Abstract. In this paper, we study \( f \)-biharmonic curves as the critical points of the \( f \)-bienergy functional \( E_2(\psi) = \int_M f \left| \tau(\psi) \right|^2 \vartheta_g \), on a Lorentzian para-Sasakian manifold \( M \). We give necessary and sufficient conditions for a curve such that has a timelike principal normal vector on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be an \( f \)-biharmonic curve. Moreover, we introduce proper \( f \)-biharmonic curves on the Lorentzian sphere \( S^4_1 \).

Keywords: \( f \)-biharmonic curves; \( f \)-bienergy functional; para-Sasakian manifold; Lorentzian sphere.

1. Introduction

Harmonic maps \( \psi : (M,g) \rightarrow (N,h) \) between Riemannian manifolds are the critical points of the energy functional defined by

\[
(1.1) \quad E(\psi) = \frac{1}{2} \int_{\Omega} \left| d\psi \right|^2 \vartheta_g,
\]

for every compact domain \( \Omega \subset M \). The Euler-Lagrange equation of the energy functional gives the harmonic equation defined by vanishing of

\[
(1.2) \quad \tau(\psi) = \text{trace} \nabla d\psi,
\]

where \( \tau(\psi) \) is called the tension field of the map \( \psi \).

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J.H. Sampson [7]. Biharmonic maps
between Riemannian manifolds $\psi : (M, g) \to (N, h)$ are the critical points of the bienergy functional

$$E_2(\psi) = \frac{1}{2} \int_{\Omega} |\tau(\psi)|^2 \, \vartheta_g,$$

for any compact domain $\Omega \subset M$.

In [3], G.Y. Jiang derived the first and the second variation formulas for the bienergy, showing that the Euler-Lagrange equation associated to $E_2$ is

$$\tau_2(\psi) = -J^\psi(\tau(\psi))$$

$$\tau_2(\psi) = -\triangle^\psi(\Psi) - \text{trace} R^N(d\psi, \tau(\psi))d\psi,$$

where $J^\psi$ is the Jacobi operator of $\psi$. The equation $\tau_2(\psi) = 0$ is called biharmonic equation. Clearly, any harmonic maps is always a biharmonic map. A biharmonic map that is not harmonic is called a proper biharmonic map.

For some recent geometric study of biharmonic maps see [14, 17, 18, 19, 24] and the references therein. Also for some recent progress on biharmonic submanifolds see [1, 2, 16, 20, 21] and for biharmonic conformal immersions and submersions see [15, 25, 27].

The concept of $f-$biharmonic maps were initiated by W.J. Lu [23]. A smooth map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is called an $f-$biharmonic map if it is a critical point of the $f-$bienergy functional defined by

$$E_{2,f}(\psi) = \frac{1}{2} \int_{\Omega} f |\tau(\psi)|^2 \, \vartheta_g,$$

for every compact domain $\Omega \subset M$.

The Euler-Lagrange equation gives the $f-$biharmonic map equation [23]

$$\tau_{2,f} = f \tau_2(\psi) + (\triangle f) \tau(\psi) + 2\nabla^\psi_{\text{grad} f} \tau(\psi)$$

$$\tau_{2,f} = 0,$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of $\psi$, respectively. Therefore, we have the following relationship among these types of maps [26]:

$$\text{Harmonic maps} \subset \text{Biharmonic maps} \subset f-\text{Biharmonic maps}.$$

From now on we will call an $f-$biharmonic map, which is neither harmonic nor biharmonic, a proper $f-$biharmonic map (see also [28]).

The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto [9]. He also introduced the notion of Lorentzian para-Sasakian manifold. In [4], I. Mihai and R. Rosca defined the same notion independently and there after many authors [5, 11, 22] studied Lorentzian para-Sasakian manifolds.

Moreover, in [17] some geometric result for spacelike and timelike curves in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be proper biharmonic were given. Motivated by this work, we introduced $f-$biharmonic curves on Lorentzian para-Sasakian manifold and Lorentzian sphere $S^4_1$. 
2. Preliminaries

2.1. $f$–Biharmonic Maps

$f$–Biharmonic maps are critical points of the $f$–bienergy functional for maps $\psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\psi) = \frac{1}{2} \int_{\Omega} f \left| \tau(\psi) \right|^2 \vartheta_g,$$

where $\Omega$ is a compact domain of $M$.

The following Theorem was proved in [23]:

**Theorem 2.1.** A map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is an $f$–biharmonic map if and only if

$$\tau_{2,f} = f \tau_2(\psi) + (\triangle f) \tau(\psi) + 2 \nabla_{\text{grad} f} \tau(\psi) = 0,$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of $\psi$, respectively. $\tau_{2,f}(\psi)$ is called the $f$–bitension field of map $\psi$.

A special case of $f$–biharmonic maps is $f$–biharmonic curves. We have the following.

**Lemma 2.1.** [26] An arclength parametrized curve $\gamma : (a, b) \to (N^m, g)$ is an $f$–biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if and only if

$$f(\nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' - R^N(\gamma', \nabla^N_{\gamma'} \gamma') \gamma') + 2f' \nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' + f'' \nabla^N_{\gamma'} \gamma' = 0.$$

2.2. Lorentzian almost paracontact manifolds

Let $M$ be an $n$-dimensional differentiable manifold with a Lorentzian metric $g$, i.e., $g$ is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in M$, the tensor $g_p : T_p M \times T_p M \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where $T_p M$ is the tangent space of $M$ at the point $p$. Then $(M, g)$ is called a Lorentzian manifold. A non-zero vector $X_p \in T_p M$ can be spacelike, null or timelike, if it satisfies $g_p(X_p, X_p) > 0$, $g_p(X_p, X_p) = 0$ or $g_p(X_p, X_p) < 0$, respectively.

Let $M$ be an $n$-dimensional differentiable manifold equipped with a structure $(\varphi, \xi, \eta)$, where $\varphi$ is a $(1, 1)$-tensor field, $\xi$ is a vector field, $\eta$ is a 1-form on $M$ such that [9]

$$\varphi^2 X = X + \eta(X) \xi,$$
The above equations imply that
\[ \eta \circ \varphi = 0, \quad \varphi \xi = 0, \quad \text{rank}(\varphi) = n - 1. \]

Then \( M \) admits a Lorentzian metric \( g \), such that
\[ g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \]
and \( M \) is said to admit a Lorentzian almost paracontact structure \( (\varphi, \xi, \eta, g) \). Then we get
\[ g(X, \xi) = \eta(X). \]

The manifold \( M \) endowed with a Lorentzian almost paracontact structure \( (\varphi, \xi, \eta, g) \) is called a Lorentzian almost paracontact manifold [9, 10]. In equations (2.4) and (2.5) if we replace \( \xi \) by \(-\xi\), we obtain an almost paracontact structure on \( M \) defined by I. Sato [6].

A Lorentzian almost paracontact manifold equipped with the structure \( (\varphi, \xi, \eta, g) \) is called a Lorentzian para-Sasakian manifold [9] if
\[ (\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \]

The conformal curvature tensor \( C \) is given by
\[
C(X, Y)W = R(X, Y)W - \frac{1}{n-2} \left\{ \frac{S(Y, W)X - S(X, W)Y}{+g(Y, W)QX - g(X, W)QY} \right\} \\
+ \frac{r}{(n-1)(n-2)} \{g(Y, W)X - g(X, W)Y\},
\]
where \( S(X, Y) = g(QX, Y) \). The Lorentzian para-Sasakian manifold is called conformally flat if conformal curvature tensor vanishes i.e., \( C = 0 \).

The quasi-conformal curvature tensor \( \hat{C} \) is defined by
\[
\hat{C}(X, Y)W = aR(X, Y)W - b \left\{ \frac{S(Y, W)X - S(X, W)Y}{+g(Y, W)QX - g(X, W)QY} \right\} \\
- \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) \{g(Y, W)X - g(X, W)Y\},
\]
where \( a, b \) constants such that \( ab \neq 0 \). Similarly the Lorentzian para-Sasakian manifold is called quasi-conformally flat if \( \hat{C} = 0 \).

We know that a conformally flat and quasi-conformally flat Lorentzian para-Sasakian manifold \( M^n \) \( (n > 3) \) is of constant curvature 1 and also a Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation \( R(X, Y) \cdot C = 0 \) holds on \( M \) [12]. For a conformally symmetric Riemannian manifold [13], we get \( \nabla C = 0 \). Thus for a conformally symmetric space the relation \( R(X, Y) \cdot C = 0 \)
\( C = 0 \) satisfies. Hence a conformally symmetric Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere [12].

Therefore, for a conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold \( M \), we have [12]

\[
R(X, Y)W = g(Y, W)X - g(X, W)Y,
\]
for any vector fields \( X, Y, W \in TM \).

3. \( f \)-Biharmonic Curves in Lorentzian Para-Sasakian Manifolds

For a Lorentzian para-Sasakian manifold \( M \), an arbitrary curve \( \gamma : I \to M \), \( \gamma = \gamma(s) \) is called spacelike, timelike or lightlike (null), if all of its velocity vectors \( \gamma'(s) \) are spacelike, timelike or lightlike (null), respectively. In this section, we give some conditions for a curve having timelike normal vector on a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold \( M \) to be an \( f \)-biharmonic curve.

**Theorem 3.1.** Let \( \gamma : I \to M \) be a curve parametrized by arclength and \( M \) be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Assume that \( \{T, N, B_1, B_2\} \) be an orthonormal Frenet frame field along \( \gamma \) such that principal normal vector \( N \) is timelike. Then \( \gamma \) is a proper \( f \)-biharmonic curve if and only if one of the following cases happens:

i) The first curvature \( \kappa_1 \) of the \( \gamma \) solves the following ordinary differential equation,

\[
3(\kappa'_1)^2 - 2\kappa_1\kappa''_1 = 4\kappa_1^4 - 4\kappa_1^2,
\]
with \( f = t_1\kappa_1^2, \kappa_2 = 0 \).

ii) The first curvature \( \kappa_1 \) of the \( \gamma \) solves the following ordinary differential equation,

\[
3(\kappa'_1)^2 - 2\kappa_1\kappa''_1 = 4\kappa_1^4 + 4\kappa_1^4t_3^2 - 4\kappa_1^2,
\]
with \( f = t_1\kappa_1^2, \kappa_2 \neq 0, \kappa_3 = 0, \kappa\kappa_3 = t_3 \).

**Proof.** Let \( \gamma \) be a curve parametrized by arclength on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold \( M \) and let \( \{T, N, B_1, B_2\} \) be an orthonormal Frenet frame field along \( \gamma \) such that principal normal vector \( N \) is timelike.

In this case for this curve, the Frenet frame equations are given by [8]

\[
\begin{bmatrix}
\nabla_T T \\
\nabla_T N \\
\nabla_T B_1 \\
\nabla_T B_2
\end{bmatrix}
= \begin{bmatrix}
0 & \kappa_1 & 0 & 0 \\
\kappa_1 & 0 & \kappa_2 & 0 \\
0 & \kappa_2 & 0 & \kappa_3 \\
0 & 0 & -\kappa_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}
\]
where $T$, $N$, $B_1$, $B_2$ are mutually orthogonal vectors and $\kappa_1$, $\kappa_2$ and $\kappa_3$ are respectively the first, the second and the third curvature of the $\gamma$.

In view of the Frenet formulas given in (3.3) and equation (2.8), we obtain

$$\nabla_T T = \kappa_1 N,$$

$$\nabla_T \nabla_T T = \kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1,$$

$$\nabla_T \nabla_T \nabla_T T = (3\kappa_1 \kappa_1') T + (\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2) N + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') B_1 + (\kappa_1 \kappa_2 \kappa_3) B_2,$$

and

$$R(T, \nabla_T T) T = -\kappa_1 N,$$

where $\kappa_1$, $\kappa_2$ and $\kappa_3$ are the first, the second and the third curvature of the $\gamma$, respectively.

Considering Theorem 2.1 and equation (2.3), we get

$$\tau_{2,f} = f \left[ \frac{(3\kappa_1 \kappa_1') T + (\kappa_1' + \kappa_1^3 + \kappa_1 \kappa_2^2 + \kappa_1 N) \kappa_1 \kappa_2 B_1 + (\kappa_1 \kappa_2 \kappa_3) B_2}{\kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1} \right] + 2f' \left[ \kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1 \right] + f'' \left[ \kappa_1 N \right] = 0.$$

Comparing the coefficients of above equation, we obtain that $\gamma$ is an $f$–biharmonic curve if and only if

(3.4) \hspace{1cm} 3\kappa_1 \kappa_1' + 2\kappa_1^2 \frac{f'}{f} = 0,

(3.5) \hspace{1cm} \kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2 + \kappa_1 + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f} = 0,

(3.6) \hspace{1cm} 2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' + 2\kappa_1 \kappa_2 \frac{f'}{f} = 0,

(3.7) \hspace{1cm} \kappa_1 \kappa_2 \kappa_3 = 0.$$

Let $\kappa_1$ be a non zero constant. Then from (3.4) we get $f$ is constant. So $\gamma$ is biharmonic. Let $\kappa_2$ be a non zero constant. From (3.4) and (3.6) one can easily see that $f$ is constant and $\gamma$ is biharmonic.
By using (3.4) - (3.7), if \( \kappa_2 = 0 \), then \( f \)-biharmonic curve equation reduces to

\[
3\kappa_1 \kappa_1' + 2\kappa_1^3 \frac{f'}{f} = 0,
\]

(3.8)

\[
\kappa_1'' + \kappa_1^3 + \kappa_1 + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f} = 0.
\]

(3.9)

Integrating the equation (3.8) we get \( f = t_1 \kappa_1^{-\frac{3}{2}} \) and using this result in (3.9), we arrive at \((i)\).

Otherwise, by use of (3.4) - (3.7), if \( \kappa_1 \neq \text{constant} \) and \( \kappa_2 \neq \text{constant} \) \( f \)-biharmonic curve the equation is equivalent to

\[
f^2 \kappa_1^3 = t_1^2,
\]

(3.10)

\[
(f \kappa_1)'' = -f \kappa_1 (\kappa_1^3 + \kappa_2^2 + 1),
\]

(3.11)

\[
f^2 \kappa_1^2 \kappa_2 = t_2,
\]

(3.12)

\[
\kappa_3 = 0.
\]

(3.13)

In view of (3.10), we find \( f = t_1 \kappa_1^{-\frac{3}{2}} \) and using this result in (3.11), we get \( \frac{\kappa_2}{\kappa_1} = t_3 \). Finally substituting these equation in (3.11), we arrive at \((ii)\).

**Proposition 3.1.** Let \( M \) be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold and \( \gamma : I \to M \) be an \( f \)-biharmonic spacelike curve parametrized by arclength such that principal normal vector is timelike. If \( \gamma \) has constant geodesic curvature then \( \gamma \) is biharmonic.

### 4. \( f \)-Biharmonic Curves on Lorentzian Sphere \( S_1^4 \)

Suppose that \( M \) is a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Since \( M \) is locally isometric to a Lorentzian unit sphere \( S_1^4 \), we give some characterizations for \( f \)-biharmonic curves in \( S_1^4 \). The Lorentzian unit sphere of radius 1 can be seen as the hyperquadric

\[
S_1^4 = \{ p \in \mathbb{R}_1^5 : < p, p > = 1 \},
\]

in a Minkowski space \( \mathbb{R}_1^5 \) with the metric

\[
< , >: -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2.
\]
Let \( \gamma : I \to S^4_1 \) be a curve parametrized by arclength. For an arbitrary vector field \( X \) along \( \gamma \), we have

\[
\nabla_T X = X' + \langle T, X \rangle \gamma,
\]

where \( \nabla \) is covariant derivative along \( \gamma \) in \( S^4_1 \).

Since \( S^4_1 \) is a Lorentzian space form of the scalar curvature 1, we have

\[
R(X,Y)W = \langle Y,W \rangle X - \langle X,W \rangle Y,
\]

for all vector fields \( X,Y,W \) in the tangent bundle of \( S^4_1 \), where \( R \) is the curvature tensor of \( S^4_1 \).

Now, we give the following:

**Proposition 4.1.** Let \( \gamma : I \to S^4_1 \) be a non-geodesic \( f \)-biharmonic curve parametrized by arclength and \( \{T, N, B_1, B_2\} \) be a Frenet frame along \( \gamma \) such that

\[
g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1, \quad g(N, N) = -1.
\]

Then, we have

\[
\gamma^{(4)} - \left( \frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1' f'}{\kappa_1 f} + \frac{f''}{f} \right) \gamma'' - \left( \frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1' f'}{\kappa_1 f} + \frac{f''}{f} + 1 \right) \gamma = 0.
\]

**Proof.** Using (3.5) and taking the covariant derivative of the second equation in (3.3), we get

\[
\nabla_T^2 N = \nabla_T (\kappa_1 T + \kappa_2 B_1)
= \kappa_1 \nabla_T T + \kappa_2 \nabla_T B_1
= (\kappa_1^2 + \kappa_2^2) N + \kappa_2 \kappa_3 B_2.
\]

Using (3.5) in (4.3), we have

\[
\nabla_T^2 N = - \left( \frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1' f'}{\kappa_1 f} + \frac{f''}{f} + 1 \right) N.
\]

On the other hand from (4.1), we arrive at

\[
\nabla_T^2 N = \nabla_T (N' + \langle T, N \rangle \gamma)
= N'' + \langle T, N' \rangle \gamma
= N'' + \langle T, \nabla_T N - \kappa_1 T + \kappa_2 B_1 \rangle \gamma
= N'' + \kappa_1 T + \kappa_2 B_1 \gamma
= N'' + \kappa_1 \gamma.
\]

From (4.3) and (4.4), we obtain

\[
\left( \frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1' f'}{\kappa_1 f} + \frac{f''}{f} + 1 \right) N = N'' + \kappa_1 \gamma.
\]
Also in view of (4.1), we have
\[ \nabla_T^T T = T' + <T, T> \gamma = \gamma'' + \gamma, \]
which yields
\[ (4.5) \quad N = \frac{1}{\kappa_1} (\gamma'' + \gamma). \]
By use of (4.5) and (4.4), we obtain (4.2). \( \square \)

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