Global structure of radial sign-changing solutions for the prescribed mean curvature problem in a ball

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\textbf{Abstract.} In this paper, we are concerned with the global structure of radial solutions, with prescribed nodal properties, to the boundary value problem

\[ \text{div}(\phi_N(\nabla v)) + \lambda f(|x|, v) = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R), \]

where \( \phi_N(y) = \frac{y}{\sqrt{1-|y|^2}}, \) \( y \in \mathbb{R}^N, \) \( \lambda \) is a positive parameter, \( B(R) = \{ x \in \mathbb{R}^N : |x| < R \}, \) and \( | \cdot | \) denote the Euclidean norm in \( \mathbb{R}^N. \) All results, depending on the behavior of nonlinear term \( f \) near 0, are obtained by using global bifurcation techniques.

\textbf{Keywords.} Mean curvature operator; Minkowski space; Radial solutions; Global bifurcation.

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1 Introduction

In this paper, we are concerned with the global structure of radial solutions, with prescribed nodal properties, to the boundary value problem

\[ \text{div}(\phi_N(\nabla v)) + \lambda f(|x|, v) = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R), \quad (1.1) \]

where \( \phi_N(y) = \frac{y}{\sqrt{1-|y|^2}}, \) \( y \in \mathbb{R}^N, \) \( \lambda \) is a positive parameter, \( B(R) = \{ x \in \mathbb{R}^N : |x| < R \}, \) and \( | \cdot | \) denote the Euclidean norm in \( \mathbb{R}^N, \) \( f \) satisfies

\((H_1)\ f : [0, R] \times (-\alpha, \alpha) \to \mathbb{R} \text{ is a continuous function, with } R < \alpha \leq \infty \text{ and such that } f(r, s)s > 0 \text{ for } r \in [0, R] \text{ and } s \in (-\alpha, 0) \cup (0, \alpha).\]

Dirichlet problem (1.1) is associated to mean curvature operator in flat Minkowski space

\[ L^{N+1} := \{ (x, t) : x \in \mathbb{R}^N, t \in \mathbb{R} \} \]

endowed with the Lorentzian metric

\[ \Sigma_{j=1}^N (dx_j)^2 - (dt)^2, \]
where \((x, t)\) are the canonical coordinates in \(\mathbb{R}^{N+1}\).

It is known (see e.g. [1, 4, 12, 29, 35]) that the study of spacelike submanifolds of codimension one in \(\mathbb{L}^{N+1}\) with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

\[
\mathcal{M}v = H(x, v) \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega, \quad \tag{1.2}
\]

where

\[
\mathcal{M}v = \text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right),
\]

\(\Omega\) is a bounded domain in \(\mathbb{R}^N\) and the nonlinearity \(H : \Omega \times \mathbb{R} \to \mathbb{R}\) is continuous.

The starting point of this type of problems is the seminal paper [12] which deals with entire solutions of \(\mathcal{M}v = 0\). The equation \(\mathcal{M}v = \text{constant}\) is then analyzed in [35], while \(\mathcal{M}v = f(v)\) with a general nonlinearity \(f\) is considered in [9]. More general sign changing nonlinearities are studied in [5].

If \(H\) is bounded, then it has been shown by Bartnik and Simon [4] that (1.2) has at least one solution \(u \in C^1(\Omega) \cap W^{2,2}(\Omega)\). Also, when \(\Omega\) is a ball or an annulus in \(\mathbb{R}^N\) and the nonlinearity \(H\) has a radial structure, then it has been proved in [6] that (1.2) has at least one classical radial solution. This can be seen as a universal existence result for the above problem in the radial case.

Very recently, Bereanu, Jebelean and Torres [7] use Leray-Schauder degree arguments and critical point theory for convex, lower semicontinuous perturbations of \(C^1\)-functionals, proved existence of classical positive radial solutions for Dirichlet problems

\[
\mathcal{M}v + f(|x|, v) = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R),
\]

under the condition \((H_1)\) and

\[
\lim_{s \to 0} \frac{f(r, s)}{\phi_1(s)} = \infty \quad \text{uniformly for} \quad r \in [0, R]. \quad \tag{1.3}
\]

Bereanu, Jebelean and Torres [8] used the upper and lower solutions and Leray-Schauder degree type arguments to study the special case of

\[
\mathcal{M}v + \lambda \mu(|x|)v^q = 0 \quad \text{in} \quad B(R), \quad v = 0 \quad \text{on} \quad \partial B(R), \quad \tag{1.4}
\]

under the condition

\((H_2)\) \(N \geq 2\) is an integer, \(R > 0\), \(q > 1\) and \(\mu : [0, \infty) \to \mathbb{R}\) is continuous, \(\mu(r) > 0\) for all \(r > 0\).

They proved that there exists \(\Lambda > 0\) such that problem (1.4) has zero, at least one or at least two positive radial solutions according to \(\lambda \in (0, \Lambda), \lambda = \Lambda\) or \(\lambda > \Lambda\). Moreover, \(\Lambda\) is strictly decreasing with respect to \(R\).

However, the existence of sign-changing radial solutions for (1.1) has been scarcely explored in the related literature, see Caprietto, Dambrosio and Zanolin [10].

When dealing with radial solutions to (1.1) on a ball, one is led to study (setting \(|x| = r\)) the BVP

\[
\begin{cases}
(r^{N-1}\phi_1(u'))' + \lambda r^{N-1}f(r, u) = 0, & r \in (0, R),
\newline
u'(0) = 0 = u(R). 
\end{cases} \tag{1.5}
\]
Capietto, Dambrosio and Zanolin [10] used a degree approach combined with a time-map technique (i.e. Abstract continuation theorem in [19]) to establish the following

**Theorem A.** [10, Theorem 3.2] Assume that

\((H_f)\) \(f(r, 0) = 0\) and\n\[
\lim_{s \to 0} \frac{f(r, s)}{\phi_1(s)} = \infty, \quad \text{uniformly in } r \in [0, R];
\]

\((H_F)\) \(F(r, s) = \int_0^s f(r, x)dx,\) and \(F(r, s)\) is differentiable with respect to \(r \in [0, R]\) and there exists a continuous function \(\omega : [0, R] \to \mathbb{R}^+\) such that
\[
\left| \frac{\partial F}{\partial r}(r, s) \right| \leq \omega(r)F(r, s), \quad r \in [0, R], \ s \in (-\alpha, \alpha).
\]

Then, there exists \(n_0\) such that for every \(n > n_0,\) (1.5)\(\lambda=1\) has at least two solutions \(u_n^+\) and \(v_n^-\) with \(u_n^+(0) > 0, \ v_n^-(0) < 0,\) all having exactly \(n\) zeros in \([0, R]\). Moreover, we have
\[
\lim_{n \to \infty} |u_n^+(r)| + |(u_n^+)'(r)| = 0 = \lim_{n \to \infty} |v_n^-(r)| + |(v_n^-)'(r)| \quad \text{uniformly in } r \in [0, R].
\]

Motivated above papers, in this paper, we investigate the global structure of radial solutions, with prescribed nodal properties, to Dirichlet problem (1.1) by the unilateral global bifurcation theory of Dancer [16-17] and López-Gómez [24, Sections 6.4, 6.5] and some preliminary results on the superior limit of a sequence of connected components due to Ma and An [26-27]. We shall use the following assumptions

(A1) \(R \in (0, \infty)\) and \(\delta \in [0, R],\) \(f : [0, R] \times (-\alpha, \alpha) \to \mathbb{R}\) is a continuous function, with \(R < \alpha \leq \infty\) and such that \(f(r, s)s > 0\) for \(r \in [0, R]\) and \(s \in (-\alpha, 0) \cup (0, \alpha).\)

(A2) \(\lim_{s \to 0} \frac{f(r, s)}{s} = m(r)\) uniformly \(r \in [\delta, R]\) for some \(m \in C[\delta, R]\) and
\[
m(r) \geq 0, \quad m(r) \neq 0 \quad \text{on any subinterval of } [\delta, R];
\]

(A3) \(f(r, 0) \equiv 0\) and
\[
f_0 := \lim_{s \to 0} \frac{f(r, s)}{\phi_1(s)} = \infty, \quad \text{uniformly in } r \in [\delta, R];
\]

(A4) \(F(r, s)\) is differentiable with respect to \(r \in [\delta, R]\) and there exists a continuous function \(\omega : [\delta, R] \to \mathbb{R}^+\) such that
\[
\left| \frac{\partial F}{\partial r}(r, s) \right| \leq \omega(r)F(r, s), \quad r \in [\delta, R], \ s \in (-\alpha, \alpha).
\]

To study the global structure of radial solutions of problem (1.1), we need to study the family of auxiliary problems
\[
\begin{cases}
(r^{N-1}\phi_1(u'))' + \lambda r^{N-1}f(r, u) = 0, & r \in (\delta, R), \\
u'(\delta) = 0 = u(R).
\end{cases}
\]

(1.6)\(\delta\)

For given \(\delta \in [0, R].\) Let
\[
X_\delta = C[\delta, R], \quad E_\delta = \{u \in C^1[\delta, R] : u'(\delta) = u(R) = 0\}
\]
be the Banach spaces endowed with the normals
\[ ||u||_{C^0[\delta, R]} = \sup_{r \in [\delta, R]} |u(r)|, \quad ||u||_{C^1[\delta, R]} = \sup_{r \in [\delta, R]} |u(r)| + \sup_{r \in [\delta, R]} |u'(r)|, \]
respectively. Denoted by \( \Sigma_\delta \) be the closure of the set
\[
\{(\lambda, u) \in [0, \infty) \times C^1[\delta, R] : u \text{ satisfies (1.6)}_\delta, \text{ and } u \neq 0\}
\]
in \( \mathbb{R} \times E_\delta \). Let \( S_{k,\delta}^+ \) denote the set of function \( u \in E_\delta \), which have exactly \( k - 1 \) non-degenerate nodal zeros in \( (\delta, R) \) and there exists \( \sigma_1 > 0 \) such that \( u \) is positive in \( (\delta, \delta + \sigma_1) \), and set \( S_{k,\delta}^- = -S_{k,\delta}^+ \), and \( S_{k,\delta} = S_{k,\delta}^+ \cup S_{k,\delta}^- \). It is clear that \( S_{k,\delta}^+ \) and \( S_{k,\delta}^- \) are disjoint and open in \( E_\delta \). Finally, let \( \Phi^+_{k,\delta} = \mathbb{R} \times S_{k,\delta}^+ \) and \( \Phi_{k,\delta} = \mathbb{R} \times S_{k,\delta} \) under the product topology. Denoted by \( \theta \) be the zero element in \( E_\delta \).

Let \( \delta \in [0, R) \) be given and let \( \lambda_k(m, \delta) \) be the \( k \)-th eigenvalue of
\[
\begin{align*}
- (r^{N-1}u')' = \lambda r^{N-1}m(r)u, & \quad r \in (\delta, R), \quad (1.7)_\delta \\
u'(\delta) = u(R) = 0.
\end{align*}
\]

The main results of the paper are as follows

**Theorem 1.1.** Assume that (A1) and (A2) hold. Then for \( \nu \in \{+, -\} \) and \( k \in \mathbb{N} \), there exists a connected component \( \zeta^\nu \in \Sigma_\delta \), such that
\[
\begin{align*}
(a) & \quad (\zeta^\nu \setminus \{\lambda_k(m, \delta), \theta\}) \subset ([0, \infty) \times \text{int } \Phi^\nu_{k,\delta}); \\
(b) & \quad \zeta^\nu \text{ joins } (\lambda_k(m, \delta), \theta) \text{ with infinity in } \lambda \text{ direction}; \\
(c) & \quad \text{Proj}_\mathbb{R} \zeta^\nu = [\lambda_*, \infty) \subset (0, \infty) \text{ for some } \lambda_* > 0.
\end{align*}
\]

**Theorem 1.2.** Let \( \delta \in [0, R) \) be given. Assume that (A1), (A3) and (A4) hold. Then for \( \nu \in \{+, -\} \) and \( k \in \mathbb{N} \), there exists a connected component \( \zeta^\nu \in \Sigma_\delta \) such that
\[
\begin{align*}
(a) & \quad (\zeta^\nu \setminus \{(0, \theta)\}) \subset ([0, \infty) \times \text{int } \Phi^\nu_{k,\delta}); \\
(b) & \quad \zeta^\nu \text{ joins } (0, \theta) \text{ with infinity in } \lambda \text{ direction}; \\
(c) & \quad \text{Proj}_\mathbb{R} \zeta^\nu = [0, \infty).
\end{align*}
\]

Obviously, as the immediate consequences of theorems 1.1-1.2, we have the following

**Corollary 1.1.** Let \( \delta \in [0, R) \) be given. Assume that (A1) and (A2) hold. Then for \( \nu \in \{+, -\} \) and \( k \in \mathbb{N} \), there exists \( \lambda^*_\nu \in (0, \lambda_k(m, \delta)] \) such that, for all \( \lambda \in (0, \lambda^*_\nu) \), the problem \((1.6)_\delta\) has no solution in \( S_{k,\delta}^\nu \) and, for all \( \lambda > \lambda_k(m, \delta) \) has at least one solution in \( S_{k,\delta}^\nu \).

**Corollary 1.2.** Let \( \delta \in [0, R) \) be given. Assume that (A1), (A3) and (A4) hold. Then for \( \nu \in \{+, -\} \) and \( n \in \mathbb{N} \), the problem \((1.6)_\delta\) has at least one solution \( u^\nu_n \in S_{n,\delta}^\nu \) for \( \lambda = 1 \). Moreover,
\[
\lim_{n \to \infty} |u^\nu_n(r)| + |(u^\nu_n)^(r)| = 0 \quad \text{uniformly in } r \in [0, R].
\]
Remark 1.1. Corollary 1.2 guarantees that for each \( n \in \mathbb{N} \), \((1.6)_{\delta}\) has two solutions \( u_n^+ \) and \( v_n^- \) with \( u_n^+(0) > 0, v_n^-(0) < 0 \), for all \( \lambda > 0 \) and \( \delta \in [0, R) \). [11, Theorem 3.2] deal with the more general problem

\[
\begin{aligned}
\text{div}(a(|\nabla u|)\nabla u) + f(|x|, u) &= 0 \quad \text{in} \ B(R), \\
u &= 0 \quad \text{on} \ \partial B(R),
\end{aligned}
\]

(1.8)

where \( a : [0, \epsilon_1] \to [0, +\infty), \ (\epsilon_1 > 0) \). However, they only proved that there exists \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \), \((1.8)\) has at least two radial solutions \( u_n^+ \) and \( v_n^- \) with \( u_n^+(0) > 0, v_n^-(0) < 0 \), all having exactly \( n \) zeros in \([0, R)\).

Remark 1.2. Notice that \((1.6)_0\) a singularity appears for \( r = 0 \). Beside this intrinsic aspect of \((1.6)_0\), assumption \((A3)\) means that \( \frac{\partial f}{\partial s} \) does not exist at \( s = 0 \). Hence, in particular, when one tries to develop some shooting argument, global existence and uniqueness to initial value problems associated to the equation in \((1.6)_{\delta}\) are not guaranteed. This is one of the reasons why few results for the case \( f_0 = \infty \) is available in the literature: we refer to the earlier works of Krasnosel’skii, Perov, Povolotskii and Zabreiko [22, Section 15], Shekhter [34, Section 15] for a more classical approach in the ODE’s case. For other results, in the PDE’s setting, we also refer to Omari and Zanolin [31], Willem [37], mainly for the case \( a(\cdot) \equiv 1 \).

The rest of the paper is organized as follows. In Section 2 we state some preliminary results on the superior limit of a sequence of connected components due to Ma and An [26-27] and on the Unilateral global bifurcation theorem of Dancer [16-17] and López-Gómez [24]. Section 3 is devoted to establish the existence of connected component of radial solutions for the prescribed mean curvature problem in an annular domain via global bifurcation technique. Finally in Section 4, we shall use the components obtained in Section 3 to construct the desired components of radial solutions for the prescribed mean curvature problem in a ball and prove Theorem 1.1-1.2 and their corollaries.

For other results concerning the problem associated to prescribed mean curvature equations in Minkowski space we refer the reader to [5, 9, 23, 29]. The existence of radial solutions satisfying various boundary conditions has been investigated by many authors, see Esteban [18], Castro and Kurepa [11], Grillakis [20], Guo [21], Cheng [13], Ambrosetti, Garcia Azorero and Peral [2], Njoku, Omari and Zanolin [30], Dai and Ma [15] and references therein.

2 Preliminary results

2.1 Unbounded connected component

Let \( M \) be a metric space and \( \{C_n \mid n = 1, 2, \cdots \} \) a family of subsets of \( M \). Then the superior limit \( \mathcal{D} \) of \( \{C_n\} \) is defined by

\[
\mathcal{D} := \limsup_{n \to \infty} C_n = \{x \in M \mid \exists \{n_k\} \subset \mathbb{N}, x_{n_k} \in C_{n_k}, \text{ such that } x_{n_k} \to x\}. \tag{2.1}
\]
Let $X$ be a Banach space with the norm $\| \cdot \|$. A *component* of a set $M \subset X$ means a maximal connected subset of $M$, see [36] for the detail.

The following results are somewhat scattered in Ma and An [26-27], Ma and Gao [28].

**Lemma 2.1 ([26, Lemma 2.4; 27, Lemma 2.2])** Let $X$ be a Banach space. Let $\{C_n\}$ be a family of closed connected subsets of $X$. Assume that

(i) there exist $z_n \in C_n$, $n = 1, 2, \cdots$ and $z_* \in X$ such that $z_n \to z_*$;
(ii) $\lim_{n \to \infty} r_n = \lim_{n \to \infty} \sup \{ \| u \| : u \in C_n \} = \infty$;
(iii) for every $R > 0$, $(\bigcup_{n=1}^\infty C_n) \cap B_R$ is a relatively compact of $X$.

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{P}$ and $z_* \in \mathcal{C}$.

### 2.2 Global alternative of Rabinowitz

In order to formulate and prove main results of this section, it is convenient to introduce Dancer [16-17] and López-Gómez’s notations [24]. Let $\mathbb{X} = \mathbb{R} \times X$. Given any $\mu \in \mathbb{R}$ and $0 < s < +\infty$, we consider an open neighborhood of $(\mu, 0)$ in $\mathbb{X}$ defined by

$$B(\mu, 0) := \{ (\mu, u) \in \mathbb{X} : \| u \| + |\mu| < s \}.$$

A mapping $G : \mathbb{X} \to X$ is said to satisfy *Assumption A* if

1. $G(0, \lambda) = 0$ for $\lambda \in \mathbb{R}$;
2. $G$ is completely continuous, $G(x, \lambda) = \lambda Lx + H(\lambda, x)$, where $L$ is a continuous linear operator on $X$;
3. $\| H(\lambda, x) \|/\| x \| \to 0$ as $\| x \| \to 0$ uniformly on bounded subsets of $\mathbb{R}$.

Define $\Phi : \mathbb{X} \to X$ by

$$\Phi(\lambda, x) = x - G(\lambda, x)$$

and

$$\mathcal{S} := \{ (\mu, u) \in \mathbb{X} : \Phi(\mu, u) = 0, u \neq 0 \}.$$

Assume that $\mu \in r(L)$ such that $\mu$ has algebraic multiplicity 1. Suppose that $\varphi \in X \setminus \{0\}$ such that

$$\varphi = \mu L \varphi.$$

Let $X_0$ be a closed subspace of $X$ such that

$$X = \text{span}\{ \varphi \} \oplus X_0.$$

Let $C_\mu$ to be the component of $\mathcal{S}$ containing $(\mu, 0)$.

**Theorem B. Global bifurcation of Rabinowitz, see López-Gómez [24, Corollary 6.3.1]**

Assume that $\mu \in r(L)$ has algebraic multiplicity 1. Then, one of the following non-excluding options occurs. Either

1. $C_\mu$ is unbounded in $\mathbb{R} \times X$.
2. There exists $\mu_1 \in r(L) \setminus \{\mu\}$ such that $(\mu_1, 0) \in C_\mu$. $\square$
2.3 Unilateral global bifurcation theorem

In this subsection, we shall introduce the unilateral global bifurcation theorem, see Dancer [16-17] and López-Gómez [24].

According to the Hahn-Banach theorem, there exists a linear functional \( l \in X^* \), here \( X^* \) denotes the dual space of \( X \), such that

\[
l(\varphi) = 1, \quad X_0 = \{ u \in X : l(u) = 0 \}.
\]

Finally, for any \( 0 < \eta < 1 \), we define

\[
K_\eta := \{ (\mu, u) \in X : |l(u)| > \eta||u|| \}.
\]

Since

\[
u t \varphi \in K_{\nu t} \quad \text{for every} \quad t > 0,
\]

we define

\[
(\mathcal{S} \setminus \{(\mu, 0)\}) \cap \bar{B}_\delta(\mu, 0) \subset K_\eta.
\]

Moreover, for each

\[
(\mu, u) \in (\mathcal{S} \setminus \{(\mu, 0)\}) \cap \bar{B}_\delta(\mu, 0),
\]

there are \( s \in \mathbb{R} \) and a unique \( y \in X_0 \) such that

\[
u t \varphi \in K_{\nu t} \quad \text{for every} \quad t > 0, \quad \nu \in \{+, -\}. \]

Applying the similar method to prove [24, Lemma 6.4.1] with obvious changes, we may obtain the following result.

**Lemma 2.2.** For every \( \eta \in (0, 1) \), there exists a number \( \delta_0 > 0 \) such that for each \( 0 < \delta < \delta_0 \),

\[
((\mathcal{S} \setminus \{(\mu, 0)\}) \cap \bar{B}_\delta(\mu, 0)) \subset K_\eta.
\]

Moreover, for each

\[
(\mu, u) \in (\mathcal{S} \setminus \{(\mu, 0)\}) \cap \bar{B}_\delta(\mu, 0),
\]

there are \( s \in \mathbb{R} \) and a unique \( y \in X_0 \) such that

\[
u t \varphi \in K_{\nu t} \quad \text{for every} \quad t > 0, \quad \nu \in \{+, -\}. \]

Furthermore, for these solutions \((\lambda, u)\)

\[
\lambda = \mu + \circ(1) \quad \text{and} \quad y = \circ(s)
\]

as \( s \to 0 \).

Let \( \delta > 0 \) be the constant from Lemma 2.2. For \( 0 < \epsilon < \delta \) we define \( \mathcal{D}_{\mu, \epsilon}^\nu \) to be the component of

\[
\{ (\mu, \theta) \} \cup (\mathcal{S} \cap \bar{B}_{\epsilon} \cap K_{\eta}) \text{ containing } (\mu, \theta), \quad C_{\mu, \epsilon}^\nu \text{ to be the component of } C_{\mu} \setminus \mathcal{D}_{\mu, \epsilon}^- \text{ containing } (\mu, \theta), \quad C_{\mu}^\nu \text{ to be the closure of } \cup_{0 < \epsilon \leq \delta} C_{\mu, \epsilon}^-.
\]

Clearly, \( C_{\mu}^\nu \) is connected. Thanks to Lemma 2.2, the definition of \( C_{\mu}^\nu \) is independent from the choice of \( \eta \) and

\[
C_{\mu} = C_{\mu}^+ \cup C_{\mu}^-.
\]
Theorem C. Unilateral global bifurcation, see Dancer [16-17]

Either $C_\mu^+$ and $C_\mu^-$ are both unbounded or $C_\mu^+ \cap C_\mu^- \neq \{(\mu, \theta)\}$. □

2.4 Uniqueness of solutions of Cauchy problem

Lemma 2.3. Let $\tau \in (\delta, R)$ be given. Let $f : [\delta, R] \times \mathbb{R} \to \mathbb{R}$ be continuous and let $f(r, u)$ be Lipschitz continuous in $u$ on bounded sets, and $f(r, 0) \equiv 0$ for $r \in [\delta, R]$. If $u$ is a solution of

$$
\begin{cases}
(r^{N-1}\phi(u'))' + \lambda r^{N-1}f(r, u) = 0, \\
u'(\tau) = 0 = u(\tau),
\end{cases}
$$

then $u \equiv 0$ in $[\delta, R]$.

Proof. (2.2) is equivalent to

$$
\begin{cases}
-(r^{N-1}u')' = r^{N-1}[\lambda f(r, u)h(u') - \frac{N-1}{r}u^q], \\
u'(\tau) = 0 = u(\tau).
\end{cases}
$$

Since the function

$$
g(r, u, p) := r^{N-1}[\lambda f(r, u)(1 - p^2)^{3/2} - \frac{N-1}{r}p^3]
$$

is continuous in $[\delta, R] \times \mathbb{R} \times [-1, 1]$ and is Lipschitz in $(u, p)$ on any bounded subset of $\mathbb{R} \times [-1, 1]$, it deduce that $u \equiv 0$ in $[\delta, R]$.

Finally, we consider the case that $f$ does not meet the Lipschitz condition at $u = 0$, i.e. we allow $f_0 = \infty$.

Lemma 2.4. [10, Lemma 2.3] Let (A4) hold. Let $\epsilon_0 = 0.9$. Then, for every $\epsilon \leq \epsilon_0$ there exists $d_\epsilon \in (0, \epsilon]$ such that if $u$ is a (local) solution of

$$
\begin{cases}
(r^{N-1}\phi(u'))' + \lambda r^{N-1}f(r, u) = 0, \\
u(\delta) = d, \\u(\delta) = 0
\end{cases}
$$

with $|d| \leq d_\epsilon$, then $u$ can be defined on $[\delta, R]$ and $||u||_{C^1[\delta, R]} \leq \epsilon$.

Lemma 2.5. [10, Lemma 2.5] Let (A4) hold. Let $u$ be a solution of

$$
\begin{cases}
(r^{N-1}\phi(u'))' + \lambda r^{N-1}f(r, u) = 0, \\
u(\delta) = u'(\delta) = 0
\end{cases}
$$

Then $u \equiv 0$ in $[\delta, R]$. □

3 Radial solutions for the prescribed mean curvature problem in an annular domain

Let $\delta \in (0, R)$ be a given constant in this section.
Let us consider the following boundary value problem

\[
\begin{cases}
\text{div}(\phi_N(\nabla v)) + \lambda f(|x|, v) = 0 & \text{in } \mathcal{A}, \\
\frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_1, \\
v = 0 & \text{on } \Gamma_2,
\end{cases}
\]  

(3.1)

where

\[\mathcal{A} = \{x \in \mathbb{R}^N : \delta < |x| < R\},\]

\[\Gamma_1 = \{x \in \mathbb{R}^N : |x| = \delta\}, \quad \Gamma_2 = \{x \in \mathbb{R}^N : |x| = R\},\]

\[\frac{\partial v}{\partial \nu}\]

and \(|\cdot|\) denote the outward normal derivative of \(v\) and the Euclidean norm in \(\mathbb{R}^N\), respectively.

Setting, as usual, \(|x| = r\) and \(v(x) = u(r)\), the above problem (3.1) reduces to

\[
\begin{cases}
-(r^{N-1} \phi_1(u'))' = \lambda r^{N-1} f(r, u), \\
u'(\delta) = 0 = u(R).
\end{cases}
\]  

(3.2)\(\delta\)

To find a radial solution of (3.1), it is enough to find a solution of (3.2)\(\delta\).

**Remark 3.1.** It is worth remarking that (3.2)\(\delta\) is equivalent to

\[
\begin{cases}
-(r^{N-1}u')' = r^{N-1}[\lambda f(r, u)h(u') - \frac{N-1}{r}u^\delta], \\
u'(\delta) = 0 = u(R).
\end{cases}
\]  

(3.3)\(\delta\)

Since the nonlinearity \(F(r, u, p) := \lambda f(r, u)h(p) - \frac{N-1}{r}p^\delta\) is singular at \(r = 0\) if \(\delta = 0\), we cannot deal with (3.3)\(0\) via the spectrum of (1.4)\(0\) directly. However, \(F(r, u, p)\) is regular at \(r = \delta\) if \(\delta > 0\), in this case, (3.3)\(\delta\) with \(\delta > 0\) can be treated via the spectrum of (1.4)\(\delta\) and the standard bifurcation technique.

This is why we firstly study the prescribed mean curvature problem in an annular domain.

**Lemma 3.1.** Let \(u \in S_{k, \delta}^+\) be a solution of

\[r^{N-1} \phi_1(u')' + r^{N-1} f(r, u) = 0, \quad r \in (\delta, R)\]  

(3.4)

Assume that (A1) holds. Then \(u(\delta) > 0\).

**Proof.** Denote by \(\tau_1\) the first positive zero of \(u\). Let \(u(\delta) := d\). Then \(u\) satisfies

\[
\begin{cases}
(r^{N-1} \phi_1(u'))' + r^{N-1} f(r, u) = 0, \\
u'(\delta) = 0, \\
u(\tau_1) = 0.
\end{cases}
\]  

(3.5)

It is easy to check that (3.5) is equivalent to

\[r^{N-1} \phi_1(u'(r)) = -\int_\delta^r t^{N-1} f(t, u(t))dt, \quad r \in [\delta, \tau_1]\]

it follows \(u' \leq 0\) because \(f(r, s) \geq 0\) for all \(r \in [\delta, R]\) and \(s \in [0, \alpha]\), so \(u\) is decreasing. Since \(u(\tau_1) = 0\), we have \(u \geq 0\) on \([\delta, \tau_1]\). As \(u\) is not identically zero, one has \(u(\delta) > 0\), and subsequently, \(u' < 0\) on \((\delta, \tau_1]\), which ensures that actually \(u\) is strictly decreasing in \([\delta, \tau_1]\) and \(u > 0\) on \([\delta, \tau_1)\). \(\square\)

**Definition 3.1.** For \(y \in S_k\), denote the zeros of \(y\) by

\[(\delta <) \tau_1 < \cdots < \tau_k (= R).\]
If \( y'(\delta) = 0 \) and, for each \( j \in \{1, \cdots, k-1\} \), there exists exactly one \( \xi_j \in (\tau_j, \tau_{j+1}) \), such that \( y'(\xi_j) = 0 \), then we call that \( y \) possesses the Property \([P(k)]\).

Let
\[
\Pi_k = \{ y \in C^3[\delta, R] : y'(\delta) = y(R) = 0, y \text{ possesses the Property } [P(k)] \}.
\]

### 3.1 Eigenvalue problem in an annular domain

Let \( \delta \in (0, R) \) be given. Let us recall the weighted eigenvalue problem
\[
\begin{align*}
- (r^{N-1}u')' &= \lambda r^{N-1}m(r)u, \quad r \in (\delta, R), \\
u'(\delta) = 0 &= u(R),
\end{align*}
\] (3.7)\( \delta \)

where

(A5) \( m \in C[\delta, R] \) and \( m(r) \geq 0, m(r) \not\equiv 0 \) on any subinterval of \([\delta, R]\).

The following result is a special case of [32, Theorem 1.5.3] when \( p = 2 \).

**Lemma 3.2.** Let (A5) hold. Then the eigenvalue problem (3.7)\( \delta \) has infinitely many simple real eigenvalues
\[
0 < \lambda_1(m, \delta) < \lambda_2(m, \delta) < \cdots < \lambda_k(m, \delta) < \cdots \to +\infty \quad \text{as} \quad k \to +\infty
\]
and no other eigenvalues. Moreover, the algebraic multiplicity of \( \lambda_k(m, \delta) \) is 1, and the eigenfunction \( \varphi_k \) corresponding to \( \lambda_k(m, \delta) \) has exactly \( k-1 \) simple zeros in \((\delta, R)\).

Let us consider the following auxiliary problem
\[
\begin{align*}
- (r^{N-1}u')' &= r^{N-1}h(r), \quad r \in (\delta, R) \text{ with } \delta > 0, \\
u'(\delta) = 0 &= u(R)
\end{align*}
\]
for a given \( h \in X_\delta \). Its Green function for \( N \geq 3 \) is explicitly given by
\[
K_\delta(t, s) = \begin{cases} 
\frac{1}{2-N}[R^{2-N} - t^{2-N}], & \delta \leq s \leq t \leq R, \\
\frac{1}{2-N}[R^{2-N} - s^{2-N}], & \delta \leq t \leq s \leq R.
\end{cases}
\]
and its Green function for \( N = 2 \) is explicitly given by
\[
K_\delta(t, s) = \begin{cases} 
\ln \frac{R}{t}, & \delta \leq s \leq t \leq R, \\
\ln \frac{R}{s}, & \delta \leq t \leq s \leq R.
\end{cases}
\]

It is well-known that for every \( h \in X_\delta \), the above auxiliary problem has a unique solution
\[
u = \int_\delta^R K_\delta(t, s)s^{N-1}h(s)ds =: \mathcal{G}_\delta(h)
\]

It is easy to check that \( \mathcal{G}_\delta : X_\delta \to E_\delta \) is continuous and compact (see [3]).

Define a linear operator \( \mathcal{L}_\delta : X_\delta \to E_\delta \) \((\mapsto X_\delta)\).
\[
\mathcal{L}_\delta(u)(r) := \mathcal{G}_\delta(mu)(r).
\]
Then \( L_\delta \) is compact and (3.7)\( _\delta \) is equivalent to

\[
u = \lambda L_\delta(u).
\]

Moreover, \( L_\delta|_{E_\delta} : E_\delta \to E_\delta \) is compact.

### 3.2 An equivalent formulation

We will use some idea in Coelho et. al. [14]. Let us define a function \( \tilde{f} : [\delta, R] \times \mathbb{R} \to \mathbb{R} \) by setting, for \( r \in [\delta, R] \),

\[
\tilde{f}(r,s) = \begin{cases} 
  f(r,s), & \text{if } 0 \leq |s| \leq R - \delta, \\
  0, & \text{if } |s| \geq (R - \delta) + 1, \\
  \text{linear}, & \text{if } R - \delta < |s| < (R - \delta) + 1.
\end{cases}
\]

Since any solution \( u \) of (3.2)\( _\delta \) satisfies

\[
||u'||_{C[\delta,R]} < 1, \quad ||u||_{C[\delta,R]} < R - \delta,
\]

it follows that (3.2)\( _\delta \) is equivalent to the same problem with \( f \) replaced by \( \tilde{f} \). Clearly, \( \tilde{f} \) satisfies all the properties assumed in the statement of the theorem. In the sequel, we shall replace \( f \) with \( \tilde{f} \); however, for the sake of simplicity, the modified function \( \tilde{f} \) will still be denoted by \( f \).

Next, let us define \( h : \mathbb{R} \to \mathbb{R} \) by setting

\[
h(y) = \begin{cases} 
  (1 - y^2)^2, & \text{if } |y| \leq 1, \\
  0, & \text{if } |y| > 1.
\end{cases}
\]

**Claim.** A function \( u \in C^1[\delta,R] \) is a solution of (3.2)\( _\delta \) if and only if it is a solution of the problem

\[
\begin{cases} 
  -(r^{N-1}u')' = \lambda r^{N-1}f(r,u)h(u') - (N-1)r^{N-2}u'^3, & r \in (\delta,R), \\
  u'(\delta) = 0 = u(R).
\end{cases}
\]

(3.8)\( _\delta \)

It is clear that a solution \( u \in C^1[\delta,R] \) of (3.2)\( _\delta \) is a solution of (3.8)\( _\delta \) as well. Conversely, suppose that \( u \in C^1[\delta,R] \) is a solution of (3.8)\( _\delta \). We aim to show that

\[
||u'||_{C[\delta,R]} < 1.
\]

Assume by contradiction that this is not the case. Then we can easily find an interval \([a, b] \subseteq [\delta, R]\) such that, either \( u'(a) = 0 \), \( 0 < |u'(r)| < 1 \) in \((a, b)\) and \( |u'(b)| = 1 \), or \( |u'(a)| = 1 \), \( 0 < |u'(r)| < 1 \) in \((a, b)\) and \( u'(b) = 0 \). Suppose the former case occurs (in the latter one the argument would be similar). The function \( u \) satisfies the equation

\[
-(r^{N-1}u')' = \lambda r^{N-1}f(r,u)
\]

in \([a, b]\). For each \( r \in (a, b) \), integrating over the interval \([a, r]\) and using (A1), we obtain

\[
|\phi_1(u'(r))| = \left| \frac{1}{r^{N-1}} \int_a^r \lambda t^{N-1} f(t,u) dt \right| \leq M
\]

and hence

\[
|u'(r)| \leq \phi_1^{-1}(M)
\]

for every \( r \in [a, b] \). Since \( \phi_1^{-1}(M) < 1 \), taking the limit as \( r \to b^- \) we get \( |u'(b)| < 1 \). This is a contradiction. Therefore \( ||u'||_{C[\delta,R]} < 1 \) and, as a consequence, \( u \) is a solution of (3.2)\( _\delta \).
3.3 Proof of Theorem 1.1-1.2 with \( \delta \in (0, R) \)

In this subsection, we shall prove Theorem 1.1-1.2 in the case \( \delta > 0 \).

**Proof of Theorem 1.1 with \( \delta \in (0, R) \).** By (A1) and (A2) we can write, for any \( r \in [\delta, R] \) and every \( s \in \mathbb{R} \),
\[
f(r, s) = (m(r) + l(r, s))s,
\]
where \( l : [\delta, R] \times \mathbb{R} \to \mathbb{R} \) is a continuous function and
\[
\lim_{s \to 0} l(r, s) = 0 \quad (3.9)
\]
uniformly in \([\delta, R]\). Let us set, for convenience,
\[
k(y) = h(y) - 1 \quad (3.9.9)
\]
for \( y \in \mathbb{R} \). We have
\[
\lim_{y \to 0} k(y) = 0 \quad (3.10)
\]
Define the operator \( \mathcal{H}_\delta : \mathbb{R} \times E_\delta \to E_\delta \) by
\[
\mathcal{H}_\delta(\lambda, u)(\cdot) = \mathcal{G}_\delta(\lambda[l(\cdot, u) + (m(\cdot) + l(\cdot, u))k(u')u - \gamma(\cdot)u^3])
\]
where \( \gamma(r) = \frac{N-1}{r} \). Clearly, \( \mathcal{H}_\delta \) is completely continuous and, by (3.9) and (3.10),
\[
\lim_{\|u\|_{C^1[\delta, R]} \to 0} \frac{\|\mathcal{H}_\delta(\lambda, u)\|_{C^1[\delta, R]}}{\|u\|_{C^1[\delta, R]}} = 0 \quad (3.11)
\]
uniformly with respect to \( \lambda \) varying in bounded intervals. Observe that, for any \( \lambda \), the couple \( (\lambda, u) \in \mathbb{R} \times E_\delta \) is a solution of the equation
\[
u = \lambda \mathcal{L}_\delta(u) + \mathcal{H}_\delta(\lambda, u) \quad (3.12)
\]
if and only if \( u \) is a solution of (3.2).\( \delta \).

Recall that \( \Sigma_\delta \subset \mathbb{R} \times E_\delta \) be the closure of the set of all nontrivial solutions \( (\lambda, u) \) of (3.12) with \( \lambda > 0 \). Note that the set \( \{u \in E_\delta \mid (\lambda, u) \in \Sigma_\delta \} \) is bounded in \( E_\delta \).

As the algebraic multiplicity of \( \lambda_k(m, \delta) \) equals 1 [25], the local index of 0 as a fixed point of \( \lambda \mathcal{L}_\delta \) changes sign as \( \lambda \) crosses \( \lambda_k(m, \delta) \). Therefore, according to a revised version of [24, Theorem 6.2.1], there exists a component, denoted by \( \mathcal{C}_k \subset \Sigma_\delta \), emanating from \( (\lambda_k(m, \delta), \theta) \).

Now, we use some notations and preliminary results on Unilateral global bifurcation.

We shall show that both \( \mathcal{C}_k^+ \) and \( \mathcal{C}_k^- \) are unbounded, and \( \mathcal{C}_k^\nu \subset \Phi_k^\nu \) for \( \nu \in \{+, -\} \).

It is easy to check that (3.12) enjoys the structural requirements for applying the unilateral global bifurcation theory of Dancer and Lópezm-Gómez (see Theorem C (by a counterexample of Dancer [16], the global unilateral theorem of Rabinowitz [33] is false as stated. So, it cannot be used). \( \mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^- \) and either both \( \mathcal{C}_k^+ \) and \( \mathcal{C}_k^- \) are unbounded, or
\[
\mathcal{C}_k^+ \cap \mathcal{C}_k^- \neq \{ (\lambda_k(m, \delta), \theta) \}.
\]

We claim that the second case can never occur.
In fact, the uniqueness of IVP guarantees that

\[ C_k^+ \subset \Phi_k^+, \quad C_k^- \subset \Phi_k^- . \tag{3.13} \]

Suppose on the contrary that \((\eta, z) \in (C_k^+ \cap C_k^-)\) for some \((\eta, z) \neq (\lambda_k(m, \delta), \theta)\). Then it follows from (3.13) that \(z = \theta\). In this case, \(\eta = \lambda_j(m, \delta)\) for some \(k \neq j\). Suppose \((\lambda_m, u_m) \to (\lambda_j(m, \delta), \theta)\) when \(m \to +\infty\) with \((\lambda_m, u_m) \in C_k\). Let \(v_m = \frac{u_m}{\|u_m\|_{C^1[\delta, R]}}\), then \(v_m\) should be a solution of the problem

\[ v_m = \lambda_m \mathcal{L}_\delta(v_m) + \frac{\mathcal{H}_\delta(\lambda, u_m)}{\|u_m\|_{C^1[\delta, R]}} \tag{3.14} \]

This together with the compactness of \(\mathcal{L}_\delta\) and \(\mathcal{H}_\delta\) imply that for some convenient subsequence \(v_m \to v_0 \neq 0\) as \(m \to +\infty\). Now \(v_0\) verifies the equation

\[-(r^{N-1}u_0')' = \lambda_j r^{N-1}m(r)v_0\]

and \(\|v_0\| = 1\). Hence \(v_0 \in S_{j,\delta}\). Since \(S_{j,\delta}\) is an open in \(E_\delta\), and as a consequence for some \(m\) large enough, \(v_m \in S_{j,\delta}\), and this is a contradiction.

Therefore, both \(C_k^+\) and \(C_k^-\) are unbounded.

Take

\[ \zeta^+ := C^+ , \quad \zeta^- := C^- . \]

Obviously, (a) is true.

(b) can be deduced from the fact that

\[ \sup\{||u'||_{C[\delta, R]} : (\lambda, u) \in \zeta^\nu\} \leq 1 , \quad \sup\{||u||_{C[\delta, R]} : (\lambda, u) \in \zeta^\nu\} \leq R - \delta. \]

(c) Let

\[ \lambda^\nu_* := \inf\{\lambda : (\lambda, u) \in \zeta^\nu\} . \]

We claim that \(\lambda^\nu_* \in (0, \infty)\).

Suppose on the contrary that \(\lambda^\nu_* = 0\). Then there exists a sequence \(\{(\mu_n, u_n)\} \subset \zeta^\nu\) satisfying \(u_n > 0\), and

\[ \lim_{n \to \infty} (\mu_n, u_n) = (0, u^*) \quad \text{in } \mathbb{R} \times X_\delta \]

for some \(u^* \geq 0\). Then it follows from

\[-(r^{N-1}\phi_1(u_n'))' = \mu_n r^{N-1}f(r, u_n) , \quad u_n'(\delta) = 0 = u_n(R) \]

that, after taking a subsequence and relabeling, if necessary,

\[ u_n \to 0 . \]

On the other hand,

\[ \begin{cases} -(r^{N-1}u_n')' = \mu_n r^{N-1}f(r, u_n)h(u_n') - (N - 1)r^{N-2}u_n^3 , & r \in (\delta, R) , \\ u_n'(\delta) = 0 = u_n(R) . \end{cases} \]
Setting, for all $n$, $v_n = u_n/\|u_n\|_{C[\delta, R]}$, we have that
\[
\begin{cases}
-(r^{N-1}v_n')' = \mu_n r^{N-1} \frac{f(u_n)}{u_n} h(u_n) v_n - (N-1)r^{N-2} u_n^2 v_n', \quad r \in (\delta, R), \\
v_n'(\delta) = 0 = v_n(R).
\end{cases}
\]  
(3.15)

Notice that
\[
r^{N-1}\phi_1(u_n(r)) = -\mu_n \int_{\delta}^{r} r^{N-1} f(\tau, u_n(\tau)) d\tau, \quad r \in [\delta, R].
\]
This together with $f(r, 0) = 0$ for $r \in [\delta, R]$ imply that
\[
\lim_{n \to \infty} \|u_n'\|_{C[\delta, R]} = 0.
\]
Combining this with (3.15) and the facts $f_0 = m(r)$, $u_n \to 0$ and $\lim_{n \to \infty} h(u_n') = 1$, it concludes that $\mu_n \to \lambda_k(m, \delta)$. This is a contradiction. \hfill \square

In the following, we will deal with the case that $f_0 = \infty$.

Define $f^{[n]} : [\delta, R] \times \mathbb{R} \to \mathbb{R}$ as follows
\[
f^{[n]}(r, s) = \begin{cases}
f(r, \frac{1}{n})s, & |s| \in [0, \frac{1}{n}], \\
f(r, s), & |s| \in (\frac{1}{n}, \infty).
\end{cases}
\]
Then $f^{[n]}$ is continuous and satisfies (A1) and
\[
(f^{[n]}_0)_n = nf(r, \frac{1}{n}) = f(r, \frac{1}{n})/(1/n) =: m^{[n]}(r) \text{ uniformly for } r \in [\delta, R].
\]

Now, let us consider the auxiliary family of the problems
\[
\begin{cases}
-(r^{N-1}u')' = \lambda r^{N-1} f^{[n]}(r, u) h(u') - (N-1)r^{N-2} u'^2, \quad r \in (\delta, R), \\
u'(\delta) = 0 = u(R).
\end{cases}
\]  
(3.16)

From the definition of $f^{[n]}$, it follows that for $r \in [\delta, R]$ and every $u \in \mathbb{R}$,
\[
f^{[n]}(r, s) = (m^{[n]}(r) + \xi^{[n]}(r, s))s,
\]
where $\xi^{[n]} : [\delta, R] \times \mathbb{R} \to \mathbb{R}$ is continuous and
\[
\lim_{s \to 0} \xi^{[n]}(r, s) = 0 \text{ uniformly for } r \in [\delta, R].
\]
Let us set, for convenience, $k(v) = h(v) - 1$ for $v \in \mathbb{R}$. We have
\[
\lim_{v \to 0} \frac{k(v)}{v} = 0. \quad (3.17)
\]

Define the operator $\mathcal{H}^{[n]}_\delta : \mathbb{R} \times E_\delta \to E_\delta$ by
\[
\mathcal{H}^{[n]}_\delta(\lambda, u) = \mathcal{G}_\delta \left( \lambda (\xi^{[n]}(\cdot, u) + [m^{[n]} + \xi^{[n]}(\cdot, u)]k(u'))u - \gamma(\cdot)u'^2 \right).
\]
Clearly, $\mathcal{H}^{[n]}_\delta$ is completely continuous and by (3.16) and (3.17), it follows that
\[
\lim_{\|u\|_{C^1[\delta, R]} \to 0} \frac{\|\mathcal{H}^{[n]}_\delta(\lambda, u)\|_{C^1[\delta, R]}}{\|u\|_{C^1[\delta, R]}} = 0,
\]
uniformly with respect to $\lambda$ varying in bounded intervals. Observe that, for any $\lambda$, the couple $(\lambda, u) \in \mathbb{R} \times E_\delta$ is a solution of the equation
\[ u = \lambda L_\delta^{[n]}(u) + H_\delta^{[n]}(\lambda, u) \tag{3.18} \]
if and only if $u$ is a solution of (3.16), here $L_\delta^{[n]}: X_\delta \to E_\delta$ be defined by $L_\delta^{[n]}(u) = G_\delta(m^{[n]}u)$.

Let $\Sigma_\delta^{[n]} \subset \mathbb{R} \times E_\delta$ be the closure of the set of all nontrivial solutions $(\lambda, u)$ of (3.18) with $\lambda > 0$. Note that the set $\{ u \in E_\delta | (\lambda, u \in \Sigma_\delta^{[n]} \}$ is bounded in $E_\delta$.

**Remark 3.2.** Note that from the compactness of the embedding $E_\delta \hookrightarrow X_\delta$, it concluded that $C_+^{[n]}$ is also an unbounded connected component in $[0, \infty) \times X_\delta$.

**Proof of Theorem 1.2 with $\delta \in (0, R)$.** Similar to the proof of Theorem 1.1 with $\delta \in (0, R)$, for each fixed $n$ and $\nu \in \{+,-\}$, there exists an unbounded component $C_{\nu,k}^{[n]} \subset \Sigma_\delta^{[n]}$ of solutions of (3.18) joining $(\lambda_k(m^{[n]}), 0) \in C_{\nu,k}^{[n]}$ to infinity in $[0, \infty) \times S_{\nu,k}$. Moreover, $(\lambda_k(m^{[n]}), 0) \in C_{\nu,k}^{[n]}$ is the a bifurcation point of (3.18) lying on a trivial solution line $u \equiv 0$ and the component $C_{\nu,k}^{[n]} \subset \Phi_\nu^{[n]}$ joins the infinity in the direction of $\lambda$ since $u$ is bounded.

It is not difficult to verify that $C_{\nu,k}^{[n]}$ satisfies all conditions in Lemma 2.1 and consequently $\limsup_{n \to \infty} C_{\nu,k}^{[n]}$ contains a component $C_{\nu,k}$ which is unbounded.

From (A3), it follows that for $r \in [\delta, R]$,
\[ \lim_{n \to \infty} \frac{f^{[n]}(r, u)}{u} = \lim_{n \to \infty} \frac{f(r, \frac{1}{n})}{1/n} = \infty, \]
and consequently,
\[ \lim_{n \to \infty} \lambda_k(m^{[n]}, \delta) = 0. \tag{3.19} \]
Thus, from (3.19), we have that the component $C_{\nu,k}$ joins $(0, \theta)$ with infinity in the direction of $\lambda$ in $[0, \infty) \times S_\nu^{[n]}$.

We claim that
\[ (C_{\nu,k} \setminus \{(0, \theta)\}) \subset (0, \infty) \times S_\nu^{[n]} . \tag{3.20} \]

Suppose on the contrary that there exists a sequence $\{(\mu_n, u_n)\} \subset C_{\nu,k}$ satisfying
\[ \lim_{n \to \infty} (\mu_n, u_n) = (\mu^*, \theta) \quad \text{in } \mathbb{R} \times X_\delta \]
for some $\mu^* > 0$. Then
\[ \begin{cases} 
- (r^{N-1} u_n')' = \mu_n r^{N-1} f^{[n]}(r, u_n) h(u_n) - (N-1) r^{N-2} u_n^3, & r \in (\delta, R), \\
u_n(\delta) = 0 = u_n(R).
\end{cases} \]

Setting, for all $n$, $v_n = u_n / \|u_n\|_{C^{[n]}[\delta,R]}$, we have that
\[ \begin{cases} 
- (r^{N-1} v_n')' = \mu_n r^{N-1} f^{[n]}(r, u_n) u_n h(u_n) v_n - (N-1) r^{N-2} u_n^2 v_n^2, & r \in (\delta, R), \\
v_n(\delta) = 0 = v_n(R). \tag{3.21}
\end{cases} \]

Notice that
\[ r^{N-1} \phi_1(u_n'(r)) = - \mu_n \int_0^r \tau^{N-1} f^{[n]}(\tau, u_n(\tau)) d\tau, \quad r \in [0, R]. \tag{3.22} \]
This together with $f^{[n]}(r, 0) = 0$ for $r \in [\delta, R]$ imply that
\[
\lim_{n \to \infty} ||u_n'||_{C[0, R]} = 0.
\] (3.23)
Combining this with (3.21) and the facts $f_0 = \infty$ and \(\lim_{n \to \infty} h(u_n') = 1\), it concludes that $\mu^* = 0$. This is a contradiction.

Therefore, (3.20) holds.  

\[\square\]

4 Radial solutions for the prescribed mean curvature problem in a ball

In this section, we shall deal with \((1.6)_{\delta}\) with $\delta = 0$.

Let
\[
g_n(r, s) = \begin{cases} 
0, & (r, s) \in (0, \frac{1}{n}] \times (-\alpha, \alpha), \\
f(r - \frac{1}{n}, s), & (r, s) \in (\frac{1}{n}, R) \times (-\alpha, \alpha). 
\end{cases}
\] (4.1)
In the following, we shall use the solutions of the family of problems
\[
\begin{cases} 
-(r^{N-1}\phi(u'))' = \lambda r^{N-1}g_n(r, u), & r \in (\frac{1}{n}, R), \\
u'(1/n) = 0 = u(R), 
\end{cases}
\] (4.2)

\[
to construct the radial solutions of the prescribed mean curvature problem in a ball
\[
\mathcal{M}v + \lambda f(|x|, v) = 0 \ \ \text{in} \ B(R), \ \ v = 0 \ \ \text{on} \ \partial B(R).
\] (4.3)

To find a radial solution of (4.3), it is enough to find a solution of the problem
\[
\begin{cases} 
-(r^{N-1}\phi(u'))' = \lambda r^{N-1}f(r, u), \\
u'(0) = 0 = u(R), 
\end{cases}
\] (4.4)

For given $n \in \mathbb{N}$, let $(\lambda, u)$ be a solution of (4.2)$_n$. For each $n$, define a function $y_n : [0, R] \to [0, \infty)$ by
\[
y_n(r) = \begin{cases} 
u(r), & \frac{1}{n} \leq r \leq R, \\
u(\frac{1}{n}), & 0 \leq r \leq \frac{1}{n}. 
\end{cases}
\] (4.5)

Then
\[
y_n \in \{w \in C^2[0, R] : w'(0) = w(R) = 0\}.
\]
Moreover, $y_n$ is a solution of the problem
\[
\begin{cases} 
-(r^{N-1}\phi(u'))' = \lambda r^{N-1}g_n(r, u), & r \in (0, R) \\
u'(0) = 0 = u(R), 
\end{cases}
\] (4.6)
i.e. $y_n$ is a solution of the problem
\[
\begin{cases} 
-(r^{N-1}u'(r))' + (N - 1)r^{N-2}[u'(r)]^3 = \lambda r^{N-1}g_n(r, u(r))h(u'(r)), & r \in (0, R), \\
u'(0) = u(R) = 0.
\end{cases}
\] (4.7)
On the other hand, if \((\lambda, y)\) is a solution of \((4.7)_n\), then \((\lambda, y|_{[\frac{1}{n}, R]}^1)\) is a solution of \((4.2)_n\).

**Lemma 4.1.** Let (A1) and (A2) hold. Let \(\hat{\lambda} : \hat{\lambda} \neq \lambda_k(m, 0)\) be given. Then there exists \(\hat{b} > 0\), such that

\[
||u||_{C[0, R]} \geq \hat{b}
\]

for any solution \((\hat{\lambda}, u) \in \Phi_{k, 0}^\nu\) of \((4.7)_n\). Here \(\hat{b}\) is independent of \(n\) and \(u\).

**Proof.** Suppose on the contrary that \((4.7)_n, n \in \mathbb{N},\) has a sequence of solutions \((\hat{\lambda}, y_j) \in \Phi_{k, 0}^\nu\) with

\[
\lim_{j \to \infty} ||y_j||_{C[0, R]} = 0.
\]

Then

\[
\begin{cases}
(r^{N-1}\phi_1(y_j(r)))' + \lambda N-1 g_n(r, y_j(r)) = 0, & r \in (0, R), \\
y_j(0) = y_j(R) = 0,
\end{cases}
\]

and consequently,

\[
r^{N-1}\phi_1(y_j(r)) = -\lambda \int_0^r \tau^{N-1} g_n(r, y_j(\tau)) d\tau, & r \in [0, R].
\]

This together with (4.8) and the fact that \(g_n(r, 0) = 0\) for \(r \in [0, R]\) imply that

\[
\lim_{j \to \infty} ||y_j'||_{C[0, R]} = 0.
\]

Recall that (4.9) can be rewritten as

\[
\begin{cases}
-(r^{N-1}y_j'(r))' + (N-1)r^{N-2}y_j'(r)^3 = \lambda r^{N-1}g_n(r, y_j(r))h(y_j(r)), \\
y_j(0) = y_j(R) = 0.
\end{cases}
\]

Setting, for all \(j, v_j = y_j/||y_j||_{C[0, R]}\), we have that

\[
\begin{cases}
-(r^{N-1}v_j'(r))' + (N-1)r^{N-2}v_j'(r)2v_j'(r) = \lambda r^{N-1}g_n(r, y_j(r))h(y_j(r))/y_j(r)v_j(r)h(y_j(r)), \\
v_j(0) = v_j(R) = 0.
\end{cases}
\]

Letting \(j \to \infty\), it follows from (4.8), (4.10) and (4.12) that there exists \(w \in C^2[0, R]\) with \(||w||_{C[0, R]} = 1\), such that

\[
\begin{cases}
-(r^{N-1}w'(r))' = \lambda r^{N-1}m(r)w(r), \\
w'(0) = w(R) = 0,
\end{cases}
\]

which implies that \(\hat{\lambda} = \lambda_k(m, 0)\). However, this contradicts the assumption \(\hat{\lambda} \neq \lambda_k(m, 0)\). \(\square\)

Using the same argument with obvious changes, we may prove the following

**Lemma 4.2.** Let (A1) and (A3) hold. Let \(\lambda \in (0, \infty)\) be given. Then there exists \(\hat{b} > 0\), such that

\[
||u||_{C[0, R]} \geq \hat{b}
\]

for any solution \((\hat{\lambda}, u) \in \Phi_{k, 0}^\nu\) of \((4.7)_n\). \(\square\)

Now, we are in the position to prove Theorem 1.1-1.2 with \(\delta = 0\).
Proof of Theorem 1.1 with $\delta = 0$. For given $n$, let $\xi_n$ be the component obtained by Theorem 1.1 with $\delta \in (0, R)$ for (4.2)$_n$. Let

$$\zeta_n := \{ (\lambda, y_n) : y_n \text{ is determined by } u \text{ via (4.5)$_n$ for } (\lambda, u) \in \xi_n \}.$$ 

Then $\zeta_n$ is a component in $[0, \infty) \times C^1[0, R]$ which joins $(\lambda_k(m^{[n]}, \frac{1}{n})), \theta)$ with infinity in the direction of $\lambda$ and

$$\sup\{ ||y||_{C^1[0, R]} : (\lambda, y) \in \zeta_n \} < M$$

for some constant $M > 0$, independent of $y$ and $n$. Here

$$m^{[n]}(r) := m(r - \frac{1}{n}), \quad \frac{1}{n} \leq r \leq R,$$

and $\lambda_k(m^{[n]}, \frac{1}{n})$ is the principal eigenvalue of the linear problem

$$\begin{cases}
- (r^{N-1}u'(r))' = \lambda r^{N-1}m^{[n]}(r)u(r), & r \in (\frac{1}{n}, R), \\
u'(\frac{1}{n}) = u(R) = 0.
\end{cases} \quad (4.15)$$

Since $\lim_{n \to \infty} \lambda_k(m^{[n]}, \frac{1}{n}) = \lambda_k(m, 0)$, it follows from Lemma 2.1 that there exists a component $\zeta$ in $\lim \sup_{n \to \infty} \zeta_n$ which joins $(\lambda_k(m, 0), \theta)$ with infinity in the direction of $\lambda$ and

$$\sup\{ ||y||_{C^1[0, R]} : (\lambda, y) \in \zeta \} \leq M. \quad (4.16)$$

Now, Lemma 4.1 ensures that

$$\zeta \cap ([0, \infty) \times \{ \theta \}) = \{ (\lambda_k(m, 0), \theta) \}. \quad \square$$

Proof of Theorem 1.2 with $\delta = 0$. It is an immediate consequence of Theorem 1.2 with $\delta > 0$ and Lemma 4.2. \quad \square

The proof of Corollary 1.1 is a direct consequence of Theorem 1.1.

Proof of Corollary 1.2. From Theorem 1.2, for each $\nu \in \{+, -\}$ and $n \in \mathbb{N}$, (1.6)$_\delta$ has a solution $u^\nu_n \in S^\nu_{n, \delta}$. We only need to show that

$$\lim_{n \to \infty} |u^\nu_n(r)| + |(u^\nu_n)'(r)| = 0 \text{ uniformly in } r \in [0, R].$$

We firstly deal with the case that $\delta > 0$. In the following, we shall replace $u^\nu_n$ with $u_n$ for fixed $\nu \in \{+, -\}$.

Step 1 We show that

$$\lim_{n \to \infty} ||u_n||_{C^1[\delta, R]} = 0. \quad (4.17)$$

Let $\tau_1(n), \cdots, \tau_n(n)$ be zeros of $u_n$ in $(0, R)$:

$$\big( \delta = \tau_0(n) < \tau_1(n) < \cdots < \tau_{n-1}(n) < \tau_n(n) (= R) \big). \quad (4.18)$$
Since \( \|u'_n\|_{C[\delta, R]} < 1 \), it follows that
\[
\|u_n\|_{C[\delta, R]} \leq \sup\{ |\tau_{j+1}(n) - \tau_j(n)| : j = 0, \ldots, n - 1 \}. \tag{4.19}
\]

Suppose on the contrary that (4.17) is not true. Then there exist a positive constant \( \sigma_0 \) and a subsequence of solution of (1.6)\( \delta \), \( \{(1, u_{n_j})\} \subseteq \{(1, u_n)\} \), such that
\[
\|u_{n_j}\|_{C[\delta, R]} \geq \sigma_0. \tag{4.20}
\]

It follows from (4.20) and (4.19) that for \( n_j \geq 6 \), there exist \( t_1(n_j), t_2(n_j) \in \{\tau_0(n_j), \tau_1(n_j), \ldots, \tau_{n_j}(n_j)\} \) satisfying
\[
t_2(n_j) - t_1(n_j) \geq \sigma_0, \tag{4.21}
\]
\[
\|u_{n_j}\|_{C[\delta, R]} = \sup\{|u_{n_j}(r)| : r \in [t_1(n_j), t_2(n_j)]\}, \tag{4.22}
\]
\[
u_{n_j}(r) \neq 0, \quad r \in (t_1(n_j), t_2(n_j)). \tag{4.23}
\]

Without loss of generality, we may assume that there exists a closed subinterval \( I_1 \) with \( \text{meas } I_1 \geq \frac{\sigma_0}{2} \), such that (after taking a subsequence and relabeling, if necessary)
\[
u_{n_j}(r) \neq 0, \quad r \in I_1, \quad j \geq j_0 \text{ for some } j_0 \in \mathbb{N}. \tag{4.23}
\]

Now, applying the facts
\[
u_{n_j}(r) = \int_{I_1}^R \phi_1^{-1}\left(\frac{1}{s^{N-1}} \int_{\delta}^{s} t^{N-1} f(t, u_{n_j}(t)) dt\right) ds \tag{4.24}
\]
and
\[
\|u'_{n_j}\|_{C[\delta, R]} < 1, \quad \|u_{n_j}\|_{C[\delta, R]} < R,
\]
and the standard argument, we deduce that after taking a subsequence and relabeling, if necessary,
\[
u_{n_j} \to u_\circ \text{ in } C^1[\delta, R],
\]
for some \( u_\circ \in C^1[\delta, R] \). Obviously,
\[
u_\circ(r) \geq \frac{\sigma_0}{2}, \quad r \in I_1, \tag{4.25}
\]
and it is easy to check that \( u_\circ \) satisfies
\[
\begin{cases}
- (r^{N-1}u'_\circ(r))' = r^{N-1} f(r, u_\circ), & r \in (\delta, R), \\
u'_\circ(\delta) = u_\circ(R) = 0.
\end{cases}
\]

On the other hand, we may take \( s_1(n_j), s_2(n_j) \in \{\tau_1(n_j), \ldots, \tau_{n_j}(n_j)\} \) with
\[
s_2(n_j) > s_1(n_j), \tag{4.26}
\]
\[
s_2(n_j), s_1(n_j) \notin [t_1(n_j), t_2(n_j)], \tag{4.27}
\]
\[
u_{n_j}(r) \neq 0, \quad r \in (s_1(n_j), s_2(n_j)), \tag{4.28}
\]
\[
s_2(n_j) - s_1(n_j) \to 0, \quad j \to \infty. \tag{4.29}
\]
For each \( n_j \geq 6 \), \( u_{n_j}(s_1(n_j)) = u_{n_j}(s_2(n_j)) = 0 \) yields
\[
u'_n(x(n_j)) = 0, \quad \text{for some } x(n_j) \in ((s_1(n_j), s_2(n_j)) \quad (4.30)
\]
After taking a subsequence and relabeling, if necessary, we may assume
\[
x(n_j) \to x_0, \quad j \to \infty.
\]
Combining this with (4.30), (4.29) and using (4.24), it follows that
\[
u_0(x_0) = u'_0(x_0) = 0 \quad \text{for some } x_0 \in [\delta, R] \quad (4.31)
\]
which means that \( u_0 \equiv 0 \) in \([\delta, R]\) (see Lemma 2.5 where \( x_0 > 0 \) is needed). However, this contradicts (4.25).

Therefore, (4.17) is true.

**Step 2** We show that
\[
\lim_{n \to \infty} ||u'_n||_{C[\delta, R]} = 0. \quad (4.32)
\]
In fact, it is an immediate consequence of (4.17) and the relation
\[
u'_n(r) = -\phi^{-1}_1\left(\frac{1}{r^{N-1}} \int_0^r t^{N-1} f(t, u_n(t)) dt\right).
\]

Next, we consider the case \( \delta = 0 \).

Let
\[
g_m(r, s) = \begin{cases} 
0, & (r, s) \in (0, \frac{1}{m}] \times (-\alpha, \alpha), \\
\frac{f(r - \frac{1}{m}, s)}{r}, & (r, s) \in (\frac{1}{m}, R) \times (-\alpha, \alpha).
\end{cases}
\]

Let \( u_m^\nu \in \mathcal{S}^{\nu}_{n, 1/m} \). Then
\[
\begin{cases} 
-(r^{N-1} \phi_1((u_m^\nu)''))' = r^{N-1} g_m(r, u_m^\nu), & r \in (\frac{1}{m}, R), \\
(u_m^\nu)'(1/m) = 0 = u_m^\nu(R).
\end{cases}
\]
For each \( n \) and \( m \), define a function \( y_m^\nu \in \mathcal{S}^{\nu}_{n, 0} \) by
\[
y_m^\nu(r) = \begin{cases} 
u_m(r), & \frac{1}{m} \leq r \leq R, \\
u_m(\frac{1}{m}), & 0 \leq r \leq \frac{1}{m}.
\end{cases}
\]

By the same argument used in proof of Theorem 1.1, with obvious changes, we may use \( y_m^\nu \) to construct two solutions, \( z_m^\nu \in \mathcal{S}^{\nu}_{n, 0} \), \( \nu \in \{+, -\} \), for (1.5)\(_{\nu=1}^\lambda\). Obviously, (4.17) and (4.32) imply that
\[
\lim_{n \to \infty} (||z_m^\nu||_{C[0, R]} + ||(z_m^\nu)'||_{C[0, R]}) = 0.
\]
\( \square \)
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