposition of the half shuffle algebra as shuffle power series in the greatest letter of the original alphabet with coefficients in a sub-algebra freely generated by a new alphabet with an infinite number of letters. This construction allows us to provide a second proof of our structure theorem.

1 Introduction

It is not too much to accept that, at least on some fine enough time scales, most instance of streamed information (text, sound, video, time series...) can be represented, as a path $\gamma : [0, 1] \to V$ with values in some finite dimensional vector space $V \simeq \mathbb{R}^d$. It was first shown by Chen in [Che57], and then explored in greater detail and generality in the context of rough path theory in [HL10, BGLY16], that any path may be faithfully represented, up to reparameterisation, by the collection of its iterated integrals known as the signature. This non-commutative exponential maps a path to a group-like element on the tensor algebra $(\mathcal{A}, \otimes)$, where $\mathcal{A}$ is the vector space spanned by words in $d$ letters and $\otimes$ is the tensor product; it is the unique solution to the control system $d\text{Sig}_\gamma(t) = \text{Sig}_\gamma(t) \otimes d\gamma(t)$ started at the identity in $\mathcal{A}$, and its range describe the set of characters $G \subset \mathcal{A}$.

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An algebra \((\mathcal{A}, \prec)\) with vector space \(\mathcal{A}\) and product \(\prec\) satisfying the identity
\[
x \prec (y \prec z) = (x \prec y) \prec z + (y \prec x) \prec z, \quad x, y, z \in \mathcal{A}
\]
is called a (left) Zinbiel algebra \([Dzh07]\). It was first shown by Schützenberger in \([Sch58]\) that the algebra \((\mathcal{A}, \prec)\) with the (left) half shuffle product \(\prec\) is free as a (left) Zinbiel algebra. In this paper we will use the terms “Zinbiel” and “half shuffle” algebra to refer to the same algebra \((\mathcal{A}, \prec)\). Any Zinbiel algebra under the anti-commutator product \(x \mp y = x \prec y + y \prec x\) is commutative and associative \([Lod95]\). In the case of the half shuffle algebra of Schützenberger, the commutative algebra \((\mathcal{A}, \mp)\) is commonly known as the shuffle algebra. Any Zinbiel algebra under the commutator product \(\text{area}(x, y) = x \prec y - y \prec x\) satisfies the so-called Tortkara identity \([Dzh07]\):
\[
\text{area}(\text{area}(x, y), \text{area}(x, z)) = \text{area}(x, \text{vol}(x, y, z))
\]
where \(\text{vol}(x, y, z) = \text{area}(\text{area}(x, y), z) + \text{area}(\text{area}(y, z), x) + \text{area}(\text{area}(z, x), y)\). An algebra satisfying this identity is called a Tortkara algebra.

In \([Dzh07]\) it was shown that the Tortkara identity is the only identity satisfied by area in arity 4 and that area does not satisfy any new identity of arity 3; in particular it does not satisfy the Jacobi identity. In this paper, we provide two new basic identities in arity 3 involving the shuffle product \(\mp\) and the product \(\text{area}\), and we show that these identities are sufficient to recover the Zinbiel and Tortkara identities. We then use them to provide a simpler proof of the main result in \([DLPR20]\) stating that polynomials in areas of areas form a generating set of the free Zinbiel algebra.

The half shuffle algebra \((\mathcal{A}, \prec)\) is the algebraic dual of the tensor algebra \((\mathcal{A}, \otimes)\) \([Reu93]\); it is automatic from this perspective to see that the restriction of linear functionals on \(\mathcal{A}\) to the range of the signature \(G\) form a unital algebra of real-valued functions that separates points \([LCL04]\). A straightforward application of the Stone-Weierstrass theorem yields that for any compact set of reparameterisation-reduced paths, linear functionals acting on their signatures are dense in the space of continuous, real-valued functions on this compact set (under a suitable choice of topology \([CT22]\)).

Because \(G\) is the set of characters, Ree’s theorem \([Ree58]\) implies that the restriction of the shuffle product of two of elements of the shuffle algebra to \(G\) is the pointwise product of the two restrictions \((x \mp y, \text{Sig}_G) = (x, \text{Sig}_G)(y, \text{Sig}_G)\), the so-called shuffle identity. This interplay between algebraic operations on the free Zinbiel algebra and calculus on paths can be extended to the half shuffle product, representing integration of a coordinate path against another \((x \prec y, \text{Sig}_G) = \int_0^1 (x, \text{Sig}_G)(s)d(y, \text{Sig}_G)(s)\), and to its commutator, the operator area, representing the area enclosed between a projection of the path onto a two dimensional subspace spanned by two of its coordinate paths and the chord connecting the two end points \((\text{area}(x, y), \text{Sig}_G) = \int_0^1 (x, \text{Sig}_G(s))d(y, \text{Sig}_G(s)) - \int_0^1 (y, \text{Sig}_G(s))d(x, \text{Sig}_G(s))\).

Thus, collectively iterated integrals provide an accurate description of the path and linear combinations of them can be determined easily by regression, making the coefficient of the signature an ideal feature set for doing machine learning on streamed data \([Lyo14] [KRPA19] [LSC21]\). However, these integrals contain some redundancy, in the sense that some can be expressed as polynomials in others. This represents a major scalability issue, particularly because the number of distinct and linearly independent iterated integrals grows exponentially with the degree of iteration in the integral. This raises a simple set of questions which we answer positively in this paper:

*Can we identify minimal sets of integrals so that each integral is an integral of two other integrals in the same class and so that every other integral can be expressed as a polynomial in them?*

The minimal sets of integrals we identify in this paper are defined hierarchically using sets of binary planar rooted trees called Hall sets \([Reu93] [Reu08]\), and can be computed recursively in a
localised way (to compute one, one must compute its ancestors but not others) which adds further
value to the results. These minimal sets of integrals fully describes the information in the stream
while the polynomials capture the nonlinearity in any function of interest. It is for this reason we
call it a structure theorem, loosely analogous to the unique factorisation of integers as products of
primes. In this way we see that identifying a basis for the space of smooth functions acting on
pathsplits the evaluation process into two parts: a) a first that engages with the underlying
stream of information, systematically extracts and packages the relevant information into atomic
objects whilst removing what’s irrelevant, b) a second that evaluates a unique polynomial function
in these expensive but informative precomputed basis elements in order to deliver the desired
function evaluation without further reference to the original stream γ.

Having established that polynomials in Hall integrals freely generate the half shuffle algebra (A, ≺),
it is natural to ask whether a similar structure theorem holds in the case where the half shuffle ≺ on
Hall trees is replace by the commutator area as basic operation. This question has been, and still
remain, of a source of conjecture, well supported by calculation, for the last decade. Nonetheless,
the search for an answer to this conjecture led us to consider an argument related to the well-known
Lazard’s elimination [Reu93] to construct a canonical, but to our knowledge, new decomposition
of the half shuffle algebra as shuffle power series in the greatest letter of the original alphabet with
coefficients in a sub-algebra freely generated by a new alphabet with an infinite number of letters.
This construction, that we refer to as elimination trick, allowed us to provide a second proof relying
on an induction argument for our structured theorem.

We now briefly outline the structure of the paper. Section 2 provides a short summary of the
algebraic background necessary for the rest of the paper. In Section 3 we make precise the interplay
between the algebraic operations ≺, /A, area on the free Zinbiel algebra and the corresponding
calculus operations on paths. We then provide two new identities in arity 3 involving the shuffle
product and the area product, and we show that these identities are sufficient to recover the Zinbiel
and Tortkara identities. In Section 4 we use these new identities to give a simpler proof of the
main result in [DLPR20], stating that polynomials in iterated areas generate the shuffle algebra. In
Section 5 we present our structure theorem for streamed information, providing a simple proof of
the main result in [Sus86] reported without proof also in [Kaw99, GK08] stating that polynomials
in Hall integrals freely generate the half shuffle algebra. Finally, using the elimination trick we
provide another even simpler proof of our structure theorem.

2 Background

First, we remind the reader in a very terse form of the general collection of objects about which
we write. Much more can be found by looking in Bourbaki [Bou08] or (and we will follow this
for the results we need) Reutenauer [Reu93]. We hope the paper is self contained, and cites what
is needed, but for the rest of this introduction, we will be very brief and assume the reader has
familiarity with the general algebraic framework.

2.1 The tensor algebra and the free Lie algebra

The starting point will be a finite alphabet A of d letters.

Definition 2.1. A word on the alphabet A is a finite sequence of letters from A, including the
empty sequence, called the empty word and denoted by e. We denote by W_A the set of all words,
including the empty word. W_A with the concatenation product is a monoid, that is free over A.
The length |w| of a word w ∈ W_A is the number of letters in w. We denote by W_n the set of
non-empty words. Finally, we denote by A the vector space spanned by all words in W_A.

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3This information extraction is done in practice via some physical integration process that responds to the under-
lying signal. Physical integration processes are intrinsically nasty as mathematical operators (controlled differential
equations in general, and in particular the integration process here, are not closable in the uniform topology on γ -
see [Lyo08]).
\( \mathcal{A} \) is graded by word length. The words of length greater than \( n \in \mathbb{N} \) span an ideal, and the quotient of \( \mathcal{A} \) by this ideal is often referred to as the truncated tensor algebra \( \mathcal{A}^{(n)} \).

**Definition 2.2.** Denote by \( (\mathcal{A}, \otimes) \) the tensor algebra over \( \mathcal{A} \), that is the free associative \( \mathbb{R} \)-algebra over \( \mathcal{A} \) with the tensor product \( \otimes \).

**Remark 2.3.** An infinite linear combination of words in \( W_\mathcal{A} \) is usually referred to as a *series*. There is a natural duality between \( \mathcal{A} \) and the associative algebra of all series \( \mathcal{A}^\infty \) given by the pairing \( (\cdot, \cdot) : \mathcal{A} \times \mathcal{A}^\infty \to \mathbb{R} \) defined as

\[
(a, b) = \sum_{\omega \in W_\mathcal{A}} a_\omega b_\omega
\]

where \( a_\omega, b_\omega \) denote the coefficients in front of the word \( \omega \) in \( a, b \) respectively. Note that this sum is finite because \( a \) is a finite linear combination of words. With this pairing, \( \mathcal{A}^\infty \) can be identified as the dual space of \( \mathcal{A} \). When restricted to \( \mathcal{A} \times \mathcal{A} \), this pairing yields a scalar product with \( W_\mathcal{A} \) as dual basis. If \( u, v \) are words with letters in \( \mathcal{A} \) then in the expression \( (u, v) \), the word \( u \) is in \( W_\mathcal{A} \), the word \( v \) is in \( \mathcal{A} \), and the action of \( u \) on \( v \) is zero unless \( u \) and \( v \) are the same word. In the sequel we allow implicit and free conversion of letters and words according to context and by a slight abuse of notation we will use the same notation \( W_\mathcal{A} \) for the basis and its dual.

\( (\mathcal{A}, [\cdot, \cdot]) \) is also a Lie algebra with Lie bracket \( [x, y] = x \otimes y - y \otimes x \) for \( x, y \in \mathcal{A} \).

**Definition 2.4.** Denote by \( (\mathcal{L}_\mathcal{A}, [\cdot, \cdot]) \) the Lie algebra generated by \( \mathcal{A} \) in \( \mathcal{A} \), i.e. the intersection of all Lie algebras in \( \mathcal{A} \) containing \( \mathcal{A} \).

**Lemma 2.5.** [Reu93, Theorem 0.5] \( (\mathcal{L}_\mathcal{A}, [\cdot, \cdot]) \) is the free Lie algebra over \( \mathcal{A} \).

**Remark 2.6.** The maps \( \exp \) and \( \log \) are classically defined as power series mapping \( \mathcal{A}^\infty \) to \( \mathcal{A}^\infty \). The truncated power series for \( \exp^{(n)} \) and \( \log^{(n)} \) provide good meaning for these operators as maps from \( \mathcal{A} \) into \( \mathcal{A} \). Those elements in \( \mathcal{A} \) that are, at each truncated level \( n \in \mathbb{N} \), in \( \mathcal{L}_\mathcal{A} \) are known as *Lie elements* and denoted by \( \mathcal{L}_\mathcal{A}^{(n)} \). Those elements in \( \mathcal{A} \) that are, at each truncated level, exponentials of Lie elements, or equivalently, whose truncated logarithm is in \( \mathcal{L}_\mathcal{A} \), are known as *grouplike elements* (and they form a group). The maps \( \log \) and \( \exp \) provide a one to one correspondence between group-like elements and Lie elements.

### 2.2 The shuffle algebra

**Definition 2.7.** The *shuffle product* \( \shuffle : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) is a bilinear form defined for any two words \( u, v \in W_\mathcal{A} \) by the following recursive relations

1. \( u \shuffle e = e \shuffle u = u \)
2. \( au \shuffle bv = a(u \shuffle bv) + b(u \shuffle v) \).

The algebra \( (\mathcal{A}, \shuffle) \) is an associative and commutative algebra also known as the *shuffle algebra*.

We report the following three classical results about the shuffle product \( \shuffle \) and the free Lie algebra \( \mathcal{L}_\mathcal{A} \): the first states that the shuffle product characterises grouplike elements [LC10, Lemma 2.17], the second provides a characterisation of Lie elements in \( \mathcal{L}_\mathcal{A} \) [Reu93, Theorem 3.1 (iv)], and the third states that the exponential of Lie elements span the tensor algebra in a way that respects degrees of truncation [DR15, Lemma 3.4].

**Theorem 2.8.** Let \( \ell \in \mathcal{L}_\mathcal{A} \) be a Lie element.

1. For any \( x_1, x_2 \in \mathcal{A} \)

\[
(x_1, \exp(\ell))(x_2, \exp(\ell)) = (x_1 \shuffle x_2, \exp(\ell))
\]

2. For any \( u, v \in W_\mathcal{A} \)

\[
(u \shuffle v, \ell) = 0
\]
The final set of objects we mention in this introduction is given by a set of trees forming the free magma over the alphabet $A$.

Definition 2.12. The free magma $\mathcal{M}_A$ is the minimal non-empty set satisfying: i) $A \subset \mathcal{M}_A$, and ii) if $t', t'' \in \mathcal{M}_A$ then $(t', t'') \in \mathcal{M}_A$. The degree of $t$ is defined recursively as $|t| = 1$ if $t \in A$, otherwise if $t', t'' \in \mathcal{M}_A$ then $|t| = |t'| + |t''|$. 

Remark 2.13. Let $V$ be a vector space. The space $\mathcal{B}$ of bilinear maps $V \times V \to V$ naturally forms a magma, via composition. For a fixed bilinear map $\phi : V \times V \to V$ and a set map $\iota : A \to V$ we abuse notation and also write $\phi : \mathcal{M}_A \to \mathcal{B}$ for the unique morphism of magmas characterized by

$$\phi(a) = \iota(a), a \in A$$
$$\phi((t', t'')) = \phi(\phi(t'), \phi(t'')).$$

Definition 2.14. The foliage map $f : \mathcal{M}_A \to W_A$ is defined on a letter $a \in A$ as $f(a) = a$ and on a tree $t = (t_1, t_2) \in \mathcal{M}_A$ as $f(t) = f(t_1)f(t_2)$ where the product is the tensor product (or concatenation of words).

Remark 2.15. As noted in [Ren93], $\mathcal{M}_A$ can be equivalently identified with the set of binary, planar, rooted trees with leaves labelled in $A$. For a given element $t \in \mathcal{M}_A$ we will refer to the collection of letters appearing in its leaves as its foliage.

This concludes the presentation of the background material.

3. For any degree of truncation $n \in \mathbb{N}$

$$\mathcal{A}^{(n)} = \text{Span}\{\exp^{(n)}(\ell) : \ell \in \mathcal{L}_A^{(n)}\}$$

In light of Theorem and of the following Lemma, the shuffle algebra $(\mathcal{A}, \cdot)$ can be identified with the algebra of $\mathbb{Q}$-polynomial functions on $\mathcal{L}_A$ with pointwise multiplication, denoted by $\mathbb{Q}[\mathcal{L}_A]$.

Lemma 2.9. For any $x \in \mathcal{A}$, the map $\ell \mapsto (x, \exp(\ell))$ is in $\mathbb{Q}[\mathcal{L}_A]$. Furthermore, the map $x \mapsto (x, \exp(\cdot))$ from $\mathcal{A}$ to $\mathbb{Q}[\mathcal{L}_A]$ is bijective.

Proof. This result is classical, so we provide only a sketch of the proof. Any element $x \in \mathcal{A}$ is a finite sum of words in $W_A$ of some maximal length $d(x)$. Fix some basis $(\ell_i)_i$ for $\mathcal{L}_A$ that respects dimension and let $\ell = \sum l_i \ell_i$. Then the map $(s, \exp(\ell)) = (s, \exp(\sum d(\ell_i) \leq d(x) l_i \ell_i))$ and the right hand side, truncated at degree $d(x)$ is clearly a polynomial in the $l_i$. The exponentials of truncated Lie elements are linearly dense in the truncated tensor algebra, therefore $x$ is completely determined by its inner product with the $(x, \exp(\ell))$ as $\ell$ varies.

Remark 2.10. It is an immediate corollary of these results, and of the Stone Weierstrass Theorem, that any finite collection of distinct grouplike elements form the vertices of a simplex, and therefore that there is a linear functional that is one on any one of the elements and zero on the others.

Remark 2.11. An analogy can be drawn with the Fourier transform seen as a change of basis for signals from time to frequency domain that turns point-wise multiplication into convolution. In our case, we can view $(\mathcal{A}, \cdot)$ as polynomial functions on $\mathcal{L}_A$ with pointwise multiplication, or as an algebra spanned by words, with the shuffle product, depending on our viewpoint.
3 Identities on Zinbiel and Tortkara algebras

In this section we discuss additional algebraic structures on \( A \) given by the half shuffle product and its commutator, we explain their connection with calculus on pathspace and provide two new basic identities involving these operations that will be used in the next section to prove one of the main result of this paper.

**Definition 3.1.** The (left) half shuffle product \( \prec : A \times A \to A \) is a bilinear form defined for any word \( v \in W_A \) and word \( u \in W_A \) given by the concatenation of \( n \) letters \( u = u_1...u_n \) by the following relations

1. \( e \prec v = e \),
2. \( u \prec v = u_1((u_2...u_n) \sqcup v) \)

It follows from this relations that for any non-empty word \( u \), \( u \prec e = u \).

### 3.1 Connections with calculus

The half shuffle product satisfies many algebraic relations that correspond to fundamental calculus operations on pathspace. Let \( \gamma, \tau, \delta : [0, 1] \to \mathbb{R} \) be three smooth paths.

**Fundamental theorem of calculus:** For any \( x \in A \)

\[ x \prec e = x - (x, e) e \]  

This is an algebraic version of the statement

\[ \int_0^t 1 \, d\gamma_s = \gamma_t - \gamma_0, \quad \forall t \in [0, 1]. \]  

**Product rule:** It is easy to check from the definition of half shuffle that for any \( x, y \in A \) the following relation is satisfied

\[ (x \sqcup y) \prec e = x \prec y + y \prec x \]  

This is an algebraic version of the statement

\[ \int_0^t 1 \, d(\gamma_s \tau_s) = \int_0^t \gamma_s d\tau_s + \int_0^t \tau_s d\gamma_s, \quad \forall t \in [0, 1]. \]  

**Remark 3.2.** It follows that the shuffle product \( \sqcup \) is the anti-commutator of the half shuffle product \( \prec \), that is for any \( x, y \in A \)

\[ x \sqcup y = x \prec y + y \prec x. \]  

**Integration by parts:** Rewriting the left hand side in the product rule gives that for any \( x, y \in A \)

\[ x \sqcup y - (x, e)(y, e)e \prec e = x \prec y + y \prec x \]  

This is an algebraic version of the statement

\[ \gamma_t \tau_t - \gamma_0 \tau_0 = \int_0^t \gamma_s d\tau_s + \int_0^t \tau_s d\gamma_s, \quad \forall t \in [0, 1] \]  

**Chain rule:** It follows directly from the definition of half shuffle that for any \( x, y, z \in A \)

\[ x \prec (y \sqcup z) = (x \prec y) \prec z \]  

This is an algebraic version of the following statement

\[ \int_0^t \gamma_s \tau_s d\delta_u = \int_0^t \gamma_s d\left( \int_0^s \tau_u d\delta_u \right). \]
3.2 The free Zinbiel algebra

The following theorem is due to Schützenberger \cite{Sch58}. A more recent proof is provided in \cite[Proposition 2]{FP13}.

**Theorem 3.3.** The algebra $(A, \prec)$ is free over $A$.

The identity of the next lemma, known in the literature as the Zinbiel identity, is a direct application of the chain rule and integration by parts.

**Lemma 3.4 (Zinbiel identity).** For any $x, y, z \in A$ such that $(x, e) = (y, e) = (z, e) = 0$ the following identity holds

$$x \prec (y \prec z) = (x \prec y) \prec z + (y \prec x) \prec z \quad (14)$$

**Proof.** A direct application of the chain rule and integration by parts yields

$$(x \prec y) \prec z = x \prec (y \oplus z)$$

$$= x \prec (y \prec z + z \prec y + (y, e)(z, e)e)$$

$$= x \prec (y \prec z) + x \prec (z \prec y) + (y, e)(z, e)(x - (x, e)e)$$

$$= x \prec (y \prec z) + x \prec (z \prec y)$$

\[\square\]

**Definition 3.5.** We will refer to the algebra $(A, \prec)$ either as the free half shuffle algebra or as the free Zinbiel algebra.

**Remark 3.6.** It follows that for any $x, y, z \in A$ such that $(x, e) = (y, e) = (z, e) = 0$ one has

$$(x \uplus y) \prec z = (x \prec y + y \prec x) \prec z \quad (15)$$

**Lemma 3.7.** For any $n \geq 2$ and any $x_1, ..., x_n \in A$ such that $(x_1, e) = ... = (x_n, e) = 0$ the following identity holds

$$x_1 \uplus ... \uplus x_n = \sum_{\sigma \in S_n} ((x_{\sigma(1)} \prec x_{\sigma(2)}) \prec ... ) \prec x_{\sigma(n)} \quad (16)$$

where $S_n$ is the symmetric group of order $n$.

The proof is left as an exercise.

3.3 A Tortkara algebra

The area operator is defined as the commutator of the half shuffle product.

**Definition 3.8.** The operator $\text{area} : A \times A \to A$ is the bilinear form defined on any two elements $x, y$ of $A$ by

$$\text{area}(x, y) = x \prec y - y \prec x \quad (17)$$

**Remark 3.9.** We note an important result obtained Dzhumadil’daev in \cite{Dzh07} stating that the area operator satisfies no further identity in arity three, but it does satisfy the so-called Tortkara identity in arity four. While the Tortkara identity will play no further role in this paper, we mention it here for completeness.

**Proposition 3.10 (Tortkara identity).** For any $x, y, z, w \in A$, we equivalently have

$$\text{area}(\text{area}(x, y), \text{area}(x, z)) = \text{area}(x, \text{vol}(x, y, z))$$
and

$$\text{area}(\text{area}(x, y), \text{area}(w, z)) + \text{area}(\text{area}(z, y), \text{area}(w, x))$$

$$= \text{area}(x, \text{vol}(y, z, w)) + \text{area}(z, \text{vol}(y, x, w))$$

where \(\text{vol}(x, y, z) = \text{area}(\text{area}(x, y), z) + \text{area}(\text{area}(y, z), x) + \text{area}(\text{area}(z, x), y)\).

We furthermore note that Tortkara algebras have been studied more in [DIMT19], where it has been shown that the span inside \(A\) of iterated areas of letters forms a free Tortkara algebra for \(|A| = 2\), while the question remains open for larger alphabets.

**Remark 3.11 (left/right areas).** In this paper area is defined as the antisymmetrization of the left half shuffle. In [DLPR20], the right half shuffle is introduced and area is defined as the as the antisymmetrization of the right half shuffle. Although closely connected, these are not identical. The left half shuffle is consistent with [Ren93] and matches the conventions for Hall basis used there (see next section). The right half shuffle is more consistent with the convention used in integration as the integrand is on the left and the integrator is on the right. The reversed order of terms within equations (12) and (13) reflect this dissonance. The proofs of our main results imply equivalent results with the other definition of area, by reversing everything.

### 3.4 Identities in arity 3 on Zinbiel and Tortkara algebras

Contrary to the Lie bracket \([\cdot, \cdot]\), area does not satisfy the Jacobi identity. However, it satisfies the following two non-trivial identities that will be leveraged to prove the first structure theorem in the next section.

**Lemma 3.12 (shuffle-pullout identity).** For any triple \(x, y, z \in A\) the following relation is satisfied

$$3 \text{area}(z, x \shuffle y) = x \shuffle \text{area}(z, y) + y \shuffle \text{area}(z, x) - x \shuffle y \shuffle z$$

$$+ \text{area}(\text{area}(z, y), x) + \text{area}(\text{area}(z, x), y)$$

**Proof.** It’s easy to check that the relation holds for letters \(a, b, c \in A\):

$$3 \text{area}(c, a \shuffle b) = -3abc - 3acb - 3bac - 3bca + 3cab + 3cba$$

$$= a \shuffle \text{area}(c, b) + b \shuffle \text{area}(c, a) - a \shuffle b \shuffle c$$

$$+ \text{area}(\text{area}(c, b), a) + \text{area}(\text{area}(c, a), b)$$

By a classical result from Schutzenberger (see Theorem 3.3) we know that \(A\) is free, as a half shuffle algebra over \(A\), therefore the above relation extends to any triple of elements in \(A\). \(\square\)

**Lemma 3.13 (area-Jacobi identity).** For any triple \(x, y, z \in A\) the following relation is satisfied

$$\text{area}(\text{area}(x, y), z) + \text{area}(\text{area}(y, z), x) + \text{area}(\text{area}(z, x), y)$$

$$= -x \shuffle \text{area}(y, z) - y \shuffle \text{area}(z, x) - z \shuffle \text{area}(x, y)$$

**Proof.** As before, it suffices to check the identity holds for letters \(a, b, c \in A\):

$$\text{area}(\text{area}(a, b), c) + \text{area}(\text{area}(b, c), a) + \text{area}(\text{area}(c, a), b)$$

$$= -abc + acb + bac - bca - cab + cba$$

$$= -a \shuffle \text{area}(b, c) - b \shuffle \text{area}(c, a) - c \shuffle \text{area}(a, b)$$.

\(\square\)
Remark 3.14. Starting only from the identities 1) \( x \uplus y = y \uplus x \), 2) \( \text{area}(x, y) = -\text{area}(y, x) \), 3) shuffle-pullout, 4) area-Jacobi, it follows from simple calculations that one can recover associativity for \( \uplus \) and the (left) Zinbiel identity for the left half shuffle \( \prec \), now defined by \( x \prec y := \frac{1}{2}(x \uplus y + \text{area}(x, y)) \). Through the Zinbiel identity one then can show the Tortkara identity for \( \text{area}(x, y) = x \prec y - y \prec x \) as usual.

4 Polynomials in iterated areas

In this section we present our first main result, namely that polynomial in iterated areas generate the free half-shuffle algebra. We note that this result already appears in [DLPR20], however our proof is significantly shorter and based on induction.

4.1 Polynomials in iterated areas are a generating set

Recalling Remark 2.13, we extend area to \( \mathcal{M}_A \).

Definition 4.1. An element \( x \) of \( A \) is an iterated area if there exists a tree \( t \in \mathcal{M}_A \) so that \( x = \text{area}(t) \).

A shuffle monomial of shuffle-degree \( n \) is the shuffle product of \( n \) iterated areas \( A_1 \uplus \ldots \uplus A_n \) \( (18) \)

The empty monomial \( e \) has shuffle-degree 0. A shuffle polynomial of shuffle-degree \( n \) is a non-degenerate linear combination of such shuffle monomials. Its shuffle-degree is the maximal shuffle-degree of the monomials in the expression.

The sequence defined in the following lemma will play a role in what follows.

Lemma 4.2. The sequence of negative rationals \( \beta_k = -(k - 1)/(k + 1) \) with \( k \geq 1 \) is monotone decreasing to \(-1\) and satisfies the following recursion

\[
\beta_1 = 0, \quad \beta_k = \frac{\beta_{k-1} - 1}{\beta_{k-1} + 3} \quad (19)
\]

Exploiting the identities we introduced in the previous section we give a short and direct proof of the main result in [DLPR20], namely that polynomials in iterated areas generate the half shuffle algebra \( (A, \prec) \).

Theorem 4.3. [DLPR20, Corollary 5.6] Any element in \( (A, \prec) \) can be written as a shuffle polynomial in iterated areas \( \{\text{area}(t) \mid t \in \mathcal{M}_A\} \).

Before reproving the theorem we establish the following fundamental re-writing rule that allows one to rewrite the area of a shuffle polynomial in iterated areas with a single iterated area as a new shuffle polynomial in iterated areas, and provides an explicit expression for the monomial of highest shuffle-degree. The proof will crucially depend on both lemmas 3.12, 3.13.

Theorem 4.4. For any \( n \geq 1 \) and any \( n + 1 \) iterated areas \( A_1, \ldots, A_n, A \), the following relation holds

\[
\text{area}(A, A_1 \uplus \ldots \uplus A_n) = \beta_n A \uplus A_1 \uplus \ldots \uplus A_n + Q \quad (20)
\]

where \( \beta_n = -(n - 1)/(n + 1) \), and \( Q \) is a shuffle polynomial in iterated areas of shuffle-degree at most \( n \).

Remark 4.5. Note that it remains an open problem whether \( \alpha = \beta_n \) is the only real number such that

\[
\text{area}(a, a_1 \uplus \ldots \uplus a_n) - \alpha a \uplus a_1 \uplus \ldots \uplus a_n
\]
can be expressed as a shuffle polynomial in iterated areas of shuffle-degree at most $n$ for any letters $a, a_1, \ldots, a_n$. This question arises due to the fact that iterated areas do not freely generate the shuffle algebra. However, for the example $n = 2$, $\beta_2 = -1/3$ is indeed the only such coefficient because the area-Jacobi identity is the only relation between iterated areas on level 3.

**Proof.** We prove the statement (20) by induction on $n$. If $n = 1$ then the statement is trivially true, with $\beta_1 = 0$ and $Q = \text{area}(A_1, A)$.

Suppose the statement (20) holds for any $n < k$. Consider $k$ iterated areas $A_1, \ldots, A_k$ and an additional iterated area $A$. We recall that the shuffle product $\shuffle$ is associative and commutative on $A$. By the shuffle-pullout identity we have

$$3 \text{area}(A, A_1 \shuffle \ldots \shuffle A_k) = A_1 \shuffle \text{area}(A, A_2 \shuffle \ldots \shuffle A_k) + A_2 \shuffle \ldots \shuffle A_k \shuffle \text{area}(A, A_1) - A_1 \shuffle \ldots \shuffle A_k \shuffle A + \text{area}(\text{area}(A, A_1), A_2 \shuffle \ldots \shuffle A_k) + \text{area}(\text{area}(A, A_2 \shuffle \ldots \shuffle A_k), A_1)$$

By induction ($n = k - 1$) we have that

$$A_1 \shuffle \text{area}(A, A_2 \shuffle \ldots \shuffle A_k) = A_1 \shuffle (\beta_{k-1} A \shuffle A_2 \shuffle \ldots \shuffle A_k + Q'_1) = \beta_{k-1} A \shuffle A_1 \shuffle \ldots \shuffle A_k + A_1 \shuffle Q'_1$$

where $A_1 \shuffle Q'_1$ is a shuffle-polynomial of shuffle-degree $k$. By definition $\text{area}(A, A_1)$ is an iterated area and so

$$Q'_3 = \text{area}(\text{area}(A, A_1), A_2 \shuffle \ldots \shuffle A_k)$$

is a shuffle monomial of shuffle-degree $k$. Similarly, the induction hypothesis implies that

$$Q'_4 = \text{area}(\text{area}(A, A_1), A_2 \shuffle \ldots \shuffle A_k)$$

is a shuffle-polynomial of shuffle-degree $k$, where $Q'_r$ is a shuffle-polynomial of shuffle-degree $k - 1$. Hence, $Q' = A_k \shuffle Q'_1 + Q'_2 + Q'_3$ is a shuffle polynomial of shuffle-degree $k$ and

$$3 \text{area}(A, A_1 \shuffle \ldots \shuffle A_k) = Q' + (\beta_{k-1} - 1) A_1 \shuffle \ldots \shuffle A_k \shuffle A + \text{area}(\text{area}(A, A_2 \shuffle \ldots \shuffle A_k), A_1)$$

It remains to consider the last term $\text{area}(\text{area}(A, A_2 \shuffle \ldots \shuffle A_k), A_1)$.

By the area-Jacobi identity and the anticommutativity of area we can rewrite this term as follows

$$\text{area}(\text{area}(A, A_2 \shuffle \ldots \shuffle A_k), A_1) = \text{area}(\text{area}(A_1, A_2 \shuffle \ldots \shuffle A_k), A) - \text{area}(\text{area}(A_1, A), A_2 \shuffle \ldots \shuffle A_k) + A \shuffle \text{area}(A_1, A_2 \shuffle \ldots \shuffle A_k) - A_2 \shuffle \ldots \shuffle A_k \shuffle \text{area}(A_1, A) - A_1 \shuffle \text{area}(A, A_2 \shuffle \ldots \shuffle A_k)$$

Again, $\text{area}(A_1, A)$ is an iterated area, and by induction the term

$$Q''_1 = - \text{area}(\text{area}(A_1, A), A_2 \shuffle \ldots \shuffle A_k)$$

is a polynomial in iterated areas of shuffle-degree at most $k$. The term

$$Q''_2 = - A_2 \shuffle \ldots \shuffle A_k \shuffle \text{area}(A_1, A)$$
is clearly a monomial in iterated areas of shuffle-degree $k$. By induction we have that

$$P_1 = \text{area}(A_1, A_2 \uplus \ldots \uplus A_k) = \beta_{k-1} A_1 \uplus \ldots \uplus A_k + P_1'$$

where $P_1'$ is a polynomial in iterated areas of shuffle-degree $k - 1$. Similarly

$$P_2 = \text{area}(A, A_2 \uplus \ldots \uplus A_k) = \beta_{k-1} A \uplus A_2 \uplus \ldots \uplus A_k + P_2'$$

where $P_2'$ is a polynomial in iterated areas of shuffle-degree $k - 1$. Therefore

$$Q_2'' = A \uplus \text{area}(A_1, A_2 \uplus \ldots \uplus A_k) = \beta_{k-1} A \uplus A_1 \uplus \ldots \uplus A_k + A \uplus P_1'$$

Similarly

$$Q_4'' = -A_1 \uplus \text{area}(A, A_2 \uplus \ldots \uplus A_k) = -\beta_{k-1} A \uplus A_1 \uplus \ldots \uplus A_k - A_1 \uplus P_2'$$

Combining terms we get a cancellation and degree reduction so that

$$Q_4'' + Q_4' = \beta_{k-1} A_1 \uplus \ldots A_k \uplus A + A \uplus P_1' - \beta_{k-1} A_1 \uplus \ldots A_k \uplus A - A_1 \uplus P_2'$$

$$= A \uplus P_1' - A_1 \uplus P_2'$$

is a polynomial in iterated areas of shuffle-degree $k$. Setting $Q'' = Q_4'' + Q_4'(\text{which is a polynomial in iterated areas of shuffle degree } k)$ and substituting in equation (21) we get

$$3 \text{area}(A, A_1 \uplus \ldots \uplus A_k) = (\beta_{k-1} - 1) A_1 \uplus \ldots \uplus A_k \uplus A$$

$$+ \text{area(area}(A_1, A_2 \uplus \ldots \uplus A_k), A) + Q' + Q''$$

$$= (\beta_{k-1} - 1) A_1 \uplus \ldots \uplus A_k \uplus A + \text{area}(P_1, A) + Q' + Q''$$

$P_1$ being a polynomial in iterated areas of shuffle-degree $k - 1$, we have by induction that $\text{area}(P_1, A)$ is a polynomial in iterated areas of shuffle-degree $k$. Hence, by construction $Q = Q' + Q'' + \text{area}(P_1, A)$ is a polynomial in iterated areas of shuffle-degree $k$. Therefore equation (22) becomes

$$3 \text{area}(A, A_1 \uplus \ldots \uplus A_k) = (\beta_{k-1} - 1) A_1 \uplus \ldots \uplus A_k \uplus A$$

$$- \beta_{k-1} \text{area}(A, A_1 \uplus \ldots \uplus A_k) + Q$$

Rearranging the terms we get the following final expression

$$\text{area}(A, A_1 \uplus \ldots \uplus A_k) = \frac{\beta_{k-1} - 1}{\beta_{k-1} + 3} A_1 \uplus \ldots \uplus A_k \uplus A + \frac{1}{\beta_{k-1} + 3} Q$$

Setting $\beta_k = \frac{\beta_{k-1} - 1}{\beta_{k-1} + 3}$ and noting that $\beta_1 = 0$ the result follows from Lemma 112.$\square$

**Proof of Theorem 4.3.** Since linear combinations of polynomials are polynomials, it suffices to prove that words in $W_A$ are polynomial in iterated areas. We prove by induction that every word $w \in W_A$ of length $|w| = n$ can be expressed as polynomial in iterated areas of shuffle-degree $n$. The result is trivial for $n = 0$. Let $n \geq 1$. We assume that $w$ is a word of length $n > 0$ and that any word of length $< n$ can be written as a polynomial in iterated areas of the appropriate degree.

Since $|w| > 0$, $w$ can be written as follows

$$w = av = a \prec v$$

(23)
where $v \in W_A$ is of word of length $|v| = n - 1$ and $a \in A \subset A$ is a letter. Moreover for any elements of $A$

\[
\begin{align*}
    a < v & = \frac{1}{2} (\text{area}(a, v) + a \shuffle v - (a, e)(v, e)e) \\
    & = \frac{1}{2} (\text{area}(a, v) + a \shuffle v)
\end{align*}
\]  

(24)

since $a$ is a letter. The length of the word $v$ in \((24)\) is equal to $n - 1$, so by induction it can be written as a polynomial in iterated areas of shuffle-degree $n - 1$. Hence, the term $a \shuffle v$ is also a polynomial in iterated areas of shuffle-degree $n$. This concludes the induction and the proof.

5 A structure theorem for streamed information

We begin this final section by recalling, Remark 2.13, that any binary operator defined on words over $A$ automatically extends to an operator acting on trees from the magma $M_A$. In particular, this extends the Lie bracket, the half shuffle $\prec$, and area operation $\text{area}$, to maps from $M_A$ to $A$.

To present our structure theorem we will need to introduce special subsets of trees in $M_A$ with a rich combinatorial structure. These sets are classically used to construct bases for the free Lie algebra $L_A$.

5.1 Hall sets

**Definition 5.1.** A total order $<$ on a subset $H$ of $M_A$ is an **ancestral order** if for any tree $t = (t', t'')$ of degree $\geq 2$ one has $t < t''$.

This definition of ancestral order makes other constructions more transparent. It is obvious that ancestral orders exist on any magma and their restrictions to a subset are also ancestral.

**Definition 5.2.** A subset $H$ of $M_A$ is a Hall set if the following conditions hold

1. $<$ is an ancestral order on $H$;
2. $A \subset H$;
3. for any tree $h = (h_1, h_2) \in M_A$ of degree $\geq 2$, $h \in H$ if and only if:
   
   \[\begin{align*}
   & (a) \ h_1, h_2 \in H \text{ and } h_1 < h_2 \\
   & (b) \text{ either } h_1 \in A \text{ or } h_2 \leq h_1' \text{ where } h_1 = (h_1', h_1'').
   \end{align*}\]

As pointed out in [Reu93, Proposition 4.1] and the surrounding discussion, Hall sets exist, any ancestral order on the full magma leads in a canonical way to a unique Hall set, and that Hall sets are closed, i.e. each subtree of a Hall tree is again a Hall tree.

**Example 5.3.** The Hall set $H$ set used in the esig package [ca10] is defined as follows: elements are ordered so that they respect degree, and for any equal-length Hall trees $h = (h_1, h_2), h' = (h_1', h_2')$ their order is defined recursively as follows: $h < h'$ if either $h_1 < h_1'$ or $h_1 = h_1'$ and $h_2 < h_2'$.

**Example 5.4.** Consider a total order on letters in $A$ and suppose that words in $W_A$ are ordered alphabetically. A **Lyndon word** on $W_A$ is a non-empty word such that for any factorisation $\omega = uv$ with $u, v \in W_A$ non-empty one has $\omega < v$. Then, the set of Lyndon words ordered alphabetically is a Hall set [Reu93, Theorem 5.1].

**Example 5.5.** Let $H_0 = A$ and order it totally. Define $H_{n+1}$ as the set of trees of the form

\[h = (...((h_1, h_2), h_3), ..., h_k)\]
where \( k \geq 2 \) and \( h_1, \ldots, h_k \in H_n \) with
\[
h_1 < h_2 \geq h_3 \geq \ldots \geq h_k.
\]
Now order \( H_{n+1} \) totally. Finally let \( H = \cup_{n \geq 0} H_n \) and extend the order in \( H_n \) to \( H \) by the condition
\[
h_1 = H_m, h_2 \in H_n, m < n \implies h_1 > h_2.
\]
Then \( H \) is a Hall set [Reu93, Theorem 5.7].

**Lemma 5.6.** [Reu93, Corollary 4.14] The number of Hall trees of degree \( n \) is equal to
\[
D_H = \frac{1}{n} \sum_{d|n} \mu(d)q^{n/d}
\]
where \( \mu \) is the Möbius function.

### 5.2 The Poincaré-Birkhoff-Witt basis and its dual

The Jacobi identities are linear relations between degree-three Lie brackets arising from associativity of the underlying group operation. They make the derivation of a basis for the free Lie algebra \( L_A \) a deep and classic challenge.

**Theorem 5.7.** [Reu93, Theorem 4.9 (i)] For any Hall set \( H \), the collection of elements \( \{[h] : h \in H \} \) form a linear basis for the free Lie algebra \( L_A \).

This basis admits a canonical extension to a basis of the tensor algebra \( (A, \otimes) \).

**Theorem 5.8.** [Reu93, Theorem 4.9] The decreasing products
\[
[h_1]^{\otimes k_1} \otimes \ldots \otimes [h_n]^{\otimes k_n}, \quad h_1 > \ldots > h_n
\]
form a basis of the tensor algebra \( (A, \otimes) \). This basis is called the Poincaré-Birkhoff-Witt (PBW) basis.

**Definition 5.9.** A word \( \omega \in W_A \) is called a Hall word if \( \omega \) is the image of a Hall tree \( h \in H \) by the foliage map, i.e.
\[
\omega = f(h).
\]

**Remark 5.10.** The foliage map is injective when restricted to a Hall set \( H \) and there are efficient algorithms for recovering the Hall tree from a Hall word [Reu93].

**Lemma 5.11.** [Reu93, Corollary 4.7] Every word \( \omega \in W_A \) can be written uniquely as a decreasing product of Hall words
\[
\omega = f(h_1)^{\otimes k_1} \otimes \ldots \otimes f(h_n)^{\otimes k_n}, \quad h_1 > \ldots > h_n
\]

**Remark 5.12.** If \( \omega \in W_A \) is a word decomposed into its unique decreasing product of Hall words according to equation (28), then \( P_\omega \) is the corresponding PBW basis element as per Theorem 5.8
\[
P_\omega = [h_1]^{\otimes k_1} \otimes \ldots \otimes [h_n]^{\otimes k_n}, \quad h_1 > \ldots > h_n
\]
\( \{P_\omega\}_{\omega \in W_A} \) is thus an enumeration of the PBW basis indexed by words. The next theorem provides exact formulae for the dual basis to the PBW basis.

**Theorem 5.13.** [Reu93, Theorem 5.3] The dual basis \( \{S_\omega\}_{\omega \in W_A} \) to the PBW basis \( \{P_\omega\}_{\omega \in W_A} \) has the following properties:

1. If \( e \) is the empty word then \( S_e = e \)
2. If \( \omega = f(h_1)^{\otimes k_1} \otimes \ldots \otimes f(h_n)^{\otimes k_n} \) is the unique factorization of the word \( \omega \) in a decreasing product of Hall trees \( h_1 > \ldots > h_n \in H \), then
\[
S_\omega = \frac{1}{k_1! \ldots k_n!} f(h_1)^{\omega_{k_1}} \sqcup \ldots \sqcup f(h_n)^{\omega_{k_n}}
\] (29)

3. If \( h \in H \), then the word \( f(h) = av \) for some letter \( a \in A \) and word \( v \in W_A \); moreover
\[
S_{f(h)} = a \otimes S_v
\] (30)

Theorem 5.13 is an important result due to Schützenberger and it is the structure theorem mentioned in the introduction. However, in the next section we provide our version of this theorem (which agrees with the version in [Sus86] but with a completely different proof) which consists of a more explicit recursive formula for the dual PBW basis elements \( \{ S_\omega \}_{\omega \in W_A} \) and identify them as Hall integrals. We note that this result is reported without proof also in [Kaw99] [GK08]

5.3 Polynomials in Hall integrals are a free generating set

Definition 5.14. An element \( x \) of \( A \) is called a Hall integral if it is the image under the operator \( \prec : M_A \to A \) of a Hall tree. That is to say, there exists a Hall tree \( h \in H \subset M_A \) so that \( x = \prec(h) \). A (shuffle) polynomial in Hall integrals is a sum of shuffles monomials in Hall integrals.

Lemma 5.15. [Reu93] Theorem 4.18] Any Hall tree \( h \in H \) can be uniquely decomposed as
\[
h = (h_1 h_2) = (\ldots ((h_1, h_2), h_2), \ldots) h_2
\] (31)
with \( h_1, h_2 \in H \), and either \( h_1 \) is a letter or \( h_1 \neq h_2 \) and where the \( h_2 \) bracketing is repeated \( k \) times. This is often referred to as the Lazard decomposition of \( h \).

Definition 5.16. If \( h = (h_1 h_2^k) \) is the Lazard decomposition of a Hall tree \( h \in H \) then we define the Lazard depth \( \alpha_h \) of \( h \) to be \( 1/k \). The accumulated Lazard depth of a Hall tree \( h \in H \) is defined recursively: \( \mathcal{A}_h = 1 \) if \( h \in A \), otherwise \( h = (h', h'') \) and \( \mathcal{A}_h = \alpha_h \mathcal{A}_{h'} \mathcal{A}_{h''} \).

The following are the main results of this section.

Theorem 5.17. For any Hall tree \( h \in H \setminus A \) one has \( h = (h', h'') \) and
\[
S_{f(h)} = \alpha_h \left( S_{f(h')} \prec S_{f(h'')} \right)
\] (32)
where \( \alpha_h \in \mathbb{Q} \) is the Lazard depth of \( h \).

Theorem 5.18. For any Hall tree \( h \in H \) one has
\[
S_{f(h)} = \mathcal{A}_h (\prec(h))
\] (33)
where \( \mathcal{A}_h \in \mathbb{Q} \) is the accumulated Lazard depth of \( h \).

Theorem 5.19. Consider all decreasing sequences \( h_i \in H \), \( h_1 > \ldots > h_n \), and strictly positive integers \( k_i > 0 \); then the elements
\[
S_\omega = \mathcal{A}_{h_1}^{k_1} \cdots \mathcal{A}_{h_n}^{k_n} (\prec(h_1))^\omega_{k_1} \sqcup \ldots \sqcup (\prec(h_n))^\omega_{k_n}
\] (34)
are the dual basis in \( A \) to the PBW basis \( \{ P_\omega = [h_1]^{\otimes k_1} \otimes \ldots \otimes [h_n]^{\otimes k_n} \}_{\omega \in W_A} \).

Before proving Theorem 5.17 we need the following combinatorial lemma.

Lemma 5.20. [Reu93] Corollary 5.14] Let \( h = (h', h'') \in H \) be a Hall tree. Now \( f(h) = av \), where \( a \in A \) and \( v \in W_A \). Let \( v = f(h_1)^{\otimes k_1} \otimes \ldots \otimes f(h_n)^{\otimes k_n} \) be the unique factorization of the word \( v \) in a decreasing product of Hall trees \( h_1 > \ldots > h_n \in H \). Then
\[
h'' = h_n
\] (35)
Theorem 5.18 allows for $i$ where we use Theorem 5.17 for the first step, the truth of the result for the proof. The specified expression for the dual basis element in terms of Hall integrals and completes the induction.

because assuming the result holds for $w$ from the chain rule (12). Note that the inner term in equation (41) can be reinterpreted as associative property of shuffle. Equation (40) follows from Lemma 5.20. Equation (41) follows definition of $\prec$. Equation (38) follows from (48) in Theorem 5.13. Equation (39) is simply the definition of $\preceq$. Equation (36) is a restatement of (30) in Theorem 5.13. Equation (37) is immediate from the proof.

Proof of Theorem 5.19. We may proceed by induction. For any Hall tree $h \in A$ one has $S_f(h) = h \in A$, $\prec(h) = h$, and $A_h = 1$ and so the theorem is true. On the other hand if $h = (h', h'')$ then, assuming the result holds for $h'$, $h''$:

$$S_f(h) = \alpha_h (S_f(h') \prec S_f(h''))$$

where we use Theorem 5.17 for the first step, the truth of the result for $h'$ and $h''$ for the second, and the recursive definitions of $\prec(h)$ and $A_h$ for the third. So the result is true for $h$. This completes the induction.

Proof of Theorem 5.18. We may proceed by induction. For any Hall tree $h \in A$ one has $S_f(h) = h \in A$, $\prec(h) = h$, and $A_h = 1$ and so the theorem is true. On the other hand if $h = (h', h'')$ then, assuming the result holds for $h'$, $h''$:

$$S_f(h) = \alpha_h (S_f(h') \prec S_f(h''))$$

where we use Theorem 5.17 for the first step, the truth of the result for $h'$ and $h''$ for the second, and the recursive definitions of $\prec(h)$ and $A_h$ for the third. So the result is true for $h$. This completes the induction.

Proof of Theorem 5.19. Recall from Schützenberger’s theorem (Theorem 5.13 in this paper) that any element $S_w$ in the dual basis to the PBW basis can be expressed uniquely as a shuffle monomial in $S_f(h)$. More precisely, consider the unique factorization of the word $w$ as a decreasing product of Hall words $w = f(h_1)^{\otimes k_1} \otimes \ldots \otimes f(h_n)^{\otimes k_n}$ where $h_1 > \ldots > h_n \in H$, then the dual basis element

$$S_w = \frac{1}{k_1! \ldots k_n!} S_f(h_1)^{\otimes k_1} \ldots \otimes S_f(h_n)^{\otimes k_n} \in \mathcal{A}$$

Theorem 5.18 allows for $i = 1 \ldots n$ the substitution of $A_{h_i}(\prec(h_i))$ for $S_f(h_i)$ in this formula which gives the specified expression for the dual basis element in terms of Hall integrals and completes the proof.

In this section we have provided formulae for the dual PBW basis elements alternative but equivalent to the ones to be found in Reutenauer’s book [Reu93].
5.4 A conjecture

Theorem 5.17 states that polynomials in Hall integrals freely generate the half shuffle algebra \((A, \prec)\). A natural question that arises is whether a similar structure theorem holds in the case where the half shuffle \(\prec\) on Hall trees is replaced by the commutator area as basic operation. This question has been, and still remain, a conjecture well supported by calculation for the last decade.

**Conjecture**  
Polynomials in Hall iterated areas \(\{\text{area}(h) : h \in H\}\) form a free generating set for \((A, \prec)\).

Trying to solve this conjecture led us to consider an argument related to the well-known Lazard’s elimination [Reu93] to construct a canonical, but to our knowledge, new decomposition of the half shuffle algebra \((A, \prec)\) as shuffle power series in the greatest letter \(c\) of the alphabet \(A\) with coefficients in a sub-algebra \(X\) freely generated by a new alphabet \(X\) with an infinite number of letters defined in terms of \(c\) and all other letters in \(A\). This construction, that we refer to as elimination trick, allows us to provide, in the next section, a second proof relying on an induction argument of our structured theorem.

5.5 Another proof of the structure theorem

The following simple and concrete observation will be expanded in this section.

If \((A, \prec)\) is the free half shuffle algebra over \(A\), and \(c \in A\), and \(X\) is the subset of \(A\) comprising \(\frac{1}{k} \prec ((ac^k)), a \in A \setminus c,\) and \(Z\) is the space spanned by words that do not begin with \(c\); then \((Z, \prec)\) is a half shuffle algebra generated by \(X\) in \(A\); moreover, \(Z\) is freely generated as a half shuffle algebra by \(X\), and therefore canonically isomorphic as a half shuffle algebra to the free half shuffle algebra \((X, \prec)\) over \(X\). In characteristic zero, \(Z\) is the half shuffle sub-algebra of \(A\) spanned by the words that do not begin with \(c\). It is complimentary to \(A/A_c\) and we have

\[
A = Z \oplus (A \cup c)
\]

and any element in \(A\) can be expressed canonically as a shuffle power series in \(c\) with coefficients in the half shuffle subalgebra \(Z\). One can repeat this process by choosing a letter \(d \in X\), and expanding every coefficient as a power series in \(d\) with coefficients in the half shuffle subalgebra generated by the elements \(\{\frac{1}{k} \prec ((ad^k)), a \in X \setminus d\}\). In what follows we will make this precise.

**Definition 5.21.** Let \(c\) be the greatest element of \(A\) with respect to an ancestral ordering \(<\). Define the subset of trees \(X = \{(ac^n), a \in A \setminus \{c\}, n \geq 0\} \subset M_A\)

With this choice of (infinite) alphabet, the following spaces and operators are automatically defined in the same way as their \(A\) counterparts:

- \(M_X\) the free magma;
- \(W_X\) the space of words in the alphabet \(X\);
- \(X\) the vector space spanned by words in \(W_X\);
- \(\otimes_X, [\cdot, \cdot]_X, \prec_X, \text{area}_X, (\cdot, \cdot)_X\) the products and pairing on these spaces;
- \(L_X\) the free Lie sub-algebra of \((X, \otimes_X)\);
- \(\exp_X, \log_X\) the tensor series for the respective maps.
Theorem 5.22. [Reu93, Theorem 0.6] The Lie algebra $L_A$ is the semi-direct product of $L_X$ and $Rc$

\[ L_A = L_X \ltimes Rc \]  

(51)

As a result of Theorem 5.22, $L_X$ is a Lie ideal and sub-algebra of co-dimension one in $L_A$ and in particular $L_A = L_X \oplus Rc$.

Next we report an important lemma from [Reu93] which provides a very simple relation between Hall sets in $M_A$ and Hall sets in $M_X$.

Lemma 5.23. [Reu93, Lemma 4.19 & Section 5.6.3] The unique homomorphism of magmas $\phi: M_X \to M_A$ that sends $x = (ac^n) \in X$ to $(ac^n) \in M_A$ is an injection of magmas and its range is the free magma over $X$. Furthermore $\phi(H_X) = H \cap \phi(M_X)$ is the Hall set in $M_X$ associated with the ordering $\prec$, $H = \{c\} \cup \phi(H_X)$, and $c$ is the greatest element of $H$.

Remark 5.24. $M_X$ is a sub-magma and inherits an ancestral ordering from $M_A$. It follows that the image by $\phi$ of the Hall set $H_X$ in $M_X$ associated to the ordering $\prec$ is $H \cap \phi(M_X)$ (Lemma 5.23).

When switching back and forth between the $X$- and $A$-spaces, the first objects one needs to have control over are letters from the two alphabets $X$ and $A$. In the next lemma we express the images under the various operators discussed so far of letters in $X$, seen as trees in $M_A$, in terms of words in $W_A$.

Lemma 5.25. For any $x \in X$, the image $\phi(x)$ in $M_A$ is of the form $(ac^n)$ for some $a \in A$ and $n \geq 0$. The image of $(ac^n)$ under the operators $[\,]$, $\prec$, area in $A$, expressed in terms of words in $W_A$ are given by

\[
[\phi(x)] = [\phi(ac^n)] = \binom{n}{0}ac...c - \binom{n}{1}cac...c + ... + (-1)^n \binom{n}{n}c...ca
\]

(52)

\[
\prec(\phi(x)) = \prec(\phi(ac^n)) = n!ac...c
\]

(53)

\[
\text{area}(\phi(x)) = \text{area}(\phi(ac^n)) = n!(ac...c - cac...c)
\]

(54)

where all the words are of length $n + 1$ and contain exactly once the letter $a$.

The proof is left as an exercise to the reader.

The next lemma tells the relationship between integrals and areas on letters from $X$.

Lemma 5.26. For any tree $(ac^n) \in M_A$ one has

\[
\prec((ac^n)) = \frac{1}{n + 1} \text{area}((ac^n)) + \frac{n}{n + 1}c \sqcup \prec((ac^{n-1}))
\]

(55)

Proof. From Lemma 5.25 we deduce the following identity

\[
\text{area}((ac^n)) + (c \sqcup n\prec((ac^{n-1}))) = (n + 1)\prec((ac^n))
\]

which after rearranging yields equation 55. \qed

Lemma 5.27. For any $(ac^n) \in M_A$ one has

\[
\prec((ac^n)) = \frac{1}{n + 1} \sum_{k=0}^{n} c^{uk} \sqcup \text{area}((ac^{n-k}))
\]

(56)

Proof. This follows immediately from Lemma 5.26 and an induction on $n$. \qed
Remark 5.28. Recall that the Lie bracket operator $[\cdot]$ is defined on $\mathcal{M}_A$ with values in $\mathcal{L}_A$. The restriction of $[\cdot]$ defined on $\mathcal{M}_A$ to $\mathcal{M}_X$ agrees with the natural definition of $[\cdot]$ on $\mathcal{M}_X$. It is also a simple exercise to prove that this compatibility between the restriction and the intrinsically defined operators holds for the tensor product and the Lie bracket.

Definition 5.29. We denote by $J_c : \mathcal{X} \to \mathcal{A}$ the unique $\prec$-homomorphism that, by freeness of $(\mathcal{X}, \prec_X)$ over $X$, extends to $\mathcal{X}$ the map

$$ (ac^n) \mapsto \frac{1}{n} (\prec ((ac^n))) \quad (n > 0) \quad (57) $$

Denote by $(Z, \prec)$ the half shuffle subalgebra of $(\mathcal{A}, \prec)$ generated by the elements

$$ \{ J_c(x) : x \in X \} $$

Next we prove that that the algebra $(Z, \prec)$ is closed under $\prec$ and provide a characterisation of $Z$ as the linear span of words in $W_A$ that do not begin with the letter $c$.

Lemma 5.30. $Z$ is the span of words in $W_A$ that do not begin with the letter $c$

$$ Z = \text{Span}\{ w = a \prec v \in W_A \mid a \neq c, a \in A, v \in W_A \} $$

In particular $Z$ is closed under $\prec$.

Proof. Let $Z'$ be the linear span in $\mathcal{A}$ of the $w \neq cv \in W_A$. It is immediate from the definitions of $\cup$ and $\prec$ on words that $Z'$ is closed under both operations. If $t = (t_1, t_2) \in \mathcal{M}_A$ and if, for $i = 1, 2$, $\prec (\phi(t_i)) \in Z'$ then $\prec (\phi (t)) = (\prec (\phi (t_1))) \prec (\prec (\phi (t_2)))$ is also in $Z'$ because $Z'$ is closed under $\prec$. Let $x \in X$, then by equation (58)

$$ \prec (\phi (x)) = \prec ((ac^n)) = n!ac...c \in Z' \quad (58) $$

We may proceed recursively to see that every $\prec (t)$ contained in $Z$ is also an element of $Z'$; since $Z$ is generated by $\{ J_c(x) : x \in X \}$ we conclude that $Z \subset Z'$. The unique decomposition of words into decreasing sequences of Hall words shows that the dimension of $Z'$ and $Z$ are equal, hence $Z = Z'$.

Lemma 5.31. The half shuffle algebra $\mathcal{A}$ has the following decomposition

$$ \mathcal{A} = Z \oplus (Z \cup c) \oplus (Z \cup c^{\cup 2}) + ... \quad (59) $$

Proof. Consider any word $w \in W_A$ beginning with $n$ number of $c$'s.

$$ w = c \prec (c \prec (... \prec (c \prec v)...) $$

where $v = av' \in Z$ is a word that doesn’t begin with $c$, i.e. $a \in A, a \neq c, v' \in W_A$. If $n = 0$ then $w \in Z$. By induction on $n$

$$ c \prec (c \prec (... \prec (c \prec v)...) = \alpha_n c^{\cup n} \cup v = L $$

where $\alpha_n \in \mathbb{R}$ and $L$ is a linear combination of words that begin with $k < n$ number of $c$'s. Hence, by induction on the number of $c$'s in front of the words, the word $w$ can be written as a shuffle polynomial in $c$ with coefficients in $Z$.

Lemma 5.32. $J_c$ maps polynomials in Hall integrals $\prec_X (h), h \in H_X$ to polynomials in Hall integrals $\prec (\phi (h))$.

Proof. This follows immediately because $J_c$ is a half shuffle (and so shuffle) homomorphism.
We now repeat our structure theorem and provide an alternative proof based on the elimination trick discussed so far in this section.

**Theorem 5.33.** The half shuffle algebra \((A, \prec)\) is freely generated by polynomials in Hall integrals \(\prec(h)\) for \(h \in H\).

**Proof.** We can assume by induction that the theorem holds for \(X\), i.e., that \((X, \prec_X)\) is freely generated by polynomials in \(\prec_X(h)\) for \(h \in \phi(H_X)\). By Lemma 5.32, \(Z\) is freely generated by polynomials in \(\prec(h)\) with \(h \in \phi(H_X)\). By Lemma 5.23, \(H = \{c\} \cup \phi(H_X)\) and with the decomposition (59) we conclude that \(A\) is freely generated by polynomials in \(\prec(h)\), \(h \in H\).

### 5.6 Scalable computations of path signatures

As mentioned in the introduction, instances of streamed information can be represented as a path \(\gamma : [0, 1] \to V\) with values on some finite dimensional vector space \(V \cong \mathbb{R}^d\), and in turns this path is faithfully represented, up to reparameterisation, by the signature \(\text{Sig}_\gamma \in (A, \otimes)\). Furthermore, because the half shuffle algebra \((A, \prec)\) is the algebraic dual of the tensor algebra \((A, \otimes)\), it is automatic to see that the restriction of linear functionals on \(A\) to the range of the signature form a unital algebra of real-valued functions that separates points. Hence, by the Stone-Weierstrass theorem linear functionals acting on the signatures are dense in the space of continuous, real-valued functions on compact sets of paths. Thus, non-linear regression on pathspace can be realised by linear regression on the terms of the signature. However, terms in the signature contain some redundancy, which represents a major scalability issue, particularly because the number of distinct and linearly independent iterated integrals grows exponentially in the truncation level. In this paper, and in particular in Theorems 5.19 and 5.33, we identified sets of Hall integrals that can be used to compute any term in the signature with a minimal amount of computations.

To illustrate this let’s consider a simple example. Let \(d = 3\) and let’s identify the 3-dimensional vector space \(V\) as the space spanned by an alphabet of two letters \(A = \{1, 2, 3\}\). Let \(\omega = 23321222211\); note that \(|\omega| = 12\). Then, computing the coefficient \((S_\omega, \text{Sig}_\gamma)\) in the signature using existing software [KL20, ca10, RG18] (based on the Chen’s relation) involve evaluating the level-12 truncated tensor exponential of increments \(\exp^{(12)}(\gamma_t - \gamma_s)\). This operation has space and time complexities of \(O(3^{12})\).

Instead, considering for example the Lyndon basis, one can precompute the factorisation of \(\omega\) into decreasing product of Lyndon words and find

\[
\omega = f(1)^{\otimes 3} \otimes f(((1, 2), 2), 2)) \otimes f(2) \otimes f((2, 3), 3))
\]

Therefore, by Theorem 5.19 one has

\[
S_\omega = \prec(1)^{\otimes 3} \mathbin{\mathcal{V}} \frac{1}{48} \prec(((1, 2), 2), 2)) \mathbin{\mathcal{V}} \prec(2) \mathbin{\mathcal{V}} \frac{1}{24} \times(((2, 3), 3))
\]

Using the interplay between algebraic operations \(\prec\) and \(\mathcal{V}\) and the rules of calculus on paths outlined in Section 3 we obtain

\[
(S_\omega, \text{Sig}_\gamma) = \frac{1}{48} \alpha_1 \alpha_2 \alpha_3 \alpha_4
\]
where

\[
\begin{align*}
\alpha_1 &= \int_0^1 d\gamma_t^{(1)} \\
\alpha_2 &= \int_0^1 \left( \int_0^u \left( \int_0^t \gamma_s^{(1)} d\gamma_s^{(2)} \right) d\gamma_u^{(2)} \right) d\gamma_t^{(2)} \\
\alpha_3 &= \int_0^1 d\gamma_t^{(2)} \\
\alpha_4 &= \int_0^1 \left( \int_0^t \gamma_s^{(2)} d\gamma_s^{(3)} \right) d\gamma_t^{(3)}.
\end{align*}
\]

6 Conclusion

In this paper, we identified the free Zinbiel algebra introduced in [Sch58] with an algebra of real-valued functions on paths. We provided two, to our knowledge, new basic identities in arity 3 involving the products $\omega$ and area, and show that these are sufficient to recover the Zinbiel and Tortkara identities introduced in [Dzh07]. We then used these identities to provide a direct proof of the main result in [DLPR20] stating that polynomials in iterated areas generate the free Zinbiel algebra. Subsequently, we introduced minimal sets of Hall integrals and showed, with two different proof techniques, that polynomial functions on these Hall integrals freely generate the half shuffle algebra. This result can be interpreted as a structure theorem for streamed information, allowing to split real valued functions on streamed data into two parts: a first that extracts and packages the streamed information into Hall integrals, and a second that evaluates a polynomial in these without further reference to the original stream.

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