Brown representability for triangulated categories with a linear action by a graded ring

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Abstract. In this paper we give necessary and sufficient conditions for a functor to be representable in a strongly generated triangulated category which has a linear action by a graded ring, and we discuss some applications and examples.

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1. Introduction. Every object $X$ in a category induces a contravariant functor into the category of sets that sends an object $Y$ to the set $\text{Hom}(Y, X)$. Any functor that is naturally isomorphic to such a functor is called representable. There are a number of results in various settings, called Brown representability, when every ‘reasonable’ functor is representable. The first such result is due to Brown; see [11].

The first Brown representability result for triangulated categories was established by Neeman [19, Theorem 3.1]. The work on hand was motivated by [10, Theorem 1.3] and [21, 4.3].

Theorem (see Theorem 2.7). Let $R$ be a $\mathbb{Z}$-graded graded-commutative noetherian ring and $\mathcal{T}$ a graded $R$-linear triangulated category, that is strongly generated, Ext-finite, and idempotent complete. Then a graded $R$-linear cohomological functor $f: \mathcal{T}^{\text{op}} \to \text{grMod}(R)$ is graded representable if and only if $f$ only takes values in $\text{grmod}(R)$.

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In contrast to the previous works, we characterize the \textit{graded representable} functors; those are the functors naturally isomorphic to
\[ \coprod_{d \in \mathbb{Z}} \text{Hom}_T(-, \Sigma^d X) \]
for some object \( X \). The result is proved in Section 2. Without the assumption that \( R \) is noetherian and \( T \) Ext-finite, we obtain necessary, through not sufficient, conditions for a functor to be graded representable; see Corollary 2.19.

The study of representability is motivated by the fact, that the characterization of representable functors in a triangulated category \( T \) yields the existence of a right adjoint functor to a functor \( S \to T \). For a nice discussion on this, see [20, Introduction]. In Section 3, we show the same holds in the graded setting.

Finally we discuss some examples where Theorem 2.7 yields new insight: When \( G \) is a finite group and \( R \) a commutative noetherian ring, then \( D_b(\text{mod}(RG)) \) is Ext-finite as a \( H^*(G, R) \)-linear category. In the second example, we consider the action of Hochschild cohomology \( HH^*(R/Q) \) on \( D_b(\text{mod}(R)) \), when \( Q \) is a regular ring and \( R = Q/(f) \) a quotient by a regular sequence.

2. \textbf{Representable functors in the graded setting.} Let \( T \) be a triangulated category with suspension functor \( \Sigma \).

2.1. For objects \( X \) and \( Y \) in \( T \), we write
\[
\text{Ext}^*_T(X, Y) := \coprod_{d \in \mathbb{Z}} \text{Hom}_T(X, \Sigma^d Y). \tag{2.1.1}
\]
When \( T = D(R) \), the derived category of modules over a ring \( R \), and \( X \) and \( Y \) are \( R \)-modules viewed as objects in \( D(R) \) via the natural embedding, then this coincides with the classical Ext-groups.

2.2. Let \( R \) be a \( \mathbb{Z} \)-graded graded-commutative ring. This means \( R \) decomposes as
\[ R = \coprod_{d \in \mathbb{Z}} R_d, \]
and the Koszul sign rule holds
\[ rs = (-1)^{de} sr \quad \text{for } r \in R_d \text{ and } s \in R_e. \]
We say \( r \in R_d \) is an homogeneous element of degree \( d \).

2.3. A triangulated category \( T \) is \textit{graded }\( R \)-\textit{linear} if
1. for any objects \( X \) and \( Y \) in \( T \), the abelian group \( \text{Ext}^*_T(X, Y) \) is a graded \( R \)-module with the grading given by the coproduct in (2.1.1), and
2. composition is \( R \)-bilinear.

This data is equivalent to a ring homomorphism \( R \to \mathbb{Z}(T) \), where
\[ \mathbb{Z}(T) := \coprod_{d \in \mathbb{Z}} \{ \eta : \text{id}_T \to \Sigma^d \eta \Sigma = (-1)^d \Sigma \eta \} \]
is the graded center of $T$. More precisely, a ring homomorphism $\varphi: R \to \mathbb{Z}(T)$ yields an $R$-action on $\text{Ext}^*_T(X,Y)$ via
\[ r \cdot - : \text{Ext}^*_T(X,Y) \to \text{Ext}^*_T(X,\Sigma^d Y) = \text{Ext}^*_T(X,Y)[d], \]
for any homogeneous element $r \in R$. Conversely, any homogeneous element $r \in R$ yields a natural transformation $\eta$ given by
\[ \eta_X := r \cdot \text{id}_X : X \to \Sigma^{|r|} X \]
for any $X \in T$. It is straightforward to check that these identifications are well-defined and mutually inverse. The graded center has been studied in a number of works; for example [8,13].

2.4. We denote by $\text{grMod}(R)$ the category of graded $R$-modules, and by $\text{grmod}(R)$ its full subcategory of finitely generated $R$-modules. The $n$th shift $M[n]$ of a graded $R$-module $M$ is given by $(M[n])_d = M_{n+d}$.

The suspension functor of a graded $R$-linear category $T$ in the first component of $\text{Ext}^*_T(-,-)$ corresponds to the negative shift in $\text{grMod}(R)$:
\[ \text{Ext}^*_T(\Sigma^n X,Y) \cong \text{Ext}^*_T(X,Y)[-n]. \]

2.5. A functor $f: T^{\text{op}} \to \text{grMod}(R)$ is graded $R$-linear if
1. the induced map $\text{Ext}^*_T(X,Y) \to \text{Ext}^*_R(f(Y),f(X))$ is a map of graded $R$-modules, and
2. the suspension becomes the negative shift under $f$, that is
\[ f(\Sigma^n X) = f(X)[-n]. \]
The functor $f$ is cohomological if $f$ applied to any exact triangle yields a long exact sequence of graded $R$-modules.

Without explicitly stating, we always assume that a natural transformation between graded $R$-linear functors respects this structure.

Definition 2.6. A functor $T^{\text{op}} \to \text{grMod}(R)$ is graded representable if it is naturally isomorphic to
\[ g_X := \text{Ext}^*_T(-,X) : T^{\text{op}} \to \text{grMod}(R) \]
for some object $X$ in $T$.

When $T$ is graded $R$-linear, then any graded representable functor is graded $R$-linear.

A graded $R$-linear functor $f: T^{\text{op}} \to \text{grMod}(R)$ is graded representable if and only if $f_d: T^{\text{op}} \to \text{Mod}(R_0)$ is representable for an (arbitrary) integer $d$. The functors $f_d$ are the degree $d$ part of $f$, that is $f_d(X) := f(X)_d$. Since $f$ is graded $R$-linear, the degree $d$ part $f_d$ for an integer $d$ encodes all the information of $f$, that is
\[ f_d(\Sigma^e X) = f(\Sigma^e X)_d = f(X)[-e]_d = f(X)_{d-e} = f_{d-e}(X). \]
**Theorem 2.7.** Let $R$ be a $\mathbb{Z}$-graded graded-commutative noetherian ring and $T$ a graded $R$-linear triangulated category, that is strongly generated, $\text{Ext}$-finite, and idempotent complete. Then a graded $R$-linear cohomological functor $f: T^{\text{op}} \to \text{grMod}(R)$ is graded representable if and only if $f$ is locally finite.

Before we give a proof, we recall some definitions and properties:

**2.8.** A $\mathbb{Z}$-graded ring $R$ is noetherian if and only if $R_0$ is noetherian and $R$ is finitely generated as an $R_0$-algebra; see for example [15, Corollaire (2.1.5)] or [12, Theorem 1.5.5]. In particular, such a ring is bounded below.

**2.9.** A graded $R$-linear triangulated category $T$ is $\text{Ext}$-finite if for all $X, Y \in T$, the graded $R$-module $\text{Ext}^\ast_T(X, Y)$ is finitely generated.

A triangulated category $T$ is idempotent complete if for every object $X$ in $T$ and every idempotent $e \in \text{End}_T(X)$, that is $e^2 = e$, there exists an object $Y$ and maps

$$i: Y \to X \quad \text{and} \quad p: X \to Y$$

such that $p \circ i = \text{id}_Y$ and $i \circ p = e$.

**2.10.** A subcategory $S \subseteq T$ is thick if it is triangulated and closed under retracts. Since the intersection of thick subcategories is thick, there exists a smallest thick subcategory of $T$ containing an object $G$, which we denote by $\text{thick}(G)$. We say $G$ finitely builds an object $X$ in $T$ when $X \in \text{thick}(G)$.

There is an exhaustive filtration of $\text{thick}_R(G)$: Let $\text{thick}^1(G)$ be the smallest full subcategory containing $G$ that is closed under finite coproducts, retracts, and suspension. Then

$$\text{thick}^n(G) := \left\{ X \in T \mid \begin{array}{l} \text{there exists } X' \in T \text{ and an exact triangle } Y \to X \oplus X' \to Z \to \Sigma Y \\
\text{such that } Y \in \text{thick}^{n-1}(G) \text{ and } Z \in \text{thick}^1(G) \end{array} \right\}.$$ 

These are full subcategories and form an exhaustive filtration of $\text{thick}(G)$; cf. [3,10]. In particular, if $X$ lies in $\text{thick}(G)$, then there exists an integer $n$ such that $X \in \text{thick}^n(G)$.

A triangulated category $T$ is strongly generated if there exists an object $G$ in $T$ and a non-negative integer $n$ such that $T = \text{thick}^n(G)$. The object $X$ is a strong generator of $T$; cf. [21].

In the remainder of this section, we give a proof of Theorem 2.7. We fix a $\mathbb{Z}$-graded graded-commutative ring $R$, a graded $R$-linear triangulated category $T$, and a graded $R$-linear cohomological functor $f: T^{\text{op}} \to \text{grMod}(R)$.

**Lemma 2.11** (Graded version of Yoneda’s lemma). For any $X \in T$, the map

$$\text{Nat}(g_X, f) \to f_0(X) \quad \text{given by} \quad \eta \mapsto \eta(X)(\text{id}_X)$$

is an isomorphism of abelian groups.

**Proof.** For $u \in f_0(X)$, we define a natural transformation

$$\eta_u: g_X \to f \quad \text{as} \quad \eta_u(Y)(f) := f(f)(u).$$
where $Y \in \mathcal{T}$ and $f \in \operatorname{Ext}^*_T(Y, X)$. Since $u$ is a degree zero element, the map $\eta_u(Y)$ is homogeneous. It is straightforward to verify that this is the inverse of the map in the claim and both are maps of abelian groups. \qed

In particular, any morphism $f: X \to Y$ corresponds to a natural transformation

$$f_*: \mathcal{g}_X \to \mathcal{g}_Y$$

given by post-composition.

2.12. Adapting the definitions in [21, Section 4] a graded $R$-linear functor $f: \mathcal{T}^{\text{op}} \to \text{grMod}(R)$ is

- **locally finitely generated** if for every $X$ in $\mathcal{T}$, there exists $Y$ in $\mathcal{T}$ and a natural transformation $\zeta: \mathcal{g}_Y \to f$ such that $\zeta(X)$ is surjective,
- **locally finitely presented** if it is locally finitely generated and the kernel of any natural transformation $\mathcal{g}_Y \to f$ is locally finitely generated, and
- **locally finite** if $f$ only takes values in $\text{grmod}(R)$.

When $f: \mathcal{T}^{\text{op}} \to \text{grMod}(R)$ is locally finitely generated or locally finitely presented, then $f_d: \mathcal{T}^{\text{op}} \to \text{Mod}(R_0)$ is locally finitely generated or locally finitely presented in the sense of [21, Section 4], respectively. The same need not hold for locally finite, for examples, see Section 3.

If $\mathcal{T}$ is Ext-finite, then any graded representable functor is locally finite. Without the assumption that $\mathcal{T}$ is Ext-finite, we can make the following statement:

**Lemma 2.13.** Any graded representable functor is locally finitely presented.

**Proof.** It is clear that a graded representable functor is locally finitely generated. Let $\mathcal{g}_X$ be a graded representable functor, and $\mathcal{g}_Y \to \mathcal{g}_X$ a natural transformation. By Yoneda’s lemma 2.11, this corresponds to a morphism $Y \to X$. If we complete this to an exact triangle $Z \to Y \to X \to \Sigma Z$, the sequence

$$\mathcal{g}_Z \to \mathcal{g}_Y \to \mathcal{g}_X$$

is exact on $\mathcal{T}$. In particular, the kernel of $\mathcal{g}_Y \to \mathcal{g}_X$ is locally finitely generated. \qed

**Lemma 2.14.** If $f$ is locally finite, then $f$ is locally finitely generated.

**Proof.** Let $X$ be an object in $\mathcal{T}$. Then the $R$-module $f(X)$ is finitely generated, and we can choose a finite set of homogeneous generators $x_1, \ldots, x_n$ of $f(X)$ in degrees $d_1, \ldots, d_n$. Set

$$Y := \prod_{j=1}^n \Sigma^{d_j} X.$$ 

For every generator $x_j$, we obtain canonical maps

$$\Sigma^{d_j} X \xrightarrow{i_j} Y \xrightarrow{\pi_j} \Sigma^{d_j} X$$
whose composition is the identity map on $\Sigma^d X$. Let $y \in f(Y)$ be the canonical element, for which
\[ x_j = f(i_j)(y) \quad \text{for } 1 \leq j \leq n. \]
Because of the suspensions introduced in the definition of $Y$, the element $y$ is homogeneous of degree 0. By Yoneda’s lemma, the element $y$ corresponds to the natural transformation $\zeta: g_Y \to f$ with $\zeta(Y)(\text{id}_Y) = y$. Then $\zeta(y)(i_j) = x_j$, and $\zeta(X)$ is surjective. That is $f$ is locally finitely generated.

In general a locally finite functor need not be locally finitely presented. This requires further assumptions on $R$ and $T$:

**Lemma 2.15.** If $R$ is noetherian and $T$ Ext-finite, then a locally finite functor $f: T^{op} \to \text{grMod}(R)$ is locally finitely presented.

**Proof.** By Lemma 2.14, the functor $f$ is locally finitely generated. Let $g_Y \to f$ be a natural transformation. We set
\[ f'(X) := \ker(g_Y(X) \to f(X)). \]
Since $T$ is Ext-finite, the $R$-module $g_Y(X)$ is finitely generated. By assumption on $f$, so is $f(X)$. Since $R$ is noetherian, the kernel $f'(X)$ is also finitely generated. Thus $f'$ is a locally finite functor and by Lemma 2.14 it is locally finitely generated. In particular, $f$ is locally finitely presented. \qed

**2.16.** Let $(f_i, \eta_i)_{i>0}$ be a direct system of cohomological functors $f_i: T \to A$ where $A$ an abelian category and natural transformations $\eta_i: f_i \to f_{i+1}$. Following [21, 4.2.2], a direct system $(f_i, \eta_i)_{i>0}$ is almost constant on a subcategory $S$ of $T$ if for every $X \in S$, the sequence
\[ 0 \to \ker(\eta_i(X)) \to f_i(X) \to \text{colim}_j f_j(X) \to 0 \]
is exact for all positive integers $i$.

A direct system $(X_i, f_i)_{i>0}$ of objects $X_i$ and morphisms $f_i: X_i \to X_{i+1}$ in $T$ is almost constant on $S$ if the induced direct system of functors $(g_{X_i}, (f_i)_*)_{i>0}$ is almost constant on $S$.

For almost constant direct systems, the following hold; see [21, Proposition 4.13].

**Facts 2.17.** Let $S \subseteq T$ be a subcategory closed under suspension, and $(f_i, \eta_i)_{i>0}$ a direct system that is almost constant on $S$. Then
1. $(f_{mi+r})_{i \geq 0}$ is almost constant on $\text{thick}^a(S)$ for any $r > 0$, and
2. $f_{n+1} \to \text{colim}_i f_i$ is split surjective on $\text{thick}^a(S)$.

If the functors $f_i$ are graded $R$-linear, the assumption that $S$ is closed under suspension is redundant.

**Proposition 2.18.** Let $T$ be a strongly generated, graded $R$-linear triangulated category and $f: T^{op} \to \text{grMod}(R)$ a cohomological graded $R$-linear functor. Then $f$ is locally finitely presented if and only if $f$ is a retract of a graded representable functor.
**Proof.** We assume \( f \) is locally finitely presented. Let \( G \in \mathcal{T} \) be a strong generator of \( \mathcal{T} \) with \( \text{thick}^d(G) = \mathcal{T} \). Then there exist \( A_1 \in \mathcal{T} \) and a natural transformation \( \zeta_1 : g_{A_1} \to f \) such that \( \zeta_1(G) \) is surjective. Inductively we construct a direct system

\[
\begin{align*}
g_{A_1} & \to g_{A_2} \to \cdots \\
\end{align*}
\]

with compatible natural transformations \( \zeta_i : g_{A_i} \to f \): Assume we have constructed \( A_i \) and \( \zeta_i \) for \( i \leq n \). Since \( f \) is locally finitely presented, there exists

\[
\begin{align*}
g_B & \to \ker(g_{A_n} \to f) \\
\end{align*}
\]

that is surjective on \( G \). This induces a natural transformation \( g_B \to g_{A_n} \), which by the graded version of Yoneda’s lemma \ref{2.11} corresponds to a morphism \( f : B \to A_n \). We complete this morphism to an exact triangle

\[
\begin{align*}
B & \to A_n \to A_{n+1} \to \Sigma B \\
\end{align*}
\]

and apply \( f_0 \), the degree 0 part of \( f \). By the graded version of Yoneda’s lemma \ref{2.11}, we obtain the exact sequence

\[
\begin{align*}
\text{Nat}(g_B, f) & \leftarrow \text{Nat}(g_{A_n}, f) \leftarrow \text{Nat}(g_{A_{n+1}}, f). \\
\end{align*}
\]

Thus by construction of \( B \), there exists a natural transformation \( \zeta_{n+1} \) whose image is \( \zeta_n \).

By this construction, we have

\[
\begin{align*}
\ker(g_{A_n}(G) \to f(G)) = \ker(g_{A_n}(G) \to g_{A_{n+1}}(G)). \\
\end{align*}
\]

Using this and that \( \zeta_1(G) \) is surjective, it is straightforward to verify that the direct system is almost constant on \( G \). Then the induced natural transformation \( \text{colim}_i g_{A_i} \to f \) is a natural isomorphism. By Facts \ref{2.17}, the natural transformation

\[
\begin{align*}
g_{A_{d+1}} & \to \text{colim}_i g_{A_i} \sim f \\
\end{align*}
\]

is split surjective, and thus \( f \) is a retract of \( g_{A_{d+1}} \) on \( \mathcal{T} \).

For the converse direction, we assume \( f \) is the retract of \( g_X \) for some object \( X \). Then we have a canonical projection and a canonical injection

\[
\begin{align*}
g_X & \to f \quad \text{and} \quad f \to g_X, \\
\end{align*}
\]

respectively. The canonical projection is surjective on \( \mathcal{T} \), the canonical injection is injective. In particular, the canonical projection yields that \( f \) is locally finitely generated. Given a natural transformation \( g_Y \to f \), its kernel coincides with the kernel of the composition \( g_Y \to f \to g_X \). By Lemma \ref{2.13}, any representable functor is locally finitely presented, and thus is \( f \).

**Corollary 2.19.** If \( \mathcal{T} \) is additionally idempotent complete, then every locally finitely presented functor is graded representable.

**Proof.** Let \( f \) be a locally finitely presented functor. By Proposition \ref{2.18}, it is a retract of a graded representable functor \( g_X \). Then the natural transformation

\[
\begin{align*}
g_X & \to f \to g_X \\
\end{align*}
\]
corresponds to an idempotent $e: X \to X$. Since $T$ is idempotent complete, there exists a retract of $Y$ of $X$ such that $e$ decomposes as the natural inclusion and projection morphism. Then $f \to g_X \to g_Y$ is a natural isomorphism, and $f$ is graded representable. \qed

Proof of Theorem 2.7. Since $T$ is Ext-finite, any graded representable functor is locally finite. For the converse, we assume $f$ is locally finite. Since $R$ is noetherian and $T$ Ext-finite, we can apply Lemma 2.15 to obtain that $f$ is locally finitely presented. Then $f$ is graded representable by Corollary 2.19. \qed

3. Applications. Adjoint functors. As explained in [20, Introduction], there is a connection between representable functors and adjoint functors. In our context, we obtain the following:

Let $R$ be a $\mathbb{Z}$-graded graded-commutative ring. A functor $f: S \to T$ between $R$-linear graded triangulated categories is graded $R$-linear if it is exact and the induced map

$$\text{Ext}^*_S(X, Y) \to \text{Ext}^*_T(f(X), f(Y))$$

is a map of graded $R$-modules.

Lemma 3.1. Let $R$ be a $\mathbb{Z}$-graded graded-commutative ring, and $S$, $T$ graded $R$-linear triangulated categories. Suppose $T$ is Ext-finite and every cohomological graded $R$-linear functor $S^{\text{op}} \to \text{grMod}(R)$, that is locally finite, is graded representable. Then every graded $R$-linear functor $f: S \to T$ has a right adjoint.

Proof. We adapt the proof of [20, Theorem 8.4.4]. Given $Y \in T$, we define a functor $h: S \to \text{grMod}(R)$ by

$$h(-) := \text{Ext}^*_T(f(-), Y).$$

This is a graded $R$-linear functor. Since $T$ is Ext-finite, this functor is locally finite. So by assumption, $h$ is graded representable, that is there exists an object $f'(Y) \in S$ such that

$$\text{Ext}^*_T(f(-), Y) \cong \text{Ext}^*_S(-, f'(Y)).$$

It remains to verify that $f'$ is a functor and this isomorphism is natural in both components. Let $f: Y \to Z$ be a morphism in $T$. Then the induced map

$$\text{Ext}^*_S(-, f'(Y)) \to \text{Ext}^*_S(-, f'(Z))$$

corresponds to a morphism $f'(Y) \to f'(Z)$ by Yoneda’s lemma 2.11. Thus $f'$ is a functor. The above isomorphism is natural by construction. So $f'$ is a right adjoint of $f$.

Corollary 3.2. Let $R$ be a $\mathbb{Z}$-graded graded-commutative noetherian ring and $S$, $T$ Ext-finite graded $R$-linear triangulated categories. Suppose $S$ is strongly generated and idempotent complete. Then every graded $R$-linear functor $f: S \to T$ has a right adjoint. \qed
Derived category. Let $R$ be a commutative noetherian ring and $A$ an $R$-algebra that is finitely generated as an $R$-module. Then $A$ is noetherian; see for example [18, Theorem 3.7]. The bounded derived category of finitely generated modules over $A$, denoted by $D_b(\text{mod}(A))$, has a canonical structure as an $R$-linear category, and the $R$-module $\text{Hom}_{D_b(\text{mod}(A))}(X,Y) = \text{Ext}_R^0(X,Y)$ is finitely generated for any $X,Y$. In general, the category $D_b(\text{mod}(A))$ need not be Ext-finite as an $R$-linear category. By [6, Corollary 2.10], the category $D_b(\text{mod}(A))$ is idempotent complete.

3.3. In general the question whether $D_b(\text{mod}(A))$ is strongly generated is rather difficult. When $A$ is artinian, then $D_b(\text{mod}(A))$ is strongly generated by [21, Proposition 7.37]. When $A = R$ is a commutative noetherian ring, then $D_b(\text{mod}(R))$ is strongly generated when $R$ is either essentially of finite type over a field or over an equicharacteristic excellent local ring; see [1, Main Theorem] and [16, Corollary 7.2].

In the following, we discuss two examples in which $D_b(\text{mod}(A))$ is Ext-finite for some cohomology ring connected to $A$.

Finite group over a commutative ring. We consider $A = RG$, the group algebra of a finite group $G$.

3.4. The group cohomology of the group algebra $RG$ with coefficients in an $RG$-complex $M$ is

$$H^*(G,M) := \text{Ext}^*_R(R,M).$$

When $M = R$, this is a $\mathbb{Z}$-graded graded-commutative ring, and every $H^*(G,M)$ is a graded $H^*(G,R)$-module. In particular, for $RG$-complexes $X, Y$, the identification

$$\text{Ext}^*_R(D_b(\text{mod}(R)), X,Y) = \text{Ext}^*_R(R,X,Y) \cong H^*(G, \text{Hom}_R(X,Y))$$

holds and the cohomology ring $H^*(G,R)$ acts on any Ext-module; see for example [9, Proposition 3.1.8]. So the bounded derived category of finitely generated $RG$-modules $D_b(\text{mod}(RG))$ is graded $H^*(G,R)$-linear.

3.5. By [14,22], the group cohomology ring $H^*(G,R)$ is noetherian, and $H^*(G,M)$ is finitely generated over $H^*(G,R)$ for every finitely generated $RG$-module $M$. In particular, the derived category $D_b(\text{mod}(RG))$ is Ext-finite as a graded $H^*(G,R)$-linear triangulated category.

Corollary 3.6. Let $R$ be a commutative noetherian ring and $G$ a finite group. If $D_b(\text{mod}(RG))$ is strongly generated, then a graded $H^*(G,R)$-linear functor

$$f: D_b(\text{mod}(RG)) \to \text{grMod}(H^*(G,R))$$

is graded representable if and only if $f$ is locally finite. □

Regular ring modulo a regular sequence. We consider $R = A$ a commutative noetherian ring.

3.7. The category $D_b(\text{mod}(R))$ is Ext-finite over $R$ if and only if the Ext-modules $\text{Ext}^*_R(X,Y)$ are bounded for all $X$ and $Y$ in $D_b(\text{mod}(R))$. That is precisely when $R$ is regular: When $R$ is regular, the Ext-modules are bounded
by definition. For the converse, for every $X$ in $D_b(\text{mod}(R))$, the Ext-module $\text{Ext}_R^*(X, R/p)$ is bounded and $X_p$ has finite projective dimension for any prime ideal $p$ of $R$. Then $X$ has finite projective dimension; see [7, Lemma 4.5] for modules, and [5, Theorem 4.1] and [17, Theorem 3.6] for complexes.

When $R$ is regular, the bounded derived category $D_b(\text{mod}(R))$ is strongly generated if and only if $R$ is a strong generator. The later holds precisely when $R$ has finite global dimension, that is $R$ has finite Krull dimension. Then Rouquier’s representability theorem [21, Corollary 4.18] applies.

3.8. Suppose $R = Q/(f)$ is the quotient of a regular ring $Q$ by a regular sequence $f = f_1, \ldots, f_c$. Then there exist cohomological operators $\chi = \chi_1, \ldots, \chi_c$ in degree 2 such that for $X, Y$ in $D_b(\text{mod}(R))$, the graded modules $\text{Ext}_R^*(X, Y)$ are finitely generated over the noetherian graded ring $R[\chi]$; see [4, Theorem (4.2)]. In particular, the category $D_b(\text{mod}(R))$ is $R[\chi]$-linear and Ext-finite.

The ring of cohomological operators coincides with the Hochschild cohomology

$$R[\chi] \cong \text{HH}^*(R/Q) := \text{Ext}_{R\otimes Q R}^*(R, R);$$

see [2, Section 3].

**Corollary 3.9.** Let $R = Q/(f)$ be the quotient of a regular ring $Q$ by a regular sequence $f = f_1, \ldots, f_c$ with cohomological operators $\chi$. If $D_b(\text{mod}(R))$ is strongly generated, then any graded $R[\chi]$-linear functor $f : D_b(\text{mod}(R)) \to \text{grMod}(R[\chi])$ is graded representable if and only if $f$ is locally finite. □

3.10. For Corollaries 3.6 and 3.9, it is crucial that the ring action on the derived category is graded since the Ext-modules need not be not bounded. In particular, Corollaries 3.6 and 3.9 are not consequences of [21, 4.3], but require Theorem 2.7.

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References

[1] Aihara, T., Takahashi, R.: Generators and dimensions of derived categories of modules. Comm. Algebra 43(11), 5003–5029 (2015)
[2] Avramov, L.L., Buchweitz, R.O.: Homological algebra modulo a regular sequence with special attention to codimension two. J. Algebra 230(1), 24–67 (2000)
[3] Avramov, L.L., Buchweitz, R.-O., Iyengar, S.B., Miller, C.: Homology of perfect complexes. Adv. Math. 223(5), 1731–1781 (2010)
[4] Avramov, L.L., Gasharov, V.N., Peeva, I.V.: Complete intersection dimension. Inst. Hautes Études Sci. Publ. Math. 86, 67–114 (1997)
[5] Avramov, L.L., Iyengar, S.B., Lipman, J.: Reflexivity and rigidity for complexes, I. Commutative rings. Algebra Number Theory 4(1), 47–86 (2010)
[6] Balmer, P., Schlöchting, M.: Idempotent completion of triangulated categories. J. Algebra 236(2), 819–834 (2001)
[7] Bass, H., Murthy, M.P.: Grothendieck groups and Picard groups of abelian group rings. Ann. Math. (2) 86, 16–73 (1967)
[8] Benson, D., Iyengar, S.B., Krause, H.: Local cohomology and support for triangulated categories. Ann. Sci. Éc. Norm. Supér. (4) 41(4), 573–619 (2008)
[9] Benson, D.J.: Representations and Cohomology. I. Basic Representation Theory of Finite Groups and Associative Algebras. Second edition. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, Cambridge (1998)
[10] Bondal, A.I., van den Bergh, M.: Generators and representability of functors in commutative and noncommutative geometry. Mosc. Math. J. 3(1), 1–36 (2003)
[11] Brown, E.H.: Cohomology theories. Ann. Math. (2) 2(75), 467–484 (1962). (Correction: Ann. of Math. (2), 78:201 (1963))
[12] Bruns, W., Herzog, J.: Cohen-Macaulay Rings, revised ed., Cambridge Stud. Adv. Math. Cambridge University Press, Cambridge (1998)
[13] Buchweitz, R.-O., Flenner, H.: Global Hochschild (co-)homology of singular spaces. Adv. Math. 217(1), 205–242 (2008)
[14] Evens, L.: The cohomology ring of a finite group. Trans. Amer. Math. Soc. 101, 224–239 (1961)
[15] Grothendieck, A.: Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. Inst. Hautes Études Sci. Publ. Math. 8, 222pp. (1961)
[16] Iyengar, S.B., Takahashi, R.: Annihilation of cohomology and strong generation of module categories. Int. Math. Res. Not. IMRN 2, 499–535 (2016)
[17] Letz, J.C.: Local to global principles for generation time over commutative Noetherian rings. Homol. Homotopy Appl. 23(2), 165–182 (2021)
[18] Matsumura, H.: Commutative Ring Theory. Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge (1989)
[19] Neeman, A.: The Grothendieck duality theorem via Bousfield’s techniques and Brown representability. J. Amer. Math. Soc. 9(1), 205–236 (1996)
[20] Neeman, A.: Triangulated Categories. Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ (2001)

[21] Rouquier, R.: Dimensions of triangulated categories. J. K-Theory 1(2), 193–256 (2008)

[22] Venkov, B.B.: Cohomology algebras for some classifying spaces (Russian). Dokl. Akad. Nauk SSSR 127, 943–944 (1959)

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