THE UNIQUE TANGENT CONE PROPERTY FOR WEAKLY HOLOMORPHIC MAPS INTO PROJECTIVE ALGEBRAIC VARIETIES

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ABSTRACT. In the present paper, we establish the uniqueness of tangent maps for general weakly holomorphic and locally approximable maps from an arbitrary almost complex manifold into projective algebraic varieties. As a byproduct of the approach and the techniques developed we also obtain the unique tangent cone property for a special class of non-rectifiable positive pseudo-holomorphic cycles. This approach gives also a new proof of the main result by C.Bellettini in \cite{Bellettini} on the uniqueness of tangent cones for positive integral \((p,p)\)-cycles in arbitrary almost complex manifolds.

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1. INTRODUCTION

1.1. Weakly holomorphic and locally approximable maps. For the purposes of this introduction, we always denote by \(M\) a connected smooth manifold without boundary and we will need to endow \(M\) with an arbitrarily chosen reference metric \(g\). Since our results are local, all our discussion will be totally independent or, in any case, not essentially effected
by such arbitrary choice. Moreover, throughout the paper we always stick to the following conventions:

1. for \( K \subset M \) compact, \( W^{1,2}(K) \) is the closure of the space \( C^\infty(K) \) with respect to the strong topology induced by the \( W^{1,2} \)-norm, given by
\[
||u||_{W^{1,2}(K)} := \left( \int_K |u|^2 \, dvol_g \right)^{1/2} + \left( \int_K |du|^2 \, dvol_g \right)^{1/2},
\]
for every \( u \in C^\infty(K) \);
2. \( W^{1,2}_{loc}(M) := \{ u \in W^{1,2}(K) \text{ for every } K \subset M \text{ compact} \} \);
3. for any \( k \in \mathbb{N} \), the space \( W^{1,2}_{loc}(M, \mathbb{R}^k) \) is the real vector space of functions \( u = (u_1, ..., u_k) \) such that \( u_j \in W^{1,2}_{loc}(M) \), \( j = 1, ..., k \);
4. given a closed (i.e. connected, compact and without boundary) smooth manifold \( N \) and a smooth isometric embedding \( N \hookrightarrow \mathbb{R}^k \), for some \( k \in \mathbb{N} \) large enough, we let
\[
W^{1,2}_{loc}(M, N) := \{ u \in W^{1,2}_{loc}(M, \mathbb{R}^k) \text{ s.t. } u(x) \in N, \text{ for } \text{vol}_g \text{-a.e. } x \in M \}.
\]

**Definition 1.1.** Let \( M \) be any even-dimensional smooth manifold without boundary and let \( J \) be a Lipschitz almost complex structure on \( M \). Let \( (N, J_N) \) be any closed smooth almost complex manifold. We say that a map \( u \in W^{1,2}_{loc}(M, N) \) is weakly \( (J, J_N) \)-holomorphic if
\[
du_x(JX) = J_N du_x(X), \quad \text{for vol}_g \text{-a.e. } x \in M, \forall X \in T_xM.
\]
Whenever we don’t need to specify which couple of complex structures is involved in the previous definition, we simply say that the map \( u \) is weakly pseudo-holomorphic or even just weakly holomorphic to lighten the notation.

Assume that \( M \) is any even-dimensional smooth manifold without boundary and let \( J \) be a Lipschitz almost complex structure on \( M \) which admits a compatible symplectic form \( \Omega \), meaning that the bilinear form \( (X, Y) \mapsto \Omega(X, JY) \) defines a Lipschitz Riemannian metric on \( M \). We will show (Lemma 3.2) that in this particular framework any weakly \( (J, J_N) \)-holomorphic map taking values into a closed smooth almost Kähler manifold \( N \) is weakly harmonic, i.e.
\[
\frac{d}{dt} \int_M |d(\pi_N \circ (\Phi \circ u + tX))|^2 \, dvol_g \bigg|_{t=0} = 0, \quad \forall X \in C^\infty (M, \mathbb{R}^k),
\]
where \( \pi_N : W \to N \) is the nearest-point projection into \( N \), defined on a suitable tubular neighbourhood \( W \) of \( N \), and \( \Phi : N \hookrightarrow \mathbb{R}^k \) denotes a smooth, isometric embedding of \( N \) into \( \mathbb{R}^k \). Nevertheless, it is well-known that no regularity is ensured for weakly harmonic maps when the dimension of the domain is larger than 2 (see [22]). Thus, we will need to prescribe some additional condition in order to get that the map \( u \) is at least stationary harmonic, i.e.
\[
\frac{d}{dt} \int_M |d(u \circ \Phi_t)|^2 \, dvol_g \bigg|_{t=0} = 0,
\]
for any smooth one-parameter family of diffeomorphisms \( \Phi_t \) of \( M \) with compact support.

We will show (see Lemma 3.3) that imposing the following local, strong approximability property with respect to the \( W^{1,2} \)-norm suffices to our purposes.

**Definition 1.2.** Let \( M \) be a smooth manifold without boundary and \( N \) be a closed smooth manifold. We say that a map \( u \in W^{1,2}_{loc}(M, N) \) is locally (strongly) approximable with
respect to the $W^{1,2}$-norm if for every open set $U \subset M$ such that $U$ is diffeomorphic to some euclidean ball there exists a sequence of smooth maps $\{u_j\}_{j \in \mathbb{N}} \subset C^\infty(U, N)$ such that $u_j \to u$ as $j \to +\infty$, strongly in $W^{1,2}(U, N)$.

If a map $u$ is locally approximable, then the following cohomological condition follows easily:

$$d(u^*\omega) = 0, \quad \text{distributionally on } M,$$

for every closed 2-form $\omega \in \Omega^2(N)$. We refer the reader to [1] for further reading concerning the deep link between local approximability and (1.1).

In the most general case that we will address, i.e. when $J$ doesn’t admit a compatible symplectic form (even locally), weakly holomorphic and locally approximable maps are not stationary harmonic. Nevertheless, they are almost stationary harmonic, in the sense that they satisfy a perturbed version of the harmonic map equation and that there exists $C > 0$ such that

$$\frac{d}{dt} \int_M |d(u \circ \Phi_t)|_g^2 \, d\text{vol}_g \bigg|_{t=0} \geq -C \int_M |X||du|^2,$$

with $X := \partial_t \Phi_t|_{t=0}$ for any smooth one-parameter family of diffeomorphisms $\Phi_t$ of $M$ with compact support. We underline that such maps were also studied before by C. Bellettini and G. Tian in [2].

1.2. Statement of the main results and previous literature. Given any $\rho \in (0, +\infty)$, we denote by $B_\rho \subset \mathbb{R}^{2m}$ the open unit ball in $\mathbb{R}^{2m}$ centred at the origin and having radius $\rho$. When we simply write $B$, we always mean the open unit ball $B_1 \subset \mathbb{R}^{2m}$. From now on, for every $\rho \in (0, 1)$ we let $\Phi_\rho : B_\rho \to B$ be given by $\Phi_\rho(x) := \rho^{-1}x$, for every $x \in B_\rho$.

Let $M$ be a smooth manifold without boundary and $N$ be a closed smooth manifold. Consider any map $u \in W^{1,2}_{loc}(M, N)$. Given a point $x_0 \in M$, pick any coordinate chart $\varphi : U \subset M \to B$ with relatively compact domain $U$ at $x_0$, i.e. such that $x_0 \in U$ and $\varphi(x_0) = 0$.

The family of the blow-ups of $u$ at the point $x_0$, denoted by $\{u_\rho\}_{\rho \in (0, 1)} \subset W^{1,2}(U, N)$, is given by $u_\rho := u \circ \varphi^{-1} \circ \Phi_\rho^{-1} \circ \varphi$, for every $\rho \in (0, 1)$. If such family is bounded in $W^{1,2}(U, N)$, by standard compactness arguments it follows that for every sequence $\rho_k \to 0^+$ as $k \to +\infty$ there exists a subsequence $\{\rho_{k_j}\}_{j \in \mathbb{N}}$ such that $u_{\rho_{k_j}} \to u_\infty \in W^{1,2}(U, N)$, weakly in $W^{1,2}(U, N)$. We say that $u_\infty$ is a tangent map for $u$ at the point $x_0$. Any tangent map at $x_0$ is meant to represent a picture of the map $u$ when one gets closer and closer to $x_0$.

Such limiting configuration may very well depend on the sequence $\{\rho_{k_j}\}_{j \in \mathbb{N}}$ that we have chosen to approach $x_0$. If this is not the case, we say that the map $u$ has a unique tangent map at the point $x_0$.

In the present paper, we aim to give a complete and self-contained proof of the following theorem.

**Theorem 1.1.** Let $m, n \in \mathbb{N}_0$ be such that $m \geq 2$. Let $M$ be a smooth $2m$-dimensional manifold without boundary and let $J$ be any Lipschitz almost complex structure on $M$. Let $N \subset \mathbb{CP}^n$ be a projective algebraic variety. Assume that $u \in W^{1,2}_{loc}(M, N)$ is weakly $(J, j_n)$-holomorphic and locally approximable, where $j_n$ stands for the standard complex structure on $\mathbb{CP}^n$ (restricted to $N$).

Then, $u$ has a unique tangent map at every point.
As we have seen before, if \( J \) admits a compatible symplectic form then weakly holomorphic and locally approximable maps are a special subclass in the much wider family of stationary harmonic maps (see previous subsection 1.1). Hence, our main result relates with the whole set of well-known facts concerning stationary harmonic maps between manifolds (see e.g. [13], [7], [24]). In particular, both the existence of tangent maps at every point and Theorem 3.1 are immediate consequences of the general theory of stationary harmonic maps. However, nothing can be said a priori about uniqueness of tangent maps to general stationary harmonic maps, since B. White (see [30]) has shown that such property might fail even for energy-minimizing maps at their singular points. Nevertheless, whenever the target manifold is analytic, uniqueness of tangent maps was proved to hold for energy-minimizing harmonic maps by L. Simon in [28]. Hence, since any projective algebraic variety is analytic, if weakly holomorphic and locally approximable maps were energy-minimizing then Theorem 1.1 would be a direct consequence of Simon’s result. Unfortunately, it’s not hard to build sequences of even holomorphic maps that converge weakly but not strongly in \( W^{1,2}_{\text{loc}}(\mathbb{R}^m, N) \) (see Example 1.1). Since for energy-minimizing harmonic maps weak convergence implies strong convergence (see [27]), this suffices to convince ourselves that weakly holomorphic and locally approximable maps are not energy-minimizing harmonic maps in general.

**Example 1.1.** Let \( N = \mathbb{S}^2 \), equipped with the standard Kähler structure. Let \( S \subseteq \mathbb{S}^2 \) be the south-pole in and let \( p_S : \mathbb{S}^2 \to \mathbb{R}^2 \) be the stereographic projection from the south-pole. For \( \lambda > 0 \), we define the map \( u_\lambda : \mathbb{R}^2 \to \mathbb{S}^2 \) as follows:

\[
u_\lambda(x) := p_S^{-1}(\lambda x) \quad \forall x \in \mathbb{R}^2.
\]

For every \( \lambda > 0 \), \( u_\lambda \) is a finite-energy, orientation-preserving conformal map from \( \mathbb{R}^2 \) to \( \mathbb{S}^2 \). Hence, \( \{u_\lambda\}_{\lambda > 0} \) is a family of holomorphic maps in \( W^{1,2}(\mathbb{R}^2, \mathbb{S}^2) \). An easy computation shows that \( u_\lambda \rightharpoonup u_\infty \equiv S \) weakly in \( W^{1,2} \) as \( \lambda \to +\infty \). Nevertheless, the convergence cannot be strong because

\[
\int_{\mathbb{R}^2} |du_\lambda|^2 d\mathcal{L}^2 \to 8\pi \neq 0 = \int_{\mathbb{R}^2} |du_\infty|^2 d\mathcal{L}^2 \quad (\lambda \to +\infty).
\]

Theorem 1.1 was already proven when the almost complex structure \( J \) on the domain is integrable by S. Sun and X. Chen in [11], thanks also to the previous contributions [17] and [16] who established the optimal bound for the Hausdorff measure of the singular set, namely

\[
\mathcal{H}^{2m-4}(\text{Sing}(u) \cap K) < +\infty, \quad \forall K \subset M \text{ compact}. \tag{1.2}
\]

Nevertheless, the proof provided by S. Sun and X. Chen in the integrable case makes an extensive use of complex holomorphic coordinates on the base manifold and of several algebraic tools that are not available in case we work in the almost complex framework.

As far as we know, the only available result concerning the non-integrable case that can be found in literature was achieved by the second author and G. Tian in [26]. In such paper, the case of a 4-dimensional domain \( M \) is completely solved, providing also the optimal size (1.2) for the singular set. Unfortunately, the proof that is given there strongly relies on positive intersection arguments that cannot be reproduced when \( m > 2 \).

**1.3. Key ideas to face the non-integrable case in higher dimensions.** In view of what we have seen in subsection 1.2, we need to think of a completely new analytic approach in
order to prove Theorem 1.1 in its full generality. From now on, we will denote by $J_0$ the standard complex structure on $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

Let $M$, $N$ and $u$ satisfy the hypotheses of Theorem 1.1. Given any point $x_0 \in M$ and a local chart $\varphi : U \subset M \to B_2$ with compact domain $U$ at $x_0$ such that $J(0) = J_0$, it’s clear that $u$ has a unique tangent map at $x_0$ if and only if the local representative $\tilde{u} := u \circ \varphi^{-1} \in W^{1,2}(B_2, N)$ of $u$ has a unique tangent map at the origin. Notice that $\tilde{u}$ is weakly $(\tilde{J}, j_n)$-holomorphic on $B_2$, where $\tilde{J} := d\varphi \circ J \circ d\varphi^{-1}$ is a Lipschitz almost complex structure on $B_2$. Moreover, a straightforward computation allows to conclude that $\tilde{u}$ is locally approximable on $B_2$. Hence, Theorem 1.1 will be proved if we just manage to show that the statement holds in case $M = B_2 \subset \mathbb{R}^{2m}$, $x_0 = 0$, $J(0) = J_0$ and $u \in W^{1,2}(B_2, N)$.

The fact that we have reduced to prove the statement on some open ball leads to a key advantage. Since $B_2$ is contractible, we can find a Lipschitz almost symplectic form $\Omega$ on $B_2$ which is compatible with $\tilde{J}$, i.e. the symmetric bilinear form $(X, Y) \mapsto g(X, Y) := \Omega(X, JY)$ defines a Lipschitz metric $g$ on $B_2$. Here and throughout the whole paper, by "almost symplectic form" we mean any non-degenerate 2-form, even if not necessarily closed (the reader should be aware that the term "almost Hermitian structure" can also be found in literature to refer to the triple $(B_2, J, \omega)$). From now on, we will assume that the domain $B_2$ is endowed with such special metric $g$ and all the computations involving scalar products will be referring to this specific choice. Notice that $\Omega^k/k!$ is a semicalibration on $B_2$ with respect to the metric $g$, for every $k = 1, \ldots, m$. This simply amounts to the fact that the comass norm of $\Omega^k/k!$ with respect to the metric $g$, given by

$$\left\| \frac{\Omega^k}{k!} \right\|_* := \sup \left\{ \frac{\Omega_x^k}{k!} \left\langle \xi, \xi \right\rangle \text{ s.t. } x \in B_2, \xi \in \wedge^k \mathbb{R}^{2m} \text{ unit simple } k\text{-vector} \right\},$$

is equal to 1. In case $\Omega^k/k!$ were also a closed form, we would say that it is a calibration on $B_2$. The notion of calibration has a long and rich history. The paper which gave its name to the corresponding general mathematical notion is the famous work of Harvey and Lawson [13] but complex analytic submanifolds and calibrated subvarieties had been introduced before. We invite the reader to consult the works of F. Morgan [20] and [21] for a more complete introduction to this important object of geometry.

Before giving the following, fundamental definition, we recall that a normal $k$-current on $B_2$ is $k$-dimensional current $T \in \mathcal{D}_k(B_2)$ such that

$$\mathcal{M}(T) \mathcal{M}(\partial T) < +\infty.$$

**Definition 1.3** (Semicalibrated currents). A normal $k$-current $T \in \mathcal{D}_k(B_2)$ for some $k \in \mathbb{N}$ is said to be semicalibrated by a given semicalibration $\omega$ on $B_2$ if one of the following equivalent conditions hold:

1. the measure theoretic orientation $\bar{T}$ of $T$ is a convex linear combination of $k$-vectors semicalibrated by $\omega$ (i.e. such that their duality with $\omega$ is unitary), $||T||$-a.e. on $B_2$ where $||T||$ stands for the total variation of $T$;
2. $\mathcal{M}(T) = \left\langle T, \omega \right\rangle$.

If $\omega$ is a calibration (i.e. if it’s closed), we say that $T$ is calibrated by $\omega$.

One of the reasons why calibrated and semicalibrated currents have been very much studied is that calibrated $k$-cycles are homologically mass-minimizing. Indeed, assume that $T \in$
$D_k(B_2)$ is a cycle, calibrated by a calibration $\omega$. If we pick any other cycle $T' \in D_k(B_2)$ in the same homology class of $T$, i.e. such that $T - T' = \partial S$ for some $S \in D_{k+1}(B_2)$, we immediately get

$$M(T) = \langle T, \omega \rangle = \langle T' + \partial S, \omega \rangle = \langle T', \omega \rangle \leq M(T'),$$

(1.3)

where we used Stokes theorem, $d\omega = 0$ and $\|\omega\|_\ast \leq 1$. Unfortunately, it happens very often that the closeness requirement in the definition of calibration is too strong to suit certain problems, such as the one we are interested in. Therefore, as initiated in [22], it is natural to study semicalibrations and semicalibrated cycles. Substantial work on the uniqueness of tangent cones was carried out for special classes of integral semicalibrated cycles. In particular, such result was already obtained by C. Bellettini together with the second author in [4] for special legendrian integral cycles in $S^5$ and C. Bellettini in [3] has proved uniqueness of tangent cone for positive integral $(p, p)$-cycles in arbitrary almost complex manifolds. The case of positive integral $(1, 1)$-cycles in arbitrary almost Kähler manifolds was previously covered by the main regularity result obtained by the second author and G. Tian in [25]. Analogous results were obtained also by C. De Lellis, E. Spadaro and L. Spolaor in [12], by exploiting the fact that any integral semicalibrated $k$-cycle $T \in D_k(B_2)$ is "almost" homologically mass-minimizing, i.e. for every $x_0 \in B_2$ there are constants $C_0, r_0, \alpha_0 > 0$ such that

$$M(T|_{B_\rho(x_0)}) \leq M((T + \partial S)|_{B_\rho(x_0)}) + C_0 \rho^{k+\alpha_0}, \quad \forall 0 < \rho < \rho_0$$

and for all $S \in D_{k+1}(B_2)$ such that $\text{spt}(S) \subset B_\rho(x_0)$ (compare with the stronger property (1.3) that holds for calibrated cycles).

Nevertheless, very little is known so far concerning uniqueness of tangent cone when the rectifiability hypothesis is dropped. In general this is not true, counter-examples have been given initially by C. O. Kiselman in [18] and then generalized in [8]. On the other hand, a positive result on this matter was obtained by C. Bellettini in [24], where the author proves that the tangent cone to normal positive $(1, 1)$-cycles is unique at any point where the density does not have a jump with respect to all of its values in a neighborhood.

As we will see below, the proof of Theorem 1.1 will be further reduced to the problem of showing uniqueness of tangent cones for a special class of non-rectifiable, semicalibrated $(2m - 2)$-cycles on $B_2$. Thus, the present paper is meant to be a contribution to this so far still fairly open class of problems.

Let $u$ satisfy the assumptions of Theorem 1.1 with $M = B_2$. We can associate to the map $u$ the $(2m - 2)$-current $T_u \in D_{2m-2}(B)$ given by

$$\langle T_u, \alpha \rangle := \int_M u^\ast \omega_{\mathbb{CP}^n} \wedge \alpha, \quad \forall \alpha \in D^{2m-2}(B).$$

We can show (see section 2) that $T_u$ satisfies the following properties:

1. $T_u$ is a cycle, i.e. $\partial T_u = 0$ in the sense of currents.
2. $T_u$ is normal, with

$$M(T_u) = \int_B |u^\ast \omega_{\mathbb{CP}^n}|_s d\text{vol}_g = \frac{1}{2} \int_B |du|^2_g d\text{vol}_g < +\infty.$$

3. $T_u$ is semicalibrated by $\frac{\Omega^{m-1}}{(m-1)!}$. 

For the reader’s convenience, we recall at this point that we say that a current \( T \) on \( B \) has a \textbf{unique tangent cone} at the origin if given any two sequences \( \{\rho_k\}_{k \in \mathbb{N}} \subset (0,1) \) and \( \{\rho'_k\}_{k \in \mathbb{N}} \subset (0,1) \) such that

1. \( \rho_k \to 0^+ \) and \( \rho'_k \to 0^+ \), as \( k \to +\infty \),
2. \( (\Phi_{\rho_k})_* T \rightharpoonup C_\infty \) and \( (\Phi_{\rho'_k})_* T \rightharpoonup C'_\infty \), as \( k \to +\infty \),

it follows that \( C_\infty = C'_\infty \).

In section 6.3 we will show that uniqueness of tangent cone for the \((2m-2)\)-dimensional cycle \( T_u \) and for its "localizations" in the target manifold can be used in order to achieve a full proof of the uniqueness of the tangent map for \( u \) at the origin. Therefore, most of our efforts will be devoted to the proof of the following statement.

\textbf{Theorem 1.2.} Let \( m, n \in \mathbb{N}_0 \) be such that \( m \geq 2 \). Let \( B \subset \mathbb{R}^{2m} \) be the open unit ball in \( \mathbb{R}^{2m} \) and let \( J \) be a Lipschitz almost complex structure on \( B_2 := 2B \) such that \( J(0) = J_0 \). Assume that \( u \in W^{1,2}(B, \mathbb{C}P^n) \) is weakly \((J,j_n)\)-holomorphic and locally approximable, where \( j_n \) stands for the standard complex structure on \( \mathbb{C}P^n \).

Then, the \((2m-2)\)-cycle \( T_u \in D_{2m-2}(B) \) has a unique tangent cone at the origin.

This last paragraph is dedicated to explain the main new ideas that we have introduced in order to prove Theorem 1.2. The whole proof is based on the fact that the level sets of any weakly \((J,j_n)\)-holomorphic and locally approximable map are rectifiable, \( J \)-holomorphic cycles. This fact is proved in section 5. By applying a slicing procedure on the right-hand-side of the monotonicity formula (2.3) (see appendix A), we get a foliation of the region of integration into rectifiable, almost \( J \)-holomorphic curves (see Definition 4.2 and Remark 4.1). By localizing properly in the target, integrating what we call the fundamental Morrey type estimate for almost \( J \)-holomorphic curves (see section 4) and passing then to the limit as the localization sets invade the codomain, we finally get uniqueness of tangent cone for the 2-dimensional current \( (T_u \mathcal{L} \pi^* \omega_{\mathbb{C}P^{m-1}}^{m-2})/(m-2)! \), where \( \pi : \mathbb{C}^m \setminus \{0\} \to \mathbb{C}P^{m-1} \) is the standard projection map (see the first paragraph of section 2). Then, the statement of Theorem 1.2 follows as shown in the conclusion of section 6.2.

1.4. \textbf{Final comments and open problems.} We would like to stress that our approach could also give an alternative proof of the uniqueness of tangent cone for integral \((p,p)\)-cycles on almost complex manifolds, which was previously obtained by C. Bellettini in [3]. This could be achieved by considering maps \( u \) that are more and more concentrated on just one rectifiable pseudo-holomorphic set in the domain (see Remark 6.1).

We also believe that the method that we have developed in this work could be useful in order to proceed further in the analysis of the singular set of weakly holomorphic and locally approximable maps. In particular, we conjecture that the optimal bound (1.2) on the size of the singular set of such maps could be achieved as a further development, also exploiting Theorem 1.1. Furthermore, an interesting open problem concerns the generalization of our result to arbitrary almost Kähler target manifolds.

Taking a wider look and abandoning the framework of weakly holomorphic maps, there are plenty of other related problems that would deserve to be studied more deeply in view of recent developments in the field. In particular, the aim is to invent new analytic techniques that are robust enough to survive the non-availability of holomorphic coordinates in the almost complex non-integrable setting. Among all these problems, for sake of brevity we
just mention uniqueness of tangent cone for triholomorphic maps in hyper-Kähler geometry (see e.g. [3]) and for Hermitian Yang-Mills connections (see [10], [9]).

2. Almost monotonicity formula and tangent cones

First, let us fix the notation that we will use throughout the present paper, whenever it won’t be differently specified. We denote by $B \subset \mathbb{R}^{2m}$ be the open unit ball in $\mathbb{R}^{2m}$, with $m \geq 2$. We let $J$ be a Lipschitz almost complex structure on $B_2 := 2B$ such that $J(0) = J_0$.

We let $\Omega$ be a Lipschitz almost symplectic form on $B_2$ which is compatible with $J$ and such that $\Omega(0) = \Omega_0$, where $\Omega_0$ stands for the standard symplectic form on $\mathbb{R}^{2m} \cong \mathbb{C}^m$. We denote by $g$ the Lipschitz metric on $B_2$ given by $g_x(v, w) := \Omega_x(v, Jw)$, for every $x \in B$ and $v, w \in \mathbb{R}^{2m}$. We indicate by $| \cdot |_g$ the norm induced by $g$. Finally, $\pi : B \smallsetminus \{0\} \to \mathbb{CP}^{m-1}$ denotes the standard projection map given by

$$\pi(x_1, y_1, \ldots, x_m, y_m) := [x_1 + iy_1 : \ldots : x_m + iy_m]$$

for every $(x_1, y_1, \ldots, x_m, y_m) \in B \smallsetminus \{0\}$.

**Remark 2.1.** Notice that the fact that $g$ is Lipschitz on $B_2$ guarantees that $| \cdot |_g$ is equivalent to the euclidean norm. Consider the function $f : \overline{B} \times S^{2m-1} \to (0, +\infty)$ given by $f(x, v) := |v|_{g(x)}^2$, where $S^{2m-1} \subset \mathbb{R}^{2m}$ is the unit sphere in $\mathbb{R}^{2m}$ with respect to the euclidean norm. Since $f$ is continuous on the compact set $\overline{B} \times S^{2m-1}$, by Weierstrass theorem and by definition of $S^{2m-1}$, there exists a constant $G > 0$ such that

$$\frac{1}{G} |v|^2 \leq f(x, v) = |v|_{g(x)}^2 \leq G |v|^2, \quad \forall x \in \overline{B}, \forall v \in S^{2m-1}.$$

By 2-homogeneity of the squared norm, it follows that

$$\frac{1}{G} |v|^2 \leq |v|_{g(x)}^2 \leq G |v|^2, \quad \forall x \in \overline{B}, \forall v \in \mathbb{R}^{2m}$$

(2.1)

and our claim follows.

Since $| \cdot |_g$ and $| \cdot |$ are equivalent, when we refer to the Sobolev spaces on $B$ we don’t need to specify which of these two norms we use in order to define them. In fact, we will use both of them according to what suits better in the context.

**Lemma 2.1.** Let $V$ be a $2m$-dimensional real vector space and $J$ a linear complex structure on $V$. Let $\Omega$ be a symplectic form on $V$ which is compatible with $J$.

Then

$$\frac{\Omega^{m-1}}{(m-1)!} \wedge \xi \wedge J\xi = |\xi|^2 \frac{\Omega^{m}}{m!}, \quad \text{for every } \xi \in V^*,$$

where $J\xi$ is given by $(J\xi)(v) := -\xi(Jv)$, for every $v \in V$.

**Proof.** Fix an $g$-orthonormal basis $\{e_j, Je_j\}_{j=1,\ldots,m}$ of $V$, so that

$$\Omega = \sum_{j=1}^m e_j^* \wedge Je_j^*.$$
First of all, notice that
\[
\frac{\Omega^n}{m!} = \frac{1}{m!} \sum_{j_1, \ldots, j_m=1}^m (e_{j_1}^* \wedge Je_{j_1}^*) \wedge \cdots \wedge (e_{j_m}^* \wedge Je_{j_m}^*)
\]
\[
= e_1^* \wedge Je_1^* \wedge \cdots \wedge e_m^* \wedge Je_m^*.
\]
Fix any \( \xi \in V^* \) and decompose it along the\( g \)-orthonormal basis denoted by \( \{e_j^*, Je_j^*\}_{j=1, \ldots, m} \) as
\[
\xi = \sum_{j=1}^m (\xi_{j1} e_j^* + \xi_{j2} Je_j^*).
\]
This in turn implies that
\[
J \xi = \sum_{j=1}^m (\xi_{j1} Je_j^* - \xi_{j2} e_j^*)
\]
and then
\[
\xi \wedge J \xi = \sum_{j,k=1}^m (\xi_{j1} \xi_{k1} e_j^* \wedge Je_k^* - \xi_{j1} \xi_{k2} e_j^* \wedge e_k^*
\]
\[
+ \xi_{j2} \xi_{k1} Je_j^* \wedge Je_k^* - \xi_{j2} \xi_{k2} Je_j^* \wedge e_k^*).
\]
Since
\[
\frac{\Omega^{m-1}}{(m-1)!} = \sum_{j_1, \ldots, j_{m-1}=1}^m (e_{j_1}^* \wedge Je_{j_1}^*) \wedge \cdots \wedge (e_{j_{m-1}}^* \wedge Je_{j_{m-1}}^*),
\]
it’s easy to see that
\[
\frac{\Omega^{m-1}}{(m-1)!} \wedge e_j^* \wedge e_k^* = \frac{\Omega^{m-1}}{(m-1)!} \wedge Je_j^* \wedge Je_k^* = 0,
\]
for every \( j, k \in \{1, \ldots, m\} \). Moreover, we notice that
\[
\frac{\Omega^{m-1}}{(m-1)!} \wedge \sum_{j,k=1}^m \xi_{j1} \xi_{k1} e_j^* \wedge Je_k^* = \left( \sum_{j=1}^m \xi_{j1}^2 \right) e_1^* \wedge Je_1^* \wedge \cdots \wedge e_m^* \wedge Je_m^*
\]
and
\[
- \frac{\Omega^{m-1}}{(m-1)!} \wedge \sum_{j,k=1}^m \xi_{j2} \xi_{k2} Je_j^* \wedge e_k^* = \left( \sum_{j=1}^m \xi_{j2}^2 \right) e_1^* \wedge Je_1^* \wedge \cdots \wedge e_m^* \wedge Je_m^*.
\]
By adding all the contributions, we get
\[
\frac{\Omega^{m-1}}{(m-1)!} \wedge \xi \wedge J \xi = \left( \sum_{j=1}^m (\xi_{j1}^2 + \xi_{j2}^2) \right) e_1^* \wedge Je_1^* \wedge \cdots \wedge e_m^* \wedge Je_m^*
\]
\[
= |\xi|^2_{g^*} e_1^* \wedge Je_1^* \wedge \cdots \wedge e_m^* \wedge Je_m^*
\]
and the statement follows. \( \square \)

**Corollary 2.1.** Let \( n \in \mathbb{N}_0 \) and let \((N^{2n}, J_N, \omega_N)\) be a compact almost Kähler manifold. Assume that \( u \in W^{1,2}(B, N) \) is weakly \((J, J_N)\)-holomorphic. Then
\[
u^* \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \frac{|du|^2_{g^2} \Omega^m}{2 m!}, \quad \mathcal{L}^{2m}\text{-a.e. on } B.
\] (2.2)
Proof. Let $E \subset B$ be the set of Lebesgue points of $du$. If $x \in E$ is such that $du_x = 0$, then equation (2.2) holds trivially. Assume then that $du(x) \neq 0$. Fix an $\omega_N$-orthonormal basis $\{\xi_i, J_i \xi_i\}_{i=1}^n$ of $T_{u(x)}N$, so that

$$ (\omega_N)_{u(x)} = \sum_{i=1}^n \xi_i^* \wedge J_i \xi_i^*. $$

Notice that, since $u$ is weakly $(J, J_N)$-holomorphic, it holds that

$$ (u^* \omega_N)_x = \sum_{i=1}^n u^* \xi_i^* \wedge u^* J_i \xi_i^* = \sum_{i=1}^n u^* \xi_i^* \wedge J u^* \xi_i^*, $$

since $(u^*(J_N \xi^*))(v) = (J(u^* \xi^*))(v)$ for every $v \in T_x B \cong \mathbb{R}^{2m}$ follows from the definition of $J_N \xi^*$ and $J(u^* \xi^*)$ (see Lemma 2.1). Thus in particular,

$$ (u^* \omega_N)_x \wedge \Omega_x^{m-1} = \sum_{i=1}^n u^* \xi_i^* \wedge J u^* \xi_i^* \wedge \Omega_x^{m-1}. $$

By applying Lemma 2.1 with $\Omega = \Omega_x$ and $\xi = u^* \xi_i$ for every $i = 1, \ldots, n$, we get that

$$ (u^* \omega_N)_x \wedge \Omega_x^{m-1} = \left( \sum_{i=1}^n \left| u^* \xi_i^* \right|^2 \right) \frac{\Omega_x^m}{m!} = \frac{|du_x|^2 \Omega_x^m}{2 m!}. $$

The statement follows immediately. \qed

Lemma 2.2. Let $n \in \mathbb{N}_0$ and let $(N^{2n}, J_N, \omega_N)$ be a closed almost Kähler smooth manifold. Assume that $u \in W^{1,2}(B, N)$ is weakly $(J, J_N)$-holomorphic and locally approximable.

Then, $T_u$ is a normal $(2m - 2)$-cycle on $B$ semicalibrated by $\frac{\Omega_x^{m-1}}{(m-1)!}.

Proof. First, we claim that $T_u$ is a cycle, i.e. that $\partial T_u = 0$. Indeed, by Stokes theorem and since $d(u^* \omega_N) = 0$ holds distributionally on $B$ by local approximability of $u$, for every fixed $\alpha \in D^{2m-3}(B)$ we get

$$ \langle \partial T_u, \alpha \rangle = \langle T_u, d\alpha \rangle = \int_B u^* \omega_N \wedge d\alpha = 0. $$

In order to conclude, we just need to show that

$$ \langle T_u, \frac{\Omega_x^{m-1}}{(m-1)!} \rangle = M(T) < +\infty. $$

Notice that, by Corollary 2.1 it holds that

$$ \langle T_u, \frac{\Omega_x^{m-1}}{(m-1)!} \rangle = \int_B u^* \omega_N \wedge \frac{\Omega_x^{m-1}}{(m-1)!} = \frac{1}{2} \int_B |du|^2_g d\text{vol}_g < +\infty, $$

since $du \in L^2(B; \mathbb{R}^{2m} \otimes u^*TN)$. We claim that

$$ M(T_u) = \frac{1}{2} \int_B |du|^2_g d\text{vol}_g. $$

Indeed, fix any Lebesgue point $x \in B$ for $du$ and let $\{e_1, J_1 e_1, \ldots, e_m, J_m e_m\}$, $\{\xi_1^*, j_1 \xi_1^*, \ldots, \xi_n^*, j_n \xi_n^*\}$ be orthonormal bases of $T_x B$ and $T_{u(x)}N$ respectively. Then, we have

$$ \langle (u^* \omega_N)_x, e_k \wedge J e_h \rangle = \sum_{i=1}^n (u^* \xi_i^* \wedge J u^* \xi_i^*)(e_k, J e_h) $$

where
Thus, for every unit and simple $2$-vector $v_1 \wedge v_2$ with $v_1, v_2 \in T_x B$ we have
\[
\langle (u^* \omega_N)_x, v_1 \wedge v_2 \rangle \leq \frac{|du_x|^2_g}{2}.
\]
Consider the unit vector
\[
v := \frac{1}{\sqrt{m}} \sum_{i \text{ odd}} e_i + \frac{1}{\sqrt{m}} \sum_{i \text{ even}} Je_i \in T_x B
\]
and notice that
\[
\langle (u^* \omega_N)_x, v \wedge Jv \rangle = \left( (u^* \omega_N)_x, \frac{\Omega_x}{m} \right) = \left( (u^* \omega_N)_x, \frac{\Omega_x^{m-1}}{(m-1)!} \right) = \frac{|du_x|^2_g}{2},
\]
By definition of comass norm and since $x \in B$ was any arbitrary Lebesgue point of $du$, we conclude that
\[
|u^* \omega_N|_x = \frac{|du_x|^2_g}{2}, \quad \text{vol}_g \text{-a.e. on } B.
\]
Moreover, since $T_u$ is the integration current induced by $u^* \omega_N$, it holds that
\[
\mathcal{M}(T_u) = \int_B |u^* \omega_N|_* d\text{vol}_g, \quad \text{for every open set } U \subset \subset B.
\]
The statement then follows.

Before stating the following fundamental proposition, we recall the following notation. Given any $x_0 \in B$, we define
\[
\Omega_{t,x_0} := (dR_{x_0} \wedge \Omega) \cdot \nu_{x_0}
\]
where $R_{x_0} := |\cdot - x_0|$, $\nu_{x_0} := (dR_{x_0})^g$ and with " $\cdot$ " we denote the interior product. We call $\Omega_{t,x_0}$ the \textbf{tangential part of $\Omega$ with respect to $x_0$}. Such notation is analogous to the one used in \cite{11}.

**Proposition 2.1** (Almost monotonicity formula). Let $n \in \mathbb{N}_0$ and let the triple $(N^{2n}, J_N, \omega_N)$ denote a closed almost Kähler smooth manifold. Assume that $u \in W^{1,2}(B, N)$ is weakly $(J, J_N)$-holomorphic and locally approximable.
Then, there exists \( A = A(\text{Lip}(\Omega)) \geq 0 \) such that
\[
e^{-A\rho}(1 + A\rho) \frac{\mathcal{M}(T_u \mathbb{L} B_\rho(x_0))}{\rho^{2m-2}} - e^{-A\sigma}(1 + A\sigma) \frac{\mathcal{M}(T_u \mathbb{L} B_\sigma(x_0))}{\sigma^{2m-2}}
\geq \int_{B_\rho(x_0) \setminus B_\sigma(x_0)} \frac{1}{|x - x_0|^{2m-2}} u^* \omega_N \wedge \frac{\Omega_{t,x_0}^{m-1}}{(m-1)!}\]
and
\[
e^{-A\rho}(1 - A\rho) \frac{\mathcal{M}(T_u \mathbb{L} B_\rho(x_0))}{\rho^{2m-2}} - e^{-A\sigma}(1 - A\sigma) \frac{\mathcal{M}(T_u \mathbb{L} B_\sigma(x_0))}{\sigma^{2m-2}}
\leq \int_{B_\rho(x_0) \setminus B_\sigma(x_0)} \frac{1}{|x - x_0|^{2m-2}} u^* \omega_N \wedge \frac{\Omega_{t,x_0}^{m-1}}{(m-1)!},
\]
for every \( x_0 \in B \) and \( 0 < \sigma < \rho \leq r_{x_0} := \text{dist}(x_0, \partial B) \).

**Proof.** A direct computation leads immediately to
\[
(\Omega_{t,x_0}^{m-1})_{t,x_0} = \Omega_{t,x_0}^{m-1}.
\]
Hence, the statement follows directly by Lemma 2.2 and [22, Proposition 1] by simply noticing that exactly the same proof works when \( \Omega \) is just Lipschitz. \( \square \)

**Remark 2.2.** Fix any \( x_0 \in B \). Notice that
\[
\ast \left( u^* \omega_N(x) \wedge \frac{\Omega_{t,x_0}^{m-1}}{(m-1)!} \right) = \left\langle \frac{\Omega_{t,x_0}^{m-1}(x)^{m-1}}{(m-1)!}, \ast u^* \omega_N \right\rangle
\]
\[
= \frac{1}{2} \left| \frac{\partial u}{\partial \nu_{x_0}} \right|_{g}^2, \quad \text{for } \mathcal{L}^{2m}\text{-a.e. } x \in B,
\]
where \( \nu_{x_0} := (dR_{x_0})^t \) as above with \( R_{x_0} := |x - x_0| \). Hence, by equation (2.3) we conclude that the function
\[
(0, r_{x_0}) \ni \rho \mapsto e^{-A\rho}(1 + A\rho) \frac{\mathcal{M}(T_u \mathbb{L} B_\rho)}{\rho^{2m-2}}
\]
is non-decreasing. As
\[
\lim_{\rho \to 0^+} e^{-A\rho}(1 + A\rho) = 1,
\]
we conclude that the limit
\[
\theta(x_0, u) := \lim_{\rho \to 0^+} \frac{\mathcal{M}(T_u \mathbb{L} B_\rho(x_0))}{\rho^{2m-2}}
\]
exists and is finite. We say that \( \theta(x_0, u) \) is the **density of the map** \( u \) **at the point** \( x_0 \).

We conclude the section by discussing the existence and the structure of tangent cones for the current \( T_u \). Let’s pick any sequence \( \rho_k \to 0^+ \) as \( k \to +\infty \) and the relative blow-up sequence \( \{T_{\rho_k} := (\Phi_{\rho_k}), T_u \}_{k \in \mathbb{N}} \). Since \( T_{\rho_k} \) is a cycle for every \( k \in \mathbb{N} \) and
\[
\mathcal{M}(T_{\rho_k}) = \frac{\mathcal{M}(T_u \mathbb{L} B_{\rho_k})}{\rho_k^{2m-2}} \leq e^{A}(1 + A)\mathcal{M}(T_u) < +\infty,
\]
by Federer-Fleming compactness theorem we know that there exists a subsequence \( \{\rho_{kj}\}_{j \in \mathbb{N}} \) of \( \{\rho_k\}_{k \in \mathbb{N}} \) such that \( T_{\rho_{kj}} \rightharpoonup C_\infty \) as \( j \to +\infty \) in the sense of currents. Moreover, by
exploiting the almost monotonicity formula, we get that any tangent cone $C_\infty$ is a $(2m - 2)$-cycle calibrated by $\Omega_0$ and invariant under dilations, i.e. $(\Phi_\rho)_*C_\infty = C_\infty$, for every $\rho \in (0, 1)$ (see e.g. [22, Section 3]).

3. Smoothness at points with small density

In the present section, we will assume that $\Omega$ is a symplectic form and we prove that weakly holomorphic and locally approximable maps are stationary harmonic in this particular case. Therefore, we conclude that such maps are smooth at points of small density via standard $\varepsilon$-regularity for stationary harmonic maps (Theorem 3.1). The almost symplectic case is completely treated in [5, Propositions 1, 3, 4], where it is shown that similar results hold in the almost stationary scenario. Hence, the conclusions of the present section hold even if $d\Omega \neq 0$. We just present here a simplified case in order to deal with less technicalities and draw some light on the key ideas and concepts.

Lemma 3.1 (Wirtinger’s inequality). Let $n \in \mathbb{N}_0$ and let $(N^{2n}, J_N, \omega_N)$ be a closed almost Kähler smooth manifold.

Then, for every map $v \in W^{1,2}(B, N)$ it holds that

$$\ast \left( v^*\omega_N \wedge \frac{\Omega_{x}^{m-1}}{(m-1)!} \right) \leq \frac{|dv|^2_g}{2}, \quad \text{vol}_g - \text{a.e. on } B. \quad (3.1)$$

Proof. Let $E \subset B$ be the set of the Lebesgue points of $dv$. If $x \in E$ is such that $dv_x = 0$, then (3.1) holds trivially. Assume then that $x \in E$ is such that $dv_x \neq 0$. Fix a $g$-orthonormal basis $\{ e_{2k-1}, e_{2k} := Je_{2k-1} \}_{k=1}^{m}$ of $T_xB$ and an $\omega_N$-orthonormal basis $\{ \xi_{2i-1}, \xi_{2i} := j\xi_{2i-1} \}_{i=1}^{n}$ of $T_{v(x)}N$, so that

$$\Omega_x = \sum_{k=1}^{m} e_{2k-1}^* \wedge e_{2k}^*$$

and

$$(\omega_N)_{v(x)} = \sum_{i=1}^{n} \xi_{2i-1} \wedge \xi_{2i}.$$ 

Then, we compute

$$\ast \left( v^*\omega_N \wedge \frac{\Omega_{x}^{m-1}}{(m-1)!} \right)$$

$$= \ast \sum_{k=1}^{m} \sum_{i=1}^{n} v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^* \wedge e_1^* \wedge e_2^* \wedge ... \wedge e_{2k-1}^* \wedge e_{2k}^* \wedge ... \wedge e_{2m-1} \wedge e_{2m}$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{n} (v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^*) (e_{2k-1} \wedge e_{2k})$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{n} (v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^*) (e_{2k-1}, e_{2k})$$

$$\leq \sum_{k=1}^{m} \sum_{i=1}^{n} |(v^*\xi_{2i-1}^* \wedge v^*\xi_{2i}^*) (e_{2k-1}, e_{2k})|_g$$
Thus, the statement follows. □

**Lemma 3.2** (Weakly holomorphic maps are weakly harmonic). Let \( n \in \mathbb{N}_0 \) and let \((N^{2n}, J_N, \omega_N)\) be a closed almost Kähler smooth manifold.

If \( u \in W^{1,2}(B, N) \) is weakly \((J, J_N)\)-holomorphic, then \( u \) is weakly harmonic.

**Proof.** Recall that we always identify \( N \) as a smooth submanifold of \( \mathbb{R}^k \), for \( k \) large enough, through the smooth isometric embedding \( \Phi : N \hookrightarrow \mathbb{R}^k \) (see Section 1). Let the map \( \pi_N : W \subset \mathbb{R}^k \to N \) be the nearest point projection from a tubular neighbourhood \( W \) of \( \Phi(N) \) onto \( N \). Fix any vector field \( X \in C^\infty_c(B; \mathbb{R}^k) \). As for every \( t \in \mathbb{R} \) the map \( \pi_N \circ (\Phi \circ u + t X) \) belongs to \( W^{1,2}(B, N) \), by exploiting Lemma 3.1 we get that

\[
\int_B \left| d(\pi_N \circ (\Phi \circ u + tX))^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right|_g^2 \, d\text{vol}_g \geq 2 \int_B (\Phi \circ u + tX)^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \, d\text{vol}_g \geq 2 \int_B (\Phi \circ u + tX)^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \, d\text{vol}_g
\]

for \( t \in \mathbb{R} \) such that \( |t| < \delta \) with \( \delta > 0 \) sufficiently small. Moreover, the equality holds for \( t = 0 \) by virtue of equation (2.2). We claim that

\[
\int_B (\Phi \circ u + tX)^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \int_B u^\ast \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!},
\]

for every \( |t| < \delta \). Indeed, it holds that

\[
\frac{d}{dt} \left( \int_B (\Phi \circ u + tX)^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right) = \int_B \frac{d}{dt} ((\Phi \circ u + tX)^\ast \pi_N \omega_N) \wedge \frac{\Omega^{m-1}}{(m-1)!} = \left\langle d(u^\ast (\pi_N \omega_N)) \mathbf{L} X, \ast \frac{\Omega^{m-1}}{(m-1)!} \right\rangle = - \int_B u^\ast (\pi_N \omega_N) \mathbf{L} X \wedge d\left( \frac{\Omega^{m-1}}{(m-1)!} \right) = 0,
\]

where \( \mathbf{L} \) stands for the interior product. Hence, equation (3.3) follows. By using together equation (3.2) and (3.3), we get

\[
\int_B \left| d(\pi_N \circ (\Phi \circ u + tX)) \right|_g^2 \, d\text{vol}_g \geq \int_B |du|_g^2 \, d\text{vol}_g \quad \text{for every } |t| < \delta,
\]

and the equality holds for \( t = 0 \). Thus, \( t = 0 \) is a global minimum for the differentiable function

\[
t \mapsto \int_B \left| d(\pi_N \circ (\Phi \circ u + tX))^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right|_g^2 \, d\text{vol}_g.
\]

Hence, we conclude that

\[
\frac{d}{dt} \left| \int_B \left| d(\pi_N \circ (\Phi \circ u + tX))^\ast \pi_N \omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right|_g^2 \, d\text{vol}_g \right|_{t=0} = 0.
\]

Since our choice of \( X \in C^\infty_c(B; \mathbb{R}^k) \) was arbitrary, the statement follows. □
Hence, we conclude that function (3.5) follows. By using together equation (3.4) and (3.5) we get that

\[
\int_B |d(u \circ (Id + tX))|^2_g \, d\text{vol}_g \geq 2 \int_B (Id + tX)^*u^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!}
\]

for every \(|t| < \delta\) and the equality holds for \(t = 0\) by virtue of equation (2.2). We claim that

\[
\int_B (Id + tX)^*u^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} = \int_B u^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!},
\]

for every \(|t| < \delta\). Indeed, as \(d(u^*\omega_N) = 0\) distributionally on \(B\), it holds that

\[
\frac{d}{dt} \left( \int_B (Id + tX)^*u^*\omega_N \wedge \frac{\Omega^{m-1}}{(m-1)!} \right) = \int_B \frac{d}{dt} ((Id + tX)^*u^*\omega_N) \wedge \frac{\Omega^{m-1}}{(m-1)!} = 0.
\]

Hence, equation (3.5) follows. By using together equation (3.4) and (3.5) we get that

\[
\int_B |d(u \circ (Id + tX))|^2_g \, d\text{vol}_g \geq \int_B |du|^2_g \, d\text{vol}_g,
\]

and the equality holds for \(t = 0\). Thus, \(t = 0\) is a global minimum for the differentiable function

\[
t \mapsto \int_B |d(u \circ (Id + tX))|^2_g \, d\text{vol}_g.
\]

Hence, we conclude that

\[
\frac{d}{dt} \int_B |d(u \circ (Id + tX))|^2_g \, d\text{vol}_g \bigg|_{t=0} = 0.
\]

Since our choice of \(X \in C_c^{\infty}(B; \mathbb{R}^k)\) was arbitrary, the statement follows. \(\square\)

The following \(\varepsilon\)-regularity statement follows immediately by Lemma 5.2 and [24, Theorem 2.1].

**Theorem 3.1** (\(\varepsilon\)-regularity for weakly holomorphic and locally approximable maps). Let \(n \in \mathbb{N}_0\) and let \((N^{2n}, J_N, \omega_N)\) be a closed almost Kähler smooth manifold.

Let \(u \in W^{1,2}(B, \mathbb{C}\mathbb{P}^n)\) be weakly \((J, J_N)\)-holomorphic and locally approximable. Then, there exists a threshold \(\varepsilon_0 = \varepsilon_0(m, n) > 0\) such that whenever \(\theta(x_0, u) < \varepsilon_0\) there exists ball \(B_\rho(x_0) \subset B\) such that \(u\) is Hölder continuous (and hence smooth) on \(B_\rho(x_0)\).
We define

$$\text{Sing}(u) := \{x_0 \in B \text{ s.t. } \theta(x_0, u) \geq \varepsilon_0\}$$

and we say that Sing\((u)\) is the \textbf{singular set} of \(u\). Moreover, by stationarity of \(u\), it follows that

$$\mathcal{H}^{2m-2}(\text{Sing}(u)) = 0.$$  

4. \textbf{The fundamental Morrey type estimate}

We aim to collect here the proofs of the (mostly technical) tools and estimates that will be used in section 5. Throughout the present paper, given any \(\mathcal{H}^k\)-rectifiable subset \(\Sigma \subset B\) equipped with an orienting \(\mathcal{H}^k\)-measurable field of unit and simple \(k\)-vectors \(\vec{\Sigma}\) we denote by \([\Sigma]\) the current of integration on \(\Sigma\), i.e. the \(k\)-dimensional current given by

$$\langle [\Sigma], \alpha \rangle := \int_\Sigma \langle \alpha, \vec{\Sigma} \rangle \ d\mathcal{H}^k, \quad \forall \alpha \in D^k(B).$$

4.1. \textbf{Some technical lemmata.}

\textbf{Definition 4.1 (J-holomorphic curves).} A locally \(\mathcal{H}^2\)-rectifiable subset \(\Sigma \subset B\) equipped with an orienting \(\mathcal{H}^2\)-measurable field of unit and simple 2-vectors \(\vec{\Sigma}\) is a \textbf{J-holomorphic curve} if \(\vec{\Sigma}(x)\) is \(J\)-invariant for \(\mathcal{H}^2\)-a.e. \(x \in \Sigma\).

Moreover, if \(\partial [\Sigma] = 0\) we say that \(\Sigma\) is \textbf{closed}.

\textbf{Definition 4.2 (Almost J-holomorphic curves).} A locally \(\mathcal{H}^2\)-rectifiable subset \(\Sigma \subset B\) equipped with an orienting \(\mathcal{H}^2\)-measurable field of unit and simple 2-vectors \(\vec{\Sigma}\) is an almost \textbf{J-holomorphic curve} if there exists some \(\mathcal{H}^2\)-measurable and \(J\)-invariant field of 2-vectors \(\vec{\Sigma}_J : \Sigma \rightarrow \bigwedge^2 \mathbb{R}^{2m}\) such that for some \(\gamma \in (0, 1]\), \(\ell \geq 0\) it holds that

$$|\vec{\Sigma}(x) - \vec{\Sigma}_J(x)| \leq \ell |x|^{\gamma}, \quad \text{for } \mathcal{H}^2\text{-a.e. } x \in \Sigma.$$  

(4.1)

Moreover, if \(\partial [\Sigma] = 0\) we say that \(\Sigma\) is \textbf{closed}.

\textbf{Remark 4.1.} Given an almost \(J\)-holomorphic curve in \(B\), we can build a 2-dimensional varifold on \(B\) associated to it in the following way.

Let \(\mathcal{G}_2(B) := B \times \text{Gr}(2, \mathbb{R}^{2m})\), where \(\text{Gr}(2, \mathbb{R}^{2m})\) is the Grassmannian of the real 2-planes in \(\mathbb{R}^{2m}\). Notice that \(\mathcal{G}_2(B)\) can be given the structure of a smooth manifold, since it is the product of two smooth manifolds. Following the notation by W.K. Allard and L. Simon (see [29, Chapter 8], [1]), a general 2-dimensional varifold on \(B\) is simply a Radon measure on \(\mathcal{G}_2(B)\). Then, we may associate to an almost \(J\)-holomorphic curve \(\Sigma \subset B\) the Radon measure on \(\mathcal{G}_2(B)\) given by

$$\mathcal{H}^2 \bigwedge \Sigma \otimes \delta_{\text{span}\{\vec{\Sigma}_J\}},$$

where by \(\otimes\) we denote the usual tensor product of measures and \(\text{span}\{\vec{\Sigma}_J\}\) denotes the field of 2-planes associated with the field of 2-vectors \(\vec{\Sigma}_J\).

Such objects are very close to being rectifiable varifolds but the almost tangent space of \(\Sigma\) is "tilted", conveniently with respect to the purposes that will be clear in the forthcoming discussion.
We point out explicitly that the form of these new objects is built (and therefore meaningful) just to work around the origin. We would need to consider a "shifted" version of almost $J$-holomorphic curves in order to work around an arbitrary point $x_0 \in B$.

Remark 4.2. All the estimates and the results that will be presented in this section concerning closed almost $J$-holomorphic curves in $B$ are still valid for closed $J$-holomorphic curves. Indeed, any $J$-holomorphic curve is trivially almost $J$-holomorphic (just pick $\Sigma_J = \Sigma$, $\ell = 0$ and $\gamma = 1$ in Definition 4.2).

Hence, in order to get the corresponding estimates for closed $J$-holomorphic curves it will always be sufficient to set $\Sigma_J = \tilde{\Sigma}$, $\ell = 0$ and $\gamma = 1$ in what follows.

From now on, we will denote by $\nu_0$ the vector field on $B \setminus \{0\}$ given by $\nu_0(x) = x/|x|$. Moreover, we notice that since $\Omega$ is Lipschitz and $\Omega(0) = \Omega_0$, there exists a constant $\tilde{L} > 0$ depending only on $\text{Lip}(\Omega)$ such that

$$|\nu - \nu_0| \leq \tilde{L} \cdot |\cdot|.$$

**Proposition 4.1 (Almost monotonicity formula).** Let $\Sigma$ be a closed almost $J$-holomorphic curve in $B$, according to Definition 4.2. Then, there exists a positive constant $A \geq 0$ depending only on the Lipschitz constant of $\Omega$ such that

$$e^{\pi \rho^2} (1 + A \rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} - e^{\pi \rho^2} (1 + A \sigma) \frac{\mathcal{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \geq \int_{\Sigma \cap (B_\rho \setminus B_\sigma)} \frac{1}{|\cdot|^2} |\Sigma_J \wedge \nu|^2 \, d\mathcal{H}^2 \tag{4.2}$$

and

$$e^{-(\pi \rho^2)} (1 - A \rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} - e^{-(\pi \rho^2)} (1 - A \sigma) \frac{\mathcal{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \leq \int_{\Sigma \cap (B_\rho \setminus B_\sigma)} \frac{1}{|\cdot|^2} |\Sigma_J \wedge \nu|^2 \, d\mathcal{H}^2, \tag{4.3}$$

Proof. Throughout this proof, $R$ will denote the smooth radial vector field on $B$ given by $R(x) := x$, for every $x \in B$. We denote by $\Omega_0$ the standard symplectic form on $B^{2m}$ and we define $\Omega_1 := \Omega - \Omega_0$. Moreover, given any arbitrary form $\alpha \in \Omega^2(B)$ we denote by $\alpha_t$ the tangential part of a form with respect to the vector field $\nu$, given by

$$\alpha_t := (d\rho \wedge \alpha) \cdot L \nu,$$

according to the notation used in Proposition 2.1.

Define the normal 2-current on $B$ given by

$$\langle [\Sigma]_J, \alpha \rangle := \int_{\Sigma} \langle \alpha, \Sigma_J \rangle \, d\mathcal{H}^2, \quad \forall \alpha \in \mathcal{D}^2(B).$$

As $[\Sigma]_J$ is semicalibrated by $\Omega$, we will apply the same method that is used in Proposition 1. Nevertheless, we need to take into account the fact that the 2-current $[\Sigma]_J$ is not a cycle (though not far from being one).

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be smooth, non-increasing and such that:

1. $\varphi \equiv 1$ on $[0, 1/2]$;
2. $|\varphi'| \leq 4$ on $[0, +\infty)$.
3. $\varphi \equiv 0$ on $[1, +\infty)$. 

Define the normal 2-current on $B$ given by

$$\langle [\Sigma]_J, \alpha \rangle := \int_{\Sigma} \langle \alpha, \Sigma_J \rangle \, d\mathcal{H}^2, \quad \forall \alpha \in \mathcal{D}^2(B).$$

As $[\Sigma]_J$ is semicalibrated by $\Omega$, we will apply the same method that is used in Proposition 1. Nevertheless, we need to take into account the fact that the 2-current $[\Sigma]_J$ is not a cycle (though not far from being one).
For every \( 0 < \rho < 1 \), define \( \varphi_{\rho}(x) := \varphi(|x|/\rho) \), for every \( x \in \mathbb{R}^m \). Notice that \( \varphi_{\rho} \equiv 1 \) on \( B_{\rho/2} \), \( \varphi_{\rho} \equiv 0 \) on \( \mathbb{R}^m \setminus B_{\rho} \) and \( |\nabla \varphi_{\rho}| \leq 4/\rho \) on \( \mathbb{R}^m \). Define

\[
I(\rho) := \int_{\Sigma} \varphi_{\rho}(\Omega, \vec{\Sigma}_j) \, d\mathcal{H}^2 = \int_{\Sigma} \varphi_{\rho} \, d\mathcal{H}^2,
\]

\[
J(\rho) := \int_{\Sigma} \varphi_{\rho}(\Omega, \vec{\Sigma}_j) \, d\mathcal{H}^2.
\]

Recall that \( \mathcal{L}_\rho \Omega_0 = 2\Omega_0 \) and compute

\[
2I(\rho) = 2 \int_{\Sigma} \varphi_{\rho}(\Omega_0, \vec{\Sigma}_j) \, d\mathcal{H}^2 + 2 \int_{\Sigma} \varphi_{\rho}(\Omega_1, \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
= \int_{\Sigma} \langle \varphi_{\rho}(\Omega_0 \mathbf{L} \cdot R), \vec{\Sigma}_j \rangle \, d\mathcal{H}^2 + 2 \int_{\Sigma} \varphi_{\rho}(\Omega_1, \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
= \int_{\Sigma} \langle d(\varphi_{\rho}(\Omega_0 \mathbf{L} \cdot R)), \vec{\Sigma} - \vec{\Sigma}_j \rangle \, d\mathcal{H}^2
\]

\[
+ \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\rho \wedge (\Omega_0 \mathbf{L} (\nu_0 - \nu)), \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
+ \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\rho \wedge (\Omega_0 \mathbf{L} \nu), \vec{\Sigma}_j) \, d\mathcal{H}^2 + 2 \int_{\Sigma} \varphi_{\rho}(\Omega_1, \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
= \int_{\Sigma} \langle d(\varphi_{\rho}(\Omega_0 \mathbf{L} \cdot R)), \vec{\Sigma} - \vec{\Sigma}_j \rangle \, d\mathcal{H}^2
\]

\[
+ \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\rho \wedge (\Omega_0 \mathbf{L} (\nu_0 - \nu)), \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
+ \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\Omega_0 - (\Omega_0)_t, \vec{\Sigma}_j) \, d\mathcal{H}^2 + 2 \int_{\Sigma} \varphi_{\rho}(\Omega_1, \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
= \int_{\Sigma} \langle d(\varphi_{\rho}(\Omega_0 \mathbf{L} \cdot R)), \vec{\Sigma} - \vec{\Sigma}_j \rangle \, d\mathcal{H}^2 + \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\Omega - (\Omega_0)_t, \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
+ \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\rho \wedge (\Omega_0 \mathbf{L} (\nu_0 - \nu)), \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
+ 2 \int_{\Sigma} \varphi_{\rho}(\Omega_1, \vec{\Sigma}_j) \, d\mathcal{H}^2 - \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\Omega_1 - (\Omega_1)_t, \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
= \rho I'(\rho) - \rho J'(\rho) + \int_{\Sigma} \langle d(\varphi_{\rho}(\Omega_0 \mathbf{L} \cdot R)), \vec{\Sigma} - \vec{\Sigma}_j \rangle \, d\mathcal{H}^2
\]

\[
+ \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\rho \wedge (\Omega_0 \mathbf{L} (\nu_0 - \nu)), \vec{\Sigma}_j) \, d\mathcal{H}^2
\]

\[
- \rho \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\Omega_1 - (\Omega_1)_t, \vec{\Sigma}_j) \, d\mathcal{H}^2 + 2 \int_{\Sigma} \varphi_{\rho}(\Omega_1, \vec{\Sigma}_j) \, d\mathcal{H}^2,
\]

which leads to

\[
-2 \frac{I(\rho)}{\rho^2} + \frac{I'(\rho)}{\rho^2} - \frac{J'(\rho)}{\rho^2} = -\frac{1}{\rho^3} \int_{\Sigma} \langle d(\varphi_{\rho}(\Omega_0 \mathbf{L} \cdot R)), \vec{\Sigma} - \vec{\Sigma}_j \rangle \, d\mathcal{H}^2
\]

\[
+ \frac{1}{\rho^2} \int_{\Sigma} \frac{\partial \varphi_{\rho}}{\partial \rho}(\rho \wedge (\Omega_0 \mathbf{L} (\nu_0 - \nu)), \vec{\Sigma}_j) \, d\mathcal{H}^2
\]
where
\[ A \leq 2 \text{Lip}(\Omega) + \tilde{L}. \]
From the previous estimate, we immediately conclude that
\[ \frac{d}{d\rho} \left( \frac{I(\rho)}{\rho^2} \right) + (A + \epsilon \rho^{-1}) \frac{I(\rho)}{\rho^2} \geq \frac{J'(\rho)}{\rho^2} - \frac{d}{d\rho} \left( A\rho \frac{I(\rho)}{\rho^2} \right) + \epsilon \rho^{-1} \left( \frac{I(\rho)}{\rho^2} - \mathcal{H}^2(\Omega \cap B_\rho) \right) \]
(4.4)
and
\[ \frac{d}{d\rho} \left( \frac{I(\rho)}{\rho^2} \right) - (A + \epsilon \rho^{-1}) \frac{I(\rho)}{\rho^2} \leq \frac{J'(\rho)}{\rho^2} + \frac{d}{d\rho} \left( A\rho \frac{I(\rho)}{\rho^2} \right) + \epsilon \rho^{-1} \left( \frac{\mathcal{H}^2(\Omega \cap B_\rho)}{\rho^2} \right. - \left. \frac{I(\rho)}{\rho^2} \right). \]
(4.5)
By letting \( \varphi \) increase to the characteristic function of the interval \([0, 1]\) in (4.4), the above estimate passes to the limit in the sense of distributions and we obtain
\[ \frac{d}{d\rho} \left( \frac{\mathcal{H}^2(\Omega \cap B_\rho)}{\rho^2} \right) + (A + \epsilon \rho^{-1}) \frac{\mathcal{H}^2(\Omega \cap B_\rho)}{\rho^2} \]
\[ \geq \frac{d}{d\rho} \left( \int_{\Sigma \cap B_\rho} |\cdot|^2 d\mathcal{H}^2 \right) - \frac{d}{d\rho} \left( A\rho \frac{\mathcal{H}^2(\Omega \cap B_\rho)}{\rho^2} \right). \]
Multiplying both of the last inequality sides by the factor $e^{A_\rho + \ell \Sigma^2}$ and taking into account the fact that the first term on the right-hand-side is non-negative, we get
\[
\frac{d}{d\rho} \left( e^{A_\rho + \ell \Sigma^2} \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} \right) \geq \frac{d}{d\rho} \left( \int_{\Sigma \cap B_\rho} \frac{\langle \Omega_t, \Sigma \rangle}{|\cdot|^2} \, d\mathcal{H}^2 \right)
\]
which turns into
\[
\frac{d}{d\rho} \left( e^{A_\rho + \ell \Sigma^2} (1 + A_\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} \right) \geq \frac{d}{d\rho} \left( \int_{\Sigma \cap B_\rho} \frac{1}{|\cdot|^2} \langle \Omega_t, \Sigma \rangle \, d\mathcal{H}^2 \right).
\]
By integration of the previous inequality, we get
\[
e^{A_\rho + \ell \Sigma^2} (1 + A_\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} - e^{A_\sigma + \ell \Sigma^2} (1 + A_\sigma) \frac{\mathcal{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \geq \int_{\Sigma \cap (B_\rho \setminus B_\sigma)} \frac{1}{|\cdot|^2} \langle \Omega_t, \Sigma \rangle \, d\mathcal{H}^2,
\]
for every $0 < \sigma < \rho < 1$. Since
\[
\langle \Omega_t, \Sigma \rangle = |\Sigma \cap v|^2_g,
\]
the estimate (4.2) follows.

By applying the same techniques to (4.5), we get (4.3) and the statement follows. \hfill \square

**Remark 4.3.** Proposition 4.1 immediately implies that the function
\[
(0, 1) \ni \rho \mapsto e^{A_\rho + \ell \Sigma^2} (1 + A_\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2}
\]
is non-decreasing. In particular, the limit
\[
\theta(0, \Sigma) := \lim_{\rho \to 0^+} \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} = \lim_{\rho \to 0^+} e^{A_\rho + \ell \Sigma^2} (1 + A_\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2}
\]
exists and it is finite.

**Lemma 4.1.** Let $\Sigma$ be a closed almost $J$-holomorphic curve in $B$, according to Definition 4.2. Then, there exist an $\mathcal{H}^2$-measurable field of unit and simple 2-vectors $\bar{\Sigma}_0 \in \Lambda^2 \mathbb{R}^{2m}$ on $\Sigma$ and a constant $L > 0$ depending only on $\ell$ and on $\text{Lip}(\Omega)$ such that for $\mathcal{H}^2$-a.e. $x \in \Sigma$ the following facts hold:

1. $\bar{\Sigma}_0(x)$ is a unit simple 2-vector calibrated by $\Omega_0$;
2. $|\bar{\Sigma}(x) - \bar{\Sigma}_0(x)| \leq L|x|^\gamma/2$;
3. if $\bar{\Sigma}(x)$ is $J_0$-invariant, then $\bar{\Sigma}_0(x) = \bar{\Sigma}(x)$;
4. if $\bar{\Sigma}(x)$ is not $J_0$-invariant, then
   \[
   |\bar{\Sigma}_0(x) \wedge v_0(x) \wedge J_0 v(x)| = \max_{v \in S_x} |v \wedge J_0 v \wedge v_0(x) \wedge J_0 v_0(x)|,
   \]
where $S_x$ denotes the unit sphere in the approximate tangent space $T_x \Sigma$.

**Proof.** Recall the definition of the vector field $v_0$, given at the beginning of the present section. If $x \in \Sigma$ is such that $\bar{\Sigma}(x)$ is $J_0$-invariant, we set $\bar{\Sigma}_0(x) := \bar{\Sigma}(x)$ and all the required
properties are satisfied. Otherwise, if \( x \in \Sigma \) is such that \( \bar{\Sigma}(x) \) is not \( J_0 \)-invariant, we first claim that there exists some \( \Omega_0 \)-orthonormal basis
\[
\{ e_1(x), J_0e_1(x), ..., e_m(x), J_0e_m(x) \}
\]
of \( \mathbb{R}^{2m} \) such that
\[
|e_1(x) \land J_0e_1(x) \land \nu_0(x) \land J_0\nu_0(x) | = \max_{v \in S_x} |v \land J_0v \land \nu_0(x) \land J_0\nu_0(x) |
\]
and we can write \( \bar{\Sigma}(x) \) as
\[
\bar{\Sigma}(x) = \cos \phi(x)e_1(x) \land J_0e_1(x) + \sin \phi(x)e_1(x) \land e_2(x).
\]
for some angle \( \phi(x) \in [0, 2\pi] \). Indeed, since \( S_x \) is compact, there exists \( e_1(x) \in S_x \) maximizing the continuous function \( v \mapsto |v \land J_0v \land \nu_0(x) \land J_0\nu_0(x) | \). We complete \( \{ e_1(x) \} \) to an \( \Omega_0 \)-orthonormal basis \( \{ e_1(x), \xi(x) \} \) of \( T_x\Sigma \), and we write \( \bar{\Sigma}(x) = e_1(x) \land \xi(x) \).

Notice that the set \( \{ e_1(x), J_0e_1(x), \xi(x) - J_0e_1(x) \} \) is linearly independent, otherwise \( \bar{\Sigma}(x) \) would be \( J_0 \)-invariant. We define \( e_2(x) \) as the unique vector such that \( \{ e_1(x), J_0e_1(x), e_2(x) \} \) is an orthonormal set and
\[
\text{span}\{ e_1(x), J_0e_1(x), e_2(x) \} = \text{span}\{ e_1(x), J_0e_1(x), \xi(x) - J_0e_1(x) \}.
\]

Eventually, notice that
\[
\bar{\Sigma}(x) = e_1(x) \land \xi(x) \in \text{span}\{ e_1(x) \land J_0e_1(x), e_1(x) \land e_2(x) \}
\]
and our initial claim follows since
\[
|e_1(x) \land J_0e_1(x) | = |e_1(x) \land e_2(x) | = 1.
\]
Then, we define \( \bar{\Sigma}_0(x) := e_1(x) \land J_0e_1(x) \). Clearly, \( \bar{\Sigma}_0(x) \) satisfies (1) and (4). For what concerns (2), notice that \( |\Omega_x - \Omega_0| \leq \text{Lip}(\Omega)|x| \). In particular, since \( \langle \bar{\Sigma}_0, \bar{\Sigma}(x) \rangle = 1 \) and \(|\bar{\Sigma}_0 - \bar{\Sigma}| \leq \ell \cdot |x|^7 \), it follows that \(|\langle \Omega_0, \bar{\Sigma}(x) \rangle - 1 | \leq C|x|^7\), for some constant \( C > 0 \) depending only on \( \ell \) and \( \text{Lip}(\Omega) \).

Then,
\[
1 + C|x|^7 \geq \langle \Omega_0, \bar{\Sigma}(x) \rangle
\]
\[
= \langle \Omega_0, \cos \phi(x)e_1(x) \land J_0e_1(x) + \sin \phi(x)e_1(x) \land e_2(x) \rangle
\]
\[
= \cos \phi(x) \geq 1 - C|x|^7.
\]

Hence, we eventually obtain
\[
|\bar{\Sigma}(x) - \bar{\Sigma}_0(x) |^2 = (1 - \cos \phi(x))^2 + \sin^2 \phi(x) = 2(1 - \cos \phi) \leq 2C|x|^7
\]
and the statement follows with \( L := \sqrt{2C} \).

For the purposed of the following lemma, we recall that with \( \pi : \mathbb{C}^m \setminus \{0\} \to \mathbb{CP}^{m-1} \) we denote the standard projection to the quotient (see section 1.3).

**Lemma 4.2.** Under the same hypothesis and notation of Lemma 4.1, let \( \tilde{r} \in (0, 1) \). Then, there exists some constant \( \Xi = \Xi(m, \tilde{r}, \text{Lip}(\Omega), \ell, \gamma) > 0 \) such that

(1) for every \( 0 < \rho \leq \tilde{r} \) and every open set \( U \subset B_\rho \) it holds that
\[
|M(\pi_*([\Sigma]_L U)) - \int_{\Sigma \cap U} |\Lambda_2d\pi(\bar{\Sigma}_0)|d\mathcal{H}^2| \leq \Xi \mathcal{H}^2(\Sigma \cap B_{\tilde{r}})\rho^{\gamma/2}.
\]
we get

\[
\int_{\Sigma \cap B_\rho} |\Lambda_2 d\pi_x(\tilde{\Sigma}_0)| d\mathcal{H}^2 \leq 2K_m G \left( e^{A_\rho + \frac{\nu_0}{\gamma}} (1 + A_\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} - \theta(0, \Sigma) \right) + \Xi \mathcal{H}^2(\Sigma \cap B_\rho) \rho^\gamma.
\]

**Proof.** First, we aim to prove (1). By Lemma 4.1 and by the definition of almost \(J\)-holomorphic curve, it follows that

\[
|\Lambda_2 d\pi_x(\tilde{\Sigma}(x))| - |\Lambda_2 d\pi_x(\tilde{\Sigma}_0(x))| \leq |\Lambda_2 d\pi_x(\tilde{\Sigma}(x)) - \Lambda_2 d\pi_x(\tilde{\Sigma}_0(x))| \leq \frac{1}{|x|^2} |\tilde{\Sigma}(x) - \tilde{\Sigma}_0(x)| \leq \frac{L}{|x|^{2-\gamma/2}},
\]

for \(\mathcal{H}^2\)-a.e. every \(x \in \Sigma \setminus \{0\}\). By integrating on \(U\) both sides in the previous inequality, we get

\[
\left| M(\pi_x(\{\Sigma\} \cup U)) - \int_{\Sigma \cap U} |\Lambda_2 d\pi_x(\tilde{\Sigma}_0)| d\mathcal{H}^2 \right| \leq L \int_{\Sigma \cap U} \frac{1}{|x|^{2-\gamma/2}} d\mathcal{H}^2 \leq L \int_{\Sigma \cap B_\rho} \frac{1}{|x|^{2-\gamma/2}} d\mathcal{H}^2.
\]

Notice that, by exploiting (4.2), we get

\[
\int_{\Sigma \cap (B_\rho \setminus B_{\rho/2})} \frac{1}{|x|^{2-\gamma/2}} d\mathcal{H}^2(x) \leq 2^{2-\gamma/2} \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^{2-\gamma/2}} \leq 4e^{A_\rho + \frac{\nu_0}{\gamma}} (1 + A_\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^{\gamma/2}} \leq 4e^{A_\rho + \frac{\nu_0}{\gamma}} (1 + A) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^{\gamma/2}} \mathcal{H}^2(\Sigma \cap B_\rho) \rho^{\gamma/2},
\]

for every \(0 < \rho \leq \tilde{\rho}\). By iteration, we obtain

\[
\int_{\Sigma \cap (B_{\rho/2} \setminus B_{\rho/2^n})} \frac{1}{|x|^{2-\gamma/2}} d\mathcal{H}^2(x) \leq 4e^{A_\rho + \frac{\nu_0}{\gamma}} (1 + A) \mathcal{H}^2(\Sigma \cap B_\rho) \left( \sum_{j=0}^{n-1} \frac{1}{(2^{\gamma/2})^j} \right) \rho^{\gamma/2}
\]

and, by passing to the limit as \(n \to +\infty\), we have

\[
\int_{\Sigma \cap B_\rho} \frac{1}{|x|^{2-\gamma/2}} d\mathcal{H}^2(x) \leq 4e^{A_\rho + \frac{\nu_0}{\gamma}} (1 + A) \mathcal{H}^2(\Sigma \cap B_\rho) \rho^{\gamma/2},
\]

(4.7)

for every \(0 < \rho \leq \tilde{\rho}\). By combining the previous estimate with (4.6), estimate (1) follows with

\[
\Xi_1 := \frac{4e^{A_\rho + \frac{\nu_0}{\gamma}} (1 + A) L}{\rho^{2}(1 - 2^{-\gamma/2})}.
\]

For what concerns (2), we simply notice that by (4.2), by point (2) in Lemma 4.1 and by [22, Lemma 1], we get

\[
\int_{\Sigma \cap B_\rho} |\Lambda_2 d\pi_x(\tilde{\Sigma}_0)| d\mathcal{H}^2 \leq K_m \int_{\Sigma \cap B_\rho} \frac{1}{|x|^{2|\tilde{\Sigma}_0 \wedge \nu_0|}} d\mathcal{H}^2
\]
\begin{align*}
&\leq 4K_m \int_{\Sigma \cap B_\rho} \frac{1}{|x|^2} |\Sigma_J \cap \nu|^2 \, d\mathcal{H}^2 \\
&\quad + 4K_m \int_{\Sigma \cap B_\rho} \frac{1}{|x|^2} |\nu - \nu_0|^2 \, d\mathcal{H}^2 \\
&\quad + 4K_m \int_{\Sigma \cap B_\rho} \frac{1}{|x|^2} \left( |\Sigma_J - \Sigma_0| \cap \nu_0 \right)^2 \, d\mathcal{H}^2 \\
&\leq 4K_m G \left( e^{A\rho + \ell^2} (1 + A\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} - \theta(0, \Sigma) \right) \\
&\quad + 4K_m \hat{L}_r \mathcal{H}^2(\Sigma \cap B_\rho) \\
&\quad + 4K_m (\ell^2 + L^2) \int_{\Sigma \cap B_\rho} \frac{1}{|x|^2-\gamma} \, d\mathcal{H}^2.
\end{align*}

By using the same method that we have used in order to prove the decay in (4.7), we can show that
\begin{equation}
\int_{\Sigma \cap B_\rho} \frac{1}{|x|^2-\gamma} \, d\mathcal{H}^2(x) \leq \frac{4e^{A+\frac{\ell^2}{\gamma}} (1 + A) \mathcal{H}^2(\Sigma \cap B_\rho)}{\hat{\rho}^2}, \tag{4.8}
\end{equation}
for very $0 < \rho \leq \hat{r}$. Moreover, we clearly have that
\begin{equation*}
\frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} \leq \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} \leq \frac{4e^{A+\frac{\ell^2}{\gamma}} (1 + A) \mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} \\
\leq \frac{4e^{A+\frac{\ell^2}{\gamma}} (1 + A) \mathcal{H}^2(\Sigma \cap B_\rho)}{\hat{\rho}^2},
\end{equation*}
for very $0 < \rho \leq \hat{r}$. Thus, we get that (2) holds with
\[\Xi_2 := \frac{4K_me^{A+\frac{\ell^2}{\gamma}} (1 + A)(\hat{L} + \ell^2 + L^2)}{\hat{\rho}^2}.\]

Hence, the statement follows with $\Xi := \max\{\Xi_1, \Xi_2\}$. \hfill \Box

**Remark 4.4.** A first remarkable consequence of Lemma 4.2 and Proposition 4.1 is that
\begin{equation}
\mathbb{M}(\pi_\ast([\Sigma] \mathcal{L} \, B_\rho)) \to 0 \quad \text{as} \quad \rho \to 0^+. \tag{4.9}
\end{equation}

**Lemma 4.3** (Good slicing). Under the same hypotheses and notation of Lemma 4.2, let $\hat{r} \in (0, 1)$. Then, for every $r \in (0, \hat{r}]$ there exist $\hat{\rho} \in [r/2, r]$ and a constant $\Theta = \Theta(m, \tilde{\rho}, \text{Lip}(\Omega), \ell, \gamma) > 0$ such that:

1. $\mathcal{H}^1(\Sigma \cap \partial B_\rho) \leq \Theta \mathcal{H}^2(\Sigma \cap B_\hat{\rho}) \hat{\rho}$;
2. $\int_{\Sigma \cap \partial B_\rho} |\bigwedge_2 d\pi(\Sigma)| \, d\mathcal{H}^1 \leq \frac{\Theta}{\hat{\rho}} \int_{\Sigma \cap (B_\hat{\rho} \setminus B_{r/2})} |\bigwedge_2 d\pi(\Sigma)| \, d\mathcal{H}^2$;
3. $\mathbb{M}(\pi_\ast \partial([\Sigma] \mathcal{L} \, B_\rho)) \leq \Theta \sqrt{K_m} \int_{r/2}^r \frac{1}{\rho} \int_{\Sigma \cap \partial B_\rho} |\Sigma_0 \cap \nu_0| \, d\mathcal{H}^1 \, d\mathcal{L}^1(\rho)$.

**Proof.** First, we notice that, by the coarea formula and the monotonicity formula (4.2), it holds that
\begin{equation*}
\int_{r/2}^r \frac{\mathcal{H}^1(\Sigma \cap \partial B_\rho)}{\rho} \, d\mathcal{L}^1(\rho) \leq \frac{2}{r} \mathcal{H}^2(\Sigma \cap B_r) \\
\leq 2e^{A\rho + \ell^2} (1 + A\rho) \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} \hat{r}.
\end{equation*}
\[ 24 \leq \frac{4e^{A+\frac{A}{2}}(1 + A)}{r^2} \mathcal{H}^2(\Sigma \cap B_r)^{\frac{7}{2}}. \]

Hence,
\[
\int_{r/2}^{r} \frac{1}{\Theta_1 \mathcal{H}^2(\Sigma \cap B_r)} \frac{\mathcal{H}^1(\Sigma \cap \partial B_r)}{\rho} d\mathcal{L}^1(\rho) \leq 1, \tag{4.10}
\]
with
\[ \Theta_1 := \frac{4e^{A+\frac{A}{2}}(1 + A)}{r^2}. \]

Moreover, again by the coarea formula, we get
\[
\int_{r/2}^{r} \rho \int_{\Sigma \cap \partial B_r} |\nabla_2 d\pi(\Sigma)| d\mathcal{H}^1 d\mathcal{L}^1(\rho)
\leq 2 \int_{r/2}^{r} \int_{\Sigma \cap \partial B_r} |\nabla_2 d\pi(\Sigma)| d\mathcal{H}^1 d\mathcal{L}^1(\rho)
= 2 \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\nabla_2 d\pi(\Sigma)| d\mathcal{H}^2 =: a,
\]
which leads to
\[
\int_{r/2}^{r} \frac{1}{a} \rho \int_{\Sigma \cap \partial B_r} |\nabla_2 d\pi(\Sigma)| d\mathcal{H}^1 d\mathcal{L}^1(\rho) \leq 1. \tag{4.11}
\]

Eventually, by [19, Lemma 7.6.1], we know that for a.e. \( \rho \in (0,1) \) the slice \( \Sigma \cap \partial B_r \) is a \( 1 \)-rectifiable subset of \( B \) and the vector field \( \overrightarrow{\Sigma}_\rho \) orienting its approximate tangent space at \( x \) belongs to \( S_x \) (see notation of Lemma 4.1). Then, by [22, Lemma 1] and by points (3) and (4) of Lemma 4.1, it follows that
\[
|\overrightarrow{\Sigma}_\rho \wedge \nu_0 \wedge J_0\nu_0|^2 = |\overrightarrow{\Sigma}_0 \wedge J_0\overrightarrow{\Sigma}_\rho \wedge \nu_0 \wedge J_0\nu_0|
\leq |\overrightarrow{\Sigma}_0 \wedge \nu_0 \wedge J_0\nu_0|
\leq K_m |\overrightarrow{\Sigma}_0 \wedge \nu_0|^2, \quad \mathcal{H}^1 \text{-a.e. on } \Sigma \cap \partial B_r.
\]
Hence, we get
\[
\mathcal{M}(\pi_* \partial([\Sigma] \llcorner B_r)) \leq \int_{\Sigma \cap \partial B_r} |d\pi(\overrightarrow{\Sigma}_\rho)| d\mathcal{H}^1
= \frac{1}{\rho} \int_{\Sigma \cap \partial B_r} |\overrightarrow{\Sigma}_\rho \wedge \nu_0 \wedge J_0\nu_0| d\mathcal{H}^1
\leq \frac{\sqrt{K_m}}{\rho} \int_{\Sigma \cap \partial B_r} |\overrightarrow{\Sigma}_0 \wedge \nu_0| d\mathcal{H}^1.
\]
Thus, by averaging the previous inequality on \([r/2, r]\), we obtain
\[
\int_{r/2}^{r} \frac{1}{b} \mathcal{M}(\pi_* \partial([\Sigma] \llcorner B_r)) d\mathcal{L}^1(\rho) \leq \frac{\sqrt{K_m}}{b} \int_{r/2}^{r} \frac{1}{\rho} \int_{\Sigma \cap \partial B_r} |\overrightarrow{\Sigma}_0 \wedge \nu_0| d\mathcal{H}^1 =: b,
\]
which leads to
\[
\int_{r/2}^{r} \frac{1}{b} \mathcal{M}(\pi_* \partial([\Sigma] \llcorner B_r)) d\mathcal{L}^1(\rho) \leq 1. \tag{4.12}
\]
By summing up the three inequalities (4.10), (4.11) and (4.12) we obtain
\[ \int_{r/2}^{r} \left( \frac{1}{\Theta_1 \mathcal{H}^2(\Sigma \cap B_\rho)} + \frac{1}{a \rho} \int_{\Sigma \cap \partial B_\rho} |\Lambda_2 d\pi(\Sigma)| \, d\mathcal{H}^1 \right) + \frac{1}{b} \mathcal{M}(\pi_* \partial([\Sigma] \mathbf{L} B_\rho)) \, dL^1(\rho) \leq 3. \]

Then, we conclude that there exists \( \rho \in [r/2, r] \) such that
\[ \frac{1}{\Theta_1 \mathcal{H}^2(\Sigma \cap B_\rho)} + \frac{1}{a \rho} \int_{\Sigma \cap \partial B_\rho} |\Lambda_2 d\pi(\Sigma)| \, d\mathcal{H}^1 \leq 3 \]
and the statement follows with \( \Theta := \max\{\Theta_1, 6\} \). \(\square\)

**Lemma 4.4** (Controlling the mass of the projected boundaries). *Under the same hypotheses and notation of Lemma 4.1, let \( \tilde{r} \in (0, 1) \). Let \( r \in (0, \tilde{r}] \) be such that
\[ \mathcal{M}(\pi_*([\Sigma] \mathbf{L} B_\rho)) < 2K_m^2 \mathcal{M}(\pi_*([\Sigma] \mathbf{L} B_{r/2})) \]
and
\[ \int_{\Sigma \cap B_r} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 > \zeta^{-1} \mathcal{H}^2(\Sigma \cap B_r) \gamma^2, \]
for some \( \zeta \in (0, 1) \). If \( \rho_j \in [r/2, r] \) is such that \( \Sigma \cap \partial B_{\rho_j} \) is a good slice of \( \Sigma \) in the sense of Lemma 4.3, then
\[ \mathcal{M}(\pi_* \partial([\Sigma] \mathbf{L} B_{\rho_j})) \leq \Lambda \sqrt{\mathcal{H}^2(\Sigma \cap B_\rho)} \sqrt{\mathcal{M}(\pi_*([\Sigma] \mathbf{L} B_{\rho}))} \]
\[ + \Lambda \mathcal{H}^2(\Sigma \cap B_\rho) \rho_j^{\gamma/4}, \]
for a constant \( \Lambda = \Lambda(m, \tilde{r}, \text{Lip}(\Omega), \ell, \gamma) > 0 \).

**Proof.** Let \( \rho_j \in [r/2, r] \) be such that \( \Sigma \cap \partial B_{\rho_j} \) is a good slice of \( \Sigma \). We apply twice the Cauchy-Schwarz inequality in the right-hand side of (3) in Lemma 4.3 and the coarea formula to get
\[ \mathcal{M}(\pi_* \partial([\Sigma] \mathbf{L} B_{\rho_j})) \leq \Theta \sqrt{K_m} \int_{r/2}^{r} \frac{1}{\rho} \int_{\Sigma \cap \partial B_\rho} |\Sigma_0 \wedge \nu_0| \, d\mathcal{H}^1 \, dL^1(\rho) \]
\[ = \Theta \sqrt{K_m} \int_{r/2}^{r} \int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho} |\Sigma_0 \wedge \nu_0| \, d\mathcal{H}^1 \, dL^1(\rho) \]
\[ \leq \Theta \sqrt{K_m} \int_{r/2}^{r} \sqrt{\mathcal{H}^1(\Sigma \cap \partial B_\rho)} \]
\[ \cdot \sqrt{\int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho^2} |\Sigma_0 \wedge \nu_0|^2 \, d\mathcal{H}^1 \, dL^1(\rho)} \]
\[ = \Theta \sqrt{K_m} \frac{2}{r} \int_{r/2}^{r} \sqrt{\mathcal{H}^1(\Sigma \cap \partial B_\rho)} \]
\[ \cdot \sqrt{\int_{\Sigma \cap \partial B_\rho} \frac{1}{\rho^2} |\Sigma_0 \wedge \nu_0|^2 \, d\mathcal{H}^1 \, dL^1(\rho)} \]
\[
\begin{align*}
\leq & \Theta \sqrt{K_m} \frac{2}{r} \sqrt{\int_{r/2}^{r} \mathcal{H}^1(\Sigma \cap \partial B_\rho) \, d\mathcal{L}^1(\rho)} \\
\cdot & \sqrt{\int_{r/2}^{r} \int_{\Sigma \cap \partial B_\rho} \frac{1}{|x|} |\Sigma_0 \cap \nu| \, d\mathcal{H}^2 \, d\mathcal{L}^1(\rho)}
\end{align*}
\]
\[
= \Theta \sqrt{K_m} \frac{2}{r} \sqrt{\int_{r/2}^{r} \mathcal{H}^1(\Sigma \cap \partial B_\rho) \, d\mathcal{L}^1(\rho)} \\
\cdot & \sqrt{\int_{\Sigma \cap (B_{r-B_{r/2}})} \frac{1}{|x|} |\Sigma_0 \cap \nu| \, d\mathcal{H}^2 \, d\mathcal{L}^1(\rho)}.
\]

We notice that, by point (1) in Lemma 4.2 and by our assumption (4.14), it holds that
\[
\left| M(\pi_*([\Sigma] \llcorner B_\rho)) - \int_{\Sigma \cap B_\rho} |\wedge_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \right| \leq \Xi \mathcal{H}^2(\Sigma \cap B_\rho) r^{7/2} \\
< \zeta \int_{\Sigma \cap B_\rho} |\wedge_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2
\]
which implies
\[
\int_{\Sigma \cap B_\rho} |\wedge_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \frac{1}{1-\zeta} M(\pi_*([\Sigma] \llcorner B_\rho)).
\]
Hence, by Lemma 1 we have
\[
\int_{\Sigma \cap (B_{r-B_{r/2}})} \frac{1}{|x|} |\Sigma_0 \cap \nu| \, d\mathcal{H}^2 \leq K_m \int_{\Sigma \cap (B_{r-B_{r/2}})} |\wedge_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \\
\leq \frac{K_m}{1-\zeta} M(\pi_*([\Sigma] \llcorner B_\rho)).
\]
Moreover, by (4.10), it follows that
\[
\sqrt{\int_{r/2}^{r} \mathcal{H}^1(\Sigma \cap \partial B_\rho) \, d\mathcal{L}^1(\rho)} \leq \sqrt{\int_{r/2}^{r} \mathcal{H}^1(\Sigma \cap \partial B_\rho) \, d\mathcal{L}^1(\rho)} \\
\leq \sqrt{2} \sqrt{2} \Theta \mathcal{H}^2(\Sigma \cap B_\rho) \sqrt{\mathcal{M}(\pi_*([\Sigma] \llcorner B_\rho))}.
\]
Thus,
\[
\mathcal{M}(\pi_*\partial([\Sigma] \llcorner B_{\rho_j})) \leq \frac{2 \Theta^{3/2} K_m}{\sqrt{1-\zeta}} \sqrt{\mathcal{H}^2(\Sigma \cap B_\rho)} \sqrt{\mathcal{M}(\pi_*([\Sigma] \llcorner B_\rho))}.
\]
By our hypothesis (4.13) and since \(\rho_j > r/2\), we obtain that
\[
\mathcal{M}(\pi_*([\Sigma] \llcorner B_\rho)) < 2K_m \mathcal{M}(\pi_*([\Sigma] \llcorner B_{r/2})) \\
\leq 2K_m \mathcal{M}(\pi_*([\Sigma] \llcorner B_{\rho_j})) + 4K_m^2 \Xi \mathcal{H}^2(\Sigma \cap B_\rho) \rho_j^{7/2}.
\]
We point out that the last inequality follows a direct application of point (1) in Lemma 4.2.

Then, the statement follows with
\[
\Lambda := \max \left\{ \frac{2 \Theta^{3/2} K_m}{\sqrt{1-\zeta}}, \frac{2 \sqrt{2} \Theta \mathcal{H}^2(\Sigma \cap B_\rho)}{\sqrt{1-\zeta}} \right\}.
\]
\[\square\]
We recall the following general fact about integral 2-currents on $\mathbb{CP}^{m-1}$ with small mass which are $\zeta$-almost semicalibrated by $\omega_{\mathbb{CP}^{m-1}}$, whose proof can be found in [22, Lemma 11].

Recall that a current $T \in \mathcal{D}^2(\mathbb{CP}^{m-1})$ is said to be $\zeta$-almost semicalibrated by $\omega_{\mathbb{CP}^{m-1}}$ for some constant $\zeta \in (0, 1)$ if

$$(1 - \zeta) | \langle T \nabla U, \omega_{\mathbb{CP}^{m-1}} \rangle | \leq M(T \nabla U) \leq (1 + \zeta) | \langle T \nabla U, \omega_{\mathbb{CP}^{m-1}} \rangle |,$$

for every open set $U \subset \mathbb{CP}^{m-1}$.

**Lemma 4.5.** Let $\zeta \in (0, 1)$. Given any couple of constants $\tilde{\Lambda} > 0$ and $\lambda > 0$, there exist $\delta > 0$ and $\varepsilon > 0$ satisfying what follows. For every integral 2-current $T \in \mathcal{D}^2(\mathbb{CP}^{m-1})$ such that

1. $T$ is $\zeta$-almost semicalibrated by $\omega_{\mathbb{CP}^{m-1}}$,
2. $M(T) + M(\partial T) < \delta$,
3. $M(\partial T) \leq \tilde{\Lambda} \sqrt{M(T)}$,

there is a complex projective $(m - 2)$-hyperplane $H \subset \mathbb{CP}^{m-1}$ and a tubular neighbourhood $H_\varepsilon \subset \mathbb{CP}^{m-1}$ of $H$ with width $\varepsilon$ such that

$$\frac{M(T \nabla H_\varepsilon)}{\varepsilon^2} \leq \lambda M(T).$$

As a last tool, we need to establish that given any complex projective $(m - 2)$-hyperplane $H \subset \mathbb{CP}^{m-1}$ and a tubular neighbourhood $H_\varepsilon \subset \mathbb{CP}^{m-1}$ of $H$ with width $\varepsilon$, we can approximate the symplectic form $\omega_{\mathbb{CP}^{m-1}}$ on $\mathbb{CP}^{m-1}$ with an exact form $d\alpha$ that coincides with $\omega_{\mathbb{CP}^{m-1}}$ on the complement of $H_\varepsilon$ and vanishes on $H_{\varepsilon/2}$. We achieve this approximation through the following lemma, whose proof is again in [22, Lemma 6].

**Lemma 4.6.** Let $H \subset \mathbb{CP}^{m-1}$ be any complex projective $(m - 2)$-hyperplane and let $H_\varepsilon \subset \mathbb{CP}^{m-1}$ be a tubular neighbourhood of $H$ with width $\varepsilon$. Then there exists a 1-form $\alpha \in \Omega^1(\mathbb{CP}^{m-1})$ and a universal constant $\kappa > 0$ such that:

1. $\omega_{\mathbb{CP}^{m-1}} = d\alpha$ on $\mathbb{CP}^{m-1} \setminus H_\varepsilon$;
2. $\alpha = 0$ on $H_{\varepsilon/2}$;
3. $||\alpha||_s \leq \kappa$;
4. $||\omega_{\mathbb{CP}^{m-1}} - d\alpha||_s \leq \frac{\kappa}{\varepsilon^2}$.

**4.2. Proof of the fundamental Morrey type estimate.** Recall that $\kappa, \Xi > 0$ are positive constants introduced in Lemma 4.3 and Lemma 4.2 respectively.

Fix any $j_0 \in \mathbb{N} \setminus \{0\}$ and let $\ell \geq 0$ be a constant depending only on $\text{Lip}(\Omega)$. Let $\delta > 0$ and $\varepsilon > 0$ be the constants given by applying Lemma 4.5 with $\tilde{\Lambda} = \Lambda(m, 2^{-j_0}, \text{Lip}(\Omega), 1/2)$ from Lemma 4.4 and $\lambda := 5(24\kappa)^{-1}$. Let $\delta' > 0$ be such that

$$\Lambda \sqrt{\delta} + \frac{\delta'}{2} + \delta' < \delta$$

and choose $\bar{r} \in (0, \min\{2^{-j_0}, \delta'(2\Xi)^{-1}\})$ such that

$$\max\{\Lambda, \Xi\} \left( e^{A + 2(1 + A)} \right)^{-1} \bar{r}^{1/4} < \frac{\delta'}{2}.$$

Assume that $\mathcal{F}$ is a family of closed almost $J$-holomorphic curves in $B$ such that every element $\Sigma \in \mathcal{F}$ satisfies the following properties:

1. $|\bar{\Sigma}(x) - \bar{\Sigma}_j(x)| \leq \ell |x|^{1/2}$, for $\mathcal{F}^2$-a.e. $x \in \Sigma$, for all $j$.
(2) $\mathcal{H}^2(\Sigma \cap B_{2^{-j_0}}) < \left( e^{A+2\ell}(1 + A) \right)^{-1}$,

(3) \[
\int_{\Sigma \cap B_{2^{-j_0}}} |\nabla_2d\pi(\Sigma_0)| \, d\mathcal{H}^2 < \frac{\delta'}{2}.
\]

Remark 4.5. For every $\Sigma \in \mathcal{F}$, the hypotheses (2) and (3) combined with point (1) in Lemma 4.2 imply that:

(1) $M(\pi_*([\Sigma] \cup B_\rho)) + M(\pi_*\partial([\Sigma] \cup B_\rho)) < \delta$,

(2) $M(\pi_*\partial([\Sigma] \cup B_\rho)) \leq \Lambda M(\pi_*([\Sigma] \cup B_\rho))$,

for every good slice $\Sigma \cap \partial B_\rho$ of $\Sigma$, where $\rho \in [r/2, r]$ with $r \in (0, \bar{r}]$ satisfying the hypotheses (4.13) and (4.14) of Lemma 4.4.

We want to show that for every $\Sigma \in \mathcal{F}$ there exist constants $C > 0$ and $0 < \alpha < 1$ depending on $m$, $j_0$, $\text{Lip}(\Omega)$ such that

\[
\left| \int_{\Sigma \cap B_\rho} \pi^*\omega_{\text{CPm-1}\Sigma} \right| \leq C\rho^\alpha, \quad \forall \rho \in (0, 2^{-j_0}). \tag{4.17}
\]

By definition of mass it holds that

\[
\left| \int_{\Sigma \cap B_\rho} \pi^*\omega_{\text{CPm-1}\Sigma} \right| = \left| (\pi_*([\Sigma] \cup B_\rho), \omega_{\text{CPm-1}\Sigma}) \right| \leq M(\pi_*([\Sigma] \cup B_\rho)), \tag{4.18}
\]

for every $\rho \in (0, 1)$. Hence, in order to prove (4.17) it is enough to show that

\[
M(\pi_*([\Sigma] \cup B_\rho)) \leq C\rho^\alpha, \quad \forall \rho \in (0, \bar{r}). \tag{4.19}
\]

Moreover, by exploiting point (1) in Lemma 4.2 we realize that if we show

\[
\int_{\Sigma \cap B_\rho} |\nabla_2d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \tilde{C}\rho^\alpha, \quad \forall \rho \in (0, \bar{r}), \tag{4.20}
\]

then (4.19) will follow with $C := \tilde{C} + \Xi$ and $\alpha := \min\{\tilde{\alpha}, 1/4\}$. Thus, we just need to show (4.20).

Fix any $\Sigma \in \mathcal{F}$. Let

\[
E(\rho) := \int_{\Sigma \cap B_\rho} |\nabla_2d\pi(\Sigma_0)| \, d\mathcal{H}^2, \quad \forall \rho \in (0, 1)
\]

and define

\[
I := \left\{ j \in \mathbb{N} \text{ s.t. } 2^{-j} \leq \bar{r} \text{ and } E(2^{-(j+1)}) \leq \frac{1}{2} E(2^{-j}) \right\},
\]

\[
J := \left\{ j \in \mathbb{N} \text{ s.t. } 2^{-j} \leq \bar{r} \text{ and } E(2^{-(j+1)}) \leq 5\Xi 2^{-(j+1)/4} \right\}.
\]

First, we claim that there exists $\theta = \theta(m, j_0, \text{Lip}(\Omega)) \in (0, 1)$ such that

\[
E(2^{-(j+1)}) \leq \theta \left( E(2^{-j}) + 2^{-j/4} \right), \tag{4.21}
\]

for every $j \in (I \cup J)^c$. Fix $j \in (I \cup J)^c$ and set $r := 2^{-j}$. Pick a radius $\rho_j \in [r/2, r]$ such that $\Sigma \cap \partial B_{\rho_j}$ is a good slice of $\Sigma$ (see Lemma 4.3). By Lemma 4.2 and Lemma 4.3 it follows that we can choose a sequence of radii $\{s_k\}_{k \in \mathbb{N}} \in (0, r/2)$ such that $s_k \to 0^+$ as $k \to +\infty$ and

\[
M(\pi_*\partial([\Sigma] \setminus B_{s_k})) \leq M(\pi_*([\Sigma] \setminus B_{r/2})), \quad \forall k \in \mathbb{N}. \tag{4.22}
\]
Since $\Sigma \in \mathcal{F}$, by Remark 4.5 and since $j \in (I \cup J)^c$ it follows that the current $T = \pi_*([\Sigma] \mathbb{L} B_{\rho_j})$ satisfies the hypotheses of Lemma 4.10 with $\zeta = 1/2$. Thus, there exists a complex projective $(m - 2)$-hyperplane $H \subset \mathbb{C}P^{m-1}$ and a tubular neighbourhood $H_\varepsilon \subset \mathbb{C}P^{m-1}$ of $H$ with width $\varepsilon > 0$ such that

$$\frac{M(\pi_*([\Sigma] \mathbb{L} B_{\rho_j}) \mathbb{L} H_\varepsilon)}{\varepsilon^2} \leq \lambda \mathbb{M}(\pi_*([\Sigma] \mathbb{L} B_{\rho_j})).$$

We let $\alpha \in \Omega^1(\mathbb{C}P^{m-1})$ be a smooth 1-form given by Lemma 4.6 relatively to $H,H_\varepsilon$. Following the proof of point (1) in Lemma 4.2, we notice that

$$\left| \int_{\Sigma \cap (B_{\rho_j} \setminus B_{s_k})} \langle \pi^*(\omega_{\mathbb{C}P^{m-1}} - d\alpha), \bar{\Sigma} \rangle d\mathcal{H}^2 \right| \leq \frac{1}{4} \int_{\Sigma \cap B_{\rho_j}} |\wedge_2 d\pi(\bar{\Sigma}_0)| d\mathcal{H}^2 + \tilde{\Xi} r^{1/4}.$$

For what concerns the second term in the last sum, by Lemmas 4.5, 4.6 and (1) in Lemma 4.2 we see that

$$\left| \int_{\Sigma \cap (B_{\rho_j} \setminus B_{s_k})} \langle \pi^*(\omega_{\mathbb{C}P^{m-1}} - d\alpha), \bar{\Sigma} \rangle d\mathcal{H}^2 \right| \leq \frac{\kappa}{\varepsilon^2} \mathbb{M}(\pi_*([\Sigma] \mathbb{L} B_{\rho_j}) \mathbb{L} H_\varepsilon).$$

Now we want to estimate

$$\left| \int_{\Sigma \cap (B_{\rho_j} \setminus B_{s_k})} \langle \pi^* d\alpha, \bar{\Sigma} \rangle d\mathcal{H}^2 \right|.$$
Thus, we have obtained
\[
\left| \int_{\Sigma \cap \partial B_{r_j}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_j}} \right| = \left| \int_{\Sigma \cap (B_{r_j} \setminus B_{r_k})} d(\pi^* \alpha) \right|_{\Sigma}
\]
\[
= \left| \int_{\Sigma \cap \partial B_{r_j}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_j}} - \int_{\Sigma \cap \partial B_{r_k}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_k}} \right|
\]
\[
\leq \left| \int_{\Sigma \cap \partial B_{r_j}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_j}} \right| + \left| \int_{\Sigma \cap \partial B_{r_k}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_k}} \right|.
\]

Since \( j \in (I \cup J)^c \), by (4.22) we get that
\[
\left| \int_{\Sigma \cap \partial B_{r_j}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_j}} \right| = |\langle \pi_* \partial (|\Sigma| \mathcal{L} B_{r_k}), \alpha \rangle| \leq \| \alpha \| \| \Sigma \| \mathcal{M}(\pi_* \partial (|\Sigma| \mathcal{L} B_{r_k}))
\]
\[
\leq 3\kappa \int_{\Sigma \cap (B_{r_j} \setminus B_{r_k})} |\wedge_2 d\pi(\Sigma)\big| d\mathcal{H}^2.
\]
Thus, we have obtained
\[
\left| \int_{\Sigma \cap (B_{r_j} \setminus B_{r_k})} \left\langle \pi^* d\alpha, \Sigma \right\rangle \right| d\mathcal{H}^2 \leq \left| \int_{\Sigma \cap \partial B_{r_j}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_j}} \right| \quad (4.25)
\]

and we just need to bound
\[
\left| \int_{\Sigma \cap \partial B_{r_j}} \pi^* \alpha \big|_{\Sigma \cap \partial B_{r_j}} \right|.
\]

To do this, we write the 1-rectifiable closed curve \( \Sigma \cap \partial B_{r_j} \) as
\[
\Sigma \cap \partial B_{r_j} = \bigcup_{i=0}^{\infty} \Gamma_i,
\]
where \( \Gamma_i \) is a Lipschitz connected closed curve in \( B \). We let \( \gamma_i : [0, \mathcal{H}^1(\Gamma_i)] \to B \) be the parametrization of \( \Gamma_i \) through its arc-length, so that \( |\gamma_i'| \equiv 1 \) a.e. on \([0, \mathcal{H}^1(\Gamma_i)]\). First, fix \( i \in \mathbb{N} \) and notice that for every smooth function \( f : B \setminus \{0\} \to \mathbb{R} \) such that \( f^i = 0 \), where
\[
\bar{f}^i := \int_{\Gamma_i} f d\mathcal{H}^1,
\]
the following Poincaré type inequality holds for \( \Gamma_i \):
\[
\left( \int_{\Gamma_i} |f|^2 d\mathcal{H}^1 \right)^{1/2} = \left( \int_{\mathcal{H}^1(\Gamma_i)} |f \circ \gamma_i|^2 |\gamma_i'| \, d\mathcal{L}^1 \right)^{1/2}
\]
\[
= \left( \int_{\mathcal{H}^1(\Gamma_i)} |f \circ \gamma_i|^2 \, d\mathcal{L}^1 \right)^{1/2}
\]
\[
\leq \mathcal{H}^1(\Gamma_i) \left( \int_{\mathcal{H}^1(\Gamma_i)} |f \circ \gamma_i|^2 \, d\mathcal{L}^1 \right)^{1/2}
\]
\[
= \mathcal{H}^1(\Gamma_i) \left( \int_{\mathcal{H}^1(\Gamma_i)} |d\gamma_i(\gamma_i')|^2 \, d\mathcal{L}^1 \right)^{1/2}
\]
where

\[ m \]

in order to get the expansion

\[ \pi^* \alpha |_{\Gamma_i} = \left\langle \pi^* \alpha, \gamma_i' \right\rangle = \sum_{a=1}^{2m-2} (\alpha_a \circ \pi) \langle dy_a, d\pi |_{\Gamma_i} \rangle. \]

Moreover, we notice that

\[
|d(\alpha_a \circ \pi)|_{\Sigma \rho_j} \leq |d\alpha_a \circ \pi|^2 |d\pi|_{\Sigma \rho_j}^2 \leq |d\alpha \circ \pi|^2 |d\pi|_{\Sigma \rho_j}^2
\]

\[
\leq \max \left\{ \left\| \omega_{\mathbb{CP}^{m-1}} \right\|_{\infty}, \frac{\kappa}{\varepsilon^2} \right\} |d\pi|_{\Sigma \rho_j}^2
\]

\[
\leq \max \left\{ \left\| \omega_{\mathbb{CP}^{m-1}} \right\|_{\infty}, \frac{\kappa}{\varepsilon^2} \right\} |d\pi|_{\Sigma \rho_j}^2
\]

\[
= M_m |d\pi|_{\Sigma \rho_j}^2,
\]

where

\[
M_m := \max \left\{ \left\| \omega_{\mathbb{CP}^{m-1}} \right\|_{\infty}, \frac{\kappa}{\varepsilon^2} \right\}
\]

depends only on \( m \). Then, by (4.26), Hölder’s inequality and point (3) in Lemma 4.1 we estimate

\[
\left| \int_{\Gamma_i} \pi^* \alpha |_{\Gamma_i} \right|
\]

\[
= \sum_{a=1}^{2m-2} \left| \int_{\Gamma_i} (\alpha_a \circ \pi) \langle dy_a, d\pi |_{\Gamma_i} \rangle \right|
\]

\[
= \sum_{a=1}^{2m-2} \left| \int_{\Gamma_i} (\alpha_a \circ \pi - \tilde{\alpha}_k) \langle dy_a, d\pi |_{\Gamma_i} \rangle \right|
\]

\[
\leq \sum_{a=1}^{2m-2} \left( \int_{\Gamma_i} |\alpha_a \circ \pi - \tilde{\alpha}_k| \langle dy, d\pi |_{\Gamma_i} \rangle \right)^2
\]

\[
\leq \sum_{a=1}^{2m-2} \left( \int_{\Gamma_i} |\alpha_a \circ \pi - \tilde{\alpha}_k| \langle dy, d\pi |_{\Gamma_i} \rangle \right)^2 \left( \int_{\Gamma_i} |d\pi|_{\Sigma \rho_j}^2 d\mathcal{H}^1 \right)^{1/2}
\]

\[
\leq \mathcal{H}^1(\Gamma_i) \sum_{a=1}^{2m-2} \left( \int_{\Gamma_i} |d(\alpha_a \circ \pi)|_{\Sigma \rho_j}^2 d\mathcal{H}^1 \right)^{1/2} \left( \int_{\Gamma_i} |d\pi|_{\Sigma \rho_j}^2 d\mathcal{H}^1 \right)^{1/2}
\]

\[
\leq M_m \mathcal{H}^1(\Gamma_i) \int_{\Gamma_i} |d\pi|_{\Sigma \rho_j}^2 d\mathcal{H}^1
\]
\[ \frac{1}{2} \int_{\Sigma \cap B_{r/2}} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \tilde{C} \left( \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 + r^{1/4} \right) \]

By letting \( k \to +\infty \) in the previous inequality, we obtain

\[ \int_{\Sigma \cap B_{r/2}} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \tilde{C} \left( \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 + r^{1/4} \right) \]

and by subtracting from both sides the quantity

\[ \frac{1}{2} \int_{\Sigma \cap B_{r/2}} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2, \]

we get

\[ \int_{\Sigma \cap B_{r/2}} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \tilde{C} \left( \int_{\Sigma \cap (B_r \setminus B_{r/2})} |\nabla_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 + r^{1/4} \right), \]
where $\bar{C} > 0$ is chosen big enough so that $\bar{C} \geq 2\bar{C}$ and 
\[
\frac{\bar{C}}{C + 1} \geq 2^{-1/4}.
\]

By the hole filling technique and recalling that $r = 2^{-j}$, we obtain 
\[
\int_{\Sigma \cap B_{2^{-j+1}} - B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \theta \int_{\Sigma \cap B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 + \theta^{j+1},
\]
with $\theta \in (0, 1)$ given by $\theta := \bar{C}/(\bar{C} + 1) \geq 2^{-1/4}$ and our claim (4.21) follows.

By (4.21) we obtain that 
\[
E(2^{-j}) \leq \theta^j E(2^{-j-1}) + \theta^j \theta^j E(2^{-j-2}) + 2\theta^j 
\leq \ldots \leq \theta^j E(2^{-j_0}) \theta^{-j} + ((j - j_0) \theta^{-j/2}) \theta^{-j/2}
\leq \hat{\theta} \theta^{-j}
\]
with $\hat{\theta} := \theta^{1/2}$ and 
\[
\hat{\theta} := \frac{\delta' \theta^j}{2} + \sup_{j \geq j_0} \{(j - j_0) \theta^{-j}\} < +\infty.
\]

In order to get (4.21), we notice that if $j \in I \cup J$ then either $j \in I$ or $j \in J \setminus I$. In the first case, we have 
\[
\int_{\Sigma \cap B_{2^{-j+1}} - B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq \frac{1}{2} \int_{\Sigma \cap B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2
\leq \theta \int_{\Sigma \cap B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2.
\]

In the second case, by definition of $J$, it holds that 
\[
\int_{\Sigma \cap B_{2^{-j+1}} - B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq 5\Xi 2^{-(j+1)/2}.
\]

By setting $\tilde{\alpha} = \min\{-\log_2 \hat{\theta}, 1/4\} \in (0, 1)$, we get 
\[
\int_{\Sigma \cap B_{2^{-j}}} |\Lambda_2 d\pi(\Sigma_0)| \, d\mathcal{H}^2 \leq 5\Xi \left(\frac{1}{2j}\right)^{\tilde{\alpha}},
\]
which leads to (4.21) with $\bar{C} := \max\{5\theta^{1/4}\Xi, \hat{\theta}\}$.

5. Almost pseudo-holomorphic foliations

Lemma 5.1. Let $m \geq 2$ and let $(X, J, \omega_X)$ be a closed, almost Kähler, smooth $(2m - 2)$-dimensional manifold. Assume that $v \in W^{1,2}(B, X)$ satisfies $v^* \text{vol}_X \in L^1(B)$ and $d(v^* \text{vol}_X) = 0$ in $\mathcal{D}'(B)$. Then, there exists a representative of $v$ such that the co-area formula holds. Moreover, given such a representative, for $\text{vol}_X$-a.e. $z \in X$ the following facts hold:

(1) $v^{-1}(z)$ is a countably $\mathcal{H}^2$-rectifiable subset of $B$;
(2) $(v^* \text{vol}_X)_x \neq 0$, for $\mathcal{H}^2$-a.e. $x \in v^{-1}(z)$;
(3) $[v^{-1}(z)]$ is a cycle of finite mass.
Proof. By Theorem 11, Theorem 12 there exists a representative of \( v \) such that both (1) and the co-area formula hold. Moreover, if we denote by \( E \subset B \) the set of all the \( x \in B \) such that \( (v^* \text{vol}_X)_x = 0 \), by the coarea formula we get

\[
0 = \int_E |v^* \text{vol}_X|_g \, d\text{vol}_g = \int_X \mathcal{H}^2(v^{-1}(z) \cap E) \, d\text{vol}_X(z),
\]

which implies that for \( \text{vol}_X\text{-a.e.} \ z \in X \) the set \( v^{-1}(z) \cap E \) has vanishing \( \mathcal{H}^2 \)-measure. Thus, (2) immediately follows.

We are just left to prove (3). By the coarea formula, it follows that

\[
\int_X \mathcal{H}^2(v^{-1}(z)) \, d\text{vol}_X(z) = \int_B |v^* \text{vol}_X| \, d\mathcal{L}^{2m} < +\infty.
\]

Hence, the function \( X \ni z \mapsto \mathcal{H}^2(v^{-1}(z)) \) belongs to \( L^1(X) \) and we know that a.e. \( z \in X \) is a Lebesgue point for \( f \) such that \( f(z) < +\infty \). Fix any such point \( z \in X \). By our choice of \( z \), it holds that \( \mathbb{M}([v^{-1}(z)]) = \mathcal{H}^2(v^{-1}(z)) = f(z) < +\infty \). Hence, just need to show that \( [v^{-1}(z)] \) is a cycle. Let \( \exp_z : \mathbb{R}^{2m-2} \to X \) be the exponential map of \( X \) at the point \( z \). Denote by \( \rho_0 \in (0, +\infty) \) the injectivity radius of \( X \) at \( z \) and we define

\[
B_\varepsilon(z) := \exp_z(B_\varepsilon(0)), \quad \text{for every } \varepsilon \in (0, \rho_0).
\]

For every \( \varepsilon \in (0, \rho_0) \), we let \( \{\varphi_{\varepsilon,k}\}_{k \in \mathbb{N}} \subset C^\infty(X) \) be a sequence of smooth functions on \( X \) such that:

1. \( \varphi_{\varepsilon,k} \equiv 0 \) on \( X \setminus B_\varepsilon(z) \);
2. \( 0 < \varphi_{\varepsilon,k} \leq (\text{vol}_X(B_\varepsilon(z)))^{-1} \) on \( B_\varepsilon(z) \);
3. it holds that
   \[
   \varphi_{\varepsilon,k} \xrightarrow{k \to \infty} \frac{1}{\text{vol}_X(B_\varepsilon(z))} \chi_{B_\varepsilon(z)}) \quad \text{vol}_X\text{-a.e. on } X.
   \]

Fix any \( \alpha \in \mathcal{D}^1(B) \). By the coarea formula, it follows that

\[
\int_B \alpha \wedge v^*(\varphi_{\varepsilon,k} \text{vol}_X) = \int_X \varphi_{\varepsilon,k}(z) \left( \int_{v^{-1}(z)} \alpha \big|_{v^{-1}(z)} \right) \, d\text{vol}_X(z).
\]

Hence, by dominated convergence, we get

\[
\lim_{k \to +\infty} \int_B \alpha \wedge v^*(\varphi_{\varepsilon,k} \text{vol}_X) = \int_{B_\varepsilon(z)} \left( \int_{v^{-1}(z)} \alpha \big|_{v^{-1}(z)} \right) \, d\text{vol}_X(z).
\]

Since \( z \) is a Lebesgue point for \( f \), we obtain

\[
\lim_{\varepsilon \to 0^+} \lim_{k \to +\infty} \int_B \alpha \wedge v^*(\varphi_{\varepsilon,k} \text{vol}_X) = \int_{v^{-1}(z)} \alpha \big|_{v^{-1}(z)} =: ([v^{-1}(z)], \alpha) \quad (5.1)
\]

Moreover, since \( v \) is such that \( d(v^* \text{vol}_X) = 0 \) distributionally on \( B \) and by the upper bound on \( \varphi_{\varepsilon,k} \), it holds that

\[
\left| \int_B \alpha \wedge v^*(\varphi_{\varepsilon,k} \text{vol}_X) \right| = \left| \int_B v^* \varphi_{\varepsilon,k} (\alpha \wedge v^* \text{vol}_X) \right| \leq \frac{1}{\text{vol}_X(B_\varepsilon(z))} \int_B \alpha \wedge v^* \text{vol}_X = 0, \quad (5.2)
\]

for every \( \varepsilon \in (0, \rho_0) \) and \( k \in \mathbb{N} \).
By (5.1) and (5.2) we get that $\langle [v^{-1}(x)], d\alpha \rangle = 0$ and, by arbitrariness of $\alpha \in \mathcal{D}^1(B)$, it follows that $\partial [v^{-1}(x)] = 0$ in the sense of currents. The statement follows.

Lemma 5.2. Let $v \in W^{1,2}(B, X)$ be a weakly $(J, J_X)$-holomorphic map such that $v^* \text{vol}_X \in L^1(B)$ and $d(v^* \text{vol}_X) = 0$ in $\mathcal{D}'(B)$. Then, there exist a representative of $u$ and a full measure set $\text{RegVal}(v) \subset X$ such that:

1. the co-area formula holds for $v$;
2. for every $z \in \text{RegVal}(u)$, the level set $v^{-1}(z)$ is a closed $J$-holomorphic curve in $B$.

Proof. By Lemma 5.1, it follows immediately that there exists a representative of $v$ such that the co-area formula holds and, for such a representative, $v^{-1}(z)$ is an $\mathcal{H}^2$-rectifiable subset of $B$ with $\partial [v^{-1}(z)] = 0$, for $\text{vol}_X$-a.e. $z \in X$. Thus, we are just left to show that $v^{-1}(z)$ is $J$-holomorphic, for a.e. $z \in X$. By the co-area formula and since $v$ is weakly $(J, J_X)$-holomorphic, for $\text{vol}_X$-a.e. $z \in X$ the form $v^* \text{vol}_X$ is non-vanishing on $v^{-1}(z)$ and $dv(Jw) = J_X dv(w)$ for every $w \in \mathbb{R}^{2m}$, up to some $\mathcal{H}^2$-negligible set. For such $z \in X$, the orienting vector field to $v^{-1}(z)$ is given by

$$\tilde{\Sigma} := \frac{*v^* \text{vol}_X}{|v^* \text{vol}_X|_g}.$$

We claim that $\tilde{\Sigma}$ is $J$-invariant for $\mathcal{H}^2$-a.e. $x \in v^{-1}(z)$. Indeed, given any $x \in v^{-1}(z)$ such that $(v^* \text{vol}_X)_x \neq 0$, we pick an orthonormal basis of $T^*_{v(x)}X$ of the form $\{\xi_1, J_X \xi_1, \ldots, \xi_{m-1}, J_X \xi_{m-1}\}$ and we notice that

$$(v^* \text{vol}_X)_x = \frac{1}{(m-1)!} v^* (\xi_1 \wedge J_X \xi_1 \wedge \ldots \wedge \xi_{m-1} \wedge J_X \xi_{m-1})$$

$$= \frac{1}{(m-1)!} v^* \xi_1 \wedge v^* J_X \xi_1 \wedge \ldots \wedge v^* \xi_{m-1} \wedge v^* J_X \xi_{m-1}$$

$$= \frac{1}{(m-1)!} v^* \xi_1 \wedge J(v^* \xi_1) \wedge \ldots \wedge J(v^* \xi_{m-1}).$$

This clearly implies that $\tilde{\Sigma}$ is $J$-invariant and the statement follows.

In the following lemma, which generalises the model situation presented in Lemma 5.2, we will adopt the notation developed in Appendix A. Moreover, we will denote by $X$ the product space $X := \mathbb{CP}^1 \times \mathbb{CP}^{n-2}$ and by $p_1 : X \rightarrow \mathbb{CP}^1$ and $p_2 : X \rightarrow \mathbb{CP}^{n-2}$ the canonical projections on the first and on the second factor respectively. We will endow $X$ with the complex structure $J_X := p_1^* j_1 + p_2^* j_{m-2}$ and with symplectic form $\omega_X := p_1^* \omega_{\mathbb{CP}^1} + p_2^* \omega_{\mathbb{CP}^{n-2}}$ in order to obtain the Kähler manifold $(X, J_X, \omega_X)$.

Lemma 5.3. Let $m, n \in \mathbb{N}_0$ be such that $m \geq 3$. Let $u \in W^{1,2}(B, \mathbb{CP}^n)$ be weakly $(J, j_n)$-holomorphic and locally approximable. If $n \geq 2$, then for a.e. $(q_1, \ldots, q_{n-1}, p) \in \mathbb{CP}^m \times \mathbb{CP}^{n-1} \times \ldots \times \mathbb{CP}^2 \times \mathbb{CP}^{n-1}$ the map $v_{q_1, \ldots, q_{n-1}, p} := (F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u, F_p \circ \pi) : B \rightarrow X$ has the following properties:

1. $v_{q_1, \ldots, q_{n-1}, p} \in W^{1,2}(B, X)$;
2. $v_{q_1, \ldots, q_{n-1}, p}^* \text{vol}_X \in L^1(B)$;
3. there exists a set $\text{RegVal}(v_{q_1, \ldots, q_{n-1}, p}) \subset X$ such that

$$\text{vol}_X (\{X \setminus \text{RegVal}(v_{q_1, \ldots, q_{n-1}, p})\}) = 0.$$
and for every \((y, z) \in \text{RegVal}(v_{q_1, \ldots, q_{n-1}, p})\) the \(\mathcal{H}^2\)-rectifiable set \(v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)\) is a closed almost J-holomorphic curve in \(B\), in the sense of Definition 3.3. Moreover, the constants \(\ell > 0\) and \(\gamma \in (0, 1]\) can be chosen as \(\ell = 2\sqrt{2} \Lip(\Omega)\) and \(\gamma = 1/2\).

If \(n = 1\), analogous properties hold for the map \(v_p := (u, F_p \circ \pi) : B \to X\) and for a.e. \(p \in \mathbb{CP}^{m-1}\).

**Proof.** Since the techniques are identical both in the case \(n = 1\) and \(n \geq 2\), we just focus on the second one.

Let \(Y := \mathbb{CP}^n \times \ldots \times \mathbb{CP}^2\). First, we want to prove (1). By Lemma \(A.1\), we know that \(p_2 \circ v_{q_1, \ldots, q_{n-1}, p}\) belongs to \(W^{1,2}(B, \mathbb{CP}^{m-2})\), for every \(p \in \mathbb{CP}^{m-1}\). We claim that \(p_1 \circ v_{q_1, \ldots, q_{n-1}, p} = F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u\) belongs to \(W^{1,2}(B, \mathbb{CP}^1)\) for a.e. \((q_1, \ldots, q_{n-1}) \in Y\). Indeed, notice that the map \(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u\) is weakly \((J, j_1)\)-holomorphic, for every \((q_1, \ldots, q_{n-1}) \in Y\). Thus, by Corollary 2.1 we have that

\[
\int_B |d(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)|_g^2 \, d\text{vol}_g = 2 \int_B (F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)^* \omega_{\mathbb{CP}^1} \wedge \frac{\Omega^{m-1}}{(m-1)!}.
\]

Hence, by Lemma \(A.2\) we obtain

\[
\begin{align*}
\int_B \varphi \left( \int_Y |d(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)|_g^2 \, d\text{vol}_Y(q_1, \ldots, q_{n-1}) \right) \, d\text{vol}_g \\
= 2 \int_B \varphi \left( \int_Y (F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)^* \omega_{\mathbb{CP}^1} \, d\text{vol}_Y(q_1, \ldots, q_{n-1}) \right) \wedge \frac{\Omega^{m-1}}{(m-1)!} \\
= 2D \int_B \varphi u^* \omega_{\mathbb{CP}^n} \wedge \frac{\Omega^{m-1}}{(m-1)!} = D \int_B \varphi |du|_g^2 \, d\text{vol}_g < +\infty,
\end{align*}
\]

where \(D := B_{n+1} \cdots B_3\), for every \(\varphi \in C_0^\infty(B)\). Thus, we get that

\[
\int_Y |d(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)|_g^2 \, d\text{vol}_Y(q_1, \ldots, q_{n-1}) = D|du|_g^2, \tag{5.3}
\]

for a.e. \(x \in B\). By integrating both sides of (5.3) on \(B\) and by Fubini’s theorem, we get

\[
\begin{align*}
\int_B \left( \int_Y |d(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)|_g^2 \, d\text{vol}_Y(q_1, \ldots, q_{n-1}) \right) \, d\mathcal{L}^{2m} \\
= \int_Y \left( \int_B |d(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)|_g^2 \, d\text{vol}_g \right) \, d\text{vol}_Y(q_1, \ldots, q_{n-1}) \\
= D \int_B |du|_g^2 \, d\text{vol}_g < +\infty. \tag{5.4}
\end{align*}
\]

Since (5.3) directly implies that

\[
\int_B |d(F_{q_{n-1}} \circ \ldots \circ F_{q_1} \circ u)|_g^2 \, d\text{vol}_g < +\infty
\]

for \(\text{vol}_Y\)-a.e. \((q_1, \ldots, q_{n-1}) \in Y\), point (1) follows.

Next, we turn to show (2). By (5.3), Lemma \(A.2\) \((2.1)\), \((A.1)\) and by Fubini’s theorem, we have

\[
\int_Y \left( \int_B |v^*_{q_1, \ldots, q_{n-1}, p} \text{vol}_X|_g \, d\text{vol}_g \right) \, d\text{vol}_{Y \times \mathbb{CP}^{m-1}}(q_1, \ldots, q_{n-1}, p)
\]
By arbitrariness of \( \varepsilon \), we get that the almost monotonicity formula (2.3), we get that estimates (5.4) and (5.6) we obtain

\[
\int_{B} |du|_{g}^{2} \cdot |d(F_{p} \circ \pi)|^{2m-4} \, d\text{vol}_{g} \leq DC^{m-2} \int_{B} |du|_{g}^{2} \cdot |d(F_{p} \circ \pi)|^{2m-4} \, d\text{vol}_{g} = DC^{m-2} \int_{B} |du|_{g}^{2} \, d\text{vol}_{g},
\]

(5.5)

where \( L_{p} \) is defined as in Appendix A. For any \( \rho \in (0, 1) \), define \( L_{p}^{\rho} := (L_{p} + B_{\rho}) \cap B \). By the almost monotonicity formula (2.3), we get that

\[
\int_{L_{p}^{\rho} \cap L_{p}^{\rho/2}} |du|_{g}^{2} \, d\text{vol}_{g} \leq \frac{2^{2m-4}}{\rho_{2m-4}} \int_{B} |du|_{g}^{2} \, d\text{vol}_{g} \leq 2^{2m-2} e^{A \rho}(1 + A \rho) \int_{B} |du|_{g}^{2} \, d\text{vol}_{g} \rho^{2} \frac{2}{4} \leq \left( 2^{2m-2} e^{A}(1 + A) \int_{B} |du|_{g}^{2} \, d\text{vol}_{g} \right) \rho^{2},
\]

for every \( \rho \in (0, 1) \). By iteration (see also the proof of Lemma 4.2), we get

\[
\int_{L_{p}^{\rho}} |du|_{g}^{2} \, d\text{vol}_{g} \leq \frac{2^{2m} e^{A}(1 + A)}{3} \left( \int_{B} |du|_{g}^{2} \, d\text{vol}_{g} \right) \rho^{2}.
\]

Combining (5.5) and (5.6), we obtain

\[
\int_{Y \times \mathbb{CP}^{m-1}} \left( \int_{B} |v_{q_{1}, \ldots, q_{n-1}, p}^{\ast} \text{vol}_{X} \right)_{g} \, d\text{vol}_{Y \times \mathbb{CP}^{m-1}}(q_{1}, \ldots, q_{n-1}, p) \leq C \int_{B} |du|_{g}^{2} \, d\text{vol}_{g} < \infty,
\]

(5.7)

where \( C > 0 \) is a constant depending on \( m, n \) and \( \text{Lip}(\Omega) \). Again, point (2) follows by Fubini’s theorem.

We are left to prove (3). First, we claim that for a.e. \( (q_{1}, \ldots, q_{n-1}, p) \in Y \times \mathbb{CP}^{m-1} \) it holds that \( d\left( v_{q_{1}, \ldots, q_{n-1}, p}^{\ast} \text{vol}_{X} \right) = 0 \) in the sense of distributions. Indeed, we already know that for \( \text{vol}_{Y} \text{-a.e.} \ (q_{1}, \ldots, q_{n-1}) \in Y \) the estimate (5.4) holds. Fix any \( \alpha \in \mathcal{D}^{1}(B) \). Notice that by estimates (5.4) and (5.6) we obtain

\[
\left| \int_{B} v_{q_{1}, \ldots, q_{n-1}, p}^{\ast} \text{vol}_{X} \land d\alpha \right| \leq C|\alpha| \int_{L_{p}^{\rho}} |du|_{g}^{2} \, d\text{vol}_{g} \leq C|\alpha| \left( \int_{B} |du|_{g}^{2} \, d\text{vol}_{g} \right) \rho^{2}, \quad \forall \rho \in (0, 1),
\]

where \( C > 0 \) is a constant depending only on \( m, n \) and \( \text{Lip}(\Omega) \). By letting \( \rho \to 0^{+} \), we get

\[
\int_{Y \times \mathbb{CP}^{m-1}} \left| \int_{B} v_{q_{1}, \ldots, q_{n-1}, p}^{\ast} \text{vol}_{X} \land d\alpha \right| \, d\text{vol}_{Y \times \mathbb{CP}^{m-1}}(q_{1}, \ldots, q_{n-1}, p) = 0.
\]

By arbitrariness of \( \alpha \in \mathcal{D}^{1}(B) \), our claim follows.

Let \( E \subset Y \times \mathbb{CP}^{m-1} \) be the set of all the \( n \)-tuples \( (q_{1}, \ldots, q_{n-1}, p) \in Y \times \mathbb{CP}^{m-1} \) such that

1. (1) and (2) hold;
2. \( d(v_{q_{1}, \ldots, q_{n-1}, p}^{\ast} \text{vol}_{X}) = 0 \) in the sense of distributions.
By what we have shown so far, we have $\text{vol}_{X \cap CP^{m-1}}(E^c) = 0$. Fix any $(q_1, \ldots, q_{n-1}, p) \in E$. We fix the representative of the map $v_{q_1, \ldots, q_{n-1}, p}$ given by Lemma 5.1. Thus, we know that the following facts hold for $\text{vol}_X$-a.e. $(y, z) \in X$:

1. The set $v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$ is $\mathcal{H}^2$-rectifiable;
2. $(v_{q_1, \ldots, q_{n-1}, p}^* \text{vol}_X)_x \neq 0$, for $\mathcal{H}^2$-a.e. $x \in v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$ and the rectifiable set $v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$ is oriented by the $\mathcal{H}^2$-measurable and unitary field $2$-vectors given by:

$$\Sigma := \left(\frac{*(v_{q_1, \ldots, q_{n-1}, p}^* \text{vol}_X)}{|v_{q_1, \ldots, q_{n-1}, p}^* \text{vol}_X|}_g\right)^\sharp$$

3. $\partial [v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)] = 0$.

Hence, we just need to show that $v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$ is almost $J$-holomorphic according to Definition 4.2. I.e. we claim that there exists some $J$-invariant and $\mathcal{H}^2$-measurable field of $g$-unitary 2-vectors $\tilde{\Sigma}_J : v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z) \to \bigwedge^2 \mathbb{R}^{2m}$ such that

$$|\tilde{\Sigma} - \tilde{\Sigma}_J| \leq \ell |\gamma|,$$

for some $\ell > 0$ and $\gamma \in (0, 1]$. In order to prove our claim, consider the following $\mathcal{H}^2$-measurable and $g$-unitary fields respectively of 2-vectors and 4-vectors on $v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$:

$$\tilde{\Sigma}^1 := \left(\frac{(p_1 \circ v_{q_1, \ldots, q_{n-1}, p})^* \omega_{CP^1}}{|(p_1 \circ v_{q_1, \ldots, q_{n-1}, p})^* \omega_{CP^1}|}_g\right)^\sharp$$

$$\tilde{\Sigma}^2 := \left(\frac{*(p_2 \circ v_{q_1, \ldots, q_{n-1}, p})^* \omega_{CP^{m-2}}}{|(p_2 \circ v_{q_1, \ldots, q_{n-1}, p})^* \omega_{CP^{m-2}}|}_g\right)^\sharp = \left(\frac{((F_p \circ \pi)^* \omega_{CP^{m-2}})}{|(F_p \circ \pi)^* \omega_{CP^{m-2}}|}_g\right)^\sharp$$

Notice that they are both well defined $\mathcal{H}^2$-a.e. on $v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$, since $(v_{q_1, \ldots, q_{n-1}, p}^* \text{vol}_X)_x \neq 0$ for $\mathcal{H}^2$-a.e. $x \in v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$. Fix any $x \in v_{q_1, \ldots, q_{n-1}, p}^{-1}(y, z)$ such that $(v_{q_1, \ldots, q_{n-1}, p}^* \text{vol}_X)_x \neq 0$, so that the subspace $W_1 := \text{span}\{\tilde{\Sigma}^1(x)\}$ is a $J_0$-holomorphic 2-plane and $W_2 := \text{span}\{\tilde{\Sigma}^2(x)\}$ is a $J_0$-holomorphic 4-plane. Let $W := \text{span}\{\tilde{\Sigma}(x)\}$ and notice that $W = W_1 \cap W_2$, $W_1 = (W_1^\perp \cap W_2) \oplus W$ and $\dim(W) = 2$. Moreover, we have that

$$4 = \dim(W_2) = \dim(W_1^\perp \cap W_2) + \dim(W) = \dim(W_1^\perp \cap W_2) + 2,$$

which implies $\dim(W_1^\perp \cap W_2) = 2$. We let $\{e_1, e_3\}$ be an $g$-orthonormal basis of $W$ and let $\{v, w\}$ be an $g$-orthonormal basis of $W_1^\perp \cap W_2$. By construction, $\{e_1, e_3, v, w\}$ is an $\Omega_0$-orthonormal basis of $W_2$ and we can write

$$\tilde{\Sigma}^2(x) := e_1 \wedge e_3 \wedge v \wedge w.$$

If $\tilde{\Sigma}^2(x)$ is $J$-invariant, we set $\tilde{\Sigma}^2_j(x) := \tilde{\Sigma}^2(x)$.

If not, notice that $\{e_1, Je_1, e_3, Je_3, v - Je_1, w - Je_3\}$ is a linearly independent set. Let $e_2, e_4$ be the unique unitary vectors such that $\{e_1, Je_1, e_3, Je_3, e_2, e_4\}$ is an $g$-orthonormal set such that

$$\text{span}\{e_1, Je_1, e_3, Je_3, v - Je_1, w - Je_3\} = \text{span}\{e_1, Je_1, e_3, Je_3, e_2, e_4\}.$$

Exactly as in the proof of Lemma 4.1, it follows that there exist two angles $\phi_1, \phi_2 \in [0, 2\pi]$ such that

$$\tilde{\Sigma}^2(x) := e_1 \wedge (\cos \phi_1 Je_1 + \sin \phi_1 e_2) \wedge e_3 \wedge (\cos \phi_2 Je_3 + \sin \phi_2 e_4).$$
Moreover, by (5.9), we have
\[ \vec{H} \]
Hence, (5.8) holds with
\[ \vec{J} \]
Eventually, we define
\[ \vec{L} \]
The same computation as in Lemma 4.1 leads to
\[ \vec{M} \]
which leads to
\[ \vec{N} \]
which leads to
\[ \vec{O} \]
and we compute
\[ \vec{P} \]
Theorem.
\[ \vec{Q} \]
Recalled now for the reader’s convenience.

This section is entirely devoted to proof Theorems 1.1 and 1.2, whose local versions will be recalled now for the reader’s convenience.

**Theorem.** Let \( m, n \in \mathbb{N}_0 \) be such that \( m \geq 2 \) and let \( B \) denote the open unit ball in \( \mathbb{R}^{2m} \). Assume that \( u \in W^{1,2}(B, \mathbb{C}^n) \) is weakly \((J, j_\nu)\)-holomorphic and locally approximable. Then, \( u \) has a unique tangent map at the origin.

**Theorem.** Let \( m, n \in \mathbb{N}_0 \) be such that \( m \geq 2 \) and let \( B \) denote the open unit ball in \( \mathbb{R}^{2m} \). Assume that \( u \in W^{1,2}(B, \mathbb{C}^n) \) is weakly \((J, j_\nu)\)-holomorphic and locally approximable. Then, the \((2m - 2)\)-cycle \( T_u \in D_{2m-2}(B) \) has a unique tangent cone at the origin.

In the first two subsections, we treat the proof of Theorem 1.2. We will first address the easy case \( m = 2, n = 1 \) in subsection 6.1, in order to clarify which will be the main ideas in order to proceed towards higher dimensions and codimensions. The general case will be discussed in subsection 6.2. Finally, in subsection 6.3 we will show how Theorem 1.1 can be obtained as a consequence of Theorem 1.2.
6.1. A model problem. Throughout all this subsection, \( B \subset \mathbb{R}^4 \) will denote the open unit ball in \( \mathbb{R}^4 \).

Let \( u \in W^{1,2}(B, \mathbb{CP}^1) \) be weakly \((J,j_1)\)-holomorphic and locally approximable. As usual, \( \pi : B \rightarrow \mathbb{CP}^1 \) denotes the Hopf map.

If \( \theta(0,u) < \varepsilon_0 \), then \( u \) is smooth in a neighbourhood of \( 0 \) by Theorem 3.1 and the statement follows. Assume then that \( \theta(0,u) \geq \varepsilon_0 \).

We use the same notations and labeling for the constants as in sections 3 and 4. Since \( u^*\omega_{\mathbb{CP}^1} \in L^1(B) \), by using Lemma 5.2 with \( X = \mathbb{CP}^1 \), we get that there exists a representative of \( u \) and a full measure set \( \text{RegVal}(u) \subset \mathbb{CP}^1 \) such that:

1. the co-area formula holds for \( u \);
2. for every \( y \in \text{RegVal}(u) \), the level set \( u^{-1}(y) \) is a closed \( J \)-holomorphic curve.

Hence, all the estimates in section 3 will be used assuming \( \Sigma_J = \Sigma \), \( \ell = 0 \) and \( \gamma = 1 \), as we stressed out in Remark 4.2.

For every \( k \in \mathbb{N}_0 \), we consider the set \( E_k \subset \mathbb{CP}^1 \) given by all the points \( y \) in \( \text{RegVal}(u) \) such that:

1. \( \mathcal{H}^2(u^{-1}(y) \cap B_{2-k}) < ((e^A(1 + A))^1) \]
2. \( \int_{u^{-1}(y) \cap B_{2-k}} |\wedge_2 d\pi(\Sigma_y^*)| d\mathcal{H}^2 < \frac{\delta'}{2} \)

where \( \delta' > 0 \) is the constant introduced in section 3.2 and \( \Sigma_y^* \) is built as shown in Lemma 4.1 starting from the \( J \)-holomorphic field of 2-vectors given by

\[
\Sigma_y^* := \frac{\ast(u^*\omega_{\mathbb{CP}^1})^2}{|u^*\omega_{\mathbb{CP}^1}|^2},
\]

which orients the closed \( J \)-holomorphic curve \( u^{-1}(y) \) for every \( y \in \text{RegVal}(u) \). We notice that \( E_{k-1} \subset E_k \) for every \( k \in \mathbb{N}_0 \). Moreover, since \( \text{RegVal}(u) \subset \mathbb{CP}^1 \) has full measure in \( \mathbb{CP}^1 \), we get

\[
\text{vol}_{\mathbb{CP}^1}(\mathbb{CP}^1 \setminus \bigcup_{k=1}^{+\infty} E_k) = 0
\]

For every \( k \in \mathbb{N}_0 \), we define the localized current \( T_k := T_u \llcorner u^{-1}(E_k) \), i.e.

\[
\langle T_k, \alpha \rangle := \int_{u^{-1}(E_k)} u^*\omega_{\mathbb{CP}^1} \wedge \alpha \quad \forall \alpha \in \mathcal{D}(B).
\]

**Claim.** We claim that every \( T_k \) has a unique tangent cone at the origin. First, notice that \( T_k \) is a normal 2-cycle on \( B \) semicalibrated by \( \Omega \). By definition of \( E_k \) and by Proposition 4.1 for every \( y \in E_k \) we get that

\[
e^{A\rho}(1 + A\rho) \frac{\mathcal{H}^2(u^{-1}(y) \cap B_{\rho})}{\rho^2} - e^{A\sigma}(1 + A\sigma) \frac{\mathcal{H}^2(u^{-1}(y) \cap B_{\sigma})}{\sigma^2}
\]

\[
\geq \int_{u^{-1}(y) \cap (B_{\rho} \setminus B_{\sigma})} \frac{1}{|\Sigma_y^*|} (\Omega, \Sigma_y^*) d\mathcal{H}^2
\]

and

\[
e^{-A\rho}(1 - A\rho) \frac{\mathcal{H}^2(u^{-1}(y) \cap B_{\rho})}{\rho^2} - e^{-A\sigma}(1 - A\sigma) \frac{\mathcal{H}^2(u^{-1}(y) \cap B_{\sigma})}{\sigma^2}
\]
for every $0 < \sigma < \rho < 1$. Since a direct computation leads to
\[
\frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2} = \frac{1}{\rho^2} \int_{E_k} \mathcal{H}^2(u^{-1}(y) \cap B_\rho) \, d\text{vol}_{CP^1}(y),
\]
by integrating on $E_k$ the two previous inequalities we get the following almost monotonicity formulas for the current $T_k$:
\[
e^{A\rho}(1 + A\rho) \frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2} - e^{A\sigma}(1 + A\sigma) \frac{\mathcal{M}(T_k \mathcal{L} B_\sigma)}{\sigma^2} \\
\geq \int_{E_k} \left( \int_{u^{-1}(y) \cap (B_\rho \setminus B_\sigma)} \frac{1}{2} \langle \Omega_t, \tilde{\nu} \rangle \, d\mathcal{H}^2 \right) \, d\text{vol}_{CP^1}(y), \quad (6.1)
\]
\[
e^{-A\rho}(1 - A\rho) \frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2} - e^{-A\sigma}(1 - A\sigma) \frac{\mathcal{M}(T_k \mathcal{L} B_\sigma)}{\sigma^2} \\
\leq \int_{E_k} \left( \int_{u^{-1}(y) \cap (B_\rho \setminus B_\sigma)} \frac{1}{2} \langle \Omega_t, \tilde{\nu} \rangle \, d\mathcal{H}^2 \right) \, d\text{vol}_{CP^1}(y), \quad (6.2)
\]
for every $0 < \sigma < \rho < 1$. Equation (6.1) immediately implies that function
\[
(0, 1) \ni \rho \mapsto e^{A\rho}(1 + A\rho) \frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2}
\]
is monotonically non-decreasing. Thus, the density of the current $T_k$ at zero, which is given by
\[
\theta(T_k, 0) := \lim_{\rho \to 0^+} \frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2} = \lim_{\rho \to 0^+} e^{A\rho}(1 + A\rho) \frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2}
\]
extists and is finite. Moreover, by (6.2), the coarea formula, (4.17) and the estimate (4.8), it follows that
\[
\left| \frac{\mathcal{M}(T_k \mathcal{L} B_\rho)}{\rho^2} - \theta(0, T_k) \right| \\
\leq C \int_{u^{-1}(E_k) \cap B_\rho} u^* \omega_{CP^1} \wedge \frac{\Omega_t}{| \cdot |^2} \\
\leq C \int_{u^{-1}(E_k) \cap B_\rho} u^* \omega_{CP^1} \wedge \pi^* \omega_{CP^1} + C \int_{u^{-1}(E_k) \cap B_\rho} u^* \omega_{CP^1} \wedge \frac{(\Omega - \Omega_0)_t}{| \cdot |^2} \\
\leq C \int_{E_k} \left( \int_{u^{-1}(y) \cap B_\rho} \pi^* \omega_{CP^1} \, d\text{vol}_{CP^1}(y) \\
+ C \int_{E_k} \int_{u^{-1}(y) \cap B_\rho} \frac{| \Omega - \Omega_0 |}{| \cdot |^2} \, d\mathcal{H}^2 \, d\text{vol}_{CP^1}(y) \right) \\
\leq C \text{vol}_{CP^1}(CP^1) \rho^\alpha + C \int_{E_k} \int_{u^{-1}(y) \cap B_\rho} \frac{1}{| \cdot |} \, d\mathcal{H}^2 \, d\text{vol}_{CP^1}(y) \\
\leq C \text{vol}_{CP^1}(CP^1) \rho^\alpha + C \text{vol}_{CP^1}(CP^1) \rho \leq C \text{vol}_{CP^1}(CP^1) \rho^\alpha,
\]
for every $\rho \in (0, \tilde{r})$, where the constant $C > 0$ and $\alpha, \tilde{r} \in (0, 1)$ all depend just on $k$ and on $\text{Lip}(\Omega)$. From the Morrey decay (6.3), uniqueness of tangent cone for $T_k$ follows by standard arguments.
Conclusion. For every $j \in \mathbb{N}_0$, consider the residual current $R_j := T_u - T_j$. By the same arguments that we have used in the proof of the previous claim, we conclude that $R_j$ is a normal 2-cycle in $B$ which is semicalibrated by $\Omega$. In particular, the quantity

$$e^{A\rho}(1 + A\rho) \frac{\mathcal{M}(R_j \mathbf{L} B_\rho)}{\rho^2}$$

is non-decreasing in $\rho \in (0, 1)$. Therefore, the limit as $\rho \to 0^+$ of the quantity (6.4) exists and it is finite. Then, since the quantity (6.4) is also non-increasing in $j \in \mathbb{N}$ and going to 0 as $j \to +\infty$, we are allowed to exchange the limits in the following chain of equalities and we get

$$\lim_{j \to +\infty} \lim_{\rho \to 0^+} \frac{\mathcal{M}(R_j \mathbf{L} B_\rho)}{\rho^2} = \lim_{j \to +\infty} \lim_{\rho \to 0^+} e^{A\rho}(1 + A\rho) \frac{\mathcal{M}(R_j \mathbf{L} B_\rho)}{\rho^2} = \lim_{\rho \to 0^+} e^{A\rho}(1 + A\rho) \lim_{j \to +\infty} \frac{\mathcal{M}(R_j \mathbf{L} B_\rho)}{\rho^2} = 0. \quad (6.5)$$

Fix any $\varepsilon > 0$. By (6.5), we can pick $j \in \mathbb{N}_0$ sufficiently large so that

$$\lim_{\rho \to 0^+} \frac{\mathcal{M}(R_j \mathbf{L} B_\rho)}{\rho^2} < \frac{\varepsilon}{2}. \quad (6.6)$$

Now assume that $\{\rho_k\}_{k \in \mathbb{N}} \subset (0, 1)$ and $\{\rho'_k\}_{k \in \mathbb{N}} \subset (0, 1)$ are two sequences converging to 0 as $k \to +\infty$ and both

$$(\Phi_{\rho_k})_* T_u \rightarrow C_\infty,$$

$$(\Phi_{\rho'_k})_* T_u \rightarrow C'_\infty,$$

where for every $\rho \in (0, 1)$ the map $\Phi_\rho$ is defined as in subsection 1.2. By further extracting subsequences if needed, we assume also that the sequences $\{(\Phi_{\rho_k})_* T_j\}_{k \in \mathbb{N}}$ and $\{(\Phi_{\rho'_k})_* T_j\}_{k \in \mathbb{N}}$ converge weakly in the sense of currents. By our previous claim, they converge to the same limit and then we have

$$C'_\infty - C_\infty = \lim_{k \to +\infty} ((\Phi_{\rho'_k})_* R_j - (\Phi_{\rho_k})_* R_j) + \lim_{k \to +\infty} (\Phi_{\rho'_k})_* T_j$$

$$- \lim_{k \to +\infty} (\Phi_{\rho_k})_* T_j$$

$$= \lim_{k \to +\infty} ((\Phi_{\rho'_k})_* R_j - (\Phi_{\rho_k})_* R_j),$$

in the sense of currents. By sequential lower semicontinuity of mass with the respect to weak convergence of currents, and by (6.6), we eventually get

$$\mathcal{M}(C'_\infty - C_\infty) \leq \liminf_{k \to +\infty} \mathcal{M}((\Phi_{\rho'_k})_* R_j - (\Phi_{\rho_k})_* R_j)$$

$$\leq \liminf_{k \to +\infty} \mathcal{M}((\Phi_{\rho'_k})_* R_j) + \liminf_{k \to +\infty} \mathcal{M}((\Phi_{\rho_k})_* R_j)$$

$$= \lim_{k \to +\infty} \frac{\mathcal{M}(R_j \mathbf{L} B_{\rho'_k})}{(\rho'_k)^2} + \lim_{k \to +\infty} \frac{\mathcal{M}(R_j \mathbf{L} B_{\rho_k})}{\rho_k^2} < \varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we obtain that $\mathcal{M}(C'_\infty - C_\infty) = 0$ and the conclusion follows.
6.2. The general case. Let $m, n \in \mathbb{N}_0$ be such that $m \geq 3$. Assume that $u \in W^{1,2}(B, \mathbb{CP}^n)$ is weakly $(J, j_n)$-holomorphic and locally approximable, where $B \subset \mathbb{R}^{2m}$ is the open unit ball in $\mathbb{R}^{2m}$. As usual, $\pi : B \to \mathbb{CP}^{m-1}$ denotes the Hopf map. If $\theta(0, u) < \varepsilon_0$, then $u$ is smooth in a neighbourhood of 0 by Theorem 3.1 and the statement follows. Assume then that $\theta(0, u) \geq \varepsilon_0$. Moreover, since the case $n = 1$ can be done exactly using the same method, we just focus on the case $n \geq 2$.

Let $T \in D_2(B)$ be the 2-current given by

$$\langle T, \alpha \rangle := \frac{1}{(m-2)!} \int_B u^* \omega_{\mathbb{CP}^n} \wedge \pi^* \omega_{\mathbb{CP}^{m-1}}^{m-2} \wedge \alpha \quad \forall \alpha \in D^2(B).$$

Notice that $T$ is well-defined and normal, since

$$|\langle T, \alpha \rangle| \leq \frac{1}{(m-2)!} |\alpha| \int_B |du|^2_{g} |\omega_{\mathbb{CP}^n}^{2m-4} \ w_{\mathbb{CP}^{m-1}}^{2} \ d\text{vol}_g$$

$$\leq C \frac{1}{(m-2)!} |\alpha| \int_B |du|^2_{g} \ w_{\mathbb{CP}^n}^{2m-4} \ d\text{vol}_g$$

$$\leq C \frac{1}{(m-2)!} |\alpha| \int_B |du|^2_{g} \ d\text{vol}_g < +\infty, \quad \forall \alpha \in D^2(B),$$

where $C = C \left( \text{Lip}(\Omega) \right) > 0$ is a constant and the last inequality follows from (2.3) exactly in the same way as estimate (5.6). Let $Y := \mathbb{CP}^m \times \ldots \times \mathbb{CP}^2 \times \mathbb{CP}^{m-1}$ and $X := \mathbb{CP}^1 \times \mathbb{CP}^{m-2}$. Notice that by Lemma A.2, Fubini’s theorem, Lemma 5.3 and the co-area formula, we can write the action of $T$ as

$$\langle T, \alpha \rangle = \frac{1}{(m-2)!} \int_Y \left( \int_B v_y \ w_X \wedge \alpha \right) \ d\text{vol}_Y(y)$$

$$= \frac{1}{(m-2)!} \int_Y \int_X \left( \int_{v_y^{-1}(z)} \langle \alpha, \Sigma_y \rangle \wedge \mathcal{H}^2 \right) \ d\text{vol}_X(z) \ d\text{vol}_Y(y),$$

for every $\alpha \in D^2(B)$, where $y := (q_1, \ldots, q_{n-1}, p) \in Y$ is any point in $Y$ such that (1), (2) and (3) of Lemma 5.3 hold, $v_y := v_{q_1, \ldots, q_{n-1}, p}$ and

$$\Sigma_y := \left( \text{vol}_{X|g} \right)^{\frac{1}{2}} \left( \text{vol}_{X|g} \right)^{\frac{1}{2}} \wedge \left( \text{vol}_{X|g} \right)^{\frac{1}{2}},$$

following the notation that is used in Lemma 5.3 is the $g$-unitary field of 2 vectors orienting $v_y^{-1}(z)$, for every $z \in \text{RegVal}(v_y) \subset X$. We define the "tilted current" $T_J \in D_2(B)$ by

$$\langle T_J, \alpha \rangle = \frac{1}{(m-2)!} \int_Y \int_X \left( \int_{v_y^{-1}(z)} \langle \alpha, \Sigma_y \rangle \wedge \mathcal{H}^2 \right) \ d\text{vol}_X(z) \ d\text{vol}_Y(y),$$

for every $\alpha \in D^2(B)$, where $\Sigma_y$ is the $J$-holomorphic field of 2-vectors that we have built in the proof of Lemma 5.3.

**Step 1.** We want to show that $T_J$ has a unique tangent cone at the origin. First, for every $k \in \mathbb{N}$ we define the set $E_k \subset Y \times X$ given by

1. points (1), (2) and (3) in Lemma 5.3 hold for the map $v_y$ and the level set $v_y^{-1}(z)$;
2. $\mathcal{H}^2(v_y^{-1}(z) \cap B_{2^{-k}}) < (e^{\mathcal{A}+2\mathcal{E}}(1 + A))^{-1};$
3. $\int_{v_y^{-1}(z) \cap B_{2^{-k}}} \lambda_{2\mathcal{D}(\Sigma_y)} \wedge \mathcal{H}^2 < \frac{\beta}{2}.$
where we are using the notation of subsection 3.2 and $\ell = \ell(\text{Lip}(\Omega)) > 0$ is the constant provided in Lemma 5.3. Notice that, by Lemma 5.3, Lemma 4.2 and Fubini’s theorem, it holds that
\[
\text{vol}_{Y \times X} \left( Y \times X \setminus \bigcup_{k \in \mathbb{N}} E_k \right). 
\]

Fix any $k \in \mathbb{N}$. Define the truncated current $T_k^j \in D_2(B)$ by
\[
\left\langle T_k^j, \alpha \right\rangle = \int_{E_k} \left( \int_{v_2^{-1}(z)} \langle \alpha, \overline{\Sigma}_j \rangle \, d\mathcal{H}^2 \right) \, d\text{vol}_{Y \times X}(y, z), \quad \forall \alpha \in D_2(B).
\]

Notice that, by Proposition 4.1 and by definition of $E_k$, for every $(y, z) \in E_k$ it holds that
\[
e^{-\left(\ell + \ell\rho^2/2\right)}(1 - A\rho) \frac{\mathcal{H}^2(v_2^{-1}(z) \cap B_\rho)}{\rho^2} 
- e^{-\left(\ell + \ell\rho^2/2\right)}(1 - A\sigma) \frac{\mathcal{H}^2(v_2^{-1}(z) \cap B_\sigma)}{\sigma^2} 
\leq \int_{v_2^{-1}(z) \cap (B_\rho \setminus B_\sigma)} \frac{1}{|z|^2} \langle \Omega, \overline{\Sigma}_j \rangle \, d\mathcal{H}^2,
\]

for every $0 < \sigma < \rho < 1$. Since a direct computation leads to
\[
\frac{\mathcal{M}(T_j \mathbb{L} B_\rho)}{\rho^2} = \frac{1}{\rho^2} \int_{E_k} \mathcal{H}^2(v_2^{-1}(z) \cap B_\rho) \, d\text{vol}_{Y \times X}(y, z), 
\]

by integrating on $E_k$ the two previous inequalities we get the following almost monotonicity formulas for the current $T_k^j$:
\[
e^{\ell + \ell\rho^2/2} (1 + A\rho) \frac{\mathcal{M}(T_k^j \mathbb{L} B_\rho)}{\rho^2} 
- e^{\ell + \ell\rho^2/2} (1 + A\sigma) \frac{\mathcal{M}(T_k^j \mathbb{L} B_\sigma)}{\sigma^2} 
\geq \int_{E_k} \left( \int_{v_2^{-1}(z) \cap (B_\rho \setminus B_\sigma)} \frac{1}{|z|^2} \langle \Omega, \overline{\Sigma}_j \rangle \, d\mathcal{H}^2 \right) \, d\text{vol}_{Y \times X}(y, z), \tag{6.7}
\]
\[
e^{-\left(\ell + \ell\rho^2/2\right)}(1 - A\rho) \frac{\mathcal{M}(T_k^j \mathbb{L} B_\rho)}{\rho^2} 
- e^{-\left(\ell + \ell\rho^2/2\right)}(1 - A\sigma) \frac{\mathcal{M}(T_k^j \mathbb{L} B_\sigma)}{\sigma^2} 
\leq \int_{E_k} \left( \int_{v_2^{-1}(z) \cap (B_\rho \setminus B_\sigma)} \frac{1}{|z|^2} \langle \Omega, \overline{\Sigma}_j \rangle \, d\mathcal{H}^2 \right) \, d\text{vol}_{Y \times X}(y, z), \tag{6.8}
\]
for every $0 < \sigma < \rho < 1$. The inequality \[6.7\] immediately implies that the function
\[
(0, 1) \ni \rho \mapsto e^{\ell + \ell\rho^2/2} (1 + A\rho) \frac{\mathcal{M}(T_j \mathbb{L} B_\rho)}{\rho^2}
\]
is monotonically non-decreasing. Thus, the density of $T^k_j$ at 0, given by

$$\theta(T^k_j, 0) := \lim_{\rho \to 0^+} \frac{\mathcal{M}(T^k_j \subseteq B_\rho)}{\rho^2} = \lim_{\rho \to 0^+} e^{A\rho + \rho^{1/2}} (1 + A\rho) \frac{\mathcal{M}(T^k_j \subseteq B_\rho)}{\rho^2}$$

exists and is finite.

We claim that $T^k_j$ has a unique tangent cone at the origin, for every given $k \in \mathbb{N}$. The fact that $T_j$ itself has a unique tangent cone at the origin will follow directly by the same method that is used in the conclusion of the previous subsection. By using (6.8), the fact that $\Omega$ is Lipschitz, point (3) in Lemma 5.3, the estimates (4.17) and (4.8), we get

$$\left| \frac{\mathcal{M}(T^k_j \subseteq B_\rho)}{\rho^2} - \theta(T^k_j, 0) \right| \leq C \int_{E_j} \left( \int_{v^{-1}_g(z) \cap B_\rho} \frac{1}{| \cdot |^2} (\Omega_\epsilon, \hat{\Sigma}_g) \, d\mathcal{H}^2 \right) \, d\text{vol}_Y \times_X (y, z)$$

$$\leq C \int_{E_j} \int_{v^{-1}_g(z) \cap B_\rho} \frac{|\Omega - \Omega_\epsilon|}{| \cdot |^2} \, d\mathcal{H}^2 \, d\text{vol}_Y \times_X (y, z)$$

$$+ C \int_{E_j} \left| \int_{v^{-1}_g(z) \cap B_\rho} \frac{\hat{\Sigma}_g}{| \cdot |^2} \, d\mathcal{H}^2 \right| \, d\text{vol}_Y \times_X (y, z)$$

$$+ C \int_{E_j} \left| \int_{v^{-1}_g(z) \cap B_\rho} \pi^* \omega_{\Sigma^g - 1} |\Sigma| \, d\mathcal{H}^2 \right| \, d\text{vol}_Y \times_X (y, z)$$

$$\leq C \text{vol}_Y \times_X (Y \times X) \rho^\alpha,$$

for every $\rho \in (0, \tilde{r})$, where the constant $C > 0$ and $\alpha, \tilde{r} \in (0, 1)$ all depend just on $k$ and on Lip$(\Omega)$. The fact that $T^k_j$ has a unique tangent cone at the origin than follows by standard arguments and step 1 is proved, due to the arbitrariness of $k \in \mathbb{N}$.  

**Step 2.** We claim $T$ has a unique tangent cone at the origin. A direct computation using the estimate in point (3) of Lemma 5.3 leads to

$$\mathcal{M}((T - T_j) \subseteq B_\rho) \leq \int_Y \int_X \int_{v^{-1}_g(z) \cap B_\rho} | \cdot |^{1/2} \, d\mathcal{H}^2 \, d\text{vol}_X (z) \, \text{vol}_Y (y).$$

Hence,

$$\frac{\mathcal{M}((T - T_j) \subseteq B_\rho)}{\rho^2} \leq \int_Y \int_X \int_{v^{-1}_g(z) \cap B_\rho} \frac{1}{| \cdot |^{3/2}} \, d\mathcal{H}^2 \, d\text{vol}_X (z) \, \text{vol}_Y (y). \quad (6.9)$$

By (4.8) (recall that $\gamma = 1/2$)), we get

$$\int_{v^{-1}_g(z) \cap B_\rho} \frac{1}{| \cdot |^{3/2}} \, d\mathcal{H}^2 \leq C \mathcal{H}^2(v^{-1}_g(z)) \rho^{1/2},$$

for every $\rho \in (0, 1)$ and for $\text{vol}_Y \times_X$-a.e. $(y, z) \in Y \times X$. By integrating the previous equality on $Y \times X$, (6.9) and (5.7), we get that

$$\frac{\mathcal{M}((T - T_j) \subseteq B_\rho)}{\rho^2} \leq C \left( \int_Y \int_B |v_g^* \text{vol}_X| \, d\text{vol}_g \right) \rho^{1/2} \leq C \left( \int_B |du|^2 \, d\text{vol}_g \right) \rho^{1/2}.$$
where $C > 0$ is a constant depending only on $m$, $n$ and $\text{Lip}(\Omega)$. This implies that the density of $T - T_J$ at 0, given by

$$\theta(T - T_J, 0) := \lim_{\rho \to 0^+} \frac{M((T - T_J) \mathbb{L} B_\rho)}{\rho^2} = 0$$

and there is a Morrey decrease of the mass ratio to the limiting density zero. Thus, $T - T_J$ has a unique tangent cone at the origin. Since by step 1 we know that $T_J$ has a unique tangent cone at the origin and $T = T_J + (T - T_J)$, our claim follows.

**Conclusion.** Notice that $T = (m - 2)!T_J$ and recall that $\pi^* \omega^m_{\text{CPm} - 1}$ is invariant under $\Phi^*_\rho$. We address the reader to [19, Section 7.2] for the definition of the standard operations "$\mathbb{L}$" and "$\wedge" when the arguments are a current and a form. Pick any two sequences of radii $\{\rho_k\}_{k \in \mathbb{N}} \subset (0, 1)$ and $\{\rho'_k\}_{k \in \mathbb{N}} \subset (0, 1)$ such that $\rho_k, \rho'_k \to 0^+$ as $k \to +\infty$ and

$$\begin{align*}
(\Phi_{\rho_k})_* T_u & \to C^*_\infty, \\
(\Phi_{\rho'_k})_* T_u & \to C'_\infty.
\end{align*}$$

Since

$$\langle (\Phi_\rho)_* T, \alpha \rangle = \langle T, (\Phi_\rho)^* \alpha \rangle = (m - 2)! \langle T_u \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}, (\Phi_\rho)^* \alpha \rangle$$

$$= (m - 2)! \langle T_u, \pi^* \omega^m_{\text{CPm} - 1} \wedge (\Phi_\rho)^* \alpha \rangle$$

$$= (m - 2)! \langle T_u, (\Phi_\rho)^*(\pi^* \omega^m_{\text{CPm} - 1} \wedge \alpha) \rangle$$

$$= (m - 2)! \langle (\Phi_\rho)_* T_u, \pi^* \omega^m_{\text{CPm} - 1} \wedge \alpha \rangle,$$

for every $\alpha \in D^2(B)$ and for every $\rho \in (0, 1)$, we get that

$$\begin{align*}
(\Phi_{\rho_k})_* T & \to (m - 2)! C^*_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1} \\
(\Phi_{\rho'_k})_* T & \to (m - 2)! C'_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}.
\end{align*}$$

Since the tangent cone to $T$ at the origin is unique, we conclude that

$$C^*_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1} = C'_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1},$$

which implies

$$C^*_\infty \pi^* \omega^m_{\text{CPm} - 1} \wedge \pi^* \omega^m_{\text{CPm} - 1} = (C'_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}) \wedge \pi^* \omega^m_{\text{CPm} - 1}. \quad (6.10)$$

Notice that

$$\begin{align*}
C^*_\infty &= (C^*_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}) \wedge \pi^* \omega^m_{\text{CPm} - 1} + (C^*_\infty \wedge \pi^* \omega^m_{\text{CPm} - 1}) \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}, \\
C'_\infty &= (C'_\infty \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}) \wedge \pi^* \omega^m_{\text{CPm} - 1} + (C'_\infty \wedge \pi^* \omega^m_{\text{CPm} - 1}) \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1}.
\end{align*}$$

Since $m \geq 3$ we have, by dimensional considerations, that

$$\begin{align*}
\pi_*(\langle C^*_\infty \wedge \pi^* \omega^m_{\text{CPm} - 1} \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1} \rangle) = 0, \quad (6.11) \\
\pi_*(\langle C'_\infty \wedge \pi^* \omega^m_{\text{CPm} - 1} \mathbb{L} \pi^* \omega^m_{\text{CPm} - 1} \rangle) = 0. \quad (6.12)
\end{align*}$$

Thus, by (6.10), (6.11) and (6.12), we get $\pi_* C^*_\infty = \pi_* C'_\infty$. Since $C^*_\infty$ and $C'_\infty$ are $J_0$-holomorphic cones, we get $C^*_\infty = C'_\infty$ and the statement of Theorem 1.2 follows.
6.3. Recovering uniqueness of tangent maps for $u$.

The case $n = 1$. Let $m \geq 3$ and let $u \in W^{1,2}(B^{2m}, \CP^1)$ be weakly $(J, j_1)$-holomorphic and locally approximable. By the methods that we have introduced in the previous subsection, it follows that uniqueness of tangent cone holds for every $(2m - 2)$-dimensional current $T_{u, \psi}$ of the form

$$
\langle T_{u, \psi}, \alpha \rangle := \int_{B^{2m}} u^*(\psi \omega_{\CP^1}) \wedge \alpha \quad \forall \alpha \in \mathcal{D}^{2m - 2}(B),
$$

with $\psi \in C^\infty(\CP^1)$.

Pick any two sequences of radii $\{\rho_k\}_{k \in \mathbb{N}} \subset (0, 1)$ and $\{\rho'_k\}_{k \in \mathbb{N}} \subset (0, 1)$ such that $\rho_k, \rho'_k \to 0^+$ as $k \to +\infty$ and

$$
u_{\rho_k} \rightharpoonup \nu, \quad \nu'_{\rho'_k} \rightharpoonup \nu',
$$

weakly in $W^{1,2}(B, \CP^1)$. By uniqueness of tangent cone for $T_{u, \psi}$ we get immediately that

$$
\int_{B^{2m}} u_{\rho_k}^*(\psi \omega_{\CP^1}) \wedge (\varphi \Omega_0) = \int_{B^{2m}} (u'_{\rho'_k})^*(\psi \omega_{\CP^1}) \wedge (\varphi \Omega_0),
$$

for every $\psi \in C^\infty(\CP^1)$, $\varphi \in C^\infty_c(B)$. Since both $u_{\rho_k}$ and $u'_{\rho'_k}$ are weakly $(J_0, j_1)$-holomorphic, by the coarea formula and by Corollary 2.1 we get

$$
\int_{\CP^1} \psi(y) \left( \int_B \varphi \chi_{u_{\rho_k}^{-1}(y)} d\mathcal{H}^{2m - 2} \right) \vol_{\CP^1}(y) \to \int_{\CP^1} \psi(y) \left( \int_B \varphi \chi_{u'_{\rho'_k}^{-1}(y)} d\mathcal{H}^{2m - 2} \right) \vol_{\CP^1}(y)
$$

for every $\psi \in C^\infty(\CP^1)$, $\varphi \in C^\infty_c(B)$. Hence,

$$
\int_{\CP^1} \psi(y) \left( \int_B \varphi \left( \chi_{u_{\rho_k}^{-1}(y)} - \chi_{(u'_{\rho'_k})^{-1}(y)} \right) d\mathcal{H}^{2m - 2} \right) \vol_{\CP^1}(y) = 0
$$

for every $\psi \in C^\infty(\CP^1)$, $\varphi \in C^\infty_c(B)$. This implies that for $\vol_{\CP^1}$-a.e. $y \in \CP^1$ the sets $u_{\rho_k}^{-1}(y)$ and $(u'_{\rho'_k})^{-1}(y)$ coincide up to $\mathcal{H}^{2m - 2}$-negligible sets. We conclude that $u_{\rho_k} = u'_{\rho'_k} \mathcal{L}^{2m}$-a.e. on $B$ and the statement of Theorem 1.1 follows.

The case $n > 1$. Let $m \geq 3, n \geq 2$ and let $u \in W^{1,2}(B^{2m}, \CP^n)$ be weakly $(J, j_n)$-holomorphic and locally approximable. By the methods that we have introduced in the previous subsection, it follows that uniqueness of tangent cone holds for every $(2m - 2)$-dimensional current $T_{u, \psi}$ of the form

$$
\langle T_{u, \psi}, \alpha \rangle := \int_{B^{2m}} (F_{q_1} \circ \cdots \circ F_{q_{n-1}} \circ u)^*(\psi \omega_{\CP^1}) \wedge \alpha \quad \forall \alpha \in \mathcal{D}^{2m - 2}(B),
$$

with $\psi \in C^\infty(\CP^n)$ and for every choice of $(q_1, \ldots, q_{n-1}) \in \CP^2 \times \cdots \times \CP^n$.

Pick any two sequences of radii $\{\rho_k\}_{k \in \mathbb{N}} \subset (0, 1)$ and $\{\rho'_k\}_{k \in \mathbb{N}} \subset (0, 1)$ such that $\rho_k, \rho'_k \to 0^+$ as $k \to +\infty$ and

$$
u_{\rho_k} \rightharpoonup \nu, \quad \nu'_{\rho'_k} \rightharpoonup \nu',
$$

weakly in $W^{1,2}(B, \CP^n)$.
Remark. The advantage of the previous approach relies in the fact we don’t get only uniqueness of tangent cone for the current $T_u$ but also for its "localizations" $T_{u,\psi}$ (see the beginning of subsection 6.3) through smooth functions $\psi \in \mathcal{C}^\infty(\mathbb{CP}^n)$. This allows more flexibility and we would like to drag the attention of the reader on the fact we could exploit such flexibility in order to get a new proof the result in [3]. Given an integer $(p,p)$-cycle $\Sigma \subset B^{2m}$, we could consider a weakly holomorphic and locally approximable map $u \in W^{1,2}(B^{2m},\mathbb{CP}^{m-p})$ such that $u(\Sigma) = \{y\} \subset \mathbb{CP}^{m-p}$. By localizing the associated cycle $T_u$ through a sequence $\{\psi_\epsilon\} \subset \mathcal{C}^\infty(\mathbb{CP}^{m-p})$ such that $\psi_\epsilon \rightarrow \delta_y$ in $\mathcal{D}'(\mathbb{CP}^{m-p})$, we could exploit our techniques to get uniqueness of tangent cone for $\Sigma$ ultimately.

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Appendix A. Slicing through singular meromorphic maps

Let $m \in \mathbb{N}$ be such that $m \geq 3$ and fix any point $p \in \mathbb{CP}^{m-1}$. As usual, let $\pi : \mathbb{C}^m \setminus \{0\} \rightarrow \mathbb{CP}^{m-1}$ be the quotient map given by

$$\pi(z_1, \ldots, z_m) := [z_1; \ldots; z_m], \quad \forall z \in \mathbb{C}^m \setminus \{0\}.$$ 

Denote by $L_p$ the complex line generated by $p$ in $\mathbb{CP}^{m-1}$ and consider the map $T_p : \mathbb{C}^m \setminus L_p \rightarrow L_p^+ \setminus \{0\}$ given by the restriction to $\mathbb{C}^m \setminus L_p$ of the standard orthogonal projection from $\mathbb{C}^m$ into $L^+_p$. Fix a complex orthonormal basis $\{e^p_1, \ldots, e^p_{m-1}\}$ of $L^+_p$ and let $\varphi_p : L^+_p \rightarrow \mathbb{C}^{m-1}$ be the following linear isomorphism:

$$\varphi_p \left( \sum_{j=1}^{m-1} \alpha_j e^p_j \right) := (\alpha_1, \ldots, \alpha_{m-1}), \quad \forall (\alpha_1, \ldots, \alpha_{m-1}) \in \mathbb{C}^{m-1}.$$ 

Let $\pi_p : L^+_p \setminus \{0\} \rightarrow \mathbb{CP}^{m-2}$ be the smooth submersion given by $\pi_p := \tilde{\pi} \circ \varphi_p$, where

$$\tilde{\pi}(\alpha_1, \ldots, \alpha_{m-1}) := [\alpha_1; \ldots; \alpha_{m-1}], \quad \forall (\alpha_1, \ldots, \alpha_{m-1}) \in \mathbb{C}^{m-1} \setminus \{0\}.$$ 

Eventually, notice that the map $F_p : \mathbb{CP}^{m-1} \setminus \{p\} \rightarrow \mathbb{CP}^{m-2}$ given by

$$F_p([z_1; \ldots; z_m]) = (\pi_p \circ T_p)(z_1, \ldots, z_m), \quad \forall [z_1, \ldots, z_m] \in \mathbb{CP}^{m-1} \setminus \{p\},$$ 

is well-defined and smooth, since the map $\pi_p \circ T_p$ is constant on the fibres of $\pi$.

Lemma A.1. Let $m \in \mathbb{N}$ be such that $m \geq 3$. Then, for every $p \in \mathbb{CP}^{m-1}$ the following facts hold:

1. the map $F_p \circ \pi$ belongs to $W^{1,2m-4}(B, \mathbb{CP}^{m-2})$;
2. $F_p \circ \pi$ is weakly $(J_0, j_{m-2})$-holomorphic, where $j_{m-2}$ is the standard complex structure on $\mathbb{CP}^{m-2}$.
(3) $F_p \circ \pi$ is such that $d((F_p \circ \pi)^* \text{vol}_{\mathbb{CP}^{m-2}}) = 0$, distributionally on $B$.

Proof. Fix any $p \in \mathbb{CP}^{m-1}$ and notice that the complex line $L_p$ is indeed a real 2-plane in $\mathbb{R}^{2m}$. Thus, $\mathcal{K}^{2m-\alpha}(L_p \cap B) = 0$, for every $\alpha \in [1, 2m - 2)$. Hence, $L_p \cap B$ has vanishing $(2m - 4)$-capacity. Since $F_p \circ \pi \in L^\infty(B) \cap C^\infty(B \setminus L_p)$ and the classical differential of $F_p \circ \pi$ on $B \setminus L_p$ can be estimated by

$$|d(F_p \circ \pi)| = |d(\pi_p \circ T_p)| \leq |\wedge_2 d\pi_p \circ T_p| \leq \frac{C}{\text{dist}(\cdot, L_p)},$$

we obtain that $d(F_p \circ \pi) \in L^{2m-4}(B \setminus L_p)$. Point (1) immediately follows.

For what concerns (2), we know that the weak differential of $F_p \circ \pi$ coincides $\mathcal{L}^{2m}$-a.e. with its classical differential on $B \setminus L_p$, where $F_p \circ \pi$ is smooth. Moreover $F_p \circ \pi = \pi_p \circ T_p$ on $B \setminus L_p$. Since both $\pi_p$ and $T_p$ are holomorphic maps, then $F_p \circ \pi$ is holomorphic on $B \setminus L_p$. Then, since $B \setminus L_p$ has full $\mathcal{L}^{2m}$-measure in $B$, the fact that the weak differential of $F_p \circ \pi$ commutes with the complex structures $J_0$ and $j_m - 2$ for $\mathcal{L}^{2m}$-a.e. $x \in B$ follows and we have proved (2).

We are just left to prove (3). Fix any $\alpha \in \mathcal{D}^3(B^{2m})$. For every any $\varepsilon > 0$ we define

$$L_p^\varepsilon := (L_p + B_\varepsilon(0)) \cap B^{2m}$$

and we notice that

$$\left| \int_{B^{2m}} (F_p \circ \pi)^* \text{vol}_{\mathbb{CP}^{m-2}} \wedge d\alpha \right| = \left| \int_{L_p} (F_p \circ \pi)^* \text{vol}_{\mathbb{CP}^{m-2}} \wedge d\alpha \right|$$

$$\leq \int_{L_p} |d\alpha| |(F_p \circ \pi)^* \text{vol}_{\mathbb{CP}^{m-2}}| d\mathcal{L}^{2m}$$

$$\leq ||d\alpha||_{L^\infty} \int_{L_p^\varepsilon} |d(F_p \circ \pi)|^{2m-4} d\mathcal{L}^{2m}$$

$$\leq ||d\alpha||_{L^\infty} ||d(F_p \circ \pi)||_{L^{2m-4}(L_p^\varepsilon)}^{2m-4/q'} \to 0$$

as $\varepsilon \to 0^+$, where $q := (2m - 3)/(2m - 4)$ and $q' = 2m - 3$ is the conjugate exponent of $q$.

By arbitrariness of $\alpha \in \mathcal{D}^3(B^{2m})$, point (3) follows. $\square$

Lemma A.2. For every $m \geq 3$, there exists a constant $B_m > 0$ such that

$$\omega_{\mathbb{CP}^{m-1}} = B_m \int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}} \, dp.$$

Proof. Throughout this proof, given any $m \geq 1$ and a unitary matrix $A \in U(m)$, we will denote by $\tilde{A} : \mathbb{CP}^{m-1} \to \mathbb{CP}^{m-1}$ the map $[z] \mapsto [Az]$.

It is well known that, up to rescalings by constant factors, the Fubini-Study metric is the only $U(m)$-invariant symplectic form on $\mathbb{CP}^{m-1}$, for every $m \geq 2$. Thus, it is enough to show that

$$\tilde{A}^* \left( \int_{\mathbb{CP}^{m-1}} F_p^* \omega_{\mathbb{CP}^{m-2}} \, dp \right) = \int_{\mathbb{CP}^{m-1}} F_{p'}^* \omega_{\mathbb{CP}^{m-2}} \, dp, \quad \forall A \in U(m).$$

Fix any $A \in U(m)$. Given $p \in \mathbb{CP}^{m-1}$, define

$$B_p := \varphi_p \circ T_p \circ A \circ S_p \circ \varphi_p^{-1} : \mathbb{CP}^{m-1} \to \mathbb{CP}^{m-1},$$

where $S_p : L_p^\perp \to \mathbb{C}^m$ is the left inverse of the orthogonal projection map $T_p : \mathbb{C}^m \to L_p^\perp$. As composition of linear and unitary maps, $B_p \in U(m - 1)$. Moreover, by construction it
holds that $F_p \circ \bar{A} = \bar{B}_p \circ F_p$.

Hence, by linearity of the integral and the definition of $F_p$, we have

$$\bar{A}^\star \left( \int_{\mathbb{CP}^{m-1}} F_p^\star \omega_{\mathbb{CP}^{m-2}} \, dp \right) = \int_{\mathbb{CP}^{m-1}} \bar{A}^\star F_p^\star \omega_{\mathbb{CP}^{m-2}} \, dp$$

$$= \int_{\mathbb{CP}^{m-1}} (F_p \circ \bar{A})^\star \omega_{\mathbb{CP}^{m-2}} \, dp$$

$$= \int_{\mathbb{CP}^{m-1}} (\bar{B} \circ F_p)^\star \omega_{\mathbb{CP}^{m-2}} \, dp$$

$$= \int_{\mathbb{CP}^{m-1}} F^\star \bar{B}^\star \omega_{\mathbb{CP}^{m-2}} \, dp$$

$$= \int_{\mathbb{CP}^{m-1}} F^\star \omega_{\mathbb{CP}^{m-2}} \, dp$$

and the statement follows by arbitrariness of $A \in U(m)$. \hfill \Box

**Lemma A.3.** Let $m \geq 3$ and pick any two points $x, y \in \mathbb{CP}^{m-1}$. For every $j = 1, \ldots, m$, let $\tilde{e}_j := \pi(e_j) \in \mathbb{CP}^{m-1}$, where $\{e_1, \ldots, e_m\}$ denotes the standard complex euclidean basis of $\mathbb{C}^m$. Assume that

$$F_{\tilde{e}_j}(x) = F_{\tilde{e}_j}(y), \quad \forall \ j = 1, \ldots, m. \quad (A.3)$$

Then, $x = y$.

**Proof.** Let $x = [x_1; \ldots; x_m]$ and $y = [y_1; \ldots; y_m]$. Fix any $j = 1, \ldots, m$. By definition of $F_{\tilde{e}_j}$, the condition $F_{\tilde{e}_j}(x) = F_{\tilde{e}_j}(y)$ implies that there exists $\lambda_j \in \mathbb{C} \setminus \{0\}$ such that

$$\lambda_j x_i = y_i, \quad \forall \ i = 1, \ldots, m \text{ with } i \neq j.$$

Hence, by enforcing (A.3) we get that there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$\lambda x_i = y_i, \quad \forall \ i = 1, \ldots, m.$$

This implies $x = y$ in $\mathbb{CP}^{m-1}$ and the statement follows. \hfill \Box

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