Lepton mixing patterns from the group $\Sigma(36 \times 3)$ with the generalized CP

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Abstract

The group $\Sigma(36 \times 3)$ with the generalized CP transformation is introduced to predict the mixing pattern of leptons. Various combinations of abelian residual flavor symmetries with CP transformations are surveyed. Six mixing patterns could accommodate the fit data of neutrinos oscillation at $3\sigma$ level. Two patterns predict the nontrivial Dirac CP phase, around ±57° or ±123°, which is in accordance with the result of the literature and the recent fit data.

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I. INTRODUCTION

The oscillation experiments of reactor neutrinos \cite{1-5} have determined the nonzero $\theta_{13}$, which opened the winder to explore unknown mixing parameters including masses ordering of neutrinos, the octant of $\theta_{23}$, and the Dirac CP phase. Among these unknowns, the nontrivial Dirac phase could provide the origin of the CP violation in the lepton sector, which is important for the interpretation of some fundamental questions in particle physics and cosmology such as the asymmetry of matter and antimatter. Although some fit data \cite{6, 7} hint that the Dirac phase may be maximal, i.e. $\sin \delta = -1$, the decisive evidence is still needed. On the theoretical respect, how to predict the CP phase is interesting. Various phenomenological schemes are proposed. One of the most popular approaches is resorting to flavor groups especially the discrete ones, see Refs. \cite{8-14} and reviews \cite{15, 16} for example. Following this approach, a general flavor group $G_f$ is assumed for the Lagrangian of the theory. Because of the nontrivial vacum expectation values of scalars, the original flavor symmetry is broken, namely $G_f$ broken to $G_e$ in the charged lepton sector and $G_\nu$ in the neutrino sector. In the direct method, the mixing matrix is completely determined by the residual flavor symmetries of leptons. However, recent systematical researches \cite{17, 18} reveal that the mixing pattern determined fully by the residual flavor groups usually predicts the trivial Dirac phase. So other strategies should be considered in the construction of the lepton mixing model. As a useful strategy, generalised CP transformations (GCP) \cite{19-38} are introduced to extract information on CP phases of the mixing matrix. Especially, in the so called semidirect method \cite{29, 37}, the initial group is $G_f \rtimes H_{CP}$, where $H_{CP}$ is the group of GCP acting on the flavor space, and the residual symmetry in the neutrino sector becomes $Z_2 \times H_{CP}^\nu$. Because of the degeneracy of the eigenvalues of the representation matrix of $Z_2$ \cite{39-41}, the mixing matrix of neutrinos is not completely determined by the residual symmetry. So there is a real parameter to coordinate the matrix, which releases the tension between the flavor group and the fit data of neutrinos. On the other hand, $H_{CP}^\nu$ could bring nontrivial CP phases in the mixing matrix.

In this article, we add the GCP to the flavor group $\Sigma(36 \times 3)$ to predict the lepton mixing pattern. The mixing pattern from the group $\Sigma(36 \times 3)$ has been studied in Ref. \cite{42}. However, without corrections, it cannot commodate the fit data at $3\sigma$ level in the direct method. We find that when the GCP is considered in the semidirect method, viable mixing angles of leptons and the nontrivial Dirac CP phase could be obtained. Since the group $\Sigma(36 \times 3)$ does not contain the Klein group $K_4$ as a subgroup. The cyclic group $Z_n$ \((n \geq 2)\) and $Z_2$ in the 3-dimensional representation
are considered as the residual flavor symmetry of the charged lepton sector and that of the neutrino sector respectively. We survey various combinations of \((Z_n, Z_2)\) with the GCP, and find that two type of combinations (up to equivalent ones) are viable at 3\(\sigma\) level of the constraints. There are two free parameters in the mixing matrix of leptons. Numerical analysis of the parameters show that the nontrivial Dirac CP phase is around \(\pm 57^\circ\) or \(\pm 123^\circ\) in a type of combinations. This result is in accordance with that obtained in Ref. [41] and the fit data [46]. The parameter space of every viable pattern is tiny. So the predicted ranges of the mixing parameters are narrow. These results are easy to examine by the future neutrinos oscillation experiments.

The outline of the article is as follows. In section II, we summarise the basic facts of the group \(\Sigma(36 \times 3)\) and the approach of employing the GCP. In section III, we survey combinations of residual flavor symmetries with the GCP and give the analytical expressions for mixing angles and the CP invariants of the viable mixing patterns. Numerical results are also shown in this section. Finally, we present a summary.

II. FRAMEWORK

In this section, we recapitulate the basic facts of the group \(\Sigma(36 \times 3)\) and summarize the approach of deriving the lepton mixing pattern on the basis of the residual flavor symmetries with the GCP in the semidirect method. Our derivations are based on the 3-dimensional representation of the flavor group and the GCP. The mass matrix of neutrinos is fixed on the Majorana type.

A. Group theory of \(\Sigma(36 \times 3)\)

The basic facts of the group \(\Sigma(36 \times 3)\) are taken from Refs. [42, 43]. This group has 108 elements. They could be expressed with the generators \(a, b, c\), which satisfy the following relations [42, 43]:

\[
a^3 = c^3 = b^4 = I, \quad ab^{-1}cb = abc^{-1}b^{-1} = (ac)^3 = I, \quad (1)
\]

where \(I\) denotes the identity element. The group has 14 irreducible representations [42, 43]:

\[
1^{(0)}, 1^{(1)}, 1^{(2)}, 1^{(3)}, 3^{(0)}, 3^{(1)}, 3^{(2)}, 3^{(3)}, (3^{(0)})^*, (3^{(1)})^*, (3^{(2)})^*, (3^{(3)})^*, 4, 4'. \quad (2)
\]
In the 3-dimensional representation \(3^{(p)}\), the generators could be expressed as \(^43\)

\[
\rho(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho(b) = \frac{i^p}{\sqrt{3}i} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\]

with \(p = 0, 1, 2, 3\), \(\omega = e^{2\pi i/3}\). Since the generators in the representation \(3^{(p)}\) satisfy the relation

\[
\rho(c^{-1}aca^{-1}) = \omega E,
\]

where \(E\) is the unit matrix, every element of the group in the 3-dimensional representation could be expressed as \(^42, 43\)

\[
\rho(g) = \omega^\alpha \rho(c^\beta \rho(a^\gamma) \rho(b^\lambda)), \quad \text{with } \alpha, \beta, \gamma = 0, 1, 2, \quad \lambda = 0, 1, 2, 3.
\]

Accordingly, the 14 conjugacy classes in the 3-dimensional representation are listed as follows \(^43\):

\[
1C_1^1 = \{ E \}, \quad 1C_2^1 = \{ \omega E \}, \quad 1C_3^3 = \{ \omega^2 E \},
\]

\[
12C_3^4 = \{ \rho(c), \omega \rho(c), \omega^2 \rho(c), \rho(c^2), \omega \rho(c^2), \omega^2 \rho(c^2), \rho(a), \omega \rho(a), \omega^2 \rho(a), \rho(a^2), \omega \rho(a^2), \omega^2 \rho(a^2) \},
\]

\[
12C_5^5 = \{ \rho(ca), \omega \rho(ca), \omega^2 \rho(ca), \rho(c^2a), \omega \rho(c^2a), \omega^2 \rho(c^2a), \rho(2a^2), \omega \rho(2a^2), \omega^2 \rho(2a^2) \},
\]

\[
9C_6^6 = \{ \rho(b^2), \rho(ab^2), \rho(a^2b^2), \rho(c^2b^2), \rho(2b^2), \rho(2^2b^2), \omega \rho(2^2ab^2), \omega^2 \rho(2^2ab^2), \omega^2 \rho(c^2a^2b^2) \},
\]

\[
9C_7^7 = \omega C_6^6, \quad 9C_8^8 = \omega^2 C_6^6,
\]

\[
9C_9^9 = \{ \rho(b), \omega \rho(cb), \omega \rho(c^2b), \omega \rho(ab), \omega \rho(a^2b), \rho(c^2ab), \rho(cb), \omega \rho(cb), \omega \rho(c^2a^2b) \},
\]

\[
9C_{10}^{10} = \omega C_9^9, \quad 9C_{11}^{12} = \omega^2 C_9^9,
\]

\[
9C_{12}^{14} = \{ \rho(b^3), \omega^2 \rho(c^2b^3), \omega^2 \rho(ab^3), \omega^2 \rho(2^2b^3), \omega^2 \rho(c^2ab^3), \omega^2 \rho(c^2a^2b^3) \},
\]

\[
9C_{13}^{12} = \omega C_{12}^{14}, \quad 9C_{14}^{12} = \omega^2 C_{12}^{14},
\]

where the notation \(iC_k^j\) denotes that the \(k\)-th conjugacy class contains \(i\) elements of order \(j\).

On the base of the conjugacy classes, we could obtain the information on the automorphism of the group. The structure of the automorphism group of \(\Sigma(36 \times 3)\) is listed as follows:

\[
Z(\Sigma(36 \times 3)) = Z_3, \quad \text{Aut}(\Sigma(36 \times 3)) \cong (Z_3 \times Z_3 \rtimes Q_8) \rtimes Z_2 \cong \Sigma(36) \rtimes K_4, \\
\text{Inn}(\Sigma(36 \times 3)) \cong (Z_3 \times Z_3) \rtimes Z_4 \cong \Sigma(36), \quad \text{Out}(\Sigma(36 \times 3)) \cong K_4 = \{ \text{id}, u_1, u_2, u_1 u_2 \},
\]

\[(15)\]
where $Z$, Aut, Inn, Out denote the centre, the automorphism group, the inner automorphism and the outer automorphism group of $\Sigma(36 \times 3)$ respectively, $\Sigma(36) \equiv \Sigma(36 \times 3)/Z_3$. An inner automorphism corresponds to the group conjugation which leaves the conjugacy classes of the group invariant. In contrast, an outer automorphism swaps conjugacy classes and representations to keep the character table of the flavor group invariant [23]. In detail, the generator $u_1$ exchanges the conjugacy classes and the representations of the group $\Sigma(36 \times 3)$ as [42]

$$1C_2^3 \leftrightarrow u_1 1C_3^3, \quad 9C_7^6 \leftrightarrow u_1 9C_8^6, \quad 9C_{10}^{12} \leftrightarrow u_1 9C_{11}^{12}, \quad 9C_{13}^{12} \leftrightarrow u_1 9C_{14}^{12},$$

(16)

$$3^{(0)} \leftrightarrow u_1 (3^{(0)})^*, \quad 3^{(2)} \leftrightarrow u_1 (3^{(2)})^*, \quad 3^{(1)} \leftrightarrow u_1 (3^{(1)})^*, \quad 3^3 \leftrightarrow u_1 (3^{(1)})^*. \quad (17)$$

Note that other conjugacy classes and representations are invariant under the action of $u_1$. These transformations could be realised through the mappings as follows [42]

$$a \rightarrow c^2 a, \quad b \rightarrow acb, \quad c \rightarrow ca.$$  

(18)

And the actions of the generator $u_2$ could be expressed as [42]

$$9C_9^4 \leftrightarrow u_2 9C_{12}^4, \quad 9C_{10}^{12} \leftrightarrow u_2 9C_{13}^{12}, \quad 9C_{11}^{12} \leftrightarrow 9C_{14}^{12},$$

(19)

$$1^{(1)} \leftrightarrow u_2 1^{(3)}, \quad 3^{(1)} \leftrightarrow u_2 3^{(3)}, \quad (3^{(1)})^* \leftrightarrow u_2 (3^{(3)})^*. \quad (20)$$

They could be realised through the mappings [42]:

$$a \rightarrow ca^2 c, \quad b \rightarrow c^2 a^2 cb^3, \quad c \rightarrow ca^2.$$  

(21)

The actions of the outer automorphism $u_1u_2$ read

$$1C_2^3 \leftrightarrow u_1u_2 1C_3^3, \quad 9C_7^6 \leftrightarrow u_1u_2 9C_8^6, \quad 9C_9^4 \leftrightarrow u_1u_2 9C_{12}^4, \quad 9C_{10}^{12} \leftrightarrow u_1u_2 9C_{13}^{12}, \quad 9C_{11}^{12} \leftrightarrow u_1u_2 9C_{14}^{12},$$

(22)

$$1^{(1)} \leftrightarrow u_1u_2 1^{(3)} = (1^{(1)})^*, \quad 3^{(0)} \leftrightarrow u_1u_2 (3^{(0)})^*, \quad 3^{(2)} \leftrightarrow u_1u_2 (3^{(2)})^*, \quad 3^{(1)} \leftrightarrow u_1u_2 (3^{(1)})^*, \quad 3^3 \leftrightarrow u_1u_2 (3^{(3)})^*. \quad (23)$$

They could be realised through the mappings:

$$a \rightarrow u_1u_2 a, \quad b \rightarrow u_1u_2 b^{-1}, \quad c \rightarrow u_1u_2 c^{-1}.$$  

(24)

As we can see, $u_1u_2$ interchanges all representations with their complex conjugations [42].
B. Semidirect approach of GCP

1. General GCP compatible with $\Sigma(36 \times 3)$

We consider a theory of leptons which satisfies the symmetry $G_f \rtimes H_{CP}$, where $G_f$ is the flavor group $\Sigma(36 \times 3)$, $H_{CP}$ is the group of GCP. The element of $H_{CP}$ satisfies the consistent condition

$$X(v)^\dagger g X^{-1}(v) = \rho(v(g)), \text{ with } v \in H_{CP}, \ g, \rho(g) \in G_f,$$

(25)

where $X, \rho$ denote the representation of $H_{CP}$ and that of $G_f$ respectively. This condition reveals that the CP transformation is a class-inverting automorphism (CIA) of the flavor group [23, 45].

For the outer automorphism group $K_4 = \{ \text{id}, u_1, u_2, u_1 u_2 \}$, the unique CIA is $u_1 u_2$. According to the action of $u_1 u_2$ (see Eq.(24)) and the 3-dimensional representation of the generators of $\Sigma(36 \times 3)$ (see Eq.(3)), $X(u_1 u_2)$ satisfies the equations as follows:

$$X(u_1 u_2)^\dagger g X^{-1}(u_1 u_2) = \rho(a) = \rho^*(a),$$

$$X(u_1 u_2)^\dagger g X^{-1}(u_1 u_2) = \rho^*(b) = \rho^*(b),$$

$$X(u_1 u_2)^\dagger g X^{-1}(u_1 u_2) = \rho^*(c) = \rho^*(c).$$

(26)

The solution (up to a global phase) is

$$X(u_1 u_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(27)

Therefore, the general GCP which is compatible with the group $\Sigma(36 \times 3)$ in the 3-dimensional representation is of the form

$$X(v) = \rho_3(g)X(u_1 u_2) = \rho_3(g) = \rho(c^i a^j b^k) \text{ with } i, j = 0, 1, 2, \ k = 0, 1, 2, 3.$$

(28)

Note that the global phase $\omega^i$ is not considered here which corresponds to the element of the centre i.e. $Z_3$.

2. Residual flavor symmetries with GCP

After leptons obtain masses through the vacua expectation values of scalars, the original symmetry $G_f \rtimes H_{CP}$ is broken to $G_e \rtimes H_{CP}^e$ in the charged lepton sector and $G_\nu \rtimes H_{CP}^\nu$ in the neutrino
sector. In the semidirect method, $G_v \rtimes H_{CP}^r$ is reduced to $Z_2 \times H_{CP}^r$. For the Majorana neutrinos mass matrix, we have following relations:

$$\rho^T(g_v)m_\nu \rho(g_v) = m_\nu, \quad X_\nu^T(v)m_\nu X_\nu(v) = m_\nu^*, \quad \text{with} \quad X_\nu(v)\rho^*(g_v)X_\nu^{-1}(v) = \rho(g_v),$$  \hspace{1cm} (29)

where $g_v \in Z_2, v \in H_{CP}^r$. Furthermore, because the masses of neutrinos are non-degenerate, the CP transformation $X_\nu$ should be a symmetric unitary matrix \[37\], i.e.

$$X_\nu = X_\nu^T, \quad X_\nu^tX_\nu = E.$$  \hspace{1cm} (30)

This type of CP transformation is the so-called Bickerstaff-Damhus automorphism \[44, 45\]. And $X_\nu$ could be decomposed as $X_\nu = \Omega_\nu \Omega_\nu^T$. On the base of this decomposition, $X_\nu$ could be transformed to a unit matrix in the new basis called CP basis, namely

$$\Omega_\nu^T X_\nu \Omega_\nu^T = \text{diag}(1, 1, 1).$$  \hspace{1cm} (31)

The mass matrix of neutrinos and the representation matrix of $Z_2$ could be converted to real-valued ones by $\Omega_\nu$, i.e.

$$(\Omega_\nu^T m_\nu \Omega_\nu)^* = \Omega_\nu^T m_\nu \Omega_\nu, \quad (\Omega_\nu^T \rho(g_v) \Omega_\nu)^* = \Omega_\nu^T \rho(g_v) \Omega_\nu.$$  \hspace{1cm} (32)

Furthermore, we can examine that the matrix $\Omega_\nu^T \rho(g_v) \Omega_\nu$ is symmetric. The derivation is given in the appendix. So it could be diagonalized by a real orthogonal matrix, namely

$$\rho^d(g_v) = O^T \Omega_\nu^T \rho(g_v) \Omega_\nu O = \text{diag}((-1)^{j_1}, (-1)^{j_2}, (-1)^{j_3}), \quad \text{with} \quad O^T O = 1, \quad O^* = O,$$  \hspace{1cm} (33)

where $j_i = 0, 1$, and two of them are equal. We should note that the decomposition of $X_\nu$ is not unique. We can introduce a new decomposition as $X_\nu = \Omega'_\nu \Omega'_\nu^T$, with $\Omega'_\nu = \Omega_\nu O$. Then $m_\nu$ could be transformed to a block-diagonal matrix by $\Omega'_\nu$. This observation could seen from the relation $\rho^T(g_v)m_\nu \rho(g_v) = m_\nu$. Substituting $\rho(g_v)$ with $\Omega'_\nu \rho^d(g_v) \Omega'_\nu^T$, we could obtain

$$\rho^d(\Omega'_\nu^T m_\nu \Omega'_\nu) \rho^d = (\Omega'_\nu^T m_\nu \Omega'_\nu).$$  \hspace{1cm} (34)

Thus, $\Omega'_\nu^T m_\nu \Omega'_\nu$ is a real block-diagonal matrix \[24\]. Therefore, the unitary matrix $U_\nu$ fulfilling $U_\nu^T m_\nu U_\nu = \text{diag}(m_1, m_2, m_3)$, could be expressed as \[24\]

$$U_\nu = \Omega'_\nu R_{\nu ij}(\theta) P_\nu,$$  \hspace{1cm} (35)

where $R_{\nu ij}(\theta)$ is a rotation matrix in the plane with $ij = 12, 23, 13$. $P_\nu$ is a phase matrix which coordinates the sign of $m_i$, i.e.

$$P_\nu = \text{diag}(1, i^{k_1}, i^{k_2}), \quad \text{with} \quad k_1, k_2 = 0, 1, 2, 3.$$  \hspace{1cm} (36)
In the following sections we denote $\Omega'$ with $\Omega_e$ for simplicity.

In the charged lepton sector, we consider two nontrivial cases, namely

Case A:

$$G_e \rtimes H^e_{CP} = Z_n \times H^e_{CP}, \text{ with } n \geq 3,$$

(37)

Case B:

$$G_e \rtimes H^e_{CP} = Z_2 \times H^e_{CP},$$

(38)

In Case A, the mass matrix of the charged leptons is constrained by the residual symmetries as

$$\rho^+(g_e)m_e^+\rho(g_e) = m_e^+m_e^+e_{e}\rho(g_e)(v)(m_e^+m_e^+)^e, \text{ with } g_e \in Z_n, \ v \in H^e_{CP}. \tag{39}$$

And the unitary matrix $U_e$ which fulfills $U^*_e m_e^+ U_e = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)$, could be obtained from the diagonalization of $\rho(g_e)$, namely

$$U^*_e \rho(g_e) U_e = \rho^d(g_e), \tag{40}$$

where $\rho^d(g_e)$ denotes the diagonal representation matrix. Since $\rho(g_e)$ is non-degenerate, $U_e$ is fixed by the residual flavor group up to permutations of columns and nonphysical phases.

In Case B, the constraints are same as those in Eq. (39) but with $g_e \in Z_2$. According to the derivations similar to those in the neutrino sector (see the appendix), $X_e$ could decomposed as $X_e = \Omega_e \Omega^T_e$, where $\Omega_e$ could diagonalise $\rho(g_e)$. So the matrix $U_e$ could be expressed as

$$U_e = \Omega_e R_{\epsilon\iota j}(\theta_2), \tag{41}$$

where $R_{\epsilon\iota j}$ is also a rotation matrix.

3. Similarity transformation

If a combination of residual flavor symmetries with the GCP, i.e. $(Z_{m(e)}, Z_{2(v)}, X_v)$ in Case A or $(Z_{2(e)}, X_e, Z_{2(v)}, X_v)$ in Case B is given, the lepton mixing matrix $U_{PMNS} = U^+_e U_v$ could be obtained (up to permutations of rows or columns). If two combinations are related by a similarity transformation, namely

$$\rho'(g_a) = \Omega_0 \rho(g_a) \Omega_0^+, \quad X'_\beta = \Omega_0 X_\beta \Omega_0^T, \tag{42}$$
with $\alpha = e, \nu$, and $\beta = e, \nu$ in Case A, $\beta = e, \nu$ in Case B, they would correspond to the same mixing matrix of leptons. Although there are $13 Z_3$ and $9 Z_4$ subgroups of $\Sigma(36 \times 3)$, we could just consider the subgroup generated by a representative in the conjugacy class. So in Case A, we consider the combination of $Z_{n(e)}$ listed in Table I and $(Z_{2(\nu)}, X_\nu)$ in Table II. Furthermore, the $Z_6$ and $Z_{12}$ subgroups in the 3-dimensional representation are different from the $Z_2$ and $Z_3$ subgroups just for the factor $\omega$. They don’t bring new mixing patterns. According to the same reason, in Case B we could fix $Z_{2(e)}$ on $Z_{2(\nu)}$ and consider the combination $(Z_{2(e)}^{c_6}, X_e, Z_{2(\nu)}, X_\nu)$.

III. RESULTS

A. Combinations in Case A

In Case A, the diagonalization matrix $U_e$ is completely determined by $Z_{n(e)}$. And a column of the matrix $U_\nu$ is determined by $(Z_{2(\nu)}, X_\nu)$. Finally, a column of the lepton mixing matrix $U_{PMNS}$ is fixed. There is a free parameter to coordinate the mixing patterns. Various of nonequivalent combinations of $(Z_{n(e)}, Z_{2(\nu)}, X_\nu)$ are surveyed. We find that neither of them could accommodate the fit data of neutrinos [46] at the 3$\sigma$ level. This observation could be seen from the fixed column of the mixing matrix. For the combinations $(Z_{3(\nu)}^{c_6}, Z_{2(\nu)}, X_\nu)$ and $(Z_{3(\nu)}^{c_6}, Z_{2(\nu)}, X_\nu)$, the magnitude of the fixed column vector (up to permutations of rows) is

$$|U_{PMNS}(\alpha, i)| = \left(0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)^T. \quad (43)$$

For the combinations $(Z_{2(\nu)}^{h}, Z_{2(\nu)}, X_\nu)$, the magnitude of the column vector includes two cases. If $Z_{2(\nu)} = Z_{2(\nu)}^{h}$, we have

$$|U_{PMNS}(\alpha, i)| = \left(0, 0, 1\right)^T. \quad (44)$$

Otherwise,

$$|U_{PMNS}(\alpha, i)| = \left(\frac{1}{2} \sqrt{\frac{2\sqrt{3} - 3}{\sqrt{3} - 1}}, \frac{1}{2} \sqrt{\frac{2\sqrt{3} + 3}{\sqrt{3} + 1}}, \frac{1}{2}\right)^T \approx \left(0.398, 0.769, 0.5\right)^T. \quad (45)$$

Anyway, neither of these columns is in the 3$\sigma$ range of the fit date of neutrinos [46].

$$|U_{PMNS}| = \begin{pmatrix} 0.801 \rightarrow 0.845 \rightarrow 0.514 \rightarrow 0.580 \rightarrow 0.137 \rightarrow 0.158 \\ 0.225 \rightarrow 0.517 \rightarrow 0.441 \rightarrow 0.699 \rightarrow 0.614 \rightarrow 0.793 \\ 0.246 \rightarrow 0.529 \rightarrow 0.464 \rightarrow 0.713 \rightarrow 0.590 \rightarrow 0.776 \end{pmatrix}. \quad (46)$$

Similar observations were obtained in Ref. [42].
TABLE I: $Z_n$ ($n \geq 3$) subgroup of $\Sigma(36 \times 3)$ generated by a representative in the conjugacy class. $Z_n^{\alpha}$ denotes the group $Z_n$ generated by the element $g_\alpha$. Other subgroups which could be obtained by the group conjugation or the factor $\omega = e^{i2\pi/3}$ are not listed here.

| $Z_n$ | $Z_3^c$ | $Z_3^ca$ | $Z_4^b$ |
|-------|---------|----------|---------|

B. Combinations in Case B

1. Viable combinations

In Case B, on the basis of the similarity transformation, the residual symmetry in the charged lepton sector is fixed as $(Z_{2e}^{b2}, X_e = E, \rho(b))$. Note that $X^{(1)}$ and $X^{(2)}$ for a given group $Z_2$ in Table II correspond to the same mixing pattern. So do $X^{(3)}$ and $X^{(4)}$. The verifications of equivalence of $X^{(i)}$ are given in the appendix. Thus, we just consider two cases of $X_e$ for a fixed $Z_2$. In the sector of neutrinos, the residual symmetry could be expressed as $(Z_{2\nu}^{b2}, X_\nu = X_\nu^{(1)}, X_\nu^{(3)})$. The 3-dimensional representation of generators of $Z_2$ subgroups and the GCP is listed in Table II. In total, there are 36 different combinations of residual symmetries in Case B. We perform a $\chi^2$ analysis on these combinations.

The $\chi^2$ function is defined as

$$\chi^2 = \sum_{ij=13,23,12} \left( \frac{\sin^2 \theta_{ij} - (\sin^2 \theta_{ij})^{bf}}{\sigma_{ij}} \right)^2,$$

where $(\sin^2 \theta_{ij})^{bf}$ are best fit values from Ref. [46], $\sigma_{ij}$ is the 1$\sigma$ error. Only 2 types of combinations can accommodate the global fit data at 3$\sigma$ level [46], namely

Type I:

$$(Z_{2e}^{b2}, E, Z_{2\nu}^{a\nu b\nu}, X^{(1)} = E), (Z_{2e}^{b2}, E, Z_{2\nu}^{ab\nu}, X^{(1)} = E),$$

$$(Z_{2e}^{b2}, E, Z_{2\nu}^{cb\nu}, X^{(1)} = \rho(c)), (Z_{2e}^{b2}, E, Z_{2\nu}^{c2b\nu}, X^{(1)} = \rho^*(c)),$$

Type V:

$$(Z_{2e}^{b2}, E, Z_{2\nu}^{a\nu b\nu}, X^{(3)} = \rho(ab^3c^2b)), (Z_{2e}^{b2}, E, Z_{2\nu}^{a2b\nu}, X^{(3)} = \rho(c^2bc^2)), (Z_{2e}^{b2}, E, Z_{2\nu}^{a2b\nu}, X^{(3)} = \rho(cbc)).$$

The mixing patterns of the combinations of Type I are equivalent. So are those of combinations of Type V. The transformation relation for the equivalence is given in the appendix. Thus, we just show the mixing pattern of the combination $(Z_{2e}^{b2}, E, Z_{2\nu}^{a2b\nu}, E)$ of Type I, and that of the combination $(Z_{2e}^{b2}, E, Z_{2\nu}^{a2b\nu}, \rho(ab^3c^2b))$ of Type V.
TABLE II: Generators of $Z_2$ subgroups of $\Sigma(36 \times 3)$ with the corresponding GCP in the 3-dimensional representation.

| $\rho(g_a)$ | $X^{(1)}$ | $X^{(2)}$ | $X^{(3)}$ | $X^{(4)}$ |
|-------------|-----------|-----------|-----------|-----------|
| $\rho(b^2)$: | \[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & \omega^2 \\ 0 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}$ | $\frac{-1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & \omega^2 \\ 0 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}$ |
| $\rho(ab^2)$: | \[
\begin{pmatrix}
0 & 0 & -1 \\
-1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\pi/6} & -i & e^{i5\pi/6} \\ -i & -i & -i \\ e^{i5\pi/6} & e^{i\pi/6} & -i \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{-i\pi/6} & i & e^{-i5\pi/6} \\ i & i & i \\ e^{-i5\pi/6} & i & e^{-i\pi/6} \end{pmatrix}$ |
| $\rho(a^2b^2)$: | \[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -\omega \\
0 & -\omega & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} -i & e^{i5\pi/6} & e^{i\pi/6} \\ e^{i5\pi/6} & -i & e^{i\pi/6} \\ e^{i\pi/6} & e^{i5\pi/6} & -i \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} i & e^{-i\pi/6} & e^{-i5\pi/6} \\ e^{-i\pi/6} & i & e^{-i5\pi/6} \\ e^{-i5\pi/6} & e^{-i\pi/6} & i \end{pmatrix}$ |
| $\rho(cb^2)$: | \[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -\omega \\
0 & -\omega & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} -i & e^{i5\pi/6} & e^{i\pi/6} \\ e^{i5\pi/6} & -i & e^{i\pi/6} \\ e^{i\pi/6} & e^{i5\pi/6} & -i \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} -i & e^{-i\pi/6} & e^{-i5\pi/6} \\ e^{-i\pi/6} & e^{-i5\pi/6} & e^{-i\pi/6} \\ e^{-i5\pi/6} & e^{-i\pi/6} & i \end{pmatrix}$ |
| $\omega \rho(ca^2b^2)$: | \[
\begin{pmatrix}
0 & -\omega & 0 \\
-\omega & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\pi/3} & e^{i\pi/6} & e^{i5\pi/6} \\ e^{i\pi/6} & e^{i5\pi/6} & -i \\ e^{i5\pi/6} & -i & e^{i\pi/6} \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{-i\pi/6} & i & e^{-i5\pi/6} \\ i & i & i \\ e^{-i5\pi/6} & i & e^{-i\pi/6} \end{pmatrix}$ |
| $\omega^2 \rho(c^2a^2b^2)$: | \[
\begin{pmatrix}
0 & -\omega & 0 \\
-\omega & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i5\pi/6} & e^{i\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & e^{i5\pi/6} & -i \\ -i & e^{i5\pi/6} & e^{i\pi/6} \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} i & i & i \\ i & i & i \\ i & i & i \end{pmatrix}$ |
| $\omega \rho(c^2ab^2)$: | \[
\begin{pmatrix}
0 & -\omega & 0 \\
-\omega & 0 & 0 \\
-\omega & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\pi/6} & e^{i\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & e^{i\pi/6} & -i \\ -i & e^{i\pi/6} & e^{i\pi/6} \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{-i\pi/6} & e^{-i5\pi/6} & e^{-i\pi/6} \\ e^{-i5\pi/6} & e^{-i\pi/6} & e^{-i\pi/6} \\ e^{-i\pi/6} & e^{-i5\pi/6} & e^{-i\pi/6} \end{pmatrix}$ |
| $\omega^2 \rho(cab^2)$: | \[
\begin{pmatrix}
0 & -\omega & 0 \\
-\omega & 0 & 0 \\
-\omega & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\] | $\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i5\pi/6} & -i & e^{i\pi/6} \\ -i & e^{i5\pi/6} & e^{i\pi/6} \\ e^{i\pi/6} & e^{i5\pi/6} & e^{i\pi/6} \end{pmatrix}$ | $\frac{1}{\sqrt{3}} \begin{pmatrix} i & i & i \\ i & i & i \\ i & i & i \end{pmatrix}$ |
2. Analytical expressions for the viable mixing patterns

In the sector of the charged leptons, the residual symmetry is $Z_2^b \times X_e$, with $X_e = E$. The unit matrix could be decomposed as $E = \Omega_e \Omega_e^T$, with $\Omega_e^T \rho(b^2) \Omega_e = diag(-1, -1, 1)$. We could choose $\Omega_e$ as

$$\Omega_{eI} = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{\frac{2}{2}} & 0 & -\sqrt{\frac{2}{2}} \\ \sqrt{\frac{2}{2}} & 0 & \sqrt{\frac{2}{2}} \end{pmatrix}. \quad (50)$$

The unitary matrix $U_e$ could be expressed as

$$U_e = \Omega_{eI} R_{e12}(\theta_2) = \begin{pmatrix} -\sin \theta_2 & \cos \theta_2 & 0 \\ \frac{\sqrt{2}}{2} \cos \theta_2 & \frac{\sqrt{2}}{2} \sin \theta_2 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \cos \theta_2 & \frac{\sqrt{2}}{2} \sin \theta_2 & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad (51)$$

where the rotation matrix $R_{12}(\theta_2)$ reads

$$R_{12}(\theta_2) = \begin{pmatrix} \cos \theta_2 & \sin \theta_2 & 0 \\ -\sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (52)$$

In the neutrino sector, the residual flavor symmetry with the GCP is $Z_2a^b \times X_\nu$, with $X_\nu = E, \rho(ab^3ab^2)$. For $X_\nu = E$, we can choose

$$\Omega_{\nuI} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \end{pmatrix}. \quad (53)$$

And for $X_\nu = \rho(ab^3ab^2)$, we choose

$$\Omega_{\nuV} = \begin{pmatrix} -e^{\frac{3\pi}{4}i} \cos \theta_1 & -e^{\frac{\pi}{4}i} \sin \theta_1 & -\frac{\sqrt{2}}{2} \\ -e^{\frac{3\pi}{4}i} \cos \theta_1 & \frac{\sqrt{2}}{2} \sin \theta_1 & -\frac{\sqrt{2}}{2} \\ e^{\frac{\pi}{4}i} \sin \theta_1 & -e^{\frac{3\pi}{4}i} \cos \theta_1 & 0 \end{pmatrix}, \text{ with } \theta_1 = \frac{1}{2} \arctan \sqrt{2}. \quad (54)$$

And the mixing matrix $U_\nu$ is obtained according to the equation $U_\nu = \Omega_{\nu\gamma} R_{\nu12}(\theta) P_\nu$, with $\gamma = I, V$. Then the lepton mixing matrix $U_{PMNS}$ is determined up to permutations of rows or columns with the matrices

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (55)$$
In detail, for the combination Type-I, we obtain the following phenomenological viable mixing patterns:

\[ U_{11} = U_e^T(\theta_2)\Omega_{e\nu}R_{12}(\theta)P_\nu S_{23} \]

\[
\begin{pmatrix}
-\frac{1}{2} \cos \theta \cos \theta_2 + \frac{\sqrt{2}}{2} \sin(\theta + \theta_2) & \frac{\sqrt{2}}{2} \cos(\theta_2 - 2\theta_1) & -\frac{1}{2} \sin \theta \cos \theta_2 - \frac{\sqrt{2}}{2} \cos(\theta + \theta_2) \\
-\frac{1}{2} \cos \theta \sin \theta_2 - \frac{\sqrt{2}}{2} \cos(\theta + \theta_2) & \frac{\sqrt{2}}{2} \sin(\theta_2 - 2\theta_1) & -\frac{1}{2} \sin \theta \sin \theta_2 + \frac{\sqrt{2}}{2} \sin(\theta + \theta_2) \\
\frac{\sqrt{2}}{2} \cos(\theta - 2\theta_1) & -\frac{1}{2} & \frac{\sqrt{2}}{2} \sin(\theta - 2\theta_1)
\end{pmatrix} P',
\]

where \( P' = S_{23}P_\nu S_{23} \). And its nonequivalent permutations:

\[ I_2 \quad U_{12} = S_{23}U_{11}, \]
\[ I_3 \quad U_{13} = U_{11}S_{23}S_{13}, \]
\[ I_4 \quad U_{14} = S_{13}S_{23}U_{11}S_{23}S_{13}. \]

Compared with the standard parametrization of the lepton mixing matrix \cite{47}

\[ U_{PMNS} = \frac{1}{2}
\begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\
s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\alpha/2} & 0 \\
0 & 0 & e^{i(\beta/2+\delta)}
\end{pmatrix},
\]

where \( s_{ij} \equiv \sin \theta_{ij}, c_{ij} \equiv \cos \theta_{ij}, \delta \) is the Dirac CP-violating phase, \( \alpha \) and \( \beta \) are Majorana Phases, the mixing angles are obtained as follows:

I1

\[
\sin^2 \theta_{13} = \frac{1}{2} [\cos(\theta + \theta_2) + \frac{1}{\sqrt{2}} \sin \theta \cos \theta_2]^2,
\]
\[
\sin^2 \theta_{23} = \frac{1}{2} [\frac{1}{\sqrt{2}} \sin \theta \sin \theta_2 + \sin(\theta + \theta_2)]^2/(1 - \sin^2 \theta_{13}),
\]
\[
\sin^2 \theta_{12} = \frac{3}{4} \cos^2(\theta_2 - 2\theta_1)/(1 - \sin^2 \theta_{13}).
\]

I2

\[
\sin^2 \theta_{13} = \frac{1}{2} [\cos(\theta + \theta_2) + \frac{1}{\sqrt{2}} \sin \theta \cos \theta_2]^2,
\]
\[
\sin^2 \theta_{23} = \frac{3}{4} \sin^2(\theta - 2\theta_1)/(1 - \sin^2 \theta_{13}),
\]
\[
\sin^2 \theta_{12} = \frac{3}{4} \cos^2(\theta_2 - 2\theta_1)/(1 - \sin^2 \theta_{13}).
\]
\[ \sin^2 \theta_{13} = \frac{1}{2} [\sin(\theta + \theta_2) - \frac{1}{\sqrt{2}} \cos \theta \cos \theta_2]^2, \]
\[ \sin^2 \theta_{23} = \frac{1}{2} [\cos(\theta + \theta_2) + \frac{1}{\sqrt{2}} \cos \theta \sin \theta_2]^2/(1 - \sin^2 \theta_{13}), \quad (61) \]
\[ \sin^2 \theta_{12} = \frac{1}{2} [\cos(\theta + \theta_2) + \frac{1}{\sqrt{2}} \sin \theta \cos \theta_2]^2/(1 - \sin^2 \theta_{13}). \]

The Dirac and Majorana CP phases are trivial for the above mixing patterns.

For the combination type-V, we obtain following mixing patterns:

\[ V1 \quad U_{V1} = U^e_\nu (\theta_2) \Omega_\nu V R_{12}(\theta) P_\nu S_{23} = (V1_1 \quad V1_2 \quad V1_3) P_\nu, \quad (63) \]

where

\[ V1_1 = \begin{pmatrix}
\frac{\sqrt{3}}{2} \cos(\theta_2 - 2\theta_1) \\
\frac{\sqrt{3}}{2} \sin(\theta_2 - 2\theta_1) \\
-\frac{1}{2} 
\end{pmatrix} \quad \text{and} \quad \sin^2 \theta_{13} = \frac{3}{4} \cos^2(\theta - 2\theta_1)/(1 - \sin^2 \theta_{13}), \quad (62) \]

\[ \sin^2 \theta_{12} = \frac{1}{2} [\sin(\theta + \theta_2) + \frac{1}{\sqrt{2}} \sin \theta \sin \theta_2]^2/(1 - \sin^2 \theta_{13}). \]

\[ V1_2 = \begin{pmatrix}
\frac{\sqrt{3}}{2} \cos(\theta_2 - 2\theta_1) \\
\frac{\sqrt{3}}{2} \sin(\theta_2 - 2\theta_1) \\
-\frac{1}{2} 
\end{pmatrix}, \quad (65) \]

\[ V1_3 = \begin{pmatrix}
\frac{1}{2} \cos(\theta - \theta_1 - \theta_2) - \frac{1}{2} \cos(\theta + \theta_1 + \theta_2) - \frac{\sqrt{3}}{4} \cos \theta_2 [\sin(\theta - \theta_1) + i \sin(\theta + \theta_1)] \\
\frac{1}{4} [(1 - i) \sqrt{3} - \sqrt{3} \cos \theta + (1 + i)(12 + 6 \sqrt{3})^{1/4} \sin \theta] 
\end{pmatrix} \]

And its permutation:

\[ U_{V2} = S_{23} U_{V1} S_{13}. \quad (67) \]

The mixing angles and CP invariants are listed as follows:

\[ V1 \]
\[ \sin^2 \theta_{13} = \frac{1}{16} [5 + \sqrt{3} \cos 2\theta + \cos 2\theta_2 + \frac{7}{\sqrt{3}} \cos 2\theta \cos 2\theta_2 - 2 \sqrt{2} \sin 2\theta_2 - 2 \sqrt{\frac{2}{3}} \cos 2\theta \sin 2\theta_2], \]
\[
\sin^2 \theta_{23} = \frac{1}{16} \left[ 5 + \sqrt{3} \cos 2\theta - \cos 2\theta_2 - \frac{7}{\sqrt{3}} \cos 2\theta \cos 2\theta_2 + 2 \sqrt{2} \sin 2\theta + 2 \sqrt{2} \cos 2\theta \sin 2\theta_2 \right] / (1 - \sin^2 \theta_{13}),
\]

\[
\sin^2 \theta_{12} = \frac{3}{4} \cos^2 (\theta_2 - 2\theta_1) / (1 - \sin^2 \theta_{13}),
\]

\[
J_{\text{cp}} = \frac{3}{32} \sin 2\theta \sin (4\theta_1 - 2\theta_2),
\]

(68)

\[
J_1 = (-1)^{k_1} \frac{\sqrt{3}}{64} \cos^2 (\theta_2 - 2\theta_1) \left[ \sqrt{3} \cos 2\theta_2 - 2 \sqrt{2} \sin 2\theta_2 + 5 \right] + 2 \sqrt{2} \sin 2\theta_2 - 7 \cos 2\theta_2 - 3, \]

\[
J_2 = \frac{(-1)^{k_1}}{768} \sin 2\theta \left[ -27 \sqrt{2} - 36 \sqrt{2} \cos 2\theta_2 + 23 \sqrt{2} \cos 4\theta_2 + 72 \sin 2\theta_2 + 52 \sin 4\theta_2 \right].
\]

Here \( J_{\text{cp}} \) denotes the Jarlskog invariant [48], \( k_i \) is the parameter introduced in Eq. (36). These invariants are defined as

\[
J_{\text{cp}} \equiv \text{Im}[U_{11} U_{13}^* U_{31}^* U_{33}] = \frac{1}{8} \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_2 \cos \theta_{13} \sin \delta.
\]

(69)

\[
J_1 \equiv \text{Im}[(U_{11}^*)^2 U_{12}^2] = \sin^2 \theta_{12} \cos^2 \theta_{12} \cos^4 \theta_{13} \sin \alpha,
\]

(70)

\[
J_2 \equiv \text{Im}[(U_{11}^*)^2 U_{13}^2] = \sin^2 \theta_{13} \cos^2 \theta_{13} \cos^2 \theta_{12} \sin \beta.
\]

(71)

\[ V2 \]

\[
\sin^2 \theta_{13} = \frac{1}{16} \left[ 5 + \cos 2\theta_2 - 2 \sqrt{2} \sin 2\theta_2 + 2(1 + \sqrt{3}) \cos 2\theta(-3 - 7 \cos 2\theta_2 + 2 \sqrt{2} \sin 2\theta_2) \right],
\]

\[
\sin^2 \theta_{23} = \frac{\sqrt{12 - 6 \sqrt{3}}}{16} \left[ 3 + \sqrt{3} + (1 + \sqrt{3}) \cos 2\theta \right] / (1 - \sin^2 \theta_{13}),
\]

\[
\sin^2 \theta_{12} = \frac{3}{4} \cos^2 (\theta_2 - 2\theta_1) / (1 - \sin^2 \theta_{13}),
\]

\[
J_{\text{cp}} = \frac{3}{32} \sin 2\theta \sin (4\theta_1 - 2\theta_2),
\]

(72)

\[
J_1 = (-1)^{k_1} \frac{\sqrt{3}}{64} \cos^2 (\theta_2 - 2\theta_1) \left[ \sqrt{3} \cos 2\theta_2 - 2 \sqrt{2} \sin 2\theta_2 + 5 \right] - 2 \sqrt{2} \sin 2\theta_2 + 7 \cos 2\theta_2 + 3, \]

\[
J_2 = \frac{(-1)^{k_1+1}}{768} \sin 2\theta \left[ -27 \sqrt{2} - 36 \sqrt{2} \cos 2\theta_2 + 23 \sqrt{2} \cos 4\theta_2 + 72 \sin 2\theta_2 + 52 \sin 4\theta_2 \right].
\]

Note that the magnitudes of the CP invariants are invariant under the permutation of columns and rows of the mixing matrix.
FIG. 1: Parameter spaces of the mixing angles for the mixing pattern I1 and V1. The left panel is for the pattern I1 and the right one for V1. Here the 3σ ranges of mixing angles are taken from the fit data in Ref. [46]. In detail, $\sin^2 \theta_{13} \in [0.0188, 0.0251]$, $\sin^2 \theta_{23} \in [0.385, 0.644]$, $\sin^2 \theta_{12} \in [0.270, 0.344]$. The parameter areas for $\theta_{12}$ at the 3σ level are the strips with blue boundaries and those for $\theta_{23}$ are the areas with dashed green boundaries. The strip of the parameters for $\theta_{13}$ is tiny, i.e. reduced to a black curve. The intersections of 3 parameter areas are short black lines signed by red dots. The parameter spaces for other viable mixing patterns are similar. So we don’t shown them here.

C. Numerical results of the viable mixing patterns

As the quantitative prediction of the viable mixing patterns, we present the numerical results here.

First, we show the parameter spaces for the mixing pattern I1 and VI in Fig. 1. As is seen from the plots, the parameter space for $\theta_{23}$ with green dashed boundary is considerable for both patterns. However, those for $\theta_{13}, \theta_{12}$ are tiny. Especially for $\theta_{13}$, the parameter strip is nearly reduced to a curve. So the common parameter spaces are rather small. They are signed by red dots in the figure. For other viable mixing patterns, the same observation also hold.

Second, we list best fit values of the leptonic mixing angles and CP phases of the viable mixing patterns in Table III. Let us give some comments on the results in the table:

(i). The best fit values of mixing angles $\theta_{ij}$ in every pattern approximate those of the global fit data in Ref. [46].

(ii). The octant of $\theta_{23}$ is not determined by the residual symmetries. Through permutations of the
TABLE III: Numerical results for the mixing patterns of type-I, type-V. NO in the table refers to the normal ordering of neutrino masses and IO denotes the inverted ordering.

| Patterns | \((\theta^d_{ij}, \theta^f_{ij})\) | \(\chi^2_{min}\) | \(\sin^2 \theta_{13}\) | \(\sin^2 \theta_{23}\) | \(\sin^2 \theta_{12}\) | \(|\sin \delta|\) | \(|\sin \alpha|\) | \(|\sin \beta|\) |
|----------|--------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| I1(NO)   | \((-0.017\pi, 0.587\pi), (0.625\pi, 0.021\pi)\) | 0.0005 | 0.0218 | 0.451 | 0.304 | 0 | 0 | 0 |
| I1(IO)   | \((-0.011\pi, 0.581\pi), (0.619\pi, 0.027\pi)\) | 15.4 | 0.0221 | 0.4634 | 0.3196 | 0 | 0 | 0 |
| I2(NO)   | \((-0.014\pi, 0.584\pi), (0.6225\pi, 0.024\pi)\) | 5.53 | 0.0219 | 0.5432 | 0.3112 | 0 | 0 | 0 |
| I2(IO)   | \((-0.018\pi, 0.589\pi), (0.626\pi, 0.019\pi)\) | 0.86 | 0.0217 | 0.5516 | 0.3004 | 0 | 0 | 0 |
| I3(NO)   | \((0.086\pi, 0.193\pi), (0.522\pi, 0.415\pi)\) | 6.84 | 0.0221 | 0.5393 | 0.3222 | 0 | 0 | 0 |
| I3(IO)   | \((0.069\pi, 0.206\pi), (0.539\pi, 0.403\pi)\) | 0.0009 | 0.0219 | 0.5796 | 0.3043 | 0 | 0 | 0 |
| I4(NO)   | \((0.074\pi, 0.202\pi), (0.534\pi, 0.406\pi)\) | 0.427 | 0.0219 | 0.4311 | 0.309 | 0 | 0 | 0 |
| I4(IO)   | \((0.0095\pi, 0.186\pi), (0.512\pi, 0.422\pi)\) | 14.9 | 0.0225 | 0.484 | 0.334 | 0 | 0 | 0 |
| V1(NO)   | \((\pm 0.548\pi, 0.021\pi)\) | 1.354 | 0.0218 | 0.405 | 0.3044 | 0.842 | 0.9996 | 0.897 |
| V1(IO)   | \((\pm 0.549\pi, 0.022\pi)\) | 31.3 | 0.0218 | 0.406 | 0.3068 | 0.850 | 0.9996 | 0.883 |
| V2(NO)   | \((\pm 0.0486\pi, 0.0214\pi)\) | 12.69 | 0.0218 | 0.5944 | 0.3054 | 0.8455 | 0.9996 | 0.8915 |
| V2(IO)   | \((\pm 0.0484\pi, 0.021\pi)\) | 0.252 | 0.0218 | 0.5945 | 0.3042 | 0.8415 | 0.9996 | 0.898 |

second and third rows of the mixing matrix, both upper and lower octant could be obtained for these viable mixing patterns.

(iii). All the CP phases are nontrivial in the mixing patterns of Type V. In detail, the Dirac CP phase \(\delta\) is around \(\pm 57^\circ\) or \(\pm 123^\circ\) which is in accordance with the result obtained in Ref. [41] and the global fit data [46]. For the Majorana phases, we have \(\alpha \sim \pm 90^\circ\), \(\beta \sim \pm 63^\circ\) or \(\pm 117^\circ\).

IV. SUMMARY

The GCP is an important approach to extract information on the CP phases of the lepton mixing pattern. We employed the group \(\Sigma(36 \times 3)\) with the GCP to predict lepton mixing patterns in a semidirect method. We first derived the general GCP which is compatible with the group \(\Sigma(36 \times 3)\). Then we surveyed various combinations of Abelian residual flavor symmetries with the GCP, i.e. \((Z_{n(e)}, Z_{2(v)}, X_v)\) and \((Z_{2(e)}, X_e, Z_{2(v)}, X_v)\). We found two viable combinations (up to those equivalent) could accommodate the fit data of neutrinos at 3\(\sigma\) level. These combinations corre-
spond to six mixing patterns among which four patterns predict trivial CP phases. Two patterns predict nontrivial CP phases. Especially, the Dirac CP phase is around $\pm 57^\circ$ or $\pm 123^\circ$ which is in accordance with the result in the recent literature.

We note that in the semidirect method the predictions for the six mixing parameters (three angles and three phases) depend on the free parameters. The physical reason for the best fit values of the free parameters is needed which is related to the open question of the origin of the lepton mixing. As a compromise, we followed the line of the literature where the mixing patterns are partially determined by the residual symmetries. So the fine-tuning of the free parameters is necessary in our work. Even though, a dynamical model for the partial pattern is still important. However, for the flavor group $\Sigma(36 \times 3)$ which contains 108 elements and 14 irreducible representations, the model would be rather complicated in the technical aspect. So a dynamical model on the basis of $\Sigma(36 \times 3)$ with the GCP is out of the scope of this paper and we will consider it in the future work.

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**Appendix**

1. $\Omega^*_v \rho(g_v) \Omega_v$ is symmetric

Employing the equation $(\Omega^*_v \rho(g_v) \Omega_v)^* = \Omega^*_v \rho(g_v) \Omega_v$, we have

$$
(\Omega^*_v \rho(g_v) \Omega_v)^T = (\Omega^*_v \rho(g_v) \Omega_v)^* = \Omega^*_v \rho^*(g_v) \Omega_v.
$$

(73)

Since $g_v \in Z_2$, namely $\rho^2(g_v) = E$, we have $\rho^*(g_v) = \rho^{-1}(g_v) = \rho(g_v)$. Therefore, we could obtain the observation $(\Omega^*_v \rho(g_v) \Omega_v)^T = \Omega^*_v \rho(g_v) \Omega_v$. 


2. Decomposition of $X_e$

Since the masses of charged leptons are non-degenerate, employing the equations
\[ X_e^+(v)m_e m_e^+ X_e(v) = (m_e m_e^+)^*, \quad U_e^+ m_e m_e^+ U_e = \text{diag}(m_e^2, m_{\mu}^2, m_{\tau}^2), \]  
we could obtain the observation
\[ U_e^+ X_e U_e^* = X_e^d, \]  
where $X_e^d$ is diagonal. Thus, $X_e$ is symmetric, namely $X_e^T = X_e$. So it could be decomposed as
\[ X_e = \Omega_e \Omega_e^T. \]  
Then we have the observation
\[ \Omega_e^T m_e m_e^+ \Omega_e = (\Omega_e^T m_e m_e^+ \Omega_e)^*. \]  
As we know, the decomposition of $X_e$ is not unique. According to the equation $X_e(v)\rho^*(i)X_e^{-1}(v) = \rho(i)$, we could diagonalize $\rho(i)$ through a special $\Omega_e$, namely
\[ \rho^d(i) = \Omega_e^* \rho(i) \Omega_e = \text{diag}((-1)^h, (-1)^j, (-1)^k). \]  
On the basis of the relation $\rho(i) m_e m_e^+ \rho(i) = m_e m_e^+$, we have could obtain the equation
\[ \rho^d(i) (\Omega_e^* m_e m_e^+ \Omega_e) \rho^d(i) = (\Omega_e^* m_e m_e^+ \Omega_e). \]  
Thus, $\Omega_e^* m_e m_e^+ \Omega_e$ is a real block-diagonal matrix.

3. Equivalence of $X^{(i)}$

The representations of $Z_2$ subgroups and the corresponding GCP are shown in Table II. For the generator $b^2$, $X^{(1)} = E$, $X^{(2)} = \rho(b^2)$. The unit matrix $E$ could be decomposed as $E = \Omega_1 \Omega_1^T$, with
\[ \Omega_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \]  
Note that $\Omega_1$ could also diagonalize $\rho(b^2)$. And the corresponding mixing matrix is $U_\nu = \Omega_1 R_{12}(\theta) P_\nu$. In contrast, $X^{(2)} = \Omega_2 \Omega_2^T$, with
\[ \Omega_2 = \begin{pmatrix} 0 & i & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \Omega_1 \cdot \text{diag}(i, i, 1). \]
And the mixing matrix is  \( U_v = \Omega_2 R_{12}(\theta) P_v = \Omega_1 R_{12}(\theta) P'_v \), with

\[
P'_v = \text{diag}(i, i, 1) P_v.
\]

Therefore, the mixing pattern of \( X^{(1)} \) is equivalent to that of \( X^{(2)} \) up to a redefinition of the phase matrix \( P_v \).

As for \( X^{(3)} \) and \( X^{(4)} \), we have \( X^{(3)} = (X^{(4)})^* \). \( X^{(3)} \) could be decomposed as \( X^{(3)} = \Omega_3 \Omega_3^T \), with

\[
\Omega_3 = \begin{pmatrix}
-\sin \theta_1 & \cos \theta_1 & 0 \\
\frac{\cos \theta_1}{\sqrt{2}} & \frac{\sin \theta_1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{\cos \theta_1}{\sqrt{2}} & \frac{\sin \theta_1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \cdot \text{diag}(e^{i\pi/4}, e^{-i\pi/4}, 1),
\]

where \( \theta_1 = \arctan \sqrt{2} \). And \( X^{(4)} \) could be decomposed as \( X^{(4)} = \Omega_4 \Omega_4^T \), with \( \Omega_4 = \Omega_3 \cdot \text{diag}(i, i, 1) \). Thus the equivalence also holds for the mixing patterns of \( X^{(3)} \) and \( X^{(4)} \).

In the above discussion, we just consider the case of \( Z_2^{b^2} \). For other subgroups \( Z_2^{a^2} \), the same observation still holds because of the similar transformation \( \rho(g_a) = \Omega_0 \rho(b^2) \Omega_0^* \). In detail, the transformation \( U_v(g_a) = \Omega_0 U_v(b^2) \) keeps the equivalence of \( X^{(i)} \) in other cases.

4. Equivalence of the mixing patterns from different residual symmetries

The 3-dimensional representation of \( \rho(g_a) \) with \( g_a \in Z_2 \) and \( X^{(i)} \) is shown in Table II. On the base of the representation, we could obtain the matrix \( \Omega_i \) which decomposes \( X^{(i)} \), namely \( X^{(i)} = \Omega_i \Omega_i^T \). With the matrices \( \Omega_i \), the equivalence of the mixing patterns on the basis of combinations of Type I or Type V could be examined. As an example, here we give transformations which relate the mixing patterns of Type V combinations.

For the combination \( (Z_2^{b^2}, E, Z_2^{a}b^2, X^{(3)} = \rho(ab^3ab^2)) \), the mixing matrix is obtained through the equation \( U_v(\theta_2, \theta) = U^*_v(\theta_2) \Omega_v \Omega_v R_{12}(\theta) P_v \) up to permutations of rows or columns, with

\[
U_v = \Omega_{cR_{12}(\theta_2)} = \begin{pmatrix}
-\sin \theta_2 & \cos \theta_2 & 0 \\
\frac{\sqrt{2}}{2} \cos \theta_2 & \frac{\sqrt{2}}{2} \sin \theta_2 & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} \cos \theta_2 & \frac{\sqrt{2}}{2} \sin \theta_2 & \frac{\sqrt{2}}{2}
\end{pmatrix}, \quad \Omega_v = \begin{pmatrix}
\frac{e^{i\pi/4} \cos \theta_1}{\sqrt{2}} & -\frac{e^{i\pi/4} \sin \theta_1}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \\
\frac{e^{i\pi/4} \cos \theta_1}{\sqrt{2}} & -\frac{e^{i\pi/4} \sin \theta_1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\
\frac{e^{i\pi/4} \sin \theta_1}{\sqrt{2}} & -\frac{e^{i\pi/4} \sin \theta_1}{\sqrt{2}} & 0
\end{pmatrix},
\]

where \( \theta_1 = \frac{1}{2} \arctan \sqrt{2} \). As for the combination \( (Z_2^{b^2}, E, Z_2^{c}b^2, \rho(c^2b^2c^2)) \), we have \( U_{v'}(\theta_2, \theta) = U^*_v(\theta_2) \Omega_{v'} \Omega_{v'} R_{12}(\theta) P_v \), where \( \Omega_{v'} \) which decomposes \( \rho(c^2b^2c^2) \) is of the form

\[
\Omega_{v'} = \begin{pmatrix}
-e^{i\pi/4} \sin \theta_1 & e^{i3\pi/4} \cos \theta_1 & 0 \\
\frac{e^{i5\pi/12} \cos \theta_1}{\sqrt{2}} & \frac{e^{i\pi/12} \sin \theta_1}{\sqrt{2}} & -\frac{\omega^2}{\sqrt{2}} \\
\frac{e^{i11\pi/12} \cos \theta_1}{\sqrt{2}} & \frac{e^{-i\pi/12} \sin \theta_1}{\sqrt{2}} & \frac{\omega}{\sqrt{2}}
\end{pmatrix},
\]

20
After a cumbersome derivation, we find the equivalent relation between them as follow

\[ U'_{\nu}(\theta_2, \theta) = \text{diag}(1, -1, -i) U_{\nu}(-\theta_2 + 2\theta_1, \pi - \theta) \text{diag}(1, -1, i). \] (86)

For the combination \((Z_{2(e)}^\nu, E, Z_{2(\nu)}^\nu, \rho(\nu))\), we have

\[ U''_{\nu}(\theta_2, \theta) = U''_{\nu}(\theta_2, \theta) \Omega_{\nu''}R_{12}(\theta)P_{\nu}, \]

with \(\Omega_{\nu''}\) of the form

\[ \Omega_{\nu''} = \begin{pmatrix}
  e^{i\pi/4} \sin \theta_1 & -e^{i3\pi/4} \cos \theta_1 & 0 \\
  -e^{-i7\pi/12} \cos \theta_1 & e^{-i7\pi/12} \sin \theta_1 & -\omega \sqrt{2} \\
  -e^{-i5\pi/12} \cos \theta_1 & e^{-i5\pi/12} \sin \theta_1 & \omega^2 \sqrt{2}
\end{pmatrix}. \] (87)

And the equivalent relation between the mixing matrices is

\[ U''_{\nu}(\theta_2, \theta) = \text{diag}(-1, -1, 1) U'_{\nu}(\theta_2, \theta). \]

Therefore, on the basis of above discussions, we find that different residual symmetries could correspond to the same mixing pattern through the redefinitions of phase matrix and free parameters.

[1] T2K collaboration, K. Abe et al., Indication of electron neutrino appearance from an accelerator-produced off-axis muon neutrino beam, Phys. Rev. Lett. 107 (2011) 041801 [arXiv: 1106.2822]
[2] MINOS collaboration, P. Adamson et al., Improved search for muon-neutrino to electron-neutrino oscillations in MINOS, Phys. Rev. Lett. 107 (2011) 181802 [arXiv: 1108.0015]
[3] DAYA-BAY collaboration, F. An et al., Observation of electron-antineutrino disappearance at Daya Bay, Phys. Rev. Lett. 108 (2012) 171803 [arXiv: 1203.1669]
[4] RENO collaboration, J. K. Ahn et al., Observation of reactor electron antineutrino disappearance in the RENO experiment, Phys. Rev. Lett. 108 (2012) 191802 [arXiv: 1204.0626]
[5] DOUBLE-CHOOZ collaboration, Y. Abe et al., Indication for the disappearance of reactor electron antineutrinos in the Double CHOOZ experiment, Phys. Rev. Lett. 108 (2012) 131801 [arXiv: 1112.6353]
[6] T2K collaboration, K. Abe et al., Observation of Electron Neutrino Appearance in a Muon Neutrino Beam, Phys. Rev. Lett. 112 (2014) 061802 [arXiv:1311.4750]
[7] D. Forero, M. Tortola, and J. Valle, Neutrino oscillations refitted, Phys. Rev. D 90 (2014) 093006 [arXiv:1405.7540]
[8] Ernest Ma, A4 symmetry and neutrinos with very different masses, Phys. Rev. D 70 (2004) 031901(R)
[9] K.S. Babu, Ernest Ma, J.W.F. Valle, Underlying A4 symmetry for the neutrino mass matrix and the quark mixing matrix, Phys. Lett. B, 552 (2003) 207-213
[10] C.S. Lam, *Mass Independent Textures and Symmetry*, Phys. Rev. D 74 (2006) 113004 [hep-ph/0611017]

[11] W. Grimus and L. Lavoura, *A model realizing the Harrison-Perkins-Scott lepton mixing matrix*, JHEP 01 (2006) 018

[12] Mu-Chun Chen and Stephen F. King, *A4 see-saw models and form dominance*, JHEP 06 (2009) 072

[13] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, *Discrete Flavour Symmetries in Light of T2K*, Phys. Lett. B 703 (2011) 447 [arXiv: 1107.3468]

[14] Gui-Jun Ding, *TFH Mixing Patterns, Large $\theta_{13}$ and $\Delta(96)$ Flavor Symmetry*, Nucl. Phys. B 862 (2012) 1 [arXiv: 1201.3279]

[15] Guido Altarelli and Ferruccio Feruglio, *Discrete flavor symmetries and models of neutrino mixing*, Rev. Mod. Phys. 82 (2010) 2701

[16] S.F. King and C. Luhn, *Neutrino Mass and Mixing with Discrete Symmetry*, Rept. Prog. Phys. 76 (2013) 056201

[17] R.M. Fonseca, W. Grimus, *Classification of lepton mixing matrices from finite residual symmetries*, JHEP 09 (2014) 033 [arXiv:1405.3678]

[18] Chang-Yuan Yao and Gui-Jun Ding, *Lepton and quark mixing patterns from finite flavor symmetries*, Phys. Rev. D 92 (2015) 096010 [arXiv: 1505.03798]

[19] P. F. Harrison and W. G. Scott, *Symmetries and Generalisations of Tri-Bimaximal Neutrino Mixing*, Phys. Lett. B 535 (2002) 163 [arXiv:hep-ph/0203209]

[20] W. Grimus, L. Lavoura, *A non-standard CP transformation leading to maximal atmospheric neutrino mixing*, Phys. Lett. B 579 (2004) 113-122 [arXiv:hep-ph/0305309]

[21] Y. Farzan and A. Yu. Smirnov, *Leptonic CP violation: zero, maximal or between the two extremes*, JHEP 01 (2007) 059 [arXiv:hep-ph/0610337 [hep-ph]].

[22] Gui-Jun Ding, Stephen F. King, Alexander J. Stuart, *Generalised CP and $A_4$ Family Symmetry*, JHEP 12 (2013) 006 [arXiv:1307.4212]

[23] M. Holthausen, M. Lindner, Michael A. Schmidt, *CP and Discrete Flavour Symmetries*, JHEP 04 (2013) 122 [arXiv:1211.6953]

[24] F. Feruglio, C. Hagedorn and R. Ziegler, *Lepton Mixing Parameters from Discrete and CP Symmetries*, JHEP 07 (2013) 027 [arXiv:1211.5560]

[25] G. J. Ding, S. F. King, C. Luhn and A. J. Stuart, *Spontaneous CP violation from vacuum alignment in $S_4$ models of leptons*, JHEP 1305 (2013) 084 [arXiv:1303.6180].
[26] F. Feruglio, C. Hagedorn and R. Ziegler, A realistic pattern of lepton mixing and masses from S4 and CP, *EPJC* 74 (2014) 2753

[27] Ivan Girardi, Aurora Meroni, S.T. Petcov, Martin Spinrath, Generalised geometrical CP violation in a $T'$ lepton flavour model, *JHEP* 02 (2014) 050

[28] Cai-Chang Li, Gui-Jun Ding Generalised CP and trimaximal $TM_1$ lepton mixing in $S_4$ family symmetry, *Nucl. Phys. B* 881 (2014) 206-232 [arXiv:1312.4401]

[29] Stephen F. King, Thomas Neder, Lepton mixing predictions including Majorana phases from $\Delta(6n^2)$ flavour symmetry and generalised CP, *Phys. Lett. B* 736 (2014) 308-316

[30] Gui-Jun Ding, Stephen F. King, Thomas Neder, Generalised CP and $\Delta(6n^2)$ family symmetry in semi-direct models of leptons, *JHEP* 12 (2014) 007 [arXiv:1409.8005]

[31] Gui-Jun Ding, Ye-Ling Zhou, Lepton Mixing Parameters from $\Delta(48)$ Family Symmetry and Generalised CP, *JHEP* 06 (2014) 023 [arXiv:1404.0592]

[32] Cai-Chang Li, Gui-Jun Ding, Lepton Mixing in $A_5$ Family Symmetry and Generalized CP, *JHEP* 05 (2015) 100 [arXiv:1503.03711]

[33] C. Hagedorn, A. Meroni, and E. Molinaro, Lepton Mixing from $\Delta(3n^2)$ and $\Delta(6n^2)$ and CP, *Nucl. Phys. B* 891 (2015) 499-557 [arXiv:1408.7118]

[34] A. Di Iura, C. Hagedorn, and D. Meloni, Lepton mixing from the interplay of the alternating group $A_5$ and CP, *JHEP* 08 (2015) 037 [arXiv:1503.04140]

[35] Peter Ballett, Silvia Pascoli, Jessica Turner, Mixing angle and phase correlations from $A_5$ with generalised CP and their prospects for discovery, *Phys. Rev. D* 92 (2015) 093008 [arXiv:1503.07543]

[36] Jessica Turner, Predictions for Leptonic Mixing Angle Correlations and Non-trivial Dirac CP Violation from $A_5$ with Generalised CP Symmetry, *Phys. Rev. D* 92 (2015) 116007 [arXiv:1507.06224]

[37] Cai-Chang Li, Gui-Jun Ding, Deviation from Bimaximal Mixing and Leptonic CP Phases in $S_4$ Family Symmetry and Generalized CP, *JHEP* 08 (2015) 017 [arXiv:1408.0785]

[38] Cai-Chang Li, Chang-Yuan Yao, Gui-Jun Ding, Lepton Mixing Predictions from Infinite Group Series $D_{9n,3n}^{(1)}$ with Generalized CP, *JHEP* 05 (2016) 007 [arXiv:1601.06393]

[39] Duane A. Dicus, Shao-Feng Ge, Wayne W. Repko, Generalized Hidden $Z_2$ Symmetry of Neutrino Mixing, *Phys. Rev. D* 83 (2011) 093007 [arXiv:1012.2571]

[40] Shao-Feng Ge, Duane A. Dicus, Wayne W. Repko, $Z_2$ Symmetry Prediction for the Leptonic Dirac CP Phase, *Phys. Lett. B* 702 (2011) 220-223 [arXiv:1104.0602]

[41] Shao-Feng Ge, Duane A. Dicus, Wayne W. Repko, Residual Symmetries for Neutrino Mixing with a
Large $\theta_{13}$ and Nearly Maximal $\delta_D$, Phys. Rev. Lett. 108 (2012) 041801 [arXiv:1108.0964]

[42] C. Hagedorn, A. Meroni, L. Vitale, Mixing Patterns from the Groups $\Sigma(n\varphi)$, J. Phys. A: Math. Theor 47 (2014) 055201 [arXiv:1307.5308]

[43] W. Grimus, P.O. Ludl, Principal series of finite subgroups of SU(3), J. Phys. A 43 (2010) 445209 [arXiv:1006.0098]

[44] R. P. Bickerstaff, T. Damhus, A necessary and sufficient condition for the existence of real coupling coefficients for a finite group, Int. J. Quant. Chem. XXVII (1985) 381

[45] Mu-Chun Chen and et al., CP violation from finite groups, Nucl. Phys. B 883 (2014) 267C305 [arXiv:1402.0507]

[46] M. C. Gonzalez-Garcia, Michele Maltoni and Thomas Schwetz, Updated fit to three neutrino mixing: status of leptonic CP violation, JHEP 11 (2014) 052 [arXiv: 1409.5439]

[47] Particle Data Group collaboration, J. Beringer et al., Review of particle physics, Phys. Rev. D 86 (2012) 010001

[48] C. Jarlskog, Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation, Phys. Rev. Lett. 55 (1985) 1039