ON STABLE SOLUTIONS OF THE FRACTIONAL HENON-LANE-EMDEN EQUATION

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Abstract. We derive a monotonicity formula for solutions of the fractional Hénon-Lane-Emden equation

$$\left(-\Delta\right)^s u = |x|^a|u|^{p-1}u \quad \mathbb{R}^n$$

where $0 < s < 2$, $a > 0$ and $p > 1$. Then we apply this formula to classify stable solutions of the above equation.

1. Introduction and Main Results

We study the classification of stable solutions of the following equation

$$\left(-\Delta\right)^s u = |x|^a|u|^{p-1}u \quad \mathbb{R}^n$$

where $(-\Delta)^s$ is the fractional Laplacian operator for $0 < s < 2$. Here is what we mean by stability.

Definition 1.1. We say that a solution $u$ of (1.1) is stable if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|\phi(x) - \phi(y)|)^2}{|x-y|^{n+2s}} \, dx \, dy - p \int_{\mathbb{R}^n} |x|^a|u|^{p-1}\phi^2 \geq 0$$

for any $\phi \in C^\infty_0(\mathbb{R}^n)$.

For the local cases $s = 1$ and $s = 2$, the classification of stable solutions is completely known for $a \geq 0$. We refer the interested readers to Farina [14] for the case of $s = 1$ and $a = 0$ and to Cowan-Fazly [6], Wang-Ye [31], Dancer-Du-Guo [7], Du-Guo-Wang [11] for the case $s = 1$ and $a > -2$. Also, for the fourth order Lane-Emden equation that is when $s = 2$ we refer to Davila-Dupaigne-Wang-Wei [10] where $a = 0$ and to Hu [20] where $a > 0$. In this note, we focus on the case of fractional Laplacian operator.

It is by now standard that the fractional Laplacian can be seen as a Dirichlet-to-Neumann operator for a degenerate but local diffusion operator in the higher-dimensional half-space $\mathbb{R}^n_{+1}$. For the case of $0 < s < 1$ this in fact can be seen as the following theorem given by Caffarelli-Silvestre [2]. See also [27].

Theorem 1.1. Take $s \in (0,1)$, $\sigma > s$ and $u \in C^{2s}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n,(1 + |t|)^{n+2s}dt)$. For $X = (x,y) \in \mathbb{R}^n_{+1}$, let

$$u_c(X) = \int_{\mathbb{R}^n} P(X,t)u(t) \, dt,$$

where

$$P(X,t) = p_{n,s} t^{2s}|X-t|^{-(n+2s)}$$

and $p_{n,s}$ is chosen so that $\int_{\mathbb{R}^n} P(X,t) \, dt = 1$. Then, $u_c \in C^2(\mathbb{R}^n_{+1}) \cap C(\overline{\mathbb{R}^n_{+1}})$, $y^{1-2s}\partial_y u_c \in C(\overline{\mathbb{R}^n_{+1}})$ and

$$\begin{cases}
\nabla \cdot (y^{1-2s}\nabla u_c) = 0 & \text{in } \mathbb{R}^n_{+1}, \\
u_c = u & \text{on } \partial \mathbb{R}^n_{+1}, \\
- \lim_{y \to 0} y^{1-2s}\partial_t u_c = \kappa_s (-\Delta)^s u & \text{on } \partial \mathbb{R}^n_{+1},
\end{cases}$$

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where
\[
\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}.
\]

From this theorem for a solution of the fractional H"{o}n-Lane-Emden equation, we get the following equation in the higher-dimensional half-space \(\mathbb{R}_+^{n+1}\),
\[
\begin{cases}
-\nabla \cdot (y^{1-2s} \nabla u_e) = 0 & \text{in } \mathbb{R}_+^{n+1} \\
-\lim_{y \to 0} y^{1-2s} \partial_y u_e = \kappa_s |x|^a |u_e|^{p-1} u_e & \text{in } \mathbb{R}^n
\end{cases}
\]
(1.4)

There are different ways of defining the fractional operator \((-\Delta)^s\) where \(1 < s < 2\), just like the case of \(0 < s < 1\). Applying the Fourier transform one can define the fractional Laplacian by
\[
\hat{(-\Delta)^s}u(\zeta) = |\zeta|^{2s} \hat{u}(\zeta)
\]
or equivalently define this operator inductively by \((-\Delta)^s = (-\Delta)^{s-1} \circ (-\Delta)\), see [26]. Recently, Yang in [29] gave a characterization of the fractional Laplacian \((-\Delta)^s\), where \(s\) is any positive, noninteger number as the Dirichlet-to-Neumann map for a function \(u_e\) satisfying a higher order elliptic equation in the upper half space with one extra spatial dimension. This is a generalization of the work of Caffarelli and Silvestre in [2] for the case of \(0 < s < 1\). We first fix the following notation then we present the Yang’s characterization. See also Case-Chang [3] and Chang-Gonzales [4] for higher order fractional operators.

**Notation 1.1.** Throughout this note set \(b := 3 - 2s\) and define the operator
\[
\Delta_b w := \Delta w + \frac{b}{y} w_y = y^{b} \text{div}(y^{-b}\nabla w).
\]

for a function \(w \in W^{2,2}(\mathbb{R}_+^{n+1}, y^b)\).

As it is shown by Yang in [29], if \(u(x)\) is a solution of (1.1) then the extended function \(u_e(x, y)\) where \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}_+^+\) satisfies
\[
\begin{cases}
\Delta_b^2 u_e = 0 & \text{in } \mathbb{R}_+^{n+1}, \\
\lim_{y \to 0} y^b \partial_y u_e = 0 & \text{in } \partial \mathbb{R}_+^{n+1}, \\
\lim_{y \to 0} y^b \partial_y \Delta_b u_e = C_{n,s} |x|^a |u|^{p-1} u & \text{in } \mathbb{R}^n
\end{cases}
\]
(1.5)

Moreover,
\[
\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = C_{n,s} \int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b u_e(x, y)|^2 dxdy
\]

Note that \(u(x) = u_e(x, 0)\) in \(\mathbb{R}^n\).

On the other hand, Herbst in [19] (see also [30]), shoed that when \(n > 2s\) the following Hardy inequality holds
\[
\int_{\mathbb{R}^n} |\xi|^{2s} |\phi|^2 d\xi > \Lambda_{n,s} \int_{\mathbb{R}^n} |x|^{-2s} \phi^2 dx
\]
for any \(\phi \in C_c^\infty(\mathbb{R}^n)\) where the optimal constant given by
\[
\Lambda_{n,s} = 2^{2s} \frac{\Gamma(\frac{n+2s}{2})^2}{\Gamma(\frac{n-2s}{2})^2}.
\]

Here we fix a constant that plays an important role in the classification of solutions of (1.1)
\[
p_S(n, a) = \begin{cases} +\infty & \text{if } n \leq 2s \\
\frac{n + 2s + 2a}{n - 2s} & \text{if } n > 2s
\end{cases}
\]
(1.6)
Remark 1.1. Note that for $p > ps(n, a)$ the function
\begin{equation}
(1.7)
  u_a(x) = A |x|^{-\frac{2s+a}{p-1}}
\end{equation}
where
\[ A^{p-1} = \lambda \left( \frac{n-2s}{2} - \frac{2s+a}{p-1} \right) \]
for constant
\begin{equation}
(1.8)
  \lambda(a) = 2^{\frac{4n}{(n-2s-2a)}} \frac{\Gamma(\frac{n+2s+2a}{2}) \Gamma(\frac{n+2s-2a}{4})}{\Gamma(\frac{n-2s-2a}{4}) \Gamma(\frac{n-2s+2a}{4})}
\end{equation}
is a singular solution of (1.1) where $0 < s < 2$. For details, we refer the interested readers to [13] for the case of $0 < s < 1$ and to [16] for the case of $1 < s < 2$.

Here is our main result

**Theorem 1.2.** Assume that $n \geq 1$ and $0 < s < \sigma < 2$. Let $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n,(1 + |y|)^{n+2s} dy)$ be a stable solution to (1.1).

- If $1 < p < ps(n, a)$ or if $ps(n, a) < p$ and
\begin{equation}
(1.9)
  \frac{\Gamma(\frac{n-2s+2a}{2}) \Gamma(s + \frac{n+2s}{p+1})}{\Gamma(\frac{n-2s+2a}{4}) \Gamma(\frac{n-2s+2a}{4})} > \frac{\Gamma(\frac{n+2s+2a}{4}) \Gamma(\frac{n+2s}{4})}{\Gamma(\frac{n+2s}{4}) \Gamma(\frac{n+2s}{4})},
\end{equation}
then $u \equiv 0$;

- If $p = ps(n, a)$, then $u$ has finite energy i.e.
\[ ||u||^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |x|^a |u|^{p+1} < +\infty. \]

If in addition $u$ is stable, then in fact $u \equiv 0$.

Note that the classification of finite Morse index solutions of (1.1) when $a = 0$ is given by Davila-Dupaigne-Wei in [9] when $0 < s < 1$ and by Fazly-Wei in [16] $1 < s < 2$.

Note also that in the absence of stability it is expected that the only nonnegative bounded solution of (1.1) must be zero for the subcritical exponents $1 < p < ps(n, a)$ where $a \geq 0$. To our knowledge not much is known about the classification of solutions when $a \neq 0$ even for the standard case $s = 1$. For the case of $s = 1$, Phan-Souplet in [23] proved that the only nonnegative bounded solution of (1.1) in three dimensions must be zero for the case of $1 < p < ps(n, a)$ and $a > -2$. Some partial results are given in [17].

2. The monotonicity formula

Here is the monotonicity formula for the case of $0 < s < 1$.

**Theorem 2.1.** Suppose that $0 < s < 1$. Let $u_e \in C^2(\mathbb{R}^{n+1}_+ \cap \mathbb{R}^{n+1})$ be a solution of (1.1) such that $y^{1-2s} \partial_y u_e \in C(\mathbb{R}^{n+1}_+ \cap \mathbb{R}^{n+1})$. For $x_0 \in \partial \mathbb{R}^{n+1}_+$, $\lambda > 0$, let
\[ E(u_e, \lambda) := \lambda^{\frac{2s(p+1)+2a}{p-1}-n+1} \left( \frac{1}{2} \int_{\mathbb{R}^{n+1}_+ \cap B_\lambda} y^{1-2s} |\nabla u_e|^2 \, dx \, dy - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_\lambda} |x|^a |u_e|^{p+1} \, dx \right) \]
\[ \text{where} \quad \lambda^{\frac{2s(p+1)+2a}{p-1}-n+1} \lambda^{1+\frac{8+2a}{p+1}} \int_{\partial B_\lambda \cap \mathbb{R}^{n+1}_+} y^{1-2s} u_e^2 \, d\sigma. \]

Then, $E$ is a nondecreasing function of $\lambda$. Furthermore,
\[ \frac{dE}{d\lambda} = \lambda^{\frac{2s(p+1)+2a}{p-1}-n+1} \int_{\partial B(x_0, \lambda) \cap \mathbb{R}^{n+1}_+} y^{1-2s} \left( \frac{\partial u_e}{\partial r} + \frac{2s+a u_e}{p-1} \right)^2 \, d\sigma. \]
Proof. Let
\begin{equation}
I(u_e, \lambda) = \lambda^{2s + \frac{4s + 2a - 2}{n - 1} - n} \left( \int_{\mathbb{R}_+^{n+1} \cap B_r} y^{1-2s} |\nabla u_e|^2 \, dx \, dy - \frac{\kappa_s}{p + 1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_r} |x|^a |u_e|^{p+1} \, dx \right)
\end{equation}

Now for \( X \in \mathbb{R}_+^{n+1} \), define
\begin{equation}
\lambda \partial u_e^\lambda = \frac{2s + a}{p - 1} u_e^\lambda + r \partial_r u_e^\lambda.
\end{equation}

Differentiating the operator (2.1) w.r.t. \( \lambda \), we find
\begin{equation}
\partial \lambda I(u_e, \lambda) = \int_{\partial B_r \cap \mathbb{R}_+^{n+1}} y^{1-2s} \nabla u_e^\lambda \cdot \nabla \partial \lambda u_e^\lambda \, dx \, dy - \kappa_s \int_{\partial \mathbb{R}_+^{n+1} \cap B_r} |x|^a |u_e^\lambda|^{p+1} \partial \lambda u_e^\lambda \, dx.
\end{equation}

Integrating by parts and then using (2.4),
\begin{align*}
\partial \lambda I(u_e, \lambda) &= \int_{\partial B_r \cap \mathbb{R}_+^{n+1}} y^{1-2s} \partial_r u_e^\lambda \partial \lambda u_e^\lambda \, dx \, dy - \frac{2s + a}{p - 1} \int_{\partial B_r \cap \mathbb{R}_+^{n+1}} y^{1-2s} \lambda^2 \partial \lambda u_e^\lambda \, dx \, dy \\
&= \lambda \int_{\partial B_r \cap \mathbb{R}_+^{n+1}} y^{1-2s} (\partial \lambda u_e^\lambda)^2 \, dx \, dy - \frac{s + a}{p - 1} \lambda \left( \int_{\partial B_r \cap \mathbb{R}_+^{n+1}} y^{1-2s} (u_e^\lambda)^2 \, dx \, dy \right).
\end{align*}

Scaling finishes the proof.

We now consider the case of \( 1 < s < 2 \) and \( a > 0 \). Note that a monotonicity formula is given for the case of \( a = 0 \) and \( s = 2 \) and \( 1 < s < 2 \) by Davila-Dupaigne-Wang-Wei in [10] and Fazly-Wei in [16], respectively. We define the energy functional
\begin{align*}
E(u_e, r) := r^{2s + \frac{4s + 2a - 2}{n - 1} - n} \left( \int_{\mathbb{R}_+^{n+1} \cap B_r} \frac{1}{2} y^{3-2s} |\Delta_h u_e|^2 \, dx \, dy - \frac{C_{n,s}}{p + 1} \int_{\partial \mathbb{R}_+^{n+1} \cap \partial B_r} |x|^a |u_e|^{p+1} \right) \\
&\quad - \frac{s + a}{p - 1} \left( \frac{p + 2s + a - 1}{p - 1} - n - b \right) r^{-3+2s+ \frac{4s + 2a - 2}{n - 1} - n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_r} y^{3-2s} u_e^2 \\
&\quad - \frac{s + a}{p - 1} \left( \frac{p + 2s + a - 1}{p - 1} - n - b \right) \frac{d}{dr} \left[ r^{4s + 2a - 2} \int_{\mathbb{R}_+^{n+1} \cap \partial B_r} y^{3-2s} u_e^2 \right] \\
&\quad + \frac{1}{2} \frac{3}{2} r^2 \left[ \frac{4s + 2a + 4s + 2s - 2}{p - 1} \int_{\mathbb{R}_+^{n+1} \cap \partial B_r} y^{3-2s} \left( \frac{2s + a}{p - 1} u_e - \frac{\partial u_e}{\partial r} \right)^2 \right] \\
&\quad + \frac{1}{2} \frac{d}{dr} \left[ r^{2s + \frac{4s + 2a}{n - 1} - n} \int_{\mathbb{R}_+^{n+1} \cap \partial B_r} y^{3-2s} \left( |\nabla u_e|^2 - \left| \frac{\partial u_e}{\partial r} \right|^2 \right) \right] \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap \partial B_r} y^{3-2s} \left( |\nabla u_e|^2 + \left| \frac{\partial u_e}{\partial r} \right|^2 \right)
\end{align*}
Theorem 2.2. Assume that $n > \frac{p + 2s + 2a - 1}{p - 1} + \frac{2a + 1}{p - 1} - b$. Then, $E(u_{c}, \lambda)$ is a nondecreasing function of $\lambda > 0$.

Furthermore,

\[
\frac{dE(\lambda, u_{c})}{d\lambda} \geq C(n, s, p) \lambda^{\frac{2(n+1)+2a}{p-2s+a-1}} \int_{\mathbb{R}^{n+1}_{+} \cap \partial B_{\lambda}} y^{3-2s} \left( \frac{2s + a - 1}{p - 1} u + \frac{\partial u_{c}}{\partial r} \right)^{2}
\]

where $C(n, s, p)$ is independent from $\lambda$.

Proof: Set,

\[
\tilde{E}(u_{c}, \lambda) := \lambda^{\frac{2(n+1)+2a}{p-2s+a-1}} \left( \int_{\mathbb{R}^{n+1}_{+} \cap \partial B_{\lambda}} \frac{1}{2} y^{b} |\Delta_{\lambda} u_{c}|^{2} dxdy - \frac{C_{n, s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_{+} \cap B_{\lambda}} |x|^{a} u_{c}^{p+1} \right)
\]

Define $v_{c} := \Delta_{\lambda} u_{c}$, $u_{c}^{\lambda}(X) := \frac{2s + a + 2}{p - 1} u_{c}(\lambda X)$, and $v_{c}^{\lambda}(X) := \lambda^{\frac{2s + a + 2}{p - 1}} v_{c}(\lambda X)$ where $X = (x, y) \in \mathbb{R}^{n+1}_{+}$. Therefore, $\Delta_{\lambda} u_{c}^{\lambda}(X) = v_{c}^{\lambda}(X)$ and

\[
\left\{
\begin{array}{ll}
\Delta_{\lambda} v_{c}^{\lambda} &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_{+}, \\
\lim_{y \to 0} y^{b} \partial_{y} v_{c}^{\lambda} &= 0 \quad \text{in} \quad \partial \mathbb{R}^{n+1}_{+}, \\
\lim_{y \to 0} y^{b} \partial_{y} v_{c}^{\lambda} &= C_{n, s} |x|^{a} (u_{c}^{\lambda})^{p} \quad \text{in} \quad \mathbb{R}^{n}
\end{array}
\right.
\]

In addition, differentiating with respect to $\lambda$ we have

\[
\Delta_{\lambda} \frac{du_{c}^{\lambda}}{d\lambda} = \frac{dv_{c}^{\lambda}}{d\lambda}.
\]

Note that

\[
\tilde{E}(u_{c}, \lambda) = \tilde{E}(u_{c}^{\lambda}, 1) = \int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} \frac{1}{2} y^{b}(v_{c}^{\lambda})^{2} dxdy - \frac{C_{n, s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_{+} \cap B_{1}} |x|^{a} u_{c}^{\lambda} u_{c}^{p+1}
\]

Taking derivate of the energy with respect to $\lambda$, we have

\[
\frac{d\tilde{E}(u_{c}^{\lambda}, 1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} y^{b} v_{c}^{\lambda} \frac{dv_{c}^{\lambda}}{d\lambda} dxdy - C_{n, s} \int_{\partial \mathbb{R}^{n+1}_{+} \cap B_{1}} |x|^{a} \frac{du_{c}^{\lambda}}{d\lambda}
\]

Using (2.7) we end up with

\[
\frac{d\tilde{E}(u_{c}^{\lambda}, 1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} y^{b} v_{c}^{\lambda} \frac{dv_{c}^{\lambda}}{d\lambda} dxdy - \int_{\partial \mathbb{R}^{n+1}_{+} \cap B_{1}} \lim_{y \to 0} y^{b} \partial_{y} v_{c}^{\lambda} \frac{dv_{c}^{\lambda}}{d\lambda}
\]

From (2.8) and by integration by parts we have

\[
\int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} y^{b} \cdot \Delta_{\lambda} u_{c}^{\lambda} \frac{dv_{c}^{\lambda}}{d\lambda} = \int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} y^{b} \Delta_{\lambda} u_{c}^{\lambda} \frac{dv_{c}^{\lambda}}{d\lambda}
\]

Note that

\[
- \int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} \nabla \Delta_{\lambda} u_{c} \cdot \nabla \frac{dv_{c}^{\lambda}}{d\lambda} = \int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} \text{div}(\nabla \Delta_{\lambda} u_{c}^{\lambda} \frac{dv_{c}^{\lambda}}{d\lambda}) - \int_{\partial (\mathbb{R}^{n+1}_{+} \cap B_{1})} y^{b} \partial_{\nu} (\Delta_{\lambda} u_{c}^{\lambda}) \frac{dv_{c}^{\lambda}}{d\lambda}
\]

Therefore,

\[
\int_{\mathbb{R}^{n+1}_{+} \cap B_{1}} y^{b} \frac{dv_{c}^{\lambda}}{d\lambda} = \int_{\partial (\mathbb{R}^{n+1}_{+} \cap B_{1})} \Delta_{\lambda} u_{c}^{\lambda} y^{b} \partial_{\nu} \left( \frac{dv_{c}^{\lambda}}{d\lambda} \right) - \int_{\partial (\mathbb{R}^{n+1}_{+} \cap B_{1})} y^{b} \partial_{\nu} (\Delta_{\lambda} u_{c}^{\lambda}) \frac{dv_{c}^{\lambda}}{d\lambda}
\]
Boundary of $\mathbb{R}_+^{n+1} \cap B_1$ consists of $\partial \mathbb{R}_+^{n+1} \cap B_1$ and $\mathbb{R}_+^{n+1} \cap \partial B_1$. Therefore,

$$
\int_{\mathbb{R}_+^{n+1} \cap B_1} y^b v^c \frac{d w^b}{d \lambda} = \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} -v^c \lim_{y \to 0} y^b \partial_y \left( \frac{d w^b}{d \lambda} \right) + \lim_{y \to 0} y^b \partial_y v^c \frac{d w^b}{d \lambda} + \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b v^c \partial_r \left( \frac{d w^b}{d \lambda} \right) - y^b \partial_r v^c \frac{d w^b}{d \lambda}
$$

where $r = |X|$, $X = (x, y) \in \mathbb{R}_+^{n+1}$ and $\partial_r = \nabla \cdot \frac{X}{|X|}$ is the corresponding radial derivative. Note that the first integral in the right-hand side vanishes since $\partial_y \left( \frac{d w^b}{d \lambda} \right) = 0$ on $\partial \mathbb{R}_+^{n+1}$. From (2.10) we obtain

$$
\frac{d \tilde{E}(u^c, 1)}{d \lambda} = \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \left( v^c \partial_r \left( \frac{d w^b}{d \lambda} \right) - \partial_r \left( v^c \right) \frac{d w^b}{d \lambda} \right)
$$

Now note that from the definition of $u^c$ and $v^c$ and by differentiating in $\lambda$ we get the following for $X \in \mathbb{R}_+^{n+1}$

$$
\frac{d u^c(X)}{d \lambda} = \frac{1}{\lambda} \left( \frac{2s + a}{p - 1} u^c(X) + r \partial_r u^c(X) \right)
$$

$$
\frac{d v^c(X)}{d \lambda} = \frac{1}{\lambda} \left( \frac{2(p + s - 1) + a}{p - 1} v^c(X) + r \partial_r v^c(X) \right)
$$

Therefore, differentiating with respect to $\lambda$ we get

$$
\lambda \frac{d^2 u^c(X)}{d \lambda^2} + \frac{d u^c(X)}{d \lambda} = \frac{2s + a u^c(X)}{p - 1} + r \partial_r \frac{d u^c(X)}{d \lambda}
$$

So, for all $X \in \mathbb{R}_+^{n+1} \cap \partial B_1$

$$
\partial_r \left( u^c(X) \right) = \lambda \frac{d u^c(X)}{d \lambda} - \frac{2s + a}{p - 1} u^c(X)
$$

$$
\partial_r \left( \frac{d u^c(X)}{d \lambda} \right) = \lambda \frac{d^2 u^c(X)}{d \lambda^2} + \frac{p - 1 - 2s - a}{p - 1} \frac{d u^c(X)}{d \lambda}
$$

$$
\partial_r \left( v^c(X) \right) = \lambda \frac{d v^c(X)}{d \lambda} - \frac{2(p + s - 1) + a}{p - 1} v^c(X)
$$

Substituting (2.15) and (2.16) in (2.11) we get

$$
\frac{d \tilde{E}(u^c, 1)}{d \lambda} = \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} y^b \left( \lambda \frac{d^2 u^c}{d \lambda^2} + \frac{p - 1 - 2s - a}{p - 1} \frac{d u^c}{d \lambda} \right) - y^b \left( \lambda \frac{d v^c}{d \lambda} - \frac{2(p + s - 1) + a}{p - 1} \frac{d v^c}{d \lambda} \right) \frac{d w^b}{d \lambda}
$$

Taking derivative of (2.12) in $r$ we get

$$
\frac{r \partial^2 u^c}{\partial r^2} + \frac{\partial u^c}{\partial r} = \lambda \frac{\partial}{\partial r} \left( \frac{d u^c}{d \lambda} \right) - \frac{2s + a}{p - 1} \frac{\partial u^c}{\partial r}
$$

So, from (2.15) for all $X \in \mathbb{R}_+^{n+1} \cap \partial B_1$ we have

$$
\frac{\partial^2 u^c}{\partial r^2} = \lambda \frac{\partial}{\partial r} \left( \frac{d u^c}{d \lambda} \right) - \frac{p + 2s + a - 1}{p - 1} \frac{\partial u^c}{\partial r}
$$

$$
= \lambda \left( \frac{d^2 u^c}{d \lambda^2} + \frac{p - 1 - 2s - a}{p - 1} \frac{d u^c}{d \lambda} \right) - \frac{p + 2s + a - 1}{p - 1} \left( \frac{d u^c}{d \lambda} - \frac{2s + a}{p - 1} u^c \right)
$$

$$
= \lambda^2 \frac{d^2 u^c}{d \lambda^2} - \frac{4s + 2a}{p - 1} \lambda \frac{d u^c}{d \lambda} + \frac{(2s + a)(p + 2s + a - 1)}{(p - 1)^2} u^c
$$
Note that
\[ v_\varepsilon^\lambda = \Delta u_\varepsilon^\lambda = y^{-b} \text{div}(y^b \nabla u_\varepsilon^\lambda) \]
and on \( \mathbb{R}_+^{n+1} \cap \partial B_1 \), we have
\[ \text{div}(v_\varepsilon^\lambda \nabla u_\varepsilon^\lambda) = (u_{rr} + (n + b)u_r)\theta_1^\varepsilon + \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \]
where \( \theta_1 = \frac{\varepsilon}{\lambda} \). From the above, (2.14) and (2.18) we get
\[ v_\varepsilon^\lambda = \lambda^2 \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} + \lambda \frac{d u_\varepsilon^\lambda}{d\lambda} (n + b - \frac{4s + 2a}{p - 1}) + u_\varepsilon^\lambda (\frac{2s + a}{p - 1}(\frac{p + 2s + a - 1}{p - 1} - n - b) + \theta_1^{-b} \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda)) \]
From this and (2.17) we get
\[
\begin{align*}
\frac{dE(u_\varepsilon^\lambda, 1)}{d\lambda} &= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{\lambda^2 d^2 u_\varepsilon^\lambda}{d\lambda^2} + \alpha \frac{d u_\varepsilon^\lambda}{d\lambda} + \beta u_\varepsilon^\lambda \right) d^2 u_\varepsilon^\lambda d\lambda^2 \\
(2.20) &+ \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{\lambda^2 d^2 u_\varepsilon^\lambda}{d\lambda^2} + \alpha \frac{d u_\varepsilon^\lambda}{d\lambda} + \beta u_\varepsilon^\lambda \right) du_\varepsilon^\lambda d\lambda \\
(2.21) &- \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} + \alpha \frac{d u_\varepsilon^\lambda}{d\lambda} + \beta u_\varepsilon^\lambda \right) d\lambda \\
(2.22) &+ \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \theta_1^{-b} \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \right) du_\varepsilon^\lambda d\lambda \\
(2.23) &+ \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \theta_1^{-b} \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \right) du_\varepsilon^\lambda d\lambda \\
(2.24) &- \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \theta_1^{-b} \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \right) du_\varepsilon^\lambda d\lambda
\end{align*}
\]
where \( \alpha := n + b - \frac{4s + 2a}{p - 1} \) and \( \beta := \frac{2a + p + 1}{p - 1} - n - b \). Simplifying the integrals we get
\[
\begin{align*}
(2.25) \frac{d\bar{E}(u_\varepsilon^\lambda, 1)}{d\lambda} &= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( 2\lambda^3 \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \right)^2 + 4\lambda^2 \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \frac{du_\varepsilon^\lambda}{d\lambda} + 2(\alpha - \beta)\lambda \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2 \right) \\
&+ \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{\beta}{2\lambda^2} \frac{d^2 \lambda}{d\lambda^2} (\lambda u_\varepsilon^\lambda)^2 \right) - \frac{1}{2} \frac{d}{d\lambda} \left( \frac{\lambda^3 \frac{d}{d\lambda} \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2}{\frac{d}{d\lambda} \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2} \right) + \beta \frac{d}{d\lambda} (u_\varepsilon^\lambda)^2 \\
&+ \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) + 3 \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \frac{du_\varepsilon^\lambda}{d\lambda} - \lambda \frac{d}{d\lambda} \left( \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \right) \frac{du_\varepsilon^\lambda}{d\lambda}
\end{align*}
\]
Note that from the assumptions we have \( \alpha - \beta - 1 > 0 \), therefore the first term in the RHS of (2.25) is positive that is
\[
2\lambda^3 \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \right)^2 + 4\lambda^2 \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \frac{du_\varepsilon^\lambda}{d\lambda} + 2(\alpha - \beta)\lambda \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2 = 2\lambda^3 \left( \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} + \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2 + 2(\alpha - \beta - 1)\lambda \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2 > 0
\]
From this we have
\[
\begin{align*}
\frac{d\bar{E}(u_\varepsilon^\lambda, 1)}{d\lambda} &= \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta_1^\varepsilon \lambda \left( \frac{\beta}{2\lambda^2} \frac{d^2 \lambda}{d\lambda^2} (\lambda u_\varepsilon^\lambda)^2 \right) - \frac{1}{2} \frac{d}{d\lambda} \left( \frac{\lambda^3 \frac{d}{d\lambda} \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2}{\frac{d}{d\lambda} \left( \frac{du_\varepsilon^\lambda}{d\lambda} \right)^2} \right) + \beta \frac{d}{d\lambda} (u_\varepsilon^\lambda)^2 \\
&+ \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \lambda \frac{d^2 u_\varepsilon^\lambda}{d\lambda^2} \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) + 3 \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \frac{du_\varepsilon^\lambda}{d\lambda} - \lambda \frac{d}{d\lambda} \left( \text{div}_{S^*}(\theta_1^\varepsilon \nabla S^* u_\varepsilon^\lambda) \right) \frac{du_\varepsilon^\lambda}{d\lambda}
\end{align*}
\]
\[ =: R_1 + R_2. \]
Theorem 3.1. Suppose that 

\[ \Gamma(\frac{n}{2} - \frac{s+\frac{a}{p}}{p-1}) > \Gamma(\frac{n-2s}{2} - \frac{s+\frac{a}{p}}{p-1}) \frac{\Gamma(\frac{n-2s}{2} - \frac{s+\frac{a}{p}}{p-1})^{2}}{\Gamma(\frac{n}{2} - \frac{s+\frac{a}{p}}{p-1})^{2}}. \]
Proof. Since $u$ satisfies (1.1), the function $\psi$ satisfies

$$ |x|^{-\frac{2n+\alpha}{p-1}} \psi^p(\theta) = \int |x|^{-\frac{2n+\alpha}{p-1}} \psi(\theta) - |y|^{-\frac{2n+\alpha}{p-1}} \psi(\sigma) \, dy $$

$$ = \int |x|^{-\frac{2n+\alpha}{p-1}} \psi(\theta) - r^{-\frac{2n+\alpha}{p-1}} t^{-\frac{2n+\alpha}{p-1}} \psi(\sigma) \, (t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}} |x|^{n-1} \, dt \, d\sigma $$

$$ = \int \frac{\psi(\theta)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt \, d\sigma $$

$$ + \int \frac{t^{-\frac{2n+\alpha}{p-1}} (\psi(\theta) - \psi(\sigma))}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} t^{n-1} \, dt \, d\sigma $$

We now drop $|x|^{-\frac{2n+\alpha}{p-1}}$ and get

$$ (3.2) \quad \psi(\theta) A_{n,s,\alpha}(\theta) + \int_{S_{n-1}} K_{\frac{2n+\alpha}{p-1}} (\psi(\theta) - \psi(\sigma)) \, d\sigma = \psi^p(\theta) $$

where

$$ A_{n,s,\alpha} := \int_0^\infty \int_{S_{n-1}} \frac{1 - t^{-\frac{2n+\alpha}{p-1}}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt \, d\sigma $$

and

$$ K_{\frac{2n+\alpha}{p-1}} (\psi(\theta) - \psi(\sigma)) := \int_0^\infty \frac{t^{n-1} - \frac{2n+\alpha}{p-1}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt $$

Note that

$$ K_{\frac{2n+\alpha}{p-1}} (\psi(\theta) - \psi(\sigma)) = \int_0^1 \frac{t^{n-1} - \frac{2n+\alpha}{p-1}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt + \int_1^\infty \frac{t^{n-1} - \frac{2n+\alpha}{p-1}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt $$

We now set $K_{\alpha}(\psi(\theta) - \psi(\sigma)) = \int_0^1 \frac{t^{n-1} - \frac{2n+\alpha}{p-1}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt$. The most important property of $K_{\alpha}$ is that $K_{\alpha}$ is decreasing in $\alpha$. This can be seen by the following elementary calculations

$$ \partial_\alpha K_{\alpha} = \int_0^1 \frac{-t^{n-1-\alpha} \ln t + t^{2s-1+\alpha} \ln t}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt $$

$$ = \int_0^1 \frac{\ln t(-t^{n-1-\alpha} + t^{2s-1+\alpha})}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, dt < 0 $$

For the last part we have used the fact that for $p > \frac{4+2\alpha+2\alpha}{n-2s}$ we have $2s - 1 + \alpha < n - 1 - \alpha$.

From (3.2) we get the following

$$ (3.3) \quad \int_{S_{n-1}} \psi^2(\theta) A_{n,s,\alpha} + \int_{S_{n-1}} K_{\frac{2n+\alpha}{p-1}} (\psi(\theta) - \psi(\sigma))^2 \, d\theta \, d\sigma = \int_{S_{n-1}} \psi^{p+1}(\theta) \, d\theta $$

We set a standard cut-off function $\eta_\epsilon \in C_c^1(\mathbb{R}_+)$ at the origin and at infinity that is $\eta_\epsilon = 1$ for $\epsilon < r < \epsilon^{-1}$ and $\eta_\epsilon = 0$ for either $r < \epsilon/2$ or $r > 2/\epsilon$. We test the stability (1.2) on the function $\phi(x) = r^{-\frac{n-2s}{2s}} \psi(\theta) \eta_\epsilon(r)$.

Note that

$$ \int_{\mathbb{R}^n} \phi(x) \, dy = \int_{S_{n-1}} \frac{r^{-\frac{n-2s}{2s}} \psi(\theta) \eta_\epsilon(r) - |y|^{-\frac{n-2s}{2s}} \psi(\sigma) \eta_\epsilon(|y|)}{(r^2 + |y|^2 - 2r|y| < \theta, \sigma >)^{\frac{n+2s}{2s}}} \, d\sigma \, d(|y|) $$
Now set \( |y| = rt \) then

\[
\int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2s}} dy = r^{-\frac{n}{2} - s} \int_0^\infty \int_{S^{n-1}} \frac{\psi(\theta) \eta_r(r) - t^{-\frac{n-2s}{2}} \psi(\sigma) \eta_r(rt)}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dtd\sigma
\]

\[
= r^{-\frac{n}{2} - s} \int_0^\infty \int_{S^{n-1}} \frac{\psi(\theta) \eta_r(r) - t^{-\frac{n-2s}{2}} \psi(\sigma) \eta_r(r) + t^{-\frac{n-2s}{2}} (\eta_r(\psi(\theta) - \eta_r(rt) \psi(\sigma))}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dtd\sigma
\]

\[
= r^{-\frac{n}{2} - s} \eta_r(r) \psi(\theta) \int_0^\infty \int_{S^{n-1}} \frac{1 - t^{-\frac{n-2s}{2}}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dtd\sigma
\]

\[+ r^{-\frac{n}{2} - s} \eta_r(r) \int_0^\infty \int_{S^{n-1}} \frac{t^{n-1} - \frac{n-2s}{2} (\eta_r(\psi(\theta) - \eta_r(rt))}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dtd\sigma
\]

Define \( \Lambda_{n,s} := \int_0^\infty \int_{S^{n-1}} \frac{1 - \frac{n-2s}{2}}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} d\sigma dt \). Therefore,

\[
\int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2s}} dy = r^{-\frac{n}{2} - s} \eta_r(r) \psi(\theta) \Lambda_{n,s}
\]

\[+ r^{-\frac{n}{2} - s} \eta_r(r) \int_{S^{n-1}} K_{\frac{n-2s}{2}}(\psi(\theta) - \psi(\sigma)) d\sigma
\]

\[+ r^{-\frac{n}{2} - s} \int_0^\infty \int_{S^{n-1}} \frac{t^{n-1} - \frac{n-2s}{2} (\eta_r(\psi(\theta) - \eta_r(rt))}{(t^2 + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}}} t^{n-1} dtd\sigma
\]

Applying the above, we compute the left-hand side of the stability inequality (1.2),

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{n+2s}} dxdy = 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\phi(x) - \phi(y))\phi(x)}{|x - y|^{n+2s}} dxdy
\]

\[= 2 \int_0^\infty r^{-1} \eta_r^2(r) dr \int_{S^{n-1}} \psi^2 \Lambda_{n,s} d\theta
\]

\[+ 2 \int_0^\infty r^{-1} \eta_r^2(r) dr \int_{S^{n-1}} K_{\frac{n-2s}{2}}(\psi(\theta) - \psi(\sigma))^2 d\sigma d\theta
\]

\[
(3.4)
\]

We now compute the second term in the stability inequality (1.2) for the test function \( \phi(x) = r^{-\frac{2s}{p-2s}} \psi(\theta) \eta_r(r) \) and \( u = r^{-\frac{2s}{p-2s}} \psi(\theta) \),

\[
p \int_0^\infty r^s |u|^{p-1} \phi^2 = p \int_0^\infty r^s r^{-p(2s+a)} r^{-(n-2s)} \psi^{p+1} \eta_r^2(r) dr
\]

\[
(3.5)
\]

Due to the definition of the \( \eta_r \), we have \( \int_0^\infty r^{-1} \eta_r^2(r) dr = \ln(2/\epsilon) + O(1) \). Note that this term appears in both terms of the stability inequality that we computed in (3.4) and (3.6). We now claim that

\[
f_\epsilon(t) := \int_0^\infty r^{-1} \eta_r(r)(\eta_r(r) - \eta_r(rt)) dr = O(\ln t)
\]
Note that $\eta(r) = 1$ for $\frac{1}{2} < r < \frac{1}{\sqrt{2}}$ and $\eta(r) = 0$ for either $r < \frac{1}{2}$ or $r > \frac{2}{\sqrt{2}}$. Now consider various ranges of value of $t \in (0, \infty)$ to compare the support of $\eta(t)$ and $\eta(rt)$. From the definition of $\eta$, we have

$$f_r(t) = \int_0^t r^{-1} \eta(r)(\eta(r) - \eta(rt)) dr$$

In what follows we consider a few cases to explain the claim. For example when $\epsilon < \frac{1}{\epsilon} < \frac{1}{\epsilon}$ then

$$f_r(t) \approx \int_0^t r^{-1} dr + \int_{\frac{t}{\epsilon}}^t r^{-1} dr \approx \ln t$$

Now consider the case $\frac{1}{\epsilon} < \frac{1}{\epsilon} < \frac{1}{\epsilon}$ then $t \approx \epsilon^2$. So,

$$f_r(t) \approx \int_0^t r^{-1} dr + \int_{\frac{t}{\epsilon}}^t r^{-1} dr \approx \ln t + \ln \epsilon \approx \ln t$$

Other cases can be treated similarly. From this one can see that

$$\int_0^\infty \left[ \int_0^\infty r^{-1}\eta(r)(\eta(r) - \eta(rt)) dr \right] \int_n^{-1} t^{-1} \ln(t^{2} + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}} \psi(\sigma) \psi(\theta) d\sigma d\theta dt$$

$$\int_{S^{n-1}} \int_{S^{n-1}} \int_0^\infty \frac{t^{n-1} - \frac{t^{2} + 1 - 2t < \theta, \sigma >}{\ln t}}{(t^{2} + 1 - 2t < \theta, \sigma >)^{\frac{n+2s}{2}} \psi(\sigma) \psi(\theta) d\sigma d\theta dt$$

Collecting higher order terms of the stability inequality we get

$$(3.9) \Lambda_{n,s} \int_{S^{n-1}} \psi^2 + \int_{S^{n-1}} K_{\frac{2s}{n}}(<\theta, \sigma >)(\psi(\theta) - \psi(\sigma))^2 d\sigma \geq p \int_{S^{n-1}} \psi^{p+1}$$

From this and (3.3) we obtain

$$(\Lambda_{n,s} - pA_{n,s,a}) \int_{S^{n-1}} \psi^2 + \int_{S^{n-1}} (K_{\frac{2s}{n}} - pK_{\frac{2s+2a}{n}})(<\theta, \sigma >)(\psi(\theta) - \psi(\sigma))^2 d\sigma \geq 0$$

Note that $K_{\alpha}$ is decreasing in $\alpha$. This implies $K_{\frac{2s}{n}} < K_{\frac{2s+2a}{n}}$ for $p > \frac{n+2s+2a}{n-2s}$. So, $K_{\frac{2s}{n}} - pK_{\frac{2s+2a}{n}} < 0$. On the other hand the assumption of the theorem implies that $\Lambda_{n,s} - pA_{n,s,a} < 0$. Therefore, $\psi = 0$.

\section{Energy Estimates}

In this section, we provide some estimates for solutions of (1.1). These estimates are needed in the next section when we perform a blow-down analysis argument. The methods and ideas provided in this section are strongly motivated by [9, 10].

**Lemma 4.1.** Let $u$ be a stable solution to (1.1). Let also $\eta \in C_c^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, define

$$\rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} dy$$

Then,

$$\int_{\mathbb{R}^n} |x|^a |u|^{p+1} \eta^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)\eta(x) - u(y)\eta(y)|^2}{|x - y|^{n+2s}} dxdy \leq C \int_{\mathbb{R}^n} u^2 \rho dx$$

**Proof.** Proof is quite similar to Lemma 2.1 in [9] and we omit it here.

\qed
Lemma 4.2. Let $m > n/2$ and $x \in \mathbb{R}^n$. Set
\[
\rho(x) = \int_{\mathbb{R}^n} \frac{|\eta(x) - \eta(y)|^2}{|x-y|^{n+2s}} dy \quad \text{where} \quad \eta(x) = (1 + |x|^2)^{-m/2}
\]
Then there is a constant $C = C(n,s,m) > 0$ such that
\[
C^{-1}(1 + |x|^2)^{-n/2-s} \leq \rho(x) \leq C(1 + |x|^2)^{-n/2-s}
\]
Proof. Proof is quite similar to Lemma 2.2 in [9] and we omit it here. \qed

Corollary 4.1. Suppose that $m > n/2$, $\eta$ given by (4.3) and $R > 1$. Define
\[
\rho_R(x) = \int_{\mathbb{R}^n} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{n+2s}} dy \quad \text{where} \quad \eta_R(x) = \eta(x/R)\phi(x)
\]
where $\phi \in C^\infty(\mathbb{R}^n) \cap [0,1]$ is a cut-off function. Then there exists a constant $C > 0$ such that
\[
\rho_R(x) \leq C\eta \left(\frac{x}{R}\right)^2 |x|^{-n-2s} + R^{-2s} \rho \left(\frac{x}{R}\right)
\]
Lemma 4.3. Suppose that $u$ is a stable solution of (1.1). Consider $\rho_R$ that is defined in Corollary 4.1 for $n/2 < m < n/2 + s(p + 1)/2$. Then there exists a constant $C > 0$ such that
\[
\int_{\mathbb{R}^n} u^2 \rho_R dx \leq CR^{n-\frac{2(p+1)+2n}{p-1}}
\]
for any $R > 1$
Proof. Note that
\[
\int_{\mathbb{R}^n} u^2 \rho_R dx \leq \left(\int_{\mathbb{R}^n} |x|^\alpha |u|^{p+1} \eta^2_R dx\right)^{\frac{1}{p+1}} \left(\int_{\mathbb{R}^n} |x|^{-\frac{2s}{p+1}} \rho^{\frac{p+1}{p+1}} \eta^{\frac{1}{p+1}}_R dx\right)^{\frac{1}{p+1}}
\]
From Lemma 4.1 we get
\[
\int_{\mathbb{R}^n} u^2 \rho_R dx \leq \int_{\mathbb{R}^n} |x|^{-\frac{2s}{p+1}} \rho^{\frac{p+1}{p+1}} \eta^{\frac{1}{p+1}}_R dx
\]
Now applying Corollary 4.1 for two different cases $|x| > R$ and $|x| < R$ one can get $\rho_R(x) \leq C(|x|^{-n-2s} + R^{-2s})$ and $\rho(x) \leq CR^{-2s}(1 + |x|^2)^{-n/2-s}$. This finishes the proof.
Note that

We are now ready to state the essential estimate on stable solutions. Since the proofs are similar to the ones given in [9], for the case of $0 < s < 1$, and in [16], for the case of $1 < s < 2$, we omit them here.

Lemma 4.4. Suppose that $p \neq \frac{n+2s+2n}{n-2s}$. Let $u$ be a stable solution of (1.1) and $u_c$ satisfies (1.5). Then there exists a constant $C > 0$ such that
(i) for $0 < s < 1$
\[
\int_{B_R} y^{1-2s} u_c^2 \leq CR^{n+2-\frac{2(p+1)+2n}{p-1}}
\]
and
(ii) for $1 < s < 2$
\[
\int_{B_R} y^{3-2s} u_c^2 \leq CR^{n+4-\frac{2(p+1)+2n}{p-1}}
\]
Lemma 4.5. Let $u$ be a stable solution of (1.1) and $u_c$ satisfies (1.5). Then there exists a positive constant $C$ such that
(i) for $0 < s < 1$

\[
\begin{aligned}
\int_{B_R \cap \partial \mathbb{R}^n_+} |x|^{a}|u_e|^{p+1} dx + \int_{B_R \cap \mathbb{R}^n_+} y^{1-2s} |\nabla u_e|^2 dy & \leq CR^{n-2(s(p+1)+2s)} \\
\end{aligned}
\]

and

(ii) for $1 < s < 2$

\[
\begin{aligned}
\int_{B_R \cap \partial \mathbb{R}^n_+} |x|^{a}|u_e|^{p+1} dx + \int_{B_R \cap \mathbb{R}^n_+} y^{3-2s} |\Delta u_e|^2 dy & \leq CR^{n-2(s(p+1)+2s)} \\
\end{aligned}
\]

5. Blow-down analysis

This section is devoted to the proof of Theorem 1.2. The methods and ideas are strongly motivated by the ones given in [9, 10].

Proof of Theorem 1.2: Let $u$ be a stable solution of (1.1) and let $u_e$ be its extension solving (1.5). For the case $1 < p \leq p_S(n,a)$ the conclusion follows from the Pohozaev identity. Note that for the subcritical case Lemma 4.5 implies that $u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Multiplying (1.1) with $u$ and doing integration, we obtain

\[
\int_{\mathbb{R}^n} |x|^a |u|^{p+1} = ||u||_{H^s(\mathbb{R}^n)}^2
\]

in addition multiplying (1.1) with $u^\lambda(x) = u(\lambda x)$ yields

\[
\int_{\mathbb{R}^n} |x|^a |u|^{p-1} u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} u^\lambda = \lambda \int_{\mathbb{R}^n} w^\lambda
\]

where $w = (-\Delta)^{s/2} u$. Following ideas provided in [10, 26] and using the change of variable $z = \sqrt{\lambda} x$ one can get the following Pohozaev identity

\[
-\frac{n + a}{p + 1} \int_{\mathbb{R}^n} |x|^a |u|^{p+1} = \frac{2s - n}{2} \int_{\mathbb{R}^n} w^2 + \frac{d}{d\lambda}|_{\lambda = 1} \int_{\mathbb{R}^n} w^{\sqrt{\lambda} w^{1/2}} dz = \frac{2s - n}{2} ||u||_{\dot{H}^s(\mathbb{R}^n)}^2
\]

This equality together and (5.1) proves the theorem for the subcritical case.

Now suppose that $p > p_S(n,a)$.

Case 1: $0 < s < 1$. We perform the proof in a few steps.

Step 1. $\lim_{\lambda \to +\infty} E(u_e, \lambda) < +\infty$. From the fact that $E$ is nondecreasing in $\lambda$, it suffices to show that $E(u_e, \lambda)$ is bounded. Write $E = I + J$, where $I$ is given by (2.1) and

\[
J(u_e, \lambda) = \lambda \int_{\mathbb{R}^n} y^{1-2s} u_e^2 d\sigma
\]

Note that Lemma 4.5 implies that $I$ is bounded. To show that $E$ is bounded we state the following argument. The nondecreasing property of $E$ yields

\[
E(u_e, \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(u, t) dt \leq C + \lambda \int_{\mathbb{R}^n_+} y^{1-2s} u_e^2
\]

From Lemma 4.4 we conclude that $E$ is bounded.

Step 2. There exists a sequence $\lambda_i \to +\infty$ such that $(u_e^{\lambda_i})$ converges weakly in $H^1_{loc}(\mathbb{R}^n_+; y^{1-2s} dy dx)$ to a function $u_e^\infty$.

This follows from the fact that $(u_e^{\lambda_i})$ is bounded in $H^1_{loc}(\mathbb{R}^n_+; y^{1-2s} dy dx)$ by Lemma 4.5.

Step 3. $u_e^\infty$ is homogeneous.
To see this, apply the scale invariance of $E$, its finiteness and the monotonicity formula: given $R_2 > R_1 > 0$,

\[
0 = \lim_{n \to +\infty} E(u_e, \lambda; R_2) - E(u_e, \lambda; R_1) = \lim_{n \to +\infty} E(u_e^\lambda, R_2) - E(u_e^\lambda, R_1) \geq \liminf_{n \to +\infty} \int_{(B_{2R_2} \setminus B_{R_1}) \cap R_2^{n+1}} y^{1-2s} e^{2-2n+\frac{4+2s}{p-1}} \left( \frac{2s + a u_e^\lambda}{p-1} \frac{\partial u_e^\lambda}{\partial r} \right)^2 \, dx dy \\
\geq \int_{(B_{2R_2} \setminus B_{R_1}) \cap R_2^{n+1}} y^{1-2s} e^{2-2n+\frac{4+2s}{p-1}} \left( \frac{2s + a u_e^\infty}{p-1} \frac{\partial u_e^\infty}{\partial r} \right)^2 \, dx dy
\]

Note that in the last inequality we only used the weak convergence of $(u_e^\lambda)$ to $u_e^\infty$ in $H^1_{loc}(\mathbb{R}_+^{n+1}; y^{1-2s} dx dy)$. So,

\[
\frac{2s + a u_e^\infty}{p-1} \frac{\partial u_e^\infty}{\partial r} = 0 \quad \text{a.e. in } \mathbb{R}_+^{n+1}.
\]

And so, $u_e^\infty$ is homogeneous.

**Step 4.** $u_e^\infty \equiv 0$. This is a direct consequence of Theorem 3.1.

**Step 5.** $(u_e^\lambda)$ converges strongly to zero in $H^1(B_R \setminus B_{r/2}; y^{1-2s} dx dy)$ and $(u_e^\lambda)$ converges strongly to zero in $L^{p+1}(B_R \setminus B_{r/2})$ for all $R > r > 0$.

From Step 2 and Step 3, we have $(u_e^\lambda)$ is bounded in $H^1_{loc}(\mathbb{R}_+^{n+1}; y^{1-2s} dx dy)$ and converges weakly to 0. Therefore, $(u_e^\lambda)$ converges strongly to zero in $L^2_{loc}(\mathbb{R}_+^{n+1}; y^{1-2s} dx dy)$. By the standard Rellich-Kondrachov theorem and a diagonal argument, passing to a subsequence, for any $B_R = B_R(0) \subset \mathbb{R}_+^{n+1}$ and $A$ of the form $A = \{(x, t) \in \mathbb{R}_+^{n+1} : 0 < t < r/2\}$, where $R, r > 0$ we obtain

\[
\lim_{i \to \infty} \int_{\mathbb{R}_+^{n+1} \cap (B_R \setminus A)} y^{1-2s} |u_e^\lambda|^2 \, dx dy = 0.
\]

By [12, Theorem 1.2],

\[
\int_{\mathbb{R}_+^{n+1} \cap B_r(x)} y^{1-2s} |u_e^\lambda|^2 \, dx dy \leq C r^2 \int_{\mathbb{R}_+^{n+1} \cap B_r(x)} y^{1-2s} |\nabla u_e^\lambda|^2 \, dx dy
\]

for any $x \in \partial \mathbb{R}_+^{n+1}$, $|x| \leq R$, with a uniform constant $C$. Applying similar arguments as [9] one can get $(u_e^\lambda)$ converges strongly to 0 in $H^1_{loc}(\mathbb{R}_+^{n+1} \setminus \{0\}; y^{1-2s} dx dy)$ and the convergence also holds in $L^{p+1}_{loc}(\mathbb{R}_+ \setminus \{0\})$.

**Step 6.** $u_e \equiv 0$.

\[
I(u_e, \lambda) = I(u_e^\lambda, 1) = \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_1} y^{1-2s} |\nabla u_e^\lambda|^2 \, dx dy - \frac{\kappa \lambda}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |x|^a |u_e^\lambda|^{p+1} \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_1} y^{1-2s} |\nabla u_e^\lambda|^2 \, dx dy - \frac{\kappa \lambda}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |x|^a |u_e^\lambda|^{p+1} \, dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_r \setminus B_1} y^{1-2s} |\nabla u_e^\lambda|^2 \, dx dy - \frac{\kappa \lambda}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_r \setminus B_1} |x|^a |u_e^\lambda|^{p+1} \, dx
\]

\[
\leq C \varepsilon^{\frac{2(p+1)+2}{p-1}} \int_{\mathbb{R}_+^{n+1} \cap B_r \setminus B_1} y^{1-2s} |\nabla u_e^\lambda|^2 \, dx dy - \frac{\kappa \lambda}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_r \setminus B_1} |x|^a |u_e^\lambda|^{p+1} \, dx
\]

\[
\leq C \varepsilon^{\frac{2(p+1)+2}{p-1}} + \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B_1} y^{1-2s} |\nabla u_e^\lambda|^2 \, dx dy - \frac{\kappa \lambda}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B_1} |x|^a |u_e^\lambda|^{p+1} \, dx
\]

Letting $\lambda \to +\infty$ and then $\varepsilon \to 0$, we deduce that $\lim_{\lambda \to +\infty} I(u_e, \lambda) = 0$. Using the monotonicity of $E$,

\[
(5.2) \quad E(u_e, \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(t) \, dt \leq \sup_{\lambda \in [\lambda, 2\lambda]} I + C \lambda^{-n-1} \epsilon^{\frac{2(p+1)+2}{p-1}} \int_{B_{2\lambda} \setminus B_\lambda} u_e^2
\]
and so \( \lim_{\lambda \to +\infty} E(u_\lambda, \lambda) = 0 \). Since \( u \) is smooth, we also have \( E(u_\epsilon, 0) = 0 \). Since \( E \) is monotone, \( E \equiv 0 \) and so \( u_\epsilon \) must be homogeneous, a contradiction unless \( u_\epsilon \equiv 0 \).

**Case 2:** \( 1 < s < 2 \). Proof of this case is very similar to Case 1. We perform the proof in a few steps.

**Step 1.** \( \lim_{\lambda \to \infty} E(u_\lambda, \lambda) < \infty \).

From Theorem 2.2, \( E \) is nondecreasing. So, we only need to show that \( E(u_\epsilon, \lambda) \) is bounded. Note that

\[
E(u_\epsilon, \lambda) \leq \frac{1}{\lambda} \int_{t}^{2\lambda} E(u_\epsilon, t) dt \leq \frac{1}{\lambda^2} \int_{t}^{2\lambda} t^{\lambda + 1} E(u_\epsilon, \gamma) d\gamma dt
\]

From Lemma 4.5 we conclude that

\[
\frac{1}{\lambda^2} \int_{t}^{2\lambda} \frac{d}{d\gamma} \left( \int_{B_{\gamma + 1} \cap B_{\gamma}} y^{2s} |\Delta u_\epsilon|^2 dydx - \frac{C_{n,s}}{p+1} \int_{\partial R^{n+1}_{+} \cap B_{\gamma}} |x|^a u_\epsilon^{p+1} dx \right) d\gamma dt \leq C
\]

where \( C > 0 \) is independent from \( \lambda \). For the next term in the energy we have

\[
\frac{1}{\lambda^2} \int_{t}^{2\lambda} \int_{B_{\gamma + 1} \cap B_{\gamma}} y^{2s} u_\epsilon^2 dydx d\gamma dt \leq \frac{1}{\lambda^2} \int_{t}^{2\lambda} \int_{B_{\gamma + 1} \cap B_{\gamma}} y^{3-2s} u_\epsilon^2 dydx dt
\]

where \( C > 0 \) is independent from \( \lambda \). In the above estimates we have applied Lemma 4.4. For the next term we have

\[
\frac{1}{\lambda^2} \left[ \int_{t}^{2\lambda} \frac{d}{d\gamma} \left( \int_{B_{\gamma + 1} \cap B_{\gamma}} y^{3-2s} \left( \frac{2s + a}{p-1} - \frac{\partial u_\epsilon}{\partial r} \right)^2 \right) dt \right] \leq C
\]

where \( C > 0 \) is independent from \( \lambda \). The rest of the terms can be treated similarly.

**Step 2.** There exists a sequence \( \lambda_i \to \infty \) such that \( (u_\lambda^\epsilon) \) converges weakly in \( L^1_{loc}(\mathbb{R}^n, y^{3-2s} dy dx) \) to a function \( u_\infty^\epsilon \).

Note that this is a direct consequence of Lemma 4.5.

**Step 3.** \( u_\infty^\epsilon \) is homogeneous and therefore \( u_\infty^\epsilon = 0 \).
To prove this claim, apply the scale invariance of $E$, its finiteness and the monotonicity formula; given $R_2 > R_1 > 0$,
\[
0 = \lim_{i \to \infty} (E(u, R_2 \lambda_i) - E(u, R_1 \lambda_i))
= \lim_{i \to \infty} (E(u^{\lambda_i}, R_2) - E(u^{\lambda_i}, R_1))
\geq \liminf_{i \to \infty} \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}^n_{+}^{+}} y^{3-2p+\frac{4s+2a}{p-1}+2s-2-n} \left( \frac{2s+a}{p-1} r^{-1} u^{\lambda_i} + \frac{\partial u^{\lambda_i}}{\partial r} \right)^2 dy dx
\geq \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}^n_{+}^{+}} y^{3-2p+\frac{4s+2a}{p-1}+2s-2-n} \left( \frac{2s+a}{p-1} r^{-1} u^{\infty} + \frac{\partial u^{\infty}}{\partial r} \right)^2 dy dx
\]
In the last inequality we have used the weak convergence of $(u^{\lambda_i})$ to $u^{\infty}$ in $H^{1}_{loc}(\mathbb{R}^n, y^{3-2s} dy dx)$. This implies
\[
\frac{2s+a}{p-1} r^{-1} u^{\infty} + \frac{\partial u^{\infty}}{\partial r} = 0 \text{ a.e. in } \mathbb{R}^n_{+}^{+}.
\]
Therefore, $u^{\infty}$ is homogeneous. Apply Theorem 3.1 we get $u^{\infty} = 0$.

**Step 5.** $(u^{\lambda_i})$ converges strongly to zero in $H^1(B_R \setminus B_\varepsilon, y^{3-2s} dy dx)$ and $(u^{\lambda_i})$ converges strongly to zero in $L^{p+1}(B_R \setminus B_\varepsilon)$ for all $R > \varepsilon > 0$.

**Step 6.** $u_\varepsilon \equiv 0$.
\[
I(u^{\varepsilon}, \lambda) = I(u^{\lambda_i}, 1)
= \frac{1}{2} \int_{\mathbb{R}^n_{+}^{+} \cap B_1} y^{3-2s} |\Delta_b u^{\lambda_i}|^2 dx dy - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^n_{+}^{+} \cap B_1} |x|^a |u^{\lambda_i}|^{p+1} dx
= \frac{1}{2} \int_{\mathbb{R}^n_{+}^{+} \cap B_1} y^{3-2s} |\Delta_b u^{\lambda_i}|^2 dx dy - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^n_{+}^{+} \cap B_1} |x|^a |u^{\lambda_i}|^{p+1} dx
\]
\[
= \varepsilon^{n-2(s(p+1)+2a)} I(u^{\varepsilon}, \lambda) + \frac{1}{2} \int_{\mathbb{R}^n_{+}^{+} \cap B_1} y^{3-2s} |\Delta_b u^{\varepsilon}|^2 dx dy - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^n_{+}^{+} \cap B_1} |x|^a |u^{\varepsilon}|^{p+1} dx
\leq C \varepsilon^{n-2(s(p+1)+2a)/p} I(u^{\varepsilon}, \lambda)
\]

Letting $\lambda \to +\infty$ and then $\varepsilon \to 0$, we deduce that $\lim_{\lambda \to +\infty} I(u^{\varepsilon}, \lambda) = 0$. Using the monotonicity of $E$,
\[
E(u^{\varepsilon}, \lambda) \leq \frac{1}{\lambda} \int_{\varepsilon}^{2\lambda} E(t) \, dt \leq \sup_{[\lambda, 2\lambda]} I + C \lambda^{-n-1} 2^{2(s(p+1)+2a)/p} \int_{B_{2\lambda} \setminus B_\lambda} u^{2} dx
\]
and so $\lim_{\lambda \to +\infty} E(u^{\varepsilon}, \lambda) = 0$. Since $u$ is smooth, we also have $E(u, 0) = 0$. Since $E$ is monotone, $E \equiv 0$ and so $u_\varepsilon$ must be homogeneous, a contradiction unless $u_\varepsilon \equiv 0$.

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