Introduction: Quantum measurements, in contradistinction to measurements in the classical regime, have a dual nature; they reveal information pertaining to a property of the quantum system being measured, while at the same time dynamically altering the quantum system’s state [1, 2]. The dynamical aspect of quantum measurements has found utility in quantum technologies, such as information-processing in measurement-based quantum computation [3, 4], inducing quenched dynamics in many-body systems [5], and powering quantum thermal machines [6–10].

The thermodynamic analysis of quantum measurements has been a key focus of the emerging field of quantum thermodynamics [11–20]. While the “average” thermodynamic properties of measurements are well understood, a full thermodynamic analysis of the measurement process requires a method for evaluating higher moments. Such an analysis is made possible by the framework of stochastic quantum thermodynamics [21–29], whereby the measurement process is decomposed into an ensemble of “quantum trajectories”\(\gamma := (m, x, n)\), with probabilities \(p(\gamma)\) induced by the Born rule. Here, \(x\) denotes the measurement outcome, while \(m\) and \(n\) label the eigenstates of the pre-measurement and post-measurement state, respectively. As such, \(\gamma\) is referred to as the (two-point) eigenstate trajectory of the measurement process. By assigning a thermodynamic property to each trajectory—namely work, heat, and entropy production—one may in principle obtain all higher moments.

However, the eigenstate trajectory approach presents a problem of ambiguity whenever the initial and final states contain degeneracies—an infinite possibility of eigen-decompositions of the quantum states leads to infinitely many sets of eigenstate trajectories \(\{\gamma\}\), which a priori may not give a unique value for the higher moments of work, heat, or entropy production. It is therefore of interest to construct more generalized, “coarse-grained”, trajectories so that the resulting moments: (i) are always uniquely defined by just the description for the measurement dynamics, as well as the initial state of the system; and (ii) agree with the moments obtained by the eigenstate trajectories in cases where the latter do not result in any ambiguities.

In the present manuscript, we address this issue for the paradigmatic case of an ideal projective measurement. Here, the full augmented measurement process involves: quenching the system Hamiltonian so that it commutes with the desired observable; performing an ideal measurement of said observable by an energy conserving interaction with the measurement apparatus; and finally quenching the Hamiltonian back to its initial configuration. Since the flow of energy into the system during the ideal measurement is compensated by an equal flow of energy out of the measurement apparatus, we shall refer to such energy as “quantum heat”, in analogy with classical heat when conservatively coupling to a thermal environment [30–32]. The remainder of the energy change is, by the first law of thermodynamics, the work done due to the full measurement process, and is associated with the Hamiltonian quenches.

We show that when the initial and final states are degenerate, then for eigenstate trajectories all moments of entropy production, as well as the first moments of work and heat, are always uniquely defined. The higher moments of work and heat, however, are only guaranteed to be unique in the commutative limit, whereby the states, Hamiltonians, and measured observable all commute. This leads to all moments of work, heat, and entropy production to vanish. However, by coarse-graining the trajectories so that only the degenerate subspaces of the initial and final states are distinguished, but not their individual eigenstates, and defining the change in energy as conditional energy changes of sequential measurements introduced in [19], we show that we always satisfy both conditions (i) and (ii). The proposed methodology thus paves the way for renewed investigations of the thermodynamical properties of degenerate quantum systems.

Stochastic quantum thermodynamics of projective measurements via coarse-grained trajectories

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A complete thermodynamic analysis of a quantum measurement process necessitates a method for obtaining higher moments of thermodynamic quantities, such as work, heat, and entropy production. While such a method is provided by the eigenstate trajectory approach of stochastic quantum thermodynamics, conceptual difficulties are encountered when the quantum system being measured is degenerate; an infinite possibility of eigen-decompositions of the quantum state results in infinitely many sets of eigenstate trajectories, thereby precluding the unique assignment of higher moments in general. In the present manuscript we provide a solution to this problem for the case of ideal projective measurements, by “coarse-graining” the eigenstate trajectories, and modifying the definition of stochastic energy change. We show that the resulting statistics of work, heat, and entropy production are always uniquely defined, and reduce to those given by the eigenstate trajectory method when the latter does not result in ambiguities. Our proposed methodology thus paves the way towards a novel approach for studying the thermodynamic properties of degenerate quantum systems.
Preliminaries: We consider a system with a finite-dimensional Hilbert space \( H \cong \mathbb{C}^d \), with \( B(H) \supseteq B_s(H) \supseteq B_p(H) \) denoting the algebra of all operators, self-adjoint operators, and positive operators on \( H \), respectively. In particular, \( O \) and \( \mathbb{I} \) are respectively the zero and identity elements of \( B(H) \). The space of quantum states on \( H \) is thus \( S(H) := \{ \rho \in B_p(H) : \text{tr}[\rho] = 1 \} \), where \( \text{tr}[\cdot] \) is the trace, and physical operations on states will be defined by the class of completely positive, trace non-increasing (CPT) maps, \( O(H) \). Moreover, for any \( \Lambda_1, \Lambda_2 \in O(H) \), the composition is \( \Lambda_2 \circ \Lambda_1 \in O(H) \), where \( \Lambda_1 \) is performed first [33].

An \( N \)-outcome observable on \( H \) is a positive operator measure (POM) \( E : \mathcal{X} \to B_p(H), \mathcal{X} \ni x \to E_x \), where \( \mathcal{X} := \{ 1, \ldots, N \} \) is the outcome set, and the effect operators satisfy \( 0 \leq E_x \leq \mathbb{I} \), and \( \sum_x E_x = \mathbb{I} \). Moreover, each observable is associated with a class of instruments \( \mathcal{I}^E : \mathcal{X} \to O(H), \mathcal{X} \ni x \to \mathcal{I}^E_x \), satisfying the probability reproducibility criterion \( p^E_x(\rho) := \text{tr}[E_x \rho] = \text{tr}[\mathcal{I}^E_x(\rho)] \) for all \( \rho \in S(H) \) [2].

The special class of sharp observables, or projective valued measures (PVMs) will be denoted \( E^O \equiv O \), whose effect operators \( E^O_x \equiv P^O_x \) are the spectral projections of \( B_s(H) \ni O \equiv \bigoplus_{x \in \mathcal{X}} \lambda_x P^O_x \). An ideal measurement of \( O \) is realized by the L"uders instruments \( L^O_x(\rho) := P^O_x \rho P^O_x \). When summing over all outcomes \( x \), these L"uders instruments induce the projection map \( \mathcal{D}^O(\rho) := \sum_{x \in \mathcal{X}} L^O_x(\rho) \), which is a completely positive, trace preserving map, or a “quantum channel”. In the special case where \( O \) is non-degenerate, with the eigenbasis \( \varphi := \{ |\varphi_x\rangle \} \in \mathcal{H} \), we shall use the short-hand notation \( \ell^O, \mathcal{D}^O, \) and \( P^O = |\varphi_x\rangle \langle \varphi_x| \).

Finally, we note the following useful properties of L"uders instruments of sharp observables, which trivially extend to their induced projection maps: (a) \( L^O \) is self-dual, i.e., \( L^O = L^O^\ast \), where \( I^\ast \) is the pre-dual of \( I \) defined as \( \text{tr}[A I(B)] = \text{tr}[I(B) A] \) for all \( A, B \in B(H) \); (b) for all \( x, y \in \mathcal{X}, L^O_x \circ L^O_y = \delta_{x,y} L^O_x \), where \( \delta \) is the Kronecker delta function; (c) Given two commuting observables, \( O \) and \( O' \), \( L^O \circ L^{O'} = L^{O'} \circ L^O \) for all \( x, x' \in \mathcal{X}' \).

Measurement scheme: Let the system have the initial Hamiltonian \( H_0 \in B_s(H) \), and be prepared in an arbitrary degenerate state \( \rho = \sum m q_m \rho_m \). Here, \( q_m \geq 0 \), \( q_m = q_{m'}, \) only if \( m = m' \), and \( \sum_m q_m = 1 \). Additionally, \( \rho_m := P^O_m/d_m \) where \( P^O_m \) are the spectral projections of \( \rho \), whose eigenstates we denote as \( |\psi_{\mu_m}\rangle \), where \( \mu_m = 1, \ldots, d_m \) characterizes the degeneracy.

We wish to measure the sharp observable \( O \) on the system by a L"uders instrument, and quantify the associated entropy production, work, and “quantum heat”. In order to define the last of these, we demand that all the energy flow into the system, due to the measurement process, must be compensated by energy flow out of the measurement apparatus – just as classical heat flowing into the system is compensated by classical heat flowing out of the thermal environment. To this end, we demand that the measurement interaction between system and apparatus conserves the total energy. However, by the Wigner-Araki-Yanase (WAY) theorem [34–41] only observables \( O \) that commute with the system Hamiltonian can be measured by a L"uders instrument, constrained by such a conservation law. Therefore, the full measurement process is defined as the following sequence of operations: Hamiltonian quench \( H_0 \mapsto H \) so that \( H \) commutes with \( O \); implementation of the L"uders instrument by an energy conserving interaction with the apparatus; Hamiltonian quench \( H \mapsto H_0 \).

After measurement, the average state of the system will be \( \rho' := \mathcal{D}^O(\rho) = \sum_n q_n' \rho_n' \), where as with the initial state, we have \( \rho_n' := P^O_n/\delta_n' \), with the eigenstates of the spectral projections \( P^O_n \) denoted as \( |\varphi_{\nu_n}\rangle \), and \( \nu_n = 1, \ldots, d_n' \) characterizing the degeneracy.

Eigensate trajectories: Given our knowledge of the eigensates of the initial and final states, \( \rho \) and \( \rho' \), we may augment the measurement scheme by performing the L"uders instruments \( L^O \) and \( L^O \) prior and posterior to the application of \( L^O \), respectively, without altering the state dynamics, since \( L^O \circ L^O \circ L^O = L^O \circ L^O \). We shall therefore introduce the sequential instrument \( IM := L^O \circ L^O \circ L^O \), which induces the POM \( M \), with outcomes \( \gamma := (\mu_m, x, \nu_n) \). \( \gamma \) is therefore the eigenstate trajectory of the measurement process for \( O \), and has the probability given by the Born rule:

\[
p^M(\gamma) := \text{tr}[IM(\rho)] = \frac{q_m}{d_m} |\langle \varphi_{\nu_n} | \psi_{\mu_m} \rangle|^2.
\]

In order to evaluate the stochastic entropy production, we must first employ a notion of time-reversal. This consists of taking \( \rho' \) as the initial state of the reversed measurement process, and performing the “dual-reverse” instruments in time-reversed order. Following [27, 42, 43], the dual-reverse of the instrument \( I : \mathcal{X} \to O(H) \) admitting an invariant state \( \pi \), i.e., satisfying \( \sum_x \mathcal{I}_x(\pi) = \pi \), is the instrument \( \tilde{I}_x(\pi) = \pi^{-1/2} \mathcal{I}_x(\pi^{-1/2} \rho \pi^{-1/2}) \pi^{-1/2} \). This ensures that for any sequence \( (x_i, \varphi_{\nu_{i}})_{i=1}^\infty, \text{tr}[\tilde{I}_{x_1} \circ \cdots \circ \tilde{I}_{x_M} (\pi)] = \text{tr}[\mathcal{I}_{x_M} \circ \cdots \circ \mathcal{I}_{x_1} (\pi)]; \) in other words, at equilibrium the time-reversed statistics will be indistinguishable from the forward statistics. Since L"uders instruments are self dual and admit the complete mixture as an invariant state, \( \pi = \mathbb{I}/d \), it follows that L"uders instruments are self dual-reverse. Consequently, the sequential instrument for the reversed measurement process is \( IM^* := IM^* = L^O \circ L^O \circ L^O \), which induces the POM \( M \) with outcomes \( \tilde{\gamma} = (\nu_n, x, \mu_m) \).

The probability of observing the time-reversed sequence of measurement outcomes \( \tilde{\gamma} \) is thus

\[
p^{IM^*}_{\rho'}(\tilde{\gamma}) := \text{tr}[IM^*(\rho')] = \frac{q_n'}{d_n'} |\langle \varphi_{\nu_n} | \psi_{\mu_m} \rangle|^2.
\]
backwards probabilities, is therefore given as

\[ S_{n.a.}(\gamma) := \ln \left( \frac{p^M_{\rho}(\gamma)}{p^M_{\rho'}(\gamma)} \right) = \ln \left( \frac{q_m}{d_m} \right) - \ln \left( \frac{q'_n}{d'_n} \right). \]  

This can equivalently be interpreted as quantifying the increase in surprisal when observing the system in its initial and final eigenstates.

Noting that for any \(a, b \in \mathbb{R}\),

\[ (a - b)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i a^{k-i} b^i, \]  

(4)

where \(\binom{k}{i} := k! / (i! (k - i)!)\) are the binomial coefficients, then by Eqs. (1), (3), and (4), we can easily show that the \(k^{th}\) moment of entropy production along all trajectories is

\[ \langle S_{n.a.}^k \rangle := \sum_{\gamma} p(\gamma) S_{n.a.}^k(\gamma) = \text{tr}[\rho(\ln(\rho) - \ln(\rho'))^k]. \]  

(5)

Surprisingly, \(\langle S_{n.a.}^k \rangle\) is independent of the eigen-decompositions of \(\rho\) and \(\rho'\), and is thus uniquely defined for all possible sets of trajectories \(\{\gamma\}\). In particular, the average and variance of entropy production read as the relative entropy \(\langle S_{n.a.} \rangle = S(\rho||\rho') \geq 0\) and relative entropy variance \(\text{Var}(S_{n.a.}) := \langle S_{n.a.}^2 \rangle - \langle S_{n.a.} \rangle^2 = V(\rho||\rho')\), respectively [44–46].

The absorbed quantum heat and increase in internal energy, for each trajectory \(\gamma\), is defined as the increase in expected values of the Hamiltonians \(H\) and \(H_0\), respectively, on the initial and final eigenstates. The work done is thus obtained by the first law of thermodynamics as the difference between these quantities:

\[ Q(\gamma) := \text{tr}[H(P_{\rho_n} - P_{\rho_m})], \]
\[ \Delta U(\gamma) := \text{tr}[H_0(P_{\rho_n} - P_{\rho_m})], \]
\[ \mathcal{W}(\gamma) := \Delta U(\gamma) - Q(\gamma) = \text{tr}[(H - \mathcal{D})(P_{\rho_n} - P_{\rho_m})], \]  

(6)

where we have introduced the work observable \(\Delta H := H_0 - H\). As before, by Eqs. (1), (4), and (6), we may compute the \(k^{th}\) moment of quantum heat absorption, and work done:

\[ \langle Q^k \rangle := \sum_{\gamma} p(\gamma) Q^k(\gamma) = \text{tr}[(H - \mathcal{D}(H))^k \rho], \]
\[ \langle \mathcal{W}^k \rangle := \sum_{\gamma} p(\gamma) \mathcal{W}^k(\gamma) = \text{tr}[(\mathcal{D}(\Delta H) - \mathcal{D}(\Delta H))^k \rho]. \]  

(7)

Already from (6) we can see that the work and heat distributions are dependent on the eigen-decompositions of \(\rho\) and \(\rho'\). However, this dependence is removed for the special case of the first moments:

\[ \langle Q \rangle = \text{tr}[(H - H_0) \rho] = 0, \quad \langle \mathcal{W} \rangle = \text{tr}[H_0(\rho' - \rho)]. \]  

(8)

This follows from the fact that the projection map is self dual. The higher moments, however, will only be uniquely defined in the commutative case, for which all moments vanish. To see this, we note that \(\mathcal{D}(\Delta H) = A\) and \(\mathcal{D}(\Delta H) = B\), where \(A, B \in B_c(H)\), for all eigen-decompositions of \(\rho\) and \(\rho'\), respectively. But when these states are degenerate, this will only be possible if \(\psi\) and \(\varphi\) are both eigenbases of \(\Delta H\), implying that \(\mathcal{D}(\Delta H) = \mathcal{D}(\Delta H) = \Delta H\), so that all moments of work vanish. Similarly, for the moments of heat to be unique, \(\psi, \varphi\) must also be eigenstates of \(H\), whereby the moments of quantum heat will vanish. Furthermore, since \(\psi, \varphi\) are eigenbases of the same observables, \(H, \Delta H\), then \(\rho\) must also commute with \(\rho'\) and \(\mathcal{O}\). Therefore, all moments of entropy production also vanish.

**Coarse-grained trajectories:** We now introduce the “coarse-grained” trajectories \(\Gamma := (m, x, n) \equiv \cup_{\mu, \nu, \gamma} \Gamma\), which only distinguish between the degenerate subspaces of \(\rho, \rho'\), but not the individual eigenstates. Operationally, we may consider such trajectories as resulting from replacing the instruments \(\mathcal{L}^\varphi\) and \(\mathcal{L}^\psi\) with \(\mathcal{L}^\phi\) and \(\mathcal{L}^\psi\), respectively, thereby constructing the sequential instruments \(\mathcal{I}^\phi := \mathcal{L}^\phi \circ \mathcal{L}^\varphi \circ \mathcal{L}^\psi\), which induces the POM \(\mathcal{M}\) with outcomes \(\Gamma\). As before, such an augmentation does not change the measurement dynamics, since \(\mathcal{D}^\phi \circ \mathcal{D}^\varphi \circ \mathcal{D}^\psi(\rho) = \mathcal{D}^\phi(\rho)\). The probabilities of the coarse-grained trajectories, as well as their time reversed counterparts, are

\[ p_{\rho}(\Gamma) = \text{tr}[\mathcal{I}^\phi(\rho)] = \frac{q_m}{d_m} \text{tr}[\mathcal{I}^\phi(1)], \]
\[ p_{\rho'}(\Gamma) = \text{tr}[\mathcal{I}^\phi(\rho')] = \frac{q'_n}{d'_n} \text{tr}[\mathcal{I}^\phi(1)], \]  

(9)

which are evaluated analogously with (1) and (2). The stochastic entropy production for the coarse grained trajectories can therefore be defined as \(S_{n.a.}(\Gamma) := \ln \left( \frac{p_{\rho}(\Gamma)/p_{\rho'}(\Gamma)}{p_{\rho}(\Gamma)/p_{\rho'}(\Gamma)} \right)\). It is immediately apparent that \(S_{n.a.}(\Gamma) = S_{n.a.}(\gamma)\), as defined in (3), for all \(\gamma \subset \Gamma\), while \(\langle S_{n.a.}^k \rangle = \langle S_{n.a.}^k \rangle\), as defined in (5). Therefore, conditions (i)-(ii) are satisfied for entropy production.

To evaluate the coarse-grained quantum heat, change in internal energy, and work, respectively denoted as \(Y = \mathcal{D}, \Delta \mathcal{W}, \mathcal{W}\), we follow Ref. [19] and obtain

\[ X(\Gamma) := \frac{\text{tr}[\mathcal{I}^\psi(\mathcal{Y}) - \mathcal{I}^\phi(\mathcal{Y})]}{\text{tr}[\mathcal{I}^\phi(1)]}, \]  

(10)

where for \(X = \mathcal{D}, \Delta \mathcal{W}, \mathcal{W}\), we have \(Y = H, H_0, \Delta H\), respectively. Here, the first term in \(X(\Gamma)\) denotes the expected value of the observable \(Y\), given the normalised state of the system after having observed outcome \(\Gamma\) of the POM \(\mathcal{M}\). The second term, meanwhile, is the (real) weak value of \(Y\), given the initial state \(\rho\), and post selected by outcome \(\Gamma\) of \(\mathcal{M}\) [47, 48].

Formally, by use of Eqs. (4), (9), and (10), we may
express each moment of $X = \mathcal{Q}, \mathcal{W}$ as
\[
\langle X^k \rangle := \sum_{\Gamma} \rho_{\gamma}^{\text{Q}}(\Gamma) X^k(\Gamma),
\]
\[
= \sum_{\Gamma} \frac{q_m}{d_m} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \frac{\text{tr}[\mathcal{I}^{\#''}(Y)]^{k-i} \text{tr}[\mathcal{I}^{\#'}(Y)]^{i}}{\text{tr}[\mathcal{I}^{\#''}(I)]^{k-i}}.
\]
(11)

Clearly, this does not depend on the eigen-decompositions of $\rho, \rho'$, and is therefore uniquely defined, while it is trivial to see that (11) reduces to (8) for $k = 1$. Moreover, we note that in the non-degenerate case, i.e., when $d_m = d'_m = 1$, (10) reduces to (6), while in the commutative case, due to properties (a)-(c) of Lüders instruments, $\mathcal{Q}(\Gamma) = \mathcal{W}(\Gamma) = \mathcal{W}(\gamma) = 0$. In both cases, therefore, (11) reduces to (7). As such, our definition for coarse-grained work and heat also satisfies conditions (i)-(ii).

While (11) generally does not admit elegant expressions analogous to (7), we find that so long as $\mathcal{L}_p^n \circ \mathcal{L}_Y^n(\Gamma) \otimes \mathcal{L}_p^n \circ \mathcal{L}_Y^n(\mathbb{I})$, an inequality can be introduced in such terms for the second moments. This condition will be satisfied for $Y = H$ if $O$ and $H$ share the same spectral projections (i.e., $O$ and $H$ are the same sharp observable), and for $Y = H, \Delta H$ if either $O$ or $\rho'$ are non-degenerate. First, let us note that the aforementioned condition ensures that $\text{tr}[\mathcal{I}^{\#''}(Y)]\text{tr}[\mathcal{I}^{\#'}(Y)] = \text{tr}[\mathcal{I}^{\#''}(I)]\text{tr}[\mathcal{D}^{\#'}(Y)\mathcal{I}^{\#'}(\mathcal{D}^{\#'}(Y))]$. Next, let us recall that $\text{tr}[\mathcal{I}^{\#'}(Y)] = \text{tr}[\mathcal{M}^{\#'}(\rho)]$, and $\text{tr}[\mathcal{I}^{\#''}(Y)] = \text{tr}[\mathcal{M}^{\#''}(\rho)].$ By the Cauchy-Schwarz inequality, we therefore have $\text{tr}[\mathcal{I}^{\#'}(Y)]^2 \leq \text{tr}[\mathcal{I}^{\#''}(\mathcal{D}^{\#'}(Y))\text{tr}[\mathcal{I}^{\#'}(I)],$ and $\text{tr}[\mathcal{I}^{\#''}(Y)]^2 \leq \text{tr}[\mathcal{I}^{\#''}(\mathcal{D}^{\#'}(Y))\text{tr}[\mathcal{I}^{\#''}(I)].$ Taken together, we arrive at the inequalities
\[
\langle X^2 \rangle \leq \text{tr}((\mathcal{D}^{\#'}(Y) - \mathcal{D}^{\#'}(Y))^2 \rho). \tag{12}
\]

For the quantum heat, we may also note that the upper bound in (12) can be expressed as $\sum_m q_m J_{\gamma_m}(H)$, where we define $J_{\gamma}(H) := \text{tr}[H^2 \rho] - \text{tr}[\rho H^2 H \rho^{-1/2}]$ as the Wigner-Yanase-Dyson skew information of $\rho$ with reference to $H$, which: is non-negative and vanishes if $\rho$ commutes with $H$; is convex in $\rho$; and satisfies $J_{\gamma}(H) = V_\gamma(H)$ whenever $\rho$ is a pure state, where $V_\gamma(H) := \text{tr}[H^2 \rho] - \text{tr}[H \rho^2]$ is the variance of $H$ in $\rho$ [49–51]. We note that $\langle Q^2 \rangle$ can also be expressed as the average skew information of the pure states $|\psi_{\mu_n}\rangle$ [32]. Furthermore, given that $\langle Q \rangle = \langle \mathcal{Q} \rangle = 0$, then by (12) and convexity of $J$ we find that
\[
\text{Var}(\mathcal{Q}) \leq \text{Var}(\mathcal{W}), \tag{13}
\]
where $\text{Var}(X) := \langle X^2 \rangle - \langle X \rangle^2$. In other words, whenever (12) holds for $X = \mathcal{Q}$, the fluctuations in coarse-grained quantum heat will be guaranteed to be smaller than that obtained for any possible set of eigenstate trajectories $\{\gamma\}$. However, (12) will not guarantee that $\text{Var}(\mathcal{W}) \leq \text{Var}(\mathcal{W})$ for all $\{\gamma\}$; this will be model-dependent, and it is possible to have $\text{Var}(\mathcal{W}) \geq \text{Var}(\mathcal{W})$ for all $\{\gamma\}$.

Let us illustrate this with a concrete example. Consider a quantum system that is composed of two qubits, with Hilbert space $\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$. We shall denote by $\{ |+\rangle, |-- \rangle \}$ an orthonormal basis of $\mathbb{C}^2$. Let the system initially have the Hamiltonian
\[
H_0 = 3\omega P_+ \otimes \mathbb{I} + 7\omega P_- \otimes P_+ + \omega P_+ \otimes P_-, \tag{14}
\]
where $P_\pm \equiv |\pm\rangle\langle \pm|$, $\omega > 0$, and prepare the state in thermal equilibrium, i.e., $\rho = e^{-\beta H_0}/\text{tr}[e^{-\beta H_0}]$, where $\beta > 0$ is the inverse temperature (we set $\hbar = k_B = 1$). Consequently, $\rho$ has only one degenerate subspace, namely the subspace projected onto by $P_+^* := P_+ \otimes \mathbb{I}$. As such, $\rho$ has infinitely many eigenstate decompositions $\{ |+\rangle \otimes |\psi_{\theta,\phi}^+\rangle, |-- \rangle \otimes |\pm\rangle \}$, where $|+\rangle \otimes |\psi_{\theta,\phi}^+\rangle$ are the eigen-
vectors of $P^\rho_{\pm}$, with

$$|\psi_{\theta,\phi}^\pm\rangle := \pm e^{\pm i \phi} \cos\left(\frac{\theta}{2}\right) |\pm\rangle + \sin\left(\frac{\theta}{2}\right) |\mp\rangle. \quad (15)$$

The degeneracy can therefore be parameterized by the point $(\theta, \phi) \in \mathbb{R}^2$, where $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$.

Now we wish to measure the system with respect to a non-degenerate sharp observable $O$, with eigenbasis $\{|i\rangle \otimes |j\rangle : i, j = 0, 1\}$, where $|1\rangle := |\psi_{\pi/3, 0}^+\rangle$ and $|0\rangle := |\psi_{-\pi/3, 0}^-\rangle$.

Therefore, we quench the Hamiltonian to one that shares the same eigenstates as $O$, which we choose as

$$H = \omega P_0 \otimes P_0 + 3 \omega P_0 \otimes P_1 + 5 \omega P_1 \otimes P_0 + 7 \omega P_1 \otimes P_1.$$  

Upon completion of the measurement process, we quench the Hamiltonian back to $H_0$. Given that the system is initially in thermal equilibrium, the average work done is positive: $\langle \mathcal{W} \rangle = \langle \mathcal{V} \rangle = \beta^{-1} \{S(\rho |\rho\rangle \rho) + S(\rho \langle\rho |\rho\rangle) \} \geq 0$.

Moreover, since the post-measurement state $\rho'$ is non-degenerate, each set of eigenstate trajectories $\{\gamma\}$ can be parameterized by $(\theta, \phi)$, while this model also satisfies the condition $L^n_{\rho'} \otimes L^n_{\rho'}(Y) \propto L^n_{\rho'} \otimes L^n_{\rho'}(1)$ for $Y = H, \Delta H$, so that the second moments of heat and work will satisfy the inequalities given in (12); indeed, for this model the equality condition of (12) is satisfied. However, while (12) guarantees (13) for quantum heat, we observe the converse relationship for work, i.e. $\text{Var}(\mathcal{W}) \geq \text{Var}(\mathcal{V})$ for all $\{\gamma\}$. This is illustrated in Fig. 1 for the parameters $\beta = \omega = 1$.

**Conclusions:** In this study we presented analytically tractable expressions for all moments of work, heat, and entropy production as a system undergoes an ideal projective measurement, by use of (two-point) eigenstate trajectories. In order to alleviate the non-uniqueness of the higher moments of energy change when initial and final states are degenerate, we “coarse-grained” the trajectories so as to only distinguish between the degenerate subspaces of these states, and not their individual eigenstates. By further generalizing the definition of stochastic energy change to the conditional change in energy given sequential measurements, we showed that the resulting statistics of coarse-grained work, heat, and entropy production constitute proper generalizations of those given by eigenstate trajectories: they reduce to the expressions obtained by eigenstate trajectories whenever these do not result in ambiguities, but provide a well-defined alternative when they do.

The present manuscript constitutes the first steps in the re-examination of statistical properties of thermodynamic processes in the presence of degeneracies, while its generalization to the case of sequential or continuous projective measurements, and indeed to the measurement of more general POMs, as well as its ramifications in regard to other research avenues within stochastic quantum thermodynamics, such as fluctuation relations, remain as open questions for further work.

**Acknowledgments:** The author wishes to thank M. Rafiee for insightful discussions, and acknowledges support from the Slovak Academy of Sciences under MoRePro project OPEQ (19MRP0027), as well as projects OPTIQUE (APVV-18-0518) and HOQIT (VEGA 2/0161/19).

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