KIRCHHOFF TYPE ELLIPTIC EQUATIONS WITH DOUBLE CRITICALITY IN MUSIELAK-SOBOLEV SPACES

SHILPA GUPTA AND GAURAV DWIVEDI

Abstract. This paper aims to establish the existence of a weak solution for the non-local problem:

\[
\begin{cases}
- a \left( \int_{\Omega} H(x, |\nabla u|) dx \right) \Delta_H u = f(x, u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subseteq \mathbb{R}^N \), \( N \geq 2 \) is a bounded and smooth domain containing two open and connected subsets \( \Omega_p \) and \( \Omega_N \) such that \( \Omega_p \cap \Omega_N = \emptyset \) and \( \Delta_H u = \text{div}(h(x, |\nabla u|)\nabla u) \) is the \( H \)-Laplace operator. We assume that \( \Delta_H \) reduces to \( \Delta_p \) in \( \Omega_p \) and to \( \Delta_N \) in \( \Omega_N \), the non-linear function \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) acts as \( |t|^{p^*} \) on \( \Omega_p \) and as \( e^{\alpha |t|^{N/(N-1)}} \) on \( \Omega_N \) for sufficiently large \( |t| \). To establish the existence results in a Musielak-Sobolev space, we use a variational technique based on the mountain pass theorem.

1. Introduction

This paper aims to establish the existence of a weak solution to the following non-local problem:

\[
\begin{cases}
- a \left( \int_{\Omega} H(x, |\nabla u|) dx \right) \Delta_H u = f(x, u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subseteq \mathbb{R}^N \) is a bounded and smooth domain, \( N \geq 2 \), \( \Delta_H u = \text{div}(h(x, |\nabla u|)\nabla u) \), \( H(x, t) = \int_0^t h(x, s) s \, ds \) and \( h : \Omega \times [0, \infty) \to [0, \infty) \) is a generalized N-function.

Problem (1.1) is known as Kirchhoff type problem as it is related to the celebrated work of Kirchhoff [34], where the author studied the equation:

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0.
\]

When \( H(x, t) = t^p \), (1.1) reduces to a Kirchhoff type problem for \( p \)-Laplace operator. Lions [39] set up an abstract framework for the study of such problems and thereafter several authors obtained existence results for \( p \)-Kirchhoff type equations, see [3][4][5][12][25][29][35][38][50][51][52] and references therein. If \( H(x, t) = t^{p(x)} \), (1.1) transforms into a Kirchhoff type problem with variable exponent and existence results are such problems are studied in the variable exponent Sobolev spaces. For some such results, one can refer to, [10][13][16][17][23][36] and references therein. When \( H(x, t) \) is independent of \( x \), the existence results for problems of type (1.1) are discussed in Orlicz-Sobolev spaces, and we refer to the work of Chaharlang and Razani [9], and Chung [11] in this direction. In the case, when \( H(x, t) \) depends

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on both $x$ and $t$, the existence of a solution for the problems of the type (1.1) is studied in Musielak-Sobolev spaces (for the definitions and properties of variable exponent Sobolev spaces and Musielak-Sobolev spaces, see Section 2). The study of Musielak spaces started in the mid-1970s with the work of Musielak [44] and Hudzik [30, 31], where the authors provide the general framework for Musielak spaces in terms of modular function. Many authors [21, 26, 42, 47] used such spaces to prove the existence of a solution for problems of the type (1.1) without the Kirchhoff term $a$. Shi and Wu [47] studied the existence result for Kirchhoff type problems in Musielak-Sobolev spaces. Recently, Alves et al. [6] developed the concept of double criticality and studied the quasilinear problem in Musielak-Sobolev spaces, and our existence results are motivated by their work.

Next, we state our hypotheses. Throughout this article, for any $r \in C([\Omega], (1, \infty))$, we denote $r^- = \min_{x \in \Omega} r(x)$ and $r^+ = \max_{x \in \Omega} r(x)$. Further, the functions $p, q, p^*, q_1 \in C([\Omega], (1, \infty))$. We consider the following assumptions on the functions $H$ and $h$:

\begin{enumerate}[(H1)]
  \item \(H(x, \cdot) \in C^1 \text{ in } (0, \infty), \forall x \in \Omega.\)
  \item \(h(x, t), \partial_t h(x, t) > 0, \forall x \in \Omega \text{ and } t > 0.\)
  \item \(p^- \leq \frac{h(x, t)}{H(x, t)} \leq q^+ \text{ for } x \in \Omega \text{ and } t \neq 0 \text{ for some } 1 < p^- \leq p(x) < N < q(x) \leq q^+ < (p^*)^- \).
  \item \(\inf_{x \in \Omega} H(x, 1) = b_1 \text{ for some } b_1 > 0.\)
  \item \(\forall t_0 \neq 0, \text{ there exists } d_0 > 0 \text{ such that } \frac{H(x, t)}{t} \geq d_0 \text{ and } \frac{H(x, t)}{t} \geq d_0 \text{ for } t \geq t_0 \text{ and } x \in \Omega, \text{ where } \tilde{H}(x, t) = \int_0^t \tilde{h}(x, s) ds, \text{ and } \tilde{h} \text{ is the complimentary function of } h \text{ which is defined as } \tilde{h}(x, t) = \sup \{s : h(x, s) \leq t\} \forall (x, t) \in \Omega \times [0, \infty).\)
\end{enumerate}

Let $S \subset \Omega$ and $\delta > 0$. The $\delta$ neighborhood of $S$ is denoted by $S_\delta$ and defined as $S_\delta = \{x \in \Omega : \text{dist}(x, S) < \delta\}$.

Assume that, we have three smooth domains $\Omega_p$, $\Omega_N$ and $\Omega_q$ with non-empty interiors such that $\Omega = \Omega_p \cup \Omega_N \cup \Omega_q$ and $(\Omega_p)_\delta \cap (\Omega_N)_\delta = \emptyset$.

Next, we define continuous functions $\psi_p, \psi_N, \psi_q : \Omega \to [0, 1]$ such that

\begin{align*}
  \psi_p(x) &= 1 \quad \forall x \in \Omega_p, & \psi_p(x) &= 0 \quad \forall x \in (\Omega_p)_\delta^c, \\
  \psi_N(x) &= 1 \quad \forall x \in \Omega_N, & \psi_N(x) &= 0 \quad \forall x \in (\Omega_N)_\delta^c, \\
  \psi_q(x) &= 1 \quad \forall x \in \Omega_q, & \psi_q(x) &= 0 \quad \forall x \in (\Omega_q)_\delta^c.
\end{align*}

We consider that the non-linear function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and of the following type:

\begin{equation}
  (f_1) \quad f(x, t) = \lambda \psi_N(x) |t|^{\beta-2} + \psi_p(x) |t|^\alpha + \psi_q(x) |t|^{p^*} \quad \forall (x, t) \in \Omega \times \mathbb{R},
\end{equation}

where $(p^*)^+ \geq p(x) \geq (p^*)^- \geq q^+ \geq q(x) \geq q^- > N > p^+ \geq p(x) \geq p^- > N/2$, $\beta > q^-$, $\lambda > 0$ and $\alpha > 0$. Moreover, $\psi_q : \Omega_q \to [0, 1], \psi : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that

\begin{align*}
  \psi_q(x) &= 1, \quad \forall x \in \Omega_q \text{ and } \tilde{\psi}_q(x) = 0, \quad \forall x \in (\Omega_q)_\delta^c, \\
  \psi_q(x) &= 1, \quad \forall x \in \Omega_q \text{ and } \tilde{\psi}_q(x) = 0, \quad \forall x \in (\Omega_q)_\delta^c.
\end{align*}
and \( \varphi(x,t) = o(|t|^{q_1(x)-1}) \) as \( t \to 0 \) uniformly on \( (\Omega_q)_\delta/2 \) for some \( q_1^+ \geq q_1(x) \geq q_1^- > q^- \) and there exists \( \lambda > q^- \) such that
\[
0 < \lambda \Phi(x,t) \leq \varphi(x,t)t, \quad \forall x \in (\Omega_q)_\delta/2,
\]
where \( \Phi(x,t) = \int_0^{|t|} \varphi(x,s)ds \).

Along with the above notations, \( h \) also satisfies the following conditions for each \( t > 0 \):
\[
\begin{align*}
(\mathcal{H}_6) & \quad h(x,t) \geq t^{N-2} \quad \forall x \in \Omega_N \text{ and } C_1 t^{N-2} \geq h(x,t) \quad \forall x \in \Omega_N \setminus (\Omega_q)_\delta \text{ for some } C_1 > 0. \\
(\mathcal{H}_7) & \quad \text{There exist a continuous function } \eta_1 : \overline{\Omega} \to \mathbb{R} \text{ such that } h(x,t) \geq \eta_1(x) t^{p(x)-2} \\
& \quad \forall x \in (\Omega_q)_\delta \text{ and } \eta_1(x) > 0, \forall x \in (\Omega_q)_\delta \text{ and } \eta_1(x) = 0, \forall x \in ((\Omega_q)_\delta)^c. \\
(\mathcal{H}_8) & \quad \text{There exist a non-negative continuous function } \eta_2 : \overline{\Omega}_p \to \mathbb{R} \text{ such that } \\
& \quad \eta_2(x) t^{p(x)-2} + C_2 t^{p(x)-2} \geq h(x,t) \geq t^{p(x)-2} \quad \forall x \in \overline{\Omega}_p \text{ and } \eta_2(x) > 0, \forall x \in (\Omega_q)_\delta \text{ and } \eta_2(x) = 0, \forall x \in (\Omega_q)_\delta \setminus (\Omega_q)_\delta, \text{ for some } C_2 > 0.
\end{align*}
\]

Next, we state our hypotheses on the nonlocal term \( a \). The continuous function \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfies following conditions:
\[
\begin{align*}
(\mathcal{A}_1) & \quad \text{There exist positive real number } a_0 \text{ such that } a(s) \geq a_0 \text{ and } a \text{ is non-decreasing } \forall s > 0. \\
(\mathcal{A}_2) & \quad \text{There exist } \theta > 1 \text{ such that } \beta > N\theta \text{ and } a(s)/s^{\theta-1} \text{ is non-increasing for } s > 0.
\end{align*}
\]

**Remark 1.1.** By \( \mathcal{A}_2 \), we have
\[
(\mathcal{A}_2') \quad \theta A(s) - a(s)s \text{ is non-decreasing, } \forall s > 0, \text{ where } A(s) = \int_0^s a(t)dt.
\]
In particular,
\[
(1.3) \quad \theta A(s) - a(s)s \geq 0, \quad \forall s > 0.
\]

Again by \( \mathcal{A}_2 \) and \( A(s) \equiv 1 \), one gets
\[
(\mathcal{A}_2'') \quad A(s) \leq s^\theta A(1) \quad \forall s \geq 1.
\]

Now, we state main result of this article:

**Theorem 1.2.** Suppose that the conditions \( (f_1), (\mathcal{H}_1) - (\mathcal{H}_8) \) and \( (\mathcal{A}_1) - (\mathcal{A}_2) \) are satisfied. Then there exists \( \lambda_1 > 0 \) such that for any \( \lambda \geq \lambda_1 \), Problem \( (1.7) \) has non trivial weak solution via mountain pass theorem.

This article is organized as follows: We discuss the definition and properties of Musielak-Sobolev spaces and the functional setup needed to prove our result in Section 2. Section 3 deals with the proof of Theorem 1.2.

### 2. PRELIMINARIES

In this section, we discuss Musielak spaces and their properties. We also provide the functional setup needed to prove our main result and discuss some helping results. Define,
\[
\mathcal{H}(x,t) = \int_0^{|t|} h(x,s)ds,
\]
where \( h : \Omega \times [0,\infty) \to [0,\infty) \). Then \( \mathcal{H}(x,t) \) is a generalized \( N \)-function. Recall that, \( \mathcal{H}(x,t) : \Omega \times [0,\infty) \to [0,\infty) \) is said to be a generalized \( N \)-function if it is
embeddings are continuous: 
and

\[ \lim_{t \to \infty} \frac{H(x,t)}{t} = \infty. \]

The Musielak space \( L^H(\Omega) \) is defined as:
\[
L^H(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable function} \mid \int_{\Omega} H(x,\tau|u|) \, dx < \infty, \text{ for some } \tau > 0 \right\}.
\]

\( L^H(\Omega) \) is a reflexive Banach space \(^{[44]}\) with the Luxemburg norm
\[
\|u\|_{L^H(\Omega)} = \inf \left\{ \tau > 0 \mid \int_{\Omega} H(x, \frac{|u|}{\tau}) \, dx \leq 1 \right\}.
\]

We say that a generalised N-function satisfies the weak \( \Delta_2 \)-condition if there exist \( C > 0 \) and a non-negative function \( k \in L^1(\Omega) \) such that
\[
H(x,2t) \leq CH(x,t) + k(x) \quad \forall (x,t) \in \Omega \times \mathbb{R}.
\]

If \( k = 0 \), then \( H \) is said to satisfy \( \Delta_2 \)-condition. Also, the function \( H \) and its complementary function \( \tilde{H} \) (defined in \((H_5)\)) satisfy the following Young’s inequality \(^{[41], Proposition \, 2.1]}\):
\[
s_1s_2 \leq H(x,s_1) + \tilde{H}(x,s_2) \quad \forall x \in \Omega, \, s_1, s_2 > 0.
\]

Further, proceeding as \(^{[24], Lemma \, A2]}\), we have
\[(2.1) \quad \tilde{H}(x,h(x,s)s) \leq H(x,2s), \quad \forall (x,s) \in \Omega \times [0, \infty).
\]

The Musielak-Sobolev space \( W^{1,H}(\Omega) \) is defined as
\[
W^{1,H}(\Omega) = \{ u \in L^H(\Omega) \mid \| \nabla u \|_{L^H(\Omega)} \leq 1 \}.
\]

\( W^{1,H}(\Omega) \) is a Banach space with the norm \(^{[44], Theorem \, 10.2]}\)
\[
\|u\|_{1,H} = \|u\|_{L^H(\Omega)} + \|\nabla u\|_{L^H(\Omega)}.
\]

The space \( W^{1,H}_{0}(\Omega) \) is defined as the closure of \( C_c^\infty(\Omega) \) in \( W^{1,H}(\Omega) \). Also, the space \( W^{1,H}_{0}(\Omega) \) is equipped with the norm \( \|u\| = \|\nabla u\|_{L^H(\Omega)} \), which is equivalent to the norm \( \|\cdot\|_{1,H} \) \(^{[26], Lemma \, 5.7]}\).

**Theorem 2.1.** \(^{[44]}\) The spaces \( L^H(\Omega) \) and \( W^{1,H}_{0}(\Omega) \) are reflexive and separable Banach spaces.

In particular, if we take \( H(x,t) = t^{p(x)} \) then we denote \( L^H(\Omega) \) as \( L^{p(x)}(\Omega) \) and \( W^{1,H}(\Omega) \) as \( W^{1,p(x)}(\Omega) \). Such spaces are called variable exponent Lebesgue and variable exponent Sobolev spaces, respectively. To know more about these spaces, one can check \(^{[15], 22, 40]}\).

Further, we have the following embedding result:

**Proposition 2.2.** \(^{[22]}\) Let \( \Omega \) be a bounded smooth domain. Then the following embeddings are continuous:

(a) \( W^{1,H}_{0}(\Omega) \hookrightarrow L^\gamma(\Omega_N), \quad 1 \leq \gamma < \infty, \)

(b) \( W^{1,H}_{0}(\Omega) \hookrightarrow L^\alpha((\Omega_p)_\delta), \) where \( s(x) \leq \frac{Np(x)}{N-p(x)} \),

(c) \( W^{1,H}_{0}(\Omega) \hookrightarrow W^{1,q^-}(\Omega_q)_\delta, \)

Moreover, the embedding
\[(2.2) \quad W^{1,H}_{0}(\Omega) \hookrightarrow C((\Omega_q)_\delta) \quad \text{is compact}.
\]
Proof. By using the conditions \((\mathcal{H}_6), (\mathcal{H}_7), (\mathcal{H}_8)\) and the definition of \(W^{1,N}_0(\Omega)\), we have continuous embeddings
\[
W^{1,N}_0(\Omega) \hookrightarrow W^{1,p}_0(\Omega_N)
\]
\[
W^{1,N}_0(\Omega) \hookrightarrow W^{1,q(x)}_0((\Omega_q)_\delta),
\]
\[
W^{1,N}_0(\Omega) \hookrightarrow W^{1,p(x)}_0((\Omega_p)_\delta).
\]
Further, \(W^{1,N}_0(\Omega_N) \hookrightarrow L^\gamma(\Omega_N)\) is continuous for any \(1 \leq \gamma < \infty\) [34, Theorem 2.4.4], which proves \((a)\).

We know that \(W^{1,p(x)}_0((\Omega_p)_\delta) \hookrightarrow L^{s(x)}((\Omega_p)_\delta)\) is continuous for \(s(x) \leq \frac{Np(x)}{N - p(x)}\) [22, Theorem 2.3]. This proves \((b)\).

For \((c)\), as \(q^- \leq q(x), W^{1,q(x)}_0((\Omega_q)_\delta) \hookrightarrow W^{1,q^-}_0((\Omega_q)_\delta)\) is continuous. Moreover, since \(q^- > N, W^{1,q^-}_0((\Omega_q)_\delta) \hookrightarrow C((\Omega_q)_\delta)\) is compact [33, Theorem 2.5.3] and this implies that \(W^{1,N}_0(\Omega) \hookrightarrow C((\Omega_q)_\delta)\) is compact. \(\Box\)

Next, we will state some results which are used to prove our main result.

**Proposition 2.3.** [21] Proposition 1.5 Let \(\mathcal{H}\) be a generalized \(N\)-function. If \((\mathcal{H}_4)\) holds then \(L^N(\Omega) \hookrightarrow L^1(\Omega)\) and \(W^{1,N}_0(\Omega) \hookrightarrow W^{1,1}_0(\Omega)\).

**Proposition 2.4.** [35] Theorem 2.1 Let \(r \in C(\overline{\Omega}, (1, \infty))\) and \(s \in C(\overline{\Omega}, (1, \infty))\) be the conjugate exponent of \(r\). Then, for any \(u \in L^r(\Omega)\) and \(v \in L^s(\Omega)\), we have
\[
\left| \int uv \, dx \right| \leq \left( \frac{1}{r^-} + \frac{1}{s^-} \right) \|u\|_{L^r(\Omega)} \|v\|_{L^s(\Omega)}.
\]

**Proposition 2.5.** [22] For any \(u \in L^{p(x)}(\Omega)\), the followings are true:
\[
\begin{align*}
(1) & \quad \|u\|_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{\mathcal{H}^+} \quad \text{whenever} \quad \|u\|_{L^{p(x)}(\Omega)} > 1, \\
(2) & \quad \|u\|_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{\mathcal{H}^+} \quad \text{whenever} \quad \|u\|_{L^{p(x)}(\Omega)} < 1, \\
(3) & \quad \|u\|_{L^{p(x)}(\Omega)} < 1(= 1; > 1) \iff \rho(u) < 1(= 1; > 1),
\end{align*}
\]
where \(\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx\).

**Proposition 2.6.** [35] Let \(r, s \in C(\overline{\Omega}, (1, \infty))\) such that \(1 < r(x)s(x) < \infty\). Then, for any \(u \in L^{r(x)}(\Omega)\), the followings are true:
\[
\begin{align*}
(1) & \quad \|u\|_{L^{r(x)}(\Omega)}^{\mathcal{H}^+} \leq \|u\|_{L^{r(x)}(\Omega)} \leq \|u\|_{L^{r(x)}(\Omega)}^{\mathcal{H}^+} \quad \text{whenever} \quad \|u\|_{L^{r(x)}(\Omega)} \geq 1, \\
(2) & \quad \|u\|_{L^{r(x)}(\Omega)}^{\mathcal{H}^+} \leq \|u\|_{L^{r(x)}(\Omega)} \leq \|u\|_{L^{r(x)}(\Omega)}^{\mathcal{H}^+} \quad \text{whenever} \quad \|u\|_{L^{r(x)}(\Omega)} \leq 1.
\end{align*}
\]
Define the function \(m : W^{1,N}_0(\Omega) \to \mathbb{R}\) as
\[
m(u) = \int_{\Omega} \mathcal{H}(x, |\nabla u|) \, dx.
\]

**Proposition 2.7.** [6] For any \(u \in W^{1,N}_0(\Omega)\), the followings are true:
\[
\begin{align*}
(1) & \quad \|u\|^p \leq m(u) \leq \|u\|^{q(x)} \quad \text{whenever} \quad \|u\| \geq 1, \\
(2) & \quad \|u\|^{q^-} \leq m(u) \leq \|u\|^p \quad \text{whenever} \quad \|u\| \leq 1.
\end{align*}
\]
In particular, \(m(u) = 1 \iff \|u\| = 1\). Moreover, if \(\{u_n\} \subset W^{1,N}_0(\Omega)\), then \(\|u_n\| \to 0 \iff m(u_n) \to 0\).
Lemma 2.8. [21] Theorem 2.2] Suppose that \((H_1) - (H_8)\) hold. If \(u_n \rightharpoonup u\) in \(W^{1,H}_0(\Omega)\) and
\[
\lim_{n \to \infty} \int_\Omega h(x, |\nabla u_n|) \nabla u_n \nabla (u_n - u) \leq 0,
\]
then \(u_n \rightharpoonup u\) in \(W^{1,H}_0(\Omega)\).

Next, we discuss some properties of the nonlinear function \(f\). We assume that the nonlinear function \(f\) has exponential type growth on \(\Omega_N\), which is motivated by the celebrated result of N. Trudinger [49]. N. Trudinger [49] proved that \(W^{1,N}_0(\Omega)\) is continuously embedded in the Orlicz space \(L_H(\Omega)\), where \(H = \exp(t^{\frac{n}{n-1}}) - 1\). The inequality of Trudinger was later sharpened by J. Moser [43] and known as Moser-Trudinger inequality. In the subsequent years, many authors improved and used the Moser-Trudinger inequality to study the problems involving exponential type non-linearities. Interested readers can refer to [1, 2, 18, 19, 23, 36, 37] and references cited therein. We will use the following version of Moser-Trudinger inequality:

Lemma 2.9. [6, Lemma 3.4] Let \(\alpha > 0\) and \(s > 1\) then \(\exists 0 < r < 1\) and \(C > 0\) such that
\[
\sup \int_\Omega e^{s\alpha |u|^N} dx \leq C,
\]
for any \(u \in W^{1,H}_0(\Omega)\) such that \(\|u\| \leq r\).

We assume that the nonlinear function \(f\) has critical growth on \(\Omega_p\), which causes a lack of compactness and hence, one can not prove the Palais-Smale condition directly. Lions established concentration compactness principle [40, Lemma 1.1] to address such issues. We use the following variable exponent version of the concentration compactness principle that was obtained by Bonder and Silva [8].

Lemma 2.10. Let \(\{u_n\}\) in \(W^{1,p(x)}_0(\Omega)\) which converges weakly to limit \(u\) such that
- \(\|\nabla u_n\|^{p(x)}\) converges weakly to a measure \(\mu\),
- \(\|u_n\|^{p(x)}\) converges weakly to a measure \(\nu\), where \(\mu\) and \(\nu\) are bounded non-negative measures on \(\Omega\).

Then there exist almost countable index set \(I\) and \((x_i)_{i \in I} \in \Omega\) such that
1. \(\nu = |u|^{p(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, \nu_i > 0\)
2. \(\mu \geq |\nabla u|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i}, \mu_i > 0\)
with \(S \nu_i^{1/p(x_i)} \leq \mu_i^{1/p(x_i)}, \forall i \in I,\)
where
\[
S = \inf_{u \in C^\infty_0(\Omega)} \left\{ \frac{\|\nabla u\|_{L^{p(x)}(\Omega)}}{\|u\|_{L^{p(x)}(\Omega)}} \right\} > 0.
\]

Next, we define a weak solution to \((1.1)\) and the corresponding energy functional.

Definition 1. We say that \(u \in W^{1,H}_0(\Omega)\) is a weak solution of \((1.1)\) if the following holds:
\[
a(m(u)) \int_\Omega h(x, |\nabla u|) \nabla u \nabla v = \int_\Omega f(x, u) v
\]
for all \(v \in W^{1,H}_0(\Omega)\).
Thus, the energy functional $J : W^{1, \mathcal{H}}_0(\Omega) \to \mathbb{R}$ corresponding to (2.3) is given by

$$J(u) = A(m(u)) - \int_\Omega F(x, u) \, dx,$$

where $F(x, t) = \int_0^t f(x, s) \, ds$ and $A(t) = \int_0^t a(s) \, ds$. It can be seen that $J$ is $C^1$ [6, Lemma 3.8] and the derivative of $J$ at any point $u \in W^{1, \mathcal{H}}_0(\Omega)$ is given by

$$J'(u)(v) = a(m(u)) \int_\Omega h(x, |\nabla u|) \nabla u \cdot \nabla v - \int_\Omega f(x, u)v$$

for all $v \in W^{1, \mathcal{H}}_0(\Omega)$. Moreover, the critical points of $J$ are the weak solutions to (1.1).

3. Proof of the Theorem

**Lemma 3.1.** There exist positive real numbers $\alpha$ and $\rho$ such that for each $\lambda \geq 1$ we have

$$J(u) \geq \alpha > 0, \quad \forall u \in W^{1, \mathcal{H}}_0(\Omega) : \|u\| = \rho.$$

**Proof.** It follows, from the definition of $f$ that

$$\int_\Omega F(x, t) \, dx = \int_{(x_0)_{s/2}} F(x, u) \, dx + \lambda \int_{\Omega \setminus (x_0)_{s/2}} F_1(x, u) \, dx + \int_{\Omega \setminus (x_0)_{s/2}} |u|^{p^*(x)} \, dx$$

where, $F_1(x, t) = \int_0^t |s|^{\beta-2} e^{ate^{\lambda|x|^{N/(N-1)}}} \, ds$. Again, from the definition of $f$, we get

$$\int_{(x_0)_{s/2}} F(x, u) \, dx \leq c_1 \int_{(x_0)_{s/2}} (|u|^q + |u|^\beta + |u|^{p^*(x)})$$

for $\|u\| = r$, where $r < 1$ is small enough and for some $c_1 > 0$. Using (2.2) and the fact that $\|u\| = r$, where $r < 1$ is small enough, one gets

$$\int_{(x_0)_{s/2}} F(x, u) \, dx \leq c_2 (\|u\|^{q_1}_{L^q (x_0)_{s/2}} + \|u\|^\beta_{L^\beta (x_0)_{s/2}} + \|u\|^\rho_{L^{\rho^*} (x_0)_{s/2}})$$

$$\leq c_3 (\|u\|^{q_1} + \|u\|^\beta + \|u\|^\rho)$$

for some $c_2, c_3 > 0$.

Next, by using the Hölder’s inequality, one gets

$$\lambda \int_{\Omega \setminus (x_0)_{s/2}} F_1(x, u) \, dx \leq \lambda \left( \int_{\Omega_N} |u|^{2\beta} \right)^{1/2} \left( \int_{\Omega_N} e^{2\alpha |u|^{N/(N-1)}} \right)^{1/2}.$$

Letting, $\|u\| = r < 1$, by Proposition 2.2 (a) and Lemma 2.9 we obtain

$$\lambda \int_{\Omega \setminus (x_0)_{s/2}} F_1(x, u) \, dx \leq c_4 \|u\|^\beta,$$

for some $c_4 > 0$.

Again, using Proposition 2.2 (b) and Proposition 2.3 we get

$$\int_{\Omega \setminus (x_0)_{s/2}} \frac{|u|^{p^*(x)}}{p^*(x)} \, dx \leq c_5 \|u\|^\rho,$$

where $\|u\| = r < 1$. Thus, we have established the existence of positive constants $\alpha$ and $\rho$ for each $\lambda \geq 1$. This completes the proof of Lemma 3.1.
for \( \|u\| = r \), where \( r < 1 \) and \( c_5 > 0 \). By the help of (3.2), (3.3), (3.4), (a1) and the Proposition 2.7, we have
\[
J(u) \geq a_0 \|u\|^q - c_6 \|u\|^q - c_7 \|u\|^\beta - c_8 \|u\|^{(p^*)^-},
\]
for some \( c_6, c_7, c_8 > 0 \). We can conclude the result by the fact that \( q^+, (p^*)^-, \beta > q^+ \).

**Lemma 3.2.** There exist \( \nu_0 \in W^{1,H}_0(\Omega) \) and \( \beta > 0 \) such that for each \( \lambda \geq 1 \), we have
\[
J(\nu_0) < 0 \text{ and } \|\nu_0\| > \beta.
\]

**Proof.** By the definition of \( f \) and as \( \lambda \geq 1 \), we get
\[
f(x, s) \geq |s|^{\beta-2}s, \quad \forall (x, s) \in (\Omega_N \setminus \Omega_\delta) \times [0, \infty).
\]
Let \( u \in C^\infty_c(\Omega_N \setminus (\Omega_\delta)^c) \{0 \} \) with \( \|u\| = 1 \), using (H6), (a0''') and (3.5), we have
\[
J(tu) = A(m(tu)) - \int_\Omega F(x, tu) \, dx
\leq A(1) \left( \int_{\Omega_N} \mathcal{H}(x, |\nabla tu|) \, dx \right)^\theta - \frac{|t|^\beta}{\beta} \int_{\Omega_N} |u|^\beta \, dx, \quad \forall t > 1,
\]
\[
\leq A(1)c_1 t^{N\theta} \left( \int_{\Omega_N} |\nabla u|^N \right)^\theta - \frac{|t|^\beta}{\beta} \int_{\Omega_N} |u|^\beta \, dx, \quad \forall t > 1,
\]
this implies that \( J(tu) \to -\infty \) as \( n \to \infty \), since \( \beta > N\theta \). Now, by setting \( \nu_0 = t_0u \) for sufficiently large \( t_0 > 1 \), we get the desired result. \( \square \)

By Lemmas 3.1 and 3.2, the geometric conditions of the mountain pass theorem are satisfied for the functional \( J \). Hence, by the version of the mountain pass theorem without (PS) condition, \( \exists \) a sequence \( \{u_n\} \subseteq W^{1,H}_0(\Omega) \) such that \( J(u_n) \to c_M \) and \( J'(u_n) \to 0 \) as \( n \to \infty \), where
\[
c_M = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t)) > 0,
\]
and
\[
\Gamma = \{ \gamma \in C([0, 1], W^{1,H}_0(\Omega)) : \gamma(0) = 0, \gamma(1) < 0 \}.
\]
Due to the lack of compactness, we are not able to prove that (PS) condition holds for \( J \) and we need some additional information about the mountain pass level \( c_M \).

**Lemma 3.3.** The \((PS)_{c_M}\) sequence is bounded in \( W^{1,H}_0(\Omega) \). Moreover, there exist \( u_0 \in W^{1,H}_0(\Omega) \) such that, up to a subsequence, we have \( u_n \rightharpoonup u_0 \) weakly in \( W^{1,H}_0(\Omega) \) and \( u_n(x) \to u_0(x) \) a.e. \( x \in \Omega \).

**Proof.** If \( \psi = \min \{ \chi, \beta, (p^*)^- \} \), we have
\[
0 < \psi F(x, t) \leq f(x, t) t, \quad \forall (x, t) \in \Omega \times (\mathbb{R} \setminus \{0\}).
\]
Since \( \{u_n\} \) is a \((PS)_{c_M}\) sequence for \( J \), we have \( J(u_n) \to c_M \) and \( J'(u_n) \to 0 \) as \( n \to \infty \), i.e.,
\[
A(m(u_n)) - \int_\Omega F(x, u_n) \, dx = c_M + \delta_n,
\]
where \( \delta_n \to 0 \) as \( n \to \infty \) and
\[
\left| a(m(u_n)) \int_\Omega h(x, |\nabla u_n|) \nabla u_n \nabla v \, dx - \int_\Omega f(x, u_n) v \right| \leq \varepsilon_n \|v\|,
\]
\( \forall v \in W^{1,H}_0(\Omega) \), where \( \varepsilon_n \to 0 \) as \( n \to \infty \). On taking \( v = u_n \), by using (3.6), (3.7) and (5.8), we obtain

\[
A(m(u_n)) - \frac{1}{\psi} a(m(u_n)) \int_\Omega h(x, |\nabla u_n|) |\nabla u_n|^2 \\
\leq c_9(1 + ||u_n||),
\]

for some \( c_9 > 0 \). It follows from (a1) that

\[
a_0 \left( 1 - \frac{q^+}{\psi} \right) m(u_n) \leq c_9(1 + ||u_n||).
\]

If \( ||u|| \geq 1 \), by Proposition 2.4, we obtain

\[
a_0 \left( 1 - \frac{q^+}{\psi} \right) ||u_n||^{p-} \leq c_9(1 + ||u_n||).
\]

This implies that \( \{u_n\} \) is bounded in \( W^{1,H}_0(\Omega) \). As \( W^{1,H}_0(\Omega) \) is a reflexive space, \( \exists u_0 \in W^{1,H}_0(\Omega) \) such that up to a subsequence, we have \( u_n \rightharpoonup u_0 \) weakly in \( W^{1,H}_0(\Omega) \). Further, by Proposition 2.3, we have \( u_n(x) \to u_0(x) \) a.e. \( x \in \Omega \). \( \square \)

**Lemma 3.4.** There exist \( \lambda_1 > 1 \) such that for each \( \lambda \geq \lambda_1 \), we have

\[
c_M \leq a_0 \left( 1 - \frac{q^+}{\psi} \right) \min \left\{ \frac{1}{N} \left( \frac{\alpha_N}{2^{N/2}} \right)^{N-1}, \frac{a_{\min}}{p^+} \right\}.
\]

where \( \psi = \min\{\chi, \beta, (p^*)^{-1} \} \) and \( a_{\min} = \min_{x \in \Omega} \min\{N/p(x)\}^{-1} \). Moreover, for any (PS)\(_{c_M} \) sequence \( \{u_n\} \), we have

\[
\limsup_{n \to \infty} ||\nabla u_n||^{N/(N-1)}_{L^N(\Omega_N)} < \frac{\alpha_N}{2^{N-1}}.
\]

**Proof.** If \( 0 \leq s \leq t_1 = \max\{q_0^+, t_0^+\} \), then \( A(s) \leq a(t_0)s \), where \( t_0 \) is defined in the proof of the Lemma 3.2. Let \( v_0 \in C_c^\infty(\Omega \setminus \{0\}) \) be as in the Lemma 3.2 and \( 0 \leq t \leq 1 \). By using (H_6), (3.3) and Proposition 2.4, we have

\[
J(tv_0) = A(m(tv_0)) - \int_\Omega F(x, tv_0)dx \\
\leq a(t_1) m(tv_0) - \frac{\lambda |t|^\beta}{\beta} \int_{\Omega_N} |v_0|^\beta dx, \\
\leq \frac{a(t_1) t^N C_1}{N} \left( \int_{\Omega_N} |\nabla v_0|^N \right) - \frac{\lambda |t|^\beta}{\beta} \int_{\Omega_N} |v_0|^\beta dx.
\]

Further, we get

\[
\max_{0 \leq t \leq 1} J(tv_0) \leq - \frac{1}{\lambda^{\frac{N}{p^+}}} \left( \frac{1}{N} \right) \frac{\left( a(t_1) t^N \right)}{\left( \int_{\Omega_N} |\nabla v_0|^N \right)^{\frac{N}{p^+}}}.
\]

On taking, \( \gamma = tv_0 \) for \( 0 \leq t \leq 1 \), we have

\[
c_M \leq \max_{0 \leq t \leq 1} J(tv_0) \leq \frac{1}{\lambda^{\frac{N}{p^+}}} \left( \frac{1}{N} \right) \frac{\left( a(t_1) t^N \right)}{\left( \int_{\Omega_N} |\nabla v_0|^N \right)^{\frac{N}{p^+}}}.
\]
Now, choosing \( \lambda_1 > 1 \) in such a way that \( \forall \lambda \geq \lambda_1, \) we have

\[
\frac{1}{\lambda^{\epsilon_0}} \left( \frac{1}{N} - \frac{1}{\beta} \right) \left( \frac{a(t_1)C_1 \| \nabla u_0 \|_{L^2(\Omega_N)}^N}{\| u_0 \|_{H^1(\Omega_N)}^\beta} \right)^{\frac{\beta-N}{\beta N}} < a_0 \left( \frac{1}{N} \psi \left( \frac{\alpha_N}{2 \epsilon_0} \right) \right)^{N-1} \min \left\{ \frac{1}{N} \left( \frac{\alpha_N}{2 N^{\frac{N-1}{N}}} \right)^{N-1} \lambda, \frac{a_{\min}}{\epsilon_0} \lambda \right\}.
\]

Therefore,

\[
(3.9) \quad c_M < a_0 \left( \frac{1}{N} \psi \left( \frac{\alpha_N}{2 \epsilon_0} \right) \right)^{N-1} \min \left\{ \frac{1}{N} \left( \frac{\alpha_N}{2 N^{\frac{N-1}{N}}} \right)^{N-1} \lambda, \frac{a_{\min}}{\epsilon_0} \lambda \right\}, \quad \forall \lambda \geq \lambda_1.
\]

Moreover, since \( \{u_n\} \) is a \((PS)_{c_M}\) sequence for \( J \), we have \( J(u_n) \to c_M \) and \( J'(u_n) \to 0 \) as \( n \to \infty \). By (3.7) and (3.8), we get

\[
A(\nu(u_n)) - \frac{1}{\psi} (\nu(u_n)) \int \Omega h(x, |\nabla u_n|) |\nabla u_n|^2 \leq \delta_n + c_M + \epsilon_n \|u_n\|.
\]

It follows from \((a_1)\) that

\[
a_0 \left( \frac{1}{N} \psi \left( \frac{\alpha_N}{2 \epsilon_0} \right) \right) m(u_n) \leq \delta_n + c_M + \epsilon_n \|u_n\|.
\]

By \((H_6)\) and (3.9), we get

\[
\limsup_{n \to \infty} \|\nabla u_n\|_{L^2(\Omega_N)}^{N/(N-1)} \leq c_M < \frac{\alpha_N}{2 N^{\frac{N-1}{N}}}.
\]

\[\square\]

**Lemma 3.5.** The functional \( J \) satisfies the \((PS)_{c_M}\) condition.

**Proof.** Define

\[
P_n = a \left( m(u_n) \right) \int \Omega h(x, |\nabla u_n|) \nabla u_n \nabla (u_n - u).
\]

Then

\[
P_n = J'(u_n) u_n + \int \Omega f(x, u_n) u_n dx - J'(u_n) u - \int \Omega f(x, u) u dx.
\]

Using the definition of \( f \), \( P_n \) can be rewritten as

\[
P_n = \int \Omega \psi_N(x) \varphi(x, u_n) u_n dx + \int \Omega \lambda |u_n|^{\beta_1} e^{\alpha |u_n|} \frac{\nabla u_n}{\nabla u_n} dx + \int \Omega \lambda \psi_N(x) |u_n|^{\beta} e^{\alpha |u_n|} \frac{\nabla u_n}{\nabla u_n} dx
\]

\[
+ \int \Omega |u_n|^{p_1(x)} dx + \int \Omega \psi(x) |u_n|^{p_1(x)} dx - \int \Omega \psi(x) \varphi(x, u_n) u dx
\]

\[
- \int \Omega \lambda |u_n|^{\beta} e^{\alpha |u_n|} \frac{\nabla u_n}{\nabla u_n} dx - \int \Omega \lambda \psi_N(x) |u_n|^{\beta} e^{\alpha |u_n|} \frac{\nabla u_n}{\nabla u_n} dx
\]

\[
- \int \Omega |u_n|^{p_1(x)-2} u_n dx - \int \Omega \psi(x) |u_n|^{p_1(x)-2} u_n dx + o_n(1).
\]
From the embedding results, we have

\[ P_n = \lambda \int_{\Omega_N} |u_n|^{p} \alpha|u_n|^{\frac{\lambda p}{N}} \, dx + \int_{\Omega_p} |u_n|^{p^*}(x) \, dx \]

\[ - \lambda \int_{\Omega_N} |u_n|^{\frac{\lambda p - 2}{2}} u_n e^{\alpha|u_n|^{\frac{\lambda}{p}}} \, dx - \int_{\Omega_p} |u_n|^{p^*}(x)^{-2} u_n u \, dx + o_n(1). \]

As proved in the [6, Lemma 3.13], we have

\[ P_n = \int_{\Omega_p} |u_n|^{p^*}(x) \, dx - \int_{\Omega_p} |u|^{p^*}(x) \, dx + o_n(1). \]

By [22, Theorem 1.14] and [52, Theorem 3.1], one gets

\[ P_n = \int_{\Omega_p} |u_n|^{p^*}(x) \, dx - \int_{\Omega_p} |u|^{p^*}(x) \, dx + o_n(1). \]

Next, we will apply the Lemma 2.10 to the sequence \( \{u_n\} \subset W^{1,p(x)}(\Omega_p) \) and will prove that

\[ (3.10) \qquad \int_{\Omega_p} |u_n|^{p^*}(x) \, dx \rightarrow \int_{\Omega_p} |u|^{p^*}(x) \, dx. \]

Since, the \( W^{1,p}(\Omega) \rightarrow C((\Omega_q)_\delta) \) is compact and \( \{u_n\} \) is bounded in \( W^{1,p}(\Omega) \), we get \( u_n \rightharpoonup u \) in \( L^{p^*}(\Omega_q)_\delta \), which implies that \( u_i \in \Omega_p(\Omega_q)_\delta \) for each \( i \in I \).

To prove (3.10), it is suffices to prove that \( I \) is finite. Further, the set \( I \) can be partitioned as \( I = I_1 \cup I_2 \), where \( I_1 = \{ i \in I : x \in \Omega_p(\Omega_q)_\delta \} \) and \( I_2 = \{ i \in I : x \in \Omega_p(\Omega_q)_\delta \} \).

First, we show that \( I_1 \) is finite. Choose a cutoff function \( \varphi_0 \in C^\infty_c(\mathbb{R}^N) \) such that

\[ \varphi_0 = 1 \text{ on } B(0, 1), \quad \varphi_0 = 0 \text{ on } B(0, 2)^c. \]

Now, for each \( \epsilon > 0 \), define \( v(x) = \varphi_0((x - x_i)/\epsilon) \forall x \in \mathbb{R}^N. \) As \( \{u_n\} \) is a (PS)\(_{c_M}\) sequence, we have

\[ a(m(u_n)) \int_{\Omega} h(x, |\nabla u_n|) \nabla u_n \nabla (v v_n) = \int_{\Omega} \tilde{\varphi}_q(x) \varphi(x, u_n) v v_n \, dx \]

\[ + \int_{\Omega} \psi_p(x)|u_n|^{p^*}(x) \, dx + o_n(1). \]

It follows from \((a_1)\) that

\[ a_0 \int_{\Omega} h(x, |\nabla u_n|) \nabla u_n \nabla (v v_n) \, dx \leq \int_{\Omega} \tilde{\varphi}_q(x) \varphi(x, u_n) v v_n \, dx + \int_{\Omega} \psi_p(x)|u_n|^{p^*}(x) \, dx + o_n(1), \]

which implies that

\[ a_0 \int_{\Omega} h(x, |\nabla u_n|) u_n \nabla u_n \nabla v \, dx \leq \int_{\Omega} \tilde{\varphi}_q(x) \varphi(x, u_n) v v_n \, dx \]

\[ + \int_{\Omega} \psi_p(x)|u_n|^{p^*}(x) \, dx - a_0 \int_{\Omega} h(x, |\nabla u_n|) |\nabla u_n|^2 \, dx + o_n(1). \]

Next, by using (2.7), \( \Delta_2\)-condition and Young’s inequality, we get

\[ (3.12) \qquad \int_{\Omega} |h(x, |\nabla u_n|)| |\nabla u_n| u_n |\nabla v| \, dx \leq \zeta m(u_n) + C \int_{\Omega} \mathcal{H}(x, |u_n| |\nabla v|) \, dx. \]
On using \((H_9)\), one gets
\[
\int_{\Omega} \mathcal{H}(x, |u_n| |\nabla v|) \, dx \leq c_{10} \left( \int_{\Omega} \eta_2(x) |u_n|^{q(x)} |\nabla v|^{q(x)} \, dx + \int_{\Omega} |u_n|^{p(x)} |\nabla v|^{p(x)} \, dx \right),
\]
for some \(c_{10} > 0\). Using generalized Hölder’s inequality \[(2.4)\] we get
\[
\int_{\Omega} \mathcal{H}(x, |u_n| |\nabla v|) \, dx \leq c_{11} \frac{\|\nabla v\|^{q(x)}}{L^{p(x) - q(x)}(\Omega)} \|u_n\|^{q(x) - \frac{q(x)}{p(x)}}(\Omega) + c_{12} \frac{\|\nabla v\|^{p(x)}}{L^{p(x) - p(x)}(\Omega)} \|u_n\|^{p(x) - \frac{p(x)}{p(x)}}(\Omega),
\]
for some \(c_{11}, c_{12} > 0\). Further, by Proposition \[(2.2)\) and Proposition \[(2.6)\), we have
\[
\int_{\Omega} \mathcal{H}(x, |u_n| |\nabla v|) \, dx \leq c_{11} \max \left\{ \left( \int_{B(x, 2\epsilon)} \frac{\|u_n\|^{q(x) - \frac{q(x)}{p(x)}}(\Omega)}{L^{p(x) - q(x)}(\Omega)} \right)^{p_1}, \left( \int_{B(x, 2\epsilon)} \frac{\|u_n\|^{q(x) - \frac{q(x)}{p(x)}}(\Omega)}{L^{p(x) - q(x)}(\Omega)} \right)^{p_2} \right\}
\]
\[
+ c_{12} \max \left\{ \left( \int_{B(x, 2\epsilon)} |\nabla v|^{p_1/\epsilon} \right)^{p_1}, \left( \int_{B(x, 2\epsilon)} |\nabla v|^{p_2/\epsilon} \right)^{p_2} \right\},
\]
where \(p_1 = \min_{x \in \Omega} \left\{ \frac{p_1(x) - q(x)}{p(x) - q(x)} \right\} \) and \(p_2 = \max_{x \in \Omega} \left\{ \frac{p_1(x) - q(x)}{p(x) - q(x)} \right\} \).

Hence, by using Proposition \[(2.2)\) \((b)\) together with the fact that \(\{u_n\}\) is bounded, we have
\[
(3.13) \lim_{\epsilon \to 0} \int_{\Omega} \mathcal{H}(x, |u_n| |\nabla v|) = 0.
\]
By using \[(3.12)\), \[(3.13)\), Proposition \[(2.7)\) and using the boundedness of \(\{u_n\}\), we get
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} h(x, |\nabla u_n|) |\nabla u_n| |u_n| |\nabla v| \, dx \leq c_{12} \zeta,
\]
for some \(c_{12} > 0\). As \(\zeta\) is arbitrary, one get
\[
(3.14) \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} \int_{\Omega} h(x, |\nabla u_n|) |\nabla u_n| |u_n| |\nabla v| \, dx \right) = 0.
\]
Consequently, by \((H_9), \) \[(3.11)\) and \[(3.14)\), we get
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \left( \int_{\Omega} \tilde{\psi}_q(x) \varphi(x, u_n) v u_n \, dx + \int_{\Omega} \psi_p(x) |u_n|^{p_1(x)} v \, dx - a_0 \int_{\Omega} |\nabla u_n|^{p(x)} v \, dx \right) \geq 0,
\]
which is
\[
\nu_i - a_0 \mu_i = \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |u_n|^{p_1(x)} v \, dx - a_0 \lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p(x)} v \, dx \geq 0.
\]
By Lemma \[(2.10)\) we get \(\nu_i \geq a_0 S^{p_1(x)} L^{p_1(x)/p(x)}, \) consequently either \(\nu_i = 0\) or \(\nu_i \geq a_0 S^{N} \). Next, we will prove that \(\nu_i \geq a_0 S^{N}\) is not possible. Let suppose
\[ \nu_i \geq a_0^{\frac{N}{p(x)}} S^N, \] then by Lemma 2.10 we get \( \mu_i \geq S^N a_0^{\frac{N}{p(x_i)}} \). Also, since \( |\nabla u_n|^p \) converges weakly to a measure \( \mu \),

\[ \lim \inf_{n \to \infty} \int_{\Omega^p} |\nabla u_n|^p \, dx \geq \mu_i, \]

and hence,

\[ (3.15) \quad \lim \inf_{n \to \infty} \int_{\Omega^p} |\nabla u_n|^p \, dx \geq S^N a_0^{\frac{N}{p(x_i)}} \geq S^N a_{\min}, \]

where, \( a_{\min} = \min_{x \in \Omega} a\left(\frac{N}{p(x)}\right) - 1 \).

Since, \( \{u_n\} \) is a \((PS)_{c_M}\) sequence for \( J \), we have \( J(u_n) \to c_M \) and \( J'(u_n) \to 0 \) as \( n \to \infty \). By (3.6), (3.7) and (3.8), we get

\[ A(m(u_n)) - \frac{1}{\psi} a(m(u_n)) \int_{\Omega} h(x, |\nabla u_n|) |\nabla u_n|^2 \, dx \]
\[ \leq \delta_n + c_M + \varepsilon_n \|u_n\|. \]

It follows from \((a_1)\) that

\[ a_0 \left(1 - \frac{q^+}{\psi}\right) m(u_n) \leq \delta_n + c_M + \varepsilon_n \|u_n\|. \]

By \((\mathcal{H}_8)\) and \( (3.9) \), we obtain

\[ a_0 \left(1 - \frac{q^+}{\psi}\right) \lim \inf_{n \to \infty} \int_{\Omega} |\nabla u_n|^p \, dx \leq c_M < a_0 \left(1 - \frac{q^+}{\psi}\right) S^N a_{\min}, \]

which is a contradiction to \( (3.15) \). Hence, \( I_1 \) is an empty set.

By using \((\mathcal{H}_8)\) and proceeding as above, one can show that \( I_2 = \emptyset \).

Therefore, we get \( P_n = a_n(1) \), and so

\[ \lim_{n \to \infty} a(m(u_n)) \int_{\Omega} h(x, |\nabla u_n|) \nabla u_n \nabla (u_n - u) = 0, \]

from which we have

\[ \lim_{n \to \infty} \int_{\Omega} h(x, |\nabla u_n|) \nabla u_n \nabla (u_n - u) \leq 0. \]

By Lemma 2.8 we have \( u_n \to u \) in \( W_0^{1,H}(\Omega) \).

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Shilpa Gupta
Department of Mathematics
Birla Institute of Technology and Science Pilani
Pilani Campus, Vidya Vihar
Pilani, Jhunjhunu
Rajasthan, India - 333031
Email address: p20180442@pilani.bits-pilani.ac.in; shilpagupta890@gmail.com

Gaurav Dwivedi
Department of Mathematics
Birla Institute of Technology and Science Pilani
Pilani Campus, Vidya Vihar
Pilani, Jhunjhunu
Rajasthan, India - 333031
Email address: gaurav.dwivedi@pilani.bits-pilani.ac.in