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Fractal-fractional mathematical modeling and forecasting of new cases and deaths of COVID-19 epidemic outbreaks in India

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Keywords:
- Fractional-order derivative-based modeling
- Fixed point technique
- SPSS program and Expert Modeler Method

Abstract

Fractional-order derivative-based modeling is very significant to describe real-world problems with forecasting and analyze the realistic situation of the proposed model. The aim of this work is to predict future trends in the behavior of the COVID-19 epidemic of confirmed cases and deaths in India for October 2020, using the expert modeler model and statistical analysis programs (SPSS version 23 & Eviews version 9). We also generalize a mathematical model based on a fractal fractional operator to investigate the existing outbreak of this disease. Our model describes the diverse transmission passages in the infection dynamics and affirms the role of the environmental reservoir in the transmission and outbreak of this disease. We give an itemized analysis of the proposed model including, the equilibrium points analysis, reproductive number \( R_0 \), and the positiveness of the model solutions. Besides, the existence, uniqueness, and Ulam-Hyers stability results are investigated of the suggested model via some fixed point technique. The fractional Adams Bashforth method is applied to solve the fractal fractional model. Finally, a brief discussion of the graphical results using the numerical simulation (Matlab version 16) is shown.

Introduction

Corona virus is the inconvenience word today that influenced commune globally. COVID-19 has caused a comprehensive loss of movement in the economic and health situation on the planet. It has made fear, distress, and anxiety and kept everyone in a situation of social isolation. Therefore, the primary treatment has become of using face masks, hand washing, and synthetic compounds delivered from the normal schedule. Governments stand weak and powerless in the face of isolation. Therefore, the primary treatment has become of using face masks, hand washing, and synthetic compounds delivered from the normal schedule. Governments stand weak and powerless in the face of isolation. Therefore, the primary treatment has become of using face masks, hand washing, and synthetic compounds delivered from the normal schedule. Governments stand weak and powerless in the face of isolation. Therefore, the primary treatment has become of using face masks, hand washing, and synthetic compounds delivered from the normal schedule.

Coronavirus is an extended family of viruses that mainly infect animals, and it can sometimes infect humans. So, these viruses are transmitted between humans through the air, secretions, or touched with contaminated bodies, especially in the winter season. The main symptoms of this virus are fever, respiratory infections, coughing, chest pain, shortness of breath at times, and digestive disorders. There are two types of coronavirus causes dangerous and sometimes fatal epidemics for humans, the first of which is the SARS virus and it caused a global epidemic that spread in 2002–2003, bats were their source and the second type is MERS virus, and it was discovered for the first time in 2012 in the Middle East. In end of the month of December 2019, a new virus was identified named coronavirus (COVID-19) in China. Then the virus mutated and transmitted to humans, and it adapted and became able to transmit from one person to another. The incubation period for the virus is believed to be seven days and may reach 14 days. So far, neither
efficient drug to fight these viruses, nor vaccine against them.

Mathematical modeling and investigation of infectious illnesses have been focal to the study of infectious disease transmission since its initiation as a discipline over a century ago [2,3]. Recently, electronic reconnaissance of infectious diseases has become widespread due to the approach of better computing, data software’s as well as the availability of potentials for distributing and preservation information over the web, and fast indicative tests and the analysis of genetic sequence using several computerized methods. Moreover, these continuous improvements have expanded the use of simulations techniques and modeling based mathematics to both the generation and testing of fundamental assumptions and introduce experimental devices for controlling the spread and transmission of the disease. Modeling, simulation and analysis of these diseases have effectively clarified before astounding perceptions and had a focal impact in numerous countries [4-10]. Both mathematical modeling and simulation play a critical role in making rapid evaluation. Hence, Simulation is additionally utilized when the expense of gathering information is restrictively costly, or there are countless experimental conditions to test. Throughout the long term, an immense number of approaches have been proposed for taking a gander at the problem from alternate points of view. These categories include three general classes:

1. Statistical strategies for observation of outbreaks and distinguishing spatial styles in real plagues.
2. Mathematical models inside the setting of dynamical frameworks used to conjecture the development of a hypothetical or on-going pandemic spread.
3. Computational intelligence techniques for the forecasting of the advancement of a progressing epidemic.

For the three classifications there are other methodologies weaving major and various previous work. Recently, from the literature we can see that several models based on differential equations were introduced for modeling and simulation of the spread of the viruses and by choosing the best parameters of these model one can control them [11-16].

Regardless, starting late the activity of fractional mathematics area that oversee fractional-order has used, also, find perceptible employment in the comprehension of many problem and phenomena from nature due to its exact simulations of hereditary qualities [19,17,21-23]. It has been noticed that differential equations involving fractional operators can be applied to modeling worldwide phenomena even more precisely. The most often utilized operators like Caputo, Caputo-Fabrizio [17] and Atangana-Baleanu [18]. Based on the complex fractal behavior many phenomena come from several areas of research and nature are simulated like, non-linear optics, quantum systems, Schrodinger equations, turbulence, biomedic, porous media, and Darcy’s law exhibit complex fractal behavior. So for this, Atangana introduced some new operators with nonlocal kernel to help in modeling of these problems [19,20].

This field of science is a very hot area of nonlinear models. Therefore, we need to present a comparison with existing papers about the COVID-19 pandemic, for e.g., Gao et al., in [24] investigated of mathematical model of coronavirus (COVID-19) in the frame of Caputo operator by natural Adomian decomposition method. In [25], the authors presented a model dynamical of COVID-19 pandemic with fractional derivative. In this work, they considered a model classical in integer-order derivative and Atangana-Baleanu operator as the best and most appropriate to simulate the dynamics of this disease. Several mathematical models are introduced for describing and simulations of the spread and transmissions of the infection COVID-19, from the human even dead and alive by many authors. For example Atangana suggested that the virus can be transferred from dead persons to another humans [26]. Gao, et al. [27] presented a new investigation for a simulation model based ordinary differential equation of the coronavirus (COVID-19) endemic and they found the optimal parameters of the model to help in controlling the spread and transmitted the virus from location to another. Recently, Qureshi and Atangana in [28] designed epidemiological model with the assist of new fractional derivative to study the spread and transmission of diarrhea virus model in Ghana. Atangana-Baleanu operator is introduced and used in many applications in science and engineering models [29-37]. Extensive review to the different epidemiological models can be seen in [38-50].

The different classical and fractional operators formulas are used in the research area of mathematical biology to comprehend the dynamics of transmission and spread of different diseases in many papers. On the other hand, the fractal fractional operators still not used especially for the dynamics of COVID-19 epidemic diseases, where this is the main significant inspiration behind the offered paper analysis. Here, we utilize the fractal fractional power-law operator in the field of mathematical epidemiology by structuring a model related to COVID-19 transmission infection occurring in India during March 15, 2020 to September 30, 2020. Then we will forecast the active cases and deaths for October.

The purpose of this paper is to inspire from [20] to generalize the epidemiological model [51,52] by a new fractional-fractional law operator to investigate the dynamic behavior of COVID-19. The predicting future trends in the behavior of the global epidemic COVID-19 of confirmed cases and deaths in India for October 2020 are calculated by using the SPSS Program and Expert Modeler Method. We also give an itemized analysis of the proposed model including, the equilibrium points analysis, reproductive number $R_0$, and the positiveness of the model solutions. Besides, we investigate the existence, uniqueness, and Ulam stability results of the proposed model by utilizing some fixed point techniques and nonlinear analysis. Finally, the fractal fractional model is solved taking into account the help of fractional Adams Bashforth technique, and also a brief discussion of the graphical results using the numerical simulation is shown.

The organization of this article is as in the following sections. In Section “Introduction” introduction is provided. The statistical analysis of the model is discussed in Section “Forecasts and statistical analysis”. Some useful fundamentals on fractional calculus Section “Basic concepts of fractal fractional calculus”. In Section “Theoretical approach”, we present the proposed epidemiological model. Further, we also prove the theoretical results of the given model. The obtained numerical results and simulation are provided in Section “Numerical solution of fractal-fractional model (3)”. In the last, a brief conclusion and future recommendations are given in Sections “Conclusion” and “Future work”.

Forecasts and statistical analysis

This section devoted to providing statistical data for the COVID-19 epidemic in India, on its accordingly, we calculate the future forecasts of infection cases and deaths by using the SPSS program and Expert Modeler method. A brief discussion of the obtained results is shown.

**Tables and Figures**

This part offers Figs. 1-6 and Table 1 that indicate the real confirmed cases and the deaths of the COVID-19 epidemic spearad in India during the studied period and also shows the possible predictions of infection and deaths from them for October 2020 as in Tables 2 and 3.

Results and discussions

The latest statistics about the COVID-19 epidemic in the state of India for the period (15/03/2020) until (30/09/2020) are shown in Figs. 1, 2. The numbers also indicate that there is a rapid and continuous increase in the number of new cases, especially in the months of July, August, and September, as the confirmed cases during the three months reached more than (5,658,105) cases with a rate (0.91) of the total number of confirmed cases until the end of the period of our study, which
Fig. 1. Confirmed cases of COVID-19 epidemic in India for the period from March 15 to the end of September 2020.

Fig. 2. Deaths cases of COVID-19 epidemic in India for the period from March 15 to the end of September 2020.

Fig. 3. Transforming the data of confirmed cases of COVID-19 epidemic in India to the first difference.
amounted to (6,224,861) cases, and there are (80,601) deaths and at rates (82.7%) from the total deaths. We note that it is on the rise, as the confirmed cases increased from (26) cases until the end of March reached (1313) with an average of (82) cases, then increased during April to reach (32,213) with an average of (1073) cases, while The number of confirmed cases during May reached (148,533), with an
with an average of (34,581) cases, also cases increased rapidly during
than the cases during the period (15/03/2020) and until (31/06/2020)
confirmed cases reached (1,072,030), which is a much greater number
also observed that during July the cases have increased rapidly and the
(384,697) cases at the end of June with an average of (12,823) cases. We
average of (4791). Then the number of cases increased to reach
-
-Test results of the predictive ability of a linear model of death in India.
Table 3

| Date       | LCL Cases Model | Predicted cases Model | UCL Cases Model |
|------------|-----------------|-----------------------|-----------------|
| 10/1/2020  | 75,391          | 81,459                | 87,700          |
| 10/2/2020  | 73,813          | 82,392                | 91,317          |
| 10/3/2020  | 72,991          | 82,846                | 93,158          |
| 10/4/2020  | 73,968          | 85,074                | 96,748          |
| 10/5/2020  | 71,455          | 83,447                | 96,117          |
| 10/6/2020  | 67,889          | 80,561                | 94,021          |
| 10/7/2020  | 70,371          | 84,184                | 98,896          |
| 10/8/2020  | 68,902          | 85,039                | 102,404         |
| 10/9/2020  | 67,724          | 85,898                | 105,630         |
| 10/10/2020 | 66,746          | 86,761                | 108,664         |
| 10/11/2020 | 65,915          | 87,628                | 111,557         |
| 10/12/2020 | 65,200          | 88,498                | 114,343         |
| 10/13/2020 | 64,577          | 89,372                | 117,044         |
| 10/14/2020 | 64,030          | 90,250                | 119,676         |
| 10/15/2020 | 63,547          | 91,132                | 122,251         |
| 10/16/2020 | 63,120          | 92,017                | 124,779         |
| 10/17/2020 | 62,741          | 92,907                | 127,267         |
| 10/18/2020 | 62,404          | 93,800                | 129,719         |
| 10/19/2020 | 62,104          | 94,696                | 132,142         |
| 10/20/2020 | 61,838          | 95,597                | 134,539         |
| 10/21/2020 | 61,602          | 96,501                | 136,913         |
| 10/22/2020 | 61,393          | 97,410                | 139,286         |
| 10/23/2020 | 61,210          | 98,322                | 141,605         |
| 10/24/2020 | 61,049          | 99,237                | 143,926         |
| 10/25/2020 | 60,910          | 100,157               | 146,234         |
| 10/26/2020 | 60,791          | 101,080               | 148,592         |
| 10/27/2020 | 60,689          | 102,007               | 150,814         |
| 10/28/2020 | 60,605          | 102,938               | 153,090         |
| 10/29/2020 | 60,536          | 103,872               | 155,357         |
| 10/30/2020 | 60,483          | 104,811               | 157,617         |
| 10/31/2020 | 60,443          | 105,753               | 159,870         |

average of (4791). Then the number of cases increased to reach
(384,697) cases at the end of June with an average of (12,823) cases. We
also observed that during July the cases have increased rapidly and the
confirmed cases reached (1,072,030), which is a much greater number
than the cases during the period (15/03/2020) and until (31/06/2020)
with an average of (34,581) cases, also cases increased rapidly during
August and September, it only reached (4,586,075) with an average
(75,181) cases. Fig. 1 illustrates this. In contrast, Fig. 2 displays the
increase in the number of deaths from 33 cases during the 15 days of
March to (97,492) deaths into (30/09/2020). There are many types of
models for time series that are useful in forecasting such as (ARIMA,
ARMA, AR, MA, ARCH, GARCH). Thus, the nature and tests of the data
for the two series under study and all the hypotheses related to them,
and the stability of the time series were verified to be used in the pre-
diction process. It is noticed that the two series have stabilized after
taking the first differences, as in Figs. 3 and 4. Therefore they can be
used in the prediction process. Statistical Analysis Software (SPSS)
version 23 was used and the expert modeler was used to predict new
daily confirmed cases and deaths at a confidence interval (95%) during
October 2020 as in Table 1 and Figs. 5 and 6.

To test the predictive ability of the model, we estimated a linear
model using the least-squares method by taking the actual values as a
dependent variable and the estimated values as an independent variable.
The closer the estimated parameter from one, the more closely the
estimated values are the actual values. Through the results of Tables 2
and 3, it is clear that the predicted parameter is close to one, and this
indicates the quality of the model in prediction and that there is a
convergence of the predicted values from the actual values and it is
statistically significant because the value of (prob. = 0.000) is less than
the level of significance (α = 0.05).

Basic concepts of fractal fractional calculus

Definition 3.1. [20] If \( u(t) \) is continuous in a closed interval \([a, b] \),
then the fractal integral of \( u \) with order \( \gamma \) is defined as
\[
I^\gamma_a u(t) = \gamma \int_a^t s^{-\gamma} u(s)ds.
\]
Definition 3.2. [20] Let \( u(t) \) be continuous in \((a, b)\) and fractal differentiable on \((a, b)\) with order \( \gamma \). Then the fractal-fractional derivative of \( u \) of order \( \gamma \) in Riemann-Liouville and Caputo sense with the power law kernel are presented by

\[
\frac{\text{FFP}}{D^\gamma_{a^+}} u(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} u'(s)ds, 0 < \gamma < 1, \theta > 0,
\]

and

\[
\frac{\text{FFP}}{D^\gamma_{a^+}} u(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} u''(s)ds, 0 < \gamma < 1, \theta > 0,
\]

respectively, where \( u''(s) = \frac{du(s)}{ds^2} \).

Definition 3.3. [20] Let \( u(t) \) be continuous in \((a, b)\). Then the fractal-fractional integral of \( u \) with order \( \gamma \) in RL sense with power law is presented by

\[
\frac{\text{FFP}}{I^\gamma_{a^+}} u(t) = \frac{\theta}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} u(s)ds.
\]

Definition 3.4. [20] Let \( f \) be continuous on \((0, +\infty)\). Then the fractal-Laplace transform of order \( \gamma \) is defined as

\[
F_{\gamma}\{f(t)\} = \int_0^\infty e^{-\theta t} f(t)dt, \theta > 0.
\]

Lemma 3.1. [20] If \( f \) is continuous on \((a, b)\), then the following fractal FDE

\[
\frac{\text{FFP}}{D^\gamma_{a^+}} u(t) = f(t)
\]

has a unique solution

\[
u(t) = u(0) + \frac{\theta}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)ds.
\]

Theoretical approach

The proposed epidemiological model with assist of fractal-fractional derivative in Caputo sense is introduced in this section. Moreover, we also give a brief analysis of the proposed model including, the positivity of the model solutions, the equilibrium analysis and reproductive number using the next generation matrix, the qualitative analysis of suggested model using some fixed point techniques.

Mathematical model

This subsection consists of the standard mathematical model which introduced Yang and Wang to study the dynamics of COVID-19 epidemic [51]. The authors gave the classical model with ordinary derivative as:

\[
\begin{align*}
\dot{S} &= \omega - \beta_S T - \beta_I I - \beta_V V - \mu S, \\
\dot{I} &= \beta_S T + \beta_I I + \beta_V V - (\alpha + \mu) I, \\
\dot{V} &= \alpha I - (\omega + \delta + \mu) V, \\
\dot{F} &= \psi_1 V + \psi_2 V - \tau F,
\end{align*}
\]

with initial conditions

\[
\begin{align*}
S(0) &= S_0, I(0) = I_0, V(0) = V_0, \\
F(0) &= F_0, \quad T(0) = T_0, \quad R(t) = R(t) = R(t).
\end{align*}
\]

where

- \( S, I, F, R \) and \( \dot{F} \) are the susceptible population, infected population, recovered population, and concentration COVID-19 in the surrounding environment, respectively,
- \( \omega \) is the population influx,
- \( \mu \) is the natural death rate,
- \( \delta \) is the period of quarantine of the infected population,
- \( \dot{F} \) are the exposed and infected population which contributing the coronavirus in the surrounding environment respectively,
- \( \omega \) and \( \dot{F} \) are the disease-induced death rate and removal rate of the COVID-19 virus from surrounding environment respectively,
- \( \beta_S \) is the rate of human to human transmission of disease between \( S \) and \( I \),
- \( \beta_I \) is the rate of human to human transmission of disease between \( I \) and \( I \),
- \( \beta_V \) is the rate of transmission disease due to environmental contact to human individuals,
- \( \beta_S, \beta_I, \beta_V \geq 0 \) and \( (\beta_S), (\beta_I), (\beta_V) \leq 0 \).

Now, we generalize the ordinary model (1) via a new fractal power law-type fractional derivative as follows:

\[
\begin{align*}
\frac{\text{FFP}}{D^\gamma_{t^+}} S(t) &= \omega - \beta_S I(t) - \beta_I I(t) - \beta_V V(t) - \mu S(t), \\
\frac{\text{FFP}}{D^\gamma_{t^+}} I(t) &= \beta_S I(t) + \beta_I I(t) + \beta_V V(t) - (\alpha + \mu) I(t), \\
\frac{\text{FFP}}{D^\gamma_{t^+}} V(t) &= \alpha I(t) - (\omega + \delta + \mu) V(t), \\
\frac{\text{FFP}}{D^\gamma_{t^+}} F(t) &= \psi_1 V(t) + \psi_2 V(t) - \tau F(t)
\end{align*}
\]

where \( \frac{\text{FFP}}{D^\gamma_{t^+}} \) denotes the fractal-power law fractional derivative of order \( 0 < \gamma < 1 \) and the fractal dimension \( \theta > 0 \).

Model (3) is a complete mathematical model involving fractal-fractional derivative in Caputo sense with power law for the transmission of COVID-19 epidemic outbreaks in India, which can depict the real-world problem to an ideal degree of exactness, offering important forecasts. Such a forecast can imply assessing the forthcoming anticipated circumstances, making one espouse controlling measures a long time before time so as to maintain a strategic distance from the direst outcome imaginable. In the first part of the current work, we give an objective approach to forecasting the continuation of the COVID-19 using Modeler Expert and SPSS version 23 based on ample historical data.

Well-Poseness and stability analysis

Positiveness of the solutions

For a positivity of the solution, we consider \( \mathbb{R}_+^5 = \{(S, I, F, R, \dot{F}) \in \mathbb{R}_+^5 : (S, I, F, R, \dot{F}) \geq 0\} \).

Theorem 4.1. The solutions \((S, I, F, R, \dot{F})\) of the given models (1)–(3) are positive and belongs to \( \mathbb{R}_+^5 \).

Proof 4.1. Because that all the proposed model parameters are positive. If \( \dot{S}(0), \dot{I}(0), \dot{F}(0), \dot{R}(0), \dot{F}(0) \) are positive, we show that \( \dot{S}(t), \dot{I}(t), \dot{F}(t), \dot{R}(t) \) are positive. Indeed,

\[
\begin{align*}
\dot{S}(t) &= \omega - \beta_S I(t) - \beta_I I(t) - \beta_V V(t) - \mu S(t) \\
&= (\beta_S I(t) + \beta_I I(t) + \beta_V V(t)) - (\alpha + \mu) I(t) \\
&= (\beta_S I(t) + \beta_I I(t) + \beta_V V(t))V(t) - (\alpha + \mu) I(t) \\
&= (\beta_S I(t) + \beta_I I(t) + \beta_V V(t))I(t) - (\alpha + \mu) I(t).
\end{align*}
\]

Thus \( \dot{S}(t) \geq 0 \) \( \exp\{-(\alpha + \mu) t\} \geq 0, \forall t \geq 0 \).

\[
\dot{I}(t) = a I(t) - (\omega + \delta + \mu) I(t) \geq 0, \forall t \geq 0.
\]

It follows that \( S(t) \geq 0, I(t) \geq 0, F(t) \geq 0, R(t) \geq 0, \forall t \geq 0 \).
which implies \( \mathcal{L}(t) \geq \mathcal{L}(t) - \mu \mathcal{L}(t) \).

Thus, by following the same arguments above along with the Laplace transform we obtain

\[
\mathcal{L}(t) = \mathcal{L}(t) - \mu \mathcal{L}(t) = \mathcal{L}(t) - \tau \mathcal{L}(t).
\]

To prove that \( \mathcal{L}(t) \) is positive, we define the norm \( \|f\|_\infty = \sup_{t \in [a, b]} |f(t)| \) for each \( f \in C[a, b] \).

\[
\mathcal{L}(t) \geq \exp(-\tau t) \geq 0, \forall t \geq 0.
\]

The nonlocal operator, we only show the positiveness for Caputo operator, because, we can describe the model (3) in terms of Caputo operator after replacing \( R_d \) with \( C_d \) as follows:

\[
\begin{align*}
C_d \mathcal{L}(t) &= \theta^{\delta-1} \left( -\sigma + \beta \mathcal{S}(t) \mathcal{I}(t) - \beta \mathcal{S}(t) \mathcal{I}(t) - \mu \mathcal{L}(t) \right), \\
C_d \mathcal{C}(t) &= \theta^{\mu-1} \left( -\sigma + \beta \mathcal{S}(t) \mathcal{I}(t) + \beta \mathcal{S}(t) \mathcal{I}(t) - (\alpha + \mu) \mathcal{L}(t) \right), \\
C_d \mathcal{C}(t) &= \theta^{\mu-1} (\delta \mathcal{S}(t) - \mu \mathcal{L}(t)), \\
C_d \mathcal{C}(t) &= \theta^{\mu-1} (\delta \mathcal{S}(t) - \mu \mathcal{L}(t)) = \theta^{\mu-1} \left( \mathcal{S}(t) + \mathcal{I}(t) - \tau \mathcal{L}(t) \right).
\end{align*}
\]

Thus, by following the same arguments above along with the Laplace transform we obtain

\[
\mathcal{L}(t) - \mathcal{L}(t) \exp(-\tau t) \geq 0, \forall t \geq 0.
\]

Equilibrium analysis

The disease equilibrium cases of the proposed model (3) are obtained by solving the following system of equations:

\[
\begin{align*}
\sigma - \beta \mathcal{S}(0) \mathcal{I}(0) - \beta \mathcal{S}(0) \mathcal{I}(0) - \mu \mathcal{L}(0) &= 0, \\
\beta \mathcal{S}(0) \mathcal{I}(0) + \beta \mathcal{S}(0) \mathcal{I}(0) &= \mathcal{L}(0), \\
(\alpha + \mu) \mathcal{L}(0) &= \delta \mathcal{S}(0) - \mu \mathcal{L}(0).
\end{align*}
\]

Hence, the disease free equilibrium case of the fractal-fractional model (3) can be written as

\[
\mathcal{E}_0 = (\sigma(0), \mathcal{S}(0), \mathcal{I}(0), \mathcal{L}(0)) = (0, 0, 0, 0).
\]

By using the basic reproduction number \( R_0 \),

The infection elements in the model are \( \mathcal{S}, \mathcal{I} \) and \( \mathcal{L} \). The matrices \( \mathcal{S} \) and \( \mathcal{V} \) represent the infection and transition matrices and can be defined by

\[
\mathcal{S} = \begin{pmatrix}
\beta \mathcal{S}(0) & \beta \mathcal{S}(0) & \beta \mathcal{S}(0)
\end{pmatrix},
\]

\[
\mathcal{V} = \begin{pmatrix}
\frac{1}{\alpha + \mu} & 0 & 0 \\
\frac{\alpha}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} & \frac{1}{\delta + \mu + \omega} & 0 \\
\frac{\omega(\delta + \mu + \omega)}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} & \frac{\tau(\delta + \mu + \omega)}{\tau(\delta + \mu + \omega)} & 1
\end{pmatrix},
\]

\[\frac{\alpha + \mu}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} = \frac{1}{\delta + \mu + \omega} = \frac{\tau(\delta + \mu + \omega)}{\tau(\delta + \mu + \omega)}.
\]

Hence, by following the same arguments above along with the Laplace transform we obtain

\[
\mathcal{L}(t) \geq \begin{pmatrix} \mathcal{L}(t) & \mathcal{L}(t) & \mathcal{L}(t) \end{pmatrix} - \begin{pmatrix} \beta \mathcal{S}(t) \mathcal{I}(t) & \beta \mathcal{S}(t) \mathcal{I}(t) & \beta \mathcal{S}(t) \mathcal{I}(t) \end{pmatrix} + \begin{pmatrix} \alpha + \mu & \beta \mathcal{S}(t) \mathcal{I}(t) & \beta \mathcal{S}(t) \mathcal{I}(t) \end{pmatrix} - \begin{pmatrix} \mathcal{L}(t) & \mathcal{L}(t) & \mathcal{L}(t) \end{pmatrix} - \begin{pmatrix} \beta \mathcal{S}(t) \mathcal{I}(t) & \beta \mathcal{S}(t) \mathcal{I}(t) & \beta \mathcal{S}(t) \mathcal{I}(t) \end{pmatrix} - \begin{pmatrix} \mathcal{L}(t) & \mathcal{L}(t) & \mathcal{L}(t) \end{pmatrix}.
\]

Equilibrium analysis

The disease equilibrium cases of the proposed model (3) are obtained from solving the following system of equations:

\[
\begin{align*}
\mathcal{S}(t) &= \beta \mathcal{S}(t) \mathcal{I}(t), \\
\mathcal{I}(t) &= \beta \mathcal{S}(t) \mathcal{I}(t), \\
\mathcal{L}(t) &= \beta \mathcal{S}(t) \mathcal{I}(t).
\end{align*}
\]

The basic reproduction number \( R_0 \) of fractal-fractional model (3) is given as in the term of the spectral radius of matrix \( \mathcal{S} \mathcal{V}^{-1} \) [53], that is

\[
R_0 = \rho(\mathcal{S} \mathcal{V}^{-1}) = \frac{\beta \mathcal{S}(0) \mathcal{I}(0) \alpha}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} \quad \frac{\beta \mathcal{S}(0) \mathcal{I}(0) \psi_1(\delta + \mu + \omega)}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)}
\]

which supplies a quantification of the disease danger. The two expressions \( \frac{\beta \mathcal{S}(0) \mathcal{I}(0) \alpha}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} \) and \( \frac{\beta \mathcal{S}(0) \mathcal{I}(0) \psi_1(\delta + \mu + \omega)}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} \) represent the contributions transmission of disease from the human to human, i.e., from \( \mathcal{S} \) to \( \mathcal{I} \) and \( \mathcal{I} \) to \( \mathcal{L} \), respectively, and the expression \( \frac{\beta \mathcal{S}(0) \mathcal{I}(0) \psi_1(\delta + \mu + \omega)}{\mu^2 + \alpha(\delta + \mu + \omega) + \mu(\omega + \delta)} \) represents the contributions transmission of disease from the environment to human, i.e., from \( \mathcal{E} \) to \( \mathcal{I} \). These three transmission modes collectively incarnation generally infection danger for the COVID-19 spread. 
In the model (4), the second equilibrium point is an endemic equilibrium which is given as follows
\[ \mathcal{X}^* = \frac{\sigma - (\alpha + \mu) S}{\mu}, \quad \mathcal{Y}^* = \left( \delta + \mu + \omega \right) \mathcal{X}^*, \quad \mathcal{Z}^* = \frac{\alpha \mathcal{X}^*}{\tau} \]
(5)

According to the first two equations of (4), the function \( \mathcal{J} \) can be expressed by a function of \( \mathcal{X} \), i.e.,
\[ \mathcal{J} = \Psi(\mathcal{X}) := \frac{\sigma - (\alpha + \mu) \mathcal{X}}{\mu}. \]

In light of the second equation of (4), and equation (5), we obtain
\[ \mathcal{J} = \Phi(\mathcal{X}) := \left( \alpha + \mu \right) \left( \mathcal{X} \right) + \frac{\alpha}{\delta + \mu + \omega} \mathcal{X}^*(\mathcal{X}^*) \left( \mathcal{X} \right) \left( \mathcal{X} \right) \left( \mathcal{X} \right), \]
and
\[ \Phi(0) = \left( \alpha + \mu \right) \left( \mathcal{X}^* \right) + \frac{\alpha}{\delta + \mu + \omega} \mathcal{X}^*(\mathcal{X}^*) \left( \mathcal{X} \right), \]
\[ \mathcal{J}(0) = \frac{\mathcal{X}(0)}{R_0}. \]

Hence, we have two cases:

(i) If \( R_0 > 1 \), then the two curves \( \Psi \) and \( \Phi \) have a unique intersection located in the inner part of \( \mathbb{R}^2_+ \), since \( \Psi(\mathcal{X}) > 0 \) and \( \Phi(\mathcal{X}) < 0 \) and \( \Phi(\mathcal{X}) > 0 \). Furthermore, at this intersection point, the Eq. (5) gives a unique endemic equilibrium \( \mathcal{X}^* = (\mathcal{X}^*, \mathcal{Y}^*, \mathcal{Z}^*) = (\mathcal{X}^*, \mathcal{Y}^*, \mathcal{Z}^*) \neq (0, 0, 0). \)

(ii) If \( R_0 < 1 \), then the two curves \( \Psi \) and \( \Phi \) have no unique intersection located in the inner part of \( \mathbb{R}^2_+ \), since \( \Psi(\mathcal{X}) > 0 \) and \( \Phi(\mathcal{X}) < 0 \). Thus, by model (4) we infer that the fractal-fractional model (3) has a unique equilibrium \( \mathcal{X}^* \) if \( R_0 > 1 \) and it has two equilibria \( \mathcal{X}_0 \) and \( \mathcal{X}^* \), if \( R_0 > 1 \).

Qualitative analysis of fractal-fractional model

In this subsection, we investigate the existence and uniqueness of solution and its stability in the concept of Ulam-Hyers for the model (3) by using some fixed point technique. Now, we rewrite the model (3) as following form:
\[ \begin{align*}
\mathcal{F} \left[ \mathcal{X} \right] &= \mathcal{F}_1(\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F} \left[ \mathcal{X} \right] &= \mathcal{F}_2(\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F} \left[ \mathcal{X} \right] &= \mathcal{F}_3(\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F} \left[ \mathcal{X} \right] &= \mathcal{F}_4(\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F} \left[ \mathcal{X} \right] &= \mathcal{F}_5(\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}),
\end{align*} \]
(6)

where
\[ \begin{align*}
\mathcal{F}_1(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}) &= \mathcal{F}_1(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F}_2(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}) &= \mathcal{F}_2(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F}_3(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}) &= \mathcal{F}_3(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F}_4(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}) &= \mathcal{F}_4(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}), \\
\mathcal{F}_5(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}) &= \mathcal{F}_5(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}),
\end{align*} \]
(7)

In view of Lemma 3.1, the fractal-fractional problem (8) is equivalent to the following fractal fractional integral equation
\[ \mathcal{F}(t) = \mathcal{F}_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t \alpha^\gamma (t - \sigma)^{\gamma - 1} \mathcal{F}(\sigma, \mathcal{X}(\sigma)) d\sigma. \]
(10)

Theorem 4.2. The kernels of model \( \mathcal{F}(t, \mathcal{X}(t)) \) satisfy the Lipschitz condition and contraction if the following inequality holds:
\[ 0 < L < 1, \text{ for } t = 1, 2, 3, 4, 5. \]

Proof 4.2. For \( \mathcal{F}_1(t, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}) \), let \( \gamma \) and \( \gamma \) be two functions in \( C([0, T], \mathbb{R}^+) \) and \( t \in [0, T] \). Then we have
where \( L_1 = \beta_2 \mathcal{A}_1 + \beta_3 \mathcal{A}_2 + \beta_4 \mathcal{A}_3 + \mu \) and \( \| \mathcal{A}_1 \|, \| \mathcal{A}_2 \|, \| \mathcal{A}_3 \| \) are bounded functions by \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \) respectively. Obviously, the Lipschitz condition is verified for \( \mathcal{F}_1 \). Moreover, \( \mathcal{F}_1 \) leads to a contraction due to \( 0 < L_1 < 1 \). Likewise, we can show that \( \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4 \) and \( \mathcal{F}_5 \) admit the contraction and Lipschitz condition, i.e.,

\[
\| \mathcal{F}_2(t, x, \gamma) - \mathcal{F}_2(x, \gamma) \| = \| L_2 \| \| x - \gamma \|,
\]

\[
\| \mathcal{F}_3(t, x, \gamma) - \mathcal{F}_3(x, \gamma) \| = \| L_3 \| \| x - \gamma \|,
\]

\[
\| \mathcal{F}_4(t, x, \gamma) - \mathcal{F}_4(x, \gamma) \| = \| L_4 \| \| x - \gamma \|,
\]

\[
\| \mathcal{F}_5(t, x, \gamma) - \mathcal{F}_5(x, \gamma) \| = \| L_5 \| \| x - \gamma \|.
\]

where \( L_2 = \beta_3 \mathcal{A}_2 + \alpha + \mu, L_3 = \omega + \delta + \mu, L_4 = \mu, L_5 = r \) and \( \| \mathcal{F}_i \| \) is function bounded by \( \mathcal{A}_i \).}

**Lemma 4.1.** \([20]\) The function \( \mathcal{F} \in C([0, T], \mathbb{R}^+) \) is a solution of the fractal-fractional problem (8) if and only if, it is a solution of the fractal-fractional integral Eq. (10).

**Theorem 4.3.** (Existence) Let \( r, \theta \in (0, 1), x_0 \in \mathbb{R}, \kappa > 0 \) and \( T^* > 0 \). Define the set

\[
D := \{(t, x) : (x - x_0)^2 < \kappa \}
\]

Let the function \( \mathcal{F} : D \rightarrow \mathbb{R} \) be continuous with \( \sup_{(t, x) \in D} |\mathcal{F}(t, x)| = \lambda \) and

\[
T^* = \left\{ t \in \mathbb{R} \mid \min \left\{ \tau \in \left[ -\frac{x_0^2}{\kappa}, 0 \right] : \frac{x_0^2}{\kappa} - \tau \leq 0 \right\} = \lambda \right\}
\]

Then, there exists a function \( \mathcal{F} \in C([0, T], \mathbb{R}^+) \) that solves the problem (9).

**Proof 4.3.** If \( \lambda = 0 \) then \( \mathcal{F} \circ (x, \gamma) = 0 \) for all \((t, x) \in D \). So, it is obvious that \( \mathcal{F} : [0, T] \rightarrow \mathbb{R}^+ \) with \( \mathcal{F}(t) = 0 \) is a solution of the problem (9). Hence a solution exists.

If \( \lambda \neq 0 \), we apply Lemma 4.1 and show that problem (8) is equivalent to Eq. (10). Let \( \mathcal{B}_x = \{ z \in C([0, T], \mathbb{R}^+) : \| z - x_0 \|_\kappa < \kappa \} \) is a closed and convex subset of \( C([0, T], \mathbb{R}^+) \). Consider the operator \( Q : C([0, T], \mathbb{R}^+) \rightarrow C([0, T], \mathbb{R}^+) \) defined by

\[
(Qz)(t) = \mathcal{F}_0 + \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} |\mathcal{F}(x, \gamma)| dx.
\]

From (10) and (12), we can write \( \mathcal{F} = Q \mathcal{F} \). Now, we need to prove that \( Q \) has a fixed point. This is achieved by the fixed point theorem of Schauder. For order and clarity, we list the proof in several steps:

**Step 1:** \( Q \mathcal{F} \in \mathcal{B}_x \), for \( \mathcal{F} \in \mathcal{B}_x \).

For \( 0 \leq t \leq T \), we have

\[
|||Q\mathcal{F}||| = \mathcal{F}_0 + \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} |\mathcal{F}(x, \gamma)| dx
\]

where

\[
\mathcal{F}_1 = \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} |(t_1-s)^{\gamma-1} - (t_2-s)^{\gamma-1}| ds,
\]

\[
\mathcal{F}_2 = \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} ds.
\]

Obvious that \( \mathcal{F}_2 = \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} (t_2 - t_1)^{\gamma-1} \). To estimate \( \mathcal{F}_1 \), consider two cases \( (\gamma = 1, \gamma < 1) \):

When \( \gamma = 1 \), then \( \mathcal{F}_1 = 0 \), whereas if \( \gamma < 1 \), we have \( (t_1 - \sigma)^{\gamma-1} > (t_2 - \sigma)^{\gamma-1} \). Hence

\[
\mathcal{F}_1 = \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} [(t_1-s)^{\gamma-1} - (t_2-s)^{\gamma-1}] ds
\]

\[
\mathcal{F}_2 = \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} ds.
\]

Thus, \( \mathcal{F} \in \mathcal{B}_x \) for each \( x \in \mathcal{B}_x \).

**Step 2:** The family \( Q\mathcal{B}_x = \{ Qz : z \in \mathcal{B}_x \} \) is relatively compact.

First, we show that \( Q\mathcal{B}_x \) is uniformly bounded. Let \( z \in Q\mathcal{B}_x \). Then for each \( t \in [0, T] \),

\[
|z(t)| = |Q(z)(t)|
\]

\[
\leq \|Q(z)\| + \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} |\mathcal{F}(x, \gamma)| dx
\]

\[
\leq \|Q(z)\| + \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} \int_0^t (t-s)^{\gamma-1} |\mathcal{F}(x, \gamma)| dx
\]

which proves the boundedness property. Next, to prove that \( Q\mathcal{B}_x \) is equicontinuous, we can easily deduce this from (13) above. In fact for \( z \in \mathcal{B}_x \) and \( 0 \leq t_1 \leq t_2 \leq T \), we have proved that

\[
|Q(z)(t_1) - Q(z)(t_2)| \leq \frac{\Gamma(\gamma)}{(\Gamma(\gamma))^2} (t_2 - t_1)^{\gamma-1} \rightarrow 0, as t_1 \rightarrow t_2.
\]

This shows an equicontinuity property. From the two properties above along with Arzela-Ascoli theorem, we infer that \( Q\mathcal{B}_x \) is relatively compact. Then the fixed point theorem of Schauder emphasizes that \( Q \) has a fixed point. This fixed point is the solution to the problem (8). This finishes the proof. □

**Theorem 4.4.** (Uniqueness) The solution of the fractal-fractional problem
\( \text{(8) is unique, provided that} \)
\[
\|Q\| = \theta \frac{\Gamma(\gamma)}{\Gamma(\gamma)} B(\gamma, \theta) < 1,
\]
where \( B(\cdot, \cdot) \) is a beta function and \( L = \max \{ l_1, l_2, l_3, l_4, l_5 \} \).

**Proof 4.4.** To prove that, we apply the Banach fixed point theorem [54]. Consider the operator \( Q \) defined by \( \text{(12)} \). According to Theorem 4.3, the operator \( Q \) is well-defined. So, we need to prove its contraction.

Let \( X_1, X_2 \in C([0, T], \mathbb{R}^+) \). Then for \( t \in [0, T] \) we have

\[
\|Q X_1(t) - Q X_2(t)\| \leq \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\gamma - 1} (t - \sigma)^{\gamma - 1} \| X_1(\sigma) - X_2(\sigma) \| d\sigma.
\]

which implies

\[
\|Q X_1 - Q X_2\| \leq \frac{\theta}{\Gamma(\gamma)} \int_0^T s^{\gamma - 1} (t - \sigma)^{\gamma - 1} \| X_1(\sigma) - X_2(\sigma) \| d\sigma.
\]

Thus, the operator \( Q \) is a contraction due to \( \text{(14)} \). Hence, the fixed point theorem of Banach shows that the problem \( \text{(8)} \) has a unique solution. \( \square \)

Now, we develop and give some recent results on the Ulam-Hyers stability of the fractal-fractional problem \( \text{(8)} \).

Consider the fractal-fractional problem \( \text{(8)} \), i.e.

\[
\text{FFP}_0^\alpha \mathcal{X}(t) = \mathcal{X}(t, \mathcal{X}(t)), \quad t \in [0, T],
\]

and for \( \epsilon > 0 \) and \( \mathcal{Y} \in C([0, T], \mathbb{R}^+) \). Consider the following inequality.

\[
\|\text{FFP}_0^\alpha \mathcal{Y}(t) \mathcal{X}(t, \mathcal{Y}(t)) - \mathcal{Y}(t, \mathcal{Y}(t))\| \leq \epsilon, \quad t \in [0, T].
\]

**Remark 1.** Let \( \epsilon > 0 \). The function \( \mathcal{Y} \in C([0, T], \mathbb{R}^+) \) satisfies \( \text{(16)} \) if and only if there exist a small perturbation \( \sigma(t) \in C([0, T], \mathbb{R}^+) \) with \( \sigma(0) = 0 \) such that

(i) \( |\sigma(t)| \leq \epsilon \), for \( t \in [0, T] \),
(ii) \( \text{FFP}_0^\alpha \mathcal{Y}(t) = \mathcal{Y}(t, \mathcal{Y}(t)) + \sigma(t) \), \( t \in [0, T] \).

**Lemma 4.2.** The solution of the perturbed problem

\[
\text{FFP}_0^\alpha \mathcal{Y}(t) = \mathcal{Y}(t, \mathcal{Y}(t)) + \sigma(t), \quad t \in [0, T],
\]

\( \mathcal{Y}(0) = \mathcal{Y}_0 > 0 \),

satisfies the following inequality

\[
\|\mathcal{Y}(t) - \mathcal{Y}(0) - \text{FFP}_0^\alpha \mathcal{Y}(t, \mathcal{Y}(t))\| \leq \Theta \epsilon,
\]

where \( \Theta := \frac{\theta}{\Gamma(\gamma)} B(\gamma, \theta) \).

**Proof 4.5.** By Lemma 3.1, the solution of perturbed problem \( \text{FFP}_0^\alpha \mathcal{X}(t, \mathcal{Y}(t)) + \sigma(t) \)

\[
= \mathcal{Y}(0) + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\gamma - 1} (t - s)^{\gamma - 1} \| \mathcal{X}(s, \mathcal{Y}(s)) + \sigma(s) \| ds.
\]

It follows from Remark 1 that

\[
\|\mathcal{Y}(t) - \mathcal{Y}(0) - \text{FFP}_0^\alpha \mathcal{Y}(t, \mathcal{Y}(t))\| \leq \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\gamma - 1} (t - s)^{\gamma - 1} \| \mathcal{X}(s, \mathcal{Y}(s)) + \sigma(s) \| ds.
\]

\[
= \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\gamma - 1} (t - s)^{\gamma - 1} \| \mathcal{Y}(s, \mathcal{Y}(s)) + \sigma(s) \| ds.
\]

\[
\leq \Theta \epsilon,
\]

\( \square \)

**Theorem 4.5.** (Ulam-Hyers Stability) Assume that hypotheses of Theorem 4.4 and \( \text{(16)} \) are satisfied. Then the fractal-fractional problem \( \text{(8)} \) is Ulam-Hyers stable.

**Proof 4.6.** Let \( \epsilon > 0 \) and \( \mathcal{Y} \in C([0, T], \mathbb{R}^+) \) be a function satisfies \( \text{(16)} \), and let \( \mathcal{X} \in C([0, T], \mathbb{R}^+) \) be a unique solution of

\[
\begin{cases}
\text{FFP}_0^\alpha \mathcal{X}(t) = X(t, \mathcal{X}(t)), \\
\mathcal{X}(0) = Y(0) > 0.
\end{cases}
\]

Lemma 3.1 gives

\[
\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \text{FFP}_0^\alpha \mathcal{X}(s, \mathcal{X}(s)) ds.
\]

Since \( \mathcal{X}(0) = Y(0) \), we get

\[
\mathcal{X}(t) = \mathcal{Y}(0) + \int_0^t \text{FFP}_0^\alpha \mathcal{X}(s, \mathcal{X}(s)) ds,
\]

It follows from Lemma 4.2 that

\[
\mathcal{X}(t) = \mathcal{Y}(0) + \int_0^t \text{FFP}_0^\alpha \mathcal{X}(s, \mathcal{Y}(s)) ds.
\]
$$\| \mathcal{Y} - \mathcal{Z} \|_{r} = \max \left( \frac{\mathcal{Y}(t) - \mathcal{Z}(t)}{\mathcal{Z}(t)} \right) = \max \left( \frac{\mathcal{Y}(0) - \mathcal{Y}(t)}{\mathcal{Y}(t)} \right)$$

$$\leq \max \left( \frac{\mathcal{Y}(t) - \mathcal{Y}(0) - \mathcal{Z}(t)}{\mathcal{Z}(t)} \right) + \max \left( \frac{\mathcal{Z}(t) - \mathcal{Z}(0)}{\mathcal{Z}(t)} \right)$$

$$\leq \Theta + \frac{\Theta}{T} \sum_{n=0}^{\infty} \left( \mathcal{Z}_n(t) - \mathcal{Z}_n(0) \right)$$

$$\leq \Theta + \frac{\Theta}{T} \sum_{n=0}^{\infty} \left( \mathcal{Z}_n(t) - \mathcal{Z}_n(0) \right)$$

which implies

$$\| \mathcal{Y} - \mathcal{Z} \|_{r} \leq C_{\mathcal{Y}} \varepsilon,$$

where

$$C_{\mathcal{Y}} = \frac{\Theta}{T}.$$

**Numerical solution of fractal-fractional model (3)**

In this part, we apply the fractional Adams Bashforth method [56,55] to obtain the numerical simulations and numerical solutions of the fractal-fractional model (3). First, from the definition of $\mathcal{FIP}_{\alpha}^{\gamma} \mathcal{Y}(t)$, we have for $r \in (0, 1)$ and $\theta > 0$

$$\mathcal{FIP}_{\alpha}^{\gamma} \mathcal{Y}(t) = \frac{1}{\Gamma(1 - r) \theta^{\alpha r}} \int_{0}^{t} (t - s)^{-r} f(s) \, ds$$

$$\Rightarrow \theta = \frac{1}{\Gamma(1 - r) \theta^{\alpha r}} \int_{0}^{t} (t - s)^{-r} f(s) \, ds$$

$$\Rightarrow \theta^{\alpha r} \mathcal{FIP}_{\alpha}^{\gamma} \mathcal{Y}(t) = \frac{1}{\Gamma(1 - r) \theta^{\alpha r}} \int_{0}^{t} (t - s)^{-r} f(s) \, ds$$

By above procedure with the assumption (7), we can express the model (6) as follows

$$\begin{align*}
\mathcal{FIP}_{\alpha}^{\gamma} \mathcal{Y}(t) &= \frac{1}{\Gamma(1 - r) \theta^{\alpha r}} \int_{0}^{t} (t - s)^{-r} f(s) \, ds \\
&= \frac{1}{\Gamma(1 - r) \theta^{\alpha r}} \int_{0}^{t} (t - s)^{-r} f(s) \, ds \\
&= \frac{1}{\Gamma(1 - r) \theta^{\alpha r}} \int_{0}^{t} (t - s)^{-r} f(s) \, ds
\end{align*}$$

By replacing $\mathcal{FIP}_{\alpha}^{\gamma} \mathcal{Y}(t)$ with $\mathcal{FIP}_{\alpha}^{\gamma} \mathcal{Y}(t)$ in order to take advantage of the initial conditions of the integer-order, then we apply the fractional Riemann-Liouville integral to both sides to get the following.

$$\begin{align*}
\mathcal{Y}(t) - \mathcal{Y}(0) &= \int_{0}^{t} \theta^{\alpha r} (t - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s) \, ds, \\
\mathcal{Z}(t) - \mathcal{Z}(0) &= \int_{0}^{t} \theta^{\alpha r} (t - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Z}(s) \, ds, \\
\mathcal{Y}(t) - \mathcal{Z}(0) &= \int_{0}^{t} \theta^{\alpha r} (t - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s) \, ds, \\
\mathcal{Z}(t) - \mathcal{Z}(0) &= \int_{0}^{t} \theta^{\alpha r} (t - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Z}(s) \, ds, \\
\mathcal{Y}(t) - \mathcal{Y}(0) &= \int_{0}^{t} \theta^{\alpha r} (t - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s) \, ds, \\
\mathcal{Z}(t) - \mathcal{Z}(0) &= \int_{0}^{t} \theta^{\alpha r} (t - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Z}(s) \, ds.
\end{align*}$$

Now, we give the numerical schemes of this model using a new approach [37] at $t_{n+1}$. The model becomes

$$\begin{align*}
\mathcal{Y}(t_{n+1}) &= \mathcal{Y}_0 + \frac{\theta}{\Gamma(1 - r)} \int_{0}^{t_{n+1}} \theta^{\alpha r} (t_{n+1} - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s) \, ds, \\
\mathcal{Z}(t_{n+1}) &= \mathcal{Z}_0 + \frac{\theta}{\Gamma(1 - r)} \int_{0}^{t_{n+1}} \theta^{\alpha r} (t_{n+1} - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Z}(s) \, ds
\end{align*}$$

Then we approximate the above acquired integrals to

$$\begin{align*}
\mathcal{Y}(t_{n+1}) &= \mathcal{Y}_0 + \frac{\theta}{\Gamma(1 - r)} \sum_{i=0}^{n} \theta^{\alpha r} (t_{n+1} - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s) \, ds, \\
\mathcal{Z}(t_{n+1}) &= \mathcal{Z}_0 + \frac{\theta}{\Gamma(1 - r)} \sum_{i=0}^{n} \theta^{\alpha r} (t_{n+1} - s)^{-1} \mathcal{I}_{\alpha}^{\gamma} \mathcal{Z}(s) \, ds
\end{align*}$$

The function $\mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s)$ can be approximated over $[t_{i-1}, t_{i+1}]$ utilizing the Lagrange interpolation polynomials with $h = t_{i+1} - t_{i-1}$ as

$$\mathcal{I}_{\alpha}^{\gamma} \mathcal{Y}(s) \approx \frac{t_{i+1} - s}{h} \mathcal{Y}_{l}(s).$$

for $i = 1, 2, 3, 4, 5$. Plugging (21) in (20), we may write (20) as
\[
\mathcal{J}(t_{n+1}) = \mathcal{J}_0 + \frac{\theta}{\Gamma(q)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{q-1} \mathcal{J}_k(\sigma) \, d\sigma,
\]
\[
\mathcal{E}(t_{n+1}) = \mathcal{E}_0 + \frac{\theta}{\Gamma(q)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{q-1} \mathcal{J}_k(\sigma) \, d\sigma,
\]
\[
\mathcal{J}(t_{n+1}) = \mathcal{J}_0 + \frac{\theta}{\Gamma(q)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{q-1} \mathcal{J}_k(\sigma) \, d\sigma,
\]
\[
\mathcal{R}(t_{n+1}) = \mathcal{R}_0 + \frac{\theta}{\Gamma(q)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{q-1} \mathcal{J}_k(\sigma) \, d\sigma,
\]
\[
\mathcal{F}(t_{n+1}) = \mathcal{F}_0 + \frac{\theta}{\Gamma(q)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{q-1} \mathcal{J}_k(\sigma) \, d\sigma.
\]

From (21) and (22), we have
\[
\begin{aligned}
\int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{-1} (t - t_k) \, d\sigma &= \frac{1}{\Gamma(q+1)}
\left[
\frac{(t_{k+1}^n - t_k^n)(t_{k+1}^{n+1} - t_k^{n+1})}{(t_{k+1} - t_k)^{q+1}}
\right]
\end{aligned}
\]

Taking \( t_k = kh \), we can deduce that
\[
\begin{aligned}
\int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{-1} (t - t_k) \, d\sigma &= \frac{p_{k+1}^{n+1}}{\Gamma(q+1)}
\left[
\frac{(n+1-k)^q(n-k+2+\gamma)}{(n-k+2+2\gamma)}
\right],
\end{aligned}
\]

\[
\begin{aligned}
\int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{-1} (t - t_k) \, d\sigma &= \frac{p_{k+1}^{n+1}}{\Gamma(q+1)}
\left[
\frac{(n+1-k)^q(n-k+2+\gamma)}{(n-k+2+2\gamma)}
\right].
\end{aligned}
\]

Using (23) and (24) into above equations, we acquire the numerical schemes for the fractal-fractional model (3) of COVID-19 epidemic as follows

\[
\int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{-1} (t - t_k) \, d\sigma
\]

and

\[
\begin{aligned}
\int_{t_k}^{t_{k+1}} (t_{k+1} - \sigma)^{-1} (t - t_k) \, d\sigma &= \frac{p_{k+1}^{n+1}}{\Gamma(q+1)}
\left[
\frac{(n+1-k)^q(n-k+2+\gamma)}{(n-k+2+2\gamma)}
\right].
\end{aligned}
\]
Fig. 7. Graphical presentation of approximate solution of susceptible class corresponding to different fractal-fractional order.

Fig. 8. Graphical presentation of approximate solution of exposed class corresponding to different fractal-fractional order.

Fig. 9. Graphical presentation of approximate solution of infected class corresponding to different fractal-fractional order.
Fig. 10. Graphical presentation of approximate solution of recovered class corresponding to different fractal-fractional order.

\[ S \approx 1 - \sum_{k=0}^{n-1} \left( \frac{\theta \Gamma}{\Gamma(y+2)} \right)^k \frac{\Gamma(y+2)}{\Gamma(y+1)} \]

\[ (n+1-k)^y (n-k+2+2 \gamma) \]

\[ -\theta \Gamma \frac{k^2}{\Gamma(y+2)} \frac{\Gamma(y+3)}{\Gamma(y+1)} \frac{\Gamma(2+y)}{\Gamma(y+1)} \]

\[ (n+1-k)^{y+1} - (n-k)^{y} (n-k+1+\gamma) \]

\[ \mathcal{J}(t_{n+1}) = \mathcal{J}_0 + \theta \frac{\Gamma}{\Gamma(y+1)} \sum_{k=0}^{n-1} (\frac{\theta \Gamma}{\Gamma(y+2)})^k \frac{\Gamma(y+2)}{\Gamma(y+1)} \frac{\Gamma(y)}{\Gamma(y+1)} \]

\[ (n+1-k)^y (n-k+2+2 \gamma) \]

\[ -\theta \Gamma \frac{k^2}{\Gamma(y+2)} \frac{\Gamma(y+3)}{\Gamma(y+1)} \frac{\Gamma(2+y)}{\Gamma(y+1)} \]

\[ (n+1-k)^{y+1} - (n-k)^{y} (n-k+1+\gamma) \]

Fig. 11. Graphical presentation of approximate solution of concentration of coronaries in surrounding corresponding to different fractal-fractional order.

\[ \mathcal{J}(t_{n+1}) = \mathcal{J}_0 + \theta \frac{\Gamma}{\Gamma(y+1)} \sum_{k=0}^{n-1} (\frac{\theta \Gamma}{\Gamma(y+2)})^k \frac{\Gamma(y+2)}{\Gamma(y+1)} \frac{\Gamma(y)}{\Gamma(y+1)} \]

\[ (n+1-k)^y (n-k+2+2 \gamma) \]

\[ -\theta \Gamma \frac{k^2}{\Gamma(y+2)} \frac{\Gamma(y+3)}{\Gamma(y+1)} \frac{\Gamma(2+y)}{\Gamma(y+1)} \]

\[ (n+1-k)^{y+1} - (n-k)^{y} (n-k+1+\gamma) \]
\begin{align*}
 F'(t_{n+1}) &= F' + \frac{\theta \Gamma}{\Gamma(\gamma+2)} \sum_{k=0}^{\infty} \left( \beta^{-1} \mathcal{F} \left( s(t_n, \mathcal{N}(t_n), \mathcal{E}(t_n), \mathcal{I}(t_n), \mathcal{R}(t_n), \mathcal{F}(t_n) \right) \right. \\
 & \left. \left( (n+1-k)^{\gamma+2} - (n-k)^{\gamma+2} \right) \right) \\
 - \frac{\mu}{\beta} \mathcal{F} \left( s(t_{n-1}, \mathcal{N}(t_{n-1}), \mathcal{E}(t_{n-1}), \mathcal{I}(t_{n-1}), \mathcal{R}(t_{n-1}), \mathcal{F}(t_{n-1}) \right) \\
 & \left. \left( (n+1-k)^{\gamma+1} - (n-k)^{\gamma+1} \right) \right). \\
 \end{align*}

(29)

**Numerical simulation**

Here, we consider some initial values for different compartments as 
\( \mathcal{S}(0) = 1353 \text{millions} \), \( \mathcal{E}(0) = 0 \text{millions} \), \( \mathcal{R}(0) = 7.06 \text{millions} \),
\( \mathcal{I}(0) = 7.08 \text{millions} \), \( \mathcal{F}(0) = 0 \text{millions} \) (assumed)
and the parameters are
\( \beta = 2.23 \text{ per day} \), \( \gamma = \frac{1}{15} \text{ per day} \), \( \sigma = 1.01 \times 10^{-4} \text{ per day} \), \( \mu = 3.01 \times 10^{-6} \text{ per day} \), \( \frac{1}{\alpha} = \frac{1}{7} \text{ per day} \),
\( \omega_1 = 2.30, \omega_2 = 0.01 \), \( \omega_3 = 0.01 \), \( \tau = 1 \),
\( \beta_E = 3.11 \times 10^{-4} \text{ per day} \), \( \beta_F = 0.62 \times 10^{-4} \text{ per day} \), \( \beta_{SF} = 1.03 \times 10^{-4} \text{ per day} \).

From Figs. 7–11, we see that in coming days the susceptibility is rapidly decreasing with different fractional order. As a result the exposed class will raise up and so the infection will go on increasing. Since the recovery rate is also rapid because the number of death and recovery from disease is also increasing. In coming days the concentration of coronavirus virus will increase in surrounding which will increase the infection.

**Conclusion**

Recently, many lethal diseases have appeared in many places of the world. In this manuscript, we have predicted future trends in the behavior of the global epidemic COVID-19 of real detected cases and deaths in India for October 2020, using the expert modeler model and statistical analysis programs SPSS. We have generalized a mathematical model based on a fractal-fractional Caputo type operator to investigate the existing outbreak of Covid-19 epidemic. This model has been used to describe the diverse transmission passages in the infection dynamics and affirms the role of the environmental reservoir in the transmission and outbreak of this disease. We have presented an itemized analysis of the proposed model including the derivation of equilibrium points endemic and disease-free, reproductive number \( R_0 \), and the positiveness of the model solutions. Moreover, the existence, uniqueness, and Ulam-Hyers stability analysis of the suggested model was also discussed via some important fixed point theorems. The proposed model is solved with the help of fractional Adams Bashforth method in the frame of a new fractal-fractional operator. Also, we have given different graphical results at the various values of the fractional-order \( \eta \) and the fractal dimension \( \theta \). The current study appears the projected scheme is strong and trustworthy in obtaining the best solution to fractal-fractional models of biological and medical systems. The obtained analytical results explore that the fractal-fractional model provides a better fit to the real data compared to the classical model. The statistical results of this work may help the government and other proxies to reconfigure their strategies according to the expected situation.

**Future work**

In mathematical modeling and simulation of biological systems, the important role played by fractional calculus cannot be avoided as it describes memory and genetics and these characteristics are characteristic of epidemiological diseases, and therefore these epidemics are more suitable for modeling in the frame of fractional operators. In the future work, the proposed model in this paper can be modified to fresh available data for various countries under fractal fractional derivatives of ABC type, where the forecasting may be compared for different countries. For prevention, the study recommends to preserve suitable measures like social distancing, repeated testing, awareness-raising campaigns, uses of muzzles and sanitizers, rewards for inventing vaccines, financial support for healthcare facilities by the government, etc.

**CRediT authorship contribution statement**

Mansour A. Abdulwasaa: Conceptualization, Investigation, Resources, Software, Validation, Writing - original draft, Writing - review & editing. Mohammed S. Abd: Investigation, Methodology, Resources, Software, Supervision, Validation, Writing - review & editing, Conceptualization, Formal analysis, Writing - review & editing. Kamal Shah: Conceptualization, Data curation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing - original draft, Writing - review & editing. Taher A. Nofal: Formal analysis, Funding acquisition, Methodology, Writing - review & editing. Satish K. Panchal: Conceptualization, Data curation. Sunil V. Kawale: Conceptualization, Data curation, Methodology, Project administration, Software, Validation, Visualization, Writing - original draft, Writing - review & editing. Abdel-Haleem Abdel-Aty: Data curation, Formal analysis, Funding acquisition, Investigation, Supervision, Writing - review & editing.

**Declaration of Competing Interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgements**

Taif University Researches Supporting Project number (TURSP-2020/031), Taif University, Taif, Saudi Arabia.

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