CHARACTERIZATION OF QUADRIC SURFACES IN TERMS OF COORDINATE FINITE TYPE GAUSS MAP

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ABSTRACT. In this article, we introduce an important class of surfaces, namely, quadrics in the Euclidean 3-space $\mathbb{E}^3$. We prove that planes, spheres and circular cylinders are the only quadric surfaces whose Gauss map $G$ satisfies a relation of the form $\Delta^I G = MG$, where $M$ is a square matrix of order 3 and $\Delta^I$ is the Laplace-Beltrami operator corresponding to the first fundamental form $I$ of the surface.

1. Introduction

Let $z : M^2 \to \mathbb{E}^3$, be the position vector field of a surface $M^2$ in the 3-dimensional Euclidean space $\mathbb{E}^3$. For any two vectors $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3) \in \mathbb{E}^3$, the inner product on $\mathbb{E}^3$ is

$$A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3.$$ 

The Euclidean vector product $A \times B$ of $A$ and $B$ is defined as follows:

$$A \times B = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The concept of surfaces of finite Chen type was born in the year 1973 and became a hot topic of interest in the field of differential geometry and geometric analysis. An Euclidean submanifold is said to be of finite Chen type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian $\Delta$ [13]. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space. In this respect, many authors, shed light on the notion of submanifolds of finite type Gauss map. See for example [1, 4, 15, 18, 19, 20]

Later, a new type of research was generated by investigating surfaces whose Gauss map $G$ satisfies a relation of the form

$$\Delta^I G = MG,$$

where $M$ is a square matrix of order 3. In [8] two classes of surfaces were studied, namely, ruled surfaces and tubes.

F. Dillen, and others in [16] studied the class of surfaces of revolution, while in [8] authors studied the Lorentz-Minkowski version for the same class. Later in [10, 11, 12] Ch. Baikoussis and L. Verstraelen studied the
translation surfaces, the helicoidal surfaces, and the spiral surfaces. In [22] authors studied translation surfaces of finite type in Sol3. H. Al-Zoubi and others investigated the tubes in $E^3$ [2, 4]. Finally, in [9] the compact and noncompact cyclides of Dupin were studied.

Following the same ideas of [17], it is interesting to study surfaces in $E^3$ whose Gauss map satisfies the relation

$$\Delta^I G = MG,$$

where $M \in \mathbb{R}^{3 \times 3}$.

In this present paper, we will firstly, create a formula for $\Delta^I r$ and $\Delta^I G$ by using Cartan’s method of the moving frame. Further, we will focus our interest by studying the class of quadrics in $E^3$. Our main theorem is

**Theorem 1.** Planes, circular cylinders and spheres are the only quadrics in $E^3$ whose Gauss map $G$ satisfying (1.2).

2. Basic concepts

Let $r = r(u, v)$ be a regular parametric representation of a surface $Q$ in $E^3$. A moving frame of the surface $Q$ can be represented by the set of vectors $\mathbb{h} = \{\xi_1(u, v), \xi_2(u, v), \xi_3(u, v)\}$, where $\det(\xi_1, \xi_2, \xi_3) = 1$. Moreover, we can choose $\xi_3$ to be the Gauss map $G$ of $Q$. Hence there exist five linear differential forms $\varpi_1, \varpi_2, \varpi_{31}, \varpi_{32}$ and $\varpi_{12}$, such that [5, 6]

$$dr = \varpi_1 \xi_1 + \varpi_2 \xi_2, \quad dG = \varpi_{31} \xi_1 + \varpi_{32} \xi_2,$$

$$d\xi_1 = \varpi_{12} \xi_2 - \varpi_{31} \xi_3, \quad d\xi_2 = -\varpi_{12} \xi_1 - \varpi_{32} \xi_3,$$

and functions $q_1, q_2, a, b, c$ of the variables $u$ and $v$ such that

$$\varpi_{31} = -a \varpi_1 - b \varpi_2, \quad \varpi_{32} = -b \varpi_1 - c \varpi_2, \quad \varpi_{12} = q_1 \varpi_1 + q_2 \varpi_2.$$

We can choose the set $h$, in such way that the principal directions of $Q$ are the vectors $\xi_1, \xi_2$. Then for the functions $a, c$ and $b$ we get $b = 0$ and $a, c$ are the principal curvatures of $Q$, hence the differential forms $\varpi_1$ and $\varpi_2$ reduce to

$$\varpi_1 = -\frac{1}{a} \varpi_{31}, \quad \varpi_2 = -\frac{1}{c} \varpi_{32}.$$

The Gauss curvature and the mean curvature of $Q$ are the following

$$K = ac, \quad H = \frac{1}{2}(a + c).$$

We consider a function $h(u, v) \in C^1$. Then $\nabla_1 h, \nabla_2 h$ denote the derivatives of Pfaff of $h$ along the curves $\varpi_2 = 0, \varpi_1 = 0$ respectively. Thus we have [21]

$$\nabla_1 \xi_1 = q_1 \xi_2 + aG, \quad \nabla_2 \xi_1 = q_2 \xi_2 + bG,$$

(2.1)
\[ \nabla_1 \xi_2 = -q_1 \xi_1 + bG, \quad \nabla_2 \xi_2 = -q_2 \xi_1 + cG, \]
\[ \nabla_1 r = \xi_1, \quad \nabla_2 r = \xi_2, \]
\[ \nabla_1 G = -a \xi_1, \quad \nabla_2 G = -c \xi_2. \]

The Mainardi-Codazzi equations are

\[ \nabla_1 c = q_2 (a - c), \quad \nabla_2 a = q_1 (a - c). \]

Let \( h \) be a sufficient differentiable function on \( Q \). The second Beltrami operator \( \Delta^I \) of \( Q \) is defined by

\[ \Delta^I h = -\nabla_1 \nabla_1 h - \nabla_2 \nabla_2 h - q_2 \nabla_1 h + q_1 \nabla_2 h. \]

For the position vector \( r \) relation (2.6) becomes

\[ \Delta^I r = -\nabla_1 \nabla_1 r - \nabla_2 \nabla_2 r - q_2 \nabla_1 r + q_1 \nabla_2 r. \]

From (2.3) we obtain

\[ \Delta^I r = -\nabla_1 \xi_1 - \nabla_2 \xi_2 - q_2 \xi_1 + q_1 \xi_2. \]

Using (2.1) and (2.2), equation (2.7) becomes

\[ \Delta^I r = -q_1 \xi_2 - aG + q_2 \xi_1 - cG - q_2 \xi_1 + q_1 \xi_2. \]

Taking into account the last equation and relation (2), we finally obtain

\[ \Delta^I r = -2HG. \]

We focus our interest now on computing \( \Delta^I G \). Inserting the position vector \( G \) in (2.6) gives

\[ \Delta^I G = -\nabla_1 \nabla_1 G - \nabla_2 \nabla_2 G - q_2 \nabla_1 G + q_1 \nabla_2 G. \]

Using equations (2.2), we find

\[ \Delta^I G = \nabla_1 (a \xi_1) + \nabla_2 (c \xi_2) + aq_2 \xi_1 - cq_1 \xi_2, \]

which becomes

\[ \Delta^I G = (\nabla_1 a) \xi_1 + a(\nabla_1 \xi_1) + (\nabla_2 c) \xi_2 + c(\nabla_2 \xi_2) + aq_2 \xi_1 - cq_1 \xi_2. \]

Taking into account equations (2.1) and (2.2), we get

\[ \Delta^I G = (\nabla_1 a) \xi_1 + aq_1 \xi_2 + a^2 G + (\nabla_2 c) \xi_2 - cq_2 \xi_1 + c^2 G + aq_2 \xi_1 - cq_1 \xi_2, \]

or

\[ \Delta^I G = (\nabla_1 a - [c - a]q_2) \xi_1 + (\nabla_2 c - [c - a]q_1) \xi_2 + [a^2 + c^2]G. \]

Using Mainardi-Codazzi equations (2.5), we get

\[ \Delta^I G = (\nabla_1 a + \nabla_1 c) \xi_1 + (\nabla_2 c + \nabla_2 a) \xi_2 + (a^2 + c^2)G. \]
Once we have
\[(\nabla_1 a + \nabla_1 c)\xi_1 + (\nabla_2 c + \nabla_2 a)\xi_2 = 2\nabla_1 H\xi_1 + 2\nabla_2 H\xi_2 = \text{grad}^I 2H,\]
and
\[4H^2 - 2K = (a^2 + c^2).\]
We finally obtain
\[\Delta^I G = \text{grad}^I 2H + (4H^2 - 2K)G.\]

3. Main result

We consider now a quadric surface \(Q\) in \(\mathbb{E}^3\). Then we have the following three cases Case I. \(Q\) is ruled, a case that has been studied in [7] and it was proved

**Theorem 2.** Among the ruled surfaces in \(\mathbb{E}^3\), the only ones whose Gauss map satisfies (1.2) are the planes, and the circular cylinders.

Case II. \(Q\) is of the form
\[z^2 = r - px^2 - qy^2, \quad p, q, r \in \mathbb{R}, \quad pq \neq 0, \quad r > 0,\]
Case III. \(Q\) is of the form
\[z = \frac{p}{2}x^2 + \frac{q}{2}y^2, \quad p, q \in \mathbb{R}, \quad p, q > 0.\]

We first prove that a surface of the form (3.1) never satisfies (1.2) unless only \(p = q = -1\), that is \(Q\) is a part of a sphere. Next we prove that a surface of the kind (3.2) is never satisfying (1.2).

**3.1. Quadrics of the first type.** This type is parameterized as follows
\[r(u, v) = (u, v, \sqrt{r + pu^2 + qv^2}).\]

For simplicity, we denote \(r + pu^2 + qv^2\) by \(\rho\). Then, using the natural frame \(\{r_u, r_v\}\) of \(Q\) defined by
\[r_u = \left(1, 0, \frac{pu}{\sqrt{\rho}}\right),\]
and
\[r_v = \left(0, 1, \frac{qv}{\sqrt{\rho}}\right),\]
the components \(g_{ij}\) of the metric \(I\) are
\[g_{11} = r_u \cdot r_u = \frac{\rho + (pu)^2}{\rho},\]
\[g_{12} = r_u \cdot r_v = \frac{pquv}{\rho},\]
\[g_{22} = r_v \cdot r_v = \frac{\rho + (qv)^2}{\rho}.\]
Hence the Laplacian $\Delta^I$ of $Q$ is

$$
\Delta^I = \frac{1}{\Phi} \left[ (\rho + q^2 v^2) \frac{\partial^2}{\partial u^2} - 2pqvw \frac{\partial^2}{\partial u \partial v} + (\rho + p^2 u^2) \frac{\partial^2}{\partial v^2} \right]
$$

\[(3.3)\]

$$
+ \frac{\Omega}{\Phi^2} \left[ \frac{pu}{\partial u} + qv \frac{\partial}{\partial v} \right],
$$

where $\Phi := \det[g_{ij}] = p(p + 1)u^2 + q(q + 1)v^2 + r$ and $\Omega := pr + qr + pq(p + 1)u^2 + pq(q + 1)v^2$.

For the normal vector $G$ of $Q$, we have

$$
G = \frac{ru \times rv}{\sqrt{\Phi}}.
$$

After a simple calculations becomes

$$
G = \frac{1}{\sqrt{\Phi}} \left(-pu, -qv, \sqrt{\rho}\right).
$$

Let $(n_1, n_2, n_3)$ the components of the vector $G$, and by $\mu_{rs}, r, s = 1, 2, 3$ the entries of the matrix $M$. From \[(1.2)\], we have

\[(3.4)\] $\Delta^I n_1 = \Delta^I \left(- \frac{pu}{\sqrt{\Phi}} \right) = \mu_{11} \left(- \frac{pu}{\sqrt{\Phi}} \right) + \mu_{12} \left(- \frac{qv}{\sqrt{\Phi}} \right) + \mu_{13} \left(\sqrt{\rho} \right)\text{,}

\[(3.5)\] $\Delta^I n_2 = \Delta^I \left(- \frac{qv}{\sqrt{\Phi}} \right) = \mu_{21} \left(- \frac{pu}{\sqrt{\Phi}} \right) + \mu_{22} \left(- \frac{qv}{\sqrt{\Phi}} \right) + \mu_{23} \left(\sqrt{\rho} \right)\text{,}

\[(3.6)\] $\Delta^I n_3 = \Delta^I \left(- \frac{\sqrt{\rho}}{\sqrt{\Phi}} \right) = \mu_{31} \left(- \frac{pu}{\sqrt{\Phi}} \right) + \mu_{32} \left(- \frac{qv}{\sqrt{\Phi}} \right) + \mu_{33} \left(\sqrt{\rho} \right)\text{.}

inserting $n_1, n_2$ of $G$ in \[(3.3)\], therefore from \[(3.4)\] and \[(3.5)\], we conclude

$$
\Delta^I \left(- \frac{pu}{\sqrt{\Phi}} \right) = \frac{pu}{\Phi^2} \left[p^2q(p + 1)^2(q + 1)u^4 + f(u, v)\right]
$$

$$
= \mu_{11} \left(- \frac{pu}{\sqrt{\Phi}} \right) + \mu_{12} \left(- \frac{qv}{\sqrt{\Phi}} \right) + \mu_{13} \left(\frac{\sqrt{\rho}}{\sqrt{\Phi}} \right),
$$

which turns into

$$
\frac{pu}{\Phi^2} \left[p^2q(p + 1)^2(q + 1)u^4 + f(u, v)\right] =
$$

\[(3.7)\]

$$
-\mu_{11} pu - \mu_{12} qv + \mu_{13} \sqrt{\rho},
$$
where
\[
f(u, v) = q^2(q + 1)^2(4p^2 - 3pq + 3p - 2q)v^4 \\
+ pr(p + 1)(2q^2 + 2q + 3p + pq)v^2 \\
+ qr(q + 1)(-3q^2 - q + 6p + 8p^2 - 2pq)v^2 \\
+ pq(p + 1)(q + 1)(-3q^2 - q + 3p + 5pq)u^2 v^2 \\
+ r^2(3p^2 + 3p + q(q + 1) + p^2 + pq).
\]

(3.8)

\[
\Delta^I \left( - \frac{qv}{\sqrt{\Phi}} \right) = - \frac{qv}{\Phi} \left[ pq^2(p + 1)(q + 1)^2 v^4 + g(u, v) \right] \\
= \mu_{21} \left( - \frac{pu}{\sqrt{\Phi}} \right) + \mu_{22} \left( - \frac{qv}{\sqrt{\Phi}} \right) + \mu_{23} \left( \frac{\sqrt{p}}{\sqrt{\Phi}} \right),
\]

which turns into
\[
- \frac{qv}{\Phi^2} \left[ pq^2(p + 1)(q + 1)^2 v^4 + g(u, v) \right] = \\
- \mu_{21} pu - \mu_{22} qv + \mu_{23} \sqrt{p},
\]

(3.9)

where
\[
g(u, v) = p^2(p + 1)^2(4q^2 - 3pq + 3q - 2p)u^4 \\
+ qr(q + 1)(2p^2 + 2p + 3q + pq)v^2 \\
+ pr(p + 1)(-3p^2 - p + 6q + 8q^2 - 2pq)v^2 \\
+ pq(p + 1)(q + 1)(-3p^2 - p + 3q + 5pq)u^2 v^2 \\
+ r^2(3q^2 + 3q + p(p + 1) + qp + q^2).
\]

(3.10)

Using \( u = 0 \) in (3.7), we obtain that
\[
- \mu_{12} qv + \mu_{13} \sqrt{1 + qv^2} = 0.
\]

(3.11)

Deriving (3.11) with respect to \( v \) gives
\[
- \mu_{12} \sqrt{1 + qv^2} + \mu_{13} v = 0.
\]

(3.12)

Considering (3.11) and (3.12) as a system in \( \mu_{12} \) and \( \mu_{13} \), and since the determinant
\[
\begin{vmatrix}
- \frac{qv}{\Phi^2} & \sqrt{1 + qv^2} \\
- \sqrt{1 + qv^2} & v
\end{vmatrix} \neq 0.
\]

Therefore we must have \( \mu_{12} = \mu_{13} = 0 \). Hence (3.7) reduces to
\[
\frac{pu}{\Phi^3} \left[ p^2 q(p + 1)^2(q + 1)u^4 + f(u, v) \right] = \mu_{11} pu,
\]

or
\[
\frac{pu}{\Phi^3} \left[ p^2 q(p + 1)^2(q + 1)u^4 + f(u, v) \right] = \mu_{11} \Phi^3.
\]

(3.13)
Using $v = 0$ in (3.13), and taking into account (3.8), we find

$$
\mu_{11} [r + p(p + 1)u^2]^3 = \left[ p^2q(p + 1)^2(q + 1)u^4 \\
+ pr(p + 1)(2q^2 + 2q + 3p + pq)u^2 \\
+ r^2 \left( 3p(p + 1) + q(q + 1) + p(p + q) \right) \right].
$$

(3.14)

Similarly, inserting $v = 0$ in (3.9), gives

$$
-\mu_{21} pu + \mu_{23} \sqrt{1 + pu^2} = 0.
$$

In the same way one can see that $\mu_{21} = \mu_{23} = 0$. Then (3.9) turns into

$$
\frac{qv}{\Phi^3} \left[ pq^2(p + 1)(q + 1)^2v^4 + g(u, v) \right] = \mu_{22} qv,
$$

or

$$
\left[ pq^2(p + 1)(q + 1)^2v^4 + g(u, v) \right] = \mu_{22} \Phi^3.
$$

(3.15)

Inserting $u = 0$ in (3.15), and taking into account (3.10), we get

$$
\mu_{22} [r + q(q + 1)v^2]^3 = \left[ pq^2(p + 1)(q + 1)^2v^4 \\
+ qr(q + 1)(2p^2 + 2p + 3pq + pq)v^2 \\
+ r^2 \left( p(p + 1) + 3q(q + 1) + q(p + q) \right) \right].
$$

(3.16)

It's clearly that relations (3.14) and (3.16) are polynomials in $u$ and $v$ respectively of degree at most 6. As $p \neq 0, q \neq 0$ and $r \neq 0$, then one can be easily obtain that $p = q = -1$. Hence $Q$ is a sphere.

Let $p = q = -1$. Then from (3.8) and (3.10) respectively, we get $f(u, v) = 2r^2$ and $g(u, v) = 2r^2$. Thus from (3.13) and (3.15) we find that $\mu_{11} = \mu_{22} = \frac{2}{r}$.

Besides, relation (3.3) reduces to

$$
\Delta^I = \frac{1}{r} \left[ \frac{u^2 - r}{\partial u^2} + 2uv \frac{\partial^2}{\partial u \partial v} + \\
(v^2 - r) \frac{\partial^2}{\partial v^2} + 2u \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v} \right].
$$

(3.17)

So relation (3.6) becomes

$$
\Delta^I (-\sqrt{\rho}) = \frac{2\sqrt{\rho}}{r} = \mu_{31} u + \mu_{32} v + \mu_{33} \sqrt{\rho}.
$$
It is easily verified that $\mu_{31} = \mu_{32} = 0$ and $\mu_{33} = \frac{2}{r}$. Thus we find that spheres are the only quadric surfaces of the kind (3.1) whose Gauss map satisfies (1.2). The resulting matrix is

$$M = \begin{bmatrix} \frac{2}{r} & 0 & 0 \\ 0 & \frac{2}{r} & 0 \\ 0 & 0 & \frac{2}{r} \end{bmatrix}.$$  

### 3.2. Quadrics of the second type.

This type of surfaces can be parameterized as follows

$$r(u, v) = \left( u, v, \frac{p}{2}u^2 + \frac{q}{2}v^2 \right).$$  

Using the natural frame $\{r_u, r_v\}$ of $Q$ defined by

$$r_u = (1, 0, pu),$$

and

$$r_v = (0, 1, qv)$$

the components $g_{ij}$ of the metric $I$ are

$$g_{11} = (pu)^2 + 1, \quad g_{12} = pquv, \quad g_{22} = (qv)^2 + 1.$$  

Therefore the Laplace operator $\Delta^I$ of $Q$ is given by

$$\Delta^I = \frac{1}{g} \left( Y \frac{\partial^2}{\partial u^2} + X \frac{\partial^2}{\partial v^2} - 2pquv - \partial^2 \partial_u \partial_v \right)$$

$$+ \left[ \frac{pY + qX}{g^2} \right] \left[ pu \frac{\partial}{\partial u} + qv \frac{\partial}{\partial v} \right],$$

where

$$X := 1 + p^2 u^2, \quad Y := 1 + q^2 v^2,$$

and

$$g := \det(g_{ij}) = p^2 u^2 + q^2 v^2 + 1$$

The Gauss map of $Q$ is

$$G = \left( -\frac{pu}{\sqrt{g}}, -\frac{qv}{\sqrt{g}}, \frac{1}{\sqrt{g}} \right).$$

We denote by $(n_1, n_2, n_3)$ the components of $G$, and by $\mu_{rs}, r, s = 1, 2, 3$ the entries of the matrix $M$. From (1.2), we get

$$\Delta^I n_1 = \Delta^I \left( -\frac{pu}{\sqrt{g}} \right) = \mu_{11} \left( -\frac{pu}{\sqrt{g}} \right) + \mu_{12} \left( -\frac{qv}{\sqrt{g}} \right) + \mu_{13} \left( \frac{1}{\sqrt{g}} \right),$$

$$\Delta^I n_2 = \Delta^I \left( -\frac{qv}{\sqrt{g}} \right) = \mu_{21} \left( -\frac{pu}{\sqrt{g}} \right) + \mu_{22} \left( -\frac{qv}{\sqrt{g}} \right) + \mu_{23} \left( \frac{1}{\sqrt{g}} \right),$$

$$\Delta^I n_3 = \Delta^I \left( -\frac{1}{\sqrt{g}} \right) = \mu_{31} \left( -\frac{pu}{\sqrt{g}} \right) + \mu_{32} \left( -\frac{qv}{\sqrt{g}} \right) + \mu_{33} \left( \frac{1}{\sqrt{g}} \right).$$
Applying the operator $\Delta_1$ to the component functions $n_1$ and $n_2$ of $G$, we find by means of (3.19)

$$\Delta_1 \left( \frac{-pu}{\sqrt{g}} \right) = -\frac{pu}{g^2} \left[ q^2 X^2 + 4p^2 Y^2 - 3q^3 v^2(qX + pY) + 5p^3 q^3 u^2 v^2 + pqg \right]$$

$$= \mu_{11} \left( -\frac{pu}{\sqrt{g}} \right) + \mu_{12} \left( -\frac{qv}{\sqrt{g}} \right) + \mu_{13} \left( \frac{1}{\sqrt{g}} \right),$$

which turns into

$$\frac{mu}{g^3} \left[ q^2 X^2 + 4p^2 Y^2 - 3q^3 v^2(qX + pY) + 5p^3 q^3 u^2 v^2 + pqg \right]$$

$$= \mu_{11} pu + \mu_{12} qv - \mu_{13}, \quad (3.20)$$

and

$$\Delta_1 \left( \frac{-qv}{\sqrt{g}} \right) = -\frac{qv}{g^2} \left[ 4q^2 X^2 + p^2 Y^2 - 3p^3 u^2(qX + pY) + 5p^3 q^3 u^2 v^2 + pqg \right]$$

$$= \mu_{21} \left( -\frac{pu}{\sqrt{g}} \right) + \mu_{22} \left( -\frac{qv}{\sqrt{g}} \right) + \mu_{23} \left( \frac{1}{\sqrt{g}} \right),$$

which turns into

$$\frac{qv}{g^3} \left[ 4q^2 X^2 + p^2 Y^2 - 3p^3 u^2(qX + pY) + 5p^3 q^3 u^2 v^2 + pqg \right]$$

$$= \mu_{21} pu + \mu_{22} qv - \mu_{23}. \quad (3.21)$$

Inserting $u = 0$ in (3.20), then the left side of the equation (3.20) vanishes. Therefore we are left to

$$\mu_{12} qv - \mu_{13} = 0,$$

which implies that $\mu_{12} = \mu_{13} = 0$. So equation (3.20) becomes

$$q^2 X^2 + 4p^2 Y^2 - 3q^3 v^2(qX + pY) + 5p^3 q^3 u^2 v^2 + pqg - \mu_{11} g^3 = 0. \quad (3.22)$$

Similarly, if we put $v = 0$ in (3.21), then the left side of (3.21) vanishes. In the same way equation (3.21) becomes

$$4q^2 X^2 + p^2 Y^2 - 3p^3 u^2(qX + pY) + 5p^3 q^3 u^2 v^2 + pqg - \mu_{22} g^3 = 0. \quad (3.23)$$

Equations (3.22) and (3.23) are nontrivial polynomials in $u$ and $v$ with constant coefficients. These two polynomials can never be zero, unless $p = q = 0$, which is clearly impossible since $p, q > 0$.

4. Conclusion

This research article was divided into three sections, where after the introduction, the needed definitions and relations regarding this interesting field of study were given. Then a formula for the Laplace operator corresponding to the first fundamental form $I$ was proved once for the position vector and another for the Gauss map of a surface $Q$ by using Cartan’s method of the moving frame. Finally, we classify the quadric surfaces $Q$ satisfying the
relation $\Delta G = MG$, for a real square matrix $M$ of order 3. An interesting study can be drawn, if this type of study can be applied to other classes of surfaces that have not been investigated yet such as spiral surfaces, or tubular surfaces.

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