Finite groups whose $n$-maximal subgroups are $\sigma$-subnormal

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Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$. A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$, for some $i \in I$, and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. A subgroup $H$ of $G$ is said to be: $\sigma$-permutable or $\sigma$-quasinormal in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and $x \in G$: $\sigma$-subnormal in $G$ if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \leq A_i$ or $A_i/(A_{i-1})A_i$ is a finite $\sigma_i$-group for some $\sigma_i \in \sigma$ for all $i = 1, \ldots, t$.

If $M_0 < M_{n-1} < \cdots < M_1 < M_0 = G$, where $M_i$ is a maximal subgroup of $M_{i-1}$, then $M_n$ is said to be an $n$-maximal subgroup of $G$. If each $n$-maximal subgroup of $G$ is $\sigma$-subnormal ($\sigma$-quasinormal, respectively) in $G$ but, in the case $n > 1$, some $(n-1)$-maximal subgroup is not $\sigma$-subnormal (not $\sigma$-quasinormal, respectively) in $G$, we write $m_\sigma(G) = n$ ($m_\sigma(G) = n$, respectively).

In this paper, we show that the parameters $m_\sigma(G)$ and $m_{\sigma q}(G)$ make possible to bound the $\sigma$-nilpotent length $l_\sigma(G)$ (see below the definitions of the terms employed), the rank $r(G)$ and the number $|\pi(G)|$ of all distinct primes dividing the order $|G|$ of a finite soluble group $G$. We also give the conditions under which a finite group is $\sigma$-soluble or $\sigma$-nilpotent, and describe the structure of a finite soluble group $G$ in the case when $m_\sigma(G) = |\pi(G)|$. Some known results are generalized.

Keywords. finite group, $n$-maximal subgroup, $\sigma$-subnormal subgroup, $\sigma$-quasinormal subgroup, $\sigma$-soluble group, $\sigma$-nilpotent group

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1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $|n|$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$. Let $A$ and $B$ be subgroups of $G$. We say that $A$ forms an irreducible pair with $B$ if $AB = BA$ and $A$ is a maximal subgroup of $AB$.

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In what follows, $\sigma = \{\sigma_i | i \in I \subseteq \mathbb{N}\}$ is some partition of $\mathbb{P}$, i.e., $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. By analogy with the notation $\pi(n)$ and $\pi(G)$, we put $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

In the mathematical practice, we often deal with the following two special partitions of $\mathbb{P}$: $\sigma = \{\{2\}, \{3\}, \ldots\}$ and $\sigma = \{\pi, \pi'\}$.

A group $G$ is called: $\sigma$-primary (see [26]) if $|\sigma(G)| \leq 1$, i.e., $G$ is a $\sigma_i$-group for some $i$: $\sigma$-nilpotent or $\sigma$-decomposable (see [24]) if $G = A_1 \times \cdots \times A_t$ for some $\sigma$-primary groups $A_1, \ldots, A_t$: $\sigma$-soluble (see [26]) if every chief factor of $G$ is $\sigma$-primary. By the $\sigma$-nilpotent length (denoted by $l_\sigma(G)$) of a $\sigma$-soluble group $G$ we mean the length of the shortest normal chain of $G$ with $\sigma$-nilpotent factors.

The group $G$ is nilpotent if and only if $G$ is $\sigma$-nilpotent where $\sigma = \{\{2\}, \{3\}, \ldots\}$: $G$ is $\pi$-decomposable, i.e., $G = O_\pi(G) \times O_{\pi'}(G)$ if and only if $G$ is $\pi$-nilpotent where $\sigma = \{\pi, \pi'\}$.

The $\sigma$-nilpotent groups have many applications in the formation theory (see [2, 4, 24, 30], and see also the recent papers [9, 11, 26] and the survey [28]), and such groups are exactly the groups in which every subgroup is $\sigma$-subnormal in the sense of the following definition (see Proposition 3.4 below).

**Definition 1.1.** A subgroup $A$ of $G$ is called $\sigma$-subnormal in $G$ (see [26]) if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1} \leq A_i$ or $A_i/(A_{i-1})_{A_i}$ is $\sigma$-primary for all $i = 1, \ldots, t$.

It should be noted that the idea of this concept originates from the paper of Kegel [15], where another generalization of nilpotency was discussed.

There are some motivations for the introduction and study of $\sigma$-subnormal subgroups. First of all, the set of all $\sigma$-subnormal subgroups of $G$ forms a sublattice of the lattice of all subgroups of $G$ (see Proposition 2.5 below). This fact is a generalization of the classical Wielandt’s result which states that the set of all subnormal subgroups of $G$ forms a sublattice of the lattice of all subgroups of $G$. Thus, the $\sigma$-subnormal subgroups are very convenient for applications. The first among such applications was obtained in [26, 27].

Recall that a set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$, for some $i \in I$, and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. A subgroup $H$ of $G$ is said to be $\sigma$-permutable or $\sigma$-quasinormal in $G$ (see [26]) if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and $x \in G$. In particular, $H$ is said to be $S$-permutable or $S$-quasinormal in $G$ (see [3, 6]) if $H$ permutes with all Sylow subgroups of $G$.

If $H$ is an $S$-quasinormal subgroup of $G$, then $H$ is subnormal in $G$ (see [15]) and $H/H_G$ is nilpotent (see [3, Theorem 1.2.14]). But in general, if $H$ is $\sigma$-quasinormal in $G$, then $H$ is not necessarily subnormal and $H/H_G$ may be non-nilpotent (see [26]). Nevertheless, in this case when $H$ is $\sigma$-quasinormal, $H$ is $\sigma$-subnormal in $G$ and $H/H_G$ is $\sigma$-nilpotent (see [26, Theorem B]).

The examples and some other applications of $\sigma$-subnormal subgroups and $\sigma$-quasinormal subgroups were discussed in [26–28]. In this paper, we consider some applications of such subgroups in the theory of $n$-maximal subgroups.

Recall that if $M_n < M_{n-1} < \cdots < M_1 < M_0 = G = (*),$ where $M_i$ is a maximal subgroup of $M_{i-1}$ for all $i = 1, \ldots, n$, then the chain $(*)$ is said to be a maximal chain of $G$ of length $n$ and $M_n$ ($n > 0$) is an $n$-maximal subgroup of $G$.

If each $n$-maximal subgroup of $G$ is $\sigma$-subnormal ($\sigma$-quasinormal, respectively) in $G$ but, in the case $n > 1$, some $(n - 1)$-maximal subgroup is not $\sigma$-subnormal (not $\sigma$-quasinormal, respectively) in $G$, we write $m_\sigma(G) = n$ ($m_{\sigma q}(G) = n$, respectively).

Note that $m_\sigma(G) = 1 = m_{\sigma q}(G)$ if and only if $G$ is $\sigma$-nilpotent by Proposition 3.4 below. We also show (see Corollaries 1.7 and 1.8 below) that $m_\sigma(G) = 2$ if and only if $G$ is a Schmidt group with abelian Sylow subgroups such that $|\sigma(G)| = |\pi(G)|$, and $m_{\sigma q}(G) = 2$ if and only if $G$ is a supersoluble group with $m_\sigma(G) = 2$. Finally, note that every group with $m_\sigma(G) = 3$ is $\sigma$-soluble (see Theorem 1.4 below), and there are non-soluble groups, for example the alternating group $A_5$ of degree 5, with $m_{\sigma q}(G) = 4$.

If $G$ is soluble, the parameters $m_\sigma(G)$ and $m_{\sigma q}(G)$ make possible to bound the $\sigma$-nilpotent length $l_\sigma(G)$, the rank $r(G)$ and the number $|\pi(G)|$ of all distinct primes dividing $|G|$. 


Recall that the rank $r(G)$ of a soluble group $G$ is the maximal integer $k$ such that $G$ has a chief factor of order $p^k$ for some prime $p$ (see [13, p.685]).

**Theorem 1.2.** Suppose that $G$ is $\sigma$-soluble and let $H$ be a complete Hall $\sigma$-set of $G$. Then the following statements hold:

(i) If $G$ is soluble but it is not $\sigma$-nilpotent and $r(H) \leq r \in \mathbb{N}$ for all $H \in H$, then $r(G) \leq m_{\sigma}(G) + r - 2$.

(ii) $l_H(G) \leq m_{\sigma}(G)$.

(iii) If $G$ is soluble but it is not $\sigma$-nilpotent, then $|\pi(G)| \leq m_{\sigma}(G)$.

Now, let us consider some applications of Theorem 1.2.

The relationship between $n$-maximal subgroups (where $n > 1$) of a group $G$ and the structure of $G$ was studied by many authors (see, in particular, the recent papers [7,16–21,23] and [6, Chapter 4]). One of the first results in this direction was obtained by Huppert [12]. In fact, Huppert [12] proved that: if every 2-maximal subgroup of $G$ is normal in $G$, then $G$ is supersoluble; if every 3-maximal subgroup of $G$ is normal in $G$, then $G$ is soluble of rank $r(G)$ at most two. The first of these two results was generalized by Agrawal [1]: If every 2-maximal subgroup $L$ of $G$ is $S$-quasinormal in $G$, then $G$ is supersoluble. In the universe of all soluble groups both Huppert’s observations and some similar results in [14] are special cases of the following general result (see [22]): If $G$ is soluble and every $n$-maximal subgroup $L$ of $G$ $(n > 1)$ is quasinormal in $G$ (i.e., $L$ permutes with all subgroups of $G$), then $r(G) \leq n - 1$.

In the case $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem 1.2(i) the following generalization of the last of these results.

**Corollary 1.3.** Suppose that $G$ is soluble and each $n$-maximal subgroup of $G$ $(n > 1)$ is $S$-quasinormal in $G$. Then $r(G) \leq n - 1$.

The following theorem allows us to obtain the above mentioned result of Agrawal [1].

**Theorem 1.4.** (i) If in every maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of $G$ of length 3, one of the subgroups $M_3$, $M_2$ and $M_1$ is $\sigma$-subnormal in $G$, then $G$ is $\sigma$-soluble.

(ii) If $1 < m_{\sigma}(G) \leq 3$, then $G$ is soluble.

**Corollary 1.5** (See [29]). If in every maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of $G$ of length 3, one of the subgroups $M_3$, $M_2$ and $M_1$ is subnormal in $G$, then $G$ is soluble.

**Corollary 1.6** (See [22]). If every 3-maximal subgroup of $G$ is subnormal in $G$, then $G$ is soluble.

Recall that $G$ is called a Schmidt group if $G$ is not nilpotent but every proper subgroup of $G$ is nilpotent.

**Corollary 1.7.** The equality $m_{\sigma}(G) = 2$ is true if and only if $G$ is a Schmidt group with abelian Sylow subgroups such that $|\pi(G)| = |\pi(G)|$.

**Corollary 1.8.** The equality $m_{\sigma}(G) = 2$ is true if and only if $G$ is a supersoluble group with $m_{\sigma}(G) = 2$.

From Corollary 1.8 and Theorem 1.4, we get the following corollary.

**Corollary 1.9** (See [1] or [31, Chapter 1, Theorem 6.5]). If every 2-maximal subgroup of $G$ is $S$-quasinormal in $G$, then $G$ is supersoluble. Moreover, if $|\pi(G)| > 2$, then $G$ is nilpotent.

From Theorem 1.2(iii) we know that for every soluble but non-$\sigma$-nilpotent group $G$ we have $|\pi(G)| \leq m_{\sigma}(G)$. In the case when $|\pi(G)| = m_{\sigma}(G)$ the structure of such a group $G$ can be described completely as follows.

**Theorem 1.10.** Suppose that $G$ is soluble. Then $m_{\sigma}(G) = |\pi(G)|$ if and only if $G$ is a group of one of the following two types:

(i) $G$ is a $p$-group for some prime $p$.

(ii) $G = D \times M$, where $D = G^{\mathbb{N}}$ is an abelian Hall subgroup of $G$, and the following hold:

(a) Every non-$\sigma$-subnormal Sylow subgroup $P_1$ of $G$ is cyclic and the maximal subgroup of $P_1$ is $\sigma$-subnormal in $G$. Moreover, if $P_1, \ldots, P_n$ is a Sylow basis of $G$, then $P_2, \ldots, P_n$ are elementary abelian and $P_1$ forms irreducible pairs with all such subgroups; if $\{H_1, \ldots, H_t\}$ is a complete Hall $\sigma$-set of $G$, $P_1 \leq H_1$ and $P_1$ is not of prime order, then $H_2, \ldots, H_t$ are normal in $G$.  


(b) Some Sylow subgroup of $M$ is not $\sigma$-subnormal in $G$. Hence $M$ acts irreducibly on every Sylow subgroup of $D$.

(c) If $G$ possesses at least two non-$\sigma$-subnormal non-isomorphic Sylow subgroups, then all non-$\sigma$-subnormal Sylow subgroups are of prime order.

(d) If $P$ is a non-$\sigma$-subnormal Sylow subgroup of $G$, $P$ is a $\sigma_i$-group and $V$ is the maximal subgroup of $P$, then $|G : N_G(V)|$ is a $\sigma_i$-number.

In this theorem $G^{\sigma}$ denotes the $\sigma$-nilpotent residual of $G$, i.e., the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$.

In the case when $\sigma = \{2, \{3\}, \ldots \}$, from Theorems 1.2 and 1.10 we get the following known result.

**Corollary 1.11** (See [22]). Suppose that $G$ is a soluble group and each $n$-maximal subgroup of $G$ is subnormal. If $n \leq |\pi(G)|$, then $G$ is either of the following type:

(a) $G$ is nilpotent.

(b) $G = HN$, where,

(i) $N$ is a normal abelian Hall subgroup, and all Sylow subgroups of $N$ are elementary abelian.

(ii) $H$ is a cyclic Hall subgroup, and $|H|$ is either a prime power or square-free number.

(iii) $|N[H]| = 1$.

(iv) If $H_q$ is a Sylow subgroup of $H$ and $N_q$ is a Sylow subgroup of $N$, then $H_q$ induces in $N_q$ an irreducible automorphism group of order $p$ or 1. In the latter case, $|N_q| = q$.

Conversely, a group of (a) or (b) has each $n$-maximal subgroup subnormal.

**Corollary 1.12** (See [22]). Suppose that $G$ is a soluble group and each $n$-maximal subgroup of $G$ is subnormal. If $n < |\pi(G)|$, then $G$ is nilpotent.

We prove Theorems 1.2, 1.4 and 1.10 in Sections 4–6, respectively. But before them, as preparatory steps, we prove in Section 2 that the set of all $\sigma$-subnormal subgroups of $G$ forms a sublattice of the lattice of all subgroups of $G$, and we collect in Section 3 some needed properties of $\sigma$-soluble and $\sigma$-nilpotent groups.

All unexplained notation and terminologies are standard. The reader is referred to [4–6] if necessary.

## 2 The lattice $L_\sigma(G)$ of all $\sigma$-subnormal subgroups

It is not difficult to show that the intersection of any two $\sigma$-subnormal subgroups of $G$ is also $\sigma$-subnormal in $G$ (see Lemmas 2.2(1) and 2.2(2) below). It is well known that any partially ordered set with 1 in which there is the greatest lower bound for each of its non-empty subsets is a lattice. Hence the set $L_\sigma(G)$ of all $\sigma$-subnormal subgroups of $G$ is a lattice. In this section, we show that $L_\sigma(G)$ is a sublattice of the lattice of all subgroups of $G$.

We use $\mathcal{S}_\sigma$ and $\mathcal{N}_\sigma$ to denote the class of all $\sigma$-soluble groups and the class of all $\sigma$-nilpotent groups, respectively.

**Lemma 2.1** (See [26, Lemma 2.5]). The class $\mathcal{N}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if $H$ is a normal subgroup of $G$ and $H/H \cap \Phi(G)$ is $\sigma$-nilpotent, then $H$ is $\sigma$-nilpotent.

In what follows, $\Pi$ is always supposed to be a non-empty subset of the set $\sigma$ and $\Pi' = \sigma \setminus \Pi$. We say that: a natural number $n$ is a $\Pi$-number if $\sigma(n) \subseteq \Pi$; $G$ is a $\Pi$-group if $|G|$ is a $\Pi$-number; $G$ is $\sigma$-perfect if $G^{\sigma_3} = G$.

We call the product of all normal $\sigma$-nilpotent subgroups of $G$ the $\sigma$-Fitting subgroup of $G$ and denote it by $F_\sigma(G)$.

**Lemma 2.2.** Let $A$, $K$ and $N$ be subgroups of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$.

1. $A \cap K$ is $\sigma$-subnormal in $K$.
2. If $K$ is a $\sigma$-subnormal subgroup of $A$, then $K$ is $\sigma$-subnormal in $G$.
3. If $N \leq K$ and $K/N$ is $\sigma$-subnormal in $G/N$, then $K$ is $\sigma$-subnormal in $G$.  

(4) If $K \subseteq A$ and $A$ is $\sigma$-nilpotent, then $K$ is $\sigma$-subnormal in $G$.

(5) If $N$ is a $\sigma_i$-group, for some $i$, then $N \leq N_G(O^{\sigma_i}(A))$.

(6) If $K \leq E \leq G$, where $K$ is $\sigma$-subnormal in $E$, then $KN/N$ is $\sigma$-subnormal in $NE/N$.

(7) If $A$ is $\sigma$-perfect, then $A$ is subnormal in $G$.

(8) Suppose that $N$ is the product of some minimal normal subgroups of $G$ and $N$ is not $\sigma$-primary. Suppose also that $G = AN$, $N$ is non-abelian and all composition factors of $N$ are isomorphic. Then $N \leq N_G(A)$.

(9) If $A$ is a $\sigma_i$-group, then $A \leq O_{\sigma_i}(G)$. Hence if $A$ is $\sigma$-nilpotent, then $A \leq F_\sigma(G)$.

**Proof.** Statements (1)–(5) follow from [26, Lemma 2.6].

(6) By hypothesis, there is a chain $K = K_0 \leq K_1 \leq \cdots \leq K_n = E$ such that either $K_{i-1} \leq K_i$ or $K_i/(K_{i-1}K_i)$ is $\sigma$-primary for all $i = 1, \ldots, n$. Consider the chain

$$KN/N = K_0N/N \leq K_1N/N \leq \cdots \leq K_nN/N = EN/N.$$ Assume that $K_{i-1}N/N$ is not normal in $K_iN/N$. Then $L = K_{i-1}$ is not normal in $T = K_i$ and so $T/L$ is $\sigma$-primary. Then

$$(T/L_T)/(L_T(T \cap N)/L_T) = (T/L_T)/((T \cap NL_T)/L_T) \cong T/(T \cap NL_T) \cong TN/LTN \cong (TN/N)/(LTN/N)$$

is $\sigma$-primary. But $LTN/N \leq (LN/N)_{TN/N}$. Hence $(TN/N)/(LN/N)_{TN/N}$ is $\sigma$-primary. This shows that $KN/N$ is $\sigma$-subnormal in $NE/N$.

By hypothesis, there is a chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ such that either $A_{i-1} \leq A_i$ or $A_i/(A_{i-1}A_i)$ is $\sigma$-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$.

(7) Assume that this assertion is false and let $G$ be a counterexample of minimal order. First we show that $A \leq M_G$. This is clear if $M$ is normal in $G$. Now assume that $G/M_G$ is a $\sigma_i$-group. Then from the isomorphism $AM_G/M_G \cong A/A \cap M_G$ and $A = A^{\sigma_i}$ we get that $A \leq O^{\sigma_i}(A) \leq A \cap M_G$, so $A \leq M_G$.

The choice of $G$ implies that $A$ is subnormal in $M$, so $A$ is subnormal in $M_G$ by Assertion (1). Therefore $A$ is subnormal in $G$.

(8) Assume that this assertion is false. First note that $N \nsubseteq M$ since $G = AN$ and $M < G$ by hypothesis.

If $M$ is not normal in $G$, then $G/M_G$ is $\sigma$-primary and so $N \cong NM_G/M_G$ is $\sigma$-primary, which contradicts the hypothesis. Hence $M$ is normal in $G$ and so $N = (N \cap M) \times N_0$, where $N \cap M$ and $N_0$ are normal in $G$. Then $N \cap M \leq Soc(M)$ by [5, Chapter A, Proposition 4.13(c)] and $M = M \cap AN = A(N \cap M)$. The choice of $G$ implies that $N \cap M \leq N_M(A) \leq N_G(A)$. On the other hand, $N_0 \cap M = 1$ and so $N_0 \leq C_G(M) \leq C_G(A)$. Thus $N \leq N_G(A)$, which leads to a contradiction. Hence we have (8).

(9) Assume that this assertion is false and let $G$ be a counterexample of minimal order. Then $1 < A < G$. Let $D = O_{\sigma_1}(G)$, $R$ be a minimal normal subgroup of $G$ and $O/R = O_{\sigma_1}(G/R)$. Then the choice of $G$ and Assertion (6) imply that $AR \leq O/R$, so $A \leq O$. Therefore $R \neq D$, so $D = 1$ and $A \cap R < R$. Suppose that $L = A \cap R \neq 1$. Assertion (1) implies that $L$ is $\sigma$-subnormal in $R$. If $R < G$, the choice of $G$ implies that $L \leq O_{\sigma_1}(R) \leq D$ since $O_{\sigma_1}(R)$ is a characteristic subgroup of $R$. But then $D \neq 1$, which leads to a contradiction. Hence $R = G$ is a simple group, which is also impossible since $1 < A < G$.

Therefore $R \cap A = 1$. If $O < G$, the choice of $G$ implies that $A \leq O_{\sigma_1}(O) \leq D = 1$. This contradiction shows that $G/R = O/R$ is a $\sigma_i$-group. Hence $R$ is the unique minimal normal subgroup of $G$. Moreover, Lemma 2.1 implies that $R \notin \Phi(G)$, so $C_G(R) \leq R$ by [5, Chapter A, Theorem 15.2].

Now we show that $A \subseteq R$. First assume that $R$ is $\sigma$-primary. Then $R$ is a $\sigma_j$-group for some $\sigma_j \in \sigma \setminus \{\sigma_1\}$ and so $O^{\sigma_j}(A) = A$. Therefore $R \leq N_G(A)$ by Assertion (5). Consequently, $A \leq C_G(R) \leq R$. Now assume that $R$ is not $\sigma$-primary. Then $R$ is not abelian. Hence $R$ is the product of some minimal normal subgroups of $RA$ by [5, Chapter A, Proposition 4.13(c)]. Hence $R \leq N_{RA}(A)$ by Assertion (8). Then $AR = A \times R$ and so also $A \leq C_G(R) \leq R$. This contradiction completes the proof of the fact that $A \leq O_{\sigma_1}(G)$. Now assume that $A$ is $\sigma$-nilpotent. Then $A = A_1 \times \cdots \times A_t$, where $A_1, \ldots, A_t$ are $\sigma$-primary
groups. Since $A$ is $\sigma$-subnormal in $G$, every factor $A_i$ is $\sigma$-subnormal in $G$. Hence $A_i$ is contained in $O_{\sigma_j}(G)$ for some $j = j(A_i)$, so $A_i \leq F_{\sigma}(G)$. Thus $A \leq F_{\sigma}(G)$.

The lemma is proved. \qed

Let

$$F_{\sigma}(G) \leq F_{1\sigma}(G) \leq \cdots \leq F_{i\sigma}(G) \leq \cdots$$

be the upper $\sigma$-nilpotent series of $G$, i.e., $F_{\infty}(G) = 1$ and

$$F_{i\sigma}(G)/F_{(i-1)\sigma}(G) = F_{\sigma}(G/F_{(i-1)\sigma}(G))$$

for $i > 0$. If $n$ is the smallest integer such that $F_{n\sigma}(G) = G$, then $n$ coincides with the $\sigma$-nilpotent length of $G$.

We use $\mathfrak{N}^n_\sigma$ to denote the class of all $\sigma$-soluble groups $G$ such that $l_\sigma(G) \leq n$ ($n > 0$).

**Lemma 2.3.** The following hold:

(i) If a non-empty class $\mathfrak{G}$ of groups is closed under taking direct products, homomorphic images and subgroups, then the class $\mathfrak{N}_\sigma \mathfrak{G}$ is also closed under taking direct products, homomorphic images and subgroups. Moreover, if $G/\Phi(G) \in \mathfrak{N}_\sigma \mathfrak{G}$, then $G \in \mathfrak{N}_\sigma \mathfrak{G}$.

(ii) The class $\mathfrak{N}^n_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if $G/\Phi(G) \in \mathfrak{N}^n_\sigma$, then $G \in \mathfrak{N}^n_\sigma$.

**Proof.** (i) This assertion can be proved by the direct calculations. For example, if $G/\Phi(G) \in \mathfrak{N}_\sigma \mathfrak{G}$ and $N/\Phi(G)$ is a normal subgroup of $G/\Phi(G)$ such that $(G/\Phi(G))/(N/\Phi(G)) \simeq G/N \in \mathfrak{G}$ and $N/\Phi(G)$ is $\sigma$-nilpotent, then $N$ is $\sigma$-nilpotent by Lemma 2.1. Hence $G \in \mathfrak{N}_\sigma \mathfrak{G}$.

(ii) In the case when $n = 1$, the assertion follows from Lemma 2.1. Now assume that $n > 1$ and the assertion is true for $n - 1$. It is not difficult to show that

$$\mathfrak{N}^n_\sigma = \mathfrak{N}_\sigma \mathfrak{N}^{n-1}_\sigma.$$

Therefore this assertion is a corollary of (i).

The lemma is proved. \qed

**Proposition 2.4.** Let $A$ be a $\sigma$-subnormal subgroup of $G$. If $A$ is $\sigma$-soluble and $l_\sigma(A) \leq n$, then $A \leq F_{n\sigma}(G)$.

**Proof.** First note that in view of Lemma 2.2(9), $F_{\sigma}(A) \leq F_{\sigma}(G)$. Hence

$$l_\sigma(AF_{\sigma}(G)/F_{\sigma}(G)) = l_\sigma(A/A \cap F_{\sigma}(G)) \leq n - 1$$

and so by induction we have $AF_{\sigma}(G)/F_{\sigma}(G) \leq F_{(n-1)\sigma}(G/F_{\sigma}(G)) = F_{n\sigma}/F_{\sigma}(G)$. Therefore $A \leq F_{n\sigma}(G)$. The lemma is proved. \qed

**Proposition 2.5.** The set of all $\sigma$-subnormal subgroups $A$ of $G$ with $l_\sigma(A) \leq n$ forms a sublattice of the lattice of all subgroups of $G$.

**Proof.** In view of Lemma 2.3, Proposition 2.4 and Statements (1) and (2) of Lemma 2.2, we need only to show that if $A$ and $B$ are $\sigma$-subnormal subgroups of $G$, then $\langle A, B \rangle$ is $\sigma$-subnormal in $G$.

Assume that this is false and let $G$ be a counterexample of minimal order. Then $A \neq 1 \neq B$ and $\langle A, B \rangle \neq G$. Let $R$ be a minimal normal subgroup of $G$.

(1) $\langle A, B \rangle R = G$. Hence $\langle A, B \rangle G = 1$.

Suppose that $L = \langle A, B \rangle R \neq G$. Lemma 2.2(1) implies that $A$ and $B$ are $\sigma$-subnormal in $L$. The choice of $G$ implies that $\langle A, B \rangle$ is $\sigma$-subnormal in $L$. On the other hand,

$$L/R = (A, B)R/R = \langle AR/R, BR/R \rangle,$$

where $AR/R$ and $BR/R$ are $\sigma$-subnormal in $G/R$ by Lemma 2.2(6), so the choice of $G$ implies that $L/R$ is $\sigma$-subnormal in $G/R$ and so $L$ is $\sigma$-subnormal in $G$ by Lemma 2.2(3). But then $\langle A, B \rangle$ is $\sigma$-subnormal in $G$ by Lemma 2.2(2). This contradiction shows that we have (1).
(2) If $S$ is a non-identity characteristic subgroup of $C$, where $C \in \{A, B\}$, then $R \nmid N_G(C)$.
Indeed, if $R \trianglelefteq N_G(C)$, then $R \nmid N_G(S)$ and so
\[
S^G = S^{(A,B)R} = S^{(A,B)} \leq C^{(A,B)} \leq \langle A, B \rangle_G = 1,
\]
which leads to a contradiction.

(3) $R$ is a $\sigma_i$-group for some $i \in I$.
Suppose that this is false. Then $R$ is non-abelian, which implies that $R$ is the product of some minimal normal subgroups of $RA$ by [5, Chapter A, Proposition 4.13(c)]. Hence $R \trianglelefteq N_{RA}(A) \nmid N_G(C)$ by Lemma 2.2(8), contrary to Claim (2). Hence we have (3).

Final contradiction. First we show that $A$ and $B$ are $\sigma_i$-groups. Indeed, since $R$ is a $\sigma_i$-group by Claim (3), $R \trianglelefteq N_G(O_{\sigma_i}(A))$ by Lemma 2.2(5). But $O^\sigma(A)$ is a characteristic subgroup of $A$, so $O^\sigma(A) = 1$ by Claim (2). Hence $A$ is a $\sigma_i$-group. Similarly one can get that $B$ is a $\sigma_i$-group. Therefore $(A, B) \leq O_\sigma(G)$ by Lemma 2.2(9). Hence $(A, B)$ is $\sigma$-subnormal in $G$ by Lemma 2.2(4). But this contradicts the choice of $G$. The proposition is proved.

**Proposition 2.6** (See [5, Chapter A, Theorem 14.4]). The set of all subnormal subgroups of $G$ forms a sublattice of the lattice of all subgroups of $G$.

### 3 Some properties of $\sigma$-soluble and $\sigma$-nilpotent groups

The direct calculations show that the following lemma is true

**Lemma 3.1.** The class $\mathcal{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, the extension of a $\sigma$-soluble group by a $\sigma$-soluble group is a $\sigma$-soluble group.

A subgroup $H$ of $G$ is said to be: a Hall $\Pi$-subgroup of $G$ if $|H|$ is $\Pi$-number and $|G : H|$ is $\Pi'$-number; a $\sigma$-Hall subgroup of $G$ if $H$ is a Hall $\Pi$-subgroup of $G$ for some $\Pi \subseteq \sigma$. If $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ such that $H_iH_j = H_jH_i$ for all $i$ and $j$, then we say that $\{H_1, \ldots, H_n\}$ is a $\sigma$-basis of $G$.

Let $A, B$ and $R$ be subgroups of $G$. Then $A$ is said to $R$-permute with $B$ (see [8]) if for some $x \in R$ we have $AB^x = B^xA$.

The following proposition gives the basic properties of $\sigma$-soluble groups.

**Proposition 3.2.** Assume $G$ is $\sigma$-soluble. Then we have the following:

(i) $|G : M|$ is $\sigma$-primary for every maximal subgroup $M$ of $G$.

(ii) For every $\sigma_i \in \sigma(G)$, $G$ has a maximal subgroup $M$ such that $|G : M|$ is a $\sigma_i$-number.

(iii) $G$ has a $\sigma$-basis $\{H_1, \ldots, H_n\}$ such that for each $i \neq j$ every Sylow subgroup of $H_i$ $\sigma$-permutes with every Sylow subgroup of $H_j$.

(iv) For any $\Pi$, $G$ has a Hall $\Pi$-subgroup and every $\sigma$-Hall subgroup of $G$ $\sigma$-permutes with every Sylow subgroup of $G$.

(v) For any $\Pi$, $G$ has a Hall $\Pi$-subgroup $E$, every $\Pi$-subgroup of $G$ is contained in some conjugate of $E$ and $E$ $\sigma$-permutes with every Sylow subgroup of $G$.

**Proof.**

(i) If $H/M_G$ is a chief factor of $G$, then $|(G/M_G) : (M/M_G)| = |G : M|$ divides $|H/M_G|$, so it is $\sigma$-primary.

(ii) Let $R$ be a minimal normal subgroup of $G$. Then $R$ is a $\sigma_k$-group, for some $k$. If $R$ is not a Hall $\sigma_k$-subgroup of $G$, then $G/R$ is a $\sigma$-soluble group such that $\sigma(G/R) = \sigma(G)$. Hence by induction, for every $\sigma_i \in \sigma(G/R)$, $G/R$ has a maximal subgroup $M/R$ such that $|(G/R) : (M/R)| = |G : M|$ is a $\sigma_i$-number. Now suppose that $R$ is a Hall $\sigma_k$-subgroup of $G$ and let $U$ be a complement to $R$ in $G$.

Then $G$ has a maximal subgroup $M$ such that $|G : M|$ divides $|R|$, so it is a $\sigma_i$-number. On the other hand, for every $\sigma_i \neq \sigma_k$, $\sigma_i \in \sigma(G/R)$ and so as above we get that $G$ has a maximal subgroup $M$ such that $|G : M|$ is a $\sigma_i$-number.

(iii)–(v) See [27, Theorems A and B]. The proposition is proved.
Let $H/K$ be a chief factor of $G$. Then we say that $H/K$ is $\sigma$-central in $G$ if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is $\sigma$-primary. Otherwise, it is called $\sigma$-eccentric in $G$.

The following lemma is well-known (see, for example, [25, Lemma 3.29]).

**Lemma 3.3.** Let $R$ be an abelian minimal normal subgroup of $G$ such that $G = RM$ for a maximal subgroup $M$ of $G$. Then $G/M_G \simeq R \rtimes (G/C_G(R))$.

It is well known that a nilpotent group can be characterized as the group in which each subgroup, or each Sylow subgroup, or each maximal subgroup is subnormal. The following result demonstrates that there is a quite similar relation between $\sigma$-nilpotency and $\sigma$-subnormality.

**Proposition 3.4.** Any two of the following conditions are equivalent:

(i) $G$ is $\sigma$-nilpotent.

(ii) Every chief factor of $G$ is $\sigma$-central in $G$.

(iii) $G$ has a complete Hall $\sigma$-set $\mathcal{H}$ such that every member of $\mathcal{H}$ is $\sigma$-subnormal in $G$.

(iv) Every subgroup of $G$ is $\sigma$-subnormal in $G$.

(v) Every maximal subgroup of $G$ is $\sigma$-subnormal in $G$.

**Proof.** Since (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are clear, it is enough to prove the implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) For every chief factor $H/K$ of $G$, where $H \leq H_i$, we have that $(H/K) \rtimes (G/C_G(H/K))$ is a $\pi(H_i)$-group. Hence $(H/K) \rtimes (G/C_G(H/K))$ is $\sigma$-primary. Now applying the Jordan-Hölder theorem (see [5, Chapter A, Theorem 3.2]), we get that every chief factor of $G$ is $\sigma$-central.

(ii) $\Rightarrow$ (v) Let $M$ be a maximal subgroup of $G$. Assume that $M_G \neq 1$. It is clear that the hypothesis holds for $G/M_G$, so $M/M_G$ is $\sigma$-subnormal in $G/M_G$ by induction. Hence $M$ is $\sigma$-subnormal in $G$ by Lemma 2.2(3). Now assume that $M_G = 1$. By [5, Chapter A, Theorem 15.2], either $G$ has a unique minimal normal subgroup $R$ or $G$ has exactly two minimal normal subgroups $R$ and $N$ and the following hold: $R$ and $N$ are isomorphic non-abelian groups, $R \cap M = 1 = N \cap M$ and $C_G(R) = N$. Let $C = C_G(R)$. Suppose that $R$ is abelian. Then $C = R$ by [5, Chapter A, Theorem 15.2], so in this case we have $G \simeq G/M_G \simeq R \rtimes (G/C_G(R))$ is $\sigma$-primary by Lemma 3.3(ii). Hence, for some $\sigma_i \in \sigma$, $G$ is a $\sigma_i$-group. But then $M$ is $\sigma$-subnormal in $G$. Similarly we get that $M$ is $\sigma$-subnormal in $G$ in the case when $C = 1$. Finally, if $C = N$, then $G/N \simeq M \simeq G/R$ is $\sigma$-primary. It follows that $G$ is a $\sigma_i$-group. Thus $M$ is $\sigma$-subnormal in $G$.

(iii) $\Rightarrow$ (i) This follows from Proposition 2.5 since every member of $\mathcal{H}$ is $\sigma$-nilpotent.

(i) $\Rightarrow$ (iv) This follows from the implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) and the evident fact that every subgroup of a $\sigma$-nilpotent group is $\sigma$-nilpotent (see Lemma 2.1).

(v) $\Rightarrow$ (i) Assume that this is false and let $G$ be a counterexample of minimal order.

First note that $G$ is $\sigma$-soluble. Indeed, for any maximal subgroup $M$ of $G$, $G/M_G$ is $\sigma$-primary and so $G/M_G$ is $\sigma$-soluble. But then $G/\Phi(G)$ is a subdirect product of some $\sigma$-soluble groups, which implies that $G/\Phi(G)$ is $\sigma$-soluble by Lemma 3.1. Hence $G$ is $\sigma$-soluble. By Proposition 3.2, $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$.

Let $H = H_i$ and $R$ be a minimal normal subgroup of $G$. We show that $H$ is normal in $G$. Assume that is false. By Lemma 2.2(6), the hypothesis holds for $G/R$, so $HR/R$ is normal in $G/R$ by the choice of $G$. Hence we can assume that $R \not\subseteq H$, so $R \cap H = 1$ since $G$ is $\sigma$-soluble. If $G$ has a minimal normal subgroup $N \neq R$, then as above we get that $HN$ is normal in $G$ and so $RH \cap NH = H(R \cap NH) = H(R \cap N) = H$ is normal in $G$. Therefore $R$ is the unique minimal normal subgroup of $G$. Moreover, in view of Lemma 2.1, we have $R \not\subseteq \Phi(G)$ since $HR/R \simeq H$ is $\sigma$-nilpotent and $HR$ is normal in $G$. Let $M$ be a maximal subgroup of $G$ such that $G = RM$. Then $M_G = 1$. But $M$ is $\sigma$-subnormal in $G$ by hypothesis and $G \simeq G/M_G$ is $\sigma$-primary, which implies that $G$ is a $\sigma_i$-group, for some $\sigma_i \in \sigma$. Therefore $H = G$. This contradiction shows that (v) $\Rightarrow$ (i).

The proposition is proved.

We say that $G$ is $\Pi$-closed if $O_{\Pi}(G)$ is a Hall $\Pi$-subgroup of $G$.

**Lemma 3.5.** Let $H$ be a normal subgroup of $G$. If $H/H \cap \Phi(G)$ is $\Pi$-closed, then $H$ is $\Pi$-closed.
Proof. See the proof of Lemma 2.5 in [26]. □

The integers \( n \) and \( m \) are called \( \sigma \)-coprime if \( \sigma(n) \cap \sigma(m) = \emptyset \).

**Lemma 3.6.** If a \( \sigma \)-soluble group \( G \) has three \( \Pi \)-closed subgroups \( A, B \) and \( C \) whose indices \( |G : A|, |G : B| \) and \( |G : C| \) are pairwise \( \sigma \)-coprime, then \( G \) is \( \Pi \)-closed.

**Proof.** Suppose that this lemma is false and let \( G \) be a counterexample of minimal order. Let \( N \) be a minimal normal subgroup of \( G \). Then the hypothesis holds for \( G/N \), so \( G/N \) is \( \Pi \)-closed by the choice of \( G \). Therefore \( N \) is not a \( \Pi \)-group. Moreover, \( N \) is the unique minimal normal subgroup of \( G \) and, by Lemma 3.5, \( N \not\subseteq \Phi(G) \). Hence \( C(G)(N) \subseteq N \). Since \( G \) is \( \sigma \)-soluble by hypothesis, \( N \) is a \( \sigma_i \)-group for some \( i \). Then \( \sigma_i \in \Pi' \).

Since \( |G : A|, |G : B| \) and \( |G : C| \) are pairwise \( \sigma \)-coprime, there are at least two subgroups, say \( A \) and \( B \), such that \( N \leq A \cap B \). Then \( O_{\Pi}(A) \leq C_G(N) \leq N \), so \( O_{\Pi}(A) = 1 \). But by hypothesis, \( A \) is \( \Pi \)-closed, hence \( A \) is a \( \Pi' \)-group. Similarly we get that \( B \) is a \( \Pi' \)-group and so \( G = AB \) is a \( \Pi' \)-group. But then \( G \) is \( \Pi \)-closed. This contradiction completes the proof of the lemma. □

**Lemma 3.7** (See [13, Chapter III, Theorem 5.2]). If \( G \) is a Schmidt group, then \( G = P \times Q \), where \( P = G^{\Pi} \) is a Sylow \( p \)-subgroup of \( G \) and \( Q = \langle x \rangle \) is a cyclic Sylow \( q \)-subgroup of \( G \). Moreover, \( \langle x^q \rangle \leq \Phi(G) \), \( P/\Phi(P) \) is a chief factor of \( G \), \( P \) is of exponent \( p \) or exponent 4 (if \( P \) is a non-abelian 2-group) and \( \Phi(P) = 1 \) if \( P \) is abelian.

**Proposition 3.8.** Let \( G \) be a \( \sigma \)-soluble group. Suppose that \( G \) is not \( \Pi \)-closed but all proper subgroups of \( G \) are \( \Pi \)-closed. Then \( G \) is a \( \Pi' \)-closed Schmidt group.

**Proof.** Suppose that this proposition is false and let \( G \) be a counterexample of minimal order. Let \( R \) be a minimal normal subgroup of \( G \) and \( \{H_1, \ldots, H_t\} \) a complete Hall \( \sigma \)-set of \( G \). Without loss of generality we can assume that \( H_i \) is a \( \sigma_i \)-group for all \( i = 1, \ldots, t \).

1. \( |\sigma(G)| = 2 \). Hence \( G = H_1H_2 \).

   It is clear that \( |\sigma(G)| > 1 \). Suppose that \( |\sigma(G)| > 2 \). Then, since \( G \) is \( \sigma \)-soluble, there are maximal subgroups \( M_1, M_2 \) and \( M_3 \) whose indices \( |G : M_1|, |G : M_2| \) and \( |G : M_3| \) are pairwise \( \sigma \)-coprime. But the subgroups \( M_1, M_2 \) and \( M_3 \) are \( \Pi \)-closed by hypothesis. Hence \( G \) is \( \Pi \)-closed by Lemma 3.6, which leads to a contradiction. Thus \( |\sigma(G)| = 2 \).

   Without loss of generality we can assume that \( \sigma_2 \in \Pi \). Then \( \Pi \cap \sigma(G) = \{\sigma_2\} \).

2. \( G \) is a \( \Pi' \)-closed Schmidt group.

   Lemma 3.5 and the choice of \( G \) imply that \( G/R \) is not \( \Pi \)-closed. On the other hand, every maximal subgroup \( M/R \) of \( G/R \) is \( \Pi \)-closed since \( M \) is \( \Pi \)-closed by hypothesis. Hence the hypothesis holds for \( G/R \). The choice of \( G \) implies that \( G/R \) is a \( \Pi' \)-closed Schmidt group.

3. \( \Phi(G) = 1 \), \( R \) is the unique minimal normal subgroup of \( G \) and \( R \leq H_1 \).

   Suppose that \( R \leq \Phi(G) \). Then \( G = G/R \) is a \( \Pi' \)-closed Schmidt group by Claim (2), so \( G = H_1 \times H_2 = P \rtimes Q \), where \( H_1 = P \) is a \( p \)-group and \( H_2 = Q \) is a \( q \)-group for some primes \( p \) and \( q \) by Lemma 3.7. Therefore, in fact, \( G \) is a \( p \)-nilpotent but every maximal subgroup of \( G \) is a \( p \)-nilpotent. Hence \( G \) is a \( \Pi' \)-closed Schmidt group by [13, Chapter IV, Theorem 5.4], which leads to a contradiction. Therefore \( \Phi(G) = 1 \).

   Now assume that \( G \) has a minimal normal subgroup \( L \neq R \). Since \( \Phi(G) = 1 \), there are maximal subgroups \( M \) and \( T \) of \( G \) such that \( LM = G \) and \( RT = G \). By hypothesis, \( M \) and \( T \) are \( \Pi \)-closed. Hence \( G/L \cong LM/L \cong M/M \cap L \) is \( \Pi \)-closed. Similarly, \( G/R \) is \( \Pi \)-closed and so \( G \cong G/L \cap R \) is \( \Pi \)-closed, which leads to a contradiction. Hence \( R \) is the unique minimal normal subgroup of \( G \) and \( R \leq H_1 \).

   Final contradiction. In view of Claim (3), \( C_G(R) \leq R \) and \( R \leq H_1 \). Hence \( |H_2| \) is a prime and \( RH_2 = G \) since every proper subgroup of \( G \) is \( \Pi \)-closed. Therefore \( R = H_1 \), so \( R \) is not abelian since \( G \) is not a \( \Pi' \)-closed Schmidt group. It is clear that for any prime \( p \) dividing \( |R| \) there is a Sylow \( p \)-subgroup \( P \) of \( G \) such that \( PH_2 = H_2P \) by Lemma 3.2(iv). But \( H_2P < G \), so \( H_2P = H_2 \times P \). Therefore \( R \leq N_G(H_2) \) and thereby \( G = R \times H_2 = H_1 \times H_2 \) is \( \sigma \)-nilpotent. This final contradiction completes the proof. □

We say that \( G \) is \( \sigma \)-fiber if \( |\sigma(G)| = |\pi(G)| \).
Corollary 3.9. Suppose that $G$ is not \(\sigma\)-nilpotent but every proper subgroup of $G$ is \(\sigma\)-nilpotent. If $G$ is \(\sigma\)-soluble, then $G$ is a \(\sigma\)-fiber Schmidt group.

Proof. It is clear that $G$ is \(\sigma\)-nilpotent if and only if $G$ is II-closed for all $\Pi \subseteq \sigma$. Hence, for some II, $G$ is not II-closed. On the other hand, every proper subgroup of $G$ is II-closed. Hence $G$ is a Schmidt group by Proposition 3.8 and clearly $|\sigma(G)| = |\pi(G)|$.

\[ \square \]

4 Proof of Theorem 1.2

In this section, we need the following lemma.

Lemma 4.1 (See [26, Lemmas 2.8, 3.1 and Theorem B(1)]). Let $H$, $K$ and $R$ be subgroups of a \(\sigma\)-soluble group $G$. Suppose that $H$ is \(\sigma\)-quasinormal in $G$ and $R$ is normal in $G$. Then we have the following:

1. \(H \leq E \leq G\), then $H$ is \(\sigma\)-quasinormal in $E$.
2. The subgroup $HR/R$ is \(\sigma\)-quasinormal in $G/R$.
3. \(R \leq H\) and $HR/R$ is \(\sigma\)-quasinormal in $G/R$, then $H$ is \(\sigma\)-quasinormal in $G$.
4. \(H\) is \(\sigma\)-subnormal in $G$.
5. If $H$ is a \(\sigma\)-group, then $O^{\sigma_i}(G) \leq N_G(H)$.

Lemma 4.2. The following statements hold:

1. $m_{\sigma}(G) \leq m_{\sigma q}(G)$.
2. If $M$ is a non-\(\sigma\)-subnormal maximal subgroup of $G$, then $m_{\sigma}(M) \leq m_{\sigma}(G) - 1$.
3. If $R$ is a normal subgroup of $G$, then $m_{\sigma}(G/R) \leq m_{\sigma}(G)$.
4. If $R$ is a normal subgroup of $G$ and $G$ is \(\sigma\)-soluble, then $m_{\sigma q}(G/R) \leq m_{\sigma q}(G)$.

Proof. (1) This follows from Lemma 4.1(4).

(2) Since $M$ is not \(\sigma\)-subnormal in $G$, $m_{\sigma q}(G) > 1$. Moreover, for $n = m_{\sigma}(G)$, each $(n - 1)$-maximal subgroup of $M$ is \(\sigma\)-subnormal in $M$ by Lemma 2.2(1). Hence $m_{\sigma}(M) \leq m_{\sigma}(G) - 1$.

(3) If each maximal chain of $G/R$ has length $r < m_{\sigma}(G)$, it is clear. Otherwise, this follows from Lemma 2.2(3) and 2.2(6).

(4) This is a corollary of Lemma 4.1(3)

The lemma is proved. \[ \square \]

The following properties of the rank of a soluble group are useful in our proof.

Lemma 4.3 (See [13, Chapter VI, Lemma 5.3]). Let $G$ be soluble. Then we have the following:

1. $r(G/R) \leq r(G)$ for all normal subgroups $R$ of $G$.
2. $r(E) \leq r(G)$ for all subgroups $E$ of $G$.
3. $r(A \times B) = \text{Max}\{r(A), r(B)\}$.

Lemma 4.4 (See [12, Lemma 11]). Let $G$ be soluble and $R$ be a minimal normal subgroup of $G$. Let $H$ be a minimal supplement to $C_G(R)$ in $G$. Then $H \cap R = 1$.

Lemma 4.5. The following statements hold:

(i) If each $n$-maximal subgroup of $G$ is \(\sigma\)-subnormal in $G$ and $n > 1$, then each $(n - 1)$-maximal subgroup of $G$ is \(\sigma\)-nilpotent.

(ii) If each $n$-maximal subgroup of $G$ is \(\sigma\)-subnormal in $G$, then each $(n + 1)$-maximal subgroup of $G$ is \(\sigma\)-subnormal in $G$.

Proof. (i) Let $H$ be an $(n - 1)$-maximal subgroup of $G$ and $K$ a maximal subgroup of $H$. Then $K$ is an $n$-maximal subgroup of $G$, so it is \(\sigma\)-subnormal in $G$. Then, by Lemma 2.2(1), $K$ is \(\sigma\)-subnormal in $H$. Therefore each maximal subgroup of $H$ is \(\sigma\)-subnormal in $H$. It follows from Proposition 3.4 that $H$ is \(\sigma\)-nilpotent.

(ii) Let $L \leq M \leq G$, where $M$ is an $n$-maximal subgroup of $G$ and $L$ is a maximal subgroup of $M$. If $n = 1$, $G$ is \(\sigma\)-nilpotent and so $L$ is \(\sigma\)-subnormal in $G$ by Proposition 3.4. On the other hand, in the case
when $n > 1$ Statement (i) implies that each $(n-1)$-maximal subgroup of $G$ is $\sigma$-nilpotent. Then $M$ is $\sigma$-nilpotent by Lemma 2.1, so $L$ is $\sigma$-subnormal in $G$ by Lemma 2.2(4).

The lemma is proved.

**Proof of Theorem 1.2.** Suppose that this theorem is false and let $G$ be a counterexample of minimal order. Let $H = \{1, H_1, \ldots, H_t\}$ be a complete Hall $\sigma$-set of $G$. Then $t > 1$.

(i) Suppose that this is false. Let $R$ be a minimal normal subgroup of $G$ and $|R| = p^n$. Without loss of generality we can assume that $R \leq H_1$. Let $n = \max(q(G))$. Since $G$ is not $\sigma$-nilpotent, some maximal subgroup $M$ of $G$ is not $\sigma$-subnormal in $G$ by Proposition 3.4 and so $M$ is not $\sigma$-quasinormal in $G$ by Lemma 4.1(4). Thus $n > 1$.

(1) $r(G/R) \leq n + r - 2$.

Assume that $r(G/R) > n + r - 2$. Note that $\{H_1/R, \ldots, H_tR/R\}$ is a complete Hall $\sigma$-set of $G/R$ and $r(H_i/R) = r(H_iH_1/R) \leq r(H_i) \leq r$ for all $i = 1, \ldots, t$ by Lemma 4.3(1). Assume that $G/R$ is $\sigma$-soluble. Then $G/R = (H_1R/R) \times \cdots \times (H_tR/R)$, so $r(G/R) \leq r \leq n + r - 2$ since $n > 1$ by Lemma 4.3(3). This contradiction shows that $G/R$ is not $\sigma$-nilpotent. Moreover, $G/R$ is $\sigma$-soluble by Lemma 3.1. By Lemmas 4.1(2) and 4.1(3), $m_{\max}(G/R) \leq m_{\max}(G) = n$. The choice of $G$ implies that $r(G/R) \leq m_{\max}(G/R) + r - 2 \leq n + r - 2$, which leads to a contradiction. Hence we have (1).

(2) $m > n + r - 2$. Hence $R$ is the only minimal normal subgroup of $G$.

First note that in view of the Jordan-Hölder theorem, Claim (1) and the choice of $G$ we have $m > n + r - 2$. If $G$ has a minimal normal subgroup $N \neq R$, then $r(G/N) \leq n + r - 2$ by Claim (1), so in view of the $G$-isomorphism $R \cong RN/N$ we get that $m \leq n + r - 2$, which leads to a contradiction. Hence $R$ is the only minimal normal subgroup of $G$.

(3) If $M$ is a proper subgroup of $G$, then $r(M) \leq n + r - 2$.

It is enough to consider the case when $M$ is a maximal subgroup of $G$. Assume that $r(M) > n + r - 2$. Then $M$ is not $\sigma$-nilpotent (see the proof of Claim (1)). Therefore $n > 2$ by Lemmas 4.1(1) and Proposition 3.4. Moreover, since $G$ is $\sigma$-soluble, $M$ possesses a complete Hall $\sigma$-set $\{M_1, \ldots, M_t\}$ such that $M_i = H_i \cap M$ for all $i = 1, \ldots, t$ by Lemma 3.2(v). Hence $r(M_i) \leq r(H_i) \leq r$ for all $i = 1, \ldots, t$ by Lemma 4.3(2). Therefore, $M$ satisfies the hypothesis, with $n - 1$ instead of $n$, by Lemma 4.1(1) and so the choice of $G$ implies that $r(M) \leq n - 1 + r - 2 \leq n + r - 2$, which leads to a contradiction. Hence we have (3).

(4) $R \not\leq \Phi(G)$.

Suppose that $R \leq \Phi(G)$. Then for a minimal supplement $H$ to $C_G(R)$ in $G$ we have $H \cap R = 1$ by Lemma 4.4, so $RH \neq G$ and $R$ is a minimal normal subgroup of $RH$. But Claim (3) implies that $r(RH) \leq n + r - 2$ and so $m \leq n + r - 2$, contrary to Claim (2). Hence we have (4).

Final contradiction for (i). Claim (4) implies that there is a maximal subgroup $M$ of $G$ such that $G = RM$ and $H_2 \leq M$. Then $M_G = 1$ by Claim (2), so $C_G(R) = C_G(R) \cap RM = R(C_G(R) \cap M) = R$. Let $H_2 = M_s$ be a member of a maximal chain $1 = M_1 < M_2 < \cdots < M_i < M_0 = M$ of $M$. Then $R$ is an $l$-maximal subgroup of $G$. First suppose that $l > n - 1$. Assume also that $H_2 \not\leq H_{n-1}$. By hypothesis $M_{n-1}$ is $\sigma$-quasinormal in $G$. Hence $H_2 \not\leq H_{n-1}$ for all $x \in G$. It follows that $(H_2)^G \leq M = 1$, so $H_2 = 1$, which leads to a contradiction. Therefore $n \leq s$, i.e., for the $n$-maximal subgroup $H = M_{n-1}$ of $G$ contained in $H_2$ we have $H \neq 1$. Then $H \leq O_{\sigma}(H_2)(G)$ by Lemma 2.2(9). But $R \cap O_{\sigma}(H_2)(G) = 1$ since $R \leq H_1$, so $1 < H \leq O_{\sigma}(H_2)(G) \leq C_G(R) = R$, which leads to a contradiction.

Therefore $n - 1 \leq l$, so $M$ possesses a maximal chain $1 = M_k < M_{k-1} < \cdots < M_1 < M_0 = M$, where $k < n$. Then $R$ is a $k$-maximal subgroup of $G$. Therefore every $l$-maximal subgroup of $R$ is a $(k + l)$-maximal subgroup of $G$. Let $R_0$ be a minimal normal subgroup of $H_1$ contained in $R$ with $|R_0| = p^a$. Let $L$ be an $(n-k)$-maximal subgroup of $R$ with $|L| = p^b$ such that $L \leq R_0$ in the case when $b < a$ and $R_0 \leq L$ if $a \leq b$. Then $L$ is an $n$-maximal subgroup of $G$, so $L$ is $\sigma$-quasinormal in $G$.

First suppose that $L \leq R_0$. Then

$$L^G = L^{H_1N_G(L)} = L^{H_1} \leq (R_0)_G = R$$

by Lemma 4.1(5). Hence $R_0 = R$. Then $m = a \leq r$ and so $m \leq r + n - 2$ since $n > 1$, contrary to (2).
Thus $R_0 \leq L$, so

$$R_0^G = R_0^{H_1 N_G(L)} = R_0^{N_G(L)} \leq L,$$

which implies that $L = R$, a contradiction also. Hence Assertion (i) is true.

(ii) Let $n = m_\sigma(G)$. Suppose that $n > l_\sigma(G)$. Then $n > 1$. Indeed, if $n = 1$, $G$ is $\sigma$-nilpotent by Proposition 3.4 and so $l_\sigma(G) = 1 = m_\sigma(G)$, which leads to a contradiction. The choice of $G$ and Lemma 4.2(3) imply that $l_\sigma(G/F_\sigma(G)) \leq m_\sigma(G/F_\sigma(G)) \leq n$. Hence $F_\sigma(G) \not\leq \Phi(G)$ by Lemma 2.3(2). Let $M$ be a maximal subgroup of $G$ such that $F_\sigma(G)M = G$. Then $l_\sigma(G/F_\sigma(G)) = l_\sigma(MF_\sigma(G)/F_\sigma(G)) = l_\sigma(M/M \cap F_\sigma(G)) = l_\sigma(G) - 1$. Lemma 2.1 implies that $M \cap F_\sigma(G) \not\leq F_\sigma(M)$. Therefore $l_\sigma(G) \leq l_\sigma(M) + 1$. Note that $m_\sigma(M) \leq m_\sigma(G) - 1$. Indeed, since $n > 1$, each $(n-1)$-maximal subgroup of $M$ is $\sigma$-quasinormal in $G$. Hence each $(n-1)$-maximal subgroup of $M$ is $\sigma$-quasinormal in $M$ by Lemma 4.1(1). Hence $m_\sigma(M) \leq n - 1$. By the choice of $G$ we have $l_\sigma(M) \leq m_\sigma(M)$, and then $l_\sigma(G) \leq l_\sigma(M) + 1 \leq m_\sigma(M) + 1 \leq m_\sigma(G)$. This contradiction completes the proof of (ii).

(iii) Suppose that $m_\sigma(G) < |\pi(G)|$. Let $P_1, \ldots, P_n$ be a Sylow basis of $G$ and $H$ a complete Hall set of $G$. Then for any $i$, $i = 1$ say, we have $P_1 < P_1P_2 < \cdots < P_1P_2 \cdots P_n = G$, so $P_1$ is at least an $(n-1)$-maximal subgroup of $G$. Therefore $P_1$ is $\sigma$-subnormal in $G$ by Lemma 4.5(ii) since $m_\sigma(G) < |\pi(G)|$. Hence every Sylow subgroup of $G$ is $\sigma$-subnormal in $G$ and so every member of $H$ is $\sigma$-subnormal in $G$ by Proposition 2.5. But then $G$ is $\sigma$-nilpotent by Proposition 3.4, which leads to a contradiction. Hence $|\pi(G)| \leq m_\sigma(G)$.

The theorem is proved.

\section{Proofs of Theorem 1.4 and Corollaries 1.7 and 1.8}

\textit{Proof of Theorem 1.4.} Let $R$ be a minimal normal subgroup of $G$.

(i) Suppose that this assertion is false and let $G$ be a counterexample of minimal order. First note that $G/R$ is $\sigma$-soluble. Indeed, if $R$ is a maximal subgroup or a 2-maximal subgroup of $G$, it is clear. Otherwise, the hypothesis holds for $G/R$ by Lemma 2.2(6), so the choice of $G$ implies that $G/R$ is $\sigma$-soluble. Hence $R$ is the unique minimal normal subgroup of $G$ by Lemma 3.1 and $R$ is not $\sigma$-primary. Hence $R$ is not abelian.

Let $p$ be any odd prime dividing $|R|$ and $R_p$ a Sylow $p$-subgroup of $R$. The Frattini argument implies that there is a maximal subgroup $M$ of $G$ such that $N_G(R_p) \leq M$ and $G = RM$. It is clear that $M_G = 1$, so $M$ is not $\sigma$-subnormal in $G$ since $G/M_G \cong G$ is not $\sigma$-primary. Let $D = M \cap R$. Then $R_p$ is a Sylow $p$-subgroup of $D$.

(1) $D$ is not nilpotent. Hence $D \not\leq \Phi(M)$ and $D$ is not a $p$-group.

Assume that $D$ is nilpotent. Then $R_p$ is normal in $M$. Hence $Z(J(R_p))$ is normal in $M$. Since $M_G = 1$, it follows that $N_G(Z(J(R_p))) = M$ and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that $R$ is $p$-nilpotent by Glauberman-Thompson’s theorem on the normal $p$-complements. But then $R$ is a $p$-group, which leads to a contradiction. Hence we have (1).

(2) $R < G$.

Suppose that $R = G$ is a simple non-abelian group. Assume that some proper non-identity subgroup $A$ of $G$ is $\sigma$-subnormal in $G$. Then there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \leq A_i$ or $A_i/(A_{i-1})A_i$ is $\sigma$-primary for all $i = 1, \ldots, n$. Without loss of generality, we can assume that $M = A_{n-1} < G$. Then $M_G = 1$ since $G = R$ is simple, so $G \cong G/1$ is $\sigma$-primary, which leads to a contradiction. Hence every proper $\sigma$-subnormal subgroup of $G$ is trivial.

Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime dividing $|G|$, and let $L$ be a maximal subgroup of $G$ containing $P$. Then, in view of [13, Chapter IV, Theorem 2.8], $|P| > p$. Let $V$ be a maximal subgroup of $P$. If $|V| = p$, then $P$ is abelian, so $P < L$ by [13, Chapter IV, Theorem 7.4]. Hence there is a 3-maximal subgroup $W$ of $G$ such that $V \leq W$. But then some proper non-identity subgroup of $G$ is $\sigma$-subnormal in $G$ by hypothesis, which leads to a contradiction. Therefore $|V| > p$, which again implies that some proper non-identity subgroup of $G$ is $\sigma$-subnormal in $G$. This contradiction shows that we have (2).
(3) $M$ is $\sigma$-soluble.

Let $L < T < M$, where $L$ is a maximal subgroup of $T$ and $T$ is a maximal subgroup of $M$. Since $M$ is not $\sigma$-subnormal in $G$, either $L$ or $T$ is $\sigma$-subnormal in $G$ and so it is $\sigma$-subnormal in $M$ by Lemma 2.2(1). Hence the hypothesis holds for $M$, so $M$ is $\sigma$-soluble by the choice of $G$.

(4) $M = D \times T$, where $T$ is a maximal subgroup of $M$ of prime order.

In view of Claim (1), there is a maximal subgroup $T$ of $M$ such that $M = DT$. Then $G = RM = R(DT) = RT$ and so, in view of (2), $T \neq 1$. Assume that $|T|$ is not a prime and let $V$ be a maximal subgroup of $T$. Since $M$ is not $\sigma$-subnormal in $G$, at least one of the subgroups $T$ or $V$ is $\sigma$-subnormal in $G$ by hypothesis. Claim (3) implies that $V$ and $T$ are $\sigma$-soluble. Consider, for example, the case when $V$ is $\sigma$-subnormal in $G$. Since $V \neq 1$ and $V$ is $\sigma$-soluble, for some $i$ we have $O_{\sigma_i}(V) \neq 1$. But $O_{\sigma_i}(V) \leq O_{\sigma_i}(G)$ by Lemma 2.2(9), so $R$ is $\sigma$-primary, which leads to a contradiction. Hence $|T|$ is a prime, so $M = D \times T$.

Final contradiction for (i). Since $T$ is a maximal subgroup of $M$ and it is cyclic, $M$ is soluble and so $|D|$ is a prime power, which contradicts (1). Hence Assertion (i) is true.

(ii) Suppose that this false. Then $2 \in \pi(G)$. Part (i) implies that $G$ is $\sigma$-soluble. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a $\sigma$-basis of $G$. Without loss of generality we can assume that $H_1$ is a $\sigma_1$-group and $2 \in \pi(H_1)$. Then $H_1$ is not soluble, so $|\pi(H_1)| > 1$. Let $p \in \pi(H_1)$.

(1) $t = 2$ and $H_2$ is a Sylow subgroup of $G$.

By Proposition 3.2(v), $G$ has a Hall $\sigma_1$-subgroup $E$ and $E$ permutes with some Sylow $p$-subgroup $P$ of $G$ for each $p \in \pi(H_1)$. It is clear that $PE < G$. We show that $PE$ is soluble. In fact, if $PE$ is $\sigma$-nilpotent, then $PE = P \times E$, where $2 \nmid |E|$, so $PE$ is soluble. Now assume that $PE$ is not $\sigma$-nilpotent. Then the hypothesis holds for $PE$, so $PE$ is soluble by the choice of $G$. Hence $PE$ has a Sylow basis $P = \{P_1, P_2, \ldots, P_t\}$. If $t > 2$ or $H_2$ is not a Sylow subgroup of $G$, then every member of $P$ is at least a 3-maximal subgroup of $G$. Hence every member of $P$ is $\sigma$-subnormal in $G$ by Lemma 2.2(9). This shows that every Sylow subgroup of $G$ is $\sigma$-subnormal in $G$. Therefore all members of $\mathcal{H}$ are normal in $G$ by Lemma 2.2(9), which implies that $G$ is $\sigma$-nilpotent. This contradiction shows that we have (1).

(2) $O_{\sigma_1}(G) \neq 1$.

Suppose that this is false. Since $G$ is $\sigma$-soluble, $R$ is $\sigma$-primary. Hence $R \leq H_2$. Let $P$ be a Sylow $p$-subgroup of $H_1$ and $P < M$, where $M$ is a maximal subgroup of $H_1$. Since $H_1$ is not soluble, $P < M < H_1 < G$ by [13, Chapter IV, Theorem 7.4]. Therefore there is a 3-maximal subgroup $W$ of $G$ such that $P < W < H_1$. Then $W$ is $\sigma$-subnormal in $G$, so $1 < W \leq O_{\sigma_1}(G)$ by Lemma 2.2(9), which leads to a contradiction.

Hence we can assume that $R \leq O_{\sigma_1}(G)$.

(3) $H_2$ is normal in $G$.

First, suppose that $R$ is a $p$-group for some prime $p$ and let $Q$ be a Sylow $q$-subgroup of $H_1$, where $q \neq p$. By Proposition 3.2, there is $Z \in G$ such that $H_2Q^Z = Q^ZH_2$. Hence we have a subgroup chain $H_2 < H_2Q^Z < RH_2Q^Z < G$. It follows from Lemma 4.5(ii) that $H_2$ is $\sigma$-subnormal in $G$, so it is normal in $G$ by Lemma 2.2(9).

Now assume that $R$ is not abelian. Then, for any odd prime $p$ dividing $|R|$, the subgroup $R$ is not $p$-nilpotent. Hence by Glauberman-Thompson’s theorem on the normal $p$-complement, we have that $P < N_R(Z(J(P))) < R$, where $P$ is a Sylow $p$-subgroup of $R$. Since $|R|, |H_2| = 1$, $H_2$ normalizes some Sylow $p$-subgroup of $R$, say $P$. Hence $H_2 \leq N_G(Z(J(P)))$. But then we have a chain $H_2 < H_2P < H_2N_R(Z(J(P))) < G$, so $H_2$ is $\sigma$-subnormal in $G$ by Lemma 4.5(ii). Consequently $H_2$ is normal in $G$ by Lemma 2.2(9).

Final contradiction for (ii). Let $P$ be a Sylow $p$-subgroup of $H_1$ and $V$ be a maximal subgroup of $P$. Since $|\pi(H_1)| > 1$, $V$ is $\sigma$-subnormal in $G$ by Lemma 4.5(ii). Suppose that $P \notin O_{\sigma_1}(G)$. Then $P \notin R$ and $P$ is not $\sigma$-subnormal in $G$ by Lemma 2.2(9), so $P$ is cyclic by Proposition 2.5. Hence $R \cap P \leq \Phi(R)$, and so $R$ is $p$-nilpotent by the Tate theorem (see [13, Chapter IV, Theorem 4.7]). But then $R$ is a $p$-group or a $p'$-group. Assume that $R$ is a $p$-group. Then $G/R$ is not $\sigma$-nilpotent. Otherwise, $H_1/R \leq G/R$. It follows from Claim (3) that $G = H_1 \times H_2$, which contradicts $m_{\sigma}(G) > 1$. Hence $G/R$ is not $\sigma$-nilpotent, and so $1 < m_{\sigma}(G/R)$. But then $G/R$ satisfies the hypothesis by Lemma 2.2(6). Hence $G/R$ is soluble by
induction. Consequently, \( G \) is soluble, which leads to a contradiction. Now assume that \( R \) is a \( p' \)-group. Then by the Frattini argument, \( P \) normalizes some Sylow \( q \)-subgroup \( Q \) of \( R \), where \( q \neq p \) divides \( R \).

Hence we have a subgroup chain \( P < PQ < H_2PQ < G \), so \( P \) is \( \sigma \)-subnormal in \( G \) by Lemma 4.5(ii). This shows that every Sylow subgroup of \( H_1 \) is \( \sigma \)-subnormal in \( G \). It follows from Lemma 2.2(9) that \( H_1 \) is \( \sigma \)-subnormal in \( G \). Thus \( H_1 \) is normal in \( G \) by Lemma 2.2(9), so \( G = H_1 \times H_2 \) is \( \sigma \)-nilpotent. It follows from Proposition 3.4 that \( m_\sigma(G) = 1 \). This final contradiction completes the proof of (ii).

The theorem is proved.

Proof of Corollary 1.7. The sufficiency is clear. We only need to prove the necessity. By Theorem 1.4, \( G \) is \( \sigma \)-soluble. Since \( m_\sigma(G) = 1 \), \( G \) is not \( \sigma \)-nilpotent by Proposition 3.4. On the other hand, if \( M \) is a maximal subgroup of \( G \), then every maximal subgroup of \( M \) is \( \sigma \)-subnormal in \( G \) and so it is \( \sigma \)-subnormal in \( M \) by Lemma 2.2(1). Therefore \( M \) is \( \sigma \)-nilpotent by Proposition 3.4. Hence \( G \) is a Schmidt group such that \( |\pi(G)| = |\sigma(G)| \) by Corollary 3.9. Then by Lemma 3.7, \( G = P \times Q \), where \( P = G^{p_n} \) is a Sylow \( p \)-subgroup of \( G \) and \( Q = \langle x \rangle \) is a cyclic Sylow \( q \)-subgroup of \( G \). Moreover, \( P \) is of exponent \( p \) or exponent 4 (if \( P \) is a non-abelian 2-group) and \( P/\Phi(P) \) is a chief factor of \( G \). If \( \Phi(P) \neq 1 \), then there exists a maximal subgroup \( M \) of \( G \) such that \( Q < M < G \). Then \( Q \) is \( \sigma \)-subnormal in \( G \) by Lemma 4.5(ii). It follows from Lemma 2.2(9) that \( Q \) is normal in \( G \). This contradiction shows that every Sylow subgroup of \( G \) is abelian.

Proof of Corollary 1.8. In view of Corollary 1.7 and Lemma 3.7, \( G = P \times Q \) is a Schmidt group with \( |\pi(G)| = |\sigma(G)| \), where \( P \) is a minimal normal subgroup of \( G \) and \( Q \) is cyclic. Let \( |P| = p^n \) and \( |Q| = q^m \). Suppose that \( n > 1 \). Then \( G \) has a 2-maximal subgroup \( L \) such that \( |G : L| = pq \). By hypothesis \( L \) is \( \sigma \)-quasinormal in \( G \), so it is, in fact, \( S \)-quasinormal in \( G \) and hence \( LQ = QL \) is a subgroup of \( G \) with \( |G : LQ| = p \). But then \( LQ \cap P \) is normal in \( LQ \) and \( |P : (LQ \cap P)| = p \), so \( LQ \cap P \) is normal in \( G \) and hence \( LQ \cap P = 1 \) in view of the minimality of \( P \). It follows that \(|P| = p \), which leads to a contradiction. Hence \(|P| = p \), so \( G \) is supersoluble. The corollary is proved.

6 Proof of Theorem 1.10

Lemma 6.1. Suppose that \( G \) is \( \sigma \)-soluble and let \( \mathcal{H} = \{H_1, \ldots, H_r\} \) be a \( \sigma \)-basis of \( G \). If \( H_i \) forms an irreducible pair with \( H_j \), then \( H_j \) is an elementary abelian Sylow subgroup of \( G \).

Proof. Without loss of generality, we can assume that \( G = H_iH_j \). By Proposition 3.2(iv), for each prime \( p \) dividing \( |H_j| \), there is a Sylow \( p \)-subgroup \( P \) of \( H_j \) such that \( H_iP = PH_i \). Hence \( G = H_iP \). Let \( R \) be a minimal normal subgroup of \( G \). If \( R \leq P \), then the maximality of \( H_i \) implies that \( R = P \) is elementary. On the other hand, if \( R \leq H_i \), then \( H_i/R \) is a maximal subgroup of \( G/R \), and so \( P \simeq PR/R \) is elementary by induction.

The following lemma can be proved similarly to [5, Chapter I, Proposition 4.16].

Lemma 6.2. Suppose that \( G \neq 1 \) is \( \sigma \)-soluble and let \( L \) be a subgroup of \( G \). Then for each \( \sigma \)-basis \( \mathcal{L} = \{L_1, \ldots, L_r\} \) of \( L \), there is a \( \sigma \)-basis \( \mathcal{H} = \{H_1, \ldots, H_r\} \) of \( G \) such that \( L_i = L \cap H_i \) for all \( i = 1, \ldots, r \).

Lemma 6.3. Suppose that \( G \) is \( \sigma \)-soluble and let \( \mathcal{H} = \{H_1, \ldots, H_n\} \) be a \( \sigma \)-basis of \( G \). If \( H_i \) forms an irreducible pair with \( H_j^x \) for all \( i > 1 \) and \( x \in G \) such that \( H_i H_j^x = H_j^x H_i \), then every subgroup \( K \) of \( G \) containing \( H_i \) is a \( \sigma \)-Hall subgroup of \( G \).

Proof. Suppose that this is false. Without loss of generality we can assume that \( H_i \) is a \( \sigma \)-group. Let \( K = \{H_1, K_2, \ldots, K_r\} \) be a \( \sigma \)-basis of \( K \). By Lemma 6.2, there is a \( \sigma \)-basis \( \{H_1, H_2^x, \ldots, H_n^x\} \) of \( G \) such that \( K_i = K \cap H_i^x \) for all \( i = 2, \ldots, r \). Hence \( H_i H_j^x \cap K = H_i(H_j^x \cap K) = H_iK_j = K_jH_i \) and so there is a subgroup chain \( H_1 \leq H_1K_j \leq H_1H_j^x \). The maximality of \( H_1 \) in \( H_1H_j^x \) implies that \( K_j = H_j^x \). Thus \( K \) is a \( \sigma \)-Hall subgroup of \( G \).

Lemma 6.4. Suppose that \( G \) is \( \sigma \)-soluble and let \( K \) be a subgroup of \( G \). If every subgroup of \( G \) containing \( K \) is a \( \sigma \)-Hall subgroup of \( G \), then \( K \) is a \( \sigma \)-maximal subgroup of \( G \), where \( k = |\sigma([G : K])| \), and \( K \) is not an \( r \)-maximal subgroup of \( G \) for all \( r > k \).
Proof. Let \( \{H_1, \ldots, H_t\} \) be a \( \sigma \)-basis of \( G \). The assertion follows from the fact that in any maximal chain \( K = M_0 < M_{k-1} < \cdots < M_k = G \), \( |M_i : M_{i+1}| \) is the order of some \( H_i \) since both \( M_i \) and \( M_{i+1} \) are \( \sigma \)-Hall subgroups of \( G \). The lemma is proved.

Proof of Theorem 1.10. Let \( \{P_1, \ldots, P_n\} \) be a Sylow basis of \( G \) and \( \{H_1, \ldots, H_t\} \) a complete Hall \( \sigma \)-set of \( G \). We can assume without loss of generality that each \( P_i \) is contained in some \( H_j \) and \( P_i \) is a \( p_i \)-group.

Necessity. First note that if \( G \) is \( \sigma \)-nilpotent, then \( m_\sigma(G) = |\pi(G)| = 1 \), and so \( G \) is a \( p \)-group for some prime \( p \). Now we show, assuming that \( G \) is not \( \sigma \)-nilpotent, then \( G \) is a group of Type (ii).

Assume that this is false and let \( G \) be a counterexample of minimal order. Let \( R \) be a minimal normal subgroup of \( G \). Without loss of generality we can assume that \( R \leq H_1 \) and \( H_k \) is a \( \sigma_{i_k} \)-group for all \( k = 1, \ldots, t \).

(1) If \( G/R \) is not \( \sigma \)-nilpotent, then \( G/R \) is a group of Type (ii).

Suppose that \( G/R \) is not \( \sigma \)-nilpotent. We show that the hypothesis holds for \( G/R \). Indeed, if \( R < P_1 \), it is clear. We may, therefore, assume that \( R = P_1 \). Then \( R \) has a complement \( M \) in \( G \) such that \( G = R \rtimes M \). Since \( |\pi(M)| = n - 1 \), \( M \) satisfies the same assumptions as \( G \), with \( n - 1 \) replacing \( n \), by Lemma 2.2(1). The choice of \( G \) implies that \( G/R \simeq M \) is a group of Type (ii).

(2) If \( V_i \) is a maximal subgroup of \( P_i \), then \( V_i \) is \( \sigma \)-subnormal in \( G \). Hence every non-\( \sigma \)-subnormal Sylow subgroup of \( G \) is cyclic.

Since \( \{P_1, \ldots, P_n\} \) is a Sylow basis of \( G \), \( V_i \) is an \( m \)-maximal subgroup of \( G \), where \( m > n \). Hence \( V_i \) is \( \sigma \)-subnormal in \( G \) by Lemma 4.5(ii). Therefore, if \( P_i \) is not \( \sigma \)-subnormal in \( G \), then it is cyclic by Proposition 2.5.

(3) If \( R \) is the only minimal normal subgroup of \( G \), then each Sylow subgroup \( P_i \) of \( H_k \) has prime order and it is not \( \sigma \)-subnormal in \( G \) for all \( k > 1 \).

Indeed, let \( V \) be a maximal subgroup of \( P_i \). Then \( V \) is \( \sigma \)-subnormal in \( G \) by Claim (2). Hence \( V \leq O_{\sigma_{i_k}}(G) \) by Lemma 2.2(9). But since \( R \leq H_1 \), we have that \( O_{\sigma_{i_k}}(G) = 1 \) and so \( V = 1 \). Moreover, if \( P_i \) is \( \sigma \)-subnormal, then \( P_i \leq O_{\sigma_{i_k}}(G) = 1 \), which leads to a contradiction. Hence we have (3).

(4) For some \( i, i = 1 \) say, \( P_1 = P_1 \) is not \( \sigma \)-subnormal in \( G \). Hence \( P_1 \) forms an irreducible pair with \( P_i \) for all \( i > 1 \).

If \( P_i \) is \( \sigma \)-subnormal in \( G \) for all \( i = 1, \ldots, n \), then \( H_k \) is normal in \( G \) for all \( k = 1, \ldots, t \) by Lemma 2.2(9) and Proposition 2.5, which means that \( G \) is \( \sigma \)-nilpotent. Hence the first assertion of (4) is true. Finally, note that if, for example, \( P_1 \) is not \( \sigma \)-subnormal in \( P_i \), then the chain \( P_1 < P_1P_2 < \cdots < P_1 \cdots P_n = G \) can be refined to a maximal chain of \( G \) of length \( n \), at least. Hence \( P_1 \) is \( \sigma \)-subnormal in \( G \) by Lemma 4.5(ii). This contradiction shows that \( P_1 \) forms an irreducible pair with \( P_i \) for all \( i > 1 \).

(5) The following assertions hold.

(a) \( P_i \) is elementary abelian for all \( i > 1 \). Hence if \( G \) possesses at least two non-\( \sigma \)-subnormal non-isomorphic Sylow subgroups, then all non-\( \sigma \)-subnormal Sylow subgroups are of prime order (this follows from Lemma 6.1 and Claims (2) and (4)).

(b) If \( P_1 \leq H_k \) and \( P_1 \) is not of prime order, then \( H_1, \ldots, H_{k-1}, H_{k+1}, \ldots, H_t \) are normal in \( G \).

Indeed, since \( P_1 \) is not \( \sigma \)-subnormal in \( G \), \( P_1 \) is cyclic by Claim (2). Hence, if \( i \neq k \) and \( P_j \leq H_i \), then \( P_j \) does not form an irreducible pair with \( P_i \) by Claim (a) and Lemma 6.1. Therefore Claim (4) implies that \( P_j \) is \( \sigma \)-subnormal in \( G \). This shows that every Sylow subgroup of \( H_i \) is \( \sigma \)-subnormal in \( G \). Hence \( H_i \) is normal in \( G \) by Lemma 2.2(9) and Proposition 2.5 for all \( i \neq k \).

(c) If \( P_1 \leq H_k \) and \( V \) is the maximal subgroup of \( P_i \), then \( |G : N_G(V)| \) is a \( \sigma_{i_k} \)-number.

If \( |P_1| \) is a prime, it is trivial. Assume that \( |P_1| \) is not a prime and let \( i \neq k \). Then \( H_i \) is normal in \( G \) by Claim (b). On the other hand, Claim (2) implies that \( V \) is \( \sigma \)-subnormal in \( G \). Hence \( H_i \leq N_G(V) \) by Lemma 2.2(5). Hence we have (c).

(6) \( D \) is a Hall subgroup of \( G \). Hence \( D \) has a complement \( M \) in \( G \).

Suppose that this is false and let \( U \) be a Sylow \( p_j \)-subgroup of \( D \) such that \( 1 < U < P_j \leq H_k \). Lemma 2.1 implies that

\[
(G/N)^{p_j} = G^{p_j} N/N = DN/N
\]

for any minimal normal subgroup \( N \) of \( G \).
Let $L$ be a minimal normal subgroup of $G$ contained in $D$. Then $G/L$ is a group of Type (ii). Indeed, assume that $G/L$ is $\sigma$-nilpotent and so $L = D$. Then $L < P_j$ and so $P_j$ does not form an irreducible pair with $P_j$. Hence $L < P_j = P_j$ by Claim (4). From Claim (5)(b) it follows that $H_1 \cdots H_{k-1}H_{k+1} \cdots H_k$ is normal in $G$, so $D \leq H_1 \cdots H_{k-1}H_{k+1} \cdots H_k$ since $G/H_1 \cdots H_{k-1}H_{k+1} \cdots H_k \simeq H_k$ is $\sigma$-nilpotent, which leads to a contradiction. Hence $G/L$ is not $\sigma$-nilpotent, and so $G/L$ is a group of Type (ii) by Claim (1).

Hence $D/L$ is a Hall subgroup of $G/L$. If $UL/L \neq 1$, then $UL/L$ is a Sylow $p_j$-subgroup of $D/L$ and so $UL/L = P_j L/L$. Hence $P_j \leq UL \leq D$ and so $U = P_j$. This contradiction shows that $UL/L = 1$, so $U = L$. Therefore $L < P_j$. But then, as above, we get that for any $i \neq k$ the subgroup $H_i$ is normal in $G$. Let $N$ be a minimal normal subgroup of $G$ contained in $H_i$. Then $G/N$ is not $\sigma$-nilpotent since $i \neq k$, and so $DN/N \simeq D$ is a Hall subgroup of $G/N$ by Claim (1), which implies that $U = P_j$ since $P_j \notin \pi(N) \subseteq \pi(H_i)$. This contradiction completes the proof of (6).

(7) Some Sylow subgroup $P_\tau^x$ of $G$ contained in $M$ is not $\sigma$-subnormal in $G$.

Suppose that every Sylow subgroup $P_\tau^x$ of $G$ contained in $M$ is $\sigma$-subnormal in $G$. Then $M$ is $\sigma$-subnormal in $G$ by Proposition 2.5. Hence there is a subgroup chain

$$M = M_0 < M_1 < \cdots < M_r = G$$

such that either $M_{i-1} \trianglelefteq M_i$ or $M_i/(M_{i-1}M_i)$ is $\sigma$-primary for all $i = 1, \ldots, r$. Since $G/D = G/G^{\Phi(G)} \simeq M$ is $\sigma$-nilpotent and $G$ is not $\sigma$-nilpotent, $M \neq G$. Hence we can assume without loss of generality that $M_1 < G$. If $M_1$ is normal in $G$, then there is a normal maximal subgroup $T$ of $G$ containing $M_1$ such that $G/T$ is nilpotent since $G$ is soluble. Then $T \nsubseteq D$, and so $G = DM = DT = T < G$, which leads to a contradiction. Therefore $G/(M_1G)$ is $\sigma$-primary and thereby it is $\sigma$-nilpotent. But then $D \leq (M_1G)$, so $G = MD \leq M_1 < G$. This contradiction shows $M$ is not $\sigma$-subnormal in $G$. Hence we have (7).

(8) $D$ is nilpotent.

Suppose that $D$ is not nilpotent. Assume that $G$ has a minimal normal subgroup $N \neq R$. Then in view of Claim (1), $D \nsubseteq D/1 = D/(R \cap N)$ is nilpotent. Therefore $R$ is the unique minimal normal subgroup of $G$ and $R \not\leq \Phi(G)$ by Lemma 2.1. It follows that $R = C_G(R) \leq D$.

Since $D/R$ is nilpotent by Claim (1), there is a normal subgroup $E/R$ of $D/R$ such that $R \leq E < D$ and $D = E \times P_i$ for some $i$. Assume that $P_i$ is $\sigma_i$-group. The Frattini argument implies that for some $x \in G$ we have $M^x \leq N_G(P_i)$.

Suppose that for each $r \neq k$ and for each Sylow subgroup $P$ of $G$ contained in $M^x$, where $P$ is $\sigma_i$-group, we have $[P, P_i] = 1$. Then $G/E \simeq P_iM^x$ is $\sigma$-nilpotent and so $D \leq E$, which leads to a contradiction. Hence $G$ has a Sylow subgroup $P_\tau^y$ satisfying the following conditions: $[P_i, P_\tau^y] \neq 1$, $P_\tau^y \leq M^x$ and $P_\tau^y$ is a $\sigma_i$-group for some $r \neq k$.

Then $P_\tau^yP_i$ is $\sigma$-fiber and so $P_\tau^y$ is not $\sigma$-subnormal by Lemma 2.2(5) since $P_\tau^y \leq N_G(P_i)$ and $[P_i, P_\tau^y] \neq 1$. Now we show that $P_\tau^y$ is of prime order. Assume that this is false. Then Claim (3) implies that $R$ is a $\sigma_i$-group, so $G$ has a minimal normal subgroup $N \neq R$ by Claim (5)(b), which leads to a contradiction. Hence $P_\tau^y$ is of prime order.

Note that since $P_\tau^y$ is not $\sigma$-subnormal, $P_\tau^y$ forms an irreducible pair with $R$ and $P_i$ by Claim (4). Then $C_R(P_\tau^y) = 1$ since $C_G(R) = R$. Note also that $C_{P_i}(P_\tau^y) = 1$. Indeed, since $P_\tau^y$ forms an irreducible pair with $P_i$ and $P_\tau^y \leq C_G(P_\tau^y)$, we have either $C_{P_i}(P_\tau^y) = P_i$ or $C_{P_i}(P_\tau^y) = 1$. But the former case is impossible since $[P_i, P_\tau^y] \neq 1$, so $C_{P_i}(P_\tau^y) = 1$.

Let $C = C_G(P_\tau^y) \cap RP_i$. Suppose that $C \neq 1$. Then $C = (R \cap C)U = U \leq P_\tau^a$ for some $a \in R$, so $U \times P_\tau^y \leq (P_\tau^aP_\tau^y)^b \simeq (P_\tau^aP_\tau^y)^b$ for some $b \in R$ since $G$ is soluble. It follows that $C = 1$. Consequently, $P_\tau^y \cap C_G(RP_i) = 1$. Hence $RP_i$ is nilpotent by the Thompson theorem (see [13, Chapter V, Theorem 8.14]). Thus $P_i \leq C_G(RP_i) = R$, which leads to a contradiction. Therefore we have (8).

Claims (2)–(8) show that the necessity is true.

Sufficiency. If $G$ is of Type (i), it is clear. Now let $G$ be a group of Type (ii). Then $G$ is not $\sigma$-nilpotent by (ii)(b) and Proposition 3.4. Hence $|\pi(G)| \leq m_\sigma(G)$ by Theorem 1.2(iii). Let $n = |\pi(G)|$.

In order to prove that $m_\sigma(G) \leq |\pi(G)|$, we only need to prove that every $n$-maximal subgroup of $G$ is $\sigma$-subnormal in $G$. Assume that this is false and let $E$ be an $n$-maximal subgroup of $G$ such that $E$ is not
σ-subnormal in \(G\). Then some Sylow subgroup \(E_1\) of \(E\) is not σ-subnormal in \(G\) by Proposition 2.5. We can assume without loss of generality that \(E_1 \leq P_1\). Then \(P_1\) is not σ-subnormal in \(G\) by Lemma 2.2(4). If \(i > 1\) and \(P_i P_i^x = P_i^x P_i\), then \(P_i\) and \(P_i^x\) are members of some Sylow basis of \(G\) (see [5, Chapter I, Proposition 4.16]). By the hypothesis, \(P_1\) forms an irreducible pair with \(P_1^x\), \(P_1\) is cyclic and the maximal subgroup of \(P_1\) is σ-subnormal in \(G\). Hence \(E_1 = P_1\), so every subgroup of \(G\) containing \(E_1\) is a Hall subgroup of \(G\) by Lemma 6.3. Then \(E\) is exactly a \(k\)-maximal subgroup of \(G\), where \(k = |\pi(|G : E|)|\), by Lemma 6.4. Hence \(k = n = |\pi(G)|\). But then \(E = 1\), so \(E\) is σ-subnormal in \(G\). This contradiction completes the proof of the sufficiency.

The theorem is proved.

7 Final remarks and some open questions

We say that \(G\) is a group of σ-Spencer height \(h_\sigma(G) = n\) if every maximal chain of \(G\) of length \(n\) contains a proper σ-subnormal entry and there exists at least one maximal chain of \(G\) of length \(n - 1\) which does not contain any proper σ-subnormal entry.

In particular, if \(\sigma = \{\{2\}, \{3\}, \ldots\}\), we write \(h(G)\) instead of \(h_\sigma(G)\).

It is clear that \(h_\sigma(G) \leq m_\sigma(G)\), and in general \(m_\sigma(G) \neq h_\sigma(G)\).

Theorems 1.2(ii) and 1.2(iii) can be improved.

Theorem 7.1 (See [10]). Suppose that \(G\) is σ-soluble. Then the following statements hold:

(i) \(l_\sigma(G) \leq h_\sigma(G)\).

(ii) If a soluble group \(G\) is not σ-nilpotent, then \(|\pi(G)| \leq h_\sigma(G)\).

Corollary 7.2 (See [29]). Suppose that \(G\) is a soluble group. Then we have the following:

(i) \(l(G) \leq h(G)\), where \(l(G)\) is the nilpotent length of \(G\).

(ii) If \(h(G) < |\pi(G)|\), then \(G\) is nilpotent.

In view of Theorems 1.2, 1.10 and 7.1, the following questions seem natural.

Question 7.1. What is the structure of a soluble group \(G\) provided \(|\pi(G)| = m_\sigma(G) + 1\)?

Question 7.2. What is the structure of a soluble group \(G\) provided \(|\pi(G)| = h_\sigma(G)\)?

Question 7.3. What is the structure of a soluble group \(G\) provided \(|\pi(G)| = h_\sigma(G) + 1\)?

Note that in the case when \(\sigma = \{\{2\}, \{3\}, \ldots\}\) the complete answers to these three questions are known (see [22, 29]).

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