Propagators for scalar bound states at finite temperature in a NJL model *

Bang-Rong Zhou
Department of Physics, The Graduate School of The Chinese Academy of Sciences
Beijing 100039, China
and CCAST (World Laboratory) P.O. Box 8730, Beijing 100080, China

Abstract

We reexamine physical causal propagators for scalar and pseudoscalar bound states at finite temperature in a chiral $U_L(1) \times U_R(1)$ NJL model, defined by four-point amputated functions subtracted through the gap equation, and prove that they are completely equivalent in the imaginary-time and real-time formalism by separating carefully the imaginary part of the zero-temperature loop integral. It is shown that the thermal transformation matrix of the matrix propagators for these bound states in the real-time formalism is precisely the one of the matrix propagator for an elementary scalar particle and this fact shows similarity of thermodynamic property between a composite and an elementary scalar particle. The retarded and advanced propagators for these bound states are also given explicitly from the imaginary-time formalism.

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1 Introduction

Finite temperature field theory has attracted much research interest of people due to its application to evolution of early universe and phase transition of the hadron matter [1-5]. However, the complete equivalence between its two formalisms - the imaginary-time and the real-time formalism [4]- has been a subtle issue. It is usually assumed that the two formalisms should give the same results. Nevertheless, in some actual problems where only amputated Green functions are involved, the calculations in the two formalisms often show different results [6]. Much work has been contributed to seeking correspondence between the two formalisms for some amputated functions [7-10]. The general conclusion is that the difference between the amputated functions obtained in the two formalisms could originate from that one actually deal with different Green functions in the two cases thus they should be used for different physical purposes [8]. In a recent research on the Nambu-Goldstone mechanism of dynamical electroweak symmetry breaking at finite temperature [11] based on a Nambu-Jona-Lasinio (NJL) model [12], we calculate the propagators for scalar bound states which show different imaginary parts in their denominators in the two formalisms. It was naturally supposed that we had calculated different Green functions in the two cases. However, this inference is open to question. The reason is that the analytic continuation used there of the Matsubara frequency to real energy, though not in the most general form, was essentially made as the way leading to a causal Green function which should just be that obtained in the real-time formalism. In addition, one notes that the propagators for scalar bound states at finite temperature in a NJL model correspond some four-point amputated functions, and the calculations of four-point amputated functions can be effectively reduced to the ones of some two-point functions. It is accepted that a two-point function should be equivalent in the two formalisms of thermal field theory [7]. Therefore, it is necessary for us to reexamine the whole calculations of a NJL model. In this paper, we will do that by means of a chiral $U_L(1) \times U_R(1)$ NJL model. After rigorous and careful calculations,
we will finally prove that the propagators for scalar bound states including the imaginary parts in their denominators are in fact identical in the two formalisms. We also find a remarkable result that, in the real-time formalism, the thermal transformation matrix of the matrix propagators for scalar bound states is precisely the one of the matrix propagator for an elementary scalar particle. The key-points to reach the above conclusions which were ignored in Ref.[12] are that, except keeping the most general form of the analytic continuation in the imaginary-time formalism, we must carefully consider and separate the imaginary part of the zero-temperature loop integral from relevant expressions. This is very similar to the case of a complete calculation of a two-point function [13], except that in the case of a NJL model we must additionally use the gap equation so as to eliminate some parts of the Green functions. We will discuss the propagators for scalar bound states in the one-loop approximation respectively in the imaginary-time and real-time formalism, then compare the derived results.

2 The imaginary-time formalism

In the imaginary-time formalism, the Lagrangian of the four-fermion interactions will be

\[ \mathcal{L}^{4F} = \frac{G}{4}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2], \]

where \( \psi \) is the fermion field with single flavor and \( N \) colors, \( G \) is the coupling constant. Assume the scalar interactions in \( \mathcal{L}^{4F} \) may lead to the formation of the fermion condensate \( \langle \bar{\psi}\psi \rangle \) at temperature \( T \), then we will obtain the gap equation which determines the dynamical fermion mass \( m(T) \equiv m \)

\[ 1 = GI, \quad I = 2N \int \frac{d^3l}{(2\pi)^3} T \sum_{n=-\infty}^{\infty} \frac{1}{(\omega_n + i\mu)^2 + \frac{\omega_l}{T} + m^2}, \]

(1)

where \( \omega_n = (2n + 1)\pi/\beta (n = 0, \pm 1, \pm 2, \ldots; \beta = 1/T) \) is the Matsubara frequency of the fermions and \( \mu \) is the chemical potential of the fermions. A scalar bound state \( \phi_S = \langle \bar{\psi}\psi \rangle \) could be formed only through the four-fermion interactions, so we must define the propagators of \( \phi_S \) as the four-point amputated function \( \Gamma^{\phi_S}(-i\Omega_m, \vec{p}) \) for the transition from \( \langle \bar{\psi}\psi \rangle \) to \( \langle \bar{\psi}\psi \rangle \), where \( \Omega_m = 2\pi m/\beta (m = 0, \pm 1, \pm 2, \ldots) \) is the Matsubara frequency corresponding to the external energy. \( \Gamma^{\phi_S}(-i\Omega_m, \vec{p}) \) submits to the the Schwinger-Dyson equation

\[ \Gamma^{\phi_S}(-i\Omega_m, \vec{p}) = \frac{G}{2}[1 + 2N(-i\Omega_m, \vec{p})\Gamma^{\phi_S}(-i\Omega_m, \vec{p})], \]

(2)

where the fermion loop contribution

\[ N(-i\Omega_m, \vec{p}) = 2N \int \frac{d^3l}{(2\pi)^3} T \sum_{n=-\infty}^{\infty} \frac{(\omega_n + i\mu)(\omega_n + i\mu + \Omega_m) + \vec{l} \cdot (\vec{l} + \vec{p}) - m^2}{[(\omega_n + i\mu)^2 + \omega_l^2][\omega_n + i\mu + \Omega_m]^2 + \omega_{l+p}^2}] \]

\[ = I + 2N \int \frac{d^3l}{(2\pi)^3} T \sum_{n=-\infty}^{\infty} \frac{-i\Omega_m^2 + \vec{p}^2 - 2m^2 - (\omega_n + i\mu)\Omega_m - \vec{l} \cdot \vec{p}^2}{[(\omega_n + i\mu)^2 + \omega_l^2][\omega_n + \Omega_m + i\mu]^2 + \omega_{l+p}^2}] \]

with the denotations \( \omega_l^2 = \vec{l}^2 + m^2 \) and \( \omega_{l+p}^2 = (\vec{l} + \vec{p})^2 + m^2 \). By means of the gap equation (1), we can write the solution of (2) as

\[ \Gamma^{\phi_S}(-i\Omega_m, \vec{p}) = -1/2N[-(\Omega_m^2 + \vec{p}^2 - 4m^2)] \int \frac{d^3l}{(2\pi)^3} A(-i\Omega_m, \vec{p}, \vec{l}), \]

\[ A(-i\Omega_m, \vec{p}, \vec{l}) = T \sum_n \frac{1}{(\omega_n + i\mu)^2 + \omega_l^2} \frac{1}{(\omega_n + \Omega_m + i\mu)^2 + \omega_{l+p}^2}, \]

(3)

where we have used the property that, owing to the Lorentz invariance, \( \Gamma^{\phi_S}(-i\Omega_m, \vec{p}) \) must be a function of \( -(\Omega_m^2 + \vec{p}^2) \), thus
Doing the sum of the Matsubara frequency $\omega_n$ in (3) by the standard procedure \[11,15\] then making the analytic continuation of the external energy $-i\Omega_m \to p^0 + i\varepsilon p^0 (\varepsilon = 0_+)$ and keeping the general form of the replacement, we may write $A(-i\Omega_m \to p^0 + i\varepsilon p^0, \vec{p}, \vec{l})$ by

\[
A(p, \vec{l}) = \frac{1}{4\omega_i\omega_{i+p}} \left\{ \frac{1-n(\omega_i+\mu)-n(\omega_{i+p}-\mu)}{-p^0 + \omega_i + \omega_{i+p} - i\varepsilon \eta(p^0)} + \frac{n(\omega_i+\mu)-n(\omega_{i+p}+\mu)}{-p^0 + \omega_i - \omega_{i+p} - i\varepsilon \eta(p^0)} \right. \\
\left. - \frac{n(\omega_i-\mu)-n(\omega_{i+p}-\mu)}{-p^0 - \omega_i + \omega_{i+p} - i\varepsilon \eta(p^0)} - \frac{1-n(\omega_i-\mu)-n(\omega_{i+p}+\mu)}{-p^0 - \omega_i - \omega_{i+p} - i\varepsilon \eta(p^0)} \right\},
\]

where $n(\omega_i \pm \mu) = 1/[e^{\beta(\omega_i \pm \mu)} + 1]$ and $\eta(p^0) = p^0/|p^0|$. For the convenience of making a comparison with the following results in the real-time formalism we express $A(p, \vec{l})$ as an integral of $l^0$ by using

\[
\sin^2 \theta(l^0, \mu) = \frac{\theta(l^0)}{\exp[\beta(l^0 - \mu)] + 1} + \frac{\theta(-l^0)}{\exp[\beta(-l^0 + \mu)] + 1}
\]

and the formula $1/(X + i\varepsilon) = X/(X^2 + \varepsilon^2) - i\pi \delta(X)$. Eventually we obtain the physical causal propagator for $\phi_S$ in the imaginary-time formalism

\[
\Gamma_{\phi^*_S\phi_S}^{\phi_S}(p) = i\Gamma_I^{\phi_S}(-i\Omega_m \to p^0 + i\varepsilon p^0, \vec{p}) \\
= -i/(p^2 - 4m^2 + i\varepsilon) [K(p) + H(p) - iS^I(p)],
\]

where

\[
K(p) = -2N \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 - m^2 + i\varepsilon)|l + p|^2 - m^2 + i\varepsilon} \\
= \frac{N}{8\pi^2} \int_0^1 dx \left( \ln \frac{\Lambda^2 + M^2}{M^2} - \frac{\Lambda^2}{\Lambda^2 + M^2} \right), \quad M^2 = m^2 - p^2 x(1-x)
\]

is the contribution from the zero temperature fermion loop with the four-dimension Euclidean momentum cut-off $\Lambda$,

\[
H(p) = 4\pi N \int \frac{d^4l}{(2\pi)^4} \left\{ \frac{(l + p)^2 - m^2}{[(l + p)^2 - m^2]^2 + \varepsilon^2} + (p \to -p) \right\} \delta(l^2 - m^2) \sin^2 \theta(l^0, \mu)
\]

and

\[
S^I(p) = \eta(p^0) 4\pi N \int \frac{d^4l}{(2\pi)^4} \delta(l^2 - m^2) \delta(|l + p|^2 - m^2) \\
\times [\sin^2 \theta(l^0, \mu) \eta(l^0 + p^0) + \sin^2 \theta(l^0 + p^0, \mu) \eta(-l^0)].
\]

It is emphasized that $S^I(p)$ does not contain any pinch singularity due to the factors $\eta(l^0 + p^0)$ and $\eta(-l^0)$ in its integrand and since

\[
\delta(l^2 - m^2) \delta(|l + p|^2 - m^2) = 0, \text{ when } 0 \leq p^2 < 4m^2,
\]

we must have $S^I(p) = 0$, when $0 \leq p^2 < 4m^2$. In addition, from (6), $K(p)$ is real when $p^2 < 4m^2$ and from (7), $H(p)$ is always real. \[16\]. Similarly, we may find out the physical causal propagator for the pseudoscalar bound state $\phi_P = (\bar{\psi} i\gamma_5 \psi)$

\[
\Gamma_{\phi^*_P\phi_P}^{\phi_P}(p) = -i/(p^2 + i\varepsilon) [K(p) + H(p) - iS^I(p)].
\]

The equations (5) and (9) show that $\phi_S$ and $\phi_P$ each have the mass $2m$ and 0 thus can be respectively identified with the massive “Higgs” scalar particle and the massless Nambu-Goldstone boson for the spontaneous symmetry breaking $U_L(1) \times U_R(1) \to U_{L+R}(1)$. This represents the Nambu-Goldstone
3 The real-time formalism

In the real-time formalism, the Lagrangian of the four-fermion interactions will be

$$\mathcal{L}_{4F}^R = \frac{G}{4} \sum_{a=1}^{2} \left\{ \left( (\bar{\psi} \gamma^\mu \psi)^{(a)} \right)^2 - \left( \bar{\psi} \gamma_5 \gamma^\mu \gamma_5 \psi \right)^{(a)} \right\} (1)^{a+1}, \quad (10)$$

where \( a = 1 \) denotes physical fields and \( a = 2 \) ghost fields. As a result of the thermal condensates \( \langle (\bar{\psi} \gamma^\mu \psi)^{(a)} \rangle_T \neq 0, (a = 1, 2) \), the gap equation becomes

$$1 = GI, \ I = 2N \int \frac{d^4l}{(2\pi)^4} \left[ \frac{i}{l^2 - m^2 + i\varepsilon} - 2\pi\delta(l^2 - m^2) \sin^2 \theta(l^0, \mu) \right], \quad (11)$$

which is actually identical to (1) obtained in the imaginary-time formalism [14].

The propagator for the scalar bound state \( \phi_S \) is now a \( 2 \times 2 \) matrix whose elements correspond to the four-point amputated functions \( \Gamma_{ba}^S(p) \) for the transition from \( (\bar{\psi} \gamma^\mu \psi)^{(a)} \) to \( (\bar{\psi} \gamma_5 \gamma_\mu \gamma_5 \psi)^{(b)} \) \( (a, b = 1, 2) \). \( \Gamma_{ba}^S(p) \) obey the Schwinger-Dyson equations

$$\Gamma_{ba}^S(p)[\delta^{ca} - GN^{ca}(-1)^a + 1] = \frac{iG}{2} \delta^{ba}(-1)^a, \quad (12)$$

where \( N^{ca}(l, m) \) is the elements of the thermal matrix propagator for the fermions [4]. After using the gap equation (11), the solution of (12) can be expressed by the matrix [16]

$$\begin{pmatrix} \Gamma_{11}^S(p) & \Gamma_{12}^S(p) \\ \Gamma_{21}^S(p) & \Gamma_{22}^S(p) \end{pmatrix} = \frac{1}{[K(p) + H(p) - iS(p)]^2 - R^2(p)} \times \begin{pmatrix} -i[K^*(p) + H(p) + iS(p)] & -(p^2 - 4m^2)R(p) \\ p^2 - 4m^2 + i\varepsilon & (p^2 - 4m^2)^2 + i\varepsilon \end{pmatrix}, \quad (13)$$

where

$$K(p) = 4\pi^2N \int \frac{d^4l}{(2\pi)^4} \delta(l^2 - m^2) [\delta((l + p)^2 - m^2)] \times [\sin^2 \theta(l^0 + p^0, \mu) \cos^2 \theta(l^0, \mu) + \cos^2 \theta(l^0 + p^0, \mu) \sin^2 \theta(l^0, \mu)] \quad (14)$$

and

$$R(p) = 2\pi^2N \int \frac{d^4l}{(2\pi)^4} \delta(l^2 - m^2) [\delta((l + p)^2 - m^2)] \sin 2\theta(l^0, \mu) \sin 2\theta(l^0 + p^0, \mu) \quad (15)$$

are the terms containing the pinch singularities. \( \Gamma_{11}^S(p) \) in (13) is of the form of a causal propagator, hence if it is identified with the physical propagator for the scalar bound state \( \phi_S \), as made in Refs. [16,18], then the main feature of \( \phi_S \) including its mass \( 2m \) could be shown. However, the expression of \( \Gamma_{11}^S(p) \) has considerable difference from \( \Gamma_{11}^F(p) \) in (5) in the imaginary-time formalism. For finding a closer correspondence between the physical propagators for \( \phi_S \) in the two formalisms, we will seek a thermal transformation matrix \( M \) which can diagonalize the matrix propagator (13) so that

$$\begin{pmatrix} \Gamma_{11}^S(p) & \Gamma_{12}^S(p) \\ \Gamma_{21}^S(p) & \Gamma_{22}^S(p) \end{pmatrix} = M^{-1} \begin{pmatrix} \Gamma_{11}^\phi_F(p) & 0 \\ 0 & \Gamma_{22}^\phi_F*(p) \end{pmatrix} M^{-1}, \quad (16)$$

where

$$M = \begin{pmatrix} \cosh \Theta & \sinh \Theta \\ \sinh \Theta & \cosh \Theta \end{pmatrix} \quad (17)$$
and \( \Gamma_{RF}^{\phi_{S}}(p) \) is now defined as the physical causal propagator for \( \phi_{S} \). It is seen from (13) that \( \Gamma_{SS}^{\phi}(p) = [\Gamma_{11}^{S}(p)]^{*} \) and \( \Gamma_{SS}^{\phi}(p) = \Gamma_{SS}^{\phi}(p) = \Gamma_{SS}^{\phi}(p)^{*} \), thus (16) may be reduced to three independent algebraic equations for \( \text{Re} \Gamma_{RF}^{\phi_{S}}(p) \), \( \text{Im} \Gamma_{RF}^{\phi_{S}}(p) \) and \( \sinh \Theta \) (or \( \cosh \Theta \))

\[
\text{Re} \Gamma_{S}^{1}(p) = (\cosh^{2} \Theta + \sinh^{2} \Theta) \text{Re} \Gamma_{RF}^{\phi_{S}}(p), \quad \text{Im} \Gamma_{S}^{11}(p) = \text{Im} \Gamma_{RF}^{\phi_{S}}(p),
\]

\[
\text{Re} \Gamma_{S}^{1}(p) = -2 \sinh \Theta \cosh \Theta \text{Re} \Gamma_{RF}^{\phi_{S}}(p).
\] (18)

Considering (13), we obtain from (18)

\[
\frac{\cosh^{2} \Theta + \sinh^{2} \Theta}{2 \sinh \Theta \cosh \Theta} = \frac{S'(p)}{R(p)}, \quad S'(p) = S(p) - \text{Im} K(p),
\] (19)

and furthermore,

\[
\cosh \Theta = \frac{1}{\sqrt{2}} \left[ \frac{S'(p)}{\sqrt{S'^{2}(p) - R^{2}(p)}} + 1 \right]^{1/2}, \quad \sinh \Theta = \frac{1}{\sqrt{2}} \left[ \frac{S'(p)}{\sqrt{S'^{2}(p) - R^{2}(p)}} - 1 \right]^{1/2}.
\] (20)

Then (18) will lead to the physical causal propagator

\[
\Gamma_{RF}^{\phi_{S}}(p) = -i/(p^{2} - 4m^{2} + i\varepsilon) \left[ \text{Re} K(p) + H(p) - i \sqrt{S'^{2}(p) - R^{2}(p)} \right].
\] (21)

Based on the interactions (10) and parallel derivation, we can give the pseudoscalar matrix propagator for the transition from \((\bar{\psi} i \gamma_{5} \psi)^{(a)} \) to \((\bar{\psi} i \gamma_{5} \psi)^{(b)} \) \((a, b = 1, 2)\) with the elements \( \Gamma_{RF}^{\phi_{P}}(p) = \Gamma_{SS}^{\phi_{P}}(p)|_{m=0} \), then diagonalize \( \Gamma_{RF}^{\phi_{P}}(p) \) by the same thermal matrix as \( M \) in (17) and obtain physical causal propagator \( \Gamma_{RF}^{\phi_{P}}(p) \) for the pseudoscalar bound state \( \phi_{P} = (\bar{\psi} i \gamma_{5} \psi) \)

\[
\Gamma_{RF}^{\phi_{P}}(p) = -i/(p^{2} + i\varepsilon) \left[ \text{Re} K(p) + H(p) - i \sqrt{S'^{2}(p) - R^{2}(p)} \right].
\] (22)

The elements (20) of the matrix \( M \) depend on \( S(p), R(p) \) and the imaginary part \( \text{Im} K(p) \) of the zero-temperature loop integral and seem to have rather complicated expressions. However, the final result is remarkable, i.e. the matrix \( M \) is identical to the thermal transformation matrix of the matrix propagator for an elementary scalar particle. In fact, from the expressions (15) and (14) and the definition (4), we can write

\[
R(p) = 2\pi^{2} N \int \frac{d^{4}l}{(2\pi)^{4}} \delta(l^{2} - m^{2}) \delta[(l + p)^{2} - m^{2}] \frac{\eta(l^{0})\eta(l^{0} + p^{0})}{\cosh[\beta(l^{0} - \mu)/2] \cosh[\beta(l^{0} + p^{0} - \mu)/2]}.
\] (23)

and

\[
S(p) = \cosh(\beta p^{0}/2)R(p) + D(p)
\] (24)

\[
D(p) = \frac{N}{16\pi^{2}} \int \frac{d^{3}l}{\omega_{l+p}} \left[ \delta(p^{0} + \omega_{l} + \omega_{l+p}) + \delta(p^{0} - \omega_{l} - \omega_{l+p}) \right].
\] (25)

On the other hand, we may calculate \( K(p) \) expressed by (6) in terms of the residue theorem and obtain

\[
K(p) = \frac{N}{16\pi^{3}} \int \frac{d^{3}l}{\omega_{l+p}} \left[ \frac{1}{p^{0} + \omega_{l} + \omega_{l+p} - i\varepsilon} - \frac{1}{p^{0} - \omega_{l} - \omega_{l+p} + i\varepsilon} \right].
\]

Hence the imaginary part of \( K(p) \)

\[
\text{Im} K(p) = D(p).
\] (26)

It is seen from (26) and (25) that, \( \text{Im} K(p) \neq 0 \) only if \( \delta|p^{0} - (\omega_{l} + \omega_{l+p})|^{2} \neq 0 \) or \( p^{2} = (\omega_{l} + \omega_{l+p})^{2} \). But the latter can be satisfied only if \( p^{2} \geq 4m^{2} \). This reproduces the former conclusion that \( K(p) \) is real when \( p^{2} < 4m^{2} \). The equations (24) and (26) show that we may separate the imaginary part \( \text{Im} K(p) \) of the zero-temperature loop integral from \( S(p) \) and this fact is essential for the following results. Substituting (26) into (24) and considering (19), we will have
then from (20) obtain
\[ \cosh \theta = [1 + n(p^0)]^{1/2}, \quad \sinh \theta = [n(p^0)]^{1/2}, \quad n(p^0) = 1/(e^{\beta|p^0|} - 1), \] (28)
which are precisely the elements of the thermal transformation matrix of the matrix propagator for an elementary scalar particle with zero chemical potential\(^4\), though now we are dealing with the scalar and pseudoscalar bound states \(\phi_S\) and \(\phi_P\) composed of fermions and antifermions in the NJL model. This implies that scalar particles, whether elementary or composite, seem always to have the same thermodynamic property.

4 Equivalence of the two formalisms and causal, retarded and advanced propagators

We will prove that \(\Gamma^{\phi_S}_{IF}(p)\) and \(\Gamma^{\phi_P}_{IF}(p)\) expressed by (5) and (9) in the imaginary-time formalism are respectively the same as \(\Gamma^{\phi_S}_{RF}(p)\) and \(\Gamma^{\phi_P}_{RF}(p)\) expressed by (21) and (22) in the real-time formalism. It is easy to see that, for this purpose, we only need prove that \(K(p) + H(p) - iS(p) = ReK(p) + H(p) - i\sqrt{S^2(p) - R^2(p)}\) or
\[ S(p) = \sqrt{S^2(p) - R^2(p)} + ImK(p) = \sinh(\beta|p^0|/2)R(p) + ImK(p), \] (29)
where (27) has been used. In fact, from (8) together with (4), (23) (25) and (26) we can obtain
\[ S(p) = \eta(p^0)\sinh(\beta p^0/2)2\pi^2N \int \frac{d^4l}{(2\pi)^4}\delta(l^2 - m^2)\delta[(l + p)^2 - m^2] \]
\[ \times \frac{\eta(l^0)\eta(l^0 + p^0)}{\cosh[\beta(l^0 - \mu)/2] \cosh[\beta(l^0 + p^0 - \mu)/2]} \]
\[ + \eta(p^0)2\pi^2N \int \frac{d^4l}{(2\pi)^4}\delta(l^2 - m^2)\delta[(l + p)^2 - m^2][\eta(l^0 + p^0) - \eta(l^0)] \]
\[ = \eta(p^0)\sinh(\beta p^0/2)R(p) \]
\[ + \eta(p^0)\frac{N}{16\pi^2} \int \frac{d^4l}{\omega_{l+l+p}}[-\delta(p^0 + \omega_l + \omega_{l+p}) + \delta(p^0 - \omega_l - \omega_{l+p})] \]
\[ = \sinh(\beta|p^0|/2)R(p) + \frac{N}{16\pi^2} \int \frac{d^4l}{\omega_{l+l+p}}[\delta(p^0 + \omega_l + \omega_{l+p}) + \delta(p^0 - \omega_l - \omega_{l+p})] \]
\[ = \sinh(\beta|p^0|/2)R(p) + ImK(p), \]
i.e. (29) is valid indeed. Here separation of \(ImK(p)\) from \(S(p)\) is also essential. Hence we can reach the conclusion that, in a NJL model, the four-point amputated functions corresponding to a scalar or pseudoscalar bound state are identical in the imaginary-time and the real-time formalism of thermal field theory. This is not surprising because the calculations of four-point amputated functions in a NJL model can be effectively reduced to the ones of usual two-point functions, but with a new feature that the propagators now discussed for the bound states, are only subtracted four-point amputated functions, rather than the whole of them, because some parts of them have been subtracted through the use of the gap equation. It is emphasized that the key-point of proving such equivalence and deriving the matrix elements (28) of \(M\) lies in that one must carefully consider and separate the imaginary part \(ImK(p)\) of the zero-temperature loop integral from relevant expressions e.g. \(K(p), S(p)\) and \(S'(p)\), which is often possibly ignored in usual calculations\(^1\).

Since the causal propagators for scalar or pseudoscalar bound state are identical in the two formalisms, we can omit the subscript "\(F\)" and "\(R\)" and uniquely express them respectively by
\[ \Gamma^{\phi_S}_{F}(p) = \Gamma^{\phi_S}_{IF}(p) = \Gamma^{\phi_S}_{RF}(p) \]
and

\[ \Gamma_{IF}^\phi(p) = \Gamma_{RF}^\phi(p) = \Gamma_{IF}^\phi(p) \]

\[ = -i/(p^2 + i\varepsilon)[\text{Re}K(p) + H(p) - i \sinh(\beta p^0/2) R(p)]. \tag{31} \]

Based on the identity of the causal propagators in the two formalisms, it is easy to obtain the retarded and the advanced propagator \( \Gamma_r^\phi(p) \) and \( \Gamma_a^\phi(p) \) (where \( \phi \) for \( \phi_S \) or \( \phi_P \)) which are also the same respectively in the two formalisms. If first in the imaginary-time formalism, then we may define

\[ \Gamma_{IF}^\phi(p) = i\Gamma_I^\phi(-i\Omega_m \to p^0 + i\varepsilon p^0), \]

\[ \Gamma_{Fr}^\phi(p) = i\Gamma_I^\phi(-i\Omega_m \to p^0 + i\varepsilon), \]

\[ \Gamma_{Ia}^\phi(p) = i\Gamma_I^\phi(-i\Omega_m \to p^0 - i\varepsilon). \tag{32} \]

From (32), we can derive

\[ \Gamma_{IF}^\phi(p) = \theta(p^0)\Gamma_{Fr}^\phi(p) + \theta(-p^0)\Gamma_{Ia}^\phi(p), \]

\[ [\Gamma_{Fr}^\phi(p)]^* = -\Gamma_{Ia}^\phi(p), \]

and furthermore,

\[ \Gamma_{IF}^\phi(p) = \theta(p^0)\Gamma_{IF}^\phi(p) - \theta(-p^0)[\Gamma_{IF}^\phi(p)]^*. \tag{33} \]

Identifying \( \Gamma_{IF}^\phi(p) \) in (33) with the common \( \Gamma_{IF}^\phi(p) \) in the two formalisms expressed by (30) and (31), we will have

\[ \Gamma_r^{\phi_S}(p) = -i/(p^2 - 4m^2 + i\varepsilon p^0)[\text{Re}K(p) + H(p) - i \sinh(\beta p^0/2) R(p)], \]

\[ \Gamma_r^{\phi_P}(p) = -i/(p^2 + i\varepsilon p^0)[\text{Re}K(p) + H(p) - i \sinh(\beta p^0/2) R(p)], \]

\[ \Gamma_a^\phi(p) = -[\Gamma_r^\phi(p)]^*, \ \phi = \phi_S \ or \ \phi_P, \tag{34} \]

which represent the retarded and the advanced propagators for the bound state \( \phi \). The result (34) can also be obtained from the matrix propagator (13) in the real-time formalism by so called the transformation in the RA basis [10], which will be discussed elsewhere.

5 Conclusions

We have proven that the physical causal (as well as retarded and advanced) propagators for scalar bound states in a chiral \( U_L(1) \times U_R(1) \) NJL model, defined by the four-point amputated functions subtracted through the use of the gap equation, are identical in the imaginary-time and real-time formalism. This result convincingly shows equivalence of the two formalisms of thermal field theory in the NJL model. The key-points to complete the above proof lie in keeping the general form of the analytic continuation of the Matsubara frequency in the imaginary-time formalism and separating carefully the imaginary part of the zero temperature loop integral from relevant expressions, and these are also certainly of crucial importance for general explicit demonstration of equivalence of the two formalisms, for instance, in the calculations to many-loop order and/or of n-point Green functions. In addition, we have found that, in the real-time formalism, the thermal transformation matrices of the matrix propagators for scalar bound states are precisely the one for an elementary scalar particle and this fact strongly indicates similarity of thermodynamic property between a composite and an elementary scalar particle.

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