Moyal product for \((n-1)\)-forms within the covariant Hamiltonian formalism for fields

Jasel Berra–Montiel\(^1\), Alberto Molgado\(^{1,2}\) and David Serrano-Blanco\(^1\)

\(^1\) Facultad de Ciencias, Universidad Autonoma de San Luis Potosi
Av. Salvador Nava S/N Zona Universitaria, San Luis Potosi, SLP, 78290 Mexico
\(^2\) Dual CP Institute of High Energy Physics, Mexico
E-mail: jberra@fc.uaslp.mx, molgado@fc.uaslp.mx, davidaerrano@fc.uaslp.mx

Abstract. In these brief notes we analyze an associative product for \((n-1)\)-forms in the polymomentum phase space which is necessary in order to complete the deformation quantization scheme within the context of the multisymplectic formalism. In particular, we identify certain conditions that the proposed product must follow to generate a Moyal bracket induced from the Gerstenhaber bracket naturally emerging within this context.

1. Introduction
The process in which a quantum theory is constructed starts in most cases as a classical system described in configuration space and then taken to its phase-space formulation. This map requires a privileged direction in the physical space to be chosen and singled out from the other coordinates in a process known as foliation of spacetime. In a physical theory this direction is often chosen as the time coordinate. A problem arises though, if one considers the principle of general covariance which states that there cannot be a privileged direction of the physical space in which a relativistic theory is described. Conventional quantization schemes based on a Hamiltonian description of a theory depend on the election of time as a special coordinate. There is, however, a less conventional formalism to get the Hamiltonian picture from the Lagrangian scheme in a completely covariant way proposed more than half century ago [1]. Indeed, while the Hamiltonian formalism possesses an intrinsic geometric symplectic structure, the covariant Hamiltonian scheme requires a generalization of this framework, commonly referred to as the multisymplectic structure, which emerges by modifying the Legendre transformation when requiring it to treat all directions of spacetime in the same way. The main purpose of this paper is to analyze the possibility of applying the scheme of deformation quantization [2] to the polysymplectic formalism. In particular we aim to study the promotion of the Gerstenhaber bracket, naturally emerging in the polysymplectic formalism, to a Moyal bracket. Our work thus focuses precisely
in characterizing an associative product for \((n - 1)\)-forms, commonly associated to observables within the polysymplectic formalism.

The rest of this work is presented as follows: In section 2 we briefly review the polysymplectic approach to the De Donder-Weyl formalism for classical field theory [3, 4]. Let us start by considering a fiber bundle \((E, \pi, M)\), with \(M\) an \(n\)-dimensional smooth manifold with local coordinates \(\{x^\mu\}_{\mu=1}^n\), \(E\) a smooth manifold of dimension \(m\) with local coordinate functions \(\{y^a\}_{a=1}^m\) and \(\pi : E \to M\) the standard projection map. Let \(\phi : M \to E\) be a local section around a point \(p \in M\), so that its local components are given by \(\phi^a := y^a \circ \phi\). We will denote the space of local sections of \(\pi\) around the point \(p\) as \(\Gamma_p(\pi)\). Now, let us define the configuration space as the first jet manifold \(J^1E\) of \(\pi\), whose elements \(j^1_\phi \in J^1E\), for each point \(p\), and every local section \(\phi\), have the local representation \(\{x^\mu, \phi^a, \pi_\mu\}\), where \(\phi_\mu^a := \partial \phi^a / \partial x^\mu\) are commonly known as the derivative coordinates [3, 5].

The polysymplectic formalism consists of some concluding remarks. In Section 3 we analyze the polysymplectic structure for \((n - 1)\)-Hamiltonian forms and propose a Moyal bracket for them. Finally, Section 4 consists of some concluding remarks.

2. Polysymplectic Geometry

In this section we succinctly review the multisymplectic approach to the De Donder-Weyl formalism for classical field theory [3, 4]. Let us start by considering a fiber bundle \((E, \pi, M)\), with \(M\) an \(n\)-dimensional smooth manifold with local coordinates \(\{x^\mu\}_{\mu=1}^n\), \(E\) a smooth manifold of dimension \(m\) with local coordinate functions \(\{y^a\}_{a=1}^m\) and \(\pi : E \to M\) the standard projection map. Let \(\phi : M \to E\) be a local section around a point \(p \in M\), so that its local components are given by \(\phi^a := y^a \circ \phi\). We will denote the space of local sections of \(\pi\) around the point \(p\) as \(\Gamma_p(\pi)\). Now, let us define the configuration space as the first jet manifold \(J^1E\) of \(\pi\), whose elements \(j^1_\phi \in J^1E\), for each point \(p\), and every local section \(\phi\), have the local representation \(\{x^\mu, \phi^a, \pi_\mu\}\), where \(\phi_\mu^a := \partial \phi^a / \partial x^\mu\) are commonly known as the derivative coordinates [3, 5]. The behaviour of a physical system is encoded by the action functional \(S\) of \(\{x^\mu, \phi^a, \pi_\mu\}\), which is an appropriate submanifold of \(M\) on which the integration is performed. Let us define the quantities \(\pi_\mu^a := \partial S / \partial \phi_\mu^a\), called the polymomenta. The triple \(\{x^\mu, \phi^a, \pi_\mu^a\}\) thus defines a local coordinate system for a new manifold \(\mathcal{P}\), known in this context as the {poly}momentum phase-space. The Lagrangian density \(L\) is associated to a smooth function \(H_{DW}\) in \(\mathcal{P}\) by the map \(\mathcal{L} : C^\infty(J^1E) \to C^\infty(\mathcal{P})\), called the covariant Legendre transformation, such that \(H_{DW}(x^\mu, \phi^a, \pi_\mu^a) := \mathcal{L}(x^\mu, \phi^a, \phi_\mu^a)\). The symbol \(\partial_\mu\) stands for the derivative with respect to the base space coordinates \(x^\mu\).

As is well known, the polymomentum phase-space \(\mathcal{P}\) is also endowed with a canonical \(n\)-form induced from the Lagrangian Cartan form \(\Theta_L\) with the local representation \(\Theta_{DW} = \pi_\mu^a d\phi^a \wedge V_\mu - H_{DW} V\), known as the Poincaré-Cartan form [3, 6]. This local representation explicitly shows the decomposition of \(\Theta_{DW}\) into two terms, its vertical and horizontal parts, respectively. As it has been shown before, the behavior of a physical system may be encoded only in the vertical term of the Poincaré-Cartan form [3, 6].

Now, given an arbitrary \(p\)-form in the polymomentum phase-space, namely, \(\tilde{F} = \frac{1}{p!} F_{M_1 \cdots M_p} dz^{M_1} \wedge \cdots \wedge dz^{M_p}\), where \(z^M := (x^\mu, \phi^a, \pi_\mu^a)\) denotes the basis of \(\mathcal{P}\), then we define its vertical derivative by \(d^V \tilde{F} := (1/p!) \partial_\mu F_{M_1 \cdots M_p} dz^v \wedge dz^{M_1} \wedge \cdots \wedge dz^{M_p}\), where \(z^v := (\phi^a, \pi_\mu^a)\) stands for the vertical coordinates in \(\mathcal{P}\). In this way, taking the vertical derivative of \(\Theta_{DW}\) we may define \(\Omega_{DW} := d^V \Theta_{DW} = d\pi_\mu^a \wedge d\phi^a \wedge V_\mu\), commonly referred to as the polysymplectic form [4]. This \((n + 1)\)-form defines an analogue of the usual symplectic structure in the polymomentum phase-space. We will call \(\tilde{X}\) a Hamiltonian multivector field of degree \(p\), if there exists a unique horizontal form \(\tilde{F}\) satisfying \(\tilde{X} \lhd \Omega_{DW} = d^{\tilde{V}} \tilde{F}\). Thus \(d^{\tilde{V}} \tilde{F}\) is called the Hamiltonian form associated
to the Hamiltonian multivector field \( \vec{X} \). Let \( \vec{F} \), and \( \vec{G} \) be \( p \) and \( q \)-Hamiltonian forms, respectively, and let \( X_F \) and \( X_G \), their associated Hamiltonian fields. The map given by
\[
\{ [ \vec{F}, \vec{G} ] \} := (-1)^{p+q} X_F \wedge X_G \wedge \Omega, \tag{1}
\]
is called the Gerstenhaber Bracket (GB) \([4, 7]\). This bracket structure results graded-commutative, and satisfies Jacobi identity and Leibniz rule under the co-exterior product, \( \bullet \), given by \( \vec{F} \bullet \vec{G} := \ast^{-1}(\ast \vec{F} \wedge \ast \vec{G}) \), with \( \ast \) being the Hodge dual defined over \( M \), and thus it is a graded-Poisson bracket. In particular, one may show that the GB closes for \((n-1)\)-forms. Besides, this co-exterior product induces a derivative operator called the total co-exterior differential over horizontal forms \([4]\). Hence, the co-exterior differential of a Hamiltonian form \( \vec{F} \) can be written as \( d \bullet \vec{F} = \{ [ H_{D\Omega}, \vec{F} ] \} + d^h \bullet \vec{F} \). Finally, using the canonical commutation relations, \( \{ [ \pi^a_{\mu} \phi^b, \phi^b ] = \delta_{\mu}^a, \{ [ \pi^a_{\mu} \phi^b, \phi^b \phi^b ] = 0, \{ [ \pi^a_{\mu}, \phi^b \phi^b ] = 0 \) \}, we obtain the De Donder-Weyl equations of motion \([3]\)
\[
\partial_{\mu} \phi^a = \{ [ H_{D\Omega}, \phi^a ] \} = \frac{\partial H_{D\Omega}}{\partial \pi^a_{\mu}}, \quad \partial_{\mu} \pi^a_{\mu} = \{ [ H_{D\Omega}, \pi^a_{\mu} ] \} = -\frac{\partial H_{D\Omega}}{\partial \phi^a}. \tag{2}
\]

It is worth noting that whenever we specifically consider \( M = \mathbb{R} \) and \( E = \mathbb{R}^n \), then all the above formalism reduces correspondingly to both standard Lagrangian and Hamiltonian formalisms for particle mechanics.

3. Moyal star-product in the polymomentum phase space

Our main aim is to apply the method of deformation quantization to the multisymplectic framework. As it is well known, the Moyal star-product in the standard symplectic case can be written in a differential representation \([2]\) as \( f * g = f \exp \left( \frac{i \hbar}{2} \left( \partial_{q} \partial_{p} - \partial_{p} \partial_{q} \right) \right) g \), where the arrows point to the function they act on. Also, within this context, we may define the Moyal bracket as \( \{ [ f, g ] \} := ( f * g - g * f ) \). This structure related to the Poisson bracket within the phase-space (together with Weyl and Wigner transforms) constitute the basic elements of the deformation quantization scheme \([8]\). Thus, the first step towards a deformation quantization scheme based on the DW-Hamiltonian, is to construct an analogue star-product to this Moyal star-product. In particular, as \((n-1)\)-forms play a privileged role under the GB structure \([1]\) we will focus on them. Therefore, in local coordinates the GB may be found by means of the product \( X_F \wedge X_G \wedge \Omega \), which explicitly reads
\[
\{ [ X_F, X_G ] \} = \sum_{\mu = 1}^{n-1} F_\mu ( \Delta^\mu _{\hat{\lambda}} \partial_\mu \phi^b_{\hat{\lambda}} - \partial_\mu \phi^b \Delta^\mu _{\hat{\lambda}} ) G^{\n-1}, \tag{3}
\]
where the operator \( \Delta^\mu _{\hat{\lambda}} := (1/(n-1)!)(\partial_{\mu_1} \cdots \partial_{\mu_{n-1}}) \partial_{\mu_{n-1}} \cdots \partial_{\mu_1} \), stands for the horizontal dual element to \( V_\mu \) (that is, \( \Delta^\mu _{\hat{\lambda}} \wedge V_\mu = \delta^\mu _{\hat{\lambda}} \)). This operator acts by appropriate contraction, either \( \Delta^\mu _{\hat{\lambda}} F_\mu = \Delta^\mu _{\hat{\lambda}} \wedge F_\mu \) or \( F_\mu \Delta^\mu _{\hat{\lambda}} = \Delta^\mu _{\hat{\lambda}} \wedge F_\mu \). In this way, by defining the differential operator \( D := \Delta^\mu _{\hat{\lambda}} \partial_\mu \phi^b - \partial_\mu \phi^b \Delta^\mu _{\hat{\lambda}} \), the GB \([3]\) can be expressed as \( F D G \). Next, we note that if we want to construct a second order differential operator from \([3]\) by bare composition of the operator \( D \), we would expect terms involving \( \Delta^\mu \hat{\Delta}^\mu \), and
so on, however, once acting over an \((n-1)\)-form, this would yield terms as \(\tilde{\Delta}^a F^{\mu \nu}\), which are not defined over the space of horizontal forms any more. To deal with this inconvenience, we need to define a product between these operators so that the degree is maintained. We start by assuming the existence of this product, and let us denote it by \(\circ\). Thus, we define the operator \(D^2 := D \circ D\), so that:

\[
\tilde{\Delta}^a F^{\mu \nu} = F(D^2 G) = F(\tilde{\Delta}^a \tilde{\Delta}^b \nabla^\sigma \nabla^\mu \nabla^\nu \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \partial_\epsilon) G .
\]

It is clear from this expression that we need to define separate cases for the product \(\circ\), namely, \(\tilde{\Delta}^a \circ \tilde{\Delta}^b := \tilde{\Delta}^c \lambda^{ab}_c\), \(\tilde{\Delta}^a \circ \tilde{\Delta}^b := \tilde{\Delta}^c \lambda^{ab}_c\), and \(\tilde{\Delta}^a \circ \tilde{\Delta}^b := \frac{1}{2} \lambda^{ab}_c (\tilde{\Delta}^c + \tilde{\Delta}^c)\), where \(\lambda^{ab}_c\) are constants to be fixed, and such that the result in every term of \(\tilde{\Delta}^a F^{\mu \nu} G\) is always an \((n-1)\)-form. With this definitions, the previous result is then simply written as:

\[
\tilde{\Delta}^a F^{\mu \nu} = \lambda^{ab}_c (\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \partial_\epsilon) G .
\]

Here, for simplicity, we used the definitions \(\tilde{\Delta}^a := (1/(n-1)!)^{\epsilon_{\nu_1 \cdots \nu_n}} F^\nu_{\nu_1 \cdots \nu_n}\), where \(\epsilon\) and \(F_{\nu_1 \cdots \nu_n}\) stand for the alternating symbol and the components of the \((n-1)\)-form \(F\), respectively, and also \(V_\mu := \partial_\mu \nu \circ V\). Noticing that the derivatives with respect to the polynomials are calculated, we need to impose the condition that the constants \(\lambda^{ab}_c\) are symmetric in the upper indices, making the product \(\circ\), a commutative operation. This last expression yields an \((n-1)\)-form, and furthermore, it is clear that repeated applications of \(D\) also result in an \((n-1)\)-form. We take advantage of the arbitrariness of the constants \(\lambda^{ab}_c\) to make this map obey Leibniz product rule under the GB. We see that a sufficient requirement for the \(\circ\) product to obey Leibniz product rule is the identity \(T^\mu_{\nu} \lambda^{\nu \alpha \beta} \nabla^\alpha \nabla^\beta = T^\mu_{\nu} \lambda^{\nu \alpha \beta} \nabla^\alpha \nabla^\beta\), for an arbitrary quantity \(T^\mu_{\nu}\). This imposition on the constants also ensures that the \(\circ\) product is associative. Having constructed an associative product between \((n-1)\)-forms, we can now define the action of \(D^k\) when \(k = 0\) as:

\[
\tilde{\Delta}^a \circ \tilde{\Delta}^b := \tilde{\Delta}^c \lambda^{ab}_c F^{\mu \nu} G .
\]

4. Conclusions
The deformation quantization scheme may provide a feasible way to obtain the quantum analogues of the DW equations. In this note we analyzed the characteristics that an associative product for \((n-1)\) forms may have in order to introduce a deformation quantization for them. As discussed, we can define the classical observables as the Hamiltonian \((n-1)\)-forms. This election can be justified by the fact that the Hamiltonian forms of this particular degree actually close under the GB. However, up until now, a closed and associative product between \((n-1)\)-forms has not been found. The proposed product was constructed in order to fit the two requirements needed to give a GB-compatible algebra which can be deformed to obtain a Moyal star-product. This led to
define the $\circ$ product of two $(n-1)$-forms as the $(n-1)$-form whose components are linear combinations of the product of the original component functions.

We found two main disadvantages in the definition of the $\circ$ product. The first is the possible lack of a geometrical meaning as this product between two $(n-1)$-forms gives no immediate geometric information. Nevertheless, the $\circ$ product could contain an intrinsic geometric structure given by the operators $\vec{\Delta}_{\mu}$ from which it has been defined as they could be associated to projection operators over the space of $(n-1)$-forms. The other disadvantage concerning the $\circ$ product is the condition that the constants $\lambda_{\sigma \mu \nu}$ must satisfy. In this sense, the interpretation of $\vec{\Delta}_{\mu}$ as projection operators could bring additional support to our choice. As a projection operator is idempotent by definition, this is $\vec{\Delta}^2 = \vec{\Delta}$, the values of the $\lambda$ constants must reduce to a generalization of a Kronecker delta, yielding an idempotent operator. The product we have proposed in these notes may be the first step towards a concrete algebraic structure needed for a deformation quantization of structures within the multisymplectic approach. Having obtained a Moyal product defined in polymomentum phase-space, the application of a deformation quantization scheme to concrete physical models can now be considered. Besides standard examples, such as scalar field and electrodynamics, the finite dimensional nature of the polymomentum phase-space makes reparametrization invariant field theories, such as gravity, perfect candidates to test our results on. A complete geometric characterization of this product is under progress, and will be addressed elsewhere.

Acknowledgments

AM acknowledges financial support from CONACYT-Mexico under project CB-2014-243433.

References

[1] T. De Donder, *Théorie Invariantive du Calcul des Variations* (Gauthier-Villars, Paris, 1935); H. Weyl, Ann. Math. (2) **36** 607–629 (1935); C. Carathéodory, Acta Sci. Math. (Szeged) **4** 193–216 (1929).
[2] L. A. Takhtajan, *Quantum Mechanics for Mathematicians*, Graduate Studies in Mathematics, Vol. 95 (American Mathematical Society, 2008); G. Esposito, G. Marmo, G. Sudarshan, *From Classical to Quantum Mechanics* (Cambridge University Press, 2004); C. K. Zachos, D. B. Fairlie and T. L. Curtright, *Quantum Mechanics on Phase Space* (World Scientific, 2005).
[3] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *New Lagrangian and Hamiltonian methods in field theory* (World Scientific, 1997); M. J. Gotay, *A multisymplectic framework for classical field theory and the calculus of variations I. Covariant Hamiltonian formalism*, in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, M. Francaviglia, ed. (North Holland, Amsterdam, 1991) pp. 203–235; H. A. Kastrup, Phys. Rep. **101** 1–167 (1983).
[4] I. V. Kanatchikov, Aachen preprint PTHA 93/41 (1993), arXiv:hep-th/9312162; Rep. Math. Phys. **40** 225–234 (1997), arXiv:hep-th/9710069; Phys. Lett. **A283** 25–36 (2001), arXiv:hep-th/0012084; Rep. Math. Phys. **43** 157–70 (1999), arXiv:hep-th/9810165; In *Current Topics in Mathematical Cosmology* (Proc. Int. Seminar, Potsdam, Germany, 1998) M. Rainer and H.-J. Schmidt eds. (World Scientific) pp. 457–67, arXiv:gr-qc/9810076.
[5] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society lecture notes series Vol. 142 (Cambridge University Press, 1989).
[6] M. Forger and L. G. Gomes, Rev. Math. Phys. **25** 1350018 (2013), arXiv:0708.1586.
[7] F. Hélein and J. Kouneiher, Adv. Theor. Math. Phys. **8** 735–777 (2004), arXiv:math-ph/0401047.
[8] C. Esposito, *Lectures on Deformation Quantization of Poisson Manifolds* (2012), arXiv:1207.3287v2 [math-ph].