A LANDSCAPE OF PEAKS: THE INTERMITTENCY ISLANDS OF THE STOCHASTIC HEAT EQUATION WITH LÉVY NOISE

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We show that the spatial profile of the solution to the stochastic heat equation features multiple layers of intermittency islands if the driving noise is non-Gaussian. On the one hand, as expected, if the noise is sufficiently heavy-tailed, the largest peaks of the solution will be taller under multiplicative than under additive noise. On the other hand, surprisingly, as soon as the noise has a finite moment of order $\frac{d}{2}$, where $d$ is the spatial dimension, the largest peaks will be of the same order for both additive and multiplicative noise, which is in sharp contrast to the behavior of the solution under Gaussian noise. However, in this case, a closer inspection reveals a second layer of peaks, beneath the largest peaks, that is exclusive to multiplicative noise and that can be observed by sampling the solution on the lattice. Finally, we compute the macroscopic Hausdorff and Minkowski dimensions of the intermittency islands of the solution. Under both additive and multiplicative noise, if it is not too heavy-tailed, the largest peaks will be self-similar in terms of their large-scale multifractal behavior. But under multiplicative noise, this type of self-similarity is not present in the peaks observed on the lattice.

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1. Introduction. Consider the stochastic heat equation (SHE)

$$
\partial_t Y(t, x) = \frac{1}{2} \Delta Y(t, x) + \sigma(Y(t, x)) \dot{\Lambda}(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d,
$$

(1.1)
driven by a space-time white noise $\dot{\Lambda}$, where either $\sigma(x) = 1$ and $Y(0, x) = 0$ (the case of additive noise) or $\sigma(x) = x$ and $Y(0, x) = 1$ (the case of multiplicative noise). In this work,
we fix $t > 0$ and explore the macroscopic behavior of $Y$ as $|x| \to \infty$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. If $d = 1$ and $\dot{\Lambda}$ is Gaussian, it is well known that for fixed $t > 0$,

$$
\begin{aligned}
\limsup_{|x| \to \infty} \frac{Y(t, x)}{(\log |x|)^{1/2}} &= \left(\frac{4t}{\pi}\right)^{1/4} \quad \text{if } \sigma(x) = 1, \\
\limsup_{|x| \to \infty} \frac{\log Y(t, x)}{(\log |x|)^{2/3}} &= \left(\frac{9t}{32}\right)^{1/8} \quad \text{if } \sigma(x) = x
\end{aligned}
$$

(1.2)

almost surely; see [15, 25, 32]. If $\dot{\Lambda}$ follows a non-Gaussian distribution, in which case $\dot{\Lambda}$ is called a Lévy noise, the results we obtain are rather unexpected. To give a flavor of them, let us suppose in this introductory part that $\dot{\Lambda}$ is a Lévy noise with Lévy measure

$$
\lambda((\infty, 1]) = 0, \quad \lambda((z, \infty)) = z^{-\alpha}, \quad z > 1,
$$

(1.3)

for some $\alpha > 0$. In this case, $\dot{\Lambda}$ is a compound Poisson noise with Pareto-distributed weights. If $\alpha$ is small, the noise is relatively heavy-tailed; if $\alpha$ is large, the noise is relatively light-tailed. In the latter case, the analogous result to (1.2) reads as follows.

**Theorem A.** Suppose that $\alpha > \frac{2}{d}$. If $f : (0, \infty) \to (0, \infty)$ is nondecreasing, then for both additive and multiplicative noise almost surely,

$$
\limsup_{x \to \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{or} \quad \limsup_{x \to \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} = 0,
$$

(1.4)

according to whether the integral

$$
\int_1^\infty x^{d-1} f(x)^{-\frac{2}{d}} \, dx
$$

(1.5)

diverges or converges.

An obvious difference to (1.2) is the fact that the spatial asymptotics of the solution are governed by an integral test. But this is not the most surprising part about Theorem A; a similar integral test has been found in [21] for the behavior of $Y$’s with $\sigma(x) = 1$ as $t \to \infty$. What is most striking in view of (1.2) is that the largest peaks of the solution at a given time $t$ are of the same order for both multiplicative and additive Lévy noise! It has been shown in [9] that the solution to the SHE with multiplicative Lévy noise is always intermittent in all dimensions $d \geq 1$, regardless of the details of $\dot{\Lambda}$. While intermittency is an asymptotic concept that describes localization on an exponential scale as $t \to \infty$, it is widely believed that the largest peaks of an intermittent process at finite times should already exceed those of a non-intermittent process (e.g., the solution to (1.1) with additive noise). Theorem A shows that this belief is incorrect in general.

This being said, if $\dot{\Lambda}$ is sufficiently heavy-tailed, multiplicative noise does produce higher peaks, even at finite times.

**Theorem B.** Suppose that $\alpha < \frac{2}{d}$ and let $\theta_{\alpha} = 1 - \frac{d}{2}(\alpha - 1)$.

(i) In the case of additive noise, we have the two possibilities in (1.4) depending on whether the following integral diverges or converges:

$$
\int_1^\infty x^{d-1} f(x)^{-\alpha} \, dx.
$$

(1.6)
(ii) In the case of multiplicative noise, there are $0 < L_s \leq L* < \infty$ such that for all $L > L*$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = 0 \quad \text{a.s.,}
\]

while for all $L < L_s$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{a.s.}
\]

So the gain in the multiplicative case is a factor roughly of order $e^{L (\log x)^{\gamma/2}}$. It follows from the previous two theorems that the highest peaks in the solution to the SHE are taller for multiplicative than for additive noise and only if $\Lambda$ has very heavy tails. If $\alpha > \frac{2}{d}$, Theorem A seems to suggest that (1.1) is subjected to multiplicative and additive noise cannot be distinguished based on the macroscopic behavior of the solution at a given time point. But this turns out to be false, too: in fact, there is a second layer of peaks, beneath the largest peaks studied in Theorem A, that is exclusive to the solution under multiplicative noise. This second layer of peaks can be observed, for example, by sampling on the discrete lattice $\mathbb{Z}^d$ instead of $\mathbb{R}^d$.

**Theorem C.** Consider $Y$ on the lattice $\mathbb{Z}^d$.

(i) If $\alpha > \frac{2}{d}$ and $\sigma(x) = 1$, then almost surely, for any nondecreasing $f : (0, \infty) \to (0, \infty)$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{or} \quad \limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = 0,
\]

according to whether the integral (1.5) diverges or converges.

(ii) If $\alpha \in \left(\frac{2}{d}, 1 + \frac{2}{d}\right)$ and $\sigma(x) = x$, then there are $0 < M_s \leq M* < \infty$ such that for all $M > M*$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = 0 \quad \text{a.s.,}
\]

while for $M < M_s$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{a.s.}
\]

(iii) If $\alpha \geq 1 + \frac{2}{d}$ and $\sigma(x) = x$, then there are $0 < M_s \leq M* < \infty$ such that for $M > M*$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = 0 \quad \text{a.s.}
\]

while for $M < M_s$,

\[
\limsup_{x \to \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{a.s.}
\]

(iv) If $\alpha < \frac{2}{d}$, then the statements of Theorem B remain valid for $\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)$.

The behavior described in part (iii) of Theorem C is particularly interesting. We do not know of any other natural model with this type of growth asymptotics.
1.1. Review of literature. Let us put Theorems A–C in the context of the existing literature. Until a few years ago, the majority of works on the SHE driven by non-Gaussian Lévy noise focused on existence, uniqueness and regularity of solutions, usually assuming strong moment assumptions on the noise or studying specific noises only (e.g., $\alpha$-stable noise); see [4, 17, 18, 19, 36, 40, 41, 43, 46]. More recently, building on [11], the paper [9] derived the most general existence and uniqueness conditions known up to date for the SHE with multiplicative Lévy noise (which are necessary and sufficient for $d = 1, 2$ and almost optimal for $d \geq 3$). Furthermore, extending [20], it was further shown in [9] that the solution to the SHE with multiplicative Lévy noise is strongly intermittent in all dimensions for all non-trivial Lévy noises. (This is another unexpected feature of (1.1), because if $d \geq 3$, intermittency does not always occur if one considers the SHE on a lattice [1, 2, 14] or the SHE with short-range correlated Gaussian noise [16, 37].) For the SHE with additive Lévy noise, the authors showed in [21] that as $t \to \infty$, $Y(t, x)$ for fixed $x$ satisfies a weak but violates a strong law of large numbers, a property referred to as additive intermittency. Finally, by showing that directed polymers in heavy-tailed environments have the SHE with Lévy noise as a scaling limit in the intermediate disorder regime, [10] established a first discrete statistical mechanics model that rescales to a Lévy-driven SHE in continuous space and time (the analogous result for convergence to the SHE with Gaussian noise was shown in [3]). For results on the stochastic wave equation with Lévy noise, we refer to [5, 6].

1.2. Overview of the remaining paper. After a rigorous introduction to the SHE with Lévy noise in Section 2, we state and prove in Section 3 tight upper and lower bounds on the probability tails of the solution $Y(t, x)$ and of its local spatial supremum $\sup_{x \in Q} Y(t, x)$, where $Q \in Q$ and $Q = \{x + (0, 1)^d : x \in \mathbb{R}^d\}$ is the collection of all unit cubes in $\mathbb{R}^d$. Theorems 3.1 and 3.5 cover the results when the tail of the noise is heaviest, while Theorems 3.2 and 3.8 contain the statement when the tail is lighter. These tail bounds are the main technical achievements of the paper; we will give more background on the approach we take to prove them in Section 3. In Section 4, we will then use these tail bounds to prove Theorems 4.1–4.3, which extend Theorems A–C to general Lévy noises. In Section 5, we further show how the tail bounds of Section 3 can be used to determine/bound the macroscopic Hausdorff and Minkowski dimensions of the peaks of $Y$. For the SHE with Gaussian noise, this program has been carried out in [32]. In the Lévy setting, multiple scales appear: in the case of additive noise, or in the case of multiplicative noise if the noise is not too heavy-tailed, we show in Theorems 5.1 and 5.2 that the largest peaks of the solution are not only multifractal in the sense of [32] but actually self-similar in terms of their multifractal behavior. At the same time, in the multiplicative case, the largest peaks under a very heavy-tailed noise or the peaks observed on the lattice $\mathbb{Z}^d$ for any Lévy noise are multifractal but not self-similar. This is a consequence of Theorems 5.1 and 5.3. Finally, the Appendix contains some technical results needed in the proofs.

Except for Section 5 where we treat both additive and multiplicative noise, we only consider and prove results for the case of multiplicative Lévy noise in Sections 3 and 4. In the case of additive noise, we can obtain exact tail asymptotics using the theory of regular variation, which is why we have deferred them to a companion paper [22]. In particular, the parts of Theorem A–C concerning additive noise also follow from [22].

In what follows, we use $C, C_1, C_2$ etc. to denote constants which do not depend on any important parameters and whose values may change from line to line. Furthermore, if we plug in a real number $x$ for an integer-valued index (e.g., $\sum_{i=1}^{x}$ or $Y^{(x)}$ if $Y^{(n)}$ is a sequence indexed by $n \in \mathbb{N}$), we always mean plugging in $\lfloor x \rfloor$, the integer-part of $x$. 
2. Preliminaries. Throughout this paper, we assume that $\hat{\Lambda}$ is a space-time white noise on $\mathbb{R}^{1+d}$, that is, a stationary random generalized function that gives independent values when applied to test functions of disjoint support. It is well known (see [26, Ch. 4.4] and [44]) that $\hat{\Lambda}$ is infinitely divisible in this case with Lévy–Itô decomposition

$$\Lambda(dt, dx) = b dt dx + v W(dt, dx) + \int_{(-1,1)} z (\mu - \nu)(dt, dx, dz)$$

$$+ \int_{(-1,1)^c} z \mu(dt, dx, dz),$$

(2.1)

where $b \in \mathbb{R}$, $v \in [0, \infty)$, $W$ is a Gaussian space-time white noise, and $\mu$ is a Poisson random measure on $\mathbb{R}^{1+d}$ with intensity measure $\nu = dt \otimes dx \otimes \lambda(dz)$, where $\lambda$, the Lévy measure of $\hat{\Lambda}$, satisfies $\int_{\mathbb{R}} (1 + z^2) \lambda(dz) < \infty$. The last two terms in (2.1) will be denoted by $\Lambda_<(dt, dx)$ and $\Lambda_>(dt, dx)$, respectively. In this paper, we assume

$$b = 0 \quad \text{and} \quad v = 0.$$  

(2.2)

The first assumption is no restriction, because $b \neq 0$ would only change the solution $Y$ to (1.1) by an additive or multiplicative constant, depending on whether $\sigma(x) = 1$ or $\sigma(x) = x$ (cf. [20, Sect. 3.3]). Regarding the second assumption, note that (1.1) has a mild solution for $v \neq 0$ only if $d = 1$, in which case (1.2) suggests—and one can modify the proofs in this paper to show this rigorously—that the macroscopic behavior of $x \mapsto Y(t, x)$ for fixed $t$ is dominated by the jump part. In addition to (2.2), we further assume that $\hat{\Lambda}$ is spectrally positive, that is,

$$\lambda((-\infty, 0)) = 0.$$  

(2.3)

The condition (2.3) is needed to guarantee positivity of the solution $Y$ in the case of multiplicative noise [9, Thm. 2.1], which is crucial for the lower bound proofs in this paper. In principle, all upper bound results remain valid if we consider signed noise, but we refrain from adding this extra bit of generality to keep the exposition simple.

From now on until the end of Section 4, we only consider the case of multiplicative noise, that is, we will assume

$$\sigma(x) = x.$$  

(2.4)

In this case, a predictable process $Y(t, x)$ is called a mild solution to (1.1) if for all $(t, x) \in (0, \infty) \times \mathbb{R}^d$,

$$Y(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) Y(s, y) \Lambda(ds, dy) \quad \text{a.s.,}$$

(2.5)

where $g(t, x) = (2\pi t)^{-d/2} e^{-|x|^2/(2t)} 1_{(t > 0)}$ is the heat kernel in dimension $d$. In Section 5, where we consider both additive and multiplicative noise again, we will use the notation

$$Y_+(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) \Lambda(ds, dy)$$

(2.6)

for the solution to (1.1) with additive Lévy noise.

Let us introduce the following truncated moments of the Lévy measure $\lambda$:

$$\mu_p(\lambda) = \int_{(0, \infty)} z^p \lambda(dz), \quad m_p(\lambda) = \int_{(0, 1)} z^p \lambda(dz), \quad M_p(\lambda) = \int_{[1, \infty)} z^p \lambda(dz)$$

and

$$m_p^{\log}(\lambda) = \int_{(0, 1)} z^p |\log z| \lambda(dz), \quad m_p^{\log}(\lambda) = \int_{(0, 1)} z^p |\log z|^p 1_{(p=1)} \lambda(dz).$$

(2.7)
Under the assumption
\[
\int_{[1,\infty)} (\log z)^{\frac{d}{2}} \lambda(dz) + m_{1+2/d}(\lambda) \mathbb{1}_{\{d \geq 2\}} < \infty,
\]
it was shown in [9] (see Thm. 2.5, Rem. 2.6 and the discussion in Sect. 3.3) that
\[
(2.10) \quad u(s, y; t, x) = g(t - s, x - y) + \sum_{N=1}^{\infty} \int_{(s,t) \times \mathbb{R}^d} \prod_{i=1}^{N+1} g(\Delta t_i, \Delta x_i) \prod_{j=1}^{N} \Lambda(dt_j, dx_j)
\]
is well defined and finite, where \(\Delta t_i = t_i - t_{i-1}, \Delta x_i = x_i - x_{i-1}, (t_{N+1}, x_{N+1}) = (t, x)\) and \((t_0, x_0) = (s, y)\). Furthermore, \(u(s, y; \cdot, \cdot)\) is a mild solution to (1.1) on \((s, \infty) \times \mathbb{R}^d\) with \(\sigma(x) = x\) and initial condition \(u(s, y; s, \cdot) = \delta_y\). In what follows, we write \(Y_<(t, x)\) and \(u_<(s, y; t, x)\) for the process obtained by substituting \(\Lambda_<\) for \(\Lambda\) in (2.5) and (2.10), respectively. We let \(u_<(s, y; t, x) = 0\) whenever \(s \geq t\) and \(Y_<(t, x) = 0\) whenever \(t < 0\). Then, similarly to [9, Eq. (8.4)], a mild solution to (1.1) is given by
\[
(2.11) \quad Y(t, x) = \sum_{N=0}^{\infty} \int_{(0,t) \times \mathbb{R}^d} Y_<(t_1, x_1) \prod_{i=2}^{N+1} u_<(t_{i-1}, x_{i-1}; t_i, x_i) \prod_{j=1}^{N} \Lambda_{\geq}(dt_j, dx_j),
\]
where the term for \(N = 0\) is \(Y_<(t, x)\). By the independence properties of \(\Lambda\), for any fixed \(t_1 < \cdots < t_N\), we have that \(Y_<(t_1, x_1), u_<(t_1; t_2, \cdot), \ldots, u_<(t_N; t, \cdot)\) are independent of each other and also independent of \(\Lambda_{\geq}\). Note that (2.9) is necessary and sufficient for the existence of solutions to (1.1) in dimensions \(d = 1, 2\) and close to optimal in dimensions \(d \geq 3\) [9].

3. Tail bounds on the solution and its local supremum. The main device to obtain Theorems A–C (and their generalizations) are sharp probability tail bounds on \(Y(t, x)\) and \(\sup_{t \in Q} Y(t, x)\), where \(Q\) is a unit cube in \(\mathbb{R}^d\). In all results, we need to distinguish between a heavy-tailed and a light-tailed scenario, which motivates the following definitions depending on a parameter \(\alpha\):

**Condition (H-\(\alpha\)).** We have (2.2) and (2.3). Moreover, we have \(m_{1+2/d}(\lambda) < \infty\) and \(\lambda([R, \infty)) \sim CR^{-\alpha}\) for some \(C \in (0, \infty)\) as \(R \to \infty\).

**Condition (L-\(\alpha\)).** We have (2.2) and (2.3). Moreover, we have \(0 < m_{1+2/d}(\lambda) + M_{\alpha}(\lambda) < \infty\).

Note that the notion of heavy- versus light-tailed is relative (and \(\alpha\)-dependent). In particular, Condition (L-\(\alpha\)) really only means that \(\Lambda\) has a finite moment of order \(\alpha\) (in which case \(\Lambda\) may still be heavy-tailed in the classical sense). In the following, we are going to prove tail bounds for two different processes (the solution and its local supremum), for each of which there will be a heavy-tailed case (Theorems 3.1 and 3.5) and a light-tailed case (Theorems 3.2 and 3.8). Each result in turn will involve an upper and a lower bound. Let us provide a short overview of the proof techniques:

- All upper bounds, except for the tail of the local supremum of \(Y\) in the light-tailed case (Theorem 3.8), are obtained by combining Markov’s inequality with sharp moment estimates and then optimizing the exponent.
- The upper bound in Theorem 3.8 cannot be obtained in this way. Instead, we first show that only “large close” jumps (in a certain sense) contribute to the tail and then use the explicit Poisson structure of the atoms to bound their tail behavior. For this part, we also use a decoupling inequality for tail probabilities (Lemma 3.7) that is of independent interest.
For the lower bounds, the level of difficulty is reversed: for the supremum in the light-tailed case (Theorem 3.8), a single (well-chosen) jump suffices to produce the tail.

In all other cases, the main strategy is to find chains of close atoms of beneficial length $N$. An optimal number $N$ has to be sufficiently large (to be able to produce a tall peak) but at the same time not too large (such that the probability of having a chain of that length is not too small). It turns out that in the heavy-tailed case, for both the solution (Theorem 3.1) and the supremum (Theorem 3.5), one needs to consider a whole range of lengths $N$, while for the solution in the light-tailed case (Theorem 3.2), considering a single length $N$ (depending on the size of the desirable peak, of course) is enough. An important observation is that for these lower bound proofs, it is crucial that we consider (1.1) on an unbounded domain. The chains of atoms that lead to a tail event have to stretch arbitrarily far into space; on a bounded domain, the tail asymptotics of the solution would be different; see Remark 3.4.

3.1. Tail bounds for the solution. Let us begin with heavy-tailed noise.

**Theorem 3.1.** Assume Condition (H-$\alpha$) for some $\alpha \in (0, 1 + \frac{2}{d})$. For every $t > 0$, there are constants $C_1, C_2 \in (0, \infty)$ such that for all $x \in \mathbb{R}^d$ and $R > 1$,

$$C_1 R^{-\alpha} e^{C_1 (\log R)^{1/(1+\theta\alpha)}} \leq \mathbb{P}(Y(t, x) > R) \leq C_2 R^{-\alpha} e^{C_2 (\log R)^{1/(1+\theta\alpha)}}.$$  

**Proof.** Step 1: Upper bound

Let $\mathbb{E}_<$ and $\mathbb{E}_\geq$ denote conditional expectation given $\Lambda_>$ and $\Lambda_<$, respectively. First suppose that $\alpha \in (0, 1]$ and let $p \in (0, \alpha)$. Because $\Lambda_>$ is a discrete measure, we can use the elementary inequality $|\sum a_i|^p \leq \sum |a_i|^p$, (2.11), and the fact that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}_>[X]]$ to obtain

$$\mathbb{E}[Y(t, x)^p] \leq \mathbb{E} \left[ \mathbb{E}_<(t, x)^p \right] + \mathbb{E} \left[ \sum_{N=1}^{\infty} \int_{(0, t) \times \mathbb{R}^d \times [1, \infty)} Y_<(t_1, x_1)^p \prod_{i=2}^{N+1} u_{<}(t_{i-1}, x_{i-1}; t_i, x_i)^p \prod_{j=1}^{N} \int_{z_j} \lambda(\mathrm{d}z_j) \right].$$

$$= \mathbb{E} \left[ \mathbb{E}_<(t, x)^p \right] + \sum_{N=1}^{\infty} M_p(\lambda)^N \int_{(0, t) \times \mathbb{R}^d} \mathbb{E} \left[ Y_<(t_1, x_1)^p \right] \prod_{i=2}^{N+1} \mathbb{E} \left[ u_{<}(t_{i-1}, x_{i-1}; t_i, x_i)^p \right] \prod_{j=1}^{N} \int_{z_j} \lambda(\mathrm{d}z_j),$$

where $M_p(\lambda) = \int_{[1, \infty)} z^p \lambda(\mathrm{d}z)$. By Jensen’s inequality,

$$\mathbb{E}[Y_<(t, x)^p] \leq \mathbb{E}[Y_<(t, x)]^p = 1,$$

$$\mathbb{E}[u_{<}(t_{i-1}, x_{i-1}; t_i, x_i)^p] \leq \mathbb{E}[u_{<}(t_{i-1}, x_{i-1}; t_i, x_i)]^p = g(\Delta t_i, \Delta x_i)^p.$$
Thus, recalling that \( \theta_p = 1 - (p - 1) \frac{d}{2} \), we have
\[
E[Y(t, x)^p] \leq 1 + \sum_{N=1}^{\infty} M_p(\lambda)^N \int_{((0, t) \times \mathbb{R}^d)^N} \prod_{i=2}^{N+1} g(\Delta t_i, \Delta x_i)^p \prod_{j=1}^{N} dt_j \, dx_j
\]
(3.3)
\[
= 1 + \sum_{N=1}^{\infty} (CM_p(\lambda))^N \int_{(0, t)^N} \prod_{i=2}^{N+1} (\Delta t_i)^{-(p-1) \frac{d}{2}} \prod_{j=1}^{N} dt_j
\]
\[
= \sum_{N=0}^{\infty} \frac{(CM_p(\lambda)\Gamma(\theta_p))^{N}}{\Gamma(1 + \theta_p N)}.
\]

The first equality follows by noting that \( g \) is a Gaussian density, while the second equality follows from [18, Lemma 3.5]. By Lemma A.1 and the fact that \( 0 < \theta_p \leq 1 + \frac{d}{2} \), we conclude that
\[
E[Y(t, x)^p] \leq Ce^{(CM_p(\lambda)\Gamma(\theta_p))^{1/\theta_p} t}
\]
for some constant that does not depend on \( p \).

The following tail bound is now an immediate consequence of Markov's inequality:
\[
P(Y(t, x) > R) \leq CR^{-p} e^{(CM_p(\lambda)\Gamma(\theta_p))^{1/\theta_p} t} \leq CR^{-p} e^{(CM_p(\lambda)\Gamma(\theta_p))^{1/\theta_p} t}
\]
for all \( p \in (0, \alpha) \). Under the tail assumption on \( \lambda \), we have that \( M_p(\lambda) \sim C(\alpha - p)^{-1} \). Inserting this expression into the previous line and choosing \( p = \alpha - (\theta_p \log R)^{-1/(1 + \theta_p)} \), we obtain that
\[
P(Y(t, x) > R) \leq CR^{-\alpha} e^{C\alpha, (\log R)^{1 - 1/(1 + \theta_p)}} e^{C, (\log R)^{1/(\theta_p + 1)}} = CR^{-\alpha} e^{C\alpha, (\log R)^{1/(1 + \theta_p)}}
\]
which completes the proof if \( \alpha \in (0, 1] \).

If \( \alpha \in (1, 1 + \frac{2}{d}] \), note that \( Y(t, x) = e^{mt} \overline{Y}(t, x) \) where \( m = \int_{(-1, 1)^d} z \lambda(\mathrm{d}z) \) is the mean of \( \Lambda \) and \( \overline{Y}(t, x) \) is the solution to (2.5) when \( \Lambda \) is replaced by \( \overline{\Lambda} = \Lambda - m \text{Leb} \). Similarly to (2.11), and with obvious notation, we have that
\[
\overline{Y}(t, x) = \sum_{N=0}^{\infty} \int_{((0, t) \times \mathbb{R}^d)^N} \overline{Y}_{<(t_1, x_1)} \prod_{i=2}^{N+1} \overline{\pi}_{<(t_{i-1}, x_{i-1}; t_i, x_i)} \prod_{j=1}^{N} \overline{\Lambda}_{\geq}(\mathrm{d}t_j, \mathrm{d}x_j),
\]
where the zeroth-order term in \( \overline{Y}_{<(t, x)} \). Thus, in dimensions \( d \geq 2 \), using the Burkholder--Davis--Gundy (BDG) inequality, we have for all \( p \in (1, \alpha) \) that
\[
E[\overline{Y}(t, x)^p]^\frac{1}{p} \leq \sum_{N=0}^{\infty} (CM_p(\lambda)^N)^N
\]
(3.6)
\[
\times \left( \int_{((0, t) \times \mathbb{R}^d)^N} E[\overline{Y}_{<(t_1, x_1)}^p] \prod_{i=2}^{N+1} E[\overline{\pi}_{<(t_{i-1}, x_{i-1}; t_i, x_i)}^p] \prod_{j=1}^{N} \mathrm{d}t_j \, \mathrm{d}x_j \right)^\frac{1}{p},
\]
where \( C \in (0, \infty) \) is a constant that can be chosen uniformly for all \( p \) close enough to \( \alpha \) \( (C \) may depend on \( \alpha \). By [9, Cor. 6.5] (combined with Minkowski's integral inequality together
with (1.17) in [9]) and its proof as well as Lemma A.1, we have that

\[ E[\mathcal{I}_{(t_i-1, x_{i-1}; t_i, x_i)}^p] \leq C \Gamma(\theta_p) \sum_{k=0}^{\infty} \left( \frac{C \Gamma(\frac{\theta_p}{3})}{\Gamma(\frac{\theta_p}{3} + k + \theta_p)} \right) g(\Delta t_i, \Delta x_i) \]

(3.7)

\[ E[\mathcal{Y}_{(t, x)}^p] \leq C \Gamma(\theta_p) \sum_{k=0}^{\infty} \left( \frac{C \Gamma(\frac{\theta_p}{3})}{\Gamma(\frac{\theta_p}{3} + k + \theta_p)} \right) \leq C \Gamma(\theta_p) e^{C t^\frac{\theta_p}{3}/\theta_p} g(\Delta t_i, \Delta x_i), \]

As \( p < \alpha < 1 + \frac{2}{d} \), \( \theta_p \) is bounded away from 0 and by [9, Cor. 6.5] we simply obtain

\[ E[\mathcal{I}_{(t_i-1, x_{i-1}; t_i, x_i)}^p] \leq C g(\Delta t_i, \Delta x_i)^p \quad \text{and} \quad E[\mathcal{Y}_{(t, x)}^p] \leq C. \]

Inserting this into (3.6), we deduce the bound

\[ E[\mathcal{Y}_{(t, x)}^p] \leq \sum_{N=0}^{\infty} (CM_p(\lambda) \Gamma(\frac{\theta_p}{3}))^N \int_0^t \int_\Omega \left( \int_\Omega g(\Delta t_i, \Delta x_i)^p dx \right) dt dy \]

(3.8)

Comparing with the estimate in (3.3), we can conclude by a similar argument.

Finally, if \( d = 1 \) and \( \alpha \in (1, 2] \), one only needs to replace the bound in (3.7) by

\[ E[\mathcal{I}_{(t_i-1, x_{i-1}; t_i, x_i)}^p] \leq C \Gamma(\theta_p) \sum_{k=0}^{\infty} \left( \frac{C \Gamma(\frac{\theta_p}{3})}{\Gamma(\frac{\theta_p}{3} + k + \theta_p)} \right) g(\Delta t_i, \Delta x_i) \leq C g(\Delta t_i, \Delta x_i), \]

(3.9)

\[ E[\mathcal{Y}_{(t, x)}^p] \leq C \Gamma(\theta_p) \sum_{k=0}^{\infty} \left( \frac{C \Gamma(\frac{\theta_p}{3})}{\Gamma(\frac{\theta_p}{3} + k + \theta_p)} \right) \leq C, \]

which also follow from the proof of [9, Cor. 6.5]. If \( d = 1, \alpha \in (2, 3) \) and \( p \in (2, \alpha) \), the bounds in [10, Prop. 6.1] do not yield optimal tail estimates, which is why we need to use a different approach. Since \( E[\mathcal{Y}_{(t, x)}^p] = E[\mathcal{Y}_{(t, 0)}^p] \) for all \( x \in \mathbb{R} \) by stationarity, we can use [39, Thm. 1] (with \( \alpha = 2 \)) and Minkowski’s integral inequality to show that

\[ E[\mathcal{Y}_{(t, 0)}^p] \leq C \left( 1 + \mu_2(\lambda) \int_0^t \int_\Omega g(t-s, x-y)^2 E[\mathcal{Y}(s, y)^{\frac{2}{p}}] ds dy \right)^{\frac{1}{2}} \]

\[ + \left( \mu_2(\lambda) \int_0^t \int_\Omega g(t-s, x-y)^p E[\mathcal{Y}(s, y)^p] ds dy \right)^{\frac{1}{p}} \]

\[ \leq C \left( 1 + \mu_2(\lambda) \int_0^t (t-s)^{-\frac{\alpha}{2}} E[\mathcal{Y}(s, 0)^p] ds \right)^{\frac{1}{2}} \]

\[ + \left( \mu_2(\lambda) \int_0^t (t-s)^{-\frac{\alpha-1}{2}} E[\mathcal{Y}(s, 0)^p] ds \right)^{\frac{1}{p}}. \]

We can absorb \( \mu_2(\lambda) \) into the constant \( C \). Moreover, by Hölder’s inequality (with respect to the measure \( (t-s)^{-1/2} ds \)),

\[ \left( \int_0^t (t-s)^{-\frac{\alpha}{2}} E[\mathcal{Y}(s, 0)^p] ds \right)^{\frac{1}{2}} \leq \left( \int_0^t (t-s)^{-\frac{\alpha}{2}} E[\mathcal{Y}(s, 0)^p] ds \right)^{\frac{1}{p}} \left( \int_0^t s^{-\frac{1}{2}} ds \right)^{\frac{1}{2}(1-\frac{2}{p})}. \]
Since \( s^{-\frac{1}{2}} \leq C s^{-(p-1)/2} \) for \( s \in (0, t] \), it follows that
\[
\mathbb{E}[\text{Y}(t, 0)^p] \leq C \left( 1 + \mu_p(\lambda) \int_0^t (t - s)^{-\frac{p-1}{2}} \mathbb{E}[\text{Y}(s, 0)^p] \, ds \right)
\]
for some constant \( C \) that may depend on \( t \). Iterating this inequality and arguing as in (3.3) and (3.4), we arrive at
\[
\mathbb{E}[\text{Y}(t, 0)^p] \leq C \sum_{N=0}^{\infty} \left( C \mu_p(\lambda) \right)^N \int_{(0, t)^N} \prod_{i=2}^{N+1} (\Delta t_i)^{-\frac{p-1}{2}} \prod_{j=1}^N dt_j
\]
(3.10)
\[
\leq C e^{(C \mu_p(\lambda) \Gamma(\theta_p))^{1/p} t}.
\]
The proof can now be completed as in the paragraph following (3.4).

**Step 2: Lower bound**

At this part it is convenient to treat the time on \((-\infty, \infty)\). Let \((\tau_0, \eta_0) = (t, x)\) and
\[
\tau_i = \sup\{ u \in (0, \tau_{i-1}) : \Lambda \geq (\{ (s, y) \in [u, \tau_{i-1}) \times \mathbb{R}^d : |\eta_{i-1} - y| \leq \sqrt{(\tau_{i-1} - s)} \}) = 1 \}
\]
for \( i \in \mathbb{N} \), where \( \eta_i \) is the spatial coordinate of the atom associated to \( \tau_i \). We denote the associated jump size by \( \zeta_i \). Note that the numbering is reversed here, since we trace atoms backwards in time, starting at \((t, x)\). Clearly, if we write \( \Delta \tau_i = \tau_{i-1} - \tau_i \) and \( \Delta \eta_i = \eta_{i-1} - \eta_i \), the events
\[
A_N = \bigcap_{i=1}^N \{ \Delta \tau_i \leq \frac{t}{N} \} \cap \{ \Delta \tau_{N+1} > \frac{t}{N} \}, \quad N \in \mathbb{N},
\]
are pairwise disjoint. Moreover, since \( u_\prec \) is nonnegative in (2.11) (see [9, Thm. 2.1]), we have
\[
Y(t, x) \geq \int_{((0, t) \times \mathbb{R}^d)^N} Y_<(t_1, x_1) \prod_{i=2}^{N+1} u_<(t_{i-1}, x_{i-1}; t_i, x_i) \prod_{j=1}^N \Lambda_>(dt_j, dx_j)
\]
\[
\geq Y_<(\tau_N, \eta_N) \prod_{i=1}^N \Lambda_>(\tau_{i-1}, \eta_{i-1}) \zeta_i
\]
on the event \( A_N \). Therefore,
\[
\mathbb{P}(Y(t, x) > R) \geq \sum_{N=1}^{\infty} \mathbb{P} \left( A_N \cap \left\{ Y_<(\tau_N, \eta_N) \prod_{i=1}^N \Lambda_>(\tau_{i-1}, \eta_{i-1}) \zeta_i > R \right\} \right)
\]
(3.11)
\[
= \sum_{N=1}^{\infty} \mathbb{P}(A_N) \mathbb{P} \left( Y_<(\tau_N, \eta_N) \prod_{i=1}^N \Lambda_>(\tau_{i-1}, \eta_{i-1}) \zeta_i > R \bigg| A_N \right).
\]
As \( \Lambda_\geq \) is a Poisson random measure, \((\Delta \tau_i)_{i \in \mathbb{N}}\) is a sequence of independent and identically distributed variables with distribution function \( 1 - e^{-C x^{1+4/2}} \), where \( C = \pi^{d/2}/\Gamma(\frac{d}{2} + 2) \). Thus,
\[
\mathbb{P}(A_N) = e^{-C t^{1+4/2}} (1 - e^{-C x^{1+4/2}})^N \geq \frac{C^N}{N (1 + \frac{2}{N})^N}.
\]
(3.12)

Next, we estimate the conditional probability in (3.11). For simplicity, we write \( \mathbb{P}_N = \mathbb{P}(\cdot \big| A_N) \), \( \mathbb{P}^\tau_\eta \) for the conditional probability given the sequences \((\tau_i)_{i \in \mathbb{N}}\) and \((\eta_i)_{i \in \mathbb{N}}\) and
\(P^\tau,\eta\) if we further condition on \(\Lambda_\tau\). Because the variables \(\zeta_i\) are independent of \(\Lambda_\tau, \tau_i\) and \(\eta_i\), Lemma 3.3 implies that

\[
P_N \left( Y_\tau(\tau_N, \eta_N) \prod_{i=1}^{N} u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \zeta_i > R \right)
= E_N \left[ P^\tau,\eta \left( \prod_{i=1}^{N} \zeta_i > \frac{R}{Y_\tau(\tau_N, \eta_N) \prod_{i=1}^{N} u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})} \right) \right]
\geq \frac{C^N}{(N-1)!} R^{-\alpha} E_N \left[ Y^{\alpha}_<(\tau_N, \eta_N) \prod_{i=1}^{N} u^\alpha_<\tau(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \right]
\times \log^{N-1} \left( \frac{R}{Y_\tau(\tau_N, \eta_N) \prod_{i=1}^{N} u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})} \right).
\]

Further restricting to the set \(\{ Y_\tau(\tau_N, \eta_N) \prod_{i=1}^{N} u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \leq \sqrt{R} \}\), we obtain that

\[
P_N \left( Y_\tau(\tau_N, \eta_N) \prod_{i=1}^{N} u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \zeta_i > R \right)
\geq \frac{(C \log R)^{N-1}}{(N-1)!} R^{-\alpha} E_N \left[ Y^{\alpha}_<(\tau_N, \eta_N) \prod_{i=1}^{N} u^\alpha_<\tau(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \right]
\times \mathbb{1}_{\{ Y_\tau(\tau_N, \eta_N) \prod_{i=1}^{N} u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \leq \sqrt{R} \}}.
\]

The last line has the form \(E[X^\alpha \mathbb{1}_{\{ X \leq \sqrt{R} \}}]\), which can be bounded from below by

\[
E[X^\alpha \mathbb{1}_{\{ X \leq \sqrt{R} \}}] = E[X^\alpha] - E[X^\alpha \mathbb{1}_{\{ X > \sqrt{R} \}}] \geq E[X^\alpha] - E[X^{\alpha \frac{p}{2}}] P(X > \sqrt{R})^{1 - \frac{p}{2}}
\geq E[X^\alpha] - E[X^{\alpha \frac{p}{2}}] R^{-\frac{p}{2} \alpha (p-1)}
\]
thanks to Hölder’s inequality and Markov’s inequality. Moreover,

\[
E_N \left[ Y^{\alpha}_<(\tau_N, \eta_N) \prod_{i=1}^{N} u^\alpha_<\tau(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \right]
= E_N \left[ E^{\tau,\eta}[Y^{\alpha}_<(\tau_N, \eta_N)] \prod_{i=1}^{N} E^{\tau,\eta}[u^\alpha_<\tau(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})] \right],
\]

where we used the independence of \(Y_\tau(\tau_N, \eta_N)\) and the variables \(u_<(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})\) as \(i\) varies under \(P^{\tau,\eta}\). Indeed, the sequence \(\tau_i\) is determined by \(\Lambda_\tau\), while \(Y_\tau\) and \(u_<\) are defined via \(\Lambda_<\).

If \(\alpha \in (0, 1)\), we use Lemma A.2 (with some fixed \(p \in (1, 1 + \frac{2}{\alpha})\)) and obtain

\[
E^{\tau,\eta}[u^\alpha_<\tau(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})] \geq C \frac{g(\Delta \tau_i, \Delta \eta_i)^{\alpha + \frac{p}{2} - 1}}{E^{\tau,\eta}[u^p_<\tau(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})]^{\frac{p}{2} - 1}} \geq C g(\Delta \tau_i, \Delta \eta_i)^{\alpha},
\]

\[
E^{\tau,\eta}[Y^{\alpha}_<(\tau_N, \eta_N)] \geq C \frac{E^{\tau,\eta}[Y^{\alpha}_<(\tau_N, \eta_N)]^{\alpha + \frac{p}{2} - 1}}{E^{\tau,\eta}[Y^p_<\tau(\tau_N, \eta_N)]^{\frac{p}{2} - 1}} \geq C,
\]

for some constant \(C\).
where the last step in both lines follows from [9, Cor. 6.5]. Thus, there is $C_1 \in (0, \infty)$ such that

$$
E_N \left[ Y_{<}(\tau_N, \eta_N) \prod_{i=1}^{N} u_{<}^{\alpha}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \right] \geq C_1^N E_N \left[ \prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i)^{\alpha} \right].
$$

(3.16)

If $\alpha \geq 1$, we can use Jensen’s inequality in (3.15) to take $\alpha$ outside of $E^{\tau, \eta}$ and obtain (3.16) as well (with $C_1 = 1$), since $E^{\tau, \eta}[Y_{<}(\tau_N, \eta_N)] = 1$ and $E^{\tau, \eta}[u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})] = g(\Delta \tau_i, \Delta \eta_i)$. At the same time, for any $\alpha \in (0, 1 + \frac{2}{d})$, upon using Jensen’s inequality if $\alpha \in (0, 1]$ and [9, Cor. 6.5] if $\alpha \in (1, 1 + \frac{2}{d})$, we have the upper bound

$$
E_N \left[ Y_{<}(\tau_N, \eta_N) \prod_{i=1}^{N} u_{<}^{\alpha}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \right] \leq C_2^N E_N \left[ \prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i)^{\alpha} \right].
$$

(3.17)

Next, we evaluate $E_N[\prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i)^{\alpha}]$. To this end, note that conditionally on $A_N$, the $\Delta \tau_i$’s are independent with density

$$
f_N(x) = \frac{C(1 + \frac{2}{d})x^{d/2}e^{-Cx^{1+d/2}}}{1 - e^{-C(\frac{d}{2} + 1)x^{1+d/2}}}, \quad x \in (0, \frac{1}{C_N}),
$$

(3.18)

for all $i = 1, \ldots, N$, while the $\Delta \eta_i$’s are independent and, conditioned on $\Delta \tau_i$’s are uniformly distributed on a centered ball with radius $\sqrt{\Delta \tau_i}$. Therefore,

$$
E_N \left[ \prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i)^{\alpha} \right] = E_N[g(\Delta \tau_1, \Delta \eta_1)^{\alpha}]^N
$$

$$
= \left( \int_{0}^{t/N} \frac{f_N(s)}{\pi^{d/2}/\Gamma(\frac{d}{2} + 1)s^{d/2}} \int_{\mathbb{R}^d} g(s, y)^{\alpha} \mathbb{1}_{\{|y| \leq \sqrt{s}\}} \, dy \, ds \right)^N
$$

$$
\leq C^N N^{(1 + \frac{d}{2})N} \left( \int_{0}^{t/N} \int_{\mathbb{R}^d} g(s, y)^{\alpha} \mathbb{1}_{\{|y| \leq \sqrt{s}\}} \, dy \, ds \right)^N
$$

$$
= C^N N^{(1 + \frac{d}{2})N} \left( \int_{0}^{t/N} s^{-\frac{d}{2}(\alpha - 1)} \, ds \right)^N = \frac{C^N N^{(1 + \frac{d}{2})N}}{N^{\theta_{\alpha}, N}}.
$$

In this calculation, we can replace $\leq$ by $\geq$ in the third line upon changing the value of $C$. As a consequence, if we combine this result with (3.13), (3.14), (3.16) and (3.17) (with $\alpha p$ in the role of $\alpha$ and $p > 1$ such that $\alpha p < 1 + \frac{2}{d}$), we obtain that

$$
P_N \left( Y_{<}(\tau_N, \eta_N) \prod_{i=1}^{N} u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \zeta_i > R \right)
$$

$$
\geq \frac{(C \log R)^{N-1} \left( N^{(1 + \frac{d}{2})N} \right)^N}{N!} R^{-\alpha} \left( \frac{C_1^N}{N^{\theta_{\alpha}, N}} - \frac{C_2^N}{N^{(1 + \theta_{\alpha})N}} R^{-\frac{2}{d}(p-1)} \right)
$$

$$
\geq \frac{(C \log R)^{N-1} \left( N^{(1 + \frac{d}{2})N} \right)^N}{N^{(1 + \theta_{\alpha})N}} R^{-\alpha} \left( 1 - \frac{(C_2/C_1)N^{\frac{d}{2}(p-1)}}{R^{\frac{2}{d}(p-1)}} \right)
$$

$$
\geq \frac{(C \log R)^{N-1} \left( N^{(1 + \frac{d}{2})N} \right)^N}{2N^{(1 + \theta_{\alpha})N}} R^{-\alpha},
$$

(3.19)
where the last step holds if \( N \leq C_0 \log R \) for some small but fixed \( C_0 > 0 \). Together with (3.11) and (3.12), we have shown that

\[
\Pi(Y(t, x) > R) \geq R^{-\alpha} (\log R)^{-1} \sum_{N=1}^{C_0 \log R} \frac{(C \log R)^N}{N(1+\theta_\alpha)}.
\]

In order to bound this sum, we use integral approximations. Because the function \( x \mapsto \varphi(x) = (C \log R)^x / x^{(1+\theta_\alpha)x} \) has a unique maximum at \( x_0 = (C \log R)^{(1/(1+\theta_\alpha))}e^{-1} \) and as \( R \to \infty \), we have, for sufficiently large \( R \),

\[
\sum_{N=1}^{C_0 \log R} \frac{(C \log R)^N}{N(1+\theta_\alpha)} \geq \frac{1}{2} \int_0^{(1+\theta_\alpha)(C_0 \log R)} \frac{(C \log R)^y}{y^{(1+\theta_\alpha)}} dy \geq \frac{1}{4(1+\theta_\alpha)} \sum_{N=0}^{(1+\theta_\alpha)(C_0 \log R)} \frac{[(C \log R)^{1/(1+\theta_\alpha)}(1+\theta_\alpha)]^N}{N!}.
\]

By Taylor’s theorem, this is further bounded from below by

\[
\frac{1}{4(1+\theta_\alpha)} e^{(C \log R)^{1/(1+\theta_\alpha)}(1+\theta_\alpha)e^{-1}} \left( 1 - \frac{[(C \log R)^{1/(1+\theta_\alpha)}(1+\theta_\alpha)e^{-1}] [(C \log R)]}{[(1+\theta_\alpha)(C_0 \log R)]} \right) \geq \frac{1}{8(1+\theta_\alpha)} e^{(C \log R)^{1/(1+\theta_\alpha)}(1+\theta_\alpha)e^{-1}} \left( 1 - \frac{(C_{\alpha})_{C_0}^{1-\theta_\alpha}}{(\log R)^{\theta_\alpha}} \right) \geq \frac{1}{8(1+\theta_\alpha)} e^{(C \log R)^{1/(1+\theta_\alpha)}(1+\theta_\alpha)e^{-1}}.
\]

This completes the proof of the lower bound in (3.1).

If the noise has lighter tails, a different slowly varying function appears in the tail.

**Theorem 3.2.** Assume Condition (H-\( \alpha \)) or (L-\( \alpha \)) with \( \alpha = 1 + \frac{2}{d} \). For every \( t > 0 \), there is \( C > 0 \) such that for all \( x \in \mathbb{R}^d \) and \( R > 1 \),

\[
C_1 R^{-\frac{1}{2} - \frac{2}{d}} e^{C_2 \frac{\log R}{\log \log R}} \leq \Pi(Y(t, x) > R) \leq C_2 R^{-\frac{1}{2} - \frac{2}{d}} e^{C_2 \frac{\log R}{\log \log R}}.
\]

**Proof.** Step 1: Upper bound

First consider \( d \geq 2 \), in which case \( 1 + \frac{2}{d} \in (1, 2] \). As in the upper bound proof of Theorem 3.1, it suffices to show the tail bound for \( Y(t, x) \). By [9, Prop. 6.3] and our assumptions on \( \lambda \), there are \( \eta > 0 \) and \( C > 0 \) such that for any \( 1 < p < 1 + \frac{2}{d} \), we have

\[
\mathbb{E}[Y(t, x)^p] \leq \sum_{N=0}^{\infty} (C L \rho)^N t^{\frac{1-\theta_\alpha}{p}} \left( \int_{(0,t)^N} 1_{\{t_1 < \cdots < t_N\}} (\Delta t_{N+1})^{\eta p - 1} \prod_{i=1}^N G_p(\Delta t_i) dt_i \right)^{\frac{1}{p}},
\]
where \( L_p = \int_{(0, a)} z^{1+2/d}(3 \log z + 1) \lambda(dz) + \int_{[0, \infty)} z^p \lambda(dz) \) and \( G_p(s) = s^{\theta_p/3-1} \) if \( s \leq 1 \) and \( G_p(s) = s^{\theta_p-1} \) if \( s \geq 1 \). On \([0, t]\), we have \( G_p(s) \leq C s^{\theta_p/3-1} \) for some \( C \) that only depends on \( t \). Furthermore, if \( m_{1+2/d}(\lambda) + M_{1+2/d}(\lambda) < \infty \), then \( L_p \leq C \) for some constant \( C \) that is independent of \( p \); if \( \lambda([R, \infty)) \sim CR^{-1-2/d} \) as \( R \to \infty \), then \( L_p \sim C \theta_p^{-1} \) as \( p \uparrow 1 + \frac{2}{d} \), for some (other) constant \( C \) that is also independent of \( p \). Therefore, in both cases,

\[
E[Y(t, x)^p] \leq \sum_{N=0}^{\infty} (C \theta_p^{-1})^N \int_{(0, t)} 1_{\{t_1 < \cdots < t_N\}} \left( \Delta t_{N+1} \right)^{\theta_p-1} \prod_{i=1}^{N} \left( \Delta t_i \right)^{\theta_p-1} dt_i.
\]

By \([10, \text{Lemma A.3}]\) and \( \text{Lemma A.1} \). Thus, by Markov’s inequality and possibly after enlarging \( C \) in the second step,

\[
P(Y(t, x) > R) \leq CR^{-p} \frac{\Gamma(\theta_p)}{\theta_p^p} \exp\{C(\theta_p^{-1} \Gamma(\theta_p))^{3/\theta_p}\} \leq CR^{-p} e^{\left(\frac{C}{\theta_p}\right)^{C/\theta_p}}.
\]

As \( p \) is close enough to \( 1 + \frac{2}{d} \) (because \( \theta_p \downarrow 0 \) as \( p \uparrow 1 + 2/d \) thus \( \Gamma(\theta_p) \sim \theta_p^{-1/2} \)).

Let \( W : (0, \infty) \to (0, \infty) \) be the (principal branch of the) Lambert \( W \) function, that is, \( W(x) \) is the unique solution on \((0, \infty)\) to the equation \( We^W = x \). We choose \( p = p(R) < 1 + \frac{2}{d} \) such that

\[
\frac{C}{\theta_p} = \exp(W(\log \log R - \log \log \log R + \log \log \log \log R)).
\]

Let us denote the expression on the right-hand side by \( z = z(R) \). Then \( p(R) = 1 + \frac{2}{d} - \frac{2C}{dz(R)} \) and (3.21) becomes

\[
P(Y(t, x) > R) \leq CR^{-1-\frac{2}{d}} R^z e^{z(R) - \log z(R)} = CR^{-1-\frac{2}{d}} R^z e^{z(R)} e^{\log z(R) \log \log R}.
\]

Note that \( z = e^{W(x)} \) satisfies \( z \log z = x \) by the definition of \( W \). Therefore,

\[
P(Y(t, x) > R) \leq CR^{-1-\frac{2}{d}} R^z e^{\frac{C}{\text{log} \log R}}. \tag{3.22}
\]

By \([42, \text{Eq. (4.13.10)}]\), there exists \( x_0 \in (0, \infty) \) such that

\[
\log x - \log \log x \leq W(x) \leq \log x - \frac{1}{2} \log \log x
\]

for all \( x \geq x_0 \). Consequently, for sufficiently large \( R \),

\[
z(R) \geq e^{W(\frac{1}{2} \log \log R)} \geq e^{\frac{1}{2} \log \log R - \log \log (\frac{1}{2} \log \log R)} = \frac{1}{2} \log \log R - \log \log (\frac{1}{2} \log \log R),
\]

which implies

\[
R^z = e^{\frac{2C}{dz(R)}} \leq e^{\frac{4C}{\text{log} \log R}}.
\]

Combining this with (3.22), we obtain (3.20) if \( d \geq 2 \). The proof essentially remains the same if \( d = 1 \): by (3.10), because \( \mu_p(\lambda) \leq m_2(\lambda) + M_p(\lambda) \leq C \theta_p^{-1} \) uniformly in \( p \in [2, 3] \),

\[
E[Y(t, x)^p] \leq C \exp\{C(\theta_p^{-1} \Gamma(\theta_p))^{1/\theta_p}\} \leq C e^{\left(\frac{C}{\theta_p}\right)^{C/\theta_p}}.
\]

With this bound, we can go back to (3.21) and complete the proof as before.
Step 2: Lower bound

Without loss of generality, we may assume that $M_0(\lambda) = \lambda([1, \infty)) > 0$. In this case, with the same notation as in the lower bound proof of Theorem 3.1, we have $\zeta_i \geq 1$ and

$$
P_N\left( Y_{<}(\tau_N, \eta_N) \prod_{i=1}^{N} u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \zeta_i > R \right)
$$

$$
\geq P_N\left( \left\{ 2^{-N-1} \prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i) > R \right\} \cap \{ Y_{<}(\tau_N, \eta_N) > \frac{1}{2} \}
\right)
$$

$$
\cap \left\{ \prod_{i=1}^{N} \left\{ u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) > \frac{1}{2} g(\Delta \tau_i, \Delta \eta_i) \right\} \right\}
$$

$$
\geq E_N \left[ 1 \left\{ 2^{-N-1} \prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i) > R \right\} P^{\tau, \eta}(Y_{<}(\tau_N, \eta_N) > \frac{1}{2})
\right]
\times \prod_{i=1}^{N} P^{\tau, \eta}(u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) > \frac{1}{2} g(\Delta \tau_i, \Delta \eta_i))
$$

where the second step follows by using the independence under $P^{\tau, \eta}$ of the variables $Y_{<}(\tau_N, \eta_N)$ and $(u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}))_{i=1, \ldots, N}$. Observe that $E^{\tau, \eta}[Y_{<}(\tau_N, \eta_N)] = 1$ and $E^{\tau, \eta}[u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1})] = g(\Delta \tau_i, \Delta \eta_i)$. Thus, by Lemma A.2 and [9, Cor. 6.5], there is a deterministic $C > 0$ such that

$$
P^{\tau, \eta}(Y_{<}(\tau_N, \eta_N) > \frac{1}{2}) > C, \quad P^{\tau, \eta}(u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) > \frac{1}{2} g(\Delta \tau_i, \Delta \eta_i)) > C
$$

for all $i = 1, \ldots, N$. Moreover, $g(\Delta \tau_i, \Delta \eta_i) \geq (2\pi \Delta \tau_i)^{-d/2} e^{-1/2} = C(\Delta \tau_i)^{-d/2}$ by the definition of $\tau_i$. Hence,

$$
P_N\left( Y_{<}(\tau_N, \eta_N) \prod_{i=1}^{N} u_{<}(\tau_i, \eta_i; \tau_{i-1}, \eta_{i-1}) \zeta_i > R \right)
$$

$$
\geq C^N P_N\left( 2^{-N-1} \prod_{i=1}^{N} g(\Delta \tau_i, \Delta \eta_i) > R \right) \geq C^N P_N\left( \prod_{i=1}^{N} (\Delta \tau_i)^{-\frac{d}{2}} > C^{-N} R \right).
$$

To evaluate this probability, recall the density of $\tau_i$ from (3.18). We have

$$
P_N\left( \prod_{i=1}^{N} (\Delta \tau_i)^{-\frac{d}{2}} > C^{-N} R \right)
$$

$$
= C^N \int_{(0, \frac{1}{C} R)^N} \frac{e^{-C \sum_{i=1}^{N} s_i^{d/2}}}{(1 - e^{-C s_i^{d/2}})^N} \prod_{i=1}^{N} s_i^{-\frac{d}{2}} > C^{-N} R \prod_{i=1}^{N} s_i^{-\frac{d}{2}} ds_i
$$

$$
\geq C^N N^{(1 + \frac{d}{2}) N} \int_{(0, \frac{1}{C} R)^N} \prod_{i=1}^{N} s_i < C^N R^{-\frac{d}{2}} \prod_{i=1}^{N} s_i^{-\frac{d}{2}} ds_i
$$

$$
= C^N N^{(1 + \frac{d}{2}) N} R^{-1 - \frac{d}{2}} \int_{0, \frac{d}{2} R^{-\frac{d}{2}} / \sqrt{N}} \prod_{i=1}^{N} u_i < 1 \prod_{i=1}^{N} u_i^{-\frac{d}{2}} du_i.
$$
Provided that \( tR^{2(N+1)} > 1 \) (i.e., \( N^N < \left( \frac{1}{t} \right)^N R^{2/d} \)), we can use Lemma A.3 (keeping only the term corresponding to \( i = N - 1 \)) to obtain
\[
P_N \left( \prod_{i=1}^{N} (\Delta \tau_i)^{-\frac{t}{2}} > C^{-N} R \right) \geq C^N N^{(1+\frac{2}{d})N} R^{-1-\frac{2}{d}} \left( \log \frac{tR^{2N}}{CN} \right)^{N-1}.
\]

We will further restrict ourselves to \( N \) such that
\[(3.25) \quad N^N \leq \left( \frac{t}{C} \right)^N R^{N/d},
\]
in which case
\[
P_N \left( \prod_{i=1}^{N} (\Delta \tau_i)^{-\frac{t}{2}} > C^{-N} R \right) \geq C^N N^{\frac{2}{d}N} R^{-1-\frac{2}{d}} \log^{N-1} R.
\]

In fact, we will consider \( N \) such that equality is attained in (3.25), that is, we choose
\[(3.26) \quad N = K e^{W(\frac{1}{\pi K} \log R)},
\]
where \( K \) is actually \( \frac{1}{t} \) from above. Recalling (3.12), we obtain
\[
P(Y(t, x) > R) \geq P(A_N) P_N \left( Y \langle \tau_N, \eta_N \rangle \prod_{i=1}^{N} u \langle \tau_i, \eta_i ; \tau_{i-1}, \eta_{i-1} \rangle \zeta_i > R \right)
\]
\[
\geq C^N N^{-N} R^{-1-\frac{2}{d}} \log^{N-1} (R)
\]
\[
= \frac{R^{-1-\frac{2}{d}} (C \log R)^K \exp(W(\frac{1}{\pi K} \log R))}{\log R (K \exp(W(\frac{1}{\pi K} \log R))) K \exp(W(\frac{1}{\pi K} \log R))}.
\]

By (3.23), for sufficiently large \( R \)
\[
P(Y(t, x) > R) \geq \frac{R^{-1-\frac{2}{d}} (C \log R)^K \exp(W(\frac{1}{\pi K} \log R))}{(\log R)^{1/2} \log R (\log(W(\frac{1}{\pi K} \log R)))^{K \exp(W(\frac{1}{\pi K} \log R))}}
\]
\[
\geq \frac{R^{-1-\frac{2}{d}} K \exp(W(\frac{1}{\pi K} \log R)) e^{\frac{1}{2} K \exp(W(\frac{1}{\pi K} \log R))}}{\log R}
\]
\[
\geq R^{-1-\frac{2}{d}} e^{\frac{1}{2} K \exp(W(\frac{1}{\pi K} \log R)) \log R}
\]
\[
\geq R^{-1-\frac{2}{d}} e^{\frac{1}{2} K \exp(W(\frac{1}{\pi K} \log R)) \log r \log R}
\]
where the last step holds for some sufficiently small \( C > 0 \).

In the previous proof, we used the following lemma, which is a uniform-in-\( N \) version of [28, Lemma 4.1 (4)].

**Lemma 3.3.** Let \( N \in \mathbb{N} \) and \( X_1, \ldots, X_N \) be independent and identically distributed such that there are \( C_0, \alpha \in (0, \infty) \) with \( P(X_1 > R) \geq C_0 R^{-\alpha} \) for all \( R \geq 1 \). Then there is \( C \in (0, \infty) \) such that for all \( N \in \mathbb{N} \) and \( R > 1 \),
\[
P\left( \prod_{i=1}^{N} X_i > R \right) \geq \frac{C^N}{(N-1)!} R^{-\alpha} \log^{N-1} R.
\]
PROOF. Conditionally on the event \( A = \bigcap_{i=1}^{N} \{ X_i > 1 \} \), the \( X_i \)'s are still independent and identically distributed and satisfy
\[
\mathbb{P}(X_1 > R \mid A) = \frac{\mathbb{P}(X_1 > R \vee 1)}{\mathbb{P}(X_1 > 1)} \geq 1_{(0,1)}(R) + \frac{C_0}{\mathbb{P}(X_1 > 1)} R^{-\alpha} 1_{[1,\infty)}(R) \\
\geq C_1 R^{-\alpha} 1_{[C_1^{-1/\alpha}, \infty)}(R)
\]
with \( C_1 = C_0 / \mathbb{P}(X_1 > 1) \), which belongs to \((0,1]\) by assumption. Let \( Y_1, \ldots, Y_N \) be independent Pareto random variables with scale parameter \( C_1^{1/\alpha} \) and shape parameter \( \alpha \) (in particular, their tail function is given by the right-hand side of the previous display). Using quantile representation, one can construct these variables in such a way that conditionally on \( A \), we have \( X_i \geq Y_i \) almost surely. Thus,
\[
\mathbb{P} \left( \bigcap_{i=1}^{N} X_i > R \right) \geq \mathbb{P}(X_1 > 1)^N \mathbb{P} \left( \prod_{i=1}^{N} X_i > R \mid A \right) \geq \mathbb{P}(X_1 > 1)^N \mathbb{P} \left( \prod_{i=1}^{N} Y_i > R \right).
\]
It is an elementary result that \( \sum_{i=1}^{N} \log \left( Y_i / C_1^{1/\alpha} \right) \) is \( \Gamma(N, \alpha) \)-distributed. Therefore,
\[
\mathbb{P} \left( \prod_{i=1}^{N} Y_i > R \right) = \mathbb{P} \left( \sum_{i=1}^{N} \log \left( \frac{Y_i}{C_1^{1/\alpha}} \right) > \log \left( \frac{R}{C_1^{N/\alpha}} \right) \right) = \frac{\alpha^N}{(N-1)!} \int_{\log(R/C_1^{N/\alpha})}^{\infty} u^{N-1} e^{-\alpha u} \, du \\
= \frac{\alpha^N}{(N-1)!} \int_{R/C_1^{N/\alpha}}^{\infty} u^{-\alpha-1} \log^{N-1}(u) \, du \\
\geq \frac{\alpha^N}{(N-1)!} \log^{N-1} \left( \frac{R}{C_1^{N/\alpha}} \right) \int_{R/C_1^{N/\alpha}}^{\infty} u^{-\alpha-1} \, du \\
= \frac{\alpha^{N-1} C_1^N}{(N-1)!} R^{-\alpha} \log^{N-1} \left( \frac{R}{C_1^{N/\alpha}} \right).
\]
By decreasing \( C_0 \) if necessary, there is no loss of generality if we assume that \( C_1 < 1 \). This implies
\[
\mathbb{P} \left( \prod_{i=1}^{N} X_i > R \right) \geq \frac{\alpha^{N-1} (C_1 \mathbb{P}(X_1 > 1))^N}{(N-1)!} R^{-\alpha} \log^{N-1} R,
\]
proving the lemma. \( \square \)

REMARK 3.4. In the lower bound proofs of both Theorem 3.1 and 3.2, it was crucial that (1.1) is considered on the whole space \( \mathbb{R}^d \). To illustrate this point, let us take a standard Poisson noise (i.e., \( \lambda = \delta_1 \)) and restrict the noise to a spatial domain \( D \) with finite and positive Lebesgue measure \( |D| \). Because \( |D| < \infty \), there is only a finite Poisson-distributed number \( L \) of points up to time \( L \). Therefore,
\[
Y(t, x) = 1 + \sum_{N=1}^{L} \int_{((0,t) \times D)^N} \prod_{i=2}^{N+1} g(\Delta t_i, \Delta x_i) \prod_{j=1}^{N} \Lambda(dt_j, dx_j).
\]
Call the \( N \)-fold integral \( I_N(t, x) \). Either by bounding the tail probability explicitly or by estimating the \( p \)th moment and then optimizing, one can show that
\[
\mathbb{P}(I_N(t, x) > R) \leq \frac{C^N}{N!} R^{-\frac{N}{2}} \log^N R
\]
for some $C > 0$ that is independent of $N$ and $R$. Therefore, by conditioning on $L$,

$$
P(Y(t, x) > R) \leq \sum_{L=1}^{\infty} \frac{e^{-t|D|/L}L}{L!} \sum_{N=1}^{L} \mathbb{P}(I_N(t, x) > R - 1)
$$

\begin{align*}
&\leq \sum_{L=1}^{\infty} \frac{e^{-t|D|/L}L}{L!} \sum_{N=1}^{L} \mathbb{P}(I_N(t, x) > R - 1) \\
&\leq \sum_{L=1}^{\infty} \frac{e^{-t|D|/L}L}{L!} \sum_{N=1}^{L} \frac{CN}{N^N} R^{-1} \approx L^{1+\frac{1}{2}} \log^N R.
\end{align*}

Since $L! \geq (L - N)!N!$ and $(t|D|) L^{1+2/d} \leq C^L = C^{L-N} C^N$, we can use Lemma A.1 to get

$$
P(Y(t, x) > R) \leq e^{-t|D|} R^{-1-2} \sum_{N=1}^{\infty} \frac{(C\log R)^N}{N!N^N} \sum_{N=1}^{\infty} \frac{C^{L-N}}{(L-N)!} \leq CR^{-1-2} e^{C(\log R)^{1/2}},
$$

which is much smaller than the tails we obtained in Theorem 3.2.

Similarly, if $\dot{\Lambda}$ has Lévy measure (1.3) with $\alpha < 1 + \frac{2}{d}$, then one can show that

$$
P(I_N(t, x) > R) \leq \frac{CN}{N(1+\theta_{\alpha})} R^{-\alpha} \log N R.
$$

Again, if $\dot{\Lambda}$ only acts on $D$, we have

$$
P(Y(t, x) > R) \leq CR^{-\alpha} \sum_{N=1}^{\infty} \frac{(C\log R)^N}{N!N(1+\theta_{\alpha})^N} \leq CR^{-\alpha} e^{C(\log R)^{1/(2+\theta_{\alpha})}},
$$

which is much lighter than the tails derived in Theorem 3.1.

Finally, let us mention [23], where the exact tail behavior of solutions to stochastic differential equations (SDEs) with multiplicative stable noise was determined. Their proof heavily relies on an exact representation of the solution as a random product of heavy-tailed terms, which is not available for the SHE. In addition, the SDE situation differs from the SHE in two aspects: first, space only consists of one point and is therefore bounded; second, the fundamental solution, unlike the heat kernel, has no singularity. This is why the tail behavior of the solution to a stable SDE is of the form given by the right-hand side of (3.27) but without $\theta_{\alpha}$ in the exponent. The reader may verify that $\theta_{\alpha}$ enters (3.27) only because the heat kernel has a singularity.

### 3.2. Tail bounds for the local supremum

We need the following assumption.

**Condition (Sup).** If $d = 1$, then $m_q(\lambda) < \infty$ for some $q \in (0, 2)$. If $d \geq 2$, we have $m_{2/d}^{(\log)}(\lambda) < \infty$.

Note that $m_2(\lambda) < \infty$ for all Lévy measures, so Condition (Sup) is rather mild in dimension $d = 1$. Also, if $m_{2/d+\varepsilon}(\lambda) = \infty$ for some small $\varepsilon > 0$, then the solution to (1.1) under additive noise is unbounded on any non-empty open subset of $\mathbb{R}^d$ at a fixed time, see [19, Theorem 3.3]. Thus, Condition (Sup) is also rather mild in dimensions $d \geq 2$. Recall that $Q$ is the set of unit cubes.
THEOREM 3.5. Assume Condition (H-$\alpha$) for some $\alpha \in (0, \frac{2}{d}]$ and Condition (Sup). For every $t > 0$, there are constants $C_1, C_2 \in (0, \infty)$ such that for all $Q \in \mathcal{Q}$ and $R > 1$,

$$C_1 R^{-\alpha} e^{C_1 (\log R)^{1/(a_\alpha)}} \leq \mathbb{P} \left( \sup_{x \in Q} Y(t, x) > R \right) \leq C_2 R^{-\alpha} e^{C_2 (\log R)^{1/(a_\alpha)}}. \tag{3.28}$$

**Proof.** We only need to prove the upper bound. The lower bound immediately follows from Theorem 3.2. Without loss of generality, assume that $Q = (0, 1)^d$. We first consider $d \geq 2$, in which case $m_1(\lambda) = \int_{(0,1)} \lambda (dz) < \infty$. Therefore, $Y(t, x) = e^{-m_1(\lambda) t} \hat{Y}(t, x)$, where $\hat{Y}(t, x)$ is the mild solution to

$$\hat{Y}(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} \int_{(0, \infty)} g(t-s, x-y) \hat{Y}(s, y) z \mu(ds, dy, dz), \tag{3.29}$$

and it suffices to prove the second inequality in (3.28) for $\hat{Y}$ instead of $Y$. Similarly to (2.11) and (3.5), we have, with obvious notation, that

$$\hat{Y}(t, x) = \sum_{N=0}^{\infty} \int_{(0,t) \times \mathbb{R}^d} \hat{Y}_N(t_1, x_1) \prod_{i=2}^{N+1} \hat{u}_N(t_{i-1}, x_{i-1}; t_i, x_i) \prod_{j=1}^N A_{\lambda}(dt_j, dx_j). \tag{3.30}$$

Therefore, using the estimate $(\sum a_i)^p \leq \sum a_i^p$ and independence, we obtain for any $0 < p < \alpha < 1$

$$\mathbb{E} \left[ \sup_{x \in Q} \hat{Y}(t, x)^p \right] \leq \sum_{N=0}^{\infty} M_p(\lambda)^N \int_{(0,t) \times \mathbb{R}^d} \mathbb{E} [\hat{Y}_N(t_1, x_1)^p] \prod_{i=2}^{N+1} \mathbb{E} [\hat{u}_N(t_{i-1}, x_{i-1}; t_i, x_i)^p] \prod_{j=1}^N dt_j \, dx_j. \tag{3.31}$$

Combining Lemma 3.6, (3.2), Lemma 3.5 in [18], and Lemma A.1, we obtain

$$\mathbb{E} \left[ \sup_{x \in Q} \hat{Y}(t, x)^p \right] \leq \frac{C}{1 - \frac{d}{2} p} \sum_{N=0}^{\infty} (CM_p(\lambda) N) \int_{(0,t) \times \mathbb{R}^d} (t - t_N)^{-\frac{d}{2} p} e^{-C|x_N|^2} \prod_{i=2}^N g(\Delta t_i, \Delta x_i)^p \prod_{j=1}^N dt_j \, dx_j$$

$$\leq \frac{C}{1 - \frac{d}{2} p} \sum_{N=0}^{\infty} (CM_p(\lambda) N) \int_0^t \prod_{i=2}^N \{ t_{i-1} < \cdots < t_N \} (t - t_N)^{-\frac{d}{2} p} \prod_{i=2}^N (\Delta t_i)^{-\frac{d}{2} p} \prod_{j=1}^N dt_j$$

$$\leq \frac{CT(1 - \frac{d}{2} p)}{1 - \frac{d}{2} p} \prod_{N=0}^{\infty} (CM_p(\lambda)^{\theta p} \Gamma(\theta)) \sum_{N=0}^{\infty} (CM_p(\lambda)^{\theta p} \Gamma(\theta)) N \Gamma(2 \sqrt{N \theta p - \frac{d}{2}}) \leq C T(1 - \frac{d}{2} p) e^{CM_p(\lambda)^{1/\theta p}}.$$

By our assumptions on $\lambda$, we have $M_p(\lambda) \sim C/(\alpha - p)$ as $p \uparrow \alpha$. Therefore, if $\alpha \in (0, \frac{2}{d})$,

$$\mathbb{P} \left( \sup_{x \in Q} \hat{Y}(t, x) > R \right) \leq R^{-p} \mathbb{E} \left[ \sup_{x \in Q} \hat{Y}(t, x)^p \right] \leq CR^{-p} e^{C(\alpha - p)^{-1/\theta p}},$$

for some constant $C$ that does not depend on $p$. If $\alpha = \frac{d}{4}$, then we obtain an extra factor $(1 - \frac{d}{2} p)^{-2}$ (since $\Gamma(x) \sim x^{-1}$ as $x \downarrow 0$) in the previous line. But this is bounded by
\( C e^{C(2/d-p)^{-1/\theta p}} \), so the last display remains valid upon enlarging the value of \( C \) in the second step. So in all cases, as in the upper bound proof of Theorem 3.1, the current proof can be completed by choosing \( p = \alpha - (\theta_{\alpha} \log R)^{-1/(1+\theta_{\alpha})} \).

If \( d = 1 \) and \( \alpha \leq 1 \), we can re-use (3.31) and the subsequent argument except that we have to replace \( \hat{Y} \) and \( u < \) by \( Y \) and \( u < \) and use Jensen’s inequality to raise the \( p \)-moments for some \( \frac{3}{2} \leq q < \theta < 2 \) before applying Lemma 3.6, where \( q \) is given in Condition (Sup). If \( 1 < \alpha \leq \frac{3}{2} \), we choose \( \theta \in (\frac{3}{2}, q) \) fixed, while if \( \alpha > \frac{3}{2} \) and then let \( \theta = p \). In both cases we let \( p \uparrow \alpha \). We first observe that applying \( \mathbf{E}_< [\cdot]^{1/\theta} \) instead of \( \mathbf{E}[\cdot]^p \) in (3.31) leads to

\[
\mathbf{E}_< \left[ \sup_{x \in Q} Y(t, x)^\theta \right]^{\frac{1}{\theta}} \leq \sum_{N=0}^{\infty} \int_{(0, t) \times \mathbb{R}^N} \mathbf{E} \left[ Y_<(t_1, x_1)^\theta \right]^{\frac{1}{\theta}} \\
\times \mathbf{E} \left[ \sup_{x \in Q} u_<(t_N, x_N; t, x)^\theta \right]^{\frac{1}{\theta}} \prod_{i=2}^{N} \mathbf{E} \left[ u_<(t_{i-1}, x_{i-1}; t_i, x_i)^\theta \right]^{\frac{1}{\theta}} \prod_{j=1}^{N} \Lambda_\geq (dt_j, dx_j).
\]

By Lemma 3.6 and [9, Cor. 6.5], the left-hand side is further bounded by

\[
\left( \frac{C}{1 - \frac{q}{2}} \right)^{\frac{1}{\theta}} \left( \sum_{N=0}^{\infty} C^N \int_{(0, t) \times \mathbb{R}^N} (t - t_N)^{-\frac{1}{2}} e^{-C^{-1}|x_N|^2} \left( \int_{\mathbb{R}^N} g(\Delta t_i, \Delta x_i) \right) \prod_{j=1}^{N} \Lambda_\geq (dt_j, dx_j) \right)^{\frac{1}{\theta}} = \left( \frac{C}{1 - \frac{q}{2}} \right)^{\frac{1}{\theta}} \left( 1 + C \int_{\mathbb{R}} (t - t_N)^{-\frac{1}{2}} e^{-C^{-1}|y|^2} Y_\geq (t_N, x_N) \Lambda_\geq (dt_N, dx_N) \right)^{\frac{1}{\theta}},
\]

where \( Y^\prime_\geq (t, x) \) is the mild solution to the stochastic heat equation with initial condition 1 and noise \( C \Lambda_\geq \). Hence, writing \( \Lambda_\geq (ds, dy) = (\Lambda_\geq (ds, dy) - M_1(\lambda) ds dy) + M_1(\lambda) ds dy \), we obtain from the conditional Jensen’s inequality, Minkowski’s integral inequality, and the BDG inequality that

\[
\mathbf{E} \left[ \sup_{x \in Q} Y(t, x)^p \right]^{\frac{1}{p}} \leq \mathbf{E} \left[ \sup_{x \in Q} Y(t, x)^\theta \right]^{\frac{1}{\theta}} \mathbf{E} \left[ Y_\geq (s, y)^p \right]^{\frac{1}{p}}
\]

\[
\leq \left( \frac{C}{1 - \frac{q}{2}} \right)^{\frac{1}{\theta}} \left( 1 + CM_1(\lambda) \int_{\mathbb{R}} (t - s)^{-\frac{1}{2}} e^{-C^{-1}|y|^2} \mathbf{E} \left[ Y_\geq (s, y)^p \right]^{\frac{1}{p}} ds dy \right.
\]
\[
+ \left. \left( CM_\mu(\lambda) \int_{\mathbb{R}} (t - s)^{-\frac{1}{2}} e^{-C^{-1}|y|^2} \mathbf{E} \left[ Y_\geq (s, y)^p \right]^{\frac{1}{p}} ds dy \right)^{\frac{1}{\theta}} \right).
\]

Since \( \mathbf{E} \left[ Y_\geq (s, y)^p \right] \leq C e^{CM_\mu(\lambda)^{1/\theta p}} \) (cf. (3.4)), it follows that

\[
\mathbf{E} \left[ \sup_{x \in Q} Y(t, x)^p \right]^{\frac{1}{p}} \leq \left( \frac{C}{1 - \frac{q}{2}} \right)^{\frac{1}{\theta}} \left( 1 + Ce^{CM_\mu(\lambda)^{1/\theta p}} + \left( \frac{CM_\mu(\lambda)}{1 - \frac{q}{2}} \right)^{\frac{1}{\theta}} e^{CM_\mu(\lambda)^{1/\theta p}} \right).
\]

Since \( \theta \) is fixed, we obtain (both when \( \alpha \in (0, \frac{2}{d}) \) and when \( \alpha = \frac{2}{d} \)) that

\[
\mathbf{E} \left[ \sup_{x \in Q} Y(t, x)^p \right]^{\frac{1}{p}} \leq e^{CM_\mu(\lambda)^{1/\theta p}},
\]

from which the second inequality in (3.28) follows as before. \( \square \)
In the proof of the previous theorem, we used some technical moment estimates on the local supremum of $Y_\leq$, $u_\leq$ and $\hat{Y}_\leq$, $\hat{u}_\leq$.

**Lemma 3.6.** Suppose that $d \geq 2$. If $m_{2/d}(\lambda) < \infty$, then, for every $T > 0$, there exists $C \in (0, \infty)$ such that for all $0 < s < t \leq T$, $Q \in \mathbb{Q}$, $y \in \mathbb{R}^d$ and $0 < p < \frac{2}{d}$,

$$
\mathbb{E} \left[ \sup_{x \in \mathbb{Q}} \hat{Y}_\leq(t, x) \right] \leq C(1 - \frac{d}{2p})^{-1},
$$

(3.32)

$$
\mathbb{E} \left[ \sup_{x \in \mathbb{Q}} \hat{u}_\leq(s, y; t, x) \right] \leq C(1 - \frac{d}{2p})^{-1}(t - s)^{-\frac{d}{2p}e^{-C^{-1}|y|^2}}.
$$

(3.33)

If $d = 1$ and there is $q \in (0, 2)$ such that $m_q(\lambda) = \int_{(0,1)} z^q \lambda(\text{d}z) < \infty$, then there exists $C > 0$ such that for all $p \in (q + \frac{2}{d}, 2)$,

$$
\mathbb{E} \left[ \sup_{x \in \mathbb{Q}} Y_\leq(t, x) \right] \leq C(1 - \frac{p}{2})^{-1},
$$

(3.33)

$$
\mathbb{E} \left[ \sup_{x \in \mathbb{Q}} u_\leq(s, y; t, x) \right] \leq C(1 - \frac{p}{2})^{-1}(t - s)^{-\frac{p}{2}e^{-C^{-1}|y|^2}}.
$$

Note the lower bound $q \sqrt{\frac{d}{2}}$ for $d = 1$ in the moment inequality. This is a minor technicality, one can extend the inequality for smaller $p$ applying Lyapunov’s inequality for moments.

**Proof.** We only prove the uniform moment bound on $\hat{u}_\leq(s, y; t, x)$ or $u_\leq(s, y; t, x)$; the bounds on $\hat{Y}_\leq(t, x)$ and $Y_\leq(t, x)$ can be shown in a similar fashion. We may and do assume that $Q = (0, 1)^d$. By definition,

$$
\hat{u}_\leq(s, y; t, x) = g(t - s, x - y) + \int_s^t \int_{\mathbb{R}^d} \int_{(0,1)} g(t - r, x - v) \hat{u}_\leq(s, y; r, v) \text{d}\mu(\text{d}r, \text{d}v, \text{d}z).
$$

If $y \in (-d - 1, d + 1)^d$, we use the bound

$$
\sup_{x \in \mathbb{Q}} \hat{u}_\leq(s, y; t, x)
$$

(3.34)

$$
\leq C \left[ (t - s)^{-\frac{d}{2}} + \int_s^t \int_{(-d - 1, d + 1)^d} \int_{(0,1)} (t - r)^{-\frac{d}{2}e^{-\frac{|y|}{2(t - r)^{1/2}}} \hat{u}_\leq(s, y; r, v) \text{d}\mu(\text{d}r, \text{d}v, \text{d}z)\right].
$$

In the second integral, we have $(t - r)^{-d/2}e^{-((|v| - \sqrt{d})^2/(2(t-r)))} \leq Ce^{-(|v| - \sqrt{d})^2/(4T)} \leq C$. Furthermore, note that there are two ways of estimating the $p$th moment (for $p \in (0, 1)$) of a Poisson integral of an adapted process $f$, namely

$$
\mathbb{E} \left[ \left( \int f \text{d}\mu \right)^p \right] \leq \int \mathbb{E}[f^p] \text{d}\nu
$$

or

$$
\mathbb{E} \left[ \left( \int f \text{d}\mu \right)^p \right] \leq \left( \int \mathbb{E}[f] \text{d}\nu \right)^p,
$$

(3.35)

depending on whether we use $(\sum a_i)^p \leq \sum a_i^p$ or Jensen’s inequality. Applying the first method to the first integral in (3.34) if $(t - r)^{-d/2} > 1$ and the second method to the first
integrals in (3.34) if \((t - r)^{-d/2} z \leq 1\) as well as to the second integral in (3.34), we derive the bound

\[
\mathbb{E}\left[\sup_{x \in Q} \tilde{u}_<(s, y; t, x)^p\right] \leq C \left[ (t - s)^{-\frac{d}{2}p} + \int_s^t \int_{(t-r)^{-d/2} z > 1} \left( t - r \right)^{-\frac{d}{2}p} z^p \mathbf{1}_{\{(t-r)^{-d/2} z \leq 1\}} \mathbb{E}[\tilde{u}_<(s, y; r, v)] \, dr \, dv \, (dz) \right]^{p}.
\]

As \(\int_{\mathbb{R}^d} \mathbb{E}[\tilde{u}_<(s, y; r, v)] \, dv \leq \int_{\mathbb{R}^d} \mathbb{E}[u_<(s, y; r, v)] \, dv \leq C(r - s)^{(1-p)d/2} \leq CT^{(1-p)d/2} \leq C\) (which remains true for \(p = 1\), we obtain

\[
\mathbb{E}\left[\sup_{x \in Q} \tilde{u}_<(s, y; t, x)^p\right] \leq C \left[ (t - s)^{-\frac{d}{2}p} + \int_s^t \int_{(t-r)^{-d/2} z > 1} \left( t - r \right)^{-\frac{d}{2}p} z^p \mathbf{1}_{\{(t-r)^{-d/2} z \leq 1\}} \mathbb{E}[\tilde{u}_<(s, y; r, v)] \, dr \, dv \, (dz) \right]^{p} + m_1(\lambda)\right].
\]

For \(0 < p < \frac{d}{2}\), the remaining integrals can be bounded by

\[
\int_s^t \int_{(t-r)^{-d/2} z > 1} \left( t - r \right)^{-\frac{d}{2}p} z^p \mathbf{1}_{\{(t-r)^{-d/2} z \leq 1\}} \mathbb{E}[\tilde{u}_<(s, y; r, v)] \, dr \, dv \leq C m_2/d(\lambda)/1 - \frac{d}{2}p,
\]

respectively, which yields the claim for \(y \in (-d - 1, d + 1)^d\). If \(y \in \mathbb{R}^d \setminus (-d - 1, d + 1)^d\), we only need to replace the uniform bound on \(g(t - s, x - y)\) by \(C e^{-(|y| - \sqrt{d})^2/4T}\).

If \(d = 1\), to ease notation, write \(v\) for the stochastic part of \(u_\prec\), that is,

\[
v(s, y; t, x) = \int_s^t \int_{\mathbb{R}^d} g(t - r, x - v) \tilde{u}_<(s, y; r, v) z \, d\mu(dr, dv, dz).
\]

We first prove that for \(p \in (\frac{3}{2} \vee q, 2)\) and all \(s, t \in [0, T]\) and \(x, x' \in Q\),

\[
\mathbb{E}[|v(s, y; t, x) - v(s, y; t, x')|^p] \leq C(t - s)^{-\frac{d}{2}p} e^{-C^{-1}|y|^2} |x - x'|^{3-p}
\]

for some constant that does not depend on \(p\), \((s, y)\) or \((t, x, x')\) (but may depend on \(q\), \(\lambda\) and \(T\)). To this end, we use the BDG inequality and [9, Cor. 6.5] to get

\[
\mathbb{E}[|v(s, y; t, x) - v(s, y; t, x')|^p] \leq C m_2/d(\lambda) \int_s^t \int_{\mathbb{R}^d} |g(t - r, x - v) - g(t - r, x' - v)|^p g(r - s, v - y) \, dr \, dv.
\]
Note that \( m_p(\lambda) \leq m_q(\lambda) \) and bound the integral by

\[
C \left( \frac{|x-x'|}{t-s} \right)^{3-p} \int_s^{s+\epsilon} \int_{\mathbb{R}} |g(t-r, x-v) + g(t-r, x'-v)|^{2p-3} (r-s)^{-\frac{3}{2}} e^{-\frac{p|v-s|^2}{2(r-s)}} \, dr \, dv
\]

(3.39)

\[+ C(t-s)^{-\frac{3}{2}} \int_s^t \int_{\mathbb{R}} |g(t-r, x-v) - g(t-r, x'-v)|^{2p-3} e^{-\frac{p|v-s|^2}{2(r-s)}} \, dr \, dv,
\]

where we used the fact that \(|g(t-s, x-y) - g(t-s, x'-y)|\) can either be simply bounded using the triangle inequality or using the mean-value theorem (with \(|\partial_x g(t, x)| \leq \frac{C}{t}\).

Let \( I_1 \) and \( I_2 \) be the two expressions in (3.39). Then

\[
I_1 \leq C \left( \frac{|x-x'|}{t-s} \right)^{3-p} \int_s^{s+\epsilon} \int_{\mathbb{R}} (t-r)^{3-p} (r-s)^{-\frac{3}{2}} \left( e^{-\frac{|x-s|^2}{C(t-r)}} + e^{-\frac{|x'-s|^2}{C(t-r)}} \right) e^{-\frac{|v-s|^2}{2(r-s)}} \, dv \, dr
\]

\[\leq C \left( \frac{|x-x'|}{t-s} \right)^{3-p} \int_s^t \int_{\mathbb{R}} (r-s)^{-\frac{3}{2}} \left( e^{-\frac{|x-s|^2}{C(t-r)}} + e^{-\frac{|x'-s|^2}{C(t-r)}} \right) \, dr,
\]

while, by distinguishing whether \( v \in (-2, 2) \) or not and by using [46, Lemme A2] and the bound \(|\partial_x g(t, x)| \leq Ce^{-|x|^2/(Ct)}\) for \(|x| > 1\), we obtain

\[
I_2 \leq C(t-s)^{-\frac{3}{2}} \left( e^{-C^{-1}|y|^2} \int_s^t \int_{\mathbb{R}} |g(t-r, x-v) - g(t-r, x'-v)|^{p} \, dv \, dr \right)
\]

\[+ \left( |x-x'| \right)^{p} \int_s^t \int_{\mathbb{R}} e^{-\frac{|v-s|^2}{C(t-r)}} \left( e^{-\frac{|x-s|^2}{C(t-r)}} + e^{-\frac{|x'-s|^2}{C(t-r)}} \right) \, dv \, dr
\]

\[\leq C(t-s)^{-\frac{3}{2}} e^{-C^{-1}|y|^2} \left( |x-x'| \right)^{3-p}.
\]

Therefore, both \( I_1 \) and \( I_2 \) are bounded by the right-hand side of (3.37), as claimed.

From here, we get a moment bound on the local supremum of \( v \) by using a quantitative version of Kolmogorov’s continuity theorem (see [24, Éq. (6.7)]) with \( m = 0 \):

\[
E \left[ \sup_{x \in Q} v(s, y; t, x)^p \right] \leq C \left( E \left[ v(s, y; t, 0)^p \right] + E \left[ \sup_{x, x' \in Q} |v(s, y; t, x) - v(s, y; t, x')|^p \right] \right)
\]

\[\leq C(t-s)^{-\frac{3}{2}} e^{-\frac{|y|^2}{2t}} + C(t-s)^{-\frac{3}{2}} e^{-C^{-1}|y|^2} \frac{2^{2p-1}/p}{1 - 2^{-2(p-1)/p}}
\]

\[\leq C(1 - \frac{\beta}{t})^{-1} (t-s)^{-\frac{3}{2}} e^{-C^{-1}|y|^2}.
\]

Thus

\[
E \left[ \sup_{x \in Q} u_{<}(s, y; t, x)^p \right] \leq C(t-s)^{-\frac{3}{2}} + E \left[ \sup_{x \in Q} v(s, y; t, x)^p \right],
\]

and the statement follows.

For the tail bounds of the local supremum when the noise is relatively light-tailed, we need a preparatory result, which is a decoupling inequality for tail probabilities.
Lemma 3.7. Let \( (\mathcal{F}_t)_{t \geq 0} \) be a filtration on a probability space \( (\Omega, \mathcal{F}, P) \) and \( \mu \) be an \( (\mathcal{F}_t)_{t \geq 0} \)-Poisson random measure on \( [0, \infty) \times E \), where \( E \) is a Polish space, with intensity measure \( \nu \). Consider a nonnegative \( (\mathcal{F}_t)_{t \geq 0} \)-adapted process \( H : \Omega \times [0, \infty) \times E \rightarrow \mathbb{R} \) and a copy \( H' \), which is defined on an additional probability space \( (\Omega', \mathcal{F}', P') \) (and therefore independent of \( (\mathcal{F}_t)_{t \geq 0} \) on the product space). Let \( \mathbf{P} = P \otimes P' \) and define the random variables

\[
X = \int_{(0,\infty) \times E} H(t, x) \mu(dt, dx), \quad X' = \int_{(0,\infty) \times E} H'(t, x) \mu(dt, dx).
\]

Then, for any \( \theta, R > 0 \),

\[
P(X > R) \leq 7\theta P(X > \frac{1}{3}R) + 2P(X' > \frac{1}{6}R) + \theta^{-1}P(X' > \frac{1}{6}\theta R).
\]

In particular, for any \( p > 0 \),

\[
\exists C > 0, \forall R > 0 : \mathbf{P}(X' > R) \leq CR^{-p} \quad \implies \quad \exists C' > 0, \forall R > 0 : P(X > R) \leq C' R^{-p}.
\]

Proof. The inequality (3.40) was shown on page 38 of [29]. Therefore, only the last statement needs a proof. If \( \mathbf{P}(X' > R) \leq CR^{-p} \) for all \( R > 0 \), (3.40) implies

\[
P(X > R) \leq 7\theta P(X > \frac{1}{3}R) + 2(6^p C) R^{-p} + 6^p C \theta^{-1} R^{-p}.
\]

Choose \( \theta < \frac{1}{3}3^{-p} \) and define \( K = 2(6^p C) + 6^p C \theta^{-1} \). Then iterating the previous equation yields, for any \( n \in \mathbb{N} \),

\[
P(X > R) \leq 7\theta P(X > \frac{1}{3}R) + K R^{-p} \leq 7\theta (7\theta P(X > \frac{1}{3^2}R) + 3^p K R^{-p}) + K R^{-p}
\]

\[
= K R^{-p} + 3^p (7\theta) K R^{-p} + (7\theta)^2 P(X > \frac{1}{3^2}R)
\]

\[
\leq (1 + 3^p (7\theta) K R^{-p} + (7\theta)^2 P(X > \frac{1}{3^2}R) + K 3^p R^{-p})
\]

\[
= (1 + 3^p (7\theta) + 3^p (7\theta)^2) K R^{-p} + (7\theta)^3 P(X > \frac{1}{3^2}R)
\]

\[
\leq \cdots \leq \left( \sum_{j=0}^{n} [3^p (7\theta)]^j \right) K R^{-p} + (7\theta)^{n+1} P(X > \frac{1}{3^{n+1}} R).
\]

Since \( \theta < \frac{1}{3}3^{-p} \), bounding the last probability by 1 and letting \( n \rightarrow \infty \), we conclude that

\[
P(X > R) \leq \frac{K}{1 - 3^p (7\theta)} R^{-p}. \quad \square
\]

Theorem 3.8. Assume Condition \( (L-\alpha) \) with \( \alpha = \frac{2}{d} \) and Condition \( (Sup) \). Then, for every \( t > 0 \), there are \( C_1, C_2 \in (0, \infty) \) such that for all \( R > 1 \) and \( Q \in Q \),

\[
C_1 R^{-\frac{d}{d}} \leq P(\sup_{x \in Q} Y(t, x)) \leq C_2 R^{-\frac{d}{d}}.
\]

Proof. Step 1: Upper bound

Without loss of generality, we may assume that \( Q = (0, 1)^d \). By assumption, \( m_1(\lambda) \) is finite if \( d \geq 2 \). Thus, if we define \( \hat{Y} = Y \), when \( d = 1 \) and \( \hat{Y} = Y^e(\mu) \) when \( d \geq 2 \), then it suffices to prove the theorem for \( \hat{Y} \) instead of \( Y \). To this end, write \( \hat{Y} \) as a sum \( 1 + Y_1 + Y_2 + Y_3 + Y_4 \mathbb{1}_{\{d=1\}} \), where

\[
Y_1(t, x) = \int_0^t \int_D \int_{(0, \infty)} g(t - s, x - y) z \mathbb{1}_{\{(t-s) - (d/2)z \geq 1\}} \hat{Y}(s, y) \mu(ds, dy, dz),
\]

\[
Y_2(t, x) = \int_0^t \int_D \int_{(0, \infty)} g(t - s, x - y) \mathbb{1}_{\{(t-s) - (d/2)z < 1\}} \hat{Y}(s, y) \mu(ds, dy, dz),
\]

\[
Y_3(t, x) = \int_0^t \int_D \int_{(0, \infty)} g(t - s, x - y) \mathbb{1}_{\{(t-s) - (d/2)z = 1\}} \hat{Y}(s, y) \mu(ds, dy, dz),
\]

\[
Y_4(t, x) = \int_0^t \int_D \int_{(0, \infty)} g(t - s, x - y) \mathbb{1}_{\{(t-s) - (d/2)z > 1\}} \hat{Y}(s, y) \mu(ds, dy, dz),
\]

\[
Y_5(t, x) = \int_0^t \int_D \int_{(0, \infty)} g(t - s, x - y) \mathbb{1}_{\{(t-s) - (d/2)z = 0\}} \hat{Y}(s, y) \mu(ds, dy, dz).
\]

Step 2: Lower bound

Step 3: Conclusion
with parameter $Y$ in the last display is replaced by $N$.

Choosing (3.42) identically distributed with density $Y$ where

By Lemma 3.7, it suffices to bound the tail of $X'$, which is the same integral except that $\tilde{Y}$ in the last display is replaced by $\tilde{Y}'$. Further observe that the number $N_t$ of atoms $(\tau, \eta, \zeta)$ that satisfy $\eta \in D$ and $(t - \tau)^{-d/2} \zeta > 1$ is Poisson distributed with parameter

$$L = 4^d \int_0^t \int_0^\infty \int_{(0,\infty)} (t - s)^{-\frac{d}{2}} z \, d\bar{s} \, d\lambda (dz) = 4^d \int_0^\infty (z^2 \lambda + t) \lambda (dz) < \infty.$$  

So conditionally on $N_t$, the corresponding atoms $(\tau_i, \eta_i, \zeta_i)_{i=1,\ldots,N_t}$ are independent and identically distributed with density $L^{-1} \bar{1}_{(s-y) \in (0,\infty)} \, dz \, d\lambda (dz)$. Therefore,

$$\mathbb{P}(X' > R) = e^{-L} \sum_{N=1}^\infty \frac{L^N}{N!} \mathbb{E}' \left[ \mathbb{P} \left( \sum_{i=1}^N (t - \tau_i)^{-\frac{d}{2}} \tilde{Y}'(\tau_i, \eta_i) \zeta_i > R \mid N_t = N \right) \right]$$

(3.42)

$$\leq e^{-L} \sum_{N=1}^\infty \frac{L^N}{N!} \mathbb{E}' \left[ \mathbb{E} \left( (t - \tau_1)^{-\frac{d}{2}} \tilde{Y}'(\tau_1, \eta_1) \zeta_1 > \frac{R}{N} \mid N_t = N \right) \right]$$

$$\leq e^{-L} \sum_{N=1}^{N-1} \frac{L^{N-1}}{(N-1)!} \int_0^t \int_D \int_0^\infty \mathbb{P}'( (t - s)^{-\frac{d}{2}} \tilde{Y}'(s, y) z > \frac{R}{N} ) \, ds \, dy \, d\lambda (dz).$$

Choosing $p \in (1 \vee \frac{2}{d}, 1 + \frac{2}{d})$ and recalling the notation $\mathbb{E}_< \text{ and } \mathbb{E}_>$ from the proof of Theorem 3.1, we can use Markov’s inequality and [9, Lemma 8.1] to obtain

$$\mathbb{P}'( (t - s)^{-\frac{d}{2}} \tilde{Y}'(s, y) z > \frac{R}{N} ) = \mathbb{P}( (t - s)^{-\frac{d}{2}} \tilde{Y}(s, y) z > \frac{R}{N} )$$

$$\leq \mathbb{E}_> \left[ C R^{-p} \mathbb{E}_< [ \tilde{Y}(s, y)^p ] N^p z^p (t - s)^{-\frac{d}{2}p} \lambda \right]$$

where $Y''$ is the solution to (2.5) driven by $\beta \Lambda_\geq$ and $\beta = \beta(p,T) > 0$ is a constant.

Since $\mathbb{E}[X \wedge 1] = \int_0^1 \mathbb{P}(X > u) \, du$,

$$\mathbb{P}'( (t - s)^{-\frac{d}{2}} \tilde{Y}'(s, y) z > \frac{R}{N} ) \leq C \int_0^1 \mathbb{P}(Y''(s, y) > \frac{R}{Nz^p} (t - s)^{\frac{d}{2}}) \, du.$$
Now observe that \( Y'' \) is a series of multiple stochastic integrals with respect to the positive measure \( \beta \Lambda_\ge \). Together with the stationarity of \( \beta \Lambda_\ge \), it follows that \( Y''(s, y) \) is stochastically dominated by \( Y''(t, 0) \). Thus, replacing \( Y''(s, y) \) by \( Y''(t, 0) \) only increases the probabilities in the last display. Making this modification and inserting the resulting bound back into (3.42), we can change variables \( s \mapsto r = (t - s)^{d/2}/2ru^{1/\alpha}/(Nz) \) to obtain

\[
P(X' > R) \leq C\mu_{2/d}(\lambda) R^{-2} \left( \sum_{N=1}^{\infty} \frac{L^{N-1}N^{2/d}}{(N-1)!} \right) \int_0^1 u^{-\frac{2}{\alpha d}} du \int_0^\infty P(Y''(t, 0) > r) C^{-1} e^{-C^{-1}|y|^2} \, dz.
\]

(3.43)

Note that the \( du \)-integral is finite since \( p > \frac{3}{2} \) and that the last integral is just a multiple of \( \mathbb{E}[Y''(t, 0)^{2/d}] \) and hence also finite because \( \mu_{2/d}(\lambda) \) is.

Next, we consider \( Y_2 \). Because \( x \in (0, 1)^d \) and \( y \notin (-2, 2)^d \), the distance \( |x - y| \) is bounded from below by 1. Therefore, \( g(t - s, x - y) \leq C\mathbb{E}e^{-C^{-1}|y|^2} \) for some \( C \in (0, \infty) \).

Since this removes the singularity of \( g \) around 0 as well as the dependence on \( x \), it is easy to show that \( \sup_{x \in Q} Y_2(x, t) \) has a uniformly bounded moment of order \( \frac{2}{d} \) on \([0, t] \). Thus, the tail of \( \sup_{x \in Q} Y_2(x, t) \) is lighter and hence negligible in (3.41).

The term \( Y_4 \) is only present if \( d = 1 \). In this case, we use the bound

\[
\mathbb{E}\left[ \sup_{x \in Q}|Y_4(t, x)|^2 \right] \leq C \int_0^t \int_{(0, \infty)} (t - s)^{-\frac{2}{d}} e^{-C^{-1}|y|^2} z \times |1_{\{s - \frac{1}{2} < z < 1\}} - 1_{\{|z| < 1\}}| \mathbb{E}|\tilde{Y}(s, y)|^2|^2 \, ds \, dy \, \lambda(dz).
\]

Evaluating the difference of indicator functions in the last line we bound this by

\[
C \int_0^t \int_{(0, \infty)} s^{-\frac{2}{d}} z 1_{\{s \leq \frac{1}{2} \}} \, ds \, \lambda(dz) + C \int_0^t \int_{(0, \infty)} s^{-\frac{2}{d}} z 1_{\{s \leq \frac{1}{2}, |z| < 1\}} \, ds \, \lambda(dz)
\]

\[
\leq C \int_1^t \int_{(1, \frac{1}{2})} z \, \lambda(dz) \, ds + C \int_0^t \int_{(0, 1)} z \, s^{-\frac{2}{d}} 1_{\{|z| < 1\}} \, ds \, \lambda(dz)
\]

\[
\leq C(M_1(\lambda) + m_2(\lambda)),
\]

which is finite. Thus, \( \sup_{x \in Q}|Y_4(t, x)| \) does not contribute to the tail in (3.41), either.

For the last remaining term \( Y_3 \), if \( d = 1 \) use [39, Thm. 1] (with \( \alpha = p = 2 \)) and Minkowski’s integral inequality to obtain

\[
\mathbb{E}[(Y_3(t, x) - Y_3(t, x'))^2] \leq C \int_0^t \int_{(0, \infty)} |g(t - s, x - y) - g(t - s, x' - y)|^2 \times \mathbb{E}[\tilde{Y}(s, y)^2] z^2 1_{\{|t - s|^{-1/2} < 1\}} \, ds \, dy \, \lambda(dz)
\]

for all \( x, x' \in \mathbb{R} \). Since \( \mathbb{E}[\tilde{Y}(s, y)^2] \) is uniformly bounded on \([0, t] \times \mathbb{R} \), it can be absorbed into the constant \( C \). Observe that \( (t - s)^{-1/2} z < 1 \) implies

\[
|g(t - s, x - y) - g(t - s, x' - y)|^2 z^2 \leq C(t - s)^{-d} z^2 \left[ e^{-\frac{|x - y|^2}{2(t - s)}} - e^{-\frac{|x' - y|^2}{2(t - s)}} \right]^2
\]

\[
\leq C(t - s)^{-d} z^2 \left[ e^{-\frac{|x - y|^2}{2(t - s)}} - e^{-\frac{|x' - y|^2}{2(t - s)}} \right]^q
\]

\[
= C|g(t - s, x - y) - g(t - s, x' - y)|^q z^q,
\]
where \( q < 2 \) is the exponent from Condition (Sup). With this estimate and again [46, Lemme A2], we conclude that

\[
E[|Y_3(t, x) - Y_3(t, x')|^2] \leq C \left( |x - x'|^{3-q} \right).
\]

Since \( 3 - q > 1 \), it follows from [30, Thm. 4.3] that

\[
E \left[ \sup_{x \in Q} Y_3(t, x)^2 \right] \leq E[|Y_3(t, 0)|^2] + E \left[ \sup_{x, x' \in Q} |Y_3(t, x) - Y_3(t, x')|^2 \right] < \infty,
\]

which shows that \( P(\sup_{x \in Q} Y_3(t, x) > R) = o(R^{-2}) \).

If \( d = 2 \), we simply bound

\[
E \left[ \sup_{x \in Q} Y_3(t, x)^2 \right] \leq C \int_0^t \int_{\mathbb{R}^d} (t - s)^{-1} e^{-C^{-1}|y|^2} z \mathbb{1}_{\{(t-s)^{-1}z < 1\}} ds dy \lambda(dz)
\]

\[
\leq \int_0^t \int_{(0,t]} z^{-1} ds \lambda(dz) \leq C \int_{(0,t^{1/2})} z(1 + |\log z|) \lambda(dz),
\]

which shows \( P(\sup_{x \in Q} Y_3(t, x) > R) = o(R^{-1}) \).

If \( d \geq 3 \), we write \( Y_3(t, x) = \sum_{i=0}^{\infty} Y_{3,i}(t, x) \) where

\[
Y_{3,0}(t, x) = \int_0^t \int_{\mathbb{R}^d \setminus (-2,2)^d} \int_{(0,\infty)} g(t-s, x-y)z \mathbb{1}_{\{(t-s)^{-1/2}z < 1\}} \tilde{Y}(s, y) \mu(ds, dy, dz),
\]

\[
Y_{3,i}(t, x) = \int_0^t \int_{(-2,2)^d} \int_{(0,\infty)} g(t-s, x-y)z \mathbb{1}_{\{(t-s)^{-1/2}z < 1\}} \tilde{Y}(s, y) \mu_i(ds, dy, dz),
\]

where the \( \mu_i \)'s are independent Poisson random measures (with intensities \( dt \, ds \, \lambda_i(dz) \)) such that \( \mu = \sum_{i=1}^{\infty} \mu_i \) and such that \( 4^d \int_0^t \int_{(0,\infty)} \mathbb{1}_{\{(t-s)^{-1/2}z < 1\}} ds \lambda_i(dz) \leq 2 \). Such a decomposition is indeed possible, see Lemma A.4. In the same way as we did for \( Y_1 \) and \( Y_2 \), one can now show that

\[
P \left( \sup_{x \in Q} Y_{3,0}(t, x) > R \right) = o(R^{-\frac{d}{4}}), \quad P \left( \sup_{x \in Q} Y_{3,i}(t, x) > R \right) \leq C \mu_{2/d}(\lambda_i) R^{-\frac{d}{4}}
\]

for some \( C > 0 \) that does not depend on \( i \). Borrowing a truncation trick from the proof of [45, Lemma 4.24], we now bound

\[
P \left( \sup_{x \in Q} \sum_{i=1}^{\infty} Y_{3,i}(t, x) > R \right)
\]

\[
\leq \sum_{i=1}^{\infty} P \left( \sup_{x \in Q} Y_{3,i}(t, x) > R \right) + P \left( \sum_{i=1}^{\infty} \sup_{x \in Q} Y_{3,i}(t, x) \mathbb{1}_{\left\{ \sup_{x \in Q} Y_{3,i}(t, x) \leq R \right\}} > R \right)
\]

\[
\leq C \mu_{2/d}(\lambda) R^{-\frac{d}{4}} + \frac{1}{R} \sum_{i=1}^{\infty} E \left[ \sup_{x \in Q} Y_{3,i}(t, x) \mathbb{1}_{\left\{ \sup_{x \in Q} Y_{3,i}(t, x) \leq R \right\}} \right].
\]

As \( E[X \mathbb{1}_{\{X \leq R\}}] \leq E[|X \cap X|] = \int_0^R P(X > u) \, du \), it follows that

\[
P \left( \sup_{x \in Q} \sum_{i=1}^{\infty} Y_{3,i}(t, x) > R \right) \leq C \mu_{2/d}(\lambda) R^{-\frac{d}{4}} + \frac{1}{R} \sum_{i=1}^{\infty} \int_0^R P \left( \sup_{x \in Q} Y_{3,i}(t, x) > u \right) du
\]

\[
\leq C \mu_{2/d}(\lambda) \left( R^{-\frac{d}{4}} + \frac{1}{R} \int_0^R u^{-\frac{d}{4}} du \right) \leq C \mu_{2/d}(\lambda) R^{-\frac{d}{4}}.
\]
Step 2: Lower bound

Without loss of generality, we may assume that \( M_0(\lambda) = \lambda([1, \infty)) > 0 \) and that \( Q = (0, 1)^d \).

By (2.11) and the positivity of \( u_<(t_{i-1}, x_{i-1}; t_i, x_i) \),

\[
\sup_{x \in Q} Y(t, x) \geq \sup_{x \in Q} \int_0^t \int \mathbb{1}_<(s, y; t, x) Y_<(s, y) \Lambda(dy) ds \geq \sup_{x \in Q} u_<(\tau, \eta; t, x) Y_<(\tau, \eta)
\]

on the event \( A = \{ \Lambda_\geq([0, t] \times Q \times [1, \infty)) = 1 \} \). Since \( P(A) = t M_0(\lambda) e^{-t M_0(\lambda)} \), we have

\[
P \left( \sup_{x \in Q} Y(t, x) > R \right) \geq t M_0(\lambda) e^{-t M_0(\lambda)} P \left( \sup_{x \in Q} u_<(\tau, \eta; t, x) Y_<(\tau, \eta) > R \right)
\]

\[
\geq t M_0(\lambda) e^{-t M_0(\lambda)} P \left( \sup_{x \in Q} g_<(t, x) > 4R, Y_<(\tau, \eta) > \frac{1}{2} \right)
\]

\[
\geq t M_0(\lambda) e^{-t M_0(\lambda)} P \left( \sup_{x \in Q} u_<(\tau, \eta; t, x) > \frac{1}{2} g(t, x, \eta) \left| A \right. \right).
\]

With a similar argument as in (3.24), it follows that

\[
P \left( \sup_{x \in Q} Y(t, x) > R \left| A \right. \right) \geq C P \left( \sup_{x \in Q} g(t, x, \eta) > 4R \left| A \right. \right)
\]

\[
= C P((2\pi(t - \tau))^{-\frac{d}{2}} > 4R \left| A \right).
\]

The lower bound in (3.41) now follows from the fact that \( \tau \) has a uniform distribution on \([0, t] \). \( \square \)

4. The spatial peaks of the solution. Armed with the probability tail bounds from the previous section, we can now state and prove extensions of Theorems A–C to general multiplicative Lévy noises.

**Theorem 4.1.** Fix \( t > 0 \) and let \( Y \) be the mild solution to (2.5). Assume Condition (Sup) and Condition (L-\( \alpha \)) with \( \alpha = \frac{\gamma}{d} \). If \( d = 1 \) and \( m_1(\lambda) = \infty \), assume Condition (L-\( \alpha \)) with some \( \alpha > 2 \). Then the statement of Theorem A remains true.

If \( d = 1 \) and \( \Lambda \) has infinite variation jumps, we need \( \lambda \) to have a finite moment of order slightly bigger than 2, in particular, in order to derive (4.9) below. We strongly believe that it is not necessary for Theorem 4.1 to hold.

**Proof of Theorem 4.1.** Let us suppose that the integral converges. For \( r > 0 \) and \( 0 < r_1 < r_2 < \infty \), let \( B_{\infty}(r) = \{ x \in \mathbb{R}^d : \max_{i=1, \ldots, d} |x_i| \leq r \} \) and \( B_{\infty}(r_1, r_2) = B_{\infty}(r_2) \setminus B_{\infty}(r_1) \). Then, for any \( K > 0 \),

\[
P \left( \sup_{x \in B_{\infty}(n, n+1)} Y(t, x) > \frac{f(n)}{K} \right) \leq C n^{d-1} \sup_{Q_0 \in Q, Q_0 \subseteq B_{\infty}(n, n+1)} P \left( \sup_{x \in Q_0} Y(t, x) > \frac{f(n)}{K} \right)
\]

\[
\leq C K^{-\frac{d}{2}} n^{d-1} f(n)^{-\frac{\gamma}{2}}, \tag{4.1}
\]
where the first step holds because the number of cubes from \( Q \) intersecting \( B_\infty(n, n+1) \) is \( O(n^{d-1}) \) and the second step follows from Theorem 3.8. By the integral test for convergence, these probabilities are summable, so by the first Borel–Cantelli lemma,

\[
\sup_{x \in B_\infty(n, n+1)} Y(t, x) \leq \frac{f(n)}{K}
\]

for all but finitely many \( n \), almost surely. This shows

\[
\limsup_{x \to \infty} \sup_{|y| \leq x} \frac{Y(t, y)}{f(x)} \leq K^{-1}
\]

and hence the claim because \( K > 0 \) was arbitrary.

For the converse, there is no loss of generality if we assume that \( \lambda([1, \infty)) > 0 \). If \( d \geq 2 \), recall that \( m_1(\lambda) < \infty \), so by (3.29), \( Y(t, x) = e^{-m_1(\lambda)t} \hat{Y}(t, x) \geq e^{-m_1(\lambda)t} \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) \Lambda(ds, dy) \), which is a multiple of the solution to the heat equation with additive Lévy noise. Hence, the result follows from [22]. The same argument applies in \( d = 1 \) if \( m_1(\lambda) < \infty \).

If \( d = 1 \) and \( m_1(\lambda) = \infty \), the proof is more technical due to infinite variation jumps. We assume without loss of generality that \( f \) is smooth. With the same notation as in the upper bound proof of Theorem 3.8, we have

\[
Y(t, x) \geq Y_0(t, x) + Y_3(t, x) + Y_4(t, x),
\]

where

\[
Y_0(t, x) = \int_0^t \int_{x-1}^{x+1} \int_0^\infty g(t-s, x-y) z \mathbb{1}_{\{(t-s)^{1/2}z \geq 1\}} Y(s, y) \mu(ds, dy, dz).
\]

Next, let \( \varphi(x) = \sqrt{x + f(x)^2} \) and \( h \) be an increasing function to be determined later and define

\[
m = m(n) = h(n), \quad \beta = \beta(n) = h(n)^{1/4}, \quad R = R(n) = K\varphi(nh(n)).
\]

Since \( x \vee f(x)^2 \leq \varphi(x)^2 \leq 2(x \vee f(x)^2) \), the divergence of the integral in (1.5) implies

\[
\int_1^\infty \frac{1}{\varphi(x)^2} \text{d}x = \infty
\]

by [21, Lemma 3.4 (2)]. We shall approximate \( Y_0 \) by

\[
Z^{(m, \beta)}(t, x) = \int_0^t \int_{x-1}^{x+1} \int_0^\infty g(t-s, x-y) z \mathbb{1}_{\{(t-s)^{1/2}z \geq 1\}} |Y^{(m, \beta)}(s, y)| \mu(ds, dy, dz),
\]

where \( Y^{(m, \beta)} \) is defined in Lemma A.5. More precisely, writing \( I_n = (nh(n) - 1, nh(n)) \), we want to prove that

\[
\sum_{n=1}^{\infty} \mathbb{P}\left( \sup_{x \in I_n} Z^{(m(n), \beta(n))}(t, x) > R(n) \right) = \infty,
\]

\[
\sum_{n=1}^{\infty} \mathbb{P}\left( \sup_{x \in I_n} |Y_0(t, x) - Z^{(m(n), \beta(n))}(t, x)| > \frac{1}{4} R(n) \right) < \infty,
\]

\[
\sum_{n=1}^{\infty} \mathbb{P}\left( \sup_{x \in I_n} |Y_3(t, x) + Y_4(t, x)| > \frac{1}{4} R(n) \right) < \infty.
\]
By the first Borel–Cantelli lemma, this implies that
\begin{equation}
(4.7) \quad \sup_{x \in I_n} |Y_0(t, x) - Z^{(m(n), \beta(n))}(t, x)| \leq \frac{1}{4} R(n), \quad \sup_{x \in I_n} |Y_3(t, x) + Y_4(t, x)| \leq \frac{1}{4} R(n)
\end{equation}
for all but finitely many \( n \), almost surely. Moreover, by (4.5) and the definition of \( Y^{(m, \beta)} \), the variables \( \sup_{x \in I_n} Z^{(m(n), \beta(n))}(t, x) \) are measurable with respect to the \( \sigma \)-field generated by the restriction of \( \Lambda \) on \( [0, t] \times (nh(n) - 2 - 2\beta(n)(tm(n))^{1/2}, nh(n) + 1 + 2\beta(n)(tm(n))^{1/2}) \), because only atoms of \( \Lambda \) that are within a distance of \( \sum_{i=1}^{m} |\Delta y_i| \leq m^{1/2}(\sum_{i=1}^{m} |\Delta y_i|)^{1/2} \) from \( Y^{(m, \beta)}(t, x) \) and because the same holds true for \( Y^{(m, \beta)}(t, x) \) and \( u^{(m, \beta)}(s, y; t, x) \). Since \( \beta(n)(tm(n))^{1/2} = o(h(n)) \), the considered variables are independent for different \( n \). By the second Borel–Cantelli lemma, (4.6) also implies that
\[
\sup_{x \in I_n} Z^{(m(n), \beta(n))}(t, x) > R(n)
\]
for infinitely many \( n \), almost surely. Combining this with (4.7), it follows that
\[
\lim_{n \to \infty} \sup_{|x| \leq nh(n)} \frac{\sup_{|x| \leq nh(n)} Y(t, x)}{f(nh(n))} \geq \lim_{n \to \infty} \frac{\sup_{|x| \leq nh(n)} Y(t, x)}{\varphi(nh(n))} \geq \frac{K}{2}
\]
almost surely. As \( K \) is arbitrary, the claim follows.

To prove (4.6), we start with the first statement and notice that
\[
P\left( \sup_{x \in I_n} Z^{(m(n), \beta(n))}(t, x) > R(n) \right)
\geq P\left( \bigvee_{i=1}^{N} (2\pi(t - \tau_i))^{-\frac{1}{2}} |Y^{(m(n), \beta(n))}(\tau_i, \eta_i)| |\zeta_i > R(n) \right),
\]
where \((\tau_i, \eta_i, \zeta_i)_{i=1}^{N}\) are atoms of \( \Lambda_1 \), where \( \Lambda_1 \) is the restriction of \( \Lambda \) to the set \( \{(s, y, z) \in (0, t) \times (x - 1, x + 1) \times (0, \infty) : (t - s)^{-1/2} z > 1\} \) (with the convention \( \bigvee_{i=1}^{0} a_i = 0 \)). By [29, Lemma 4.1],
\[
P\left( \sup_{x \in I_n} Z^{(m(n), \beta(n))}(t, x) > R(n) \right)
\geq \frac{1}{2} P \otimes P\left( \bigvee_{i=1}^{N} (2\pi(t - \tau_i))^{-\frac{1}{2}} |Y^{r(m(n), \beta(n))}(\tau_i, \eta_i)| |\zeta_i > R(n) \right),
\]
where \( Y^{r(m, \beta)} \) is a copy of \( Y^{(m, \beta)} \) that is independent of \( \Lambda \) (and defined on an auxiliary probability space \( (\Omega', F', P') \)). Hence, the right-hand side of the previous display is bounded from below by
\[
\frac{1}{2} P(N \geq 1) P \otimes P\left( (2\pi(t - \tau_1))^{-\frac{1}{2}} \zeta_1 > 2R(n), \ |Y^{r(m(n), \beta(n))}(\tau_1, \eta_1)| > \frac{1}{2} \right).
\]
As in (3.24), we have \( P'(|Y^{r(m, \beta)}(t, x)| > \frac{1}{2}) > C \) locally uniformly in \( t \) and \( x \) (and uniformly in \( m \) and in \( \beta \) outside a neighborhood of 0). Therefore,
\[
P\left( \sup_{x \in I_n} Z^{(m(n), \beta(n))}(t, x) > R(n) \right) \geq \frac{1}{2} C P\left( (2\pi(t - \tau_1))^{-\frac{1}{2}} \zeta_1 > 2R(n) \right) \geq CR(n)^{-2},
\]
where the last step follows by a quick computation. Thus, the first relation in (4.6) is satisfied if
\begin{equation}
\sum_{n=1}^{\infty} \frac{1}{\varphi(nh(n))^2} = \infty \iff \int_{1}^{\infty} \frac{1}{\varphi(xh(x))^2} \, dx = \infty.
\end{equation}

Let us move on to the second relation in (4.6). By Lemma A.6,
\begin{equation}
P \left( \sup_{x \in I_n} |Y_0(t, x) - Z^{(m(n), \beta(n))}(t, x)| > \frac{1}{4} R(n) \right) \leq 16 C_{m(n), \beta(n)} R(n)^{-2},
\end{equation}
where \(C_{m, \beta}\) is the constant from the lemma. With our choices of \(m\) and \(\beta\), we have \(C_{m(n), \beta(n)} = O(e^{-C^{-1}h(n)^{1/4}})\), so the second line in (4.6) is implied by
\begin{equation}
\sum_{n=1}^{\infty} \frac{e^{-C^{-1}h(n)^{1/4}}}{\varphi(nh(n))^2} < \infty \iff \int_{1}^{\infty} \frac{e^{-C^{-1}h(x)^{1/4}}}{\varphi(xh(x))^2} \, dx < \infty.
\end{equation}

Lastly, because \(\sup_{x \in I_n} |Y_4(t, x)|\) has uniformly (in \(n\)) bounded moments of order \(\alpha > 2\), we have that the last relation in (4.6) is implied by
\begin{equation}
\sum_{n=1}^{\infty} \frac{1}{\varphi(nh(n))^\alpha} < \infty \iff \int_{1}^{\infty} \frac{1}{\varphi(xh(x))^\alpha} \, dx < \infty.
\end{equation}

And this is true, because \(\varphi(xh(x)) \geq \varphi(x) \geq \sqrt{x}\). Consequently, in order to complete the proof, it remains to choose \(h\) such that (4.8) and (4.10) are satisfied.

To this end, we will restrict our choice of \(h\) to the class of increasing smoothly varying functions of index 0 (see [12, Ch. 1.8]). In this case, if we change the variable \(x = g(x) = xh(x)\), then there exist \(y_0 > 0\) and a smoothly varying function \(h^\#\) with index 0 (the de Bruijn conjugate of \(h\)) such that \(g^{-1}(y) = yh^\#(y)\) for all \(y > y_0\) (see [12, Thm. 1.8.9]). Moreover, for \(y > y_0\), we have \((g^{-1})'(y) = h^\#(y) + y(h^\#)'(y)\) and \(y(h^\#)'(y) = o(h^\#(y))\) (see [12, p. 44]). Therefore the conditions in (4.8) and (4.10) are equivalent to having both
\begin{equation}
\int_{y_0}^{\infty} \frac{h^\#(y)}{\varphi(y)^2} \, dy = \infty \quad \text{and} \quad \int_{y_0}^{\infty} \frac{e^{-C^{-1}h^\#(y)^{-1/4}} h^\#(y)}{\varphi(y)^2} \, dy < \infty.
\end{equation}

Note that \(h^\#\) is decreasing (as \(h\) is increasing). Thus, the previous line is implied by
\begin{equation}
\int_{y_0}^{\infty} \frac{h^\#(y)}{\varphi(y)^2} \, dy = \infty \quad \text{and} \quad \int_{y_0}^{\infty} \frac{h^\#(y)^2}{\varphi(y)^2} \, dy < \infty.
\end{equation}

To achieve this, we choose \(h\) as the de Bruijn conjugate of
\begin{equation}
h^\#(y) = \left( \int_{\frac{y}{2}}^{y} \varphi(u)^{-2} \, du \right)^{-1}, \quad y > 1.
\end{equation}

Indeed, \(h^\#\) is smoothly varying with index 0 as it has the Karamata representation \(h^\#(y) = c \exp(-\int_{y}^{\infty} \varepsilon(t) \, dt)\), where
\begin{equation}
\lim_{t \to \infty} \varepsilon(t) = \lim_{t \to \infty} \frac{t}{\varphi(t)^2 \int_{1}^{t} \varphi(u)^{-2} \, du} = \lim_{t \to \infty} \frac{1}{1 + (f(t)^2) \int_{1}^{t} \varphi(u)^{-2} \, du + 1} = 0
\end{equation}
by (4.4). Finally, \(h^\#\) satisfies (4.11) by (4.4) and the Abel–Dini theorem [35, p. 290]. \(\square\)

**Theorem 4.2.** Fix \(t > 0\) and let \(Y\) be the mild solution to (2.5). If Condition (H-\(\alpha\)) for some \(\alpha \in (0, \frac{3}{4}]\) and Condition (Sup) are satisfied, then Theorem B (ii) remains true.
PROOF. Define (with the convention $\inf \emptyset = \infty$)

\[(4.12) \quad L_0 = \inf \left\{ L \in (0, \infty) : \limsup_{R \to \infty} R^{\alpha} e^{-L (\log R)^{1/(1+\theta^*)}} P \left( \sup_{x \in (0,1)^d} Y(t,x) > R \right) < \infty \right\} \]

and $L^* = \alpha^{-1} \left( \frac{d}{\alpha} \right)^{1/(1+\theta^*)} L_0$. By Theorem 3.5, we have $L_0 < \infty$. For any $L > L^*$ and any fixed $\varepsilon > 0$ (to be chosen later), combining the upper bound in Theorem 3.5 with a similar argument to (4.1) shows that

\[
P \left( \sup_{x \in B_{\infty}(n,n+1)} Y(t,x) > \frac{n^{d/\alpha} e^{L (\log n)^{1/(1+\theta^*)}}}{K} \right) \leq C n^{-1} \exp \left( -L \alpha (\log n)^{1/(1+\theta^*)} + (L_0 + \varepsilon) \left[ \log(K^{-1} n^{d/\alpha} e^{L (\log n)^{1/(1+\theta^*)}}) \right]^{1/(1+\theta^*)} \right) \leq C n^{-1} \exp \left( -[L \alpha - (L_0 + \varepsilon) (\frac{d}{\alpha} + \varepsilon)^{1/(1+\theta^*)}] (\log n)^{1/(1+\theta^*)} \right)
\]

for sufficiently large $n$ (note that $C$ in the previous display may depend on $\varepsilon$, $L$ and $K$ but not on $n$). By our assumption on $L$, if $\varepsilon$ is small enough, we have $L \alpha - (L_0 + \varepsilon) (\frac{d}{\alpha} + \varepsilon)^{1/(1+\theta^*)} > 0$. Therefore, the probabilities in the previous display are summable, so by the Borel–Cantelli lemma,

\[
\limsup_{x \to \infty} \sup_{|y| \leq x} \frac{Y(t,y)}{x^{d/\alpha} e^{L (\log x)^{1/(1+\theta^*)}}} \leq \frac{1}{K}
\]

almost surely. Therefore, (1.7) follows by letting $K \to \infty$. Equation (1.8) is a direct consequence of Theorem 4.3 (i) (with $L^* = M_*$), which we state and prove next. \(\square\)

**Theorem 4.3.** Fix $t > 0$ and let $Y$ be the mild solution to (2.5).

(i) If Condition (H-α) holds for some $\alpha \in (0, 1 + \frac{2}{d})$, then the statement of Theorem C (ii) remains valid.

(ii) If Condition (H-α) or (L-α) holds with $\alpha = 1 + \frac{2}{d}$, then the statement of Theorem C (iii) remains valid.

**Proof.** We assume without loss of generality that $\lambda([1, \infty)) > 0$. Let $C_0$ be the number from (3.19) in the case of (i) and $C_0 = 1$ in the case of (ii). Furthermore, recall the definition of $Y(N)(t, x)$ from Lemma A.5, which satisfies $Y(t,x) \geq Y(N)(t, x)$. If $\lambda$ satisfies the conditions of part (i), define (with the convention $\sup \emptyset = 0$)

\[(4.13) \quad M_1 = \sup \left\{ M \in (0, \infty) : \liminf_{R \to \infty} R^{\alpha} e^{-M (\log R)^{1/(1+\theta^*)}} P \left( Y([C_0 \log R]) (t,0) > R \right) > 0 \right\},
\]

\[M_2 = \inf \left\{ M \in (0, \infty) : \limsup_{R \to \infty} R^{\alpha} e^{-M (\log R)^{1/(1+\theta^*)}} P \left( Y(t,0) > R \right) < \infty \right\}
\]

and let $M_* = \alpha^{-1} \left( \frac{d}{\alpha} \right)^{1/(1+\theta^*)} M_1$ and $M^* = \alpha^{-1} \left( \frac{d}{\alpha} \right)^{1/(1+\theta^*)} M_2$; if $\lambda$ satisfies the conditions of part (ii), define

\[(4.14) \quad M_1 = \sup \left\{ M \in (0, \infty) : \liminf_{R \to \infty} R^{1+\frac{2}{d}} e^{-M (\log R)^{\log \log R} / \log log R} P \left( Y(C_0 \log R) (t,0) > R \right) > 0 \right\},
\]

\[M_2 = \inf \left\{ M \in (0, \infty) : \limsup_{R \to \infty} R^{1+\frac{2}{d}} e^{-M (\log R)^{\log \log R} / \log log R} P \left( Y(t,0) > R \right) < \infty \right\}
\]
and let $M_* = \frac{d^3}{(d+2)^2} M_1$ and $M^* = \frac{d^3}{(d+2)^2} M_2$ instead. Both (1.9) and (1.11) can be shown similarly to (1.7), so we leave the details to the reader. Also, the proofs of (1.10) and (1.12) are similar, so we only give the details for the former and assume Condition (H-$\alpha$) for some $\alpha \in (0, 1 + \frac{2}{d})$.

In the lower bound proof of Theorem 3.1, we have seen that $M_1 > 0$. Let $N(R) = C_0 \log R$, $m(R) = \log R$ and $\beta(R) = (\log R)^2$. Then, by Lemma A.5, we also have

$$\liminf_{R \to \infty} R^p e^{-M_0 (\log R)^{1/(1+\theta_\alpha)}} \mathbb{P}(Y (N(R), m(R), \beta(R)) (t, x) > R) > 0$$

for all $0 < M_0 < M_1$ and $x \in \mathbb{R}^d$. Similarly to what we observed in the proof of Theorem 4.1, the variable $Y (N(R), m(R), \beta(R)) (t, x)$ is measurable with respect to the $\sigma$-algebra generated by $\Lambda$ restricted to a ball of radius $(t N(R))^{1/2} + \beta(R) (m(R))^{1/2}$ around $x$. Therefore, if we let $f(n) = n^{d/\alpha} e^{M (\log n)^{1/(1+\theta_\alpha)}}$ (with $M < M_*$) and $R = R(n) = K f(n \log^3 n)$ for $n, K \in \mathbb{N}$ and distribute $k(n) = c n^{d-1} \log n$ points, $c > 0$, from $\mathbb{Z}^d$ to the annulus $B_0 \log n - 1, n \log n^3$ such that these points, say, $x_1^{(n)}, \ldots, x_k^{(n)}$ are at least $\log^3 n$ apart from each other, then all but finitely many of the variables

$$\left\{ Y_i^{(n)} = Y (N(R(n)), m(R(n)), \beta(R(n))) (t, x_i^{(n)}) : i = 1, \ldots, k(n), n \in \mathbb{N} \right\}$$

are independent of each other. Moreover, by (4.15), for any $\varepsilon > 0$ there is $C > 0$ such that

$$\sum_{n=1}^\infty \sum_{i=1}^{k(n)} \mathbb{P}(Y_i^{(n)} > R(n)) \geq \frac{C}{K} \sum_{n=1}^\infty n^{d-1} R(n)^{-\alpha} e^{M (1 - \varepsilon)/(\log R(n))^{1/(1+\theta_\alpha)}}$$

$$\geq \sum_{n=1}^\infty n^{-1} (\log n)^{-3d} e^{-M_0 (\log n \log^3 n)^{1/(1+\theta_\alpha)}} e^{M (1 - \varepsilon)/(\log R(n))^{1/(1+\theta_\alpha)}}.$$

As $M < M_*$, if $\varepsilon$ is small enough, the last series is infinite, so by the second Borel–Cantelli lemma, $Y_i^{(n)} > R(n)$ for infinitely many $n$ and $i$. At the same time, by Lemma A.5, our choice of $\beta = (\beta(R(n))$ and $m = m(R(n))$ and the first Borel–Cantelli lemma, the events $\{ |Y (N(R(n))) (t, x_i^{(n)}) - Y (N(R(n)), m(R(n)), \beta(R(n))) (t, x_i^{(n)})| > \frac{1}{2} R(n) \}$ only occur finitely many times, which implies

$$\limsup_{n \to \infty} \sup_{x \in \mathbb{Z}^d, |y| \leq n \log^3 n} \frac{Y(t, x)}{f(n \log^3 n)} \geq \frac{K}{2}$$

almost surely. Because $K \in \mathbb{N}$ was arbitrary, this implies (1.9).

5. Macroscopic dimension of peaks. As another application of the tail estimates of Section 3, we determine the macroscopic Hausdorff and Minkowski dimensions of the peaks of the solution to (1.1), both in the case of additive and multiplicative noise. In the case where $\Lambda$ is a Gaussian noise in dimension 1, a similar program has been carried out by [32]; see also [33]. Let us briefly review the relevant definitions, first introduced by [7, 8] for subsets of $\mathbb{Z}^d$ and extended to subsets of $\mathbb{R}^d$ by [32, 33] and [34]. Writing $Q_1 = \{ Q(x, r) = x + (0, r)^d : x \in \mathbb{R}^d, r \geq 1 \}$ for the collection of cubes with side length $\text{side}(Q(x, r)) = r \geq 1$, we define, for $E \subseteq \mathbb{R}^d$, $\rho > 0$ and $n \in \mathbb{N}$,

$$\nu^n_\rho (E) = \inf \left\{ \sum_{i=1}^m \left( \frac{\text{side}(Q_i)}{e^n} \right)^\rho : Q_1, \ldots, Q_m \in Q_1, E \cap S_n \subseteq \bigcup_{i=1}^m Q_i \right\},$$

$$\nu^n_\rho (E) = \inf \left\{ \sum_{i=1}^m \left( \frac{\text{side}(Q_i)}{e^n} \right)^\rho : Q_1, \ldots, Q_m \in Q_1, E \cap S_n \subseteq \bigcup_{i=1}^m Q_i \right\}.$$
where $S_n = B_\infty(e^{n-1}, e^n)$. Letting $\log_+(x) = \log(x \vee e)$, we define

$$\text{Dim}_H(E) = \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} n^\rho \|E\| < \infty \right\},$$

(5.1)

$$\text{Dim}_M(E) = \limsup_{n \to \infty} \frac{1}{n} \log_+ \left| \{ q \in \mathbb{Z}^d : E \cap Q(q, 1) \neq \emptyset \} \right|.$$

5.1. The multifractal nature of peaks. We shall determine the macroscopic dimensions of the largest peaks observed on $\mathbb{R}^d$ and $\mathbb{Z}^d$. Recall the convention that $Y_+$ denotes the solution to (1.1) with $\alpha(x) = 1$, while $Y$, as before, is the solution with $\alpha(x) = x$. For $\gamma \in [0, \infty)$, we consider

$$\mathcal{E}_\gamma^{+,c} = \{ x \in \mathbb{R}^d : Y_+(t, x) \geq |x|^\gamma \}, \quad \mathcal{E}_\gamma^{x,c} = \{ x \in \mathbb{R}^d : Y(t, x) \geq |x|^\gamma \},$$

$$\mathcal{E}_\gamma^{+,d} = \{ x \in \mathbb{Z}^d : Y_+(t, x) \geq |x|^\gamma \}, \quad \mathcal{E}_\gamma^{x,d} = \{ x \in \mathbb{Z}^d : Y(t, x) \geq |x|^\gamma \}.$$

(5.2)

**Theorem 5.1.** Let $\gamma \in [0, \infty)$. In the following, $\text{Dim}_{H|M}$ means one can take $\text{Dim}_H$ or $\text{Dim}_M$ in the statement. Also, $\text{Dim}_{H|M}(A) < 0$ means that $A$ is a bounded set.

(i) Assume Condition (Sup) and Condition (L-\alpha) with $\alpha = \frac{2}{d}$. If $d = 1$ and $m_1(\lambda) = \infty$, assume Condition (L-\alpha) with some $\alpha > 2$. Then almost surely,

$$\text{Dim}_{H|M}(\mathcal{E}_\gamma^{x,c}) = \text{Dim}_{H|M}(\mathcal{E}_\gamma^{+,c}) = d - \frac{2}{\gamma}.$$

(5.3)

(ii) If Condition (Sup) and Condition (H-\alpha) hold with $\alpha \in (0, \frac{2}{d}]$, then almost surely,

$$\text{Dim}_{H|M}(\mathcal{E}_\gamma^{+,c}) = \text{Dim}_{H|M}(\mathcal{E}_\gamma^{x,c}) = d - \alpha \gamma.$$

(5.4)

(iii) If Condition (L-\alpha) holds with $\alpha = 1 + \frac{2}{d}$, then almost surely,

$$\text{Dim}_{H|M}(\mathcal{E}_\gamma^{+,d}) = \text{Dim}_{H|M}(\mathcal{E}_\gamma^{x,d}) = d - (1 + \frac{2}{d}) \gamma.$$

(5.5)

(iv) If Condition (H-\alpha) holds with $\alpha \in (0, 1 + \frac{2}{d}]$, then almost surely,

$$\text{Dim}_{H|M}(\mathcal{E}_\gamma^{x,d}) = \text{Dim}_{H|M}(\mathcal{E}_\gamma^{+,d}) = d - \alpha \gamma.$$

(5.6)

**Proof.** The statements when the right-hand sides of (5.3)–(5.6) are negative follow from Theorems 4.1 and 4.3 in the case of multiplicative noise and from [22, Theorems 6 and 7] in the case of additive noise. In the following, we only give the full details for the proof of $\text{Dim}_M(\mathcal{E}_\gamma^{x,c}) \leq d - \frac{2}{\gamma} \gamma$ (Step 1) and the proof of $\text{Dim}_H(\mathcal{E}_\gamma^{x,c}) \geq d - \frac{2}{\gamma} \gamma$ (Step 2), both under Condition (Sup) and Condition (L-\alpha) with $\alpha = \frac{2}{d}$ and the assumption $\gamma \leq d^2/2$. For both parts, the proofs are inspired by ideas from [32]. By [8, Lemma 3.1], Steps 1 and 2 imply $\text{Dim}_H(\mathcal{E}_\gamma^{x,c}) = \text{Dim}_M(\mathcal{E}_\gamma^{x,c}) = d - \frac{2}{\gamma} \gamma$. We explain towards the end of the proof (Step 3) why all other equalities in (5.3)–(5.6) can be shown analogously.

**Step 1: $\text{Dim}_M(\mathcal{E}_\gamma^{x,c}) \leq d - \frac{2}{\gamma} \gamma$**

Clearly,

$$\mathbb{E} \left[ \left| \{ q \in \mathbb{Z}^d : \mathcal{E}_\gamma^{x,c} \cap Q(q, 1) \neq \emptyset \} \right| \right] = \sum_{q \in \mathbb{Z}^d \cap S_n} \mathbb{P}(\mathcal{E}_\gamma^{x,c} \cap Q(q, 1) \neq \emptyset)$$

$$\leq \sum_{q \in \mathbb{Z}^d \cap S_n} \mathbb{P} \left( \sup_{x \in Q(q, 1)} Y(t, x) > (|q| - 1)^\gamma \right).$$
Since $|Z^d \cap S_n| \leq Ce^{nd}$ for some $C > 0$ and $|q| \geq e^{n-1}$ for all $q \in Z^d \cap S_n$, Theorem 3.8 implies

$$E\left[\{q \in Z^d \cap S_n : \mathcal{E}_{\gamma}^{\times,c} \cap Q(q, 1) \neq \emptyset\}\right] \leq Ce^{nd}(e^{n-1} - 1)^{-\frac{2}{d}} \leq Ce^{n(d - \frac{2}{d})}$$

for all $n \geq 2$. By Markov’s inequality, $P(\{q \in Z^d \cap S_n : \mathcal{E}_{\gamma}^{\times,c} \cap Q(q, 1) \neq \emptyset\} > e^{\theta n})$ is summable for all $\theta > d - \frac{2}{d}\gamma$. According to the first Borel–Cantelli lemma, for all $\theta$ in this range,

$$\limsup_{n \to \infty} \frac{1}{n} \log E[\{q \in Z^d \cap S_n : \mathcal{E}_{\gamma}^{\times,c} \cap Q(q, 1) \neq \emptyset\}] \leq \theta$$

almost surely. The upper bound on $\dim_M(\mathcal{E}_{\gamma}^{\times,c})$ follows by letting $\theta \downarrow d - \frac{2}{d}\gamma$.

**Step 2**: $\dim_H(\mathcal{E}_{\gamma}^{\times,c}) \geq d - \frac{2}{d}\gamma$

We can assume that $d - \frac{2}{d}\gamma > 0$. As in the proof of Theorem 4.1, the case $d \geq 2$ is easier because the jumps are summable. So starting with $d \geq 2$, we choose $\theta \in (2\gamma/d^2, 1)$ and consider the grid

$$\{x^n_k : k = 1, \ldots, K^n\} = \{e^{n-1} + i e^{\theta n} : i \in \mathbb{N}, 1 \leq i \leq e^{n(1-\theta)} - e^{n(1-\theta)-1}\}$$

in $S_n$ and, within each of the cubes $Q(x^n_k, e^{\theta n})$, the subgrid

$$\{z^n_{k,\ell} : \ell = 1, \ldots, L^n_{k}\} = x^n_k + \{j \in \mathbb{N} : 1 \leq j \leq e^{\theta n}\}.$$

For every $k = 1, \ldots, K^n$ and $\ell = 1, \ldots, L^n_{k}$, we introduce the random fields

$$Y^n_{k,\ell}(t, x) = \int_0^t \int_{Q(z^n_{k,\ell},1)} g(t - s, x - y) \Lambda(ds, dy), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

which are independent for different values of $k$ and $\ell$ and satisfy $Y^n_{k,\ell}(t, x) \leq \tilde{Y}(t, x) = e^{m_1(\lambda)t}Y(t, x)$. Therefore, for all $n \in N$ and $k = 1, \ldots, K^n$,

$$P\left(\sup_{x \in Q(x^n_k, e^{\theta n})} \frac{Y(t, x)}{|x|^{\gamma}} < 1\right) \leq P\left(\max_{\ell=1,\ldots,L^n_k} \sup_{x \in Q(z^n_{k,\ell},1)} Y(t, x) < e^{\theta n}\right)$$

$$\leq P\left(\max_{\ell=1,\ldots,L^n_k} \sup_{x \in Q(z^n_{k,\ell},1)} Y^n_{k,\ell}(t, x) < e^{m_1(\lambda)t+n\gamma}\right)$$

$$= \prod_{\ell=1}^{L^n_k} P\left(\sup_{x \in Q(z^n_{k,\ell},1)} Y^n_{k,\ell}(t, x) < e^{m_1(\lambda)t+n\gamma}\right).$$

By Theorem 3.8, the last probability is less than or equal to $1 - C e^{-2n\gamma/d}$. Applying the bound $1 - x \leq e^{-x}$ and noticing that $\frac{1}{2} e^{\theta n} \leq L^n_k \leq e^{\theta n}$ by construction, we have

$$P\left(\sup_{x \in Q(x^n_k, e^{\theta n})} \frac{Y(t, x)}{|x|^{\gamma}} < 1\right) \leq \exp(-CL^n_k e^{-\frac{2}{d}\gamma n}) \leq \exp(-\frac{1}{2} C e^{(\theta d - \frac{2}{d}\gamma)n}).$$

Since $K^n \leq e^{(1-\theta)nd}$ and $\theta d - \frac{2}{d}\gamma > 0$ by our choice of $\theta$, we conclude that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{K^n} P\left(\sup_{x \in Q(x^n_k, e^{\theta n})} \frac{Y(t, x)}{|x|^{\gamma}} < 1\right) \leq \sum_{n=1}^{\infty} \exp((1-\theta)nd - \frac{1}{2} C e^{(\theta d - \frac{2}{d}\gamma)n}) < \infty.$$
So the Borel–Cantelli lemma implies that the following holds with probability one: except for finitely many \( n \), the intersection \( Q(x_k^n, e^{\theta n}) \cap E_{\gamma}^{n, \epsilon} \) is nonempty for all \( k = 1, \ldots, K^n \). In other words, the set \( E_{\gamma}^{n, \epsilon} \) is almost surely \( \theta \)-thick in the sense of [32, Def. 4.3]. Thus, \( \text{Dim}_H(E_{\gamma}^{n, \epsilon}) \geq d(1 - \theta) \) almost surely by [32, Prop. 4.4] and the lower bound follows by letting \( \theta \downarrow 2\gamma/d^2 \).

For \( d = 1 \), recall the processes \( Y_0, Y_3, Y_4 \) and \( Z^{(m, \beta)} \) from (4.2) and (4.5). This time, we let \( m(n) = n, \beta(n) = n^2 \) and consider the subgrid \( \{ \hat{z}_{k, \ell} = x_k^n + \ell n^3 : \ell \in \mathbb{N}, 1 \leq \ell \leq \hat{L}_k = [n^{-3} e^{\theta n}] \} \). Any two points in \( \{ \hat{z}_{k, \ell} : k = 1, \ldots, K^n, \ell = 1, \ldots, \hat{L}_k \} \) are at least \( Cn^3 \) apart from each other, where \( C \) is a positive number. Therefore, for large \( n \) and \( k = 1, \ldots, K^n \), the variables

\[
\left\{ \sup_{x \in (\hat{z}_{k, \ell} - 1, \hat{z}_{k, \ell})} Z_{(m(n), \beta(n))}(t, x) : \ell = 1, \ldots, \hat{L}_k \right\}
\]

are independent of each other; cf. the paragraph after (4.7). Thus, for any \( \theta > 2\gamma \),

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{K^n} \sum_{\ell=1}^{\hat{L}_k} \mathbb{P}\left( \max_{t=1, \ldots, L_k} \sup_{x \in (\hat{z}_{k, \ell} - 1, \hat{z}_{k, \ell})} \left| \frac{Z_{(m(n), \beta(n))}(t, x)}{|x|^\gamma} \right| < 3 \right) \leq \sum_{n=1}^{\infty} \exp\left( (1 - \theta)n - \frac{1}{7} Cn^{-3} e^{(\theta - 2\gamma)n} \right) < \infty.
\]

At the same time, by Lemma A.6,

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{K^n} \sum_{\ell=1}^{\hat{L}_k} \mathbb{P}\left( \sup_{x \in (\hat{z}_{k, \ell} - 1, \hat{z}_{k, \ell})} \left| \frac{|Y_0(t, x) - Z_{(m(n), \beta(n))}(t, x)|}{|x|^\gamma} \right| > 1 \right) \leq \sum_{n=1}^{\infty} e^{(1-\theta)n} e^{\theta n} n^{-3} (e^{-n^2} + Cn^{-\frac{1}{\gamma}}) e^{-2\gamma n^\gamma} < \infty,
\]

which shows that \( E_{\gamma}^{n, \epsilon, 0} = \{ x \in \mathbb{R}^d : Y_0(t, x) \geq 2|x|^\gamma \} \) is \( \theta \)-thick, whence \( \text{Dim}_H(E_{\gamma}^{n, \epsilon, 0}) \geq 1 - 2\gamma \).

In addition, combining Step 1 with how we estimated \( Y_3 \) and \( Y_4 \) in the proof of Theorem 4.1, we have that \( E_{\gamma}^{n, \epsilon, 34} = \{ x \in \mathbb{R}^d : |Y_3(t, x) + Y_4(t, x)| \geq |x|^\gamma \} \) satisfies \( \text{Dim}_H(E_{\gamma}^{n, \epsilon, 34}) \leq \text{Dim}_M(E_{\gamma}^{n, \epsilon, 34}) \leq 1 - \alpha \gamma < 1 - 2\gamma \). Since \( E_{\gamma}^{x, \epsilon} \supseteq E_{\gamma}^{n, \epsilon, 0} \setminus E_{\gamma}^{n, \epsilon, 34} \) and \( \text{Dim}_H(A) \) remains unchanged when a set of lower dimension is removed (see [8, Property (viii), p. 128]), we conclude that \( \text{Dim}_H(E_{\gamma}^{x, \epsilon}) \geq 1 - 2\gamma \).

**Step 3: The remaining equalities**

Steps 1 and 2 show that \( \text{Dim}_H(E_{\gamma}^{x, \epsilon}) = \text{Dim}_M(E_{\gamma}^{x, \epsilon}) = d - \frac{2}{d} \gamma \) under Condition (Sup) and Condition (L-\( \alpha \)) with \( \alpha = \frac{2}{d} \). With the same methods, all remaining equalities in (5.3)–(5.6) can be deduced from the tail estimates in Theorems 3.1, 3.2 and 3.5 in the case of multiplicative noise and from the analogous results [22, Theorems 2 and 5] in the case of additive noise. Note that the slowly varying functions in the tail estimates in Theorems 3.1, 3.2 and 3.5 are negligible on the scale of sets \( E_{\gamma}^{x, \epsilon} \). This is why the macroscopic dimensions of \( E_{\gamma}^{x, \epsilon} \) are the same for both additive and multiplicative noise. \( \square \)
5.2. Self-similarity of intermittency islands (or lack thereof). While $\mathcal{E}_{d/2}$ is almost surely unbounded by Theorem 4.1, both its Minkowski and Hausdorff dimensions are zero as the previous theorem asserts. Loosely speaking, the peaks that contribute to $\mathcal{E}_{d/2}$ are too rare under the standard scale to have a positive macroscopic fractal dimension. However, under additive noise or under a multiplicative noise that is not too heavy-tailed, we can show that after appropriate changes of scale, these peaks will again exhibit a multifractal structure that is reminiscent of the peaks studied so far. In fact, as we shall show, there exist infinitely many layers of peaks which, despite being defined on different scales, all share the same multifractal behavior. In these cases, we conclude that the spatial peaks form large-scale self-similar multifractals.

**Theorem 5.2.** Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and $x^q = \frac{1}{|x|^q}$ for $x \neq 0$ and $q > 0$. Furthermore, for $n \in \mathbb{N}$, let $\log^{(n)}(r) = \log((\log^{(n-1)}(r))$ for $r \in \mathbb{R}$ (with $\log^{(0)}(r) = r$) and define $\log^{(n)}(x) = \frac{1}{|x|} \log^{(n)}(|x|)$ for $x \in \mathbb{R}^d \setminus \{0\}$.

(i) Assume Condition (Sup) and Condition (L-$\alpha$) with $\alpha = \frac{2}{3}$. If $d = 1$ and $m_1(\lambda) = \infty$, assume Condition (L-$\alpha$) with some $\alpha > 2$. For $N \in \mathbb{N}$ and $\gamma > 0$, consider

$$
\mathcal{E}_{\gamma}^{(x,c,N)} = \left\{ x \in \mathbb{R}^d : Y(t,x) \geq |x|^{\frac{d}{2}} \left( \prod_{p=1}^{N-1} \| \log^{(p)}(x) \|^{\frac{1}{\gamma}} \right) \right\},
$$

(5.11)

$$
\mathcal{E}_{\gamma}^{(+,c,N)} = \left\{ x \in \mathbb{R}^d : Y_+(t,x) \geq |x|^{\frac{d}{2}} \left( \prod_{p=1}^{N-1} \| \log^{(p)}(x) \|^{\frac{1}{\gamma}} \right) \right\}.
$$

Then almost surely,

(5.12) $\dim_{H,M} \left( \left\{ \log^{(N)}(\mathcal{E}_{\gamma}^{(x,c,N)}) \right\} \right) = d - \frac{2}{3} \gamma$.

(ii) Assume Condition (Sup) and Condition (H-$\alpha$) with $\alpha \in (0, \frac{2}{3})$. For $N \in \mathbb{N}$ and $\gamma > 0$, let

$$
\mathcal{E}_{\gamma}^{(+,c,N)} = \left\{ x \in \mathbb{R}^d : Y_+(t,x) \geq |x|^{\frac{d}{2}} \left( \prod_{p=1}^{N-1} \| \log^{(p)}(x) \|^{\frac{1}{\gamma}} \right) \right\}.
$$

(5.13)

Then almost surely,

(5.14) $\dim_{H,M} \left( \left\{ \log^{(N)}(\mathcal{E}_{\gamma}^{(+,c,N)}) \right\} \right) = d - \alpha \gamma$.

(iii) Assume Condition (H-$\alpha$) with $\alpha \in (0, 1 + \frac{2}{3})$ or Condition (L-$\alpha$) with $\alpha = 1 + \frac{2}{3}$. In both cases, consider for $N \in \mathbb{N}$ and $\gamma > 0$ the sets

$$
\mathcal{E}_{\gamma}^{(+,d,N)} = \left\{ x \in \mathbb{R}^d : Y_+(t,x) \geq |x|^{\frac{d}{2}} \left( \prod_{p=1}^{N-1} \| \log^{(p)}(x) \|^{\frac{1}{\gamma}} \right) \right\}.
$$

(5.15)

Then almost surely,

(5.16) $\dim_{H,M} \left( \left\{ \log^{(N)}(\mathcal{E}_{\gamma}^{(+,d,N)}) \right\} \right) = d - \alpha \gamma$.

**Proof of Theorem 5.2.** The proofs of (5.12), (5.14) and (5.16) are completely analogous. We therefore only show the part in (5.12) concerning $\mathcal{E}_{\gamma}^{(x,c,N)}$. We begin with a technicality: the result in (5.12) does not depend on the choice of the norm $\|\cdot\|$. Indeed, let $\|\cdot\|$ be another norm on $\mathbb{R}^d$ and write $\log^{(N)}(\|\cdot\|)$ and $\log^{(N)}(\|\cdot\|)$ and, similarly, $\mathcal{E}_{\gamma,\|\cdot\|}$ and $\mathcal{E}_{\gamma,\|\cdot\|}$ as well.
as $(\cdot)^r = (\cdot)^r_{||\cdot||}$ and $(\cdot) = (\cdot)_{||\cdot||}$ to emphasize the dependence on the chosen norm. For $K > 0$, further let $\mathcal{E}^{(x,c,N)}(K)$ be the right-hand side of the first line of (5.11) when $Y(t, x)$ is replaced by $Y(t, x)/K$, and define $\mathcal{E}^{(x,c,N)}(K)$ analogously. By the equivalence of norms on $\mathbb{R}^d$, there is $C \geq 1$ such that $\mathcal{E}^{(x,c,N)}(CK) \subseteq \mathcal{E}^{(x,c,N)}(K) \subseteq \mathcal{E}^{(x,c,N)}(C^{-1} K)$ for all $K > 0$. Thus,

$$\left(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(CK))\right)^{1/d}_{||\cdot||} \subseteq \left(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(K))\right)^{1/d}_{||\cdot||} = f\left(\left(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(K))\right)^{1/d}_{||\cdot||}\right)$$

with the function $f$ from (A.9). This function is bounded and Lipschitz continuous outside a ball containing the origin according to Lemma A.8. Since the macroscopic Hausdorff and Minkowski dimensions are monotone and insensitive to adding or deleting bounded subsets,

$$\text{Dim}_{H|M}(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(CK)))^{1/d}_{||\cdot||} \leq \text{Dim}_{H|M}(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(K)))^{1/d}_{||\cdot||}$$

by Lemma A.7. By symmetry, this inequality also holds if we switch the role of the two norms. So the part in (5.12) concerning $\mathcal{E}^{(x,c,N)}(\gamma)$ is proved if we show that

$$\text{Dim}_{H|M}(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(K)))^{1/d}_{||\cdot||} \leq d - \frac{2}{d} \gamma,$$

$$\text{Dim}_{H|M}(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(K)))^{1/d}_{||\cdot||} \geq d - \frac{2}{d} \gamma$$

for every $K > 0$. In order to simplify notation, we omit all subscripts $|\cdot|$ and $||\cdot||$ in the following and write $|\cdot|$ for both norms, with the agreement that $|\cdot|$ is the supremum norm in Step 1 and the Euclidean norm in Step 2 below.

Next, let $\exp^{(n)}(x) = \frac{1}{n!} \exp^{(n)}(|x|)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}^d \setminus \{0\}$, where $\exp^{(n)}$ is the iterated exponential defined in Lemma A.8. A moment’s thought reveals that

$$\text{Dim}_{H|M}(\log^d_{||\cdot||} (\mathcal{E}_{\gamma,\cdot}(K)))^{1/d}_{||\cdot||} = \text{Dim}_{H|M}(\overline{\mathcal{E}_{\gamma,\cdot}}(K)),$$

where for $K > 0$,

$$\overline{\mathcal{E}_{\gamma,\cdot}}(K) = \left\{ x \in \mathbb{R}^d : Y(t, \exp^{(N)}(x)^d) > K |\exp^{(N)}(x)^d|^{\frac{1}{\gamma}} \prod_{p=1}^{N-1} |\exp^{(p)}(x)^d|^{\frac{1}{\gamma}} |x|^\gamma \right\}.$$

As the statement of the theorem for $\gamma > d^2/2$ follows from Theorem 4.1, we may (and will) assume $d - \frac{2}{d} \gamma \geq 0$ in the following. By [8, Lemma 3.1], it is enough to prove that $\text{Dim}_{H}(\overline{\mathcal{E}_{\gamma,\cdot}}(K)) \leq d - \frac{2}{d} \gamma$ and $\text{Dim}_{H}(\overline{\mathcal{E}_{\gamma,\cdot}}(K)) \geq d - \frac{2}{d} \gamma$ almost surely.

**Step 1: Upper bound**

Let $k(n) \in \mathbb{N}$ and $n - 1 =: r_0^{(n)} < \cdots < r_{k(n)-1}^{(n)} < n + 1 \leq r_k^{(n)}$ be defined via the relations

$$\exp^{(N)}(\exp(dr_i^{(n)})) - \exp^{(N)}(\exp(dr_{i-1}^{(n)})) = 1, \quad i = 1, \ldots, k(n),$$

and let $S_n(i) = B_\infty(\exp(r_i^{(n)}), \exp(r_{i-1}^{(n)}))$ for $i = 1, \ldots, k(n)$. We required $r_k^{(n)} \geq n + 1$ and not just $r_k^{(n)} \geq n$ in order that $\bigcup_{q \in \mathbb{Z}^d \cap S_n} Q(q, 1) \subseteq \bigcup_{i=1}^{k(n)} S_n(i)$. For every $n$ and $i$, the image of $S_n(i)$ under the mapping $\exp^{(N)}(\cdot)^d$ can be covered with unit cubes $(Q_j^{n,i} : j = 1, \ldots, \ell_n(i))$, where

$$\ell_n(i) \leq C[(\exp^{(N)}(\exp(dr_i^{(n)})))^d - (\exp^{(N)}(\exp(dr_{i-1}^{(n)})))^d]$$
for some finite $C > 0$ independent of $n$ and $i$. Denoting the pre-image of $Q_{j_i}^{n,i}$ under the same mapping by $\widetilde{Q}_{j_i}^{n,i}$ and assuming $\widetilde{Q}_{j_i}^{n,i} \cap \lbrace -\exp(r_{i-1}^{(n)}), \exp(r_{i-1}^{(n)}) \rbrace^d = \emptyset$ without loss of generality, we have

$$|\{q \in \mathbb{Z}^d \cap S_n : \mathcal{E}^{(N)}_{\gamma} \cap Q(q, 1) \neq \emptyset\}|$$

$$\leq \sum_{i=1}^{k(n)} \sum_{j=1}^{\ell_i} 1 \left\{ \exists x \in \widetilde{Q}_{j_i}^{n,i} : Y(t, \exp^{(N)}(x^d)) > K|\exp^{(N)}(x^d)|^\frac{a^2}{2} \right.$$  

$$\left. \times \left( \prod_{p=1}^{N-1} |\exp(p)(x^d)|^\frac{a}{p} \right) |x|^{\gamma} \right\}$$

$$\leq \sum_{i=1}^{k(n)} \sum_{j=1}^{\ell_i} 1 \left\{ \exists x \in \widetilde{Q}_{j_i}^{n,i} : Y(t, \exp^{(N)}(x^d)) > K(\exp^{(N)}(\exp(dr_{i-1}^{(n)})))^\frac{a^2}{2} \right.$$  

$$\left. \times \left( \prod_{p=1}^{N-1} (\exp(p)(\exp(dr_{i-1}^{(n)})))^\frac{a}{p} \right) \exp(\gamma r_{i-1}^{(n)}) \right\}$$

$$= \sum_{i=1}^{k(n)} \sum_{j=1}^{\ell_i} 1 \left\{ \sup_{x \in \widetilde{Q}_{j_i}^{n,i}} Y(t, x) > K(\exp^{(N)}(\exp(dr_{i-1}^{(n)})))^\frac{a^2}{2} \right.$$  

$$\left. \times \left( \prod_{p=1}^{N-1} (\exp(p)(\exp(dr_{i-1}^{(n)})))^\frac{a}{p} \right) \exp(\gamma r_{i-1}^{(n)}) \right\}.$$  

Taking expectation and using (5.17) and Theorem 3.8, we obtain

$$E\left[ |\{q \in \mathbb{Z}^d \cap S_n : \mathcal{E}^{(N)}_{\gamma} \cap Q(q, 1) \neq \emptyset\}| \right]$$

$$\leq C \sum_{i=1}^{k(n)} \left[ \exp(d \exp^{(N)}(dr_{i}^{(n)})) - \exp(d \exp^{(N)}(dr_{i-1}^{(n)})) \right] \exp(-d \exp^{(N)}(dr_{i-1}^{(n)}))$$

$$\times \left( \prod_{p=1}^{N-1} \exp(- \exp(p)(dr_{i-1}^{(n)})) \right) \exp(-\frac{2}{3} \gamma r_{i-1}^{(n)}).$$

Applying the mean-value theorem to the difference in brackets and noticing that the derivative of $r \mapsto \exp(d \exp^{(N)}(dr))$ increases in $r$, we further deduce that

$$E\left[ |\{q \in \mathbb{Z}^d \cap S_n : \mathcal{E}^{(N)}_{\gamma} \cap Q(q, 1) \neq \emptyset\}| \right]$$

$$\leq C d^2 \sum_{i=1}^{k(n)} (r_{i}^{(n)} - r_{i-1}^{(n)}) \exp(d \exp^{(N)}(dr_{i}^{(n)})) \left( \prod_{p=1}^{N} \exp(p)(dr_{i}^{(n)}) \right)$$

$$\times \exp(-d \exp^{(N)}(dr_{i-1}^{(n)})) \left( \prod_{p=1}^{N-1} \exp(- \exp(p)(dr_{i-1}^{(n)})) \right) \exp(-\frac{2}{3} \gamma r_{i-1}^{(n)}).$$

By construction, $\exp(\exp^{(N)}(dr_{i}^{(n)}) = 1 + \exp(\exp^{(N)}(dr_{i-1}^{(n)})) \leq 2 \exp(\exp^{(N)}(dr_{i-1}^{(n)})).$

Taking logarithm consecutively on both sides, we also get $\exp(p)(dr_{i}^{(n)}) \leq 2 \exp(p)(dr_{i-1}^{(n)})$
for \( p = 1, \ldots, N \). Thus, we can simplify the estimate in the previous display to
\[
E\left[ \left\{ q \in \mathbb{Z}^d \cap S_n : \mathcal{E}^{(N)}_\gamma \cap Q(q, 1) \neq \emptyset \right\} \right] \leq \sum_{i=1}^{k(n)} r_i^{(n)} - r_i^{(n-1)} \exp((d - \frac{2\gamma}{\pi}) r_i^{(n)})
\]
\[
\leq \sum_{i=1}^{k(n)} e^{(n+1)(d-\frac{2\gamma}{\pi})} r_i^{(n)} - r_i^{(n-1)}
\]
\[
\leq \sum_{i=1}^{k(n)} e^{(n+1)(d-\frac{2\gamma}{\pi})}
\]
\[
\leq C_d 2^{N+1} e^{(n+1)(d-\frac{2\gamma}{\pi})}
\]

This estimate is analogous to the bound (5.7) in the proof of Theorem 5.1, so the proof can be completed just like there.

**Step 2: Lower bound**

As in the proof of Theorem 5.1, our strategy will be to show that \( \mathcal{E}^{(N)}_\gamma (K) \) is \( \theta \)-thick for all \( \theta \in (4\gamma/d^2, 1) \), assuming \( d - \frac{2\gamma}{\pi} > 0 \) without loss of generality. We again consider the grid \( \{ x_k^n : k = 1, \ldots, K^n \} \) from (5.8) and the associated cubes \( Q(x_k^n, e\rho^n) \). Unfortunately, if \( d \geq 2 \), we do not have sufficient control over the shape of the images that we obtain from applying the mapping \( \exp^{(N)} \) to these cubes. This is why in \( d \geq 2 \), we will inscribe some auxiliary geometric solids that are easier to analyze in those cubes. For the remaining proof, we only consider the case \( d \geq 2 \); the one-dimensional situation is geometrically much simpler and is therefore left to the reader (the potential existence of infinite variation jumps can be addressed as in the proof of Theorem 5.1).

For \( d \geq 2 \), we consider geometric shapes that we call (spherical) shell sectors; see Figure 1. These are obtained by intersecting a shell (i.e., the set difference of two balls with the same center) with a cone that has this center as apex. Equivalently, a shell sector is the difference of two concentric sectors. (A sector results from cutting a ball into two parts by a hyperplane and taking the union of the smaller part with the cone formed by the intersection, an \( (n-1) \)-dimensional ball, as base and the center of the cut ball as apex; “concentric” here means that both sectors have the same apex and the same axis of revolution.)

A shell sector \( S = S(A, O, \rho, s) \) (see Figure 2 for illustration) is uniquely parametrized by four parameters: its apex \( A \) (i.e., the joint apex of the two sectors), its suspension point \( O \) (i.e., the center of the base of the larger sector), its base radius \( \rho \) (i.e., the radius of the base of the larger sector), and its side length \( s \) (i.e., the difference of the radii of the two balls). Several other characteristics of \( S \) will be important to us: its inner radius \( r \) and outer radius \( R \) (i.e., the radius of the smaller and the larger ball, respectively), its inner vertex \( v \) and outer vertex \( V \) (i.e., the point on the boundary of the smaller and larger ball, respectively, that is collinear with \( A \) and \( O \)), its height \( h \) (i.e., the distance between the base center of the smaller sector and \( V \)), its angle \( \phi \) (i.e., the largest possible angle between the half-lines \( AO \) and \( AP \), where \( P \) is a boundary point of \( S \)), and its direction \( w = (O - A)/(O - A) \).

Simple geometric considerations yield the following relations:

\[
R = r + s, \; \sin \phi = \frac{\rho}{R}, \; \tan \phi = \frac{\rho}{\rho_0}, \; h = s \cos \phi + \sqrt{\rho_0^2 + \rho^2 - \rho_0},
\]

where \( \rho_0 = |O - A| \). As a result, another way of parametrization is \( S = S[A, w, r, R, \phi] \). Moreover, \( S \) can be inscribed in a box with one side of length \( h \) and all other sides of length \( 2\rho \). This box has diameter \( \sqrt{h^2 + 4(d - 1)\rho^2} \), which, in particular, implies

\[
\max_{P \in S} \text{dist}(O, P) \leq \sqrt{h^2 + 4(d - 1)\rho^2}.
\]
Back to the cubes $Q(x^n_k, e^\theta n)$, let $z^n_k$ be the center of these cubes and consider the shell sectors
\begin{equation}
S^n_k = S(0, z^n_k, \frac{1}{4\sqrt{d}} e^{\theta n}, \frac{1}{\sqrt{d}} e^{\theta n}).
\end{equation}

By (5.18) and the elementary inequality $\sqrt{x + y} - \sqrt{x} \leq \sqrt{y}$, the height of $S^n_k$ is bounded by $\frac{1}{2\sqrt{d}} e^{\theta n}$. Together with (5.19), it follows that $\text{dist}(z^n_k, P) \leq \frac{1}{2} e^{\theta n}$ for any point $P$ in $S^n_k$. The important conclusion is that
\begin{equation}
S^n_k \subseteq Q(x^n_k, e^\theta n).
\end{equation}

For later reference, let us also give an estimate on $\phi^n_k$, the angle of $S^n_k$: since $e^{n-1} \leq |z^n_k| \leq \sqrt{de^n}$ and $\frac{1}{2}x \leq \arctan x \leq x$ for small $x > 0$, (5.18) implies
\begin{equation}
\frac{1}{8d} e^{(\theta-1)n} \leq \arctan \frac{e^{\theta n}}{4de^n} \leq \phi^n_k \leq \arctan \frac{e^{\theta n}}{4\sqrt{de^{n-1}}} \leq \frac{e}{4\sqrt{d}} e^{(\theta-1)n}
\end{equation}
for large $n$.

The reason why we have introduced the shell sectors $S^n_k$ at all is that $\overline{S^n_k} = \exp^{(N)}((S^n_k)^d)$ are again shell sectors. In fact,
\begin{equation}
\overline{S^n_k} = S[0, w^n_k, \exp^{(N)}((r^n_k)^d), \exp^{(N)}((R^n_k)^d), \phi^n_k],
\end{equation}
where $w^n_k = z^n_k/|z^n_k|$ is the direction and $r^n_k$ and $R^n_k$ are the inner and outer radius of $S^n_k$, respectively. Given $n$ and $k$, we now define $u^{n,k}_0 < \cdots < u^{n,k}_{\ell_k}$ by setting $u^{n,k}_{\ell_k} = r^n_k$ and requiring $\ell_k$ be the maximal number such that
\begin{equation}
\exp^{(N)}((u^{n,k}_{\ell_k})^d) - \exp^{(N)}((u^{n,k}_{\ell-1})^d) = 1
\end{equation}
for all $\ell = 1, \ldots, \ell_k$ and $u^{n,k}_{\ell_k} \leq R^n_k$. By construction and the first identity in (5.18),
\begin{equation}
e^{n-1} \leq r^n_k = u^{n,k}_{\ell_k} < \cdots < u^{n,k}_{\ell_k} \leq R^n_k \leq \sqrt{de^n},
\end{equation}
\begin{equation}
\frac{1}{8\sqrt{d}} e^{\theta n} \leq \frac{1}{2}(R^n_k - r^n_k) \leq u^{n,k}_{\ell_k} - u^{n,k}_{\ell_k} \leq R^n_k - r^n_k = \frac{1}{4\sqrt{d}} e^{\theta n}.
\end{equation}
Next, given \( n, k \) and \( \ell \), consider
\[
\mathcal{S}_{\ell}^{n,k} = S(0, w_k^n, \exp^{(N)}((u_{\ell-1}^{n,k})^d), \exp^{(N)}((u_{\ell}^{n,k})^d), \phi_k^n).
\]
By a simple calculation (cf. [27, Sect. V]), there is a constant \( C_d > 0 \) that only depends on \( d \) such that, with obvious notation,
\[
\text{Leb}(\mathcal{S}_{\ell}^{n,k}) = C_d \left( (R_{\ell}^{n,k})^d - (r_{\ell}^{n,k})^d \right) \int_0^\phi_k^n (\sin t)^{d-2} dt
\]
\[\geq \frac{C_d}{2^{d-2} (d-1)} \left( (R_{\ell}^{n,k})^d - (r_{\ell}^{n,k})^d \right) (\phi_k^n)^{d-1}
\]
for large \( n \). As a consequence of the Vitali covering theorem (see [13, Thm. 5.5.2]), there are \( \varepsilon > 0 \) and pairwise disjoint cubes \( Q_{\ell,1}^{n,k}, \ldots, Q_{\ell,m_{\ell,k}}^{n,k} \subseteq \mathcal{S}_{\ell}^{n,k} \) with side length within \((\varepsilon, 1] \) such that
\[
m_{\ell,k}^{n,k} \geq \sum_{m=1}^{m_{\ell,k}^{n,k}} \text{Leb}(Q_{\ell,m}^{n,k}) \geq \frac{C_d}{2^{d-1} (d-1)} \left( (R_{\ell}^{n,k})^d - (r_{\ell}^{n,k})^d \right) (\phi_k^n)^{d-1}.
\]
We are now ready for the final (probabilistic) part of the proof. Whenever \( n \) is sufficiently large, we deduce from (5.21) and (5.25) that for all \( k = 1, \ldots, K^n \),
\[
P \left( \sup_{x \in Q(x_{\ell,m}^n, \epsilon^n)} \frac{Y(t, \exp^{(N)}(x^d))}{\exp^{(N)}(x^d) \left( \prod_{p=1}^{N-1} \exp^{(p)}(x^d)^{\frac{d}{p}} \right) x^{\gamma}} \leq K \right)
\]
\[
\leq \prod_{\ell=1}^{\ell_{\ell,k}} \prod_{m=1}^{m_{\ell,k}^{n,k}} \left( \sup_{x \in Q_{\ell,m}^{n,k}} Y(t, x) \leq K (\exp^{(N)}((u_{\ell}^{n,k})^d)^{\frac{d^2}{2}} \times \left( \prod_{p=1}^{N-1} \exp^{(p)}((u_{\ell}^{n,k})^d)^{\frac{d}{p}} \right) (u_{\ell}^{n,k})^{\gamma} \right).
\]
Let \( Y_{\ell,m}^{n,k} \) be defined in the same way as \( Y_{\ell,m}^{n,k} \) in (5.9) but with \( Q(z_{\ell,m}^n, 1) \) replaced by \( Q_{\ell,m}^{n,k} \). By construction, the latter are mutually disjoint for different values of \( \ell \) and \( m \). Therefore, \( \{ Y_{\ell,m}^{n,k} : \ell = 1, \ldots, \ell_{\ell,k}, m = 1, \ldots, m_{\ell,k} \} \) is a family of independent random fields. In addition, they clearly satisfy \( Y_{\ell,m}^{n,k}(t, x) \leq \hat{Y}(t, x) = e^{m_{\ell,k}} Y(t, x) \), so
\[
P \left( \sup_{x \in Q(x_{\ell,m}^n, \epsilon^n)} \frac{Y(t, \exp^{(N)}(x^d))}{\exp^{(N)}(x^d) \left( \prod_{p=1}^{N-1} \exp^{(p)}(x^d)^{\frac{d}{p}} \right) x^{\gamma}} \leq K \right)
\]
\[
\leq \prod_{\ell=1}^{\ell_{\ell,k}} \prod_{m=1}^{m_{\ell,k}^{n,k}} P \left( \sup_{x \in Q_{\ell,m}^{n,k}} Y_{\ell,m}^{n,k}(t, x) \leq K e^{-m_{\ell,k}} (\exp^{(N)}((u_{\ell}^{n,k})^d)^{\frac{d^2}{2}} \times \left( \prod_{p=1}^{N-1} \exp^{(p)}((u_{\ell}^{n,k})^d)^{\frac{d}{p}} \right) (u_{\ell}^{n,k})^{\gamma} \right)
\]
\[
\leq \exp \left( -C \sum_{\ell=1}^{\ell_{\ell,k}} m_{\ell,k}^{n,k} (\exp^{(N)}((u_{\ell}^{n,k})^d))^{-d} \left( \prod_{p=1}^{N-1} \exp^{(p)}((u_{\ell}^{n,k})^d)^{\frac{d}{p}} - \frac{d}{p} \right) \right),
\]
where we used (3.41) and the estimate \(1 - x \leq e^{-x}\) for the last step. One detail is worth mentioning: The bounds in (3.41) were proved for cubes \(Q\) of side length 1. The reader may easily verify that the same bound holds uniformly for all cubes of side length larger than \(\varepsilon\), except that the values of the limit inferior and superior in (3.41) now depend on \(\varepsilon\). This is why the constant \(C\) in the previous display may depend on \(\varepsilon\) but not on \(n, k, \ell \) or \(m\).

By (5.22), (5.23) (which implies \(\exp(p)((u_{\ell_{k}}^{n,k})^d) - \exp(p)((u_{\ell_{k}+1}^n)^d) \leq 1\) for all \(p = 1, \ldots, N\), (5.24) and (5.26) together with the mean-value theorem and the bound \(x \geq \frac{1}{2}(x + 1)\) for \(x > 1\), we deduce that

\[
\sum_{\ell = 1}^{\ell_k} m_{\ell}^{n,k}(\exp(N)((u_{\ell_{k}}^{n,k})^d) - \exp(N)((u_{\ell_{k}+1}^n)^d)) \leq C \sum_{\ell = 1}^{\ell_k} \left(\exp(N)((u_{\ell_{k}}^{n,k})^d) - \exp(N)((u_{\ell_{k}+1}^n)^d)) \right) \leq C \exp(\theta(d-\frac{2}{\alpha}\gamma)) n e^{(\theta(d-\frac{2}{\alpha}\gamma)n)}.
\]

In summary,

\[
P\left(\sup_{x \in Q(x_{\ell_{k}}^{n,k}) \exp(N)(x_{\ell_{k}}^{n,k})^d) \left(\frac{\exp(N)(x_{\ell_{k}}^{n,k})^d}{\prod_{\ell = 1}^{N}(\exp(N)(x_{\ell_{k}}^{n,k})^d)}\right) \leq K \right) \leq \exp(-C e^{(\theta(d-\frac{2}{\alpha}\gamma)n)}).
\]

This bound is analogous to (5.10) in the proof of Theorem 5.1. The subsequent arguments apply in our current situation as well and complete the proof of Step 2.

By contrast, under a multiplicative noise, if we consider the peaks of the solution to (1.1) on a lattice or if we consider the peaks on \(\mathbb{R}^d\) and the noise is sufficiently heavy-tailed, they are not self-similar in terms of their multifractal behavior. Given the tail estimates of Section 3, the proof is very similar to that of the previous theorem (with \(N = 1\), which is why we omit it.

**Theorem 5.3.** Let \(M \in [0, \infty)\).

(i) Assume Condition \((\text{Sup})\) and Condition \((\text{H-}\alpha)\) with some \(\alpha \in (0, \frac{2}{\alpha})\). Define the sets

\[
\mathcal{F}_{M}^{(\alpha)} = \left\{ x \in \mathbb{R}^d : Y(t,x) \geq |x|^{\frac{2}{\alpha}} x^{M(\exp(|x|)^{1/(1+\theta\alpha)})} \right\}.
\]

If \(L_0\) is the number from (4.12) and \(M_1\) is the number from (4.13), then almost surely,

\[
\dim_{\text{H-M}} \left(\exp(\log(\mathcal{F}_{M}^{(\alpha)}))\right) \leq L_0 \left(\frac{\alpha}{\alpha} \right)^{1/\theta\alpha} - \alpha M,
\]

\[
\dim_{\text{H-M}} \left(\exp(\log(\mathcal{F}_{M}^{(\alpha)}))\right) \geq M_1 \left(\frac{\alpha}{\alpha} \right)^{1/\theta\alpha} - \alpha M.
\]
(ii) Assume Condition \((H-\alpha)\) with some \(\alpha \in (0, 1 + \frac{2}{d})\). Define the sets

\[
(5.29) \quad \mathcal{F}^{(x,d)}_{M} = \left\{ x \in \mathbb{Z}^d : Y(t, x) \geq \frac{d}{\alpha} e^{M(\log|x|)^{1/(1+\theta_0)}} \right\}. 
\]

If \(M_1 \) and \(M_2 \) are the numbers from (4.13), then almost surely,

\[
(5.30) \quad \text{Dim}_{H[M]} \left( \exp(1) \left( \frac{(\log(1)(\mathcal{F}^{(x,d)}_{M}))}{\alpha N} \right) \right) \leq M_2 \left( \frac{d}{\alpha} \right)^{1/(1+\theta_0)} - \alpha M, 
\]

\[
\text{Dim}_{H[M]} \left( \exp(1) \left( \frac{(\log(1)(\mathcal{F}^{(x,d)}_{M}))}{\alpha N} \right) \right) \geq M_1 \left( \frac{d}{\alpha} \right)^{1/(1+\theta_0)} - \alpha M. 
\]

(iii) Assume Condition \((L-\alpha)\) with \(\alpha = 1 + \frac{2}{d}\). Define the sets

\[
(5.31) \quad \mathcal{F}^{(x,d)}_{M} = \left\{ x \in \mathbb{Z}^d : Y(t, x) \geq \frac{d^2}{\alpha} e^{M(\log|x|)(\log \log |x|)/\log |x|} \right\} 
\]

and the function \(H : \mathbb{R}^d \to \mathbb{R}^d, H(x) = \exp(1) \frac{(\log(1)(\mathcal{F}^{(x,d)}_{M}))}{\alpha N} \). If \(M_1 \) and \(M_2 \) are the numbers from (4.14), then almost surely,

\[
(5.32) \quad \text{Dim}_{H[M]} \left( H(\mathcal{F}^{(x,d)}_{M}) \right) \leq M_2 \frac{d^2}{2 + \frac{d}{2}} - (1 + \frac{2}{d})M, 
\]

\[
\text{Dim}_{H[M]} \left( H(\mathcal{F}^{(x,d)}_{M}) \right) \geq M_1 \frac{d^2}{2 + \frac{d}{2}} - (1 + \frac{2}{d})M. 
\]

**APPENDIX: TECHNICAL RESULTS**

In this appendix, we state and prove some technical results.

**LEMMA A.1.** For every \(\alpha, \beta, \gamma > 0\), there is \(C_{\alpha,\gamma} \in (0, \infty)\) such that for all \(z \geq 0\),

\[
\sum_{N=0}^{\infty} \frac{z^N}{\Gamma(\alpha N + \beta)^{1/\gamma}} \leq \frac{\gamma}{\alpha} C_{\alpha,\gamma} e^{-\gamma z^{\alpha/\gamma}}. 
\]

One can choose \(C_{\alpha,\gamma}\) such that it is locally bounded in \(\alpha\) and \(1/\gamma\) and independent of \(\beta\).

**PROOF.** In this proof, we use \(C_{\alpha,\gamma}\) to denote a positive constant that is locally bounded in \(\alpha\) and \(\gamma\), and whose value may change from line to line. Let \(z_0\) be the unique minimum of the gamma function on the positive real line. Then \(\Gamma(\alpha N + \beta) \geq \Gamma((\alpha N + \beta) \vee z_0) \geq \Gamma(\alpha N \vee z_0)\), so by Stirling’s formula for gamma functions, there is \(C \in (0, \infty)\) such that

\[
\sum_{N=0}^{\infty} \frac{z^N}{\Gamma(\alpha N + \beta)^{1/\gamma}} \leq C \sum_{N=0}^{\infty} \frac{1}{\Gamma(\alpha N \vee z_0)^{1/\gamma}} \left( \frac{e^{\alpha/\gamma} z}{N} \right)^{\alpha N / \gamma} 
\]

\[
\leq C \sum_{N=0}^{\infty} \frac{1}{\Gamma(\alpha N \vee z_0)^{1/\gamma}} \left( \frac{e^{\alpha/\gamma} z}{N} \right)^{\alpha N / \gamma} e^{N(\gamma e - \gamma)} \leq C_{\alpha,\gamma} \sum_{N=0}^{\infty} \frac{(C_{\alpha,\gamma} z)^N}{N^{\alpha N / \gamma}}, 
\]

where we used the bound \(\alpha^{\alpha N / \gamma} \geq e^{-N(\gamma e - \gamma)}\) for the second step. The function \(x \mapsto (C_{\alpha,\gamma} z)^x / x^{\alpha x / \gamma}\) has a unique maximum at \(x = (C_{\alpha,\gamma} z)^{\gamma / \alpha} e^{-1}\). Thus, by integral approximation, a change of variable \((y = \alpha x / \gamma)\) and a Riemann sum approximation,

\[
\sum_{N=0}^{\infty} \frac{z^N}{\Gamma(\alpha N + \beta)^{1/\gamma}} \leq C_{\alpha,\gamma} \left( \int_0^{\infty} \frac{(C_{\alpha,\gamma} z)^x}{x^{\alpha x / \gamma}} \, dx + e^{\alpha (C_{\alpha,\gamma} z)^{\gamma / \alpha} / (\gamma e)} \right) 
\]

\[
\leq C_{\alpha,\gamma} e^{C_{\alpha,\gamma} z^{\gamma / \alpha}} + C_{\alpha,\gamma} \int_0^{\infty} \frac{(\gamma / \alpha)^{y-1} (C_{\alpha,\gamma} z)^{\gamma y / \alpha}}{y^y} \, dy 
\]
\[ \leq C_{\alpha,\gamma} e^{C_{\alpha,\gamma} z^{\gamma/\alpha}} + C_{\alpha,\gamma} \left( \sum_{N=1}^{\infty} \frac{(\alpha/\gamma)^{N-1}(C_{\alpha,\gamma} z)^{\gamma N/\alpha}}{N^N} + \frac{\gamma}{\alpha} e^{\alpha(C_{\alpha,\gamma} z)^{\gamma/\alpha}}(\gamma e) \right) \]

\[ \leq \frac{\gamma}{\alpha} C_{\alpha,\gamma} e^{C_{\alpha,\gamma} z^{\gamma/\alpha}}. \]

\textbf{Lemma A.2.} For \( \alpha, \delta \in (0, 1) \), \( p > 1 \) and a positive random variable \( X \) with \( 0 < \mathbb{E}[X^p] < \infty \), we have

\[ \mathbb{P}(X > \delta \mathbb{E}[X]) \geq (1 - \delta)^{1/p} \mathbb{E}[X^q]^{1/p}, \quad \mathbb{E}[X^q] \geq 2^{1-q/p} \mathbb{E}[X]^{q/p}. \]

\textbf{Proof.} Both inequalities are variants of the classical Paley–Zygmund inequality. The first one was proved in [31, Lemma 7.3] (the assumption \( p \geq 2 \) in the mentioned reference was not needed in the proof). The second follows from the first by Markov’s inequality.

\textbf{Lemma A.3.} For \( R > 1 \), \( \alpha > -1, \beta > -1 \),

\[ H_{N;\alpha,\beta}(R) = \int_0^R \cdots \int_0^R (y_1 \cdots y_N)^{\alpha} \left( \log \frac{1}{y_1 \cdots y_N} \right)^{\beta} \mathbb{1}_{\{y_1 \cdots y_N \leq 1\}} \, dy_1 \cdots dy_N 
\]

\[ = \sum_{i=0}^{N-1} c_{N,i}(\log R)^i, \]

where

\[ c_{N,i} = \frac{\Gamma(N - i + \beta)}{(N - i - 1)!((\alpha + 1)^N - i + \beta).} \]

\textbf{Proof.} To ease notation, we suppress the subscripts \( \alpha \) and \( \beta \). Changing variables \( u_i = y_N^{1/(N-i)} y_i \) for \( i = 1, \ldots, N - 1 \), we obtain

\[ H_N(R) = \int_0^R \left( \int_{[0, R y^{1/(N-1)}]_{N-1}}^{u_1 \cdots u_{N-1}} \right) \left( \log \frac{1}{u_1 \cdots u_{N-1}} \right)^{\beta} \mathbb{1}_{\{u_1 \cdots u_{N-1} \leq 1\}} y_N^{-1} \, du_1 \cdots du_{N-1} \right) \, dy_N 
\]

(A.1)

\[ = \int_0^R \frac{1}{y} H_{N-1}(R y^{1/(N-1)}) \, dy \]

for all \( R > 0 \).

We prove the lemma by induction. For \( N = 1 \), the statement is clear. Assume that the statement holds for \( N \geq 1 \). Since

\[ \int_1^{R^{N+1}} \frac{1}{u} \log^i u \, du = \frac{(N + 1)^{i+1}}{i + 1} \log^{i+1} R \]
for \( i \geq 0 \), we can use (A.1), a change of variables \( u = yR^\lambda \) and the induction hypothesis to obtain

\[
H_{N+1}(R) = \int_0^R \frac{1}{y} H_N(Ry^{\frac{1}{\lambda}}) \, dy = \int_0^1 \frac{1}{u} H_N(u^{\frac{1}{\lambda}}) \, du + \sum_{i=0}^{N-1} c_{N,i} \int_1^{R^{N+1}} \frac{1}{u} (\log u^{\frac{1}{\lambda}})^i \, du \\
= \int_0^1 \frac{1}{u} H_N(u^{\frac{1}{\lambda}}) \, du + \sum_{i=0}^{N-1} c_{N+1,i+1}(\log R)^{i+1}.
\]

Thus, it remains to show

(A.2) \[ c_{N+1,0} = \int_0^1 \frac{1}{u} H_N(u^{\frac{1}{\lambda}}) \, du. \]

We claim that

(A.3) \[ c_{N+1,0} = \int_0^1 \frac{(\log u^{-1})^j}{j! u} H_{N-j}(u^{1/(N-j)}) \, du, \quad j = 0, \ldots, N-1. \]

When \( j = 0 \), this becomes (A.2). Noting that \( \frac{d}{du} H_k(u) = ku^{-1}H_{k-1}(u^{1/(k-1)}) \) for \( u \in (0, 1) \), one can show (A.3) using integration by parts and a backwards induction argument.

Thus, it remains to verify (A.3) at the base case \( j = N-1 \):

\[
\int_0^1 \frac{(\log u^{-1})^{N-1}}{(N-1)! u} H_1(u) \, du = \int_0^1 \frac{(\log u^{-1})^{N-1}}{(N-1)! u} \int_0^u y^\alpha (\log y^{-1})^\beta \, dy \, du \\
= \int_0^1 \frac{(\log u^{-1})^N}{N!} u^\alpha (\log u^{-1})^\beta \, du \\
= \frac{\Gamma(N+1+\beta)}{N!(\alpha+1)^{N+1+\beta}} = c_{N+1,0}. \]

\[
\square
\]

**Lemma A.4.** For any \( t > 0 \), the Poisson random measure \( \mu \) can be decomposed into \( \mu = \sum_{i=0}^{\infty} \mu_i \) such that

- the \( \mu_i \)'s are independent Poisson random measures,
- \( \mu_0 \) is the restriction of \( \mu \) to \([0, \infty) \times (\mathbb{R}^d \setminus (-2,2)^d) \times (0, \infty)\),
- \( \mu_i \) has intensity \( dt \mathbb{1}_{(-2,2)^d}(x) \, dx \lambda_i(dz) \) and \( m_0(\lambda_i) \leq 2 \).

**Proof.** We first construct a decomposition \( \lambda = \sum_{i=1}^{\infty} \lambda_i \) into pieces satisfying \( m_0(\lambda_i) \leq 2 \) for all \( i \), assuming that \( \lambda((0, \infty)) = \infty \) (if \( \lambda((0, \infty)) < \infty \), the construction is similar, with all but finitely many \( \lambda_i \)'s equal to 0). Define \( z_0 = \infty \) and \( z_\nu = \sup\{z > 0 : \lambda(z) > \nu\} \) for \( \nu \in \mathbb{N} \), where \( \lambda(z) = \lambda((z, \infty)) \). Clearly, \( \lambda(z_n) \leq \lambda(z_{n+1}) \). Let \( \lambda_1 = \lambda(0, z_1) \), and assume that \( z_{k+1} < z_k \) and that \( \lambda_{k+1} \) has already been defined for some \( k \geq 0 \). If \( z_{k+2} < z_{k+1} \), put \( \lambda_{k+2} = \lambda_{k+2}(z_{k+2}, z_{k+1}) \). Since \( z_{k+1} < z_k \) implies \( \lambda(z_{k+1}) \geq k \), we have that

\[
\lambda_{k+2}((0, \infty)) = \lambda((z_{k+2}, z_{k+1})) = \lambda(z_{k+2}) - \lambda(z_{k+1}) \leq 2.
\]

If \( z_{k+1} = z_{k+2} \), then let \( \ell \geq 2 \) be the number for which \( z_{k+1} = z_{k+\ell} > z_{k+\ell+1} \). Then let

\[
\lambda_{k+2} = \cdots = \lambda_{k+\ell} = \delta_{z_{k+1}}, \quad \lambda_{k+\ell+1} = \lambda(z_{k+\ell+1}, z_{k+1}) + (\lambda(z_{k+1}) - (\ell - 1))\delta_{z_{k+1}}.
\]
where $\delta_x$ stands for the Dirac delta at $x$. Note that $\ell - 1 \leq \lambda(\{z_{k+1}\}) \leq \ell$, so $\lambda_{k+\ell+1}$ is a positive measure. Furthermore, we have
\[
\lambda((z_{k+\ell+1}, z_{k+1})) + \lambda(\{z_{k+1}\}) \leq \lambda(x, y) \leq \lambda((z_{k+1}, z_{k+1})) - k + \ell + 1 = \ell + 1,
\]
which implies that $\lambda_{k+\ell+1}((0, \infty)) \leq 2$. This completes the construction of the decomposition $\lambda = \sum_{i=1}^{\infty} \lambda_i$. The $\mu_i$'s can now be obtained by restrictions and thinnings of $\mu$ (see [38, Sect. 5]).

**Lemma A.5.** For $m, N \in \mathbb{N}$ and $\beta > 0$, let
\[
Y^{(0)}(t, x) = Y^{(0, m, \beta)}(t, x) = Y_{<}(t, x),
\]
\[
Y^{(0,\beta)}(t, x) = Y^{(0, \beta)}(t, x) = 1, \quad u^{(0,\beta)}(s, y; t, x) = g(t - s, x - y)
\]
and introduce the following processes inductively:
\[
Y^{(N)}(t, x) = Y_{<}(t, x) + \int_{0}^{t} \int_{\mathbb{R}^{d}} u_{<}(s, y; t, x) \frac{1}{\sqrt{1 - s}} Y^{(N-1)}(s, y) \Lambda_{>}(ds, dy),
\]
\[
Y^{(m,\beta)}(t, x) = 1 + \int_{0}^{t} \int_{\mathbb{R}^{d}} g(t - s, x - y) \frac{1}{\sqrt{1 - s}} Y^{(m-1,\beta)}(s, y) \Lambda_{>}(ds, dy),
\]
\[
u^{(m,\beta)}(s, y; t, x) = g(t - s, x - y) + \int_{0}^{t} \int_{\mathbb{R}^{d}} g(t - r, x - w)
\]
\[
\times 1_{\{|x-w| \leq \beta \sqrt{1 - s}|Y^{(m-1,\beta)}(s, y) \Lambda_{>}(dr, dw),
\]
\[
Y^{(N,m,\beta)}(t, x) = Y^{(m,\beta)}(t, x) + \int_{0}^{t} \int_{\mathbb{R}^{d}} u^{(m,\beta)}(s, y; t, x)
\]
\[
\times 1_{\{|x-y| \leq \beta \sqrt{1 - s}|Y^{(N-1,m,\beta)}(s, y) \Lambda_{>}(ds, dy),
\]
\[
Y^{(m,\beta)}(t, x) = Y^{(m,\beta)}(t, x) + \int_{0}^{t} \int_{\mathbb{R}^{d}} u^{(m,\beta)}(s, y; t, x)
\]
\[
\times 1_{\{|x-y| \leq \beta \sqrt{1 - s}|Y^{(m-1,\beta)}(s, y) \Lambda_{>}(ds, dy),
\]
For every $t > 0$, there is a constant $C \in (0, \infty)$ such that for all $p \in (1, 1 + \frac{2}{\beta})$, $s \in (0, t)$, $x, y \in \mathbb{R}^{d}$, and $m \in \mathbb{N}$ with $\theta m > 1$,
\[
E[|Y_{<}(t, x) - Y^{(m,\beta)}(t, x)|^{p}] \leq C \left( e^{-C^{-1} \beta} + (C \theta p^{-\frac{2}{\beta}})^{m} \frac{m}{\theta p} m \right) e^{(C/\theta p)^{3/\beta}},
\]
(A.4)
\[
E[|u_{<}(s, y; t, x) - u^{(m,\beta)}(s, y; t, x)|^{p}] \leq C \left( e^{-C^{-1} \beta} + (C \theta p^{-\frac{2}{\beta}})^{m} \frac{m}{\theta p} m \right) e^{(C/\theta p)^{3/\beta}} g(t - s, x - y).
\]
One can further choose $C$ in such a way that
(A.5)
\[
E[|Y(t, x) - Y^{(m,\beta)}(t, x)|^{p}] \leq C \left( e^{-C^{-1} \beta} + (C \theta p^{-\frac{2}{\beta}})^{m} \frac{m}{\theta p} m \right) e^{(C M_{p}(\lambda)/\theta p)^{3/\beta}},
\]
\[
E[|Y^{(N)}(t, x) - Y^{(N,m,\beta)}(t, x)|^{p}] \leq C \left( e^{-C^{-1} \beta} + (C \theta p^{-\frac{2}{\beta}})^{m} \frac{m}{\theta p} m \right) e^{(C M_{p}(\lambda)/\theta p)^{3/\beta}}.
\]
PROOF. We start with the first inequality in (A.4). Let $Y_{<}^{(m)}(t, x) = Y_{<}^{(m,\infty)}(t, x)$. Because

$$
\mathbb{E}[|Y_{<}(t, x) - Y_{<}^{(m,\beta)}(t, x)|^p]^{\frac{1}{p}} \leq \mathbb{E}[||Y_{<}(t, x) - Y_{<}^{(m)}(t, x)|^p]^{\frac{1}{p}} + \mathbb{E}[||Y_{<}^{(m)}(t, x) - Y_{<}^{(m,\beta)}(t, x)|^p]^{\frac{1}{p}},
$$

we can bound the two terms on the right-hand side separately. Upon noticing that $Y_{<}^{(m)}(t, x)$ is, in fact, the sum of the first $m+1$ terms in the chaos expansion of $Y_{<}(t, x)$, we infer from (3.7) that for $d \geq 2$ and $p \in (1, 1 + \frac{2}{d})$,

$$
\mathbb{E}[|Y_{<}(t, x) - Y_{<}^{(m)}(t, x)|^p]^{\frac{1}{p}} \leq C \theta_p \frac{1}{p} \sum_{k=m+1}^{\infty} \left( \frac{\Gamma(\theta p / \theta p k + \theta p / 1/\theta p)}{k} \right) \left( \frac{\Gamma(\theta p / \theta p k + \theta p / 1/\theta p)}{k} \right) \mathbb{E}[|Y_{<}(t, x) - Y_{<}^{(m)}(t, x)|^p]^{\frac{1}{p}}
$$

(A.6)

where we used Lemma A.1, Stirling’s formula for gamma functions and the property $\Gamma(x) \sim x^{-1}$ as $x \to 0$. By (3.9) and (3.10), the last bound remains true if $d = 1$.

Next, observe that

$$
Y_{<}^{(m)}(t, x) - Y_{<}^{(m,\beta)}(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) \mathbb{I}_{\{|x-y|>\beta\sqrt{t-s}\}} Y_{<}(m-1)(s, y) \Lambda_{<}(ds, dy)
$$

$$
+ \int_0^t \int_{\mathbb{R}^d} g(t - s, x - y) \mathbb{I}_{\{|x-y|\leq\beta\sqrt{t-s}\}} Y_{<}(m-1)(s, y) - Y_{<}(m-\beta)(s, y) \Lambda_{<}(ds, dy).
$$

Iterating this $m$ times, denoting $(t, x) = (t_{k+1}, x_{k+1})$, we derive the identity

$$
Y_{<}^{(m)}(t, x) - Y_{<}^{(m,\beta)}(t, x) = \sum_{k=1}^{m} \int_{(0, t) \times \mathbb{R}^d} \left( \prod_{i=1}^{k+1} g(\Delta t_i, \Delta x_i) \mathbb{I}_{\{|\Delta x_i|\leq\beta\sqrt{\Delta t_i}\}} \right)
$$

$$
\times g(\Delta t_2, \Delta x_2) \mathbb{I}_{\{|\Delta x_2|>\beta\sqrt{\Delta t_2}\}} Y_{<}^{(m-k)}(t_1, x_1) \prod_{j=1}^{k} \Lambda_{<}(dt_j, dx_j).
$$

Representing $Y_{<}^{(m-k)}(t_1, x_1)$ itself in a series, we obtain

(A.7)

$$
Y_{<}^{(m)}(t, x) - Y_{<}^{(m,\beta)}(t, x) = \sum_{k=1}^{m} \sum_{\ell=1}^{m-k-1} \int_{(0, t) \times \mathbb{R}^d} \prod_{i=2}^{k+\ell} g_{i,k,\ell}(\Delta t_i, \Delta x_i) \prod_{j=1}^{m} \Lambda_{<}(dt_j, dx_j),
$$

where $g_{i,k,\ell}(t, x) = g(t, x)$ if $i = 2, \ldots, \ell$, $g_{i,k,\ell}(t, x) = g(t, x) \mathbb{I}_{\{|x|>\beta\sqrt{t}\}}$ if $i = \ell + 1$ and $g_{i,k,\ell}(t, x) = g(t, x) \mathbb{I}_{\{|x|\leq\beta\sqrt{t}\}}$ if $i = \ell + 2, \ldots, k + \ell$. An important observation is now that the moment bounds on $Y_{<}(t, x)$ obtained in [9, Proposition 6.1] or through the series of arguments leading to (3.10) (if $d = 1$ and $p \in (2, 3)$) are, first of all, obtained by estimating each term in a series expansion of $Y_{<}(t, x)$ separately and, second of all, can only increase if the kernels $g_{i,k,\ell}$ are replaced by something larger. Therefore, bounding $g_{i,k,\ell}(t, x) \leq g(t, x) \leq C g(2t, x)$ if $i \neq \ell + 1$ and

$$
g_{i,k,\ell}(t, x) \leq (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} e^{-\frac{t}{4}} \leq C e^{-C^{-1} \beta} g(2t, x)
$$
if $i = \ell + 1$, we conclude that

$$E[|Y^{(m)}_x(t, x) - Y^{(m, \beta)}_x(t, x)|^p] \leq C e^{-C^{-1}\beta} e^{(C/\theta_p)^{3/\nu}}.$$  

Together with (A.6), this shows the first inequality in (A.4); the proof of the second inequality in (A.4) and the proof of (A.5) are similar and therefore skipped.

**Lemma A.6.** Let $d = 1$, $\beta > 0$, $m \in \mathbb{N}$ and consider the processes $Y_0$ and $Z^{(m, \beta)}$ defined in (4.3) and (4.5), respectively. Assume Condition (Sup) and Condition (L-\alpha) with $\alpha = 2$. Then there exists a constant $C > 0$ such that for any interval $I$ of length 1 and $R > 1$, we have that

$$P\left(\sup_{x \in I} |Y_0(t, x) - Z^{(m, \beta)}(t, x)| > R\right) \leq C_{m, \beta} R^{-2} \log R,$$

where $C_{m, \beta} = C(e^{-C^{-1}\beta} + C^m m^{-m/12})$. If $M_{\alpha}(\lambda) < \infty$ for some $\alpha > 2$, the factor $\log R$ can be omitted.

**Proof.** The proof is very similar to how we dealt with $Y_1$ in the proof of Theorem 3.8. In fact, we only need to estimate the last integral in (3.42) with $\tilde{Y}'(s, y)$ replaced by (a copy of) $|Y(s, y) - Y^{(m, \beta)}(s, y)|$. By Markov's inequality and Lemma A.5, we have that

$$P((t - s)^{-\frac{1}{d}}|Y(s, y) - Y^{(m, \beta)}(s, y)| > \frac{R}{\lambda}) \leq C_{m, \beta} R^{-2} N^2 z^2 (t - s)^{-1} \wedge 1 \leq C_{m, \beta} N^2 (R^{-2} z^2 (t - s)^{-1} \wedge 1).$$

Therefore,

$$\int_0^t \int_{(-2, 2)} \int (0, \infty) P((t - s)^{-\frac{1}{d}}|Y(s, y) - Y^{(m, \beta)}(s, y)| > \frac{R}{\lambda}) \, ds \, dy \, \lambda(dz)$$

$$\leq 4N^2 C_{m, \beta} \int (0, \infty) \left( R^{-2} z^2 \int_{-2}^{-2} s^{-1} \, ds + R^{-2} z^2 \int_{2}^{t} s^{-1} \, ds \right) \lambda(dz)$$

$$\leq 4N^2 C_{m, \beta} (C \mu_2(\lambda) R^{-2} \log R + 2m_2^2 \log(\lambda) R^{-2}),$$

which yields the desired bound since $m_2^{\log}(\lambda) < \infty$ by Condition (Sup). If $M_{\alpha}(\lambda) < \infty$ for some $\alpha > 2$, we can get rid of the logarithmic factor by using power $\alpha$ in Markov's inequality above.

**Lemma A.7.** Let $E \subseteq \mathbb{R}^d$ and $f: E \rightarrow \mathbb{R}^p$ be a Lipschitz continuous function such that

$$\varepsilon = \liminf_{x \in E, |x| \rightarrow \infty} \frac{|f(x)|_\infty}{|x|_\infty} > 0.$$  

Then $\text{Dim}_H(f(E)) \leq \text{Dim}_H(E)$ and $\text{Dim}_M(f(E)) \leq \text{Dim}_M(E)$.

**Proof.** The statement for the Hausdorff dimension is exactly [32, Lemma 2.4]. In order to obtain the statement concerning the Minkowski dimension, we notice that

$$\text{Dim}_M(E) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \{q \in \mathbb{Z}^d : E \cap Q(0, e^n) \cap Q(q, 1) \neq \emptyset\},$$

which can be easily deduced from [34, Prop. 2.5]. Let $A_E(n)$ be the set whose cardinality is counted in the previous line. Then $E \cap Q(0, e^m) \subseteq \bigcup_{q \in A_E(m)} Q(q, 1)$ by definition and hence, $f(E \cap Q(0, e^m)) \subseteq \bigcup_{q \in A_E(m)} f(Q(q, 1))$ for every $m \in \mathbb{N}$.  

If \( C \in \mathbb{N} \) is larger than \( \log e^{-1} \) and \( n \) is large enough, then \( f(x) \in Q(0,e^n) \) for some \( x \in E \) implies \( x \in Q(0,e^{n+C}) \) by the growth assumption on \( f \). Therefore, \( f(E) \cap Q(0,e^n) \subseteq f(E) \cap Q(0,e^{n+C}) \subseteq \bigcup_{Q \in A_E(n+C)} f(Q(q,1)) \). If \( L \) is the Lipschitz constant of \( f \) with respect to the supremum norm, then \( f(Q(q,1)) \) has at most diameter \( L \) (in the same norm) and can therefore be covered by \( L^p \) unit cubes, or \( (L + 1)^p \) unit cubes with integer corners. In total, we need at most \( |A_E(n + C)|(L + 1)^p \) such cubes to cover \( f(E) \cap Q(0,e^n) \). Thus,

\[
\dim_M(f(E)) = \limsup_{n \to \infty} \frac{1}{n} \log_+ |A_f(E)(n)| \leq \limsup_{n \to \infty} \frac{1}{n} \log_+ (|A_E(n + C)|(L + 1)^p) = \limsup_{n \to \infty} \frac{1}{n} \log_+ |A_E(n + C)| = \dim_M(E).
\]

**Lemma A.8.** Let \( |\cdot| \) and \( \|\cdot\| \) be norms on \( \mathbb{R}^d \) and \( \exp^{(n)}(r) = \exp(\exp^{(n-1)}(r)) \) for \( n \in \mathbb{N} \) and \( r \in \mathbb{R} \) (with \( \exp^{(0)}(r) = r \)). Then, for every \( N \in \mathbb{N} \), the function

\[
f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d, \quad x \mapsto \frac{x}{|x|} \left( \log(N) \left( \frac{|x|}{\|x\|} \exp^{(N)}(\|x\|^d) \right) \right)^{1/d},
\]

satisfies (A.8) and is Lipschitz continuous on \( \{x \in \mathbb{R}^d : |x| \geq s\} \) for some \( s > 0 \).

**Proof.** Because all norms are equivalent on \( \mathbb{R}^d \), we have \( C^{-1} \leq |x|/\|x\| \leq C \) for some \( C > 1 \). Consider the mapping \( h(r,s) = (\log(N)(r \exp^{(N)}(s^d)))^{1/d} \) for \( r \in (C^{-1},C) \) and \( s > 0 \). For sufficiently large \( s \) (so that \( C^{-1} \exp^{(N)}(s^d) > e \)), its partial derivatives are given by

\[
\frac{\partial}{\partial r} h(r,s) = \frac{d^{-1} (\log(N)(r \exp^{(N)}(s^d)))^{1/d-1}}{r \prod_{p=1}^{N-1} \log(p) (r \exp^{(N)}(s^d))},
\]

\[
\frac{\partial}{\partial s} h(r,s) = \frac{s^{d-1} (\log(N)(r \exp^{(N)}(s^d)))^{1/d-1} \prod_{p=1}^{N-1} \exp(p)(s^d)}{\prod_{p=1}^{N-1} \log(p) (r \exp^{(N)}(s^d))}.
\]

By induction on \( p \), one can easily verify that \( \log(p) (r \exp^{(N)}(s^d)) \geq \frac{1}{2} \frac{1}{s^d} \exp^{(N-p)}(s^d) \) as soon as \( s \) is large enough so that \( \frac{1}{2} s^d > \log 2 \vee \log C \). This shows (A.8) on the one hand and that the partial derivatives of \( h \) are uniformly bounded for \( r \in (C^{-1},C) \) and large \( s \) on the other hand.

Moreover, by elementary estimates,

\[
\frac{|x|}{|y|} - \frac{|y|}{|x|} = \frac{|x - y||y| + y(|y| - |x|)}{|x||y|} \leq \frac{2|x - y|}{|x|},
\]

\[
\frac{|x|}{|y|} - \frac{|y|}{|x|} = \frac{|(|x| - |y|)||y| + |y|(|y| - |x|)}{|x||y||y|} \leq \frac{|x - y|}{|x|} + \frac{|y|}{|x||y||y|} |x - y|.
\]

By writing

\[
f(x) - f(y) = \left( \frac{x}{|x|} - \frac{y}{|y|} \right) h\left( \frac{|x|}{|x||y|}, |x| \right) + \frac{y}{|y|} h\left( \frac{|x|}{|x||y|}, |x| \right) - h\left( \frac{|y|}{|y||y|}, |y| \right),
\]

the Lipschitz property of \( f \) now follows from the previous estimates and a straightforward application of the mean-value theorem to the second difference above.

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