WANDERING LAKES OF WADA

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Abstract. We construct a transcendental entire function for which infinitely many Fatou components share the same boundary. This solves the long-standing open problem whether Lakes of Wada continua can arise in complex dynamics, and answers the analogue of a question of Fatou from 1920 concerning Fatou components of rational functions. Our theorem also provides the first example of an entire function having a simply connected Fatou component whose closure has a disconnected complement, answering a recent question of Boč Thaler. Using the same techniques, we give new counterexamples to a conjecture of Eremenko concerning curves in the escaping set of an entire function.

1. Introduction

Let \( f \) be either a rational self-map of the Riemann sphere \( \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \) or a transcendental entire function \( f : \mathbb{C} \to \mathbb{C} \). The Fatou set \( F(f) \) is the set of those values whose orbits under \( f \) remain stable under small perturbations. More formally, \( F(f) \) is the largest open set on which the iterates of \( f \) are equicontinuous with respect to spherical distances; equivalently, it is the largest forward-invariant open set that omits at least three points of the Riemann sphere [Ber93, Section 2.1]. The complement \( J(f) := \hat{\mathbb{C}} \setminus F(f) \) is called the Julia set.

These two sets are named after Pierre Fatou and Gaston Julia, who independently laid the foundations of one-dimensional complex dynamics in the early 20th century. In his seminal memoir on the iteration of rational functions, Fatou posed the following question concerning the structure of the connected components of \( F(f) \), which are today called the Fatou components of \( f \).

Question 1.1 ([Fat20, p. 51–52]). If \( f \) has more than two Fatou components, can two of these components share the same boundary?\(^1\)

Fatou asked this question in the context of rational self-maps of the Riemann sphere, but it makes equal sense for transcendental entire functions, whose dynamical study Fatou initiated in 1926 [Fat26]. We give a positive answer to Question 1.1 in this setting.

Theorem 1.2 (Fatou components with a common boundary). There exists a transcendental entire function \( f \) and an infinite collection of Fatou components of \( f \) that all share the same boundary. In particular, the common boundary of these Fatou components is a Lakes of Wada continuum.

\(^1\)“S’il y a plus de deux régions contiguës distinctes (et par suite une infinité), deux régions contiguës peuvent-elles avoir la même frontière, sans être identiques?”
Here a Lakes of Wada continuum is a compact and connected subset of the plane that is the common boundary of three or more disjoint domains (see Figure [1]). That such continua exist was first shown by Brouwer [Bro10, p. 427]; the name “Lakes of Wada” arises from a well-known construction that Yoneyama [Yon17, p. 60] attributed to his advisor, Takeo Wada (1882–1944); compare [HY61, p. 143] or [HO95, Section 8]. In this construction, one begins with the closure of a bounded finitely connected domain, which we may think of as an island in the sea. We think of the complementary domains, which we assume to be Jordan domains, as bodies of water: the sea (the unbounded domain) and the lakes (the bounded domains). One then constructs successive canals, which ensure that the maximal distance from any point on the remaining piece of land to any body of water tends to zero, while keeping the island connected; see Figure 2. The land remaining at the end of this process is the boundary of each of its complementary domains; in particular, if there are at least three bodies of water, it is a Lakes of Wada continuum. One may even obtain infinitely many complementary domains, all sharing the same boundary, by introducing new lakes throughout the construction.

Lakes of Wada continua may appear pathological, but they occur naturally in the study of dynamical systems in two real dimensions; see [KY91]. For example, Figure 1 is (a projection of) the Plykin attractor [Ply74] (see also [Cou06]), which is the attractor of a perturbation of an Anosov diffeomorphism of the torus. Hubbard and Oberste-Vorth [HO95, Theorem 8.5] showed that, under certain circumstances, the basins of attraction of Hénon maps in \( \mathbb{R}^2 \) form Lakes of Wada.

Theorem 1.2 provides the first example where such continua arise in one-dimensional complex dynamics, answering a long-standing open question. The second author learned of this problem for the first time in an introductory complex dynamics course by Bergweiler in Kiel during 1998–99; compare [Ber13, p. 27].

A Fatou component \( U \) is called periodic if \( f^p(U) \subseteq U \) for some \( p \in \mathbb{N} \), and \( U \) is preperiodic if there exists \( q \in \mathbb{N} \) such that \( f^q(U) \) is contained in a periodic Fatou component; otherwise \( U \) is called a wandering domain of \( f \). Rational maps have no wandering domains by a famous theorem of Sullivan [Sul85]. If \( f \) is a polynomial, a bounded periodic
Fatou component $U$ is either an immediate basin of a finite attracting or parabolic periodic point, or a Siegel disc, on which the dynamics is conjugate to an irrational rotation. In the former case, it is known that $U$ is a Jordan domain \cite{RY08}. Dudko and Lyubich have announced a proof that the boundary of a Siegel disc of a quadratic polynomial is also a Jordan curve. This would rule out Lakes of Wada boundaries, and provide a negative answer to Question 1.1 for maps in this family. Both questions remain open for polynomials of degree at least three, but it seems reasonable to expect that, for polynomials and rational maps, the answer to Question 1.1 is negative. We note that the question whether the entire Julia set can be a Lakes of Wada continuum is related to the Makienko conjecture concerning completely invariant domains and buried points of rational functions; compare \cite{SY03} and \cite{CMMR09}.

In contrast, the Fatou components in Theorem 1.2 are wandering domains; in fact, on these components the iterates of $f$ tend to $\infty$ uniformly. Our construction is closely related to a result of Boc Thaler \cite{Boc21}. Let $U \subseteq \mathbb{C}$ be a bounded domain such that $\mathbb{C} \setminus U$ is connected, and such that furthermore $U$ is regular in the sense that $U = \text{int}(\overline{U})$. Then Boc Thaler proves that there is an entire function $f$ for which $U$ is a wandering domain on which the iterates of $f$ tend to $\infty$ \cite[Theorem 1]{Boc21}. He then asks the following.

**Question 1.3** (\cite[p. 3]{Boc21}). Is it true that the closure of any bounded simply connected Fatou component, of an entire function, has a connected complement?

If $U$ is a bounded domain such that $\partial U$ is a Lakes of Wada continuum, then the complement of $\overline{U}$ has at least two connected components by definition. Hence the closure of the Fatou component from Theorem 1.2 has a disconnected complement, answering Question 1.3 in the negative. In fact, the proof of our theorem is similar to Boc Thaler’s, which uses approximation theory. Instead of beginning with a simply connected domain and approximating its closure, as in \cite{Boc21}, we start the construction with a full compact set $K \subseteq \mathbb{C}$; i.e., $K$ has connected complement. This small change of point of view results in the following, significantly stronger result, which implies Theorem 1.2.

**Theorem 1.4** (Entire functions with wandering compacta). Let $K \subseteq \mathbb{C}$ be a full compact set. Then there exists a transcendental entire function $f$ such that $f^n|_{K} \rightarrow \infty$ uniformly.
as \( n \to \infty \), \( \partial K \subseteq J(f) \), and \( f^n(K) \cap f^m(K) = \emptyset \) for \( n \neq m \). Every connected component of \( \text{int}(K) \) is a wandering domain of \( f \).

A continuum is a non-empty compact and connected metric space. A non-degenerate continuum \( K \subseteq J(f) \) with the property that \( f^n(K) \cap f^m(K) = \emptyset \) for \( n \neq m \) is called a wandering continuum of \( f \). Note that in Theorem 1.4, every connected component of \( \partial K \) is a wandering continuum of \( f \). As far as we are aware, in all cases of polynomials or rational maps where the topology of wandering continua is known, these are either arcs or simple closed curves. (See [CPT16] concerning wandering Jordan curves for certain post-critically finite rational maps; [Che17] implies the existence of wandering arcs for certain quadratic polynomials with irrationally indifferent fixed points. Cheraghi and Pedramfar have also announced the existence of wandering arcs for certain classes of infinitely renormalisable quadratic polynomials; compare [CP19].) In contrast, the existence of wandering continua with non-trivial topology for certain transcendental entire functions has long been known; compare e.g. [DJ02, Rem07, RRRS11, Wor17, Rem16, BR21]. Theorem 1.4 extends this list considerably, with a simpler proof than the cited results. In particular, the following is a curious application of Theorem 1.4. Write \( \text{fill}(X) \) for the fill of a compact set \( X \subseteq \mathbb{C} \); i.e., \( \text{fill}(X) \) is the union of \( X \) and its bounded complementary components.

Corollary 1.5 (Polynomial Julia sets appear in transcendental dynamics). Let \( P \) be a polynomial of degree at least two. Then there exists a transcendental entire function \( f \) such that:

(a) \( J(P) \subseteq J(f) \).

(b) Every bounded Fatou component of \( P \) is a simply connected wandering domain of \( f \).

(c) Every connected component of \( J(P) \) is a wandering continuum of \( f \).

(d) \( f^n\big|_{K(P)} \to \infty \) uniformly as \( n \to \infty \).

In view of Theorem 1.4 we may ask the following modification of Question 1.3.

Question 1.6. Suppose that \( U \) is a bounded simply connected Fatou component of a transcendental entire function, and let \( K = \text{fill}(U) \). Is it true that \( \partial U = \partial K \)?
**Question 1.7.** Let $f$ be a transcendental entire function and suppose that $U$ is an invariant Fatou component of $f$. Must every connected component of $\mathbb{C} \setminus U$ intersect the Julia set $J(f)$?

A positive answer to Question 1.7 would imply, in particular, that a transcendental entire function has at most one completely invariant Fatou component; compare see [RS19]. The above-mentioned partial results in the polynomial case motivate the following strengthening of Question 1.7 for bounded $U$.

**Question 1.8.** Let $f$ be an entire function and suppose that $U$ is a bounded invariant Fatou component of $f$. Must $\partial U$ be a simple closed curve?

For the entire function $f$ in Theorem 1.4, the set $K$ is in the escaping set

$$I(f) := \{ z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \}. $$

The escaping set has been the subject of considerable attention recently (compare [Obe99]). In 1989, Eremenko [Ere89] proved that for a transcendental entire function $f$, the connected components of $I(f)$ are all unbounded, and he asked whether the same holds for the connected components of $I(f)$ itself; this is known as the Eremenko conjecture, and is still an open question. (Rippon and Stallard [RS11] proved the weaker result that $I(f) \cup \{ \infty \}$ is always connected.) A stronger version of the conjecture was also stated in [Ere89]: Can every point of $I(f)$ be joined to $\infty$ by a curve of points in $I(f)$? This strong Eremenko conjecture was disproved by Rottenfußer, Rückert, Schleicher and the second author [RRRS11, Theorem 1.1]. On the other hand, they also show that the conjecture does hold for a large class of entire functions (we refer to [RRRS11, Theorem 1.2] for the precise statement). Using Theorem 1.4, we may give a new, and much simpler, disproof of the strong Eremenko conjecture.

**Theorem 1.9** (Counterexamples to the strong Eremenko conjecture). Let $X \subseteq \mathbb{C}$ be a full continuum. Then there exists a transcendental entire function $f$ such that every path-connected component of $X$ is a path-connected component of the escaping set $I(f)$. In particular, no point of $X$ can be connected to $\infty$ by a curve in $I(f)$.

We note that the counterexample $f$ in [RRRS11] has a number of additional properties:

(a) $f$ belongs to the Eremenko-Lyubich class $B$; that is, the closure $S(f)$ of the set of critical and asymptotic values of $f$ is bounded.

(b) The Fatou set $F(f)$ is connected and contains $S(f)$.

(c) The escaping set $I(f)$ contains no curves to $\infty$. In fact, $f$ can be constructed so that $I(f)$ contains no arc [RRRS11, Theorem 8.4]; compare also [BR21].

This places considerable restrictions on the structure of $I(f)$. For example, by (a), $I(f) \subseteq J(f)$, and hence must have empty interior by [EL92, Theorem 1]. Moreover, for any $f$ satisfying (a) and (b), any continuum $X \subseteq J(f)$ has span zero [Rem16, Theorem 1.4], which is a rather restrictive topological property. Theorem 1.9 thus yields a much richer class of examples; however, since $f$ is constructed using approximation theory, we have no control over its global dynamics, its set of singular values, or the global structure of its escaping set.
Notation. Throughout the paper, if \( K \subseteq \mathbb{C} \) is a compact set, we denote by \( \text{fill}(K) \) the union of \( K \) with its bounded complementary components. For a set \( X \subseteq \mathbb{C} \), \( \partial X \), \( \text{int}(X) \), and \( X^\circ \) denote, respectively, the boundary, the interior, and the closure of \( X \) in \( \mathbb{C} \). If \( z \in \mathbb{C} \) and \( X \subseteq \mathbb{C} \), we write \( \text{dist}(z,X) \) for the Euclidean distance of \( z \) to \( X \). The Euclidean disc of radius \( \delta > 0 \) around \( z \in \mathbb{C} \) is denoted \( D(z,\delta) \), and the unit disc is denoted \( D := D(0,1) \).

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2. Proof of Theorem 1.4

Recall that Theorem 1.4 states that, for every full compact set \( K \subseteq \mathbb{C} \), there exists a transcendental entire function \( f \) for which every connected component of \( \partial K \) is a wandering continuum, and every component of the interior of \( K \) is a simply connected wandering domain of \( f \).

The main tool to prove Theorem 1.4 is Runge’s theorem (see [Gai87, Theorem 2 on p. 76]). In contrast to Boc Thaler, who uses a modified version of the theorem in which the approximating function takes exact prescribed values at finitely many points, we use the classical form of Runge’s theorem.

Theorem 2.1 (Runge’s theorem). Let \( A \subseteq \mathbb{C} \) be a compact set such that \( \mathbb{C} \setminus A \) is connected. Suppose that \( \varepsilon > 0 \) and \( g: \Omega \to \mathbb{C} \) is an analytic function, where \( \Omega \) is an open neighbourhood of \( A \). Then there exists a polynomial \( f \) such that \( |f(z) - g(z)| \leq \varepsilon \) for \( z \in A \).

In the following, let \( K \subseteq \mathbb{C} \) be a full compact set, as in Theorem 1.4. Applying an affine transformation, we may assume without loss of generality that \( K \subseteq \mathbb{D} \). Choose sequences \((K_j)_{j=0}^\infty\) and \((L_j)_{j=1}^\infty\) of compact sets such that the following hold:

(a) \( K_0 \subseteq \mathbb{D} \).
(b) \( L_j \subseteq \text{int}(K_{j-1}) \) and \( K_j \subseteq \text{int}(L_j) \) for all \( j \geq 1 \).
(c) \( \bigcap_{j=0}^\infty K_j = \bigcap_{j=1}^\infty L_j = K \).
(d) Each \( L_j \) and each \( K_j \) is the fill of a finite union of pairwise disjoint simple closed curves.

The existence of such sequences is a consequence of standard plane topology; compare e.g. [Why42, p. 108]. If \( K \) is a connected set, then we may take the sets \( L_j \) and \( K_j \) to be closed Jordan domains, bounded by equipotential lines of the corresponding Green’s function; see Figure 3.

We may also choose a sequence \((p_j)_{j=1}^\infty\) with \( p_j \in K_{j-1} \setminus L_j \) for \( j \geq 1 \), in such a way that every point of \( \partial K \) is an accumulation point of the sequence \((p_j)_{j=1}^\infty\). For \( j \geq -1 \), we define the discs

\[
D_j := D(3j,1) = \{ z \in \mathbb{C} : |z - 3j| < 1 \}.
\]

Our main goal now is to prove the following proposition, which implies Theorem 1.4.
Proposition 2.2. There is a transcendental entire function $f$ with the following properties:

(a) $f(D_{-1}) \subseteq D_{-1}$.
(b) $f^j(p_j) \in D_{-1}$ for all $j \geq 1$.
(c) $f^j$ is injective on $K_j$ for all $j \geq 0$, with $f^j(K_j) \subseteq D_j$.

Proof of Theorem 1.4 using Proposition 2.2. Let $j \geq 0$; then $K \subseteq K_j$ and hence we have $f^j(K) \subseteq D_j$. In particular, $f^j|_K \to \infty$ uniformly as $j \to \infty$, and $\text{int}(K) \subseteq F(f)$ by Montel’s theorem.

On the other hand, we have $f^k(p_j) \in D_{-1}$ for $k \geq j$. Since every point $z \in \partial K \subseteq K$ is the limit of a subsequence of $(p_j)$, it follows that the family $(f^k)_{k=1}^\infty$ is not equicontinuous at $z$, and thus $z \in J(f)$. In particular, every connected component of $\text{int}(K)$ is a wandering Fatou component. This concludes the proof of Theorem 1.4.

To prove Proposition 2.2, we use two simple facts concerning approximation. The first is a version of Hurwitz’s theorem, while the second is an elementary exercise. For the reader’s convenience, we include the short proofs.

Lemma 2.3 (Approximation of univalent functions). Let $U, V \subseteq \mathbb{C}$ be simply connected domains, and let $\varphi: U \to V$ be a conformal isomorphism.

For every compact set $A \subseteq U$, there is $\varepsilon > 0$ with the following property. If $f: U \to \mathbb{C}$ is holomorphic with $|f(z) - \varphi(z)| \leq \varepsilon$ for all $z \in U$, then $f$ is injective on $A$, with $f(A) \subseteq V$.

Proof. Let $\gamma \subseteq U$ be a Jordan curve surrounding $A$. If

$$\varepsilon < \text{dist}(\varphi(\gamma), \varphi(A) \cup \partial V)/2,$$
and \( f \) is as in the statement of the lemma, then \( f(\gamma) \subseteq V \), and in particular \( f(A) \subseteq V \). Furthermore, \( f(A) \cap f(\gamma) = \emptyset \). By the argument principle, the number
\[
N(f, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{f(\zeta) - f(z)} d\zeta
\]
counts the number of solutions of \( f(\zeta) = f(z) \), where \( z \in A \), in the region enclosed by \( \gamma \). Since \( N(\varphi, z) = 1 \) and \( N(f, z) \) depends continuously on \( f \), it follows that \( N(f, z) = 1 \) for all \( f \) as above and all \( z \in A \).

\[\Box\]

**Lemma 2.4** (Approximation of iterates). Let \( U \subseteq \mathbb{C} \) be open, and let \( g: U \to \mathbb{C} \) be continuous. Suppose that \( K \subseteq U \) is compact and \( n \geq 1 \) is such that \( g^n(K) \) is defined and a subset of \( U \) for all \( k < n \).

Then for every \( \varepsilon > 0 \) there is \( \delta > 0 \) with the following property. If \( f: U \to \mathbb{C} \) is continuous with \( |f(z) - g(z)| < \delta \) for all \( z \in E \), then
\[
|f^k(z) - g^k(z)| < \varepsilon
\]
for all \( z \in K \) and all \( k \leq n \).

**Proof.** For every \( n \geq 1 \), we prove the existence of a function \( \delta_n \) such that \( \delta = \delta_n(\varepsilon) \) has the desired property. The proof proceeds by induction on \( n \); for \( n = 1 \) we may set \( \delta_1(\varepsilon) = \varepsilon \). Suppose that the induction hypothesis holds for \( n \), and let \( \varepsilon > 0 \). By the continuity of \( g \) and the compactness of \( K \), there is \( \delta' < \text{dist}(g^n(K), \partial U) \) such that \( |g(z) - g(\omega)| < \varepsilon/2 \) whenever \( \omega \in g^n(K) \) and \( |\zeta - \omega| < \delta' \). Define
\[
\delta_{n+1}(\varepsilon) = \min\{\delta_n(\varepsilon), \delta_n(\delta'), \varepsilon/2\}
\]
and let \( z \in K \). Then \( (i) \) holds for \( k \leq n \) by the induction hypothesis. Setting \( \zeta := f^n(z) \) and \( \omega := g^n(z) \), we have \( |\zeta - \omega| < \delta' \) by the induction hypothesis, and in particular \( \zeta \in U \). Thus
\[
|f^{n+1}(z) - g^{n+1}(z)| \leq |f(\zeta) - g(\zeta)| + |g(\zeta) - g(\omega)| < \varepsilon,
\]
as required.

\[\Box\]

**Proof of Proposition 2.2.** We construct \( f \) as the limit of a sequence of polynomials \( (f_j)_{j=0}^\infty \), which are defined inductively using Runge’s theorem. More precisely, for \( j \geq 1 \) the function \( f_j \) approximates a function \( g_j \), defined and holomorphic on a neighbourhood of a compact subset \( A_j \subseteq \mathbb{C} \), up to an error of at most \( \varepsilon_j \), where \( \varepsilon_j > 0 \). The function \( g_j \) in turn is defined in terms of the previous function \( f_{j-1} \).

Define
\[
\Delta_j := \overline{D(-3, 1 + 3j)} \supseteq D_{j-1}
\]
for \( j \geq 0 \). The inductive construction ensures the following properties:

(i) For every \( j \geq 0 \), \( f_j \) is injective on \( K_j \) and \( f_j^j(K_j) \subseteq D_j \).
(ii) For \( j \geq 1 \), \( \Delta_{j-1} \subseteq A_j \subseteq \Delta_j \).
(iii) \( \varepsilon_1 < 1/4 \) and \( \varepsilon_j \leq \varepsilon_{j-1}/2 \) for \( j \geq 2 \).
(iv) \( \bigcup_{j=0}^\infty A_j = \mathbb{C} \).

To anchor the induction, we set \( f_0(z) := (z - 3)/2 \) for \( z \in \mathbb{C} \). Then \( (i) \) holds trivially for \( j = 0 \).
Let $j \geq 1$ and suppose that $f_{j-1}$ has been defined, and that $\varepsilon_{j-1}$ has been defined if $j \geq 2$. Set $q_j := f_{j-1}^{-1}(p_j)$. By (i), $q_j \in D_{j-1}$ and

$$r_j := \min \left\{ \frac{\text{dist}(q_j, \partial D_{j-1})}{2}, \frac{\text{dist}(q_j, f_{j-1}^{-1}(L_j))}{2} \right\} > 0.$$ 

Define $B_j := D(q_j, r_j)$ and set

$$A_j := \Delta_{j-1} \cup B_j \cup f_{j-1}^{-1}(L_j).$$

We have $B_j \subseteq D_{j-1}$ by choice of $r_j$, and $f_{j-1}^{-1}(L_j) \subseteq f_{j-1}^{-1}(K_{j-1}) \subseteq D_{j-1}$ by the inductive hypothesis (i). In particular, they do not intersect $\Delta_{j-1}$, and they do not intersect each other by choice of $r_j$. It follows that $A_j$ satisfies (ii) and the hypotheses of Runge’s theorem. We define

$$g_j: A_j \rightarrow \mathbb{C}; \quad z \mapsto \begin{cases} f_{j-1}(z), & \text{if } z \in \Delta_{j-1}, \\ z - q_j - 3, & \text{if } z \in B_j, \\ z + 3, & \text{if } z \in f_{j-1}^{-1}(L_j). \end{cases}$$

(See Figure 4.) By definition, the function $g_j$ extends analytically on a neighbourhood of $A_j$. Observe that $g_j(B_j) \subseteq D_{j-1}$, and that $g_j^j$ is defined and univalent on int$(L_j)$ by (i).

Now choose $\varepsilon_j$ according to (iii) and sufficiently small that any entire function $f$ with $|f(z) - g_j(z)| \leq 2\varepsilon_j$ on $A_j$ satisfies:

1. $f_{j-1}(p_j) \in B_j$.
2. $f(B_j) \subseteq D_{j-1}$.
3. $f$ is injective on $K_j \subseteq \text{int}(L_j)$.
4. $f(K_j) \subseteq D_j$. 

Figure 4. The construction of $g_j$ in the proof of Proposition 2.2, for $j = 2$. 

...
Here \([3] \text{ and } [4]\) are possible by Lemmas 2.3 and 2.4. We now let \(f_j : \mathbb{C} \to \mathbb{C}\) be a polynomial approximating \(g_j\) up to an error of at most \(\varepsilon_j\), according to Theorem 2.1. This completes the inductive construction.

Condition \((iii)\) implies that \((f_j)_{j=k}^\infty\) forms a Cauchy sequence on every set \(A_k\), so by \((iv)\) the functions \(f_j\) converge to an entire function \(f\). For \(1 \leq k \leq j\),

\[
|f_j(z) - g_k(z)| \leq \varepsilon_k + \cdots + \varepsilon_j \leq 2\varepsilon_k, \quad \text{for all } z \in A_k.
\]

Hence the limit function \(f\) satisfies \(|f(z) - g_k(z)| \leq 2\varepsilon_k\) for all \(z \in A_k\) and \(k \geq 1\).

Since \(g_1(D_{-1}) = f_0(D_{-1}) = D(-3, 1/2)\), and \(2\varepsilon_1 < 1/2\), it follows that \(f(D_{-1}) \subseteq D_{-1}\). Moreover, \(f^j(p_j) \in D_{-1}\) for \(j \geq 1\) by \((1)\) and \((2)\). Finally, by \((3)\) and \((4)\) \(f^j\) is injective on \(K_j\) and \(f^j(K_j) \subseteq D_j\). This completes the proof of Proposition 2.2. ■

Proof of Theorem 1.2 Let \(X \subseteq \mathbb{C}\) be a continuum such that \(\mathbb{C} \setminus X\) has infinitely many connected components and the boundary of every such component coincides with \(X\). As mentioned in the introduction, such a continuum was first constructed by Brouwer [Bro10, p. 427], and can be obtained using the construction described by Yoneyama [Yon17, p. 60]. Let \(U\) be the unbounded connected component of \(\mathbb{C} \setminus X\).

The set \(K := \text{fill}(X) = \mathbb{C} \setminus U\) satisfies \(\partial K = \partial U = X\). Apply Theorem 1.4 to the full continuum \(K\) to obtain a transcendental entire function \(f\) for which \(X = \partial K \subseteq J(f)\), and every connected component of \(\text{int}(K) \subseteq F(f)\) is a simply connected wandering domain. Since there are infinitely many such components, each of which is bounded by \(X\), Theorem 1.2 is proved. ■

3. Proof of Theorem 1.9

We now construct our counterexamples to the strong Eremenko conjecture. To do so, we require an additional property of the function from Proposition 2.2 which can be achieved by applying the following version of Runge’s theorem in its proof.

**Proposition 3.1** (Modified Runge’s theorem). Suppose that \(A\), \(g\), and \(\varepsilon\) are as in Runge’s theorem, and suppose that \(w \in \mathbb{C}\) is such that \(g(z) \neq w\) for all \(z \in A\). Then there is an entire function \(f\) that omits \(w\) and satisfies \(|f(z) - g(z)| \leq \varepsilon\) for all \(z \in A\).

**Proof.** Post-composing with a translation, we may assume without loss of generality that \(w = 0\). Since \(\mathbb{C} \setminus A\) is connected and \(g\) omits zero, we may define a continuous branch \(G : A \to \mathbb{C}\) of \(\log g\). Set \(M := \max_{z \in A}|g(z)|\) and \(\tilde{\varepsilon} := \log(1 + \varepsilon/M)\). Apply Runge’s theorem to \(G\) with error bound \(\tilde{\varepsilon}\) to obtain a polynomial \(F\). The function \(f(z) := \exp(F(z))\) omits zero and satisfies

\[
|f(z) - g(z)| = |g(z)| \cdot \left| \frac{f(z)}{g(z)} - 1 \right| \leq M \cdot |\exp(F(z) - G(z))| - 1 | \leq M \cdot (\exp(\tilde{\varepsilon}) - 1) = \varepsilon
\]

for all \(z \in A\). ■

**Corollary 3.2.** The function \(f\) in Proposition 2.2 may be chosen to omit the value \(w = -2\).
Figure 5. A continuum \( X' \) having a singleton path-connected component. \( X' \) is obtained as a countable union of progressively smaller copies of the \( \sin(1/x) \) continuum, accumulating on a single point (the left-most point in the figure).

Proof. The function \( f_0 \) defined in the proof of Proposition 2.2 omits \(-2\) on \( \Delta_0 = \overline{D}_{-1} \). Moreover, for every \( j \geq 1 \), the function \( g_j \) omits \(-2\) on \( A_j \setminus \Delta_{j-1} \). It follows inductively that we may use Proposition 3.1 instead of Theorem 2.1 to obtain the function \( f_j \) in the inductive step of the construction. Then the functions \( f_j \) omit \(-2\), and by Hurwitz’s theorem the limit function \( f \) does also.

We also require the following well-known property of the Julia set of a transcendental entire function.

**Theorem 3.3** (Blowing-up property [Ber93, p. 156]). Let \( f \) be a transcendental entire function. Suppose that \( U \subseteq \mathbb{C} \) is an open set which meets \( J(f) \), and that \( K \subseteq \mathbb{C} \) is a compact set that does not contain any exceptional point of \( f \). Then there exists \( n_0 = n_0(K, U) \in \mathbb{N} \) such that \( f^n(U) \supseteq K \) for all \( n \geq n_0 \).

We are now ready to prove Theorem 1.9, which is the main result of this section.

**Proof of Theorem 1.9.** If \( X \) is a singleton, then it suffices to consider a continuum \( X' \) for which \( X \) is a path-connected component and continue the proof with \( X' \) instead of \( X \). (For example, we may take \( X' \) to be as shown in Figure 5.) From now on we therefore suppose that \( X \) is not a singleton.

We define \( K \) to be the union of \( X \) together with a ray that limits on \( X \), in such a way that no point of \( \partial X \) is accessible from \( \mathbb{C} \setminus K \) (see Figure 6). For example, let \( K = X \cup \{ \psi(\exp(-r + i/r)) : r \in (0, 1] \} \), where \( \psi : \mathbb{D} \to \hat{\mathbb{C}} \setminus X \) is a conformal isomorphism with \( \psi(0) = \infty \). Note that since \( X \) is full, \( K \) is also full. As in Section 2 suppose without loss of generality that \( K \subseteq \mathbb{D} \), and let \((K_j)_{j=0}^\infty\), \((L_j)_{j=1}^\infty\), and \((p_j)_{j=1}^\infty\) be as before. Moreover, let \( \varphi : \mathbb{D} \to \hat{\mathbb{C}} \setminus K \) be a conformal isomorphism with \( \varphi(0) = \infty \). We may clearly choose \( (p_j) \) in such a way that the set of accumulation points of \( (\varphi^{-1}(p_j)) \) coincides with the entire unit circle.

Apply Proposition 2.2 according to Corollary 3.2 the resulting function \( f \) may additionally be chosen to omit \(-2\). We claim that there is no curve in \( I(f) \) that connects any point of \( \partial X \) to \( \infty \). Otherwise, there is an arc \( \gamma : [0, 1] \to \hat{\mathbb{C}} \) with \( \gamma(0) \in \partial X, \gamma(1) = \infty \), and \( \gamma(t) \in \mathbb{C} \setminus X \) for \( 0 < t < 1 \). Consider the set \( A := \gamma^{-1}(K) \subseteq [0, 1] \). Since neither \( K \setminus X \) nor \( \mathbb{C} \setminus K \) contains an arc that tends to a single point of \( \partial X \), it follows that \( 0 \) is an accumulation point of both \( A \) and \( [0, 1] \setminus A \). In particular, \([0, 1] \setminus A \) has a connected component \( (a, b) \) with \( a, b \in A \). So \( C := \gamma([a, b]) \) is a cross-cut of \( \hat{\mathbb{C}} \setminus K \) that is contained...
entirely in $I(f)$. Let $V$ be the unique bounded component of $\mathbb{C} \setminus (K \cup C)$. Note that $\psi^{-1}(C)$ is a cross-cut of $\mathbb{D}$; hence $V$ contains one of the points $p_j$. Since $p_j$ is in an attracting basin, the boundary of the Fatou component containing $p_j$ cannot be entirely contained in $I(f)$ [RS11, Theorem 1.2]. Since $\partial V \subseteq I(f)$, it follows that $V \cap J(f) = \emptyset$.

For every $R > 0$, the blowing up property (see Theorem 3.3) implies that there exist $n_0 = n_0(R)$ such that $f^n(V) \supseteq \partial D(0, R)$ for all $n \geq n_0$. Now $-2$ is an omitted value of $f$, and belongs to the attracting basin of $-3$. It follows that this basin has an unbounded connected component, which hence must intersect $\partial f^n(V)$. But this is a contradiction because $\partial f^n(V) \subseteq f^n(\partial V) \subseteq I(f)$. Theorem 1.9 is proved. ■

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