Spin 1/2 as propagation on a lattice with symmetries modulo gauge transformations

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Abstract

Relativistic spin 1/2, as represented by Susskind’s 1977 discretization of the Dirac equation on a spatial lattice, is shown to follow from basic, not typically relativistic but essentially quantum theoretic assumptions: that position eigenstates propagate to nearest neighbours while respecting lattice symmetries modulo gauge transformations.

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Appendix 12
1 Introduction

The mutual consistency of quantum mechanics and special relativity has remained a nontrivial issue, particularly with respect to locality [1] and quantum measurements [2], but also in describing a free particle. Quantum mechanics, on the one-particle level, makes a fundamental distinction between the roles of space and time which, it seems, can be overcome only in the framework of quantum field theory. But even in quantum field theory, in the usual line of argument, relativity is something that needs to be enforced. Moreover, if it is enforced by postulating unitary representations of the Poincaré group, there appear theoretical possibilities [3] which are never observed in Nature, such as continuous spin\(^1\) or tachyons.

In this paper I suggest an ahistorical route to relativistic quantum mechanics as represented by the Dirac equation. I derive the equation, and with it the Lorentz invariance, from seemingly “non”-relativistic quantum theory. In fact, Dirac himself came to the conclusion in the 1950s [4] that the Michelson-Morley experiment, in view of symmetries present in quantum but not classical mechanics, had been overinterpreted as a support of special relativity. As for a derivation of Lorentz invariance from a “mechanism”, there is a well-known precedent: Maxwell [5], in a balance of working hypothesis and actual belief [6], utilized mechanistic ideas of electromagnetic fields which did not enforce relativity but got it right automatically. More recently, in constructing cellular automata, Bialynicki-Birula [7] noted that an automaton simulating the Weyl equation would require only very general conditions: a two-component wave function, an evolution that is linear and unitary, and (a vague remnant of relativity) that a wave function constant in space be also constant in time.

By the technicalities used (not by the direction of argument) the present paper is based on a discretization of the Dirac equation devised by Susskind in 1977 [8]. The issue was to compensate for the doubling of the degrees of freedom encountered in replacing a derivative with an antihermitian difference

\[
\frac{\Delta f}{\Delta x} = \frac{f(x + a) - f(x - a)}{2a}
\]

A zero difference function, for example, is not only obtained from \(f = \text{const}\) but also from an alternating constant \((-1)^{x/a}\) on the lattice sites. Susskind

\(^1\)Continuous spin arises if, in terms of induced representations, the little group of a light-like four-momentum is represented non-trivially in all of its components.
showed that spinorial degrees of freedom can be consistently assigned to
different sites on a 3-dimensional lattice, thus thinning out the degeneracy
of energy levels on a given lattice by a factor of four. The discretized Dirac
equation resulting in this way is, in case of zero mass,

\[ i\dot{\psi}(x, y, z, t) = i(\psi(x + 1, y, z, t) - \psi(x - 1, y, z, t)) \]
\[ + i(\psi(x, y + 1, z, t) - \psi(x, y - 1, z, t)) (\psi(x, y, z + 1, t) - \psi(x, y, z - 1, t)) (-1)^x \]
\[ + (-1)^x \psi \]

(1)

where \( \psi(x, y, z, t) \) is a one-component wave function. Thus spin 1/2, usually
thought of as “internal” to a point particle, can be encoded in a spatial
arrangement of hopping amplitudes for a particle without internal structure.

In fact, equation (1) is the unique consequence of basic, not typically
relativistic assumptions on the propagation of quantum particles living on
the sites (as opposed to links or plaquettes) of a cubic spatial lattice:

- Locality: immediate propagation to nearest neighbours only
- Lattice symmetries are realised modulo gauge transformations

Also, time evolution will be assumed to be linear and unitary. There will
remain two kinematical options, one of which will be discarded because it
is infinitely slower than the other. Remarkably, the slow option is the one
that would realise lattice symmetries in a strict sense, without accompanying
gauge transformations.

In Section 2 the assumptions are specified; they include a general equation
for linear, unitary propagation as it was already proposed by this author
\[ \text{[9]} \]. In Section 3, the assumed invariances of the equation of propagation are
evaluated, and equation (1) is derived. As for introducing particle mass, it is
pointed out that an alternative to the standard term due to Susskind exists
which avoids species doubling on infinite lattices. In Section 4, I present
my Conclusions. In the Appendix, some omissions and simplifications antici-
 patched in Sections 2 and 3 are justified.

2 Specifying the assumptions

2.1 Nearest-neighbour hopping

We assume, as in \[ \text{[9]} \], that a quantum particle, initially in a position eigen-
state, will “move” by gradually (differentiably in \( t \)) forming superpositions
of nearest-neighbour eigenstates. Then a general state vector, given as a superposition of eigenstates with coefficients \( \psi(\vec{s}, t) \), will evolve according to

\[
i\dot{\psi}(\vec{s}, t) = \sum_{\vec{n}} \kappa(\vec{s}, \vec{n}) \psi(\vec{s} + \vec{n}, t)
\]

where the sum runs over nearest neighbours, and where \( \kappa(\vec{s}, \vec{n}) \) are complex hopping amplitudes whose properties are to be determined. The sum also includes an on-site hopping amplitude represented by \( \vec{n} = 0 \).

Since the distances between nearest neighbours are all the same, we assume that all hopping amplitudes are of the same magnitude. With a suitable rescaling of time we thus assume

\[
|\kappa(\vec{s}, \vec{n})| = 1 \quad \text{for all } \vec{s} \text{ and all } \vec{n} \neq 0
\]

Unitarity of time evolution and hermiticity of the Hamiltonian will be taken for granted. Using the standard scalar product of wave functions,

\[
\langle \psi \mid \varphi \rangle = \sum_{\vec{s}} \psi(\vec{s}) \varphi(\vec{s})
\]

the linear operator acting on the RHS of (2) is hermitian if and only if

\[
\kappa(\vec{s}, -\vec{n}) = \overline{\kappa(\vec{s} - \vec{n}, \vec{n})}
\]

\[2.2 \quad \text{Invariances modulo gauge transformations}\]

For a free particle, the equation of motion should be “the same” at all times and locations, as well as after a rotation. Quantum mechanically, the arbitrariness of the phases of position eigenstates allows to interpret “the same” as “gauge equivalent”.

In a local gauge transformation, the wave function at each space-time point is multiplied by a phase factor. Thus

\[
\psi(\vec{s}, t)_{\text{old}} = g(\vec{s}, t) \psi(\vec{s}, t)_{\text{new}} \quad |g(\vec{s}, t)| = 1
\]

In terms of the new wave function, equation (2) involves the hopping amplitudes

\[
\kappa(\vec{s}, \vec{n})_{\text{new}} = g(\vec{s} + \vec{n}, t) \kappa(\vec{s}, \vec{n})_{\text{old}} g(\vec{s}, t)^{-1} \quad \vec{n} \neq 0
\]

\[
\kappa(\vec{s}, 0)_{\text{new}} = \kappa(\vec{s}, 0)_{\text{old}} - ig(\vec{s}, t)g(\vec{s}, t)^{-1}
\]
If $S$ is a symmetry operation on the lattice (translation or rotation) the hopping amplitudes of equation (2) would in general change according to

$$\kappa(\vec{s}, \vec{n})_{\text{new}} = \kappa(S^{-1}\vec{s}, S^{-1}\vec{n})_{\text{old}}$$

Our assumption is that $\kappa_{\text{new}}$ is a local gauge transform of $\kappa_{\text{old}}$:

$$\kappa(\vec{s}, \vec{n})_{\text{new}} = g(\vec{s} + \vec{n}) \kappa(\vec{s}, \vec{n})_{\text{old}} g(\vec{s})^{-1}$$

Expressing this entirely in terms of $\kappa_{\text{old}}$, and dropping the index, we have

$$g(\vec{s} + \vec{n}) \kappa(\vec{s}, \vec{n}) g(\vec{s})^{-1} = \kappa(S^{-1}\vec{s}, S^{-1}\vec{n})$$ (7)

Since $g$ depends on $S$ we eventually write $g(\vec{s}, S)$.

### 2.3 Maximal gauge fixing

Working in a particular gauge will greatly facilitate the evaluation of symmetries up to gauge transformations. Following the procedure of maximal gauge fixing as devised in [10] for Hamiltonian lattice gauge theories, let us choose to have

$$\kappa(x, y, z, \hat{1}) = 1 \text{ for all } x, y, z \quad (8)$$

This is accomplished, using (3), by a gauge transformation with a suitable behaviour in the $\hat{1}$ direction:

$$g(x + 1, y, z, t) = g(x, y, z, t) \kappa(x, y, z, \hat{1})_{\text{old}}^{-1}$$

The values of $g$ on a plane with a constant $x$ coordinate are still free, and can be used to fix $\kappa(\vec{s}, \hat{2})$ on that plane. Let us choose to have

$$\kappa(0, y, z, \hat{2}) = 1 \text{ for all } y, z \quad (9)$$

which requires

$$g(0, y + 1, z, t) = g(0, y, z, t) \kappa(0, y, z, \hat{2})_{\text{old}}^{-1}$$

Finally, the values of $g$ along the line $x = y = 0$ can be chosen so that

$$\kappa(0, 0, z, \hat{3}) = 1 \quad (10)$$

Any further gauge transformation $g$ that is not constant throughout the lattice will destroy at least one of the conditions (8)-(10).
3  Propagation on a simple cubic lattice

3.1  Symmetries used

For a translation by a vector $\vec{a}$ we have

$$S^{-1}\vec{n} = \vec{n} \quad S^{-1}\vec{s} = \vec{s} - \vec{a} \quad (11)$$

For a rotation by $90^\circ$ about the $\hat{1}$ axis,

$$S^{-1}\hat{1} = \hat{1} \\
S^{-1}\hat{2} = -\hat{3} \\
S^{-1}\hat{3} = \hat{2} \quad S^{-1}(x,y,z) = (x,z,-y) \quad (12)$$

For a rotation by $90^\circ$ about the $\hat{3}$ axis,

$$S^{-1}\hat{1} = -\hat{2} \\
S^{-1}\hat{2} = \hat{1} \\
S^{-1}\hat{3} = \hat{3} \quad S^{-1}(x,y,z) = (y,-x,z) \quad (13)$$

3.2  Determining the hopping amplitudes

3.2.1  Evaluating translations

Let $S$ in equation (7) be a translation as specified in (11). Thus

$$g(\vec{s} + \vec{n},\vec{a}) \kappa(\vec{s},\vec{n}) g(\vec{s},\vec{a})^{-1} = \kappa(\vec{s} - \vec{a},\vec{n}) \quad \vec{n} \neq 0 \quad (14)$$

Putting $\vec{n} = \hat{1}$ in (14) and using (8) we see that $g(\vec{s},\vec{a})$ must be independent of the coordinate $x$,

$$g(x,y,z,\vec{a}) = g(y,z,\vec{a}) \quad \text{for all } \vec{a}$$

Now putting $\vec{n} = \hat{2}$ and $\vec{a} = \hat{1}$ in (14), we find

$$\kappa(x - 1,y,z,\hat{2}) = \kappa(x,y,z,\hat{2}) \left(g(y + 1,z,\hat{1}) g(y,z,\hat{1})^{-1}\right)$$

Solving the recursion in $x$ and using gauge condition (9), we have

$$\kappa(x,y,z,\hat{2}) = e^{ix\alpha(y,z)} \text{ where } e^{i\alpha(y,z)} = g(y,z,\hat{1}) g(y + 1,z,\hat{1})^{-1}$$
In fact, $\alpha$ must be independent of $y$ and $z$ since by re-inserting the last equation in (14) (with $\vec{n} = \hat{2}$) and considering $\vec{a} = \hat{2}, \hat{3}$ we encounter an $x$ dependence of $\exp(ix\alpha(y, z))$ on the LHS and $\exp(ix\alpha(y-1, z))$ or $\exp(ix\alpha(y, z-1))$, respectively, on the RHS. Hence, $\alpha(y - 1, z) = \alpha(y, z - 1) = \alpha(y, z)$ so that

$$\kappa(x, y, z, \hat{2}) = e^{i\alpha x}$$

(15)

Using (13), equation (14) with $\vec{n} = \hat{3}$ and $\vec{a} = \hat{1}, \hat{2}$ can now be read as a recursion relation determining the gauge transformations $g(\vec{s}, \hat{1})$ and $g(\vec{s}, \hat{2})$ up to their values at $x = y = 0$. We find

$$g(y, z, \hat{1}) = g(0, z, \hat{1}) e^{-i\alpha y}$$

(16)

$$g(y, z, \hat{2}) = g(0, z, \hat{2})$$

(17)

To obtain restrictions on the hopping amplitude in the $\hat{3}$ direction, we now insert (10) and (17) into (14), using $\vec{n} = \hat{3}$ and $\vec{a} = \hat{1}, \hat{2}$. Thus

$$g(0, z + 1, \hat{1}) \kappa(x, y, z, \hat{3}) g(0, z, \hat{1})^{-1} = \kappa(x - 1, y, z, \hat{3})$$

$$g(0, z + 1, \hat{2}) \kappa(x, y, z, \hat{3}) g(0, z, \hat{2})^{-1} = \kappa(x, y - 1, z, \hat{3})$$

Taking into account the gauge condition (10) the recursions are readily resolved, yielding

$$\kappa(x, y, z, \hat{3}) = e^{ix\beta(z)} e^{iy\gamma(z)}$$

(18)

where

$$e^{i\beta(z)} = g(0, z + 1, \hat{1})^{-1} g(0, z, \hat{1})$$

$$e^{i\gamma(z)} = g(0, z + 1, \hat{2})^{-1} g(0, z, \hat{2})$$

3.2.2 Evaluating unitarity

Applying (4) to (8), (15), (18) we obtain

$$\kappa(x, y, z, -\hat{1}) = 1$$

$$\kappa(x, y, z, -\hat{2}) = e^{-i\alpha x}$$

$$\kappa(x, y, z, -\hat{3}) = e^{-ix\beta(z-1)} e^{-iy\gamma(z-1)}$$

(19)
3.2.3 Evaluating 90° rotations about the x axis

Let \( S \) in equation (7) be the rotation specified by (12). Putting \( \vec{n} = \hat{1}, \hat{2}, \hat{3}, \)

\[
\begin{align*}
g(x + 1, y, z) \kappa(x, y, z, \hat{1}) \quad & g(x, y, z)^{-1} = \kappa(x, z, -y, \hat{1}) \\
g(x, y + 1, z) \kappa(x, y, z, \hat{2}) \quad & g(x, y, z)^{-1} = \kappa(x, z, -y, -\hat{3}) \\
g(x, y, z + 1) \kappa(x, y, z, \hat{3}) \quad & g(x, y, z)^{-1} = \kappa(x, z, -y, \hat{2})
\end{align*}
\]

whence, using (8),(13), (18) and (19),

\[
\begin{align*}
g(x + 1, y, z) \quad & g(x, y, z)^{-1} = 1 \\
g(x, y + 1, z) e^{i\alpha x} \quad & g(x, y, z)^{-1} = e^{-i \beta z} e^{-i \gamma y} \\
g(x, y, z + 1) e^{i \beta z} e^{i \gamma y} \quad & g(x, y, z)^{-1} = e^{i \alpha x}
\end{align*}
\]

By the first of these equations, \( g \) must not depend on \( x \). Thus, in the third and second equation, the only \( x \) dependence occurs in the exponentials, implying

\[
e^{i \beta z} = e^{i \alpha} = e^{-i \beta (-y - 1)} \quad \text{for all } y, z
\]

Thus \( \beta = \text{const} \), and there remain two possibilities,

\[
e^{i \alpha} = e^{i \beta} = \pm 1
\]

3.2.4 Evaluating 90° rotations about the z axis

Now let \( S \) in equation (7) be the rotation specified by (13). Putting \( \vec{n} = \hat{1}, \hat{2}, \)

\[
\begin{align*}
g(x + 1, y, z) \kappa(x, y, z, \hat{1}) \quad & g(x, y, z)^{-1} = \kappa(y, -x, z, -\hat{2}) \\
g(x, y + 1, z) \kappa(x, y, z, \hat{2}) \quad & g(x, y, z)^{-1} = \kappa(y, -x, z, \hat{1})
\end{align*}
\]

Using (8),(13), (19) we obtain

\[
\begin{align*}
g(x + 1, y, z) \quad & g(x, y, z)^{-1} = e^{-i \alpha y} \\
g(x, y + 1, z) \quad & g(x, y, z)^{-1} = e^{-i \alpha x}
\end{align*}
\]

The solution to these recursion relations is

\[
g(x, y, z) = e^{-i \alpha y} g(0, 0, z)
\]

The \( xy \) dependent factor drops out when inserted in (7) with \( \vec{n} = \hat{3}, \) leaving

\[
g(0, 0, z + 1) \kappa(x, y, z, \hat{3}) \quad g(0, 0, z)^{-1} = \kappa(y, -x, z, \hat{3})
\]
Using (18) we arrive at
\[ g(0, 0, z + 1) e^{i\alpha x} e^{iy\gamma(z)} g(0, 0, z)^{-1} = e^{i\alpha y} e^{-ix\gamma(z)} \]

Considering the \( x \) and \( y \) dependences we obtain
\[ e^{i\gamma} = e^{i\alpha} \] (22)

Thus, using (21), the hopping amplitudes are determined up to the choice of \( \alpha = 0 \),
\[ \kappa(x, y, z, \hat{n}) \equiv 1 \] (23)

or \( \alpha = \pi \),
\[ \kappa(x, y, z, \hat{1}) = \kappa(x, y, z, -\hat{1}) = 1 \\
\kappa(x, y, z, \hat{2}) = \kappa(x, y, z, -\hat{2}) = (-1)^x \\
\kappa(x, y, z, \hat{3}) = \kappa(x, y, z, -\hat{3}) = (-1)^{x+y} \] (24)

### 3.3 Staticity of the scalar solution

Option (23) for the hopping amplitudes would also result from postulating strict invariance under the lattice symmetries, as already studied in [9]. Its continuum limit was found to be the nonrelativistic Schrödinger equation of a scalar particle. However, the time scale on which the wave functions would evolve was found to be \( \lambda^2/\kappa a^2 \), where \( \kappa \) denotes the nearest-neighbour hopping amplitude, and \( \lambda \) is a length scale of the wave function. \( \lambda^2/\kappa a^2 \) is also the time scale of unitary cellular automata simulating scalar particles [11]. In contrast, the time scale of option (24) can be seen to be \( \lambda/\kappa a \) from the initial choice of scale in (3) and its modification in (27); again, \( \kappa \) denotes the nearest-neighbour hopping amplitude. If \( a \) is very small (like the Planck length), we obviously have
\[ (\lambda/a)^2 \gg \lambda/a \]

so option (23) tends to a static (non-kinetic and, in this sense, non-particle) scenario relative to (24).

\( ^2 \kappa \) is an inverse time by equation (2). The lattice spacing \( a \) emerges from Taylor expansions of next-neighbour terms; hence, its dimension is always cancelled by that of a spatial derivative. The derivatives act on wave functions in the continuum limit, so they are independent of \( a \).
3.4 Recovering the massless Dirac equation

This section reviews standard procedure with lattice fermions. Using (24) the hopping equation (2) reads

\[ i \dot{\psi}(x, y, z, t) = (\psi(x + 1, y, z, t) + \psi(x - 1, y, z, t)) \]
\[ + (\psi(x, y + 1, z, t) + \psi(x, y - 1, z, t)) (-1)^x \]
\[ + (\psi(x, y, z + 1, t) + \psi(x, y, z - 1, t)) (-1)^y \]

Equation (1) is recovered by the gauge transformation \( \psi_{\text{old}} = i^{x+y+z} \psi_{\text{new}} \).

For the new \( \psi \),

\[ i \dot{\psi}(x, y, z, t) = i (\psi(x + 1, y, z, t) - \psi(x - 1, y, z, t)) \]
\[ + i (\psi(x, y + 1, z, t) - \psi(x, y - 1, z, t)) (-1)^x \]
\[ + i (\psi(x, y, z + 1, t) - \psi(x, y, z - 1, t)) (-1)^y \]

Since the alternating sign factors are strongly fluctuating when viewed on a length scale much larger than the lattice spacing \( a \), there can be no smooth solution to equation (1). However, the equation is solved by a superposition of wave functions of the form

\[ \psi_{00}(x, y, z, t) \]
\[ \psi_{01}(x, y, z, t)(-1)^x \]
\[ \psi_{10}(x, y, z, t)(-1)^y \]
\[ \psi_{11}(x, y, z, t)(-1)^{y+x} \]

(25)

where \( \psi_{AB}(x, y, z, t) \) is assumed to be smooth in the sense that it varies from a lattice site to the next in \( O(a) \) at most. As suggested by the double index, the space of solutions is a tensor product. Multiplication by \((-1)^x\), for example, interchanges the presence/absence of that factor in the wave function, hence it is represented by the Pauli matrix \( \sigma_1 \) acting on the first index, and by \( \sigma_1 \otimes 1 \) acting on both indices. Similarly, differentiating along the \( x \) direction gives an extra minus sign depending on whether the factor \((-1)^x\) is present or absent; this corresponds to the action of \( \sigma_3 \otimes 1 \). Thus the right-hand side of (1) combines matrix factors and spatial differences into

\[ \sigma_3 \otimes 1 \ i \Delta_x + \sigma_1 \otimes \sigma_3 \ i \Delta_y + \sigma_1 \otimes \sigma_1 \ i \Delta_z \]

(26)

The tensor products are readily seen to satisfy the algebraic relations of the Dirac \( \alpha \) matrices. The difference operations asymptotically tend to \( 2a \partial / \partial x \), \( 2a \partial / \partial y \), \( 2a \partial / \partial z \) in the continuum limit \( a \rightarrow 0 \). We may absorb the factor of \( 2a \) in a redefinition of the time parameter, thus recovering the massless Dirac equation

\[ i \frac{\partial \psi}{\partial t'} = i \alpha_k \frac{\partial \psi}{\partial x_k} \quad t' = 2at \]

(27)
### 3.5 Mass terms

In spatial continuum, massless Dirac particles have chiral symmetry. This symmetry gets broken if a mass term is introduced. Therefore it is not unsatisfactory to find that mass terms on a lattice may require the breaking of a lattice symmetry. Susskind’s mass term is an on-site hopping amplitude \( \mu(-1)^{x+y+z} \) so that equation (1) becomes

\[
i \psi(x, y, z, t) = \left( i \Delta_x + i(-1)^x \Delta_y + i(-1)^{x+y} \Delta_z + \mu(-1)^{x+y+z} \right) \psi(x, y, z, t)
\]

The last term breaks the invariance (modulo gauge transformations) under translations by one lattice unit, while invariance under translations by two units is preserved. Susskind’s mass term requires a doubling of the dimension of the space of solutions, since the functions (25) need to be complemented by analogous functions with an extra factor of \((-1)^z\). This expands the tensor products (26) of the Hamiltonian to

\[
\sigma_3 \otimes 1 \otimes 1 \ i \Delta_x + \sigma_1 \otimes \sigma_3 \otimes 1 \ i \Delta_y + \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \ i \Delta_z + \mu \sigma_1 \otimes \sigma_1 \otimes \sigma_1
\]

The enlarged space of states is also recovered by acting on (25) with a symmetry of equation (1), Susskind’s version of the parity operation

\[
\psi(x, y, z, t) \rightarrow (-1)^{x+y+z} \psi(-x, -y, -z, t)
\]

In Monte Carlo simulations, which can only use lattices with a finite number of sites, there is no natural distinction between smooth and strongly fluctuating wave functions. Thus the degeneracy related to the above parity operation can only be suppressed at the expense of some arbitrariness.

On infinite lattices, however, an extra factor of \((-1)^z\) to the functions (25) does make a difference. It may therefore be of interest to note that Susskind’s term is not the only possibility of introducing mass. For example, we may allow for a variation of the magnitude of the hopping amplitude in the \(x\) direction so as to violate (3) while keeping (4),

\[
\kappa(x, y, z, \hat{1}) = i + i \mu (-1)^x \quad \kappa(x, y, z, -\hat{1}) = -i + i \mu (-1)^x
\]

Physically this would correspond to an alternating variation of the lattice spacing. The additional term in hopping equation (4) is

\[
i \mu (-1)^x \left( \psi(x+1, y, z, t) + \psi(x-1, y, z, t) \right)
\]
Repeating the arguments that lead to (26) we obtain the operator

\[ \sigma_3 \otimes 1 + i \Delta_x + \sigma_1 \otimes \sigma_3 i \Delta_y + \sigma_1 \otimes \sigma_1 i \Delta_z + 2i \mu (\sigma_1 \sigma_3) \otimes 1 \]

Since \( i \sigma_1 \sigma_3 = \sigma_2 \) and since \( \sigma_2 \otimes 1 \) anticommutes with the first three tensor products we recover the Dirac matrices in the form

\[ \sigma_3 \otimes 1 = \alpha_1 \quad \sigma_1 \otimes \sigma_3 = \alpha_2 \quad \sigma_1 \otimes \sigma_1 = \alpha_3 \quad \sigma_2 \otimes 1 = \beta \]

4 Conclusions

We have derived Susskind’s discretization of the Dirac equation from assumptions which apparently do not anticipate special relativity. While time was assumed to run continuously, spatial coordinates were confined to a lattice (reminiscent of a stack of particle detectors). Locality, too, was imposed in an unrelativistic sense, assuming that propagation from some position will, within a short interval of time, reach the nearest neighbours only. The intrinsically quantum-mechanical assumption was that the amplitudes of propagation will respect the symmetries of the lattice to the extent they have to in quantum theory, namely up to phase shifts of position eigenstates.

On the basis of these assumptions, a non-relativistic and a relativistic option appeared at the same stage in section 3.3. As it happened, this was simultaneously the alternative between strict symmetry and symmetry modulo gauge transformations. It was a matter of kinematical speed, rather than principle, that the non-relativistic option was discarded.

What insight do we gain by this route to the Dirac equation? I think it explains the preferred role of spin 1/2 in the Standard Model, since no internal structure of a particle (of a kind living on lattice sites) was assumed, and yet the Dirac equation resulted. In particular, in the case of zero mass where continuous spin is a possibility consistent with Poincaré invariance, it was just the massless version of the Dirac equation which emerged. More generally, I think the unity rather than mere consistency of special relativity and quantum theory—even in a “non”-relativistic formulation of the latter—is emerging here. Finally, gauge transformations turn out to be as fundamental to the propagation of free particles as to particle interactions.

It would be interesting to determine the “internal” degrees of freedom on lattices with other than simple cubic structure, especially with some of the infinitely many close-pack structures [12].
References

[1] For a strictly quantum-theoretical perspective, cf. H. P. Stapp, *From Einstein Nonlocality to Von Neumann Reality*, quant-ph/0003064.

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Appendix

Time-dependence ruled out

To simplify the notation, hopping amplitudes were so far considered as functions of the spatial coordinates only. Here we show that in the gauge we were using there is, in fact, no other possibility consistent with time-translation invariance modulo gauge transformations.
Let \( g(\vec{s}, t) \) be the gauge transformation accomplishing the shift \( S \) of the hopping amplitudes by a time \( \delta t \). Equation (7) then reads

\[
g(\vec{s} + \vec{n}, t) \kappa(\vec{s}, \vec{n}, t) g(\vec{s}, t)^{-1} = \kappa(\vec{s}, \vec{n}, t - \delta t)
\] (28)

Putting \( \vec{n} = \hat{1} \) and using our gauge condition (8), which holds at all times, we obtain \( g(x + 1, y, z, t) = g(x, y, z, t) \) and hence

\[
g(x, y, z, t) = g(y, z, t)
\]

Now putting \( \vec{n} = \hat{2} \) and \( x = 0 \) in (28) and using gauge condition (9) we moreover obtain

\[
g(y, z, t) = g(z, t)
\]

Finally, putting \( \vec{n} = \hat{3} \) and \( x = y = 0 \) and using gauge condition (10) we see that the gauge factor can only be a function of the time parameter:

\[
g(z, t) = g(t)
\]

But then \( g \) has no effect at all in equation (28), and the nearest-neighbour hopping amplitudes must be strictly invariant under a time shift:

\[
\kappa(\vec{s}, \vec{n}, t) = \kappa(\vec{s}, \vec{n}, t - \delta t)
\]

**On-site hopping gauged away**

On-site hopping amplitudes consistent with the spatial symmetries (modulo gauge transformations) can always be gauged away, as we now show. Thus it was justified to omit them in the previous sections (excluding section 3.5 where translational symmetry was partially broken).

The analogue of equation (7), using gauge transformation (6), would be

\[
\kappa(\vec{s}, 0, t) - i \dot{g}(\vec{s}, t)g(\vec{s}, t)^{-1} = \kappa(S^{-1}\vec{s}, 0, t)
\] (29)

Let us reconsider the gauge factors which accomplished spatial translations of the nearest-neighbour amplitudes (section 3.2.1), taking into account the final expressions of the amplitudes as given by (24). Equation (14) with \( \vec{n} \) put equal to \( \hat{1}, \hat{2}, \hat{3} \) can then be read as a recursion relation determining the \( x, y, z \) dependence of \( g(x, y, z, \vec{a}) \) for a given shift vector \( \vec{a} \). The arguments leading to those expressions did anticipate that \( \kappa(x, y, z, \hat{n}) \) would be independent of time, but this was justified in the previous section.
Those recursion relations implicit in (14) determine \( g(x, y, z, \vec{a}) \), at each instant of time, up to a global phase factor. That factor could depend on \( t \), allowing for an expression of the form
\[
g(x, y, z, t, \vec{a}) = g_0(x, y, z, \vec{a}) \cdot h(t, \vec{a})
\]
Inserting this in (29) the time-independent \( g_0 \) drops out, leaving
\[
\kappa(\vec{s}, 0, t) - i\dot{h}(t, \vec{a})h(t, \vec{a})^{-1} = \kappa(\vec{s} - \vec{a}, 0, t)
\]
In particular, putting \( \vec{a} = \hat{1}, \hat{2}, \hat{3} \) and defining
\[
\vec{c} = (c_1, c_2, c_3) \quad \quad \quad c_n(t) = i\dot{h}(t, \vec{n})h(t, \vec{n})^{-1}
\]
we see that on-site hopping amplitudes can at most take the form
\[
\kappa(\vec{s}, 0, t) = \kappa(\vec{0}, 0, t) + \vec{s} \cdot \vec{c}(t)
\]
But \( \vec{c}(t) \) must vanish due to rotational symmetry (modulo gauge transformations) by an argument similar to the above for translations. Reconsidering section 3.2.3 we find that equations (20) in conjunction with (24) determine the gauge factor up to a time-dependent global factor,
\[
g(x, y, z, t, R_x) = g_0(x, y, z, R_x) \cdot h(t, R_x)
\]
In equation (29), again, the time-independent \( g_0 \) drops out, leaving
\[
\kappa(\vec{s}, 0, t) - i\dot{h}(t, R_x)h(t, R_x)^{-1} = \kappa(R_x^{-1} \vec{s}, 0, t)
\]
Taken at the origin \( \vec{s} = 0 \) the equation implies that \( \dot{h} \) must vanish. Hence, \( \kappa(\vec{s}, 0, t) \) must be strictly invariant under the 90° rotation about the \( x \) axis, which implies that \( \vec{c}(t) \) can at most have an \( x \) component. This latter possibility can finally be ruled out by reconsidering the 90° rotation about the \( z \) axis as in section 3.2.4.

The remaining term of (30) can be removed by a gauge transformation only dependent on time, satisfying \( \dot{g}(t) = -i\kappa(\vec{0}, 0, t)g(t) \).