A Besicovitch set in $AG(n, q)$ is a set of points containing a line in every direction. The Kakeya problem is to determine the minimal size of such a set. We solve the Kakeya problem in the plane, and substantially improve the known bounds for $n > 4$.

1. Introduction

We denote by $\pi_q$ the projective plane $PG(2, q)$ over the Galois field $GF(q)$ with $q$ elements, $q > 2$ a prime power.

Let $\ell$ be a line in $\pi_q$ and, for every point $P$ on $\ell$, let $\ell_P$ be a line on $P$ other than $\ell$. The set

\[ K = \left( \bigcup_{P \in \ell} \ell_P \right) \setminus \ell \]

is called a Kakeya set, or a minimal Besicovitch set. The finite plane Kakeya problem asks for the smallest size $k(q)$ of a Kakeya set; it is the two-dimensional version of the finite field Kakeya problem posed by T. Wolff in his influential paper [11] of 1996.

In the following, unless explicitly mentioned otherwise, we will use the same notation of (1) for the lines defining a Kakeya set $K$.

Let $\Omega$ be a set of $q + 2$ points in $\pi_q$. A point $P \in \Omega$ is said to be an internal nucleus of $\Omega$ if every line through $P$ meets $\Omega$ in exactly one other point. Internal nuclei of $(q + 2)$—sets were first considered by A. Bichara and G. Korchnáros in [1]; here they proved the following result.
Proposition 1 (1982). Let $q$ be an odd prime-power. Every set of $q + 2$ points in $\pi_q$ has at most two internal nuclei.

The $q + 2$ lines defining a Kakeya set in $\pi_q$ can be viewed as a set of $q + 2$ points with an internal nucleus in the dual plane $\pi_q^*$. More precisely, if $K$ is a Kakeya set in $\pi_q$, the lines $\ell$ and $\ell_P$, $P \in \ell$, give rise in $\pi_q^*$ to a set $\Omega(K)$ of $q + 2$ points with $\ell$ as an internal nucleus. Vice versa, every set of $q + 2$ points with an internal nucleus in $\pi_q$ defines in an obvious way a Kakeya set in $\pi_q^*$. Thanks to this duality, the finite plane Kakeya problem is equivalent to ask for the smallest number $k^*(q)$ of lines in $\pi_q$ meeting a set of $q + 2$ points with an internal nucleus; to be precise, we have

$$k^*(q) = 1 + q + k(q).$$

2. Old and New Results in the Plane

Let us start by recalling that the first author and A. A. Bruen studied in [2] the smallest number of lines intersecting a set of $q + 2$ points in $\pi_q$; here no assumption on the existence of internal nuclei is made. Nevertheless the dual of the theorem 1.3 of [2] contains the following result as a special case.

Proposition 2 (1989). If $q \geq 7$ is odd, then

$$|K| \geq \frac{q(q + 1)}{2} + \frac{q + 2}{3},$$

for every Kakeya set $K$.

Example 1. Assume $q$ is even and consider in $\pi_q$ a dual hyperoval $\mathcal{H}$, i.e. a $(q + 2)$—set of lines, no three of which are concurrent. Fix a line $\ell \in \mathcal{H}$ and, for every point $P \in \ell$, let $\ell_P$ the line of $\mathcal{H}$ on $P$ other than $\ell$. Then the Kakeya set

$$K(\mathcal{H}, \ell) = \left( \bigcup_{P \in \ell} \ell_P \right) \setminus \ell$$

is said to be associated to $\mathcal{H}$ and $\ell$ and it is of size

$$|K(\mathcal{H}, \ell)| = \frac{q(q + 1)}{2}. \quad \blacksquare$$