BIHARMONIC HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN SPACE FORMS

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Abstract. Let $M^n$ be a biharmonic hypersurface with constant scalar curvature in a space form $M^{n+1}(c)$. We show that $M^n$ has constant mean curvature if $c > 0$ and $M^n$ is minimal if $c \leq 0$, provided that the number of distinct principal curvatures is no more than 6. This partially confirms Chen’s conjecture and Generalized Chen’s conjecture. As a consequence, we prove that there exist no proper biharmonic hypersurfaces with constant scalar curvature in Euclidean space $\mathbb{E}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$ for $n < 7$.

1. Introduction

In 1983, Eells and Lemaire [15] introduced the concept of biharmonic maps in order to generalize classical theory of harmonic maps. A biharmonic map $\phi$ between an $n$-dimensional Riemannian manifold $(M^n, g)$ and an $m$-dimensional Riemannian manifold $(N^m, h)$ is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g,$$

where $\tau(\phi) = \text{trace} \nabla d\phi$ is the tension field of $\phi$ that vanishes for a harmonic map. More clearly, the Euler-Lagrange equation associated to the bienergy is given by

$$\tau_2(\phi) = -\Delta \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi))d\phi = 0,$$

where $R^N$ is the curvature tensor of $N^m$ (e.g. [24]). We call $\phi$ to be a biharmonic map if its bitension field $\tau_2(\phi)$ vanishes.

Biharmonic maps between Riemannian manifolds have been extensively studied by some geometers. In particular, many authors investigated a special class of biharmonic maps named biharmonic immersions. An immersion $\phi : (M^n, g) \rightarrow (N^m, h)$ is biharmonic if and only if its mean curvature vector field $\vec{H}$ fulfills the fourth-order semi-linear elliptic equations (e.g. [4])

$$\Delta \vec{H} + \text{trace} R^N(d\phi, \vec{H})d\phi = 0. \quad (1.1)$$

It is well-known that any minimal immersion (satisfying $\vec{H} = 0$) is harmonic. The non-harmonic biharmonic immersions are called proper biharmonic.

We should mention that biharmonic submanifolds in a Euclidean space $\mathbb{E}^m$ were independently defined by B. Y. Chen in the middle of 1980s (see [8]) with the geometric condition $\Delta \vec{H} = 0$, or equivalently $\Delta^2 \phi = 0$. Interestingly, both biharmonic submanifolds and biharmonic immersions in Euclidean spaces coincide with each other.

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In recent years, the classification problem of biharmonic submanifolds has attracted a great attention in geometry. In particular, there is a longstanding conjecture on biharmonic submanifolds due to B. Y. Chen [8] in 1991:

**Chen’s conjecture:** Every biharmonic submanifold in Euclidean space $\mathbb{E}^m$ is minimal.

Until now, Chen’s conjecture remains open, even for hypersurfaces. Only partial answers to Chen’s conjecture have been obtained for more than three decades, e.g. [1], [2], [10], [32]. In the case of hypersurfaces, Chen’s conjecture is true for the following special cases:

- surfaces in $\mathbb{E}^3$ [8], [24];
- hypersurfaces with at most two distinct principal curvatures in $\mathbb{E}^m$ [14];
- hypersurfaces in $\mathbb{E}^4$ [19] (see also [13]);
- $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces in $\mathbb{E}^m$ [11];
- weakly convex hypersurfaces in $\mathbb{E}^m$ [25];
- hypersurfaces with at most three distinct principal curvatures in $\mathbb{E}^m$ [20];
- generic hypersurfaces with irreducible principal curvature vector fields in $\mathbb{E}^m$ [17];
- invariant hypersurfaces of cohomogeneity one in $\mathbb{E}^m$ [27].

In 2001, Caddeo, Montaldo and Oniciuc [6] proposed the following generalized Chen’s conjecture:

**Generalized Chen’s conjecture:** Every biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

Recently, Ou and Tang in [34] constructed a family of counter-examples that the generalized Chen’s conjecture is false when the ambient space has non-constant negative sectional curvature. However, the generalized Chen’s conjecture remains open when the ambient spaces have constant sectional curvature. For more recent developments of the generalized Chen’s conjecture, we refer to [9], [10], [26], [30], [28], [33].

In general, the classification problem of proper biharmonic submanifolds in Euclidean spheres is rather rich and interesting. The first example of proper biharmonic hypersurfaces is a generalized Clifford torus $S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \hookrightarrow S^{n+1}$ with $p \neq q$ and $p + q = n$ given by Jiang [23]. The complete classifications of biharmonic hypersurfaces in $S^3$ and $S^4$ were obtained in [6], [5]. Moreover, biharmonic hypersurfaces with at most three distinct principal curvatures in $S^n$ were classified in [5], [24]. For more details, we refer the readers to Balmus, Caddeo, Montaldo, Oniciuc et al.’s work [3], [18], [29], [30], [16].

In general, the classification problem of proper biharmonic hypersurfaces in space forms becomes more complicated when the number of distinct principal curvatures is four or more.

In view of the above aspects, it is reasonable to study biharmonic submanifolds with some geometric conditions. In geometry, hypersurfaces with constant scalar curvature have been intensively studied by many geometers for the rigidity problem and classification problem, for instance, see the well-known paper of Cheng-Yau [12]. Some estimate for scalar curvature of compact proper biharmonic hypersurfaces with constant scalar curvature in spheres was obtained in [4]. Recently, it was proved in [22] that a biharmonic hypersurface with constant scalar curvature in the 5-dimensional space forms $M^5(c)$ necessarily has constant mean curvature.
Motivated by above results, in this paper we consider biharmonic hypersurfaces $M^n$ with constant scalar curvatures in a space form $\mathbb{M}^n(c)$. More precisely, we obtain:

**Theorem 1.1.** Let $M^n$ be an orientable biharmonic hypersurface with at most six distinct principal curvatures in $\mathbb{M}^{n+1}(c)$. If the scalar curvature $R$ is constant, then $M^n$ has constant mean curvature.

In general, it is difficult to deal with the biharmonic immersion equation (1.1) due to its high nonlinearity. In order to prove Theorem 1.1, we use some new ideas to overcome the difficulty of treating the equation of a biharmonic hypersurface. More precisely, we transfer the problem into a system of algebraic equations (see Lemma 3.3), so we can determine the behavior of the principal curvature functions by investigating the solution of the system of algebraic equations (see Lemma 3.4). Then, we are able to prove that a biharmonic hypersurface with constant scalar curvatures in a space form $\mathbb{M}^n(c)$ must have constant mean curvature, provided that the number of distinct principal curvature is no more than six. We would like to point out that our approach in this paper is different from those in [21], [22], [13], [5].

**Remark 1.2.** Balmus-Montaldo-Oniciuc in [4] conjectured that the proper biharmonic hypersurfaces in $S^{n+1}$ must have constant mean curvature. Theorem 1.1 with $c = 1$ gives a partial answer to this conjecture.

We should point out that the complete classification of proper biharmonic hypersurfaces with constant mean curvature in a sphere is still open for the case that the number of distinct principal curvatures is more than three (cf. [30]).

Moreover, combining these results with the biharmonic equations in Section 2, we have:

**Corollary 1.3.** Any biharmonic hypersurface with constant scalar curvature and with at most six distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$ is minimal.

Thus, this result gives a partial answer to Chen’s conjecture and the generalized Chen’s conjecture.

Furthermore, as a direct consequence, we get the following characterization result:

**Corollary 1.4.** Any biharmonic hypersurface with constant scalar curvature in Euclidean space $\mathbb{E}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$ for $n < 7$ has to be minimal.

**Remark 1.5.** We could replace or weaken the condition constant scalar curvature in Theorem 1.1 by constant length of the second fundamental form or linear Weingarten type, i.e. the scalar curvature $R$ satisfying $R = aH + b$ for some constants $a$ and $b$. In fact, the discussion is extremely similar to the proof of Theorem 1.1 and the same conclusion holds true as well.

The paper is organized as follows. In Section 2, we recall some necessary background for theory of hypersurfaces and equivalent conditions for biharmonic hypersurfaces. In Section 3, we prove some useful lemmas (Lemma 3.1-Lemma 3.6), which are crucial to prove the main theorem. Finally, in Section 4, we give a proof of Theorem 1.1.
2. Preliminaries

In this section, we recall some basic material for the theory of hypersurfaces immersed in a Riemannian space form.

Let \( \phi : M^n \to M^{n+1} \) be an isometric immersion of a hypersurface \( M^n \) into a space form \( M^{n+1} \) with constant sectional curvature \( c \). Denote the Levi-Civita connections of \( M^n \) and \( M^{n+1} \) by \( \nabla \) and \( \tilde{\nabla} \), respectively. Let \( X \) and \( Y \) denote the vector fields tangent to \( M^n \) and let \( \xi \) be a unit normal vector field. Then the Gauss and Weingarten formulas (cf. [10]) are given respectively by

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),
\]

\[
\tilde{\nabla}_X \xi = -AX,
\]

where \( h \) is the second fundamental form and \( A \) is the Weingarten operator. Note that the second fundamental form \( h \) and the Weingarten operator \( A \) are related by

\[
\langle h(X,Y),\xi \rangle = \langle AX,Y \rangle.
\]

The mean curvature vector field \( \overrightarrow{H} \) is defined by

\[
\overrightarrow{H} = \frac{1}{n} \text{trace} \, h.
\]

Moreover, the Gauss and Codazzi equations are given respectively by

\[
R(X,Y)Z = c(\langle Y,Z \rangle X - \langle X,Z \rangle Y) + \langle AX,Y \rangle AX - \langle AX,Z \rangle AY,
\]

\[
(\nabla_X A)Y = (\nabla_Y A)X,
\]

for all \( X,Y,Z \) tangent to \( M^n \).

Assume that \( \overrightarrow{H} = H\xi \) and \( H \) denotes the mean curvature.

By identifying the tangent and the normal parts of the biharmonic condition (1.1) for hypersurfaces in a space form \( M^{n+1} \), the following characterization result for \( M^n \) to be biharmonic was obtained (see also [7], [5]).

**Proposition 2.1.** The immersion \( \phi : M^n \to M^{n+1} \) of a hypersurface \( M^n \) in an \( n+1 \)-dimensional space form \( M^{n+1} \) is biharmonic if and only if

\[
\begin{cases}
\Delta H + H \text{trace} \, A^2 = ncH, \\
2A \text{grad} H + nH \text{grad} H = 0.
\end{cases}
\]

The Laplacian operator \( \Delta \) on \( M^n \) acting on a smooth function \( f \) is given by

\[
\Delta f = -\text{div}(\nabla f) = -\sum_{i=1}^{n} <\nabla_{e_i}(\nabla f), e_i> = -\sum_{i=1}^{n} (e_i e_i - \nabla_{e_i} e_i)f.
\]

The following result was obtained in [21].

**Theorem 2.2.** Let \( M^n \) be an orientable proper biharmonic hypersurface with at most three distinct principal curvatures in \( M^{n+1} \). Then \( M^n \) has constant mean curvature.
3. SOME LEMMAS

We now consider an orientable biharmonic hypersurface $M^n$ ($n > 3$) in a space form $M^{n+1}(c)$.

In general, the set $M_A$ of all points of $M^n$, at which the number of distinct eigenvalues of the Weingarten operator $A$ (i.e. the principal curvatures) is locally constant, is open and dense in $M^n$. Since $M^n$ with at most three distinct principal curvatures everywhere in a space form $M^{n+1}(c)$ is CMC, i.e. the mean curvature is constant (Theorem 2.2), one can work only on the connected component of $M_A$ consisting by points where the number of principal curvatures is more than three (by passing to the limit, $H$ will be constant on the whole $M^n$). On that connected component, the principal curvature functions of $A$ are always smooth.

Suppose that, on the component, the mean curvature $H$ is not constant. Thus, there is a point $p$ where $\text{grad} \, H(p) \neq 0$. In the following, we will work on an neighborhood of $p$ where $\text{grad} \, H(p) \neq 0$ at any point of $M^n$.

The second equation of (2.6) shows that $\text{grad} \, H$ is an eigenvector of the Weingarten operator $A$ with the corresponding principal curvature $-nH/2$. We may choose $e_1$ such that $e_1$ is parallel to $\text{grad} \, H$, and with respect to some suitable orthonormal frame $\{e_1, \ldots, e_n\}$, the Weingarten operator $A$ of $M$ takes the following form

$$A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_i$ are the principal curvatures and $\lambda_1 = -nH/2$. Therefore, it follows from (2.4) that

$$\sum_{i=1}^{n} \lambda_i = nH,$$

and hence

$$\sum_{i=2}^{n} \lambda_i = -3\lambda_1. \quad (3.2)$$

Denote by $R$ the scalar curvature and by $B$ the squared length of the second fundamental form $h$ of $M$. It follows from (3.1) that $B$ is given by

$$B = \text{trace} \, A^2 = \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=2}^{n} \lambda_i^2 + \lambda_1^2. \quad (3.3)$$

From the Gauss equation, the scalar curvature $R$ is given by

$$R = n(n-1)c + n^2H^2 - B = n(n-1)c + 3\lambda_1^2 - \sum_{i=2}^{n} \lambda_i^2. \quad (3.4)$$

Hence

$$\sum_{i=2}^{n} \lambda_i^2 = n(n-1)c - R + 3\lambda_1^2. \quad (3.5)$$

Since $\text{grad} \, H = \sum_{i=1}^{n} e_i(H)e_i$ and $e_1$ is parallel to $\text{grad} \, H$, it follows that

$$e_1(H) \neq 0, \quad e_i(H) = 0, \quad 2 \leq i \leq n,$$

and hence

$$e_1(\lambda_1) \neq 0, \quad e_i(\lambda_i) = 0, \quad 2 \leq i \leq n. \quad (3.6)$$

Put $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_i^j e_k$ ($1 \leq i, j \leq n$). A direct computation concerning the compatibility conditions $\nabla_{e_i} (e_i, e_i) = 0$ and $\nabla_{e_k} (e_i, e_j) = 0$ ($i \neq j$) yields respectively
that
\begin{equation}
\omega_{ki} = 0, \quad \omega_{ki} + \omega_{kj} = 0, \quad i \neq j.
\end{equation}
(3.7)

The Codazzi equation could yield to
\begin{equation}
e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j,
\end{equation}
(3.8)
\begin{equation}
(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j
\end{equation}
(3.9)
for distinct $i, j, k$.

Moreover, from (3.6) we have
\begin{equation}
[e_i, e_j](\lambda_1) = 0,
\end{equation}
which yields directly
\begin{equation}
\omega_{ij}^1 = \omega_{ji}^1, \quad 2 \leq i, j \leq n \text{ and } i \neq j.
\end{equation}
(3.10)

**Lemma 3.1.** Let $M^n$ be an orientable biharmonic hypersurface with non-constant mean curvature in $M^{n+1}(c)$. Then the multiplicity of the principal curvature $\lambda_1$ ($= -nH/2$) is one, i.e. $\lambda_j \neq \lambda_1$ for $2 \leq j \leq n$.

*Proof.* If $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (3.8) we get
\begin{equation}
0 = (\lambda_1 - \lambda_j)\omega_{ji}^j = e_1(\lambda_j) = e_1(\lambda_1),
\end{equation}
which contradicts to (3.6). \hfill \square

**Lemma 3.2.** The smooth real-valued functions $\lambda_i$ and $\omega_{ii}^1$ ($2 \leq i \leq n$) satisfy the following differential equations
\begin{equation}
e_1e_1(\lambda_1) = e_1(\lambda_1)\left(\sum_{i=2}^{n} \omega_{ii}^1\right) + \lambda_1\left(n(n-2)c - R + 4\lambda_1^2\right),
\end{equation}
(3.11)
\begin{equation}
e_1(\lambda_i) = \lambda_i\omega_{ii}^1 - \lambda_1\omega_{ii}^1,
\end{equation}
(3.12)
\begin{equation}
e_1(\omega_{ii}^1) = (\omega_{ii}^1)^2 + \lambda_1\lambda_i + c.
\end{equation}
(3.13)

*Proof.* Substituting $H = -2\lambda_1/n$ into the first equation of (2.6), and using (2.7), (3.6), (3.3) and (3.5), we get (3.11). By putting $i = 1$ in (3.8), combining this with (3.9) gives (3.12).

Next, we will prove equation (3.13).

For $j = 1$ and $i \neq 1$ in (3.8), by (3.6) we have $\omega_{1i}^1 = 0$ ($i \neq 1$). Combining this with (3.7), we have
\begin{equation}
\omega_{ii}^1 = 0 \quad \text{for } 1 \leq i \leq n.
\end{equation}
(3.14)

For $j = 1$, and $k, i \neq 1$ in (3.9) we have
\begin{equation}
(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1,
\end{equation}
which together with (3.10) yields
\begin{equation}
\omega_{ki}^1 = 0, \quad k \neq i, \quad \text{if } \lambda_k \neq \lambda_i.
\end{equation}
(3.15)

For $i \neq j$ and $2 \leq i, j \leq n$, if $\lambda_i = \lambda_j$, then by putting $k = 1$ in (3.9) we have
\begin{equation}
(\lambda_1 - \lambda_i)\omega_{ij}^1 = 0,
\end{equation}
which together with Lemma 3.1, (3.15) and (3.7) yields
\begin{equation}
\omega_{ij}^1 = 0, \quad i \neq j, \quad \text{and } 2 \leq i, j \leq n.
\end{equation}
(3.16)
From the Gauss equation and (3.1), we have \( \langle R(e_1, e_i) e_1, e_i \rangle = -\lambda_1 \lambda_i - c \). On the other hand, the Gauss curvature tensor \( R \) is defined by \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \). Using (3.14), (3.16) and (3.7), a direct computation gives

\[
\langle R(e_1, e_i) e_1, e_i \rangle = -e_1(\omega^1_{ii}) + (\omega^1_{ii})^2.
\]

Therefore, we obtain differential equation (3.13), which completes the proof of Lemma 3.2.

Consider an integral curve of \( e_1 \) passing through \( p = \gamma(t_0) \) as \( \gamma(t) \), \( t \in I \). Since \( e_i(\lambda_1) = 0 \) for \( 2 \leq i \leq n \) and \( e_1(\lambda_1) \neq 0 \), it is easy to show that there exists a local chart \( (U, t = x^1, x^2, \ldots, x^m) \) around \( p \), such that \( \lambda_1(t, x^2, \ldots, x^m) = \lambda_1(t) \) on the whole neighborhood of \( p \).

In the following, we begin our arguments under the assumption that the scalar curvature \( \lambda_1 \) is always constant. The following system of algebraic equations is important for us to proceed further.

**Lemma 3.3.** Assume that \( \lambda_1 \) is constant. We have

\[
(\omega^1_{ii})^k = f_k(t), \text{ for } k = 1, \ldots, 5,
\]

where \( f_k(t) \) are some smooth real-valued functions with respect to \( t \).

**Proof.** Since \( e_1(\lambda_1) \neq 0 \), \( \lambda_1 = \lambda_1(t) \) and \( \lambda_1 \) is constant, (3.11) becomes

\[
\sum_{i=2}^n \omega^1_{ii} = f_1(t),
\]

where

\[
f_1(t) = \frac{e_1e_1(\lambda_1) - \lambda_1(n(n-2)c + 4\lambda_1^2 - R)}{e_1(\lambda_1)}.
\]

Taking the sum of (3.13) and (3.12) for \( i \) and taking into account (3.2) and (3.18) respectively, we have

\[
\sum_{i=2}^n (\omega^1_{ii})^2 = f_2(t),
\]

(3.20)

\[
\sum_{i=2}^n \lambda_i\omega^1_{ii} = g_1(t),
\]

where \( f_2 = 3\lambda_1^2 - (n-1)c + e_1(f_1) \) and \( g_1(t) = \lambda_1 f_1 - 3e_1(\lambda_1) \).

Multiplying \( \omega^1_{ii} \) on both sides of equation (3.13), we have

\[
\frac{1}{2}e_1((\omega^1_{ii})^2) = (\omega^1_{ii})^3 + \lambda_1\lambda_i\omega^1_{ii} + c\omega^1_{ii}.
\]

Taking the sum of the above equation and using (3.18)-(3.20), we obtain

\[
\sum_{i=2}^n (\omega^1_{ii})^3 = f_3(t),
\]

where \( f_3 = \frac{1}{2}e_1(f_2) - \lambda_1 g_1 - cf_1 \).
Differentiating (3.20) with respect to $e^1$ and using (3.12) and (3.13), we have

\[
(3.22) \quad e_1(g_1) = 2 \sum_{i=2}^{n} \lambda_i (\omega^1_{ii})^2 + \lambda_1 \sum_{i=2}^{n} \lambda_i^2 + c \sum_{i=2}^{n} \lambda_i - \lambda_1 \sum_{i=2}^{n} (\omega^1_{ii})^2.
\]

Hence, from (3.2), (3.5) and (3.19) that (3.22) yields

\[
(3.23) \quad \sum_{i=2}^{n} \lambda_i (\omega^1_{ii})^2 = g_2(t),
\]

where

\[
g_2 = \frac{1}{2} \{ e_1(g_1) - \lambda_1 (n(n-1)c - R + 3\lambda_1^2) + 3c\lambda_1 + \lambda_1 f_2 \}.
\]

Multiplying $(\omega^1_{ii})^3$ on both sides of equation (3.13), we have

\[
\frac{1}{3} e_1((\omega^1_{ii})^3) = (\omega^1_{ii})^4 + \lambda_1 \lambda_i (\omega^1_{ii})^2 + c(\omega^1_{ii})^2.
\]

Taking the sum of the above equation for $i$ and applying (3.19), (3.21) and (3.23), we obtain

\[
(3.24) \quad \sum_{i=2}^{n} (\omega^1_{ii})^4 = f_4(t),
\]

where

\[
f_4 = \frac{1}{4} e_1(f_4) - \lambda_1 g_2 - c f_2.
\]

Multiplying $\lambda_i$ on both sides of equation (3.12) gives

\[
\lambda_i^2 \omega^1_{ii} = \frac{1}{2} e_1 (\lambda_i^2) + \lambda_1 \lambda_i \omega^1_{ii},
\]

which together with (3.5) and (3.20) yields

\[
(3.25) \quad \sum_{i=2}^{n} \lambda_i^2 \omega^1_{ii} = g_3(t),
\]

where

\[
g_3 = 3\lambda_1 e_1(\lambda_1) + \lambda_1 g_1.
\]

Differentiating (3.23) with respect to $e^1$ and using (3.12)-(3.13), we have

\[
(3.26) \quad e_1(g_2) = 3 \sum_{i=2}^{n} \lambda_i (\omega^1_{ii})^3 - \lambda_1 \sum_{i=2}^{n} (\omega^1_{ii})^3 + 2\lambda_1 \sum_{i=2}^{n} \lambda_i^2 \omega^1_{ii} + 2c \sum_{i=2}^{n} \lambda_i \omega^1_{ii}.
\]

Substituting (3.20), (3.21) and (3.25) into (3.26) gives

\[
(3.27) \quad \sum_{i=2}^{n} \lambda_i (\omega^1_{ii})^3 = g_4(t),
\]

where

\[
g_4 = \frac{1}{3} (e_1(g_2) + \lambda_1 f_3 - 2\lambda_1 g_3 - 2c g_1).
\]

Multiplying $(\omega^1_{ii})^4$ on both sides of equation (3.13), we have

\[
\frac{1}{4} e_1((\omega^1_{ii})^4) = (\omega^1_{ii})^5 + \lambda_1 \lambda_i (\omega^1_{ii})^3 + c(\omega^1_{ii})^3.
\]

After taking the sum of the above equation for $i$, using (3.21), (3.24) and (3.27) we have

\[
(3.28) \quad \sum_{i=2}^{n} (\omega^1_{ii})^5 = f_5(t),
\]

where

\[
f_5 = \frac{1}{4} e_1(f_4) - \lambda_1 g_4 - c f_3.
\]

At this moment, the proof of Lemma 3.3 has been completed. \(\Box\)
Lemma 3.4. Assume that $R$ is constant. If the number $m$ of distinct principal curvatures satisfies $m \leq 6$, then $e_i(\lambda_j) = 0$ for $2 \leq i, j \leq n$, i.e. all principal curvature $\lambda_i$ depend only on one variable $t$.

Proof. Since the number $m$ of distinct principal curvatures satisfies $m \leq 6$, there are at most five distinct principal curvatures for $\lambda_i (2 \leq i \leq n)$ except $\lambda_1$. It follows easily from (3.12) and (3.13) that

$$\lambda_i \neq \lambda_j \iff \omega_{ii}^1 \neq \omega_{jj}^1.$$ 

We now distinguish the following two cases:

Case A. Suppose that $m = 6$. We denote by $\tilde{\lambda}_i$ the five distinct principal curvatures with the corresponding multiplicities $n_i$ for $1 \leq i \leq 5$. Note that here $n_i$ are positive integers and $\sum_{i=1}^{5} n_i = n-1$ (see Lemma 3.1). According to (3.12), let

$$u_i := \frac{e_1(\tilde{\lambda}_i)}{\lambda_i - \lambda_1}.$$ 

Thus, $u_i$ are mutually different for $1 \leq i \leq 5$.

In this case, the system of polynomial equations (3.17) becomes

$$\begin{align*}
&n_1 u_1 + n_2 u_2 + n_3 u_3 + n_4 u_4 + n_5 u_5 = f_1, \\
&n_1 u_1^2 + n_2 u_2^2 + n_3 u_3^2 + n_4 u_4^2 + n_5 u_5^2 = f_2, \\
&n_1 u_1^3 + n_2 u_2^3 + n_3 u_3^3 + n_4 u_4^3 + n_5 u_5^3 = f_3, \\
&n_1 u_1^4 + n_2 u_2^4 + n_3 u_3^4 + n_4 u_4^4 + n_5 u_5^4 = f_4, \\
&n_1 u_1^5 + n_2 u_2^5 + n_3 u_3^5 + n_4 u_4^5 + n_5 u_5^5 = f_5.
\end{align*}$$

(3.29)

Since $e_i(f_1) = 0$ for $2 \leq i \leq n$, differentiating both sides of equations in (3.29) with respect to $e_i$ $(2 \leq i \leq n)$, we obtain

$$\begin{align*}
&n_1 e_i(u_1) + n_2 e_i(u_2) + n_3 e_i(u_3) + n_4 e_i(u_4) + n_5 e_i(u_5) = 0, \\
&n_1 u_1 e_i(u_1) + n_2 u_2 e_i(u_2) + n_3 u_3 e_i(u_3) + n_4 u_4 e_i(u_4) + n_5 u_5 e_i(u_5) = 0, \\
&n_1 u_1^2 e_i(u_1) + n_2 u_2^2 e_i(u_2) + n_3 u_3^2 e_i(u_3) + n_4 u_4^2 e_i(u_4) + n_5 u_5^2 e_i(u_5) = 0, \\
&n_1 u_1^3 e_i(u_1) + n_2 u_2^3 e_i(u_2) + n_3 u_3^3 e_i(u_3) + n_4 u_4^3 e_i(u_4) + n_5 u_5^3 e_i(u_5) = 0, \\
&n_1 u_1^4 e_i(u_1) + n_2 u_2^4 e_i(u_2) + n_3 u_3^4 e_i(u_3) + n_4 u_4^4 e_i(u_4) + n_5 u_5^4 e_i(u_5) = 0.
\end{align*}$$

(3.30)

Now consider this system of five linear equations with five unknowns $e_i(u_k)$ for $1 \leq k \leq 5$.

According to Cramer’s rule in linear algebra, for any $k$, $e_i(u_k) \equiv 0$ holds true if and only if the determinant of the coefficient matrix of (3.30) is not vanishing, i.e.

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{vmatrix} \neq 0.$$
We note that the determinant in (3.31) is the famous Vandermonde determinant with order 5 and hence

\[
(3.32) \quad \begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
u_1 & u_2 & u_3 & u_4 & u_5 \\
u_1^2 & u_2^2 & u_3^2 & u_4^2 & u_5^2 \\
u_1^3 & u_2^3 & u_3^3 & u_4^3 & u_5^3 \\
u_1^4 & u_2^4 & u_3^4 & u_4^4 & u_5^4
\end{vmatrix} = \prod_{1 \leq j < i \leq 5} (u_i - u_j).
\]

Since \(u_i\) are mutually different for \(1 \leq i \leq 5\), (3.32) implies that (3.31) holds true identically. Hence, we have \(e_i(u_k) = 0\) for any \(1 \leq k \leq 5\) and \(2 \leq i \leq n\).

Therefore, by using \(e_i(u_k) = 0\) and

\[
e_i e_1(u_k) - e_1 e_i(u_k) = [e_i, e_1](u_k) = \sum_{j=2}^{n} (\omega_{i1}^j - \omega_{1i}^j)e_j(u_k),
\]

we get

\[
e_i e_1(u_k) = 0.
\]

Noting that with the notation \(u_k\), (3.13) becomes

\[
e_1(u_k) = (u_k)^2 + \lambda_1 \lambda_k + c.
\]

Differentiating the above equation with respect to \(e_i\), by taking into account \(e_i(u_k) = 0\) and \(e_i e_1(u_k) = 0\) we derive

\[
e_i(\lambda_k) = 0
\]

for any \(1 \leq k \leq 5\) and \(2 \leq i \leq n\).

**Case B.** Suppose \(m \leq 5\). Denote by \(\tilde{\lambda}_i\) the distinct principal curvatures with the corresponding multiplicities \(n_i\) for \(1 \leq i \leq 4\). Then the number of different \(u_i\) is less than or equal to four. In the case that four ones of \(u_i\) are mutually different, it is needed only to consider the system (3.17) for \(k = 1, 2, 3, 4\). A similar discussion as in Case A could yield the conclusion. If three ones or less of \(u_i\) are mutually different, then the conclusion follows by some similar arguments as above.

Thus, we conclude Lemma 3.4. \(\square\)

**Lemma 3.5.** For arbitrary three distinct principal curvatures \(\lambda_i, \lambda_j, \lambda_k\) \((2 \leq i, j, k \leq n)\), we have the following relations:

\[
(3.33) \quad \omega_{ij}^k (\lambda_j - \lambda_k) = \omega_{ji}^k (\lambda_i - \lambda_k) = \omega_{kj}^i (\lambda_j - \lambda_i),
\]

\[
(3.34) \quad \omega_{ij}^k \omega_{ji}^k + \omega_{jk}^i \omega_{kj}^i + \omega_{ki}^j \omega_{ik}^j = 0,
\]

\[
(3.35) \quad \omega_{ij}^k (\omega_{jj}^1 - \omega_{kk}^1) = \omega_{ji}^k (\omega_{ii}^1 - \omega_{kk}^1) = \omega_{kj}^i (\omega_{jj}^1 - \omega_{ii}^1).
\]

**Proof.** We recall in the beginning part of this section that the number \(m\) of distinct principal curvatures satisfies \(m \geq 4\). Hence, by taking into account the second expression of (3.7) and (3.9) for three distinct principal curvatures \(\lambda_i, \lambda_j\) and \(\lambda_k\) \((2 \leq i, j, k \leq n)\), we obtain (3.33) and (3.34) immediately.

Let us consider (3.35). It follows from the Gauss equation that

\[
(R(e_i, e_j) e_k, e_1) = 0.
\]

Moreover, since \(\omega_{ij}^1 = 0\) for \(i \neq j\) from (3.7) and (3.16), from the definition of the curvature tensor we have

\[
(3.36) \quad \omega_{ij}^k (\omega_{jj}^1 - \omega_{kk}^1) = \omega_{ji}^k (\omega_{ii}^1 - \omega_{kk}^1).
\]
Proof. In the following, we consider the case that the number of distinct principal curvatures is 6.

\[
\omega^1_{ii}\omega^1_{jj} - \sum_{k=2, k \neq l(i,j)}^{n} 2\omega^k_{ij}\omega^k_{ji} = -\lambda_i\lambda_j - c, \quad \text{for } \lambda_i \neq \lambda_j,
\]

which together with (3.7) and (3.36) gives (3.35).

\[\Box\]

Lemma 3.6. Under the assumptions as above, we have

\[
(3.37) \quad \omega^1_{ii}\omega^1_{jj} - \sum_{k=2, k \neq l(i,j)}^{n} 2\omega^k_{ij}\omega^k_{ji} = -\lambda_i\lambda_j - c,
\]

where \(l(i,j)\) stands for the indexes satisfying \(\lambda_{l(i,j)} = \lambda_i\) or \(\lambda_j\).

Proof. In the following, we consider the case that the number \(m\) of distinct principal curvatures is 6.

Without loss of generality, except \(\lambda_1\), we assume that \(\lambda_p, \lambda_q, \lambda_r, \lambda_u, \lambda_v\) are the five distinct principal curvatures in sequence with the corresponding multiplicities \(n_1, n_2, n_3, n_4, n_5\) respectively, i.e.
\[
\begin{aligned}
\lambda_1, & \lambda_p, \ldots, \lambda_p, \\
& \lambda_q, \lambda_r, \ldots, \lambda_r, \\
& \lambda_u, \lambda_u, \ldots, \lambda_u, \\
& \lambda_v, \ldots, \lambda_v.
\end{aligned}
\]

We now compute \(\langle R(e_p, e_q)e_p, e_q \rangle\). On one hand, it follows from the Gauss equation and (3.1) that

\[
(3.38) \quad \langle R(e_p, e_q)e_p, e_q \rangle = -\lambda_p\lambda_q - c.
\]

On the other hand, since
\[
\begin{align*}
\nabla_{e_p} \nabla_{e_q} e_p & = \sum_{k=1}^{n} e_p (\omega^k_{qp}) e_k + \sum_{k=1}^{n} \omega^k_{qpp} \omega^l_{pk} e_l, \\
\nabla_{e_q} \nabla_{e_p} e_p & = \sum_{k=1}^{n} e_q (\omega^k_{pp}) e_k + \sum_{k=1}^{n} \omega^k_{ppq} \omega^l_{qpk} e_l, \\
\nabla_{[e_p, e_q]} e_p & = \sum_{k=1}^{n} (\omega^k_{pq} - \omega^k_{qp}) \sum_{l=1}^{n} \omega^l_{kp} e_l,
\end{align*}
\]

it follows that

\[
(3.39) \quad \langle R(e_p, e_q)e_p, e_q \rangle = e_p (\omega^q_{qp}) + \sum_{k=1}^{n} \omega^k_{qpq} \omega^q_{pk} - e_q (\omega^q_{pp})
\]

\[-\sum_{k=1}^{n} \omega^k_{qpq} \omega^q_{pk} - \sum_{k=1}^{n} (\omega^k_{pq} - \omega^k_{qp}) \omega^q_{kp}.
\]

Since \(\lambda_p \neq \lambda_q\), from (3.8), (3.7) and Lemma 3.4 we have

\[
(3.40) \quad \omega^q_{qp} = \omega^q_{pq} = \omega^q_{pp} = 0, \quad \text{and} \quad \sum_{k=2}^{n} \omega^k_{ppq} \omega^q_{qk} = 0.
\]

Moreover, if \(2 \leq k \leq n_1 + 1\), then \(\lambda_k = \lambda_p\), by the second expression of (3.7) and (3.9) we get
\[
(\lambda_p - \lambda_k)\omega^k_{qp} = (\lambda_q - \lambda_k)\omega^k_{pq}, \quad \text{and} \quad (\lambda_k - \lambda_q)\omega^q_{pk} = (\lambda_p - \lambda_q)\omega^q_{kp},
\]

which imply that

\[
(3.41) \quad \omega^q_{pq} = \omega^q_{pk} = \omega^q_{kp} = 0.
\]

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Similarly, if \( n_1 + 2 \leq k \leq n_1 + n_2 + 1 \), we also have

\[
\omega^k_{pq} = \omega^q_{kp} = \omega^q_{kk} = 0.
\]

Hence, by taking into account (3.40)-(3.42), (3.39) becomes

\[
\langle R(e_p, e_q) e_p, e_q \rangle = \omega_{pp}^1 \omega_{qq}^1 + \sum_{k=n_1+n_2+2}^{n_2} \left\{ \omega_{qp}^k \omega_{pk}^q - \left( \omega_{pq}^k - \omega_{qp}^k \right) \omega_{kp}^q \right\},
\]

which together with (3.38), (3.7) and (3.34) gives

\[
\omega_{pp}^1 \omega_{qq}^1 - \sum_{k=n_1+n_2+2}^{n_2} 2\omega_{qp}^k \omega_{qp}^k = -\lambda_p \lambda_q - c.
\]

Similarly, we could deduce other equations for different pairs \( \omega_{rr}^1 \omega_{rr}^1, \omega_{uu}^1 \omega_{uu}^1, \ldots \).

Hence we get equation (3.37).

In the case that the number \( m \) of distinct principal curvatures satisfies \( m = 4 \), or 5, a very similar argument gives (3.37) as well. \( \square \)

### 4. Proof of Theorem 1.1

Assume that the mean curvature \( H \) is not constant.

Differentiating (3.2) with respect to \( e_1 \) and using (3.12)-(3.13), we obtain

\[
3e_1(\lambda_1) = \sum_{i=2}^{n} (\lambda_1 - \lambda_i) \omega_{ii}^1.
\]

Following the previous section, we only deal with the case that the number of distinct principal curvatures is 6, i.e. \( m = 6 \). In fact, the proofs for the cases that \( m = 5, 4 \) are very similar, so we omit it here without loss of generality.

According to Lemma 3.5, we consider the following cases:

**Case A.** \( \omega_{pq}^r \neq 0, \omega_{pq}^u \neq 0, \) and \( \omega_{pq}^v \neq 0 \). Since \( \lambda_p, \lambda_q, \lambda_r, \lambda_u, \lambda_v \) are mutually different, equations (3.33) and (3.35) reduce to

\[
\frac{\omega_{pp}^1 - \omega_{qq}^1}{\lambda_p - \lambda_q} = \frac{\omega_{pp}^1 - \omega_{rr}^1}{\lambda_p - \lambda_r} = \frac{\omega_{qq}^1 - \omega_{rr}^1}{\lambda_q - \lambda_r} = \frac{\omega_{pp}^1 - \omega_{uu}^1}{\lambda_p - \lambda_u} = \frac{\omega_{qq}^1 - \omega_{uu}^1}{\lambda_q - \lambda_u} = \frac{\omega_{pp}^1 - \omega_{vv}^1}{\lambda_p - \lambda_v} = \frac{\omega_{qq}^1 - \omega_{vv}^1}{\lambda_q - \lambda_v}.
\]

Thus, there exist two smooth functions \( \varphi \) and \( \psi \) depending on \( t \) such that

\[
\omega_{ii}^1 = \varphi \lambda_i + \psi.
\]

Differentiating with respect to \( e_1 \) on both sides of equation (4.2), and using (3.12) and (3.13) we get

\[
e_1(\varphi) = \lambda_1(\varphi^2 + 1) + \varphi \psi,
\]

\[
e_1(\psi) = \psi(\lambda_1 \varphi + \psi) + c.
\]

Taking into account (4.2), and using (3.2), (3.5) one has

\[
\sum_{i=2}^{n} \omega_{ii}^1 = -3\lambda_1 \varphi + (n - 1) \psi,
\]
and (4.1) and (3.11) respectively become

\begin{align}
(4.5) \quad 3e_1(\lambda_1) &= (R - n(n - 1)c - 6\lambda_1^2)\varphi + (n + 2)\lambda_1\psi, \\
(4.6) \quad e_1e_1(\lambda_1) &= e_1(\lambda_1)(-3\lambda_1\varphi + (n - 1)\psi) + \lambda_1(n(n - 2)c - R + 4\lambda_1^2).
\end{align}

Differentiating (4.5) with respect to \( e_1 \), we may eliminate \( e_1e_1(\lambda_1) \) by (4.6). Using (4.3), (4.4) and (4.6) we have

\begin{align}
(4.7) \quad 3(n - 4)e_1(\lambda_1)\psi &= \lambda_1 (6R - (4n^2 - 12n - 3)c - 27\lambda_1^2).
\end{align}

Note here that \( n > 4 \) since the number of distinct principal curvatures is six.

Eliminating \( e_1(\lambda_1) \) between (4.5) and (4.7) gives

\begin{align}
(4.8) \quad (n - 4)\left\{(R - n(n - 1)c - 6\lambda_1^2)\varphi + (n + 2)\lambda_1\psi^2\right\} &= \lambda_1 (6R - (4n^2 - 12n - 3)c - 27\lambda_1^2).
\end{align}

Moreover, differentiating (4.7) with respect to \( e_1 \), by (4.4), (4.6), (4.7) we have

\begin{align}
(4.9) \quad (432\lambda_1^4 + a_1\lambda_1^2 + a_2)\varphi + \left\{-54(n + 3)\lambda_1^3 + a_3\lambda_1\right\}\psi &= 12(n - 4)\lambda_1^3 + a_4\lambda_1,
\end{align}

where

\begin{align*}
a_1 &= (97n^2 - 111n + 60)c - 105R, \\
a_2 &= ((4n^2 - 9n + 9)c - 6R)(n(n - 1)c - R), \\
a_3 &= 12R - (4n^2 - 6n + 21)c, \\
a_4 &= 3n(n - 4)(n - 2)c.
\end{align*}

Differentiating (4.9) with respect to \( e_1 \) and using (4.3)-(4.4), we get

\begin{align*}
(1728\lambda_1^3 + 2a_1\lambda_1)\varphi e_1(\lambda_1) + (432\lambda_1^1 + a_1\lambda_1^2 + a_2)\left\{\lambda_1(\varphi^2 + 1) + \varphi\psi\right\} \\
+ \left\{-162(n + 3)\lambda_1^3 + a_3\right\}\psi e_1(\lambda_1) + \left\{-54(n + 3)\lambda_1^3 + a_3\lambda_1\right\}\left\{\psi(\lambda_1\varphi + \psi) + c\right\} \\
= (36(n - 4)\lambda_1^2 + a_4)e_1(\lambda_1).
\end{align*}

Multiplying \( 3(n - 4) \) on both sides of the above equation and using (4.5) and (4.7) we have

\begin{align}
(4.10) \quad (n - 4)(1728\lambda_1^3 + 2a_1\lambda_1)\varphi \left\{(R - n(n - 1)c - 6\lambda_1^2)\varphi + (n + 2)\lambda_1\psi\right\} \\
+ 3(n - 4)(432\lambda_1^1 + a_1\lambda_1^2 + a_2)\left\{\lambda_1(\varphi^2 + 1) + \varphi\psi\right\} \\
+ \lambda_1 \left\{-162(n + 3)\lambda_1^3 + a_3\right\}\left\{6R - (4n^2 - 12n - 3)c - 27\lambda_1^2\right\} \\
+ 3(n - 4)\left\{-54(n + 3)\lambda_1^3 + a_3\lambda_1\right\}\left\{\psi(\lambda_1\varphi + \psi) + c\right\} \\
= (n - 4)(36(n - 4)\lambda_1^2 + a_4)\left\{(R - n(n - 1)c - 6\lambda_1^2)\varphi + (n + 2)\lambda_1\psi\right\}.
\end{align}

Note that equation (4.10) could be rewritten as

\begin{align}
q_1(\lambda_1)\varphi^2 + q_2(\lambda_1)\varphi\psi + q_3(\lambda_1)\psi^2 + q_4(\lambda_1)\varphi + q_5(\lambda_1)\psi + q_6(\lambda_1) = 0,
\end{align}
where \( q_i \) are non-trivial polynomials concerning function \( \lambda_1 \) and given by:

\[
\begin{align*}
q_1 &= (n - 4)(1728\lambda_1^3 + 2a_1\lambda_1)(R - n(n - 1)c - 6\lambda_1^2) + 3(n - 4)(432\lambda_1^4 + a_1\lambda_1^2 + a_2)\lambda_1, \\
q_2 &= (n - 4)(n + 2)\lambda_1(1728\lambda_1^3 + 2a_1\lambda_1) + 3(n - 4)(432\lambda_1^4 + a_1\lambda_1^2 + a_2) + 3(n - 4)\{ -54(n + 3)\lambda_1^3 + a_3\lambda_1 \} \lambda_1, \\
q_3 &= 3(n - 4)\{ -54(n + 3)\lambda_1^3 + a_3\lambda_1 \}, \\
q_4 &= (n - 4)(36(n - 4)\lambda_1^2 + a_4)(R - n(n - 1)c - 6\lambda_1^2), \\
q_5 &= -(n - 4)(n + 2)(36(n - 4)\lambda_1^2 + a_4)\lambda_1, \\
q_6 &= -3(n - 4)(432\lambda_1^4 + a_1\lambda_1^2 + a_2)\lambda_1 + \lambda_1\{ -162(n + 3)\lambda_1^3 + a_3 \} \{ 6R - (4n^2 - 12n - 3)c - 27\lambda_1^2 \} + 3c(n - 4)\{ -54(n + 3)\lambda_1^3 + a_3\lambda_1 \}. \\
\end{align*}
\]

In the same manner, (4.8) and (4.9) could be also rewritten respectively as:

\[
\begin{align*}
p_1(\lambda_1)\varphi + p_2(\lambda_1)\psi^2 &= p_4(\lambda_1), \\
h_1(\lambda_1)\varphi + h_2(\lambda_1)\psi &= h_3(\lambda_1),
\end{align*}
\]

where \( p_i, h_i \) \((i = 1, 2)\) are polynomials concerning function \( \lambda_1 \) and given by

\[
\begin{align*}
p_1 &= (n - 4)(R - n(n - 1)c - 6\lambda_1^2), \\
p_2 &= (n - 4)(n + 2)\lambda_1, \\
p_3 &= \lambda_1(6R - (4n^2 - 12n - 3)c - 27\lambda_1^2), \\
h_1 &= 432\lambda_1^4 + a_1\lambda_1^2 + a_2, \\
h_2 &= -54(n + 3)\lambda_1^3 + a_3\lambda_1, \\
h_3 &= 12(n - 4)\lambda_1^3 + a_4\lambda_1.
\end{align*}
\]

Multiplying \( h_1^2 \) on both sides of the equation (4.11), by taking into account (4.14) we may eliminate \( \varphi \) and get

\[
P_1\psi^2 + P_2\psi = P_3,
\]

where

\[
\begin{align*}
P_1 &= q_1h_2^2 - q_2h_1h_2 + q_3h_1^2, \\
P_2 &= -2q_1h_2h_3 + q_2h_1h_3 - q_4h_1h_2 + q_5h_1^2, \\
P_3 &= -q_1h_3^2 - q_4h_1h_3 - q_6h_1^2.
\end{align*}
\]

Similarly, eliminating \( \varphi \) in (4.13) by using (4.14) yields

\[
Q_1\psi^2 + Q_2\psi = Q_3,
\]

where

\[
\begin{align*}
Q_1 &= p_2h_1 - p_1h_2, \\
Q_2 &= p_1h_3, \\
Q_3 &= p_3h_1.
\end{align*}
\]

Moreover, multiplying \( Q_1 \) and \( P_1 \) on both sides of the equations (4.16) and (4.18) respectively, after eliminating the \( \psi^2 \) part we obtain

\[
(P_2Q_1 - P_1Q_2)\psi = P_3Q_1 - P_1Q_3.
\]
Multiplying $P_1\psi$ on (4.20) and then combining this with (4.16) give
(4.21) \[ P_1(P_3Q_1 - P_1Q_3) + P_2(P_2Q_1 - P_1Q_2)\psi = P_3(P_2Q_1 - P_1Q_2). \]

At last, after eliminating $\psi$ between (4.20) and (4.21) we get
(4.22) \[ P_1(P_3Q_1 - P_1Q_3)^2 + P_2(P_2Q_1 - P_1Q_2)(P_3Q_1 - P_1Q_3) = P_3(P_2Q_1 - P_1Q_2)^2. \]

We observe from (4.12), (4.15), (4.17) and (4.19) that both $P_i$ and $Q_i$ ($1 \leq i \leq 3$) are polynomials concerning $\lambda$ with constant coefficients. Hence, it follows that

\begin{align*}
P_1 &= -10077696(n-4)(n+3)(n-1)\lambda_1^{11} + \cdots, \\
P_2 &= -839808(n-4)^2(11n+5)\lambda_1^{11} + \cdots, \\
P_3 &= -69984(19n+113)\lambda_1^{13} + \cdots, \\
Q_1 &= 108(n-4)(n-1)\lambda_1^5 + \cdots, \\
Q_2 &= -72(n-4)^2\lambda_1^5 + \cdots, \\
Q_3 &= -11664\lambda_1^7 + \cdots,
\end{align*}

where we only need to write the highest order terms of $\lambda_1$.

By substituting $P_i$ and $Q_i$ into equation (4.22), we get a polynomial equation concerning $\lambda_1$ with constant coefficients $c_i = c_i(n, c, R)$:

(4.23) \[ \sum_{i=0}^{47} c_i\lambda_1^i = 0, \]

where the coefficient $c_{47}$ of the highest order term satisfies

\[ c_{47} = -10077696(n-4)^2(n+3)(n-1)^2 \left[ 69984 \times 108(19n+113) \right. \]
\[ \left. + 10077696 \times 11664(n+3) \right]^2 \neq 0. \]

Therefore, $\lambda_1$ has to be constant and $H = -2\lambda_1/n$ is a constant, which is a contradiction.

**Case B.** $\omega_{pq}^r \neq 0, \omega_{pq}^u \neq 0$, and $\omega_{pq}^k \neq 0$ for all other distinct triplets $\{i, j, k\}$ and distinct principal curvatures $\lambda_i, \lambda_j, \lambda_k$. Then, (3.37) implies that

(4.24) \[ \omega_{pq}^i \omega_{iv}^j = -\lambda_p \lambda_v - c, \]
(4.25) \[ \omega_{pq}^i \omega_{uv}^j = -\lambda_q \lambda_v - c, \]
\[ \omega_{qr}^i \omega_{iv}^j = -\lambda_r \lambda_v - c, \]
\[ \omega_{uv}^i \omega_{iv}^j = -\lambda_u \lambda_v - c. \]

Similar to Case A, since $\omega_{pq}^r \neq 0, \omega_{pq}^u \neq 0$, (3.33) and (3.35) imply that

(4.26) \[ \omega_{ii}^1 = \varphi \lambda_i + \psi, \quad \text{for } i = p, q, r, u, \]

where $\varphi$ and $\psi$ satisfy the differential equations (4.3) and (4.4).

Substituting (4.26) into (4.24) and (4.25), we obtain

(4.27) \[ \omega_{iv}^1 = -\frac{1}{\varphi} \lambda_v, \]
(4.28) \[ \lambda_v \psi = c\varphi, \]

which means that $\omega_{iv}^1$ and $\lambda_v$ are determined completely by $\varphi$ and $\psi$. 

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Substitute (4.26)-(4.28) into (4.1), and then differentiate it with respect to \( e_1 \). By using (4.3), (4.4) and (3.11), a similar discussion as Case A could give a polynomial concerning function \( \lambda_1 \) with constant coefficients. Hence, \( \lambda_1 \) has to be constant, which yields a contradiction as well.

**Case C.** \( \omega_{pq}^r \neq 0 \) (or \( \omega_{pq}^r = 0 \)), and all the \( \omega_{ij}^k = 0 \) for distinct triplets \( \{i, j, k\} \) and distinct principal curvatures \( \lambda_i, \lambda_j, \lambda_k \). Then, (3.37) implies that

\[
\begin{align*}
\omega_{pp}^1 & = -\lambda_p \lambda_u - c, \quad \omega_{pu}^1 \omega_{pu}^1 = -\lambda_p \lambda_v - c, \\
\omega_{pq}^1 & = -\lambda_q \lambda_u - c, \quad \omega_{pq}^1 \omega_{pq}^1 = -\lambda_q \lambda_v - c, \\
\omega_{pu}^1 & = -\lambda_r \lambda_u - c, \quad \omega_{pu}^1 \omega_{pu}^1 = -\lambda_r \lambda_v - c, \\
\omega_{uv}^1 & = -\lambda_u \lambda_v - c.
\end{align*}
\]

We first consider \( \lambda_i \neq 0 \) for \( i = p, q, r, u, v \). Consequently, (4.29)-(4.32) reduce to

\[
\begin{align*}
\frac{\omega_{pp}^1}{\lambda_p} & = \frac{\omega_{pq}^1}{\lambda_q} = \frac{\omega_{pu}^1}{\lambda_r} = \frac{\lambda_u - \lambda_v}{\omega_{uu}^1 - \omega_{vv}^1}, \\
\frac{\omega_{uu}^1}{\lambda_u} & = \frac{\omega_{vv}^1}{\lambda_v} = \frac{-\lambda_p - \lambda_q}{\omega_{pp}^1 - \omega_{qq}^1},
\end{align*}
\]

and hence

\[
\begin{align*}
\frac{\omega_{pp}^1}{\lambda_p} & = \frac{\omega_{pq}^1}{\lambda_q} = \frac{\omega_{pu}^1}{\lambda_r} = \varphi, \\
\frac{\omega_{uu}^1}{\lambda_u} & = \frac{\omega_{vv}^1}{\lambda_v} = \psi
\end{align*}
\]

for two functions \( \varphi \) and \( \psi \).

Substituting (4.33) and (4.34) back to (4.29) gives

\[
\begin{align*}
(1 + \varphi \psi) \lambda_p \lambda_u & = -c, \\
(1 + \varphi \psi) \lambda_p \lambda_v & = -c,
\end{align*}
\]

which imply that \( \lambda_u = \lambda_v \). This is impossible.

If \( \lambda_p = 0 \), then (3.12) and (4.29) imply that \( \omega_{pp}^1 = 0 \) and \( c = 0 \). Then (4.30) and (4.31) yield

\[
\frac{\omega_{uu}^1}{\lambda_u} = \frac{\omega_{uv}^1}{\lambda_v} = \gamma
\]

for some function \( \gamma \). However, combining (4.35) with (4.32) gives \( \gamma^2 = -1 \). Hence it is a contradiction.

At last, we consider \( \lambda_u = 0 \). Then (3.12) and (4.29) reduce to \( \omega_{uu}^1 = c = 0 \). The second equations of (4.29)-(4.31) show that

\[
\begin{align*}
\frac{\omega_{pp}^1}{\lambda_p} & = \frac{\omega_{pq}^1}{\lambda_q} = \frac{\omega_{pu}^1}{\lambda_r} = \varphi, \\
\frac{\omega_{uu}^1}{\lambda_u} & = -\frac{1}{\varphi}
\end{align*}
\]

By taking into account (4.36) and (4.37) together with (3.11) and (4.1), a very similar and direct computation as Case A also gives a polynomial concerning function \( \lambda_1 \) with constant coefficients. Hence, this is a contradiction and the mean curvature \( H \) has to be constant.
In conclusion, we complete the proof of Theorem 1.1.

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