Superconductivity in a Luttinger liquid mediated by spin fluctuations

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We study spin-fluctuation mediated divergent superconducting fluctuations in a Luttinger liquid proximity-coupled to a spin chain. Our study provides insight into how spin-fluctuations can induce superconductivity in a strongly correlated non-Fermi liquid with repulsive electronic interactions only. The electrons in the system are governed by the Extended Hubbard Hamiltonian and are coupled to a chain of localized spins modeled by the spin-$\frac{1}{2}$ XX Hamiltonian. Using a multi-channel Luttinger liquid approach, we determine the phase diagram of the metal chain. We find that spin-polarized triplet superconducting correlations persist for repulsive electronic interactions for sufficiently large interchain couplings.

Introduction. During the last decade, experimental progress in nanoengineering has allowed for unprecedented control over structures with pronounced physical properties in the quantum domain [1–4]. Many of the nanostructures are interesting in their own right, as several long-standing predictions have been probed directly [5–7]. However, with the ability to use nanostructures as fundamental building blocks, one can also construct complex heterostructure where the emergent physics is richer than the sum of its constituent parts. The advent of a significant array of experimentally realizable low-dimensional structures, motivates renewed efforts on the theoretical side to determine new avenues to be pursued.

One such avenue is low-dimensional hybrid structures involving gapless fermionic surface states, magnetic insulators, and superconductors. These systems have received much attention already both theoretically [8–11] and experimentally [12–14], especially over the last decade, following the discovery of metallic surface states in topological insulators [15–17]. In search of Majorana fermions, suggested as fundamental building blocks in topological quantum computers, one-dimensional hybrid structures in particular, have been the subject of intense investigations [18–27]. For two-dimensional systems, heterostructures of this type have been investigated in the context of obtaining spin-polarized supercurrents with potential applications to superconducting spintronics [28, 29]. Furthermore, planar interfaces consisting of, on the one hand, metals or metallic surface states of topological insulators, and ferromagnetic or antiferromagnetic insulators, on the other hand, have been studied in the context of magnon-mediated unconventional superconductors [30–36].

In many studies on low-dimensional hybrid systems, the metallic states are mostly modeled using a well-defined single-particle physics picture for the electrons that are proximity-coupled to other states in the heterostructure. On the other hand, it is well-known that in one-dimensional systems, any amount of two-body scattering suffices to destroy the one-to-one correspondence between the interacting and non-interacting low-energy excitations. The resulting fixed-point, the Luttinger liquid [37, 38], is one where low-energy fermionic excitations of the on-interacting case are replaced by well-defined bosons [39], describing collective density fluctuations in the spin- and charge sectors. In the context of magnon-mediated unconventional superconductivity in low-dimensional heterostructures, it is thus of some interest to consider the fate of the superconducting state when it no longer arises out of a Fermi liquid. Similar issues need to be considered in the context of high-\(T_c\) superconductivity in cuprate oxides [40, 41]. In this paper, we therefore revisit the question of if and how superconductivity arises when a one-dimensional interacting fermion chain with gapless fermions interacts with a one-dimensional chain of localized spins, using simple lattice models of both components of the hybrid structure.

Microscopic model. To model the fermion chain, we use the extended Hubbard (EHB) Hamiltonian, \(H_{\text{EHB}}\), which has been extensively studied in one dimension [42–47]. In terms of annihilation and creation operators \(c^\dagger_{\sigma, i}\) and \(c_{\sigma, i}\) for electrons on site \(i\) with spin \(\sigma\), \(H_{\text{EHB}}\) can be expressed as

\[
\begin{align*}
H_{\text{EHB}} = -t \sum_{i, \sigma} c^\dagger_{\sigma, i} c_{\sigma, i+1, \sigma} - \mu \sum_{i} n_i + U \sum_{i} n_{i\uparrow} n_{i\downarrow} + V \sum_{i} n_i n_{i+1},
\end{align*}
\]

where \(n_i = c^\dagger_{\sigma, i} c_{\sigma, i}\), \(n_{i\sigma} = n_{i\uparrow} + n_{i\downarrow}\), \(t\) is the hopping amplitude between adjacent sites, \(\mu\) is the chemical potential, \(U\) is the on-site interaction, and \(V\) is the interaction between electrons situated on adjacent sites. The quantum spin operators \(S^z_i = (S^x_i, S^y_i, S^z_i)^\dagger\), satisfying the commutation relation \([S^\mu_i, S^\nu_i] = i\hbar \epsilon_{\alpha\beta\gamma} S^\gamma_i\), are used to describe the spin chain, modeled by the spin-$\frac{1}{2}$ XX Hamiltonian, \(H_{\text{FMI}}\)

\[
\begin{align*}
H_{\text{FMI}} = -J_{xy} \sum_{i} (S^x_i S^x_{i+1} + S^y_i S^y_{i+1}),
\end{align*}
\]

where \(J_{xy}\) is the ferromagnetic exchange coupling. The interchain coupling, denoted \(H_{\text{int}}\), is parametrized by \(J\), and is given by

\[
\begin{align*}
H_{\text{int}} = -2J \sum_{i} (c^\dagger_{i\uparrow}, c^\dagger_{i\downarrow}) \tau (c_{i\uparrow}, c_{i\downarrow})^T \cdot S_i,
\end{align*}
\]

inspired by [35], where \(\tau\) is a vector of the Pauli matrices, acting on the electron spin degree of freedom. The Hamiltonian describing the entire system is \(H = H_{\text{EHB}} + H_{\text{FMI}} + H_{\text{int}}\). Note that because \(J_{xy}\) is zero, \(H\) is not \(SU(2)\) symmetric. For convenience, we employ natural units \(\hbar = 1\) and use \(t\) the unit of energy in \(H\).
For our purposes, the most suitable approach to the problem is to employ the Jordan-Wigner transformation [48], as it allows a unified treatment of both chains. By introducing $S_i^± = (S_i^x + iS_i^y)$, and the spinless fermion operators $d_i^±$, the well-known mappings $S_i^± = d_i^1 e^{±iπΣ_{j=1}^∞ d_j^+d_j}$ and $S_i^z = d_i^+d_i - 1/2$ are established. By extracting the cubic terms from $H_{\text{int}} = H'_{\text{int}} + H_{\text{FMI}}$, the string operator vanishes when inserting the fermion operators into $H'_{\text{int}}$ and $H_{\text{FMI}}$

$$H_{\text{FMI}} = -J_{xy} \sum_i d_i^1 d_{i+1}^\dagger \quad (4)$$
$$H'_{\text{int}} = -2\bar{J} \sum_i (c_i^\dagger c_{i+1}^\dagger - c_i^\dagger c_i)(d_i^+d_i - \frac{1}{2}) \quad (5)$$

From equation (5), it follows that $\bar{J}$ acts as both the strength of the chain coupling and as an effective magnetic field in the z-direction felt only by the metal chain. The latter is accounted for by introducing a spin-dependency in the chemical potential $\mu_\sigma = \mu - \sigma J$. Equation (4) shows that $J_{xy}$ plays the role of a hopping parameter in the spin chain. $H'_{\text{int}}$ will be discussed further in the next section.

All three species of fermions have the same kinetic structure. Their dispersion relations are $\epsilon_s(k) = -2t_1 \cos(k) - \mu_l$, with $l$ being the species index $l = (\uparrow, \downarrow, S)$. From this, one finds the Fermi momentum and Fermi velocity, $k_F = \arccos(-\mu_l/(2t_1))$ and $v_F^l = 2t_1 \sin(k_F^l)$, respectively. The spin chain has $v_F^S = 2J_{xy}$ and $k_F^S = \pi/2$, physically corresponding to the absence of any net magnetization in the z-direction arising due to terms in $H_{\text{FMI}}$. $k_F^S$ is dependent on $\mu_\sigma$, such that $\mu$ determines $k_F$ in the absence of any chain coupling, while $J$ controls the extent of the spin splitting.

**Continuum limit field theory.** To describe the low-energy physics of our system, we use bosonization [49]. The low-energy excitations are described by linearizing the spectrum of the non-interacting case around the two Fermi points $\pm k_F^\uparrow$. Annihilation operators can then be written as $c_{\downarrow} = \sum_x \psi_{\downarrow}(x = ia)$ where $\psi_{\downarrow}(x)$ destroys a fermion of species $l$ on the branch $s = \pm$. By extending the linearized spectrum to $\pm \infty$, using a soft cutoff, and taking the continuum limit, the following operator identity holds [39]

$$\psi_{\downarrow}(x) = \lim_{\alpha \to 0} \frac{U_{\alpha l}}{2\pi \alpha} e^{i(k_F^\uparrow - \pi/L)x} e^{-i(s\phi_1(x) - \theta_1(x))}. \quad (6)$$

Here, $U_{\alpha l}$ is a Klein factor which has the effect of ensuring correct fermionic anticommutation relations and moreover of raising or lowering the number of fermions in the system [49], $\alpha$ is a cutoff ensuring finite bandwidth, and $\phi$ and $\theta$ are bosonic fields. The details of the construction of $\phi$ and $\theta$ and their explicit representation can be found in several reviews on abelian bosonization [50–52], and will not be repeated here. Due to the relations $\nabla \phi(x) = -\pi(n_R(x) + n_L(x))$ and $\nabla \theta(x) = -\pi(n_L(x) - n_R(x))$, $\phi$ and $\theta$ can be interpreted as density and current fields, respectively.

For repulsive $U$, using renormalization-group theory one finds that the backscattering term is a priori relevant and gaps the spin sector. In the presence of a sufficiently strong magnetic field, backscattering is however suppressed. This readmits a TL representation [54]. The absence of large momentum transfers can be attributed to the Fermi momentum mismatch between opposite-spin electrons, $\delta k_F = k_F^\uparrow - k_F^\downarrow$. With $\bar{J}$ acting as an effective magnetic field. The same will hold for the system we consider, especially since we will focus on the parameter regime where $\bar{J}$, and thus also $\delta k_F$, is large. Furthermore, we use bosonic fields $\phi_\sigma$ and $\theta_\sigma$ associated to the spin chain to represent $S_i^\sigma$ [52]. It then follows that the terms in $H'_{\text{int}}$ are a product of two complex exponentials. The first is a linear combination of slowly varying fields, while the other is $e^{i(\delta k_F + nk_F^\uparrow)x}$ with $n = 0, 2$. Since $\delta k_F > 0$, the latter exponential oscillates rapidly. Thus, when integrating over the length of the system, the cubic terms average to zero and may be neglected.

By the preceding argument, it follows that only terms quadratic in the fields remain in interactions between different fermion species, as they have different Fermi momenta. The same-spin interactions between nearest neighbors require more care. In the weak-coupling regime they take the form [55]

$$V \sum_i n_{i+1,\sigma} n_{i,\sigma} = \int dx \frac{1}{\pi^2} V(1 - \cos(2k_F^\sigma a)) (\nabla \phi_\sigma(x))^2. \quad (7)$$

In general we will avoid half-filling, and any accidental Umklapp scattering in the metal chain arising if either spin band is at half-filling is neglected. Because $J_{xy} = 0$, there is no Umklapp scattering in the spin chain either, yielding a purely quadratic theory describing a TL liquid.

We next introduce in standard fashion bosonic fields associated with the charge and spin densities in the TL liquid originating with $H_{\text{EBB}}$, $n_{\rho} = (n_{\uparrow} + n_{\downarrow})/\sqrt{2}$ and $n_{\sigma} = (n_{\uparrow} - n_{\downarrow})/\sqrt{2}$, respectively. This will also accentuate the magnetic nature of the interchain coupling in equation (5). Employing the bases $\phi = (\phi_\uparrow, \phi_\downarrow, \phi_\sigma)^T$ describing densities in the three channels ($\rho, \sigma, S$) and $\theta = (\theta_\uparrow, \theta_\downarrow, \theta_\sigma)^T$ describing currents in the same three channels, one obtains from (7) the following expression for $H$

$$H = \frac{1}{2\pi} \int dx \partial_x (\phi^T \theta^T) \begin{pmatrix} V_{\phi} & 0 \\ V_{\theta} & 0 \end{pmatrix} \partial_x \begin{pmatrix} \phi \\ \theta \end{pmatrix}. \quad (8)$$

The symmetric matrices $V_{\phi}$ and $V_{\theta}$ contain all microscopic details of the model. $V_{\phi}$ is

$$V_{\phi} = \begin{pmatrix} v_F^\uparrow & \delta v_F^\uparrow & 0 \\ \delta v_F^\uparrow & v_F^\uparrow & 0 \\ 0 & 0 & v_F^S \end{pmatrix}, \quad (9)$$

where $v_F^\sigma \equiv (v_F^\uparrow + v_F^\downarrow)/2$ and $\delta v_F^\sigma \equiv (v_F^\uparrow - v_F^\downarrow)/2$. The expression for $V_{\theta}$ is more complicated

$$V_{\theta} = \begin{pmatrix} \delta v_F^\uparrow & \delta v_F^\uparrow & 0 \\ \delta v_F^\uparrow & v_F^\uparrow & 0 \\ 0 & 0 & 2v_F^S \end{pmatrix} \frac{2\sqrt{2l}}{\pi} v_F^\downarrow \begin{pmatrix} v_F^\uparrow \\ v_F^\uparrow \\ v_F^S \end{pmatrix}, \quad (10)$$

For $U < 0$, the backscattering term is a priori relevant and gaps the spin sector. In the presence of a sufficiently strong magnetic field, backscattering is however suppressed. This readmits a TL representation [54]. The absence of large momentum transfers can be attributed to the Fermi momentum mismatch between opposite-spin electrons, $\delta k_F = k_F^\uparrow - k_F^\downarrow$. With $\bar{J}$ acting as an effective magnetic field. The same will hold for the system we consider, especially since we will focus on the parameter regime where $\bar{J}$, and thus also $\delta k_F$, is large. Furthermore, we use bosonic fields $\phi_\sigma$ and $\theta_\sigma$ associated to the spin chain to represent $S_i^\sigma$ [52].
with

\[ U_\sigma = U + V(\cos(2k_\phi^x a) + \cos(2k_\theta^x a)) \]  
\[ U_\sigma = U + V(\cos(2k_\phi^y a) + \cos(2k_\theta^y a)) \]  
\[ \delta V = V(\cos(2k_\phi^z a) - \cos(2k_\theta^z a)). \]

From equations (9) and (10), the influence of the intrachannel coupling is seen to be twofold. Firstly, the effective magnetic field \( J \) destroys the spin-charge separation normally present in the EHB model, since the coupling between the electron spin and charge channels is nonzero. Secondly, \( J \) also acts as an interchannel coupling between the electron spin channel and the channel describing the spin chain.

**Multi-channel Luttinger liquids and correlation functions.**

From the relation \( [\phi_i(x_1), \partial_x \phi_m(x_2)] = i\delta_m \delta(x_1 - x_2)/\pi \) and equation (8), the action of the system is obtained

\[ S[\phi, \theta] = \frac{1}{2\pi} \int dx d\tau \left( \phi^T \right)^T \left[ \begin{array}{cc} 0 & I_3 \\ I_3 & 0 \end{array} \right] i\partial_\tau + \left( \begin{array}{cc} V_\phi & 0 \\ 0 & V_\theta \end{array} \right) \partial_x \left( \begin{array}{c} \phi \\ \theta \end{array} \right), \]

where \( \tau \) is imaginary time, \( I \) is the identity matrix, and the differential operators inside the square bracket act to the left.

Equation (12) describes the action of a multi-channel Luttinger liquid. Such systems are often considered when studying the Fermi-Luttinger liquid crossover and when introducing disorder to a microscopic model [56–58]. In that setting, a method for mapping the case of interchannel interactions, back to the well known case of diagonal interaction matrices has been devised [59–61]. Introducing the matrix \( M \) with the properties \( M^T V_\phi M = M^{-1} V_\theta M^{-T} = u \), where \( u \) is a diagonal matrix congruent to both \( V_\phi \) and \( V_\theta \). Introducing the transformed fields, \( \tilde{\phi} = M^{-1} \phi \) and \( \tilde{\theta} = M^T \theta \), the first term in equation (12) is left invariant, while the second term is diagonalized

\[ S[\tilde{\phi}, \tilde{\theta}] = \frac{1}{2\pi} \int dx d\tau \left( \tilde{\phi}^T \right)^T \left[ \begin{array}{cc} 0 & I_3 \\ I_3 & 0 \end{array} \right] i\partial_\tau + \left( \begin{array}{cc} u & 0 \\ 0 & u \end{array} \right) \partial_x \left( \begin{array}{c} \tilde{\phi} \\ \tilde{\theta} \end{array} \right). \]

\( M \) is constructed using the procedure presented in [62]. Similar approaches are used in [63, 64] for the two-channel case. The entries in \( u \) are the velocities of the three types of collective excitations in the system. Since the relation between \( (\phi, \theta) \) and \( (\tilde{\phi}, \tilde{\theta}) \) is known, calculating correlation functions is effected by a change of basis and using equation (13).

To this end, we introduce the symmetric Luttinger matrix \( K = MM^T \), which will play a role corresponding to the Luttinger parameter for the single-channel case, i.e. the entries are determined by the parameters of the model.

By using equation (13), it is straightforward to calculate correlation functions in the form

\[ I = \langle \exp(i \sum_i A_i^T \phi(r_i) + B_i^T \theta(r_i)) \rangle. \]
in terms of $K_{11}$, $K_{12}$ and $K_{22}$. In general, the two OPs in each expression $R_q$ cannot be distinguished, with the polarized TS being the exception as one can use the appropriate sign in front of $K_{12}$. This is easily understood for $R_{SS/TS^0}$. With $J = 0$, $V = 0$, and $U < 0$ the system is gapped in the spin sector, and the dominant phase is a SS. With an effective magnetic field, the SS is converted into a FFLO state [66, 67] with center of mass momentum $\pm \delta k_F$ [54, 68, 69]. In $R_{CDW/SDW}$ one can observe two distinct density waves, with wave numbers $2k_F^\downarrow$ and $2k_F^\uparrow$, each wave carrying net spin and charge. Finally, we note that the density-density correlations have an additional $k_F$-independent term, which always exhibits Fermi liquid decay with $x^{-2}$ independent of the microscopic details.

Results and discussion. Figure 1 presents three phase diagrams for our model. The dominant phase is found by calculating $K$ for every set of microscopic parameters. The smallest $\nu_q$ is subsequently determined using equation (17). This identifies the dominant divergence and the most favorable phase. The colors of the figures are associated with different OPs. No divergent $\chi$ indicates that the Fermi liquid decay in the density correlations dominates. The unstable region indicates that one of the velocities in $\nu$ is imaginary. This may be indicative of a phase transition [63, 70], sometimes referred to as a Wentzel-Bardeen (WB) singularity [71, 72]. When $J$ becomes the dominant interaction, it is possible that the WB singularity arises because the system becomes phase separated, as both chains are separated into regions with equal polarization, similar to the $t$-$J$-model [73]. The lightly shaded region marks the area where $\delta k_F$ is not large enough to safely discard large momentum transfer terms. We choose the value $\delta k_F a = 0.2$ to bound this region. Since changing the sign of $J$ is equivalent to flipping the quantization axis, all OPs insensitive to this operation are symmetric with $J$, while spin-polarized OPs are mapped to their spin-flipped counterpart.

In Figures 1 (a, b), $V = 0$, hence the system is similar to the Hubbard models studied in [53, 54], with an additional channel due to the spin chain, resulting in a richer phase diagram. Despite having a different Fermi velocity in the spin chain, the two systems exhibit the same qualitative traits for attractive $U$, with the preferred state being the FFLO state. However, once $|J|$ becomes large enough, the polarized TS is preferred, with the sign of $J$ determining the polarization of the state. Note that this occurs in regions where the quadratic theory is valid. This transition can be understood as competing electron pairing mechanisms. An attractive $U$ favors on-site pairing with opposite spin, but this pairing is suboptimal when including interchain interactions. For large $J$, the optimal placing of the electrons is such that they are always adjacent to a localized spin with the same spin-polarization, avoiding double occupancy of a site.

The competing interactions can be further understood by considering the properties of the Luttinger matrix $K$. While the explicit expression for $K$ in terms of the underlying microscopic parameters rapidly becomes intractable as the number of channels increases, the two-channel case has been studied in detail [61, 63] and provides valuable insight into the physics of the present three-channel case. An important property for two-channel systems is that density (current) inter-channel interactions are found to enhance (suppress) the diagonal elements in $K$ [60]. In the present three-channel case, the charge channel does not couple to the spin chain, so to a first approximation, we may consider the electron spin-charge block and the spin-spin block as two distinct two-channel systems, and use the insight derived for the two-channel case [64] on each system separately. We emphasize that our results for $K_{ij}$ are obtained using the full interaction matrices $V_0$ and $V_\phi$. The spin-charge block describes a metal chain subject to a magnetic field, and the density and current inter-channel interactions neutralize each other. However, in the spin-spin block, since $V_0^{0\delta} = 0$, the density interaction in $V_\phi^{23}$ enhances $K_{22}$.
The (off-)diagonal elements of both $K$ are symmetric in $J$, and thus the same phase diagram. Considering a narrow interaction matrices by a multiplicative factor yields the same congruence relation $KV = VK$. Due to the congruence relation $K_{12} = K_{12}$, the increase of $K_{12}$, is plotted in Figure 2 (b), and exhibits small oscillations for $J < 0.5$, revealing that there is no large distinction between the two CDW decay rates. We also note that that $V > 0$ introduces a region without any divergent response functions for intermediate values of $J$, since most diagonal entries in both $K$ and $K^{-1}$ are larger than one in Figure 2 (b). Lastly, we again observe that the spin-polarized TS is dominant close to the unstable region, well inside the repulsive region of the phase diagram.

**Outlook.** Our results indicate that spin-polarized triplet correlations in a metal chain coupled to a spin chain persist despite including repulsive interactions between electrons. This suggests that fluctuations in the spin chain provide a mechanism for superconductivity, as has been observed in similar planar interfaces [31, 35]. Our findings correspond well with the results found when coupling a metal chain to acoustical phonons, as strong electron-phonon coupling may induce superconductivity [74], particularly near the unstable region of the phase diagram. Moreover, the spin fluctuations, unlike phonons, may change the spin structure of the electronic pairing since they couple to the spin channel of the electrons.

The system presented here is modeled using a TL description. There are several effects one could consider in future work, which would require an RG treatment. Among them are systematically accounting for Umklapp scattering, using a $SU(2)$ symmetric model, or removing the effective magnetic field by placing the metal chain between two spin chains. However, the emergent physics in our relatively simple, one-dimensional system still offers obvious parallels to magnon-mediated superconductivity in heterostructures of higher dimensions. Our main point is that we have demonstrated that spin-electron coupling provides a mechanism for driving superconducting instabilities even in non-Fermi liquids.

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**FIG. 2.** Elements of the Luttinger matrix, $K$, as a function of the interchain coupling $J$. The interaction strengths $U$ and $V$ are different in the two plots, while the parameters $v_{F}^{\parallel} = 0.5$ and $k_{F}a/\pi = 0.45$ are the same. Only the matrix relevant for the electron correlation function exponents are considered.
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S1. Calculation of Correlation Functions in Multi-Channel Luttinger Liquids

This Supplemental material is meant to elucidate the calculation of correlation functions for multi-channel Luttinger liquids using the Luttinger matrix formalism. Starting from the action describing a general $n$-channel Luttinger liquid

$$S[\phi, \theta] = \frac{1}{2\pi} \int \! dx \! d\tau \left( \phi^T \cdot \theta^T \right) \left[ \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} i \partial_x + \begin{pmatrix} V_\phi & 0 \\ 0 & V_\theta \end{pmatrix} \partial_x \right] \partial_x \begin{pmatrix} \phi \\ \theta \end{pmatrix},$$  \hfill (S1)

where the $n \times n$ interaction matrices $V_\phi$ and $V_\theta$ are not diagonal, we will demonstrate how to calculate correlation functions in the form

$$I = \langle \exp(i \sum_i A_i^T \phi(r_i) + B_i^T \theta(r_i)) \rangle.$$  \hfill (S2)

To serve as an example, consider the correlation function for the spin-polarized triplet state, $R_{TS}$. In terms of the bosonic fields, $R_{TS}(x, \tau = 0)$ is given by

$$R_{TS}(x) = \frac{1}{(2\pi \beta)^2} \langle e^{\sqrt{2}i(\theta_{\phi}(x) - \theta_{\phi}(0)) - \sqrt{2}i(\theta_{\phi}(x) - \theta_{\phi}(0))} \rangle.$$  \hfill (S3)

Calculating $R_{TS}$ is thus equivalent to calculating $I$ with spatial coordinates $x_1 = x$, $x_2 = 0$, $\tau_1 = \tau_2 = 0$, and coefficients $B_1^x = \sqrt{2}, B_1^\phi = -\sqrt{2}, B_2^x = -\sqrt{2}, B_2^\phi = \sqrt{2}$. The other coefficients in $B$ as well as all coefficients in $A$ are zero.

As described in the main text, one can find a matrix $M$ relating the original fields $\phi$ and $\theta$ to new fields $\tilde{\phi}$ and $\tilde{\theta}$ which diagonalize equation (S1). By employing the following transformation

$$\phi = M \tilde{\phi},$$  \hfill (S4a)

$$\theta = M^{-T} \tilde{\theta},$$  \hfill (S4b)

and inserting into equation (S1), we obtain

$$S[\tilde{\phi}, \tilde{\theta}] = \frac{1}{2\pi} \int \! dx \! d\tau \left( \tilde{\phi}^T \cdot \tilde{\theta}^T \right) \left[ \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} i \partial_x + \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \partial_x \right] \partial_x \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \end{pmatrix},$$  \hfill (S5)

where $u$ is a diagonal matrix.

We will now calculate $I$ using equations (S4) and (S5). The calculation follows the single-channel case in [52] closely, the main point is to show how one accounts for the basis change required when $V_\phi$ and $V_\theta$ are not diagonal. We consider the case where only $B$ contains nonzero values, and insert the aforementioned coefficients associated with $R_{TS}$ at the end of calculation. This specific configuration of $I$ is thus $I_0 = \langle e^{i \sum \sum B_i^\phi \theta(r_i)} \rangle$. The calculation of other correlation functions is analogous. The first step of the calculation is to rewrite the fields in their Fourier representation

$$\sum_i B_i^T \theta(r_i) = \frac{1}{2\beta \Omega} \sum_k \left( (B^*)^T k \theta(k) + \text{h.c.} \right),$$  \hfill (S6)

with $k$ as the shorthand notation for $k = (q, \omega_n)$ where $q$ is a wave vector and $\omega_n$ is a bosonic Matsubara frequency. $\Omega$ and $\beta$ denote the system size and inverse temperature, respectively. We also introduced

$$B(k) \equiv \sum_i e^{-i r_i k} B_i.$$  \hfill (S7)

Since $S$ is not diagonal in the original fields we must rewrite equation (S6) in terms of $\tilde{\theta}$ by using equation (S4b)

$$\frac{1}{2\beta \Omega} \sum_k \sum_i B_i^*(k) \theta_i(k) + B_i(k) \theta_i^*(k) = \frac{1}{2\beta \Omega} \sum_k \sum_i \left[ B_i^*(k)(k) \sum_m M_i^m - T \tilde{\theta}_m(k) + B_i(k) \sum_m M_i^m - T \tilde{\theta}_m^*(k) \right].$$  \hfill (S8)

Because $I_0$ is independent of $\phi$, one can integrate out the density fields in $S[\tilde{\phi}, \tilde{\theta}]$, akin to the textbook procedure used when considering the one-channel case. The resulting action is

$$S[\tilde{\theta}] = \frac{1}{2\pi} \int \! dx \! d\tau \sum_m \frac{1}{u_m} ((\partial_r \tilde{\theta}_m)^2 + (u_m \partial_x \tilde{\theta}_m)^2),$$  \hfill (S9)
where \( u_m \) is the \( m \)'th diagonal element in \( u \). Using the standard formula for integrating a multivariate Gaussian integral, one can integrate over the density sector by using equations (S8) and (S9)

\[
I_\theta = \frac{1}{Z_\theta} \int \mathcal{D}\theta \exp\left(-\frac{1}{2\beta} \sum_k \sum_m \left[ \frac{\hat{\theta}_m(k) \hat{\theta}_m^*(k)}{u_m} - i\pi \sum_l B^*(k) M_{lm} - \hat{\theta}_m^*(k) \sum_{l'} B^{l'}(k) M_{l'm}^\dagger \right] \right)
\]

\[
= \exp\left(-\frac{\pi}{2\beta} \sum_{l,l'} \sum_m M_{lm}^\dagger M_{l'm} \sum_{i,j} \sum_k B_i^l B_j^{l'} \frac{u_m}{\omega_n^2 + u_m^2 q^2} \right)
\]

\[
= \exp\left(-\frac{\pi}{2\beta} \sum_{l,l'} \sum_m M_{lm}^\dagger M_{l'm} \sum_{i,j} \sum_k B_i^l B_j^{l'} \frac{u_m}{\omega_n^2 + u_m^2 q^2} \right)
\]

where we reinserted equation (S7) in going from the second to third line. \( Z_\theta \) is the partition function for the system after integrating out the density fields. By employing the same trick as in [S2], adding and subtracting the same quantity in the exponent, we obtain

\[
I_\theta = \exp\left(-\frac{\pi}{2\beta} \sum_{l,l'} \sum_m M_{lm}^\dagger M_{l'm} \sum_{i,j} \sum_k B_i^l B_j^{l'} \frac{u_m}{\omega_n^2 + u_m^2 q^2} \right)
\]

\[
\exp\left(-\frac{\pi}{2\beta} \sum_{l,l'} \sum_m M_{lm}^\dagger M_{l'm} \sum_{i,j} \sum_k B_i^l B_j^{l'} \frac{u_m}{\omega_n^2 + u_m^2 q^2} \right).
\]

The last exponential can be further simplified

\[
\exp\left(-\frac{\pi}{2\beta} \sum_{l,l'} \sum_m M_{lm}^\dagger M_{l'm} \sum_{i,j} \sum_k B_i^l B_j^{l'} \frac{u_m}{\omega_n^2 + u_m^2 q^2} \right) = \exp\left(-\frac{\pi}{2\beta} \sum_m \left( \sum_{l} M_{lm}^\dagger B_l^i \right)^2 \sum_k \frac{u_m}{\omega_n^2 + u_m^2 q^2} \right).
\]

Unless the quantity \( \sum_{l} M_{lm}^\dagger B_l^i \) is zero, the exponent diverges, thus exponentially suppressing \( I_\theta \). This gives rise to the criterion stated in the main text, namely that \( \sum_{l} M_{lm}^\dagger B_l^i = 0 \). Assuming that the criterion is fulfilled, \( I_\theta \) the Matsubara sum and the wave vector integral is calculated following [S2] in the zero-temperature limit, and we finally arrive at the general expression

\[
I_\theta = \prod_m \frac{\alpha^2}{(x_i - x_j)^2 + (u_m |r_i - r_j| + \alpha)^2} \left( \frac{-4 \sum_{l,l'} \sum_{i,j} M_{lm}^\dagger M_{ml'}^\dagger A_i^l A_j^{l'}}{\alpha^2 \sum_{i,j} M_{ml}^\dagger M_{ml'}^\dagger A_i^l A_j^{l'}} \right).
\]

By inserting the coefficients used in \( R_{TS^+} \), we obtain

\[
R_{TS^+}(x) = \frac{1}{(2\pi \alpha)^2} \left( \frac{1}{x^2 + K_{11}^2 + K_{22}^2 + K_{12}^2} \right),
\]

as stated in the main text. Computations of the remaining correlation functions can be carried out in a similar manner.