Anomalous Hall effect in chiral superconductors from impurity superlattices

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Unlike anomalous quantum Hall insulators, clean single-band chiral superconductors do not exhibit intrinsic Hall effect at the one-loop approximation. Finite ac Hall conductance was found to emerge beyond one-loop, such as with vertex corrections associated with extrinsic random impurity scatterings. In this paper, we investigate the effect of impurities embedded in chiral superconductors in a superlattice pattern, instead of in random distributions. The impurity-induced Bogoliubov quasiparticle bound states hybridize to form subgap bands, constituting an emergent low-energy effective theory whose anomalous Hall effect can be studied with ease. We demonstrate that the occurrence of the Hall effect depends on the superlattice geometry and the parity of the chiral pairing. Fundamentally, due to the composite particle-hole character of the subgap states, the Hall conductance arises at the one-loop level of the current-current correlator in our effective model. Generalized to random impurities, our theory provides a deeper insight into the physics of impurity-induced anomalous Hall conductivity and Kerr rotation in chiral superconductors.

Introduction. Topological chiral superconductors are classified by a topological invariant – the Chern number, and they exhibit protected chiral edge modes. Odd-parity chiral superconductors (e.g. chiral p-, f-wave, etc) can further support half-quantum vortices that host Majorana zero modes. These excitations obey non-Abelian braiding statistics and could therefore be utilized for topological quantum computation.

The time-reversal symmetry breaking of the chiral pairings can be detected in polar Kerr effect measurements, where a linearly polarized light normally incident on the superconductor is reflected with a rotated polarization. Signatures of Kerr rotation have been reported in several unconventional superconductors, including Sr2RuO4, UPt3 and URu2Si2. Such an effect is closely related to the anomalous Hall effect. However, unlike in an anomalous Hall insulator, the effect is not expected in a clean and uniform single-band chiral superconductor. This could be understood in the following simple terms. The pairing potential Δ\(k\), whose \(k\)-dependence describes the relative motion between the paired electrons, does not generate center-of-mass motion for the Cooper pair. Thus the current operator of a superconductor contains no contribution originating from \(\Delta_k\). Consequently, the Hall conductance is not directly related to the Berry curvature of the Bogoliubov quasiparticles and it in fact vanishes at the one-loop approximation.

Nonetheless, vertex corrections, such as those arising from extrinsic impurity scatterings and certain intrinsic superconducting collective modes, have both been shown to induce finite Hall conductance. Separate intrinsic mechanisms exist for multiband superconductors, but those involve either interband chiral pairing or odd-frequency pairing. Thus far, whether these effects could quantitatively explain the observed Kerr rotation is still debated.

In previous studies, the leading order impurity effects are captured by the so-called skew-scattering diagrams. However, some important microscopic details of the impurity-induced physics are absent in this diagramatic treatment. In particular, individual impurities are known to induce subgap quasiparticle bound states. How such low-energy states influence the electromagnetic response of the system remains largely unexplored and is the focus of the present study.

To facilitate our discussions, we imagine depositing impurities on the underlying chiral superconductors in a superlattice pattern. Due to the chiral nature of the pairing, the bound states from different impurity sites hybridize in a ubiquitous fashion that depends on their relative position. We construct a low-energy effective theory of the emergent subgap bands on the superlattice and study the resultant anomalous Hall effect. Despite having similar appearance, the new effective Hamiltonian differs from the original BdG Hamiltonian in a fundamental way, that the components of the new spinor-basis are no longer purely electron or hole, but rather a linear superposition of both. This leads to profound consequences on the current operators, which allow for a transparent interpretation of finite Hall conductance at the one-loop level.
We consider several representative impurity superlattice geometries, and show that the resultant physics is model-dependent. For example, while the Hall conductance is generically nonvanishing on a honeycomb superlattice in any underlying chiral pairing, it vanishes for square and triangular superlattices embedded in even-parity chiral superconductors, such as chiral d- and g-wave. When generalized to randomly distributed impurities, our theory provides a physically intuitive understanding for the appearance of the anomalous Hall effect in all chiral superconductors.

Impurity states and impurity superlattices. In the Nambu spinor basis \( \hat{\varphi}(r) = (c_{r\uparrow}, c_{r\downarrow})^T \), the underlying chiral superconducting state is described by the continuum Bogoliubov-de Gennes (BdG) Hamiltonian \( \hat{H}_\text{BdG}^{(\text{bulk})} = \int d^3r' \hat{\varphi}^\dagger(r) \hat{H}_\text{BdG}^{(\text{bulk})}(r, r') \hat{\varphi}(r') + \text{H.c.} \), in which

\[
\hat{H}_\text{BdG}^{(\text{bulk})}(r, r') = \left( \begin{array}{cc}
\delta_{r,r'}(\frac{\nabla^2}{2m_e} - \mu) & \Delta(r, r') \\
\Delta^*(r, r') & \delta_{r,r'}(\frac{\nabla^2}{2m_e} + \mu)
\end{array} \right)
\]  

(1)

where \( c_r \) stands for the electron annihilation (creation) operators, \( m_e \) and \( \mu \) are the electron mass and the chemical potential, respectively. The off-diagonal term \( \Delta(r, r') = g(|r - r'|)e^{i\theta_{r'-r'}} \) is the chiral pairing potential, where \( \theta_r \) is the azimuthal angle of \( r \), and \( g(|r - r'|) \), assumed to be a certain (unimportant) decaying function of \(|r - r'|\), describes the spatial profile of the Cooper pair wavefunction. Here \( \delta \) denotes the order of the chiral pairing, i.e., the Cooper pair angular momentum, which takes the values \( 1, 2, \cdots \) for \( \delta = 1, 2, \cdots \). Notice that we have assumed a uniform order parameter independent of the Cooper pair center-of-mass position, \( (r + r')/2 \). Consideration of spatial variations around impurities does not qualitatively alter our conclusion.

Impurities in chiral superconductors are known to induce bound states. Consider first a single-impurity at \( \mathbf{R} = 0 \), described by a delta-function-like potential \( U \delta (r - \mathbf{R}) \tau_3 \) where \( U \) is the impurity strength and \( \tau_3 \) is the third component of the Pauli matrices operating in the Nambu space. The bound state wavefunctions take the forms \( \psi_{\pm}(r) = (u(r), v(r))^T = (u_r, e^{-i\theta_r}v_r)^T \) and \( \psi_{-}(r) = (-v^*(r), u^*(r))^T \) [27]. Here the ‘\( + \)’ and ‘\( - \)’ designate, respectively, the state with subgap energy \( +E_0 \) and the other with \(-E_0 \), where \( E_0 < \Delta_0 \) and \( \Delta_0 \) denotes the superconducting gap. These two states are related by particle-hole symmetry, but the detailed forms of \( u_r \) and \( v_r \) are model-dependent and are not constrained by any other symmetry, except that they shall in general decay as \( e^{-r/\xi} / \sqrt{\xi_{\text{Qr}}} \) sufficiently far away from the impurity center. Here \( k_F \) is the Fermi momentum and \( \xi \) the superconducting healing length. In the following, we shall assume a sizable impurity strength such that \( E_0 \ll \Delta_0 \) [29], under which circumstance the low-energy theory associated with these bound states are well separated from the continuum spectrum.

On an impurity lattice where the interlattice spacing \( R_0 \) is larger than \( \xi \), the above-described bound state wavefunctions on each single site still constitute a good approximation. States from neighboring impurity sites ‘hybridize’ via the microscopic kinetic hopping and pairing in the original Hamiltonian Eq. (1). Written in the second quantized form where \( c_{i, \pm}^\dagger \) denote the creation (annihilation) of the respective bound states on each site, an emergent low-energy tight-binding model on the superlattice reads

\[
\hat{H}_\text{eff} = \sum_{i,j} H_{i,j} \Psi_i^T \left[ E_0 \delta_{ij} \sigma_3 + \hat{h}_{i,j}(1 - \delta_{ij}) \right] \Psi_j + \text{H.c.},
\]  

with the Pauli \( \sigma \)-matrices operating in the space spanned by \( \Psi_i = (c_{i,+}, c_{i,-})^T \), and

\[
\hat{h}_{i,j} = \left( \begin{array}{cc}
t_{ij}^{++} & t_{ij}^{+-} \\
t_{ij}^{-+} & t_{ij}^{--}
\end{array} \right).
\]  

(2)

in which

\[
t_{ij}^{\mu
u} = \int d^3r d^3r' \psi_{\mu}^\dagger(r - \mathbf{R}) \hat{H}_\text{BdG}(r, r') \psi_{\nu}(r' - \mathbf{R}_j),
\]  

(3)

It is obvious that the hopping of the bound states could arise from both the kinetic and pairing terms in the underlying microscopic Hamiltonian. A detailed analysis of the hopping integrals can be found in the Supplementary [27], which we summarize below and in Fig. 2 (a) and (b). The hybridization between the ‘\( + \)’ (‘\( - \)’) states satisfy \( t_{ij}^{++} = -t_{ij}^{--} = \lambda_{ij} \), where \( \lambda_{ij} \) is a real constant determined by the separation \(|\mathbf{R}_j - \mathbf{R}_i|\). On the other hand, the integral between ‘\( + \)’ and ‘\( - \)’ states has the relation \( t_{ij}^{+-} = (t_{ij}^{-+})^* = \eta_{ij} \), where \( \eta_{ij} = |\eta_{ij}|e^{i\theta_{\mathbf{R}_j} - \mathbf{R}_i} \) [28]. It thus depends on both the relative position between the two sites and the order of the chiral pairing.

Our later analyses of the current operators require distinguishing in Eq. (3) contributions originating from the pairing, the electron and hole kinetic hopping processes, i.e. \( \lambda_{ij} = \lambda_{ij}^e + \lambda_{ij}^h \) and \( \eta_{ij} = \eta_{ij}^e + \eta_{ij}^h \). The kinetic part of \( \eta_{ij}^e + \eta_{ij}^h \), deserves special attention. Written explicitly,

\[
\eta_{ij}^e + \eta_{ij}^h = \int d^3r d^3r' \left[ -u_{r - \mathbf{R}_i} \delta_{r,r'} \left( -\nabla^2_e/2m_e - \mu \right) e^{i\theta_r} u_{r' - \mathbf{R}_j} + e^{i\theta_r} u_{r - \mathbf{R}_i} \delta_{r,r'} \left( \nabla^2_e/2m_e + \mu \right) u_{r' - \mathbf{R}_j} \right]
\]  

(4)

where we have performed partial integration and substitution of variables to obtain the second line, and the two terms in
1 + e^{i\mu x}$ are associated with $\eta^e_{ij}$ and $\eta^h_{ij}$, respectively. The relation $|\eta^e_{ij}| = |\eta^h_{ij}|$ is a consequence of the particle-hole symmetry between the ‘$+$’ and ‘$-$’ states. For $l$ odd, $\eta^e_{ij} = -\eta^h_{ij}$, hence the kinetic contribution vanishes if the underlying chiral pairing has odd-parity; for $l$ even, by contrast, $\eta^e_{ij} = \eta^h_{ij}$. We shall later see that the corresponding current operators have the opposite even and odd $l$-dependence. Finally, it is easy to check that the relation $\eta_{ij} = |\eta_{ij}|e^{i \theta_{ij}} - \kappa,$ also holds for the individual constituents of $\eta_{ij}.$

As a concrete example, in a chiral $p$-wave superconductor, a square impurity superlattice with up to nearest-neighbor hopping has the following effective Hamiltonian,

$$\hat{H}_{\text{eff}} = \mathcal{E}_{3k} \sigma_3 + \mathcal{E}_{1k} \sigma_1 - \mathcal{E}_{2k} \sigma_2$$

where we have set $R_0 = 1$ for brevity, $\mathcal{E}_{3k} = 2\lambda \cos k_x + \cos k_y + E_0,$ $\mathcal{E}_{1k} = 2\eta \sin k_x$ and $\mathcal{E}_{2k} = 2\eta \sin k_y$. Here, $\lambda$ denotes the nearest-neighbor hopping integrals of $\lambda_{ij},$ and $\eta$ the corresponding counterpart of $|\eta_{ij}|.$ Notice the implicit decomposition such as $\eta = \eta^\Delta + \eta^e + \eta^h$ (although $\eta^e + \eta^h = 0$ for chiral $p$-wave). Due to the angle dependence of the complex off-diagonal hopping $\eta_{ij},$ Eq. [5] resembles the form of the underlying chiral $p$-wave Hamiltonian. The band topology could be engineered by controlling parameters such as the impurity potential and the superlattice constant [29,30]. These hold for higher order chiral superconductors, although further neighbor hybridizations must be considered to make the band topology transparent. In like manner, impurity chains immersed in odd-parity chiral states support an emergent 1D $p$-wave model and may give rise to isolated Majorana zero modes at the ends of the chains.

**Current operators.** The composite particle-hole nature of each of the spinor component in $\hat{\Psi}$ (i.e. each impurity bound state) has a profound consequence on the particle current operators. Foremost, the portion of the hopping integrals originating from the underlying Cooper pairing, i.e. $\lambda^\Delta$ and $\eta^\Delta,$ shall have no contribution, as in the case of clean superconductors. The only contribution stems from the mutually ‘canceling’ electron hopping ($\lambda^e$ and $\eta^e$) and hole hopping ($\lambda^h$ and $\eta^h$). Understandably, if the ‘$+$’ state is purely electron-like and the ‘$-$’ state purely hole-like, $\eta^e = \eta^h = 0,$ and the resultant current operators resemble those of a clean superconductor.

For the model given in [5], the current operators $\hat{J}^+_i$ and $\hat{J}^-_i$ defined on the superlattice bonds are sketched in Fig. 2(c) and (d). The properties of the $\eta^e_{ij}$’s imply the following general relation: $\hat{J}^+_i = -(\hat{J}^-_i)^*$ and $\hat{J}^+_i = (\hat{J}^-_i)^*.$ Specifically, to the model in (5), the $x$-component of the current operator reads,

$$\hat{J}^{\text{eff}}_{xk} = \hat{J}_{3k} \sigma_3 + \hat{J}_{1k} \sigma_1 + \hat{J}_{2k} \sigma_2 \, ,$$

where $\hat{J}_{3k} = -2(\lambda^e - \lambda^h) \sin k_x,$ $\hat{J}_{1k} = 2(\eta^e - \eta^h) \cos k_x,$ and $\hat{J}_{2k} = 0.$ Note that $\hat{J}_{2k}$ could be nonzero if further neighbor hoppings are considered. The $y$-component follows similarly and can be found in the Supplementary [27]. The cancellation between the electron and hole contributions is evident in these expressions. Notably, although $\eta^e + \eta^h = 0$ for odd-parity pairing, the corresponding kinetic contribution to the particle current is finite and scales as $\eta^e - \eta^h = 2\eta^e,$ such as in $\hat{J}_{1k}.$ In the case of underlying even-parity pairing, however, since $\eta^e = \eta^h,$ $\hat{J}^x_{ij} \propto \eta^e - \eta^h = 0$—suggesting a perfect cancellation between the electron and hole transport. Hence $\hat{J}_{1x(y)}$ and $\hat{J}_{2x(y)}$ must both vanish in this case.

**Anomalous Hall conductivity.** We are now in position to study the anomalous Hall conductance of our low-energy theory. Within linear response theory, it is given by the antisymmetric part of the $x \rightarrow y$ correlation function $\pi_{xy}(\mathbf{q},\omega)$ [32],

$$\sigma_{xy}(\mathbf{q} = 0, i\nu_m) = T \sum_{\mathbf{k},i\nu_n} \text{Tr} \left[ \hat{J}^{\text{eff}}_{y\mathbf{k}} \hat{G}(\mathbf{k}, i\omega_n + i\nu_m) \right. \right.$$ 

where, at the one-loop approximation,

$$\pi_{xy}(\mathbf{q} = 0, i\nu_m) = T \sum_{\mathbf{k},i\nu_n} \text{Tr} \left[ \hat{J}^{\text{eff}}_{y\mathbf{k}} \hat{G}(\mathbf{k}, i\omega_n + i\nu_m) \right. \right.$$ 

where $T$ is the temperature, $\omega_n = (2n + 1)\pi T$ and $\nu_m = 2m\pi T$ are, respectively, the fermionic and bosonic Matsubara frequencies, and $\hat{G}(\mathbf{k}, i\omega_n) = (i\omega_n - \hat{H}^{\text{eff}}_{\mathbf{k}})^{-1}$ stands for the impurity-band Green’s function. For the square lattice model introduced above, we arrive at the following,

$$\sigma_{xy}(\omega + i\delta) = \sum_{\mathbf{k}} \frac{\hat{f}_{\mathbf{k}}}{E_{\mathbf{k}}[(\omega + i\delta)^2 - 4E_{\mathbf{k}}^2]},$$

where $E_{\mathbf{k}} = \sqrt{\mathcal{E}_{1k}^2 + \mathcal{E}_{2k}^2 + \mathcal{E}_{3k}^2}$ is the dispersion of the impurity subgap band, and

![Figure 2. (a) (b) Tight-binding construction of a square impurity superlattice immersed in a chiral p-wave superconductor. Note the relation $\lambda = \lambda^\lambda + \lambda^\lambda + \eta^\lambda$ and $\eta = \eta^\lambda + \eta^\lambda + \eta^\lambda$. (c) (d) The current operator on the superlattice. The ‘$+$’ and ‘$-$’ symbols on the sites label the impurity bound states, and arrows indicate the reference direction of hopping or current flow.](attachment:figure2.png)
\[ f_k = \sum_{m,n,s} \frac{\epsilon_{mns}}{2} [J_{mzk} J_{nyk} - J_{myk} J_{nzk}] \mathcal{E}_{nk}, \] (10)

where \( \epsilon_{mns} \) denotes the Levi-Civita tensor with indices \( m, n, s = 1, 2, 3 \). Obviously, \( \sigma_H \) vanishes for any underlying even-parity chiral pairing, as their current operators contain only \( J_{3z(y)} \), even when further neighbor hoppings are included. In contrast, odd-parity pairings shall in general see a finite Hall conductance. This distinction applies to any superlattice configuration with no sublattice degree of freedom, including triangular lattices (see Table II and Ref. [27]).

There are several features worth remarking. Firstly, the magnitude of \( \sigma_H \) is determined by the above-defined hopping integrals which describe the hybridization between the impurity-bound states. Secondly, the minimal frequency at which the imaginary part of \( \sigma_H \) becomes nonzero is set by the gap between the impurity bands. By contrast, as the skew-scattering diagramatic analysis captures only the continuum state contributions, the minimal frequency identified there is the superconducting gap 2\( \Delta_0 \) [9,12]. Finally, unlike the proposals which require particle-hole asymmetric normal state electron dispersion to obtain finite \( \sigma_H \) [9,19], our low-energy theory has no such restriction.

**Honeycomb superlattice.** The composite particle-hole nature of the impurity subgap bands implies that there exists no fundamental symmetry constraints for the appearance of anomalous Hall effect, i.e. the vanishing of \( \sigma_H \) in some of the models above must be accidental. Given that those models are characterized by single-sigma-matrix current operators, looking for systems that exhibit more structured current operators may be a promising route to obtain finite \( \sigma_H \). One possibility is to introduce sublattice degrees of freedom. We turn below to a honeycomb lattice model for illustration [Fig. 1(b)].

Consider up to nearest-neighbor terms, in the basis \( \Psi_i = (c_{i,+}, c'_{i,+}, c_{i,-}, c'_{i,-})^T \) where \( c \) and \( c' \) represent the two sublattices, the emergent tight-binding Hamiltonian has the form [27].

\[
\hat{H}_{\text{eff}} = \begin{bmatrix}
E_0 & 0 & \lambda_k & 0 \\
0 & E_0 & (-1)^i \eta_{-k} & 0 \\
\lambda_k^* & (-1)^i \eta_{-k}^* & -E_0 & -\lambda_k \\
0 & 0 & -\lambda_k^* & -E_0
\end{bmatrix}
\] (11)

where \( \lambda_k = \sum_\delta e^{ik \cdot R_\delta} \lambda \) and \( \eta_k = \sum_\delta e^{ik \cdot R_\delta} e^{il \theta_{R_\delta}} \eta_i \), and \( R_\delta \) \((\delta = 1, 2, 3)\) designate the three shortest vectors connecting sublattice \( c \) to \( c' \). Interestingly, at \( E_0 = 0 \), the model resembles a low-energy theory proposed for the Moiré superlattice in twisted bilayer graphene [33].

As we have seen, in the case of even-parity pairing, the hopping between the ‘+’ and ‘−’ states on different sites does not generate particle current. However, the inter-sublattice hopping between the ‘+’ (or ‘−’) states introduces two off-diagonal components in the current operators. For example, in the present model,

\[
\hat{j}_{\text{eff}} = J_{1zk} \theta_1 \otimes \sigma_3 + J_{2zk} \theta_2 \otimes \sigma_3,
\] (12)

in which \( \theta_i \) \((i = 1, 2, 3)\) are the Pauli matrices operating in the sublattice manifold, and \( J_{1zk} = -3(\lambda^e - \lambda^h) \sin(\frac{3k_z}{2}) \cos(\frac{\sqrt{3}k_y}{2}) \) and \( J_{2zk} = 3(\lambda^e - \lambda^h) \cos(\frac{3k_z}{2}) \cos(\frac{\sqrt{3}k_y}{2}) \). A lengthy calculation for \( \sigma_H \) presented in the Supplementary [27] leads to an integral form involving \( J_{1zk} J_{2yk} - J_{2zk} J_{1yk} \left| \theta_k \right|^2 - |\eta_{-k}|^2 \) in the numerator of the integrand. The integral is generically finite, in contrast to the square and triangular superlattice scenarios. For odd-parity pairings, an additional contribution to the current operators arises from the inter-sublattice hopping between the ‘+’ and ‘−’ states, and the Hall conductance is again finite.

| superlattice structure | p-wave \((l = 1)\) | d-wave \((l = 2)\) | f-wave \((l = 3)\) | g-wave \((l = 4)\) |
|------------------------|-----------------|-----------------|-----------------|-----------------|
| Square                 | ✓               | ×               | ✓               | ×               |
| Triangular             | ✓               | ×               | ✓               | ×               |
| Honeycomb              | ✓               | ✓               | ✓               | ✓               |
| Random                 | ✓               | ✓               | ✓               | ✓               |
| (skew scattering)      | ✓               | ✓               | ✓               | ✓               |

**Concluding remarks.** The foregoing analyses, especially the fact that honeycomb impurity lattice exhibits finite \( \sigma_H \) for all chiral states, implicate that the anomalous Hall effect must also be generically expected for the scenarios of random impurities. In striking contrast, analyses based on skew-scattering diagram calculations predicted that, in the continuum limit, the effect is only present in chiral p-wave [12]. We summarize our main results in Table I.

Besides ignoring the contribution from the quasiparticle continuum, we also stopped short of discussing the magnitude of Hall conductance in our effective theory and have evaded the question whether our results could connect to more experimentally relevant scenarios. For example, normally impurity concentration and impurity strength must reach certain level to produce a Hall conductance that matches the measured Kerr rotation. Yet excessive strong impurities would have, on the other hand, significantly suppressed the superconductivity. Nevertheless, since the hopping/hybridization between bound states on neighboring impurity sites grows exponentially as a function of the impurity spacing, there could well be a range of impurity concentration, which leaves the superconducting pairing more or less intact, but is sufficient to sustain a sizeable Hall conductance.

Against the backdrop of the above deficiencies, the intuitive and transparent argument we present, that the composite particle-hole character of the impurity bands lies at the heart of the appearance of finite Hall conductance in our effective model, and that the conductance emerges at the one-loop level, provides a novel perspective towards the impurity-
induced anomalous Hall effect in chiral superconductors.

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Supplemental Material for “Anomalous Hall effect in chiral superconductors from impurity superlattices”

I. IMPURITY-INDUCED BOUND STATES IN CHIRAL SUPERCONDUCTORS

The BdG Hamiltonian for a two-dimensional chiral superconductor with impurities can be written as

\[ H_{\text{BdG}} = H_{\text{BdG}}^{(\text{bulk})} + H_{\text{BdG}}^{(\text{imp})}, \]  

which is expressed in the Nambu space spanned by the spinor \( \phi_k = (c_k, c_{-k}^\dagger)^T \), where \( c_k \) \( (c_k^\dagger) \) is the electron annihilation (creation) operator. In the continuum limit, the bulk Hamiltonian in the momentum space, \( H_k^{(\text{bulk})} \), can be expanded in terms of the Pauli matrices \( \tau_i \) \((i = 1, 2, 3)\) as

\[ H_k^{(\text{bulk})} = \int \frac{dk}{(2\pi)^2} H_k^{(\text{bulk})} = \int \frac{dk}{(2\pi)^2} [\epsilon_k \tau_3 + \text{Re} (\Delta_k) \tau_1 - \text{Im} (\Delta_k) \tau_2]. \]  

\( \epsilon_k = k^2/2m - \varepsilon_F \) is the dispersion of electrons relative to the Fermi energy \( \varepsilon_F \). \( \Delta_k = \Delta e^{i\theta_k} \) is the gap function of the chiral pairing, where \( \theta_k \) is the azimuthal angle of \( k \), and \( l \) and \( \Delta \) represent, respectively, the Cooper pair angular quantum number and the \( k \) independent gap magnitude.

We first solve a single-impurity problem, with a delta-function potential with strength \( U \) located at the origin,

\[ H_{\text{BdG}}^{(\text{imp})} (r) = U \tau_3 \delta (r). \]  

The equation to be solved is \( H_{\text{BdG}} \psi (r) = E \psi (r) \), where \( E \) is the eigenvalue. Performing a Fourier transformation \( \psi (r) = \int \frac{dk}{(2\pi)^2} e^{ikr} \psi_k \), one obtains

\[ \left[ E - H_k^{(\text{bulk})} \right] \psi_k = U \tau_3 \psi (0). \]  

Transformed back into the real space, the wavefunction becomes

\[ \psi (r) = U G (E, r) \tau_3 \psi (0), \]  

where \( G (E, r) \) is the bulk Green’s function,

\[ G (E, r) = \int \frac{dk}{(2\pi)^2} e^{i k r} \left[ E - H_k^{(\text{bulk})} \right]^{-1} = \int \frac{dk}{(2\pi)^2} e^{i k r} \frac{\epsilon_k \tau_3 + \text{Re} (\Delta_k) \tau_1 - \text{Im} (\Delta_k) \tau_2}{E^2 - \epsilon_k^2 - | \Delta_k |^2} = X_0 \tau_0 + X_1 \tau_3 + i X_2^+ \tau_+ + i X_2^- \tau_-, \]  

in which \( \tau_{\pm} = (\tau_1 \pm i \tau_2) / 2 \), and \( X_0, X_1, X_2^\pm \) are given by [1][2],

\[ X_0 (E, r) = -\int \frac{dk}{(2\pi)^2} \frac{E e^{i k r}}{\epsilon_k^2 + | \Delta_k |^2 - E^2} \approx -\frac{2N_F E}{\sqrt{\Delta^2 - E^2}} \text{Im} \{ K_0 [(\kappa - i) k_F r] \}, \]  

\[ X_1 (E, r) = -\int \frac{dk}{(2\pi)^2} \frac{\epsilon_k e^{i k r}}{\epsilon_k^2 + | \Delta_k |^2 - E^2} \approx -2N_F \text{Re} \{ K_0 [(\kappa - i) k_F r] \}, \]  

\[ X_2^\pm (E, r) = \pm \int \frac{dk}{(2\pi)^2} \frac{i \Delta e^{i \theta_k} e^{i k r}}{\epsilon_k^2 + | \Delta_k |^2 - E^2} \approx \pm e^{\pm i (\kappa + 1) \pi / 2} \frac{2N_F \Delta}{\sqrt{\Delta^2 - E^2}} e^{\pm i \theta_k} \text{Im} \{ K_1 [(\kappa - i) k_F r] \}. \]

In these expressions, \( N_F \) is the density of states at \( \varepsilon_F, \kappa \equiv \sqrt{\Delta^2 - E^2} / (k_F r) \), and the function \( K_n (x) \) represents the modified Bessel functions of the second kind of order \( n \). Far from the impurity, \( X_0, X_1, X_2^\pm \) all decay as \( e^{-r/\xi} / \sqrt{k_F r} \), in which \( \xi = v_F / \sqrt{\Delta^2 - E^2} \) is the effective coherence length. The above equations are valid for \( |E| < \Delta \) and \( \kappa k_F r = r / \xi \gtrsim 1 \). Note
that the Bessel functions involved diverge at $r = 0$. Right at $r = 0$, the $k$ integral can be performed without resorting to Bessel functions, leading to

$$X_0(E, 0) = - \frac{\pi N_F E}{\sqrt{\Delta^2 - E^2}}, \quad X_1(E, 0) = 0, \quad X_2^\pm(E, 0) = 0. \quad (S10)$$

An ultraviolet energy cut off is needed to regulate the divergence in the $k$ integrals, in order to obtain the correct behavior of the Green’s function at $0 < r/\xi \lesssim 1$. However, we will ignore this short-distance behavior since it is not important for our following discussions.

At $r = 0$, i.e., right at the impurity site, the eigenvalue equation becomes

$$\left[1 - U G(E, 0) \tau_3 \right] \psi(0) = 0. \quad (S11)$$

Using Eqs. (S10) and (S6) we obtain the impurity induced subgap state energies as $E = \pm E_0$ with $E_0 = \Delta/\sqrt{1 + \beta^2}$, where $\beta = \pi N_F U$. The two energies are symmetric with respect to $E = 0$, which is not the case in general if the particle-hole asymmetry of the normal state energy dispersion is introduced; also, the expression of $E_0$ is independent of the sign of $U$, which needs to be modified if the $k$-dependence of the gap function is included. However, considering more general cases does not alter the conclusions obtained in the main text. We denote the two eigenvectors corresponding to $E = \pm E_0$ as $\psi_+(r)$ and $\psi_-(r)$, respectively, and consider the $U > 0$ (repulsive) and $U < 0$ (attractive) cases separately in the following.

1) For $U > 0$ the two eigenvectors at $r = 0$ are $\psi_+(0) = (1, 0)^T$ (particle-like) and $\psi_-(0) = (0, 1)^T$ (hole-like). At $r \neq 0$

$$\psi_+(r) = \frac{1}{\mathcal{N}} U G (+E_0, r) \tau_3 \psi_+(0)$$

$$= \frac{1}{\sqrt{X_0(E_0, r) + X_1(E_0, r)}^2 + |X_2^+(E_0, r)|^2} \begin{pmatrix} X_0(E_0, r) + X_1(E_0, r) \\ iX_2^+(E_0, r) \end{pmatrix} \equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}, \quad (S12)$$

where $\mathcal{N}$ is a normalization coefficient and, similarly,

$$\psi_-(r) = \frac{1}{\sqrt{-X_0(-E_0, r) + X_1(-E_0, r)}^2 + |X_2^+(E_0, r)|^2} \begin{pmatrix} -iX_2^+(E_0, r) \\ -X_0(-E_0, r) + X_1(-E_0, r) \end{pmatrix} \equiv \begin{pmatrix} -v^*(r) \\ u^*(r) \end{pmatrix}. \quad (S13)$$

Note that $X_0(E, r)$ is odd in $E$, while $X_1(E, r)$ and $X_2^+(E, r)$ are both even in $E$. From Eqs. (S7)–(S9) we see that $u(r)$ is real for the given $\psi_+(0)$ and $\psi_-(0)$, and we can write $\psi_+(r) = (u(r), v(r))^T = (u_r, e^{-i\theta_r + i\alpha} v_r)^T$, where $u_r$ and $v_r$ are two real functions of $r$ only, and $\alpha$ is an $r$-independent phase. For notational simplicity, we will set $\alpha = 0$, which will not qualitatively affect our conclusions.

2) The eigenvectors for $U < 0$ can be obtained similarly. At $r = 0$, $\psi_+(0) = (0, 1)^T$ (hole-like) and $\psi_-(0) = (1, 0)^T$ (particle-like). At $r \neq 0$,

$$\psi_+(r) = \frac{1}{\sqrt{-X_0(E_0, r) + X_1(E_0, r)}^2 + |X_2^+(E_0, r)|^2} \begin{pmatrix} -iX_2^+(E_0, r) \\ -X_0(E_0, r) + X_1(E_0, r) \end{pmatrix} \equiv \begin{pmatrix} -v^*(r) \\ u^*(r) \end{pmatrix}, \quad (S14)$$

$$\psi_-(r) = \frac{1}{\sqrt{X_0(-E_0, r) + X_1(-E_0, r)}^2 + |X_2^+(E_0, r)|^2} \begin{pmatrix} X_0(-E_0, r) + X_1(-E_0, r) \\ iX_2^+(E_0, r) \end{pmatrix} \equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}. \quad (S15)$$

Again, $u(r)$ is real, and we can write $\psi_-(r) = (u^r(r), v^r(r))^T = (u_r^e, e^{-i\theta_r + i\alpha} v_r)^T$, where $u_r^e$ and $v_r$ are real functions of $r$, and $\alpha$ is again a constant phase we will set to be zero without altering our conclusions.

In the main text and in the following discussions, we only consider the case with repulsive $U$. The attractive-$U$ scenario produces similar physics.

**II. LOW-ENERGY EFFECTIVE MODEL OF THE IMPURITY SUPERLATTICE**

In an impurity lattice, the bound states from different impurity sites hybridize through the kinetic hopping and Cooper pairing in the original microscopic BdG Hamiltonian, forming subgap bands. Treating the ‘+’ and ‘−’ bound states on each impurity
site as two independent orbitals, we now construct an effective tight-binding Hamiltonian for the subgap states on an impurity lattice. In the second-quantization formulation, the creation (annihilation) of the orbitals are denoted by the operators $c^\pm_i$ ($c^\pm_{\pm}$).

We first consider a two-impurity system with impurities located at $\mathbf{R}_i$ and $\mathbf{R}_j$. In the basis $\Psi_i = (c_{i+, c_{i-})}^T$ where $i$ is the site index, the emergent effective Hamiltonian reads $H = \sum_{ij} \hat{E}_0 \delta_{ij} \sigma_3 + \hat{h}_{ij} (1 - \delta_{ij}) \hat{\Psi}_j + \text{H.c.}$, in which

$$h_{ij} = \left(\begin{array}{cc} t_{ij}^{++} & t_{ij}^{+-} \\ t_{ij}^{-+} & t_{ij}^{--} \end{array}\right),$$

where

$$t_{ij}^{\mu\nu} = \int d\mathbf{r} d\mathbf{r}' \psi^\dagger_{\mu} (\mathbf{r} - \mathbf{R}_i) H_{\text{bulk}}^{(\text{bulk})} (\mathbf{r}, \mathbf{r}') \psi_{\nu} (\mathbf{r}' - \mathbf{R}_j),$$

and $\mu, \nu = +, -$. Explicitly,

$$t_{ij}^{++} = \int d\mathbf{r} d\mathbf{r}' \left\{ u_{1 \mathbf{r} - \mathbf{R}_i} \left[ \delta_{\mathbf{r}, \mathbf{r}'} \left( -\frac{\nabla^2_{\mathbf{r}'}}{2m_e} - \mu \right) \right] u_{1 \mathbf{r}' - \mathbf{R}_j} + u_{1 \mathbf{r} - \mathbf{R}_i} \Delta (\mathbf{r} - \mathbf{r}') e^{-i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r}' - \mathbf{R}_j} \right\},$$

$$t_{ij}^{+-} = \int d\mathbf{r} d\mathbf{r}' \left\{ -u_{1 \mathbf{r} - \mathbf{R}_i} \left[ \delta_{\mathbf{r}, \mathbf{r}'} \left( -\frac{\nabla^2_{\mathbf{r}'}}{2m_e} - \mu \right) \right] e^{i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r}' - \mathbf{R}_j} + u_{1 \mathbf{r} - \mathbf{R}_i} \Delta (\mathbf{r} - \mathbf{r}') u_{1 \mathbf{r}' - \mathbf{R}_j} \right\},$$

$$t_{ij}^{-+} = \int d\mathbf{r} d\mathbf{r}' \left\{ -e^{-i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r} - \mathbf{R}_i} \left[ \delta_{\mathbf{r}, \mathbf{r}'} \left( -\frac{\nabla^2_{\mathbf{r}'}}{2m_e} - \mu \right) \right] e^{i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r}' - \mathbf{R}_j} - e^{-i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r} - \mathbf{R}_i} \Delta (\mathbf{r} - \mathbf{r}') u_{1 \mathbf{r}' - \mathbf{R}_j} \right\},$$

$$t_{ij}^{--} = \int d\mathbf{r} d\mathbf{r}' \left\{ e^{i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r} - \mathbf{R}_i} \left[ \delta_{\mathbf{r}, \mathbf{r}'} \left( -\frac{\nabla^2_{\mathbf{r}'}}{2m_e} - \mu \right) \right] e^{-i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r}' - \mathbf{R}_j} - e^{-i\theta_{\mathbf{r}', \mathbf{r}}} u_{1 \mathbf{r} - \mathbf{R}_i} \Delta (\mathbf{r} - \mathbf{r}') u_{1 \mathbf{r}' - \mathbf{R}_j} \right\}.$$  

From these expressions, one can easily obtain the relations, $t_{ij}^{++} = - (t_{ij}^{--})^* \equiv \lambda_{ij}$, $t_{ij}^{+-} = (t_{ij}^{+-})^T \equiv \eta_{ij}$. The hybridization has three distinct origins: electron-electron hopping, hole-hole hopping and Cooper pairing. Hence we decompose the hopping terms as $\lambda_{ij} = \lambda^e_{ij} + \lambda^h_{ij} + \lambda^c_{ij}$, and $\eta_{ij} = \eta^e_{ij} + \eta^h_{ij} + \eta^c_{ij}$, the details of which we provide below.

**Symmetry aspects of the hybridization matrix elements**

By changing the variables, one can easily find that $\lambda_{ij}$ and $\eta_{ij}$ depend on the relative position of $\mathbf{R}_i$ and $\mathbf{R}_j$, i.e., $\lambda_{ij} \equiv \lambda (\mathbf{R}_i - \mathbf{R}_j)$, $\eta_{ij} \equiv \eta (\mathbf{R}_i - \mathbf{R}_j)$. Define $\mathbf{R}_\delta = \mathbf{R}_j - \mathbf{R}_i$, the expressions for $\lambda (\mathbf{R}_\delta)$ and $\eta (\mathbf{R}_\delta)$ can be reduced as

$$\lambda (\mathbf{R}_\delta) = \lambda^e (\mathbf{R}_\delta) + \lambda^h (\mathbf{R}_\delta) + \lambda^A (\mathbf{R}_\delta),$$

$$\eta (\mathbf{R}_\delta) = \eta^e (\mathbf{R}_\delta) + \eta^h (\mathbf{R}_\delta) + \eta^A (\mathbf{R}_\delta).$$
To inspect the dependence of \( \lambda (R_\delta) \) and \( \eta (R_\delta) \) on the orientation of \( R_\delta \), let us perform a rotation \( (\hat{R}) \) of arbitrary angle \( \phi \). Then,

\[
\begin{align*}
\lambda(\hat{R}_\delta R_\delta) &= \int d\mathbf{r} d\mathbf{r}' \left\{ u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} - \hat{R}_\delta R_\delta \left( -\frac{\nabla_r^2}{2m_\varepsilon} - \mu \right) \right] u_{r'} + e^{i\theta_r \mathbf{v}_r} \delta_{\mathbf{r},\mathbf{r}'} R_\delta, r \left( \frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) e^{-i\theta_{r'} \mathbf{v}_{r'}} \right. \\
&\left. + 2 \text{Re} \left[ \Delta (\mathbf{r} - \mathbf{r}') - \hat{R}_\delta R_\delta \right] e^{-i\theta_{r'} \mathbf{v}_{r'}} \right\} u_r u_{r'} \\
= &\int d\mathbf{r} d\mathbf{r}' \left\{ u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} \hat{R}_\delta R_\delta \left( -\frac{\nabla_r^2}{2m_\varepsilon} - \mu \right) \right] u_{r'} + e^{i\theta_r \mathbf{v}_r} \delta_{\mathbf{r},\mathbf{r}'} R_\delta, r \left( \frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) \right. \\
&\left. \times e^{-i\theta_{r'} \mathbf{v}_{r'}} u_{r'} + 2 \text{Re} \left[ \Delta (\hat{R}_\delta (\mathbf{r} - \mathbf{r}') - R_\delta \mathbf{r}) \right] e^{-i\theta_{r'} \mathbf{v}_{r'}} \right\} u_r u_{r'} \\
= &\int d\mathbf{r} d\mathbf{r}' \left\{ u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} \hat{R}_\delta (\mathbf{r} - \mathbf{r}') - R_\delta \mathbf{r} \right] u_{r'} + e^{i\theta_r \mathbf{v}_r} \delta_{\mathbf{r},\mathbf{r}'} R_\delta, r \left( \frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) \right. \\
&\left. \times e^{-i\theta_{r'} \mathbf{v}_{r'}} u_{r'} + 2 \text{Re} \left[ \Delta (\mathbf{r} - \mathbf{r}' + R_\delta) e^{-i\theta_{r'} \mathbf{v}_{r'}} \right] u_r u_{r'} \right\} \\
= &\lambda(R_\delta), \quad (S24)
\end{align*}
\]

which is independent of the orientation of \( R_\delta \), i.e., \( \lambda (R_\delta) = \lambda (|R_\delta|) \). And

\[
\begin{align*}
\eta(\hat{R}_\delta R_\delta) &= \int d\mathbf{r} d\mathbf{r}' \left\{ \Delta (\hat{R}_\delta (\mathbf{r} - \mathbf{r}') - R_\delta) u_r u_{r'} - e^{i\theta_{r} \mathbf{v}_r} \Delta^* (\hat{R}_\delta (\mathbf{r} - \mathbf{r}') - R_\delta) u_{r'} u_r \right. \\
&\left. \times (1 + e^{i\pi}) \int d\mathbf{r} d\mathbf{r}' u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} \hat{R}_\delta R_\delta \left( -\frac{\nabla_r^2}{2m_\varepsilon} - \mu \right) \right] e^{i\theta_{r'} \mathbf{v}_{r'}} \right\} u_r u_{r'} \\
= &\int d\mathbf{r} d\mathbf{r}' \left\{ \Delta (\hat{R}_\delta (\mathbf{r} - \mathbf{r}') - R_\delta) u_r u_{r'} - e^{i\theta_{r} \mathbf{v}_r} \Delta^* (\hat{R}_\delta (\mathbf{r} - \mathbf{r}') - R_\delta) u_{r'} u_r \right. \\
&\left. \times (1 + e^{i\pi}) \int d\mathbf{r} d\mathbf{r}' u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} \hat{R}_\delta R_\delta \left( -\frac{\nabla_r^2}{2m_\varepsilon} - \mu \right) \right] e^{i\theta_{r'} \mathbf{v}_{r'}} \right\} u_r u_{r'} \\
= &\int d\mathbf{r} d\mathbf{r}' \left\{ e^{i\theta_{r} \mathbf{v}_r} \Delta^* (\mathbf{r} - \mathbf{r}' + R_\delta) u_r u_{r'} - \hat{R}_\delta R_\delta \right. \\
&\left. \times (1 + e^{i\pi}) \int d\mathbf{r} d\mathbf{r}' u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} \hat{R}_\delta R_\delta \left( -\frac{\nabla_r^2}{2m_\varepsilon} - \mu \right) \right] e^{i\theta_{r'} \mathbf{v}_{r'}} \right\} u_r u_{r'} \right\} \\
= &\eta(R_\delta). \quad (S25)
\end{align*}
\]

Thus the off-diagonal matrix element \( t^+ \) inherits the rotational symmetry property of the chiral pairing in the original bulk BdG Hamiltonian.

Furthermore, in the hybridization between the ‘+’ and ‘−’ states, the contribution from the electron and hole kinetic processes, \( \eta_{ij}^e + \eta_{ij}^h \), are sensitive to the parity of the Cooper pairing: \( \eta_{ij}^e + \eta_{ij}^h \) vanishes in odd-parity pairing and is finite in even-parity pairing. This is more obvious in the following expression,

\[
\begin{align*}
\eta_{ij}^e + \eta_{ij}^h &= \int d\mathbf{r} d\mathbf{r}' \left\{ -u_{|R_\delta|} \left[ \delta_{\mathbf{r},\mathbf{r}'} \left( -\frac{\nabla_r^2}{2m_\varepsilon} - \mu \right) \right] e^{i\theta_{r'} - R_j} u_{|R_\delta|} + e^{i\theta_{r} - R_j} u_{|R_\delta|} \left[ \delta_{\mathbf{r},\mathbf{r}'} \left( -\frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) \right] u_{|R_\delta|} \right\} \\
&= \int d\mathbf{r} d\mathbf{r}' \left\{ u_{|R_\delta|} \left[ \delta_{\mathbf{r},\mathbf{r}'} \left( -\frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) \right] e^{i\theta_{r'} - R_j} u_{|R_\delta|} + e^{i\theta_{r'} + R_j} u_{|R_\delta|} \left[ \delta_{\mathbf{r},\mathbf{r}'} \left( -\frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) \right] u_{-|R_\delta|} \right\} \\
&\quad = (1 + e^{i\pi}) \int d\mathbf{r} d\mathbf{r}' u_r \left[ \delta_{\mathbf{r},\mathbf{r}'} + R_j - R_i \left( -\frac{\nabla_r^2}{2m_\varepsilon} + \mu \right) \right] e^{i\theta_{r'} \mathbf{v}_{r'}}, \quad (S26)
\end{align*}
\]

which vanishes for odd \( l \)'s. To obtain the second equation, we made a substitution of variables, \( \mathbf{r} \rightarrow \mathbf{R}_\delta - (\mathbf{r}' - \mathbf{R}_j) \) and \( \mathbf{r}' \rightarrow \mathbf{R}_j - (\mathbf{r} - \mathbf{R}_i) \). Pictorially, the two terms in the integrand of the second line are depicted in Fig. \[S1\]. The final expression was obtained after a partial integration and a substitution of variable.

III. EFFECTIVE TIGHT-BINDING HAMILTONIAN & ANOMALOUS HALL CONDUCTIVITY

We are now in position to formally construct the effective tight-binding Hamiltonian for square, triangular and honeycomb superlattices, and study their anomalous Hall effects.
Figure S1. Schematic diagram showing the relation between the integrand of Eq. (4) at two sets of variables: \((r, r')\) indicated by solid arrows and \([R_i - (r' - R_i), R_j - (r - R_j)]\) in dashed arrows. These two sets are related by a 180° rotation about \((R_1 + R_2)/2\).

A). Square impurity superlattice

Let us first consider the case with underlying chiral \(p\)-wave pairing. Following Fig. 2(a) and (b) and by Fourier transformation, in square superlattice, the hybridization matrix in the momentum space can be expressed as 
\[
\hat{h}_k = \sum_\delta e^{i k R_\delta} \hat{h}(R_\delta),
\]
in which the matrix elements with only considering the nearest-neighbor terms are expressed as
\[
\lambda_k = 2\lambda (\cos k_x + \cos k_y), \quad \eta_k = 2\eta (\sin k_x + i \sin k_y), \quad (S27)
\]
where \(\lambda \equiv \lambda(R_0), \eta \equiv \eta(R_0\hat{y})\) are real constants. The decomposition \(\lambda = \lambda^e + \lambda^h\) and \(\eta = \eta^e + \eta^h\) are implicit. Note that, for brevity, we have suppressed \(R_0\) in \(\lambda_k, \eta_k\) and hereafter. Then, the effective Hamiltonian for the impurity superlattice follows as,
\[
H_{\text{eff}}^k = \mathcal{E}_{3k} \sigma_3 + \mathcal{E}_{1k} \sigma_1 - \mathcal{E}_{2k} \sigma_2, \quad (S28)
\]
in which
\[
\mathcal{E}_{3k} = E_0 + 2\lambda (\cos k_x + \cos k_y), \quad \mathcal{E}_{1k} = 2\eta \sin k_x, \quad \mathcal{E}_{2k} = 2\eta \sin k_y, \quad (S29)
\]
This effective Hamiltonian resembles the original chiral \(p\)-wave model. As a side remark, the idea to design topological band structure through super-modulations of the order parameter is not new. Besides the present model which also appeared in Ref. [4], a superlattice of magnetic impurities in a conventional superconductor with Rashba spin-orbit coupling has also been shown to support subgap bands with high Chern numbers [5]. In another context, a pair-density-wave of a chiral \(p\)-wave order parameter was shown to generate topologically protected low-energy excitations [6].

Similarly, following the argument given in the main text as well as Fig. 2(c) and (d), the matrix elements for effective current operators along the \(x\)- and \(y\)-directions take the following forms:
\[
J_{xk}^{++} = -2 (\lambda^e - \lambda^h) \sin k_x, \quad J_{xk}^{--} = 2(\eta^e - \eta^h) \cos k_x = 4\eta^e \cos k_x, \quad (S30)
\]
\[
J_{yk}^{++} = -2 (\lambda^e - \lambda^h) \sin k_y, \quad J_{yk}^{--} = 2(\eta^e - \eta^h) \cos k_y = 4\eta^e \cos k_y, \quad (S31)
\]
Hence the \(i\)-th component of the current operators in terms of the Pauli matrices can be written as:
\[
\hat{J}_{ik}^{\text{eff}} = \mathcal{J}_{3ik} \sigma_3 + \mathcal{J}_{1ik} \sigma_1 + \mathcal{J}_{2ik} \sigma_2, \quad (S32)
\]
in which
\[
\mathcal{J}_{3xk} = -2 (\lambda^e - \lambda^h) \sin k_x, \quad \mathcal{J}_{1xk} = 4\eta^e \cos k_x, \quad \mathcal{J}_{2xk} = 0, \quad (S33)
\]
for \(x\)-direction, and
\[
\mathcal{J}_{3yk} = -2 (\lambda^e - \lambda^h) \sin k_y, \quad \mathcal{J}_{1yk} = 0, \quad \mathcal{J}_{2yk} = -4\eta^e \cos k_y, \quad (S34)
\]
for \(y\)-direction. Within linear-response theory, the transverse current-current correlation function at one-loop level is given by
\[
\pi_{xy}(q, i\omega_n) = T \sum_{k, i\omega_n} \text{Tr} \left[ \hat{J}_{2k}^{\text{eff}} \hat{G}(k + q, i\omega_n + i\nu_m) \hat{J}_{3k}^{\text{eff}} \hat{G}(k, i\omega_n) \right], \quad (S35)
\]
where $T$ is the temperature, $\omega_n = (2n+1)\pi T$ and $\nu_m = 2m\pi T$ are the fermionic and bosonic Matsubara frequencies, respectively. $\hat{G}\left(k, i\omega_n\right)$ is the single-particle Green’s function which can be written as

$$\hat{G}\left(k, i\omega_n\right) = \left(i\omega_n\sigma_0 - H_{\text{eff}}^\ast\right)^{-1} = \left(i\omega_n\sigma_0 + \hat{E}_{3k}\sigma_3 + \hat{E}_{1k}\sigma_1 + \hat{E}_{2k}\sigma_2\right) / \left(i\omega_n^2 - E_k^2\right),$$ (S36)

where $E_k = \sqrt{\hat{E}_{3k}^2 + \hat{E}_{1k}^2 + \hat{E}_{2k}^2}$ is the quasiparticle dispersion.

The Hall conductivity is given by the antisymmetric part of the transverse current correlator. After some algebra and an analytical continuation $i\nu_m \to \omega + i\delta$, we arrive at the following,

$$\sigma_{xy}(\omega + i\delta) = \frac{i}{2\omega} \lim_{\mathbf{q} \to 0} \left[ \pi_{xy}(\mathbf{q}, \omega + i\delta) - \pi_{yx}(\mathbf{q}, \omega + i\delta) \right] = \sum_{\mathbf{k}} \frac{f(\mathbf{k})}{E_k \left(\omega + i\delta\right)^2 - 4E_k^2},$$ (S37)

in which

$$f(\mathbf{k}) = \sum_{s,m,n} \frac{\epsilon_{mn}}{2} \left[ \hat{J}_{xk}\hat{J}_{myk} - \hat{J}_{yk}\hat{J}_{myk}\right] \epsilon_{nk}. $$ (S38)

Substituting the expressions, we see that a non-zero anomalous Hall conductivity emerges in the impurity superlattice embedded in a chiral p-wave superconductor.

We now turn to the case of underlying chiral $d$-wave pairing. We find that, a full description of low-energy model requires a consideration of up to the next-nearest neighboring terms shown in Fig. S2 after which we obtain,

$$\lambda_k = 2\lambda \left(\cos k_x + \cos k_y\right) + 4\tilde{\lambda} \cos k_x \cos k_y, \quad \eta_k = -2\eta \left(\cos k_x - \cos k_y\right) - 4\tilde{\eta} \sin k_x \sin k_y, $$ (S39)

in which $\tilde{\lambda} \equiv \lambda(\sqrt{2}R_0)$ and $\tilde{\eta} \equiv \eta(\sqrt{2}R_0)$ are hopping integrals associated with the next-nearest neighboring contributions. Written in the form of Eq. (S28), the corresponding $\hat{E}_{nk}$ are given by

$$\hat{E}_{3k} = E_0 + 2\lambda \left(\cos k_x + \cos k_y\right) + 4\tilde{\lambda} \cos k_x \cos k_y, \quad \hat{E}_{1k} = -2\eta \left(\cos k_x - \cos k_y\right), \quad \hat{E}_{2k} = 4\tilde{\eta} \sin k_x \sin k_y. $$ (S40)

Turning to the current operators, we have $\hat{J}_{xk}^+ = \hat{J}_{yk}^- = 0$ on account of the parity constraints ($\eta^c - \eta^h = 0$ for underlying even-parity pairing) discussed in the previous section. Thus $\hat{j}_{nk}^{\text{eff}} = \hat{J}_{3nk}\sigma_3 = -2\lambda \cos k_x \sigma_3$ with $i = x, y$. A straightforward calculation shows that the resultant model generate no anomalous Hall conductivity at the one-loop calculation.
B). Triangular impurity superlattice

In the case of triangular impurity superlattices, the anomalous Hall conductivity has the same form as in the case of a square superlattice, but with slight modifications. Consider only the nearest-neighbor hoppings, one obtains,

$$\mathcal{E}_{3k} = E_0 + 2\lambda \left( \cos k_x + 2 \cos \frac{k_x}{2} \cos \frac{\sqrt{3}k_y}{2} \right),$$  \hspace{1cm} (S41a)

$$\mathcal{E}_{1k} = \begin{cases} 2\eta \left( \sin k_x + \frac{1}{2} \cos \frac{\sqrt{3}k_y}{2} \right), & (l = 1) \\ 2\eta \left( \sin k_x - \cos \frac{\sqrt{3}k_y}{2} \right), & (l = 2) \end{cases},$$ \hspace{1cm} (S41b)

$$\mathcal{E}_{2k} = \begin{cases} 2\sqrt{3}\eta \cos \frac{k_x}{2} \sin \frac{\sqrt{3}k_y}{2}, & (l = 1) \\ 2\sqrt{3}\sin \frac{k_x}{2} \sin \frac{\sqrt{3}k_y}{2}, & (l = 2) \end{cases},$$ \hspace{1cm} (S41c)

and the associated components of the current operators are,

$$\mathcal{J}_{3xk} = -2 \left( \lambda^e - \lambda^h \right) \left( \sin k_x + \frac{1}{2} \cos \frac{\sqrt{3}k_y}{2} \right),$$ \hspace{1cm} (S42a)

$$\mathcal{J}_{1xk} = \begin{cases} 4\eta^e \left( \cos k_x + \frac{1}{2} \cos \frac{\sqrt{3}k_y}{2} \right), & (l = 1) \\ 0, & (l = 2) \end{cases},$$ \hspace{1cm} (S42b)

$$\mathcal{J}_{2xk} = \begin{cases} -6\eta^e \sin \frac{k_x}{2} \cos \frac{\sqrt{3}k_y}{2}, & (l = 1) \\ 0, & (l = 2) \end{cases},$$ \hspace{1cm} (S42c)

and

$$\mathcal{J}_{3yk} = -2\sqrt{3} \left( \lambda^e - \lambda^h \right) \frac{k_x}{2} \sin \frac{\sqrt{3}k_y}{2},$$ \hspace{1cm} (S43a)

$$\mathcal{J}_{1yk} = \begin{cases} -6\eta^e \sin \frac{k_x}{2} \sin \frac{\sqrt{3}k_y}{2}, & (l = 1) \\ 0, & (l = 2) \end{cases},$$ \hspace{1cm} (S43b)

$$\mathcal{J}_{2yk} = \begin{cases} 6\eta^e \cos \frac{k_x}{2} \cos \frac{\sqrt{3}k_y}{2}, & (l = 1) \\ 0, & (l = 2) \end{cases},$$ \hspace{1cm} (S43c)

It can thus be seen that the both the effective tight-binding Hamiltonian and the effective current operators follows the same overall structure as those in the square superlattice models. One thus expects the same outcome as far as the anomalous Hall effect is concerned.

C). Honeycomb impurity superlattice

The honeycomb impurity superlattice is very different from the previous two cases, since the enlargement of the Hilbert space due to an added sublattice degree of freedom. Consider a basis $$\Psi_i = (c_{i,+}, c'_{i,+}, c_{i,-}, c'_{i,-})^T$$, with $$c$$ and $$c'$$ representing the two sublattices, a general effective Hamiltonian for impurity superlattice with nearest-neighbor hoppings has the following form

$$\hat{H}_{eff}^k = \begin{bmatrix} E_0 & \lambda_k^e & 0 & \eta_k^e \\ \lambda_k^e & E_0 & (-1)^l \eta_k^{e,-} & 0 \\ 0 & (-1)^l \eta_k^{e,-} & -E_0 & -\lambda_k^e \\ \eta_k^e & 0 & -\lambda_k^e & -E_0 \end{bmatrix},$$ \hspace{1cm} (S44)

where the matrix elements are given by,

$$\lambda_k = \sum_\delta e^{ik\cdot R_\delta} \lambda \left( 1 + 2e^{-i\frac{\sqrt{3}k_y}{2}} \cos \frac{\sqrt{3}k_y}{2} \right),$$ \hspace{1cm} (S45)

$$\eta_k = \sum_\delta e^{ik\cdot R_\delta} e^{il\theta_{R_\delta}} \eta \left[ 1 + 2e^{-i\frac{\sqrt{3}k_y}{2}} \cos \left( \frac{\sqrt{3}k_y}{2} - \frac{2\pi}{3} \right) \right].$$ \hspace{1cm} (S46)
in which we have eliminated a prefactor $e^{ik_x}$ by a standard gauge transformation similar to the treatment for monolayer graphene. Note also that in the last expression, we have explicitly taken the example of $l = 2$ for underlying chiral d-wave, and the same below. The current operator follows as,

$$
\hat{j}_{xk} = \begin{pmatrix}
0 & J_{xk}^{++} & 0 & J_{xk}^{-+} \\
J_{xk}^{++} & 0 & 0 & J_{xk}^{+-} \\
0 & -J_{x-k}^{+-} & 0 & -J_{x-k}^{--} \\
J_{x-k}^{-+} & 0 & -J_{x-k}^{--} & 0
\end{pmatrix}
= \mathcal{J}_{1xk}\theta_1 \otimes \sigma_3 + \mathcal{J}_{2xk}\theta_2 \otimes \sigma_3 + \mathcal{J}_{3xk}\theta_1 \otimes \sigma_1 + \mathcal{J}_{4xk}\theta_1 \otimes \sigma_2 + \mathcal{J}_{5xk}\theta_2 \otimes \sigma_1 + \mathcal{J}_{6xk}\theta_2 \otimes \sigma_2,
$$

or, equivalently,

$$
\hat{j}_{yk} = \mathcal{J}_{1yk}\theta_1 \otimes \sigma_3 + \mathcal{J}_{2yk}\theta_2 \otimes \sigma_3 + \mathcal{J}_{3yk}\theta_1 \otimes \sigma_1 + \mathcal{J}_{4yk}\theta_1 \otimes \sigma_2 + \mathcal{J}_{5yk}\theta_2 \otimes \sigma_1 + \mathcal{J}_{6yk}\theta_2 \otimes \sigma_2,
$$

where

$$
J_{xk}^{++} = \sum_\delta e^{ik_R}\mathcal{J}_{xk}^{++}(\mathbf{R}_\delta) = -3 (\lambda^e - \lambda^h) \left( \sin \frac{3k_x}{2} \cos \frac{\sqrt{3}k_y}{2} + i \cos \frac{3k_x}{2} \cos \frac{\sqrt{3}k_y}{2} \right),
$$

$$
J_{yk}^{++} = \sum_\delta e^{ik_R}\mathcal{J}_{yk}^{++}(\mathbf{R}_\delta) = -\sqrt{3} (\lambda^e - \lambda^h) \left( \cos \frac{3k_x}{2} \sin \frac{\sqrt{3}k_y}{2} - i \sin \frac{3k_x}{2} \sin \frac{\sqrt{3}k_y}{2} \right),
$$

$$
J_{xk}^{+-} = \sum_\delta e^{i k_R} \mathcal{J}_{xk}^{+-}(\mathbf{R}_\delta) = -3 (\eta^e - \eta^h) \left( \sin \frac{3k_x}{2} + i \cos \frac{3k_x}{2} \cos \frac{\sqrt{3}k_y}{2} \right),
$$

$$
J_{yk}^{+-} = \sum_\delta e^{ik_R}\mathcal{J}_{yk}^{+-}(\mathbf{R}_\delta) = -\sqrt{3} (\eta^e - \eta^h) \left( \cos \frac{3k_x}{2} - i \sin \frac{3k_x}{2} \sin \frac{\sqrt{3}k_y}{2} \right),
$$

and $\mathcal{J}_{1ik} = \text{Re}(\mathcal{J}_{ik}^{++}), \mathcal{J}_{2ik} = -\text{Im}(\mathcal{J}_{ik}^{++}), \mathcal{J}_{3ik} = \text{Re}(\mathcal{J}_{ik}^{+--} - \mathcal{J}_{i-k}^{+-})/2, \mathcal{J}_{4ik} = -\text{Im}(\mathcal{J}_{ik}^{+--} - \mathcal{J}_{i-k}^{--})/2, \mathcal{J}_{5ik} = -\text{Im}(\mathcal{J}_{ik}^{++} + \mathcal{J}_{i-k}^{+-})/2, \mathcal{J}_{6ik} = -\text{Re}(\mathcal{J}_{ik}^{++} + \mathcal{J}_{i-k}^{--})/2$, with $i = x, y$.

The Green’s function $G(\mathbf{k}, i\omega_n) = (i\omega_n\sigma_0 - H^{\Gamma}_K)^{-1}$ acquires the following form,

$$
\hat{G}(\mathbf{k}, i\omega_n) = \sum_{i,j=0,1,2,3} g_{ij} \delta_i \otimes \sigma_j,
$$

where $g_{00} = -i\omega_n(\omega_n^2 + E_0^2 + |\lambda_k|^2 + |\eta_k|^2 - |\eta_{-k}|^2), g_{03} = -E_0(\omega_n^2 + E_0^2 - |\lambda_k|^2 + |\eta_{-k}|^2 - |\eta_k|^2), g_{33} = -i\omega_n(\omega_n^2 - |\eta_{-k}|^2 - \frac{1}{2} |\lambda_k|^2), g_{30} = -E_0(\omega_n^2 - |\lambda_k|^2), g_{11} = -\frac{1}{2} \text{Re}[\eta_{-k}(\omega_n^2 + E_0^2 + |\lambda_k|^2 + |\eta_k|^2 + \lambda_k^2 + \eta_k^2 + \lambda_k^2 + \eta_k^2)], g_{12} = -\frac{1}{2} \text{Im}[\eta_{-k}(\omega_n^2 + E_0^2 - |\eta_{-k}|^2 + |\lambda_k|^2 + \eta_k^2 + \lambda_k^2 + \eta_k^2 + \lambda_k^2 + \eta_k^2)], g_{22} = -\frac{1}{2} \text{Re}[\eta_{-k}(\omega_n^2 + E_0^2 + |\eta_{-k}|^2 - |\lambda_k|^2)], g_{21} = -\frac{1}{2} \text{Im}[\eta_{-k}(\omega_n^2 + E_0^2 + |\eta_{-k}|^2 - |\lambda_k|^2)], g_{23} = -\frac{1}{2} \text{Im}[\lambda_k(\omega_n^2 + E_0^2 + |\eta_{-k}|^2 - |\lambda_k|^2 - \eta_k^2)], g_{32} = -i\omega_n \text{Re}[\lambda_k^* \eta_{-k}^* - \lambda_k^* \eta_{-k}^*], g_{33} = i\omega_n \text{Im}[\lambda_k^* \eta_{-k} - \lambda_k \eta_{-k}^*], g_{01} = E_0 \text{Re}[\lambda_k^* \eta_{-k}^* + \lambda_k \eta_{-k}], g_{13} = E_0 \text{Im}[\lambda_k^* \eta_{-k}^* - \lambda_k \eta_{-k}], g_{02} = E_0 \text{Re}[\lambda_k \eta_{-k}^* + \lambda_k^* \eta_{-k}], g_{23} = E_0 \text{Im}[\lambda_k \eta_{-k}^* - \lambda_k^* \eta_{-k}], g_{32} = -E_0 \text{Im}[\lambda_k \eta_{-k}^* + \lambda_k^* \eta_{-k}],
$$

and

$$
E_{\pm,k} = \sqrt{E_0^2 + |\lambda_k|^2 + \frac{1}{2} (|\eta_{-k}|^2 + |\eta_k|^2)} \pm \frac{1}{4} \left( |\lambda_k|^2 E_0^2 + |\lambda_k|^2 |\eta_{-k}|^2 + \frac{1}{2} (|\eta_k|^2 - |\eta_{-k}|^2)^2 \right).
$$

We mainly focus on the case with chiral d-wave (even-parity) pairing in which $\mathcal{J}_{3i} = \mathcal{J}_{4i} = \mathcal{J}_{3j} = \mathcal{J}_{4j} = 0 (i = x, y)$, and study its anomalous Hall conductivity. A lengthy calculation leads to,

$$
\pi_{xy}(\mathbf{q}) = 0, i\nu_m
$$

$$
= \sum_k \frac{\nu_m}{E_+ - E_-} \left\{ \frac{f(E_+ - f(E_-)}{(E_{+k} - E_{-k})^2 + \nu_m^2} + \frac{1 - f(E_+ - f(E_-)}{(E_{+k} + E_{-k})^2 + \nu_m^2} \right\}
\times E_0 \left| \eta_k^2 - |\eta_{-k}|^2 \right|^2 \left[ \mathcal{J}_{12}(\mathbf{k}) \mathcal{J}_{25}(\mathbf{k}) - \mathcal{J}_{22}(\mathbf{k}) \mathcal{J}_{15}(\mathbf{k}) \right].
$$
It returns the following zero-temperature anomalous Hall conductivity at real frequency:

\[
\sigma_H(\omega + i\delta) = \sum_k \frac{E_0 \left( |\eta_k|^2 - |\eta_{-k}|^2 \right) \left[ \mathcal{J}_{1x}(k) \mathcal{J}_{2y}(k) - \mathcal{J}_{2x}(k) \mathcal{J}_{1y}(k) \right]}{2E_{+,k}E_{-,k} \left( E_{+,k} + E_{-,k} \right) \left[ (E_{+,k} + E_{-,k})^2 - (\omega + i\delta)^2 \right]}
\]

(S55)

This quantity is generically finite. Hence, distinct from cases of square and triangular superlattices, the anomalous Hall conductivity for chiral \(d\)-wave and other even-parity chiral states is finite. One can further check that the honeycomb superlattice models with underlying odd-parity chiral pairings also support finite Hall conductance.

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