QUANTUM PRINCIPAL BUNDLES
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Abstract: A noncommutative-geometric generalization of the theory of principal bundles is sketched. A differential calculus over corresponding quantum principal bundles is analysed. The formalism of connections is presented. In particular, operators of covariant derivative and horizontal projection are described and analysed. Quantum counterparts for the Bianchi identity and the Weil’s homomorphism are found. Illustrative examples are considered.

I INTRODUCTION

The purpose of this letter is to present the most important elements of a quantum generalization of the theory of principal bundles, in which quantum groups play the role of structure groups, and quantum spaces the role of base manifolds. All considerations are performed within the conceptual scheme of non-commutative differential geometry [C1,2]. A detailed exposition of the theory is given in papers [D1,2].

The paper is organized as follows. Section II begins with a definition of quantum principal bundles. Then, questions related to differential calculus are discussed. Section III is devoted to the formalism of connections. In Section IV a generalization of the Weil’s theory of characteristic classes is sketched. Finally, in Section V some examples of quantum principal bundles are considered.

Before passing to quantum principal bundles we shall fix the notation, and introduce the relevant quantum group entities. Here, we shall deal with compact matrix quantum groups [W2]. Let \( G \) be such a group. The algebra of ‘polynomial functions’ on \( G \) will be denoted by \( \mathcal{A} \). The group structure on \( G \) is determined by the comultiplication \( \phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \), the counit \( e: \mathcal{A} \rightarrow C \), and the antipode \( k: \mathcal{A} \rightarrow \mathcal{A} \). The result of the action of an \( n \)-fold comultiplication on elements \( a \in \mathcal{A} \) will be symbolically written as \( a^{(1)} \otimes \ldots \otimes a^{(n)} \). We shall denote by \( ad: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \) the adjoint action of \( G \) on itself. Explicitly, this action is given by

\[
ad(a) = a^{(2)} \otimes k(a^{(1)})a^{(3)}.
\]

Let \((\Gamma, d)\) be a first-order differential calculus [W3] over \( G \), and let

\[
\Gamma^\wedge = \bigoplus_{k \geq 0} \Gamma^\wedge_k
\]

be the universal differential envelope of \((\Gamma, d)\) [D1] (with \( \Gamma^\wedge_0 = \mathcal{A} \) and \( \Gamma^\wedge_1 = \Gamma \)). For each \( k \geq 0 \) let \( p_k: \Gamma^\wedge \rightarrow \Gamma^\wedge_k \) be the corresponding projection. Further, let

\[
\Gamma^\otimes = \bigoplus_{k \geq 0} \Gamma^\otimes_k
\]

be the tensor bundle algebra over \( \Gamma \) (\( \Gamma^\otimes_k = \Gamma \otimes \mathcal{A} \ldots \otimes \mathcal{A} \Gamma \) (\( k \)-times) and \( \Gamma^\otimes_0 = \mathcal{A} \)). Let us assume that \((\Gamma, d)\) is left-covariant. We shall denote by \( \Gamma_{\text{inv}} \) the space of left-invariant elements of \( \Gamma \) while \( \mathcal{R} \subseteq \text{ker}(e) \) will be the right \( \mathcal{A} \)-ideal which canonically, in the sense of [W3] corresponds to \((\Gamma, d)\). The map \( \pi: \mathcal{A} \rightarrow \Gamma_{\text{inv}} \) given by

\[
\pi(a) = k(a^{(1)})da^{(2)}
\]

is surjective, and \( \text{ker}(\pi) = C1 \oplus \mathcal{R} \). Because of this, there exists a natural isomorphism

\[
\Gamma_{\text{inv}} = \ker(e)/\mathcal{R}.
\]

The above isomorphism induces a right \( \mathcal{A} \)-module structure on \( \Gamma_{\text{inv}} \), which will be denoted by \( \circ \). Explicitly,

\[
\pi(a) \circ b = \pi(ab),
\]

for each \( a \in \ker(e) \) and \( b \in \mathcal{A} \). The tensor product of \( k \) copies of \( \Gamma_{\text{inv}} \) will be denoted by \( \Gamma^\otimes_{\text{inv}} \). The tensor algebra over \( \Gamma_{\text{inv}} \) will be denoted by \( \Gamma^\otimes_{\text{inv}} \). It is naturally isomorphic to the space of left-invariant elements of \( \Gamma^\otimes \). The subalgebra of left-invariant elements of \( \Gamma^\wedge \) will be denoted by \( \Gamma^\wedge_{\text{inv}} \). We have

\[
\Gamma^\wedge_{\text{inv}} = \bigoplus_{k \geq 0} \Gamma^\wedge_k_{\text{inv}}
\]

where \( \Gamma^\wedge_k_{\text{inv}} \) consists of left-invariant elements of \( k \)-th degrees. For each \( k \geq 0 \) let \( \Pi_k: \Gamma^\wedge_k \rightarrow \Gamma^\wedge_k_{\text{inv}} \) be the projection onto left-invariant elements (characterized by \( \Pi_k(a\vartheta) = e(a)\vartheta \) for each \( a \in \mathcal{A} \) and \( \vartheta \in \Gamma^\wedge_k_{\text{inv}} \)), and let \( \mathcal{P}_{\text{inv}}: \Gamma^\wedge_{\text{inv}} \rightarrow \Gamma^\wedge_{\text{inv}} \) be the standard projection map. The following natural isomorphism holds

\[
\Gamma^\wedge_{\text{inv}} = \Gamma^\otimes_{\text{inv}}/\mathcal{P}_{\text{inv}}.
\]
Here $I_{\text{inv}}^\wedge \subseteq \Gamma^\wedge_{\text{inv}}$ is the ideal generated by elements of the form $\pi(a^{(1)}) \otimes \pi(a^{(2)})$ where $a \in \mathcal{R}$. The right $\mathcal{A}$-module structure $\circ$ can be uniquely extended from $\Gamma_{\text{inv}}$ to $\Gamma^\wedge_{\text{inv}}$ such that
\[
1 \circ a = e(a)1 \quad (\vartheta \eta) \circ a = (\vartheta \circ a^{(1)}) (\eta \circ a^{(2)})
\]
for each $\vartheta, \eta \in \Gamma^\wedge_{\text{inv}}$ and $a \in \mathcal{A}$.

Let us now assume that the calculus $(\Gamma, d)$ is bicovariant, and let $\tilde{ad} : \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \mathcal{A}$ be the adjoint action of $G$ on $\Gamma_{\text{inv}}$ (coinciding with the restriction of the right action of $G$ on $\Gamma_{\text{inv}}$). We have
\[
\tilde{ad} \pi = (\pi \otimes \text{id}) \tilde{ad}.
\]
In the following, we shall denote by $\sim \otimes , \sim \otimes k , \sim^\wedge , \sim^\wedge k$ the adjoint actions of $G$ on the corresponding spaces.

The map $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ admits the unique extension to the homomorphism $\hat{\phi} : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$ of (graded) differential algebras.

II QUANTUM PRINCIPAL BUNDLES AND THE CORRESPONDING DIFFERENTIAL CALCULUS

The aim of this section is to introduce quantum principal bundles, and to describe differential calculus over them.

Let $M$ be a quantum space, represented by a $\ast$-algebra $\mathcal{V}$. The elements of $\mathcal{V}$ play the role of appropriate 'functions' on $M$.

**DEFINITION 2.1.** A quantum principal $G$-bundle over $M$ is a triplet of the form $P = (\mathcal{B}, i, F)$ where $\mathcal{B}$ is a $\ast$-algebra, while $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ and $i : \mathcal{V} \to \mathcal{B}$ are unital $\ast$-homomorphisms such that

(i) The following identities hold
\[
id = (id \otimes e)F \\
(id \otimes \phi)F = (F \otimes id)F.
\]

(ii) The map $i : \mathcal{V} \to \mathcal{B}$ is injective and $b \in i(\mathcal{V})$ iff $F(b) = b \otimes 1$, for each $b \in \mathcal{B}$.

(iii) A linear map $X : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ defined by
\[
X(a \otimes b) = aF(b)
\]
is surjective.

The map $F$ plays the role of the dualized right action of $G$ on $P$. Condition (i) justifies this interpretation. The map $i : \mathcal{V} \to \mathcal{B}$ can be interpreted as the dualized projection on $M$. Condition (ii) says that $M$ can be identified with the corresponding 'orbit space' of $P$. Finally, condition (iii) is an effective quantum counterpart of the classical requirement that $G$ acts freely on $P$.

Let $P = (\mathcal{B}, i, F)$ be a quantum principal $G$-bundle over $M$.

We are going to construct a graded differential algebra representing verticalized differential forms on $P$. Let us fix a bicovariant first-order differential $\ast$-calculus $(\Gamma, d)$ over $G$. The $\ast$-involution naturally extends from $\Gamma$ to $\Gamma^\wedge$ (such that $(\vartheta \eta)^* = (-1)^{\deg(\vartheta) \deg(\eta)} \eta^* \vartheta^*$ for each $\vartheta, \eta \in \Gamma^\wedge$). Algebras $\Gamma^\wedge_{\text{inv}} \subseteq \Gamma^\wedge$ are $\ast$-invariant.

Let us consider the (graded) vector space $\text{ver}(P) = \mathcal{B} \otimes \Gamma^\wedge_{\text{inv}}$.

**LEMMA 2.1.** The formulas
\[
(q \otimes \vartheta)(b \otimes \eta) = \sum_k q b_k \otimes (\vartheta \circ c_k) \eta \\
(b \otimes \eta)^* = \sum_k b^*_k \otimes (\eta^* \circ c_k^*) \\
d_v(b \otimes \eta) = b \otimes d \eta + \sum_k b_k \otimes \pi(c_k) \eta
\]
where $F(b) = \sum_k b_k \otimes c_k$ determine the structure of a graded differential $\ast$-algebra on $\text{ver}(P)$. As a differential algebra, $\text{ver}(P)$ is generated by $\mathcal{B} = \text{ver}^0(P)$. □
We shall assume that a differential calculus over the bundle \( P \) is specified by a graded differential \(*\)-algebra \( \Omega(P) \) such that

(i) The differential algebra \( \Omega(P) \) is generated by \( \mathcal{B} = \Omega^0(P) \).

(ii) The map \( F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A} \) is extendable to a homomorphism

\[
\hat{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge
\]

of (graded) differential algebras.

The map \( \hat{F} \) is uniquely determined by the above conditions. We have

\[
(\hat{F} \otimes \text{id})\hat{F} = (\text{id} \otimes \hat{\phi})\hat{F}.
\]

The formula

\[
F^\wedge = (\text{id} \otimes p_0)\hat{F}
\]

defines the action \( F^\wedge : \Omega(P) \to \Omega(P) \otimes \mathcal{A} \) of \( G \) on differential forms (extending the action \( F \)). The map \( F^\wedge \) is a \(*\)-homomorphism and

\[
(id \otimes \eta)F^\wedge = \text{id}
\]

\[
(F^\wedge \otimes \text{id})F^\wedge = (id \otimes \phi)F^\wedge
\]

\[
F^\wedge d = (d \otimes \text{id})F^\wedge.
\]

Let us construct a quantum analog of the verticalising homomorphism. For each \( w \in \Omega^k(P) \) the element

\[
(id \otimes \Pi_{inv}^k)\hat{F}(w)
\]

belongs to \( \mathcal{B} \otimes \Gamma^\wedge_{inv} = \text{ver}^k(P) \). Hence, the formula \( \pi_v(w) = (id \otimes \Pi_{inv}^k)\hat{F}(w) \) defines a linear grade-preserving map

\[
\pi_v : \Omega(P) \to \text{ver}(P).
\]

**Lemma 2.2.** The map \( \pi_v \) is an epimorphism of graded differential \(*\)-algebras.\( \square \)

Now, horizontal forms will be defined. Intuitively speaking, they can be characterized as forms possessing trivial differential properties along vertical fibers.

**Definition 2.2.** The elements of the graded \(*\)-subalgebra \( \text{hor}(P) = \hat{F}^{-1}[\Omega(P) \otimes \mathcal{A}] \) of \( \Omega(P) \) are called horizontal forms.

Horizontal forms \( w \) satisfying \( F^\wedge(w) = w \otimes 1 \) are interpretable as differential forms on \( M \). They constitute a graded differential \(*\)-subalgebra \( \Omega(M) \subseteq \Omega(P) \), with \( \Omega^0(M) = i(V) \).

**III THE FORMALISM OF CONNECTIONS**

Before introducing connections in the game, we shall define (pseudo)tensorial forms.

Let \( \psi(P) \) be the space of linear maps \( f : \Gamma_{inv} \to \Omega(P) \) satisfying

\[
F^\wedge f = (f \otimes \text{id})\widetilde{ad}.
\]

This space is naturally graded. The elements of \( \psi^k(P) \) are imaginable as pseudotensorial \( k \)-forms on \( P \), with values from the 'lie algebra' of \( G \). Further, \( \psi(P) \) is closed with respect to compositions with \( d : \Omega(P) \to \Omega(P) \).

Let

\[
\tau(P) = \{ f \in \psi(P) : f(\Gamma_{inv}) \subseteq \text{hor}(P) \}
\]

be the graded subspace of \( \psi(P) \) consisting of tensorial forms.

The formula

\[
f^*(\vartheta) = f(\vartheta^*)^*
\]

determines a \(*\)-involution on \( \psi(P) \) (and \( \tau(P) \)).

**Definition 3.1.** A connection on \( P \) is a linear map \( \omega : \Gamma_{inv} \to \Omega^1(P) \) such that

\[
\hat{F}\omega(\vartheta) = (\omega \otimes \text{id})\widetilde{ad}(\vartheta) + 1 \otimes \vartheta
\]

\[
\omega(\vartheta^*) = \omega(\vartheta)^* ,
\]

for each \( \vartheta \in \Gamma_{inv} \).
Connections can be equivalently defined as hermitian pseudotensorial one-forms \( \omega \) satisfying
\[
\pi_v \omega(\vartheta) = 1 \otimes \vartheta
\]
for each \( \vartheta \in \Gamma_{\text{inv}} \).

**Lemma 3.1.** The bundle \( P \) admits at least one connection. □

Let \( \text{con}(P) \) be the set of all connections on \( P \). This is is a real affine subspace of \( \psi^1(P) \). The corresponding vector space consists of hermitian tensorial 1-forms.

Let us fix a linear map \( \delta : \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \) with the following properties
\[
\delta \ast = -(\ast \otimes \ast) \delta \\
(\delta \otimes \text{id}) \widetilde{\text{ad}} = \widetilde{\text{ad}} \otimes \delta \\
d\vartheta = \sum_k \vartheta^1_k \vartheta^2_k
\]
where \( \vartheta \in \Gamma_{\text{inv}} \) and \( \delta(\vartheta) = \sum_k \vartheta^1_k \otimes \vartheta^2_k \).

For given linear maps \( \varphi, \eta : \Gamma_{\text{inv}} \to \Omega(P) \) let us define new linear maps \( < \varphi, \eta >, [\varphi, \eta] : \Gamma_{\text{inv}} \to \Omega(P) \) by
\[
< \varphi, \eta > = m_\Omega(\varphi \otimes \eta) \\
[\varphi, \eta] = m_\Omega(\varphi \otimes \eta) c^T
\]
where \( c^T = (id \otimes \pi) \widetilde{\text{ad}} : \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \) and \( m_\Omega : \Omega(P) \otimes \Omega(P) \to \Omega(P) \) are the 'transposed commutator' [W3] and the multiplication map.

If \( \varphi \in \psi^i(P) \) and \( \eta \in \psi^j(P) \) then \( < \varphi, \eta >, [\varphi, \eta] \in \psi^{i+j}(P) \).

For each \( \omega \in \text{con}(P) \) let us consider a map
\[
R_\omega = d\omega - < \omega, \omega >.
\]

**Lemma 3.2.** We have \( R_\omega = \text{ad}(R_\omega) \).

In other words, \( R_\omega \) is a tensorial hermitian 2-form. □

**Definition 3.2.** The map \( R_\omega \) is called the curvature of \( \omega \).

It is worth noticing that \( R_\omega \) depends on the choice of \( \delta \). This dependence disappears if \( \omega \) satisfies the following multiplicativity property.

**Definition 3.3.** A connection \( \omega \) is called multiplicative iff
\[
\omega \pi(\alpha^{(1)}) \omega \pi(\alpha^{(2)}) = 0
\]
for each \( \alpha \in R \).

If \( \omega \) is multiplicative then it can be uniquely extended, by multiplicativity, to a unital \( (\ast, \ast) \) homomorphism \( \omega^\wedge : \Gamma_{\text{inv}}^\wedge \to \Omega(P) \). Another interesting class of connections consists of those having the following regularity property.

**Definition 3.4.** A connection \( \omega \) is called regular iff
\[
\omega \pi(\alpha^{(1)}) \omega \pi(\alpha^{(2)}) = 0
\]
for each \( \alpha \in R \).

Let \( \sigma : \Gamma_{\text{inv}}^\otimes \otimes \Gamma_{\text{inv}}^\otimes \) be the canonical flip-over operator [W3].
LEMMA 3.3. If $\omega \in \text{conr}(P)$ then
\[ m_\Omega(\omega \otimes \varphi) = (-1)^k m_\Omega(\varphi \otimes \omega)\sigma \]
for each $\varphi \in \tau^k(P)$. $\Box$

Now, we are going to introduce the operator of covariant derivative. This operator will be first defined on a restricted domain consisting of horizontal forms. After introducing the operator of horizontal projection, the domain of covariant derivative will be extended to the whole algebra $\Omega(P)$.

For each $\omega \in \text{con}(P)$ and $\varphi \in \text{hor}(P)$ let us define a new form
\[ D_\omega(\varphi) = d\varphi - (-1)^{\deg(\varphi)} \sum_k \varphi_k \omega(c_k), \]
where $F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$.

The form $D_\omega(\varphi)$ is horizontal, too.

DEFINITION 3.5. A linear map $D_\omega : \text{hor}(P) \to \text{hor}(P)$ is called the covariant derivative associated to $\omega$.

PROPOSITION 3.4.
(i) The map $D_\omega$ intertwines the action $(F^\wedge | \text{hor}(P)) : \text{hor}(P) \to \text{hor}(P) \otimes A$ with itself.
(ii) If $\omega$ is multiplicative then $D_\omega^2(\varphi) = -\sum_k \varphi_k R_\omega \pi(c_k)$,
for each $\varphi \in \text{hor}(P)$.
(iii) If $\omega$ is regular then $D_\omega(\varphi \psi) = D_\omega(\varphi) \psi + (-1)^{\deg(\varphi)} \varphi D_\omega(\psi)$
\[ D_\omega(\varphi^*) = D_\omega(\varphi)^* \]
for each $\varphi, \psi \in \text{hor}(P)$.
(iv) If $\varphi \in \Omega(M)$ then $D_\omega(\varphi) = d\varphi$. $\Box$

The space $\tau(P)$ is closed under taking compositions with $D_\omega$. This fact enables us to define the action of covariant derivative in the space of tensorial forms.

LEMMA 3.5. We have
\[ D_\omega(\varphi) = d\varphi - (-1)^{\deg(\varphi)} [\varphi, \omega] \]
for each $\varphi \in \tau(P)$. $\Box$

Let us consider a linear map $q_\omega : \psi(P) \to \psi(P)$ defined by
\[ q_\omega(\varphi) = <\omega, \varphi> - (-1)^{\deg(\varphi)} <\varphi, \omega> - (-1)^{\deg(\varphi)} [\varphi, \omega]. \]

We have then
\[ \hat{F}q_\omega(\varphi) = (q_\omega \otimes id) F^\wedge(\varphi) \]
for each $\varphi \in \tau(P)$. In particular, $q_\omega \tau(P) \subseteq \tau(P)$. Moreover, if $\omega \in \text{con}(P)$ then $(q_\omega | \tau(P)) = 0$.

The following lemma gives the quantum counterpart for the classical Bianchi identity.

LEMMA 3.6. We have
\[ (D_\omega - q_\omega)(R_\omega) = <\omega, <\omega, \omega>> - <\omega, \omega>, \omega > \]
for each $\omega \in \text{con}(P)$. $\Box$

If the connection $\omega$ is multiplicative, then the right hand side of the above equality vanishes. On the other hand, if $\omega$ is regular then the second summand in the left hand side vanishes. It is worth noticing that regular connections are not necessarily multiplicative. However, there exists a common obstruction to multiplicativity for all regular connections, so that they are multiplicative, or not, at the same time.
In general, the lack of multiplicativity of the connection \( \omega \) is measured by a map \( r_\omega : \mathcal{R} \to \Omega^2(P) \) given by \( r_\omega(a) = \omega \pi(a^{(1)}) \omega \pi(a^{(2)}) \).

**LEMMA 3.7.**

(i) The following identities hold

\[
  r_\omega(k(a)^*) = -r_\omega(a)^*
\]

\[
  \pi_\nu r_\omega = 0
\]

\[
  \hat{F} r_\omega(a) = (r_\omega \otimes id) ad(a).
\]

In particular, \( r_\omega(a) \) is horizontal for each \( a \in \mathcal{R} \).

(ii) The map \( \omega \mapsto r_\omega \) is constant on cosets from the space \( \text{con}(P)/\text{conr}(P) \). If \( \omega \in \text{conr}(P) \) then

\[
  r_\omega(a) \varphi = \sum_k \varphi_k r_\omega(ac_k)
\]

for each \( a \in \mathcal{R} \) and \( \varphi \in \text{hor}(P) \), where \( F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k \). Further,

\[
  dr_\omega(a) = \langle \omega, \omega > \pi(a^{(1)}) \omega \pi(a^{(2)}) - \omega \pi(a^{(1)}) < \omega, \omega > \pi(a^{(2)})\). \]

Let us assume that \( P \) admits regular connections, and let \( J(P) \) be the ideal in \( \Omega(P) \) generated by the space \( r_\omega(\mathcal{R}) \), for some \( \omega \in \text{conr}(P) \). The previous lemma implies

\[
  J(P)^* = J(P)
\]

\[
  \hat{F} J(P) \subseteq J(P) \otimes \Gamma^\wedge
\]

\[
  \pi_\nu J(P) = \{0\}
\]

\[
  dJ(P) \subseteq J(P).
\]

Consequently, it is possible to project the whole formalism on the factor algebra \( \Omega(P)/J(P) \). In the framework of this projected calculus regular connections become multiplicative.

The last topic in this section is the construction and the analysis of horizontal projection operators. Let us fix a splitting of the form

\[
  \Gamma^\wedge_{\text{inv}} = \Gamma^\wedge_{\text{inv}} \oplus I^\wedge_{\text{inv}}
\]

in which \( \Gamma^\wedge_{\text{inv}} \) is realized as a complement of the space \( \Gamma^\wedge_{\text{inv}} \), with the help of a grade-preserving hermitian section \( \iota : \Gamma^\wedge_{\text{inv}} \to \Gamma^\wedge_{\text{inr}} \), intertwining the adjoint actions. Further, let us assume that \( \delta(\vartheta) = id(\vartheta) \). Finally, let us consider a linear map \( m_\omega : \text{hor}(P) \otimes \Gamma^\wedge_{\text{inv}} \to \Omega(P) \) given by

\[
  m_\omega(\varphi \otimes \vartheta) = \varphi \omega^\wedge \iota(\vartheta).
\]

Here, \( \omega^\wedge : \Gamma^\wedge_{\text{inv}} \otimes \Omega(P) \) is the unital multiplicative extension of \( \omega \). It is worth noticing that in the special case of multiplicative connections the map \( m_\omega \) is \( \iota \)-independent, because \( \omega^\wedge \iota = \omega^\wedge \) in this case.

**PROPOSITION 3.8.**

(i) The map \( m_\omega \) is bijective. It intertwines the product of actions \( (F^\wedge \mid \text{hor}(P)) \) and \( \tilde{\iota}^\wedge \), with the action \( F^\wedge \).

(ii) If \( \omega \) is regular and if \( J(P) = \{0\} \) then \( m_\omega \) is an isomorphism of *-algebras. Here, it is assumed that \( \text{hor}(P) \otimes \Gamma^\wedge_{\text{inv}} \) is endowed with a (graded) *-algebra structure specified by

\[
  (\psi \otimes \eta)(\varphi \otimes \vartheta) = \sum_k (-1)^{\deg(\varphi) \deg(\eta)} \psi \varphi_k \otimes (\eta \circ c_k) \vartheta
\]

\[
  (\varphi \otimes \vartheta)^* = \sum_k \varphi_k^* \otimes \vartheta^* \circ c_k^*,
\]

where \( F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k \). \]

The horizontal projection operator \( h_\omega : \Omega(P) \to \text{hor}(P) \) can be now defined as follows

\[
  h_\omega = (id \otimes p^0_{\text{inv}}) m_\omega^{-1}
\]

Clearly, \( h_\omega \) projects \( \Omega(P) \) onto \( \text{hor}(P) \).
With the help of $h_\omega$, the domain of the covariant derivative can be extended to the whole algebra $\Omega(P)$. Indeed, the map $D_\omega : \Omega(P) \to \text{hor}(P)$ given by

$$D_\omega = h_\omega d$$

extends the previously defined covariant derivative.

**PROPOSITION 3.9.**

(i) The maps $h_\omega, D_\omega$ intertwine the actions $F^\wedge$ and $(F^\wedge \mid \text{hor}(P))$.

(ii) If $\omega \in \text{conr}(P)$ then $h_\omega$ is a $*$-homomorphism, and

$$D_\omega(ww') = D_\omega(w)h_\omega(w') + h_\omega(w)D_\omega(w')$$

$$D_\omega(w^*) = D_\omega(w)^*$$

for each $w, w' \in \Omega(P)$. □

Compositions of pseudotensorial forms with the covariant derivative are tensorial. Consequently, it is possible to define the covariant derivative $D_\omega : \psi(P) \to \tau(P)$. The following lemma gives an equivalent, more geometrical, description of the curvature.

**LEMMA 3.10.** We have

$$R_\omega = D_\omega(\omega)$$

for each $\omega \in \text{con}(P)$. □

**IV CHARACTERISTIC CLASSES**

In this section we shall sketch a quantum generalization of the Weil’s theory of characteristic classes. We shall assume that the bundle $P$ admits regular connections, and that $J(P) = \{0\}$. For each $k \geq 0$ let $\text{Inv}^k \subseteq \Gamma^\otimes k_{\text{inv}}$ be the subspace of $ad$-invariant elements, and let $\text{Inv}$ be the direct sum of all these spaces. Clearly, $\text{Inv}$ is a unital $*$-subalgebra of the tensor algebra $\Gamma^\otimes k_{\text{inv}}$. Let $H(M)$ be the graded $*$-algebra of cohomology classes associated to $\Omega(M)$.

Let us consider a connection $\omega$. There exists the unique unital homomorphism $R^\otimes_\omega : \Gamma^\otimes \text{inv} \to \Omega(P)$ extending the curvature $R_\omega$. The map $R^\otimes_\omega$ is $*$-preserving, and intertwines $ad^\otimes$ and $F^\wedge$.

**PROPOSITION 4.1.**

(i) If $\vartheta \in \text{Inv}^k$ then $R^\otimes_\omega(\vartheta) \in \Omega^{2k}(M)$.

(ii) If $\omega \in \text{conr}(P)$ then $dR^\otimes_\omega(\vartheta) = 0$ for each $\vartheta \in \text{Inv}$.

(iii) The cohomological class of $R^\otimes_\omega(\vartheta)$ in $\Omega(M)$ is independent on the choice of a regular connection $\omega$, for each $\vartheta \in \text{Inv}$.

(iv) The map $W : \text{Inv} \to H(M)$ given by $W(\vartheta) = [R^\otimes_\omega(\vartheta)]$ is a unital $*$-homomorphism. □

The homomorphism $W$ plays the role of the Weil’s homomorphism in classical differential geometry [KN]. In fact, in classical geometry the domain of the Weil’s homomorphism is restricted on the algebra of symmetric invariant elements of the corresponding tensor algebra. However, besides simplifying the domain of $W$, such a restriction gives nothing new: the image of the Weil’s homomorphism will be the same.

A similar situation holds in the noncommutative case. Let $\text{Sym}(\sigma)$ be the $*$-algebra obtained from $\Gamma^\otimes_{\text{inv}}$ by factorising through the ideal $I(\sigma)$ generated by $\text{Inv}(I - \sigma) \subseteq \Gamma^\otimes_{\text{inv}}$. The algebra $\text{Sym}(\sigma)$ plays the role of polynomials over the ‘lie algebra’ of $G$. The adjoint action $ad^\otimes$ is naturally projectable on $\text{Sym}(\sigma)$. Let $\text{Inv}(\sigma) \subseteq \text{Sym}(\sigma)$ be the subalgebra of elements invariant under the projected action (playing the role of invariant polynomials). Clearly, $\text{Inv}(\sigma) = \text{Inv}/(I(\sigma) \cap \text{Inv})$.

**PROPOSITION 4.2.** If $\omega \in \text{conr}(P)$ then

$$R^\otimes_\omega(\vartheta) = R^\otimes_{\omega}(\vartheta)$$

for each $\vartheta \in \Gamma^\otimes_{\text{inv}}$. □

The above statement implies that $W$ and $R^\otimes_\omega$ can be factorised through the ideal $I(\sigma)$.
V EXAMPLES AND REMARKS

(A) All quantum phenomena characteristic for the presented theory of quantum principal bundles already figure in a special version of this theory dealing with bundles over classical smooth manifolds. The theory of principal bundles of this kind is developed in [D1].

The main structural result is that $G$-bundles $P$ over a classical manifold $M$ are in a natural correspondence with classical bundles $P_{cl}$ over the same manifold, with the structure group $G_{cl}$ consisting of classical points of $G$. More precisely, the elements of $G_{cl}$ are *-characters $g : A \to C$. The product and the inverse in $G_{cl}$ are given by

$$gg' = (g \otimes g')\phi$$
$$g^{-1} = gk,$$

while the counit $e : A \to C$ is the neutral element. The correspondence $P \leftrightarrow P_{cl}$ can be roughly described as follows. The bundle $P_{cl}$ consists of classical points of $P$ (*-characters on $B$). Conversely, if $P_{cl}$ is given then $P$ can be recovered by applying an analog of the classical construction of extending structure groups.

In developing a differential calculus on such semiclassical bundles $P$ it is natural to assume that all local trivializations of the bundle locally trivialize the calculus, too. This requirement, together with the specification of the calculus $\Gamma^\circ$ over $G$, uniquely fixes the algebra $\Omega(P)$. However, the calculus $(\Gamma, d)$ can not be chosen arbitrarily. It must satisfy specific consistency requirements, interpretable as compatibility properties with certain retrivialization maps’ of the bundle. Such differential calculi are called ‘admissible’ in [D1]. It turns out that a left-covariant calculus $(\Gamma, d)$ is admissible iff $(X \otimes id)ad(R) = \{0\}$, for each $X \in lie(G_{cl})$. Here, the Lie algebra of $G_{cl}$ is understood as the space of hermitian functionals $X$ on $A$ satisfying $X(ab) = e(a)X(b) + e(b)X(a)$, for each $a, b \in A$.

There exists the minimal admissible left-covariant calculus: it is based on the right-ideal $R \subseteq ker(e)$ consisting of elements killed by all operators $(X \otimes id)ad$. This calculus is also *-covariant and right-covariant. If $G$ is an ordinary compact matrix group then the minimal admissible calculus coincides with the usual one (based on differential forms). However, small quantum deformations of the classical group structure may cause drastic changes at the level of the minimal admissible calculus. For example [D1], if $G = SU_\mu(2)$ [W1] and $\mu \in (-1, 1) \setminus \{0\}$ then the space $\Gamma_{inv}$ is infinite-dimensional, and can be naturally identified with the algebra of polynomial functions on the quantum 2-sphere $S^2_{\mu}$.

(B) Classical principal bundles provide a natural mathematical framework for the study of gauge theories. It is interesting to see what will be the counterparts of these theories, in the context of quantum principal bundles [D3] ($M$ playing the role of the space-time). Properties of such ‘quantum gauge’ theories essentially depend (besides on the ‘symmetry group’ $G$), on the following two prespecifications:

As first, it is necessary to fix a (bicovariant *) calculus $(\Gamma, d)$ over $G$. This determines kinematical degrees of freedom. Secondly, we have to choose a map $\delta : \Gamma_{inv} \to \Gamma_{inv}^{\otimes 2}$. This influences dynamical properties of the theory, because $\delta$ implicitly figures in the expression for the curvature.

Closely related with problematics of quantum gauge theories is the question of ‘gauge transformations’. If $M$ is a classical smooth manifold then the most direct way of defining gauge transformations as automorphisms of the bundle $P$ gives nothing new, because of the inherent geometrical inhomogenity of the bundle $P$. More precisely, automorphism groups of $P$ and its classical part $P_{cl}$ are isomorphic. However, a proper quantum generalization of gauge transformations can be introduced via the concepts of quantum (infinitesimal) gauge bundles [D2,3]. These are bundles associated to $P$, relative to the adjoint actions of $G$ on $G$ and $\Gamma_{inv}$ respectively.

(C) Interesting examples of quantum principal bundles can be obtained from quantum homogeneous spaces. A general construction is this [D2]. Let $G'$ be a compact matrix quantum group. Entities related to $G'$ will be endowed with a prime. Let us assume that $G$ is a subgroup of $G'$. At the formal level, this presumes a specification of an *-epimorphism $q : A' \to A$ such that

$$(q \otimes q)\phi' = \phi q$$
$$kq = q k'.$$

The *-homomorphism $F : A' \to A' \otimes A$ given by

$$F = (id \otimes q)\phi'$$

is interpretable as the right action of $G$ on $G'$. Let $M$ be the corresponding ‘orbit space’. This space is represented by the fixed point *-subalgebra $\mathcal{V}$. Let $i : \mathcal{V} \hookrightarrow A'$ be the inclusion map. The triplet $P = (A', i, F)$ is a quantum principal $G$-bundle over $M$. Because of $\phi'(V) \subseteq A' \otimes V$ there exists a natural left action of $G'$ on $M$, represented by $\phi' i : \mathcal{V} \to A' \otimes \mathcal{V}$ ($M$ is a quantum homogeneous $G'$-space).
Let \((\Gamma', d')\) be a bicovariant first-order \(*\)-calculus over \(G'\), and \(\mathcal{R}' \subseteq \ker(e')\) the corresponding right \(\mathcal{A}'\)-ideal. Let us assume that

\[
q(\mathcal{R}') \subseteq \mathcal{R},
\]

where \(\mathcal{R} \subseteq \ker(e)\) is the right \(\mathcal{A}\)-ideal determining the calculus \((\Gamma, d)\) over \(G\). Then the map \(q : \mathcal{A}' \to \mathcal{A}\) can be (uniquely) extended to the \((\ast\)\)-homomorphism \(\hat{q} : \Gamma'^{\wedge} \to \Gamma^{\wedge}\) of differential algebras.

Let \(\Omega(P)\) be an arbitrary graded differential \(*\)-algebra built over \((\Gamma', d')\), satisfying (ii) of Section II. The differential algebra \(\Gamma'^{\wedge}\) possesses this property. In this particular case \(\hat{F} = (id \otimes \hat{q})\hat{d}'\).

Let us assume that a linear map \(\epsilon : \Gamma_{inv} \to \Gamma'_{inv}\) is given, such that the following identities are satisfied:

\[
(id \otimes q)\tilde{a}\epsilon = (\epsilon \otimes id)\tilde{a}
\]

\[
\hat{q}\epsilon(\vartheta) = \vartheta.
\]

Then the map \(\omega : \Gamma_{inv} \to \Omega^{1}(P)\), obtained by composing \(\epsilon\) with the canonical inclusion \(\Gamma_{inv}^{\wedge} \hookrightarrow \Omega^{1}(P)\), is a connection on \(P\). Further, if

\[
\epsilon(\vartheta \circ q(a)) = \epsilon(\vartheta) \circ a
\]

then \(\epsilon \otimes \epsilon\) intertwines cannonical flip-over operators. Finally, if in addition

\[
d'\epsilon(\vartheta) = 0
\]

then the commutation property

\[
\omega(\vartheta)\varphi = (-1)^{\deg(\varphi)} \sum_{k} \varphi_{k}\omega(\vartheta \circ c_{k})
\]

holds for each \(\varphi \in \Omega(P)\). In particular, \(\omega\) is regular.

As a concrete illustration, let us briefly consider the case \(G' = SU_{\mu}(2) (\mu \in (-1, 1) \setminus \{0\})\) and \(G = G_{SU} = U(1)\). This gives the quantum Hopf fibering \(SU_{\mu}(2) \to S^{2}_{\mu}\). Let us assume that the calculus on \(G\) is the standard one, based on differential forms. A bicovariant calculus \((\Gamma', d')\) over \(G'\) satisfies \(q(\mathcal{R}') \subseteq \mathcal{R}\) if it is admissible, in the sense mentioned in (A). Let us assume that \((\Gamma', d')\) is a minimal admissible calculus, and let \(\Gamma'_{inv}\) be identified with the polynomial algebra over \(S^{2}_{\mu}\). Let \(\tau \in \Gamma'_{inv}\) be the element corresponding to the unit function on \(S^{2}_{\mu}\) and let us define

\[
\Omega(P) = \Gamma'^{\wedge}/\{d'\tau = 0\}.
\]

The space \(\Gamma_{inv}\) is 1-dimensional and generated by \(\hat{q}(\tau)\). A map \(\epsilon\) can be defined by \(\epsilon\hat{q}(\tau) = \tau\). It is worth noticing that the curvature of the corresponding connection \(\omega\) vanishes.

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