Interval edge-colorings of composition of graphs

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An edge-coloring of a graph $G$ with consecutive integers $c_1, \ldots, c_t$ is called an \textit{interval $t$-coloring} if all colors are used, and the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. The set of all interval colorable graphs is denoted by $\mathcal{N}$. In 2004, Giaro and Kubale showed that if $G, H \in \mathcal{N}$, then the Cartesian product of these graphs belongs to $\mathcal{N}$. In the same year they formulated a similar problem for the composition of graphs as an open problem. Later, in 2009, the first author showed that if $G, H \in \mathcal{N}$ and $H$ is a regular graph, then $G[H] \in \mathcal{N}$. In this paper, we prove that if $G \in \mathcal{N}$ and $H$ has an interval coloring of a special type, then $G[H] \in \mathcal{N}$. Moreover, we show that all regular graphs, complete bipartite graphs and trees have such a special interval coloring. In particular, this implies that if $G \in \mathcal{N}$ and $T$ is a tree, then $G[T] \in \mathcal{N}$.

Keywords: edge-coloring, interval coloring, composition of graphs, complete bipartite graph, tree.

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. For a graph $G$, by $\overline{G}$ we denote the complement of the graph $G$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of $G$ by $\Delta(G)$, and the chromatic index of $G$ by $\chi'(G)$. The terms and concepts that we do not define can be found in [3, 8, 20, 35].

A proper edge-coloring of a graph $G$ is a coloring of the edges of $G$ such that no two adjacent edges receive the same color. A proper edge-coloring of a graph $G$ with consecutive integers $c_1, \ldots, c_t$ is an \textit{interval $t$-coloring} if all colors are used, and the colors of edges incident to each vertex of $G$ are form an interval of integers. A graph $G$ is

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interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by \( \mathcal{N} \). The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [1] in 1987. In [1], they proved that if \( G \in \mathcal{N} \), then \( \chi'(G) = \Delta(G) \). Asratian and Kamalian also proved [1, 2] that if a triangle-free graph \( G \) admits an interval t-coloring, then \( t \leq |V(G)| - 1 \). In [16, 17], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph \( K_{m,n} \) has an interval t-coloring if and only if \( m + n - \gcd(m, n) \leq t \leq m + n - 1 \), where \( \gcd(m, n) \) is the greatest common divisor of \( m \) and \( n \). In [24], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved that if \( n \leq t \leq \frac{n(n+1)}{2} \), then the hypercube \( Q_n \) has an interval t-coloring. Later, in [27], it was shown that the hypercube \( Q_n \) has an interval t-coloring if and only if \( n \leq t \leq \frac{n(n+1)}{2} \). In [31], Sevast’janov proved that it is an NP-complete problem to decide whether a bipartite graph has an interval coloring or not. In papers [11, 2, 6, 7, 9, 16, 17, 20, 24, 26, 27, 28, 31], the problems of existence, construction and estimating the numerical parameters of interval colorings of graphs were investigated. Surveys on this topic can be found in some books [3, 15, 20].

Graph products [8] were first introduced by Berge [5], Sabidussi [30], Harary [10] and Vizing [32]. In particular, Sabidussi [30] and Vizing [32] showed that every connected graph has a unique decomposition into prime factors with respect to the Cartesian product. In the same direction there are also many interesting problems of decomposing of the different products of graphs into Hamiltonian cycles. In particular, in [4] it was proved Bermond’s conjecture that states: if two graphs are decomposable into Hamiltonian cycles, then their composition is decomposable, too. A lot of work was done on various topics related to graph products, on the other hand there are still many questions open. For example, it is still open Hedetniemi’s conjecture [12], Vizing’s conjecture [33] and the conjecture of Harary, Kainen and Schwenk [11].

There are many papers [13, 14, 19, 21, 22, 23, 29, 34] devoted to proper edge-colorings of various products of graphs, however very little is known on interval colorings of graph products. Interval colorings of Cartesian products of graphs were first investigated by Giaro and Kubale [6]. In [7], Giaro and Kubale proved that if \( G, H \in \mathcal{N} \), then \( G \square H \in \mathcal{N} \). In 2004, they formulated [20] a similar problem for the composition of graphs as an open problem. In 2009, the first author [25] showed that if \( G, H \in \mathcal{N} \) and \( H \) is a regular graph, then \( G[H] \in \mathcal{N} \). Later, Yepremyan [28] proved that if \( G \) is a tree and \( H \) is either a path or a star, then \( G[H] \in \mathcal{N} \). Some other results on interval colorings of various products of graphs were obtained in [20, 25, 26, 27, 28].

In this paper, we prove that if \( G \in \mathcal{N} \) and \( H \) has an interval coloring of a special type, then \( G[H] \in \mathcal{N} \). Moreover, we show that all regular graphs, complete bipartite graphs and trees have such a special interval coloring. In particular, this implies that if \( G \in \mathcal{N} \) and \( T \) is a tree, then \( G[T] \in \mathcal{N} \).

2. Notations, Definitions and Auxiliary Results

We use standard notations \( C_n \) and \( K_n \) for the simple cycle and complete graph on \( n \)
vertices, respectively. We also use standard notations $K_{m,n}$ and $K_{m,n,t}$ for the complete bipartite and tripartite graph, respectively, one part of which has $m$ vertices, the other part has $n$ vertices and the third part has $l$ vertices.

For two positive integers $a$ and $b$ with $a \leq b$, we denote by $[a,b]$ the interval of integers $\{a, \ldots, b\}$.

Let $L = (l_1, \ldots, l_k)$ be an ordered sequence of nonnegative integers. The smallest and largest elements of $L$ are denoted by $\underline{L}$ and $\overline{L}$, respectively. The length (the number of elements) of $L$ is denoted by $|L|$. By $L(i)$, we denote the $i$th element of $L$ ($1 \leq i \leq k$). An ordered sequence $L = (l_1, \ldots, l_k)$ is called a continuous sequence if it contains all integers between $\underline{L}$ and $\overline{L}$. If $L = (l_1, \ldots, l_k)$ is an ordered sequence and $p$ is nonnegative integer, then the sequence $(l_1 + p, \ldots, l_k + p)$ is denoted by $L \oplus p$. Clearly, $(L \oplus p)(i) = L(i) + p$ for any $p \in \mathbb{Z}_+$.

Let $G$ and $H$ be two graphs. The composition (lexicographic product) $G[H]$ of graphs $G$ and $H$ is defined as follows:

\[
V(G[H]) = V(G) \times V(H),
\]
\[
E(G[H]) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G) \lor (u_1 = u_2 \land v_1v_2 \in E(H))\}.
\]

A partial edge-coloring of $G$ is a coloring of some of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a proper edge-coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ (spectrum of a vertex $v$) denotes the set of colors appearing on edges incident to $v$. The smallest and largest colors of $S(v, \alpha)$ are denoted by $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$, respectively. A proper edge-coloring $\alpha$ of $G$ with consecutive integers $c_1, \ldots, c_t$ is called an interval $t$-coloring if all colors are used, and for any $v \in V(G)$, the set $S(v, \alpha)$ is an interval of integers. A graph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the smallest and the largest values of $t$ for which it has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively.

In [1, 2], Asratian and Kamalian obtained the following result.

**Theorem 1** If $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Moreover, if $G$ is a regular graph, then $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.

In [16], Kamalian proved the following result on complete bipartite graphs.

**Theorem 2** For any $m, n \in \mathbb{N}$, the complete bipartite graph $K_{m,n}$ is interval colorable, and

1. $w(K_{m,n}) = m + n - \gcd(m,n)$,
2. $W(K_{m,n}) = m + n - 1$,
3. if $w(K_{m,n}) \leq t \leq W(K_{m,n})$, then $K_{m,n}$ has an interval $t$-coloring.

In [18], König proved the following result on bipartite graphs.

**Theorem 3** If $G$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.
Let $\alpha$ be a proper edge-coloring of $G$ and $V' = \{v_1, \ldots, v_k\} \subseteq V(G)$. Consider the sets $S(v_1, \alpha), \ldots, S(v_k, \alpha)$. For a coloring $\alpha$ of $G$ and $V' \subseteq V(G)$, define two ordered sequences $LSE(V', \alpha)$ (Lower Spectral Edge) and $USE(V', \alpha)$ (Upper Spectral Edge) as follows:

$$LSE(V', \alpha) = (S(v_{i_1}, \alpha), S(v_{i_2}, \alpha), \ldots, S(v_{i_k}, \alpha)),$$

where $S(v_{i_l}, \alpha) \leq S(v_{i_{l+1}}, \alpha)$ for $1 \leq l \leq k-1$, and

$$USE(V', \alpha) = (S(v_{j_1}, \alpha), S(v_{j_2}, \alpha), \ldots, S(v_{j_k}, \alpha)),$$

where $S(v_{j_l}, \alpha) \leq S(v_{j_{l+1}}, \alpha)$ for $1 \leq l \leq k-1$.

$$G:$$

Figure 1. The graph $G$ with its coloring $\alpha$ and with $LSE(V(G), \alpha) = (1, 1, 2, 2, 4)$, $USE(V(G), \alpha) = (2, 2, 3, 4, 4)$.

For example, if we consider the graph $G$ with its coloring $\alpha$ shown in Fig. 1, then $LSE(V(G), \alpha) = (1, 1, 2, 2, 4)$ and $USE(V(G), \alpha) = (2, 2, 3, 4, 4)$. Moreover, the sequence $(1, 1, 2, 2, 4)$ is not continuous, but the sequence $(2, 2, 3, 4, 4)$ is continuous.

Recall that for ordered sequences $LSE(V', \alpha)$ and $USE(V', \alpha)$, the number of elements in $LSE(V', \alpha)$ and $USE(V', \alpha)$ is denoted by $|LSE(V', \alpha)|$ and $|USE(V', \alpha)|$, respectively. Clearly, $|LSE(V(G), \alpha)| = |USE(V(G), \alpha)| = |V(G)|$.

We also need the following lemma.

**Lemma 4** If $K_{n,n}$ is a complete bipartite graph with bipartition $(U, V)$, then for any continuous sequence $L$ with length $n$, $K_{n,n}$ has an interval coloring $\alpha$ such that

$LSE(U, \alpha) = LSE(V, \alpha) = L$.

**Proof.** Let $K_{n,n}$ be a complete bipartite graph with bipartition $(U, V)$, where $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. Also, let $L = \left(\underbrace{l_1, \ldots, l_1}_{n_1}, \underbrace{l_2, \ldots, l_2}_{n_2}, \ldots, \underbrace{l_k, \ldots, l_k}_{n_k}\right)$ be a continuous sequence with length $n \left(\sum_{i=1}^{k} n_i = n\right)$. Clearly, $l_{i+1} = l_i + 1$ for $1 \leq l \leq k-1$.

First we define a partial edge-coloring $\alpha$ of $K_{n,n}$ as follows:
Figure 2. The interval coloring $\gamma$ of $K_{5,5}$ with $LSE(U, \gamma) = LSE(V, \gamma) = (2, 2, 3, 4, 4)$

1) for $1 \leq i \leq k - 1$ and $p + q = 1 + \sum_{j=1}^{i} n_j$, let $\alpha(u_pv_q) = l_i$;

2) for $1 \leq i \leq k - 1$ and $p + q = n + 1 + \sum_{j=1}^{i} n_j$, let $\alpha(u_pv_q) = l_i + n$.

Define a subgraph $G$ of $K_{n,n}$ as follows:

$$V(G) = V(K_{n,n})$$

and

$$E(G) = \{e : e \in E(K_{n,n}) \land \alpha(e) \in [l_1, l_{k-1}] \cup [l_1 + n, l_{k-1} + n]\}.$$

By the definition of $\alpha$, $G$ is a spanning $(k - 1)$-regular bipartite subgraph of $K_{n,n}$. Next we define a subgraph $G'$ of $K_{n,n}$ as follows:

$$V(G) = V(K_{n,n})$$

and

$$E(G') = E(K_{n,n}) \setminus E(G).$$

Clearly, $G'$ is a spanning $(n - k + 1)$-regular bipartite subgraph of $K_{n,n}$. By Theorem $\text{3} \chi'(G') = \Delta(G') = n - k + 1$. Let $\beta$ be a proper edge-coloring of $G'$ with colors $l_k, l_{k+1}, \ldots, l_k + n - k$. By the definition of $\beta$, for each vertex $v \in V(K_{n,n})$, $S(v, \beta) = [l_k, l_k + n - k]$.

Now we are able to define an edge-coloring $\gamma$ of $K_{n,n}$.

For every $e \in E(K_{n,n})$, let

$$\gamma(e) = \begin{cases} 
\alpha(e), & \text{if } e \in E(G), \\
\beta(e), & \text{if } e \in E(G'). 
\end{cases}$$

Let us prove that $\gamma$ is an interval $(l_k + n - 1)$-coloring of $K_{n,n}$ such that $S(u_i, \gamma) = S(v_i, \gamma)$ and $S(u_i, \gamma) = S(v_i, \gamma) = l_i$ for $1 \leq i \leq n$.

By the definition of $\gamma$, for $1 \leq i \leq n$, we have
For a special type, then Theorem 5 says that if \( G \) is continuous, then

\[
S(u_i, \gamma) = S(v_i, \gamma) = [l_1, l_1 + n - 1] \text{ if } i \in [1, n_1],
\]

\[
S(u_i, \gamma) = S(v_i, \gamma) = [l_2, l_2 + n - 1] \text{ if } i \in [n_1 + 1, n_1 + n_2],
\]

\[
S(u_i, \gamma) = S(v_i, \gamma) = [l_k, l_k + n - 1] \text{ if } i \in \left[ \sum_{j=1}^{k-1} n_j + 1, \sum_{j=1}^{k} n_j \right].
\]

This implies that \( \gamma \) is an interval \((l_k + n - 1)\)-coloring of \( K_{n,n} \) and \( LSE(U, \gamma) = LSE(V, \gamma) = L \). \( \square \)

Fig. 2 shows the interval coloring \( \gamma \) of \( K_{5,5} \) described in the proof of Lemma 4.

3. The Main Result

Here, we prove our main result which states that if \( G \in \mathfrak{R} \) and \( H \) has an interval coloring of a special type, then \( G[H] \in \mathfrak{R} \).

**Theorem 5** If \( G \in \mathfrak{R} \) and \( H \) has an interval coloring \( \alpha_H \) such that \( USE(V(H), \alpha_H) \) is continuous, then \( G[H] \in \mathfrak{R} \). Moreover, if \( |V(H)| = n \) and \( L = USE(V(H), \alpha_H) \), then

\[
w(G[H]) \leq w(G) \cdot n + L \quad \text{and} \quad W(G[H]) \geq W(G) \cdot n + L.
\]

**Proof.** Let \( V(G) = \{u_1, \ldots, u_m\} \), \( V(H) = \{w_1, \ldots, w_n\} \) and

\[
V(G[H]) = \left\{ v_j^{(i)} : 1 \leq i \leq m, 1 \leq j \leq n \right\}
\]

\[
E(G[H]) = \left\{ v_p^{(i)} v_q^{(j)} : u_i u_j \in E(G), 1 \leq p \leq n, 1 \leq q \leq n \right\} \cup \bigcup_{i=1}^{m} E^i,
\]

where \( E^i = \left\{ v_p^{(i)} v_q^{(j)} : w_p w_q \in E(H) \right\} \).

Let \( \alpha_G \) be an interval \( t \)-coloring of \( G \) and \( L \) be a continuous sequence with length \( n \) such that \( L = USE(V(H), \alpha_H) \). Without loss of generality we may assume that vertices of \( H \) are numbered so that \( \overline{S}(w_i, \alpha_H) = L(i) \) for \( 1 \leq i \leq n \). Let us consider the graph \( K_2[H] \). Clearly, \( K_2[H] \) is isomorphic to \( K_{n,n} \). Let \( V(K_2[H]) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) and \( E(K_2[H]) = \{x_i y_j : 1 \leq i \leq n, 1 \leq j \leq n\} \). Since \( L \) is a continuous sequence, \( L \oplus 1 \) is a continuous sequence, too. By Lemma 4, \( K_2[H] \) has an interval coloring \( \beta \) such that \( \overline{S}(x_i, \beta) = \overline{S}(y_i, \beta) = L(i) + 1 \) for \( 1 \leq i \leq n \).

Now we are able to define an edge-coloring \( \alpha_{G[H]} \) of \( G[H] \).

1) For \( 1 \leq i \leq m \) and \( v_p^{(i)} v_q^{(i)} \in E^i \) \( (p, q = 1, \ldots, n) \), let

\[
\alpha_{G[H]} \left( v_p^{(i)} v_q^{(i)} \right) = \left( \overline{S}(u_i, \alpha_G) - 1 \right) n + \alpha_H(w_p w_q).
\]

2) For \( 1 \leq i < j \leq m \) and \( v_p^{(i)} v_q^{(j)} \in E(G[H]) \) \( (p, q = 1, \ldots, n) \), let
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Moreover, \( w(G[H]) \leq w(G) \cdot n + \overline{L} \) and \( W(G[H]) \geq W(G) \cdot n - \overline{L} \). \( \square \)

**Corollary 6** If \( G, H \in \mathcal{R} \) and \( H \) is an \( r \)-regular graph, then \( G[H] \in \mathcal{R} \). Moreover, if \( |V(H)| = n \), then

\[
 w(G[H]) \leq w(G) \cdot n + r \text{ and } W(G[H]) \geq W(G) \cdot n + r. 
\]

**Proof.** Since \( H \in \mathcal{R} \) and \( H \) is an \( r \)-regular graph, by Theorem 4 \( \chi'(H) = \Delta(H) = r \). This implies that \( H \) has a proper edge-coloring \( \alpha_H \) with colors \( 1, \ldots, r \). Hence, for every \( v \in V(H) \), \( S(v, \alpha_H) = [1, r] \). Clearly, \( \alpha_H \) is an interval \( r \)-coloring and \( USE(V(H), \alpha_H) = (r, \ldots, r) \) is continuous, so, by Theorem 5 \( G[H] \in \mathcal{R} \). Moreover, if \( |V(H)| = n \), then \( w(G[H]) \leq w(G) \cdot n + r \) and \( W(G[H]) \geq W(G) \cdot n + r. \) \( \square \)

**Corollary 7** If \( G \in \mathcal{R} \), then \( G[\overline{K}_n] \in \mathcal{R} \) for any \( n \in \mathbb{N} \). Moreover, \( w(G[\overline{K}_n]) \leq w(G) \cdot n \) and \( W(G[\overline{K}_n]) \geq W(G) \cdot n. \)

**Proof.** We may assume that \( \overline{K}_n \) has an interval coloring \( \alpha \) such that \( USE(V(\overline{K}_n), \alpha) = (0, \ldots, 0) \). Since \( USE(V(\overline{K}_n), \alpha) = (0, \ldots, 0) \) is continuous, by Theorem 5 \( G[\overline{K}_n] \in \mathcal{R} \). Moreover, \( w(G[\overline{K}_n]) \leq w(G) \cdot n \) and \( W(G[\overline{K}_n]) \geq W(G) \cdot n. \) \( \square \)

Fig. 3 shows the interval 14-coloring \( \alpha_{P_3[H]} \) of \( P_3[H] \) described in the proof of Theorem 5.
4. Applications of the Main Result

This section is devoted to applications of the main result from the previous section for some classes of graphs. We first consider complete bipartite graphs.

**Theorem 8** If \( G \in \mathfrak{N} \), then \( G[K_{m,n}] \in \mathfrak{N} \) for any \( m, n \in \mathbb{N} \). Moreover, for any \( m, n \in \mathbb{N} \), we have

\[
\begin{align*}
    w(G[K_{m,n}]) & \leq (w(G) + 1)(m + n) - 1 \\
    W(G[K_{m,n}]) & \geq (W(G) + 1)(m + n) - 1.
\end{align*}
\]

**Proof.** Let \((U, V)\) be a bipartition of \( K_{m,n} \), where \( U = \{u_1, \ldots, u_m\} \) and \( V = \{v_1, \ldots, v_n\} \). Define an edge-coloring \( \alpha \) of \( K_{m,n} \) as follows: for each edge \( u_iv_j \in E(K_{m,n}) \), let \( \alpha(u_iv_j) = i + j - 1 \), where \( 1 \leq i \leq m, 1 \leq j \leq n \). Clearly, \( \alpha \) is an interval \((m + n - 1)\)-coloring of \( K_{m,n} \). Moreover, \( S(u_i, \alpha) = [i, i + n - 1] \) for \( 1 \leq i \leq m \) and \( S(v_j, \alpha) = [j, j + m - 1] \) for \( 1 \leq j \leq n \). This implies that \( \text{USE}(U, \alpha) = (n, n + 1, \ldots, m + n - 1) \) and \( \text{USE}(V, \alpha) = (m, m + 1, \ldots, m + n - 1) \). Since \( \text{USE}(V(K_{m,n}), \alpha) \) is the union of \( \text{USE}(U, \alpha) \) and \( \text{USE}(V, \alpha) \), we obtain \( \text{USE}(V(K_{m,n}), \alpha) \) is a continuous sequence. By Theorem 5, \( G[K_{m,n}] \in \mathfrak{N} \). Moreover, \( w(G[K_{m,n}]) \leq w(G) \cdot (m + n) + m + n - 1 \) and \( W(G[K_{m,n}]) \geq W(G) \cdot (m + n) + m + n - 1 \). □

Next, we consider complete graphs of even order. Here we need one result on interval colorings of complete graphs of even order. In [24], it was proved the following result.
Theorem 9 For any \( n \in \mathbb{N} \), \( K_{2n} \) has an interval \((3n-2)\)-coloring \( \alpha \) such that for each \( i \in [1,n] \), there are vertices \( v'_i, v''_i \in V(K_{2n}) \) \((v'_i \neq v''_i)\) with \( S(v'_i, \alpha) = S(v''_i, \alpha) = i \).

Now we are able to prove our result on complete graphs of even order.

Theorem 10 If \( G \in \mathcal{K} \), then \( G[K_{2n}] \in \mathcal{K} \) for any \( n \in \mathbb{N} \). Moreover, for any \( n \in \mathbb{N} \), we have

\[
w(G[K_{2n}]) \leq (2 \cdot w(G) + 2)n - 1 \quad \text{and} \quad W(G[K_{2n}]) \geq (2 \cdot W(G) + 3)n - 2.
\]

Proof. By Corollary 6 if \( G \in \mathcal{K} \), then \( G[K_{2n}] \in \mathcal{K} \) and \( w(G[K_{2n}]) \leq w(G) \cdot 2n + 2n - 1 \) for any \( n \in \mathbb{N} \).

Now we show that \( W(G[K_{2n}]) \geq (2 \cdot W(G) + 3)n - 2 \). By Theorem 9 \( K_{2n} \) has an interval \((3n-2)\)-coloring \( \alpha \) such that for each \( i \in [1,n] \), there are vertices \( v'_i, v''_i \in V(K_{2n}) \) \((v'_i \neq v''_i)\) with \( S(v'_i, \alpha) = S(v''_i, \alpha) = [i, i + 2n - 2] \). This implies that \( USE(V(K_{2n}), \alpha) = (2n - 1, 2n - 1, 2n, 2n, \ldots, 3n - 2, 3n - 2) \), which is a continuous sequence. By Theorem 6 \( G[K_{2n}] \in \mathcal{K} \) and \( W(G[K_{2n}]) \geq W(G) \cdot 2n + 2n - 2 \).

A similar result also can be obtained for even cycles.

Theorem 11 If \( G \in \mathcal{K} \), then \( G[C_{2n}] \in \mathcal{K} \) for any integer \( n \geq 2 \). Moreover, for any integer \( n \geq 2 \), we have

\[
w(G[C_{2n}]) \leq 2w(G) \cdot n + 1 \quad \text{and} \quad W(G[C_{2n}]) \geq (2 \cdot W(G) + 1)n + 1.
\]

Proof. By Corollary 6 if \( G \in \mathcal{K} \), then \( G[C_{2n}] \in \mathcal{K} \) and \( w(G[C_{2n}]) \leq w(G) \cdot 2n + 2 \) for any \( n \geq 2 \).

Now we show that \( W(G[C_{2n}]) \geq (2 \cdot W(G) + 1)n + 1 \). Let \( V(C_{2n}) = \{v_1, \ldots, v_{2n}\} \) and \( E(C_{2n}) = \{v_i v_{i+1} : 1 \leq i \leq 2n - 1\} \cup \{v_1 v_{2n}\} \). Define an edge-coloring \( \alpha \) of \( C_{2n} \) as follows: for \( 1 \leq i \leq n \), let \( \alpha(v_i v_{i+1}) = \alpha(v_{2n+1-i} v_{2n-i}) = i + 1 \) and \( \alpha(v_1 v_{2n}) = 1 \). Clearly, \( \alpha \) is an interval \((n+1)\)-coloring of \( C_{2n} \) such that for each \( i \in [1,n] \), \( S(v_i, \alpha) = S(v_{2n+1-i}, \alpha) = [i, i + 1] \). This implies that \( USE(V(C_{2n}), \alpha) = (2, 2, 3, 3, \ldots, n + 1, n + 1) \), which is a continuous sequence. By Theorem 5 \( G[C_{2n}] \in \mathcal{K} \) and \( W(G[C_{2n}]) \geq W(G) \cdot 2n + n + 1 \).

Finally, we show that every tree \( T \) has an interval coloring \( \alpha \) such that \( USE(V(T), \alpha) \) is continuous.

Theorem 12 If \( T \) is a tree, then it has an interval coloring \( \alpha \) such that \( USE(V(T), \alpha) \) is continuous.

Proof. Let \( T \) be a tree with \(|V(T)| = n \ (n \geq 2)\). We prove the theorem by induction on \(|E(T)|\). We will construct tree \( T \) starting from some \( v_1 v_2 \) edge and adding a new leaf on each step. For \( 1 \leq i \leq n - 1 \), we denote by \( T_i \) the tree obtained on step \( i \) and by \( \alpha_i \) its edge-coloring. For a tree \( T_i \) and its edge-coloring \( \alpha_i \ (1 \leq i \leq n - 1) \), define numbers \( a_i \) and \( b_i \) as follows:

\[
a_i = \min_{e \in E(T_i)} \alpha_i(e) \quad \text{and} \quad b_i = \max_{e \in E(T_i)} \alpha_i(e).
\]
We show that in each step $T_i$ and $\alpha_i$ satisfy the following two conditions:

(1) for each $v \in V(T_i)$, $S(v, \alpha_i)$ is an interval of integers;

(2) each color of the interval $[a_i, b_i]$ appears in $USE(V(T_i), \alpha_i)$.

Let $V(T_1) = \{v_1, v_2\}$ and $E(T_1) = \{v_1, v_2\}$. Define an edge-coloring $\alpha_1$ of $T_1$ as follows: $\alpha_1(v_1v_2) = |E(T)|$. Since $S(v_1, \alpha_1) = S(v_2, \alpha_1) = \{|E(T)|\}$, we have $a_1 = b_1 = |E(T)|$ and $USE(V(T_1), \alpha_1) = \{|E(T)|, |E(T)|\}$. This implies that (1) and (2) hold for $T_1$. Suppose that $n \geq 3$, (1) and (2) are satisfied for a tree $T_{m-1}$ and its edge-coloring $\alpha_{m-1}$, and prove that (1) and (2) are also satisfied for a tree $T_m$ and its edge-coloring $\alpha_m$ ($2 \leq m \leq n - 1$). Let $u$ be the pendant vertex that should be added to $T_{m-1}$ to get $T_m$. Also, let $uw \in E(T_m)$, where $w \in V(T_{m-1})$.

Define an edge-coloring $\alpha_m$ of $T_m$ as follows: for every $e \in E(T_m)$, let

$$\alpha_m(e) = \begin{cases} 
\alpha_{m-1}(e), & \text{if } e \in E(T_{m-1}), \\
\overline{S}(w, \alpha_{m-1}) - 1, & \text{if } e = uw.
\end{cases}$$

By the definition of $\alpha_m$, we have:

1) for each $v \in V(T_m)$, $S(v, \alpha_m)$ is an interval of integers;

2) for $v \in V(T_{m-1})$, $\overline{S}(v, \alpha_m) = \overline{S}(v, \alpha_{m-1})$ and $USE(V(T_m), \alpha_m)$ is the union of $USE(V(T_{m-1}), \alpha_{m-1})$ and $(\alpha_m(uw))$;

3) $a_m = \min\{a_{m-1}, \alpha_m(uw)\}$, $b_m = b_{m-1}$ and $\alpha_m(uw) = \overline{S}(w, \alpha_{m-1}) - 1 \geq a_{m-1} - 1$.

By 1), 2) and 3), and taking into account that each color of the interval $[a_{m-1}, b_{m-1}]$ appears in $USE(V(T_{m-1}), \alpha_{m-1})$, we obtain that each color of the interval $[a_m, b_m]$ appears in $USE(V(T_m), \alpha_m)$. This implies that (1) and (2) also hold for $T_m$. So, taking $m = n - 1$, we get that $T = T_{n-1}$. Finally, define an edge-coloring $\alpha$ of $T$ as follows: for every $e \in E(T)$, let $\alpha(e) = \alpha_{n-1}(e) - a_{n-1} + 1$. It is not difficult to see that $\alpha$ is an interval $|E(T)| - a_{n-1} + 1$-coloring of $T$ such that $USE(V(T), \alpha)$ is continuous. □

Corollary 13 If $G \in \mathfrak{R}$ and $T$ is a tree, then $G[T] \in \mathfrak{R}$.

5. Concluding Remarks

In the previous sections it was proved that if $G \in \mathfrak{R}$ and $H$ has an interval coloring $\alpha_H$ such that $USE(V(H), \alpha_H)$ is continuous, then $G[H] \in \mathfrak{R}$. Unfortunately, not all interval colorable graphs have such a special interval coloring. For example, if we consider the complete tripartite graph $K_{1,1,2n}$ ($n \geq 2$), then it is not difficult to see that for every interval coloring $\alpha$ of $K_{1,1,2n}$ ($n \geq 2$), $USE(V(K_{1,1,2n}), \alpha)$ is not continuous. This implies that the problem on interval colorability of the composition of interval colorable graphs still remains open.
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