The Quantization of Anomalous Gauge Field Theory and BRST-invariant Models of Two Dimensional Quantum Gravity

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ABSTRACT

We analyze the problems with the so called gauge invariant quantization of the anomalous gauge field theories originary due to Faddeev and Shatashvili (FS). Our analysis bring to a generalization of FS method which allows to construct a series of classically equivalent theories which are non equivalent at quantum level. We prove that these classical theories are all consistent with the BRST invariance of the original gauge symmetry with suitably augmented field content. As an example of such a scenario, we discuss the class of physically distinct models of two dimensional induced gravity which are a generalization of the David-Distler-Kawai model.
1. Introduction

The consistent quantization of (classical) gauge invariant field theory requires the complete cancellation of anomalies [1] [2]. Here, ”consistent” means that we want not only to require renormalizability (perturbative finiteness), but also unitarity of S-matrix, non-violation of Lorentz invariance etc. Moreover, in physical 4d world, anomaly cancellation condition itself often leads to the physical predictions. The well known example is the equality of numbers of quarks and leptons in the Standard Model of Weinberg and Salam.

On lower dimensional (eg. \(d = 2\)) field theory, the cancellation of anomalies is still the crucial ingredient for the model building. The critical string dimension \(d = 26\) is often quoted [3] as the consequence of anomaly free condition for bosonic string (although in this example the cancellation of anomaly does not guarantee full consistency of the model in above sense, due to the presence of tachyons).

In the case of lower dimensional field theory \((d < 4)\), one often tries to quantize a gauge field theory when there is no way of cancelling its anomaly. The classical example of this situation is the attempt to the quantization of chiral Schwinger model by Jackiw and Rajaraman [4] [5]. They have shown that the model can be consistently quantized (=free field theory) even when the gauge invariance is broken through anomaly.

In general there seem to be two ways for attempting the quantization of anomalous gauge field theory:

1) *Gauge non invariant method*

One ignores the breaking of gauge symmetry and try to show that the theory can be quantized even without the gauge invariance. The example of this approach is the above Jackiw-Rajaraman quantization of the chiral Schwinger model. The problem here is that it is not easy to develop the general technics covering wide
class of physically relevant models with anomaly.

2) Gauge invariant method

In this case, one first tries to recover gauge invariance by introducing new degrees of freedom. The theory is anomalous when one can not find local counter term to cancel the gauge non invariance due to the one loop "matter" integrals in presence of gauge fields, by making use exclusively of the degrees of freedom (fields) already present in the classical action.

In ref. [6], Faddeev and Shatashvili (FS) have tried to justify the introduction of new degrees of freedom which are necessary to construct the anomaly cancelling counter term. Their argument is based on the idea of projective representation of gauge group. They observe that the appearance of anomaly does not mean the simple breakdown of (classical) gauge symmetry, but it rather signals that the symmetry is realised projectively (this is related to the appearance of anomalous commutators of relevant currents). Such a realization, through projective representations, necessitates the enlargement of physical Hilbert space. Thus they argued that the introduction of new fields in the model is not an ad hoc (and largely arbitrary) construction.

Independently of their "philosophy", the FS method gives the gauge invariant action at the price of introducing the extra degrees of freedom (generally physical). The serious problem of this method is, however, that the gauge invariance thus "forced" upon the theory, does not automatically guarantee the consistency of the theory. This is in contrast with our experience with some 4d models such as the Standard Model.

For example, one may apply the FS method to the celebrated case of chiral Schwinger model [4] [5] [5A]. In this case, we have the classical action

\[
S_0 = \int \frac{dz \wedge d\bar{z}}{2i} \left[ \bar{\psi}_R \gamma_z (\bar{\partial} + R) \psi_R + \bar{\psi}_L \gamma_z \partial \psi_L + \frac{1}{4} Tr F^2 \right]
\]
where
\[ \psi_{R/L} = \frac{1 \pm \gamma_5}{2} \psi \]
\[ R/L = A_1 \pm iA_2, \quad F = \partial L - \partial R + [R, L] \]
(we are using the euclidian notation).

So is invariant under the gauge transformation
\[ \psi_R \rightarrow \psi_R^g = S(g)\psi_R \]
\[ \psi_L \rightarrow \psi_L \]
\[ A_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1} \]
for any \( g(z, \bar{z}) \in G \).

The theory is anomalous because the one loop integral
\[ e^{-W_R(R)} = \int \mathcal{D}\psi_R \mathcal{D}\bar{\psi}_R \text{exp} - \int \bar{\psi}_R \gamma_{\bar{z}}(\partial + R)\psi_R \]
is not gauge invariant under
\[ R \rightarrow gRg^{-1} + g\bar{\partial}g^{-1} \]
(for any choice of the regularization).

Following FS'technic (see next section) however, one can introduce the local counter term \( \Lambda(R, L; g), \quad (g(z, \bar{z}) \in G) \) so that the gauge variation of \( \Lambda \) cancels the non invariance of \( W_R(R) \).

There is certain arbitrariness in the choice of \( \Lambda \) but the convenient one is
\[ \Lambda(R, L; g) = - \left( \alpha_L(L, g) + \frac{1}{4\pi} \int Tr(RL) \right) \]
where

\[ \alpha_L(L, g) = \frac{1}{4\pi} \left[ -\int \frac{dz \wedge d\bar{z}}{2i} Tr(g^{-1} \partial g, L) + \frac{1}{2} \int \frac{dz \wedge d\bar{z}}{2i} Tr(g \partial g^{-1}, g \bar{\partial} g^{-1}) \right. \]

\[ \left. - \frac{1}{2} \int_0^1 dt \int \frac{dz \wedge d\bar{z}}{2i} Tr(g' \partial_t g'^{-1}, [g' \partial g'^{-1}, g' \bar{\partial} g'^{-1}]) \right] \]

\[ g'(0, z, \bar{z}) = 1, \quad g'(1, z, \bar{z}) = g(z, \bar{z}) \]

is the Wess-Zumino-Novikov-Witten action corresponding to the anomaly of left fermion \( \psi_L, \bar{\psi}_L \) (\( \alpha_L \) is not globally a local action but it is so for "small" \( g \simeq 1 + i\xi \)). That is, one can write

\[ \alpha_L(L, g) = W_L(L^g) - W_L(L) \]

where

\[ e^{-W_L(L)} = \int D\psi'_L D\bar{\psi}'_L e^{\int \bar{\psi}'_L \gamma_z(\partial + L) \psi'_L} \]

(note that \( \psi'_L, \bar{\psi}'_L \) have nothing to do with \( \psi_L, \bar{\psi}_L \) in \( S_0 \)).

With this choice of counter term, one can show that the theory is equivalent to a) free decoupled fermion \( \psi_L, \bar{\psi}_L \) and b) the vector Schwinger model. In fact, the added bosonic degree of freedom \( g(z, \bar{z}) \in G \) can be "fermionized" to act as missing \( \psi'_L, \bar{\psi}'_L \) with the right coupling to the left component \( L \) of gauge field.

However, there is still a point missing in this story. In fact, after introducing the new degree of freedom \( g \), there is no reason to exclude the other type of invariant local counter term such as

\[ \frac{a}{4\pi} \int Tr(L^g R^g) = \frac{a}{4\pi} \int Tr(L g^{-1} + g \partial g^{-1}, R g^{-1} + g \bar{\partial} g^{-1}) \]

(one can also attribute it to indefinite - regularization dependent - part of fermionic integral, i.e. \( W_R(R) + W_L(L) + \frac{a}{4\pi} \int Tr(R L) \)).
It is well known [4] that the arbitrary constant $a$ enters the physical spectrum. For abelian case, $G = U(1)$, the mass square of massive boson is given by

$$m^2 = \frac{e^2 a^2}{a - 1}$$

thus, for $a < 1$, the theory is not consistent although the requirement of gauge invariance is satisfied.

In the fermionized version of the theory [5A], $a$ enters the charges of left and right fermions as

$$e_{R/L} = \frac{e}{2} \left( \sqrt{a - 1} \pm \frac{1}{\sqrt{a - 1}} \right)$$

This means that the condition $a > 1$ is necessary also for the real coupling constant, or the hermitian hamiltonian.

In general, the consistence of the theory can be proved if one can set up the BRST scheme with certain physical conditions at the start, such as hermiticity of the hamiltonian [11].

In what follows, we discuss the possibility of recasting the FS method into BRST formalism, thus facilitating the analysis of the consistency of the theory.

2. Faddeev-Shatashvili method

a) Path integral formalism

We shall briefly describe the Faddeev-Shatshvili (FS) method of quantizing anomalous gauge field theory in the path integral formalism, following the work of Harada and Tsutsui [7], Babelon, Shaposnik and Vialet [8].

Let us take a generic gauge field theory described by the classical action

$$S_0(A, X) = S_G(A) + S_M(X; A)$$

where $\{A(x)\}$ and $\{X(x)\}$ represent respectively gauge fields and "matter fields", gauge invariantly coupled to the former.
The total action $S_0$ as well as the pure gauge part $S_G$ and the matter part $S_M$ are invariant under the local gauge transformation

$$A \rightarrow A' = A^g$$
$$X \rightarrow X' = X^g \quad g(x) \in G.$$  \hspace{1cm} (2)

Being anomalous generally means that the one loop matter integral (assumed that $S_M(X, A)$ is quadratic in $X$)

$$\int \mathcal{D}X e^{-S_M(X, A)} \equiv e^{-W(A)}$$ \hspace{1cm} (3)

cannot be regularized in such a way as to preserve the gauge invariance of the functional $W(A)$,

$$W(A^g) - W(A) = \alpha(A; g) \neq 0$$  \hspace{1cm} (4)

Naturally, $\alpha(A; g)$ depends on the regularization used, but there is no way of cancelling it completely by adding some local counter term $\Lambda(A, X)$ to the action.

One can understand eq. (4) as the non invariance of the path integral measure, $\mathcal{D}X$:

$$\mathcal{D}X^g \neq \mathcal{D}X$$ \hspace{1cm} (5)

In fact, as shown by Fujikawa [9], one can write the "anomaly equation"

$$W(A^g) - W(A) = \alpha(A; g)$$
$$\det \left( \frac{\mathcal{D}X^g}{\mathcal{D}X} \right) = e^{-\alpha(A; g)} = e^{\alpha(A; g^{-1})}$$ \hspace{1cm} (6)

In this situation, clearly one can not hope that the usual Faddeev-Popov (FP) ansatz to quantize the theory may go through.
If one inserts the $\delta$ function identity

$$1 = \Delta(A) \int \mathcal{D}g \delta(F(A^g))$$

(7)

where $F(A)$ is a gauge fixing function, into the path integral expression for the partition function

$$Z = \int \mathcal{D}A \int \mathcal{D}X e^{-\left(S_G(A) + S_M(X;A)\right)}$$

then one obtains

$$Z = \int \mathcal{D}A \int \mathcal{D}X \Delta(A) e^{-\left[S_G(A) + S_M(X;A)\right]} \int \mathcal{D}g \delta(F(A^g))
\quad = \int \mathcal{D}A \int \mathcal{D}g \Delta(A) e^{-S_G(A^g)} \int \mathcal{D}X e^{-S_M(X^g;A^g)} \delta(F(A^g))$$

(8)

the second equality follows from the gauge invariance of the classical action: $S_0(A^g; X^g) = S_0(A; X)$.

In the case of usual gauge field theory, such as chiral Schwinger model, we can make a series of assumptions on the remaining functional measures $\mathcal{D}A$ and $\mathcal{D}g$.

First, we assume

$$\mathcal{D}A = \mathcal{D}A^g$$

(9)

then with the change of variable $A^g \rightarrow A$ and $X^g \rightarrow X$ in (7), we get

$$Z = \int \mathcal{D}g \int \mathcal{D}A \Delta(F(A)) \Delta(A^g) e^{-S_G(A)} \int \mathcal{D}X e^{-\left[S_M(X;A) + \alpha(A;g^{-1})\right]}$$

(10)

where we have used eq. (5), i.e. $\mathcal{D}X = \mathcal{D}X^g e^{-\alpha(A;g^{-1})}$. 

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Further, one can assume for the usual gauge group, the invariance of Haar measure $\mathcal{D}g$, i.e. for any $h$ in $G$

$$\mathcal{D}(gh) = \mathcal{D}(hg) = \mathcal{D}g \quad (11)$$

which results, as is well known, in the invariance of the FP factor $\Delta(A)$

$$\Delta(Ag^{-1}) = \Delta(A) \quad (12)$$

Thus, we get the expression for $Z$ proposed in ref.s [6] and [7]

$$Z = \int \mathcal{D}g \int \mathcal{D}A \Delta(A) \delta(F(A)) \int \mathcal{D}X e^{-S_{eff}(X,A,g)} \quad (13)$$

with

$$S_{eff}(X,A,g) = S_0(X,A,g) + \alpha(A;g^{-1}) \quad (14)$$

As one can see from eq. (4) the effect of the counter term $\alpha(A;g^{-1})$ is to transform the one loop path integral $W(A)$, eq. (3), to $W(Ag^{-1})$, which is trivially gauge invariant under the extended gauge transformation

$$A \rightarrow A^h, \quad X \rightarrow X^h$$

$$g \rightarrow hg \quad (15)$$

and thus the model is invariant up to one loop level.

We have repeated here the above well known manipulations [7] to emphasize the relevance of the invariance conditions 1) and 2) (eq.s (9) and (11)).

In many familiar example, such as the chiral Schwinger model, these conditions are trivially satisfied.
One well known case where these conditions become problematic is the 2d induced gravity or off-critical string. In this case, if one fixes the path integral measures $\mathcal{D}\phi$ for the Weyl factor of metric and $\mathcal{D}\sigma$ for the Weyl group element by the invariance under the diffeomorphisms of 2d manifold, then they are not invariant under the translations, e.g., $\sigma \rightarrow \sigma + \alpha$ (i.e., Weyl transformation). Thus, the path integral measure (i.e., $\mathcal{D}A\mathcal{D}g$) can never be invariant under the whole gauge group

$$G = Diff \otimes Weyl$$

b) BRST [10] quantization

A more rigorous strategy to have a consistent formulation of a gauge field theory is to recast it in the BRST formalism. In this way, one may discuss the physically important questions such as the unitarity of S-matrix [11].

In the simpler example like the chiral gauge field theory where the invariance of the measure $\mathcal{D}g\mathcal{D}A$, eqs (9) and (11), under the gauge transformations are respected, there is no difficulty in setting up the BRST procedure once the anomaly has been removed.

One replaces the "heuristic" FP factor

$$\Delta(A)\delta(F(A)) = \det \left( \frac{\delta F(A^h)}{\delta h} \right)_{h=1} \delta(F(A))$$

with BRST gauge fixing term

$$exp - \int \hat{s}(\bar{c}F(A)) = exp - \int \left[ BF(A) - \bar{c} \frac{\delta F(A^h)}{\delta h} \right]_{h=1} c$$

where $c, \bar{c}$ are the BRST ghosts corresponding to the gauge group $G$ while $B$ ("Lagrange multiplier") is the Nakanishi-Lautrup field. Under the BRST operator
\begin{equation}
\tilde{s}c = B \\
\tilde{s}B = 0 \\
(\tilde{s}^2 = 0)
\end{equation}

The counter term $\alpha(A;g)$ cancelling the one-loop anomaly, one can show easily the validity of the Slavnov-Taylor identity

$$
\frac{\delta\tilde{\Gamma}}{\delta A} \frac{\delta\tilde{\Gamma}}{\delta K} + \frac{\delta\tilde{\Gamma}}{\delta \Phi_i} \frac{\delta\tilde{\Gamma}}{\delta K_i} + \frac{\delta\tilde{\Gamma}}{\delta c} \frac{\delta\tilde{\Gamma}}{\delta L}
$$

\begin{equation}
(\tilde{\Gamma} \ast \tilde{\Gamma} = 0)
\end{equation}

up to one loop.

$\tilde{\Gamma}$ is the generating functional of the one particle irreducible part $\Gamma$ (with added external source for composite operators) minus the ”gauge fixing term” (in (16), $A$ and $c$ are the classical counter parts of the gauge fields $A$ and ghost $c$, while $\{\Phi_i\}$ are the classical fields for the matter $X$ and newly introduced field $g$; $K, K_i$ and $L$ are the usual external sources for the gauge variations $\hat{\delta}A, \hat{\delta}\Phi_i$ and $\delta c$ respectively). One then hopes that it is possible to chose the higher order local counter term in such a way that eq. (16) is satisfied to all orders.

Let us now imagine, however, that the invariance conditions 1) and 2) for the measure $DADg$ (eq.s (9) and (11)) are not satisfied [11A]. This means that one should take account of one or both of the following situations:

(1') the condition (1) is not satisfied, i.e. $DA \neq DA^g = DAe^{-\alpha'(A;g)}$, where $\alpha'(A;g)$ is the ”Fujikawa determinant” associated with the non gauge invariance of measure over gauge field itself.

(2') the condition (2) is not satisfied, i.e. $\Delta(A^g) \neq \Delta(A)$.

First of all, the non invariance property 2’) means that the factor $\Delta(A)\delta(F(A))$ in eq. (11) must be replaced by $\Delta(A^{g^{-1}})\delta(F(A))$. 

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Thus, instead of a BRST gauge fixing term (14) one ends up with

$$\int \hat{s}(\bar{c}F(A) + \ln \left( \frac{\Delta(A g^{-1})}{\Delta(A)} \right))$$

(17)

The trouble is that one cannot transform $-\ln \Delta(A)$ into a BRST invariant local term in the action. In fact, the BRST gauge fixed action would appear something like

$$S_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \ln \left( \frac{\Delta(A g^{-1})}{\Delta(A)} \right) + \int \hat{s}(\bar{c}F(A))$$

(18)

The extra one loop term $\alpha'(A; g)$ does not cause any trouble for the BRST scheme to work at least in the example we are interested. One way to push through the BRST scheme may be to replace eq. (18) with

$$S'_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \int \hat{s}(\bar{c}F(A))$$

(19)

It is likely that the effective action (19) leads to a consistent BRST quantization. One may only add that it does not correspond to the path integral method of ref.s [7] and [8] when $\Delta(A g) \neq \Delta(A)$.

To reconcile the "path integral" formulation of FS method with BRST scheme, we propose another possibility.

It must be realized that once the new degree of freedom $g$ is admitted in the theory then there is no reason to exclude new local counter terms of the right dimension which are BRST invariant and which may also depend on $g$. Naturally this will change the model and its "physics", but nevertheless it can remain consistent, in so far as the BRST invariance is maintained.
Let us then introduce the following counter term in our theory

\[ \tilde{\Lambda}_G(A, g; c, \bar{c}, c', \bar{c}', B) = \left[ B G(A g^{-1}) \right] - \left[ B G(A) \right] \]  

where the second pair of "ghosts" \( c', \bar{c}' \) are defined as the BRST singlet

\[ \hat{d} c' = 0 \]
\[ \hat{s} c' = 0 \]  

and \( G(A) \) is the "pseudo gauge fixing" which is generally different from \( F(A) \).

The first term in \( \tilde{\Lambda}_G \) is trivially BRST invariant since all the fields involved are either gauge invariant by themselves or appear as invariant combinations. The second term, on the other hand, can be written as

\[ \hat{s}(\bar{c} G(A)) \]

so it is invariant too.

The effective action now takes the form

\[ S_{eff} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \int \tilde{\Lambda}_G(A, g; c, \bar{c}, c', \bar{c}', B) + \int \hat{s}(\bar{c} F(A)) \]  

Note that the gauge freedom of the BRST invariant theory (22) is represented by the (arbitrary) gauge fixing function \( F(A) \) while each different choice of "pseudo gauge function" \( G(A) \) defines a new model.

Each choice of \( G(A) \) then results in a gauge invariant model which must then be gauge fixed by choosing a particular form for \( F(A) \). In the limit of singular gauge

\[ F(A) \rightarrow G(A) \]  

the effective action (22) gives the series of models depending on \( G(A) \) alone. The
corresponding effective action can be formally written

\[ S_{\text{eff}} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \left[ B G (A^{-1}) - c' \frac{\delta G (A^{-1} h)}{\delta h} \right]_{h=1} \]  

(24)

Note that in (24) the gauge is already fixed (with a singular gauge). To see the gauge invariance property of the model (24), one must go back to eq. (22) with (20), i.e.

\[ S^{\text{inv}}_{\text{eff}} = S_0 + \alpha(A; g^{-1}) + \alpha'(A; g^{-1}) + \left[ B G (A^{-1}) - c' \frac{\delta G (A^{-1} h)}{\delta h} \right]_{h=1} \]  

\[ - \int [B[F(A) - G(A)] - c' \frac{\delta}{\delta h} [F(A^h) - G(A^h)]]_{h=1} \]  

(25)

We have seen in this way that the FS method of formulating an anomalous theory within the path integral formalism apparently generates a series of physically distinct and BRST invariant gauge fields theories.

We will discuss the possible candidate for such a scenario in the next section.

3. Two dimensional induced gravity

In this section we would like to apply the FS method of §1 to analyze the quantization problem of 2d gravity [13] (off critical string) in conformal gauge [14]. The theory at classical level is defined in term of the Polyakov action

\[ S_0 = \sum_{\mu=1}^{d} \int d^2 x \sqrt{g} g^{ab} \partial_a X_{\mu} \partial_b X^\mu \]  

(26)

where \( \{X^\mu(x)\}_{\mu=1,d} \) are the bosonic matter fields coupled to the 2d metric \( g_{ab} \) (in the string language, the string is immersed in a \( d \)-dimensional target space).
We use euclidian metric and introduce the complex coordinates

\[ z = x_1 + ix_2 \]
\[ \bar{z} = x_1 - ix_2 \]

The invariant line element can be written as

\[ ds^2 = g_{ab}dx^a dx^b = e^\phi |dz + \mu d\bar{z}|^2 \tag{27} \]

Thus, one can conveniently parametrize the metric as

\[ g_{zz} = \mu e^\phi, \quad g_{\bar{z}\bar{z}} = \mu e^\phi \]
\[ g_{z\bar{z}} = g_{\bar{z}z} = \frac{1 + \mu \bar{\mu}}{2} e^\phi \]

In term of the parameters \( \mu, \bar{\mu} \) and \( \phi \) the classical action (26) takes the form [15]

\[ S_0 = \sum_{\mu=1}^{d} \int dz \wedge d\bar{z} \frac{\bar{\partial} - \mu \partial} {2i} X_{\mu} (\bar{\partial} - \bar{\mu} \partial) X^{\mu} \]

It is understood that \( \mu \) and \( \bar{\mu} \) are constrained by

\[ |\mu|^2 < 1 \]

The classical action \( S_0 \) is invariant under the gauge group \( G \) which is the semidirect product of Diffeomorphisms (general coordinate transformations) and Weyl transformations. These symmetry groups imply respectively:
1) the symmetry under the general coordinate transformation

\[ z \to z' = f(z, \bar{z}) \]
\[ \bar{z} \to \bar{z}' = \bar{f}(z, \bar{z}) \quad \text{(28)} \]

where the relevant fields transform as follows

\[ X^\mu(z, \bar{z}) \to X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}) \quad \text{(scalar)} \]
\[ \mu(z, \bar{z}) \to \mu'(z', \bar{z}') = -\frac{\partial f - \mu \partial \bar{f}}{\partial f - \mu \partial \bar{f}}(z, \bar{z}) \quad \text{(29)} \]
\[ \phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \phi(z, \bar{z}) + \ln \frac{(\partial f - \mu \partial \bar{f})(\partial f - \bar{\mu} \partial \bar{f})}{D_f^2} \]

where

\[ D_f = \text{det} \begin{pmatrix} \partial f & \partial \bar{f} \\ \partial \bar{f} & \partial \bar{f} \end{pmatrix} \]

2) The symmetry under the local rescaling of the 2d metric

\[ g_{ab} \to e^\sigma g_{ab} \]

or in term of the \( \mu, \bar{\mu} \) and \( \phi \) variables

\[ \mu \to \mu, \quad \bar{\mu} \to \bar{\mu}, \quad \phi \to \phi + \sigma \quad \text{(30)} \]

It is well known that the theory is anomalous, i.e. one can not regularize the path integral in a way that conserves the whole \( G = \text{Diffeo} \times \text{Weyl} \) group.

One can see this easily, examining the matter integral measure \( \mathcal{D}X^\mu \). With the simplest (translationally invariant or "flat") regularization \( \mathcal{D}_0 X^\mu \), one has

\[ \prod_{\mu=1}^d \int \mathcal{D}_0 X^\mu e^{S_0(X^\mu, \bar{\mu})} = \exp - \frac{d}{24\pi} [W(\mu) + \bar{W}(\bar{\mu})] \quad \text{(31)} \]

where \( W(\mu) \) is the Polyakov’s "light cone gauge" action [13].
This is naturally Weyl invariant (\( S_0 \) does not contain the variable \( \phi \)). On the other hand, it is equally clear that one has lost diffeomorphism’s invariance, since the invariance under general coordinate transformations means

\[
\delta W(\mu) = 0
\]  

(32)

under \( \delta \mu = (\bar{\partial} - \mu \partial + \partial \mu)(\epsilon + \mu \bar{\epsilon}) \), which corresponds to the infinitesimal version of eq.s (29) with \( f(z, \bar{z}) = \epsilon(z, \bar{z}), \tilde{f}(z, \bar{z}) = \bar{\epsilon}(z, \bar{z}) \).

Eq. (32) is equivalent to the functional differential equation

\[
(\bar{\partial} - \mu \partial - 2\partial \mu) \frac{\delta W}{\delta \mu(z, \bar{z})} = 0
\]

A well known computation [16] gives, instead,

\[
(\bar{\partial} - \mu \partial - 2\partial \mu) \frac{\delta W}{\delta \mu(z, \bar{z})} = \partial^3 \mu \neq 0
\]  

(33)

Thus, \( D_0X^\mu \) can not be invariant under diffeomorphisms. One can define the diffeomorphisms invariant measure \( D_{Diffeo}X^\mu \) by introducing the local counter term

\[
\Lambda(\mu, \bar{\mu}, \phi) = -\frac{1}{2} \int \frac{dz \wedge d\bar{z}}{2\pi} \left[ \frac{1}{1 - \mu \bar{\mu}} [(\bar{\partial} - \bar{\mu} \partial)\phi(\bar{\partial} - \mu \partial)\phi
- 2(\bar{\partial}\bar{\mu}(\bar{\partial} - \mu \partial) + \partial \mu(\partial - \bar{\mu} \bar{\partial}))\phi] + F(\mu, \bar{\mu}) \right]
\]  

(34)

where \( F(\mu, \bar{\mu}) \) is a local function of \( \mu \) and \( \bar{\mu} \) only. We do not need the explicit form of \( F \) [17].

The new effective action

\[
W_{cov}(\mu, \bar{\mu}, \phi) = W(\mu) + \tilde{W}(\bar{\mu}) + \Lambda(\mu, \bar{\mu}, \phi)
\]

is invariant under diffeomorphisms.
One can write $W_{cov}(\mu, \bar{\mu}, \phi)$ compactly in the form

$$W_{cov}(\mu, \bar{\mu}, \phi) = \int \frac{dz \wedge d\bar{z}}{2\pi} \frac{(\partial - \bar{\mu}\bar{\partial})\Phi(\bar{\partial} - \mu\partial)\Phi}{1 - \mu\bar{\mu}} = \int d^2x \sqrt{g}g^{ab}\partial_a\Phi\partial_b\Phi$$

(35)

where $\Phi = \phi - \ln\partial\zeta\bar{\partial}\zeta$ and $\mu = \frac{\partial\zeta}{\zeta}$ (Beltrami differentials). Non local (with respect to $\mu$ and $\bar{\mu}$) parameter $\zeta(z, \bar{z})$ is Polyakov meson field (13) in 2$d$ gravity.

One characterizes the diffeomorphisms invariant measure $D_{Diffeo}X^\mu$ by

$$\prod_{\mu=1}^{d} \int D_{Diffeo}X^\mu e^{S_0(X,\mu,\bar{\mu})} = exp - \frac{d}{24\pi} W_{cov}(\mu, \bar{\mu}, \phi)$$

(36)

(One can understand the appearance of $\phi$ field, which is absent in the classical action, as due to the introduction of a covariant regularization: $\Lambda_{cov}$, $ds^2 \sim e^\phi|dz|^2 > \Lambda_{cov}^2$).

Following for instance DDK [14], in what follows we consistently make use of the diffeomorphisms invariant measure. Thus, except when indicated explicitly otherwise,

$$D_X^\mu \equiv D_{Diffeo}X^\mu$$

(37)

and more generally $D\varphi \equiv D_{Diffeo}\varphi$ for any other filed $\varphi$.

Evidently, the diffeomorphisms invariant measure $DX^\mu$ can not be invariant under the Weyl transformation

$$\phi \rightarrow \phi + \sigma$$

Thus, one establishes that the theory is $G$ anomalous.

(Faddeev-Shatashvili method)
Having seen that our model for 2d gravity is anomalous, one would like to apply to it the FS method of "gauge invariant" quantization of §1. As in §1, we "preestablish" the gauge choice for the full group \( G = \text{Diffeo} \times \text{Weyl} \)

\[
\begin{align*}
\mu &= \mu_0 \\
\bar{\mu} &= \bar{\mu}_0 \quad \text{diffeomorphisms} \\
F(\phi) &= 0 \quad \text{Weyl}
\end{align*}
\]

Since our regularization preserves the diffeomorphisms we assume that the gauge fixing problem (with relevant "b, c" ghosts) for diffeomorphisms has been already taken care for.

To deal with anomalous Weyl symmetry, we have to introduce an extra degree of freedom, a scalar field \( \sigma(z, \bar{z}) \), corresponding to the element of Weyl symmetry group \( g = e^{\sigma(z, \bar{z})} \).

The anomaly cancelling counter term suggested by FS is then given by

\[
\alpha(\mu, \bar{\mu}, \phi; -\sigma) = W_{\text{cov}}(\mu, \bar{\mu}, \phi - \sigma) - W_{\text{cov}}(\mu, \bar{\mu}, \phi) =
\]

\[
= -\frac{1}{2} \int \frac{dz \wedge d\bar{z}}{2i} \frac{1}{1 - \mu \bar{\mu}} \left[ (\partial - \bar{\mu} \bar{\partial}) \sigma(\bar{\partial} - \mu \partial) \phi + 2(\partial - \bar{\mu} \bar{\partial}) \sigma(\bar{\partial} - \mu \partial) \phi \\
- 2(\bar{\partial} \bar{\mu} (\bar{\partial} - \mu \partial) + \partial \mu (\partial - \bar{\mu} \bar{\partial})) \phi \right]
\]

Note that the non local part of \( W_{\text{cov}} \) is cancelled and \( \alpha(\mu, \bar{\mu}, \phi; -\sigma) \) is perfectly local. Naturally, one needs the counter term \( \alpha \) for each covariant one loop integral corresponding not only to the matter field \( \{X^\mu\}_{\mu=1}^d \), but also to the diffeomorphism ghosts, \( b, c \) and \( \bar{b}, \bar{c} \), as well as to the \( \phi \) field contained in \( W_{\text{cov}}(\mu, \bar{\mu}, \phi) \).

Thus, the effective action in sense of §2 is given by

\[
S_{\text{eff}} = S_0(X, \mu, \bar{\mu}) + S_{\text{gf}}^{(d)}(b, c, \bar{b}, \bar{c}, B, \bar{B}, \mu, \bar{\mu}) + \gamma' \alpha(\mu, \bar{\mu}, \phi; -\sigma)
\]

where \( S_{\text{gf}}^{(d)} \) is the gauge fixing term with respect to the non anomalous diffeomorphism symmetry.
As explained above, the coefficient $\gamma'$ is contributed by all the relevant fields, that is $\{X^\mu\}_{\mu=1}^d \Rightarrow d, (b, c, \bar{b}, \bar{c}) \Rightarrow -26, \phi \Rightarrow 1$, which gives $\gamma' = \frac{d-26+1}{24\pi} = \frac{d-25}{24\pi}.$

Note that the contribution of $\phi$ field is due to the fact that $D_{Diff} \phi \neq D_0 \phi$, or in the terminology of §2, that one needs the "second" FS counter term "$\alpha'(\phi; \sigma)$".

One can now write down the partition function $Z$ with the FS prescription (within the path integral formalism of ref. [7], see eq. (8) of §1). Integrating out the "matter fields" $\{X^\mu, b, c, \bar{b}, \bar{c}\}$, one has

$$Z \sim \int D\sigma D\phi \left[ \exp -\gamma' \int \frac{dz \wedge d\bar{z}}{2i} \frac{1}{1-\mu_0 \bar{\mu}_0} \left( (\partial - \bar{\mu}_0 \bar{\partial})(\phi - \sigma)(\bar{\partial} - \mu_0 \partial)(\phi - \sigma) 
- 2(\bar{\partial} \mu_0(\bar{\partial} - \mu_0 \partial) + \partial \mu_0(\partial - \bar{\mu}_0 \bar{\partial}))(\phi - \sigma) \right) \Delta(\phi - \sigma) \delta(F(\phi)) \right]$$

where the local action in the exponential is essentially a Liouville action $S_L'(\phi')$, $(\phi' = \phi - \sigma)$. The last two factors come from the $\delta$ function insertion

$$\Delta(\phi) \int D\sigma \delta(F(\phi + \sigma)) = 1$$

Note that, since $D\sigma \equiv D_{Diff} \sigma \neq D_0 \sigma$ ($D_0 \sigma$ "flat" measure)

$$\Delta(\phi - \sigma) \neq \Delta(\phi)$$

Formally, one can write the $\Delta(\phi - \sigma)$ factor as a local action with the help of the "Weyl ghosts" $\psi$ and $\bar{\psi}$

$$\Delta(\phi - \sigma) = \int D\psi D\bar{\psi} \exp - \int \bar{\psi} \frac{\delta F(\phi - \sigma)}{\delta \phi} \psi$$

(BRST procedure)

The path integral argument of §1 is at best heuristic. It may suggest the possible models but one can not prove in this way their consistency.
As it has been argued in §1, one may start a more precise discussion after setting up the BRST quantization procedure. The BRST properties of the type of models we are dealing with here, have been studied in details for the critical case, i.e. for \( d = 26 \), where the theory is not anomalous. In ref. [15], the BRST transformation properties of the fields are given. They may be used to study our (off critical) model.

One has (see eq. (29))

\[
\begin{align*}
\hat{\delta}X^\mu &= (\xi \cdot \partial)X^\mu \\
\hat{\delta}\mu &= (\bar{\partial} - \mu \partial + \partial\mu)c \\
\hat{\delta}\phi &= \psi + (\xi \partial)\phi + (\partial\xi) + \mu \partial\xi + \bar{\mu} \bar{\partial}\xi \\
\hat{\delta}\xi &= (\xi \cdot \partial)\xi \\
\hat{\delta}c &= c\partial c \\
\hat{\delta}\psi &= (\xi \cdot \partial)\psi
\end{align*}
\] (45)

where \( \xi \cdot \partial \) means \( \xi \partial + \bar{\xi} \bar{\partial} \).

Here \( \hat{\delta} \) stands for the both Weyl and diffeomorphism symmetries. The diffeomorphism ghosts \( c, \bar{c} \) are related to the original \( (\xi, \bar{\xi}) \) (corresponding to \( \delta z = \epsilon(z, \bar{z}), \delta \bar{z} = \bar{\epsilon}(z, \bar{z}) \)) by

\[
\begin{align*}
c &= \xi + \mu \bar{\xi} \\
\bar{c} &= \bar{\xi} + \bar{\mu} \xi
\end{align*}
\] (46)

To eq. (45), we must add the transformation of the auxiliary field \( \sigma(z, \bar{z}) \). Since \( \sigma \) must be a scalar with respect to diffeomorphisms one has

\[
\hat{\delta}\sigma = \psi + (\xi \cdot \partial)\sigma
\] (47)

Together with the formulae in eq.s (45) to (47), one consistently finds

\[
\hat{\delta}^2 = 0
\] (48)

One should add also the diffeomorphisms anti ghost \((b, \bar{b})\) and Weyl anti ghost
\( \bar{\psi} \) with the corresponding Nakanishi-Lantrup fields \( B \) and \( D \). Their transformation properties are

\[
\begin{align*}
\hat{s}b &= B, \quad \hat{s}\bar{b} = \bar{B}, \quad \hat{s}\bar{\psi} = D \\
\hat{s}B &= \hat{s}\bar{B} = \hat{s}D = 0
\end{align*}
\]

(49)

We have seen, however, that the Faddeev-Popov factor \( \Delta(\phi) \) is not Weyl invariant (43). Thus, according to the result of §1, one needs to correct the effective action \( S_{\text{eff}} \) by modifying the factor \( \Delta(\phi - \sigma)\delta(F(\phi)) \) into a BRST gauge fixing term. As we have seen in §1, such a prescription is not unique. Formally, any action of the form

\[
BRST(\text{invariant}) + \hat{s}(\psi F(\phi))(BRST\text{exact})
\]

will do the job.

Now the factor \( \Delta(\phi - \sigma)\delta(F(\phi)) \) can be rewritten in the form

\[
\exp \left( - \int \left( D\bar{\psi} + \bar{\psi}\delta F \delta(\phi - \sigma)\psi' \right) \right)
\]

Thus, in order to follow this expression as close as possible, we suggest to add a counter term of the form of eq. (20) in §1

\[
\tilde{\Lambda}_G(\phi, \sigma; \psi, \bar{\psi}, \psi', \bar{\psi}', D) = \left[ DG(\phi - \sigma) + \bar{\psi}' \frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \right] - \left[ DG(\phi) + \bar{\psi} \frac{\delta G}{\delta \phi}(\phi)\psi \right]
\]

(50)

where we have introduced the function \( G(\phi) \) to distinguish it from the true gauge fixing term \( s(\bar{\psi} F(\phi)) \). The new fields \( \psi' \) and \( \bar{\psi}' \) in eq. (50) (\( c' \) and \( \bar{c}' \) in eq. (20) ) are Weyl singlet and transform as

\[
\begin{align*}
\hat{c}' &= 0 \\
\hat{\bar{c}}' &= (\xi \cdot \partial)\psi'
\end{align*}
\]

(51)
With the addition of the counter term \( \tilde{\Lambda}_G \), the effective action now reads

\[
\tilde{S}_{\text{eff}} = S''_L(\phi - \sigma) + \int \tilde{\Lambda}_G(\phi, \sigma; \psi, \bar{\psi}, \psi', \bar{\psi}', D) + \int \tilde{s}(\bar{\psi} F(\phi))
\]

\[
= S''_L(\phi - \sigma) + \int \left[ DG(\phi - \sigma) + \psi' \frac{\delta G}{\delta \phi}(\phi - \sigma) \psi' \right] + \int \tilde{s}(\bar{\psi}(F - G)(\phi))
\]

(52)

The expression for \( \tilde{S}_{\text{eff}} \) contains two arbitrary functions \( F(\phi) \) and \( G(\phi) \). Their roles are completely different. While \( F(\phi) \) is a genuine gauge fixing function, each choice of \( G(\phi) \) actually defines a new model.

Naturally, the "series" of models (at arbitrary gauge) includes the familiar cases. For example, if one fix the model by choosing

\[ G = 0 \]

one reproduces the physically equivalent formulations of DDK model.

Alternatively, for any given \( G \), one may consider the singular gauge limit

\[ F \rightarrow G \]

In this limit the model formally corresponds to the action

\[
\tilde{S}_{\text{eff}} = S''_L(\phi - \sigma) + \int \left[ DG(\phi - \sigma) + \psi' \frac{\delta G}{\delta \phi}(\phi - \sigma) \psi' \right]
\]

(53)

This is the type of model treated in ref. [18]. One may further add the BRST invariant term \( -\frac{1}{2} \lambda \int D^2 \) and transform \( \tilde{S}_{\text{eff}} \) into

\[
\tilde{S}'_{\text{eff}} = S''_L(\phi - \sigma) + \int \left[ \frac{1}{2\lambda} G^2(\phi - \sigma) + \psi' \frac{\delta G}{\delta \phi}(\phi - \sigma) \psi' \right]
\]

(54)

Eq. (53) (or (54)) seems to be the closest BRST quantizable approximation to
the consequence of FS prescription, i.e. the insertion

$$1 = \Delta(\phi) \int D\sigma \delta(G(\phi + \sigma)) \quad (55)$$

In ref. [18], and in some later works, the choice

$$G(\phi) = R(\phi) - R_0 \quad (56)$$

with $R$ the scalar curvature, has been made. Using (56), the effective action (54) becomes

$$\tilde{S}''_{eff}((\phi' = \phi - \sigma), \psi', \bar{\psi}, \bar{\psi}') = S''_{L}(\phi') + \int \left[ \frac{1}{2\lambda}(R(\phi') - R_0)^2(\phi - \sigma) + \bar{\psi}' \frac{\delta R}{\delta \phi}(\phi - \sigma)\psi' \right] \quad (57)$$

Note that the model defined by (57) is fully interacting. In particular a) the presence of propagating $\psi'$ and $\bar{\psi}'$ fields and b), more importantly, the presence of $\psi', \bar{\psi}'$ and $\phi'$ (Yukawa) interaction in (57), change the parameters in the Liouville type action $S''_{L}(\phi')$. Such a change, which affects the low energy dynamics of (57), can not be calculated exactly. It is not easy even to develop a systematic perturbation expansion [20]. We believe [18] [19] that the modification represented by eq. (57) may result in deviations from the classical DDK result, when one uses (57) to calculate such physical quantities as string tension and anomalous dimension.

Lastly, it must be mentioned that the BRST invariant term

$$\int \bar{\psi}' \frac{\delta G}{\delta \phi}(\phi - \sigma)\psi' \quad (58)$$

in (53) could also be obtained from the alternative gauge fixing

$$S_{gf} = \int \hat{s} \left[ \bar{\psi}' \frac{\delta G}{\delta \phi}(\phi - \sigma)\sigma \right] \quad (59)$$

In this case, one can dispense with the extra BRST invariant (for Weyl trans-
formation) $\psi'$ and $\bar{\psi}'$ degrees of freedom. The gauge fixing function is

$$F(\phi, \sigma) = \frac{\delta G(\phi - \sigma)}{\delta \phi} (\phi - \sigma) \sigma \quad (60)$$

It looks as if this model is gauge equivalent to the DDK model, since the gauge choice $G(\phi) = \phi$ gives the effective action

$$S_{eff} = S'_{L}(\phi - \sigma) + \int (\bar{\psi} \psi + D\sigma) \sim S'_L(\phi) \quad (\sigma = 0) \quad (61)$$

The Liouville action $S'_L$ here is identical to eq. (42) without further renormalization (eq. (57) is a free field action).

In the next section, we apply the DDK [14] type consistency arguments to analyze the consequences of the model (eq. (57)), paying attention to the influence of the Yukawa term $(\bar{\psi}' \psi' \phi)$ in (57).

4. Physical consequences (modified KPZ-DDK model)

After reading the last section, one may wonder if the counter term such as

$$DG(\phi - \sigma) + \bar{\psi}' \frac{\delta G}{\delta \phi} (\phi - \sigma) \psi' \quad (62)$$

(eq. (50) of §3) may indeed influence the physics in any way. In fact, it is very probable that such an influence is washed away for large class of "pseudo gauge functions" $G(\phi)$ by the renormalization group argument.

However, for the specific choice of ref. [18], i.e. (eq. (56) of §3)

$$G(\phi) = R(\phi) - R_0$$

it gives actually the possibility of modifying the classical KPZ results on the string tension and anomalous conformal dimensions.
The effective action which corresponds to the above choice of $G(\phi)$ is given by eq. (57). As it has been remarked previously, this action is equivalent to the well known Kawai-Nakayama $R^2$ model [21], if one omits precisely the "fake" FP term

$$\int \bar{\psi}' \frac{\delta R}{\delta \phi} (\phi - \sigma) \psi'$$

(63)

in eq. (57).

Now, in ref. [21], it has been shown that the Kawai-Nakayama model with $R^2$ (or $(R - R_0)^2$) term gives the same scaling behaviour for large distance as the original DDK model. That is, for the fixed area partition function

$$Z(A) = \int \mathcal{D}(\text{fields})(\text{Jacobians}) \exp - S_{eff} \times \delta \left( \int dx^2 \sqrt{g} - A \right) \sim \text{const} A^{-\Gamma(h) - 3}$$

(64)

as $A \to \infty$, except for $\frac{1}{A}$ correction in the exponent. The string tension is the same as the KPZ result ($h$ is the genus)

$$\Gamma(h) = (1 - h) \frac{25 - d + \sqrt{(1 - d)(25 - d)}}{12}$$

(65)

It is not a simple matter to calculate the possible change with respect to this result in the presence of pseudo FP term (63). The difficulty is due to the fact that we have here the genuine interacting theory instead of effective gaussian model such as the original DDK case [14].

Here we present the approximate analysis which is, at best, valid for the low energy (large distance) regime.

Writing down the pseudo FP term (63) in detail, we have

$$S'' = \int \frac{dz \wedge d\bar{z}}{2i} \sqrt{g} \bar{\psi}' (-\bar{\partial} \bar{\partial} + \partial \bar{\partial}(\phi - \sigma) + \bar{R}) \psi'$$

(66)

(in the conformal gauge where $\mu = \mu_0 = 0, \bar{\mu} = \bar{\mu}_0 = 0$).
In $S''$, the free part for $\bar{\psi}'$ and $\psi'$ has the structure of the so called $bc$ ghost system

$$ S''(\text{free}) = \int b \bar{\psi}' \partial c \frac{dz \wedge d\bar{z}}{2i} \quad (67) $$

if one identifies

$$ b \equiv \partial \bar{\psi}' $$

$$ c \equiv \psi' \quad (68) $$

with the stress energy tensor and ghost number current given by

$$ T = -b \partial c $$

$$ J = bc \quad (69) $$

(note that the conformal dimension of $b$ and $c$ here are respectively 1 and 0).

Then one can give an equivalent bosonic system with

$$ T' = -\frac{1}{2}((\partial \varphi)^2 + Q' \partial^2 \varphi) $$

$$ J' = i \partial \varphi \quad (70) $$

which reproduces the same algebraic structure as the system (69) for the suitable value of $Q'(=i)$, if the new scalar field $\varphi$ satisfies

$$ \varphi(z) \varphi(w) \sim -\log |z - w|^2 $$

One should still take account of the interaction (Yukawa) term in (66). To do so, we write

$$ \partial \bar{\psi}' \psi' = iA \partial \varphi + ... \quad (71) $$

where the ... represent the higher order corrections.
Thus, the low energy equivalent of (66) is

\[ S'' \sim \int \sqrt{g}( -\varphi \partial \bar{\partial} \varphi - i(1 + A)\varphi \hat{R} + i\alpha A\varphi \partial \bar{\partial} \phi') \] (72)

(\phi' = \phi - \sigma, see §3).

The undetermined constant A represents the first order correction due to the interaction. The constant \(\alpha\) is the usual gravitational correction \((g_{ab} = e^\phi \hat{g}_{ab} \rightarrow e^{\alpha\phi} \hat{g}_{ab})\).

To avoid the imaginary coupling constant in (72) (problem of unitarity in BRST approach), we "Wick rotate" \(\varphi, \varphi \rightarrow i\varphi\).

Then, with the redefinition of constants in (72), one can write a low energy approximation as

\[ S'' = \frac{1}{8\pi} \int \sqrt{\hat{g}}(\varphi \partial \bar{\partial} \varphi - 2B\varphi \partial \bar{\partial} \phi' + \tilde{Q} \varphi \hat{R}) \] (73)

Putting this together with the rest of the effective action in eq. (57), our low energy approximation consists of taking a gaussian model with two scalars

\[ S'_{eff} = \frac{1}{8\pi} \int dz \wedge d\bar{z} \frac{2i}{2i} \sqrt{\hat{g}}(-M_{ij}\Phi^i \Delta g \Phi^j - Q_i \hat{R} \Phi^i) \] (74)

where

\[ \Phi^i \equiv (\Phi^1, \Phi^2) = (\phi', \varphi) \]

\[ Q_i \equiv (Q_1, Q_2) = (Q, -\tilde{Q}) \]

and

\[ M_{ij} = \begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix} \]

Such a 2-bosonic system with "Lorentzian" metric as a kind of improvement over the standard Liouville type 2d gravity (DDK model) has been suggested in
the past [18] [23]. More recently, Cangemi, Jackiw and Zwiebach have given the thorough field theoretical analysis of such a system for $B = 0$, treating it as the "dilatation gravity" [24]. Here, however, the presence of $\varphi - \varphi'$ coupling term ($B \neq 0$) is crucial for the possible modification of KPZ-DDK result.

The origin of such a term is, of course, the Yukawa coupling in the original counter term (62).

At this point, one can in principle apply the technics of ref. [20] to get the perturbative estimate of the constant $A$ (i.e. $B$). We leave such an analysis for further publication and content ourselves with repeating the original DDK consistency arguments to indicate that indeed one has the possibility of changing the KPZ-DDK result.

Thus, we would like to apply the effective action (74) to estimate

a) string tension $\Gamma(h)$, and

b) renormalization $\Delta_0 \to \Delta$ of the conformal dimension of a primary operator $\mathcal{O}$.

From the effective action (74), one can derive the expression for the gravitational stress energy tensor

$$T_{grav} = -\frac{1}{2}(M_{ij}\partial\Phi^i\partial\Phi^j + BQ_i\partial^2\Phi^i)$$  \hspace{1cm} (75)

which contributes to the central charge by the amount

$$c_{grav} = 2 + 3M_{ij}Q_iQ_j$$  \hspace{1cm} (76)

where $M^{ik}M_{kj} = \delta^i_j$.

a) String tension

The detail of how to generalize DDK argument to get the string tension $\Gamma(h)$ in our model is given in Ref. [18]. We limit ourselves, therefore, to give the more relevant results.
The consistency conditions lead to the determination of $Q_i$’s as

$$
Q_1 = -\frac{1}{\sqrt{3}} \left( B \sqrt{1 + Bd} - \sqrt{1 + B^2 + (B - 1)d} \right)
$$

$$
Q_2 = \sqrt{\frac{1 + Bd}{3}}
$$

(77)

Then the string tension $\Gamma(h)$ is given, just as in ref. [14], by

$$
\Gamma(h) = \chi(h) \frac{Q_1}{\alpha} + 2
$$

($\chi(h) = 2(1 - h)$ is the Euler index). $\alpha$ can be calculated again as in ref. [14] from

$$
dim(e^{\alpha e^\phi \sqrt{\bar{g}}}) = 1
$$

which gives

$$
\alpha = -\frac{\sqrt{1 + B^2}}{2\sqrt{3}} \left( \sqrt{25 + (B - 1)d} - \sqrt{1 + (B - 1)d} \right)
$$

(78)

Thus, one obtains the string tension in our model as the function of $B$

$$
\Gamma(h) = \frac{2(1 - h) B \sqrt{1 + Bd} - \sqrt{(1 + B^2)(25 + (B - 1)d)}}{\sqrt{1 + B^2} \sqrt{25 + (B - 1)d} - \sqrt{1 + (B - 1)d}} + 2
$$

(79)

which reduces to the KPZ expression if $B = 0$ (i.e. $d \leq 1$). Note that $\Gamma(h)$ is real for $B \geq 1 - \frac{1}{d}$ with an arbitrary positive $d$. In view of excellent agreement between KPZ formula and ”experiments” for $d < 1$, we might expect some sort of phase transition behaviour

$$
B \propto \theta(d - 1)
$$

(where $\theta(x)$ is the step function) but it would be very hard to show such a behaviour by the limited technics available [20]. For the various ”improvements” and applications to statistical mechanic of (79) we refer to [19].
b) Anomalous dimension

The calculation of the renormalization of conformal dimension is more straightforward.

Let the base conformal dimensions of an operator $\mathcal{O}$ be $(\Delta_0, \tilde{\Delta}_0)$. We would like to construct the globally defined operators $\int e^{\alpha \phi} \sqrt{g}$ and $\int e^{\beta \phi} \sqrt{\tilde{g}} \mathcal{O}$. This requirement implies

$$\dim(e^{\alpha \phi} \sqrt{g}) = (1, 1)$$
$$\dim(e^{\beta \phi} \sqrt{\tilde{g}} \mathcal{O}) = (1, 1)$$

These conditions give rise to the two equations

$$\alpha^2 + (Q_1 + BQ_2)\alpha + 2(1 + B^2) = 0$$
$$\beta^2 + (Q_1 + BQ_2)\beta + 2(1 + B^2)(1 - \Delta_0) = 0$$

(80)

The renormalized dimension $\Delta$ of the operator $\mathcal{O}$ can be read off from the asymptotic formula

$$F_{\mathcal{O}}(A) = \int \mathcal{D}\phi' \mathcal{D}\varphi e^{-\tilde{S}_{adj}} \left( \int e^{\alpha' \phi} \sqrt{g} - A \right) \int \mathcal{O} e^{\beta' \phi} \sqrt{\tilde{g}} / Z(A) \sim K_{\mathcal{O}} A^{1-\Delta}$$

This gives, just as in ref. [14],

$$1 - \Delta = \frac{\beta}{\alpha}$$

(81)

From eq.s (80) and (81), one gets the equation determining $\Delta$

$$\Delta - \Delta_0 = -\frac{1}{2} \frac{1}{1 + B^2} \alpha^2 \Delta(\Delta - 1)$$

Needless to say, this too reduces to KPZ result when $B = 0$.

5. Conclusion

In this note, we have tried to analyze further consequences of the Faddeev-Shatshvili method of quantizing anomalous gauge fields theories.
In contrast with other authors [5A], we did not try to show the equivalence with the "gauge non invariant" method of which the Jackiw-Rajaraman treatment of the chiral Schwinger model is a distinguished example. On the contrary, we have argued that, in certain cases of physical interest, the FS method can be used to generates new models.

The series of "new" 2d gravity models proposed here includes the models in ref.s [18] [19] as well as the Kawai-Nakayama type \((R - R_0)^2\) (or \(R^2\)) models [21] [22].

The analysis presented in §4 with respect to the model eq. (57) is at best heuristic and we certainly cannot (and do not) claim to have "solved" the famous \(d = 1\) barrier problem in 2-d gravity. We merely indicate possible ways to modify the original DDK model.

To see if the possibility of enlarging in this way the 2d (induced) gravity models really throws some light on the problem of the \(d = 1\) barrier in 2d gravity, we need a more thorough analysis of the consistency of these models as well as a better understanding of their physical consequences.

One would like to end by mentioning a further peculiarity about the anomalous Diffeomorphism-Weyl gauge symmetry of 2d gravity.

It is natural to ask whether, instead of somehow trying to conserve the entire gauge symmetry of the anomalous classic model, one may still have a physical consistent quantum model by keeping only the "maximal" anomalous free part of the classical symmetry (up to local-counterterms).

Recently precisely such a suggestion has been made by R. Jackiw and others [25]. They counter the conventional argument favouring the Diffeomorphism symmetry (over Weyl) by pointing out the even greater difficulty of conserving the whole Diffeomorphism symmetry in the quantum canonical hamiltonian approach [26].

Thus, in ref. [25], it has been suggested to conserve Weyl symmetry plus area
(volume) preserving diffeomorphim (i.e. Diffeomorphism \( x^\mu \to x'^\mu = f^\mu(x) \) with the constrain \( \det \left[ \frac{\partial f^\mu}{\partial x^\nu} \right] = 1 \).)

Jackiw’s formalism can be generalized to the series of models which are symmetric under the modified Diffeomorphisms \( D^{(k)} \):

\[
x^\mu \to x'^\mu = f^{(k)}_\mu(x)
g_{\mu\nu}(x) \to g_{\mu'\nu'}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \left[ \det \left( \frac{\partial x^\eta}{\partial x'^\lambda} \right) \right]^{\frac{k-1}{k}},
\]

where \( 0 < k \leq 1 \). While \( k = 1 \) corresponds to the usual Diffeomorphism invariant DDK like model, the limit \( k \to 0 \) can be shown to give the improved Weyl invariant model of Jackiw et al.

Superficially, these models parameterized by \( k \) corresponds to different gauge symmetries and, in particular, one might expect a drastic change of the physics between the two limits \( k = 1 \) (Diffeomorphisms) and \( k \to 0 \) (Weyl and Area preserving diffeomorphisms).

However, there are reasons to believe that they actually correspond to the same physics.

(1) One can move formally from the "standard" \( D^{(1)} \) invariant model to the \( D^{(k\neq1)} \) defined throught Eq. (82) by a simple changing of variables. In terms of Beltrami parameterization of the 2d metric in section 3; this changing of variables is given by

\[
\begin{align*}
\mu &\to \mu^{(k)} = \mu \\
\bar{\mu} &\to \bar{\mu}^{(k)} = \bar{\mu} \\
\phi &\to \phi^{(k)} = k\phi + (1-k) \log \left( \frac{1}{1-\bar{\mu}\mu} \right) = \phi + (k-1) \log \sqrt{-g}.
\end{align*}
\]

At quantum level, Eq. (83) amounts to different choice of local counter-term.
(2) In ref. [27], the two dimensional Hawking radiation has been calculated using Jackiw’s Weyl invariant model as well as the general $D^{(k)}$ invariant model. In either case, the result is identical with the standard ($D^{(1)}$ invariant) model. This fact means that, at least the black-hole thermodynamics is independent from the parameter $k$.

If one conjectures from these facts that the choice of invariant gauge group is in some sense irrelevant (at least for 2d gravity), the implication for the anomalous gauge field theory is not clear.

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