Level-rank duality of $SU(2)_k$ Chern-Simons theory, and of hypergraph and magic states

Howard J. Schnitzer

Martin Fisher School of Physics, Brandeis University, Waltham, Massachusetts 02453, USA

E-mail: schnitzr@brandeis.edu

ABSTRACT: The level-rank duality of $SU(2)_k$ Chern-Simons theory is discussed, and applied to graph, hypergraph, and magic states.
1 Introduction

Level-rank duality of Chern-Simons theory and WZW theories has a long history, [1–6] with many applications. It generally involves transposition of the Young tableaux of the representations for a pair of theories, linked by the duality. However, in general level-rank duality is not a one-to-one map between the tableaux of the two theories. For example, $SU(2)_3$ has four states, while $SU(3)_2$ has six. However, using cominimal equivalent or simple current maps relating states, [5] a one-to-one map of the representations is possible. Note that $SU(2)_3/Z_3$ and $SU(3)_2/Z_2$ have just two states related one-to-one by transposition of the respective Young tableaux. The generalization of this theme is central to this paper.

It is the purpose of this paper to discuss the level-rank duality of $SU(2)_k$ and its relationship to $SU(k)_2$. The generalized Pauli group can be defined for each side of the duality, which can then be used to construct graph and hypergraph states for the level-rank pairs. Since a sub-set of hypergraph states describe magic states, level-rank duals of magic states also results.

2 Level-rank dualities of $SU(2)_k$

The integral representations of $SU(2)_k$ Chern-Simons theory can be described by a single-row Young tableau with $0 \leq k$ boxes, while that of $SU(k)_2$ is described by a Young tableau with two columns, $l_1$ and $l_2$, where $l_1 \geq l_2$, and $l_1$ ranges from 0 to $k - 1$. Thus, $SU(2)_k$ has $k + 1$ states, while $SU(k)_2$ has $\frac{1}{2}k(k + 1)$ states. Therefore a transpose map between the representations of $SU(2)_k$ and $\tilde{SU}(k)_2$ is not one-to-one, where $\tilde{SU}$ indicates the transpose of the Young tableau. However, making use of the orbits of cominimal maps or equivalently simple current maps, [5] one can obtain level-rank dual maps which are one-to-one for the Young tableaux. The two cases that we focus on are

$$SU(2)_{2k} = S\tilde{U}(2k)_2/Z_k$$  \hspace{1cm} (2.1)

which has $2k + 1$ states, and

$$SU(2)_{2k+1}/Z_2 = S\tilde{U}(2k + 1)_2/Z_{2k+1}$$  \hspace{1cm} (2.2)

which has $k + 1$ states. In the next subsection we consider the generalized Pauli group for these two cases.
2.1 Generalized Pauli group

The qudit Pauli group for a $d$-dimensional system, for both $d$ odd and even, is described by Farinholt [7]. One defines the operators

$$ x = \sum_{x \in \mathbb{Z}_d} |x + 1\rangle\langle x| $$

(2.3)

and

$$ Z = \sum_{x \in \mathbb{Z}_d} \omega^x |x\rangle\langle x| $$

(2.4)

where $\omega = \exp \left( \frac{2\pi i}{d} \right)$ is a primitive root of unity. We focus on $d$-odd for simplicity for the dual pairs (2.1) and (2.2). For the dual pairs of (2.1), $d$ is odd for any integer $k$. However, for (2.2), $d$ odd requires $k$ even.

With these restrictions, the operators $X$ and $Z$ are defined for the level-rank dual pairs (2.1) and (2.2). Given the level-rank dual pairs (2.1) and (2.2), and the restriction to $d$ odd, identify $X = \tilde{X}$ and $Z = \tilde{Z}$ for the Pauli operators of both sides of (2.1) and (2.2). Thus the dual pairs of $X = \tilde{X}$ and $Z = \tilde{Z}$ enables one to define level-rank dual pairs of graphs and hypergraph states, which we consider in the next sub-section.

2.2 Graph and hypergraph states

Graph states There are many equivalent constructions of graph states [8–12]. We follow arxiv:1612.06418 for a definition of qudit graph states. The multigraph is $G = (V, E)$, with vertices $V$ and edges $E$, where an edge has multiplicity $m_e \in \mathbb{Z}_d$. To $G$ associate a state $|G\rangle$ such that to each vertex $i \in V$, there is a local state

$$ |+\rangle = |p_0\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle $$

(2.5)

Define

$$ S^* |0\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle $$

$$ = |+\rangle = |p_0\rangle. $$

(2.6)

To each edge $e = \{i, j\}$ apply the unitary

$$ Z_e^{m_e} = \sum_{q_i=0}^{d-1} |q_i\rangle\langle q_i| \otimes (Z_j^{m_e})^{q_i} $$

(2.7)
to the state

\[ |+\rangle^V = \bigotimes_{i \in V} |+\rangle_i \tag{2.8} \]

The graph state is

\[ |G\rangle = \prod_{e \in E} Z_{m_e} |+\rangle^V \tag{2.9} \]
\[ = \prod_{e \in E} Z_{m_e} \bigotimes_{i \in V} |+\rangle_i \tag{2.10} \]

Every stabilizer state is LC equivalent to a graph state, while the Clifford group enables conversion between different multigraphs [8–12].

**Hypergraph states** We again follow arxiv:1612.06418 for the construction of qudit multi-hypergraph states. Given a multi-hypergraph \( H = (V, E) \), associate a quantum state \( |H\rangle \), with \( m_e \in \mathbb{Z}_d \) the multiplicity of the hyperedge \( e \). To each vertex \( i \in V \), associate a local state

\[ |+\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle \tag{2.11} \]

To each hyperedge \( e \in E \), with multiplicity \( m_e \), apply the controlled unitary \( Z_{m_e} \) to the state

\[ |+\rangle^V = \bigotimes_{i \in V} |+\rangle_i \tag{2.12} \]

The hypergraph state is

\[ |H\rangle = \prod_{e \in E} Z_{m_e} |+\rangle^V \tag{2.13} \]

and the elementary hypergraph state is

\[ |H\rangle = \sum_{q=0}^{d-1} |q\rangle\langle q| \otimes \left( Z_{m_e} \right)^q |+\rangle^V \tag{2.14} \]

For \( d \) prime, all \( n \)-elementary hypergraph states are equivalent under SLOCC.

Hypergraph and graph states admit a representation in terms of Boolean functions,

\[ |H\rangle = \sum_{q=0}^{d-1} \omega_f(q) |q\rangle \tag{2.15} \]
with \( f : \mathbb{Z}_d^n \to \mathbb{Z}_d \), where
\[
f(x) = \sum_{i_1, \ldots, i_k \in V} x_{i_1} \cdots x_{i_k} \tag{2.16}
\]
For graph states, \( f(x) \) is quadratic, i.e.
\[
f(x) = \sum_{i_1, i_2 \in V} x_{i_1} x_{i_2} \tag{2.17}
\]
while for \( f(x) \) cubic or higher, \(|H\rangle\) is a hypergraph state. Therefore, for quadratic \( f(x) \), one has a representation of stabilizer states, up to LC equivalence. For \( f(x) \) cubic or higher, \(|H\rangle\) represents hypergraph states which contain “magic” states. Examples of magic states are the CCZ state and Toffoli states, constructed from appropriate gates. Thus
\[
\text{CCZ}|x_1 x_2 x_3\rangle = \omega^{x_1 x_2 x_3} |x_1 x_2 x_3\rangle \tag{2.18}
\]
with
\[
|\text{CCZ}\rangle = \text{CCZ}|+^{\otimes 3}\rangle \tag{2.19}
\]
as an example of a magic hypergraph state. Similarly
\[
|\text{Toff}\rangle = \text{Toff}|+^{\otimes 3}\rangle \tag{2.20}
\]
Explicitly,
\[
\text{Toff}|i, j, k\rangle = |i, j, ij + k, \mod d\rangle \tag{2.21}
\]

### 2.3 Level-rank duality

The level-rank duality of the Pauli operators \( X, \tilde{X}, Z, \tilde{Z} \) in 2.1 allows one to express the graph and hypergraph states (2.7), (2.9), (2.10), (2.13), (2.14) as the level-rank duals of graph and hypergraph states for \( SU(2)_{2k} \) and for \( SU(2)_{2k+1}/\mathbb{Z}_2 \) \((k \text{ even})\). Since the hypergraph states contain magic states, there is a one-to-one map between such magic states and their level-rank duals.

It has been shown, using level-rank duality, that a universal topological quantum computer based on Chern-Simons theory for \( SU(2)_2 \) [13] also implies an analogous universal quantum computer based on \( SU(3)_2 \) [14]. However, this result depends on the level-rank duality of the Jones representation of the braid group, which differs from the duality discussed in this paper.

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