THE FIRST $L^p$-COHOMOLOGY OF SOME GROUPS WITH ONE END

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ABSTRACT. Let $p$ be a real number greater than one. In this paper we study the vanishing and nonvanishing of the first $L^p$-cohomology space of some groups that have one end. We also make a connection between the first $L^p$-cohomology space and the Floyd boundary of the Cayley graph of a group. We apply the result about Floyd boundaries to show that there exists a real number $p$ such that the first $L^p$-cohomology space of a nonelementary hyperbolic group does not vanish.

1. Introduction

In this paper $G$ will always be a finitely generated infinite group with identity 1 and symmetric generating set $S$. Let $\mathcal{F}(G)$ denote the set of all real-valued functions on $G$. Let $1 \leq p \in \mathbb{R}$ and set

$$D^p(G) = \{ f \in \mathcal{F}(G) \mid \sum_{g \in G} |f(gs^{-1}) - f(g)|^p < \infty \text{ for all } s \in S \}.$$ 

The set $D^p(G)$ is known as the set of $p$-Dirichlet finite functions on $G$. Observe that the constant functions are in $D^p(G)$. We define a norm on $D^p(G)$ by

$$\| f \|_{D^p(G)} = \left( \sum_{s \in S} \sum_{g \in G} |f(gs^{-1}) - f(g)|^p + |f(1)|^p \right)^{1/p}.$$ 

Under this norm $D^p(G)$ is a Banach space. We now define an equivalence relation on $D^p(G)$ by $f_1 \sim f_2$ if and only if $f_1 - f_2$ is a constant function. Identify the constant functions by $\mathbb{R}$. Now $D^p(G)/\mathbb{R}$ is a Banach space under the norm induced from $D^p(G)$. That is, if $[f]$ is an equivalence class from $D^p(G)/\mathbb{R}$ then

$$\| [f] \|_{D^p(G)/\mathbb{R}} = \left( \sum_{s \in S} \sum_{g \in G} |f(gs^{-1}) - f(g)|^p \right)^{1/p}.$$ 

We shall write $\| f \|_{D^p(G)}$ for $\| [f] \|_{D^p(G)/\mathbb{R}}$. The norm for $D^p(G)$ and $D^p(G)/\mathbb{R}$ depends on the symmetric generating set $S$, but the underlying topology does not. If $A \subseteq D^p(G)$, then $\overline{A}_{D^p(G)}$ will denote the closure of $A$ in $D^p(G)$. Similarly if $B \subseteq D^p(G)/\mathbb{R}$, then $\overline{B}_{D^p(G)/\mathbb{R}}$ will denote the closure of $B$ in $D^p(G)/\mathbb{R}$. Let $L^p(G)$ be the set that consists of functions on $G$ for which $\sum_{g \in G} |f(g)|^p$ is finite. Observe

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that $L^p(G)$ is contained in $D^p(G)/\mathbb{R}$. The main object of study in this paper is the space

$$\tilde{H}_1^p(G) = D^p(G)/(L^p(G) \oplus \mathbb{R})_{D(p)}.$$ 

The space $\tilde{H}_1^p(G)$ is known as the first reduced $L^p$-cohomology space of $G$. This paper was inspired by the paper [2].

It is well known that if $G$ has two ends then $\tilde{H}_1^p(G) = 0$ for $1 < p \in \mathbb{R}$. It is also well known that if $G$ has infinitely many ends then $\tilde{H}_1^p(G) \neq 0$ for $1 \leq p \in \mathbb{R}$, see [14, Corollary 4.3] for a proof. A reasonable question to ask is: What can we say about $\tilde{H}_1^p(G)$ if $G$ has one end? It was shown in [14, Corollary 3.6] that if $G$ has polynomial growth, then $\tilde{H}_1^p(G) = 0$ for $1 < p \in \mathbb{R}$. In [1, Theorem 2] it was shown that if $G$ is a properly discontinuous subgroup of isometries of a proper CAT($-1$) space with finite critical exponent and if the limit set of $G$ has at least three points, then $\tilde{H}_1^p(G) \neq 0$ for $p > \max\{1, \text{critical exponent of } G\}$. Another result concerning groups with one end was given in [13] where it was shown that if $G$ is a co-compact lattice in $Sp(n,1)$, then $\tilde{H}_1^p(G) \neq 0$ exactly for $p > 4n + 2$.

Before we state our first result we need to define what it means for a Riemannian manifold to be rotationally symmetric. Let $M_n$ be a simply connected, $n$-dimensional Riemannian manifold with all sectional curvatures bounded above by a negative constant. Now fix a point on $M_n$ and use the exponential map at this point to transfer the polar coordinates on $R^n$ to the manifold. So the Riemannian metric on $M_n$ can be written as $dx^2 = dr^2 + f(r)^2d\theta^2$ where $d\theta^2$ is the usual metric on the unit sphere $S^{n-1}, n \geq 2$. If the submanifolds $r = k$, where $k$ is a constant, are spheres of constant curvature then we shall say that $M_n$ is rotationally symmetric. In this paper we will prove:

**Theorem 1.1.** Let $M_n$ be a complete, simply connected, $n$-dimensional Riemannian manifold with all sectional curvatures bounded above by a negative constant. Furthermore assume that $M_n$ is rotationally symmetric. Suppose that $G$ acts properly discontinuously on $M_n$ by isometries and that the action is cocompact and free. Then for $1 < p \leq n - 1, \tilde{H}_1^p(G) = 0$.

What happens if $p > n - 1$? Let $H^n$ denote hyperbolic $n$-space. By combining Theorem 2 of [1] with Theorem 1.6.1 of Nicholls [12] we obtain the following:

**Theorem 1.2.** Suppose that $G$ acts properly discontinuously on $H^n$ by isometries and that the action is cocompact and free. If the limit set of $G$ has at least three points, then $\tilde{H}_1^p(G) \neq 0$ for $p > n - 1$.

One of the hypothesis for Theorem 1.2 is that the limit set of $G$, which is a subset of the $(n - 1)$-dimensional unit sphere, contain at least three points. Thus a possible first step in trying to determine whether $\tilde{H}_1^p(G)$ vanishes or does not vanish for groups with one end is to use a boundary for $G$ that is finer than the end boundary. One such boundary is the Floyd boundary. In Section 3 we will prove

**Theorem 1.3.** Let $G$ be a finitely generated group and let $F$ be a Floyd admissible function on $G$. If the Floyd boundary of $G$ with respect to $F$ is nontrivial and if $\sum_{g \in G}(F(|g|)) < \infty$, then $\tilde{H}_1^p(G) \neq 0$.

All concepts in Theorem 1.3 that are unfamiliar to the reader will be explained in Section 3. We will conclude Section 3 by proving the following consequence, which appears to be known to Gromov (see pages 257-258 of [4]), of Theorem 1.3.
Corollary 1.4. Let $G$ be a nonelementary hyperbolic group, then there exists a real number $p$ such that $\tilde{H}^1_{(p)}(G) \neq 0$.

Let $f$ be an element of $\mathcal{F}(G)$ and let $g \in G$. Let $1 < p \in \mathbb{R}$ and define

$$(\Delta_p f)(g) := \sum_{s \in S} |f(gs^{-1}) - f(g)|^{p-2}(f(gs^{-1}) - f(g)).$$

In the case $1 < p < 2$, we make the convention that $|f(gs^{-1}) - f(g)|^p = (f(gs^{-1}) - f(g))^p$.

Let $f \in \mathcal{F}(G)$ and let $g \in G$. Let $H^D_{(p)}(G)$ be the set of $p$-harmonic functions on $G$.

Theorem 1.5. Let $1 < p \in \mathbb{R}$ and suppose $(L^p(G))_{H^D_{(p)}(G)} \neq D^p(G)$. Then for $f \in D^p(G)$, we can write $f = u + h$, where $u \in (L^p(G))_{H^D_{(p)}(G)}$ and $h \in H^D_{(p)}(G)$. This decomposition is unique up to a constant function.

It follows from the theorem that each nonzero class in $\tilde{H}^1_{(p)}(G)$ can be represented by a nonconstant function from $H^D_{(p)}(G)$. This gives us the following:

Corollary 1.6. Let $G$ be a finitely generated group

1. If $G$ satisfies the hypothesis of Theorem 1.4, then $H^D_{(p)}(G) = \mathbb{R}$ for $1 < p \leq n - 1$.
2. If $G$ satisfies the hypothesis of Theorem 1.2 then $H^D_{(p)}(G)$ contains a nonconstant function for $p > n - 1$.
3. If $G$ is a nonelementary hyperbolic group, then there exist a real number $p$ such that $H^D_{(p)}(G)$ contains a nonconstant function.

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2. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. We will begin by giving some definitions and other preliminaries needed for the proof of the theorem. Let $g \in G$ and let $f \in \mathcal{F}(G)$. Convolution of $f$ by $g - 1$, denoted by $f * (g - 1)$, is the function $(f * (g - 1))(x) = f(xg^{-1}) - f(x)$ for $x \in G$. Observe that $f * (g - 1) \in L^p(G)$ when $f \in D^p(G)$. The right translation of $f$ by $g$ is the function defined by $f_g(x) = f(xg^{-1})$. We will denote by $C_0(G)$ the set of those $f \in \mathcal{F}(G)$ for which the set $\{g \mid |f(g)| > \epsilon \}$ is finite for each $\epsilon > 0$. The $L^p(G)$-norm for functions $f$ in $L^p(G)$ is denoted by $\|f\|_p$ and is given by $\|f\|_p^p = \sum_{g \in G} |f(g)|^p$.

Let $M_n$ be a $n$-dimensional Riemannian manifold that satisfies the hypothesis of Theorem 1.1. The action of $G$ on $M_n$ will be denoted by $xg^{-1}$, where $x \in M_n$ and $g \in G$. The space $L^p(M_n)$ will consist of all real-valued functions on $M_n$ for which $\int_{M_n} |f(x)|^pdx < \infty$, where $1 \leq p \in \mathbb{R}$. We now proceed to prove the theorem.
Suppose $\tilde{H}^1_{(p)}(G) \neq 0$ for some $p$ that satisfies $1 < p \leq n - 1$. Then by the remark following Theorem 11.3 there exists a nonconstant $p$-harmonic function $h$ in $HD^p(G)$ that represents a nonzero class in $\tilde{H}^1_{(p)}(G)$. Define an affine isometric action of $G$ on $L^p(G)$ by $gf = f_g + h \ast (g - 1)$. Let $(x_1, f_1)$ and $(x_2, f_2)$ be elements of the direct product $M_n \times L^p(G)$. We shall say that $(x_1, f_1)$ is related to $(x_2, f_2)$ if and only if there exists a $g \in G$ for which $x_2 = x_1 g^{-1}$ and $f_2 = g f_1$. It is an easy exercise to show that this relation is an equivalence relation. Denote the quotient space of this equivalence relation by $M_n \times_G L^p(G)$. We now have a fibre bundle $M_n \times_G L^p(G) \xrightarrow{\pi} M_n / G$, where $\pi$ denotes the projection map. Let $s$ be a smooth section of this bundle. Then $s(x) = (x, f) = (x g^{-1}, g f)$ where $\pi(x, f) = x$. We now define a smooth map $\hat{s} : M_n \rightarrow L^p(G)$ by $\hat{s}(x) = f$. Observe that $\hat{s}(x g^{-1}) = g \hat{s}(x)$ since $(x, f) = (x g^{-1}, g f)$. We now define a real-valued function on $M_n$ by $f(x) := (\hat{s}(x) + h)(1)$. If $g \in G$ and $x \in M_n$ then $f(x g^{-1}) = (g \hat{s}(x) + h)(1) = (\hat{s}(x) + h \ast (g - 1) + h)(1) = \hat{s}(x)(g^{-1}) + h(g^{-1}) = (\hat{s}(x) + h)(g^{-1})$. Since $df(x) = \hat{s}(x)(1)$ it now follows that $df(x g^{-1}) = \hat{s}(x)(g^{-1})$, where $df$ is the differential of $f$. Due to the compactness of $M_n / G$ there exists a constant $C$ such that $\sum_{g \in G} |\hat{s}(x)(g^{-1})|^p = \| \hat{s}(x) \|_p < C$ for all $x \in M_n$. Thus $\int_{M_n} |df(x)|^p dx = \int_{M_n/G} \sum_{g \in G} |\hat{s}(x)(g^{-1})|^p dV \leq C(\text{volume}(M_n/G))$. Hence $df \in L^p(M_n)$ by the compactness of $M_n / G$. Using the canonical identification of $df$ with $\nabla f$, page 160 of [3], where $\nabla f$ is the gradient of $f$, we see that $\nabla f \in L^p(M_n)$. By [3] Theorem 5.8 there exists a constant $c$ such that $f - c \in L^p(M_n)$. Thus $\int_{M_n/G} \sum_{g \in G} |(f - c)(x g^{-1})|^p dV = \int_{M_n} |(f - c)(x)|^p dx < \infty$. Hence, $\sum_{g \in G} |(f - c)(x g^{-1})|^p < \infty$ for a fixed $x \in M_n$. Consequently $\hat{s}(x) + h \ast c \in L^p(G)$. Thus the $p$-harmonic function $h - c \in L^p(G) \subseteq C_0(G)$ since $\hat{s}(x) \in L^p(G)$. Lemma 6.1 of [14] tells us that $h - c = 0$ on $G$, contradicting the fact that $h$ is nonconstant. Therefore, $\tilde{H}^1_{(p)}(G) = 0$ for $1 < p \leq n - 1$. This concludes the proof of Theorem 11.3

3. FLOYD BOUNDARIES

Let $(X, S)$ be the Cayley graph of $G$ with respect to the generating set $S$. Thus the vertices of $(X, S)$ are the elements of $G$, and $g_1, g_2 \in G$ are joined by an edge if and only if $g_1 = g_2 s^{\pm 1}$ for some generator $s$. For the rest of this paper we will denote $(X, S)$ by $X$. We can make $X$ into a metric space by assigning length one to each edge, and defining the distance $d_s(g, h)$ between any two vertices $g, h$ in $X$ to be the length of the shortest path between $g$ and $h$. The metric $d_s$ on $X$ is known as the word metric. For the rest of this paper we will drop the use of the subscript $s$ and $d(x, y)$ will always denote the distance between $x$ and $y$ in the word metric. We will denote $d(1, g)$ by $|g|$ for $g \in G$. If $A$ is a set of vertices from $X$, then $|A| = \inf_{a \in A} d(1, a)$. Let $F$ be a function from the natural numbers $\mathbb{N}$ into the positive real numbers $\mathbb{R}^+$. We shall say that $F$ is a Floyd admissible function if it is monotonically decreasing, summable and for which there is a positive constant $L$ that satisfies $F(n + 1) \leq F(n) \leq LF(n + 1)$ for all $n \in \mathbb{N}$. We will now show how to construct a Floyd boundary for $X$ with respect to $F$. First we use $F$ to define a new metric on $X$. The new length of an edge joining $g$ and $h$ is $F(|\{g, h\}|)$. Let $\alpha = \{g_i\}$ be a path in $X$. The length $L_F$ of $\alpha$ is given by $\sum_i F(|\{g_i, g_{i+1}\}|)$ and the new distance between $x$ and $y$ in $X$ is $d_F(x, y) := \inf_{\alpha} L_F(\alpha)$, where the infimum is taken over all paths $\alpha$ connecting $x$ and $y$. It is straight forward to verify that $d_F$ is a metric on $X$. Let $(\bar{X}^F, \bar{d}_F)$ denote the completion of $(X, d_F)$ in the sense
of metric spaces. The Floyd boundary of $X$ is the set $\partial_F X = \overline{X}^F \setminus X$. We shall say that $\partial_F X$ is trivial if it consists of only 0, 1 or 2 points. Lots of information about Floyd boundaries can be found in [9, 10, 11]. If $A$ is a set, then the cardinality of $A$ will be denoted by $\#(A)$.

We now prove Theorem 1.3. Let $f$ be a continuous functions from $\overline{X}^F$ into $\mathbb{R}$ that satisfies a Lipshitz condition. Thus

$$\sum_{g \in G} \sum_{s \in S} |f(gs^{-1}) - f(g)|^p \leq \sum_{g \in G} \sum_{s \in S} C(d_F(gs^{-1}, g))^p$$

$$\leq \sum_{g \in G} \sum_{s \in S} C(F(|g|))^p$$

$$= #(S)C \sum_{g \in G} (F(|g|))^p < \infty.$$ 

So $f$ restricted to $G$ is an element of $D^p(G)$. Let $\xi \in \partial X_F$. Define a real-valued function on $\overline{X}^F$ by $f(x) := d_F(x, \xi)$. Now $f \in D^p(G)$ since it satisfies a Lipshitz’s condition. Since $d_F$ is a metric $f$ is nonconstant on $\partial X_F$. By continuity of $f$ it follows that $f(|g|)$ does not tend towards a constant number as $|g|$ goes to infinity in $X$. Thus $f$ is not an element of $L^p(G) \oplus \mathbb{R}$. Also $G$ is nonamenable since $\partial X_F$ contains more than two points, [10 Corollary 2]. Hence $L^p(G) \oplus \mathbb{R}$ is closed in $D^p(G)$ [5, Corollary 1], also see [14 Theorem 4.1] for a proof. Therefore $f$ represents a nonzero class in $\overline{R}(G)$ and the proof of Theorem 1.3 is now complete.

We will now apply Theorem 1.3 to a class of hyperbolic groups. For the rest of this section assume that $G$ is a hyperbolic group with hyperbolic constant $\delta$. Let $X$ be the Cayley graph of $G$. The Gromov inner product with basepoint $1$ in $X$ is defined to be

$$(x \mid y) = \frac{1}{2}(||x|| + ||y|| - d(x, y))$$

where $x$ and $y$ are vertices in $X$. Let $(x_n)$ be a sequence in $X$. We shall say that $(x_n)$ converges to infinity if $\lim_{n,m \to \infty} (x_n \mid x_m) = \infty$. Let $S_\infty(X)$ be the set of all sequences on $X$ which converge to infinity. We shall also say that two sequences, $(x_n)$ and $(y_n)$ in $S_\infty(X)$ are related if and only if $\lim_{n \to \infty} (x_n \mid y_n) = \infty$. This relation is an equivalence relation since $G$ is hyperbolic. The sequential boundary of $G$, denoted by $\partial G$, is the set of equivalence classes of sequences under the above relation. A hyperbolic group is nonelementary if there are more than two elements in $\partial G$.

Choose $a > 0$ such that $e^{3a} - 1 < \sqrt{2} - 1$. Define a Floyd admissible function $F$ from $\mathbb{N}$ into $\mathbb{R}^+$ by $F(n) = e^{-an}$. Let $g \in G$ and $s \in S$. Then $g$ and $gs^{-1}$ are neighbors in $X$ and $F(|(g, gs^{-1})|) = e^{-a|g||gs^{-1}|}$. Thus for $x$ and $y$ in $X$,

$$d_F(x, y) = \inf\{\sum_{i=1}^{\infty} e^{-a|x_i|x_{i+1}} \mid n \geq 1, x = x_0, x_1, x_2, \ldots, x_n = y \in X\},$$

where $x_0, x_1, \ldots, x_n$ is a path from $x$ to $y$ in $X$. Now let $(x_n)$ and $(y_n)$ be sequences in $X$. By Proposition 22.8 of [16] we have the following inequality

$$(3 - 2e^{3a})e^{-a|x_n|y_n} \leq d_F(x_n, y_n) \leq e^{-a|x_n|y_n}.$$ 

Thus $\lim_{n \to \infty} d_F(x_n, y_n) = 0$ if and only if $\lim_{n \to \infty} (x_n \mid y_n) = \infty$. Hence, the cardinality of $\partial X_F$ equals the cardinality of $\partial G$. 
Since $G$ is finitely generated it has at most exponential growth. Consequently, there exists a real number $p$ such that
\[
\sum_{g \in G} |F(g)|^p = \sum_{g \in G} e^{-a|g|^p} \\
\leq 2k \sum_{n=1}^{\infty} (2k - 1)^{n-1} e^{-anp} \\
< \infty,
\]
where $2k$ is the cardinality of $S$. Now apply Theorem 1.3 to obtain Corollary 1.4.

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