Estimates of Potential functions of random walks on \( \mathbb{Z} \) with zero mean and infinite variance and their applications

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Abstract

We consider an irreducible random walk on the one dimensional integer lattice with zero mean, infinite variance and i.i.d. increments \( X_n \) and obtain certain asymptotic properties of the potential function, \( a(x) \), of the walk; we especially show that as \( x \to \infty \)

\[
a(x) \asymp \frac{x}{m_-(x)} \quad \text{and} \quad \frac{a(-x)}{a(x)} \to 0 \quad \text{if} \quad \lim_{x \to +\infty} \frac{m_+(x)}{m_-(x)} = 0,
\]

where \( m_\pm(x) = \int_0^x dy \int_y^\infty P[\pm X_1 > u]du \). The results are applied in order to obtain a sufficient condition for the relative stability of the ladder height and estimates of some escape probabilities from the origin.

Key words: recurrent random walk; potential function; zero mean; infinite variance; escape probability; relatively stable; ladder height

AMS MSC (2010): Primary 60G50, Secondary 60J45.

1 Introduction and Statements of Results

We study asymptotic properties of the potential function \( a(x) \) of a recurrent random walk on the integer lattice \( \mathbb{Z} \) with i.i.d. increments. Denote by \( F \) the common distribution function of the increments and let \( X \) be a random variable having the distribution \( F \). Let \( (\Omega, \mathcal{F}, P) \) be the probability space on which \( X \) as well as the random walk is defined and \( E \) the expectation by \( P \). Then \( a(x) \) is defined by

\[
a(x) = \sum_{n=0}^{\infty} (P[S_n^0 = 0] - P[S_n^0 = -x]),
\]

where \( S_n^0 \) denotes the random walk started at 0 (see Section 5); the infinite series on the RHS is convergent, and if \( \sigma^2 = EX^2 < \infty \) then \( a(x+1) - a(x) \sim \pm 1/2\sigma^2 \) as \( x \to \pm \infty \) (\( \sim \) designates that the ratio of two sides of it tends to 1), whereas in case \( \sigma^2 = \infty \), the known result on the behaviour of \( a(x) \) for large values of \( |x| \) is less precise: it says only that

\[
a(x+1) - a(x) \to 0
\]

and

\[
\bar{a}(x) := (a(x) + a(-x))/2 \to \infty
\]  

(1.1)
as \(|x| \to \infty\) (cf. \cite{15} Sections 28 and 29\), unless \(F\) satisfies some specific condition like regular variation of its tails. In this paper we suppose that the walk is irreducible,

\[
EX = 0 \quad \text{and} \quad \sigma^2 = \infty
\]

unless the contrary is stated explicitly, and obtain an expression of the growth order of \(\bar{a}\) in terms of a certain functional of \(F\) for a large class of \(F\) and observe that if the right-hand tail of \(F\) is asymptotically negligible ‘in average’ in comparison to the left-hand tail, then \(a(x)/a(-x) \to 0\). The potential function \(a\) plays a fundamental role for the analysis of the walk, especially for the walk killed on hitting the origin, the Green function of the killed walk being given by \(g(x,y) = a(x) + a(-y) - a(x-y)\). We shall give some applications of the results based on this fact mostly in case when one tail of \(F\) is negligible relative to the other.

In order to state the results we need the following functionals of \(F\) that also bear a great deal of relevance to our analysis: for \(x \geq 0\)

\[
\mu_-(x) = P[X < -x], \quad \mu_+(x) = P[X > x], \quad \mu(x) = \mu_-(x) + \mu_+(x);
\]

\[
\eta_\pm(x) = \int_x^\infty \mu_\pm(y)dy, \quad \eta(x) = \eta_-(x) + \eta_+(x);
\]

\[
m_\pm(x) = \int_0^x \eta_\pm(t)dt, \quad m(x) = m_+(x) + m_-(x);
\]

\[
\alpha_\pm(t) = \int_0^\infty \mu_\pm(y) \sin ty dy, \quad \alpha(t) = \alpha_+(t) + \alpha_-(t);
\]

\[
\beta_\pm(t) = \int_0^\infty \mu_\pm(y)(1 - \cos ty)dy, \quad \beta(t) = \beta_+(t) + \beta_-(t);
\]

\[
\psi(t) = Ee^{itX}.
\]

Here \(x\) designates a real number, whereas \(x\) is always an integer in \(a(x)\); this duplicity will cause little confusion. The following condition will be required for an upper bound of \(\bar{a}\):

\[
(H) \quad \delta_H := \lim \inf_{t \to 0} \left| \frac{1 - \psi(t)}{t\beta(t)} \right| > 0.
\]

Note that \(\{1 - \psi(t)\}/t = \alpha(t) + i(\beta_+(t) - \beta_-(t))\).

**Theorem 1.** (i) For some universal constant \(C_+ > 0\)

\[
\bar{a}(x) \geq C_+x/m(x) \quad \text{for all sufficiently large } x \geq 1.
\]

(ii) Suppose condition \((H)\) holds. Then for some constant \(C^*_H\) depending only on \(\delta_H\),

\[
\bar{a}(x) \leq C^*_H x/m(x) \quad \text{for } x \geq 1.
\]

According to Theorem 1 if \((H)\) holds, then \(\bar{a}(x) \asymp x/m(x)\) as \(x \to \infty\) (i.e., the ratio of two sides of it is bounded away from zero and infinity), which entails some regularity of \(a(x)\) like the boundedness of \(\bar{a}(y)/\bar{a}(x)\) for \(1 \leq x \leq y \leq 2x\), while in general \(\bar{a}(x)\) may behave very irregularly as will be exhibited by an example (see Section 6.2).

Condition \((H)\) is implied by each of the following three conditions

\[
m_+(x)/m_-(x) \to 0 \quad \text{as} \quad x \to +\infty; \quad (1.3)
\]
\[
\limsup_{x \to \infty} \mu_+(x)/\mu_-(x) < 1; \quad (1.4)
\]

\[
\limsup_{x \to \infty} \frac{x \eta(x)}{m(x)} < 1.
\] (1.5)

The first two conditions entail asymmetry of the distribution of \(X\), whereas the third one is irrelevant to any symmetry property of \(X\). That (1.4) is sufficient for (H) is immediate in view of \(\lim_{t \downarrow 0} \beta(t)/t = \infty\). The sufficiency of the other two will be verified in Section 2 (see (2.7) and Corollary 2.1). (1.3) seems to be much stronger for (H) to be valid but its sufficiency is not immediate and is needed for the next theorem.

**Theorem 2.** Suppose (1.3) to hold. Then \(\delta_H = 1\) in (H) and

\[
\frac{a(-x)}{a(x)} \to 0 \quad \text{as} \quad x \to +\infty.
\]

Theorems 1 and 2 are applied to the asymptotic estimate of the upwards overshoot distribution of the walk over a high level, \(R\) say, as well as of the probability of its escaping the origin and going beyond the level \(R\) or \(\pm R\). The overshoot distribution is related to the relative stability of the ladder height variable which has been studied in [14], [9], [5] etc. when \(X\) is in the domain of attraction of a stable law, whereas the escape probability of this kind seems to have rarely been investigated. It seems hard to have a definite result in general for these subject. Under condition (1.3), however, Theorems 1 and 2 are effectively used to yield natural results: we shall obtain a sufficient condition for the relative stability of the ladder height (Proposition 5.1) and the asymptotic forms of the escape probabilities (Proposition 5.2 (one-sided) and Propositions 5.3 and Corollary 5.4 (two-sided)). As a byproduct of the latter we deduce under (1.3) the asymptotic monotonicity of \(a(x), x > 0\) by showing that

\[
P[\sigma^0_{[R,\infty)} < \sigma^0_{[0]}] \sim 1/a(R)
\]

(see Corollary 5.2) as well as the following result on the classical two-sided exit problem that has not been satisfactorily answered in case \(\sigma^2 = \infty\); for any \(\varepsilon > 0\), uniformly for \(\varepsilon R < x < R,

\[
P[\sigma^x_{[R,\infty)} < \sigma^x_{(-\infty,0]}] \sim a(x)/a(R) \quad (1.6)
\]

as \(R \to \infty\) (Corollary 5.5), where \(\sigma^x_B\) denotes the first entrance time into a set \(B\) of the walk starting at \(x\); we shall show \(P[\sigma^x_{[R,\infty)} < \sigma^x_{(-\infty,0]}] \sim P[\sigma^x_{(R)} < \sigma^x_{(0)}]\), which would explain the appearance of \(a\) on the RHS in (1.6). There is an identity for Lévy processes having no upwards jumps (cf. [6, Section 9.4]), of which the analogue for the walk should be

\[
P[\sigma^x_{[R,\infty)} < \sigma^x_{(-\infty,0]}] \sim V_{ds}(x)/V_{ds}(R) \quad (1 \leq x < R),
\] (1.7)

where \(V_{ds}(x)\) denotes the renewal function for the weak descending ladder height process of the walk. The upper bound always true: for all \(x \geq 1,

\[
P[\sigma^x_{[R,\infty)} < \sigma^x_{(-\infty,0]}] \leq V_{ds}(x)/V_{ds}(R),
\] (1.8)

and if (1.3) is assumed, then (1.7) seems to hold quite generally: in fact, if the walk \(S\) is attracted to a stable process of index \(1 < \alpha \leq 2\) with no positive jump, then (1.7) holds (cf. [19, Appendix (C)]); also if the first ascending ladder height variable, \(Z\) say,
has finite expectation, then (1.7) holds uniformly for \(1 \leq x \leq R\)—this fact conforms to (1.6) since \(a(x) \sim V_{ds}(x)/EZ (x \to \infty)\) (cf. [19, Theorem 1(ii)]).

In addition to the signs \(\sim\) and \(\asymp\) that have already been introduced we use \(\wedge\) and \(\vee\) to denote the minimum and maximum of two terms on their two sides. By \(C, C', C_1, etc.\) we denote the generic positive finite constants whose values may change from line to line.

In the next section we derive some fundamental facts about \(a(x)\) as well as the functionals introduced above, which incidentally yield (ii) of Theorem 1 (see Lemma 2.4) and the sufficiency for (H) of (1.3) and (1.5) mentioned above; also the lower bound in Theorem 1 is verified under a certain side condition. The proof of (i) of Theorem 1 is more involved and given in Section 3. Theorem 2 is proved in Section 4. Applications are discussed in Sections 5. In Section 6 we give two examples: for the first one the walk is supposed in the domain of attraction for a stable law of exponent \(1 \leq \alpha \leq 2\) and some precise asymptotic forms of \(a(x)\) as \(|x| \to \infty\) are exhibited, while the second one reveals how \(a(x)\) can irregularly behave for large values of \(x\).

2 Preliminaries and proof of Theorem 1(ii)

In this section we first present some easy facts most of them are taken from [16], and then give several lemmas, in particular Lemmas 2.4 and 2.6 which together assert that \(\bar{a}(x) \asymp x/m(x)\) under (1.5) and whose proofs involve typical arguments that are implicitly used in Sections 3 and 4. Because of our moment condition, i.e., \(EX = 0, t/(1 - \psi(t))\) is integrable about the origin and \(a\) is certainly expressed as

\[
a(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \frac{1 - e^{ixt}}{1 - \psi(t)} \, dt. \tag{2.1}
\]

As in [16] we bring in the following functionals of \(F\) in addition to those introduced in Section 1 (the notation of [16] is adopted):

\[
c(x) = \int_0^x y\mu(y) \, dy, \tag{2.2}
\]

\[
\tilde{c}(x) = \frac{1}{x} \int_0^x y^2\mu(y) \, dy, \quad \tilde{m}(x) = \frac{2}{x} \int_0^x y\eta(y) \, dy,
\]

\[
h_\varepsilon(x) = \int_0^{\varepsilon x} y[\mu(y) - \mu(\pi x + y)] \, dy \quad (0 < \varepsilon \leq 1);
\]

also \(c_\pm(x), \tilde{c}_\pm(x)\) and \(\tilde{m}_\pm(x)\) are defined with \(\mu_\pm\) in place of \(\mu\), so that \(c(x) = c_-(x) + c_+(x), etc.\) Our problem involves the difference

\[
\gamma(t) = \beta_+(t) - \beta_-(t).
\]

By our basic hypothesis (1.2) \(x\eta(x), c(x)\) and \(h_\varepsilon(x)\) tend to infinity as \(x \to \infty\); \(\eta, c\) and \(h_\varepsilon\) are monotone. It is noted that \(\Im \psi(t) = -t\gamma(t)\) (not \(t\gamma(t)\)) as is easily checked by integrating by parts the integral \(\int (\sin tx - tx) dP[X \leq x]\). Here and throughout the rest of the paper \(x \geq 0\) and \(t \geq 0\). We shall be concerned with the behaviour of these functions as \(x \to \infty\) or \(t \downarrow 0\) and therefore omit “\(x \to \infty\), “\(t \downarrow 0\)” when it is obvious.
The function \( m \) admits the decomposition
\[
m(x) = x\eta(x) + c(x).
\]

\( m \) is a rather tractable function: increasing and concave, hence subadditive and
\[
m(kx) \leq km(x) \quad \text{for any } k > 1,
\]
while \( c \), though increasing, may vary quite differently. The ratio \( c(x)/m(x) \) may converges to 0 or to 1 as \( x \to \infty \) depending on \( \mu \) and possibly oscillates between 0 and 1 (cf. Section 6.2). Similarly
\[
\tilde{m}(x) = x\eta(x) + \tilde{c}(x)
\]
and \( \tilde{m} \) is increasing and concave.

It follows that \( 1 - \psi(t) = t\alpha(t) + it\gamma(t) \). Hence
\[
\frac{1}{1 - \psi(t)} = \frac{\alpha(t) - i\gamma(t)}{\alpha^2(t) + \gamma^2(t)} \cdot \frac{1}{t},
\]
and from (2.1) it follows that
\[
\bar{a}(x) = \frac{1}{\pi} \int_0^\pi \frac{\alpha(t)}{[\alpha^2(t) + \gamma^2(t)]t}(1 - \cos xt)dt.
\]

Note that \( \alpha_\pm(t) \) and \( \beta_\pm(t) \) are all positive (for \( t > 0 \)); by Fatou’s lemma \( \liminf \alpha(t)/t = \liminf t^{-2} \int_0^\infty (1 - \cos tx)d(-\mu(x)) \geq \frac{1}{2}\sigma^2 \), so that \( \alpha(t)/t \to \infty \) under the present framework.

In order to find asymptotics of \( a \) we need to know asymptotics of \( \alpha(t) \) and \( \gamma(t) \) as \( t \downarrow 0 \) (which entail those of \( \alpha_\pm \) and \( \beta_\pm \) as functionals of \( \mu_\pm \)). Although the arguments given below are virtually the same as in [16], we give the full proofs since some of constants in [16] are wrong and need to be corrected—the values of the constants involved are not significant in [16] but turn out to be things of crucial importance in our proof of Theorem 1(i).

**Lemma 2.1.** For \( 0 < \varepsilon \leq 1 \),
\[
\varepsilon^{-1}(\sin \varepsilon)h_\varepsilon(1/t) \leq \alpha(t)/t \leq [\pi^2 c(1/t)] \wedge m(1/t),
\]
where \( s \wedge t = \min\{s,t\} \).

**Proof.** By monotonicity of \( \mu \) it follows that
\[
\alpha(t) > \left( \int_0^{\varepsilon/t} + \int_\pi^{(\pi+\varepsilon)/t} \right) \mu(z) \sin tz \, dz = \int_0^{\varepsilon/t} [\mu(z) - \mu(\pi/t + z)] \sin tz \, dz,
\]
which by \( \sin tz \geq \varepsilon^{-1}(\sin \varepsilon)tz \ (tz \leq \varepsilon) \) shows the first inequality of the lemma. The second inequality follows by observing that
\[
\alpha(t) < \int_0^{\pi/t} \mu(z) \sin tz \, dz \wedge [tc(1/t) + \eta(1/t)] \leq t[c(\pi/t) \wedge m(1/t)]
\]
and \( c(Mx) = M^2 \int_0^x \mu(Mu)udu \leq M^2 c(x) \) for \( M > 1 \). \( \square \)
Lemma 2.2.
\[
\begin{cases}
(a) \quad \frac{1}{2} \tilde{m}(1/t)t \leq \beta(t) \leq 2\tilde{m}(1/t)t, \\
(b) \quad \frac{1}{3} m(1/t)t \leq \alpha(t) + \beta(t) \leq 3m(1/t)t.
\end{cases}
\] (2.5)

Proof. Integrating by parts and using the inequality \(\sin u \geq (2/\pi)u\) \((u < \frac{1}{2}\pi)\) in turn we see
\[
\beta(t) = t \int_0^\infty \eta(x) \sin tx \, dx \geq \frac{2t}{\pi} \int_0^{\pi/2t} \eta(x)x \, dx + t \int_0^\infty \eta(x) \sin tx \, dx.
\]
Observing that the first term of the last member is equal to \(\frac{1}{2}t\tilde{m}(\pi/2t) \geq \frac{1}{2}t\tilde{m}(1/t)\) and the second one is to \(-\int_\pi^{2t} \mu(x) \cos txdx > 0\) we obtain the left-hand inequality of (2.5a). As for the right-hand one of (2.5a) we deduce from the definition that \(\beta(t) \leq \frac{1}{2}t\tilde{c}(1/t) + 2\eta(1/t) \leq 2\tilde{m}(1/t)\). The upper bound of (2.5b) is immediate from those in (1.3) and (a) just proved since \(\tilde{m}(x) \leq m(x)\). To verify the lower bound use (2.4) and the inequalities \(h_1(x) \geq c(x) - \frac{1}{2}x^2 \mu(x)\) and \(\sin 1 > 5/6\) to obtain
\[
[x(t) + \beta(t)]/t > (5/6)[c(1/t) - \mu(1/t)/2t^2] + 2^{-1}[\tilde{c}(1/t) + \eta(t)/t].
\]
By \(x^2 \mu(x) \leq [2c(x)] \wedge [3\tilde{c}(x)]\) it follows that \(\frac{5}{6} \mu(1/t)/2t^2 \leq \frac{1}{2} c(1/t) + \frac{1}{2} \tilde{c}(1/t)\) and hence \(\alpha(t) + \beta(t) > \frac{2}{6} c(1/t) + \frac{1}{2} \eta(1/t)/t > \frac{1}{3} m(1/t)t\) as desired. \(\Box\)

For \(t > 0\) define
\[
f(t) = \frac{1}{t^2 m^2(1/t)} \quad \text{and} \quad f^\circ(t) = \frac{1}{\alpha^2(t) + \gamma^2(t)}.
\]
Observe
\[
\left(\frac{x}{m(x)}\right)' = \frac{c(x)}{m^2(x)},
\]
(2.6) hence \(x/m(x)\) is increasing and \(f(t)\) is decreasing. By (2.5b) applied to \(\alpha_\pm + \beta_\pm\) in place of \(\alpha + \beta\) it follows that if \(m_+(x)/m(x) \to 0\), then
\[
\lim_{t \downarrow 0} \frac{\alpha_+(t) + \beta_+(t)}{\alpha_-(t) + \beta_-(t)} = 0
\]
(the converse is also true), which entails
\[
\alpha(t) + \beta_-(t) - \beta_+(t) \sim \alpha(t) + \beta(t) \sim m(1/t)t,
\]
and therefore \(f^\circ(t) \asymp f(t)\). Thus we obtain the following

Lemma 2.3. Suppose \(m_+(x)/m(x) \to 0\). Then \(f(t) \asymp 1/[\alpha^2(t) + \gamma^2(t)] (t \downarrow 0)\) so that
\[
\bar{a}(x) \asymp \int_0^\pi \frac{f(t)\alpha(t)}{t}(1 - \cos xt)dt \quad (x \to +\infty),
\]
(2.8) where the constants involved in \(\asymp\) are universal.

It is now easy to obtain the upper bound asserted in Theorem (ii). Note that (2.8) holds if \(|\gamma(t)/\beta(t)|\) is bounded away from zero.

Lemma 2.4. If (2.8) holds, then \(\bar{a}(x) \leq C x/m(x)\) for some constant \(C\).
Proof. We break the integral on the RHS of (2.8) into two parts

\[ J(x) = \int_{0}^{\pi/2x} \frac{f(t)\alpha(t)}{t} (1 - \cos xt) \, dt \quad \text{and} \quad K(x) = \int_{\pi/2x}^{\pi} \frac{f(t)\alpha(t)}{t} (1 - \cos xt) \, dt. \]

By (2.8) \( \bar{a}(x) \simeq J(x) + K(x) \). Using \( \alpha(t) \leq \pi^2 c(1/t) \) we observe

\[ \frac{K(x)}{\pi^2} \leq \int_{\pi/2x}^{\pi} \frac{2c(1/t)}{t^2 m^2(1/t)} \, dt = \int_{1/\pi}^{2\pi/\pi} \frac{2c(y)}{m^2(y)} \, dy = \left[ \frac{2y}{m(y)} \right]_{y=1/\pi}^{x} \leq \frac{x}{m(x)}. \quad (2.9) \]

Similarly

\[ \frac{J(x)}{\pi^2} \leq x^2 \int_{0}^{\pi/2x} f(t)c(1/t) t^2 \, dt = x^2 \int_{2\pi/\pi}^{\infty} \frac{c(y)}{y^2 m^2(y)} \, dy \]

and, observing

\[ \int_{x}^{\infty} \frac{c(y)}{y^2 m^2(y)} \, dy \leq \int_{x}^{\infty} \frac{dy}{y^2 m(y)} \leq \frac{1}{x m(x)}, \quad (2.10) \]

we apply \( m(2x/\pi) \geq m(x)2/\pi \) to have \( J(x) \leq \pi^4 x/m(x) \), finishing the proof.

If there exists a constant \( A > 0 \) such that

\[ \alpha(t) \geq Ac(1/t) \quad \text{for} \quad 0 < t \leq \pi, \]

then the lower bound for \( \bar{a} \) is easily deduced as we see shortly. Unfortunately condition (2.11) may fail to hold in general: in fact the ratio \( \alpha(t)/[c(1/t)] \) may oscillate between \( 1 - \varepsilon \) and \( \varepsilon \) infinitely many times for any \( 0 < \varepsilon < 1 \) (cf. Section 6.2). To cope with the situation the following lemma will be used.

Lemma 2.5. For all \( x > 0 \),

\[ h_u(x) \geq c(\varepsilon x) - (2\pi)^{-1} \varepsilon^2 x [\eta(\varepsilon x) - \eta(\pi x + \varepsilon x)]. \]

Proof. On writing \( h_u(x) = c(\varepsilon x) - \int_{0}^{\varepsilon x} u\mu(\pi x + u) \, du \), the integration by parts yields

\[ h_u(x) - c(\varepsilon x) = \int_{0}^{\varepsilon x} \left[ \eta(\pi x + u) - \eta(\pi x + \varepsilon x) \right] \, du, \]

by monotonicity and convexity of \( \eta \) it follows that if \( 0 < u \leq \varepsilon x \) (entailing \( 0 \leq \varepsilon x - u < \pi x \)),

\[ \eta(\pi x + u) - \eta(\pi x + \varepsilon x) \leq \frac{\varepsilon x - u}{\pi x} [\eta(\varepsilon x) - \eta(\pi x + \varepsilon x)]. \]

and substitution readily leads to the inequality of the lemma.

Corollary 2.1. Put \( \delta := \lim \inf c(x)/m(x) \) and suppose \( \delta > 0 \). Then there exists a positive constant \( A \) depending only on \( \delta \) such that \( \alpha(t) > A t\delta/(1/t) \) for all sufficiently small \( t \).

Proof. We can choose \( \varepsilon \) in Lemma 2.5 small enough that the premise of the lemma implies \( h_u(x) \geq c(\varepsilon x) - (2\pi)^{-1} \varepsilon m(\varepsilon x) \geq C_1 \varepsilon m(\varepsilon x) \geq C_1 \varepsilon m(x) \) for all \( x \) large enough and hence the assertion of the lemma follows from Lemma 2.1.
Lemma 2.6. Let $\delta > 0$ as in Corollary 2.1. Then for some constant $C > 0$ depending only on $\delta$,

$$ C^{-1}x/m(x) \leq \bar{a}(x) \leq Cx/m(x) \quad \text{for all } x \text{ large enough.} $$

Proof. By Corollary 2.1 condition (H) is satisfied under $\delta > 0$. Although this combined with Theorem 1 which will be shown independently of Lemma 2.6 in the next section, we here provide a direct proof.

Let $K(x)$ be as in the proof of Lemma 2.4. By Corollary 2.1 we may suppose that $\alpha(t)/t \geq Ac(1/t)$ with $A > 0$. Since both $c(1/t)$ and $f(t)$ are decreasing and hence so is their product, we see

$$ K(x)/A \geq \int_{\pi/2x}^{\pi} f(t)c(1/t)(1 - \cos xt)dt \geq \int_{\pi/2x}^{\pi} \frac{c(1/t)}{t^2m^2(1/t)}dt, $$

from which we deduce, as in (2.9), that

$$ K(x)/A \geq \left[ \frac{y}{m(y)} \right]_{y=\pi}^{2x/\pi} \geq \frac{2}{\pi} \cdot \frac{x}{m(x)} - \frac{1}{\eta(\pi)}. \quad (2.12) $$

Thus the assertion of Lemma 2.6 follows in view of Lemma 2.4.

3 Proof of Theorem 1(i)

By virtue of the right-hand inequality of (2.5b) $f^\circ(t) \geq \frac{1}{\ln x} f(t)$ and for the present purpose it suffices to bound the integral in (2.8) from below by a positive multiple of $x/m(x)$. We take for a lower bound of it the contribution from the interval $\pi/2x < t < 1$. We also employ the lower bound $\alpha(t) \geq \int_0^{2\pi/t} \mu(z) \sin tz \, dz$ and write down the resulting inequality as follows:

$$ \frac{\bar{a}(x)}{A} \geq \int_{\pi/2x}^{1} \frac{f(t)\alpha(t)}{t} (1 - \cos xt)dt $$

$$ \geq \int_{\pi/2x}^{1} \frac{f(t)}{t} (1 - \cos xt)dt \int_0^{2\pi/t} \mu(z) \sin tz \, dz $$

$$ = K_I(x) + K_{II}(x) + K_{III}(x), $$

where $A$ is a universal positive constant,

$$ K_I(x) = \int_{\pi/2x}^{1} \frac{f(t)}{t} \int_0^{\pi/2t} \mu(z) \sin tz \, dz, $$

$$ K_{II}(x) = \int_{\pi/2x}^{1} \frac{f(t)}{t} \int_{\pi/2t}^{2\pi/t} \mu(z) \sin tz \, dz $$

and

$$ K_{III}(x) = \int_{\pi/2x}^{1} f(t)(-\cos xt)\frac{dt}{t} \int_0^{2\pi/t} \mu(z) \sin tz \, dz.$$

Lemma 3.1.

$$ K_I(x) \geq \frac{5}{3\pi} \cdot \frac{x}{m(x)} - \frac{1}{m(1)}. $$
Proof. By \( \sin 1 \geq \frac{5}{6} \) it follows that \( \int_{0}^{\pi/2t} \mu(z) \sin tz \, dz > \int_{0}^{1/t} \mu(z) \sin tz \, dz \geq \frac{5}{6}tc(1/t) \), and hence
\[
K_I(x) > \frac{5}{6} \int_{\pi/2x}^{1} \frac{c(1/t)}{t^2m^2(1/t)} \, dt \geq \frac{5}{6} \int_{1}^{2x/\pi} \frac{c(z)}{m^2(z)} \, dz \geq \frac{5}{6m(1)} \cdot \frac{x}{m(2x/\pi)} - \frac{5}{6m(1)},
\]
implying the inequality of the lemma because of the monotonicity of \( m \).

**Lemma 3.2.** \( K_{III}(x) \geq -2\pi f(1/2)/x \) for all sufficiently large \( x \).

**Proof.** We claim that the function \( g(t) := t^{-1} \int_{0}^{2\pi/t} \mu(z) \sin tz \, dz \) is also decreasing. Observe that
\[
g'(t) = -\frac{1}{t^2} \int_{0}^{2\pi/t} \mu(z)(\sin tz - t \cos tz) \, dz,
\]
and that the integrand of the integral above has unique zero, say \( z_0 \), in the open interval \((0, 2\pi/t)\) (note that \( \sin u > u \cos u \) for \( 0 < u < \pi \)). Then it follows that
\[
g'(t) \leq -\mu(z_0) \frac{1}{t^2} \int_{0}^{2\pi/t} (\sin tz - t \cos tz) \, dz = 0,
\]
as claimed. Now \( f \) being decreasing, it follows that \(-K_{III}(x) \leq \int_{1}^{(2n+\frac{1}{2})\pi/x} f(t) \, dt \) for any integer \( n \) such that \((2n+\frac{1}{2})\pi/x \leq 1\), hence the inequality of the lemma.

**Lemma 3.3.**
\[
K_{II}(x) \geq -\frac{4}{3\pi} \cdot \frac{x}{m(x)} - \frac{1}{m(1)}.
\]

**Proof.** Since \( \mu \) is non-increasing, we have \( \int_{\pi/2t}^{2\pi/t} \mu(z) \sin tz \, dz \geq \int_{3\pi/2t}^{2\pi/t} \mu(z) \sin tz \, dz \), so that
\[
K_{II}(x) \geq \int_{\pi/2x}^{1} f(t) \frac{dt}{t} \int_{1/2}^{2\pi/t} \mu(z) \sin tz \, dz.
\]
We wish to make integration by \( t \) first. Observe that the region of the double integral is included in
\[
\{3\pi/2 \leq z \leq 4x; 3\pi/2 < t < 2\pi/z\},
\]
and hence
\[
K_{II}(x) \geq \int_{1}^{4x} \mu(z) \, dz \int_{3\pi/2z}^{2\pi/z} f(t) \frac{\sin tz}{t} \, dt,
\]
the integrand of the inner integral being negative. Put
\[
\lambda = 3\pi/2.
\]
Then, since \( f \) is non-increasing, the RHS is further bounded below by
\[
\int_{1}^{4x} \mu(z) f(\lambda/z) \, dz \int_{3\pi/2z}^{2\pi/z} \frac{\sin tz}{t} \, dt.
\]
The inner integral being equal to \( \int_{3\pi/2z}^{2\pi/z} \sin u t/u \) which is larger than \(-2/3\pi = -1/\lambda\), after change of variable we obtain
\[
K_{II}(x) \geq -\int_{1/\lambda}^{4x/\lambda} \mu(\lambda z) f(1/z) \, dz \geq -\int_{1/\lambda}^{x} \mu(\lambda z) f(1/z) \, dz.
\]
Recall \( f(1/x) = x^2/m^2(x) \). Since \( \int_{x}^{\infty} \mu(\lambda z)dz = \lambda^{-1}\eta(\lambda x) \) and, by integration by parts,
\[
-\int_{1/\lambda}^{x} \mu(\lambda z)f(1/z)dz = \frac{1}{\lambda} \left[ \eta(\lambda z)z^2 \right]_{z=1/\lambda}^{x} - \frac{2}{\lambda} \int_{1/\lambda}^{x} z\eta(\lambda z)c(z)z^2dz
\]
\[
\geq -\frac{2}{\lambda} \int_{1/\lambda}^{x} z\eta(z)c(z)dz - \frac{\eta(1)}{\lambda^2 m^2(1/\lambda)}.
\]
Noting \( x\eta(x) = m(x) - c(x) \) we have
\[
\frac{z\eta(z)c(z)}{m^3(z)} = \frac{c(z)}{m^2(z)} - \frac{c^2(z)}{m^3(z)}.
\]
Since \( \lambda m(1/\lambda) > m(1) > \eta(1) \), we conclude
\[
K_{II}(x) \geq -\frac{2}{\lambda} \int_{1/\lambda}^{x} \frac{c(z)}{m^2(z)}dz - \frac{1}{\lambda^2 m(1/\lambda)} \geq -\frac{4}{3\pi} \frac{x}{m(x)} - \frac{1}{m(1)}
\]
as desired. \( \square \)

**Proof of Theorem 1(i).** Combining Lemmas 3.1 through 3.3 and Lemma 2.3 we obtain
\[
\bar{a}(x)/A \geq \int_{\pi/2x}^{1} f(t)\alpha(t)\frac{1-\cos xt}{t}dt \geq \frac{1}{3\pi} \cdot \frac{x}{m(x)} - C
\]
with a constant \( C \) that may depend on \( 1/m(1) \), and hence Theorem 1(i).

### 4 Proof of Theorem 2

By (2.1) it follows that
\[
a(x) = \frac{1}{\pi} \int_{0}^{\pi} \frac{\alpha(t)(1-\cos xt) - \gamma(t) \sin xt}{[\alpha^2(t) + \gamma^2(t)]t}dt.
\]
Recalling \( \gamma(t) = \beta_+(t) - \beta_-(t) \), we put
\[
b_{\pm}(x) = \frac{1}{\pi} \int_{0}^{\pi} \frac{\beta_{\pm}(t) \sin xt}{[\alpha^2(t) + \gamma^2(t)]t}dt
\]
so that
\[
a(x) = \bar{a}(x) + b_-(x) - b_+(x). \tag{4.1}
\]

On choosing a number \( N \) such that \( E[|X|; |X| > N] \leq P[|X| \leq N] \), define a sequence \( p_*(x) \) on \( \mathbb{Z} \) by \( p_*(1) = E[|X|; |X| > N] \), \( p_*(0) = P[|X| \leq N] - p_*(1) \) and
\[
p_*(k) = \begin{cases} 
  p(k) + p(-k) & \text{if } k < -N, \\
  0 & \text{if } -N \leq k \leq -1 \text{ or } k \geq 2,
\end{cases}
\]
where \( p(k) = P[X = k] \). Then \( p_* \) is a probability distribution on \( \mathbb{Z} \) with zero mean. Denote the corresponding functions by \( a_*, b_{\pm*}, \alpha_*, \alpha_{\pm*} \), etc. Since \( p_*(z) = 0 \) for \( z \geq 2 \),
and $\sigma_\ast^2 = \infty$, we have for $x > 0$, $a_\ast(x) = 0$ so that $a_\ast(x) = a_\ast(x)/2 + b_-(-x) - b_+(x)$, hence
\[
a_\ast(x) = 2^{-1}a_\ast(x) = b_-(-x) - b_+(x).
\]
We shall show that if $m_+(x)/m(x) \to 0$, then
\[
\bar{a}(x) \sim \bar{a}_+(x), \quad |b_+(x)| + |b_-(x)| = o(\bar{a}(x)) \tag{4.2}
\]
and
\[
b_+(x) = b_-(x) + o(\bar{a}(x)). \tag{4.3}
\]
These together with (4.1) yield
\[
a(x) = \bar{a}(x)\{1 + o(1)\} + b_-(x) = 2^{-1}a_\ast(x)\{1 + o(1)\} + b_-(x) \sim a_\ast(x) \sim 2\bar{a}(x)
\]
and therefore $a(-x)/a(x) \to 0$.

The rest of this section is devoted to the proof of (4.2) and (4.3). It is easy to see
\[
\alpha_\ast(t) = \alpha(t) + O(t)
\]
\[
\beta_+(t) = \beta_- (t) + \beta_+(t) + O(t^2)
\]
\[
\beta_+(t) = \beta_+(1(1 - \cos t) = O(t^2)
\]
as $t \downarrow 0$. Let $D(t)$, $t > 0$ denote the difference
\[
D(t) := f^\circ(t) - f^\circ_*(t) = \frac{1}{\alpha^2(t) + \gamma^2(t)} - \frac{1}{\alpha^2_*(t) + \gamma^2_*(t)} = \left\{(\alpha^2_\ast - \alpha^2)(t) + (\gamma^2_\ast - \gamma^2)(t)\right\}f^\circ(t)f^\circ_*(t).
\]
Observe that $(\gamma_\ast + \gamma)(t) = -2\beta_- (t) + O(t^2)$, $(\gamma_\ast - \gamma)(t) = -2\beta_+(t) + O(t^2)$ and that
\[
(\alpha^2_\ast - \alpha^2)(t) = 2\alpha(t) \times O(t) \quad \text{and} \quad (\gamma^2_\ast - \gamma^2)(t) = 4\beta_- (t)\beta_+(t) + O(t^2). \tag{4.4}
\]
Now we suppose $m_+(x)/m(x) \to 0$. Then by (2.5) it follows that $f(t) \asymp f^\circ(t) \asymp f^\circ_*(t)$ and $\beta_- (t)\beta_+(t)/[\alpha^2(t) + \beta^2(t)] \to 0$, and hence that $D(t) = o(f(t))$, which implies that $\bar{a}(x) \sim \bar{a}_\ast(x)$, the first relation of (4.2). The proofs of the second relation in (4.2) and of (4.3) are somewhat involved since we need to take advantage of the oscillating nature of the integrals defining $\beta_\pm(t)$. First we dispose of the non-oscillatory parts of these integrals.

**Lemma 4.1.** If $m_+(x)/m(x) \to 0$, then
\[
\int_0^{1/x} f(t)\beta_+(t)dt = o(1/m(x)).
\]

**Proof.** By (2.5), (2.7), a change of variable and the concavity of $m$ the assertion of the lemma is the same as
\[
\int_x^\infty \frac{m_+(y)}{ym^2(y)}dy = o(1/m(x)).
\]
Putting $g(x) = \int_x^\infty dy/y^2m(y)$ and integrating by parts we have
\[
\int_x^\infty \frac{m_+(y)}{ym^2(y)}dy = -\left[ g(y)\frac{ym_+}{m}\right]_{y=x}^\infty + \int_x^\infty g(y)\left(\frac{ym_+}{m}\right)'dy
\]
as well as
\[ g(y) = \frac{1}{ym(y)} - \int_y^\infty \frac{\eta(u)}{um^2(u)} du. \]

On observing
\[ \left( \frac{y\tilde{m}_+}{m} \right)' = \frac{2y\eta_+}{m} - \frac{y\tilde{m}_+\eta}{m^2} \leq \frac{2y\eta_+}{m}, \]
substitution leads to
\[ \int_x^\infty \frac{\tilde{m}(y)}{ym^2(y)} dy \leq \frac{\tilde{m}_+(x)}{m^2(x)} + \int_x^\infty \frac{2\eta_+(y)}{m^2(y)} dy. \]

The first term on the RHS is \( o(1/m(x)) \) owing to the assumption of the lemma since \( \tilde{m}_+ \leq m_+ \). On the other hand on integrating by parts again
\[ \int_x^\infty \frac{\eta_+(y)}{m^2(y)} dy = -\frac{m_+(x)}{m^2(x)} + \int_x^\infty \frac{m_+(y)\eta(y)}{m^3(y)} dy \leq \sup_{y \geq x} \frac{m_+(y)}{m(y)} \int_x^\infty \frac{\eta(y)}{m^2(y)} dy = o(1/m(x)). \]

The proof is complete. \( \square \)

**Lemma 4.2.** If \( m_+(x)/m(x) \to 0 \),
\[ \int_1^x \frac{c_+(x)}{m^2(x)} dx = o\left( \frac{x}{m(x)} \right). \]

**Proof.** The assertion of the lemma follows from the following identity for primitive functions
\[ \int \frac{c_+(x)}{m^2(x)} dx = 2 \int \frac{c(x)}{m^2(x)} \cdot \frac{m_+(x)}{m(x)} dx - \frac{xm_+(x)}{m^2(x)}. \quad (4.5) \]
This identity may be verified by differentiation as well as derived by integration by parts, the latter giving
\[ \int \frac{c_+(x)}{m^2(x)} dx = \frac{x}{m_+(x)} \cdot \frac{m_+(x)}{m(x)} - 2 \int \frac{x}{m_+(x)} \cdot \frac{(\eta_+c - c_+\eta)m_+(x)}{m^3} dx, \]
from which we deduce (4.5) by an easy algebraic manipulation. \( \square \)

The following lemma is crucial (see also 4.6) in order to handle the oscillating part.

**Lemma 4.3.** For \( 0 < t < s \leq \pi \)
\[ |\alpha(t) - \alpha(s)| \lor |\beta(t) - \beta(s)| \leq |s - t|c(\pi/t) + \pi \mu(\pi/2t)/t \]
where \( s \lor t = \max\{s,t\} \).

**Proof.** By definition \( \beta(t) - \beta(s) = \int_0^\infty \mu(z)(\cos sz - \cos tz) dz \). Put
\[ I_t = \int_0^{3\pi/2t} \mu(z) \cos tz \, dz \quad \text{and} \quad J_t = \int_0^{\pi/2t} \mu(z) \cos tz \, dz. \]

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Then for \( t > 0 \),
\[
I_t \leq \int_0^\infty \mu(z) \cos tz \, dz \leq J_t.
\]
Let \( s > t \). Then
\[
|\beta(t) - \beta(s)| \leq (J_t - I_s) \vee (J_s - I_t).
\]
If \( J_t > I_s \),
\[
J_t - I_s \leq \int_0^{\pi/2t} \mu(z) |\cos tz - \cos sz| \, dz + \int_{\pi/2t}^{3\pi/2s} \mu(z)(-\cos sz) \, dz
\leq |t - s| c(\pi/2t) + \mu(\pi/2t)(\pi/t).
\]
If \( J_s > I_t \), then, on noting \( J \) is decreasing, \( 0 < J_s - I_t \leq J_t - I_t = -\int_{\pi/2t}^{3\pi/2t} \mu(z) \cos tz \, dz \leq \pi \mu(\pi/2t)/t \), thus
\[
|\beta(t) - \beta(s)| \leq |t - s| c(\pi/2t) + \pi \mu(\pi/2t)/t.
\]
In a similar way we see \( |\alpha(t) - \alpha(s)| \leq |t - s| c(\pi/t) + \pi \mu(\pi/t)/t. \)

Since \( c(\pi x) \leq \pi^2 c(x) \) and \( \frac{1}{4} x^2 \mu(x) \leq c(x) \) so that \( \mu(\pi/2t) \leq 2(2t/\pi)^2 c(\pi/2t) \), from the lemma 4.3 we may conclude that
\[
|\alpha(t) - \alpha(s)| \vee |\beta(t) - \beta(s)| \leq \kappa c(1/t) t \quad \text{if} \quad t \geq \pi/x \text{ and } s = t + \pi/x \tag{4.6}
\]
(with a universal constant \( \kappa \)). We shall apply Lemma 4.3 in this form in the sequel.

Proof of (4.6). We prove \( b_+(x) = o(a(x)) \) only, \( b_+ \) being dealt with in the same way. In view of Theorem 14 and Lemma 4.1 it suffices to show that
\[
\int_{\pi/x}^{\pi} \frac{f^\circ(t) \beta_+(t)}{t} \sin xt \, dt = o\left(\frac{x}{m(x)}\right) \quad \text{where} \quad f^\circ(t) := \frac{1}{\alpha^2(t) + \gamma^2(t)}.	ag{4.7}
\]
We make the decomposition
\[
2 \int_{\pi/x}^{\pi} \frac{(f^\circ \beta_+)(t)}{t} \sin xt \, dt = \int_{\pi/x}^{\pi} \frac{(f^\circ \beta_+)(t)}{t} \sin xt \, dt - \int_0^{\pi-\pi/x} \frac{(f^\circ \beta_+)(t + \pi/x)}{t + \pi/x} \sin xt \, dt
= I(x) + II(x) + III(x) + r(x)
\]
where
\[
I(x) = \int_{\pi/x}^{\pi} \frac{f^\circ(t) - f^\circ(t + \pi/x)}{t} \beta_+(t) \sin xt \, dt,
\]
\[
II(x) = \int_{\pi/x}^{\pi} \frac{f^\circ(t + \pi/x)}{t} \beta_+(t) - \beta_+(t + \pi/x) \sin xt \, dt,
\]
\[
III(x) = \int_{\pi/x}^{\pi} f^\circ(t + \pi/x) \beta_+(t + \pi/x) \left(\frac{1}{t} - \frac{1}{t + \pi/x}\right) \sin xt \, dt
\]
and
\[
r(x) = - \int_{\pi/x}^{2\pi/x} \frac{(f^\circ \beta_+)(t)}{t} \sin xt \, dt + \int_{\pi-\pi/x}^{\pi} \frac{(f^\circ \beta_+)(t)}{t} \sin xt \, dt.
\]
From (4.6) (applied not only with \( \mu \) but with \( \mu_+ \) in place of \( \mu \)) we obtain
\[
|\beta_+(t + \pi/x) - \beta_+(t)| \leq \kappa c_+(1/t) t
\]
and

\[|f^\circ(t + \pi/x) - f^\circ(t)| \leq C_1 c(1/t)[f(t)]^{3/2}t\]

for \(t > \pi/x\). From the last inequality together with \(f^{3/2}(t) = 1/t^2m^3(t), \beta_+(t) \leq C_3\tilde{m}_+(1/t)t\) and \(\tilde{m}_+(x) \leq m_+(x) = o(m(x))\) we infer that

\[|I(x)| \leq C \int_{1/2}^{x/2} \frac{\tilde{m}_+(y)}{m(y)} \cdot \frac{c(y)}{m^2(y)} dy = o\left(\frac{x}{m(x)}\right)\]

Similarly

\[|II(x)| \leq C \int_{1/2}^{x/2} \frac{c_+(y)}{m^2(y)} dy = o\left(\frac{x}{m(x)}\right)\]

and

\[|III(x)| \leq \frac{C}{x} \int_{1/\pi}^{x/2\pi} \frac{\tilde{m}_+(y)y}{m^2(y)} dy = o\left(\frac{x}{m(x)}\right),\]

where the equalities follow from Lemma 4.2 and the monotonicity of \(y/m(y)\) in the bounds of \(|II(x)|\) and \(|III(x)|\), respectively. Finally

\[|r(x)| \leq \int_{x/2\pi}^{x/\pi} \frac{\tilde{m}_+(y)}{m^2(y)} dy = o\left(\frac{x}{m(x)}\right).\]

Thus we have verified (4.7) and accordingly (4.2).

\[\square\]

**Proof of (4.3).** Recalling \(D(t) = f^\circ(t) - f^\circ_*\) we have

\[\pi\{b_- - b_\ast\} = \int_0^{\pi} D(t) \frac{\beta_-(t)}{t} \sin xt\, dt + \int_0^{\pi} f^\circ_* \frac{\beta_-(t) - \beta_\ast(t)}{t} \sin xt\, dt = J + K\]

(say).

Suppose \(m_+/m \to 0\). Since \(\beta_\ast(t) - \beta_-(t) = \beta_+(t) + O(t^2)\) and \(f^\circ_*\) is essentially of the same regularity as \(f^\circ\), the proof of (4.7) and Lemma 4.1 applies to \(K\) on the RHS above to yield \(K = o(x/m(x))\). As for \(J\) we first observe that in view of (4.4)

\[|D(t)| \leq C[f(t)]^{3/2}(t + \beta_+(t))\]

so that the integral defining \(J\) restricted to \([0, \pi/x]\) is \(o(x/m(x))\) in view of Lemma 4.1.

It remains to show that

\[\int_{\pi/x}^{\pi} D(t) \frac{\beta_-(t)}{t} \sin xt\, dt = o\left(\frac{x}{m(x)}\right).\]

We decompose \(D(t) = D_1(t) + D_2(t)\) where

\[D_1(t) = \{(\alpha_*^2 - \alpha^2)(t) + (\gamma_*^2 - \gamma^2)(t) - 4(\beta_\ast(t) - \beta_+(t))\} f^\circ(t) f^\circ_*(t)\]

and

\[D_2(t) = 4(\beta_\ast(t) - \beta_+(t)) f^\circ(t) f^\circ_*(t)\]

By (4.4) \(|D_1(t)\beta_-(t)/t| \leq C_1(\alpha(t) + t)[f(t)]^{3/2} \leq C_2 c(1/t)/[t^2m^3(1/t)]\) and

\[\left|\int_{\pi/x}^{\pi} D_1(t) \frac{\beta_-(t)}{t} \sin xt\, dt\right| \leq C_2 \int_{\pi/x}^{\pi} \frac{c(y)}{m^3(y)} dy = o(x/m(x)).\]
as is easily verified. For the integral involving $D_2$ we proceed as in the proof of (4.7). To this end it suffices to evaluate the integrals corresponding to $I(x)$ and $II(x)$, namely
\[ J_I(x) := \int_{\pi/x}^\pi \frac{D_2(t) - D_2(t + \pi/x)}{t} \beta_-(t) \sin xt \, dt, \]
and
\[ J_{II}(x) := \int_{\pi/x}^\pi D_2(t + \pi/x) \beta_-(t) - \beta_-(t + \pi/x) \sin xt \, dt, \]
the other integrals being easily dealt with as before. By (4.6) the integrand for $J_{II}(x)$ is dominated in absolute value by a constant multiple of $[f(t)]^{3/2} \beta_+(t)c(1/t)$ from which it follows immediately that $J_{II}(x) = o(x/m(x))$. For the evaluation of $J_I(x)$, observe that
\[ |(\beta_\beta_\beta_\beta(t) - (\beta_\beta_\beta_\beta(t) + \pi/x)| \leq C_1 \{ \beta_+(t)c(1/t) + \beta_-(t)c_+(1/t)t \}
\]
and
\[ |(f^o f)(t) - (f^o f)(t + \pi/x)| \beta_-(t) \leq C_1 c(1/t)[f(t)]^2 \]
so that
\[ \left| \frac{D_2(t) - D_2(t + \pi/x)}{t} \beta_-(t) \right| \leq C \{ \beta_+(t)c(1/t) + \beta_-(t)c_+(1/t)t \} [f(t)]^{3/2} \]
\[ \leq C' \frac{\tilde{\mu}(t)c(1/t)}{m(1/t)} + C' \frac{c_+(1/t)}{m^2(1/t)} \].

The integral of the first term of the last member is immediately evaluated and that of the second by Lemma 4.2, showing
\[ \left| \int_{\pi/x}^\pi D_2(t) \frac{\beta_-(t)}{t} \sin xt \, dt \right| \leq C_2 \int_{1/\pi}^\infty \left[ \frac{\tilde{\mu}(y)c(y)}{m^3(y)} + \frac{c_+(y)}{m^2(y)} \right] dy = o \left( \frac{x}{m(x)} \right). \]

The proof of (4.3) is complete. \( \square \)

5 Applications

Let $S^x$ be a random walk on $\mathbb{Z}$ started at $x$, where $X_1, X_2, X_3, \ldots$ are independent and have the same distribution as $X$. For $B \subset \mathbb{Z}$ let $\sigma^x_B$ denotes the first hitting time of $B$ by $S^x$ after time 0 so that we have always $\sigma^x_B \geq 1$. We write $S^x_{\sigma_B}$ or sometimes $S^x_{\sigma_B}$ for $S^x_{\sigma_B}$ and $\sigma^x_y$ for $\sigma^x_{S^x}$, in order to simplify the notation. Let $g_B(x, y) = \sum_{n=0}^\infty P[S^x_{\sigma_B} = y, \sigma^x_B > n]$, the Green function of the walk killed on $B$. Our definition of $g_B$ is not standard: $g_B(x, y) = \delta_{x,y} + E[g_B(S^x_{\sigma_B}, y) ; S^x \notin B]$ not only for $x \notin B$ but for $x \in B$, while $g_B(x, y) = \delta_{x,y}$ for all $x \in \mathbb{Z}, y \in B$. (Here $\delta_{x,y}$ designates Kronecker’s delta kernel.) The function $g_B(\cdot, y)$ restricted on $B$ equals the hitting distribution of $B$ by the dual walk started at $y$, in particular
\[ g_{\{0\}}(0, y) = 1 \quad (y \in \mathbb{Z}). \]

(The usual Green function of the walk killed on 0 is defined by $g(x, y) = g_{\{0\}}(x, y) - \delta_{0,x}$ so that $g(0, \cdot) = g(\cdot, 0) = 0$.) The potential function $a$ bears relevance to it through the identity
\[ g_{\{0\}}(x, y) = a^T(x) + a(-y) - a(x - y) \]
(cf. [15] p.328), which entails
\[
P[\sigma_y^x < \sigma_0^x] = \frac{a^\dagger(x) + a(-y) - a(x - y)}{2a(y)} \quad (y \neq 0, x). \tag{5.1}
\]

Here we put \(a^\dagger(x) = \delta_{x,0} + a(x)\). If the walk is left-continuous (i.e., \(P[X \leq -2] = 0\)), then \(a(x) = x/\sigma^2\) for \(x > 0\); analogously \(a(x) = -x/\sigma^2\) for \(x < 0\) for right-continuous walks; \(a(0) = 0\) and \(a(x) > 0\) for all \(x > 0\) except for the left-continuous walks.

To simplify the expression we write the condition \(\lim_{x \to \infty} m_+/m(x) = 0\); we also write \(\eta_+/\eta_- \to \infty, m_+ \asymp m\) etc. to indicate corresponding conditions. We shall use the letter \(R\) to denote a (large) positive integer without exception. We suppose \(P[X \geq 2] > 0\), the right-continuous walks being not interesting for the discussion in this section.

5.1. SOME ASYMPOTIC ESTIMATES OF \(P[\sigma^x_R < \sigma_0^x]\).

The potential function satisfies the functional equation
\[
\sum_{y = -\infty}^{\infty} p(y - x)a(y) = a^\dagger(x),
\tag{5.2}
\]
(cf. [15] p.352), which restricted on \(x \neq 0\) states that \(a\) is harmonic there so that the process \(M_n := a(S_{\sigma_0^x \land n})\) is a martingale for each \(y \neq 0\) and by the optional sampling theorem
\[
a(y) \geq E[a(S_{\sigma_0^x \land \sigma_y})] = a(x)P[\sigma_y^x < \sigma_0^x] \quad (y \neq 0). \tag{5.3}
\]

Lemma 5.1. For all \(x, y \in \mathbb{Z}\),
\[
-\frac{a(x)}{a(-x)} a(y) \leq a(x + y) - a(x) \leq a(y) \quad \text{if} \quad a(-x) \neq 0. \tag{5.4}
\]

Proof. Comparing (5.1) and (5.3) (with variables suitably chosen) we have
\[
\frac{a(y) + a(x) - a(x + y)}{a(x) + a(-x)} \leq \frac{a(y)}{a(-x)} (a(-x), y \neq 0),
\]
which, after rearrangement, becomes the left-hand inequality of (5.4), the case \(y = 0\) being obvious. The second one is the same as \(g_{\{0\}}(y, -x) \geq 0\).

Remark 5.1. (a) The right-hand inequality of (5.4) is the well known subadditivity of \(a\). The left-hand one, which seems much less familiar, will play a significant role in the sequel. On using [15] Theorem 30.2 it can be shown to be the same as \(\lim_{|w| \to \infty} P[\sigma^w_y < \sigma_0^w \land \sigma_{-x}^w] \geq 0\).

(b) The left-hand inequality of (5.4) may yield useful upper as well as lower bounds of the middle term. Here we write down such an example:
\[
-\frac{a(x - R)}{a(R - x)} \cdot \frac{a(-x)}{a^\dagger(x)} \leq \frac{a(-R) - a(x - R)}{a^\dagger(x)} \leq \frac{a(-R)}{a(R)}, \tag{5.5}
\]
where we substitute in (5.4) \(-R + x\) and \(-x\) for \(x\) and \(y\), respectively and divide by \(a^\dagger(x)\) for the lower bound and \(-R\) and \(x\) for \(x\) and \(y\), respectively and divide by \(-a^\dagger(x)\) for the upper bound. (5.5) is efficient in case \(a(-R)/a(R) \to 0\) and will be used later.

(c) By (5.1) and the subadditivity of \(a\)

\[
P[\sigma^x_0 < \sigma^x_y] = \frac{a(x) - a(y)}{2a(y)} \leq \frac{\bar{a}(x)}{a(y)} \quad (y \neq 0, x).
\]

**Lemma 5.2.** (i) If \(\lim_{z \to \infty} a(-z)/a(z) = \infty\), then uniformly for \(x \geq R\) as \(R \to \infty\)

\[
a(x) - a(x - R) \to 0 \quad \text{and} \quad P[\sigma^x_R < \sigma^x_0] \to 1.
\]

(ii) If \(\lim_{z \to \infty} a(-z)/a(z) = 0\), then uniformly for \(-M < x < R\) with any fixed \(M > 0\), as \(R \to \infty\)

\[
a(R) - a(x - R)/a^\dagger(x) \to 0 \quad \text{and} \quad P[\sigma^x_R < \sigma^x_0] = \frac{a^\dagger(x)}{a(R)}\{1 + o(1)\}.
\]

**Proof.** Suppose \(\lim_{z \to \infty} a(-z)/a(z) = 0\). This excludes the possibility of the left-continuity of the walk so that \(a^\dagger(x) > 0\) for all \(x\) and the first relation of (ii) follows immediately from (5.5) and implies the second one in view of (5.1). (i) is deduced from (5.4) in a similar way (substitute \(x - R\) and \(R\) for \(x\) and \(y\) respectively for the lower bound; use subadditivity of \(a\) for the upper bound).

Let \(Q\) and \(R\) be positive integers and put \(A = \{S^x_{\sigma(-Q)+} \text{ hits } R \text{ before } 0\}\), the event that \(S^x\), after its hitting \(-Q\), visits \(R\) before 0. Since

\[
\{\sigma^x_{-Q} < \sigma^x_R < \sigma^x_0\} = \{\sigma^x_{-Q} < \sigma^x_{\{0,R\}}\} \cap A \subset \{\sigma^x_{-Q} < \sigma^x_0\} \cap A,
\]

we have

\[
P[\sigma^x_{-Q} < \sigma^x_R < \sigma^x_0] \leq P[\sigma^x_{-Q} < \sigma^x_0]P[\sigma^x_R < \sigma^x_0].
\]

By the right-hand inequality in (5.5) (with \(x = -R\) and \(Q\) in place of \(R\))

\[
2\bar{a}(R)P[\sigma^x_{-Q} < \sigma^x_0] = a(-Q) + a(-R) - a(-Q - R)
\]

\[
\leq a(-Q) + \frac{a(-Q)}{a(Q)}a(-R).
\]

Combining these together verifies that if \(P[\sigma^x_{-Q} < \sigma^x_0] \sim a^\dagger(x)/a(Q)\), then

\[
P[\sigma^x_{-Q} < \sigma^x_R < \sigma^x_0] \leq \frac{a^\dagger(x)}{a(R)}\left\{\frac{a(-Q)a(-R)}{[a(Q)]^2}\{1 + o(1)\} + o(1)\right\}.
\]

5.2. Overshoots I.

Let \(Z\) stand for \(S^0_{\sigma(1,\infty)}\), the strictly ascending ladder height of the walk \(S^0\). In case \(E\bar{Z} < \infty\), according to a standard renewal theory the law of the overshoot \(S^0_{\sigma(1,\infty)} - R\) itself converges weakly as \(R \to \infty\) to a proper probability distribution \([8\ (XI.3.10)]\), whereas in case \(E\bar{Z} = \infty\), according to Kesten \([10\ Section 4]\) \(\limsup_{n \to \infty} Z_n/[Z_1 + \ldots + Z_n] \to \infty\) a.s. \(\text{as } n \to \infty\).
\[
\cdots Z_{n-1} = \infty \text{ a.s., where } Z_k \text{ are i.i.d. copies of } Z, \text{ or equivalently } \limsup_{R \to \infty} S_{\sigma[R,\infty]}^0/R = \infty \text{ a.s. It in particular follows that }
\]
\[
\lim_{R \to \infty} S_{\sigma[R,\infty]}^0/R = 1 \text{ a.s. if and only if } EZ < \infty.
\]

In this subsection we are interested in the convergence in probability which is often more significant than the a.s. convergence and holds true under a much weaker condition. In this respect the following result due to Rogozin [14] is relevant:

\[
Z \text{ is relatively stable if and only if } S_{\sigma[R,\infty]}^0/R \xrightarrow{P} 1; \text{ and for this to be the case it is sufficient that } X \text{ is positively relatively stable. (5.8)}
\]

Here an \(R\)-valued random variable \(\xi\) is called relatively stable if there exists a deterministic sequence \((B_n)\) such that \((\xi_1 + \cdots + \xi_n)/B_n \xrightarrow{P} 1\), where \(\xi_1, \xi_2, \ldots\) are i.i.d. random variables with the same distribution as \(\xi\) (the symbol "\(\xrightarrow{P}\)" designates convergence in probability); if \(B_n > 0\) in the above, we call \(\xi\) positively relatively stable according to [12].

By combining our Theorem 2 and a known criterion for relative stability of \(X\) (cf. [12] (see (1.15) in it), [11]) we obtain a reasonably fine sufficient condition for \(Z\) to be relatively stable. For condition (C1) in the following result we do not assume \(EX = 0\).

**Proposition 5.1.** For relative stability of \(Z\) each of the following conditions is sufficient.

(C1) \[
\lim_{x \to \infty} \int_0^x \frac{\mu_+(y) - \mu_-(y)}{x \mu(x)} dy = \infty.
\]

(C2) \[
EX = 0, \lim_{x \to \infty} x\eta_+(x)/m(x) = 0 \text{ and condition (H) holds.}
\]

**Proof.** According to [11] (see around (1.15) in it) condition (C1) is equivalent to the positive relative stability of \(X\) and hence it is a sufficient condition for relative stability of \(Z\) in view of (5.8). As for (C2), expressing \(P[S_{\sigma[R,\infty]}^0 > R + \varepsilon R]\) as the infinite series

\[
\sum_{w>0} g_{[R,\infty)}(0, R-w) P[X > \varepsilon R + w],
\]

one observes first that \(g_{[R,\infty)}(0, R-w) < g_{[0]}(-R, -w) \leq g_{[0]}(-R, -R) = 2\bar{a}(R)\), and then that if (H) holds (so that \(\bar{a}(x) \asymp x/m(x)\) by Theorem [11], for any \(\varepsilon > 0\)

\[
P[S_{\sigma[R,\infty]}^0 > R + \varepsilon R] \leq 2\bar{a}(R) \sum_{w>0} P[X > \varepsilon R + w] \asymp \frac{R\eta_+(\varepsilon R)}{m(R)} \leq \frac{R\eta_+(\varepsilon R)}{m(\varepsilon R)}
\]

and hence (C2) implies \(S_{\sigma[R,\infty]}^0/R \xrightarrow{P} 1\), concluding the proof in view of (5.8) again. \(\square\)

It is noted that the positive relative stability implies \(P[S_n] \to 1\), hence by what is mentioned in the above proof (C1) holds only if this is the case.

**Remark 5.2.** Suppose \(EX = 0\). Then:

(a) Condition (C1) is rephrased as \(\lim\{\eta_-(x) - \eta_+(x)\}/x \mu(x) = \infty\), which implies that \(\eta_-(x) \sim \eta(x)\), hence that both \(\eta\) and \(\eta_-\) are slowly varying at infinity by Karamata’s theorem [8 Theorem VIII.9.1(a)].
(b) Condition (C2) is satisfied if \( m_+/m \to 0 \). The converse is of course not true ((C2) may be fulfilled even if \( m_+/m \to \infty \)) and there does not seem to be any simpler substitute for (C2). Under the restriction \( m_+(x) \asymp m(x) \), however, (C2) holds if and only if \( m_+ \) is slowly varying (which is the case if \( x^2 \mu_+(x) \asymp L(x) \) with a slowly varying \( L \)). Indeed, under this same restriction the condition \( \eta_+/m \to 0 \) is the same as \( \eta_+/m_+ \to 0 \), which is equivalent to the slow variation of \( m_+ \) and implies \( c_+ \sim m_+ \asymp m \) so that (H) holds.

(c) If \( X \) belongs to the domain of attraction of a stable law and Spitzer’s condition holds (namely \( n^{-1} \sum P[S_n^0 > 0] \) converges), then that either (C1) or (C2) holds is also necessary for the relative stability of \( Z \) as will be discussed in Section 6.1.2.

The following result, used in the next subsection, concerns an overshoot estimate for the walk conditioned on avoiding the origin.

**Lemma 5.3.** (i) Suppose \( a(x) \asymp x/m(x) \ (x \geq 1) \) and let \( \delta \) be a positive number. Then for \( z > 0 \) and \( x < R \) satisfying \( P[\sigma_{[R,\infty]}^x < \sigma_0^x] \geq \delta \bar{a}^\dagger(x)/\bar{a}(R) \),

\[
P[S_{\sigma_{[R,\infty]}^x}^x > R + z \mid \sigma_{[R,\infty]}^x < \sigma_0^x] \leq 2\delta^{-1} \eta_+(z) \bar{a}(R).
\] (5.9)

(ii) If \( m_+/m \to 0 \), for each \( \epsilon > 0 \), uniformly for \( 0 \leq x < R \),

\[
P[S_{\sigma_{[R,\infty]}^x}^x > R + \epsilon R \mid \sigma_{[R,\infty]}^x < \sigma_0^x] \to 0 \ (R \to \infty).
\] (5.10)

The condition \( P[\sigma_{[R,\infty]}^x < \sigma_0^x] \geq \delta \bar{a}^\dagger(x)/\bar{a}(R) \) may be replaced by an obviously stronger restriction \( P[\sigma_{[R]}^x < \sigma_0^x] \geq \delta \bar{a}^\dagger(x)/\bar{a}(R) \), and the latter, easier to check and valid for any \( x \) fixed, will be used in our applications.

**Proof.** Suppose \( a(x) \asymp x/m(x) \ (x \geq 1) \) and put

\[
r(z) = P[S_{\sigma_{[R,\infty]}^x}^x > R + z, \sigma_{[R,\infty]}^x < \sigma_0^x].
\]

Plainly \( g_{(0)\cup [R,\infty]} \leq g_{(0)} \) and \( g_{(0)}(x, z) \leq 2\bar{a}^\dagger(x) \) (where \( 2\bar{a}^\dagger(x) = a^\dagger(x) + a^\dagger(-x) \)), hence

\[
r(z) = \sum_{w>0} g_{(0)\cup [R,\infty]}(x, R-w) P[X > z + w] \leq 2\bar{a}^\dagger(x) \sum_{w>0} P[X > z + w].
\]

Hence by \( \sum_{w>0} P[X > z + w] = \eta_+(z+1) \) it follows that if \( P[\sigma_{[R,\infty]}^x < \sigma_0^x] \geq \delta \bar{a}^\dagger(x)/\bar{a}(R) \),

\[
r(z) \leq 2\bar{a}^\dagger(x) \eta_+(z) \leq P[\sigma_{[R,\infty]}^x < \sigma_0^x] \times 2\delta^{-1} \bar{a}(R) \eta_+(z)
\]

and dividing by \( P[\sigma_{[R,\infty]}^x < \sigma_0^x] \) we find (5.9).

Suppose \( m_+/m \to 0 \). Then \( a(x) \asymp x/m(x) \) owing to Theorem 1 and for \( 0 \leq x < R \), \( a(-x)/a(x) \to 0 \) by virtue of Theorem 2. We can accordingly apply Lemma 5.2 to see that \( P[\sigma_R^x < \sigma_0^x] = \bar{a}^\dagger(x)/\bar{a}(R) \{1 + o(1)\} \) uniformly for \( 0 \leq x < R \), so that (5.9) obtains on the one hand. On the other hand recalling \( \eta_+(z) \sim m_+(z)/z \) we deduce that

\[
\bar{a}(R) \eta_+(z) \leq C \frac{m_+(z) \bar{a}(R)}{m(z) \bar{a}(z)},
\]

(5.11)
of which the last member with \( z = \epsilon R \) tends to zero. Thus (5.10) follows. \( \square \)
We shall need an estimate of overshoots as the walk exits from the half line \((-\infty,-R]\) after its entering into it. Put
\[
\tau^x(R) = \inf\{n > \sigma^x_{(-\infty,-R]} : S^x_n \notin (-\infty,-R]\},
\]
the first time when the walk exits from \((-\infty,-R]\) after once entering it.

**Lemma 5.4.** Suppose \(m_+/m \to 0\). Then for each constant \(\varepsilon > 0\), uniformly for \(x > -R\) satisfying \(P[\sigma_{(-\infty,-R]}^x < \sigma_0^x] \geq \varepsilon\tilde{a}(x)/\bar{a}(R)\), as \(R \to \infty\)
\[
P[S^x_{\tau(R)} > -R + \varepsilon R \mid \sigma_{(-\infty,-R]}^x < \sigma_0^x] \to 0.
\]

**Proof.** If the family of random variables \(S_{\sigma_{(-\infty,-R]}^x}^x/R\) is tight, then the assertion is immediate from the preceding lemma. To deal with the general case, we write down
\[
P[S^x_{\tau(R)} > -R + \varepsilon R \mid A^x] = \sum_{w \leq -R} P[S^x_{\sigma_{(-\infty,-R]}^x} = w \mid A^x]P[S^w_{\sigma_{(-R,\infty)}^w} > \varepsilon R - R],
\]
where put \(A^x = \{\sigma_{(-\infty,-R]}^x < \sigma_0^x\}\). If \(P[A^x] \geq \varepsilon\tilde{a}(x)/\bar{a}(R)\), by Lemma 5.3(i) (applied for \(-S^x\))
\[
P[S_{\sigma_{(-\infty,-R]}^x}^x < -R - z \mid A^x] \leq 2\varepsilon^{-1}m(z)/z.
\]
Given \(\delta > 0\) (small enough) we define \(\zeta = \zeta(\delta, R) (R > 0)\) by the equation
\[
\frac{m(\zeta)R}{m(R)\zeta} = \delta
\]
(unequally determined since \(x/m(x)\) is increasing), so that on using \(2\bar{a}(R) < CR/m(R)\)
\[
P[S_{\sigma_{(-\infty,-R]}^x}^x < -R - \zeta \mid A^x] \leq 2\varepsilon^{-1}m(\zeta)\bar{a}(R)/\zeta \leq (C\varepsilon^{-1})\delta.
\]
For \(-\zeta - R \leq w \leq -R,\)
\[
P[S_{\sigma_{(-R,\infty)}^w}^w > -R + \varepsilon R] = \sum_{y > -R + \varepsilon R} \sum_{z \leq -R} g(-R,\infty)(w, z)p(y - z)
\]
\[
= \sum_{y > \varepsilon R} \sum_{z \leq 0} g(1,\infty)(w + R, z)p(y - z)
\]
\[
\leq C_1 \bar{a}(\zeta - R)\eta_+(\varepsilon R)
\]
\[
\leq \delta^{-1}C'R\eta_+(\varepsilon R)/m(R),
\]
where the first inequality follows from \(g(1,\infty)(w + R, z) \leq g(1)(w + R, z) \leq \bar{a}(w + R - 1)\) and the second from \(\bar{a}(\zeta) \leq C_2\zeta/m(\zeta)\) and the definition of \(\zeta\). Now, reverting to (5.13) we apply the bounds derived above to see
\[
P[S_{\tau(R)}^x > -R + \varepsilon R \mid \sigma_{(-\infty,-R]}^x < \sigma_0^x] \leq (C\varepsilon^{-1})\delta + \delta^{-1}C'R\eta_+(\varepsilon R)/m(R),
\]
Since \(R\eta_+(\varepsilon R)/m(R) \to 0\) and \(\delta\) may be arbitrarily small, this concludes the proof. \(\square\)

If \(\tau\) is a stopping time of the walk \(S^0\) and \(A\) is a measurable event depending only on \(\{S^0_n, n \geq \tau\}\), then by strong Markov property \(P[A, \sigma^0_0 < \tau] = P(A)P[\sigma^0_0 < \tau]\) so that \(A\) is stochastically independent of \(\{\sigma^0_0 < \tau\}\) and hence of \(\{\sigma^0_0 > \tau\}\). In particular
\[
P[S_{\tau(R)}^0 = x] = P[S_{\tau(R)}^0 = x \mid \sigma_{(-\infty,-R]}^0 < \sigma_0^0]
\]
so that Lemma 5.4 yields the following
Corollary 5.1. Suppose $m_+/m \to 0$. Then for any $\varepsilon > 0$, as $R \to \infty$

$$P[S^0_{\tau(R)} > -R + \varepsilon R] \to 0.$$ 

If $F$ is attracted to the spectrally negative stable law of exponent one, then $S^0_{\sigma(-\infty,-R]/R} \overset{P}{\to} -\infty$ (cf. [?]) but, according to the above corollary, still $S^0_{\tau(R)}/R \overset{P}{\to} 0$, hence for a suitable function $M(R) \to \infty$, $S^{-R\text{M}(R)}/R \overset{P}{\to} 0$.

5.3. Comparison between $\sigma^x_R$ and $\sigma^y_{(R,\infty)}$.

The results obtained in this section are thought of as examining to what extent the results which are simple for the right-continuous walks to those satisfying $m_+/m_- \to 0$. The following result gives the asymptotic form in terms of $a$ of the probability of one-sided escape of the walk killed on hitting 0.

Proposition 5.2. (i) If $\lim_{z \to \infty} a(-z)/a(z) = \infty$, then as $R \to \infty$

$$P[\sigma^x_R < \sigma^x_0]/P[\sigma^x_{[R,\infty)} < \sigma^x_0] \to 1 \quad \text{uniformly for } x \in \mathbb{Z}.$$ 

(ii) Suppose $m_+/m \to 0$. Then $\lim_{z \to \infty} a(-z)/a(z) = 0$ and as $R \to \infty$

$$P[\sigma^x_{[R,\infty)} < \sigma^x_0] \sim P[\sigma^x_R < \sigma^x_0] \sim \frac{a^1(x)}{a(R)} \quad \text{uniformly for } 0 \leq x < R. \quad (5.15)$$ 

Proof. Let $A^x_R$ stand for the event $\{\sigma^x_{[R,\infty)} < \sigma^x_0\}$. The assertion of (i) then may be rephrased as

$$\lim_{R \to \infty} \inf_{x \in \mathbb{Z}} P[\sigma^x_R < \sigma^x_0 \mid A^x_R] = 1. \quad (5.16)$$ 

Make the decomposition

$$P[\sigma^x_R < \sigma^x_0 \mid A^x_R] = \sum_{z \geq R} P[S^x_\sigma_{[R,\infty) = z} \mid A^x_R]P[\bar{\sigma}^x_R < \sigma^x_0],$$

where $\bar{\sigma}^x_R$ is defined to be zero if $z = R$ and agree with $\sigma^x_R$ otherwise. By Lemma 5.2(i) $\inf_{z \geq R} P[\bar{\sigma}^x_R < \sigma^x_0] \to 1$ as $R \to \infty$ under the premise of (i) and we conclude (5.16).

Proof of (ii): Suppose $m_+/m \to 0$. Given $\varepsilon > 0$, we choose $u > 0$ so that

$$u/m(u) = \varepsilon R/m(R)$$

and let $z = z(R,\varepsilon)$ be the integer determined by $u - 1 < z \leq u$. Since $\bar{a}(x) \asymp x/m(x)$, this entails

$$\varepsilon C' \leq \bar{a}(z)/\bar{a}(R) < \varepsilon C''$$

for $R$ large enough with some universal positive constants $C'$, $C''$. Now define $h_{R,\varepsilon}$ via

$$P[\sigma^x_R < \sigma^x_0 \mid A^x_R] = h_{R,\varepsilon} + \sum_{R \leq y \leq R+z} P[S^x_\sigma_{[R,\infty) = y} \mid A^x_R]P[\bar{\sigma}^y_R < \sigma^y_0]$$

with $\bar{\sigma}^y_R$ defined as in the proof of (i). Then by Lemma 5.3(i) (see also (5.11))

$$h_{R,\varepsilon} \leq C \frac{m_+(z)}{m(z)} \frac{\bar{a}(R)}{\bar{a}(z)} \leq [C/\varepsilon C'] \frac{m_+(z)}{m(z)},$$

and still $h_{R,\varepsilon} \to 0$ as $R \to \infty$. This concludes the proof of (ii).
whereas \(1 - P[\sigma_R^y < \sigma_0^y] \leq \bar{a}(y-R)/\bar{a}(R) < \varepsilon C_1\) for \(R < y < R + z\) (see Remark 5.1(c) for the first inequality). Since \(m_+(z)/m(z) \to 0\) so that \(h_{R,\varepsilon} \to 0\), we conclude

\[
\liminf_{R \to \infty} \inf_{0 \leq x < R} P[\sigma^x_R < \sigma^0_0 | \sigma^x_{[R,\infty)} < \sigma^0_{[0,\infty)}] > 1 - \varepsilon C,
\]

hence \(P[\sigma^x_R < \sigma^0_0 | \sigma^x_{[R,\infty)} < \sigma^0_{[0,\infty)}] \to 1\), for \(\varepsilon\) can be made arbitrarily small. This verifies the first equivalence in (5.15). The second one follows from Lemma 5.2.

Letting \(x = 0\) in (5.15) gives that as \(R \to \infty\)

\[
P[\sigma^0_{[R,\infty)} < \sigma^0_0] \sim 1/a(R).
\]

Since the probability on the LHS is monotone this yields

**Corollary 5.2.** If \(m_+/m \to 0\), then \(\bar{a}(x)\) is asymptotically increasing in the sense that there exists an increasing function \(f(x)\) such that \(\bar{a}(x) = f(x)\{1 + o(1)\}\) \((x \to \infty)\).

Using the stochastic independence of the events \(\{S^0_{[x,\infty)} = x\}\) and \(\{\sigma^0_{[x,\infty)} < \sigma^0_0\}\) (cf. (5.14)), one derives that if \(f(x) := P[\sigma^0_{[x,\infty)} < \sigma^0_0]\), then

\[
H^{-x}_{[0,\infty)}(0) = P[S^0_{[x,\infty)} = x] = [f(x) - f(x + 1)]/f(x),
\]

which combined with Proposition 5.2 yields the following

**Corollary 5.3.** If \(m_+/m \to 0\), then uniformly for \(y > x\), as \(x \to \infty\)

\[
\sum_{x \leq z \leq y} H^{-z}_{[0,\infty)}(0) \sim \log \left[\frac{a(y)}{a(x)}\{1 + o(1)\}\right].
\]

Corollary 5.2 is used to extend the range of validity of (5.15) to negative values of \(x\) to some extent. We bring in the condition

\[
[a(-R) - a(-R + x)]/a^1(x) \to 0 \quad (x < R),
\]

which holds automatically for \(0 \leq x < R\) under \(m_+/m \to \infty\) owing to Lemma 5.2(ii).

**Lemma 5.5.** Suppose \(m_+/m \to 0\). Then as \(R \to \infty\)

(i) \(P[\sigma^x_{[R,\infty)} < \sigma^0_0] \leq [a^1(x)/a(R)]\{1 + o(1)\}\) uniformly for \(x < R\),

(ii) \(P[\sigma^x_{[R,\infty)} < \sigma^0_0] \sim P[\sigma^x_R < \sigma^0_0] \sim [a^1(x)/a(R)]\) uniformly for \(x < 0\) subject to condition (5.17) as well as for \(0 \leq x < R\).

**Proof.** Since \(a(S^x_{[0,\infty)})\) is a martingale, applying the optional stopping theorem and Fatou’s lemma one obtains

\[
E[a(S^x_{[0,\infty)} \wedge \sigma_{[R,\infty)})] \leq a^1(x).
\]

The expectation on the LHS is bounded below by \(a(R)P[\sigma^x_{[R,\infty)} < \sigma^0_0] \{1 + o(1)\}\) owing to Corollary 5.2 (applicable under \(m_+/m \to 0\)), hence (i) follows.

Under the constraint (5.17) the second relation in (ii) holds owing to (5.17), and (ii) then follows from (i) in view of the trivial inequality \(P[\sigma^x_{[R,\infty)} < \sigma^0_0] \geq P[\sigma^x_R < \sigma^0_0].\)
Remark 5.3. In view of (5.5), \( \sup_{x < R} [a(-R) - a(-R + x)]/a(x) \to 0 \) so that the upper bound for (5.17) is always valid while the lower bound is problematic. The extent of \( x \) for which (5.17) holds depends on \( F \); if \( a(-z) \) is bounded for \( z < 0 \) it holds for \( x < R \), otherwise it fails to hold if \( -x \approx R \) in most cases of \( F \), while it holds if \( x = o(R) \) at least under some regularity of the tails of \( F \) (cf. Proposition 5.1).

5.4. Escape into \( |x| > R \).

Let \( Q \) as well as \( R \) be a positive integer. Here we consider the event \( \sigma_{x < \infty, -Q}^{\infty - Q \cup \{0, \infty \}} < \sigma_0^R \), the escape into \((-\infty, -Q) \cup (R, \infty)\) from the killing at \( 0 \). The next result is essentially a corollary of Proposition 5.2 and Lemma 5.5(ii).

**Proposition 5.3.** If \( m_+ / m \to \infty \), then uniformly for \( -Q < x < R \) subject to (5.17), as \( Q \wedge R \to \infty \)

\[
P[\sigma_{\infty, -Q}^{\infty - Q} < \sigma_0^R] = P[\sigma_{x < Q}^R < \sigma_0^R] \{1 + o(1)\}. \tag{5.18}
\]

*Proof.* Put \( \tau_+ = \sigma_{\infty, -Q}^R \) and \( \tau_0^x = \sigma_{\infty, R}^R \). It suffices to show that

\[
\frac{P[\tau_+ \wedge \tau_0^x < \sigma_0^R]}{P[\tau_0^x < \sigma_0^R]} \to 0.
\]

The numerator of the ratio above is less than

\[
P[\tau_+ < \sigma_0^R] + P[\tau_0^x < \sigma_0^R] = P[\sigma_{x}^{-Q} \times o(1) + P[\sigma_{x}^R < \sigma_0^R] \times o(1),
\]

where Lemma Proposition 5.2 and Lemma 5.5(ii) are applied for the bounds of the first and second terms on the LHS, respectively. Hence it is of small order of the denominator.

For any subset \( B \) of \( \mathbb{R} \) such that \( B \cap \mathbb{Z} \) is non-empty, define

\[
H_B^R(y) = P[S_\sigma^R = y] \quad (y \in B), \tag{5.19}
\]

the hitting distribution of \( B \) for the walk \( S^\tau \). Let \( B(Q, R) = \{-Q, 0, R\} \). Then (5.18) is rephrased as

\[
P[\sigma_{\infty, -Q}^{\infty - Q} < \sigma_0^R] \sim 1 - H_B^{R(Q,R)}(0). \tag{5.20}
\]

By using Theorem 30.2 of Spitzer [15] one can compute an explicit expression of \( H_B^{R(Q,R)}(0) \) in terms of \( a(\cdot) \). The following result however is derived without using it.

**Lemma 5.6.** Suppose \( m_+ / m \to 0 \). Then uniformly for \( -Q < x < R \), as \( Q \wedge R \to \infty \)

\[
P[\sigma_{x}^{-Q} < \sigma_0^R] \geq 1 - H_B^{R(Q,R)}(0) - P[\sigma_{x}^R < \sigma_0^R] P[\sigma_{x}^{-Q} < \sigma_0^R] \geq P[\sigma_{x}^{-Q} < \sigma_0^R] P[\sigma_{x}^{-Q} < \sigma_0^R].
\]

*Proof.* Write \( B \) for \( B(Q, R) \). Then plainly we have

\[
1 - H_B^R(0) = P[\sigma_{x < \infty, -Q}^x < \sigma_0^R] = P[\sigma_{x < \infty, -Q}^x < \sigma_0^R] + P[\sigma_{x < \infty, -Q}^x < \sigma_0^R] - P[\sigma_{x < \infty, -Q}^x \wedge \sigma_{x < \infty, -Q}^R < \sigma_0^R]. \tag{5.21}
\]

Let \( A^x \) denote the event \( \{\sigma_{x < \infty, -Q}^x \wedge \sigma_{x < \infty, -Q}^R < \sigma_0^R \} \). Then

\[
A^x \subset \{\sigma_{x < \infty, -Q}^R < \sigma_0^R \} \quad \text{and} \quad \{\sigma_{x < \infty, -Q}^x \wedge \sigma_{x < \infty, -Q}^R < \sigma_0^R \} \subseteq A^x \subset \{\sigma_{x < \infty, -Q}^x < \sigma_0^R \}. \]
By (5.6) \( P[\sigma_{z_Q} < \sigma_{R}^z < \sigma_{0}^z] \leq P[\sigma_{z,Q}^z < \sigma_{0}^Q]P[\sigma_{R}^Q < \sigma_{0}^{-Q}] \). It therefore follows that

\[
0 \leq P[\sigma_{z,Q}^z \vee \sigma_{R}^z < \sigma_{0}^z] - P(A^x) \leq P[\sigma_{z,Q}^z < \sigma_{0}^z]P[\sigma_{R}^Q < \sigma_{0}^{-Q}],
\]

which together with \( P(A^x) = P[\sigma_{R}^z < \sigma_{0}^z]P[\sigma_{R}^Q < \sigma_{0}^Q] \) substituted into (5.21) yields the relation of the lemma.

Proof. On taking

\[
H^x_B(y) = u_B(y) + \sum_{z \in B^x(y)} a(x-z)H^x_B(y) - a^\dagger(x-y)(1 - H^x_B(y)) \quad (y \in B, x \in \mathbb{Z}),
\]

(5.22)

and accordingly

\[
1 - H^x_B(0) = [a(-R) - a(x-R)]H^R_B(0) + [a(Q) - a(x+Q)]H^{-Q}_B(0) + a^\dagger(x)(1 - H^0_B(0)).
\]

(5.23)

Noting \( P[\sigma_{z}^y < \sigma_{z}^x] = 1/2\bar{a}(x-y) \) is symmetric we see that \( H^R_{B(Q,R)}(0) \vee H^{-Q}_{B(Q,R)}(0) \leq P[\sigma_{R}^0 < \sigma_{0}^0] \vee P[\sigma_{-Q}^0 < \sigma_{0}^0] \leq 1 - H^0_{B(Q,R)}(0) \). It therefore follows that uniformly for \( -Q < x < R \) subject to the condition

\[
|a(-R) - a(x-R)| + |a(Q) - a(x+Q)| = o(a^\dagger(x))
\]

(5.24)

as \( R \rightarrow \infty \)

\[
1 - H^x_{B(Q,R)}(0) \sim a^\dagger(x)(1 - H^0_{B(Q,R)}(0)).
\]

(5.25)

(Identity (5.22) and hence what are mentioned right above hold true for every recurrent random walk irreducible on \( \mathbb{Z} \).) Condition (5.24) (to be understood to entail \( a^\dagger(x) \neq 0 \)—always satisfied for each \( x \) (fixed) with \( a^\dagger(x) \neq 0 \)—is necessary and sufficient for the following to hold:

\[
P[\sigma_{z,Q}^z < \sigma_{0}^z] \sim a^\dagger(x)/a(Q) \quad \text{and} \quad P[\sigma_{R}^z < \sigma_{0}^z] \sim a^\dagger(x)/a(R).
\]

(5.26)

Corollary 5.4. Suppose \( m_+/m \rightarrow 0 \). Then uniformly for \( -Q < x < R \) subject to condition (5.24), as \( Q \wedge R \rightarrow \infty \)

\[
P[\sigma_{(-\infty,-Q]\cup[R,\infty)}^z < \sigma_{0}^z] \sim a^\dagger(x)(1 - H^0_{B(Q,R)}(0))) \sim a^\dagger(x)\frac{a(Q + R)}{a(Q)a(R)}. \]

(5.27)

Proof. On taking \( x = 0 \) in the formula of Lemma 5.6 its right-hand inequality yields

\[
1 - H^0_{B(Q,R)}(0) \geq \frac{a(Q + R) + a(-Q) - a(R)}{4a(R)a(Q)} + \frac{a(-Q - R) + a(R) - a(-Q)}{4a(Q)a(R)}
\]

\[
= \frac{a(Q + R) + a(-Q - R)}{4a(Q)a(R)}
\]

\[
= \frac{a(Q + R)}{a(Q)a(R)}\{1 + o(1)\},
\]

and the left-hand one gives the corresponding lower bound, showing the second relation of (5.27). Proposition 5.3 combined with (5.25) (valid under condition (5.24)) verifies the first relation of (5.27).
Remark 5.4. (a) If \( \liminf m_+(x)/m(x) > 0 \) (and \( \sigma^2 = \infty \)), (5.20) would fail to be true (for each \( x \)) for a large class of \( F \) (see Remark 6.2).

The following condition is relevant in the next result: as \( Q \land R \to 0 \)
\[
a(-Q)a(-R)/[a(Q)]^2 \to 0. \tag{5.28}
\]

**Proposition 5.4.** Suppose \( m_+ / m \to 0 \). Then as \( Q \land R \to \infty \), for \( -Q < x < R \) subject to (5.24),

(i) under \( (5.28) \), \( H^x_{B(Q,R)}(R) \sim a^+(x)/a(R) \) and
\[
P[\sigma^x_R < \sigma^x_Q \land \sigma^x_{(-Q,R)} < \sigma^x_0] \sim a(Q)/a(Q + R). \tag{5.29}
\]

(ii) if \( 0 < \lim \inf Q/R \leq \lim \sup Q/R < \infty \) in addition,
\[
P[\sigma^x_{(-Q,R),\infty} < \sigma^x_{(-\infty,-Q)} \land \sigma^x_{(-\infty,-Q) \cup (R,\infty)} < \sigma^x_0] \sim a(Q)/a(Q + R).
\]

**Proof.** By decomposing
\[
\{\sigma^x_R < \sigma^x_0\} = \{\sigma^x_R < \sigma^x_Q < \sigma^x_0\} + \{\sigma^x_R < \sigma^x_Q \land \sigma^x_0\} \tag{5.30}
\]
(+ designates the disjoint union) it follows that
\[
H^x_{B(Q,R)}(R) = P[\sigma^x_R < \sigma^x_Q \land \sigma^x_0] = P[\sigma^x_R < \sigma^x_0] - P[\sigma^x_Q < \sigma^x_0].
\]

Owing to (5.24) we have (5.26), and by (5.7)
\[
P[\sigma^x_Q < \sigma^x_R < \sigma^x_0] \leq \frac{a^+(x)}{a(R)} \left\{ \frac{a(-Q)a(-R)}{[a(Q)]^2} \{1 + o(1)\} + o(1) \right\}.
\]

Hence if (5.28) holds, \( H^x_{B(Q,R)}(R) \sim a^+(x)/a(R) \). Now (i) follows immediately from (5.24) and (5.27).

For the proof of (ii) let \( \tau^x(Q) \) be the first time \( S^x \) exits from \((-\infty,-Q)\) after its entering this half line (see (5.12)) and \( A^x \) denote the event \( \{\sigma^x_{(-\infty,-Q)} < \sigma^x_0\} \). Then
\[
P[\sigma^x_{(-\infty,-Q)} < \sigma^x_{(-\infty,-Q)} < \sigma^x_0] \leq \sum_{y > -Q} P[S^y_{\tau^x(Q)} = y, A^x] P[\sigma^y_{(-Q,R)} < \sigma^y_0] \]
\[
= \sum_{0 > y > -Q} P[S^y_{\tau^x(Q)} = y, A^x] P[\sigma^y_{(-Q,R)} < \sigma^y_0] + P(A^x) \times o(1),
\]

where \( o(1) \) is due to Lemma 5.4 applied with \( -Q \) in place of \( -R \).

We claim that if \( \lim \sup Q/R < \infty \), \( P[\sigma^y_{(-Q,R)} < \sigma^y_0] \to 0 \) uniformly for \( -Q < y < 0 \), which combined with the bound above yields
\[
P[\sigma^x_{(-\infty,-Q)} < \sigma^x_{(-\infty,-Q)} < \sigma^x_0] = P(A^x) \times o(1). \tag{5.31}
\]

Since for any \( \varepsilon > 0 \), \( P[S^y_{\sigma^y_{(-Q,R)}} \geq \varepsilon Q] \to 0 \) uniformly for \( -Q < y < 0 \) owing to Proposition 5.1 it follows that under \( \lim \sup Q/R < \infty \)
\[
P[\sigma^y_{(-Q,R)} < \sigma^y_0] \leq \sum_{0 < z < R} P[S^y_{\sigma^y_{(-Q,R)}} = z] P[\sigma^z_{(-Q,R)} < \sigma^z_0] + o(1) \leq P[\sigma^y_R < \sigma^y_0] + o(1) \to 0,
\]

where Proposition 5.2(ii) is used for the second inequality. Thus the claim is verified.

For the rest we can proceed as in the proof of (i) above with an obvious analogue of (5.30). Under (5.24), by Lemma 5.5(ii) \( P[\sigma^x_{(-Q,R)} < \sigma^x_0] \sim a^+(x)/a(R) \) and by Proposition 5.2(i) \( P(A^x) \sim a^+(x)/a(Q) \). Using (5.27) together with (5.31) we now obtain the asserted result as in the same way as above. The proof of Proposition 5.4 is finished. \( \square \)
Corollary 5.5. Suppose \( m_+ / m \to 0 \). Then as \( Q \to \infty \) under \( \lim \inf Q / R > 0 \),
\[
P[\sigma^0_{[R,\infty]} < \sigma^0_{(-\infty,-Q)}] \sim P[\sigma^0_{R} < \sigma^0_{-Q}] \sim a(Q) / a(Q + R).
\]

Proof. If \( \lim Q / R < \infty \) is supposed in addition, the result follows from Proposition 5.4 as Corollary 5.1 did from Lemma 5.4. Thus it only remains to address the case \( Q / R \to \infty \), in which however the asserted formula is trivial since each member in it tends to one as is deduced directly for the second and third ones and then from the monotonicity for the first. \( \square \)

5.5. Comparison between \( g_{[0]}(x, y) \) and \( g_B(x, y) \).

In the proof of Proposition 5.1 we have replaced \( g_{[0,\infty)} \) by its upper bound \( g_{[0]} \) in order to make use of some estimates that are available. It may be worth to evaluate the error of the replacement for consideration of in what situation it neglects a significant quantity or not.

For a subset \( B \subset \mathbb{Z} \) such that \( 0 \in B \)
\[
g_{[0]}(x, y) = g_B(x, y) + E[g(S^r_{\sigma_B}, y)] \quad (x, y \in \mathbb{Z} \setminus B),
\]
where \( g(x, y) = a(x) + a(-y) - a(x - y) \). In below we suppose \( a(-x) / a(x) \to 0 \) \((x \to \infty)\).

Let \( \hat{z} \) denote the first strictly descending ladder height variable. The supposition made right above implies \( E|\hat{z}| = \infty \) and hence by [15] \( H^r_{(-\infty,0)} \{z\} \to 0 \ (x \to \infty) \) for all \( z < 0 \). For if \( E|\hat{z}| < \infty \), then \( m_{-}(x) / m_{x} \to 0 \) [16] and hence \( a(-x) / a(x) \to \infty \), a contradiction.

Let \( B = (-\infty, 0] \). Then, uniformly for \( y > 0 \), as \( x \to \infty \)
\[
E[g(S^r_{\sigma_{(-\infty,0]}}, y)] \leq \{1 + o(1)\} a(-y),
\]
for \( S^r_{\sigma_{(-\infty,0)}} \) \( \xrightarrow{P} -\infty \) as \( x \to \infty \) and by Lemma 5.1 \( g(z, y) \leq \{1 + a(z) / a(-z)\} a(-y) \) for \( z < 0 \). On the other hand
\[
g_{[0]}(x, y) \sim g_{[0]}(x, x) \quad \text{as} \quad x \to \infty \quad \text{uniformly for} \quad y > x
\]
Indeed by Lemma 5.1 again \( a(-y) - a(x - y) \geq -[a(x - y) / a(y - x)] a(-x) \), of which the RHS divided by \( a(x) \) tends to zero. Consequently, for \( y > x \), the replacement of \( g_B(x, y) \) by \( g_{[0]}(x, x) \) would yield no significant error for \( y > x \) as far as \( a(-y) \ll a(x) \).

In case \( B = [0, \infty) \), uniformly as \( x \vee y \to -\infty \)
\[
E[g(S^r_{\sigma_{[0,\infty]}}, y)] = \begin{cases} 
    o(a(x)) & \text{if} \ EZ < \infty, \\
    a(x) \{1 + o(1)\} & \text{if} \ EZ = \infty.
\end{cases}
\]
Indeed, uniformly for \( z > 0 \), \( g(z, y) \leq 2a(z) = a(z) \{1 + o(1)\} \) as \( z \to \infty \) and the equality above follows from \( E[a(S^r_{\sigma_{[0,\infty]}})] = o(a(x)) \) or \( a(x) \{1 + o(1)\} \) according as \( EZ \) is finite or infinity (cf. [19, Corollary 1]). Using Lemma 5.1 we deduce as before that \( g_{[0]}(x, y) = a(-y) + O(a(y)) \) for \( x < y < 0 \). Hence for \( x < y < 0 \), the replacement of \( g_B(x, y) \) by \( g_{[0]}(y, y) \) may therefore be made with little error as far as \( a(x) = o(a(-y)) \).
6 Examples

Here we give two examples of different nature. The first one is the case when the law of $X$ belongs to the domain of attraction of a stable law; we compute the exact asymptotic form of $a(x)$ and describe behaviour of the (one-sided) overshoot. The second one exhibits how $a(x)$ and/or $c(x)/m(x)$ can behave in irregular ways.

6.1. Distributions in domains of attraction.

In this subsection we suppose (in addition to (1.2)) that $X$ belongs to the domain of attraction of a stable law with exponent $1 \leq \alpha \leq 2$, or equivalently

\begin{align*}
(a) \quad \int_{-\infty}^{x} y^2 dF(y) &\sim 2L_\alpha(x) \quad \text{if } \alpha = 2 \\
(b) \quad \mu_+(x) &\sim (1-p)x^{-\alpha}L(x) \quad \mu_-(x) \sim px^{-\alpha}L(x) \quad \text{if } 1 \leq \alpha < 2 \tag{6.1}
\end{align*}

as $x \to \infty$. Here and in the sequel $L$ and $L_\alpha$ are always positive and slowly varying at infinity and $0 \leq p \leq 1$. Note that (6.1b) is equivalent to $\int_{0}^{x} y\mu(y) dy \sim L_\alpha(x)$ and holds with $L_\alpha(x) = \int_{0}^{x} L(y) dy/y$ if $\mu(x) \sim L(x)/x^2$ (the converse is not true in general). Let $Y$ be the limiting stable variable whose characteristic function $\Psi(t) = Ee^{itY} = e^{-\Phi(t)}$ is given by

$$\Phi(t) = \begin{cases} \frac{C_\Phi|t|^\alpha}{1 - i\text{sgn}(t)(p - q) \tan \frac{1}{2} \alpha \pi} & \text{if } 1 \leq \alpha < 2, \\
\frac{C_\Phi|t|\left(\frac{1}{2} \pi + i\text{sgn}(t)(p - q) \log |t|\right)} & \text{if } \alpha = 1,
\end{cases}$$

where $q = 1 - p$, $C_\Phi$ is some positive constants that depend on the scaling factors and $\text{sgn} t = t/|t|$ (cf. [8, (XVII.3.18-19)]). It is also supposed that $\int_{1}^{\infty} L(x)x^{-1}dx < \infty$ if $\alpha = 1$ and $\lim L(x) = \infty$ if $\alpha = 2$, so that $E|X| < \infty$ (so as to conform to $EX = 0$ and $EX^2 = \infty$.

6.1.1. Asymptotics of $a(x)$.

If $\alpha = 2$, condition (6.1b) implies $x^2\mu(x) = o(L_\alpha(x)) \tag{8 XVII.5.16])$ and hence is equivalent to $c(x) \sim L_\alpha(x)$ (as one sees by integrating $y^2dF(y)$ by parts), which together show

\[ \tilde{m}(x) = x\eta(x) + \tilde{c}(x) = o(m(x)) \quad \text{if } \alpha = 2. \tag{6.2} \]

Put $L^*(x) = \int_{x}^{\infty} y^{-1}L(y)dy$. Then

\[ \eta(x) = \begin{cases} o(c(x)/x) & \text{if } \alpha = 2, \\
(\alpha - 1)^{-1}x^{1-\alpha}L(x)\{1 + o(1)\} & \text{if } 1 < \alpha < 2, \\
L^*(x)\{1 + o(1)\} & \text{if } \alpha = 1
\end{cases} \tag{6.3} \]

and

\[ c(x) = \begin{cases} L_\alpha(x)\{1 + o(1)\} & \text{if } \alpha = 2, \\
(2 - \alpha)^{-1}x^{2-\alpha}L(x)\{1 + o(1)\} & \text{if } 1 < \alpha < 2, \\
L_\alpha(x) & \text{if } \alpha = 1\tag{6.4}
\end{cases} \]

where $c(x)$ is defined by (2.2); accordingly

\[ m(x) \sim \begin{cases} L_\alpha(x) & \text{if } \alpha = 2, \\
x^{2-\alpha}L(x)/[(2 - \alpha)(\alpha - 1)] & \text{if } 1 < \alpha < 2, \\
xL^*(x) & \text{if } \alpha = 1.
\end{cases} \]

The derivation is straightforward.
Asymptotics of $\alpha(t)$ and $\beta(t)$ as $t \to 0$ are given as

$$
\alpha(t) \sim \begin{cases} 
  tL_e(1/t) & \text{if } \alpha = 2, \\
  \kappa'_{\alpha}t^{\alpha-1}L(1/t) & \text{if } 1 < \alpha < 2, \\
  \frac{1}{2}\pi L(1/t) & \text{if } \alpha = 1,
\end{cases}
$$

and

$$
\beta(t) = \begin{cases} 
  o(\alpha(t)) & \text{if } \alpha = 2, \\
  \kappa''_{\alpha}t^{\alpha-1}L(1/t)\{1 + o(1)\} & \text{if } 1 < \alpha < 2, \\
  L^*(1/t)\{1 + o(1)\} & \text{if } \alpha = 1,
\end{cases}
$$

where $\kappa'_{\alpha} = \Gamma(1 - \alpha)\cos \frac{1}{2}\pi\alpha$ and $\kappa''_{\alpha} = -\Gamma(1 - \alpha)\sin \frac{1}{2}\pi\alpha$; in particular for $1 < \alpha < 2$,

$$
1 - Ee^{itx} = t[\alpha(t) + i(\beta_+(t) - \beta_-(t))] \sim (\kappa'_{\alpha} + i(2p - 1)\kappa''_{\alpha})t^\alpha L(1/t) \quad (t \downarrow 0). \quad (6.5)
$$

For verification see [13, 2] Theorems 4.3.1-2] if $1 \leq \alpha < 2$. In case $\alpha = 1$ we shall need the following estimate:

$$
\beta(t) = L^*(1/t) + O(L(1/t)), \quad (6.6)
$$

which follows from the bounds $\int_0^{1/t}\mu(y)(1 - \cos ty)dy \leq CL(1/t)$ and $\int_{1/t}^\infty\mu(y)\cos ty\,dy = O(L(1/t))$, both being easy to see. The estimate in case $\alpha = 2$ is deduced from (6.2).

Indeed, uniformly for $\varepsilon > 0$

$$
\alpha(t) = \int_0^{\varepsilon/t}\mu(x)\sin tx\,dx + O(\eta(\varepsilon/t)) = tc(\varepsilon/t)\{1 + O(\varepsilon^2) + o(1)\},
$$

so that $\alpha(t) \sim tc(1/\varepsilon t) \sim tL_e(1/t)$; as for $\beta(t)$ use (2.5a).

**Proposition 6.1.** Suppose that (6.1) is satisfied and that $p \neq 1/2$ if $\alpha = 1$. Then as $x \to \infty$

(i) \quad $\bar{a}(x) \sim \kappa^{-1}_\alpha x/m(x)$,

where $\kappa_\alpha = \kappa_\alpha(p) = 2\left\{ \cos^2 \frac{1}{2}\pi\alpha + (p - q)^2\sin^2 \frac{1}{2}\pi\alpha \right\}\Gamma(\alpha)\Gamma(3 - \alpha)$;

(ii) if $1 \leq \alpha < 2$,

$$
\begin{cases} 
  a(-x) \sim 2p\bar{a}(x), \\
  a(x) \sim 2q\bar{a}(x),
\end{cases} \quad (6.7)
$$

where the sign ‘$\sim$’ is interpreted in the obvious way if $pq = 0$; and

(iii) if $\alpha = 2$, and $\eta_+(x) \sim pL(x)/x$ and $\eta_-(x) \sim qL(x)/x \quad (x \to \infty) \quad (p + q = 1)$, then (6.7) holds true.

[The assumption in (iii) is the same as assuming the second relation of (6.1) for $\alpha = 2$ (hence entailing the first of it) according to the monotone density theorem [2].]

**Proof.** (i) and (ii) are known [1, Lemma 3.3] except for the case $\alpha = 1$ (cf. also [20, Lemma 3.1]). Suppose $\alpha = 1$ and $p \neq q$. Then

$$
\bar{a}(x) = \int_0^\pi \frac{1}{2}L(1/t)\{1 + o(1)\}

\frac{\frac{1}{2}L(1/t)\{1 + o(1)\}}{\left[ \frac{1}{2}\pi L(1/t) \right]^2 + [(p - q)L^*(1/t)]^2(1 - \cos xt)}\,dt \quad t.
$$

Let $\tilde{L}(1/t)$ denote the ratio in the integrand which is slowly varying. By virtue of the assumption $p \neq q$, we have $\tilde{L}(1/t) \sim \frac{1}{2}(p - q)^{-2}L(1/t)/[L^*(1/t)]^2$. It is easy to see that
the integral restricted to \((0,1/x)\) is \(O(\tilde{L}(x))\). Since \(1 - \cos xt \geq 0\), \(\tilde{L}(x)\) can be replaced by any asymptotic equivalent of it and hence we may suppose that there exist constants \(\delta, \delta' \in (0,1)\) such that \(h(t) := \tilde{L}(1/t)t^{-\delta}\) is decreasing on \(0 < t < \delta'\) so that

\[
\int_{1/x}^{\delta'} \tilde{L}(1/t) \cos xt \frac{dt}{t} = \int_{1/x}^{\delta'} dh(t) \int_{t}^{\infty} u^{-1+\delta} \cos xu \, du + O(\tilde{L}(x) \lor 1) = O(\tilde{L}(x) \lor 1) \quad (6.8)
\]

(for the first equality integrate the repeated integral by parts). From these bounds together with the relations \(x/m(x) \sim 1/\mathcal{L}(x) \gg \tilde{L}(x) \lor 1\) and \(c(x) \sim xL(x)\) we deduce

\[
\bar{a}(x) \sim \int_{1/x}^{\pi} \tilde{L}(1/t) dt \sim \frac{1}{2(p-q)^2} \int_{1}^{x} \frac{L(y)dy}{[\mathcal{L}(y)]^2 y} \sim \frac{1}{\kappa_1} \int_{1}^{x} \frac{c(y)dy}{m^2(y)} \sim \frac{x}{\kappa_1 m(x)},
\]

showing (i) for \(\alpha = 1\).

For the proof of (6.7) recall that \(a(x) = \bar{a}(x) + b_-(x) - b_+(x)\), where

\[
b_{\pm}(x) = \frac{1}{\pi} \int_{0}^{\pi} \frac{\beta_{\pm}(t)}{[\alpha^2(t) + \gamma^2(t)]} \cdot \frac{\sin xt}{t} \, dt
\]

(see (41)). Let \(\alpha = 2\). By the assumption of (iii) \(\tilde{L}(x) := x\eta_+(x) \sim pL(x)\) and \(\tilde{L}'(x) = \eta_+(x) - x\mu_+(x) = o(L(x)/x) \quad (x \to \infty)\), which allows us to derive

\[
\beta_+(t) = t \int_{0}^{\infty} \eta_+(y) \sin ty \, dy = \int_{0}^{\infty} \tilde{L}(y) \frac{\sin ty}{y} \, dy \sim \frac{1}{2} \pi pL(1/t) t\quad (6.9)
\]

by a standard way (cf. [22, Theorem V.2.6]). Thus as above we see

\[
b_+(x) = \frac{p}{2} \int_{0}^{\pi} \frac{L(1/t)[1 + o(1)]}{[tL_*(1/t)]^2} \sin xt \, dt = \int_{1/2}^{x} \frac{L(u)}{L_*(u)} \, du.
\]

For \(\varepsilon > 0\), on using \(|\sin xt| \leq 1\), the contribution from \(t > \varepsilon/x\) to the above integral is dominated in absolute value by a constant multiple of

\[
\int_{\varepsilon/x}^{\pi} \frac{L(1/t)}{t^2[L_*(1/t)]^2} \, dt = \int_{1/\pi}^{x/\varepsilon} \frac{L(y)}{[L_*(y)]^2 y} \, dy \sim \frac{x}{\varepsilon [L_*(x)]^2} = o\left(\frac{x}{L_*(x)}\right).
\]

while the remaining integral may be written as

\[
x \int_{0}^{\varepsilon/x} \frac{L(1/t)[1 + o_{\varepsilon}(1)]}{[tL_*(1/t)]^2} \, dt = x \int_{x/\varepsilon}^{\infty} \frac{L(y)}{[L_*(y)]^2 y} \, \left(1 + o_{\varepsilon}(1)\right) = \frac{x}{L_*(x)} \left[1 + o_{\varepsilon}(1)\right].
\]

Thus \(b_{\pm}(x) \sim \frac{1}{2} p x / L_*(x)\). In the same way \(b_-(x) \sim \frac{1}{2} q x / L_*(x)\) and by (i) \(\bar{a}(x) \sim \frac{1}{2} x / L_*(x)\). Consequently \(a(x) \sim q x / m(x) \sim 2qa(x)\).

For \(\alpha = 1\) we put \(L_{\pm}(x) = x\mu_{\pm}(x), L_1^\pm(x) = \eta_{\pm}(x) = \int_{x}^{\infty} L_{\pm}(y)dy/y\) and \(H(x) = L_1^+(x) - L_1^-(x)\). Then by (6.6) \(\beta_+(t) = L_1^+(1/t) + O(L_1^+(1/t))\) (valid even if \(pq = 0\), and if \(p \neq q\), \(H(x) = (p-q)L_1^*(x) + O(L(1/t))\) and

\[
\frac{\beta_{\pm}(t)}{[\alpha^2(t) + \gamma^2(t)]} = \frac{L_1^+(1/t) + O(L(1/t))}{H^2(1/t)}.
\]

Note \(L_1^*(x)/H^2(x) \sim (2p/\kappa_1)/L_1^*(x) \quad (\kappa_1 = 2(p-q)^2)\) and \(H'(x) = \mu_-(x) - \mu_+(x) \sim (q-p)L(x)/x\) and observe

\[
\frac{d}{dt} L_1^*(1/t) \sim \frac{L(1/t)}{[L_1^*(1/t)]^2 t} \quad (t \downarrow 0).
\]
Then in a manner analogous to that deriving (6.9) we obtain

\[ b_+(x) = \frac{1}{\pi} \int_0^\pi \frac{L^*_+(1/t)}{H^2(1/t)} \cdot \frac{\sin x t}{t} dt + O(1) = \frac{p}{\kappa_1 L^*_+(x)} \{1 + o(1)\}. \]

In the same way \( b_-(x) \sim [q/\kappa_1]/L^*(x) \). This concludes the required formulae for \( a(\pm x) \).

The proof of Proposition 6.1 is complete. \( \square \)

**Remark 6.1.** (a) Let \( \alpha = 1 \). In Proposition 6.1 we have assumed \( p \neq q \) to have \( \alpha(t)/\gamma(t) \to 0 \) and \( \kappa_\alpha = 2(p - q)^2 > 0 \). If \( p = q \), we need to know the second order terms for \( \mu_\pm(x) \) in order to compare \( \alpha(t) \) and \( \gamma(t) \), but in any case we have for each \( M > 0 \)

\[ \frac{M}{L^*(x)} \leq \frac{\bar{a}(x)}{1 + o(1)} \leq \frac{2}{\pi^2} \int_1^x \frac{dy}{L(y)y}; \]

the second inequality sign may be replaced by the equality sign in case \( \gamma(t)/\alpha(t) \to 0 \); we also have

\[ \gamma(t) = \int_0^\infty [\mu_-(x) - \mu_+(x)] \cos tx \, dx = \int_0^{1/t} [\mu_-(x) - \mu_+(x)] dx + O(L(1/t)), \]

as is verified by \( \int_1^\infty \mu_\pm(x) \cos tx \, dx = O(L(1/t)) \) and \( \int_0^{1/t} \mu_\pm(x)(\cos tx - 1)dx = O(L(1/t)) \).

(b) Let \( 1 < \alpha \leq 2 \) and determine a real number \( \gamma = \gamma(p, \alpha) \) by \( |\gamma| \leq 2 - \alpha \) and \( \tan \frac{1}{2} \gamma \pi = -(p - q) \tan \frac{1}{2} \alpha \pi \). Then for \( \alpha \neq 2 \), \( \kappa_\alpha^{-1} \) in Proposition 6.1 is rewritten as

\[ \frac{1}{\kappa_\alpha} = \frac{-\tan(\frac{1}{2} \alpha \pi) \cos^2(\gamma \pi/2)}{\pi(\alpha - 1)(2 - \alpha)}. \]

Suppose \( m_+(x)/m(x) \to 0 \). Then \( E(-\hat{Z}) = \infty \) and we have [19, Corollary 1]

\[ a(-x) = \sum_{y=1}^\infty H_{[0,\infty)}^{-x}(y)a(y), \quad x > 0. \]

(6.11)

Applying this identity we are going to derive the asymptotic form of \( a(-x) \) as \( x \to \infty \) when \( \mu_+(x) \) varies regularly at infinity. Recall that \( V_{ds}(x) \) denotes the renewal function for the weakly descending ladder height process of the walk. Let \( U_{as}(x) \) be the renewal function for the strictly ascending one. Put

\[ \ell^*_+(x) = \int_0^x P[Z > t] dt \quad \text{and} \quad \ell^*_-(x) = \int_0^x P[-\hat{Z} > t] dt. \]

We know that \( \ell^*_+(x) \) is slowly varying as \( x \to \infty \) for all \( 1 \leq \alpha \leq 2 \) and so are \( \ell^*_-(x) \) for \( \alpha = 2 \) and \( P[-\hat{Z} > x] \) for \( \alpha = 1 \) and that

\[ U_{as}(x) \sim \frac{x}{\ell^*_+(x)} \quad \text{and} \quad V_{ds}(x) \sim \begin{cases} 
\frac{c_0^{-1}x/\ell^*_+(x)}{\kappa_\alpha^{-1/\alpha}\ell^*_+(x)/L(x)} & \alpha = 2, \\
\frac{c_0^{-1}x^\alpha/\ell^*_+(x)/L(x) \kappa_\alpha^{-1/\alpha}}{\ell^*_+(x)/L(x)} & 1 < \alpha < 2, \\
\frac{c_0^{-1}/P[-\hat{Z} > x]}{\kappa_\alpha^{-1/\alpha}} & \alpha = 1, 
\end{cases} \]

where \( \kappa_\alpha' = -\pi^{-1} \sin \alpha \pi \) and \( c_0 = e^{-\sum p^k(0)/k} \) (provided (6.11) holds): see [14] Theorems 2, 3 and 9], [20, Lemma 8.8] except for the slow variation of \( P[-\hat{Z} > x] \) which is shown in [18] (see the last mentioned result in (iii) of Subsection 6.1.2).
Proposition 6.2. Suppose that (6.1) holds and \( m_+(x)/m(x) \to 0 \) and that either \( \mu_+(x) \) is regularly varying at infinity with index \(-\beta\) or \( \sum_{x=1}^{\infty} \mu_+(x) [a(x)]^2 < \infty \). Then

\[
a(-x) \sim \begin{cases} 
CU_\alpha(x) \sum_{z=x}^{\infty} \frac{\mu_+(z) V_{ds}(z) a(z)}{z} & \text{if } \alpha = \beta = 2, \\
CU_\alpha(x) \frac{x}{x} \sum_{z=1}^{x} \mu_+(z) V_{ds}(z) a(z) & \text{otherwise}
\end{cases}
\]

as \( x \to \infty \), where \( C \) a constant that is positive unless the walk is right-continuous.

[Explicit expressions of the right side are given in the proof: see (6.10) to (6.19).]

Proof. Put \( u_{as}(x) = U_{as}(x) - U_{as}(x-1) \), \( x > 0 \) and \( u_{as}(0) = U_{as}(0) = 1 \). Then \( G(x_1, x_2) := u_{as}(x_2 - x_1) \) is the Green function of the strictly increasing ladder process killed on its exiting the half line \((-\infty, 0]\) and by the last exit decomposition we obtain

\[
H_{[0,\infty)}^{-x}(y) = \sum_{k=1}^{x} u_{as}(x-k) P[Z = y + k] \quad (x \geq 1, y \geq 0). \quad (6.13)
\]

Suppose the conditions of the proposition to hold and let \( \mu_+(x) \sim L_+(x)/x^\beta \) with \( \beta \geq \alpha \) and \( L^+ \) slowly varying at infinity. Then

\[
P[Z > y] = \sum_{k=-\infty}^{0} g_{[1,\infty)}(0, k) \mu_+(y-k) = \sum_{z=0}^{\infty} v_{ds}(z) \mu_+(y+z)
\]

\[
\sim \beta \sum_{z=0}^{\infty} V_{ds}(z) \frac{\mu_+(y+z)}{y+z+1}
\]

\[
\sim C_0 V_{ds}(y) \mu_+(y) \quad (y \to \infty),
\]

where \( v_{ds}(z) = V_{ds}(z) - V_{ds}(z-1) \) and \( C_0 = \beta \int_{0}^{\infty} t^{\alpha-1} (1+t)^{-\beta-1} dt \).

If \( \beta > 2\alpha - 1 \), then \( \sum \mu_+(x) [a(x)]^2 < \infty \) and \( a(-x) \) converges to a constant that is positive if the walk is not right-continuous (cf. [19, Theorem 2]). Hence we may consider only the case \( \alpha \leq \beta \leq 2\alpha - 1 \).

Let \( \alpha > 1 \). Then substituting the above equivalence into (6.13), returning to (6.11) and performing summation by parts lead to

\[
a(-x) \sim (\alpha - 1) \sum_{y=1}^{\infty} \sum_{k=0}^{x} u_{as}(x-k) P[Z \geq y+k] \frac{a(y)}{y}
\]

\[
\sim (\alpha - 1) C_0 \sum_{y=1}^{\infty} \sum_{k=0}^{x} u_{as}(x-k) V_{ds}(y+k) \mu_+(y+k) \frac{a(y)}{y}. \quad (6.14)
\]

It follows [19] Lemma 7.1] that

\[
u_{as}(x) \sim 1/\ell_+^*(x) \quad (x \to \infty),
\]

which shows that one can replace \( u_{as}(x-k) \) by \( 1/\ell_+^*(x) \) in (6.14), the inner sum over \((1-\varepsilon)x < k \leq x\) being negligible as \( x \to \infty \) and \( \varepsilon \to 0 \). After changing the variables by \( z = y+k \) and \( y = z-j \) the last double sum restricted to \( y+k \leq x \) then becomes

\[
\frac{1}{\ell_+^*(x)} \sum_{z=1}^{x} V_{ds}(z) \mu_+(z) \sum_{j=0}^{z-1} \frac{a(z-j)}{z-j} \sim \frac{1}{(\alpha - 1)\ell_+^*(x)} \sum_{z=1}^{x} V_{ds}(z) \mu_+(x) a(z). \quad (6.15)
\]
If $\beta = 2\alpha - 1$ (entailing $\ell^*_+(x) \to EZ < \infty$), then the last sum varies slowly and the remaining part of the double sum on the right side of (6.14) is negligible, showing

$$a(-x) \sim \frac{C_0}{\ell^*_+(x)} \sum_{z=1}^{x} V_{ds}(z)\mu_+(z)a(z) \sim \frac{C_0}{\ell^*_+(x)} \sum_{z=1}^{x} \frac{L_+(z)}{\ell(z)x}, \quad (6.16)$$

where

$$\ell(z) = \begin{cases} 
\ell^*_+(z)L_+(z) & \alpha = 2, \\
(\kappa_\alpha/2\kappa'_\alpha)[|L(z)|^2/\ell^*_+(z)] & 1 < \alpha < 2.
\end{cases}$$

Let $\alpha \leq \beta < 2\alpha - 1$. If $(2 - \alpha)(\beta - \alpha) \neq 0$, then the outer sum in (6.14) over $y > Mx$ get negligibly small as $M$ becomes large and one can easily infer that

$$a(-x) \sim \frac{C_2L_+(x)x^{2\alpha-1-\beta}}{\ell^*_+(x)\ell(x)} \sim \frac{C_3}{\ell^*_+(x)} \sum_{z=1}^{x} V_{ds}(z)\mu_+(z)a(z), \quad (6.17)$$

where $C_2 = (\alpha - 1)C_0 \int_0^1 \alpha |s - t|^{-\beta+\alpha-1}dt$, $C_3 = (2\alpha - 1 - \beta)C_2$ and $\ell$ is as above. If $\alpha = \beta = 2$, then we have for each $M > 2$ there exist positive constants $C', C''$ such that for any $x > 1$ and $0 \leq k \leq x$

$$\sum_{y=x}^{\infty} V_{ds}(y+k)\mu_+(y+k)\frac{a(y)}{y} \geq C' \sum_{y=x}^{\infty} \frac{L_+(y+k)}{(y+k)\ell^*_+(y+k)L_+(y)} \geq C'' \frac{L_+(x)}{\ell(x)} \log \frac{M}{2},$$

while on recalling (6.15) $\sum_{k=1}^{\infty} \sum_{y=1}^{x} V_{ds}(y+k)\mu_+(y+k)a(y)/y \leq C''xL_+(x)/\ell(x)$. From these two bounds we conclude that

$$a(-x) \sim \frac{Cx}{\ell^*_+(x)} \sum_{y=x}^{\infty} V_{ds}(y)\mu_+(y)\frac{a(y)}{y} \sim \frac{Cx}{\ell^*_+(x)\ell(y)} \sum_{y=x}^{\infty} \frac{L_+(y)}{\ell(y)y}, \quad (6.18)$$

where $C = (\alpha - 1)C_0$. The relations (6.16) to (6.18) together show those of the lemma in case $1 < \alpha \leq 2$ since $U_{ds}(x) \sim x/\ell^*(x)$.

It remains to deal with the case $\alpha = \beta = 1$. In place of (6.14) we have

$$a(-x) \sim \sum_{y=1}^{\infty} \sum_{k=0}^{x} \frac{u_{as}(x-k)\mu_+(y+k)}{c_0P[-\tilde{Z} > y+k]} \cdot \frac{1}{dy} \frac{1}{L^*(y)}.$$ 

Note $P[-\tilde{Z} > x]$ is slowly varying and $\frac{dy}{dy}[1/L^*(y)] = L(y)/y[L^*(y)^2]$. Then one sees that the above double sum restricted to $y + k \leq x$ is asymptotically equivalent to

$$\frac{1}{\ell^*_+(x)} \sum_{z=1}^{x} \frac{L_+(z)}{c_0zP[-\tilde{Z} > z]L^*(z)}, \quad (6.19)$$

hence slowly varying, while the outer sum over $y > x$ is negligible. It therefore follows that the above formula represents the asymptotic form of $a(-x)$ and may be written alternatively as $x^{-1}U_{as}(x) \sum_{y=1}^{x} V_{ds}(y)\mu_+(y)a(y)$ as required.

### 6.1.2 Relative stability.

We continue to suppose (6.11) (with $L$ satisfying the conditions stated after it) and examine the behaviour of the overshoot

$$Z_R := S_{\sigma(R,\infty)} - R.$$
It will in particular be observed that the sufficient condition of Proposition 5.1 is also necessary for $Z$ to be relatively stable under (6.1) except for the case $\alpha = 1, p = 1/2$.

In case $1 < \alpha \leq 2$ let $\rho = P[Y > 0]$ that equals Spitzer’s constant given in Remark 6.1

(i) If $p = 0$ or $\alpha = 2$, then $x\eta_+(x) = o(m(x))$ and it follows from Proposition 5.1 that $Z_R/R \to 0$ as $R \to \infty$. (In this case we have $\alpha\rho = 1$ and the same result also follows from Theorems 9 and 2 of [14].)

(ii) If $1 < \alpha < 2$ and $p > 0$, then $0 < \alpha\rho < 1$, which implies that $P[Z > x]$ is regularly varying of index $\alpha\rho$ [14] and the distribution of $Z_R/R$ converges weakly to the probability law determined by the density $C_{\alpha\rho}/x^{\alpha\rho}(1 + x), x > 0$ ([14], [8] Theorem XIV.3]).

(iii) Let $\alpha = 1$. If $p = q = 1/2$ we suppose that Spitzer’s condition is satisfied, namely, there exists

$$\lim \frac{1}{n} \sum_{k=1}^{n} P[S_k > 0] = r,$$

which necessarily holds if $p \neq 1/2$ with $r = 0$ or $1$ according as $p > 1/2$ or $p < 1/2$. If $p < 1/2$, then $\int_{0}^{x}\mu_{+}(y) - \mu_{-}(y)dy = \eta_{-}(x) - \eta_{+}(x) \sim L^{*}(x) \gg x\mu(x)$; and also in case $p = 1/2$, if $r = 1$, then $\eta_{-}(x) - \eta_{+}(x) \gg x\mu(x)$ as we see shortly. Thus in case $r = 1$ (C1) of Proposition 5.1 holds, entailing that $X$ is positively relatively stable and hence so is $Z$ in view of (5.8) so that $Z_R/R \stackrel{P}{\to} 0$. If $0 < r < 1$ (entailing $p = \frac{1}{2}$), then the law of $Z_R/R$ converges to a probability law with the density $[\pi\sqrt{x}(1 + x)]^{-1}, x > 0$. If $r = 0$ (i.e., either $p > 1/2$ or $p = 1/2$ with $r = 0$), then $P[Z > x]$ is slowly varying at infinity and $Z_R/R \stackrel{P}{\to} \infty$.

Thus in case $1 < \alpha \leq 2$ condition (C2) works as a criterion for the relative stability of $Z$, while if $\alpha = 1$, $Z$ can be relatively stable under $x\eta_+(x) \approx m(x)$ (so that (C2) does not hold) and condition (C1) must be employed for the criterion.

For (iii) an explanation is needed. Let $\alpha = 1$. We recall the fact that $\varepsilon_n S_n - nb_n$ converges in law to the stable variable $Y$, if

$$n\varepsilon_n^2 E[X^2; |X| \leq 1/\varepsilon_n] \to C_{\Phi} \quad (n \to \infty) \quad \text{and} \quad b_n = E[\sin(\varepsilon_n X)] = -\varepsilon_n \gamma(\varepsilon_n) \quad (6.21)$$

[8] Theorem XVII.5.3]. It follows that $\gamma(t) \sim (p - q)\beta(t)$ so that the relation above implies

$$n\varepsilon_n \sim C_{\Phi}/L(1/\varepsilon_n) \quad \text{and} \quad b_n \sim -(p - q)\varepsilon_n L^{*}(1/\varepsilon_n),$$

respectively. Since $L^{*}(x)/L(x) \to \infty$, $nb_n$ diverges to $-\infty$ if $p > q$ and $+\infty$ if $p < q$. In case $p = q$ the supposed Spitzer’s condition implies

$$\lim P[S_n > 0] = r$$

(according to [8]). Let $M_r \in [-\infty, +\infty]$ be determined by $P[Y \leq M_r] = 1 - r$. On writing $P[S_n > 0] = P[\varepsilon_n S_n - nb_n > -nb_n]$, this means that $M_r = -\lim nb_n$. Since by (6.21) $nb_n \sim -C\gamma(\varepsilon_n)/L(1/\varepsilon_n)$, we obtain

$$\frac{C_{\Phi}\gamma(\varepsilon_n)}{L(1/\varepsilon_n)} \to \begin{cases} M_r \in (-\infty, +\infty) & (0 < r < 1) \\ -\infty & (r = 1), \\ +\infty & (r = 0), \end{cases}$$

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Using (6.10) we see that if \( r = 1 \), (C1) holds and hence \( Z_R/R \xrightarrow{P} 0 \). On the other hand if \( 0 < r < 1 \), \( \varepsilon_n S_n^0 \) converges in law, implying that the asserted convergence in law of \( Z_R/R \) holds (cf. [14, Section 4]). In case \( r = 0 \) (including the case \( p > q \)) the result stated in (iii) is shown in [18].

**Remark 6.2.** The equality (5.18) holds if \( \alpha = 2 \) but does not in case \( 1 \leq \alpha < 2 \) with \( 0 < p < 1 \) and Spitzer’s condition (6.20) supposed. This is verified from (6.3), (6.4), what are mentioned in (i), (ii) and (iii) and the fact that the conditioning \( \sigma_0^r > \sigma_R^r \land \sigma_{-R}^r \) does not change the probabilities in question if \( x = 0 \). If \( \alpha \neq 1 \) or \( \alpha = 1 \) with \( p = 1/2 \) and \( 0 < r < 1 \), verification is easy and omitted. Let \( \alpha = 1 \) with \( r = 1 \) (including the case \( 0 < p < 1/2 \)), the case \( r = 0 \) being similar. Since then the probability of \( S^0_n \) exiting \([-R, R]\) through \( R \) tends to 1 (cf. [6, Section 7.3], it suffices to show that \( P[\sigma^0_R < \sigma^0_{-R}] \sim q \). Let \( \lambda_\pm = P[\sigma^0_R < \sigma^0_{\pm R}, \sigma^0_{\mp R}] \). By using Proposition 6.1 one observes that \( P[\sigma^0_R < \sigma^0_{-R}] \to 1 - p \) and \( P[\sigma^0_R < \sigma^0_{-R}] \to p \) so that

\[
P[\sigma^0_R < \sigma^0_{-R}] = P[\sigma^0_R < \sigma^0_{-R}] \sim \lambda_- + q\lambda_+ \sim \lambda_+ + p\lambda_-,
\]

showing \( \lambda_-/\lambda_+ \sim p/q \). Hence \( P[\sigma^0_R < \sigma^0_{-R}] = \lambda_+/(\lambda_- + \lambda_+) \sim q \).

**6.2. An example exhibiting irregular behaviour of \( a(x) \).**

We construct a recurrent symmetric walk such that \( E[|X|^\alpha / \log(|X| + 2)] < \infty \), \( 1 < \alpha < 2 \) and

\[
\tilde{a}(x_n/2)/\tilde{a}(x_n) \longrightarrow \infty \quad \text{as} \quad n \to \infty \quad (6.22)
\]

for some sequence \( x_n \uparrow \infty \), \( x_n \in 2\mathbb{Z} \), which provides a counter example for the fact mentioned right after Theorem 1. In fact for the walk given below it holds that there exists positive constants \( \delta \) and \( c_\ast \) such that for all sufficiently large \( n \),

\[
\tilde{a}(x)/\tilde{a}(x_n) \geq c_\ast n \quad \text{for all integers} \quad x \quad \text{satisfying} \quad 2^{-\delta^\ast} x_n < x \leq 2^{-1} x_n. \quad (6.23)
\]

(Actually \( \tilde{a} \) diverges to \( \infty \) fluctuating with relatively steep ups and very long downs.)

Put

\[
x_n = 2^n, \quad \lambda_n = x_n^{-\alpha} = 2^{-\alpha n^2} \quad (n = 0, 1, 2, \ldots),
\]

\[
p(x) = \begin{cases} A\lambda_n & \text{if} \quad x = \pm x_n \quad (n = 0, 1, 2, \ldots), \\ 0 & \text{if} \quad x \notin \{\pm x_n : n = 0, 1, 2, \ldots\}, \end{cases}
\]

where \( A \) is the constant chosen so as to make \( p(\cdot) \) a probability. Denote by \( \eta_{n,k,t}^{(1)} \) the value of \( 1 - 2(1 - \cos u)/u^2 \) at \( u = x_{n-k} \) so that uniformly for \( |t| < 1/x_{n-1} \) and \( k = 1, 2, \ldots, n, \)

\[
\frac{\lambda_n - k [1 - \cos(x_{n-k} t)]}{1 - \eta_{n,k,t}^{(1)}} = \frac{\lambda_n - k x_n^2}{2x_n^2} (x_n t)^2 = \frac{1}{2} 2^{-2(2n-k)k} \lambda_n(x_n t)^2
\]

and

\[
\eta_{n,k,t}^{(1)} = o(1) \quad \text{if} \quad k \neq 1 \quad \text{and} \quad 0 \leq \eta_{n,1,t}^{(1)} < (x_{n-1} t)^2/12.
\]

Then

\[
\sum_{x=1}^{x_{n-1}} p(x) (1 - \cos xt) = A\varepsilon_n \lambda_n(x_n t)^2 \{1 - \eta_{n,k,t}^{(1)} + o(1)\}, \quad |t| < \frac{1}{x_{n-1}} \quad (6.24)
\]

with

\[
\varepsilon_n = 2^{1-\alpha} 2^{-2(2-\alpha)n},
\]

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for the last term \( p(x_{n-1})(1 - \cos x_{n-1}t) \) of the series is dominant over the rest. On the other hand for \( x > x_{n-1} \),

\[
\mu(x) = \sum_{y > x} p(y) = \begin{cases} 
A\lambda_n + O(\lambda_{n+1}) & (x_{n-1} \leq x < x_n) \\
o(\lambda_n\varepsilon_n) & (x \geq x_n).
\end{cases}
\] (6.25)

Thus, uniformly for \( |t| < 1/x_{n-1} \) and \( k = 1, 2, \ldots, n \),

\[
1 - \psi(t) = t\alpha(t) = 2A\lambda_n \left[ 1 - \cos x_n t + \varepsilon_n(x_n t)^2(1 - \eta_{n,k,t}^{(1)}) \right] + o(\lambda_n\varepsilon_n). \quad \text{(6.26)}
\]

Also, \( 2A\lambda_n(1 - \cos x_n t) = A\lambda_n(x_n t)^2(1 - \eta^{(2)}(n, t)) \) with \( 0 \leq \eta^{(2)} \leq 1/2 \) for \( |t| < 1/x_n \). It is remembered that

\[
\bar{a}(x) = \frac{1}{\pi} \int_0^\pi \frac{1 - \cos xt}{1 - \psi(t)} \, dt.
\]

First we compute the upper bound of \( \bar{a}(x_n) \). To this end we break the above integral into three parts

\[
(\pi A)\bar{a}(x_n) = \left( \int_0^{1/x_n} + \int_{1/x_n}^{1/x_{n-1}} + \int_{1/x_{n-1}}^\pi \right) \frac{1 - \cos x_n t}{|1 - \psi(t)|/A} \, dt
= I + II + III \quad \text{(say)}.
\]

On using the trivial inequality \( 1 - \psi(t) \geq 2p(x_n)(1 - \cos x_n t) \)

\[
I \leq 1/2\lambda_n x_n = x_n^{\alpha-1}/2.
\]

By (6.24) it follows that for sufficiently large \( n \),

\[
II \leq 2 \int_{1/x_n}^{1/x_{n-1}} \frac{1 - \cos x_n t}{2\lambda_n(1 - \cos x_n t) + \varepsilon_n \lambda_n(x_n t)^2} \, dt
= 2x_n^{\alpha-1} \int_{1}^{x_n/x_{n-1}} \frac{1 - \cos u}{2(1 - \cos u) + \varepsilon_n u^2} \, du
\leq 2 \frac{1}{\lambda_n x_n} \sum_{k=0}^{2n} \int_{-\pi}^{\pi} \frac{1 - \cos u}{2(1 - \cos u) + \varepsilon_n (u + 2\pi k)^2} \, du.
\]

Observing

\[
\int_{0}^{\infty} dk \int_{0}^{\pi} \frac{u^2}{u^2 + \varepsilon_n k^2} \, du = \frac{\pi/2}{\sqrt{\varepsilon_n}} \int_{0}^{\pi} \frac{u}{\sqrt{\varepsilon_n}} \, du = \frac{\pi^3/4}{\sqrt{\varepsilon_n}},
\]

we obtain

\[
II \leq C_1 x_n^{\alpha-1}/\sqrt{\varepsilon_n}.
\]

For the evaluation of III we deduce that

\[
\int_{1/x_{n-1}}^{1/x_n} \frac{1 - \cos x_n t}{|1 - \psi(t)|/A} \, dt \leq \int_{1/x_n}^{1/x_{n-1}} \frac{4}{2\lambda_n-1(1 - \cos x_{n-1} t) + \varepsilon_{n-1} \lambda_{n-1}(x_{n-1} t)^2} \, dt
\leq \frac{4}{\lambda_n x_{n-1}} \int_{1}^{x_n/x_{n-1}} \frac{1}{2(1 - \cos u) + \varepsilon_{n-1} u^2} \, du.
\]
The last integral is less than
\[
\int_{-\pi}^{\pi} \frac{1}{u^2/3 + \varepsilon_{n-1}u^2} du + \sum_{k=1}^{2n} \int_{-\pi}^{\pi} \frac{1}{u^2/3 + \varepsilon_{n-1}(\pi + 2\pi k)^2} du \leq C_1 \sum_{k=1}^{2n} \frac{1}{\sqrt{\varepsilon_n k}} \leq C_2 \frac{n}{\sqrt{\varepsilon_n}}.
\]

Since \(\lambda_n/\lambda_{n-1} = 2^{-(\alpha-1)(2n-1)}\) and the remaining parts of III is smaller,

\[
III \leq C_3 n 2^{-(\alpha-1)n} \frac{x_n^{\alpha-1}}{\sqrt{\varepsilon_n}}.
\]

Consequently

\[
\bar{a}(x_n) \leq C x_n^{\alpha-1}/\sqrt{\varepsilon_n}. \tag{6.27}
\]

The lower bound of \(\bar{a}(x_n/2)\). Owing to (6.26), for \(n\) sufficiently large,

\[
[1 - \psi(t)]/A \leq 2\lambda_n (1 - \cos x_n t) + 4\varepsilon_n \lambda_n (x_n t)^2 \quad \text{for } 1/\sqrt{x_n} < |t| < 1/\sqrt{x_{n-1}},
\]

hence

\[
(\pi A)\bar{a}(x_n/2) \geq \int_{1/x_n}^{1/x_{n-1}} \frac{1 - \cos(x_n t/2)}{2\lambda_n (1 - \cos x_n t) + 4\varepsilon_n \lambda_n (x_n t)^2} dt
\]

\[
= \frac{1}{2\lambda_n x_n} \int_{1/x_n}^{x_n/x_{n-1}} \frac{1 - \cos(u/2)}{1 - \cos u + 2\varepsilon_n u^2} du
\]

\[
\geq \frac{x_n^{\alpha-1}}{2} \sum_{k=1}^{2n/2\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(\frac{u}{2} + k\pi)}{1 - \cos u + 2\varepsilon_n (u + 2\pi k)^2} du \tag{6.28}
\]

\[
\geq \frac{x_n^{\alpha-1}}{2} \sum_{k=1}^{2n/4\pi} \int_{-\pi}^{\pi} \frac{1}{u^2 + 4\varepsilon_n (u + 4\pi k)^2} du,
\]

where for the last inequality we have restricted the first sum to odd \(k\)'s. On using the bound \(\int_{-\pi}^{\pi} \frac{du}{u^2 + b^2} \geq 1/(b \vee b^2) \ (b > 0)\), the last sum is bounded from below by a constant multiple of

\[
\sum_{k=1}^{2n/4\pi} \frac{1}{\sqrt{\varepsilon_n k} \vee \varepsilon_n k^2} \leq \sum_{k=1}^{1/\sqrt{\varepsilon_n}} \frac{1}{\sqrt{\varepsilon_n k}} \geq \frac{(2 - \alpha)n \log 2}{\sqrt{\varepsilon_n}}.
\]

Hence \(\bar{a}(x_n/2)/\bar{a}(x_n) \geq C n\) if \(n\) is large enough, showing (6.22).

The same proof shows (6.23). In fact we modify it from the line (6.28) on to see that if \(3 \leq j < (2 - \alpha)n\),

\[
(\pi A)\bar{a}(x_n/2^j) \geq \frac{1}{2\lambda_n x_n} \sum_{k=2-j}^{2n/2\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(2^{-j} u + 2^{-j+1} k\pi)}{1 - \cos u + 2\varepsilon_n (u + 2\pi k)^2} du
\]

\[
\geq c_1 \frac{(2 - \alpha)n - j}{\lambda_n x_n \sqrt{\varepsilon_n}} = c_1 \frac{(2 - \alpha)n - j}{\sqrt{\varepsilon_n}} x_n^{\alpha-1},
\]

which together with (6.27) shows (6.23) with any \(\delta < 2 - \alpha\).
This example also exhibits that \( c(x)/m(x) \) oscillates between 0 and 1 and \( \bar{a}(x) \) does not behave like \( x/m(x) \). Indeed without difficulty one can see that for \( x_{n-1} \leq x \leq x_n \),
\[
c(x)/A \sim \frac{1}{2}(x^2 - x_{n-1}^2)\lambda_n + \frac{1}{2}x_{n-1}^2\lambda_{n-1},
\]
\[
x\eta(x)/A \sim x(x_n - x)\lambda_n + xx_{n+1}\lambda_{n+1},
\]
and
\[
m(x)/A \sim x(x_{n-1} - x)\lambda_n + \frac{1}{2}x_{n-1}^2\lambda_{n-1},
\]
and then that \( x\eta(x)/c(x) \) tends to zero for \( x \) with \(-o(x_{n-1}) < x - x_{n-1} < o(2^{(\alpha-1)}x_{n-1})\) and diverges to infinity for \( x \) satisfying \( x_{n-1}2^{(\alpha-1)n} << x << x_n \). From (6.26) one infers that \( \alpha(t)/[tc(1/t)] \) oscillates between 8 and \( 4\epsilon_n t^2x_n^2\{1 - o(1)\} \) about \( M/2\pi \) times when \( t \) ranges over the interval \([1/x_n, M/x_n]\) and that \( \alpha(2\pi/x_n)/\alpha(\pi/x_n) = O(\epsilon_n) \).

References

[1] B. Belkin, An invariance principle for conditioned recurrent random walk attracted to a stable law, Zeit. Wharsch. Verw. Gebiete 21 (1972), 45-64.

[2] N.H. Bingham, G.M. Goldie and J.L. Teugels, Regular variation, Cambridge Univ. Press, Cambridge, 1989.

[3] J. Bertoin and R. A. Doney, Spitzer’s condition for random walks and Lévy processes, Ann. Inst. Henri Poincaré, 33 (1997), 167-178

[4] R. A. Doney, Spitzer’s condition and ladder variables for random walks, Probab. Theor. Rel. Fields, 101 (1995), 577-580.

[5] R.A. Doney and R. A. Maller, The relative stability of the overshoot for Lévy processes, Ann. Probab. 30 (2002), 188-212.

[6] R. A. Doney, Fluctuation theory for Lévy processes, Lecture Notes in Math. 1897 (2007). Springer, Berlin.

[7] K.B. Erickson, Strong renewal theorems with infinite mean, Trans. Amer. Math. Soc. 151 (1970), 263-291.

[8] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, 2nd edn. John Wiley and Sons, Inc. NY. (1971)

[9] P. E. Greenwood, E. Omey and J.I Teugels, Harmonic renewal measures, Z. Wahrsch. verw. Geb. 59 (1982), 391-409.

[10] H. Kesten, The limit points of a normalized random walk. Ann. Math. Statist. 41 (1970) 1173-1205.

[11] H. Kesten and R. A. Maller, Stability and other limit laws for exit times of random walks from a strip or a half line, Ann. Inst. Henri Poincaré, 35 (1999), 685-734.

[12] H. Kesten and R. A. Maller, Infinite limits and infinite limit points of random walks and trimmed sums, Ann. Probab., 22 (1994), 1473-1513

[13] E.J. G. Pitman, On the behaviour of the characteristic function of a probability distribution in the neighbourhood of the origin, JAMS(A) 8 (1968), 422-43.
[14] B.A. Rogozin, On the distribution of the first ladder moment and height and fluctuations of a random walk, Theory Probab. Appl. 16 (1971), 575-595.

[15] F. Spitzer, Principles of Random Walks, Van Nostrand, Princeton, 1964.

[16] K. Uchiyama, A note on summability of ladder heights and the distributions of ladder epochs for random walks, Stoch. Proc. Appl. 121 (2011), 1938-1961.

[17] K. Uchiyama, Asymptotic behaviour of a random walk killed on a finite set, Potential Anal. 46(4), (2017), 689-703.

[18] K. Uchiyama, On the ladder heights of random walks attracted to stable laws of exponent 1, Electron. Commun. Probab. 23 (2018), no. 23, 1-12. doi.org/10.1214/18-ECP122

[19] K. Uchiyama, The potential function and ladder variables of a recurrent random walk on \( \mathbb{Z} \) with infinite variance. (preprint: available at: [http://arxiv.org/abs/1805.03971](http://arxiv.org/abs/1805.03971))

[20] K. Uchiyama, Asymptotically stable random walks of index \( 1 < \alpha < 2 \) killed on a finite set. (preprint)

[21] V. M. Zolotarev, Mellin-Stieltjes transform in probability theory, Theor. Probab. Appl. (1957), 433-460.

[22] A. Zygmund, Trigonometric series, vol. 2, 2nd ed., Cambridge Univ. Press (1959)