Spin dependent observable effect for free particles using the arrival time distribution

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The mean arrival time of free particles is computed using the quantum probability current. This is uniquely determined in the non-relativistic limit of Dirac equation, although the Schroedinger probability current has an inherent non-uniqueness. Since the Dirac probability current involves a spin-dependent term, an arrival time distribution based on the probability current shows an observable spin-dependent effect, even for free particles. This arises essentially from relativistic quantum dynamics, but persists even in the non-relativistic regime.

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I. INTRODUCTION

The treatment of time in quantum mechanics is a much debated question. A testimony to this is the proliferation of recent papers on the problems of tunneling time, decay time, dwell time and the arrival time. In this paper we are specifically concerned with the issue of arrival time.

In classical mechanics, a particle follows a definite trajectory; hence the time at which a particle reaches a given location is a well defined concept. On the other hand, in standard quantum mechanics, the meaning of arrival time is rather problematic. Indeed, there exists an extensive literature on the treatment of arrival time distribution in quantum mechanics. A straightforward procedure would be to try to construct a self-adjoint operator for the arrival time in quantum mechanics which is conjugate to the Hamiltonian, but then it is seen that the operator does not have a basis of orthogonal eigenstates.

Using the Born interpretation, \(|\psi(x, t_1)|^2, |\psi(x, t_2)|^2...\) give the position probability distributions at different instants \(t_1, t_2,...\) Now, the question posed is that if we fix the positions at \(x=X_1, X_2,...\), can the functions \(|\psi(X_1, t)|^2, |\psi(X_2, t)|^2...\) give the time probability distributions at different locations \(X_1, X_2,...?\) Note that if at any instant \(t = t_i\), \(\int_{t_i}^{\infty} |\psi(x, t = t_i)|^2 d^2x = 1\), the probability of finding the particle anywhere at that instant is unity. But if we fix the position at, say, \(x = X_1\) and \(t\) is varied, the value of the integral \(\int_{0}^{\infty} |\psi(x = X_1, t)|^2 dt\) is not equal to 1. In this case what may be pictured is that at a given point, say, \(X_1\) the relevant probability changes with time and this change of probability is governed by the following continuity equation which suggests a “flow of probability”

\[
\frac{\partial}{\partial t} |\psi(x, t)|^2 + \nabla \cdot \mathbf{J}(x, t) = 0 \tag{1}
\]

where \(\mathbf{J}(x, t) = \frac{\hbar}{2mi} (\psi \nabla \psi^* - \psi^* \nabla \psi)\) is the probability current density.

Different approaches for analysing the problem of arrival time distribution have been suggested using the path integrals and positive-operator-valued measures. Delgado and Muga proposed an interesting approach by constructing a self-adjoint operator having dimensions of time which is relevant to the arrival time distribution, but then its conjugate Hamiltonian has an unbounded spectrum. The implications of this approach have been studied in detail by Delgado.

In this paper we take recourse to the definition of arrival time distribution in terms of the quantum probability current density \(\mathbf{J}(x = X, t)\). Interpreting the equation of continuity in terms of the physical probability flow, the Born interpretation for the modulus of the wave function and its time derivative seems to imply that the mean arrival time of the particles reaching a detector located at \(X\) is given by

\[
\bar{\tau} = \frac{\int_{0}^{\infty} |\mathbf{J}(x = X, t)| dt}{\int_{0}^{\infty} |\mathbf{J}(x = X, t)| dt} \tag{2}
\]

Here, it should be emphasized that the definition of the mean arrival time used in Eq. (2) is not a unique result within standard quantum mechanics. We also note that \(\mathbf{J}(x, t)\) can be negative, hence one needs to take the modulus sign in order to use the above definition. However, the Bohmian model of quantum mechanics in terms of the causal trajectories of individual particles leads to the above expression for the mean arrival time in a rather conceptually elegant way.

The quantum probability current interpreted as the streamlines of a conserved flux has been used for studying the tunneling times of Dirac electrons. However, it
is easily seen that in non-relativistic quantum mechanics the form of the probability current density is not unique, a point which has already been discussed by a number of authors [10–12]. If we replace J by J’ in Eq.(1) where J’=J+δJ, with ∇·δJ=0, J’ satisfies the same probability conservation given by Eq.(1). Then this new current density J’ can lead to a different distribution function for the arrival time [12]. Hence the question arises how one can uniquely fix the arrival time distribution via the quantum probability current in the regime of non-relativistic quantum mechanics?

In order to address the above question, we take a vital clue from the interesting result Holland [13] had shown in the context of analysing the uniqueness of the Bohmian model of quantum mechanics, viz. that the Dirac equation implies a unique expression for the probability current density for spin 1/2 particles in the non-relativistic regime. In Section II we highlight the feature that the uniqueness of the probability current density is a generic consequence of any relativistic equation of quantum dynamics. In Section III, the particular case of spin dependent probability current density as derived from the Dirac equation is discussed in detail. Subsequently, in Section IV, using the non-relativistic limit of the Dirac current density, we compute the effect of spin on the arrival time distribution of free particles for an initial Gaussian wave packet. Such a line of investigation has not been explored sufficiently; to the best of our knowledge, it seems that only Leavens [14] has studied this specifically in terms of the Bohmian causal model of spin-1/2 particles.

II. UNIQUENESS OF THE PROBABILITY CURRENT DENSITY FOR ANY RELATIVISTIC WAVE EQUATION

The current obtained from any consistent relativistic quantum wave equation will have to satisfy the covariant form of the continuity equation ∂j/∂t=0, where the zeroth component of jμ will be associated with the probability density. Now, let us replace jμ by \( \tilde{j}^\mu \) which should again be conserved, i.e. ∂μ\( \tilde{j}^\mu \)=0, where \( \tilde{j}^0 = j^\mu + a^\mu \), \( \tilde{a}^\mu \) being an arbitrary 4-vector. But then the zeroth component \( \tilde{j}^0 \) will have to reproduce the same probability density \( j^0 \), and hence \( a^0=0 \). This current as seen from another Lorentz frame is \( j'^\mu = j^\mu + a'^\mu \). Then in this frame \( j'^0 = 0 \), and again from the previous argument \( a'^0=0 \). But we know that the only 4-vector whose fourth component vanishes in all frames is the null vector. Hence \( a^\mu=0 \). Thus, for any consistent relativistic quantum wave equation, satisfying the covariant form of the continuity equation, the relativistic current is uniquely fixed. Unique expressions for the conserved currents have been explicitly derived by Holland [15] for the Dirac equation, the Klein-Gordon equation, and also for the coupled Maxwell-Dirac equations.

Now, an interesting point is that this uniqueness will be preserved in the non-relativistic regime. Hence, given any relativistic wave equation, one can calculate the unique form of the current which can be applied in the non-relativistic regime. Thus using the (normalized) modulus of the probability current density as the arrival time distribution, if one calculates the mean arrival time, it can be used to empirically test any consistent relativistic wave equation such as the relativistic Kemmer equation [16] for the massive spin 0 and spin 1 bosons. Of late, the unique form of the probability current density expressions has been derived in the non-relativistic limit of the relativistic Kemmer equation for spin-0 and spin-1 particles [17]. This general scheme for testing relativistic quantum wave equation in terms of the arrival time distribution is not contingent on any specific form of the relativistic wave equation. However, in the following detailed study we specifically use the Dirac equation for spin-1/2 particles.

III. SPIN DEPENDENT EFFECT ON THE ARRIVAL TIME DISTRIBUTION USING DIRAC EQUATION

The Dirac equation for a free particle is

\[
i\hbar \frac{\partial \psi}{\partial t} = \left( \frac{\hbar c}{i} \sigma \cdot \frac{\partial}{\partial x^i} + \beta m_0 c^2 \right) \psi \tag{3}
\]

\[\alpha_i = \left( \begin{array}{cc} 0 & \sigma_i \\ \sigma_i & 0 \end{array} \right), \beta = \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right), \psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)\]

\(\psi\) is a four component column matrix and \(\sigma_i\) are the Pauli matrices. Choosing a representation where \(\psi_1\) and \(\psi_2\) are two component spinors, one gets two coupled equations

\[
\frac{\partial \psi_1}{\partial t} = -c \sigma_i \frac{\partial \psi_2}{\partial x^i} - \frac{i m_0 c^2}{\hbar} \psi_1 \tag{4}
\]

\[
\frac{\partial \psi_2}{\partial t} = -c \sigma_i \frac{\partial \psi_1}{\partial x^i} + \frac{i m_0 c^2}{\hbar} \psi_2 \tag{5}
\]

Combining Eqs.(4) and (5) one gets

\[
\frac{\partial}{\partial t} (\psi_1^\dagger \psi_1) = -c \psi_1^\dagger \sigma_i \frac{\partial \psi_2}{\partial x^i} - c \frac{\partial \psi_2^\dagger \sigma_i \psi_1}{\partial x^i} \tag{6}
\]

For positive energies, one can take \(\psi_2 \propto \exp(-iEt/\hbar)\). In the non-relativistic limit, \(E\) is the rest mass energy and \(E + m_0 c^2 = (m + m_0)c^2 \approx 2m_0c^2\). Thus using Eq.(5) one can write

\[
\psi_2 = -\frac{i \hbar}{E + m_0 c^2} \sigma_i \frac{\partial \psi_1}{\partial x^i} = -\frac{i \hbar}{2m_0 c} \sigma_i \frac{\partial \psi_1}{\partial x^i} \tag{7}
\]
Putting this value of $\psi_2$ in Eq.(6), one gets
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \]  
(8)
where $\mathbf{J}$ is the Dirac current in the non-relativistic limit that can be decomposed into two terms as was shown by Holland [13,15], as
\[ \mathbf{J} = -\frac{i\hbar}{2m} \left[ \psi_1^\dagger \sigma (\nabla \psi_1) - (\nabla \psi_1^\dagger) \sigma \psi_1 \right] \]  
(9)
and $\rho = \psi_1^\dagger \psi_1$. $\psi_1$ is a two component spinor which can be written for a particle in a spin eigenstate as
\[ \psi_1 = \psi(x,t) \chi = \left[ R(x,t) \exp \left( \frac{iS(x,t)}{\hbar} \right) \right] \chi \]  
(10)
Here $\psi(x,t)$ is the Schroedinger wavefunction and $\chi$ is a spin eigenstate. Putting this form of $\psi_1$ in the expression for current in Eq.(9) one gets
\[ \mathbf{J} = \frac{1}{m} \rho \nabla S + \frac{1}{m} \left( \nabla \rho \times \mathbf{s} \right) \]  
\[ \equiv \mathbf{J}_i + \mathbf{J}_s \]  
(11)
with
\[ \mathbf{s} = (\hbar/2) \chi^\dagger \sigma \chi, \quad \rho = R^2, \quad \chi^\dagger \chi = 1 \]

The first term ($\mathbf{J}_i$) in Eq.(11) is independent of spin, while the second term ($\mathbf{J}_s$) is the contribution of the spin of a free particle to the unique conserved vector current in the non-relativistic limit. It is then clear that the mean arrival time given by Eq.(2) can be computed by using the unique expression for $\mathbf{J}$ in Eq.(11). Thus we can obtain a spin-dependent contribution in the expression for the mean time of arrival for free particles, which could be an experimentally measurable quantity. On the other hand, by ignoring the spin-dependent term one would obtain the mean arrival time given by
\[ \bar{\tau} = \int_0^\infty |\mathbf{J}_i| t dt \]  
(12)
In the following section IV we study the situations where the difference between the actual magnitudes of $\bar{\tau}$ and $\bar{\tau}_i$ is significant, thereby enhancing the feasibility of experimentally testing the specific spin-dependent effect.

**IV. THE COMPUTED EFFECTS ON THE ARRIVAL TIME DISTRIBUTION**

We consider a freely evolving Gaussian wave packet in the two separate cases (A and B) corresponding to an initially symmetric and an asymmetric wave packet respectively.

**Case A: Symmetric wave packet**

Let us consider a Gaussian wave packet for a free spin 1/2 particle of mass $m$ centered at the point $x = 0$, $y = 0$, and $z = 0$. We choose the spin to be directed along the z-axis, i.e., $(s = \frac{1}{2})$.
\[ \psi(x,t = 0) = \frac{1}{(2\pi \sigma_0^2)^{3/4}} \exp(\frac{i\mathbf{k} \cdot \mathbf{x}}{\hbar}) \exp \left( - \frac{x^2}{4\sigma_0^2} \right) \]  
(13)
The time evolved wave function can be written as
\[ \psi(x,t) = R(x,t) \exp \left[ \frac{iS(x,t)}{\hbar} \right] \]  
(14)
where
\[ R(x,t) = (2\pi \sigma^2)^{-3/4} \exp \left[ \frac{-(x-ut)^2}{4\sigma^2} \right] \]  
(15)
and
\[ S(x,t) = -\frac{\hbar}{2} \tan^{-1} \left( \frac{\hbar t}{2m \sigma_0^2} \right) \]  
(16)
with $(\mathbf{u} = h\mathbf{k}/m)$ the initial group velocity taken along the x-axis, and
\[ \sigma = \sigma_0 \left[ 1 + \frac{\hbar^2 t^2}{4m^2 \sigma_0^4} \right]^{1/2} \]  
(17)
The total current density can be calculated using Eq.(11) to be (we set $m = 1$, $\hbar = 1$)
\[ \mathbf{J} = \rho \left[ \left( u + \frac{(x-ut)t}{4\sigma_0^2 \sigma^2} \right) \hat{x} + \left( \frac{yt}{4\sigma_0^2 \sigma^2} \right) \hat{y} + \left( \frac{zt}{4\sigma_0^2 \sigma^2} \right) \hat{z} \right] \]  
\[ + \rho \left[ -\left( \frac{y}{2\sigma^2} \right) \hat{x} + \frac{(x-ut)}{2\sigma^2} \hat{y} \right] \]  
(18)
where the contribution of spin is contained in the second term only.

We can now compute $\bar{\tau}$ and $\bar{\tau}_i$ numerically by substituting Eq.(18) in Eqs.(2) and (12) respectively. It is instructive to examine the behaviour of the contribution of spin-dependent term towards the mean arrival time. For this purpose, we define a quantity
\[ \bar{\tau}_s = \int_0^\infty |\mathbf{J}_s| t dt \int_0^\infty |\mathbf{J}_s| dt \]  
(19)
We first compute $\bar{\tau}_s$ for a range of the initial velocity $u$ in units of $m=1$, and $\hbar=1$. We find that the spin of a free particle contributes towards altering its mean arrival time for a wide range of initial velocities. This feature holds generally, except for very small magnitudes of velocity.
where the spin-dependent contribution may be negligible depending on the location of the detector vis-a-vis the direction of the initial group velocity \( \mathbf{u} \). This feature is shown in Figure 1 where we plot the variation of \( \bar{\tau}_x \) with \( \mathbf{u} \). The initial wave packet is peaked at the origin with \( \sigma_0 = 0.01 \). The detector position is chosen at \((x = 1, y = 1, z = 1)\). We also find that the difference of magnitude between \( \tau \) and \( \bar{\tau}_i \) can be increased by choosing asymmetric detector positions as well as asymmetric spread for the initial wave packet, an example of which we highlight below.

![Image of \( \tau \) vs \( \mathbf{u} \)]

**FIG. 1.** The spin-dependent contribution to the mean arrival time computed at the point \( x=1, y=1, z=1 \) is plotted against the initial group velocity of the packet along the x-axis.

**Case B:** Asymmetric wave packet

We consider an initial free particle wave packet in three dimensions which is centered at the point \( x = -x_1, y = 0, z = 0 \).

\[
\psi(x, y, z, t = 0) = \left( \frac{1}{\pi^{3/2} \alpha \beta \gamma} \right)^{1/4} \exp(ikx) \exp\left[ -\frac{(x + x_1)^2}{2\alpha^2} \right] \exp\left[ -\frac{y^2}{2\beta^2} \right] \exp\left[ -\frac{z^2}{2\gamma^2} \right]
\]

where \( \alpha, \beta, \gamma \) are positive constants. (Such a form for the wave function was considered by Finkelstein [12] in the context of arrival time distributions.) The particle is given an initial velocity in the direction represented by \( \mathbf{u} = \frac{\hbar k}{m} \). The time evolved wave function is given by

\[
\psi(x, y, z, t) = \left( \frac{\alpha \beta \gamma}{\pi^{3/2}} \right)^{1/4} \frac{\exp[i(kx - \alpha \beta \gamma t/2)]}{\alpha \beta \gamma} \exp\left[ -\frac{(x + x_1 - kt)^2}{2\alpha^2} \right] \exp\left[ -\frac{y^2}{2\beta^2} \right] \exp\left[ -\frac{z^2}{2\gamma^2} \right]
\]

where \( \alpha = (a^2 + it)^{1/2}; \beta = (b^2 + it)^{1/2}; \gamma = (c^2 + it)^{1/2}. \)

Writing the wave function as

\[
\psi(x, y, z, t) = R(x, y, z, t)\exp\left[ \frac{iS(x, y, z, t)}{\hbar} \right]
\]

one obtains

\[
R(x, y, z, t) = \left( \frac{a^2 b^2 c^2}{\pi^3} \right)^{1/4} \frac{1}{(p^2 + q^2)^{1/4}} \exp\left[ -\frac{a^2(x + x_1 - kt)^2}{2(a^4 + t^2)} \right] \exp\left[ -\frac{b^2y^2}{2(b^4 + t^2)} \right] \exp\left[ -\frac{c^2z^2}{2(c^4 + t^2)} \right]
\]

and

\[
S(x, y, z, t) = \hbar k x - \frac{\hbar k^2 t}{2} - \frac{\hbar y^2}{2} \tan^{-1}(q/p) + \frac{\hbar y^2}{2(a^4 + t^2)} + \frac{\hbar y^2}{2(b^4 + t^2)} + \frac{\hbar z^2}{2(c^4 + t^2)}
\]

with

\[
p = a^2 b^2 c^2 - a^2 t^2 - b^2 t^2 - c^2 t^2
\]

\[
q = a^2 b^2 t + a^2 c^2 t + b^2 c^2 t - t^3
\]

Considering again a spin-1/2 particle with spin directed along z-axis (s = \( \frac{1}{2} \)), the total current density defined in Eq.(11) is given by (in units of \( \hbar = 1 = m \))

\[
\mathbf{J} = \rho \left[ \left( u + \frac{(x + x_1 - ut)t}{a^4 + t^2} \right) \hat{x} + \frac{yt}{b^4 + t^2} \hat{y} + \frac{zt}{c^4 + t^2} \hat{z} \right] + \rho \left[ \frac{b^2 y}{(b^4 + t^2)} \hat{x} + \frac{a^2(x + x_1 - ut)}{(a^4 + t^2)} \hat{y} \right]
\]

where the second term represents the spin-dependent contribution to the current.
We compute numerically the arrival times $\bar{\tau}$ and $\bar{\tau}_i$. Figure 2 shows the variation of $\bar{\tau}$ and $\bar{\tau}_i$ with the initial group velocities ($u_i$) of the wave packet. Here we choose the parameters as $x_1 = 0, a = 0.001, b = 0.4, c = 0.01$. Accordingly the mean arrival time is computed at the position $x = 1.0, y = 2.0, z = 1.0$. One sees that the difference in the magnitudes of $\bar{\tau}$ and $\bar{\tau}_i$ can suitably be enhanced by a judicious choice of asymmetric initial spreads and detector positions.

V. CONCLUDING REMARKS

Let us now summarise the salient features of our scheme. For measuring the spin of a particle, it is usually subjected to an external field, like in a Stern-Gerlach apparatus. But the scheme we have suggested would enable to detect, a spin-dependent effect without using any external field. Such an observable effect thus highlights the feature that the spin of a particle is an intrinsic property and is not contingent on the presence of an external field. As demonstrated in this paper, the spin-dependent term in the Dirac probability current density contributes significantly to the computed mean arrival time for a range of suitably chosen parameters of the Gaussian wave packet. Thus if the arrival time distribution can be measured, this predicted spin-dependent effect would be empirically verifiable.

Another way of perceiving the significance of such an effect is as follows. The dynamical properties of free particles like position, momentum, and energy can of course be measured. However, one cannot usually measure the static or innate particle properties such as charge without using any external field. Nevertheless, the scheme we have proposed shows that the magnitude of total spin can be measured without subjecting the particle to an external field.

Another interesting implication of the measurement of the spin dependent arrival times for free particles could be to view this as implying a fundamental difference between the magnitude of total spin of a particle and its other static properties such as mass and charge in the following sense. The property of spin seems to be crucially contingent on the fundamentally relativistic nature of the dynamical equation of the wave function of the particle so that the wave function is essentially 4-component, or 2-component in the non-relativistic limit.

Here we would like to stress that the spin-dependent term which contributes significantly to the arrival time distribution we have computed in the nonrelativistic regime originates from the relativistic Dirac equation and hence this provides a rather rare example of an empirically detectable manifestation of a relativistic dynamical equation in the nonrelativistic regime, an effect which cannot be derived uniquely from the Schroedinger dynamics.

A further line of investigation as an offshoot of this paper could be to explore the possibilities of using the relativistic quantum mechanical wave equations of particles with spins other than spin 1/2 (such as using the Kemmer equation [16,17] for spin 0 and spin 1 particles) in order to compute the spin-dependent terms in the probability current densities and their effects on the arrival time distribution. Such a study seems worthwhile because then the arrival time distribution may provide a means of checking the validity of the various suggested relativistic quantum mechanical equations which have otherwise eluded any empirical verification.

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