Numerical investigation of alternating-direction method for Poisson equation with weighted integral conditions

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Abstract. The present paper deals with a generalization of the alternating-direction implicit (ADI) method for a two dimensional Poisson equation in a rectangle domain with a weighted integral boundary condition in one coordinate direction. We consider the alternating direction method for a system of difference equations that approximates Poisson equation with weighed integral boundary conditions with the fourth-order accuracy. Sufficient conditions of stability for ADI method are investigated numerically. An analysis of results of computational experiments is presented.

Keywords: elliptic equation, nonlocal integral conditions, finite-difference method, alternating-direction method, convergence of iterative method.

Introduction

Various numerical finite difference schemes have been proposed to solve boundary value problems of elliptic partial differential equations. The demand for better and more efficient methods has grown as the range of applications has increased. Mathematical models involving elliptic partial differential equations arise in diverse applications such as heat conduction, electrostatics, mechanical engineering and theoretical physics [1]. The simplest discretization method for elliptic equations is the classical finite difference method then equation are discretized and reduced to finite difference equations, and then linear systems with large sparse matrices are solved using appropriate numerical techniques. But many difficulties arise in the case of more complicated differential operators, boundary conditions, and computational domains. Problems with nonlocal boundary conditions have been intensively studied during last time. Second-order finite-difference methods for elliptic equations with nonlocal conditions were considered in [3, 6, 10], where the main aspect was the convergence of methods. In [7, 5], a difference scheme of fourth-order accuracy was considered for elliptic equation with nonlocal integral conditions. Necessary and sufficient conditions and a convergent iterative solution method were found for the system of difference equations in the case of a unique solution. In this paper, we consider the alternating direction method for a system of difference equations that approximates Poisson equation with weighed integral boundary conditions with the fourth-order accuracy.
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The matrix of a system of difference equations with nonlocal conditions is always nonsymmetric, and its spectrum can be rather complicated, i.e., there may exist negative, multiple, or complex eigenvalues [8]. The purpose of the present work is to perform a numerical investigation of the influence of the integral boundary conditions on the ADI method applicability to solution of the elliptic differential equations with nonlocal conditions [5].

This paper is organized as follows. In Section 1, we formulate a difference problem and write the fourth-order ADI method [4]. In Section 2, we present results of numerical experiment. Section 3 contains some brief conclusions and comments.

1 Statement of a difference problem

We consider two-dimensional elliptic equation with weighted integral conditions:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega, \]  
\[ u(x, 0) = \gamma_0 \int_0^{L_x} \alpha(x, y)u(x, y) \, dx + v^l(y), \quad y \in [0, L_y], \]  
\[ u(L_x, y) = \gamma_1 \int_0^{L_x} \beta(x, y)u(x, y) \, dx + v^r(y), \quad y \in [0, L_y], \]  
\[ u(x, 0) = v^l(x), \quad u(L_x, 0) = v^r(x), \quad x \in [0, L_x]. \]

where \( \Omega = (0, L_x) \times (0, L_y) \) is a rectangular domain, \( \gamma_0 \) and \( \gamma_1 \) are given constants. Finite-difference methods of fourth order accuracy for elliptic equations with nonlocal conditions \( (\alpha(x, y) = \beta(x, y) = 1) \) were considered and investigated in [7]. The ADI method for this problem was considered in [5, 9].

In the domain \( \Omega \) we consider the grid \( \mathcal{U}^h = \mathcal{W}_x^h \times \mathcal{W}_y^h \) with steps \( h_x = \frac{L_x}{n} \) and \( h_y = \frac{L_y}{m} \).

Equations (1)–(4) are replaced with a finite-difference equations:

\[- \left( \delta_x^2 + \delta_y^2 + \frac{h_x^2 + h_y^2}{12} \delta_x \delta_y \right) u_{ij} = T_{ij}, \]  
\[ T_{ij} = f_{ij} + \frac{h_x^2}{12} \delta_x f_{ij} + \frac{h_y^2}{12} \delta_y f_{ij}, \quad (x_i, y_j) \in \mathcal{W}^h, \]  
\[ u_{0j} = \gamma_0 \sum_{i=0}^{n} \alpha_{ij} w_{ij}^l + v_{j}^l, \quad y_j \in \mathcal{W}_y^h, \]  
\[ u_{nj} = \gamma_1 \sum_{i=0}^{n} \beta_{ij} w_{ij}^r + v_{j}^r, \quad y_j \in \mathcal{W}_y^h, \]  
\[ u_{i0} = w_{i}^l, \quad u_{im} = w_{i}^r, \quad x_i \in \mathcal{W}_x^h, \]

where \( \delta_x^2 \) and \( \delta_y^2 \) are second order central difference operators. We assume the comparability of the boundary conditions

\[ w_{0}^l = \gamma_0 \sum_{i=0}^{n} \alpha_{0i} w_{i}^l + v_{0}^l, \quad w_{n}^r = \gamma_1 \sum_{i=0}^{n} \beta_{ni} w_{i}^r + v_{n}^r. \]
ADI method for Poisson equation with weighted integral conditions

We approximate integral conditions (2)–(4) using the Simpson formula with the fourth order approximation error: \( \rho_i = (3 + (-1)^{i-1})/3, i = 1, \ldots, n-1, \rho_0 = \rho_n = 1/3, n \) is even.

We consider the corresponding discrete Sturm–Liouville problem

\[
-\delta_i^2 v_i = \lambda v_i, \quad x_i \in \omega_x^h, \quad (9)
\]

\[
v_{0j} = \gamma_0 \sum_{i=0}^n \alpha_i v_i \rho_i h_x, \quad v_{nj} = \gamma_1 \sum_{i=0}^n \beta_i v_i \rho_i h_x, \quad (10)
\]

where \( \alpha_i, \beta_i, \rho_i \in F(\omega_x^h) \). Problem (9)–(10) can be interpreted as difference analogue of differential eigenvalue problem:

\[
\frac{d^2 v}{dx^2} = \lambda v, \quad 0 < x < L_x, \quad (11)
\]

\[
v(0) = \gamma_0 \int_0^{L_x} \alpha_0(x) v(x) \, dx, \quad v(L_x) = \gamma_1 \int_0^{L_x} \alpha_1(x) v(x) \, dx. \quad (12)
\]

Stationary problems with different nonlocal conditions are investigated in paper [2] and the following theorem is valid.

**Theorem 1.** \( \lambda = 0 \) is eigenvalue of matrix \(-A_x\) if and only if

\[
\det K \cdot \gamma_0 \gamma_1 - \gamma_0 k_{00} - \gamma_1 k_{11} + 1 = 0, \quad (13)
\]

where

\[
K = \begin{pmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{pmatrix}, \quad k_{00} = \int_0^{L_x} \alpha_i(x) x \, dx, \quad k_{10} = \int_0^{L_x} \alpha_i(x) (1 - x) \, dx, \quad i = 0, 1.
\]

Namely, \( \lambda = 0 \) is eigenvalue of matrix \(-A_x\) if and only if point \((\gamma_0, \gamma_1)\) belongs to the second order curve.

- If \( \det K \neq 0, k_{01} k_{10} \neq 0 \) it is hyperbola with vertical and horizontal asymptotes;
- If \( \det K \neq 0, k_{01} k_{10} = 0 \) it is a pair of perpendicular lines (the vertical line and the horizontal line);
- If \( \det K = 0, k_{01} k_{10} \neq 0 \) it is a line.

2 Numerical experiment

We consider a problem (1)–(4) in unit square domain \([0, 1] \times [0, 1]\) with weights

\[
\alpha(x) = \frac{2(1 - bx)}{2 - b}, \quad \beta(x) = \frac{2(1 + bx)}{2 + b}.
\]

This gives a qualitatively different regions of convergence for the ADI method.

The exact solution of this test problem is given by

\[
u(x, y) = x^6 + y^6.
\]

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The right-hand side function \( f(x, y) \), initial and boundary conditions were prescribed to satisfy the given exact solution \( u(x, y) \). We consider uniform grids with different mesh sizes and analyze the convergence and accuracy of the computed solution from the present ADI scheme. We compute the maximum norm of the error of the numerical solution with respect to the exact solution, which is defined as

\[
\|\varepsilon\|_{C,h} = \max_{j=1,...,m} \max_{i=1,...,n} |u(x_i, y_j) - u_{ij}|.
\]

Test problems were solved with different values of parameters \( b, \gamma_1 \) and \( \gamma_2 \). Numerical experiments let us to determine domains where real parts of all eigenvalues are positive and domains where they are negative or zero. We fix values of the parameter \( b \) (\( b = -2.5, b = -3, \) or \( b = -3.5 \)) and study the influence of weight functions on the stability of solutions given by ADI method depending on \( \gamma_1 \) and \( \gamma_2 \).

In all cases there exists a region where ADI method gives stable solutions. Fig. 1 demonstrate the regions where the real part of eigenvalues for operator of the ADI method are positive or negative for different weights \( \alpha \) and \( \beta \). All eigenvalues are positive in \( \Omega_1 \), so ADI method is converged. There exists negative eigenvalue in \( \Omega_2 \), but ADI method is converged in \( \Omega_2 \), too. ADI method isn’t converged in other parts of domain \( \Omega \). Therefore, \( \Omega_1 \) is the domain of the stability of the solution finding by theoretical result based on spectral stability while \( \Omega_2 \) is the region of the stability of the solution established by numerical experiment.

In the case \( b = -3.5 \) according to theoretical results connected with eigenvalue problem stability region for ADI method is region \( \Omega_1 \) under the hyperbola in the third quarter (Fig. 1(2)). So convergence region for problem with \( b = -3.5 \) where positive eigenvalues exist located under the hyperbola, but the method converges in a much larger area with a negative eigenvalue. Table 1 presents the performances of the algorithm for various weights depends on \( b = -3.5 \). Errors for the discrete solution on the grids \( 32 \times 32 \), minimal eigenvalues, convergence rates and number of iterations are presented for different \( \gamma_1, \gamma_2 \). The results show that the method can be applied even when there is a negative or zero eigenvalue. The method has the fourth-order convergence.

For \( b = -3 \) stability region of positive eigenvalues is angle between a pair of perpendicular lines. (Fig. 1(1)).

For \( b = -2.5 \) region of positive eigenvalues is located between a pair of hyperbolas (Fig. 1(3)). Table 2 presents the performances of the algorithm for various weights depends on \( b = -2.5 \). Note that for large values of \( \gamma_0, \gamma_1 \) error increases.
Table 1. Errors for the discrete solution for problem solved with different $\gamma_1, \gamma_2$ for $b = -3.5$.

| $\gamma_0$ | $\gamma_1$ | $h$ | $\lambda_{\text{min}}$ | $\varepsilon_k$ | $\varepsilon_{k-1}/\varepsilon_k$ | Number of iter. |
|------|------|-----|----------------|---------|----------------|----------------|
| 1.0  | 1.0  | 0.125 | 0.000 | 0.002822 | 15.61 | 18 |
|      |      | 0.0625 | 1.8071 ⋅ 10^{-4} | 15.90 | 23 |
|      |      | 0.03125 | 1.1363 ⋅ 10^{-5} | 15.97 | 27 |
|      |      | 0.015625 | 7.1129 ⋅ 10^{-7} | 15.97 | 31 |
| 10.0 | 0.0  | 0.125 | -7.80 | 0.01918 | 15.52 | 23 |
|      |      | 0.0625 | 1.2358 ⋅ 10^{-3} | 15.88 | 28 |
|      |      | 0.03125 | 7.7792 ⋅ 10^{-5} | 15.97 | 32 |
|      |      | 0.015625 | 4.8705 ⋅ 10^{-6} | 15.97 | 36 |
| 4.0  | 2.0  | 0.125 | -3.91 | 0.03858 | 15.7895 | 20 |
|      |      | 0.0625 | 2.4979 ⋅ 10^{-3} | 15.90 | 24 |
|      |      | 0.03125 | 1.5729 ⋅ 10^{-4} | 15.97 | 29 |
|      |      | 0.015625 | 9.8482 ⋅ 10^{-6} | 15.97 | 33 |

Table 2. Results for $b = -2.5$.

| $\gamma_0$ | $\gamma_1$ | $h$ | $\lambda_{\text{min}}$ | $\varepsilon_k$ | $\varepsilon_{k-1}/\varepsilon_k$ | Number of iter. |
|------|------|-----|----------------|---------|----------------|----------------|
| 4.5  | 0.3  | 0.125 | -3.1618 | 0.0095502 | 15.5525 | 19 |
|      |      | 0.0625 | -3.1650 | 6.1406 ⋅ 10^{-4} | 15.89 | 24 |
|      |      | 0.03125 | -3.1652 | 3.8637 ⋅ 10^{-5} | 15.9736 | 28 |
|      |      | 0.015625 | -3.1653 | 2.4188 ⋅ 10^{-6} | 15.9721 | 33 |
| 0.0  | 0.95 | 0.125 | -8.0803 | 0.06771 | 15.7895 | 23 |
|      |      | 0.0625 | -8.0594 | 4.2881 ⋅ 10^{-3} | 15.9006 | 28 |
|      |      | 0.03125 | -8.0581 | 2.6968 ⋅ 10^{-4} | 15.9721 | 32 |
|      |      | 0.015625 | -8.0580 | 1.6885 ⋅ 10^{-5} | 15.9721 | 37 |
| -10  | -10 | 0.125 | 32.7539 | 0.0025521 | 10.0200 | 14 |
|      |      | 0.0625 | 32.7801 | 0.0002651 | 10.0200 | 18 |
|      |      | 0.03125 | 32.7818 | 2.6459 ⋅ 10^{-5} | 10.0200 | 22 |
|      |      | 0.015625 | 32.7819 | 2.4178 ⋅ 10^{-6} | 10.0200 | 27 |
| 2.0  | 0.5  | 0.125 | 4.4570 | 0.0027356 | 15.6044 | 17 |
|      |      | 0.0625 | 4.4597 | 1.7531 ⋅ 10^{-4} | 15.9024 | 26 |
|      |      | 0.03125 | 4.4599 | 1.1024 ⋅ 10^{-5} | 15.9773 | 30 |
|      |      | 0.015625 | 4.4599 | 6.8999 ⋅ 10^{-7} | 15.9773 | 30 |
| 10.0 | 1.0  | 0.125 | 37.6440 | 0.0060 | 15.0682 | 13 |
|      |      | 0.0625 | 37.6874 | 3.9819 ⋅ 10^{-4} | 15.0682 | 18 |
|      |      | 0.03125 | 37.6885 | 3.9388 ⋅ 10^{-5} | 15.0682 | 22 |
|      |      | 0.0156 | 37.6902 | 2.7557 ⋅ 10^{-4} | 15.0682 | 26 |

3 Conclusions and remarks

The ADI method can be used for the Poisson equation with weighted integral condition with nontrivial weights $\alpha(x)$ and $\beta(x)$. Nonlocal integral conditions never make more problems than classical conditions both in number of iterations and precision of solution. But this conditions affect the region of convergence of the method. Convergence domain depends essentially on the weight functions. The values of parameters $\gamma_1$ and $\gamma_2$ in NBC are essential for the stability of the ADI method. The results of numerical experiment are in good agreement with existing theoretical results for two dimensional Poisson equation in a rectangle domain with a weighted integral
boundary condition in one coordinate direction [5]. But numerical experiments show that this problem is complicated and additional investigation is needed.

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