THREE-LAYER HELE-SHAW DISPLACEMENT 
WITH AN INTERMEDIATE NON-NEWTONIAN FLUID

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Abstract: We study the displacement of two Stokes immiscible fluids in a porous medium, approximated by the Hele-Shaw horizontal model. An intermediate non-Newtonian polymer-solute, whose viscosity is depending on the velocity, is considered between the initial fluids. The linear stability problem of this three-layer displacement does not make sense. If the intermediate viscosity depends on velocity and on the polymer concentration, we can obtain a minimization of the Saffman-Taylor instability.

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1. Introduction

We consider a displacing fluid (water) with constant viscosity $\mu_W > 0$, which is pushing a second immiscible fluid (oil) with constant viscosity $\mu_O > \mu_W$, in a horizontal Hele-Shaw cell. A well-known result is given by Saffman and Taylor [15]: the flow is unstable.

An intermediate non-Newtonian liquid, with a variable viscosity depending on the velocity, is considered between water and oil. We study the effect of the intermediate liquid on the flow stability.

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The Hele-Shaw cell is parallel with the fix plane $x_1Oy$. The flow is from
the left to the right, due to the water velocity $U$ far upstream ($x_1 \to -\infty$). In
the moving coordinate system $x = x_1 - Ut$, the displacing and displaced fluids
are contained in the regions $x < -L$ and $x > 0$. The intermediate region is the
interval $-L < x < 0$ and contains the non-Newtonian liquid. The $Oz$ axis is
orthogonal on the plates. The gravity effects are neglected.

As usual in the Hele-Shaw model, between the above three immiscible fluids
we have two sharp interfaces. We prove that the system which governs the linear
stability of the interfaces of this three-layer displacement does not make sense.

Some previous results were obtained when the middle region is filled by
a polymer solution with variable viscosity, which is function of the polymer
concentration $c$, or is verifying a diffusion equation. In the first case - see [3],
[6], [7], [10] - it was proved that an “optimal” viscosity profile exists, which can
give a significant improvement of the stability. In the second case - see [4], [5],
[11] - it was proved that the diffusion is improving the stability, compared with
the Saffman-Taylor case, when the viscosity profile is verifying some specific
conditions.

In [14] we proved that several intermediate Stokes liquids with constant
viscosities, inserted between the initial fluids, can not minimize the Saffman-
Taylor instability.

The use of polymer-solute in the middle region is related with liquids which
are non-Newtonian. The relation between tangential stress and deformation
tensor is not linear, or/and the viscosity in the middle region depends on some
parameters of the flow.

When an Oldroyd-B fluid is displaced by air or two Oldroyd-B fluids are
displacing in a Hele-Shaw cell, the interfaces are much more unstable compared
with the Saffman-Taylor displacement - see [12] and [16].

In Section 2 we consider an intermediate polymer-solute whose viscosity is
depending only on the displacing velocity $U$ far upstream. The linear stability
problem does not make sense.

In Section 3 we consider an intermediate viscosity depending on $U$ and
on the polymer concentration $c$. With some additional conditions, we get a
stabilizing effect, compared with the Saffman-Taylor case.

2. When $\mu$ is depending on $U$

A non-Newtonian model of a liquid with viscosity depending on the velocity was
proposed in [8]. A Darcy’s law type was obtained for a shear-rate dependent
viscosity. This model can be used to justify the flows studied in [1], [2].

Consider a horizontal Hele-Shaw cell with the gap \( b \) between the plates. The pressure, the averaged velocities (across the gap \( b \)) and the viscosity are denoted by \( p, (u, v, w), \mu \). The component \( w \) is neglected. The following equations are considered (in the cited above papers) for the flow in the middle region:

\[
\begin{align*}
\nabla_{x,y} p &= -\mu(u^2/b^2)u, \quad p_z = 0, \\
u &= (u, v), \quad u_x + v_y = 0, \\
\mu(u^2) &= \mu_S \frac{1 + au^2 \tau}{1 + u^2 \tau} \cdot \frac{b^2}{12}, \\
d\mu/d\theta &= \mu_S \frac{(a - 1) \tau (1 + \theta \tau)^2}{(1 + \theta \tau)^{12}}, \quad \theta = u^2,
\end{align*}
\]

where \( \tau \) is a characteristic time. The parameter \( a \) is governing the sign of \( d\mu/d\theta \). We have shear thinning for \( a < 1 \) and shear thickening for \( a > 1 \).

In [8] is given the following explanation for the Darcy type law (1). A dependence of \( \mu \) on the square of the trace of the rate-of-strain-tensor is assumed. For a Poiseuille flow, that means \( \mu \) is function of \( u_z^2 \).

The first step is to use the general form of the flow equations - see for example [9]. Thus the unsteady flow is governed by the equations

\[
\begin{align*}
\mathbf{u}_t &= -\nabla p + [\mu(\phi)u_z]_z = -\nabla p + \eta u_{zz}, \\
\eta(\phi) &= [\mu(\phi) + 2\phi \frac{d\mu}{d\phi}], \quad \phi = u_z^2,
\end{align*}
\]

where \( \eta \) is the “effective viscosity“. We need a positive \( \eta \). To this end we impose the condition

\[
\mu(\phi) + 2\phi \frac{d\mu}{d\phi} > 0.
\]

Therefore, in the steady 2D case, the general flow equations with viscosity depending on the rate-of-strain tensor are

\[
\nabla^2 p = \{\mu(u_z^2)u_z\}_z, \quad \mathbf{u} = (u, v), \quad u_x + v_y = 0,
\]

where the index 2 on the \( \nabla \) operator stands for \( x,y \) derivatives. We integrate with respect to \( z \) and get (recall \( p \) is not depending on \( z \))

\[
z \nabla^2 p = \mu(u_z^2)u_z.
\]

The second step is to find \( u_z \) as function of \( \nabla^2 p \). The invertibility of the above relation near \( u_z = 0 \), by using the implicit function theorem, is possible only if

\[
h(\psi)(0) > 0, \quad h(\psi) = \psi \mu(\psi^2), \quad \psi = u_z.
\]
We have $h_\psi = \mu(\psi^2) + 2\psi^2\mu_\psi(\psi^2)$, therefore the invertibility condition is exactly the inequality (4). The inversion procedure is performed in [8]. The last step for obtaining the Darcy type law (1) is the average procedure of the velocity across the Hele-Shaw gap. Thus $u$ is obtained in terms of the pressure gradient.

We consider the basic solution with two straight interfaces:

$$x_L = -L, \quad x_R = 0;$$

$$u = U, v = 0; \quad P_x = -\mu U; P_y = 0;$$

$$\mu = \mu_W, x < -L; \quad \mu = \mu_O, x > 0;$$

$$\mu_W < \mu = \mu(U^2) < \mu_O, \quad x \in (-L, 0).$$

(7)

In this section we study the linear stability of the solution (7).

The small perturbations are denoted by $u', v', p', \mu'$.

The viscosity perturbation $\mu'$ is obtained as follows. In the frame of the linear stability analysis, we neglect $u'^2, v'^2$ and get

$$E := \mu((U + u')^2 + v'^2) \approx \mu(U^2 + 2Uu').$$

The first-order Taylor expansion is giving

$$E = \mu(U^2) + 2Uu' \cdot \mu_\theta(U^2) := \mu(\theta) + \mu',$$

$$\mu' := 2U\mu_\theta u', \quad \theta = U^2, \quad \mu_\theta = d\mu/d\theta.$$

(8)

We insert $u', v', p', \mu'$ in the flow equations (1) and obtain

$$(P + p')_x = -(\mu + \mu')(U + u'),$$

$$(P + p')_y = -(\mu + \mu')(v'),$$

$$(u + u')_x + (v + v')_y = 0 \Rightarrow$$

$$u'_x + v'_y = 0,$$

(10)

$$p'_x = -\mu(\theta)u' - 2\theta\mu_\theta u',$$

(11)

$$p'_y = -\mu(\theta)v'.$$

(12)

We use a Fourier decomposition for the velocity perturbation

$$u'(x, y, t) = f(x) \exp(iky + \sigma t)$$
and from (10) and (12) we get
\[ v' = -\left[ f_x/ik \right] \exp(iky + \sigma t), \]
\[ p' = -\mu(\theta) f_x/k^2 \exp(iky + \sigma t). \]  
(13)

By cross derivation of the pressure perturbations (11) and (12) it follows the amplitude equation:
\[ \left[ \mu(\theta) u' + 2\theta \mu_0 u' \right]_y = \left[ \mu(\theta) v' \right]_x \Rightarrow -\mu f_{xx} + k^2 (\mu + 2\theta \mu_0) f = 0. \]  
(14)

The above equation is quite similar with the relation (11) of [14], but the coefficient of \( f \) is \( k^2 (\mu + 2\theta \mu_0) \) instead of \( k^2 \).

The relation (14) holds for all \( x \in \mathbb{R} \ x \neq -L, x \neq 0 \).

We use the relation (2) and get
\[ \mu + 2\theta \mu_0 = \frac{1 + a\theta \tau}{1 + \theta \tau} + 2\theta \frac{\tau(a - 1)}{(1 + \theta \tau)^2} \frac{\mu_0 b^2}{12} > 0 \]
\[ \Leftrightarrow (\theta \tau)^2 a + (\theta \tau)(3a - 1) + 1 > 0, \forall(\theta \tau). \]  
(15)

The last above inequality is verified only if we impose the condition
\[ \Delta_1 = (3a - 1)^2 - 4a < 0 \Leftrightarrow 9a^2 - 10a + 1 < 0 \]
\[ \Leftrightarrow a \in (1/9, 1). \]  
(16)

Therefore we get the following result:
\[ \mu + 2\theta \mu_0 > 0 \Leftrightarrow a \in (1/9, 1). \]  
(17)

The inequality \( \mu + 2\theta \mu_0 > 0 \) is quite similar with the restriction (4) of [8]. In this paper we consider the viscosity profiles which verify the condition (17).

We introduce the notation
\[ \gamma = k \left( 1 + 2\theta \mu_0/\mu \right)^{1/2}, \]  
(18)
thus from (17) it follows \( \gamma > 0 \). The relation (14) becomes
\[ -f_{xx} + \gamma^2 f = 0, \quad x \neq \{-L, 0\}. \]

We need far decay perturbations, thus by using again the condition (17) (that means \( \gamma > 0 \)) we obtain
\[ f(x) = f(-L) \exp[(x + L)\gamma], \quad x < -L, \]
where the indices $^-,^+$ stands for “left” and “right” limits.

The solution of (14) inside the intermediate region is

$$f(x) = Ae^{\gamma x}, \quad (20)$$

because a term of the form $Be^{-\gamma x}$ becomes very large for large positive $\gamma$ (recall $x < 0$) and the amplitudes $f$ must be small in the frame of the linear stability analysis.

We study now the perturbed interfaces in $x = 0, x = -L$. Near the point $x = 0$ we consider the perturbed interface denoted by $g(x, y, t)$. As the interface is a material one, in the first approximation we get

$$g_t = u' \Rightarrow g(0, y, t) = \left[ f(0)/\sigma \right] \exp(iky + \sigma t). \quad (21)$$

The perturbed pressure is obtained by using the Darcy law (1), the relation (13) and the first-order Taylor expansion for the basic pressure $P$ near $x = 0$:

$$p^{+}(0) = P(0, y, t) + P^{+}_{x}(0, y, t) \cdot g(0, y, t) + p^{'+}(0, y, t)$$

$$= P(0) - \mu^{+}(0) \left[ \frac{Uf(0)}{\sigma} + \frac{f^{+}_{x}(0)}{k^2} \right] \exp(iky + \sigma t). \quad (22)$$

The same procedure is used to get the left limit of the pressure in $x = 0$ and it follows

$$p^{-}(0) = P(0) - \mu^{-}(0) \left[ \frac{Uf(0)}{\sigma} + \frac{f^{-}_{x}(0)}{k^2} \right] \exp(iky + \sigma t). \quad (23)$$

A similar relation can be obtained for the point $x = -L$, which will be used in the relation (27) below.

On the interfaces $x = 0, x = -L$ we consider the surface tensions $T(0), T(-L)$. We use the Laplace law and get

$$p^{+}(0) - p^{-}(0) = T(0)g_{yy}(0, y, t),$$

$$p^{+}(-L) - p^{-}(-L) = T(-L)g_{yy}(-L, y, t). \quad (24)$$

For simplicity, we will use notations

$$f_0 = f(0), \quad f_L = f(-L), \quad T_0 = T(0), \quad T_L = T(-L),$$

$$f^{+,-}_{xo} = f^{+,-}_{x}(0), \quad f^{+,-}_{xL} = f^{+,-}_{x}(-L),$$
\[ \mu^+ - \mu^- = \mu^+ - \mu^-(0), \quad \mu^+_L = \mu^+ - (-L). \]  

From relations (22) - (24) it follows

\[ \mu^- \left[ \frac{U f_0}{\sigma} + \frac{f^-}{k^2} \right] - \mu^+ \left[ \frac{U f_0}{\sigma} + \frac{f^+}{k^2} \right] = -T_0 \frac{f_0}{\sigma} k^2, \]  

\[ \mu^-_L \left[ \frac{U f_L}{\sigma} + \frac{f^-}{k^2} \right] - \mu^+_L \left[ \frac{U f_L}{\sigma} + \frac{f^+_L}{k^2} \right] = -T_L \frac{f_L}{\sigma} k^2. \]

By direct calculations, as in [14], we obtain

\[ \mu^-_0 f^-(0) - \mu^+_0 f^+(0) = \frac{k^2 U [\mu^+_0 - \mu^-_0] - k^3 T_0 f_0}{\sigma}, \]

\[ \mu^-_L f^-(-L) - \mu^+_L f^+(L) = \frac{k^2 U [\mu^+_L - \mu^-_L] - k^3 T_L f_L}{\sigma}. \]

Therefore the interfaces stability is governed by the equation (14) and the boundary conditions (28), (29). We recall that \( \mu \) is not depending on \( x \); we consider

\[ \mu_W < \mu^-(0) = \mu^+(L) = \mu(U) < \mu_O. \]

All growth rates \( \sigma \) must verify the equations (28) - (29).

We use the far field conditions (19) for \( f \). Thus all limit values of \( f_x \) in \( x = -L, x = 0 \) contains the factor \( \gamma \) given by (18).

We compute \( \sigma \) from both equations (28) - (29), we simplify with \( \gamma \) in the denominators, and obtain

\[ \frac{k U (\mu - \mu_W) - k^3 T_0}{\mu W - \mu} = \frac{k U (\mu_O - \mu) - k^3 T_L}{\mu + \mu_O}. \]

We equate the coefficient of \( k, k^3 \), thus it follows

\[ \mu_O = 0, \quad \mu = \mu_W T_L / (T_L + T_0). \]

Both above relations are in contradiction with our hypothesis:

i) The oil viscosity must be strictly positive.

ii) \( \mu \) is depending on \( U \) and is not depending on \( T_0, T_L \).

As a consequence, the growth rates \( \sigma \) can not exists. The stability problem (14), (28), (29) does not make sense.
3. When \( \mu \) is depending on \( U \) and \( c \)

Let us consider a polymer-solute in the middle region, with a variable viscosity \( \mu \) of the type

\[
\mu_W < \mu = \mu(c, U^2) < \mu_O,
\]

where \( c = c(x) \) is the polymer concentration. In [13] was pointed out that, for diluted polymer-solutes, \( \mu \) is invertible with respect to \( c \). Thus the viscosity \( \mu \) in the middle region is depending also on \( x \).

The perturbed viscosity is obtained as follows. We have

\[
\mu(c + c', (U + u')^2 + v'^2) \approx \mu(c, U^2) + \mu_c c' + \mu_\theta 2Uu',
\]

(32)

therefore

\[
\mu' = \mu_c c' + \mu_\theta 2Uu',
\]

(33)

where \( c \) is the basic concentration profile and \( c' \) is the perturbed concentration.

As in [6], [7], we consider the following equation for \( c \) in the fix coordinate system \( x_1Oy \)

\[
c_t + Uc_{x_1} = 0.
\]

(34)

We use the moving reference \( x = x_1 - Ut \) and in the first approximation we get

\[
c'_t = -c_x u', \quad c' = -\frac{1}{\sigma} u' c_x,
\]

(35)

(see [6], relations (2.8) and (2.12) with \( c \) instead of \( \mu \)). The equations (33) and (35) give us

\[
\mu' = \{-\mu_c c_x/\sigma + +\mu_\theta 2U\} u'
\]

\[
= \{-\mu_x/\sigma + +\mu_\theta 2U\} u'.
\]

(36)

The equations (9)-(10) still hold. We use the expression (36) and obtain

\[
p'_x = -\mu u' - U \mu'
\]

\[
= -\mu u' - U(-\mu_x/\sigma + 2\mu_\theta U)u',
\]

\[
p'_y = -\mu v'.
\]

(37)

The relation \( (p'_x)_y = (p'_y)_x \) gives us the amplitude equation inside the middle region \((-L, 0)\):

\[
k^2(-\mu + U \mu_x/\sigma - 2\theta \mu_\theta)f = -(\mu f_x)_x,
\]

(38)
\[-(\mu f_x)_x + k^2(\mu + 2\theta \mu_\theta)f = k^2 U \mu_x f/\sigma. \tag{39}\]

This time we do not know the exact expression of \(f\) inside the middle region, but we have the same relations (19) for \(f\) in the far field (where \(\mu_x = 0\)). The boundary conditions (26), (27) still hold.

We multiply the relation (39) with \(f\), we integrate on \((-L, 0)\) and obtain

\[-(\mu^- f^- f)(0) + (\mu^+ f^+ f)(-L) + \int \mu f_x^2 dx\]
\[+ k^2 \int (\mu + 2\theta \mu_\theta) dx = k^2 \frac{U}{\sigma} \int \mu_x f^2 dx, \tag{40}\]

where we used the notations

\[(FGH)(x) = F(x)H(x)F(x), \quad \int F dx = \int_{-L}^0 F(x) dx.\]

The boundary conditions (28), (29) and (40) give us:

\[-[kE_0 f_0^2/\sigma - \gamma \mu_0^+ f_0^2] + [\gamma \mu_L^- f_L^2 - kE_L f_L^2/\sigma]\]
\[+ \int \mu f_x^2 dx + k^2 \int (\mu + 2\theta \mu_\theta) dx = k^2 \frac{U}{\sigma} \int \mu_x f^2 dx, \]
\[\mu_0^{+,-} = \mu^{+,-}(0), \quad \mu_L^{-+} = \mu^{-+}(-L), \]
\[E_0 = kU [\mu_W - \mu_0^+] k^3 T_0, \]
\[E_L = kU [\mu_L^- - \mu_W] - k^3 T_L. \tag{41}\]

From the above relations (41) we obtain

\[\sigma = \frac{kE_0 f_0^2 + kE_L f_L^2 + k^2 U \int \mu_x f^2 dx}{\mu_0 \gamma f_0^2 + \mu_W \gamma f_L^2 + \int \mu f_x^2 dx + k^2 \int (\mu + 2\theta \mu_\theta) f^2 dx}. \tag{42}\]

We consider now a continuous viscosity and zero surfaces tensions, that means

\[E_0 = E_L = T_0 = T_L = 0. \tag{43}\]

For a Newtonian intermediate liquid depending only on \(c\), the following estimate of the growth rates (say, \(\sigma_c\)) is given in formula (44) of [13]:

\[\sigma_c \leq \frac{U \int \mu_x f^2 dx}{\int \mu f^2 dx}. \tag{44}\]
From (42) we get the upper estimate
\[
\sigma \leq \frac{U \int \mu_x f^2 dx}{\int (\mu + 2\theta \mu) f^2 dx}. \quad (45)
\]

The upper limit (45) is less than the upper bound (44) when \(\mu_\theta > 0\). Therefore we get an improved stability in the non-Newtonian case, if the hypothesis (43) is fulfilled.

**Remark.** For small enough \(\mu_x\) and large enough \(\mu_\theta\) (recall \(\theta = U^2\)), the upper bound (45) becomes arbitrary small. Thus we can almost suppress the Saffman-Taylor instability, even if the surface tensions are zero. We recall the Saffman-Taylor growth rate \(\sigma_{ST}\) for water displacing oil with a surface tension \(T\):
\[
\sigma_{ST} = \frac{kU(\mu_O - \mu_W) - k^3T}{\mu_O + \mu_W}.
\]

We have
\[
T = 0 \Rightarrow \lim_{k \to \infty} \sigma_{ST} = \infty.
\]

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