Higher spin holography for SYM in $d$ dimensions

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Abstract

We derive the spectrum of gauge invariant operators for maximally supersymmetric Yang-Mills theories in $d$ dimensions. After subtracting the tower of BPS multiplets, states are shown to fall into long multiplets of a hidden $SO(10,2)$ symmetry dressed by thirty-two supercharges. Their primaries organize into a universal, i.e. $d$-independent pattern. The results are in perfect agreement with those following from (naive) KK reduction of type II strings on the warped $AdS \times S$ near-horizon geometry of Dp-branes.

1 Introduction

Holography between type II strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills (SYM) theory implies that there is a point in the string parameter space at which the symmetry enhances to the infinite dimensional higher spin algebra — dual to SYM theory at vanishing coupling constant $g_{YM} = 0$. In [1] a proposal for the string spectrum at this symmetry enhancement point was put forward. The results rely on the assumption that the massive

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string spectrum organizes in $S^5$ Kaluza-Klein (KK) towers on top of an $SO(6)$ gauged version of the five-dimensional theory that follows from dimensional reduction of type II strings on a flat torus. This is similar to what happens for supergravity states which fall into an infinite tower of BPS multiplets \[2\] whose ground floor shares the physical degrees of freedom with ten-dimensional type II supergravity. The full string spectrum is then obtained by standard KK techniques \[3\]. Masses of the string states were determined in \[4\] by extrapolating pp-wave frequencies down to finite $J$ along the line of vanishing gauge coupling $g_{YM} = 0$. The resulting spectrum perfectly matches that of gauge invariant operators in the dual free $\mathcal{N} = 4$ SYM theory, providing strong support in favor of these (naive) assumptions.

On the gauge theory side, gauge invariant SYM operators beyond the BPS bound generically organize into long representations of the supersymmetry algebra. At $g_{YM} = 0$ some of these long multiplets are shortened \[5\] and regroup into infinite multiplets of the higher spin (HS) algebra, manifesting the HS symmetry enhancement \[6\] (see \[7\] for recent state of the art reports and complete lists of references on HS gauge theories). On the string side this corresponds to an infinite tower of string states coming down to zero mass.\[1\] Still, by far not all the string states become massless, as one would have concluded from a naive tensionless limit, but only those corresponding to the conserved currents which realize the HS algebra. This minimal HS gauge theory on AdS$_5$ realizes the $hs(2,2|4)$ higher spin algebra and was first studied in \[8\]. HS currents are associated to doubletons in SYM$_4$. The remaining states organize in either Goldstone multiplets, required for the Higgsing of the HS symmetry, or in genuinely massive multiplets of the superconformal algebra. Reversely, when gauge interactions are turned on ($g_{YM} \neq 0$) the HS symmetry is broken. In the bulk this corresponds to massless higher spin fields acquiring mass via the pantagruelic Higgs mechanism (“grande bouffe”), in which an infinite tower of Higgs particles is eaten by the infinite tower of massless multiplets in order to become massive.

Independently of conformal symmetry a similar picture should be realized by any gauge theory which arises as the holographic counterpart of some bulk theory. Here we consider the simplest nonconformal scenario: strings on near horizon Dp-brane geometries. The boundary theories in these cases are maximally supersymmetric SYM$_d$ theories in $d = p+1$ dimensions \[2\] and for $p \neq 3$, they are nonconformal at $g_{YM} \neq 0$. On the string side this corresponds to the fact that the Dp-brane near horizon geometry is not a direct but a warped product AdS$_{d+1} \times S^{9-d}$. Accordingly, the ground state in the $(p+2)$-dimensional effective bulk theory does not correspond to a pure AdS geometry but rather to a domain wall solution. So far, only the supergravity content of these so called domain wall/QFT correspondences \[10\] has been explored. Supergravity in warped AdS$x \times S$ spaces can be studied with similar techniques like those used for pure AdS$x \times S$ geometries. The spectrum of chiral primaries \[11\] and certain two-point functions \[12\] have been determined and the

\[1\]Mass here is always understood in the sense of the AdS$_5$ background.
results have been shown to agree with what is expected from holography.

The aim of this letter is to extend the match between QFT/domain wall spectra beyond
the supergravity level. In particular, we apply the KK algorithm developed in [1, 4] to
strings moving on warped AdS×S near horizon Dp-brane geometries. Masses of the string
states again follow from extrapolation of the pp-wave limits. String theories on Dp-brane
plane waves are solvable for any p and have been studied in detail in [13]. We show
that the results are in perfect agreement with the spectrum of charges and multiplicities
of gauge invariant operators in d-dimensional SYM theory. On the gauge theory side,
operators are counted via Polya theory following [1] (see also [14]). Recent applications
of Polya theory to the study of N = 4 SYM can be found in [15].

The letter is organized as follows. In section 2, we briefly review the proposal of [1, 4]
for the string spectrum on AdS5 × S5 at the higher spin enhancement point and show how
the proposal naturally extends to the string spectra on (warped) AdSd+1×S9−d geometries.
In section 3, we compute the spectrum of gauge invariant single-trace SYM operators in
d dimensions. We show that after subtracting the tower of BPS multiplets, the SYM
spectra take a universal form, manifestly covariant under an SO(10, 2) symmetry dressed
with 32 supercharges. Signs of this underlying structure in string theory have appeared in
various contexts [16] but it still remains to be fully elucidated. Comparing to the string
spectra on the warped geometries we find perfect agreement. Two appendices contain the
technical details about the partition functions and the BPS multiplets.

2 Strings on Dp-brane geometries

In this section we consider the spectrum of KK descendants of massive string excitations
on warped AdSd+1 × S9−d spaces. It has been argued in [1] that the string spectrum on
AdS5 × S5 at the higher spin symmetry enhancement point may be put into the simple form

\[ Z_{\text{string}} = Z_{\text{sugra}} + T_{\text{susy}} \times T_{\text{KK}} \times Z_{\text{flat}}, \]  

(2.1)

where \( Z_{\text{sugra}} \) is the BPS tower comprising the massless string spectrum,

\[ Z_{\text{flat}} = \sum_{\ell=1}^{\infty} (\text{vac}_{\ell} \times \text{vac}_{\ell}) , \]

denotes the massive string spectrum in flat space after dividing out the 2^{16}-dimensional
long multiplet of the ten-dimensional type II superalgebra, and

\[ T_{\text{susy}} = \frac{(1 - t^{1/2})^{16}}{(1 - t)^4}, \quad T_{\text{KK}} = \frac{1 - t^2}{(1 - t)^6}, \]  

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denote the fundamental long multiplet of $PSU(2,2\mid4)$ and the KK tower of completely symmetric $SO(6)$ vector representations, respectively. The variable $t$ here labels the quantum number $\Delta$ in $PSU(2,2\mid4)$ corresponding to the mass of the string states. The massive string spectrum thus manifestly organizes into long multiplets while the massless part comes in the infinite tower of BPS multiplets of $PSU(2,2\mid4)$. The factors $\mathcal{T}_{\text{susy}}$ and $\mathcal{T}_{\text{KK}}$ organize the supersymmetry and KK descendants, respectively, and combine into the $SO(10,2)$ covariant expression $\mathcal{T}_{SO(10,2)} \equiv \mathcal{T}_{\text{susy}} \times \mathcal{T}_{\text{KK}}$. In other words, spacetime derivatives and KK descendants combine into $\mathcal{T}_{SO(10,2)}$ to reconstruct the ten-dimensional momentum. The factor $Z_{\text{flat}}$ in (2.1) is obtained from the flat space string spectrum after an appropriate lift to $SO(10) \times SO(2)$. The extra quantum number $\Delta$ is obtained by the BMN inspired mass formula $\Delta - J = \nu$ with $J$ the light cone charge and $\nu$ the string occupation number (see [4] for details and explicit expressions of $Z_{\text{flat}}$ for the first string levels). Later on, $\Delta$ will be related to the naive dimensions of gauge invariant operators in SYM$_d$. Together, this shows that the massive part of the string spectrum (2.1) takes the manifestly $SO(10) \times SO(2)$ covariant form $\mathcal{T}_{SO(10,2)} \times Z_{\text{flat}}$.

The same line of argument now applies to the computation of KK descendants of massive string states on $S^{9-d}$. The natural proposal for the string spectrum on the warped $AdS_{d+1} \times S^{9-d}$ background thus is

$$Z_{\text{string}}^{(d)} = Z_{\text{sugra}}^{(d)} + \mathcal{T}_{\text{susy}}^{(d)} \times \mathcal{T}_{\text{KK}}^{(d)} \times Z_{\text{flat}} = Z_{\text{sugra}}^{(d)} + \mathcal{T}_{SO(10,2)} \times Z_{\text{flat}} .$$

Interestingly, of the full spectrum only the supergravity part is sensitive to the dimension of the sphere $S^{9-d}$ while for the massive part of the spectrum different values of $d$ simply correspond to different decompositions of the universal multiplet $\mathcal{T}_{SO(10,2)}$ into superderivatives and KK descendants. Specifically, these towers are given by

$$\mathcal{T}_{\text{susy}}^{(d)} = \frac{(1 - t^{1/2})^{16_c}}{(1 - t)^d} , \quad \mathcal{T}_{\text{KK}}^{(d)} = \frac{(1 - t^2)}{(1 - t)^{10-d}} .$$

Here and in the following we denote by $d$ and $10-d$ the vector representations of $SO(d)$ and $SO(10-d)$, respectively. The completely symmetric/antisymmetric tensor products of these representations are generated by the expansions

$$(1 - t)^d \equiv 1 - d \ t + (d \times d)_A \ t^2 + \ldots ,$$
$$\frac{1}{(1 - t)^d} \equiv 1 + d \ t + (d \times d)_S \ t^2 + \ldots ,$$

and likewise for the $10-d$. The $16_c$ representation in (2.3) is understood as branching of the corresponding $SO(10)$ spinor representation under $SO(10-d) \times SO(d)$.

In summary, the full spectrum of massive string excitations on $AdS_{d+1} \times S^{9-d}$ at the HS enhancement point can be written in the $SO(10) \times SO(2)$ covariant form (2.2)
and is independent of the dimension \(d\). The dependence on \(d\) exclusively arises from the supergravity or BPS part \(Z_{\text{sugra}}^{(d)}\) of the spectrum. From the holographic perspective this is a priori surprising. It implies that the total SYM\(_d\) spectrum after subtracting its BPS part assumes a universal, \(d\)-independent form organized in \(SO(10) \times SO(2)\) representations. In the next section we show that this is indeed the case.

### 3 SYM\(_d\) cyclic words

We now consider maximally supersymmetric \(U(N)\) gauge theories in \(d \leq 10\) dimensions defined by dimensional reduction of \(\mathcal{N} = 1\) SYM in \(d = 10\). The elementary fields are the gauge field \(A_\mu\), \((10 - d)\) scalar fields \(\phi_i\) and fermions \(\psi_{\alpha,a}\), \(\bar{\psi}_{\dot{\alpha},\dot{a}}\) all in the adjoint representation of the \(U(N)\) gauge group. The gauge field and the scalars transform in the vector representations of the Lorentz \(SO(d-\,1,\,1)\) and \(R\)-symmetry \(SO(10 - d)\) group, respectively. The fermions are spinors of plus-plus or minus-minus chirality with respect to the two symmetry groups, such that they combine into a single irreducible representation of the ten-dimensional \(SO(9,\,1)\). To facilitate later comparison with the string spectrum we will consider the SYM theory on \(\mathbb{R} \times S^{d-1}\) and organize the spectrum according to the \(SO(d)\) isometry group of the sphere.

Together with their derivatives and modulo their field equations the elementary fields can be encoded in the one-letter partition function \(Z_1^{(d)}(t)\) to which we also refer as the singleton representation: the set of “letters”. The variable \(t\) is counting the dimension of the SYM letters. In general, gauge invariant operators are given in terms of multi-trace combinations, i.e. “sentences” built from “words” (single-trace) made from an “alphabet” of these letters. Here we focus on the spectrum of single-trace operators, and therefore we count cyclic “words”.

The various contributions to \(Z_1^{(d)}(t)\) can be written as

\[
\begin{align*}
\mathcal{D}^s \phi_i : \quad & t \sum_{s = 0}^{\infty} \left( \underbrace{\quad \cdots \quad} - \text{traces} \right) = \frac{t (1 - t^2)}{(1 - t)^d} (10 - d), \\
\mathcal{D}^{s-1} F : \quad & \sum_{s = 1}^{\infty} \left( \underbrace{\quad \cdots \quad} - \text{traces} \right) = 1 - \frac{1 - dt (1 - t^2) - t^4}{(1 - t)^d}, \\
\mathcal{D}^s \psi, \mathcal{D}^s \bar{\psi} : \quad & \sum_{s = 0}^{\infty} \left( \underbrace{\quad \cdots \quad} \times \underbrace{\quad} - \text{traces} \right) = -\frac{16_s t^{3/2} - 16_s t^{5/2}}{(1 - t)^d},
\end{align*}
\]

(3.1)

with the boxes \(\boxed{}\) and \(\blacksquare\) representing the vector and spinorial representation, respectively, of \(SO(10 - d)\), and denominators generate spacetime derivatives according to (2.4). More
precisely, the towers \(1/(1-t)^d\) account for \(SO(d)\) descendants (derivatives along \(S^{d-1}\)) while \((1 - t^2)\) removes the traces associated to \(D^2\) terms. Finally, subtracting the \(16_c t^{5/2}\) term imposes the Dirac equation \(D\psi = 0\) on the fermionic modes. Collecting all the terms from (3.1), the singleton partition function in \(d\) dimensions takes the simple form

\[
Z_1^{(d)}(t) = 1 - \frac{Z_S(t)}{(1-t)^d}, \tag{3.2}
\]

in terms of the \(d\)-independent characteristic function

\[
Z_S(t) \equiv 1 - 10 t + 16_s t^{3/2} - 16_c t^{5/2} + 10 t^3 - 1 t^4. \tag{3.3}
\]

The function \(Z_S(t)\) carries a natural \(SO(10) \times SO(2)\) structure. This hidden symmetry is broken in the singleton partition function (3.2) only by explicit insertion of the \(SO(d)\) descendants.

The spectrum of single-trace operators can then be determined by counting the number of cyclic words via Polya’s formula

\[
Z_{\text{SYM}}^{(d)}(t) = - \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log \left[ 1 - Z_1^{(d)}(t^m) \right], \tag{3.4}
\]

with Euler’s totient function \(\varphi(m)\). Plugging (3.2) into (3.4) one finds for the SYM partition function

\[
Z_{\text{SYM}}^{(d)}(t) = - \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log [Z_S(t^m)] + \frac{t \partial_t (1-t)^d}{(1-t)^d} = Z_{\text{SYM}}^{(0)}(t) + \frac{t \partial_t (1-t)^d}{(1-t)^d}. \tag{3.5}
\]

Notice that according to (2.4), the last term contains only a finite number of completely antisymmetric forms and their \(SO(d)\) descendants. Remarkably, this is the only dependence of the total partition function on the space-time dimension \(d\) while

\[
Z_{\text{SYM}}^{(0)}(t) = - \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log [Z_S(t^m)] , \tag{3.6}
\]

comes with a manifest \(SO(10) \times SO(2)\) structure.

According to our discussion above, holography would require that the \(d\)-dependent part in (3.5) originates exclusively from BPS states. We will now show that this is indeed the case. The BPS multiplet in \(d\) dimensions is constructed by acting on the completely
symmetric chiral primary \((n0000) t^n\) with the 8 unbroken supercharges and spacetime derivatives

\[
Z_{\text{BPS}}^{(d)}(t) = \frac{1}{(1-t)^d} \sum_{\epsilon_s=0,1} \dim [(n0000) + \epsilon_s Q_s^{\text{BPS}}] t^{n+1/2 \sum_s \epsilon_s} , \tag{3.7}
\]

with \(Q_s^{\text{BPS}} \in \{(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \mid \text{with even number of } +\}\) .

Here \((w_1, w_2, w_3, w_4, w_5)\) denote the weights of the corresponding \(SO(10-d) \times SO(d)\) representation. The explicit results in the various dimensions (in the Dynkin basis) are given in tables I–V in Appendix B below. It is interesting to note that, omitting energies and \(SO(2) \subset SO(10)\) charges along the light cone plane, the result in any dimension can be compactly written as \((8_v - 8_s)(8_v - 8_s) \times [n000]\) with the appropriate branching of the \(SO(8)\) representations. This was used in [11] to show that the BPS part of the SYM spectrum indeed reproduces the supergravity spectrum \(Z_{\text{sugra}}(d)\) on \(\text{AdS}_{d+1} \times S^{9-d}\).

The total BPS spectrum is then obtained by summing over \(n\)

\[
Z_{\text{BPS}}^{(d)}(t) = \sum_{n=1}^{\infty} Z_{\text{BPS}}^{(d)}(t) . \tag{3.8}
\]

The result can be resumed in the remarkably simple relation among BPS towers in different spacetime dimensions:

\[
Z_{\text{BPS}}^{(d)}(t) = Z_{\text{BPS}}^{(0)}(t) + t \frac{\partial_t (1-t)^d}{(1-t)^d} . \tag{3.9}
\]

This relation is checked using the explicit form of BPS towers in Appendix B. Combining (3.5) and (3.9), the total SYM spectrum can be written as

\[
Z_{\text{SYM}}^{(d)}(t) = Z_{\text{BPS}}^{(d)}(t) + Z_{\text{long}}^{\text{SYM}}(t) , \tag{3.10}
\]

with

\[
Z_{\text{long}}^{\text{SYM}}(t) \equiv Z_{\text{SYM}}^{(d)}(t) - Z_{\text{BPS}}^{(d)}(t) = - \sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log [Z_s(t^m)] - Z_{\text{BPS}}^{(0)}(t) , \tag{3.11}
\]

a \(d\)-independent and manifestly \(SO(10) \times SO(2)\) covariant function describing the non-BPS part of the SYM spectrum. This is exactly the form predicted by holography from

\[2\text{Here, for convenience we include in the BPS tower the } n = 1 \text{ singleton multiplet associated to } \mathcal{N} = 4 \text{ SYM multiplet living on the AdS boundary.}\]
the string spectrum. More precisely, comparing the expressions (2.2) and (3.10) leads to the prediction

$$Z_{\text{SYM}}^{\text{long}} = T_{\text{SO}(10,2)} \times Z_{\text{flat}},$$  \hspace{1cm} (3.12)

directly in terms of $SO(10) \times SO(2)$ representations, which no longer explicitly depends on the spacetime dimensions $d$. The first few terms in the expansions of the functions appearing in this relation are given in Appendix A. In [4] the two sides of this equation were shown to agree for $d = 4$ till $\Delta = 10$. Since this relation is $d$-independent, the agreement for $d \neq 4$ finally is a direct consequence of the $d = 4$ match.

We have thus shown that the spectrum of free SYM theory in $d$ dimensions takes the form (3.10) of a $d$-dependent tower of BPS multiplets and a universal $d$-independent part that falls into $SO(10) \times SO(2)$ representations. We have then used the results of [1, 4] to show that this spectrum precisely agrees with the spectrum of massive string excitations on the warped background $\text{AdS}_{d+1} \times S^{9-d}$. The explicit match is given in Appendix A till $\Delta = 6$.

Let us finish by showing that the universal part $Z_{\text{SYM}}^{\text{long}}$ of the SYM spectrum indeed shares the factor $(1 - t^{1/2})^{16e}$ with the long multiplet $T_{\text{SO}(10,2)}$. For this one needs to work with the full character polynomial

$$ (1 - t^{1/2})^{16e} \equiv \prod_{\alpha=1}^{16} (1 - y^{q_\alpha} \cdot t^{1/2}), \quad y^{q_\alpha} = \prod_i y_i^{2q_{i\alpha}},$$ \hspace{1cm} (3.13)

and show that $Z_{\text{SYM}}(t, y)$ has zeros at $t = y^{-2q_\alpha}$ for any $\alpha = 1, \ldots, 16$. The following observation is then crucial: at the particular values $t = y^{-2q_\alpha}$ the character polynomial of the $SO(10) \times SO(2)$ function $Z_S(t, y)$ from (3.3) factorizes according to

$$Z_S(t, y) \big|_{t=y^{-2q_\alpha}} = \prod_{i=1}^{5} \left( 1 - y_i^{2q_{i\alpha}} y^{-2q_\alpha} \right).$$ \hspace{1cm} (3.14)

As a consequence, the infinite sum in (3.11) at these points can be explicitly evaluated as

$$Z_{\text{SYM}}^{(0)}(t, y) \big|_{t=y^{-2q_\alpha}} = -\sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log [Z_S(t^m, y^m)] \bigg|_{t=y^{-2q_\alpha}} = \sum_{i=1}^{5} \frac{y_i^{2q_{i\alpha}}}{y^{2q_\alpha} - y_i^{2q_{i\alpha}}}.$$  

With the full character polynomial of the $d=0$ BPS tower from table I, one then verifies that

$$Z_{\text{SYM}}^{\text{long}}(t=y^{-2q_\alpha}, y) = Z_{\text{SYM}}^{(0)}(t=y^{-2q_\alpha}, y) - Z_{\text{BPS}}^{(0)}(t=y^{-2q_\alpha}, y) = 0,$$ \hspace{1cm} (3.15)
i.e. the non-BPS part $Z_{\text{long}}^{\text{SYM}}$ of the SYM spectrum indeed organizes into long multiplets composed of $2^{16} \times \dim(\text{hws})$ states and their $SO(10,2)$ descendants. Note that in the counting we have not marked the length $L$ of SYM words. Indeed, the long multiplets in general combine SYM words of length $L, L+1, \text{and } L+2$. Accordingly, there is no trace of this “quantum number” in the string side. This is not surprising since the length $L$ is not a real quantum number after interactions are turned on.

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Appendix

A Partition functions

Here we list the blind partition functions appearing in both sides of equation (3.12). The left hand side is defined in terms of the $d=0$ SYM spectrum according to (3.11). The BPS$_0$ polynomial follows from table I using the SO(10) multiplicity formula (B.1) and summing over all $n$. The SYM$_0$ partition function is given by (3.6). Together, one finds

$$Z_{\text{SYM}}^{0} = 10 t - 16 t^{3/2} + 55 t^2 - 144 t^{5/2} + 450 t^3 - 1440 t^{7/2} + 4735 t^4 - 15616 t^{9/2} +$$

$$+ 52354 t^5 - 177840 t^{11/2} + 608655 t^6 - \ldots$$

$$Z_{\text{BPS}}^{(0)} = \frac{t \left(10 + 64 \sqrt{t} + 196 t + 352 t^{3/2} + 406 t^2 + 304 t^{5/2} + 145 t^3 + 40 t^{7/2} + 5 t^4\right)}{(1-t) \left(1+\sqrt{t}\right)^9}. \quad (A.1)$$

The right hand side of (3.12) counts KK and supersymmetric descendants of the type II string spectrum. The on-shell spectrum of the ten-dimensional string in flat space has been written in a manifestly $SO(10) \times SO(2)$ covariant form in Appendix B of [4]. Masses were derived by extrapolating BMN frequencies down to finite $J$. Evaluating the resulting partition function one finds

$$\sum_{\ell} \left| \langle \text{vac} \rangle_\ell \right|^2 = t^2 + \left(10 t^2 - t^3\right)^2 + \left(-16 t^{5/2} + 54 t^3 - 10 t^4\right)^2 +$$

$$+ \left(45 t^3 - 144 t^{7/2} + 210 t^4 + 16 t^{9/2} - 54 t^5\right)^2 + \left(t^3 + \ldots\right)^2 + \ldots,$$

$$T_{\text{susy}}^{(d)} T_{\text{KK}}^{(d)} = \frac{(1-t^{1/2})^{16}c(1-t^2)}{(1-t)^{10}}. \quad (A.2)$$
In (A.1), (A.2) we listed the expansions relevant for comparisons $\Delta \leq 6$ (but the agreement was checked all the way till $\Delta = 10$ in [4]).

## B BPS multiplets

In this appendix we summarize the BPS multiplets in $d=2k$ even spacetime dimensions computed from (3.7) and organized under the group $SO(10-d) \times SO(d) \times SO(2)$. From their explicit form it is straightforward to verify the relation (3.9) among BPS towers in different dimensions.

| $\Delta$ | $n$ |
|----------|-----|
| $n$      | $[n, 0000]$ |
| $n+\frac{1}{2}$ | $[n-1, 0001]$ |
| $n+1$    | $[n-2, 0100]$ |
| $n+\frac{3}{2}$ | $[n-3, 0101]$ |
| $n+2$    | $[n-3, 0200] + [n-4, 0000]$ |
| $n+\frac{5}{2}$ | $[n-4, 1010]$ |
| $n+3$    | $[n-4, 0100]$ |
| $n+\frac{7}{2}$ | $[n-4, 0001]$ |
| $n+4$    | $[n-4, 0000]$ |

Table I: $d=0$ BPS multiplet $[n, 0000]$ under $SO(10)$.

| $\Delta$ | $n$ |
|----------|-----|
| $n$      | $[n, 0000](0)$ |
| $n+\frac{1}{2}$ | $[n-1, 0001](+1) + [n-1, 0100](-1)$ |
| $n+1$    | $[n-2, 1001](+2) + [n-2, 0000](0) + [n-2, 0100](0) + [n-2, 1100](0)$ |
| $n+\frac{3}{2}$ | $[n-3, 0011](+3) + [n-3, 0101](+1) + [n-3, 1101](+1) + [n-3, 0100](0) + [n-3, 1010](0) + [n-3, 1001](-1)$ |
| $n+2$    | $[n-4, 0011](+4) + [n-4, 0100](0) + [n-4, 0001](0) + [n-4, 1001](0) + [n-4, 1000](0) + [n-4, 0000](0)$ |
| $n+\frac{5}{2}$ | $[n-5, 0000](0)$ |

Table II: $d=2$ BPS multiplet $[n, 0000](0)$ under $SO(8) \times SO(2)$. 

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Table III: $d = 4$ BPS multiplet $[n,00](00)$ under $SO(6) \times SO(4)$.

| $\Delta$ | $[n,n](000)$ |
|----------|----------------|
| $n$      | $[n,n](000)$   |
| $n+\frac{1}{2}$ | $[n,n-1](001) + [n-1,n](010)$ |
| $n+1$    | $[n,n-2](100) + [n-2,n](100) + [n-1,n-1](011) + [n-1,n-1](000)$ |
| $n+\frac{3}{2}$ | $[n-2,n-1](100) + [n-1,n-2](110) + [n-1,n-2](001) + [n-2,n-1](010)$ |
| $n+2$    | $[n-2,n-2](000) + [n-2,n-2](200) + [n-2,n-2](011) + [n-3,n-1](002)$ |
| $n+\frac{5}{2}$ | $[n-3,n-2](101) + [n-2,n-3](110) + [n-2,n-3](001) + [n-3,n-2](010)$ |
| $n+3$    | $[n-2,n-4](100) + [n-4,n-2](100) + [n-3,n-3](000) + [n-3,n-3](011)$ |
| $n+\frac{7}{2}$ | $[n-3,n-4](001) + [n-4,n-3](010)$ |
| $n+4$    | $[n-4,n-4](000)$ |

Table IV: $d = 6$ BPS multiplet $[n,n](000)$ under $SO(4) \times SO(6)$.

| $\Delta$ | $[+2n](0000) + [-2n](0000)$ |
|----------|-------------------------------|
| $n$      | $[+2n-1](0001) + [-2n+1](0010)$ |
| $n+\frac{1}{2}$ | $[+2n-2](0100) + [-2n+2](0100)$ |
| $n+1$    | $[+2n-3](1010) + [-2n+3](1001)$ |
| $n+\frac{3}{2}$ | $[+2n-4](0020) + [-2n+4](0002) + [+2n-4](2000) + [-2n+4](2000)$ |
| $n+2$    | $[+2n-5](1010) + [-2n+5](1001)$ |
| $n+\frac{5}{2}$ | $[+2n-6](0100) + [-2n+6](0100)$ |
| $n+3$    | $[+2n-7](0001) + [-2n+7](0010)$ |
| $n+\frac{7}{2}$ | $[+2n-8](0000) + [-2n+8](0000)$ |

Table V: $d = 8$ BPS multiplet $[+2n](0000)$ under $SO(2) \times SO(8)$. 

11
The dimensions of $SO(10-d) \times SO(d)$ representations follow (upon reduction) from the $SO(10)$ multiplicity formula \(^3\)

$$\dim[n_1, n_2, n_3, n_4, n_5] = \frac{1}{7! 5! 3! 4!} (1+n_1)(1+n_2)(1+n_3)(1+n_4)(1+n_5) \times$$

\[
\begin{align*}
&\times (2+n_2+n_3)(2+n_1+n_2)(2+n_3+n_4)(2+n_3+n_5)(3+n_1+n_2+n_3) \\
&\times (3+n_2+n_3+n_4)(3+n_2+n_3+n_5)(3+n_3+n_4+n_5)(4+n_1+n_2+n_3+n_4) \\
&\times (4+n_1+n_2+n_3+n_5)(4+n_2+n_3+n_4+n_5)(5+n_1+n_2+n_3+n_4+n_5) \\
&\times (5+n_2+2n_3+n_4+n_5)(6+n_1+n_2+2n_3+n_4+n_5)(7+n_1+2n_2+2n_3+n_4+n_5) .
\end{align*}
\]

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\(^3\)More precisely $SO(2m)$ multiplicities are given by suppressing any term in the product (B.1) involving $n_i$ with $i > m$ and the corresponding factorials.
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