Framed 4-valent Graph Minor Theory II: Special Minors and New Examples

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Abstract

In the present paper, we proceed the study of framed 4-graph minor theory initiated in [8] and justify the planarity theorem for arbitrary framed 4-graphs; besides, we prove analogous results for embeddability in $\mathbb{R}P^2$.

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1 Introduction. Basic Notions

Some years ago, a milestone in graph theory was established: as a result of series of papers by Robertson, Seymour (and later joined by Thomas) [10] proved the celebrated Wagner conjecture [12] which stated that if a class of graphs (considered up to homeomorphism) is minor-closed (i.e., it is closed under edge deletion and edge contraction), then it can be characterized by a finite number of excluded minors. For a beautiful review of the subject we refer the reader to L.Lovász [4]. This conjecture was motivated by various evidences for concrete natural minor-closed properties of graphs, such as knotless or linkless embeddability in $\mathbb{R}^3$, planarity or embeddability in a standardly embedded $S_g \subset \mathbb{R}^3$. Framed 4-valent graphs (see definition below) are a very important class of graphs which arise as medial graphs of arbitrary graphs drawn on 2-surfaces. In some sense, they approximate arbitrary graphs; in particular, a new proof of the Pontrjagin-Kuratowski planarity criterion was recently found by Nikonov [9]. The two smoothing operations for framed 4-valent medial graph $M$ of a

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In the present paper, we proceed the study of minor closed properties for framed 4-valent graphs and go on proving theorems that for some minor closed properties the number of minimal minor obstructions is finite. Here a property is called minor closed if whenever \( P' \) is a minor of \( P \) and \( P \) possesses the property, so does \( P' \); a graph \( Q \) is called a minimal minor obstruction for some property if \( Q \) does not possess the desired property and all minors of \( Q \) do; in [8] and in the present paper we deal with various definitions of minors; they lead to definitions of minor closed properties.

In [8], we introduced the class of framed 4-graphs, 4-valent graph minor theory and proved a planarity criterion for framed 4-graphs admitting a source-sink structure (see below). Whenever drawing a framed four-valent graph on the plane, we shall indicate its vertices by solid dots, (self)intersection points of edges will be encircled, and the framing is assumed to be induced from the plane: those half-edges which are drawn opposite in \( \mathbb{R}^2 \) are thought to be opposite. Half-edges of a framed four-valent graph incident to the same vertex are which are not opposite, are called adjacent. According to the main theorem of [8], the only graph which was an obstruction to planarity for framed 4-valent graphs, is the \( \Delta \)-graph, see Fig. ??.

This graph \( \Delta \), in turn, successfully turns out to be the unique obstruction (unique forbidden minor) for immersibility in \( \mathbb{R}^2 \) with no more than 2 crossings, for linkless embeddability in \( \mathbb{R}^3 \) (in a proper sense). However, all these properties hold upon an important obstruction imposed on framed 4-graphs:
Figure 2: Edge deletion and edge contraction yield smoothing the existence of a source-sink structure.

We recall that a regular 4-graph is called framed if each vertex of it is endowed with a framing: for the four emanating edges, we indicate two pairs of opposite edges. Non-opposite edges are called adjacent. Besides graphs in the proper sense we also allow 4-graphs to have circular components.

At every vertex $V$ of a framed 4-graph $\Gamma$, there are two ways of pairing the four edges into two pairs of adjacent ones. For each of these two pairings, we define the smoothing of $\Gamma$ at $V$ as the graph obtained by breaking $\Gamma$ at $\times$, and pasting together one pair of the four edges edges into one edge and the other pair of edges into the other edge; $\times \rightarrow \Box$ and $\times \rightarrow \Diamond$.

**Definition 1.** A source-sink structure of a framed 4-graph is an orientation of all its circular edges together with an orientation of all its non-circular edges such that at every crossing two opposite half-edges are incoming, and the other two are emanating.

**Remark 1.** Whenever talking about an embedding or an immersion of a framed 4-graph into any 2-surface we always assume its framing to be preserved: opposite edges at every crossing should be locally opposite on the surface.

It follows obviously from the definition that for a connected framed 4-graph there exist no more than two source-sink structures: starting with an orientation of an edge, we can orient all edges of the corresponding connected component.

Moreover, the smoothing operation at crossings, agrees with the source-sink orientation: if the initial graph admits a source-sink structure, then every smoothing of it inherits this source-sink structure, see Fig.3.
Definition 2. A framed 4-valent graph $G'$ is a minor of a framed 4-valent graph $G$ if $G'$ can be obtained from $G$ by a sequence of smoothing operations ($\times \rightarrow \otimes$ and $\otimes \rightarrow \bigcirc$) and deletions of connected components.

Now, let us look at the planarity problem. There exists a very simple one-vertex graph $\Gamma$ having one vertex $X$ and two half-edges $a$ and $b$ both adjacent to $X$ and each being opposite to itself at $X$, see Fig. 4.

Note that $X$ admits no source-sink structures, thus, $X$ is not a minor of any framed 4-graph admitting a source-sink structure.

This framed 4-graph is obviously non-planar: we have two cycles with exactly one transverse intersection point [5].

Besides, we can see in Fig. 5 that $\Gamma$ sits inside $\Delta$: in Fig. 5 $\Gamma$ is drawn as a subgraph of $\Delta$ in red.

So, $\Delta$ is a non-planar graph because of the graph $\Gamma$ “sitting inside” it.
Thus, besides minors there should be another (more general) notion expressing that \( \Gamma \) “sits inside \( \Delta \)”. The notion described below will not be local. Let us now introduce \( s \)-minors of framed 4-graphs.

**Definition 3.** A framed 4-graph \( \Gamma' \) is an \( s \)-minor of a framed 4-graph \( \Gamma \) if it can be obtained from \( \Gamma \) by a sequence of two operations:

1. Passing to a subgraph with all vertices of even valency and deleting all vertices of valency 2;
2. Removing connected components.

**Remark 2.** By definition, if \( P \) is a minor of \( Q \), then \( P \) is an \( s \)-minor of \( Q \); the inverse statement is wrong: \( \Gamma \) is an \( s \)-minor of \( \Delta \), but is not a minor of \( \Delta \).

As in [8], we shall use a way of coding framed 4-graphs by chord diagrams. Unlike [8], we have to encode all framed 4-graphs, which will require a more general notion of a framed chord diagram.

**Definition 4.** By a rotating circuit of a connected framed 4-graph not homeomorphic to a circle we mean a surjective map \( f : S^1 \rightarrow \Gamma \) which is a bijection everywhere except preimages of crossings of \( \Gamma \) such that at every crossing \( X \) the neighbourhoods \( V(Y_1) \) and \( V(Y_2) \) of the two preimages \( Y_1, Y_2 \) of \( X \) belong to unions of adjacent half-edges each. In other words, the circuit “passes” from a half-edge to a non-opposite half-edge.
For a framed 4-graph homeomorphic to the circle, the circuit is a homeomorphism of the circle and the graph.

A circuit \( f \) is called good at a vertex \( X \) of \( P \) if for the two inverse images \( Y_1, Y_2 \in S^1 \) of \( X \), the neighbourhoods of the small segments \( (Y_1 - \varepsilon, Y_1] \) and \( (Y_2 - \varepsilon, Y_2] \) of the circle are taken by \( f \) to a pair of opposite half-edges at \( X \).

Otherwise the rotating circuit is called bad at \( X \).

The rotating circuit is good if it is good at every vertex.

Rotating circuits play a crucial role in the study of embeddings of framed 4-valent graphs, see [5, 6, 7, 3].

An easy exercise (see, e.g. [5]) shows that every connected framed 4-graph admits a rotating circuit.

Usually, we shall denote a circuit by a small letter (say, \( f \)) when we want to consider it as a map, and by a capital letter (say, \( C \)) when we want to deal with its image as a subgraph.

Remark 3. It follows from the definition that if a connected non-circular framed 4-graph \( P \) admits a source-sink structure then every rotating circuit of \( P \) is good at every vertex.

The opposite statement is true as well; moreover, if there exists one rotating circuit which is good at every vertex, then the framed 4-graph \( P \) admits a source-sink structure.

Definition 5. By a chord diagram we mean either an oriented circle (empty chord diagram) or a cubic graph \( D \) consisting of an oriented cycle (the core) passing through all vertices of \( D \) such that the complement to it is a disjoint union of edges (chords) of the diagram.

A chord diagram is framed if every chord of it is endowed with a framing 0 or 1.

Definition 6. We say that two chords \( a, b \) of a chord diagram \( D \) are linked if the ends of the chord \( b \) belong to two different components of the complement \( Co \setminus \{a_1, a_2\} \) to the endpoints of \( a \) in the core circle \( Co \) of \( D \).

Having a circuit \( C \) of a framed connected 4-graph \( G \), we define the framed chord diagram \( DC(G) \), as follows. If \( G \) is a circle, then \( DC(G) \) is empty. Otherwise, think of \( C \) as a map \( f : S^1 \to D \); then we mark by points on \( S^1 \) preimages of vertices of \( G \). Thinking of \( S^1 \) as a core circle and connecting the preimages by chords, we get the desired cubic graph.

The framing of good vertices is set to be equal to 0, the framing of bad vertices is set to be equal to 1.

Remark 4. (Framed) chord diagrams are considered up to combinatorial equivalence.

The opposite operation (of restoring a framed 4-graph from a chord diagram) is obtained by removing chords from the chord diagram and approaching two endpoints of each chord towards each other. For every chord, we create a crossing, and for every chord with framing zero, we create a small twist, as shown in Fig. 6.
Figure 6: Restoring a framed 4-graph from a chord diagram

**Definition 7.** A (framed) chord diagram $D'$ is called a subdiagram of a chord diagram $D$ if $D$ can be obtained from $D$ by deleting some chords and their endpoints (with framing respected).

It follows from the definition that the removal of a chord from a framed chord diagram results in a smoothing of a framed 4-graph. Consequently, if $D'$ is a subdiagram of $D$, then the resulting framed 4-graph $G(D')$ is a minor of $G(D)$.

## 2 The Planarity Criterion

Note that planarity is a minor closed and s-minor closed property; thus, it makes sense to look for minimal planarity obstructions.

When dealing with all framed 4-graphs (not necessarily admitting a source-sink structure), we obtain the following

**Theorem 1.** A graph $P$ is non-planar if and only if it admits either $\Gamma$ or $\Delta$ as a minor.

Alternatively, $P$ is non-planar if and only if it admits $\Gamma$ as an $s$-minor.

**Proof.** This proof goes along the lines of [3]: the second statement of the theorem actually repeats the main statement of [3]: a framed 4-graph is non-planar iff it contains two cycles sharing no edges and having exactly one transverse intersection point. These two cycles form exactly an $s$-minor $\Gamma$ inside $P$.

Let us now prove the first statement of the theorem. If a graph $P$ admits a source-sink structure then it admits $\Delta$ as a minor, as proved in [3]. Otherwise, let us construct a rotating circuit and a chord diagram of $P$. By definition, this chord diagram will have at least one chord of framing one. This means that $G(D_1)$ is a minor of $P$, where $D_1$ is the chord diagram with the unique chord of framing 1. But one can easily see that $G(D_1)$ is isomorphic to $\Gamma$.

This exactly means that $P$ contains $\Gamma$ as a minor. $\square$
3 Checkerboard Embeddings and $\mathbb{R}P^2$

Definition 8. An embedding of a graph $P$ in a 2-surface $\Sigma$ is cellular if the complement $\Sigma \setminus P$ is a union of 2-cells.

When talking of embeddings, one usually deals with cellular ones. For example, when defining the minimal embedding genus for a given graph $P$, one certainly means the genus of a cellular embedding. Nevertheless, the cellular embeddability into a surface of a given genus $g$ is not a minor closed property (it is not so for arbitrary graphs and minors in the usual sense; neither it is so for framed 4-graphs); the reason is that if $P$ is embeddable into $\Sigma$, then it yields an embedding of any minor $P'$ of $P$ into $\Sigma$; however, it may well happen that the complement to the image $P$ is a union of 2-cells, whence the complement to the image of $P'$ is not.

Thus, we shall not restrict ourselves to just cellular embeddings; thus, for instance, every planar framed 4-graph is embeddable into any 2-surface.

Having a framed 4-graph $P$ and a 2-surface $\Sigma$, we may consider embeddings of $P$ into $\Sigma$. Among all embeddings, we draw special attention to checkerboard embedding.

Definition 9. A checkerboard embedding $f : P \to \Sigma$ is an embedding such that the connected components of the complement $\Sigma \setminus P$ can be colored in black and white in a way such that every two components sharing an edge have different colours.

One can easily see that checkerboard embeddability into any fixed 2-surface $\Sigma$ is a minor closed property: if the complement to the image of a framed 4-graph $P$ is checkerboard colourable, then so is the complement to the image of $P'$, where $P'$ is obtained from $P$ by a smoothing at a vertex. Certainly, the connected components to the image of $P'$ might not be homeomorphic to 2-cells.

For more about checkerboard embeddings of graphs, see [1].

Note that in the case when $\Sigma$ is $\mathbb{R}^2$ (or $S^2$), all embeddings are checkerboard. It can be shown ([6]) that an embedding is checkerboard if and only if the image of the graph viewed as an element of $H_1(\Sigma, \mathbb{Z}_2)$ is zero.

Theorem 2 ([6]). If a framed 4-graph $P$ admits a source-sink structure and a cellular checkerboard embedding into a closed 2-surface $\Sigma$, then $\Sigma$ is orientable. If $P$ admits no source-sink structure but admits a cellular checkerboard embedding into $\Sigma$ then $\Sigma$ is not orientable.

It turns out (see [6]) that checkerboard embeddings are very convenient to deal with: they lead to a splitting of the surface into the black part and the white part, as follows.

Given a framed 4-graph $P$; let us consider a rotating circuit $C$ of it. A checkerboard embedding $g : P \to \Sigma$ leads to the composite map $f \circ g : S^1 \to \Sigma$; this map is bijective everywhere except those points mapped to images of
crossings of $P$. At every crossing, we can slightly deform the map $f \circ g$ to get an embedding.

Denote the resulting embedding by $f'$. Note that $f(S^1)$ splits the surface $\Sigma$ into the black part $\Sigma_B$ and the white part $\Sigma_W$. Moreover, chords of the chord diagram $D_C(P)$ can be naturally thought of as “black” ones and “white” ones: small segments in neighbourhoods of vertices shown in Fig. 7 can be thought of as images of chords of the chord diagram.

Now, if the surface $\Sigma$ is homeomorphic to $\mathbb{R}P^2$ then one of the two parts $\Sigma_B, \Sigma_W$ is homeomorphic to the Möbius band, and the other part is homeomorphic to the disc.

This means that the chord diagram $D_C(P)$ has a very specific form. Namely, in [6], the following theorem is proved.

**Theorem 3.** A framed 4-graph $P$ is embeddable in $\mathbb{R}P^2$ if and only if the for some rotating circuit $C$ there is a way to split all chords of $D_C(P)$ into two families in such a way that the resulting subdiagrams $D_1$ and $D_2$ are as follows:

1. All chords of $D_1$ having framing 0 are pairwise unlinked;
2. All chords of $D_2$ of framing 1 are pairwise linked; all chords of framing 0 of $D_2$ are pairwise unlinked with all other chords of $D_2$.

In [6], it is shown that if the condition of Theorem 3 holds for some rotating circuit of $P$ then it holds for any rotating circuit of $P$. The point is that if the condition of Theorem 3 holds for some rotating circuit, then this means that one family of chords leads to the subdiagram which corresponds to the disc, and the other family leads to the subdiagram which corresponds to the Möbius band. For more details, see [6].

Now, let us denote by $\hat{D}$ the framed chord diagram with two unlinked chords of framing 1; let $\Gamma_1$ be the corresponding framed 4-graph.
Now, we are ready to formulate the main theorem of the present section.

**Theorem 4.** A framed 4-graph $P$ is not checkerboard-embeddable in $\mathbb{R}P^2$ if and only if it contains one of the subgraphs $\Delta, \Gamma_1$ as a minor.

More precisely, if $P$ admits a source-sink structure then checkerboard-embeddability in $\mathbb{R}P^2$ is equivalent to checkerboard embeddability into $\mathbb{R}^2$. If $P$ does not admit a source-sink structure then the only obstruction to checkerboard embeddability into $\mathbb{R}P^2$ is the existence of $\Gamma_1$ as a minor.

**Proof.** Note that if a checkerboard embedding of $\iota : P \to \mathbb{R}P^2$ is not cellular, then one component of the complement $\mathbb{R}P^2 \setminus \iota(P)$ is homeomorphic to the Möbius band without boundary; removing this Möbius band and pasting its boundary component by a disc, we see that $P$ is actually planar.

Thus, according to Theorem 2 if $P$ admits a checkerboard embedding to $\mathbb{R}P^2$ then $P$ is either planar or it does not admit a source-sink structure.

We know that if $P$ admits a source-sink structure then $\Delta$ is the only planarity obstruction.

Now, assume $P$ does not admit a source-sink structure. Take a rotating circuit $C$ and consider a framed chord diagram $D_C(P)$.

According to our assumption, this chord diagram has at least one chord of framing 1.

Let us look at the obstruction from Theorem 2. We shall try to split all chords into two families $D_1$ and $D_2$ and see when it is impossible.

Let $H$ be the following chord diagram graph: vertices of $H$ are in one-to-one correspondence with chords of $D_C(P)$, and two vertices are connected by an edge if either one of the corresponding chords has framing zero and the chords are linked or both chords have framing 1 and they are unlinked.

It is easy to see that the existence of two families $D_1$ and $D_2$ as in Theorem 2 means exactly that $H$ is bipartite. The obstruction for $H$ to be bipartite is an existence of an odd cycle. Consider such a cycle with the smallest possible number of vertices; denote them subsequently by $v_1, \ldots, v_{2k+1}$ and the corresponding chords $c_1, \ldots, c_{2k+1}$ of $D_C(P)$. If all these chords have framing 0, then the corresponding diagram has a $(2k+1)$-gon as a subdiagram; hence, $P$ has $\Delta$ as a minor.

Now, assume there is at least one chord of framing 1 in the cycle $v_1, \ldots, v_{2k+1}$. If there are 3 chords of framing one among $c_j$, one can easily find a shorter cycle with the same property. Thus, there are either exactly two chords of framing 1 or exactly one chord of framing 1. If we have two chords of framing 1 and they are linked, we may find a shorter cycle in $H$ with the required property.

If we have two unlinked chords of framing 1, then they form a subdiagram $D'$ such that the minor corresponding to $D'$ is isomorphic to $\Gamma_1$.

Thus, it remains to consider the case when we have exactly one chord of framing 1 in our odd cycle. Without loss of generality, assume the only chord of framing 1 is $c_1$; consider the chords $c_2$ and $c_3$ are linked chords of framing 0.

Consider the framed 4-graph $P_{2k+1}$ corresponding to this cycle; it is a minor of $P$. Let us show now that $\Gamma_1$ is a minor of $P_{2k+1}$. First, we shall show that
$P_{2k-1}$ is a minor of $P_{2k+1}$ for every $k \geq 1$ in a way similar to the proof of Theorem[1]. Finally, we shall show that $\Gamma_1$ is a minor of $P_3$. Denote the chords of $P_{2k+1}$ by the same letters as those of $P$.

At each chord of framing 0, there are two ways of smoothing of the corresponding vertex: one way gives rise to the graph corresponding to the subdiagram obtained from the initial diagram by deleting the chord, and the other.

Let us change the rotating circuit of $P_{2k+1}$ at vertices corresponding to $v_2, v_3$.

By abuse of notation, denote by $c_j$ the chord of the new circuit corresponding to the vertex which corresponds to $c_j$ in the initial circuit. Then we see that in the chain formed by all chords except $c_2, c_3$ the incidences changes only for the pair $(c_1, c_4)$; thus, we a $(2k-1)$-gon.

Here we see the $(2k-1)$-gon which shows that $P_{2k-1}$ is a minor of $P_{2k+1}$.

Finally, if we look at $P_3$ and change the circuit at the vertex $v_1$ as shown in Fig. 8 we see that $P_3$ contains $\Gamma_1$ as a minor.

This completes the proof of the Theorem.

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