WEAK MIXING PROPERTIES OF VECTOR SEQUENCES

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Dedicated to the memory of our colleague Gert K. Pedersen

Abstract. Notions of weak and uniformly weak mixing (to zero) are defined for bounded sequences in arbitrary Banach spaces. Uniformly weak mixing for vector sequences is characterized by mean ergodic convergence properties. This characterization turns out to be useful in the study of multiple recurrence, where mixing properties of vector sequences, which are not orbits of linear operators, are investigated. For bounded sequences, which satisfy a certain domination condition, it is shown that weak mixing to zero is equivalent with uniformly weak mixing to zero.

1. Introduction

We recall that the upper density \( D^*(A) \) and the lower density \( D_*(A) \) of some \( A \subset \mathbb{N} := \{0, 1, 2, \ldots \} \) are defined by
\[
D^*(A) := \lim_{n \to \infty} \frac{1}{n+1} \text{card} (A \cap [0, n]), \quad D_*(A) := \lim_{n \to \infty} \frac{1}{n+1} \text{card} (A \cap [0, n])
\]
(see e.g. [7], Chapter 3, §5 or [12], §2.3). If upper and lower densities coincide then \( A \) is called having density \( D(A) := D^*(A) = D_*(A) \).

Clearly, for \( A \subset \mathbb{N}^* := \mathbb{N} \setminus \{0\} = \{1, 2, \ldots \} \) we can use also the formulas
\[
D^*(A) = \lim_{n \to \infty} \frac{1}{n} \text{card} (A \cap [1, n]), \quad D_*(A) = \lim_{n \to \infty} \frac{1}{n} \text{card} (A \cap [1, n]) .
\]

The upper (resp. lower) density of a sequence \((k_j)_{j \geq 1}\) in \( \mathbb{N}^* \) means the upper (resp. lower) density of the subset \( \{k_j : j \geq 1\} \) of \( \mathbb{N}^* \). It is easy to see that the lower density of a strictly increasing \((k_j)_{j \geq 1}\) is \( > 0 \) if and only if \( \sup_{j \geq 1} \frac{k_j}{j} < +\infty \).

Let \( X \) be a Banach space with dual space \( X^* \). We shall say that a sequence \((x_k)_{k \geq 1}\) in \( X \) is weakly mixing to zero if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} | \langle x^*, x_k \rangle | = 0 \quad \text{for all} \quad x^* \in X^* ,
\]
and we shall say that it is uniformly weakly mixing to zero if
\[
\lim_{n \to \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} | \langle x^*, x_k \rangle | ; \ x^* \in X^* , \ \|x^*\| \leq 1 \right\} = 0 .
\]

A linear operator \( U : X \to X \) is usually called weakly mixing to zero at \( x \in X \) if the orbit \((U^k(x))_{k \geq 1}\) is weakly mixing to zero.
The following characterization of weak mixing to zero for power bounded linear operators, which is a counterpart of the Blum-Hanson theorem \[1\] for weak mixing, was proved by L. K. Jones and M. Lin \[10\]:

**Theorem 1.1.** Let \( U \) be a power bounded linear operator on a Banach space \( X \), \( x \in X \), and \( x_k = U^k(x), \ k \geq 1 \). Then the following conditions are equivalent:

(i) The sequence \((x_k)_{k \geq 1}\) is weakly mixing to zero.

(j) The sequence \((x_k)_{k \geq 1}\) is uniformly weakly mixing to zero.

(jj) For every sequence \( k_1 < k_2 < \ldots \) in \( \mathbb{N}^\ast \) of lower density \( > 0 \),

\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=1}^{n} x_{k_j} \right\| = 0.
\]

One main goal of this paper is to prove in the next section that conditions (jj) and (jjj) in Theorem 1.1 are equivalent for any bounded sequence \((x_k)_{k \geq 1}\) in the Banach space \( X \), not only for the points of an orbit of some power bounded linear operator on \( X \) (Theorem 2.2). Therefore, for any bounded sequence in a Banach space, uniformly weak mixing to zero is equivalent with the mean ergodic convergence property from (jj) (in particular, for bounded sequences in Hilbert spaces, our notion of “uniformly weak mixing to zero” coincides with the notion of “weak mixing” considered in \[2\]).

We notice that this result was used by C. Niculescu, A. Strôh, and L. Zsidó to prove that if \( \Phi \) is a *-endomorphism of a \( C^\ast \)-algebra \( A \), leaving invariant a state \( \varphi \) of \( A \), whose support in \( A^{\ast \ast} \) belongs to the centre of \( A^{\ast \ast} \), and \( \Phi \) is weakly mixing with respect to \( \varphi \), then \( \Phi \) is automatically weak mixing of order 2 (\[13\], Theorem 1.3): this is a partial extension to the non-commutative \( C^\ast \)-dynamical systems of a classical result of H. Furstenberg, according to which every weakly mixing measure preserving transformation of a probability measure space is weakly mixing of any order (\[7\], Theorem 4.11).

For general bounded sequences in Banach spaces (or even in Hilbert spaces), condition (i) in Theorem 1.1 does not imply the equivalent conditions (j) and (jjj) (Examples 3.1 and 5.2). Nevertheless, we shall prove in Section 5 that (i) implies (j) and (jjj) provided that the sequence satisfies some appropriate domination condition, called “convex shift-boundedness”, which of course holds if the sequence is an orbit of some power bounded linear operator. Actually it will be proved that if a convex shift-bounded sequence \((x_k)_{k \geq 1}\) in the Banach space \( X \) is weakly mixing to zero, then

\[
\lim_{a, b \in \mathbb{N}^\ast} \sup_{b-a \to \infty} \left\{ \frac{1}{b-a+1} \sum_{k=a}^{b} |\langle x^*, x_k \rangle| ; \ x^* \in X^\ast, \ |x^*| \leq 1 \right\} = 0
\]

(Theorem 4.2). Its proof depends upon a structure theorem for sets of natural numbers of non-zero upper Banach density (Theorem 4.2), which is of interest for itself. We notice that if \((x_k)_{k \geq 1}\) is an orbit of some power bounded linear operator on \( X \), then (1.3) is an immediate consequence of (1.2).

Finally, in Section 6 it will be shown that in uniformly convex Banach spaces the above implication holds for sequences which satisfy a condition weaker than convex shift-boundedness (Theorem 6.3).

We notice that a short investigation of the ergodicity, that is of the Cesaro norm-convergence to zero, of convex shift-bounded sequences is postponed in an appendix.

2. **Uniformly weak mixing to zero**

A subset \( N \) of \( \mathbb{N}^\ast \) is called relativelty dense if there exists \( L > 0 \) such that every interval of natural numbers of length \( \geq L \) contains some element of \( N \). In this case holds clearly \( D_\ast(N) \geq \frac{1}{L} \), so relatively dense sets are of lower density \( > 0 \).

A sequence \((k_j)_{j \geq 1}\) in \( \mathbb{N}^\ast \) is called relatively dense if the subset \( \{ k_j ; j \geq 1 \} \) of \( \mathbb{N}^\ast \) is relatively dense. It is easy to see that a strictly increasing sequence \((k_j)_{j \geq 1}\) is relatively dense if and only if \( \sup_{j \geq 1} (k_{j+1} - k_j) < +\infty \).
The proof of the following lemma is immediate and we give it only for the sake of completeness:

**Lemma 2.1.** For any sequence \((x_k)_{k \geq 1}\) in a Banach space and any sequence \((k_j)_{j \geq 1}\) in \(\mathbb{N}^*\) of lower density \(> 0\) we have

\[
\left\| \frac{1}{n} \sum_{j=1}^{n} x_k \right\| \rightarrow 0 \iff \left\| \frac{1}{n} \sum_{k \in \{k_1, k_2, \ldots\} \cap [k_n, n]} x_k \right\| \rightarrow 0.
\]

**Proof.** For \(\Rightarrow\): with \(n \geq k_1\), defining \(j(n) \in \mathbb{N}^*\) by \(k_n \leq n < k_{j(n)} + 1\), we have

\[
\left\| \frac{1}{n} \sum_{k \leq n} x_k \right\| = \left\| \frac{1}{n} \sum_{j=1}^{j(n)} x_{k_j} \right\| = \frac{j(n)}{n} \left\| \sum_{j=1}^{j(n)} x_{k_{j(n)}} \right\| \rightarrow 0.
\]

The converse implication \(\Leftarrow\) follows by using

\[
\left\| \frac{1}{n} \sum_{j=1}^{n} x_k \right\| = \left\| \frac{1}{n} \sum_{k \leq k_n} x_k \right\| = \frac{k_n}{n} \left\| \sum_{k \in \{k_1, k_2, \ldots\} \cap [k_n, n]} x_k \right\|.
\]

\(\square\)

The next lemma is the main ingredient in the proof of the main result of the section:

**Lemma 2.2.** Let \(\Omega\) be a compact Hausdorff topological space, and \(f_1, f_2, \ldots\) continuous complex functions on \(\Omega\) of uniform norm \(\|f_k\|_{\infty} \leq 1\). If

\[
\left\| \frac{1}{n} \sum_{j=1}^{n} f_{k_j} \right\|_{\infty} \rightarrow 0 \quad \text{for every relatively dense } (k_j)_{j \geq 1} \subset \mathbb{N}^*
\]

then

\[
\left\| \frac{1}{n} \sum_{k=1}^{n} |f_k| \right\|_{\infty} \rightarrow 0.
\]

**Proof.** Without loss of generality we can assume that the functions \(f_k\) are real. Furthermore, since \(|f_k| = 2f_k^+ - f_k\), it is enough to prove that

\[
\left\| \frac{1}{n} \sum_{k=1}^{n} f_k^+ \right\|_{\infty} \rightarrow 0.
\]

Let us assume the contrary, that is the existence of some \(\varepsilon_o > 0\) for which

\[
\mathcal{J} := \left\{ n \geq 1 ; \left\| \frac{1}{n} \sum_{k=1}^{n} f_k^+ \right\|_{\infty} \geq \varepsilon_o \right\}
\]

is infinite.

For every \(n \in \mathcal{J}\) there exists \(\omega_n \in \Omega\) such that

the cardinality of \(\mathcal{N}_n := \{1 \leq k \leq n ; f_k^+ (\omega_n) \geq \frac{\varepsilon_o}{2}\}\) is

\[
\geq \frac{n \varepsilon_o}{2}.
\]

Indeed, if \(\omega_n \in \Omega\) is chosen such that

\[
\frac{1}{n} \sum_{k=1}^{n} f_k^+ (\omega_n) = \left\| \frac{1}{n} \sum_{k=1}^{n} f_k^+ \right\|_{\infty} \geq \varepsilon_o
\]
then
\[
\varepsilon_o \leq \frac{1}{n} \left( \sum_{k \in N_o} f_k^+ (\omega_n) + \sum_{1 \leq k \leq n} f_k^+ (\omega_n) \right) \leq
\]
\[
\leq \frac{1}{n} \left( \text{card} (N_n) + \frac{\varepsilon_o}{2} (n - \text{card} (N_n)) \right) \leq
\]
\[
\leq \frac{1}{n} \text{card} (N_n) + \frac{\varepsilon_o}{2}.
\]

Denoting now the least element of \( J \) by \( k_1 \), we can construct recursively a sequence \( k_1 < k_2 < \ldots \) in \( J \) such that

the cardinality of \( N_{k_{j+1}}^j := \{ k \in N_{k_{j+1}}; k > k_j \} \) is \( \geq \frac{k_{j+1} \cdot \varepsilon_o}{4}, \quad j \geq 1 \).

For it is enough to choose \( k_{j+1} \geq \frac{4k_j}{\varepsilon_o} \), because then
\[
\text{card} (N_{k_{j+1}}^j) \geq \text{card} (N_{k_{j+1}}) - k_j \geq \frac{k_{j+1} \cdot \varepsilon_o}{2} - k_j \geq \frac{k_{j+1} \cdot \varepsilon_o}{4}.
\]

Putting
\[
N' := \bigcup_{j \geq 2} N_{k_j}^j,
\]
we have for every \( j \geq 2 \)
\[
N \cap (k_{j-1}, k_j] = N_{k_j}^j \subset N_{k_j},
\]
in particular,
\[
k \in N', k_{j-1} < k \leq k_j \implies f_k^+ (\omega_{k_j}) \geq \frac{\varepsilon_o}{2}
\]
\[
\implies f_k (\omega_{k_j}) = f_k^+ (\omega_{k_j}) \geq \frac{\varepsilon_o}{2} .
\]

Let us choose some integer \( p \geq \frac{16}{\varepsilon_o} \). Since
\[
N'(p) := N \cup \{ p, 2p, 3p, \ldots \} \subset N^*
\]
is relatively dense, by the assumption on the functions \( f_k \) and by Lemma 2.1 there exists \( m_0 \geq 1 \) such that
\[
m \geq m_0 \implies \left\| \frac{1}{m} \sum_{k \in N'(p) \atop k \leq m} f_k \right\|_\infty \leq \frac{\varepsilon_o^2}{34}.
\]

Then we get for any \( j \geq 2 \) with \( k_{j-1} \geq m_0 \)
\[
\frac{\varepsilon_o^2}{17} = \frac{2}{34} \varepsilon_o^2 \geq \left\| \frac{1}{k} \sum_{k \in N'(p) \atop k \leq k_j} f_k \right\|_\infty + \left\| \frac{1}{k_{j-1}} \sum_{k \in N'(p) \atop k \leq k_{j-1}} f_k \right\|_\infty \geq
\]
\[
\geq \left\| \frac{1}{k_j} \sum_{k \in N'(p) \atop k_{j-1} < k \leq k_j} f_k \right\|_\infty \geq \left\| \frac{1}{k_j} \sum_{k \in N'(p) \atop k_{j-1} < k \leq k_j} f_k (\omega_{k_j}) \right\|_\infty \geq
\]
\[
\geq \left\| \frac{1}{k_j} \sum_{k \in N \atop k_{j-1} < k \leq k_j} f_k (\omega_{k_j}) \right\| - \left\| \frac{1}{k_j} \sum_{k \in N \atop k_{j-1} < k \leq k_j \text{ k multiple of } p} f_k (\omega_{k_j}) \right\| \geq
\]
\[
\geq \frac{1}{k_j} \cdot \varepsilon_o \cdot \text{card} (N_{k_j}) - \frac{1}{k_j} \cdot \text{card} \left( \{ 1 \leq k \leq k_j ; k \text{ multiple of } p \} \right) \geq
\]
\[
\geq \frac{1}{k_j} \cdot \varepsilon_o \cdot \frac{k_j \cdot \varepsilon_o}{4} - \frac{1}{k_j} \cdot \frac{k_j}{p} = \varepsilon_o^2 \cdot \frac{1}{8} - \frac{\varepsilon_o^2}{16},
\]
which is absurde.

Now we can characterize uniformly weak mixing to zero for bounded sequences in Banach spaces by mean ergodic convergence properties:

**Theorem 2.3** (Mean ergodic description of uniformly weak mixing). For a bounded sequence $(x_k)_{k \geq 1}$ in a Banach space $X$, the following conditions are equivalent:

(i) $(x_k)_{k \geq 1}$ is uniformly weakly mixing to zero, that is

$$\lim_{n \to \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} |\langle x^*, x_k \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\} = 0.$$

(ii) For every sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}^*$ of lower density $> 0$, $\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=1}^{n} x_{k_j} \right\| = 0$.

(iii) For every relatively dense sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}^*$, $\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=1}^{n} x_{k_j} \right\| = 0$.

**Proof.** Implication (i) $\Rightarrow$ (ii) follows immediately from Lemma 2.1 and (ii) $\Rightarrow$ (iii) is trivial.

For (iii) $\Rightarrow$ (i) we recall that the closed unit ball $B_{X^*}$ of $X^*$ is weak*-compact and the evaluation functions $f_x : B_{X^*} \ni x^* \mapsto \langle x^*, x \rangle$, $x \in X$ are weak*-continuous. Since

(j) means $\frac{1}{n} \sum_{k=1}^{n} |f_{x_k}|$ uniformly $\to 0$ and

(jjj) means that, for every relatively dense sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}^*$,

$$\frac{1}{n} \sum_{k=1}^{n} f_{x_{k_j}} \xrightarrow{\text{uniformly}} 0,$$

implication (iii) $\Rightarrow$ (i) follows from Lemma 2.2.

□

Theorem 2.3 yields a similar characterization of weak mixing to zero:

**Corollary 2.4.** For a bounded sequence $(x_k)_{k \geq 1}$ in a Banach space $X$ and $x^* \in X^*$, the following conditions are equivalent:

(i) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\langle x^*, x_k \rangle| = 0$.

(ii) For every sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}^*$ of lower density $> 0$,

$$\lim_{n \to \infty} \left\langle x^*, \frac{1}{n} \sum_{j=1}^{n} x_{k_j} \right\rangle = 0.$$

(iii) For every relatively dense sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}^*$,

$$\lim_{n \to \infty} \left\langle x^*, \frac{1}{n} \sum_{j=1}^{n} x_{k_j} \right\rangle = 0.$$

**Proof.** We have just to apply Theorem 2.3 to the bounded scalar sequence $(\langle x^*, x_k \rangle)_{k \geq 1}$.

□
3. Comparison of weak and uniformly weak mixing to zero

Let us first give an example of a bounded sequence in the Banach space $C([0,1])$ of all continuous functions on $[0,1]$, which satisfies (i) but not (j) in Theorem 1.3. $\| \cdot \|_\infty$ will stand for the uniform norm on $C([0,1])$ and $\text{supp}(f)$ will denote the support of $f \in C([0,1])$.

**Example 3.1.** Let $1 = n_1 < n_2 < \ldots$ be a sequence in $\mathbb{N}^*$ such that

$$\frac{n_j - 1}{n_{j+1} - 1} \leq \frac{1}{2}, \quad j \geq 1$$

(for example, $n_1 = 1$, $n_2 = 2$ and $n_{j+1} = 2n_j - 1$ for $j \geq 2$),

$$1 > t_1 > t_2 > \ldots > 0, \quad t_j \to 0$$

real numbers, and $g_j : [0,1] \to [0,1]$, $j \geq 1$, continuous functions such that $\text{supp}(g_j) \subset [t_{j+1}, t_j]$ and $\|g_j\|_\infty = 1$ for all $j \geq 1$.

If we set

$$f_k = g_j \quad \text{for} \quad n_j \leq k < n_{j+1},$$

then $(f_k)_{k \geq 1}$ is a bounded sequence in $C([0,1])$, which is weakly convergent to zero, and so is weakly mixing to zero, but which is not uniformly weakly mixing to zero.

**Proof.** Since $0 \leq f_k \leq 1$ for every $k \geq 1$, according to the Riesz representation theorem and the Lebesgue dominated convergence theorem, the weak convergence of $(f_k)_{k \geq 1}$ to zero is equivalent to

(3.1)

$$f_k \overset{\text{pointwise}}{\to} 0,$$

while (3.2) for $(f_k)_{k \geq 1}$ is equivalent with

(3.2)

$$\frac{1}{n} \sum_{k=1}^{n} f_k \overset{\text{uniformly}}{\to} 0.$$

For (3.1) let $t \in [0,1]$ be arbitrary. If $t = 0$ then (3.1) holds obviously because $f_k(0) = 0$ for all $k \geq 1$. On the other hand, if $0 < t \leq 1$ then there exists some $j \geq 1$ with $t_j < t$ and so $f_k(t_j = 0, \quad n \geq n_j$.

Now, by the positivity of the functions $g_j$ and $f_k$, we have for every $j \geq 1$:

$$\frac{1}{n_{j+1} - 1} \sum_{k=1}^{n_{j+1} - 1} f_k \geq \frac{1}{n_{j+1} - 1} \sum_{k=n_j}^{n_{j+1} - 1} f_k = \frac{n_{j+1} - n_j}{n_{j+1} - 1} g_j = \left(1 - \frac{n_j - 1}{n_{j+1} - 1}\right) g_j$$

$$\geq \frac{1}{2} g_j.$$

Consequently

$$\left\| \frac{1}{n_{j+1} - 1} \sum_{k=1}^{n_{j+1} - 1} f_k \right\|_\infty \geq \frac{1}{2} \|g_j\|_\infty = \frac{1}{2} \quad \text{for all} \quad j \geq 1$$

and so (3.2) does not hold.

\[\square\]

A similar counterexample can be given also in the Hilbert space $L^2([0,1])$, whose inner product and norm will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|_2$, respectively:

**Example 3.2.** Let $1 = n_1 < n_2 < \ldots$ be a sequence in $\mathbb{N}^*$ such that

$$\frac{n_j - 1}{n_{j+1} - 1} \leq \frac{1}{2}, \quad j \geq 1$$

(for example, $n_1 = 1$, $n_2 = 2$ and $n_{j+1} = 2n_j - 1$ for $j \geq 2$),
Example 3.3. Let us define the sequence \((f_k)_{k \geq 1}\) in \(L^2([0, 1])\) by setting for every \(k \in \mathbb{N}^+\) with \(k \equiv 1 \pmod{4}\)
\[
f_k(t) := t^k, \quad f_{k+1}(t) := t^{k + \frac{1}{4(k+2)}}, \quad f_{k+2}(t) := t^{k+1}, \quad f_{k+3}(t) := t^{k+1 + \frac{1}{2}}.
\]
Then \((f_k)_{k \geq 1}\) is convex shift-bounded, but there exists no bounded linear operator \(U : L^2([0, 1]) \to L^2([0, 1])\) such that

\[
1 > t_1 > t_2 > \ldots > 0, \quad t_j \to 0
\]
real numbers, and \(g_j : [0, 1] \to [0, +\infty), \quad j \geq 1,\) continuous functions such that \(\text{supp} (g_j) \subset [t_{j+1}, t_j]\) and \(\|g_j\|_2 = 1\) for all \(j \geq 1\).

If we set \(f_k = g_j\) for \(n_j \leq k < n_{j+1}\), then \((f_k)_{k \geq 1}\) is a bounded sequence in \(L^2([0, 1])\), which is weakly convergent to zero, and so is weakly mixing to zero, but which is not uniformly weakly mixing to zero.

**Proof.** Since the functions \(g_j\) are mutually orthogonal, by the Bessel inequality we have for every \(f \in L^2([0, 1])\):
\[
\sum_{j=1}^{\infty} |(g_j|f)|^2 \leq \|f\|_2^2 < +\infty, \quad \text{hence } (g_j|f) \to 0.
\]
Therefore \(f_k \xrightarrow{\text{weakly}} 0\).

On the other hand, for every \(j \geq 1\),
\[
\left\| \frac{1}{n_{j+1} - 1} \sum_{k=1}^{n_{j+1}-1} f_k \right\|_2^2 = \frac{1}{(n_{j+1} - 1)^2} \left\| \sum_{l=1}^j \sum_{k=n_l}^{n_{l+1}-1} f_k \right\|_2^2
\[
= \frac{1}{(n_{j+1} - 1)^2} \sum_{l=1}^j (n_{l+1} - n_l)^2
\[
\geq \left( \frac{n_{j+1} - n_j}{n_{j+1} - 1} \right)^2 = \left( 1 - \frac{n_j - 1}{n_{j+1} - 1} \right)^2 \geq \frac{1}{4}.
\]

Consequently, \(\left\| \frac{1}{n} \sum_{k=1}^n f_k \right\|_2 \to 0\), so \(\|f_k\|_2 \to 0\) does not hold for \((f_k)_{k \geq 1}\).

\[\square\]

In spite of the above examples, Theorem 1.1 entails that for orbits of power bounded linear operators weak mixing to zero and uniformly weak mixing to zero are equivalent. We are now looking for a larger class of vector sequences, for which weak mixing to zero and uniformly weak mixing to zero are still equivalent.

Let us call a sequence \((x_k)_{k \geq 1}\) in a Banach space \(X\) **convex shift-bounded** if there exists a constant \(c > 0\) such that

\[
(p \sum_{j=1}^p \lambda_j x_{j+k} \leq c \sum_{j=1}^p \lambda_j x_j), \quad k \geq 1
\]
holds for any choice of \(p \in \mathbb{N}^+\) and \(\lambda_1, \ldots, \lambda_p \geq 0\). Clearly:

- the convex shift-boundedness of a sequence implies its boundedness;
- if \(U : X \to X\) is a power bounded linear operator and \(x \in X\), then the sequence \((U^k(x))_{k \geq 1}\) is convex shift-bounded.

We notice that not every convex shift-bounded sequence, even in a Hilbert space, is the orbit of a bounded linear operator:

**Example 3.3.** Let us define the sequence \((f_k)_{k \geq 1}\) in \(L^2([0, 1])\) by setting for every \(k \in \mathbb{N}^+\) with \(k \equiv 1 \pmod{4}\)
\[
f_k(t) := t^k, \quad f_{k+1}(t) := t^{k + \frac{1}{4(k+2)}}, \quad f_{k+2}(t) := t^{k+1}, \quad f_{k+3}(t) := t^{k+1 + \frac{1}{2}}.
\]

Then \((f_k)_{k \geq 1}\) is convex shift-bounded, but there exists no bounded linear operator \(U : L^2([0, 1]) \to L^2([0, 1])\) such that
\[ f_k = U^k(f), \quad k \geq k_o \]

for some \( f \in L^2([0, 1]) \) and \( k_o \in \mathbb{N}^* \).

**Proof.** First of all, if \( 0 < \alpha_1 < \alpha_2 < \ldots \) are real numbers and \( g_k \in L^2([0, 1]) \) is defined by \( g_k(t) := t^{\alpha_k} \), then the sequence \( (g_k)_{k \geq 1} \) is convex shift-bounded. Indeed, for any \( p \in \mathbb{N}^* \) and \( \lambda_1, \ldots, \lambda_p \geq 0 \), the function

\[ N^* \ni k \mapsto \left\| \sum_{j=1}^p \lambda_j g_{j+k} \right\|^2_2 = \frac{1}{\alpha_{j+k} + \alpha_{j' + k} + 1} \]

is decreasing. In particular, the sequence \( (f_k)_{k \geq 1} \) is convex shift-bounded.

On the other hand, if \( \alpha, \varepsilon > 0 \) and we define \( h \in L^2([0, 1]) \) by \( h(t) := t^\alpha - t^{\alpha + \varepsilon} \), then

\[ \|h\|_2^2 = \int_0^1 (t^{2\alpha} + t^{2\alpha + 2\varepsilon} - 2 t^{2\alpha + \varepsilon}) \, dt = \frac{2\varepsilon^2}{(2\alpha + 1)(2\alpha + \varepsilon + 1)(2\alpha + 2\varepsilon + 1)}. \]

It is easy to verify that

\[ \|h\|_2^2 \leq \frac{1}{2(2\alpha + 1)} \left( \frac{\varepsilon}{\alpha + \varepsilon} \right)^2 \leq \frac{1}{2(2\alpha + 1)} \left( \frac{\varepsilon}{\alpha} \right)^2 \quad \text{if} \quad \varepsilon \leq 1, \]

\[ \|h\|_2^2 \geq \frac{1}{4(2\alpha + 1)} \left( \frac{\varepsilon}{\alpha + \varepsilon} \right)^2 \geq \frac{1}{4(2\alpha + 1)} \left( \frac{\varepsilon}{\alpha + 1} \right)^2 \quad \text{if} \quad \varepsilon \leq 1, \alpha \geq 2. \]

Now let \( k \in \mathbb{N}^* \) be arbitrary such that \( k \equiv 1 (\text{mod} \ 4) \). Then we have by (3.4)

\[ \|f_k - f_{k+1}\|_2 \leq \frac{1}{2(2k + 1)} \left( \frac{1}{4(k + 2)} \right)^2 \leq \frac{1}{32k^2(2k + 2)^2(2k + 1)}, \]

while (3.5) yields

\[ \|f_{k+2} - f_{k+3}\|_2 \geq \frac{1}{4(2k + 3)} \left( \frac{1}{2(k + 2)} \right)^2 = \frac{1}{16(k + 2)^2(2k + 3)}. \]

Consequently \( \|f_{k+2} - f_{k+3}\|_2^2 \geq k^2 \|f_k - f_{k+1}\|_2^2 \), and so

\[ \|f_{k+2} - f_{k+3}\|_2 \geq k \|f_k - f_{k+1}\|_2. \]

Let us assume that there is a bounded linear operator \( U : L^2([0, 1]) \rightarrow L^2([0, 1]) \) such that

\[ f_k = U^k(f), \quad k \geq k_o \]

for some \( f \in L^2([0, 1]) \) and \( k_o \in \mathbb{N}^* \). Then, for every \( k \geq k_o \) with \( k \equiv 1 (\text{mod} \ 4) \), (3.6) yields

\[ k \|f_k - f_{k+1}\|_2 \leq \|f_{k+2} - f_{k+3}\|_2 = \|U^2(f_k - f_{k+1})\|_2 \leq \|U\|^2 \|f_k - f_{k+1}\|_2, \]

hence \( \|U\| \geq \sqrt{\lambda} \). But this contradicts the boundedness of \( U \).

\[ \square \]

We shall prove (in this section in the realm of reflexive Banach spaces and in Section 5 in full generality) that weak mixing to zero is equivalent with uniformly weak mixing to zero for any convex shift-bounded sequence. First we prove an easy implication of weak mixing to zero:

**Lemma 3.4.** Let \( (x_k)_{k \geq 1} \) be a bounded sequence in a Banach space \( X \), which is weakly mixing to zero, and \( A \subset \mathbb{N}^* \) with \( D^*(A) > 0 \). Then the norm-closure of the convex hull \( \text{conv}\{\{x_k : k \in A\}\} \) of \( \{x_k : k \in A\} \) contains \( 0 \).

**Proof.** Let us assume that \( 0 \) is not in the norm-closure of \( \text{conv}\{\{x_k : k \in A\}\} \). Then the Hahn-Banach theorem yields the existence of some \( \varepsilon_o > 0 \) and \( x^* \in X^* \) such that

\[ \Re\{x^*, x_k\} \geq \varepsilon_o, \quad k \in A. \]
Further, by a classical result of B. O. Koopman and J. von Neumann (see e.g. [12], Chapter 2, (3.1) or [13], Lemma 9.3), there is a zero density set $E \subset \mathbb{N}^*$ such that

$$\lim_{\varepsilon \not\in k \to \infty} \langle x^*, x_k \rangle = 0.$$  

Then $A \setminus E$ is infinite, because otherwise we would get the contradiction

$$0 < D^*(A) \leq D^*(A \setminus E) + D^*(E) = 0.$$  

Let $k_1 < k_2 < \ldots$ be the elements of $A \setminus E$. Then (3.3) implies that $\langle x^*, x_k \rangle \to 0$, in contradiction with (3.4).

For weakly relatively compact sequences a stronger statement holds, which is essentially [4], Corollary 2:

**Lemma 3.5.** A weakly relatively compact sequence $(x_k)_{k \geq 1}$ in a Banach space $X$ is weakly mixing to zero if and only if there exists a zero density set $E \subset \mathbb{N}^*$ such that

$$\lim_{\varepsilon \not\in k \to \infty} x_k = 0 \text{ with respect to the weak topology of } X.$$  

**Proof.** An inspection of the proof of [9], Corollary 2 shows that it works for any weakly relatively compact sequence in a Banach space, not only for those, which are orbits of power bounded linear operators. □

We notice that, if $(x_k)_{k \geq 1}$ is a weakly relatively compact sequence in a Banach space $X$, which is weakly mixing to zero, and $E \subset \mathbb{N}^*$ is as in Lemma 3.5 then, according to the classical Mazur theorem about the equality of the weak and norm closure of a convex subset of $X$, the norm-closure of the convex hull of every infinite subset of $\mathbb{N}^* \setminus E$ contains $0$. In particular, for any $A \subset \mathbb{N}^*$ with $D^*(A) > 0$, the norm-closure of the convex hull of the infinite set $A \setminus E$ contains $0$.

Now we prove a consequence of the negation of uniformly weak mixing to zero (cf. the first part of the proof of [5], Theorem IV):

**Lemma 3.6.** Let $(x_k)_{k \geq 1}$ be a sequence in the closed unit ball of a Banach space $X$, which is not uniformly weakly mixing to zero. Then there exist

$$0 < \varepsilon_0 \leq 1,$$

$B \subset \mathbb{N}^*$ with $D^*(B) \geq \varepsilon_0$,

$$k_1, k_2, \ldots \in \mathbb{N}^* \text{ with } k_j - k_{j-1} > j,$$

$$x_1^*, x_2^*, \ldots \in X^* \text{ with } \|x_j^*\| \leq 1,$$

such that

$$B \cap \bigcup_{j \geq 2} (k_{j-1}, k_{j-1} + j) = \emptyset,$$

$$\Re \langle x_j^*, x_k \rangle \geq 2 \varepsilon_0, \quad k \in B \cap (k_{j-1} + j, k_j], \ j \geq 2.$$  

**Proof.** For any complex number $z$ we shall use the notations

$$\Re^+ z := \begin{cases} \Re z & \text{if } \Re z \geq 0, \\ 0 & \text{if } \Re z \leq 0 \end{cases}, \quad \Re^- z := \begin{cases} 0 & \text{if } \Re z \geq 0, \\ -\Re z & \text{if } \Re z \leq 0 \end{cases}.$$  

Then $\Re z = \Re^+ z - \Re^- z = \Re^+ z - \Re^+ (-z)$. Since $(x_k)_{k \geq 1}$ is not uniformly weakly mixing to zero, there is $0 < \varepsilon_0 \leq 1$ such that

$$J := \left\{ n \geq 1 : \sup \left\{ \frac{1}{n} \sum_{k=1}^n |\langle x^*, x_k \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\} > 16 \varepsilon_0 \right\}$$  

is infinite. Using (in the complex case) $\langle x^*, x_k \rangle = \Re \langle x^*, x_k \rangle - i \Re \langle i x^*, x_k \rangle$, it follows that also
\[ \mathcal{J}_\mathbb{R} := \left\{ n \geq 1; \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} \left| \Re \langle x^*, x_k \rangle \right|; x^* \in X^*, \|x^*\| \leq 1 \right\} > 8 \varepsilon_o \right\} \]

is infinite. Now, since \( \Re \langle x^*, x_k \rangle = \Re^+ \langle x^*, x_k \rangle - \Re^+ \langle -x^*, x_k \rangle \), we obtain that

\[ \mathcal{J}_+ := \left\{ n \geq 1; \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} \Re^+ \langle x^*, x_k \rangle; x^* \in X^*, \|x^*\| \leq 1 \right\} > 4 \varepsilon_o \right\} \]

is infinite.

Let \( n \in \mathcal{J}_+ \) be arbitrary. Then there exists \( y^*_n \in X^* \) with \( \|y^*_n\| \leq 1 \) such that

\[ \frac{1}{n} \sum_{k=1}^{n} \Re^+ \langle y^*_n, x_k \rangle > 4 \varepsilon_o. \]

Denoting \( B_n := \{ 1 \leq k \leq n; \Re^+ \langle y^*_n, x_k \rangle > 2 \varepsilon_o \} \), we have

\[ 4 \varepsilon_o < \frac{1}{n} \left( \sum_{k \in B_n} \Re^+ \langle y^*_n, x_k \rangle + \sum_{1 \leq k \leq n; k \notin B_n} \Re^+ \langle y^*_n, x_k \rangle \right) \leq \frac{1}{n} \text{card} (B_n) + 2 \varepsilon_o, \]

hence \( \text{card} (B_n) \geq 2 n \varepsilon_o \).

Denoting now by \( k_j \) the least element of \( \mathcal{J}_+ \), we can construct recursively a sequence \( k_1, k_2, \ldots \in \mathcal{J}_+ \) such that, for every \( j \geq 2 \),

\[ k_j - k_{j-1} > j \quad \text{and} \quad \text{the cardinality of } B_{k_j} := \{ k \in B_{k_j}; k > k_{j-1} + j \} \geq k_j \varepsilon_o. \]

Indeed, if we choose \( k_j \) in the infinite set \( \mathcal{J}_+ \) such that \( k_j > \frac{k_{j-1} + j}{\varepsilon_o} \), then

\[ \text{card} (B_{k_j}) \geq \text{card} (B_{k_j}) - (k_{j-1} + j) \geq 2 k_j \varepsilon_o - (k_{j-1} + j) > k_j \varepsilon_o. \]

Putting

\[ \mathcal{B} := \bigcup_{j \geq 2} B_{k_j}, \]

we have for every \( j \geq 2 \)

\[ \mathcal{B} \cap (k_{j-1}, k_{j-1} + j] = \emptyset, \quad \mathcal{B} \cap (k_{j-1} + j, k_j] = B_{k_j}' \subset B_{k_j}, \]

and so

\[ \Re \langle y^*_{k_j}, x_k \rangle = \Re^+ \langle y^*_{k_j}, x_k \rangle > 2 \varepsilon_o \quad \text{for all } k \in \mathcal{B} \cap (k_{j-1} + j, k_j]. \]

On the other hand,

\[ D^*(\mathcal{B}) = \lim_{n \to \infty} \frac{1}{n} \text{card} (\mathcal{B} \cap [1, n]) \geq \lim_{j \to \infty} \frac{1}{k_j} \text{card} \left( \underbrace{\mathcal{B} \cap (k_{j-1} + j, k_j]}_{= B_{k_j}'} \right) \geq \varepsilon_o. \]

Therefore, setting \( x^*_j := y^*_{k_j} \), the proof is complete.

\[ \square \]

We recall the following lemma of L. K. Jones on sequences of integers (see [3], Lemma 3 or [4], Lemma):

**Lemma 3.7.** Let \( \mathcal{A}_o, \mathcal{B} \) be subsets of \( \mathbb{N}^* \) with \( D^* (\mathcal{A}_o) = 1 \) and \( D^* (\mathcal{B}) > 0 \). Then there exists an infinite subset \( \mathcal{I} \subset \mathcal{A}_o \) such that

\[ D^* (\{ k \in \mathcal{B}; \mathcal{I} + k \subset \mathcal{B} \}) > 0 \quad \text{for any finite } \mathcal{F} \subset \mathcal{I}. \]

\[ \square \]

Now, using the idea of the proof of [3], Theorem IV, we can prove that weak mixing to zero and uniformly weak mixing to zero are equivalent for any convex shift-bounded sequence in a reflexive Banach space.
Proposition 3.8. For a convex shift-bounded sequence in a reflexive Banach space, weak mixing to zero is equivalent to uniformly weak mixing to zero.

Proof. Let \( (x_k)_{k \geq 1} \) be a convex shift-bounded sequence in the closed unit ball of a reflexive Banach space \( X \), which is weakly mixing to zero, and let us assume that it is not uniformly weakly mixing to zero. Let \( c > 0 \) be such that (3.3) holds for any choice of \( p \in \mathbb{N}^* \) and \( \lambda_1, \ldots, \lambda_p \geq 0 \).

By Lemma 3.5 there exist

\[
0 < \varepsilon_0 \leq 1, \quad \mathcal{B} \subset \mathbb{N}^* \text{ with } D^*(\mathcal{B}) \geq \varepsilon_0, \quad k_1, k_2, \ldots \in \mathbb{N}^* \text{ with } k_j - k_{j-1} > j, \quad x_1^*, x_2^*, \ldots \in X^* \text{ with } \|x_j^*\| \leq 1,
\]

such that

\[
\mathcal{B} \cap \bigcup_{j \geq 2} (k_{j-1}, k_{j-1} + j] = \emptyset, \quad \Re(x_j^*, x_k) > 2\varepsilon_o, \quad k \in \mathcal{B} \cap (k_{j-1} + j, k_j], j \geq 2.
\]

On the other hand, since any bounded set in a reflexive Banach space is weakly relatively compact, by Lemma 3.6 there exists \( \mathcal{A}_0 \subset \mathbb{N}^* \) with \( D_o(\mathcal{A}_0) = 1 \) such that \( \lim_{n \to \infty} x_n = 0 \) with respect to the weak topology of \( X \).

Finally, by Lemma 3.7 there exists an infinite subset \( I \subset \mathcal{A}_0 \) such that

\[
D^*(\{ k \in \mathcal{B} : \mathcal{F} + k \subset \mathcal{B} \}) > 0 \text{ for any finite } \mathcal{F} \subset I.
\]

Since \( \lim_{n \to \infty} x_n = 0 \) with respect to the weak topology of \( X \), there are \( p \in \mathbb{N}^* \), \( n_1 < \ldots < n_p \) in \( I \) and \( \lambda_1, \ldots, \lambda_p \geq 0 \), \( \lambda_1 + \ldots + \lambda_p = 1 \), such that

\[
\left\| \sum_{j=1}^p \lambda_j x_{n_j} \right\| \leq \frac{\varepsilon_0}{c}, \quad k \geq 1.
\]

By (3.3) it follows that

\[
\left\| \sum_{j=1}^p \lambda_j x_{n_j+k} \right\| \leq c \left\| \sum_{j=1}^p \lambda_j x_{n_j} \right\| \leq \varepsilon_0, \quad k \geq 1.
\] (3.9)

Now set \( j_o := \max(n_p - n_1, 2) \). Since the set \( \{ k \in \mathcal{B} : \{ n_1, \ldots, n_p \} + k \subset \mathcal{B} \} \) has strictly positive upper density and so is infinite, it contains some \( k \) such that \( n_1 + k \geq j_o \). Then there is a unique \( j_1 \in \mathbb{N}^* \) with \( k_{j_1-1} < n_1 + k \leq k_{j_1} \), for which we have \( k_{j_o} \leq n_1 + k \leq k_{j_1} \), hence \( j_o \leq j_1 \).

We claim that

\[
k_{j_1-1} + j_1 < n_1 + k \leq n_p + k \leq k_{j_1}.
\] (3.10)

Indeed, \( k_{j_1-1} < n_1 + k \) and \( n_1 + k \in \mathcal{B} \) and \( \mathcal{B} \cap (k_{j_1-1}, k_{j_1-1} + j_1) = \emptyset \) imply that \( k_{j_1-1} + j_1 < n_1 + k \).

Similarly, \( n_p + k = n_1 + k + (n_p - n_1) \leq k_{j_1} + j_1 \) and \( n_p + k \in \mathcal{B} \) and \( \mathcal{B} \cap (k_{j_1}, k_{j_1} + j_1 + 1) = \emptyset \) yield \( n_p + k \leq k_{j_1} \).

By (3.10) we have \( n_j + k \in \mathcal{B} \cap (k_{j_1-1} + j, k_{j_1}], 1 \leq j \leq p \), so

\[
\Re(x_j^*, x_{n_j+k}) > 2\varepsilon_o, \quad 1 \leq j \leq p.
\]

Since \( \|x_j^*\| \leq 1 \), it follows that

\[
\left\| \sum_{j=1}^p \lambda_j x_{n_j+k} \right\| \geq \Re \left( \sum_{j=1}^p \lambda_j x_{n_j+k} \right) = \sum_{j=1}^p \lambda_j \Re(x_j^*, x_k) > 2\varepsilon_o
\]

in contradiction with (3.3). \( \square \)
If in Lemma 3.7 the set \( I \) would be not only infinite, but with \( D^*(I) > 0 \), then in the proof of Proposition 3.8 we could use Lemma 3.3 instead of Lemma 3.6 and so we would get a proof of Proposition 3.8 without the reflexivity assumption. In the next section we shall prove a result like Lemma 3.7 (Theorem 3.2), which implies that for every \( \mathcal{B} \subset \mathbb{N}^* \) with \( D^*(\mathcal{B}) > 0 \) there is a set \( \mathcal{A} \subset \mathbb{N}^* \) with \( D^*(\mathcal{A}) > 0 \) such that any finite subset of \( \mathcal{A} \) has infinitely many translates contained in \( \mathcal{B} \). This result will enable us to eliminate the reflexivity condition in Proposition 3.8.

4. Sets of Non-Zero Upper Banach Density

We recall that the upper Banach density \( BD^*(\mathcal{B}) \) of some \( \mathcal{B} \subset \mathbb{N} = \mathbb{N}^* \cup \{0\} \) is defined by

\[
BD^*(\mathcal{B}) := \lim_{a,b \to \infty} \frac{1}{b-a+1} \text{card} (\mathcal{B} \cap [a,b]) = \lim_{a,b \to \infty} \frac{1}{b-a+1} \text{card} (\mathcal{B} \cap [a,b])
\]

(see e.g. [7], Chapter 3, §5). For any \( \mathcal{B} \subset \mathbb{N}^* \) we have \( BD^*(\mathcal{B}) \geq D^*(\mathcal{B}) \), but it is easily seen that \( BD^*(\mathcal{B}) > D^*(\mathcal{B}) \) can happen. In this section we investigate the structure of the sets \( \mathcal{B} \subset \mathbb{N}^* \) with \( BD^*(\mathcal{B}) > 0 \) by proving a precise version of the theorem of R. Ellis [7], Theorem 3.20. The proof is based on the ergodic theoretical methods of H. Furstenberg exposed in [4], Chapter 3, §5.

Let us consider \( \Omega := \{0,1\}^\mathbb{N} \) and endow it with the metrizable compact product topology of the discrete topologies on \( \{0,1\} \). We shall denote the components of \( \omega \in \Omega \) by \( \omega_k \), so that \( \omega = (\omega_k)_{k \in \mathbb{N}} \).

For every \( \mathcal{B} \subset \mathbb{N} \) we define \( \omega(\mathcal{B}) \in \Omega \) by setting

\[
\omega(\mathcal{B})_k := \begin{cases} 1 & \text{if } k \in \mathcal{B} \\ 0 & \text{if } k \not\in \mathcal{B} \end{cases}
\]

In other words, \( \omega(\mathcal{B}) \) is the characteristic function of \( \mathcal{B} \), considered an element of \( \Omega \). Clearly, \( \mathcal{B} \mapsto \omega(\mathcal{B}) \) is a bijection of the set of all subsets of \( \mathbb{N} \) onto \( \Omega \).

Let \( s_- \) denote the backward shift on \( \Omega \), defined by

\[
s_-((\omega_k)_{k \in \mathbb{N}}) = (\omega_{k+1})_{k \in \mathbb{N}},
\]

and set, for every \( \mathcal{B} \subset \mathbb{N} \),

\[
\Omega(\mathcal{B}) := \{ s_n(\omega(\mathcal{B})) : n \geq 0 \}.
\]

Clearly, \( s_-(\Omega(\mathcal{B})) \subset \Omega(\mathcal{B}) \).

The following result is the one-sided version of [7], Lemma 3.17 and it establishes a link between upper Banach density and the ergodic theory of the dynamical system \((\Omega, s_-)\). Its proof is almost identical to the proof of [7], Lemma 3.17 and we sketch it only for the sake of completeness:

**Lemma 4.1.** For every \( \mathcal{B} \subset \mathbb{N} \) and every \( \varepsilon > 0 \) there exists an ergodic \( s_- \)-invariant probability Borel measure \( \mu \) on \( \Omega(\mathcal{B}) \) such that

\[
\mu(\{ \omega \in \Omega(\mathcal{B}) : \omega_0 = 1 \}) > BD^*(\mathcal{B}) - \varepsilon.
\]

**Proof.** Choose some \( a_1, b_1, a_2, b_2, \ldots \in \mathbb{N} \) with \( b_j - a_j \geq j, j \geq 0 \), such that

\[
\frac{1}{b_j - a_j + 1} \text{card} (\mathcal{B} \cap [a_j, b_j]) \to BD^*(\mathcal{B}).
\]

Passing to a subsequence if necessary, we can assume that, for any continuous function \( f \in C(\Omega(\mathcal{B})) \),

\[
\lim_{j \to \infty} \frac{1}{b_j - a_j + 1} \sum_{n=a_j}^{b_j} f(s_n(\omega(\mathcal{B}))) \text{ exists.}
\]

Then \( I \) is a positive linear functional on \( C(\Omega(\mathcal{B})) \) and \( I(1) = 1 \). Moreover,

\[
I(f \circ s_-) = I(f), \quad f \in C(\Omega(\mathcal{B})).
\]

Indeed, for every \( f \in C(\Omega(\mathcal{B})) \),

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By the Riesz representation theorem there exists a probability Borel measure \( \nu_f \) on \( \Omega^B \) such that

\[
I(f) = \int_{\Omega^B} f(\omega) \, d\nu_f(\omega), \quad f \in C(\Omega^B).
\]

Property \((\text{4.1})\) of \( \Omega \) implies that \( \nu_f \) is \( s_- \)-invariant. Moreover, since the characteristic function \( \chi \) of \( \{ \omega \in \Omega^B : \omega_0 = 1 \} \) is continuous, we have

\[
\nu_f(\{ \omega \in \Omega^B : \omega_0 = 1 \}) = \int_{\Omega^B} \chi(\omega) \, d\nu_f(\omega) = \lim_{j \to \infty} \frac{1}{b_j - a_j + 1} \sum_{n=a_j}^{b_j} \chi(s_n(\omega^B))
\]

\[
= \lim_{j \to \infty} \frac{1}{b_j - a_j + 1} \text{card}(B \cap [a_j, b_j]) = BD^*(B).
\]

The convex set \( \mathcal{P}^{s_-}(\Omega^B) \) of all \( s_- \)-invariant probability Borel measures on \( \Omega^B \), considered imbedded in the dual space of \( C(\Omega^B) \), is weak*-compact and its extreme points are the ergodic measures in \( \mathcal{P}^{s_-}(\Omega^B) \) (see, for example, [7], Proposition 3.4). According to the Krein-Milman theorem, it follows that \( \nu_f \) is a weak*-limit of convex combinations of ergodic measures in \( \mathcal{P}^{s_-}(\Omega^B) \). Therefore, since \( \nu_f(\{ \omega \in \Omega^B : \omega_0 = 1 \}) = BD^*(B) \), we conclude that there exists an ergodic measure \( \mu \in \mathcal{P}^{s_-}(\Omega^B) \) such that \( \mu(\{ \omega \in \Omega^B : \omega_0 = 1 \}) > BD^*(B) - \varepsilon \).

Now we prove the announced extension of [7], Theorem 3.20:

**Theorem 4.2.** If \( B \subset \mathbb{N} \) and \( 0 < \varepsilon < BD^*(B) \), then there exist

\[ A \subset \mathbb{N} \text{ having density } D(A) > BD^*(B) - \varepsilon, \]

\[ 0 \leq m_1 < m_2 < \ldots \text{ and } 0 \leq n_1 < n_2 < \ldots \text{ in } \mathbb{N}, \]

for which

\[ A \cap [0, m_j] = \{ k \in [0, m_j] : k + n_j \in B \}, \quad j \geq 1, \]

that is

\[ k \in A \iff k + n_j \in B \text{ whenever } 0 \leq k \leq m_j, \quad j \geq 1. \]

**Proof.** For every \( \omega \in \Omega \) we set \( A_\omega = \{ k \in \mathbb{N} : \omega_k = 1 \} \), so that \( \omega = \omega(A_\omega) \). Clearly, \( A_{\omega_0} = B \).

By Lemma [13] there exists an ergodic \( s_- \)-invariant probability Borel measure \( \mu \) on \( \Omega^B \) such that

\[ \mu_B := \mu(\{ \omega \in \Omega^B : \omega_0 = 1 \}) > BD^*(B) - \varepsilon. \]

Let \( \chi \) denote the characteristic function of \( \{ \omega \in \Omega^B : \omega_0 = 1 \} \subset \Omega^B \). Then, by the Birkhoff ergodic theorem, for \( \mu \)-almost every \( \omega \in \Omega^B \) we have

\[
\frac{1}{n+1} \text{card}(A_\omega \cap [0, n]) = \frac{1}{n+1} \sum_{k=0}^{n} \chi(s_k(\omega)) \to \mu_B.
\]

Let \( \Omega^B_{\text{Birkhoff}} \) be the set of all \( \omega \in \Omega^B \), for which \((\text{13})\) holds. Then \( A_\omega \) has density \( D(A_\omega) = \mu_B > BD^*(B) - \varepsilon \) for every \( \omega \in \Omega^B_{\text{Birkhoff}} \), \( \Omega^B_{\text{Birkhoff}} \) is \( \mu \)-measurable and \( \mu(\Omega^B \setminus \Omega^B_{\text{Birkhoff}}) = 0 \).
Case 1: there exists \( \omega \in \Omega^{(B)}_{\text{Birkhoff}} \setminus \{s^n(\omega(B)) ; n \geq 0\} \).

Set \( A := \mathcal{A}_\omega \) and choose some \( m_1 \geq 0 \). Since

\[
(4.3) \quad \omega \in \Omega^{(B)} = \left\{ s^n(\omega(B)) ; n \geq 0 \right\},
\]

there exists a smallest \( n_1 \geq 0 \) such that

\[
\omega_k = s^{n_1}(\omega(B)) = \omega_k^{(B)} = \begin{cases} 1 & \text{if } k + n_1 \in B \\ 0 & \text{if } k + n_1 \notin B \end{cases}, \quad 0 \leq k \leq m_1,
\]

that is \( A \cap [0, m_1] = \{ k \in [0, m_1] ; k + n_1 \notin B \} \).

Next \( \omega \neq s^{n_1}(\omega(B)) \) implies that \( \omega_{m_2} \neq s^{n_1}(\omega(B))_{m_2} \) for some \( m_2 \in \mathbb{N} \). Since \( \omega_k = s^{n_1}(\omega(B))_k \) for all \( 0 \leq k \leq m_1 \), we have \( m_1 < m_2 \). Now, again by (4.3), there exists a smallest \( n_2 \geq 0 \) such that

\[
\omega_k = s^{n_2}(\omega(B)) = \omega_k^{(B)} = \begin{cases} 1 & \text{if } k + n_2 \in B \\ 0 & \text{if } k + n_2 \notin B \end{cases}, \quad 0 \leq k \leq m_2,
\]

that is \( A \cap [0, m_2] = \{ k \in [0, m_2] ; k + n_2 \notin B \} \). By the minimality property of \( n_1 \) we have \( n_1 \leq n_2 \), while \( \omega_{m_2} \neq s^{n_1}(\omega(B))_{m_2} \) yields \( n_1 \neq n_2 \). Therefore \( n_1 < n_2 \).

By induction we obtain \( m_1 < m_2 < \ldots \) and \( n_1 < n_2 < \ldots \) in \( \mathbb{N} \) such that

\[
A \cap [0, m_j] = \{ k \in [0, m_j] ; k + n_j \in B \} \quad \text{for all } j \geq 1.
\]

Case 2: \( \Omega^{(B)}_{\text{Birkhoff}} \subset \{s^n(\omega(B)) ; n \geq 0\} \).

We claim that there exists a smallest \( n_o \in \mathbb{N}^* \) such that \( s^{n_o}(\omega(B)) = \omega(B) \).

For let us assume that all \( s^n(\omega(B)) \) are different. Then, for every \( n \geq 0 \), since

\[
\{s^n(\omega(B))\} \subset s^{n-1}(s^{n+1}(\omega(B))) \subset \{s^n(\omega(B))\} \cup \left( \Omega^{(B)} \setminus \Omega_{\text{Birkhoff}}^{(B)} \right),
\]

by the \( s^- \)-invariance of \( \mu \) we obtain \( \mu\left(\{s^n(\omega(B))\}\right) = \mu\left(\{s^{n+1}(\omega(B))\}\right) \). Thus

\[
\mu(\Omega^{(B)}) = \sum_{n=0}^{\infty} \mu(\{s^n(\omega(B))\}) = \begin{cases} 0 & \text{if } \mu(\{\omega(B)\}) = 0 \\ +\infty & \text{if } \mu(\{\omega(B)\}) > 0 \end{cases},
\]

in contradiction with \( \mu(\Omega^{(B)}) = 1 \).

Now \( s^{n_o}(\omega(B)) = \omega(B) \) means that \( k \in \mathbb{N} \) belongs to \( B \) if and only if \( k + n_o \in B \). Therefore, with \( A := B \), any \( 0 \leq m_1 < m_2 < \ldots \) and \( n_j := j n_o \), we have

\[
D(A) = \frac{1}{n_o} \text{ card } (B \cap [0, n_o - 1]) = BD^*(B),
\]

\[
A \cap [0, m_j] = \{ k \in [0, m_j] ; k + n_j \in B \}, \quad j \geq 1.
\]

\[ \square \]

We recall that a celebrated theorem of E. Szemerédi (answering a conjecture of P. Erdős) states that if \( B \subset \mathbb{N}^* \) has non-zero upper Banach density, then it contains arbitrarily long arithmetic progressions \[14\]. H. Furstenberg gave a new ergodic theoretical proof of Szemerédi’s theorem by deducing it from a far-reaching multiple recurrence theorem \[8\] (see also \[7\], Chapter 3, §7). It is interesting to notice, even if it looks not to be relevant, that via Theorem 4.2 the proof of Szemerédi’s theorem can be reduced to the case when \( B \) has non-zero density.

The above theorem implies the following counterpart of Lemma 2.6:

**Corollary 4.3.** Let \( \mathcal{A}_o, \mathcal{B} \) be subsets of \( \mathbb{N}^* \) with \( D^*(\mathcal{A}_o) = 1 \) and \( 0 < \varepsilon < BD^*(\mathcal{B}) \). Then there exists \( \mathcal{I} \subset \mathcal{A}_o \) with \( D^*(\mathcal{I}) > BD^*(\mathcal{B}) - \varepsilon \), such that

\[
\{ k \in \mathbb{N} ; F + k \subset B \} \text{ is infinite for any finite } F \subset \mathcal{I}.
\]
Lemma 5.1. Let \( \mathcal{A} \subset \mathbb{N} \) having density \( D(\mathcal{A}) > BD^*(\mathcal{B}) - \varepsilon \), as well as \( 0 \leq m_1 < m_2 < \ldots \) and \( 0 \leq n_1 < n_2 < \ldots \) in \( \mathbb{N} \), such that
\[
\mathcal{A} \cap [0, m_j] = \{ k \in [0, m_j] ; k + n_j \in \mathcal{B} \}, \quad j \geq 1.
\]
Set \( \mathcal{I} := \mathcal{A} \cap \mathcal{A}_o \). Since
\[
1 = D^*(\mathcal{A}_o) \leq D^*(\mathcal{A} \cap \mathcal{A}_o) + D^*(\mathbb{N} \setminus \mathcal{A}) = D^*(\mathcal{I}) + 1 - D(\mathcal{A}),
\]
we have \( D^*(\mathcal{I}) \geq D(\mathcal{A}) > BD^*(\mathcal{B}) - \varepsilon \). On the other hand, for any \( j \geq 1 \), the set
\[
\{ k \in \mathbb{N} ; (\mathcal{I} \cap [0, m_j]) + k \in \mathcal{B} \}
\]
contains \( \{ n_j, n_j+1, \ldots \} \), hence is infinite. \( \square \)

5. Weak mixing to zero for convex shift-bounded sequences

Using Theorem 4.2 in this section we show that Proposition 3.8 holds without the reflexivity assumption. Actually we shall prove a slightly more general result, stating that any convex shift-bounded sequence in a Banach space, which is weakly mixing to zero, satisfies 1.3. For the proof we shall use the following counterpart of Lemma 3.6 for the sequences not satisfying 1.3:

**Lemma 5.1.** Let \( (x_k)_{k \geq 1} \) be a sequence in the closed unit ball of a Banach space \( X \), such that, for some \( a_1, b_1, a_2, b_2, \ldots \in \mathbb{N}^* \) with \( b_j - a_j \geq j, j \geq 1 \), we have
\[
\lim_{j \to \infty} \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} |\langle x^*, x_k \rangle| ; x^* \in X^*, \|x^*\| \leq 1 \right\} > 0.
\]
Then there exist
\[
0 < \varepsilon \leq 1,
\]
\( \mathcal{B} \subset \mathbb{N}^* \) with \( BD^*(\mathcal{B}) \geq \varepsilon \),
\[
j_1 < j_2 < \ldots \text{ in } \mathbb{N}^* \text{ with } b_{j_n} - b_{j_{n-1}} > n,
\]
\( x^*_1, x^*_2, \ldots \in X^* \) with \( \|x^*_n\| \leq 1 \), such that
\[
\mathcal{B} \cap \bigcup_{n \geq 2} \{ b_{j_{n-1}}, b_{j_{n-1}} + n \} = \emptyset,
\]
\( \Re \langle x^*_n, x_k \rangle > 2\varepsilon \), \( k \in \mathcal{B} \cap \{ b_{j_{n-1}} + n, b_{j_n} \} \), \( n \geq 2 \).

**Proof.** We shall proceed similarly as in the proof of Lemma 3.6.
Let \( 0 < \varepsilon \leq 1 \) be such that
\[
0 < 16 \varepsilon \leq \lim_{j \to \infty} \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} |\langle x^*, x_k \rangle| ; x^* \in X^*, \|x^*\| \leq 1 \right\}.
\]
Then
\[
\mathcal{J} := \left\{ j \geq 1 ; \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} |\langle x^*, x_k \rangle| ; x^* \in X^*, \|x^*\| \leq 1 \right\} > 16 \varepsilon \right\}
\]
is infinite. Using (in the complex case) \( \langle x^*, x_k \rangle = \Re \langle x^*, x_k \rangle - i \Im \langle x^*, x_k \rangle \), it follows that also
\[
\mathcal{J}_{\Re} := \left\{ j \geq 1 ; \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} |\Re \langle x^*, x_k \rangle| ; x^* \in X^*, \|x^*\| \leq 1 \right\} > 8 \varepsilon \right\}
\]
is infinite. Now, since \( \Re \langle x^*, x_k \rangle = \Re^+ \langle x^*, x_k \rangle - \Re^+ \langle -x^*, x_k \rangle \), we obtain that
\[
\mathcal{J}^+ := \left\{ j \geq 1 ; \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} \Re^+ \langle x^*, x_k \rangle ; x^* \in X^*, \|x^*\| \leq 1 \right\} > 4 \varepsilon \right\}
\]
is infinite.

Let \( j \in J_+ \) be arbitrary. Then there exists \( y_j^* \in X^* \) with \( \|y_j^*\| \leq 1 \) such that

\[
\frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} \mathbb{R}^+(y_j^*, x_k) > 4 \varepsilon_o.
\]

Denoting \( B_j := \{ a_j \leq k \leq b_j ; \mathbb{R}^+(y_j^*, x_k) > 2 \varepsilon_o \} \), we have

\[
4 \varepsilon_o < \frac{1}{b_j - a_j + 1} \left( \sum_{k \in B_j} \mathbb{R}^+(y_j^*, x_k) + \sum_{a_j \leq k \leq b_j \setminus B_j} \mathbb{R}^+(y_j^*, x_k) \right)
\]

\[
\leq \frac{1}{b_j - a_j + 1} \operatorname{card}(B_j) + 2 \varepsilon_o,
\]

hence \( \operatorname{card}(B_j) \geq 2 (b_j - a_j + 1) \varepsilon_o \).

Denoting now by \( j_1 \) the least element of \( J_+ \), we can construct recursively a sequence \( j_1 < j_2 < \ldots \) in \( J_+ \) such that, for every \( n \geq 2 \),

\[ b_{j_n} - b_{j_n-1} > n \]

the cardinality of \( B_{j_n}' := \{ k \in B_{j_n} ; k > b_{j_n-1} + n \} \) is \( > (b_{j_n} - a_{j_n}) \varepsilon_o \).

Indeed, if we choose \( j_n \) in the infinite set \( J_+ \) such that \( j_n > j_{n-1} \) and

\[ b_{j_n} - a_{j_n} + 1 \geq j_n + 1 > \frac{b_{j_{n-1}} + n}{\varepsilon_o} \geq b_{j_{n-1}} + n, \]

then \( b_{j_n} - b_{j_{n-1}} > n \) and

\[
\operatorname{card}(B_{j_n}') \geq \operatorname{card}(B_{j_n}) - (b_{j_n-1} + n) \geq 2 (b_{j_n} - a_{j_n} + 1) \varepsilon_o - (b_{j_n-1} + n)
\]

\[
> (b_{j_n} - a_{j_n} + 1) \varepsilon_o.
\]

Putting

\[ B := \bigcup_{n \geq 2} B_{j_n}, \]

we have for every \( n \geq 2 \)

\[ B \cap (b_{j_n-1}, b_{j_n-1} + n] = \emptyset, \quad B \cap (b_{j_n-1} + n, b_{j_n}] = B_{k_n}' \subset B_{k_n}, \]

and so

\[ \mathbb{R}(y_{j_n}^*, x_k) = \mathbb{R}^+(y_{j_n}^*, x_k) > 2 \varepsilon_o \] for all \( k \in B \cap (b_{j_n-1} + n, b_{j_n}] \).

On the other hand,

\[
BD^*(B) \geq \lim_{n \to \infty} \frac{1}{b_{j_n} - a_{j_n} + 1} \operatorname{card}(B \cap [a_{j_n}, b_{j_n}])
\]

\[
\geq \lim_{n \to \infty} \frac{1}{b_{j_n} - a_{j_n} + 1} \operatorname{card}(B_{j_n}' \cap [a_{j_n}, b_{j_n}]) \geq \varepsilon_o.
\]

Therefore, setting \( x_n^* := y_{j_n}^* \), the proof is complete.

\[ \square \]

For the proof of the next theorem we adapt the proof of Proposition \ref{prop:exp} in which instead of Lemmas \ref{prop:exp}, \ref{prop:exp-2} and \ref{prop:exp-3} we use Lemma \ref{lem:exp} Theorem \ref{thm:exp} and Lemma \ref{lem:exp}:

**Theorem 5.2** (Weak mixing for convex shift-bounded sequences). For a convex shift-bounded sequence \((x_k)_{k \geq 1}\) in a Banach space \(X\), the following conditions are equivalent:

(i) \((x_k)_{k \geq 1}\) is weakly mixing to zero, that is

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\langle x^*, x_k \rangle| = 0 \] for all \( x^* \in X^* \).
(j) \((x_k)_{k \geq 1}\) is uniformly weakly mixing to zero, that is

\[
\lim_{n \to \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} |\langle x^*, x_k \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\} = 0.
\]

(jw) holds, that is

\[
\lim_{b \to a \to \infty} \sup_{b-a \in \mathbb{N}^*} \left\{ \frac{1}{b-a+1} \sum_{k=a}^{b} |\langle x^*, x_k \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\} = 0.
\]

Proof. The implications (jw) \(\Rightarrow\) (j) \(\Rightarrow\) (i) are trivial. For (i) \(\Rightarrow\) (jw) we shall show that (i) and the negation of (jw) lead to a contradiction.

Let \(c > 0\) be a constant such that (5.3) holds for any choice of \(p \in \mathbb{N}^*\) and \(\lambda_1, \ldots, \lambda_p \geq 0\). Since (5.3) does not hold, there exist \(a_1, b_1, a_2, b_2, \ldots \in \mathbb{N}^*\) with \(b_j - a_j \geq j, j \geq 1\), such that

\[
\lim_{j \to \infty} \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} |\langle x^*, x_k \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\} > 0.
\]

By Lemma 5.1 there exist

\[
0 < \varepsilon_0 \leq 1, \quad B \subset \mathbb{N}^* \quad \text{with} \quad BD^*(B) \geq \varepsilon_0,
\]

\[
\text{for which} \quad B \cap \bigcup_{n \geq 2} \left( (b_{j_{n-1}}, b_{j_n} + n) \right) = \emptyset,
\]

\[
\Re \langle x_n^*, x_k \rangle > 2 \varepsilon_0, \quad k \in B \cap \left( b_{j_{n-1}} + n, b_{j_n} \right), \quad n \geq 2.
\]

Further, by Theorem 1.2 there exist

\[
\mathcal{A} \subset \mathbb{N}^* \quad \text{having density} \quad D(\mathcal{A}) > 0,
\]

\[
1 \leq m_1 < m_2 < \ldots \quad \text{and} \quad 1 \leq n_1 < n_2 < \ldots \quad \text{in} \quad \mathbb{N}^*,
\]

such that

\[
\mathcal{A} \cap [1, m_j] = \{ k \in [1, m_j] : k + n_j \in B \}, \quad j \geq 1.
\]

Finally, (i) and Lemma 3.4 entail that there are \(p \in \mathbb{N}^*, k_1 < \ldots < k_p \) in \(\mathcal{A}\) and \(\lambda_1, \ldots, \lambda_p \geq 0, \lambda_1 + \ldots + \lambda_p = 1\), such that

\[
\left\| \sum_{j=1}^{p} \lambda_j x_{k_j} \right\| \leq \frac{\varepsilon_0}{c}.
\]

By (5.3) it follows that

\[
(5.1) \quad \left\| \sum_{j=1}^{p} \lambda_j x_{k_j+n} \right\| \leq c \left\| \sum_{j=1}^{p} \lambda_j x_{k_j} \right\| \leq \varepsilon_0, \quad n \geq 1.
\]

Now let \(q \in \mathbb{N}^*\) be such that \(k_1, \ldots, k_p \leq m_q\). Then

\[
k_1 + n_j, \ldots, k_p + n_j \in B, \quad j \geq q.
\]

Choose \(j_{*} \geq q\) with \(k_1 + n_{j*} \geq b_{j_{*}q}\) and define \(n \in \mathbb{N}^*\) by \(b_{j_{n-1}} < k_1 + n_j \leq b_{j_n}\). Since \(b_{j_{*}q} \leq k_1 + n_j \leq b_{j_n}\), and the sequence \((b_{j_{n-1}})_{n \geq 1}\) is increasing, we have \(m_q \leq n\). We claim that

\[
(5.2) \quad b_{j_{n-1}} + n < k_1 + n_j \leq k_p + n_j \leq b_{j_n}.
\]

Indeed, \(b_{j_{n-1}} < k_1 + n_j\), \(k_1 + n_j \in B\) and \(B \cap (b_{j_{n-1}}, b_{j_{n-1}} + n] = \emptyset\) imply \(b_{j_{n-1}} + n < k_1 + n_j\). Further, \(k_p + n_j = k_1 + n_j + (k_p - k_1) \leq b_{j_n} + m_q < b_{j_n} + n + 1, k_p + n_j \in B\) and \(B \cap (b_{j_n}, b_{j_n} + n + 1] = \emptyset\) yield \(k_p + n_j \leq b_{j_n}\).

By (5.2) we have \(k_1 + n_j, \ldots, k_p + n_j \in B \cap (b_{j_{n-1}} + n, b_{j_n}]\), so
that\[ \Re \langle x_n^*, x_{k_j+n_j} \rangle > 2 \varepsilon_o, \quad 1 \leq j \leq p. \]

Since \( \|x_n^*\| \leq 1 \), it follows that
\[
\left\| \sum_{j=1}^{p} \lambda_j x_{k_j+n} \right\| \geq \Re \left\langle x_n^*, \sum_{j=1}^{p} \lambda_j x_{k_j+n} \right\rangle = \sum_{j=1}^{p} \lambda_j \Re \langle x_n^*, x_{k_j+n_j} \rangle > 2 \varepsilon_o,
\]
in contradiction with (5.1).

\[ \square \]

6. Weak mixing to zero for Cesaro shift-bounded sequences

If \( X \) is a uniformly convex Banach space, then Theorem 5.2 holds under a milder assumption on \((x_k)_{k \geq 1}\) than convex shift-boundedness.

For we shall use that in uniformly convex Banach spaces the classical Mazur theorem about the equality of the weak and norm closure of a convex subset holds in the following sharper form:

**Theorem 6.1** (Mazur type theorem in uniformly convex Banach spaces). Let \( S \) be a bounded subset of a uniformly convex Banach space \( X \), and \( x \) an element of the weak closure of \( S \). Then there exists a sequence \((x_k)_{k \geq 1} \subset S\), such that
\[
\lim_{n \to \infty} \left\| x - \frac{1}{n} \sum_{k=1}^{n} x_k \right\| = 0.
\]

**Proof.** Uniformly convex Banach spaces are reflexive (see e.g. [4], page 131), so \( S \) is weakly relatively compact. Consequently, since normed linear spaces are angelic in their weak topology (see e.g. [5], 3.10,(1)), there exists a sequence \((y_j)_{j \geq 1}\) in \( S \), which is weakly convergent to \( x \). Now, according to the validity of the “Banach-Saks Theorem” [4] in uniformly convex Banach spaces, due to S. Kakutani [11] (see also [4], Chapter VIII, Theorem 1), there exists a subsequence \((y_{k_j})_{j \geq 1}\) such that
\[
\lim_{n \to \infty} \left\| x - \frac{1}{n} \sum_{k=1}^{n} y_{k_j} \right\| = 0.
\]

\[ \square \]

Let us call a sequence \((x_k)_{k \geq 1}\) in a Banach space \( X \) **Cesaro shift-bounded** if there exists a constant \( c > 0 \) such that \( \|x\| \) holds for any choice of \( p \in \mathbb{N}^* \) and \( \lambda_1, \ldots, \lambda_p \in \{0, 1\} \), that is
\[
\left\| \sum_{j=1}^{p} x_{n_j+n} \right\| \leq c \left\| \sum_{j=1}^{p} x_{n_j} \right\|, \quad n \geq 1
\]
for any \( p \in \mathbb{N}^* \) and \( n_1, \ldots, n_p \in \mathbb{N}^* \) with \( n_1 < \ldots < n_p \).

Clearly, every convex shift-bounded sequence in \( X \) is Cesaro shift-bounded, but the converse does not hold, even in Hilbert spaces:

**Example 6.2.** Let \( H \) be an infinite-dimensional Hilbert space and choose, for every \( k \in \mathbb{N} \), three vectors \( u_k, v_k, w_k \in H \) such that
\[
\|u_k\| = \|v_k\| = \|w_k\| = 1, \quad 0 < \|u_k - v_k\| < \frac{1}{k+3}, \quad w_k \perp \{u_k, v_k\} \quad \text{for all} \quad k \in \mathbb{N},
\]
\[
\{u_k, v_k, w_k\} \perp \{u_l, v_l, w_l\} \quad \text{whenever} \quad k \neq l.
\]

Let us define the sequence \((x_n)_{n \geq 1}\) by
\[
x_{3k+1} := 2u_k, \quad x_{3k+2} := -v_k, \quad x_{3k+3} := w_k \quad \text{for even} \quad k \in \mathbb{N},
\]
\[
x_{3k+1} := 2u_k, \quad x_{3k+2} := w_k, \quad x_{3k+3} := -v_k \quad \text{for odd} \quad k \in \mathbb{N}.
\]

Then \((x_n)_{n \geq 1}\) is Cesaro shift-bounded, but not convex shift-bounded.
Proof. For every $k \in \mathbb{N}$ we denote by $\mathcal{V}_k$ the set of the vectors

$$
\begin{align*}
x_{3k+1}, x_{3k+2}, x_{3k+3} \\
x_{3k+1} + x_{3k+2}, x_{3k+1} + x_{3k+3}, x_{3k+2} + x_{3k+3} \\
x_{3k+1} + x_{3k+2} + x_{3k+3}.
\end{align*}
$$

It is easy to verify that $2/3 \leq \|x\| \leq \sqrt{5}$ for all $x \in \mathcal{V}_k$.

Now let $p \in \mathbb{N}^*$, $n_1, \ldots, n_p \in \mathbb{N}^*$ with $n_1 < \ldots < n_p$, and $n \in \mathbb{N}^*$ be arbitrary. Let $q$ denote the number of all $\mathcal{V}_k$ which contain some $x_{n_j}$. Then

\begin{equation}
\left\| \sum_{j=1}^p x_{n_j} \right\| \geq \sqrt{q \left( \frac{2}{3} \right)^2} = \sqrt{\frac{4}{9}} q. \tag{6.1}
\end{equation}

On the other hand, since the number of all $\mathcal{V}_k$ which contain some $x_{n_j + n}$ is $\leq 2q$, we have

\begin{equation}
\left\| \sum_{j=1}^p x_{n_j + n} \right\| \leq \sqrt{2q \left( \sqrt{5} \right)^2} = \sqrt{10} q. \tag{6.2}
\end{equation}

Now (6.2) and (6.1) entail that

\[ \left\| \sum_{n=1}^p \lambda_n x_{n+m} \right\| \leq \sqrt{\frac{45}{2}} \left\| \sum_{j=1}^p x_{n_j} \right\| \]

and we conclude that the sequence $(x_n)_{n \geq 1}$ is Cesaro shift-bounded.

To show that $(x_n)_{n \geq 1}$ is not convex shift-bounded, let us assume the contrary, that is the existence of some constant $c > 0$ such that

\[ \left\| \sum_{n=1}^p \lambda_n x_n \right\| \leq c \left\| \sum_{n=1}^p \lambda_n x_n \right\|, \quad m \geq 1 \]

for any $p \in \mathbb{N}^*$ and $\lambda_1, \ldots, \lambda_p \geq 0$. Let $k$ be an arbitrary even number in $\mathbb{N}^*$ and set $p := 3k + 2$, $\lambda_n := 0$ for $1 \leq n \leq 3k$, $\lambda_{3k+1} := 1$ and $\lambda_{3k+2} := 2$. Then

\[ \left\| \sum_{n=1}^p \lambda_n x_n \right\| = \left\| 2u_k - 2v_k \right\| < \frac{2}{k + \frac{3}{2}}, \]

while

\[ \left\| \sum_{n=1}^p \lambda_n x_{n+3} \right\| = \left\| 2u_{k+1} + 2w_{k+1} \right\| = 2 \sqrt{2}. \]

It follows that $2 \sqrt{2} \leq \frac{2c}{k + \frac{3}{2}}$, what is not possible for any even $k \in \mathbb{N}^*$.

□

Using Theorem 6.1, we can adapt the proof of Theorem 5.2 to the case of Cesaro shift-bounded sequences in uniformly convex Banach spaces:

**Theorem 6.3 (Weak mixing for Cesaro shift-bounded sequences).** For a Cesaro shift-bounded sequence $(x_k)_{k \geq 1}$ in a uniformly convex Banach space $X$, the following conditions are equivalent:

(i) $(x_k)_{k \geq 1}$ is weakly mixing to zero,

(j) $(x_k)_{k \geq 1}$ is uniformly weakly mixing to zero,

(jw) 1.14 holds.
Proof. Since the implications \((jw) \Rightarrow (j) \Rightarrow (i)\) are trivial, to complete the proof we need only to prove that \((i) \Rightarrow (jw)\).

Let \(c > 0\) be a constant such that \((6.3)\) holds for any choice of \(p \in \mathbb{N}^*\) and \(\lambda_1, \ldots, \lambda_p \in \{0, 1\}\). Since \((6.3)\) does not hold, there exist \(a_1, b_1, a_2, b_2, \ldots \in \mathbb{N}^*\) with \(b_j - a_j \geq j, j \geq 1\), such that

\[
\lim_{j \to \infty} \sup \left\{ \frac{1}{b_j - a_j + 1} \sum_{k=a_j}^{b_j} |\langle x^*, x_k \rangle| : x^* \in X^*, \|x^*\| \leq 1 \right\} > 0.
\]

By Lemma \(5.3\) there exist

- \(0 < \varepsilon_o \leq 1\),
- \(B \subset \mathbb{N}^*\) with \(BD^*(B) \geq \varepsilon_o\),
- \(j_1 < j_2 < \ldots\) in \(\mathbb{N}^*\) with \(b_{j_n} - b_{j_{n-1}} > n\),
- \(x_1^*, x_2^*, \ldots \in X^*\) with \(\|x_n^*\| \leq 1\),

for which

\[
B \cap \bigcup_{n \geq 2} (b_{j_{n-1}} + n, b_{j_{n-1}} + n] = \emptyset,
\]

\[
\Re \langle x_n^*, x_k \rangle > 2 \varepsilon_o, \quad k \in B \cap (b_{j_{n-1}} + n, b_{j_n}], n \geq 2.
\]

On the other hand, since \(X\) is reflexive and any bounded set in a reflexive Banach space is weakly relatively compact, by Lemma \(5.3\) there exists \(A_o \subset \mathbb{N}^*\) with \(D_o(A_o) = 1\) such that \(\lim_{A_o \ni k \to \infty} x_k = 0\) in the weak topology of \(X\).

Finally, by Corollary \(3.3\) there exists \(I \subset A_o\) with \(D^*(I) > 0\), such that

\[
\{n \in \mathbb{N}^* : F + n \subset B\} \text{ is infinite for any finite } F \subset I.
\]

Then \(\lim_{I \ni k \to \infty} x_k = 0\) with respect to the weak topology of \(X\) and by Theorem \(6.1\) there are \(p \in \mathbb{N}^*\) and \(k_1 < \ldots < k_p\) in \(I\) such that

\[
\left\| \frac{1}{p} \sum_{j=1}^{p} x_{k_j} \right\| \leq \frac{\varepsilon_o}{c}.
\]

By \((6.3)\) it follows that

\[
\left\| \frac{1}{p} \sum_{j=1}^{p} x_{k_j+n} \right\| \leq c \left\| \frac{1}{p} \sum_{j=1}^{p} x_{k_j} \right\| \leq \varepsilon_o, \quad n \geq 1.
\]

Now set \(m := \max \{k_p - k_1, 2\}\). Since the set \(\{k \in \mathbb{N}^* : \{k_1, \ldots, k_p\} + n \subset B\}\) is infinite, it contains some \(n_o\) such that \(k_1 + n_o \geq b_{j_{n_o}}\). We define \(n_1 \in \mathbb{N}^*\) by \(b_{j_{n_1-1}} < k_1 + n_o \leq b_{j_{n_1}}\). Since \(b_j \leq k_1 + n_o \leq b_{j_{n_1}}\) and the sequence \((b_{j_n})_{n \geq 1}\) is increasing, we have \(m \leq n_1\). We claim that

\[
(6.4) \quad b_{j_{n_1-1}} + n_1 \leq k_1 + n_o < k_p + n_o \leq b_{j_{n_1}}\).
\]

Indeed, \(b_{j_{n_1-1}} < k_1 + n_o \leq b_{j_{n_1}}\) and \(B \cap (b_{j_{n_1-1}} + n_1, b_{j_{n_1-1}} + n_1] = \emptyset\) imply \(b_{j_{n_1-1}} + n_1 < k_1 + n_o\). Further, \(k_p + n_o = k_1 + n_o + (k_p - k_1) \leq b_{j_{n_1}} + m < b_{j_{n_1}} + n_1 + 1, k_p + n_o \in B\) and \(B \cap (b_{j_{n_1}}, b_{j_{n_1}} + n_1 + 1] = \emptyset\) yield \(k_p + n_o \leq b_{j_{n_1}}\).

By \((6.3)\) we have \(k_1 + n_o, \ldots, k_p + n_o \in B \cap (b_{j_{n_1-1}} + n_1, b_{j_{n_1}}]\), so

\[
\Re \langle x_{n_1}^*, x_{k_j+n_o} \rangle > 2 \varepsilon_o, \quad 1 \leq j \leq p.
\]

Since \(\|x_{n_1}^*\| \leq 1\), it follows that

\[
\left\| \frac{1}{p} \sum_{j=1}^{p} x_{k_j+n_o} \right\| \geq \Re \left\langle x_{n_1}^*, \frac{1}{p} \sum_{j=1}^{p} x_{k_j+n_o} \right\rangle = \frac{1}{p} \sum_{j=1}^{p} \Re \langle x_{n_1}^*, x_{k_j+n_o} \rangle > 2 \varepsilon_o,
\]

in contradiction with \((6.3)\).
7. Appendix: Ergodic theorem for convex shift-bounded sequences

We can define also ergodicity of a bounded sequence \((x_k)_{k \geq 1}\) in a Banach space \(X\) by requiring that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k = 0
\]

(cf. [2], Section 3). Clearly, if \((x_k)_{k \geq 1}\) is uniformly weak mixing, then it is ergodic. In this section we complete our knowledge about convex shift-bounded sequences by proving a mean ergodic theorem for them (Corollary 7.2).

Let \(l^\infty(X)\) denote the vector space of all bounded sequences \((x_j)_{j \geq 1}\) in a Banach space \(X\), endowed with the uniform norm \(\| (x_j) \|_\infty = \sup_j \| x_j \|\), and let \(\sigma_\cdot\) be the backward shift on \(l^\infty(X)\), defined by

\[
\sigma_\cdot ((x_j)_{j \geq 1}) = (x_{j+1})_{j \geq 1}
\]

**Theorem 7.1** (Mean Ergodic Theorem for sequences). For a bounded sequence \((x_k)_{k \geq 1}\) in a Banach space \(X\), the following conditions are equivalent:

\[
\begin{align*}
&\text{(e)} \quad \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=m+1}^{n} x_k \right\| = 0, \\
&\text{(ee)} \quad \text{The norm-closure of the convex hull}
\end{align*}
\]

\[
\conv \left\{ \sigma_\cdot^k ((x_j)_{j \geq 1}) \mid k \geq 0 \right\} \subset l^\infty(X)
\]

contains the zero sequence.

**Proof.** Without loss of generality we can assume that \(\| x_k \| \leq 1\) for all \(k \geq 1\).

The proof of (e) \(\Rightarrow\) (ee) is immediate. Indeed, if \(\varepsilon > 0\) and \(n_\varepsilon \in \mathbb{N}^\ast\) are such that

\[
\left\| \sum_{k=m+1}^{n} x_k \right\| \leq (n-m) \varepsilon, \quad 0 \leq m < n, \quad n-m \geq n_\varepsilon,
\]

then we have for every \(n \geq n_\varepsilon\):

\[
\left\| \frac{1}{n} \sum_{k=1}^{n} \sigma_\cdot^k ((x_j)_{j \geq 1}) \right\| = \left\| \frac{1}{n} \sum_{k=1}^{n} (x_{j+k})_{j \geq 1} \right\| = \sup_{j \geq 1} \left\| \frac{1}{n} \sum_{k=1}^{n} x_{k+j} \right\| \leq \varepsilon.
\]

Conversely, let us assume that (ee) is satisfied and let \(\varepsilon > 0\) be arbitrary. Then there exist \(p \in \mathbb{N}^\ast\) and \(\lambda_1, \ldots, \lambda_p \geq 0\), \(\lambda_1 + \ldots + \lambda_p = 1\), such that

\[
\sup_{k \geq 1} \left\| \sum_{j=1}^{p} \lambda_j x_{j+k} \right\| = \left\| \sum_{j=1}^{p} \lambda_j \sigma_\cdot^j ((x_k)_{k \geq 1}) \right\| \leq \frac{\varepsilon}{2}.
\]

On the other hand, we have for every \(0 \leq m < n\) with \(n-m \geq p\):

\[
\frac{1}{n-m} \sum_{k=m+1}^{n} x_k - \frac{1}{n-m} \sum_{k=m+1}^{n} \left( \sum_{j=1}^{p} \lambda_j x_{j+k} \right) =
\]

\[
= \frac{1}{n-m} \sum_{k=m+1}^{n} \sum_{j=1}^{p} \lambda_j (x_k - x_{j+k}) = \frac{1}{n-m} \sum_{j=1}^{p} \lambda_j \sum_{k=m+1}^{n} (x_k - x_{j+k}) =
\]

\[
= \frac{1}{n-m} \sum_{j=1}^{p} \lambda_j \left( \sum_{k=m+1}^{m+j} x_k - \sum_{k=m+1}^{n+j} x_k \right),
\]

hence

\[
\left\| \frac{1}{n-m} \sum_{k=m+1}^{n} x_k - \frac{1}{n-m} \sum_{k=m+1}^{n} \left( \sum_{j=1}^{p} \lambda_j x_{j+k} \right) \right\| \leq \frac{2p}{n-m}.
\]

(7.2)
Now (7.1) and (7.2) yield
\[ 0 \leq m < n, \ n - m \geq 4 \frac{p}{\varepsilon} \implies \left\| \frac{1}{n-m} \sum_{k=m+1}^{n} x_k \right\| \leq \varepsilon. \]

For convex shift-bounded vector sequences the statement of Theorem 7.1 can be strengthened:

**Corollary 7.2** (Mean Ergodic Theorem for convex shift-bounded sequences). For a convex shift-bounded sequence \((x_k)_{k \geq 1}\) in a Banach space \(X\), the following conditions are equivalent:

1. \[ \lim_{0 \leq m < n \to \infty} \left\| \frac{1}{n-m} \sum_{k=m+1}^{n} x_k \right\| = 0. \]
2. \[ \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=1}^{n} x_k \right\| = 0. \]
3. The weak closure of the convex hull \(\text{conv} \left\{ (x_k ; k \geq 1) \right\} \subset X\) contains 0.

**Proof.** The implications (e) \(\Rightarrow\) (f) \(\Rightarrow\) (ff) are trivial.

Since the weak closure of \(\text{conv} \left\{ (x_k ; k \geq 1) \right\}\) is equal to its norm closure, (ff) implies that, for any \(\varepsilon > 0\), there are \(p \in \mathbb{N}^*\) and \(\lambda_1, \ldots, \lambda_p \geq 0\), \(\lambda_1 + \ldots + \lambda_p = 1\), such that
\[ \left\| \sum_{j=1}^{p} \lambda_j x_j \right\| \leq \varepsilon. \]

Using (ee), it follows that
\[ \left\| \sum_{j=1}^{p} \lambda_j \sigma_{\epsilon}^j \left( (x_k)_{k \geq 1} \right) \right\| = \sup_{k \geq 1} \left\| \sum_{j=1}^{p} \lambda_j x_{j+k} \right\| \leq c \varepsilon. \]

By the above (ff) implies condition (ee) in Theorem 7.1, hence (e).

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