Classical invariants of Legendrian knots in the 3-dimensional torus

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Abstract

All knots in $\mathbb{R}^3$ possess Seifert surfaces, and so the classical Thurston-Bennequin and rotation (or Maslov) invariants for Legendrian knots in a contact structure on $\mathbb{R}^3$ can be defined. The definitions extend easily to null-homologous knots in any 3-manifold $M$ endowed with a contact structure $\xi$. We generalize the definition of Seifert surfaces and use them to define these invariants for all Legendrian knots, including those that are not null-homologous, in a contact structure on the 3-torus $T^3$. We show how to compute the Thurston-Bennequin and rotation invariants in a tight oriented contact structure on $T^3$ using projections.

Keywords:
Legendrian knots, Thurston-Bennequin invariant, Maslov invariant, contact structures, 3-torus $T^3$, Seifert surfaces.

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1. Introduction and Statement of Results

Let $\xi$ be an oriented contact structure on a smooth oriented 3-manifold $M$, i.e., a 2-plane field that locally is the kernel of a totally non-integrable 1-form $\omega$, so that locally $\xi = \ker(\omega)$ with the induced orientation and $\omega \wedge d\omega$ is non-vanishing. A knot in a smooth 3-manifold $M$ is a smooth embedding $\alpha : S^1 \to M$. A knot in the contact manifold $(M, \xi)$ is Legendrian if $\alpha$ is
everywhere tangent to $\xi$. Two Legendrian knots $\alpha_0$ and $\alpha_1$ are Legendrian homotopic if there is a smooth 1-parameter family of Legendrian knots $\alpha_t$, $t \in [0, 1]$, that connects them.

The Thurston-Bennequin and rotation (or Maslov) numbers are well-known classical invariants of null-homologous oriented Legendrian knots in $(M^3, \xi)$ that depend only on their Legendrian homotopy class and, for the rotation invariant, on a fixed Legendrian vector field $\mathbf{Z}$ [1, 5]. Given a Seifert surface $\Sigma$ for the Legendrian knot $\alpha$ in $(M^3, \xi)$, the Thurston-Bennequin invariant $\text{tb}(\alpha)$ is defined to be the number of times the contact plane $\xi$ rotates relative to the tangent plane to $\Sigma$ in one circuit of $\alpha$. The rotation invariant $r(\alpha)$ is the number of times the tangent vector $\alpha'$ rotates in $\xi$ relative to a fixed Legendrian vector field $\mathbf{Z}$ in a single circuit of $\alpha$. For the standard contact structure $\xi_{std} = \ker(dx - ydz)$ on $\mathbb{R}^3$, both invariants of a generic Legendrian knot can be calculated using the front and Lagrangian projections of $\mathbb{R}^3$ to $\mathbb{R}^2$ (see Sections 3.3 and 4.1).

On the 3-dimensional torus $T^3$, we define generalized Seifert surfaces for knots that are not null-homologous (Definition 2.1) and use them to extend the definition of the Thurston-Bennequin invariant to all Legendrian knots in an arbitrary contact structure $\xi$ on $T^3$. Let $\alpha : S^1 = \mathbb{R}/\mathbb{Z} \to T^3$ be a knot in $T^3$ and let $\tilde{\alpha} : \mathbb{R} \to \mathbb{R}^3$ be a lift of $\alpha$ to the universal cover $\tilde{T}^3 = \mathbb{R}^3$, where $T^3$ is identified with $\mathbb{R}^3/\mathbb{Z}^3$. If a component of $\tilde{\alpha}$ is compact, then it is a knot on $\mathbb{R}^3$ and the usual definition of Seifert surfaces applies. Hence we usually assume the following General Hypothesis, and then prove the following Proposition.

**General Hypothesis 1.1.** Each component of $\tilde{\alpha}$ is assumed to be non-compact. (Equivalently, $\alpha$ is not contractible in $T^3$.)

Note that the components of $\tilde{\alpha}$ are homeomorphic to each other, so if one component is non-compact, then all of them are. In this case $\tilde{\alpha}$ will be periodic, say with smallest period $(p, q, r) \in \mathbb{Z}^3$. We define a connected oriented surface $\Sigma \subset \mathbb{R}^3$ to be a covering Seifert surface for such a knot $\alpha$ in $T^3$ if $\Sigma$ is $(p, q, r)$-periodic, $\partial \Sigma$ is one component of $\tilde{\alpha}$, and outside a tubular neighborhood of that component, $\Sigma$ coincides with an affine half-plane in $\mathbb{R}^3$. Then we call $\hat{\Sigma} = \Sigma/\mathbb{Z}(p, q, r)$ a (generalized) Seifert surface for $\alpha$ (See Definition 2.1). The arguments using covering Seifert surfaces $\Sigma \subset \mathbb{R}^3$ and Seifert surfaces $\hat{\Sigma} \subset \mathbb{R}^3/\mathbb{Z}(p, q, r)$ are parallel and equivalent, so some attention is needed to distinguish the two types of Seifert surfaces.
Proposition 1.2. Let $\alpha$ be a smooth knot on $T^3$ whose lift $\tilde{\alpha}$ has all components non-compact and let $\hat{\alpha} = \tilde{\alpha}/\mathbb{Z}(p,q,r)$. Then

1. There is a covering Seifert surface $\Sigma$ with corresponding (generalized) Seifert surface $\hat{\Sigma}$ for the knot $\alpha$;
2. If $\Sigma_1$ and $\Sigma_2$ are both covering Seifert surfaces for the knot $\alpha$, then the relative rotation number $\rho(\Sigma_1, \Sigma_2)$, defined to be the number of times $X_2$ rotates relative to $X_1$ in the normal plane to $\tilde{\alpha}$ in one circuit of $\alpha$, where $X_i$ ($i = 1,2$) is a unit vector field tangent to $\Sigma_i$ and orthogonal to $\tilde{\alpha}'$, is zero.

Using these Seifert surfaces, we extend the definition of the Thurston-Bennequin invariant to $(T^3, \xi)$, as follows. The Thurston-Bennequin invariant $tb(\alpha)$ for a Legendrian knot $\alpha$ in $(T^3, \xi)$ is defined to be the rotation number of the contact plane $\xi$ with respect to a (generalized) Seifert surface $\hat{\Sigma}$ for $\alpha$ in one circuit of $\alpha$ (See Definition 3.1).

Theorem 1. 
1. If $\alpha$ is a Legendrian knot in $(T^3, \xi)$ satisfying the General Hypothesis 1.1, then the Thurston-Bennequin invariant $tb(\alpha)$ is well-defined.
2. If $\alpha$ satisfies 1.1 but is null-homologous, then our definition of $tb(\alpha)$ agrees with the standard definition of $tb(\alpha)$ in $T^3$.
3. If $\alpha$ does not satisfy 1.1, so that $\alpha$ is contractible, then the standard definitions of $tb(\alpha)$ and $tb(\tilde{\alpha})$ coincide, where $\tilde{\alpha}$ and $\tilde{\xi}$ are the lifts of $\alpha$ and $\xi$ to the universal cover $(\tilde{T}^3, \tilde{\xi})$ of $(T^3, \xi)$.

This shows that Definition 3.1 extends the usual definition in $\mathbb{R}^3$. Recall that Kanda [9] defines a Thurston-Bennequin invariant for quasilinear Legendrian knots on $T^3$, i.e., those that are isotopic to knots with constant slope, using an incompressible torus containing the knot to replace the Seifert surface. In the universal cover, the torus lifts to a plane, half of which is isotopic to our covering Seifert surface, so the following result holds.

Proposition 1.3. For quasilinear knots in $T^3$, our definition of $tb(\alpha)$ agrees with the definition of Kanda [9].

We shall be especially interested in the tight contact structures

$$\xi_n = \ker(\cos(2\pi nz)dx + \sin(2\pi nz)dy)$$
on $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$, where $n$ is a positive integer. Kanda \cite{9} has shown that for every tight contact structure $\xi$ on $T^3$ there is a contactomorphism (i.e., a diffeomorphism that preserves the contact structure) from $\xi$ to $\xi_n$, for some $n > 0$.

Define projections $p_{xy}, p_{xz} : T^3 \to T^2$ by setting $p_{xy}(x, y, z) = (x, y)$ and $p_{xz}(x, y, z) = (x, z)$, where $x, y, z$ are the coordinates modulo 1 in $T^3$ and in $T^2$. The projection $p_{xy}$ is called the front projection, for if we identify $T^3$ with the space of co-oriented contact elements on $T^2$, then the wave fronts of the propagation of a wave on $T^2$ are images under $p_{xy}$ of Legendrian curves in $T^3$. Then a knot in $T^3$ is generic relative to both projections $p_{xy}$ and $p_{xz}$ if its curvature vanishes only at isolated points and the only singularities of the projected knot are transverse double points and cusps. Note that every Legendrian knot can be made generic by an arbitrarily small Legendrian homotopy. The following Theorem shows how to calculate both invariants of generic Legendrian knots for $\xi_n$ using the projections $p_{xy}, p_{xz} : T^3 \to T^2$.

**Theorem 2.** Let $\alpha$ be a generic oriented Legendrian knot in $(T^3, \xi_n)$.

1. For the projection $p_{xy}$ of $\alpha$, $tb(\alpha) = P - N + C/2$, where $P$ and $N$ are the numbers of positive and negative crossings and $C$ is the number of cusps for $p_{xy} \circ \alpha$ in one circuit of $\alpha$;
2. For the projection $p_{xz}$ of $\alpha$, $tb(\alpha) = P - N$, where $P$ and $N$ are the numbers of positive and negative crossings for $p_{xz} \circ \alpha$ in one circuit of $\alpha$, and there are no cusps;
3. For the projection $p_{xy}$ of $\alpha$, the rotation invariant relative to the Legendrian vector field $Z = \partial / \partial z$ is $r(\alpha) = 1/2(C_+ - C_-)$, where $C_+$ and $C_-$ are the numbers of positive and negative cusps of $p_{xy} \circ \alpha$ in one circuit of $\alpha$;
4. Let $V = \{ t \in S^1 \mid (x'(t), y'(t)) = (0, 0) \}$ and suppose that $2nz(t) \notin \mathbb{Z}$ for every $t \in V$. Then for the projection $p_{xz}$ of $\alpha$, the rotation invariant relative to $Z = \partial / \partial z$ is $r(\alpha) = 1/2 \sum_{t \in V} a(t)b(t)$, where $a(t) = (-1)^{2nz(t)}$ and $b(t) = \pm 1$ according to whether $p_{xz} \circ \alpha'(t)$ is turning in the positive or negative direction in the $xz$-plane.

Generalized Seifert surfaces for knots in $T^3$ will be defined and studied in §2. The Thurston-Bennequin invariant $tb(\alpha)$ and the rotation invariant $r(\alpha)$ for a generic oriented Legendrian knot in $T^3$ will be treated in §3 and §4, respectively. The proofs of the four assertions of Theorem 2 are given
in the proofs of the Propositions 3.5, 3.3, 4.6, and 4.5, respectively, in the Subsections 3.3 and 4.1. In the last Section 5 we calculate the invariants for quasilinear Legendrian knots in $\left( T^3, \xi \right)$ and observe that the Bennequin inequality for null-homologous Legendrian knots in a tight contact structure has to be modified in this case. Finally we make a conjecture about the extension of the Bennequin inequality for tight contact structures on $T^3$.

This paper is a continuation of the work of the second author in his masters thesis [11] at the Pontifícia Universidade Católica of Rio de Janeiro under the direction of the first author.

2. Seifert surfaces in $T^3$

In this section we consider smooth knots in $T^3$ and their Seifert surfaces, without reference to any contact structure, as a preparation for studying Legendrian knots and their Thurston-Bennequin and rotation invariants in the next two sections.

Figure 1: Inserting a twisted strip at a crossing.

Recall that a *Seifert surface* for an oriented knot (or link) $\alpha$ in $\mathbb{R}^3$ is a compact connected oriented surface $\Sigma$ whose boundary is $\alpha$ with the induced orientation. Every knot and link $\alpha$ has Seifert surfaces, and there is a well-known method of constructing one using a regular projection of $\alpha$ in the plane (2, pp. 16-18). Each crossing is replaced by a non-crossing that respects the orientation, the resulting circles are capped off by disjoint embedded disks, and then a twisted interval is inserted at each crossing, as in Figure 1. Finally the *Seifert circles*, the boundary of the resulting surface, are capped off by disjoint embedded disks.

Let $\alpha : S^1 = \mathbb{R}/\mathbb{Z} \to T^3$ be a knot in $T^3$ and let $\widetilde{\alpha} : \mathbb{R} \to \mathbb{R}^3$ be a lift of $\alpha$ to the universal cover $\widetilde{T^3} = \mathbb{R}^3$, where $T^3$ is identified with $\mathbb{R}^3/\mathbb{Z}^3$. Let $(p, q, r) \in \mathbb{Z}^3$ be a generator of the cyclic group of translations that preserve $\widetilde{\alpha}$. Note that $(p, q, r)$ is determined up to multiplication by $\pm 1$ and we choose
the sign so that $\tilde{\alpha}(t+1) = \tilde{\alpha}(t) + (p, q, r)$. Any subset of $\mathbb{R}^3$ that is invariant under this group will be said to be $(p, q, r)$-periodic. The following definition adapts the classical concept of Seifert surfaces for knots in $\mathbb{R}^3$ to the present context.

**Definition 2.1.** Let $\alpha$ be a knot in $T^3$ whose lift $\tilde{\alpha}$ has non-compact components and let $(p, q, r)$ generate the cyclic group of translations that preserve $\tilde{\alpha}$. A smooth surface $\Sigma \subset \mathbb{R}^3$ is a covering Seifert surface for $\alpha$ if it satisfies the following conditions:

1. $\Sigma$ is connected, orientable, properly embedded in $\mathbb{R}^3$, and $(p, q, r)$-periodic;
2. $\partial \Sigma = \tilde{\alpha}$; and
3. There is an affine half-plane $P_+ \subset \mathbb{R}^3$ with boundary a straight line $S$ such that $\Sigma$ coincides with $P_+$ outside a $\delta$-neighborhood $N$ of $S$, for some sufficiently large $\delta$.

In this case we say that $\hat{\Sigma} = \Sigma/\mathbb{Z}(p, q, r)$ is a (generalized) Seifert surface for $\alpha$. If the components of $\tilde{\alpha}$ are compact, then a Seifert surface for one of the components can be translated by the action of $\mathbb{Z}^3$ to give a periodic covering Seifert surface.

![Figure 2: A $(p, q, r)$-periodic knot projected into an affine plane.](image)

In this section we shall usually deal with covering Seifert surfaces, but the same properties could be developed for Seifert surfaces, and there is a complete correspondence.
Clearly the half-plane $P_+$ and its boundary $S$ are also $(p, q, r)$-periodic. It is convenient to choose $P_+$ to be disjoint from $\tilde{\alpha}$ and such that $P_+ \subset \Sigma$. Given a covering Seifert surface $\Sigma$ of $\alpha$, we define the associated vector field $X = X(\alpha, \Sigma)$ along $\tilde{\alpha}$ in $\mathbb{R}^3$ to be the $(p, q, r)$-periodic unit vector field along $\tilde{\alpha}$ that is orthogonal to $\tilde{\alpha}$, tangent to $\Sigma$, and directed towards the interior of $\Sigma$.

Given two covering Seifert surfaces $\Sigma_i$ with associated vector fields $X_i$, $i = 1, 2$, the relative rotation number $\rho(\Sigma_1, \Sigma_2)$ of $\Sigma_2$ with respect to $\Sigma_1$ is defined to be the number of revolutions that $X_2$ makes with respect to $X_1$ in the positive direction in the normal plane field to $\tilde{\alpha}$ along $\tilde{\alpha}$ from a point on $\tilde{\alpha}$ to its first $(p, q, r)$-translate in the positive direction, i.e., in one circuit of $\alpha$. Note that this number is independent of the orientation of $\alpha$, since changing the orientation of $\alpha$ also changes the orientation of the normal plane.

**Proof of Proposition 1.2.** First we construct a covering Seifert surface for a knot $\alpha$ in $T^3$ with period $(p, q, r)$. Choose a $(p, q, r)$-periodic plane $P$ meeting the proper curve $\tilde{\alpha}$ such that the orthogonal projection of $\tilde{\alpha}$ onto $P$ is regular (i.e., the only singularities are transverse double points). Fix an orientation of $\tilde{\alpha}$. The $(p, q, r)$-periodicity of both $\tilde{\alpha}$ and the affine plane $P$ permits us to adapt the classical construction of the Seifert surface of a knot in $\mathbb{R}^3$ (see [2], pp. 16-18) in a $(p, q, r)$-periodic fashion. At each crossing of the image of $\tilde{\alpha}$ in $P$, which we call the knot diagram, replace the crossing by two arcs, respecting the orientation of $\tilde{\alpha}$, and insert a twisted strip, as in Figure 1. Do this so that the resulting collection of “Seifert curves” is pairwise disjoint and $(p, q, r)$-periodic. The Seifert curves that are simple closed curves are capped off in a periodic fashion by mutually disjoint disks meeting $P$ only in their boundaries. Then there will be a number of non-compact proper $(p, q, r)$-periodic Seifert curves left over. It is easy to check that this number will be odd, say $2k + 1$, with $k + 1$ of them oriented in the positive direction of $\tilde{\alpha}$ and the other $k$ in the opposite direction. (To see this, consider a plane perpendicular to the direction $(p, q, r)$ that meets $\tilde{\alpha}$ transversely and examine the sign of the intersections of $\tilde{\alpha}$ with this plane.) These curves can be capped off in pairs with opposite orientations by disjoint oriented periodic infinite strips, starting with a pair whose projections are adjacent in the plane $P$. This process will leave one $(p, q, r)$-periodic Seifert curve which can be joined to a half plane $P_+$ contained in $P$ by another infinite periodic strip so that the result is embedded. The whole construction is done so as to preserve $(p, q, r)$-periodicity. The result is a covering Seifert surface for $\alpha$. 


that coincides with $P_+$ outside a sufficiently large tubular neighborhood $N$ of the line $S$.

Now suppose that $\Sigma_1$ and $\Sigma_2$ are two covering Seifert surfaces for $\alpha$. Take a $(p, q, r)$-periodic line $S$ in $\Sigma_1$ and a tubular neighborhood $N$ of $S$ sufficiently large so that the parts of $\Sigma_1$ and $\Sigma_2$ outside $N$ are half-planes. Remove these half-planes and add an infinite periodic strip in $\partial \bar{N}$ to connect $\Sigma_1$ and $\Sigma_2$, if necessary. Thus we obtain a new proper $(p, q, r)$-periodic surface $\Sigma$ which agrees with the union of $\Sigma_1$ and $\Sigma_2$ inside $N$. This surface $\Sigma$ will be a proper immersed surface contained in $\bar{N}$. Note that $\Sigma$ projects onto a compact oriented immersed surface on $T^3$. The following lemma will complete the proof, since by definition $\rho(\Sigma_1, \Sigma_2) = \rho(X_1, X_2)$. □

**Lemma 2.2.** Let $\Sigma$ be a properly immersed $(p, q, r)$-periodic oriented surface in $\mathbb{R}^3$ whose boundary has two components, one being $\bar{\alpha}$ with the positive orientation and the other $\tilde{\alpha}$ with the negative orientation, and which projects to a compact surface in $\widehat{T}^3 = \mathbb{R}^3/\mathbb{Z}(p, q, r)$. Let $X_1$ and $X_2$ be the vector fields associated to the two boundary components of $\Sigma$. Then the rotation number $\rho(X_1, X_2)$ of $X_2$ relative to $X_1$ along $\tilde{\alpha}$ is zero.

**Proof.** The quotient mappings $\pi' : \mathbb{R}^3 \to \widehat{T}^3$ and $\pi : \widehat{T}^3 \to T^3$ are projections of covering spaces. Note that $\widehat{T}^3$ is diffeomorphic to $\mathbb{R}^2 \times S^1$. The curve $\tilde{\alpha}$ projects under $\pi'$ to a compact knot $\hat{\alpha}$ in $\widehat{T}^3$. Let $V$ be a small closed tubular neighborhood of $\hat{\alpha}$ (so that $V$ is diffeomorphic to $S^1 \times D^2$, where $D^2$ is the closed unit disk in the plane) and set $M = \widehat{T}^3 \setminus \text{Int} \ V$. Let $\alpha_1$ and $\alpha_2$ be the loops on the torus $\partial V = \partial M$ obtained by isotoping $\alpha$ in the directions of the vector fields $X_1$ and $X_2$. We claim that their homology classes satisfy $[\alpha_1] = [\alpha_2] \in H_1(\partial V)$, which implies that the mutual rotation number $\rho(X_1, X_2)$ vanishes, as claimed.

To see this claim, note that there is a compact oriented surface $\hat{\Sigma}$ immersed in $M$ obtained from the projection of $\Sigma$ into $\widehat{T}^3$ by a small isotopy so that its boundary $\partial \hat{\Sigma}$ is the union of $\alpha_1$ and $\alpha_2$ with opposite orientations. Consequently $i''([\alpha_1] - [\alpha_2]) = 0 \in H_1 M$, where $i'' : \partial V \to M$ is the inclusion. Now let $\ell$ and $m$ be the oriented longitude and meridian of $\partial V$, so that there are integers $n_1$ and $n_2$ such that the homology classes of $\alpha_1$ and $\alpha_2$ on $\partial V$ satisfy $[\alpha_r] = [\ell] + n_r[m]$, $r = 1, 2$. Since $m$ is contractible on the solid torus $V$, $i'_*[([\alpha_1] - [\alpha_2])] = 0 \in H_1 V$, where $i' : \partial V \to V$ is the inclusion. In the Mayer-Vietoris exact sequence

$$\cdots \to H_2 \widehat{T}^3 \xrightarrow{\partial} H_1 \partial V \xrightarrow{i'_*[\cdot]} H_1 V \oplus H_1 M \xrightarrow{j_*} H_1 \widehat{T}^3 \to \cdots$$
\(i_* = i'_* + i''_*\) so \(i_*([\alpha_1] - [\alpha_2]) = 0.\) Since \(H_2(\mathbb{T}^3) \approx H_2(\mathbb{R}^3/\mathbb{Z}(p,q,r)) = 0,\) \(i_*\) is injective, so \([\alpha_1] = [\alpha_2],\) as claimed. \qed

3. The Thurston-Bennequin invariant

In this section, we extend the classical definition of the Thurston-Bennequin invariant \(tb(\alpha)\) to all Legendrian knots for an arbitrary contact structure on \(T^3,\) and we show how to compute the invariant \(tb(\alpha)\) of Legendrian knots in \((T^3, \xi)\) using projections.

3.1. The Thurston-Bennequin invariant for null-homologous knots.

First we recall the definition of \(tb(\alpha)\) for an oriented null-homologous Legendrian knot \(\alpha\) relative to a contact structure \(\xi\) on an oriented 3-manifold \(M^3.\) Since \(\alpha\) is null-homologous it has a Seifert surface \(\Sigma,\) which by definition is an oriented compact connected surface embedded in \(M^3\) with oriented boundary \(\alpha.\) Let \(X\) and \(Y\) be unit vector fields orthogonal to \(\alpha\) (with respect to a metric on \(M\)), with \(X\) tangent to \(\Sigma\) and \(Y\) tangent to \(\xi.\) Then \(tb(\alpha)\) is defined to be the algebraic number of rotations of \(Y\) relative to \(X\) in the normal plane field \(\alpha^\perp,\) which is oriented by the orientations of \(M^3\) and \(\alpha,\) as we make one circuit of \(\alpha\) in the positive direction. If we let \(\alpha^+\) be a knot obtained by pushing \(\alpha\) a short distance in the direction \(Y,\) then it is easy to see that \(tb(\alpha)\) is the intersection number of \(\alpha^+\) with \(\Sigma.\) This is just the linking number of \(\alpha^+\) with \(\alpha\) because \(\Sigma\) is a compact oriented surface with boundary \(\alpha.\) An argument analogous to the proof of Lemma 2.2, taking \(\Sigma\) to be the disjoint union of two Seifert surfaces \(\Sigma_1\) and \(\Sigma_2\) for \(\alpha\) with opposite orientations, shows that \(tb(\alpha)\) is independent of the choice of the Seifert surface.

3.2. The Thurston-Bennequin invariant in \(T^3.\)

Now consider \(T^3\) with an oriented contact structure \(\xi\) and let \(\alpha\) be a Legendrian knot in \(T^3\) with a covering Seifert surface \(\Sigma\) for \(\alpha\) using the lift \(\tilde{\alpha}\) to the universal cover \(\mathbb{R}^3.\) We can define the rotation number of the lifted contact structure \(\tilde{\xi}\) with respect to \(\Sigma\) to be the number of rotations of one of the two unit orthogonal vector fields \(Y\) along \(\tilde{\alpha}\) that is tangent to \(\tilde{\xi}\) with respect to a unit orthogonal vector field \(X\) along \(\tilde{\alpha}\) that is tangent to \(\Sigma,\) in one circuit of \(\alpha.\)
Definition 3.1. The Thurston-Bennequin invariant $tb(\alpha)$ for a Legendrian knot $\alpha$ in $T^3$ is the rotation number of the contact structure $\xi$ with respect to a covering Seifert surface $\Sigma$ for $\alpha$.

It is clear that, on $T^3 = \mathbb{R}^3/\mathbb{Z}(p,q,r)$, $tb(\alpha)$ is the rotation number of the induced contact structure $\tilde{\xi}$ with respect to a (generalized) Seifert surface $\tilde{\Sigma}$ for $\alpha$ in one circuit of $\tilde{\alpha}$. As for knots in $\mathbb{R}^3$, $tb(\alpha)$ will be the intersection number of $\tilde{\alpha}^+$, the lifted knot $\tilde{\alpha}$ pushed a short distance in a direction transverse to the lifted contact structure $\tilde{\xi}$, with the covering Seifert surface $\Sigma$, in one circuit of $\alpha$. In this case, however, if $\alpha$ is not null-homologous, then $tb(\alpha)$ is not a linking number, since $\Sigma$ will not be compact.

We note that Definition 3.1 is an extension of the above definition of $tb(\alpha)$ for a null-homologous Legendrian knot $\alpha$ in an oriented 3-manifold $M^3$ endowed with a contact structure $\xi$. As in the null-homologous case, the following holds.

Lemma 3.2. The Thurston-Bennequin invariant for a Legendrian knot $\alpha$ in $T^3$ is independent of the choice of the covering Seifert surface and the orientation of $\alpha$.

Proof. According to Proposition 1.2, the rotation number of one covering Seifert surface for the knot $\alpha$ with respect to another one is zero. Hence the rotation numbers of the two covering Seifert surfaces with respect to the contact structure coincide. Given an orientation of $\alpha$, we choose the orientation of the plane field orthogonal to $\alpha$ such that the orientations of $\alpha$ and the plane field determine the standard orientation of $T^3$, so reversing the orientation of $\alpha$ reverses the orientation of the plane orthogonal field as well and the rotation number does not change.

It is worth remarking that our extended definition of $tb(\alpha)$ continues to satisfy the usual properties: it does not change if we replace the vector field $Y$ tangent to $\xi$ by $-Y$ or by a vector field $Y^{\text{tr}}$ transverse to $\xi$, or if we use $-X$ or a vector field $X^{\text{tr}}$ transverse to $\Sigma$ in place of $X$.

3.3. Computation of $tb$ using projections

In this subsection we compute the Thurston-Bennequin invariant $tb(\alpha)$ of an oriented Legendrian knot $\alpha$ in $T^3$ relative to Kanda’s tight contact structure $\xi_n$, $n > 0$, using the projections $p_{xy}, p_{xz}: T^3 \to T^2$ defined in the Introduction and a covering Seifert surface $\Sigma$ for $\alpha$, as defined in Section 2.
First, we recall how to do this for a generic oriented Legendrian knot $\alpha$ in $\mathbb{R}^3$ with the standard contact structure $\xi_{\text{std}}$ utilizing its front and Lagrangian projections. The front (resp., Lagrangian) projection of a Legendrian knot $\alpha$ in $(\mathbb{R}^3, \xi_{\text{std}})$ is the map $\bar{\alpha} = \text{pr}_F \circ \alpha$ (resp., $\bar{\alpha} = \text{pr}_L \circ \alpha$) where the map $\text{pr}_F : \mathbb{R}^3 \to \mathbb{R}^2$ (resp., $\text{pr}_L : \mathbb{R}^3 \to \mathbb{R}^2$) is defined by $\text{pr}_F(x, y, z) = (x, z)$ (resp., $\text{pr}_L(x, y, z) = (x, y)$).

![Front and Lagrangian projections of a Legendrian unknot and trefoil for $\xi_{\text{std}}$ on $\mathbb{R}^3$.](image)

![Positive and negative crossings.](image)

**Computation of $tb$ using projections of $\mathbb{R}^3$.** Let $\bar{\alpha} = \text{pr}_F \circ \alpha$ be the front projection of $\alpha$. The vector field $Y = \partial / \partial z$ is transverse to $\xi_{\text{std}} = \ker(dz - ydx)$ along $\alpha$, and we let $\alpha^+$ be a knot obtained by shifting $\alpha$ slightly in the direction $Y$. Then, as observed above, $tb(\alpha)$ is the intersection number of $\alpha^+$ with the Seifert surface $\Sigma$, and this is the definition of the linking number of $\alpha^+$ with $\alpha$. This linking number is known to be half of the algebraic number of crossings of $\bar{\alpha}$ and $\bar{\alpha}^+$, where a crossing is positive if it is right handed and negative if it is left handed (see Figure 4). One can check this directly by observing the intersections of $\alpha^+$ and $\Sigma$, if $\alpha^+$ is chosen.
to be slightly above the \((x, y)\)-plane, and the part of \(\Sigma\) near to \(\bar{\alpha}\) is chosen to be in this plane. Each crossing of \(\bar{\alpha}\) will yield one intersection point and contribute \(+1\) if the crossing is positive and \((-1)\) if it is negative.

Figure 5: The pieces of the curves \(\bar{\alpha}\) and \(\bar{\alpha}^+\).

The crossings and cusps of \(\bar{\alpha}\) and \(\bar{\alpha}^+\) in the front projection are shown in Figure 5 with \(\bar{\alpha}\) in black and \(\bar{\alpha}^+\) in gray. Each cusp of \(\bar{\alpha}\) pointing to the left contributes 0 to the intersection of \(\alpha\) and \(\Sigma\) since \(\alpha^+\) does not meet \(\Sigma\) near the cusp, and a cusp pointing to the right contributes \((-1)\), so two adjacent cusps contribute \((-1)\). Hence if \(P\) and \(N\) are the number of positive and negative crossings of the front projection \(\bar{\alpha}\), respectively, and \(C\) is the number of cusps, we conclude that

\[
\text{tb}(\alpha) = P - N - C/2.
\] (1)

In the Lagrangian projection \(pr_L(x, y, z) = (x, y)\) the knots \(\alpha\) and \(\alpha^+\) project to the same diagram, since \(\alpha^+\) is obtained by moving \(\alpha\) a small distance in the \(Y = \partial/\partial z\)-direction. We can see that \(\text{tb}(\alpha)\), the linking number of \(\alpha\) and \(\alpha^+\), is the algebraic number of the positive and negative crossings of the Lagrangian projection of \(\alpha\), \(\text{tb}(\alpha) = P - N\), as in [5], p. 13.

**Computation of tb for Legendrian knots in** \((T^3, \xi_n)\). Now consider an oriented Legendrian knot \(\alpha\) in \((T^3, \xi_n)\) for a fixed \(n > 0\). Let

\[
\hat{p}_{xy}, \hat{p}_{xz} : T^2 \times \mathbb{R} \to T^2
\]

be the lifts to the covering space \(T^2 \times \mathbb{R}\) of the projections \(p_{xy}, p_{xz} : T^3 = T^2 \times S^1 \to T^2\), where \(\mathbb{R} \to S^1\) is the universal cover of the circle. We shall show how to compute \(\text{tb}(\alpha)\) using the lifted projections \(\hat{p}_{xy}, \hat{p}_{xz}\) in a similar
way to the case of \((\mathbb{R}^3, \xi_{std})\) treated above. We cannot use a linking number here, since the lifted knot \(\hat{\alpha}\) does not bound a compact surface, but we can use the intersection number of a perturbed lifted knot \(\hat{\alpha}^+\) with the image \(\hat{\Sigma}\) in \(T^2 \times \mathbb{R}\) of the covering Seifert surface of \(\Sigma\) of \(\alpha\) in \(\mathbb{R}^3\) in one circuit of \(\alpha\).

**Proposition 3.3.** For a generic Legendrian knot \(\alpha\) in \((T^3, \xi_n)\) and the projection \(p_{xy}\),

\[
\text{tb}(\alpha) = P - N + C/2
\]

where \(P\) is the number of positive crossings, \(N\) is the number of negative crossings, and \(C\) is the number of cusps of \(p_{xy} \circ \alpha\), which must be even, in one circuit of \(\alpha\).

As before, since \(\alpha\) is generic, the only singularities are transverse double points and isolated cusps. To determine which crossings are positive and which are negative we use a single component of the lift \(\hat{\alpha}\) of \(\alpha\) to \(T^2 \times \mathbb{R}\) to see which strand of this component is above and which one is below; then following \(\hat{\alpha}\) from the double point on one arc to the same double point on the other arc, the change in the vertical coordinate \(z\) determines which arc is above the other, and hence whether the crossing is positive or negative (see Figure 4). Note that each crossing in the projection corresponds to exactly one pair of strands in \(\hat{\alpha}\), and crossings involving two different components do not contribute anything. The statement of the Proposition 3.3 could just as well be formulated in terms of one period of the lifted curve \(\hat{\alpha}\).

![Diagram](image)

Figure 6: The curve \(\hat{\alpha}^+\), in gray, intersects \(\hat{\Sigma}\) positively and negatively close to positive and negative crossings, respectively.
Proof of Proposition 3.3. Lift the contact structure $\xi_n$ to the contact structure $\hat{\xi}_n$ on $T^2 \times \mathbb{R}$. The perpendicular vector field $\hat{Y} = (\cos 2\pi nz, \sin 2\pi nz, 0)$ determines the orientation of $\hat{\xi}_n$. Let $\hat{\alpha}^+$ be a copy of $\hat{\alpha}$ obtained by shifting $\hat{\alpha}$ slightly in the positive direction of $\hat{Y}$. By Definition 3.1 the Thurston-Bennequin invariant of $\alpha$ is equal to the signed intersection number of $\hat{\alpha}^+$ with a Seifert surface $\hat{\Sigma}$ of $\alpha$ in one circuit of $\hat{\alpha}$. It is convenient to choose the Seifert surface $\hat{\Sigma}$ to descend vertically near the Seifert curves, except near the cusp points on $T^2$, where the covering Seifert surface must move out horizontally for a small distance before descending. Then it is easy to check that the contribution of a crossing will be $(+1)$ for a positive crossing and $(-1)$ for a negative crossing, since the upper strand of $\hat{\alpha}^+$ near the crossing will not meet $\hat{\Sigma}$, and the lower strand will pierce $\hat{\Sigma}$ just once, with the appropriate orientation, as shown in Figure 6 for certain typical values of $z$.

The contribution of a cusp point of $\hat{\rho}_{xy} \circ \hat{\alpha}$ is illustrated in Figure 7, which shows $\hat{\alpha}$ and $\hat{\alpha}^+$ on $T^2 \times \mathbb{R}$. The arrows show the direction in which the vertical coordinate $z$ increases. If the vector field $Y$ points to the left of $\hat{\alpha}$ as $\hat{\alpha}$ approaches the cusp point, as in Figure 7 (a), then the knot $\hat{\alpha}^+$ perturbed in the direction $Y$ will be above $\hat{\Sigma}$, so there is no intersection and the contribution will be 0. If, on the other hand, $Y$ points toward the right as $\hat{\alpha}$ approaches the cusp point, then there will be a single intersection point $p$ where the perturbed knot $\hat{\alpha}^+$ pierces $\hat{\Sigma}$, and the contribution will be $+1$, as the orientations in Figure 7 (b) show.

![Figure 7](image_url)

Figure 7: In part (a), the perturbed (gray) curve $\hat{\alpha}^+$ does not intersect $\hat{\Sigma}$ and, in part (b), $\hat{\alpha}^+$ intersects $\hat{\Sigma}$ positively.

The following lemma will complete the proof of the Proposition. ∎
Lemma 3.4. The contributions of the cusps alternate between $+1$ and $0$, so the total contribution of the cusps is $C/2$, where $C$ is the number of cusps.

Proof. If the direction of increasing $z$ is the same from a cusp with value $+1$ to the next cusp, then $Y$ will point to the left as the next cusp is approached and its contribution will be 0. On the other hand, if the direction of increasing $z$ reverses, then again the contribution will be 0, since the direction of increasing $z$ will be reversed, so $Y$ will point to the right leaving the next cusp in the direction of increasing $z$. In a similar manner, if a cusp has contribution 0, the next cusp will contribute $+1$.  

The projection onto the $xz$-plane. Now we shall compute $tb(\alpha)$ using the projection $p_{xz}$ of a generic Legendrian knot $\alpha$. Set $\alpha(t) = (x(t), y(t), z(t))$ and note that by a small Legendrian perturbation of $\alpha$ we can suppose that

$$2nz(t) \notin \mathbb{Z} \text{ whenever } (x'(t), y'(t)) = (0, 0).$$

In other words, for these values of $t$ the vertical component of $\alpha'(t)$ is not zero. For other values of $t$, the plane $\xi_n$ of the contact structure projects onto the tangent plane of $T^2$ under $p_{xz}$, and so the image $p_{xz} \circ \alpha$ is a smooth non-singular curve.

The argument used for the projection $p_{xy}$ shows the following result. By analogy with the previous analysis, we let the covering Seifert surface move off the knot $\hat{\alpha}$ in the direction of the $y$-axis, instead of the $z$-axis. To determine whether a crossing is positive or negative, lift the knot to $S^1 \times \mathbb{R} \times S^1$, where the order of the arcs passing through a double point is well defined.

Proposition 3.5. For a generic Legendrian knot $\alpha$ in $(T^3, \xi_n)$ that satisfies (2),

$$tb(\alpha) = P - N$$

where $P$ is the number of positive crossings and $N$ is the number of negative crossings of $p_{xz} \circ \alpha$ in one circuit of $\alpha$.

It is possible to calculate $tb(\alpha)$ for a generic Legendrian knot $\alpha$ that does not satisfy (2) using its projection in the $xz$-plane, but the formula is more complicated, so we omit it.

These calculations prove the first two assertions of Theorem [2] which relate to the Thurston-Bennequin invariant.
4. The Rotation Number

Recall that a null-homologous oriented Legendrian knot $\alpha$ in a 3-manifold $M$ with an oriented contact structure $\xi$ has an invariant $r(\alpha)$, called the rotation (or Maslov) number, which depends on the choice of a non-vanishing section $Z$ of $\xi$. If $\alpha$ is null-homologous, then it has a covering Seifert surface $\Sigma$, and the vector field $Z$ can be determined (up to Legendrian homotopy) by requiring that it extend to a non-vanishing section of $\xi$ over $\Sigma$.

**Definition 4.1.** The **rotation number** (or Maslov number) of the oriented Legendrian knot $\alpha$, $r(\alpha)$, is the algebraic number of rotations of the tangent vector $\alpha'$ with respect to $Z$ in the plane field $\xi$ in a single circuit of $\alpha$.

**Proposition 4.2.** If $\alpha$ is a null homologous oriented Legendrian knot, the rotation number $r(\alpha)$ does not depend on the section $Z$. Furthermore, two Legendrian knots that are isotopic through Legendrian knots have the same rotation number with respect to the same section $Z$.

**Proof.** The second affirmation in obvious, since the rotation number is an integer that varies continuously as the Legendrian knot varies.

Now let $Z'$ be another global section of $\xi$ and let $f : M \to S^1$ be the function which gives the angle from $Z$ to $Z'$. Since $\alpha$ is null homologous in $M$, the image $f_\ast[\alpha]$ of its homology class must vanish in $H_1(S^1)$, so the mutual rotation number of $Z'$ relative to $Z$ is 0. Hence the rotation numbers are the same.

Consequently $r(\alpha)$ for a null-homologous knot $\alpha$ depends only on the orientations of $\alpha$ and $\xi$. Reversing one of these orientations changes the sign of $r(\alpha)$. As in the case of the definition of the Thurston-Bennequin invariant, the rotation number can also be defined for non-null homologous oriented Legendrian knots, but then it does depend on the choice of the section $Z$ of $\xi$.

This dependence holds, in particular, when $M = T^3$ (see [7]), but for Kanda’s tight contact structure $\xi_n$, we can use the covering Seifert surface of $\alpha$ to determine $Z|_\alpha$ up to Legendrian homotopy.

**Lemma 4.3.** Let $\alpha$ be an oriented Legendrian knot for the contact structure $\xi_n$ on $T^3$, and let $\Sigma \subset \mathbb{R}^3 = \tilde{T}^3$ be a covering Seifert surface for $\alpha$ containing an affine half-plane $P \subset \mathbb{R}^3$. Then there is a non-vanishing Legendrian vector field $Z$ in $\xi_n|\Sigma$ whose restriction $Z|_P$ is a section of $\xi_n \cap P$. Furthermore along $\tilde{\alpha}$, the restriction $Z|_{\tilde{\alpha}}$ is unique up to periodic Legendrian homotopy.
Proof. If the vertical Legendrian vector field $\partial/\partial z$ is in $P$, then along $P$, $Z = \partial/\partial z$ is a section of $\xi_n \cap P$. Furthermore, $\xi_n$ and $P$ are transverse except along isolated values of $z$, so by continuity the section $Z$ is determined up to multiplication by a non-vanishing function. If $\partial/\partial z$ is not in $P$, then $\xi_n$ and $P$ are transverse, so again $Z$ is determined as a section of the line field $\xi_n \cap P$ on $P$. Now extend $Z$ arbitrarily to a periodic and non-vanishing vector field tangent to $\Sigma$. Clearly $Z$ is determined up to periodic Legendrian homotopy on $P$, and the usual argument shows that the restriction $Z|_{\tilde{\alpha}}$ is also determined up to periodic Legendrian homotopy along $\tilde{\alpha}$. \hfill $\square$

It is interesting to note that on $P$ the restriction $Z|_P$ is periodically Legendrian homotopic to the vertical Legendrian vector field $\partial/\partial z$, since, in the second case of the above proof, the angle between them is never $\pi$.

**Definition 4.4.** The rotation number (or Maslov number) of an oriented Legendrian knot $\alpha$, $r(\alpha)$, on the contact manifold $(T^3, \xi_n)$ is the algebraic number of rotations in the plane field $\xi_n$ of the tangent vector $\alpha'$ with respect to the vector field $Z$ given by Lemma 4.3, in a single circuit of $\alpha$.

**4.1. Computation of the rotation invariant using projections**

**Computation of $r$ for Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.** Let $\alpha$ be an oriented generic Legendrian knot in the standard contact structure $\xi_{std} = \ker(dz - ydx)$ on $\mathbb{R}^3$. In order to calculate the rotation number $r(\alpha)$ we fix the Legendrian vector field $Y = \partial/\partial y$. Then $r(\alpha)$ is the algebraic number of times the field of tangent vectors $\alpha'$ rotates in $\xi_{std}$ relative to $Y$, so $r(\alpha)$ can be obtained by counting how many times $\alpha'$ and $\pm Y$ point in the same direction. The sign is determined by whether $\alpha'$ passes $\pm Y$ counterclockwise $(+1)$ or clockwise $(-1)$, and then we must divide by two, since in one rotation $\alpha'$ passes both $Y$ and $-Y$.

![Figure 8: Up cusps and down cusps.](image)
If $\tilde{\alpha}$ denotes the front projection of $\alpha$, the field of tangent vectors to $\tilde{\alpha}$, $\tilde{\alpha}'$, points in the direction of $\pm Y = \pm \partial / \partial y$ at the cusps, which are horizontal in the $xz$-plane. Let us analyze the upwards left-pointing cusp, the first of the four cusps in Figure 8. The value of $y$ is just the slope of $\tilde{\alpha}'$, so $y$ is negative before the cusp and becomes positive, and thus at the cusp $y(t)$ is increasing so $y'(t)$ is positive and $\alpha'$ passes $+Y$ at the cusp point. Before the cusp, $x$ is decreasing, so $x'(t)$ passes from negative to positive at the cusp. Thus the vector $\tilde{\alpha}'(t)$ turns in the negative direction, and the contribution is $(-1)$. By a similar analysis of the other three cases, we see that a cusp going upwards (the first two cusps in the figure) contributes $(-1)$, while a cusp going downwards (the third and fourth cusps in the figure) contributes $(+1)$. Therefore we have shown that the rotation number of $\alpha$ in the front projection is

$$r(\alpha) = 1/2(C_d - C_u),$$

where $C_u$ is the number of up cusps and $C_d$ is the number of down cusps in the front projection of $\alpha$. Since $\alpha$ is null-homologous, $r(\alpha)$ does not depend on the choice of the vector field $Y$, as we observed above.

In the Lagrangian projection $pr_L(x, y, z) = (x, y)$, the vector field $Y$ projects to $\partial / \partial y$, thus the rotation number of $\alpha$ is simply the winding number of the field of tangent vectors of the Lagrangian projection $pr_L \circ \alpha$ of $\alpha$ in $\xi_{std}$,

$$r(\alpha) = \text{winding}(pr_L(\alpha)).$$

**Computation of $r$ for Legendrian knots in $(T^3, \xi_n)$** Let $\alpha$ be an oriented generic Legendrian knot in $T^3$ for the tight contact structure $\xi_n$. We shall calculate the rotation invariant $r(\alpha)$ relative to the vertical vector field $Z = \partial / \partial z \in \xi_n$.

![Figure 9: Values of $b(t)$ in the projection $p_{xz}$](image)

The projection $p_{xz}$. First we use the projection $p_{xz} : T^3 \to T^2$. By a small Legendrian perturbation, if necessary, we guarantee that if $2nz(t) \in \mathbb{Z}$
then the tangent vector $\alpha'(t)$ is not vertical, i.e., $(x'(t), y'(t)) \neq (0, 0)$. The only contributions to the rotation number $r(\alpha)$ occur for points where $\alpha'(t)$ is vertical, and then since $2nz(t) \notin \mathbb{Z}$ the projection $p_{xz}$ takes $\xi_n$ onto the tangent plane to $T^2$. Near to where $\alpha'(t)$ is vertical the tangent vector, which must be non-zero, will be turning in either the positive direction with respect to the orientation of the $xz$-plane and pass the vertical line in the positive direction, and then we set $b(t) = +1$ (as in the last two cases of Figure 9), or in the negative direction (as in the first two cases) where we set $b(t) = -1$. Let $a(t) = (-1)^{[2nz(t)]}$, where the brackets indicate the largest integer function, so that $a(t)$ is positive where the projection $p_{xz}$ of $\xi_n$ onto the tangent $xz$-plane preserves the orientation and negative where the orientation is reversed, except when $2nz(t) \in \mathbb{Z}$, but we have guaranteed that then $\alpha'$ will not be vertical. Thus the contribution of a point where $\alpha'$ is vertical is half the product of $a(t)$ and $b(t)$. We have shown the following.

**Proposition 4.5.** The rotation invariant $r(\alpha)$ of a generic oriented knot $\alpha$ in $T^3$ with respect to the projection $p_{xz} : T^3 \to T^2$ is

$$r(\alpha) = \frac{1}{2} \sum_{t \in V} a(t)b(t)$$

where $V = \{ t \in S^1 \mid (x'(t), y'(t)) = (0, 0) \}$ in one circuit of $\alpha$, provided that $(x'(t), y'(t)) \neq (0, 0)$ whenever $2nz(t) \in \mathbb{Z}$.

**The projection $p_{xy}$.** For the projection $p_{xy} : T^3 \to T^2$, we must count how many times the tangent field of $\tilde{\alpha} = p_{xy} \circ \alpha$ and $Z = \partial/\partial z$ point in the same direction, and this will happen where $\tilde{\alpha}$ has a cusp since the projection $\tilde{\alpha}$ will have velocity $\tilde{\alpha}'(t_0) = 0$ at such a point. Observe that the horizontal normal vector $Y = (\cos 2\pi nz, \sin 2\pi nz, 0)$, which determines the orientation of the perpendicular contact plane $\xi_n$, projects to a vector $\tilde{Y} = p_{xy}(Y) = (\cos 2\pi nz, \sin 2\pi nz)$ perpendicular to the line tangent to the cusp in the $xy$-plane. The slope of this line, determined by the value of $z(t_0)$ at the cusp, may have any value.
Consider the orientation of $\alpha$ and the direction of $Y$ in Figure 10. Since the tangent vector $\vec{\alpha}'(t)$ is turning in the positive direction in the $xy$-plane, $z'(t_0) > 0$ at the cusp. Before the cusp $\vec{\alpha}'(t)$ is directed toward the cusp, and afterwards, it is directed away from the cusp. Hence it is clear that $\alpha'(t)$ passes the vertical vector $Z$ in the positive direction in the contact plane $\xi_n$, so in this case the contribution of the cusp is $+1$, and we call the cusp \textit{positive}. In this case the projection of $\vec{\alpha}'(t)$ onto the line through $Y$ has the same direction at $Y$, both before and after the cusp point. The result is the same if the diagram in Figure 10 is rotated in the $xy$-plane.

Now it is clear that if the orientation of $\alpha$ or the direction of $Y$ is reversed, the sign of the contribution of the cusp changes. It follows that in all four cases of the orientation of $\alpha$ and the perpendicular direction of $Y$, the contribution of the cusp is $+1$ and the cusp is \textit{positive} if the projection of $\vec{\alpha}'(t)$ onto the line through $Y$ both before and after the cusp has the same direction as $Y$, and the cusp is \textit{negative}, with contribution $-1$, if the direction is opposite to $Y$, as shown in Figure 11.

Thus we have shown the following.
Proposition 4.6. The rotation number of a generic oriented knot $\alpha$ in $(T^3, \xi_n)$ with respect to the projection $p_{xy}$ is
\[ r(\alpha) = \frac{1}{2}(C_+ - C_-) \]
where $C_+$ is the number of positive cusps and $C_-$ is the number of negative cusps of $p_{xy} \circ \alpha$ in one circuit of $\alpha$.

This completes the calculation of the rotation invariant using the projections $p_{xz}$ and $p_{xy}$ as in Theorem 2, so its proof is complete.

5. Does a Bennequin inequality hold?

For a null-homologous Legendrian knot $\alpha$ on a tight contact 3-manifold $(M, \xi)$ with Seifert surface $\Sigma$, the Thurston-Bennequin inequality
\[ tb(\alpha) + |r(\alpha)| \leq -\chi(\Sigma) \] (3)
gives an upper bound on $tb(\alpha)$, provided that $\chi(\Sigma) \leq 0$ [3, 5]. It is natural to ask (and we thank the referee for suggesting this) whether this inequality remains valid for our extension of these invariants. The following proposition gives an example which shows that the inequality must be modified. It also motivates a conjecture as to what ought to hold. Recall that according to Kanda [9], a Legendrian knot $\alpha$ in $(T^3, \xi_n)$ is quasilinear if $\alpha$ is isotopic on $T^3$ to a knot which lifts to a straight line in the universal cover $\tilde{T}^3 = \mathbb{R}^3$.

Proposition 5.1. For any $(p, q, r) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$ there is a quasilinear Legendrian knot $\alpha$ for the tight contact structure $(T^3, \xi_n)$ with a $(p,q,r)$-periodic lift to $\mathbb{R}^3 = \tilde{T}^3$ such that $\alpha$ satisfies the following equation:
\[ tb(\alpha) + r(\alpha) = -\chi(\hat{\Sigma}) + rn. \] (4)

Proof. First, we note that for every $n > 0$ and $(p, q, r) \in \mathbb{Z}^3$ there exists a $(p, q, r)$-periodic Legendrian knot $\alpha$ in $(T^3, \xi_n)$, i.e., such that $\tilde{\alpha}(t+1) = \tilde{\alpha}(t) + (p, q, r)$, where $\tilde{\alpha}(t) = (x(t), y(t), z(t))$ is the lift of $\alpha$ to $\mathbb{R}^3$. Furthermore, we may construct $\alpha$ so that for all $t \in \mathbb{R}$ if $r > 0$ (respectively, $r = 0$ or $r < 0$) we have $z'(t) > 0$ (respectively, $z'(t) = 0$ or $r'(t) < 0$). We construct such a piecewise linear Legendrian knot and then smooth it out by a small isotopy. If $r > 0$, take $t_0, t_1 \in [0,1]$ with $t_0 < t_1$ such that $\xi_n(t_0)$ is parallel to the $x$-axis and $\xi_n(t_1)$ is parallel to the $y$-axis. Then define a piecewise
linear \((p, q, r)\)-periodic Legendrian knot by letting \(z(t)\) increase linearly on
the intervals \([0, t_0]\), \([t_0 + \epsilon, t_1]\), and \([t_1 + \epsilon, 1]\) modulo 1 (for sufficiently small \(\epsilon > 0\), with \(x(t)\) and \(y(t)\) both constant on these intervals, while on the
interval \([t_0, t_0 + \epsilon]\) \(x(t)\) increases by \(p\) and on \([t_1, t_1 + \epsilon]\) \(y(t)\) increases by \(q\),
with \(z(t)\) constant. Next, by a small Legendrian isotopy, deform this PL knot
to a smooth Legendrian knot \(\alpha\) so as to preserve the property that \(z'(t) > 0\)
for every \(t\). The case \(r < 0\) is similar. For the case \(r = 0\) we may take \(\alpha\) to
be the linear Legendrian knot \(\alpha(t) = (pt, qt, z_0), t \in [0, 1]\), where \(z_0\) is such
that the vector \(Y = (p, q, 0) \in \xi_n(x, y, z_0)\) for every \(x, y \in \mathbb{R}\).

For such a Legendrian knot \(\alpha\) with \(r > 0\) let us calculate the invariants.
Since \(z'(t) > 0\), the rotation number of the tangent vector \(\alpha'\) with respect
to the constant Legendrian vector field \(Z = \partial/\partial z\) in \(\xi_n\) is \(r(\alpha) = 0\). Next,
take a vector \(X \in \mathbb{R}^3\) orthogonal to \((p, q, r)\) and construct a Seifert surface
\(\hat{\Sigma} \subset \hat{T}^3\) such that along \(\hat{\alpha} X\) is tangent to \(T\hat{\Sigma}\) and points inwards towards
\(\hat{\Sigma}\). Then it follows that \(tb(\alpha) = rn\), since as \(z(t)\) increases by \(r\) the contact
structure \(\xi_n\) rotates exactly \(rn\) times. The Seifert surface \(\hat{\Sigma}\) can be taken to
be homeomorphic to \(S^1 \times [0, \infty)\), so \(\chi(\hat{\Sigma}) = 0\). Thus equation \((4)\) holds in
this case. For the case that \(z < 0\), consider \(-\alpha\), the knot \(\alpha\) with the reversed
orientation, and apply the case \(r > 0\). The signs of \(r(\alpha)\) and \(r\) are reversed,
while \(tb(\alpha)\) and \(\chi(\hat{\Sigma})\) continue to vanish, so the same formula holds. The

case \(r = 0\) is analogous, with \(r(\alpha) = tb(\alpha) = 0\). \(\square\)

These examples motivate the following conjecture.

**Conjecture 5.2.** For any Legendrian knot \(\alpha\) on \((T^3, \xi_n)\) that is \((p, q, r)\)-
periodic (in the positive direction of \(\alpha\)), there is a Bennequin inequality
\[
\text{tb}(\alpha) + r(\alpha) \leq -\chi(\hat{\Sigma}) + rn. \tag{5}
\]

We note that if \(\alpha\) is a null-homologous Legendrian knot in \((T^3, \xi_n)\), then
\((p, q, r) = (0, 0, 0)\) and \((5)\) is equivalent to the classical Bennequin inequality \([3]\). Furthermore, since every tight contact structure on \(T^3\) is contactomor-
phic to some \(\xi_n\), the conjecture implies that a similar inequality should hold
for Legendrian knots in any tight contact structure \(\xi\) on \(T^3\), provided \(Z\) is
taken to correspond to \(\partial/\partial z\) under the contactomorphism.

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