Quantum Field Theories in Nonextensive Tsallis Statistics

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Within the framework of Tsallis statistics with \( q \simeq 1 \), we construct a perturbation theory for treating relativistic quantum field systems. We find that there appear initial correlations, which do not exist in the Boltzmann-Gibbs statistics. Applying this framework to a quark-gluon plasma, we find that the so-called thermal masses of quarks and gluons are smaller than in the case of Boltzmann-Gibbs statistics.

§1. Introduction

It is expected that a quark-gluon plasma will soon be produced in ultrarelativistic heavy-ion collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC). The CERN Large Hadron Collider (LHC) will also soon be ready for experiments. A few observations regarding quark-gluon plasmas are in order.

1) According to hot QCD (statistical QCD at high temperature),\(^1\) in the chromoelectric sector of gluons in a quark-gluon plasma, a Debye screening mass develops, and as a consequence the chromoelectric-gluon exchange interaction is short range. By contrast, a Debye-like mass does not appear (at least to one-loop order) in the chromomagnetic sector of gluons, and therefore the chromomagnetic-gluon exchange interaction is long range.\(^{\ast\ast\ast}\)

2) The size of a region containing a quark-gluon plasma produced in the collision of heavy ions in RHIC experiments is not very large. In fact, the radius of heavy ions of mass number \( M \approx 200 \) is \( R \sim 7.9 \times 10^{-15} \) m. At the highest RHIC energy, \( \sqrt{s_{NN}} = 200 \) GeV, the Lorentz-contraction factor is \( \sim 1/100 \), and therefore the longitudinal size of the system just after the collision is \( L_l \sim 7.9 \times 10^{-17} \) m, and it increases due to the expansion of the system. Contrastingly, the chromoelectric Debye mass, given by\(^1\) \( m_D \sim 1.2gT \) (with \( g \) the QCD coupling constant and \( T \) the temperature of the plasma) for three quark flavors, is \( \sim 2.8 \) GeV for \( T = 1 \) GeV (where \( g^2/4\pi \simeq 0.43 \)), which corresponds to the Debye screening length\(^1\) \( l_D \sim 6.9 \times 10^{-17} \) m. As mentioned above, the chromomagnetic “Debye mass”, if it exists, is at most of \( O(g^2T) \). Thus the “Debye screening length” in the chromomagnetic sector is much larger than \( l_D \).

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\(^{\ast\ast}\) It is worth mentioning here that, according to hot QED, the magnetic mass does not appear in the “transverse-photon” sector to all orders in perturbation theory.\(^2\) Therefore, a magnetic-photon exchange interaction in an electron-positron-photon plasma is long range.

\(^2\) The quark counterpart\(^1\) of the Debye screening length is much longer, specifically, \( l_D \sim 2.1 \times 10^{-16} \) m for \( T = 1 \) GeV.
3) The single-particle transverse momentum distribution of hadrons, which are produced by heavy ion collisions, exhibits a power-law tail.

Boltzmann-Gibbs statistics is valid for the systems that have the following properties: (i) compared to the size of the system, both the interactions and the memories are of short range, and (ii) the spacetime in which the system evolves is nonfractal. From the above observations 1) and 2), we find it doubtful that Boltzmann-Gibbs statistics can be applied in a strict sense to a quark-gluon plasma, at least in the early stages after a collision. It has been argued that the above observation 3) may be a sign of the fact that the object (a quark-gluon plasma or a hadronic fire ball) produced just after a heavy ion collision does not obey Boltzmann-Gibbs statistics (see, e.g., Ref. 3) and references therein).

For a realistic description of a systems with long-range interactions, long-range memories and/or fractal structure, nonextensive generalization of Boltzmann-Gibbs statistics is essential. Such a generalization was proposed by Tsallis\(^4\) seventeen years ago (so called Tsallis statistics). Since then, a variety of works have appeared concerning theoretical aspects of this form as well as its applications to various nonextensive systems. The most important problem among the former is to determine the extent to which Tsallis statistics is unique among other nonextensive statistics. A short review of this problem is given in §2.1. Tsallis statistics has been successfully applied to a number of nonextensive systems. Examples\(^\ast\) are Lévy-type anomalous diffusion,\(^6\) Euler turbulence,\(^7\) the specific heat of the hydrogen atom,\(^8\) peculiar velocities in galaxies,\(^9\) and self-gravitating systems and related matters.\(^7\),\(^10\) Some theoretical frameworks in Boltzmann-Gibbs statistics have been generalized to the case of Tsallis statistics, e.g., linear response theory,\(^11\) the Green function method\(^12\) and path integrals.\(^13\) Some approximation schemes, such as the \((1-q)\) expansion,\(^14\),\(^15\) factorization approximation,\(^15\),\(^16\) perturbation theory\(^17\) and the semi-classical expansion.\(^18\)

Tsallis statistics contains a (real) parameter \(q\), which is a measure of nonextensiveness of the system, or more precisely, the range of interactions acting among the constituents of the system.\(^19\) For short-range interactions, standard Boltzmann-Gibbs statistics is realized, which corresponds to Tsallis statistics with \(q = 1\).

With the above motivation, in this paper we treat quantum field systems on the basis of Tsallis statistics assuming that \(q\) differs from 1 by a very small amount. More precisely, our region of interest is that satisfying

\[
|1 - q| \leq O(1/(VT^3)^2) \ll 1, \quad (1.1)
\]

where \(T\) is the “temperature” and \(V\) is the volume of the system.\(^\ast\)* (For \(|1 - q|\) expansions in different contexts, see, e.g., Refs. 14) and 15), together with references therein.)

\(^\ast\) For a comprehensive list of references, see Ref.5).

\(^\ast\)* The volume of a heavy ion with mass number \(\sim 200\) is \(V \simeq 2.7 \times 10^{-4} \text{ (MeV}/hc)^{-3}\). (As mentioned above, at early stages after the collision, due to the Lorentz contraction, the volume of the produced quark-gluon plasma is much smaller than this \(V\).) For \(T = 200, 500\) and \(1000\) MeV, \(VT^3 \simeq 2.1 \times 10^3, 3.3 \times 10^4\) and \(2.7 \times 10^5\), respectively.
In §2, brief introductions to Tsallis statistics and the closed time-path (CTP) formalism\cite{1},\cite{20},\cite{21} for treating quantum field systems are given. In §3, we deduce the form for the CTP propagators in Tsallis statistics. In §4, we discuss physical implications of the results and present a procedure for computing higher-order corrections to the propagators.

§2. Preliminaries

2.1. Tsallis statistics

Throughout this paper, we use units in which $k_B = \hbar = c = 1$. Tsallis postulates\cite{4} the following form for the generalized entropy (the Tsallis entropy):

$$S_q[\hat{\rho}] = \frac{(1 - \Tr \hat{\rho}^q)}{q - 1}, \quad (2.1)$$

Here $q$ is a real-number parameter and $\hat{\rho}$ is the density operator ($\Tr \hat{\rho} = 1$). In the limit $q \to 1$, Eq. (2.1) reduces to the standard Boltzmann-Gibbs-Shannon entropy $S_1 = -\Tr \hat{\rho} \ln \hat{\rho}$. However, contrast to the case of Boltzmann-Gibbs statistics, the form for the entropy in the case we consider presently cannot be uniquely deduced. The Tsallis entropy (2.1) meets the requirement of “concavity”, the requirement which any entropy should satisfy.\cite{22} Since the introduction of the Tsallis entropy, dos Santos\cite{23} has demonstrated the uniqueness of the form appearing in Eq. (2.1) assuming pseudo-additivity, $S_{A+B} = S_A + S_B + (1-q)S_A S_B$ (where ‘$A$’ and ‘$B$’ represent the systems in question, and ‘$A + B$’ represents the composite system of ‘$A$’ and ‘$B$’), together with several other conditions. Hotta and Joichi\cite{24} have shown that Eq. (2.1) can be “derived” from less restrictive requirements, namely, the composability condition, $S_{A+B} = \Omega(S_A, S_B)$ (with $\Omega$ being some function), and the ansatz $S[\hat{\rho}] = C + \Tr \phi(\hat{\rho})$ (with $C$ a constant and $\phi$ some function of $\hat{\rho}$), together with a few other conditions. Modified Tsallis entropies have also been proposed by several authors (see, e.g., Refs. 24 – 27). The connection between $S_q$ and the theory of quantum groups has been pointed out and discussed, e.g., in Refs. 26 and 28).

The form of $\hat{\rho}$ is determined by generalizing the procedure employed in Boltzmann-Gibbs statistical mechanics. There had been some disagreement regarding the definition of the expectation value of an operator $\hat{A}$, but this issue has been settled, and it is now known that this expectation value is given by the following:\cite{29}

$$\langle \hat{A} \rangle = \frac{\Tr \hat{A} \hat{\rho}}{\Tr \hat{\rho}^q}, \quad (2.2)$$

which is called the $q$-expectation value and preserves various desirable properties. The form of the density operator $\hat{\rho}$ is determined by maximizing $S_q[\hat{\rho}]$ with the constraints $\Tr \hat{\rho} = 1$ and $\langle \hat{H} \rangle = E$, where $\hat{H}$ is the Hamiltonian. Introducing Lagrange multipliers, we easily carry out the maximization and obtain

$$\hat{\rho} = Z_q^{-1} \left[ 1 - (1 - q)\hat{H}/T \right]^{1/(1-q)}, \quad (2.3)$$
\[ Z_q = \text{Tr}' \left[ 1 - (1 - q)\hat{H}/T \right]^{1/(1-q)}. \] (2.4)

Here, \( \text{Tr}' \) means that \( \text{Tr} \) is taken over the energy eigenstates with \( 1 - (1 - q)E/T \geq 0 \) with \( E \) the eigenvalue of \( \hat{H} \). For \( 1 - q < 0 \), this restriction obviously does not apply. As mentioned in §1, \( q = 1 \) corresponds to Boltzmann-Gibbs statistics, \( \hat{\rho} \propto e^{-\beta \hat{H}} \) with \( T \equiv 1/\beta \) the temperature. In the following, we simply refer to \( T \) in Eq. (2.4) as the temperature. (For a thorough discussion of the temperature of nonextensive systems, see, e.g., Ref. 30) and references therein.)

Before moving on, for convenience, we rewrite the formula (2.2) for the expectation value as

\[ \langle \hat{A} \rangle = \text{Tr}' \hat{\rho}' \hat{A}, \quad \hat{\rho}' = \frac{\hat{\sigma}^q}{\text{Tr} \hat{\sigma}^q}, \] (2.5)

\[ \hat{\sigma}^q = \left[ 1 - (1 - q)\hat{H}/T \right]^{q/(1-q)} = \left[ 1 - \epsilon \hat{H}/T \right]^{(1-\epsilon)/\epsilon} \quad (\epsilon = 1 - q) \] (2.6)

\[ \tilde{\epsilon} \equiv \frac{\epsilon}{1 - \epsilon}, \quad \tilde{\beta} = (1 - \epsilon)/T. \] (2.7)

### 2.2. Closed time-path formalism

For treating quantum field systems, we employ the closed time-path (CTP) formalism.\(^{1,20,21}\) In the single-time representation of the CTP formalism, every field becomes two fields: \( \phi \rightarrow (\phi_1, \phi_2) \). Here, \( \phi_i \ (i = 1, 2) \) is called the type-\( i \) field, and \( \phi_1 \) is called the physical field. Then, the \( n \)-point Green function consists of \( 2^n \) components. At the very end of calculation \( \phi_1 \) and \( \phi_2 \) are set equal.

In the following, we employ the complex scalar field theory governed by the Lagrangian density \( \mathcal{L} = -\phi^\dagger (\partial^2 + m^2)\phi + \mathcal{L}_{\text{int}} \) with \( \mathcal{L}_{\text{int}} = -\lambda/(4)(\phi^\dagger \phi)^2 \). Generalization to other field theories is straightforward. We restrict our consideration to the case in which the density operator is electrically neutral.

#### 2.2.1. Single-time representation

The \( 2n \)-point Green function, which consists of \( 2^{2n} \) components, is defined by

\[ G_{i_1 \cdots i_n, j_1 \cdots j_n}(x_1, \cdots, x_n; y_1, \cdots, y_n) = i(-)^n \text{Tr} \left[ \prod_{l=1}^{n} \phi_{i_l}^{(H)}(x_l) \prod_{m=1}^{n} \phi_{j_m}^{(H)}(y_m) \right] \tilde{\rho}, \] (2.8)

where the operators \( \phi^{(H)} \) and \( \phi^{(H)}\dagger \) are the Heisenberg field operators, and \( \tilde{\rho} \) is the density operator. Note that, for Tsallis statistics, we have \( \tilde{\rho} = \hat{\rho}' \) [Eq. (2.5)]. Here, \( T_c \) is the “ordering” operator with the following properties: i) move the type-2 fields to the left of the type-1 fields, ii) rearrange the operators \( \phi_1^{(H)} \) according to a time-ordering \( (T_c \rightarrow T) \), and iii) rearrange the operators \( \phi_2^{(H)} \) according to an anti-time-ordering \( (T_c \rightarrow \bar{T}) \).

Here we summarize the Feynman rules for computing \( G \) perturbatively. The rules are the same as in the vacuum theory, except that the (bare) propagators and vertices take the following forms.
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1. Propagators (bare two-point functions)

1a) Two-point propagators:

\[ \begin{align*}
    i\Delta_{11}(x - y) &= \langle T \left( \phi_1(x)\phi_1^\dagger(y) \right) \rangle_{\phi_1 = \phi}, \\
    i\Delta_{22}(x - y) &= \langle T \left( \phi_2(x)\phi_2^\dagger(y) \right) \rangle_{\phi_2 = \phi}, \\
    i\Delta_{12}(x - y) &= \langle \phi_1^\dagger(y)\phi_1(x) \rangle_{\phi_1 = \phi_2 = \phi}, \\
    i\Delta_{21}(x - y) &= \langle \phi_2(x)\phi_1^\dagger(y) \rangle_{\phi_1 = \phi_2 = \phi},
\end{align*} \]

where \( \phi \) and \( \phi^\dagger \) are the interaction-picture fields and \( \langle \cdots \rangle \equiv \text{Tr} \cdots \hat{\rho}_0 \) with \( \hat{\rho}_0 \) the “bare” density operator.

1b) “2n-point propagators” (\( 2 \leq n \)): The 2n-point propagators consist of \( 2^{2n} \) components, all of which are the same:

\[ \begin{align*}
    C_n(x_1, \cdots, x_n; y_1, \cdots, y_n), \\
    &\equiv \langle : \phi(x_1) \cdot \cdots \cdot \phi(x_n)\phi^\dagger(y_1) \cdots \cdot \phi^\dagger(y_n) : \rangle_c, \quad (2 \leq n)
\end{align*} \]

where : \( \cdots : \) represents the operation of taking a normal product and ‘c’ stands for the contribution from the connected diagrams. \( C_n \) is called the initial correlation.

2. Vertices

The vertex factor for a vertex at which type-1 fields meet is the same as in the vacuum theory, i.e. \( -i\lambda \), while the vertex factor at which type-2 fields meet is \( i\lambda \).

2.2.2. Physical representation

Let us introduce \( \phi_c \) and \( \phi_\Delta \) (and their hermitian conjugates) through the relations

\[ \phi_c = \frac{1}{2}(\phi_1 + \phi_2), \quad \phi_\Delta = \phi_1 - \phi_2. \]

Using these relations, the single-time representation outlined above can be transformed into the representation written in terms of \( \phi_c, \phi_\Delta, \phi_c^\dagger \) and \( \phi_\Delta^\dagger \), which is called the physical representation.

The 2n-point Green function is defined by

\[ \tilde{G}_{c_1\cdots c_{2n}}(x_1, \cdots, x_n; y_1, \cdots, y_n) = i(-)^n \text{Tr} \left[ T_c \left( \prod_{i=1}^{m_1} \phi_c^{(H)}(x_i) \prod_{j=m_1+1}^{n} \phi_\Delta^{(H)}(x_j) \prod_{k=1}^{m_2} \phi_c^{(H)}(y_k) \prod_{l=m_2+1}^{n} \phi_\Delta^{(H)}(y_l) \right) \hat{\rho} \right], \]

\[ (2.11) \]

The propagators and vertices in the Feynman rules are as follows.

\(^*\) The normalization of \( \tilde{G} \) here is different from that of its counterpart in Ref. 21).
1. Propagators

1a) Two-point propagators:

\[ i\tilde{\Delta}_{\Delta\Delta} = 0, \]
\[ i\tilde{\Delta}_{cc} = i(\Delta_{11} + \Delta_{22})/2 \equiv i\Delta_c/2, \]
\[ i\tilde{\Delta}_{\Delta c} = i(\Delta_{11} - \Delta_{21}) \equiv i\Delta_A, \]
\[ i\tilde{\Delta}_{c\Delta} = i(\Delta_{11} - \Delta_{12}) \equiv i\Delta_R, \] (2.12)

where \( \Delta_{R(A)} \) is the retarded (advanced) propagator, which, in momentum space, reads

\[ \Delta_{R(A)}(P) = \frac{1}{P^2 - m^2 \pm ip_0} \] (2.13)

1b) Initial correlations: Among the \( 2^n \) components of the \( 2^n \)-point propagators \( \tilde{C}_n \) (\( 2 \leq n \)), only the \( (cc \cdots c; cc \cdots c) \)-components are nonzero;

\[ \left( \tilde{C}_n \right)_{c\cdots c; c\cdots c}(x_1, \cdots, x_n; y_1, \cdots, y_n) = C_n(x_1, \cdots, x_n; y_1, \cdots, y_n), \] (2.14)

where \( C_n \) is as in Eq. (2.10).

2. Vertices

The vertex factors for the vertices \( \phi^\dagger c \phi c \phi c \phi, \phi^\dagger c \phi^\dagger \phi \phi \phi \) and \( \phi^\dagger \phi^\dagger \phi^\dagger \phi^\dagger \phi \phi \) are \( -i\lambda, -i\lambda, -i\lambda/4 \) and \( -i\lambda/4 \), respectively. All other vertices vanish.

2.2.3. Forms of the propagators

From this point, we restrict our attention to the case \( \rho_0 = \rho_0(\hat{H}_0) \), with \( \hat{H}_0 \) the free Hamiltonian. Then, the system under consideration is spacetime-translation invariant, and therefore can go to momentum space. To obtain the forms of the propagators, we first construct single-particle wave functions by adopting a fixed volume \( V \) quantization with discrete momenta, \( p = \left( \frac{2\pi}{V^{1/3}} \right) n \), where \( n = (n_1, n_2, n_3) \), with \( n_1, n_2, n_3 \) integers. The complex scalar fields \( \phi(x) \) are decomposed by using the plane-wave basis constructed in this manner,

\[ \phi(x) = \sum_p \frac{1}{(2E_p V)^{1/2}} \left[ a_p e^{-i(E_p x_0 - p \cdot x)} + \tilde{b}_p e^{i(E_p x_0 - p \cdot x)} \right], \] (2.15)

\[ \left[ a_p, a_{p'}^\dagger \right] = \left[ b_p, \tilde{b}_{p'}^\dagger \right] = \delta_{p,p'}, \] (2.16)

where \( E_p = \sqrt{p^2 + m^2} \) is the single-particle energy. Then, \( \hat{H}_0 \) becomes

\[ \hat{H}_0 = \sum_p E_p \left( a_p^\dagger a_p + \tilde{b}_p^\dagger \tilde{b}_p \right). \] (2.17)

The form of \( \Delta_{ij}(P) \) is obtained by substituting Eq. (2.15) and its hermitian conjugate for \( \phi \) and \( \phi^\dagger \), respectively, into Eq. (2.17). Then, taking the large-volume limit, \( V \to \infty \), we obtain (after taking the Fourier transform)

\[ \Delta_{ij}(P) = \Delta_{ij}^{(0)}(P) + \Delta_{ij}(P), \]
\[ \Delta^{(0)}_{11}(P) = -\left( \Delta^{(0)}_{22}(P) \right)^* = \frac{1}{P^2 - m^2 + i0^+}, \]
\[ \Delta^{(0)}_{12(21)}(P) = -2\pi i\theta(\mp p_0)\delta(P^2 - m^2), \]
\[ \Delta_\beta(P) = -2\pi iN(p_0)\delta(P^2 - m^2), \tag{2.18} \]

where \( P^2 = p_0^2 - p^2 \) and \( N(p_0) \) is the number-density function,
\[ N(p_0) = \lim_{V \to \infty} \left[ \theta(p_0)\text{Tr}_p^a a_p \hat{\rho}_0 + \theta(-p_0)\text{Tr}_p^b b_p \hat{\rho}_0 \right] \]
\[ \equiv \text{Tr} \hat{N}(p_0). \tag{2.19} \]

Similarly, we obtain the form for \( C_n \) [Eq. (2.10)] as follows:
\[ C_n = \int \left( \prod_{i=1}^n \frac{d^4 P_i}{(2\pi)^4} 2\pi\delta(P_i^2)\theta(p_0^i) \right) \sum_{l=0}^n \prod_{i=1}^l \hat{N}(p_0) \prod_{j=l+1}^n \hat{N}(-p_j) \right)_c \]
\[ \times \left\{ \left\{ e^{-iP_1(x_1 - y_1)} \cdots e^{-iP_l(x_l - y_l)} \right\} + \text{perms.} \right\} \]
\[ \times \left\{ e^{-iP_{l+1}(x_1' - y_1')} \cdots e^{-iP_n(x_n' - y_n')} \right\} + \text{perms.} \right\}. \tag{2.20} \]

Here, \( P \cdot (x - y) = p_0(x_0 - y_0) - p \cdot (x - y) \) and \( \sum_{i_1 < \cdots < i_l} \left( \sum_{j_1 < \cdots < j_l} \right) \) is the summation over all possible choices of \( i_1 \cdots i_l(j_1 \cdots j_l) \) from among the values \( 1, 2, \cdots n \), subject to the conditions \( i_1 < i_2 < \cdots < i_l \) \((j_1 < j_2 < \cdots < j_l)\). \( j'_1 < j'_2 < \cdots < j'_{n-l} \) \((j'_1 < j'_2 < \cdots < j'_{n-l})\) is obtained from \( j_1 < j_2 < \cdots < j_l \) \((j_1 < j_2 < \cdots < j_l)\) by removing \( i_1 < i_2 < \cdots < i_l \) \((j_1 < j_2 < \cdots < j_l)\). The first ‘perms.’ indicates that all permutations among \((j_1, j_2, \cdots, j_l)\) are taken and the second ‘perms.’ indicates that all permutations among \((j'_1, j'_2, \cdots, j'_{n-l})\) are taken.

For Tsallis statistics [cf. Eq. (2.5) and (2.6)], we have
\[ \hat{\rho}_0 = \frac{\hat{\sigma}_0^q}{\text{Tr} \hat{\sigma}_0^q} = \left[ \frac{1 - \hat{\varepsilon}\beta \hat{H}_0}{\text{Tr} \left[ 1 - \hat{\varepsilon}\beta \hat{H}_0 \right]^{1/\epsilon}} \right]^{1/\epsilon} \tag{2.21} \]

This is invariant under charge conjugation, and therefore we have \( N(p_0) = N(|p_0|) \).

Equation (2.20) is simplified as
\[ C_n = \int \left( \prod_{i=1}^n \frac{d^4 P_i}{(2\pi)^4} 2\pi\delta(P_i^2) \right) \sum_{l=0}^n \prod_{i=1}^l \hat{N}(|p_0|) \right)_c \]
\[ \times \left\{ e^{-iP_1(x_1 - y_1)} \cdots e^{-iP_n(x_n - y_n)} \right\} + \text{perms.} \right\}, \tag{2.22} \]

where ‘perms.’ indicates that all permutations among \((y_1, y_2, \cdots, y_n)\) are taken.

2.2.4. Gibbs ensemble

A standard Gibbs ensemble with temperature \( T (= 1/\beta) \) and vanishing chemical potential is described by \( \hat{\rho}_0 = e^{-\beta \hat{H}_0}/\text{Tr}e^{-\beta \hat{H}_0} \). The initial correlation \( C_n (2 \leq n) \), given in Eq. (2.10), vanishes.
From Eq. (2.17) with Eq. (2.16), in the limit \( V \to \infty \), we obtain

\[
\text{Tr}e^{-\beta H_0} = e^{P_0 \beta V} = \lim_{V \to \infty} \prod_p (1 - e^{-\beta E_p})^{-2} = \exp \left[ -2 \times V \int \frac{d^3p}{(2\pi)^3} \ln \left( 1 - e^{-\beta E_p} \right) \right], \tag{2.23}
\]

\[
N(|p_0| \big| p_0| = E_p = \lim_{V \to \infty} \frac{\text{Tr}p_p e^{-\beta H_0} \text{Tr}e^{-\beta H_0}}{\text{Tr}p_p e^{-\beta H_0}} = \lim_{V \to \infty} \frac{\text{Tr}b_p p_p e^{-\beta H_0}}{\text{Tr}b_p p_p e^{-\beta H_0}} = \frac{1}{e^{\beta E_p} - 1} \equiv N_{BE}(E_p). \tag{2.24}
\]

Here \( P_0 \) is the pressure of the free complex scalar field system and \( N_{BE} \) is the familiar Bose distribution function. The factor of 2 in Eq. (2.23) corresponds to the number of degrees of freedom of the complex scalar field. A straightforward manipulation of Eq. (2.23) yields, for \( m\beta << 1 \),

\[
\text{Tr}e^{-\beta H_0} = e^{2 \times C_{BE} V/(3\beta^3)}, \tag{2.25}
\]

\[
C_{BE} = \frac{\pi^3}{30} \left( 1 - \frac{15}{4\pi^2}(m\beta)^2 + \cdots \right). \tag{2.26}
\]

More generally, for a system that consists of single or several kinds of bosons and/or fermions, we have

\[
\text{Tr}e^{-\beta H_0} = \exp \left[ \left( \sum_i n_{df}^{(i)} C_{BE}^{(i)} + \sum_j n_{df}^{(j)} C_{FD}^{(j)} \right) V/(3\beta^3) \right], \tag{2.27}
\]

where \( i \) and \( j \) label the kinds of bosons and fermions, respectively. The quantity \( n_{df}^{(i)} (n_{df}^{(j)}) \) is the number of degrees of freedom of the \( i \)th kind of boson (\( j \)th kind of fermion) and

\[
C_{BE}^{(i)} = \frac{\pi^2}{30} \left( 1 - \frac{15}{4\pi^2}(m_i\beta)^2 + \cdots \right),
\]

\[
C_{FD}^{(j)} = \frac{\pi^2}{30} \left( \frac{7}{8} - \frac{15}{8\pi^2}(m_j\beta)^2 + \cdots \right). \tag{2.28}
\]

It is worth mentioning that when we replace \( \text{Tr} \cdots \) in Eq. (2.23) with \( \text{Tr}' \cdots \) (for \( \epsilon > 0 \) [cf. Eq. (2.21)], terms of \( O(\epsilon^{-1}/\epsilon) \) and \( O((\beta m/\epsilon)^2) \) appear in Eqs. (2.24) and (2.28). Such terms can safely be ignored for \( \epsilon \ll 1 \) [cf. Eq. (1.1)].

\section*{§3. Computation of the propagators}

In this section, we compute the propagators on the basis of Tsallis statistics.
3.1. Preliminaries

\( \tilde{\sigma}_0^q \), appearing in Eq. (2.21), is expanded as follows:

\[
\tilde{\sigma}_0^q = \exp \left[ \frac{1}{\epsilon} \ln \left( 1 - \tilde{\epsilon} \beta \tilde{H}_0 \right) \right] \\
= e^{-\beta \tilde{H}_0} \exp \left[ -\sum_{l=1}^{\infty} \frac{\tilde{\epsilon}^l (\beta \tilde{H}_0)^{l+1}}{l+1} \right] \\
= e^{-\beta \tilde{H}_0} \prod_{l=1}^{\infty} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\tilde{\epsilon}^l (\beta \tilde{H}_0)^{l+1}}{l+1} \right)^n \right].
\]

(3.2)

The region of interest is that defined by Eq. (1.1). In the following, we compute field propagators up to and including \( O(\epsilon) \) terms, with \( \epsilon \leq O(1/(VT^3)^2) \). For this purpose, for \( \tilde{\sigma}_0^q \), we should employ the form

\[
\tilde{\sigma}_0^q = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{-\tilde{\epsilon}}{2} \right)^l \left( \beta \tilde{H}_0 \right)^{2l} - \frac{\tilde{\epsilon}^2}{3} \beta^3 \partial^2 + \frac{\epsilon^4}{18} \beta^6 \partial^6 - \frac{\tilde{\epsilon}^3}{4} \beta^4 \partial^4 + \ldots \right) e^{-\beta \tilde{H}_0}
\]

\[
\equiv D_\beta e^{-\beta \tilde{H}_0}
\]

(3.3)

and for \( \beta^N \partial^N \left( \text{Tr} e^{-\beta \tilde{H}_0} \right) / \partial \beta^N \), the form

\[
\beta^N \frac{\partial^N}{\partial \beta^N} \text{Tr} e^{-\beta \tilde{H}_0} = \beta^N \frac{\partial^N}{\partial \beta^N} e^{2C_{BE}V/(3\beta^3)}
\]

\[
= \left( \frac{-1}{\beta^3N} \right) \left[ (2C_{BE}V)^N + 2N(N-1)(2C_{BE}V)^{N-1} \beta^3 \\
+ 2N(N-1)(N-2)(3N-4)(2C_{BE}V)^{N-2} \beta^6 + \ldots \right] e^{2C_{BE}V/(3\beta^3)}.
\]

(3.4)

Here, the terms represented by “\( \ldots \)” yield at most \( O(\epsilon^{3/2}) \) contribution to the propagators.

In the reminder of the paper, for simplicity, we restrict ourselves to the massless case, \( m = 0 \), and therefore we have \( C_{BE} = \pi^2/30 \) and \( E_p = |p| = p \). Generalization to the massive case is straightforward.

Equation (3.3) and the remark made at the end of §2 guarantee that \( \text{Tr} \tilde{\sigma}_0^q \simeq \text{Tr} \hat{\sigma}_0^q \). Then, using Eqs. (3.3), (2.25) and (3.4), we get

\[
\text{Tr} \tilde{\sigma}_0^q = D_\beta \text{Tr} e^{-\beta \tilde{H}_0}
\]
\[ \begin{align*} 
&= \sum_{l=0}^{\infty} \frac{1}{l!} (-\tilde{y})^l \left[ 1 + 4\{2l(l-1) + l\} \left( \frac{\tilde{\epsilon}}{2\tilde{y}} \right)^{1/2} + \frac{8}{3}l(l-1)(6l-2)(l-3) + 23(l-2) + 12 \right] \frac{\tilde{\epsilon}}{\tilde{y}} \\
&\quad - \frac{4\tilde{y}}{3} \left( \frac{\tilde{\epsilon} \tilde{y}}{2} \right)^{1/2} - \frac{8\tilde{y}}{3} \{2l(l-1) + 7l + 3\} \tilde{\epsilon} + \frac{4\tilde{\epsilon}^3}{9} \tilde{y}^2 - \tilde{y}^2 \tilde{\epsilon} + \cdots \right] e^{2C_{BE}V/(3\tilde{\beta}^2)} \\
&= \left[ 1 + \frac{2}{3} \sqrt{2\tilde{y}^2}(5\tilde{y} - 3) + \tilde{\epsilon} \tilde{y} \left( \frac{100}{9} \tilde{y}^2 - \frac{131}{3} \tilde{y} + 24 \right) + \cdots \right] e^{-\tilde{y}} e^{2C_{BE}V/(3\tilde{\beta}^2)}, 
\end{align*} \]

where

\[ \tilde{y} \equiv 2C_{BE}^2 \tilde{\epsilon} \left( V\tilde{T}^2 \right)^{1/2}. \quad (\tilde{T} = 1/\tilde{\beta}) \]  

3.2. Two-point propagator \( \Delta_{ij} \) (\( i,j = 1,2 \))

The propagator \( \Delta_{ij}(P) \) is given by Eq. (2.18), with \( N(|p_0|) = N(p) \big|_{p = |p_0|} \) for the number-density function \( N(p_0) \). Then, from Eqs. (2.23) and (2.24), we have

\[ \lim_{V \to \infty} \text{Tr} a^\dagger_P a_P e^{-\beta \hat{H}_0} = \lim_{V \to \infty} \text{Tr} b^\dagger_P b_P e^{-\beta \hat{H}_0} = \frac{1}{e^{\beta p} - 1} e^{2C_{BE}V/(3\tilde{\beta}^2)} = N_{BE}(p) e^{2C_{BE}V/(3\tilde{\beta}^2)}. \]  

Using Eqs. (3.3), (2.25) with \( m = 0 \) and (3.7), we obtain

\[ \lim_{V \to \infty} \text{Tr} a^\dagger_P a_P \sigma_0^\theta = D_{\tilde{\beta}} \left( \tilde{N} e^{2C_{BE}V/(3\tilde{\beta}^2)} \right) \]

\[ = \tilde{N} \sum_{l=0}^{\infty} \frac{1}{l!} (-\tilde{y})^l \left[ 2l\tilde{\beta}p(1 + \tilde{N}) \left( \frac{\tilde{\epsilon}}{2\tilde{y}} \right)^{1/2} + \frac{1}{2} \left\{ 8l(l-1)(2l-1)\tilde{\beta}p(1 + \tilde{N}) + l(2l-1)\tilde{\beta}p(1 + \tilde{N})(1 + 2\tilde{N}) \right\} \frac{\tilde{\epsilon}}{\tilde{y}} \right. \]

\[ - \frac{2}{3} (2l + 3)\tilde{\beta}p(1 + \tilde{N})\tilde{\epsilon} \tilde{y} + \cdots \right] e^{2C_{BE}V/(3\tilde{\beta}^2)} + \tilde{N} \text{Tr} \sigma_0^\theta \]

\[ = \tilde{N} e^{-\tilde{y}} \left[ -2\tilde{\beta}p(1 + \tilde{N}) \left( \frac{\tilde{\epsilon} \tilde{y}}{2} \right)^{1/2} + 4\tilde{\beta}p(1 + \tilde{N})\tilde{y}(3 - 2\tilde{y})\tilde{\epsilon} \right. \]

\[ + \frac{(\tilde{\beta}p)^2}{2} (2\tilde{N}^2 + 3\tilde{N} + 1)(2\tilde{y} - 1)\tilde{\epsilon} - \frac{2}{3} \tilde{\beta}p(1 + \tilde{N})\tilde{y}(3 - 2\tilde{y})\tilde{\epsilon} + \cdots \right] e^{2C_{BE}V/(3\tilde{\beta}^2)} + \tilde{N} \text{Tr} \sigma_0^\theta, \]  

(3.8)
where \( \tilde{N} = 1/(e^{\tilde{\beta}p} - 1) \). From Eqs. (3.5) and (3.8), we finally obtain for the number-density function,

\[
N(p) = \lim_{V \to \infty} \frac{\text{Tr} \tilde{p}^\dagger \tilde{p} \tilde{\sigma}_0^q}{\text{Tr} \tilde{\sigma}_0^q}
\]

\[
= \tilde{N} \left[ 1 - \sqrt{2\tilde{\beta}p} (1 + \tilde{N}) + e \left\{ 6\tilde{\beta}p (1 + \tilde{N}) + \frac{(\tilde{\beta}p)^2}{2} (2\tilde{N}^2 + 3\tilde{N} + 1) (2\tilde{y} - 1) \right\} + \cdots \right]
\]

\[
= N_{BE} \left[ 1 - \sqrt{2\tilde{y}e \tilde{\beta}p (1 + N_{BE})} + \epsilon \left\{ (1 + 6y)\beta p (1 + N_{BE}) + \frac{2y - 1}{2} (\beta p)^2 (1 + N_{BE}) (1 + 2N_{BE}) \right\} + O(\epsilon^{3/2}) \right],
\]

(3.9)

where use has been made of Eq. (2.7). In Eq. (3.9), we have used \( N_{BE} = 1/(e^{\beta p} - 1) \) and

\[
y = \frac{\pi^4}{450} \epsilon (VT^3)^2.
\]

(3.10)

Generalization to other fields is straightforward. For a system of bosons and/or fermions, we have, for the number-density function,

\[
N_{b/f}(p) = N_{BE/FD} \left[ 1 - \sqrt{2\tilde{y}e \tilde{\beta}p (1 \pm N_{BE/FD})} + e \left\{ (1 + 6y)\beta p (1 \pm N_{BE/FD}) + \frac{2y - 1}{2} (\beta p)^2 (1 \pm N_{BE/FD}) (1 \pm 2N_{BE/FD}) \right\} + O(\epsilon^{3/2}) \right],
\]

(3.11)

\[
y = \frac{\pi^4}{1800} \epsilon (VT^3)^2. \quad (\leq O(1))
\]

(3.12)

Here ‘b’ (‘f’) indicates a boson (fermion), \( C^{(i)/(j)}_{BE/FD} \) is as in Eq. (2.28), with \( m = 0 \), and \( N_{FD} = 1/(e^{\beta p} + 1) \).

Equation (3.11) is valid for \( |y| \leq O(1) \). We note that, from Eq. (3.12), the quantities \( y (\propto \epsilon) \) and \( \epsilon \) are of the same sign. Then, from Eq. (3.11), we see that the \( O(\sqrt{\epsilon}) \) term is negative and invariant under \( \epsilon \to -\epsilon \), so that \( N_{BE/FD} - N_{b/f} \) is the same for \( \epsilon = |\epsilon| \) as for \( \epsilon = -|\epsilon| \). The difference between the cases \( \epsilon > 0 \) and \( \epsilon < 0 \) arises at \( O(\epsilon) \).

Low-temperature limit: \( \beta \to \infty \)

In the limit \( \beta p \gg 1 \), we have \( N_{BE/FD} \sim e^{-\beta p} \), and then

\[
N_{b/f}(p) = \left[ 1 - \sqrt{2ye \beta p} + (1 + 6y)e \beta p + \frac{1}{2} (2y - 1) e (\beta p)^2 + O(\epsilon^{3/2}) \right] e^{-\beta p}.
\]
The series in the square brackets on the R.H.S. here seems to be nonconvergent for
\[ 1/\sqrt{\epsilon} \lesssim \beta p \text{ or } 1/\sqrt{\epsilon y} \lesssim \beta p. \] (3-13)
However, due to the factor \( e^{-\beta p} \), the region (3-13) is unimportant. We see that
\[ [N_{b/f}(p)]_{c>0} < [N_{b/f}(p)]_{c=-|c|}. \]

High-temperature limit: \( \beta \to 0 \)

In the limiting case \( \beta p \simeq 0 \), we have \( N_{BE} \sim 1/(\beta p) \) and \( N_{FD} \simeq 1/2 - \beta p/4 \). Then, we obtain
\[
N_b(p) = \frac{1}{\beta p} \left[ 1 - \sqrt{2y\epsilon} + 8y\epsilon + O(\epsilon^{3/2}) \right],
N_f(p) = \frac{1}{2} - \frac{1}{4}\beta p \left[ 1 + \sqrt{2y\epsilon} - (1 + 6y)\epsilon + O(\epsilon^{3/2}) \right].
\] (3-14)
In this case, we see that \([N_{b/f}(p)]_{c>0} > [N_{b/f}(p)]_{c=-|c|}\).

3.3. Initial correlations (2n-point propagators) \( C_n \)

Here, we compute \( \langle \prod_{i=1}^n N(p_i) \rangle \) (\( |p_i| = p_i \)), appearing in Eq. (2.22), in the content of complex scalar field theory. A procedure similar to that above leading to Eq. (3.11) in the present case yields [cf. Eq. (3.17)]
\[
\langle \prod_{i=1}^n \tilde{N}(p_i) \rangle = \frac{D_\beta \left( \prod_{i=1}^n (e^{\beta p_i} - 1)^{-1} e^{2C_{BE}V/(3\beta^3)} \right)}{D_\beta e^{2C_{BE}V/(3\beta^3)}}
= \left( \prod_{i=1}^n N_{BE}(p_i) \right) \left[ 1 - \sqrt{2y\epsilon} \sum_{i=1}^n \beta p_i (1 + N_{BE}(p_i)) \right.
\left. + \epsilon \sum_{i=1}^n \left( (1 + 6y)\beta p_i (1 + N_{BE}(p_i)) + \frac{2y-1}{2} (\beta p_i)^2 (1 + N_{BE}(p_i))(1 + 2N_{BE}(p_i)) \right.ight.
\left. + \sum_{j(i \neq i)} (2y - 1)\beta^2 p_i p_j (1 + N_{BE}(p_i))(1 + N_{BE}(p_j)) \right) + O(\epsilon^{3/2}) \right].
\] (3.15)
We are now in a position to compute the connected contribution, \( \langle \prod_{i=1}^n \tilde{N}(p_i) \rangle_c \).
For the sake of generality, from this point, we consider a system of bosons and/or fermions. We start with \( n = 2 \), in which case we have
\[
\langle \tilde{N}(p_1)\tilde{N}(p_2) \rangle_c = \langle \tilde{N}(p_1)\tilde{N}(p_2) \rangle - N_{b/f}(p_1)N_{b/f}(p_2).
\] (3.16)
Then, using Eqs. (3.11) and (3.14), we obtain
\[
\langle \tilde{N}(p_1)\tilde{N}(p_2) \rangle_c = -\epsilon \beta^2 p_1 p_2 N_{BE/FD}(p_1)N_{BE/FD}(p_2) \left( 1 \pm N_{BE/FD}(p_1) \right)
\times (1 \pm N_{BE/FD}(p_2)) + O(\epsilon^{3/2}),
\] (3.17)
which is of $O(\epsilon)$. For $n = 3$, we have

$$\langle \hat{N}(p_1)\hat{N}(p_2)\hat{N}(p_3) \rangle_c = \langle \hat{N}(p_1)\hat{N}(p_2)\hat{N}(p_3) \rangle - N_{b/f}(p_1)N_{b/f}(p_2)N_{b/f}(p_3)$$

$$- \langle \hat{N}(p_1)\hat{N}(p_2) \rangle_c N_{b/f}(p_3) - \langle \hat{N}(p_2)\hat{N}(p_3) \rangle_c N_{b/f}(p_1)$$

$$- \langle \hat{N}(p_1)\hat{N}(p_3) \rangle_c N_{b/f}(p_2) = O(\epsilon^{3/2}).$$

(3.18)

In a similar manner, for $4 \leq n$, we obtain

$$\langle \prod_{i=1}^{n} \hat{N}(p_i) \rangle_c = \langle \prod_{i=1}^{n} \hat{N}(p_i) \rangle - \prod_{i=1}^{n} N_{b/f}(p_i)$$

$$- \sum_{\text{perm.}} \langle \hat{N}(p_1)\hat{N}(p_2) \rangle_c \prod_{i=3}^{n} N_{b/f}(p_i) + O(\epsilon^{3/2})$$

$$= O(\epsilon^{3/2}).$$

(3.19)

Thus, we conclude that, to $O(\epsilon)$, $C_n = 0$ for $3 \leq n$. $C_2$ is given by Eq. (3.17), which is of $O(\epsilon)$.

Finally, we point out that in the case of Boltzmann-Gibbs statistics, we have

$$\langle \prod_{i=1}^{n} \hat{N}(p_i) \rangle = \prod_{i=1}^{n} \langle \hat{N}(p_i) \rangle,$$

and hence $C_n = 0$.

§4. Physical implications

4.1. Hard thermal loops in hot QCD

According to QCD at high temperature (hot QCD), among the formally higher-order amplitudes are those that are of the same order of magnitude as their lowest-order counterparts.\(^1\) This is the case for classes of amplitudes whose external momenta are all soft, i.e., $P^\mu = O(gT)$ with $g$ the QCD coupling constant. The relevant diagrams are the one-loop diagrams with hard loop momenta $[Q_{\text{loop}} = O(T)]$, and for this reason, these are called hard thermal loops (HTL). Each HTL amplitude is proportional to the characteristic mass, the gluon thermal mass ($m_g$), or the quark thermal mass ($m_q$):

$$m_g^2 = \frac{1}{2}g^2 T^2 (1 + N_f/6) = \frac{3g^2}{\pi^2} (1 + N_f/6)\mathcal{I}_{\text{BG}},$$

$$m_q^2 = \frac{1}{6}g^2 T^2 = \frac{g^2}{\pi^2} \mathcal{I}_{\text{BG}},$$

(4.1)

where $N_f$ is the number of active (massless) quark flavors. In Eq. (4.1), we have introduced $\mathcal{I}_{\text{BG}}$ (where ‘BG’ means ‘Boltzmann-Gibbs’), which is defined by

$$\mathcal{I}_{\text{BG}} \equiv \int_0^{\infty} dp p N_{\text{BE}}(p) = 2 \int_0^{\infty} dp p N_{\text{FD}}(p) = \pi^2 T^2/6.$$ 

(4.2)
For Tsallis statistics, the above $I_{BG}$ is modified. From Eq. (3.11), we can compute the Tsallis statistics counterpart of Eq. (4.2), $I_T$:

$$I_T = \int_0^\infty dp \rho_b(p) = 2 \int_0^\infty dp \rho_i(p) = \left[ 1 - 2\sqrt{2ye} + \epsilon(18y - 1) + O(\epsilon^{3/2}) \right] I_{BG}.$$  (4.3)

Thus, we obtain for the gluon and quark thermal masses in Tsallis statistics,

$$\frac{\langle m_{g/q}^2 \rangle_T}{\langle m_{g/q}^2 \rangle_{BG}} = 1 - 2\sqrt{2ye} + \epsilon(18y - 1) + O(\epsilon^{3/2}),$$  (4.4)

where $\langle m_{g/q}^2 \rangle_{BG}$ is as in Eq. (4.1) and $y$ is as in Eq. (3.12), with $n_{df} = 16$ and $n_{df'} = 12N_f$. We find that, to $O(\sqrt{\epsilon})$, $\langle m_{g/q}^2 \rangle_T < \langle m_{g/q}^2 \rangle_{BG}$.

4.2. Manifestation of the initial correlations

Here we give some examples of quantities in which $C_\alpha$ participates. We consider a complex scalar field system. (Generalization to other field systems is straightforward.) First, we note that each of the average values $\langle \phi_c \phi_c^\dagger \rangle (\sim i\Delta_{cc})$, $\langle \phi_c \phi_c^\dagger \phi_c^\dagger \phi_c^\dagger \rangle (\sim i\Delta_{cccc})$, $\cdots$ that we know represents information that we possess concerning the statistical properties of the system.

The linear response theory is formulated in terms of the so-called linear response function of the $n$-point correlation with the external field that couples to the quantity under consideration. The linear response functions are given by the appropriate Green functions, $G$, in the CTP formalism.$^{21}$ As an example, we take the field $\phi^\dagger (\phi)$ itself as such a quantity. Then, the linear response function of the $(n - 1)$-point correlation ($n = 2, 4, \cdots$) is given by $G_{cc\cdots c\cdots c\Delta}$ ($G_{c\cdots \Delta c\cdots c}$). Similarly, the nonlinear response functions are also given by the appropriate Green functions in the CTP formalism. For example, the second-order response functions of the two-point correlations are given by $G_{cc\Delta\Delta}$, $G_{c\Delta\Delta}$ and $G_{\Delta\Delta cc}$.

The initial correlations, given in Eq. (2.14), contribute to the above-mentioned response functions as well as to the average correlations $G_{cc\cdots c\cdots c}$, and hence they can be "measured" in principle. For example, the diagrams that include one vertex and one four-point propagator $C_2$ contribute to the four-point Green functions $G_{cccc}$, $G_{cccc\Delta}$ and $G_{cc\Delta cc}$. Explicit computation of the contribution to, e.g., $G_{cccc\Delta}$, using the form for $C_2$ obtained in §3.3 yields

$$G_{cccc\Delta}(x_1, x_2; y_1, y_2) = \frac{\lambda}{2} \int d^4z \left\{ 2C_2(z, x_1; z, y_1)S_R(z - y_2)S_R(x_2 - z) + (x_1 \leftrightarrow x_2) \right\}
+ C_2(x_1, x_2; z, z)S_R(z - y_2)S_R(y_1 - z)
$$

$$= \frac{\lambda}{2} \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4P_1}{(2\pi)^4} \int \frac{d^4P_2}{(2\pi)^4} 2\pi\delta(P_1^2)2\pi\delta(P_2^2) \left\{ (N(p_1)N(p_2))_T \right\}_c 
\times \left\{ 2e^{-i\hat{P}(x_2 - y_2) + P_2(x_1 - y_1)} (S_R(P))^2 \right\}.$$
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\[ + 2 e^{-i[P(x_2 - y_2) + P_1(x_2 - y_1) - P_2(x_2 - x_1)]} S_R(P) S_R(P + P_1 - P_2) \]
\[ + e^{-i[P(x_1 - y_2) + P_1(x_1 - y_1) + P_2(x_2 - y_1)]} S_R(P) S_R(P - P_1 - P_2) \]
\[ + \{x_1 \leftrightarrow x_2\} \]  \hspace{1cm} (4.5)

Note that \( \tilde{G}_{cc\Delta}(x_1, x_2; y_1, y_2) \) can also be rewritten in the form\(^{21}\)

\[ \tilde{G}_{cc\Delta}(x_1, x_2; y_1, y_2) \]
\[ = \frac{i}{4} \sum_{\text{perms.}} \left[ \theta(x_1, x_2, y_1, y_2) \langle \left\{ \phi(x_1), \phi(x_2) \right\}, \phi^\dagger(y_1) \rangle \right] \]
\[ + \theta(x_1, x_2, y_2, y_1) \langle \left\{ \phi(x_1), \phi(x_2) \right\}, \phi^\dagger(y_2) \rangle \theta(y_1) \]
\[ + \theta(x_1, y_2, x_2, y_1) \langle \left\{ \phi(x_1), \phi(y_2) \right\}, \phi^\dagger(y_1) \rangle \theta(y_1) \]
\[ + \theta(x_1, y_2, x_2, y_1) \langle \left\{ \phi(x_1), \phi(y_2) \right\}, \phi^\dagger(y_1) \rangle \theta(y_1) \]  \hspace{1cm} (4.6)

where ‘perms.’ indicates that all permutations among \((x_1, x_2, y_1)\) are taken, and \(\theta(x, y, u, v) \equiv \theta(x_0 - y_0) \theta(y_0 - u_0) \theta(u_0 - v_0)\). From this formula, we see that \( \tilde{G}_{cc\Delta} = 0 \) for \(y_20 > x_10, x_20, y_10\), as should be the case. As must be the case, \( \tilde{G}_{cc\Delta} \) in Eq. (4.5) satisfies this property.

4.3. Remarks on higher-order corrections

Higher-order corrections to the CTP propagators come from the following two sources.

1. Corrections due to the interactions, which, as in the case of Boltzmann-Gibbs statistics, are obtained by computing the vertex-insertion diagrams.
2. Corrections arising from the corrections to the pressure \(P\) computed within the Boltzmann-Gibbs statistics [cf. Eqs. (223) and (224)]. For hot QCD, the corrections to \(P\) are known\(^{31}\) up to \(O(g^5 \ln(1/g))\).

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