The null-timelike boundary problems of linear wave equations in asymptotically anti-de Sitter spacetime

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Received 3 October 2019, revised 15 April 2020
Accepted for publication 1 May 2020
Published 28 July 2020

Abstract
In this paper, we study the initial-boundary problem of the massive wave equation which is conformal regular on an asymptotically anti-de Sitter spacetime, where the initial-boundary data are given on an outgoing null hypersurface and a timelike hypersurface, and the asymptotic information is given on the future null infinity. After a conformal rescaling, this problem will reduce to a null-timelike boundary problem of linear wave equation on the conformal compactified spacetime, where the initial data is given on a null hypersurface and the boundary data are given on two timelike hypersurfaces. We demonstrate well posedness for the associated null-timelike boundary problems. The proofs rely on energy estimates (weighted energy estimates for null-timelike boundary problems) and local existence of solution for initial-boundary problems. This is a toy model motivated by AdS/CFT correspondence.

Keywords: initial-boundary value problem, asymptotic AdS space-time, linear wave equation

1. Introduction

Initial value problem is one of central problems of mathematical physics equations. Since the requirements of different physical problems, several kinds of initial value problems have been studied. For example, characteristic initial value problems [1–4], initial-boundary value problems [5]. In this paper, null-timelike initial-boundary value problem will be considered, i.e. boundary data is imposed on a light cone and a timelike boundary. This problem was first considered by Friedlander [6, 7] and the motivation was to study the radiation field of wave equations which could be seen as a toy model of gravitational waves [8, 9]. In 1997,
Bartnik tried to solve Einstein equations by imposing boundary data on a timelike cylinder and an outgoing light cone which started from a space-like two sphere \[10\]. One year later, Balean solved a simple model of such problem, i.e. null-timelike boundary problem for linear wave equations in Minkowski spacetime \[11\]. In following paper, Balean and Bartnik generalized Balean’s result to Maxwell field in Minkowski spacetime \[12\]. Since their method strongly depends on Minkowski back ground, it is nontrivial to generalize their work to general back ground. In 1999, Frittelli and Lehner considered the existence and uniqueness of characteristic evolution in Bondi–Sachs coordinates in general relativity \[13\]. After that, Frittelli also studied the characteristic problem of the linearized Einstein equations \[14\]. Similar topic also has been studied by Kreiss and Winicour \[15, 16\]. Recently, this work has been generalised to asymptotically flat spacetime for linear wave equations \[17\] and for Maxwell equations and spin-2 field equations \[18\]. One motivation of our paper is to consider this problem on asymptotically anti-de Sitter back ground.

Physical motivation of this paper is to understand AdS/CFT correspondence \[18\] in terms of initial-boundary value problem. It is well-known that AdS/CFT correspondence is a great breakthrough in theoretical physics. At the beginning of AdS/CFT correspondence, Witten \[19\] has pointed out that the Euclidean version of this conjecture could be understood in terms of boundary problem of elliptic partial differential equations. In a recent work \[20\], Witten reconsidered this problem and gave some concrete conjectures on how to choose suitable boundary conditions. In past decades, more and more research work has been focused on how to use AdS/CFT conjecture solving practical physical problems. For example, problems from QCD (AdS/QCD theory) \[21, 22\], problems from condense matter theory (AdS/condense matter theory) \[23–25\] and problems from hydrodynamics (fluid/gravity correspondence) \[26, 27\], . . . . The main idea of these work is considering dynamical perturbations on asymptotic AdS black hole back ground and simulating associated physical properties by using AdS/CFT correspondence. Since the perturbation is introduced from a time-like boundary or conformal boundary of asymptotic AdS spacetime \[26\], the null-timelike boundary value problem will give a mathematical foundation for those work.

Our main idea is following the method used in \[17\], consider the existence of analytic case first\(^3\), then extend our result to the existence of general solution by using energy estimates. Roughly speaking, this method can be explained as following: one can construct sequences of equations, such that the coefficients and boundary data of any equation in such sequence are all analytic functions. Further more, the boundary data and equation coefficients also converge to the original one. With the existence of analytic case, each equation in this sequence has a solution \(u'\). With help of energy estimates, one can prove that the sequence \(\{u'\}\) will converge to a function \(u\) and this function will solve the original null-timelike boundary value problem. This is a quite standard method to consider well-posedness of initial-boundary value problem \[28, 29\].

This paper is organized as following: in next section, some back ground knowledge and statement of main result is given. In section 3, some necessary notations and conformal transformations are introduced, which will be used in the proof. Section 4 focus on energy-estimates. These results are key tools for our proof. With theorems of section 4, we finish the proof in section 5. We consider the local existence of analytic solution first. After that, with the help of energy-estimate’s result in section 6, we prove the local existence of general solution by Schauder method (theorem 5.1). The final step in this section is to prove the global existence

\(^3\) The analytic case means the equation takes the same form of the equation which we want to solve, but boundary data and coefficients in equation are all analytic functions.
of solution by using the standard bootstrap method (theorems 5.4 and 5.5). Section 6 contains some discussions on our results.

2. Background and statement of main theorem

Let \((\mathcal{M}, \hat{g})\) be a Lorentzian manifold, where \(\mathcal{M}\) is an orientable \((1 + 3)\)-dimensional manifold and \(g\) is an asymptotically anti-de Sitter metric. Consider a domain \(M \subset \mathcal{M}\), and we assume there is a global Bondi–Sachs coordinate system \(\{\tau, r, \theta\}\) on \(M\) \([8, 9]\), i.e.,

\[ M = \{ (\tau, r, \theta) | \tau \in [0, +\infty), r \in [R, +\infty), \theta \in S^2 \}, \]

for some positive constant \(R\). Here, \(\Sigma_\tau = \{ \tau = \text{const}\}\) is an outgoing null hypersurface in \(M\), and \(\theta = \{ x^2, x^3 \}\) are local coordinates of the topological two-sphere \(S_{\tau, r}\), i.e. the two-dimensional surfaces with \(\tau = \text{const}\) and \(r = \text{const}\), and \(4\pi r^2 = \text{Area}(S_{\tau, r})\). In this coordinates, \(\hat{g}\) is assumed to have the form

\[ \hat{g} = -V e^{2\eta} d\tau^2 - 2 e^{2\eta} d\tau dr + r^2 h_{AB} (dx^A - U^A d\tau) (dx^B - U^B d\tau), \quad (2.1) \]

where \(V, \eta, U^A, h_{AB}\) are functions of \(\tau, r\) and \(x^A\) with \(A = 2, 3\), and \(\det(h_{AB}) = 1\). The boundary of \(M\) is denoted by \(\Sigma_0 \cup T\), where

\[ \Sigma_0 = \{ (\tau, r, \theta) | \tau = 0, r \geq R \}, \]
\[ T = \{ (\tau, r, \theta) | \tau \geq 0, r = R \}. \]

It is clear that \(T\) is a timelike hypersurface and \(\Sigma_0\) is an outgoing light cone with respect to \(\hat{g}\). The spacetime region where we consider and the boundary of the spacetime region are showed in figure 1.

We assume that \(\hat{g}\) is asymptotically anti-de Sitter, i.e., as \(r \to +\infty\),

\[ V = 1 - \frac{\Lambda}{3} r^2 + O\left(\frac{1}{r}\right), \quad \text{uniformly on } [0, T] \times S^2, \quad (2.2) \]

\[ \lim_{r \to \infty} r U^A = \lim_{r \to \infty} \eta = 0, \quad \zeta = O\left(\frac{1}{r}\right), \quad \zeta - 2\eta = O\left(\frac{1}{r^2}\right) \quad \text{uniformly on } [0, T] \times S^2, \quad (2.3) \]

and

\[ h = h_{AB} dx^A dx^B \to g_{S^2}, \quad \text{uniformly on } [0, T] \times S^2, \quad (2.4) \]

for any fixed constant \(T\), and where \(\Lambda < 0\) is cosmological constant, and \(g_{S^2}\) is the standard round metric on two-sphere \(S^2\) with radius 1. We also require that \(\hat{g}\) satisfies the conformal regular condition in the following section and satisfies the non-degenerate assumption in section 4.

In this paper, we study the linear wave equation

\[ \Box_{\hat{g}} u - \frac{2\Lambda}{3} u = 0, \quad (2.5) \]
with the initial-boundary condition as follows\(^4\):

\[
\begin{align*}
\left. u \right|_{\Sigma_0} &= \frac{\varphi e^{-\eta}}{r}, \\
\left. u \right|_{\mathcal{T}} &= \frac{\psi_1 e^{-\eta}}{R},
\end{align*}
\]

where \(\varphi\) is a function on \(\Sigma_0\) and \(\lim_{r \to \infty} \varphi(r, \cdot)\) exists on \(S^2\), \(\psi_1\) is a function on \(\mathcal{T}\) and satisfies \(\psi_1 = \varphi\) on \(\Sigma_0 \cap \mathcal{T}\). And we give an asymptotic condition of \(u\) at conformal infinity \(\mathcal{I}\), i.e.,

\[
\lim_{r \to \infty} e^{\eta} r u(\tau, r, \theta) = \psi_2(\tau, \theta), \\
\] on \([0, \infty) \times S^2\), (2.7)

\(^5\)where \(\psi_2\) is a function on \([0, \infty) \times S^2\), and satisfies \(\psi_2(0, \cdot) = \lim_{r \to \infty} \varphi(r, \cdot)\) on \(S^2\).

We will prove the following result,

**Theorem 2.1.** For some integer \(k \geq 1\), let \(g\) be a Lorentz metric given by (2.1) on \(M\) which is asymptotically anti-de Sitter and satisfies a conformal regularity condition (definition 3.1) and a non-degenerate assumption (hypothesis 4.3). Suppose \(g\) sufficiently regular, \(\psi_1 \in H^{2k-1}([0, T] \times S^2)\), \(\psi_2 \in H^k([0, T] \times S^2)\) and \(\varphi \in H^{2k-1}([R, \infty) \times S^2)\), where

\[
\|\varphi\|_{\tilde{H}^{2k-1}([R, \infty) \times S^2)} = \sum_{|\alpha| \leq 2k-1} \left( \int_{[R, \infty) \times S^2} \left| \frac{\partial^\alpha \varphi}{\partial \tau^{\alpha_0} \partial \theta^{\alpha_3}} \right|^2 r^2 \sin \theta \, dr \, d\sigma \right)^{\frac{1}{2}},
\]

Then, there exists a unique solution \(u \in \tilde{H}^k([0, T] \times [R, \infty) \times S^2)\) of (2.5)–(2.7), and

\[
\|e^{\eta} ru\|_{\tilde{H}^k([0, T] \times [R, \infty) \times S^2)} \leq C \left\{ \|\varphi\|_{\tilde{H}^{2k-1}([R, \infty) \times S^2)} + \|\psi_1\|_{H^{2k-1}([0, T] \times S^2)} + \|\psi_2\|_{H^k([0, T] \times S^2)} \right\},
\]

\(^4\)The initial data which is given on \(\Sigma_0\) has the decay of \(\frac{1}{r}\) at conformal infinity.

\(^5\)More generally, one can see [34] and its related references for the asymptotic conditions on conformal infinity \(\mathcal{I}\) of the massive wave equations in asymptotically AdS spacetimes.
where,
\[
\|e^{g}ru\|_{\mathfrak{H}^{k}_{\mathfrak{I}}([0,T] \times [\mathcal{R}, \infty) \times \mathbb{S}^{2})} = \sum_{|\alpha| \leq k, \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)} \left( \int_{0}^{T} \int_{\mathcal{R}} \int_{\mathbb{S}^{2}} \frac{\left| \partial^{\alpha}(e^{g}ru) \right|^{2}_{\mathfrak{I}} \text{d} \sigma \text{d} r \text{d} \tau} {r^{2+4\alpha_1}} \right)^{1/2},
\]
and \(C\) is a positive constant depending only on \(g, R, T\) and \(k\).

In fact, the equation (2.5) is a massive wave equation with the specified mass depending on the cosmological constant \(\Lambda\). The special value of the mass is just so called BF bound [24, 30]. The reason why we are interested on that equation is that it is the simplest toy model of holographic condensed matter theory [23, 24].

3. Conformal transformation

In this section, we will consider conformal compactification of \(M\). The general definition of conformal compactification can be found in [32]. After that, we re-express null-timelike boundary problem on the un-physic spacetime. As [6, 7], we introduce a new coordinate
\[
z = \frac{1}{r}.
\]
In the new coordinates \((\tau, z, x^2, x^3)\), we have
\[
\hat{g} = -V e^{2\eta} d\tau^2 + \frac{2}{z^2} e^{2\eta} d\tau dz + \frac{1}{z^2} h_{AB}(dx^A - U^A d\tau)(dx^B - U^B d\tau).
\]  (3.1)
With the choice of conformal factor \(\Xi = z e^{-\eta}\), the un-physical metric \(g\) is
\[
g = \Xi^2 \hat{g} = -z^2 V d\tau^2 + 2 d\tau dz + h_{AB} e^{-2\eta}(dx^A - U^A d\tau)(dx^B - U^B d\tau).
\]  (3.2)

Definition 3.1 (Conformal regular condition). We say \(g\) satisfies the conformal regular condition, if the conformal metric \(\hat{g}\) can be smoothly\(^6\) extended to the boundary \(\{z = 0\}\).

Set \(v = \Xi^{-1} u\), then
\[
\square_v u = \frac{1}{(-\text{det} \hat{g})^{1/2}} \sum_{i=0}^{3} \partial_i \left[ (-\text{det} \hat{g})^{1/2} \hat{g}^{ij} \partial_j u \right] = \Xi^3 \square_{\hat{g}} v + [\Xi^3 \square_{\hat{g}} \Xi - 2 \Xi \hat{g}^{ij} \partial_i \Xi \partial_j \Xi] v.
\]  (3.3)
Since \(\partial_i \Xi = \delta_i e^{-\eta} - \Xi \partial_i \eta\), one has
\[
\hat{g}^{ij} \partial_i \Xi \partial_j \Xi = \Xi^2 V - 2 \Xi e^{-\eta} \hat{g}^{ij} \partial_j \eta + \Xi^2 \hat{g}^{ij} \partial_i \eta \partial_j \eta,
\]
\[
\square_{\hat{g}} \Xi = \partial_i \hat{g}^{ij} e^{-\eta} - 4 e^{-2\eta} \hat{g}^{ij} \partial_i \eta \partial_j \eta - \Xi (\square_{\hat{g}} \eta - \hat{g}^{ij} \partial_i \eta \partial_j \eta),
\]
where \(\text{det}(g) = -e^{-4\eta}\) has been used. With the asymptotic behavior of \(V\),
\[
\Xi^2 \square_{\hat{g}} \Xi - 2 \Xi e^{-\eta} \hat{g}^{ij} \partial_i \Xi \partial_j \Xi
\]
\[
= \Xi^3 [ -2 V + \Xi^{-1} e^{-\eta} \hat{g}^{ij} \partial_i \eta - \square_{\hat{g}} \eta - \hat{g}^{ij} \partial_i \eta \partial_j \eta ]
\]
\(^6\)In fact, we only need \(C^k\)-extensions, for some appropriate \(k\). We need to point out that this is not a new definition but an analog of regularity assumption on the conformal compactification of null infinity, see [32].
\[ = \Xi^3[ -2 + \frac{2\Lambda}{3}e^{\xi}z^{-2} + O(z) + \Xi^{-1}e^{-\eta}\partial_i g^{ij} - \Box \eta - g^{ij}\partial_i \eta \partial_j \eta ] \]
\[ = \Xi^3(\frac{2\Lambda}{3}e^{\xi}z^{-2} + \bar{\omega}), \tag{3.4} \]

where \[ \bar{\omega} = -2 + \Xi^{-1}e^{-\eta}\partial_i g^{ij} - \Box \eta - g^{ij}\partial_i \eta \partial_j \eta + O(z). \]

Then equation (2.5) can be re-written as
\[ \Box u - \frac{2\Lambda}{3}u = \Xi^{-1}\{ \Box v + \omega v \}, \]
where
\[ \omega = \frac{2\Lambda}{3}e^{2\eta}(e^{-\xi} - 2 + \bar{\omega}). \tag{3.5} \]

**Remark 3.2.** From the asymptotic behavior of \( V, U, \eta \) and \( \zeta \) (equations (2.2) and (2.3)), it is clear that \( \omega \) and \( \bar{\omega} \) are all regular on \( \{ z = 0 \} \).

The final form of wave equation (2.5) in terms of \( v \) becomes
\[ \Box v + \omega v = 0. \]

The un-physical spacetime \( (\bar{M}, g) \)
\[ \bar{M} = \{(\tau, z, \theta) | \tau \in [0, +\infty), z \in [0, \frac{1}{R}], \theta \in S^2 \} \]
has topology \([0, +\infty) \times [0, \frac{1}{R}] \times S^2\). Its boundary contains three parts \( \Sigma_0, \mathcal{T}, \mathcal{I} \),
\[ \Sigma_0 = \{(\tau, z, \theta) | \tau = 0, 0 \leq z \leq \frac{1}{R}, \theta \in S^2 \}, \]
\[ \mathcal{T} = \{(\tau, z, \theta) | \tau \geq 0, z = \frac{1}{R}, \theta \in S^2 \}, \]
\[ \mathcal{I} = \{(\tau, z, \theta) | \tau \geq 0, z = 0, \theta \in S^2 \}, \]
where \( \mathcal{I} \) is the conformal infinity of the physical spacetime \( (M, \hat{g}) \). From (2.2) and (2.3), it is easy to see that \( \mathcal{I} \) is a timelike boundary of \( (\bar{M}, g) \). It is worth to emphasis that \( (\bar{M}, g) \) is not globally hyperbolic.

Let \( S_{\tau, z} \) denote the coordinate two-sphere with constant \( \tau \) and \( z \). \( H_{\mu} \) denotes the ingoing light cone starting from the two-sphere \( S_{\lambda, \mu} \) on \( \Sigma_0 \). The generator \( \gamma(\lambda) \) of \( H_{\lambda, \mu} \) satisfies \( \gamma(0) = p \in S_{0, \mu} \) and \( \gamma(0) = N_{z}(p) = (\partial_\tau + \frac{1}{2}z^2\nabla \partial_z + g^{ij}\partial_\lambda)_{|p} \), where \( N_{z} \) is the ingoing null vector field which will be introduced in (4.4). Specially, \( H \) means \( H_{0} \).

We denote the maximal determined domain with respect to \( \Sigma_0 \) and \( \mathcal{T} \) by \( \mathcal{M}_1 \), i.e.,
\[ \mathcal{M}_1 = \bigcup_{\mu \in [0, \frac{1}{R}]} H_{\mu}, \]
and \( \mathcal{M}_1 \) has boundary \( \Sigma_0 \cup \mathcal{T} \cup \mathcal{H} \). And then we define
\[ \mathcal{M}_2 = \bar{M} \backslash \mathcal{M}_1, \]
where \( \mathcal{M}_2 \) has boundary \( \mathcal{H} \cup \mathcal{I} \). Figure 1 is a diagram of all these definitions.
For any fixed $T > 0$, we set
\[ \Omega_T = \{ (\tau, z, \theta) | 0 \leq \tau \leq T, 0 \leq z \leq \frac{1}{R}, \theta \in S^2 \}, \]
\[ \Sigma_T = \{ (\tau, z, \theta) | \tau = T, 0 \leq z \leq \frac{1}{R}, \theta \in S^2 \}, \]
which will be used later.

4. Energy estimate

In this section, we work on the un-physical spacetime $(\mathcal{M}, g)$. Moreover, we assume, for some positive constants $\lambda_1$ and $\lambda_2$,
\[ \lambda_1 g_{\mathbb{S}^2} \leq g_{AB} \leq \lambda_2 g_{\mathbb{S}^2} \quad \text{in } \mathcal{M}, \tag{4.1} \]
where $g_{AB}$ is the induced metric on $S_{\tau, z}$. Recall the wave equation on $(\mathcal{M}, g)$
\[ \Box_g v + \omega v = 0. \tag{4.2} \]
By $v = re^\theta u$ and (2.6) and (2.7), we have following equivalent initial-boundary condition of $v$, i.e.,
\[ v|_{\Sigma_0} = \varphi, \]
\[ v|_{\mathcal{T}} = \psi_1, \]
\[ v|_{\mathcal{I}} = \psi_2, \tag{4.3} \]
where $\varphi$, $\psi_1$ and $\psi_2$ are given in (2.6) and (2.7).

**Remark 4.1.** We can give a rough view for the way of solving the initial-boundary problem (4.2) and (4.3). Step 1, by the initial-boundary data $\varphi$ and $\psi_1$ on $\Sigma_0$ and $\mathcal{T}$, we can solve the wave equation (4.2) in $\mathcal{M}_1$, and also we can get the solution of $v$ on $\mathcal{H}$. Step 2, we solve the wave equation (4.2) with initial-boundary data on $\mathcal{H}$ and $\psi_2$ on $\mathcal{I}$, thus we can get the solution of $v$ in $\mathcal{M}_2$.

We introduce two future-directed null vector fields
\[ N_1 = -\nabla \tau = -\partial_\tau, \quad N_2 = \nabla z - \frac{1}{2} \varepsilon^2 V \nabla \tau = \partial_\tau + \frac{1}{2} \varepsilon^2 V \partial_\theta + g^{AB} \partial_A. \tag{4.4} \]
It is easy to check
\[ g(N_1, N_1) = g(N_2, N_2) = 0, \quad g(N_1, N_2) = -1. \]
Furthermore, we choose null tetrad as
\[ \{N_1, N_2, e_3, e_4\}, \tag{4.5} \]
where $e_3, e_4$ are tangent vector fields of $S_{\tau, z}$ and satisfy
\[ g(e_3, e_3) = g(e_4, e_4) = 1, \quad g(N_1, e_3) = g(N_1, e_4) = g(N_2, e_3) = g(N_2, e_4) = g(e_3, e_3) = 0. \]
For a $C^1$-function $\phi$, the associated energy momentum tensor $Q[\phi]$\(^7\) is a symmetric two-tensor defined by, for any vector fields $X$ and $Y$,

$$Q[\phi](X, Y) = (X\phi)(Y\phi) - \frac{1}{2}g(X, Y)|\nabla\phi|^2 - g(X, Y)\phi^2.$$  

**Lemma 4.2.** Let $\phi$ be a given $C^2$-function and $Q[\phi]$ be the associated energy momentum tensor. Set $X = a_1N_1 + a_2N_2$ and $Y = a_3N_1 + a_4N_2$, for some $a_i$, $i = 1, 2, 3, 4$. Then,

$$Q[\phi](X, Y) = a_1a_3(\partial_z\phi)^2 + a_2a_4(N_2\phi)^2 + (a_1a_4 + a_2a_3)(\frac{1}{2}g^{AB}\partial_A\phi\partial_B\phi + \phi^2).$$  

Moreover, if $a_i \geq 0$, $i = 1, 2, 3, 4$, then $Q[\phi](X, Y) \geq 0$.

**Proof.** First, we have

$$Q[\phi](X, Y) = a_1a_3Q[\phi](N_1, N_1) + a_2a_4Q[\phi](N_2, N_2) + (a_1a_4 + a_2a_3)Q[\phi](N_1, N_2).$$

Next,

$$|\nabla\phi|^2 = 2\partial_\tau\phi\partial_\tau\phi + z^2V(\partial_z\phi)^2 + 2g^{AB}\partial_A\phi\partial_B\phi + g^{AB}\partial_A\phi\partial_B\phi,$$

and then

$$Q[\phi](N_1, N_1) = (\partial_z\phi)^2, \quad Q[\phi](N_2, N_2) = (N_2\phi)^2,$$

$$Q[\phi](N_1, N_2) = \frac{1}{2}g^{AB}\partial_A\phi\partial_B\phi + \phi^2.$$  

A simple substitution yields the desired results. $\square$

**4.1. Energy on $H$**

As we mentioned in remark 4.1, we need the information of $v$ on $H$ at step 2, so the energy density on $H$ will be studied in this section. It is worth to emphasis that we only know $H$ is ingoing null hypersurface. To analyze the energy density, we now turn our attention to the null normal vector field $L$ of $H$. For any null vector $L$, it can be expressed as \(^8\)

$$L = l_1N_1 + l_2N_2 + l_3e_3 + l_4e_4, \quad (4.6)$$

where the coefficients satisfy the following equality,

$$-2l_1l_2 + l_3^2 + l_4^2 = 0, \quad (4.7)$$

and we have

$$L|_{S_{0,0}} = N_2|_{S_{0,0}} = \partial_\tau - \frac{A}{6}\partial_z. \quad (4.8)$$

**Hypothesis 4.3** (Non-degenerate assumption). We assume that the coefficient $l_2$ in (4.6) satisfies $l_2 \neq 0$ on entire $H$, which means that any outgoing null hypersurface $\Sigma_r$ traverses the ingoing null hypersurface $H$.

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\(^7\) We would like to emphasis that the energy-momentum tensor here is not the standard energy-momentum tensor of massive scalar field. It is just a mathematical tool.

\(^8\) For our fixed background metric $g$, the tetrad vector fields (4.5) are globally defined on $\overline{M}$ as a frame vector fields.
Furthermore we assume $L$ points toward the future, i.e., we set
\[ l_1 \geq 0, \quad l_2 > 0. \]
Then, for the energy density on $\mathcal{H}$, we have following lemma,

**Lemma 4.4.** Let $Y = b_1 N_1 + b_2 N_2$ be a vector field with $b_1 \geq 0$ and $b_2 \geq 0$, then we have $Q[\phi](L, Y) \geq 0$.

**Proof.** In fact, by (4.7), we have
\[
Q[\phi](L, Y) = b_1 l_1 Q[\phi](N_1, N_1) + b_2 l_2 Q[\phi](N_2, N_2) + (b_2 l_1 + b_1 l_2)Q[\phi](N_1, N_2)
\]
\[ + l_1 \phi_1^2 + l_2 \phi_2^2 + l_4 \phi_4^2 \]
\[ = \frac{1}{2} b_1 l_1 \left[ \phi_1^2 + \phi_2^2 + \phi_4^2 \right] + \frac{1}{2} b_2 l_2 \phi_2^2 + \frac{1}{2} l_1 \phi_3 + \frac{1}{2} l_2 \phi_4.
\]

hence, we know $Q[\phi](L, Y) \geq 0$.

We define three new vector fields on $\mathcal{H}$,
\[ E_1 = \frac{l_1}{l_2} N_1 + e_3, \quad E_2 = \frac{l_4}{l_2} N_1 + e_4, \quad E_3 = N_2 + \frac{l_5}{l_2} e_3 + \frac{l_4}{l_2} e_4. \]
A straightforward calculation yields
\[ g(L, E_3) = g(L, E_2) = g(L, E_3) = 0, \]
which means that $E_1, E_2, E_3$ tangent to $\mathcal{H}$. It is easy to show that they are linearly independent. By simple calculation, we have
\[ \frac{l_3}{l_2} e_4 - \frac{l_4}{l_2} e_3 = \frac{l_3}{l_2} e_2 - \frac{l_4}{l_2} e_1, \]
and
\[ L = l_2 E_3 + \frac{l_1}{2} E_1 + \frac{l_2}{2} E_2. \]

**Remark 4.5.** We definite
\[ E[\phi, l_2, b_1, b_2] = \frac{b_1 l_1}{2} (E_1 \phi)^2 + (E_2 \phi)^2 + b_2 l_2 (E_3 \phi)^2 + b_1 l_2 \phi^2, \]

hence, following from lemma 4.4, we have
\[ Q[\phi](L, Y) \geq E[\phi, l_2, b_1, b_2]. \]
Further more, if $b_1 > 0, b_2 > 0$, then $Q[\phi](L, Y)$ is positive definite on $\mathcal{H}$. For convenience, we set
\[ E[\phi] := E[\phi, 1, 1, 1] = \frac{1}{2} [(E_1 \phi)^2 + (E_2 \phi)^2 + (E_3 \phi)^2 + \phi^2]. \]
4.2. $H^1$-estimates in $\Omega_T \cap M_1$

In this subsection, we will get the energy estimates in $\Omega_T \cap M_1$. First, we derive the $H^1$-estimate in $\Omega_T \cap M_1$. In this paper, we use the weighted energy estimates so that interior terms can control all derivatives, which has been used in [17]. For completeness, we will state some results in [17] and give simple proof.

For simplicity, we also use the coordinates $(x^0, x^1, x^2, x^3)$ with $x^0 = \tau, x^1 = \varsigma$, and $(x^2, x^3)$ are local coordinates on $S_c$. For the needs of higher derivative estimates, in this section we consider more general wave equation

$$Lv = \Box_g v + a_i \partial_i v + bv = f,$$  

(4.12)

for some given functions $b, f$ and $a_i, i = 0, 1, 2, 3$ on $\overline{M}$.

For any $C^3$-function $h$ and vector field $X$, we have

$$\text{div}(hP(\phi, X)) = (\Box_g \phi) (hX\phi) + \frac{1}{2} hQ(\phi)\alpha^\alpha \pi^\alpha_\beta + Q(\phi)(\nabla h, X) - 2g(X, \nabla\phi)h\phi,$$  

(4.13)

where

$$P(\phi, X) = Q(\phi)X^\beta, \quad (X)_{\alpha\beta} = \partial^\alpha X^\beta + \partial^\beta X^\alpha - X(g^\alpha\beta).$$  

(4.14)

We will prove following weighted $H^1$-estimate in $M_1 \cap \Omega_T$.

**Theorem 4.6.** For some fixed $T > 0$, and $v$ is a solution of (4.12). There exists constants $q_0 > 0$ and $l > 0$ depending on $\lambda_1, \lambda_2, \|g^{ij}\|_{C^4(\Omega_T \cap M_1)}$, $|a^i|_{L^\infty(\Omega_T \cap M_1)}$ and $|b|_{L^\infty(\Omega_T \cap M_1)}$. such that for any $q \geq q_0$ and $h = e^{-q\tau + q\varsigma}$, we have

$$q^2 \|v\|_{H^1(\Omega_T \cap M_1)} + \|\partial TV\|_{L^2_1(\Omega_T \cap M_1)} + \|V\|_{H^2(\Omega_T \cap M_1)} + \left( \int_{\Omega_T \cap M_1} hE_H[v] d\mathcal{H} \right)^\frac{1}{2} \leq C \left\{ \|v\|_{H^1(\Sigma_0)} + \|V\|_{H^1(\Omega_T \cap M_1)} + \|V\|_{L^2_1(\Omega_T \cap M_1)} + \|f\|_{L^2_1(\Omega_T \cap M_1)} \right\}$$  

(4.15)

where, $\partial \mathcal{H}$ is the area element on $\mathcal{H}$ and $C$ is a positive constant depending only on $\lambda_1, \lambda_2, \|g^{ij}\|_{C^4(\Omega_T \cap M_1)}$, $|a^i|_{L^\infty(\Omega_T \cap M_1)}$, $|b|_{L^\infty(\Omega_T \cap M_1)}$ and $\inf_{T/2\Omega_T} \{Z^2 \Omega \}$. In particular, $C$ does not depend on $q$.

**Proof.** For some constants $m, p, q > 0$ to be chosen later, we choose a timelike vector field

$$Y = mN_1 + N_2,$$  

(4.16)

and functions

$$w = -pr + q\varsigma, \quad h = e^w.$$  

(4.17)

For a hypersurface $\Sigma$, we can define the $H^p(\Sigma)$ space and the corresponding $H^p(\Sigma)$-norm, with derivatives taken only with respect to variables on $\Sigma$. And, let $b$ the positive function, define

$$\|u\|_{H^p(\Sigma)} = \sum_{l=0}^{p} \|b^l \nabla^l u\|_{L^2(\Sigma)}.$$

and similarly for $\|u\|_{H^p(\Sigma)}$.
Set $P_{\alpha} = Q[v](\partial_{\alpha}, Y)$ and $P^{\alpha} = g^{\alpha\beta}P_{\beta}$. For the vector field $P$, Stokes’ theorem over $\Omega_T \cap M_1$ yields
\[
\int_{\Omega_T \cap M_1} \text{div}(hP) \, d\Omega = \int_{\partial(\Omega_T \cap M_1)} (hP)_{\alpha} n^\alpha \, dS,
\] (4.18)
where $d\Omega$ is the volume element on $\mathcal{M}$, $d\Omega = e^{-2h}d\tau d\sigma_{\mathcal{M}}$ and $dS$ is the induced volume element on boundary, $n^\alpha$ is the normal vector of boundary. By (4.13) and (4.18), we obtain
\[
\int_{\Omega_T \cap M_1} \{Q[v](\nabla h, Y) + \frac{1}{2} hQ[v]_{\alpha\beta}(v)\pi^{\alpha\beta} - 2g(Y, \nabla v)hv + (f - a^\alpha \partial_{\alpha} v - bv)h(Y v)\} \, d\Omega \\
= \int_{\Sigma_T \cap M_1} hQ[v](\nabla \gamma, Y)e^{-2h}d\sigma_{\mathcal{M}} - \int_{\Sigma_T \cap M_1} hQ[v](L, Y) \, d\Sigma \\
- \int_{\Sigma_T \cap M_1} hQ[v](\nabla \gamma, Y)e^{-2h}d\gamma d\sigma_{\mathcal{M}},
\] (4.19)
where $d\Sigma$ is the induced volume element on $\Sigma$. By the expression of $h$, we have
\[
\nabla h = h(-p \nabla Y + q \nabla Z) = h((p - \frac{1}{2} z^2 V q) N_1 + q N_2).
\]
By choosing the appropriate constant $p$ and $q$ such that $\nabla h$ is timelike, then $Q[v](\nabla h, Y)$ can control all derivatives of $v$. In fact
\[
Q[v](\nabla h, Y) = Q[v]((p - \frac{1}{2} z^2 V q) N_1 + q N_2, m N_1 + N_2) \\
= q(\partial_{\alpha} v)^2 + |m| + \frac{1}{2} q^2 V (\frac{1}{2} z^2 V - m))(\partial_{\beta} v)^2 + (mq + p - \frac{1}{2} q^2 V) \\
\times \left( \frac{1}{2} g^{AB} \partial_{\alpha} v \partial_{\beta} v + v^2 \right) + q^2 V \partial_{\alpha} v \partial_{\beta} v + 2q g^{1A} \partial_{\alpha} v \partial_{\beta} v \\
= q^2 V g^{1A} \partial_{\alpha} v \partial_{\beta} v + q(g^{1A} \partial_{\alpha} v)^2.
\]
By choosing $p = l q$, for some constant $l > 0$ sufficiently large and $l$ depends on $m$, $\lambda_1$, $\lambda_2$, $|z^2 V|_{L^\infty(\Omega_T \cap M_1)}$ and $|g^{1A}|_{L^\infty(\Omega_T \cap M_1)}$, then we have
\[
Q[v](\nabla h, Y) \geq \frac{1}{2} q \left( (\partial_{\alpha} v)^2 + (\partial_{\beta} v)^2 + g^{AB} \partial_{\alpha} v \partial_{\beta} v + v^2 \right).
\]
We now analyze the expression on the left-hand side of (4.19). By (4.14) and the definition of $Q$, we have
\[
Q[v]_{\alpha\beta}(v)\pi^{\alpha\beta} = \sum_{i, j = 0}^{3} a_{ij} \partial_{\alpha} v \partial_{\beta} v + cv^2,
\]
where $a_{ij}$ and $c$ depend on $m$, $g$ and $\partial g$. Hence, by choosing $q$ sufficiently large, depending on $|g^{ij}|_{C^1(\Omega_T \cap M_1)}$, $|a^{ij}|_{C^1(\Omega_T \cap M_1)}$, and $|h|_{L^\infty(\Omega_T \cap M_1)}$, we obtain
\[
\int_{\Omega_T \cap M_1} \left\{ Q[v](\nabla h, Y) + \frac{1}{2} hQ[v]_{\alpha\beta}(v)\pi^{\alpha\beta} - 2g(Y, \nabla v)hv + (f - a^\alpha \partial_{\alpha} v - bv)h(Y v) \right\} \, d\Omega \\
\geq \frac{1}{4} \int_{\Omega_T \cap M_1} h \left[ (\partial_{\alpha} v)^2 + (\partial_{\beta} v)^2 + g^{AB} \partial_{\alpha} v \partial_{\beta} v + v^2 \right] - f^2 \, d\Omega.
\] (4.20)
We now analyze the boundary integrals on the right-hand side of (4.19). By simple calculation, we have
\[
Q[v](\nabla \tau, Y) = -Q[v](N_1, mN_1 + N_2) = -\left[ m(\partial_v v)^2 + \frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2 \right],
\]
and
\[
Q[v](\nabla z, Y) = Q[v](N_2 - \frac{1}{2} z^2 V N_1, m N_1 + N_2) = (N_2 v)^2 - \frac{1}{2} z^2 V m(\partial_v v)^2 + (m - \frac{1}{2} z^2 V)(\frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2)
\]
\[= (\partial_v v + g^{AB} \partial_A v)^2 + (m - \frac{1}{2} z^2 V)(\frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2)\]
\[- \frac{1}{2} z^2 V(m - \frac{1}{2} z^2 V)(\partial_v v)^2 + \frac{1}{2} z^2 V \partial_v v \partial_v v + \frac{1}{2} z^2 V g^{AB} \partial_A v \partial_B v.
\]
(4.22)

As we know, the boundary hypersurface $\mathcal{T}$ is timelike, hence the energy density $Q[v](\nabla z, Y)$ is not positive on $\mathcal{T}$, one can also see this from (4.22). From the boundary data which is given on $\mathcal{T}$, then $v$, $\partial_v v$ and $\nabla \phi v$ are considered to be known functions, but the boundary data does not yield any information on $\partial_v v|_{\mathcal{T}}$. Hence, we will choose appropriate $m$, such that the coefficient of the term $(\partial_v v)^2$ in (4.22) has a good sign, so that this term can be controlled. In fact, by choosing $m$ large, such that $m - \frac{1}{2} z^2 V \geq 1$ on $\mathcal{T} \cap \Omega_1$, we get
\[
q \int_{\Omega_1 \cap \Omega_1} h[(\partial_v v)^2 + (\partial_v v)^2 + g^{AB} \partial_A v \partial_B v + v^2] \mathrm{d}\Omega + \int_{\mathcal{T} \cap \Omega_1} h(\partial_v v)^2 \mathrm{d}r \mathrm{d}S_{\mathcal{S}}
\]
\[+ \int_{\mathcal{T} \cap \Omega_1} h[(\partial_v v)^2 + g^{AB} \partial_A v \partial_B v + v^2] \mathrm{d}z \mathrm{d}S_{\mathcal{S}} + \int_{\mathcal{T} \cap \Omega_1} hQ[v](L, Y) \mathrm{d}\mathcal{H} \leq C \left\{ \int_{\Omega_1 \cap \Omega_1} h f^2 \mathrm{d}\Omega + \int_{\mathcal{S}_0} h[(\partial_v v)^2 + g^{AB} \partial_A v \partial_B v + v^2] \mathrm{d}z \mathrm{d}S_{\mathcal{S}}
\]
\[+ \int_{\mathcal{T} \cap \Omega_1} h[(\partial_v v)^2 + g^{AB} \partial_A v \partial_B v + v^2] \mathrm{d}r \mathrm{d}S_{\mathcal{S}} \right\},
\]
where $C$ is a positive constant depending on $\lambda_1$, $\lambda_2$, $|d|_{L^\infty(\Omega_1 \cap \Omega_1)}$, $|d|_{L^\infty(\Omega_1 \cap \Omega_1)}$, $|b|_{L^\infty(\Omega_1 \cap \Omega_1)}$, $\inf_{\mathcal{T} \cap \Omega_1} \{z^2 V\}$.

Finally, we complete the proof by remark 4.5, (4.1) and the boundedness of $\eta$. □

We have another version of $H^1$-estimates. In fact, by (4.22) we take $m$ small enough, such that $m - \frac{1}{2} z^2 V \leq 0$ on $\mathcal{T} \cap \Omega_1$, hence the sign of the coefficient of the term $\frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2$ in (4.22) is negative, then we have,
\[
q^\frac{1}{2} \| v \|_{H^1(\Omega_1 \cap \Omega_1)} + \| \partial_v v \|_{L^2(\mathcal{T} \cap \Omega_1)} + \| v \|_{H^1(\mathcal{S}_0 \cap \Omega_1)} + \left( \int_{\mathcal{T} \cap \Omega_1} h E_h[v] \mathrm{d}\mathcal{H} \right)^{\frac{1}{2}} \leq C \left\{ \| v \|_{H^1(\mathcal{S}_0)} + \| N_2 v \|_{L^2(\mathcal{T} \cap \Omega_1)} + \| f \|_{L^2(\mathcal{T} \cap \Omega_1)} \right\},
\]
(4.23)
where $C$ is a positive constant depending on $\lambda_1, \lambda_2$, $|g^{ij}|_{C^1(\Omega_T \cap M_1)}$, $|\alpha^i|_{L^\infty(\Omega_T \cap M_1)}$, $|b|_{L^\infty(\Omega_T \cap M_1)}$ and $\inf_{T \cap \Omega_T} \{z^2V\}.

**Corollary 4.7.** Suppose $\varphi \in H^1(\Sigma_0)$ and $\psi_1 \in H^1(\partial T \cap \Omega_T)$. Let $v$ be a $C^2(\Omega_T \cap M_1)$-solution of (4.2) and (4.3). Then,

$$
\|v\|_{H^1(\partial T \cap M_1)} + \|\partial^a v\|_{L^2(\partial T \cap M_1)} + \left( \int_{\partial T \cap M_1} E_H[v] d\mathcal{H} \right)^{1/2} \leq C \left\{ \|\varphi\|_{H^1(\Sigma_0)} + \|\psi_1\|_{H^1(\partial T \cap M_1)} \right\},
$$

where $C$ is a positive constant depending on $R, T, \lambda_1, \lambda_2$, $|g^{ij}|_{C^1(\Omega_T \cap M_1)}$, $|\alpha^i|_{L^\infty(\Omega_T \cap M_1)}$ and $\inf_{T \cap \Omega_T} \{z^2V\}.

### 4.3. Higher order estimates in $\Omega_T \cap M_1$

In this subsection, we derive $H^k$-estimates of $v$, for $k \geq 2$. To this end, we need to differentiate the equation (4.2). In fact, let $X$ be a vector field with a deformation tensor $\pi = (X)_\pi$, then (see [31]),

$$
[X, \partial] = \pi^{\alpha \beta} \nabla^2 \phi_{\alpha \beta} + \nabla_\alpha \pi^{\alpha \beta} \partial_\beta \phi - \frac{1}{2} \theta^{\alpha \beta} (\text{tr} \pi) \partial_\alpha \phi.
$$

In the following, we denote the multi-indices $\alpha, \beta \in \mathbb{Z}_+^d$ by $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$, etc. Take an arbitrary multi-index $\alpha$ with $|\alpha| = p$, then

$$
\Box_p \partial^p v = \sum_{|\beta| = p+1} c_{\alpha \beta} \partial^\beta v + f_\alpha,
$$

where

$$
f_\alpha = \sum_{|\beta| \leq p} c_{\alpha \beta} \partial^\beta v.
$$

We attempt to apply the derived weighted $H^1$-estimate to (4.24) and get an estimate of the $H^1$-norm of $\partial^p v$. There are two issues we need to resolve. First, the right hand contains derivatives of $v$ of order $|\alpha| + 1$, not all of which can be written as $\partial_i \partial^p v$, for some $i = 0, 1, 2, 3$. If we simply apply the derived $H^1$-estimates, there are derivatives of order $|\alpha| + 1$ in the right-hand side, which are not yet controlled. Second, we need to determine the initial values of $\partial^p v$ on $\Sigma_0$ and the boundary values on $\Sigma_1$.

**Theorem 4.8.** For any integers $p \geq 1$, suppose $\varphi \in H^{2p+1}(\Sigma_0)$ and $\psi_1 \in H^{2p+1}(\partial T \cap \Omega_T)$. Let $v$ be a $C^{2p+1}(\Omega_T \cap M_1)$-solution of (4.2) and (4.3). Then,

$$
\|v\|_{H^{2p+1}(\partial T \cap M_1)} + \sum_{|\alpha| = p+1} \|\partial^\alpha v\|_{L^2(\partial T \cap M_1)} + \sum_{|\beta| = p} \left( \int_{\partial T \cap M_1} E_H[\partial^\beta v] d\mathcal{H} \right)^{1/2} \leq C \left\{ \|\varphi\|_{H^{2p+1}(\Sigma_0)} + \|\psi_1\|_{H^{2p+1}(\partial T \cap M_1)} \right\},
$$

where $C$ is a positive constant depending only on $p, R, T, \lambda_1, \lambda_2$, $|g^{ij}|_{C^{p+1}(\Omega_T \cap M_1)}$, $|\alpha^i|_{L^\infty(\Omega_T \cap M_1)}$, and $\inf_{T \cap \Omega_T} \{z^2V\}.
Proof. Step 1. Main estimates. Take an arbitrary multi-index $\alpha$ with $|\alpha| = p$, and applying theorem 4.6 to (4.24), we have,

$$
q^2 \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{1})} + \| \partial_r \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{1})} + \left( \int_{\mathcal{H} \cap \Omega_{\gamma}} h E_{\mathcal{H}}[\partial^\alpha v] d\mathcal{H} \right)^{\frac{1}{2}} \\
\leq C \left\{ \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{0})} + \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{1})} + \sum_{|\beta|=p+1} \| \partial^\beta v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \| f_{\alpha} \|_{L^2(\Omega_{\gamma} \cap M_{1})} \right\}.
$$

(4.27)

Take a fixed $r$ with $0 \leq r \leq p$ and consider $\alpha$ with $|\alpha| = p$ and $\alpha_1 = r$. Then by (4.27), we have,

$$
\sum_{|\alpha|=p+1, \alpha_1=r+1} \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{r})} \leq C \left\{ M + \sum_{|\alpha|=p+1, \alpha_1=r} \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{r})} \right\},
$$

(4.28)

where

$$
M = \sum_{|\alpha|=p} \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{0})} + \sum_{|\alpha|=p+1, \alpha_1=0} \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \sum_{|\beta|=p+1} \| \partial^\beta v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \sum_{|\alpha|=p} \| f_{\alpha} \|_{L^2(\Omega_{\gamma} \cap M_{1})}.
$$

A simple iteration of (4.28) on $r = 0, 1, \ldots, p$ yields

$$
\sum_{|\alpha|=p+1} \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{r})} \leq CM.
$$

(4.29)

By (4.27) and (4.29), we get

$$
\sum_{|\alpha|=p} \left[ q^2 \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \| \partial \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \left( \int_{\mathcal{H}} h E_{\mathcal{H}}[\partial^\alpha v] d\mathcal{H} \right)^{\frac{1}{2}} \right] \leq CM.
$$

(4.30)

Note the domain integrals of derivatives of $v$ of order $p+1$ in $M$ as the same as those on the left-hand side. By choosing $q$ sufficiently large, we can absorb those terms in $M$. For such a fixed $q$, we can remove $h$ from all integrals. Hence, we have

$$
\sum_{|\alpha|=p+1} \left[ \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{0})} + \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \sum_{|\beta|=p} \left( \int_{\mathcal{H}} E_{\mathcal{H}}[\partial^\beta v] d\mathcal{H} \right)^{\frac{1}{2}} \right] \leq C \left\{ \sum_{|\alpha|=p} \| \partial^\alpha v \|_{H^1(\Omega_{\gamma} \cap M_{0})} + \sum_{|\alpha|=p+1, \alpha_1=0} \| \partial^\alpha v \|_{L^2(\Omega_{\gamma} \cap M_{1})} + \sum_{|\alpha|=p} \| f_{\alpha} \|_{L^2(\Omega_{\gamma} \cap M_{1})} \right\}.
$$

(4.31)
A simple iteration on the last term on the right-hand side of (4.31), we have

\[
\sum_{|\alpha|=p+1} \left[ \| \partial^\alpha v \|_{L^2(\Omega \cap M_{1})} + \| \partial^\alpha v \|_{L^2(\mathcal{T} \cap \Omega_{1})} \right] + \sum_{|\beta|=p} \left( \int_{\mathcal{T}} E_{H}[\partial^\beta v] d\mathcal{H} \right)^{\frac{1}{2}} \\
\leq C \left\{ \sum_{|\alpha| \leq p} \| \partial^\alpha v \|_{H^1(\Omega_{0})} + \| v \|_{H^{p+1}(\mathcal{T} \cap \Omega_{1})} \right\}. \tag{4.32}
\]

Step 2. Estimates on \( \Sigma_{0} \). To complete the proof, we need to control the first term on the right-hand side of (4.32). In fact, we can prove that\(^{10}\) for any multi-index \( \alpha \) with \( |\alpha| = p \), and \( 0 \leq \alpha_{0} = l \leq p \), we have

\[
\| \partial^\alpha v \|_{L^2(\Sigma_{0})} \leq C \left\{ \| v \|_{H^{p+1}(\Sigma_{0})} + \| \psi \|_{H^{p+1}(\mathcal{T} \cap \Sigma_{0})} \right\}, \tag{4.33}
\]

where \( C \) is a positive constant depending only on \( p \), \( |g_{ij}| \leq C p \), \( l \), and \( |\omega| \leq C p \).

By restricting the equation (4.2) to \( \Sigma_{0} \), we have

\[
2 \partial_{l}(\partial_{z} v) + a^{0} \partial_{z} v = f_{l}, \tag{4.34}
\]

where

\[
f_{l} = \sum_{|\beta| \leq l+1, \alpha_{0} = 0} a_{\beta} \partial^\beta v.
\]

We point out that no derivatives of \( v \) with respect to \( \tau \) appear in \( f_{l} \). We view (4.34) as an ODE of \( \partial_{z} v \) in \( z \) on \( \Sigma_{0} \) with the initial value given by

\[
\partial_{z} v = \partial_{z} \psi_{1} \quad \text{on} \quad \Sigma_{0} \cap \mathcal{T}.
\]

Then,

\[
\partial_{z} v = \partial_{z} \psi_{1} e^{-\frac{1}{2} \int_{z_{0}}^{z} a_{0} dz'} - \frac{1}{2} \int_{z_{0}}^{z} f_{1} e^{\frac{1}{2} \int_{z_{0}}^{z'} a_{0} dz''} dz''
\]

on \( \Sigma_{0} \).

Therefore,

\[
\| \partial_{z} v \|_{L^2(\Sigma_{0})} \leq C \left\{ \| v \|_{H^{p+1}(\Sigma_{0})} + \| \partial_{z} \psi \|_{L^2(\Sigma_{0} \cap \mathcal{T})} \right\}. \tag{4.35}
\]

For \( l \geq 2 \), by applying \( \partial_{z}^{l-1} \) to (4.34), we obtain

\[
2 \partial_{z}(\partial_{z}^{l} v) + a^{0} \partial_{z}^{l} v = f_{l},
\]

where

\[
f_{l} = \sum_{|\beta| \leq l+1, \alpha_{0} \leq l-1} c_{\beta} \partial^\beta v.
\]

Similarly, we view this as an ODE of \( \partial_{z}^{l} v \) in \( z \) on \( \Sigma_{0} \) with the initial value given by

\[
\partial_{z}^{l} v = \partial_{z}^{l} \psi_{1} \quad \text{on} \quad \Sigma_{0} \cap \mathcal{T}.
\]

\(^{10}\) The prove same as in [17].
Then,
\[ \partial_t^j v = \partial_t^j \psi_1 e^{-\frac{j}{2} L_0^\alpha} - \frac{1}{2} \int_x^T \frac{1}{2} f e^{-\frac{j}{2} L_0^\alpha} dz' \]
\[ \text{on} \ \Sigma_0. \]

For \( \alpha = (l, \alpha_1, \alpha_2, \alpha_3) \), we write \( \alpha' = (0, \alpha_1, \alpha_2, \alpha_3) \). Then,
\[ \partial^\alpha v = \partial^\alpha' \left\{ \partial_t^j \psi_1 e^{-\frac{j}{2} L_0^\alpha} - \frac{1}{2} \int_x^T \frac{1}{2} f e^{-\frac{j}{2} L_0^\alpha} dz' \right\} \]
\[ \text{on} \ \Sigma_0. \]

Hence, for \( \alpha \) with \( |\alpha| = p \) and \( \alpha_0 = l' \),
\[ \| \partial^\alpha v \|_{L^2_0(\Sigma_0)} \leq C \left\{ \| \partial_t^j \psi_1 \|_{H^{p+1} (\Sigma_0 \cap T)} + \sum_{|\beta| \leq p} \| \partial^\beta v \|_{L^2_0(\Sigma_0)} \right\}. \]

By the trace theorem, we have
\[ \| \partial^\alpha v \|_{L^2_0(\Sigma_0)} \leq C \left\{ \| \psi_1 \|_{H^{p+1} (\Sigma_0 \cap T)} + \sum_{|\beta| \leq p} \| \partial^\beta v \|_{L^2_0(\Sigma_0)} \right\}. \]

We note that in the summation above, the highest degree of derivatives increases by 1 but the highest degree of derivatives with respect to \( \tau \) decreases by 1. So we can iterate this inequality \( l \) times and obtain the desired result.

We substitute (4.33) in (4.32), then we finish the prove. \( \square \)

4.4. Energy estimates in \( \Omega_T \cap \mathcal{M}_2 \)

In this subsection, we will get the energy estimates in \( \mathcal{M}_2 \cap \Omega_T \). First, we have following weighted energy estimates in \( \Omega_T \cap \mathcal{M}_2 \).

**Theorem 4.9.** For some fixed \( T > 0 \), and \( v \) is a solution of (4.12). There exists constants \( q_0 > 0 \) and \( l > 0 \) depending on \( \lambda_1, \lambda_2, |g^{ij}|_{\mathcal{C}_0(\Omega_T \cap \mathcal{M}_2)}, \ |a'|_{L^{\infty}(\Omega_T \cap \mathcal{M}_2)} \) and \( |b|_{L^{\infty}(\Omega_T \cap \mathcal{M}_2)} \), such that for any \( q \geq q_0 \) and \( h = e^{-h} + q \), we have
\[ q^\frac{1}{2} \| v \|_{H^l_0(\Omega_T \cap \mathcal{M}_2)} + \| v \|_{H^l_0(\Sigma_0 \cap \Omega_T)} + \| \partial_t v \|_{L^2_0(\Sigma_0 \cap \Omega_T)} \]
\[ \leq C \left\{ \left( \int_{R^+ \cap T} h E_h[v] dH \right)^{\frac{1}{2}} + \| v \|_{H^l_0(\Sigma_0 \cap \Omega_T)} + \| f \|_{L^2_0(\mathcal{M}_2 \cap \Omega_T)} \right\}, \]
\[ (4.36) \]

where \( C \) is a positive constant depending only on \( \lambda_1, \lambda_2, |g^{ij}|_{\mathcal{C}_0(\Omega_T \cap \mathcal{M}_2)}, |a'|_{L^{\infty}(\Omega_T \cap \mathcal{M}_2)}, |b|_{L^{\infty}(\Omega_T \cap \mathcal{M}_2)} \) and \( -\Delta \).

**Proof.** The proof is similar to theorem 4.6, so we just point out the key differences in the prove. We also take \( Y = mN_1 + N_2 \) and \( h = e^{-h} + q \), then we have,
\[ \int_{\mathcal{M}_2 \cap \Omega_T} Q[v](\nabla h, Y) + \frac{1}{2} h Q[v]_{\alpha_3} (\nabla r)^{\alpha_3} - 2g(Y, \nabla v)h + (f - a^\alpha \partial_4 v - b v)h(Yv) \]
\[ = \int_{\mathcal{M}_2 \cap \Sigma_T} h Q[v](\nabla \tau, Y)e^{2h} d\sigma g^2 + \int_{\Omega_T \cap \Omega_T} h Q[v](L, Y) dH \]
\[ - \int_{\Omega_T \cap \Omega_T} h Q[v](\nabla z, Y)e^{2h} d\sigma g^2. \]
\[ (4.37) \]
The domain integrals on the left-hand side of (4.37) and the first term on right-hand side of (4.37) can be analyzed in the same way as in theorem 4.6. Hence, we only need to analyze the last term on the right-hand side of (4.37). In fact, we have $N_2 = \partial_r - \frac{A}{6} \partial_z$ on $\mathcal{I}$, hence

$$Q[v](\nabla z, Y) = Q[v](N_2 + \frac{A}{6} N_1, m N_0 + N_2)$$

$$= (N_2 v)^2 + \frac{A}{6} m (\partial_r v)^2 + (m + \frac{A}{6}) \frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2]$$

$$= (\partial_r v)^2 + \frac{A}{6} (\partial_z v)^2 + (m + \frac{A}{6}) \frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2] - \frac{A}{3} \partial_r v \partial_z v$$

$$\geq - (\partial_r v)^2 + \frac{A}{6} (\partial_z v)^2 + (m + \frac{A}{6}) \frac{1}{2} g^{AB} \partial_A v \partial_B v + v^2]. \quad (4.38)$$

We take $m = -\frac{A}{27}$, then we can get the desired result. \hfill \square

**Corollary 4.10.** Suppose $\psi_2 \in H^1(\mathcal{I} \cap \Omega_T)$. Let $v$ be a $C^2(\Omega_T \cap \mathcal{M}_2)$-solution of (4.2) and (4.3). Then,

$$\|v\|_{H^1(\mathcal{I} \cap \Omega_T)} + \|\partial_r v\|_{L^2(\mathcal{I} \cap \Omega_T)} \leq C \left\{ \left( \int_{\mathcal{H}^2} E_H[v] \|d\mathcal{H} \right)^{\frac{1}{2}} + \|\psi_2\|_{H^2(\mathcal{I} \cap \Omega_T)} \right\},$$

where $C$ is a positive constant depending on $R$, $T$, $\lambda_1$, $\lambda_2$, $|g^{ij}|_{C^1(\mathcal{I} \cap \mathcal{M}_1)}$, $|\omega|_{L^\infty(\mathcal{I} \cap \mathcal{M}_1)}$ and $-\Lambda$.

Similarly, we have higher-order energy estimates in $\Omega_T \cap \mathcal{M}_2$.

**Theorem 4.11.** For any integers $p \geq 2$, suppose $\psi_2 \in H^p(\mathcal{I} \cap \Omega_T)$. Let $v$ be a $C^p(\Omega_T \cap \mathcal{M}_1)$-solution of (4.2) and (4.3). Then,

$$\|v\|_{H^p(\mathcal{I} \cap \Omega_T)} + \sum_{|\alpha| = p} \|\partial_\alpha v\|_{L^2(\mathcal{I} \cap \Omega_T)}$$

$$\leq C \left\{ \sum_{|\alpha| = p} \left( \int_{\mathcal{H}^2} E_H[\partial_\alpha v] \|d\mathcal{H} \right)^{\frac{1}{2}} + \|\psi_2\|_{H^2(\mathcal{I} \cap \Omega_T)} \right\}, \quad (4.39)$$

where $C$ is a positive constant depending only on $p$, $R$, $T$, $\lambda_1$, $-\Lambda \lambda_2$, $|g^{ij}|_{C^p(\mathcal{I} \cap \mathcal{M}_2)}$ and $|\omega|_{C^{p-1}(\mathcal{I} \cap \mathcal{M}_2)}$.

Following form theorems 4.8 and 4.11, we have

**Corollary 4.12.** For any integers $k \geq 1$, suppose $\varphi \in H^{2k-1}(\mathcal{S}_0)$, $\psi_1 \in H^{2k-1}(\mathcal{J} \cap \Omega_T)$ and $\psi_2 \in H^k(\mathcal{I} \cap \Omega_T)$. Let $v$ be a $C^{2k-1}(\Omega_T)$-solution of (4.2) and (4.3), then,

$$\|v\|_{H^k(\mathcal{I} \cap \Omega_T)} \leq C \left\{ \|\varphi\|_{H^{2k-1} (\mathcal{S}_0)} + \|\psi_1\|_{H^{2k-1}(\mathcal{J} \cap \Omega_T)} + \|\psi_2\|_{H^k(\mathcal{I} \cap \Omega_T)} \right\}, \quad (4.40)$$

where $C$ is a positive constant depending only on $k$, $R$, $T$, $-\Lambda$, $\lambda_1$, $\lambda_2$, $|g^{ij}|_{C^{2k-1}(\mathcal{I} \cap \Omega_T)}$, $|\omega|_{C^{2k-2}(\mathcal{I} \cap \Omega_T)}$ and $\inf_{\mathcal{J} \cap \Omega_T} \{z^2 V\}$. 

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5. Existence of solutions

In this section, we will consider the existence of solution of the initial-boundary problem (4.2) and (4.3). We first prove the local existence of null-timelike problems. As the same as in [17], we first prove the existence for the analytic case following the method in [28], and then obtain the existence for general case by approximations.

Now we prove the local existence near $\Sigma_0 \cap T$.

**Theorem 5.1** (Local existence of null − timelike problems). Let $U \subset M$ be a neighborhood of $\Sigma_0 \cap T$, suppose $g^{ij} \in C^{2k-1}(U \cap M)$, $\omega \in C^{2k-2}(U \cap M)$, $\varphi \in H^{2k-1}(\Sigma_0 \cap U)$ and $\psi \in H^{2k-1}(T \cap U)$. Then, there exists an interior point $O \in U \cap M$, and the causal past of $O$ is denoted by $J^-(O)$, such that, there exists a unique solution $v \in H^k(J^-(O) \cap U \cap M)$ of (4.2) satisfying $v|_{J^-(O) \cap \Sigma_0} = \varphi$ and $v|_{J^-(O) \cap T} = \psi$.

**Proof.** We write the equation in (4.2) in the form
\[
2 \partial_z v + z^2 V \partial_z v + 2 g^{1A} \partial_A v + g^{AB} \partial_B v + a^1 \partial_1 v + a^A \partial_A v + \omega v = 0
\]
(5.1)

We first assume that $g^{ij}, a^i, \omega$ and $\varphi, \psi$ are real analytic in $U$ and hence can be expanded as a power series in $\tau$. For example, we have
\[
\psi = \sum_{i=0}^{\infty} \psi_i(\theta) \tau^i.
\]

Here and hereafter, we denote by $\theta = (x^2, x^3)$ coordinates on $S^2$. We define
\[
u^0 = v = \sum_{i=0}^{\infty} u^0_i(z, \theta) \tau^i,
\]
\[
u^1 = \partial_z v = \sum_{i=0}^{\infty} u^1_i(z, \theta) \tau^i,
\]
\[
u^A = \partial_A v = \sum_{i=0}^{\infty} u^A_i(z, \theta) \tau^i, \quad A = 2, 3,
\]
and
\[
\omega = \partial_\tau v = \sum_{i=0}^{\infty} \omega_i(z, \theta) \tau^i.
\]

For $\nu^0, \nu^1, \nu^2, \nu^3$, and $\omega$, we have
\[
\partial_\tau \nu^0 = \omega,
\]
\[
2 \partial_z \nu^1 = -z^2 V \partial_z \nu^1 - 2 g^{1A} \partial_A \nu^A - g^{AB} \partial_B \nu^A - \omega \nu^0 - a^1 \nu^1 - a^A \nu^A - a^0 \omega,
\]
\[
\partial_\tau \nu^A = \partial_A \omega,
\]
and
\[
2 \partial_z \omega = -z^2 V \partial_z \nu^1 - 2 g^{1A} \partial_A \nu^A - g^{AB} \partial_B \nu^A - \omega \nu^0 - a^1 \nu^1 - a^A \nu^A - a^0 \omega.
\]

11 The prove same as in [17].
Therefore,

\[(i + 1)u^0_{i+1} = w_i,\]

\[2(i + 1)u^1_{i+1} = \sum_{k \leq i} L_4[\partial u^0_k, \partial u^1_k, \partial u^2_k, \partial u^3_k, w_k],\]

\[(i + 1)u^A_{i+1} = \partial_A w_i,\]  

and

\[2\partial_z w_i + dw_i = \sum_{k \leq i} L_4[\partial u^0_k, \partial u^1_k, \partial u^2_k, \partial u^3_k] + \sum_{k \leq i-1} L_4[w_k].\]

Note

\[u^0_0 = \varphi, \quad u^1_0 = \partial_z \varphi, \quad u^A_0 = \partial_A \varphi.\]

The equation (5.3) is an ODE of \(w_i\) with respect to \(z\) and the initial value is given by

\[w_i(\frac{1}{R}) = (i + 1)\psi_{i+1}\] on \(S^2\).

For some \(i \geq 0\), assume we already know \(u^l_0, \ldots, u^l_i\), for \(l = 0, 1, 2, 3\), and \(w_0, \ldots, w_{i-1}\), then we can find \(w_i\) by solving (5.3) and find \(u^l_{i+1}\), for \(l = 0, 1, 2, 3\), by (5.2).

For simplicity, we assume

\[u^0_0 = u^1_0 = u^2_0 = u^3_0 = 0, \quad w_0|_{z=\frac{1}{R}} = 0.\]

Otherwise, we set

\[\tilde{u}^0 = u^0 - \varphi, \quad \tilde{u}^1 = u^1 - \partial_z \varphi, \quad \tilde{u}^A = u^A - \partial_A \varphi, \quad \tilde{w} = w - \partial_z \psi.\]

We now consider a given point on \(\Sigma_0 \cap T\), say \((0, \frac{1}{R}, 0, 0)\). Set

\[s = a^2 \tau + a(z - \frac{1}{R}) + x_2 + x_3.\]

In a neighborhood of \((0, \frac{1}{R}, 0, 0)\), take \(K > 0, \rho > 0, \text{ and } a > 1\), such that the function

\[F(s) = \frac{K}{1 - \rho^{-1} \left( a^2 \tau + a(z - \frac{1}{R}) + x_2 + x_3 \right)}\]

is a majorizing function of \(g^0, a, \omega \text{ and } \varphi, \psi\). Then,

\[\partial_z u^l = F(s) \{ \partial_z u^l + \sum_{A,B=2,3} [\partial_A u^A + \partial_B u^B + \partial_A w] + \sum_{j=0}^3 u^j + w + 1 \},\]

\[\partial_z w = F(s) \{ \partial_z u_0^l + \sum_{A,B=2,3} [\partial_A u^A + \partial_B u^B] + \sum_{j=0}^3 u^j + w + 1 \}.\]
forms a majorizing system. We now treat \( s \) as an independent variable. To construct a special solution \( u' = U(s) \), \( i = 0, 1, 2, 3 \), and \( w = W(s) \) of (5.4), we consider a system of linear ordinary differential equations given by

\[
\begin{aligned}
\{a^2 - F(s)(3a + 4)\} \frac{dU}{ds} - 2F(s) \frac{dW}{ds} &= F(s)(4U + W + 1), \\
-F(s)(3a + 4) \frac{dU}{ds} + a \frac{dW}{ds} &= F(s)(4U + W + 1),
\end{aligned}
\]

(5.5)

with \( U(0) = W(0) = 0 \). Take \( \lambda \) small such that

\[
\begin{bmatrix}
a^2 - F(s)(3a + 4) & -2F(s) \\
-F(s)(3a + 4) & a
\end{bmatrix}
\]

is positive definite. Then, we can solve (5.5) and its solutions \( U \) and \( W \) are real analytic in the domain of \( F(s) \). Therefore, the domain where \( u' \) and \( w \) exist and are real analytic is the same as the domain where all coefficients and initial values are real analytic. Similarly as in [28], by \( a^2 - F(s)(3a + 4) > 0 \), \( -2F(s) < 0 \), \( -F(s)(3a + 4) < 0 \), \( a > 0 \), the coefficients in the series of \( U(s) \) and \( W(s) \) are nonnegative, provided that \( U(0) \) and \( W(0) \) are 0. This proves the existence of an analytic solution \( v \) in \( U \).

Take an interior point \( O \in U \cap M \), such that we can find analytic sequences \( g_j \) and \( \omega_j \) which satisfy \( \lim_{j \to \infty} g_j = g \) and \( \lim_{j \to \infty} \omega_j = \omega \) uniformly in \( C^{2k-1}(\overline{J}(O) \cap U \cap M) \)-norm. And we can find sequences of polynomials \( P_j \) and \( Q_j \), such that \( \lim_{j \to \infty} P_j = \varphi \) in \( H^{2k}(\overline{J}(O) \cap \Sigma_0) \)-norm, and \( \lim_{j \to \infty} Q_j = \psi \) in \( H^{2k}(\overline{J}(O) \cap T) \)-norm.

Denote by \( v' \) the solution of \( \Box_g v_j + \omega_j v_j = 0 \), with the initial value and the boundary value given by \( P_j \) and \( Q_j \), respectively. By the \( H^2 \)-estimates provided by theorem 4.8, we find that the \( v' \) converges, as \( j \to \infty \), to a solution \( v \in H^4(\overline{J}(O) \cap U \cap M) \) of (4.2) with the initial value and the boundary value given by \( \varphi \) and \( \psi \), respectively.

We state the local existence theorem of the characteristic initial value problem for wave equations, see [33].

**Theorem 5.2** (Local existence of characteristic initial value problem). Let \( N_1, N_2 \) be the transversely intersecting null hypersurfaces with respect to \( g \), and \( \Omega \) represent the region bounded by \( N_1, N_2 \). Let \( \varphi \) smooth on \( N_1 \), \( \psi \) smooth on \( N_2 \), and \( \varphi = \psi \) on \( N_1 \cap N_2 \). Then there exists an open neighborhood \( \mathcal{N}_1 \cap \mathcal{N}_2 \), and a unique \( v \in C^\infty(U \cap \Omega) \) solve (4.2) and satisfies the boundary condition \( v|_{N_1 \cap \Omega} = \varphi \) and \( v|_{N_2 \cap \Omega} = \psi \).

We now prove the existence of solution in \( \Omega_T \cap M_1 \). We define

\[
S_t = \{ (\tau, z, \theta) | \tau = \tau(z), \theta \in S^2 \} \cap \Omega_T \cap M_1, \quad t \geq 0,
\]

where the function \( \tau(z), z \in [0, \frac{1}{R}] \) satisfies \( \tau(z) = t \) and \( 0 < \tau'(z) \leq \frac{1}{2z} \left( \tau(z) - \frac{1}{R} \right) \). In fact, \( S_t \) is a spacelike hypersurface.

**Remark 5.3.** Following from the proof of \( H^2 \)-estimates in \( \Omega_T \cap M_1 \), we can easily prove

\[
\sum_{|\alpha| = k} \| \partial^\alpha v \|_{L^2(S_t)} \leq C \left\{ \| \varphi \|_{H^{2k-1}(\Sigma_0)} + \| \psi \|_{H^{2k-1}(\tau \cap \Omega_T)} \right\},
\]

(5.6)

where \( C \) is a positive constant depending only on \( k, R, T, \lambda_1, \lambda_2, |g^{ij}|_{C^{2k-1}(\Omega_T \cap M_1)} \) and \( |\omega|_{C^{2k-2}(\Omega_T \cap M_1)} \) and \( \inf_{T \cap \Omega_T} \{ z^2 V \} \).
Theorem 5.4 (Existence of solution in $\mathcal{M}_1 \cap \Omega_T$). For some integer $k \geq 1$, suppose $\varphi \in H^{2k-1}(\Sigma_0)$ and $\psi_1 \in H^{2k-1}(\tau \cap \Omega_T)$. Then, there exists a unique solution $v \in H^k(\mathcal{M}_1 \cap \Omega_T)$ of (4.2) and (4.3) in $\Omega_T \cap \mathcal{M}_1$.

Proof. By theorem 5.1 there exists solution near $\Sigma_0 \cap \tau$, hence, there exists $\epsilon_0$ sufficiently small such that the solution exists in $\bigcup_{l < \epsilon} S_l$. Suppose we does not have a global solution in $\mathcal{M}_1 \cap \Omega_T$, and then let

$$t^* = \sup \left\{ t : \text{the solution exists in } \bigcup_{l < t} S_l \right\}.$$ 

We take $\epsilon$ sufficiently small, then solution exists in $\bigcup_{l \leq t^* - \epsilon} S_l$. By remark 5.3, we have

$$\sum_{|\alpha|=k} \left\| \partial^\alpha v \right\|_{L^2(S_{t^* - \epsilon})} \leq C \left\{ \left\| \varphi \right\|_{H^{2k-1}(\Sigma_0)} + \left\| \psi_1 \right\|_{H^{2k-1}(\tau \cap \Omega_T)} \right\}, \tag{5.7}$$

where $C$ does not depend on $t^*$ and $\epsilon$.

There are three cases of $t^*$ (see figure 2). For case I, First, by the standard theory of linear wave equations, there exists local solution for (4.2) in $\mathcal{D}^+(S_{t^*-\epsilon})$ the domain of dependence of $S_{t^*-\epsilon}$. And then, by the local existence of null-timelike problem, there exists local solution in the domain which is bounded by $\tau$ and $\partial(\mathcal{D}^+(S_{t^*-\epsilon}))$. Last, by the local existence of characteristic initial value problem, there exists local solution in the region which is bounded by $\partial(\mathcal{D}^+(S_{t^*-\epsilon}))$ and $\Sigma_0$. Hence, for the $\epsilon$ sufficiently small, there exists $\epsilon' > \epsilon$, such that the solution exists in $\bigcup_{l \leq t^* - \epsilon'} S_l$. For case II, we only need the standard theory and the local existence of characteristic initial value problem. For case III, by the standard theory we can get the same result. Hence, we have global existence solution in $\Omega_T \cap \mathcal{M}_1$. \hfill $\square$
Following from theorems 5.4 and 4.8, then there exists solution \( v \in H^p(H \cap \Omega_T) \) on \( H \cap \Omega_T \). And then by the same method, we can prove the existence of solution in \( M_2 \cap \Omega_T \). Hence, we have

**Theorem 5.5.** For some integer \( k \geq 1 \), suppose \( \varphi \in H^{2k-1}(\Sigma_0) \), \( \psi_1 \in H^{2k-1}(T \cap \Omega_T) \) and \( \psi_2 \in H^k(I \cap \Omega_T) \). Then there exists a unique solution \( v \in H^p(\Omega_T) \) of (4.2) and (4.3).

### 6. Discussion

In this paper, we consider the null-timelike boundary value problem of linear wave equation (2.5) in general asymptotic AdS spacetime. We show that the solution is globally existence and unique if the data is give on the time-like, null and conformal boundary. In fact, the main part of our paper is solving the conformal rescaled linear wave equation on the conformal compactificated spacetime. We obtain spacetime weighted energy estimates, and then by the local existence of solution for null-timelike boundary problem, we get the well-posedness of our problem.

As we have explained in introduction, the physical motivation of our paper is to understand AdS/CFT correspondence in terms of initial-boundary value problem. The wave equation (2.5) is a simple toy model of AdS/condense matter theory which is one of the most active branch of AdS/CFT correspondence. Another toy models of AdS/condense matter theory is the null-timelike boundary problem of Maxwell field in asymptotic AdS back ground. The null-timelike boundary problem of Maxwell field and spin-2 field in general asymptotic flat spacetime have been studied in [18]. Following our method in this paper, we also find similar result also exist for Maxwell field in asymptotic AdS spacetime and the paper is under preparing. In fact, the standard holographic model for condense matter theory contains scalar field, Maxwell field and gravity field [23], so we need to consider the coupled system of scalar field, Maxwell field and linearized gravity. This is a very important and interesting mathematical problem and will be considered in future work.

### Acknowledgment

This work is supported by the Natural Science Foundation of China (NSFC) under Grant Nos. 11575286 and 11731001.

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