On averages of sums over regular integers modulo $n$

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Abstract

In this paper, we provide formulas for partial sums of weighted averages over regular integers modulo $n$ of the gcd-sum function with any arithmetic function. Many interesting applications of the results are also given.

1 Introduction and results

Let $\gcd(k, n)$ be the largest common divisor of the positive integers $k$ and $n$. An integer $k$ is called regular (mod $n$) if there exists an integer $x$ such that $k^2 x \equiv k \pmod{n}$, namely, the residue class of $k$ is a regular element of the ring $\mathbb{Z}_n$ of residue classes (mod $n$). Then, $k$ is regular (mod $n$) if and only if $\gcd(k, n)$ is a unitary divisor of $n$. We recall that $d$ is a unitary divisor of $n$ if $d | n$ and $\gcd(d, n/d) = 1$, notation $d || n$. The set $\text{Reg}_n$ is defined by

$$\text{Reg}_n = \{k \in \mathbb{N} : 1 \leq k \leq n, \ k \text{ is regular (mod } n\}$$

and $\rho(n)$ denotes the number of the elements in $\text{Reg}_n$. Throughout the paper we use the notations: Let $f$ and $g$ be two arithmetic functions. We define the Dirichlet convolution and the unitary convolution of $f$ and $g$ by $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$ and $(f \star g)(n) = \sum_{d|n} f(d)g(n/d)$, for every positive integer $n$, respectively. The arithmetic functions $1(n)$ and $\text{id}(n)$ are defined by $1(n) = 1$ and $\text{id}(n) = n$ for all $n$. Let $\phi(n)$ be the Euler totient function and let $\tau(n)$ be number of distinct divisors of $n$, namely $\tau(n) = \sum_{d|n} 1$. Let $\sigma_k(n)$ denote the sum of $k$-th powers of divisors of $n$, namely $\sigma_k(n) = \sum_{d|n} d^k$. The Jordan totient function is defined as $\phi_s(n) = n^s \prod_{p|n} (1 - 1/p^s)$ or $\sum_{d|n} d^s \mu(n/d)$ which is a generalization of the Euler totient function. The Möbius function is defined as follows

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes}; \\ 0 & \text{if } n \text{ is divisible by the square of a prime}. \end{cases}$$

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The letter $p$ always stands for a prime and $\zeta$ is the Riemann zeta function. The gcd-sum function, which is also known as Pillai’s arithmetic function, is defined by

$$P(n) = \sum_{k=1}^{n} \gcd(k, n).$$

In 1937, Pillai [7] derived the identity $P(n) = \sum_{d|n} d\phi(n/d)$. For a nice survey on the gcd-sum function see [13]. The gcd-sum function over regular integers modulo $n$ was first introduced by Tóth [12] showing that $\tilde{P}(n) = \sum_{k \in \text{Reg}_{n}} \gcd(k, n)$. He proved that

$$\tilde{P}(n) = \sum_{d|n} d\phi\left(\frac{n}{d}\right), \quad (1)$$

and that

$$\sum_{n \leq x} \tilde{P}(n) = \frac{x^2}{2\zeta(2)} (K_1 \log x + K_2) + O\left(x^{3/2} \exp\left(-C \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right), \quad (2)$$

where $C$ is a positive constant and

$$K_1 = \prod_{p} \left(1 - \frac{1}{p(p+1)}\right), \quad (3)$$

$$K_2 = K_1 \left(2\gamma - \frac{1}{2} - 2 \frac{\zeta'(2)}{\zeta(2)}\right) - \sum_{n \geq 1} \frac{\mu(n)}{n} \frac{(\log n - \alpha(n) + 2\beta(n))}{n \psi(n)}, \quad (4)$$

with $\gamma$ is the Euler constant and

$$\alpha(n) = \sum_{p|n} \frac{\log n}{p - 1}, \quad \beta(n) = \sum_{p|n} \frac{\log p}{p^2 - 1}.$$

Here the function $\psi(n)$ is well known as the Dedekind function defined by $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$. Recently, the error term in Eq (2) has been improved by Zhang and Zhai [16], under the Riemann hypothesis. They proved that

$$\sum_{n \leq x} \tilde{P}(n) = \frac{x^2}{2\zeta(2)} (K_1 \log x + K_2) + O\left(x^{15/11+\varepsilon}\right),$$

for any sufficiently small positive number $\varepsilon > 0$. In [11], Apostol and Tóth introduced the multidimensional generalization of $\rho$. They established identities for the power sums of regular integers (mod $n$), involving the Bernoulli polynomials, the Gamma function and the cyclotomic polynomials. More recently, Kiuchi and Matsuoka [5, Theorem 3.1] proved that, for any fixed positive integer $r$ and any arithmetic function $f$, we have

$$\frac{1}{n^r} \sum_{k \in \text{Reg}_n} f(\gcd(k, n))k^r = \frac{f(n)}{2} + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{d|n} f\left(\frac{n}{d}\right) \phi_{1-2m}(d),$$

where $B_m$ is the $m$th Bernoulli number.
where $B_m$ are the Bernoulli numbers. In the present paper, we give formulas for the partial sums of weighted averages over regular integers (mod $n$) of the gcd-sum function with any arithmetic function $f$. We prove that

**Theorem 1.1.** For any arithmetic function $f$, any sufficiently large positive number $x > 2$ and fixed positive integer $r$, we have

$$U_f(x) := \sum_{n \leq x} \frac{1}{n^{r+1}} \sum_{k \in \text{Reg}_n} f(\gcd(k,n)) k^r =$$

$$\frac{1}{2} \sum_{n \leq x} f(n) + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{d \leq x \atop \gcd(d,\ell)=1} \frac{f(\ell) \phi_{1-2m}(d)}{\ell} \ .$$

When $f$ satisfies the condition $f(n) \neq 0 \ \forall n \in \mathbb{N}$, we have

$$V_f(x) := \sum_{n \leq x} \frac{1}{n^r f(n)} \sum_{k \in \text{Reg}_n} f(\gcd(k,n)) k^r =$$

$$\frac{x}{2} - \frac{\theta(x)}{2} - \frac{1}{4} + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{d \leq x \atop \gcd(d,\ell)=1} \frac{f(\ell) \phi_{1-2m}(d)}{f(d\ell)},$$

where $\theta(x) = x - \lfloor x \rfloor - 1/2$.

This theorem is actually easily deduced from the results in [5], but the advantage of it is that this theorem is a source of various interesting number-theoretic formulas. Next theorem provides applications of the above results for various multiplicative functions such as $\text{id}, \mu, \tau$ and $\phi_2$.

**Theorem 1.2.** Under the hypotheses of Theorem 1.1 we have

$$U_{\text{id}}(x) = \frac{K_1}{(r+1)\zeta(2)} x \log x$$

$$+ \left( \frac{1}{2} - \frac{K_1}{2(r+1)\zeta(2)} + \frac{K_2}{(r+1)\zeta(2)} + \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} C_m \right) x$$

$$+ O_r \left( x^{1/2} \exp \left( -C \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right) \right) ,$$

where

$$C_m := \prod_p \left( 1 - \frac{(p-1)(p^{2m-1} - 1)}{p(p^{2m+1} - 1)} \right) ,$$

and $K_1, K_2$ are defined by Eqs (3) and (4) respectively,
\[ U_r(x) = \frac{x}{(r+1)\zeta(2)} \prod_p \left(1 + \frac{(2p^2-1)p}{(p-1)^2(p+1)^3}\right) + O_r \left((\log x)^5\right). \tag{9} \]

Moreover, we have
\[ V_{\phi_s}(x) = \left(\frac{1}{2} + \frac{1}{r+1} \prod_p \left(1 + \frac{1}{(p+1)^2}\right)\right)x \]
\[ + \frac{x}{r+1} \sum_{m=0}^{[r/2]} \left(\frac{r+1}{2m}\right) \cot^{2m} \prod_p \left(1 - \frac{p(p^{2m-1} - 1)}{(p+1)(p^{2m^2} - 1)}\right) + O_r \left((\log x)^4\right). \tag{10} \]

## 2 Auxiliary results

Before we proceed the proof of our results, we need some auxiliary lemmas.

**Lemma 2.1.** For any positive integer \( t > 1 \), we have
\[ \sum_{1 \leq n \leq x \atop (n,t)=1} 1 = \frac{\phi(t)}{t} x - \sum_{d \mid t} \mu(d) \left(\frac{x}{d}\right), \tag{11} \]
where \( \theta(x) = x - \lfloor x \rfloor - 1/2 \). Let \( \tau^* \) be the number of square-free divisors of \( t \). Then, we have
\[ \sum_{1 \leq n \leq x \atop (n,t)=1} \frac{\phi(n)}{n} = \frac{t\phi(t)}{\zeta(2)\phi_2(t)} x + O \left(\tau^*(t) \log x\right). \tag{12} \]

For any positive integer \( s \geq 2 \), we have
\[ \sum_{1 \leq n \leq x \atop (n,t)=1} \frac{\phi_{-s}(n)}{n} = O \left(\frac{\phi_s(t)}{t^s} \log x + \frac{\sigma(t)}{t} \log x\right). \tag{13} \]
and
\[ \sum_{1 \leq n \leq x \atop (n,t)=1} \frac{\phi_{-1}(n)}{n} = O \left(\frac{\phi(t)}{t} (\log x)^2 + \frac{\phi(t)}{t} \log \log(3t) \log x + \frac{\phi(t)}{t} \log x + \tau(t) \log x\right). \tag{14} \]

**Proof.** For Eq. (11) see [5, Lemma 2.1]. The proof of Eq. (12) can be found in [10] or [9] Chapter 1, Section 1.24]. Eqs. (13) and (14) follow by the Abel identity and using the fact that
\[ \sum_{1 \leq n \leq x \atop (n,t)=1} \phi_{-s}(n) = O \left(\frac{\phi_s(t)}{t^s} x + \frac{\sigma(t)}{t}\right), \quad (s \geq 2) \]
and
\[\sum_{n \leq x} \phi_1(n) = O\left(\frac{\phi(t)}{t} x \log x + x \frac{\phi(t)}{t} \log \log(3t) + x\tau(t)\right).\]

The proof of the above two sums can be found in \[5, Lemma 2.2, Lemma 2.3\].

\textbf{Lemma 2.2.} For \(x \geq 2\), we have
\[\sum_{n \leq x} \mu(n) \frac{n}{n} = O\left(\exp\left(-B(\log x)^{\frac{3}{5}}(\log \log x)^{-\frac{1}{5}}\right)\right),\]  
(15)
\[\sum_{n \leq x} \phi(n) \frac{n}{n^2} = \frac{1}{\zeta(2)} \log x + \frac{1}{\zeta(2)} \left(\gamma - \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(\frac{\log x}{x}\right),\]  
(16)

where \(B > 0\) and \(\gamma\) is Euler’s constant. Furthermore, we have
\[\sum_{n \leq x} \tau(n) \frac{n}{n} = \frac{1}{2}(\log x)^2 + 2\gamma \log x + (2\gamma - 1) + O\left(x^{285/416}(\log x)^{26947/8320}\right),\]  
(17)
\[\sum_{n \leq x} \tau^*(n) \frac{n}{n} = \frac{1}{2\zeta(2)}(\log x)^2 + \left(2\gamma + \frac{\zeta'(2)}{\zeta(2)}\right) \frac{\log x}{\zeta(2)} + \left(2\gamma - 1 + \frac{\zeta'(2)}{\zeta(2)}\right) \frac{1}{\zeta(2)} + O\left(x^{-1/2} \exp\left(A(\log x)^{3/5}(\log \log x)^{-1/5}\right)\right),\]  
(18)

and
\[\sum_{n \leq x} \tau^2(n) \frac{n}{n} = \frac{(\log x)^4}{4\pi^2} + \left(\frac{B_1}{3} + \frac{1}{\pi^2}\right) (\log x)^3 + B_1 (\log x)^2 + C \log x + O(1),\]  
(19)

where \(A > 0\) and \(B_1, C\) are constants.

\textbf{Proof.} For Eqs. (15) and (16) see \[3\] and \[4, Lemma 2.1\]. Eq. (17) follows at once by Abel’s identity and the following formula
\[\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x),\]  
(20)

with \(\Delta(x) = O\left(x^{\theta + \varepsilon}\right)\). It is known that \(1/4 \leq \theta \leq 131/416\), and that the best upper bound we have for \(\Delta(x)\) is \(O\left(x^{131/416}(\log x)^{26947/8320}\right)\), which has been given by Huxley \[2\].

Eqs. (18) and (19) follow from the Abel identity and the formulas
\[\sum_{n \leq x} \tau^*(n) = \frac{x}{\zeta(2)} \left(\log x + 2\gamma - 1 + \frac{\zeta'(2)}{\zeta(2)}\right) + O\left(x^{1/2} \exp\left(-A_1(\log x)^{3/5}(\log \log x)^{-1/5}\right)\right)\]
and
\[ \sum_{n \leq x} \tau^2(n) = \frac{1}{\pi^2} x (\log x)^3 + B_1 x (\log x)^2 + C x + D + O \left( x^{1/2 + \epsilon} \right), \quad (21) \]

where \( A_1 > 0 \) and \( B_1, C, D \) are constants and \( \epsilon > 0 \). The proof of the last two sums can be found in [11], [15] and [9, Chapter 2, Section II.13].

**Lemma 2.3.** For any sufficiently large positive number \( x > 2 \), we have
\[
\sum_{\ell > x} \frac{\tau(\ell)}{\ell^2} = O \left( \frac{\log x}{x} \right),
\]
\[
\sum_{\ell > x} \frac{\tau^2(\ell)}{\ell^2} = O \left( \frac{(\log x)^3}{x} \right),
\]
\[
\sum_{\ell > x} \frac{\tau(\ell)}{\ell^3} = O \left( \frac{(\log x)^2}{x^2} \right).
\]

**Proof.** In order to prove Eq. (22), we write
\[
\sum_{\ell > x} \frac{\tau(\ell)}{\ell^2} = 2 \sum_{\ell > x} \tau(\ell) \int_{\ell}^{\infty} \frac{dt}{t^3} = 2 \int_{x}^{\infty} \sum_{x < t \leq \ell} \frac{\tau(\ell)}{t^3} dt.
\]
From Eq. (20), we get the desired result. The treatment of Eqs. (23) and (24) is similar as in the above and with the help of Eqs. (20) and (21).

### 3 Proofs

#### 3.1 Proof of Theorem 1.1

We recall that
\[
\frac{1}{n^r} \sum_{k \in \text{Reg}_n} f(\gcd(k, n))k^r = \frac{f(n)}{2} + \frac{1}{r + 1} \sum_{m=0}^{[r/2]} \binom{r + 1}{2m} B_{2m} \sum_{d | n} f \left( \frac{n}{d} \right) \phi_{1 - 2m}(d).
\]

Then, we have
\[
U_f(x) = \frac{1}{2} \sum_{n \leq x} \frac{f(n)}{n} + \frac{1}{r + 1} \sum_{m=0}^{[r/2]} \binom{r + 1}{2m} B_{2m} \sum_{n \leq x} \frac{1}{n} \sum_{d | n} f \left( \frac{n}{d} \right) \phi_{1 - 2m}(d).
\]
Here, the condition \( d \mid | n \) can be translated to \( d \mid n \) and \( \gcd(d, n/d) = 1 \). Then the last sum on the right-hand side above becomes

\[
\sum_{n \leq x} \frac{1}{n} \sum_{d \mid n} f\left(\frac{n}{d}\right) \phi_{1-2m}(d) = \sum_{n \leq x} \frac{1}{n} \sum_{d \mid n, \gcd(d,n/d)=1} f\left(\frac{n}{d}\right) \phi_{1-2m}(d)
\]

\[
= \sum_{n \leq x} \frac{1}{n} \sum_{\ell = d, \gcd(d,\ell)=1} f(\ell) \phi_{1-2m}(d)
\]

\[
= \sum_{d \ell \leq x, \gcd(d,\ell)=1} \frac{1}{d \ell} f(\ell) \phi_{1-2m}(d).
\]

Hence, we get

\[
U_f(x) = \frac{1}{2} \sum_{n \leq x} \frac{f(n)}{n} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{d \ell \leq x, \gcd(d,\ell)=1} \frac{f(\ell) \phi_{1-2m}(d)}{d}.
\]

This completes the proof of Eq. (\ref{eq:5}). Similarly, if \( f(n) \neq 0 \) for all \( n \in \mathbb{N} \), we find that

\[
V_f(x) = \frac{1}{2} \sum_{n \leq x} 1 + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{d \ell \leq x, \gcd(d,\ell)=1} \frac{f(\ell) \phi_{1-2m}(d)}{f(d \ell)}.
\]

Using the fact that

\[
\sum_{n \leq x} 1 = x - \theta(x) - \frac{1}{2}, \quad (25)
\]

where \( \theta(x) = x - [x] - 1/2 \), we deduce

\[
V_f(x) = \frac{x}{2} - \frac{\theta(x)}{2} - \frac{1}{4} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{d \ell \leq x, \gcd(d,\ell)=1} \frac{f(\ell) \phi_{1-2m}(d)}{f(d \ell)},
\]

as desired.

### 3.2 Proof of Theorem 1.2

By Eq. (\ref{eq:5}) of Theorem 1.1 with \( f \) replaced by \( \text{id} \), we find that

\[
U_{\text{id}}(x) = \frac{1}{2} \sum_{n \leq x} 1 + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{d \ell \leq x, \gcd(d,\ell)=1} \frac{\phi_{1-2m}(d)}{d}.
\]
Using Eq. (25), we get

\[
U_{id}(x) = \frac{x}{2} - \frac{\theta(x)}{2} - \frac{1}{4} + \frac{1}{r+1} \sum_{d \leq x} \frac{\phi(d)}{d} \sum_{\ell \leq x \atop \gcd(d,\ell) = 1} \frac{\phi(\ell)}{\ell} + \frac{1}{r+1} \sum_{m=1}^{\lfloor r/2 \rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{d \leq x \atop \gcd(d,\ell) = 1} \frac{\phi_{1-2m}(d)}{d}.
\]

The first summation on the right-hand side above is just the case \(m = 0\) in the second summation, because \(B_0 = 1\). Put

\[
A_0(x) := \sum_{d \leq x \atop \gcd(d,\ell) = 1} \frac{\phi(d)}{d}, \quad A_m(x) := \sum_{d \leq x \atop \gcd(d,\ell) = 1} \frac{\phi_{1-2m}(d)}{d},
\]

and note that \(A_0(x)\) can be rewritten as follows

\[
A_0(x) = \sum_{n \leq x} \sum_{d \mid n} \frac{\phi(d)}{d} \sum_{n \leq x} \left( 1 \ast \frac{\phi}{id} \right) (n) = \sum_{n \leq x} \hat{P}(n) / n,
\]

where the symbol \(\ast\) is the unitary convolution and \(\hat{P}(n)\) is the gcd-sum function over regular integers modulo \(n\) defined by Eq. (1). Using Eq. (2) and Abel’s identity, we get

\[
A_0(x) = \frac{x}{\zeta(2)} (K_1(\log x - 1/2) + K_2) + O \left( x^{1/2} \exp \left( -C \frac{(\log x)^3/5}{(\log \log x)^{1/5}} \right) \right). \tag{26}
\]

Moreover, \(A_m(x)\) is written as

\[
A_m(x) = \sum_{d \leq x} \frac{\phi_{1-2m}(d)}{d} \sum_{\ell \leq x/d \atop \gcd(\ell,d) = 1} 1.
\]

By Eq. (11), we get

\[
A_m(x) = x \sum_{d \leq x} \frac{\phi_{1-2m}(d) \phi(d)}{d^3} - \sum_{d \leq x} \frac{\phi_{1-2m}(d)}{d} \sum_{k \mid d} \mu(k) \theta \left( \frac{x}{kd} \right). \tag{27}
\]

Since the ordinary product of two multiplicative functions is a multiplicative function and \(\sum_{d \geq 1} |\phi_{1-2m}(d) \phi(d)|d^{-3} < \infty\), we can write the first term on the right-hand side of Eq. (27) by Euler’s product representation, namely

\[
\sum_{d \leq x} \frac{\phi_{1-2m}(d) \phi(d)}{d^3} = \prod_{p} \left( 1 - \frac{(p-1)(p^{2m-1} - 1)}{p(p^{2m+1} - 1)} \right) + O \left( x^{-1} \right).
\]

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because
\[\sum_{d \geq 1} \frac{\phi_{1-2m}(d) \phi(d)}{d^3} = \prod_p \left( 1 - \frac{1}{p} \left( 1 - \frac{p^{2m-1}}{p^{2m+1}-1} \right) \right),\]

and
\[\sum_{d > x} \frac{\phi_{1-2m}(d) \phi(d)}{d^3} = O \left( x^{-1} \right).\]

As for the second term on the right-hand side of Eq. (27), we have
\[\left| \sum_{d \leq x} \frac{\phi_{1-2m}(d)}{d} \sum_{k \mid d} \mu(k) \theta \left( \frac{x}{kd} \right) \right| \leq \sum_{d \leq x} \frac{\phi_{1-2m}(d) \tau(d)}{d}.\]

Then, we get
\[\sum_{d \leq x} \frac{\phi_{1-2m}(d)}{d} \sum_{k \mid d} \mu(k) \theta \left( \frac{x}{kd} \right) = O_m \left( (\log x)^2 \right),\]
where we used Eq. (17) and the fact \( \phi_{1-2m}(d) \ll 1. \) From the above, we obtain
\[A_m(x) = \prod_p \left( 1 - \frac{1}{p} \left( 1 - \frac{p^{2m-1}}{p^{2m+1}-1} \right) \right) x + O_m((\log x)^2). \quad (28)\]

From Eqs. (26) and (28), we obtain the desired formula (7).

Next, we take \( f = \mu \) into Eq. (5) of Theorem 1.1 to get
\[U_\mu(x) = \frac{1}{r+1} \left( r + 1 \right) \sum_{m=0}^{\left\lfloor r/2 \right\rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{\substack{d \leq x \\text{gcd}(d,\ell) = 1}} \frac{\phi_{1-2m}(d) \mu(\ell)}{d \ell} + O \left( \exp \left( -B(\log x)^{3/2}(\log \log x)^{-1/2} \right) \right).\]

Using Eq. (11) of Lemma 2.2, the above function is rewritten as
\[U_\mu(x) = \frac{1}{r+1} \sum_{m=0}^{\left\lfloor r/2 \right\rfloor} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{\substack{d \leq x \\text{gcd}(d,\ell) = 1}} \frac{\phi_{1-2m}(d) \mu(\ell)}{d \ell} + O \left( \exp \left( -B(\log x)^{3/2}(\log \log x)^{-1/2} \right) \right).\]

We call the inner sum on the right-hand side above \( L_m(x) \), namely
\[L_m(x) := \sum_{\substack{d \ell \leq x \\text{gcd}(d,\ell) = 1}} \frac{\phi_{1-2m}(d) \mu(\ell)}{d \ell}.\]

For \( m = 0 \), we have
\[L_0(x) = \sum_{\substack{d \ell \leq x \\text{gcd}(d,\ell) = 1}} \frac{\phi(d) \mu(\ell)}{d \ell} = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \sum_{\substack{d \leq x/\ell \\text{gcd}(d,\ell) = 1}} \frac{\phi(d)}{d}.\]
Applying Eq (12) of Lemma 2.1 to the inner sum, we get

\[ L_0(x) = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \left( \frac{\ell \phi(\ell)}{\zeta(2) \phi_2(\ell)} \frac{x}{\ell} + O \left( \tau^*(\ell) \log \left( \frac{x}{\ell} \right) \right) \right) \]

\[ = \frac{x}{\zeta(2)} \sum_{\ell \leq x} \frac{\mu(\ell) \phi(\ell)}{\ell \phi_2(\ell)} + O \left( \log x \sum_{\ell \leq x} \frac{\mu(\ell) \tau^*(\ell)}{\ell} \right). \] (29)

Since \( \mu, \phi \) and \( \phi_2 \) are multiplicative functions and \( \sum_{\ell \geq 1} |\mu(\ell)\phi(\ell)/\phi_2(\ell)| \ell^{-1} < \infty \), the first sum of Eq (29) is rewritten as

\[ \sum_{\ell \leq x} \frac{\mu(\ell) \phi(\ell)}{\ell \phi_2(\ell)} = \prod_p \left( 1 - \frac{1}{p(p+1)} \right) + O \left( \frac{\log x}{x} \right), \]

because

\[ \sum_{\ell \geq 1} \frac{\mu(\ell) \phi(\ell)}{\ell \phi_2(\ell)} = \prod_p \left( 1 - \frac{1}{p(p+1)} \right) = 0.704442 \ldots, \quad \sum_{\ell > x} \frac{\mu(\ell) \phi(\ell)}{\ell \phi_2(\ell)} = O \left( \frac{\log x}{x} \right). \]

This latter is estimated by using the facts \( \phi(\ell) \ll \ell, \phi_2(\ell) \geq \ell^2/\tau(\ell) \), and Eq (22). Using Eq (18), the error term of \( L_0(x) \) is estimated by \( (\log x)^3 \). Thus

\[ L_0(x) = \frac{x}{\zeta(2)} \prod_p \left( 1 - \frac{1}{p(p+1)} \right) + O \left( (\log x)^3 \right). \]

For \( m = 1 \), we note that

\[ L_{-1}(x) = \sum_{\substack{d \leq x \\gcd(d,\ell) = 1}} \frac{\phi^{-1}(d) \mu(\ell)}{d} = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \sum_{\substack{d \leq x/\ell \\gcd(d,\ell) = 1}} \frac{\phi^{-1}(d)}{d}. \]

Using Eq (14) of Lemma 2.1, we get

\[ L_{-1}(x) = O \left( \sum_{\ell \leq x} \frac{\mu(\ell) \phi(\ell)}{\ell^2} (\log x/\ell)^2 + \sum_{\ell \leq x} \frac{\mu(\ell) \phi(\ell)}{\ell^2} \log(3\ell) \log(x/\ell) \right) \]

\[ + O \left( \sum_{\ell \leq x} \frac{\mu(\ell) \phi(\ell)}{\ell^2} \log(x/\ell) + \sum_{\ell \leq x} \frac{\mu(\ell) \tau(\ell)}{\ell} \log(x/\ell) \right). \]

From Eqs (16) and (17), the first two summations and the last one are estimated by \( (\log x)^3 \). While the third one is estimated by \( (\log x)^2 \). Hence, we deduce

\[ L_{-1}(x) = O \left( (\log x)^3 \right). \]

---

The inequality \( \phi_2(\ell) \geq \ell^2/\tau(\ell) \) can be found in [1, Chapter 1, Section I.4] or in [8].
As for \( L_m(x) \) with \( m \geq 2 \), we have

\[
L_m(x) = \sum_{d \ell \leq x} \frac{\phi_{1-2m}(d) \mu(\ell)}{d \ell} = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \sum_{d \ell \leq x} \frac{\phi_{1-2m}(d)}{d}.
\]

Applying Eq (13) of Lemma 2.1 to the inner sum, we get

\[
L_m(x) = O \left( \sum_{\ell \leq x} \frac{\mu(\ell) \phi_{2m-1}(\ell)}{\ell^{2m}} \log(x/\ell) + \sum_{\ell \leq x} \frac{\mu(\ell) \sigma(\ell)}{\ell^2} \log(x/\ell) \right).
\]

Using the following formulas, see [4, Lemma 2.1] and [14],

\[
\sum_{n \leq x} \frac{\phi_s(n)}{n} = \frac{x^s}{s \zeta(s+1)} + O_s \left( x^{s-1} \right), \quad (s \geq 2)
\]

\[
\sum_{n \leq x} \frac{\sigma(n)}{n^2} = \frac{\pi^2}{6} x - \frac{1}{2} \log x + O \left( (\log x)^{2/3} \right),
\]

and the Abel identity, we obtain

\[
\sum_{n \leq x} \frac{\phi_s(n)}{n^{s+1}} = \frac{\log x}{\zeta(s+1)} + O_s \left( 1 \right),
\]

and

\[
\sum_{n \leq x} \frac{\sigma(n)}{n^2} = \frac{\pi^2}{6} \log x + \frac{\pi^2}{6} - \frac{1}{2} + O \left( \frac{(\log x)^{2/3}}{x} \right).
\]

It follows that

\[
L_m(x) = O_m( (\log x)^2 ).
\]

Putting everything together, we therefore conclude that

\[
U_\mu(x) = \frac{x}{(r+1)\zeta(2)} \prod_p \left( 1 - \frac{1}{p(p+1)} \right) + O_r \left( (\log x)^3 \right).
\]

This completes the proof of Eq (8).

Now, we proceed the proof as before. We let \( f = \tau \) in Eq (5) of Theorem 1.1. Then, we get

\[
U_\tau(x) = \frac{1}{2} \sum_{n \leq x} \frac{\tau(n)}{n} + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{d \ell \leq x} \frac{\phi_{1-2m}(d) \tau(\ell)}{d \ell}.
\]
Using Eq \((17)\), we find that

\[
U_\tau(x) = \frac{1}{4} (\log x)^2 + \gamma \log x + \frac{(2\gamma - 1)}{2} + \frac{1}{r + 1} \sum_{m=0}^{[r/2]} \binom{r + 1}{2m} B_{2m} \sum_{\substack{d \leq x \\ \gcd(d,\ell) = 1}} \frac{\phi_{1-2m}(d) \tau(\ell)}{d} + O \left( x^{-285/416} (\log x)^{26947/8320} \right).
\]

Put

\[
G_m(x) = \sum_{\substack{d \leq x \\ \gcd(d,\ell) = 1}} \frac{\phi_{1-2m}(d) \tau(\ell)}{d} \ell.
\]

For \(m = 0\), we have

\[
G_0(x) = \sum_{\substack{d \leq x \\ \gcd(d,\ell) = 1}} \frac{\phi(d) \tau(\ell)}{d} = \sum_{\ell \leq x} \frac{\tau(\ell)}{\ell} \sum_{\substack{d \leq x/\ell \\ \gcd(d,\ell) = 1}} \frac{\phi(d)}{d}.
\]

Using Eq \((12)\) of Lemma 2.1, we infer

\[
G_0(x) = \frac{x}{\zeta(2)} \sum_{\ell \leq x} \frac{\tau(\ell) \phi(\ell)}{\ell \phi_2(\ell)} + O \left( \log x \sum_{\ell \leq x} \frac{\tau(\ell) \tau^*(\ell)}{\ell} \right).
\]

(30)

Repeating the previous argument in the preceding case, when \(f = \mu\), with \(L_0(x)\), but using Eq \((23)\) instead of Eq \((22)\), the first sum on the right-hand side is

\[
\sum_{\ell \leq x} \frac{\tau(\ell) \phi(\ell)}{\ell \phi_2(\ell)} = \prod_p \left( 1 + \frac{(2p^2 - 1)p}{(p - 1)^2(p + 1)^3} \right) + O \left( \frac{(\log x)^3}{x} \right).
\]

By Eq \((19)\), the error term in Eq \((30)\) is estimated by \((\log x)^5\). From the above, we deduce that

\[
G_0(x) = \frac{x}{\zeta(2)} \prod_p \left( 1 + \frac{(2p^2 - 1)p}{(p - 1)^2(p + 1)^3} \right) + O \left( (\log x)^5 \right).
\]

As for the function \(G_{-1}(x)\). By an argument similar to the above, we write

\[
G_{-1}(x) = \sum_{\substack{d \leq x \\ \gcd(d,\ell) = 1}} \frac{\phi_{-1}(d) \tau(\ell)}{d} \ell = \sum_{\ell \leq x} \frac{\tau(\ell)}{\ell} \sum_{\substack{d \leq x/\ell \\ \gcd(d,\ell) = 1}} \frac{\phi_{-1}(d)}{d}.
\]

Applying Eq \((14)\) of Lemma 2.1 to the latter sum above, we get

\[
G_{-1}(x) = O \left( \sum_{\ell \leq x} \frac{\tau(\ell) \phi(\ell)}{\ell^2} (\log(x/\ell))^2 + \sum_{\ell \leq x} \frac{\tau^2(\ell)}{\ell} \log(x/\ell) \right).
\]
It is easy to check that the first sum above is estimated by \((\log x)^4\), while the second is \(O((\log x)^5)\), where we used Eq (17) and Eq (19). Thus

\[
G_{-1}(x) = O \left( (\log x)^5 \right).
\]

For \(G_m(x)\) with \(m \geq 2\), it may be written as

\[
G_m(x) = \sum_{d \ell \leq x, \gcd(d, \ell) = 1} \frac{\phi_{1-2m}(d)}{d} \frac{\tau(\ell)}{\ell} = \sum_{\ell \leq x} \tau(\ell) \sum_{d \leq x/\ell, \gcd(d, \ell) = 1} \frac{\phi_{1-2m}(d)}{d}.
\]

By Eq (13), we get

\[
G_m(x) = O \left( \sum_{\ell \leq x} \frac{\tau(\ell)\phi_{2m-1}(\ell)}{\ell^{2m}} \log(x/\ell) + \sum_{\ell \leq x} \frac{\tau(\ell)\sigma(\ell)}{\ell^2} \log(x/\ell) \right).
\]

The first sum above is estimated at once by \((\log x)^3\), by using Eq (17) and \(\phi_{2m-1}(\ell) \ll \ell^{2m-1}\) for \(m \geq 1\). Since \(\sigma(\ell) + \phi(\ell) \leq \ell \tau(\ell)\) for \(\ell \geq 2\), see [6] or [9, Chapter 1, Section I.3], we have

\[
\sum_{\ell \leq x} \frac{\tau(\ell)\sigma(\ell)}{\ell^2} \log(x/\ell) \ll \log x \left( \sum_{\ell \leq x} \frac{\tau^2(\ell)}{\ell} + \sum_{\ell \leq x} \frac{\tau(\ell)\phi(\ell)}{\ell} \right).
\]

From Eqs (17) and (19) it follows that

\[
G_m(x) = O_m((\log x)^5).
\]

Putting everything together, we therefore conclude

\[
U_r(x) = \frac{x}{(r + 1)\zeta(2)} \prod_p \left( 1 + \frac{(2p^2 - 1)p}{(p - 1)^2(p + 1)^3} \right) + O_r \left( (\log x)^5 \right).
\]

Which completes the proof of Eq (9).

In order to prove Eq (10), we replace \(f\) by \(\phi_2\) in Eq (6) of Theorem 1.1

\[
V_{\phi_2}(x) = \frac{1}{2} \sum_{n \leq x} 1 + \frac{1}{r + 1} \sum_{m = 0}^{[r/2]} \binom{r + 1}{2m} B_{2m} \sum_{d \ell \leq x, \gcd(d, \ell) = 1} \frac{\phi_{1-2m}(d)}{\phi_2(d)}.
\]

Using Eq (25) for the first sum on the right-hand side, we get

\[
V_{\phi_2}(x) = \frac{x}{2} - \frac{\theta(x)}{2} - \frac{1}{4} + \frac{1}{r + 1} \sum_{m = 0}^{[r/2]} \binom{r + 1}{2m} B_{2m} \sum_{d \ell \leq x, \gcd(d, \ell) = 1} \frac{\phi_{1-2m}(d)}{\phi_2(d)}.
\]
where \( \theta(x) = x - \lfloor x \rfloor - 1/2 \). Put

\[
F_m(x) = \sum_{d \leq x, \gcd(d, d) = 1} \frac{\phi_{1-2m}(d)}{\phi_2(d)}, \quad F_0(x) = \sum_{d \leq x} \frac{\phi(d)}{\phi_2(d)} \sum_{\ell \leq x/d, \gcd(d, \ell) = 1} 1.
\]

Using Eq (11) of Lemma 2.1 we find that

\[
F_0(x) = x \sum_{d \leq x} \frac{\phi(d)^2}{d^2\phi_2(d)} - \sum_{d \leq x} \frac{\phi(d)}{\phi_2(d)} \sum_{k \mid d} \mu(k) \theta \left( \frac{x}{dk} \right). \quad (31)
\]

By arguing in the same manner as before and using \((\phi(d))^2 \leq \phi_2(d)\), the first sum of Eq. (31) is written

\[
\sum_{d \leq x} \frac{(\phi(d))^2}{d^2\phi_2(d)} = \prod_p \left( 1 + \frac{1}{(p+1)^2} \right) + O(x^{-1}),
\]

because

\[
\sum_{d \geq 1} \frac{(\phi(d))^2}{d^2\phi_2(d)} = \prod_p \left( 1 + \frac{1}{(p+1)^2} \right), \quad \sum_{d > x} \frac{(\phi(d))^2}{d^2\phi_2(d)} = O(x^{-1}).
\]

On the other hand, the second sum is estimated by \((\log x)^4\), because

\[
\sum_{d \leq x} \frac{\phi(d)}{\phi_2(d)} \sum_{k \mid d} \mu(k) \theta \left( \frac{x}{dk} \right) \ll \sum_{d \leq x} \frac{(\phi(d)\tau^2(d)}{d^2} \ll (\log x)^4.
\]

where we again used the inequality \(\phi_2(d)\tau(d) \geq d^2\), Eq (19) and \(\phi(d) \ll d\). This yields

\[
F_0(x) = \prod_p \left( 1 + \frac{1}{(p+1)^2} \right) x + O((\log x)^4). \quad (32)
\]

For the function \(F_m(x)\) with \(m \geq 1\), we write

\[
F_m(x) = x \sum_{d \leq x} \frac{\phi_{1-2m}(d)\phi(d)}{d^2\phi_2(d)} - \sum_{d \leq x} \frac{\phi_{1-2m}(d)}{\phi_2(d)} \sum_{k \mid d} \mu(k) \theta \left( \frac{x}{dk} \right). \quad (33)
\]

Again the first sum on the right-hand side is

\[
\sum_{d \leq x} \frac{\phi_{1-2m}(d)\phi(d)}{d^2\phi_2(d)} = \prod_p \left( 1 - \frac{p(p^{2m-1} - 1)}{(p+1)(p^{2m+2} - 1)} \right) + O_m \left( \frac{\log x}{x^2} \right),
\]

because

\[
\sum_{d \geq 1} \frac{\phi_{1-2m}(d)\phi(d)}{d^2\phi_2(d)} = \prod_p \left( 1 - \frac{p(p^{2m-1} - 1)}{(p+1)(p^{2m+2} - 1)} \right), \quad \sum_{d > x} \frac{\phi_{1-2m}(d)\phi(d)}{d^2\phi_2(d)} = O_m \left( \frac{\log x}{x^2} \right).
\]

This latter follows from Eq (24). By Abel’s identity of Eq (19), the second sum of Eq (33) is estimated by \((\log x)^3/x\). Thus

\[
F_m(x) = \prod_p \left( 1 - \frac{p(p^{2m-1} - 1)}{(p+1)(p^{2m+2} - 1)} \right) x + O_m \left( \frac{(\log x)^3}{x} \right). \quad (34)
\]

From Eq (32) and Eq (34), Eq (10) is proved.
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