Positive periodic solutions for multiparameter nonlinear differential systems with delays

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Abstract

We establish several criteria for the existence of positive periodic solutions of the multi-parameter differential systems

\[
\begin{align*}
    u'(t) + a_1(t)g_1(u(t))u(t) &= \lambda b_1(t)f(u(t - \tau_1(t)), v(t - \zeta_1(t))), \\
    v'(t) + a_2(t)g_2(v(t))v(t) &= \mu b_2(t)g(u(t - \tau_2(t)), v(t - \zeta_2(t))),
\end{align*}
\]

where the functions \(g_1, g_2 : [0, \infty) \rightarrow [0, \infty]\) are assumed to be unbounded. The analysis in the paper relies on the classical fixed point index theory. Our main findings improve and complement some existing results in the literature.

MSC: 34B15

Keywords: Positive periodic solutions; Existence; Multiparameter systems; Fixed point

1 Introduction

Let \(\omega > 0\) be a constant. In this article we shall seek some criterion to guarantee that the multiparameter system

\[
\begin{align*}
    u'(t) = a_1(t)g_1(u(t))u(t) - \lambda b_1(t)f(u(t - \tau_1(t)), v(t - \zeta_1(t))), \\
    v'(t) = a_2(t)g_2(v(t))v(t) - \mu b_2(t)g(u(t - \tau_2(t)), v(t - \zeta_2(t)))
\end{align*}
\]

(1.1)

admits a positive \(\omega\)-periodic solution, where the functions \(a_i, b_i, \tau_i, \zeta_i \in C([0, \infty), [0, \infty])\) are \(\omega\)-periodic, and \(g_i \in C([0, \infty) \times [0, \infty], [0, \infty])\) are unbounded, \(i = 1, 2\). In addition, we assume that the nonlinear terms \(f, g \in C([0, \infty), [0, \infty])\) and \(\lambda, \mu\) are positive parameters.

Here a positive periodic solution of (1.1) means a solution \((u, v) \in E := X^2\) of (1.1) satisfying \(u > 0, v > 0\) on \([0, \omega]\), where

\[
X = \{x \in C([0, \omega], \mathbb{R}) : x(t + \omega) = x(t)\}.
\]
For this purpose, we assume studied the spectral problem
\[ L \] with positive parameter \( \lambda \) in the current paper, to further improve and generalize the results in the literature.

Reasons, we shall concentrate on the existence of positive periodic solutions for system (1.1) for the special case \( a_i(u) \equiv 1, i = 1, 2 \), where nonlinearities \( f(u, v) \) and \( g(u, v) \) were assumed to be nondecreasing, and only the case \( f(0, 0) > 0, g(0, 0) > 0 \) was treated. Therefore, we want to know whether or not (1.1) has a positive periodic solution under more relaxed assumption \( f(0, 0) = 0, g(0, 0) = 0 \). In view of above reasons, we shall concentrate on the existence of positive periodic solutions for system (1.1) in the current paper, to further improve and generalize the results in the literature.

For this purpose, we assume
\begin{itemize}
  \item[(C1)] \( a_i, b_i, \tau_i, \zeta_i \in C(\mathbb{R}, [0, \infty)) \) are \( \omega \)-periodic with \( \int_0^\omega a_i(t) \, dt > 0, \int_0^\omega b_i(t) \, dt > 0, i = 1, 2 \).
  \item[(C2)] There is \( l_i > 0 \) such that \( 0 < l_i \leq g_i(s) < \infty, s \in [0, \infty) \).
  \item[(C3)] \( f, g \in C([0, \infty) \times [0, \infty), [0, \infty)) \) with \( f(u, v) > 0, g(u, v) > 0 \) for \( (u, v) \neq (0, 0) \).
\end{itemize}
Remark 1.1 For other research work on periodic solutions of functional differential equations and systems, we refer the readers to [15–17] and references therein.

The remainder of the paper is arranged as follows. In Sect. 2, we introduce some preliminaries needed in our proof. Section 3 is devoted to stating and proving our main findings. Meanwhile, some related results and remarks will be given.

2 Preliminaries
Recall that $E = X^2$ is the Banach space defined as in Sect. 1. We first give the following lemma.

Lemma 2.1 Assume (C1)–(C3). If $(u, v) \in E$ is a solution of (1.1), then

- \begin{align*}
  u(t) &= \lambda \int_t^{t+\omega} G_1(t, s)b_1(s)f(u(s-\tau_1(s)), v(s-\zeta_1(s))) \, ds, \\
  v(t) &= \mu \int_t^{t+\omega} G_2(t, s)b_2(s)g(u(s-\tau_2(s)), v(s-\zeta_2(s))) \, ds,
\end{align*}

where

- \begin{align*}
  G_1(t, s) &= \frac{e^{-\int_s^t a_1(\theta)g_1(u(\theta))) \, d\theta}}{1 - e^{-\int_0^\tau a_1(\theta)g_1(u(\theta))) \, d\theta}}, \quad G_2(t, s) = \frac{e^{-\int_s^t a_2(\theta)g_2(u(\theta))) \, d\theta}}{1 - e^{-\int_0^\tau a_2(\theta)g_2(u(\theta))) \, d\theta}}, \quad t \leq s \leq t + \omega.
\end{align*}

Proof Multiplying the both sides of the first equation of (1.1) with $e^{-\int_s^t a_i(\theta)g_i(u(\theta))) \, d\theta}$, we can obtain

- \begin{align*}
  (u(t)e^{-\int_s^t a_1(\theta)g_1(u(\theta))) \, d\theta})' &= -\lambda b_1(t)f(u(t-\tau_1(t)), v(t-\zeta_1(t))) \cdot e^{-\int_s^t a_1(\theta)g_1(u(\theta))) \, d\theta}.
\end{align*}

Integrating above equation from $t$ to $t + \omega$ and by elementary calculation, we can easily get

- \begin{align*}
  u(t) &= \lambda \int_t^{t+\omega} G_1(t, s)b_1(s)f(u(s-\tau_1(s)), v(s-\zeta_1(s))) \, ds.
\end{align*}

Similar evaluation shows

- \begin{align*}
  v(t) &= \mu \int_t^{t+\omega} G_2(t, s)b_2(s)g(u(s-\tau_2(s)), v(s-\zeta_2(s))) \, ds. \quad \square
\end{align*}

Let $q > 0$ be a fixed constant. Then we can establish a series of lemmas required in the subsequent discussion.

Lemma 2.2 Assume (C1)–(C3). Let $\sigma_i = e^{-\int_0^\tau a_i(\theta) \, d\theta}$, $i = 1, 2$. Then for any $(u, v) \in E$ satisfying $(u, v) \geq (0, 0)$ and $\|(u, v)\| \leq q$,

- \begin{align*}
  0 < \frac{\sigma_i^{\sigma_i(q)}}{1 - \sigma_i^{\sigma_i(q)}} \leq G_i(t, s) \leq \frac{1}{1 - \sigma_i^{\sigma_i(q)}}, \quad i = 1, 2,
\end{align*}

(2.1)
where
\[
g_i^*(q) = \max_{0 \leq s \leq q} g_i(s), \quad g_{i*}(q) = \min_{0 \leq s \leq q} g_i(s), \quad i = 1, 2.
\]

**Proof** Clearly, for \((u, v) \in E \) with \((u, v) \geq (0, 0) \) and \(\| (u, v) \| \leq q \), we have \(0 \leq u \leq \| u \| \leq q \).

Thus,
\[
g_{i*}(q) \leq g_i(u) \leq g_{i*}(q),
\]
and then simple estimation shows (2.1) holds for \(i = 1 \). The case \(i = 2 \) is similar. \(\square\)

Defining for \(i = 1, 2,\)
\[
m_i(q) = \frac{\sigma_{i}^* g_i^*(q)}{1 - \sigma_{i}^* g_i^*(q)}, \quad M_i(q) = \frac{1}{1 - \sigma_{i}^* g_i^*(q)}, \quad \eta_i(q) = \frac{m_i(q)}{M_i(q)},
\]

Then it is not hard to verify \(\eta_i(q) \in (0, 1)\), and accordingly,
\[
\eta(q) := \min \{ \eta_1(q), \eta_2(q) \} \in (0, 1).
\]

Set
\[
P = \{(u, v) \in E : u(t) \geq 0, v(t) \geq 0, t \in [0, \omega]\},
\]
\[
K_q = \{(u, v) \in P : u(t) + v(t) \geq \eta(q) \| (u, v) \|, t \in [0, \omega]\},
\]

and for \(r > 0,\)
\[
\Omega_r = \{(u, v) \in K_q : \| (u, v) \| < r\}, \quad \partial \Omega_r = \{(u, v) \in K_q : \| (u, v) \| = r\}.
\]

Then \(P\) and \(K_q\) are cones in \(E\).

**Lemma 2.3** Assume \((C1)–(C3)\). Let \(0 < r \leq q\). Then for any \((u, v) \in \Omega_r,\)
\[
\frac{\sigma_{i}^* g_i^*(q)}{1 - \sigma_{i}^* g_i^*(q)} \leq G_i(t, s) \leq \frac{1}{1 - \sigma_{i}^* g_i^*(r)} \leq \frac{1}{1 - \sigma_{i}^* g_i^*(r)}, \quad i = 1, 2.
\]

**Proof** Similar to the proof of Lemma 2.2, we obtain for \(t \leq s \leq t + \omega,\)
\[
\frac{\sigma_{i}^* g_i^*(r)}{1 - \sigma_{i}^* g_i^*(r)} \leq G_i(t, s) \leq \frac{1}{1 - \sigma_{i}^* g_i^*(r)}, \quad i = 1, 2.
\]

Moreover, since \(\varphi(t) := \frac{\sigma_{i}^*}{1 - \sigma_{i}^*}\) and \(\psi(t) := \frac{1}{1 - \sigma_{i}^*}\) are strictly decreasing on \([0, \infty)\), one can easily see that (2.2) holds true. \(\square\)

Define, for given \((u, v) \in E,\)
\[
T_{\lambda, \mu}(u, v)(t) = (A_{\lambda}(u, v)(t), B_{\mu}(u, v)(t))
\]
where

\[ A_\lambda(u, v)(t) = \lambda \int_t^{t+\omega} G_1(t, s)b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s))) \, ds \]

and

\[ B_\mu(u, v)(t) = \mu \int_t^{t+\omega} G_2(t, s)b_2(s)g(u(s - \tau_2(s)), v(s - \xi_2(s))) \, ds. \]

Then we have

**Lemma 2.4** Assume (C1)–(C3) and \( 0 < r \leq q \). Then \( T_{\lambda, \mu}(\Omega_r) \subseteq K_q \) and \( T_{\lambda, \mu} : \Omega_r \to K_q \) is completely continuous.

**Proof** For \((u, v) \in \Omega_r\), we can deduce from Lemma 2.3 that

\[
A_\lambda(u, v)(t) = \lambda \int_t^{t+\omega} G_1(t, s)b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s))) \, ds \\
\leq \lambda \frac{1}{1 - \sigma_1^{\sigma_1(r)}} \int_0^\omega b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s))) \, ds,
\]

which yields

\[
\|A_\lambda(u, v)\| \leq \lambda \frac{1}{1 - \sigma_1^{\sigma_1(r)}} \int_0^\omega b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s))) \, ds.
\]

Meanwhile, (2.2) implies

\[
A_\lambda(u, v)(t) \geq \lambda \frac{\sigma_1^{\sigma_1(r)}}{1 - \sigma_1^{\sigma_1(r)}} \int_0^\omega b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s))) \, ds \\
= \lambda \frac{\sigma_1^{\sigma_1(r)} (1 - \sigma_1^{\sigma_1(r)})}{1 - \sigma_1^{\sigma_1(r)}} \int_0^\omega b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s))) \, ds \\
\geq \frac{\sigma_1^{\sigma_1(r)} (1 - \sigma_1^{\sigma_1(r)})}{1 - \sigma_1^{\sigma_1(r)}} \|A(u, v)\| \\
\geq \eta_1(q) \|A_\lambda(u, v)\| \\
\geq \eta(q) \|A_\lambda(u, v)\|. \tag{2.3}
\]

In an analogous manner, we get

\[
B_\mu(u, v)(t) \geq \eta(q) \|B_\mu(u, v)\|, \quad (u, v) \in \Omega_r.
\]

Hence \( T_{\lambda, \mu}(\Omega_r) \subseteq K_q \). The completely continuity of \( T_{\lambda, \mu} \) is obvious. \( \square \)

It is obvious that if \((u, v)\) is a fixed point of the completely continuous operator \( T_{\lambda, \mu} \) in \( K_q \), then \((u, v)\) is a positive periodic solution of (1.1). We conclude this section by giving the main tool employed in proving our main results.
Lemma 2.5 ([18, 19]) Assume $E$ is a Banach space and $K \subseteq E$ is a cone. For $r > 0$, let $K_r = \{ u \in K : \|u\| < r \}$ and $\partial K_r = \{ u \in K : \|u\| = r \}$. Suppose $T : \bar{K} \rightarrow K$ is a completely continuous operator satisfying $Tu \neq u$, $u \in \partial K_r$. Then

(i) If $\|Tu\| < \|u\|$, $u \in \partial K_r$, then $i(T, \bar{K}_r, K) = 1$;
(ii) If $\|Tu\| > \|u\|$, $u \in \partial K_r$, then $i(T, \bar{K}_r, K) = 0$.

3 Main results

Let

$$f_0 = \lim_{(u,v) \rightarrow 0} \frac{f(u,v)}{u + v}, \quad g_0 = \lim_{(u,v) \rightarrow 0} \frac{g(u,v)}{u + v}.$$  

Theorem 3.1 Assume (C1)–(C3) hold and $f_0 = 0 = g_0$. Then for every $q > 0$, there is a constant $\gamma_q > 0$ such that for all $\lambda, \mu > \gamma_q$, system (1.1) admits a positive periodic solution $(u, v)$ satisfying $\|u, v\| \leq q$.

Proof Choose $r_1 = q$ and define

$$\psi_f(q) = \min \{f(u,v) : \eta(q)q \leq u + v \leq q\},$$

$$\psi_g(q) = \min \{g(u,v) : \eta(q)q \leq u + v \leq q\}.$$  

Take

$$\gamma_q = q \cdot \max \left\{ \frac{1}{2\psi_f(q)m_1(q)\int_0^1 b_1(s)\,ds}, \frac{1}{2\psi_g(q)m_2(q)\int_0^1 b_2(s)\,ds} \right\}.$$  

By Lemma 2.4, we know $T_{\lambda,\mu}(\bar{Q}_q) \subseteq K_q$ and $T_{\lambda,\mu} : \bar{Q}_q \rightarrow K_q$ is completely continuous. Fix $\lambda, \mu > \gamma_q$. Then for $(u,v) \in \partial Q_q$, we have $\eta(q)q \leq u + v \leq q$, and so

$$A_\lambda(u,v)(t) = \lambda m_1(q) \int_0^1 b_1(s)f(u(s - \tau_1(s)), v(s - \xi_1(s)))\,ds$$

$$= \lambda m_1(q)\psi_f(q) \int_0^1 b_1(s)\,ds$$

$$> \frac{q}{2} = \frac{\|u,v\|}{2},$$

which implies

$$\|A_\lambda(u,v)\| > \frac{\|u,v\|}{2}, \quad (u,v) \in \partial Q_q.$$  

Similarly,

$$\|B_\mu(u,v)\| > \frac{\|u,v\|}{2}, \quad (u,v) \in \partial Q_q.$$  

Hence $\|T_{\lambda,\mu}(u,v)\| > \|u,v\|$ on $\partial Q_q$, and then Lemma 2.5 gives $i(T_{\lambda,\mu}, \bar{Q}_q, K_q) = 0$.

On the other hand, since $f_0 = g_0 = 0$, there exists a constant $r_2$ with $0 < r_2 < q$, such that for $(u,v)$ satisfying $0 < u + v \leq r_2$,

$$f(u,v) \leq \varepsilon(u + v), \quad g(u,v) \leq \varepsilon(u + v),$$
where $\varepsilon > 0$ is a constant satisfying
\[
\frac{2\lambda \varepsilon}{1 - \sigma_1^{\frac{1}{q}(q)}} \int_0^\infty b_1(s) \, ds < 1, \quad \frac{2\mu \varepsilon}{1 - \sigma_2^{\frac{1}{q}(q)}} \int_0^\infty b_2(s) \, ds < 1.
\] (3.1)

For $(u, v) \in \partial \Omega_{r_2}$, we can deduce by (2.2) and (3.1) that
\[
A_\lambda(u, v)(t) \leq \lambda \frac{1}{1 - \sigma_1^{\frac{1}{q}(q)}} \int_t^{t+\tau_0} b_1(s)f(u(s - \tau_1(s)), v(s - \zeta_1(s))) \, ds
\]
\[
\leq \lambda \frac{\varepsilon}{1 - \sigma_1^{\frac{1}{q}(q)}} \cdot \int_0^\infty b_1(s) \, ds \cdot \|u, v\|
\]
\[
< \frac{\|u, v\|}{2},
\]
and hence
\[
\|A_\lambda(u, v)\| < \frac{\|u, v\|}{2}, \quad (u, v) \in \partial \Omega_{r_2}.
\]

In an analogous way, we get
\[
\|B_\mu(u, v)\| < \frac{\|u, v\|}{2}, \quad (u, v) \in \partial \Omega_{r_2}.
\]

Thus $\|T_{\lambda, \mu}(u, v)\| < \|u, v\|$ on $\partial \Omega_{r_2}$. Lemma 2.5 ensures $i(T_{\lambda, \mu}, \bar{\Omega}_{r_2}, K_q) = 1$.

Consequently, $i(T_{\lambda, \mu}, \bar{\Omega}_q \setminus \Omega_{r_2}, K_q) = -1$. Therefore, $T_{\lambda, \mu}$ possesses a fixed point $(u, v)$ in $\bar{\Omega}_q \setminus \Omega_{r_2}$, and system (1.1) has a positive periodic solution $(u, v)$ with $\|u, v\| \leq q$. \hfill \Box

**Theorem 3.2** Assume (C1)–(C3) hold and $f_0 = \infty$. Then for every $q > 0$, there is a constant $\gamma_q > 0$ such that for all $\lambda, \mu < \gamma_q$, system (1.1) admits a positive periodic solution $(u, v)$ satisfying $\|u, v\| \leq q$.

**Proof** Fix $r_1 = q$ and set
\[
\Psi_f(q) = \max \{f(u, v) : \eta(q)q \leq u + v \leq q\},
\]
\[
\Psi_g(q) = \max \{g(u, v) : \eta(q)q \leq u + v \leq q\}.
\]

Define
\[
\gamma_q = q \cdot \min \left\{ 1, \frac{1}{2\Psi_f(q)M_1(q) \int_0^\infty b_1(s) \, ds}, \frac{1}{2\Psi_g(q)M_2(q) \int_0^\infty b_2(s) \, ds} \right\}.
\]

By Lemma 2.4, $T_{\lambda, \mu}(\bar{\Omega}_q) \subseteq K_q$ and $T_{\lambda, \mu} : \bar{\Omega}_q \rightarrow K_q$ is completely continuous. Thus, for fixed $\lambda, \mu < \gamma_q$ and $(u, v) \in \partial \Omega_q$,
\[
A_\lambda(u, v)(t) \leq \lambda M_1(q) \int_0^\infty b_1(s)f(u(s - \tau_1(s)), v(s - \zeta_1(s))) \, ds
\]
\[
= \lambda M_1(q) \psi_f(q) \cdot \int_0^\infty b_1(s) \, ds
\]
\[
< \frac{q}{2} = \frac{\|u, v\|}{2},
\]
and then
\[ \|A_\lambda(u, v)\| < \frac{\|u, v\|}{2}, \quad (u, v) \in \partial \Omega_q. \]

By a similar argument, we can also obtain
\[ \|B_\mu(u, v)\| < \frac{\|u, v\|}{2}, \quad (u, v) \in \partial \Omega_q. \]

Therefore, \( \|T_\lambda, \mu(u, v)\| < \|u, v\| \) for \((u, v) \in \partial \Omega_q\). Using Lemma 2.5 again, we can easily get
\[ \iota(T_\lambda, \mu, \bar{\Omega}_{r_2}, K_q) = 1. \]

By the assumption \( f_0 = \infty \), there exists a constant \( r_2 \in (0, q) \), such that for \((u, v)\) satisfying \( 0 < u + v \leq r_2 \),
\[ f(u, v) \geq \Upsilon(u + v), \]

where \( \Upsilon > 0 \) satisfies
\[ \lambda \Upsilon \eta(q) \frac{\sigma_1^{q_1(q)}}{1 - \sigma_1^{q_1(q)}} \int_0^\infty b_1(s) ds > 1. \tag{3.2} \]

Thus for \((u, v) \in \partial \Omega_{r_2}\), we get by (2.2) and (3.2) that
\[
A_\lambda(u, v)(t) \geq \lambda \frac{\sigma_1^{q_1(r_2)}}{1 - \sigma_1^{q_1(r_2)}} \int_0^\infty b_1(s)f(u(s - \tau_1(s)), v(s - \zeta_1(s))) ds
\]
\[ \geq \lambda \Upsilon \eta(q) \frac{\sigma_1^{q_1(q)}}{1 - \sigma_1^{q_1(q)}} \int_0^\infty b_1(s) ds \cdot \|u, v\| \]
\[ > \|u, v\|, \]

which means \( \|A_\lambda(u, v)\| > \|u, v\| \) on \( \partial \Omega_{r_2} \). Hence
\[
\|T_\lambda, \mu(u, v)\| \geq \|A_\lambda(u, v)\| > \|u, v\|, \quad \text{for} \quad (u, v) \in \partial \Omega_{r_2},
\]

and Lemma 2.5 again implies \( \iota(T_\lambda, \mu, \bar{\Omega}_{r_2}, K_q) = 0 \).

Consequently, \( \iota(T_\lambda, \mu, \bar{\Omega}_q \setminus \Omega_{r_2}, K_q) = 1 \). Thus, \( T_\lambda, \mu \) has a fixed point \((u, v)\) in \( \bar{\Omega}_q \setminus \Omega_{r_2} \), and (1.1) has a positive periodic solution \((u, v)\) with \( \|u, v\| \leq q \). \( \square \)

Similarly to Theorems 3.1 and 3.2, we can prove the following

**Theorem 3.3** Assume (C1)–(C3) and \( g_0 = \infty \). Then for every \( q > 0 \), there is a constant \( \gamma_q > 0 \) such that for all \( \lambda, \mu < \gamma_q \), system (1.1) admits a positive periodic solution \((u, v)\) satisfying \( \|u, v\| \leq q \).

**Remark 3.1** Clearly, the results of Theorems 3.1–3.3 generalize and complement the corresponding ones in [7, 9, 12–14].
To illustrate our main findings, we may choose \( \omega = 2\pi \) and \( \tau_i \equiv 0, \zeta_i \equiv 0 \) \((i = 1, 2)\) in the subsequent discussion. Let

\[
\begin{align*}
  a_1(t) &= \sin t + 1, \quad a_2(t) = \sin t + 2, \quad t \in [0, 2\pi], \\
  b_1(t) &= \cos t + 2, \quad b_2(t) = \cos t + 1, \quad t \in [0, 2\pi].
\end{align*}
\]

Then it is not hard to check that (C1) is satisfied. Moreover, define

\[
  g_1(s) = \exp^t, \quad g_2(s) = 2\exp^t, \quad s \in [0, \infty),
\]

then there are constants \( l_1 = 1 \) and \( l_2 = 2 \) such that

\[
  0 < 1 = l_1 \leq g_1(s) < \infty, \quad \quad 0 < 2 = l_2 \leq g_2(s) < \infty, \quad s \in [0, \infty).
\]

Hence (C2) is also satisfied.

**Example 3.1** For \((u, v) \in [0, \infty) \times [0, \infty)\), let

\[
  f(u, v) = 3(u + v)^2(u^2 + v^2 + 1)^2, \quad g(u, v) = 2(u + v)^4(u^2 + v^2 + 5)^2.
\]

Then \( f, g \in C([0, \infty) \times [0, \infty), [0, \infty)) \) with \( f(u, v) > 0, g(u, v) > 0 \) for \((u, v) \neq (0, 0)\). Thus (C3) holds true. Furthermore, simple calculation gives \( f_0 = 0 = g_0 \). Consequently, the results of Theorem 3.1 are valid.

**Example 3.2** We shall follow the same notations and definitions as before. Let us redefine

\[
  f(u, v) = \sqrt{u + v} \cdot (u^2 + v^2 + 1)^2, \quad (u, v) \in [0, \infty) \times [0, \infty).
\]

Clearly, \( f \) verifies (C3). Moreover, it is not difficult to see \( f_0 = \infty \), and accordingly the results of Theorem 3.2 are also valid.

At the end of the section, we list some related results and remarks.

Let us consider the multiparameter differential systems

\[
\begin{align*}
  u'(t) &= -a_1(t)g_1(u(t))u(t) + \lambda b_1(t)f(u(t - \tau_1(t)), v(t - \zeta_1(t))), \\
  v'(t) &= -a_2(t)g_2(v(t))v(t) + \mu b_2(t)g(u(t - \tau_2(t)), v(t - \zeta_2(t))),
\end{align*}
\]

where \( \lambda, \mu > 0 \) are parameters. Under the same assumptions as before, one can check that system (3.3) is equivalent to

\[
\begin{align*}
  u(t) &= \int_t^{t+\omega} G_1(t, s)b_1(s)f(u(s - \tau_1(s)), v(s - \zeta_1(s)))\, ds, \\
  v(t) &= \int_t^{t+\omega} G_2(t, s)b_2(s)g(u(s - \tau_2(s)), v(s - \zeta_2(s)))\, ds,
\end{align*}
\]

where

\[
\begin{align*}
  G_1(t, s) &= \frac{e^{\int_t^s a_1(\theta)g_1(u(\theta))\, d\theta}}{e^{\int_0^{t+\omega} a_1(\theta)g_1(u(\theta))\, d\theta} - 1}, \quad G_2(t, s) = \frac{e^{\int_t^s a_2(\theta)g_2(v(\theta))\, d\theta}}{e^{\int_0^{t+\omega} a_2(\theta)g_2(v(\theta))\, d\theta} - 1}, \quad t \leq s \leq t + \omega.
\end{align*}
\]
Furthermore, by a similar argument as above, it is not difficult to see that the results of Theorems 3.1–3.3 remain true for system (3.3).

**Remark 3.2** It is worth remarking that, under some reasonable assumptions, the results of the paper are still valid for the more general coupled systems

\[ u_i'(t) + a_i(t)g_i(u_i(t))u_i(t) = \lambda_i b_i(t)f_i(u_1(t - \tau_{1i}(t)), \ldots, u_n(t - \tau_{ni}(t))), \quad i = 1, 2, \ldots, n \]

and

\[ u_i'(t) - \lambda_i b_i(t)f_i(u_1(t - \tau_{1i}(t)), \ldots, u_n(t - \tau_{ni}(t))), \quad i = 1, 2, \ldots, n. \]

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Not applicable.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

RC analyzed and proved the main results, and was a major contributor in writing the manuscript. XL checked the English grammar and typing errors in the full text. All authors read and approved the final manuscript.

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