AN OKA PRINCIPLE FOR A PARAMETRIC INFINITE TRANSITIVITY PROPERTY

FRANK KUTZSCHEBAUCH AND ALEXANDRE RAMOS-PEON

Abstract. It is an elementary fact that the action by holomorphic automorphisms on $\mathbb{C}^n$ is infinitely transitive, that is, $m$-transitive for any $m \in \mathbb{N}$. The same holds on complex manifolds with the holomorphic density property. We study a parametrized case, where the points depend holomorphically on a parameter on any Stein manifold. This new property is shown to enjoy an Oka principle, to the effect that the obstruction to a holomorphic solution is of a purely topological nature.

1. Introduction

Let $X$ and $W$ be complex manifolds. Let $Y_{X,N}$ be the configuration space of ordered $N$-tuples of points in $X$: $Y_{X,N} = X^N \setminus \Delta$, where

$$\Delta = \{(z^1, \ldots, z^N) \in X^N; z^i = z^j \text{ for some } i \neq j\}$$

is the diagonal. Consider a holomorphic map $x : W \to Y_{X,N}$, that is, $N$ holomorphic maps $x^j : W \to X$ such that for each $w \in W$, the $N$ points $x^1(w), \ldots, x^N(w)$ are pairwise distinct. Interpreting $x : W \to Y_{X,N}$ as a parametrized collection of points, the following property can be thought of as a strong type of parametric infinite transitivity.

Definition. Fix $N$ pairwise distinct points $z^1, \ldots, z^N$ in $X$. We say that the parametrized points $x^1, \ldots, x^N$ are simultaneously standardizable if there exists an element $\alpha \in \text{Aut}^W(X)$, where

$$\text{Aut}^W(X) = \{\alpha \in \text{Aut}_{\text{hol}}(W \times X); \alpha(w, z) = (w, \alpha^w(z))\},$$

with

$$\alpha^w(x^j(w)) = z^j$$

for all $w \in W$ and $j = 1, \ldots, N$.

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This notion was introduced by the first author and S. Lodin in [KL13], where it is shown that for $X = \mathbb{C}^n$ and $W = \mathbb{C}^k$, if $k < n - 1$, then any collection of parametrized points $W \to Y_{X,N}$ is simultaneously standardizable. Our main theorem is the following.

**Theorem 1.** Let $W$ be a Stein manifold and $X$ a Stein manifold with the holomorphic density property, and let $x : W \to Y_{X,N}$ be a holomorphic map. Then the parametrized points $x^1, \ldots, x^N$ are simultaneously standardizable by an automorphism lying in the path-connected component of the identity $(\text{Aut}^W(X))_0$ of $\text{Aut}^W(X)$ if and only if $x$ is null-homotopic.

When $W = \mathbb{C}^k$, which is contractible, any map $W \to Y_{X,N}$ is null-homotopic, so we recover the result of [KL13], without any restrictions on the dimension of $W$. Moreover, Theorem 1 reduces the problem of simultaneous standardization to a purely topological problem as explained in Section 6.

In what follows the dependence of an automorphism on a parameter is always understood to be a holomorphic dependence as just described. We will drop the index hol in $\text{Aut}_{\text{hol}}$ since we will not consider the algebraic category. When clear from the context, we also drop the subindices in $Y_{X,N}$. A homotopy connecting two maps $f_0$ and $f_1$ between any two complex spaces $W \to X$ is only assumed to be a continuous function $f : W \times [0, 1] \to Y$, and we write as usual $f_t : W \to X$. If each $f_t$ is holomorphic, we speak of a homotopy through holomorphic maps, and if furthermore the function is $C^k$ (resp. $C^\infty$) in the $t$ variable, it is a $C^k$ (resp. smooth) homotopy between $f_0$ and $f_1$. Finally if the variable $t$ is allowed to vary in a complex disc $D_r \subset \mathbb{C}$ ($r > 1$), and $f$ is holomorphic, we speak of an analytic homotopy.

The paper is organized as follows. In Section 2 we recall the definition of manifolds with the density property, and we collect a number of preliminary facts about them, in particular about the existence of automorphisms and the control that we can impose. In Section 3 we establish that $Y$ is elliptic in Gromov’s sense and hence an Oka-Grauert-Gromov h-principle applies to maps $W \to Y$. This will allow us to use a variant of the Andersén-Lempert theorem to construct automorphisms of $X$ depending on a parameter. The proof then consists in defining countably many such automorphisms, in such a way that their composition converges —Section 4 provides sufficient conditions for this—and maps $x$ to a constant map $\hat{x}$. Section 5 contains the core of the proof, preceded by a number of preparatory technicalities. Finally in Section 6 we make explicit a homotopic-theoretical point of view and consider examples: in the first, simultaneous standardization is always
possible, independently of null-homotopy; in the second, the topological obstruction does prevent from simultaneous standardization. These concluding remarks make precise the claim following Theorem 1.

2. Density property and the Andersén-Lempert theorem

Let \( X \) be a complex manifold, \( \Omega \subset X \) an open set, and \( k \in \mathbb{N} \cup \{\infty\} \). A \( \mathcal{C}^k \) isotopy of injective holomorphic maps is a \( \mathcal{C}^k \) map \( F : \Omega \times [0,1] \to X \) such that \( F_0 \) is the inclusion and that for each fixed \( t \in [0,1] \), the map \( F_t : \Omega \to X \) is an injective holomorphic map. The main theorem in [FR93] states that given a \( \mathcal{C}^2 \) isotopy of injective biholomorphic maps \( F_t : \Omega_0 \to \Omega_t \) between Runge domains in \( \mathbb{C}^n \), then all the maps \( F_t \) can be approximated uniformly on compacts by automorphisms of \( \mathbb{C}^n \). In the same paper, approximation “near polynomially convex sets” is proved, in [For94] regularity with respect the parameter is obtained, and in [Kut05] a parametric version is shown to hold (see also [For03], where it is used to prove an approximation result for holomorphic submersions). Combining this we obtain:

**Theorem** (Andersén-Lempert Theorem). Let \( n \geq 2 \) and \( \Omega \) be an open set in \( \mathbb{C}^k \times \mathbb{C}^n \). Let \( F_t \) be a \( \mathcal{C}^p \) (\( p \geq 0 \)) isotopy of injective holomorphic maps from \( \Omega \) into \( \mathbb{C}^k \times \mathbb{C}^n \) of the form

\[
F_t(w,z) = (w,F^w_t(z)), \quad (w,z) \in \Omega.
\]

Suppose \( K \) is a compact polynomially convex subset of \( \mathbb{C}^k+n \) contained in \( \Omega \), and assume that \( F_t(K) \) is polynomially convex in \( \mathbb{C}^k \times \mathbb{C}^n \) for each \( t \in [0,1] \). Then for all \( t \in [0,1] \), \( F_t \) can be approximated uniformly on \( K \) (in the \( \mathcal{C}^p \)-norm) by automorphisms \( \alpha_t \in \text{Aut}^\mathcal{C}^k(\mathbb{C}^n) \); moreover \( \alpha_t \) depends smoothly on \( t \), and \( \alpha_0 \) can be chosen to be the identity.

We refer to the cited references for a complete proof of this theorem. However it is convenient to introduce here the ideas involved, which are best understood in the language of vector fields and their flows. In what follows and in the rest of this paper we drop the adjective “holomorphic” when it comes to vector fields, as we will only consider holomorphic objects.

The isotopy can be interpreted as the flow of a time-dependent vector field, which has no component in the \( w \) direction. The polynomial convexity is used to construct a Runge neighborhood of \( K \) and of its images, and the Runge property is used to approximate the vector field by a globally defined vector field (without components in the \( w \) direction). The main point is that this global field is approximated by a finite sum of complete fields, that is, fields for which the flow is assumed to exist for all complex times and initial conditions. Then the
flow of such a complete field is an automorphism depending on \( w \). A version of this was proven in the early 90's by Andersén and Lempert [AL92]; the association of their names to this theorem has since then been established in the literature.

In [Var01] D. Varolin introduced a class of manifolds where the “main point” above holds. Namely, a complex manifold \( X \) is said to have the density property if the Lie algebra generated by the complete vector fields is dense in the algebra of all vector fields on \( X \) in the compact-open topology. The relevant fact is that the flow of a vector field on such a manifold can be approximated by flows of complete fields. In a recent paper by T. Ritter [Rit13] there is a detailed proof of a general version of the Andersén-Lempert theorem for manifolds with the density property, where polynomial convexity must be replaced by \( \mathcal{O}(X) \)-convexity: given a complex manifold \( X \), a compact \( K \subset X \) is said to be \( \mathcal{O}(X) \)-convex if \( K = \tilde{K}_{\mathcal{O}(X)} \) where

\[
\tilde{K}_{\mathcal{O}(X)} = \left\{ x \in X ; |f(x)| \leq \sup_{y \in K} |f(y)| \forall f \in \mathcal{O}(X) \right\}.
\]

If we now allow the maps \( F_t \) to depend holomorphically on a parameter \( w \) in a Stein manifold \( W \), then we can closely follow the proof in [Rit13] by carrying a parameter. The only apparent difficulty arises when the density property is used to construct a vector field in the Lie algebra generated by complete fields, for the holomorphic dependence of these new fields on \( w \) is not obvious. However Lemma 3.5 in [Var01] shows precisely that if \( V_w \) is a vector field on \( X \) depending holomorphically on a Stein parameter \( w \), then \( V_w \) can be approximated locally uniformly on \( W \times X \) by Lie combinations of complete vector fields which depend holomorphically on the parameter. This proves the following parametric version of the Andersén-Lempert theorem in manifolds with the density property.

**Theorem 2.** Let \( W \) be a Stein manifold and \( X \) a Stein manifold with the density property. Let \( \Omega \subset W \times X \) be an open set and \( F_t : \Omega \to W \times X \) be a smooth isotopy of injective holomorphic maps of the form

\[
F_t(w,x) = (w,F_t^w(x)), \quad (w,x) \in \Omega.
\]

Suppose \( K \subset \Omega \) is a compact set such that \( F_t(K) \) is \( \mathcal{O}(W \times X) \)-convex for each \( t \in [0,1] \). Then for all \( t \in [0,1] \), \( F_t \) can be approximated

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This is usually called the “holomorphic” density property but we drop this adjective, as we do not consider the algebraic category.
uniformly on $K$ (with respect to any distance function on $X$) by auto-
morphisms $\alpha_t \in \text{Aut}^W(X)$ which depend smoothly on $t$, and moreover
we can choose $\alpha_0 = id$.

We use the notation $\text{VF}(U) = \{\text{vector fields on } U \subset X\}$ and $\text{CVF}(X) = \{\text{complete vector fields on } X\}$. It will be useful to consider objects in $X$ and in its configuration space $Y = Y_{X,N}$ simultaneously. For this we introduce the linear map $\oplus^N : \text{VF}(X) \to \text{VF}(Y)$

defined as follows: for each $V \in \text{VF}(X)$ let $\oplus^N V \in \text{VF}(X^N)$ be the vector field in $X^N$ defined by $\oplus^N V(z^1, \ldots, z^N) = (V(z^1), \ldots, V(z^N)) \in T_{(z^1, \ldots, z^N)}X^N$. In fact this field is tangent to $\Delta$, so $\oplus^N V \in \text{VF}(Y)$. It is clear that $\oplus$ restricts to a map between complete fields. Similarly, we have an obvious map $\oplus : \text{Aut}(X) \to \text{Aut}(Y)$, where we have dropped the index $N$, since we assume it fixed throughout.

**Lemma 3.** Let $X$ be a Stein manifold with the density property. Then there exist complete vector fields $V_1, \ldots, V_m$ on $X$ such that

$$T_y Y = \text{Span}\{\oplus V_j(y)\}_j, \quad \forall y \in Y.$$ 

In particular, the statement for $N = 1$ holds. That case is treated in [KK11]. We adapt that proof to this more general situation.

**Proof.** Let $x^1, \ldots, x^N \in X$ be $N$ pairwise distinct points in $X$. Since $X$ is Stein we can pick a Runge open set around $\{x^1\} \cup \cdots \cup \{x^N\}$ of the form $U = \bigcup_{j=1}^N U^j$, so small that a chart $U^j \to \mathbb{C}^n$ exists for each $j$, where $n$ is the dimension of $X$. By pulling back the coordinate vector fields in $\mathbb{C}^n$ we obtain, for each $j = 1, \ldots, N$,

$$V^j_1, \ldots, V^j_n \in \text{VF}(U^j)$$

such that $\text{Span}\{V^j_i(x^j)\}_i = T_{x^j} X$.

For each fixed $j$, define $n$ vector fields on $U$ as follows: for $i = 1, \ldots, n$, let $\Theta^j_i \in \text{VF}(U)$ be the trivial extension of $V^j_i$ to $U$ (that is, extend it as the zero field outside of $U^j$). Consider the vector fields $\oplus \Theta^j_i$ defined on $U^1 \times \cdots \times U^N \subset Y$. They span the tangent space to $y_0 = (x^1, \ldots, x^N)$:

$$T_{y_0} Y = \text{Span}\{\oplus \Theta^j_i(y_0)\}_{i,j}.$$ 

Since $U$ is Runge in $X$, there exists $\eta^j_i \in \text{VF}(X)$ approximating $\Theta^j_i$ on $\overline{U}$, because by Cartan’s theorem A, sections of coherent sheaves on Stein manifolds can be approximated by global sections. By continuity
of the \( \oplus \) operator this implies that \( \oplus \eta^j_i \) approximates \( \oplus \Theta^j_i \), so we can assume that

\[
T_y Y = \text{Span}\{\oplus \eta^j_i(y)\}_{i,j}
\]

holds for all \( y \) in a neighborhood of \( y_0 \) in \( Y \). By the density property, we can further approximate each \( \eta^j_i \) by a finite sum of complete vector fields \( \eta^j_{i,k} \) on \( X \). Indeed, given \( V,W \in \text{CVF}(X) \), \( [V,W] = \lim_{t \to 0^+} \frac{(V^t)^*W - W}{t} \), where \( V^t \) is the time-\( t \) map of the flow of \( V \); observe that multiplication by \( 1/t \) and the pullback by a global automorphism preserves the completeness of a field. Let \( \eta_k \in \text{CVF}(X) \) be the collection of the complete fields just obtained. Then the fields \( V_k = \oplus \eta_k \in \text{CVF}(Y) \) span \( T_y Y \) for all \( y \) in a neighborhood of \( y_0 \).

We now enlarge this family in order to generate the tangent spaces at any \( y \in Y \). Notice that the fields \( V_k \) span \( T_y Y \) on \( Y \), minus a proper analytic set \( A \), which we decompose into its (possibly countably many) irreducible components \( A_i \) \((i \geq 1)\). It suffices to show that there exists \( \Psi \in \text{Aut}(X) \) such that \( (\oplus \Psi)(Y \setminus A) \cap A_i \neq \emptyset \) for all \( i \). Indeed, this would imply that the family \( \{(\oplus \Psi)_*(V_k)\}_k \) of complete vector fields spans \( T_{y^*_i} Y \) (where \( a_i \in (\oplus \Psi)(Y \setminus A) \cap A_i \) for each \( i \)), so the enlarged finite collection \( \{(\oplus \Psi)_*(V_k)\}_k \cup \{V_k\}_k \) of complete fields would fail to span the tangent space in an exceptional variety of lower dimension. The conclusion follows from the finite iteration of this procedure. To obtain this automorphism, consider

\[
B_i = \{\Psi \in \text{Aut}(X); (\oplus \Psi)(Y \setminus A) \cap A_i \neq \emptyset\}.
\]

Each \( B_i \) is clearly an open set. To verify that it is also dense, let \( \alpha \in \text{Aut}(X) \) and \( y^* \in A_i \). As above, there are finitely many complete fields \( \Theta_k \) on \( X \) such that \( \Theta \Theta_k \) span the tangent of \( Y \) at \( y^* \). So there is some \( V \in \text{CVF}(X) \) such that \( \Theta V \) is not tangent to \( A_i \). Thus \( V_t \circ \alpha \) is an element in \( B_i \) for small \( t \) (where \( V_t \) is the flow of the field \( V \)). By the Baire category theorem there exists \( \Psi \in \bigcap B_i \) and we are done. \( \square \)

Given complex manifolds \( M \) and \( X \), we say that a map \( \Psi : M \to \text{Aut}(X) \) is holomorphic if the mapping \( M \times X \to X \) given by \((m, x) \mapsto \Psi(m)(x) \) is holomorphic. The following lemma is crucial.

**Lemma 4.** Let \( X \) be a Stein manifold with the density property and fix a metric \( d \) on it. Let \( y_0 = (x^1, \ldots, x^N) \in Y \), \( \epsilon > 0 \) and a compact \( K \subset X \) containing each \( x^j \) be given. Then there is a neighborhood \( U \) of \( y_0 \) in \( Y \) with the following property: given a complex manifold \( W \) and an analytic homotopy \( f : W \times D_r \to Y \) \((r > 1)\) satisfying

\[
f_t(W) \subset U \text{ for all } t \in D_r,
\]

This applies here because \( \text{Aut}(X) \) is a complete metric space: see [KK11].
there exists a holomorphic map $\Psi : D_r \to \text{Aut}^W(X)$ such that $\Psi^w_0 = id_X$ and for all $(w, t) \in W \times D_r$,

(1) $d(\Psi^w_t, id) < \epsilon$ and $d((\Psi^w_0)^{-1}, id) < \epsilon$ on $K$;

(2) and $(\oplus \Psi^w_t) \circ f_0(w) = f_t(w)$.

Proof. By the previous lemma, there are complete vector fields $V_1, \ldots, V_m$ on $X$ such that $\{\oplus V_j(y_0)\}_{j}$ span $T_{y_0}Y$. By discarding linearly dependent elements of the generating set, we can assume that $m = nN$, where $n$ is the dimension of $X$. Let $\phi_j$ be the flow of $V_j$. By completeness its time-$t$ map, denoted $\phi^t_j$, is an automorphism of $X$. Define two holomorphic maps $\phi, \phi_\ast : \mathbb{C}^m \times X \to X$ by

$$
\phi(t, z) = \phi^{t_1}_1 \circ \cdots \circ \phi^{t_m}_m(z)
$$

$$
\phi_\ast(t, z) = \phi^{-t_m}_m \circ \cdots \circ \phi^{-t_1}_1(z).
$$

and consider the holomorphic map $\varphi : \mathbb{C}^m \to \text{Aut}(X)$ given by

$$
\varphi(t) = \phi(t, \cdot) : X \to X;
$$

define $\varphi_\ast$ analogously. We have that $\varphi(0) = id$ and since both $\varphi$ and $\varphi_\ast$ are continuous for the compact-open topology on $\text{Aut}(X)$, there exists a ball $B_R \subset \mathbb{C}^m$ around 0 such that for each $t \in B_R$,

$$
d(\varphi(t), id) < \epsilon/2 \quad \text{and} \quad d(\varphi_\ast(t), id) < \epsilon/2 \quad \text{on} \quad K^c,
$$

where $K^c$ is a compact containing $\{x \in X; d(x, K) < \epsilon\}$. Consider now the map $s : \mathbb{C}^m \to Y$ defined by

$$
s(t) = (\phi(t, x^1), \ldots, \phi(t, x^N)).
$$

Then $s(0) = y_0$ and, for all $j = 1, \ldots, m$,

$$
\frac{\partial s}{\partial t_j}(0) = (V_j(x^1), \ldots, V_j(x^N)) = \oplus V_j(y_0).
$$

Since $\text{Span}\{\oplus V_j(y_0)\}_{j} = T_{y_0}Y$, by the implicit function theorem $s$ is locally biholomorphic on a neighborhood (which we assume contained in $B_R$) of 0 onto a neighborhood $U$ of $y_0$ in $Y$. Since all the considered functions are holomorphic, so is $\psi = \varphi \circ s^{-1} : U \to \text{Aut}(X)$. It clearly satisfies, for each $y \in U$,

1'. $d(\psi(y), id) < \epsilon/2 \quad \text{and} \quad d(\psi^{-1}(y), id) < \epsilon/2 \quad \text{on} \quad K^c$;

2'. $(\oplus \psi(y))(y_0) = y$.

Now set

$$
\tilde{\Psi}^w_t(x) = \psi(f_t(w))(x).
$$

Then $\Psi : D_r \to \text{Aut}^W(X)$, where $\Psi_t(w, x) = \Psi^w_t(x)$ and

$$
\Psi^w_t = \tilde{\Psi}^w_t \circ (\tilde{\Psi}^w_0)^{-1},
$$
satisfies the desired properties. \(\square\)

We call such map \(\Psi : D_r \to \text{Aut}^L(X)\) satisfying \(\Psi_0^w = id\) an **analytic isotopy of parametrized automorphisms**. Of course an analogous result holds if \(f_t\) is a homotopy with \(t\) varying smoothly in \([0,1]\) instead of a complex disc: we obtain a **smooth** (in fact real analytic) isotopy of parametrized automorphisms \(\Psi : [0,1] \to \text{Aut}^L(X)\) satisfying the same properties.

**Remark.** Suppose \(W\) is compact and a single map \(f : W \to Y\) satisfies \(f(W) \subset U\). Consider the map \(s\) as in the proof above. Since \(s^{-1}(f(W))\) is compact in \(B_R\), for \(\eta > 0\) small enough and \(r = 1 + \eta\), the function

\[
    f_t(w) = s(t \cdot (s^{-1} \circ f(w))), \quad (w,t) \in W \times D_r
\]

takes values in \(U\) and defines an analytic homotopy between the constant \(f_0 = y_0\) and \(f : W \to Y\).

### 3. The Oka Property

Stein manifolds with the density property are of interest not only in view of the Andersén-Lempert approximation described previously, but also because they enjoy a flexibility property now referred in the literature as the “Oka property”.

There are several equivalent characterizations of Oka-Forstnerič manifolds; the survey [FL11] gives a detailed account. For our purposes, we define \(Y\) to be an **Oka-Forstnerič manifold** if it enjoys the following property, called in the literature the **Basic Oka Property with approximation and interpolation**:

**Property.** Let \(T\) be a closed complex submanifold of a Stein manifold \(S\), and \(K\) be a \(\mathcal{O}(S)\)-convex compact subset of \(S\). Let \(f : S \to Y\) be a continuous map such that \(f\) is holomorphic in a neighborhood of \(K\), and \(f\) is holomorphic on \(T\). Then there is (a homotopy joining \(f\) to) some holomorphic \(g : S \to Y\) such that \(g = f\) on \(T\) and \(g\) is uniformly close to \(f\) on \(K\).

**Theorem 5.** If \(X\) is a Stein manifold with the density property, then \(Y_{X,N}\) is an Oka-Forstnerič manifold for any \(N\).

**Proof.** In Lemma 3 we showed that there exist finitely many complete vector fields on \(Y\) spanning the tangent space everywhere. This provides a dominating spray on \(Y\), and so \(Y\) is an “elliptic” manifold. It is a theorem of Gromov that such manifolds enjoy the Oka properties. See [For11] for details on this theory. \(\square\)
Corollary 6. Let $W$ be a Stein manifold and $Y$ and Oka-Forstnerič manifold. If $f : W \times [0, 1] \to Y$ is a homotopy between two holomorphic maps $f_0$ and $f_1$, then they are in fact homotopic via an analytic homotopy. In particular they are smoothly homotopic through holomorphic maps.

Proof. Let $r > 1$ and $R : D_r \to [0, 1] \subset \mathbb{C}$ be any continuous retraction of the disc $D_r \subset \mathbb{C}$ onto the interval. Then

$$F : W \times D_r \to Y$$

$$(w, t) \mapsto f_{R(t)}(w)$$

is a continuous map extending $f$ from $W \times [0, 1]$ to $W \times D_r$. Now $T = W \times \partial [0, 1]$ is a closed complex submanifold of the Stein manifold $S = W \times D_r$. The map $F$ is holomorphic when restricted to $W \times \partial [0, 1]$, so according to the Basic Oka Property with interpolation (but no approximation) it can be deformed to a holomorphic map $H : W \times D_r \to Y$, which equals $F$ on $W \times \partial [0, 1]$. □

Observe that this proof does not allow to obtain additionally approximation over a $\mathcal{O}(W)$-convex compact piece $L$, since the Oka property requires holomorphicity on some neighborhood of $L \times [0, 1] \subset W \times D_r$, which we cannot ensure because of the retraction. However in the case that $Y$ is the configuration space of a Stein manifold with the density property, something can be said if the homotopy is of a special type and connects a holomorphic map to a constant. See Theorem 9 in Section 5.

4. Composition of automorphisms

We will use the following criterion for the convergence of an infinite composition of automorphisms. In [For99, Prop. 5.1] this result appears for $X = \mathbb{C}^n$ and $W = \{\emptyset\}$.

Lemma 7. Let $X$ be a Stein manifold equipped with a distance function $d$ and $W$ be any manifold. Suppose $W$ is exhausted by compact sets $L_j$ ($j \geq 1$), and $X$ by compacts $K_j$ ($j \geq 0$). For each $j \geq 1$, let $\epsilon_j$ be a real number such that

$$0 < \epsilon_j < d(K_{j-1}, X \setminus K_j) \quad \text{and} \quad \sum \epsilon_j < \infty.$$

For each $j \geq m \geq 1$, let $\alpha_j \in \text{Aut}^W(X)$, and let $\beta^w_{j,m} \in \text{Aut}(X)$ be defined by

$$\beta^w_{j,m} = \alpha_j^w \circ \cdots \circ \alpha_m^w.$$
Assume that for each \( w \in L_j \setminus L_{j-1} \) (take \( L_0 = \emptyset \)),

\[
\begin{align*}
(1) \quad & d(\alpha^w_j, id) < \epsilon_j \quad \text{on } K_j \\
(2) \quad & d(\alpha^w_{j+s}, id) < \epsilon_{j+s} \quad \text{on } K_{j+s} \cup \beta^w_{j+s-1,j}(K_{j+s}), \quad \forall s \geq 1.
\end{align*}
\]

Then \( \beta = \lim_{m \to \infty} \beta_{m,1} \) exists uniformly on compacts and defines an element in \( \text{Aut}^W(X) \).

**Proof.** Let \( w \in L_1 \). The remark which is the content of [Rit13, Prop. 1] shows that if (1) holds for all \( j \), then the limit \( \beta^w \) is a Fatou-Bieberbach map onto \( X \) defined on the set which consists exactly of the points \( z \) in \( X \) such that the sequence \( \{ \beta^w_{m,j}(z); m \in \mathbb{N} \} \) is bounded. If we assume furthermore that

\[
d(\alpha^w_s, id) < \epsilon_s \quad \text{on } K_s \cup \beta^w_{s-1,1}(K_s), \quad \forall s \geq 2,
\]

which is equation (2) for \( j = 1 \), we can ensure that the set of convergence for \( \beta^w \) is \( X \). Hence \( \{ \beta^w_{m,1} \}_m \) converges to an automorphism of \( X \) if \( w \in L_1 \). For \( w \in L_j \setminus L_{j-1} \) and \( j \geq 2 \), the same reasoning shows that \( \lim_{m \to \infty} \beta^w_{m+j,j} \) is an automorphism and we obtain \( \beta^w \in \text{Aut}(X) \) by precomposing it with the automorphism \( \beta^w_{j-1,1} \). It is clear from the construction that \( \beta \) depends holomorphically on \( w \), since the convergence is uniform on compacts. \( \Box \)

In practice we will construct the automorphisms \( \alpha_j \) for \( j \geq 1 \) inductively. Observe that when defining \( \alpha_j \), there are only \( j \) constraints to satisfy: \( d(\alpha^w_j, id) < \epsilon_j \) should hold

- on \( K_j \) if \( w \in L_j \setminus L_{j-1} \), according to equation (1);
- on \( K_j \cup \beta^w_{j-1,m}(K_j) \) if \( w \in L_m \setminus L_{m-1} \) (\( 1 \leq m \leq j-1 \)), according to (2).

5. **Proof of Theorem 1**

Let \( X, Y \) and \( W \) be as in Theorem 1. Fix from now on a distance function \( d \) on \( X \), an \( N \)-tuple \( \hat{x} = (\hat{x}^1, \ldots, \hat{x}^N) \in Y \), a holomorphic map \( x_0 = (x^1, \ldots, x^N) : W \to Y \), and a homotopy \( x : W \times [0, 1] \to Y \) between \( x_0 \) and \( x_1 = \hat{x} \). The metric \( d \) induces a natural distance in \( Y \): for \( \eta, \zeta \in Y \), let

\[
d_Y(\eta, \zeta) = \max_{j=1, \ldots, N} d(\eta^j, \zeta^j).
\]

For \( \Sigma \subset W \) and \( f = (f^1, \ldots, f^N) : W \to Y \), define

\[
\Gamma_\Sigma(f) = \bigcup_{j=1}^N \{ (w, f^j(w)) ; w \in \Sigma \} \subset W \times X,
\]
the graph of \( f \) in \( X \) over \( \Sigma \). We drop the \( \oplus \) sign in the following case to ease the notation: if \( f : W \to Y \) and \( \alpha \in \text{Aut}(X) \) (or \( \text{Aut}^W(X) \)), we let \( \Gamma_\Sigma(\alpha \circ f) = \Gamma_\Sigma((\oplus \alpha) \circ f) \).

The rest of this section is the iterative construction of maps \( \alpha_j \in \text{Aut}_W(X) \) and the verification of their convergence to an element in \( \text{Aut}_W(X) \). As a first iteration we want to use the Andersén-Lempert theorem to approximate the “motion of following the paths \( x_j \) simultaneously” by an automorphism in \( \text{Aut}_W(X) \). As we will repeatedly do this, we record the following remark.

**Lemma 8.** Let \( f : W \times [0, 1] \to Y \) be a smooth homotopy through holomorphic maps between \( f_0 \) and \( f_1 \). If \( L \subset W \) is an \( \mathcal{O}(W) \)-convex compact, then given \( \epsilon > 0 \) there exists \( A_t \in \text{Aut}_W(X) \), with \( A_0 = \text{id} \), depending smoothly on \( t \) such that

\[
d_Y((\oplus A_t^w) \circ f_0(w), f_t(w)) < \epsilon, \quad \forall (w, t) \in L \times [0, 1].
\]

**Proof.** It suffices to show that there exists a neighborhood \( \Omega \) of \( \Gamma_W(f_0) \) in \( W \times X \) and a smooth isotopy of injective holomorphic maps \( F_t : \Omega \to W \times X \) of the form

\[
F_t(w, z) = (w, F_t^w(z))
\]

with the property that

\[
(\oplus F_t^w) \circ f_0(w) = f_t(w), \quad \forall t \in [0, 1],
\]

and moreover, that the compact set \( F_t(\Gamma_L(f_0)) \) is \( \mathcal{O}(W \times X) \)-convex for all \( t \). Indeed, the lemma will follow from an application of Theorem 2 to \( F \), which yields \( A_t \in \text{Aut}_W(X) \) such that:

\[
d(A_t^w(z), F_t^w(z)) < \epsilon, \quad \forall (w, z) \in \Gamma_L(f_0) \text{ and } t \in [0, 1].
\]

The proof of the “moreover” part is simple: \( F_t(\Gamma_W(f_0)) = \Gamma_W(f_t) \) is an analytic set in \( W \times X \), and \( \mathcal{O}(W \times X) \)-convexity easily follows from the Cartan extension theorem on Stein manifolds.

Fix now \( w \in W \). Consider an embedding \( \iota : X \hookrightarrow \mathbb{C}^l \) and the time-dependent vector fields \( \Theta^j_t \) on \( \mathbb{C}^l \)

\[
\Theta^j_t(z) = \left. \frac{\partial (\iota \circ f_t^j(w))}{\partial t} \right|_{t=s}
\]

which are defined for all \( z \in \mathbb{C}^l \). The exact sequence of holomorphic vector bundles \( 0 \to TX \to T\mathbb{C}^l_{\mid X} \to N_{\mathbb{C}^l/X} \to 0 \) has a holomorphic splitting by Cartan’s theorem B (see for example [For11, 2.4.5]). In particular there is a projection

\[
\pi_x : T_{\iota(x)} \mathbb{C}^l \to T_x X
\]
which depends holomorphically on $x \in X$. By projecting $\Theta^i$ to $TX$, we obtain a time-dependent vector field on $X$ coinciding with the tangent vectors
\[
\left. \frac{\partial (f^i_t(w))}{\partial t} \right|_{t=s} \in T_{f^i_s(w)}X
\]
at $f^i_s(w)$. Let $\phi^w_{j,t}$ be the time-$t$ map of its local flow starting at $f^i_0(w)$. By the local existence of solutions to ODE’s, the field may be flown for a time strictly greater than 1. Since this lifespan time is a lower semi-continuous function on $X$, there exists a neighborhood $\Omega^i(w)$ of $f^i_0(w)$ such that $\phi^w_{j,t}(z)$ exists for all $t \in [0, 1]$ and $z \in \Omega^i(w)$. We may shrink each $\Omega^i(w)$ so that, for each $i \neq j$,
\[
\phi^w_{j,t}(\Omega^j(w)) \cap \phi^w_{i,t}(\Omega^i(w)) = \emptyset, \quad \forall t \in [0, 1],
\]
and such that $\Omega = \bigcup_{w \in W} \{w\} \times \Omega(w)$ is open. Then $\Omega$ and $F : \Omega \times [0, 1] \to W \times X$ defined by $F_t(w, z) = (w, \phi^w_{j,t}(z))$ for $z \in \Omega^j(w)$ clearly satisfy the required conditions.

Fix a compact exhaustion of $W \times X$, of the form $W = \bigcup_{j=1}^{\infty} L_j$ and $X = \bigcup_{j=0}^{\infty} K_j$, where each $L_j$ and $K_j$ are $\mathcal{O}(W)$ (resp. $\mathcal{O}(X)$)-convex compact sets, as well as real numbers $\epsilon_j$ ($j \geq 1$) such that $0 < \epsilon_j < d(K_{j-1}, X \setminus K_j)$ and $\sum \epsilon_j < \infty$. We can assume that $K_0$ contains $\hat{x}^j$ for all $j = 1, \ldots, N$.

By Corollary 6, we can assume that $x_0$ and $\hat{x}$ are smoothly homotopic through holomorphic maps $x_t : W \to Y$. So by Lemma 8, for any $\epsilon > 0$ there exists $\alpha \in \text{Aut}^W(X)$ such that
\[
d_Y(\oplus \alpha^w \circ x_0(w), \hat{x}) < \epsilon, \quad \forall w \in L_1.
\]
To continue we would like to construct automorphisms approximating the motion which is given by a “small homotopy” over $L_1$ and by the abstract homotopy on the rest of $W$. However to use Theorem 2 we need to glue these homotopies and obtain holomorphic data. This is the content of the next theorem.

**Theorem 9.** Let $f : W \times [0, 1] \to Y$ be a smooth homotopy through holomorphic maps connecting $f_0$ to some constant $f_1 = \hat{x}$. Given a $\mathcal{O}(W)$-convex compact $L$ and $\epsilon > 0$ small enough, there exists $\alpha \in \text{Aut}^W(X)$ and a smooth homotopy through holomorphic maps $h : W \times [0, 1] \to Y$ with $h_0 = \oplus \alpha \circ f_0$, $h_1 = \hat{x}$, and
\[
d_Y(h_t(w), \hat{x}) < \epsilon, \quad \forall (w, t) \in L \times [0, 1].
\]
We prove this in two steps. First, a topological observation.
Lemma 10. Let $M$ be a smooth manifold with metric $d$, $W$ a compact manifold and $f : W \times [0, 1] \to M$ a smooth homotopy between a smooth map $f_0$ and a constant $f_1 = c$. Then there exists an $\epsilon > 0$ with the following property: for any homotopy $F : W \times [0, 1] \to M$ such that $F_t = f_{2t-1}$ for $t \geq 1/2$ and
\[
d(F_t(w), F_{1-t}(w)) < \epsilon, \quad \forall (w, t) \in W \times [0, 1],
\]
and for any smooth homotopy $\sigma : W \times [0, 1] \to M$ with $\sigma_0 = F_0$, $\sigma_1 = c$ and
\[
d(\sigma_t(w), c) < \epsilon, \quad \forall (w, t) \in W \times [0, 1],
\]
there exists a “homotopy of homotopies” between $\sigma$ and $F$ (fixing the endpoints).

Proof. The image under the continuous map $f$ of $W \times [0, 1]$ is a compact set in $M$, on which the injectivity radius for the metric $d$ is bounded from below by a positive constant $\epsilon$.

Consider $w \in W$ fixed. For each $s \in [0, 1]$ there is a unique geodesic path in $M$ from $F_s(w)$ to $F_{1-s}(w)$. By following it at constant speed, the parametrization $\gamma_s^w : [0, 1] \to M$ is uniquely determined. Let $h_s^w : [0, 1]_s \times [0, 1]_t \to M$ be defined by
\[
h_s^w(t) = \begin{cases} 
F_t(w) & \text{if } 0 \leq t \leq s/2 \\
\gamma_{s/2}^w(l(t)) & \text{if } s/2 \leq t \leq 1 - s/2 \\
F_t(w) & \text{if } 1 - s/2 \leq t \leq 1,
\end{cases}
\]
where $l$ is the linear function of $t$ taking values $0$ at $s/2$ and $1$ at $1 - s/2$. This is a well-defined homotopy between $F$ and the geodesic segment $h_0^w$ going from $F_0(w)$ to the constant $c$; it is uniquely defined for each $w$. By letting $w$ vary in $W$, all the elements in the definition of $h_s(t)$ vary continuously, so $h : W \times [0, 1] \times [0, 1] \to M$ provides a homotopy of homotopies between $F$ and the geodesic segment $h_0$.

Now it suffices to connect $\sigma : W \times [0, 1] \to M$ to $h_0 : W \times [0, 1] \to M$. This is achieved in a similar way, and we conclude by composing the previous homotopy with this one. \hfill \Box

Proposition 11. Let $L$ be a $O(W)$-convex compact set, and $f_t : W \to Y$ be a smooth homotopy between some holomorphic map $f_0$ and the constant $f_1 = \hat{x}$. Then there exists an $\epsilon > 0$ with the following property: for every smooth $F : W \times [0, 1] \to Y$ with $F_t = f_{2t-1}$ for $t \geq 1/2$ satisfying
\[
d_Y(F_t(w), F_{1-t}(w)) < \epsilon/2, \quad \forall (w, t) \in L \times [0, 1],
\]
and every $L^*$ a $O(W)$-convex compact such that $L^* \subset \text{int}(L)$, there exists an analytic homotopy $H : W \times D_r \to Y$ between $F_0$ and $\hat{x}$ such
that
\[ d_Y(H_t(w), \hat{x}) < \epsilon, \quad \forall (w, t) \in L^* \times D_r. \]

**Proof.** Apply the previous lemma to \((Y, d_Y)\) and \(L\) as the compact parameter space and let \(\epsilon > 0\) be its output. If necessary, make \(\epsilon\) smaller so that the remark following Lemma 4 can apply: there exists an analytic homotopy \(S : L \times D_R \to Y\) between \(S_0 = F_0\) and \(S_1 = \hat{x}\) such that
\[ d_Y(S_t(w), \hat{x}) < \epsilon/2, \quad \forall (w, t) \in L \times D_R. \]

Denote by \(\sigma\) its restriction to \(L \times [0, 1]\). Now Lemma 10 gives a continuous \(h : L \times [0, 1]_s \times [0, 1]_t \to Y\) such that
\[
\begin{align*}
  h(w, 0, t) &= F_t(w) \quad h(w, s, 0) = \sigma_0(w) \\
  h(w, 1, t) &= \sigma_t(w) \quad h(w, s, 1) = \sigma_1(w).
\end{align*}
\]

Let \(\rho : D_R \to [0, 1] \to D_R\) be a homotopy between the identity \(\rho_0\) and a continuous retraction \(\rho_1 : D_R \to [0, 1]\). Set \(\tilde{F}(w, t) = F_{\rho_1(t)}(w)\) for \((w, t) \in W \times D_R\) and extend \(h\) to \(L \times [0, 1]_s \times D_R\) by defining
\[
H(w, s, t) = \begin{cases} 
  \tilde{F}(w, \rho_3(s)(t)) & \text{if } 0 \leq s \leq 1/3 \\
  h(w, 3s - 1, \rho_1(t)) & \text{if } 1/3 \leq s \leq 2/3 \\
  S(w, \rho_{3-3s}(t)) & \text{if } 2/3 \leq s \leq 1.
\end{cases}
\]

Let \(U\) be a neighborhood of \(L^*\) such that \(\overline{U} \subset \text{int}(L)\). Then there exists a smooth function \(\chi : W \to [0, 1]\) such that \(\chi|_U = 1\) and \(\chi|_{W \setminus L} = 0\). Define \(\tilde{H} : W \times D_R \to Y\) by
\[
\tilde{H}(w, t) = \begin{cases} 
  H(w, \chi(w), t) & \text{if } w \in L, \\
  \tilde{F}(w, t) & \text{if } w \notin L.
\end{cases}
\]

Consider the inclusion of the closed complex submanifold \(T = W \times \partial[0, 1]\) into the Stein manifold \(\mathcal{S} = W \times D_R\). The map \(\tilde{H}\) is continuous on \(\mathcal{S}\), restricts to the holomorphic maps \(\sigma_0\) and \(\sigma_1\) on \(T\), and is equal to the holomorphic \(S\) on a neighborhood of the \(\mathcal{O}(\mathcal{S})\)-convex set \(L^* \times D_r\) (for some \(1 < r < R\)). By the Oka Property, there is a holomorphic map \(H : \mathcal{S} \to Y\) which restricts to \(\tilde{H}\) on \(T\) and approximates \(\tilde{H}\) on \(L^* \times D_r\). \(\square\)

**Proof of Theorem 9.** Pick a \(\mathcal{O}(W)\)-convex compact \(L^+\) such that \(\text{int}(L) \subset L^+\). By Lemma 8, given \(\epsilon > 0\) there is some \(A_t \in \text{Aut}^W(X)\) depending smoothly on \(t\) such that \(A_0 = \text{id}\) and
\[ d_Y(\oplus A_t^w \circ f_0(w), f_t(w)) < \epsilon/2, \quad \forall (w, t) \in L^+ \times [0, 1]. \]
Let $\alpha = A_1$. Define $F : W \times [0, 1] \to Y$ by
\[
F_t(w) = \begin{cases} 
\oplus A_{1-2t}^\nu \circ f_0(w) & \text{if } t \leq 1/2 \\
 f_{2t-1}(w) & \text{if } t \geq 1/2.
\end{cases}
\]
This is a smooth homotopy between the holomorphic map $\oplus \alpha \circ f_0$ and $\hat{x}$. By the above inequality $F_t$ satisfies (4), hence by Proposition 11 for $\epsilon > 0$ small enough there exists (in particular) a smooth homotopy through holomorphic maps $h : W \times [0, 1] \to Y$ between $h_0 = \oplus \alpha \circ f_0$ and $h_1 = \hat{x}$ with approximation
\[
d_Y(h_t(w), \hat{x}) < \epsilon, \quad \forall (w, t) \in L \times [0, 1].\]

Next we show how to extend the homotopy obtained in Theorem 9 to an isotopy of holomorphic maps on a larger compact in $W$ containing $L_1$ and with suitable approximation over some large compact $L_1 \times K$. We begin with a simple observation.

**Lemma 12.** Let $L_1 \subset L_2$ be $\mathcal{O}(W)$-convex subsets of $W$, $K$ a $\mathcal{O}(X)$-convex set, and $A$ an analytic subvariety in $W \times X$. Then $P = (L_1 \times K) \cup (A \cap (L_2 \times X))$ is $\mathcal{O}(W \times X)$-convex.

**Proof.** Let $(w_0, x_0)$ be in the complement of $P$. If $w_0 \in W \setminus L_2$, define $f(w, x) = g(w)$ where $g : W \to \mathbb{C}$ takes value 1 at $w_0$ and $|g|_{L_2} < 1$. Then $|f|_P < 1$ and $f(w_0, x_0) = 1$. If $w_0 \in L_2 \setminus L_1$, extend the function which is 0 on $A$ and 1 on $(w_0, x_0)$ to a holomorphic function $h$ on $W \times X$ (Cartan’s theorem) and pick $g \in \mathcal{O}(W)$ with $g(w_0) = 1$ and $|g|_{L_1} < 1$. For large $N$, the holomorphic function $h(w, x) \cdot g(w)^N$ is smaller than 1 in norm on $L_1 \times K$, vanishes on $A$, and takes value 1 at $(w_0, x_0)$. Finally if $w_0 \in L_1$ a similar reasoning allows us to conclude. \hfill \square

**Proposition 13.** Let $\eta > 0$ and $K$ a compact in $X$ containing each $\hat{x}^j$ be given. Then there exists a real number $\delta(K, \eta) > 0$ with the following property. If $h : W \times [0, 1] \to Y$ is a smooth homotopy through holomorphic maps, with $h_1 = \hat{x}$ and approximation
\[
d_Y(h_t(w), \hat{x}) < \delta(K, \eta), \quad \forall (w, t) \in P_1 \times [0, 1],
\]
where $P_1 \subset W$ is a $\mathcal{O}(W)$-convex compact, then:
1) There exists a smooth isotopy of parametrized automorphisms $\Psi : [0, 1] \to \text{Aut}^W(X)$, such that for all $(w, t) \in P_1 \times [0, 1],$
\[
\oplus \Psi^w_t \circ h_0(w) = h_t(w),
\]
(5)
\[
d(\Psi^w_t, \text{id}) < \eta \text{ on } K.
\]
2) Given $\epsilon > 0$, $P_2$ a $\mathcal{O}(W)$-convex compact containing $P_1$, and a $\mathcal{O}(X)$-convex compact $C$, there exists a smooth isotopy $A_t \in \text{Aut}^W(X)$
such that
\begin{equation}
\label{eq:6}
d(A_t^w(z), \Psi_t^w(z)) < \eta, \quad \forall (w, z, t) \in P_1 \times C \times [0, 1]
\end{equation}
and
\[d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \epsilon \text{ on } P_2 \times [0, 1].\]

In practice we will use a constant $\delta$ smaller than the one described above, in fact smaller than the constant referred to in Proposition 11. We have preferred, for clarity, to leave this minor detail out of the statement of the proposition.

Proof. 1) The existence of $\delta(K, \eta)$ with these properties follows immediately from Lemma 4.

2) Define a time-dependent vector field on $P_1 \times \mathcal{X}$ by
\[\Theta_s(w, x) = \left. \frac{\partial}{\partial t} \Psi_t^w((\Psi_t^w)^{-1}(x)) \right|_{t=s}.\]

It satisfies, for each $j = 1, \ldots, N$ and $s \in [0, 1]$,
\[\Theta_s(w, h_t^j(w)) = \left. \frac{\partial}{\partial t} h_t^j(w) \right|_{t=s},\]
which implies that $\Theta_s(w, x)$ is a vector field on $(P_1 \times \mathcal{X}) \cup \Gamma_P(\hat{x})$. We will show that, for each $s$, this field can be extended to $P_2 \times \mathcal{X}$ with approximation on $P_1 \times C$.

There is a smooth isotopy of parametrized automorphisms $\beta_t \in \text{Aut}^P_2(\mathcal{X})$ such that $\beta_1 = \text{id}$ and $(\oplus \beta_t) \circ h_t = \hat{x}$. Indeed, observe that Lemma 8 applied to $h_t$ provides $\tilde{B}_t \in \text{Aut}^W(\mathcal{X})$, depending smoothly on $t$, with the property that $\tilde{B}_t = \tilde{B}_1 \circ \tilde{B}_t^{-1}$ maps $\Gamma_{P_2}(h_t)$ arbitrarily close to $\Gamma_{P_2}(\hat{x})$. Hence Lemma 4 applied to $\oplus B_t \circ h_t : W \to Y$ gives elements $\Phi_t \in \text{Aut}^P(\mathcal{X})$, depending smoothly on $t$, such that
\[\oplus (\Phi_t^w \circ B_t^w) \circ h_t(w) = \oplus B_0^w \circ h_0(w), \quad \forall w \in P_2.\]

Then $\Phi_1^{-1} \circ \Phi_t \circ B_t \in \text{Aut}^P_2(\mathcal{X})$ is the desired $\beta_t$.

The push-forwards $(\beta_t)_*(\Theta_t)$ define together a time-dependent vector field on $(P_1 \times \mathcal{X}) \cup \Gamma_{P_2}(\hat{x})$. By a classical result of Grauert and Cartan (see Thm 2.1bis in [Car58], or [For11]) it can be extended from the analytic subvariety $\Gamma_{P_2}(\hat{x})$ to $P_2 \times \mathcal{X}$ with arbitrary approximation on a large $O(W \times \mathcal{X})$ compact of the form $P_1 \times \tilde{K}$ containing
\begin{equation}
\label{eq:7}
\beta_{[0,1]}^{P_1}(C) = \{ \beta_t^w(x); w \in P_1, x \in C, t \in [0, 1] \} \subset \mathcal{X}.
\end{equation}

Its pull-back is an extension of the time-dependent vector field $\Theta$ above, whose flow provides, as in the proof of Lemma 8, an isotopy of injective
holomorphic maps $F_t : \Omega \to W \times X$, where $\Omega$ is a neighborhood of $\Gamma_{P_2}(h_0)$ containing $P_1 \times C$, and such that

\begin{align*}
(8) \quad d(F_t^w(z), \Psi_t^w(z)) < \eta/2, \quad \forall (w, t, z) \in P_1 \times [0, 1] \times C \\
(9) \quad \oplus F_t^w \circ h_0(w) = h_t(w), \quad \forall (w, t) \in P_2 \times [0, 1].
\end{align*}

By Lemma 12, we may apply Theorem 2 to $F_t$ and obtain $A_t \in \text{Aut}^W(X)$ such that

$$d(A_t^w(z), F_t^w(z)) < \min(\epsilon, \eta/2)$$
on $(P_1 \times C) \cup \Gamma_{P_2}(h_0)$. This and (8) show that (6) hold. Furthermore, by (9),

$$d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \epsilon, \quad \forall (w, t) \in P_2 \times [0, 1]. \quad \square$$

**Proof of Theorem 1.** By Corollary 6, $x_0$ and $\hat{x}$ are smoothly homotopic through holomorphic maps. Hence Theorem 9 gives $\alpha_0 \in \text{Aut}^W(X)$ and a smooth homotopy of holomorphic maps $h : W \times [0, 1] \to Y$ between $h_0 = \oplus \alpha_0 \circ x_0$ and $h_1 = \hat{x}$ with

$$d_Y(h_t(w), \hat{x}) < \delta(K_1, \epsilon_1/2), \quad \forall (w, t) \in L_1 \times [0, 1].$$

Apply the first part of Proposition 13 to $h_t$: we obtain $\Psi : [0, 1] \to \text{Aut}^{L_1}(X)$. Consider the compact

$$\Psi_{[0, 1]}^{L_1}(K_2)$$

(recall the notation from equation (7)) and define $C_1$ to be a $O(X)$-convex compact containing its $(\epsilon_1/2)$-envelope (the compact containing the points in $X$ at distance no greater than $\epsilon_1/2$ to it). By part 2) of Proposition 13, we obtain a smooth isotopy of automorphisms $A_t \in \text{Aut}^W(X)$ such that

$$d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \delta(C_1, \epsilon_2/2)/2, \quad \forall (w, t) \in L_2^+ \times [0, 1],$$

where $L_2^+$ is a $O(W)$-convex compact containing $L_2$. Now by combining (5) and (6), we get

$$d(A_t^w(z), z) < \epsilon_1, \quad \forall (w, t, z) \in L_1 \times [0, 1] \times K_1.$$ We let $\alpha_1 = A_1$. Then in particular

$$d(\alpha_1^w, id) < \epsilon_1$$
on $L_1 \times K_1$.

Thus $\alpha_1$ satisfies the only condition imposed by Lemma 7. Observe finally that by (6), $\alpha_1^w(K_2) \subset C_1$ for $w \in L_1$.

We are now ready to construct inductively $\alpha_j$ for $j \geq 2$. Fix $j \geq 2$ and assume that we have defined $C_k \subset X$ and $\alpha_k \in \text{Aut}^W(X)$, for all $1 \leq k \leq j$, such that the following conditions hold (recall that $\beta_j^w = \alpha_j^w \circ \cdots \circ \alpha_1^w$ and $\beta_j = \beta_j^0$):
We have just verified that these conditions hold for $j$. Indeed, by condition 4, Lemma 7 implies that $\beta = \lim_{t \to 1} \beta_j \circ x_0$ and $\hat{x}$, (and $L^+_{j+1} \supset L_{j+1}$ is $O(W)$-convex);

(3) $C_j$ contains $K_{j+1}$, and $\{\beta^w_{j,m}(K_{j+1}) ; w \in L_m \setminus L_{m-1}\}$ for every $1 \leq m \leq j$;

(4) and every $A^w_t$ satisfies the $j$ conditions of Lemma 7, that is, for every $1 \leq m \leq j$, if $w \in L_m \setminus L_{m-1}$, then $d(A^w_t \circ id) < \epsilon_j$ on $K_j \cup \beta^w_{j-1,m}(K_j)$.

We have just verified that these conditions hold for $j = 1$. It now suffices to show that $\alpha_{j+1}$ and $C_{j+1}$ can be constructed satisfying the above conditions: indeed, by condition 4, Lemma 7 would imply that $\beta = \lim_{j \to \infty} \beta_{j,1} \in \text{Aut}^W(X)$ exists, and by construction it maps $\Gamma_W(\alpha_0 \circ x_0)$ to $\hat{x}$, so $\beta \circ \alpha_0 \in \text{Aut}^W(X)$ would be the simultaneous standardization.

Pick a $O(W)$-convex compact $L^+_{j+1}$ containing the interior of $L_{j+1}$. By the comment following Proposition 13, we can apply Proposition 11 to the mapping $F : W \times [0, 1] \to Y$ given by

$$F_t(w) = \begin{cases} +A_{t-2}^w \circ h_0(w) & \text{if } t \leq 1/2 \\ h_{2t-1}(w) & \text{if } t \geq 1/2, \end{cases}$$

where $A$ and $h$ are as in conditions 1 and 2 at step $j$. We obtain a smooth homotopy through holomorphic maps $H : W \times [0, 1] \to Y$, such that $H_0 = +\beta_j \circ x$ and $H_1 = \hat{x}$ and for all $t \in [0, 1]$,

$$d_Y(H_t(w), \hat{x}) < \delta(C_j, \epsilon_{j+1}/2), \quad \forall w \in L_{j+1}.$$  

By the first part of Proposition 13 there is a smooth isotopy

$$\Psi : [0, 1] \to \text{Aut}^{L_{j+1}}(X)$$

with $+\Psi_t^w \circ H_0(w) = H_t(w)$ and

$$(10) \quad d(\Psi_t^w \circ id) < \epsilon_{j+1}/2 \text{ on } L_{j+1} \times C_j.$$  

Define $C_{j+1}$ to be a $O(X)$-convex compact containing the $(\epsilon_{j+1}/2)$-envelope of

$$C_j \cup \Psi_{[0, 1]}^{L_{j+1}}(K_{j+2}) \cup \bigcup_{1 \leq m \leq j} \Psi_{[0, 1]}^{L_m}(\beta_{j,m}(K_{j+2})).$$
By the second part of Proposition 13, there are $A^w_t \in \text{Aut}^W(X)$ smoothly depending on $t$ such that
\begin{align}
(12) \quad d(A^w_t, \Psi^w_t) < \epsilon_{j+1}/2 \text{ on } L_{j+1} \times C_{j+1} \times [0,1] \\
d_Y(\oplus A^w_t \circ H_0(w), H_t(w)) < (C_{j+1}, \epsilon_{j+2}/2)/2 \text{ on } L_{j+2}^+ \times [0,1],
\end{align}
which means that condition 2 of the induction is met at step $j + 1$; condition 1 is obviously satisfied for $\alpha_{j+1} = A_1 \in \text{Aut}^W(X)$.

Let us check that condition 4 holds at step $j + 1$. Note that by (12) and (10),
\[d(A^w_t, id) < d(A^w_t, \Psi^w_t) + d(\Psi^w_t, id) < \epsilon_{j+1} \text{ on } L_{j+1} \times C_j.\]
If $w \in L_{j+1} \setminus L_j$, then clearly $d(A^w_t, id) < \epsilon_{j+1}$ on $K_{j+1}$. Now let $w \in L_m \setminus L_{m-1}$, where $1 \leq m \leq j$. Since by hypothesis $C_j$ contains $K_{j+1} \cup \beta^w_{j,m}(K_{j+1})$, $d(A^w_t, id) < \epsilon_{j+1}$ holds on $K_{j+1} \cup \beta^w_{j,m}(K_{j+1})$.

It remains to show that $C_{j+1}$ satisfies condition 3. Since $\Psi_0 = id$, it contains $K_{j+2}$. Let $1 \leq m \leq j + 1$, $w \in L_m \setminus L_{m-1}$ and $z \in K_{j+2}$. By the definition of $C_{j+1}$, it suffices to check that
\[(13) \quad d(\beta_{j+1,m}^w(z), z') < \epsilon_{j+1}/2\]
where $z'$ is some element of the compact (11). If $m = j + 1$, pick $z' = \Psi^w_t(z)$. Then (13) follows from (12). If $m < j + 1$, let $z' = \Psi^w_t(\beta_{j,m}^w(z))$, which belongs to (11). Then
\[d(\beta_{j+1,m}^w(z), z') = d(\alpha_{j+1}(\beta_{j,m}^w(z)), \Psi^w_t(\beta_{j,m}^w(z))) < \epsilon_{j+1}/2\]
where the inequality again follows from (12), since $\beta_{j,m}^w(z) \in C_{j+1}$. The induction is complete.

To conclude, observe that by conditions 1 and 4, the parametric automorphism $\beta \circ \alpha_0$ is homotopic to the identity via
\[\lim_{j \to \infty} A_t^{(j)} \circ \cdots \circ A_t^{(1)} \circ A_t^{(0)} \in \text{Aut}^W(X),\]
where each $A_t^{(j)}$ is the isotopy of condition 1 at step $j$. Since the convergence is uniform on compacts, these objects depend continuously on $t$. Hence $\beta \circ \alpha_0$ lies in the path-connected component of the identity in $\text{Aut}^W(X)$. □

6. Concluding Remarks

In this section we change slightly our point of view. With $W$ and $X$ as before, we consider $\text{Hol}(W, Y_{X,N})$, the space of $N$ parametrized points in $X$. We identify the group $\text{Aut}^W(X)$ with the group of holomorphic mappings from $W$ to $\text{Aut}(X)$, which we denote by $G = \text{Hol}(W, \text{Aut}(X))$. We naturally get an identification between $G_0$, the
path-connected component of the identity in $G$, with $\text{Aut}^W(X)_0$. The group $G$ acts on the space $\text{Hol}(W,Y_{X,N})$ by
\[
\alpha \cdot x = (\oplus \alpha(w)) \circ x(w)
\]
where $x = (x^1, \ldots, x^N) \in \text{Hol}(W,Y_{X,N})$ as before. It also acts on the space of homotopy classes (or path-connected components), which we denote here by $[\text{Hol}(W,Y_{X,N})]$. Since the path-connected component $G_0$ of the identity in $G$ acts trivially, we get an action of $G/G_0$, the space of homotopy classes $[\text{Hol}(W,\text{Aut}(X))]$ of holomorphic maps from $W$ to $\text{Aut}(X)$, on $[\text{Hol}(W,Y_{X,N})]$. Then an immediate consequence of Theorem 1 can be phrased as follows.

**Corollary 14.** Any $x \in \text{Hol}(W,Y_{X,N})$ is simultaneously standardizable if and only if $G/G_0$ acts transitively on $[\text{Hol}(W,Y_{X,N})]$.

By Theorem 5, $Y_{X,N}$ is an Oka-Forstnerič manifold. Hence the Oka principle, or weak homotopy equivalence principle (see e.g. [For11, 5.4.8]) applies: $[\text{Hol}(W,Y_{X,N})]$ is isomorphic to the space of homotopy classes $[\text{Cont}(W,Y_{X,N})]$ of continuous maps from $W$ to $Y_{X,N}$. Thus we deduce:

**Corollary 15.** Any $x \in \text{Hol}(W,Y_{X,N})$ is simultaneously standardizable if and only if $G/G_0$ acts transitively on $[\text{Cont}(W,Y_{X,N})]$.

Let us consider the special case $X = \mathbb{C}^n$, $n > 1$. The group of holomorphic automorphisms $\text{Aut}(\mathbb{C}^n)$ admits a strong deformation retract onto $\text{GL}_n(\mathbb{C})$. Therefore
\[
[\text{Hol}(W,\text{Aut}(\mathbb{C}^n))] \cong [\text{Hol}(W,\text{GL}_n(\mathbb{C}))]
\]
\[
[\text{Cont}(W,\text{Aut}(\mathbb{C}^n))] \cong [\text{Cont}(W,\text{GL}_n(\mathbb{C}))].
\]
By the Oka principle (since $\text{GL}_n(\mathbb{C})$ is Oka-Forstnerič),
\[
[\text{Hol}(W,\text{GL}_n(\mathbb{C}))] \cong [\text{Cont}(W,\text{GL}_n(\mathbb{C}))].
\]
As a consequence, the following purely topological characterization of simultaneous standardization can be deduced from our main theorem.

**Corollary 16.** Any $x \in \text{Hol}(W,Y_{\mathbb{C}^n,N})$ is simultaneously standardizable if and only if $[\text{Cont}(W,\text{GL}_n(\mathbb{C}))]$ acts transitively on $[\text{Cont}(W,Y_{\mathbb{C}^n,N})]$.

We also see from equations (14) to (16) that
\[
[\text{Cont}(W,\text{Aut}(\mathbb{C}^n))] \cong [\text{Hol}(W,\text{Aut}(\mathbb{C}^n))].
\]
which is a partial Oka principle of the infinite-dimensional manifold \( \text{Aut}(\mathbb{C}^n) \). We can ask the following question: is it true that for any Stein manifold \( X \) with the density property, we have that

\[
[\text{Cont}(W, \text{Aut}(X))] \cong [\text{Hol}(W, \text{Aut}(X))].
\]

In other words, is there an “Oka theory” for infinite-dimensional manifolds, and are the groups \( \text{Aut}(X) \) for \( X \) a Stein manifold with the density property, Oka-Forstnerič manifolds in any such sense?

We end this section with two examples. The first shows that even if the map \( x \in \text{Hol}(W,Y_{\mathbb{C}^2,2}) \) is not null-homotopic, there may be a simultaneous standardization. The second is an example where the topological obstruction does prevent from simultaneous standardization, i.e., in this example \([\text{Cont}(W,\text{GL}_n(\mathbb{C}))]\) does not act transitively on \([\text{Cont}(W,Y_{\mathbb{C}^n,2})]\).

**Example.** Let \( W \) be any Stein manifold. Then any \( x \in \text{Hol}(W,Y_{\mathbb{C}^2,2}) \) is simultaneously standardizable.

**Proof.** Let

\[
x = (x_1(w), x_2(w)) = \left( \begin{pmatrix} z_1(w) \\ \eta_1(w) \end{pmatrix}, \begin{pmatrix} z_2(w) \\ \eta_2(w) \end{pmatrix} \right),
\]

and define

\[
\alpha_1^w(z, \eta) = (z - z_1(w), \eta - \eta_1(w))
\]

\[
\alpha_2^w(z, \eta) = \left( \begin{array}{c} z_2(w) - z_1(w) \\ \eta_2(w) - \eta_1(w) \end{array} \right) \cdot \left( \begin{array}{c} f(w) \\ g(w) \end{array} \right)
\]

Observe that \( z_2(w) - z_1(w) \) and \( \eta_2(w) - \eta_1(w) \) have no common zeros. Hence, since \( W \) is Stein, Cartan’s theorem B implies that there are \( f, g \in \mathcal{O}(W) \) such that

\[
\left( \begin{array}{c} z_2(w) - z_1(w) \\ \eta_2(w) - \eta_1(w) \end{array} \right) \in \text{SL}_2(\mathbb{C}).
\]

Hence \( \alpha_1, \alpha_2 \in \text{Aut}^W(\mathbb{C}^2) \) and \( (\alpha_2^{-1})^w \circ \alpha_1^w \) maps \( x_1(w) \) to \((0,0)\) and \( x_2(w) \) to \((1,0)\), which gives the simultaneous standardization. \( \square \)

As a consequence, by Corollary 16, \([\text{Cont}(W,\text{GL}_2(\mathbb{C}))]\) acts transitively on \([\text{Cont}(W,Y_{\mathbb{C}^2,2})]\). Now consider the special case \( W = \text{SL}_2(\mathbb{C}) \). Then there exists a non null-homotopic \( x \in \text{Hol}(\text{SL}_2(\mathbb{C}),Y_{\mathbb{C}^2,2}) \) which can be standardized with an element not in \((\text{Hol}(\text{SL}_2(\mathbb{C}),\text{Aut}(\mathbb{C}^2)))_0\).

Indeed, the holomorphic map \( \text{SL}_2(\mathbb{C}) \to Y_{\mathbb{C}^2,2} \) given by

\[
A \mapsto \left( A \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)
\]
induces the identity mapping on the 3-sphere (by projection to the first factor of $Y_{C^2,2}$), so is not a null-homotopic map.

**Example.** Let $W$ be a small (so that the map below gives pairwise different points) Grauert tube around $SU_2$, i.e., a Stein neighborhood of $SU_2$ in $SL_2(C)$ which contracts onto the 3-sphere $SU_2$. Then $x \in \text{Hol}(W, Y_{C^2,3})$ defined by

$$A \mapsto \begin{pmatrix} A \frac{1}{0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is not simultaneously standardizable.

**Proof.** Consider the map $\phi: Y_{C^2,3} \to S^3 \times S^3$ given by

$$(x_1, x_2, x_3) \mapsto \left( \frac{x_2 - x_3}{|x_2 - x_3|}, \frac{x_1 - x_2}{|x_1 - x_2|} \right)$$

Since $W$ contracts to $SU_2 \cong S^3$ the composition $\phi \circ x: W \to S^3 \times S^3$ gives a map from $S^3 \to S^3 \times S^3$. It has bidegree $(0, 1)$ and applying any element in $\text{Hol}(W, \text{Aut}(C^2)) \cong [\text{Cont}(W, GL_2(C))]$ to it, changes both degrees by the same amount, so the corresponding bidegree will never be $(0, 0)$. Therefore no application of an element in $\text{Hol}(W, \text{Aut}(C^2))$ to $x$ can lead to a null-homotopic map. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BERN, SIDLERSTRASSE 5, CH-3012. BERN, SWITZERLAND.

E-mail address: frank.kutzschebauch@math.unibe.ch

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BERN, SIDLERSTRASSE 5, CH-3012. BERN, SWITZERLAND.

E-mail address: alexandre.ramos@math.unibe.ch