Spinning squashed extra dimensions, chiral gauge theory and hierarchy from $\mathcal{N} = 4$ SYM

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Abstract

New solutions of $SU(N) \mathcal{N} = 4$ SYM on $\mathbb{R}^4$ interpreted as spinning self-intersecting extra dimensions are discussed. Remarkably, these backgrounds lead to a low-energy sector with 3 generations of chiral fermions coupled to scalar and gauge fields, with standard Lorentz-invariant kinematics. This sector arises from zero modes localized in the extra dimensions, which are oblivious to the background rotation at low energies. In addition there is a sector of “heavy” excitations which is not described by a Lorentz-invariant field theory, but is argued to be suppressed at low energies assuming that resonances can be avoided. Depending on the rotation frequencies, some of the low-energy scalar fields acquire a VEV, and large hierarchies can naturally be stabilized by the background. We identify configurations which may lead to a low-energy physics not far from the broken phase of the standard model.
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# 1 Introduction

Simplicity has always been a central guiding principle in theoretical physics. In the theory of fundamental interactions, this leads naturally to the idea of grand unified models. Another idea is that a simple higher-dimensional theory, such as string theory, might explain the rich low-energy phenomenology in terms of some compactification $R^4 \times K$. Remarkably, both ideas can be combined and realized within 4-dimensional gauge theory, through a geometrical version of the Higgs mechanism. For example, starting with $\mathcal{N} = 4$ super-Yang-Mills (SYM) amended by a suitable cubic potential, the behavior of a higher-dimensional theory on $R^4 \times K_N$ may emerge in a non-trivial vacuum, within a certain range of energies. Here $K_N$ is some approximation to a compact space $K \subset R^6$, such as a fuzzy sphere, or some more complicated fuzzy manifold. This mechanism of dynamically generating fuzzy extra dimensions has been studied in various guises and examples, see e.g. [1][2]. It allows to obtain the attractive features and the structure of a higher-dimensional theory starting from a simple 4-dimensional gauge theory.
Although in basic examples the resulting low-energy physics is somewhat academic, progress has been made recently towards more interesting low-energy behavior. In particular, chiral fermions can be obtained for backgrounds which locally span all 6 internal dimensions in $\mathcal{N} = 4$ SYM, e.g. via intersecting branes \cite{13, 14}. A particularly interesting and non-trivial example was found in \cite{15}, where $\mathcal{K}_N \cong \mathcal{C}_N[\mu]$ is a fuzzy version of a squashed (or projected) self-intersecting coadjoint orbit of $SU(3)$. This leads to 3 generations of chiral fermions in the low-energy theory, localized at the origin of the extra dimensions. These solutions can be stabilized by a cubic soft SUSY breaking term in the potential, which is added by hand. This may seem unavoidable, but it spoils the simplicity and the special role of $\mathcal{N} = 4$ SYM. Analogous solutions arise also in the IKKT or IIB matrix model \cite{16}, where the cubic terms are more problematic since logarithmic UV divergences typically lead to UV/IR mixing.

In the present paper, we show that the $\mathbb{R}^4 \times \mathcal{C}_N[\mu]$ solutions arise even in pure vanilla $\mathcal{N} = 4$ SYM without any SUSY-breaking potential, if the compact space is allowed to spin. This corresponds to a condensate of $SO(6)$ currents with maximal rank. There are of course many spinning background solutions in $\mathcal{N} = 4$ SYM and related matrix models (see e.g. \cite{17, 18}), however they typically lead to significant breaking of Lorentz invariance with unacceptable low-energy kinematics, at least for interesting cases with chiral fermions at intersecting branes \cite{19}. The miraculous property of the squashed branes $\mathcal{C}_N[\mu]$ is that they lead to a low-energy sector consisting of three generations of bosonic and fermionic chiral zero modes, which are oblivious to the background rotation, and governed by a standard Poincare-invariant 4-dimensional field theory at low energies. This “miracle” can be understood by the fact that the chiral fermionic and bosonic zero modes arise at the intersection of the different sheets of the fuzzy brane at the origin \cite{15}. In addition, there is a sector of fluctuation modes which couple to the background rotation. This sector is not described by an ordinary field theory. We provide a formalism for an appropriate description, and argue that it is stable and suppressed at low energies, provided no resonances arise.

Studying the fluctuations around these backgrounds leads to further intriguing results. If the three rotation frequencies $\omega_i$ along the 3 internal rotation axes coincide, there are a number of bosonic zero modes (and no negative modes), which form chiral supermultiplets with the fermionic zero modes. However if the $\omega_i^2$ slightly differ, then some of these bosonic would-be zero modes acquire a negative mass, pointing towards further spontaneous symmetry breaking. We indeed find some non-trivial exact solutions of the interacting system where some of these modes are switched on. The scale of their vacuum expectation value (VEV) is set by the difference of the $\omega_i$, which is a classical quantity imprinted in the background. This mechanism also applies in the presence of the soft SUSY breaking flux term in \cite{15}. A further hierarchy is introduced by the quantum numbers of the background brane $\mathcal{C}_N[\mu]$, which may be large. These are solid mechanisms for introducing a low scale into the theory, which are clearly protected from quantum corrections.

To get some insight into the emerging low-energy physics, we start exploring the (rather complicated) structure of the Yukawa couplings between the low-energy Higgs modes and the chiral fermions, as well as the couplings between the low-energy fermionic currents and the gauge fields. This is particularly interesting for stacks of such branes. The resulting pattern of fermions linking a stack of point branes attached to $\mathcal{C}_N[\mu]$ branes leads quite naturally to the matter content of the standard model coupled to the appropriate gauge fields, with all the correct quantum numbers. It turns out that only a special subset of the Kaluza-Klein tower of gauge fields couples to the low-energy currents, which includes in particular a pair
of “chiral” gauge fields with distinct coupling to different chiral fermion modes attached to
the brane, independent of generation. Additional fields such as various gauginos and higher
Kaluza-Klein modes also arise, and no claim on the physical viability can be made at this
point. Nevertheless, it is striking that one can arrive quite naturally in the vicinity of the
standard model (in the broken phase), reproducing precisely all its odd quantum numbers and
even “predicting” the number of generations.

Although the underlying assumption of a spinning background may seem strange at first,
it is in a sense very generic. In fact, the assumption of having precisely zero background
currents is non-generic, hence the present discussion is quite natural. The solutions under
consideration arise naturally in the presence of a $SO(6)$ $R$-current condensate with full rank,
and appear to be at least meta-stable. However, the stability of the background with its heavy,
time-dependent excitations needs to be analyzed in more detailed to establish a reliable basis
for further work. In any case, the accumulation of little miracles in the most symmetric of all
field theories $\mathcal{N} = 4$ SYM is certainly remarkable, and lends to the hope that these solutions
may point the way towards actual physics.

2 Rotating branes in $\mathcal{N} = 4$ SYM

The action of $\mathcal{N} = 4$ $SU(N)$ SYM is organized most transparently in terms of 10-dimensional
SYM reduced to 4 dimensions:

$$S_{\text{YM}} = \int d^4x \frac{1}{4g^2} \text{tr} \left( - F^{\mu\nu} F_{\mu\nu} - 2D^\mu \Phi^a D_\mu \Phi_a + [\Phi^a, \Phi^b][\Phi_a, \Phi_b] \right)$$

$$+ \text{tr} \left( \bar{\Psi} \gamma^\mu (i\partial_\mu + [A_\mu, .]) \Psi + \bar{\Psi} \Gamma^a [\Phi_a, \Psi] \right).$$

(2.1)

Here $F_{\mu\nu}$ is the field strength, $\Phi^a$, $a \in \{1, 2, 4, 5, 6, 7\}$ are 6 scalar fields
$\Psi$ is a matrix-valued Majorana-Weyl spinor of $SO(9,1)$ dimensionally reduced to 4-dimensions, and $\Gamma^a$ arise from
the 10-dimensional gamma matrices. All fields transform in the adjoint of the $SU(N)$ gauge
symmetry, and the coupling constant $g$ is absorbed in the scalar fields $\Phi$. It will be useful to
work with dimensionless scalar fields labeled by the roots $\pm \alpha_i$ of $\mathfrak{su}(3)$,

$$\Phi_\alpha = m X_\alpha, \quad \alpha \in \mathcal{I} = \{\pm \alpha_i, \ i = 1, 2, 3\}$$

(2.2)

where $m$ has the dimension of a mass. Explicitly,

$$X_1^\pm = \frac{1}{2} (X_4 \pm iX_5) \equiv X_{\pm \alpha_1},$$

$$X_2^\pm = \frac{1}{2} (X_6 \mp iX_7) \equiv X_{\pm \alpha_2},$$

$$X_3^\pm = \frac{1}{2} (X_1 \pm iX_2) \equiv X_{\pm \alpha_3}.$$

(2.3)

The potential for the scalar fields can then be written as

$$V[X] = -\frac{m^4}{4g^2} \text{tr} \left( \sum_{\alpha, \beta \in \mathcal{I}} [X_\alpha, X_\beta] [X^\alpha, X^\beta] \right)$$

(2.4)

\footnote{The unusual numbering of the indices will become clear below.}
where
\[ X^\alpha = 2X_{-\alpha}. \] (2.5)

Including variations around some background
\[ X_\alpha \rightarrow X_\alpha + \phi_\alpha \] (2.6)

the potential can be written as
\[
V(X + \phi) = \frac{m^4}{g^2} \text{tr} \left( -\frac{1}{4}[X_\alpha, X_\beta][X^\alpha, X^\beta] + \phi_\alpha \Box_X X^\alpha + X_\alpha \Box_\phi \phi^\alpha \\
+ \frac{1}{2} \phi^\alpha \left( \Box_X \delta^\beta_\alpha + 2[[X_\alpha, X^\beta]],.] \right) \phi_\beta - \frac{1}{2} f^2 + \frac{1}{4} [\phi_\alpha, \phi_\beta][\phi^\alpha, \phi^\beta] \right). \] (2.7)

Here
\[ \Box_X = \sum_{a \in \mathcal{I}} [X_\alpha, [X^\alpha, . ]] = 2([X^+_j, [X^-_j, . ]] + [X^-_j, [X^+_j, . ]]) \] (2.8)

and similarly \( \Box_\phi \), and
\[ f = i[\phi_\alpha, X^\alpha] \] (2.9)

can be viewed as gauge-fixing function in extra dimensions. Therefore the equations of motion (eom) are
\[
(\Box_4 + m^2 \Box_X)X_\alpha = 0 \] (2.10)

where \( \Box_4 = -\partial_\mu \partial^\mu \). The quadratic terms \( V_2(\phi) \) governing the fluctuations in \( \phi^\alpha \) is
\[
V_2(\phi) = \frac{m^4}{g^2} \text{tr} \left( \phi_\alpha \left( \Box_X \delta^\alpha_\beta + 2[[X_\alpha, X^\beta]],.] \right) \phi_\beta \right). \] (2.11)

Here we dropped the gauge-fixing function \( f \), as discussed below.

### 2.1 Rotating squashed brane solutions

We recall the construction of the squashed fuzzy coadjoint orbits \( C_N[\mu] \) with singular embedding in \( R^6 \)[13]. Let \( T_a, \ a = 1, ..., 8 \) be generators of the Lie algebra \( \mathfrak{su}(3) \), with structure constants
\[
[T_a, T_b] = i\epsilon_{abc}T_c \] (2.12)

canonically normalized with respect to the Killing form
\[
\kappa_{ab} = \langle T_a, T_b \rangle = \text{tr}(T_a T_b) = 2\delta_{ab}. \] (2.13)
The Cartan subalgebra is spanned by the two Cartan generators $H_3 \equiv T_3$ and $H_8 \equiv T_8$. The remaining generators $T_\alpha$, $\alpha \in \mathcal{I}$ are combined into the ladder or root generators

\[ T^\pm_1 \equiv \frac{1}{2}(T_4 \pm iT_5) \equiv T_{\pm\alpha_1}, \]
\[ T^\pm_2 \equiv \frac{1}{2}(T_6 \mp iT_7) \equiv T_{\pm\alpha_2}, \]
\[ T^\pm_3 \equiv \frac{1}{2}(T_1 \pm iT_2) = \pm[T_2^\pm, T_3^\pm] \equiv T_{\pm\alpha_3}, \tag{2.14} \]

where $\alpha_1, \alpha_2$ are the simple roots and $\alpha_3 = \alpha_1 + \alpha_2$, see figure \[1\]. Then the Lie algebra relations can be written in the Cartan–Weyl form as

\[ [T_\alpha, T_\beta] = \pm T_{\alpha+\beta}, \quad 0 \neq \alpha + \beta \in \mathcal{I} \]
\[ [T_\alpha, T_{-\alpha}] = H_\alpha, \]
\[ [H, T_\alpha] = \alpha(H)T_\alpha \tag{2.15} \]

or explicitly

\[ [T^+_i, T^-_i] = H_{\alpha_i}, \quad i = 1, 2, 3 \]
\[ [T^+_1, T^+_2] = T^+_3 \]
\[ [T^+_1, T^-_3] = -T^-_2 \]
\[ [T^-_2, T^-_3] = T^-_1 \]
\[ [T^+_1, T^-_2] = [T^+_2, T^-_3] = [T^+_3, T^+_2] = 0. \tag{2.16} \]

We recall that the Cartan generators are canonically associated to weights $\alpha \in \mathfrak{h}_0^*$, such that $H_\alpha|M\rangle = (\alpha, M)|M\rangle$ for weight states $|M\rangle$ in any representation. We also recall the Weyl group $\mathcal{W}$, which is generated by the reflections in weight space along the roots $\alpha_i$.

Let $\mathcal{H}_\mu$ be the irreducible representation with highest weight $\mu = n_1\Lambda_1 + n_2\Lambda_2$, where $\Lambda_i$ are the fundamental weights of $\mathfrak{su}(3)$. For any $n_1, n_2 \in \mathbb{N}$, this provides us with 6 hermitian matrices $X_\mu^a = \pi_\mu(T^a)$, or equivalently

\[ X_{\pm\alpha_i} \equiv X^\pm_i = \pi_\mu(T^\pm_i), \quad i = 1, 2, 3 \tag{2.17} \]

(we will drop the subscript $\mu$ from now on). Note that the Cartan generators are not included in the $X^a$. As explained in \[15\], the $X^a$ can be viewed as non-commutative embedding functions $X^a \sim x^a : \mathcal{C}[\mu] \hookrightarrow \mathbb{R}^8 \rightarrow \mathbb{R}^6$, where $\mathcal{C}[\mu]$ is a coadjoint orbit of $SU(3)$. This defines squashed fuzzy $\mathcal{C}_N[\mu]$, interpreted as noncommutative brane with squashed embedding in $R^6$. For $n_1 = n_2$, these are projections of fuzzy $\mathbb{C}P^2_N [20]$.

As they stand, these matrices $X^a$ are of course not solutions of $\mathcal{N} = 4$ SYM. One possibility is to add a cubic soft SUSY breaking term to the potential, such that they are solutions $[15]$. Here we pursue a different possibility, following the observation $[17]$ that such branes can often be stabilized by rotation. Thus consider the following ansatz corresponding to rotating squashed fuzzy $\mathcal{C}_N[\mu]$ branes

\[ X^\pm_i = r_i e^{\pm\omega_0 x^a} \pi_\mu(T^\pm_i) \tag{2.18} \]

\(^3\) The fundamental representation with $\mu = \Lambda_1$ corresponds to the Gell-Mann matrices $\lambda_\alpha = \pi_{(1,0)}(T_\alpha)$. 


for $i = 1, 2, 3$, and $ωx ≡ ω_{i,μ}x^μ$. We allow for different (dimensionless) radii $r_i$ and rotation frequency vectors $ω_{i,μ}$. This amounts to a $x$-dependent $SO(6)$ rotation. Using the Lie algebra relations, we compute

$$\frac{1}{2} □ X_i^± = \sum_j ([X_j^+, [X_j^-, X_i^±]] + [X_j^-, [X_j^+, X_i^±]])$$

$$= r_i^2 ([T_i^+, [T_i^-, X_i^±]] + \sum_{j \neq i} r_j^2 [T_j^-, [T_j^+, X_i^±]])$$

$$= (2r_i^2 + \sum_{j \neq i} r_j^2) X_i^±.$$

(2.19)

Similarly,

$$□ X_i^± = 2(2r_i^2 + r_j^2) X_i^±,$$

$$□ X_2^± = 2(r_1^2 + 2r_2^2 + r_3^2) X_2^±,$$

$$□ X_3^± = 2(r_1^2 + r_2^2 + 2r_3^2) X_3^±.$$

(2.20)

Hence the eom (2.10) become

$$ω_1^2 = -2m^2(2r_1^2 + r_2^2 + r_3^2)$$

$$ω_2^2 = -2m^2(r_1^2 + 2r_2^2 + r_3^2)$$

$$ω_3^2 = -2m^2(r_1^2 + r_2^2 + 2r_3^2)$$

(2.21)

or

$$\begin{pmatrix} r_1^2 \\ r_2^2 \\ r_3^2 \end{pmatrix} = \frac{1}{8m^2} \begin{pmatrix} -3ω_1^2 + ω_2^2 + ω_3^2 \\ ω_1^2 - 3ω_2^2 + ω_3^2 \\ ω_1^2 + ω_2^2 - 3ω_3^2 \end{pmatrix}$$

(2.22)

Note that these equations are independent of the representation $π_μ$. Therefore the above rotating branes are solutions of $N = 4$ SYM for suitable time-like vectors $ω_i$, as long as the rhs of (2.22) is positive. The scale of the background fields $Φ_±^i$ is then set by $m^2 r^2 = -\frac{1}{3} ω^2$. Similarly one obtains solutions of the IKKT matrix model on the quantum plane $R^2_0$.

### 3 Excitation modes on rotating branes

Consider internal fluctuations around this rotating brane background corresponding to 4-dimensional scalar fields,

$$X_{α} = \bar{X}_{α} + φ_{α}$$

(3.1)

or more explicitly

$$X_i^± = r_i e^{±iω_ix} T_i^± + φ_i^±.$$

(3.2)

\(^4\text{Note that the sign of } ω_i \text{ (forward or backward in time) is undetermined.}\)
The action (2.11) leads to the equation of motion\(^5\) for the fluctuations

\[
\left(\frac{\Box}{m^2} + \Box X + 2\hat{\mathcal{D}}_{ad}\right)\phi = 0,
\]

(3.3)
in terms of the “adjoint” Dirac operators

\[
(\hat{\mathcal{D}}_{ad}\phi)_{\alpha} = [[X_{\alpha}, \bar{X}_{\beta}], \phi_{\beta}] = \hat{\mathcal{D}}_{\text{mix}} + \hat{\mathcal{D}}_{\text{diag}},
\]

\[
(\hat{\mathcal{D}}_{\text{mix}}\phi)_{\alpha} = \sum_{\beta \neq \alpha} [[X_{\alpha}, \bar{X}_{\beta}], \phi_{\beta}]
\]

(3.4)

\[
(\hat{\mathcal{D}}_{\text{diag}}\phi)_{\alpha} = 2[[X_{\alpha}, \bar{X}_{\bar{\alpha}}], \phi_{\alpha}]
\]

with \(X^{\alpha} \equiv 2X_{-\alpha}\). Here \(\hat{\mathcal{D}}_{\text{mix}}\) mixes the polarizations \(\alpha\) (in the adjoint of \(\text{su}(3)\)) and is therefore time-dependent, while \(\hat{\mathcal{D}}_{\text{diag}}\) preserves the polarizations and is static. Using the time-dependent commutation relations for the background (cf. (2.15))

\[
[[X_{\alpha}, \bar{X}_{\beta}], \phi_{\gamma}] = g^{\alpha\beta}_{\gamma\gamma}X_{\gamma} + c^i_{\alpha\beta}r_i H_i
\]

(3.5)

where \(g^{\alpha\beta}_{i\gamma} = \epsilon^{(\omega_\alpha + \omega_\beta - \omega_\gamma)x} r_\alpha r_\beta r_\gamma^{-1} c^i_{\alpha\beta}\) is \(x\)-dependent and \(c^i_{\alpha\beta}\) is constant, these are explicitly

\[
(\hat{\mathcal{D}}_{\text{mix}}\phi)_{\alpha} = 2\gamma_{\alpha, -\beta}[X_{\gamma}, \phi_{\beta}] = \pm 2 \sum_{\beta \neq \alpha} \epsilon^{(\omega_\alpha - \omega_\beta - \omega_\gamma)x} r_\alpha r_\beta r_\gamma^{-1}[X_{\alpha - \beta}, \phi_{\beta}]
\]

\[
(\hat{\mathcal{D}}_{\text{diag}}\phi)_{\alpha} = 2r_\alpha H_i[\phi_{\alpha}]
\]

(3.6)

Only terms with \(\alpha - \beta \in I\) contribute in the first line. We note that

\[
\tau \hat{\mathcal{D}}_{\text{mix}} = -\hat{\mathcal{D}}_{\text{mix}} \tau,
\]

(3.7)

where \(\tau = \text{diag}(i, i, -i)\) is the generator of \(U(1) \subset SO(6)\) corresponding to the simultaneous rotations of the \(\phi_{\alpha}\) along the \(\alpha_1, \alpha_2\) and \(-\alpha_3\) directions (cf. figure [1]). The equations of motion separate into time-independent and time-dependent terms,

\[
\left(\frac{\Box}{m^2} + \Box X + 2\hat{\mathcal{D}}_{\text{diag}}\right)\phi + 2\hat{\mathcal{D}}_{\text{mix}}\phi = 0.
\]

(3.8)

Let us spell out these equations in the complex root basis, starting with (3.3)

\[
\left(\frac{\Box}{m^2} + \Box X\right)\phi_i^+ + 4 \sum_j r_j r_j e^{i\omega_j x}\left( e^{-i\omega_j x}[T_j^+, T_j^-], \phi_j^+\right) + e^{i\omega_j x}[T_j^+, T_j^-, \phi_j^+] = 0
\]

(3.9)

along with the 3 conjugate equations. Using the \(\text{su}(3)\) relations \([T_i^-, T_j^+] = \delta_i^j H_i\) for \(i = 1, 2, T_1^+, T_3^- = -T_2^-\) and \([T_2^+, T_3^-] = T_1^-\), the explicit form of (3.8) is

\[
\left(\frac{\Box}{m^2} + \Box X + 4r_1 H_1\right)\phi_1^+ + 4r_1 e^{i\omega_1 x}\left(r_2 e^{i\omega_2 x}[T_3^+, \phi_2^-] - r_3 e^{-i\omega_3 x}[T_2^-, \phi_3^+]\right) = 0
\]

\[
\left(\frac{\Box}{m^2} + \Box X + 4r_2 H_2\right)\phi_2^+ + 4r_2 e^{i\omega_2 x}\left(r_3 e^{-i\omega_3 x}[T_1^-, \phi_3^-] - r_1 e^{i\omega_1 x}[T_3^+, \phi_3^-]\right) = 0
\]

\[
\left(\frac{\Box}{m^2} + \Box X + 4r_3 H_3\right)\phi_3^+ + 4r_3 e^{i\omega_3 x}\left(r_2 e^{-i\omega_2 x}[T_1^+, \phi_2^-] - r_1 e^{-i\omega_1 x}[T_2^+, \phi_1^-]\right) = 0
\]

(3.10)

\(^5\)Recall that we added a gauge-fixing term \(f^2\) to the potential following [15], which removes the unphysical pure gauge modes from the massless spectrum, thus simplifying the analysis of the vector fluctuations.
as well as the conjugate relations. Note that the first terms are time-independent and diagonal in the polarization, while the second terms are time-dependent and mix the polarizations. It turns out that the solutions to these equations fall into two classes: First, there is a set of (would-be) zero modes which are oblivious to the background rotations, as the second terms in (3.10) vanish. All other modes couple to the background rotation, but are argued to be heavy and hence suppressed at low energies.

3.1 Zero modes

The equations of motion admit a set of (would-be) zero modes $\phi^{(0)}$, which are oblivious to the explicit $x^\mu$-dependence of the background due to $\mathcal{D}_{\text{mix}}\phi^{(0)} = 0$. In view of (3.6) or (3.10), such modes must satisfy

$$T_{\alpha - \beta} \triangleright \phi^{(0)} = [T_{\alpha - \beta}, \phi^{(0)}] = 0 \quad \forall \alpha, \beta \text{ with } \alpha - \beta \in \mathcal{I}. \tag{3.11}$$

or equivalently

$$T_\alpha \triangleright \phi^{(0)} = [T_\alpha, \phi^{(0)}] = 0 \quad \text{for } \alpha + \beta \in \mathcal{I} \tag{3.12}$$

Here $\triangleright$ denotes the $\mathfrak{su}(3)$ action on

$$\phi_\alpha \in \text{End}(\mathcal{H}) = \oplus V_\Lambda. \tag{3.13}$$

It is easy to find all solutions of (3.11): These are two equations for each $\phi_\beta$, which state that $\phi_\beta$ is an extremal weight vector in $V_\Lambda$ with weight in the Weyl chamber opposite to $\beta$. For example, take $\alpha = -\alpha_3$. Then $T_1^{+} \triangleright \phi^{(0)}_{-\alpha_3} = 0 = T_2^{+} \triangleright \phi^{(0)}_{-\alpha_3}$ is tantamount to the statement that $\phi_{-\alpha_3}$ is a highest weight vector in $V_\Lambda$,

$$\phi^{(0)}_{-\alpha_3} = |\Lambda, \Lambda\rangle \tag{3.14}$$

Analogous modes are obtained by acting with the Weyl group $\mathcal{W}$ on $\phi^{(0)}_{-\alpha_3}$, denoted as

$$\phi^{(0)}_{-\alpha_i} = |w_\alpha \Lambda, \Lambda\rangle, \quad i = 1, 2, 3 \tag{3.15}$$

where $w_\alpha$ maps the fundamental Weyl chamber into that of $\alpha$, i.e. $w_\alpha \cdot \alpha_3 = \alpha$. Particular examples of such modes are given by

$$\phi^{\pm(0)}_i = (X_i^{\pm})^n, \tag{3.16}$$

for $n$ below some cutoff. They all satisfy (3.11) and thus are oblivious to the background rotation, and

$$\mathcal{D}_{\text{mix}}\phi^{(0)}_\alpha = 0. \tag{3.17}$$

---

$^6$A weight $\Lambda$ in a (finite-dimensional) representation $V$ is called extremal if it is related by some element $w \in \mathcal{W}$ to a highest weight of $V$.

$^7$We recall that the Weyl group $\mathcal{W}$ – or more precisely a certain covering $\tilde{\mathcal{W}}$ in the braid group – can be viewed as a discrete subgroup of $SU(3)$, and therefore acts on any integrable representation of $\mathfrak{su}(3)$. 

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As a consequence, the equations of motion reduce to

\[
\left( \frac{\Box}{m^2} + \Box_X + 2\mathcal{D}_{\text{diag}} \right) \phi^{(0)} = 0. 
\] (3.18)

We evaluate \( \Box_X \) explicitly for \( \phi_{-\alpha_3}^{(0)} = |\Lambda, \Lambda \rangle \):

\[
\frac{1}{2} \Box_X |\Lambda, \Lambda \rangle = \sum_i [X_i^+, [X_i^-, \cdots]|\Lambda, \Lambda \rangle = \sum_i r_i^2 T_i^+ T_i^- |\Lambda, \Lambda \rangle \\
= \sum_i r_i^2 [T_i^+, T_i^-] |\Lambda, \Lambda \rangle = \sum_i r_i^2 H_i |\Lambda, \Lambda \rangle \\
= (r_1^2 \alpha_1 + r_2^2 \alpha_2 + r_3^2 \alpha_3, \Lambda) |\Lambda, \Lambda \rangle.
\] (3.19)

Furthermore,

\[
\mathcal{D}_{\text{diag}} \phi_{-\alpha_3}^{(0)} = 2r_3^2 [T_3^-, T_3], \phi_{-\alpha_3}^{(0)} = -2r_3^2 [H_3, \phi_{\alpha_3}^{(0)}] \\
= -2r_3^2 (\alpha_3, \Lambda) \phi_{-\alpha_3}^{(0)}
\] (3.20)

and the eigenvalue on \( \phi_{-\alpha_3}^{(0)} \) is the same. Therefore we have obtained the \( \alpha = -\alpha_3 \) case of

\[
(\Box_X + 2\mathcal{D}_{\text{diag}}) \phi_{\alpha}^{(0)} = M_{\phi}^\phi \phi_{\alpha}^{(0)},
\] (3.21)

with

\[
M_{\phi}^\phi_{\alpha_3} = 2(r_1^2 \alpha_1 + r_2^2 \alpha_2 - r_3^2 \alpha_3, \Lambda).
\] (3.22)

For \( \phi_{\alpha_3}^{(0)} = |\Omega \Lambda, \Lambda \rangle \) where \( \Omega \Lambda \) is the extremal weight in the opposite Weyl chamber to \( \Lambda \), we obtain

\[
M_{\phi}^\phi_{\alpha_3} = 2( - r_1^2 \alpha_1 - r_2^2 \alpha_2 + r_3^2 \alpha_3, \Omega \Lambda)
\] (3.23)

and \( M_{-\alpha_3}^\phi = M_{\alpha_3}^\phi \) if \( \Omega \Lambda = -\Lambda \). In general, the mass of \( \phi_{\alpha}^{(0)} \) is given by

\[
M_{\alpha}^\phi = 2(\sigma_1 r_1^2 \alpha_1 + \sigma_2 r_2^2 \alpha_2 + \sigma_3 r_3^2 \alpha_3, \Lambda')
\] (3.24)

where \( \sigma_i \) is the sign of the Weyl group element relating \( \alpha_i \) with the extremal weight \( \Lambda' \) (which is in the Weyl chamber opposite to \( \alpha \)). This becomes more transparent in terms of the over-complete weight labels

\[
n_1 = (\Lambda, \alpha_1), \quad n_2 = (\Lambda, \alpha_2), \quad n_3 = (\Lambda, -\alpha_3) = -(n_1 + n_2)
\] (3.25)

Then the eigenvalues can be written as

\[
\begin{pmatrix}
M_{\alpha_3}^\phi \\
M_{-\alpha_3}^\phi \\
M_{-\alpha_2}^\phi
\end{pmatrix} = 2
\begin{pmatrix}
n_1 & n_2 & n_3 \\
n_2 & n_3 & n_1 \\
n_3 & n_1 & n_2
\end{pmatrix}
\begin{pmatrix}
r_1^2 \\
r_2^2 \\
r_3^2
\end{pmatrix}.
\] (3.26)
and similarly for the opposite roots. While the individual masses can have either sign, they satisfy the sum rule

$$M_{\phi,\alpha_1} + M_{\phi,\alpha_2} + M_{\phi,\alpha_3} = 0$$ (3.27)

and similarly for the opposite roots. For \( r_1 = r_2 = r_3 \), we recover \( M_{\phi} = 0 \), and all \( \phi_{\alpha}^{(0)} \) are exact zero modes.

Conversely, we show in appendix A that for \( r_i = r \), all zero modes of the time-independent potential \((\square_x + 2\rho_{\text{diag}})\) are indeed given by the above modes \( \phi_{\alpha}^{(0)} \), which in turn are zero modes of \( \rho_{\text{mix}} \). Therefore we have properly identified the massless sector, and shown that it obeys a standard Lorentz-invariant kinematics. Since the \( \phi_{\alpha}^{(0)} \) modes decouple from the rotation also for different \( r_i \), we will continue to denote them as (would-be) zero modes, even though they may acquire non-vanishing masses \( M_{\phi}^2 \) proportional to \( \Lambda \). Thus among the negative (would-be zero) modes, those with maximal \( \Lambda \) should be the most significant ones.

Finally we need to take into account the reality condition \( (\phi^{(0)}_+ + i) = (\phi^{(0)}_-) \), mapping the extremal weights \( \Lambda \) to \( -\Lambda \). For real representations \( V_{\Lambda} \), the reality condition relates the two opposite extremal modes, so that

$$\left( \phi^{(0)}_{+} \right)^{\dagger} = -w^{-i}_{\Lambda, \Lambda} = |w^{+i}_{\Lambda, \Lambda} \rangle = \phi^{(0)}_{-}. \quad \text{(3.28)}$$

For complex representation, which arise in particular for links between different branes, the reality condition relates zero modes with opposite extremal weights in the conjugate representations \( V_{\Lambda} \) and \( V_{-\Lambda} = V_{-\Omega \Lambda} \), where \( \Omega \) is the longest element in the Weyl group.

To summarize, we have found 6 real or 3 complex zero modes \( \phi_{i}^{\pm(0)}(k)e^{ikx} \) on \( \mathbb{R}^4 \), corresponding to the extremal weights \( \mathcal{W} \Lambda \) in the decomposition \( \text{End}(\mathcal{H}) = \bigoplus V_{\Lambda} \). As discussed in [15], they can be interpreted as oriented strings linking the 6 coincident sheets of \( \mathcal{C}_N[\mu] \) with opposite flux. This is most obvious for the maximal weight state \( |\Lambda, \Lambda \rangle = |\mu \rangle \langle \Omega \mu | \) in \( \text{End}(\mathcal{H}_\mu) \), since the \( |\mu \rangle \) can be interpreted as coherent state on a sheet of \( \mathcal{C}_N[\mu] \) localized at the origin. Then conjugate modes correspond to reversed links, which are identified by the hermiticity constraint. We will see that for negative mass \( M_{\phi}^2 < 0 \), the mode is stabilized by the quartic terms in the interaction, leading to a non-trivial vacuum expectation value \( \langle \phi^{(0)}_{i} \rangle \neq 0 \). This introduces a rigid low-energy scale into the massless sector, as discussed below.

### 3.2 The heavy or mixing sector

Now we are interested in the generic solutions of the equation of motion for the fluctuations

$$\left( \square + m^2(\square_x + 2\rho_{\text{diag}}) \right)\phi + 2m^2 \rho_{\text{mix}}(x)\phi = 0 \quad \text{(3.29)}$$

where \( \square = -\partial^\mu \partial_\mu \). The explicit \( x^\mu \)-dependence of the mixing term is indicated by writing \( \rho_{\text{mix}}(x) \). Our time-dependent background admits a modified translation invariance \( \mathcal{T} \cong \mathbb{R}^4 \), acting as

$$x^\mu \to x^\mu + c^\mu, \quad X_\alpha \to (\exp(i\omega t \gamma_i)X)_\alpha \quad \text{(3.30)}$$

assuming that \( \mu \) has no stabilizer in \( \mathcal{W} \); otherwise they link the 3 sheets of squashed \( \mathbb{C}P^2_N \). [15].
with generators
\[ V_{\mu} = -i \partial_{\mu} - \omega_{\mu} \tau_i . \] (3.31)
Here \( \tau_i \) generates the appropriate \( U(1)_i \subset SO(6) \) rotation of the scalar fields. Clearly \( V_{\mu} \) annihilate the background \( X_\alpha \). We can therefore organize the fluctuation modes into unitary irreducible representations of \( T \),
\[ V_{\mu} \phi^{(n)}_k (x) = k_{\mu} \phi^{(n)}_k (x) \] (3.32)
or equivalently
\[ \phi^{(n)}_k (x + a) = e^{i \omega \tau_i x} \phi^{(n)}_k (x) . \] (3.33)
Thus the generalized momentum \( V_{\mu} \) resp. \( k_{\mu} \) is conserved, but not the usual momentum \( p_{\mu} \) associated to \( \partial_{\mu} \). This signals a potentially dangerous situation. However, the results for the zero modes discussed above imply that these \( \phi^{(0)}_k (x) \) are not only eigenfunctions of \( V_{\mu} \) but simultaneously of the \( \tau_i \) and \( \partial_{\mu} \), with
\[ \partial_{\mu} \phi^{(0)}_k (x) = i p_{\mu} \phi^{(0)}_k (x), \quad p_{\mu} = k_{\mu} + \omega \tau . \] (3.34)
We will see that all quadratic and quartic interactions of the zero modes \( \phi^{(0)}_k \) (and also for the fermionic zero modes and the gauge bosons) respect the \( U(1)_i \) rotations, so that the usual momentum \( p \) is also conserved. For cubic interactions of the scalar modes with the background such as in (4.1) this is no longer the case, and there may be some Lorentz-violating effects. However those arise only with large momentum transfer, which is suppressed by the high-energy scale, and they occur only for the Higgs fluctuations. This should effectively save Poincare invariance of the low-energy theory.

Now we turn to the generic solutions of (3.29). These are not eigenfunctions of \( \tau_i \), so that Poincare invariance is broken and replaced by \( T \) invariance. This may be acceptable for the heavy modes. Although the concept of mass is no longer well-defined, the spectrum of \( p = k + \omega \tau \) still determines the decay properties of the 2-point functions according to (3.33). The background rotation can be taken into account using the ansatz
\[ \tilde{\phi}^{(n)}_k (x) = e^{-i \omega \tau_i x \tau_i} \phi^{(n)}_k (x) \]
\[ -i \partial_{\mu} \tilde{\phi}^{(n)}_k (x) = e^{-i \omega \tau_i x} V_{\mu} \phi^{(n)}_k (x) = k_{\mu} \tilde{\phi}^{(n)}_k (x) . \] (3.35)
Therefore the \( x \)-dependence of the eigenmodes is given by
\[ \tilde{\phi}^{(n)}_k (x) = e^{ikx} \tilde{\phi}^{(n)}_k , \quad \tilde{\phi}^{(n)}_k (x) = e^{i(k+\omega \tau_i)x} \tilde{\phi}^{(n)}_k \] (3.36)
and we have
\[ -i \partial_{\mu} \phi^{(n)}_k (x) = e^{i \omega \tau_i} P \tilde{\phi}^{(n)}_k (x) = e^{i \omega \tau_i (k + \omega \tau_i)} \tilde{\phi}^{(n)}_k (x) \] (3.37)
where
\[ P := -i \partial + \omega \tau_i . \] (3.38)
Thus $\phi^{(n)}_k(x)$ is a superposition of plane waves with wave numbers given by $p = k + \omega \tau$. Then (3.29) turns into an algebraic equation

$$((k + \omega \tau)^2 + m^2 \mathcal{O}_V) \tilde{\phi}^{(n)}_k = 0$$

with

$$\mathcal{O}_V = \Box x + 2 \mathcal{D}_{\text{diag}} + 2 \mathcal{D}_{\text{mix}},$$

where $\mathcal{D}_{\text{mix}}$ is independent of $x$. This is a finite-dimensional linear equation, which for any fixed $\vec{k}$ has a complete set of eigenmodes $\tilde{\phi}^{(n)}_k$ with generalized momenta $k = (k_0, \vec{k})$, where $k_0$ is determined by the roots of the polynomial $\det((k + \omega \tau)^2 + m^2 \mathcal{O}_V) = 0$. This determines deformed “mass shells” $k^{(n)}_0(\vec{k})$. The non-trivial issue is whether all these $k_0$ are real; an imaginary part of $k_0$ would indicate an instability due to some resonance with the background. For large $\vec{k}$, the characteristic polynomial is dominated by the first term, and there are clearly distinct branches with positive and negative generalized “energy” $k^\pm_0(\vec{k})$. However in general this is not clear, and depends on the detailed structure of $\mathcal{O}_V$. We show in appendix A based on a result in [15] that $\mathcal{O}_V$ has a strictly positive spectrum apart from the ("regular") zero modes identified above, and a set of exceptional zero modes. All modes except for the regular zero modes are superpositions of different eigenstates of $\tau_i$, and hence of different wave-numbers $p = k + \omega \tau_i$. Even though we do not have any solid justification, we assume that resonances can be avoided, and separate branches with real $k_0^\pm(\vec{k})$ arise. It may help if the three $\omega_i$ have different space components $\omega_{i\mu} = (\omega_0, \vec{\omega}_i)$. Quantum corrections may also be important here.

Finally, it is plausible that the cubic “flux” term

$$\text{tr}(ig_{abc}X^aX^bX^c) = -12\text{tr}([X^+_1, X^+_2]X^-_3 + \text{h.c.})$$

arises in the effective action (or it may be added by hand), which is invariant under rotations with $\omega_1 + \omega_2 - \omega_3 = 0$ and can stabilize the present type of background as shown in [15]. In any case, the stability of the rotating background should be studied in more detail elsewhere.

Even if stability holds in the sense that all $k_0(\vec{k})$ are real, it remains to be shown that the $\tilde{\phi}^{(n)}_k$ modes are massive (or massless, but not tachyonic). Since all wave-numbers $p = k + \omega \tau_i$ associated to $k$ have the same group velocity $\nabla k_0(\vec{k})$, this question can be answered once the dispersion relation $k^{(n)}_0(\vec{k})$ is known, and these modes can have a perfectly nice physical meaning as propagating heavy particles. It is illuminating to observe here the relation with Brillouin zones. The rotating background defines a periodic structure on the non-compact space $\mathcal{M}^4 = \mathbb{R}^4$. Assuming that the three momenta $\omega_i = \omega_i^\mu$ are independent, we can introduce a reciprocal basis $a_i$ for the subspace spanned by these momenta supplemented by a vector $b$ such that

$$\omega^i \cdot a_j = 2\pi \delta^i_j, \quad \omega^i \cdot b = 0$$

w.r.t. the standard Minkowski metric. Then the $X_i^\pm$ are invariant under the discrete translation (sub)group defined by

$$x^\mu \rightarrow x^\mu + n_i a_i^\mu, \quad x^\mu \rightarrow x^\mu + c b^\mu = 0$$

Although rotations with $\omega_1 + \omega_2 - \omega_3 = 0$ can be gauged away and traded with Wilson lines as shown in section 4.3, they still provide a mechanism for further symmetry breaking as discussed in section 4.1.
for \( n_i \in \mathbb{Z} \) and \( c \in \mathbb{R} \), and the \( \phi_k(x) \) satisfy
\[
\phi_k^{(n)}(x + a_i) = \phi_k^{(n)}(x)e^{ika_i}, \quad \phi_k^{(n)}(x + cb) = \phi_k^{(n)}(x)e^{ikcb}
\] (3.44)

Thus the \( \phi_k^{(n)}(x) \) can be viewed as Bloch wave functions, and the momenta \( p = k \pm \omega \tau \) involved are identical modulo the Brillouin zone defined by \(-\pi \leq ka_i \leq \pi\). In particular, these modes behaves like ordinary particles from a low-energy point of view, with dispersion relation \( k_0(\vec{k}) \). While this “effective” momentum \( p \) is conserved only modulo the Brillouin zone, the generalized momentum \( k \) is preserved as a vector in \( \mathbb{R}^4 \). It is clear that this lattice will generically break Lorentz invariance. The remarkable property of our particular background is that Lorentz invariance is preserved for the (would-be) zero modes.

For illustration, a two-point function made from these modes will have the schematic structure
\[
\phi^{(n)}(x, y) = \int d^3k e^{i(k+\omega \tau)x} \phi_k^{(n)}(x) \phi_{-k}^{(n)}(y) e^{-i(k+\omega \tau)y}
\] (3.45)

This is clearly invariant under the generalized translations, and also under ordinary lattice translations \( \mathcal{T} \). For \( x, y \) on the lattice \( \mathcal{T} \), the \( \omega_i \) drop out, and
\[
\phi^{(n)}(x, y) = \int d^3k \phi_k^{(n)} \phi_{-k}^{(n)} e^{ik(x-y)}
\] (3.46)

This is now an ordinary 2-point function governed by the (presumably “massive”) dispersion relation \( k_0(\vec{k}) \), and Lorentz invariance is broken except for the zero modes.

### 3.3 Fermions

The spectrum of 4-dimensional fermions and their masses is governed by the Dirac operator on squashed \( \mathcal{C}_N[\mu] \). In the time-dependent background, it can be written as
\[
\mathcal{D}_{(6)} \Psi = \sum_{a \in \mathcal{I}} \Delta_a [X_a, \Psi] = 2 \sum_{i=1}^3 \left( \Delta^-_i [X^+_i, \cdot] + \Delta^+_i [X^-_i, \cdot] \right)
\]
\[
= 2 \sum_{i=1}^3 r_i \left( \Delta^-_i e^{i\omega_i x} \sigma_i + \Delta^+_i e^{-i\omega_i x} \sigma_i \right)
\] (3.47)

where the spinorial ladder operators
\[
2\Delta^-_1 = \Delta_4 - i\Delta_5, \quad 2\Delta^+_1 = \Delta_4 + i\Delta_5,
\]
\[
2\Delta^-_2 = \Delta_6 + i\Delta_7, \quad 2\Delta^+_2 = \Delta_6 - i\Delta_7,
\]
\[
2\Delta^-_3 = \Delta_1 - i\Delta_2, \quad 2\Delta^+_3 = \Delta_1 + i\Delta_2,
\] (3.48)
satisfy
\[
\{\Delta^-_i, \Delta^+_j\} = \delta_{ij}.
\] (3.49)

In particular, the partial chirality operator on \( \mathbb{R}^2 \) is given by
\[
\chi_i = -2(\Delta^+_i \Delta^-_i - \frac{1}{2}),
\] (3.50)
acting on the spin-$\frac{1}{2}$ irreducible representation. Using the form (3.47), we can easily find the zero modes as in [15]: let $Y^\Lambda$ be the highest weight vector of $H^\Lambda \subset \text{End}(H\mu)$. Then
\[ \mathcal{D}_4 \Psi^\Lambda = 0 \quad \text{for} \quad \Psi^\Lambda = |\uparrow\uparrow\uparrow\rangle Y^\Lambda. \] (3.51)
This follows immediately from the decomposition (3.47) of the Dirac operator, noting that
\[ \Delta^+_i |\uparrow\uparrow\uparrow\rangle = 0. \] (3.52)
Analogous zero modes $\mathcal{D}_4 \Psi^\Lambda = 0$ are obtained for any extremal weight $\Lambda' \in \mathcal{W}^\Lambda$ in $V^\Lambda$:
\[ \Psi^\Lambda|w_i\rangle = |s_1, s_2, s_3\rangle Y^\Lambda|w_i\rangle \] (3.53)
for suitable spin states $|s_1, s_2, s_3\rangle$. They can be obtained successively by applying a (generalized) Weyl reflection relating extremal weights $\Lambda'$ in adjacent Weyl chambers as follows:
\[ \Psi^\Lambda|w_i\rangle = \tilde{\omega}_i \cdot \Psi^\Lambda := (\omega_i |\uparrow\uparrow\uparrow\rangle) (w_i \cdot Y^\Lambda|w_i\rangle), \] (3.54)
provided $\Lambda'$ and $w_i\Lambda'$ are extremal weights of $V^\Lambda$ in adjacent Weyl chambers. Here $\omega_i$ implements the Weyl reflection $w_i$ on the internal spinor space $\mathbb{C}^2 \otimes^3$ associated to $\Delta^+_1, \ldots, \Delta^+_6$ as follows:
\[ \omega_i \Sigma(i) \omega_i^{-1} = -\Sigma(i), \quad i = 1, 2, 3 \]
\[ \omega_j \Sigma(j) \omega_i^{-1} = \Sigma(j), \quad j \neq i \]
\[ \Sigma(i) = \frac{1}{2} [\Delta^-_i, \Delta^+_i] = \frac{1}{2} \chi_i. \] (3.55)
For example if $\Lambda = \alpha_3$, then $\Psi^\Lambda|w_3\rangle = \tilde{\omega}_3 \cdot \tilde{\omega}_2 \cdot \tilde{\omega}_1 \cdot \Psi^\Lambda$; note that the $\tilde{\omega}_i$ do not satisfy the Weyl group relations, but this is not a problem here. These spin states are visualized in figure [1] and fall into chirality classes $C_L$ and $C_R$ with well-defined internal chirality
\[ \chi^\Lambda \Psi^\Lambda = (-1)^{\text{wt}} \Psi^\Lambda. \] (3.56)
These zero modes have definite chirality on $\mathbb{R}^4$, because $\Psi$ is subject to the Majorana-Weyl condition $\Psi^C = \Psi = \Gamma \Psi$. Moreover, it was shown in [15] that the extremal modes $\Psi^\Lambda$ and $\Psi^{-\Lambda}$ are related by (internal) charge conjugation and have opposite chirality,
\[ C^{(6)} \Psi^\Lambda = \Psi^{-\Lambda}. \] (3.57)
As long as $\Lambda$ has no stabilizer in $\mathcal{W}$, we can denote them as $\Psi_i^\pm$, identifying $\Lambda$ with the root $\alpha_i^\pm$ in the same Weyl chamber. Taking into account the Majorana-Weyl condition, this implies that the corresponding solutions of the full Dirac operator have the form [15]
\[ \Psi_i(x) = \Psi_i^\pm \otimes \psi_i^\pm(x) + \Psi_i^\mp \otimes \psi_i^\mp(x), \] (3.58)
where the four-dimensional spinors $\psi_i^\pm$ satisfy
\[ \mathcal{D}_4 \psi_i^\pm(x) = 0, \quad \gamma_5 \psi_i^\pm(x) = \pm \psi_i^\pm(x), \quad (\psi_i^\pm(x))^C = \psi_i^\mp(x). \] (3.59)
This means that the $\psi^i_\pm$ are not independent, as $\psi_+^i(x)$ determines $\psi_-^i(x)$. We can expand the general solution in terms of plane wave Weyl spinors $\psi_\pm^{i;k}(x)$ on $R^4$ with momentum $k$,

$$\Psi_i(x) = \int \frac{d^3k}{\omega_k} (\psi_+^{i;k}(x)\Psi^i_+ + \psi_-^{i;k}(x)\Psi^-_i), \quad i = 1, 2, 3. \quad (3.60)$$

This can be viewed either in terms of three 4-dimensional Majorana spinors $\psi^+_i + \psi^-_i$, or three Weyl spinors $\psi^+_i$ in 4D.

To summarize, the fermionic zero modes are in one-to-one correspondence with extremal weights $\Lambda$, with $\Lambda$ related to $-\Lambda$ by charge conjugation precisely as the scalar zero modes. Hence these modes can be succinctly collected in terms of chiral $N = 1$ multiplets labeled by positive extremal $\Lambda$. Moreover, the chirality (3.56) of the fermionic zero modes is distinguished by their charges under the noncommutative gauge fields on squashed $\mathbb{C}P^3 \mathbb{C}_N^\mu$ linked by $\Psi_{w\Lambda}$. This can be seen in figure 1 for the gauge field modes $A_\mu^{(3,8)} (x) T_{3,8}$ corresponding to the Cartan generators of $su(3)$, which are among the lowest non-trivial gauge field modes according to section 3.4. It is even more obvious for the gauge fields corresponding to $\chi_{L,R}$, as discussed in sections 4.3 and 6. In other words, different chiralities have different gauge couplings, which is a signature of a chiral model. Of course the total index $tr \gamma_5$ of the zero modes vanishes and the model is guaranteed to be anomaly free, suggesting some left-right symmetric model. However, more interesting behavior is possible in suitable brane configurations, as explained in section 6.

We have also seen that in the case $r_i \neq r_j$, some of the scalar fields acquire a negative mass, which will lead to non-trivial vacuum expectation values and mass terms for these fermions.

If $\Lambda$ has non-trivial stabilizer $w_i \in W$, the above analysis goes through, but with two fermionic zero modes for each (positive) extremal weight $\Lambda$. There is a similar doubling for the scalar zero modes, since then $\Lambda$ is in the Weyl chamber opposite to two roots. Therefore the fermionic and bosonic zero modes can still be grouped into chiral supermultiplets. Finally, the $\Lambda = 0$ modes correspond to a $N = 4$ supermultiplet of trivial $\Lambda = 0$ modes.
3.4 Gauge bosons

The 4-dimensional gauge bosons $A_{\mu}(x)$ decompose into eigenmodes of the scalar Laplacian $\Box_X$. Since the scalar Laplacian is time-independent, all these modes are governed by a standard 4-dimensional kinematics, in contrast to the heavy Higgs modes discussed above. It is easy to compute its spectrum for equal $r_i=r$, observing that $\Box_X = r^2(C_2 - [H_3, [H_3, \cdot]] - [H_8, [H_8, \cdot]])$, where $C_2$ is the quadratic Casimir of $SU(3)$. This gives

$$\Box_X |M, \Lambda\rangle = 2r^2 \lambda_{\Lambda,M}|M, \Lambda\rangle, \quad \lambda_{\Lambda,M} = (\Lambda, \Lambda + 2\rho) - (M, M) \quad (3.61)$$

so that the $|M, \Lambda\rangle$ states are indeed eigenstates of $\Box_X$. As a check, we recover

$$\Box_X X^\pm_i = 2r^2(\Lambda, 2\rho)X^\pm_i = 8r^2X^\pm_i \quad (3.62)$$

since $X^\pm_i \in \{\pm \Lambda, \Lambda\}$ with $\Lambda = \rho = \alpha_3$ and $(\alpha_3, \alpha_3) = 2$. As shown in [15], it follows that the only zero modes are those with $M = \Lambda = 0$, while the lowest non-vanishing eigenvalues arise for $M = w\Lambda$ with eigenvalues $(\Lambda, \rho) \geq r^2$. Via the Higgs effect elaborated in (3.31), these modes acquire a mass

$$m_A^2 = 2m^2r^2\lambda_{\Lambda,M} \approx -\frac{1}{4}\omega^2\lambda_{\Lambda,M} \quad (3.63)$$

using (2.22) and assuming that $\omega_i^2 \approx \omega^2$.

We will see in section 4.3 that only the the $M = 0$ gauge modes couple to the currents arising from the low-energy sector of fermions linking point branes with $C[\mu]$; these will be of primary interest from the particle physics perspective. The lowest such modes are the two weight zero modes in $\Lambda = (1,1)$ given by the Cartan generators $H_3, H_8 \in \mathfrak{su}(3)$, with mass $m_A^2 = 12m^2r^2$. However let us focus on the next-lowest modes given by the $M = 0$ modes in $(0,3)$ and $(3,0)$, which lead to particularly interesting chiral gauge fields. They arise in the algebra of functions on $C_N[\mu]$ as follows: Consider the functions

$$E_{ab} = \varepsilon^{(8)}_{abcdefgh}X^aX^dX^eX^fX^gX^h = \frac{1}{8}\varepsilon^{(8)}_{abcdefgh}[X^c, X^d][X^e, X^f][X^g, X^h] = -E_{ba}$$

$$= h_{abcde}T^cT^dT^e \quad (3.64)$$

with $\mathfrak{su}(3)$ indices in $\{1, \ldots, 8\}$, which form a multiplet $(8) \wedge (8) = (3,0) \oplus (0,3) \oplus (1,1)$. It is not hard to see that the $(1,1)$ components do not occur, so that the $E_{ab}$ decompose into $(3,0) \oplus (0,3)$. We focus on the two $M = 0$ modes in $(3,0)$ and $(0,3)$. Having weight zero means that they commute with the Cartan generators $H_a$, and diagonalize on the extremal weight states in $\text{End}(\mathcal{H})$. One hermitian combination of these is given by the orientation form $\Theta$, defined by

$$\Theta := iE_{3,8} = i\varepsilon^{(6)}_{abcdef}X^aX^bX^cX^dX^eX^f = \frac{1}{8}\varepsilon^{(6)}_{abcdef}[X^c, X^d][X^e, X^f]$$

$$\sim \frac{1}{8}\varepsilon^{(6)}_{abcdef}\{x^a, x^b\}\{x^e, x^f\}\{x^c, x^d\} = \text{Pf}\theta \quad (3.65)$$

with indices restricted to the roots. This is invariant under the $Z_3$ subgroup of Weyl rotations, odd under reflections, and reduces in the semi-classical limit to the (symplectic) orientation.
form $Pf \theta$ on the left- and right-handed sheets $C_L$ and $C_R$ of the 6-dimensional branes $C_N[\mu]$ \cite{15}. Therefore $\Theta$ takes eigenvalues $\pm c$ on the left- and right-handed maximal states $\Psi_{\Lambda}'$. The other hermitian combination is given by any of the three forms

$$\Xi := E_{\alpha_1,-\alpha_1} \sim E_{\alpha_2,-\alpha_2} \sim -E_{\alpha_3,-\alpha_3}$$

which must be invariant under the full Weyl group (since e.g. $w_1 E_{\alpha_1,-\alpha_1} = E_{\alpha_1,-\alpha_1}$). It turns out that if acting on $(1,1)$, $\Xi$ acts non-trivially on the weight zero states but annihilates all extremal states. For generic representations however, the polynomial function on $\mathcal{C}[\mu]$ defined by $\Xi$ in the semi-classical limit cannot vanish identically along the edges of the representations, hence we expect that $\Xi$ will assume a non-vanishing eigenvalue on the extremal weight states. By taking suitable linear combinations of $\Theta$ and $\Xi$, we can then define characteristic functions $\chi_L$ and $\chi_R$ on the chirality classes $C_L$ resp. $C_R$ of the extremal fermions $\Psi_{\Lambda}'$. The corresponding gauge fields couple only to left resp. right-handed fermionic zero modes, which is a crucial ingredient of a chiral gauge theory.

Finally for different $r_i$, it is clear that the trivial zero modes $M = \Lambda = 0$ persist as above, and the massive modes will get somewhat deformed, but preserving the qualitative features as long as $r_i \approx r$. The $M = \Lambda = 0$ modes become interesting in the case of $n$ coincident branes, where the turn into nonabelian $U(n)$ gauge bosons.

4 Interactions

4.1 Scalar self-interactions and nontrivial Higgs

Now consider the interactions of the scalar zero modes discussed in section 3.1. The quartic and cubic self-interactions are easily obtained from (2.7),

$$V_{\text{int}}(\phi) = \frac{m^4}{g^4} \text{tr} X_\alpha \Box \phi^\alpha + \frac{1}{4} [\phi_\alpha, \phi_\beta] [\phi^\alpha, \phi^\beta]$$

(4.1)

The quartic term will stabilize the (would-be) zero modes, since they always have some non-vanishing commutator $[\phi_\alpha, \phi_\beta] \neq 0$. The cubic term introduces an explicit time dependent interaction, however the momentum conservation is only satisfied if at least one particle has large energy of order $\omega$. Moreover, the cubic interaction term is suppressed since the mass scale $\omega$ associated with $X_\alpha = r_\alpha T_\alpha e^{i\omega_\alpha x}$ is much higher than that of the (would-be) zero modes $\phi_\alpha^{(0)}$. Therefore for low energies, the quartic interaction should be the dominant one.

Now assume that some of these $\phi_\alpha$ acquire a non-vanishing time-independent VEV, denoted as “non-trivial Higgs” solutions. We should accordingly obtain the equations of motion in the presence of such $\phi_\alpha \neq 0$, separating the time-dependent $X_\alpha$ from the time-independent $\phi_\alpha$. Consider first the variation of the cubic terms:

$$\text{tr} X_\alpha \delta \Box \phi^\alpha = \text{tr} X_\alpha \left( [\delta \phi_\beta, [\phi_\beta, \phi^\alpha]] + [\phi_\beta, [\delta \phi^\beta, \phi^\alpha]] \right)$$

$$= \text{tr} \delta \phi_\beta \left( -2 [X_\alpha, [\phi_\beta, \phi^\alpha]] - [\phi^\beta, [\phi^\alpha, X_\alpha]] \right)$$

(4.2)

using the Jacobi identity. Now the following term vanishes

$$[X^\alpha, \phi_\alpha^{(0)}] = 0$$

(4.3)
since $\phi_\alpha$ is a zero mode on $C_N[\mu]$; this follows from the extremal weight property discussed in section 3.1 (in particular, the gauge condition $f = 0$ holds). Therefore for such backgrounds,

$$\delta_\phi V_{\text{int}}[\phi] = \frac{m^4}{g^4} \text{tr} \delta \phi^\alpha \left( 2[[\phi_\alpha, \phi^\beta], X_\beta] + \Box_\phi X_\alpha + \Box_\phi \phi_\alpha \right). \quad (4.4)$$

Thus the full equation of motion for our background with non-vanishing Higgs can written compactly as

$$\left( \frac{\Box_4}{m^2} + \Box_X + \Box_\phi + 2 \delta^\phi \right) X_\alpha + \left( \frac{\Box_4}{m^2} + \Box_X + \Box_\phi + 2 \delta^X \right) \phi_\alpha = 0 \quad (4.5)$$

where $\delta^X \equiv \delta_\phi$, and we introduce

$$(\delta^\phi X)_\alpha = \sum_\beta [[[\phi_\alpha, \phi^\beta], X_\beta] = ((\delta^\phi_{\text{mix}} + \delta^\phi_{\text{diag}}) X)_\alpha$$

$$(\delta^\phi_{\text{mix}} X)_\alpha := \sum_{\beta \neq \alpha} [[[\phi_\alpha, \phi^\beta], X_\beta]$$

$$(\delta^\phi_{\text{diag}} X)_\alpha = 2[[\phi_\alpha, \phi_{-\alpha}], X_\alpha] \quad \text{(no sum)} \quad (4.6)$$

in analogy to section 3. To maintain the decoupling between the rotating background $X_\alpha$ and the low-energy Higgs $\phi_\alpha$, we should therefore require that

$$\delta^\phi_{\text{mix}} X_\alpha = 0 \quad (4.7)$$

in addition to (3.17). Then the above equations of motion take the symmetric form

$$\left( \frac{\Box_4}{m^2} + \Box_X + \Box_\phi + 2 \delta^\phi_{\text{diag}} \right) X_\alpha = 0$$

$$\left( \frac{\Box_4}{m^2} + \Box_\phi + 2 \delta^X_{\text{diag}} \right) \phi_\alpha = 0 \quad (4.8)$$

For illustration, we elaborate one non-trivial solution based on the lowest zero modes of type (3.16), given by

$$\phi_\alpha = \varphi_\alpha T_{-\alpha}. \quad (4.9)$$

with real-valued $\varphi_\alpha = \varphi_{-\alpha}$. Then $\delta^\phi_{\text{mix}} X_\alpha = 0$ holds for precisely the same reason as $\delta^X_{\text{mix}} \phi_\alpha = 0$, as discussed in section 3.1. By literally repeating the computation leading to $M^2_\alpha$ in (3.22), we obtain

$$(\Box_\phi + 2 \delta^\phi_{\text{diag}}) X_{\pm \alpha_1} = M^X_{\pm \alpha_1}, \quad (4.10)$$

with

$$M^X_3 = 2(\varphi_1^2 \alpha_1 + \varphi_2^2 \alpha_2 - \varphi_3^2 \alpha_3, \alpha_3) = 2(\varphi_1^2 + \varphi_2^2 - 2 \varphi_3^2),$$

$$M^X_2 = 2(\varphi_1^2 \alpha_1 + \varphi_2^2 \alpha_2 - \varphi_3^2 \alpha_3, -\alpha_2) = 2(\varphi_1^2 - 2 \varphi_2^2 + \varphi_3^2),$$

$$M^X_1 = 2(\varphi_1^2 \alpha_1 + \varphi_2^2 \alpha_2 - \varphi_3^2 \alpha_3, -\alpha_1) = 2(-2 \varphi_1^2 + \varphi_2^2 + \varphi_3^2) \quad (4.11)$$
the equations of motion become

\[
\left( \frac{\Box}{m^2} + \Box X + M^X \right) X_i^\pm = 0, \\
\left( \frac{\Box}{m^2} + \Box \phi + M^\phi \right) \phi_i^\pm = 0 
\] (4.12)

where \( M^\phi = M^\phi_{\pm \alpha} \) is the eigenvalue of \( (\Box X + 2 D^X) \) given by (3.26). Using also (2.20) and its analog

\[
\Box \phi_i^\pm = 2 (2 \phi_1^2 + \phi_2^2 + \phi_3^2) \phi_i^\pm, \\
\Box \phi_1^\pm = 2 (\phi_1^2 + 2 \phi_2^2 + \phi_3^2) \phi_1^\pm, \\
\Box \phi_3^\pm = 2 (\phi_1^2 + \phi_2^2 + 2 \phi_3^2) \phi_3^\pm. 
\] (4.13)

we obtain the eom

\[
\begin{pmatrix}
2 & 1 & 1 & -2 & 1 & 1 \\
1 & 2 & 1 & 1 & -2 & 1 \\
1 & 1 & 2 & 1 & 1 & -2 \\
-2 & 1 & 1 & 2 & 1 & 1 \\
1 & -2 & 1 & 1 & 2 & 1 \\
1 & 1 & -2 & 1 & 1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
r_1^2 \\
r_2^2 \\
r_3^2 \\
\varphi_1^2 \\
\varphi_2^2 \\
\varphi_3^2 \\
\end{pmatrix}
= -\frac{1}{2m^2}
\begin{pmatrix}
\omega_1^2 \\
\omega_1^2 \\
\omega_1^2 \\
\omega_3^2 \\
\omega_3^2 \\
\omega_3^2 \\
\end{pmatrix} 
\] (4.14)

Assuming that \( \varphi_i^2 > 0 \) for all \( i \), the last three equations lead to an inconsistency, which reflects the sum rule \( M_1^\phi + M_2^\phi + M_3^\phi = 0 \) (3.27). Therefore either one or two of the \( \varphi_i \) must vanish.

It turns out that there is no solution with two non-vanishing \( \varphi_i \), but there is a solution with \( \varphi_3 \neq 0 \) and \( \varphi_1 = 0 = \varphi_2 \), given by

\[
\begin{pmatrix}
r_1^2 \\
r_2^2 \\
r_3^2 \\
\varphi_1^2 \\
\varphi_2^2 \\
\varphi_3^2 \\
\end{pmatrix}
= \frac{1}{8m^2}
\begin{pmatrix}
-2\omega_1^2 + 2\omega_2^2 - \omega_3^2 \\
2\omega_1^2 - 2\omega_2^2 - \omega_3^2 \\
-\omega_1^2 - \omega_2^2 + 2\omega_3^2 \\
-\omega_1^2 + \omega_2^2 + \omega_3^2 \\
-\omega_1^2 + 2\omega_2^2 + \omega_3^2 \\
-\omega_1^2 - \omega_2^2 + 2\omega_3^2 \\
\end{pmatrix} 
\] (4.15)

The \( r_i^2 \) are indeed positive if all \( \omega_{i, \mu} \) are approximately equal and timelike, and

\[
\varphi_3^2 = \frac{1}{8m^2} (2\omega_3^2 - \omega_1^2 - \omega_2^2) = -\frac{1}{4} (r_1^2 + r_2^2 - 2r_3^2) = -\frac{1}{4} M_3^\phi > 0 
\] (4.16)

must also be positive, which is consistent with \( M_3^\phi < 0 \) (3.26). This holds for suitable frequencies \( \omega_i \), which we assume here. Putting back the dimensions, the scale of the Higgs \( \varphi \) is set by the difference of the rotation frequencies

\[
m^2 \varphi^2 \sim \frac{1}{4} \Delta \omega_i^2. 
\] (4.17)

This is a classical quantity imprinted in the background, which is naturally small. This provides a solid mechanism for introducing a low scale into the theory.
It is not clear at this point if the above solution (4.9) is stable. Indeed according to (3.22), other extremal modes with weight in the same Weyl chamber will also acquire a negative mass in the unbroken phase. The above non-linear equations of motion will couple these modes, and modify also the ansatz for \( X_\alpha \). Thus finding a stable Higgs configuration is clearly a challenging task, which should be addressed elsewhere. However let us single out the following Higgs modes (3.16) for later use

\[
\phi_{i}^{\pm(n)} = Y_{\pm n_{i}}^{\text{max}} \sim (X_{i}^{\mp})^{n}. 
\]

They satisfy the decoupling condition (4.7), but lead to a modification of the background \( X_{\alpha} \).

### 4.2 Higgs background and Yukawa couplings

Now consider the interaction of the fermionic zero modes with the above Higgs zero modes. We shall denote them as “would-be zero modes” henceforth, since they may acquire a mass in the presence of a Higgs. The Yukawa couplings between fermions and Higgs modes arise from

\[
m \text{Tr} \gamma_{5} \bar{\Psi}(6) \Psi = m \text{Tr} \bar{\Psi} \gamma_{5} \Delta^{\alpha}[\phi_{\alpha}, \Psi] \\
= 2m \sum_{i=1}^{3} \bar{\Psi} \gamma_{5} \left( \Delta^{-}[\phi_{i}^{+}, .] + \Delta^{+}[\phi_{i}^{-}, .] \right) \Psi 
\]

where \( \Psi \) is subject to the Majorana-Weyl condition \( \Psi^{C} = \Psi = \Gamma \Psi \). We focus on the contribution from the fermionic zero modes

\[
\Psi_{\Lambda'} = |s_{1}s_{2}s_{3}\rangle Y_{\Lambda'}^{A} 
\]

as in (3.53). Here \( Y_{\Lambda'}^{A} \) is an extremal weight state in \( V_{A} \). Now assume that due to a negative mass term, some of the following Higgs modes on \( C_{N}[\mu] \) are switched on, such as

\[
\phi_{i}^{\pm(n)} = \varphi_{i}^{(n)}(X_{i}^{\mp})^{n} 
\]

as discussed above. For suitable \( n \) (which may depend on \( i, \Lambda \)), these \( \phi_{i}^{\pm(n)} \) relate extremal weights \( \Lambda' \in \mathcal{W}A \) of \( V_{A} \) in adjacent Weyl chambers,

\[
[(X_{i}^{\mp})^{n}, Y_{\Lambda'}^{A}] \sim w_{i} \cdot Y_{\Lambda'}^{A} = Y_{w_{i}A'}^{A} 
\]

where \( w_{i} \) is the Weyl reflection along \( \alpha_{i} \). Moreover, \( \Delta_{i}^{\pm} \) implements the same Weyl reflection of the spinors as defined in (3.55), e.g.

\[
\Delta_{i}^{-}[\uparrow\uparrow\uparrow\uparrow] = \tilde{\omega}_{i} \cdot [\uparrow\uparrow\uparrow\uparrow], \quad i = 1, 2, 3. 
\]

It is then easy to see that

\[
(\Delta_{i}^{-}[\phi_{i}^{+}, .] + \Delta^{+}_{i}[\phi_{i}^{-}, .]) \Psi_{\Lambda'} \sim \varphi_{i}^{(n)} \tilde{\omega}_{i} \cdot \Psi_{\Lambda'} 
\]

for fermionic zero modes \( \Psi_{\Lambda'} \) and appropriate \( n \) (and only for those). As discussed in section 3.3 \( \tilde{\omega}_{i} \cdot \Psi_{\Lambda'} \) is the zero mode \( \Psi_{w_{i}A'} \) if \( w_{i}A' \) is adjacent to \( \Lambda' \), but not otherwise; for example if
\Lambda = \alpha_3$, then \( \tilde{\omega}_3 \cdot \Psi_A \not= \Psi_{w_3 A} = \tilde{\omega}_2 \cdot \tilde{\omega}_1 \cdot \Psi_A \), hence \( \tilde{\omega}_3 \cdot \Psi_A \) is not a zero mode. Therefore we obtain Yukawa couplings

\[
m \text{tr} \bar{\Psi} \gamma_5 D_{(6)} \Psi \sim 2m \sum_{i,n,\Lambda'' = w_i \Lambda'} \text{tr} \bar{\Psi}_\Lambda' \gamma_5 \varphi^{(n)}_i \tilde{\omega}_i \cdot \Psi_{\Lambda''}
\]

relating adjacent fermionic zero modes with the appropriate \( \varphi^{(n)}_i \). There are also terms coupling the fermionic zero modes to various heavy fermionic modes, which presumably can be neglected at low energies.

To summarize, extremal fermionic zero modes in adjacent Weyl chambers related by a Weyl reflection acquire a mass in the presence of a suitable Higgs mode \( \langle \varphi^{(n)}_1 \rangle \not= 0 \), which links their weights. Taking into account the results of previous section, this mass is expected to be of order

\[
m^2_{\Psi} \sim \Delta \omega_i^2.
\]

This will give mass to the off-diagonal standard-model-like fermions in section 6. Similar mass terms may arise more generally for fermionic zero modes in adjacent Weyl chambers linked by other Higgs fields, provided the corresponding cubic matrix elements \( \text{tr} \bar{Y}_{w_i \Lambda'} [\phi, Y^{\Lambda}_w] \) are non-vanishing; this boils down to a representation-theoretic question. Thus on a given large \( C[\mu] \) brane, there are many possible Yukawas coupling the fermionic zero modes via the various Higgs, which suggests that they tend to be heavy. The detailed structure is clearly complicated.

### 4.3 Fermionic currents and coupling to gauge bosons

The interaction of the fermionic zero modes with the gauge bosons is given by

\[
S_A(\Psi) = \text{Tr} \bar{\Psi} \gamma^\mu [A_\mu, \Psi] = \text{Tr} J^{\mu A}_M A^{\Lambda, M}_\mu
\]

where

\[
J^{\mu A}_M(x) = \bar{\Psi}(x) \gamma^\mu [Y^\Lambda_M, \Psi(x)]
\]

and we decompose the gauge field into the \( |M, \Lambda \rangle \) modes in \( \text{End}(\mathcal{H}_\mu) \),

\[
A_\mu(x) = \sum A^{\Lambda, M}_\mu(x) Y^\Lambda_M
\]

We are primarily interested in the currents which arise within the low-energy sector, due to fermionic (would-be) zero modes \( \Psi_{\Lambda'} \) with extremal weights \( \Lambda' \) in \( \text{End}(\mathcal{H}) \) as discussed above. Since \( J^{\mu A}_M \) does not involve any internal Clifford generators, both fermions must have the same internal spins, hence their extremal weights \( \Lambda, \Lambda' \) must be in the same Weyl chamber (see figure 4); for example, both must be highest weights. Thus \( J^{\mu A}_M \not= 0 \) only if either \( M = 0 \), or if \( M \) relates different extremal weights \( \Lambda' \) in the same Weyl chamber. For the fermions on a given \( C[\mu] \) brane, this may hold for various currents \( J^{\mu A}_M \) with \( M \not= 0 \), if the corresponding cubic matrix elements \( \text{tr} \bar{Y}_{w_i \Lambda'} [Y_M, Y^\Lambda_w] \) are non-vanishing. For large branes \( C[\mu] \), this leads to a complicated structure of mutually coupled gauge and scalar fields.
However, we will be mainly interested in fermions linking a $C[\mu]$ brane with a point brane, as discussed below. Then there is only one extremal weight $\mu$ in each Weyl chamber, so that only currents $J_{M=0}$ with $M = 0$ can arise. They couple only to gauge bosons $A_\mu \sim Y^0_0$ with weight $M = 0$. The mass of these gauge bosons is given by $m_A^2 = 2m^2 r^2 (\Lambda, \Lambda + 2\rho)$ using (3.63), assuming $r_i \approx r$. The most interesting such modes are the $\chi_L$ and $\chi_R$ modes identified in section 3.4, which couple to left resp. right-handed fermionic zero modes $C_L$ resp. $C_R$ of the extremal fermions $\Psi_\Lambda$. This is a crucial ingredient of a chiral gauge theory.

### 4.4 Higgs – gauge bosons coupling

The interaction of the scalar fields with the gauge bosons arises from their kinetic term, via

$$D_\mu \Phi^\pm_\mu = m(\partial_\mu - i[A_\mu, \cdot])(X^\pm_\mu + \phi^\pm_\mu) = m(i\omega_\mu X^\pm_\mu + \partial_\mu \phi^\pm_\mu - i[A_\nu, X^\pm_\mu + \phi^\pm_\mu]).$$

(4.30)

Keeping only the quadratic terms in the fields, this leads to

$$-\int d^4x \frac{1}{g^2} \text{tr} D_\mu \Phi^+_\mu D^\mu \Phi^-_\mu = -\int d^4x \frac{m^2}{g^2} \text{tr}(\omega^2 X^+ X^- + \partial_\mu \phi^+ \partial^\mu \phi^- - A^\mu H_\mu + \frac{1}{2} A_\mu \Box X A^\mu)$$

(4.31)

after some straightforward algebra, using the decoupling property $[X^+, \phi^-] = 0 = [X^-, \phi^+]$ (cf. (4.3)) as well as $\text{tr} X^+ \phi^- = 0 = \text{tr} X^- \phi^+$ (since $X^+ \phi^- \phi^+$ has non-zero weight). Here

$$\lambda_\mu = 2r_i^2 \omega_i \alpha_i = \sum_{\alpha \in I} \omega^2 \alpha_{i,\mu} \alpha$$

(4.32)

can be viewed as vector-valued functional on $g^*_0$, where $\omega_{\pm \alpha_i} = \pm \omega_i$ and $r_{\pm \alpha_i} = r_i$. The linear term in $A_\mu$ is an unphysical pure gauge term, and can be removed by a non-abelian gauge transformation $U = e^{ic^\beta x H_\beta}$. This preserves the weight structure and therefore the gauge condition $[X^\alpha, \phi_\alpha] = 0$, and acts on the embedding as

$$X_\alpha \rightarrow U X_\alpha U^{-1} = e^{ic^\beta x H_\beta} X_\alpha, \quad A_\mu \rightarrow U A_\mu U^{-1} + c_\mu^\beta H_\alpha$$

(4.33)

It amounts to a shift of the frequency vectors\footnote{In the IIB matrix model, such a gauge transformation would turn the stack of $R_5^4$ branes into a stack of transversally shifted copies of $R_5^4$.}

$$\omega_\alpha \rightarrow \omega_\alpha - c^\beta (\beta, \alpha)$$

(4.34)

leaving $\omega_1^+ + \omega_2^+ + \omega_3^-$ unchanged. Under this transformation, $\lambda_\mu$ changes as

$$\lambda_\mu \rightarrow \lambda_\mu - r^2 \omega^\beta (\beta, \alpha)$$

(4.35)

hence we can always choose the gauge $\lambda_\mu = 0$ since the $c^\beta_\mu$ are arbitrary. Thus in general, this gauge choice may amount to switching on a non-trivial gauge field or a Wilson line (4.33). However for simple backgrounds, this linear term vanishes anyway, recalling that we are free to choose the sign of $\omega_i$. E.g. for $\omega_1 = \omega_2 = -\omega_3 = \omega$, we have

$$\lambda_\mu = \omega_\mu (r_1^2 \alpha_1 + r_2^2 \alpha_2 - r_3^2 \alpha_3) = 0$$

(4.36)
even for $A_\mu = 0$. Dropping this term, the action governing the gauge bosons and scalars in the rotating background is

$$-\int d^4x \frac{1}{g^2} \text{tr} D_\mu \Phi^+ \Phi^- = -\int d^4x \frac{m^2}{g^2} \text{tr} \left( \omega_i^2 X^+_i X^-_i + \partial_\mu \phi^+_i \partial^\mu \phi^-_i + \frac{1}{2} A_\mu \Box X A^\mu \right)$$

(4.37)

up to higher order terms. The canonically rescaled scalar fields are hence given by

$$\tilde{\phi}_i^\pm = \frac{m}{g} \phi_i^\pm.$$ 

(4.38)

The first term is the kinetic energy of the rotating background. The last term leads to the mass $m^2$ (3.63) of the gauge fields in terms of the eigenvalues of $m^2 \Box X$, as anticipated. As usual, the cubic and quartic terms above will lead to interactions of Higgs fluctuations and the gauge fields, and the rotation drops out. In the presence of a non-trivial Higgs VEV $\phi_\alpha$, the higher-order terms in $\phi_\alpha$ must of course be included, however this is beyond the scope of this paper.

5 Stacks of branes

Now consider fermions in a background which consists of a sum of different branes of the above type, such as

$$X^a = \begin{pmatrix} X^a_{(1)} & 0 \\ 0 & X^a_{(2)} \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}.$$ 

(5.1)

Although a priori the different blocks might have independent parameters, stability suggests that the rotation vectors $\omega_i$ are the same for all branes, and for simplicity we assume that also the $r_i$ are the same. Then the above $X^a$ can be written in the familiar form

$$X^\pm_i = r_i e^{\pm i \omega_i \cdot x} T^\pm_i$$

(5.2)

where $T^a$ acts on the reducible representation $H_{\Lambda_1} \oplus H_{\Lambda_2}$. Some blocks may be trivial, with $H_0 = \mathbb{C}$ and $T^a = 0$, corresponding to point branes located at the origin. The fermions on the diagonal blocks behave as discussed above. On a point brane, there are only trivial fermionic zero modes $\Lambda = 0$ with degenerate spin states. The off-diagonal fermions can be written as $\Psi_{12} = \sum |s_i\rangle \langle \mu_1, \Lambda_1 | \langle \mu_2, \Lambda_2|$, and live in $H_{12} = H_1 \otimes H_2$. Similarly $\Psi_{21}$ lives in $H_{21} = H_2 \otimes H_1$, but this is redundant due to the M-W constraint $\Psi_{21} = \Psi_{12}^C$. Therefore all our results from section 3.3 on fermionic and bosonic zero modes carry over directly to the off-diagonal blocks: The (would-be) zero modes can be collected into $N = 4$ supermultiplets of trivial $\Lambda = 0$ modes, as well as chiral supermultiplets labeled by the extremal weights $\Lambda$ in $\text{End}(V_\mu)$. On a stack of squashed $C_N[\mu]$ branes, this leads to a quiver gauge theory, with gauge group $U(n_\mu)$ on each vertex $\mu_i$ and arrows labeled by $\Lambda$ corresponding to the chiral supermultiplets in $\text{Hom}(V_{\mu_i}, V_{\mu_j})$ with extremal weights $\Lambda$. These fields transform in the bi-fundamental $(n_\mu) \otimes (\bar{n}_\mu)$ of $U(n_\mu) \times U(\bar{n}_\mu)$. Charge conjugation relates the modes $\Lambda$ and $\Lambda^\dagger$ with reversed arrow, i.e. $\text{Hom}(V_{\mu_i}, V_{\mu_j})$ with $\text{Hom}(V_{\mu_j}, V_{\mu_i})$, so that these fields are not independent.

The case of a squashed $C_N[\mu]$ brane and a point brane is particularly interesting:
5.1 Point branes attached to squashed brane

Consider a background consisting of a $C_N[\mu]$ brane and a point brane at the origin:

$$X_i^\pm = \begin{pmatrix} X_{i(1)}^\pm & 0 \\ 0 & 0 \end{pmatrix}. \quad (5.3)$$

The first block is given by $End(H_1)$ with $H_1 \cong V_\mu$ a highest weight module with highest weight $\mu$. The off-diagonal modes live in $H_{12} = Hom(C, V_\mu) \cong V_\mu$ and $H_{21} = Hom(V_\mu, C) \cong V_\mu^*$, with extremal weights $\Lambda \in W_\mu$ resp. $\Lambda \in -W_\mu$. This leads to 6 independent off-diagonal fermionic and bosonic zero modes.

Let us discuss the fermionic zero modes $\Psi_{12}^\mu$ in more detail. Using the above results, they have the form

$$\Psi_{12}^\mu = |\uparrow\uparrow\uparrow\rangle |\mu\rangle_1 \psi_{12}^\mu, \quad (5.4)$$

along with 5 additional zero modes $\Psi_{12}^{\mu \cdot w}$ obtained by acting on $\Psi_{12}^\mu$ with suitable reflections. Their chirality is determined by the sign $|w|$ of the Weyl group element $w$. This leads to 3 left-handed fermions denoted by $C_L$, and 3 right-handed fermions called $C_R$, as in figure 1.

Let us estimate the energy scale of the heavy off-diagonal $\Psi_{12}^\mu$. Since they all live in the irreducible representation $V_\mu$, the mass terms arising from $m/D$ should lead to

$$m_{\Psi_{12}^{\text{heavy}}}^2 \approx \frac{1}{8} |\mu|^2 \omega^2. \quad (5.5)$$

which is very large for large branes $C[\mu]$. Then the scale of the off-diagonal heavy fermions is much larger than the mass of the relevant gauge bosons in $(0,3)$ resp. $(3,0)$ discussed in section 4.3 of order $|\omega|$. This is important in the approach to the standard model discussed below.

Now assume that some Higgs modes $\phi_i^{\pm(n)} = \varphi_i^{(n)}(X_i^\pm)^n$ on the brane $C_N[\mu]$ are switched on. Using the results of section 4.2, this leads to Yukawa couplings (4.25) relating adjacent fermionic zero modes $\Psi^\mu_{12}$. Thus the off-diagonal fermionic zero modes acquire a mass $m_{\Psi}^2 \sim \Delta \omega^2$ in the presence of a suitable Higgs mode $\langle \varphi^{(n)} \rangle \neq 0$ which links adjacent modes. No other fields on $C_N[\mu]$ can induce such a mass term, and the couplings between the fermionic zero modes and various heavy fermionic modes can presumably be neglected at low energies. Hence the off-diagonal sector is relatively simple.

5.1.1 Off-diagonal Higgs $S_{12}$

The off-diagonal scalar modes are parametrized as

$$X_i^\pm = X_i^\pm + \phi_i^\pm = \begin{pmatrix} X_{i(1)}^\pm & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & S_{i(12)}^\pm \\ S_{i(21)}^\pm & 0 \end{pmatrix}, \quad (5.6)$$

with $\phi_i^\pm = (\phi_i^\mp)^\dagger$. Since $H_{12} = V_\mu$ and $H_{21} = V_\mu^*$, we can apply the general results in section 3.1. There are 6 complex scalar (would-be) zero modes denoted as $S_{12}^{\mu \pm}$.

$$S_{i(12)}^\pm = h_i^\pm |\mu_i^\pm\rangle. \quad (5.7)$$

\footnote{The notation $S$ is used to emphasize the analogy with the $S$ Higgs in \cite{13}.}
with extremal weights $\mu_i^\pm \in W_\mu$ in the Weyl chamber opposite to $\pm \alpha_i$. The other off-diagonal modes are then determined by the hermiticity constraint,

$$S_{\mu(21)}^\pm = (S_{\mu(12)}^\pm)^\dagger = (h_i^\pm)^*(\mu_i^\pm).$$

This has the extremal weight $-\mu_i^\pm$ in the Weyl chamber opposite to $\pm \alpha_i$ (just like $\mu_i^\pm$), corresponding to the 6 (would-be) zero modes in $\mathcal{H}_{(21)} = V_\mu^* = V_\mu$. Therefore the off-diagonal low-energy modes correspond to the extremal weights in complex representations $V_\mu \oplus V_\bar{\mu}$, subject to the above hermiticity constraint.

If the $\omega_i$ are different, some of these off-diagonal modes acquire again a negative mass. To find solutions with $\langle S_{\mu}^\pm \rangle \neq 0$, we can use the results in section 4.1 for the matrix $\phi_{\mu}^\pm$. The extremal weight properties again imply $\mathcal{D}_{\text{mix}}^\nu X = 0 = \mathcal{D}_{\text{mix}}^\phi \phi$, so that the eom (4.8) applies. In analogy to the solution in section 4.1, we assume that $\phi_{\mu}^\pm = 0$ and $\phi_{\bar{\mu}}^\pm \neq 0$. However, the hermiticity conditions now relates different off-diagonal modes, and it is possible to have $S_{\mu}^\pm \neq 0$ while $S_{\bar{\mu}}^\pm = 0$; this will be important in section 6. Thus we make the ansatz

$$\phi_{\bar{\mu}}^3 = \varphi \begin{pmatrix} 0 \\ 0 \\ \langle \mu_3^\pm \rangle \end{pmatrix}, \quad \phi_{\mu}^3 = \varphi^* \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and all other $\phi_i^\pm = 0$. Then

$$[\phi_{\mu}^3, \phi_{\bar{\mu}}^3] = |\varphi|^2 \begin{pmatrix} |\mu_3^\pm\rangle \langle \mu_3^\pm | \\ 0 \\ -1 \end{pmatrix} = -[\phi_{\mu}^3, \phi_{\bar{\mu}}^3]$$

using $\langle \mu_3^\pm | \mu_3^\pm \rangle = 1$, and all other $[\phi_{\alpha}, \phi_{\beta}] = 0$. Therefore the equation of motion (4.8) are

$$(\square_{4} + \square_{X} + \square_{\phi}) X_i^\pm = 0, \quad i = 1, 2$$

$$(\square_{4} + \square_{X} + \square_{\phi} + 2 \mathcal{D}_{\text{diag}}^\phi) X_3^\pm = 0,$$

$$(\square_{4} + \square_{\phi} + \square_{X} + 2 \mathcal{D}_{\text{diag}}^X) \phi_3^\pm = 0$$

where

$$\mathcal{D}_{\text{diag}}^\phi X_3^\pm = \pm 2|\varphi|^2 [P_+, X_3^\pm], \quad P_+ = \begin{pmatrix} |\mu_3^\pm\rangle \langle \mu_3^\pm | \\ 0 \\ 0 \end{pmatrix},$$

$$\square_{\phi} \phi_3^\pm = 4|\varphi|^2 \phi_3^\pm.$$  (5.12)

We also know $(\square_{X} + 2 \mathcal{D}_{\text{diag}}^X) \phi_3^\pm = M_3^\phi \phi_3^\pm$ from section 3.1 with

$$M_3^\phi = 2(r_1^2 \alpha_1 + r_2^2 \alpha_2 - r_3^2 \alpha_3, \mu).$$

(5.13)

Therefore the last equation in (5.11) would be easy to handle. On the other hand, the first equations in (5.11) together with

$$\square_{\phi} X_i^\pm = 2|\varphi|^2 (P_+ X_i^\pm + X_i^\pm P_+)$$

require a modification of the matrix elements of $X_i^\pm$. The ladder structure of $X_i^\pm$ and hence the crucial decoupling properties should be preserved, since $P_+$ is diagonal in weight space.
Finding a solution should therefore not be too hard, but we postpone this to future work. An additional complication arises in the presence of non-vanishing Higgses $\phi_i^{(0)}$ on $C_N[\mu]$. The important point is that it is possible to have Higgs configurations with $S_{(12)}^+ \neq 0$ but $S_{(12)}^- = 0$, involving $|\mu_3^+\rangle$ but not $|\mu_3^-\rangle$. Since the $|\mu_3^\pm\rangle$ states are localized on one specific sheet $C_R$ (or $C_L$) of $C_N[\mu]$ with specific orientation, such a configuration breaks the symmetry between the opposite chiralities, leading to a chiral low-energy gauge theory with fermions of different chirality behaving differently. We will briefly discuss in section 6 how one might obtain in this way an effective low-energy behavior resembling the standard model.

5.1.2 Yukawas for off-diagonal fermions and Higgs

Finally we want to understand the impact of an off-diagonal Higgs $S^\alpha_{(12)}$ as in (5.6) to the off-diagonal fermions $\Psi^{12}_\mu$. We compute

$$
[S_{(12)}^+, \Psi_{\mu}^{12}] = (h_i^+)^* \left[ \langle \mu_1^- |, \uparrow\uparrow\uparrow \rangle |\mu_1\rangle \right] \\
= (h_i^+)^* \left( \langle \mu_1^- |, \mu \rangle \mathbb{1}_2 - |\mu_1\rangle \langle \mu_1^- | \right) \uparrow\uparrow\uparrow \rangle \psi_{\mu}^{12}
$$

and similarly for $\phi^-$. Consider first the Yukawa couplings with the fermionic zero modes $\Psi_{22}$ on the point brane, which arise from the terms $\sim \mathbb{1}_2$ above. Assuming that $\mu$ is in the dominant Weyl chamber, only $i = 3$ contributes in the Yukawa coupling

$$
m \text{tr} \bar{\Psi}_{22} \gamma_5 \left( \Delta^- [S_{(12)}^+, \Psi_{\mu}^{12}] + \Delta^+ [S_{(12)}^-, \Psi_{\mu}^{21}] \right)
$$

leading to

$$
m (h_3^-)^* \langle \mu_3^- |, \mu \rangle \bar{\Psi}_{22} \gamma_5 \Delta^- \uparrow\uparrow\uparrow \rangle \psi_{\mu}^{12}.
$$

This is non-vanishing, since $\Psi_{22}$ is a trivial $\Lambda = 0$ mode. More generally, the chiral zero mode $\Psi_{\mu}^{12}$ with $\mu$ equal to the weight $\mu_3^\pm$ of some non-vanishing $S_{(12)}^{\pm}$ is combined with the corresponding polarization of $\Psi_{22}$ into a massive Dirac fermion, which are thereby raised into a massive sector. If the Higgs configuration is such that $S_{(12)}^+ \neq 0$ but $S_{(12)}^- = 0$, then a chiral low-energy gauge theory emerges at low energies comprising only the remaining unpaired chiral modes in $\Psi_{\mu}^{12}$. In the context of the standard-model-like configuration discussed below, this may provide a mechanism to remove $\nu_R$ from the low-energy spectrum.

The second terms in (5.15) lead to Yukawa couplings of $\Psi^{12}_\mu$ with the fermionic zero modes $\Psi_{11}$ on $C[\mu]$. Assume that $\mu' = \mu$ is the highest weight in $V_\mu$, we obtain the Yukawas

$$
-m (h_i^-)^* \text{tr} (\bar{\Psi}_{11} |\mu_1\rangle \langle \mu_1^- |) \Delta^- \gamma_5 \uparrow\uparrow\uparrow \rangle \psi_{\mu}^{12}
$$

Now the $i = 3$ contribution can be ignored since then $\mu_3^- = \mu$ and $|\mu_1\rangle \langle \mu_1^- |$ has weight zero, but there is no $\Psi_{11}$ zero mode with vanishing weight. For $i = 1, 2$ the weight $\mu_i^-$ is adjacent to $\mu$ so that the polarizations match, but the overlap $\text{tr}(Y_{N'}^L |\mu_1\rangle \langle \mu_1^- |)$ with the extremal mode in $Y_{N'}^L \in \text{End}(\mathcal{H}_\mu)$ with $\Lambda' = \mu - \mu_i^-$ may be small. Thus there may be some suppressed Yukawa couplings between the $\Psi_{12}$ and $\Psi_{11}$ zero modes, which however requires more work.

\footnote{This is not in contradiction with the index theorem, since the other chirality is in the conjugate sector, which is however redundant due to the MW condition.}
6 Towards the standard model

As an example of the above configurations, consider two congruent 6-dimensional branes $C_N[\mu]$ denoted by $D_u$ resp. $D_d$, and one extra point brane denoted by $D_l$. Let us denote by $\mu_i^U$ the 3 extremal weights related to $\mu$ by a Weyl rotation, and by $\mu_i^L$ the remaining 3 extremal weights; in other words, $C_L = \{\mu_i^L\}$ and $C_R = \{\mu_i^R\}$ for $i = 1, 2, 3$, cf. figure \[1\]. As discussed before, off-diagonal chiral fermionic zero modes $\Psi^{(12)}_{\Lambda}$ arise, connecting $D_l$ with the points $C_L$ and $C_R$ in $D_u$ resp. $D_d$. We denote these off-diagonal fermions as follows:

$$l^i_L = \begin{pmatrix} \nu^i_L \\ \epsilon^i_L \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(ul)}_{\mu_i^L} \\ \Psi^{(dl)}_{\mu_i^L} \end{pmatrix}, \quad l^i_R = \begin{pmatrix} \nu^i_R \\ \epsilon^i_R \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(ur)}_{\mu_i^R} \\ \Psi^{(dr)}_{\mu_i^R} \end{pmatrix}. \quad (6.1)$$

These play the role of the leptons, with the appropriate chirality. Here $i = 1, 2, 3$ corresponds to the generations. This background admits an unbroken $U(2) \times U(1)_t$ gauge symmetry, where $SU(2)$ mixes $D_u$ and $D_d$ so that $l_L$ and $l_R$ are doublets. The $U(1)_t$ arises from $D_l$.

Now assume that there is an off-diagonal Higgs $S_{ab}^{(ul)} \neq 0$ connecting $D_l$ with $D_u$, attached to one (or several) of the extremal weight states $\mu_i^R$ of $C_R$. As discussed in section \[5.1.1\] we may and will assume that the corresponding $S_{a-i}^{(ul)} = 0$. This means that $S_{a-i}^{(ul)}$ transforms like $\nu^i_R$. Then the above symmetry is broken to $U(1)_{ul} \times U(1)_d$, where $U(1)_{ul}$ arises on the $D_u \cup D_l$ branes. In particular, $\nu_R$ becomes a singlet.

To obtain baryons, we add three extra baryonic point branes $D_{bj}$, $j = 1, 2, 3$. Now assume that there is no Higgs linking these baryonic branes and the leptonic ones.\[^{14}\] We denote the fermionic zero modes linking $D_{bj}$ with $D_u$ resp. $D_d$ according to chirality as

$$q^i_L = \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(ub)}_{\mu_i^L} \\ \Psi^{(db)}_{\mu_i^L} \end{pmatrix}, \quad q^i_R = \begin{pmatrix} u^i_R \\ d^i_R \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(ub)}_{\mu_i^R} \\ \Psi^{(db)}_{\mu_i^R} \end{pmatrix}. \quad (6.2)$$

The unbroken symmetry can then be organized as $U(1)_{uL} \times U(1)_b \times U(1)_{tr} \times SU(3)_c$, where the electric charge generator

$$Q = \frac{1}{2}(1_{ul} - 1_{d} - \frac{1}{3}1_{b}) \quad (6.3)$$

gives the correct charge assignment via the adjoint (cf. \[15\]), and $U(1)_{tr}$ is the overall trace-$U(1)$. Here $1_{ul} = 1_u + 1_l$, and $Q$ is traceless. The $U(1)_{tr}$ decouples in $\mathcal{N} = 4$ SYM, but it acquires an interesting role related to gravity in the IKKT matrix model \[22\].

Now consider the lowest massive gauge bosons. As explained in sections \[4.3\] only those gauge bosons with weight $M = 0$ couple to the low-energy currents arising from the off-diagonal fermions. In the absence of $S^{ul}$, the lowest among those are the $M = 0$ modes in $(1,1) \otimes u_2$ where $u_2$ denotes the $su(2)$ algebra acting on the coincident branes $D_u \cap D_d$. Let us however first discuss the next-to-lowest modes, given the $\chi_L^{ud}$ and $\chi_R^{ud}$ generators in $((3,0) + (0,3)) \otimes u_2$ discussed in section \[4.3\] which couple to the $C_L$ and $C_R$ states on $D_u$ and $D_d$, respectively. Thus they generate $U(2)_{dL}$ acting on $l_L$ and $q_L$, and $U(2)_{dR}$ acting on $l_R$ and

\[^{14}\]One possibility to ensure this might be to let the point branes rotate in a suitable way; however it is not clear if this can be consistently implemented.
$q_R$; this would lead to a left-right-symmetric model. However \( \chi_R^u \) acquires a mass from the off-diagonal Higgs \( S^{(ul)} \), which we assume to be large. This breaks \( U(2)_R \) to \( U(1)^d_R \) generated by \( \chi_R^d \), leaving the electroweak \( SU(2)_L \) generated by the \( \chi_L^{ud} \), and \( \chi_R^d \). Since the latter is anomalous upon restricting to the off-diagonal zero modes\(^{15}\), it seems more natural to replace it by the combination

\[
Y = I_{uL} - \frac{1}{3} I_b - (\chi_L^u + \chi_R^d)
\]

(6.4)
corresponding to weak hypercharge, which is traceless and anomaly free in the off-diagonal low-energy sector. We simply assume here that this happens through some unknown mechanism. It is plausible then that \( Y \) inherits the mass of \( \chi_R^d \), which is also the mass of the \( W \) bosons

\[
m_Y^2 \approx m_W^2 \sim 18m^2r^2 \approx \frac{9}{4} \omega^2.
\]

(6.5)

For the zero modes, this leads to the Gell-Mann-Nishijima formula

\[
2Q - Y = \chi_L^u - \chi_L^d = T_3^L.
\]

(6.6)

Now we come back to the \( M = 0 \) gauge boson modes in \((1,1)\) on \( C[\mu] \), given by the Cartan generators \( H_{3,8}^u \). These don’t fit into the standard model, because they are sensitive to the generation. Those on \( D_\alpha \) acquire a mass due to \( S^{(ul)}_\alpha \). There are candidates for Higgs modes on \( C[\mu] \) which give mass to all \( H_{3,8} \) modes but not to the \( \chi_{L,R} \) modes, notably the \( \Lambda = (3,0) \) and \( \Lambda = (0,3) \) modes which relate the weights within \( C_L \) resp. \( C_R \). If those are switched on, then the \( H_{3,8} \) modes would become massive, and leave the “chiral” \( \chi_{L,R} \) as lowest massive gauge bosons. However this is simply an assumption at this point. In addition, there is a Kaluza-Klein tower of gauge bosons on \( C[\mu] \) with roughly equidistant masses above \( m_W \). They are expected to acquire additional mass terms form various Higgses, which hopefully raise them sufficiently far above the \( W \) scale.

We have thus identified all the off-diagonal (would-be) fermionic zero modes with standard model fermions, plus \( \nu_R \). They are expected to acquire a mass of order \((4.26)\)

\[
m_\nu^2 \sim \Delta \omega_i^2
\]

(6.7)
due to the Higgs fields \( \phi^{(0)}_\alpha \) on \( D_{u,d} \). Since in the (illustrative) solution \((4.16)\) only one polarization \( \phi^{(0)}_\alpha \) is switched on, it seems likely that one generation is much heavier that the others; this is also suggested by the sum rule \((3.27)\). On the other hand, additional fermionic (would-be) zero modes arise on the branes \( D_{u,d} \) and \( D_{l,b} \). Some fermions on \( D_l \) will pair via \( S^{(ul)} \) with \( \nu_R \) to form a massive Dirac fermion, according to section \((5.1.2)\). The fermions on \( D_b \) are superpartners of the gluons in a \( N = 4 \) supermultiplet. Finally the fermions on \( D_{u,d} \) are mixed by many possible Yukawa couplings via the various Higgs modes, as discussed in section \((4.2)\). This suggests that they tend to be heavier than the off-diagonal fermions, which have a rather clean structure of Yukawas. The details are clearly complicated and require much more work.

\(^{15}\)Any anomaly which may arise upon restricting to a subset of fermionic zero modes must ultimately cancel, since the underlying model is anomaly free.
We also recall that for branes \( C[\mu] \) with large \( \mu \), the heavy off-diagonal fermions \( 5.5 \) are much heavier than the \( W \) mass. This is an important improvement over the configuration in \[13\]. While the “internal” sector of zero modes on the \( D_{u,d} \) is much simpler and cleaner on minimal branes \( C[(1,1)] \), the heavy off-diagonal fermions would be roughly at the same scale as the \( W \) mass. Hence the case of large \( \mu \) seems more attractive from the particle physics point of view.

It is important to note that up to now, we discussed only the “upper-diagonal” \( \Psi_{12} \)-type modes between the branes. However this gives indeed the full story, since the conjugate \( \Psi_{21} \) modes are fixed by the MW condition and do not represent independent physical modes. We emphasize again that it is the assumption that \( S^{(ul)} \) is connected to \( C_R \) but not \( C_L \) which is responsible for turning a left-right symmetric \( SU(2)_L \times SU(2)_R \) model into the above chiral standard-like model, starting from \( \mathcal{N} = 4 \) SYM.

We content ourselves with these qualitative observations here, leaving a more detailed analysis to future work. No claim on the viability of such a scenario can be made. However, given the special status of \( \mathcal{N} = 4 \) SYM (and the IKKT matrix model), it is certainly very intriguing that one can arrive quite naturally in the vicinity of the standard model (in the broken phase), reproducing all its odd quantum numbers and even “predicting” the number of generations.

### 7 Energy and current

Let us compute the energy density of the background, which in our conventions is given by

\[
E = T_{00} = \frac{1}{4g^2} \text{tr} \left( 2D_0 \Phi^a D_0 \Phi_a + D_\mu \Phi^a D^\mu \Phi_a + \Phi_a \Box \Phi \Phi_b \right).
\]

Assuming \( r_i = r \) and \( \omega_i = \omega \) and using \( \Box \Phi_a = -\omega^2 \Phi_a \), we obtain

\[
E = \frac{1}{2g^2} \text{tr} D_0 \Phi^a D_0 \Phi_a = \frac{1}{2g^2} \omega^2 m^2 r^2 \text{tr} \left( \sum \alpha T^\alpha T_\alpha \right)
\]

using \( 2.22 \). Here \( \text{tr} \left( \sum \alpha T^\alpha T_\alpha \right) \) can easily be evaluated in a given representation, cf. \[15\]. This is clearly a very large energy density, set by the UV scale of the model. Similarly, the total \( R \)-current corresponding to the sum of the 3 commuting \( U(1)_i \) rotations with frequency \( \omega \) is

\[
J_\mu = \frac{1}{g^2} \text{tr} \left( \partial_\mu \Phi^a (t_1 + t_2 + t_3)_{ab} \Phi^b \right) = \frac{1}{g^2} m^2 r^2 \omega \text{tr} \left( \sum \alpha T^\alpha T_\alpha \right)
\]

where \( t_i \) generates \( U(1)_i \). Hence the energy per \( R \)-charge is

\[
\frac{E}{J_0} = \frac{1}{2} \omega_0.
\]

The above expressions for \( E \) and \( J_\mu \) apply quite generally for various rotating brane solutions. For example, there are rotating fuzzy sphere solutions given by \( X_i^\pm = r e^{\pm i\omega x} J_i^{(l)} \) where \( J_i^{(l)} \) are the spin \( l \) generators of \( \mathfrak{su}(2) \). All these solutions have the same energy per \( R \) charge ratio for given \( \omega \) (the formulae apply even for a rotating plane wave solution \( X_i^\pm \sim mre^{\pm i\omega x} \)).
with light-like current $J_\mu \sim \omega_\mu$). Therefore any of these solutions costs the same energy per given $R$-charge density, and there is no clear “ground state” with lowest energy per $R$ charge. In particular, there is no obvious instability of the squashed brane background under consideration in this paper.

8 Discussion and conclusion

We have studied certain generalized “vacua” of $\mathcal{N} = 4$ SYM which are not Poincare-invariant but admit a generalized translational invariance (3.30), due to spinning scalar fields. We found a remarkable class of such backgrounds leading to a low-energy sector of excitations with Poincare-invariant kinematics, which is largely oblivious to the background rotation. These backgrounds have a geometrical interpretation as spinning fuzzy squashed coadjoint orbits of $SU(3)$ introduced in [15], embedded in 6 extra dimension. We started to explore the physics of the low-lying excitations, which includes fermionic zero modes with distinct chiralities, analogous scalar zero modes, and a tower of gauge bosons. If the 3 rotation frequencies $\omega_i$ of the background are slightly different, then some of the scalar (would-be) zero modes acquire a non-trivial VEV, with scale set by the difference of the frequencies. This provides a solid mechanism for introducing a low scale into the theory, with a stable hierarchy between the lowest fermions and the rotation scale.

In addition to these zero modes, the branes lead to generalized Kaluza-Klein towers of “heavy” modes, which couple to the rotating background. We assume that there are no instabilities due to resonances, for suitable choices of the three rotation frequencies. A finite-dimensional equation governing these heavy modes is found, which should allow in principle to determine all the required properties of the heavy sector. The stability assumption appears to be reasonable, but needs to be addressed in more detail elsewhere.

On stacks of such branes, gauge theories of quiver type arise, retaining part of the underlying supersymmetry structure. While the detailed content of these low-energy theories is rather complicated and needs further elaboration, it can be studied effectively using group-theoretical tools. The structure of the Yukawa couplings is investigated, leading to a intricate pattern of interactions with the low-energy VEVs of the various Higgs candidates.

Finally, we identified simple brane configurations which lead to a low-energy sector close to the standard model, reproducing all the quantum numbers and even correctly “predicting” the number of generations. These standard model matter fields arise as strings connecting different branes. Additional fields arise within the branes, whose detailed structure and physics remains to be clarified. Therefore no claim on physical viability can be made, but it is certainly very intriguing that one can arrive quite naturally in the vicinity of the standard model (in the broken phase), starting from pure $\mathcal{N} = 4$ SYM.

The results and constructions of this paper generalize immediately to the IKKT or IIB matrix model [16], which can be viewed as a non-commutative $\mathcal{N} = 4$ SYM. While the noncommutative field aspects discussed here should have a smooth commutative limit [21], the trace-$U(1)$ sector acquires an interesting role related to gravity [22]. The present results are therefore also very encouraging towards considering matrix models as fundamental theories. Furthermore, given the much-studied relation between $\mathcal{N} = 4$ SYM and string theory on $AdS^5 \times S^5$ [23], it is natural to ask about a dual description in terms of supergravity or string theory. In the present paper, the effective spectral geometry of the extra dimensions
was exhibited; its description could be made more explicit along the lines of [22]. On the other hand, the relation with supergravity should be seen upon taking into account quantum corrections. Our solutions are reminiscent of the story of giant gravitons [24], which are dual to operators with non-vanishing $R$-charge [25]. Therefore the solutions might be interpreted as a condensate of some sort of giant gravitons in supergravity, however we leave this to future work.

It is clear that the present paper is just the beginning of a new approach to obtain interesting low-energy physics from $\mathcal{N} = 4$ SYM. There are countless issues to be studied in more detail, and many variations are conceivable, with or without explicit flux terms such as in [15]. If the stability of the rotating background and the “heavy sector” can be reliably established, this would greatly expand the scope of $\mathcal{N} = 4$ SYM, moving it much closer to actual physics than previously conceived. How close one can get of course remains to be seen.

**Acknowledgments.** This work is supported by the Austrian Fonds für Wissenschaft und Forschung under grant P24713. I would like to thank in particular J. Zahn for related collaboration and discussions. This work also benefited from useful discussions with H. Aoki, A. Chatzitavrakidis, C-S. Chu, H. Kawai, J. Nishimura, and A. Tsuchyia.

### A Structure of the potentials and mixing

In this appendix, we identify the eigenmodes of the static part of the potential assuming $r_i = r$, and show that it is strictly positive apart from the zero modes identified before. Its eigenstates have definite polarization $\alpha \in \mathcal{I}$ and weights $M$ in $\mathcal{H}_\Lambda$. Moreover, we will establish positivity of the operator $O_V$ (3.40) based on results in [15].

It is convenient to collect the fluctuation modes as

$$\phi = \sum_{\alpha \in \mathcal{I}} \lambda^\alpha \otimes \phi_\alpha \in (8) \otimes \mathcal{H}_\Lambda$$

(A.1)

for $\alpha \in \mathcal{I}$, subject to $\phi^\dagger = \phi$. We can restrict ourselves to a given highest weight irrep $V_\Lambda$, with basis $|M\rangle_\Lambda$. Then the quadratic potential takes a simple form, separated into static potential

$$V_{\text{stat}}[\Phi] = \text{tr} \Phi \left( \frac{1}{2} \Box_X + \mathcal{D}_{\text{diag}} \right) \Phi$$

(A.2)

and rotating potential

$$V_{\text{rot}}[\Phi] = \text{tr} \Phi \mathcal{D}_{\text{mix}} \Phi,$$

(A.3)

noting that $\text{tr} \lambda_\alpha \lambda_\beta = 2 \kappa_{\alpha\beta}$. Consider first the static potential. It is not hard to see that in the representation (A.1), we can write $\mathcal{D}_{\text{diag}}$ as

$$\mathcal{D}_{\text{diag}} = -\left( [\lambda_3,] [H_3,] + [\lambda_8,] [H_8,] \right).$$

(A.4)

Clearly, both $\Box_X$ and $\mathcal{D}_{\text{diag}}$ are diagonal on the product states $|L, M\rangle \equiv |L\rangle_{(8)} \otimes |M\rangle_\Lambda \in (8) \otimes \mathcal{H}_\Lambda$ with weights $L$ resp. $M$. Using the assumption that the radii $r_i = r$ are all the same, we can write

$$\frac{1}{r^2} \Box_X = \Box_T - \left( [H_3, [H_3,]] + [H_8, [H_8,]] \right)$$

(A.5)
where □_T = \sum_{a=1}^{8} [T_a, [T_a, .]] is the quadratic Casimir on V_\Lambda. Therefore
\[ \mathcal{D}_{\text{diag}} |L, M\rangle = 2(M, L)|L, M\rangle \]
\[ \Box_X |L, M\rangle = 2((\Lambda, \Lambda + 2\rho) - (M, M))|L, M\rangle \] (A.6)

where \( \rho = \alpha_1 + \alpha_2 = \alpha_3 \) is the Weyl vector of su(3). As a consistency check, we recover \( \Box_X X^a = 8r^2X^a \). Therefore the \( \Phi_{LM} = |L, M\rangle \) states are indeed the eigenstates of \( \frac{1}{2}\Box_X + \mathcal{D}_{\text{diag}} \).

On these states, we obtain
\[ V_{\text{stat}}[\Phi_{LM}] = r^2 \text{tr} \Phi_{LM}^\dagger \left( (\Lambda, \Lambda + 2\rho) - (M, M) \right) \Phi_{LM} \]
\[ \geq r^2 \text{tr} \Phi_{LM}^\dagger \left( (\Lambda, \Lambda + 2\rho) - (\Lambda, \Lambda) - (M, 2L) \right) \Phi_{LM} \]
\[ = 2r^2 \text{tr} \Phi_{LM}^\dagger \left( (\Lambda, \rho) - (M, L) \right) \Phi_{LM}. \] (A.7)

This is clearly non-negative, and vanishes only if \((M, L) = (\Lambda, \rho)\) and its images under \( \mathcal{W} \). But these are precisely the extremal zero modes \( \phi_{\alpha}^{(0)} \) discussed above, with fixed polarization \( \alpha \). Thus \( V_{\text{stat}}[\Phi] \) is positive for all the non-zero modes, with eigenvalues of order one or larger.

Now consider the mixing operator \( \mathcal{D}^{\text{mix}} \), for fixed background. The corresponding bilinear from can be written in the following transparent way\[^{[15]}\]
\[ \frac{1}{2} \text{Tr} \Phi^\dagger \mathcal{D}^{\text{mix}} \Phi = \sum_{\alpha, \beta, \gamma I} ig_{\alpha \gamma \beta} \text{tr}(\Phi^{\dagger \alpha}[X^\beta, \Phi^\gamma]) \]
\[ = -\frac{1}{2} \sum_{\alpha, \beta, \gamma I} \text{tr}([\lambda_\alpha, [\lambda_\beta, \lambda_\gamma]] \Phi^{\dagger \alpha}[X^\beta, \Phi^\gamma]) \]
\[ = -\frac{1}{2} \sum_{\beta I} \text{tr}(\Phi^{\dagger \beta}[\lambda_\beta, [X^\beta, \Phi]]) \]
\[ = -\frac{1}{2} \text{tr}(\Phi^\dagger \tilde{\mathcal{D}}^{\text{mix}} \Phi) \] (A.8)

where we introduce\[^{[16]}\]
\[ \tilde{\mathcal{D}}^{\text{mix}} \Phi = -\sum_{\beta I} [\lambda_\beta, [X^\beta, \Phi]] \]
\[ \tilde{\mathcal{D}}_{\text{ad}} \Phi = -\sum_{b=1}^{8} [\lambda_b, [X_b, \Phi]] = \tilde{\mathcal{D}}^{\text{mix}} \Phi + \mathcal{D}_{\text{diag}} \Phi \] (A.9)

acting on \( \Phi \in (8) \otimes \mathcal{H}_\Lambda \). This implies that
\[ -2 \tilde{\mathcal{D}}_{\text{ad}} = \Box_{T+L} - \Box_T - 6 \] (A.10)

where
\[ L_a = [\lambda_a, .], \quad \sum_{a=1}^{8} L_a L_a = 12. \] (A.11)

\[^{[16]}\]This form is valid to compute the bilinear form, but note that \( \mathcal{D}^{\text{mix}} \Phi \) differs from the definition (??) by terms with \( \lambda_8 \otimes \Phi_8 + \lambda_8 \otimes \Phi_8 \); these drop out in the bilinear form.
We can simultaneously diagonalize $M$ and $\Lambda$ (but not $\Lambda'$) acting on $\Phi \in (8) \otimes \mathcal{H}_\Lambda$. Then $\not\!\!D_{ad}$ is diagonal on the fluctuation modes corresponding to combined irreps with highest weight $\Lambda'$,

$$\Phi \in (8) \otimes V_\Lambda = \oplus V_{\Lambda'}.$$  \hfill (A.12)

with eigenvalues

$$2\not\!\!D_{ad} = -2(\Lambda', \Lambda' + 2\rho) + 2(\Lambda, \Lambda + 2\rho) + 12$$  \hfill (A.13)
on $V_{\Lambda'}$. Thus we can write

$$O_V = \Box_X + 2\not\!\!D_{ad} = \Box_X - \Box_{T+L} + \Box_T + 12$$

$$= 2\Box_T - \Box_{T+L} - ([H_3, [H_3, .]] + [H_8, [H_8, .]]) + 12$$  \hfill (A.14)

It is not easy to compute the spectrum of $O_V$, since these operators do not commute. However, we observe that $O_V$ differs from the static potential (B.7) in \cite{15} only by the sign of $\not\!\!D_{mix}$, and the two are in fact unitarily equivalent. Indeed using $\tau \not\!\!D_{mix} = -\not\!\!D_{mix} \tau$ (3.7), we have

$$\tau O_V \tau = \Box_X - 2\not\!\!D_{mix} + 2\not\!\!D_{diag}.$$  \hfill (A.15)

This is precisely the potential considered in \cite{15}, which was shown to be positive definite except for the above zero modes, and a set of exceptional zero modes with $\Lambda' = m\Lambda_1$ and $\Lambda' = m\Lambda_2$. This implies that $O_V$ is also positive semi-definite. We can recover the (regular) zero modes of $O_V$ for $\Lambda' = \Lambda + \rho$ and $M = \Lambda$ directly from (A.14), where the operators commute, with eigenvalue

$$4(\Lambda, \Lambda + 2\rho) - 2(\Lambda', \Lambda' + 2\rho) - 2(M, M) + 12 = 0.$$  \hfill (A.16)

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