Equivariant multiplicities of Coxeter arrangements and invariant bases

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Abstract

Let $\mathcal{A}$ be an irreducible Coxeter arrangement and $W$ be its Coxeter group. Then $W$ naturally acts on $\mathcal{A}$. A multiplicity $m : \mathcal{A} \to \mathbb{Z}$ is said to be equivariant when $m$ is constant on each $W$-orbit of $\mathcal{A}$. In this article, we prove that the multi-derivation module $D(\mathcal{A}, m)$ is a free module whenever $m$ is equivariant by explicitly constructing a basis, which generalizes the main theorem of [T2002]. The main tool is a primitive derivation and its covariant derivative. Moreover, we show that the $W$-invariant part $D(\mathcal{A}, m)^W$ for any multiplicity $m$ is a free module over the $W$-invariant subring.

1 Introduction

Let $V$ be an $\ell$-dimensional Euclidean space with an inner product $I : V \times V \to \mathbb{R}$. Let $S$ denote the symmetric algebra of the dual space $V^*$ and $F$ be its quotient field. Let $\text{Der}_S$ be the $S$-module of $\mathbb{R}$-linear derivations from $S$ to itself. Let $\Omega^1_S$ be the $S$-module of regular 1-forms. Similarly define $\text{Der}_F$ and $\Omega^1_F$ over $F$. The dual inner product $I^* : V^* \times V^* \to \mathbb{R}$ naturally induces an $F$-bilinear form $I^* : \Omega^1_F \times \Omega^1_F \to F$. Then one has an $F$-linear bijection

$I^* : \Omega^1_F \to \text{Der}_F$

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defined by \([I^*(\omega)](f) := I^*(\omega, df)\) for \(f \in F\).

Let \(\mathcal{A}\) be an irreducible Coxeter arrangement with its Coxeter group \(W\). For each \(H \in \mathcal{A}\), choose \(\alpha_H \in V^*\) with \(H = \ker(\alpha_H)\). Let \(Q = \prod_{H \in \mathcal{A}} \alpha_H \in S\).

Recall the \(S\)-module of logarithmic forms

\[\Omega^1(\mathcal{A}, \infty) = \{\omega \in \Omega^1_F \mid Q^N \omega \text{ and } (Q/\alpha_H)^N \omega \wedge d\alpha_H \text{ are both regular for any } H \in \mathcal{A} \text{ and } N \gg 0\}\]

and the \(S\)-module of logarithmic derivations

\[D(\mathcal{A}, -\infty) = I^*(\Omega^1(\mathcal{A}, \infty))\]

from \([AT2010Z]\). A map \(m : \mathcal{A} \to \mathbb{Z}\) is called a multiplicity. For an arbitrary multiplicity, let

\[D(\mathcal{A}, m) = \{\theta \in D(\mathcal{A}, -\infty) \mid \theta(\alpha_H) \in \alpha_H^{m(H)} S_{(\alpha_H)} \text{ for all } H \in \mathcal{A}\},\]

\[\Omega^1(\mathcal{A}, m) = (I^*)^{-1}D(\mathcal{A}, -m),\]

where \(S_{(\alpha_H)}\) is the localization of \(S\) at the prime ideal \((\alpha_H)\). These two modules were introduced in \([Sa1980]\) (when \(m\) is constantly equal to one), in \([Z1989]\) (when \(\text{im}(m) \subset \mathbb{Z}_{\geq 0}\)), and in \([A2008, AT2010Z, AT2009]\) (when \(m\) is arbitrary). A derivation \(0 \neq \theta \in \text{Der}_F\) is said to be homogeneous of degree \(d\), or \(\deg \theta = d\), if \(\theta(\alpha) \in F\) is homogeneous of degree \(d\) whenever \(\theta(\alpha) \neq 0\) \((\alpha \in V^*)\). A multiarrangement \((\mathcal{A}, m)\) is called to be free with exponents \(\exp(\mathcal{A}, m) = (d_1, \ldots, d_\ell)\) if \(D(\mathcal{A}, m) = \oplus_{i=1}^\ell S \cdot \theta_i\) with a homogeneous basis \(\theta_i\) such that \(\deg(\theta_i) = d_i\) \((i = 1, \ldots, \ell)\). A multiplicity \(m : \mathcal{A} \to \mathbb{Z}\) is said to be equivariant when \(m(H) = m(wH)\) for any \(H \in \mathcal{A}\) and any \(w \in W\), i.e., \(m\) is constant on each orbit. In this article we prove

**Theorem 1.1** For any irreducible Coxeter arrangement \(\mathcal{A}\) and any equivariant multiplicity \(m\), the multiarrangement \((\mathcal{A}, m)\) is free.

For a fixed arrangement \(\mathcal{A}\), we say that a multiplicity \(m\) is free if \((\mathcal{A}, m)\) is free. Although we have a limited knowledge about the set of all free multiplicities for a fixed irreducible Coxeter arrangement \(\mathcal{A}\), it is known that there exist infinitely many non-free multiplicities unless \(\mathcal{A}\) is either one- or two-dimensional \([ATY2009]\). Theorem 1.1 claims that any equivariant multiplicity is free for any irreducible Coxeter arrangement.

When the \(W\)-action on \(\mathcal{A}\) is transitive, an equivariant multiplicity is constant and a basis was constructed in \([ST1998, Z2002, AY2007, AT2010Z]\). So we may assume, in order to prove Theorem 1.1 that the \(W\)-action on \(\mathcal{A}\) is not transitive. In other words, we may only study the cases when \(\mathcal{A}\) is of the
type either $B_\ell$, $F_4$, $G_2$ or $I_2(2n)$ ($n \geq 4$). In these cases, $\mathcal{A}$ has exactly two $W$-orbits: $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. The orbit decompositions are explicitly given by: $B_\ell = A_1^\ell \cup D_\ell$, $F_4 = D_4 \cup D_4$, $G_2 = A_2 \cup A_2$ or $I_2(2n) = I_2(n) \cup I_2(n)$ ($n \geq 4$). Note that $A_1^\ell$ is not irreducible.

When $\mathcal{A}$ is irreducible, the primitive derivations play the central role to define the Hodge filtration introduced by K. Saito. (See [Sa2003] for example.) For $R := S^W$, let $D$ be an element of the lowest degree in $\text{Der}_R$, which is called a primitive derivation corresponding to $\mathcal{A}$. Then $D$ is unique up to a nonzero constant multiple. A theory of primitive derivations in the case of non-irreducible Coxeter arrangements was introduced in [AT2009]. Thus we may consider a primitive derivation $D_i$ corresponding with the orbit $\mathcal{A}_i$ ($1 \leq i \leq 2$). We only use $D_1$ because of symmetricity. Note that $D_1$ is not unique up to a nonzero multiple when $\mathcal{A}_1 = A_1^\ell$ (non-irreducible). Denote the reflection groups of $\mathcal{A}_i$ by $W_i$ ($i = 1, 2$). The Coxeter arrangements $B_\ell, F_4, G_2$ and $I_2(2n)$ ($n \geq 4$) are classified into two cases, that is, (1) the primitive derivation $D_1$ can be chosen to be $W$-invariant for $B_\ell$ and $F_4$ (the first case) while (2) $D_1$ is $W_2$-antiinvariant for $G_2$ and $I_2(2n)$ ($n \geq 4$) (the second case) as we will see in Section 4. Since the second cases are two-dimensional, Theorem 1.1 holds true. Thus the first case is the only remaining case to prove Theorem 1.1.

Let

$$\nabla : \text{Der}_F \times \text{Der}_F \to \text{Der}_F$$

$$(\theta, \delta) \mapsto \nabla_\theta \delta$$

be the Levi-Civita connection with respect to the inner product $I$ on $V$. We need the following theorem for our proof of Theorem 1.1.

**Theorem 1.2** ([AT2010Z, AT2009]) Let $D(\mathcal{A}, -\infty)^W$ be the $W$-invariant part of $D(\mathcal{A}, -\infty)$. Then

$$\nabla_D : D(\mathcal{A}, -\infty)^W \overset{\sim}{\longrightarrow} D(\mathcal{A}, -\infty)^W$$

is a $T$-linear automorphism where $T := \{f \in R \mid Df = 0\}$. When $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition,

$$\nabla_{D_1} : D(\mathcal{A}_1, -\infty)^{W_1} \overset{\sim}{\longrightarrow} D(\mathcal{A}_1, -\infty)^{W_1}$$

is a $T_1$-linear automorphism where

$$R_1 := S^{W_1}, \quad T_1 := \{f \in R_1 \mid D_1f = 0\}.$$
Let $E$ be the Euler derivation characterized by the equality $E(\alpha) = \alpha$ for every $\alpha \in V^*$. Suppose that $A = A_1 \cup A_2$ is the orbit decomposition and that the primitive derivation $D_1$ is $W$-invariant. Define

$$E^{(p,q)} := \nabla_D^{-q} \nabla_{D_1}^{-p} E$$

for $p, q \in \mathbb{Z}$. Here, thanks to Theorem 1.2, we may interpret \(\nabla_D^m = (\nabla_D^{-1})^{-m}\) and $\nabla_{D_1}^m = (\nabla_{D_1}^{-1})^{-m}$ when $m$ is negative. Denote the equivariant multiplicity $m$ by $(m_1, m_2)$ when $m(H) = m_1$ ($H \in A_1$) and $m(H) = m_2$ ($H \in A_2$).

Let $x_1, \ldots, x_\ell$ be a basis for $V^*$ and $P_1, \ldots, P_\ell$ be a set of basic invariants with respect to $W$: $R = \mathbb{R}[P_1, \ldots, P_\ell]$. Let $P_1^{(i)}(\ldots, P_\ell^{(i)})$ be a set of basic invariants with respect to $W_i$: $R_i = \mathbb{R}[P_1^{(i)}, \ldots, P_\ell^{(i)}]$ ($i = 1, 2$). Define

$$d_j := \deg P_j, \ d_j^{(i)} := \deg P_j^{(i)} (i = 1, 2, 1 \leq j \leq \ell).$$

We assume

$$d_1 \leq d_2 \leq \cdots \leq d_\ell, \ d_1^{(i)} \leq d_2^{(i)} \leq \cdots \leq d_\ell^{(i)} (i = 1, 2).$$

Then $h := d_\ell$ is called the Coxeter number of $W$. We call $h_i := \deg P_i^{(i)}$ the Coxeter number of $W_i$ ($i = 1, 2$). We use the notation

$$\partial_{x_j} := \partial/\partial x_j, \ \partial_{P_j} := \partial/\partial P_j, \ \partial_{P_j^{(i)}} := \partial/\partial P_j^{(i)} (1 \leq j \leq \ell, 1 \leq i \leq 2).$$

The following theorem gives an explicit construction of a basis:

**Theorem 1.3** Let $A$ be an irreducible Coxeter arrangement. Suppose that $A = A_1 \cup A_2$ is the orbit decomposition and that the primitive derivation $D_1$ is $W$-invariant. Then

1. the $S$-module $D(A, (2p - 1, 2q - 1))$ is free with $W$-invariant basis

$$\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}$$

with $\deg \nabla_{\partial_{P_i}} E^{(p,q)} = ph_1 + qh_2 - d_i + 1$ for $i = 1, \ldots, \ell$,

2. the $S$-module $D(A, (2p - 1, 2q))$ is free with basis

$$\nabla_{\partial_{P_1^{(i)}}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell^{(i)}}} E^{(p,q)}$$

with $\deg \nabla_{\partial_{P_i^{(i)}}} E^{(p,q)} = ph_1 + qh_2 - d_i^{(1)} + 1$ for $i = 1, \ldots, \ell$. 

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(3) the $S$-module $D(\mathcal{A}, (2p, 2q - 1))$ is free with basis
\[ \nabla_{\partial^{(2)}_{p_i}} E^{(p,q)}, \ldots, \nabla_{\partial^{(2)}_{\ell}} E^{(p,q)} \]
with $\deg \nabla_{\partial^{(2)}_{p_i}} E^{(p,q)} = ph_1 + qh_2 - d_i^{(2)} + 1$ for $i = 1, \ldots, \ell$.

(4) the $S$-module $D(\mathcal{A}, (2p, 2q))$ is free with basis
\[ \nabla_{\partial x_1} E^{(p,q)}, \ldots, \nabla_{\partial x_\ell} E^{(p,q)} \]
with $\deg \nabla_{\partial x_i} E^{(p,q)} = ph_1 + qh_2$ for $i = 1, \ldots, \ell$.

The existence of the primitive decomposition of $D(\mathcal{A}, (2p-1, 2q-1))^W$ is proved by the following theorem:

**Theorem 1.4** Under the same assumption of Theorem 1.3 define
\[ \theta_i^{(p,q)} := \nabla_{\partial x_i} \nabla_{\partial x_i} \nabla_{\partial x_i}^{-p} E \quad (1 \leq i \leq \ell) \]
for $p, q \in \mathbb{Z}$. Then the set
\[ \{ \theta_i^{(p+k,q+k)} \mid k \geq 0, 1 \leq i \leq \ell \} \]
is a $T$-basis for $D(\mathcal{A}, (2p - 1, 2q - 1))^W$. Put
\[ \mathcal{G}^{(p,q)} := \bigoplus_{i=1}^\ell T : \theta_i^{(p,q)}. \]
Then we have a $T$-module decomposition (called the primitive decomposition)
\[ D(\mathcal{A}, (2p - 1, 2q - 1))^W = \bigoplus_{k \geq 0} \mathcal{G}^{(p+k,q+k)}. \]

We will also prove

**Theorem 1.5** For any irreducible Coxeter arrangement $\mathcal{A}$ and any multiplicity $m$, the $R$-module $D(\mathcal{A}, m)^W$ is free.

Theorems 1.1 and 1.3 are used to prove the freeness of Shi-Catalan arrangements associated with any Weyl arrangements in [AT2010].

The organization of this article is as follows. In Section 2 we prove Theorem 1.3 when $q \geq 0$. In Section 3 we prove Theorem 1.4 to have the primitive decomposition, which is a key to complete the proof of Theorem 1.3 at the end of Section 3. In Section 4 we verify that the primitive derivation $D_1$ can be chosen to be $W$-invariant when $\mathcal{A}$ is a Coxeter arrangement of either the type $B_\ell$ or $F_4$. We also review the cases of $G_2$ and $I_2(2n)$ ($n \geq 4$) and find that the primitive derivation $D_1$ is $W_2$-antiinvariant. In Section 5, combining Theorem 1.3 with earlier results in [T2002, AT2010Z, W2010], we finally prove Theorems 1.1 and 1.5.
2 Proof of Theorem 1.3 when \(q \geq 0\)

In this section we prove Theorem 1.3 when \(q \geq 0\).

Recall \(R = S^W = \mathbb{R}[P_1, \ldots, P_\ell]\) is the invariant ring with basic invariants \(P_1, \ldots, P_\ell\) such that \(2 = \deg P_1 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell = h\), where \(h\) is the Coxeter number of \(W\). Put \(D = \partial P_\ell \in \text{Der} R\) which is a primitive derivation. Recall \(T = \ker(D : R \to R) = \mathbb{R}[P_1, \ldots, P_{\ell-1}]\). Then the covariant derivative \(\nabla_D\) is \(T\)-linear. For \(P := [P_1, \ldots, P_\ell]\), the Jacobian matrix \(J(P)\) is defined as the matrix whose \((i, j)\)-entry is \(\frac{\partial P_j}{\partial x_i}\). Define \(A := [I^*(dx_i, dx_j)]_{1 \leq i, j \leq \ell}\) and \(G := [I^*(dP_i, dP_j)]_{1 \leq i, j \leq \ell} = J(P)^T AJ(P)\).

**Definition 2.1** ([Y2002, W2010]) Let \(m : A \to Z\) and \(\zeta \in D(A, -\infty)^W\). We say that \(\zeta\) is \(m\)-universal when \(\zeta\) is homogeneous and the \(S\)-linear map

\[
\Psi_\zeta : \text{Der}_S \longrightarrow D(A, 2m)
\]

\[
\theta \longmapsto \nabla_\theta \zeta
\]

is bijective.

**Example 2.2** The Euler derivation \(E\) is \(0\)-universal because \(\Psi_E(\delta) = \nabla_\delta E = \delta\) and \(D(A, 0) = \text{Der}_S\).

Recall the \(T\)-automorphisms

\[
\nabla^k_D : D(A, -\infty)^W \cong D(A, -\infty)^W (k \in \mathbb{Z})
\]

from Theorem 1.2. Recall the following two results concerning the \(m\)-universality:

**Theorem 2.3** ([W2010, Theorem 2.8]) If \(\zeta\) is \(m\)-universal, then \(\nabla^{-1}_D \zeta\) is \((m + 1)\)-universal.

**Proposition 2.4** ([W2010, Proposition 2.7]) Suppose that \(\zeta\) is \(m\)-universal. Let \(k : A \to \{+1, 0, -1\}\). Then an \(S\)-homomorphism

\[
\Phi_\zeta : D(A, k) \to D(A, k + 2m)
\]

defined by

\[
\Phi_\zeta(\theta) := \nabla_\theta \zeta
\]

gives an \(S\)-module isomorphism.
We require that assumption of Theorem 1.3 is satisfied in the rest of this section: Suppose that \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \) is the orbit decomposition and that \( D_1 \), a primitive derivation with respect to \( \mathcal{A}_1 \) in the sense of [AT2009, Definition 2.4], is \( W \)-invariant. Let \( W_i, R_i, P_i, T_i, D_i \) \((i = 1, 2)\) are defined as in Section 1. Even when \( \mathcal{A}_1 \) is not irreducible, we may consider a \( T_1 \)-isomorphism

\[
\nabla_{D_1}^k: D(\mathcal{A}_1, -\infty)^{W_1} \xrightarrow{\sim} D(\mathcal{A}_1, -\infty)^{W_1} (k \in \mathbb{Z})
\]

from Theorem 1.2.

**Proposition 2.5** Suppose \( q \geq 0 \). The derivation \( E^{(p,q)} := \nabla_D^{-q} \nabla_{D_1}^{-p} E \) is \((p,q)\)-universal.

**Proof.** When \( \mathcal{A}_1 \) is irreducible, [AY2007] and [AT2010Z] imply that \( \nabla_{D_1}^{-p} E \) is \((p - q, 0)\)-universal. When \( \mathcal{A}_1 \) is not irreducible, \( \nabla_{D_1}^{-p} E \) is \((p - q, 0)\)-universal because of [AT2009]. Thus \( E^{(p,q)} = \nabla_D^{-q} \nabla_{D_1}^{-p} E \) is \((p,q)\)-universal by Theorem 2.3. \( \square \)

Since \( E^{(p,q)} \) is \((p,q)\)-universal, Proposition 2.4 yields the following:

**Proposition 2.6** Let \( q \geq 0 \) and \( m: \mathcal{A} \to \{+1, 0, -1\} \). Then an \( S \)-homomorphism

\[
\Phi_{p,q}: D(\mathcal{A}, m) \to D(\mathcal{A}, (2p, 2q) + m)
\]

defined by

\[
\Phi_{p,q}(\theta) := \nabla_\theta E^{(p,q)}
\]

gives an \( S \)-module isomorphism.

**Proof of Theorem 1.3** \((q \geq 0)\). We may apply Proposition 2.6 because

1. \( \partial P_1, \ldots, \partial P_\ell \) form a basis for \( D(\mathcal{A}, (-1, -1)) \),
2. \( \partial_{P_1(1)}, \ldots, \partial_{P_\ell(1)} \) form a basis for \( D(\mathcal{A}, (-1, 0)) \),
3. \( \partial_{P_1(2)}, \ldots, \partial_{P_\ell(2)} \) form a basis for \( D(\mathcal{A}, (0, -1)) \), and
4. \( \partial_{x_1}, \ldots, \partial_{x_\ell} \) form a basis for \( D(\mathcal{A}, (0, 0)) \). \( \square \)

### 3 Primitive decompositions

In this section we first prove Theorem 1.4 to define the primitive decomposition of \( D(\mathcal{A}, (2p - 1, 2q - 1))^W \). Next we prove Theorem 1.3.
Proposition 3.1 Let $\zeta$ be $m$-universal. Then

1. The set \( \{ \nabla_{\partial_{P_j}} \nabla_D^{-k} \zeta \mid 1 \leq j \leq \ell, k \geq 0 \} \) is linearly independent over \( T \).

2. Define \( G^{(k)} \) to be the free \( T \)-module with basis \( \{ \nabla_{\partial_{P_j}} \nabla_D^{-k} \zeta \mid 1 \leq j \leq \ell \} \) for \( k \geq 0 \). Then the Poincaré series \( \text{Poin}(\bigoplus_{k \geq 0} G^{(k)}, t) \) satisfies:

\[
Poin(\bigoplus_{k \geq 0} G^{(k)}, t) = \left( \prod_{i=1}^{\ell} \frac{1}{1 - t^{d_i}} \right) \left( \sum_{j=1}^{\ell} t^{p-d_j} \right),
\]

where \( p = \deg \zeta \) and \( d_j = \deg P_j \) (1 \( \leq j \leq \ell \)).

3. \( D(\mathcal{A}, 2m - 1)W = \bigoplus_{k \geq 0} G^{(k)}. \)

Proof. Let \( k \in \mathbb{Z}_{\geq 0} \). By Theorem 2.3, \( \zeta^{(k)} := \nabla_D^{-k} \zeta \) is \( (m + k) \)-universal, where the “\( k \)” in the \( (m + k) \) stands for the constant multiplicity \( k \) by abuse of notation. Thus by Proposition 2.4 we have the following two bases:

\[ \nabla_{\partial_{P_1}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k)}, \]

for the \( S \)-module \( D(\mathcal{A}, 2m + 2k - 1) \) and

\[ \nabla_{\partial_{I^*(dP_1)}} \zeta^{(k)}, \ldots, \nabla_{\partial_{I^*(dP_\ell)}} \zeta^{(k)}, \]

for the \( S \)-module \( D(\mathcal{A}, 2m + 2k + 1) \). Note that the two bases are also \( R \)-bases for \( D(\mathcal{A}, 2m + 2k - 1)^W \) and \( D(\mathcal{A}, 2m + 2k + 1)^W \) respectively. Since the \( T \)-automorphism

\[ \nabla_D : D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W \]

in Theorem 1.2 induces a \( T \)-linear bijection

\[ \nabla_D : D(\mathcal{A}, 2m + 2k + 1)^W \xrightarrow{\sim} D(\mathcal{A}, 2m + 2k - 1)^W \]

as in [AT2009, Theorem 4.4], we may find an \( \ell \times \ell \)-matrix \( B^{(k)} \) with entries in \( R \) such that

\[
\nabla_D \left( \begin{bmatrix} \nabla_{\partial_{P_1}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k)} \end{bmatrix} \right) G = \nabla_D \left[ \nabla_{\partial_{I^*(dP_1)}} \zeta^{(k)}, \ldots, \nabla_{\partial_{I^*(dP_\ell)}} \zeta^{(k)} \right] = \begin{bmatrix} \nabla_{\partial_{P_1}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_\ell}} \zeta^{(k)} \end{bmatrix} B^{(k)},
\]

The degree of \((i, j)\)-th entry of \( B^{(k)} \) is \( m_i + m_j - h \leq h - 2 < h \). In particular, the degree of \( B_{i,i+1-i}^{(k)} \) is 0 and \( B_{i,j}^{(k)} = 0 \) if \( i + j < \ell + 1 \). Hence each entry
of $B^{(k)}$ lies in $T$ and $\det B^{(k)} \in \mathbb{R}$. Since $D$ is a derivation of the minimum degree in $\text{Der}_R$, one gets $[D, \partial_{P_i}] = 0$. Thus $\nabla_D \nabla_{\partial_{P_i}} = \nabla_{\partial_{P_i}} \nabla_D$. Operate $\nabla^{-1}_D$ on both sides of the equality above, and get

$$\left[\nabla_{\partial_{P_i}} \zeta^{(k)} + \cdots, \nabla_{\partial_{P_k}} \zeta^{(k)}\right] G = \left[\nabla_{\partial_{P_i}} \zeta^{(k+1)} + \cdots, \nabla_{\partial_{P_k}} \zeta^{(k+1)}\right] B^{(k)}.$$

This implies that $\det B^{(k)} \in \mathbb{R}^\times$ because $\nabla_{\partial_{P_i}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_k}} \zeta^{(k)}$ are linearly independent over $S$. Inductively we have

$$\left[\nabla_{\partial_{P_i}} \zeta^{(k+1)} + \cdots, \nabla_{\partial_{P_k}} \zeta^{(k+1)}\right] = \left[\nabla_{\partial_{P_i}} \zeta^{(k)}, \ldots, \nabla_{\partial_{P_k}} \zeta^{(k)}\right] G(B^{(k)})^{-1}
= \left[\nabla_{\partial_{P_i}} \zeta, \ldots, \nabla_{\partial_{P_k}} \zeta\right] G(B^{(0)})^{-1} G(B^{(1)})^{-1} \cdots G(B^{(k)})^{-1}
= \left[\nabla_{\partial_{P_i}} \zeta, \ldots, \nabla_{\partial_{P_k}} \zeta\right] G_{k+1},$$

where $G_i = G(B^{(0)})^{-1} G(B^{(1)})^{-1} \cdots G(B^{(i-1)})^{-1}$ ($i \geq 0$). Note that $G$ appears $i$ times in the definition of $G_i$. For $M = (m_{ij}) \in M_{\ell}(F)$, define $D[M] = (D(m_{ij})) \in M_\ell(F)$. Then $D^j[G_i] = 0$ when $j > i$ and $\det D^j[G_i] \neq 0$ because $\det D[G] \neq 0$ and $D^2[G] = O$ (e.g., see [Sa1993, AT2009]).

1) Suppose that $\left\{\nabla_{\partial_{P_j}} \zeta^{(k)} \mid 1 \leq j \leq \ell, k \geq 0\right\}$ is linearly dependent over $T$. Then there exist $\ell$-dimensional column vectors $g_0, g_1, \ldots, g_{q} \in T^{\ell}(q \geq 0)$ with $g_0 \neq 0$ such that

$$0 = \sum_{i=0}^{q} \left[\nabla_{\partial_{P_i}} \zeta^{(i)}, \ldots, \nabla_{\partial_{P_k}} \zeta^{(i)}\right] g_i = \left[\nabla_{\partial_{P_i}} \zeta, \ldots, \nabla_{\partial_{P_k}} \zeta\right] \left(\sum_{i=0}^{q} G_i g_i\right).$$

Since $\nabla_{\partial_{P_i}} \zeta, \ldots, \nabla_{\partial_{P_k}} \zeta$ are linearly independent over $R$, one has

$$0 = \sum_{i=0}^{q} G_i g_i.$$

Applying the operator $D$ on both sides $q$ times, we get $D^q[G_q]g_q = 0$. Thus $g_q = 0$ which is a contradiction. This proves (1).

2) Compute

$$\text{Poin}(\bigoplus_{k \geq 0} G^{(k)}, t) = \sum_{k \geq 0} \left(\prod_{i=1}^{\ell-1} \frac{1}{1-t^{d_i}}\right) \left(\sum_{j=1}^{\ell} t^{p-d_j + kd_i}\right) = \left(\prod_{i=1}^{\ell-1} \frac{1}{1-t^{d_i}}\right) \left(\sum_{k \geq 0} t^{kd_i}\right) \left(\sum_{j=1}^{\ell} t^{p-d_j}\right) = \left(\prod_{i=1}^{\ell-1} \frac{1}{1-t^{d_i}}\right) \left(\sum_{j=1}^{\ell} t^{d_j}\right).$$
We have
\[ D(A, 2m - 1)^W \supseteq \bigoplus_{k \geq 0} G^{(k)} \]
by (1). So it suffices to prove
\[ \text{Poin}(D(A, 2m - 1)^W, t) = \text{Poin}(\bigoplus_{k \geq 0} G^{(k)}, t). \]
Since \( D(A, 2m - 1)^W \) is a free \( R \)-module with a basis
\[ \nabla_{\partial P_1} \zeta, \ldots, \nabla_{\partial P_\ell} \zeta, \]
we obtain
\[ \text{Poin}(D(A, 2m - 1)^W, t) = \left( \prod_{i=1}^\ell \frac{1}{1 - t^p_i d_i} \right) \left( \sum_{i=1}^\ell t^{p_i - d_i} \right) = \text{Poin}(\bigoplus_{k \geq 0} G^{(k)}, t), \]
which completes the proof. \( \square \)

We require that the assumption of Theorem 1.3 is satisfied in the rest of this section.

Proof of Theorem 1.4. Suppose \( q \geq 0 \) to begin with. Then, by Proposition 3.4, \( E^{(p,q)} \) is \((p,q)\)-universal. Apply Proposition 3.1 for \( \zeta = E^{(p,q)} \) and \( m = (p,q) \), and we have Theorem 1.4:
\[ D(A, (2p - 1, 2q - 1))^W = \bigoplus_{k \geq 0} G^{(p+k,q+k)} \]
when \( q \geq 0 \). Send the both handsides by \( \nabla_D \), and we get
\[ D(A, (2p - 3, 2q - 3))^W = \bigoplus_{k \geq 0} G^{(p+k-1,q+k-1)} \]
because \( \nabla_D \left( D(A, (2p - 1, 2q - 1))^W \right) = D(A, (2p-3, 2q-3))^W \) as in [AT2009, Theorem 4.4] and \( \nabla_D(\theta_i^{(p,q)}) = \theta_i^{(p-1,q-1)} \). Apply \( \nabla_D \) repeatedly to complete the proof for all \( q \in \mathbb{Z} \). \( \square \)

Note that we do not assume \( p \geq 0 \) in the following proposition:

Proposition 3.2 For \( p, q \in \mathbb{Z} \), the \( S \)-module \( D(A, (2p - 1, 2q - 1)) \) has a \( W \)-invariant basis.
Proof. Recall that
\[ \nabla_{\partial P_1} E^{(p,q)}, \nabla_{\partial P_2} E^{(p,q)}, \ldots, \nabla_{\partial P_\ell} E^{(p,q)}, \]
which are $W$-invariant, form an $S$-basis for $D(A, (2p-1, 2q-1))$ when $q \geq 0$ by Theorem 1.3 (1). It is then easy to see that they are also an $R$-basis for $D(A, (2p-1, 2q-1))^W$ for $q \geq 0$. By [A2008] [AT2010Z], there exists a $W$-equivariant nondegenerate $S$-bilinear pairing
\[ (\ , \ ) : D(A, (2p-1, 2q-1)) \times D(A, (-2p+1, -2q+1)) \rightarrow S, \]
characterized by
\[ (I^*(\omega), \theta) = \langle \omega, \theta \rangle \]
where $\omega \in \Omega^1(A, (-2p+1, -2q+1))$ and $\theta \in D(A, (-2p+1, -2q+1))$. Let $\theta_1, \ldots, \theta_\ell$ denote the dual basis for $D(A, (-2p+1, -2q+1))$ satisfying
\[ \left( \nabla_{\partial P_i} E^{(p,q)}, \theta_j \right) = \delta_{ij} \]
for $1 \leq i, j \leq \ell$. Then $\theta_1, \ldots, \theta_\ell$ are $W$-invariant because the pairing $(\ , \ )$ is $W$-equivariant. \[ \square \]

Although the following lemma is standard and easy, we give a proof for completeness.

Lemma 3.3 Let $M$ be an $S$-submodule of $\operatorname{Der}_F$. The following two conditions are equivalent:

1. $M$ has a $W$-invariant basis $\Theta$ over $S$.
2. The $W$-invariant part $M^W$ is a free $R$-module with a basis $\Theta$ and there exists a natural $S$-linear isomorphism
\[ M^W \otimes_R S \simeq M. \]

Proof. It suffices to prove that (1) implies (2) because the other implication is obvious. Suppose that $\Theta = \{\theta_\lambda\}_{\lambda \in \Lambda}$ is a $W$-invariant basis for $M$ over $S$. Since it is linearly independent over $S$, so is over $R$. Let $\theta \in M^W$. Express
\[ \theta = \sum_{i=1}^n f_i \theta_i, \]
with $f_i \in S$ and $\theta_i \in \Theta$ ($i = 1, \ldots, n$). Let $w \in W$ act on the both handsides. Then we get
\[ \theta = \sum_{i=1}^n w(f_i) \theta_i. \]
This implies \( f_i = w(f_i) \) for every \( w \in W \). Hence \( f_i \in R \) for each \( i \). Therefore \( \Theta \) is a basis for \( M^W \) over \( R \). This is (2).

**Proposition 3.4** For any \( p, q \in \mathbb{Z} \), \( E^{(p,q)} \) is \((p, q)\)-universal.

**Proof.** By Theorem 1.4 we have the decomposition:

\[
D(A, (2p - 1, 2q - 1))^W = \bigoplus_{k \geq 0} G^{(p+k,q+k)}
\]

for \( p, q \in \mathbb{Z} \). As we saw in Proposition 3.1 (2), we have

\[(3.1) \quad \text{Poin}(D(A, (2p - 1, 2q - 1))^W, t) = \text{Poin}(\bigoplus_{k \geq 0} G^{(p+k,q+k)}, t) = \left(\prod_{i=1}^\ell \frac{1}{1-t^{d_i}}\right) \left(\sum_{i=1}^\ell t^{m-d_j}\right),\]

where \( m := \deg E^{(p,q)} \). Recall that the \( S \)-module \( D(A, (2p - 1, 2q - 1)) \) has a \( W \)-invariant basis \( \theta_1, \ldots, \theta_\ell \) by Proposition 3.2. By Lemma 3.3 we know that \( \theta_1, \ldots, \theta_\ell \) form a basis for the \( R \)-module \( D(A, (2p - 1, 2q - 1))^W \). Thanks to (3.1) we may assume that \( \deg \theta_j = m - d_j = \deg \partial_{\partial x_j} E^{(p,q)} \). Therefore there exists \( M \in M_\ell(R) \) such that

\[
[\theta_1, \ldots, \theta_\ell]M = [\partial_{\partial x_1} E^{(p,q)}, \ldots, \partial_{\partial x_\ell} E^{(p,q)}]
\]

with \( \det M \in \mathbb{R} \). Since

\[
\max_{1 \leq i, j \leq \ell} |\deg \theta_i - \deg \partial_{\partial x_j} E^{(p,q)}| = d_\ell - d_1 < \deg P_\ell,
\]

we get \( M \in M_\ell(T) \). Since \( \partial_{\partial x_1} E^{(p,q)}, \ldots, \partial_{\partial x_\ell} E^{(p,q)} \) are linearly independent over \( T \) by Proposition 3.1 (1), we have \( \det M \in \mathbb{R}^\times \). Thus

\[
\partial_{\partial x_1} E^{(p,q)}, \ldots, \partial_{\partial x_\ell} E^{(p,q)}
\]

form an \( S \)-basis for \( D(A, (2p - 1, 2q - 1)) \). Since

\[
\left[\partial_{\partial x_1} E^{(p,q)}, \ldots, \partial_{\partial x_\ell} E^{(p,q)}\right] J(P)^T = \left[\partial_{\partial x_1} E^{(p,q)}, \ldots, \partial_{\partial x_\ell} E^{(p,q)}\right],
\]

we may apply the multi-arrangement version of Saito’s criterion [Sa1980, Z1989, A2008] to prove that \( \partial_{\partial x_1} E^{(p,q)}, \ldots, \partial_{\partial x_\ell} E^{(p,q)} \) form an \( S \)-basis for \( D(A, (2p, 2q)) \) for any \( p, q \in \mathbb{Z} \). This shows that \( E^{(p,q)} \) is \((p, q)\)-universal for any \( p, q \in \mathbb{Z} \). □

**Proof of Theorem 1.3** \( (q \in \mathbb{Z}) \). Theorem 2.3 and Proposition 3.4 complete the proof by the same argument as that in Section 2 for \( q \geq 0 \). □
4 The cases of $B_\ell$, $F_4$, $G_2$ and $I_2(2n)$

• The case of $B_\ell$

The roots of the type $B_\ell$ are:

$$\pm x_i, \pm x_i \pm x_j \quad (1 \leq i < j \leq \ell)$$

in terms of an orthonormal basis $x_1, \ldots, x_\ell$ for $V^*$. Altogether there are $2\ell^2$ of them. Define

$$Q_1 := \prod_{i=1}^{\ell} x_i, \quad Q_2 := \prod_{1 \leq i < j \leq \ell} (x_i \pm x_j), \quad Q = Q_1 Q_2.$$

Then the arrangement $A_1$ defined by $Q_1$ is of the type $A_1 \times \cdots \times A_1 = A_1^{\ell}$. The arrangement $A_2$ defined by $Q_2$ is of the type $D_\ell$. The arrangement $A$ defined by $Q$ is of the type $B_\ell$ and $A = A_1 \cup A_2$ is the orbit decomposition. Note that $A_1^{\ell}$ is not irreducible. Define

$$D_1 := \sum_{i=1}^{\ell} \frac{1}{x_i} \partial x_i$$

which is a primitive derivation in the sense of [AT2009]. Obviously $D_1$ is $W$-invariant. Let $P_j = \sum_{i=1}^{\ell} x_i^{2j}$ ($j \geq 1$). Then $P_1, \ldots, P_\ell$ form a set of basic invariants under $W$ while $Q_1, P_1, \ldots, P_{\ell-1}$ form a set of basic invariants under $W_2$. Define a primitive derivation $D_2$ with respect to $A_2$ so that

$$D_2(Q_1) = D_2(P_j) = 0 \quad (j = 1, \ldots, \ell - 2), \quad D_2(P_{\ell-1}) = 1.$$

Thus

$$(wD_2)(P_{\ell-1}) = D_2(w^{-1} P_{\ell-1}) = D_2(P_{\ell-1}) = 1 \quad (w \in W).$$

This implies that $D_2$ is $W$-invariant.

• The case of $F_4$

The roots of the type $F_4$ are:

$$\pm x_i, \ (\pm x_1 \pm x_2 \pm x_3 \pm x_4)/2, \ \pm x_i \pm x_j \quad (1 \leq i < j \leq 4)$$

in terms of an orthonormal basis $x_1, x_2, x_3, x_4$ for $V^*$. Altogether there are 48 of them. Define

$$Q_1 := \prod_{1 \leq i < j \leq 4} (x_i \pm x_j), \quad Q_2 := \prod_{i=1}^{4} x_i \prod_{1 \leq i < j \leq 4} (x_1 \pm x_2 \pm x_3 \pm x_4), \quad Q = Q_1 Q_2.$$
The arrangement $\mathcal{A}_i$ defined by $Q_i$ is of the type $D_4$ ($i = 1, 2$). Then the arrangement $\mathcal{A}$ defined by $Q$ is of the type $F_4$ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition. Define

$$P_1^{(1)} = \sum_{i=1}^{4} x_i^2, \quad P_2^{(1)} = \sum_{i=1}^{4} x_i^4, \quad P_3^{(1)} = x_1 x_2 x_3 x_4, \quad P_4^{(1)} = \sum_{i=1}^{4} x_i^6 + 5 \sum_{i \neq j} x_i x_j^4.$$

Compute

$$P_4^{(1)} = -4 \sum_{i=1}^{4} x_i^6 + 5 P_1^{(1)} P_2^{(1)}.$$

Thus $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, P_4^{(1)}$ are a set of basic invariants under $W_1$. The reflection $\tau$ with respect to $x_1 + x_2 + x_3 + x_4 = 0$ is given by

$$\tau(x_i) = \frac{2x_i - \sum_{j=1}^{4} x_j}{2} \quad (i = 1, 2, 3, 4).$$

A calculation shows that $P_4^{(1)}$ is $\tau$-invariant. Let $s_i$ denote the reflection with respect to $x_i = 0$ ($1 \leq i \leq 4$). Since the Coxeter group $W_2$ is generated by $\tau$ and $s_i$ ($1 \leq i \leq 4$), we know that $P_4^{(1)}$ is $W_2$-invariant thus $W$-invariant. Define a primitive derivation $D_1$ with respect to $\mathcal{A}_1$ so that

$$D_1(P_j^{(1)}) = 0 \quad (j = 1, 2, 3), \quad D_1(P_4^{(1)}) = 1.$$ 

Thus

$$(wD_1)(P_4^{(1)}) = D_1(w^{-1}P_4^{(1)}) = D_1(P_4^{(1)}) = 1 \quad (w \in W).$$

This implies that $D_1$ is $W$-invariant. We conclude that $D_2$ is also $W$-invariant because an orthonormal coordinate change

$$x_1 = \frac{y_1 - y_2}{\sqrt{2}}, \quad x_2 = \frac{y_1 + y_2}{\sqrt{2}}, \quad x_3 = \frac{y_3 - y_4}{\sqrt{2}}, \quad x_4 = \frac{y_3 + y_4}{\sqrt{2}}$$

switches $\mathcal{A}_1$ and $\mathcal{A}_2$.

- **The cases of $G_2$ and $I_2(2n)$ ($n \geq 4$)**

The arrangement $\mathcal{A}$ of the type $G_2$ has exactly two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is of the type $A_2$. Let $n \geq 4$. Then the arrangement $\mathcal{A}$ of the type $I_2(2n)$ has exactly two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is of the type $I_2(n)$. In both cases, by [W2010], one may choose

$$D_1 = Q_2 D, \quad D_2 = Q_1 D.$$ 

Since $Q_2$ is $W_2$-antiinvariant and $D$ is $W$-invariant, $D_1$ is $W_2$-antiinvariant. Similarly $D_2$ is $W_1$-antiinvariant.
5 Proofs of Theorems 1.1 and 1.5

Assume that $A$ is an irreducible Coxeter arrangement in the rest of the article.

Proof of Theorem 1.1 If $A$ has the single orbit, then the result in [T2002, AY2007, AT2010Z] completes the proof. If not, then $A$ has exactly two orbits. If $A$ is of the type either $G_2$ or $I_2(2n)$ with $n \geq 4$, then $D(A, m)$ is a free $S$-module because $A$ lies in a two-dimensional vector space. For the remaining cases of the type $B_l$ and $F_4$, Section 4 allows us to apply Theorem 1.3 to complete the proof.

A multiplicity $m : A \to \mathbb{Z}$ is said to be odd if its image lies in $1 + 2\mathbb{Z}$.

Proposition 5.1 If $m$ is equivariant and odd, then $D(A, m)$ has a $W$-invariant basis over $S$.

Proof. When $A$ has the single orbit, $m$ is constant. In this case Proposition was proved in [T2002, AY2007, AT2010Z]. If $A$ is of the type either $G_2$ or $I_2(2n)$ ($n \geq 4$), then Proposition was verified in [W2010]. For the remaining cases of $B_l$ and $F_4$, Proposition 3.2 completes the proof.

Recall the $W$-action on $A$:

$$W \times A \to A$$

by sending $(w, H)$ to $wH$ ($w \in W$, $H \in A$). For any multiplicity $m : A \to \mathbb{Z}$, define a new multiplicity $m^*$ by

$$m^*(H) := \max_{w \in W} \left(2 \cdot \left\lfloor m(wH)/2 \right\rfloor + 1\right),$$

where $\lfloor a \rfloor$ stands for the greatest integer not exceeding $a$. Then $m^*$ is obviously equivariant and odd.

Proposition 5.2 For any irreducible Coxeter arrangement $A$ and any multiplicity $m$,

$$D(A, m)^W = D(A, m^*)^W.$$

Proof. Since $m(H) \leq m^*(H)$ for any $H \in A$, we have

$$D(A, m)^W \supseteq D(A, m^*)^W.$$

We will show the other inclusion. Let $H \in A$ and $\theta \in D(A, m)^W$. It suffices to verify the following two statements:

(A) $\theta(\alpha_H) \in \alpha_H^{m(wH)} S_{\alpha_H}$ for any $w \in W$,
(B) \( \theta(\alpha_H) \in \alpha_{2m}^H S(\alpha_H) \) implies \( \theta(\alpha_H) \in \alpha_{2m+1}^H S(\alpha_H) \) for any \( m \in \mathbb{Z} \).

For \( w \in W \) let \( w^{-1} \) act on the both sides of

\[ \theta(\alpha_{wH}) \in \alpha_{m(wH)}^H S(\alpha_{wH}) \]

to get

\[ \theta(\alpha_H) \in \alpha_{m(wH)}^H S(\alpha_H). \]

This verifies (A).

Fix \( H \in A \). Let \( s \) be the orthogonal reflection through \( H \). Then \( s(\alpha_H) = -\alpha_H \). Suppose that \( \theta(\alpha_H) = \alpha_H^{2m} p \) with \( p \in S(\alpha_H) \). Let \( s \) act on the both handsides and we have \( \theta(-\alpha_H) = (-\alpha_H)^{2m} s(p) \). This implies \( -p = s(p) \). Since \( s(p) = p \) on \( H \), one has \( p = 0 \) on \( H \), which implies \( p \in \alpha_H S(\alpha_H) \). This verifies (B). \( \square \)

**Proof of Theorem 1.5.** Thanks to Proposition 5.2 we may assume that \( m \) is equivariant and odd. Apply Proposition 5.1 and Lemma 3.3. \( \square \)

**Corollary 5.3**

\[ D(A, m)^W \otimes_R S \simeq D(A, m^*). \]

**Proof.** Apply Proposition 5.1 and Lemma 3.3 to get

\[ D(A, m^*)^W \otimes_R S \simeq D(A, m^*). \]

Then Proposition 5.2 completes the proof. \( \square \)

The following corollary shows that the converse of Proposition 5.1 is true.

**Corollary 5.4** The \( S \)-module \( D(A, m) \) has a \( W \)-invariant basis if and only if \( m \) is odd and equivariant.

**Proof.** Assume that \( D(A, m) \) has a \( W \)-invariant basis over \( S \). Then, by Lemma 3.3 we get

\[ D(A, m)^W \otimes_R S \simeq D(A, m). \]

Compare this with Corollary 5.3 and we know that there exists a common \( S \)-basis for both \( D(A, m) \) and \( D(A, m^*) \). By the multi-arrangement version of Saito’s criterion \( [Sa1980, Z1989, A2008] \), we have \( m = m^* \). \( \square \)
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