THE DISCRETE MODULE CATEGORY FOR THE RING OF $K$-THEORY OPERATIONS

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Abstract. We study the category of discrete modules over the ring of degree zero stable operations in $p$-local complex $K$-theory. We show that the $K_{(p)}$-homology of any space or spectrum is such a module, and that this category is isomorphic to a category defined by Bousfield and used in his work on the $K_{(p)}$-local stable homotopy category [2]. We also provide an alternative characterisation of discrete modules as locally finitely generated modules.

1. Introduction

An explicit description of the topological ring $A$ of degree zero stable operations in $p$-local complex $K$-theory was given in [5]. Here we consider the category of ‘discrete modules’ over this ring. We focus attention on discrete modules because the $K_{(p)}$-homology of any space or spectrum is such a module; see Proposition 2.6.

In Section 2 we recall some results from [5] about the ring $A$, define discrete $A$-modules, and show how the category $\mathcal{D} A$ of such modules is easily seen to be a cocomplete abelian category.

Our first main result comes in Section 3, where we provide an interesting alternative characterisation of discrete $A$-modules: an $A$-module is discrete if and only if it is locally finitely generated. We also show that the category $\mathcal{D} A$ is isomorphic to the category of comodules over the coalgebra $K_0(K)_{(p)}$ of which $A$ is the dual.

The category $\mathcal{D} A$ arose in disguised form in [2]. There Bousfield introduced a certain category $\mathcal{A}(p)$ as the first step in his investigation of the $K_{(p)}$-local stable homotopy category. In Section 4 we recall Bousfield’s (rather elaborate) definition, and we prove that his category is isomorphic to the category of discrete $A$-modules.

This allows us to simplify and clarify in Section 5 some constructions in Bousfield’s work. In particular, there is a right adjoint to the forgetful functor from $\mathcal{A}(p)$ to the category of $\mathbb{Z}_{(p)}$-modules. For Bousfield, this functor has to be constructed in an ad hoc fashion, separating cases. In our context, it is revealed as simply a continuous $\text{Hom}$.
functor. We give a construction in this language of a four-term exact sequence involving the right adjoint.

Bousfield’s aim was to give an algebraic description of the $K(p)$-local stable homotopy category. He succeeded at the level of objects, and in Section 5 we translate his main result into our language of discrete $\mathcal{A}$-modules.

Of course, $p$-local $K$-theory splits as a sum of copies of the Adams summand. We have chosen to write the main body of this paper in the non-split context, but very similar results hold in the split setting. We record these in an appendix.

We note that a full algebraic description of the $K(p)$-local stable homotopy category has been given by Franke [6]. The interested reader may wish to consult [12, 8]. In a different direction, we note the further work of Bousfield, building a unified version of $K$-theory in order to combine information from different primes [3].

Throughout this paper $p$ will denote an odd prime.

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2. DISCRETE $\mathcal{A}$-MODULES

Let $p$ be an odd prime and let $A = K^0_0(K(p))$ be the ring of degree zero stable operations in $p$-local complex $K$-theory. This ring can be described as follows; see [5]. Choose an integer $q$ that is primitive modulo $p^2$, and let $\Psi^q \in A$ be the corresponding Adams operation. Let

$$q_i = q^{(-1)^i[i/2]},$$

and define polynomials $\Theta_n(X)$, for each integer $n \geq 0$, by $\Theta_n(X) = \prod_{i=1}^{n}(X-q_i).$ Then let the operation $\Phi_n \in A$ be given by $\Phi_n = \Theta_n(\Psi^q)$. For example, $\Phi_4 = (\Psi^q-1)(\Psi^q-q)(\Psi^q-q^{-1})(\Psi^q-q^2).$ These operations have been chosen so that any infinite sum $\sum_{n \geq 0} a_n \Phi_n,$ with coefficients $a_n$ in the $p$-local integers $\mathbb{Z}_p$, converges. The following theorem says that any operation can be written uniquely in this form.

**Theorem 2.2.** [5, Theorem 6.2] The elements of $A$ can be expressed uniquely as infinite sums $\sum_{n \geq 0} a_n \Phi_n,$ where $a_n \in \mathbb{Z}_p$. □

For each $m \geq 0$, we define

$$A_m = \left\{ \sum_{n \geq m} a_n \Phi_n : a_n \in \mathbb{Z}_p \right\} \subseteq A,$$

so that $A_m$ is the ideal of operations which annihilate the coefficient groups $\pi_{2i}(K(p))$ for $-m/2 < i < (m+1)/2$, and thus it does not
depend on the choice of primitive element \( q \). We obtain a decreasing filtration

\[ A = A_0 \supset A_1 \supset \cdots \supset A_m \supset A_{m+1} \supset \cdots. \]

We use this filtration to give the ring \( A \) a topology in the standard way (see, for example, Chapter 9 of [11]): the open sets are unions of sets of the form \( y + A_m \), where \( y \in A \) and \( m \geq 0 \). We note that \( A \) is complete with respect to this topology. Indeed, another way to view the topological ring \( A \) is as the completion of the polynomial ring \( \mathbb{Z}(p)[\Psi^q] \) with respect to the filtration by the principal ideals \( \Phi_m \mathbb{Z}(p)[\Psi^q] = \mathbb{Z}(p)[\Psi^q] \cap A_m \). (By way of warning, this filtration is not multiplicative in the sense of [11], and, since \( A \) is not Noetherian [5, Theorem 6.10], it is not the completion of a polynomial ring with respect to any multiplicative filtration.)

**Definition 2.3.** A discrete \( A \)-module \( M \) is an \( A \)-module such that the action map

\[ A \times M \to M \]

is continuous with respect to the discrete topology on \( M \) and the resulting product topology on \( A \times M \).

In practice we use the following criterion to recognise discrete modules.

**Lemma 2.4.** An \( A \)-module \( M \) is discrete if and only if for each \( x \in M \), there is some \( n \) such that \( A_n x = 0 \).

*Proof.* Fixing \( x \in M \), the map sending \( \alpha \in A \) to \( (\alpha, x) \in A \times M \) is continuous. Thus if \( M \) is discrete, the map \( \alpha \mapsto \alpha x \in M \) is continuous, which implies that its kernel contains \( A_n \) for some \( n \).

Suppose now that for each \( x \in M \), there exists \( n \) such that \( A_n x = 0 \). If \( x, y \in M \) are such that \( \alpha x = y \), and \( A_n x = 0 \), then \( (\alpha + A_n) x = y \) so that \( (\alpha + A_n) \times \{ x \} \) is an open neighbourhood of \( (\alpha, x) \) in the preimage under the action map of \( \{ y \} \). This shows that \( M \) is a discrete module. \( \square \)

**Remark 2.5.** For \( n \geq 1 \), the principal ideal \( \Phi_n A \) is strictly contained in \( A_n \); see Proposition 3.3. Thus, if an \( A \)-module \( M \) has the property that for each \( x \in M \) there is some \( n \geq 0 \) with \( \Phi_n x = 0 \), then it does not follow that \( M \) is a discrete \( A \)-module.

For example, the \( A \)-module \( A/\Phi_1 A \) is not discrete, although every element is annihilated by \( \Phi_1 \). For suppose the element \( 1 + \Phi_1 A \in A/\Phi_1 A \) were annihilated by the ideal \( A_n \), i.e., \( A_n \subseteq \Phi_1 A \). Then \( A/\Phi_1 A \) would be finitely generated over \( \mathbb{Z}(p) \), being a quotient of \( A/A_n \) which has rank \( n \) over \( \mathbb{Z}(p) \). However, \( A/\Phi_1 A \) contains the submodule \( A_1/\Phi_1 A \) which is isomorphic to \( \mathbb{Z}_p/\mathbb{Z}(p) \), where \( \mathbb{Z}_p \) denotes the \( p \)-adic integers. The proof of this is analogous to that of the corresponding result for connective \( K \)-theory which is given in [6, Corollary 3.7].
We will see in Section 3 that if \( Ax \) is finitely generated over \( \mathbb{Z}_p \), then \( \Phi_n x = 0 \) does imply \( A_n x = 0 \).

The motivating example for the concept of a discrete \( A \)-module is the \( p \)-local \( K \)-homology of a spectrum.

**Proposition 2.6.** The degree zero \( p \)-local \( K \)-homology \( K_0(X; \mathbb{Z}_p) \) of a spectrum \( X \) is a discrete \( A \)-module with the action map given by

\[
A \otimes K_0(X; \mathbb{Z}_p) \to A \otimes K_0(K) \otimes K_0(X; \mathbb{Z}_p) \to K_0(X; \mathbb{Z}_p),
\]

in which \( K_0(K) \) denotes the ring of degree zero cooperations in \( p \)-local \( K \)-theory, the first map arises from the coaction map, and the second comes from the Kronecker pairing.

**Proof.** Consider the Kronecker pairing \( \langle - , - \rangle : A \otimes K_0(K) \to \mathbb{Z}_p \). We claim that for each element \( f \in K_0(K) \), there is some \( m \) such that \( \langle A_m, f \rangle = 0 \). Indeed this is clear, since \( A = \text{Hom}_{\mathbb{Z}_p}(K_0(K), \mathbb{Z}_p) \), with the \( \Phi_n \) being the dual basis to a particular \( \mathbb{Z}_p \)-basis of \( K_0(K) \); see [5, Theorem 6.2].

Now let \( X \) be a spectrum, and let \( x \in K_0(X; \mathbb{Z}_p) \). Under the coaction \( K_0(X; \mathbb{Z}_p) \to K_0(K) \otimes K_0(X; \mathbb{Z}_p) \), the image of \( x \) is a finite sum \( \sum_{i=1}^{k} f_i \times x_i \) for some elements \( f_i \in K_0(K) \) and \( x_i \in K_0(X; \mathbb{Z}_p) \). By the preceding paragraph, for each \( f_i \), there is some \( m_i \) such that \( \langle A_{m_i}, f_i \rangle = 0 \). Hence, if \( m = \max \{ m_1, \ldots, m_k \} \), we have \( \langle A_{m}, f_i \rangle = 0 \) for all \( i \). Thus \( A_m x = \sum_{i=1}^{k} \langle A_{m_i}, f_i \rangle x_i = 0 \). \( \square \)

If an \( A \)-module is a discrete \( A \)-module, then the \( A \)-action is determined by the action of \( \Phi_1 \), or, equivalently, by the action of \( \Psi_1 = 1 + \Phi_1 \).

**Lemma 2.7.** If \( M \) and \( N \) are discrete \( A \)-modules, and \( f : M \to N \) is a \( \mathbb{Z}_p \)-homomorphism which commutes with the action of \( \Psi^q \), then \( f \) is a homomorphism of \( A \)-modules.

**Proof.** Since \( \Phi_i = \Theta_i(\Psi^q) \), the map \( f \) must also commute with \( \Phi_i \) and hence with all finite linear combinations \( \sum_{i=0}^{k} a_i \Phi_i \). Now let \( \alpha = \sum_{i \geq 0} a_i \Phi_i \in A \) and \( x \in M \). Since \( M \) and \( N \) are discrete \( A \)-modules, for some \( n \), we have \( A_n x = 0 \) and \( A_n f(x) = 0 \). Then

\[
f \left( \sum_{i \geq 0} a_i \Phi_i x \right) = f \left( \sum_{i=0}^{n-1} a_i \Phi_i x \right) = \sum_{i=0}^{n-1} a_i \Phi_i f(x) = \sum_{i \geq 0} a_i \Phi_i f(x).
\]

Hence \( f \) commutes with all elements of \( A \), i.e., it is an \( A \)-homomorphism. \( \square \)

We end this section by making some remarks about the category of discrete \( A \)-modules.

**Definition 2.8.** We define the category of discrete \( A \)-modules \( \mathcal{DA} \) to be the full subcategory of the category of \( A \)-modules whose objects are discrete \( A \)-modules.
Lemma 2.9. The category $\mathcal{D}A$ is closed under submodules and quotients.

Proof. It is clear that a submodule of a discrete $A$-module is discrete. Let $M$ be a discrete $A$-module and suppose that $N$ is an $A$-submodule of $M$. Consider the quotient $A$-module $M/N$. For $x \in M$, there is some $n$ such that $A_n x = 0$. Then $A_n(x + N) = 0$, and so $M/N$ is a discrete $A$-module. □

Lemma 2.10. The category $\mathcal{D}A$ has arbitrary direct sums.

Proof. Let $M_i$ be a discrete $A$-module for each $i$ in some indexing set $\mathcal{I}$. Each element of $\bigoplus_{i \in \mathcal{I}} M_i$ is a finite sum $x_1 + \cdots + x_k$, where each $x_j$ belongs to some $M_i$. For each $j$, there is some $n_j$ such that $A_{n_j} x_j = 0$. Hence $A_n(x_1 + \cdots + x_k) = 0$ if $n \geq \max\{n_1, \ldots, n_k\}$. So the direct sum $\bigoplus_{i \in \mathcal{I}} M_i$ is discrete. □

Corollary 2.11. $\mathcal{D}A$ is a cocomplete abelian category.

Proof. That $\mathcal{D}A$ is abelian follows directly from Lemmas 2.9 and 2.10 and the fact that $A$-modules form an abelian category. An abelian category with arbitrary direct sums is cocomplete; see, for example, [13, Proposition 2.6.8]. □

When combined with Theorem 4.6 below, the following result corresponds to 10.5 of [2].

Proposition 2.12. $\mathcal{D}A$ is isomorphic to the category of $K_0(K)_{(p)}$-comodules.

Proof. Let $G_n(w) = q^{[n/2]}F_n(w) \in K_0(K)_{(p)}$, where $F_n(w)$ is as defined in the proof of Theorem 6.2 in [5]. That proof shows that the $\Phi_n$ are dual to the $G_n(w)$, i.e., $\langle \Phi_n, G_i(w) \rangle = \delta_{ni}$.

If $M$ is a discrete $A$-module, define $\varphi_M : M \to K_0(K)_{(p)} \otimes M$ by $\varphi_M(x) = \sum_{n \geq 0} G_n(w) \otimes \Phi_n x$, in which only finitely many terms are non-zero because $M$ is discrete. It is easily checked that this makes $M$ into a $K_0(K)_{(p)}$-comodule and that an $A$-module homomorphism between discrete $A$-modules is also a $K_0(K)_{(p)}$-comodule homomorphism.

If $M$ is a $K_0(K)_{(p)}$-comodule with coaction $\varphi_M : M \to K_0(K)_{(p)} \otimes M$, then, just as in the proof of Proposition 2.6, $M$ is a discrete $A$-module with action given by

$$A \otimes M \xrightarrow{1 \otimes \varphi_M} A \otimes K_0(K)_{(p)} \otimes M \xrightarrow{(-,-) \otimes 1} M.$$ 

It is clear that this construction is functorial.

It is routine to check that the two constructions are mutually inverse. □
3. Locally Finitely Generated $A$-modules

**Definition 3.1.** An $A$-module $M$ is *locally finitely generated* if, for every $x \in M$, the submodule $Ax$ is finitely generated over $\mathbb{Z}(p)$.

This section is devoted to proving the following theorem.

**Theorem 3.2.** An $A$-module $M$ is discrete if and only if it is locally finitely generated.

We need to prove first a number of preliminary results.

**Proposition 3.3.** For all $n \geq 1$, the quotient $A_n/\Phi_n A$ is a rational vector space.

**Proof.** The result will follow if we can show that $A_n/\Phi_n A$ is divisible and torsion-free. We thus need to show:

1. For any $\alpha \in A_n$, there exists $\beta \in A$ such that $\Phi_n \beta - \alpha \in p A_n$;
2. if $\alpha \in A_n$ and $p \alpha \in \Phi_n A$, then $\alpha \in \Phi_n A$.

Recall from Proposition 6.8 of [5] that

$$\Phi_n \Phi_j = \sum_{k = \max(j,n)}^{j+n} \frac{c_{j,n}^k \Phi_k}{p}$$

for certain coefficients $c_{j,n}^k \in \mathbb{Z}(p)$ (denoted $A_{j,n}^k$ in [5]).

Suppose $\alpha = \sum_{k \geq n} a_k \Phi_k \in A_n$, then $\beta = \sum_{j \geq 0} b_j \Phi_j \in A$ will satisfy $\Phi_n \beta - \alpha \in p A_n$ if and only if the congruences

$$\sum_{j = k-n}^{k} c_{j,n}^k b_j \equiv a_k \mod p \quad (k \geq n)$$

can be solved for $(b_j)_{j \geq 0}$. We will verify (1) by showing that these congruences can always be solved, and (2) by showing that the solution is unique modulo $p$.

It follows from Propositions A.2 and A.4 of [5] that, with $q_k$ as defined in (2.1),

$$c_{j,n}^k = (q_{k+1} - q_n) c_{j,n-1}^k + c_{j,n-1}^{k-1} - 1$$

where $c_{j,n}^k = 0$ unless $j, n \leq k \leq j + n$, and $c_{j,n}^k = 1$ if $k = j + n$. We claim that for any integer $s \geq 1$,

$$c_{j,n}^k \equiv 0 \mod p \quad \text{for} \quad (2p - 2)s \leq j \leq (2p - 2)s + n.$$  

This follows by induction on $n$ from (3.5) and the periodicity of the sequence $(q_i)_{i \geq 1}$ modulo $p$: $q_i \equiv q_{i+2p-2} \mod p$.

It follows from (3.6) that for $k = (2p - 2)s + n - 1$ the congruence (3.4) is $b_{(2p - 2)s} \equiv a_{(2p - 2)s + n - 1} \mod p$, and that for $k < (2p - 2)s + n - 1$ the congruences have the form

$$b_{k-n} + \text{terms involving } b_j \text{ for } k - n < j < (2p - 2)s \equiv a_k \mod p.$$
It is now clear that the congruences (3.4) have a unique solution modulo \( p \) for the \( b_j \).

**Corollary 3.7.** If \( M \) is a locally finitely generated \( A \)-module and \( x \in M \) satisfies \( \Phi_n x = 0 \) for some \( n \geq 0 \), then \( A_n x = 0 \).

*Proof.* By hypothesis the map \( A_n \to Ax \) sending \( \alpha \) to \( \alpha x \) factors through the \( \mathbb{Q} \)-module \( A_n / \Phi_n A \). Hence, since \( Ax \) is a finitely generated \( \mathbb{Z}(p) \)-module, the map must be zero. \( \square \)

We now need to show that, for an element \( x \) in a locally finitely generated \( A \)-module \( M \), we have \( \Phi_n x = 0 \) for some \( n \geq 0 \). The next results establish this, first for \( \mathbb{Z}(p) \)-torsion modules, then for \( \mathbb{Z}(p) \)-free modules and finally in the general case.

**Proposition 3.8.** If \( M \) is a finite \( A \)-module, then \( \Phi_n M = 0 \) for some \( n \).

*Proof.* Let \( x \in M \) be non-zero, and define
\[
I = \{ f(X) \in \mathbb{Z}(p)[X] : f(\Psi^q)x = 0 \},
\]
which is clearly an ideal of \( \mathbb{Z}(p)[X] \).

Let \( f(X) \) be any element of \( I \) such that its reduction \( \bar{f}(X) \in \mathbb{F}_p[X] \) is not zero. Since the elements \( \Psi^r x \) for \( r \geq 0 \) cannot be distinct, we may, for example, take an \( f(X) \) of the form \( X^{r_1} - X^{r_2} \) with \( r_1 \neq r_2 \).

Suppose that
\[
\bar{f}(X) = \bar{g}(X) \prod_{k=1}^{p-1} (X - k)^{e_k} \quad (e_k \geq 0),
\]
where \( \bar{g}(X) \) has no roots in \( \mathbb{F}_p^\times \), and thus we can write
\[
f(X) = g(X) \prod_{k=1}^{p-1} (X - k)^{e_k} + ph(X),
\]
for some \( g(X), h(X) \in \mathbb{Z}(p)[X] \), where \( g(k) \in \mathbb{Z}(p)_k \) for \( k = 1, 2, \ldots, p-1 \).

We recall now from [10] and [5, §6] that if \( \varphi(X) \in \mathbb{Z}(p)[X] \), the element \( \varphi(\Psi^q) \) is a unit in \( A \) if and only if \( \varphi(q_i) \) is a unit in \( \mathbb{Z}(p) \) for all \( i \geq 1 \). Since the \( q_i \) take the values \( 1, 2, \ldots, p-1 \) modulo \( p \), it follows that \( g(\Psi^q) \) is a unit in \( A \). Since \( f(\Psi^q) \) cannot be a unit, it follows that \( e_k > 0 \) for at least one value of \( k \).

We have
\[
g(\Psi^q) \prod_{k=1}^{p-1} (\Psi^q - k)^{e_k} x = -ph(\Psi^q)x,
\]
and thus
\[
\prod_{k=1}^{p-1} (\Psi^q - k)^{e_k} x = p\alpha x,
\]
where \( \alpha = -g(\Psi^q)^{-1}h(\Psi^q) \in A \).
As $M$ is finite, there is some $s$ such that $p^s x = 0$, so that
$$
\prod_{k=1}^{p-1} (q - k)^{c_k s} = p^s \alpha^s x = 0.
$$

But it is clear that $\prod_{k=1}^{p-1} (X - k)^{c_k s}$ is a factor modulo $p^s$ of $\Theta_n(X)$ for sufficiently large $n$, and thus $\Phi_n x = 0$ for such $n$.

By choosing the maximum such $n$ over all non-zero $x \in M$, we have $\Phi_n M = 0$. □

**Proposition 3.9.** If $M$ is an $A$-module which is free of finite rank over $\mathbb{Z}(p)$, then $\Phi_n M = 0$ for some $n$.

**Proof.** Let $x \in M$. The action map $\eta_x : A \to M$ given by $\eta_x(\alpha) = \alpha x$ is a homomorphism of $A$-modules. In particular, it is a homomorphism of abelian groups. By [7, Theorem 95.3], the target is a slender group, so there is some $m$ such that $\eta_x(\Phi_m) = 0$, i.e., $\Phi_m x = 0$.

If $x_1, \ldots, x_r$ is a $\mathbb{Z}(p)$-basis of $M$ and $\Phi_m x_i = 0$ for $i = 1, \ldots, r$, then $\Phi_n M = 0$ where $n = \max\{m_i : 1 \leq i \leq r\}$. □

Since $A$ is not an integral domain, we must exercise some care with quotients. However, if $n > m$ the polynomial $\Theta_m(X)$ is a factor of $\Theta_n(X)$, so we may let $\Phi_n/\Phi_m$ denote the value of the polynomial $\Theta_n(X)/\Theta_m(X)$ at $\Psi^q$.

**Lemma 3.10.** If $n > m$ and $n - m$ is divisible by $2p - 2$, then
$$
\frac{\Phi_n}{\Phi_m} \equiv \Phi_{n-m} \mod p^{1+\nu_p(n-m)}.
$$

**Proof.** It is easy to verify that whenever $n > m > 0$,
$$
\frac{\Phi_n}{\Phi_m} = \frac{\Phi_{n-1}}{\Phi_{m-1}} + (q_m - q_n) \frac{\Phi_{n-1}}{\Phi_m}.
$$

Expanding the first term on the right in the same way, and repeating this process, leads to the equation
$$
\frac{\Phi_n}{\Phi_m} = \Phi_{n-m} + \sum_{i=0}^{m-1} (q_m - q_n) \frac{\Phi_{n-i-1}}{\Phi_{m-i}}.
$$

If $n - m = 2k$, then $q_m - q_n$ is divisible by $q^k - 1$ for all $i$. Moreover, if $k$ is divisible by $(p - 1)p^{r-1}$, then $q^k - 1 \equiv 0 \mod p^r$. □

**Proof of Theorem 3.2.** First suppose that $M$ is a discrete $A$-module. Then, for $x \in M$, there is some $n$ such that $Ax = (A/A_n)x$. But $A/A_n$ is free of finite rank over $\mathbb{Z}(p)$, so $M$ is locally finitely generated.

Now suppose that $M$ is locally finitely generated. By Corollary 3.7, it is enough to show that for each $x \in M$ there is some $n$ such that $\Phi_n x = 0$.

Let $x \in M$, and write $N = Ax$. By hypothesis, $N$ is finitely generated over $\mathbb{Z}(p)$. Let $T \subseteq N$ be the $A$-submodule of $N$ consisting of
the \( \mathbb{Z}_p \)-torsion elements. So there is some \( s \) such that \( p^s T = 0 \). The quotient \( N/T \) is an \( A \)-module which is free of finite rank over \( \mathbb{Z}_p \).

By Proposition 3.9 there is some \( k \) such that \( \Phi_k (N/T) = 0 \), and so \( \Phi_k x \in T \). Then, by Proposition 3.8 there is some \( r \) such that \( \Phi_r \Phi_k x = 0 \).

By increasing \( r \) if necessary, we may arrange that \( r \) is divisible by \( (2p - 2)p^s \), so that, by Lemma 3.10, we have

\[
\frac{\Phi_{r+k}}{\Phi_k} = \Phi_r + p^s \theta,
\]

for some \( \theta \in A \). Hence

\[
\Phi_{r+k} x = \frac{\Phi_{r+k}}{\Phi_k} \Phi_k x = \Phi_r \Phi_k x + p^s \theta \Phi_k x = 0 + \theta (p^s \Phi_k x) = 0.
\]

\[\square\]

4. Bousfield’s Category of \( K \)-theory modules

In this section we relate the category \( DA \) of discrete \( A \)-modules to a category considered by Bousfield in his work on the \( K_p \)-local stable homotopy category.

Let \( R = \mathbb{Z}_p[\mathbb{Z}_p^{\times}] \) be the group-ring of the multiplicative group of units in \( \mathbb{Z}_p \), with coefficients in \( \mathbb{Z}_p \) itself. For clarity, as well as to reflect the topological applications, we write \( \Psi^j \in R \) for the element \( j \in \mathbb{Z}_p^{\times} \). Hence elements of \( R \) are finite \( \mathbb{Z}_p \)-linear combinations of the \( \Psi^j \).

**Definition 4.1.** [2] Bousfield’s category \( A(p) \) is the full subcategory of the category of \( R \)-modules whose objects \( M \) satisfy the following conditions. For each \( x \in M \),

(a) the submodule \( Rx \subseteq M \) is finitely generated over \( \mathbb{Z}_p \),

(b) for each \( j \in \mathbb{Z}_p^{\times} \), \( \Psi^j \) acts on \( Rx \otimes \mathbb{Q} \) by a diagonalisable matrix whose eigenvalues are integer powers of \( j \),

(c) for each \( m \geq 1 \), the action of \( \mathbb{Z}_p^{\times} \) on \( Rx/p^m Rx \) factors through the quotient homomorphism \( \mathbb{Z}_p^{\times} \to (\mathbb{Z}/p^k \mathbb{Z})^{\times} \) for sufficiently large \( k \).

We call the objects of this category **Bousfield modules**.

If \( M \) is a Bousfield module which is finitely generated over \( \mathbb{Z}_p \), then condition (a) holds automatically, and the rational diagonalisability and \( p \)-adic continuity conditions of (b) and (c) hold globally for \( M \), as well as for each submodule \( Rx \subseteq M \).

Note that \( R \subseteq A \), so an \( A \)-module can be considered as an \( R \)-module by restricting the action. (An explicit formula expressing each \( \Psi^j \), for \( j \in \mathbb{Z}_p^{\times} \), in terms of the \( \Phi_n \) is given in [3] Proposition 6.6.)
Theorem 4.2. If $M$ is a discrete $A$-module, then $M$ is a Bousfield module with the $R$-action given by the inclusion $R \subset A$.

Proof. Suppose $A_nx = 0$, then it is clear that $Rx = Ax = (A/A_n)x$, which is finitely generated over $\mathbb{Z}(p)$. The matrix representing $\Psi^g$ on $Rx \otimes \mathbb{Q}$ is annihilated by the polynomial $\Theta_n(X)$, and thus its minimal polynomial is a factor of $\Theta_n(X)$. Since $\Theta_n(X)$ has distinct rational roots, the matrix can be diagonalised over $\mathbb{Q}$. The eigenvalues are roots of $\Theta_n(X)$, which are integer powers of $q$.

Proposition 6.6 of \cite{5} shows that

$$\Psi^g = \sum_{i \geq 0} g_i(j)\Phi_i,$$

if $j \in \mathbb{Z}^\times_{(p)}$, where $g_i(w)$ is a certain Laurent polynomial (given explicitly in \cite{5}) satisfying $g_i(j) \in \mathbb{Z}(p)$ for all $j \in \mathbb{Z}^\times_{(p)}$. Since $A_nx = 0$, this shows that $\Psi^g$ acts on $Rx$ as the polynomial in $\Psi^q$

$$P_j(\Psi^q) = \sum_{i = 0}^{n-1} g_i(j)\Theta_i(\Psi^q).$$

It follows that the eigenvalues of the matrix of the action of $\Psi^j$ on $Rx \otimes \mathbb{Q}$ are $P_j(q^r)$, where $q^r$ is an eigenvalue of the action of $\Psi^q$. By considering the action on the coefficient group $\pi_2(K(p))$, we see that $P_j(q^r) = j^r$. Hence condition (b) of Definition 4.1 is satisfied.

The Laurent polynomials $g_i(j)$ are uniformly $p$-adically continuous functions of $j \in \mathbb{Z}^\times_{(p)}$, i.e., for each $m \geq 1$ there is an integer $K_i$ such that $g_i(j) \equiv g_i(j + p^k a) \mod p^m$ whenever $k \geq K_i$. Hence $\Psi^j x \equiv \Psi^{j+p^k a} x \mod p^m$ for $k \geq \max\{K_0, K_1, \ldots, K_{n-1}\}$. This shows that condition (c) of Definition 4.1 holds.

Since each $\Phi_n$ is a polynomial in $\Psi^q$, any finite linear combination of the $\Phi_n$ can be considered as an element of $R$. In order to show that a Bousfield module can be given the structure of an $A$-module, we need to specify how an infinite sum $\sum_{n \geq 0} a_n \Phi_n$ acts. We need a preliminary lemma.

Lemma 4.3. Let $M$ be a Bousfield module and $x \in M$. There is some $k \geq 1$ such that $\Phi_n x \equiv 0 \mod p$ for all $n \geq p^k(p-1)$.

Proof. Proposition 6.5 of \cite{5} gives an explicit formula for the expansion of $\Phi_{p^k(p-1)}$ as a finite $\mathbb{Z}((p)$-linear combination of the $\Psi^q$. The coefficient of $\Psi^q$ has as a factor the $q$-binomial coefficient $[p^k(p-1)]$, which is divisible by $p$ for $0 < j < p^k(p-1)$. Thus

$$\Phi_{p^k(p-1)} \equiv \Psi^{p^k(p-1)} + q^{p^k(p-1)/2} \mod p$$

$$\equiv \Psi^{p^k(p-1)} - 1 \mod p.$$
Now condition (c) of Definition 4.1 ensures that \((\Psi^q x^{(p-1)} - 1)x \equiv 0 \mod p\) for sufficiently large \(k\). Thus \(\Phi_{p^k} x^{(p-1)} \equiv 0 \mod p\), and consequently \(\Phi_n x \equiv 0 \mod p\) for all \(n \geq p^k(p-1)\). □

**Theorem 4.4.** If the \(R\)-module \(M\) is a Bousfield module, then the \(R\)-action extends uniquely to an \(A\)-action in such a way as to make \(M\) a discrete \(A\)-module.

**Proof.** Let \(x \in M\). By condition (b) of Definition 4.1, the minimal polynomial of the action of \(\Psi^q\) on \(Rx \otimes \mathbb{Q}\) has the form \(\prod_{i=1}^t (X - q^k_i)\), where \(k_i \in \mathbb{Z}\). For sufficiently large \(m\), this polynomial is a factor of \(\Theta_m(X)\), so that \(\Phi_m x = 0\) in \(Rx \otimes \mathbb{Q}\). This means that \(\Phi_m x \in T\), the \(\mathbb{Z}(\mathbb{p})\)-torsion submodule of \(Rx\). As \(T\) is finitely generated, there is some exponent \(e\) such that \(p^e T = 0\).

Let \(k\) be as in Lemma 4.3 and let \(n = (2p-2)p^r\), where \(r \geq k\). Then \(\Phi_n x \equiv 0 \mod p\), and so \(\Phi_n^\ell x \equiv 0 \mod p^\ell\) for any \(\ell \geq 1\).

On the other hand, iterating Lemma 4.10 shows that \(\Phi_{2^s n} \equiv \Phi_{2^s n}^{2^s} \mod p^{1+r}\). It is thus clear that by choosing \(r\) and \(s\) sufficiently large we may ensure firstly that \(\Phi_{2^s n} x \in T\) and then that \(\Phi_{2^s n} x \equiv 0 \mod p^e\), which means that \(\Phi_{2^s n} x = 0\). There is thus some integer \(N\) such that \(\Phi_N x = 0\). Given \(\alpha = \sum_{k \geq 0} a_k \Phi_k x \in A\), let \(\alpha x = \sum_{k=0}^{N-1} a_k \Phi_k x\). It is clear that this gives \(M\) the structure of a discrete \(A\)-module. The uniqueness of this structure follows from Lemma 2.7. □

**Remark 4.5.** We have chosen to give a direct algebraic proof. This can, of course, be replaced by appealing to Bousfield’s topological results: the non-split analogue of [2, Proposition 8.7] shows that any object in \(A(\mathbb{p})\) can be expressed as \(K_0(X; \mathbb{Z}_{(p)})\) for some spectrum \(X\). Proposition 2.6 shows that \(K_0(X; \mathbb{Z}_{(p)})\) is a discrete \(A\)-module.

Assembling the main results of this section, we have proved the following.

**Theorem 4.6.** Bousfield’s category \(A(\mathbb{p})\) is isomorphic to the category \(DA\) of discrete \(A\)-modules. □

5. COFREE OBJECTS AND INJECTIVE RESOLUTIONS

To illustrate the utility of our point of view, we consider cofree objects. Bousfield showed the existence of a right adjoint functor \(U\) to the forgetful functor from his category \(A(\mathbb{p})\) to the category of \(\mathbb{Z}_{(p)}\)-modules. To do this, he had to give different descriptions in two special cases and then deduce the existence of such a functor in the general case without constructing it explicitly.

In our context, the functor \(U\) is just a continuous Hom functor, right adjoint to the forgetful functor from \(DA\) to \(\mathbb{Z}_{(p)}\)-modules. So, not only do we not need to treat separate cases, but our uniform description
is conceptually simple and fits into a standard framework for module categories.

If $S$ is a unital, commutative algebra over a commutative ring $R$ with 1, then there is a right adjoint to the forgetful functor from the category of $S$-modules to the category of $R$-modules, given by $\text{Hom}_R(S, -)$; see [13, Lemma 2.3.8]. For an $R$-module $L$, $\text{Hom}_R(S, L)$ is an $S$-module via $(sf)(t) = f(ts)$ for $s, t \in S$.

We will need to modify this construction since $\text{Hom}_{\mathbb{Z}(p)}(A, L)$ need not be a discrete $A$-module.

Example 5.1. The $A$-module $\text{Hom}_{\mathbb{Z}(p)}(A, \mathbb{Q})$ is not discrete. To see this, note that, since $\mathbb{Q}$ is not slender [7, Section 94], there exists $f \in \text{Hom}_{\mathbb{Z}(p)}(A, \mathbb{Q})$ such that $f(\Phi_n) \neq 0$ for all $n$. Then $(\Phi_n f)(1) = f(\Phi_n) \neq 0$ for all $n$, and so there is no $n$ such that $A_n f = 0$.

Notice that we can make $\mathbb{Q}$ into a discrete $A$-module via $A \xrightarrow{\varphi} \mathbb{Z}(p) \hookrightarrow \mathbb{Q}$, where $A \xrightarrow{\varphi} \mathbb{Z}(p)$ is the augmentation given by $\varepsilon(\sum_{k \geq 0} a_k \Phi_k) = a_0$. Thus $\text{Hom}_{\mathbb{Z}(p)}(A, L)$ need not be a discrete $A$-module, even when $L$ is such a module.

Definition 5.2. If $L$ is a $\mathbb{Z}(p)$-module, let $\text{Hom}_{\mathbb{Z}(p)}^{\text{cts}}(A, L)$ denote the $A$-submodule of $\text{Hom}_{\mathbb{Z}(p)}(A, L)$ consisting of the homomorphisms which are continuous with respect to the filtration topology on $A$ and the discrete topology on $L$.

Note that a homomorphism $A \to L$ is continuous if and only if its kernel contains $A_n$ for some $n$. Example 5.1 shows that $\text{Hom}_{\mathbb{Z}(p)}^{\text{cts}}(A, L)$ may be a proper submodule of $\text{Hom}_{\mathbb{Z}(p)}(A, L)$ even if $L$ is a discrete $A$-module.

Proposition 5.3. The functor $U$ from $\mathbb{Z}(p)$-modules to the category $\mathcal{DA}$ of discrete $A$-modules given by $UL = \text{Hom}_{\mathbb{Z}(p)}^{\text{cts}}(A, L)$ is right adjoint to the forgetful functor.

Proof. For any $A$-module $N$, we define the discrete heart of $N$ to be the largest $A$-submodule which is a discrete $A$-module:

$$N^{\text{disc}} = \{ x \in N : A_n x = 0 \text{ for some } n \}.$$  

It is easy to show that the ‘discrete heart functor’ is right adjoint to the forgetful functor from discrete $A$-modules to $A$-modules, hence $\text{Hom}_{\mathbb{Z}(p)}(A, -)^{\text{disc}}$ is right adjoint to the forgetful functor from discrete $A$-modules to $\mathbb{Z}(p)$-modules. The proof is completed by observing that $\text{Hom}_{\mathbb{Z}(p)}(A, L)^{\text{disc}} = \text{Hom}_{\mathbb{Z}(p)}^{\text{cts}}(A, L)$ for any $\mathbb{Z}(p)$-module $L$. \qed

Proposition 5.4. For any $\mathbb{Z}(p)$-module $L$, there is a natural isomorphism of $A$-modules $UL \cong K_0(K)_{(p)} \otimes L$, where $K_0(K)_{(p)} \otimes L$ is an $A$-module via the $A$-module structure on $K_0(K)_{(p)}$. 

Theorem 5.5. The functor $U$ is exact, and preserves direct sums and direct limits.

Proof. As a right adjoint, $U$ is left exact. On the other hand, if $f : L_1 \to L_2$ is a $\mathbb{Z}(p)$-module epimorphism, then $Uf : UL_1 \to UL_2$ is an epimorphism of discrete $A$-modules, for if $g \in \text{Hom}^{cts}_{\mathbb{Z}(p)}(A, L_2)$, then we can define $h \in \text{Hom}^{cts}_{\mathbb{Z}(p)}(A, L_1)$ with $g = Uf(h)$ as follows. For each $n \geq 0$, we can find some $y_n \in L_1$ such that $f(y_n) = g(\Phi_n)$, and, since $g(\Phi_n) = 0$ for $n \gg 0$, we may choose $y_n$ to be 0 for $n \gg 0$. Thus we can define $h : A \to L_1$ by $h(\sum_{n \geq 0} a_n \Phi_n) = \sum_{n \geq 0} a_n y_n$.

That $U$ preserves direct sums is immediate from the definition, and an exact functor which preserves direct sums automatically preserves direct limits.

Corollary 5.6. If $D$ is an injective $\mathbb{Z}(p)$-module, then $UD$ is injective. Hence $DA$ has enough injectives.

Proof. The injectivity of $UD$ follows from adjointness. As each $\mathbb{Z}(p)$-module can be embedded in an injective $\mathbb{Z}(p)$-module $D$, so any discrete $A$-module can be embedded as a $\mathbb{Z}(p)$-module in such a $D$. Then, using adjointness and left-exactness of $U$, any discrete $A$-module can be embedded as an $A$-module in some $UD$.

Bousfield introduced in [2 (7.4)] a four-term exact sequence which underlies the fact, due to Adams and Baird [1], that all $\text{Ext}^{>2}$ groups are zero. Bousfield’s construction applies to the version of his category which corresponds to the split summand of $K$-theory; we briefly discuss this category in the Appendix. We end this section by constructing the corresponding exact sequence in $DA$, using the functor $U$.

Theorem 5.7. For any $M$ in $DA$, there is an exact sequence in $DA$:

$$0 \to M \xrightarrow{\alpha} UM \xrightarrow{\beta} UM \xrightarrow{\gamma} M \otimes \mathbb{Q} \to 0,$$

where $UM$ denotes the discrete $A$-module obtained by applying $U$ to the $\mathbb{Z}(p)$-module underlying $M$ (i.e., applying a forgetful functor to $M$ before applying $U$).
Proof. The map $\alpha$ is adjoint to the identity map $M \to M$; explicitly $\alpha x$ maps $\theta \in A$ to $\theta x$. It is clear that $\alpha$ is a monomorphism. For any $x \in M$, the map $\alpha x : A \to M$ is $A$-linear. Moreover, any $A$-linear map $f : A \to M$ is determined by $f(1)$, and $f = \alpha(f(1))$. Thus the image of $\alpha$ is exactly the subset $\text{Hom}_A(A, M)$ of $A$-linear maps in $\text{UM} = \text{Hom}^{cts}_{\mathbb{Z}(p)}(A, M)$ (an $A$-linear map into a discrete $A$-module is automatically continuous).

The map $\beta$ is given by $\beta f = \Psi q \circ f - f \circ \Psi q$, where $f : A \to M$. Here $\Psi q \circ f$ uses the $A$-module structure of $M$, not that of $\text{UM}$, in other words, $\beta f : \theta \mapsto \Psi q f(\theta) - f(\Psi q \theta)$. It is straightforward to check that $\beta$ is an $A$-homomorphism. A continuous $\mathbb{Z}(p)$-homomorphism from $A$ into a discrete $A$-module which commutes with $\Psi q$ is an $A$-module homomorphism, hence $\ker \beta = \text{Hom}_A(A, M) = \text{im} \alpha$.

We define $\gamma$ as follows. Let $\Theta_n^{(j)}(X) = \prod_{i=1 \atop i \neq j}^n (X - q_i)$, and let $\Phi_n^{(j)} = \Theta_n^{(j)}(\Psi q) \in A$. Note that $\Phi_n^{(j)} = \Phi_n$ if $j > n$, since $\Theta_n^{(j)}(X) = \Theta_n(X)$ in that case. If $x \in M$ and $\Phi_n x = 0$, then each $\Phi_n^{(j)} x$ is an eigenvector of $\Psi q$ with eigenvalue $q_j$. Moreover we have

\begin{equation}
1 = \Phi_0 = \sum_{j=1}^n \frac{\Phi_n^{(j)}}{\Theta_n^{(j)}(q_j)}
\end{equation}

in $A \otimes \mathbb{Q}$ for all $n \geq 1$.

If $f \in \text{Hom}^{cts}_{\mathbb{Z}(p)}(A, M)$, choose $n$ such that $f(\Phi_k) = 0$ for all $k \geq n$ and $\Phi_n f(\Phi_k) = 0$ for $0 \leq k < n$. Then let

$$
\gamma f = \sum_{k \geq 0} \sum_{j=1}^{k+1} \frac{\Phi_n^{(j)} f(\Phi_k)}{\Theta_n^{(j)}(q_j) \Theta_n^{(j)}(q_{k+1})(q_j)} \in M \otimes \mathbb{Q}.
$$

Note that we may reverse the order of summation to obtain

\begin{equation}
\gamma f = \sum_{j=1}^n \frac{\Phi_n^{(j)} x_j}{\Theta_n^{(j)}(q_j)},
\end{equation}

where

$$
x_j = \sum_{k=j-1}^{n-1} \frac{f(\Phi_k)}{\Theta_n^{(j)}(q_{k+1})(q_j)}.
$$

In this form it is apparent that the formula for $\gamma f$ is independent of the choice of $n$. For if $n \leq m$ and $1 \leq j \leq m$, it follows from
The discrete module category for $K$-theory operations

\[ \Psi^q \Phi_n^{(j)} x_j = q_j \Phi_n^{(j)} x_j \]

\[ \frac{\Phi_n^{(j)} x_j}{\Theta_n^{(j)}(q_j)} = \frac{\Phi_n^{(j)} x_j}{\Theta_n^{(j)}(q_j)} \]

Suppose now that \( f = \beta g \), with \( g(\Phi_k) = 0 \) for all \( k \geq n \) and \( \Phi_n g(\Phi_k) = 0 \) for \( 0 \leq k < n \). Then

\[ x_j = \sum_{k=j-1}^{n-1} \frac{(\Psi^q - q_{k+1})g(\Phi_k) - g(\Phi_{k+1})}{\Theta_n^{(j)}(q_{k+1})} \]

But, since \( \Theta_n^{(j)}(q_{k+1}) = (q_j - q_{k+1})\Theta_k^{(j)}(q_j) \) for \( 1 \leq j \leq k \), and \( g(\Phi_n) = 0 \),

\[ x_j = \sum_{k=j-1}^{n-1} \frac{(\Psi^q - q_j)g(\Phi_k)}{\Theta_n^{(j)}(q_{k+1})} \]

It follows from (5.9) that \( \gamma \beta g = 0 \), since \( \Phi_n^{(j)}(\Psi^q - q_j) = \Phi_n \).

To prove that \( \ker \gamma \subseteq \im \beta \) we need first the following lemma.

**Lemma 5.10.** Suppose \( f \in \ker \gamma \), where \( M \) is a discrete \( A \)-module and \( n \) is chosen as above. Then \( \sum_{k=0}^{n-1}(\Phi_n/\Phi_{k+1})f(\Phi_k) \) is a \( \mathbb{Z}(\rho) \)-torsion element of \( M \).

**Proof.** Since \( \gamma f = 0 \), in equation (5.9) each \( \Phi_n^{(j)} x_j = 0 \) in \( M \otimes \mathbb{Q} \), since otherwise \( \Psi^q \) would have linearly dependent eigenvectors with distinct eigenvalues. Thus for each \( j = 1, \ldots, n \),

\[ 0 = \sum_{k=j-1}^{n-1} \frac{\Phi_n^{(j)} f(\Phi_k)}{\Theta_n^{(j)}(q_{k+1})} \]

Now if \( j \leq k + 1 \leq n \), \( \Phi_n^{(j)} = \Phi_{k+1}^{(j)}(\Phi_n/\Phi_{k+1}) \) in \( A \). Hence (5.11) becomes

\[ 0 = \sum_{k=j-1}^{n-1} \frac{\Phi_{k+1}^{(j)}(\Phi_n/\Phi_{k+1})f(\Phi_k)}{\Theta_n^{(j)}(q_{k+1})} \]

in \( M \otimes \mathbb{Q} \). Let \( d_n \) denote the least common multiple of the \( \Theta_n^{(j)}(q_j) \) for \( j \leq k + 1 \leq n \). Then multiplying by \( d_n \) yields \( 0 = z_j \) in \( M \otimes \mathbb{Q} \), where

\[ z_j := \sum_{k=j-1}^{n-1} \left( (d_n/\Theta_n^{(j)}(q_j))\Phi_{k+1}^{(j)}(\Phi_n/\Phi_{k+1})f(\Phi_k) \right) \in M. \]
Hence $z_j$ is a $\mathbb{Z}_{(p)}$-torsion element in $M$, and so is

$$\sum_{j=1}^{n} z_j = \sum_{k=0}^{n-1} \left( \sum_{j=1}^{k+1} (d_n/\Theta_{k+1}^{(j)}(q_j)) \Phi_{k+1}^{(j)} \right) \left( \Phi_n/\Phi_{k+1} \right) f(\Phi_k)$$

$$= d_n \sum_{k=0}^{n-1} (\Phi_n/\Phi_{k+1}) f(\Phi_k),$$

where we use (5.8). The result follows.

Now if $f \in \text{Hom}_{\mathbb{Z}_{(p)}}^{\text{cts}}(A, M)$ and $r \geq 1$, define $\tilde{f}_r \in \text{Hom}_{\mathbb{Z}_{(p)}}^{\text{cts}}(A, M)$ by

$$\tilde{f}_r(\Phi_k) = \begin{cases} 
\sum_{i=0}^{k-1} (\Phi_k/\Phi_{i+1}) f(\Phi_i), & \text{if } 1 \leq k \leq r, \\
0, & \text{if } k = 0 \text{ or } k > r.
\end{cases}$$

It is a simple calculation that

$$(\beta \tilde{f}_r + f)(\Phi_k) = \begin{cases} 
0, & \text{if } k < r, \\
\sum_{i=0}^{r} (\Phi_{r+1}/\Phi_{i+1}) f(\Phi_i), & \text{if } k = r, \\
f(\Phi_k), & \text{if } k > r.
\end{cases}$$

Suppose now that $f \in \text{Ker } \gamma$, that $f(\Phi_k) = 0$ for all $k \geq n$ and $\Phi_n f(\Phi_k) = 0$ for $0 \leq k < n$. Then if $r > n$,

$$(\beta \tilde{f}_r + f)(\Phi_k) = (\Phi_{r+1}/\Phi_n) y,$$

where $y = \sum_{i=0}^{n-1} (\Phi_n/\Phi_{i+1}) f(\Phi_i)$ is, by Lemma 5.10, a $\mathbb{Z}_{(p)}$-torsion element of $M$. But now Lemma 3.10 shows that for $r$ sufficiently large $(\Phi_{r+1}/\Phi_n) y = 0$, in which case $f = \beta(-\tilde{f}_r)$. Hence we have shown that $\text{Ker } \gamma \subseteq \text{Im } \beta$.

It remains to show that $\gamma$ maps onto $M \otimes \mathbb{Q}$. Let $x \in M$, and suppose $\Phi_n x = 0$. Letting $d$ denote the least common multiple of the $\Theta_n^{(k)}(q_k)$ for $k = 1, \ldots, n$, define $f \in \text{Hom}_{\mathbb{Z}_{(p)}}^{\text{cts}}(A, M)$ by $f(\Phi_k) = (d\Theta_{k+1}^{(k)}(q_k)/\Theta_n^{(k)}(q_k)) \Phi_n^{(k)} x$. Since $\Phi_n^{(j)} \Phi_n^{(k)} x = \Theta_n^{(j)}(q_k) \Phi_n^{(k)} x$ is zero unless $j = k$, it is a simple calculation using (5.8) that $\gamma f = dx$ in $M \otimes \mathbb{Q}$.

The proof that $\gamma$ is an epimorphism is completed by showing that if $f \in \text{Hom}_{\mathbb{Z}_{(p)}}^{\text{cts}}(A, M)$, there exists $g \in \text{Hom}_{\mathbb{Z}_{(p)}}^{\text{cts}}(A, M)$ such that $f - pg \in \text{Ker } \gamma$.

To see this, choose a multiple $m$ of $2p - 1$ such that $f(\Phi_k) = 0$ for all $k \geq m$ and $\Phi_m f(\Phi_k) = 0$ for all $k \geq 0$. By Lemma 3.10 for each $k \geq m$ there is a $\theta_k \in A$ such that $(\Phi_{k+1}/\Phi_{k-m+1}) = \Phi_m + p\theta_k$. We
define $g$ as follows
\[
g(\Phi_k) = \begin{cases} 
\theta_k f(\Phi_{k-m}), & \text{if } k \geq m, \\
0, & \text{otherwise}.
\end{cases}
\]

For suitably large $n$, let
\[
y_j = \sum_{k=j-1}^{n-1} \frac{pg(\Phi_k)}{\Theta^{(j)}_{k+1}(q_j)} = \sum_{k=j-m-1}^{n-m-1} \frac{(\Phi_{k+m+1}/\Phi_{k+1}) f(\Phi_k)}{\Theta^{(j)}_{k+m+1}(q_j)},
\]
so that $p\gamma g = \sum_{j=1}^{n} \Phi_n^{(j)} y_j / \Theta_n^{(j)}(q_j)$. It is now clear that $p\gamma g = \gamma f$. For
\[
(\Phi_{k+m+1}/\Phi_{k+1})^{(j)} f(\Phi_k) = \left( \prod_{i=k+2}^{k+m+1} (q_j - q_i) \right) \Phi_n^{(j)} f(\Phi_k),
\]
which is zero if $k + 2 \leq j \leq k + m + 1$, while
\[
\Theta^{(j)}_{k+m+1}(q_j) = \Theta^{(j)}_{k+1}(q_j) \prod_{i=k+2}^{k+m+1} (q_j - q_i),
\]
if $1 \leq j \leq k + 1$. \[\square\]

6. BOUSFIELD’S MAIN RESULT

We translate Bousfield’s main result, giving an algebraic classification of $K(p)$-local homotopy types, into our language of discrete $A$-modules. We claim no originality here, but we believe it is useful to summarise Bousfield’s results in our language.

There are several steps in Bousfield’s construction: we need a graded version of our discrete module category, we need to understand $\text{Ext}$ groups in this category, and we need to see $k$-invariants associated to $K(p)$-local spectra as elements in such $\text{Ext}$ groups. We outline these steps without proof. The proofs can be easily adapted from those in [2].

1) The category $DA_*$

For $i \in \mathbb{Z}$, there is an automorphism $T^i : DA \to DA$ with $T^i M$ equal to $M$ as a $\mathbb{Z}(p)$-module but with $\Psi^i : T^i M \to T^i M$ equal to $q^i \Psi^i : M \to M$. An object of $DA_*$ is a collection of objects $M_n \in DA$ for $n \in \mathbb{Z}$ together with isomorphisms $u : TM_n \cong M_{n+2}$ in $DA$ for all $n$. A morphism $f : M \to N$ in $DA_*$ is a collection of morphisms $f_n : M_n \to N_n$ in $DA$ such that $uf_n = f_{n+2}u$ for all $n \in \mathbb{Z}$.

The point of this construction is that $K_*(X; \mathbb{Z}(p))$ is an object of $DA_*$ for any spectrum $X$.

2) Ext groups in $DA_*$

The category $DA_*$ has enough injectives, allowing the definition of (bigraded) Ext groups. The groups $\text{Ext}_{DA*}^{s,t}(-,-)$ vanish for $s > 2$.\[\square\]
essentially as a consequence of the exact sequence of Theorem 5.7.

There is an Adams spectral sequence with

\[ E_2^{s,t}(X, Y) = \text{Ext}_{DA^*}^{s,t}(K_{(p)}^*(X), K_{(p)}^*(Y)), \]

converging strongly to \([X_{K_{(p)}}, Y_{K_{(p)}}]^*].

3) \(k\)-invariants and the category \(kDA^*_*\).

To each \(K_{(p)}\)-local spectrum \(X\) is associated a \(k\)-invariant \(k_X \in \text{Ext}_{DA^*}^{2,1}(K_{(p)}^*(X), K_{(p)}^*(X)).\) The only non-trivial differential \(d_2\) in the Adams spectral sequence can be expressed in terms of these \(k\)-invariants. We form the additive category \(kDA^*_*\) whose objects are pairs \((M, \kappa)\), with \(M \in DA^*_*\) and \(\kappa \in \text{Ext}_{DA^*}^{2,1}(M, M)\), and whose morphisms from \((M, \kappa)\) to \((N, \lambda)\) are morphisms \(f : M \rightarrow N\) in \(DA^*_*\) with \(\lambda f = f\kappa \in \text{Ext}_{DA^*}^{2,1}(M, N)\).

This now allows us to translate the main result of [2] into our setting.

**Theorem 6.1.** Homotopy types of \(K_{(p)}\)-local spectra are in one-to-one correspondence with isomorphism classes in \(kDA^*_*\). □

7. **Appendix: The Split Setting**

In this section we summarise the Adams summand analogues of our results. We omit proofs as these are easy adaptations of those given in the preceding sections.

For a fixed odd prime \(p\), we denote the periodic Adams summand by \(G\), and we write \(B\) for the ring of stable degree zero operations \(G^0(G)\). As usual, we choose \(q\) primitive modulo \(p^2\), and we set \(\hat{q} = q^{p-1}\). Let \(\hat{\Phi}_n = \prod_{i=1}^n (\Psi_q - \hat{q}^{-i} \psi i/2) \in B\). The elements \(\hat{\Phi}_n\), for \(n \geq 0\), form a topological \(\mathbb{Z}_{(p)}\)-basis for \(B\); see [5, Theorem 6.13]. Just as for \(A\), the topological ring \(B\) can be viewed as a completion of the polynomial ring \(\mathbb{Z}_{(p)}[\Psi_q]\). The results in the two cases are formally very similar, differing only in that in many formulas \(q\) must be replaced by \(\hat{q}\).

Let \(DB\) denote the category of discrete \(B\)-modules, defined in the obvious way by analogy with Definition 2.3. There is a corresponding graded version \(DB^*_*\). Then the additive category \(kDB^*_*\) is formed just as we did in the non-split setting. Its objects are pairs \((M, \kappa)\), where \(M \in DB^*_*\) and \(\kappa \in \text{Ext}_{DB^*_*}^{2,1}(M, M)\).

**Theorem 7.1.**

1. \(DB\) is a cocomplete abelian category with enough injectives.
2. For any spectrum \(X\), \(G_0(X)\) is an object of \(DB\), and \(G_*(X)\) is an object of \(DB^*_*\).
3. \(DB\) is isomorphic to Bousfield’s category \(B(p)\).
4. The functor \(U(-) = \text{Hom}_{\mathbb{Z}_{(p)}^*}(B, -)\) from \(\mathbb{Z}_{(p)}\)-modules to \(DB\) is right adjoint to the forgetful functor.
(5) For any $M$ in $\mathcal{DB}$, there is an exact sequence in $\mathcal{DB}$

$$0 \to M \xrightarrow{\alpha} UM \xrightarrow{\beta} UM \xrightarrow{\gamma} M \otimes \mathbb{Q} \to 0.$$  

(6) The groups $\text{Ext}_{\mathcal{DB}}^{s,t}(-, -)$ vanish for $s > 2$.

(7) Homotopy types of $G$-local spectra are in one-to-one correspondence with isomorphism classes in $k\mathcal{DB}_*$. □

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