COMPACTNESS ESTIMATES FOR THE $\overline{\partial}$ - NEUMANN PROBLEM IN WEIGHTED $L^2$ - SPACES.

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Abstract.
In this paper we discuss compactness estimates for the $\overline{\partial}$-Neumann problem in the setting of weighted $L^2$-spaces on $\mathbb{C}^n$. For this purpose we use a version of the Rellich - Lemma for weighted Sobolev spaces.

1. Introduction.

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. We consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0,$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on $\Omega$ with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial} \left( \sum_{J} a_J \, d\zeta_J \right) = \sum_{j=1}^{n} \sum_{J} \frac{\partial a_J}{\partial \overline{\zeta}_j} \, d\overline{\zeta}_j \wedge d\zeta_J,$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices $J$. The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$. The complex Laplacian $\Box = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial}$ acts as an unbounded selfadjoint operator on $L^2_{(0,q)}(\Omega)$, $1 \leq q \leq n$, it is surjective and therefore has a continuous inverse, the $\overline{\partial}$-Neumann operator $N_q$. If $v$ is a $\overline{\partial}$-closed $(0,q+1)$-form, then $u = \overline{\partial}^* N_{q+1} v$ provides the canonical solution to $\overline{\partial} u = v$, namely the one orthogonal to the kernel of $\overline{\partial}$ and so the one with minimal norm (see for instance [ChSh]).

A survey of the $L^2$-Sobolev theory of the $\overline{\partial}$-Neumann problem is given in [BS].

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The question of compactness of $N_q$ is of interest for various reasons. For example, compactness of $N_q$ implies global regularity in the sense of preservation of Sobolev spaces $[KN]$. Also, the Fredholm theory of Toeplitz operators is an immediate consequence of compactness in the $\bar{\partial}$-Neumann problem $[HI], [CD]$. There are additional ramifications for certain $C^*$-algebras naturally associated to a domain in $\mathbb{C}^n$ $[SSU]$. Finally, compactness is a more robust property than global regularity - for example, it localizes, whereas global regularity does not - and it is generally believed to be more tractable than global regularity.

A thorough discussion of compactness in the $\bar{\partial}$-Neumann problem can be found in $[FS1]$ and $[FS2]$.

The study of the $\bar{\partial}$-Neumann problem is essentially equivalent to the study of the canonical solution operator to $\bar{\partial}$:

The $\bar{\partial}$-Neumann operator $N_q$ is compact from $L^2_{(0,q)}(\Omega)$ to itself if and only if the canonical solution operators

$$\bar{\partial}^* N_q : L^2_{(0,q-1)}(\Omega) \to L^2_{(0,q)}(\Omega) \quad \text{and} \quad \bar{\partial}^* N_{q+1} : L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega)$$

are compact.

Not very much is known in the case of unbounded domains. In this paper we continue the investigations of $[HaHe]$ concerning existence and compactness of the canonical solution operator to $\bar{\partial}$ on weighted $L^2$-spaces over $\mathbb{C}^n$, where we applied ideas which were used in the spectral analysis of the Witten Laplacian in the real case, see $[HeNi]$.

Let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic $C^2$-weight function and define the space

$$L^2(\mathbb{C}^n, \varphi) = \{ f : \mathbb{C}^n \to \mathbb{C} : \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < \infty \},$$

where $\lambda$ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and the space $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. Let

$$\langle f, g \rangle_\varphi = \int_{\mathbb{C}^n} f \overline{g} e^{-\varphi} d\lambda$$

denote the inner product and

$$\| f \|_\varphi^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda$$

the norm in $L^2(\mathbb{C}^n, \varphi)$.

We consider the weighted $\bar{\partial}$-complex

$$L^2(\mathbb{C}^n, \varphi) \xrightarrow{\bar{\partial}^*_{\varphi}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \xrightarrow{\bar{\partial}^*_{\varphi}} L^2_{(0,2)}(\mathbb{C}^n, \varphi),$$

where $\bar{\partial}^*_{\varphi}$ is the adjoint operator to $\bar{\partial}$ with respect to the weighted inner product. For $u = \sum_{j=1}^n u_j d\bar{z}_j \in \text{dom}(\bar{\partial}^*_{\varphi})$ one has

$$\bar{\partial}^*_{\varphi} u = - \sum_{j=1}^n \left( \frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) u_j.$$
The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_{\varphi} := \overline{\partial} \partial^*_{\varphi} + \overline{\partial}^* \partial_{\varphi},$$

where the symbol $\Box_{\varphi}$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $C^\infty_0$, i.e., the space of smooth functions with compact support. $\Box_{\varphi}$ is a selfadjoint and positive operator, which means that

$$\langle \Box_{\varphi} f, f \rangle_{\varphi} \geq 0 \, , \text{ for } f \in \text{dom}(\Box_{\varphi}).$$

The associated Dirichlet form is denoted by

$$Q_{\varphi}(f, g) = \langle \partial f, \partial g \rangle_{\varphi} + \langle \overline{\partial}_{\varphi} f, \overline{\partial}_{\varphi} g \rangle_{\varphi},$$

for $f, g \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_{\varphi})$. The weighted $\overline{\partial}$-Neumann operator $N_{\varphi}$ is - if it exists - the bounded inverse of $\Box_{\varphi}$.

There is an interesting connection between $\overline{\partial}$ and the theory of Schrödinger operators with magnetic fields, see for example [Ch], [B], [FS3] and [ChF] for recent contributions exploiting this point of view.

Here we use a Rellich - Lemma for weighted Sobolev spaces to establish compactness estimates for the $\overline{\partial}$-Neumann operator $N_{\varphi}$ on $L^2(0,1)(\mathbb{C}^n, \varphi)$ and we use this to give a new proof of the main result of [HaHe] without spectral theory of Schrödinger operators.

2. **Weighted basic estimates.**

In the weighted space $L^2(0,1)(\mathbb{C}^n, \varphi)$ we can give a simple characterization of $\text{dom}(\overline{\partial}_{\varphi})$:

**Proposition 2.1.** Let $f = \sum f_j d\tau_j \in L^2(0,1)(\mathbb{C}^n, \varphi)$. Then $f \in \text{dom}(\overline{\partial}_{\varphi})$ if and only if

$$\sum_{j=1}^n \left( \frac{\partial f_j}{\partial \tau_j} - \frac{\partial \varphi}{\partial \tau_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi).$$

**Proof.** Suppose first that $\sum_{j=1}^n \left( \frac{\partial f_j}{\partial \tau_j} - \frac{\partial \varphi}{\partial \tau_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi)$, which equivalently means that $e^\varphi \sum_{j=1}^n \frac{\partial}{\partial \tau_j} (f_j e^{-\varphi}) \in L^2(\mathbb{C}^n, \varphi)$. We have to show that there exists a constant $C$ such that $|\langle \overline{\partial} g, f \rangle_{\varphi}| \leq C\|g\|_{\varphi}$ for all $g \in \text{dom}(\overline{\partial})$. To this end let $(\chi_R)_{R \in \mathbb{N}}$ be a family of radially symmetric smooth cutoff funtions, which are identically one on $\mathbb{B}_R$, the ball with radius $R$, such that the support of $\chi_R$ is contained in $\mathbb{B}_{R+1}$, $\text{supp}(\chi_R) \subset \mathbb{B}_{R+1}$, and such that furthermore all first order derivatives of all functions in this family are uniformly bounded by a constant $M$. Then for all $g \in C^\infty_0(\mathbb{C}^n)$:

$$\langle \overline{\partial} g, \chi_R f \rangle_{\varphi} = \sum_{j=1}^n \frac{\partial g}{\partial \tau_j} \chi_R f_j \varphi = -\int \sum_{j=1}^n g \frac{\partial}{\partial \tau_j} (\chi_R f_j e^{-\varphi}) d\lambda,$$
by integration by parts, which in particular means

$$|\langle \partial g, f \rangle_{\varphi}| = \lim_{R \to \infty} |\langle \partial g, \chi_R f \rangle_{\varphi}| = \lim_{R \to \infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^{n} g \frac{\partial}{\partial z_j} (\chi_R f_j e^{-\varphi}) \, d\lambda \right|.$$ 

Now we use the triangle inequality, afterwards Cauchy–Schwarz, to get

$$\lim_{R \to \infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^{n} g \frac{\partial}{\partial z_j} (\chi_R f_j e^{-\varphi}) \, d\lambda \right| \leq \lim_{R \to \infty} \left| \int_{\mathbb{C}^n} \chi_R g \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \, d\lambda \right| + \lim_{R \to \infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^{n} f_j g \frac{\partial \chi_R}{\partial z_j} e^{-\varphi} \, d\lambda \right|$$

$$\leq \lim_{R \to \infty} \| \chi_R g \|_{\varphi} \left\| e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \right\|_{\varphi} + M \| g \|_{\varphi} \| f \|_{\varphi}$$

$$= \| g \|_{\varphi} \left\| e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \right\|_{\varphi} + M \| g \|_{\varphi} \| f \|_{\varphi}.$$

Hence by assumption,

$$|\langle \partial g, f \rangle_{\varphi}| \leq \| g \|_{\varphi} \left\| e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \right\|_{\varphi} + M \| g \|_{\varphi} \| f \|_{\varphi} \leq C \| g \|_{\varphi}$$

for all $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, and by density of $\mathcal{C}_0^\infty(\mathbb{C}^n)$ this is true for all $g \in dom(\bar{\partial})$. Conversely, let $f \in dom(\bar{\partial}_{\varphi})$ and take $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. Then $g \in dom(\bar{\partial})$ and

$$\langle g, \bar{\partial}_{\varphi} f \rangle_{\varphi} = \langle \bar{\partial} g, f \rangle_{\varphi}$$

$$= \sum_{j=1}^{n} \langle \partial g \frac{\partial}{\partial z_j}, f_j \rangle_{\varphi}$$

$$= - \langle g, \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \rangle_{L^2}$$

$$= - \langle g, e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \rangle_{\varphi},$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n, \varphi)$, we conclude that

$$\bar{\partial}_{\varphi} f = -e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f e^{-\varphi}),$$

which in particular implies that $e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \in L^2(\mathbb{C}^n, \varphi).$ \qed

The following Lemma will be important for our considerations.
Lemma 2.2. **Forms with coefficients in** $C^\infty_0(\mathbb{C}^n)$ **are dense in** $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi)$ **in the graph norm** $f \mapsto (\|f\|^2 + \|\overline{\partial} f\|^2 + \|\overline{\partial}_\varphi f\|^2)^{\frac{1}{2}}$.

**Proof.** First we show that compactly supported $L^2$-forms are dense in the graph norm. So let $\{\chi_R\}_{R \in \mathbb{N}}$ be a family of smooth radially symmetric cutoffs identically one on $\mathbb{B}_R$ and supported in $\mathbb{B}_{R+1}$, such that all first order derivatives of the functions in this family are uniformly bounded in $R$ by a constant $M$.

Let $f \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi)$. Then, clearly, $\chi_R f \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi)$ and $\chi_R f \to f$ in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ as $R \to \infty$. As observed in Proposition 2.1, we have

$$\overline{\partial}_\varphi^* f = -\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} (f_j e^{-\varphi}),$$

hence

$$\overline{\partial}_\varphi (\chi_R f) = -\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} (\chi_R f_j e^{-\varphi}).$$

We need to estimate the difference of these expressions

$$\overline{\partial}_\varphi^* f - \overline{\partial}_\varphi (\chi_R f) = \overline{\partial}_\varphi^* f - \chi_R \overline{\partial}_\varphi^* f + \sum_{j=1}^n \frac{\partial}{\partial z_j} \chi_R f_j,$$

which is by the triangle inequality

$$\|\overline{\partial}_\varphi^* f - \overline{\partial}_\varphi (\chi_R f)\|_\varphi \leq \|\overline{\partial}_\varphi^* f - \chi_R \overline{\partial}_\varphi^* f\|_\varphi + M \sum_{j=1}^n \int_{B_R} |f_j|^2 e^{-\varphi} d\lambda.$$}

Now both terms tend to 0 as $R \to \infty$, and one can see similarly that also $\overline{\partial} (\chi_R f) \to \overline{\partial} f$ as $R \to \infty$.

So we have density of compactly supported forms in the graph norm, and density of forms with coefficients in $C^\infty_0(\mathbb{C}^n)$ will follow by applying Friedrich’s Lemma, see appendix D in [ChSh], see also [Jo].

As in the case of bounded domains, the canonical solution operator to $\overline{\partial}$, which we denote by $S_{\varphi}$, is given by $\overline{\partial}_\varphi N_{\varphi}$. Existence and compactness of $N_{\varphi}$ and $S_{\varphi}$ are closely related. At first, we notice that equivalent weight functions have the same properties in this regard.

Lemma 2.3. Let $\varphi_1$ and $\varphi_2$ be two equivalent weights, i.e., $C^{-1}\|\cdot\|_{\varphi_1} \leq \|\cdot\|_{\varphi_2} \leq C\|\cdot\|_{\varphi_1}$ for some $C > 0$. Suppose that $S_{\varphi_2}$ exists. Then $S_{\varphi_1}$ also exists and $S_{\varphi_1}$ is compact if and only if $S_{\varphi_2}$ is compact.

An analog statement is true for the weighted $\overline{\partial}$-Neumann operator.

**Proof.** Let $\iota$ be the identity $\iota : L^2_{(0,1)}(\mathbb{C}^n, \varphi_1) \to L^2_{(0,1)}(\mathbb{C}^n, \varphi_2)$, $\iota f = f$, let $j$ be the identity $j : L^2_{\varphi_2} \to L^2_{\varphi_1}$ and let furthermore $P$ be the orthogonal projection onto ker$(\overline{\partial})$ in $L^2_{\varphi_1}$.

Since the weights are equivalent, $\iota$ and $j$ are continuous, so if $S_{\varphi_2}$ is compact, $j \circ S_{\varphi_2} \circ \iota$ gives a solution operator on $L^2_{(0,1)}(\mathbb{C}^n, \varphi_1)$ that is compact. Therefore the canonical solution operator $S_{\varphi_1} = P \circ j^{-1} \circ S_{\varphi_2} \circ \iota$ is also compact. Since the problem
is symmetric in $\varphi_1$ and $\varphi_2$, we are done.

The assertion for the Neumann operator follows by the identity

$$N_\varphi = S_\varphi S_\varphi^* + S_\varphi^* S_\varphi.$$  

Note that whereas existence and compactness of the weighted $\overline{\partial}$-Neumann operator is invariant under equivalent weights by Lemma 2.3, regularity is not. For examples on bounded pseudoconvex domains, see for instance [ChSh], chapter 6.

Now we suppose that the lowest eigenvalue $\lambda_\varphi$ of the Levi - matrix

$$M_\varphi = \left( \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right)_{jk}$$

of $\varphi$ satisfies

$$\lim \inf_{|z| \to \infty} \lambda_\varphi(z) > 0. \quad (*)$$

Then, by Lemma 2.3, we may assume without loss of generality that $\lambda_\varphi(z) > \epsilon$ for some $\epsilon > 0$ and all $z \in \mathbb{C}^n$, since changing the weight function on a compact set does not influence our considerations. So we have the following basic estimate

**Proposition 2.4.**

For a plurisubharmonic weight function $\varphi$ satisfying $(*)$, there is a $C > 0$ such that

$$\|u\|_{\varphi}^2 \leq C(\|\overline{\partial} u\|_{\varphi}^2 + \|\overline{\partial}^* u\|_{\varphi}^2)$$

for each $(0,1)$-form $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial^*})$.

**Proof.** By Lemma 2.2 and the assumption on $\varphi$ it suffices to show that

$$\int_{\mathbb{C}^n} \sum_{j,k=1}^n \left( \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right) u_j u_k e^{-\varphi} \ d\lambda \leq \|\overline{\partial} u\|_{\varphi}^2 + \|\overline{\partial}^* u\|_{\varphi}^2,$$

for each $(0,1)$-form $u = \sum_{k=1}^n u_k \, d\bar{z}_k$ with coefficients $u_k \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, for $k = 1, \ldots, n$. For this purpose we set $\delta_k = \frac{\partial}{\partial \bar{z}_k} - \frac{\partial \varphi}{\partial \bar{z}_k}$ and get since

$$\overline{\partial} u = \sum_{j<k} \left( \frac{\partial u_j}{\partial \bar{z}_k} - \frac{\partial u_k}{\partial \bar{z}_j} \right) \ d\bar{z}_j \wedge d\bar{z}_k$$

that

$$\|\overline{\partial} u\|_{\varphi}^2 + \|\overline{\partial}^* u\|_{\varphi}^2 = \int_{\mathbb{C}^n} \sum_{j,k=1}^n \left( \frac{\partial u_j}{\partial \bar{z}_k} - \frac{\partial u_k}{\partial \bar{z}_j} \right)^2 e^{-\varphi} \ d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n \delta_j u_j \overline{\delta_k u_k} e^{-\varphi} \ d\lambda$$

$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( \frac{\partial u_j}{\partial \bar{z}_k} \right)^2 e^{-\varphi} \ d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( \delta_j u_j \overline{\delta_k u_k} - \frac{\partial u_j}{\partial \bar{z}_k} \overline{\frac{\partial u_k}{\partial \bar{z}_j}} \right) e^{-\varphi} \ d\lambda$$

$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( \frac{\partial u_j}{\partial \bar{z}_k} \right)^2 e^{-\varphi} \ d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( \delta_j, \overline{\delta_k} \right) u_j \overline{u_k} e^{-\varphi} \ d\lambda,$$
where we used the fact that for $f, g \in C_0^\infty(\mathbb{C}^n)$ we have

$$\langle \frac{\partial f}{\partial z_k}, g \rangle_\varphi = -(f, \delta_k g)_\varphi.$$

Since

$$\left[ \delta_j, \frac{\partial}{\partial z_k} \right] = \frac{\partial^2 \varphi}{\partial z_j \partial z_k},$$

and $\varphi$ satisfies (*) we are done (see also [H]). □

Now it follows by Proposition 2.4 that there exists a uniquely determined $(0, 1)$-form $N_\varphi u \in \text{dom } (\overline{\partial}) \cap \text{dom } (\overline{\partial^*})$ such that

$$\langle u, v \rangle_\varphi = Q_\varphi (N_\varphi u, v) = \langle \overline{\partial} N_\varphi u, \overline{\partial} v \rangle_\varphi + \langle \overline{\partial^*} N_\varphi u, \overline{\partial^*} v \rangle_\varphi,$$

and again by 2.4 that

$$\| \overline{\partial} N_\varphi u \|_\varphi^2 + \| \overline{\partial^*} N_\varphi u \|_\varphi^2 \leq C_1 \| u \|_\varphi^2$$

as well as

$$\| N_\varphi u \|_\varphi^2 \leq C_2(\| \overline{\partial} N_\varphi u \|_\varphi^2 + \| \overline{\partial^*} N_\varphi u \|_\varphi^2) \leq C_3 \| u \|_\varphi^2,$$

where $C_1, C_2, C_3 > 0$ are constants. Hence we get that $N_\varphi$ is a continuous linear operator from $L^2_{(0, 1)}(\mathbb{C}^n, \varphi)$ into itself (see also [H] or [ChSh]).

### 3. Weighted Sobolev spaces

We want to study compactness of the weighted $\overline{\partial}$-Neumann operator $N_\varphi$. For this purpose we define weighted Sobolev spaces and prove, under suitable conditions, a Rellich - Lemma for these weighted Sobolev spaces. We will also have to consider their dual spaces, which already appeared in [BDH] and [KM].

**Definition 3.1.**

For $k \in \mathbb{N}$ let

$$W^k(\mathbb{C}^n, \varphi) := \{ f \in L^2(\mathbb{C}^n, \varphi) : D^\alpha f \in L^2(\mathbb{C}^n, \varphi) \text{ for } |\alpha| \leq k \},$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} y_n}$ for $(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$ with norm

$$\| f \|_{k, \varphi}^2 = \sum_{|\alpha| \leq k} \| D^\alpha f \|_\varphi^2.$$

We will also need weighted Sobolev spaces with negative exponent. But it turns out that for our purposes it is more reasonable to consider the dual spaces of the following spaces.

**Definition 3.2.**

Let

$$X_j = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \text{ and } Y_j = \frac{\partial}{\partial y_j} - \frac{\partial \varphi}{\partial y_j},$$

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for $j = 1, \ldots, n$ and define
\[ W^1(\mathbb{C}^n, \varphi, \nabla \varphi) = \{ f \in L^2(\mathbb{C}^n, \varphi) : X_j f, Y_j f \in L^2(\mathbb{C}^n, \varphi), j = 1, \ldots, n \}, \]
with norm
\[ \| f \|_{W^1(\mathbb{C}^n, \varphi, \nabla \varphi)}^2 = \| f \|_{\varphi}^2 + \sum_{j=1}^{n} (\| X_j f \|_{\varphi}^2 + \| Y_j f \|_{\varphi}^2). \]

In the next step we will analyze the dual space of $W^1(\mathbb{C}^n, \varphi, \nabla \varphi)$.
By the mapping $f \mapsto (f, X_j f, Y_j f)$, the space $W^1(\mathbb{C}^n, \varphi, \nabla \varphi)$ can be identified with a closed product of $L^2(\mathbb{C}^n, \varphi)$, hence each continuous linear functional $L$ on $W^1(\mathbb{C}^n, \varphi, \nabla \varphi)$ is represented (in a non-unique way) by
\[ L(f) = \int_{\mathbb{C}^n} f(z) g_0(z) e^{-\varphi(z)} \, d\lambda(z) + \sum_{j=1}^{n} \int_{\mathbb{C}^n} (X_j f(z) g_j(z) + Y_j f(z) h_j(z)) e^{-\varphi(z)} \, d\lambda(z), \]
for some $g_j, h_j \in L^2(\mathbb{C}^n, \varphi)$.
For $f \in C_0^\infty(\mathbb{C}^n)$, it follows that
\[ L(f) = \int_{\mathbb{C}^n} f(z) g_0(z) e^{-\varphi(z)} \, d\lambda(z) - \sum_{j=1}^{n} \int_{\mathbb{C}^n} f(z) \left( \frac{\partial g_j(z)}{\partial x_j} + \frac{\partial h_j(z)}{\partial y_j} \right) e^{-\varphi(z)} \, d\lambda(z). \]

Since $C_0^\infty(\mathbb{C}^n)$ is dense in $W^1(\mathbb{C}^n, \varphi, \nabla \varphi)$ we have shown

**Lemma 3.3.**
Each element $u \in W^{-1}(\mathbb{C}^n, \varphi, \nabla \varphi) := (W^1(\mathbb{C}^n, \varphi, \nabla \varphi))'$ can be represented in a non-unique way by
\[ u = g_0 + \sum_{j=1}^{n} \left( \frac{\partial g_j}{\partial x_j} + \frac{\partial h_j}{\partial y_j} \right), \]
where $g_j, h_j \in L^2(\mathbb{C}^n, \varphi)$.
The dual norm $\| u \|_{-1, \varphi, \nabla \varphi} := \sup\{ |u(f)| : \| f \|_{\varphi, \nabla \varphi} \leq 1 \}$ can be expressed in the form
\[ \| u \|_{-1, \varphi, \nabla \varphi}^2 = \inf\{ \| g_0 \|^2 + \sum_{j=1}^{n} (\| g_j \|^2 + \| h_j \|^2) \}, \]
where the infimum is taken over all families $(g_j, h_j)$ in $L^2(\mathbb{C}^n, \varphi)$ representing the functional $u$.

(see for instance [1])

In particular each function in $L^2(\mathbb{C}^n, \varphi)$ can be indentified with an element of $W^{-1}(\mathbb{C}^n, \varphi, \nabla \varphi)$.

**Proposition 3.4.**
Suppose that the weight function satisfies
\[ \lim_{|z| \to \infty} (\theta |\nabla \varphi(z)|^2 + \Delta \varphi(z)) = +\infty, \]
for some $\theta \in (0, 1)$, where
\[ |\nabla \varphi(z)|^2 = \sum_{k=1}^{n} \left( \frac{\partial \varphi}{\partial x_k} \right)^2 + \left( \frac{\partial \varphi}{\partial y_k} \right)^2. \]
Then the embedding of \( W^1(\mathbb{C}^n, \varphi, \nabla \varphi) \) into \( L^2(\mathbb{C}^n, \varphi) \) is compact.

**Proof.** We adapt methods from \([BDH]\) or \([Jo]\), Proposition 6.2., or \([KM]\). For the vector fields \( X_j \) from 3.2 and their formal adjoints \( X_j^* = -\frac{\partial}{\partial x_j} \), we have

\[
(X_j + X_j^*) f = -\frac{\partial \varphi}{\partial x_j} f \quad \text{and} \quad [X_j, X_j^*] f = -\frac{\partial^2 \varphi}{\partial x_j^2} f,
\]

for \( f \in C_0^\infty(\mathbb{C}^n) \), and

\[
\langle [X_j, X_j^*] f, f \rangle_\varphi = \|X_j^* f\|_\varphi^2 - \|X_j f\|_\varphi^2,
\]

\[
\|(X_j + X_j^*) f\|_\varphi^2 \leq (1 + 1/\epsilon)\|X_j f\|_\varphi^2 + (1 + \epsilon)\|X_j^* f\|_\varphi^2
\]

for each \( \epsilon > 0 \). Similar relations hold for the vector fields \( Y_j \). Now we set

\[
\Psi(z) = |\nabla \varphi(z)|^2 + (1 + \epsilon)\Delta \varphi(z).
\]

It follows that

\[
\langle \Psi f, f \rangle_\varphi \leq (2 + \epsilon + 1/\epsilon) \sum_{j=1}^n (\|X_j f\|_\varphi^2 + \|Y_j f\|_\varphi^2).
\]

Since \( C_0^\infty(\mathbb{C}^n) \) is dense in \( W^1(\mathbb{C}^n, \varphi, \nabla \varphi) \) by definition, this inequality holds for all \( f \in W^1(\mathbb{C}^n, \varphi, \nabla \varphi) \). If \( (f_k)_k \) is a sequence in \( W^1(\mathbb{C}^n, \varphi, \nabla \varphi) \) converging weakly to 0, then \( (f_k)_k \) is a bounded sequence in \( W^1(\mathbb{C}^n, \varphi, \nabla \varphi) \) and our the assumption implies that

\[
\Psi(z) = |\nabla \varphi(z)|^2 + (1 + \epsilon)\Delta \varphi(z)
\]

is positive in a neighborhood of \( \infty \). So we obtain

\[
\int_{\mathbb{C}^n} |f_k(z)|^2 e^{-\varphi(z)} d\lambda(z) \leq \int_{|z| < R} |f_k(z)|^2 e^{-\varphi(z)} d\lambda(z) + \int_{|z| \geq R} \frac{\Psi(z)|f_k(z)|^2}{\inf\{\Psi(z) : |z| \geq R\}} e^{-\varphi(z)} d\lambda(z)
\]

\[
\leq C_{\varphi, R} \|f_k\|_{L^2(\mathbb{B}_R)}^2 + \frac{C_{\epsilon} \inf\{\Psi(z) : |z| \geq R\}}{\inf\{\Psi(z) : |z| \geq R\}}.
\]

Hence the assumption and the fact that the injection \( W^1(\mathbb{B}_R) \hookrightarrow L^2(\mathbb{B}_R) \) is compact (see for instance \([T]\)) show that a subsequence of \( (f_k)_k \) tends to 0 in \( L^2(\mathbb{C}^n, \varphi) \).

\[ \square \]

**Remark 3.5.** It follows that the adjoint to the above embedding, the embedding of \( L^2(\mathbb{C}^n, \varphi) \) into \( (W^1(\mathbb{C}^n, \varphi, \nabla \varphi))' = W^{-1}(\mathbb{C}^n, \varphi, \nabla \varphi) \) (in the sense of \([3,3]\)) is also compact.

**Remark 3.6.** Note that one does not need plurisubharmonicity of the weight function in Proposition 3.3. If the weight is plurisubharmonic, one can of course drop \( \theta \) in the formulation of the assumption.
4. Compactness estimates

The following Proposition reformulates the compactness condition for the case of a bounded pseudoconvex domain in $\mathbb{C}^n$, see [BS], [Str]. The difference to the compactness estimates for bounded pseudoconvex domains is that here we have to assume an additional condition on the weight function implying a corresponding Rellich - Lemma.

**Proposition 4.1.**
Suppose that the weight function $\phi$ satisfies (*) and
\[
\lim_{|z| \to \infty} (\theta|\nabla \phi(z)|^2 + \Delta \phi(z)) = +\infty,
\]
for some $\theta \in (0,1)$, then the following statements are equivalent.

(1) The $\overline{\partial}$-Neumann operator $N_{1,\phi}$ is a compact operator from $L^2_{(0,1)}(\mathbb{C}^n, \phi)$ into itself.

(2) The embedding of the space $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial^*})$, provided with the graph norm $\|u\| = (\|\overline{\partial} u\|^2 + \|\overline{\partial^*} u\|^2)^{1/2}$, into $L^2_{(0,1)}(\mathbb{C}^n, \phi)$ is compact.

(3) For every positive $\epsilon$ there exists a constant $C_\epsilon$ such that
\[
\|u\|^2 \leq \epsilon (\|\overline{\partial} u\|^2 + \|\overline{\partial^*} u\|^2) + C_\epsilon \|u\|_{-1,\phi,\nabla}\phi,
\]
for all $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial^*})$.

(4) The operators
\[
\overline{\partial} \phi, N_{1,\phi} : L^2_{(0,1)}(\mathbb{C}^n, \phi) \cap \ker(\overline{\partial}) \to L^2(\mathbb{C}^n, \phi)
\]
and
\[
\overline{\partial} \phi, N_{2,\phi} : L^2_{(0,2)}(\mathbb{C}^n, \phi) \cap \ker(\overline{\partial}) \to L^2_{(0,1)}(\mathbb{C}^n, \phi)
\]
are both compact.

**Proof.** First we show that (1) and (4) are equivalent: suppose that $N_{1,\phi}$ is compact. For $f \in L^2_{(0,1)}(\mathbb{C}^n, \phi)$ it follows that
\[
\|\overline{\partial} \phi, N_{1,\phi} f\|^2 \leq \langle f, N_{1,\phi} f \rangle \leq \epsilon \|f\|^2 + C_\epsilon \|N_{1,\phi}\|^2
\]
by Lemma 2 of [CD]. Hence $\overline{\partial} \phi, N_{1,\phi}$ is compact. Applying the formula
\[
N_{1,\phi} - (\overline{\partial} \phi, N_{1,\phi})^* (\overline{\partial} \phi, N_{1,\phi}) = (\overline{\partial} \phi, N_{2,\phi})(\overline{\partial} \phi, N_{2,\phi})^*,
\]
(see for instance [ChSh]), we get that also $\overline{\partial} \phi, N_{2,\phi}$ is compact. The converse follows easily from the same formula.

Now we show (4) $\implies$ (3) $\implies$ (2) $\implies$ (1). We follow the lines of [Str], where the case of a bounded pseudoconvex domain is handled.

Assume (4): if (3) does not hold, then there exists $\epsilon_0 > 0$ and a sequence $(u_n)_n$ in $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial^*})$ with $\|u_n\|_{\phi} = 1$ and
\[
\|u_n\|^2 \geq \epsilon_0 (\|\overline{\partial} u_n\|^2 + \|\overline{\partial^*} u_n\|^2) + n \|u_n\|^2_{-1,\phi,\nabla}\phi.
\]
for each \( n \geq 1 \), which implies that \( u_n \to 0 \) in \( W_{(0,1)}^{-1}(C^n, \varphi, \nabla \varphi) \). Since \( u_n \) can be written in the form

\[
\begin{align*}
u_n &= (\overline{\partial}_\varphi N_{1,\varphi})^* \overline{\partial}_\varphi u_n + (\overline{\partial}_\varphi N_{2,\varphi}) \overline{\partial} u_n,
\end{align*}
\]

(4) implies there exists a subsequence of \( (u_n)_n \) converging in \( L^2_{(0,1)}(C^n, \varphi) \) and the limit must be 0, which contradicts \( \|u_n\|_\varphi = 1 \).

To show that (3) implies (2) we consider a bounded sequence in \( \text{dom (\overline{\partial})} \cap \text{dom (\overline{\partial}_\varphi)} \). By (2) this sequence is also bounded in \( L^2_{(0,1)}(C^n, \varphi) \). Now (3) implies that it has a subsequence converging in \( W_{(0,1)}^{-1}(C^n, \varphi, \nabla \varphi) \). Finally use (3) to show that this subsequence is a Cauchy sequence in \( L^2_{(0,1)}(C^n, \varphi) \), therefore (2) holds.

Assume (2) : by (2) and the basic facts about \( N_{1,\varphi} \), it follows that

\[
N_{1,\varphi} : L^2_{(0,1)}(C^n, \varphi) \longrightarrow \text{dom (\overline{\partial})} \cap \text{dom (\overline{\partial}_\varphi)}
\]

is continuous in the graph topology, hence

\[
N_{1,\varphi} : L^2_{(0,1)}(C^n, \varphi) \longrightarrow \text{dom (\overline{\partial})} \cap \text{dom (\overline{\partial}_\varphi)} \hookrightarrow L^2_{(0,1)}(C^n, \varphi)
\]

is compact.

\[\square\]

**Remark 4.2.**

Suppose that the weight function \( \varphi \) is plurisubharmonic and that the lowest eigenvalue \( \lambda_{\varphi} \) of the Levi - matrix \( M_{\varphi} \) satisfies

\[
\lim_{|z| \to \infty} \lambda_{\varphi}(z) = +\infty . \quad (**)
\]

This condition implies that \( N_{1,\varphi} \) is compact [HaHe].

It also implies that the condition of the Rellich - Lemma [3.4] is satisfied.

This follows from the fact that we have for the trace \( \text{tr}(M_{\varphi}) \) of the Levi - matrix

\[
\text{tr}(M_{\varphi}) = \frac{1}{4} \Delta \varphi,
\]

and since for any invertible \((n \times n)\)-matrix \( T \)

\[
\text{tr}(M_{\varphi}) = \text{tr}(TM_{\varphi}T^{-1}),
\]

it follows that \( \text{tr}(M_{\varphi}) \) equals the sum of all eigenvalues of \( M_{\varphi} \). Hence our assumption on the lowest eigenvalue \( \lambda_{\varphi} \) of the Levi - matrix implies that the assumption of Proposition [3.4] is satisfied.

In order to use Proposition [4.1] to show compactness of \( N_{\varphi} \) we still need

**Proposition 4.3. (Gårding’s inequality)** Let \( \Omega \) be a smooth bounded domain. Then for any \( u \in W^1(\Omega, \varphi, \nabla \varphi) \) with compact support in \( \Omega \)

\[
\|u\|_{1,\varphi, \nabla \varphi} \leq C(\Omega, \varphi) \left( \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi u\|_\varphi^2 + \|u\|_\varphi^2 \right).
\]
Proof. The operator $-\partial_\varphi$ is strictly elliptic since its principal part equals the Laplacian. Now $-\partial_\varphi = -(\overline{\partial} + \overline{\partial}_\varphi)^* \circ (\overline{\partial} + \overline{\partial}_\varphi)$, so from general PDE theory follows that the system $\overline{\partial} + \overline{\partial}_\varphi$ is elliptic. This is, because a differential operator $P$ of order $s$ is elliptic if and only if $-1\rangle^s P^* \circ P$ is strictly elliptic. So because of ellipticity, one has on each smooth bounded domain $\Omega$ the classical Gårding inequality

$$\|u\|^2 \leq C(\Omega) \left( \|\partial u\|^2 + \|\partial_\varphi u\|^2 + \|u\|^2 \right)$$

for any $(0,1)$-form $u$ with coefficients in $C_0^\infty$. But our weight $\varphi$ is smooth on $\overline{\Omega}$, hence the weighted and unweighted $L^2$-bounded domain $\Omega$ the classical Gårding inequality

$$\|u\|^2_{1,\varphi,\nabla \varphi} \leq C_1(\|u\|^2_{1,\varphi} + \|u\|^2_{\varphi}) \leq C_2(\|u\|^2_{1,\varphi} + \|u\|^2) \leq C_3(\|\partial u\|^2 + \|\partial_\varphi u\|^2 + \|u\|^2) \leq C_4(\|\partial u\|^2 + \|\partial_\varphi u\|^2 + \|u\|^2)$$

$$\|u\|^2 \leq C(\Omega) \left( \|\partial u\|^2 + \|\partial_\varphi u\|^2 + \|u\|^2 \right)$$

We are now able to give a different proof of the main result in [HaHe].

Theorem 4.4. Let $\varphi$ be plurisubharmonic. If the lowest eigenvalue $\lambda_\varphi(z)$ of the Levi matrix $M_\varphi$ satisfies (**), then $N_\varphi$ is compact.

Proof. By Proposition 3.4 and Remark 4.2, it suffices to show a compactness estimate and use Proposition 4.1. Given $\epsilon > 0$ we choose $M \in \mathbb{N}$ with $1/M \leq \epsilon/2$ and $R$ such that $\lambda(z) > M$ whenever $|z| > R$. Let $\chi$ be a smooth cutoff function identically one on $B_R$. Hence we can estimate

$$M\|f\|^2 \leq \sum_{j,k} \int_{\mathbb{C}^n \setminus B_R} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \tilde{f}_j \tilde{f}_k \chi e^{-\varphi} d\lambda + M\|\chi f\|^2$$

$$\leq Q_\varphi(f, f) + M(\chi f, f)_\varphi$$

$$\leq Q_\varphi(f, f) + M\|\chi f\|^2_{1,\varphi,\nabla \varphi}$$

$$\leq Q_\varphi(f, f) + Ma\|\chi f\|^2_{1,\varphi,\nabla \varphi} + a^{-1}M\|f\|^2_{1,\varphi,\nabla \varphi} + a^{-1}M\|f\|^2_{1,\varphi,\nabla \varphi}$$

where $a$ is to be chosen a bit later. Now we apply Gårding’s inequality 4.3 to the second term, so there is a constant $C_R$ depending on $R$ such that

$$M\|f\|^2_{1,\varphi} \leq Q_\varphi(f, f) + MaC_R(Q_\varphi(f, f) + \|f\|^2_{1,\varphi}) + a^{-1}M\|f\|^2_{1,\varphi,\nabla \varphi}$$

By Proposition 2.4 and after increasing $C_R$ we have

$$M\|f\|^2_{1,\varphi} \leq Q_\varphi(f, f) + MaC_R Q_\varphi(f, f) + a^{-1}M\|f\|^2_{1,\varphi,\nabla \varphi}$$

Now choose $a$ such that $aC_R \leq \epsilon/2$, then

$$\|f\|^2_{1,\varphi} \leq aQ_\varphi(f, f) + a^{-1}\|f\|^2_{1,\varphi,\nabla \varphi}$$

and this estimate is equivalent to compactness by 4.1.

Remark 4.5. Assumption (***) on the lowest eigenvalue of $M_\varphi$ is the analog of property (P) introduced by Catlin in [Ca] in case of bounded pseudoconvex domains. Therefore the proof is similar.
Remark 4.6. We mention that for the weight $\varphi(z) = |z|^2$ the $\bar{\partial}$-Neumann operator fails to be compact (see [HaHe]), but the condition
\[
\lim_{|z| \to \infty} (\theta |\nabla \varphi(z)|^2 + \Delta \varphi(z)) = +\infty
\]
of the Rellich - Lemma is satisfied.

Remark 4.7. Denote by $W_m^{\text{loc}}(\mathbb{C}^n)$ the space of functions which locally belong to the classical unweighted Sobolev space $W^m(\mathbb{C}^n)$. Suppose that $\Box \varphi v = g$ and $g \in W_m^{\text{loc}}(0,1)$. Then $v \in W_m^{m+2}(0,1)$ (in particular, if there exists a weighted $\bar{\partial}$-Neumann operator $N_\varphi$, it maps $C^\infty(0,1) \cap L^2(0,1)$ into itself).

\[\Box \varphi\] is strictly elliptic, and the statement in fact follows from interior regularity of a general second order elliptic operator. The reader can find more on elliptic regularity for instance in [Ev], chapter 6.3.

An analog statement is true for $S_\varphi$. If there exists a continuous canonical solution operator $S_\varphi$, it maps $C^\infty(0,1) \cap L^2(0,1)$ into itself. This follows from ellipticity of $\bar{\partial}$.

Although $\Box \varphi$ is strictly elliptic, the question whether $S_\varphi$ is globally or exactly regular is harder to answer. This is, because our domain is not bounded and neither are the coefficients of $\Box \varphi$. Only in a very special case the question is easy - this is, when $A_\varphi^2$ (the weighted space of entire functions) is zero. In this case, there is only one solution operator to $\bar{\partial}$, namely the canonical one, and if $f \in W_k^0(0,1)$ and $u = S_\varphi f$, it follows that $\bar{\partial}D^\alpha u = D^\alpha f$, since $\bar{\partial}$ commutes with $\frac{\partial}{\partial x_j}$. Now $S_\varphi$ is continuous, so $\|D^\alpha u\|_\varphi \leq C\|D^\alpha f\|_\varphi$, meaning that $u \in W_k^k$. So in this case $S_\varphi$ is a bounded operator from $W_k^k(0,1) \to W_k^k$.

Remark 4.8. Let $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$ denote the space of $(0,1)$-forms with holomorphic coefficients belonging to $L^2(\mathbb{C}^n, \varphi)$.

We point out that assuming (***) implies directly - without use of Sobolev spaces - that the embedding of the space
\[A^2_{(0,1)}(\mathbb{C}^n, \varphi) \cap \text{dom } (\bar{\partial}_\varphi^+)\]
provided with the graph norm $u \mapsto (\|u\|^2_\varphi + \|\bar{\partial}_\varphi u\|^2_\varphi)^{1/2}$ into $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$ is compact. Compare [4.7] (2).

For this purpose let $u \in A^2_{(0,1)}(\mathbb{C}^n, \varphi) \cap \text{dom } (\bar{\partial}_\varphi^+)$. Then we obtain from the proof of [2.4] that
\[
\|\bar{\partial}_\varphi u\|_\varphi^2 = \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} u_j \bar{u}_k e^{-\varphi} d\lambda.
\]
Let us for \( u = \sum_{j=1}^{n} u_j \partial_{\bar{z}_j} \) identify \( u(z) \) with the vector \((u_1(z), \ldots, u_n(z)) \in \mathbb{C}^n\). Then, if we denote by \( \langle \cdot, \cdot \rangle \) the standard inner product in \( \mathbb{C}^n \), we have
\[
\langle u(z), u(z) \rangle = \sum_{j=1}^{n} |u_j(z)|^2 \quad \text{and} \quad \langle M_\varphi u(z), u(z) \rangle = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi(z)}{\partial z_j \partial \bar{z}_k} u_j(z) \bar{u}_k(z).
\]

Note that the lowest eigenvalue \( \lambda_\varphi \) of the Levi - matrix \( M_\varphi \) can be expressed as
\[
\lambda_\varphi(z) = \inf_{u(z) \neq 0} \frac{\langle M_\varphi u(z), u(z) \rangle}{\langle u(z), u(z) \rangle}.
\]

So we get
\[
\int_{\mathbb{C}^n} \langle u, u \rangle e^{-\varphi} \, d\lambda \leq \int_{\mathbb{B}_R} \langle u, u \rangle e^{-\varphi} \, d\lambda + \left[ \inf_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z) \right]^{-1} \int_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z) \langle u, u \rangle e^{-\varphi} \, d\lambda
\]
\[
\leq \int_{\mathbb{B}_R} \langle u, u \rangle e^{-\varphi} \, d\lambda + \left[ \inf_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z) \right]^{-1} \int_{\mathbb{C}^n} \langle M_\varphi u, u \rangle e^{-\varphi} \, d\lambda.
\]

For a given \( \epsilon > 0 \) choose \( R \) so large that
\[
\left[ \inf_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z) \right]^{-1} < \epsilon,
\]
and use the fact that for Bergman spaces of holomorphic functions the embedding of \( A^2(\mathbb{B}_{R_1}) \) into \( A^2(\mathbb{B}_{R_2}) \) is compact for \( R_2 < R_1 \). So the desired conclusion follows.

**Remark 4.9.** Part of the results, in particular Theorem 4.4, are taken from [Ga]. We finally mention that the methods used in this paper can also be applied to treat unbounded pseudoconvex domains with boundary, see [Ga].

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