Classification of Equivariant Star Products on Symplectic Manifolds

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July 2015

Abstract

In this note we classify invariant star products with quantum momentum maps on symplectic manifolds by means of an equivariant characteristic class taking values in the equivariant cohomology. We establish a bijection between the equivalence classes and the formal series in the second equivariant cohomology, thereby giving a refined classification which takes into account the quantum momentum map as well.

Contents

1 Introduction 2
2 Preliminaries 3
3 Fedosov Construction 5
4 Classification 10
5 Characteristic Class 12
References 14

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1 Introduction

The classification of formal star products \([2]\) up to equivalence is well-understood, both for the symplectic and the Poisson case, see e.g. the textbook \([25]\) for more details on deformation quantization. While the general classification in the Poisson case is a by-product of the formality theorem of Kontsevich \([15,16]\), the symplectic case can be obtained easier by various different methods \([4,6,9,11,20,26]\).

The result is that in the symplectic case there is an intrinsically defined characteristic class

\[
c(\mathbf{\star}) \in \left[\frac{\omega}{\nu} + H^2_{\text{dR}}(M, \mathbb{C})\right][\nu]
\]

for every star product \(\mathbf{\star}\) which is a formal series in the second de Rham cohomology. By convention, one places the symplectic form as reference point in order \(\nu^{-1}\). Then \(\mathbf{\star}\) and \(\mathbf{\star}'\) are equivalent iff \(c(\mathbf{\star}) = c(\mathbf{\star}')\). Moreover, if we denote by \(\text{Def}(M, \omega)\) the set of equivalence classes of star products quantizing \((M, \omega)\), the characteristic class induces a bijection

\[
c: \text{Def}(M, \omega) \ni [\mathbf{\star}] \mapsto c(\mathbf{\star}) \in \left[\frac{\omega}{\nu} + H^2_{\text{dR}}(M, \mathbb{C})\right][\nu].
\]

If one has in addition a symmetry in form of a group action of a Lie group by symplectic or Poisson diffeomorphisms, one is interested in invariant star products. Again, the classification is known both in the symplectic and in the Poisson case, at least under certain assumptions on the action. The general Poisson case makes use of the equivariant formality theorem of Dolgushev \([7]\), which can be obtained whenever there is an invariant connection on the manifold. Such an invariant connection exists e.g. if the group action is proper, but also in far more general situations. In the easier symplectic situation one can make use of Fedosov’s construction of a star product \([8]\) and obtain the invariant characteristic class \(c_{\text{inv}}\), now establishing a bijection

\[
c_{\text{inv}}: \text{Def}^{\text{inv}}(M, \omega) \ni [\mathbf{\star}] \mapsto c(\mathbf{\star}) \in \left[\frac{\omega}{\nu} + H^2_{\text{dR}}(M, \mathbb{C})\right][\nu].
\]

such that \(\mathbf{\star}\) and \(\mathbf{\star}'\) are invariantly equivalent iff \(c_{\text{inv}}(\mathbf{\star}) = c_{\text{inv}}(\mathbf{\star}')\), see \([3]\). Here one uses the more refined notion of invariant equivalence of invariant star products, where the equivalence \(S = \text{id} + \sum_{r=1}^{\infty} \nu^r S_r\) with \(S(f \mathbf{\star} g) = Sf \mathbf{\star}' g Sg\) for \(f, g \in C^\infty(M)[[\nu]]\) is now required to be invariant. Moreover, \(H^2_{\text{dR}}(M, \mathbb{C})\) denotes the invariant second de Rham cohomology, i.e. invariant closed two-forms modulo differentials of invariant one-forms. Again, one needs an invariant connection for this to work.

In symplectic and Poisson geometry, the presence of a symmetry group is typically not enough: one wants the fundamental vector fields of the action to be Hamiltonian by means of an \(\text{Ad}^*\)-equivariant momentum map \(J: M \rightarrow \mathfrak{g}^*\), where \(\mathfrak{g}\) is the Lie algebra of the group \(G\) acting on \(M\). The notion of a momentum map has been transferred to deformation quantization in various flavours, see e.g. the early work \([1]\). The resulting general notion of a quantum momentum map is due to \([27]\) but was already used in examples in e.g. \([5]\). We will follow essentially the conventions from \([17,18]\), see also \([12,13]\): a quantum momentum map is a formal series \(J \in C^1(\mathfrak{g}, C^\infty(M))[[\nu]]\) such that for all \(\xi \in \mathfrak{g}\) the function \(J(\xi)\) generates the fundamental vector field by \(\mathbf{\star}\)-commutators and such that one has the equivariance condition that \([J(\xi), J(\eta)]_\mathbf{\star} = \nu J([\xi, \eta])\) for \(\xi, \eta \in \mathfrak{g}\). Here the zeroth order is necessarily an equivariant momentum map in the classical sense. We assume the zeroth order to be fixed once and for all.

The classification we are interested in is now for the pair of a \(G\)-invariant star product \(\mathbf{\star}\) and a corresponding quantum momentum map \(J\) with respect to the equivalence relation determined by equivariant equivalence: we say \((\mathbf{\star}, J)\) and \((\mathbf{\star}', J')\) are equivariantly equivalent if there is a \(G\)-invariant equivalence transformation \(S\) relating \(\mathbf{\star}\) to \(\mathbf{\star}'\) as before and \(SJ = J'\). In most interesting cases, the group \(G\) is connected and hence invariance is equivalent to infinitesimal invariance under the Lie
algebra action by fundamental vector fields. Thus it is reasonable to assume a Lie algebra action from the beginning, whether it actually comes from a corresponding Lie group action or not. The set of equivalence classes for this refined notion of equivalence will then be denoted by Def$_g(M,\omega)$. The main result of this work is then the following:

**Main Theorem (Equivariant equivalence classes):** Let $(M,\omega)$ be a connected symplectic manifold with a strongly Hamiltonian action $g \ni \xi \mapsto X_\xi \in \Gamma^\infty(TM)$ by a real finite-dimensional Lie algebra $g$. Suppose there exists a $g$-invariant connection. Then there exists a characteristic class

$$c_g : \text{Def}_g(M,\omega) \longrightarrow \frac{[\omega - J_0]}{\nu} + H^2_\text{inv}(M)[\nu]$$

(1.4)

establishing a bijection to the formal series in the second equivariant cohomology. Under the canonical map $H^2_g(M) \longrightarrow H^2_\text{inv}(M)$ the class becomes the invariant characteristic class.

The main idea is to base the construction of this class on the Fedosov construction. This is where we need the invariant connection in order to obtain invariant star products and quantum momentum maps. The crucial and new aspect compared to the existence and uniqueness statements obtained earlier is that we have to find an invariant equivalence transformation $\tilde{S}$ for which we can explicitly compute $S\tilde{J}$ in order to compare it to $J'$.

Of course, the above theorem only deals with the symplectic situation which is substantially easier than the genuine Poisson case. Here one can expect similar theorems to hold, however, at the moment they seem out of reach. The difficulty is, in some sense, to compute the effect of invariant equivalence transformations on quantum momentum maps by means of some chosen equivariant formality, say the one of Dolgushev. On a more conceptual side, this can be seen as part of a much more profound equivariant formality conjecture stated by Nest and Tsygan [19, 24]. From that point of view, our result supports their conjecture.

One of our motivations to search for such a characteristic class comes from the classification result of $G$-invariant star products up to equivariant Morita equivalence [13], where a reminiscent of the equivariant class showed up in the condition for equivariant Morita equivalence.

The paper is organized as follows: in Section 2 we collect some preliminaries on invariant star products and the existence of quantum momentum maps. Section 3 contains a brief reminder on those parts of Fedosov’s construction which we will need in the sequel. It contains also the key lemma to prove our main theorem. In Section 4 we establish a relative class which allows to determine whether two given pairs of star products and corresponding quantum momentum maps are equivariantly equivalent. In the last Section 5 we define the characteristic class and complete the proof of the main theorem.

## 2 Preliminaries

Throughout this paper let $(M,\omega)$ denote a connected, symplectic manifold, $\{\cdot,\cdot\}$ the corresponding Poisson bracket, $\Omega^\cdot(M)$ the differential forms on $M$, $Z^\cdot(M)$ the closed forms, $g$ a real finite-dimensional Lie algebra, and $\nabla$ a torsion-free, symplectic connection on $M$. Then every anti-homomorphism $g \longrightarrow \Gamma^\infty_{\text{symp}}(TM) : \xi \mapsto X_\xi$ from $g$ into the symplectic vector fields on $M$ gives rise to a representation of $g$ on $\mathcal{C}^\infty(M)$ via $\xi \mapsto (f \mapsto -\mathcal{L}_{X_\xi}f)$ where $\mathcal{L}$ denotes the Lie derivation. For convenience, we will abbreviate $\mathcal{L}_{X_\xi}$ to $\mathcal{L}_\xi$ and analogously for the insertion $i_\xi$. In most cases, the Lie algebra action arises as the infinitesimal action of a Lie group action by a Lie group $G$ acting symplectically on $M$. In the case of a connected Lie group, we can reconstruct the action of $G$ as usual. However, we do not assume to have a Lie group action since in several cases of interest the vector fields $X_\xi$ might not have complete flows.

Since the main focus here will be on the interaction between symmetries conveyed by $g$ and formal star products on $M$, we shall briefly recall the relevant basic definitions, following notations
and conventions from [22], see also [25] Sect. 6.4 for a more detailed introduction. Let $\mathcal{C}^\infty(M)[\nu]$ be the space of formal power series in the formal parameter $\nu$ with coefficients in $\mathcal{C}^\infty(M)$. A star product on $(M, \omega)$ is a bilinear map

$$\ast: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[\nu]: (f, g) \mapsto f \ast g = \sum_{k=0}^{\infty} \nu^k C_k(f, g),$$

(2.1)

such that its $\nu$-bilinear extension to $\mathcal{C}^\infty(M)[\nu] \times \mathcal{C}^\infty(M)[\nu]$ is an associative product, $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \{f, g\}$ holds for all $f, g \in \mathcal{C}^\infty(M)$, and $C_k: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a bidifferential operator vanishing on constants for all $k \geq 1$. Such a star product is called $g$-invariant if the $\nu$-linear extension of $\mathcal{L}_\xi$ to $\mathcal{C}^\infty(M)[\nu]$ is a derivation of $\ast$ for all $\xi \in g$.

Recall further that a linear map $J_0: g \to \mathcal{C}^\infty(M)$ is called a (classical) Hamiltonian for the action if it satisfies $\mathcal{L}_\xi f = -\{J_0(\xi), f\}$ for all $\xi \in g$ and $f \in \mathcal{C}^\infty(M)$. It is called a (classical) momentum map, if in addition $J_0(\{\xi, \eta\}) = \{J_0(\xi), J_0(\eta)\}$ holds for all $\xi, \eta \in g$. We can adapt a similar notion for star products on $M$ by replacing the Poisson bracket with the $\ast$-commutator, compare also [27]. We will generally adopt the notation $\text{ad}_\ast(f) g = [f, g]$, with $[\cdot, \cdot]_\ast$ being the commutator with respect to the product $\ast$. Finally, we will, for any vector space $V$, denote by $\mathcal{C}^k(g, V)$ the space of $V$-valued, $k$-multilinear, alternating forms on $g$. The following definition is by now the standard notion [18][27].

**Definition 2.1 (Quantum momentum map)** Let $\ast$ be a $g$-invariant star product. A map $J \in C^1(g, \mathcal{C}^\infty(M))[\nu]$ is called a quantum momentum map if

$$\mathcal{L}_\xi = -\frac{1}{\nu} \text{ad}_\ast(J(\xi)) \quad \text{and} \quad J(\{\xi, \eta\}) = \frac{1}{\nu}[J(\xi), J(\eta)],$$

(2.2)

hold for all $\xi, \eta \in g$. If only the first equality is satisfied, we will call $J$ a quantum Hamiltonian.

Evaluating the above equations in zeroth order in $\nu$ for any quantum Hamiltonian (quantum momentum map) $J$, one can readily observe that $J_0 = J |_{\nu=0}$ is a classical Hamiltonian (momentum map). Conversely, we will say that $J$ deforms the Hamiltonian (momentum map) $J_0$. Having the previous definitions at hand, one can define various flavours of equivalences between star products.

**Definition 2.2 ($g$-Equivariant equivalence)** Let $\ast$ and $\ast'$ be star products on $M$.

i.) They are called equivalent if there is a formal series $T = \text{id} + \sum_{k=1}^{\infty} \nu^k T_k$ of differential operators $T_k: \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that

$$f \ast g = T^{-1}(T(f) \ast' T(g)) \quad \text{and} \quad T(1) = 1$$

(2.3)

for all $f, g \in \mathcal{C}^\infty(M)[\nu]$. In this case $T$ is called an equivalence from $\ast$ to $\ast'$.

ii.) If $\ast$ and $\ast'$ are $g$-invariant star products we will call an equivalence transformation $T$ from $\ast$ to $\ast'$ a $g$-invariant equivalence if $\mathcal{L}_\xi T = T \mathcal{L}_\xi$ holds for all $\xi \in g$.

iii.) If in addition $J$ and $J'$ are quantum momentum maps of $\ast$ and $\ast'$ respectively, we will call the pairs $(\ast, J)$ and $(\ast', J')$ equivariantly equivalent if there is a $g$-invariant equivalence $T$ from $\ast$ to $\ast'$ such that $TJ = J'$.

The first two versions of equivalence [11][13][14][19][20][26] and invariant equivalence [4] were discussed in the literature already extensively, leading to the well-known classification results. In this work we will deal with the third version.

For the concluding classification result, we will need the equivariant cohomology (in the Cartan model) on $M$ with respect to $g$, for more details, see e.g. the monograph [10]. Since we are mainly
interested in the Lie algebra case, the underlying complex is the complex of equivariant differential forms on \( M \), that is
\[
\Omega^k_g(M) = \bigoplus_{2i+j=k} (S^i(\mathfrak{g}^*) \otimes \Omega^j(M))^{\text{inv}},
\]
where \( \text{inv} \) denotes the space of \( \mathfrak{g} \)-invariants with respect to the coadjoint representation on \( S^*\mathfrak{g}^* \) and \( -\mathcal{L}_\xi \) on \( \Omega^*(M) \). Equivalently, one can view the elements \( \alpha \in \Omega^k_g(M) \) as \( \Omega^k(M) \)-valued polynomials \( \mathfrak{g} \rightarrow \Omega^*(M) \) subject to the equivariance condition
\[
\alpha([\xi, \eta]) = -\mathcal{L}_\xi \alpha(\eta) \quad \text{for all } \xi, \eta \in \mathfrak{g}.
\]
(2.5)

The differential \( d_g: \Omega^k_g(M) \rightarrow \Omega^{k+1}(M) \) is defined to be
\[
(d_g \alpha)(\xi) = d(\alpha(\xi)) + i_\xi \alpha(\xi) \quad \text{or} \quad d_g \alpha = d\alpha + i_\alpha \alpha,
\]
where \( d \) denotes the de Rham differential on \( M \) and \( i_\xi \) the insertion of \( X_\xi \) into the first argument. The equivariant cohomology on \( M \) with respect to \( \mathfrak{g} \) is then, as usual, defined by \( H^*_g(M) = \ker d_g / \text{im} d_g \) and we will denote the equivariant class of the representative \( \alpha \in \Omega^*_g(M) \) by \([\alpha]_g\).

3 Fedosov Construction

Since our central classification result will make heavy use of the Fedosov construction \[8\], we shall collect and recall the relevant results here briefly. As in the previous section, we will follow the exposition in \[22\], see also \[25\] Sect. 6.4 for further details. Let us start by defining the formal Weyl algebra
\[
\mathcal{W} \otimes \Lambda(M) = \prod_{k=0}^{\infty} \left( \mathbb{C} \otimes \Gamma^\infty \left( \mathfrak{g}^k T^* M \otimes \Lambda^* M \right) \right) \left[ \nu \right],
\]
(3.1)
where \((M, \omega)\) is a symplectic manifold. Then \( \mathcal{W} \otimes \Lambda \) (we will frequently drop the reference to \( M \)) is obviously an associative graded commutative algebra with respect to the pointwise symmetrized tensor product in the first factor and the \( \Lambda \)-product in the second factor. The resulting product we will denote by \( \mu \). We can additionally observe that \( \mathcal{W} \otimes \Lambda \) is graded in various ways and define the corresponding degree maps on elements of the form \( a = (X \otimes \alpha)^\nu K \) with \( X \in S^k T^* M \) and \( \alpha \in \Lambda^m T^* M \) as
\[
\text{deg}_a a = \ell a, \quad \text{deg}_a a = ma, \quad \text{and} \quad \text{deg}_a a = ka,
\]
(3.2)
extend them as derivations to \( \mathcal{W} \otimes \Lambda \), and refer to the first two as symmetric and antisymmetric degree, respectively. Finally, the so called total degree \( \text{Deg} = \text{deg}_a + 2 \text{deg}_\nu \) will be needed later on.

We can then proceed and define another associative product on \( \mathcal{W} \otimes \Lambda \) first locally in a chart \((U, x)\) by
\[
a \circ_F b = \mu \circ \exp \left( \frac{\nu}{2} \omega_{ij} d x^i \otimes d x^j \right) \left( a \otimes b \right),
\]
(3.3)
with \( a, b \in \mathcal{W} \otimes \Lambda \), where \( \omega \big|_U = \frac{1}{2} \omega_{ij} d x^i \wedge d x^j \) and \( \omega^{ik} \omega_{jk} = \delta^i_j \), and with \( i_\nu(\partial_i) \) being the insertion of \( \partial_i \) into the first argument on \( \mathcal{W} \). The tensorial character of the insertions then shows that this is actually globally well-defined and yields an associative product since the insertions in the symmetric tensors are commuting derivations.

It is easy to see that \( \circ_F \) is neither \( \text{deg}_{a^*} \) nor \( \text{deg}_{\nu^*} \)-graded but that it is \( \text{deg}_{a^*} \) and \( \text{Deg} \)-graded. We can additionally use the latter to obtain a filtration of \( \mathcal{W} \otimes \Lambda \). To that end, let \( \mathcal{W}_k \otimes \Lambda \) denote those elements of \( \mathcal{W} \otimes \Lambda \) whose total degree is greater than or equal to \( k \). We then have
\[
\mathcal{W} \otimes \Lambda = \mathcal{W}_0 \otimes \Lambda \supseteq \mathcal{W}_1 \otimes \Lambda \supseteq \cdots \supseteq \{0\} \quad \text{and} \quad \bigcap_{k=0}^{\infty} \mathcal{W}_k \otimes \Lambda = \{0\},
\]
(3.4)
We will frequently use this filtration together with Banach’s fixed point theorem to find unique solutions to equations of the form \( a = L(a) \) for \( a \in \mathcal{W} \otimes \Lambda \) and \( L : \mathcal{W} \otimes \Lambda \rightarrow \mathcal{W} \otimes \Lambda \) such that \( L \) is contracting with respect to the total degree. For details see e.g. [23, Sect. 6.2.1].

Essential for the Fedosov construction are then the following operators on \( \mathcal{W} \otimes \Lambda \)

\[
\delta = (1 \otimes dx^i) i_\nu \partial_i \quad \delta^* = (dx^i \otimes 1) i_\nu \partial_i \quad \nabla = (1 \otimes dx^i) \nabla_{\partial_i},
\]

where \( \nabla \) is any torsion-free, symplectic connection on \( M \). Again, it is clear that these definitions yield chart-independent operators. From the definition of \( \delta \) and \( \delta^* \) one can easily calculate that \( \delta \) is a graded derivation of \( \mathcal{O}_F \) and \( \delta^2 = (\delta^*)^2 = 0 \). With the help of the projection \( \sigma : \mathcal{W} \otimes \Lambda \rightarrow C^\infty(M) [\nu] \) onto symmetric and antisymmetric degree 0 as well as a normalized version of \( \delta^* \) which is defined on homogeneous elements \( a \in \mathcal{W}^k \otimes \Lambda^\ell \) as

\[
\delta^{-1} a = \begin{cases} \frac{1}{k + \ell} \delta^* a & \text{for } k + \ell \neq 0 \\ 0 & \text{for } k + \ell = 0, \end{cases}
\]

one finds that

\[
\text{id}_{\mathcal{W} \otimes \Lambda} - \sigma = \delta \delta^{-1} + \delta^{-1} \delta.
\]

For \( \nabla \) on the other hand, one can check that \( \nabla \) is a graded derivation of \( \mathcal{O}_F \) and that

\[
\nabla^2 = -\frac{1}{\nu} \text{ad}_\nu(R)
\]

with \( R \in \mathcal{W}^2 \otimes \Lambda^2 \) being the curvature tensor of \( \nabla \). The ingenious new element of Fedosov is then a graded derivation of \( \mathcal{O}_F \), which is subject of the following theorem.

**Theorem 3.1 (Fedosov)** Let \( \Omega \in \nu Z^2(M)[\nu] \) be a series of closed two forms on \( M \). Then there exists a unique \( r \in \mathcal{W}_2 \otimes \Lambda^1 \) such that

\[
r = \delta^{-1} \left( \nabla r - \frac{1}{\nu} r \mathcal{O}_F r + R + 1 \otimes \Omega \right).
\]

The Fedosov derivation

\[
\mathfrak{D} = -\delta + \nabla - \frac{1}{\nu} \text{ad}_\nu(r)
\]

is then a graded \( \mathcal{O}_F \)-derivation of antisymmetric degree 1 with \( \mathfrak{D}^2 = 0 \).

One finds that on elements \( a \in \mathcal{W} \otimes \Lambda^{\geq 1} \) with positive antisymmetric degree there is a homotopy operator corresponding to \( \mathfrak{D} \), which is given by

\[
\mathfrak{D}^{-1} a = -\delta^{-1} \left( \text{id} - \left[ \delta^{-1}, \nabla - \frac{1}{\nu} \text{ad}_\nu(r) \right] \right) \frac{1}{\text{id} - \left[ \delta^{-1}, \nabla - \frac{1}{\nu} \text{ad}_\nu(r) \right]} \quad \text{such that} \quad \mathfrak{D} \mathfrak{D}^{-1} a + \mathfrak{D}^{-1} \mathfrak{D} a = a
\]

where \( \mathfrak{D} \) is constructed from \( r \) according to the previous theorem. Using \( \mathfrak{D}^{-1} \) one can show that there is a unique isomorphism of \( \mathbb{C}[\nu] \)-vector spaces \( \tau : C^\infty(M)[\nu] \rightarrow \ker \mathfrak{D} \cap \mathcal{W} \otimes \Lambda \), called the Fedosov-Taylor series, with inverse being the projection \( \sigma \) restricted to the codomain of \( \tau \). The Fedosov-Taylor series \( \tau \) can explicitly be written as \( \tau(f) = f - \mathfrak{D}^{-1}(1 \otimes df) \). We can now proceed to define the Fedosov star product \( \ast_{\Omega} \) on \( C^\infty(M)[\nu] \) as the pullback of \( \mathcal{O}_F \) with \( \tau \),

\[
f \ast_{\Omega} g = \sigma(\tau(f) \mathcal{O}_F \tau(g))
\]

for all \( f, g \in C^\infty(M)[\nu] \), where we explicitly referenced the formal series of two-forms \( \Omega \) from which \( r \) in the Fedosov derivation \( \mathfrak{D} \) has been constructed. Of course, in this very brief discourse we omitted
numerous details, which are however fully displayed in [9], see also [23 and 25 Sect. 6.4]. As a final remark let us briefly note that the construction obviously depends on the choice of the torsion-free, symplectic connection $\nabla$. However, as mentioned in the previous section, we fix one such connection and will mostly omit any explicit mention.

Instead, let us focus on another aspect of the Fedosov construction, namely on its connection with symmetries in the form of representations of a Lie algebra $\mathfrak{g}$ on $\mathcal{C}^\infty(M)[[\nu]]$. The first result we would like to cite from [18] clarifies under which circumstances the above construction yields a $\mathfrak{g}$-invariant star product.

**Proposition 3.2 (Müller-Bahns, Neumaier)** The Fedosov star product obtained from a torsion-free, symplectic connection $\nabla$ and $\Omega \in \nu Z^2(M)[[\nu]]$ is $\mathfrak{g}$-invariant if and only if

$$\left[\nabla, \mathcal{L}_\xi\right] = 0 \quad \text{and} \quad \mathcal{L}_\xi \Omega = 0$$

(3.13)

for all $\xi \in \mathfrak{g}$.

In other words, both ingredients, the symplectic connection and the series of closed two-forms, have to be $\mathfrak{g}$-invariant. Therefore we shall assume from now on that we have an invariant, torsion-free, symplectic connection $\nabla$ fixed once and for all. Its existence can be guaranteed under various assumptions on the action of $\mathfrak{g}$. One rather simple option is to assume that the Lie algebra action integrates to a proper action of $G$, for which one has invariant connections. However, having an invariant connection is far less restrictive than having a proper action.

One crucial ingredient in the proof of the previous statement, which will also come in handy for our purposes later on, is an expression of the Lie derivative on $W \otimes \Lambda$ and the Fedosov derivation, the so-called deformed Cartan formula. To formulate it one uses, for each $X \in \Gamma^\infty_{\text{sympl}}(TM)$, the one-form

$$\theta_X = i_X \omega$$

(3.14)

and the symmetrized covariant derivative acting on $W \otimes \Lambda$, explicitly given by

$$D = [\delta^*, \nabla] = (dx^i \otimes 1) \nabla \partial_i.$$

(3.15)

The deformed Cartan formula is as follows [22][23] for the Fedosov derivation $\mathcal{D}$ based on $r$ as in (3.9):

**Lemma 3.3** Let $X \in \Gamma^\infty_{\text{sympl}}(M, \omega)$ be a symplectic vector field. Then the Lie derivative on $W \otimes \Lambda$ and the Fedosov derivation $\mathcal{D}$ are related as follows:

$$\mathcal{L}_X = \mathcal{D} i_a(X) + i_a(X) \mathcal{D} - \frac{1}{\nu} \text{ad}_\nu \left( \theta_X \otimes 1 + \frac{1}{2} D\theta_X \otimes 1 - i_a(X) r \right).$$

(3.16)

As it turns out, there is also a very convenient expression for the Fedosov-Taylor series of any quantum Hamiltonian $J$ of a Fedosov star product $\star_{\Omega}$. Later on, the crucial part in the following lemma will be that for once, $\tau(J)$ only depends on $J$ in symmetric and antisymmetric degree 0 and secondly, that the only dependence on $\Omega$ lies in the summand $i_a r$. The remaining parts only depend on the symplectic 2-form $\omega$ and the symplectic, $\mathfrak{g}$-invariant connection $\nabla$ on $(M, \omega)$. From [18] we recall the following formulation:

**Lemma 3.4** Let $\star_{\Omega}$ be a Fedosov star product constructed from $\Omega \in \nu Z^2(M)[[\nu]]$ with quantum Hamiltonian $J$. Then the Fedosov-Taylor series of $J$ is given by

$$\tau(J(\xi)) = J(\xi) + \theta_\xi \otimes 1 + \frac{1}{2} D\theta_\xi \otimes 1 + i_a(\xi) r$$

(3.17)

for all $\xi \in \mathfrak{g}$ where $\theta_\xi = i_\xi \omega$. 

7
where we again exploited the fact that \( \text{ad} \).  

Again, the inverse of \( T \) is a graded derivation of \( A \).  

Lemma 3.5 Let \( \ast_{\Omega} \) and \( \ast_{\Omega'} \) be two Fedosov star products constructed from \( \Omega, \Omega' \in \nu Z^2(M)[\nu] \) respectively and let additionally \( \Omega - \Omega' = dC \) for a fixed \( C \in \nu \Omega^1(M)[\nu] \). Then there is an equivalence \( T_C \) from \( \ast_{\Omega} \) to \( \ast_{\Omega'} \) given by 

\[
T_C = \sigma \circ A_h \circ \tau,
\]
where \( A_h = \exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} \) and \( h \in \mathcal{W}_3 \) is obtained as the unique solution of 

\[
h = C \otimes 1 + \delta^{-1} \left( \nabla h - \frac{1}{\nu} \text{ad}_{\Omega}(r)h - \frac{1}{\nu} \frac{\text{ad}_{\Omega}(h)}{\exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} - \text{id}} (r' - r) \right)
\]

with \( \sigma(h) = 0 \). Furthermore, the Fedosov derivation of \( h \) is given by 

\[
\mathcal{D} h = -1 \otimes C + \frac{\frac{1}{\nu} \text{ad}_{\Omega}(h)}{\exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} - \text{id}} (r' - r).
\]

Proof: In \cite{22} Sect. 3.5.1.1], the case of Fedosov star products of Wick type was considered. The argument transfers immediately to the more general situation we need here. Nevertheless, for convenience we sketch the proof. First of all, let us consider the map \( A_h : W \otimes \Lambda \rightarrow W \otimes \Lambda \) given by \( A_h = \exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} \) for any \( h \in \mathcal{W}_3 \otimes \Lambda^0 \). Counting degrees, one finds that \( \frac{1}{\nu} \text{ad}_{\Omega}(h) \) increases the total degree by at least one, what guarantees that \( A_h \) is well-defined. Furthermore, since \( \frac{1}{\nu} \text{ad}_{\Omega}(h) \) is a graded derivation of \( \Omega \), we have \( A_h(\alpha \circ \beta) = A_h(\alpha) \circ \Omega F A_h(\beta) \) for all \( \alpha, \beta \in \mathcal{W} \otimes \Lambda \) and thus \( A_h \) is actually an algebra automorphism of \( (\mathcal{W} \otimes \Lambda, \Omega F) \) with inverse given by \( A_h^{-1} = A_{-h} \).

Next, we propose that \( S_h = \sigma \circ A_h \circ \tau \) is an equivalence from \( \ast_{\Omega} \) to \( \ast_{\Omega'} \) if 

\[
\mathcal{D}' = A_h \circ \mathcal{D} \circ A_{-h}
\]
holds, where we denoted by \( \mathcal{D}' \) the Fedosov derivation constructed from \( \Omega' \). Here we quickly note that, since \( \tau \) maps functions into \( \ker \mathcal{D} \), we have \( \mathcal{D} \tau(f) = 0 \) for all \( f \in \mathcal{C}^\infty(M)[\nu] \) and hence also \( \mathcal{D}'A_h \tau(f) = A_h \mathcal{D} \tau(f) = 0 \) because of (3.21). Additionally, with the help of the Fedosov-Taylor series \( \tau' \) constructed from \( \mathcal{D}' \), one can easily observe that \( A_h \tau(f) = (\tau' \circ \sigma)(A_h \tau(f)) = \tau'(S_h(f)) \), which finally enables us to show 

\[
S_h(f \ast_{\Omega} g) = (\sigma \circ A_h \tau(f) \circ \Omega F A_h \tau(g)) = (\tau'(S_h(f)) \circ \Omega F \tau'(S_h(g))) = S_h(f) \ast_{\Omega'} S_h(f).
\]

Again, the inverse of \( S_h \) is obviously given by \( S_h^{-1} = \sigma \circ A_{-h} \circ \tau' \).

Given these preliminary considerations, the goal of this proof will be to solve (3.21) for \( h \). To this end, let us rewrite said equation by using the definition of \( A_h \), which results in 

\[
\mathcal{D}' = A_h \mathcal{D} A_{-h} = \mathcal{D} - \frac{1}{\nu} \text{ad}_{\Omega}(h) \left( \frac{\exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} - \text{id}}{\exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} - \text{id}} \right) (\mathcal{D} h),
\]

where we again exploited the fact that \( \text{ad}_{\Omega}(h) \) and \( \mathcal{D} \) are graded \( \Omega \)-derivations and thus \([\text{ad}_{\Omega}(h), \mathcal{D}] = -\text{ad}_{\Omega}(\mathcal{D} h)\]. Comparing \( \mathcal{D} \) and \( \mathcal{D}' \) from (3.10) we see that the above equation is satisfied if 

\[
r' - r - \frac{\exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} - \text{id}}{\exp \left\{ \frac{1}{\nu} \text{ad}_{\Omega}(h) \right\} - \text{id}} (\mathcal{D} h) = 1 \otimes C,
\]

(3.22)
where $C$ is the series of 1-forms with $dC = \Omega - \Omega'$ from the prerequisites. We further claim that this $h$ can be obtained as the unique solution of (3.19) with $\sigma(h) = 0$. First of all, from counting the involved degrees we know that (3.19) has indeed a unique solution. We will now proceed to use this solution to define

$$B = \frac{1}{\nu \text{ad}_{op}(h)} \exp\left\{ \frac{1}{\nu \text{ad}_{op}(h)} \right\} - \text{id} (r' - r) - \mathcal{D} h - 1 \otimes C.$$

At this point we will merely cite a technical result from [22 Sect. 3.5.1.1], which essentially is only a tedious calculation, concerning the Fedosov derivation of $B$. One obtains

$$\mathcal{D} B = \frac{1}{\nu \text{ad}_{op}(h)} \exp\left\{ \frac{1}{\nu \text{ad}_{op}(h)} \right\} - \text{id} \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{t=0}^{s-2} \text{ad}_{op}(h)^{t} \text{ad}_{op}(B) \text{ad}_{op}(h)^{s-2-t} \times\left( r' - r \right) = R_{h,r',r}(B),$$

where we denote the right hand side as a linear operator $R_{h,r',r}(B)$ acting on $B$. Applying $\delta^{-1}$ to both sides and using $\delta^{-1}B = 0$, $\sigma(h) = 0$ and (3.7) as well as (3.10), we arrive at

$$B = \delta^{-1}\left( \nabla B - \frac{1}{\nu \text{ad}_{op}(r)} B - R_{h,r',r}(B) \right).$$

Yet again, by counting degrees, we observe that the above equation has a unique solution and that $B = 0$ is this solution. From here it is easy to see that $B = 0$ is equivalent to $h$ satisfying (3.22), which completes the construction of $T_C$ as $T_C = S_{h(C)}$.

**Corollary 3.6** Let $\Omega, \Omega' \in \nu \mathbb{Z}^2(M)^{\text{inv}}[\nu]$ with $\Omega - \Omega' = dC$ for $C \in \nu \Omega^1(M)^{\text{inv}}[\nu]$ and $h$ as in (3.19).

i.) For all $\xi \in \mathfrak{g}$ we have

$$\mathcal{L}_{\xi} h = 0. \quad (3.23)$$

ii.) For all $\xi \in \mathfrak{g}$ we have

$$\mathcal{L}_{\xi} \circ T_C = T_C \circ \mathcal{L}_{\xi}, \quad (3.24)$$

i.e. the equivalence $T_C$ is $\mathfrak{g}$-invariant.

We come now to the key lemma needed for the proof of our main theorem. If we are interested in the equivariant classification, we need to know the effect of an invariant equivalence transformation on quantum momentum maps. For the particular equivalences from [Lemma 3.5] we have the following result:

**Lemma 3.7** Let $\Omega \in \nu \mathbb{Z}^2(M)^{\text{inv}}[\nu]$ and $C \in \nu \Omega^1(M)^{\text{inv}}[\nu]$. Then for any quantum Hamiltonian $J$ of the Fedosov star product $\ast_{\Omega}$ and the $\mathfrak{g}$-invariant equivalence $T_C$ obtained from $C$ via [Lemma 3.5] we have

$$J(\xi) + i_\xi C - T_C J(\xi) = 0. \quad (3.25)$$

**PROOF:** This proof is essentially a straightforward calculation using (3.16), (3.17), (3.20) and (3.23) as well as the fact that $\text{deg}_a h = 0$ and thus $i_a(\xi) h = 0$ for the unique solution $h$ of (3.19). We have

$$\frac{1}{\nu} \text{ad}_{op}(h) r(J(\xi)) = - \frac{1}{\nu} \text{ad}_{op}(J(\xi) + \theta \otimes 1 + \frac{1}{2} D \theta \otimes 1 - i_a(\xi) r) h = (\mathcal{L}_\xi - \mathcal{D}) i_a(\xi) + i_a(\xi) \mathcal{D} h.$$
Then there exists a quantum momentum map if and only if there is an element 
with the previous sections as preparations we can proceed towards our central classification result. 
where we denoted by \( \tau \) the Fedosov-Taylor series corresponding to, and by \( J' \) any quantum Hamiltonian of the Fedosov star product constructed from \( \Omega - dC \) (one might, for example, choose \( J' = T_C J \)). Next, applying \((\exp\{\frac{1}{\nu} \text{ad}_{\Omega}(h)\} - \text{id})/\frac{1}{\nu} \text{ad}_{\nu}(h)\) to the above equation yields 
\[
\left(\exp\left\{\frac{1}{\nu} \text{ad}_{\nu}(h)\right\} - \text{id}\right) \tau(J(\xi)) + 1 \otimes i_\xi C = (J'(\xi) - \tau'(J'(\xi)) - J(\xi) + \tau(J(\xi))).
\] (3.26)
Finally, we can apply \( \sigma \), observe that the right hand side cancels out entirely and that the left hand side results in the desired terms after using (3.18). \( \Box \)

4 Classification

With the previous sections as preparations we can proceed towards our central classification result. Namely, we will demonstrate that pairs of Fedosov star products and quantum momentum mappings of those star products, respectively, are equivariantly equivalent if and only if a certain class in the second equivariant cohomology vanishes. Said class will turn out to be \([\Omega - J - (\Omega' - J')]_g\) where \( \Omega \) and \( \Omega' \) are the series of closed two forms from which the Fedosov star products have been constructed and \( J \) and \( J' \) respective quantum momentum maps. To this end we will firstly employ two results from [18] that will guarantee that our classes are in fact well defined:

**Lemma 4.1** A \( g \)-invariant Fedosov star product for \((M, \omega)\) obtained from \( \Omega \in \nu Z^2(M)^{\text{inv/} \nu} \) admits a quantum Hamiltonian if and only if there is an element \( J \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))_{/\nu} \) such that 
\[
dJ(\xi) = i_\xi(\omega + \Omega)
\] (4.1) 
for all \( \xi \in \mathfrak{g} \). We then have \( \mathcal{L}_\xi = -\frac{1}{\nu} \text{ad}_\xi(J(\xi)) \).

Note that since quantum Hamiltonians for the same star product differ only by an element in 
\( C^1(\mathfrak{g}, \mathcal{C}^\infty(M))_{/\nu} \), (4.1) holds for every quantum Hamiltonian of \( \ast_\Omega \). The second result from [18] is the following consequence:

**Corollary 4.2** Let \( \ast_\Omega \) be a \( g \)-invariant Fedosov star product constructed from \( \Omega \in \nu Z^2(M)^{\text{inv/} \nu} \). Then there exists a quantum momentum map if and only if there is an element \( J \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))_{/\nu} \) such that 
\[
i_\xi(\omega + \Omega) = dJ(\xi) \quad \text{and} \quad (\omega + \Omega)(X_\xi, X_\eta) = J([\xi, \eta]).
\] (4.2) 
The following little calculation, which is valid for any quantum momentum map \( J \), 
\[
J([\xi, \eta]) = \frac{1}{\nu} \text{ad}_{\xi}(J(\xi))J(\eta) = -i_\xi dJ(\eta) = (\omega + \Omega)(X_\xi, X_\eta)
\]
then shows that any quantum momentum map necessarily satisfies (4.2). And vice versa, any quantum Hamiltonian satisfying (4.2) is in fact a quantum momentum map. However, we will only use the above results in the following capacity, namely to show that first, any quantum momentum map \( J \) is an element of \( \Omega^2_M(\mathfrak{g})_{/\nu} \). For this we have to demonstrate that the equivariance condition holds, which reads \( J([\xi, \eta]) = -i_\xi dJ(\eta) \) and is obviously fulfilled as shown by the previous calculation.
Second, let us show that the equivariant cochain $\omega + \Omega - J \in \Omega^2_{g}(M)[\nu]$, with $\Omega \in \nu Z^2(M)^{\text{inv}}[\nu]$ and $J$ being a quantum momentum map for the Fedosov star product $\star_{\Omega}$, is $d_{g}$-closed. Indeed,

$$d_{g}(\omega + \Omega - J)(\xi) = i_{\xi}(\omega + \Omega) - dJ(\xi) = 0.$$ 

Using this observation we can restate the above condition on the existence of a quantum momentum map as follows: there exists a quantum momentum map if and only if $J^{\prime} - J$ is a $d_{g}$-cocycle.

**Lemma 4.3** Let $J$ and $J^{\prime}$ be quantum momentum maps of a $g$-invariant star product $\star$ deforming the same momentum map $J_{0}$. Then there exists a $g$-invariant self-equivalence $A$ of $\star$ with $AJ = J^{\prime}$ if and only if $J^{\prime} - J$ is a $d_{g}$-cocycle.

**Proof:** First of all, from the defining property $\mathcal{L}_{\xi} = -\frac{1}{\nu} \text{ad}_{\nu}(J) = -\frac{1}{\nu} \text{ad}_{\nu}(J^{\prime})$ it is clear that $j = J^{\prime} - J$ is central, hence a constant function on $M$ for all $\xi \in g$ and consequently a $d_{g}$-coboundary. Here we use that $M$ is connected.

Now assume that there is a $\theta \in \nu \Omega^{1}(M)^{\text{inv}}[\nu]$ with $d_{g}\theta = j$, which is equivalent to $i_{\xi}\theta = j(\xi)$ and $d\theta = 0$. Consequently the self-equivalence $A = \exp\{\frac{1}{\nu} \text{ad}_{\nu}(\theta)\}$ is well defined: on sufficiently small open subsets $U \subseteq M$ we have $\theta|_{U} = dt_{U}$. Hence we can calculate locally

$$\frac{1}{\nu} \text{ad}_{\nu}(t_{U})J(\xi)|_{U} = \mathcal{L}_{\xi}t_{U} = j(\xi)|_{U} \quad \text{and hence} \quad \left(\frac{1}{\nu} \text{ad}_{\nu}(t_{U})\right)^{k}J(\xi)|_{U} = 0$$

for all $k \geq 2$. This allows to compute

$$A\cdot J(\xi)|_{U} = \exp\left\{\frac{1}{\nu} \text{ad}_{\nu}(t_{U})\right\}J(\xi)|_{U} = J(\xi)|_{U} + j(\xi)|_{U} = J^{\prime}(\xi)|_{U}.$$ 

For the second part, assume that there is a $g$-invariant self-equivalence $A$ of $\star$ with $AJ = J^{\prime}$. Then there is a closed, $g$-invariant one-form $\theta \in \nu \Omega^{1}(M)^{\text{inv}}[\nu]$ with $A = \exp\{-\frac{1}{\nu} \text{ad}_{\nu}(\theta)\}$, see e.g. [25, Thm. 6.3.18] for the case without invariance. The invariance of $\theta$ is clear from the invariance of $A$. We can again calculate locally

$$j(\xi)|_{U} = A\cdot J(\xi)|_{U} - J(\xi)|_{U} = \left(\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{1}{\nu} \text{ad}_{\nu}(t_{U})\right)^{k-1}\right)\frac{1}{\nu} \text{ad}_{\nu}(J(\xi)|_{U})t_{U},$$

where $\theta|_{U} = dt_{U}$. Since the term in the brackets is a power series starting with $\text{id}$ it is invertible and its inverse is again a power series in $\frac{1}{\nu} \text{ad}_{\nu}(t_{U})$ starting with $\text{id}$. Applying the inverse to both sides, using that $j(\xi)|_{U}$ is constant and $d\theta = 0$, we arrive at

$$j(\xi)|_{U} = \mathcal{L}_{\xi}t_{U} = d_{g}\theta(\xi)|_{U}. \quad \Box$$

Using this result we can now phrase the first classification result of invariant star products with quantum momentum maps on a connected symplectic manifold. To this end, we define the **equivariant relative class**

$$c_{g}(\Omega^{\prime}, J^{\prime}; \Omega, J) = \left[(\Omega^{\prime} - J^{\prime}) - (\Omega - J)\right]_{g}$$

(4.3) of two Fedosov star products built out of the data of the closed two-forms and the quantum momentum maps. Note that we use for both star products the **same** $g$-invariant symplectic connection.
Proposition 4.4 Let $\Omega, \Omega' \in \nu\Omega^2(M)^{\text{inv}}[\nu]$ and $\star_{\Omega}, \star_{\Omega'}$ their corresponding Fedosov star products. Let furthermore $J$ and $J'$ be quantum momentum maps of $\star_{\Omega}$ and $\star_{\Omega'}$ respectively, deforming the same momentum map $J_0$. Then there exists a $g$-invariant equivalence $S$ from $\star_{\Omega}$ to $\star_{\Omega'}$ such that $SJ = J'$ if and only if

$$c_g(\Omega', J'; \Omega, J) = 0. \quad (4.4)$$

Proof: First, as a preliminary step, we need to show that $c_g$ is well-defined at all, i.e. that $(\Omega' - J') - (\Omega - J)$ is $d\nu$-closed which, however, is nothing more than a simple application of Lemma 4.1. Next, we assume we are given such an equivalence $S$. This necessarily implies that $\Omega - \Omega' = dC$ for some $C \in \nu\Omega^1(M)^{\text{inv}}[\nu]$ and hence we can obtain another equivalence $T_C$ from Lemma 3.5. Additionally, from Lemma 4.3 we can deduce that $[T_C J - S J]_g = 0$. Therefore we are able to calculate with the help of Lemma 3.7 that

$$c_g(\Omega', J'; \Omega, J) = \left[ (\Omega' - J') - (\Omega - J) \right]_g$$

$$= \left[ (\Omega' - SJ) - (\Omega - J) - (J + i\xi C - T_C J) \right]_g$$

$$= \left[ (T_C J - SJ) - (dC + i\xi C) \right]_g$$

$$= 0.$$

On the other hand, assume that $c_g(\Omega', J'; \Omega, J) = 0$. Its representatives exterior degree-two part is just $\Omega' - \Omega$ and thus we know that there exists a $C \in \nu\Omega^1(M)^{\text{inv}}[\nu]$ such that $\Omega - \Omega' = dC$. This again allows us to obtain a $g$-invariant equivalence $T_C$ from $\star_{\Omega}$ to $\star_{\Omega'}$ with the help of Lemma 3.5. As before, we use Lemma 3.7 to calculate

$$0 = c_g(\Omega', J'; \Omega, J) = \left[ (\Omega' - J') - (\Omega - J) - (J + i\xi C - T_C J) \right]_g = [T_C J - J']_g.$$

Thus we obtain a $g$-invariant self-equivalence $A$ of $\star_{\Omega'}$ from Lemma 4.3 with $AT_C J = J'$. Hence arrive at the desired equivalence $S = A \circ T_C$. \hfill \Box

5 Characteristic Class

From the classification of star products and $g$-invariant star products due to [4, 6, 20, 21, 26] and [3] we already know that two Fedosov star products (invariant Fedosov star products) $\star_{\Omega}, \star_{\Omega'}$ are equivalent if and only if the relative class $c(\star_{\Omega}, \star_{\Omega'}) = [\Omega - \Omega']$ ($c^{\text{inv}}(\star_{\Omega}, \star_{\Omega'}) = [\Omega - \Omega']^{\text{inv}}$) in the de Rham (invariant de Rham) cohomology vanishes. Proposition 4.3 from the previous section is then the specialization of those to equivariant star products. However, there are slightly stronger results for the two previously known cases, namely there are bijections $c: \text{Def}(M, \omega) \rightarrow H^2_{\text{de Rham}}(M)[\nu]$ and $c^{\text{inv}}: \text{Def}^{\text{inv}}(M, \omega) \rightarrow H^{\text{inv}, 2}_{\text{de Rham}}(M)[\nu]$, respectively, between equivalence classes of star products (invariant star products) to the second de Rham (invariant de Rham) cohomology which is defined on Fedosov star products by

$$c(\star_{\Omega}) = \frac{1}{\nu}[\omega + \Omega] \in \frac{[\omega]}{\nu} + H_{\text{de Rham}}(M)[\nu] \quad \text{and} \quad c^{\text{inv}}(\star_{\Omega}) = \frac{1}{\nu}[\omega + \Omega]^{\text{inv}} \in \frac{[\omega]^{\text{inv}}}{\nu} + H^{\text{inv}, 2}_{\text{de Rham}}(M)[\nu],$$

respectively, and extended to all star products by the fact that every star product (invariant star product) is equivalent (invariantly equivalent) to a Fedosov star product (invariant Fedosov star product). The aforementioned relative class is then precisely the difference of the images of those maps (up to a normalization factor), i.e.

$$\frac{1}{\nu}c(\star_{\Omega}, \star_{\Omega'}) = c(\star_{\Omega}) - c(\star_{\Omega'}) \quad \text{and} \quad \frac{1}{\nu}c^{\text{inv}}(\star_{\Omega}, \star_{\Omega'}) = c^{\text{inv}}(\star_{\Omega}) - c^{\text{inv}}(\star_{\Omega'}).$$
In the following, we will similarly define a bijection $c_g: \text{Def}_g(M, \omega) \rightarrow \frac{1}{\nu}[\omega - J_0]_g + H^2_{\text{dR}}(M)[\nu]$ from the equivalence classes of equivariant star products to the equivariant cohomology. In view of the classification result (4.4) it is tempting to define the class simply by taking the equivariant class of $\Omega$ and $J$. However, it is not completely obvious that this is only depending of $\ast$ and $J$ as we have to control the behaviour of $J$ under invariant self-equivalences. Nevertheless, with the previous results this turns out to be correct. Hence we can state the following definition:

**Definition 5.1 (Equivariant characteristic class)** Let $\ast_\Omega$ be the Fedosov star product constructed from $\Omega \in \nu Z^2(M)[[\nu]]$ and $J$ a quantum momentum map of $\ast_\Omega$. Then the equivariant characteristic class of $(\ast, J)$ is defined by

$$c_g(\ast_\Omega, J) = \frac{1}{\nu}[(\omega + \Omega) - J]_g \in \frac{[\omega - J_0]}{\nu} + H^2_{\text{dR}}(M)[\nu].$$

(5.1)

Here we need to verify that $c_g(\ast_\Omega, J)$ is well-defined by showing that $(\omega + \Omega) - J$ is $d_g$-closed, which is equivalent to Lemma 4.1. Using this equivariant class we can reformulate Proposition 4.4 slightly:

**Theorem 5.2** Let $\Omega, \Omega' \in \nu Z^2(M)[[\nu]]$ and $\ast_\Omega, \ast_\Omega'$ their corresponding Fedosov star products. Let furthermore $J$ and $J'$ be quantum momentum maps of $\ast_\Omega$ and $\ast_\Omega'$ respectively, deforming the same momentum map. Then there exists a $g$-invariant equivalence $S$ from $\ast_\Omega$ to $\ast_\Omega'$ such that $S J = J$ if and only if

$$c_g(\ast_\Omega, J') = c_g(\ast_\Omega, J).$$

(5.2)

Finally, we wish to extend Definition 5.1 from only Fedosov star products to all star products on $M$ and their corresponding quantum momentum maps. To do so, we first cite a result from [3] stating that for every $g$-invariant star product $\ast$ there is a $g$-invariant equivalence $S$ to a $g$-invariant Fedosov star product $\ast_\Omega$. Given a quantum momentum map $J$ of $\ast$ we can use $S$ to assign the equivariant class $c_g(\ast, J) := c_g(\ast_\Omega, SJ)$ to the pair $(\ast, J)$. This class obviously does not depend on the choice of either $S$ or $\Omega$, since, given another $g$-invariant equivalence $T$ to another Fedosov star product $\ast_{\Omega'}$, we immediately acquire a $g$-invariant equivalence $T \circ S^{-1}$ between $\ast_\Omega$ and $\ast_{\Omega'}$ with $(T \circ S^{-1})SJ = TJ$, showing (with the help of Theorem 5.2) that $c_g(\ast_\Omega, SJ) = c_g(\ast_{\Omega'}, TJ)$. In conclusion, $c_g$ defines a map

$$c_g: \text{Def}_g(M, \omega) \rightarrow \frac{[\omega - J_0]}{\nu} + H^2_{\text{dR}}(M)[\nu]$$

from the set $\text{Def}_g(M, \omega)$ of equivalence classes of star products on $M$ with quantum momentum maps to the second equivariant cohomology $H^2_{\text{dR}}(M)[\nu]$. The map $c_g$ is then easily recognized to be invertible with inverse given as

$$\frac{1}{\nu}[\omega + \Omega - J]_g \mapsto [\ast_\Omega, J]_g,$$

once we remember that $\Omega^2_g(M) = \Omega^2(M)^{\text{inv}} \oplus S^1(g)^{\text{inv}}$, which completes the proof of our main theorem.

As a final remark, let us note that the three classification results for star products, invariant star products and equivariant star products are connected by the sequence of maps

$$H^2_{\text{dR}}(M) \rightarrow H^2_{\text{inv}}(M) \rightarrow H^2_{\text{inv}}(M),$$

where the first map is the projection of $H^2_{\text{dR}}(M)$ onto the first summand and the second map is the natural inclusion of invariant differential forms into the differential forms. This shows in particular that equivariantly equivalent star products are invariantly equivalent and likewise invariantly equivalent star products are equivalent.
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