Dihedral and reflexive modules with $\infty$-simplicial faces and dihedral and reflexive homology of involutive $A_\infty$-algebras over unital commutative rings.

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Abstract

The concepts of a dihedral and a reflexive module with $\infty$-simplicial faces are introduced. For each involutive $A_\infty$-algebra, the dihedral and the reflexive tensor modules with $\infty$-simplicial faces are constructed. On the basis of dihedral and reflexive modules with $\infty$-simplicial faces that defined by an involutive $A_\infty$-algebra the constructions of the dihedral and the reflexive homology of involutive $A_\infty$-algebras over any unital commutative rings are given. The conception of an involutive homotopy unital $A_\infty$-algebra is introduced. A long exact sequence that connecting the dihedral and the reflexive homology of involutive homotopically unital $A_\infty$-algebras over any unital commutative rings is constructed.

Dihedral homology of involutive associative algebras over fields of characteristic zero was first defined in [1] as the homology of the complex of coinvariants of the action of the dihedral group on the Hochschild complex of these involutive associative algebras. After that in [2] (see also [3]) the theory of dihedral simplicial modules and in particular the theory of dihedral modules with simplicial faces was developed. Further on the basis of the combinatorial technique of dihedral modules with simplicial faces the dihedral homology theory of involutive associative algebras over any unital commutative rings was constructed. In [2] also was show that over fields of characteristic zero the definitions from [1] and [2] are equivalent. Constructed in [2] the dihedral homology theory proved to be a very useful tool in the study of the hermitian algebraic $K$-theory of involutive unital associative algebras, algebraic and homotopy properties of the hermitian algebraic $K$-theory of topology spaces and also in the study of the rational homotopy type of groups of diffeomorphisms of smooth manifolds (see, g.e., the survey [4]).

On the other hand in [5] the dihedral homology of involutive $A_\infty$-algebras over fields of characteristic zero was defined as the homology of the complex of coinvariants of the action of the dihedral group on the Hochschild complex of these involutive $A_\infty$-algebras. In this regard gives rise to the important and very interesting problem of constructing the dihedral homology theory of involutive $A_\infty$-algebras over any unital commutative rings, which generalizes developed in [2] the dihedral homology
theory of involutive unital associative algebras over any unital commutative rings. The interest to this problem is caused, mainly, by the important question about the possible of constructing the hermitian algebraic $K$-theory of involutive homotopy unital $A_{\infty}$-algebras over any unital commutative rings by the analogue to how it was done in [6] for involutive unital associative algebras over any unital commutative rings. Moreover, great interest to constructing of the dihedral homology theory of involutive $A_{\infty}$-algebras over any unital commutative rings is caused also the possible of applying of this theory to the study of Kontsevich graph-complexes and the cohomology of moduli spaces by the analogue to how it was done in [7] for the cyclic homology of $A_{\infty}$-algebras.

The present paper is devoted to the solution of the above-mentioned problem, namely, to constructing on the basis of the combinatorial technique of dihedral modules with $\infty$-simplicial faces the dihedral homology theory of involutive $A_{\infty}$-algebras over any unital commutative rings. The paper consists of three paragraphs. In first paragraph, on the basis of the conceptions of a differential module with $\infty$-simplicial faces [8]-[14] and a $D_{\infty}$-differential module [15]-[23] we introduce the notions of a dihedral module with $\infty$-simplicial faces and a reflexive module with $\infty$-simplicial faces. After that, the notions of the dihedral homology of dihedral modules with $\infty$-simplicial faces and the reflexive homology of reflexive modules with $\infty$-simplicial faces are given. In second paragraph, we construct the dihedral module with $\infty$-simplicial faces for each involutive $A_{\infty}$-algebra over any unital commutative ring (see Theorem 2.1). Then, we define the dihedral homology of an involutive $A_{\infty}$-algebra over any unital commutative ring as the dihedral homology of the dihedral module with $\infty$-simplicial faces determined by the given involutive $A_{\infty}$-algebra. Since each dihedral module with $\infty$-simplicial faces can be view as the reflexive module with $\infty$-simplicial faces, the reflexive homology of involutive $A_{\infty}$-algebras over any unital commutative rings always are defined. Next, we show that over fields of characteristic zero the definition of the dihedral homology of involutive $A_{\infty}$-algebras introduced here is equivalent to that proposed in [5] (see Corollary 2.1). Also we consider properties of the reflexive homology of involutive $A_{\infty}$-algebras over fields of characteristic zero (see Corollary 2.2). In the third paragraph, we introduce the conception of a involutive homotopy unital $A_{\infty}$-algebra, which is the involutive analogue of the conception of a homotopy unital $A_{\infty}$-algebra [24] (see also [13]). Next, we construct an exact sequence that connecting the dihedral and the reflexive homology of involutive homotopy unital $A_{\infty}$-algebras over any unital commutative rings (see Theorem 3.1). This exact sequence generalizes the well-known Krasauskas-Lapin-Solov’ev exact sequence [2] in the dihedral homology theory of involutive unital associative algebras.

All modules and maps of modules considered in this paper are, respectively, $K$-modules and $K$-linear maps of modules, where $K$ is any unital (i.e., with unit) commutative ring.

§ 1. Dihedral and reflexive modules with $\infty$-simplicial faces

In what follows, by a bigraded module we mean any bigraded module $X = \{X_{n,m}\}$, $n \geq 0$, $m \geq 0$, and by a differential bigraded module, or, briefly, a differential module
(X, d), we mean any bigraded module X endowed with a differential \(d : X_{\bullet, \bullet} \to X_{\bullet, \bullet-1}\) of bidegree \((0, -1)\).

Recall that a differential module with simplicial faces is defined as a differential module \((X, d)\) together with a family of module maps \(\partial_i : X_{n, \bullet} \to X_{n-1, \bullet}\), \(0 \leq i \leq n\), which are maps of differential modules and satisfy the simplicial commutation relations \(\partial_i \partial_j = \partial_j \partial_i\), \(i < j\). The maps \(\partial_i : X_{n, \bullet} \to X_{n-1, \bullet}\) are called the simplicial face operators or, more briefly, the simplicial faces of the differential module \((X, d)\).

Now, we recall the notion of a differential module with \(\infty\)-simplicial faces \([9]\) (see also \([10]-[14]\)), which is a homotopy invariant analogue of the notion of a differential module with simplicial faces.

Let \(\Sigma_k\) be the symmetric group of permutations on a \(k\)-element set. Given an arbitrary permutation \(\sigma \in \Sigma_k\) and any \(k\)-tuple of nonnegative integers \((i_1, \ldots, i_k)\), where \(i_1 < \ldots < i_k\), we consider the \(k\)-tuple \((\sigma(i_1), \ldots, \sigma(i_k))\), where \(\sigma\) acts on the \(k\)-tuple \((i_1, \ldots, i_k)\) in the standard way, i.e., permutes its components. For the \(k\)-tuple \((\sigma(i_1), \ldots, \sigma(i_k))\), we define a \(k\)-tuple \((\sigma(i_1), \ldots, \sigma(i_k))\) by the following formulae

\[
\sigma(i_s) = \sigma(i_s) - \alpha(\sigma(i_s)), \quad 1 \leq s \leq k,
\]

where each \(\alpha(\sigma(i_s))\) is the number of those elements of \((\sigma(i_1), \ldots, \sigma(i_s), \ldots \sigma(i_k))\) on the right of \(\sigma(i_s)\) that are smaller than \(\sigma(i_s)\).

A differential module with \(\infty\)-simplicial faces or, more briefly, an \(F_\infty\)-module \((X, d, \tilde{\partial})\) is defined as a differential module \((X, d)\) together with a family of module maps

\[
\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)} : X_{n, \bullet} \to X_{n-k, \bullet+k-1}\}, \quad 1 \leq k \leq n,
\]

\[
i_1, \ldots, i_k \in \mathbb{Z}, \quad 0 \leq i_1 < \ldots < i_k \leq n,
\]

which satisfy the relations

\[
d(\partial_{(i_1, \ldots, i_k)}) = \sum_{\sigma \in \Sigma_k} \sum_{I_\sigma} (-1)^{\text{sign}(\sigma)+1} \partial_{(\sigma(i_1), \ldots, \sigma(i_m))} \partial_{(\sigma(i_{m+1}), \ldots, \sigma(i_k))},
\]

(1.1)

where \(I_\sigma\) is the set of all partitions of the \(k\)-tuple \((\sigma(i_1), \ldots, \sigma(i_k))\) into two tuples \((\sigma(i_1), \ldots, \sigma(i_m))\) and \((\sigma(i_{m+1}), \ldots, \sigma(i_k))\), \(1 \leq m \leq k-1\), such that the conditions \(\sigma(i_1) < \ldots < \sigma(i_m)\) and \(\sigma(i_{m+1}) < \ldots < \sigma(i_k)\) holds.

The family of maps \(\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)}\}\) is called the \(F_\infty\)-differential of the \(F_\infty\)-module \((X, d, \tilde{\partial})\). The maps \(\partial_{(i_1, \ldots, i_k)}\) that form the \(F_\infty\)-differential of an \(F_\infty\)-module \((X, d, \tilde{\partial})\) are called the \(\infty\)-simplicial faces of this \(F_\infty\)-module.

It is easy to show that, for \(k = 1, 2, 3\), relations (1.1) take, respectively, the following view

\[
d(\partial_{(i)}) = 0, \quad i \geq 0, \quad d(\partial_{(i,j)}) = \partial_{(i-1,j)} \partial_{(i)} - \partial_{(i)} \partial_{(j)}, \quad i < j,
\]

\[
d(\partial_{(i_1, i_2, i_3)}) = -\partial_{(i_1)} \partial_{(i_2, i_3)} - \partial_{(i_1, i_2)} \partial_{(i_3)} - \partial_{(i_3-2)} \partial_{(i_1, i_2)} - \partial_{(i_2-1, i_3-1)} \partial_{(i_1)} + \partial_{(i_2-1)} \partial_{(i_1, i_3)} + \partial_{(i_1, i_3-1)} \partial_{(i_2)}, \quad i_1 < i_2 < i_3.
\]
Simplest examples of $F_\infty$-modules are differential modules with simplicial faces. Indeed, given any differential module with simplicial faces $(X, d, \partial)$, we can define an $F_\infty$-differential $\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)}\} : X \to X$ by setting $\partial_{(i)} = \partial_i$, $i > 0$, and $\partial_{(i_1, \ldots, i_k)} = 0$, $k > 1$, thus obtaining the $F_\infty$-module $(X, d, \tilde{\partial})$.

It is worth mentioning that the notion of an $F_\infty$-module specified above is a part of the general notion of a differential $\infty$-simplicial module introduced in [10] by using the homotopy technique of differential Lie modules over curved colored coalgebras.

Now, we proceed to the notion of a dihedral module with $\infty$-simplicial faces. By a dihedral bigraded module $(X, t, r, d)$, we mean any bigraded module $X$ together with two families of module maps $t = \{t_n : X_{n, \bullet} \to X_{n, \bullet}\}$, $r = \{r_n : X_{n, \bullet} \to X_{n, \bullet}\}$, $n \geq 0$, satisfying the conditions

$$t_{n+1} = 1_{X_{n, \bullet}}, \quad r_n^2 = 1_{X_{n, \bullet}}, \quad r_n t_n = t_n^{-1} r_n, \quad n \geq 0.$$ 

In other words, on each graded module $X_{n, \bullet}$, $n \geq 0$, the dihedral group of order $2(n + 1)$ with generators $t_n$ and $r_n$ acts on the left.

In what follows, we use the term dihedral differential module for any quadruple $(X, t, r, d)$, where $(X, t, r)$ is a dihedral bigraded module, $(X, d)$ is a differential module, and the conditions $dt_n = t_n d$, $dr_n = r_n d$, $n \geq 0$, holds.

Now, recall that a dihedral module with simplicial faces [2] is defined as a dihedral differential module $(X, t, r, d)$ together with a family of maps $\tilde{\partial}_i : X_{n, \bullet} \to X_{n-1, \bullet}$, $0 \leq i \leq n$, with respect to which the triple $(X, d, \tilde{\partial}_i)$ is a differential module with simplicial faces and, moreover, the relations

$$\partial_i t_n = t_{n-1} \partial_{i-1}, \quad 0 < i \leq n, \quad \partial_i t_n = \partial_n, \quad \partial_i r_n = r_{n-1} \partial_{n-i}, \quad 0 \leq i \leq n,$$

for each $n \geq 0$ are true.

Note that if in the definition of a dihedral module with simplicial faces we remove the family of automorphisms $r_n : X_{n, \bullet} \to X_{n, \bullet}$, $n \geq 0$, and the relations $\partial_i t_n = t_{n-1} \partial_{n-i}$, $0 \leq i \leq n$, $n \geq 0$, then we obtain the definition of a cyclic module with simplicial faces [25].

**Definition 1.1.** A dihedral module with $\infty$-simplicial faces or, more briefly, a $DF_\infty$-module, is any five-tuple $(X, t, r, d, \tilde{\partial})$, where $(X, t, r, d)$ is a dihedral differential module and $(X, d, \tilde{\partial})$ is a differential module with $\infty$-simplicial faces related by

$$\partial_{(i_1, \ldots, i_k)} t_n = \begin{cases} t_{n-k} \partial_{(i_1-1, \ldots, i_k-1)} & i_1 > 0, \\ (-1)^{k-1} \partial_{(i_1-1, \ldots, i_k-1, n)} & i_1 = 0, \end{cases}$$

$$\partial_{(i_1, \ldots, i_k)} r_n = (-1)^{k(k-1)/2} r_{n-k} \partial_{(n-i_k, \ldots, n-i_1)}.$$ \hspace{1cm} (1.2)

In what follows, we refer to the family of maps $\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)} : X_{n, \bullet} \to X_{n-k, \bullet+k-1}\}$ as the $F_\infty$-differential of the $DF_\infty$-module $(X, t, r, d, \tilde{\partial})$. The maps $\partial_{(i_1, \ldots, i_k)}$ that form the $F_\infty$-differential of a $DF_\infty$-module $(X, t, r, d, \tilde{\partial})$ are called the $\infty$-simplicial faces of this $DF_\infty$-module.

Now, we note that if in the definition 1.1 we remove the family of automorphisms $r_n : X_{n, \bullet} \to X_{n, \bullet}$, $n \geq 0$, and the relations (1.3) then we obtain the definition of a
cyclic module with \(\infty\)-simplicial faces \(^{26}\) or, more briefly, \(CF_{\infty}\)-module. Therefore each \(DF_{\infty}\)-module \((X, t, r, d, \tilde{\partial})\) always defines the \(CF_{\infty}\)-module \((X, t, d, \tilde{\partial})\).

Simple examples of \(DF_{\infty}\)-modules are dihedral modules with simplicial faces. Indeed, given any dihedral module with simplicial faces \((X, t, r, d, \tilde{\partial})\), we can define an \(F_{\infty}\)-differential \(\tilde{\partial} = \{\partial(i, \ldots, i_k)\}\) by setting \(\partial(j) = \partial_i, i \geq 0, \) and \(\partial(i, \ldots, i_k) = 0, k > 1,\) thus obtaining the \(DF_{\infty}\)-module \((X, t, r, d, \tilde{\partial})\).

Now, we make preparations to introduce the notion of the dihedral homology of a dihedral module with \(\infty\)-simplicial faces.

First, recall that a \(D_{\infty}\)-differential module \(^{15}\) (see also \([16]-[23]\) or, more briefly, a \(D_{\infty}\)-module \((X, d^i)\) is defined as a module \(X\) together with a family of module maps \(\{d^i : X \to X \mid i \in \mathbb{Z}, i \geq 0\}\) satisfying the relations

\[
\sum_{i+j=k} d^i d^j = 0, \quad k \geq 0. \tag{1.4}
\]

It is worth noting that a \(D_{\infty}\)-module \((X, d^i)\) can be equipped with any \(\mathbb{Z}^{\times n}\)-grading, i.e., \(X = \{X_{k_1,\ldots,k_n}, (k_1, \ldots, k_n) \in \mathbb{Z}^{\times n}, n \geq 1,\) and maps \(d^i : X \to X, i \geq 0,\) can have any \(n\)-degree \((l_1(i), \ldots, l_n(i)) \in \mathbb{Z}^{\times n}, \) i.e., \(d^i : X_{k_1,\ldots,k_n} \to X_{k_1+l_1(i),\ldots,k_n+l_n(i)}\). For \(k = 0,\) the relations (1.4) have the form \(d^0 d^0 = 0,\) and hence \((X, d^0)\) is a differential module. In \([15]\) was established the homotopy invariance of the structure of a \(D_{\infty}\)-differential module over any unital commutative ring under homotopy equivalences of differential modules. Later, it was shown in \([27]\) that the homotopy invariance of the structure of the module over fields of characteristic zero can be established by using the Koszul duality theory.

A \(D_{\infty}\)-module \((X, d^i)\) is said to be stable if, for each \(x \in X,\) there exists a number \(k = k(x) \geq 0\) such that \(d^i(x) = 0, i > k.\) Any stable \(D_{\infty}\)-module \((X, d^i)\) determines the differential \(\overrightarrow{d} : X \to X\) defined by \(\overrightarrow{d} = (d^0 + d^1 + \ldots + d^i + \ldots).\) The map \(\overrightarrow{d} : X \to X\) is indeed a differential, because relations (1.4) imply the equality \(\overrightarrow{d} \overrightarrow{d} = 0.\) It is easy to see that if the stable \(D_{\infty}\)-module \((X, d^i)\) is equipped with a \(\mathbb{Z}^{\times n}\)-grading \(X = \{X_{k_1,\ldots,k_n}, k_1 \geq 0, \ldots, k_n \geq 0,\) and maps \(d^i : X \to X, i \geq 0,\) have \(n\)-degree \((l_1(i), \ldots, l_n(i))\) satisfying the condition \(l_1(i) + \ldots + l_n(i) = -1,\) then there is the chain complex \((\overrightarrow{X}, \overrightarrow{d})\) defined by the following formulae:

\[
\overrightarrow{X}_m = \bigoplus_{k_1+\ldots+k_n=m} X_{k_1,\ldots,k_n}, \quad \overrightarrow{d} = \sum_{i=0}^\infty d^i : X_m \to \overrightarrow{X}_{m-1}, \quad m \geq 0.
\]

It was shown in \([11]\) that any \(F_{\infty}\)-module \((X, d, \tilde{\partial})\) determines the sequence of stable \(D_{\infty}\)-modules \((X, d_q^i), q \geq 0,\) equipped with the bigrading \(X = \{X_{n,m}, n \geq 0, m \geq 0,\) and defined by the following formulae:

\[
d_q^0 = d, \quad d_q^k = \sum_{0 \leq i_1 < \ldots < i_k \leq n-q} (-1)^{i_1+\ldots+i_k} \partial(i_1,\ldots,i_k) : X_{n,\bullet} \to X_{n-k,\bullet+k-1}, \quad k \geq 1. \tag{1.5}
\]

Let us recall \([26]\) the construction of the chain bicomplex \((C(\overrightarrow{X}), \delta_1, \delta_2)\) that is defined by the cyclic module with \(\infty\)-simplicial faces \((X, t, d, \tilde{\partial})\). Given any \(CF_{\infty}\)-module \((X, t, d, \tilde{\partial})\), consider the two \(D_{\infty}\)-modules \((X, d_q^0)\) and \((X, d_q^1)\) defined by (1.5) for...
Consider also the two families of maps

\[ T_n = (-1)^n t_n : X_{n, \bullet} \to X_{n, \bullet}, \quad n \geq 0, \]

\[ N_n = 1 + T_0 + T_1 + \ldots + T_n : X_{n, \bullet} \to X_{n, \bullet}, \quad n \geq 0. \]

Obviously, the condition \( t_n^{n+1} = 1 \), \( n \geq 0 \), implies the relations

\[ (1 - T_n) N_n = 0, \quad N_n (1 - T_n) = 0, \quad n \geq 0. \quad (1.6) \]

Moreover, in [26, 27] it was shown that the families of module maps \( \{ T_n : X_{n, \bullet} \to X_{n, \bullet} \} \), \( \{ N_n : X_{n, \bullet} \to X_{n, \bullet} \} \), \( \{ d_0^i : X_{s-i, \bullet+i-1} \to X_{s-i, \bullet+i-1} \} \) and \( \{ d_1^i : X_{s, \bullet} \to X_{s-i, \bullet+i-1} \} \) are related by

\[ d_0^i (1 - T_n) = (1 - T_{n-i}) d_1^i, \quad d_1^i N_n = N_{n-i} d_0^i, \quad i \geq 0, \quad n \geq 0. \quad (1.7) \]

For example, for \( i = 2 \) and \( n = 3 \), we have the following equalities:

\[ d_0^3 (1 - T_3) = d_0^2 (1 + t_3) = (-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(0,3)} - \partial_{(1,2)} + \partial_{(1,3)} - \partial_{(2,3)})(1 + t_3) = \]

\[ \quad = \partial_{(0,1)} + \partial_{(0,2)} - \partial_{(0,3)} - \partial_{(1,2)} + \partial_{(1,3)} - \partial_{(2,3)} - \partial_{(0,1)} t_3 + \partial_{(0,2)} t_3 - \partial_{(0,3)} t_3 - \]

\[ \quad - \partial_{(1,2)} t_3 + \partial_{(1,3)} t_3 - \partial_{(2,3)} t_3 + \partial_{(0,1)} + \partial_{(0,2)} - \partial_{(0,3)} - \partial_{(1,2)} + \partial_{(1,3)} - \]

\[ \quad - \partial_{(0,3)} - (-1)^2 - \partial_{(1,2)} - (-1)^2 - \partial_{(2,3)} - t_1 \partial_{(0,1)} + t_1 \partial_{(0,2)} - t_1 \partial_{(1,2)} = \]

\[ = (1 + t_1)(-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(1,2)}) = (1 + t_1) d_1^2 = (1 - T_1) d_1^2. \]

\[ d_1^3 N_3 = d_1^2 (1 + T_3 + T_2 + T_1) = \]

\[ \quad = d_1^2 (1 + t_3 + t_2^2 t_3^2 + t_3^3 - t_3^3 - t_3^3) = (-\partial_{(0,1)} + \partial_{(0,2)}) (1 - t_3 + t_2^2 + t_3^2 - t_3^3) = \]

\[ = (-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(1,2)}) - (-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(1,2)}) t_3 + (-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(1,2)}) t_3^2 - \]

\[ \quad - (-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(1,2)}) t_3^2 = (-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(1,2)}) - \partial_{(0,3)} - \partial_{(1,3)} - t_1 \partial_{(0,1)} + \]

\[ \quad + (-\partial_{(2,3)} - t_1 \partial_{(0,2)} + t_1 \partial_{(0,3)}) - (-t_1 \partial_{(1,2)} + t_1 \partial_{(1,3)} - t_1 \partial_{(2,3)}) = \]

\[ = (1 - t_1)(-\partial_{(0,1)} + \partial_{(0,2)} - \partial_{(0,3)} - \partial_{(1,2)} + \partial_{(1,3)} - \partial_{(2,3)}) = (1 - t_1) d_0^2 = \]

\[ = (1 + T_1) d_0^2 = N_1 d_0^2. \]

Now, we consider the chain complexes \((\mathcal{X}, b)\) and \((\mathcal{X}, b')\) corresponding to the \( D_{\infty} \)-modules \((X, d_0^i)\) and \((X, d_1^i)\) specified above; here

\[ \mathcal{X}_n = \bigoplus_{k=0}^n X_{k,n-k}, \quad b = d_0 = \sum_{i=0}^n d_0^i : \mathcal{X}_n \to \mathcal{X}_{n-1}, \]

\[ b' = d_1 = \sum_{i=0}^n d_1^i : \mathcal{X}_n \to \mathcal{X}_{n-1}, \quad n \geq 0. \]

Consider also the two families of maps

\[ \mathcal{T}_n = \sum_{k=0}^n T_k : \mathcal{X}_n \to \mathcal{X}_n, \quad \mathcal{N}_n = \sum_{k=0}^n N_k : \mathcal{X}_n \to \mathcal{X}_n, \quad n \geq 0. \]
It is seen from (1.6) and (1.7) that

\[(1 - T_n)\overline{N}_n = 0, \quad \overline{N}_n(1 - T_n) = 0, \quad n \geq 0,
\]

\[b(1 - T_n) = (1 - T_{n-1})b', \quad b'\overline{N}_n = \overline{N}_{n-1}b, \quad n \geq 0.
\]

It follows from these relations that any \(CF_\infty\)-module \((X, t, d, \widehat{\partial})\) determines the chain bicomplex

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
X_{n+1} & X_{n} & X_{n+1} & \overline{X}_{n} & \overline{X}_{n+1} & \overline{X}_{n} & \overline{X}_{n+1} & \ldots \\
b & 1 - T_{n+1} & -b' & b & 1 - T_{n+1} & -b' & b & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X_{n} & X_{n} & X_{n} & X_{n} & \overline{X}_{n} & \overline{X}_{n} & \overline{X}_{n} & \ldots \\
b & 1 - T_{n} & -b' & b & 1 - T_{n} & -b' & b & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X_{n-1} & X_{n-1} & X_{n-1} & X_{n-1} & \overline{X}_{n-1} & \overline{X}_{n-1} & \overline{X}_{n-1} & \ldots \\
b & 1 - T_{n-1} & -b' & b & 1 - T_{n-1} & -b' & b & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & & & \\
\end{array}
\]

We denote this chain bicomplex by \((C(X), \delta_1, \delta_2)\), where \(C(X)_{n,m} = \overline{X}_n\), \(n \geq 0\), \(m \geq 0\), \(\delta_1 : C(X)_{n,m} \to C(X)_{n-1,m}\), \(\delta_2 : C(X)_{n,m} \to C(X)_{n,m-1}\), and

\[
\delta_1 = \begin{cases} b, & m \equiv 0 \text{ mod}(2), \\ -b', & m \equiv 1 \text{ mod}(2) \end{cases}, \quad \delta_2 = \begin{cases} 1 - T_n, & m \equiv 1 \text{ mod}(2), \\ \overline{N}_n, & m \equiv 0 \text{ mod}(2) \end{cases}
\]

For the chain complex associated with the chain bicomplex \((C(X), \delta_1, \delta_2)\), we use the notation \((\text{Tot}(C(X)), \delta)\), where \(\delta = \delta_1 + \delta_2\).

Recall [26] that the cyclic homology \(HC(X)\) of a \(CF_\infty\)-module \((X, t, d, \widehat{\partial})\) is defined as the homology of the chain complex \((\text{Tot}(C(X)), \delta)\) associated with the chain bicomplex \((C(X), \delta_1, \delta_2)\).

Given any \(DF_\infty\)-module \((X, t, r, d, \widehat{\partial})\), consider the specified above \(D_\infty\)-modules \((X, d_0^i)\) and \((X, d_1^i)\), and consider the family of maps

\[R_n = (-1)^{n(n+1)/2}r_n : X_{n,\bullet} \to X_{n,\bullet}, \quad n \geq 0.
\]

It is easily verified that the relations \(t_n^{n+1} = r_n^2 = 1\), \(r_nt_n = t_n^{-1}r_n\), \(n \geq 0\), implies the equalities

\[(1 - T_n)(R_nT_n) = -R_n(1 - T_n), \quad N_nR_n = (R_nT_n)N_n, \quad n \geq 0. \quad (1.8)
\]
For any collection $0 \leq i_1 < \ldots < i_k \leq n$, the relations (1.3) implies the equality
\[
(−1)^{i_1+\ldots+i_k} \partial_{(i_1,\ldots,i_k)} R_n = (−1)^{(n−i_k)+\ldots+(n−i_1)} R_{n−k} \partial_{(n−i_k,\ldots,n−i_1)}.
\]
Moreover, for any collection $0 \leq i_1 < \ldots < i_k \leq n−1$, by using the relations (1.2) and (1.3) we obtain the equality
\[
(−1)^{i_1+\ldots+i_k} \partial_{(i_1,\ldots,i_k)} (R_n T_n) = (−1)^{(n−i_k−1)+\ldots+(n−i_1−1)} (R_{n−k} T_{n−k}) \partial_{(n−i_k−1,\ldots,n−i_1−1)}.
\]
For specified above families $\{d_0^i : X_{s,•} \to X_{s−i,•+i−1}\}$ and $\{d_1^i : X_{•,•} \to X_{•−i,•+i−1}\}$, the last two equalities implies the relations
\[
d_0^i R_n = R_{n−i} d_0^i, \quad d_1^i (R_n T_n) = (R_{n−i} T_{n−i}) d_1^i, \quad i \geq 0, \quad n \geq 0. \tag{1.9}
\]
Consider the chain complexes $(\overline{X}, b)$ and $(\overline{X}, b')$ that corresponds to the $D_\infty$-modules $(X, d_0^i)$ and $(X, d_1^i)$. Moreover, consider the family of maps
\[
\overline{R}_n = \sum_{m=0}^{n} R_m : \overline{X}_n \to \overline{X}_n, \quad n \geq 0.
\]
By using the formulae (1.8) and (1.9) we obtain the following equalities:
\[
\begin{align*}
(1−\overline{T}_n)(\overline{R}_n \overline{T}_n) &= −\overline{R}_n (1−\overline{T}_n), \quad (\overline{N}_n \overline{R}_n) = (\overline{R}_n \overline{T}_n) \overline{N}_n, \quad n \geq 0, \\
\overline{b} \overline{R}_n &= \overline{R}_{n−1} \overline{b}, \quad \overline{b} (\overline{R}_n \overline{T}_n) = (\overline{R}_{n−1} \overline{T}_{n−1}) \overline{b}', \quad n \geq 0. \tag{1.10}
\end{align*}
\]
Since the $DF_\infty$-module $(X, t, r, d, \tilde{\partial})$ always defines the $CF_\infty$-module $(X, t, d, \vartheta)$, for the $DF_\infty$-module $(X, d, t, r, \tilde{\partial})$, the chain bicomplex $(C(\overline{X}), \delta_1, \delta_2)$ always is defined. The equalities (1.10) say us that there is a left action of the group $\mathbb{Z}_2 = \{1, \vartheta | \vartheta^2 = 1\}$ on the chain bicomplex $(C(\overline{X}), \delta_1, \delta_2)$ of any $DF_\infty$-module $(X, t, r, d, \tilde{\partial})$. This left action is defined by means of the automorphism $\vartheta : C(\overline{X})_{s,•} \to C(\overline{X})_{s,•}$ of the order two, which at any element $x \in C(\overline{X})_{n,m}$ is given by the following rule:
\[
\vartheta (x) = \begin{cases} 
(−1)^k \overline{T}_n (x), & m = 2k, \\
(−1)^{k+1} \overline{T}_n (x), & m = 2k + 1.
\end{cases}
\]

**Definition 1.2.** We define the dihedral homology $HD(X)$ of a dihedral module with $\infty$-simplicial faces $(X, t, r, d, \tilde{\partial})$ as the hyperhomology $H(Z_2; (C(\overline{X}), \delta_1, \delta_2))$ of the group $\mathbb{Z}_2$ with coefficients in $(C(\overline{X}), \delta_1, \delta_2)$ relative to the specified above action of the group $\mathbb{Z}_2$ on the chain bicomplex $(C(\overline{X}), \delta_1, \delta_2)$.

The hyperhomology $H(Z_2; (C(\overline{X}), \delta_1, \delta_2))$ is defined as the homology of the chain complex that associated with the triple chain complex $(\mathcal{P}(\mathbb{Z}_2) \otimes K[\mathbb{Z}_2] C(\overline{X}), \delta_1, \delta_2, \delta_3)$, where $K[\mathbb{Z}_2]$ is a groups algebra of the group $\mathbb{Z}_2 = \{1, \vartheta | \vartheta^2 = 1\}$, the chain complex $(\mathcal{P}(\mathbb{Z}_2), d)$ is any projective resolvent of the trivial $K[\mathbb{Z}_2]$-module $K$, and the differential $\delta_3$ is defined by
\[
\delta_3 = (−1)^{n+m} d \otimes 1 : \mathcal{P}(\mathbb{Z}_2) t \otimes K[\mathbb{Z}_2] C(\overline{X})_{n,m} \to \mathcal{P}(\mathbb{Z}_2) t−1 \otimes K[\mathbb{Z}_2] C(\overline{X})_{n,m}.
\]
If we take as the projective resolvente \((P(Z_2), d)\) the standard free resolvente 
\[
(S(Z_2), d) : K[Z_2] \xleftarrow{1-\vartheta} K[Z_2] \xleftarrow{1+\vartheta} K[Z_2] \xleftarrow{1-\vartheta} K[Z_2] \xleftarrow{1+\vartheta} \cdots,
\]
then we obtain that the dihedral homology \(HD(X) = H(Z_2; (C(X), \delta_1, \delta_2))\) of any \(DF_\infty\)-module \((X, t, r, d, \tilde{\partial})\) is the homology of the chain complex associated with the triple chain complex \((D(X), \delta_1, \delta_2, \delta_3)\), where \(D(X)_{n,m,l} = C(X)_{n,m} = X_n, n \geq 0, m \geq 0, l \geq 0\), the differentials

\[
\delta_1 : D(X)_{n,m,l} = C(X)_{n,m} \to C(X)_{n,m-1} = D(X)_{n,m-1,l},
\]

\[
\delta_2 : D(X)_{n,m,l} = C(X)_{n,m} \to C(X)_{n-1,m} = D(X)_{n-1,m,l}
\]

were defined above, and the differential \(\delta_3 : D(X)_{n,m,l} \to D(X)_{n,m,l-1}\) is given by

\[
\delta_3 = \begin{cases} 
(-1)^n(1 + (-1)^l T_n), & m \equiv 0 \mod(4), \\
(-1)^{n+1}(1 + (-1)^{l+1} T_n), & m \equiv 1 \mod(4), \\
(-1)^n(1 + (-1)^l T_n), & m \equiv 2 \mod(4), \\
(-1)^{n+1}(1 + (-1)^{l+1} T_n), & m \equiv 3 \mod(4).
\end{cases}
\]

The chain complex associated with the triple chain complex \((D(X), \delta_1, \delta_2, \delta_3)\) we denote by \((\text{Tot}(D(X)), \tilde{\delta})\), where \(\tilde{\delta} = \delta_1 + \delta_2 + \delta_3\).

Note that if a \(DF_\infty\)-module \((X, t, r, d, \tilde{\partial})\) is a dihedral module with simplicial faces \((X, t, r, d, \partial_i)\), then the triple chain complex \((D(X), \delta_1, \delta_2, \delta_3)\) coincides with well-known \([2]\) the triple chain complex, which computes the dihedral homology of the dihedral module with simplicial faces \((X, t, r, d, \partial_i)\).

Now, let us describe the convenient method of a computation of the dihedral homology of dihedral modules with \(\infty\)-simplicial faces over fields of characteristic zero.

Given an arbitrary \(DF_\infty\)-module \((X, t, r, d, \tilde{\partial})\), consider the corresponding chain complexes \((X, b)\) and \((\bar{X}, -b)\). It easily follows from the mentioned above relations 
\[
b(1 - T_n) = (1 - \bar{T}_{n-1})b, \quad bR_n = \bar{R}_{n-1}b, \quad n \geq 0,
\]
that the chain complex \((M(X), b)\), where

\[
M(X)_n = X_n / \left(\text{Im}(1 - T_n) + \text{Im}(1 - \bar{T}_n)\right), \quad n \geq 0,
\]

is well defined.

The following assertion describes the convenient method of a computation of the dihedral homology of dihedral modules with \(\infty\)-simplicial faces over fields of characteristic zero.

**Theorem 1.1.** The dihedral homology \(HD(X)\) of any \(DF_\infty\)-module \((X, t, r, d, \tilde{\partial})\) over a field of characteristic zero is isomorphic to the homology of the chain complex \((M(X), b)\).

**Proof.** First, given any \(DF_\infty\)-module \((X, t, r, d, \tilde{\partial})\), we note that the specified above action of the group \(Z_2\) on the chain bicomplex \((C(X), \delta_1, \delta_2)\) induces the action of the group \(Z_2\) on the chain complex \((\text{Tot}(C(X)), \delta)\). This action at any element \((x_0, \ldots, x_n) \in \text{Tot}(C(X))_n\) is given by \(\vartheta(x_0, \ldots, x_n) = (\vartheta(x_0), \ldots, \vartheta(x_n))\), where \(x_i \in C(X)_{i,n-1}\) for \(0 \leq i \leq n\). It is easy see that \(HD(X) = H(Z_2; (\text{Tot}(C(X)), \delta))\). Now,
given any $DF_\infty$-module $(X, t, r, d, \tilde{\partial})$, consider chain complexes $(\overline{X}, b)$ and $(\overline{X}, -b')$. It follows from the relations $b(1 - T_n) = (1 - T_{n-1})b'$, $n \geq 0$, that the chain complex $(L(\overline{X}), b)$, where $L(\overline{X})_n = \overline{X}_n/\text{Im}(1 - T_n)$ for $n \geq 0$, is well defined. The first and third equalities in (1.10) imply that on the chain complex $(L(\overline{X}), b)$ correctly acting the group $\mathbb{Z}_2$ by means of the automorphism $\vartheta : L(\overline{X})_* \to L(\overline{X})_*$ of the order two, which at any element $x \in L(\overline{X})_n$, $n \geq 0$, is given by $\vartheta(x) = \overline{T}_n(x)$. Now, we consider the chain bicomplex

$$(P'(\overline{X}), d_0, d_1) = (S(\mathbb{Z}_2) \otimes_{K[\mathbb{Z}_2]} \text{Tot}(C(\overline{X})), d_0, d_1), \quad d_0 = 1 \otimes \delta,$$

$$d_1(a \otimes x) = (-1)^m d(a) \otimes x, \quad a \otimes x \in S(\mathbb{Z}_2)_n \otimes_{K[\mathbb{Z}_2]} \text{Tot}(C(\overline{X}))_m, \quad n \geq 0, \quad m \geq 0,$$

where $(S(\mathbb{Z}_2), d)$ is the specified above standard free resolvente. Moreover, consider the chain bicomplex

$$(P''(\overline{X}), d_0, d_1) = (S(\mathbb{Z}_2) \otimes_{K[\mathbb{Z}_2]} L(\overline{X}), d_0, d_1), \quad d_0 = 1 \otimes b,$$

$$d_1(a \otimes x) = (-1)^m d(a) \otimes x, \quad a \otimes x \in S(\mathbb{Z}_2)_n \otimes_{K[\mathbb{Z}_2]} L(\overline{X})_m, \quad n \geq 0, \quad m \geq 0,$$

where $(S(\mathbb{Z}_2), d)$ is the same as above. It was shown in [26] that over a field of characteristic zero the map of differential modules

$$g : (\text{Tot}(C(\overline{X})), \delta) \to (L(\overline{X}), b), \quad g(x_0, \ldots, x_n) = [x_n] = x_n + \text{Im}(1 - T_n),$$

where $(x_0, \ldots, x_n) \in \text{Tot}(C(\overline{X}))_n$, induces the isomorphism of homology modules. Clearly that the chain map $g$ is an $\mathbb{Z}_2$-equivariant map and, consequently, defines the map of chain bicomplexes

$$G = 1 \otimes g : (P'((\overline{X}), d_0, d_1) \to (P''(\overline{X}), d_0, d_1).$$

Now, we consider the spectral sequences $\{(E'_i, d_i)\}_{i \geq 0}$ and $\{(E''_i, d_i)\}_{i \geq 0}$ that corresponds to the bicomplexes $(P'(\overline{X}), d_0, d_1)$ and $(P''(\overline{X}), d_0, d_1)$, where $(E'_0, d_0) = (P'(\overline{X}), d_0)$ and $(E''_0, d_0) = (P''(\overline{X}), d_0)$. Since $G$ is a map of chain bicomplexes, this map induces the map of spectral sequences $\{G_i : E'_i \to E''_i\}_{i \geq 0}$, where $G_0 = 1 \otimes g$. The map $g$ over a field characteristic zero induces the isomorphism of homology modules. Therefore the map $G_0 = 1 \otimes g$ induces the isomorphism of bigraded homology modules. It implies that the map $G_1 : E'_{i} \to E''_{i}$ is an isomorphism of bigraded modules. By using the comparison theorem of spectral sequences we obtain that all maps $G_i : E'_i \to E''_i$, $1 \leq i \leq \infty$, are isomorphisms of bigraded modules. Now, we note that over any field the limit term $E'_\infty$ of the spectral sequence $\{(E'_i, d_i)\}_{i \geq 0}$ is isomorphic to the homology of the chain complex that associated with the chain bicomplex $(P'(\overline{X}), d_0, d_1)$. Similarly we have that over any field the limit term $E''_\infty$ of the spectral sequence $\{(E''_i, d_i)\}_{i \geq 0}$ is isomorphic to the homology of the chain complex that associated with the chain bicomplex $(P''(\overline{X}), d_0, d_1)$. It follows that the dihedral homology $HD(X)$ and the homology of the chain complex that associated with the chain bicomplex $(P'(\overline{X}), d_0, d_1)$ are isomorphic. Thus we have the isomorphism

$$HD(X) = H(\mathbb{Z}_2; (L(\overline{X}), b)).$$
Let us show that there is the isomorphism \( H(\mathbb{Z}_2; (L(X), b)) = H(M(X), b) \). Since we have the isomorphism of modules \( P''(\overline{X})_{n,m} = S(\mathbb{Z}_2)_n \otimes_{K[\mathbb{Z}_2]} L(\overline{X})_m = L(\overline{X})_{m,n} \), \( n \geq 0, m \geq 0 \), the chain bicomplex \( (P''(\overline{X}), d_0, d_1) \) can be identified with the chain bicomplex \( (N(\overline{X}), d_0', d_1') \) that is given by

\[
N(\overline{X})_{n,m} = L(\overline{X})_n, \quad n \geq 0, \quad m \geq 0,
\]

\[
d_0' = 1 + (-1)^m R_n : N(\overline{X})_{n,m} \to N(\overline{X})_{n,m-1}, \quad d_1' = b : N(\overline{X})_{n,m} \to N(\overline{X})_{n-1,m}.
\]

Consider the spectral sequence \( \{E_i, d_i\} \) of the chain bicomplex \( (N(\overline{X}), d_0', d_1') \), where \( (E_0, d_0) = (N(\overline{X}), d_0') \). Let us calculate \( E_1)_{n,m} = H_m(N(\overline{X})_{n,\bullet}, d_0') \). Clearly, we have \( H_0(N(\overline{X})_{n,\bullet}, d_0') = M(\overline{X})_n \). Let us show that \( H_m(N(\overline{X})_{n,\bullet}, d_0') = 0 \) for all \( m \geq 1 \). Since we have the equality of chain complexes \( (N(\overline{X})_{n,\bullet}, d_0') = (N(\overline{X})_{n,\bullet + 1}, d_0) \), it suffices to show that

\[
H_1(N(\overline{X})_{n,\bullet}, d_0') = \ker(1 - R_n)/\text{im}(1 + R_n) = 0.
\]

Suppose that an element \( a \in N(\overline{X})_{n,1} \) satisfies the condition \( d_0'(a) = (1 - R_n)(a) = 0 \). Consider the element \( c = (1/2)a \in N(\overline{X})_{n,2} = L(\overline{X})_n \). Since \( R_n(a) = a \), we have the equality \( d_0'(c) = (1/2)(1 + R_n)(a) \) which means that \( H_1(N(\overline{X})_{n,\bullet}, d_0') = 0 \). Thus, we have \( (E_1)_{n,0} = M(\overline{X})_n, n \geq 0 \), and \( (E_1)_{n,m} = 0, n \geq 0, m \geq 1 \). It is clear that the differential \( d_1 : (E_1)_{n,0} \to (E_1)_{n-1,0} \) induced by the differential \( d_1 : N(\overline{X})_{n,0} \to N(\overline{X})_{n-1,0} \) coincides with the differential \( b : M(\overline{X})_n \to M(\overline{X})_{n-1} \). This, together with the fact that the limit term \( E_\infty = E_2 = H(M(\overline{X}), b) \) of the spectral sequence \( \{E_i, d_i\} \) over a field is isomorphic to the homology of the chain complex that is associated with the chain bicomplex \( (N(\overline{X}), d_0', d_1') \), implies that there is the isomorphism \( H(\mathbb{Z}_2; (L(X), b)) = H(M(\overline{X}), b) \). Thus, \( HD(X) = H(\mathbb{Z}_2; (L(X), b)) = H(M(\overline{X}), b) \) and, consequently, we obtain the required isomorphism \( HD(X) = H(M(\overline{X}), b) \).}

Now, we proceed to the notion of a reflexive module with \( \infty \)-simplicial faces.

By a reflexive bigraded module \( (X, r) \) we mean any bigraded module \( X \) together with a family of module maps \( r = \{r_n : X_{n,\bullet} \to X_{n,\bullet}\} \), \( n \geq 0 \), satisfying the conditions

\[
r_n^2 = 1_{X_{n,\bullet}}, \quad n \geq 0.
\]

In other words, on each graded module \( X_{n,\bullet}, n \geq 0 \), the group \( \mathbb{Z}_2 \) of order 2 with generators \( r_n \) acts on the left.

In what follows, we use the term reflexive differential module for any triple \( (X, r, d) \), where \( (X, r) \) is a reflexive bigraded module, \( (X, d) \) is a differential module, and the conditions \( dr_n = r_n d \) for all \( n \geq 0 \) holds.

Now, let us recall that a reflexive module with simplicial faces [2] is defined as a reflexive differential module \( (X, r, d) \) together with a family of module maps \( \partial_i : X_{n,\bullet} \to X_{n-i,\bullet}, 0 \leq i \leq n \), with respect to which the triple \( (X, d, \partial_i) \) is a differential module with simplicial faces and, moreover, the relations

\[
\partial_i r_n = r_{n-i} \partial_{n-i}, \quad 0 \leq i \leq n,
\]

for each \( n \geq 0 \) are true.
It is easy to see that if in the definition of a dihedral module with simplicial faces we remove the family of automorphisms $t_n : X_{n, \bullet} \to X_{n, \bullet}$, $n \geq 0$, and the relations $\partial_i t_n = t_{n-1} \partial_{i-1}$, $0 < i \leq n$, $\partial_0 t_n = \partial_n$, $n \geq 0$, then we obtain the definition of a reflexive module with simplicial faces.

**Definition 1.3.** A reflexive module with $\infty$-simplicial faces or, more briefly, a $\text{RF}_\infty$-module, is any quadruple $(X, r, d, \tilde{\partial})$, where $(X, r, d)$ is a reflexive differential module and $(X, d, \tilde{\partial})$ is a differential module with $\infty$-simplicial faces related by the relations (1.3).

In what follows, we refer to the family of maps $\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)} : X_{n, \bullet} \to X_{n-k, \bullet+k-1}\}$ as the $F_\infty$-differential of the $\text{RF}_\infty$-module $(X, r, d, \tilde{\partial})$. The maps $\partial_{(i_1, \ldots, i_k)}$ that form the $F_\infty$-differential of a $\text{RF}_\infty$-module $(X, r, d, \tilde{\partial})$ are called the $\infty$-simplicial faces of this $\text{RF}_\infty$-module.

It is obvious that if in the definition 1.1 we remove the family of automorphisms $t_n : X_{n, \bullet} \to X_{n, \bullet}$, $n \geq 0$, and the relations (1.2) then we obtain the definition 1.3. Therefore each $D_{\text{RF}}$-module $(X, t, r, d, \tilde{\partial})$ always defines the $\text{RF}_\infty$-module $(X, r, d, \tilde{\partial})$.

Simple examples of $\text{RF}_\infty$-modules are reflexive modules with simplicial faces. Indeed, given any reflexive module with simplicial faces $(X, r, d, \partial)$, we can define an $F_\infty$-differential $\tilde{\partial} = \{\partial_{(i_1, \ldots, i_k)}\}$ by setting $\partial_{(i)} = \partial_i$, $i \geq 0$, and $\partial_{(i_1, \ldots, i_k)} = 0$, $k > 1$, thus obtaining the $\text{RF}_\infty$-module $(X, r, d, \tilde{\partial})$.

Given any reflexive module with $\infty$-simplicial faces $(X, r, d, \tilde{\partial})$, consider the specified above $D_\infty$-module $(X, d^0)$ and the family of maps

$$R_n = (-1)^{n(n+1)/2} r_n : X_{n, \bullet} \to X_{n, \bullet}, \quad n \geq 0.$$  

In the same way as it was made in the case of $D_{\text{RF}}$-modules we obtain the relations $d^0_i R_n = R_{n-i} d^0_i$, $i \geq 0$, $n \geq 0$. Consider the chain complex $(X, b)$, which corresponds to the $D_\infty$-module $(X, d^0)$, and the family of maps

$$\overline{R}_n = (R_0 + R_1 + \ldots + R_n) : X_n \to X_n, \quad n \geq 0.$$  

By using $d^0_i R_n = R_{n-i} d^0_i$, $i \geq 0$, $n \geq 0$, we obtain the relations $b \overline{R}_n = \overline{R}_{n-1} b$, $n \geq 0$. This relations say us that there is a left action of the group $\mathbb{Z}_2 = \{1, \vartheta \mid \vartheta^2 = 1\}$ on the chain complex $(X, b)$ of any $\text{RF}_\infty$-module $(X, r, d, \tilde{\partial})$. This left action is defined by means of the automorphism $\vartheta : X \to X$ of the order two, which at any element $x \in X_n$ is given by $\vartheta(x) = \overline{R}_0(x)$.

**Definition 1.4.** We define the reflexive homology $HR(X)$ of a reflexive module with $\infty$-simplicial faces $(X, r, d, \tilde{\partial})$ as the hyperhomology $H(\mathbb{Z}_2; (X, b))$ of the group $\mathbb{Z}_2$ with coefficients in $(X, b)$ relative to the specified above action of the group $\mathbb{Z}_2$ on the chain complex $(X, b)$.

It is easy to see that if calculate the hyperhomology $H(\mathbb{Z}_2; (X, b))$ by using the specified above standard free resolvente $(S(\mathbb{Z}_2), d)$, then we obtain that the reflexive homology $HR(X) = H(\mathbb{Z}_2; (X, b))$ of any $\text{RF}_\infty$-module $(X, r, d, \tilde{\partial})$ is the homology of the chain complex associated with the chain bicomplex $(R(X), b, D)$ that is defined by

$$R(X)_{n,m} = X_n, \quad n \geq 0, \quad m \geq 0,$$  

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\[
D = (-1)^n(1 + (-1)^mR_n) : R(X)_{n,m} \to R(X)_{n,m-1},
\]
\[
b : R(X)_{n,m} = X_n \to X_{n-1} = R(X)_{n-1,m}.
\]
The chain complex associated with the chain bicomplex \((R(X), b, D)\) we denote by \((\text{Tot}(R(X)), \hat{D})\), where \(\hat{D} = b + D\).

Note that if a \(RF_{\infty}\)-module \((X, r, d, \bar{\partial})\) is a reflexive module with simplicial faces \((X, r, d, \partial)\), then the chain bicomplex \((R(X), b, D)\) coincides with well-known \([2]\) the chain bicomplex, which computes the reflexive homology of the reflexive differential module with simplicial faces \((X, r, d, \partial)\).

Consider the above chain complex \((X, b)\) that is determined by any \(RF_{\infty}\)-module \((X, r, d, \bar{\partial})\). It follows from the mentioned above relations \(bR_n = \bar{R}_{n-1}b, n \geq 0\), that chain complex \((N(X), b)\), where
\[
N(X)_n = X_n / \text{Im}(1 - \bar{R}_n), \quad n \geq 0,
\]
is well defined.

The following assertion describes the convenient method of a computation of the reflexive homology of reflexive modules with \(\infty\)-simplicial faces over fields of characteristic zero. This assertion proved similarly to the theorem 1.1.

**Theorem 1.2.** The reflexive homology \(HR(X)\) of any \(RF_{\infty}\)-module \((X, r, d, \bar{\partial})\) over a field of characteristic zero is isomorphic to the homology of the chain complex \((N(X), b)\).

\[\blacksquare\]

§ 2. Dihedral and reflexive homology of involutive \(A_{\infty}\)-algebras

First, following \([28]\) (see also \([29]\)), we recall that an \(A_{\infty}\)-algebra \((A, d, \pi_n)\) is any differential module \((A, d)\) with \(A = \{A_n\}, n \in \mathbb{Z}, n \geq 0, d : A_* \to A_{*-1}\), equipped with a family of maps \(\{\pi_n : (A^\otimes(n+2))_* \to A_{*-n}\}, n \geq 0\), satisfying the following relations for any integer \(n \geq 1\):
\[
d(\pi_{n-1}) = \sum_{m=1}^{n-1} \sum_{t=1}^{m+1} (-1)^{t(n-m)+n+1} \pi_{m-1}(1 \otimes \ldots \otimes 1 \otimes \pi_{n-m-1} \otimes 1 \otimes \ldots \otimes 1), \quad (2.1)
\]
where \(d(\pi_{n-1}) = d\pi_{n-1} + (-1)^n \pi_{n-1}d\). For example, at \(n = 1, 2, 3\) the relations (2.1) take the forms
\[
d(\pi_0) = 0, \quad d(\pi_1) = \pi_0(\pi_0 \otimes 1) - \pi_0(1 \otimes \pi_0),
\]
\[
d(\pi_2) = \pi_0(\pi_1 \otimes 1 + 1 \otimes \pi_1) - \pi_1(\pi_0 \otimes 1 \otimes 1 - 1 \otimes \pi_0 \otimes 1 + 1 \otimes 1 \otimes \pi_0).
\]

Now, we recall the notion of an involutive \(A_{\infty}\)-algebra \([5]\), which generalizes the notion of a differential associative algebra with an involution. An involutive \(A_{\infty}\)-algebra \((A, d, \pi_n, \ast)\) is defined as an \(A_{\infty}\)-algebra \((A, d, \pi_n)\) together with the automorphism of graded modules \(\ast : A_* \to A_*\) (the notation \(\ast(a) = a^\ast\) for \(a \in A\)), which at any elements \(a \in A\) and \(a_0, a_1, \ldots, a_n, a_{n+1} \in A\) satisfies the conditions
\[
a^{**} = a, \quad d(a^\ast) = d(a)^\ast,
\]
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\[
\pi_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1})^* = (-1)^\varepsilon \pi_n(a_{n+1}^* \otimes a_n^* \otimes \ldots \otimes a_1^* \otimes a_0^*),
\]
(2.2)
\[
\varepsilon = \frac{n(n+1)}{2} + \sum_{0 \leq i < j \leq n+1} |a_i||a_j|, \quad n \geq 0,
\]
where \(|a| = q\) means that \(a \in A_q\).

Given any involutive \(A_\infty\)-algebra \((A, d, \pi_n, \ast)\), consider the dihedral differential module \((\varepsilon M(A), t, r, d)\), where \(q = \pm 1\), defined by
\[
\begin{align*}
\varepsilon M(A) & = \{\varepsilon M(A)_{(n,m)}\}, \quad \varepsilon M(A)_{(n,m)} = (A^{\otimes (n+1)})_{(m)}, \quad n \geq 0, \quad m \geq 0, \\
t_n(a_0 \otimes \ldots \otimes a_n) & = (-1)^{|a_0|(|a_0|+\ldots+|a_{n-1}|)}a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}, \\
r_n(a_0 \otimes \ldots \otimes a_n) & = q(-1)^{\sum_{0 < i < j < n} |a_i||a_j|} a_0^* \otimes a_n^* \otimes a_{n-1}^* \ldots \otimes a_1^*, \\
d(a_0 \otimes \ldots \otimes a_n) & = \sum_{i=0}^n (-1)^{|a_0|+\ldots+|a_{i-1}|} a_0 \otimes \ldots \otimes a_{i-1} \otimes d(a_i) \otimes a_{i+1} \otimes \ldots \otimes a_n.
\end{align*}
\]

It is easy to verify that maps \(t_n\) and \(r_n\) satisfy the conditions \(t_n^{n+1} = 1\), \(r_n^2 = 1\), \(t_nr_n = r_nt_n^{-1}\) for all \(n \geq 0\). Therefore, \((\varepsilon M(A), t, r, d)\) is indeed a dihedral differential module. Now, consider the family of maps \(\tilde{\partial} = \{\tilde{\partial}_{(i_1, \ldots, i_k)} : \varepsilon M(A)_{n,p} \to \varepsilon M(A)_{n-k,p+k-1}\}, \quad 0 \leq i_1 < \ldots < i_k \leq n, \quad n \geq 0, \quad p \geq 0\), defined by
\[
\tilde{\partial}_{(i_1, \ldots, i_k)} = \begin{cases} 
(-1)^{k(p-1)} \otimes_{i_1}^j \pi_{k-1} \otimes 1^{\otimes (n-k-j)}, & \text{if } 0 \leq j \leq n-k \text{ and } (i_1, \ldots, i_k) = (j, j+1, \ldots, j+k-1); \\
(-1)^{q(k-1)} \partial_{(0,1,\ldots,k-1)} t_n^q, & \text{if } 1 \leq q \leq k \\
0, & \text{otherwise.}
\end{cases}
\]
(2.3)

Theorem 2.1. For any involutive \(A_\infty\)-algebra \((A, d, \pi_n, \ast)\), the specified above five-tuple \((\varepsilon M(A), t, r, d, \tilde{\partial})\) is a dihedral module with \(\infty\)-simplicial faces.

Proof. In [23] it was shown that the quadruple \((\varepsilon M(A), t, r, d, \tilde{\partial})\) is a cyclic differential module with \(\infty\)-simplicial faces. Therefore, it remains only to check that the maps \(\tilde{\partial}_{(i_1, \ldots, i_k)} \in \tilde{\partial}\) defined by (2.3) satisfy the relations (1.3). Consider several cases.  

1. Suppose that \((i_1, \ldots, i_k) = (j, j+1, \ldots, j+k-1)\), where \(1 \leq j \leq n-k\). In this case, on the one hand, we have at any element \(a_0 \otimes \ldots \otimes a_n \in (A^{n+1})_p\) the equalities
\[
\begin{align*}
\partial_{(j, j+1, \ldots, j+k-1)} r_n(a_0 \otimes \ldots \otimes a_n) & = \partial(-1)\alpha \partial(\otimes_{j+1}^{j+k-1}) (a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^*) = \\
& = \partial(-1)^{a+k(p-1)} (1 \otimes \pi_{k-1} \otimes 1^{\otimes (n-k-j)}) (a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^*) = \\
& = \partial(-1)^{a+k(p-1)+\beta} a_0^* \otimes a_n^* \otimes \ldots \otimes a_{n-j-2}^* \otimes \\
& \otimes \pi_{k-1} (a_{n-j+1}^* \otimes \ldots \otimes a_{n-j-k+1}^*) \otimes a_{n-j-k}^* \otimes \ldots \otimes a_1^*;
\end{align*}
\]
where \(\alpha = \sum_{0 < i < j \leq n} |a_i||a_j|\), \(\beta = (k-1)(|a_0| + |a_n| + \ldots + |a_{n-j+2}|)\).
On the other hand, we have the equalities

\[
(\mathbf{1})^{n-k} r_{n-k} \partial_{(n-j-k+1,n-j-k+2,\ldots,n-j)}(a_0 \otimes \ldots \otimes a_n) = \\
= (\mathbf{1})^{n-k} r_{n-k} (1 \otimes (n-j-k+1) \otimes \pi_{k-1} \otimes 1 \otimes (j-1)) (a_0 \otimes \ldots \otimes a_n) = \\
= (\mathbf{1})^{n-k} r_{n-k} (a_0 \otimes \ldots \otimes a_{n-j-k} \otimes \pi_{k-1} (a_{n-j-k+1} \otimes \ldots \otimes a_{n-j+1}) \otimes \\
\otimes a_{n-j+2} \otimes \ldots \otimes a_n) = \\
= \partial (-1)^{(\mathbf{1})^{n-k}/2+k(p-1)+\gamma+\delta} a_0^* \otimes \ldots \otimes a_{n-j+2} \otimes \\
\otimes \pi_{k-1} (a_{n-j-k+1} \otimes \ldots \otimes a_{n-j+1})^* \otimes a_{n-j-k} \otimes \ldots \otimes a_1^* = \\
= \partial (-1)^{(\mathbf{1})^{n-k}/2+k(p-1)+\gamma+\mu} a_0^* \otimes \ldots \otimes a_{n-j+2} \otimes \\
\otimes \pi_{k-1} (a_{n-j+1}^* \otimes \ldots \otimes a_{n-j-k+1}^*) \otimes a_{n-j-k} \otimes \ldots \otimes a_1^*,
\]

where

\[
\gamma = (k-1)(|a_0| + |a_1| + \ldots + |a_{n-j-k}|),
\]

\[
\delta = \sum_{0 \leq i < j \leq n} |a_i||a_j| - \sum_{n-j-k+1 \leq s \leq t \leq n-j+1} |a_s||a_t| + \\
+ (k-1)(|a_1| + \ldots + |a_{n-j-k}| + |a_{n-j+2}| + \ldots + |a_n|),
\]

\[
\mu = \frac{k(k-1)}{2} + \sum_{n-j-k+1 \leq s < t \leq n-j+1} |a_s||a_t|.
\]

Since \(\alpha + \beta \equiv (k-1)/2 + \gamma + \delta + \mu \mod (2)\), we obtain the required relation

\[
\partial_{(j,j+1,\ldots,j+k-1)} r_n = (-1)^{k(k-1)/2} r_{n-k} \partial_{(n-j-k+1,n-j-k+2,\ldots,n-j)}.
\]

2). Suppose that \((i_1, \ldots, i_k) = (0, 1, \ldots, k-1)\). In this case, on the one hand, we have at any element \(a_0 \otimes \ldots \otimes a_n \in (A^{n+1})_p\) the equalities

\[
\partial_{(0,1,\ldots,k-1)} r_n (a_0 \otimes \ldots \otimes a_n) = \partial (-1)^{\alpha} \partial_{(0,1,\ldots,k-1)} (a_0^* \otimes a_1^* \otimes \ldots \otimes a_1^*) = \\
= \partial (-1)^{\alpha+k(p-1)} (\pi_{k-1} \otimes 1 \otimes (n-k)) (a_0^* \otimes a_1^* \otimes \ldots \otimes a_1^*) = \\
= \partial (-1)^{\alpha+k(p-1)} \pi_{k-1} (a_0^* \otimes a_1^* \otimes \ldots \otimes a_{n-k+1}) \otimes a_{n-k} \otimes \ldots \otimes a_1^*,
\]

where \(\alpha\) is the same as in (1). On the other hand, since the second equality in (2.3) at \(q = k\) becomes the equalities \(\partial_{(n-k-1,k-1)} = \partial_{(0,1,\ldots,k-1)} t_n^k\), we have the equality

\[
(-1)^{(k-1)/2} r_{n-k} \partial_{(n-k+1,n-k+2,\ldots,n)} (a_0 \otimes \ldots \otimes a_n) = \\
= (-1)^{(k-1)/2} r_{n-k} \partial_{(0,1,\ldots,k-1)} (a_0 \otimes \ldots \otimes a_n) = \\
= (-1)^{(k-1)/2} r_{n-k} \partial_{(0,1,\ldots,k-1)} (a_0 \otimes \ldots \otimes a_n) = \\
= (-1)^{(k-1)/2} \nu r_{n-k} \partial_{(0,1,\ldots,k-1)} (a_{n-k+1} \otimes \ldots \otimes a_{n-k}) = \\
= (-1)^{(k-1)/2} \nu + k(p-1) r_{n-k} (\pi_{k-1} \otimes 1 \otimes (n-k)) (a_{n-k+1} \otimes \ldots \otimes a_{n-k}) = \\
= (-1)^{(k-1)/2} \nu + k(p-1) r_{n-k} (\pi_{k-1} (a_{n-k+1} \otimes \ldots \otimes a_{n-k}) \otimes a_0^* \otimes a_1^* \otimes \ldots \otimes a_1^* = \\
= \partial (-1)^{(k-1)/2} \nu + k(p-1) + \delta \pi_{k-1} (a_{n-k+1} \otimes \ldots \otimes a_{n-k})^* \otimes a_{n-k}^* \otimes \ldots \otimes a_1^*.
\]

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\[ q(-1)^{k(k-1)/2+\nu+k(p-1)+\vartheta} \pi_{k-1}(a_0^* \otimes a_n^* \otimes a_{n-k+1}^* \otimes a_{n-k}^* \otimes \ldots \otimes a_1^*), \]

where

\[ \nu = (|a_{n-k+1}| + \ldots + |a_n|)(|a_0| + \ldots + |a_{n-k}|), \quad \vartheta = \sum_{0<i<j\leq n-k} |a_i||a_j|, \]

\[ \lambda = \frac{k(k-1)}{2} + \sum_{n-k+1\leq i<j\leq n} |a_i||a_j| + (|a_{n-k+1}| + \ldots + |a_n|)|a_0|. \]

Since \((k(k-1)/2 + \nu + \vartheta + \lambda \equiv \alpha \mod(2),\) we obtain the required relation

\[ \partial(0,1,\ldots,k-1) r_n = (-1)^{k(k-1)/2} r_{n-k} \partial(n-k+1,n-k+2,\ldots,n). \]

3). Suppose that \((i_1, \ldots, i_k) = (0, 1, \ldots, k - q - 1, n - q + 1, n - q + 2, \ldots, n),\)
where \(1 \leq q \leq k.\) The second equality in (2.3) follows that

\[ \partial(0,1,\ldots,k-q-1,n-q+1,n-q+2,\ldots,n) = (-1)^q \partial(0,1,\ldots,k-1) t_n^q. \]

By using \(t_n^q r_n = r_n t_n^q = r_n t_n^{q+1} = r_n t_n^{q+k-1}, \)
we have

\[ \partial(0,1,\ldots,k-q-1,n-q+1,n-q+2,\ldots,n) r_n = (-1)^q \partial(0,1,\ldots,k-1) t_n^q r_n = \]

\[ = (-1)^q \partial(0,1,\ldots,k-1) r_n t_n^{k-1} = \]

\[ = (-1)^q (k-1) \partial(n-k+1,n-k+2,\ldots,n) t_n^{k-1} = \]

\[ = (-1)^q (k-1) \partial(0,1,\ldots,k-1) t_n^{k-1}. \]

The second equality in (2.3) follows that

\[ \partial(0,1,\ldots,k-1) t_n^{k-1} = (-1)^{k-q} \partial(0,1,\ldots,q-1,n-k+q+1,\ldots,n). \]

Since \(q(k-1) + (k(k-1)/2) + (k-q)(k-1) \equiv (k(k-1)/2) \mod(2),\) we obtain the required relation

\[ \partial(0,1,\ldots,k-q-1,n-q+1,n-q+2,\ldots,n) r_n = (-1)^{k(k-1)/2} r_{n-k} \partial(0,1,\ldots,q-1,n-k+q+1,\ldots,n). \]

Thus, all maps \(\partial(i_1,\ldots,i_k) \in \tilde{\partial}\) satisfy the conditions (1.3) and, consequently, the five-tuple \((\epsilon M(A), t, r, d, \tilde{\partial})\) is a \(DF_{\infty}\)-module. \(\blacksquare\)

Note that if an involutive \(A_{\infty}\)-algebra \((A, d, \pi_n, \ast)\) is a differential involutive associative algebra \((A, d, \pi, \ast)\), where \(\pi = \pi_0\) and \(\pi_n = 0, n > 0,\) then the \(DF_{\infty}\)-module \((\epsilon M(A), t, r, d, \tilde{\partial})\) coincides with the well-known [2] dihedral module with simplicial faces that defined by a differential involutive associative algebra \((A, d, \pi, \ast)\).

**Definition 2.1.** We define the dihedral homology \(\epsilon HD(A)\) of an arbitrary involutive \(A_{\infty}\)-algebra \((A, d, \pi_n, \ast)\) as the dihedral homology \(HD(\epsilon M(A))\) of the \(DF_{\infty}\)-module \((\epsilon M(A), t, r, d, \tilde{\partial})\).
Thus, the dihedral homology $^e HD(A)$ of an involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ is the homology of the chain complex $(\text{Tot}(D(\hat{\varepsilon}M(A))), \hat{\delta})$ associated with the triple chain complex $(D(\hat{\varepsilon}M(A)), \delta_1, \delta_2, \delta_3)$.

Note that if an involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ is a differential involutive associative algebra $(A, d, \pi, \ast)$, where $\pi = \pi_0$ and $\pi_n = 0$, $n > 0$, then the triple chain complex $(D(\hat{\varepsilon}M(A)), \delta_1, \delta_2, \delta_3)$ coincides with the well-known [2] triple chain complex that computes the dihedral homology of a differential involutive associative algebra $(A, d, \pi, \ast)$.

If apply the theorem 1.1 to the $DF_\infty$-module $(^e M(A), t, r, d, \tilde{\partial})$, then we obtain the following assertion.

**Corollary 2.1.** The dihedral homology $^e HD(A)$ of any involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ over a field of characteristic zero is isomorphic to the homology of the chain complex $(M(^e M(A)), b)$. ■

Note that the chain complex $(M(^e M(A)), b)$ was defined in [5] without employing dihedral modules with $\infty$-simplicial faces. In [5], where the ground ring is a field of characteristic zero, the homology of this chain complex was referred to as the dihedral homology of the involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$. Thus, the corollary 2.1 implies that, over fields of characteristic zero, the definition 2.1 of the dihedral homology of an involutive $A_\infty$-algebra is equivalent to that given in [5].

As it was said above, an arbitrary $DF_\infty$-module always can be considered as a $RF_\infty$-module. Therefore, the $DF_\infty$-module $(^e M(A), t, r, d, \tilde{\partial})$ that defined by any involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ always determines the $RF_\infty$-module $(^e M(A), r, d, \tilde{\partial})$.

Note that if an involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ is a differential involutive associative algebra $(A, d, \pi, \ast)$, where $\pi = \pi_0$ and $\pi_n = 0$, $n > 0$, then the $RF_\infty$-module $(^e M(A), r, d, \tilde{\partial})$ coincides with the well-known [2] reflexive module with simplicial faces that defined by a differential involutive associative algebra $(A, d, \pi, \ast)$.

**Definition 2.1.** We define the reflexive homology $^e HR(A)$ of an arbitrary involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ as the reflexive homology $HR(M(A))$ of the $RF_\infty$-module $(^e M(A), r, d, \tilde{\partial})$.

Thus, the reflexive homology $^e HR(A)$ of an involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ is the homology of the chain complex $(\text{Tot}(R(\hat{\varepsilon}M(A))), \hat{D})$ associated with the chain bicomplex $(R(\hat{\varepsilon}M(A)), b, D)$.

Note that if an involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ is a differential involutive associative algebra $(A, d, \pi, \ast)$, where $\pi = \pi_0$ and $\pi_n = 0$, $n > 0$, then the chain bicomplex $(R(\hat{\varepsilon}M(A)), b, D)$ coincides with the well-known [2] chain bicomplex that computes the reflexive homology of a differential involutive associative algebra $(A, d, \pi, \ast)$.

If apply the theorem 1.2 to the $RF_\infty$-module $(^e M(A), r, d, \tilde{\partial})$, then we obtain the following assertion.

**Corollary 2.2.** The reflexive homology $^e HR(A)$ of any involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$ over a field of characteristic zero is isomorphic to the homology of the chain complex $(N(^e M(A)), b)$. ■

§ 3. An exact sequence for the dihedral and the reflexive homology of involutive homotopy unital $A_\infty$-algebras
To introduce the notion of an involutive homotopy unital $A_\infty$-algebra, we need definitions and constructions related to the notions of a (nonsymmetric) operad and an algebra over an operad in the category of differential modules (see, e.g., [27]).

A differential family or, more briefly, a family, $\mathcal{E} = \{\mathcal{E}(j)\}_{j \geq 0}$ is any family of differential modules $(\mathcal{E}(j), d)$, $j \geq 0$. A morphism of families $f : \mathcal{E}' \to \mathcal{E}''$ is any family of maps $\alpha = \{\alpha(j) : (\mathcal{E}'(j), d) \to (\mathcal{E}''(j), d)\}_{j \geq 0}$ of differential modules. For any families $\mathcal{E}'$ and $\mathcal{E}''$, the family $\mathcal{E}' \times \mathcal{E}''$ is defined by

$$(\mathcal{E}' \times \mathcal{E}'')(j) = \bigoplus_{j_1 + \cdots + j_k = j} \mathcal{E}'(k) \otimes \mathcal{E}''(j_1) \otimes \cdots \otimes \mathcal{E}''(j_k), \quad j \geq 0.$$  

Clearly, the $\times$-product of families thus defined is associative i.e. given any families $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$, there is an isomorphism of families $\mathcal{E} \times (\mathcal{E}' \times \mathcal{E}'') \approx (\mathcal{E} \times \mathcal{E}') \times \mathcal{E}''$.

A (nonsymmetric) operad $(\mathcal{E}, \gamma)$ is any family $\mathcal{E}$ together with a morphism of families $\gamma : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ satisfying the condition $\gamma(\gamma \times 1) = \gamma(1 \times \gamma)$. Moreover, there exists an element $1 \in \mathcal{E}(1)_0$ such that $\gamma(1 \otimes e_j) = e_j$ for each $e_j \in \mathcal{E}(j)$, $j \geq 0$, and $\gamma(e_j \otimes 1 \otimes \cdots \otimes 1) = e_j$ for each $e_j \in \mathcal{E}(j)$, $j \geq 1$. In what follows, we write elements of the form $\gamma(e_k \otimes e_j \otimes \cdots \otimes e_{j_k})$ as $e_k(e_j) \otimes \cdots \otimes e_{j_k}$.

A morphism of operads $f : (\mathcal{E}', \gamma) \to (\mathcal{E}'', \gamma)$ is a morphism of families $f : \mathcal{E}' \to \mathcal{E}''$ satisfying the condition $f \gamma = \gamma(f \times f)$.

A canonical example of an operad is the operad $(\mathcal{E}_X, \gamma)$ defined by

$$((\mathcal{E}_X(j), d) = (\text{Hom}(X^\otimes j, X), d), \quad \gamma(f_k \otimes f_{j_1} \otimes \cdots \otimes f_{j_k}) = f_k(f_{j_1} \otimes \cdots \otimes f_{j_k})$$

for any differential module $(X, d)$.

An algebra over an operad $(\mathcal{E}, \gamma)$ or, more briefly, an $\mathcal{E}$-algebra $(X, d, \alpha)$ is defined as a differential module $(X, d)$ together with a fixed morphism of operads $\alpha : \mathcal{E} \to \mathcal{E}_X$.

An important example of an operad is the Stasheff operad $(A_\infty, \gamma)$. As a graded operad this is the free operad with generators $\pi_n \in A_\infty(n + 2)_n$, $n \geq 0$; at each generator $\pi_{n-1}$, $n \geq 1$, the differential is defined by (2.1).

It is easy to see that, given a differential module $(A, d)$, where $A = \{A_n\}$, $n \geq 0$, $d : A \to A_{*-1}$ defining the structure of an algebra $(A, d, \alpha)$ over the operad $(A_\infty, \gamma)$ on the differential module $(A, d)$ is equivalent to specifying a family of maps

$$\{\pi_n = \alpha(\pi_n) : (A^\otimes(n+2))_\bullet \to A_{\bullet + n}\}, \quad n \geq 0,$$

for which the relations (2.1) hold, i.e., to endowing $(A, d)$ with the structure of the $A_\infty$-algebra $(A, d, \pi_n)$.

Now, following [24] (see also [13]), we recall the notion of a homotopy unital $A_\infty$-algebra. Consider the operad $(A_u^\text{su} < u, h >, \gamma)$ introduced in [24]. As a graded operad, $(A_u^\text{su} < u, h >, \gamma)$ has the generators

$$\pi_n \in (A_u^\text{su} < u, h >)(n + 2)_n, \quad n \geq 0, \quad 1^\text{su} \in (A_u^\text{su} < u, h >)(0)_0,$$

$$u \in (A_u^\text{su} < u, h >)(0)_0, \quad h \in (A_u^\text{su} < u, h >)(0)_1,$$

which satisfy the relations

$$\pi_0(1^\text{su} \otimes 1) = 1, \quad \pi_0(1 \otimes 1^\text{su}) = 1, \quad \pi_n(1^\otimes k \otimes 1^\text{su} \otimes 1^\otimes(n-k+1)) = 0, \quad n > 0, \quad (3.1)$$
where \(0 \leq k \leq n+1\); the differential is defined at the generators by the formulae (2.1) and
\[
d(1^u) = 0, \quad d(u) = 0, \quad d(h) = u - 1^u.
\]

The operad \((A_{\infty}^u < u, h>, \gamma)\) contains the suboperad \((A_{\infty}^{hu}, \gamma)\) with generators
\[
\tau_0^0 = u \in (A_{\infty}^u < u, h>)(0)_0, \quad \tau_n^0 = \pi_{n-1} \in (A_{\infty}^u < u, h>)(n+1)_{n-1}, \quad n \geq 1,
\]
\[
\tau_n^{j_1, \ldots, j_l} = \prod_{j_1}^{j_2} \prod_{j_l \ldots j_q} (\tau_0^0 \otimes 1^u \otimes \ldots \otimes 1^u \otimes h) \in (A_{\infty}^u < u, h>)(n-q+1)_{n+q-1}, \quad n \geq 1, \quad q \geq 1, \quad n \geq j_q \geq \ldots \geq j_1 \geq 0, \quad n_s \geq 0, \quad 1 \leq s \leq q+1, \quad n_1 + \ldots + n_{q+1} = n - q + 1.
\]

It is worth mentioning for clarity that each \(j_k, 1 \leq k \leq q\), is equal to the number of all tensor factors on the left of the \(k\)th occurrence of \(h\), counting from the beginning of the tensor stack to the right. For example, according to this rule, we have
\[
\tau_4^{3,1} = \pi_3(1 \otimes h \otimes 1 \otimes h \otimes 1), \quad \tau_3^{3,2} = \pi_2(1 \otimes 1 \otimes h \otimes h), \quad \tau_5^{5,1,0} = \pi_4(h \otimes h \otimes 1 \otimes 1 \otimes 1 \otimes h).
\]

Note that, in [24], an element \(\tau_n^{j_1, \ldots, j_l}\) is denoted by \(m_{n_1,n_2,\ldots,n_{q+1}}\), where the numbers \(n_1, \ldots, n_{q+1}\) are the same as in the above expression for \(\tau_n^{j_1, \ldots, j_l}\). At the generators \(\tau_n^{j_1, \ldots, j_l}\), \(n \geq 0, q \geq 0, n + q \geq 1\), where \(\tau_n^{j_1, \ldots, j_l} = \tau_0^0 = \pi_{n-1}\) for \(q = 0\) and \(n \geq 1\), the differential is completely determined by the formulae (2.1), (3.2) and the relations (3.1).

Algebras \((A, d, \alpha)\) over the operad \((A_{\infty}^{hu}, \gamma)\), i.e., \(A_{\infty}^{hu}\)-algebras, are called homotopy unital \(A_{\infty}\)-algebras.

It is easy to see that defining the structure of an \(A_{\infty}^{hu}\)-algebra \((A, d, \alpha)\) on a differential module \((A, d)\), where \(A = \{A_n\}, n \in \mathbb{Z}, n \geq 0, d : A_n \to A_{n+1}\), is equivalent to specifying a family of maps
\[
\{\alpha(\tau_n^{j_1, \ldots, j_l}) : (A^\otimes(n-q+1))_\bullet \to A_{n+q-1} \mid n \in \mathbb{Z}, \quad n \geq 0, \quad q \geq 0, \quad n + q \geq 1\},
\]

which satisfy the relations
\[
d(\alpha(\tau_n^{j_1, \ldots, j_l})) = \alpha(d(\tau_n^{j_1, \ldots, j_l})), \quad (3.3)
\]
where \(d(\alpha(\tau_n^{j_1, \ldots, j_l})) = d\alpha(\tau_n^{j_1, \ldots, j_l}) + (-1)^{n+q} \alpha(\tau_n^{j_1, \ldots, j_l})d\). In what follows, we denote the map \(\alpha(\tau_n^{j_1, \ldots, j_l})\) by \(\tau_n^{j_1, \ldots, j_l}\). For example, by (3.3) we have the following relations:
\[
d(\tau_2^0) = \tau_1^0 \pi_0 + \pi_0(\tau_1^0 \otimes 1) - \pi_1(\tau_0^0 \otimes 1 \otimes 1),
\]

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\[ d(\tau_2^0) = -\tau_1^0 \tau_1^0 - \tau_1^0 \tau_1^0 - \tau_2^0 (\tau_1^0 \otimes 1) + \tau_2^0 (1 \otimes \tau_1^0), \]
\[ d(\tau_3^0) = \tau_1^0 (1 \otimes \tau_2^0) - \tau_2^0 (1 \otimes \tau_1^0) - \tau_1^0 \tau_2^0 - \tau_2^0 (\tau_1^0 \otimes 1) + \tau_2^0 (1 \otimes \tau_1^0) + \tau_2^0 (1 \otimes \tau_2^0). \]

Note that, for \( q = 0 \), the relations (3.3) in which the maps \( \alpha(\tau_n^q) \), \( n \geq 1 \), are denoted by \( \pi_{n-1} \) transforms into the relations (2.1). Thus, each \( A^\text{hu}_n \)-algebra \( (A, d, \tau_n^{j_0, \ldots, j_1}) \) is an \( A^\infty \)-algebra \( (A, d, \pi_n) \). In what follows, we write \( A^\text{hu}_n \)-algebras \( (A, d, \tau_n^{j_0, \ldots, j_1}) \) in the form \( (A, d, \pi_n, \tau_n^{j_0, \ldots, j_1}) \), where \( \pi_n = \pi_n^{j_0, \ldots, j_1}, n \geq 0 \).

Note also that, by virtue of (3.2) and (3.3), we have
\[ d(\tau_0^0) = 0, \quad d(\tau_1^0) = \pi_0(\tau_0^0 \otimes 1) - 1, \quad d(\tau_1^1) = \pi_0(1 \otimes \tau_0^0) - 1, \]
for any \( A^\infty \)-algebra \( (A, d, \pi_n, \tau_n^{j_0, \ldots, j_1}) \); thus, the map \( \tau_0^0 : K \to A \) or, to be more precise, the element \( \tau_0^0(1) \in A_0 \) is up to homotopy the unit of the differential homotopy associative algebra \( (A, d, \pi_0) \). Moreover, it was shown in [26] that in an explicit form the relations (3.3), for \( q = 1 \) and \( j_q = j_1 = n \geq 2 \), are written as the following equalities:
\[ d(\tau_n^1) = \sum_{m=1}^{n-1} \sum_{t=0}^{m-1} (-1)^{t(m-n)+n} \pi_m^n (1 \otimes \ldots \otimes 1 \otimes \pi_{n-m-1} \otimes 1 \otimes \ldots \otimes 1) + \sum_{m=1}^n (-1)^{m+1} \pi_{m-1} (1 \otimes \ldots \otimes 1 \otimes \tau_{n-m}^m). \]

Now we introduce the notion of an involutive homotopy unital \( A^\infty \)-algebra, which generalizes the notion of a differential unital associative algebra with a involution.

**Definition 3.1.** We define the involutive homotopy unital \( A^\infty \)-algebra, briefly, the involutive \( A^\text{hu}_n \)-algebra \( (A, d, \pi_n, \tau_n^{j_0, \ldots, j_1}, \ast) \) as any five-tuple \( (A, d, \pi_n, \tau_n^{j_0, \ldots, j_1}, \ast) \) that satisfy the following conditions:

1. \( (A, d, \pi_n, \ast) \) is an involutive \( A^\text{hu}_n \)-algebra;
2. \( (A, d, \pi_n, \tau_n^{j_0, \ldots, j_1}) \) is a homotopy unital \( A^\text{hu}_n \)-algebra;
3. For any elements \( a_0, \ldots, a_{q-n} \in A, n \geq 1, q \geq 1 \), there are the relations
\[ \tau_n^{j_0, \ldots, j_1}(a_0 \otimes a_1 \otimes \ldots \otimes a_{q-n-1} \otimes a_{q-n})^* = \]
\[ = (-1)^\varepsilon \tau_n^{j_0, \ldots, j_1}(a_{q-n}^* \otimes a_{n-q-1}^* \otimes \ldots \otimes a_1^* \otimes a_0^*), \]
\[ n \geq j_q > \ldots > j_1 \geq 0, \quad \varepsilon = \frac{n(n-1)}{2} + \frac{q(q-1)}{2} + \sum_{0 \leq i < j \leq n-q} |a_i||a_j|, \]
where \( |a| = q \) means that \( a \in A_q \).

Given an \( A^\text{hu}_n \)-algebra \( (A, d, \pi_n, \tau_n^{j_0, \ldots, j_1}, \ast) \) consider the family of maps
\[ s_k^{-1} : (A^\otimes(n+1))_p \to (A^\otimes(n-k+2))_{p+k}, \quad n \geq 0, \quad p \geq 0, \quad 0 \leq k \leq n+1, \]
defined by the following formulae:
\[ s_k^{-1} = (-1)^\varepsilon \frac{1 \otimes \ldots \otimes 1 \otimes \tau_k^k}{n-k+1}, \]
Consider also the $DF_\infty$-module $(\varepsilon M(A), t, r, d, \ddot{\partial})$ corresponding to this $A_\infty^{hu}$-algebra. As mentioned above, this $DF_\infty$-module determines the $D_\infty$-module $(\varepsilon M(A), d^i)$ according to (1.4) with $q = 1$. In [26], it was shown that, given any integer $k \geq 0$, the families of maps

$$\{d^i_1 : \varepsilon M(A)_{s-i} \to \varepsilon M(A)_{s-i+1}\} \quad \text{and} \quad \{s_1^{i-1} : \varepsilon M(A)_{s-i} \to \varepsilon M(A)_{s-i+1}\}$$

are related by

$$\sum_{i+j=k} d^i_1 s^{j-1} + s^{j-1} d^i_1 = \begin{cases} 1_{\varepsilon M(A)}, & k = 1, \\ 0, & k \neq 1. \end{cases} \quad (3.6)$$

Given a stable $D_\infty$-module $(\varepsilon M(A), d^i_1)$, consider the corresponding chain complex $(\varepsilon M(A), b')$, where $b' = (d^0_1 + d^1_1 + \ldots + d^i_1 + \ldots) : \varepsilon M(A) \to \varepsilon M(A)_{s-1}$. It follows from (3.6) that the map

$$s = (s^{-1} + s^0 + s^1 + \ldots + s^i + \ldots) : \varepsilon M(A) \to \varepsilon M(A)_{s+1}$$

satisfies the condition $b' s + s b' = 1_{\varepsilon M(A)}$, which means that $s$ is a contracting homotopy for the chain complex $(\varepsilon M(A), b')$.

Given an arbitrary involutive $A_\infty^{hu}$-algebra $(A, d, \pi_n, \tau_{j_1}^n \ldots j_1, \ast)$, the relations (1.9) follows that if there is a left act of the group $\mathbb{Z}_2 = \{1, \vartheta \mid \vartheta^2 = 1\}$ on the chain complex $(\varepsilon M(A), b')$. This left act is defined by means of the module automorphism $\vartheta : \varepsilon M(A) \to \varepsilon M(A)$ of the order two, which at any element $x \in \varepsilon M(A)_n$ is given by $\vartheta(x) = \overline{R_n T_n (x)}$.

Consider the chain bicomplex $(Q(\varepsilon M(A)), -b', D')$ defined by

$$Q(\varepsilon M(A))_{n,m} = \varepsilon M(A)_{n-m}, \quad n \geq 0, \quad m \geq 0,$$

$$D' = (-1)^{n+1} + (-1)^{m+1} \overline{R_n T_n} : D' : Q(\varepsilon M(A))_{n,m} \to Q(\varepsilon M(A))_{n,m-1},$$

$$-b' : Q(\varepsilon M(A))_{n,m} \to \varepsilon M(A)_{n-1,m} = Q(\varepsilon M(A))_{n-1,m}.$$

For the chain complex associated with the chain complex $(Q(\varepsilon M(A)), -b', D')$, we use the notation $(\text{Tot}(Q(\varepsilon M(A))), \dot{D})$, where $\dot{D} = -b + D'$. If we consider the spectral sequence of the chain bicomplex $(Q(\varepsilon M(A)), -b', D')$ and use the the contractibility of the chain complex $(\varepsilon M(A), b')$, then we obtain $H_n(\text{Tot}(Q(\varepsilon M(A))), \dot{D}^i) = 0$ for all $n \geq 0$.

In what follows, by the dihedral homology $\varepsilon HD(A)$ of any involutive $A_\infty^{hu}$-algebra $(A, d, \pi_n, \tau_{j_1}^n \ldots j_1, \ast)$ we mean the dihedral homology $\varepsilon HD(A)$ of the corresponding involutive $A_\infty$-algebra $(A, d, \pi_n, \ast)$. Similarly, by the reflexive homology $\varepsilon HR(A)$ of any involutive $A_\infty^{hu}$-algebra $(A, d, \pi_n, \tau_{j_1}^n \ldots j_1, \ast)$ we mean the reflexive homology $\varepsilon HR(A)$ of the corresponding $A_\infty$-algebra $(A, d, \pi_n, \ast)$.

Now, using the acyclicity of the chain complex $(\text{Tot}(Q(\varepsilon M(A))), \dot{D})$, we find a relationship between the dihedral and the reflexive homology of involutive homotopy unital $A_\infty$-algebras.
Theorem 3.1. For any involutive $A_{\infty}^{h\mu}$-algebra $(A, d, \pi_n, \tau_{n, j_0 \ldots j_l}, \ast)$, there is the long exact sequence

$$\cdots \xrightarrow{\delta} \partial_{HR_n}(A) \xrightarrow{i} \partial HD_n(A) \xrightarrow{p} \partial HD_{n-2}(A) \xrightarrow{\delta} \partial_{HR_{n-1}}(A) \xrightarrow{i} \cdots,$$

(3.7)

where maps $i_{\ast}$ and $p_{\ast}$ are induced respectively by the inclusion and by the projection, and the map $\delta_{\ast}$ is induced by the connecting homomorphism.

Proof. Given the triple chain complex $(D(\varepsilon M)(A), \delta_1, \delta_2, \delta_3)$, consider the chain bicomplex $(\varepsilon M(A), \delta_1, \delta_2)$ defined by

$$D(\varepsilon M(A))_{n,m} = \bigoplus_{k+l=n} D(\varepsilon M(A))_{k,m,l}, \quad n \geq 0, \quad m \geq 0, \quad \delta_1 = \delta_1 + \delta_3, \quad \delta_2 = \delta_2.$$ 

The chain complex associated with the chain bicomplex $(D(\varepsilon M(A)), \delta_1, \delta_2)$ we denote by $(\text{Tot}(D(\varepsilon M(A))), \delta)$, where $\delta = \delta_1 + \delta_2$. It is easy to see that the chain complex $(\text{Tot}(D(\varepsilon M(A))), \delta)$ is isomorphic to the chain complex $(\text{Tot}(\varepsilon M(A)), \delta)$ and, consequently, we have the isomorphism $H(\text{Tot}(D(\varepsilon M(A))), \delta) = \partial HD(A)$. Consider in the chain complex $(\text{Tot}(D(\varepsilon M(A))), \delta)$ the chain subcomplex $(P(\varepsilon M(A)), \delta)$ defined by the following formulæ:

$$P(\varepsilon M(A))_n = \varepsilon M(A)_{n-1,1} \oplus \varepsilon M(A)_{n,0}, \quad n \geq 0,$$

$$\delta(x_{n-1}, x_n) = (\tilde{\delta}_1(x_{n-1}), \tilde{\delta}_1(x_n) + \tilde{\delta}_2(x_{n-1}), \quad (x_{n-1}, x_n) \in P(\varepsilon M(A))_n.$$ 

It is easy to see that there are the equalities of modules

$$\varepsilon M(A)_{n-1,1} = \text{Tot}(Q(\varepsilon M(A)))_{n-1}, \quad \varepsilon M(A)_{n,0} = \text{Tot}(R(\varepsilon M(A)))_n.$$ 

Consider the short exact sequence of chain complexes

$$0 \rightarrow (P(\varepsilon M(A)), \delta) \xrightarrow{j} (\text{Tot}(\varepsilon M(A)), \delta) \xrightarrow{p} \Sigma^{-2}(\text{Tot}(\varepsilon M(A)), \delta) \rightarrow 0,$$

$$j((x_{n-1}, x_n)) = (0, \ldots, 0, x_{n-1}, x_n), \quad n \geq 0,$$

$$\Sigma^{-2}(\text{Tot}(\varepsilon M(A)))_{\bullet}, \delta) = (\text{Tot}(\varepsilon M(A)))_{\bullet-2, \delta},$$

$$p((x_0, \ldots, x_{n-2}, x_{n-1}, x_n)) = (x_0, \ldots, x_{n-2}), \quad n \geq 0.$$ 

In the homology, this short exact sequence induces the long exact sequence

$$\cdots \xrightarrow{\delta} H_n(P(\varepsilon M(A))) \xrightarrow{i} \partial HD_n(A) \xrightarrow{p}$$

$$-\partial HD_{n-2}(A) \xrightarrow{\delta} H_{n-1}(P(\varepsilon M(A))) \xrightarrow{i} \cdots.$$ 

(3.8)

Now, let us prove that the graded modules $H(P(\varepsilon M(A)))$ and $HR(A)$ are isomorphic. To this purpose we consider the short exact sequence of chain complexes

$$0 \rightarrow (\text{Tot}(R(\varepsilon M(A)))_{\bullet}, \hat{D}) \xrightarrow{\alpha} (P(\varepsilon M(A))_{\bullet}, \delta) \xrightarrow{\beta} (\text{Tot}(Q(\varepsilon M(A)))_{\bullet-1}, \hat{D'}) \rightarrow 0,$$
where \( \alpha(x_n) = (0, x_n) \), \( \beta(x_{n-1}, x_n) = x_{n-1}, (x_{n-1}, x_n) \in (P(\mathcal{M}(A)))_n \). In the homology, this short exact sequence induces the long exact sequence. Since the equality \( H_{n-1}(\text{Tot}(Q(\varepsilon M(A)) \to D')) = 0 \) is true, this long exact sequence implies that the map \( \alpha_* : \varepsilon HR_*(A) \to H_*(P(\varepsilon M(A))) \) is an isomorphism. Now, replacing the modules \( H_*(P(\varepsilon M(A))) \) in the long exact sequence (3.8) by \( \varepsilon HR_*(A) \) and the maps \( j_* \) and \( \delta \) by \( i_* = j_* \alpha_* \) and \( \delta_* = \alpha_*^{-1} \delta \), respectively, we obtain the long exact sequence (3.7).

In conclusion we mention that if an involutive \( A_{\infty}^u \)-algebra \( (A, d, \pi, \tau_{j_1}, \ldots, \tau_{j_n}, \ast) \) is a differential involutive unital associative algebra \( (A, d, \pi, u, \ast) \), where \( \pi = \pi_0, u = \tau_0 \) and \( \tau_{j_1}, \ldots, \tau_{j_n} = 0 \) in all other cases, then the exact sequence (3.7) coincides with the well-known Krasauskas-Lapin-Solov’ev exact sequence that connecting the dihedral and the reflexive homology of an algebra \( (A, d, \pi, u, \ast) \) over any unital commutative ring.

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