The superfluidity of fermions coupled to gravity

G.V. Vlasov*

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Abstract

We investigate superfluidity of the relativistic fermi-gas with gravitational interaction. The excitation spectrum is obtained within the linearized theory. While superfluidity may take place at a definite ratio of the Fermi momentum, rest mass and coupling constant, the metric coefficients play predominant role forming the gap of excitation spectrum.

1 Introduction

The superfluidity (superconductivity, in the strict sense) of relativistic fermi-systems is under scientific attention since Bailin and Love [6] initiated investigation in this field. Several significant papers were dedicated to superfluidity in nuclear systems [2] with direct applications to a neutron star matter. Among them the paper by Kucharek and Ring [1] and, especialy, Ref. [3] and [4] should be noted as the basic works where the general approach to relativistic superfluidity is outlined. Besides, Capelle and Gross [5] have shown how to find the excitation spectrum.

In the present paper we shall discuss superfluidity of a fermi-system with gravitational interaction between particles. This kind of fermi gas attracts the interst in recent years [7].

*E-mail: vs@landau.ac.ru
In general the investigation of superfluidity includes several steps. 1) Specifying the Lagrangian \( L \) or Hamiltonian \( H \); 2) specifying the anomalous terms there; 3) the Bogolubov transformation; 4) obtaining the equations of motion; 5) derivation of the excitation spectrum from equations of motion truncated up to the Hartree approximation.

### 1.1 The interacting terms

For a given Lagrangian \( L \) of the many-fermion interacting system the canonical formalism

\[
H = L - \dot{\phi} \pi^i \quad \pi^i = \frac{\partial L}{\partial \dot{\phi}^i}
\]

allows to find the Hamiltonian

\[
H = T + U
\]

consisting of the kinetic energy of free fermions

\[
T = \int d^3r \left( i \bar{\psi} \gamma^k \partial_k \psi + m \right)
\]

and the interaction term

\[
U = \int d^3r \bar{\psi} \Sigma \psi \quad \Sigma = f_i \Gamma^i A_i
\]

which represents the interaction with field \( A_i \) corresponding to vertex \( \Gamma^i \) and coupling constant \( f_i \). If \( \Gamma \) includes no derivative coupling we can extract the interacting term immediately from the Lagrangian \( L \).

For instance

\[
\Gamma^\omega = \gamma^\mu \quad \Gamma^\sigma = 1
\]

are the vertices of \( \sigma - \omega \) model, while the vertex of QCD is

\[
\Sigma = f \gamma^\mu \lambda_a A^a_\mu
\]

In order to present the Hamiltonian of great canonical ensemble, the chemical potential \( \mu \) is introduced as an additional term

\[
- \mu \psi^\dagger \psi
\]
Indeed,
\[ \int d^3r \mu \psi \psi \equiv \mu N \] (8)

This term stands in (4) implicitly, or we can extract \(-\mu\) directly from the scalar potential:
\[ \Sigma \rightarrow \Sigma - \gamma^0 \mu \] (9)

The Hamiltonian of superfluid fermi-system includes besides (4) the anomalous term
\[ W = \bar{\psi} \beta \Delta \psi + \bar{\psi} \beta \Delta \psi^\dagger \quad \Delta = f_i \Gamma^i a_i \] (10)
constructed from (4), where \(a_i\) is the anomalous field (do not mix it with \(A_i\)), while \(\bar{\psi}_c = -\psi^T C\) and \(\psi_c = C \bar{\psi}^T = C \beta \psi^*\) are the charge-conjugated spinors and
\[ C = i \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} \] (11)
is the matrix of charge conjugation. It is clear that
\[ -C^T = C = \beta C \beta \quad C^2 = -1 \] (12)
\[ C \beta = \beta C = i \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} \] (13)
and \(\bar{\psi}_c \psi\) is replaced by \(\varphi_\uparrow \varphi_\downarrow\) in the non-relativistic limit.

1.2 The Bogolubov transformation

Substituting
\[ \hat{\psi} = \sum_p \phi_p (r) \hat{b}_p^\dagger \] (14)
we find a common field equation:
\[ \{-i \vec{\gamma} \cdot \nabla + m + \Sigma\} \varphi_p = \beta \varepsilon_p \varphi_p \] (15)
(no summation over \(p\) in the right side) that is, briefly,
\[ \hat{h} \varphi = \varepsilon \varphi \quad \hat{h} = -i \vec{\alpha} \cdot \nabla + \beta m + \beta \Sigma \] (16)
As soon as the anomalous term (10) appears we must use the Bogolu-
bov transformation
\[ \psi = \sum_p u_p(r) \hat{b}_p + v_p^*(r) \hat{b}_p \quad |u_p|^2 + |v_p|^2 = 1 \] (17)
instead of (14). For short we shall omit index \( p \); for instance
\[ \psi = u \hat{b}^\dagger + v^* \hat{b} \psi^\dagger = u^* \hat{b} + v \hat{b}^\dagger \] (18)
Thereby, we get two equations
\[
\begin{aligned}
\{ -i \vec{\gamma} \cdot \nabla + m + \Sigma \} u &= \beta \varepsilon u - \Delta^\dagger \beta C v \\
\{ -i \vec{\gamma} \cdot \nabla + m + \Sigma \} v &= -\beta \varepsilon v + C \beta \Delta u
\end{aligned}
\] (19)
for functions \( u \) and \( v \) instead of one for \( \phi \). Therefore,
\[
\begin{pmatrix}
\hat{h} + \Sigma & \beta \Delta^\dagger \beta C \\
-C \Delta & \hat{h} + \Sigma
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix} u \\ v \end{pmatrix}
\end{pmatrix}
= E
\begin{pmatrix}
\begin{pmatrix} u \\ v \end{pmatrix}
\end{pmatrix}
\] (20)
Capelle and Gross [5] found two branches of excitations. It is the anal-
logue of the massive and acoustic modes in relativistic bose-condensate [11]. The
subject of our interest is the massless mode whose excitation spectrum within
the Hartree approximation looks as
\[ E^2 = (\varepsilon - \mu)^2 + \|\Delta\|^2 \quad \varepsilon^2 = p^2 + m^2 \] (21)
or
\[ E^2 (\xi) = \xi^2 + \|\Delta\|^2 (\xi) \] (22)
where the gap
\[ \|\Delta\|^2 = -C \Delta \beta \Delta^\dagger \beta C \] (23)
is determined immediately from (20). Particularly, for the scalar field we get
from (23) the result of Capelle and Gross [5] \( \|\Delta\|^2 = |a|^2 \).
Discussion in the frames of Bogolu-bov-Hartree-Fock approximation, for
the scalar-vector and pseudoscalar-isovector vertices, is performed in [1, 3].
The basis of exact solution in the frames of density functional theory is not
emphasized yet.
2 Coupling with the gravitational field

The Lagrangian of fermions coupled to the gravitational field \[8\] reads as

\[
\Lambda = \sqrt{-g} L = \det g_{\mu\nu} \tag{24}
\]

where \(g_{\mu\nu}\) is the metric tensor and

\[
L = i \bar{\psi} \gamma^\mu \nabla_\mu \psi + m \bar{\psi} \psi \tag{25}
\]

has the form of usual free Lagrangian containing, however, gamma-matrices \(\gamma^\mu\) and covariant derivative

\[
\nabla_\mu = \partial_\mu + \Omega_\mu \tag{26}
\]

with properties

\[
\begin{align*}
\Omega_\mu &= \frac{1}{4} \gamma^\nu \gamma_\nu = -\frac{1}{4} \Omega_{\mu ab} \Sigma_{ab} \\
\Omega_{\mu ab} &= \partial_\mu g_{ab} g^{ab} - \Gamma^a_{\mu b} g_{aa} g^{ab} \\
\Sigma_{ab} &= \frac{1}{2} (\gamma_a \gamma_b - \gamma_b \gamma_a) \\
\Sigma_{ab}^2 &= 1 \\
\gamma^a \Sigma_{ab} &= 0
\end{align*} \tag{27}
\]

The Lagrangian includes the nonlinear terms (with respect to field \(g_{\mu\nu}\)) which differ sufficiently from the usual Lagrangians in flat space whose interacting terms are presented in a simple current-field form \([4]\).

3 The linearized theory

This linearized approximation \([8]\):

\[
\begin{align*}
g_{aa} &\approx \eta_{aa} + \frac{1}{2} h_{aa} \\
g_{\mu\nu} &= g_{\mu a} g_{\nu a} \approx \eta_{\mu\nu} + \frac{1}{2} (h_{\mu\nu} + h_{\nu\mu}) \\
\sqrt{-g} &\approx 1 + \frac{1}{2} h_{aa} = 1 + \frac{1}{2} h \\
\Omega_{\mu ab} &\approx \frac{1}{4} \{ \partial_\mu h_{ba} + \partial_\mu h_{ab} - \partial_b h_{\mu a} - \partial_b h_{ab} + \partial_a h_{\mu b} + \partial_a h_{\mu b} \} \\
\Omega_\mu &\approx -\frac{1}{8} \Sigma_{ab} \partial_\mu h_{b\mu} \equiv \frac{1}{4} \Sigma_{ab} \partial_b h_{a\mu}
\end{align*} \tag{28}
\]

(where \(\eta_{\mu\nu}\) is the Minkowsky metric) with respect to weak field \(h_{\mu\nu}\) allows to simplify Lagrangian \((23)\) and split

\[
L = \left[ 1 + \frac{1}{2} h \right] i \bar{\psi} \gamma^a [\eta^\mu_a + h^\mu_a] [\partial_\mu + \Omega_\mu] \psi + + \left[ 1 + \frac{1}{2} h \right] m \bar{\psi} \psi \tag{29}
\]
into a sum

$$L = L_0 + \bar{L}$$

(30)

of free Lagrangian

$$L_0 = \bar{\psi} \left( i \bar{\psi} \gamma^\mu \partial_\mu + m \right) \psi$$

(31)

and the coupling term

$$\bar{L} = \bar{\psi} \left\{ i \left( \frac{1}{2} h \gamma^\mu + \gamma^a h_a^\mu \right) \partial_\mu + i \gamma^\mu \Omega_\mu + \frac{1}{2} hm \right\} \psi$$

(32)

4 Hamiltonian of the linearized theory

Having

$$\varphi_i = \{ \psi; h_{\mu \nu} \}$$

(33)

as dynamical variables, we find Hamiltonian (1) as a sum

$$H = H_0 + U$$

(34)

of free Hamiltonian

$$H_0 = \bar{\psi} \left( i \gamma^k \partial_k + m \right) \psi$$

(35)

and interacting term

$$U = \bar{\psi} \left\{ \frac{1}{2} hm + i \left( \frac{1}{2} h \gamma^k + \gamma^a h_a^k \right) \partial_k + i \gamma^\mu \Omega_\mu \right\} \psi$$

(36)

On account of derivative coupling, the term (3) differs from (25).

According to formula (10) the gap matrix is

$$\Delta = i \left( \frac{1}{2} h \gamma^k + \gamma^a h_a^k \right) \partial_k + i \gamma^\mu \Omega_\mu + \frac{1}{2} hm$$

(37)

In general the calculation of gap (10) involves tedious arithmetics that can be omitted in several particular cases.

5 Particular Example: a simplified metric of the rotating massive body

For a rotating massive body whose metric is

$$h_a^\mu = \frac{1}{4} h \delta_a^\mu \quad h_0^k = \tilde{h} \quad h, \tilde{h} \simeq \text{const} \Rightarrow \Omega = 0$$

(38)
the coefficient (28)
\[ \Omega_\mu = \frac{1}{4} \Sigma_{ij} \partial_j h_{0\mu} + \frac{1}{4} \Sigma_{ij} \partial_j h_{i\mu} \] (39)
vanishes. Hence, according to (37), the gap matrix is
\[ \Delta = \left( \frac{3}{4} h \gamma + \gamma_0 \right) \cdot \vec{p} + \frac{1}{2} hm \] (40)
and
\[ \Delta^\dagger = \left( \frac{3}{4} h \gamma - \gamma_0 \right) \cdot \vec{p} + \frac{1}{2} hm \]
\[ \beta \Delta^\dagger \beta = -\left( \frac{3}{4} h \gamma + \gamma_0 \right) \cdot \vec{p} + \frac{1}{2} hm \] (41)
where
\[ \vec{p} \equiv -i \nabla \] (42)
Thereby,
\[ \Delta \beta \Delta^\dagger \beta = \frac{1}{4} h^2 \left[ m^2 + \frac{9}{4} \vec{p}^2 \right] - \left( \vec{h} \cdot \vec{p} \right)^2 \] (43)
and, finally,
\[ |\Delta|^2 = -C \Delta \beta \Delta^\dagger \beta C = \frac{1}{4} h^2 \left[ m^2 + \frac{9}{4} \vec{p}^2 \right] - \left( \vec{h} \cdot \vec{p} \right)^2 \] (44)
Note that coefficient \( \frac{9}{4} \) results from the tensor nature of gravitational coupling: a pure scalar coupling \( \bar{\psi} \frac{1}{2} h m \psi \) added to the free Lagrangian (31) leads merely to the gap
\[ |\Delta|^2 = \frac{1}{4} h^2 m^2 \] (45)
which determines the ordinary excitation spectrum of BCS theory [10] with quadratic dependence of excitation energy (22) on \( \xi \).
Substituting the expression
\[(\xi + \mu)^2 - m^2 = \vec{p}^2 \quad \mu > m \] (46)
in (44), we get
\[ |\Delta|^2 (\xi, \chi) = \frac{1}{4} h^2 \left[ \frac{9}{4} (\xi + \mu)^2 - \frac{5}{4} m^2 \right] - \vec{h}^2 \left[ (\xi + \mu)^2 - m^2 \right] \cos^2 \chi \] (47)
Superfluidity takes place if the right side of (47) is positive at any value of \( \xi \).
6 Summary

6.1 Conclusion for a pure spherical metric

Since the Fermi energy $\varepsilon_F \equiv \mu > m$, the gap (47) corresponding to metric of non-rotating body ($\vec{h} = 0$) is positive at any $\xi$ for

$$\|\Delta\|^2 (\xi) = \frac{1}{4} \hbar^2 \left[ \frac{9}{4} (\xi + \mu)^2 - \frac{5}{4} m^2 \right]$$

implies occurrence of superfluidity. Note that Eq. (48) does not duplicate the BCS gap (45) of pure scalar coupling but immediately reduces to

$$\|\Delta\|^2 (\xi) = \frac{1}{4} \hbar^2 m^2$$

in the non-relativistic limit $\mu \to m$. Indeed, the general formula (47) also tends to (48) for a non-relativistic system. It should be noted that the ultra-relativistic, or massless, Fermi-gas also has a non-zero gap

$$\|\Delta\|^2 (\xi) = \frac{9}{16} \hbar^2 (\xi + \mu)^2$$

due to tensor gravitational interaction.

6.2 Conclusion for a metric with rotation

While the gap corresponding to a pure spherical metric (48) does not convey the qualitative difference from the ordinary phenomenon of superfluidity with scalar coupling, the gravitational coupling whose metric includes rotation (finite $\vec{h}$) makes up the gap (47) depending both on momentum $|\vec{p}|$ (or $\xi$) and angle $\chi$ between $\vec{p}$ and $\vec{h}$. The latter dependence is a specific property of the gravitational coupling: it is not known in BCS with electromagnetic or scalar coupling. Meanwhile, we have not considered pairing with the non-zero orbital momentum (like in He-3) wherein one may expect new possibilities.

Eq. (47) implies that superfluidity may be forbidden at definite $\chi$. The requirement of positive gap $\|\Delta\|^2 (\xi, \chi)$ at arbitrary $\xi$ and $\chi$ determines the condition

$$\frac{1}{4} \hbar^2 \left[ \frac{9}{4} (\xi + \mu)^2 - \frac{5}{4} m^2 \right] - \vec{h}^2 \left[ (\xi + \mu)^2 - m^2 \right] > 0$$
or
\[
\left( \frac{9}{16} h^2 - h^2 \right) (\xi + \mu)^2 + \left( h^2 - \frac{5}{16} h^2 \right) m^2 > 0
\]
(52)
necessary for the occurrence of superfluidity. It is satisfied at arbitrary \( \xi \) (it can be negative) if
\[
h^2 > \frac{16}{9} \bar{h}^2
\]
(53)
The same requirement is imposed on a massless field whose gap is
\[
\| \Delta \|^2 (\xi, \chi) = (\xi + \mu)^2 \left( \frac{9}{16} h^2 - \bar{h}^2 \cos^2 \chi \right)
\]
(54)
After all, we note that the gravitational coupling with metric tensor \( h^\mu_\nu = \{ h^k_0 \} \) does not admit superfluidity at all.

The excitation spectrum obtained, can be applied to the calculation of thermodynamic functions as it is used in the usual theory of superconductivity \[10, 3\], i.e. internal energy density, temperature dependence of gap etc.

7 Appendix: Equation of motion for a non-superfluid system

The equations of motion are
\[
\left\{ -i \beta \gamma^k \cdot \partial_k + \beta \left[ m + \Sigma^\# \right] \right\} u_p = \epsilon_p u_p \tag{55}
\]
\[
\left\{ -i \beta \gamma^k \cdot \partial_k + \beta \left[ m + g_i \Gamma^i A^\#_i \right] \right\} \varphi_p (r) = \epsilon_p \varphi_p (r) \tag{56}
\]
where we have used the notation \( \hat{\psi} (r) = \sum_p \varphi_p (r) \hat{b}_p^\dagger \) with index \( p = \{ p\sigma \} \) related to single-particle baryon densities; also \( \hat{\psi} (r) = \sum_p u_p (r) \hat{b}_p + v^*_p (r) \hat{b}_p^\dagger \).

The self-consistent local potentials are defined as
\[
A^\#_i (r) = A_i (r) + \frac{\delta R [n]}{\delta n^i (r)} \tag{57}
\]
where the interacting energy \( R [n] = E_H [n] + E_x [n] + E_c [n] \) includes the Hartree, exchange and correlation contribution. Particularly, \( A^\#_H (r) = \int dt d^3 r_2 \Delta_i (t, r_2 - r_1) n^i (r_2) \).
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