PENCIL-BASED ALGORITHMS FOR TENSOR RANK DECOMPOSITION
ARE NOT STABLE

CARLOS BELTRÁN, PAUL BREIDING, AND NICK VANNIEUWENHOVEN

Abstract. We prove the existence of an open set of \( n_1 \times n_2 \times n_3 \) tensors of rank \( r \) on which a popular and efficient class of algorithms for computing tensor rank decompositions based on a reduction to a linear matrix pencil, typically followed by a generalized eigendecomposition, is arbitrarily numerically forward unstable. Our analysis shows that this problem is caused by the fact that the condition number of the tensor rank decomposition can be much larger for \( n_1 \times n_2 \times 2 \) tensors than for the \( n_1 \times n_2 \times n_3 \) input tensor. Moreover, we present a lower bound for the limiting distribution of the condition number of random tensor rank decompositions of third-order tensors. The numerical experiments illustrate that for random tensor rank decompositions one should anticipate a loss of precision of a few digits.

1. Introduction

We study the numerical stability of one of the most popular and effective class of algorithms for computing the tensor rank decomposition, or canonical polyadic decomposition (CPD), of a tensor. Recall that a rank-1 tensor is represented by a multidimensional \( n_1 \times n_2 \times \cdots \times n_d \) array \( \mathcal{B} = (b_{i_1 i_2 \cdots i_d})_{1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d} \) whose elements satisfy the following property:

\[
b_{i_1 i_2 \cdots i_d} = b_1^{(1)} i_2^{(2)} \cdots b_d^{(d)},
\]

where \( b_i^{(k)} \) is the tensor rank decomposition for computing the petitividad under projects MTM2017-83816-P and MTM2017-90682-REDT (Red ALAMA), as well as by the Postdoctoral Fellowship of the Research Foundation–Flanders (FWO).

For brevity, one writes \( \mathcal{B} = b^1 \otimes b^2 \otimes \cdots \otimes b^d \). The CPD of \( \mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d} \) was proposed by Hitchcock [26]. It expresses \( \mathcal{A} \) as a minimum-length linear combination of rank-1 tensors:

\[
\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_r, \quad \text{where } \mathcal{A}_i = a_i^1 \otimes a_i^2 \otimes \cdots \otimes a_i^d \text{ and } a_i^k \in \mathbb{R}^{n_k}
\]

for all \( i = 1, \ldots, r \) and \( k = 1, \ldots, d \). The number \( r \) in (1.1) is called the rank and \( d \) is the order of \( \mathcal{A} \). It is often convenient to consider the factor matrices \( A_1, \ldots, A_d \), where \( A_k := [a_i^k]_{i=1}^{R_k} \).

Mainly due to its simplicity and uniqueness properties [12, 30], the CPD has found application in a diverse set of scientific fields; see [8, 14, 15, 28, 29, 39, 40]. A rank-\( r \) tensor \( \mathcal{A} \) is called r-identifiable if the set of rank-1 tensors \( \{ \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \} \) whose sum is \( \mathcal{A} \), as in (1.1), is uniquely determined given \( \mathcal{A} \). A classic result states that the Kruskal rank \( k_M \) of a matrix \( M \): \( k_M \) is the largest integer \( k \) such that every subset of \( k \) columns of \( M \) has rank equal to \( k \).

Lemma 1.1 (Kruskal’s criterion). Let \( \mathcal{A} = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i \) be a tensor with factor matrices \( A = [a_i], B = [b_i], \text{ and } C = [c_i] \). A sufficient condition for the \( r \)-identifiability of \( \mathcal{A} \) is

\[
r \leq \frac{1}{2}(k_A + k_B + k_C - 2) \text{ and } k_A, k_B, k_C > 1.
\]

Most low-rank tensors satisfy Kruskal’s criterion; more precisely, there is an open dense subset of the set of rank-\( r \) tensors in \( \mathbb{R}^{n_1 \times n_2 \times n_3} \), \( n_1 \geq n_2 \geq n_3 \geq 2 \), where \( r \)-identifiability holds, provided that \( r \leq n_1 + \min\{\frac{3}{2}, \delta\} \) with \( \delta := n_2 + n_3 - n_1 - 2 \).

The computational problem of recovering the set of rank-1 tensors \( \{ \mathcal{A}_1, \ldots, \mathcal{A}_r \} \) whose sum is \( \mathcal{A} \) is called the tensor rank decomposition problem (TDP). When the rank of a third-order tensor is sufficiently small, there are efficient, numerical, direct algorithms for solving the TDP.
such as those in [18–20, 33, 34, 37, 38]. All of these algorithms involve the computation of a generalized eigendecomposition (GEVD) of a linear matrix pencil constructed from the low-rank input tensor. An algorithm for solving TDPs that involves such a reduction to a matrix pencil will subsequently be called a pencil-based algorithm (PBA). This will be given a precise meaning in Definition 5.1, where we rigorously define the class of PBAs.

A prototypical example of a PBA is presented next. The essential idea is to project a given tensor $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, $n_1 \geq n_2 \geq r$, to a tensor of format $n_1 \times n_2 \times 2$ and recover the first factor matrix from the latter. The input $A = \sum_{i=1}^r a_i \otimes b_i \otimes c_i$ is assumed to admit a unique rank-$r$ CPD with $\|a_i\| = 1$ for all $i = 1, \ldots, r$. Let $Q \in \mathbb{R}^{n_2 \times 2}$ be a matrix with orthonormal columns. Then, contracting $A$ along the third mode by $Q^T$, which is a special type of multilinear multiplication [17, 28], yields the tensor

$$\mathcal{B} = (I_{n_1}, I_{n_2}, Q^T) \cdot A := \sum_{i=1}^r a_i \otimes b_i \otimes z_i \in \mathbb{R}^{n_1 \times n_2 \times 2},$$

and $I_m$ denotes the $m \times m$ identity matrix. Let $Q_1 \in \mathbb{R}^{n_1 \times r}$, respectively $Q_2 \in \mathbb{R}^{n_2 \times r}$, be a matrix with orthonormal columns that form a basis for $\{a_i\}$, respectively $\{b_i\}$. The following is then a specific orthogonal Tucker decomposition [42] of $\mathcal{B}$:

$$\mathcal{B} := (Q_1, Q_2, I) \cdot S := \sum_{i=1}^r (Q_1 x_i') \otimes (Q_2 y_i') \otimes z_i,$$

where $x_i' = Q_1^T a_i$ and $y_i' = Q_2^T b_i$.

Let $X = [x_{i,j}]_{1 \leq i \leq r}$ and $Y = [y_{i,j}]_{1 \leq i \leq r}$. Then it follows from the properties of multilinear multiplication that the core tensor $S = (Q_1^T Q_2, I)$, $S \in \mathbb{R}^{r \times r \times 2}$ has the following two 3-slices:

$$S_j := ((I, e_j) Y^T \cdot S := \sum_{i=1}^r \lambda_{j,i} \cdot x_i \otimes y_i = \sum_{i=1}^r \lambda_{j,i} \cdot x_i y_i = X \text{diag} (\lambda_j) Y^T, \quad j = 1, 2,$$

where $\lambda_j := [z_i, ||x_i'||, ||y_i'||]_{i=1}$. Whenever $S_1$ and $S_2$ are nonsingular, we have

$$S_j S_j^{-1} = X \text{diag}(\lambda_j)^{-1} X^{-1};$$

thus $X$ is the matrix of eigenvectors of the GEVD of the nonsingular matrix pencil $(S_1, S_2)$. As long as the eigenvalues are distinct, the matrix $X$ is uniquely determined and it follows that $A = Q_1 X$. Finally, the rank-1 tensors $\mathcal{A}_i = a_i \otimes b_i \otimes c_i$ are recovered by the following well-known property [28, 39] of the 1-flattening: $\mathcal{A}_{(1)} = A(B \circ C)^T$, where $M \circ N := [m_i \otimes n_i]_{i=1}^r \in \mathbb{R}^{mn \times r}$ is the Khatri–Rao product of $M \in \mathbb{R}^{m \times r}$ and $N \in \mathbb{R}^{n \times r}$. Then, we see that

$$A \circ (A^T \mathcal{A}_{(1)})^T = A \circ (B \circ C) = A \circ B \circ C = [\mathcal{A}_1 \mathcal{A}_2 \cdots \mathcal{A}_r],$$

where $X^T$ is the Moore–Penrose pseudoinverse of $X$. This procedure thus solves the TDP.

The above algorithm and those in [18–20, 33, 34, 37, 38] have the major advantage that the CPD can be computed via a sequence of numerically stable and efficient linear algebra algorithms for solving classic problems such as linear system solving, linear least-squares and generalized eigendecomposition problems. In light of the plentiful indications that computing a CPD is a difficult problem—the NP-completeness of tensor rank [27], the ill-posedness of the corresponding approximation problem [17], and the potential (average) ill-conditioning of the TDP [4, 5]—the existence of aforementioned algorithms is almost too good to be true. We show that there is a price to be paid in the currency of the achievable precision by establishing the following result.

**Theorem 1.2.** Let $n_1 \geq n_2 \geq n_3 > r+1 \geq 2$. For every pencil-based algorithm, there exists an open set of the rank-$r$ tensors in $\mathbb{R}^{n_1 \times n_2 \times n_3}$ for which it is unstable.

The instability in the theorem is with respect to the standard model of floating-point arithmetic [25], namely

$$fl(a) = (1 + \delta) a \quad \text{and} \quad fl(a \circ b) = (1 + \delta) (a \circ b), \quad |\delta| \leq \epsilon_u, \quad \text{where} \quad \circ \in \{+, -, \cdot, /\},$$

where $fl(a)$ denotes the floating-point representation of $a$, and $\epsilon_u$ is the unit roundoff. In IEEE double-precision floating-point arithmetic $\epsilon_u \approx 1.11 \cdot 10^{-16}$ [25, Chapter 2].

In practice, Theorem 1.2 covers the algorithms from [20, 33, 34, 37, 38], cpd_gevd from Tensorlab v3.0 [45], [18, Algorithm 2], and the foregoing prototypical PBA. Algorithm 1 of [18], as well as both algorithms in [19], are likely also unstable because they use an unstable algorithm in intermediate steps; a more thorough analysis would be required to show this rigorously.
Remark 1.3. For higher-order tensors $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ with $d \geq 4$ it is a common practice to reshape them into a third-order tensor $\mathcal{A}_{(j,k,l)} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by choosing a partition of the indices $\{1, \ldots, d\} = \{j_1, \ldots, j_d\} \cup \{k_1, \ldots, k_2\} \cup \{l_1, \ldots, l_u\}$ with $m_1 = j_1 \cdots j_d$, $m_2 = k_1 \cdots k_l$, and $m_3 = l_1 \cdots l_u$. Under the conditions of section 7 of [13], the CPD of $\mathcal{A}_{(j,k,l)}$, i.e., the set of rank-1 tensors, can be reshaped back into a set of order-$d$ tensors in $\mathbb{R}^{n_1 \times \cdots \times n_d}$ yielding the CPD of $\mathcal{A}$. According to Theorem 1.2 this strategy employs an unstable algorithm as intermediate step, so we should a priori expect that the resulting algorithm is also unstable. This can be proved rigorously for $u = \|l\| = 1$ by a slight generalization of the argument in Section 6. We leave a general proof as an open question.

It is important to mention that the stabilities of algorithms employed in the intermediate steps of a PBA are not the reason why PBAs are unstable. In the above prototypical PBA, all individual steps can be implemented using numerically stable algorithms, but the resulting algorithm is nevertheless unstable. The instability in Theorem 1.2 is caused by a large difference between the condition numbers of the TDPs in $\mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathbb{R}^{n_1 \times n_2 \times 2}$.

The condition number of the TDP was studied in [4]. Let us denote the set of $n_1 \times \cdots \times n_d$ tensors of rank 1 by $S$. This set is actually a smooth manifold, called the Segre manifold; see Section 4.1. Tensors of rank at most $r$ are obtained as the image of the addition map $\Phi_r : S^r \times \mathbb{R}^{n_1 \times \cdots \times n_d} \rightarrow S^r$, $(A_1, \ldots, A_r) \rightarrow A_1 + \cdots + A_r$. The condition number of the TDP at a rank-$r$ tensor $A$ with ordered CPD $a = (A_1, \ldots, A_r)$ is

$$\kappa(A, (A_1, \ldots, A_r)) = \lim_{\epsilon \rightarrow 0} \sup_{\|\mathbf{A} - \mathbf{B}\|_F < \epsilon, \mathbf{A} \text{ has rank } r, \mathbf{B} \text{ has rank } r} \frac{\|\Phi_a^{-1}(\mathbf{A}) - \Phi_a^{-1}(\mathbf{B})\|_F}{\|\mathbf{A} - \mathbf{B}\|_F},$$

where $\Phi_a^{-1}$ is the local inverse function of $\Phi_a$ that satisfies $\Phi_a^{-1}(A) = (A_1, \ldots, A_r)$; see [4]. The norms are the Euclidean norms on the ambient spaces of domain and image of $\Phi_a$, which is naturally identified with the Frobenius norms of tensors, i.e., the square root of the sum of squares of the elements. It follows from the spectral characterization in [4, Theorem 1.1] that $\mathcal{A}$ depends uniquely on the (unordered) CPD $(\mathcal{A}_1, \ldots, \mathcal{A}_r)$; therefore we often write $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ for the condition number. If such a local inverse does not exist, we have $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) = \infty$.

In Section 4.1 we discuss in more detail the existence of this local inverse function; it will be shown in Proposition 4.7 that “most tensors have a finite condition number.”

While the proof of Theorem 1.2 is not straightforward, the main intuition that led us to its conclusion is the observation that there appears to be a gap in the expected value of the condition number of TDPs in $\mathbb{R}^{m_1 \times m_2 \times m_3}$ and other spaces $\mathbb{R}^{m_1 \times m_2 \times m_3}$, $m_1 \geq m_2 \geq m_3 \geq 3$, as we observed in [5]. Here, we derived a further characterization of the distribution of the condition number of random CPDs, based on a result of Cai, Fan, and Jiang [10] about the distribution of the minimum distance between random points on spheres.

Theorem 1.4. Let $a_1, \ldots, a_r \in \mathbb{R}^{m_1}$, $b_1, \ldots, b_r \in \mathbb{R}^{m_2}$ be arbitrary and fixed, while we assume that $c_1, \ldots, c_r \in \mathbb{R}^{m_3}$ are independent random vectors with standard normal entries. Consider the random rank-1 tensors $\mathcal{A}_s = a_s \otimes b_s \otimes c_s \in \mathbb{R}^{m_1 \times m_2 \times m_3}$. Then, for any $\alpha > 0$ we have

$$P \left[ \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) \geq \alpha \frac{n_1^{d-1}}{n_1} \right] \geq T_{r,\alpha}, \quad \text{where } \lim_{r \rightarrow \infty} T_{r,\alpha} = 1 - e^{-K\alpha^{1-m_3}};$$

herein, $K = \frac{2^{(m_3-3)!}}{\sqrt{\pi} \Gamma(\frac{m_3}{2})}$, where $\Gamma$ is the gamma function. In particular, if $m_3 = 2$ we have

$$P \left[ \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) \geq \alpha^2 \right] \geq T_{r,\alpha}, \quad \text{where } \lim_{r \rightarrow \infty} T_{r,\alpha} = 1 - e^{-\alpha^2} \approx \frac{1}{\sqrt{2 \pi \alpha}}.$$

This theorem suggests that as $m_3$ increases, very large condition numbers become increasingly unlikely. The worst case thus seems to occur for $m_3 = 2$, which is exactly the space from which PBAs try to recover the CPD. For example, if $m_3 = 2$ and $r$ is large we can expect that the condition number is greater than $4r^2$ with probability at least (around) 5%.

---

1A condition number of the different problem of computing the factor matrices was considered in [43].
Outline. The next section recalls some preliminary material. As Theorem 1.4 provides the main intuition for the main result, we will treat it first in Section 3. Before proving Theorem 1.2, we need a precise definition of a PBA. This definition relies on the notion of $r$-nice tensors that we study in Section 4; these rank-$r$ tensors have convenient differential-geometric properties. Then, in Section 5 we define the class of PBAs. Section 6 is dedicated to the proof of Theorem 1.2. Numerical experiments validating the theory and illustrating typical behavior for random CPDs are presented in Section 7. Finally, Section 8 presents our main conclusions.

Notation. The following notational conventions are observed throughout this paper: scalars are typeset in lower-case letters (a), vectors in bold-face lower-case letters (a), matrices in upper-case letters (A), tensors in a calligraphic font (A), and varieties and manifolds in an alternative calligraphic font (A). The unit sphere over a set $V \subseteq \mathbb{R}^m$ is $S(V) := \{v \mid v \in V, \|v\| = 1\}$. The Moore-Penrose pseudoinverse of a matrix $A \in \mathbb{R}^{m \times n}$ is denoted by $A^\dagger$. The symmetric group of permutations on r elements is denoted by $\mathcal{S}_r$. $P_r$ denotes the $r \times r$ permutation matrix representing the permutation $\pi \in \mathcal{S}_r$. The standard Euclidean inner product on $\mathbb{R}^m$ is $(x,y) := x^\top y$ for $x, y \in \mathbb{R}^m$.

Acknowledgements. We thank Vanni Noferini and Leonardo Robol for interesting discussions on the definition of numerical instability.

2. Preliminaries

Some elementary definitions from multilinear algebra and differential geometry are recalled.

2.1. Multilinear algebra. The tensor product $\otimes$ of vector spaces $V_1, \ldots, V_d$ is denoted by $\otimes$; see [21, Chapter 1]. As the tensor product is unique up to isomorphisms of the vector spaces $V_1 \times \cdots \times V_d$ and $V_1 \otimes \cdots \otimes V_d$, we will be particularly liberal between the interpretations $\mathbb{R}^{a_1} \otimes \cdots \otimes \mathbb{R}^{a_d} \simeq \mathbb{R}^{a_1 \times \cdots \times a_d} \simeq \mathbb{R}^{a_1 \cdots a_d}$. Elements in the first space are abstract order-$d$ tensors, in the second space they are $d$-arrays, while in the last space they are long vectors. We do not use a “vectorization” operator to indicate the natural bijection between the last two spaces.

The tensor product of linear maps is also well defined [21, Chapter 1]. We use this definition in expressions $M_1 \otimes \cdots \otimes M_d$, where $M_k = [m_{ij}^k]_{i \in \mathbb{R}^{n_k \times n_k}}$, whose columns are $m_{ij}^k \otimes \cdots \otimes m_{ij}^d$, the order will not be relevant wherever it is used. The multilinear multiplication of a tensor $A = \sum_{i_1, \ldots, i_d} a_{i_1, \ldots, i_d} e_{i_1}^1 \otimes \cdots \otimes e_{i_d}^d \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ with the above matrices $M_k$ is

$$(M_1, \ldots, M_d) \cdot A := (M_1 \otimes \cdots \otimes M_d)(A) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} a_{i_1, \ldots, i_d} (M_1 e_{i_1}^1) \otimes \cdots \otimes (M_d e_{i_d}^d).$$

This also entails the following well-known formula for the inner product between rank-1 tensors:

$$(a_1 \otimes \cdots \otimes a_d, b_1 \otimes \cdots \otimes b_d) = \prod_{k=1}^d (a_k, b_k);$$

see, e.g., [22, Section 4.5]. The Khatri-Rao product of the matrices $M_k = [m_{ij}^k]_{i \in \mathbb{R}^{n_k \times r}}$ is $M_1 \odot \cdots \odot M_d := [m_{ij}^1 \otimes \cdots \otimes m_{ij}^d]_{i \in \mathbb{R}^{n_1 \cdots n_d \times r}}$. Note that it is a subset of columns from the tensor product $M_1 \otimes \cdots \otimes M_d$.

2.2. Differential geometry. The following elementary definitions are presented here only for submanifolds of Euclidean spaces; see, e.g., [32] for the general definitions. By a smooth (C$^\infty$) manifold we mean a topological manifold with a smooth structure, in the sense of [32]. The tangent space at $x$ to an n-dimensional smooth submanifold $M \subset \mathbb{R}^N$ can be defined as

$$T_x M = \left\{ v \in \mathbb{R}^N \mid \exists \gamma \text{ a smooth curve } \gamma(t) \subset M \text{ with } \gamma(0) = x : \frac{d}{dt}\bigg|_{t=0} \gamma(t) \right\}.$$

It is a vector subspace whose dimension coincides with the dimension of $M$. Moreover, at every point $x \in M$, there exist open neighborhoods $V \subset M$ and $U \subset T_x M$ of $x$, and a bijective smooth map $\phi : V \to U$ with smooth inverse. The tuple $(V, \phi)$ is a coordinate chart of $M$. A smooth map between manifolds $F : M \to N$ is a map such that for every $x \in M$ and coordinate chart $(V, \phi)$ containing $x$, and every coordinate chart $(W, \psi)$ containing $F(x)$, we have that $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(F(U))$ is a smooth map. The derivative of $F$ can be defined as the linear
map \( d_x F : T_x \mathcal{M} \rightarrow T_{F(x)} \mathcal{N} \) taking the tangent vector \( v \in T_x \mathcal{M} \) to \( \frac{d}{dt}|_{t=0} F(\gamma(t)) \in T_{F(x)} \mathcal{N} \) where \( \gamma(t) \subset \mathcal{M} \) is a curve with \( \gamma(0) = x \) and \( \gamma'(0) = v \).

A Riemannian manifold \((\mathcal{M}, g)\) is a smooth manifold \(\mathcal{M}\) equipped with a Riemannian metric \(g\), which is an inner product \(g_x(\cdot, \cdot)\) on the tangent space \(T_x \mathcal{M}\) that varies smoothly with \(x \in \mathcal{M}\). If \(\mathcal{M} \subset \mathbb{R}^m\), then the inherited Riemannian metric from \(\mathbb{R}^m\) is \(g_x(x, y) = \langle x, y \rangle\) for every \(x \in \mathcal{M}\).

The length of a smooth curve \(\gamma : [0, 1] \rightarrow \mathcal{M}\) is defined by

\[
\text{length}_{\mathcal{M}}(\gamma) = \int_0^1 g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2} \, dt,
\]

and the distance \(\text{dist}_{\mathcal{M}}(x, y)\) between two points \(x, y \in \mathcal{M}\) is the length of the minimal curve with extremes \(x\) and \(y\).

In Section 1, we denoted the Segre manifold of rank-1 tensors in \(\mathbb{R}^{n_1 \times \cdots \times n_d}\) by \(\mathcal{S}\). To emphasize the format, we sometimes write \(\mathcal{S}_{n_1, \ldots, n_d}\) instead. Section 1 also defined the addition map

\[
\Phi_r : \mathcal{S}^{\times r} \rightarrow \mathcal{S}^{n_1 \times \cdots \times n_d}, \quad (\mathcal{A}_1, \ldots, \mathcal{A}_r) \mapsto \mathcal{A}_1 + \cdots + \mathcal{A}_r.
\]

Tensors of rank (at most) \(r\) are denoted by

\[
\sigma_r = \sigma_r(\mathcal{S}_{n_1, \ldots, n_d}) = \Phi_r((\mathcal{S}_{n_1, \ldots, n_d})^{\times r}) = \left\{ \sum_{i=1}^{r} a_i^1 \otimes \cdots \otimes a_i^r \mid a_i^k \in \mathbb{R}^{n_k} \right\}.
\]

It is a semi-algebraic set by the Tarski–Seidenberg principle [3], because it is the projection of an algebraic variety, namely the graph of \(\Phi_r\) [32]. Recall that this means that \(\sigma_r\) can be described as the locus of points that satisfy a system of polynomial equations and inequalities; see [3]. The dimension of \(\sigma_r\) equals the dimension of the smallest \(\mathbb{R}\)-variety \(\overline{\sigma_r}\) containing it [3, Chapter 2].

### 2.3. Numerical analysis.

For a smooth map \(f : \mathcal{M} \rightarrow \mathcal{N}\) between Riemannian manifolds \((\mathcal{M}, g)\) and \((\mathcal{N}, h)\) there is a standard definition of the condition number [2, 9, 36], which generalizes the classic case of smooth maps between Euclidean spaces, namely

\[
\kappa[f](x) = \max_{t_x \in T_x \mathcal{M}} \frac{\| (d_x f)(t_x) \|_{\mathcal{N}, f(x)}}{\| t_x \|_{\mathcal{M}, x}},
\]

where \(d_x f : T_x \mathcal{M} \rightarrow T_{F(x)} \mathcal{N}\) is the derivative of \(f\), and \(\| t_x \|_{\mathcal{M}, x} := \sqrt{g_x(t_x, t_x)}\) for \(t_x \in T_x \mathcal{M}\) (resp. \(\| t_y \|_{\mathcal{N}, y} := \sqrt{h_y(t_y, t_y)}\) for \(t_y \in T_y \mathcal{N}\)) is the norm on the tangent space \(T_x \mathcal{M}\) (resp. \(T_y \mathcal{N}\)) induced by the Riemannian metric \(g\) (resp. \(h\)).

### 3. Estimating the distribution of the condition number.

We start by proving the second main result, Theorem 1.4, because little technical machinery is required. In the proof, we use the following identification of the condition number with the inverse of the smallest singular value of an auxiliary matrix: for \(1 \leq i \leq r\) let \(U_i\) be a matrix whose columns form an orthonormal basis of \(T_{\mathcal{M}} \mathcal{S}\). Then, by [4, Theorem 1.1],

\[
\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) = \frac{1}{\sigma_{\min}(d_{(\mathcal{A}_1, \ldots, \mathcal{A}_r)} \Phi_r)} = \frac{1}{\sigma_{\min}(\{ t_{U_i} - t_{U_j} \})},
\]

where \(\sigma_{\min}\) denotes the smallest singular value. The smallest singular value \(\sigma_{\min}(d_{(\mathcal{A}_1, \ldots, \mathcal{A}_r)} \Phi_r)\) is actually equal to the \(r(n_1 + \cdots + n_d - d + 1)\)th singular value of the Jacobian matrix of \(\Phi_r\) seen as a \(C^\infty\) map from \(\mathbb{R}^{n_1 \cdots n_d}\) to \(\mathbb{R}^{n_1 \cdots n_d}\). Moreover, from (3.1) it follows that the condition number is scale invariant: for all \(t_1, \ldots, t_r \in \mathbb{R}\) \(\{0\}\) we have \(\kappa(t_1 \mathcal{A}_1, \ldots, t_r \mathcal{A}) = \kappa(\mathcal{A}_1, \ldots, \mathcal{A})\). Cai, Fan, and Jiang [10] proved tail probabilities for the maximal pairwise angle of an independent sample of uniformly distributed points on the sphere. The idea for using their results in the proof of Theorem 1.4 is to lower bound the condition number by such a maximal angle. This we do next.

**Lemma 3.1.** For \(i = 1, \ldots, r\) let \(\mathcal{A}_i = t_i \mathbf{a}_i \otimes \mathbf{b}_i \otimes \mathbf{c}_i \in \mathbb{R}^{m_1 \times m_2 \times m_3}\) be fixed rank-1 tensors with \(t_i \in \mathbb{R}\) \(\{0\}\) and \(\| \mathbf{a}_i \| = \| \mathbf{b}_i \| = \| \mathbf{c}_i \| = 1\) for all \(i\). Then, we have

\[
\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) \geq \left( \frac{1}{\max_{1 \leq i \neq j \leq r} \sqrt{1 - |\langle \mathbf{c}_i, \mathbf{c}_j \rangle|}} \right)^{1/2}.
\]
Proof. Without restriction we can assume that the maximum is attained for \( i = 1 \) and \( j = 2 \). By (3.1), the condition number is the inverse of the least singular value of the matrix \( T = [U_i]_{i=1}^r \) where \( U_i \) is any orthonormal basis for \( T_A S \). In particular, the following orthonormal bases can be chosen for \( T_A S \) and \( T_B S \) (see, e.g., [4, Section 5.1]):

\[
U_1 = [I_{n_i} \otimes b_1 \otimes c_1 \ a_1 \otimes Q^2_1 \otimes c_1 \ a_1 \otimes b_1 \otimes Q^1_1]
\]

\[
U_2 = [Q^2_1 \otimes b_2 \otimes c_2 \ a_2 \otimes I_{n_2} \otimes c_2 \ a_2 \otimes b_2 \otimes Q^2_2],
\]

for \( Q^1_1, Q^2_1 \) being orthonormal bases for \( a_1^+, b_1^+, c_1^+ \), respectively. Observe that \( U_1 \) contains the tangent vector \( a_2 \otimes b_1 \otimes c_1 \) and \( U_2 \) contains the tangent vector \( a_2 \otimes b_1 \otimes c_2 \) as columns. Then, using the computation rules for inner products from (2.1), we find that the least singular value of \( T \) is smaller than

\[
\frac{\|a_2 \otimes b_1 \otimes c_1 - a_2 \otimes b_1 \otimes c_2\|}{\sqrt{\|a_2 \otimes b_1 \otimes c_1\|^2 + \|a_2 \otimes b_1 \otimes c_2\|^2}} = \frac{\sqrt{2 - 2\langle a_2 \otimes b_1 \otimes c_1,a_2 \otimes b_1 \otimes c_2\rangle}}{\sqrt{2}} = \frac{\sqrt{1 - \langle c_1,c_2\rangle}}{\sqrt{1 - \langle c_1,c_2\rangle}}.
\]

Repeating the argument for the tangent vector \( -a_2 \otimes b_1 \otimes c_2 \) in \( U_2 \) we get

\[
\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) \geq \max_{1 \leq i \neq j \leq r} \max \left\{ \frac{1}{\sqrt{1 - \langle c_i,c_j\rangle}}, \frac{1}{\sqrt{1 + \langle c_i,c_j\rangle}} \right\} = \max_{1 \leq i \neq j \leq r} \frac{1}{\sqrt{1 - |\langle c_i,c_j\rangle|}},
\]

concluding the proof.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Recall that for a random vector with i.i.d. standard normal entries \( x \), the normalized vector \( \|x\|^{-1} x \) is uniformly distributed in the sphere. From the invariance of the condition number under scaling, we can assume that the entries of \( c_i \), \( 1 \leq i \leq d \), are uniformly distributed in \( S(\mathbb{R}^{m_i}) \). This and Lemma 3.1 show that

\[
P \left[ \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r) \geq \alpha r^{-2/m_3} \right] \geq P \left[ \max_{1 \leq i \neq j \leq r} \frac{1}{\sqrt{1 - |\langle c_i,c_j\rangle|}} \geq \alpha r^{-2/m_3} \right] \geq P \left[ \alpha^{-2} \left( 1 - \max_{1 \leq i \neq j \leq r} |\langle c_i,c_j\rangle| \right) \leq \alpha^{-2} \right]
\]

From [10, Proposition 17], for any fixed \( \alpha > 0 \), this last expression has limit \( 1 - e^{-K \alpha^{1-m_3}} \). This concludes the proof.

Theorem 1.4 is illustrated in Figure 3.1 for \( 15 \times 15 \times n \) tensors of rank 15 for \( n = 2, 3, 5, 10, 15 \). Every solid line represents a limiting complementary cumulative distribution function (ccdf) \( \lim_{n \to \infty} T_{r,n} \), which provide asymptotic lower bounds on the ccdfs of the condition numbers of random rank-\( r \) CPDs. The dashed lines in Figure 3.1 show the empirical ccdfs of the condition number based on two different Monte Carlo experiments.

In the first set of experiments, visualized in Figure 3.1(A), we generated \( 10^5 \) random rank-15 tensors \( \mathcal{A} = \sum_{i=1}^{15} a_i \otimes b_i \otimes c_i \) by independently sampling the entries of the factor matrices \( A = [a_i] \in \mathbb{R}^{15 \times 15}, B = [b_i] \in \mathbb{R}^{15 \times 15} \) and \( C = [c_i] \in \mathbb{R}^{n \times 15} \) from a standard normal distribution. It is observed that the limiting distribution of Theorem 1.4 seems to approximate the shape of the distribution of the condition numbers reasonably well. However, the lower bound seems rather weak for \( n = 2 \). One of the main observations, which is also evident from the formula of the limiting distribution, is that as \( n \) increases the probability of sampling tensors with a high condition number decreases. As is evident from the empirical ccdf in Figure 3.1(A), \( n = 2 \) admits the worst distribution by far: there is a 10% probability of sampling a condition number greater than 10, and still a 0.1% chance to encounter a condition number greater than 10^8. On the other hand, for \( n = 15 \), all sampled tensors had a condition number less than 10.

In the second set of experiments, shown in Figure 3.1(B), we generated \( 10^5 \) random rank-15 tensors of size \( 15 \times 15 \times n \) in a different way in order to illustrate the quality of the lower bound in Theorem 1.4. This time, after sampling the factor matrices \( (A, B, C) \) as above, we perform Gram–Schmidt orthogonalization of \( A \) and \( B \). As can be seen in Figure 3.1(B), the empirical ccdfs here are close to the corresponding limiting distributions.
PENCIL-BASED ALGORITHMS FOR CPD ARE UNSTABLE

(a) A, B, and C i.i.d. standard normal entries.

(b) Arbitrary orthogonal matrices A and B; C i.i.d. standard normal entries.

Figure 3.1. The empirical complementary cumulative distribution function of the condition number for rank-15 tensors of size $15 \times 15 \times n$ is shown in dashed lines. The corresponding solid lines show the lower bound from Theorem 1.4.

The tensors $A = \sum_{i=1}^{15} a_i \otimes b_i \otimes c_i$ were generated by randomly sampling factor matrices $A \in \mathbb{R}^{15 \times 15}$, $B \in \mathbb{R}^{15 \times 15}$ and $C \in \mathbb{R}^{n \times 15}$, as indicated.

We had one additional reason to treat Theorem 1.4 first: on a fundamental level, a PBA solves the TDP for $n_1 \times n_2 \times n_3$ tensors by transforming it into a TDP for $n_1 \times n_2 \times 2$ tensors. The above experiments clearly show that the latter problem has a much worse distribution of condition numbers than the original problem. In other words, from the viewpoint of sensitivity, PBAs try to solve an easy problem via the solution of a significantly more difficult problem. This approach is nearly guaranteed to end in instability.

4. The Manifold of $r$-nice Tensors

While the instability of PBAs is already plausible from Figure 3.1, proving Theorem 1.2 is substantially more complicated. In order to prove it, we should first formalize what we mean by “solving a TDP.” This problem is rife with subtleties.

For example, what should the solution of a TDP be if the input tensor $A$ is the generic rank-11 tensor in $\mathbb{C}^{11 \times 6 \times 3}$? This tensor has 352,716 isolated CPDs [24]. Computing all of them seems computationally infeasible. Nevertheless, all of them are well-behaved because each one of these will vary smoothly in a small open neighborhood of $A$ in $\mathbb{C}^{11 \times 6 \times 3}$. On the other hand, the generic rank-6 tensor of multilinear rank $(4, 4, 4)$ $B$ in $\mathbb{C}^{6 \times 6 \times 6}$ behaves erratically. It has 2 isolated decompositions [11, Theorem 1.3], but a generic rank-6 tensor close to $B$ has only one decomposition that can be moved around continuously such that its limit is a decomposition of $B$. This process works for both of $B$’s decompositions, because the rank-6 tensors have two smooth folds meeting in $B$ [12, Example 4.2]. What should an algorithm compute in this case?

For an $r$-identifiable tensor $A$ there is an unambiguous answer to the above question. Namely, the solution is the unique set of rank-1 tensors $\{A_1, \ldots, A_r\}$ whose sum is $A$. The goal of this section is to carefully define a tensor decomposition map $\tau_{r, n_1, \ldots, n_d}$ in Definition 4.8 whose computation solves the TDP for a subset of rank-$r$ tensors. The domain where the smooth function $\tau_{r, n_1, \ldots, n_d}$ is well defined deserves its own definition, Definition 4.1 below; we call it the
manifold of \( r \)-nice tensors \( \mathcal{N} \subset \sigma_r \). In Proposition 4.7 we prove that \( \mathcal{N} \) is a Zariski open dense subset of the set of rank-\( r \) tensors, so that “almost all tensors are \( r \)-nice.”

Before defining \( \mathcal{N} \), we first need the following two standard definitions. If for a collection of \( r \) vectors \( p_1, \ldots, p_r \in \mathbb{R}^n \) every subset of \( \min\{r,n\} \) many vectors is linearly independent, then the vectors are said to lie in \emph{general linear position} (GLP). We say that a collection of \( r \) rank-1 tensors \( \{a_1^i \cdots \otimes a_k^i\} \) is in \emph{super general linear position} (SGLP) if for every \( 1 \leq s \leq d \) and every \( h \subset \{1, \ldots, d\} \) with \( |h| = s \), the set \( \{a_1^{h_1} \cdots \otimes a_k^{h_k}\} \) is in GLP.

**Definition 4.1** (\( r \)-nice tensors). Recall from (2.3) the definition of rank-\( r \) tensors \( \sigma_r \) and its closure \( \sigma_r \). Then, \( \mathcal{M}_{r,n_1,\ldots,n_d} \subset \mathcal{S}_{n_1,\ldots,n_d}^{\times r} \) is defined to be the set containing all the rank-1 tuples \( a = (\mathcal{A}_1, \ldots, \mathcal{A}_r) \) satisfying the following properties:

(i) \( \Phi_r(a) \) is a smooth point of \( \sigma_r \),
(ii) \( \Phi_r(a) \) is \( r \)-identifiable, and, thus, has rank equal to \( r \),
(iii) \( a \) has finite condition number,
(iv) \( a \) is in SGLP, and
(v) for all \( i \) the \( (1,1,\ldots,1) \)-entry of \( \mathcal{A}_i \) is not equal to zero.

The \( r \)-nice tensors \( \mathcal{N}_{r,n_1,\ldots,n_d} \) are defined to be the image of \( \mathcal{M}_{r,n_1,\ldots,n_d} \) under the addition map \( \Phi_r \) from (2.2):

\[
\mathcal{N}_{r,n_1,\ldots,n_d} := \Phi_r(\mathcal{M}_{r,n_1,\ldots,n_d}).
\]

If it is clear from the context we drop the subscript from both \( \mathcal{M}_{r,n_1,\ldots,n_d} \) and \( \mathcal{N}_{r,n_1,\ldots,n_d} \) and simply write \( \mathcal{M} \) and \( \mathcal{N} \).

**Remark 4.2.** The reason for the last requirement, (v), is that under this restriction we can define a parametrization of rank-1 tensors that is a \emph{diffeomorphism}; see the next subsection for details.

### 4.1 Elementary results.
Before proceeding, we need a few elementary results related to the differential geometry of CPDs, which we did not find in the literature.

The rank-1 tensors in \( \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d} \), i.e., \( \mathcal{S} = \{a_1 \cdots \otimes a_d \mid a_k \in \mathbb{R}^{n_k}\} \setminus \{0\} \), form the affine cone over a smooth projective variety (see, e.g., [31]) and, hence, \( \mathcal{S} \) is an analytic submanifold of \( \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d} \). Its dimension is \( 1 + \sum_{k=1}^{d} (n_k - 1) \) [31]. The map

\[
\Psi_{n_1,\ldots,n_d} : \mathbb{R} \setminus \{0\} \times \mathbb{S}(\mathbb{R}^{n_1}) \times \cdots \times \mathbb{S}(\mathbb{R}^{n_d}) \to \mathcal{S}, \quad (\lambda, u_1, \ldots, u_d) \mapsto \lambda u_1 \otimes \cdots \otimes u_d
\]

is a surjective \emph{local diffeomorphism}: every point in the domain has an open neighborhood such that \( \Psi_{n_1,\ldots,n_d} \) restricted to this neighborhood is an open, smooth (\( C^\infty \)), bijective map with smooth inverse [32, p. 79]. Indeed, it can be verified that the derivative is injective at every point; see, e.g., [4, Section 5.1]. Note that the fiber of \( \Psi_{n_1,\ldots,n_d} \) at \( \lambda u_1 \otimes \cdots \otimes u_d \) is exactly the set \( \{ (\omega_1 \lambda, \omega_1 u_1, \ldots, \omega_d u_d) \mid \omega_1 \cdots \omega_d = 1, \omega_1 \in \{-1,1\} \} \), which has \( 2^d \) elements. Moreover, \( \Psi_{n_1,\ldots,n_d} \) is a proper map so that it is a \( 2^d \)-sheeted smooth covering map [32, p. 91-95].

Let \( \mathbb{S}^+ = \{ u \in \mathbb{S}(\mathbb{R}^n) \mid u_1 > 0 \} \) be the “upper” half of the unit sphere; it is a submanifold in the subspace topology on \( \mathbb{R}^n \). Let us define the following restriction of \( \Psi \):

\[
\Psi_{n_1,\ldots,n_d}^+ : \mathbb{R} \setminus \{0\} \times \mathbb{S}^+(\mathbb{R}^{n_1}) \times \cdots \times \mathbb{S}^+(\mathbb{R}^{n_d}) \to \mathcal{S}, \quad (\lambda, u_1, \ldots, u_d) \mapsto \lambda u_1 \otimes \cdots \otimes u_d.
\]

It follows from the foregoing that \( \Psi_{n_1,\ldots,n_r}^+ \) is a bijective local diffeomorphism onto its image, so it is a \emph{(global) diffeomorphism} onto its image. Let \( \mathcal{S}_{n_1,\ldots,n_r}^+ \) be the image of \( \Psi_{n_1,\ldots,n_r}^+ \):

\[
\mathcal{S}_{n_1,\ldots,n_r}^+ := \Psi_{n_1,\ldots,n_r}^+(\mathbb{R} \setminus \{0\} \times \mathbb{S}^+(\mathbb{R}^{n_1}) \times \cdots \times \mathbb{S}^+(\mathbb{R}^{n_d})).
\]

When it is clear from the context we drop the subscripts from \( \Psi_{n_1,\ldots,n_d}^+ \), \( \mathcal{S}_{n_1,\ldots,n_d}^+ \) and \( \mathcal{S}_{n_1,\ldots,n_d}^+ \). The foregoing explains part (v) in Definition 4.1: we wish to work with a parametrization of \( \mathcal{S} \) that is a diffeomorphism, so we restrict ourselves to \( \mathcal{S}^+ \) and use \( \Psi^+ \). We will show in the proof of Proposition 4.5 that \( \mathcal{S}^+ \) is open in the Zariski topology and, hence, open and dense in the Euclidean topology.

Finally, we consider the subset \( \mathcal{S}_{r,n} \subset (\mathbb{S}^+(\mathbb{R}^n))^{\times r} \) defined as

\[
\mathcal{S}_{r,n} := \{ (s_1, \ldots, s_r) \in (\mathbb{S}^+(\mathbb{R}^n))^{\times r} \mid [s_1, \ldots, s_r] \in \mathbb{R}^{n \times r} \text{ has full rank} \}.
\]

Note that \( \mathcal{S}_{r,n} \) is an open submanifold, because the locus of points not satisfying the rank condition is closed in the Zariski topology. We also have the following result.

---

\( ^2 \)The following items are most naturally considered in projective space, but in order to avoid as much technicalities as is feasible we prefer to present the results concretely as subspaces of Euclidean spaces.
Lemma 4.3. Let $\mathfrak{S}_r$ be the symmetric group on $r$ elements. Then, $\widehat{S}_{r,n} := S_{r,n}/\mathfrak{S}_r$ is a manifold. Moreover, the projection $\pi : S_{r,n} \to \widehat{S}_{r,n}, (x_1, \ldots, x_r) \mapsto \{x_1, \ldots, x_r\}$ is a local diffeomorphism.

Proof. $\mathfrak{S}_r$ is a discrete Lie group acting smoothly [32, Example 7.22(e)]. The group action is also free because $S \in S_{r,n}$ can be a fixed point of some permutation only if $s_i, s_j \in S$ with $i \neq j$ are equal. It can be verified that the conditions in [32, Lemma 21.11] hold, so that the action is proper. The result follows by the quotient manifold theorem [32, Theorem 21.10].

4.2. Differential geometry of $r$-nice tensors. Recall that a Segre manifold $S \subset \mathbb{R}^{n_1 \times \cdots \times n_d}$ is said to be generically $r$-identifiable if all tensors in a Zariski-open subset of $\overline{S}$ are identifiable; see [12,13] for the state of the art. In the context of PBAs, the following standard result suffices.

Lemma 4.4. Let $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$. If $r \leq n_2$, then $S_{n_1, \ldots, n_d}$ is generically $r$-identifiable.

Proof. This follows, for example, from the effectiveness of Kruskal’s criterion; see [13]. □

Next, we prove an important property of the set $M_{r,n_1,\ldots,n_d}$ from Definition 4.1.

Proposition 4.5. Let $S_{n_1,\ldots,n_d}$ be generically $r$-identifiable. Then, $M_{r,n_1,\ldots,n_d}$ is a Zariski-open submanifold of $S_{n_1,\ldots,n_d}^{*r}$.

Proof. Let $S = S_{n_1,\ldots,n_d}$ and $M = M_{r,n_1,\ldots,n_d}$ for brevity. We show that the set of tuples not satisfying either of the conditions in Definition 4.1 is contained in a union of five Zariski-closed subsets of $M$: these subsets are denoted by $B_{(i)}, B_{(ii)}, B_{(iii)}, B_{(iv)}$ and $B_{(v)}$. Taking

$$M = S^{*r} \setminus (B_{(i)} \cup B_{(ii)} \cup B_{(iii)} \cup B_{(iv)} \cup B_{(v)}),$$

would then prove the assertion.

Recall that generic $r$-identifiability implies nondefectivity of $\sigma_r$; see [31, Chapter 5, specifically Corollary 5.3.1.3]. Hence, $\dim \sigma_r = \dim \overline{S} = \dim S^{*r} = r \dim S$. The subvariety $\Sigma \subset \overline{S}$ of singular points is proper and closed in the Zariski topology by definition [23]. This means that in addition to the polynomials that vanish on the $\mathbb{R}$-variety $\overline{S}$, there are $k \geq 1$ additional nontrivial polynomial equations with coefficients over $\mathbb{R}$ such that $f_1(y) = \cdots = f_k(y) = 0$ for all $y \in \Sigma$. If $y$ has a preimage $x \in S^{*r}$ under $\Phi_r$, then $f_1(\Phi_r(x)) = \cdots = f_k(\Phi_r(x)) = 0$. Hence, the locus $B_{(i)}$ of decompositions not satisfying condition (i) in Definition 4.1, which maps into the singular locus $\Sigma$ under $\Phi_r$ is a Zariski-closed set. It is also a proper subset, because otherwise $\Phi_r(S^{*r}) = \sigma_r \subset \Sigma$, which is a contradiction as $\dim \Sigma < \dim \overline{S} = \dim \sigma_r$.

The set of tensors in $\overline{S}$ with several decompositions is closed in the Zariski topology by assumption. We can apply the same argument as in the previous paragraph to conclude that the variety of decompositions $B_{(ii)} \subset S^{*r}$ that map to points of $\overline{S}$ that are not $r$-identifiable is a proper Zariski closed subset in $S^{*r}$.

The subset $B_{(iii)} \subset S^{*r}$ of decompositions with condition number $\infty$, is contained in a Zariski-closed set if the $r$-secont variety $\overline{S}$ is nondefective by [6, Lemma 5.3].

The set of points $B_{(iv)} \subset S^{*r}$ not satisfying (iv) is Zariski-closed by [13, Lemma 4.4].

For the last point, observe that condition (v) of Definition 4.1 is equivalent to $p \in (S^r)^{*r}$. By definition of $S^r$ in (4.2), the set of points in $S \setminus S^r$ is the intersection of $S$ with the union of the following linear varieties: $L_k = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_k-1} \otimes \mathbb{R}^{n_k}/(e_1) \otimes \mathbb{R}^{n_{k+1}} \otimes \cdots \otimes \mathbb{R}^{n_d}$, where $\mathbb{R}^{n_k}/(e_1) = (e_2, \ldots, e_{n_k})$ and $e_i$ is the $i$th standard basis vector of $\mathbb{R}^{n_k}$. In fact,

$$S \setminus S^r = S \cap \bigcup_{k=1}^{d} L_k = \bigcup_{k=1}^{d} (S \cap L_k) \simeq \bigcup_{k=1}^{d} S_{n_1,\ldots,n_k-1,n_k-1,n_k+1,\ldots,n_d},$$

which is thus a Zariski-closed set because $\dim S_{n_1,\ldots,n_k-1,n_k-1,n_k+1,\ldots,n_d} < \dim S$. Therefore, taking $B_{(v)} = \bigcup_{k=1}^{d} S^{*r} (i-1) \setminus S^r \times S^{*r} (r-i)$ yields the Zariski-closed variety of points not satisfying (v). This concludes the proof. □

The definition of $M_{r,n_1,\ldots,n_d}$ is nice, because the addition map $\Phi_r$ from (2.2) restricted to $M_{r,n_1,\ldots,n_d}$ is a local diffeomorphism. However, we wish to work with global diffeomorphisms and therefore need the following proposition.

Proposition 4.6. If $S_{n_1,\ldots,n_d}$ is generically $r$-identifiable, then $\mathcal{M}_{r,n_1,\ldots,n_d} = M_{r,n_1,\ldots,n_d}/\mathfrak{S}_r$ is a manifold. Moreover, the projection $\pi : \mathcal{M}_{r,n_1,\ldots,n_d} \to M_{r,n_1,\ldots,n_d}, (A_1, \ldots, A_r) \mapsto \{A_1, \ldots, A_r\}$ is a local diffeomorphism.
Proof. Combine the proof of Lemma 4.3 with the fact that \( r \)-identifiability implies that the rank-1 tensors in a decomposition \((A_1, \ldots, A_r) \in \mathcal{M}_{r, n_1, \ldots, n_d}\) are pairwise distinct. \(\square\)

It is clear that the addition map \(\Phi_r\) is constant on \(\mathcal{S}_r\)-orbits in \(\mathcal{M}_{r, n_1, \ldots, n_d}\). Therefore, \(\Phi_r\) is well defined on \(\widetilde{\mathcal{M}}_{r, n_1, \ldots, n_d}\). Now, we have the following crucial result.

**Proposition 4.7.** Let \(N_{r, n_1, \ldots, n_d} \subset \sigma_r\) be the set of \(r\)-nice tensors. If \(S_{n_1, \ldots, n_d}\) is generically \(r\)-identifiable, then

\[
\Phi_r : \widetilde{\mathcal{M}}_{r, n_1, \ldots, n_d} \to N_{r, n_1, \ldots, n_d}, \quad \{A_1, \ldots, A_r\} \mapsto A_1 + \cdots + A_r
\]

is a diffeomorphism. Moreover, \(N_{r, n_1, \ldots, n_d}\) is an open dense submanifold of \(\sigma_r\).

**Proof.** As before, for brevity, we drop all subscripts. Let \(a = (A_1, \ldots, A_r) \in \mathcal{M}\). By definition, \(a\) has a finite condition number. This means, by [4, Theorem 1.1], that the derivative of \(\Phi_r\) at \(a\) is injective. Hence, \(\Phi_r\) is a smooth immersion [32, p. 78]. By the generic \(r\)-identifiability assumption, it follows that the \(r\)-secant variety \(\overline{\sigma}_r\) is not defective so that \(\dim(\overline{\sigma}_r) = r \dim(S)\). Moreover, by Proposition 4.5, we have \(r \dim S = \dim \mathcal{M}_{r, n_1, \ldots, n_d}\) and, by construction, we have \(\dim \mathcal{M}_{r, n_1, \ldots, n_d} = \dim \widetilde{\mathcal{M}}_{r, n_1, \ldots, n_d}\). As \(\Phi_r\) is injective by generic \(r\)-identifiability and by having taken the particular quotient in Proposition 4.6, then [32, Proposition 4.22(d)] entails that \(\Phi_r\) is a smooth embedding. The first conclusion follows by [32, Proposition 5.2].

The foregoing already shows that \(N_{r, n_1, \ldots, n_d} \subset \sigma_r\) is open. We show that it is dense. Let \(A \in \sigma_r \setminus N_{r, n_1, \ldots, n_d}\) with decomposition \(A = \Phi_r(a) = A_1 + \cdots + A_r\). By Proposition 4.5, there exist a sequence

\[(A^{(i)}_1, \ldots, A^{(i)}_r) \in \mathcal{M}_{r, n_1, \ldots, n_d}\]

such that \(\lim_{j \to \infty} (A^{(j)}_1, \ldots, A^{(j)}_r) = (A_1, \ldots, A_r)\).

Note that this is convergence in the usual Euclidean topology that \(\mathcal{M}_{r, n_1, \ldots, n_d}\) inherits from the ambient space \((\mathbb{R}^{n_1 \times \cdots \times n_d})^r\). Consequently, the components also converge individually: \(\lim_{j \to \infty} A^{(j)}_i = A_i\), \(i = 1, \ldots, r\). The result follows from the fact that adding the above convergent sequences results in a convergent sequence in \(N_{r, n_1, \ldots, n_d}\) with limit \(A\). Hence, \(A \in \overline{\sigma}_{n_1, \ldots, n_d}\) so that \(N_{r, n_1, \ldots, n_d}\) is dense in \(\sigma_r\), concluding the proof. \(\square\)

From Proposition 4.7, \(\Phi_r\) has a smooth inverse, which solves the TDP on \(N_{r, n_1, \ldots, n_d} \subset \mathbb{R}^{n_1 \times \cdots \times n_d}\). We finally arrive at the goal of this section.

**Definition 4.8.** The inverse of \(\Phi_r\) on the manifold of \(r\)-nice tensors is

\[
\tau_{r, n_1, \ldots, n_d} : N_{r, n_1, \ldots, n_d} \to \widetilde{\mathcal{M}}_{r, n_1, \ldots, n_d}, \quad A_1 + \cdots + A_r \mapsto \{A_1, \ldots, A_r\}.
\]

We call this mapping the tensor decomposition map.

**Remark 4.9.** One way to interpret the construction in this section is that near \(A \in N_{r, n_1, \ldots, n_d}\) we locally have the identification \(\tau_{r, n_1, \ldots, n_d} = \tilde{\pi} \circ \Phi^{-1}_r\), where \(a = (A_1, \ldots, A_r)\) is any ordered \(r\)-nice decomposition of \(A\), \(\Phi^{-1}_r\) is the local inverse in (1.2), and \(\tilde{\pi}\) is as in Proposition 4.6.

**4.3. Implications for the condition number.** Let \(a = (A_1, \ldots, A_r)\) be any ordered \(r\)-nice decomposition in \(\mathcal{M}_{r, n_1, \ldots, n_d}\). For the \(r\)-nice tensor \(A = A_1 + \cdots + A_r \in N_{r, n_1, \ldots, n_d}\), we will relate the condition number \(\kappa(\tau_{r, n_1, \ldots, n_d})(A)\), as in (2.4), to the condition number of the CPD \(\kappa(A_1, \ldots, A_r)\) from [4]. We have the following result.

**Lemma 4.10.** Let us choose the Riemannian metrics on \(N_{r, n_1, \ldots, n_d}\) and \(\mathcal{M}_{r, n_1, \ldots, n_d}\) inherited from their respective ambient spaces. Then, the mapping \(\tilde{\pi}\) from Proposition 4.6 induces a natural Riemannian metric on \(\widetilde{\mathcal{M}}_{r, n_1, \ldots, n_d}\) with the following properties:

1. \(\tilde{\pi}\) is a local isometry;
2. for all \(A = A_1 + \cdots + A_r \in N_{r, n_1, \ldots, n_d}\), we have \(\kappa(A_1, \ldots, A_r) = \kappa(\tau_{r, n_1, \ldots, n_d})(A)\); and
3. for any \(\{A_1, \ldots, A_r\}, \{B_1, \ldots, B_r\} \in \mathcal{M}\) we have

\[
\text{dist}_{\mathcal{M}}(\{A_1, \ldots, A_r\}, \{B_1, \ldots, B_r\}) = \min_{\pi \in \mathcal{E}_r} \left( \text{dist}_{\mathcal{M}}((A_1, \ldots, A_r), \pi(B_1, \ldots, B_r)) \right).
\]

Here, \(\text{dist}_{\mathcal{M}}\) and \(\text{dist}_{\mathcal{M}}\) are the respective Riemannian distances.
Because of the equality of condition numbers in Lemma 4.10 and (1.2), we find that for every 
\( a = (A_1, \ldots, A_r) \in M_{r,n_1,\ldots,n_d} \) we have
\[
\kappa(\tau)[A] = \lim_{\epsilon \to 0} \frac{\sup_{\|A-B\|_F \leq \epsilon} \frac{\delta_{A,B}(\tau(A), \tau(B))}{\|A-B\|_F}}{\sup_{\|A-B\|_F \leq \epsilon} \min_{\pi \in \Pi_{r}} \frac{\|\Phi^{-1}(A) - \pi \circ \Phi^{-1}(B)\|_F}{\|A-B\|_F}},
\]
where \( \tau = \tau_{r,n_1,\ldots,n_d} \) and the last equality follows from (3) in Lemma 4.10. This above equality is
very significant because it allows us to make sense of the distance between two unordered CPDs, i.e., sets of rank-1 tensors, \( \{A_1, \ldots, A_r\} \) and \( \{B_1, \ldots, B_r\} \). As a consequence, we get an instance of the well-known rule of thumb in numerical analysis:
\[
\min_{\pi \in \Pi_{r}} \frac{\|A-BP_\pi\|_F}{\kappa(\tau)[A]} \lesssim \frac{\|A-B\|_F}{\kappa(\tau)[A]},
\]
for nearby \( A = A_1 + \cdots + A_r \) and \( B = B_1 + \cdots + B_r \); herein, \( A = [A_i]_i \in \mathbb{R}^{n_1 \times \cdots \times n_d \times r} \) (resp. \( B = [B_i]_i \in \mathbb{R}^{n_1 \times \cdots \times n_d \times r} \)) is a matrix that contains the vectorized rank-1 tensors \( A_i \) (resp. \( B_i \)) as columns, and \( P_\pi \) is the \( r \times r \) permutation matrix representing the permutation \( \pi \). The notation \( \lesssim \) indicates that the bound is asymptotically sharp for infinitesimal \( \|A-B\|_F \).

5. Pencil-based algorithms for the CPD

We start by specifying a very general class of numerical algorithms to which the analysis in
Section 6 applies. The construction may seem a bit abstract at first sight, so it is useful to keep
in mind that the prototypical algorithm from the introduction is an example of a PBA.

As it suffices, in principle, to present a single input for which an algorithm is unstable, we
can choose a well-behaved subset of \( r \)-nice tensors \( N^* \subset N_{r,n_1,n_2,n_3} \subset \mathbb{R}^{n_1 \times n_2 \times n_3} \) (for the exact choice of \( N^* \) see Definition 5.1 below) and specify what a PBA should compute for such inputs. If
the numerical instability already occurs on this subset, then it is also unstable on larger domains.
We recall from Section 4 that by considering only \( r \)-nice tensors \( N_{r,n_1,n_2,n_3} \), the TDP consists of computing the action of the function \( \tau_{r,n_1,n_2,n_3} \) from Definition 4.8. PBAs compute this map in a particular way, via the four transformations described below.

The input of a PBA is assumed to be the multidimensional array \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \). The first
transformation is the multilinear multiplication \( pq \) that maps \( n_1 \times n_2 \times n_3 \) tensors to format
\( n_1 \times n_2 \times 2 \) via the matrix \( Q \in \mathbb{R}^{n_3 \times 2} \) with orthonormal columns:
\[
\rho_Q : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_1 \times n_2 \times 2}, \ \ A \mapsto (I_{n_1}, I_{n_2}, Q^T) \cdot A.
\]

The second transformation, \( \hat{\theta} \), computes the set of unit-norm columns of the first factor matrix
\( A \) of the CPD when restricted to \( N_{r,n_1,n_2,2} \):
\[
\hat{\theta}|_{N_{r,n_1,n_2,2}} : N_{r,n_1,n_2,2} \to \hat{S}_{r,n_1}, \ \ \mathbf{b} = \sum_{i=1}^{r} a_i \otimes b_i \otimes z_i \mapsto \{a_1, \ldots, a_r\}.
\]
Herein, \( \hat{S}_{r,n_1} = S_{r,n_1} / \mathbb{S}_r \), where \( S_{r,n_1} \) is as in (4.3). Note the curious definition of \( \hat{\theta} \) involving the restriction to \( N_{r,n_1,n_2,2} \). The reason for this formulation is that a PBA will be executed using floating-point arithmetic. It is unlikely that the floating point representation \( \hat{\mathbb{f}}(\mathbb{b}) \in \mathbb{R}^{n_1 \times n_2 \times 2} \) is exactly in \( N_{r,n_1,n_2,2} \subset \mathbb{R}^{n_1 \times n_2 \times 2} \), even when \( \mathbb{b} \in N_{r,n_1,n_2,2} \). Therefore, a minimal additional demand is placed on \( \hat{\theta} \): For every \( \mathbb{b} \in N_{r,n_1,n_2,2} \), \( \hat{\theta} \) must be defined for \( \hat{\mathbb{f}}(\mathbb{b}) \).

The third transformation, \( v \), when restricted to
\[
\mathcal{R}_{r,n_1,n_2,3} := \{(A, A) \mid A = (a_1, \ldots, a_r) \in S_{r,n_1} \text{ and } A = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i \in N_{r,n_1,n_2,3}\},
\]
essentially computes the Khatri–Rao product \( B \odot C \) of the remaining factor matrices, namely
\[
v|_{\mathcal{R}_{r,n_1,n_2,3}} : \mathcal{R}_{r,n_1,n_2,3} \to S_{r,n_2,3}^{\otimes r}, \ \ (A = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i, (a_1, \ldots, a_r) \mapsto (b_1 \otimes c_1, \ldots, b_r \otimes c_r).
For the proof of instability in Section 6, it will not matter if or how \( v \) is defined outside of \( \mathcal{R}_{r,n_1,n_2,n_3} \), so we impose no further constraints. The final step computes the (unordered) Khatri–Rao product of two ordered sets of vectors:

\[
\hat{\circ} : \mathbb{R}^{p \times r} \times \mathbb{R}^{q \times r} \to \mathcal{S}_{p,q}^r / \mathcal{S}_r, \quad ((x_1, \ldots, x_r), (y_1, \ldots, y_r)) \mapsto \{x_1 \hat{\circ} y_1, \ldots, x_r \hat{\circ} y_r\}.
\]

Applied to \( A \) and \( B \circ C \), this yields the set of rank-1 tensors solving the TDP.

We will define a PBA to be an algorithm composing the above functions. The input space for a PBA is thus \( \mathcal{N}^* := \rho_Q^{-1}(\mathcal{N}_{r,n_1,n_2,2}) \cap \mathcal{N}_{r,n_1,n_2,3} \) (it is the subset \( \mathcal{N}^* \) mentioned at the start of this section). Hence, we arrive at the definition of the class of PBAs for solving a TDP whose input is in \( \mathcal{N}^* \subset \mathbb{R}^{n_1 \times n_2 \times n_3} \).

**Definition 5.1** (Pencil-based algorithm). A pencil-based algorithm for solving the TDP is an algorithm that computes the tensor decomposition map \( \tau_{r,n_1,n_3} \) when given the \( n_1 \times n_2 \times n_3 \) input array \( A = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i \in \mathcal{N}^* \), where \( a_i \in \mathcal{S}^r(\mathbb{R}^{n_1}) \) and \( \mathcal{N}^* = \rho_Q^{-1}(\mathcal{N}_{r,n_1,n_2,2}) \cap \mathcal{N}_{r,n_1,n_2,3} \), by performing the following steps:

S1. \( B \leftarrow \rho_Q(A) \);

S2. \( \{a_1, \ldots, a_r\} \leftarrow \hat{\theta}(B) \);

S3.a Choose an order \( A := (a_1, \ldots, a_r) \);

S3.b \( (b_1 \otimes c_1, \ldots, b_r \otimes c_r) \leftarrow v(A, A) \);

S4. output \( \leftarrow \hat{\circ}((a_1, \ldots, a_r), (b_1 \otimes c_1, \ldots, b_r \otimes c_r)) \).

6. PENCIL-BASED ALGORITHMS ARE UNSTABLE

We continue by showing that PBAs are numerically forward unstable for solving the TDP for third-order tensors. For \( A \in \mathcal{N}^* \subset \mathbb{R}^{n_1 \times n_2 \times n_3} \) let \( \{\hat{A}_1, \ldots, \hat{A}_r\} \) be the CPD returned by a PBA in floating-point representation. The overall goal in the proof of Theorem 1.2 is showing that for all small \( \epsilon > 0 \) there exists an open neighborhood \( \Omega_\epsilon \subset \mathcal{N}_{r,n_1,n_2,n_3} \) of \( r \)-nice tensors such that for \( A = \hat{A}_1 + \cdots + \hat{A}_r \) in that neighborhood the excess factor

\[
\omega(A) := \frac{\min_{\tau \in \mathcal{S}_r} \sqrt{\sum_{i=1}^{r} \|A_i - \tilde{A}_{\tau(i)}\|_F^2}}{\kappa_{\mathcal{R}_{r,n_1,n_2,n_3}}(A) \cdot \|A - \hat{\theta}(A)\|_F}
\]

is at least a constant times \( \epsilon^{-1} \). The exact statement is in Theorem 6.1 below.

We call \( \omega \) the excess factor because it measures by how much the forward error\(^3\) produced by the numerical algorithm, as measured by the numerator, exceeds the forward error that one can expect from solving the TDP (which is equivalent to computing the map \( \tau_{r,n_1,n_2,n_3} \)), as measured by the denominator. Showing that the excess factor can become arbitrarily large on the domain of \( \tau_{r,n_1,n_2,n_3} \) is essentially equivalent to the standard definition of numerical forward instability of an algorithm for computing \( \tau_{r,n_1,n_2,n_3} \) [25]. In fact, the excess factor can be interpreted as a quantitative measure of the forward numerical instability of an algorithm on a particular input. Ideally, \( \omega \) is bounded by a small constant, but for numerically unstable algorithms \( \omega \) is “too large” relative to the problem dimensions. The next result is a more precise version of Theorem 1.2 which states that for all \( A \in \Omega_\epsilon \), a PBA becomes arbitrarily unstable as \( \epsilon \to 0 \), irrespective of the problem size.

**Theorem 6.1.** There exist a constant \( k > 0 \) and a tensor \( A \in \mathcal{N}_{r,n_1,n_2,n_3} \) with the following properties: For all sufficiently small \( \epsilon > 0 \), there exists an open neighborhood \( \Omega_\epsilon \) of \( A \) such that for all tensors \( A \in \Omega_\epsilon \) we have

1. \( A \in \mathcal{N}^* \) is a valid input for a PBA, and
2. \( \omega(A) \geq k\epsilon^{-1} \).

Herein, \( \mathcal{N}^* \) is as in Definition 5.1.

6.1. The key ingredients. The key observation is that for computing the tensor decomposition map \( \tau_{r,n_1,n_2,n_3} \) every PBA computes \( \hat{\theta} \) in S2. We will show that the condition number of \( \hat{\theta} \) is comparable to the condition number of \( \tau_{r,n_1,n_2,2} \). Combining this result with the observations from Section 3 and [6], which both demonstrated that the condition number of the tensor decomposition map \( \tau_{r,n_1,n_2,2} \) for \( n_1 \times n_2 \times 2 \) tensors can be much worse than the one of \( \tau_{r,n_1,n_2,n_3} \) for \( n_1 \times n_2 \times n_3 \) tensors, motivated our proof of Theorems 1.2 and 6.1.

\(^3\)Recall its definition from (4.4).
Let us consider the relation between the tensor decomposition map for $n_1 \times n_2 \times 2$-tensors and $\hat{\theta}$. For brevity, we denote the manifold of r-nice tensors in $\mathbb{R}^{n_1 \times n_2 \times 2}$ by

$$\mathcal{N} := \mathcal{N}_{r,n_1,n_2}.$$ 

The main intuition underpinning the proof of Theorem 6.1 is the following diagram:

$$\begin{array}{c}
\mathcal{N} \\
\downarrow \quad \text{Id}_\mathcal{N} \times \hat{\theta}
\end{array}$$

$$\begin{array}{c}
\mathcal{N} \\
\downarrow \quad \tau_{r,n_1,n_2}
\end{array}$$

$$\begin{array}{c}
\hat{\eta} \\
\uparrow \quad \hat{S}_{r,n_1}
\end{array}$$

$$\begin{array}{c}
\hat{\mathcal{M}}_{r,n_1,n_2}
\end{array}$$

(6.2)

Herein, $\hat{\eta}$ is any map so that $\tau_{r,n_1,n_2} = \hat{\eta} \circ (\text{Id}_\mathcal{N} \times \hat{\theta})$. For example, we could take the map $\hat{\eta} = \tau_{r,n_1,n_2} \circ \pi_1$, where $\pi_1(x,y) = x$ projects onto the first factor. For clearly conveying the main idea, let us imagine for a moment that $\text{Id}_\mathcal{N} \times \hat{\theta}$, $\hat{\eta}$, and $\tau_{r,n_1,n_2}$ were smooth ($C^\infty$) multivariate functions between Euclidean spaces. For any such functions $f$, $g$, we have that $\kappa(f)(x) = \|J_f(x)\|_2$, where $J_f(x)$ is the Jacobian matrix of $f$ at $x$; see, e.g., [9, Proposition 14.1]. Consequentially, for the composite function $g \circ f$, we get

$$\kappa[g \circ f](x) = \|J_g(f(x))J_f(x)\|_2 \leq \|J_g(f(x))\|_2\|J_f(x)\|_2 = \kappa[g](f(x)) \cdot \kappa[f](x).$$

(6.3)

It thus seems feasible to obtain lower bounds on the condition number of $f = \text{Id}_\mathcal{N} \times \hat{\theta}$ in function of the condition numbers of $g \circ f = \tau_{r,n_1,n_2}$ and $g = \hat{\eta}$. The key insight is that $\hat{\eta}$ should be chosen in such a way that it has a condition number bounded by a constant, so that $\kappa[\text{Id}_\mathcal{N} \times \hat{\theta}](\mathcal{N})$ would be comparable in magnitude to $\kappa[\tau_{r,n_1,n_2}](\mathcal{N})$.

Using the above ideas, we will rigorously prove the next lemma in the appendix, which states that the condition number of $\hat{\theta}$ can be bounded from below by the condition number of the tensor decomposition map $\tau_{r,n_1,n_2}$ in some cases.

**Lemma 6.2.** Let $\nu > 0$ be sufficiently small. Let $\mathcal{B} = \sum_{i=1}^r a_i \otimes b_i \otimes z_i$ be an element of $\mathcal{N}$. Assume that $\|a_i\|_1 = 1$ and $\|b_i \otimes z_i\| < 1 + \nu$ for $i = 1, \ldots, r$. Let $A = [a_i]$. If there exists a matrix $A' \in \mathbb{R}^{n_1 \times r}$ with orthonormal columns such that $\|A - A'\|_F \leq \nu$, then

$$\kappa[\hat{\theta}](\mathcal{N})(\mathcal{B}) \geq \frac{\kappa[\tau_{r,n_1,n_2}](\mathcal{N})(\mathcal{B})}{10r} - 1.$$ 

This shows that in some circumstances, the condition number of $\hat{\theta}|_{\mathcal{N}}$ is proportional to the condition number of $\tau_{r,n_1,n_2}$ in $\mathbb{R}^{n_1 \times n_2 \times 2}$. Unfortunately, the errors in the computation of $\hat{\theta}|_{\mathcal{N}}$ cannot always be corrected, as we prove the following result in the appendix.

**Lemma 6.3.** Let $\nu > 0$ be sufficiently small. Let $A = \sum_{i=1}^r a_i \otimes b_i \otimes c_i \in A^r$ with $\|a_i\|_1 = 1$ and factor matrices $A, B, C$. Let $\hat{A} \in \mathbb{R}^{n_1 \times r}$ be a fixed matrix with unit-norm columns and let $\delta := \min_{\pi \in \mathcal{E}_r} \|A - \hat{A}\|_F$. If $\|b_i \otimes c_i\| \geq 1 - \nu$ for $i = 1, \ldots, r$, $\delta < 1$, and there exists a matrix $A' \in \mathbb{R}^{n_1 \times r}$ with orthonormal columns such that $\|A - A'\|_F \leq \nu$, then for every $\hat{B} \in \mathbb{R}^{n_2 \times r}$ and every $\hat{C} \in \mathbb{R}^{n_3 \times r}$ we have

$$\min_{\pi \in \mathcal{E}_r} \|A \otimes B \otimes C - (\hat{A} \otimes \hat{B} \otimes \hat{C})\Pi_\pi\|_F \geq \sqrt{2}(1 - \nu)\delta.$$ 

This result implies that, even if steps S3 and S4 of a PBA could perfectly recover the rank-1 terms, the PBA would not be able to compensate the error introduced in the computation of $\hat{\theta}$ in step S2. Moreover, under the assumptions of Lemma 6.2, the condition number of $\hat{\theta}$ is proportional to the condition number of $\tau_{r,n_1,n_2}$. This indicates that the magnification of an input perturbation of a PBA will be roughly proportional to the condition number of the TDP for $n_1 \times n_2 \times 2$ tensors. However, we recall from Section 3 and [5] that there is a great discrepancy between the distribution of the condition numbers of the TDPs for $n_1 \times n_2 \times 3$ and $n_1 \times n_2 \times 2$ tensors, the latter being much larger than the former on average. This will then imply that the excess factor $\omega$ in (6.1) is large. In the next subsections, we exploit Lemmata 6.2 and 6.3 for showing that $\omega$ is actually unbounded.
6.2. Constructing a bad tensor. The role of \( \mathcal{O} \) in Theorem 6.1 will be played by the following tensor. Let \( A' = [a'_i]_{i=1}^n \in \mathbb{R}_{n_1 \times r} \) and \( B' = [b'_i]_{i=1}^n \in \mathbb{R}_{n_2 \times r} \) be matrices with orthonormal columns. Let \( U \) be the \( n_3 \times n_3 \) matrix \( U = [q^+ q] \), where \( q^+ \) is an \( n_3 \times (n_3 - 2) \) matrix whose columns form an orthonormal basis of the complement of the columns of \( Q \). Define the matrix with \( r \) columns

\[
C' := U \left( I_{n_3 \times r} - \frac{2}{n_3} \mathbf{1}_{n_3} \mathbf{1}^T_r \right) \text{diag}(1, -1, \ldots, -1) = \frac{2}{n_3} U \begin{bmatrix}
  \frac{\mathbf{1}^T - 1}{\mathbf{1}} & 1 & 1 \\
  -1 & \frac{\mathbf{1}^T - 1}{\mathbf{1}} & 1 \\
  -1 & 1 & \frac{\mathbf{1}^T - 1}{\mathbf{1}} \\
  \vdots & \vdots & \vdots \\
  -1 & 1 & 1
\end{bmatrix},
\]

where \( \mathbf{1}_k \in \mathbb{R}^k \) is the vector of ones, and \( I_{m \times n} = [e_i]_{i=1}^n \), where \( e_i \) is the \( i \)th standard basis vector of \( \mathbb{R}^m \). By construction, \( C' \) is orthonormal columns. The orthogonally decomposable (odeco) tensor associated with these factor matrices is

\[
\mathcal{O} := \sum_{i=1}^r a'_i \otimes b'_i \otimes e'_i.
\]

It will satisfy the requirements in Theorem 6.1 and complete the proof of instability of PBAs.

It is a very bad omen that \( \mathcal{O} \) is not a valid input for PBAs. This is because the projected tensor \( \rho_Q(\mathcal{O}) \) has a positive-dimensional family of decompositions, implying \( \kappa_{m,r,n_1,n_2} = \infty \). Indeed, we have \( Q^T e'_i = \frac{2}{n_3} [-1, -1, \ldots, -1]_r \) and, since \( n_3 > r + 1 \), for all \( 2 \leq i \leq r \) we have \( Q^T e'_i = \frac{2}{n_3} [1, 1, \ldots, 1]_r \), so that the projected tensor is

\[
\rho_Q(\mathcal{O}) = -\frac{2}{n_3} a'_i \otimes b'_i \otimes [1]_r + \frac{2}{n_3} \sum_{i=2}^r a'_i \otimes b'_i \otimes [1]_r.
\]

By Lemma 3.1, [4, Corollary 1.2], or [13, Lemma 6.5] the condition number of \( \rho(Q) \) is infinite. By taking a neighborhood of \( \mathcal{O} \) the proof of Theorem 6.1 will be completed.

Let \( (\mathcal{O}_1, \ldots, \mathcal{O}_r) \in \mathcal{S}^{x^r} \) be an ordered CPD of \( \mathcal{O} \). Then, the next lemma states that most of the tensors that have a decomposition in

\[
\mathcal{U}_\epsilon = \{ (\mathcal{A}_1, \ldots, \mathcal{A}_r) \in \mathcal{M}_{r,n_1,n_2,n_3} \in \mathcal{S}^{x^r} \mid \| \mathcal{A}_i - \mathcal{O}_i \|_F < \epsilon, i = 1, \ldots, r \}
\]

are valid inputs for a PBA, where \( \mathcal{M}_{r,n_1,n_2,n_3} \) is as in Proposition 4.5. The lemma is proved in the appendix.

**Lemma 6.4.** \( \mathcal{O}_\epsilon = \Phi_{r}(\mathcal{U}_\epsilon) \cap \mathcal{N}^r \) is an open subset of \( \sigma_r \) with \( \mathcal{O} \in \overline{\mathcal{O}_\epsilon} \).

The next result allows us to apply Lemmata 6.2 and 6.3 for tensors in \( \mathcal{O}_\epsilon \).

**Lemma 6.5.** Let \( \epsilon > 0 \) be sufficiently small, and let \( A', B', C' \) be as in the definition of \( \mathcal{O} \) in (6.4). Then, there exists a constant \( S > 0 \) so that for all \( (\mathcal{A}_1, \ldots, \mathcal{A}_r) \in \mathcal{O}_\epsilon \), with factor matrices \( A, B, C \), where both \( A \) and \( B \) have unit-norm columns, the following bounds holds:

\[
\| A - A' \|_F \leq S \epsilon, \quad \| B - B' \|_F \leq S \epsilon \quad \text{and} \quad \| C - C' \|_F \leq S \epsilon.
\]

Moreover, the columns of \( B \odot C \) satisfy \( 1 - S \epsilon \leq \| b_i \otimes c_i \|_F \leq 1 + S \epsilon \).

This lemma is proved in the appendix. Combining these two lemmata with Lemmata 6.2 and 6.3, we get the following important corollary.

**Corollary 6.6 (An r-nice bad tensor).** Let \( \epsilon > 0 \) be sufficiently small. Let \( \mathcal{A} = \sum_{i=1}^r a_i \otimes b_i \otimes c_i \) be an element of \( \mathcal{O}_\epsilon \), such that the factor matrices \( A \in \mathbb{R}_{n_1 \times r} \) and \( B \in \mathbb{R}_{n_2 \times r} \) have unit-norm columns. Then,

1. \( \mathcal{A} \in \mathcal{N}^r \), i.e., \( \mathcal{A} \) is r-nice and its projection \( \mathcal{B} = \rho_Q(\mathcal{A}) \) is also r-nice;
2. there exists an \( \mathcal{A}' \in \mathbb{R}_{n_1 \times r} \) with orthonormal columns, such that \( \| A - A' \|_F \leq S \epsilon \); and
3. \( B \odot C \in \mathbb{R}_{n_3 \times r} \) has columns whose norms are bounded by \( 1 - S \epsilon \leq \| b_i \otimes c_i \| \leq 1 + S \epsilon \).
6.3. Proof of Theorem 6.1. Let $A \in \mathcal{O}$, be as in Corollary 6.6. Its floating-point representation is $\bar{A} := \text{fl}(A)$. We show that the excess factor $\omega(A)$ from (6.1) is proportional to $\epsilon^{-1}$.

We assume that the output of step S1 is the best possible numerical result when providing $A$ as input, namely the floating-point representation of $B = \rho_Q(A) = \sum_{i=1}^{n_2} a_i \otimes b_i \otimes (Q^T c_i)$, i.e., $\bar{B} = \text{fl}(B) = \text{fl}(\rho_Q(A))$. For streamlining the analysis, we ignore further compounding of roundoff errors, assuming the best possible case in which the PBA manages to execute steps S2, S3 and S4 exactly (perhaps by invoking an oracle). Let $\{a_1, \ldots, a_n\} = \bar{\theta}(\bar{B})$ and $A := [a_i]$. Then, by the same construction as in Section 4.3, we have

$$\min_{\pi \in \mathbb{O},} \| A - \bar{A} \|_F \leq \kappa(\bar{B}) \cdot \| B - \bar{B} \|_F,$$

In fact, a small component in the direction of the worst perturbation is expected: From the concentration-of-measure phenomenon, assuming that the perturbation $B - \bar{B}$ is random with no preferred direction, it follows with high probability that the component of the perturbation in the worst direction is of size comparable to $(r \dim S_{n_1, n_2, 2})^{-\frac{1}{2}} = (r(n_1 + n_2))^{-\frac{1}{2}}$. See Armentano’s work [1] for an analysis of the impact of this consideration in the observed value of the so-called stochastic condition number. It follows that there exists a number $1 \geq \beta_1 > 0$ such that

$$\min_{\pi \in \mathbb{O},} \| A - \bar{A} \|_F = \beta_1 \cdot \kappa(\bar{B}) \cdot \| B - \bar{B} \|_F \geq \beta_1 \cdot \kappa(\bar{B}) \cdot \| B - \bar{B} \|_F,$$

where the last inequality is by definition of condition numbers and restrictions of maps. Applying Lemma 6.2 and using the properties from Corollary 6.6, yields

$$\min_{\pi \in \mathbb{O},} \| A - \bar{A} \|_F \geq \frac{\beta_1}{10\gamma} \cdot \kappa(\bar{B}) \cdot \| B - \bar{B} \|_F,$$

where $Z := Q^T C$. Assume that the left-hand side is bounded from above by 1. Regardless of the particular $\{b_i \otimes \hat{c}_i\}$, that the PBA computes in step S3, invoking Lemma 6.3 shows that after completion of step S4 the forward error satisfies

$$\min_{\pi \in \mathbb{O},} \| A \circ B \circ C - (\bar{A} \circ \bar{B} \circ \bar{C}) \|_F \geq \frac{(1 - S\epsilon)\beta_1}{10\gamma} \cdot \kappa(\bar{B}) \cdot \| B - \bar{B} \|_F.$$

Dividing both sides of this expression by $\kappa(\bar{B}) \cdot \| A - \bar{A} \|_F$ gives the excess factor $\omega(A)$:

$$\omega(A) \geq \frac{(1 - S\epsilon)\beta_1}{10\gamma} \cdot \frac{1}{\kappa(\bar{B}) \cdot \| A - \bar{A} \|_F} \cdot \| B - \bar{B} \|_F.$$

We continue by bounding the factor $\| B - \bar{B} \|_F \cdot \| A - \bar{A} \|_F^{-1}$ in this inequality. Since $B = \text{fl}(B)$ and $\bar{A} = \text{fl}(A)$, we have in the standard model of floating-point arithmetic,

$$\| A - \bar{A} \|_F = \| \delta \|_F = \| \delta \|_F = \sum_{i_1, i_2, i_3 = 1}^{n_1, n_2, n_3} (a_{i_1, i_2, i_3} - (1 + \delta_{i_1, i_2, i_3})a_{i_1, i_2, i_3})^2 \leq \epsilon_u \| A \|_F^2,$$

where $|\delta_{i_1, i_2, i_3}| \leq \epsilon_u$, and

$$\| B - \bar{B} \|_F = \sum_{i_1, i_2, i_3 = 1}^{n_1, n_2, n_3} (b_{i_1, i_2, i_3} - (1 + \tilde{\delta}_{i_1, i_2, i_3})b_{i_1, i_2, i_3})^2 \leq \sum_{i_1, i_2, i_3 = 1}^{n_1, n_2, n_3} (\tilde{\delta}_{i_1, i_2, i_3})^2 \leq \epsilon_u \| B \|_F^2,$$

where $|\tilde{\delta}_{i_1, i_2, i_3}| \leq \epsilon_u$. There exists a $\beta_2 \geq 0$ so that $\| B - \bar{B} \|_F = \beta_2 \epsilon_u \| B \|_F$. While a detailed analysis of the value of $\beta_2$ is outside of the scope of this work, it is reasonable to assume that $\beta_2$ is not too small, say $\beta_2 \geq 10^{-1}$. Hence, we need bounds on the norms of $A$ and $B$. To this end,
the following well-known result is useful.

\[ \| \mathbf{o} \|_F = \left\| (A' \otimes B' \otimes C') \sum_{i=1}^{r} \mathbf{e}_i \otimes \mathbf{e}_i \right\|_F = \left\| \sum_{i=1}^{r} \mathbf{e}_i \otimes \mathbf{e}_i \right\|_F = \sqrt{r}, \]

where \( \mathbf{e}_i \) is the \( i \)th standard basis vector of \( \mathbb{R}^r \). Let \( \mathcal{E}_i := \mathcal{A}_i - \mathcal{O}_i \). Note that \( \| \mathcal{E}_i \|_F \leq \epsilon \). The norms of \( \mathcal{A} \) and \( \mathcal{B} \) are then estimated as follows:

\[ \| \mathcal{A} \|_F = \left\| \sum_{i=1}^{r} \mathcal{A}_i \right\|_F = \left\| \sum_{i=1}^{r} (\mathcal{O}_i + \mathcal{E}_i) \right\|_F \leq \| \mathcal{O} \|_F + \sum_{i=1}^{r} \| \mathcal{E}_i \|_F = \sqrt{r}(1 + \sqrt{r}/\epsilon), \]

Exploiting the linearity of the multilinear multiplication \( \rho_Q \) we also have

\[ \| \mathcal{B} \|_F = \| \rho_Q(\mathcal{A}) \|_F = \left\| \rho_Q(\mathcal{O}) + \sum_{i=1}^{r} \rho_Q(\mathcal{E}_i) \right\|_F \geq \| \rho_Q(\mathcal{O}) \|_F - \sum_{i=1}^{r} \| \rho_Q(\mathcal{E}_i) \|_F \geq \| Q^T C' \|_F - r\epsilon, \]

where we used that the 3-flattening of \( \rho_Q(\mathcal{O}) \) is \((Q^T C')(A' \otimes B')^T \) and, since \( A' \) and \( B' \) have orthonormal columns, that \( \| \rho_Q(\mathcal{O}) \|_F = \| Q^T C' \|_F \). By construction, \( Q^T C' = \frac{2}{n_3} \begin{bmatrix} -1 & \ldots & -1 \end{bmatrix} \), so that we have \( \| Q^T C' \|_F = \sqrt{2} \frac{2}{n_3} \). We have thus shown that

\[ \omega(\mathcal{A}) \geq \frac{\kappa(\tau_{r,n_1,n_2})}{\kappa(\tau_{r,n_1,n_2})} \frac{(1 - S\epsilon)\beta_2\sqrt{3}}{10r} \frac{2\sqrt{2r}}{n_3 \sqrt{r}(1 + \sqrt{r}/\epsilon)}. \]

The condition number \( \kappa(\tau_{r,n_1,n_2})(\mathcal{A}) \) is bounded as follows. Let \( \mathbf{a} = (\mathcal{A}_1, \ldots, \mathcal{A}_r) \in \mathcal{S}^{r \times r} \) be an ordered CPD of \( \mathcal{O} \). By Lemma 4.10 (2), for all \( \mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r \) in \( \Phi_r(\mathcal{U}) \), we have \( \kappa(\tau_{r,n_1,n_2})(\mathcal{A}) = \kappa(\mathbf{a}) \), where \( \mathbf{a} = (\mathcal{A}_1, \ldots, \mathcal{A}_r) \) and \( \kappa(\mathbf{a}) \) being the condition number of the tensor rank decomposition from (1.2). Furthermore, \( \kappa(\mathbf{a}) = 1 \) by [4, Proposition 5.2]. From this [4, Theorem 1.1] implies that \( \kappa(\mathbf{a}) \) is the classic spectral 2-norm of the derivative of \( \Phi_{n_3}^{-1} \). Since \( \Phi_{n_3}^{-1} \) is smooth and the spectral norm is a Lipschitz-continuous function with Lipschitz constant 1, it follows that there exists a Lipschitz constant \( \ell > 0 \) such that for sufficiently small \( \epsilon > 0 \), we have \( |\kappa(\tau_{r,n_1,n_2})(\mathcal{A}) - \kappa(\tau_{r,n_1,n_2})(\mathcal{O})| \leq \ell \| \mathcal{A} - \mathcal{O} \|_F \) for all \( \mathcal{A} \in \mathcal{O} \). Hence,

\[ \kappa(\tau_{r,n_1,n_2})(\mathcal{A}) \leq 1 + \ell \epsilon. \]

Finally, we bound \( \kappa(\tau_{r,n_1,n_2})(\mathcal{B}) \). Let \( d_r = r \dim S_{n_1,n_2} \), and recall that \( \mathbf{z}_i := Q^T \mathbf{c}_i \). Recall \( T \) from the proof of Lemma 3.1, applying it to \( \mathcal{B} \)’s CPD. Consider the next submatrix of \( T \),

\[ T' := [\mathbf{I}_{n_1} \otimes \mathbf{b}_1 \otimes \mathbf{e}_{11} \mathbf{a}_2 \otimes \mathbf{I}_{n_2} \otimes \mathbf{e}_{12}] \quad \text{and set} \quad \mathbf{v}' := \begin{bmatrix} \mathbf{z}_1 \mathbf{a}_2 \\ \mathbf{z}_2 \mathbf{b}_1 \end{bmatrix}. \]

Note that \( \| \mathbf{v}' \| = \| \mathbf{z}_1 \|^2 + \| \mathbf{z}_2 \|^2 \). From the identification of condition numbers from Lemma 4.10 and from the steps in the proof of Lemma 3.1 it follows that

\[ \kappa(\tau_{r,n_1,n_2})(\mathcal{B}) = \left( \min_{\mathbf{v}' \in \mathbb{R}^{d_r}} \frac{\| \mathbf{v}' \|}{\| \mathbf{v}' \|} \right)^{-1} \geq \| \mathbf{v}' \| = \frac{\| \mathbf{v}' \|}{\| \mathbf{a}_2 \otimes \mathbf{b}_1 \otimes (\mathbf{z}_1 + \mathbf{z}_2) \|} = \frac{\| \mathbf{v}' \|}{\| \mathbf{z}_1 + \mathbf{z}_2 \|}. \]

We already showed above that \( \| Z - Q^T C' \|_F \leq S\epsilon \) and that \( \mathbf{z}_i' := Q^T \mathbf{c}_i' = \frac{2}{n_3}(-1)^i \begin{bmatrix} \ldots & -1 \end{bmatrix} \) for \( i = 1, 2 \). Note that \( \mathbf{z}_1' + \mathbf{z}_2' = 0 \). Consequently, we get the bounds

\[ \| \mathbf{z}_1 + \mathbf{z}_2 \| = \| (\mathbf{z}_1' + \mathbf{z}_2') + (\mathbf{z}_1 - \mathbf{z}_1') + (\mathbf{z}_2 - \mathbf{z}_2') \| \leq \| \mathbf{z}_1 - \mathbf{z}_1' \| + \| \mathbf{z}_2 - \mathbf{z}_2' \| \leq \sqrt{2} S\epsilon, \]

\[ \| \mathbf{v}' \|^2 = \| \mathbf{z}_1 \|^2 + \| \mathbf{z}_2 \|^2 \geq (\max(0, \| \mathbf{z}_1 \| - \| \mathbf{z}_1 - \mathbf{z}_1' \|))^2 + (\max(0, \| \mathbf{z}_2 \| - \| \mathbf{z}_2 - \mathbf{z}_2' \|))^2. \]

Note that in the last inequality we can bound \( \| \mathbf{z}_i' \| - \| \mathbf{z}_i - \mathbf{z}_i' \| \geq \frac{2}{n_3} S\epsilon - S\epsilon \) for \( i = 1, 2 \). Assuming that \( \epsilon \) is sufficiently small, we obtain \( \| \mathbf{v}' \| \geq \frac{2}{n_3} S\epsilon - \frac{1}{2} = O(\epsilon^{-1}) \). Plugging all of these into (6.8) yields

\[ \kappa(\tau_{r,n_1,n_2})(\mathcal{B}) \geq \frac{2}{n_3} S\epsilon^{-1} - \frac{1}{2} = O(\epsilon^{-1}). \]

Plugging (6.7) and (6.9) into (6.6), the proof of Theorem 6.1 is concluded. This ultimately completes the proof of Theorem 1.2.
Remark 6.7. It is important to observe that the construction of the open set \( \mathcal{O} \) depends on the projection operator \( \rho_Q \) and, hence, on \( Q \in \mathbb{R}^{n_3 \times 2} \). That is, we have shown that regardless of a choice of \( Q \) that is independent of \( \mathcal{A} \), there exists an open set such where the PBA with projection \( \rho_Q \) is unstable. The above construction does not automatically apply to situations where \( Q \) is chosen as a function of the input \( \mathcal{A} \).

7. Numerical experiments

We present the results of some numerical experiments in Matlab R2017b for supporting the main result and exemplifying the behavior of PBAs on third-order random CPDs. They were performed on a computer system consisting of two Intel Xeon E5-2697 CPUs (12 cores, 2.6GHz each) with 128GB of main memory.

Three PBAs are considered in the experiments below, which we refer to as \( \text{cpd}_\text{pba} \), \( \text{cpd}_\text{pba2} \) and \( \text{cpd}_\text{gevd} \), respectively.\(^5\) The first, \( \text{cpd}_\text{pba} \), is an ordinary implementation of the prototypical PBA discussed in Section 1, using ST-HOSVD [44] as orthogonal Tucker compression. \( \text{cpd}_\text{pba2} \) computes the CPD by randomly projecting the input tensor \( \mathcal{A} \) with \( \rho_Q \), then employing the cpd function from Tensorlab v3.0 to recover the two factor matrices \( A \) and \( B \), and finally computing \( A \odot B \odot (\langle A \odot B \rangle^T A_{(3)}^T)^T \) to obtain a representative of the set of rank-1 tensors. The last PBA we consider is the \( \text{cpd}_\text{gevd} \) function from Tensorlab v3.0. The analysis in Section 6 does not strictly apply to the default settings\(^6\) of \( \text{cpd}_\text{gevd} \), because it chooses the projection matrix \( Q \) as a function of the input tensor \( \mathcal{A} \) in \( \mathbb{R}^{n_1 \times n_2 \times n_3} \). Specifically, if \((U_1, U_2, U_3) \cdot S = \mathcal{A} \) is the HOSVD [16], then \( \text{cpd}_\text{gevd} \) chooses \( Q \) as the first two columns of \( U_3 \).

Throughout these experiments, the forward error of the TDP is evaluated as follows. If \( \mathcal{A} = \sum_{i=1}^r a_i \odot b_i \odot c_i \) and \( \mathcal{A}' = \sum_{i=1}^r a_i' \odot b_i' \odot c_i' \), then we recall from (4.4) that

\[
\text{err}_{\text{forward}} := \min_{\pi \in \Theta_r} \| A \odot B \odot C - (A' \odot B' \odot C') P_{\pi} \|_F,
\]

is the forward error. Evaluating all \( r! \) permutations is a Herculean task when \( r \gg 10 \). Fortunately, when \( \mathcal{A} \) and \( \mathcal{A}' \) are very close, the optimal permutation can be found heuristically by solving the linear least-squares problem \( \min_{X \in \mathbb{R}^{r \times r}} \| A \odot B \odot C - (A' \odot B' \odot C') X \|_F \) and then projecting the minimizer to the set of permutation matrices by setting the largest value in every row to 1 and the rest to zero. In all experiments, the forward error is computed in this manner.

7.1. The bad odeco tensor

We start with an experiment to support the analysis of Section 6. Let \( \rho_Q = \text{Id} \odot \text{Id} \odot Q^T \), where \( Q \in \mathbb{R}^{n_3 \times 2} \), be the projection operator of the PBA. Let \( \mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be an \( r \)-nice tensor whose CPD is \( \epsilon \)-close to the odeco tensor (6.4), i.e., \( \mathcal{A} \in \mathcal{O} \), where the latter is as in Lemma 6.4. According to the analysis in Section 6, the excess factor \( \omega(\mathcal{A}) \) of a PBA with projection operator \( \rho_Q \) should behave like \( \mathcal{O}(\epsilon^{-1}) \).

We consider \( 89 \times 29 \times 11 \) tensors. \( Q \in \mathbb{R}^{11 \times 2}, A' \in \mathbb{R}^{89 \times 10} \) and \( B' \in \mathbb{R}^{29 \times 10} \) were respectively generated by computing the \( Q \)-factor of the QR-decomposition of a matrix with i.i.d. standard normal entries. The matrix \( C' \in \mathbb{R}^{11 \times 10} \) was constructed as in the definition of (6.4). Then, \( Q_i := a_i' \odot b_i' \odot c_i' \) for \( i = 1, \ldots, 10 \). For \( k = 1, \ldots, 50 \), we constructed the randomly perturbed

\(^5\)Both \( \text{cpd}_\text{pba} \) and \( \text{cpd}_\text{pba2} \) are provided in the ancillary files to this manuscript; they require some functionality from Tensorlab v3.0 [45].

\(^6\)There is an option to use a random orthonormal projection, in which case the theory of this paper applies.
tensors $\mathbf{X}_{k,i} = \mathbf{A}_k + 2^{-k} \mathbf{X}_{k,i} \mathbf{f}_i$, where $\mathbf{X}_{k,i}$ has i.i.d. standard normal entries. Using the cpd function with default settings from Tensorlab, we then computed the rank-1 approximations $\mathbf{A}_{k,i}$ of $\mathbf{X}_{k,i}$. Let $e_k := \max \{ \| \mathbf{A}_{k,i} - \mathbf{X}_{k,i} \|_F \}$, and then the corresponding tensor is $\mathbf{A}_k = \sum_{i=1}^{10} \mathbf{A}_{k,i}$, so that $\mathbf{A}_k \in \mathcal{O}_n$, with probability $1$. Let $\mathbf{a}^*_k = \{ \mathbf{A}_{k,1}, \ldots, \mathbf{A}_{k,10} \}$ denote the true CPD. A rank-10 CPD $\mathbf{a}_0 \in S^{\times r} / S_r$ of $\mathbf{A}$ was computed numerically using cpd_pba and the forward error relative to $\mathbf{a}^*_k$ was computed. We also applied cpd with default settings to $\mathbf{A}_k$ for numerically computing another rank-10 CPD $\mathbf{a}_0'$. The forward error between $\mathbf{a}^*_k$ and $\mathbf{a}_0'$ was recorded.

The results of the above experiment are shown in Figure 7.1. cpd attains a forward error of approximately $4 \times 10^{-16}$ in all cases. As the random tensors are very close to the edeco tensor, their condition numbers are approximately 1. A forward error equal to a small multiple of the machine precision $1.11 \times 10^{-16}$ is thus anticipated from a stable algorithm. The situation is dramatically different for cpd_pba. Since the edeco tensor was chosen to behave badly with respect to the projection $\rho_{O_{\mathbb{R}^3}}$, we expect from Section 6 that the forward error of the PBA grows like the excess factor $\omega = \mathcal{O}(\epsilon^{-1})$. The dashed line in Figure 7.1 shows the result of fitting the model $k\epsilon^{-1}$ to the data with $\epsilon > 10^{-14}$. As can be seen, the experimental data match the predictions from the theory in Section 6 very well, specifically with regard to the growth rate of the excess factor.

7.2. Distribution of the excess factors. The previous experiment illustrated the forward error in worst possible case that we know of, mainly to illustrate Theorem 1.2. Based on the construction in Section 6, it is not reasonable to expect that this will correspond to the typical behavior. However, the next experiment shows that, unfortunately, one should typically expect a loss of precision of at least a few digits.

The setup is as follows. For each tested tensor shape $n_1 \times n_2 \times n_3$, we generated $10^5$ random rank-$n_2$ CPDs $\{ \mathbf{a}_1 \otimes \mathbf{b}_i \otimes \mathbf{c}_1, \ldots, \mathbf{a}_1 \otimes \mathbf{b}_i \otimes \mathbf{c}_r \}$ by sampling the entries of the vectors $\mathbf{a}_i \in \mathbb{R}^{n_1}$, $\mathbf{b}_i \in \mathbb{R}^{n_2}$ and $\mathbf{c}_i \in \mathbb{R}^{n_3}$ i.i.d. from a standard normal distribution. The corresponding tensor $\mathcal{A} = \sum_{i=1}^{n_r} \mathbf{a}_i \otimes \mathbf{b}_i \otimes \mathbf{c}_i$ was then constructed. We used the three PBAs as well as Tensorlab’s cpd function to compute the CPD from $\mathcal{A}$, recording the forward error. The results are displayed in Figure 7.2. The plots on the left show the empirical cdfs of the forward errors of the four algorithms. The plots on the right show the excess factors of the PBAs.

Recall that cpd by default will use the PBA cpd_gevd as initialization and will then refine its output by running a quasi-Newton method; see [41, 45]. The stopping criterion for cpd was set to $\| \mathcal{A} - \sum_{i=1}^{n_r} \mathcal{A}_i \|_F \leq 2 \sqrt{10^6} \epsilon_u$, where $\epsilon_u \approx 1.1 \times 10^{-16}$ is the unit roundoff of standard double precision floating point arithmetic, and $\mathcal{A}_i$ are the rank-1 tensors. The forward error of cpd will thus be bounded approximately by $2 \sqrt{10^6} \mathcal{A}(\mathcal{A}_1, \ldots, \mathcal{A}_r) \cdot \epsilon_u$. Recalling the shape of the cdfs of the condition number from Figure 3.1, we again note that as $n_3$ increases, the likelihood of large condition numbers diminishes. In fact, most of the generated TDPs were well-conditioned, as can be inferred from the figure by noting that the forward error of cpd is always less than $10^{-11}$.

The loss of precision of the two PBAs is very pronounced in Figure 7.2. Although cpd_gevd is not strictly a PBA, because its projection operator depends on the tensor, its loss of precision in Figure 7.2 asymptotically matches that of the PBAs. Note the seemingly asymptotic log-linear relationship between the probability $\mathbb{P}(\omega > x)$ and $x$ in the right plots in Figure 7.2; that is, it seems plausible that asymptotically $\mathbb{P}(\omega > x) \approx ax^{-1}$ for some $a > 0$. A possible explanation of this behavior follows from our geometrical interpretation of the causes of instability. The inputs $\mathcal{A}$ for which we expect $\omega(\mathcal{A}) \geq x$ with large $x$ are those such that $\mathbf{c}_i \neq \mathbf{c}_j$ and yet $Q^T \mathbf{c}_i \approx Q^T \mathbf{c}_j$ for some $i \neq j$. This is more likely to happen if $n_3$ is large, since $\mathbf{c}_i \in S(\mathbb{R}^{n_3})$ and $Q^T \mathbf{c}_i \in S(\mathbb{R}^3)$. Indeed, the extreme case $Q^T \mathbf{c}_i = Q^T \mathbf{c}_j$, for some $i \neq j$, corresponds to a hypersurface $\mathcal{L}$ of $S(\mathbb{R}^{n_3})$.

We proved in Theorem 1.2 that popular pencil-based algorithms for computing the CPD of low-rank third-order tensors are numerically unstable. Moreover, not only do there exist inputs for which such algorithms are unstable, the numerical experiments suggest that for certain random CPDs the loss of precision is roughly $\mathcal{O}(1 / \log_{10}(\epsilon))$ with probability $\epsilon$. In addition to these results, we bounded the distribution of condition numbers of random CPDs, in Theorem 1.4.
The main conclusion of our work is this: \textit{PBAs should be handled with care,} as the numerical experiments in Section 7 demonstrated that an excess loss of precision is probable. When the most accurate result is sought, we advise to apply a Newton-based refinement procedure to the output of a PBA. This is in fact the default strategy pursued by the \texttt{cpd} function from Tensorlab v3.0. While this strategy is certainly advisable when the input is perturbed only by roundoff errors, it is not clear to us whether employing a PBA for generating a starting point for an iterative method is more effective than a random initialization in the presence of significant (measurement) errors in the input data, both for reasons of conditioning (Theorem 1.4) and stability (Theorem 1.2). We believe that a further study on this point is required.

We hope that the construction of inputs for which PBAs are unstable, in Section 6, offers insights that can help in the design of numerically stable algorithms for computing CPDs. Our
analysis suggests that methods partly recovering the rank-1 tensors from a matrix pencil are numerically unstable in the neighborhood of some adversarially chosen inputs.

Finally, we emphasize that the reason why PBAs are numerically unstable is caused by transforming the tensor decomposition problem into a more difficult computational problem that is nevertheless perceived to be easier to solve, probably because there are direct algorithms for solving them. Here is thus a decidedly positive message that we wish to stress: computing a CPD can be easier, from a numerical point of view, than solving the generalized eigendecomposition problem for a projected tensor. We hope that these observations may (re)invigorate the search for numerically stable algorithms for computing CPDs.

Appendix A. Proof of the Lemmata

The proofs of the technical Lemmata 4.10 and 6.2 to 6.5 are presented.

A.1. Proof of Lemma 4.10. For brevity, let

\[ \mathcal{M} = M_{r; n_1, \ldots, n_d}, \quad \tilde{\mathcal{M}} = \tilde{M}_{r; n_1, \ldots, n_d}, \quad \mathcal{N} = N_{r; n_1, \ldots, n_d}, \quad \text{and} \quad \tau = \tau_{r; n_1, \ldots, n_d}. \]

For (1) we just refer to [35, Section 2.3] which covers our case since the group $\mathcal{G}_r$ acts by isometries on $\mathcal{M}$. Therefore, the induced metric $\tilde{g}$ on $\tilde{\mathcal{M}}$ is the pushforward $\tilde{g} := \hat{\pi}_* g$ of the Riemannian metric $g$ on $\mathcal{M}$ that is inherited from the standard product of inner products on the ambient Euclidean space (namely $(\mathbb{R}^{n_1 \times \cdots \times n_d})^r$ of $\mathcal{M} \subset \mathcal{S} \subset (\mathbb{R}^{n_1 \times \cdots \times n_d})^r$. We denote by $\hat{a}$ the metric on $\mathcal{N}$ which is given by the standard Euclidean inner product $(\cdot, \cdot)$ that $\mathcal{N}$ inherits from the ambient space $\mathbb{R}^{n_1 \times \cdots \times n_d}$.

It will be insightful to describe the metric $\tilde{g}$ on $\tilde{\mathcal{M}}$ more concretely. Let $a = (a_1, \ldots, a_r) \in \mathcal{M}$ be an arbitrary ordered $r$-nice decomposition, and let $\tilde{a} := \hat{\pi}(a)$ denote the corresponding CPD. Let $\hat{\pi}_a^{-1}$ be the smooth local section with $(\hat{\pi}_a^{-1} \circ \hat{\pi})(a) = a$. The pushforward $\tilde{g} = \hat{\pi}_* g$ is defined (see [32, p. 183]) as the map satisfying $\tilde{g}(\hat{\pi}(t), \hat{\pi}(t)) := g_a(d\hat{\pi}_a^{-1}(\hat{a}), d\hat{\pi}_a^{-1}(1))$ for all $\hat{a}, \hat{t} \in T_{\hat{a}} \tilde{\mathcal{M}} \simeq T_{\hat{a}} \mathcal{M}$ where $g_a(b, c) := \sum_{i=1}^r (b_i, c_i)$ with $b_i, c_i \in T_{a_i} \mathcal{S}$. Using the identification $T_{\hat{a}} \tilde{\mathcal{M}} \simeq T_{\hat{a}} \mathcal{M}$ which is given by the isometry $d\hat{\pi}$ we can define $\hat{t} = (t_1, \ldots, t_r)$ with $t_i \in T_{a_i} \mathcal{S}$. Similarly, we can write $\hat{\pi} = \{s_1, \ldots, s_r\}$ with $s_i \in T_{a_i} \mathcal{S}$. Then, it follows that

\[ g_{\tilde{a}}(\hat{t}, \hat{t}) = \sum_{i=1}^r (t_i, s_i) \quad \text{and} \quad \|t\|_{\tilde{\mathcal{M}}, \tilde{a}}^2 = \sum_{i=1}^r \|t_i\|^2 = \left\| t_1, \ldots, t_r \right\|_F^2, \]

where $\|t\|_{\tilde{\mathcal{M}}, \tilde{a}}$ is the induced norm on $T_{\tilde{a}} \tilde{\mathcal{M}}$.

From the foregoing discussion it indeed follows for every choice of $a \in \hat{\pi}^{-1}(\tau(\mathcal{A}))$ that

\[ \kappa[\tau(\mathcal{A})] = \max_{t \in T_{a_1} \mathcal{N}} \frac{\|d\tau(t)\|_{T_{\hat{a}} \tilde{\mathcal{M}}, \tau(\mathcal{A})}}{\|t\|_F} = \max_{t \in T_{a_1} \mathcal{N}} \frac{\|d\hat{\pi}_a^{-1}(d\tau(t))\|_F}{\|t\|_F} = \frac{\max_{t \in T_{a_1} \mathcal{N}} \|d\tau(t)\|_F}{\|t\|_F} = \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_r), \]

where the second equality is by the definition of the metric, the third by the linearity of derivatives, and the final equality is precisely Theorem 1.1 of [4]. This finishes the proof of (2).

Finally, (3) follows from the fact that $\hat{\pi}$ is a local isometry and thus preserves the lengths of curves. Given any curve joining two elements in $\tilde{\mathcal{M}}$, its lift through the covering $\hat{\pi}$ thus has the same length. Since we are free to choose the representative, we thus choose one that minimizes the length of the lifted curve.

\[ \square \]

A.2. Proof of Lemma 6.2. For brevity, we drop all subscripts:

\[ \mathcal{N} = N_{r; n_1, n_2, 2}, \quad \tilde{\mathcal{M}} = \tilde{M}_{r; n_1, n_2, 2}, \quad \mathcal{M} = M_{r; n_1, n_2, 2}, \quad \tilde{S} = \tilde{S}_{r; n_1}, \quad S = S_{r; n_1}, \quad \text{and} \quad \tau = \tau_{r; n_1, n_2, 2}. \]

Consider again the diagram from (6.2). Note that $\mathcal{N}$, $\tilde{\mathcal{M}}$, and $\mathcal{N} \times \tilde{S}$ are manifolds. We claim that $\Theta = \text{Id}_\mathcal{N} \times \theta_{\mathcal{N}}$ and $\tilde{\eta}$ are smooth maps between manifolds. We can explicitly write $\tilde{\eta}$ as

\[ \tilde{\eta} : \mathcal{N} \times \tilde{S} \to \tilde{\mathcal{M}}, \quad (\mathcal{B}_r, \{a_1, \ldots, a_r\}) \mapsto \hat{\pi}(\mathcal{A} \circ (A^T g_{\{1\}} T)), \]

where $A = [a_i]_{r \times 1} \in S$ is a $n_1 \times r$ matrix with the $a_i$’s as columns in any order; $g_{\{1\}} = A(B \otimes Z)^T$ is the 1-flattening [31] of $\mathcal{B} = \sum_{i=1}^r a_i \otimes b_i \otimes z_i$; and with a minor abuse of notation $\hat{\pi}$ is the smooth map that takes a matrix and sends it to the set of its columns. By assumption $r \leq n_1$
so that $S$ is the manifold of matrices with linearly independent unit-norm columns. Therefore, $A^1 = (A^T A)^{-1} A^T$ for all $A \in S$, which is a smooth map. Consequently, $\tilde{\eta}$ is a smooth map, by [32, Proposition 2.10 (d)]. Let $\Psi^*_{n_1, \ldots, n_d}$ be the map from (4.1). Then, we have

$$\tilde{\theta}|_N = \pi \circ (\pi_2 \circ (\Psi^*_{n_1, n_2})^{-1}) \circ \tau,$$

where $\pi_2 : \mathbb{R} \times \{0\} \times S^+(\mathbb{R}^{n_1}) \times S^+(\mathbb{R}^{n_2}) \times \mathbb{R}^{n_1} \to S^+(\mathbb{R}^{n_1})$ projects onto the second factor. The projection $\pi$ is a local diffeomorphism by Lemma 4.3, the coordinate projection $\pi_2$ is smooth, $\Psi^*_{n_1, n_2}$ is a diffeomorphism, and $\tau$ is a diffeomorphism by Proposition 4.7. Therefore, $\theta|_N$ is smooth, by [32, Proposition 2.10(d)], and so $\Theta$ is smooth.

Recall that the spectral norm of a linear operator $F : V \to W$, where $V$ and $W$ are normed vector spaces with respective norms $\| \cdot \|_V$ and $\| \cdot \|_W$, is $\|F\|_{V,W} := \max_{t \in V} \frac{\|F(t)\|_W}{\|t\|_V}$. For composable maps, the foregoing spectral norms are submultiplicative. Since $\tau = \Theta \circ \tilde{\eta}$ is a composition of smooth maps between manifolds, we have that $d_\tau \tau = d_{\Theta \circ \tilde{\eta}} \circ d_{\tilde{\eta}}$. Therefore,

$$\kappa[\tau](\mathcal{A}) := \|d_\tau \tau\|_{N \times \tilde{S}, \Theta}(\mathcal{A}) = \|d_{\Theta \circ \tilde{\eta}}(\mathcal{A})\|_{N \times \tilde{S}, \Theta}(\mathcal{A}) = \|d_{\tilde{\eta}} \Theta(\mathcal{A})\|_{N \times \tilde{S}, \Theta}(\mathcal{A}) = \kappa[\tilde{\eta}](\Theta(\mathcal{A})), $$

where the last step follows from the definition in (2.4). Note that this generalizes (6.3).

We can write the condition number of $\Theta$ as a function of the condition number of $\theta|_N$. Indeed, let $t \in T_{\mathcal{A}}$ be arbitrary, and observe that

$$\|d_{\tilde{\eta}} \Theta(t)\|_{N \times \tilde{S}, \Theta}(\mathcal{A}) = \|d_{\tilde{\eta}} \Theta(t)\|_{N \times \tilde{S}, \Theta}(\mathcal{A}) = \frac{\|d_{\tilde{\eta}}(\Theta(t))\|_{N \times \tilde{S}, \Theta}(\mathcal{A})}{\|d_{\tilde{\eta}} \Theta(t)\|_{N \times \tilde{S}, \Theta}(\mathcal{A})} \leq 1.$$

As a result, we find

$$\frac{\kappa[\tau](\mathcal{A})}{\kappa[\tilde{\eta}](\Theta(\mathcal{A}))} \leq 1 \leq \kappa[\tilde{\eta}](\Theta(\mathcal{A})).$$

The proof will be completed by bounding $\kappa[\tilde{\eta}](\Theta(\mathcal{A}))$ from above. As Riemannian metric on $N \times \tilde{S}$ we choose the product metric of the natural Riemannian metric on $N$, which is inherited from the ambient $\mathbb{R}^{n_1 \times n_2^2} \simeq \mathbb{R}^{n_1 n_2^2}$, and the Riemannian metric that is the pushforward of the standard Euclidean inner product that $S$ inherits from $\mathbb{R}^{n_1 \times n_2}$ via the map $\pi : S \to \tilde{S}$, which is also a local isometry by the same arguments as in the proof of Lemma 4.10. Let $A = [a_i]_{i \in S}$ be a factor matrix of $\mathcal{B} = \sum_{i=1}^{n_1} a_i \otimes b_i \otimes z_i$, which thus imposes an order on the $a_i$’s. Let us denote the other two factor matrices by $B = [b_i]_{i \in S}$, $b_i \in \mathbb{R}^{n_2 \times r}$ (the $b_i$’s are in GLP) and $Z = [z_i]_{i \in \mathbb{R}^{2 \times r}}$. Since $N \times \tilde{S}$ is locally isometric to $N \times S$, there is a local section $\pi_{\mathcal{A}}$ of $\pi$. As $\tilde{\mathcal{M}}$ is locally isometric to $\mathcal{M}$ via $\tilde{\pi}$, there is also a local section $\tilde{\pi}_{\mathcal{A}}^{-1}$ that is consistent with $A$ in the sense that

$$(\tilde{\pi}_{\mathcal{A}}^{-1} \circ \tilde{\eta})(\mathcal{B}, \{a_1, \ldots, a_r\}) = \eta(\mathcal{B}, \pi_{\mathcal{A}}^{-1}(\{a_1, \ldots, a_r\})),$$

where $\eta(\mathcal{B}, A) := A \circ (A^T \mathcal{B})^T$. We have that $\kappa[\eta](\mathcal{B}, A) = \kappa[\tilde{\eta}](\mathcal{B}, \{a_1, \ldots, a_r\})$ because of the local isometries. Hence, we can study $\kappa[\eta](\mathcal{B}, A)$ instead.

The derivative of $\eta$ is computed as follows. We note that

$$(dA^\dagger)(\mathcal{A}) = (d(A^T A)^{-1} A^T)(\mathcal{A}) = (A^T A)^{-1} A^T + (A^T A)^{-1}(A^T A + A^T \hat{A})(A^T A)^{-1} A^T$$

$$= (A^T A)^{-1}(A^T + A^T \hat{A})(A^T A)^{-1},$$

where $\hat{A}$ is a tangent vector in $T \mathcal{A} S_{r, n_1}$. We find that

$$(d_{\mathcal{B}, A}) \eta(\mathcal{B}, \mathcal{A}) = A \circ (A^T \mathcal{B}_{(1)})^T + \hat{A} \circ (A^T \mathcal{B}_{(1)})^T$$

$$+ A \circ ((A^T A)^{-1}(A^T + (A^T A)(A^T A)) \mathcal{B}_{(1)})^T.$$ 

Now, by definition of the Riemannian metrics

$$(A.2) \quad \kappa[\eta](\mathcal{B}, A) = \max_{\|\mathcal{B}\|_{\mathcal{B}}^2 + \|A\|_{\mathcal{A}}^2 = 1} \|d_{\mathcal{B}, A} \eta(\mathcal{B}, \mathcal{A})\|.$$
Let \((\mathcal{B}, \hat{A})\) be a maximizer of \((A.2)\). Note that \(\|\mathcal{B}\|_F \leq 1\) and \(\|\hat{A}\|_F \leq 1\). Since \(A \odot (A^T R_{11})^T\) is a submatrix of \(A \odot (A^T R_{11})^T\), it follows that \(\|A \odot (A^T R_{11})^T\|_F \leq \|A\|_F \|A^T R_{11}\|_F\). Exploiting this inequality and the triangle inequality a few times, we obtain

\[
\kappa[\mathcal{B}, A] \leq \|A\|_F \|A^T R_{11}\|_F + \|A^T R_{11}\|_F + \|A\|_F \|A^T A\|^{-1} \left( \hat{A}^T + (\hat{A}^T A + A^T \hat{A}) A \right) R_{11}\|_F.
\]

The right-hand side is a Lipschitz continuous function in \((\mathcal{B}, A) \in \mathbb{R}^{n \times n_{22}} \times \mathbb{R}^{n_{11} \times r}\), say with Lipschitz constant \(\kappa > 0\).

By assumption there is a matrix \(A' = [a'_{ij}]\), with orthonormal columns with \(\|A - A'\|_F < \nu\). Let \(\mathcal{B}'\) be the tensor with factor matrices \(A', B, Z\); that is, \(\mathcal{B}' := \sum_{i=1}^r a' \otimes b_i \otimes z_i\). Then, by the triangle inequality and the computation rules for inner products of rank-1 tensors from \((2.1)\),

\[
\|\mathcal{B}' - \mathcal{B}\|_F \leq \sum_{i=1}^r \|a_i - a'_i\| \otimes b_i \otimes z_i = \sum_{i=1}^r \|a_i - a'_i\|_F \|b_i \otimes z_i\|_F \leq \nu \sqrt{1 + \nu},
\]

where the last step is because \(\|b_i \otimes z_i\|_F < 1 + \nu\) for each \(i\). This shows that

\[
\|(\mathcal{B}, A) - (\mathcal{B}', A')\|_F \leq \sqrt{\nu^2 (1 + \nu)^2 + \nu^2} = \nu \sqrt{(1 + \nu)^2 + 1}.
\]

Assume that \(\nu \leq 1\) and let us write \(L := \kappa(\mathcal{B}, A) = \nu \sqrt{(1 + \nu)^2 + 1} + 1\). Then, using the Lipschitz continuity from \((A.1)\) we find

\[
\kappa[\mathcal{B}, A] \leq \sqrt{\nu^2 (1 + \nu)^2 + \nu^2} = \sqrt{(1 + \nu)^2 + 1}.
\]

Recall that for matrices \(X, Y\) we have the inequality \(\|XY\|_F \leq \min\{\|X\|_2 \|Y\|_F, \|X\|_F \|Y\|_2\}\). Observe that \(\|A'\|_F = 1\) and \(\|(\mathcal{B}_1')\|_2 = 1\). Exploiting these we obtain

\[
\kappa[\mathcal{B}, A] \leq \sqrt{(1 + \nu)^2 + 1} = \nu \sqrt{(1 + \nu)^2 + 1}.
\]

Finally, we have \(\|A'\|_F \leq \|A\|_F\). Hence, \(\|A'\|_F \leq 1\). Then, since \(\mathcal{B}_1' = \mathcal{B}'(B \otimes Z)^T\), we also have \(\|\mathcal{B}'_1\|_2 \leq \|B \otimes Z\|_2 \leq \|B\|_F \|Z\|_F \leq \sqrt{(1 + \nu)}\). This shows

\[
\kappa[\mathcal{B}, A] \leq \sqrt{(1 + \nu)^2 + 1} \leq \nu \sqrt{(1 + \nu)^2 + 1}.
\]

where in the last step we assumed that \(\nu \leq 1\). Plugging this into \((A.1)\) finishes the proof.

### A.3. Proof of Lemma 6.3

Observe that \(\mathcal{B} \odot \mathcal{C}\) can naturally be regarded as a matrix in the space \(\mathbb{R}^{n_{22} \times n_{33} \times r}\). Therefore,

\[
\varepsilon := \min_{\pi \in \mathcal{S}} \|A \odot B \odot C - (\hat{A} \odot \hat{B} \odot \hat{C}) P_\pi\|_F \geq \min_{\pi \in \mathcal{S}} \min_{M \in \mathbb{R}^{n_{22} \times n_{33} \times r}} \|A \odot B \odot C - (\hat{A} \odot M) P_\pi\|_F,
\]

where \(P_\pi\) is the permutation matrix corresponding to \(\pi\). Let \(\pi \in \mathcal{S}\) be any permutation. Then,

\[
\min_{M \in \mathbb{R}^{n_{22} \times n_{33} \times r}} \|A \odot B \odot C - (\hat{A} \odot M) P_\pi\|_F = \min_{M \in \mathbb{R}^{n_{22} \times n_{33} \times r}} \|A \odot B \odot C - (\hat{A} P_\pi) \odot M\|_F,
\]

where the last step is because of the definition of the Khatri–Rao product, and because every \(M \in \mathbb{R}^{n_{22} \times n_{33} \times r}\) can be factored as \((MP_\pi^{-1}) P_\pi\) since \(P_\pi\) is invertible. Let \(m_1, \ldots, m_r\) be the columns of \(M\). Then, we have that

\[
(A.3) \quad \|A \odot B \odot C - (\hat{A} P_\pi) \odot M\|_F^2 = \sum_{i=1}^r \|a_i \otimes (b_i \otimes c_i) - \tilde{a}_{\pi_i} \otimes m_i\|_F^2
\]

is a sum of squares, so that we can minimize each \(m_i\) separately. The first-order necessary optimality conditions are

\[
(\tilde{a}_{\pi_i} \otimes m_i) \odot (b_i \otimes c_i) - \tilde{a}_{\pi_i} \otimes m_i = 0, \quad i = 1, \ldots, r.
\]

Solving for \(m_i\) yields the unique solution \(m_i = (\tilde{a}_{\pi_i}, a_i) \otimes b_i \otimes c_i\). Plugging this minimizer into the right-hand side of \((A.3)\), we find

\[
\|a_i - (\tilde{a}_{\pi_i}, a_i) \tilde{a}_{\pi_i}\|_F^2 = \|a_i - (\tilde{a}_{\pi_i}, a_i) \tilde{a}_{\pi_i}\|_F^2 \|b_i \otimes c_i\|_F^2 \geq (1 - \nu^2) \|a_i - (\tilde{a}_{\pi_i}, a_i) \tilde{a}_{\pi_i}\|_F^2,
\]

where we used the computation rules for inner products from \((2.1)\) in the first step, and the assumption that \(\|b_i \otimes c_i\|_F \geq 1 - \nu\) in the last step. From this it follows that

\[
\min_{M \in \mathbb{R}^{n_{22} \times n_{33} \times r}} \|A \odot B \odot C - (\hat{A} P_\pi) \odot M\|_F^2 \geq (1 - \nu^2) \|A - \hat{A} P_\pi \text{ diag}(\tilde{a}_{\pi_1}, a_1, \ldots, (\tilde{a}_{\pi_r}, a_r))\|_F^2.
\]
Let us define $\zeta_\pi := \|A - \tilde{A}P_\pi \text{diag}(\tilde{a}_{\pi}, a_1, \ldots, \tilde{a}_{\pi}, a_r)\|_F$. We claim that the minimizer of $\min_{\pi \in \Theta_\pi} \zeta_\pi$ equals the minimizer $\pi^*$ of $\min_{\pi \in \Theta_\pi} \|A - \tilde{A}P_\pi\|_F$. To prove this, we show that $\zeta_\pi > \min_{\pi \in \Theta_\pi} \zeta_\pi$ by exhibiting an upper bound for $\zeta_\pi$ that is smaller than a lower bound for $\zeta_\pi$ with $\pi \neq \pi^*$. Note that $\langle \tilde{a}_{\pi^*}, a_i \rangle = \langle a_i - f_i, a_i \rangle = 1 - \langle f_i, a_i \rangle$. Hence,

$$\zeta_\pi = \|A - \tilde{A}P_\pi + \tilde{A}P_\pi \text{diag}(\tilde{f}_1, a_1, \ldots, \tilde{f}_i, a_i)\|_F \leq \|A - \tilde{A}P_\pi\|_F + \|\tilde{A}P_\pi \text{diag}(\tilde{f}_1, a_1, \ldots, \tilde{f}_i, a_i)\|_F \leq \delta + \|\tilde{A}\|_F \|P_\pi\|_2 \|\text{diag}(\tilde{f}_1, a_1, \ldots, \tilde{f}_i, a_i)\|_2 = \delta + \sqrt{r} \max_{1 \leq i \leq r} \|\langle f_i, a_i \rangle\| \leq \delta(1 + \sqrt{r}),$$

where the last step is due to the Cauchy–Schwarz inequality. Next, we lower bound $\zeta_\pi$ with $\pi' \neq \pi^*$. In this case, there is always some $k$ such that $\pi_k = \pi_{j_k}$ with $j \neq k$. Then,

$$\|a_k - (\tilde{a}_{\pi'}, a_k)\tilde{a}_{\pi'}\|^2 = \|a_k - \tilde{a}_{\pi'}\|^2 \geq \|a_k - (\tilde{a}_{\pi'}, a_k)\tilde{a}_{\pi'}\|^2 = 1 - (\tilde{a}_{\pi'}, a_k)^2.$$ 

Note that for all $i = 1, \ldots, r$ we have that

$$0 \leq \|a_i' - \tilde{a}_{\pi'}\| = \|a_i' - (a_i + f_i)\| \leq \|a_i' - a_i\| + \|f_i\| \leq \nu + \delta,$$

where $f_i := a_i - \tilde{a}_{\pi'}$, and where we used $\delta_i := \|f_i\| = \|a_i - \tilde{a}_{\pi'}\| \leq \|A - \tilde{A}P_\pi\|_F = \delta$ in the last step. Therefore, we have

$$\|\tilde{a}_{\pi'}\| = \|a_i' + f_i, a_k' + (a_k - a_k')\| \leq \|a_i' + f_i, a_k' + (a_k - a_k')\| \leq \|a_i' + f_i, a_k' + (a_k - a_k')\| \leq 0 + \|a_k - a_k'\| \|f_i\| \|f_j\| \|a_k - a_k'\| \leq \nu + \delta.$$ 

It follows that we have the following lower bound

$$\zeta_\pi^2 = \sum_{i=1}^r \|a_i - (\tilde{a}_{\pi'}, a_i)\tilde{a}_{\pi'}\|^2 \geq \|a_j - (\tilde{a}_{\pi'}, a_j)\tilde{a}_{\pi'}\|^2 = 1 - (\tilde{a}_{\pi'}, a_j)^2 \geq 1 - (\nu + \delta)^2.$$ 

When both $\nu$ and $\delta$ are sufficiently small, we have

$$\zeta_\pi < (1 + \sqrt{\delta}) \delta < \sqrt{1 - (\nu + \delta)^2} \leq \zeta_\pi$$

for all $\pi' \neq \pi^*$. This indeed proves that $\pi^*$ is also the minimizer of $\min_{\pi \in \Theta_\pi} \zeta_\pi$.

Combining the foregoing results, we find

$$\varepsilon^2 \geq (1 - \nu)^2 \min_{\pi \in \Theta_\pi} \zeta_\pi^2 = (1 - \nu)^2 \sum_{i=1}^r \|a_i - (\tilde{a}_{\pi'}, a_i)\tilde{a}_{\pi'}\|^2.$$ 

As before we have $\|a_i - (\tilde{a}_{\pi'}, a_i)\tilde{a}_{\pi'}\|^2 = 1 - (\tilde{a}_{\pi'}, a_i)^2$. By the law of cosines $(\tilde{a}_{\pi'}, a_i) = 1 - \frac{1}{4} \delta_i^2$, so that $1 - (\tilde{a}_{\pi'}, a_i)^2 = \delta_i^2(1 - \frac{1}{4} \delta_i^2)$. Since $\delta_i \leq \delta < 1$, we find

$$\varepsilon^2 \geq \min_{\pi \in \Theta_\pi} \|A \otimes B \otimes C - (\tilde{A} \otimes \tilde{B} \otimes \tilde{C})\|_F^2 \geq (1 - \nu)^2 \sum_{i=1}^r \delta_i^2(1 - \frac{4}{4} \delta_i^2) \geq \frac{3}{4} (1 - \nu)^2 \delta^2,$$

because $\delta^2 = \sum_{i=1}^r \delta_i^2$. This concludes the proof. 

\[ \Box \]

**A.4. Proof of Lemma 6.4.** Recall that $\rho_Q = \text{Id}_{\mathbb{R}^{n_1}} \otimes \text{Id}_{\mathbb{R}^{n_2}} \otimes QT$. Both $\sigma_r(S_{n_1, n_2, n_3})$ and $\sigma_r(S_{n_1, n_2, n_3})$ are generically $r$-identifiable by Lemma 4.4 because of the assumption on $r$. The image $\Phi_r(\mathbb{U}_r/\Theta_r)$ is open because $\Phi_r$ is a diffeomorphism onto its image and $\mathbb{U}_r/\Theta_r \subset M^r_{n_1, n_2, n_3}$ is an open submanifold by construction. The key step consists of showing that

$$N^r = \rho_Q^{-1}(N_{r, n_1, n_2, n_3}) \cap N_{r, n_1, n_2, n_3}$$

is open dense in $\sigma_r(S_{n_1, n_2, n_3})$. By Proposition 4.7, we already know that $N_{r, n_1, n_2, n_3}$ is open dense, so that it suffices to prove that $\rho_Q^{-1}(N_{r, n_1, n_2, n_3})$ is dense in $\sigma_r(S_{n_1, n_2, n_3})$. We show this next.

Let $\mathcal{A} \in \sigma_r(S_{n_1, n_2, n_3})$ be arbitrary. We let $\mathcal{B} := \rho_Q(\mathcal{A})$ and write

$$\mathcal{A} = \sum_{i=1}^r a_i \otimes b_i \otimes c_i, \text{ and } \mathcal{B} = \sum_{i=1}^r a_i \otimes b_i \otimes z_i,$$

where $a_i \in \mathbb{R}^{n_1}, b_i \in \mathbb{R}^{n_2}, c_i \in \mathbb{R}^{n_3}, z_i \in \mathbb{R}^2$. 

Let us decompose $c_i = Qz_i + Q^\perp z_i'$ where $Q^\perp \in \mathbb{R}^{n_3 \times (n_3-2)}$ is a matrix whose columns form an orthonormal basis of the orthogonal complement of the space spanned by the columns of $Q$ and $z_i' \in \mathbb{R}^{n_3-2}$. Consider a generic sequence such that

$$\lim_{k \to \infty} a_i^{(k)} = a_i, \quad \lim_{k \to \infty} b_i^{(k)} = b_i,$$

and $z_i^{(k)} = z_i$.

Note that $\mathcal{S}_i^{(k)} := a_i^{(k)} \otimes b_i^{(k)} \otimes z_i^{(k)}$ lives in $\mathcal{S}_{n_1,n_2,2}$ by construction. As the sequence is arbitrary and $\mathcal{M}_{r; n_1, n_2, 2}$ is open dense in $\mathcal{S}_{n_1,n_2,2}$ by Proposition 4.5, we can assume that the sequence is restricted so that all $(\mathcal{S}_i^{(k)}, \ldots, \mathcal{S}_i^{(k)}) \in \mathcal{M}_{r; n_1, n_2, 2}$. Taking the quotient with the symmetric group $\mathcal{S}_r$, we get by Proposition 4.6: $\{\mathcal{S}_i^{(k)}, \ldots, \mathcal{S}_i^{(k)}\} \in \mathcal{M}_{r; n_1, n_2, 2}$. Note that $\Phi_\ast (\{\mathcal{S}_i^{(k)}, \ldots, \mathcal{S}_i^{(k)}\}) = \sum_{i=1}^r \mathcal{S}_i^{(k)} \in \mathcal{N}_{r; n_1, n_2, 2}$ by Proposition 4.7. Now, let

$$\mathcal{A}_i^{(k)} := a_i^{(k)} \otimes b_i^{(k)} \otimes (Qz_i^{(k)} + Q^\perp z_i').$$

Then, $\rho Q(\mathcal{A}_i^{(k)}) = \mathcal{S}_i^{(k)}$ so that $\mathcal{A}_i^{(k)} \in \rho_Q^{-1}(\mathcal{N}_{r; n_1, n_2, 2})$. Now observe that $\lim_{k \to \infty} \sum_{i=1}^r \mathcal{A}_i^{(k)} = \mathcal{A}$; in other words, $\mathcal{A} \in \rho^{-1}(\mathcal{N}_{r; n_1, n_2, 2})$. Since it was arbitrary, this proves the claim.

A.5. Proof of Lemma 6.5. Recall from (4.1) the map $\Psi_{n_1,n_2,n_3}$ and that it is a diffeomorphism. There is a natural isomorphism between $\mathbb{R} \setminus \{0\} \times \mathbb{S}^r(\mathbb{R}^{n_1})$ and $\mathbb{R}^{n_3} \times \{0\}$, so that

$$\Psi^* : \mathbb{S}^r(\mathbb{R}^{n_1}) \times \mathbb{S}^r(\mathbb{R}^{n_2}) \times \{0\} \to \mathbb{S}(\mathbb{R}^{n_3}) \ni (x,y,z) \mapsto x \otimes y \otimes z$$

also is a diffeomorphism. The reason for introducing $\Psi^*$ is that it is difficult to ensure that the tensor $\mathcal{O}$ lies in the image of $\Psi^*$. Nevertheless, $\mathcal{O}$ lies in the image of $\Psi^*$. Since $\Psi^*$ is a diffeomorphism, there is a Lipschitz constant $\ell$ so that for all $i = 1, \ldots, r$, we have

$$\|a_i - b_i'\|_F \leq \ell, \quad \|b_i - b_i'\|_F \leq \ell, \quad \|c_i - c_i'\|_F \leq \ell,$$

where the norm on the left-hand side is the standard product norm of the Euclidean norms on $\mathbb{S}(\mathbb{R}^{n_1})$, $\mathbb{S}(\mathbb{R}^{n_2})$, and $\mathbb{R}^{n_3}$. In particular, this implies:

$$\|A - A'\|_F < \sqrt{\ell} \epsilon, \quad \|B - B'\|_F < \sqrt{\ell} \epsilon, \quad \|C - C'\|_F < \sqrt{\ell} \epsilon. $$

Hence, for $S \geq \sqrt{\ell} \ell$ the first part of the lemma holds. For the second part, we write $\Delta b_i := b_i - b_i'$ and $\Delta c_i := c_i - c_i'$. Then, we have

$$b_i \otimes c_i = b_i' \otimes c_i' + b_i' \otimes \Delta c_i + \Delta b_i \otimes c_i' + \Delta b_i \otimes \Delta c_i.$$

By the definition of the odeco tensor $\mathcal{O}$ in (6.4), we have $\|b_i\| = \|c_i\| = 1$. Using the triangle inequality and the computation rules for inner products from (2.1), we get

$$\|b_i \otimes c_i\|_F - \|b_i' \otimes c_i'\|_F \leq \|b_i' \otimes \Delta c_i\|_F + \|\Delta b_i \otimes c_i'\|_F + \|\Delta b_i \otimes \Delta c_i\|_F = \|b_i'\| \|\Delta c_i\|_F + \|\Delta b_i\| \|c_i'\|_F + \|\Delta b_i\| \|\Delta c_i\|_F \leq 2 \ell + \ell^2 \ell.$$ 

Since $\|b_i' \otimes c_i'\| = 1$, taking $S \geq \max \{ (\ell + 1) \ell, \sqrt{\ell} \ell \}$ finishes the proof. \hfill \Box

References

[1] D. Armentano, Stochastic perturbations and smooth condition numbers, J. Complexity 26 (2010), no. 2, 161–171.
[2] L. Blum, F. Cucker, M. Shub, and S. Smale, Complexity and real computation, Springer-Verlag, New York, 1998. MR 1479636 (99a:68070)
[3] J. Bochnak, M. Coste, and M. Roy, Real Algebraic Geometry, Springer–Verlag, 1998.
[4] P. Breiding and N. Vannieuwenhoven, The condition number of join decompositions, SIAM J. Matrix Anal. Appl. 39 (2018), no. 1, 287–309.
[5] , On the average condition number of tensor rank decompositions, arXiv:1801.01673 (2018), submitted.
[6] , A Riemannian trust region method for the canonical tensor rank approximation problem, SIAM J. Optim. (2018), accepted.
[7] R. P. Brent, On the precision attainable with various floating-point number systems, IEEE Trans. Computers C-22 (1973), no. 6, 601–607.
[8] P. Bürgisser, M. Clausen, and M. A. Shokrollahi, Algebraic Complexity Theory, Grundlehren der mathematischen Wissenschaften, vol. 315. Springer, Berlin, Germany, 1997.
[9] P. Bürgisser and F. Cucker, Condition: The Geometry of Numerical Algorithms, Grundlehren der mathematischen Wissenschaften, vol. 349, Springer–Verlag, 2013. MR 3098452
[10] T. Cai, J. Fan, and T. Jiang, Distributions of angles in random packing on spheres, J. Mach. Learn. Res. 14 (2013), 1837–1864.
