Field equations of electromagnetic and gravitational fields

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The paper studies the validity of Maxwell equation in the case for coexistence of electromagnetic field and gravitational field. With the algebra of quaternions, the Newton’s law of gravitation is the same as that in classical theory of gravitational field. Meanwhile the Maxwell equation is identical with that in classical theory of electromagnetic field. And the related conclusions can be spread to the case for coexistence of electromagnetic field and gravitational field by the algebra of octonions. The study claims that Maxwell equation keeps unchanged in the case for coexistence of gravitational field and electromagnetic field, except for the direction of displacement current. The paper attempts to find out why Maxwell equation keeps the same in most cases, even for this special situation. The paper studies the validity of Maxwell equation in the case for coexistence of electromagnetic field and gravitational field by the algebra of octonions.

II. GRAVITATIONAL FIELD

The feature of gravitational field can be described by the algebra of quaternions. In the quaternion space, the coordinates are \( r_0, r_1, r_2, \) and \( r_3 \), with the basis vector \( E_g = (1, i_1, i_2, i_3) \). The radius vector \( R_g = \Sigma(r_i i_i) \), and the velocity \( V_g = \Sigma(v_i i_i) \), with \( i_0 = 1 \). Where, \( r_0 = v_0 t; \) \( v_0 \) is the speed of gravitational intermediate boson, and \( t \) is the time. \( j = 1, 2, 3; i = 0, 1, 2, 3 \).

The gravitational potential is,
\[
A_g = \Sigma(a_i i_i)
\]

and the strength \( B_g = \Sigma(b_i i_i) \) of gravitational field is
\[
B_g = \bigcirc A_g .
\]

where, the \( \bigcirc \) denotes the quaternion multiplication; the operator \( \bigcirc = \Sigma(i_i \partial_i); \) \( \partial_i = \partial/\partial r_i \).

The gravitational strength \( B_g \) covers two components, \( g/v_0 = \partial_0 a + \nabla a_0 \) and \( b = \nabla \times a \).

\[
\begin{align*}
\mathbf{g}/v_0 &= i_1(\partial_0 a_1 + \partial_1 a_0) + i_2(\partial_0 a_2 + \partial_2 a_0) \\
&\quad + i_3(\partial_0 a_3 + \partial_3 a_0) \\
\mathbf{b} &= i_1(\partial_2 a_3 - \partial_3 a_2) + i_2(\partial_3 a_1 - \partial_1 a_3) \\
&\quad + i_3(\partial_1 a_2 - \partial_2 a_1)
\end{align*}
\]

where, the gauge equation \( \partial_0 = \partial_0 a_0 + \nabla \cdot a = 0 \); the vectorial potential \( a_0 = \Sigma(a_i i_i); \nabla = \Sigma(i_i \partial_i) \).

The linear momentum density, \( \mathbf{P} = m \mathbf{V}_g \), is the source density \( \mathbf{S}_g \) of gravitational field. The latter one is defined from the gravitational strength \( \mathbf{B}_g \).
\[
\bigcirc^\ast B_g = -\mu_g S_g
\]

where, \( m \) is the mass density; \( \ast \) denotes the quaternion conjugate; \( \mu_g = 4\pi G/v_0^2 \) is one coefficient, and \( G \) is the gravitational constant.

| TABLE I: The quaternion multiplication table. |
|---|---|---|---|
| 1 | 1 | \( i_1 \) | \( i_2 \) | \( i_3 \) |
| \( i_1 \) | \( i_1 \) | -1 | \( i_3 \) | -\( i_2 \) |
| \( i_2 \) | \( i_2 \) | -\( i_1 \) | -1 | \( i_1 \) |
| \( i_3 \) | \( i_3 \) | \( i_2 \) | -\( i_1 \) | -1 |

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The strength $b$ is too weak to detect. For example, the mass $m_1, m_2$ move with the speed $v_1, v_2$ respectively. They move parallel, and there will be two forces $f_1, f_2$ between them. Wherein, $f_1 = m_2g_1$, with $g_1 = Gm_2/r^2$; and $f_2 = m_2b_1v_2$, with $b_1 < (\mu_g/4\pi)m_1v_1/r^2$. In most cases, $f_2/f_1 \approx v_1v_2c^2 \ll 1$, therefore we will neglect $f_2$ generally. In the Newtonian gravity theory, there are $a = 0, b = 0$, and $f_2 = 0$ specially.

A. Newton’s law of gravitation

In the gravitational field, the scalar part $s_0$ of source $S_g$ in Eq.(5) is rewritten as follows.

$$\nabla^* \cdot h = -\mu_g s_0$$

(6)

where, $h = \Sigma(b_j i_j); s_0 = p_0 = mv_0$.

Further, the above is reduced to

$$\nabla^* \cdot (g/v_0 + b) = -\mu_g m v_0.$$  

(7)

Eqs.(2) and (4) yield the equation,

$$\nabla \cdot b = 0$$

(8)

and then, we have Newton’s law of gravitation.

$$\nabla^* \cdot g = -m/\varepsilon_g$$

(9)

where, the coefficient $\varepsilon_g = 1/(\mu_g v_0^2)$.

The above states that the gravitational potential $a$ has an influence on the Newton’s law of gravitation, although the term $\nabla \cdot \partial_0 a$ is very tiny. Meanwhile Newton’s law of gravitation is an invariant under Galilean transformation and Lorentz transformation, and the definition of gravitational strength can be extended from the steady state to movement state.

B. Ampere’s law of gravitation

In the quaternion space, the vectorial part $s$ of linear momentum density $S_g$ can be decomposed from Eq.(5).

$$\partial_0 h + \nabla^* \times h = -\mu_g s$$

(10)

where, $s = \Sigma(s_j i_j); s_j = p_j = mv_j$.

The above can be rewritten as follows.

$$\partial_0 (g/v_0 + b) + \nabla^* \times (g/v_0 + b) = -\mu_g s$$

(11)

Eqs.(2) – (4) yield the equation,

$$\partial_0 b + \nabla^* \times g/v_0 = 0$$

(12)

and then, we obtain Ampere’s law of gravitation.

$$\partial_0 g/v_0 + \nabla^* \times b = -\mu_g s$$

(13)

The above means the Newton’s law of gravitation in the quaternion space is the same as that in classical gravitational theory. In the quaternion space, the masses in either steady state or movement state can exert the gravity on other objects. The linear momentum yields the gravitational strength $b$, which may be quite weak. And the strength $b$ and $g$ can be induced each other in the gravitational field from Eq.(12).

III. ELECTROMAGNETIC FIELD

The feature of electromagnetic field can be represented by the algebra of quaternions as well. In the quaternion space, the coordinates are $r_0, r_1, r_2, r_3$, with the basis vector $\mathbf{e}_q = (1, i_1, i_2, i_3)$. The radius vector $\mathbf{r}_q = \Sigma(r_i i_i)$, and the velocity $\mathbf{v}_q = \Sigma(v_i i_i)$. Where, $r_0 = v_0t; v_0$ is the speed of electromagnetic intermediate boson, and $t$ is the time.

The electromagnetic potential is,

$$\mathbf{A}_q = \Sigma(A_i i_i)$$

(14)

and the electromagnetic strength $\mathbf{B}_q = \Sigma(B_i i_i)$ is

$$\mathbf{B}_q = \cdots \cdot \mathbf{A}_q.$$  

(15)

where, $\mathbf{A} = \Sigma(A_j i_j)$.

The electromagnetic strength $\mathbf{B}_q$ includes two parts, $\mathbf{E}_q/v_0 = \partial_0 \mathbf{A} + \nabla \mathbf{A}$ and $\mathbf{B}_q = \nabla \times \mathbf{A}$.

$$\mathbf{E}_q/v_0 = i_1 (\partial_0 A_1 + \partial_1 A_0) + i_2 (\partial_0 A_2 + \partial_2 A_0) + i_3 (\partial_0 A_3 + \partial_3 A_0)$$

(16)

$$\mathbf{B}_q = i_1 (\partial_2 A_3 - \partial_3 A_2) + i_2 (\partial_3 A_1 - \partial_1 A_3) + i_3 (\partial_1 A_2 - \partial_2 A_1)$$

(17)

where, the gauge equation $\mathbf{B}_q = \partial_0 \mathbf{A} + \nabla \cdot \mathbf{A} = 0$.

The electric current density $\mathbf{S}_q = q \mathbf{v}_q$ is the source density of electromagnetic field, and is defined from the electromagnetic strength $\mathbf{B}_q$.

$$\cdots \cdot \mathbf{B}_q = -\mu_q \mathbf{S}_q$$

(18)

where, $q$ is the electric charge density; $\mu_q$ is the electromagnetic constant.

The above equation is same as that in classical theory of electromagnetic field.
A. Gauss’s law

In the electromagnetic field, the scalar part $S'_q$ of the source $S_q$ in Eq.(18) is rewritten as follows.

$$\nabla^* \cdot \mathbf{H}_q = -\mu_q S'_q$$

(19)

where, $\mathbf{H}_q = \Sigma(B_j \mathbf{i}_j)$; $S'_q = qv_0$.

Further, the above is reduced to

$$\nabla^* \cdot (\mathbf{E}_q/v_0 + \mathbf{B}_q) = -\mu_q qv_0.$$  \hspace{1cm} (20)

Eqs.(15) and (17) yield the Gauss’s law for magnetism,

$$\nabla \cdot \mathbf{B} = 0$$ \hspace{1cm} (21)

and then, we have Gauss’s law as follows.

$$\nabla^* \cdot \mathbf{E}_q = -q/\varepsilon_q$$ \hspace{1cm} (22)

where, the coefficient $\varepsilon_q = 1/(\mu_q v_0^2)$.

By comparison with the Maxwell equation, we find that Eqs.(21) and (22) are the same as that in Maxwell equation of classical electromagnetic theory, although the definition of the gauge equation $B_0 = 0$ is different.

B. Ampere-Maxwell law

In the quaternion space, the vectorial part $\mathbf{S}_q$ of electric current density $S_q$ can be decomposed from Eq.(18).

$$\partial_0 \mathbf{H}_q + \nabla^* \times \mathbf{H}_q = -\mu_q \mathbf{S}_q$$ \hspace{1cm} (23)

where, $\mathbf{S}_q = \Sigma(S'_j \mathbf{i}_j)$; $S'_j = qv_j$.

The above can be rewritten as follows.

$$\partial_0(\mathbf{E}_q/v_0 + \mathbf{B}_q) + \nabla^* \times (\mathbf{E}_q/v_0 + \mathbf{B}_q) = -\mu_q \mathbf{S}_q$$ \hspace{1cm} (24)

Eqs.(15) – (17) yield Faraday’s law of induction,

$$\partial_0 \mathbf{B}_q + \nabla^* \times \mathbf{E}_q/v_0 = 0$$ \hspace{1cm} (25)

and then, we obtain Ampere-Maxwell law in the electromagnetic field as follows.

$$\partial_0 \mathbf{E}_q/v_0 + \nabla^* \times \mathbf{B}_q = -\mu_q \mathbf{S}_q$$ \hspace{1cm} (26)

The above means that Eqs.(21) and (22) are combined with Eqs.(25) and (26) to become Maxwell equation. By contrast with Maxwell equation, we find that Eqs.(25) and (26) are the same as that in classical electromagnetic theory, except for the direction of displacement current.

IV. ELECTROMAGNETIC FIELD AND GRAVITATIONAL FIELD

The feature of gravitational field and electromagnetic field can be described simultaneously by the octonion space, which is consist of two quaternion spaces.

| TABLE III: The octonion multiplication table. |
|---------------------------------------------|
| $\mathbf{I}_1$ | $\mathbf{I}_2$ | $\mathbf{I}_3$ |
| $\mathbf{I}_1$ | $\mathbf{I}_2$ | $\mathbf{I}_3$ |
| $\mathbf{I}_2$ | $\mathbf{I}_3$ | $\mathbf{I}_1$ |
| $\mathbf{I}_3$ | $\mathbf{I}_1$ | $\mathbf{I}_2$ |

In the quaternion space for the gravitational field, the basis vector is $\mathbf{E}_g$, the radius vector is $\mathbf{R}_g$, and the velocity is $\mathbf{v}_g$. In the quaternion space for the electromagnetic field, the basis vector is $\mathbf{E}_e = (\mathbf{I}_0, \mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)$, the radius vector is $\mathbf{R}_e = \Sigma(\mathbf{R}_i \mathbf{I}_i)$, and the velocity is $\mathbf{v}_e = \Sigma(\mathbf{v}_i \mathbf{I}_i)$. The $\mathbf{E}_e$ is independent of the $\mathbf{E}_g$, with $\mathbf{E}_e = \mathbf{E}_g \odot \mathbf{I}_0$.

These two quaternion spaces can be combined together to become one octonion space, with the octonion basis vector $\mathbf{E} = (1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{I}_0, \mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)$. The radius vector in the octonion space is $\mathbf{R} = \Sigma(r_i \mathbf{i}_i + \mathbf{r}_i \mathbf{I}_i)$, the octonion velocity is $\mathbf{v} = \Sigma(v_i \mathbf{k}_i + \mathbf{v}_i \mathbf{I}_i)$.

When the electric charge is combined with the mass to become the electron or the proton etc., we obtain the relation, $\mathbf{R}_e \mathbf{I}_i = r_i \mathbf{k}_i \mathbf{I}_i$, and $\mathbf{v}_e \mathbf{I}_i = v_i \mathbf{k}_i \mathbf{I}_i$. Meanwhile, the gravitational intermediate boson and electromagnetic intermediate boson may be combined together to become the photon etc. Here, the symbol $\odot$ denotes the octonion multiplication.

The potential of gravitational field and electromagnetic field are $\mathbf{A}_g = \Sigma(a_i \mathbf{i}_i)$ and $\mathbf{A}_e = \Sigma(a_i \mathbf{I}_i)$ respectively. They are combined together to become the potential in the octonion space.

$$\mathbf{A} = \mathbf{A}_g + k_{eg} \mathbf{A}_e$$ \hspace{1cm} (27)

where, $k_{eg}$ is a coefficient; $\mathbf{A}_e = \mathbf{A}_g \odot \mathbf{I}_0$.

The strength $\mathbf{B}$ consists of the gravitational strength $\mathbf{B}_g$ and electromagnetic strength $\mathbf{B}_e$. The selecting gauge equations are $b_0 = 0$ and $B_0 = 0$.

$$\mathbf{B} = \diamond \odot \mathbf{A} = \mathbf{B}_g + k_{eg} \mathbf{B}_e$$ \hspace{1cm} (28)

The gravitational strength $\mathbf{B}_g$ in Eq.(2) includes two components, $\mathbf{g} = (g_01, g_02, g_03)$ and $\mathbf{b} = (g_23, g_31, g_12)$, while the electromagnetic strength $\mathbf{B}_e$ involves two parts, $\mathbf{E} = (B_01, B_02, B_03)$ and $\mathbf{B} = (B_23, B_31, B_12)$.

$$\mathbf{E}/V_0 = \mathbf{I}_1(\partial_0 \mathbf{A}_1 + \partial_1 \mathbf{A}_0) + \mathbf{I}_2(\partial_0 \mathbf{A}_2 + \partial_2 \mathbf{A}_0) + \mathbf{I}_3(\partial_0 \mathbf{A}_3 + \partial_3 \mathbf{A}_0)$$ \hspace{1cm} (29)

$$\mathbf{B} = \mathbf{I}_1(\partial_3 \mathbf{A}_2 - \partial_2 \mathbf{A}_3) + \mathbf{I}_2(\partial_0 \mathbf{A}_3 - \partial_3 \mathbf{A}_1) + \mathbf{I}_3(\partial_0 \mathbf{A}_1 - \partial_1 \mathbf{A}_2)$$ \hspace{1cm} (30)

The electric current density $\mathbf{S}_e = q\mathbf{v}_e$ is the source of electromagnetic field in the octonion space. While the linear momentum density is the source of gravitational
field still. And the source $S$ was devised to consistently describe the sources of electromagnetism and gravitation.

$$\hat{\bigcirc}^* \mathbf{B} = -\mu S = -(\mu_g S_g + k_{eg} \mu_e S_e) \quad (31)$$

where, $\mu$ is a coefficient; $\mu_e = \mu_q; k_{eg}^2 = \mu_g / \mu_e$; $*$ denotes the conjugate of octonion.

From Eq.(31), we have,

$$\hat{\bigcirc}^* \mathbf{B}_g + k_{eg} \hat{\bigcirc}^* \mathbf{B}_e = -(\mu_g S_g + k_{eg} \mu_e S_e) \quad (32)$$

and the above equation can be decomposed as follows, according to the basis vectors and multiplication.

$$\hat{\bigcirc}^* \mathbf{B}_g = -\mu_g S_g \quad (33)$$
$$\hat{\bigcirc}^* \mathbf{B}_e = -\mu_e S_e \quad (34)$$

In the octonion space, Eq.(33) is the same as Eq.(5) for the gravitational field. And Eq.(34) is for the electromagnetic field.

The above description states that the electromagnetic field is independent of the gravitational field absolutely, and the conjugate of octonion.

The inferences in gravitational field or electromagnetic field can be represented in the quaternion space too. By means of the octonion algebra, the gravitational field and electromagnetic field is opposite. With the relation, $\mathbf{A}_e = \mathbf{A}_g \circ \mathbf{I}_0$, these equations can be reduced to Maxwell equation in the quaternion space.

### A. Gauss’s law

In the electromagnetic field, the part $S_0$ of the source $S_e$ in Eq.(34) is rewritten as,

$$\nabla^* \cdot \mathbf{H} = -\mu_e S_0$$

where, $S_0 = S_0 \mathbf{I}_0$, $S_0 = qV_0$, $\mathbf{H} = \Sigma B_j \mathbf{I}_j$.

Further, the above is reduced to

$$\nabla^* \cdot (\mathbf{E}/V_0 + \mathbf{B}) = -\mu_e qV_0 \mathbf{I}_0$$

Eqs.(27), (29), and (30) yield the Gauss’s law of magnetism,

$$\nabla \cdot \mathbf{B} = 0 \quad (37)$$

and then, we have the Gauss’s law as follows.

$$\nabla^* \cdot \mathbf{E} = -(q/\varepsilon) \mathbf{I}_0 \quad (38)$$

where, the coefficient $\varepsilon_e = 1/(\mu_e V_0^2)$.

The above states that the electromagnetic potential has an influence on Gauss’s law of electromagnetism. By far, we have two equations, Eqs.(37) and (38), for the Maxwell equation, although the gauge equation $B_0 = 0$ is different to that in classical electromagnetic theory.

### B. Ampere-Maxwell law

In the octonion space, the vectorial part $\mathbf{S}$ of electromagnetic source $S_e$ can be decomposed from Eq.(34).

$$\partial_0 \mathbf{H} + \nabla^* \times \mathbf{H} = -\mu_e \mathbf{S} \quad (39)$$

where, $\mathbf{S} = \Sigma (S_j \mathbf{I}_j); S_j = qV_j$.

The above can be rewritten as follows.

$$\partial_0 (\mathbf{E}/V_0 + \mathbf{B}) + \nabla^* \times (\mathbf{E}/V_0 + \mathbf{B}) = -\mu_e \mathbf{S} \quad (40)$$

Eqs.(27), (29), and (30) yield the Faraday’s law,

$$\partial_0 \mathbf{B} + \nabla^* \times \mathbf{E}/V_0 = 0 \quad (41)$$

and then, we obtain Ampere-Maxwell law in the electromagnetic field as follows.

$$\partial_0 \mathbf{E}/V_0 + \nabla^* \times \mathbf{B} = -\mu_e \mathbf{S} \quad (42)$$

The above means that the electric charge in either steady state or movement state can exert the electric force and magnetic force on other charges. The strength $\mathbf{B}$ and $\mathbf{E}$ can be induced each other in the electromagnetic field from Eq.(41). And Eqs.(41) and (42) are combined with Eqs.(37) and (38) to become Maxwell equation in the octonion space. With the relation, $\mathbf{A}_e = \mathbf{A}_g \circ \mathbf{I}_0$, these equations can be reduced to Maxwell equation in the quaternion space.

### V. CONCLUSIONS

The feature of gravitation field can be described with quaternion spaces. Making use of the algebra of quaternions, we obtain the Newton's law of gravitation etc. In the gravitational field, there may exist the gravitational strength part $\mathbf{b}$, although the strength part $\mathbf{b}$ may be quite weak and difficult to detect.

In the electromagnetic field, some characteristics can be represented in the quaternion space too. By means of the algebra of quaternions, we attain Maxwell equations etc. Making use of the algebra of quaternions, we attain Maxwell equations etc. By comparison with classical electromagnetic theory, we find the definition of gauge equation is different, and the direction of displacement current in Ampere-Maxwell equation is opposite.

With the octonion algebra, the gravitational field and electromagnetic field can be described simultaneously. The inferences in gravitational field or electromagnetic field can be spread to the case for coexistence of electromagnetic field and gravitational field. We achieve same conclusions as that in the quaternion space, including the Maxwell equation etc.
It should be noted the study for the validity of Maxwell equation in the electromagnetic field and gravitational field examined only some simple cases in quaternion and octonion spaces. Despite its preliminary characteristics, this study can clearly indicate that the Newton’s law of gravitation can be derived with quaternions. And that Maxwell equation of electromagnetic field can be deduced with the algebra of quaternions as well. In the octonion space, some equations of the two fields can be drawn out simultaneously. For the future studies, the research will concentrate on only some predictions about the strong strength \(b\) in gravitational field as well as the direction of displacement current in electromagnetic field.

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[1] W. R. Hamilton, Elements of Quaternions (Longmans, Green & Co., London, 1866).
[2] S. L. Adler, Quaternionic Quantum Mechanics and Quantum Fields (Oxford University Press, Oxford, UK, 2001).
[3] J. C. Maxwell, A Treatise on Electricity and Magnetism (Dover Publications Inc., New York, 1954).
[4] P. A. M. Dirac, Proc. Roy. Irish Academy, 50 (1945) 261.
[5] W. Jackson, Life and work of Oliver Heaviside, Nature, 165 (1950) 991-3.
[6] M. Gogberashvili, Octonionic electrodynamics, Journal of Physics A, 39 (2006) 7099–7104.
[7] V. Dzhunushaliev, Toy Models of a Nonassociative Quantum Mechanics, Advances in High Energy Physics, 2007 (2007) 12387.
[8] Z.-H. Weng and Y. Weng, Variation of Gravitational Mass in Electromagnetic Field, in Proceedings of Progress in Electromagnetics Research Symposium 2009 in Beijing, Beijing, China, March 2009, 105–107.
[9] I. Newton, The Mathematical Principles of Natural Philosophy (Dawsons of Pall Mall, London, 1968).
[10] S. Weinberg, Gravitation and cosmology (John Wiley & Sons., New York, 1972).
[11] A. Cayley, The Collected Mathematical Papers (Johnson Reprint Co., New York, 1963).
[12] J. C. Baez, The octonions, Bulletin of the American Mathematical Society, 39 (2002) 145–205.