A theory of non-Abelian superfluid dynamics

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We write down a theory for non-Abelian superfluids with a partially broken (semisimple) Lie group. We adapt the offshell formalism of hydrodynamics to superfluids and use it to comment on the superfluid transport compatible with the second law of thermodynamics. We find that the second law can be also used to derive the Josephson equation, which governs dynamics of the Goldstone modes. In the course of our analysis, we derive an alternate and mutually distinct parametrization of the recently proposed classification of hydrodynamic transport and generalize it to superfluids.

Hydrodynamics is the study of universal low energy fluctuations of a quantum system near its ground state. Any quantum system in this regime, called a fluid, can be characterized by a set of transport coefficients such as pressure, viscosity and conductivity. When a part of the global symmetry of the microscopic theory is spontaneously broken in the ground state, low energy fluctuations can also contain massless Goldstone modes [1] corresponding to the broken symmetry. Therefore the associated fluid, commonly known as a superfluid [2, 3], contains many new transport coefficients in its spectrum.

Superfluidity with a broken U(1) was first observed in liquid 4He [4, 5], which since then has been well explored in the literature, at least up to the first order in derivatives (see e.g. [4, 5]). In recent years, non-Abelian superfluids have also started to attract some attention (see [9] and references therein) in relation to the p-wave superfluidity observed in liquid 3He [10, 11]. On a different front, entire transport of an ordinary fluid compatible with the second law of thermodynamics has been classified [12, 13], and a good amount of progress is being made towards writing down a Wilsonian effective action describing the entire ordinary hydrodynamics [13–16].

The goal of this note is to set up a theory for superfluids with an arbitrarily broken internal symmetry, and explore the constraints imposed upon it by the second law of thermodynamics. In particular, we will show how the Josephson equation, which governs dynamics of the Goldstone modes, naturally emerges in our formalism as a consequence of the second law. While addressing these questions, we will propose a natural and mutually distinct parametrization of the classification mentioned above [12, 13].

**SPONTANEOUS SYMMETRY BREAKING**

Let us start with a quick recap of the spontaneous symmetry breaking; details can be found in §19 of [17]. Consider a microscopic theory invariant under spacetime translations and action of a spacetime invariant semisimple Lie group $G$ (with Lie algebra $\mathfrak{g}$). Let $\psi$ be a field in the theory transforming under some unitary representation $\mathcal{D}(G)$ of $G$, i.e. under a $g \in G$ transformation $\psi \to \mathcal{D}(g)\psi$. $\psi$ is said to spontaneously break the symmetry from $G$ to its Lie subgroup $H \subset G$ (with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$), if its ground state expectation value $\langle \psi \rangle$ is only invariant under $H$, i.e. $\mathcal{D}(h)\langle \psi \rangle = \langle \psi \rangle$ if and only if $h \in H$. $\mathcal{D}(g)\langle \psi \rangle$ with $g \notin H$ are “other” ground states system could have spontaneously chosen from. Around $\langle \psi \rangle$, the field $\psi$ can be expressed as group transformation of a reference field $\tilde{\psi}$, i.e. $\psi = \mathcal{D}(\gamma)\tilde{\psi}$, defined by,

$$\tilde{\psi} \mathcal{D}(g)\langle \psi \rangle = \tilde{\psi} \mathcal{D}(\gamma)\langle \psi \rangle, \quad \forall \, g \in G. \quad (1)$$

Roughly speaking, $\gamma$ corresponds to fluctuations of $\psi$ which takes us to the nearby ground states with no energy cost, while $\tilde{\psi}$ contains genuine excitations of $\psi$. Note that eqn. (1) is invariant under $\tilde{\psi} \to \mathcal{D}(h)\tilde{\psi}$ with $h \in H$ and hence determines $\gamma$ only up to a coset equivalence $\gamma \sim h\gamma$. Let us pick a representative from each coset $\gamma = \gamma(\varphi)$ parametrized by a field $\varphi$ living in the Lie algebra quotient $\mathfrak{g}/\mathfrak{h}$, which can be identified as the Goldstone modes of the broken symmetry. Under a $g \in G$ transformation,

$$\gamma(\varphi) \to g\gamma(\varphi)h(\varphi, g)^{-1}, \quad \tilde{\psi} \to \mathcal{D}(h(\varphi, g))\tilde{\psi}, \quad (2)$$

for some $h(\varphi, g) \in H$, such that $\psi \to \mathcal{D}(g)\psi$ and eqn. (1) remains invariant. From these transformation properties, it is clear that the theory cannot contain a mass term for $\varphi$, rendering it massless. It follows that $\varphi$ substantially affects the low energy fluctuations of the theory and must be taken into account in the superfluid description. A quick comparison can be made with the Abelian case, where $G = U(1)$ is broken down to $H = \{1\}$, with $\gamma(\varphi) = e^{-i\varphi}$. Under a $e^{i\Lambda} \in U(1)$ transformation $\varphi \to \varphi - \Lambda$, which is well known in the Abelian superfluid literature.

For notational purposes, let us introduce a set of generators $\{t_\alpha\} = \{t_i, t_3\}$ of $G$ such that the subset $\{t_i\}$ generates $H$. We orthonormalize these generators by choosing $t_\alpha \cdot t_\beta = \text{Tr}[t_\alpha t_\beta] = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is a diagonal matrix with entries $\pm 1$. Given an $X = X^\alpha t_\alpha \in \mathfrak{g}$, under a $g \in G$ transformation $X \to \text{Ad}_g(X) = (\text{Ad}_g)^\alpha \beta X^\beta t_\alpha = gXg^{-1}$.

While dealing with partially broken symmetries, we are confronted with an obstacle: the quotient $\mathfrak{g}/\mathfrak{h}$ is not a Lie Algebra and hence $\varphi$ does not transform “nicely” under the action of $G$, which poses a difficulty while formulating...
superfluids. We circumvent this problem by introducing a pair of projection operators \( P, \bar{P} : \mathfrak{g} \to \mathfrak{g} \) as,

\[
P(X) = P^\beta_{\alpha} X^\beta t_\alpha = ((\text{Ad}_T)_\gamma^\alpha (\text{Ad}^{-1}_T)_\beta) X^\beta t_\alpha, \\
\bar{P}(X) = \bar{P}_{\beta} X^\beta t_\alpha = ((\text{Ad}_T)_\alpha^\alpha (\text{Ad}^{-1}_T)_\beta) X^\beta t_\alpha.
\]

They transform covariantly under the action of \( G \), i.e. under a \( g \in G \) transformation \( P(X) \to \text{Ad}_g(P)(X), \text{Ad}_g(\bar{P})(X) \). Using these we can re-bundle the information in \( \varphi \) into \( \bar{\delta}_\gamma = \bar{P}(i \partial_\gamma \gamma(\varphi)(\varphi)^{-1}) \in \mathfrak{g} \) which transforms “nicely” in the Adjoint representation of \( G \). Introducing the operators \( P, \bar{P} \) will also considerably simplify the notation in the following non-Abelian superfluid analysis, resulting in a pleasant resemblance with the better known Abelian results. As an added benefit, we can revert back to ordinary fluids at any point by setting \( \bar{P} = 0, P = \text{id}_\mathfrak{g} \) (identity in \( \mathfrak{g} \)).

**SUPERFLUID DYNAMICS**

We are interested in studying low energy fluctuations of a theory with a spontaneously broken internal symmetry. As eluded before, any such description must contain the Goldstone modes \( \varphi \) as a dynamical field, with dynamics provided by a \( \dim(\mathfrak{g}/\mathfrak{h}) \)-component equation,

\[
K = 0 \in \bar{P}(\mathfrak{g}). \tag{4}
\]

Here \( K \) depends on the details of the microscopic theory. Allowing for an arbitrary dynamical equation for \( \varphi \) is a novel feature of our formalism, which in the conventional treatment of superfluids is taken to be the “Josephson equation” by hand (see e.g. [7]). For us however, this will follow as a constraint from the second law of thermodynamics. A theory invariant under spacetime translations and \( G \) transformations must also contain an associated conserved energy-momentum tensor \( T^\mu{}_{\nu} \) and a \( \mathfrak{g} \)-valued charge current \( J^\mu \) in its spectrum. To probe these observables we couple the theory to a slowly varying metric \( g_{\mu\nu} \), and a gauge field \( A_\mu \). We denote the covariant derivative associated with the Levi-Civita connection \( \Gamma^\lambda_{\mu\nu} \) by \( \nabla_\mu \), while the gauge covariant derivative associated with \( A_\mu \) and \( \Gamma^\lambda_{\mu\nu} \) is denoted by \( D_\mu \). In presence of these external sources, respective conservation laws take the form,

\[
\nabla_\nu T^\nu{}_{\mu} = F^\nu{}_{\mu} \cdot J_\nu + \xi^\nu \cdot K + T^\nu{}_{\mu \perp}, \quad D_\mu J^\mu = J^\mu_{\perp} - K, \tag{5}
\]

where we have allowed for \( \varphi \) to go offshell (\( K \neq 0 \)). \( F^\mu{}_{\nu} = 2\partial_\mu A_\nu - i[A_\mu, A_\nu] \in \mathfrak{g} \) is the gauge field strength and \( \xi_\mu = \bar{P}(A_\mu) + \bar{\delta}_\mu \varphi \in \bar{P}(\mathfrak{g}) \) is called the superfluid velocity. The Hall currents \( T^\mu{}_{\mu \perp}, J^\mu_{\perp} \) represent the contribution from possible gravitational and flavor anomalies in the microscopic theory respectively. If the conservation laws [9] are unfamiliar to the reader, one way to derive them is to consider a field theory effective action \( S[g_{\mu\nu}, A_\mu, \varphi] \), and parametrize its infinitesimal variation as,

\[
\delta S = \int (dx^\mu) \sqrt{-g} \left[ \frac{1}{2} T^\mu{}_{\nu} \delta g_{\mu\nu} + J^\mu \cdot \delta A_\mu + K \cdot \delta \varphi \right], \tag{6}
\]

where \( g = \det g_{\mu\nu} \) and \( \delta \varphi = \bar{P}(i \partial_\gamma \gamma(\varphi)(\varphi)^{-1}) \). Given this setup, one can check that the conservation laws [9] are merely the Ward identities corresponding to infinitesimal diffeomorphisms and \( G \) gauge transformations.

The conservation laws [9] can provide dynamics for a theory formulated in terms of the hydrodynamic fields: normalized 4-velocity \( u^\mu \) (with \( u^\mu u_\mu = -1 \)), temperature \( T \) and chemical potential \( \mu \in \mathfrak{g} \), in addition to the Goldstone modes \( \varphi \). It should be noted however that these are merely some fields chosen to describe the system, and like in any field theory, can admit an arbitrary redefinition; we will return to this issue later. In general, the observables \( T^{\mu\nu}, J^\mu, K \) appearing in eqns. (4) and (5) can have an arbitrary dependence on the fields \( \Psi = \{u^\mu, T, \mu, g_{\mu\nu}, A_\mu, \xi_\mu\} \). In hydrodynamics however, we are only interested in the low energy fluctuations of the constituent fields \( \Psi \), which can be translated as the configurations of \( \Psi \) that admit a perturbative expansion in derivatives. This allows us to write down the most generic allowed expressions for \( T^{\mu\nu}, J^\mu, K \) in terms of \( \Psi \) truncated up to a finite order in derivatives, called the superfluid constitutive relations. At a given order, constitutive relations will contain all the possible tensor structures allowed by symmetry (modulo field redefinitions) called data, multiplied with arbitrary scalars called transport coefficients. The explicit functional form of these transport coefficients depends on the underlying microscopic theory, and can be computed using the Kubo formula [18] in linear response theory. Even without knowledge of the microscopic theory however, we can put some stringent constraints on the transport coefficients by imposing some physical requirements such as a local version of the second law of thermodynamics,

> “Given a set of constitutive relations \( T^{\mu\nu}, J^\mu, K \), there must exist an entropy current \( J^S \) whose divergence is non-negative, i.e. \( \nabla_\mu J^S_\mu \geq 0 \), for all the superfluid configurations satisfying the conservation laws [9].”

It is worth pointing out that this statement is slightly stronger than the one used previously in the superfluid literature (e.g. [7]), as it is imposed even when \( \varphi \) is off-shell. This extra information fixes eqn. (6) to be the Josephson equation, as we will now illustrate.

**Ideal superfluids.**—Consider the most generic constitutive relations and entropy current of a superfluid at zero derivative order,

\[
T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \xi^\mu \cdot \xi^\nu, \quad J^\mu = qu^\mu + \rho_s \cdot \xi^\mu, \quad J^S_\mu = su^\mu + s_s \cdot \xi^\mu, \tag{7}
\]

along with a scalar \( K \). We have fixed the ideal order definition of \( u^\mu \) by eliminating a term like \( e_s \cdot \xi(\mu u^\mu) \) from
of the conservation laws (5) to it is possible to extend the second law to cases where the higher in the derivative expansion, because at a given order in derivatives we are required to use the lower order combinations of the independent hydrostatic data, modulo the total results in ordinary fluids \[22, 23\], imposes strict equality constraints in the hydrostatic sector, while in the non-hydrostatic sector it only gives a few inequalities at the first order in derivatives and none thereafter. We will present a quick proof of this statement; in the hydrostatic sector we will closely follow \[13\] with appropriate modifications for superfluids, while in the non-hydrostatic sector our presentation will be independent and simpler. **Hydrostatic sector.**—Consider the most generic constitutive relations \( \mathcal{C} = \mathcal{C}_{\text{hydrostatic}} \) which are made up of the hydrostatic data. For these, every independent term in the RHS of eqn. \(12\) will contain exactly one bare (isn’t acted upon by a derivative) \( \delta_{\text{B}} \). Hence the associated \( N^\mu \) also must contain the hydrostatic data only, otherwise \( \nabla_\mu N^\mu \) will either be void of a bare \( \delta_{\text{B}} \) or will contain multiple \( \delta_{\text{B}} \). The most generic \( N^\mu \) in the hydrostatic sector can therefore be written as, \( N^\mu_{\text{hydrostatic}} = (N\beta^\mu + \Theta^\mu_N) + \mathfrak{n}^{\mu} \)

where \( \Delta \) is a positive definite quadratic form. To make the notation compact we have introduced, \( \mathcal{C} = (T^{\mu \nu} J^\mu K), \quad \Phi = (\frac{1}{2} \delta_{\text{B}} g_{\mu \nu} \delta_{\text{B}} A_\mu \delta_{\text{B}} \varphi) \), (13) which are vectors in the composite space \( \mathfrak{F} = (\text{sym. tensor}) \oplus (\mathfrak{g} \times \text{vector}) \oplus \overline{\mathfrak{P}}(\mathfrak{g}) \). “\( \delta_{\text{B}} \)” denotes an infinitesimal diffeomorphism and \( G \) gauge transformation with parameters \( \mathcal{B} = \{ \beta^\mu, \Lambda_\beta = \nu - A_\mu \beta^\mu \} \),

\[
\delta_{\text{B}} g_{\mu \nu} = \mathcal{E}_{\beta} g_{\mu \nu} = 2\mathcal{E}_{\beta} \nu, \quad \delta_{\text{B}} A_\mu = \mathcal{E}_{\beta} A_\mu + \partial_\mu \Lambda_\beta - i[A_\mu, \Lambda_\beta] = \mathcal{E}_{\beta} \nu + \beta^\nu F_{\mu \nu}, \quad \mathcal{E}_{\beta} \varphi = \mathcal{P} (i \mathcal{E}_{\beta} \gamma(\varphi) \gamma(\varphi)^{-1}) = \mathcal{P} (i \mathcal{E}_{\beta} \gamma(\varphi) \gamma(\varphi)^{-1} + \Lambda_\beta) = \beta^\mu \xi_\mu - \mathcal{P} (\nu).
\]

One can check that the ideal order definitions of \( u^\mu \), \( T^\mu \), \( \mu \) (given around eqn. \(8\)) imply the relations \( \beta^\mu = u^\mu/T \), \( \nu = \mu/T \) at ideal order. We fix the remaining ambiguity in the fluid fields by assuming these relations to hold at all orders in the derivative expansion. Having done that, the allowed superfluid constitutive relations are the most generic expressions with \( T^{\mu \nu}, J^\mu, K \) in terms of \( \mathcal{F} \) which satisfy eqn. \(12\) for some \( N^\mu \) and \( \Delta \geq 0 \).

Note that it is always possible to write down terms \( N^\mu_2 \in N^\mu \) whose divergence is either zero or is balanced by some counter terms \( \Delta_2 \in \Delta \), i.e \( \nabla_\mu N^\mu_2 = \Delta_2 \). We refer to these terms as Class S. They are not genuine (super)fluid transport, instead they parametrize the multitude of entropy currents which satisfy the second law for the same set of constitutive relations.

We split the tensor structures that can appear in the constitutive relations into two sectors: “non-hydrostatic data” (independent data that contains at least one instance of “\( \delta_{\text{B}} \)” and “hydrostatic data” (largest collection of independent data with no non-hydrostatic linear combination). The second law, similar to the known results in ordinary fluids \[22, 23\], imposes strict equality constraints in the hydrostatic sector, while in the non-hydrostatic sector it only gives a few inequalities at the first order in derivatives and none thereafter. We will present a quick proof of this statement; in the hydrostatic sector we will closely follow \[13\] with appropriate modifications for superfluids, while in the non-hydrostatic sector our presentation will be independent and simpler. **Hydrostatic sector.**—Consider the most generic constitutive relations \( \mathcal{C} = \mathcal{C}_{\text{hydrostatic}} \) which are made up of the hydrostatic data. For these, every independent term in the RHS of eqn. \(12\) will contain exactly one bare (isn’t acted upon by a derivative) \( \delta_{\text{B}} \). Hence the associated \( N^\mu \) also must contain the hydrostatic data only, otherwise \( \nabla_\mu N^\mu \) will either be void of a bare \( \delta_{\text{B}} \) or will contain multiple \( \delta_{\text{B}} \). The most generic \( N^\mu \) in the hydrostatic sector can therefore be written as, \( N^\mu_{\text{hydrostatic}} = (N\beta^\mu + \Theta^\mu_N) + \mathfrak{n}^{\mu} \)

where \( \mathfrak{n}^{\mu} u_\mu = 0 \). \( N \) is the most generic scalar made out of the independent hydrostatic data, modulo the total

\( \nabla_\mu T^{\mu \nu} = 0 \), \( \nabla_\mu J^\mu = 0 \), \( \nabla_\mu (\nabla_\mu T^{\mu \nu} - F^{\mu \nu} \cdot J_\nu - \xi^\mu \cdot K - T^{\mu \nu}_H \) \( + \nu \cdot (\mathcal{D}_\mu J^\mu + K - J_H^\mu) \geq 0 \).

Here \( \beta^\mu, \nu \) are some arbitrary fields. Let us define \( N^\mu = J^\mu_2 + \beta^\mu T^{\mu \nu} + \nu \cdot J^\mu \) and \( N_H^\mu = T^{\mu \nu}_H + \nu \cdot J_H^\mu \). In terms of these, eqn. \(11\) can be recasted in a more useful form, \( \nabla_\mu N^\mu - N_H^\mu - \Delta = \Phi \cdot \mathcal{C} \), (12)
derivative terms. \( \mathcal{E}^{\mu}_N \) is a \( N \) dependent non-hydrostatic vector defined via,
\[
\nabla_\mu (N^\beta \mu) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta \phi} \left( \sqrt{-g} N \phi \right) = \Phi \cdot \mathcal{E}_{H_S} = \nabla_\mu \Theta^{\mu}_N, \tag{15}
\]
which ensures that \( \nabla_\mu (N^\beta \mu + \Theta^{\mu}_N) \) has a bare \( \delta_B \). Eqn. (13) also defines the constitutive relations \( \mathcal{E}_{H_S} \) associated with \( \mathcal{N} \), called Class \( H_S \). \( N^\mu \) on the other hand is the most generic hydrostatic vector transverse to \( u^\mu \), such that \( \nabla_\mu N^\mu - N^\mu_T \) has exactly one bare \( \delta_B \). This requirement happens to completely determine \( N^\mu \) up to some constants, which includes the terms responsible for anomalies. The easiest way to find \( N^\mu \) is using a (transcendental) anomaly polynomial \( \Phi \) which is written only in terms of the curvature \( R^\mu_{\nu \rho \sigma} \), field strength \( F^\mu_{\nu} \) and an auxiliary \( U(1)_T \) field strength \( F^T_{\mu \nu} = 2 \partial_\mu A_\nu^T \). It follows that \( N^\mu \) is independent of \( \phi \) and hence is ignorant of the fluid being in the superfluid phase. It allows us to directly import \( N^\mu \) and the respective Class \( H_V \cup A \) constitutive relations \( \mathcal{E}_{H_V} + \mathcal{E}_A \) from the ordinary fluid literature \( \textbf{[13]} \), where Class \( A \) is the contribution from anomalies. \( \mathcal{E}_{\text{hydrostatic}} = \mathcal{E}_{H_S} + \mathcal{E}_{H_V} + \mathcal{E}_A \) are therefore the most generic hydrostatic constitutive relations compatible with the second law. Comparing these to the most generic expressions allowed by symmetry, we can read out the equality constraints. It is worth pointing our that these constraints can also be generated using an equilibrium effective action \( \textbf{[14]} \).

**Non-hydrostatic sector.**—This sector of hydrodynamics contains constitutive relations \( \mathcal{E} = \mathcal{E}_{\text{non-hydrostatic}} \) which are purely made of the non-hydrostatic data. Since every non-hydrostatic data has at least one \( \delta_B \), it can be written as a differential operator acting on \( \Phi \) defined in eqn. (13). Introducing a symmetric covariant derivative operator \( D^n = D(\mu_1 \cdots D_{\mu_n}) \) (anti-symmetric derivatives can be represented by curvature and field strength), the most generic non-hydrostatic constitutive relations can therefore be written in a compact form,
\[
\mathcal{E}_{\text{non-hydrostatic}} = - \sum_{n=0}^{\infty} \frac{1}{2} \left[ \mathcal{E}_n \cdot (D^n \Phi) + D^n (\mathcal{E}_n \cdot \Phi) \right]. \tag{16}
\]
\( \mathcal{E}_n \in \mathfrak{U} \times \mathfrak{U} \) are matrices with additional \( n \) symmetric indices to be contracted with \( D^n \). The last term in eqn. (16) is taken purely for convenience and can be absorbed into the first via differentiation by parts. Let us factor \( \mathcal{E}_{\text{non-hydrostatic}} \) into a dissipative (Class \( D \)) and a non-dissipative (Class \( \overline{D} \)) part parametrized by,
\[
\mathcal{D}_n = \frac{1}{2} \left( \mathcal{E}_n + (-)^n \mathcal{E}_n^T \right), \quad \overline{\mathcal{D}}_n = \frac{1}{2} \left( \mathcal{E}_n - (-)^n \mathcal{E}_n^T \right), \tag{17}
\]
respectively. The nomenclature can be justified by multiplying eqn. (16) with \( \Phi \) giving us (see also \( \textbf{22, 23} \)),
\[
\Phi \cdot \mathcal{E}_D = - \Delta_D + \nabla_\mu N^\mu_D, \quad \Phi \cdot \mathcal{E}_{\overline{D}} = \nabla_\mu N^\mu_{\overline{D}}, \tag{18}
\]
where \( N^\mu_D, N^\mu_{\overline{D}} \) are some vectors gained via successive differentiation by parts. \( \Delta_{\overline{D}} \) however is given as,
\[
\Delta_D = (\mathcal{Y} \Phi) \cdot \mathcal{D}^{(0)}_0 \cdot (\overline{\mathcal{Y}} \Phi), \tag{19}
\]
where \( \mathcal{Y} = \sum_{d=0}^{\infty} \mathcal{Y}_d : \mathfrak{U} \rightarrow \mathfrak{U} \) is a differential operator defined by \( \mathcal{D}^{(0)}_0 \) is the part of \( \mathcal{D}_0 \) with \( n \) number of derivatives, and \( \overline{\mathcal{Y}} \) denotes the conjugate of a differential operator: \( \Phi_1 \cdot \mathcal{O} \Phi_2 = (\overline{\Phi}_1) \cdot \Phi_2 + \nabla_\mu (\cdots)^\mu \),
\[
\mathcal{Y}_{d+1}^{(0)} \bigg|_{d=1} = - (\mathcal{D}^{(0)}_0)^{-1} \left[ \sum_{k=1}^{d-1} \mathcal{Y}_k + \frac{1}{2} \mathcal{Y}_d \right] \left( \mathcal{D}^{(0)}_0 \cdot \mathcal{Y}_d \right), \quad \mathcal{Y}_0 = 1, \quad \mathcal{Y}_1 = \frac{1}{2} (\mathcal{D}^{(0)}_0)^{-1} \sum_{n=1}^{\infty} (\mathcal{D}^{(0)}_n + \mathcal{D}_n D^n). \tag{20}
\]
Comparing eqns. (12) and (18), we can see that Class \( \overline{D} \) constitutive relations satisfy the second law with \( N^\mu = N^\mu_{\overline{D}} \) and \( \Delta = 0 \), hence the name non-dissipative. On the other hand, dissipative Class \( D \) constitutive relations satisfy the second law with \( N^\mu = N^\mu_D \) and \( \Delta = \Delta_D \). The condition \( \Delta \geq 0 \) implies that all the eigenvalues of the zero derivative matrix \( \mathcal{D}^{(0)}_0 = \mathfrak{U} \times \mathfrak{U} \) are non-negative. It follows that the only constraints imposed by the second law in non-hydrostatic sector are some inequalities in Class \( D \) at the first order in derivatives.

At the end of the day, we are only interested in describing the superfluid and not its surroundings, hence the constitutive relations only differing by combinations of the conservation laws must be identified. It can be verified that for the constitutive relations satisfying eqn. (12), the conservation laws \( \textbf{[5]} \) are purely non-hydrostatic. Hence without loss of generality, we can use them to eliminate a vector \( u^\mu \delta_S g_{\mu \nu} \) and a \( g \)-valued scalar \( u^\mu \delta_S A_\mu \) from the non-hydrostatic data. The upshot of this is that we can drop the respective terms from \( \mathcal{E}_D \) and \( \mathcal{E}_{\overline{D}} \). Had we eliminated any other data using the conservation laws, the respective constitutive relations would be related to the current ones, at most, by a field redefinition.

**CLASSIFICATION**

In our quest of finding the constraints, we have classified the entire (super)fluid transport compatible with the second law of thermodynamics into 5 mutually distinct classes: \( A \) (anomalies), \( H_S \) (hydrostatic scalars), \( H_V \) (hydrostatic vectors), \( \mathcal{D} \) (non-hydrostatic non-dissipative) and \( D \) (dissipative), along with a Class \( S \) worth of arbitrariness in the associated entropy current.

To compare with the classification of \( \textbf{[12]} \), we decompose Class \( S \) into a part with \( \Delta_S = 0 \) (Class \( C \)) and remaining (Class \( S_0 \)). In the ordinary fluid limit, Classes \( A, C, H_S, H_V \) of \( \textbf{[12]} \) are same as ours by definition, while their Class \( D \) is Class \( D \cup S_D \) for us. A major difference between the two classifications is that our Class \( \overline{D} \) contains (but is not equal to) their Classes \( B \cup \mathfrak{H}_S \cup \mathfrak{H}_V \).
For completeness, [12] introduced a “Class B with Υ operators” which can be shown to be equal to our Class D (and hence containing their own Classes B \( \cup \mathbb{H}_S \cup \mathbb{H}_V \)), but parametrized very differently. It is evident therefore, that our classification eliminates some of the redundancies inherent in the classification of [12]. In the dissipative sector, unlike [12] our parametrization allows us to isolate the “true dissipation” from mere entropy current redundancies. Additionally, our parametrization in eqn. (18) of Classes D \( \cup \mathbb{D} \) allows us to easily eliminate constitutive relations related to each other by combinations of equation of motion (interpreted as “residual field redefinitions” in [12]).

OUTLOOK

This completes our analysis of the (non-Abelian) superfluid constitutive relations compatible with the second law of thermodynamics. The results can also be applied to an ordinary fluid, seen as a special case of a superfluid where no symmetry is broken. Similar to an ordinary fluid, we find that the second law gives no constraints in the non-dissipative non-hydrostatic sector, while it only gives inequalities at the first derivative order in the dissipative sector. In the hydrostatic sector however, we get equality-type constraints at every derivative order, as inherent in the classification of [12]. In the derivative expansion.

In this note we concentrated on fluids with broken in-

ternal symmetries. The procedure can also be extended to the breaking of spacetime symmetries, interpreted as introducing space-time boundaries/surfaces in the (super)fluid \([28]\). It will be interesting to see how the second law constrains the surface transport coefficients in (super)fluids, and if there is a natural extension of the presented classification to surface transport.

Finally, all of the results presented here can easily be extended to Galilean superfluids using the null fluid formalism of \([27,28]\). In a companion paper \([30]\), we will use “null superfluids” to work out the constraints on Abelian Galilean superfluid transport up to first order in the derivative expansion.

ACKNOWLEDGEMENTS

The author would like to thank Nabamita Banerjee, Jytirmoy Bhattacharya, Suvankar Dutta and Felix Haehl for extensive discussions on various points presented in this work. Author also wishes to acknowledge helpful conversations with Michael Appels, Jácime Armas, Leopoldo Cuspinera and Ruth Gregory during the course of this project. AJ is financially supported the Durham Doctoral Scholarship offered by Durham University.
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