Statistical theory of hierarchical avalanche ensemble

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The statistical ensemble of avalanche intensities is considered to investigate diffusion in ultrametric space of hierarchically subordinated avalanches. The stationary intensity distribution and the steady-state current are obtained. The critical avalanche intensity needed to initiate the global avalanche formation is calculated depending on noise intensity. The large time asymptotic for the probability of the global avalanche appearance is derived.

I. INTRODUCTION

In recent years considerable study has been given to the theory of self-organized criticality (SOC) that explains avalanche dynamics in vast variety of systems type of sandpile model [1], intermittency in biological evolution [2], earthquakes and propagation of forest-fires, depinning transitions in random medium and so on (see [3]). Without any special tuning of their external parameters, such systems evolve to a critical state that displays universality in both spatiotemporal behaviour and avalanche distribution over size $s$ and intensity $f$ . So, the latter is given by homogeneous function

$$P(s, f) = s^{-\tau} g(x), \quad x \equiv s/s_c,$$

where a cutoff size $s_c$ is determined as

$$s_c \sim (f_c - f)^{-1/\sigma},$$

$f_c$ is a critical intensity, $\tau$, $\sigma$ are critical exponents. Such relations are obtained as result of making use of scaling-type arguments supplemented with extensive computer simulations [4] as well as within framework of the theory of branching processes [5]. These methods allow to find the magnitudes of the parameters $f_c$, $\tau$, $\sigma$, but as to the function $g(x)$ in Eq.(1), it is known only that this function is a monotonically falling down.

This work is devoted to determination of the scaled function $g(x)$ within the framework of statistical theory that deals with avalanche ensemble in the course of SOC progressing. The cornerstone of our approach is hierarchical representation of the avalanche ensemble that is embedded into ultrametric space corresponding to hierarchical landscape of the system energy [6].

The paper is organized as follows. Sec.II deals with the theory of hierarchical coupling between elementary avalanches arising from the complexity of the phase space landscape of self-organized system. The time dependence of the probability of the global avalanche appearance is studied in Sec.III. Sec.IV contains discussion of obtained results.

II. INTENSITY DISTRIBUTION IN HIERARCHICAL AVALANCHE ENSEMBLE

Let the maximum number of hierarchically subordinated avalanches $N$ be on the bottom hierarchical level $s = 0$ where the avalanche intensity equals $f$. Correspondingly, there is the only avalanche of the intensity $F \gg f$ on the top level $s = s_c \gg 1$. The problem is to find the dependencies $N(s)$ and $F(s)$ that define the distribution of avalanche number and their intensity over hierarchical levels $s \in [0, s_c - 1]$.

The first part of the problem can be approached in terms of geometry by representing the avalanche ensemble as a hierarchical Cayley tree [7,8]. The basic types of the trees are shown in Fig.1: regular tree with integer branching ratio $j$, regular Fibonacci tree with fractional one $j = \tau \approx 1.618$, degenerate tree with the only branching node per level and the tree of our primary concern – irregular tree. Let $k$ be the numbering index for the levels, so that $k$ increases from the top level to the bottom one. The variable

$$s = s_c - k$$

(3)
then defines the distance in the ultrametric space. Geometrically, objects of this space correspond to the nodes of the bottom level \(k = s_c\) of a Cayley tree. Since the distance between the nodes is defined by the number of steps to a common ancestor, the distance is eventually the level number \(\text{(3)}\), counted from the bottom.

As it can be seen in Fig.1a, in the simplest case of regular tree with integer branching ratio \(j\) the number of avalanches \(N_k = j^k\) exponentially decays to zero with the distance \(s\) between them:

\[
N(s) = N \exp(-s \ln j), \quad N \equiv j^{s_c}. \quad (4)
\]

In Eq. (4) the equality (3) is used and the avalanche number \(N\) is related to the total number of levels \(s_c\). For the Fibonacci tree (see Fig. 1b), where \(N_k = q\tau^k\), \(q \approx 1.171\), \(\tau \approx 1.618\), we have

\[
N(s) = N \exp(-s \ln \tau), \quad N \equiv q\tau^{s_c}. \quad (5)
\]

When Eq. (5) is compared with Eq. (4), it is clear that the exponential decay remains unaltered in the case of fractional branching ratio and characterizes the regularity of tree.

For the degenerate tree (see Fig.1c) \(N_k = (j - 1)k + 1\) and Eq.(3) provides the following linear dependence

\[
N(s) = N - (j - 1)s, \quad N \equiv (j - 1)s_c + 1. \quad (6)
\]

It can be shown that in the case of irregular tree, displayed in Fig.1d, the power law dependence is realized:

\[
N_k = k^a, \quad a > 1. \quad (7)
\]

Indeed, the latter can be regarded as an intermediate case between the exponential Eqs.(4), (3) and linear Eq.(3) obtained for the limiting cases of regular and degenerate trees, respectively. Formally, the approximation (7) means that a function \(N(x)\) defined on the self-similar set of hierarchically subordinated avalanches is homogeneous, \(N(kx) = k^a N(x)\). It is convenient to rewrite Eq.(7) in term of the distance:

\[
N_k = N(1 - s/s_c)^a, \quad N \equiv s_c^a, \quad a > 1. \quad (8)
\]

Now, let us define \(F_k\) as total intensity of avalanches on the \(k\)-th level. If one considers the value \(F_k\) as an effective density of some particles and the level number \(k\) as a coordinate, then corresponding density of hierarchical current can be taken in the Onsager-type form:

\[
j_k = -D(F_k) \frac{dF_k}{dk}. \quad (9)
\]

Here, within the multiplicative noise approach, the effective diffusion coefficient has form

\[
D(F) = D F^{-\alpha} \quad (10)
\]

to depend on the constant \(D > 0\) and the exponent \(\alpha\).

The basic assumption of this section is that the total current \(J\) of all avalanches at given hierarchical level is independent on the level number \(k\):

\[
j_k N_k = \text{const} \equiv J. \quad (11)
\]

Inserting Eqs.(8)-(10) into Eq.(11) gives the total avalanches intensity on the level \(k\)

\[
F_k = F_k^{-b}, \quad F^{1-\alpha} = \frac{1 - \alpha}{a - 1} J, \quad b \equiv \frac{a - 1}{1 - \alpha} > 0 \quad (12)
\]

normalized by the maximum value \(F \equiv F_{k=1}\). Introducing the distance (3), we obtain

\[
F(s) = f(1 - s/s_c)^{-b}, \quad (13)
\]

where the intensity at the bottom level \(s = 0\) is

\[
f \equiv F s_c^{-b} = F N^{-b/a}. \quad (14)
\]

After generalizing Eqs.(13), (14), the following scaling relation can be assumed
\[ F_k = N^{b/a} k^{-b} f_k \]  
\[ \frac{dx}{d\kappa} = -\frac{\partial V}{\partial x} \]  
\[ \kappa \equiv \ln k^b, \quad x \equiv f_k/f_c, \quad f_c^{1-\alpha} \equiv (J/bD) N^{-(a-1)/a} \]  

where \( f_k \) is a slowly varying function. According to Eqs. (9)-(11), this function obeys the Landau-Khalatnikov equation:

\[ \frac{dx}{d\kappa} = -\frac{\partial V}{\partial x} \]  

where one denotes

\[ \kappa \equiv \ln k^b, \quad x \equiv f_k/f_c, \quad f_c^{1-\alpha} \equiv (J/bD) N^{-(a-1)/a} \]

and the effective potential is introduced

\[ V = \frac{x^{1+\alpha}}{1+\alpha} - \frac{x^2}{2} \]  

As indicated in Fig. 2, the potential \( V \) reaches its maximum value \( V_c = (1 - \alpha)/2(1 + \alpha) \) at \( x = 1 \) and decreases indefinitely at \( x > 1 \). So, in order to initiate the global avalanche formation, a low intensity avalanche with \( f < f_c \) at the bottom level needs to penetrate the barrier \( V_c \) of the potential (18). It implies fluctuation mechanism for the SOC regime progressing provided that \( x \) is a stochastic variable and we proceed with Langevin-type equation derived from Eq. (16) by adding a Gaussian white noise to the right-hand side:

\[ \frac{dx}{d\kappa} = -\frac{\partial V}{\partial x} + \zeta, \]  

where the noise intensity \( D \) equals the diffusion coefficient in Eq. (10).

The usual way to study a set of solutions to the stochastic equation (19) is to introduce distribution function \( g(\kappa, x) \) associated with the probability of solution’s realization. It is known that \( g(\kappa, x) \) obeys the Fokker-Planck equation [11]:

\[ \frac{\partial g}{\partial \kappa} + \frac{\partial j}{\partial x} = 0, \quad j \equiv -g \frac{\partial V}{\partial x} - D \frac{\partial g}{\partial x}. \]  

Since there is no current at the equilibrium state \( (j = 0) \), the corresponding distribution function of avalanche intensities at the bottom level

\[ g_0(x) \propto \exp \left( -\frac{V(x)}{D} \right) \]  

is dictated by the potential (18). In the case of non-equilibrium steady state the probability density \( g \) does not depend on the hierarchical level variable \( \kappa \) and the current \( j \) being constant, in compliance with the conservation law (11), can take a non-zero value. From Eq. (21) the stationary distribution then is expressed in terms of the equilibrium distribution \( g_0(x) \) and the current \( j \) [12]:

\[ \frac{g(f)}{g_0(f)} = \frac{j}{D} \int_{f/f_c}^{\infty} \frac{dx}{g_0(x)}, \]  

where the boundary condition \( g \to 0 \) as \( f \to \infty \) is taken into account.

Given the intensity \( f \) equation (23) allows the current \( j \) to be found. In trying to do it, special consideration should be given to the fact that the intensity \( f \) is bounded from below, \( f > G \), by a gap \( G \) that is inherent in hierarchical ensemble of avalanches [4]. Indeed, after merging of avalanches within a hierarchical cluster of the size \( s_g \), all \( s \), such that \( s < s_g \), are appeared to be dropped out the consideration as well as low intensities with \( f < f(s_g) \equiv G \) (see Fig.1). The expression for the current \( j \) then can be derived from Eq. (23) with the second boundary condition \( g(G) = g_0(G) \). The result reads

\[ j = 2DW \left\{ 1 + \text{erf} \left[ \sqrt{\frac{1-a}{2D}} \left( 1 - \frac{G}{f_c} \right) \right] \right\}^{-1}, \]  

3
where the factor

\[ W \propto \exp(-V_c/D), \quad V_c = \frac{1}{2} \frac{1 - \alpha}{1 + \alpha} \]  

(25)
gives the probability that fluctuation will surmount the barrier \( V_c \) of the potential \( (28) \). Equation \( (24) \) shows that in the case of small gap, \( G \ll f_c \), the current \( j \) is about \( WD \), but the current is doubled under \( G = f_c \). It can be understood if we picture the effect of the gap as a mirror that reflects diffusing particles at the point \( f = G \): if \( G < f_c \) a particle penetrating the barrier can move along both directions, but in the case of \( G = f_c \) the mirror is placed at the point corresponding to the top of the barrier and all particles go down the side where the intensity \( f \) grows indefinitely.

Given the current \( j \) the stationary distribution function \( g(f) \) is defined by Eq.\( (23) \), according to which, \( g(f) \approx g_0(f) \) in the subcritical region \( f < f_c \), while in the supercritical range \( f \gg f_c \) we have \( g_0(f) \gg g(f) \) due to indefinite increase of \( g_0(f) \). As far as the stationary distribution is concerned, it can be derived from the current definition \( (21) \), where the last diffusion term is negligible for supercritical intensities: \( j \approx -(\partial V/\partial x)g \). The result is that the probability \( g(f) \) remains almost unaltered \( (g(f) \approx g(f_c)) \) in the range from the critical value \( f_c \) up to the boundary one \( f_g \) and \( g(f) \approx 0 \) at \( f \gg f_g \) (see \( (22) \)). The growth of \( f_g \) is governed by the equation

\[ \frac{df_g}{d\kappa} = D \frac{f_g - f_c}{f_g^2} \]  

(26)

that is a counterpart of the known gap-equation presenting the system behaviour below the critical value \( f_c \) \( \equiv \).

Since the above picture is essentially statistical, it enables the critical avalanche intensity \( f_c \) for the transition point to be found. Indeed, when the definition of the macroscopic current \( J \) in Eq.\( (1) \) is compared to that of the microscopic current \( j \) in Eq.\( (21) \), it is apparent that they differ from one another only by the factor \( N^{(a-1)/a} \equiv s_c^{-1} \) dependent on the total number of avalanches \( N \). On this basis, the last expression of Eq.\( (17) \) and Eq.\( (24) \) at \( G = 0, D \ll 1 \) gives the desired result:

\[ f_c = F \exp(-D_0/D), \quad D_0^{-1} = 2(1 + \alpha), \]  

(27)

where the pre-exponent factor \( F \) determines the probability of the barrier penetrating and cannot be calculated within the framework of the presented approach. Equation \( (27) \) bears a resemblance to the well-known result of the superconductivity BCS theory for the temperature of the phase transition and predicts the slow growth of the critical intensity \( f_c \) of elementary avalanche with the hierarchical diffusion coefficient \( D \) that plays the role of the parameter of effective interaction.

**III. KINETICS OF THE GLOBAL AVALANCHE FORMATION**

Since the ensemble of hierarchically subordinated avalanches represents a self-similar set, the probability distribution \( P(s, f) \) of avalanches in the course of SOC process is a homogeneous function \( (1) \), where \( g(s/s_c(f)) = g(f) \) is the stationary distribution of elementary avalanches considered in the previous section. Physically, Eq.\( (1) \) implies that the total intensity \( F \), being measured by the scale \( N^{b/a} \), equals the intensity of an elementary avalanche \( f \) in accordance with Eq.\( (14) \).

In this section we are aimed to describe kinetics of the global avalanche formation produced by virtue of the hierarchical coupling between elementary avalanches. As it has been clarified in Sec.\( \( \text{I} \), this process can be conceived of as diffusion in ultrametric space that makes the distribution \( (1) \) mounted. In order to find the conditional probability \( \overline{P}(t) \) that no global avalanche will appear at time \( t \) one has to integrate over \( s \) the distribution \( (1) \) weighted with the function

\[ p_s(t) = \exp(-t/t(s)), \quad t(s) = t_0 \exp(F(s)/D) \]  

(28)
descriptive of Debay relaxation with the time \( t(s) \) governed by the barrier height \( F(s) \) \( (t_0 \) is a microscopic time). By using the steepest descent method, it is not difficult to derive the late time \( (t \to \infty) \) asymptotic formula

\[ \overline{P}(t) = \left( \frac{f}{D} \right)^{\tau/b} \left[ 1 - \left( \frac{D}{f} \ln \frac{t}{t_{ef}} \right)^{-1/b} \right]^{-\tau}, \quad t_{ef} \equiv \frac{\tau b}{\alpha} \left( \frac{f}{D} \right)^{1/b} t_0. \]  

(29)

This equation has been obtained by assuming that the condition \( 1 \ll s_m \leq s_c \) is met, where \( s_m \) denotes the location of the maximum of integrand and obeys the equation
\[
\frac{D\tau (1 - y)^{1+b}}{b f y} = \frac{t}{t_0} \exp \left(-\frac{f}{D}(1 - y)^{-b}\right), \quad y = \frac{s_m}{s_c}.
\] (30)

Taking into consideration the scaling relation (2) for the number of hierarchical levels \(s_c\), which is the cut-off parameter, we readily come to the conclusion that the condition is satisfied provided

\[
f - f_c \ll \frac{t}{t_0} \exp \left((f_c/D)^{-1/b} - 1\right)^{-b}. \quad (31)
\]

So, the intensity \(f\) in Eqs. (29), (30) can be replaced by the critical value \(f_c\). In accordance with Eq. (29), the probability \(P(t) = 1 - \mathcal{P}(t)\) of the global avalanche appearance logarithmically increases in time up to the value \(P = 1 - (f_c/D)^{\tau/b}\).

In order for the probability \(P\) to be non-negative, the factor \(F\) in Eq. (27) must be equal \(F_0 \equiv (e/2)(1 + \alpha)^{-1}\), whereas the effective diffusion coefficient \(D\) must be bounded from above by the value \(D_0 \equiv (1/2)(1 + \alpha)^{-1}\).

**IV. DISCUSSION**

According to the presented picture, the initiated elementary avalanches form statistical ensemble of hierarchically subordinated objects, characterized by intensity \(f\) and distance in ultrametric space \(s\) (the latter corresponds avalanche size (4)). Since the global avalanche formation is caused by effective diffusion in the space, then, similar to Brownian particle with coordinate \(f\) at time \(s\), the ensemble can be described by Langevin equation (19) subjected to the noise Eq. (20) with \(D\) being the effective diffusion coefficient and corresponding to the Fokker-Planck equation (21). The stationary intensity distribution and the steady-state current are given by Eqs. (23), (24). The condition of current conservation Eq. (11) yields the avalanche intensity distribution (13) over hierarchical clusters in the ultrametric space.

The ensemble of elementary avalanches, being weakly dependent on \(s\), is governed by the effective potential (18) that reaches its maximum at the critical intensity (27) (see Fig. 2). So, the global avalanche generation requires supercritical elementary avalanche intensity, \(f > f_c\), to surmount the barrier \(V_c\) with the characteristic time (cf. Eq. (25))

\[
T \approx t_0 \exp \left(V_c/D\right), \quad V_c = \frac{1}{2} \frac{1 - \alpha}{1 + \alpha}.
\] (32)

This picture bears some resemblance with the formation process of supercritical embryo in the theory of the first-order phase transitions (12), where in the course of transformation the next stage is the diffusion growth of the embryo. Analogously, in the case under consideration the above growth implies an increase of the supercritical avalanche in intensity \(F(s)\), Eq. (13), due to the diffusion growth of hierarchical cluster in ultrametric space. As a result of the total cluster formation, we have the logarithmically slow large time asymptotic for the probability of the global avalanche appearance:

\[
\mathcal{P}(t) = 1 - \mathcal{P} \left[1 - \mathcal{P}^{1/\tau} \left(\ln \frac{t - T}{t_{ef}}\right)^{-1/b}\right]^{-\tau}, \quad t_{ef} \equiv (\tau/b)\mathcal{P}^{1/\tau} t_0,
\] (33)

where time \(t\) is counted from the instant \(T\), Eq. (32), and \(\mathcal{P}\) is the maximum probability that no global avalanche will occur

\[
\mathcal{P} = \left(\frac{D_0}{D}\right)^{\tau/b} \exp \left[-\frac{\tau}{b} \left(\frac{D_0}{D} - 1\right)\right].
\] (34)

From Eq. (34) the probability is determined by the ratio of the noise intensity \(D\) (see Eq. (20)) and its maximum value \(D_0 = (1/2)(1 + \alpha)^{-1}\). The key point is that the maximum probability \(\mathcal{P} \equiv 1 - \mathcal{P}\) of the global avalanche appearance is completely suppressed under high intensity variance in ensemble of elementary avalanches (see Fig. 3).
FIGURE CAPTIONS

FIG. 1. Different types of hierarchical trees (the level number is indicated at left, corresponding number of nodes – at right): a) regular tree with $j = 2$; b) Fibonacci tree; c) degenerate tree with $j = 3$; d) irregular tree.

FIG. 2. The effective potential (18) as a function of $f/f_c$ at $\alpha = 0.1$.

FIG. 3. The dependence of the maximum probability $P \equiv 1 - \overline{P}$ of the global avalanche appearance on the intensity variance $D$ in ensemble of elementary avalanches.
