ON SUMS AND PRODUCTS ALONG THE EDGES, II

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Abstract. This note is a continuation of an earlier paper of the authors [1]. We describe improved constructions addressing a question of Erdős and Szemerédi on sums and products of real numbers along the edges of a graph. We also add a few observations about related versions of the problem.

1. Introduction

In this note, we describe an improved construction addressing a question of Erdős and Szemerédi about sums and products along the edges of a graph. We also mention some related problems. The main improvement is obtained by a simple modification of the construction in [1] which works for real numbers, instead of the integers considered there.

In their original paper Erdős and Szemerédi [5] considered sum and product along the edges of graphs. Let $G_n$ be a graph on $n$ vertices, $v_1, v_2, \ldots, v_n$, with $n^{1+c}$ edges for some real $c > 0$. Let $A$ be an $n$-element set of real numbers, $A = \{a_1, a_2, \ldots, a_n\}$. The sumset of $A$ along $G_n$, denoted by $A + G_n A$, is the set $\{a_i + a_j | (i, j) \in E(G_n)\}$. The product set along $G_n$ is defined similarly, $A \cdot G_n A = \{a_i \cdot a_j | (i, j) \in E(G_n)\}$.

The Strong Erdős-Szemerédi Conjecture, which was refuted in [1], is the following.

Conjecture 1. [5] For every $c > 0$ and $\epsilon > 0$, there is a threshold, $n_0$, such that if $n \geq n_0$ then for any $n$-element subset of reals $A \subset \mathbb{R}$ and any graph $G_n$ with $n$ vertices and at least $n^{1+c}$ edges $|A + G_n A| + |A \cdot G_n A| \geq |A|^{1+c-\epsilon}$.

Now the question is to find dense graphs with small sumset and product set along the edges. Here we extend the construction in [1]. The improvement follows by considering real numbers, instead of integers only.

2. Constructions

2.1. Sum-product along edges with real numbers. Here we extend our earlier construction so that we get better bounds in a range of edge densities. In our previous paper for arbitrary large $m_0$, we constructed a set of integers, $A$, and a graph on $|A| = m \geq m_0$ vertices, $G_m$, with $\Omega(m^{2/3} / \log^{1/3} m)$ edges such that $|A + G_m A| + |A \cdot G_m A| = O\left((|A| \log |A|)^{4/3}\right)$.

Thus we had a graph on $m$ vertices and roughly $m^{2-c}$ edges with roughly $m^{2-2c}$ sums and products along the edges for $c = 1/3$. In the following construction, we show a similar bound in a range covering all $0 \leq c \leq 2/5$. In what follows, it is convenient to ignore the logarithmic terms. We thus use now the common notation $f = \tilde{O}(g)$ for two functions $f(n)$ and $g(n)$ to denote that there are absolute positive constants $c_1, c_2$ so that $f(n) \leq c_1 g(n)(\log g(n))^{c_2}$ for all admissible values of $n$. The notation $f = \tilde{\Omega}(g)$ means that $g = \tilde{O}(f)$ and $f = \tilde{\Theta}(g)$ denotes that $f = \tilde{\Omega}(g)$ and $g = \tilde{O}(f)$.

Research supported in part by NSF grant DMS-2154082.
Research supported in part by an OTKA NK 133819 grant.
Research supported in part by an NSERC and an OTKA NK 133819 grant.
Theorem 2. For arbitrary large $m_0$, and parameter $\alpha$, where $0 \leq \alpha \leq 1/5$, there is a set of reals, $A$, and a graph on $|A| = m \geq m_0$ vertices, $G_m$, with $$\widetilde{\Omega}(m^{2-2\alpha})$$ edges such that $$|A + G_m A| + |A \cdot G_m A| = \tilde{O}(|A|^{2-4\alpha}).$$

Proof: It is easier to describe the construction using prime numbers only. We get a slightly larger exponent in the hidden logarithmic factor, but we are anyway ignoring these factors here. The set of primes is denoted by $\mathbb{P}$ here. We define the set $A$ first and then the graph using the parameter $\alpha$.

$$A := \left\{ \frac{uw\sqrt{v}}{\sqrt{w}} \mid u, v, w \in \mathbb{P} \text{ distinct and } v, w \leq n^\alpha, u \leq n^{1-2\alpha} \right\}.$$ 

It is clear that distinct choices of 3-tuples $u, v, w$ lead to distinct reals. Thus with this choice of parameters, the size of $A$ is $\Theta(n)$. We are going to define a graph $G_m$ with vertex set $A$, where $|A| = m = \Theta(n)$. Two elements, $a, b \in A$ are connected by an edge if in the definition of $A$ above $a = \frac{uw\sqrt{v}}{\sqrt{w}}$ and $b = \frac{zw\sqrt{v}}{\sqrt{w}}$. Since the degree of every vertex here is $\tilde{O}(n^{1-2\alpha})$ the number of edges is $$\widetilde{\Omega}(m^{2-2\alpha}).$$

The products of pairs of elements of $A$ along an edge of $G_m$ are integers of size at most $$n^{2-4\alpha} = \tilde{O}(m^{2-4\alpha}).$$

The sums along the edges are of the form

$$\frac{uw\sqrt{v}}{\sqrt{w}} + \frac{zw\sqrt{v}}{\sqrt{w}} = \frac{wu + vz}{\sqrt{vw}}.$$ 

The number of possibilities for the denominator is at most $n^{2\alpha}$ and the numerator is a positive integer of size at most $2n^{1-\alpha}$, hence the number of sums is, at most $$O(n^{1+\alpha}) = \tilde{O}\left(m^{2-(1-\alpha)}\right).$$

The sum is asymptotically smaller than the product set, as long as $1 - \alpha > 4\alpha$, i.e. $\alpha < 1/5$.

Based on this construction, one can easily get examples of sparser graphs, simply taking smaller copies of $G_m$ and leaving other vertices isolated.

Theorem 3. For every parameters $0 \leq \nu \leq 3/5$ and $n_0$ there are $n > n_0$, an $n$-element set of reals, $A \subset \mathbb{R}$, and a graph $H_n$ with $\tilde{\Omega}(n^{1+\nu})$ edges such that $$|A + H_n A| + |A \cdot H_n A| = \tilde{O}\left(|A|^{3(1+\nu)/4}\right).$$

Proof: The construction of Theorem 2 with $\alpha = 1/5$ supplies a set of $m$ reals and a graph with $\tilde{\Omega}(m^{8/5})$ edges so that the number of sums and products along the edges is at most $\tilde{O}(m^{6/5})$. Take this construction with $m = n^{5(1+\nu)/8}(\leq n)$ and add to it $n - m$ isolated vertices assigning to them arbitrary distinct reals that differ from the ones used already.

A similar statement holds for integers too.

Theorem 4. For every parameters $0 \leq \nu \leq 2/3$ and $n_0$ there are $n > n_0$, an $n$-element set of integers $A$, and a graph $H_n$ with $\tilde{\Omega}(n^{1+\nu})$ edges such that $$|A + H_n A| + |A \cdot H_n A| = \tilde{O}\left(|A|^{4(1+\nu)/5}\right).$$

This follows as in the real case by starting with the construction of [H] that gives a set of $m$ integers and a graph with $\tilde{\Omega}(m^{5/3})$ edges so that the number of sums and products along the edges is at most $\tilde{O}(m^{4/3})$. This construction with $m = n^{3(1+\nu)/5} \leq n$ together with $n - m$ isolated vertices with arbitrary $n - m$ new integers implies the statement above.
2.2. Matchings. A particular variant of the sum-product problem for integers is the following:

**Problem 5.** Given two \( n \)-element sets of integers, \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \), let us define a sumset and a product set as
\[
S = \{a_i + b_i | 1 \leq i \leq n\} \quad \text{and} \quad P = \{a_i \cdot b_i | 1 \leq i \leq n\}.
\]
Erdős and Szemerédi conjectured that
\[
|P| + |S| = \Omega(n^{1/2+c})
\]
for some constant \( c > 0 \).

The best-known lower bound is due to Chang [4], who proved that
\[
|P| + |S| \geq n^{1/2} \log^{1/48} n.
\]
It was shown recently in [9] that under the assumption of a special case of the Bombieri-Lang conjecture [2], one can take \( c = 1/10 \) in equation (1), i.e. \( |P| + |S| = \Omega(n^{3/5}) \), even for multisets.

**Theorem 6.** [9] Let \( M = \{(a_i, b_i) | 1 \leq i \leq n\} \) be a set of distinct pairs of integers. If \( P \) and \( S \) are defined as above, then under the hypothesis of the Bombieri-Lang conjecture \( |P| + |S| = \Omega(n^{1/2+c}) \) with \( c = 1/10 \).

If multisets are allowed, and the only requirement is that the pairs assigned to distinct edges of the matching are distinct, then any construction of a graph with \( n \) edges yields a construction of a matching of size \( n \). It thus follows from [1, Theorem 3] (or from Theorem 4 here) that for the multiset version there is, for arbitrarily large \( n \), an example of a matching \( M \) of size \( n \) as above, with \( n \) distinct pairs of integers \( (a_i, b_i) \), so that \( |P| + |S| = O(n^{1/5}) \). This shows that the statement of Theorem 6 cannot be improved beyond an extra \( 1/5 \) in the exponent.

3. Lower bounds

In [1], we followed Elekes’ method using point-line incidence bounds to give a lower bound on the sum-product problem along the edges of a graph. For sparser graphs, Oliver Roche-Newton improved our bound, extending the range where a non-trivial bound can be established. He proved the following

**Theorem 7** (Theorem 6.1 in [6]). For arbitrary set of reals, \( A \), and a graph on \( |A| = m \) vertices, \( G_m \), with
\[
\tilde{\Omega}(m^{2-2\alpha})
\]
edges the following bound holds:
\[
|A + G_m A| + |A \cdot G_m A| = \tilde{\Omega}(|A|^{9^{-12\alpha}}).
\]

The result follows from applying an Elekes-Szabó type bound on the intersection size of polynomials and Cartesian products. Roche-Newton used the bound from [7], however, a better result follows from the recent improvement in [10].

**Theorem 8.** (Theorem 1.4 in [10]) Let \( f \in \mathbb{C}[x,y,z] \) be an irreducible polynomial. Then at least one of the following is true.

(A) For all finite sets \( A, B, C \subset \mathbb{R} \) with \( |A| \leq |B| \leq |C| \), we have
\[
|(A \times B \times C) \cap Z(f)| = \tilde{O}(|A||B||C|)^{4/7} + |B||C|^{1/2},
\]
where the implicit constant depends on the degree of \( f \).

(B) After possibly permuting the coordinates \( x, y, z \), we have \( f(x, y, z) = g(x, y) \), for some bivariate polynomial \( g \).

(C) \( f \) encodes additive group structure.\(^\dagger\)

\(^\dagger\)When \( f(x, y, z) \) is of the special form \( h(x, y) - z \), then \( f \) encodes additive structure if and only if \( h \) has the form \( h(x, y) = p(q(x) + r(y)) \) or \( h(x, y) = p(q(x)r(y)) \) for univariate polynomials \( p, q, r \).
Now we state a new lower bound on the size of the sumset and product set along the edges of a graph.

**Theorem 9.** For arbitrary set of reals, \( A \), and a graph on \(|A| = m\) vertices, \( G_m \), with \( m^{2-2\alpha} \)

edges the following bound holds:

\[
|A + G_m \cdot A| + |A \cdot G_m \cdot A| = \tilde{\Omega} \left( |A|^{\frac{5-7\alpha}{4}} \right).
\]

**Proof:** For the proof we can follow the arguments in [6] and use the new Elekes-Szabó type bound from Theorem 8. We consider the zero set of the polynomial

\[
f(x, y, z) = x(y - x) - z,
\]

and its intersection with the Cartesian product \( A \times \{A + G_m \cdot A\} \times \{A \cdot G_m \cdot A\} \). Every edge in \( G_m \) which connects vertices \( a \) and \( b \) determines an intersection point, by \( x = a, y = a + b \) and \( z = ab \).

This is the polynomial variant of Elekes’ original sum-product bound in [4] where he considered lines \( \alpha(X - \beta) - Y = 0 \) with \( \alpha, \beta \in A \) and \( X \in A + A, Y \in AA \). As it was shown in [6], for this polynomial Part A applies from Theorem 8. From that, we have the bound

\[
m^{2-2\alpha} = \tilde{O} \left( (|A||A + G_m \cdot A||A \cdot G_m \cdot A|)^{4/7} + |A + G_m \cdot A||A \cdot G_m \cdot A|^{1/2} \right)
\]

which implies

\[
|A + G_m \cdot A| + |A \cdot G_m \cdot A| = \tilde{\Omega} \left( |A|^{\frac{5-7\alpha}{4}} \right).
\]

\[\square\]

4. Remarks

There is still a gap between the lower bound and our construction. It is inevitable as long as the original sum-product conjecture is open. Our construction goes to the conjectured optimum as the graph is getting denser. The lower bound approaches Elekes’ bound [4].

![Figure 1](image.png)

**Figure 1.** The exponents in the upper and lower bounds when the number of edges is \( m^{2-2\alpha} \) (top line) and \( 0 < \alpha < 1/5 \)

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