Orbits of $Z \circ (2.O_8^+(2).2)$ in Dimension 8

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Abstract

Groups of structure $2.O_8^+(2)$ have an irreducible representation of degree 8 which can be realized over $\mathbb{Z}$ and any prime field $\mathbb{F}_p$. This representation extends to a group of structure $2.O_8^+(2).2$. Any subgroup $Z \leq \mathbb{F}_p^\times$ acts by scalar multiplication on this module over $\mathbb{F}_p$.

In this short note we determine for which primes $p > 7$ and which $Z$ the central products $Z \circ (2.O_8^+(2)$ and $Z \circ (2.O_8^+(2).2$ have a regular orbit on the 8-dimensional $\mathbb{F}_p$-module.

This work was triggered by an omission in the paper [KP01] by Köhler and Pahlings, a paper which is used in various places in work on the $k(GV)$-problem.

1 Introduction

In [KP01] Köhler and Pahlings investigated the following problem:

Let $p$ be a prime, $G$ be a finite group whose order is not divisible by $p$ and $V$ be a finite dimensional faithful $\mathbb{F}_pG$-module of dimension $n$. Furthermore, assume that $G$ has a quasisimple normal subgroup $E$ which also acts irreducibly on $V$. Does $G$ have a regular orbit on $V$?

In most cases the answer to this question is yes, but there is a list of exceptions. The main result of [KP01, Theorem 2.2] is the table of these exceptions.

For a fixed quasisimple group $E$ and $\mathbb{F}_pE$-module $V$ the possible groups $G$ (as groups of endomorphisms of $V$) are generated by a subgroup of $\text{Aut}(E)$ and a subgroup $Z \leq \mathbb{F}_p^\times$ of scalar matrices, see [KP01, Section 3].

From now we consider the specific case $E = 2.O_8^+(2)$ and its irreducible representation of degree $n = \dim(V) = 8$, hence $p > 7$. In this case $\text{Aut}(E)$
has the structure $2.O_8^+(2).2$ and the representation extends in two ways to this group, see [CCN+85, p.85]. The authors of [KP01] determine for which $p$ the groups of form $Z \circ E$ have a regular orbit on $V$ by an elaborate computation with the table of marks of $O_8^+(2)$. But they forgot (in statement and proof) to handle the groups of form $Z \circ E$.2. In this short note we will close this gap. We will also recover (with a slight correction) their result for the groups $Z \circ E$ with an easier argument.

2 The groups $2.O_8^+(2).2$ and its 8-dimensional representations over $\mathbb{F}_p$

There are two isomorphism types of groups with structure $2.O_8^+(2).2$ which are isoclinic, see [CCN+85, Ch.6, Sec.7]. For one type the (Brauer)-characters of the 8-dimensional representations have values in the rational integers and for the other type the character values generate $\mathbb{Z}[i]$ ($i^2 = -1$). In the latter case the 8-dimensional representations can only be realized over $\mathbb{F}_p$ if the field contains primitive fourth roots of unity, that is if $p \equiv 1 \mod 4$. If $Z \leq \mathbb{F}_p^\times$, $|Z| = 4$, and $G$ is one group of type $2.O_8^+(2).2$ we find the isoclinic group as subgroup of index 2 in $Z \circ G$ (exchange the generators $x$ of $G$ which are not in the derived subgroup $G'$ with $i \cdot x$).

Now let $W$ be the Weyl group of type $E_8$. It has the structure $2.O_8^+(2).2$, see [CCN+85, p.85]. Since Weyl groups have rational character values it is clear which one of the isoclinic groups this is. We denote $\tilde{W}$ the isoclinic group.

Now we can state our result.

**Theorem 2.1.** A group of type $Z \circ (2.O_8^+(2))$ has no regular orbit on its 8-dimensional irreducible $\mathbb{F}_pG$-module if and only if $p \leq 23$ or $p = 31$ and $|Z| > 2$.

A group $G$ of type $Z \circ W$ has no regular orbit on its 8-dimensional irreducible $\mathbb{F}_pG$-modules, if and only if $p \leq 29$ or $p = 31$ and $|Z| > 2$.

In case $p \equiv 1 \mod 4$ a group $G$ of type $Z \circ \tilde{W}$ has no regular orbit on its 8-dimensional irreducible $\mathbb{F}_pG$-modules if and only if one of the following holds

- $p < 29$,
- $p = 29$ and $4 \mid |Z|$,
\* \( p = 31 \) and \(|Z| > 2\).

3 The proof

From one of the irreducible representations of \( 2.O_8^+(2).2 \) of dimension 8 we get the other one by tensoring with the 1-dimensional representation with kernel \( 2.O_8^+(2) \). So the action of elements outside the derived subgroup only differs by scalar multiplication with \(-1\). Since the central element, which is contained in the derived subgroup, also acts by the scalar \(-1\), the orbits on \( V \) are the same for both module structures. Therefore, it is enough to consider one of the 8-dimensional modules.

We now consider the Weyl group \( W \) of type \( E_8 \). It can be described as follows as subgroup of \( GL_8(\mathbb{Z}) \), and this way we get for any prime \( p \) an 8-dimensional representation of \( W \) over \( \mathbb{F}_p \) by reducing the matrix entries modulo \( p \).

Let \( Y = \mathbb{Z}^8 \) and \( X \) be the dual \( \mathbb{Z} \)-lattice. We describe a root datum of type \( E_8 \). For this we take the standard basis vectors of \( Y \) as set of simple coroots \( \alpha_j^\vee \), \( 1 \leq j \leq 8 \), and as corresponding simple roots \( \alpha_j \in X \) the rows of the following matrix (written with respect to the \( \mathbb{Z} \)-basis of \( X \) which is dual to the simple coroots, the elements of this basis are also called fundamental weights):

\[
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{pmatrix}
\]

For \( 1 \leq j \leq 8 \) we can use \( \alpha_j \) and \( \alpha_j^\vee \) to define the following reflection on \( X \):

\[
s_j : X \to X, \quad x \mapsto x - \alpha_j^\vee(x) \cdot \alpha_j.
\]

The group generated by these reflections \( W = \langle s_j \mid 1 \leq j \leq 8 \rangle \) is the Weyl group of type \( E_8 \), it is a Coxeter group with the \( s_j \), \( 1 \leq j \leq 8 \), as set of Coxeter generators. The orbit \( \alpha_1 W \) is called the root system of type \( E_8 \), it contains 240 roots. The dual construction on \( Y \) yields the corresponding coroots and the highest coroot (the one with is componentwise \( \geq \) the
coordinates of all other coroots) is

\[ \alpha_0^\vee = (2 3 4 6 5 4 3 2). \]

The corresponding root is \( \alpha_0 = (0 0 0 0 0 0 0 1) \) and this defines a reflection \( s_0 : X \to X \) as above.

For any prime \( p > 7 \) we are interested in the orbits of the action of \( W \) on \( X \) modulo \( pX \) (identifying \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) and \( \mathbb{F}_p^8 = X/pX \)).

The subgroup of bijections \( X \to X \), generated by \( W \) and the translations by elements of \( pX \) is called the affine Weyl group \( W_p \) of type \( E_8 \), see [Hum90, 4.3, 4.8] (we use that the \( \mathbb{Z} \)-span of the roots is all of \( X \), and we scale the translations in the reference by a factor \( p \)).

The action of \( W_p \) on \( X \) can be \( \mathbb{R} \)-linearly extended to the vector space \( X \otimes \mathbb{R} \). We consider the following subset of \( X \otimes \mathbb{R} \) which is called the bottom alcove:

\[ A_0 := \{ x \in X \otimes \mathbb{R} \mid \alpha_j^\vee(x) > 0 \text{ for } 1 \leq j \leq 8, \alpha_0^\vee(x) < p \}, \]

and will use the following theorem, see [Hum90, 4.8].

**Theorem 3.1.** The closure \( \bar{A}_0 \) of \( A_0 \) is a fundamental domain for the action of \( W_p \) on \( X \otimes \mathbb{R} \). If \( x \in \bar{A}_0 \) then the stabilizer of \( x \) in \( W \) is generated by those \( s_j \), \( 1 \leq j \leq 8 \), with \( \alpha_j^\vee(x) = 0 \) together with \( s_0 \) in case \( \alpha_0^\vee(x) = p \).

In particular, every \( W_p \)-orbit on \( X \otimes \mathbb{R} \) has a unique representative \( x \in \bar{A}_0 \) and the orbit is regular if and only if \( x \in A_0 \).

Restricting this to points \( x = (x_1, \ldots, x_8) \in X \) we conclude: There exists a regular \( W \)-orbit on \( X \) modulo \( pX \) if and only if \( A_0 \) contains a point with integer coordinates. That is, all \( x_j \in \mathbb{Z}_{>0} \) (because \( \alpha_j^\vee(x) = x_i \) for \( 1 \leq j \leq 8 \) and \( \alpha_0^\vee(x) < p \). Since the coordinates of \( \alpha_0^\vee \) are all positive, such an \( x \) exists if and only if \( \rho := (1 1 1 1 1 1 1 1) \in A_0 \) if and only if \( \alpha_0^\vee(\rho) = 29 < p \).

It is an easy programming exercise to enumerate for moderate \( p \) all integral points \( x \in A_0 \), and to read off their stabilizers in \( W \). For all \( p < 29 \) these stabilizers all have order \( > 2 \). This shows that also the orbits of the derived subgroup \( W' = 2.O_8^+(2) \) of \( W \) are never regular.

The case \( p = 29 \). Here all integral \( x \in A_0 \) with \( x \neq \rho \) have stabilizer of order \( > 2 \). But \( \rho \in A_0 \) and its stabilizer is generated by \( s_0 \) and is of order 2. Since the reflections are a single conjugacy class of \( W \) and generate \( W \) we see that \( s_0 \not\in W' \). So, the orbit of \( \rho \) is (the only) regular \( W' \)-orbit.

We have shown our Theorem 2.1 for \( G = W' \) and \( G = W \).
Action of scalars

Now we consider the action of scalars. We will see that we find all information we need by considering the orbit of \( \rho \in X \) modulo \( pX \).

We want to know for all primes \( p \) all \( c \in \mathbb{Z} \) modulo \( p \) such that there is an element \( w \in W \) with \( \rho w \equiv c \rho \mod pX \). The center of \( W' = 2.O_8^+ (2) \) acts as scalar \(-1\). So, all \( W'\)-orbits on \( X \) will be closed under multiplication with \(-1\).

Fixing \( w \in W \) and setting \( y = (y_1, \ldots, y_8) := \rho w \) we want to know for which \( p \) there is a \( c \) such that \( y \equiv c \rho \mod pX \). This relation is equivalent to the condition \( \gcd(y_1 - c, \ldots, y_8 - c) \equiv 0 \mod p \) for some \( c \in \mathbb{Z} \). We use

\[
\gcd(y_1 - c, \ldots, y_8 - c) = \gcd(y_1 - c, y_2 - y_1, \ldots y_8 - y_1) \mid \gcd(y_2 - y_1, \ldots, y_8 - y_1)
\]

and compute the latter expression.

The possible \( p \) are the prime divisors of \( \gcd(y_2 - y_1, \ldots, y_8 - y_1) \). And it is clear that for each such prime there is some \( c \) (unique modulo \( p \)) such that \( y_1 - c \) is also divisible by \( p \).

Using a computer and GAP [GAP19] we computed the full \( W \)-orbit of \( \rho \) and the described \( \gcd \)'s. For this we used the explicit construction of the representation given above. The computation took about 45 minutes on the authors notebook.

The prime divisors of these numbers are all \( \leq 31 \). So, for \( p > 31 \) the only \( \mathbb{F}_p \)-multiple of \( \rho \) occurring in the orbit of \( \rho \) modulo \( pX \) is \(-\rho \), and therefore the orbit of \( \rho \) is also regular for any central product \( Z \circ W \) and so for \( Z \circ W' \), \( Z \leq \mathbb{F}_p^\times \).

The case \( p = 31 \). It is easy to see that \( \rho \) is the only integral point in \( A_0 \) for \( p = 31 \), so there is only one regular \( W \)-orbit modulo 31, namely the orbit of \( \rho \). Therefore it is not surprising that this \( W \)-orbit contains all scalar multiples of \( \rho \). Looking closer at the set of \( w \in W \) which yield multiples of \( \rho \) modulo 31 in our \( \gcd \)-computations we notice that they are all of even length, so that already the \( W'\)-orbit of \( \rho \) contains all multiples of \( \rho \) modulo 31.

This proves the exceptions for \( p = 31 \) in Theorem 2.1.

The case \( p = 29 \). Our \( \gcd \)-computations show that in the orbit of \( \rho \) modulo 29 the only multiples of \( \rho \) are \( \pm \rho \). The orbit is not regular, and \( \rho w = \rho \) modulo 29 for \( w = 1 \) and \( w = s_0 \). Now we consider the isoclinic group \( \tilde{W} \) which is generated by the \( is_j, 1 \leq j \leq 8 \) where \( i \in \mathbb{F}_{29} \) is of order 4. The orbit of \( \rho \) under \( \tilde{W} \) contains powers of \( i \) scalar multiples of the vectors.
in the orbit under \( W \). The element \( s_0 \in W \) is of length 57, so that the corresponding product of generators of \( \tilde{W} \) will map \( \rho \) to \( i\rho \). This shows that

the orbit of \( \rho \) under \( \tilde{W} \) contains all four multiples \( i^k \rho \), \( 0 \leq k \leq 3 \). So \( \rho \tilde{W} \) is twice as long as \( \rho W \), hence a regular orbit.

We have shown all statements in Theorem 2.1 concerning the cases with \( p = 29 \). This finishes our proof.

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### References

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of Finite Groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[GAP19] GAP – Groups, Algorithms, and Programming, Version 4.10.2. [https://www.gap-system.org](https://www.gap-system.org), Jun 2019.

[Hum90] J. E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[KP01] Ch. Köhler and H. Pahlings. Regular Orbits and the \( k(GV) \)-Problem. In *Groups and Computation, III (Columbus, OH, 1999)*, volume 8 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 209–228. de Gruyter, Berlin, 2001.

[Lee20] Melissa Lee. Regular Orbits of Quasisimple Linear Groups I. [https://arxiv.org/abs/1911.05785](https://arxiv.org/abs/1911.05785), 2020.