Abstract. A $Q$-manifold $M$ is a supermanifold endowed with an odd vector field $Q$ squaring to zero. The Lie derivative $L_Q$ along $Q$ makes the algebra of smooth tensor fields on $M$ into a differential algebra. In this paper, we define and study the invariants of $Q$-manifolds called characteristic classes. These take values in the cohomology of the operator $L_Q$ and, given an affine symmetric connection with curvature $R$, can be represented by universal tensor polynomials in the repeated covariant derivatives of $Q$ and $R$ up to some finite order. As usual, the characteristic classes are proved to be independent of the choice of the affine connection used to define them. The main result of the paper is a complete classification of the intrinsic characteristic classes, which, by definition, do not vanish identically on flat $Q$-manifolds. As an illustration of the general theory we interpret some of the intrinsic characteristic classes as anomalies in the BV and BFV-BRST quantization methods of gauge theories. An application to the theory of (singular) foliations is also discussed.

1. Introduction

By definition, a $Q$-manifold is a pair $(M, Q)$, where $M$ is a smooth supermanifold equipped with an odd vector field $Q$ satisfying the integrability condition $[Q, Q] = 0$. Every such $Q$ is called a homological vector field. Equivalently, one can think of a $Q$-manifold as a smooth supermanifold whose structure sheaf of supercommutative algebras of functions is endowed with the differential $Q$. The action of $Q$ is naturally extended from $C^\infty(M)$ to the whole tensor algebra of $M$ endowed with the usual tensor operations: the tensor product, contraction and permutation of indices. The Lie derivative $L_Q$ along $Q$ respects the tensor operations and makes the tensor algebra into a differential tensor algebra. We define the group of $Q$-cohomologies (with tensor coefficients) as the quotient $H_Q(M) = \ker(L_Q)/\text{Im}(L_Q)$. The tensor operations in $\mathcal{T}(M)$ induce those in $H_Q(M)$; hence, we can speak of the tensor algebra of $Q$-cohomology.

Let us mention the examples of $Q$-manifolds, which are important in mathematics and physics.

Example 1.1. An odd tangent bundle $\Pi TN$ of an ordinary manifold $N$. This is obtained by applying the parity reversing functor $\Pi$ to the tangent space of $N$. The algebra of smooth functions $C^\infty(\Pi TN)$ is naturally isomorphic to the exterior algebra of differential forms on $N$ and the role of $Q$ is played by the exterior differential.

Example 1.2. Replacing the tangent bundle of $N$ by a general Lie algebroid $E \to N$, we come to the $Q$-manifold $\Pi E$. The differential algebra of smooth functions on $\Pi E$ is modeled on $\Gamma(\bigwedge^\bullet E^*)$, the exterior algebra of $E$-forms, with the differential $Q$ being the Lie algebroid...
differential \( d_E : \Gamma(\wedge^n E^*) \to \Gamma(\wedge^{n+1} E^*) \). For a more detailed discussion of the relationship between Lie algebroids and homological vector fields see [1].

**Example 1.3.** Any \( L_\infty \)-algebra can be thought of as a formal \( Q \)-manifold with the homological vector field \( Q \) vanishing at the origin [2]. In a particular case of Lie algebras we have a \( Q \)-manifold \( \Pi G \), where \( G \) is a Lie the algebra and \( Q \) is given by the Chevalley-Eilenberg differential on \( \wedge^* G^* \simeq C^\infty(\Pi G) \). One can also view \( \Pi G \) as a linear \( Q \)-manifold coming from the Lie algebroid \( G \to \{\ast\} \) over a single point set.

**Example 1.4.** In the theory of gauge systems, the homological vector field generates the BRST symmetry [4], [5]. To any variational gauge system one associates either a symplectic or an anti-symplectic manifold (according to which formalism, Lagrangian BV or Hamiltonian BFV, is used) together with an odd, selfcommuting, (anti-)Hamiltonian vector field \( Q \). The corresponding (anti-)symplectic two-form, being \( \omega \) is obviously invariant under the action of \( L_\varphi \) and thus it is a differential subalgebra of \( A \). The algebra \( A \) is generated by the repeated covariant derivatives of \( Q \) and \( R \) is the curvature of \( \nabla \). The tensor algebra \( A \subset T(M) \) of all local covariants associated to \( Q \) and \( \nabla \) is generated by the repeated covariant derivatives of \( Q \) and \( R \). The algebra \( A \) is obviously invariant under the action of \( L_Q \) and thus it is a differential subalgebra of \( T(M) \). We say that a \( Q \)-closed local covariant \( C \in A \) is a **universal cocycle** if the closedness equation \( L_Q C = 0 \) follows from the integrability condition \([Q, Q] = 0\) regardless of any specificity of \( Q \), \( \nabla \) and \( M \). In other words, the universal cocycles are \( Q \)-invariant tensor polynomials in \( \nabla^n Q \) and \( \nabla^m R \) that can be attributed to any \( Q \)-manifold with connection.

The characteristic classes of \( Q \)-manifolds can now be defined as the elements of \( H_Q(M) \) represented by universal cocycles. It can be shown (Theorem 2.1) that the \( Q \)-cohomology classes of universal cocycles do not depend on the choice of symmetric connection and hence they are invariants of the \( Q \)-manifold as such.

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1 In the context of classical differential geometry, this statement is known as the second reduction theorem [12] p.165, and the elements of \( A \) are called **differential concomitants** of \( \nabla \) and \( Q \), see also [13].
The simplest examples of the universal cocycles are the tensor powers of the homological vector field \( Q^{\otimes n} \). Obviously, these cocycles exhaust all the universal cocycles that do not involve the connection. A less trivial example of a universal cocycle is obtained by taking the complete contraction of the 2n-form representing the Pontryagin characters of the tangent bundle \( TM \) with the contravariant tensor \( Q^{\otimes 2n} \). The result is the sequence of \( Q \)-invariant functions \( P_n = \text{Str}(R_n^{QQ}) \), where \( R_{QQ} = [\nabla_q, \nabla_q] \) is a (1,1)-tensor defining an endomorphism of \( TM \) and the exponent \( n \) means the \( n \)th power of the endomorphism.

The cocycles \( Q^{\otimes n} \) and \( P_n \) are members of two complementary sets of universal cocycles: intrinsic and vanishing. A universal cocycle is called vanishing if it vanishes identically upon setting the curvature \( R \) to zero. From the viewpoint of the \( Q \)-structure the most interesting are intrinsic cocycles. The intrinsic cocycles survive on flat \( Q \)-manifolds, that is why their cohomology classes are closely related to the structure of the homological vector field rather than the topology of \( M \). This motivates us to introduce the notion of intrinsic characteristic classes. To give their formal definition we recall that the tensor algebra \( A \) of local covariants associated to \( Q \) and \( \nabla \) is constructed from two infinite sequences of tensors \( \{ \nabla^n Q \} \) and \( \{ \nabla^m R \} \).

Let \( \mathcal{R} \) denote the ideal of \( A \) generated by \( \{ \nabla^m R \} \). Since \( L_Q \mathcal{R} \subset \mathcal{R} \), we have the short exact sequence of complexes

\[
0 \rightarrow \mathcal{R} \xrightarrow{i} A \xrightarrow{p} A/\mathcal{R} \rightarrow 0
\]

giving rise to the exact triangle in cohomology

\[
\begin{array}{cccc}
H(\mathcal{R}) & \xrightarrow{i_*} & H(A) & \xrightarrow{p_*} H(A/\mathcal{R}) \\
\downarrow & & \downarrow \partial & \\
H(A/\mathcal{R})
\end{array}
\]

Here \( H(A) \) is the tensor algebra of characteristic classes and \( \partial \) is the connecting homomorphism. By definition, the space of intrinsic characteristic classes is identified with the subspace \( \text{Im} \ p_* = \text{Ker} \ \partial \subset H(A/\mathcal{R}) \). Geometrically, one can view the group \( H(A/\mathcal{R}) \) as the space of characteristic classes of a flat \( Q \)-manifold. The universal cocycles of a flat \( Q \)-manifold are built out of the \( (n,1) \)-tensors \( \nabla^n Q \), symmetric in lower indices. If \( \partial \) is nonzero, not any characteristic class can be extended from flat to arbitrary \( Q \)-manifolds and the obstruction to extendability is controlled by the elements of \( \text{Im} \ \partial = \text{Ker} \ i_* \subset H(\mathcal{R}) \). In Sec. 6, we show that the space \( \text{Im} \ \partial \) is nonempty and spanned by the functions \( P_n \).

The main result of this paper is a complete classification of the intrinsic characteristic classes. In the case of flat \( Q \)-manifolds this is done in Sec. 5, where we construct a multiplicative (w.r.t. the tensor product) basis in \( H(A) \). The universal cocycles representing the basis elements are assembled into three infinite series \( A, B \), and \( C \). The elements of \( A \)-series are represented by odd functions on \( M \), while the elements of \( B \)- and \( C \)-series are represented by tensor fields of types \( (n,1) \) and \( (n,0) \), respectively, one for each \( n \in \mathbb{N} \). The surprising thing is that one can always choose the basis cocycle to be tensor polynomials in \( \nabla Q \) and \( \nabla^2 Q \) alone, i.e., no derivatives higher than two of the homological vector field are needed to define all the characteristic classes. This resembles the situation with the characteristic classes of framed foliations or, more generally, with the Gelfand-Fuks cohomology \([14]\), and this is more than just an analogy. In fact, with the concept of classifying space, we show that the enumeration problem for all independent characteristic classes of flat \( Q \)-manifolds amounts to computation of the stable cohomology of the Lie algebra of formal vector fields with tensor coefficients, the problem that was actually solved by D.B. Fuks \([15]\). The stable cohomology groups under
consideration are also known as graph cohomology groups, because they can be calculated via certain complex of finite graphs. The graph complexes were introduced by M. Kontsevich in the beginning of the nineties [16], [17] and since then they have appeared in various contexts of differential geometry and topology as well as in the topological quantum field theory. It was stressed in [18] that various invariants (characteristic classes) of differential geometric structures can be defined as the image of the graph cohomology in the $Q$-cohomology of an appropriate $Q$-manifold (perhaps, with an additional structure). So the $Q$-manifolds provide a general framework for most of the known constructions of characteristic classes (Lie algebroids, vector bundles, foliations, complex structures, knots, strongly homotopical algebras, rational homotopy types, etc.). The further development of this framework was one of our motives for writing this paper.

In Sec. 6 we extend the $A$-, $B$-, and $C$-series of characteristic classes from flat to arbitrary $Q$-manifolds. The extension is quite straightforward for the characteristic classes of series $B$ and $C$, but it is up against some topological obstructions for $A$-series. Roughly, only half of $A$-series’ characteristic classes can be defined for general $Q$-manifolds, and the definition involves a special choice of the symmetric affine connection. The obstructions for extendability of the other half are explicitly identified with the aforementioned set of $Q$-invariant functions $\{P_n\}$.

In Sec. 7 we focus on the exterior algebra of local covariants with values in forms. The algebra has the structure of bicomplex with respect to the exterior differential $d$ and the Lie derivative $L_Q$. Under assumption that the Pontryagin classes of $TM$ vanish, we prove that for any characteristic class represented by a differential form there exists a $d$-exact representative.

In Sec. 8 the general construction is illustrated with some examples from quantum field theory and theory of foliations. Namely, we interpret the first term of $A$-series (the modular class) and the second term of $C$-series as lower-order anomalies of gauge symmetries in the BRST quantization approach. We briefly discuss a relationship between characteristic classes of $Q$-manifolds, Lie algebroids, and (singular) foliations and give an example of a regular foliation with nontrivial modular class.

Some auxiliary results are proved in Appendices A, B, and C.

Terminology and notation. We use the standard language and notation of supermanifold theory [19], [20]. Our tendency, however, is to omit the prefix “super” whenever possible. So the terms like manifolds, vector bundles, smooth functions etc., will usually mean the corresponding notions of supergeometry. Given a smooth supermanifold $M$ with $\dim M = p|q$, we set $\dim M = p + q$. We let $\mathcal{T}(M)$ denote the algebra of tensor fields, $\mathfrak{X}(M)$ the Lie algebra of vector fields, $\Omega(M)$ the exterior algebra of differential forms, and $\mathfrak{A}(M)$ the associative algebra of endomorphisms of $TM$. The algebra $\mathfrak{A}(M)$, being a $\mathbb{Z}_2$-graded algebra over $C^\infty(M)$, possesses a natural trace $\text{Str} : \mathfrak{A}(M) \to C^\infty(M)$. Sometimes it will be convenient to treat the space $\mathfrak{X}(M)$ as a right $\mathfrak{A}(M)$-module and the space $C^\infty(M)$ as a left $\mathfrak{X}(M)$-module. All the partial or covariant derivatives are assumed to act from the left.

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2. Q-VECTOR BUNDLES

We begin with a collection of definitions and simple facts concerning the concept of equivariant vector bundles [21]. Let $G$ be a Lie group (possibly zero or infinite dimensional). A $G$-manifold is a smooth manifold $M$ together with a group homomorphism $G \to \text{Diff}(M)$. In other words, the Lie group $G$ acts smoothly on $M$. A $G$-equivariant map between $G$-manifold $M'$ and $M''$ is a smooth map $f : M' \to M''$ such that $f(gx) = gf(x)$ for all $g \in G$ and $x \in M'$. In what follows we will refer to the $G$-equivariant maps as $G$-maps.

Analogously, consider a vector bundle $E$ over a smooth manifold $M$ and let $\text{Aut}(E)$ denote the group of bundle automorphisms, i.e., the group of fiberwise linear diffeomorphisms $f : E \to E$ mapping fibers to fibers. A $G$-structure on $E$ is determined by a homomorphism $G \to \text{Aut}(E)$. Given a $G$-structure, we refer to $E$ as a $G$-equivariant vector bundle or just a $G$-bundle for short. Note that both the total space $E$ and the base $M$ of a $G$-bundle are $G$-manifolds and the canonical projection $p : E \to M$ is a $G$-map.

The $G$-bundles form a category (for a given $G$) whose morphisms are $G$-equivariant bundle homomorphisms. Upon restricting to the $G$-bundles over a fixed base manifold, we get a subcategory for which all the usual operations on vector bundles are naturally defined: the direct sum, tensor product, and dualization. The parity reversion functor $E \mapsto \Pi E$, being compatible with bundle automorphisms, is one more natural operation in the category of the $G$-vector bundles.

An even (odd) section $s : M \to (\Pi)E$ of a $G$-vector bundle is called equivariant if it is a $G$-map. The equivariant sections form a vector subspace $\Gamma^G(E) \subset \Gamma(E)$ in the space of all smooth sections. They can also be viewed as fixed points of the natural action of $G$ on $\Gamma(E)$.

$G$-vector bundles are of frequent occurrence. Here are some examples.

Example 2.1. Any tangent bundle $TM$ can be regarded as a $\text{Diff}(M)$-equivariant vector bundle and so are all the associated tensor bundles. If $M$ is a $G$-manifold, then $TM$ carries the canonical $G$-structure induced by the homomorphisms $G \to \text{Diff}(M) \to \text{Aut}(TM)$.

Example 2.2. Let $M$ be a $G$-manifold and $\sigma : G \to GL(V)$ be a representation of $G$ in a vector space $V$. Then $E = M \times V$ is a trivial $G$-bundle; here $G$ acts on $E$ by $g(x, v) = (gx, \sigma(g)v)$.

Example 2.3. If $E \to M$ is a vector bundle, then the $n$-fold tensor product $E^\otimes n$ is an $S_n$-bundle, where $S_n$ is the symmetric group permuting the factors of the product and $M$ is regarded as a trivial $S_n$-manifold.

In this paper, we mostly consider equivariant vector bundles associated to the Lie group $\mathbb{R}^{0,1}$. If $\pi : E \to M$ is such a vector bundle, then the infinitesimal action of $\mathbb{R}^{0,1}$ on $E$ and $M$ is generated by some homological vector fields $Q_E$ and $Q_M$, respectively; in so doing, $Q_M$ appears to be a unique vector field $\pi$-related to $Q_E$. For this reason, we will refer in sequel to $\mathbb{R}^{0,1}$-equivariant vector bundles as $Q$-bundles or vector bundles endowed with a $Q$-structure. Let $T(E) = \oplus_{n,m} T^{n,m}(E)$ denote the algebra of the $E$-tensor fields on $M$; by definition, $T^{n,m}(E)$ is the $C^\infty(M)$-module of sections of $(E^*)^\otimes n \otimes E^\otimes m$. Then the action of $Q_E$ on $E$, being fiberwise linear, endows $T(E)$ with the structure of a differential tensor algebra. The differential $\delta : T(E) \to T(E)$ is completely determined by its action on local coordinate

\footnote{It is usually assumed that $f|_M = \text{id}_M$. In our definition of the group $\text{Aut}(E)$ we admit a nontrivial action of $f$ on the base $M$.}
functions \( \{x^i\} \) on \( U \subset M \) and a frame of sections \( \{s_a\} \) in \( E|_U \):

\[
\delta x^i = Q^i_M(x), \quad \delta s_a = -(-1)^a \Lambda^b_a(x) s_b.
\]

Here \( Q_M = \pi_*(Q_E) \) is the aforementioned homological vector field on the base \( M \), and the odd matrix \( \Lambda \) is called the twisting element. It follows from the relation \( \delta^2 = 0 \) that in each trivializing chart the twisting element obeys the Maurer-Cartan equation

\[
Q_M \Lambda = \Lambda^2.
\]

If \( \{y^a\} \) are the fiber coordinates on \( E \) dual to the frame sections \( \{s_a\} \), then the homological vector field \( Q_E \) can be locally written as

\[
Q_E = Q^i_M(x) \frac{\partial}{\partial x^i} + y^a \Lambda^b_a(x) \frac{\partial}{\partial y^b}.
\]

Writing \( C^\infty_{lin}(E) \) for the space of smooth functions on \( E \) that are linear in the fiber coordinates, we can invariantly characterize \( Q_E \) as a homological vector field whose action preserves the subspace \( C^\infty_{lin}(E) \subset C^\infty(E) \) (and, as a consequence, the subalgebra \( C^\infty(M) \subset C^\infty(E) \)).

Since \( Q_E \) is odd, its action is always integrable to the action of \( \mathbb{R}^{0|1} \) and so there is a one-to-one correspondence between the \( \mathbb{R}^{0|1} \)-equivariant vector bundles and the vector bundles endowed with the action of the linear homological vector field \( Q \). From this perspective, a morphism of \( Q \)-vector bundles is just a fiberwise linear map \( \phi : E_1 \to E_2 \) such that \( Q_2 = \phi_*(Q_1) \). In what follows, it will be convenient to refer to a \( Q \)-vector bundle as a pair \( (E, \delta) \), where \( E \) is a vector bundle and \( \delta \) is a differential on the algebra of \( E \)-tensor fields. \( Q \)-invariant \( E \)-tensors (i.e., \( \mathbb{R}^{0|1} \)-equivariant sections of the associated tensor bundle) are by definition cocycles of the differential \( \delta \). Clearly, the corresponding group of \( \delta \)-cohomology, defined by \( H(E, \delta) = \ker \delta/\text{im} \delta \), inherits the structure of tensor algebra, \( H(E, \delta) = \bigoplus H^{n,m}(E, \delta) \).

Let us give some examples of \( Q \)-vector bundles.

**Example 2.4.** If \( E \) is a vector bundle, then \( (E, 0) \) is a \( Q \)-vector bundle with trivial differential.

**Example 2.5.** The tangent bundle \( TM \) of a \( Q \)-manifold has a canonical \( Q \)-structure defined by the Lie derivative \( L_Q \) along \( Q \).

**Example 2.6.** Let \( E \) be a vector bundle over a \( Q \)-manifold and suppose \( E \) to admit a flat connection \( \nabla \). Then \( \nabla^2_Q = 0 \), and we have the \( Q \)-vector bundle \( (E, \nabla_Q) \).

Any morphism \( \phi : E_1 \to E_2 \) of \( Q \)-vector bundles induces a homomorphism on sections\footnote{We simply identify \( \Gamma(E^*) \) with \( C^\infty_{lin}(E) \), then \( \phi_* \) is given by the pullback of \( \phi \).}:

\[
\phi_* : \Gamma(E^*_2) \to \Gamma(E^*_1),
\]

where \( E^*_{1,2} \) are the \( Q \)-vector bundles dual to \( E_{1,2} \). The homomorphism \( \phi_* \) in its turn gives rise to a homomorphism of the cohomology groups \( \ker \delta/\text{im} \delta \) of the differential \( \delta \) on the space of sections \( \Gamma(E^*_{1,2}) \). Generally there is no natural way to extend \( \phi_* \) to the full algebras of \( E \)-tensor fields except when \( \phi \) is a fiberwise isomorphism. In this last case, we have a unique homomorphism \( \tilde{\phi}_* : \Gamma(E_2) \to \Gamma(E_1) \) such that

\[
\langle \phi_*(u), \tilde{\phi}_*(v) \rangle_1 = \langle u, v \rangle_2 \circ \phi|_{M_1}, \quad \forall u \in \Gamma(E^*_2), \quad \forall v \in \Gamma(E_2).
\]

Here the triangle brackets \( \langle \cdot, \cdot \rangle \) stand for pairing between the spaces \( \Gamma(E) \) and \( \Gamma(E^*) \). The pair \( \varphi = (\phi_*, \tilde{\phi}_*) \) defines a homomorphism \( \varphi : \mathcal{T}(E_2) \to \mathcal{T}(E_1) \) of differential tensor algebras. Thus,
we get a homomorphism of $\delta$-cohomology groups
\[ \varphi_* : H(E_2, \delta_2) \to H(E_1, \delta_1). \]

We emphasize that the last homomorphism can be defined only when $\text{rank}E_1 = \text{rank}E_2$ and $\phi : E_1 \to E_2$ is a fiberwise isomorphism.

Let us now specify the constructions above to the tangent bundle of a $Q$-manifold. In this case, we have a canonical $Q$-structure on $TM$ identified with the operator of Lie derivative $\delta = L_Q$. The differential tensor algebra of $M$ was denoted by $\mathcal{T}(M)$ and the $Q$-cohomology group was denoted by $H_Q(M)$ (see the previous section). Upon choosing a symmetric affine connection $\nabla$ on $M$, we can define the algebra of local covariants $\mathcal{A} \subset \mathcal{T}(M)$. As a tensor algebra $\mathcal{A}$ is generated by two sequences of tensor fields $\{\nabla^n Q\}$ and $\{\nabla^n R\}$, $R$ being the curvature of $\nabla$. Since $\delta \mathcal{A} \subset \mathcal{A}$, $\mathcal{A}$ is a differential subalgebra of $\mathcal{T}(M)$ with the cohomology group $H(\mathcal{A})$. The natural inclusion
\[ i : \mathcal{A} \to \mathcal{T}(M) \]
induces the homomorphism in cohomology
\[ i_* : H(\mathcal{A}) \to H_Q(M). \]

In the previous section, we gave a preliminary definition of the characteristic classes of $Q$-manifolds as elements of $\text{Im} i_*$ that are represented by the so-called universal cocycles. The adjective “universal” implies that the $\delta$-closedness condition is satisfied by virtue of the integrability condition $[Q, Q] = 0$ alone. With this interim definition, we can readily show the independence of the characteristic classes of the choice of $\nabla$.

**Theorem 2.1.** The characteristic classes of a $Q$-manifold do not depend on the choice of symmetric connection and hence they are invariants of the $Q$-manifold itself.

**Proof.** Let $\mathcal{C}_0[Q]$ and $\mathcal{C}_1[Q]$ be two universal cocycles that differ only by the choice of the connection. Consider the direct product of $M$ and the linear superspace $\mathbb{R}^{1|1}$ with one even coordinate $t$ and one odd coordinate $\theta$. The product structure of $\widetilde{M} = M \times \mathbb{R}^{1|1}$ induces the decomposition of the linear space of tensor fields:
\[ \mathcal{T}(\widetilde{M}) = \mathcal{T}'(\widetilde{M}) \oplus \mathcal{T}''(\widetilde{M}), \]
where $\mathcal{T}'(\widetilde{M})$ is the space of sections of the vector bundle
\[ T^{**}M \times \mathbb{R}^{1|1} \to \widetilde{M}. \]

Simply stated, the elements of $\mathcal{T}'(\widetilde{M})$ are the smooth families of tensor fields on $M$ parameterized by “points” of $\mathbb{R}^{1|1}$. So, we have two natural projections
\[ \pi' : \mathcal{T}(\widetilde{M}) \to \mathcal{T}'(\widetilde{M}), \quad \pi'' : \mathcal{T}(\widetilde{M}) \to \mathcal{T}''(\widetilde{M}). \] (5)

Equip the supermanifold $\widetilde{M} = M \times \mathbb{R}^{1|1}$ with the homological vector field $\widetilde{Q} = Q + \theta \partial_t$ and an adapted connection $\widetilde{\nabla}$. The latter is completely specified by the covariant derivatives:
\[ \widetilde{\nabla}_\mu V = \partial_\mu V, \quad \widetilde{\nabla}_a V = \partial_a V, \quad \widetilde{\nabla}_X V = \nabla_X (\pi' V) + X(\pi'' V), \]
where $V \in \mathfrak{X}(\widetilde{M})$, $X \in \mathfrak{X}(M)$ and $\nabla^t = t\nabla_1 + (1 - t)\nabla_0$ is the one-parameter family of connections on $M$. Clearly, the operator $\delta = L_{\widetilde{Q}}$ commutes with the projectors (5):
\[ \tilde{\delta} \pi' = \pi' \delta, \quad \tilde{\delta} \pi'' = \pi'' \delta. \] (6)
Consider now the universal cocycle $\mathcal{C}_V[\tilde{Q}]$. Due to the specific structure of $\tilde{Q}$ and $\tilde{V}$ we have

$$\pi'(\mathcal{C}_V[\tilde{Q}]) = \mathcal{C}_V[Q] + \theta \Psi$$

for some $\Psi \in \mathcal{T}'(\tilde{M})$ obeying $\partial_0 \Psi = 0$. Since $\tilde{\delta} \mathcal{C}_V[\tilde{Q}] = 0$, it follows from (6) and (7) that

$$\tilde{\delta} \pi' \mathcal{C}_V[\tilde{Q}] = \tilde{\delta}(\mathcal{C}_V[Q] + \theta \Psi) = 0.$$ 

The last equation is equivalent to the following ones:

$$\delta(\mathcal{C}_V[Q]) = 0, \quad \partial_i \mathcal{C}_V[Q] = \delta \Psi.$$ 

Integrating the second equation by $t$ from 0 to 1, we get

$$\mathcal{C}_{V_1}[Q] - \mathcal{C}_{V_0}[Q] = \delta \int_0^1 dt \Psi.$$ 

Thus, the $\delta$-cohomology class of the universal cocycle $\mathcal{C}_V[Q]$ does not depend on the choice of symmetric connection. \hfill \Box

3. The classifying $Q$-space

Let $V$ be a finite-dimensional superspace with coordinates $y^i$. Denote by $L_0(V)$ the Lie algebra of formal vector fields on $V$ vanishing at the origin. The generic element of $L_0(V)$ reads

$$v = \sum_{n=1}^{\infty} y^i y^{1 \ldots i_n} \frac{\partial}{\partial y^i}.$$ 

One can regard the expansion coefficients $v^j_{1 \ldots i_n} \in \mathbb{R}$ as coordinates in the infinite-dimensional superspace $L_0(V) = L^0(V) \oplus L^1(V)$.

As usual, we can associate to $L_0(V)$ the linear manifold $\mathbb{M} = \Pi L_0(V)$ with coordinates $\{c^i_{1 \ldots i_n}\}$. By definition, $\epsilon(c^i_{1 \ldots i_n}) = \epsilon(v^j_{1 \ldots i_n}) + 1$. The Lie algebra structure on $L_0(V)$ is then encoded by the homological vector field

$$\mathbb{Q} = \sum_{n=1}^{\infty} \sum_{i_1=1}^{n} \begin{pmatrix} n \\ l \end{pmatrix} (-1)^{\epsilon_i} c^{i_{1 \ldots i_l}}_{i_{n+1 \ldots i_n}} c^{j}_{m_{i_{n+1 \ldots i_n}}} \frac{\partial}{\partial c^j_{i_{1 \ldots i_n}}}$$

on $\mathbb{M}$. Besides, $\mathbb{M}$ is provided with the natural action of $GL(V)$. Since $\mathbb{Q}$ is obviously invariant under the $GL(V)$-transformations, one can think of it as an equivariant section of $GL(V)$-vector bundle $T\mathbb{M}$.

Taking $V$ as typical fiber, consider the trivial vector bundle $E = \mathbb{M} \times V$ with the diagonal action of $GL(V)$. We can assign $E$ with a $Q$-structure starting with the homological vector field $Q$ on the base. To this end, we need to specify a twisting element $\Lambda \in \mathcal{T}^{1,1}(E)$ satisfying the Maurer-Cartan equation (2). Using the natural frame $v_i = \partial/\partial y^i$ in $V$, we set

$$\delta v_i = -(-1)^{\epsilon_i} c^j_i v_j.$$ 

Since the action of the homological vector field commutes with the general linear transformations, we can regard $E$ as an $\mathbb{R}^{0|1} \times GL(V)$-equivariant vector bundle. For reasons clarified below we refer to $(E, \delta)$ as a classifying $Q$-space.

Associated to $E$ is the differential algebra $\mathcal{T}(E) = \bigoplus \mathcal{T}^{n,m}(E)$ of $E$-tensor fields. Denote by $\mathcal{T}(E)^{\text{inv}} \subset \mathcal{T}(E)$ the tensor subalgebra of $GL(V)$-equivariant sections of $T^{\bullet \bullet} E$ or, what is the same, $GL(V)$-invariant $E$-tensors. The subalgebra $\mathcal{T}(E)^{\text{inv}}$ is also invariant under the action of the differential $\delta$. Therefore we can speak about the differential tensor algebra $(\mathcal{T}(E)^{\text{inv}}, \delta)$.
and the corresponding algebra of $\delta$-cohomology $H(\mathbb{E})^{\text{inv}}$. It should be emphasized that we treat $\mathcal{M}$ as a formal manifold, so the components of $\mathbb{E}$-tensors are given by formal power series in the coordinates $\{c_{i_1\ldots i_n}\}$. As an example, consider the following sequence of $GL(V)$-invariant tensor fields with linear dependence of coordinates:

$$C_n = dy^{i_n} \otimes \ldots \otimes dy^{i_1} \frac{\partial}{\partial y^{j}} \in \mathcal{T}^{n,1}(\mathbb{E})^{\text{inv}}.$$  \hspace{1cm} (10)

It follows from the first main theorem of invariant theory \cite{22, 14} that $\mathbb{E}$-tensors (10) constitute a multiplicative basis of $\mathcal{T}(\mathbb{E})^{\text{inv}}$ so that any $GL(V)$-invariant tensor is made up algebraically of $\{C_n\}$. The space $\mathcal{T}(\mathbb{E})^{\text{inv}}$ is naturally graded by the subspaces $\mathcal{T}_r(\mathbb{E})^{\text{inv}}$ consisting of the homogeneous tensor polynomials of degree $r$ in $\{C_n\}$. This grading makes the space of $GL(V)$-invariant tensor fields into the cochain complex:

$$\delta : \mathcal{T}_r(\mathbb{E})^{\text{inv}} \to \mathcal{T}_{r+1}(\mathbb{E})^{\text{inv}}.$$  

Besides, we have an increasing filtration of $\mathcal{T}(\mathbb{E})^{\text{inv}}$ by the sequence of subcomplexes

$$0 \subset F_1\mathcal{T}(\mathbb{E})^{\text{inv}} \subset F_2\mathcal{T}(\mathbb{E})^{\text{inv}} \subset \ldots \subset F_\infty\mathcal{T}(\mathbb{E})^{\text{inv}} = \mathcal{T}(\mathbb{E})^{\text{inv}},$$ \hspace{1cm} (11)

where the $\mathbb{E}$-tensors from $F_n\mathcal{T}(\mathbb{E})^{\text{inv}}$ are generated by $C_1, \ldots, C_n$.

Observe that the tensor algebra $\mathcal{T}(\mathbb{E})^{\text{inv}}$ is not freely generated by $\{C_n\}$ and it is not hard to write some tensor polynomials in $C$’s that vanish identically for some values of $\dim V$. Denote the space of all such polynomials by $I$. Then, the second main theorem of invariant theory \cite{22, 14} ensures that $I \cap F_k\mathcal{T}(\mathbb{E})^{\text{inv}} = 0$ for $\dim V \gg k, r$. Informally speaking, the algebra $\mathcal{T}(\mathbb{E})^{\text{inv}}$ becomes free as the dimension of $V$ goes to infinity. We will return to this point in Sec. 5.

4. The characteristic map

Let $M$ be a flat $Q$-manifold, i.e., a smooth manifold endowed with a homological vector field $Q$ and a flat symmetric connection $\partial$. We can assume without loss of generality that $M$ is simply connected (otherwise replace $(M,Q)$ by its universal covering $Q$-manifold $(\tilde{M}, \tilde{Q})$, where $\tilde{Q}$ is the lift of $Q$ with respect to the covering map $\tilde{p} : \tilde{M} \to M$). Under these assumptions $M$ is a parallelizable manifold and the Lie derivative $\delta = \partial Q$ makes $TM$ into a trivial $Q$-vector bundle endowed with a flat connection.

In this section, we use the data above to construct a characteristic map of $TM$ to the classifying $Q$-space $\mathbb{E} = \mathbb{M} \times V$, where $V$ is the typical fiber of $TM$. The construction is not completely canonical and depends on the choice of a trivialization $\varphi : TM \to \mathbb{M} \times V$. Nonetheless, we show that the pullback of the characteristic map gives rise to a well-defined homomorphism in the cohomology $H(\mathbb{E})^{\text{inv}} \to H_Q(M)$ that depends neither on the flat connection nor on the choice of trivialization. The construction goes as follows.

Let $\varphi : TM \to \mathbb{M} \times V$ be a trivialization and $\varphi_\ast : \Gamma(M \times V) \to \mathfrak{X}(M)$ is the induced homomorphism on sections. Given a flat symmetric connection $\partial$, we say that the trivialization $\varphi$ is compatible with $\partial$, if the pullback of any constant section $v \in \Gamma(M \times V)$ is a covariantly constant section of $TM$, i.e., $\partial_X \varphi_\ast(v) = 0$ for all $X \in \mathfrak{X}(M)$. Clearly, the pullback of constant sections defines a global frame in $TM$, which is completely determined by its value at any

\footnote{To define the universal covering of a supermanifold $M$, we identify $M$ with the total space of an odd vector bundle $E \to M_B$, where $M_B$ is the body of $M$. If $\tilde{M}_B$ is a universal covering of $M_B$ and $p : \tilde{M}_B \to M_B$ is the corresponding projection, then the universal covering of $M$ is, by definition, the supermanifold $\tilde{M}$ associated to the total space of the pullback bundle $p_\ast(E)$.}
single point $p \in M$. (The frame in $T_q M$ results from the parallel transport of a frame in $T_p M$ along any path joining $p$ to $q$.) Since any two frames in $T_p M$ are related to each other by an invertible linear transformation, we can identify the set of all the compatible trivializations with the group $GL(V)$.

The isomorphism $\varphi$ is naturally prolonged to the isomorphism of associated tensor bundles

$$\varphi : T^{n,m} M \to M \times V^{n,m}.$$ 

Here $V^{n,m} = V^\otimes n \otimes (V^*)^\otimes m$ is the standard $GL(V)$-module. Given an even (odd) section $s : M \to (\Pi) T^{n,m} M$, one can define the so-called Gauss map $G : M \to (\Pi) V^{n,m}$ through the composition of maps

$$M \xrightarrow{s} (\Pi) T^{n,m} M \xrightarrow{\varphi} M \times (\Pi) V^{n,m} \xrightarrow{p} (\Pi) V^{n,m},$$

with $p$ being the projection onto the second factor. Given the homological vector field $Q$ and the flat symmetric connection $\partial$, we can build the sequence of tensors $\partial^n Q \in T_{n,1}^n (M)$. Taken together the tensor fields $\{\partial^n Q\}_{n=1}^\infty$ define the Gauss map $G : M \to (\Pi) E$. Note that the Gauss map $G$ depends on the trivialization $\varphi$, even though we do not indicate this explicitly. Let $\{x^i\}$ be a local coordinate system on $M$ adapted to the flat connection $\partial$ in the sense that the Christoffel symbols of $\partial$ vanish and let $V$ spans the coordinate vector fields $\{\partial/\partial x^i\}$. Then the coordinate expression of the map (12) is

$$c_{i_1 \ldots i_n}^j = \partial_{i_1} \cdots \partial_{i_n} Q^j(x), \quad n = 1, 2, \ldots.$$ (13)

Further, we can trivially extend the Gauss map $G$ to the bundle map

$$\hat{G} : TM \to E$$

by setting

$$\hat{G} : TM \xrightarrow{\varphi} M \times V \xrightarrow{G \times id} M \times V = E.$$ (15)

Again, the map $\hat{G}$ depends on the chosen trivialization $\varphi$. Since $\hat{G}$ is a fiberwise isomorphism, it gives rise to the pullback map on sections, which then extends to the homomorphism of the full tensor algebras

$$\hat{G}_* : \mathcal{T}(E) \to \mathcal{T}(M).$$

**Theorem 4.1.** The map $\hat{G}$ defined by Eq. (15) is a morphism of $Q$-vector bundles.

**Proof.** We only need to show that the homomorphism (15) obeys the chain property

$$\delta \circ \hat{G}_* = \hat{G}_* \circ \delta.$$ (17)

In view of the Leibniz rule, it is enough to check the last operatorial identity on the coordinate functions on $M$ and the frame sections of $E$. This can be done directly by making use of the coordinate description of $\hat{G}$ and $\delta$ given by Eqs. (8), (9), (13). Applying (17) to the coordinate functions $c_{i_1 \ldots i_n}^j \in C^\infty(M)$, we get

$$Q^m \partial_m \partial_{i_1} \cdots \partial_{i_n} Q^j = \sum_{l=1}^n \binom{n}{l} (-1)^{\varepsilon_{i_1} + \cdots + \varepsilon_{i_l} + 1} \partial_{(i_1} \cdots \partial_{i_l} Q^m \partial_{i_{l+1}} \cdots \partial_{i_n)} Q^j,$$ (18)
where the parentheses around indices denote symmetrization in the graded sense. But the last equality is just the differential consequence of the integrability condition for the homological vector field:
\[ \partial_{i_1} \cdots \partial_{i_n} Q^2 = 0. \]

If now \( \{ y^i \} \) are linear coordinates in \( V \) such that \( \hat{G}_*(\partial/\partial y^i) = \partial/\partial x^i \), then
\[
(\delta \circ \hat{G}_*)(\frac{\partial}{\partial y^i}) = -(-1)^{\epsilon_i} \frac{\partial Q^j}{\partial x^i} \frac{\partial}{\partial x^j},
\]
and the same expression results from the right hand side of (17):
\[
(\hat{G}_* \circ \delta)(\frac{\partial}{\partial y^i}) = \hat{G}_*\left( (-1)^{1+\epsilon_i} c^j_i \frac{\partial}{\partial y^j} \right)
\]
\[ = \hat{G}_*((-1)^{1+\epsilon_i} c^j_i) \cdot \hat{G}_*\left( \frac{\partial}{\partial y^j} \right) = (-1)^{1+\epsilon_i} \frac{\partial Q^j}{\partial x^i} \frac{\partial}{\partial x^j}. \]

\[ \square \]

Of course, the proof above is rather technical and involves a local coordinate consideration. A more “conceptual” proof of Theorem 4.1 is given in Appendix A.

Given a flat \( Q \)-manifold \( M \), we call (14) a characteristic map.

To simplify our notation we will denote the restriction of \( \hat{G}_* \) to \( T(E)^{\text{inv}} \subset T(E) \) by
\[
\chi : T(E)^{\text{inv}} \to T(M) .
\]

As a differential tensor algebra, \( T(E)^{\text{inv}} \) is generated by the linear \( GL(V) \)-invariant tensor fields (10). The homomorphism \( \chi \) assigns to each generator \( C_n \subset T^{n,1}(E)^{\text{inv}}, n > 0 \), the tensor field
\[
\chi(C_n) = \partial^n Q \in T^{n,1}(M) \quad (20)
\]
given by the covariant derivatives of the homological vector field. Supplementing the tensors (20) by the homological vector field itself, we get the full set of generators for the differential tensor algebra of local covariants \( A \subset T(M) \) associated to a flat \( Q \)-manifold (see Sec. 1). The homomorphism (19) induces a well-defined homomorphism in cohomology
\[
\chi_* : H(E)^{\text{inv}} \to H_Q(M) . \quad (21)
\]

**Theorem 4.2.** The homomorphism (21) is independent of a flat connection and a compatible trivialization.

**Proof.** The homomorphism \( \chi_* \) takes a \( \delta \)-cocycle \( A \in T(E)^{\text{inv}} \) to the universal \( \delta \)-cocycle \( \chi_*(A) \in A \) and the independence of a flat connection follows from Theorem 2.1.

If now \( M \times V \) and \( M \times V' \) are two compatible trivializations of \( TM \), then \( V' = g(V) \) for some linear isomorphism \( g \). Thus the group \( GL(V) \) acts transitively on the set of all trivializations compatible with a given flat connection. This action then translates to the natural action of \( GL(V) \) on \( E \) and, by definition, leaves invariant the tensor fields of \( T(E)^{\text{inv}} \).

**Remark 4.1.** Actually, the homomorphism (19) makes sense for arbitrary flat \( Q \)-manifolds, not necessarily simply connected: It just defines a way of constructing local covariants from the elementary ones \( \{ \partial^n Q \} \). The chain property \( \chi \circ \delta = \delta \circ \chi \) is apparently a local condition following from the identity (13). However, for a simply connected \( Q \)-manifold \( M \) we have a nice geometric interpretation for (19), (21) as homomorphisms induced by a morphism of \( Q \)-vector bundles.
To give the final definition of the characteristic classes we need some facts concerning the structure of the tensor algebra $A$. Let $A' \subset A$ denote the subalgebra generated by the homological vector field $Q$ (as a linear space, $A'$ is spanned by the tensor powers $\{Q \otimes_n\}$) and let $A'' \subset A$ be the subalgebra generated by the elementary covariants $\partial^n Q$ with $n > 0$.

**Proposition 4.1.** With the definitions above,

(a) $A'' = \text{Im} \chi$;
(b) both $A'$ and $A''$ are differential subalgebras of $A$;
(c) $A = A' \otimes A''$, and hence $H(A) = H(A') \otimes H(A'')$.

**Proof.** The equality $(a)$ is obvious and the subalgebra $A'' \subset A$ is closed under the action of $\delta$ as the homomorphic image of the differential tensor algebra $T(E)^{\text{inv}}$. It is also clear that $\delta A' = 0$ and $(b)$ is proved.

Next we note that the elements of $A$ are obtained by applying the tensor operations to the generators $\{\partial^n Q\}$; in so doing, we can ignore the elements involving contractions of the homological vector field $Q$ with $\partial^n Q$. Indeed, as is seen from Eq. (18), any such contraction is equal to an element of $A''$ and we are led to conclude that each element of $A$ is uniquely represented by a linear combination of tensor products $u \otimes w$, where $u \in A'$ and $w \in A''$. Applying the Künneth formula to the tensor product of complexes $A' \otimes A''$ completes the proof of $(c)$. □

As is seen, the contribution of the homological vector field per se to the algebra of local covariants as well as its cohomology can trivially be factor out, so that the most interesting universal cocycles (i.e. ones involving derivatives of the homological vector field) are centered in $A''$. This motivates us to identify the characteristic classes of flat $Q$-manifolds with the group $H(A'')$ rather than the whole group $H(A)$ as it was done in the Introduction. Then we have the following

**Definition 4.1.** The characteristic classes of a flat $Q$-manifold are the $Q$-cohomology classes belonging to the image of the homomorphism $\chi$.  

5. **Stable characteristic classes and graph complexes**

The characteristic classes will carry a valuable piece of information about the structure of $Q$-manifolds provided that the $\delta$-cohomology groups $H_{r}^{m,n}(E)^{\text{inv}}$ of the classifying space are simultaneously wide enough and effectively computable. The next logical step is thus to find an explicit description for the equivariant cohomology of $E$. Unfortunately, the computation of the groups $H_{r}^{m,n}(E)^{\text{inv}}$ for arbitrary $m,n,r$ and $\dim V$ appears to be a hard problem yet to be solved. The problem, however, becomes much simpler in the so-called *stable range of dimensions*, where by stability we mean $|\dim V| \gg r, n$. For an even vector space $V$ the corresponding stable cohomologies were computed by Fuks [15] (see also [23], for evaluation of the lower bound of stable dimensions). It turns out that the method of [15] applies well to an arbitrary superspace $V$, not necessarily even. For completeness sake, below we re-expose Fuks’ results for general superspaces in a form convenient for our subsequent discussion.

The first step of our computation consists in the reinterpretation of the $GL(V)$-equivariant cohomology of $E$ as the cohomology of a certain graph complex. We consider the graphs satisfying the following special properties:

(a) each edge is equipped with a direction;
(b) each vertex has exactly one outgoing and at least one incoming edge;
(c) we admit outgoing and incoming legs, i.e., edges bounded by a vertex from one side and having a “free end” on the other;
(d) the vertices, the incoming and outgoing legs are numbered (numbering each of these three sets we define a decoration of a graph).

The graphs need not be connected and loops are allowed.

The genus of a graph $\Gamma$ is the first Betty number of its geometric realization as a one-dimensional cell complex (with one extra 0-cell added for each leg). Accordingly, the zero Betty number counts the number of connected components of $\Gamma$. Two graphs are considered to be equivalent if they are isomorphic as cell complexes and the corresponding isomorphism respects both the decoration and orientation of edges.

It is easy to see that the restriction (b) imposed on the vertices of our graphs implies that the genus of each connected graph is either 0 or 1. We refer to the graphs of these two groups as tree and cyclic, respectively, see Fig. 1. Notice that each tree graph has the only outgoing leg, while a cyclic graph has none.

The algebra $\mathcal{T}(\mathbb{E})^{\text{inv}}$ admits a very helpful visualization. Namely, to each generator
\begin{equation}
C_n = dy^{i_n} \otimes dy^{i_{n-1}} \otimes \cdots \otimes dy^{i_1} c_{i_1 \ldots i_n}^{j_n} \frac{\partial}{\partial y^j}
\end{equation}
of $\mathcal{T}(\mathbb{E})^{\text{inv}}$ we associate a one-vertex planar graph
\begin{equation}
\gamma_n =
\end{equation}
called corolla, whose incoming legs $1, 2, \ldots, n$ symbolize the covariant indices $i_1, \ldots, i_n$, while the unique outgoing leg corresponds to the contravariant index $j$. A linear basis in $\mathcal{T}(\mathbb{E})^{\text{inv}}$ is obtained from the generators (22) by means of the tensor operations. These have obvious graphical counterparts. The tensor product of $k$ generators
\begin{equation}
C = C_{n_1} \otimes C_{n_2} \otimes \cdots \otimes C_{n_k}
\end{equation}
is represented by the ordered disjoint union
\begin{equation}
\Gamma = \gamma_{n_1} \sqcup \gamma_{n_2} \sqcup \cdots \sqcup \gamma_{n_k}
\end{equation}
of the corresponding corollas. The lexicographical ordering in writing the disjoint union assigns the unique vertex of $\gamma_{n_i} \subset \Gamma$ with the number $i$, and the numberings of the incoming and outgoing legs of $\gamma_{n_i}$ are shifted by $n_1 + \cdots + n_{i-1}$ and $i - 1$, respectively. In such a way the graph $\Gamma$ gets a natural decoration.
To describe a contraction of the tensor (24), say, one corresponding to a covariant index \( i \) and a contravariant index \( j \), we are just gluing the free end of the outgoing leg \( j \) of \( \Gamma \) with the free end of the incoming leg \( i \) to produce an oriented edge joining a pair of vertices; in so doing, the uncontracted legs are renumbered by consecutive numbers according to their order in \( \Gamma \).

Finally, a permutation of indices in (24) results in exchange of numbers among the corresponding legs.

It is quite clear that any of the graphs can be produced from the corollas (23) with the help of elementary operations above and to any such graph \( \Gamma \) one can assign a unique tensor \( C \in T(E)_{\text{inv}} \). The correspondence \( \Gamma \mapsto C \) defines a map, denoted by \( R \), from the set of all graphs obeying (a)-(d) to the set of \( GL(V) \)-invariant \( E \)-tensors.

**Example 5.1.** By way of illustration, let us apply the correspondence map \( R \) to the graph \( \Gamma \) of the form

\[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
1 & \rightarrow & 2
\end{array}
\end{array}
\]

To write down the tensor \( R(\Gamma) \in T(E)_{\text{inv}} \) we first cut the graph into the ordered disjoint union of two corollas \( \gamma_3 \sqcup \gamma_2 \); the order is defined by the numeration of the vertices. To each corolla \( \gamma_n, \ n = 2, 3, \) we associate a generator \( C_n \in T_n(E)_{\text{inv}} \) such that the map \( R \) takes \( \gamma_3 \sqcup \gamma_2 \) to the tensor product \( C_3 \otimes C_2 \). Then we glue the cut edge down to produce the original graph \( \Gamma \), graphically

\[
\begin{array}{ccc}
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
1 & \rightarrow & 2
\end{array}
\end{array}
\]

contracting simultaneously the corresponding indices of \( C_3 \otimes C_2 \). The tensors \( \{C_n\} \) being symmetric in lower indices, it does not matter which incoming leg of \( \gamma_2 \) is glued to the outgoing leg of \( \gamma_3 \). Finally, we rearrange the indices according to the order of legs. The result is given by

\[
A = (dy^k \otimes dy^j \otimes dy^i c^n_{ijk} \partial_n) \otimes (dy^l \otimes dy^s c^m_{sl} \partial_m)
= dy^s \otimes dy^k \otimes dy^j \otimes dy^i a^m_{ijks} \partial_m ,
\]

where

\[
a^m_{ijks} = (-1)^{\epsilon_s c^n_{ijk} c^m_{ns}} .
\]

Our convention for the pairing of vectors and covectors is

\[
\partial_i \otimes dy^j := \langle \partial_i, dy^j \rangle = \delta_i^j ,
\]

and we follow the usual sign rule: if two object of parities \( \epsilon_1 \) and \( \epsilon_2 \) are interchanged then the factor \((-1)^{\epsilon_1 \epsilon_2} \) is inserted.
Observe that applying the correspondence map \( \mathcal{R} \) to the graph (25) with altered order of vertices (but the same order of legs) yields

\[
(dy^l \otimes dy^s c^m_{ul} \partial_m) \otimes (dy^k \otimes dy^j \otimes dy^i c^n_{ijk} \partial_n) = -A.
\]

Clearly, using the cut-and-paste algorithm above, one can unambiguously reconstruct a tensor field by a graph with an arbitrary set of vertices.

Let us now introduce the real vector space \( \tilde{\mathcal{G}} \) spanned by all the graphs satisfying the conditions (a)-(d) above. The tensor operations and the correspondence map \( \mathcal{R} \) extend to the space \( \tilde{\mathcal{G}} \) by linearity. The space \( \tilde{\mathcal{G}} \), however, is not what we actually need, since the homomorphism \( \mathcal{R} : \tilde{\mathcal{G}} \to T(\mathbb{E})^{\text{inv}} \) is far from being a bijection. For example, if \( \Gamma' \) is the graph obtained from a graph \( \Gamma \) by transposition of two labels on vertices, then

\[
\mathcal{R}(\Gamma') = -\mathcal{R}(\Gamma).
\]

The minus sign is due to the fact that the elementary covariants \( \{C_n\} \) depicted by the corollas \( \{\gamma_n\} \) are Grassman odd (c.f. Example 5.1). This motivates us to introduce the quotient space \( \mathcal{G} = \tilde{\mathcal{G}} / \sim \) with respect to the equivalence relation \( \Gamma' \sim -\Gamma \). The tensor operations or, more precisely, their graphical representation pass through the quotient making \( \mathcal{G} \) into an abstract tensor algebra \( \mathcal{G} \) freely generated by the corollas (23). Concerning the general concept of an “abstract tensor calculus” we refer the reader to the recent papers [24], [25]. As a linear space, \( \mathcal{G} \) splits into the direct sum of finite-dimensional subspaces:

\[
\mathcal{G} = \bigoplus_{n,m,k} \mathcal{G}^{n,m}_k,
\]

where the superscripts \( n \) and \( m \) refer to the number of incoming and outgoing legs of a graph, while the subscript \( k \) indicates the number of the vertices. In addition to this trigrading the algebra \( \mathcal{G} \) possesses an increasing filtration

\[
0 \subset F_1 \mathcal{G} \subset F_2 \mathcal{G} \subset \cdots \subset F_\infty \mathcal{G} = \mathcal{G}
\]
mimicking the filtration (11) of \( T(\mathbb{E})^{\text{inv}} \). By definition, the subalgebra \( F_n \mathcal{G} \subset \mathcal{G} \) is generated by the corollas (23) with at most \( n \) incoming legs.

The map \( \mathcal{R} \) induces the map from \( \mathcal{G} \) to \( T(\mathbb{E})^{\text{inv}} \), which we will denote by the same letter \( \mathcal{R} \). Passing to the quotient \( \tilde{\mathcal{G}} / \sim \) does not kill the kernel of \( \mathcal{R} \) completely. For example, if \( \dim V = 1 \mid 0 \), then

\[
\mathcal{R}\left(\begin{array}{c}
1 \\
2
\end{array}\right) = 0.
\]

The last equality, however, could not take place if the space \( V \) were big enough. More precisely, the second main theorem of invariant theory [22], [14] ensures that the map

\[
\mathcal{R} : F_r \mathcal{G}^{n,m} \to F_r T^{n,m}(\mathbb{E})^{\text{inv}}
\]

is an isomorphism of vector spaces provided that \( | \dim V | \gg r, n. \)

Thus, in the stable range of dimensions the homomorphism \( \mathcal{R} \) allows one to replace the tensor algebra \( T(\mathbb{E})^{\text{inv}} \) by the graph algebra \( \mathcal{G} \). Then, the pullback of the differential \( \delta \) in \( T(\mathbb{E})^{\text{inv}} \) via

\[\text{Unfortunately, there is no commonly accepted name for this object. A more expressive term would be desirable. As we will see in a moment \( \mathcal{G} \) is not just an algebra but a cochain complex, whose coboundary operator respects the tensor operations. So, an appropriate name might be the differential algebra of graphs (DAG).}\]
the isomorphism (26) gives $G$ the structure of cochain complex with respect to the coboundary operator

$$\partial = R^{-1} \delta R : G^{n,m}_k \to G^{n,m}_{k+1},$$

in so doing, the stable cohomology groups of $T(E)^{inv}$ appear to be isomorphic to the graph cohomology groups $H(G) = \text{Ker} \partial / \text{Im} \partial$. Since $\partial$ differentiates the tensor product and commutes with the contraction it is sufficient to describe its action on corollas. Translating formulas (66) and (67) into the graph language, we get

$$\partial \left( \begin{array}{c} m \\ m+1 \end{array} \right) = \begin{array}{c} m \\ m+1 \end{array}$$

(27)

and

$$\partial \left( \begin{array}{c} m \\ j \\ j' \\ \ldots \\ m+1 \\ j'' \end{array} \right) = \sum_{j' \sqcup j'' = j \atop |j'| > 0, |j''| > 1} \left( \begin{array}{c} m \\ j' \\ j'' \end{array} \right)$$

(28)

for $|J| > 1$. A decoration on $\Gamma \in G$ induces a decoration on each summand of $\partial \Gamma$ as follows: splitting of the $m$th vertex produces a pair of new vertices that are labelled by $m$ and $m+1$, the labels less than $m$ remain intact while the labels greater than $m$ increase by 1.

A remarkable property of the differential $\partial$ is that it neither permutes the connected components of a decorated graph nor changes their number. More precisely, the complex $G^{n,m}$ is decomposed into a direct sum of its subcomplexes $G^{n,m}_{A_1,\ldots,A_k;B_1,\ldots,B_k}$, where $\{A_1,\ldots,A_k\}$ and $\{B_1,\ldots,B_k\}$ are partitions of the sets $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$, and $G^{n,m}_{A_1,\ldots,A_k;B_1,\ldots,B_k}$ is generated by graphs with $k$ connected components, the $l$th of which contains incoming and outgoing legs labelled by the elements of $A_l$ and $B_l$, respectively. Notice that the sets $\{A_1,\ldots,A_k\}$ and $\{B_1,\ldots,B_k\}$ are defined up to simultaneous permutations of $A_i$ with $A_j$ and $B_i$ with $B_j$, and some of the sets $A_1,\ldots,A_k, B_1,\ldots,B_k$ may be empty. Let $\tilde{G} = \bigoplus G^{n,m}$ denote the subcomplex of connected graphs. Then it is clear that

$$G^{n,m}_{A_1,\ldots,A_k;B_1,\ldots,B_k} \cong \bigotimes_{l=1}^k \tilde{G}^{\{A_l\}|\{B_l\}},$$

and by the Künneth formula the computation of the graph cohomology boils down to the computation of the group $H(\tilde{G})$.

The complex $\tilde{G}$ in its turn splits into the direct sum of three subcomplexes:

$$\tilde{G} = \tilde{G}' \oplus \tilde{G}'' \oplus \tilde{G}''',$$

Here $\tilde{G}'$ is spanned by the bivalent graphs build from $\gamma_1$. The subalgebra $\tilde{G}''$ is generated by the corollas $\gamma_n$ with $n > 1$, that is the valency of each vertex is at least three. Finally, $\tilde{G}'''$ spans the graphs with at least one bivalent vertex and at least one vertex of valency greater than two. The invariance of each of these three subspaces under the action of the coboundary operator is obvious.

The following statement can be viewed as a special case of Losik’s lemma [26] (see also [14, Theorem 2.2.8]).
Proposition 5.1. \( H(\bar{G}''') = 0 \), hence \( H(\bar{G}) = H(\bar{G}') \oplus H(\bar{G}'') \).

Proof. Observe that for each graph \( \Gamma \in \bar{G}''' \) there exists a unique, up to decoration, graph \( \tilde{\Gamma} \in \bar{G}'' \) such that \( \Gamma \) is obtained from \( \tilde{\Gamma} \) by putting bivalent vertices on the edges of the latter. The edges of \( \tilde{\Gamma} \), equipped with bivalent vertices, will be called branches. The length of a branch is just the number of bivalent vertices inserted. Denote by \( \gamma_n \) the branch of length \( n \) with the special decoration

\[
\gamma_n = \begin{array}{c}
\bullet \\
\bullet \\
\vdots \\
\bullet
\end{array}
\]

and consider a decreasing filtration

\[
\bar{G}''' = F_1\bar{G}''' \supset F_2\bar{G}''' \supset \cdots \supset F_\infty\bar{G}''' = 0.
\] (29)

By definition, the graph \( \Gamma \) belongs to \( F_n\bar{G}''' \), if the underlying graph \( \tilde{\Gamma} \in \bar{G}'' \) contains at least \( n \) vertices. Clearly, the differential \( \partial \) preserves the filtration. Associated to the filtration (29) is the first quadrant spectral sequence \( \{ E_r, d_r \} \) with

\[
E_{p,q}^0 = F_p\bar{G}'''.
\]
Geometrically, the zeroth differential \( d_0 : E_0 \rightarrow E_0 \) acts only on the bivalent vertices of \( \Gamma \) according to the general rule (27), lengthening the branches of odd length by 1 and annihilating the branches of even length. For instance

\[
d_0\gamma_n = \begin{cases} 
\gamma_{n+1}, & \text{for } n \text{ odd;} \\
0, & \text{otherwise.}
\end{cases}
\] (30)

We claim that \( E_1 = \text{Ker}d_0/\text{Im}d_0 = 0 \). Indeed, consider the operator \( h : E_0 \rightarrow E_0 \), successively acting on \( \gamma_n \) by the rule

\[
h\gamma_n = \begin{cases} 
\gamma_{n+1}, & \text{for } n \text{ even;} \\
0, & \text{otherwise.}
\end{cases}
\] (31)

In so doing, the labels on the other vertices that do not belong to \( \gamma_n \subset \Gamma \) increase by 1. The action of \( h \) and \( d_0 \) on an arbitrary decorated branch is easily reconstructed from (31) and (30), if one notes that the change of decoration can always be compensated by an overall sign factor.

Writing \( \Delta = hd_0 + d_0h \), we see that \( \Delta(\Gamma) = n\Gamma \), where \( n > 0 \) is the number of branches of nonzero length. The operator \( \Delta \) being invertible, we can take \( \Delta^{-1}h : E_0 \rightarrow E_0 \) to be a contracting homotopy. Thus the complex \( (E_0, d_0) \) is acyclic and so is \( \bar{G}''' \). \( \square \)

The complex of bivalent graphs splits as

\[
\bar{G}' = \bar{G}^{0,0} \oplus \bar{G}^{1,1},
\]
where the subcomplexes \( \bar{G}^{1,1} \) and \( \bar{G}^{0,0} \) span, respectively, the straight line and polygon graphs.

Proposition 5.2. The complex \( \bar{G}^{1,1} \) is acyclic and

\[
H_n(\bar{G}^{0,0}) = \begin{cases} 
\mathbb{R}, & \text{if } n = 2m-1; \\
0, & \text{if } n = 2m.
\end{cases}
\]

The corresponding nontrivial cocycles are given by the polygons

\[
\begin{array}{c}
5 \\
4 \\
3 \\
2 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
5 \\
4 \\
3 \\
2m-1 \\
2m \\
\end{array}
\] (32)
with an odd number of vertices.

Proof. The proof is straightforward.

We now turn to computation of the group \( H(\tilde{G}'') \). The complex \( \tilde{G}'' \) breaks up into the direct sum of various subcomplexes

\[
\tilde{G}'' = \left( \bigoplus_{n=2}^{\infty} (\tilde{G}'')_{n,1} \right) \oplus \left( \bigoplus_{n=1}^{\infty} (\tilde{G}'')_{n,0} \right).
\]

Here the spaces \((\tilde{G}'')_{n,1}\) and \((\tilde{G}'')_{n,0}\) are spanned, respectively, by the tree and cyclic graphs with \( n \) incoming legs and no bivalent vertices (see Fig. 1). To ease the notation, we will write

\[
T^n_m = (\tilde{G}'')_{m+1,1} \quad \text{and} \quad C^n_m = (\tilde{G}'')_{m,0},
\]

where the lower index \( m \) points to the number of vertices of a graph, while the upper index \( n \) counts the difference between its incoming and outgoing legs. Since the valency of each vertex is assumed to be greater than 2, it is easy to see that the following inequality holds for any graph from \( T^n_m \) or \( C^n_m \):

\[
n = \sum_{v \in V} (n_v - 2) \geq m,
\]

\( n_v \) being the valency of a vertex \( v \in V \). So, for \( m > n \) the spaces \((33)\) are assumed to be zero, and for any \( n \in \mathbb{N} \) the graph complexes \( T^n_m = \{T^n_m\} \) and \( C^n_m = \{C^n_m\} \) are given by finite sequences of finite-dimensional vector spaces.

**Theorem 5.1** (D.B. Fuks [15]).

\[
\dim H_m(T^n) = \dim H_m(C^n) = \begin{cases} (n-1)! & \text{if } n=m; \\ 0 & \text{otherwise}. \end{cases}
\]

Proof. Here we reproduce the original Fuks’ proof. Another proof can be found in [27, Sec.4].

Consider first the case of cyclic graphs with \( n \) legs. A typical cyclic graph is depicted in Fig. [1]. It is given by a collection of legs and trees attached to the vertices of the cycle. Following Fuks, we introduce the complex dual to the complex \( C^n \). Fixing basis in \( C^n \) allows one to identify \( C^n \) with its dual space \((C^n)^*\). Then the differential \( \partial^* : C^n_{k-1} \to C^n_k \) in the dual complex has an extremely simple description. The graph \( \partial^* \Gamma \) is given by a signed sum of decorated graphs which can be obtained from \( \Gamma \) by collapsing a single edge; in so doing, the loops remain intact. To specify the signs and decorations we can assume that the initial vertex of a collapsed edge \( e \) is labelled by 1 and the terminal vertex is labelled by 2. (The general case reduces to that by altering the order of vertices with appropriate sign factors.) Then, the vertex which results from collapsing \( e \) is numbered by 1 and the labels of all other vertices are reduced by 1. The labels of the legs remain the same.

Since the groups \( \{C^n_m\} \) are just finite-dimensional vector spaces,

\[
H_m(C^n, \partial) \simeq H_m(C^n, \partial^*),
\]

and we may focus upon computation of the \( \partial^* \)-cohomology.

We filter \((C^n, \partial^*)\) by the length of the cycle, i.e., let \( F_pC^n \) be the subcomplex of \( C^n \) spanned by graphs with at most \( p \) cyclic vertices (or edges). In view of \((34)\) the filtration is finite:

\[
0 = F_0C^n \subset F_1C^n \subset \cdots \subset F_nC^n = C^n.
\]

Let \( \{E^r, d^r\} \) be the spectral sequence associated to the filtration. Notice that \( E^0_{p,q} = F_pC^n_{p+q} = 0 \) for \( q < 0 \) and we have a first quadrant spectral sequence. The coboundary operator \( \partial^* : C^n_k \to
Characteristic classes of $Q$-manifolds is given by the sum $\partial^* = \partial^*_c + \partial^*_{nc}$, where the operator $\partial^*_c$ collapses only cyclic edges, and the operator $\partial^*_nc$ collapses only non-cyclic edges. By definition $d^0 = \partial^*_{nc}$.

To compute $E^1 = H(E^0, d^0)$, we arrange the cyclic vertices into three groups labelled by the letters $a$, $b$, and $c$. The vertices of type $a$ are trivalent vertices incident to one leg and two edges, the vertices of type $b$ are trivalent vertices incident to three edges, and the other cyclic vertices (of valency greater than 3) belong to the type $c$ (see Fig. 1). With this partition we define a homotopy $h : C^n_k \to C^n_{k+1}$. The operator $h$ acts as a differentiation on the $c$-type vertices. Graphically,

$$h \left( \begin{array}{c}
\text{tree} \\
c
\end{array} \right) = \left( \begin{array}{c}
\text{tree} \\
c
\end{array} \right)$$

A straightforward consideration shows that the homotopy $h$ connects 0 with the endomorphism that multiplies each cyclic graph by the total number of cyclic vertices of types $b$ and $c$. Thus the subspace of graphs with at least one vertex of $b$ or $c$ type is acyclic.

On the other hand, the graphs with only $a$-type vertices have the form

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} + \gamma_n \quad (35)$$

and differ from one another only by decoration. Being free of non-cyclic edges, the graphs (35) are automatically closed and represent nontrivial classes of $d^0$-cohomology group $E^1_{n,0}$. The group $S_n$ acts on (35) by permuting incoming legs. Since only the cyclic permutations lead to isomorphic graphs, the number of different decorations or, what is the same, the dimension of the space $E^1_{n,0}$ is equal to $(n-1)!$.

For reasons of dimension, the spectral sequence degenerates on the first page and we get $E^1_{p,0} \simeq H_p(C^n)$. This proves the theorem for the case of the cyclic graph complex $C^n$.

The case of tree graphs may be handled in much the same way, with the only difference that the filtration of the complex $T^n$ is now defined by the length of the unique path joining the first incoming leg of a connected tree graph to its outgoing leg. This path plays the role of the cycle in the previous consideration. The corresponding spectral sequence degenerates from $E^1$ yielding the only nontrivial group $E^1_{n,0}$. The space $E^1_{n,0}$ is spanned by the trivalent graphs of the form

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_{n-1} + \gamma_n \quad (36)$$

By construction, the leftmost leg is labelled by 1 and the labels on the other $n-1$ incoming legs can be prescribed arbitrarily. Hence $\dim E^1_{n,0} = (n-1)!$.

\[\square\]

Remark 5.1. We can also give an explicit description for basis cocycles whose classes of $\partial$-cohomology generate the groups $H_n(C^n)$ and $H_n(T^n)$. Relation (34) implies that the spaces $T^n_n$ and $C^n_n$ are spanned by the trivalent graphs. Since $\partial \gamma_2 = 0$, any trivalent graph is automatically $\partial$-closed. To extract a basis of nontrivial cocycles notice that the subspace of $\partial$-coboundaries
is generated by graphs $\partial \Gamma$, where $\Gamma$ has $n-2$ trivalent vertices and one vertex of valency four. In accordance with (28), the action of the differential on the 4-valent vertex reads

$$\frac{1}{2} \partial \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) = \begin{array}{c} \text{vertex graph} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} + \begin{array}{c} \text{vertex graph} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} + \begin{array}{c} \text{vertex graph} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Thus adding a coboundary amounts to the following equivalence transformation for a pair of nearby vertices of a trivalent graph:

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \sim \begin{array}{c} 2 \\ 3 \\ 1 \end{array} + \begin{array}{c} 3 \\ 1 \\ 2 \end{array}$$

(37)

It is easy to verify that applying this transformation properly time and again, one can bring any connected trivalent graph to a linear combinations of the graphs (35) or (36). For dimensional reasons, the graphs (35) and (36) must then span a basis in the space of nontrivial $\partial$-cocycles.

Interestingly enough that the equivalence relation (37) coincides in form with the defining relation for the quadratic operad of Lie algebras $\mathcal{L}ie$. Our computations then show that the graph complex $\mathcal{G}_{\text{tree}} = \bigoplus_{n>1} \mathcal{T}^n$, being an algebra under the grafting of trees, realizes the minimal resolution of the Lie algebra operad, that is, $\mathcal{L}ie_{\infty} = (\mathcal{G}_{\text{tree}}, \partial)$. For more details on the operadic interpretation of the graph cohomology, including the case of cyclic graph complex $\mathcal{G}_{\text{cycle}} = \bigoplus_{n>0} \mathcal{C}^n$, we refer the reader to [27], [28].

Let us return to $Q$-manifolds. We define the stable characteristic classes of a flat $Q$-manifold as the image of the stable cohomology classes of $\mathcal{T}(\mathfrak{E})^{\text{inv}}$ under the homomorphism (19). Using the explicit formulas (13) for the characteristic map together with the correspondence map $R$, one can easily convert the basis cocycles of the graph complex into the universal cocycles of a flat $Q$-manifold.

Given a homological vector field $Q$ and a flat symmetric connection $\nabla$, we write $\Lambda$ for the odd endomorphism defined by the rule $\Lambda(X) = \nabla_X Q$ for $\forall X \in \mathfrak{X}(M)$. The covariant derivative of $\Lambda$ yields a $\mathfrak{X}(M)$-module homomorphism from $\mathfrak{X}(M)$ to $\mathfrak{A}(M)$. Graphically, the tensors $\Lambda \in \mathfrak{X}(M)$-module homomorphism from $\mathfrak{X}(M)$ to $\mathfrak{A}(M)$, we can write

$$A_n = \text{Str}(\Lambda^{2n-1}).$$

(38)

The invariance of $A_n$ follows immediately from the identity $\nabla_Q \Lambda = -\Lambda^2$ and the cyclic property of the supertrace. The cocycles (36) give rise to the sequence of $Q$-invariant tensors $B_n \in \mathcal{T}^{n+1,1}(M)$. Identifying the $(n+1, 1)$-tensors with homomorphisms from $\mathfrak{X}(M)^{\otimes n}$ to $\mathfrak{A}(M)$, we can write

$$B_n(X_1, X_2, \ldots, X_n) = \nabla_{X_1} \Lambda \nabla_{X_2} \Lambda \cdots \nabla_{X_n} \Lambda.$$

(39)
Taking now the trace, we get the universal cocycles corresponding to the cyclic graphs (35):
\[ C_n(X_1, X_2, ..., X_n) = \text{Str} B_n(X_1, X_2, ..., X_n) \].
(40)

By construction, \( C_n \in T_{n,0}(M) \). The \( Q \)-invariance of the tensor fields \( \{ B_n \} \) and \( \{ C_n \} \) can be checked directly. In fact, it is enough to check the invariance of \( B_1 = \nabla \Lambda \) as the other cocycles are made of \( B_1 \) by means of tensor operations. We have
\[
(-1)^c(X)(L_Q B_1)(X) = L_Q B_1(X) - B_1([Q, X]) = \nabla Q B_1(X) + [\Lambda, B_1(X)] - B_1([Q, X]) = (-1)^c(X) \nabla X B_1(Q) + [\Lambda, B_1(X)] = -(-1)^c(X) \nabla X \Lambda^2 - [\Lambda, \nabla X \Lambda] = 0.
\]
(41)

We will refer to the \( \delta \)-cohomology classes of the cocycles (38), (39) and (40) as the characteristic classes of \( A\)-, \( B\)-, and \( C\)-series. The results of this section can now be summarized as follows.

**Theorem 5.2.** In the stable range of dimensions, the characteristic classes of a flat \( Q \)-manifold are generated by the characteristic classes of \( A\)-, \( B\)-, and \( C\)-series by means of tensor products and permutations of indices.

6. **Intrinsic characteristic classes**

So far we have dealt with construction and classification of the characteristic classes associated to flat \( Q \)-manifolds. In this section, we are going to extend the above consideration from the flat to arbitrary \( Q \)-manifold. To begin with we note that the straightforward substitution of an arbitrary, non-flat, symmetric connection to formulas (38-40) fails to produce the universal cocycles. Taking for example \( A_1 \) we get
\[
\delta \text{Str}(\Lambda) = \nabla Q \text{Str}(\nabla Q) = \frac{1}{2} \text{Str}(R_{QQ}) \neq 0;
\]

where
\[
R_{XY} = [\nabla X, \nabla Y] - \nabla_{[X,Y]} \in \mathfrak{X}(M) \quad \forall X, Y \in \mathfrak{X}(M)
\]
is the curvature of \( \nabla \). We can then try to restore the \( \delta \)-closedness of the tensors (38,40), where \( \nabla \) is now an arbitrary symmetric connection, by adding to them appropriate curvature-dependent terms. Most easily this can be done for the characteristic cocycles of \( B \)- and \( C \)-series. Define the \((2,1)\)-tensor field \( B_1 \) as a \( C^\infty(M)\)-linear homomorphism from \( \mathfrak{X}(M) \) to \( \mathfrak{A}(M) \):
\[
X \mapsto B_1(X) = \nabla_X \Lambda - R_{XQ}.
\]

A calculation similar to (41) shows that \( B_1 \) is a \( Q \)-invariant tensor for an arbitrary (i.e., not necessary flat) connection \( \nabla \). Therefore by making replacement \( B_1 = \nabla \Lambda \mapsto B_1 \) in (39) and (40), we get two infinite series of universal cocycles \( \{ B_n \} \) and \( \{ C_n \} \) that generalize \( B \)- and \( C \)-series to arbitrary \( Q \)-manifolds. Theorem 2 ensures that the \( \delta \)-cohomology classes of these cocycles do not depend on the choice of symmetric connection, so that we have two well-defined series of characteristic classes \( [B_n], [C_n] \in H_Q(M) \). We call these characteristic classes **intrinsic** to stress the fact that they may well be nontrivial even for a flat \( Q \)-manifold. In other words, the intrinsic characteristic classes are intrinsically related to the structure of the homological
depth critical changes for coherence and fluency.
vector field rather than to the topology of the underlying manifold. The latter may prevent the existence of a flat connection and give rise to universal cocycles that essentially involve the curvature and vanish in the zero-curvature limit. These last cocycles are called \textit{vanishing} ones.

The typical examples of vanishing universal cocycles are the following $Q$-invariant functions:

$$P_n = \text{Str}(R^n),$$

where $R = R_{QQ} \in \mathfrak{A}(M)$. One can also view the curvature of $\nabla$ as a two-form $R$ with values in $\mathfrak{A}(M)$ and define the function $P_n$ as the value of the $2n$-form $P_n = \text{Str}(R^n)$ on the homological vector field $P_n = P_n(Q)$. Then the $\delta$-closedness of $P_n$ immediately follows from the $d$-closedness of $P_n$:

$$\delta P_n = (dP_n)(Q) = 0.$$  

The de Rham classes of the $2n$-forms $P_n$ are known as the Pontryagin characters of the tangent bundle $TM$. For a deeper discussion of the Chern-Weyl theory of characteristic classes in the category of supervector bundles we refer to [29] and [30]. The latter paper contains also a very natural generalization of the Chern-Weyl construction to the category of $Q$-bundles.

The next theorem establishes a one-to-one correspondence between the vanishing cocycles (42) and the universal cocycles of $A$-series (38).

\textbf{Theorem 6.1 \cite{10}}. Let $M$ be a $Q$-manifold with symmetric connection. Then for each $n \in \mathbb{N}$ there exists an invariant matrix polynomial $A_n(\Lambda, R) \in C^\infty(M)$ in $\Lambda$ and $R$ such that

$$A_n(\Lambda, 0) = \text{Str}(\Lambda^{2n-1}) \text{ and } \delta A_n(\Lambda, R) = \binom{2n-1}{n}\text{Str}(R^n).$$

\textbf{Corollary 1}. The vanishing cocycles $P_n = \text{Str}(R^n)$ are trivial.

\textbf{Corollary 2}. If $P_n = 0$, then the function $A_n(\Lambda, R)$ is a $\delta$-cocycle.

\textit{Proof}. We start with some auxiliary algebraic constructions, which model our geometric situation. Namely, consider a DGA-algebra $W = \bigoplus_{n \geq 0} W_n$ over $\mathbb{R}$ freely generated by one element $a$ of degree 1 and one element $b$ of degree 2. The elements of $W$ are just linear combinations of words associated to the binary alphabet $\{a, b\}$ and multiplication is given by concatenation of words. Denote by $|w|$ the degree of the word $w$. The action of the differential $d : W_n \to W_{n+1}$ on the generators reads

$$da = a^2 + b, \quad db = [a, b],$$

The homological vector field being odd, the complete contraction of $P_n$ with $Q^\otimes 2n$ is not equal to zero identically.
and extends to the whole $W$ by the Leibniz rule:
\[ d(w_1 w_2) = dw_1 w_2 + (-1)^{|w_1|}w_1 dw_2. \]

We define the cyclic space $\hat{W}$ as the quotient $\hat{W} = W/[W,W]$, where the subspace $[W,W] \subset W$ is generated by commutators
\[ w_1 w_2 - (-1)^{|w_1||w_2|}w_2 w_1. \]

The action of the differential (44) passes through the quotient making the cyclic space $\hat{W}$ into a cochain complex. Explicitly,
\[ \hat{d}a = -a^2 + b, \quad \hat{d}b = 0. \]

The complex $(\hat{W}, \hat{d})$ is acyclic. This can be easily seen from the filtration of $\hat{W}$ by the length of words:
\[ W_n = F_1\hat{W}_n \supset F_2\hat{W}_n \supset \cdots \supset F_{n+1}\hat{W}_n = 0. \]

The zero differential of the corresponding spectral sequence $\{E_n, d_n\}$ acts on the generators as
\[ d_0 a = b, \quad d_0 b = 0. \]

So, the complex $(E_0, d_0)$ is obviously acyclic and $H(\hat{W}) = 0$.

Let us now introduce one more grading on the space $\hat{W}$ by prescribing the following degrees to the generators:
\[ \deg a = 0, \quad \deg b = 2. \]

Associated to this grading is a decreasing filtration
\[ \hat{W}_n = F_0\hat{W}_n \supset F_1\hat{W}_n \supset \cdots \supset F_{n+1}\hat{W}_n = 0. \]

(45)

Here $F_p\hat{W} = \bigoplus_{n>p} \hat{W}^{(n)}$ and $\deg \hat{W}^{(n)} = n$. Since the differential (44) preserves the filtration (45), we have the first quadrant spectral sequence $\{E_r, d_r\}_{r \geq 0}$ converging to $H(\hat{W}) = 0$. By definition, $E_{0,q}^p = F_p\hat{W}_{p+q}/F_{p+1}\hat{W}_{p+q}$. The zero differential of this spectral sequence is completely defined by its action on the generators of $W$:
\[ d_0 a = -a^2, \quad d_0 b = 0. \]

(46)

Notice that $E_{0,q}^0 = 0$, if $p$ is odd. The complex $(E_0, d_0)$ splits into the direct product $E_0 = A \oplus B \oplus C$ of three subcomplexes. The space $A = \bigoplus_{q>0} E^{0,q}$ is generated by the odd powers of the letter $a$, $B = \bigoplus_{p>0} E^{p,0}$ is given by polynomials in $b$, and $C = \bigoplus_{p,q>0} E^{p,q}$ is spanned by cyclic words containing at least one syllable $ab$. It is clear from (46) that $H(A) \simeq A$ and $H(B) \simeq B$. The remaining complex $C$ turns out to be acyclic. Indeed, any word $w \in C$ is decomposed into a product $w = a_{n_1}a_{n_2} \cdots a_{n_k}$ of syllables $a_n = a^n b$ and this decomposition is unique up to a cyclic permutation of factors. We have
\[ d_0 a_n = \begin{cases} -a_{n+1}, & \text{if } n \text{ is odd}; \\ 0, & \text{otherwise}. \end{cases} \]

Define a differentiation $h : C \to C$ by its action on syllables:
\[ ha_n = \begin{cases} -a_{n-1}, & \text{if } n > 0 \text{ is even}; \\ 0, & \text{otherwise}. \end{cases} \]

Applying the operator $\Delta = d_0 h + hd_0$ to a word $w \in C$, we get $\Delta w = mw$, where $m > 0$ is the number of syllables $a_n$ with $n > 0$ the word $w$ consists of. Thus, the operator $\Delta$ is invertible and $\Delta^{-1} h : C \to C$ is a contracting homotopy. The nonzero entries of the group $E_1$ are depicted in Fig.2a. Since $E_r \Rightarrow H(W) = 0$, each nonzero element of the fiber space is transgressive and
“kills” some element on the base. In other words, the maps \( d_r : E_r^{0,r-1} \to E_r^{r,0}, \ r = 2, 4, \ldots \), are isomorphisms of one-dimensional vector spaces, so that \( d_2a^{2n-1} = \alpha_n b^n \) for some \( \alpha_n \neq 0 \). The last equation amounts to the existence of a sequence of elements \( c_m \in E_0^{2m,2(n-m)-1} \) such that

\[
d(a^{2n-1} + c_1 + c_2 + \cdots + c_{n-1}) = \alpha_n b^n. \tag{47}
\]

With a little work one can also find that \( \alpha_n = \binom{2n-1}{n} \).

Comparing \((14)\) with the identities

\[
\delta \Lambda = \Lambda^2 + \frac{1}{2} R, \quad \delta R = [\Lambda, R],
\]

we see that the homomorphism \( h : W \to \mathfrak{a}(M) \) given by

\[
a \mapsto \Lambda, \quad b \mapsto \frac{1}{2} R \tag{48}
\]

is a representation of DGA-algebra \((W,d)\) as a subalgebra of \((\mathfrak{a}(M), \delta)\). In view of the cyclic property of the trace, the composition \( \text{Str} \circ h : W \to C^\infty(M) \) descends to the cyclic space \( \hat{W} \) defining a homomorphism of complexes. Applying this homomorphism to both sides of Eq.(47), we conclude that the invariant matrix polynomial

\[
A_n(\Lambda, R) = \text{Str} \circ h(a^{2n-1} + c_1 + c_2 + \cdots + c_{n-1})
\]

satisfies Eqs.(13). It is not hard to write explicit expressions for the first polynomials:

\[
A_1 = \text{Str}(\Lambda),
\]

\[
A_2 = \text{Str}(\Lambda^3 + \frac{3}{2} R \Lambda),
\]

\[
A_3 = \text{Str}(\Lambda^5 + \frac{5}{2} R \Lambda^3 + \frac{15}{4} R^2 \Lambda),
\]

\[
A_4 = \text{Str}(\Lambda^7 + \frac{7}{2} R \Lambda^5 + \frac{35}{4} R^2 \Lambda^3 + \frac{7}{4} R \Lambda R \Lambda^2 + \frac{35}{8} R^3 \Lambda).
\]

\[\square\]

In fact, Theorem 6.1 identifies the trivial \( \delta \)-cocycles \( P_n \) with obstructions to extendability of \( A \)-series’ characteristic classes to the non-flat \( Q \)-manifolds. To prove this, we only need to show that the \( \delta \)-cocycles \( P_n \) become nontrivial when considered as elements of the subcomplex of vanishing covariants \( \mathcal{R} \subset \mathcal{A} \) (see comments after (1)). The proof of the last fact is not particularly interesting and we relegate it to Appendix B. Summarizing, it appears impossible to define the construction of \( A \)-series \((38)\) in the full category of \( Q \)-manifolds, i.e., without any conditions on the geometry of \( M \). Of course, the assumption that \( M \) admits a flat connection is unduly restrictive; instead, one may only suppose the triviality of the \( n \)th Pontryagin character of the tangent bundle \( TM \). In the latter case there exists a form \( F_n \in \Omega^{2n-1}(M) \) such that \( \mathbf{P}_n = dF_n \). The form \( F_n \) is not uniquely determined by the connection \( \nabla \) as one is free to add to \( F_n \) any closed \((2n-1)\)-form. If \( F_n = F_n(Q) \), then \( \delta F_n = P_n \) and by Theorem 6.1 we have the \( Q \)-invariant function

\[
A_n^F = A_n(\Lambda, R) - \binom{2n-1}{n} F_n. \tag{49}
\]

If \( F'_n \) is another potential for the Pontryagin form, i.e., \( \mathbf{P}_n = dF'_n \), then the difference \( K_n = \binom{2n-1}{n}(F_n - F'_n) \) is closed and defines the de Rham class \( [K_n] \in H_d^{2n-1}(M) \). We have

\[
A_n^{F'} - A_n^F = K_n, \tag{50}
\]
where $K_n = K_n(Q)$. In case $K_n = dW_n$, the r.h.s. of (50) is given by the coboundary $\delta W_n(Q)$, so that the $\delta$-cocycles $A''_n$ and $A'_n$ appear to be cohomologous whenever $[K_n] = 0$. This motivates us to consider the homomorphism

$$h_Q : \Omega(M) \to C^\infty(M)$$

that evaluates an exterior form on the homological vector field $Q$. Since $h_Q d + \delta h_Q = 0$, $h_Q$ induces the homomorphism in cohomology

$$h_Q^* : H_d(M) \to H_\delta(M).$$

Passing in (50) to the $\delta$-cohomology classes, we can write

$$[A'_n] - [A''_n] \in \text{Im} h_Q^*.$$

The last formula just amounts to saying that the class $[A'_n] + \text{Im} h_Q^*$ does not actually depend on the choice of $F_n$. Thus, we have proved the following

**Theorem 6.2.** Let $M$ be a $Q$-manifold. Assume that the $n$th Pontryagin character of the tangent bundle $TM$ is trivial. Then the $Q$-invariant function (49) represents a well-defined element of $H_Q(M)/\text{Im} h_Q^*$.

It is well known that all the Pontryagin characters $[P_{2m+1}]$ vanish for the real vector bundles over the usual (even) manifolds, and the same statement holds true in the category of supervector bundles. This means the existence of classes $[A''_{2m+1}] + \text{Im} h_Q^*$ for all $Q$-manifolds. Moreover, each class $[A''_{2m+1}] + \text{Im} h_Q^*$ contains a canonical representative $[A^0_{2m+1}] \in H_Q(M)$, which can be viewed as an invariant of the $Q$-manifold itself. The existence of such a representative is established by the next two propositions whose proofs can be found in Appendix C.

**Proposition 6.1.** Every supermanifold $M$ admits a symmetric affine connection $\nabla$ with $P_{2m+1} = 0$ for all $m \geq 0$.

Since the construction of $\nabla$ satisfying the property above utilizes two auxiliary metrics, we will refer to $\nabla$ as the metric connection. Using this connection, we can define the series of $\delta$-cohomology classes $[A^0_{2m+1}] \in H_Q(M)$ represented by the functions (49) with $F_{2m+1} = 0$. One can view these classes as a proper generalization for the half of the scalar characteristic classes of $A$-series (38) to the case of arbitrary (i.e., non-flat) $Q$-manifolds. The construction of the metric connection involves a great deal of ambiguity concerning the choice of the metrics, and it is important to check that this ambiguity does not affect on the classes $[A^0_{2m+1}]$.

**Proposition 6.2.** The class $[A^0_{2m+1}] \in H_Q(M)$ is independent of the choice of metric connection.

In the particular case of homological vector fields coming from the Lie algebroids (see Example 1.2) the corresponding characteristic classes $[A^0_{2m+1}]$ add up to the secondary characteristic classes of Lie algebroids that were introduced and studied by Fernandes [31] within the framework of classical differential geometry. The class $[A^0_1]$, called the modular class of $Q$-manifolds [10], unifies and generalizes the well-known constructions of the modular classes of (complex) Poisson manifolds [32], [33], Lie algebroids [34], and the Lie-Rinehart algebras [35]; hence the name. Let us look at it more closely.

Given a metric connection $\nabla$, the modular class is represented by the covariant divergence of the homological vector field

$$A^0_1 = \text{Str}(\Lambda) = \nabla_i Q^i.$$
Equivalently, it can be defined in terms of a nowhere vanishing density \( \rho \), instead of the metric connection:

\[
A^0_1 = \text{div}_\rho Q = \rho^{-1} \partial_i (\rho Q^i) .
\]

For instance, one can take \( \rho = (\det g^0)^\frac{2}{n} (\det g^1)^\frac{2}{n} \), where \( g^0 \) and \( g^1 \) are metrics entering the definition of \( \nabla \). If \( \rho' \) is another density on \( M \), then \( A^0_1 \) is obviously cohomologous to \( A^0_1' \):

\[
A^0_1' - A^0_1 = \delta f , \quad f = \ln(\rho'/\rho) \in C^\infty(M) .
\]

Writing \( A^0_1 = \rho^{-1}L_Q \rho \) we see that the vanishing modular class \( [A^0_1] \in H_Q(M) \) is the necessary and sufficient condition for \( M \) to admit a \( Q \)-invariant nowhere vanishing density.

We conclude this section with the following extension of Theorem 5.2 to non-flat \( Q \)-manifolds.

**Theorem 6.3.** Given a \( Q \)-manifold with metric connection, the tensor algebra of intrinsic characteristic classes is generated by the \( \delta \)-cohomology classes of the universal cocycles \( \{ A^0_{2m+1} \} \), \( \{ B_n \} \), and \( \{ C_n \} \) with the help of tensor products and permutations of indices. The intrinsic characteristic classes depend only on the \( Q \)-manifold, not on the choice of the metric connection.

### 7. Characteristic classes with values in forms

If \( M \) is a \( Q \)-manifold, then the algebra of exterior forms \( \Omega(M) \) carries a pair of commuting differentials: the exterior differential \( d \) and the Lie derivative \( \delta = L_Q \). In this section, we discuss an interesting interplay between both the differentials in the context of characteristic classes. For this purpose, let us consider the bigraded, bidifferential, associative algebra \( A = A \cap \Omega(M) \) whose elements are form-valued local covariants associated to the homological vector field \( Q \) and (not necessary metric) connection \( \nabla \). The first grading in \( A = \bigoplus A^{n,m} \) is just the form degree, while the second one is the degree of homogeneity of an element \( f \in A \) as a function of \( Q \):

\[
A^{n,m} \ni f \iff f(tQ) = t^m f(Q) \quad \forall t \in \mathbb{R} .
\]

The commuting differentials \( d \) and \( \delta \) increase the respective degrees by one, making \( A \) into a multiplicative bicomplex.

The intrinsic part of the \( \delta \)-cohomology of the differential algebra \( A \) has been in fact computed in the previous section. It follows from Theorem 5.3 that the multiplicative basis of intrinsic \( \delta \)-cocycles is given by the forms

\[
C_n = \text{Str}(B^n_1) \in A^{n,n}, \quad n \in \mathbb{N} , \tag{51}
\]

where

\[
B_1 = \nabla A - i_Q R \in \Omega^1(M) \otimes A(M) .
\]

Hereafter we treat the connection as a first-order differential operator \( \nabla : \Omega^\bullet(M) \otimes \mathcal{F}(M) \to \Omega^{\bullet+1}(M) \otimes \mathcal{F}(M) \) satisfying the Leibniz rule:

\[
\nabla (\omega \otimes u) = d\omega \otimes u + (-1)^{n+1} \omega \otimes \nabla u , \quad \forall \omega \in \Omega^n(M), \quad \forall u \in \mathcal{F}(M) .
\]

The action of \( \nabla \) is then canonically extended from \( \Omega(M) \otimes \mathcal{F}(M) \) to \( \Omega(M) \otimes \mathcal{T}(M) \) by usual formulas of differential geometry. The curvature of the connection is the matrix-valued two-form \( \nabla^2 = R \in \Omega^2(M) \otimes A(M) \). One can view \( \Omega(M) \otimes A(M) \) as an associative graded superalgebra over \( \Omega(M) \) with trace. Multiplication in \( \Omega(M) \otimes A(M) \) is given by the exterior product in \( \Omega(M) \) and the composition of endomorphisms in \( A(M) \), and the supertrace entering the definition (51)
is defined in a natural way as the $\Omega(M)$-linear map $\text{Str} : \Omega(M) \otimes \mathfrak{A}(M) \rightarrow \Omega(M)$ vanishing on commutators.

As is seen from the definition, the $Q$-invariant forms $\{C_n\}$ are given by the totally antisymmetric part of the corresponding $\delta$-cocycles [10].

The next theorem is the main result of this section.

**Theorem 7.1.** Let $M$ be a $Q$-manifold with $[P_n] = 0$. Then the $\delta$-cohomology class $[C_n] \in H^Q(M)$ contains a $d$-exact representative.

Our proof will be based on comparing two spectral sequences canonically associated to some bicomplex $V \subset \mathfrak{A}$.

Consider first the graded subalgebra $\bar{W} = \langle a_0, a_1, \cdots, a_6 \rangle \subset \Omega(M) \otimes \mathfrak{A}(M)$ generated by seven tensor fields:

$$a_0 = \Lambda, \quad a_1 = i_q^2 R, \quad a_2 = i_q R, \quad a_3 = R,$$
$$a_4 = \nabla a_0 + a_2, \quad a_5 = \nabla a_1, \quad a_6 = \nabla a_2.$$

Note that $\nabla a_3 = 0$ in virtue of the Bianchi identity. Let $W$ denote the image of $\bar{W}$ under the map $\text{Str} : \Omega(M) \otimes \mathfrak{A}(M) \rightarrow \Omega(M)$. Since $\nabla^2 a_i = [a_i, a_i] \in W$, the subalgebra $W \subset \Omega(M) \otimes \mathfrak{A}(M)$ is invariant under the action of $\nabla$ and we have the commutative diagram

$$\begin{array}{ccc}
W^p & \overset{\nabla}{\longrightarrow} & W^{p+1} \\
\text{Str} \downarrow & & \downarrow \text{Str} \\
W^p & \overset{d}{\longrightarrow} & W^{p+1}
\end{array}$$

Therefore $(W, \delta)$ is a subcomplex of the de Rham complex of $M$. Furthermore, $W$ is invariant under the action of the Lie derivative $\delta = L_Q$. Using the general relation $\delta a_i = \nabla_Q a_i + [a_0, a_i]$ and the Bianchi identity for the curvature tensor one can readily check that

$$\delta a_0 = a_0^2 - \frac{1}{2} a_1, \quad \delta a_1 = [a_0, a_1], \quad \delta a_2 = [a_0, a_2] - \frac{1}{2} a_5,$$
$$\delta a_3 = [a_0, a_3] - a_6, \quad \delta a_4 = 0, \quad \delta a_5 = [a_0, a_5] - [a_2, a_1],$$
$$\delta a_6 = [a_0, a_6] - \frac{1}{2} [a_3, a_1].$$

Thus, $W \subset \mathfrak{A}$ is a bicomplex. The following theorem computes the $d$- and $\delta$-cohomology groups of $W$.

**Lemma 7.1.**

$$H^Q_\delta(W^p, \bullet) = \begin{cases} \mathbb{R}, & \text{if } p = q; \\
0, & \text{otherwise}. \end{cases}$$
$$H^Q_\delta(W^p, \bullet, q) = \begin{cases} \mathbb{R}, & \text{if } p \text{ is even and } q = 0; \\
0, & \text{otherwise}. \end{cases}$$

The corresponding nontrivial $\delta$- and $d$-cocycles can be chosen as

$$C_n = \text{Str} a^n_4 \in W^{n, n}, \quad P_n = \text{Str} a^n_3 \in W^{2n, 0}.$$

**Proof.** The cocycles $C_n$ and $P_n$ are obviously nontrivial. To prove that these span the space of all nontrivial cocycles we will construct two homotopy operators $h_1$ and $h_2$ such that

$$h_1 \delta + \delta h_1 = \Delta_1, \quad h_2 d + dh_2 = \Delta_2,$$
$$\text{Ker } \Delta_1 = \text{span}(C_1, C_2, \ldots), \quad \text{Ker } \Delta_2 = \text{span}(P_1, P_2, \ldots).$$
Define the following pair of odd differentiations of the algebra $\tilde{W}$:
\[
\begin{align*}
\bar{h}_1 a_0 &= 0, & \bar{h}_1 a_1 &= -2a_0, & \bar{h}_1 a_2 &= 0, & \bar{h}_1 a_3 &= 0, \\
\bar{h}_1 a_4 &= 0, & \bar{h}_1 a_5 &= -2a_2, & \bar{h}_1 a_6 &= -a_3; \\
\bar{h}_2 a_0 &= 0, & \bar{h}_2 a_1 &= 0, & \bar{h}_2 a_2 &= 0, & \bar{h}_2 a_3 &= 0, \\
\bar{h}_2 a_4 &= a_0, & \bar{h}_2 a_5 &= a_1, & \bar{h}_2 a_6 &= a_2.
\end{align*}
\]
(52)

Writing
\[
\bar{\Delta}_1 = \bar{h}_1 \delta + \delta \bar{h}_1, \quad \bar{\Delta}_2 = \bar{h}_2 \nabla + \nabla \bar{h}_2,
\]
we find
\[
\bar{\Delta}_1 a_0 = a_0, \quad \bar{\Delta}_1 a_1 = a_1 + 2a_0^2, \quad \bar{\Delta}_1 a_2 = a_2, \quad \bar{\Delta}_1 a_3 = a_3, \\
\bar{\Delta}_1 a_4 = 0, \quad \bar{\Delta}_1 a_5 = a_5 + 2[a_0, a_2], \quad \bar{\Delta}_1 a_6 = a_6 + [a_0, a_3]; \\
\bar{\Delta}_2 a_i = a_i, \quad \forall i \neq 3, \\
\bar{\Delta}_2 a_3 = 0.
\]
(53)

Now for any $\bar{b} = a_{i_1} \cdots a_{i_k} \in \tilde{W}$ and $b = \text{Str}(\bar{b}) \in W$ we set
\[
h_1 b = \text{Str}(\bar{h}_1 \bar{b}), \quad h_2 b = \text{Str}(\bar{h}_2 \bar{b}).
\]
It is clear that applying the operator $\Delta_2 = [d, h_2]$ to $b$ yields
\[
\Delta_2 b = \text{Str}(\bar{\Delta}_2 b) = nb,
\]
where $n$ is the number of letters $a_i$ in the word $\bar{b} = a_{i_1} \cdots a_{i_k}$ which are different from $a_3$. So the elements $P_n = \text{Str} a_3^n$ do span the kernel of $\Delta_2$.

To compute the kernel of $\Delta_1 = [\delta, h_1]$ consider the filtration of the space $W$ by the length of words in $a_i$:
\[
W^{(n)} \ni b \iff b = \sum_{k=n}^{\infty} b_k, \quad b_k \in \text{span}(\text{Str}(a_{i_1} \cdots a_{i_k})).
\]
As is seen from (53) the operator $\Delta_1$ preserves the filtration and the equality $\Delta_1 b = 0$ implies $b_n = \beta_n C_n$ for some $\beta_n \in \mathbb{R}$. Since $C_n \in \text{Ker} \Delta_1$, we have $\Delta_1 (b - b_n) = 0$ and hence $b_{n+1} = \beta_{n+1} C_{n+1}$ for some $\beta_{n+1} \in \mathbb{R}$. Proceeding by induction, we deduce that $b = \sum_{k=n}^{\infty} \beta_k C_k$. \hfill \square

Suppose now that the $n$th Pontryagin character of the tangent bundle $TM$ is trivial. To model this situation algebraically we extend $W$ by the bigraded space
\[
\tilde{K} = \text{span}(F^0_n, F^1_n, \ldots, F^{2n-1}_n; \bar{F}^0_n, \bar{F}^1_n, \ldots, \bar{F}^{2n-2}_n),
\]
where
\[
F^k_n = i^k_Q F^0_n, \quad \bar{F}^k_n = \delta F^k_n, \quad d F^0_n = P_n = \text{Str}(R^n).
\]
(54)

It follows immediately from the definition that the extended space $V = W \oplus \tilde{K}$ is invariant under the action of $d$ and $\delta$. Indeed, using the Cartan formula $\delta = d i_Q + i_Q d$, we find
\[
d F^k_n = k F^{k-1}_n + i^k_Q P_n, \quad d \bar{F}^k_n = -\frac{1}{k+1} d i^{k+1} P_n,
\]
\footnote{In the previous section, the $(2n - 1)$-form $F^0_n$ was denoted by $F_n$, and the function $F^{2n-1}_n$ was denoted by $F_n$.}
Lemma 7.2. \[ \delta \mathbf{F}_n^{2n-1} = i_Q^n \mathbf{P}_n = P_n. \] Thus, \((V, d, \delta)\) is a bicomplex having \((W, d, \delta)\) as a subcomplex.

\[ H^q_\delta(V^p, \bullet) = \begin{cases} \mathbb{R}, & \text{if } p = q \text{ or } p = 0 \text{ and } q = 2n - 1; \\ 0, & \text{otherwise.} \end{cases} \]

\[ H^p_d(V^{\bullet, q}) = \begin{cases} \mathbb{R}, & \text{if } q = 0 \text{ and } p \text{ is even and not equal to } 2n; \\ 0, & \text{otherwise.} \end{cases} \]

In plain English, the lemma says that vanishing of the \(n\)th Pontryagin character “kills” one class of \(d\)-cohomology in degree \(2n\), giving simultaneously birth to a new \(\delta\)-cohomology class in degree \(2n - 1\).

\textbf{Proof.} Define the quotient complex \(K = V/W\). As a linear space \(K \simeq \tilde{K}\). The short exact sequence

\[ 0 \rightarrow W \rightarrow i \rightarrow V \rightarrow p \rightarrow K \rightarrow 0, \]

gives rise to a long exact sequence in cohomology.

For the \(d\)-cohomology we have

\[ \cdots \rightarrow H^{2n-1}_d(K) \xrightarrow{\partial} H^{2n}_d(W) \xrightarrow{i_*} H^{2n}_d(V) \xrightarrow{p_*} H^{2n}_d(K) \rightarrow \cdots \]

It follows from (54) that \(F^n_0\) is a nontrivial \(d\)-cocycle of \(K^{2n-1,0}\) and \(\partial[F^n_0] = [P_n]\), where \([P_n]\) spans \(H^{2n}_d(W) \simeq \mathbb{R}\). Hence \(\partial\) is epic and \(i_* = 0\). Since \(H^{2n}_d(K) = 0\), we conclude that \(H^{2n}_d(V) = 0\).

Consider now the long exact sequence for the \(\delta\)-cohomology groups:

\[ \cdots \rightarrow H^{2n-2}_\delta(K) \xrightarrow{\partial^r} H^{2n-1}_\delta(W) \xrightarrow{i_*} H^{2n-1}_\delta(V) \rightarrow \]

\[ \xrightarrow{p_*} H^{2n-1}_\delta(K) \xrightarrow{\partial^r} H^{2n}_\delta(V) \rightarrow \cdots \]

Eq. (55) implies that \(H^{2n-1}_\delta(K) \simeq \mathbb{R}\) and \(\partial[F^n_{2n-1}] = [P_n]\). By Theorem \ref{thm:6.1} \(P_n\) is proportional to \(\delta \mathbb{A}_n(A, R)\). Hence \([P_n] = 0\) and \(\partial = 0\). Since \(H^{2n-2}_\delta(K) = 0\) and \(H^{2n-1}_\delta(W) \simeq \mathbb{R}\), we infer that \(H^{2n-1}_\delta(V) \simeq \mathbb{R}^2\). As a vector space the group \(H^{2n-1}_\delta(V)\) is generated by the \(\delta\)-cohomology classes \([C_n]\) and \([A^F]\).

Considering the other segments of the long exact sequences above, one can easily verify that \(H^m_d(W) = H^m_d(V)\) and \(H^{m-1}_d(W) = H^{m-1}_d(V)\) for all \(m \neq 2n\). The details are left to the reader. \(\square\)

Now we are in position to prove Theorem \ref{thm:7.1}. Consider the total complex \(\tilde{V} = \text{Tot}V\) of the bicomplex \(V\):

\[ \tilde{V}^n = \bigoplus_{p+q=n} V^{p,q}, \quad D = d + \delta : \tilde{V}^n \rightarrow \tilde{V}^{n+1}. \]

Let \(\mathcal{E} = \{E^p_q, d^r_{\cdot}\}\) and \(\mathcal{E}' = \{E'^p_q, d'^r_{\cdot}\}\) denote two spectral sequences associated to the first and second filtrations

\[ F'_p\tilde{V}^n = \bigoplus_{s \geq p} V^{s,n-s}, \quad F^m_p\tilde{V}^n = \bigoplus_{s \geq p} V^{n-s,s} \]

of the total complex \(\tilde{V}\). By definition, \(\mathcal{E}^p_q \simeq H^q_\delta(V^{p, \bullet})\) and \(\mathcal{E}'^p_q = H^p_d(V^{\bullet, q})\). Both the spectral sequences lie in the first quadrant and converge to the common limit \(H_D(V)\). By Lemma \ref{lem:7.2} the term \(\mathcal{E}^1\) is supported on the \(p\)-line and \(\mathcal{E}'^{2n-1,0} = 0\) for all \(m \in \mathbb{N}\). Hence \(\mathcal{E}^1 \simeq \mathcal{E}_2 \simeq \mathcal{E}_\infty \).
and $H^{2n}_D(\bar{V}) \simeq H^{2n}_d(V) = 0$. On the other hand, all but one of the nonzero entries of $\{E_{1,p,q}^0\}$ are centered on the diagonal $p = q$, as it is shown in Fig.2b. The only nonzero off-diagonal group is given by $E_{0,2n-1}^0 \simeq \mathbb{R}$. Consequently, $E_1 \simeq E_n$. The triviality of the group $H^{2n}_D(V) = 0$ implies that $d_n : E_{n,2n-1}^{0,2n-1} \to E_{n,n}^{n,n}$ is an isomorphism of one-dimensional vector spaces. Explicitly, one may readily see that the space $E_{n,2n-1}^{0,2n-1}$ is generated by the $\delta$-cocycle $A_n^F$ and the map $d_n$ takes this cocycle to the $\delta$-cocycle $(\binom{2n-1}{n-1}) C_n$, which generates $E_{n,n}^{n,n}$. The last fact amounts to the existence of a sequence of elements $c_k \in V^{k,2n-1-k}$ such that

\[
\begin{align*}
\delta c_n + dc_{n-1} &= (\binom{2n-1}{n-1}) C_n, \\
\delta c_{n-1} + dc_{n-2} &= 0, \\
&\quad \ldots \\
\delta c_1 + dA_n^F &= 0, \\
\delta A_n^F &= 0.
\end{align*}
\]

Multiplying the left and right hand sides of the upper line by $(\binom{2n-1}{n-1})^{-1}$, we get the statement of Theorem 7.1.

Here are the explicit expressions for the $d$-exact representatives of the first three classes $[C_1]$, $[C_2]$, and $[C_3]$:

\[
\begin{align*}
C_1 &= \frac{1}{2} \delta F_1^0 = d \left[ \text{Str}(a_0) + F_1^1 \right], \\
C_2 &= \delta \left[ \text{Str}(a_3a_0) + \frac{1}{4} F_2^1 \right] = d \left[ \text{Str}(a_0a_4 - a_0a_2) + \frac{1}{8} F_2^2 \right], \\
C_3 &= \frac{3}{4} \delta \left[ \text{Str}(a_3a_0a_4 + a_3a_4a_0 - a_3a_0a_2 - a_3a_2a_0) - \frac{1}{12} F_3^2 \right] \\
&= d \left[ \text{Str}(a_0a_4^2 - \frac{1}{2}a_0a_4a_2 - \frac{1}{2}a_0a_2a_4 + a_0a_2^2 + \frac{1}{2}a_3a_0a_4 \right. \\
&\quad + \frac{3}{8}a_3a_1a_0 + \frac{3}{8}a_1a_3a_0) - \frac{1}{48} F_3^3 \right].
\end{align*}
\]

Since all the Pontryagin characters $[P_{2m+1}]$ are known to be zero, the $d$-exact representatives exist for all classes $[C_{2m+1}]$.

8. Applications and Interpretations

8.1. Quantum anomalies. It is a common knowledge that the anomalies appearing in quantum field theory have a topological nature. In a wide sense, the term anomaly refers to breaking of a classical gauge symmetry upon quantization. As a practical matter, the anomalies manifest themselves as nontrivial BRST cocycles in ghost number 1 or 2 depending on which formalism, Lagrangian BV or Hamiltonian BFV, is used. These cocycles represent cohomological obstructions to the solvability of quantum master equations. Below we interpret the modular class $[A_0^1]$ of the BRST differential as the first obstruction to the existence of quantum master action. Examining the existence problem for the quantum BRST charge we encounter the universal cocycle $C_2$, whose association with anomalies, however, is more complicated.

Throughout this section, we assume that the reader is familiar with basics of the BRST theory. The general reference here is [4]. A comprehensive review of quantum anomalies and renormalization in BV formalism can be found in [36].
8.1.1. One-loop anomalies in the BV formalism. In the Batalin-Vilkovisky approach to the quantization of gauge systems, one usually deals with the odd cotangent bundle $\Pi^* M$ of an (infinite-dimensional) supermanifold $M$. The local coordinates $\{x^i\}$ on $M$ are called fields and the linear coordinates $\{x^*_i\}$ on the fibers of $\Pi^* M$ are called antifields. The total space of $\Pi^* M$, denoted below by $\mathcal{M}$, is endowed with the canonical antibracket\(^9\)

\[ (f, g) = (-1)^{\epsilon(x^i) + 1} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^*_i} - (-1)^{\epsilon(x^*_i + 1)} \frac{\partial f}{\partial x^*_i} \frac{\partial g}{\partial x^i} \]

for all $f, g \in C^\infty(\mathcal{M})$. Besides the Grassman parity, the structure sheaf of functions on $\mathcal{M}$ is endowed with an additional $\mathbb{Z}$-grading, called the ghost number, so that $\text{gh}(x^*_i) = -\text{gh}(x^i) - 1$. The presence of the extra $\mathbb{Z}$-grading is inessential, for our consideration and we will not mention it below.

The Feynman probability amplitude $e^{\hbar S}$ on the space of fields and antifields is defined by the quantum master action $S \in C^\infty(\mathcal{M}) \otimes \mathbb{C}[[\hbar]]$. The latter is given by a formal power series in $\hbar$ with smooth even coefficients and obeys the quantum master equation

\[ (S, S) = 2i\hbar \Delta S \quad \Leftrightarrow \quad \Delta e^{\hbar S} = 0. \tag{56} \]

Here $\Delta : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ is a second-order differential operator, called the odd Laplacian. It is defined in terms of a nowhere zero density $\rho$ on $\mathcal{M}$ by the rule

\[ \Delta f = \frac{1}{2} (-1)^{\epsilon(f)} \text{div}_\rho X_f, \]

where $X_f = (f, \cdot)$ is the Hamiltonian vector field associated to $f \in C^\infty(\mathcal{M})$. The density $\rho$ is supposed to be chosen in such a way that $\Delta^2 = 0$. For example, if $\sigma$ is a nowhere zero density on $M$, then $\rho = \sigma^2$ is an appropriate density on $\mathcal{M}$.

A fundamental property of the odd Laplacian is that it differentiates the antibracket:

\[ \Delta(f, g) = (\Delta f, g) + (-1)^{\epsilon(f) + 1} (f, \Delta g). \]

This relation is easily derived from the following one identifying the antibracket as the defect of the odd Laplacian to be a differentiation of the commutative algebra of functions:

\[ \Delta(f \cdot g) = \Delta f \cdot g + (-1)^{\epsilon(f)} (f, g) + (-1)^{\epsilon(f)} f \cdot \Delta g. \]

All the above properties of the antibracket and the odd Laplace operator allow one to look upon the quantum master equation (56) as a sort of Maurer-Cartan equation.

By definition, the quantum master action $S = S_0 + \hbar S_1 + \cdots$ can be regarded as a formal deformation of the classical one $S_0 \in C^\infty(\mathcal{M})$. Expanding (56) in powers of $\hbar$ yields the sequence of equations

\[ (S_0, S_0) = 0, \]

\[ (S_0, S_1) = i\Delta S_0, \]

\[ (S_0, S_n) = i\Delta S_{n-1} + \sum_{k=1}^{n-1} (S_k, S_{n-k}), \quad n \geq 2. \tag{57} \]

The first equation is known as a classical master equation. Its solution $S_0$ is completely determined (under some properness and regularity conditions) by the classical gauge theory and can be systematically constructed by means of the homological perturbation theory \[4\]. Given a

\[ \text{Another name is the odd Poisson bracket. Upon identification } C^\infty(\mathcal{M}) \text{ with the space of polyvector fields on } M, \text{ the antibracket becomes the standard Schouten-Nijenhuis bracket.} \]
classical master action \( S_0 \), the actual problem is to iterate the higher-order quantum corrections \( S_n \) satisfying (57).

The classical master equation ensures that the Hamiltonian action of \( S_0 \) defines a homological vector field \( Q = (S_0, \cdot) \) on \( M \), called a classical BRST differential. With the BRST differential Eqs. (57) take the cohomological form

\[
\delta S_n = B_n(S_0, \ldots, S_{n-1}).
\]

By induction on \( n \), one can see that the function \( B_n \) is \( \delta \)-closed, provided that \( S_0, \ldots, S_{n-1} \) satisfy the first \( n \)th equations (57). Thus the existence problem for the quantum master action appears to be equivalent to the vanishing of a certain sequence of the \( \delta \)-cohomology classes \([B_n]\) in which the \( n \)th cohomology class is defined provided that all the previous classes vanish. In particular, the second equation in (57) expresses the triviality of the modular class \([A_0]\), where \( A_0 = \Delta S_0 = \frac{1}{2} \text{div}_\rho Q \), This allows us to identify the modular class of the classical BRST differential with the first cohomological obstruction to the solvability of the quantum master equation.

8.1.2. Two-loop anomalies in the BFV formalism. The Hamiltonian analog of the BV field-antifield formalism is known as the BFV formalism. This time all the information about the gauge structure of a theory is encoded by the quantum BRST charge \( \Omega \). In the deformation quantization approach [37], one regards \( \Omega \) as an odd element of the associative algebra \((C^\infty(M) \otimes \mathbb{C}[[\hbar]], \ast)\), where \( M \) is a supermanifold endowed with a non-degenerate Poisson bracket \{\cdot, \cdot\} and the \( \ast \)-product on \( C^\infty(M) \otimes \mathbb{C}[[\hbar]] \) is defined as a local, \( \mathbb{C}[[\hbar]] \)-linear deformation in \( \hbar \) of the ordinary function multiplication satisfying the correspondence principle

\[
f \ast g - (-1)^{(\epsilon(f)+1)(\epsilon(g)+1)} g \ast f = i\hbar \{f, g\} + \mathcal{O}(\hbar^2) \quad \forall f, g \in C^\infty(M).
\]

(Again, we leave aside the ghost grading on \( M \).) By definition, the charge \( \Omega \) obeys the quantum master equation

\[
\Omega \ast \Omega = 0. \tag{58}
\]

It is well known [38] that all the inequivalent \( \ast \)-products on a given symplectic manifold \((M, \omega)\) are in one-to-one correspondence with the elements of the affine space \([\omega]/\hbar + H^2(M) \otimes \mathbb{C}[[\hbar]]\). The points of this space are called characteristic classes of the \( \ast \)-product and are denoted by \( \text{cl}(\ast) \). Below we set for definiteness \( \text{cl}(\ast) = [\omega]/\hbar \). The explicit recurrent formulas for Fedosov’s \( \ast \)-product on symplectic supermanifolds can be found in [39], [40]. One more fact about the deformation quantization we need below is that any \( \ast \)-product on a symplectic manifold is equivalent to one with the property

\[
f \ast_n g = (-1)^{(n+\epsilon(f)+1)(\epsilon(g)+1)} g \ast_n f,
\]

where \( \ast_n \) stands for the bidifferential operator determining the \( n \)-th order in the \( \hbar \)-expansion of \( \ast \).

Substituting the general expansion \( \Omega = \sum_{n \geq 0} \hbar^n \Omega_n \) in (58), we get a (possibly infinite) sequence of equations

\[
\{\Omega_0, \Omega_n\} = - \sum_{k+l+m=n+1 \atop l, m < n} \Omega_l \ast_k \Omega_m. \tag{59}
\]

The first equation of this sequence, \( \{\Omega_0, \Omega_0\} = 0 \), is called the classical master equation for the classical BRST charge \( \Omega_0 \). The charge \( \Omega_0 \) is completely determined by the first-class constraints of the original Hamiltonian theory and, similar to the classical master action, it can always be
constructed by means of the homological perturbation theory. The classical BRST differential \( \delta \) is defined now by the homological vector field \( \mathcal{Q} = \{ \Omega_0, \cdot \} \) on \( M \). Then the next equation in (59) takes the form

\[
\delta \Omega_1 = 0.
\]

It identifies the first quantum correction to the classical BRST charge with a BRST cocycle. Let \( \{ f, g \} = [\Pi, df \wedge dg] \), where the triangle brackets denote the natural pairing between bivectors and two-forms. Then the second-order correction \( \Omega_2 \) is defined by the equation

\[
\delta \Omega_2 = [\Pi, \frac{1}{18} \mathcal{C}_2 - \frac{1}{2} d\Omega_1 \wedge d\Omega_1],
\]

with \( \mathcal{C}_2 \) being here the second universal cocycle of the C-series associated to the classical BRST differential \( \mathcal{Q} \). If \( \Gamma \) are the local one-forms determining a symplectic connection \( \nabla \), then \( \mathcal{C}_2 = \text{Str}(\delta \Gamma \wedge \delta \Gamma) \).

Since the homological vector field \( \mathcal{Q} \) is Hamiltonian, the Poisson bivector is \( Q \)-invariant, \( \delta \Pi = 0 \), and the right hand side of (61) is obviously \( \delta \)-closed (but not \( \delta \)-exact in general). Thus the class \( [\mathcal{C}_2] \) can obstruct the solvability of the quantum master equation. More precisely, we have the following

**Proposition 8.1.** Let \( (M, \omega) \) be a symplectic supermanifold endowed with the Hamiltonian action of a classical BRST differential \( \delta \). If \( [\mathcal{C}_2] = 0 \), then there exists a \( * \)-product on \( M \) such that the quantum master equation (58) is solvable up to order three in \( \hbar \).

In case \( [\mathcal{C}_2] = 0 \) we can set \( \Omega_1 = 0 \) and take \( \Omega_2 = [\Pi, \delta^{-1} \mathcal{C}_2] \). It should be noted that the above analysis of low-order anomalies is not complete as we have restricted ourselves to a particular class of \( * \)-products. For a general \( * \)-product with \( \text{cl}(\cdot) = [\Pi^{-1}] / \hbar + [\omega_0] + \hbar[\omega_1] + \cdots \) the right hand sides of equations (60) and (61) will involve additional \( \delta \)-closed terms proportional to \( \omega_0 \) and \( \omega_1 \).

**8.2. Characteristic classes of foliations.** Given a regular foliation \( \mathcal{F} \) of an ordinary (even) manifold \( N \), denote by \( T\mathcal{F} \subset TN \) the subbundle of tangent spaces to the leaves of \( \mathcal{F} \). Since \( T\mathcal{F} \) is integrable, the inclusion map \( T\mathcal{F} \to TN \) defines a regular Lie algebroid over \( N \). Thus, there is a one-to-one correspondence between the categories of regular foliations and injective Lie algebroids. On the other hand, to any Lie algebroid \( E \to TN \) one can associate a homological vector field on \( \Pi E \) (see Example 1.2) together with the corresponding characteristic classes. When the Lie algebroid comes from a regular foliation, these characteristic classes can be attributed to the foliation itself. Furthermore, the construction of characteristic classes, being insensitive to the regularity of the Lie algebroid structure, can also be used to the study of singular foliations. The question of whether every singular foliation corresponds to the characteristic foliation of a Lie algebroid remains open. Below we give an example of regular foliation with nontrivial modular class.

Let \( SL(2, \mathbb{R}) \) denote the group of \( 2 \times 2 \)-matrices with real entries and determinant 1. It is well known that this group admits discrete subgroups \( \Gamma \) such that the right quotient space \( N = SL(2, \mathbb{R})/\Gamma \) is a compact manifold. Since \( SL(2, \mathbb{R}) \) has dimension three, so does \( N \).

Let \( \{ e_a \} \) be Weyl’s basis in the space of right invariant vector fields on \( SL(2, \mathbb{R}) \):

\[
[e_{-1}, e_{1}] = 2e_0, \quad [e_0, e_{1}] = e_1, \quad [e_0, e_{-1}] = -e_{-1}.
\]

The canonical projection \( \pi : SL(2, \mathbb{R}) \to N \) takes \( \{ e_a \} \) to the vector fields \( e_a = \pi_*(e_a) \) on \( N \) satisfying the same commutation relations. (The vector fields \( e_a \) and \( e_b \) are said to be \( \pi \)-related.) The pair \( \{ e_0, e_1 \} \) generates the action of the Borel subalgebra \( B \subset sl(2, \mathbb{R}) \) on \( N \). Hence, we have a two-dimensional foliation \( \mathcal{F} \) of \( M \). The inclusion map \( \rho : T\mathcal{F} \to TN \) defines
the transformation Lie algebroid $E = N \times B$ over $N$. If $x^i$ are local coordinates on $N$ and $c^a$ are odd coordinates on $\Pi B$, then the homological vector field on $N \times \Pi B$ reads

$$Q = c^0 e^i \frac{\partial}{\partial x^i} + c^1 e^i \frac{\partial}{\partial x^i} - c^0 c^1 \frac{\partial}{\partial c^1}. $$

Notice that the dual to the three-vector $e_{-1} \wedge e_0 \wedge e_1$ is a $SL(2, \mathbb{R})$-invariant volume form $\sigma$ on $N$. Therefore, $\text{div}_e e_0 = 0$. Taking $\rho = \sigma dc^0 dc^1$ to be a nowhere zero integration density on $N \times \Pi B$, we get the nontrivial $\delta$-cocycle

$$A_1^0 = \text{div}_\rho Q = c^0. \quad (62)$$

If $A_1^0$ were trivial there would be a function $f \in C^\infty(N)$ such that $e^0 f = 1$. But every function on a compact manifold has a critical point $p$ at which $(e^0 f)(p) = 0$. Thus the modular class of the foliation $\mathcal{F}$ is nontrivial.

It is interesting to note that the foliation $\mathcal{F}$ is the one that was originally used by Roussarie to demonstrate the nontriviality of the Godbillon-Vey class [11], [14]. It is not an accident that both the modular and Godbillon-Vey classes of $\mathcal{F}$ are nonzero. One can see that the modular class (62) coincides in fact with the Reeb class of $\mathcal{F}$. The latter takes value in $H^1_\mathcal{F}(N)$, the first group of the leafwise cohomology. It is well known that the vanishing of the Reeb class results in the vanishing of the Godbillon-Vey class but not vice versa in general. Thus, we have the implication $GV[\mathcal{F}] \neq 0 \Rightarrow A_1^0 [\mathcal{F}] \neq 0$. A detailed discussion of the relationship between the modular and Reeb classes of regular Poisson manifolds can found in [42].

8.3. Lie algebras. Let $\mathcal{G}$ be a Lie algebra with a basis $\{t_a\}$ and the commutation relations

$$[t_a, t_b] = f^d_{ab} t_d. $$

Then the homological vector field on $\Pi \mathcal{G}$ is given by

$$Q = \frac{1}{2} c^b c^a f^d_{ab} \frac{\partial}{\partial c^d}. \quad (63)$$

The linear space of functions on $\Pi \mathcal{G}$ endowed with the differential $\delta$ gives us a model for the Chevalley-Eilenberg complex of the Lie algebra $\mathcal{G}$. With a flat connection on $\Pi \mathcal{G}$ we see that the characteristic classes of $A$-series (38) are nothing but the primitive elements of the Lie algebra cohomology:

$$A_n = \text{tr}(\text{ad}_{a_1} \cdots \text{ad}_{a_{2n-1}})c^{a_1} \cdots c^{a_{2n-1}} \quad \forall n \in \mathbb{N}, $$

with $\text{ad}_a = (f^d_{ab})$ being the matrices of the adjoint representation of $\mathcal{G}$.

The universal cocycles of $B$- and $C$-series are then identified with $\text{Ad}$-invariant tensors on $\Pi \mathcal{G}$:

$$B_n = (\text{ad}_{a_1} \cdots \text{ad}_{a_n})^b_{a_{n+1}} dc^{a_1} \otimes \cdots \otimes dc^{a_n} \otimes dc^{a_{n+1}} \otimes \frac{\partial}{\partial c^b}, $$

$$C_n = \text{tr}(\text{ad}_{a_1} \cdots \text{ad}_{a_n}) dc^{a_1} \otimes \cdots \otimes dc^{a_n} \quad \forall n \in \mathbb{N}. $$

Since any coboundary of (63) is necessarily proportional to $c^a$, the tensor cocycles are either zero or nontrivial. In case $\mathcal{G}$ is semi-simple, for instance, the one-form $C_1$ is zero, while the two-form $C_2$ is non-degenerate (the Killing metric).
Appendix A. The exponential map

In this Appendix, we reformulate the chain property (17) of the characteristic map \( \hat{\varphi} : TM \to E \) as the integrability condition for some homological vector field on \( TM \). As a by-product, we get an explicit expression for the action of \( \delta \) on the basis covariants \( \{ \partial^a Q \} \) of a flat \( Q \)-manifold.

Given a smooth manifold \( M \), we can treat the total space of the tangent bundle \( TM \) as a “partially formal” manifold, that is, formal in the directions of fibers. Concerning the general theory of partially formal supermanifolds we refer the reader to \([3, 43]\). Let \( \mathcal{F} \) denote the commutative algebra of formal functions on \( TM \); the elements of \( \mathcal{F} \) are formal power series in fiber coordinates with smooth coefficients:

\[
\mathcal{F} \ni f(x, y) = \sum_{n=0}^{\infty} y^{i_1} \cdots y^{i_n} f_{i_1 \cdots i_n}(x).
\]

The expansion coefficients \( f_{i_1 \cdots i_n}(x) \) are covariant symmetric tensor fields on \( M \). We define the formal vector fields on \( TM \) to be the derivations of the commutative algebra \( \mathcal{F} \) and denote the Lie algebra of all the derivations by \( \text{Der}(\mathcal{F}) \).

The natural inclusion \( i : C^\infty(M) \to \mathcal{F} \) identifies \( C^\infty(M) \) with the subalgebra of \( y \)-independent functions of \( \mathcal{F} \). Suppose \( M \) admits an affine connection \( \partial \) with zero torsion and curvature. Then we can define one more homomorphism \( \phi : C^\infty(M) \to \mathcal{F} \) known as the exponential map. The exponential map takes a smooth function \( f \in C^\infty(M) \) to the formal function

\[
\phi(f) = f(x + y) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} y^{i_1} \cdots y^{i_n} \partial_{i_1} \cdots \partial_{i_n} f(x).
\]

It is clear that \( \phi(f \cdot g) = \phi(f) \cdot \phi(g) \).

Associated to the exponential map of functions is the exponential map of vector fields

\[
\phi' : \mathfrak{X}(M) \to \text{Der}(\mathcal{F}).
\]

This is defined as follows. Consider \( C^\infty(M) \) and \( \mathcal{F} \) as left modules over the Lie algebras \( \mathfrak{X}(M) \) and \( \text{Der}(\mathcal{F}) \), respectively. Then for any vector field \( X \in \mathfrak{X}(M) \) there is a unique formal vector field \( \tilde{X} = \phi'(X) \in \text{Der}(\mathcal{F}) \) such that the following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\hat{\varphi}} & \mathcal{F} \\
i & & \downarrow i
\end{array} \quad \begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{F} \\
C^\infty(M) \xrightarrow{X} C^\infty(M) & & C^\infty(M) \xrightarrow{X} C^\infty(M)
\end{array}
\]

In terms of local coordinates adapted to \( \partial \) we have

\[
\tilde{X} = X^i(x) \left( \frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i} \right) + X^i(x + y) \frac{\partial}{\partial y^i} \\
= X^i(x) \frac{\partial}{\partial x^i} + \sum_{n=1}^{\infty} \frac{1}{n!} y^{i_1} \cdots y^{i_n} \partial_{i_1} \cdots \partial_{i_n} X^i(x) \frac{\partial}{\partial y^i}.
\]

The pair \( (\phi', \phi) \) defines a homomorphism of the \( \mathfrak{X}(M) \)-module \( C^\infty(M) \) to the \( \text{Der}(\mathcal{F}) \)-module \( \mathcal{F} \):

\[
\phi'([X, Y]) = [\phi'(X), \phi'(Y)], \quad \phi'(X)\phi(f) = \phi(X f).
\]
Now let us apply the exponential map (64) to a homological vector field $Q$ on $M$. The result is a formal homological vector field $\tilde{Q}$ on $TM$. The latter can be expanded in the sum of homogeneous components

$$\tilde{Q} = \sum_{n=0}^{\infty} \tilde{Q}_n, \quad [N, \tilde{Q}_n] = n\tilde{Q}_n,$$

where $N = y^i\partial/\partial y^i$ is a vertical vector field on $TM$ called sometimes the Euler vector field. The integrability of $\tilde{Q}$ implies that

$$2[\tilde{Q}_0, \tilde{Q}_m] = -\sum_{k=1}^{m-1} [\tilde{Q}_{m-k}, \tilde{Q}_k] \quad \forall m \in \mathbb{N}. \quad (65)$$

In particular,

$$\tilde{Q}_0 = Q^i \frac{\partial}{\partial x^i} + y^j \frac{\partial Q^i}{\partial y^j}$$

is a smooth homological vector field defining a $Q$-structure on the tangent bundle $TM$. The other homogeneous components $\{\tilde{Q}_n\}_{n=1}^{\infty}$ of $\tilde{Q}$ are vertical vector fields, which are naturally identified with the elementary local covariants of the flat $Q$-manifold $M$. Upon this identification Eq. (65) compactly expresses the action of $\delta$ on the elementary local covariants $\{\partial^n Q\}$ with $n > 1$. Namely,

$$\delta \tilde{Q}_m = \sum_{k=1}^{m-1} [\tilde{Q}_{m-k}, \tilde{Q}_k] \quad \forall m \in \mathbb{N}. \quad (66)$$

For the remaining two generators – the homological vector field $Q$ itself and its first covariant derivative $\Lambda := \partial Q \in \mathfrak{A}(M)$ – we have

$$\delta Q = 0, \quad \delta \Lambda = \Lambda^2. \quad (67)$$

As is seen, Eq. (66) coincides exactly with Eq. (18) characterizing the Gauss map (12) as a morphism of $Q$-manifolds.

**Appendix B. Scalar characteristic classes**

Let $M$ be a $Q$-manifold endowed with a symmetric affine connection $\nabla$ and let $\mathfrak{A}$ be the differential tensor algebra of the local covariants associated to $Q$ and $\nabla$. As a tensor algebra, $\mathfrak{A}$ is generated by the repeated covariant derivatives of the homological vector field and the curvature tensor. A suitable generating set of $\mathfrak{A}$ is given by the following elementary covariants:

$$Q^i, \quad Q^i_j = \nabla_i Q^j,$$

$$Q^i_{j_1\cdots j_{n+2}} = \nabla_{(j_1} \cdots \nabla_{i_{n+2})} Q^j - \nabla_{(j_1} \cdots \nabla_{i_n R^j_{i_{n+1}i_{n+2})k}} Q^k,$$

$$R^i_{j_1\cdots i_{n+3}} = \nabla_{i_1} \cdots \nabla_{i_n R^j_{i_{n+1}i_{n+2}i_{n+3}}}. \quad (68)$$

The round brackets denote symmetrization of the enclosed indices. Notice that the generators are not free and satisfy an infinite set of tensor relations coming from the integrability condition for the homological vector field, the Bianchi identities for the curvature tensor, and all their differential consequences. Of particular importance for our analysis will be the following identities:

$$\nabla_Q Q = 0, \quad \nabla_Q \Lambda = -\Lambda^2 + \frac{1}{2} R_{QQ}, \quad (69)$$
\[ R_{QX}(Q) = -(-1)^{\varepsilon(X)e(Q)} R_{XQ}(Q) = \frac{1}{2} R_{QQ}(X), \]

\[ R_{QQ}(Q) = 0 , \quad \nabla_Q R_{QQ} = 0 . \]

The action of the differential \( \delta = L_Q \) is given by
\[
\delta Q^j = 0 ,
\]
\[
\delta Q^j_i = Q^j_k Q^i_k + \frac{1}{2} R^j_{QQ} i ,
\]
\[
\delta Q^j_{i_1 \cdots i_{n+2}} = - \sum_{l=2}^{n+1} \binom{n+2}{l} Q^m_{i_l \cdots i_l} Q^j_{i_1 \cdots i_{n+3}} ,
\]
\[
\delta R^j_{i_1 \cdots i_{n+3}} = Q^i R^j_{i_1 \cdots i_{n+3}} - R^j_{i_1 \cdots i_{n+3}} Q^j_i + \sum_{l=1}^{n+3} (-1)^{\varepsilon_l + \cdots + \varepsilon_{l-1}} Q^j_{i_l} R^j_{i_1 \cdots i_{n+3}} .
\]

As is seen the generators in the second and third lines of (68), taken separately, generate two differential ideals of \( \mathcal{A} \), which we denote respectively by \( \mathcal{Q} \) and \( \mathcal{R} \). The quotient \( \mathcal{A} / \mathcal{Q} \) is naturally isomorphic to the differential subalgebra \( \mathcal{Q}' \subset \mathcal{A} \) generated by \( Q^j \), \( \mathcal{Q}^j \), and \( \{ R^j_{i_1 \cdots i_{n+3}} \} \), so that the complex \( \mathcal{A} \) breaks up into the direct sum of subcomplexes
\[ \mathcal{A} = \mathcal{Q} \oplus \mathcal{Q}' . \]

Notice that \( B_n, C_n \in \mathcal{Q} \), while \( P_n \in \mathcal{Q}' \). Yet another decomposition is due to the tensor type of cochains the complex \( \mathcal{A} \) consists of:
\[ \mathcal{A} = \bigoplus_{n,m} \mathcal{A}^{n,m} , \quad \mathcal{A}^{n,m} = \mathcal{A} \cap T^{n,m}(M) . \]

The elements of \( \mathcal{A} \) can also be depicted graphically as directed, decorated graphs with “black” and “white” vertices. The graphs are constructed by gluing together incoming and outgoing legs of the corollas
\[ Q^j_{i_1 \cdots i_n} = \quad \text{and} \quad R^j_{i_1 \cdots i_{n+3}} = \]
by the general rules discussed in Sec. 5. The planarity of the corollas allows us to order the incoming legs by reading them anticlockwise with respect to the vertex so that the first incoming leg appears to be the nearest one to the outgoing leg from the left. This order, however, is only crucial for the correspondence between the white corollas and the \( R \)-generators, as the \( Q \)-generators are fully symmetric in the lower indices.

Let us now clarify the structure of the differential subalgebra \( \mathcal{A}^{0,0} \subset \mathcal{A} \). Denote by \( \bar{\mathcal{A}}^{0,0} = \mathcal{A}^{0,0} / (\mathcal{A}^{0,0})^2 \) the subspace of indecomposable elements of \( \mathcal{A}^{0,0} \). Then \( \mathcal{A}^{0,0} = \bigoplus_{k \in \mathbb{N}} (\bar{\mathcal{A}}^{0,0})^k \) and we have the following

**Proposition B.1.** In the stable range of dimensions, the complex \( \bar{\mathcal{A}}^{0,0} \) is isomorphic to the complex \( \bar{\mathcal{W}} \) of cyclic words from the proof of Theorem B.7.

**Proof.** The elements of \( \mathcal{A}^{0,0} \) correspond to linear combinations of connected graphs without legs. If \( \Gamma \) is such a graph, then it has the same number of vertices and edges. Therefore, \( \Gamma \) contains exactly one cycle. The arrows of this cycle are all directed into one side, and the remaining edges are directed towards the cycle, forming trees growing from the cyclic vertices
(possibly several from a single vertex). The “crown” of each tree is made of univalent black vertices that may join either a cyclic or a non-cyclic vertex. In the latter case $\Gamma$ contains one of the following subgraphs:

\begin{align*}
\text{and}
\end{align*}

But in view of the symmetry of the $Q$-generators in lower indices and the deferential consequences of the identities (69), (70) all these subgraphs correspond to zero elements of $\mathcal{A}$. Thus the non-vanishing elements of $\mathcal{A}^{0,0}$ are represented by graphs having no non-cyclic vertices other than univalent black vertices. The cyclic vertices can also produce vanishing subgraphs if the algebraic identities (69), (70) are allowed for. Relations (69) and the symmetry of $Q$-type generators force us to set

\begin{align*}
\frac{1}{2} - 2 - 1 + \ldots = 0 .
\end{align*}

Further, taking the iterated covariant derivatives of relations (70), we arrive at the following graph equalities:

\begin{align*}
\ldots = - \ldots = \frac{1}{2} \ldots = 0 , \\
\ldots = 0 .
\end{align*}

All the relations above severely restrict the possible form of a cyclic vertex, leaving in fact only two nontrivial options:

\begin{align*}
\ldots , \\
\ldots .
\end{align*}

These vertices correspond to the tensors $\Lambda$ and $R = R_{QQ}$, which are already algebraically independent in the stable range of dimensions. Identifying these tensors with the generators of the cyclic space (48), we get the desired isomorphism between the complexes $\mathcal{A}^{0,0}$ and $\mathcal{W}$. \hfill \Box

As immediate corollaries from the proposition above we have

**Corollary 3.** The complex of scalar covariants $\mathcal{A}^{0,0}$ is acyclic.

Indeed, from the proof of Theorem 6.1 we know that the complex $\mathcal{W} \simeq \mathcal{A}^{0,0}$ is acyclic. Applying the Künneth formula to $\mathcal{A}^{0,0} = \bigoplus_{k \in \mathbb{N}} (\mathcal{A}^{0,0})^{\otimes k}$ yields the statement.

**Corollary 4.** The functions $P_n$ are nontrivial $\delta$-cocycles of the subcomplex $\mathcal{R} \subset \mathcal{A}$.

Suppose the statement were false. Then we could find a function $f_n \in \mathcal{R}$ such that $\delta f_n = P_n$. By Theorem 6.1 the function

\begin{align*}
\Lambda_n(\Lambda, R) - (2^{n-1}) f_n
\end{align*}

would be then a nontrivial intrinsic $\delta$-cocycle, which contradicts acyclicity of $\mathcal{A}^{0,0}$. 
The proof of Proposition 6.1. Due to the fundamental classification theorem for smooth supermanifolds [44], we can identify $\mathcal{M}$ with the total space of an odd vector bundle $\pi : E \to M_0$, where $M_0$ is the body of $M$. Consider the triple $(g^0, g^1, \nabla^1)$ consisting of a Riemannian metric $g^0$ on $M_0$, a Euclidean metric $g^1$ on $\Pi E$, and a connection $\nabla^1$ on $E$. Without loss in generality we can assume $\nabla^1$ to be compatible with $g^1$ (otherwise replace $\nabla^1$ by $\nabla^1 - (1/2)(g^1)^{-1}\nabla^1 g^1$). Notice also that the metric $g^1$ on $\Pi E$ defines and is defined by a fiberwise symplectic structure on $E$.

Let $\{x^i\}$ be a coordinate system in a trivializing chart $U \subset M_0$ and let $\{\theta^a\}$ be odd coordinates dual to some frame $\{e_a\}$ in $E|_U$. If $\nabla^1(e_a) = dx^i\Gamma^b_{ia}e_b$, then the local vector fields

$$v_a = \frac{\partial}{\partial \theta^a} \quad \text{and} \quad h_i = \frac{\partial}{\partial x^i} - \theta^a\Gamma^b_{ia}\frac{\partial}{\partial \theta^b},$$

span, respectively, the subspaces of vertical and horizontal vector fields of $T(E|_U)$. Define the affine connection $\tilde{\nabla}$ on the total space of $E|_U$ by setting

$$\tilde{\nabla}_h h_j = \Gamma^k_{ij} h_k, \quad \tilde{\nabla}_h v_a = \Gamma^b_{ia} v_b, \quad \tilde{\nabla}_v_a h_i = 0, \quad \tilde{\nabla}_v_a v_b = 0,$$

where $\Gamma^k_{ij}$ are the Christoffel symbols of a unique symmetric connection $\nabla^0$ compatible with the metric $g^0$. Relations (72) are obviously form invariant under the coordinate changes

$$x'^i = x^i, \quad \theta'^a = \theta^a(x)\theta^b,$$

on all nonempty intersections $E|_U \cap E|_{U'}$; hence $\tilde{\nabla}$ is a well-defined affine connection on the whole $M$. The connection $\tilde{\nabla}$ is not symmetric unless $\nabla^1$ is flat. The nonzero components of the torsion tensor $T$ are given by

$$T_{h_i h_j} = \tilde{\nabla}_h h_j - \tilde{\nabla}_h j_i - [h_i, h_j] = \theta^a R^b_{ija} v_b,$$

where $\{R^b_{ija}\}$ is the curvature tensor of $\nabla^1$. Subtracting torsion from $\tilde{\nabla}$ yields a symmetric connection $\nabla = \tilde{\nabla} - (1/2)T$ on $M$.

In the frame (71), the nonzero components of the curvature tensor of $\nabla$ are collected to the following supermatrices:

$$R_{h_i h_j} = \left( \begin{array}{c|c} R^i_{ijk} & \frac{1}{2}\theta^a \nabla_k R^b_{ija} \\ \hline \hline 0 & R^b_{ija} \end{array} \right), \quad R_{v_a h_i} = \left( \begin{array}{c} 0 \\ R^b_{ija} \end{array} \right),$$

where $\nabla = \nabla^0 \oplus \nabla^1$ is the connection on $TM_0 \oplus E$ and $\{R^i_{ijk}\}$ is the curvature tensor of $\nabla^0$.

Since the supermatrices have the block-triangular form, we readily get

$$P^\nabla_{2m+1} = \pi_*(P^\nabla^0_{2m+1} - P^\nabla^1_{2m+1}) = 0.$$

Here we used the definition of the supertrace and the fact that both $\nabla^0$ and $\nabla^1$ are metric connections. Thus, $\nabla$ is a desired connection.

The proof of Proposition 6.2. Let $\nabla$ and $\tilde{\nabla}$ be two metric connections associated to the triples $(g^0, g^1, \nabla^1)$ and $(g^0, \tilde{g}^1, \tilde{\nabla}^1)$. Define the triple $(g^0_t, g^1_t, \nabla^1_t)$, where

$$g^0_t = (1-t)g^0 + t\tilde{g}^0, \quad g^1_t = (1-t)g^1 + t\tilde{g}^1,$$

and

$$\nabla^1_t = (1-t)\nabla^1 + t\tilde{\nabla}^1 - \frac{1}{2}[(1-t)\nabla^1 + t\tilde{\nabla}^1]g^1_t.$$
is a one-parameter family of connections interpolating between $\nabla^1$ and $\nabla^1$ as $t$ runs the interval $[0, 1]$. By definition, $\dot{\nabla}^1 g^t_1 = 0$. Denote by $\dot{\nabla}^0$ the symmetric connection compatible with $g^0_t$, and let $\nabla$ be the symmetric connection associated to the data $(g^0_t, g^1_t, \nabla^1)$.

The rest of the proof runs in much the same way as the proof of Theorem 2.1. Namely, we introduce the product manifold $\tilde{M} = M \times \mathbb{R}^{1|1}$ endowed with the homological vector field $\tilde{Q} = Q + \theta \partial_t$ and the connection $\tilde{\nabla} = \dot{\nabla} \oplus \nabla'$, where $\nabla'$ is the standard flat connection on $\mathbb{R}^{1|1}$. The matrices $\tilde{\Lambda} = \tilde{\nabla} \tilde{Q}$ and $\tilde{R} = [\tilde{\nabla}_Q, \tilde{\nabla}_{\tilde{Q}}]$ have the form

$$\tilde{\Lambda} = \begin{pmatrix} \Lambda_t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} R_t + \theta \Psi_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\Lambda_t = \dot{\nabla} Q, \quad \Psi_t = \partial_t (\dot{\nabla} Q), \quad R_t = [\dot{\nabla} Q, \dot{\nabla} Q].$$

Taking into account the identities

$$\partial_t R_t = \dot{\nabla} Q \Psi_t, \quad \dot{\nabla} Q R_t = 0, \quad P^\dot{\nabla}_{2m+1}(Q) = 0,$$

one can easily check that

$$P^\dot{\nabla}_{2m+1}(\tilde{Q}) = \tilde{Q} F_{2m+1}, \quad F_{2m+1} = (2m+1) \int_0^t \text{Str}((R_s)^{2m} \Psi_s) ds.$$

Similar to the curvature (73) the supermatrix $\Psi_t$ has the block-triangular form

$$\Psi_t = \begin{pmatrix} \Psi^0_t \\ * \end{pmatrix}, \quad \Psi^a_t = \partial_t (\dot{\nabla}^a Q), \quad a = 0, 1.$$

This allows us to rewrite the function $F_{2m+1}$ as

$$F_{2m+1} = (2m+1) \int_0^t [\text{tr}((R_s)^{2m} \Psi^0_s) - \text{tr}((R_s^{1})^{2m} \Psi^1_s)] ds,$$

where $R^a_t = [\dot{\nabla}^a Q, \dot{\nabla}^a Q]$. Denoting $I^a_t = (g^a_t)^{-1} \partial_t g^a_t$ and using the obvious identities

$$(g^a_t)^{-1} \partial_t g^a_t = -\dot{\nabla}^a Q I^a_t, \quad (g^a_t)^{-1} R^a_t g^a_t = -R^a_t, \quad \dot{\nabla}^a Q R^a_t = 0,$$

(the first one is obtained by differentiating the identity $\dot{\nabla}^a Q g^a_t = 0$), we get

$$F_{2m+1} = \int_0^t U_{2m+1}(s) ds, \quad U_{2m+1}(t) = (2m+1) \left[\text{tr}((R^0_t)^{2m} I^0_t) - \text{tr}((R^1_t)^{2m} I^1_t)\right].$$

Now substituting $\tilde{Q}$ and $\tilde{\nabla}$ to the general formula (49), we obtain the following $\tilde{Q}$-invariant function on $\tilde{M}$:

$$\tilde{\Lambda}_{2m+1}^F = A_{2m+1}(\tilde{\Lambda}, \tilde{R}) - (4m+1)F_{2m+1}$$

$$= A_{2m+1}(\Lambda_t, R_t + \theta \Psi_t) - (4m+1)F_{2m+1}$$

$$= A_{2m+1}(\Lambda_t, R_t) - (4m+1)F_{2m+1} + \theta W_{2m+1}.$$

In view of equation (74), the identity $\tilde{Q} \tilde{\Lambda}_{2m+1}^F = 0$ amounts to
$QA_{2m+1}(A_t,R_t) = 0,$

$\partial A_{2m+1}(A_t,R_t) = Q \left[ W_{2m+1} + \frac{(4m+1)}{2m+1} U_{2m+1} \right].$

Integrating the last equality with respect to $t$ from 0 to 1, we conclude that $\delta$-cocycles $A_{2m+1}(A_0,R_0)$ and $A_{2m+1}(A_1,R_1)$ associated to the metric connections $\nabla$ and $\tilde{\nabla}$ are cohomologous, and the proof is complete.

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