The covariance of GPS coordinates and frames

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Received 15 December 2005, in final form 10 March 2006
Published 28 April 2006
Online at stacks.iop.org/CQG/23/3531

Abstract
We explore, in the general relativistic context, the properties of the recently introduced global positioning system (GPS) coordinates, as well as those of the associated frames and coframes that they define. We show that they are covariant and completely independent of any observer. We show that standard spectroscopic and astrometric observations allow any observer to measure (i) the values of the GPS coordinates at his position, (ii) the components of his 4-velocity and (iii) the components of the metric in the GPS frame. This provides this system with a unique value both for conceptual discussion (no frame dependence) and for practical use (involved quantities are directly measurable): localization, motion monitoring, astrometry, cosmography and tests of gravitation theories. We show explicitly, in the general relativistic context, how an observer may estimate his position and motion, and reconstruct the components of the metric. This arises from two main results: the extension of the velocity fields of the probes to the whole (curved) spacetime, and the identification of the components of the observer’s velocity in the GPS frame with the (inversed) observed redshifts of the probes. Specific cases (non-relativistic velocities, Minkowski and Friedmann–Lemaître spacetimes, geodesic motions) are studied in detail.

PACS numbers: 04.20.—q, 04.80.—y, 91.10.—v, 95.40.+s

(Some figures in this article are in colour only in the electronic version)

1. Introduction and notation

In general relativity, ‘real’ quantities are represented by covariant objects, i.e., defined independently of any frame or system of coordinates. A moving object (hereafter, a probe) is represented by its worldline, and the associated 4-velocity, 4-acceleration vectors and so on. All these entities are covariant. An observer, as a particular case of moving object,
is also represented by its (covariant) worldline, velocity and acceleration. An observation requires an observer and an observed system, both described by covariant quantities. Any observable quantity is a scalar covariant combination (usually a tensorial contraction) of covariant quantities associated with the observer and the observed system.

Most often, calculations in general relativity cannot be performed without the auxiliary introduction of specific frames (or coordinates). This perfectly justified procedure introduces however a risk of misinterpretation of the intermediary frame-dependent (thus, non-covariant) quantities introduced. In addition, one has to distinguish between observer-dependent and observer-independent results. Thus, one has usually the choice between covariant and observable quantities on the one hand, and frame-dependent quantities on the other hand, which can be more easy to calculate but require special care to be converted into really observable quantities.

This motivates the introduction of the following global positioning system (GPS) coordinates which are covariant and observable. Let us emphasize that the quasi totality of coordinate systems used in general relativity calculations does not offer these advantages. In addition, this system of coordinates is completely observer independent. This allows easy comparison of measurements made by different observers, or by the same observer at different locations. This makes it possible to know what another observer would observe in the same situation. Finally, even for a well-defined observer, there is no global frame canonically defined (see, e.g., [4]), so that any observer-dependent frame involves an undesirable arbitrariness, which is removed with the GPS coordinates.

GPS coordinates have recently been introduced by Rovelli [7] and also, independently, in [2] and [6] under the name emission coordinates. They are covariant; they represent directly measurable quantities, and they are independent of the observer. They show some similarities with the optical coordinates introduced by [8], which however are observer dependent. The denomination ‘GPS coordinates’ comes from the fact that their definition assumes the existence of four reference probes, a situation which corresponds to the well-known global positioning system (GPS; see for instance [5]). However, their use is much more general, since they can be defined from any system of ‘probes’, which we define here as objects emitting a signal. They can be part of the system of artificial satellites (like the GPS), but also stars, pulsars, galaxies, etc. This system of probes constitutes what is called a generic and (almost) immediate location system of [2]. The only requirement is that an observer is able to measure the direction and the arrival time of the emitted signal (in his proper time) and its redshift. It is shown below how this allows us to construct the GPS coordinates and different additional quantities. This makes GPS coordinates very convenient for localization, astrometry, tests of gravitation theories and cosmography. Note that the situation can be improved in especially designed experiments, where the probes send additional information, like for instance the instant (in any time coordinate) of the signal emission. Moreover, a further improvement (auto-located positioning systems, [2]) occurs when the probes also transmit the values of the proper time information that they receive from the other probes. But we will not consider explicitly such cases in this paper.

In [7], Rovelli has introduced several fundamental properties of the GPS coordinates system and derived their explicit calculation in Minkowski spacetime (see also [2] and [6]). Here we extend these calculations and introduce the frames and coframes defined by the GPS coordinates. We show that they are also covariant in the general relativistic meaning, they do not depend on any observer and their components can be measured in a manner that we specify explicitly. Our calculations apply to an arbitrary spacetime with curvature. They are valid in the context of any metric theory (including general relativity) and thus allow potential applications for checking gravitation theories.
Observer’s frames. On the other hand, it may be convenient for an observer to choose a preferred frame for making his calculation, expressing and interpreting his observational results. The timelike direction defined by his 4-velocity allows a local (strictly speaking, infinitesimal) space + time splitting. We will indicate precisely the passage between an arbitrary frame (for instance one linked to an observer) and the GPS frame. We show the nice property that the (inverse) velocity components of the observer, in the GPS frame, identify with the set of the redshifts he receives from the probes.

After recalling some geometrical results of differential geometry (section 1.1), section 2 first considers the question of the observation of a probe by an observer in a metric theory. Then, in section 3, we study the coordinates, frames and coframes defined by a set of four such probes. In section 4, we present potential applications, and finally in section 5, we apply our results to the simplified case of Minkowski spacetime and recover some results already found in the literature.

1.1. Geometrical preliminaries

At any point $m$ of a differential manifold $M$ (here, spacetime), the tangent space $T_mM$ is dual to the cotangent space $T^*_mM$. The duality is however not canonical in the sense that there is no natural association of a 1-form with a vector. There is however a natural way to define the dual of a frame $(e_\mu)$ as the coframe $(e^\mu)$ defined through

$$\langle e^\mu, e_\nu \rangle = \delta^\mu_\nu.$$

Each $e_\mu$ is a vector; each $e^\mu$ is a 1-form, and the brackets represent the action of 1-forms on vectors. Note that this frame duality does not depend on any metric.

We assume now a (Lorentzian) metric $g$. A null frame $(e_\mu)$ is such that

$$g_{\mu\mu} \equiv g(e_\mu, e_\mu) \equiv e_\mu \cdot e_\mu = 0, \quad \forall \mu.$$

(no sum over indices). A null coframe is such that

$$g^{\mu\mu} \equiv g(e^\mu, e^\mu) \equiv e^\mu \cdot e^\mu = 0, \quad \forall \mu.$$

Note that the dual of a null coframe is not a null frame, and vice versa.

With any vector $V$, the musical isomorphism (or ‘canonical isomorphism’) associates the 1-form $\flat V$ defined through

$$g(V, W) \equiv V \cdot W = \langle \flat V, W \rangle, \quad \forall W.$$

Similarly, with a 1-form $\theta$, one associates the vector $\sharp \theta$ such that $\langle \flat (\sharp \theta), V \rangle = \theta$.

With a given frame $(e_\mu)$, the musical isomorphism associates the coframe $(\flat e_\mu)$. In general, it does not coincide with the dual coframe $(e^\mu)$. Only when the frame is orthogonal (as it is common in standard calculations), the two coframes coincide up to constants. This is not the case for null frames or coframes.

In the following, we will consider not only the holonomic coframe $(\omega^\alpha = ds^\alpha)$ associated with the GPS coordinates $(s^\alpha)$ and its dual frame $(\omega_\alpha = \frac{\partial}{\partial s^\alpha})$, but also the musically transformed frame $(\Omega_\alpha \equiv \flat s^\alpha)$ and its dual coframe $(\Omega^\alpha \equiv \flat \omega_\alpha)$.

Let us also recall that a frame is holonomic when it is linked to (local) coordinates. In this case, one can write (locally)

$$\omega^\alpha = ds^\alpha, \quad \omega_\alpha = \frac{\partial}{\partial s^\alpha} = \partial_\alpha.$$
2. Observing a probe

2.1. Probes

Very generally, we call a probe any object emitting an observable signal, which can be used as a reference. The only requirement is that an observer can measure the arrival time of a signal (in his proper time) and its redshift. As we will see below, this gives direct access to the proper time of the signal emission by the probe, which is at the basis of the formalism. Typically, the probe can be a GPS satellite, hence the denomination. It can also be a star (in particular a pulsar) sending its radiation, which may constitute an efficient astrometric tool. This can also be a distant galaxy, or a quasar, in cosmology. Under some circumstances, the probe can be a member of a specially designed system of satellites (e.g., the GPS system or the LISA experiment). In this case, these satellites can send some additional information, offering additional possibilities, including the tests of metric theories. In this paper, we do not explore such opportunities, and only consider the basic case where we can measure the arrival time and direction, as well as the redshift of the signals, from which results the (probe’s) proper time of emission, s.

The motion of the probe (assumed to be given) is defined by its timeline \( s \mapsto P(s) \), with s its proper time. We do not assume geodesic motion. Although, in general, the observer has no direct access to s, he can in fact monitor its value by integration of the redshift, as shown below. Thus we consider s as a measurable quantity. The (normalized) velocity of the probe, \( u \equiv \frac{d}{ds} P \), defines a vector field along its worldline. It verifies \( u \cdot u = 1 \). We will show below how to extend this velocity field to the whole spacetime.

The proper time of the probe, s, is defined along the worldline P only. We extend it as a function \( x \mapsto s(x) \) defined ‘everywhere’ (hereafter, ‘everywhere’ means the region of spacetime where these coordinates are well defined; see the discussions in [7] and [6]): at any event \( x \) in spacetime, \( s(x) \) is defined as the proper time of the probe, when it emitted the light ray reaching \( x \) (see the discussion in [7] for the case where many light rays reach the observer). In other words, the hypersurface \( \Sigma_{s_0} \) of the equation \( s = s_0 \) is the future light cone of the probe at its proper time \( s_0 \) (figure 1). This is a null hypersurface (for the properties of null hypersurfaces, see [3]). It is a fundamental fact that s is an observable quantity.

Each \( \Sigma_{s_0} \) admits a one-dimensional vector space of vectors orthogonal to it at any point. These vectors are all proportional to \( \frac{d}{ds} \). Since the surface is null, these vectors are also tangent to it, and they are null vectors (\( \frac{d}{ds} \cdot \frac{d}{ds} = 0 \)). Their integral lines are null geodesics which constitute the null generators of \( \Sigma_{s_0} \). They are the future directed light rays emanating from the event \( P(s) \). In a null surface, the null vectors cannot be normalized as for a spatial surface. However, one may select here a unique orthogonal vector, from the form \( \omega \equiv ds \), which is well defined over \( \Sigma_{s_0} \) (except on the line \( P \) itself, where \( \Sigma_{s_0} \) becomes singular). The canonical isomorphism (generated by the metric g of spacetime) transforms the 1-form \( \omega \) to the vector \( \Omega \equiv g(\omega) \) such that

\[
(\omega, \Omega) = g(\Omega, \Omega) \equiv \Omega \cdot \Omega \equiv g(\omega, \omega) = 0
\]

(see the detailed proofs in [3]). This implies

\[
\Omega \cdot \nabla \Omega \equiv \nabla \Omega \Omega = \Omega \cdot \nabla \omega = 0,
\]

which means that \( \Omega \) and \( \omega \) are parallelly transported by \( \Omega \) and that the null lines generated by \( \Omega \) are geodesic, with \( \Omega \) an affine vector along them. The vector \( \Omega \) is the frequency vector [1] of the light ray (here normalized so that the emitted frequency is unity).

Note that \( \Omega \) and \( \omega \) are not defined on the worldline \( P \) itself. However, for each light ray, it is possible to extend them by continuity so that they are parallelly transported. This allows us to consider parallel transport along the null geodesics, including their intersections with \( P \).
Note that the function \( s(x) \) obeys the equation
\[
\mathcal{W}[P(s(x)), x] = 0,
\]
where \( \mathcal{W}(x, y) \) is the world function [8], defined as half the geodesic distance between the two points of spacetime \( x \) and \( y \).

2.2. Extending the velocity of the probe

Now let us define ‘everywhere’ the timelike vector field \( u \) obeying the following requirements.

- It is normalized: \( u \cdot u = 1 \).
- It is parallelly transported along the null generators:
  \[
  \nabla_\Omega u = 0.
  \]

Note that we have extended above the parallel transport up to the line \( P(s) \) itself, and we require that \( u \) is parallelly transported in this way.

- It coincides with the velocity of the probe \( P \) along its worldline. Thus, it may be seen as the velocity of the probe transported everywhere by the null generators.

Since \( \Omega \) is also parallelly transported by itself, this implies that \( \langle u, \Omega \rangle = u \cdot \Omega \) is constant along the null generators (the light rays from \( P \)).

To apply the third condition, we consider the 2-surface \( \mathcal{H} \) generated by the probe worldline, and the light rays reaching the observer, at successive moments (see figure 2). It also contains the observer’s worldline. The vector field \( u - V \) is well defined on the observer’s worldline. It represents the relative 4-velocity of the probe with respect to the observer.

The two vector fields \( u \) and \( \Omega \) are well defined on \( \mathcal{H} \) including the worldline \( P \) thanks to the mentioned extension. Moreover, \( \Omega \) remains tangent to \( \mathcal{H} \). On \( P \), the scalar product
Figure 2. The surface \( \mathcal{H} \) contains the worldline of the probe and of the observers, as well as the light rays from the former to the latter.

\[ \Omega \cdot u = 1 \] because there the velocity \( u = \frac{d}{ds} \). Since this product is preserved by the null generators, as shown above, this implies that it keeps the value 1 on \( \Sigma_t \):

\[ \Omega \cdot u = 1 \] everywhere,

the main result of this paper. This allows the decomposition

\[ \Omega = u + \nu, \]

where the (spacelike) vector \( \nu \) verifies \( \nu \cdot \nu = -1, \nu \cdot u = 0. \)

2.3. Observers and redshifts

An observer is defined by his worldline \( \sigma \mapsto O(\sigma) \), along which flows his proper time \( \sigma \). His normalized velocity \( V \equiv \frac{d}{d\sigma} \), with \( V \cdot V = 1 \). By definition, he sees the probe with a redshift

\[ 1 + z = \frac{d\sigma}{ds}. \]

along his worldline.

Since he can monitor this redshift as a function of his proper time, \( z(\sigma) \), this gives him access (up to an additional constant) to the proper time of the probe at the moment of emission:

\[ s(\sigma) = \int d\sigma \frac{ds}{d\sigma} = \int d\sigma \frac{1}{1 + z(\sigma)} + C \text{te}. \]

This makes \( s(\sigma) \) an observable quantity as announced.

On the other hand, it is easy to show that

\[ 1 + z = \frac{u(P) \cdot \Omega(P)}{V \cdot \Omega(O)} = \frac{1}{V \cdot \Omega(O)} = \frac{1}{\langle \omega, V \rangle(O)}, \]

where \( (O) \) and \( (P) \) mean that the quantities are evaluated at the observer’s and probe’s positions, respectively. Note that \( z(\sigma) \) is a perfectly measurable and covariant quantity. But
it is observer dependent (it depends on his velocity). Through this integral, the covariant quantity $s(\sigma)$ is also measurable up to a constant (we can imagine an optimal situation where this value is directly sent by the probe). The covariant spacetime function $s(x)$ is observer independent.

It is a fundamental fact that the observer is able to estimate the value of $s$ by monitoring the redshift as a function of his proper time, $z(s)$, and by integrating this relation. (In some specifically designed experiments, as in a future version of the GPS system, the probe may emit explicitly the value of $s$, which would allow additional possibilities.)

By projecting the velocity $V$ of the observer onto the vector $\Omega$, one obtains

$$\Omega = \frac{1}{1+z}(V - n), \quad n \cdot n = -1, \quad n \cdot V = 0, \quad V \cdot V = 1.$$  

The vector $n$ represents the spatial (unit) direction in which the observer sees the probe. For the observer, this vector is purely spatial in the sense that it is orthogonal to his velocity $V$, which defines the time direction for him. Note that the observer is able to monitor this direction, which is also a covariant (but observer dependent) quantity.

Taking into account $u \cdot \Omega = 1$, this relation implies

$$1 + z = u \cdot V - u \cdot n.$$  

We emphasize that these relations hold in any spacetime, for any observer, and for any metric theory.

### 2.4. Non-relativistic motion

All formulae above hold for arbitrary velocities. If we work in the solar system, or in the galaxy (i.e., for instance, with pulsars), the velocities involved are most often non-relativistic, which corresponds to $z \ll 1$. All scalar quantities can be developed in this small parameter.

We have for instance

$$\Omega \cdot V = \frac{1}{1+z} \simeq 1 - z \quad \Rightarrow \quad \Omega \cdot (u - V) \approx z.$$  

This relation was obtained in Minkowski spacetime by [9], and is at the basis of their analysis of pulsar timing. We have shown here that it remains true in an arbitrary spacetime.

When the probe (e.g., a pulsar) has an intrinsic period $T$, the observer monitors a period

$$T_{\text{obs}} = T(1+z) = 1 - u \cdot n + O(z^2).$$

To continue, we may assume that the pulsar has an intrinsic variation $\dot{T}$ of its period. Then, by writing (3) at two successive instants, one obtains the observed period variation

$$T_{\text{obs}} \equiv \frac{dT_{\text{obs}}}{d\sigma} = T(1+z) + T \frac{dz}{d\sigma}.$$  

This is an exact formula. However, the expression of $\frac{dz}{d\sigma}$ is quite complicated. Reference [9] has given an approximation of it, which is valid for small redshifts, and in Minkowski spacetime.

### 2.5. Application to Minkowski spacetime

The case of Minkowski spacetime is particularly simple. Without loss of generality, we can place the observer at the centre of spatial coordinates: $O = (t = \sigma, 0, 0, 0)$, so that his velocity has components $V = (1, 0, 0, 0)$. A radial light ray arriving to the observer has a (future directed) affine tangent vector $\Omega$, with components $(K, -K, 0, 0)$, with $K > 0$. And
the probe has a velocity \((u', u', u\theta, u\phi)\), with \(u' > 0\) for a probe moving away from the source. Then it is trivial to form the scalar products \(\Omega \cdot V = K\) and \(\Omega \cdot u = K (u' + u')\), from which it results
\[
1 + z = u' + u' \approx 1 + u',
\]
where the latter approximation holds for non-relativistic motion.

### 2.6. Application to cosmology

In a Friedmann–Lemaître universe, with metric in the Robertson–Walker form
\[
g = dt^2 - a(t)^2 [dr^2 + f(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2)],
\]
the observer is assumed at the centre of spatial coordinates so that his velocity has components \(V = (1, 0, 0, 0)\). The null radial vector \(\Omega^r = \Omega^r \frac{\partial}{\partial r} - \frac{\partial}{\partial t}\), with metric dual \(\omega = \Omega^r dt + a(t)\Omega^r dr\). The requirement to be an affine vector implies \(\Omega^r = K/a(t)\), with \(K\) a constant along the light ray. The value of the latter is fixed by the condition
\[
1 = \Omega^r \cdot V = K/a(t)\left[u'(P) + a(P)u'(P)\right],
\]
where \((P)\) means evaluated at the position of the probe. This allows us to find the expression for the extension of the velocity field of the probe everywhere. In the case where the velocity is purely radial \((u\theta = u\phi = 0)\), we obtain
\[
\begin{align*}
u_1' &= \frac{1}{2} \left(\frac{K}{a} + \frac{a}{K}\right), \\
u_2' &= \frac{1}{2a} \left(\frac{a}{K} - \frac{K}{a}\right),
\end{align*}
\]
where we defined
\[
K \equiv aP \left[u'(P) - aP u'(P)\right] = \frac{aP}{u'(P) + aP u'(P)}.
\]
Using \(\Omega \cdot V = \Omega' = K/a(t)\), we obtain the redshift
\[
1 + z = \frac{a(t)}{K} = \frac{a(t)}{a(P)}\left[u'(P) + a(P)u'(P)\right],
\]
which reduces to the usual formula \(1 + z = \frac{a(t)}{a(P)}\) for a comoving galaxy, defined by \(u'(P) = 0, u'(P) = 1\).

### 3. GPS coordinates and frames

#### 3.1. Coframes and frames

Now we assume four probes \(P^\alpha (\alpha = 1, 2, 3, 4)\) and adopt, as in [7], the four corresponding functions \(s^\alpha\) as coordinates (see [7] and [6] for discussions about the range of validity of these coordinates, to which we refer as ‘everywhere’). As emphasized above, these coordinates are measurable. They define the coframe \((\omega^\alpha \equiv ds^\alpha)\). By definition, the contravariant components of the metric tensor are given by
\[
g^{\alpha\beta} = \omega^\alpha \cdot \omega^\beta.
\]
As we have seen in (2), each \(s^\alpha\) is a null function so that
\[
g^{\alpha\alpha} = \omega^\alpha \cdot \omega^\alpha = 0.
\]
This means that \((\omega^\alpha)\) is a null coframe.
Figure 3. Three probes (which would become four in four-dimensional spacetime) provide GPS coordinates.

Table 1. Frames and coframes linked to the GPS coordinates.

| $\omega^\alpha$ | $\omega_\alpha = \partial_\alpha$ | $\Omega^\alpha = \theta^{\alpha}_{\beta} \partial_{\beta}$ | $\Omega_\alpha = \theta^\alpha_{\beta} \partial_{\beta}$ |
|-----------------|-----------------|-----------------|-----------------|
| Null            | Holonomic       | Coframe         | Frame           |
| Holonomic       | Null            | Coframe         | Frame           |

This coframe is holonomic. Its dual frame, given by $\omega_\alpha \equiv \partial_\alpha \equiv \frac{\partial}{\partial s_\alpha}$, obeys the duality relations $(\omega^\alpha, \partial_\beta) = \delta^\alpha_{\beta}$. We have by definition

$$g_{\alpha\beta} = \partial_\alpha \cdot \partial_\beta.$$ 

The covariant components $g_{\alpha\beta}$ of the metric are defined, as usual, from the relations $g_{\alpha\beta} = g^{\gamma\delta} g_{\gamma\alpha} g_{\delta\beta}$. Note that, in general, $g_{\alpha\alpha} \neq 0$, so that $(\partial_\alpha)$ is not a null frame.

With the system of coordinates $(s^\alpha)$, are naturally associated the (holonomic) frame and coframes $\partial_\alpha$ and $\omega^\alpha$. These frames involve only covariant quantities. They are defined ‘everywhere’. They are completely independent of any observer.

The musical isomorphism (see section 1) defines the vectors $\theta^{\omega^\alpha} = \Omega_\alpha$ (despite the upper index, the $\theta^{\omega^\alpha}$ are vectors) through

$$\langle \omega^\alpha, \Omega_\beta \rangle = \omega^\alpha \cdot \omega_\beta = \Omega_\alpha \cdot \Omega_\beta = g^{\alpha\beta}.$$ 

Since $g^{\alpha\alpha} = 0$, $(\Omega_\alpha)$ form a null frame.

As vectors, the $\Omega_\alpha$ can be expanded in the $(\partial_\alpha)$ basis:

$$\Omega_\alpha = g^{\alpha\beta} \partial_\beta \quad \Rightarrow \quad (\Omega_\alpha)^\beta = (\Omega_\beta)^\alpha = g^{\alpha\beta}.$$ 

See table 1 and figure 3 for illustrations of these frames and coframes.
3.2. Measurements

Measurements are performed by an observer, which we assume given by his worldline \( \sigma \mapsto O(\sigma) \), \( \sigma \) being his proper time, that he is able to read on his clock. His (normalized) velocity vector, \( V \equiv \frac{d}{d\sigma} \), defines a natural time direction for him, at each point of his worldline and, by orthogonality, a set of spacelike directions. In the following, we will assume that the observer used this natural time + space splitting to define (locally) an ON frame \( (e_I) \equiv (e_0 \equiv V, e_i) \), with \( I = 0, 1, 2, 3; i = 1, 2, 3 \). We do not care about the way the spatial part of this frame (the \( e_i \)) is defined.

By spectroscopic observations, the observer is able to estimate the redshifts corresponding to the signals emitted by the four probes, \( z^\alpha \). As we will see, these observed quantities play a fundamental role. As any vector, the observer’s velocity can be expanded in the basis as \( V = V^\alpha \partial_\alpha \). From (2), it results the very important relation

\[ V \cdot \Omega_\alpha = \langle \omega^\alpha, V \rangle = \frac{1}{1 + z^\alpha}. \]

This implies

\[ V = \frac{1}{1 + z^\alpha} \partial_\alpha. \]

The inverse components of the observer’s velocity reduce to the redshifts with respect to the probes. It results that, from the observed redshifts, the observer can derive the components of his velocity in the GPS frame, and thus monitor his motion. Note that the four redshifts are linked by the relation

\[ g_{\alpha \beta} \frac{1}{1 + z^\alpha} \frac{1}{1 + z^\beta} = 1. \]

The frames and coframes defined by these coordinates are also covariant and independent of the observers. However, observations do not give direct access to the components of vectors, forms or tensors. Those are generally evaluated in a given reference frame, chosen for convenience (although the physics should be independent of such a choice).

3.3. The reconstruction of the metric

All the components of the metric tensor (in the GPS frame) identify with covariant quantities (in general, a tensor is a covariant quantity, but not its components). In addition, they are (as they must) completely observer independent. Now we show that these components are directly measurable.

The observer is able to project any quantity (like a vector or a tensor), at his position, parallelly and orthogonally to his velocity vector \( V \). He will interpret the projected quantities as temporal and spatial components. Note that this decomposition, although of course observer dependent, remains perfectly covariant (does not require the introduction of any frame). In particular,

\[ \Omega_\alpha = V^\alpha (V - n_\alpha), \]

and its metric dual (musical) version

\[ \omega^\alpha = V^\alpha (\flat V - \flat n_\alpha), \]

where

\[ n_\alpha \cdot n_\alpha = \flat n_\alpha \cdot \flat n_\alpha = -1, \quad V \cdot n_\alpha = 0, \quad V \cdot V = 1. \]
The spatial normalized vectors, \( n_\alpha \), represent the (spatial) directions of arrival of the signals of the four probes. The \( n_\alpha \) are the spatial normalized 1-forms, which represent the corresponding wavefronts.

The observer is able to monitor these directions (in any frame), and thus to estimate the scalar products \( K_{\alpha\beta} \equiv n_\alpha \cdot n_\beta \) (obeying \( K_{\alpha\alpha} = -1 \)). These measurable quantities are frame independent.

Some easy algebra shows
\[
n_\alpha = -V^\beta K_{\alpha\beta} \partial_\beta \quad \text{(sum on \( \beta \))}
\]

Also, from the relations above,
\[
g^{\alpha\beta} = \Omega_\alpha \cdot \Omega_\beta = \frac{1}{1 + z^\alpha} \frac{1}{1 + z^\beta} (1 + K_{\alpha\beta})
\]
(no sum on indices). It results that all the metric coefficients \( g^{\alpha\beta} \) are measurable by the observer. By matrix inversion, he may also reconstitute the matrix \( g_{\alpha\beta} \). This possibility of reconstructing the metric has also been considered in the two-dimensional case in [2].

**Observer dependent frames**

The observer may select three of these vectors, \( \Omega_i \), that we label now with Latin indices \( i, j = 1, 2, 3 \). With his velocity \( V \), they constitute a perfectly valid frame, although observer dependent. It was introduced by Synge [8] under the qualification of ‘optical’. Also, the three spatial vectors \( n_i \), with \( V \), constitute a frame of a more usual form, with one timelike and three spacelike vectors. In this case, we will label \( V \) by the index 0, and it is easy to establish that, in this frame,
\[
g_{00} = 1, \quad g_{0i} = 0, \quad g_{ij} = K_{ij}.
\]

Thus, the observer is also perfectly able to reconstruct the metric coefficients in this frame.

**4. Potential applications**

Via the measured redshifts (or directly in appropriate situations), the observer can monitor \( s^\alpha \) as a function of his proper time. It is also possible to measure the scalar products \( K_{\alpha\beta} \), and thus to reconstruct completely the metric as indicated above. Since the GPS frame is defined everywhere, it is possible to perform the same measurements from other places: directly, following the observer’s motion moving in spacetime, or indirectly, using measurements from other probes, which play the role of additional observers. The procedure above provides the metric coefficients in the different corresponding points of spacetime. By derivations, this would give access to the components of the connection and of the curvature.

In any case, the observer can monitor the redshifts along his worldline, i.e., with respect to his proper time \( \sigma \). Then,
\[
\frac{ds^\alpha}{d\sigma} = A \cdot \Omega_\alpha + V \cdot \frac{d\omega_\alpha}{d\sigma}.
\]

This relation allows him to calculate his own acceleration \( A \). This could be at the basis of experimental tests of gravitation theories.

If the probes send additional information, in the form of a variation of the period of a signal emitted by the probe (this is the case from a pulsar), relation (4) brings then more constraints to the parameters of the system. They can be used for instance to determine the acceleration of the earth, or of the solar system (see [9]).
4.1. Localization

The observer may wish to use such coordinates for his localization. In some sense, this is a triviality since \( s^\alpha \) are already bona fide coordinates; their measurements provide formally a perfect localization. In general, however, the observer is not interested in localization with respect to the probes, but rather with respect to a more usual system of coordinates \( x^\mu \), as for instance a terrestrial one. In such a case, he would renounce to covariance without interest for his purpose.

This requires no more than a simple conversion of variables \( (s^\alpha) \rightarrow (x^\mu) \). This possibility is guaranteed by the fact that \( s^\alpha \) are true coordinates. Explicitly, it requires the explicit knowledge of the ephemeris \( P^\mu_{(a)}(s^\alpha) \) of the four probes, in his favourite coordinate system. For a system of artificial probes, they are perfectly known, and the conversion is simply a matter of calculations, although there is no general analytic expression for them in non-flat spacetime. In practice, the ephemeris has to be inserted in the world equation.

This provides the formulae for the change of coordinates, \( s^\alpha(x^\mu) \), and its inverse \( x^\mu(s^\alpha) \). From this, the Jacobian provides the transformation matrices for the change of frames:

\[
E^\alpha_\mu \equiv \frac{\delta s^\alpha}{\delta x^\mu}, \quad E^\mu_\alpha \equiv \frac{\delta x^\mu}{\delta s^\alpha}.
\]

This allows us to express the vectors (1-forms) of this frame in the other arbitrary frame (coframe),

\[
\omega^\alpha = ds^\alpha = E^\mu_\alpha e^\mu, \quad \partial_\alpha = E^\mu_\alpha e^\mu, \\
e^\mu = E^\mu_\alpha \omega^\alpha, \quad e_\mu = E_\mu^\alpha \partial_\alpha.
\]

Note that the last formulae apply even when the new frame is not holonomic (i.e., does not correspond to a system of coordinates).

These formulae would be useful when an observer wishes to evaluate tensorial components in his personal frame (at the price of loosing covariance and observer’s independence). Note that the knowledge of the explicit correspondence requires us to know the ephemeris of the probes.

5. Application to Minkowski spacetime

In flat Minkowski spacetime, the properties of parallel transport imply that the four velocities \( u^\alpha \) are vectors constant on each light cone. The positions \( X \equiv O(\sigma) \) of the observer and \( P^\alpha \) of the probe may be considered as vectors, and we define the separation vectors (between the probes and the observer)

\[
D^\alpha \equiv P^\alpha - X.
\]

Note that \( D^\alpha \cdot D^\alpha = 0 \).

It is easy to establish the relations

\[
\therefore ds^\alpha \equiv \Omega_\alpha = \frac{D^\alpha}{D^\alpha \cdot u^\alpha}, \\
V^\alpha = 1 + z^\alpha = \frac{D^\alpha \cdot V}{D^\alpha \cdot u^\alpha}, \\
\therefore n^\alpha = V - \frac{D^\alpha}{D^\alpha \cdot V}.
\]

It results

\[
g^{\alpha\beta} = \frac{D^\alpha \cdot D^\beta}{(D^\alpha \cdot u^\alpha)(D^\beta \cdot u^\beta)}.
\]
If the motions of the four probes are geodesic, we have $P^\alpha = P^\alpha_0 + u^\alpha s^\alpha$, where $P^\alpha_0$ represents the origin of the geodesic motion of the probe $\alpha$. Defining the four vectors $\xi^\alpha = X - P^\alpha_0$, we obtain an explicit expression for the GPS coordinates,

$$s^\alpha = \xi^\alpha \cdot u^\alpha - \sqrt{\left(\xi^\alpha \cdot u^\alpha\right)^2 - \xi^\alpha \cdot \xi^\alpha}.$$

This expression has been found in [7], in the case where the four probes start at the same origin ($P^\alpha_0 = 0$).

### 5.1. Localization in Minkowski spacetime

The observer monitors the GPS coordinates $(s^\alpha)$, and wishes to know the coordinates $x^\mu$ in his favourite frame, like the terrestrial one. The conversion requires us to know the ephemeris of the four probes, $P^\mu_{(\alpha)}(t)$, where $t$ is an arbitrary parameter that we take to be the time function in the terrestrial frame. Each ephemeris may be made explicit, along the worldline of the probe, as a function of its proper time, giving after conversion $P^\mu_{(\alpha)}(t) = \Pi^\mu_{(\alpha)}(s^\alpha)$. Then, we have to simply solve the system of four equations,

$$\left[X^0 - \Pi^0_{(\alpha)}(s^\alpha)\right]^2 = \left[X^1 - \Pi^1_{(\alpha)}(s^\alpha)\right]^2 + \left[X^2 - \Pi^2_{(\alpha)}(s^\alpha)\right]^2 + \left[X^3 - \Pi^3_{(\alpha)}(s^\alpha)\right]^2,$$

for the four values of $\alpha$. Here, $\Pi^\mu_{(\alpha)}(s^\alpha)$ are the functions derived from the ephemeris, and the unknown are the four $X^\mu$ which represent the coordinates of the observer in the desired frame.

Of course, the system becomes more complicated when there is curvature, but without problem to be solved numerically.

### 6. Conclusion and discussion

The GPS coordinates and corresponding frames offer a series of advantages. First, they are covariant quantities. It is not common in general relativity calculations to dispose of coordinates or frames with this property. This provides them an absolute and intrinsic character, which is however balanced by the fact that they depend on the choice of a set of probes.

Second, they are completely independent of the observer. This allows different observers, or an observer at different locations in spacetime, to compare directly different observational results and to interpret them, without involving different coordinates systems and frames.

We have shown how their use, with that of the associated frames and coframes, allows the complete reconstruction of the metric and, under some circumstances, of the curvature and connection.

In the previous analysis, we have considered four probes and an observer. But the treatment of the observer is perfectly identical to those of the probes (the velocity $V$ of the observer is analogous to $u$ of a probe; his proper time $\sigma$ is analogous to $s$ of the probe). Thus the observer can perfectly be a probe of the same type, i.e., a fifth satellite. Conversely, one of the probes can be chosen to be the terrestrial observer. In this case, one loses of course the independence w.r.t. the observer, but this case allows the easy reconstruction of a frame with more usual properties, involving timelike (for instance $V$) and spacelike vectors.

The possible applications are numerous. This allows not only the (relative) localization of the terrestrial observer, but also the monitoring of its motion (velocity and acceleration), as well as the reconstruction of the metric, curvature and connection components. The latter possibilities provide an opportunity to combine the different relations established above (which remain valid in any metric theory, not only in general relativity) to provide checks of the gravitation theories.
As we have remarked, the properties of a system of pulsars show many similarities with those of a GPS system. This allows us to use such a system for precise astrometry in the general relativistic context (see for instance the discussions in [9]).

Finally, the situation also applies to cosmography. Although the metric of a Friedmann–Lemaître universe model is generally put in the Robertson–Walker form, it is perfectly possible to express it by using GPS coordinates defined from a set of four distant galaxies, which may be assumed (or not) to be in free fall. This is feasible in different manners, which correspond to different choices of probes.

More generally, any metric can be expressed in a GPS frame. This can be done directly by searching for the coefficients expressing the change of frames. More easily, this could result from the choice of a set of convenient probes, from which the coordinates and frames are defined as above.

The exploration of such possibilities, for systems of artificial satellites, for cosmology and for the Schwarzschild metric, is in progress.

Acknowledgments

I thank B Coll and J M Pozo for useful comments on an earlier version of this paper.

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