PAPER

Shared purity and concurrence of a mixture of ground and low-lying excited states as indicators of quantum phase transitions

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Abstract

We investigate the efficacy of shared purity, a measure of quantum correlation that is independent of the separability-entanglement paradigm, as a quantum phase transition indicator in comparison with concurrence, a bipartite entanglement measure. The order parameters are investigated for thermal states and pseudo-thermal states, of the systems considered. In the case of the one-dimensional J_{1} − J_{2} Heisenberg quantum spin model and the one-dimensional transverse-field quantum Ising model, shared purity turns out to be as effective as concurrence in indicating quantum phase transitions. In the two-dimensional J_{1} − J_{2} Heisenberg quantum spin model, shared purity indicates the two quantum phase transitions present in the model, while concurrence detects only one of them. Moreover, we find diverging finite-size scaling exponents for the order parameters near the transitions in odd- and even-sized systems governed by the one-dimensional J_{1} − J_{2} model, as had previously been reported for quantum spins on odd- and even-legged ladders. It is plausible that the divergence is related to a Möbius strip-like boundary condition required for odd-sized systems, while for even-sized systems, the usual periodic boundary condition is sufficient.

1. Introduction

Interacting quantum spin Hamiltonians are useful models for studying phenomena in a variety of physical systems. Somewhat recently, it was discerned that these Hamiltonians can also be realized in artificial materials, in particular by using ultracold atoms in optical lattices and ion-traps [1, 2]. Realization of these models in artificial materials have, among other things, helped experimenters to reach unprecedented levels of control.

Quantum information and computation is an emerging field at the cross-roads of several fields including physics and information technology. It has recently been realized that quantum information concepts can be utilized to obtain a fresh perspective on many-body phenomena [1–3]. The study of quantum phase transitions of spin models using quantum entanglement [4–7] started about two decades ago, when bipartite entanglement was used as an indicator of quantum phase transition in the transverse-field quantum Ising model [8, 9]. Since then, a significant body of work has appeared in the area, connecting quantum critical phenomena of interacting quantum Hamiltonians with typical concepts that are useful in quantum information, like bipartite and multipartite entanglement, fidelity, etc. Along with providing a different perspective and potentially useful understanding of cooperative physical phenomena, such studies are also useful in understanding the potential of a physical substrate for realizing information processing tasks.

In this paper, we primarily focus on frustrated Heisenberg spin models. Quantum phase transitions in the J_{1} − J_{2} Heisenberg quantum spin models in one- and two-dimensional lattices have been investigated using entanglement properties of individual energy eigenstates of the systems [10–13]. The ground state bipartite as well as genuine multipartite entanglement are unable to conclusively indicate the presence of quantum phase transitions in these spin models [11] but bipartite entanglement of the first excited state indicates their presence.
We approach the same problem from a different perspective. A different observable, namely, shared purity [14], a quantum correlation measure independent of the entanglement-separability paradigm, is measured for thermal states and pseudo-thermal states of these systems. We also calculate the concurrence [15, 16], a measure of bipartite entanglement, of the same states for comparison.

The reason behind the consideration of a mixture of eigenstates instead of the ground state are that concurrence and shared purity of the ground state can not conclusively detect quantum phase transitions of the one- and two-dimensional $J_1 - J_2$ models [11], and it is experimentally difficult to create the ground state of a system. We investigate whether signatures of the quantum phase transitions are still present at low temperatures, wherein the mixing of low-lying eigenstates is inevitable. Therefore, the method employed searches for signatures of quantum phase transitions in the ground state by looking at properties of low-temperature states of the same systems.

The one-dimensional $J_1 - J_2$ Heisenberg quantum spin system lies in the spin fluid phase in the parameter range $\alpha = J_2/J_1 \lesssim 0.25$ [17–19]. The spin fluid phase is usually observed in frustrated magnets [20, 21]. The spin model lies in the dimer phase for $\alpha = J_2/J_1 \gtrsim 0.25$ [17–19]. The increase of the antiferromagnetic next-nearest coupling coefficient $J_2$ enables the formation of dimers [22]. We show that shared purity and concurrence (a measure of bipartite entanglement) of the pseudo-thermal state detect this spin fluid to dimer quantum phase transition. Moreover, we find that the odd- and even-sized systems have different finite-size scaling exponents.

The two-dimensional $J_1 - J_2$ Heisenberg model lies in the ordinary Néel order phase for $\alpha = J_2/J_1 \lesssim 0.40$ and lies in the collinear Néel order phase for $\alpha = J_2/J_1 \gtrsim 0.60$. The Néel order phase is an antiferromagnetic phase below a sufficiently low temperature (known as the Néel temperature) [23, 24]. Above the Néel temperature, materials are usually found in the paramagnetic phase. In between $\alpha \approx 0.40$ and $\alpha \approx 0.60$, there is an intermediate phase, and although there remains some uncertainty, it has been predicted to be a plaquette or columnar dimer phase in the literature [25–30].

Shared purity ($Sp$) is a relatively new quantum correlation measure based on fidelity, which is a measure of the closeness of two quantum states. It is defined as the difference between global and local fidelities [14, 31], and is given, for a bipartite quantum state, by

$$Sp = F_G - F_L. \tag{1}$$

The global fidelity ($F_G$) calculates the closeness of the quantum state in the argument with all pure states, while the local fidelity ($F_L$) calculates the closeness of the same quantum state with pure product states. The computation of the global fidelity is easy, as it is the largest eigenvalue of the density matrix corresponding to the quantum state. However, the optimization over the set of pure product states, required to compute the local fidelity, is often difficult to handle analytically and may have to be done numerically [14]. In this paper, to calculate the local fidelity, we minimize the distance of the state under investigation from a set of $10^6$ Haar uniformly generated random pure product states [32, 33]. The convergence of the optimization is checked by using an independent Haar uniform preparation of $10^6$ states. For a pure quantum state, shared purity is the same as the geometric measure of entanglement [34], but it is a different observable for a mixed quantum state [14]. Note that shared purity is an indicator derived from fidelity, and the latter is a single-shot metric that can be estimated—in principle—through a single experiment (although several, in principle, infinite, runs would be required). However, shared purity is a difference between two fidelities, and for its determination, one must gauge the maximum fidelity or overlap separately with pure states and with pure product states, necessitating multiple experimental set-ups. It is known that both entanglement and fidelity are connected to quantum phase transitions [8–13]. Therefore, the shared purity of a mixed state may also be linked to the detection of quantum phase transitions.

Concurrence is a measure of entanglement and is usually defined for a two-qubit quantum state. For a two-qubit density matrix $\rho$, it is defined as

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{2}$$

where the $\lambda_i$’s are square roots of the eigenvalues of $\rho \tilde{\rho}$ in descending order. Here $\tilde{\rho}$ is the spin-flipped $\rho$: $\tilde{\rho} = (\sigma_x \otimes \sigma_x) \rho^\dagger (\sigma_x \otimes \sigma_x), \rho^\dagger$ is the complex conjugate of $\rho$ in the computational basis [15, 16].

Concurrence is a monotonically increasing function of the entanglement of formation, which quantifies the necessary amount of singlets to create the bipartite state $\rho$ by local operation and classical communication [35, 36].

In the upcoming sections 2, 3 and 4, we analyze the concurrence and shared purity of thermal states and pseudo thermal states for the one-dimensional $J_1 - J_2$ Heisenberg quantum spin model, the one-dimensional transverse-field quantum Ising model, and the two-dimensional $J_1 - J_2$ Heisenberg quantum spin model respectively. We present a conclusion in section 5.
2. One-dimensional antiferromagnetic $J_1 - J_2$ Heisenberg model

The Hamiltonian for the one-dimensional antiferromagnetic $J_1 - J_2$ Heisenberg spin model may be written as

$$H_{J_1,J_2} = J_1 \sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + J_2 \sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+2},$$  \hspace{1cm} (3)$$

where $\vec{\sigma} = \sigma^x \hat{x} + \sigma^y \hat{y} + \sigma^z \hat{z}$ with $\sigma^x, \sigma^y$ and $\sigma^z$ being the Pauli spin matrices and $N$ is the number of sites in the spin chain. $J_1$ and $J_2$ are both positive and represent the nearest neighbor and the next-nearest neighbor coupling coefficients respectively. The spin system may be imagined as a zig-zag ‘chain’ as in figure 1, where the dotted lines represent the nearest neighbor interactions and the solid lines represent the next-nearest neighbor interactions. In this sense, the one-dimensional antiferromagnetic $J_1 - J_2$ Heisenberg model is a quasi-two-dimensional spin model. In trying to mimic the infinite-size system in a finite-size one, we use periodic boundary conditions, i.e., we assume $\vec{\sigma}_{N+1} = \vec{\sigma}_1$. The spin system undergoes a quantum phase transition from the gapless phase to the gapped phase at $\alpha = J_2/J_1 \approx 0.24$. [17–19].

In an experiment, it is typically difficult to prepare the ground state of a system of interacting sub-systems and it is already known that the concurrence and shared purity of the ground state alone cannot detect quantum phase transitions of the one- and two-dimensional Heisenberg $J_1 - J_2$ models [11, 12]. At some low but non-zero temperature, it may however be easier to obtain a mixture of the ground state with low-lying energy levels of the same system. To investigate along this direction, we analyze three separate cases, as follows. (i) First of all, we consider the thermal state (the canonical equilibrium state) at a particular temperature. (ii) Next, we consider a ‘pseudo-thermal’ state, where we take the ground state and the first three excited states with Maxwell-Boltzmann probabilities, normalized to unity. (iii) In the final case, we consider a pseudo-thermal state with only the ground and first excited states.

We therefore begin with the investigation of detection of quantum phase transition by using concurrence and shared purity of the thermal state for the one-dimensional $J_1 - J_2$ Heisenberg quantum spin model. A thermal state at a nonzero temperature may be written as

$$\rho(\beta) = \frac{1}{Z} \sum_k e^{-\beta E_k} \rho_k,$$  \hspace{1cm} (4)$$

where $E_k$ is the $k^{th}$ eigen energy, and $\rho_k = \frac{1}{Z} e^{\beta E_k}$ is the equally mixed density matrix of the $k^{th}$ energy level having $d$-fold degeneracy with $|E_k| \leq k_B T$ being the individual degenerate orthonormal eigenstates spanning that eigenspace. The summation is over all the energy states of the system. The factor in the denominator, $Z = \sum_k e^{-\beta E_k}$, is called the partition function and $\beta = \frac{1}{k_B T}$ is proportional to the inverse temperature, where $k_B$ is the Boltzmann constant. We calculate the concurrence and shared purity of the thermal state with $J_1 \beta = 1$ for the one-dimensional antiferromagnetic $J_1 - J_2$ Heisenberg spin model. As seen from the left panel of the figure 2, the concurrence and shared purity of the thermal state are unable to detect the quantum phase transition.

In the right panel of the figure 2, we consider a pseudo-thermal state with contributions up to the third excited state, i.e., the summation over $k$ runs from 0 to 3 in equation (4) for and for the expression of the partition function $Z$, which in this case may be called a pseudo-partition function. We observe that if we consider a mixture of a few low-lying states with weights as in the thermal state, we find no indication for the quantum phase transition. The result is shown for both shared purity and concurrence for both odd $N$ and even $N$ spin chains. Apparently, the result does not depend on the inverse temperature $\beta$, as we do not find any indication of quantum phase transition by varying $J_1/\beta$. We denote shared purity and concurrence of system size $N$ by $S_{PN}$ and $C_{N}$, respectively, in the $y$-axis labels of the subfigures of figure 2 and figure 3. We can, however, detect the quantum phase transition if we consider a mixture of the ground and the first excited states only, with weights as...
in the thermal state, i.e., if summation over $k$ runs from 0 to 1 in equation (4) and in the expression of the pseudo-partition function $Z$. As seen in figure 3, we observe discontinuities in the value of shared purity and concurrence indicating the quantum phase transition in the one-dimensional $J_1 - J_2$ model. Therefore, the considered order parameters when applied to the pseudo-thermal state with contributions from the ground and the first excited states only, detect the quantum phase transition. The positions and quanta of discontinuities are different for odd $N$ and even $N$ spin chains. The discontinuities corresponding to odd $N$ chains are further away towards the right from the phase transition point, compared with those of even $N$ chains. The positions of these discontinuities are tabulated in table 1. In the left panel of figure 2, we have studied the shared purity and concurrence of two-body reduced density matrices of the thermal state of the one-dimensional $J_1 - J_2$ Hamiltonian. Quantum phase transition being a zero temperature phenomenon which is generally manifested by the measurement of some physical properties associated with the ground state of the many-body states.
Hamiltonian, the results evident from the studies depicted in the left panel of figure 2 are on expected lines. The figure has been used to demonstrate what is already known. Moreover, it has already been shown in some previous investigations \([10–12]\) that physical properties of the ground state of the Hamiltonian in question has failed to indicate the known quantum phase transition in the system. In some of these papers \([12, 13, 37–39]\) it has also been shown that investigation of the properties (concurrence, fidelity) of the reduced first excited state, as well as the asymptotic energy gap between the first excited state and the ground state have provided an estimate of the quantum phase transition point. We know that absolute zero temperature is an idealization which cannot be achieved practically. It may be argued that the mixtures of quantum states contemplated in cases (ii) and (iii) are artificial. We accept that we are unable to identify a precise experimental setup that can realize the mixtures in cases (ii) and (iii). We however note that these mixtures are valid quantum states of the systems being considered. And we envisage a situation where an experimentalist tries to prepare the system in its ground state. However, it is plausible that the prepared state turns out to be a mixture of the ground state and a few low-lying excited states due to environment-induced ‘leaks’ from the ground to the first few excited states. However, to stop at the first few levels or the first excited level, we must assume that the so-called leak is not thermal in nature, as that would lead to the thermal (canonical equilibrium) state. We have arbitrarily chosen these probabilities to be the ones associated with a (renormalized) thermal state. In reality, they will depend on the character of the ‘leak’, but just like for thermal states, the higher excited states will have a subdued presence for weak leaks. It appears that the importance of the weights of the excited states in the mixture (when more than one excited state is considered) is paramount for efficient detection of QPT using the proposed method.

It may be noted that the discontinuities in the plots of shared purity versus the driving parameter \(\alpha\) coincide with those in the plots of concurrence versus the driving parameter. Asymptotically, the positions of all these discontinuities in the \(\alpha\)-axis should converge towards the point \(\alpha_c \approx 0.2412\), where a gapless to gapped quantum phase transition from spin fluid phase to dimer phase exists \([10–13, 37–39]\). Concurrence and shared purity of ground state does not indicate this quantum phase transition while concurrence of the 1st excited state does indicate it \([12]\). We do the finite-size scaling analysis of these discontinuities in figure 4 and all of them allow straight line fits on a log-log scale, given by the following equation:

\[
\ln (\alpha_c^N - \alpha_c) = \delta \ln N + \text{constant.}
\]  

\[ \text{(5)} \]
is interesting to note that a mixture of the ground and thus, the order parameters applied to the reduced pseudo-thermal state detect the phase transition point. It is also observed that the order parameter and 2.013 for shared purity as the same. The derivatives of the concurrence and shared purity with respect to \( c \) and \( p \) are given by equations (7) and (8), respectively:

\[
\ln(\lambda_C^N - \lambda) = -0.6207 \times \ln N + \text{constant.}
\]

The slopes of these straight lines are the scaling exponents which tell us how quickly the finite-size critical points converge to the quantum critical point \( \lambda_c \) for the corresponding order parameter. The scaling exponents are given by \( \delta^{\text{odd}} = -2.084 \) and \( \delta^{\text{even}} = -1.993 \).

Although both the lines in figure 4 reach the same quantum critical point, the scaling exponents are different for odd- and even-site spin chains. Note that the boundary sites of the even spin chains become natural neighbors on application of the periodic boundary condition, while the boundary sites of the odd spin chains can be properly aligned by forming a Möbius strip of the quasi two-dimensional spin chain (refer to figure 1). One remembers here of different scaling exponents obtained for the generalised geometric measure [40], for odd- and even-legged Heisenberg ladders in [41] (see also [42–46]). The spin chains investigated here are much simpler systems manifesting a similar phenomenon.

### 3. One-dimensional transverse-field quantum Ising model

The Hamiltonian of the one-dimensional transverse-field quantum Ising model can be written as

\[
H_{\text{IS}} = \frac{\lambda}{2} \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^{N} \sigma_i^z,
\]

where \( \sigma^x \) and \( \sigma^z \) are Pauli spin matrices and \( \lambda \) is the ratio of the xx coupling constant to the transverse magnetic field strength. We now discuss the concurrence and shared purity of a mixture of the ground and low-lying excited states of the transverse-field quantum Ising model, whose ground state concurrence successfully detects the quantum phase transition present in the system [8]. For \( \lambda < 1 \) and \( \lambda > 1 \) the spin system lies in gapped phases. But at \( \lambda = 1 \) the spin model has gapless excitations, and the system undergoes a quantum phase transition \([8, 47]\).

In this case, to look for the quantum phase transition in the model, we consider and present the profiles of the order parameters (concurrence and shared purity) in the case for which the quantum phase transition in the one-dimensional \( J_1 - J_2 \) model was successfully detected. That is the case (iii) of section 2. Accordingly we consider the concurrence and shared purity of a mixture of the ground and first excited states with weights as in the thermal state, and a subsequent renormalization. They are shown in figures 6 and 7 respectively. The plots change their curvature close to the quantum phase transition point and phase transition points of the finite-size systems shift towards the actual phase transition point for larger system sizes. The extremum points of the derivatives of the concurrence and shared purity with respect to \( \lambda \) is tabulated in the tables 2 and 3 respectively. Therefore, the order parameters applied to the reduced pseudo-thermal state detect the phase transition point. It is interesting to note that a mixture of the ground and (one or more) low-lying excited states with rapidly decreasing weights also detect the QPT point in this system as well. Note that although the order parameters applied to the reduced ground state detect the QPT, it may not be experimentally possible to prepare the system in its ground state and mixing with low-lying excited states may be inevitable.

Finite-size scaling analyses for both plots are shown in the corresponding insets. We fit a straight line through the tabulated data points, on a log-log scale, and the scaling exponents are 0.6207 for concurrence as the order parameter and 2.013 for shared purity as the same. The fitted lines for concurrence and shared purity are given by equations (7) and (8), respectively:

\[
\ln(\lambda_C^N - \lambda) = -0.6207 \times \ln N + \text{constant.}
\]

### Table 2. System size \( N \) versus positions of minima in the plots of derivative of the concurrence with respect to \( \lambda \), for the transverse-field Ising model.

| \( N \) | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| \( \lambda_c^N \) | 1.0611 | 1.0572 | 1.0541 | 1.0516 | 1.0496 |

### Table 3. System size \( N \) versus positions of maxima in the plots of derivative of the shared purity with respect to \( \lambda \), for the transverse-field Ising model.

| \( N \) | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| \( \lambda_{sp}^N \) | 0.9652 | 0.9712 | 0.9774 | 0.9801 | 0.9819 |

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The Hamiltonian of the one-dimensional transverse-field quantum Ising model has gapless excitations, and the system undergoes a quantum phase transition [8]. For \( \lambda < 1 \) and \( \lambda > 1 \) the spin system lies in gapped phases. But at \( \lambda = 1 \) the spin model has gapless excitations, and the system undergoes a quantum phase transition [8].
Note that the scaling behaviors of the phase transition points as functions of number of qubits, described by equations (5), (7), and (8) are similar in nature and differ only in their respective scaling exponents. However, there is a difference between the scaling behavior of the one-dimensional antiferromagnetic $J_1 - J_2$ Heisenberg model and the one-dimensional transverse-field quantum Ising model. For the former spin model, odd- and even-sized spin chains have different scaling exponents while there is no such difference in the latter. Furthermore, it has been shown while studying the transverse field XY Hamiltonian in [8] that while the classical correlation length diverges at the phase transition point, entanglement generally remains short ranged. However, the derivative of entanglement with the driving parameter diverged at the phase transition point in the asymptotic limit. In the case of the one-dimensional $J_1 - J_2$ Hamiltonian, we note a discontinuity in entanglement as a function of the driving parameter near the phase transition. The same behavior of entanglement as a function of the driving parameter has been observed in [48] for the anisotropic one-dimensional $J_1 - J_2$ Hamiltonian, confirming the universality class. However, we are unable to describe either the one-dimensional $J_1 - J_2$ Hamiltonian or the two-dimensional $J_1 - J_2$ Hamiltonian by some universal scaling functions.
We now consider the two-dimensional $J_1 - J_2$ Heisenberg model on a $4 \times 4$ square lattice, whose Hamiltonian can be written as

$$H_{2-D} = J_1 \sum_i \vec{\sigma}_i \cdot \vec{\sigma}_j + J_2 \sum_i \vec{\sigma}_i^\prime \cdot \vec{\sigma}_k$$

where the first sum runs over nearest neighbor pairs while the second runs over nearest neighbor diagonal pairs. $J_1$ and $J_2$, the coupling coefficients of nearest neighbor and diagonal neighbor spin site pairs respectively are both positive. We impose the periodic boundary condition for this two-dimensional model as well. Earlier studies with this model have predicted that there are two quantum phase transitions present—one from ordinary Néel order phase to a plaquette or columnar dimer phase at $\alpha \approx 0.4$ and another one from the plaquette or columnar dimer phase to a collinear Néel phase at $\alpha \approx 0.6$.

In this case again as in section 3, to look for the quantum phase transitions in the model, we consider the profiles of the order parameters (concurrence and shared purity) in the case for which the quantum phase transition in the one-dimensional $J_1 - J_2$ model was successfully detected, that is case (iii) of section 2. Accordingly we investigate the concurrence and shared purity of the reduced pseudo-thermal state with contributions from the ground and first excited states only. The results obtained have been plotted in figure 8.
We observe that the concurrence and shared purity of the mixture of the ground state and the first excited state with weights as in the thermal state can detect the quantum phase transitions of the two-dimensional $J_1 - J_2$ Heisenberg spin model. The quantum phase transition point from ordinary Néel order phase to a plaquette or columnar dimer phase is indicated by a sharp drop at $\alpha_{16} \approx 0.4078$. Interestingly, a mixture of the ground and low-lying excited states (one or more) with rapidly decreasing weights, detect this QPT point in this case as well. Bipartite entanglement of the ground state cannot detect this quantum phase transition conclusively [11]. The quantum phase transition point from the plaquette or columnar dimer phase to a collinear Néel phase is detected by shared purity when it changes its curvature near the quantum phase transition point. Note that concurrence does not see this phase transition. Apparently it looks like concurrence exhibits a sudden change at $\alpha \approx 0.56$. However, actually its value decreases continuously and reaches zero at that point. We fit a cubic polynomial in the region of interest through the data points corresponding to shared purity versus the driving parameter $\alpha$. The derivative of the polynomial shows a minimum at $\alpha_{16} \approx 0.6239$, which we interpret as indicating the second quantum phase transition point.

5. Conclusions

Quantum phase transitions [49] are zero-temperature cooperative phenomena, but cooling a physical system to near the absolute zero temperature can be difficult. Hence, low-lying excited states of many-body systems in general and quantum spin systems in particular may form an important component of experimental realization of a physical phenomenon, especially when it is cooperative. It is therefore reasonable to consider mixtures of ground and low-lying excited states for analyzing and characterizing cooperative physical phenomena.

We used two physical quantities as order parameters for detecting and characterizing quantum phase transitions in certain frustrated quantum spin models as well as the non-frustrated transverse quantum Ising model. The two quantities are the shared purity and the concurrence. While the latter is a measure of entanglement, the former, although a measure of quantum correlation, is independent of the separability-entanglement paradigm. The frustrated spin models studied are the $J_1 - J_2$ Heisenberg models in one and two lattice dimensions.

The concurrence and shared purity in the different models have been investigated in the reduced thermal states and certain reduced pseudo-thermal states, for the detection of quantum phase transition points in the models. We found that the order parameters, when applied to the thermal states fail to conclusively detect the phase transition points in the one-dimensional $J_1 - J_2$ spin system. The same happens for certain pseudo-thermal states also. However, when the same order parameters are considered for pseudo-thermal states with only the ground and first excited states, the detection of the critical points in the one-dimensional $J_1 - J_2$ model is successful. We subsequently utilized the state corresponding to the successful case for the detection of quantum phase transitions in the one-dimensional transverse Ising model and the two-dimensional $J_1 - J_2$ model using the same order parameters. Note that the order parameters applied to mixtures of the ground and (one or more) low-lying excited states with rapidly decreasing weights, also detect the QPT points in all the three systems considered. It appears that the weights of the excited states in the mixture is of paramount importance in the detection of the QPT points.

We found that shared purity detects the quantum phase transition point from columnar dimer to collinear Néel phase in the two-dimensional $J_1 - J_2$ Heisenberg spin model, which is untraceable using concurrence. Furthermore, a higher finite-size scaling exponent for detection of the quantum phase transition point by shared purity in comparison to concurrence, in the case of the transverse-field Ising model, makes shared purity a promising tool for probing the connection between many-body physical systems and quantum information science. Interestingly, we found different scaling exponents of quantum phase transition points for odd- and even-site spin chains governed by the one-dimensional Heisenberg $J_1 - J_2$ spin model. In the case of the one-dimensional transverse field Ising model, where no next-nearest neighbor interactions are present, the quantum phase transition points show the same scaling behavior for odd- and even-site spin chains. Diverging behavior of scaling exponents for odd- and even-sized systems have been previously reported for Heisenberg ladders.

Note that we primarily displayed the results throughout the paper considering thermal and pseudo-thermal states, keeping $J_1, \beta = 1$, corresponding to a low temperature. However, we also investigated the cases by varying the $J_1, \beta$. From figure 5 it is evident that altering the parameter $J_1, \beta$ to other values like $J_1, \beta = 2$, 0.5 does not affect the location of the abrupt decline in shared purity concerning $\alpha$.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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