Properties of the sums of resonances
found in the singularity analysis of certain classes
of ordinary differential equations

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Abstract. In the singularity analysis of ordinary differential equations it often happens that there
are various possible leading-order behaviours. It is observed that the sums of the resonances in
some instances are the same. We illustrate this with certain examples from differential sequences
and higher-order equations of Bureau symbol $P_2$. For the Riccati Differential Sequence we prove
two generic properties dealing with the occurrence of the coefficients of the leading-order term and
the variation of the sum of the resonances. An explanation of observed discrepancies in the cases
of fourth- and fifth-order equations of Bureau symbol $P_2$ is presented.

1. Introduction
There are three elements to the singularity analysis of ordinary differential equations. They are
the determination of the leading-order behaviour, the determination of the resonances and the
establishment of consistency at successive resonances and with the terms in the equation which are
not included in the leading-order elements. See for example the standard references of Tabor [1] and
Ramani et al [2]. By way of example consider the equation $y'' = y^2 + 1$. We substitute $y = \alpha(x-x_0)^p$
to determine the leading-order behaviour and obtain $\alpha p(p-1)(x-x_0)^{p-2} = \alpha^2(x-x_0)^{2p} + 1$ from
which it is evident that balance is obtained when $p = -2$ and $\alpha = 6$ with the first and the second
terms. To determine the resonances, i.e. the values of the exponents at which the second arbitrary
constant of integration appears, we substitute $y = 6(x-x_0)^{-2} + \mu(x-x_0)^{-2+s}$ into the dominant
terms and find that $\mu$ is arbitrary for $s = -1, 6$. The substitution of the series up to the resonance
at 6 into the full equation shows that there is consistency. The resonance, $s = -1$, is generic and
is associated with the arbitrary location of the pole at $x = x_0$.

When one makes a study of a number of equations with various possibilities for the coefficient
of the leading-order term, there are some occasions for which the sum of the resonances follows
some pattern or is even the same. Here we examine several systems of equations for which one or
other of these phenomena occurs. One example is found in the Riccati Differential Sequence [3]
and another set of examples comprises higher-order equations of Painlevé-type as discussed by
Cosgrove [4]. In Section 2 we investigate these properties for members of the Riccati Differential
Sequence and in Section 3 we examine the same properties for fourth- and fifth-order equations
of Bureau symbol $P_2$, i.e. the singularity is a second-order pole. For the former we infer general
properties based on the examples of earlier members of the sequence, formulate conjectures based on these inferences and then prove them for general $n$. In the latter examples we find that there are some inconsistencies and we explain how they arise.

2. The Riccati Differential Sequence

The Riccati Differential Sequence, $R_n = 0$, is defined through the Rodrigues Formula \[5\]

$$R_n = \exp\left[-\int y(x)dx\right] D^n \left\{ \exp\left[\int y(x)dx\right] \right\}, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (1)

This leads to a recursion relation. We write the Rodrigues Formula, (1), as

$$R_{n+1} = \exp\left[-\int y(x)dx\right] \left\{ D^{n+1} \left( \exp\left[\int y(x)dx\right] \right) \right\}$$
$$= \exp\left[-\int y(x)dx\right] D \left\{ D^n \left( \exp\left[\int y(x)dx\right] \right) \right\}$$
$$= \exp\left[-\int y(x)dx\right] D \left\{ R_n \left( \exp\left[\int y(x)dx\right] \right) \right\}$$
$$= \exp\left[-\int y(x)dx\right] \{ R'_n + yR_n \} \left( \exp\left[\int y(x)dx\right] \right)$$
$$= R'_n + yR_n,$$

i.e. the recursion relation is

$$R_{n+1} = R'_n + yR_n$$

and the recursion operator is $D + y$.

An alternate definition makes use of the recursion operator, $D + y$, with the same seed function. We list the first few members of the Sequence, omitting the zeroth and first elements as they are rather trivial,

$$R_2 : \quad y' + y^2 = 0,$$
$$R_3 : \quad y'' + 3yy' + y^3 = 0,$$
$$R_4 : \quad y^{(3)} + 4yy'' + 3y'^2 + 6y^2y' + y^4 = 0,$$
$$R_5 : \quad y^{(4)} + 10y'y'' + 10y^2y' + 5yy'^3 + 15yy'^2 + 10y^3y' + y^5 = 0,$$
$$R_6 : \quad y^{(5)} + 6yy^{(4)} + 15y^2y^{(3)} + 10y''^2 + 60yy'y'' + 20y^3y'' + 15y'^3 + 45y^2y'^2$$
$$+ 15y^4y' + y^6 = 0,$$

where in a standard notation $y^{(n)}$ denotes the $n$th derivative of $y(x)$ with respect to the independent variable, $x$. We recognise $R_2$ as the Riccati Equation \[6\] and $R_3$ as the Painlevé–Ince Equation \[7\] which has been the subject of many studies due to its myriad and interesting properties.

In Table 1 we list the properties relevant to the singularity analysis and the topic of this paper.

Evidently one could continue indefinitely, but the pattern of results has become already quite obvious. Firstly we note that the conventional Painlevé Expansion, that of a series ascending from the singularity occurs only for one possibility of the value of the coefficient of the leading-order term. The possibility of negative resonances was discussed by Fordy and Pickering \[8\] in terms of a perturbative expansion. Feix et al \[9\] recognised the possibility of ascending and descending series and termed the expressions “Left Painlevé Series” and “Right Painlevé Series”. Subsequently Andriopoulos et al \[10\] explained all in terms of the three possible Laurent expansions one could have in the complex plane and supported the explanation with concrete examples.
Table 1. Possible leading-order coefficients (with the exception of a zero), their corresponding resonances and the sum of the resonances. In all cases the singularity is given by \( p = -1 \).

|   | \( \alpha \) | \( s \) | \( \Sigma \) |
|---|---|---|---|
| 2 | 1  | -1 | -1 |
| 3 | 1  | -1, 1 | 0 |
|   | 2  | -1, -2 | -3 |
| 4 | 1  | -1, 1, 2 | 2 |
|   | 2  | -2, -1, 1 | -2 |
|   | 3  | -3, -2, -1 | -6 |
| 5 | 1  | -1, 1, 2, 3 | 5 |
|   | 2  | -2, -1, 1, 2 | 0 |
|   | 3  | -3, -2, -1, 1 | -5 |
|   | 4  | -4, -3, -2, -1 | -10 |
| 6 | 1  | -1, 1, 2, 3, 4 | 9 |
|   | 2  | -2, -1, 1, 2, 3 | 3 |
|   | 3  | -3, -2, -1, 1, 2 | -3 |
|   | 4  | -4, -3, -2, -1, 1 | -9 |
|   | 5  | -5, -4, -3, -2, -1 | -15 |
| 7 | 1  | -1, 1, 2, 3, 4, 5 | 14 |
|   | 2  | -2, -1, 1, 2, 3, 4 | 7 |
|   | 3  | -3, -2, -1, 1, 2, 3 | 0 |
|   | 4  | -4, -3, -2, -1, 1, 2 | -7 |
|   | 5  | -5, -4, -3, -2, -1, 1 | -14 |
|   | 6  | -6, -5, -4, -3, -2, -1 | -21 |

From the contents of Table 1 it is evident that for higher members of the Sequence the Left and Right Series are simply the extremes of a number of intermediate series which are indicative of a complete Laurent expansion over some annulus centred upon the singularity. For example in the case of \( R_7 \) there are four intermediate series and each one has its own annulus.

If one observes the Check Sum for each particular value of \( n \) for which this makes a sensible activity, it is obvious that the decrease is by a common amount for each of the possible coefficients for the leading-order term. Furthermore this constant decrease increases by one as the value of \( n \) increases. On the other hand the commencing value of \( \Sigma \), i.e. the value of the Check Sum for the Right Painlevé Series, increases by unity as \( n \) increases by unity. Similarly the magnitude of the final value of \( \Sigma \), which is the result for the Left Painlevé Series with the exception of \( n = 2 \), increases by 1 for increasing \( n \).

The observations of the properties listed in Table 1 lead to two conjectures.

**Conjecture 1:** The coefficients of the leading-order term of \( R_n \) for \( n > 0 \) take the nonzero values \( \alpha = 1, 2, \ldots, n - 1 \).

**Conjecture 2:** The sum of the resonances for each value of the leading-order coefficient, \( \alpha \), is given by \( \sum_{i=1}^{n} s_i = \frac{1}{2} (n(n + 1) - \alpha \beta) \).

**Proof of Conjecture 1.** In terms of the degree of the derivative, \( y' \), the ultimate and penultimate terms in \( R_n \) are \( y^n \) and \( \beta_n y^{n-2} y' \), where \( \beta_n \) is some numerical coefficient. From the recursion relation the corresponding terms in \( R_{n+1} \) are \( y^{n+1} \) and \( n y^{n-1} y' + \beta_n y^{n-1} y' \). Thus the penultimate
term is \((\beta_n + n)y^{n-1}y'\). When the substitution for the leading-order term, \(y \rightarrow \alpha\chi^p\), is made and \(p\) is identified as \(-1\), the two terms are \(\alpha^n\chi^{-n} - \alpha^{n-1}(\beta_n + n - 1)\chi^{-n}\) so that

\[ \alpha_{n+1} = -(\beta_n + n - 1), \]

i.e. the sum of the roots of the polynomial determining the values of \(\alpha\) increases by \(n - 1\).

From Table 1 we see that the increase for the first few members of the Sequence is indeed \(n - 1\) and that this increase expresses itself as the value of the additional root. Consequently for \(R_n\) the sum of the roots is given by

\[ \sum_{i=1}^{n} \alpha_i = \frac{1}{2}(n - 2)(n - 1). \]

**Proof of Conjecture 2.** We use the highest and second-highest terms with respect to the degree of the derivative. Without knowing the coefficient we write these two terms for \(R_n\) as

\[ y^{(n)} + \gamma yy^{(n-1)}, \]

where \(\gamma\) is a numerical coefficient. We make the substitution

\[ y(x) \rightarrow \alpha\chi^{-1} + m\chi^{s-1}, \]

where, as before, \(\alpha\) is the value of the leading-order coefficient. After some rearrangement the coefficient of \(m\) in the highest and second-highest terms in the order of the derivative is

\[ s^n - \left[ \frac{1}{2}n(n + 1) - \alpha\gamma \right] s^{n-1}. \]

Consequently the sum of the resonances is given by

\[ \sum_{i=1}^{n} s_i = \frac{1}{2} \left(n(n + 1) - \alpha\gamma\right), \]

which result is, in a sense, independent of the individual values of the resonances. \(\Box\)

### 3. Fourth- and fifth-order equations of Bureau Symbol P2

Cosgrove has made an extensive study of the singularity properties of higher-order equations. In particular he reported on equations with the so-called Painlevé Property in the Polynomial Class 1 with Bureau Symbol P2 [4]. We examine some of those equations in the spirit of the analysis above.

#### 3.1. Equations of the fourth order

The fourth-order equations which we consider are

- **T-I:** \(y^{(4)} - 12yy'' - 12y'^2 - (\alpha x + \beta)y' - 2\alpha y + \frac{1}{6}(\alpha x + \beta)^2 = 0\),
- **T-II:** \(y^{(4)} - 3yy'' + 4y'^2 = 0\),
- **T-III:** \(y^{(4)} - 15yy'' - \frac{45}{8} y'^2 + 15y^3 - \alpha y - \beta = 0\),
- **T-IV:** \(y^{(4)} - 30yy'' + 60y^3 - \alpha y - \beta = 0\),
- **T-V:** \(y^{(4)} - 20yy'' - 10y'^2 + 40y^3 - \alpha y - \kappa x - \beta = 0\) and
- **T-VI:** \(y^{(4)} - 18yy'' - 9y'^2 + 24y^3 - \alpha y^2 - \frac{1}{5}\alpha^2 y - \kappa x - \beta = 0\),

where the nomenclature is that of Cosgrove.

In all cases the dominant terms in the equation possess the two Lie point symmetries \(\Gamma_1 = \partial_x\) and \(\Gamma_2 = x\partial_x - 2y\partial_y\) with the algebra \(A_2\). The structure of the self-similar symmetry, \(\Gamma_2\), indicates that the leading-order behaviour is \(\chi^{-2}\). We can immediately proceed to the next step of the
Table 2. All of the six equations have leading-order exponent $-2$. We list the coefficient(s) of the leading-order terms, the resonances associated with each coefficient and the Check Sum. Note that we use $a$ as the coefficient of the leading-order terms as Cosgrove uses $\alpha$ as a parameter in some of the equations.

| Equation | $a$ | $s$ | $\Sigma$ |
|----------|-----|-----|-----|
| T-I      | 1   | 1, 4, 5, 6 | 14 |
| T-II     | 60  | 3, $-2$, $-1$, 20 | 14 |
| T-III    | 1   | 1, 3, 5, 7 | 14 |
|          | 8   | 7, $-1$, 10, 12 | 14 |
| T-IV     | 1   | 1, 2, 3, 10 | 14 |
|          | 2   | 2, $-1$, 5, 12 | 14 |
| T-V      | 1   | 1, 2, 5, 8 | 14 |
|          | 3   | 3, $-1$, 8, 10 | 14 |
| T-VI     | 1   | 1, 3, 4, 8 | 14 |
|          | 5   | 5, $-1$, 8, 12 | 14 |

singularity analysis by writing $y \rightarrow a(x-x_0)^{-2} + \mu(x-x_0)^{s-2}$. In Table 2 we provide the possible coefficients of the leading-order term, resonances and Check Sum.

The sum of the resonances is 14 in each case. Cosgrove excludes T-II as a possible candidate for being integrable in the sense of singularity and in the cases of T-III to T-VI allows only the first set of resonances as indicating integrability as he follows the criterion advanced by Fordy et al. In the spirit of Feix et al and as developed by Andriopoulos et al all cases are integrable either as a Right Painlevé Series over a punctured disc centred upon the singularity or a full Laurent Expansion in an annulus centred upon the singularity. In no case is there the possibility of a Left Painlevé Series. In fact Cosgrove describes T-II as being a fourth-order equivalent to the famous Chazy Barrier Equation [11]. In a comprehensive numerical investigation of the equations governing the Mixmaster Universe Model Bountis and Drossos [12] demonstrated the nature of the barrier in that model. To make the numerical integration one needed to fix the location of the singularity as part of the initial conditions. When one integrates outwards from near the singularity, a barrier of complex structure is certainly found. However, in the case of an equation such as T-II one would expect to have a problem with an integration begun near the singularity but not if one began within the annulus. The surmise is that as one approaches either boundary of the annulus that the numerical integration would show a similar complexity. The problem is where to locate the annulus in the complex plane even if the location of the singularity is fixed.
3.2. Equations of the fifth order
The corresponding equations of the fifth order as listed by Cosgrove are

\[
\begin{align*}
P-I : & \quad y^{(5)} - 15yy'''' - \frac{75}{2}yy'' + 45y^2y' - (\lambda x + \alpha)y' - 2\lambda y = 0, \\
P-II : & \quad y^{(5)} - 30yy''''' - 30y'y'' + 180y^2y' - (\lambda x + \alpha)y' - 2\lambda y = 0, \\
P-IIIa : & \quad y^{(5)} - 20yy''''' - 40y'y'' + 120y^2y' - (\lambda x + \alpha)y' - 2\lambda y - \kappa = 0, \\
P-IIIb : & \quad y^{(5)} - 20yy''''' - 40y'y'' + 120y^2y' - \alpha y' \\
& \quad -(y^{(4)} - 20yy'' - 10y^2 + 40y^3 - \alpha y - \beta)/x = 0, \\
P-IVa : & \quad y^{(5)} - 18yy''''' - 36y'y'' + 72y^2y' - 3\lambda y'' \\
& \quad -(y^{(4)} - 18yy' - 9y^2 + 24y^3 - 3\lambda y + \kappa)/x \\
& \quad -\frac{1}{2}\lambda x(5y'''' - 36yy') + \frac{1}{2}\lambda^2x(2xy' + y) = 0, \\
P-IVb : & \quad y^{(5)} - 18yy''''' - 36y'y'' + 72y^2y' - 2\alpha yy' - \frac{7}{5}\alpha^2y' - \kappa = 0 \quad \text{and} \quad \\
P-IVc : & \quad y^{(5)} - 18yy''''' - 36y'y'' + 72y^2y' - 2\alpha yy' - \frac{7}{5}\alpha^2y' \\
& \quad -(y^{(4)} - 18yy'' - 9y^2 + 24y^3 - \alpha y^2 - \frac{7}{5}\alpha^2y - \beta)/x = 0.
\end{align*}
\]

The similarity symmetry again indicates that the value of \( p \) is \(-2\).

**Table 3.** We list the coefficient(s) of the leading-order terms, the resonances associated with each
coefficient and their Check Sum.

| Equation | \( a \) | \( s \) | \( \Sigma \) |
|----------|-------|-------|-------|
| P-I      | 1, 1, 3, 5, 6, 7 | 8, 7, -1, 6, 10, 12 | 20, 20 |
| P-II     | 1, 1, 2, 3, 6, 10 | 2, 2, -1, 5, 6, 12 | 20, 20 |
| P-IIIa   | 1, 1, 2, 5, 6, 8 | 3, 3, -1, 6, 8, 10 | 20, 20 |
| P-IIIb   | 1, 1, 2, 5, 7, 8 | 3, 3, -1, 7, 8, 10 | 21, 21 |
| P-IVa    | 1, 1, 3, 4, 7, 8 | 5, 5, -1, 7, 8, 12 | 21, 21 |
| P-IVb    | 1, 1, 3, 4, 6, 8 | 5, 5, -1, 6, 8, 12 | 20, 20 |
| P-IVc    | 1, 1, 3, 4, 7, 8 | 5, 5, -1, 7, 8, 12 | 21, 21 |

We note that the unity of results evinced by the fourth-order equations is not matched by this
set of fifth-order equations. The Check Sum is either 20 or 21. Each equation has a Right Painlevé
Series and a full Laurent Expansion. For all the Check Sum is unaffected by the nature of the series.
The explanation of the two possible case is quite simple when they occur. In the equations for which
the Check Sum is 21, there is an additional term contributing to the leading-order behaviour. If we take \( P-IVc \) as an example, there is the term \(-y^{(4)}/x\) and this adds one to the sum of the resonances either as a Right Painlevé Series or as a full Laurent Expansion. Were the coefficient a number other than \(-1\), the number by which the Check Sum would change would be minus that number.

4. Discussion

The Riccati Differential Sequence represents an example of a set of differential equations of ever-increasing order generated by the Rodrigues Formula (1). An alternate mode of generation is to use the recursion operator \( R = D + y \) which is the real form of Dirac’s creation operator for the solution of the Schrödinger equation for the simple harmonic oscillator. All members of this sequence are integrable in terms of elementary functions [13]. Here we have been concerned with the singularity analysis of members of this sequence. Admittedly we have presented the analysis of only a very finite selection of equations taken from the lower members of the sequence. Nevertheless the properties are so regular that one may infer the properties of elements of the sequence in general. These properties have been summarised in Table 1 and the chief of these properties were proven in the proofs of Conjectures 1 and 2.

The second class of equations examined is taken from those equations of the fourth- and fifth-order which possess the Painlevé Property with the particular property that the Bureau Symbol is \( P2 \), i.e. the singularity identified from the leading-order terms is a second-order pole. Here we saw behaviour differing from that observed for the elements of the Riccati Differential Sequence in that across the equations of the same order belonging to a given class the Check Sum was the same no matter the mix of resonances. The exceptions in the class of fifth-order equations were easily explained in terms of subdominant terms being rendered dominant by a multiplier comprising a suitably singular function of \( x \). It is the order of the singularity in this function and not its location which is the critical feature [14].

Finally we emphasise that the singularity analysis does not deal only with a Laurent Expansion commencing at the power of the singularity and thereafter in increasing powers. The nature of the expansion depends upon the region of the complex plane where the expansion is valid. This does open the possibility of numerical experiments to determine “natural barriers” not only from expansion about the singular point but also advancing towards the singularity from afar (Left Painlevé Series) or going both inwards and outwards towards the bounding circles of an annulus (full Laurent Expansion).

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References

[1] Tabor M 1989 Chaos and Integrability in Nonlinear Dynamics. An Introduction (New York: John Wiley & Sons)
[2] Ramani A, Grammaticos B and Bountis T 1989 The Painlevé property and singularity analysis of integrable and nonintegrable systems Phys. Rep. 180:3 159
[3] Euler M, Euler N and Leach P G L 2007 The Riccati and Ermakov–Pinney hierarchies J. Nonlinear Math. Phys. 14:2 290
[4] Cosgrove C M 2000 Higher-order Painlevé equations in the polynomial class. I. Bureau Symbol P2 Stud. Appl. Math. 104:1 1
[5] Rodrigues O 1816 De l’attraction des sphéroïdes (Correspondence sur l’École Impériale Polytechnique vol 3) (Paris) p 361
[6] Riccati J 1764 Opere (Lucca: Jacopo Guisti) p 83
[7] Ince E L 1927 Ordinary Differential Equations (London: Longmans, Green and co. ltd)
[8] Fordy A and Pickering A 1991 Analysing negative resonances in the Painlevé test Phys. Lett. A 160:4 347
[9] Feix M R, Géronimi C, Cairó L, Leach P G L, Lemmer R L and Bouquet S É 1997 On the singularity analysis of ordinary differential equations invariant under time translation and rescaling. J. Phys. A 30:21 7437
[10] Andriopoulos K and Leach P G L 2006 An interpretation of the presence of both positive and negative nongeneric resonances in the singularity analysis Phys. Lett. A 359:3 199
[11] Chazy J 1911 Sur les équations différentielles du troisième ordre et d’ordre supérieur dont l’intégrale générale a ses points critiques fixes Acta Math. 34:1 317
[12] Bountis T C and Drossos L B 1997 Evidence of a natural boundary and nonintegrability of the mixmaster universe model Nonlinear Science 7:1 45
[13] Andriopoulos K, Leach P G L and Maharaj A 2011 On differential sequences Appl. Math. Inf. Sci. 5:3 525
[14] Andriopoulos K and Leach P G L 2011 Singularity analysis for autonomous and nonautonomous differential equations Appl. Anal. Discrete Math. 5:2 230