Study of superradiance in the Lambert-W potential barrier

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We present a new potential barrier that presents the phenomenon of superradiance when the reflection coefficient $R$ is greater than one. We calculated the transmission and reflection coefficients for three different regions. The results are compared with those obtained for the hyperbolic tangent potential barrier and the step potential barrier. We also present the solution of the Klein-Gordon equation with the Lambert-W potential barrier in terms of the Heun Confluent functions.

Keywords: Heun Confluent function, Lambert-W potential, scattering solutions.

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1. Introduction

The Klein-Gordon equation is used to describe spin-0 particles. The scattering of a spin-0 particle by a one-dimensional potential barrier is a typical problem that appears in relativistic quantum mechanics and has been the subject of much interest in recent years.\textsuperscript{1–10} Consider an incoming wave from left to right; the common situation is that the wave loses energy because of its interaction with the potential barrier; therefore, the incoming amplitude is greater than the amplitude of the reflected wave.\textsuperscript{11} The amplitude of the incident wave, $T$, is called the transmission coefficient, and the amplitude of the reflected wave, $R$, is called the reflection coefficient. Another phenomenon, called superradiance,\textsuperscript{12} where the energy is extracted from the barrier, also appears in relativistic quantum mechanics when the Klein-Gordon equation is applied for an abrupt or smooth potential barrier. In this case, the amplitude of the reflected wave is larger than the amplitude of the incoming wave, which means that more particles are reflected than those that are incident on the potential barrier, or there is a particle–antiparticle pair creation. The superradiance phenomenon has been widely discussed in the literature for the Dirac equation\textsuperscript{12}\textsuperscript{15} and for the Klein-Gordon equation.\textsuperscript{5}\textsuperscript{16} This phenomenon also appears in astrophysics in the scattering of scalar waves by rotating black holes.\textsuperscript{11,18}
In this article, we study the phenomenon of superradiance, when the reflection coefficient $R$ is greater than one, for the Lambert-W potential barrier. The Lambert-W potential barrier is an asymmetric potential barrier that is solvable for the Schrödinger equation for bound states\textsuperscript{19} and scattering states\textsuperscript{20} The behavior of the reflection $R$ and transmission $T$ coefficients is studied for three different regions of energy: $m < E < V_0 - m$, $V_0 - m < E < V_0 + m$, and $E > V_0 + m$. We have observed that for the region $m < E < V_0 - m$, $R > 1$ and $T < 0$; therefore, the phenomenon of superradiance also is observed in the Lambert-W potential barrier. In addition, we compare the reflection $R$ and transmission $T$ coefficients for the Lambert-W potential barrier with those for the hyperbolic tangent potential barrier and the step potential barrier.

The remainder of this paper is organized as follows. In section 2, we present the Lambert-W potential barrier, the hyperbolic tangent potential barrier, and the step potential barrier. In Section 3, we find the solution of the Klein-Gordon equation with each of the potential barriers, as well as the reflection $R$ and transmission $T$ coefficients for each case. In section 4, we discuss our results; finally, in section 5, our conclusions are presented.

2. Potentials

2.1. Case 1: the step potential barrier

The step potential barrier is given by

$$V_{SP}(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geq 0 \end{cases}$$

where $V_0$ is the height of the barrier.

2.2. Case 2: the hyperbolic tangent potential

The hyperbolic tangent potential barrier is defined as

$$V_{HT}(x) = \frac{V_0}{2}[\tanh(bx) + 1],$$

where $V_0$ represents the height of the potential and $b$ gives the smoothness of the curve.

2.3. Case 3: the Lambert-W potential barrier

The asymmetric Lambert-W potential barrier is defined as
V_{LW}(x) = \frac{V_0}{1 + W(e^{-x/\sigma})},  \tag{3}

where $V_0$ is the height of the barrier and $\sigma$ gives the smoothness of the potential barrier.

The forms of these potentials barrier are shown in Fig. 1. We can observe that as $b \to \infty$, the hyperbolic tangent potential reduces to the step potential. Because it is an asymmetric potential, the Lambert-W potential barrier does not reduce to the step potential for $\sigma \ll 1$.

![Fig. 1. Comparison between potential barriers; in all cases, $V_0 = 3$. Lambert-W potential barrier with $\sigma = 0.15$ (dotted line), hyperbolic tangent potential barrier with $b = 0.5$ (dashed line), and step potential barrier with $V_0 = 3$ (solid line).](image)

3. Reflection and Transmission coefficients

3.1. Case 1: the step potential barrier

The reflection and transmission coefficients for the step potential barrier are given by

\begin{align*}
R_{SP} &= \left| \frac{\mu - \nu}{\mu + \nu} \right|^2, \tag{4} \\
T_{SP} &= \frac{\mu}{\nu} \left| \frac{2\nu}{\mu + \nu} \right|^2. \tag{5}
\end{align*}
where \( \nu = \sqrt{E^2 - m^2} \) and \( \mu = \sqrt{(E - V_0)^2 - m^2} \).

The behavior of the reflection and transmission coefficients for the step potential barrier is represented in Fig. 2(a) and Fig. 2(b).

![Fig. 2](image)

Fig. 2. Reflection \( R \) and transmission \( T \) coefficients with varying energy \( E \) for the relativistic step potential barrier with \( V_0 = 3 \) and \( m = 1 \).

### 3.2. Case 2: the hyperbolic tangent potential barrier

To consider the scattering solutions, we solve the differential equation

\[
\frac{d^2 \phi(x)}{dx^2} + \left\{ \left[ E - \frac{V_0}{2} \{ \tanh(bx) + 1 \} \right]^2 - m^2 \right\} \phi(x) = 0. \tag{6}
\]

Writing the hyperbolic tangent potential barrier in terms of the exponential functions, we obtain

\[
\frac{d^2 \phi(x)}{dx^2} + \left[ \left( E - \frac{V_0 e^{2bx}}{1 + e^{2bx}} \right)^2 - m^2 \right] \phi(x) = 0. \tag{7}
\]

By substituting \( y = -e^{2bx} \), Eq. (7) becomes

\[
4b^2 y \frac{d}{dy} \left[ y \frac{d\phi(y)}{dy} \right] + \left[ \left( E + \frac{V_0 y}{1 - y} \right)^2 - m^2 \right] \phi(y) = 0. \tag{8}
\]

Substituting \( \phi(y) = y^\alpha (1 - y)^\beta f(y) \), Eq. (8) reduces to the hypergeometric differential equation.
\[ y(1 - y)f'' + [(1 + 2\alpha) - (2\alpha + 2\beta + 1)y]f' - (\alpha + \beta - \gamma)(\alpha + \beta + \gamma)f = 0, \tag{9} \]

where the primes denote the derivative with respect to \( y \); the parameters \( \alpha, \beta, \) and \( \gamma \) are

\[
\alpha = iv \quad \text{with} \quad v = \frac{\sqrt{E^2 - m^2}}{2b},
\]
\[
\beta = \lambda \quad \text{with} \quad \lambda = \frac{b + \sqrt{b^2 - V_0^2}}{2b},
\]
\[
\gamma = i\mu \quad \text{with} \quad \mu = \sqrt{(E - V_0)^2 - m^2} 2b.
\]

Eq. (9) has a general solution in terms of Gauss hypergeometric functions

\[
\phi(y) = C_1 y^\alpha (1 - y)^\beta _2F_1 (\alpha + \beta - \gamma, \alpha + \beta + \gamma, 1 + 2\alpha; y) + C_2 y^{-\alpha} (1 - y)^\beta _2F_1 (-\alpha + \beta - \gamma, -\alpha + \beta + \gamma, 1 - 2\alpha; y). \tag{13} \]

In terms of the variable \( x \), Eq. (13) becomes

\[
\phi(x) = c_1 (e^{2bx})^{-iv} (1 + e^{2bx})^\lambda _2F_1 (iv + \lambda - i\mu, iv + \lambda + i\mu, 1 + 2iv; -e^{2bx}) + c_2 (e^{2bx})^{-iv} (1 + e^{2bx})^\lambda _2F_1 (-iv + \lambda + i\mu, -iv + \lambda - i\mu, 1 - 2iv; -e^{2bx}). \tag{14} \]

From Eq. (14), the incident and reflected waves are

\[
\phi_{inc}(y) = d_1 (1 + e^{2bx})^\lambda e^{2ibux} _2F_1 (iv + \lambda - i\mu, iv + \lambda + i\mu, 1 + 2iv; -e^{2bx}). \tag{15} \]
\[
\phi_{ref}(y) = d_2 (1 + e^{2bx})^\lambda e^{-2ibux} _2F_1 (-iv + \lambda + i\mu, -iv + \lambda - i\mu, 1 - 2iv; -e^{2bx}). \tag{16} \]

Using the relationship

\[
_2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} (-z)^{(-a)} _2F_1(a, 1 - c + a, 1 - b + a; z^{-1}) + \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} (-z)^{(-b)} _2F_1(b, 1 - c + b, 1 - a + b; z^{-1}). \tag{17} \]

The transmitted wave becomes
\[ \phi_{\text{trans}}(x) = d_3 e^{-2b\lambda x} \left( 1 + e^{2bx} \right)^\lambda e^{2ib\mu x} {}_2F_1 \left( i\nu + \lambda - i\mu, -i\nu + \lambda - i\mu, 1 - 2i\mu; -e^{-2bx} \right). \] (18)

As the incident wave is equal to the sum of the transmitted wave and the reflected wave,
\[ \phi_{\text{inc}}(x) = A \phi_{\text{trans}}(x) + B \phi_{\text{ref}}(x). \] (19)

We use the relationship (17) and the equation for \( \phi_{\text{trans}}(x) \) to find
\[ \phi_{\text{inc}}(x) = A \left( 1 + e^{2bx} \right)^\lambda e^{2ib\nu x} {}_2F_1 \left( i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2bx} \right). \] (20)

\[ \phi_{\text{ref}}(x) = B \left( 1 + e^{2bx} \right)^\lambda e^{-2ib\nu x} {}_2F_1 \left( -i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2bx} \right). \] (21)

where the coefficients \( A \) and \( B \) in Eqs. (20) and (21) can be expressed in terms of the Gamma function as
\[ A = \frac{\Gamma(1 - 2i\mu)\Gamma(-2i\nu)}{\Gamma(-i\nu + \lambda - i\mu)\Gamma(1 - i\nu - \lambda - i\mu)}. \] (22)
\[ B = \frac{\Gamma(1 - 2i\mu)\Gamma(2i\nu)}{\Gamma(i\nu + \lambda - i\mu)\Gamma(1 + i\nu - \lambda - i\mu)}. \] (23)

As \( x \to \pm\infty \), \( V \to \pm a \) and the asymptotic behavior of Eqs. (18), (20), and (21) are plane waves with the relationship of dispersion \( \nu \) and \( \mu \),
\[ \phi_{\text{inc}}(x) = Ae^{2ib\nu x}, \] (24)
\[ \phi_{\text{ref}}(x) = Be^{-2ib\nu x}, \] (25)
\[ \phi_{\text{trans}}(x) = e^{2ib\mu x}. \] (26)

To find \( R \) and \( T \), we used the definition of the electrical current density for the one-dimensional Klein-Gordon equation
\[ \vec{j} = \frac{i}{2} \left( \phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^* \right) \] (27)

The current as \( x \to -\infty \) can be decomposed to \( j_L = j_{\text{inc}} - j_{\text{refl}} \), where \( j_{\text{inc}} \) is the incident current and \( j_{\text{refl}} \) is the reflected one. Analogously, on the right side, as \( x \to \infty \), the current is \( j_R = j_{\text{trans}} \), where \( j_{\text{trans}} \) is the transmitted current.
The reflection coefficient $R$, and the transmission coefficient $T$, in terms of the incident $j_{\text{inc}}$, reflected $j_{\text{ref}}$, and transmission $j_{\text{trans}}$ currents are

$$R = \frac{j_{\text{ref}}}{j_{\text{inc}}} = \frac{|B|^2}{|A|^2}. \quad (28)$$

$$T = \frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{\mu}{\nu} \frac{1}{|A|^2}. \quad (29)$$

The behavior of reflection and transmission coefficients for the hyperbolic tangent potential barrier is represented in Fig. 3(a) and Fig. 3(b).

![Graph](image)

Fig. 3. Reflection $R$ and transmission $T$ coefficients with varying energy $E$ for the relativistic hyperbolic tangent potential barrier with $V_0 = 3$, $b = 0.5$, and $m = 1$.

### 3.3. Case 3: the Lambert-W potential barrier

The one-dimensional Klein-Gordon equation to be solved is

$$\frac{d^2 \phi(x)}{dx^2} + \left\{ \left[ E - V(x) \right]^2 - m^2 \right\} \phi(x) = 0, \quad (30)$$

where $E$ is the energy, $V(x)$ is the potential, and $m$ is the mass of the particle.

To consider the scattering solutions, we solve the differential equation

$$\frac{d^2 \phi(x)}{dx^2} + \left\{ \left[ E - \frac{V_0}{1 + W(e^{-x/\sigma})} \right]^2 - m^2 \right\} \phi(x) = 0. \quad (31)$$
On making the substitution $y = -W(e^{-x/\sigma})$, Eq. (31) becomes

$$\frac{d^2 \phi(y)}{dy^2} + \frac{1}{y(1-y)} \frac{d\phi(y)}{dy} + \sigma^2 \left\{ \frac{[E(1-y) - V_0]^2 - m^2 (1-y)^2}{y^2} \right\} = 0.$$  

(32)

We propose the following solution to Eq. (32):

$$\phi(y) = e^{\alpha/2} y^\beta f(y),$$  

(33)

where $f(y)$ is the solution of the confluent Heun equation.

Using the variable change Eq. (41), the differential equation (32) becomes

$$\frac{d^2 f(y)}{dy^2} - \frac{\alpha y^2 + (-\beta + \alpha - \gamma - 2)y + \beta + 1}{y(y-1)} \frac{d f(y)}{dy} \left\{ \frac{((-\beta - \gamma - 2)\alpha - 2\delta)y + (\beta + 1)\alpha + (-\gamma - 1)\beta - 2\eta - \gamma}{2y(y-1)} \right\} f = 0.$$  

(34)

where $\alpha, \beta, \gamma, \delta,$ and $\eta$ are given by:

$$\alpha = 2\sigma \sqrt{m^2 - E^2},$$  

(35)

$$\beta = 2\sigma \sqrt{m^2 - E^2 + 2EV_0 - V_0^2},$$  

(36)

$$\gamma = -2,$$  

(37)

$$\delta = 2\sigma^2 (m^2 - E^2 + EV_0),$$  

(38)

$$\eta = 1 - 2\sigma^2 (m^2 - E^2 + EV_0).$$  

(39)

Eq. (34) has a general solution in terms of the confluent Heun functions:

$$f(y) = c_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, y) + c_2 y^{-\beta} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, y),$$  

(40)

then Eq. (41) becomes,

$$\phi(y) = c_1 e^{\alpha/2} y^{\beta/2} \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, y) + c_2 e^{\alpha/2} y^{-\beta/2} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, y).$$  

(41)

Finally, in terms of the variable $x$, the solution of Eq. (51) becomes

$$\phi(x) = c_1 e^{-x/\sigma} W(e^{-x/\sigma}) W(e^{-x/\sigma})^{\beta/2} \text{HeunC}[\alpha, \beta, \gamma, \delta, \eta, -W(e^{-x/\sigma})]$$

$$+ c_2 e^{-x/\sigma} W(e^{-x/\sigma})^{-\beta/2} \text{HeunC}[\alpha, -\beta, \gamma, \delta, \eta, -W(e^{-x/\sigma})].$$  

(42)
For the Lambert-W potential barrier, A. M. Ishkhanyan has found the following reflection coefficient\(^{20}\)

\[ R_{LW} = e^{-2\pi\sigma\mu} \frac{\sinh \left[ \frac{\pi\sigma}{2}(\nu - \mu)^2 \right]}{\sinh \left[ \frac{\pi\sigma}{2}(\nu + \mu)^2 \right]}, \quad (43) \]

where for a relativistic particle, \( \nu = \sqrt{E^2 - m^2} \) and \( \mu = \sqrt{(E - V_0)^2 - m^2} \).

The behavior of the reflection and transmission coefficients for the Lambert-W potential barrier is represented in Fig. 4(a) and Fig. 4(b).

![Fig. 4](image)

Fig. 4. Reflection \( R \) and transmission \( T \) coefficients with varying energy \( E \) for the relativistic Lambert-W potential barrier with \( V_0 = 3, \sigma = 0.15 \), and \( m = 1 \).

4. Results

The dispersion relations \( \nu'^2 = E^2 - m^2 \) and \( \mu'^2 = (E + V_0)^2 - m^2 \) do not determine the sign of \( \nu' \) and \( \mu' \). These relations must be positive because they correspond to an incident particle moving from left to right, and their sign depends on the group velocity, which is calculated by taking the derivative of each dispersion relation with respect to the energy \( E \)

\[ \frac{dE}{d\nu'} = \frac{\nu'}{E} \geq 0, \quad (44) \]

\[ \frac{dE}{d\mu'} = \frac{\mu'}{E - V_0} \geq 0. \quad (45) \]

For these potentials, we have three different regions:

\[ \text{Region I: } \nu' < 0, \mu' < 0 \]

\[ \text{Region II: } \nu' > 0, \mu' < 0 \]

\[ \text{Region III: } \nu' > 0, \mu' > 0 \]
4.1. Region 1: $V_0 - m > E > m$

In the region $\mu' < 0$ and $\nu' > 0$, both terms are real and the quotient $\mu'/\nu' < 0$, which implies that the transmitted wave oscillates in region 1. Because $\mu' < 0$, in this region, $T \leq 0$ and $R \geq 1$, the reflected current is greater than the incident current; therefore, superradiance occurs. The unitary relation $T + R = 1$ is satisfied.

4.2. Region 2: $V_0 + m > E > V_0 - m$

In this region, the dispersion relations $\mu$ and $\nu$ are purely imaginary, and the transmitted wave is exponentially damped down; therefore, $T = 0$ and $R = 1$; thus, $T + R = 1$.

4.3. Region 3: $E > V_0 + m$

In this region, $\mu' > 0$ and $\nu' > 0$, both terms are real and the quotient $\mu'/\nu' > 0$, which implies that the transmitted wave oscillates in region 3. Because $\mu' > 0$, and $R > 0$ and $T > 0$, both are positive and satisfy the unitary relation $R + T = 1$.

Figs. 2(a) - 4(b) show the reflection $R$ and transmission $T$ coefficients with various parameters for each potential. It is clear in the figures that, in the region $V_0 - m > E > m$, the reflection coefficient $R$ is greater than one, whereas the transmission coefficient $T$ becomes negative; therefore, superradiance is observed. In the region $V_0 + m > E > V_0 - m$, the reflection coefficient becomes equal to 1 and the transmission coefficient is equal to 0. In the region $E > V_0 + m$, we observe that $1 \geq R \geq 0$ and $1 \geq T \geq 0$. In all regions, the coefficients $R$ and $T$ satisfy the unitary condition $T + R = 1$.

5. Conclusion

In this study, we investigated the phenomenon of superradiance in the Lambert-W potential barrier. We compared its solution with those of the hyperbolic tangent potential barrier and the step potential barrier. We compared the reflection $R$ and transmission $T$ coefficients for these three different potentials, showing that for the region where $V_0 - m > E > m$, the phenomenon of superradiance occurs.

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7. References

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