OPTIMAL REGULARITY OF SOLUTION TO A DEGENERATE
ELLIPIC SYSTEM ARISING IN ELECTROMAGNETIC FIELDS

HONG-MING YIN
Department of Mathematics
Washington State University
Pullman, WA 99163

Abstract. In this paper we prove a fundamental estimate for the weak solution of a degenerate elliptic system:
∇× \left[ \rho(x) \nabla \times H \right] = F, \nabla \cdot H = 0 \text{ in a bounded domain in } \mathbb{R}^3,
where \( \rho(x) \) is only assumed to be in \( L^\infty \) with a positive lower bound. This system is the steady-state of Maxwell’s system for the evolution of a magnetic field \( H \) under the influence of an external force \( F \), where \( \rho(x) \) represents the resistivity of the conductive material. By using Campanato type of techniques, we show that the weak solution to the system is Hölder continuous, which is optimal under the assumption. This result solves the regularity problem for the system under the minimum assumption on the coefficient. Some applications arising in inductive heating are presented.

1. Introduction. It is well known that DeGiorgi-Nash’s estimate ([5, 14]) plays an essential role in the study of elliptic and parabolic equations. This estimate is not true in general for an elliptic or parabolic system, unless the system is in diagonal form (see [5, 7, 11]). Counterexamples can be founded in [7]. In this paper we study the regularity of weak solution to a class of nondiagonal elliptic systems and show that the weak solution to the system is indeed Hölder continuous under the minimal assumption on the coefficient.

To see the motivation of our study, we recall Maxwell’s equations in an electromagnetic field. Let \( E \) and \( H \) be the electric and the magnetic fields in a connected region \( \Omega \), respectively. Throughout this paper a bold letter represents a vector in \( \mathbb{R}^3 \). It is well known that \( E \) and \( H \) satisfy the classical Maxwell system ([10]):
\begin{align}
\varepsilon \frac{\partial E_i}{\partial t} + \sigma E_i &= \nabla \times H, \\
\mu \frac{\partial H_i}{\partial t} + \nabla \times E &= F, \\
\nabla \cdot H &= 0,
\end{align}
where \( \varepsilon \) is the permittivity of the electric field, \( \mu \) the permeability of the magnetic field and \( \sigma \) the conductivity of the material.

For a conductive material, the current displacement is negligible since it is very small in comparison of Eddy currents \( J = \sigma E \) (see [10, 12]). This leads to the following evolution system:
\begin{align}
\mu \frac{\partial H_i}{\partial t} + \nabla \times \left[ \rho(x, t) \nabla \times H \right] &= F, \\
\nabla \cdot H &= 0,
\end{align}

1991 Mathematics Subject Classification. 35J45, 35J70.
Key words and phrases. Maxwell's equations in anisotropic medium, optimal regularity, and applications in inductive heating.
where
\[ \rho(x, t) = \frac{1}{\sigma(x, t)} \]
represents the resistivity of the material.

It is easy to see that the system (1.4) will become a heat equation in vector form if \( \rho \) is a constant. However, when the conductive material is anisotropic, the resistivity \( \rho(x, t) \) strongly depends on the medium. Note that the system (1.4) is degenerate by the classical definition ([11]). This system has been studied in [17] where the existence of a unique weak solution is obtained. Moreover, optimal regularity of the weak solution is established when \( \rho(x, t) \) is of class \( C^\alpha(\bar{\Omega}) \) or assumed to be continuous. However, when \( \rho(x, t) \) is assumed only to be bounded with a positive lower bound, the regularity problem remains open. The best possible regularity for this case is that the weak solution to the system (1.4) is Hölder continuous.

The present paper will give a positive answer to the above open question for the steady-state system (1.4). The main technique is based on Campanato type of estimates for elliptic equations with measurable coefficients ([3, 15]). This technique has been used by several researchers for various regularity problems ([5, 7, 15, 17] and the references therein).

The result obtained in this paper has many applications. As an example, we will use the result to study the following system
\[
-\nabla \times [\rho(u) \nabla \times H] = 0, x \in \Omega, t > 0.
\]
\[
u_t - \Delta u = \rho(u) |\nabla \times H|^2,
\]
where the resistivity \( \rho \) depends on the temperature \( u(x, t) \). The problem arises in inductive heating ([1, 2, 16, 18]). The reader may consult [12] for many more relevant references. Note that if \( H \) is a scalar function, then the system is the mathematical model of a thermistor, which has been studied by many researchers ([1, 2, 13]).

The main result and the proof are given in Section 2. Some applications are presented in Section 3. Various Sobolev spaces and Campanato spaces used in this paper are the same as in [15].

2. The Fundamental Estimate. Consider the following steady-state system:
\[
\nabla \times [\rho(x) \nabla \times H] = \nabla \times G + F, \quad x \in \Omega, \quad (2.1)
\]
\[
\nabla \cdot H(x) = 0, \quad x \in \Omega, \quad (2.2)
\]
\[
H(x) = 0, \quad x \in \partial \Omega. \quad (2.3)
\]

H(2.1): Let \( \rho(x) \) be measurable in \( \Omega \) and \( 0 < \rho_0 \leq \rho(x) \leq \rho_1 \) in \( \Omega \).
H(2.2): Let \( G(x) \in L^{2+\mu}(\Omega) \) for \( \mu \in (1, 3) \). Let \( F \) be of class \( H^1(\Omega)^3 \) (the product space \( H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \) with usual product norm). Moreover, the consistency condition holds:
\[
\nabla \cdot F = 0, \quad x \in \Omega.
\]

Because of the gauge invariance, we must fix the gauge to assume that
\[
F = \nabla \times \Phi, \nabla \cdot \Phi = 0, x \in \Omega.
\]
For the completeness, we give the definition of a weak solution to (2.1)-(2.3). Let

\[ B^d(\Omega) = \{ \Psi \in H^1_0(\Omega)^3, \text{div}\Psi = 0 \} \]

**Definition 2.1**: A vector field \( H \in B^d(\Omega) \) is said to be a weak solution of the problem (2.1)-(2.3), if the following integral identity holds:

\[
\int_{\Omega} [\rho(x) (\nabla \times H) \cdot (\nabla \times \Psi)] \, dx = \int_{\Omega} [G \cdot \nabla \times \Psi + F \cdot \Psi] \, dx \tag{2.4}
\]

for all \( \Phi(x) \in B^d(\Omega) \).

The existence of a unique weak solution is established in [17]. We state the following proposition ([3]), which is fundamental in the proof of the main result.

**Proposition**: Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \), then for any \( \mu \in (3,5) \) the Hölder space \( C^\alpha(\bar{\Omega}) \) is isomorphism to the Campanato space \( L^{2,\mu}(\Omega) \) with \( \alpha = \frac{\mu - 3}{2} \).

Our main result in this paper is the following regularity and the a priori estimate.

**Theorem 2.1**: Under the assumptions \( H(2.1)-(2.2) \), the weak solution of the system (2.1)-(2.3) is Hölder continuous. Moreover, the following estimate holds:

\[ ||H||_{C^\alpha(\Omega)} \leq C[||G||_{L^{2,\mu}(\Omega)} + ||\Phi||_{L^{2,\mu}(\Omega)} + ||F||_{L^2(\Omega)}], \]

where the constant \( C \) depends only on \( \Omega, \rho_0, \rho_1 \) and \( \mu \).

**Proof**: Without loss of generality, we may assume that the solution is smooth since we derive an a priori estimate.

First of all, from [17] we have the following a priori estimate for the weak solution \( H \) in the Sobolev space \( H^1(\Omega)^3 \):

\[ ||H||_{H^1(\Omega)} \leq C[||G||_{L^2(\Omega)} + ||F||_{L^2(\Omega)}], \]

where \( C \) depends only on \( \rho_0, \rho_1 \) and \( \Omega \).

By assumption \( H(2.2) \), there exists a vector field \( \Phi \) such that

\[ F = \nabla \times \Phi, \nabla \cdot \Phi = 0. \]

It follows that the system (2.1) can be written

\[
\nabla \times [\rho(x) \nabla \times H] - \nabla \times [G + \Phi] = 0, \quad x \in \Omega, \tag{2.5}
\]

On the other hand, for a vector field \( V \) if \( \nabla \times V = 0 \) in a connected domain, then the vector field \( V \) can be expressed as the gradient of a potential function in the domain. Using the above fact for the system (2.5) implies that there exists a potential function \( \psi(x) \) such that

\[
\rho(x) \nabla \times H - [G + \Phi] = \nabla \psi, \quad x \in \Omega. \tag{2.6}
\]

It is clear that the potential function \( \psi \) is not unique since \( \psi(x) + C \) for any constant \( C \) will satisfies the same condition. However, since \( H(x) = 0 \) on the boundary \( \partial \Omega \), from the definition of weak solution, we see that

\[ \psi(x) = 0, \quad x \in \partial \Omega. \]

With the above boundary condition, the potential \( \psi \) is uniquely determined from (2.6).
From (2.6), we obtain
\[
\nabla \times H = \frac{1}{\rho(x)} \nabla \psi + \frac{1}{\rho(x)}[\Phi(x) + G(x)], \quad x \in \Omega. \quad (2.7)
\]

Note that for any smooth vector field \( V \), the following identity holds:
\[
\nabla \cdot (\nabla \times V) = 0.
\]

It follows from (2.7) that
\[
\nabla \left( \frac{1}{\rho(x)} \nabla \psi \right) = -
\nabla \cdot \left( \frac{1}{\rho(x)}(G + \Phi) \right), \quad x \in \Omega. \quad (2.8)
\]

Now Campanato estimate for the elliptic equation (2.8) ([15]) implies that there exist constants \( C \) and \( \mu \in (1, 3) \) (recall the space dimension is 3) such that
\[
\| \nabla \psi \|_{L^{2\mu}(\Omega)} \leq C \left[ \| G \|_{L^{2\mu}(\Omega)} + \| \Phi \|_{L^{2\mu}(\Omega)} + \| \psi \|_{H^1(\Omega)} \right], \quad (2.9)
\]
where the constant \( C \) depends only on the \( \rho_0, \rho_1, \Omega \).

It follows by the Sobolev imbedding that \( \psi(x) \in L^{2\alpha+2}(\Omega) \). By the proposition, we see that the potential function
\[
\psi(x) \in C^{\alpha}(\bar{\Omega}),
\]
where \( \alpha = \frac{\mu-1}{2} \). Moreover,
\[
\| \nabla \psi \|_{C^{\alpha}(\Omega)} \leq C \| \nabla \psi \|_{L^{2\alpha}(\Omega)} \leq C \left[ \| G \|_{L^{2\alpha}(\Omega)} + \| \Phi \|_{L^{2\alpha}(\Omega)} + \| \psi \|_{H^1(\Omega)} \right].
\]

On the other hand, from Eq. (2.6), we see that for \( \mu \in (1, 3) \)
\[
\| \nabla \times H \|_{L^{2\mu}(\Omega)} \leq C \left[ \| \nabla \psi \|_{L^{2\mu}(\Omega)} + \| G \|_{L^{2\mu}(\Omega)} + \| \Phi \|_{L^{2\mu}(\Omega)} \right].
\]

Since \( H(x) = 0 \) on the boundary of \( \Omega \) and \( \nabla \cdot H = 0 \), we find
\[
\sum_{k=1}^3 \| \nabla h_k \|_{L^2(\Omega)} = \| \nabla \times H \|_{L^2(\Omega)},
\]
where \( H = \{ h_1(x), h_2(x), h_3(x) \} \).

It follows that
\[
\sum_{k=1}^3 \| \nabla h_k \|_{L^{2\mu}(\Omega)} = \| \nabla \times H \|_{L^{2\mu}(\Omega)},
\]
for any \( \mu > 0 \).

Again by the Sobolev embedding, we see that
\[
H \in L^{2\mu+2}(\Omega)
\]
and
\[
\| H \|_{L^{2\mu+2}(\Omega)} \leq C \left[ \| G \|_{L^{2\mu}(\Omega)} + \| \Phi \|_{L^{2\mu}(\Omega)} + \| \psi \|_{H^1(\Omega)} \right].
It follows from the proposition that we see that $h_k(x) \in C^\alpha(\bar{\Omega})$ and
\[ \sum_{k=1}^{3} ||h_k||_{C^\alpha(\bar{\Omega})} \leq C\left(||G||_{L^2,\nu(\Omega)} + ||\Phi||_{L^2,\nu(\Omega)} + ||\psi||_{H^1(\Omega)}\right), \]
where $C$ depends only on $\rho_0, \rho_1$ and $\Omega$.

Finally, to obtain the desired estimate in Theorem 2.1 we only need to estimate $||\psi||_{H^1(\Omega)}$. From (2.6)
\[ ||\psi||_{H^1(\Omega)} \leq C||H||_{H^1(\Omega)} + ||G||_{L^2(\Omega)} + ||\Phi||_{L^2(\Omega)} + ||F||_{L^2(\Omega)} \]
\[ \leq C\left(||G||_{L^2,\nu(\Omega)} + ||\Phi||_{L^2,\nu(\Omega)} + ||F||_{L^2(\Omega)}\right) \]
where we have used the $H^1(\Omega)$-estimate for $H$.

Q.E.D.

**Remark 2.1:** When the boundary condition (2.2) is replaced by the following nonhomogeneous one:
\[ H(x) = K(x), \quad x \in \partial \Omega, \]
with the consistency condition (since $\text{div} H = 0$)
\[ K \cdot N = 0, \quad x \in \partial \Omega, \]
where $N$ is the outward normal direction on $x \in \partial \Omega$. Let
\[ H^* = H - K. \]
Then $H^*$ will satisfy a system similar to (2.1) along with a homogeneous boundary condition.

**Remark 2.2:** Although the Campanato theory are available (see [4, 19]), the optimal regularity of the weak solution to the evolution system (1.4) is still open.

### 3. Applications

Similar to many nonlinear problems, the crucial step to establish the existence is to derive an a priori estimate in an appropriate space. Then one uses some type of fixed-point theorem to prove the existence. In this section, we will focus on how to derive such an a priori estimate for two nonlinear problems as examples.

As a first application of Theorem 2.1, we consider the following system:
\[ \nabla \times \left[a(x, |H|)\nabla \times H\right] = \nabla \times G + F, \quad x \in \Omega, \quad (3.1) \]
\[ \nabla \cdot H = 0, \quad x \in \Omega, \quad (3.2) \]
\[ H = 0, \quad x \in \partial \Omega. \quad (3.3) \]

$H(3.1)$: Suppose that $a(x, s)$ is differentiable with respect to $x$ and $s$ and there exist positive constants $a_0$ and $a_1$ such that
\[ 0 < a_0 \leq a(x, s) \leq a_1. \]
Moreover, $G$ is of class $C^{1+\alpha}(\bar{\Omega})$ and $F$ is of class $C^{\alpha}(\Omega)$.

**Theorem 3.1:** Under the assumptions $H(2.2)$ and $H(3.1)$, the problem (3.1)-(3.3) has a unique classical solution.
Proof: The proof is just a consequence of Theorem 2.1. Indeed, from Theorem 2.1, we know that $H$ is Hölder continuous and there exists a constant $C_1$ such that

$$
\|H\|_{C^{\alpha}(\bar{\Omega})} \leq C_1,
$$

where $C_1$ depends only on known data.

It follows that $a(x, |H|)$ is Hölder continuous and

$$
\|a(x, |H|)\|_{C^{\alpha}(\bar{\Omega})} \leq C_2,
$$

where $C_2$ depends only on known data.

From the regularity theory of [17], we see that $H$ is of class $C^{1+\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ and

$$
\|H\|_{C^{1+\alpha}(\bar{\Omega})} \leq C_3,
$$

where $C_3$ depends only on known data.

Note that $\nabla \cdot H = 0$, it follows that

$$
\nabla \times \nabla \times H = -\Delta H.
$$

Consequently, the system (3.1) is equivalent to

$$
-a(x, |H|)\Delta H + \nabla a(x, |H|) \times (\nabla \times H) = \nabla \times G + F, \quad x \in \Omega.
$$

We use the classical theory for elliptic systems ([8]) to obtain

$$
\|H\|_{C^{2+\alpha}(\bar{\Omega})} \leq C_4,
$$

where $C_4$ depends only on known data.

With the above apriori estimate in hand, one can easily use Schauder or Leray-Schauder's fixed point theorem (see [8]) to establish the existence. We shall skip this step.

Q.E.D.

Remark 3.1: It would be of interesting to study the following more general system

$$
\nabla \times [a(x, |H|, |\nabla \times H|)\nabla \times H] = F(x, H, \nabla \times H), \quad x \in \Omega.
$$

As the second application, we consider the following problem arising in inductive heating ([1, 2, 12, 13, 16, 18]):

$$
\begin{align*}
-\nabla \times [\rho(u)\nabla \times H] &= 0, \quad x \in \Omega, t > 0, \\
\rho(u)\nabla \times H &= 0, \quad x \in \Omega, t > 0., \\
\text{div} H &= 0, \quad x \in \Omega, t > 0,
\end{align*}
$$

subject to appropriate initial and boundary conditions.

For simplicity, we assume that initial and boundary conditions are given as follows:

$$
\begin{align*}
H(x) &= F(x), \quad x \in \Omega, \\
u(x, 0) &= u_0(x), \quad x \in \Omega, \\
u(x, t) &= g(x, t), \quad (x, t) \in \partial \Omega \times (0, T).
\end{align*}
$$

To obtain the classical solution for the problem (3.7)-(3.9), one needs to impose the following basic conditions for the known data.

H(3.3): (a) Suppose that $F \in C^{2+\alpha}(\Omega)$ and $F \cdot N = 0.$
(b) There exists a function $G(x, t) \in C^{2+\alpha,1+\alpha}(\bar{Q}_T)$ such that

$$G(x, 0) = u_0(x) \text{ for all } x \in \Omega \text{ and}$$

$$G(x, t) = g(x, t) \text{ for all } (x, t) \in \partial\Omega \times [0, T].$$

**Theorem 3.2:** Under the assumption $H(3.3)$, if $\rho(u)$ is differential and $0 < \rho_0 \leq \rho(u) \leq \rho_1$, then the problem (3.4)-(3.9) has a unique classical solution.

**Proof:** Again we focus on how to derive an a priori estimate in $C^{2+\alpha,1+\alpha}(\bar{Q}_T)$.

Since $0 < \rho_0 \leq \rho(u) \leq \rho_1 < \infty$, by Theorem 2.1 we see that $H \in C^{\alpha}(\bar{\Omega})$ and $\nabla \times H \in L^{2,\mu}(\Omega)$ for any $\mu \in (1,3)$. Moreover,

$$||H||_{C^\alpha(\bar{\Omega})} + ||\nabla \times H||_{L^{2,\mu}(\Omega)} \leq C,$$

where $C$ depends only on known data.

Now, we use the system (3.4) to rewrite the nonlinear term in (3.5) as follows:

$$\rho(u)|\nabla \times H|^2 = \nabla[\rho(u)H \cdot (\nabla \times H)].$$

It follows that $u(x, t)$ satisfies the following equation:

$$u_t - \Delta u = \rho(u)|\nabla \times H|^2$$

$$= \nabla[\rho(u)H \cdot (\nabla \times H)], \quad x \in \Omega, t > 0. \quad (3.10)$$

Since $H \in C^\alpha(\bar{\Omega})$ and $\nabla \times H \in L^{2,\mu}(\Omega)$, we apply $L^{2,\mu}(Q_T)$-theory for parabolic equations (see [19]) that

$$||u||_{C^{\alpha+\eta}(\bar{\Omega})} + ||\nabla u||_{L^{2,\mu+\eta}(Q_T)} \leq C,$$

where $C$ depends only on the known data.

Next we go back to the system (3.4). Since $\rho(u)$ is Hölder continuous, by the results of [17] we obtain

$$||H||_{C^{1+\alpha}(\bar{\Omega})} \leq C,$$

where $C$ depends only on the known data.

By applying Schauder’s estimate to the equation (3.5), we obtain

$$||u||_{L^{2,\mu+\eta}(Q_T)} \leq C||H||_{C^{1+\alpha}(\bar{\Omega})} + ||g||_{L^{2,\mu+\eta}(Q_T)} \leq C,$$

where $C$ depends only on known data.

Once we have the above a priori estimate, the existence of a solution can be established by Leray-Schauder’s fixed point theorem. Again we shall not give the detail here.

Q.E.D.

**Addendum:** The paper was completed in December 2000 and was published as the technical report 01-1, January 2001, at Washington State University. During the refereeing process, Professor S. Kim informed the author that he and Professor K. Kang also proved a similar result [20].
REFERENCES

[1] G. Cimatti and G. Prodi, Existence results for a nonlinear elliptic system modelling a temperature dependent electric resistor, Ann. Mat. Pura Appl. 62 (1988), 227–236.

[2] G. Cimatti, On two problems of electrical heating of conductors, Quarterly of Applied Mathematics, Vol. XLIX, No.4 (1991), 729–740.

[3] S. Campanato, Equazioni ellittiche del secondo ordine e spazi $L^{2,\alpha}$, Ann. Mat. Pura Appl., 69 (1965), 321–382.

[4] S. Campanato, Equazioni paraboliche del secondo ordine e spazi $L^{2,\alpha}$, Ann. Mat. Pura Appl., 73 (1965), 55–102.

[5] E. De Giorgi, Sulla differenziabilita e analiticita delle estremali degli integral multipli regolari, Mem. Accad. Sci. Torino Cl. Sci Fis. Mat. Natur., 3 (1957), 25–43.

[6] E. DiBenedetto, “Degenerate Parabolic Equations,” Springer-Verlag, New York, 1993.

[7] M. Giaquinta, “Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems,” Annals of Mathematics Studies, Princeton Press, New Jersey, 1983.

[8] D. Gilbarg and N. Trudinger, “Elliptic Partial Differential Equations of Second Order,” Springer-Verlag, Berlin, 2nd ed., 1983.

[9] R. Glassey and Hong-Ming Yin, On Maxwell’s equations with a temperature effect, Communications in Mathematical Physics, Vol. 194 (1998), 343–358.

[10] L. D. Landau and E.M. Lifshitz, “Electrodynamics of Continuous Media,” Pergamon Press, New York, 1960.

[11] O.A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural’ceva, “Linear and Quasi-linear Equations of Parabolic Type,” AMS Transl., 23, Providence., R.I., 1968.

[12] A.C. Metaxas and R.J. Meredith, “Industrial Microwave Heating, London,” Per Peregrinus Ltd., 1983.

[13] A. Metaxas, “Foundations of Electroheat, a unified approach,” John Wiley and Sons, New York, 1996.

[14] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958), 931–945.

[15] G. Troianiello, “Elliptic Differential Equations and Obstacle Problems,” Plenum Press, New York, 1987.

[16] H. M. Yin, Global Solutions of Maxwell’s equations in an electromagnetic field with the temperature-dependent electrical conductivity, European Journal of Applied Mathematics, 5 (1994), 57–64.

[17] H.M. Yin, Regularity of solutions to Maxwell’s system in quasi-stationary electromagnetic fields and applications, Comm. in P.D.E., 22 (1997), 1029–1053.

[18] H.M. Yin, On Maxwell’s equations in an electromagnetic field with the temperature effect, SIAM Journal of Mathematical Analysis, 29 (1998), 637–651.

[19] H.M. Yin, $L^p$-estimates for parabolic equations and applications, J. Part. Diff. Equ., 10 (1997), 31–44.

[20] K. Kang and S. Kim, On the Hölder continuity of solutions of a certain system related to Maxwell’s equations, IMA preprint series, #1805, August, 2001.

Received February 2001; revised June 2001.

E-mail address: hyin@wsu.edu