HOMOLOGICAL LIE BRACKETS ON MODULI SPACES AND PUSHFORWARD OPERATIONS IN TWISTED K-THEORY

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Abstract. We develop a general theory of pushforward operations for principal $G$-bundles equipped with a certain type of orientation.

In the case $G = BU(1)$ and orientations in twisted $K$-theory we construct two pushforward operations, the projective Euler operation, whose existence was conjectured by Joyce, and the projective rank operation. We classify all stable pushforward operations in this context and show that they are all generated by the projective Euler and rank operation.

As an application, we construct a graded Lie algebra structure on the homology of a commutative $H$-space with a compatible $BU(1)$-action and orientation. These play an important role in the context of wall-crossing formulas in enumerative geometry.

1. Introduction

Many traditional coarse moduli spaces $P$ are obtained from a moduli stack that has a scalar action by $\mathbb{C}^*$ on the morphism sets. Examples are moduli spaces of coherent sheaves, connections, or quiver representations. Topologically, the action leads to a principal $BU(1)$-bundle $P \to B = P/BU(1)$, where $BU(1)$ is a topological group model for the classifying space of complex line bundles. Enumerative geometry studies the intersection theory of virtual fundamental classes in homology. These usually depend on auxiliary parameters and the resulting virtual fundamental classes should then be related by wall-crossing formulas. Joyce [5] describes a comprehensive, partly conjectural, new theory of wall-crossing formulas in which the relationship between the homology $H_*(P)$ of the coarse moduli space and the homology $H_*(B)$ in the sense of stacks plays a key role. The main purpose of this paper is to clarify this relationship.

A principal $BU(1)$-bundle $P \to B$ determines a cohomology class in $H^3(B; \mathbb{Z})$. We say that $P$ is rationally trivial if its image $\eta_P$ in $H^3(B; \mathbb{Q})$ vanishes. In this case there is a Künneth decomposition $H^*(P; \mathbb{Q}) \cong H^*(B; \mathbb{Q}) \otimes \mathbb{Q}[c_1]$, where $c_1$ is the generator of the cohomology of $BU(1) \cong \mathbb{CP}^\infty$. In particular, there are many projections from $H^*(P)$ to $H^*(B)$. In the general case we will show that one can still construct a certain combination of these projections using additional data, called an orientation. If $P$ is trivial, an orientation amounts to a class $\vartheta \in K(B)$ in complex $K$-theory. Let $r$ be the rank of $\vartheta$. In this case the projective Euler operation is given by

$$H^*(P; \mathbb{Q}) \xrightarrow{\pi^*} H^{*+2r+2}(B; \mathbb{Q}), \quad \alpha \times c_1^r \mapsto \alpha \cup c_{r+1}(\vartheta), \quad (1.1)$$

where $c_j(\vartheta)$ denotes the $j$th Chern class of $\vartheta$. We will prove that this globalizes to a construction for general $P$, provided $\vartheta$ is replaced by a class $\theta \in K_P(B)$ in twisted complex $K$-theory, as we explain after a brief digression on twisted K-theory.

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Let $P \to B$ be a principal $BU(1)$-bundle. Let $B = \bigcup_{i \in I} U_i$ be an open cover and write $U_{ij} = U_i \cap U_j$, etc. Given sections $\sigma_i$ of $P|_{U_i}$, the transition functions $\gamma_{ij}^P \colon U_{ij} \to BU(1)$ satisfy $\sigma_i|_{U_{ij}} \cdot \gamma_{ij}^P = \sigma_j|_{U_{ij}}$ and classify complex line bundles $L_{ij}$ over $U_{ij}$. The cocycle identity for $\gamma_{ij}^P$ yields isomorphisms $\gamma_{ij}^P : L_{ij}|_{U_{ijk}} \otimes L_{jk}|_{U_{ijk}} \to L_{ik}|_{U_{ijk}}$. Each collection $\theta = \{(\gamma_{ij}^P)_{ij \in I}\}$ of complex vector bundles $\gamma_{ij}$ over $U_{ij}$ and isomorphisms $\gamma_{ij}^P : \gamma_{ij}|_{U_{ij}} \otimes L_{ij} \to \theta|_{U_{ij}}$ satisfying the twisted cocycle identity $\gamma_{jk}^P \circ (\gamma_{ij}^P \otimes \text{id}_{L_{jk}}) = \gamma_{ik}^P \circ (\text{id}_{\gamma_{ij}} \otimes \gamma_{ij}^P)$ determines a class in twisted K-theory $K_P(B)$. If $B$ is a compact Hausdorff space, we could define $K_P(B)$ in this way as a group completion; the official definition is given in §2. Using the universal complex line bundle $V(1) \to BU(1)$, we can construct vector bundles $\theta \boxtimes V(1)$ over $U_i \times BU(1)$ for which the isomorphisms $\{\gamma_{ij}^P\}$ can be viewed as descent data, yielding a vector bundle $\theta$ over $P$. Therefore, every twisted K-theory class $\theta \in K_P(B)$ has an underlying complex K-theory class $\bar{\theta} \in K(P)$. If $P$ is trivial, then $\theta$ amounts to a class $\bar{\theta} \in K(B)$ in ordinary K-theory and $\theta = \bar{\theta} \boxtimes V(1)$.

According to Atiyah–Segal [2, Prop. 8.8], characteristic classes for twisted K-theory classes of rank $r$ are in bijection with cohomology classes in $H^*(BU \times \{r\})$ that are invariant under the $BU(1)$-action. For a (virtual) complex vector bundle $\theta$ of rank $r$ and a complex line bundle $L$, we have

$$c_j(\theta \otimes L) = \sum_{k + \ell = j} \binom{r - \ell}{k} c_\ell(\theta) \cup c_1(L)^k.$$  

In particular, $c_{r+1}(\theta \otimes L) = c_{r+1}(\theta)$, so $c_{r+1}(\theta)$ is invariant under the $BU(1)$-action and therefore $c_{r+1}(\theta)$ is well-defined also for twisted K-theory classes $\theta$ of rank $r$.

The projective Euler class is similar to a transfer map in the sense of [11, §15]: the pull-push $\pi^\theta \circ \pi^*$ is the multiplication by an ‘Euler class’, which here is the characteristic class $\pi^\theta(1_P) = c_{r+1}(\theta)$ of Atiyah–Segal.

A key observation is that [12] implies that a change of trivialization $\gamma : B \to BU(1)$ in [11] leads to an automorphism of $H^*(P)$ under which $\pi^\theta_\gamma$ becomes $\pi^{\theta \otimes L}_\gamma$, where $L$ is the line bundle corresponding to $\gamma$. This explains why orientations in twisted K-theory are needed.

Our first theorem summarizes the properties of the projective Euler class. To state it, we need more notation. A pullback diagram of principal $BU(1)$-bundles

$$\begin{array}{ccc}
P_1 & \xrightarrow{\Phi} & P_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
B_1 & \xrightarrow{\phi} & B_2
\end{array}$$

(1.3)

determines a pullback morphism $\Phi^* : K_{P_1}(B_2) \to K_{P_1}(B_1)$ in twisted K-theory.

The classifying space $BU(1)$ will always be taken to be a topological abelian group $G$. We can then define the dual principal G-bundle $\hat{P} \to B$ by precomposing the action by the inversion in $G$. Taking duals of vector bundles defines a map $K_P(B) \to K_{\hat{P}}(B)$, $\theta \mapsto \hat{\theta}$, of twisted K-theory groups. The tensor product $P_1 \otimes_G P_2$ of principal $G$-bundles $P_1 \to B$ and $P_2 \to B$ is the quotient of the fiber product $P_1 \times_B P_2$ by the $G$-action $\rho_{P_1 \times_B P_2}((p_1, p_2), g) = (p_1 g^{-1}, p_2 g)$. Write $p_1 \otimes_G p_2$ for the orbit of $(p_1, p_2)$ under this action. The action on the orbit space $P_1 \oplus P_2$ is defined by $\rho_{P_1 \oplus P_2}((p_1, p_2), g) = (p_1 g) \otimes (p_2 g)$.

Under the obvious homeomorphisms $\pi_2^\circ(P_1) \cong P_1 \times_B P_2 \cong \pi_1^\circ(P_2)$ the map $\kappa_3(p_1, p_2) = p_1 \otimes_G p_2$ may be regarded in two ways as a morphism of principal bundles, $\pi_1^\circ(P_2) \to P_3$ or $\pi_2^\circ(P_1) \to P_3$, or as a principal $G$-bundle $P_1 \times_B P_2 \to P_3$. The projection $\kappa_1(p_1, p_2) = p_1$ can similarly be viewed as a principal $G$-bundle $\pi_1^\circ(P_2) \to P_1$ or as a morphism $\pi_2^\circ(P_1) \to P_1$ or $P_1 \times_B P_2 \to P_3$; similarly for $\kappa_2$. 

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Theorem 1.1. For each principal \( BU(1) \)-bundle \( \pi: P \rightarrow B \) and orientation \( \theta \in K_P(B) \) in twisted K-theory there is a projective Euler operation
\[
\pi^\theta_\pi: H^*(P; \mathbb{Q}) \rightarrow H^{*+2r+2}(B; \mathbb{Q}),
\]
where \( r \) is the rank of \( \theta \), uniquely determined by the following properties.

(a) Naturality: For every pullback diagram \([1.3]\) and for the pullback orientation \( \theta_1 = \Phi^\ast(\theta_2) \) on \( P_1 \) of rank \( r \), there is a commutative square
\[
\begin{array}{ccc}
H^*(P_1; \mathbb{Q}) & \xleftarrow{\Phi^\ast} & H^*(P_2; \mathbb{Q}) \\
\downarrow{(\pi_1)}_r^\ast & & \downarrow{(\pi_2)}_r^\ast \\
H^{*+2r+2}(B_1; \mathbb{Q}) & \xleftarrow{\phi^\ast} & H^{*+2r+2}(B_2; \mathbb{Q}).
\end{array}
\]

(b) Stability: For the pullback orientation \( \theta \times S^1 = \pi^\ast_P(\theta) \) on \( P \times S^1 \), where \( \pi_P: P \times S^1 \rightarrow P \) is the projection, there is a commutative square
\[
\begin{array}{ccc}
H^*(P; \mathbb{Q}) & \xrightarrow{\times[\mathbb{S}^1]} & H^{*+1}(P \times S^1; \mathbb{Q}) \\
\downarrow{\pi^\ast} & & \downarrow{(\pi \times \text{id}_{S^1})^\ast} \\
H^{*+2r+2}(B; \mathbb{Q}) & \xrightarrow{\times[\mathbb{S}^1]} & H^{*+2r+3}(B \times S^1; \mathbb{Q}).
\end{array}
\]

(c) Normalization: If \( P = B \times BU(1) \) is trivial, \( \pi^\ast_\pi(1_B \times c_1^\ast) = c_{r+1}(\tilde{\theta}) \), where \( \tilde{\theta} \in K(B) \) is the ordinary K-theory class corresponding to \( \theta \).

(d) Euler class: \( \pi^\ast_P(1_P) = c_{r+1}(\theta) \) is the characteristic class of Atiyah-Segal.

(e) Base-linearity: For all \( \alpha \in H^*(P; \mathbb{Q}) \) and \( \beta \in H^*(B; \mathbb{Q}) \), we have
\[
\pi^\ast_\pi(\alpha \cup \pi^\ast(\beta)) = \pi^\ast(\alpha) \cup \beta.
\]

In particular, the pull-push formula \( \pi^\ast_\pi \circ \pi^\ast(\beta) = c_{r+1}(\tilde{\theta}) \cup \beta \) holds.

(f) Push-pull formula: For the underlyng complex K-theory class \( \tilde{\theta} \in K(P) \) of the orientation \( \theta \) and using the operation ‘to’ from Definition \([3.4]\), we have \( \pi^\ast \circ \pi^\ast_\pi(\alpha) = \sum_i \lambda_i^{c_i+1} \tilde{\theta} \cup \alpha \) for all \( \alpha \in H^*(P; \mathbb{Q}) \).

(g) Duality: For the dual principal \( BU(1) \)-bundle \( \tilde{\pi}: \tilde{P} \rightarrow B \) and the dual orientation \( \tilde{\theta} \in K_P(B) \), we have \( \tilde{\pi}^\ast_\tilde{\pi} = (-1)^{r+1} \pi^\ast \).

(h) Composition: Let \( \pi_1: P_1 \rightarrow B \) and \( \pi_2: P_2 \rightarrow B \) be principal \( BU(1) \)-bundles and let \( P_3 = P_1 \oplus_{BU(1)} P_2 \). Let \( \theta_k \in K_{P_k}(B) \) for \( 1 \leq k \leq 3 \). Then
\[
(\pi_2)^\ast \circ (\kappa_{21})_h^{\ast \theta_1 + \ast \theta_3} - (\pi_1)^\ast \circ (\kappa_{12})_h^{\ast \theta_2 + \ast \theta_3} = (-1)^{r_1} (\pi_3)^\ast_3 \circ (\kappa_{3})_h^{\ast \theta_1 + \ast \theta_2}
\]
for the pushforwards around the three routes in the following commutative diagram of bundle projections of principal \( BU(1) \)-bundles:
\[
\begin{array}{ccc}
P_3 & \xrightarrow{\pi_3} & P_1 \\
\downarrow{\pi_2} & & \downarrow{\kappa_1} \\
P_2 & \xrightarrow{\kappa_2} & P_1
\end{array}
\]

Here, \( r_1 \) is the rank of \( \theta_1 \), we use the obvious homeomorphisms \( \pi^\ast_2(P_1) \cong P_1 \times_B P_2 \cong \pi^\ast_1(P_2) \) to identify all the domain cohomology groups in \([1.7]\), and the orientation \( \kappa_1^\ast \theta_1 + \kappa_2^\ast \theta_3 \) on \( \pi^\ast_3(P_3) \) is constructed by pullback along the morphisms \( \kappa_1: \pi^\ast_3(P_3) \rightarrow P_1 \), and \( \kappa_2: \pi^\ast_3(P_3) \rightarrow P_3 \); similarly for the orientation \( \kappa_2^\ast \theta_2 + \kappa_3^\ast \theta_3 \) on \( \pi^\ast_1(P_1) \) and for \( \kappa_1^\ast \theta_1 + \kappa_2^\ast \theta_2 \) on \( P_1 \times_B P_2 \).

(i) For all \( \alpha \in H^*(P) \), \( \pi^\ast_\pi(\alpha) \cup \eta_P = 0 \) for the characteristic class \( \eta_P \) of \( P \).
Theorem 1.2. For every principal BU(1)-bundle $\pi: P \to B$ and orientation $\theta \in K_P(B)$ in twisted K-theory, there is a projective rank operation
\[ s_\theta^r: H^*(P; \mathbb{Q}) \to H^*(B; \mathbb{Q}), \]
uniquely determined by the following properties.

(a) Naturality: For every pullback diagram $1.3$ and the pullback orientation $\theta_1 = \Phi^*(\theta_2)$, we have $\phi^* \circ s_{\theta_2}^r = s_{\theta_1}^r \circ \Phi^*$.
(b) Stability: $(s \times \id_{S^1})_\theta \times_S (\alpha \times [S^1^n]) = s_\alpha^r(\alpha) \times [S^1]$ for all $\alpha \in H^*(P; \mathbb{Q})$.
(c) Normalization: If $P$ is trivial, $s_\theta^r(1_B \times c_1^i) = (-1)^i \id \cdot \chi_i(\theta)$ for all $i \geq 0$, where $\theta \in K(B)$ is the ordinary K-theory class corresponding to $\theta$.
(d) Euler class: $s_\theta^r(1_P) = r 1_B$, where $r$ is the rank of $\theta$.
(e) Base-linearity: For all $\alpha \in H^*(P; \mathbb{Q})$ and $\beta \in H^*(B; \mathbb{Q})$, we have $s_\alpha^r(\alpha \cup \pi^*(\beta)) = s_\alpha^r(\alpha) \cup \beta$. In particular, $s_\theta^r \circ \pi^*(\beta) = r\beta$.
(f) Push-pull formula: For the underlying complex K-theory class $\tilde{\theta} \in K(P)$ of the orientation, we have $\pi^* \circ s_{\tilde{\theta}}^r(\alpha) = \sum_{i \geq 0} (-1)^i (t^i \alpha) \cup \chi_i(\tilde{\theta})$.

We prove Theorem 1.2 in §3.2.

Remark 1.3. Following Atiyah–Segal [1], the projective unitary group $PU(H) = U(H)/U(1)$ of a separable complex Hilbert space is a non-abelian topological group model for $BU(1)$. Let $L \to PU(H)$ be the universal complex line bundle. If $P \to B$ is a principal $PU(H)$-bundle, a twisted complex K-theory class is represented by an equivariant family of Fredholm operators $\theta = \{\theta_p \mid p \in P\}$ on $H$ satisfying
\[ \theta_{pg} = \theta_p \otimes L_{|g|}, \quad \forall p \in P, g \in PU(H). \] (1.9)
In this model, the rank of $\theta$ becomes the index. The one-dimensional complex vector spaces $\Lambda(\theta) = \Lambda^{\text{top}}(\text{Ker} \theta) \otimes \Lambda^{\text{top}}(\text{Coker} \theta)^*$ are the fibers of the determinant line bundle $\Lambda(\theta) \to P$. From (1.9) we find $\Lambda(\theta_{fg}) \cong \Lambda(\theta_p \otimes L_{|g|}) \cong \Lambda(\theta_p) \otimes (L_{|g|})^\oplus \text{ind} \theta_p$. If $\text{ind} \theta_p = 0$, it follows that $\Lambda(\theta)$ descends to a complex line bundle on $B$ whose first Chern class is the Atiyah–Segal characteristic class $c_1(\theta)$. If $\text{ind} \theta_p = 1$, the determinant line bundle is $PU(H)$-equivariant and its classifying map $f_{\Lambda(\theta)}: P \to PU(H)$ determines a global trivialization $(\pi, f_{\Lambda(\theta)}) : P \to B \times PU(H)$ of $P$. The projective rank operation $s_\theta^r$ is the pullback along the corresponding global section of $P$, thus explaining the notation.

Our next result states that $\pi_\theta^r$ and $s_\theta^r$ are the basic generators of all operations of this type. To explain this, we need more terminology. In §2 we will study in a general context a new algebro-topological tool, roughly the bundle version of traditional cohomology operations. Let $G$ be a topological group. For every principal $G$-bundle $\pi: P \to B$ with a certain type of orientation $\theta$, a pushforward operation of type $(m, n)$ determines a map
\[ \varphi_{m,n}^{P|\theta} : H^m(P) \to H^n(B), \] (1.10)
natural in $P$ and $\theta$. In general, these are hard to classify. This simplifies once we introduce stable pushforward operations whose maps are defined for all $m$ with fixed degree $k = n - m$ and which satisfy the analogue of (1.5).

For $G = BU(1)$ and $r \in \mathbb{Z}$ let $\Pi_{BU_\times \{r\}}$ be the group of stable pushforward operations $H^*(P; \mathbb{Q}) \to H^{*+k}(B; \mathbb{Q})$ of degree $k$ for principal $BU(1)$-bundles with orientations in twisted complex K-theory of rank $r$. Given a pushforward operation,
we can construct new pushforward operations by composing with endomorphisms of $H^*(P; \mathbb{Q})$ and $H^*(B; \mathbb{Q})$. Let $\rho_P : P \times G \to P$ denote the principal action and let $t \in H_2(BU(1))$ be the class dual to $c_1$. The ring $\mathbb{Q}[t]$ acts on $H^*(P; \mathbb{Q})$ by $t \cdot \alpha = \iota \cdot \rho_P(\alpha)$, see Definition \[3.3.\] Moreover, $\mathbb{Q}[c_1, c_2, \ldots]$ acts by multiplication with the Chern classes of the underlying complex K-theory class $\theta \in K(P)$. This determines on $\Pi_{BU \times \{r\}}$ a projective Euler operation $\Phi$:

Theorem 3.21. Let $M$ be interesting to determine all of the relations in the $\s$-module $\Pi_{BU \times \{r\}}$ with the Chern classes of the underlying complex K-theory class $\theta \in K(P)$. This determines on $\Pi_{BU \times \{r\}}$ a projective Euler operation $\Phi$:

Theorem 1.5. Let $(M, \Phi, \Psi)$ be a commutative H-space with $BU(1)$-action, orientations $\theta_{\alpha, \beta}$, and signs $\epsilon_{\alpha, \beta}$ satisfying Assumption \[3.4.\] Define a grading on $H_*(M/BU(1); \mathbb{Q}) = \bigoplus_{\alpha, \beta} H_*(M_\alpha/BU(1); \mathbb{Q})$ by $|\zeta| = a + 2 - \chi(\alpha, \beta)$ for $\zeta \in H_*(M_\alpha/BU(1))$. There is a graded Lie bracket on $H_*(M/BU(1); \mathbb{Q})$ defined by

$$[\zeta, \eta] = \epsilon_{\alpha, \beta}(-1)^{\chi(\beta, \beta)}(\Phi_{\alpha, \beta}/BU(1))_*(\pi_{\alpha, \beta})^0_{\alpha, \beta}(\zeta \times \eta)$$  (1.11)

for $\zeta \in H_*(M_\alpha/BU(1))$ and $\eta \in H_*(M_{\beta}/BU(1))$.

We prove Theorem \[1.5.\] in \[4.1.\] In the rationally trivial case, this result is due to Joyce \[5, \S 3.4.\] where the reader may also find applications of Theorem \[1.5.\]. For instance, the input data for Theorem \[1.5.\] arises naturally from an additive $\mathbb{C}$-linear dg-category $\mathcal{A}$: the direct sum of objects defines the H-space operation on the topological realization of $\mathcal{A}$, the $BU(1)$-action comes from scaling morphisms by phase, and the orientations are obtained from the Ext-complexes.

The paper ends with an Appendix \[A\] in which we prove two technical but elementary results that are used in the text.

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2. Pushforward operations and stability

In this section, we study pushforward operations for general groups. These are defined in §2.1 and we also prove an elementary classification result there. In §2.2 we define the notion of stability and prove a better classification result for stable pushforward operations. The behaviour with respect to the multiplicative structure is studied in §2.3. In §2.4 we prove a technical result that will be useful in §3. Throughout, we always use singular cohomology with rational coefficients.

2.1. Unstable pushforward operations. Let $G$ be a topological group with a left action $\ell_F: G \times F \to F$ on a topological space $F$.

**Definition 2.1.** (a) Let $P \to B$ be a principal $G$-bundle with principal action $\rho_P: P \times G \to P$. The associated fiber bundle $P \otimes_G F$ is the quotient of $P \times F$ by the anti-diagonal $G$-action $\rho_P \times F((p, f), g) = (\rho_P(p, g), \ell_F(g^{-1}, f))$. The orbit of $(p, f)$ is denoted by $p \otimes_G f$.

(b) An $F$-orientation on a principal $G$-bundle $P \to B$ is a homotopy class $\theta$ of $G$-equivariant maps $f_\theta: P \to F$, satisfying $f_\theta(p \otimes_G p, g) = \ell_F(g^{-1}, f_\theta(p))$. If $\Phi: P_1 \to P_2$ is a morphism of principal $G$-bundles and $\theta_2$ is an $F$-orientation on $P_2$, there is a pullback orientation $\theta_1 = \Phi^*(\theta_2)$ on $P_1$.

(c) A pushforward operation $\Xi^{m|n}_{P,\theta}$ of type $(m, n)$ assigns to each principal $G$-bundle $P \to B$ with $F$-orientation $\theta$ a possibly non-linear map

\[ \Xi^{m|n}_{P,\theta} : H^m(P) \to H^n(B). \]

For every morphism $\Phi: P_1 \to P_2$ and $F$-orientations satisfying $\theta_1 = \Phi^*(\theta_2)$, we require a commutative naturality diagram

\[ \begin{array}{ccc}
H^m(P_2) & \xrightarrow{\Phi^*} & H^m(P_1) \\
\downarrow & & \downarrow \\
H^n(B_2) & \xrightarrow{\phi^*} & H^n(B_1).
\end{array} \]

(2.2)

(d) A pushforward operation $\Xi^{m|n}_{P,\theta}$ is pointed if $\Xi^{m|n}_{P,\theta}(0_p) = 0_B$ for all $P \to B$ and $\theta$; it is linear if each $\Xi^{m|n}_{P,\theta}$ is a linear map.

In this paper, the main example of such a setup will be twisted K-theory; our presentation follows Atiyah–Segal [1] and Freed–Hopkins–Teleman [3].

**Definition 2.2.** Let $V(r) \to BU(r)$ be the universal complex vector bundle of rank $r$ and let $BU = \text{colim} \, BU(r)$ be the classifying space for stable complex vector bundles. The external tensor product $V(1)^* \boxtimes V(r)$ by the dual of the universal complex line bundle is classified by a map $BU(1) \times BU(r) \to BU(r)$. Embed $BU(r) \subset BU \times \{r\} \subset BU \times \mathbb{Z}$. In suitable models for $BU(1)$ and $BU$, this construction can be made into a topological group action $\ell_{BU \times \mathbb{Z}}$ of a topological abelian group $BU(1)$ on $BU \times \mathbb{Z}$ that is compatible with taking direct sums in $BU \times \mathbb{Z}$. From this point of view, twisted K-theory $K_P(B)$, where $P \to B$ is a principal $BU(1)$-bundle, is the set of all homotopy classes of $BU(1)$-equivariant maps $f_\theta: P \to BU \times \mathbb{Z}$. Hence, in this case $F$-orientations are just classes in twisted K-theory. The direct sum operation on $BU \times \mathbb{Z}$ makes $K_P(B)$ into an abelian group. By ignoring the equivariance of $\theta$, every twisted K-theory class determines an underlying complex K-theory class $\bar{\theta} \in K(P)$ satisfying $\rho_P(\bar{\theta}) = \bar{\theta} \boxtimes V(1)$, where $\rho_P: P \times BU(1) \to P$ denotes the principal action (we avoid the dual of $V(1)$ here by using the dual of the usual action of $BU(1)$ on $BU \times \mathbb{Z}$; this leads to a simpler...
formula in Theorem 11(c)). Combined with (1.2),
\[ \rho_p^*(c_j(\hat{\theta})) = \sum_{k+\ell = j} \binom{r-\ell}{k} c_k(\hat{\theta}) \times c_\ell, \quad r = \text{rank}(\hat{\theta}), \] (2.3)
which will be useful later. If \( P \) is trivial, then we can identify \( f_\theta \) with a homotopy class of maps \( B \to BU \times \mathbb{Z} \) and thus an ordinary K-theory class \( \theta \in K(B) \).

We next review background from [10, Def. 2.1] on stable homotopy theory necessary for the construction and classification of pushforward operations.

**Definition 2.3.** Let \( G \) be a topological group. A \( G \)-spectrum \( \{ F_m, \varphi_m \} \) is a sequence of pointed \( G \)-spaces \( (F_m, *_{F_m}) \) for all \( m \geq 0 \) and \( G \)-equivariant based connecting maps \( \varphi_m : \Sigma(F_m) \to F_{m+1} \) on the reduced suspensions. If all the adjoint maps \( \varphi_m^\dagger : F_m \to \Omega(F_{m+1}) \) are homeomorphisms, we call \( \{ F_m, \varphi_m \} \) an \( \Omega \)-\( G \)-spectrum.

**Example 2.4.** The Eilenberg–Mac Lane \( \Omega \)-spectrum \( \{ H_m, \eta_m \} \) is characterized by \( \pi_k(H_m) = 0 \) for \( k \neq m \) and \( \pi_m(H_m) = \mathbb{Q} \). Here, \( G \) is trivial. As in [11], there are natural isomorphisms \( [X, H_m] \cong H^m(X) \) to the cohomology groups under which the connecting maps correspond to the suspension isomorphism. Putting \( X = H_m \), we obtain the tautological class \( [\text{id}_{H_m}] = j_m \in H^m(H_m) \). Hence every \( \alpha \in H^m(X) \) determines a unique homotopy class of maps \( h_\alpha : X \to H_m \) such that \( \alpha = h_\alpha(j_m) \).

**Example 2.5.** The mapping spectrum \( \text{Map}(G, H_m) \) is equipped with the connecting maps \( \mu_m = (\eta_m)^{-1} : \text{Map}(G, H_m) \to \text{Map}(G, \Omega H_{m+1}) = \Omega \text{Map}(G, H_{m+1}) \) and is an \( \Omega \)-\( G \)-spectrum. Write \( \mu_m : \Sigma \text{Map}(G, H_m) \to \text{Map}(G, H_{m+1}) \) for the adjoint. Our convention for the action of \( g \in G \) on \( \mu \in \text{Map}(G, H_m) \) is
\[ \ell_{\text{Map}} : G \times \text{Map}(G, H_m) \to \text{Map}(G, H_m), \quad \ell_{\text{Map}}(g, \mu)(x) = \mu(g^{-1}x). \] (2.4)

The base-point \( *_{\text{Map}} \) is the constant map to the base-point \( *_{H_m} \).

Let \( EG \to BG \) be the universal principal \( G \)-bundle with base-points \( *_{EG} \) and \( *_{BG} \) and let \( \rho_{EG} \) be the principal action.

**Definition 2.6.** Let \( P_m^{\text{cla}} = EG \times F \times \text{Map}(G, H_m) \) and define a \( G \)-action on \( P_m^{\text{cla}} \) by \( \rho_{P_m^{\text{cla}}}((e, f, \mu), g) = (\rho_{EG}(e, g), \ell_F(g^{-1}f), \ell_{\text{Map}}(g^{-1}, \mu)) \). Then \( P_m^{\text{cla}} \) is the total space of a principal \( G \)-bundle with quotient space \( B_m^{\text{cla}} = EG \otimes_G (F \times \text{Map}(G, H_m)) \).

There is a natural \( F \)-orientation \( \theta_m^{\text{cla}} \) on \( P_m^{\text{cla}} \) represented by \( f_{\theta_m^{\text{cla}}} : (e, f, \mu) \mapsto f \) and a cohomology class \( \alpha_m^{\text{cla}} \in H^m(P_m^{\text{cla}}) \) represented by \( h_{\alpha_m^{\text{cla}}}(e, f, \mu) = \mu(1) \).

The following proposition shows that \( (P_m^{\text{cla}}, \theta_m^{\text{cla}}, \alpha_m^{\text{cla}}) \) is the universal example of the data \( (P, \theta, \alpha) \). This will lead to the classification of pushforward operations.

**Proposition 2.7.** For every triple \( (P, \theta, \alpha) \) of a principal \( G \)-bundle \( P \to B \) with \( F \)-orientation \( \theta \) and \( \alpha \in H^m(P) \) there is a morphism of principal \( G \)-bundles \( \Phi_{P, \theta, \alpha} : P \to P_m^{\text{cla}} \), unique up to homotopy through such morphisms, satisfying \( \theta = \Phi_{P, \theta, \alpha}(\theta_m^{\text{cla}}) \) and \( \alpha = \Phi_{P, \theta, \alpha}(\alpha_m^{\text{cla}}) \). In particular, given \( (P_k, \theta_k, \alpha_k) \) for \( k = 1, 2 \) and a morphism \( \Phi : P_1 \to P_2 \) of principal \( G \)-bundles such that \( \theta_1 = \Phi^*(\theta_2) \) and \( \alpha_1 = \Phi^*(\alpha_2) \), the morphisms \( \Phi_{P_1, \theta_1, \alpha_1} \) and \( \Phi_{P_2, \theta_2, \alpha_2} \circ \Phi \) are homotopic.

**Proof.** The morphism \( \Phi_{P, \theta, \alpha} \) has three components: a morphism \( \Phi_P : P \to EG \) classifying the principal \( G \)-bundle, for which there is a contractible choice; a \( G \)-equivariant map \( f_\theta : P \to F \) with homotopy class \( \theta \); and a \( G \)-equivariant map \( P \to \text{Map}(G, H_m) \), which is equivalently a non-equivariant map \( h_{\alpha} : P \to H_m \) representing \( \alpha \). The choices \( f_\theta \) and \( h_\alpha \) are thus unique up to homotopy, proving the uniqueness part. For existence, define
\[ \Phi_{P, \theta, \alpha}(p) = (\Phi_P(p), f_\theta(p), h_{\alpha}(p)), \] (2.5)
where $\mu_{h,a} \in \text{Map}(G,H_m)$ is defined by $\mu_{h,a}(g) = h_a(p_g)$. Then $\Phi_{P,\theta,\alpha}$ is $G$-equivariant and satisfies $\Phi_{P,\theta,\alpha}(f_{p_m}) = f_0$ and $\Phi_{P,\theta,\alpha}(h_{\alpha_{\text{cla}}}) = h_\alpha$.

The morphism $\Phi_{P,\theta,\alpha}$ in Proposition 2.4 is called a classifying morphism for $(P,\theta,\alpha)$ and we write $\phi_{P,\theta,\alpha} : B \rightarrow B_{\text{cla}}$ for the quotient classifying map. Recall here that every morphism $\Phi : P_1 \rightarrow P_2$ of principal $G$-bundles induces a quotient map $\phi : B_1 \rightarrow B_2$ of the base spaces, leading to a pullback square as in (1.3).

Observe that each space $B_{\text{cla}}^n = EG \otimes_G (F \times \text{Map}(G,H_m))$ is a fiber bundle over $EG \otimes_G F$ with a natural section $i_m : EG \otimes_G F \rightarrow B_{\text{cla}}^n$, $e \otimes_G f \mapsto e \otimes_G (f,\star_{\text{Map}})$.

**Proposition 2.8.** (a) Pushforward operations $\Xi^{m|n}$ for principal $G$-bundles with $F$-orientation are in 1-1 correspondence with classes $\Xi^{m|n}_{\text{cla}} \in H^n(B_{\text{cla}}^{m|n})$.

This correspondence is defined by

\[ \Xi^{m|n}_{P,\theta} (\alpha) = \phi_{P,\theta,\alpha}^{*} (\Xi^{m|n}_{\text{cla}}), \]  

\[ \Xi^{m|n}_{\text{cla}} = \Xi^{m|n}_{P,\theta,\alpha} (\alpha_{\text{cla}}). \]  

(b) $\Xi^{m|n}$ is pointed if and only if $i_m^{*} (\Xi^{m|n}_{\text{cla}}) = 0$.

(c) If $P$ is trivial, $\phi_{P,\theta,\alpha}$ maps into the fiber $F \times \text{Map}(G,H_m)$ of $B_{\text{cla}}^n \rightarrow BG$ over $*_{BG}$ and the orientation corresponds to a homotopy class of maps $f_0 : B \rightarrow F$. Let $\alpha \in H^m(P)$ and write $h_1 : B \rightarrow \text{Map}(G,H_m)$ for the adjoint of a representing map $h_0 : B \times G \rightarrow H_m$. If $\Xi^{m|n}_{\text{cla}} |_{F \times \text{Map}(G,H_m)} = \xi \times x$ for $\xi \in H^*(F)$ and $x \in H^*(\text{Map}(G,H_m))$, then

\[ \Xi^{m|n}_{P,\theta} (\alpha) = f_0^{*} (\xi) \cup (h_{\alpha})^{*} (x). \]  

**Proof.** (a) By the naturality of the classifying maps in Proposition 2.4, the construction (2.3) indeed defines a stable pushforward operation for each class $\Xi^{m|n}$ in $H^n(B_{\text{cla}}^{m|n})$. By the uniqueness part of Proposition 2.4, $\phi_{P,\theta,\alpha} : \Xi^{m|n} \rightarrow \Xi^{m|n}_{\text{cla}}$, from which it follows that the class of this operation is the original class $\Xi^{m|n}_{\text{cla}}$. Conversely, given a pushforward operation $\Xi^{m|n}$, define $\Xi^{m|n}_{\text{cla}} = \Xi^{m|n}_{P,\theta,\alpha} (\alpha_{\text{cla}})$. Then the construction (2.3) recovers the original operation $\Xi^{m|n}$, since

\[ \phi_{P,\theta,\alpha} (\Xi^{m|n}_{\text{cla}}) = \phi_{P,\theta,\alpha} \circ \Xi^{m|n}_{P,\theta,\alpha} (\alpha_{\text{cla}}) = \Xi^{m|n}_{P,\theta} (\Phi_{P,\theta,\alpha}^{*} (\alpha_{\text{cla}})) = \Xi^{m|n}_{P,\theta} (\alpha). \]

(b) Let $Q_{P,\theta} = EG \times F$ and define an action $\rho_{Q_{P,\theta}}((e,f),g) = (\rho_G (e,g), \ell_f (g^{-1}, f))$. Then $Q_{P,\theta}^{m|n}$ is the total space of a principal $G$-bundle with base space $EG \otimes_G F$. There is a natural $F$-orientation $\theta_{Q_{P,\theta}}^{m|n}$ on $Q_{P,\theta}^{m|n}$ represented by $f_{Q_{P,\theta}} : (e,f) \mapsto f$. Let $\alpha = 0$ and take $h_0$ to be the constant map onto $*_{H_m}$. Then (2.5) shows that $\phi_{Q_{P,\theta}}^{*} (\theta_{Q_{P,\theta}}^{m|n}) = i_m$. If $\Xi^{m|n}$ is pointed, $i_m^{*} (\Xi^{m|n}_{\text{cla}}) = \phi_{Q_{P,\theta}}^{*} (\theta_{Q_{P,\theta}}^{m|n}) (\Xi^{m|n}_{\text{cla}}) = \Xi^{m|n}_{P,\theta} (\alpha) = 0$.

Conversely, suppose that $i_m^{*} (\Xi^{m|n}_{\text{cla}}) = 0$ and let $P \rightarrow B$ be a principal $G$-bundle with $F$-orientation $\theta$. Observe that $Q_{P,\theta}^{m|n}$ is the universal example of a principal $G$-bundle with $F$-orientation, by the same argument as for Proposition 2.4. Choose a classifying morphism $\Phi_{P,\theta} : P \rightarrow Q_{P,\theta}^{m|n}$ with $\theta = \Phi_{P,\theta}^{*} (\theta_{Q_{P,\theta}}^{m|n})$ and quotient map $\phi_{P,\theta}$. Since $\Phi_{P,\theta}^{*} (\alpha) = 0$, the uniqueness part of Proposition 2.4 implies that $\phi_{P,\theta,\alpha}$ is homotopic to $\Phi_{P,\theta}^{*} (\theta_{Q_{P,\theta}}^{m|n}) \circ \phi_{P,\theta} = i_m \circ \phi_{P,\theta}$. Therefore,

\[ \Xi^{m|n}_{P,\theta} (\alpha) = \phi_{P,\theta,\alpha}^{*} (\Xi^{m|n}_{\text{cla}}) = \phi_{P,\theta,\alpha}^{*} (\Xi^{m|n}_{\text{cla}}) = 0. \]

(c) The trivial bundle is classified by $\Phi : P \rightarrow EG$, $(b,g) \mapsto \rho_{EG} (*,EG,g)$. Taking $p = (b,1)$ in (2.3) gives $\phi_{P,\theta,\alpha} (b) = *_{EG \otimes_G (f_0 \in G, h_{\alpha_{\text{cla}}}})$, where $\mu_{h_1(b,1)} = h_1(b)$. Identifying the fiber with $F \times \text{Map}(G,H_m)$, we find $\phi_{P,\theta,\alpha} = (f_0, h_{\alpha})$ and hence $\Xi^{m|n}_{P,\theta} (\alpha) = \phi_{P,\theta,\alpha}^{*} (\xi \times x) = f_0^{*} (\xi) \cup (h_{\alpha})^{*} (x)$. \(\square\)
Homological Lie brackets and pushforward operations

2.2. Stable pushforward operations.

**Definition 2.9.** Let \( \Sigma^{m+1}|n+1 \) be a pointed pushforward operation for principal \( G \)-bundles with \( F \)-orientation of type \( (m+1, n+1) \). The desuspension is the pointed pushforward operation \( \sigma(\Sigma^{m+1}|n+1) \) of type \( (m, n) \) defined as follows. Let \( P \to B \) be a principal \( G \)-bundle and equip the pullback \( P \times S^1 \to B \times S^1 \) along the projection \( \pi_B: B \times S^1 \to B \) with the pullback \( F \)-orientation \( \theta = \pi_B^* \theta \), where \( \pi_P: P \times S^1 \to P \) is the projection. The long exact sequence of the pair \( (P \times S^1, P \times \{1\}) \) splits as

\[
\begin{array}{cccccc}
0 & \to & H^m(P) & \xrightarrow{\varepsilon_P} & H^{m+1}(P \times S^1) & \to & H^{m+1}(P) & \to & 0 \\
0 & \to & H^*(B) & \xrightarrow{\varepsilon_B} & H^{n+1}(B \times S^1) & \to & H^{n+1}(B) & \to & 0.
\end{array}
\]

Here, \( \varepsilon_X: H^*(X) \cong \tilde{H}^{n+1}(\Sigma X^+) \to H^{*+1}(X \times S^1) \) denotes the suspension isomorphism followed by the inclusion. The right-hand square commutes by naturality \( (2.9) \).

Define \( \sigma(\Sigma^{m+1}|n+1) \) as the restriction indicated in \( (2.9) \), using exactness and \( \Sigma^{m+1}|n+1(0) = 0 \).

**Definition 2.10.** A stable pushforward operation \( \Sigma^{m+k} \) of degree \( k \in \mathbb{Z} \) for principal \( G \)-bundles with \( F \)-orientation is a sequence of pointed pushforward operations \( \Sigma^{m+k} \) of type \( (m, m+k) \) for all \( m \geq 0 \) such that \( \sigma(\Sigma^{m+1+k}) = \Sigma^{m+k} \). We sometimes omit the superscripts and write \( \sigma_P \) as the restriction indicated in \( (2.9) \), using exactness and \( \Sigma^{m+1+k}(0) = 0 \).

The fiber bundles \( B_{\text{cl}} \to EG \otimes G F \) from above with their base-point sections \( i_m \) form a parameterized spectrum as in May–Sigurdsson [11]. This means that on the fiberwise suspensions \( \Sigma^j B_{\text{cl}}^m = EG \otimes_G (F \times \Sigma \text{Map}) \) there are maps

\[
\beta_m: \Sigma^j B_{\text{cl}}^m \to B_{\text{cl}}^{m+1}, \quad e \otimes_G (f, \mu \times t) \mapsto e \otimes_G (f, \eta_m(\mu(-), t)),
\]

where \( e \in EG \), \( f \in F \), \( \mu \in \text{Map}(G, H_m) \), \( t \in S^1 \), and where \( \eta_m \) is the connecting map of the Eilenberg–Mac Lane spectrum. Note that \( \Sigma^j B_{\text{cl}}^m \) is again a fiber bundle over \( EG \otimes_G F \) with a natural base-point section; the usual suspension \( \Sigma^j B_{\text{cl}}^m \) is obtained from \( \Sigma^j B_{\text{cl}}^m \) by collapsing this section. The maps \( B_{\text{cl}}^m \to \Omega B_{\text{cl}}^{m+1} \) adjoint to \( \beta_m \) into the fiberwise loop space are homeomorphisms.

**Definition 2.11.** Let \( E_m = EG \otimes_G (F \times \text{Map}(G, H_m))/EG \otimes_G (F \times \{\ast \text{Map}\}) \) be the quotient \( q_m: B_{\text{cl}}^m \to E_m \) obtained by collapsing the base-point section \( i_m \). The connecting map \( \epsilon_m: \Sigma E_m \to E_m+1 \) is induced by \( \beta_m \), so \( q_m+1 \circ \beta_m = \epsilon_m \circ \Sigma q_m \).

Let \( F_m = (F \times \text{Map}(G, H_m))/\Sigma \text{Map} \) and let \( k_m: F_m \to E_m \) be the inclusion given by \( (f, \mu) \mapsto \ast EG \otimes_G (f, \mu) \). The maps \( \epsilon_m \) restrict to connecting maps \( \varphi_m: \Sigma F_m \to F_m+1 \) such that \( \epsilon_m \circ (\Sigma k_m) = k_m+1 \circ \varphi_m \). With the action \( \ell g, (f, \mu) = (\ell_f g, f), \ell \text{Map}(g, \mu), \{F_m, \varphi_m\} \) is a G-spectrum.

The long exact sequences in cohomology of the pairs \( (E_m, F_m) \) and \( (\Sigma E_m, \Sigma F_m) \) and the suspension isomorphisms yield a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
H^{n+1}(E_{m+1}, F_{m+1}) & \to & \tilde{H}^{n+1}(E_{m+1}) & \to & \tilde{H}^{n+1}(F_{m+1}) & \\
\downarrow \epsilon_m & & \downarrow \sigma_m & & \downarrow \sigma_m & \\
H^{n+1}(\Sigma E_m, \Sigma F_m) & \to & \tilde{H}^{n+1}(\Sigma E_m) & \to & \tilde{H}^{n+1}(\Sigma F_m) & \\
\cong H^n(E_m, F_m) & & \cong \tilde{H}^n(E_m) & & \cong \tilde{H}^n(F_m) &
\end{array}
\]

The vertical maps determine inverse systems for each \( m \). Since taking the inverse limit preserves exactness over \( Q \), we obtain for all \( m, k \) an exact sequence:

\[
\cdots \to \lim H^{m+k}(E_m, F_m) \to \lim \tilde{H}^{m+k}(E_m) \to \lim \tilde{H}^{m+k}(F_m) \to \cdots
\]
Since the section \( i_m \) has a retraction, the exact sequence of \((B_m^{\text{cla}}, EG \otimes_G F)\) in cohomology splits into short exact sequences
\[
0 \longrightarrow \tilde{H}^n(E_m) \xrightarrow{\eta_m} H^n(B_m^{\text{cla}}) \xrightarrow{i_m^*} H^n(EG \otimes_G F) \longrightarrow 0. \quad (2.12)
\]
It follows that the class of a pointed pushforward operation lies in the image of \( q_m^* \).

In this case we can regard \( \Xi_{\text{cla}}^{m|n} \) as a class in the subspace \( \tilde{H}^n(E_m) \).

We can now state the classification result for stable pushforward operations.

**Theorem 2.12.** The class of the desuspension \( \sigma(\Xi_{\text{cla}}^{m+1|n+1}) \in \tilde{H}^n(E_m) \) of a pointed pushforward operation \( \Xi_{\text{cla}}^{m+1|n+1} \) is the image of \( \Xi_{\text{cla}}^{m|n+1} \) under the middle vertical map in (2.10). In particular, stable pushforward operations of degree \( k \) for principal \( G \)-bundles with \( F \)-orientation correspond bijectively to elements \( \Xi_{\text{cla}}^k = \{ \Xi_{\text{cla}}^{m+k} \}_{m \geq 0} \) in the inverse limit \( \tilde{H}^{m+k}(E_m) \), viewed as a subset of the Cartesian product \( \prod_n \tilde{H}^{m+k}(E_m) \).

**Proof.** Using the notation of Definition 2.6 equip the pullback bundle \( P_m^{\text{cla}} \times S^1 \) with the pullback \( F \)-orientation \( \rho_m^{\text{cla}} \times S^1 \) and the cohomology class \( \varsigma_P^{\text{cla}}(\alpha_m^{\text{cla}}) \in H^{m+1}(P_m^{\text{cla}} \times S^1) \). Let \( \Gamma_m = \Phi_{P_m^{\text{cla}} \times S^1, \rho_m^{\text{cla}} \times S^1, \varsigma_P^{\text{cla}}(\alpha_m^{\text{cla}})} \) be the classifying morphism from Proposition 2.7 which satisfies \( \Gamma_m(\alpha_m^{\text{cla}}) = \varsigma_P^{\text{cla}}(\alpha_m^{\text{cla}}) \) and \( \Gamma_m^*(\rho_m^{\text{cla}}) = \theta_m \times S^1 \). Let \( \gamma_m \) be the quotient map of \( \Gamma_m \). We then compute:
\[
\gamma_m^*(\Xi_{\text{cla}}^{m+1|n+1}) = \gamma_m^*(\Xi_{P_m^{\text{cla}} \times S^1}^{m+1|n+1})(\alpha_m^{\text{cla}}) = \Xi_{P_m^{\text{cla}} \times S^1}^{m+1|n+1}(\Gamma_m(\alpha_m^{\text{cla}})) \quad \text{by } 2.7
\]
\[
= \Xi_{P_m^{\text{cla}} \times S^1}^{m+1|n+1}(\Gamma_m(\alpha_m^{\text{cla}})) \quad \text{by nat. } 2.2
\]
\[
= \Xi_{P_m^{\text{cla}} \times S^1}^{m+1|n+1}(\varsigma_P^{\text{cla}}(\alpha_m^{\text{cla}})) \quad \text{by } \text{nat. } 2.9
\]
\[
= \varsigma_{P_m^{\text{cla}}}(\sigma(\Xi_{\text{cla}}^{m+1|n+1})) \quad \text{by } 2.9
\]
Hence \( \gamma_m^*(\Xi_{\text{cla}}^{m+1|n+1}) = \varsigma_{P_m^{\text{cla}}}(\sigma(\Xi_{\text{cla}}^{m+1|n+1})) \) holds in \( H^{n+1}(B_m^{\text{cla}} \times S^1) \). There is a commutative diagram
\[
H^{n+1}(B_m^{\text{cla}}) \xrightarrow{\gamma_m^*} H^{n+1}(B_m^{\text{cla}} \times S^1) \xleftarrow{\varsigma_{P_m^{\text{cla}}}} H^n(B_m^{\text{cla}})
\]
\[
\tilde{H}^{n+1}(E_m) \xrightarrow{\eta_m^*} \tilde{H}^{n+1}(\Sigma E_m) \xleftarrow{\gamma_m^*} \tilde{H}^n(E_m),
\]
whose maps on the sides are injective, by (2.12). Since the suspension at the bottom is bijective, the middle vertical map is injective as well. We conclude that \( \epsilon_m^*(\Xi_{\text{cla}}^{m+1|n+1}) \) is the suspension of \( \Xi_{\text{cla}}^{m|n} \) in the group \( \tilde{H}^{n+1}(\Sigma E_m) \), as required. \( \square \)

**Theorem 2.13.** Every stable pushforward operation is automatically \( Q \)-linear.

**Proof.** Recall from (2.12) that \( \Xi_{\text{cla}}^{m|n+k} \in \tilde{H}^{m+k}(E_m) \) for a pointed pushforward operation. Therefore, we can represent these classes by maps \( h_m : B_m^{\text{cla}} \rightarrow H_{m+k} \) such that \( h_m \circ i_m = \tau_{H_{m+k}} \) and \( \Xi_{\text{cla}}^{m|n+k} = h_m^*(J_{m+k}) \) for the tautological class from Example 2.4. Observe that there is a homotopy-commutative diagram
\[
\begin{align*}
\Sigma E_m & \xrightarrow{\beta_m} B_m^{\text{cla}} \xrightarrow{\gamma_m^*} H_m^{m+k} \xrightarrow{h_{m+1}} E_{m+1} \\
\Sigma H_{m+k} & \xrightarrow{\eta_{m+k}} H_{m+1+k}
\end{align*}
\]
since by Theorem 2.12 the class $\Xi_{m+m}^{||k}$ is the image of $\Xi_{m+m}^{m+1+k}$ under the middle vertical map in (2.13), meaning that the outer square commutes up to homotopy; then precompose the homotopy with $\Sigma^l q_m$.

Let $*$ be the loop composition and $\star$ the fiberwise loop composition on the fiberwise loop space $\Omega^l$. Let $P \to B$ be a principal $G$-bundle and let $\alpha, \beta \in H^m(P)$. Then the representative map $h_{\alpha+\beta}$ for $\alpha + \beta$ can be taken to be the composite

$P \xrightarrow{(h_\alpha, h_\beta)} H_m \times H_m \xrightarrow{\eta_m \times \eta_m} \Omega H_{m+1} \times \Omega H_{m+1} \xrightarrow{\star} \Omega H_{m+1} (\eta_m)^{-1} H_m.$

Hence $\phi_{P,\theta,\alpha+\beta}$ is the composition of $(\phi_{P,\theta,\alpha}, \phi_{P,\theta,\beta}) : B \to B_{\alpha,\beta} \times C$ where $C = EG \otimes_G F$, with the upper row of the homotopy-commutative diagram

\[
\begin{array}{ccc}
B_{\alpha,\beta} \times C & \xrightarrow{\beta_{\alpha,\beta} \times C} & \Omega^l B_{\alpha,\beta} \times C \\
(h_{\alpha,\beta}) \times \eta_{\alpha,\beta} \downarrow & & \downarrow \Omega^l (h_{\alpha,\beta}) \times \eta_{\alpha,\beta} \\
H_{m+k} \times H_{m+k} & \xrightarrow{\star} & \Omega H_{m+1+k} \times \Omega H_{m+1+k} \xrightarrow{\eta_{m+k} \times \eta_{m+k}} H_{m+k}.
\end{array}
\]

Here, the outer two squares homotopy-commute by the adjoint of the inner square in (2.13). The lower row represents the addition $a : H_{m+k} \times H_{m+k} \to H_{m+k}$, so $a^*(b_{m+k}) = \pi_1^*(b_{m+k}) + \pi_2^*(b_{m+k})$ for the two projections $\pi_1$ and $\pi_2$. Compute

\[
\Xi_{P,\theta}^{\{m+k\}} (a + \beta) = \phi_{P,\theta,\alpha+\beta}^* (\Xi_{\alpha,\beta}^{\{m+k\}}) = \phi_{P,\theta,\alpha}^* h_m^* (\eta_{m+k}) = (\phi_{P,\theta,\alpha}, \phi_{P,\theta,\beta})^* (h_{\alpha,\beta}) = \phi_{P,\theta,\alpha}^* h_m^* (\eta_{m+k}) + \phi_{P,\theta,\beta}^* h_m^* (\eta_{m+k}) = \phi_{P,\theta,\alpha}^* (\Xi_{\alpha,\beta}^{\{m+k\}}) + \phi_{P,\theta,\beta}^* (\Xi_{\alpha,\beta}^{\{m+k\}}) = \Xi_{\alpha,\beta}^{\{m+k\}} (a) = \Xi_{\alpha,\beta}^{\{m+k\}} (\beta).
\]

This proves that $\Xi_{P,\theta}^{\{m+k\}} : H^m(P) \to H^{m+k}(B)$ is an additive map between $\mathbb{Q}$-vector spaces. All such maps are automatically $\mathbb{Q}$-linear.

2.3. Multiplicative properties. Recall that the Eilenberg-MacLane ring spectrum is equipped with product maps $\pi_{m_1, m_2} : H_{m_1} \times H_{m_2} \to H_{m_1+m_2}$ characterized by $\pi_{m_1, m_2} (j_{m_1} \times j_{m_2}) = j_{m_1+m_2}$ for the tautological classes from Example 2.4. Define $\mu_{m_1, m_2} : \text{Map}(G, H_{m_1}) \times H_{m_2} \to \text{Map}(G, H_{m_1+m_2})$ by $\mu_{m_1, m_2} (\text{ev} \circ \text{id}_{H_{m_1+m_2}}) = j_{m_1+m_2} \circ (\text{ev} \times \text{id}_{H_{m_1}})$ using the evaluation map. In the notation of Definition 2.6 there is then a pullback diagram

\[
P_{\alpha,\beta} \times H_{m_2} \xrightarrow{id_{\alpha,\beta} \times \text{id}_{H_{m_2}}} P_{\alpha,\beta} \times H_{m_2} \\
P_{\alpha,\beta} \times H_{m_2} \xrightarrow{\mu_{m_1, m_2}} P_{\alpha,\beta} \times H_{m_2} \xrightarrow{\rho_{m_1, m_2}} P_{\alpha,\beta}
\]

where $\rho_{m_1, m_2}$ denotes the quotient map and we observe that the upper horizontal map is equivariant. The orientation $\rho_{m_1, m_2}$ pulls back to $\pi_{m_1, m_2}^* (\rho_{m_1, m_2})$, where $\pi_{m_1, m_2} : B_{\alpha,\beta} \times H_{m_2} \to B_{\alpha,\beta}$ denotes the projection.

Proposition 2.14. Let $\Xi^{\{m+k\}}\{m+k\}$ be a stable pushforward operation with class $\Xi_{\alpha,\beta}^{\{m+k\}} = \{\Xi_{\alpha,\beta}^{\{m+k\}}\}$, where $\Xi_{\alpha,\beta}^{\{m+k\}} \in H^{m+k}(B_{\alpha,\beta})$. Then $\Xi^{\{m+k\}}\{m+k\}$ is $H^* (B)$-linear, meaning

\[
\Xi_{\alpha,\beta}^{m_1+m_2} (\alpha \cup \beta) = \Xi_{\alpha,\beta}^{m_1+m_2} (\alpha) \cup \beta
\]

for all $\alpha \in H^{m_1}(P)$ and $\beta \in H^{m_2}(B)$, if and only if for all $m_1, m_2$ we have

\[
\rho_{m_1, m_2}^* (\Xi_{\alpha,\beta}^{m_1+m_2}) = \Xi_{\alpha,\beta}^{m_1+m_2}
\]
Proof. If \((2.16)\) holds, then \((2.13)\) follows by expanding both sides using \((2.6)\) and \(\rho_{m_1,m_2} \circ (\phi_{P,\theta,\alpha},\iota) = \phi_{P,\theta,\alpha} \circ \iota\). Note that \((id_{EG} \times id_F \times \tau_{m_1,m_2})^*(\alpha_{m_1+m_2}) = M_{m_1,m_2} \circ (\alpha_{m_1} \times id_{m_2})\). Hence \((id_{EG} \times id_F \times \tau_{m_1,m_2})^*(\alpha_{m_1+m_2}) = \alpha_{m_1} \times j_{m_2}^*\). Assuming \((2.16)\), we deduce \((2.10)\) by applying naturality to \((2.14)\), as follows.

\[
\rho_{m_1,m_2}(\xi_{m_1+m_2}^{m_1+m_2+k}) = \rho_{m_1,m_2}(\xi_{m_1+m_2}^{m_1+m_2+k}\alpha_{m_1+m_2}^* \circ \iota_{m_1} \times j_{m_2}) \quad \text{by nat.} \tag{2.22}
\]

\[
\Xi_{m_1+m_2}^{m_1+m_2+k} = \Xi_{m_1+m_2}^{m_1+m_2+k}(\iota_{m_1} \times j_{m_2}) \quad \text{by nat.} \tag{2.22}
\]

Definition 2.15. Let \(\xi^G_0 \in H^n(EG \otimes_G F)\). Let \(P \to B\) be a principal \(G\)-bundle with \(F\)-orientation \(\theta\). Write \(S_{BG} : P \to P \otimes_G F\) for the section associated to \(f_0 : P \to F\), defined by \(S_B(b) = p \otimes f\) if \(f_0(p) = f\). Let \(\Phi : P \to EG\) be a classifying morphism. Define the characteristic class \(\xi^G_0(\theta) = S^*_B(\Phi_P \otimes_G id_F)^*(\xi^G_0) \in H^n(B)\).

Let \(\pi_F\) and \(\pi_{Map}\) be the projections of \(B_0^{cl} = EG \otimes_G (F \times Map(G,H))\) onto \(EG \otimes_G F\) and \(EG \otimes_G Map(G,H)\). Let \(j : Map(G,H) \to EG \otimes_G Map(G,H)\) be the inclusion \(j(\mu) = \mu \otimes g\). Let \(ev : G \times Map(G,H) \to H\) be the evaluation map and let \(t_0 \in H_0(G)\) be such that \((t_0,1_G) = 1\).

Proposition 2.16. Let \(G\) be a connected topological group. Suppose that \(\xi^G_0 = \pi^G_{cl} \cup \pi^G_{Map}(x)\) for \(\xi^G_0 \in H^n(EG \otimes_G F)\) and \(x \in H^0(EG \otimes_G Map(G,H))\) and that \(j^*(x) = ev^*(t_0)/t_0\). Then \(\xi^G_0(1_P) = \xi^G_0(\theta)\) for every principal \(G\)-bundle \(\pi : P \to B\) with \(F\)-orientation \(\theta\).

Proof. The inclusion \(\kappa : H_0 \to Map(G,H)\) of the constant maps is a homeomorphism since \(H_0\) is discrete. Let \(h_{1_P} : B \to H_0\) and \(h_{1_P} \circ \pi = h_{1_P}\). Then \(\pi_{Map} \circ \phi_{P,\theta,1_P} = j \circ \kappa \circ h_{1_P}\) and \(\pi \circ \phi_{P,\theta,1_P} = (\Phi_P \otimes_G id_F) \circ S_B\). Let \(\pi H_0 : G \times H_0 \to H_0\) be the projection. Then

\[
\Xi^G_0(1_P) = \phi_{P,\theta,1_P}(\xi_{cl}^G(1_P)) = \phi_{P,\theta,1_P}(\pi^G_{cl}(\xi^G_0(\theta) \cup \pi^G_{Map}(x)))
\]

\[
\Xi^G_0(\theta) \cup h_{1_P}^* \kappa^*(ev^*(t_0))/t_0 \quad \text{by Def.} \tag{2.10}
\]

\[
= \xi^G_0(\theta) \cup h_{1_P}^* \kappa^*(ev^*(t_0))/t_0 \quad \text{nat. of} \ 'j' \tag{2.22}
\]

\[
= \xi^G_0(\theta) \cup h_{1_P}^*(1_G \times j_0)/t_0 \quad \text{by ev(id}_G \times \kappa = \pi_{H_0})
\]

\[
= \xi^G_0(\theta) \cup h_{1_P}^*(t_0) = \xi^G_0(\theta) \cup 1_B = \xi^G_0(\theta) \quad \text{by} \ (t_0,1_G) = 1. \tag{2.17}
\]

2.4. A technical result. By Theorem 2.12 stable pushforward operations are in bijection with classes in \(\lim H^{m+k}(F_m)\). To compute these, the inverse limit groups of the fiber spectra \(H^{m+k}(F_m)\) play an important role, so we describe these next.

Assume that \(H_*(G)\) is of finite type and fix a basis \(\{b_i\}_{i \in I}\) of homogeneous elements. Using the tautological classes, define slant product classes

\[
x_i^{(m)} = ev^*(t_0)/b_i \in H^{m-|b_i|}(Map(G,H_m)). \tag{2.17}
\]

Theorem 2.17. (a) \(\hat{H}^*(Map(G,H_m))\) is a free graded-commutative \(Q\)-algebra on \(x_i^{(m)}\) for those \(b_i\) with \(|b_i| \leq m\). Use the notation of Example 2.3 and the suspension isomorphism \(\sigma^{-1} : \hat{H}^{*+1}(\Sigma Map(G,H_m)) \cong \hat{H}^*(Map(G,H_m))\).
to define \( \mu^\dagger_m = \sigma^{-1} \circ \mu^\ast_m : \tilde{H}^{*+1}(\text{Map}(G, H_{m+1})) \to \tilde{H}^*(\text{Map}(G, H_m)) \). Then

\[
\mu^\dagger_m(x_i^{(m+1)}) = \begin{cases} 
  x_i^{(m)} & \text{if } |b_i| \leq m, \\
  0 & \text{if } |b_i| = m + 1,
\end{cases}
\]

and \( \mu^\dagger_m \) annihilates monomials of degree \( \geq 2 \) in the \( x_i^{(m+1)} \).

(b) Let \( F \) be a \( G \)-space and let \( F_m = (F \times \text{Map}(G, H_m))/(F \times \{ \text{Map} \}) \) be as in Definition 2.11. Then there is an isomorphism of inverse systems \( \tilde{H}^*(F_m) \cong H^*(F) \otimes H^*(\text{Map}(G, H_m)), \) where the structure maps are given by (2.10) and by \( \varphi^\ast_m = \text{id} \otimes H^*(F) \otimes \mu^\dagger_m, \) respectively.

(c) The inverse limit \( \Gamma^\ast_F = \lim_{\to} \tilde{H}^{m+s}(F_m) \cong H^*(F) \langle x_i \mid i \in I \rangle \) is isomorphic to a free graded \( H^*(F) \)-module on generators \( x_i = (x_i^{(m)})_{m \geq 0} \) of degree \( -|b_i| \), where we view the inverse limit as a subset of \( \prod_{m \geq 0} \tilde{H}^{m+s}(F_m) \).

Proof. (a) Represent \( x_i^{(m)} \) by a pointed map \( \phi_i^{(m)} : \text{Map}(G, H_m) \to H_{m-|b_i|} \). Taken together, these determine a map

\[
\phi^{(m)} : \text{Map}(G, H_m) \to \prod_{I \subseteq I, |b| \leq m} H_{m-|b|}.
\]

Identifying \( \pi_k(\text{Map}(G, H_m)) \cong [G^+ \wedge S^k, H_m] = \tilde{H}^m(G^+ \wedge S^k) \) and \( \pi_k(H_m-|b|) = \tilde{H}^m-|b|((S^k)) \), we have \( \pi_k(\phi_i^{(m)}) (\alpha) = \alpha/|b_i| \) for \( \alpha \in H^m(G^+ \wedge S^k) \). Using the basis dual to \( b_i \), we can further identify \( \prod_{I \subseteq I, |b| \leq m} \tilde{H}^m-|b|((S^k)) \cong H^m(S^k) \otimes \tilde{H}^k(S^k) \) and then \( \pi_k(\phi^{(m)}(\alpha)) \) becomes the Künneth map \( \tilde{H}^m(G^+ \wedge S^k) \to H^m(S^k) \otimes \tilde{H}^k(S^k), \) which is an isomorphism over \( \mathbb{Q} \). This holds for all \( k \), so (2.19) is a weak equivalence.

The fact that \( H^*(H_m-|b|) \) is the free graded-commutative \( \mathbb{Q} \)-algebra on \( J_{m-|b|} \) and (2.19) then imply that \( \tilde{H}^*(\text{Map}(G, H_m)) \) is the free graded-commutative \( \mathbb{Q} \)-algebra on \( (\phi_i^{(m)})(J_{m-|b|}) = x_i^{(m)}. \)

We compute \( \mu^\dagger_m \). Using naturality of the slant product, one verifies \( \mu^\dagger_m(x_i^{(m+1)}) = \sigma(x_i^{(m)}) \), which vanishes if \( |b_i| = m + 1 \), giving (2.18). Finally, \( \mu^\dagger_m \) vanishes on monomials of degree \( \geq 2 \) because cup products vanish on suspensions.

Part (b) is clear. For (c) we use that, by (a), the image of the \( H^*(F) \)-linear map \( \varphi^\ast_m \) is \( H^*(F) \langle x_i^{(m)} \mid |b_i| \leq m \rangle. \) Hence the inverse limit is \( H^*(F) \langle x_i \mid i \in I \rangle. \)

3. Projective Euler operations

In this section, we compute the group \( \Pi_{BU \times \mathbb{Z}} \) of stable pushforward operations for principal \( BU(1) \)-bundles and orientations in twisted K-theory. The first step in 4.1 is to place the group \( \Pi_{BU \times \mathbb{Z}} \) in an exact sequence. This allows us to construct the projective Euler operation and the projective rank operation in 3.2, where we prove Theorem 3.1(a)–(I) and Theorem 3.2. Parts (g)–(i) of Theorem 3.1 are proven later in 3.3. In 3.3 we introduce the module structure on \( \Pi_{PU \times \mathbb{Z}} \) used in the classification problem and Theorem 3.4 is proven in 3.4. The homological version of the projective Euler operation is constructed in 3.6.

Assumption 3.1. Specialize the situation of 2.2 to the following setup of spaces.

(a) A topological abelian group model \( G = BU(1) \) for the classifying space of complex line bundles with the tensor product as the group operation.

(b) A classifying space \( BU \times \mathbb{Z} \) for (virtual) complex vector bundles with the direct sum operation, making it a commutative \( H \)-space.

(c) An involution \( (\cdot) \) on \( BU \times \mathbb{Z} \) given by taking the dual vector bundle. Composing \( (\cdot) \) with an orientation \( \theta \) defines the dual orientation \( \tilde{\theta} \).
(d) A topological group action $\ell_{BU \times \mathbb{Z}}: BU(1) \times (BU \times \mathbb{Z}) \to BU \times \mathbb{Z}$ given by taking the tensor product of the dual of a line bundle with a vector bundle; see Definition 2.2 for an explanation of our convention.

We can fix the rank $r \in \mathbb{Z}$ and restrict (d) to an operation $\ell_{BU \times \{r\}}$ on $F = BU \times \{r\}$. Since pushforward operations make sense for fixed rank of $\theta$, we have $\Pi^*_{BU \times \{r\}} = \prod_{r \in \mathbb{Z}} \Pi^*_{BU \times \{r\}}$ and we can consider one rank at a time.

3.1. Exact classification sequence. Let $G = BU(1)$ and $F = BU \times \{r\}$. Recall from Definition 2.11 the spaces $F_m = (F \times \text{Map}(G, H_m))/\langle F \times \{\text{Map} \rangle$ and $E_m$, obtained by collapsing the base-point section of the principal $G$-bundle $B_m^\text{cla} = EG \circ G (F \times \text{Map}(G, H_m))$ over $EG \circ G F$. By Theorem 2.12 and Definition 2.2 we have $\Pi^*_{BU \times \{r\}} = \lim \tilde{H}^{m+*}(E_m)$ is the group of stable pushforward operations with orientations in twisted K-theory of rank $r$.

Depending on how one identifies the hemispheres, there are two possibilities differing by an inversion; our convention is fixed by the sign in (5.7). We then have a pullback fiber bundle $\phi_{B^2}(B_m^\text{cla}) = \phi_{B^2}(EG) \circ G (F \times \text{Map}(G, H_m))$ and a quotient space, written $\phi_{B^2}(E_m)$, obtained by collapsing the base-point section $\phi_{B^2}(EG) \circ G F \to \phi_{B^2}(B_m^\text{cla})$ to a point. The maps $e_m$ from Definition 2.11 pull back to connecting maps $\Sigma \phi_{B^2}(E_m) \to \phi_{B^2}(E_{m+1})$ which determine an inverse system $\tilde{H}^{n+1}(\phi_{B^2}(E_{m+1})) \to \tilde{H}^{n+1}(\Sigma \phi_{B^2}(E_m)) \cong H^*(\phi_{B^2}(E_m))$ as in (2.10). Since $BBU(1) = K(\mathbb{Z}, 3)$ is a rational homotopy sphere and the transition function $\gamma|_{\mathbb{CP}^1}$ is a generator of $\pi_2(BU(1)) \cong \pi_2(BBU(1)) = \mathbb{Z}$, we obtain the following.

Proposition 3.2. We have $\lim \tilde{H}^{m+*}(E_m; \mathbb{Q}) \cong \lim \tilde{H}^{m+*}(\phi_{B^2}(E_m); \mathbb{Q})$.

Recall that $H^*(BU(1))$ is the polynomial ring $\mathbb{Q}[c_1]$ on the first Chern class of the universal complex line bundle. Let $t \in H_2(BU(1))$ be the homology class dual to $c_1$. The group operation $m_{BU(1)}$ on $BU(1)$ induces a product on $H_*(BU(1))$ for which the $i$th divided power $t^i/i!$ becomes dual to $c_1^i$. Recall from [12, Thm. 14.5] that $H^*(BU \times \{r\})$ is a free commutative algebra $\mathbb{Q}[c_1, c_2, \ldots]$ on the Chern classes of the universal complex vector bundle. We may also use the Chern character components $c_0, j > 0$, as the free generators. By convention, $c_0 = r$ and $c_0 = 1$.

Definition 3.3. Let $\partial$ be the algebra derivation on $H^*(BU \times \{r\})$ defined by $\partial(c_j) = (r - j + 1)c_{j-1}$ on the polynomials generators.

Lemma 3.4. We have $\partial(c_{j+1}) = c_j$ for all $j \geq 0$ and $\partial(c_0) = 0$.

Proof. By the splitting principle, $c_j$ is characterized by its additivity under direct sums and by $c_j(L) = \frac{1}{j!}c_1(L)^j$ for complex line bundles. As $\partial$ is linear, $\partial(c_{j+1})$ is again additive under direct sums. If $L$ is a complex line bundle, then

$$
\partial(c_{j+1}(L)) = \partial \left( \frac{c_1(L)^{j+1}}{(j+1)!} \right) = \frac{(j+1)c_1(L)^j \partial c_1(L)}{(j+1)!} = \frac{c_1(L)^j}{j!} = c_j(L).
$$

Define the group $\Gamma_{BU}^r = \lim \tilde{H}^{m+*}(F_m)$, which is independent of $r$. Apply Theorem 2.11(c) to the basis $b_i = t^i$, $i \geq 0$, to see that $\Gamma_{BU}^r$ is a free $H^*(BU)$-module on generators $x_i$ of degree $-2i$ for all $i \geq 0$ with components $x_i^{(m)} = cv^*(c_1)/t^i$. 

Theorem 3.5. There is an exact sequence

\[
0 \longrightarrow \Pi^2_{BU}(r) \longrightarrow \Gamma^2_{BU} \xrightarrow{\delta_r} \Gamma^2_{BU} \xrightarrow{\delta_r} \Pi^2_{BU}(r) \xrightarrow{\delta_r} 0,
\]

where \(\delta_r\) is the unique \(\mathbb{Q}\)-linear map such that for all \(\xi \in H^*(BU)\) and \(i \geq 0\),

\[
\delta_r(\xi x_i) = \partial(\xi x_i + \xi x_{i+1}).
\]

In particular, rational stable pushforward operations of degree \(2e\) correspond to series \(\mathbb{Z}_{\mathbb{Q}}^2 = \sum_{i \geq 0} \xi_i x_i\), where \(\xi_i \in H^{2e+i}(BU \times \{r\})\) are cohomology classes with \(\partial \xi_0 = 0\) and \(\partial \xi_{i+1} = -\xi_i \quad (\forall i \geq 0)\).

Moreover, odd-degree rational stable pushforward operations correspond to elements in the cokernel of \(\delta_r\).

**Proof.** Following [2], consider the Wang exact sequence

\[
\cdots \xrightarrow{-\delta_r} \hat{H}^{*-3}(F_m) \longrightarrow \hat{H}^*(\phi^*_3)(E_m)) \longrightarrow \hat{H}^*(F_m) \xrightarrow{-\delta_r} \cdots
\]

This sequence is obtained from the long exact sequence of the pair \((\phi^*_3(E_m), F_m)\) by using the identification \(H^*(\phi^*_3(E_m), F_m) \cong \hat{H}^{*-3}(F_m)\) induced by the homomorphism \(\phi^*_3(E_m)/F_m \cong \Sigma^3(F_m)\). Hence, after taking the limit, we are just considering the sequence (2.11). As \(\Gamma^*_{BU}\) is concentrated in even degrees, the long exact sequence reduces to (3.1). Alternatively, the exact sequence (3.1) is obtained from the Serre spectral sequence, which degenerates at the \(E_2\)-page with only two non-zero columns and which therefore reduces to a long exact sequence. Then \(\delta_r\) becomes the differential \(d_3\) of the Serre spectral sequence. In particular, \(\delta_r\) is a derivation of the algebra \(H^*(F_m)\). It remains to verify the formula (3.2). By (2.12), it suffices to verify this in the cohomology of \(\phi^*_3(B^3_{\mathbb{Q}})\), where \(B^3_{\mathbb{Q}}\) is the associated fiber bundle of \(EBU(1) \to BBU(1)\) with fiber \(BU \times Map(BU(1), H_m)\) using the product \(\ell_{BU \times Map}(t)\) and \(\ell_{Map}\) from (2.4). Hence

\[
\pi_{Map} \circ \ell_{BU \times Map} = \ell_{Map}(\text{id}_{BU(1)} \times \pi_{Map}),
\]

\[
\pi_{BU} \circ \ell_{BU \times Map} = \ell_{BU \times \{r\}}(\text{id}_{BU(1)} \times \pi_{BU})
\]

for the two projections \(\pi_{BU}\) and \(\pi_{Map}\) of \(BU \times Map(BU(1), H_m)\). It follows that the pullback bundle \(\phi^*_3(B^3_{\mathbb{Q}})\) over \(S^3\) is obtained by clutching a pair of trivial bundles over \(S^2 \cong \mathbb{CP}^1\) using the transition function \(\ell_{BU \times Map}(\gamma_{\mathbb{CP}^1} \times \text{id}_{BU(1)})\). According to [2] (3.8), this leads to

\[
\delta_r(z) = -(\gamma_{\mathbb{CP}^1} \times \text{id}_{BU(1)} \times \pi_{Map})(x^{(m)})/[(\mathbb{CP}^1)]
\]

for the differential. We verify (3.2) on the generators \(\pi_{BU}(e_j)\) and \(\pi_{Map}(x^{(m)})\). For \(\hat{m}_{BU(1)} : BU(1) \to BU(1) \to BU(1), (g, h) \mapsto g^{-1}h\) we have \(\text{ev}(\text{id}_{BU(1)} \times \ell_{Map}) = \text{ev}(\hat{m}_{BU(1)} \times \text{id}_{Map})\). Then we compute:

\[
\delta_r(\pi_{Map}(x^{(m)})) = -(\gamma_{\mathbb{CP}^1} \times \pi_{Map})(x^{(m)})/[(\mathbb{CP}^1)]
\]

by (3.5) and (3.6).

\[
= -\pi_{Map}(\ell_{Map}(\text{ev}^*(J_m)/t))/t
\]

by properties of \('j'\);

\[
= -\pi_{Map}(\ell_{Map}(\text{id}_{BU(1)} \times \ell_{Map})*\text{ev}^*(J_m)/t)/t
\]

by (3.7).

Notice that (1.2) and our convention for the action in Assumption 3.1(d) implies

\[
\ell^e_{BU \times \{r\}}(e_j) = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} e_j \times e_{r^k}.
\]
To verify (3.2) on $\pi_B^*(c_j)$, we compute:

$$\delta_r(\pi_B^*(c_j)) = -((\gamma_{\mathbb{CP}^1} \times \pi_B^*)\ell_{\mathbb{CP}^1}(c_j))/[\mathbb{CP}^1]$$

by (3.3) and (3.7)

$$= -((\text{id}_{\mathbb{CP}^1} \times \pi_B^*)\ell_{\mathbb{CP}^1}(c_j))/t$$

by $\gamma_{\mathbb{CP}^1}(\mathbb{CP}^1) = t$

$$= -\pi_B^*(\ell_{\mathbb{CP}^1}(c_j))/t$$

nat. of $'$

$$= (r - j + 1)\pi_B^*(c_{j-1})$$

by (3.8)

$\square$

Let $\Xi_1^{i+2c}$ be a stable pushforward operation with class $\Xi_{\text{cla}}^{2c} = \sum_{i \geq 0} \xi_i x_i$ in $\Pi_{BU \times \{r\}}^{2c}$. We call $\xi_0$ the constant term of $\Xi_{\text{cla}}^{2c}$. As in the proof of Theorem 3.5, there is an exact sequence

$$0 \to H^{2c}(EBU(1) \otimes_{BU(1)} BU) \to H^{2c}(BU \times \{r\})$$

$$\to H^{2c}(BU \times \{r\}) \to H^{2c+2}(EBU(1) \otimes_{BU(1)} BU) \to 0.$$  

(3.9)

Since $\xi_0 = \xi_{BU(1)}^{0}$ is in the kernel of $\partial$, it determines a characteristic class for twisted K-theory classes of rank $r$ as in Definition 2.15. This is the same argument as for [2, Prop. 8.8] and (3.9) is just [2 (3.5)].

**Definition 3.6.** Let $P \to B$ be a principal $BU(1)$-bundle with principal action $\rho_P : P \times BU(1) \to P$. Recall that $t \in H_2(BU(1))$ is dual to $c_1$. Define $t \cdot \alpha = \langle t, \rho_P^*(\alpha) \rangle$. Since each $t \cdot \alpha$ is a nilpotent endomorphism of $H^*(P)$, the action extends by the universal property of the formal power series ring to an action of the algebra $\mathbb{Q}[t]$.

Suppose that the bundle has a section $s : B \to P$. Then we define $\gamma_{P,s} : P \to G$ by $\gamma_{P,s}(s(b)g) = g$ for all $b \in B$ and $g \in G$. Moreover, the underlying K-theory class $\bar{t} \in K(P)$ of $\theta \in K(P)$ pulls back to a K-theory class $s^*(\bar{t})$ on $B$.

**Corollary 3.7.** Let $\Xi_{\text{cla}}^{2c} = \sum_{i \geq 0} \xi_i x_i$ in $\Pi_{BU \times \{r\}}^{2c}$ be the class of an even-degree stable pushforward operation $\Xi_1^{i+2c}$. Let $\pi : P \to B$ be a principal $BU(1)$-bundle with orientation $\theta \in K_P(B)$ of rank $r$.

(a) If $P = B \times BU(1)$ is trivial, then $\Xi_{P,\theta}(1) \times c_1^i / i!$ is the characteristic class $\xi_0 \in H^{2i}(BU \times \{r\})$ evaluated at the K-theory class $\bar{t} = (\text{id}_{BU}, 1_{BU(1)})^*\bar{t}$ in $K(B)$. More generally, if $P$ admits a section $s$, we have

$$\Xi_{P,\theta}^*(\gamma_{P,s}^*(\frac{c_1^i}{i!})) = \xi^s(\xi^s(\bar{t})).$$

(3.10)

In particular, even-degree stable pushforward operations are determined by their values on the trivial bundle with arbitrary orientations $\theta$.

(b) We have $\Xi_{P,\theta}(\alpha \cup \pi^*(\beta)) = \Xi_{P,\theta}(\alpha) \cup \beta$ for all $\alpha \in H^*(P)$ and $\beta \in H^*(B)$.

(c) The image $\Xi_{P,\theta}(1)$ of the unit is the characteristic class $\xi^{BU(1)}_0(\bar{t})$. Hence for all $\beta \in H^*(B)$ the pull-push formula $\Xi_{P,\theta} \circ \pi^*(\beta) = \xi^{BU(1)}_0(\bar{t}) \cup \beta$ holds.

(d) Push-pull formula: $\pi^* \circ \Xi_{P,\theta}(\alpha) = \sum_{i \geq 0} (t \cdot \alpha) \cup \xi_i(\bar{t})$ for $\alpha \in H^*(P)$.

(e) Let $A$ be a topological space and let $\pi_B : A \times B \to B$ and $\pi_P : A \times P \to P$ be the projections. For the bundle $A \times P \cong \pi_B^*(P)$ and the pullback orientation $A \times \bar{t} = \pi_P^*(\bar{t})$, we have $\Xi_{A \times P, A \times \theta}^* = \text{id}_{H^*(A)} \times \Xi_{P,\theta}$.

**Proof.** (a) Apply Proposition 2.2(c) to $\alpha = 1_B \times c_1^i$. Let $h_{c_1^i} \in \text{Map}(BU(1), H_{2i})$ represent the class $c_1^i \in H^{2i}(BU(1))$ and define $h_{\alpha} = h_{c_1^i} \circ \pi_{BU(1)}$ using the projection $\pi_{BU(1)} : BU(1) \times B \to BU(1)$. Then $h_{\alpha}^1 : B \to \text{Map}(BU(1), H_{2i})$ takes the
constant value $h_{c_1}$. Using $\ev \circ (\id_{\BU(1)} \times h_{c_1}^t) = h_{c_1} \circ \pi_{\BU(1)}$, we compute:

$$
\Xi_{P,\theta}(1_B \times c_1^t) = \sum_{j \geq 0} \vartheta^*(\xi_j) \cup (h_{c_1}^t)^*(\ev^*(J_{c_1}))/t_j
\quad \text{by (2.8) and (2.17)}
\quad = \sum_{j \geq 0} \vartheta^*(\xi_j) \cup (h_{c_1} \circ \pi_{\BU(1)}^t)^*(J_{c_1})/t_j
\quad = \sum_{j \geq 0} \vartheta^*(\xi_j) \cup ((c_1^t \times 1_B)/t_j) = \delta_i \vartheta^*(\xi_i)
\quad \text{by } \pi_{\BU(1)}(c_1^t) = c_1^t \times 1_B
\quad (c_1^t \times 1_B)/t_j = \delta_{i,j} t_1^t 1_B
$$

This proves (3.10) for $P$ trivial; the general case follows by naturality, using the isomorphism $(\pi, \gamma_{P,\theta}) : P \rightarrow B \times \BU(1)$. The last claim in (a) is obvious, as $\Xi_{cla}^{2e}$ is given by its coefficients $\xi_i$ which are determined entirely by their values on arbitrary K-theory classes, which are equivalently orientations on the trivial bundle.

(b) We apply Proposition 2.14. Recall from [2.3] the maps $\overline{\Xi}_{m_1,m_2}$ characterized by $\ev \circ (\id_{\BU(1)} \times \overline{\Xi}_{m_1,m_2}) = M_{m_1,m_2} \circ (\ev \times \id_{H_{m_2}})$. As in Theorem 2.17(c), the components of $\Xi_{cla}^{2e}$ are $\Xi_{cla}^{m_1+m_2} = \sum_{i \geq 0} \xi_i x_i^{(m)}$ for all $m$. As a preliminary, compute:

$$
\overline{\Xi}_{m_1,m_2}^{x_i^{(m_1+m_2)}}((1_B \times c_1^t)) = \overline{\Xi}_{m_1,m_2}^{ev^*(J_{m_1+m_2})/t^t} \quad \text{by (2.17)}
\quad = (\id_{\BU(1)} \times \overline{\Xi}_{m_1,m_2})^* \ev^*(J_{m_1+m_2})/t^t \quad \text{nat. of } \ev^*
\quad = (\ev \times \id_{H_{m_2}})^* M_{m_1,m_2}^*(J_{m_1+m_2})/t^t
\quad = (\ev^*(J_{m_1} \times J_{m_2})/t^t = x_i^{(m)} \times J_{m_2} \quad \text{by } M_{m_1,m_2}^*(J_{m_1+m_2}) = J_{m_1} \times J_{m_2}
$$

Applying this result to the expressions $\Xi_{cla}^{m_1+m_2|m_1+m_2+2e} = \sum_{i \geq 0} \xi_i x_i^{(m_1+m_2)}$ and $\Xi_{cla}^{m_2|m_1+m_2+2e} = \sum_{i \geq 0} \xi_i x_i^{(m_2)}$, verifies (2.16), keeping (2.14) in mind.

(c) follows from Proposition 2.16. The push-pull formula then follows from (b).

(d) Let $\rho_P : P \times \BU(1) \rightarrow P$ be the principal action. Using that $c_1^t$ is dual to $t^t/\delta!$, one checks $\rho_P^*(\alpha) = \sum_{i \geq 0} (\xi_i^t \circ \alpha) \times c_1^t$. The pullback $\pi^*(P)$ has the section $s(p) = (p, p)$. The two coordinate projections $\pi_1$ and $\pi_2$ fit into a pullback square:

$$
\begin{array}{ccc}
\pi^*(P) & \xrightarrow{\pi_2} & P \\
\downarrow & & \downarrow \pi \\
\quad & \pi_1 & \\
P & \xrightarrow{\pi} & B.
\end{array}
$$

Moreover, the pullback orientation $\pi_2^*(\theta)$ pulls back under the section $s$ to the class $\theta$ on the base $P$. Pulling back the formula for $\rho_P^*(\alpha)$ along $(\pi_1, \gamma_{P,\theta}(s))$ and using $\rho_P \circ (\pi_1, \gamma_{P,\theta}(s)) = \pi_2$ implies $\pi_2^*(\alpha) = \sum_{i \geq 0} \xi_i^t (t^i \circ \alpha)$, hence:

$$
\pi^* \circ \Xi_{P,\theta}(\alpha) = \Xi_{P,\theta}^{\pi_2^*(\alpha)} = \Xi_{P,\theta}^{\pi_2^*(\alpha)}(\pi_2^*(\alpha))
\quad = (b) \sum_{i \geq 0} (t^i \circ \alpha) \cup \Xi_{P,\theta}^{\pi_2^*(\alpha)}(\gamma_{P,\theta}(s)(c_i^t))
\quad = \sum_{i \geq 0} \xi_i (t^i \circ \alpha).
$$

(e) Let $\alpha \in H_\ast(A)$ and $\beta \in H_\ast(P)$. By the Künneth theorem, it suffices to verify the claimed formula at $\alpha \times \beta$. Using (b) and naturality (2.2), we compute:

$$
\Xi_{A \times P,\alpha \times \beta}(\alpha \times \beta) = \Xi_{A \times P,\alpha \times \beta}(\id_A \times \pi)^*(\alpha \times 1_B) \cup \pi^*_{P,\beta}(\beta)
\quad = (\alpha \times 1_B) \cup \Xi_{A \times P,\alpha \times \beta} \circ \pi^*_{P,\beta}(\beta) = (\alpha \times 1_B) \cup \pi^*_{P,\beta} \circ \Xi_{P,\theta}(\beta) = \alpha \times \Xi_{P,\theta}(\beta).
\square
3.2. Proof of Theorem 1.1(a)–(f) and Theorem 1.2

**Definition 3.8.** (a) For each $r \in \mathbb{Z}$ the *projective Euler operation* is the rational stable pushforward operation with class $\Xi^{PE}_{cl} = \sum_{i \geq 0} \frac{\epsilon_{i+1} x_i}{i+1} \in \Pi_{BU}^{2r+1}(r)$. For a principal $BU(1)$-bundle $\pi: P \to B$ with orientation $\theta$ of rank $r$, we write $\pi^\theta = \Xi^{PE}_{cl}$ for the corresponding pushforward operation.

(b) The *projective rank operation* is the rational stable pushforward operation of degree 0 with class $\Xi^{rk}_{cl} = \sum_{i \geq 0} (-1)^i ch_i$. We simply write $s^\theta_0$ for $\Xi^{rk}_{cl}$.

Using Definition 3.3 and Lemma 3.4, one checks that the coefficients of $\Xi^{PE}_{cl}$ and $\Xi^{rk}_{cl}$ satisfy (3.3), so the existence of the rational stable pushforward operations $\pi^\theta$ and $s^\theta_0$ is a consequence of Theorem 3.5. Parts (a)–(f) of Theorems 1.1 and 1.2 follow from Corollary 3.7.

3.3. Module structure on group of pushforward operations. The next step in the classification is to make $\Pi_{BU}^{2r+1}(r)$ a module over a large ring.

**Definition 3.9.** Let $P \to B$ be a principal $BU(1)$-bundle with orientation $\theta$. Recall the endomorphism $t \circ -$ of $H^* (P)$ from Definition 3.6. Let $\tilde{\theta} \in K(P)$ be the underlying K-theory class of $\theta$. For each $j$, multiplication by the Chern character $ch_j (\tilde{\theta}) \in H^{2j} (P)$ defines further endomorphisms of $H^* (P)$.

Let $\Xi^{*+k}$ be a stable pushforward operation of degree $k$. We can then construct new operations $ch_j (\Xi)$ of degree $k + 2j$ and $t (\Xi)$ of degree $k - 2$ by

$$ch_j (\Xi)_{P, \theta} (\alpha) = \Xi_{P, \theta} (\alpha \cup ch_j (\tilde{\theta})), \quad t (\Xi)_{P, \theta} (\alpha) = \Xi_{P, \theta} (t \circ \alpha). \quad (3.11)$$

**Proposition 3.10.** If $\Xi^{*+k}$ is a stable pushforward operation, then $ch_j (\Xi)$ and $t (\Xi)$ are again stable pushforward operations.

**Proof.** We check naturality. As in Definition 3.1(c), let $(\phi, \Phi): P_1 \to P_2$ be a morphism of principal $BU(1)$-bundles such that $\tilde{\theta}_1 = \Phi^* (\tilde{\theta}_2)$ for the orientations.

Then $\tilde{\theta}_1 = \Phi^* (\tilde{\theta}_2)$ for the associated K-theory classes, so $ch_j (\tilde{\theta}_1) = \Phi^* (ch_j (\tilde{\theta}_2))$ and

$$ch_j (\Xi)_{P_1, \theta_1} (\Phi^* (\alpha)) = \Xi_{P_1, \theta_1} (\Phi^* (\alpha) \cup ch_j (\tilde{\theta}_1)) = \Xi_{P_1, \theta_1} (\Phi^* (\alpha) \cup ch_j (\tilde{\theta}_2)) \quad \text{(3.2)}$$

$$\phi^* \circ \Xi_{P_2, \theta_2} (\alpha \cup ch_j (\tilde{\theta}_2)) = ch_j (\Xi)_{P_2, \theta_2} (\phi^* (\alpha)).$$

Similarly, using $\Phi \circ \rho_{P_1} = \rho_{P_2} \circ (\Phi \times id_{BU(1)})$ for the principal actions,

$$t (\Xi)_{P_1, \theta_1} (\Phi^* (\alpha)) = \Xi_{P_1, \theta_1} (t (\Phi \times id_{BU(1)}) \circ \rho_{P_1} (\alpha)) = \Xi_{P_1, \theta_1} (\Phi^* (t \circ \rho_{P_1} (\alpha)))$$

$$\phi^* \circ \Xi_{P_2, \theta_2} (t \circ \rho_{P_2} (\alpha)) = \phi^* \circ t (\Xi)_{P_2, \theta_2} (\alpha).$$

Moreover, $ch_j (\Xi)(0) = 0$ and $t (\Xi)(0) = 0$ show that both are pointed pushforward operations. To verify stability for $ch_j (\Xi)$, let $c_{P_1}, c_{P_2}$ be the suspensions as in (2.3). Using the projection $\pi_P: P \times S^1 \to P$, we compute:

$$ch_j (\Xi)_{P \times S^1, \theta \times S^1} (c_P (\alpha)) = \Xi_{P \times S^1, \theta \times S^1} (c_P (\alpha) \cup \pi_P^* (ch_j (\tilde{\theta}))) \quad \text{(3.3)}$$

$$= \Xi_{P \times S^1, \theta \times S^1} (c_P (\alpha) \cup c_P (\theta)) \circ \Xi_{P, \theta} (\alpha \cup ch_j (\tilde{\theta})) = \Xi_{P \times S^1, \theta \times S^1} (\Xi_{P, \theta} (\alpha \cup ch_j (\tilde{\theta}))) \quad \text{(3.4)}$$

The verification for $t (\Xi)$ is similar, using $t (\rho_{P \times S^1} (c_P (\alpha))) = c_P (t \circ \rho_{P} (\alpha)). \quad \Box$
Proposition 3.11. Let $\Xi^{1+2c}$ be an even-degree rational stable pushforward operation with $\Xi^{2c}_{\text{cla}} = \sum_{i \geq 0} \xi_i x_i$. Setting $\xi_1 = 0$, we have

$$ch_j(\Xi^{2c}_{\text{cla}}) = \sum_{i \geq 0} \left( \frac{i+k}{k} \xi_{i+k} ch_{j-k} \right) x_i, \quad (3.12)$$

$$t(\Xi^{2c}_{\text{cla}}) = \sum_{i \geq 0} \xi_{i-1} x_i, \quad (3.13)$$

Proof. By Corollary 3.7(a) it suffices to verify these assertions for the trivial bundle $P$, where the orientation is given by a class $\vartheta \in K(B)$ and $\vartheta = \vartheta \boxtimes V(1)$. Using $\text{ch}_j(\vartheta \boxtimes V(1)) = \sum_{k \geq 0} \text{ch}_{j-k}(\vartheta) \times \text{ch}_k(V(1)) = \sum_{k \geq 0} \text{ch}_{j-k}(\vartheta) \times c_1^k$, we compute:

$$\text{ch}_j(\Xi)_{P,\vartheta}(1_B \times c_1^i) = \Xi_{P,\vartheta}(1_B \times c_1^i) \cup \text{ch}_j(\vartheta)$$

$$= \Xi_{P,\vartheta} \left( \sum_{k \geq 0} \frac{1}{i!k!} \text{ch}_{j-k}(\vartheta) \times c_1^{i+k} \right)$$

$$= \sum_{k \geq 0} \frac{1}{i!k!} \text{ch}_{j-k}(\vartheta) \cup \Xi_{P,\vartheta}(1_B \times c_1^{i+k}) \quad \text{by Cor. 3.7(b)}$$

$$= \sum_{k \geq 0} \frac{(i+k)!}{i!k!} \text{ch}_{j-k}(\vartheta) \cup \xi_{i+k} \vartheta \quad \text{by Cor. 3.7(a)}$$

This implies (3.12). The proof of (3.13) is similar, using $t \circ c_1 = i c_1^{i-1}$. \hfill \Box

The commutator of (3.12) with (3.13) is

$$t(\text{ch}_j(\Xi^{2c}_{\text{cla}})) - \text{ch}_j(t(\Xi^{2c}_{\text{cla}})) = \text{ch}_{j-1}(\Xi^{2c}_{\text{cla}}). \quad (3.14)$$

Definition 3.12. Let $S$ be the graded abelian group $\mathbb{Q}[\![t]\!] \otimes H^*(BU \times \{r\})$, where $t$ is a power series variable of degree $-2$, with a product that commutes with infinite series and satisfies the relation $t \text{ch}_j - \text{ch}_j t = \text{ch}_{j-1}$, where $\text{ch}_0 = r$.

As the relation is homogeneous, $S$ is a graded ring. By (3.13), the two constructions in Definition 3.9 make $\Pi^*_{BU \times \{r\}}$ into a graded module over $S$.

3.4. Proof of Theorem 1.4. To determine the structure of $\Pi^*_{BU \times \{r\}}$ as a module over $S$, we first introduce an auxiliary ring $R = \mathbb{Q}[z_0, z_1, z_2, \ldots]$ with a bigraded $|z_k| = (1, k)$ and derivation given by

$$\partial(z_k) = z_{k-1} \quad (\forall k \geq 1), \quad \partial(z_0) = 0.$$

Note that $\partial$ is not a differential, as $\partial \circ \partial \neq 0$. The derivation property means that $\partial(ab) = \partial(a)b + a\partial(b)$, $\partial(1) = 0$.

Introduce a unital algebra homomorphism $\epsilon : R \rightarrow \mathbb{Q}$ by $\epsilon(z_k) = 0$ for $k \geq 0$ and the $\mathbb{Q}[z_0]$-linear map (since $\partial$ is element-wise nilpotent, the infinite sum makes sense)

$$\gamma = \sum_{k \geq 0} (-1)^k z_k \partial^k.$$

The motivation for introducing $\gamma$ is that it encodes the action of $H^*(BU)$ on $\Pi^2_{BU \times \{r\}}$, see Lemma 3.15, and is useful to construct preimages under $\partial$.

Clearly $\epsilon \circ \partial = 0$ on generators and $\partial \circ \gamma = 0$ is also easy to check. Write $R^{d,e} \subset R$ for the vector subspace of elements of bidegree $(d,e)$. Then $\partial$ and $\gamma$ have bidegrees $(0, -1)$ and $(1, 0)$. We have a chain complex (with $\mathbb{Q}[0, 0]$ put in bidegree $(0, 0)$)

$$R^{d-1,e} \xrightarrow{\gamma} R^{d,e} \xrightarrow{\partial} R^{d,e-1} \xrightarrow{\epsilon} \mathbb{Q}[0, 0]^{d,e-1} \longrightarrow 0. \quad (3.15)$$

Lemma 3.13. For all $(d,e) \neq (0, 0)$ the sequence (3.15) is exact.
Proof. Using \( R^{0,e} = \{0\} \) for \( e \neq 0 \), that \( R^{1,e} = \mathbb{Q}z_e \) is one-dimensional, and \( R^{d,0} = \mathbb{Q}z_0^d \), the cases \((d,e) \in \{(1,0),(1,1),(1,2),(2,0)\}\) and \((d,e) = (0,e)\) for all \( e \geq 1 \) are easily checked by direct inspection. Moreover, when \( d < 0 \) or \( e < 0 \) the entire sequence vanishes, so we restrict to \( d,e \geq 0 \).

We proceed by induction on the total degree \( n = d+e \). We have already discussed that the base case \( n = 2 \), namely \((d,e) \in \{(2,0),(1,1),(0,2)\}\), holds true. For the inductive step, we must prove that (3.15) is exact for every bidegree \((d,e)\) with \( n = d+e > 2 \). By induction, (3.15) is exact for all \( d+e < n \). Since we have already discussed the case \( d = 0 \), we may also assume \( d > 0 \). Define an algebra epimorphism \( \phi: R \to R \) by \( \phi(z_k) = z_{k-1} \) for \( k \geq 1 \) and \( \phi(z_0) = 0 \). Then \( \phi \) maps \( R^{d,e} \) to \( R^{d,e-d} \).

Observe that there is the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & R^{d-2,e} & \xrightarrow{\gamma} & R^{d-1,e} & \xrightarrow{\phi} & R^{d-1,e-d+1} & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | & | \\
0 & \longrightarrow & R^{d-1,e} & \xrightarrow{\gamma} & R^{d,e} & \xrightarrow{\phi} & R^{d,e-d} & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | & | \\
0 & \longrightarrow & R^{d-1,e-1} & \xrightarrow{\gamma} & R^{d,e-1} & \xrightarrow{\phi} & R^{d,e-d-1} & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | & | \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q}[0,0]^{d,e-1} & \longrightarrow & \mathbb{Q}[0,0]^{d,e-d-1} & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | & | \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

This short exact sequence of chain complexes induces a long exact sequence

\[
\cdots \longrightarrow H_1(\text{Column 1}) \longrightarrow H_1(\text{Column 2}) \longrightarrow H_1(\text{Column 3}) \longrightarrow \cdots.
\]

We will prove the exactness of Column 2 by showing that Columns 1 and 3 are exact. Column 1 is exact by induction, as \((d-1) + e < n\) and \((d-1, e) \neq (0, 0)\). Also, \( \mathbb{Q}[0,0]^{d-1,e-1} = \{0\} \) since \((d,e) \neq (1,1)\). For Column 3 we know that

\[
R^{d-1,e-d} \xrightarrow{\gamma} R^{d,e-d} \xrightarrow{\partial} R^{d,e-d-1} \xrightarrow{\epsilon} \mathbb{Q}[0,0]^{d,e-d-1} \longrightarrow 0 \quad (3.16)
\]

is exact by induction, as \( d+e-d = n-d < n \) and \((d,e-d) \neq (0,0)\). By induction we also have that \( \partial: R^{d-1,e-d} \to R^{d-1,e-d-1} \) is surjective since \((d-1)+e-d+1 = n-d < n\) and \((d-1,e-d+1) \neq (0, 0)\) and since for \((d,e) \neq (1,1)\) the cokernel is \( \mathbb{Q}[0,0]^{d-1,e-d-1} = \{0\} \). Therefore we may replace \( \gamma \) by \(-\gamma \circ \partial\) in (3.16) and retain exactness, leading to the required exactness of Column 3.

The additional variable \( z_0 \) corresponds to the rank \( r \in \mathbb{Z} \). Define an algebra epimorphism \( \psi: R \to H^r(\mathbb{B}U \times \{r\}) \) by \( \psi(z_0) = r \) and \( \psi(z_k) = \text{ch}_k \) for \( k \geq 1 \).

Setting \( R^e = \bigoplus_{k \geq 0} R^{d,e} \), the map \( \psi: R^e \to H^{2e}(\mathbb{B}U \times \{r\}) \) doubles the degree and has the graded ideal \( \langle z_0 - r \rangle \) as its kernel. Let

\[
\gamma_r = r \cdot \text{id}_{H^r(\mathbb{B}U \times \{r\})} + \sum_{k \geq 1} (-1)^k \text{ch}_k \partial^k. \quad (3.17)
\]

Then for \( e > 0 \) we have a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle z_0 - r \rangle \cap R^e & \xrightarrow{\gamma} & R^e & \xrightarrow{\psi} & H^{2e}(\mathbb{B}U \times \{r\}) & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | & | \\
0 & \longrightarrow & \langle z_0 - r \rangle \cap R^e & \xrightarrow{\gamma} & R^e & \xrightarrow{\psi} & H^{2e}(\mathbb{B}U \times \{r\}) & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | & | \\
0 & \longrightarrow & \langle z_0 - r \rangle \cap R^{e-1} & \xrightarrow{\gamma} & R^{e-1} & \xrightarrow{\psi} & H^{2e-2}(\mathbb{B}U \times \{r\}) & \longrightarrow & 0
\end{array}
\]
The middle column is exact by Lemma 3.13 provided \( e > 1 \). Then the left column is also exact for \( e > 1 \). For example, if \( 0 = \partial((z_0 - r) \cdot r) = (z_0 - r) \cdot (\partial r) \) for \( r \in \mathbb{R}^e \), then \( \partial r = 0 \) and by the exactness of the middle column we can write \( \lambda = \gamma(\xi) \) for some \( \xi \in \mathbb{R}^e \). Hence \( \gamma((z_0 - r) \cdot r) = (z_0 - r) \cdot \lambda \). Similarly for the surjectivity of \( \partial \).

Hence the left column is exact and, as above, the long exact sequence in homology implies the exactness of the right vertical sequence. In case \( e = 1 \) we note that \( \partial: H^2(\mathbb{U} \times \{r\}) \to H^0(\mathbb{U} \times \{r\}) \) from Definition 3.13 is surjective for \( r \neq 0 \) and else has image the polynomials in \( H^*(\mathbb{U} \times \{0\}) \) with zero constant term. The case \( e = 0 \) being trivial, we summarize our discussion as follows.

**Lemma 3.14.** (a) For \( r \neq 0 \), there is an exact sequence

\[
H^2e(\mathbb{U} \times \{r\}) \xrightarrow{\gamma} H^2e(\mathbb{U} \times \{r\}) \xrightarrow{\partial} H^{2e-2}(\mathbb{U} \times \{r\}) \rightarrow 0.
\]

(b) For \( r = 0 \), there is an exact sequence

\[
H^2e(\mathbb{U} \times \{0\}) \xrightarrow{\gamma} H^2e(\mathbb{U} \times \{0\}) \xrightarrow{\partial} H^{2e-2}(\mathbb{U} \times \{0\}) \xrightarrow{\epsilon} \mathbb{Q}[0]^{2e-2} \rightarrow 0.
\]

We now compute the groups of stable pushforward operations \( \Pi_{\mathbb{U} \times \{r\}} \) and complete the proof of Theorem 1.4. Recall that the groups \( \Pi_{\mathbb{U} \times \{r\}} \) are part of the exact sequence (3.1) and hence determined by the kernel and cokernel of the morphism \( \delta_1: \Gamma^{2e+2}_{\mathbb{U}} \to \Gamma^{2e}_{\mathbb{U}} \) defined in (3.2), where \( \Gamma^{2e}_{\mathbb{U}} \) is the free module over \( H^*(\mathbb{U}) \) on generators \( x_i \) of degree \(-2i\) for all \( i \geq 0 \).

**Proposition 3.15.** (a) If \( r \neq 0 \), then \( \delta_1 \) is onto. Therefore, \( \Pi_{\mathbb{U} \times \{r\}}^{\text{odd}} = 0 \).

(b) If \( r = 0 \), the cokernel of \( \delta_0 \) is one-dimensional and spanned by the image of \( x_0 \) under the projection \( \Gamma^0_{\mathbb{U}} \to \Pi^1_{\mathbb{U} \times \{0\}} \) in (3.1). Hence \( \Pi_{\mathbb{U} \times \{0\}}^{\text{odd}} = \mathbb{Q} \eta \) for \( \eta \in \Pi^1_{\mathbb{U} \times \{0\}}[0] \).

**Proof.** An element of the codomain \( \Gamma^0_{\mathbb{U}} \) can be written \( \Lambda = \sum_{i > 0} \lambda_i x_i \) for \( \lambda_i \in H^*(\mathbb{U}) \). By (3.2), \( \delta_1(\sum_{i > 0} \xi_i x_i) = \sum_{i > 0} \lambda_i x_i \) is equivalent to

\[
\partial \delta_0 = 0, \quad \partial \xi_i = \lambda_i - \xi_{i-1} \quad (\forall i \geq 1).
\]

(a) If \( r \neq 0 \), then \( \partial \) is surjective by Lemma 3.14(a). Hence we can always solve \( \partial \delta_0 = 0 \) and, given \( \xi_0, \ldots, \xi_{i-1} \), find a preimage \( \xi_i \) satisfying (3.15). This recursion shows that every \( \Lambda \in \Gamma^0_{\mathbb{U}} \) has a preimage under \( \delta_1 \), which is therefore onto.

(b) If \( r = 0 \), then by Lemma 3.14(b) the image of \( \partial \) is the set of all polynomials in \( \mathbb{Q}[c_1, c_2, \ldots] \) with zero constant term. Hence the equation \( \partial \delta_0 = 0 \) has a one-dimensional cokernel spanned by \( \lambda_0 = 1 \). Suppose that for \( i \geq 1 \) we have already constructed \( \xi_0, \ldots, \xi_{i-1} \) satisfying (3.15). Solving \( \partial \xi_i = \lambda_i - \xi_{i-1} \) is only possible if \( \epsilon(\lambda_i - \xi_{i-1}) = 0 \), but we can change \( \xi_i \) to \( \xi_i = \xi_{i-1} + \epsilon(\lambda_i - \xi_i) \) and ensure \( \epsilon(\lambda_i - \xi_{i-1}) = 0 \). Also, \( \partial \xi_{i-1} = \partial \xi_{i-2} \) shows that \( \xi_0, \ldots, \xi_{i-2}, \xi_{i-1} \) still solves (3.15).

Hence we can construct a preimage of \( \Lambda \) under \( \delta_0 \) if and only if \( \epsilon(\lambda_0) = 0 \) and the cokernel \( \Pi_{\mathbb{U} \times \{0\}}^{\text{odd}} \) of \( \delta_0 \) is spanned by the image of \( x_0 \in \Gamma^0_{\mathbb{U}} \) in \( \Pi^1_{\mathbb{U} \times \{0\}} \). \( \square \)

Proposition 3.15 completes the computation of \( \Pi_{\mathbb{U} \times \{r\}}^{\text{odd}} \). To compute the even part, recall from Definition 4.8 the projective Euler class \( \Xi^{\text{PE}}_{\text{ch}} = \sum_{i > 0} \frac{c_i + 1}{i} \) and the projective rank class \( \Xi^{\text{rk}}_{\text{ch}} = \sum_{i > 0} (-1)^i \text{ch}_i x_i \).

**Lemma 3.16.** For \( s \in H^*(\mathbb{U}) \subset C \) let \( s(\Xi^{\text{rk}}_{\text{ch}}) = \sum_{i > 0} \sigma_i x_i \) be the class of the pushforward operation obtained by the action of \( s \), see Definition 4.12. Write \( s(\Xi^{\text{rk}}_{\text{ch}})^{(n)} \) for the \( i \)th term \( \sigma_i \). Then \( s(\Xi^{\text{rk}}_{\text{ch}})^{(n)} = \sum_{k > 0} (-1)^{n+k} (\gamma^k) \text{ch}_k \partial^k(s) \). In particular, \( s(\Xi^{\text{rk}}_{\text{ch}})^{(0)} = \gamma(s) \) for the constant term.
Proof. It suffices to consider monomials $s = ch_{j_1} \cdots ch_{j_n}$. Our proof proceeds by induction on $n$. The base case $n = 1$ follows from (3.12) and $\partial^k ch_j = ch_{j-k}$ for the inductive step, recall that the general Leibniz rule for derivations states that

$$\partial^m(ch_{j_1} \cdots ch_{j_n}) = \sum_{k \geq 0} \binom{m}{k} \partial^k(ch_j)\partial^{m-k}(ch_{j_2} \cdots ch_{j_n}).$$

(3.19)

Of course, the binomial coefficients $\binom{m}{k}$ are zero unless $0 \leq k \leq m$. Using the associativity of the action of $S$ on $\Pi^*_B U \times_r$, we compute:

$$s(\Xi_{\text{cl}}^r)^{(i)} = (ch_{j_1}(ch_{j_2} \cdots ch_{j_n}(\Xi_{\text{cl}}^r)))^{(i)}$$

$$= \sum_{k \geq 0} \binom{i+k}{k} ch_{j_1-k}(ch_{j_2} \cdots ch_{j_n})(\Xi_{\text{cl}}^r)^{(i+k)} \quad \text{by (3.12)}$$

$$= \sum_{k \geq 0} \binom{i+k}{k} ch_{j_1-k}$$

$$\bigcup_{\ell \geq 0} (-1)^{i+k+\ell} \binom{i+k+\ell}{\ell} ch_{i+k+\ell} ch_{j_2} \cdots ch_{j_n} \quad \text{induction}$$

$$= \sum_{m,k \geq 0} (-1)^i m \binom{i+m}{m-k} ch_{j_1-k} ch_{i+m} \partial^{m-k}(ch_{j_2} \cdots ch_{j_n}) \quad \text{reindex}$$

$$= \sum_{m,k \geq 0} (-1)^i m \binom{i+m}{i} ch_{i+m} ch_{j_1-k} \partial^{m-k}(ch_{j_2} \cdots ch_{j_n}) \quad \text{by (3.13)}$$

$$= \sum_{m \geq 0} (-1)^i m \binom{i+m}{i} ch_{i+m} \partial^m(ch_{j_2} \cdots ch_{j_n}) \quad \text{by (3.19)}$$

This completes the proof of the inductive step $n - 1 \mapsto n$. \hfill \qed

Lemma 3.17. Let $r \neq 0$. For all $\Xi^r_{\text{cl}} \in \Pi^r_{B U \times r}$ there exist $s \in H^{2c}(BU) \subset S$ and $\Lambda^{2c+2}_{\text{cl}} \in \Pi^{2c+2}_{B U \times r}$ such that $\Xi^r_{\text{cl}} = s(\Xi^r_{\text{cl}}) + t(\Lambda^{2c+2}_{\text{cl}})$.

Proof. Let $\Xi^r_{\text{cl}} = \sum_{i \geq 0} x_i$. Then $\partial \xi_0 = 0$ by (3.3), exactness of Lemma 3.14(a) implies that there exists $s \in H^{2c}(BU)$ with $\gamma_r(s) = \xi_0$. Now Lemma 3.16 implies that $\Xi^r_{\text{cl}} - s(\Xi^r_{\text{cl}})$ has vanishing constant term and can therefore be written as $t(\Lambda^{2c+2}_{\text{cl}})$ for some $\Lambda^{2c+2}_{\text{cl}} \in \Pi^{2c+2}_{B U \times r}$, see (3.13). \hfill \qed

Lemma 3.18. Let $r \in \mathbb{Z}$. For all $\Xi^r_{\text{cl}} \in \Pi^r_{B U \times r}$ with $e \neq 0$ there exist $j \geq 0$, $\Theta^{2c-2j}_{\text{cl}} \in \Pi^{2c-2j}_{B U \times r}$, and $\Lambda^{2c+2}_{\text{cl}} \in \Pi^{2c+2}_{B U \times r}$ such that $\Xi^r_{\text{cl}} = ch_j(\Theta^{2c-2j}_{\text{cl}}) + t(\Lambda^{2c+2}_{\text{cl}})$.

Proof. Let $\Xi^r_{\text{cl}} = \sum_{i \geq 0} \xi_i$. If $\xi_0 = 0$, then $\Xi^r_{\text{cl}} = t(\Lambda^{2c+2}_{\text{cl}})$ for some $\Lambda^{2c+2}_{\text{cl}}$ and we are done. If $\xi_0 \neq 0$, we must have $e \geq 0$, since $\xi_0 \in H^{2c}(BU)$. As $\partial \xi_0 = 0$, we may apply Lemma 3.14 to $\xi_0$ and find $\omega \in H^{2c}(BU)$ such that $\xi_0 = \gamma_r(\omega)$. For $j \geq 0$ such that $\partial^j \omega = 0$ we can write this as

$$\xi_0 = ch_0 \omega + ch_1 \partial \omega + ch_2 \partial^2 \omega + \ldots + (-1)^j ch_j \partial^j \omega.$$

Define $\omega_j = (-1)^j \omega_j$, $\omega_{j-1} = \partial \omega_j$, $\ldots$, $\omega_0 = \partial^j \omega_j$. As $\omega_j$ has degree $2c > 0$, we have $\omega_j = 0$. Hence Lemma 3.14 allows us to pick $\omega_{j+1} \in H^{2c}(BU)$ with $\partial \omega_{j+1} = \omega_j$. Continuing in this way, we obtain $\omega_{j+2}, \omega_{j+3}, \ldots$ such that $\Theta^{2c-2j}_{\text{cl}} = \sum_{i \geq 0} (-1)^i \omega_i x_i \in \Pi^{2c-2j}_{B U \times r}$ is in the kernel of $\delta_r$. Moreover, by (3.12) the constant term of $ch_j(\Theta^{2c-2j}_{\text{cl}})$ is $\omega_j ch_{-1} \omega_{j-1} + \ldots + (-1)^j \omega_j ch_0 = \xi_0$. Hence $\Xi^r_{\text{cl}} - ch_j(\Theta^{2c-2j}_{\text{cl}})$ has vanishing constant term and can thus be written as $t(\Lambda^{2c+2}_{\text{cl}})$. \hfill \qed
Proof. If \( e < 0 \), then \( \xi_0 = 0 \) and we can write \( \Xi_{\text{ cla}}^e = t(\Lambda_{\text{ cla}}^{2e+2}) \) and put \( s = 0 \). If \( e = 0 \), then \( \xi_0 \) is a constant, but the existence of \( \xi_1 \) with \(-\partial \xi_1 = \xi_0 \) from (3.3) and the exactness of Lemma 3.19 implies that the constant must be zero. Hence we again have \( \Xi_{\text{ cla}}^e = t(\Lambda_{\text{ cla}}^{2e+2}) \) and put \( s = 0 \). For \( e \geq 1 \) we proceed by induction on \( e \).

In the base case \( e = 1 \) we have \( H^2(BU) = \mathbb{Q}_{c_1} \), so we can write \( \xi_0 = sc_1 \) with \( s \in \mathbb{Q} \). Then \( \Xi_{\text{ cla}}^e = s \Xi_{\text{ cla}}^{PE} \) has vanishing constant term, hence \( \Xi_{\text{ cla}}^e = s \Xi_{\text{ cla}}^{PE} + t(\Lambda_{\text{ cla}}^{2e+2}) \) for some \( \Lambda_{\text{ cla}}^{2e+2} \). For the inductive step \( e - 1 \mapsto e \), use Lemma 3.19 to write

\[
\Xi_{\text{ cla}}^{2e} = c_1(\Theta_{\text{ cla}}^{2e-2j}) + t(\Lambda_{\text{ cla}}^{2e+2})
\]

for some \( j \geq 0 \), \( \Theta_{\text{ cla}}^{2e-2j} \in \Pi_{BU}^{2e-2j}(0) \), and \( \Lambda_{\text{ cla}}^{2e+2} \in \Pi_{BU}^{2e+2}(0) \). We will show by reverse induction on \( k = j, j-1, \ldots, 0 \) that there exist \( s_k \in S \), \( \Theta_k \in \Pi_{BU}^{2k-2k}(0) \), and \( \Lambda_k \in \Pi_{BU}^{2k+2}(0) \) such that

\[
\Xi_{\text{ cla}}^{2e} = s_k(\Xi_{\text{ cla}}^{PE}) + ch_k(\Theta_k) + t(\Lambda_k).
\]

For \( k = 0 \) we will then have \( ch_k = r = 0 \), so (3.21) completes the inductive step.

As to the induction over \( k \), in the base case \( k = j \) we set \( s_j = 0 \), \( \Theta_j = \Theta_{\text{ cla}}^{2e-2j} \), and \( \Lambda_j = \Lambda_{\text{ cla}}^{2e+2} \) using the classes from (3.20). For the inductive step \( k \mapsto k - 1 \) with \( k - 1 \geq 0 \), we begin with \( s_k, \Theta_k, \Lambda_k \) satisfying (3.21). Since \( \Theta \) has degree \( 2e - 2k < 2e \), we apply the inductive hypothesis (for the outer induction on \( e \)) to \( \Theta_k \) and find \( \tilde{s} \in S \) and \( \tilde{\Lambda} \in \Pi_{BU}^{2e-2k+2}(0) \) such that \( \Theta_k = s(\Xi_{\text{ cla}}^{PE}) + t(\tilde{\Lambda}) \). Hence

\[
\Xi_{\text{ cla}}^{2e} = s_k(\Xi_{\text{ cla}}^{PE}) + ch_k(\tilde{s}(\Xi_{\text{ cla}}^{PE}) + t(\tilde{\Lambda})) + t(\Lambda_k)
\]

by (3.21)

\[
= (sk + ch_k \tilde{s})(\Xi_{\text{ cla}}^{PE}) + t(ch_k(\tilde{\Lambda}) + \Lambda_k) + ch_{k-1}(\Lambda_k)
\]

by (3.4).

Setting \( s_{k-1} = sk + ch_k \tilde{s}, \Theta_{k-1} = \tilde{\Lambda}, \) and \( \Lambda_{k-1} = ch_k(\tilde{\Lambda}) + \Lambda_k \) completes the inductive step \( k \mapsto k - 1 \) and also concludes the inductive step \( e - 1 \mapsto e \). \( \square \)

To complete the proof of Theorem 1.1, it remains to prove the following.

**Proposition 3.20.** (a) If \( r \neq 0 \), \( \Pi_{BU}^{2e}(r) \) is generated as an \( S \)-module by \( \Xi_{\text{ cla}}^{r} \).

(b) If \( r = 0 \), then \( \Pi_{BU}^{2e}(0) \) is generated as an \( S \)-module by \( \Xi_{\text{ cla}}^{r} \).

**Proof.** The proofs of (a) and (b) are the same, except that one uses the projective rank class and Lemma 3.17 for (a) and the projective Euler class and Lemma 3.19 for (b). We only spell out the proof of (b). Let \( \Xi_{\text{ cla}}^{0} \in \Pi_{BU}^{0}(0) \) and write \( \Xi_{\text{ cla}}^{0} = \Xi_{\text{ cla}}^{2e} \) by applying Lemma 3.19 recursively, for all \( n \geq 0 \) there exist \( s_n \in S \) and \( \Xi_{\text{ cla}}^{n} \in \Pi_{BU}^{2n}(0) \) such that \( \Xi_{\text{ cla}}^{n} = s_n(\Xi_{\text{ cla}}^{PE}) + t(\Xi_{\text{ cla}}^{n+1}) \). Define \( s = \sum_{n \geq 0} t^n s_n \in S \).

Then \( \Xi_{\text{ cla}}^{2e} = s(\Xi_{\text{ cla}}^{PE}) \), since for every \( n \) the first \( n \) terms of \( \Xi_{\text{ cla}}^{2e} \) and \( s(\Xi_{\text{ cla}}^{PE}) \) agree:

\[
\Xi_{\text{ cla}}^{2e} = s_0(\Xi_{\text{ cla}}^{PE}) + t(\Xi_{\text{ cla}}^{1}) = (s_0 + ts_1)(\Xi_{\text{ cla}}^{PE}) + t^2(\Xi_{\text{ cla}}^{2})
= \cdots = (s_0 + ts_1 + \ldots + t^n s_n)(\Xi_{\text{ cla}}^{PE}) + t^{n+1}(\Xi_{\text{ cla}}^{n+1})
\]

(3.5. Proof of Theorem 1.1(g)–(i). We have already checked Theorem 1.1(a)–(f) in 3.2. It remains to prove Parts (g)–(i).

(g) By Corollary 3.7(a), it suffices to treat the case of a trivial principal \( BU(1) \)-bundle with a global section \( s: B \to P \), where the orientation amounts to an ordinary K-theory class \( s^*(\theta) \in K(B) \) which, for simplicity, we write in this section without the tilde as \( s^*(\theta) \). Recall the map \( \gamma_P: s\theta: P \to BU(1) \) defined by \( \gamma_P(s\theta)(p) = s(p) \) for all \( p \in B \) and \( g \in BU(1) \). Recall from Corollary 3.7 and Theorem 1.1(a) that the projective Euler operation is normalized by

\[
\pi_1^*(\gamma_P s\theta) = c_{1+r+1}(s^*(\theta)).
\]
For the dual bundle \( \hat{\mathcal{P}} \to B \) the action is inverted and using the same section \( \hat{s} := s \), we have \( \gamma_{\hat{s} \circ s} = u \circ \gamma_{s} \) for the inversion \( u : BU(1) \to BU(1) \). For the dual orientation \( \hat{\theta} \) we have \( c_{i+r+1}(\hat{s}^{*}(\hat{\theta})) = (-1)^{i+r+1}c_{i+r+1}(s^{*}\theta) \). Also, \( \gamma_{\hat{s} \circ s}(c_{i}^{1}) = \gamma_{s}^{*}(u^{*}c_{i}^{1}) = (-1)^{i}\gamma_{s}^{*}(c_{i}^{1}) \) and therefore

\[
\hat{s}^{\theta}_{1}(\gamma_{s}^{*}(c_{i}^{1})) = (-1)^{i}\hat{s}^{\theta}_{1}(\gamma_{s}^{*}(c_{i}^{1})) = (-1)^{i}c_{i+r+1}(\hat{s}^{*}(\hat{\theta})) = (-1)^{r+1}\pi_{1}(\gamma_{s}^{*}(c_{i}^{1})).
\]

(h) Just as in Theorem 5.5, a rational stable pushforward operation of even degree for principal \( (BU(1) \times BU(1)) \)-bundles is also uniquely determined by its values on trivial bundles (the key point is that \( BU(1) \times BU(1) \) has no odd cohomology). Let \( \pi_{1} : P_{1} \to B \) and \( \pi_{2} : P_{2} \to B \) be principal \( BU(1) \)-bundles with sections \( s_{1} : B \to P_{1} \) and \( s_{2} : B \to P_{2} \). These determine further sections

\[
\begin{align*}
B & \xrightarrow{s_{3}} P_{1} \otimes_{BU(1)} P_{2} = P_{3}, & b \mapsto s_{1}(b) \otimes_{BU(1)} s_{2}(b), \\
P_{3} & \xrightarrow{\sigma_{3}} P_{1} \times_{B} P_{2}, & s_{1}(b)g \otimes_{BU(1)} s_{2}(b) \mapsto (s_{1}(b)g, s_{2}(b)), \\
P_{1} & \xrightarrow{\sigma_{1}} \pi_{1}^{*}(P_{2}), & p_{1} \mapsto (p_{1}, s_{2}(\pi_{1}(p_{1}))), \\
P_{2} & \xrightarrow{\sigma_{2}} \pi_{2}^{*}(P_{1}), & p_{2} \mapsto (s_{1}(\pi_{2}(p_{2})), p_{2}),
\end{align*}
\]

which fit into a commutative diagram

\[
\begin{array}{ccc}
P_{1} & \xrightarrow{\sigma_{1}} & P_{2} \\
\downarrow{\pi_{1}} & & \downarrow{\pi_{2}} \\
B & \xrightarrow{s_{3}} & P_{1} \otimes_{BU(1)} P_{2} = P_{3} & \xrightarrow{\sigma_{3}} & B
\end{array}
\]

Define also \( \gamma_{(s_{1}, s_{2})} : P_{1} \times_{B} P_{2} \to BU(1) \times BU(1), (s_{1}g_{1}, s_{2}g_{2}) \mapsto (g_{1}, g_{2}) \). Then \( H^{*}(P_{1} \times_{B} P_{2}) \) is generated as an \( H^{*}(B) \)-module by \( \gamma_{(s_{1}, s_{2})}(c_{1}^{i} \times c_{1}^{j}) \) and it suffices to check (4.7) on these classes. We calculate the first term in (4.7). Using \( \gamma_{(s_{1}, s_{2})} = (\gamma_{s_{2}}(P_{1}), \gamma_{s_{1}}(P_{2}), \sigma_{1}, \sigma_{2}) \), we have \( \gamma_{(s_{1}, s_{2})}(c_{1}^{i} \times c_{1}^{j}) = \gamma_{s_{2}}(P_{1})(\gamma_{s_{1}}(P_{2}))(c_{1}^{i} \times c_{1}^{j}) \cup \kappa_{3}^{*}(\gamma_{s_{2}}(P_{1})), \) so by using linearity over \( \kappa_{3}^{*} \), see Theorem 1.1, and the normalization of the projective Euler operation, we get

\[
(\kappa_{3}^{*}\gamma_{s_{1}, s_{2}}(c_{1}^{i} \times c_{1}^{j})) = c_{i+r+1+r+1}(\sigma_{2}^{2}\kappa_{1}^{*}+\sigma_{2}^{2}\kappa_{3}^{*}\theta_{3})(\gamma_{s_{2}}(P_{2})(\gamma_{s_{1}}(P_{2})))
\]

(3.22)

(Here and in the following calculations, we omit the cup product symbol.) Since

\[
\begin{align*}
c_{i+r+1+r+1}(\sigma_{2}^{2}\kappa_{1}^{*}+\sigma_{2}^{2}\kappa_{3}^{*}\theta_{3}) &= \sum_{a+q} c_{a}(\sigma_{2}^{2}\kappa_{1}^{*}+\sigma_{2}^{2}\kappa_{3}^{*}\theta_{3})c_{q}(\sigma_{2}^{2}\kappa_{3}^{*}\theta_{3}) & \text{by Whitney sum formula} \\
&= \sum_{a+q} \pi_{2}^{a}c_{a}(s_{1}^{*}\theta_{1})c_{q}(s_{3}^{*}\pi_{2}^{*}\pi_{2}^{*}\gamma_{s_{2}}(P_{2}))^{*}\rho_{P_{1}}(\theta_{3}) & \text{by } \kappa_{1}\sigma_{2} = s_{1}^{*}\pi_{2} \\
&= \sum_{a+q} \pi_{2}^{a}c_{a}(s_{1}^{*}\theta_{1}) \sum_{b+c=q} \left( t^{3} - b \right) \pi_{2}^{c}c_{b}(s_{3}^{*}\theta_{3})\gamma_{P_{2}, s_{2}}^{*}(c_{1}^{i}) & \text{by } 23,
\end{align*}
\]
we can rewrite (5.22) as \[ \sum_{a+b+c=m} \binom{r_3-c}{b} \pi_2^a \pi_2^b \pi_2^c (s_1^r \theta_3) \gamma_{p_{3,3,3}} (c_1^i \times c_1^j) \] Now apply \((\pi_2)^{\theta_2}\) and use linearity over \(\pi_2\) to get

\[
(\pi_2)^{\theta_2} \left( \kappa_3^i \gamma_{(s_1, s_3)} (c_1^i \times c_1^j) \right)
\]

(3.23)

\[
= \sum_{a+b+c=m} \binom{r_3-c}{b} \pi_2^a \pi_2^b \pi_2^c (s_1^r \theta_3) \gamma_{p_{3,3,3}} (c_1^i \times c_1^j)
\]

where in the last step we subtract \(j + r_2 + 1\) from the index \(b\). Symmetrically, the substitution \((1, i, j, a, b) \rightarrow (2, j, i, b, a)\) yields

\[
(\pi_1)^{\theta_1} \left( \kappa_1^i \gamma_{(s_1, s_3)} (c_1^i \times c_1^j) \right)
\]

(3.24)

\[
= \sum_{a+b+c=m} \binom{r_3-c}{b} \pi_1^a \pi_1^b \pi_1^c (s_1^r \theta_3) \gamma_{p_{3,3,3}} (c_1^i \times c_1^j)
\]

To compute the middle route through \((3, 2)\), let \(\hat{\mu}_{BU(1)} : BU(1) \times BU(1) \rightarrow BU(1)\), \(\hat{\mu}_{BU(1)}(g, h) = gh^{-1}\). Recalling the convention for the action \(\rho_{p_{1,3}}\) from Page 2 we have \(\gamma_{p_{3,3}} = (\hat{\mu}_{BU(1)} \circ (\gamma_{p_{1,3}}, \gamma_{p_{1,3}}) \circ (\gamma_{p_{1,3}}, \gamma_{p_{1,3}}))\). Combined with \(\hat{\mu}_{BU(1)}(c_1^i) = (c_1 \times 1 - 1 \times c_1)^i\) and with the binomial theorem, we then find

\[
\gamma_{p_{3,3}} (c_1^i \times c_1^j) = \sum_{m=0}^{\infty} \binom{i}{m} \kappa_3^m \gamma_{p_{3,3}} (c_1^m) \gamma_{p_{3,3}} (c_1^{i-m}) \gamma_{p_{3,3}} (c_1^j).
\]

Set \(n = i + j - m\). Using linearity over \(\kappa_3^m\), the image under \((\kappa_3)\gamma_{p_{3,3}} (c_1^i \times c_1^j)\) is

\[
\sum_{n+m=i+j} (-1)^{n-m} \binom{i}{m} \gamma_{p_{3,3}} (c_1^m) \gamma_{p_{3,3}} (c_1^{i-m}) \gamma_{p_{3,3}} (c_1^j).
\]

Inserting this into the calculation

\[
c_{n+r_1+r_2+1} (\sigma_3^m \kappa_1^i \gamma_{p_{3,3}} (c_1^i \times c_1^j))
\]

\[
= \sum_{n+m+i+j} \binom{i}{m} (-1)^{n-m+k} \binom{r_1-a}{k} \pi_3^k \pi_3^c (s_1^r \theta_3) \gamma_{p_{3,3}} (c_1^i \times c_1^j)
\]

where at the last step we used \(\gamma_{p_{3,3}} = u \circ \gamma_{p_{3,3}}\), and \(u^*(c_1^i) = (-1)^k c_1^i\), we obtain

\[
\sum_{n+m+i+j} \binom{i}{m} (-1)^{n-m+k} \binom{r_1-a}{k} \pi_3^k \pi_3^c (s_1^r \theta_3) \gamma_{p_{3,3}} (c_1^{i+m+k}) \pi_3^c (s_1^r \theta_3) \gamma_{p_{3,3}} (c_1^i \times c_1^j).
\]
Now reindex $m$ by $q = m + k$, use the formula \[ \sum_k \binom{n-k}{k} \binom{m+n}{q} \] for binomial coefficients to eliminate the sum over $k$, and use $c_a(s^*_\theta \beta) = (-1)^n c_a(s^*_\theta \gamma)$ to get
\[
\sum_{i+j+r_1+r_2+1} (-1)^{i-q-a} \binom{r_1 - a + i}{q} \pi^*_3 c_a(s^*_\theta \beta_1) \pi^*_3 c_a(s^*_\theta \beta_2) \gamma^*_3 c(s^*_\theta \gamma^2).
\]

Applying the map $(\pi_3)^\theta$, we then find
\[
(\pi_3)^\theta \left( (\kappa_3)^{\gamma^2} \theta, \gamma^2 \right) (\gamma_3 \times c^i) = \sum_{a+b+q=0} (-1)^{i-q-a} \binom{r_1 - a + i}{q} c_a(s^*_\theta \beta_1) c_b(s^*_\theta \beta_2) c_{q+r_1+r_2+1} (s^*_\theta \beta_3)
\]
\[
\sum_{a+b+c} (-1)^{b-j-r_1-r_2-1} \binom{r_1 - a + i}{q} c_a(s^*_\theta \beta_1) c_b(s^*_\theta \beta_2) c_c(s^*_\theta \beta_3),
\]
where in the last step we have reindexed by $c = q + r_3 + 1$ and used the equation for the summation to rewrite the sign. The composition property \[ (1.7) \] now follows by comparing \[ (3.23), \] the negative of \[ (3.24), \] and \[ (3.25) \], while noticing that
\[
\left( b - j - r_2 - 1 \right) - \left( a - i - r_1 - 1 \right) = (-1)^{b-j-r_2-1} \binom{r_1 - a + i}{q} c_a(s^*_\theta \beta_1) c_b(s^*_\theta \beta_2) c_c(s^*_\theta \beta_3),
\]
for each term $(a, b, c)$ with $a + b + c = i + j + r_1 + r_2 + r_3 + 2$ in the sum. Indeed, putting $n = r_3 - c$ and $k = b - j - r_2 - 1$ we can rewrite this as the identity \[ \binom{n}{n-1} = (-1)^{k-n-1} \], which is proven in the appendix as \[ (A.6) \]. This completes the proof of the composition property Theorem \[ (1.1) \].

(i) If $r \neq 0$, $\eta \neq 0$ and the claim is obvious. If $r = 0$, this follows from Theorem \[ (1.2) \] as $\alpha \mapsto \pi^\theta_3 (\alpha) \cup \eta \pi$ is a stable pushforward operation in $\Pi_3^{BU \times \{0\}} = \{0\}$.

\[ \square \]

3.6. Projective Euler operation in homology.

**Theorem 3.21.** For every principal $BU(1)$-bundle $\pi : P \to B$ and orientation $\theta \in K_{P}(B)$ in twisted complex $K$-theory of rank $r$, there is a projective Euler operation $\pi^\theta_3 : H_*(B; \mathbb{Q}) \to H_{*-2r-2}(P; \mathbb{Q})$, uniquely characterized by $(\alpha, \pi^\theta_3(v)) = (\pi^\theta_3(\alpha), v)$ for all $\alpha \in H^{*+2r-2}(P; \mathbb{Q})$ and $v \in H_*(B; \mathbb{Q})$.

In particular, the duals of all of the properties stated in Theorem \[ (1.1) \] hold for $\pi^\theta_3$. Thus, for every pullback diagram as in Theorem \[ (1.1) \] (a) and $\theta_1 = \Phi^*(\theta_2)$, the square
\[
\begin{array}{ccc}
H_{*-2r-2}(P; \mathbb{Q}) & \overset{\Phi^*}{\longrightarrow} & H_{*-2r-2}(P; \mathbb{Q}) \\
(\pi_3)^\theta_1 & \downarrow & (\pi_3)^\theta_2 \\
H_*(B_1; \mathbb{Q}) & \overset{\phi^*}{\longrightarrow} & H_*(B_2; \mathbb{Q})
\end{array}
\]

commutes. The homological version of Theorem \[ (1.1) \] (g) states that
\[ \pi^\theta_3 = (-1)^{r+1} \pi^\theta. \]

Moreover, in the situation of Theorem \[ (1.1) \] (h), we have
\[ (\pi_3)^\theta \circ (\pi_3)^\theta = (\pi_3)^\theta \circ (\pi_3)^\theta = (-1)^{r+1} (\pi_3)^\theta \circ (\pi_3)^\theta. \]

**Proof.** We apply Lemma \[ (A.3) \] to $V = H_*(B; \mathbb{Q})$, $W = H_{*-2r-2}(P; \mathbb{Q})$, and $\phi = \pi^\theta_3$ for a fixed degree $a \in \mathbb{Z}$, using the universal coefficient theorem \[ (1.3) \] 3.6.6 to identify $H^*(B; \mathbb{Q}) \cong V^*$ and $H^{*+2r-2}(P; \mathbb{Q}) \cong W^*$. To verify \[ (A.3) \], let $v \in H_*(B; \mathbb{Q})$. As singular chains have compact support, we can pick a finite CW-complex $B$, map
Assumption 4.1. Let $G$ dualizing the corresponding statements for the cohomological version. The claimed properties of the homological projective Euler operation follow by naturality we find that

$$\langle \phi(\beta), v \rangle = \langle \pi^0(\beta), i_*(v) \rangle = \langle i^*(\beta), v \rangle = \langle \pi^0 I^*(\beta), \bar{v} \rangle = 0.$$ 

The claimed properties of the homological projective Euler operation follow by dualizing the corresponding statements for the cohomological version. \qed

4. HOMOLOGICAL LIE BRACKETS ON MODULI SPACES

We first list the additional data needed for the construction of a Lie bracket.

**Assumption 4.1.** Let $BU$ and $BU(1)$ be as in Assumption 3.1 and abbreviate $G = BU(1)$. Let $M$ be a topological space and assume the following.

(a) $\Phi: M \times M \to M$ is an operation that is associative and commutative up to homotopy. The set $\pi_0(M)$ of path-components of $M$ is then a commutative monoid. Write $M_\alpha \subset M$ for the path-component corresponding to $\alpha \in \pi_0(M)$ (hence $\alpha = M_\alpha$) and let $\Phi_{\alpha, \beta} = \Phi|_{M_\alpha \times M_\beta}$.

(b) $\Psi: M \times G \to M$ is a group action such that $M \to M/G$ is a principal $G$-bundle. The action commutes with $\Phi$, meaning $\Psi(\Phi(m_1, m_2), g) = \Phi(\Psi(m_1, g), \Psi(m_2, g))$ for all $m_1, m_2 \in M$ and $g \in G$.

(c) Let $e_1, \ldots, e_q \in \pi_0(M)$ and let $\pi^{e_1, \ldots, e_q}: M_{e_1} \times \cdots \times M_{e_q} \to M_{e_1 \times \cdots \times M_{e_q}} (m_1, \ldots, m_q) \to (m_1, \ldots, m_q)$. Then (a) implies

$$\Phi_{\alpha + \beta, \gamma} \circ (\Phi_{\alpha, \beta} \times \text{id}_{M_\gamma}) \simeq \Phi_{\alpha, \beta + \gamma} \circ (\text{id}_{M_\alpha} \times \Phi_{\beta, \gamma}), \quad \Phi_{\alpha, \beta} \simeq \Phi_{\beta, \alpha} \circ \tau_{\alpha, \beta}.$$ 

(d) Let $\Delta^n G$ be the diagonal in $G^n$. Let $P_{\alpha, \beta, \gamma} = (M_\alpha \times M_\beta \times M_\gamma)/\Delta^3 G$ and $R_{\alpha, \beta} = (M_\alpha \times M_\beta)/\Delta^2 G$ be the quotient spaces by the diagonal actions. Let $Q_{\alpha, \beta, \gamma} = (M_\alpha \times M_\beta)/\Delta^2 G \times M_\gamma/G$. Equip these with the $G$-actions

$$\rho_{P_{\alpha, \beta, \gamma}}((m_\alpha, m_\beta, m_\gamma)\Delta^3 G, g) = (m_\alpha g, m_\beta g, m_\gamma)\Delta^3 G,$$

$$\rho_{Q_{\alpha, \beta, \gamma}}((m_\alpha, m_\beta)\Delta^2 G, m_\gamma, g) = ((m_\alpha g, m_\beta)\Delta^2 G, m_\gamma g),$$

$$\rho_{R_{\alpha, \beta}}((m_\alpha, m_\beta)\Delta^2 G, g) = (m_\alpha g, m_\beta)\Delta^2 G.$$ 

The quotient projections $\kappa_{\alpha, \beta, \gamma}: P_{\alpha, \beta, \gamma} \to Q_{\alpha, \beta, \gamma}$, $Q_{\alpha, \beta, \gamma} \to R_{\alpha, \beta}$, $R_{\alpha, \beta} \to M_\alpha/G \times M_\beta/G \times M_\gamma/G$ and $\pi_{\alpha, \beta}: R_{\alpha, \beta} \to M_\alpha/G \times M_\beta/G \times M_\gamma/G$ are then principal $G$-bundles. The maps $\tau^{e_1, \ldots, e_q}$ from (c) induce maps on the various quotient spaces. For example, using this notation, $Q_{\alpha, \beta, \gamma}$ is isomorphic to the pull-back of $R_{\alpha, \beta}$ along $\tau_{\alpha, \beta, \gamma}: M_\alpha/G \times M_\beta/G \times M_\gamma/G \to M_\alpha/G \times M_\beta/G \times M_\gamma/G$. We abbreviate $B_{e_1, \ldots, e_q} := M_{e_1}/G \times \cdots \times M_{e_q}/G$.

(e) For all $\alpha, \beta \in \pi_0(M)$ there are orientations $\theta_{\alpha, \beta} \in K_{R_{\alpha, \beta}}(R_{\alpha, \beta})$ such that

$$\langle (\tau^{e_1, \ldots, e_q})_{\alpha, \beta}(\theta_{\beta, \alpha}), \theta_{\alpha, \beta} \rangle. \quad (4.1)$$ 

The map $\Phi_{\alpha, \beta} \times \text{id}_{M_\gamma}$ induces a map $(\Phi_{\alpha, \beta} \times \text{id}_{M_\gamma})/G: P_{\alpha, \beta, \gamma} \to R_{\alpha, \beta, \gamma}$. For the orientations we require that

$$\langle ((\Phi_{\alpha, \beta} \times \text{id}_{M_\gamma})/G)_{\alpha, \beta, \gamma}(\theta_{\alpha, \beta}), \theta_{\alpha, \beta} \rangle = \langle \tau^{e_1, \ldots, e_q}_{\alpha, \beta, \gamma}(\theta_{\alpha, \beta}), \theta_{\alpha, \beta} \rangle + \langle (\tau^{e_1, \ldots, e_q})_{\alpha, \beta, \gamma}(\theta_{\alpha, \beta}), \theta_{\alpha, \beta} \rangle. \quad (4.2)$$ 

In particular, the Euler form $\chi(\alpha, \beta) = \text{rk} \Phi_{\alpha, \beta}$ is symmetric and bi-additive.

(f) There are signs $\epsilon_{\alpha, \beta} \in \{\pm 1\}$ for all $\alpha, \beta \in \pi_0(M)$ satisfying

$$\epsilon_{\alpha, \beta} \cdot \epsilon_{\beta, \alpha} = (-1)^{\chi(\alpha, \beta)} \chi(\alpha, \alpha) \chi(\beta, \beta), \quad \epsilon_{\alpha, \beta} \cdot \epsilon_{\alpha + \beta, \gamma} = \epsilon_{\beta, \gamma} \cdot \epsilon_{\alpha, \beta + \gamma}. \quad (4.3)$$
Remark 4.2. Usually, the signs $\epsilon_{\alpha,\beta}$ are determined geometrically and correspond to orientations; see [5] §8.3. These orientation problems are solved in the series [6–9] using the excision technique of [13].

4.1. Proof of Theorem 1.5. Clearly, the Lie bracket defined in (1.11) by

$$[\zeta, \eta] = \epsilon_{\alpha,\beta}(-1)^{\alpha\chi(\beta,\beta)}(\Phi_{\alpha,\beta}/G)_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}(\zeta \times \eta)$$ (4.4)

for $\zeta \in H\alpha(M\alpha/G)$ and $\eta \in H\beta(M\beta/G)$ is bilinear. We prove skew symmetry. To compute $[\eta, \zeta]$, note that the pullback principal $G$-bundle $(\tau_{\alpha,\beta})^*(R\beta,\alpha)$ is isomorphic to the dual bundle $\tilde{R}_{\alpha,\beta}$ with the action inverted. By (4.1), the pullback orientation on $\tilde{R}_{\alpha,\beta}$ is $\tilde{\theta}_{\alpha,\beta}$. Hence

$$(\pi_{\beta,\alpha})^\dagger_{\theta_{\beta,\alpha}} \circ (\tau_{\beta,\alpha})_* = (\tau_{\beta,\alpha})_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}$$

by naturality (3.26)

$$= (-1)^{\chi(\alpha,\beta)+1}(\tau_{\beta,\alpha})_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}$$

by duality (3.27).

Combining this with $\Phi_{\alpha,\beta} \simeq \Phi_{\beta,\alpha} \circ \tau_{\beta,\alpha}^\dagger$ and the skew symmetry of the cross product, $\eta \times \zeta = (-1)^{ab}(\tau_{\beta,\alpha})_* (\zeta \times \eta)$, we obtain

$$[\eta, \zeta] = \epsilon_{\alpha,\beta}(-1)^{b\chi(\alpha,\alpha)+a\chi(\alpha,\beta)+1}(\Phi_{\alpha,\beta}/G)_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}(\zeta \times \eta).$$

This differs from $[\zeta, \eta]$ by the sign

$$\epsilon_{\alpha,\beta}(-1)^{\alpha\chi(\beta,\beta)}(\Phi_{\alpha,\beta}/G)_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}(-1)^{b\chi(\alpha,\alpha)+a\chi(\alpha,\beta)+1}$$

$$= (-1)^{\chi(\alpha,\beta)+1}(\tau_{\beta,\alpha})_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}},$$

where we recall the grading $\langle \zeta \rangle' = a + 2 - \chi(\alpha,\alpha)$. This proves skew symmetry.

It remains only to prove the Jacobi identity

$$(-1)^{|\xi'|^\lambda'}[[\zeta, \eta], \lambda] + (-1)^{|\eta'|^\zeta'}[[\eta, \lambda], \zeta] + (-1)^{|\lambda'|^\eta'}[[\lambda, \zeta], \eta] = 0$$ (4.5)

for $\zeta \in H\alpha(M\alpha/G)$, $\eta \in H\beta(M\beta/G)$, and $\lambda \in H\gamma(M\gamma/G)$. Expanding the definition,

$$[[\zeta, \eta], \lambda] = \epsilon_{\alpha,\beta}(-1)^{\alpha\chi(\beta,\beta)}(\Phi_{\alpha,\beta}/G)_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}(\zeta \times \eta), \lambda$$

$$= \epsilon_{\alpha,\beta}(-1)^{\alpha\chi(\beta,\beta)}\epsilon_{\alpha+\beta,\gamma}(\Phi_{\alpha+\beta,\gamma}/G)_* \circ (\pi_{\alpha+\beta,\gamma})^\dagger_{\theta_{\alpha+\beta,\gamma}}(\Phi_{\alpha+\beta,\gamma}/G)_* \circ (\pi_{\alpha+\beta,\gamma})^\dagger_{\theta_{\alpha+\beta,\gamma}}((\Phi_{\alpha+\beta,\gamma}/G) \times (\pi_{\alpha+\beta,\gamma})^\dagger_{\theta_{\alpha+\beta,\gamma}}(\zeta \times \eta) \times \lambda).$$

To continue this calculation, observe that there is a pullback square

$$P_{\alpha,\beta,\gamma} \xrightarrow{(\Phi_{\alpha,\beta}/\text{id}_{M\gamma}/G)} R_{\alpha+\beta,\gamma}$$

$$\xrightarrow{\kappa_{\alpha,\beta,\gamma}} Q_{\alpha,\beta,\gamma} \xrightarrow{(\Phi_{\alpha+\beta,\gamma}/\text{id}_{M\gamma}/G)} M\alpha+\beta,\gamma \times M\gamma/G$$

and that the pullback orientation on $\kappa_{\alpha,\beta,\gamma}$ is $(\tau_{\alpha,\gamma})^*(\theta_{\alpha,\gamma}) + (\tau_{\beta,\gamma})^*(\theta_{\beta,\gamma})$ by (4.2). By the dual of Corollary 3.7(e), we have $(\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}} \times \text{id}_{M\gamma/G} = (\pi_{\alpha,\beta} \times \text{id}_{M\gamma/G})^\dagger_{(\pi_{\alpha,\beta} \times \text{id}_{M\gamma/G})^*(\theta_{\alpha,\beta})}$. Using these facts and naturality (3.26), we get

$$(-1)^{|\xi'|^\lambda'}[[\zeta, \eta], \lambda] = (-1)^{a\chi(\alpha,\beta)+b\chi(\gamma,\gamma)+c\chi(\alpha,\alpha)+\chi(\alpha,\alpha)\chi(\gamma,\gamma)}\epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma}$$

$$((\Phi_{\alpha+\beta,\gamma}/G)_* \circ (\Phi_{\alpha,\beta} \times \text{id}_{M\gamma}/G)_* \circ (\pi_{\alpha,\beta} \times \text{id}_{M\gamma/G})^\dagger_{(\pi_{\alpha,\beta} \times \text{id}_{M\gamma/G})^*(\theta_{\alpha,\beta})}((\Phi_{\alpha+\beta,\gamma}/G)_* \circ (\pi_{\alpha,\beta})^\dagger_{\theta_{\alpha,\beta}}(\zeta \times \eta) \times \lambda).$$ (4.6)
To evaluate the terms in (3.28), we must identify the involved bundles and pullback we will derive from the composition property (3.28). Consider

\begin{align}
(\Phi_{\beta,\gamma,a}/G)_* \circ ( (\Phi_{\beta,\gamma} \times \text{id}_{M_a})/G)_* \circ (\kappa_{\beta,\gamma,a})^!_{(\tau_{\gamma,\alpha}^\beta a)}*(\theta_{\beta,a}) + (\tau_{\gamma,\alpha}^\beta a )^* \gamma^\beta a )^*(\theta_{\gamma,a}) \tag{4.7}
\end{align}

and

\begin{align}
(\Phi_{\gamma,\alpha,b}/G)_* \circ ( (\Phi_{\gamma,\alpha} \times \text{id}_{M_b})/G)_* \circ (\kappa_{\gamma,\alpha,b})^!_{(\tau_{\gamma,\alpha}^\gamma b)}*(\theta_{\gamma,b}) + (\tau_{\gamma,\alpha}^\gamma b )^* \gamma^\gamma b )^*(\theta_{\gamma,b}) \tag{4.8}
\end{align}

We have computed all of the individual terms of the Jacobi identity (4.5), which we will derive from the composition property (3.28). Consider

\begin{align}
P_1 &= Q_{a,b,\gamma} = (M_a \times M_b)/G \times M_\gamma/G \\
\text{with G-action } \rho_{P_1} (((m_a, m_b)\Delta^2 G, m_\gamma G), g) &= ((m_a g, m_b)\Delta^2 G, m_\gamma G),
\end{align}

and the obvious projections \( \pi_1 \) and \( \pi_2 \) to \( B_{a,b,\gamma} \). We define the orientations on \( P_1 \cong (\tau_{a,b}^{\gamma}\alpha)^* (R_{a,b}) \) and \( P_2 \cong (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (R_{\gamma,\alpha}) \) by \( \theta_1 = (\tau_{a,b}^{\gamma}\alpha)^* (\theta_{a,b}) \) and \( \theta_2 = (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\gamma,\alpha}) \). To define \( \theta_3 \) on \( P_3 = P_1 \otimes G P_2 \), use the pullback square

\begin{align}
\begin{array}{ccc}
\hat{Q}_{\gamma,\alpha,\beta} & \xrightarrow{\mu} & P_1 \otimes G P_2 \\
\downarrow \pi_{\gamma,\alpha} \times \text{id}_{M_b}/G \qquad & & \downarrow \pi_3 \\
B_{\gamma,\alpha,\beta} & \xrightarrow{\tau_{\gamma,\alpha}^{\beta}\gamma} & B_{a,b,\gamma},
\end{array}
\end{align}

Observe here that \( \mu \) is well-defined and \( G \)-equivariant. Define the orientation \( \theta_3 \) on \( \pi_3 \) by requiring its pullback to \( \hat{Q}_{\gamma,\alpha,\beta} \) to be \( (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\gamma,\alpha}) \). In particular, by naturality \( \text{(3.26)} \) and duality \( \text{(3.27)} \), we have

\begin{align}
(\pi_3)^!_{\theta_3} \circ (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\gamma,\alpha}) = (-1)^{\chi(\alpha,\gamma)+1} \mu_* \circ (\pi_{\gamma,\alpha} \times \text{id}_{M_b}/G)^!_{(\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\gamma,\alpha})} \tag{4.9}
\end{align}

To evaluate the terms in \( \text{(3.28)} \), we must identify the involved bundles and pullback orientations. There are pullback diagrams of principal \( G \)-bundles

\begin{align}
\begin{array}{ccc}
Q_{\beta,\gamma,a} & \xrightarrow{\text{id}_{P_1}} & P_2 \\
\downarrow \pi_{\beta,\gamma} \times \text{id}_{M_a}/G \quad \uparrow \pi_2 & & \downarrow \pi_2 \\
B_{\beta,\gamma,a} & \xrightarrow{\tau_{\beta,\gamma}^\alpha a} & B_{a,b,\gamma},
\end{array}
\end{align}

where \( \xi_1 ((m_\beta, m_\gamma, m_\alpha)\Delta^2 G) \) has fiber coordinate \( ((m_\alpha, m_\beta)\Delta^2 G, m_\gamma G) \) in \( P_1 \) and base coordinate \( ((m_\alpha, m_\gamma)\Delta^2 G, m_\beta G) \) in \( P_2 \). Hence by naturality \( \text{(3.26)} \)

\begin{align}
(\pi_2)^!_{\xi_1} \circ (\tau_{\beta,\gamma}^\alpha a )^* (\theta_{\beta,a}) + (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\gamma,a}) \tag{4.10}
\end{align}

In \( \text{(4.10)} \), we use that the pullback of the orientation \( \kappa_1^* \theta_1 \) along \( \xi_1 \) is \( \xi_1^* \kappa_1^* \theta_1 = (\tau_{\alpha,\beta}^{\gamma}\alpha)^* (\theta_{\alpha,\beta}) = (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\alpha,\beta}) \); similarly, the pullback of \( \kappa_3^* \theta_3 \) along \( \xi_1 \) is \( (\tau_{\beta,\gamma}^{\alpha} a )^* (\theta_{\beta,a}) \). Using duality \( \text{(3.27)} \) and \( \text{(4.11)} \), \( \text{(4.11)} \) becomes

\begin{align}
(\kappa_2)^!_{\xi_1} \circ (\tau_{\beta,\gamma}^{\alpha} a )^* (\theta_{\alpha,\beta}) + (\tau_{\gamma,\alpha}^{\beta}\gamma)^* (\theta_{\gamma,a}) \tag{4.11}
\end{align}
Combining this with (4.10) gives

\[(\kappa_1)_{\xi_1 + \kappa_2 \theta_2} \circ (\pi_3)_{\theta_3} \circ (\tau_{\alpha,\beta})_{\gamma} = (-1)\chi(\alpha,\beta) + \chi(\alpha,\gamma) + 1\]

which relates the 1st term of (3.28) with (4.7). For the 2nd term, there is a pullback

\[P_{\alpha,\beta,\gamma} \xrightarrow{\xi_2} \pi_1(P_2) \]

\[\downarrow \kappa_{\alpha,\beta,\gamma} \]

\[Q_{\alpha,\beta,\gamma} \xrightarrow{id_{P_1}} P_1,\]

where \(\xi_2((m_\alpha, m_\beta, m_\gamma)\Delta^3G)\) has fiber coordinate \(((m_\beta, m_\gamma)\Delta^2G, m_\alpha G)\) in \(P_2\) and base coordinate \(((m_\alpha, m_\beta)\Delta^2G, m_\gamma G)\) in \(P_1\). Hence naturality (3.20) implies

\[(\kappa_1)_{\xi_2} \kappa_2 \theta_2 \circ (\pi_3)_{\theta_3} (\pi_1)_{\xi_1} = (\xi_2)_{\kappa_{\alpha,\beta,\gamma}} (\tau_{\alpha,\beta})_{\gamma} \circ (\pi_{\alpha,\beta} \times \text{id}_{M_\alpha/G})_{\tau_{\alpha,\beta}} (\pi_{\alpha,\beta})\]

This relates the 2nd term of (3.28) with (1.6). Finally, there is a pullback square

\[\tilde{P}_{\gamma,\alpha,\beta} \xrightarrow{\xi} P_1 \times_B P_2 \]

\[\downarrow \kappa_{\gamma,\alpha,\beta} \]

\[\tilde{Q}_{\gamma,\alpha,\beta} \xrightarrow{\mu} P_1 \otimes_G P_2,\]

where \(\xi((m_\gamma, m_\alpha, m_\beta)\Delta^3G) = (((m_\alpha, m_\beta)\Delta^2G, m_\gamma G), ((m_\beta, m_\gamma)\Delta^2G, m_\alpha G))\), recalling the convention for \(P_1 \times_B P_2\) from Page 2 to see that \(\xi\) is equivariant. Since \(\kappa_1 = \kappa_2 \theta_2\) pulls back to \((\tau_{\gamma,\alpha,\beta})^*(\theta_{\alpha,\beta}) + (\tau_{\gamma,\alpha,\beta})^*(\theta_{\beta,\gamma})\), we have

\[(\kappa_3)_{\xi_1 + \kappa_2 \theta_2} \circ \mu_\gamma = \xi_\gamma \circ (\kappa_{\gamma,\alpha,\beta})_{\tau_{\gamma,\alpha,\beta}} (\theta_{\alpha,\beta}) + (\tau_{\gamma,\alpha,\beta})_{\gamma} (\theta_{\beta,\gamma}) \quad \text{by (3.20)}\]

\[= (-1)\chi(\alpha,\beta) + \chi(\alpha,\gamma) + 1 \quad \text{by (3.27) and (4.11)}\]

Combined with (4.9), we can relate the 3rd term of (3.28) with (1.8):

\[(\kappa_3)_{\xi_1 + \kappa_2 \theta_2} \circ (\pi_3)_{\theta_3} \circ (\tau_{\gamma,\alpha,\beta}) = (-1)\chi(\alpha,\gamma) + \chi(\alpha,\gamma) + 1\]

Recall that in (3.28) the (non-equivariant) homeomorphisms \(\pi_3(P_1) \cong P_1 \times_B P_2 \cong \pi_1(P_2)\) are implicit. The maps \(\xi_1, \xi_2,\) and \(\xi\) fit into a commutative diagram

\[
\begin{array}{ccc}
\tilde{P}_{\beta,\gamma,\alpha} & \xrightarrow{\pi_{\beta,\gamma,\alpha}} & \tilde{P}_{\gamma,\alpha,\beta} \\
\downarrow{\xi_2} & & \downarrow{\xi_2} \\
\pi_2(P_1) & \xrightarrow{\cong} & P_1 \times_B P_2 \xrightarrow{\cong} \pi_1(P_2) \\
\end{array}
\]
Finally, use (4.3) to check that all of the appearing signs agree, that is, the Jacobi identity (4.5) follows.

Using (4.12), (4.13), and (4.14), we find that the composition of (3.28) with $(\zeta_2)^{-1}$ multiplied by $(-1)^{\chi(\alpha,\gamma)}$ is

\[
(-1)^{\chi(\alpha,\gamma)+1} (\tau^\beta_{\alpha,\beta,\gamma})_* \circ (\kappa_{\beta,\gamma,\alpha})_1 \circ (\pi_{\beta,\gamma} \times \text{id}_{M_\alpha/G})_1 \circ (\tau^\gamma_{\alpha,\beta,\gamma})_*^{-1} \\
+ (-1)^{\chi(\alpha,\gamma)} \circ (\tau^\beta_{\alpha,\beta,\gamma})_* \circ (\pi_{\alpha,\beta} \times \text{id}_{M_\gamma/G})_1
\]

\[
= (-1)^{\chi(\beta,\gamma)(\tau^\gamma_{\alpha,\beta,\gamma})_* \circ (\kappa_{\alpha,\beta,\gamma})_1 \circ (\pi_{\gamma,\alpha} \times \text{id}_{M_\beta/G})_1 \circ (\tau^\gamma_{\alpha,\beta,\gamma})_*^{-1},
\]

omitting the orientations from the notation from now on. Post-compose this expression with the maps induced by $(\Phi_{\alpha,\beta} \times \text{id}_{M_\gamma})/G: P_{\alpha,\beta,\gamma} \to (M_{\alpha,\beta,\gamma}/G)$ and $(\Phi_{\alpha,\beta,\gamma}/G): (M_{\alpha,\beta,\gamma}/G) \to M_{\alpha,\beta,\gamma}/G$ in homology and evaluate at the homology class $\zeta \times \eta \times \lambda$. Using the formulas

\[
(\tau^\beta_{\alpha,\beta,\gamma})_*^{-1}(\zeta \times \eta \times \lambda) = (-1)^{a(b+c)}(\eta \times \lambda \times \zeta),
\]

\[
(\tau^\beta_{\alpha,\beta,\gamma})_*^{-1}(\zeta \times \eta \times \lambda) = (-1)^{(a+b)+c}(\lambda \times \zeta \times \eta),
\]

we obtain

\[
(-1)^{\chi(\alpha,\beta)+1+a(b+c)} (\Phi_{\beta,\gamma,\alpha}/G)_* ((\Phi_{\beta,\gamma} \times \text{id}_{M_\alpha})/G)_* (\kappa_{\beta,\gamma,\alpha})_1 (\pi_{\beta,\gamma} \times \text{id}_{M_\alpha/G})_1 (\eta \times \lambda \times \zeta)
\]

\[
+ (-1)^{\chi(\alpha,\gamma)} (\Phi_{\alpha,\beta,\gamma}/G)_* ((\Phi_{\alpha,\beta} \times \text{id}_{M_\gamma})/G)_* (\kappa_{\alpha,\beta,\gamma})_1 (\pi_{\alpha,\beta} \times \text{id}_{M_\gamma/G})_1 (\zeta \times \eta \times \lambda)
\]

\[
= (-1)^{\chi(\beta,\gamma)+(a+b)+c} (\Phi_{\gamma,\alpha,\beta}/G)_* ((\Phi_{\gamma,\alpha} \times \text{id}_{M_\beta})/G)_* (\kappa_{\gamma,\alpha,\beta})_1 (\pi_{\gamma,\alpha} \times \text{id}_{M_\beta/G})_1 (\lambda \times \zeta \times \eta).
\]

Bring all terms to the right-hand side, multiply by $(-1)^{ac+\chi(\beta,\gamma)+b\chi(\gamma,\alpha)+c\chi(\alpha,\beta)}$ and use (4.6), (4.7), and (4.8) to get

\[
(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\beta)\chi(\beta,\gamma)\epsilon_{\beta,\gamma,\epsilon_{\beta,\gamma,\alpha}}(1)\eta(1)[\epsilon_{\beta,\gamma,\alpha},(1)\eta][\epsilon_{\beta,\gamma,\alpha}]]
\]

\[
+ (-1)^{\chi(\alpha,\gamma)+\chi(\alpha,\alpha)\chi(\gamma,\gamma)\epsilon_{\alpha,\beta,\gamma}(1)\eta(1)\eta(1)[\epsilon_{\alpha,\beta,\gamma},(1)\eta][\epsilon_{\alpha,\beta,\gamma}]]
\]

\[
+ (-1)^{\chi(\beta,\gamma)+\chi(\beta,\beta)\chi(\gamma,\gamma)\epsilon_{\gamma,\alpha,\beta}(1)\eta(1)\eta(1)[\epsilon_{\gamma,\alpha,\beta},(1)\eta][\epsilon_{\gamma,\alpha,\beta}]] = 0.
\]

Finally, use (4.3) to check that all of the appearing signs agree, that is,

\[
(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\beta)\chi(\beta,\gamma)\epsilon_{\beta,\gamma,\epsilon_{\beta,\gamma,\alpha}}(1)\eta(1)\eta(1)[\epsilon_{\beta,\gamma,\alpha},(1)\eta][\epsilon_{\beta,\gamma,\alpha}]]
\]

\[
= (-1)^{\chi(\beta,\gamma)+\chi(\beta,\beta)\chi(\gamma,\gamma)\epsilon_{\gamma,\alpha,\beta}(1)\eta(1)\eta(1)[\epsilon_{\gamma,\alpha,\beta},(1)\eta][\epsilon_{\gamma,\alpha,\beta}]]
\]

The Jacobi identity (4.5) follows.
APPENDIX A. Elementary technical facts

The first goal of this appendix is to prove an unfamiliar identity for the binomial coefficients \( \binom{n}{k} \) with integer \( n \) and \( k \). Recall the standard properties

\[
\begin{align*}
  & k < 0 \text{ or } 0 \leq n < k \quad \iff \quad \binom{n}{k} = 0, \\
  & k \in \mathbb{Z} \text{ and } 0 \leq n \quad \implies \quad \binom{n}{k} = \binom{n}{n-k}, \\
  & k \in \mathbb{Z} \text{ and } n \in \mathbb{Z} \quad \implies \quad \binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \\
  & 0 \leq h + k \text{ and } 0 \leq n \quad \implies \quad \binom{n-h}{h} = \binom{n-h-k}{k}.
\end{align*}
\]

(A.1) (A.2) (A.3) (A.4)

The following result may be viewed as an extension of (A.2) to integer \( n \).

Proposition A.1. For all integers \( n \) and \( k \) we have

\[
(-1)^k \binom{k-n-1}{n-1} = \begin{cases} 
\binom{n}{k} - \binom{n}{n-k} & \text{if } n \geq 0, \\
-\binom{n}{n-k} & \text{if } n < 0 \text{ and } k \leq n, \\
\binom{n}{k} & \text{if } n < 0 \text{ and } k > n.
\end{cases}
\]

(A.5)

Hence by (A.1) in every case we have

\[
(-1)^k \binom{k-n-1}{n-1} = \binom{n}{k} - \binom{n}{n-k}.
\]

(A.6)

Proof. Suppose \( n \geq 0 \). Then the left hand side of (A.5) vanishes by (A.1) and the right hand side vanishes by (A.2). Suppose \( n < 0 \) and \( k \leq n \). Then

\[
\binom{n}{n-k} \overset{(A.5)}{=} (-1)^{n-k} \binom{-k-1}{n-k} \overset{(A.2)}{=} (-1)^{n-k} \binom{-k-1}{-n-1} \overset{(A.3)}{=} (-1)^{k+1} \binom{k-n-1}{-n-1}.
\]

Finally, suppose \( n < 0 \) and \( k > n \). Then

\[
\binom{n}{k} \overset{(A.3)}{=} (-1)^k \binom{k-n-1}{k} \overset{(A.2)}{=} (-1)^k \binom{k-n-1}{-n-1}.
\]

\( \square \)

The second goal of the appendix is to explain how to define a morphism of possibly infinite-dimensional vector spaces over a field \( \mathbb{K} \) in terms of its dual. The following is the key result. Let \( \text{ev}_v = \langle -, v \rangle \) denote the evaluation at \( v \).

Lemma A.2. Let \( V \) be a vector space over \( \mathbb{K} \). The image of the embedding \( \text{ev}_v : V \to V^{**}, v \mapsto \text{ev}_v, \) into the double dual of \( V \) is the set of all morphisms \( \epsilon : V^* \to \mathbb{K} \) for which there exists a finite-dimensional subspace \( V_\epsilon \subseteq V \) such that

\[
\alpha \in V^*, \text{ev}_v \alpha |_{V_\epsilon} = 0 \implies \epsilon(\alpha) = 0.
\]

(A.7)

Proof. For \( \epsilon = \text{ev}_v \), the property (A.7) holds for the span \( V_\epsilon \) of \( v \). Conversely, given \( V_\epsilon \) satisfying (A.7), pick a basis \((v_i)_{i=1,\ldots,n}\) of \( V_\epsilon \) and let \((v_i^*)_{i=1,\ldots,n}\) be the dual basis of \( V_\epsilon^* \). Extend \( v_i^* \) arbitrarily to \( V \). Define \( v = \sum_{i=1}^n \epsilon(v_i^*) v_i \). We have \( \epsilon = \text{ev}_v \), since for all \( \alpha \in V^* \)

\[
\langle \epsilon - \text{ev}_v \rangle (\alpha) = \epsilon(\alpha) - \sum_{i=1}^n \epsilon(v_i^*)(\alpha(v_i)) = \epsilon \left( \alpha - \sum_{i=1}^n \alpha(v_i) v_i^* \right) \overset{(A.7)}{=} 0.
\]

(\( \square \))

Lemma A.3. Let \( V \) and \( W \) be vector spaces over \( \mathbb{K} \) and let \( \phi : V^* \to W^* \) be a morphism. There exists a (unique) morphism \( f : V \to W \) with \( \phi = f^* \) if and only if for every \( v \in V \) there exists a finite-dimensional subspace \( W_v \subseteq W \) such that

\[
\beta \in W^*, \beta |_{W_v} = 0 \implies \langle \phi(\beta), v \rangle = 0.
\]

(A.8)
Proof. If $\phi = f^*$ and $v \in V$, then (A.8) holds for the span $W_v$ of $f(v)$. Conversely, given (A.8) we can construct $f$ in the commutative square

$$
\begin{array}{ccc}
V^{**} & \xrightarrow{\phi^*} & W^{**} \\
\ev & & \ev \\
V & \xrightarrow{f} & W
\end{array}
$$

by showing for each $v \in V$ that $\epsilon = \phi^*(\ev_v)$ is in the image of $W$. Define $W_v = W_v$ and check (A.7): If $\beta \in W^*$ satisfies $\beta|_{W_v} = 0$, then $\epsilon(\beta) = \langle \phi(\beta), v \rangle$ (A.8) $= 0$. □

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