Deep Bellman Hedging
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Abstract
We discuss an actor-critic-type reinforcement learning algorithm for solving the problem of hedging a portfolio of financial instruments such as securities and over-the-counter derivatives using purely historic data.
Essentially we present a practical implementation for solving a problem of the structure

\[ V^*(\text{current state}) \overset{\dagger}{=} \sup_{\text{action } a} U[\text{discount factor} \cdot V^*(\text{future state}(a)) + \text{rewards}(a)] \]

for the optimal value function \( V^* \) where \( U \) represents a risk-adjusted return metric.
The key characteristics of our approach are: the ability to hedge with derivatives such as forwards, swaps, futures, options; incorporation of trading frictions such as trading cost and liquidity constraints; applicability to any reasonable portfolio of financial instruments; realistic, continuous state and action spaces; and formal risk-adjusted return objectives. We do not impose a boundary condition on some future maturity.
Most importantly, the trained model provides an optimal hedge for arbitrary initial portfolios and market states without the need for re-training.
We also prove existence of finite solutions to our Bellman equation, and show the relation to our vanilla Deep Hedging approach [BGTW19]
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1 Introduction

This note discusses a model-free, data-driven method of managing a portfolio of financial instruments such as stock, FX, securities and derivatives with reinforcement learning “AI” methods. It is a dynamic programming “Bellman” version of our Deep Hedging approach [BGTW19].

Quant Finance 2.0

The motivation for the work presented in this article – and of the Deep Hedging framework in general – is to build financial risk management models which “learn” to trade from historic data and experiences. Today, portfolios of derivatives, securities and other instruments are managed using the traditional quantitative finance engineering paradigm borne out of the seminal work by Black, Scholes & Merton. However, practical experience on any trading desk is that such models do not perform sufficiently well to be automated directly. To start with, they do not take into account trading frictions such as cost and liquidity constraints. Even beyond that they suffer from the underlying engineering approach which prioritizes a focus on interpolating hedging instruments such as forwards, options, swaps over realistic market dynamics. It is an indication of the state of affairs that standard text books on financial engineering in quantitative finance do not discuss real data out-of-sample performance of the models proposed.

As a result, the prices and risk management signals (“greeks”) are overly simplistic and do not capture important real-life dynamics. A typical trader will therefore need to adjust the prices and trading signals from such as standard models using their own heuristics.

From Deep Hedging to Deep Bellman Hedging

Our Deep Hedging framework [BGTW19], [WWP+21], [BMPW21], [BMPW22] takes a different approach and focuses on robust performance under real-life dynamics, enabled by the use of modern machine learning techniques. Its key advantages above other proposed methods for hedging a portfolio of financial instruments are:

- hedging with any number of reasonable derivatives such as forwards, swaps, futures, options;
- incorporation of trading frictions such as trading cost and liquidity or risk constraints;
- applicability to any reasonable portfolio of financial instruments, not just portfolios of primary assets;
- realistic, continuous state and action spaces; and
- provision for formal risk-adjusted return objectives.

1 See also the discussion in [BMPW22].
Our original approach \cite{BGTW19} solved this problem for a given initial portfolio and market state. That means it needs to be re-trained, say, daily to reflect changes in our trading universe or the market. The method proposed here, on the other hand, attempts to solve the optimal hedging problem over an for any portfolio and market state which is close to past experiences. It does not impose a boundary restriction for some terminal maturity.

Our main Bellman equation \eqref{eq:bellman} for the optimal value function \( V^* \) has the structural form

\[
V^* (\text{current state}) \equiv \sup_{\text{action } a} U \left[ \text{discount factor} \cdot V^* (\text{future state}(a)) + \text{rewards}(a) \right]
\]

where \( U \) represents a risk-adjusted return metric.

The work presented here is an extension of our patent application \cite{BMW20}. The main contribution is to provide a numerical implementation method for the practical problem of being able to represent arbitrary portfolios of derivatives as states using purely historic data. This means it does not require the development of a market simulator, c.f. \cite{WWP+21}.

The current article is the closest attempting to mimicking a trader’s real life behaviour in that here we will give an AI the same historic “experience” a real trader would have. Of course, our model will still be limited by the coverage of historic scenarios used to train it. Hence, human oversight is still required to cater for abrupt changes in market scenarios or starkly adverse risk scenarios.

As a fundamental contribution we also clarify conditions under which the corresponding Bellman equations are well-posed and admit unique finite solutions.

The website \url{http://deep-hedging.com} gives an overview over available material on the wider topic.

Related Works

There a few related works concerning the use of machine learning methods for managing portfolios of financial instruments which include derivatives, starting with our own \cite{BGTW19}. There, we solved the optimal trading problem for a fixed initial portfolio with a given terminal maturity and a fixed initial market state using periodic policy search, a method akin to “American Monte Carlo”.

In \cite{DJK+20} the authors discuss the use of Bellman methods for this task, namely using DQN and a number of similar methods. They also use risk-adjusted returns in the form of a mean-variance objective. However, in their work the state and action spaces are finite which is not realistic in practise. Moreover, their parametrization of the derivative portfolio is limited to single vanilla options. They also do not cover derivatives as hedging instruments. The method relies on a fixed terminal maturity.

In \cite{Hal17} the authors also develop a discrete state approach, where the problem is solved for each derivative position separately. The authors focus in their first work on vanilla options and minimize the terminal variance of the delta-hedged position. The maturity of the problem is fixed. In their later \cite{Hal19} the authors present methods to smooth the state space. In neither account are derivatives as hedging instruments supported.

A forthcoming ICML contribution \cite{MWB+22} we solve the Bellman equation associated with the Deep Hedging problem for a fixed maturity, using continuous states and the entropy as risk-adjusted return.

There is a larger literature on the application of AI methods for managing portfolio risks in the context of perpetual primary assets such as stock and FX portfolios whose distributions
might reasonably be approximated by Gaussian variables. See the summary [KR19] for an overview, where they also cover the related topic of trade execution with AI methods.

Underlying our work is the use of dynamic risk measures, a topic with a wide literature. We refer the interested reader to [DS05] and [PP10] among many others.

2 Deep Bellman Hedging

In this note we will use a notation much more similar to standard reinforcement learning literature, chiefly [SB18]. That means in particular that we will formulate our approach essentially as a continuous state Markov Decision Process (MDP) problem. We will make a decision from some point in time to another. That would typically be intraday or from day to day. To simplify our discussion we will assume we are making a decision “today” and then again “tomorrow.” Variables which are valid tomorrow will be indicated by a ′. We will strive to use bold letters for vectors. A product of two vectors is element wise, while “·” represents the dot product. We will use small letters for instances of data, and capital letters for random variables.

We denote by \( m \) the market state today. The market contains all information available to us today such as current market prices, time, past prices, bid/asks, social media feeds and the like. The set of all market states is denoted by \( \mathcal{M} \subset \mathbb{R}^N \). All quantities observed today are a function of the market state. Past market states are also known today. The market tomorrow is a random variable \( \mathbf{M}' \) whose distribution is assumed to only depend on \( \mathbf{m} \), and not on our trading activity. In terms of notation, think \( \mathbf{m} \equiv \mathbf{m}_t \) and \( \mathbf{M}' \equiv \mathbf{M}_{t+1} \). The expectation operator of a function \( f \) of \( \mathbf{M}' \) conditional on \( \mathbf{m} \) is written as \( \mathbb{E}[f(\mathbf{M}')|\mathbf{m}] := \int f(\mathbf{m}') p[dm'|\mathbf{m}] \).

We will trade financial instruments such as securities, OTC derivatives or currencies. We will loosely refer to them as “derivatives” as the most general term, even if we explicitly include primary asset such as stocks and currencies. We use \( \mathcal{X} \) to refer to the space of these instruments. For \( x \in \mathcal{X} \) we denote by \( r(x, \mathbf{m}) \in \mathbb{R} \) the cashflows arising from holding \( x \) today, aggregated into our accounting currency. Cashflows here cover everything from expiry settlements, coupons, dividends, to payments arising from borrowing or lending an asset. For a vector \( \mathbf{x} \) of instruments we use \( r(\mathbf{x}; \mathbf{m}) \) to denote the vector of their cashflows.

An instrument changes with the passage of time: an instrument \( x \in \mathcal{X} \) today becomes \( x' \in \mathcal{X} \) tomorrow, representing only cashflows from tomorrow onwards. If the expiry of the instrument is today, then \( x' = 0 \).

Every instrument \( x \) we may trade has a book value in our accounting currency which we denote by \( B(x, \mathbf{m}) \). The book value of a financial instrument is its official mark-to-market, computed using the prevailing market data \( \mathbf{m} \). This could be a simple closing price, a weighted mid-price, or the result of running a more complex standard derivative model. Following our notation \( B(x', \mathbf{M}') \) denotes the book value of the instrument tomorrow. We use \( B(\mathbf{x}, \mathbf{m}) \) for the vector of book values if \( \mathbf{x} \) is a vector of instruments. We like to stress that contrary to our other work [BMPW22] here the book value is with respect only to today’s and future cashflows, not past cashflows.

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[2] Mathematically, we say that \( \mathbf{m} \) generates today’s \( \sigma \)-algebra
[3] This means the sequence of market states generates a filtration.
[4] See the lecture notes [BH22] for an example of incorporating market impact.
[5] This implies implies that spot-FX transactions are frictionless.
In order to take into account the value of money across time, we will also assume are given a bank account – usually called the numeraire - which charges the same overnight interest rate for credits and deposits. The respective one-day discount factor from tomorrow to today is denoted by $\beta(m)$ and we will assume that there is some $\beta^*$ such that $\beta(m) \leq \beta^* < 1$. Contrary to [BGTW19] we do not assume that cashflows are discounted using the numeraire.

The discounted profit-and-loss (P&L) for a given instrument $x \in X$ is the random variable

$$\text{dB}(x, m, M') := B(x, M') - B(x, m) + r(x, m).$$

If $x \in X$ is a vector, then $d\text{B}(x, m, M')$ denotes the vector of P&Ls.

Trading

A trader is in charge of a portfolio $z \in X$ – also called “book” – of financial instruments such as currencies, securities and over-the-counter (OTC) derivatives. We call the combined $s := (z, m)$ our state today which takes values in $S := X \times M$. We will switch in our notation between writing functions in both variables $(z, m)$ and only in $(s)$ depending on context.

In order to risk manage her portfolio, the trader has access to $n$ liquid hedging instruments $h \equiv h(m) \equiv h(s) \in X^n$ in each time step. These are any liquid instruments such as forwards, options, swaps etc. Across different market states they will usually not be the contractually same fixed-strike fixed-maturity instruments; instead, they will usually be defined relative the prevailing market in terms of time-to-maturities and strikes relative to at-the-money. See [BGTW19] for details.

The action of buying $a \in \mathbb{R}^n$ units of our hedging instruments will incur transaction cost $c(a; z, m)$ on top of the book value. Transaction cost as function of $a$ is assumed to be normalized to $c(0; s) = 0$, non-negative, and convex. The convex set of admissible actions is given as $A(z, m) := \{a \in \mathbb{R}^n : c(a; z, m) < \infty\}$. Making cost dependent on both the current portfolio and the market allows modelling trading restrictions based on our current position such as short-sell constraints, or restrictions based on risk exposure.

Given $z$ and an action $a$ today the new portfolio tomorrow is $z' + a \cdot h' \in X$.

A trading policy $\pi$ is a function $\pi$ which determines the next action based on our current state, i.e. simply $a = \pi(z, m)$. We use $P := \{\pi : X \times M \to A(z, m)\}$ to refer to the convex set of all admissible policies.

A trader will usually manage her book by referring to the change in book values plus any other cashflows, most notably cashflows and the cost of hedging. The associated reward for taking an action $a$ per time step is given as

$$R(a; z, m, M') := dB(z, m, M') + a \cdot dB(h, m, M') - c(a; z, m).$$

For the rewards of a policy $\pi$ we will use the convention

$$R(\pi; z, m, M') := R(\pi(z, m); z, m, M').$$

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6Selling amounts to purchasing a negative quantity.
7Convexity excludes fixed transaction cost.
2.1 The Bellman Equation for Monetary Utilities

Standard reinforcement learning as discussed for example in [SB18] usually aims to maximize the discounted expected future rewards of running a given policy. Essentially, the optimal value function $V^*$ is stipulated to satisfy a Bellman equation of the form

$$V^*(z; m) = \sup_{a \in A(z,m)} \mathbb{E} \left[ \beta(m) V^*(z' + a \cdot h'; M') + R(a; z, m, M') \mid m \right].$$

Instead of using the expectation it is more natural in finance to choose an operator $U$ which takes into account risk aversion: this roughly means that if two events have the same expected outcome, then we prefer the one with the lower uncertainty.

**Definition 1** The Deep Bellman Hedging problem is finding a value function $V^*$ which satisfies

$$V^*(z; m) \overset{!}{=} (TV^*)(z, m)$$

$$(Tf)(z, m) := \sup_{a \in A(z,m)} \mathbb{E} \left[ \beta(m) f(z' + a \cdot h'; M') + R(a; z, m, M') \mid m \right].$$

(The action $a$ in above sup operator is a function of the state $s = (z, m).$)

We would like to stress that the value function here represents the “excess value” of a portfolio over its book value: if $V^*$ were zero, that would mean the optimal risk-adjusted value for a portfolio were given as its book value. Remark 4 makes this statement explicit.

There are many different reasonable risk-adjusted return metrics $U$ used in finance, most notably mean-volatility, mean-variance and their downside versions. We refer the reader to the seminal [Mar52]. Mean-volatility in particular remains a popular choice for many practical applications. However, it is well known that mean-volatility, mean-variance and their downside variants are not monotone, which means that even if $f(s) \geq g(s)$ for all states $s = (z, m)$ it is not guaranteed that $U[f(S')] \geq U[g(S')]$, c.f. [Bue17]. The lack of monotonicity means that standard convergence proofs for the Bellman equation do not apply; see section 5.

We will here take a more formal route and focus on monetary utilities. A functional $U$ is called a concave (diversification works) and cash-invariant. The latter means that $U[f(S', s) + y(s)] = U[f(S', s)] + y(s)$ for any function $y(s)$. The intuition behind this property is if we add a cash amount $y$ to our portfolio, then its monetary utility increases by this amount. An important implication of cash-invariance is that our optimal actions do not depend on our current wealth. We will also assume that our monetary utilities are risk-averse with respect to $\mathbb{P}$ in the sense that $\mathbb{E}[f(S')] = U[\mathbb{E}[f(S)]] \geq U[f(S')]$. The negative of a monetary utility is called a convex risk measure, c.f. [FS16]. See also [DS05] on the topic of dynamic and time-consistent risk measures.

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8 If $f \geq g$ then $U[f(S')] \geq U[g(S')]$.

9 For $X = f(S')$, $Y = g(S')$ and $\alpha \in [0, 1]$ we have $U[\alpha X + (1 - \alpha) Y] \geq \alpha U[X] + (1 - \alpha) U[Y]$.

10 We have shown in [Bue17] that cash-invariance is equivalent to being able to write-off parts of our portfolio for the worst possible outcome.

11 We note that concavity of $U$ does not imply risk aversion w.r.t. $\mathbb{P}$ in general. As an example chose a measure $\mathbb{Q} \not\equiv \mathbb{P}$ and set $U(f(S')) := \mathbb{E}_Q[f(S')]$. 

6
As in [BMPW22] we will focus on monetary utilities given as optimized certainty equivalents (OCE) of a utility function, introduced by [BTT07].

**Definition 2** Let \( u : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^1 \) utility function which means it is monotone increasing and concave. We normalize it to \( u(0) = 0 \) and \( u'(0) = 1 \). The respective OCE monetary utility is then defined by

\[
U[ f(S') \mid s ] := \sup_{y(s) \in \mathbb{R}} \mathbb{E}[ u( f(S') + y(s) ) \mid s ] - y(s)
\]

The function \( y \) in our definition will be modelled as a neural network.

Examples of OCE utility functions are

- **Expectation** (risk-neutral): \( u(x) := x \).
- **Worst Case**: \( u(x) := \inf x \).
- **CVaR or Expected Short Fall**: \( u(x) := (1 + \lambda) \min \{0, x\} \).
- **Entropy**: \( u(x) := (1 - e^{-\lambda x})/\lambda \) in which case \( U[f(S')]\{s\} = -\frac{1}{\lambda} \mathbb{E}[\exp(-\lambda f(S'))\{s\}] \). The entropy reduces to mean-variance if the variables concerned are normal. It has many other desirable properties, but it also penalizes losses rather harshly: an unhedgable short position in a Black&Scholes stock has negative infinite entropy.
- **Truncated Entropy**: to avoid the harsh penalties for short positions imposed by the exponential utility we might instead use \( u(x) := (1 - e^{-\lambda x})/\lambda 1_{x > 0} + (x - \frac{1}{2} \lambda x^2)1_{x < 0} \).
- **Vicky**: the following functional was proposed in [HH09]: \( u(x) := \frac{1}{\lambda} \left( 1 + \lambda x - \sqrt{1 + \lambda^2 x^2} \right) \).
- **Normalized quadratic utility**: \( u(x) := -\frac{1}{2} \lambda (x - \frac{1}{\lambda})^2 1_{x < \frac{1}{\lambda}} + \frac{1}{\lambda} x \).

We call a monetary utility coherent if \( U[n(s) f(S')]\{s\} = n(s) U[f(S')]\{s\} \). An OCE monetary utility is coherent if \( u(n x) = n u(x) \). Coherence is not a particularly natural property: it says that the risk-adjusted value of a position grows linearly with position size. Usually, we would assume that risk increases superlinearly. The practical relevance of this property for us is that if \( U \) is coherent, then we can move the discount factor \( \beta \) in and out of our monetary utility:

\[
\beta(s) U[f(S')]\{s\} = U[\beta(s) f(S')]\{s\}.
\]

We say \( U \) is time-consistent if iterative application lead to the same monetary utility in the sense that

\[
U[U[f(S'')]\{s\}]\{s\} = U(f(S'')\{s\}).
\]

The only time-consistent OCE monetary utilities are the entropy and the expectation, c.f. [KSO99].

We may now present the first key result of this article: recall the definition of our rewards \(^2\).

We say that rewards are finite if

\[
\sup_{a \in A(z, m)} U[ R(a; z, m, M') \mid m ] < \infty.
\]

for all \( z \in \mathcal{X} \), \( m \in \mathcal{M} \). This can be achieved for example if \( A(z, m) \) is bounded.

**Theorem 1** Assume rewards are finite and that \( U \) is an OCE monetary utility.

Then the Bellman equation \(^3\) has a unique finite solution.
The proof can be found in section 5. It relies on monotonicity and cash-invariance of the monetary utility, which means that while it does not apply to mean-variance, mean-volatility and related operators, it does apply to $U$ being VaR, e.g. the percentile function $U(X) := \mathbb{P}[X \leq \cdot]^{-1}(1 - \alpha)$ where $\alpha \in [0, 1)$ is a confidence interval. This is a good alternative to mean-volatility as it has a similar interpretation, as long as VaR’s lack of concavity is not a concern.

Remark 1 (Using only Cashflows as Rewards) Our definition of our rewards \[ (2) \] as the full mark-to-market of the hedged portfolio is in so far unusual as the reward term contains future variables, namely the book value of the hedged book tomorrow.

A more classic approach would be to let the rewards represent only actual cashflows, e.g. \[ \hat{R}(a, z, m) := r(z, m) - a \cdot B(h, m) - c(a, z, m), \] where we used \[ the discount factors \] steps. Let \[ \beta := \prod_{e=1}^{n} \beta(M(e-1)) \] and set

\[ (T_n f)(z, m) := \sup_{\pi} U \left[ \beta_n f\left(z^{(n)} + A^{(n)} \cdot H^{(n)}; M^{(n+1)} \right) + \sum_{i=1}^{n} \beta_{i-1} R(A^{(i)}, z^{(i)}, M^{(i)}, M^{(i+1)}) | M^{(i)} = m \right] \]

where we used \[ A^{(n)} := \pi(z^{(n)}, M^{(n)}). \] Our indexing scheme means that $T_1 = T$.

It is straight forward to formally extend \[ (3) \] to multiple time steps. Let $S^{(1)} := s$ and $S^{(i+1)} = S^{(i)}$. We use the state notation

\[ V_n \neq T^n V^* \text{ unless } U \text{ is the expectation} \]

We show the claim for $n = 2$. Let $R^{(n)} := R(A^{(n)}, S^{(n)}, M^{(n+1)})$. We use the state notation $s = (z, m)$.

\[ T^2 f(s) = \sup_{\pi} U \left[ \beta(M^{(1)}) \left\{ \sup_{\pi} U \left[ \beta(M^{(2)}) f(\cdots, M^{(3)}; R^{(2)} | S^{(2)}) \right] + R^{(1)} \right| s \right\] \]
3 Numerical Implementation

We now present an algorithm which will iteratively approach an optimal solution of our Bellman equation \((13)\). This is an extension over the entropy case presented in [BMW20].

Recall that \(V^*(z, m) = 0\) is the optimal solution if the book value of the portfolio \(z\) is indeed its value function. We therefore use as initial guess \(V^{(0)}(z, m) := 0\) for all portfolios and states \(s = (z, m)\).\(^{13}\)

We then solve our problem iteratively using \(V^{(n)}, \pi^{(n)}\) and \(y^{(n)}\) for \((n - 1) \to n\) with the following actor-critic scheme:

1. **Actor**: given \(V^{(n-1)}\) we wish to find an optimal neural network policy \(\pi^{(n)} : \mathcal{X} \times \mathcal{M} \to \mathbb{R}^n\) which maximizes for all states \(s = (z, m) \in \mathcal{S}\) the expression

\[
(TV^{(n-1)})(z, m) := \sup_{\pi} \left\{ U \left[ \beta(m) V^{(n-1)}(z' + \pi(z, m) \cdot h'; M') + R(\pi; z, m, M') \right] m \right\}.
\]

(8)

We recall that \(c(a; z, m) = \infty\) whenever \(a \notin \mathcal{A}(z, m)\). Hence, any finite solution \(\pi^{(n)}\) will be a member of \(\mathcal{P}\).

In the case of our OCE monetary utility, we will need to find both a network \(\pi^{(n)}\) and a network \(y^{(n)}\) to jointly maximize for all states \(s = (z, m)\) i.e:

\[
\sup_{\pi, y} \left\{ E \left[ \beta(m) u \left( V^{(n-1)}(z' + \pi(z, m) \cdot h'; M') + y(z, m) \right) - y(z, m) + R(\pi; z, m, M') \right] m \right\}.
\]

(9)

Following [SB18] will approach this by stipulating that we have a density \(Q\) over all samples from \(\mathcal{S} = \mathcal{X} \times \mathcal{M}\). In practise, \(\mathcal{S}\) is a finite sample set, and \(Q\) could be a uniform distribution. We note that under \(Q\) the current portfolio and the current market are random variables \(Z\) and \(M\), respectively. The associated unconditional expectation is denoted by \(E[\cdot] = \int Q[ds] E[\cdot|s]\).

We then solve

\[
\sup_{\pi, y} \left\{ E \left[ u \left( \beta(M) V^{(n-1)}(Z + \pi(Z, M) \cdot H'; M') + y(Z, M) \right) - y(Z, M) + R(\pi; Z, M, M') \right] \right\}.
\]

(10)

Under \(Q\) the current market state, the portfolio and the hedging instrument representation are random variables, hence we have referred to them with capital letters.

The choice of \(Q\) is not trivial: it represents the probability of possible portfolio and market states.

2. **Critic (Interpolation)**: as next step, we estimate a new value function \(V^{(n)}\) given \(\pi^{(n)}\) and \(y^{(n)}\). This means fitting a neural network \(V^{(n)}\) such that

\[
V^{(n)}(z, m) \equiv (TV^{(n-1)})(z, m)
\]

(11)

The equation \((\ast)\) is true if \(\pi\) is found over the entire set of measurable functions, and if the state contains all previous state information. Moreover, \((\ast\ast)\) is an equality for the expectation or any other coherent monetary utility. For all others convexity and \(U(0) = 0\) imply the stated inequality. The final equality \((\ast\ast)\) is only true of \(U\) is time-consistent which means either the entropy or the expectation.

\(^{13}\)It is not a good idea to initialize a network with zero to achieve this as all gradients will look rather the same. Assume \(N(\theta; x)\) is a neural network initialized by random weights \(\theta_0 \in \mathbb{R}^W\). Then use the Buehler-zero network \(N(\theta; x) := N(\theta; x) - \eta N(\theta_0; x)\) and learn \((\theta, \eta)\) where \(\eta \in [0, 1]\).
We note that solving (11) numerically with packages like TensorFlow or PyTorch will also yield samples $TV^{(n-1)}(z, m)$ for all $(z, m) \in S$. Assuming this is the case we may find network weights for $V^{(n)}$ by solving an interpolation problem of the form

$$
\inf_{V} : \mathbb{E}\left[ d \left( -V(Z, M) + (TV^{(n-1)})(Z, M) \right) \right]
$$

for some distance $d$ over our discrete sample space. An example is $d(\cdot) = | \cdot |$. Instead of using neural networks for the last step we may also consider classic interpolation techniques such as kernel interpolators.

This scheme is reasonably intuitive as it iteratively improves the estimation of the monetary utility $V^{(n)}$ and the optimal action $a^{(n)}$. There is a question on how many training epochs to use when solving, in each step, for the action and the value function. In [SB18] there is a suggestion that using just one step is sufficient. The authors call this the actor-critic method. There are several other discussions on the viability of such methods, see also [MBM+16] and the references therein.

**Remark 3** In some applications we may not be able to use samples of $TV^{(n-1)}$ to solve (11), but make use of trained $a^{(n)}$ and $y^{(n)}$ directly. We therefore may solve

$$
\inf_{V} : \mathbb{E}\left[ \left( -V(Z; M) + \mathbb{E}[\beta(M) u\left( V^{(n-1)}(\cdots; M') + y^{(n)}(Z, M) \right) - y^{(n)}(Z, M) + R(\pi^{(n)}; Z, M, M') \big| Z, M) \right)^2 \right].
$$

The nested expectation is numerically sub-optimal. In order to address this, we solve instead the unconditional

$$
\inf_{V} : \mathbb{E}\left[ \left( -V(Z; M) + \beta(M) u\left( V^{(n-1)}(\cdots, M') + y^{(n)}(Z, M) \right) - y^{(n)}(Z, M) + R(\pi^{(n)}; Z, M, M') \right)^2 \right],
$$

which has the same gradient in $V$, and therefore the same optimal solution.\footnote{\textit{Proof} – Assume that $V(\theta) \equiv V(\theta; s)$ where $\theta$ are our network parameters. Denote by $\partial_{\theta}$ the derivative with respect to the $i$th parameter. Our equation then has the form $\inf_{\theta} f(\theta)$ where

$$f(\theta) := \mathbb{E}\left[ (V(\theta; S) + E[h(S')|S] + g(S))^2 \right]$$

The gradient is

$$\partial_{\theta} f'(\theta) = 2 \mathbb{E}\left[ \partial_{\theta} V(\theta; S)(V(\theta; S) + E[h(S')|S] + g(S)) \right] = 2 \mathbb{E}\left[ \partial_{\theta} V(\theta; S) V(\theta; S) + h(S') + g(S) \right]$$

Therefore $f$ has the same gradient as $\theta \mapsto \mathbb{E}\left[ (V(\theta; S) + h(S') + g(S))^2 \right]$. ∎}

### 3.1 Representing Portfolios

The most obvious challenge when applying the approach presented in section 3 is the need to represent our portfolio in some numerically efficient way. The following is an extension of the patent [BMW20] where we proposed using a more cumbersome signature representation of our trader instruments a’la [LNA19].

Assume that we are given historic market data $m_t$ at time points $\tau_0, \ldots, \tau_N$. Further assume that at each point $\tau_j$ we had in our book instruments $x^{j} = (x^{j,1}, \ldots, x^{j,m_t})$ with $x^{j,i} \in X$.

As $x$ were actual historic instruments, we have for each $x^{j,i}$ a vector $f_{j,i}^{x} \in \mathbb{R}^L$ of historic risk metrics computed in $t$, such as the book value, a range of greeks, scenarios and other calculations
made in $\tau_t$ to assist humans in their risk management decisions. We assume that those metrics $\mathbf{f}_t = (\mathbf{f}_t^1, \ldots, \mathbf{f}_t^{m_u})$ are also available for the same instruments at the next time step $\tau_{t+1}$, denoted by $\mathbf{f}_{t+1}$. Instrument which expire between $\tau_t$ and $\tau_{t+1}$ will have their book value and all greeks and scenario values set to zero.

It is a reasonable assumption that those metrics $\mathbf{f}$ have decent predictive power for the behaviour of our instruments; after all this is what human traders use to drive their risk management decisions. Hence we will use them as instrument features. We will here only consider linear features such that for any weight vector $\mathbf{w} \in \mathbb{R}^m$ the feature vector (the greeks, scenarios etc) of the weighted instrument $\mathbf{w} \cdot \mathbf{x}^t$ is correctly given as $\mathbf{w} \cdot \mathbf{f}_t$, so there is no need to recompute it later.\footnote{We note that this linearity is satisfied for all common risk metric calculations except VaR and counterparty credit calculations.}

We further denote by $r_{t}^i$ the historic aggregated cashflows of $x^{t,i}$ over the period $[\tau_t, \tau_{t+1})$, all in our accounting currency. We set $\mathbf{r}_t := (r_{t}^1, \ldots, r_{t}^{m_u})$. The aggregated cashflows of a weighted instrument $\mathbf{w} \cdot \mathbf{x}^t$ are $\mathbf{w} \cdot \mathbf{r}_t$. Similarly, we use $\mathbf{B}_u = (\mathbf{B}_{u}^1, \ldots, \mathbf{B}_{u}^{m_u})$ to refer to the book values of our instruments in $u \in \{t, t+1\}$, respectively.

We also assume that we have for all our hedging instruments access to their respective feature vectors $\mathbf{f}_{t,t}^{h,i}$ for both $\tau_t$ and $\tau_{t+1}$. It is important to recall that the greeks $\mathbf{f}_{t+1,t}^{h,i}$ refer to the features of the $i$th hedging instrument traded at $\tau_t$, but computed at $\tau_{t+1}$. That means in particular $\mathbf{f}_{\tau_{t+1}}^{h,i} \neq \mathbf{f}_{\tau_{t+1}+1}^{h,i}$ as the instrument mechanics changes between time steps. We also denote by $\mathbf{b}_{u,t}^h$ the book values of our hedging instruments for $u \in \{t, t+1\}$.

In addition to our instrument features, we also assume that we chose a reasonable subset of market features at each time step $\tau_t$. We continue to use the symbol $\mathbf{m}$ for those features even though in practise we will not use the entire available state vector.

We will now generate random scenarios as follows

1. Randomly choose $t \in \{0, \ldots, N-1\}$, which determines the market states $\mathbf{m} := \mathbf{m}_t$ and $\mathbf{m}^\prime := \mathbf{m}_{t+1}$.

2. Identify the hedging instruments $\mathbf{h}$ with their finite Markov representation

| Terminal FMR of hedging instruments | $\mathbf{h}^\prime$ | $\mathbf{h}^{t+1}$ |
|---|---|---|
| Book values for our hedging instruments | $\mathbf{B}(\mathbf{h}, \mathbf{m})$ | $\mathbf{b}_{t}^{h,t}$ |
| | $\mathbf{B}(\mathbf{h}, \mathbf{m}^\prime)$ | $\mathbf{b}_{t+1}^{h,t}$ |
| Cashflows of our hedging instruments | $\mathbf{r}(\mathbf{h}, \mathbf{m})$ | $\mathbf{r}_{t}^{h,t}$ |
| Cost | $c(\mathbf{a}; z, \mathbf{m})$ | $\mathbf{s}_{t}, \mathbf{f}_{t}^{h}$ |

The concrete implementation of the last line depends on the specifics of the cost function. For example, proportional transaction cost on net traded feature exposure are implemented using a weight vector $\mathbf{\gamma} \in \mathbb{R}^F$ and setting $c(\mathbf{a}; z, \mathbf{m}) := |\mathbf{a} \cdot (\mathbf{\gamma} \mathbf{f}_t^h)|$.\footnotemark
3. Choose a random weight vector $w \in \mathbb{R}^m$ and define a sample portfolio as $z := w \cdot x$ with

| Initial and terminal FMR of the portfolio | $z$ | $z'$ |
|-----------------------------------------|-----|-----|
|                                         | $:= w \cdot f^t$ | $:= w \cdot f'^{t+1}$ |

| Book value of our portfolio | $B(z, m)$ | $B(z, m')$ |
|-----------------------------|-----------|-----------|
|                             | $:= w \cdot b^t$ | $:= w \cdot b'^{t+1}$ |

| Cashflows of the portfolio | $r(z, m)$ |
|----------------------------|-----------|
|                            | $:= w \cdot x_t$ |

The construction of a reasonable randomization of the weight vector is important: if the samples are too different from likely portfolios, then the resulting model will underperform. However, if only historic portfolios are used, then the model is less able to learn handling deviations. More importantly, though, generating portfolios increases sample size.

This approach allows us training our actor-critic model with real data scenarios without the need of a market simulator. Indeed, the use of a market simulator is difficult for this particular set up as it would require computing book values and greeks for simulated data for a large number of made up derivative instruments. An open research topic is whether it is feasible to write a simulator for the data sets generated above, e.g. synthetically generating returns of market data jointly with the feature vectors of instruments.

4 Relation to Vanilla Deep Hedging

We will now discuss the relation of equation (3) to the solution of a corresponding vanilla Deep Hedging Problem.

We start by stating our original Deep Hedging problem [BGTW19] adapting the notation used here so far. We fix some initial time $t = 0$ with market state $m \equiv m_0 \equiv M_0$. Subsequent market states are denoted by $M_{t+1} := M'_t$. We use a similar notation for all other variables. We use $T(M_t) := t$ as the operator which extracts from a state $M_t$ current calendar time. We also define the stochastic discount factor to zero as $\beta_t := \beta(M_{t-1}) \beta_{t-1}$ starting with $\beta_0 := 1$.

We note that $\beta_1 = \beta(m)$ and $\beta_{T(M_t)} = \beta(M_{t-1}) \beta_{T(M_{t-1})}$.

For this part we will need to assume that every hedging instrument has a time-to-maturity less than $\tau^*$ in the sense that if we by $a \cdot h^t$ at time $t$, then all the book value and all cashflows from the portfolio beyond $t + \tau^*$ are zero. This assumption excludes perpetual assets such as shares or currencies. We will need to trade those with futures or forwards in the current setup.

Assume then that we are starting with an initial portfolio $z$ and follow a trading policy $\pi$. Accordingly, $Z_0 := z$ and $Z_{t+1} := Z'_t + A_t \cdot h^t$ where $A_t := \pi(Z_t, M_t)$. We also use $S_t := (Z_t, M_t)$ where convenient. Assume the portfolio has maturity $T^*$ beyond which all cashflows are zero.
The total gains from trading \( \pi \) starting in \( z \) are given as
\[
G^\pi(z) := \sum_{t=0}^{\infty} \beta_t R(A_t; Z_t, M_t, M_{t+1}) \tag{14}
\]
\[
= -B(z,s_0) + \sum_{t=0}^{T^*} \beta_t r(z,M_t)
\]
\[+ \sum_{t=0}^{\infty} \beta_t \left( -A_t \cdot B(h^t,M_t) - c(A_t,M_t) + A_t \cdot \sum_{u=t}^{t+\gamma} \beta_u r(h^u,M_u) \right) \]
Cost of trading \( A_t \cdot h^t \) in \( t \)
\[\text{P&L from } z \]
\[\text{All future rewards from trading } A_t \cdot h^t \text{ in } t \]
We now introduce the discounted cashflow rewards
\[
\hat{R}(a;z,m) := \beta_T(m) \hat{R}(a;z,m) \tag{15}
\]
such that
\[
G^\pi(z) = \sum_{t=0}^{\infty} \hat{R}(A_t; Z_t, M_t) \tag{16}
\]
We say that the market has only finite statistical arbitrage if we cannot make an infinite amount of money by trading from an empty portfolio in our hedging instruments. Formally,
\[
\infty > \gamma := \sup_{\pi} E[G^\pi(0)]
\]

**Proposition 1** If the market has only finite statistical arbitrage, then \( U[G^\pi(z)] \leq E[G^0(z)] + \gamma < \infty \) for all integrable \( z \in X \) and all trading policies \( \pi \in \mathcal{P} \). \[\square\]

That means the following definition makes sense:

**Definition 3** The value function of the Vanilla Deep Hedging problem for an infinite trading horizon expressed as in \[\text{[BGTW19]}\] in units of the underlying numeraire is given as the finite
\[
U^*(z,m) := \sup_{\pi} U \left[ \sum_{t=0}^{\infty} \hat{R}(\pi;Z_t,M_t) \mid Z_0 = z, M_0 = m \right] \tag{15}
\]

The actual cash value function is given as
\[
U^*(z,m) := \frac{U^*(z,m)}{\beta_T(m_0)} \tag{16}
\]
Simple arithmetic shows that the value function for Deep Hedging solves a Bellman equation:
\[\square\]

\[\text{Since } U \text{ is risk-averse we have } U[G^\pi(z)] = U[\sum_{t=0}^{\infty} \beta_t R(A_t,S_t)] \leq E[\sum_{t=0}^{\infty} \beta_t R(A_t,S_t)] = E[G^0(z)] + \gamma < \infty. \]

\[\text{We may write out} \]
\[
U^*(z,m) = \sup_{\pi} U \left[ \sum_{t=0}^{\infty} \hat{R}(\pi;Z_t,M_t) \mid Z_0 = z, M_0 = m \right] \]
\[
= \sup_{a \in A} U \left[ \left. \sum_{t=1}^{\infty} \hat{R}(\pi;Z_t,M_t) \mid Z_1 = M_1 \right) + \hat{R}(a,Z_0,M_0) \mid Z_0 = z, M_0 = m \right] \]
\[
= \sup_{a \in A} U \left[ U^*(Z_1,M_1) + \hat{R}(a,Z_0,M_0) \mid Z_0 = z, M_0 = m \right] \]
Theorem 2 (Vanilla Deep Hedging Bellman Equation) Assume that the market is strictly free of statistical arbitrage, and that $U$ is time-consistent (i.e. it is the entropy or the expectation). Then the value function $\hat{U}^*$ relative to the underlying numeraire satisfies the discounted dynamic programming equation

$$
\begin{aligned}
\hat{U}^*(z; m) & = (\hat{T}\hat{U}^*)(z, m) \\
(\hat{T}f)(z, m) & := \sup_{a \in A(z, m)} \left[ U[f(z' + a \cdot h'; M')] \bigg| m \right] + \hat{R}(a; z, m).
\end{aligned}
$$

for discounted rewards $\hat{R}$.

The cash value function $U^*(z, m) := \hat{U}^*(z, m)/\beta_T(m)$ satisfies

$$
\begin{aligned}
U^*(z; m) & = (\hat{T}U^*)(z, m) \\
(\hat{T}f)(z, m) & := \sup_{a \in A(z, m)} \left[ \frac{1}{\beta_T(m)} U[\beta_T(M')f(z' + a \cdot h'; M')] \bigg| m \right] + \hat{R}(a; z, m).
\end{aligned}
$$

(note the presence of the non-discounted rewards $\hat{R}$ instead of $\hat{R}$).

The first observation we make is that (18) only reduces to our original (3) if $U$ is coherent. In this case, we can move $\frac{1}{\beta_T(m)}$ inside $U$. However, we have already seen that the value function of the vanilla Deep Hedging problem only solves (18) if $U$ is the entropy or the expectation. Since the entropy is not coherent, this means

Corollary 1 The cash value function $U^*$ is only a solution to the Deep Bellman Hedging problem (3) if $U = E$.

4.1 Solutions to the Vanilla Deep Hedging Bellman Equation

As in theorem 1, we now reverse the situation and ask under which circumstances (18) has a finite solution: recall that $\beta_T(M_t) = \beta(M_{t-1})\beta_T(M_{t-1})$

Theorem 3 (Existence of finite unique Solutions for the Vanilla Deep Hedging Bellman Equation)
Assume that rewards $\hat{R}$ are finite and that $U$ is a monetary utility.

Then, the Vanilla Deep Hedging Bellman Equation (18) has a unique finite solution.

If $U$ is the entropy or the expectation, then this optimal solution coincides with the cash value function (16) of the Vanilla Deep Hedging problem.

The proof for the existence of a unique finite solution is presented in section 5.1.

In light of this result it is evident that both the Deep Bellman Hedging equation (3) and the Vanilla Deep Hedging Bellman equation (18) are reasonable candidates for solving the generic hedging problem. However, the latter requires us to essentially fix an initial point point $t = 0$, upon which the numeraire $\beta_t$ is based. In order to maintain consistence across time, that initial time point would need to be kept constant and therefore in the distant past. We therefore recommend using (3) as presented.
5 Existence of a Unique Finite Solution for Deep Bellman Hedging

We will now prove with theorem 1 convergence of our Deep Bellman hedging equations. This is easiest understood when the space \( Z \) of future cashflows is parameterized in \( \mathbb{R}^{\left| \mathcal{Z} \right|} \) with a finite Markov representation. However, in more generality we may assume that \( Z \) represents the set of suitably integrable adapted stochastic processes with values in \( \mathbb{R} \). Therefore, we may just assume that \((S, \mathcal{Q})\) with \( S = Z \times \mathcal{M} \) is a measure space. In the following we will consider the function space \( F \) of the \( \mathcal{Q} \)-equivalence classes of functions \( f : S \to \mathbb{R} \).

Let as before \( \beta(a) \) for \( a = f(s - g(s)) \leq \beta ||f - g|| \). This means that \( f(s) - g(s) \leq ||f - g|| \). Monotonicity and cash-invariance of the operator \( T \) yield

\[
(Tf)(s) \leq T(g + ||f - g||)(x) \leq (Tg)(s) + \beta ||f - g||
\]

Similarly,

\[
(Tg)(s) \leq T(f + ||f - g||)(x) \leq (Tf)(s) + \beta ||f - g||
\]

Jointly this gives

\[
||Tf - Tg|| \leq \beta ||f - g||
\]

Applying the Banach fixed-point theorem then yields the desired result. □

5.1 Vanilla Deep Hedging Bellman Equation

We now focus on (18) i.e. the operator

\[
(\tilde{T}f)(z, m) := \sup_{a \in A(z, m)} \frac{1}{\beta_T(m)} U \left[ \beta_T(M') f \left( z' + a \cdot h', M' \right) \right] = \tilde{R}(a; z, m).
\]

For illustration, we sketch a simple proof: Chose \( f_0 \) and let \( f_n := T f_{n-1} \) such that \( f_n = T^n f_0 \). We know that \( ||Tf_1 - T f_0|| \leq \beta ||f_1 - f_0|| \) and therefore iteratively \( ||T f_n - T f_{n-1}|| \leq \beta^n ||f_n - f_{n-1}|| \). Triangle inequality implies \( ||T f_n - T f_m|| \leq \sum_{i=m+1}^{n} ||T f_i - T f_{i-1}|| \leq \sum_{i=m+1}^{n} \beta^{n-i} \cdot 0 \). This means \( T f_n \) is a Cauchy sequence and therefore converges to a unique point \( f_n \to f \).

To show that \( f \) is a fixed point note that \( ||f - f|| \leq ||Tfg - f|| + ||f - f|| \leq \beta ||f - f_n|| + ||f_n - f|| \downarrow 0 \).
Proof of theorem\[3\] With our previous assumptions is clear that $\tilde{T}0 < \infty$. We now show that $\tilde{T}$ admits to a discounted form of cash invariance in the sense that

$$T(f(\cdot) + y(z, m))(z, m) = (Tf)(z, m) + \beta(m)y(z, m)$$

To this end, recall that $\beta_{T(M')} = \beta(m)\beta_T(m)$. Hence,

$$\begin{align*}
(T(f(\cdots) + y(z, m))(\cdot)(z, m)) &= \sup_{a \in A(z, m)} \frac{1}{\beta_T(m)} U[ \beta(m)\beta_{T(m)} (f(\cdots) + y(z, m)) \mid m ] + \bar{R}(a; z, m) \\
&= (Tf)(z, m) + \beta(m)y(z, m).
\end{align*}$$

The last equality follows from cash-invariance of the operator $U$.

Hence, just as in the preceding proof,

$$(\tilde{T}f)(s) \leq \tilde{T}(g + \|f - g\|)(x) \leq (\tilde{T}g)(s) + \beta^*\|f - g\|$$

which shows that $\tilde{T}$ is also a contraction. Applying the Banach fixed-point theorem yields the existence of a unique solution. \[\square\]

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