Capacity Outer Bound and Degrees of Freedom of Wiener Phase Noise Channels with Oversampling

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Abstract—The discrete-time Wiener phase noise channel with an integrate-and-dump multi-sample receiver is studied. A novel outer bound on the capacity with an average input power constraint is derived as a function of the oversampling factor. This outer bound yields the degrees of freedom for the scenario in which the oversampling factor grows with the transmit power $P$ as $P^\alpha$. The result shows, perhaps surprisingly, that the largest pre-log that can be attained with phase modulation at high signal-to-noise ratio is at most $1/4$.

I. INTRODUCTION

In the discrete-time Wiener phase noise (WPN) channel, the channel input is affected by both additive white Gaussian noise (AWGN) and multiplicative WPN. The Wiener phase process can be used to model a number of random phenomena: from imperfections in the oscillator circuits at the transceivers, to slow fading effects in wireless environments or laser imperfections in optical communications. For the WPN channel, the sampled output of the filter matched to the transmit filter does not always represent a sufficient statistic [1], and oversampling does help in achieving higher rates over the continuous-time channel [2], [3]. For this reason, it is of interest to study the effect of oversampling on the maximum achievable rates [4]. In this paper, we study the discrete-time Wiener phase noise with oversampling channel, a model obtained by sampling the output of the continuous-time phase noise channel faster than the symbol frequency. For this model, we determine a novel outer bound on capacity and provide the generalized degrees of freedom (GDoF) for the scenario in which the oversampling factor grows to infinity as $P^\alpha$ where $P$ is the transmit power. This result shows that, even in the high signal-to-noise ratio (SNR) regime, it is not possible to attain more than a pre-log factor of $1/4$ through phase modulation.

State of the Art: The literature on channel affected by both additive noise and phase noise considers three models: (i) the continuous-time model, (ii) the discrete-time model and (iii) the discrete-time model with oversampling.

For the continuous-time case, the joint effect of phase noise and additive white Gaussian noise is first considered in [5]. In [7], the authors investigate white (Gaussian) phase noise for which they observe a “spectral loss” phenomenon induced by white phase noise. The continuous-time channel in the presence of white noise is proposed and discussed in [11]. Here it is shown that, for linear modulation, the output of the baud-sampled filter matched to the shaping waveform represents a sufficient statistic. Bounds on the SNR penalty for the case of Wiener phase noise affecting the channel input are developed in [7].

The discrete-time phase noise channel is obtained by considering a discrete-time WPN process sampled at symbol frequency.

This model was first studied in [8]; here the high SNR capacity is derived using duality arguments. The authors of [9] propose a numerical method of precise evaluation of information rate bounds for this model. In [10] the authors derive closed-form approximations to capacity which are shown to be tight through numerical evaluations.

Finally, in the discrete-time model with oversampling multiple samples for every input symbol are obtained in output. This model was first considered in [11] where it is shown that, if the number of samples per symbol grows with the square root of the SNR, the capacity pre-log is at least $3/4$. The result in [11] is extended in [12] to consider all scaling of the oversampling coefficient of the form $P^\alpha$. Further simulations to compute lower bounds on the information rates achieved by the multi-sample receiver have been recently shown in [13], [14].

Contribution: In this paper, we investigate the capacity of the point-to-point channel corrupted by Wiener phase noise and additive white Gaussian noise with an integrate-and-dump multi-sample receiver, which we refer to as oversampled Wiener phase noise (OWPN) channel. Our main contributions are described as follows:

- **Sec. IV-Capacity outer bound**: Using the I-MMSE relation [15] and a lower bound on the minimum mean-square error (MMSE) estimate expressed through a recursive equation [16], we obtain a novel outer bound on the capacity of the OWPN channel.

- **Sec. V-Degrees of Freedom**: The outer bound in Sec. IV is shown to be tight at high SNR; more specifically we derive the GDoF for the model in which the transmit power $P$, and the oversampling factor $L$, grow large for $L = [P^\alpha]$. 

Paper Organization: The channel model is presented in Sec. II while the known results in the literature are presented in Sec. III. Outer bounds are derived in Sec. IV while the degrees of freedom analysis is shown in Sec. V. Conclusions are drawn in Sec. VI.
II. SYSTEM MODEL

We consider the OWPN, that is the point-to-point channel corrupted by Wiener phase noise and additive white Gaussian noise with an integrate-and-dump multi-sample receiver. The main assumptions are as in [12], [17], that is (i) the amplitude fading component, obtained by low-pass filtering the continuous time Wiener process, is neglected by setting it to one, and (ii) the complex envelope of the transmitted waveform is constant in each symbol time interval. Under these assumptions, the channel output for this model is obtained as

\[ Y_{mL+t} = X_m e^{j \Theta_{mL+t}} + W_{mL+t} \quad (1) \]

for \( m \in [1, \ldots, M] \) and \( t \in [0, \ldots, L-1] \) and where \( W_j \sim \mathcal{CN}(0,2) \) is the additive noise, and \( L \in \mathbb{N} \) is the oversampling factor, that is equal to the reciprocal of the sampling time by assuming that the symbol time is unitary. The Wiener phase noise process \( \{ \Theta_j \} \) is defined as

\[
\begin{align*}
\Theta_{L-1} & \sim U([0,2\pi]) \quad (2a) \\
\Theta_{k+1} & = \Theta_k + N_k \quad (2b) \\
N_k & \sim \mathcal{N}(0,\sigma^2L^{-1}), \quad k \in [L,\ldots,\infty) \quad (2c)
\end{align*}
\]

where \( U(I) \) indicates the uniform distribution over the set \( I \). The channel input is subject to the power constraint

\[
E[|X_m|^2] \leq L^{-1} P, \quad (3)
\]

which correspond to an average power constraint of \( P \) for the transmitted waveform. As a consequence, the SNR is equal to \( P/2 \). Define \( Y_m = Y_{mL}^{(m+1)L-1} = [Y_{mL}, \ldots, Y_{(m+1)L-1}] \) and \( W_m = W_{mL}^{(m+1)L-1} \); with this notation, the capacity of the channel in (1) can be expressed as

\[
C(P, \sigma^2, L) = \lim_{M \to \infty} \frac{1}{M} \sup M \sup \{ Y^M; X^M \} \quad (4)
\]

where the supremum is over all the distributions of \( X^M \) such that the power constraint \( P \) is satisfied and for an oversampling factor equal to \( L \).

We also consider the high-SNR asymptotics of the expression in (4) which are described by the GDoF, defined as

\[
D(\alpha) = \lim_{P \to \infty} \frac{C(P, \sigma^2, \lfloor P^\alpha \rfloor)}{\log(P)}, \quad (5)
\]

that is, the capacity pre-log factor when \( P \) grows to infinity while \( L = \lfloor P^\alpha \rfloor \).

Note that, in the above formulation, the additive noise variance is not affected by the oversampling factor, while the transmit power of a sample is. The detailed derivation of the discrete-time model in (1) from the continuous-time one is presented in [13]. Since \( P \) and SNR are directly related, the degrees of freedom formulation in (5) correctly captures the asymptotic behaviour of capacity at high SNR.

III. KNOWN RESULTS

The OWPN encompasses the classic discrete-time Wiener phase noise (WPN) channel as the special case in which \( L = 1 \). We have recently derived the capacity of the WPN channel to within a small additive gap.

Theorem III.1. Capacity bounds on the WPN channel [18]. The capacity of the WPN channel is upper-bounded as

\[
C \leq \frac{1}{2} \log(1 + P/2) + \begin{cases} 
\frac{1}{4} \log(4\pi e) + 2 & \text{if } \sigma^2 > \frac{2\pi}{e} \\
\frac{1}{2} \log(1 + (2\pi)^{1/2}) + \log(2e) + 2 & \text{if } P^{-1} \leq \sigma^2 \leq \frac{2\pi}{e} \\
2 & \text{if } P^{-1} > \sigma^2,
\end{cases} 
\]

and the exact capacity is to within \( \mathcal{G} \) bpcu from the outer bound in (6), where

\[
\mathcal{G} \leq \begin{cases} 
4 & \text{if } \sigma^2 > \frac{2\pi}{e} \\
7.36 & \text{if } P^{-1} \leq \sigma^2 \leq \frac{2\pi}{e} \\
1.8 & \text{if } P^{-1} > \sigma^2.
\end{cases} \quad (7)
\]

The result in Th. III.1 is interesting at it shows that the capacity of the WPN channel can be sub-divided in three regimes: (i) for large values of the frequency noise variance \( \sigma^2 \), the channel behaves similarly to a channel with circularly uniform iid phase noise; (ii) when the frequency noise variance is small, the effect of the additive noise dominates over that of the phase noise, while (iii) for intermediate values of the frequency noise variance, the transmission rate over the phase modulation channel has to be reduced due to the presence of phase noise.

A lower bound on the GDoF of the OWPN channel for \( \alpha = 1/2 \) is obtained in [4] and is later extended in [12] to yield an inner bound to the GDoF region.

Theorem III.2. GDoF lower bound [4], [12]. The function \( D(\alpha) \) in (5) can be lower-bounded as

\[
D(\alpha) \geq \begin{cases} 
\frac{1+\alpha}{3\pi} & 0 \leq \alpha < \frac{1}{2} \\
0 & \alpha \geq \frac{1}{2}
\end{cases} \quad (8)
\]

The inner bound in Th. III.2 is obtained by letting the channel input have uniformly distributed phase in \([0, 2\pi]\) while the amplitude has a shifted exponential distribution. At the receiver, the statistic used for detecting \( |X_k| \) is \( |Y_k| \), and the one used for detecting \( \angle X_k \) is \( \angle \left(Y_{KL} (Y_{KL-1} e^{-j\angle X_{K-1}}) \right) \).

An outer bound on the capacity of the OWPN channel is derived in [12] which, together with Th. III.2, yields the exact GDoF expression for \( \alpha \in [0, 1/2] \).

Theorem III.3. OWPN channel outer bound [12]. The capacity of the OWPN channel is upper-bounded as

\[
C \leq \frac{1}{2} \log \left( 1 + \frac{P}{2} \right) + \frac{1}{2} \log \left( \frac{2\pi}{e\sigma^2L^{-1}} \right) + O(1). \quad (9)
\]

\[1\]In the following, we indicate the dependency of \( C \) to \([P, \sigma^2, L]\) only when necessary.

\[2\]The original result is derived for \( \alpha \in [0, 1/2] \) but can be easily extended to the case of \( \alpha \in \mathbb{R}^+ \).
Combining the results in [4] and [12], for \( \alpha \in [0, 1/2] \) we obtain that
\[
D(\alpha) = \frac{1 + \alpha}{2}.
\]
(10)

In the next section, we derive an outer bound tighter than that in Th. III.3 which yields the GDoF region for any \( \alpha \in \mathbb{R}^+ \).

IV. OUTER BOUND

We begin by deriving an outer bound on the capacity which improves over the result in Th. III.3. Specifically, we provide a better estimate of the transmission rate that can be attained through phase modulation of the channel input.

**Theorem IV.1. Capacity Outer bound.** The capacity of the OWPN channel is upper-bounded as
\[
C \leq \frac{1}{2} \log \left( 1 + \frac{P}{2} \right) + \log (2\pi) + \frac{1}{2} \log \left( \frac{1}{\sigma^2 L^2 P} + 1 \right) \tag{11}
\]

Proof: Let us begin by upper bounding the information rate in (3) as
\[
I(X_1^M; Y_1^M) = \sum_{k=1}^M I(X_1^M; Y_k | Y_1^{k-1}) \leq \sum_{k=1}^M I(X_1^M, \Theta_{kL-1}; Y_k | Y_1^{k-1}) \tag{12}
\]

where (12) follows from the Markov chain \( Y_k \rightarrow X_k, \Theta_{kL-1} \rightarrow Y_1^{k-1} \). Since the noise is circularly symmetric, a circularly distributed input is capacity achieving: accordingly we have
\[
I(\Theta_{kL-1}; Y_k | Y_1^{k-1}) \leq I(\Theta_{kL-1}; \Theta_{kL-1} + \sum_{i=0}^{L-1} N_{kL+i-1}) | X_k, W_k | Y_1^{k-1} = I(\Theta_{kL-1}; \Theta_{kL-1} + \sum_{i=0}^{L-1} N_{kL+i-1}, | X_k, W_k | Y_1^{k-1}) = 0 \tag{13}
\]

where (13) follows from the fact that the input is circularly symmetric, so that \( |X_k| \perp X_k \), and independent of the phase \( \Theta_{kL-1} \). Similarly to [12], Eq. (19)), we note that the term \( I(X_k; Y_k | \Theta_{kL-1}) \) can be divided into two contributions: one from the channel input amplitude and the other from channel input phase. In fact, using (13), we can write
\[
\frac{1}{M} I(X_1^M; Y_1^M) \leq \frac{1}{M} \sum_{k=1}^M I(X_k; Y_k | \Theta_{kL-1}) = I(X_1; Y_1 | \Theta_{L-1}) + I(X_1; Y_1 | \Theta_{L-1}, | X_1) \tag{14}
\]

where the first equality follows from stationarity of the processes, and the last step by polar decomposition of \( X_1 \). In the following, we refer to \( I(X_1; Y_1 | \Theta_{L-1}) \) as the rate of the amplitude channel and \( I(X_1; Y_1 | \Theta_{L-1}, | X_1) \) as the rate of the phase channel. Analogously to [12] Eq. (20), the rate of the amplitude channel rate can be bounded as
\[
I(X_1; Y_1 | \Theta_{L-1}) \leq I(X_1; Y_1, \Theta_{L-1}^2 \mid \Theta_{L-1}) = I(X_1; \sqrt{L}X_1 + \frac{1}{\sqrt{L}} \sum_{t=0}^{L-1} W_{L+t}) \tag{15a}
\]

where (15b) follows from the result in [13] Thm. IV.1 which bounds the entropy of a non-central chi-square random variable.

The rate in the phase modulation channel can be written as
\[
I(X_1; Y_1 | \Theta_{L-1}, | X_1) = h(X_1 | \Theta_{L-1}, | X_1) - h(X_1 | \Theta_{L-1}, | X_1, Y_1) \tag{16a}
\]

which follows from the fact that \( X_k \sim U([0, 2\pi]) \). The entropy term in (16a) can be rewritten as
\[
h(X_1 | \Theta_{L-1}, | X_1) = -h(X_1 | X_1, Y_1 e^{-j\Theta_{L-1}}) \tag{17a}
\]

where (17a) follows from the fact that \( X_1 \sim U([0, 2\pi]) \) and independent of all other variables. In (17b) we let \( \tilde{X}_1 = \Theta_{L-1} \): this substitution is to stress the fact that \( X_1 \) is independent of all other variables and is thus statistically equivalent to \( \Theta_{L-1} \). In other words, the entropy of the phase of \( X_1 \) given the knowledge of \( \Theta_{L-1} \) and \( Y_{L-1} \) is equivalent to the entropy of \( \Theta_{L-1} \) given \( Y_{L-1}, | X_1 \). Finally, (17c) follows from the “conditioning reduces entropy” property.

From the I-MMSE relationship [13] Eq. (182)], we have
\[
h(\Theta_{L-1} | X_1, \tilde{Y}_L) = \frac{1}{2} \log(2\pi e \text{Var}[\Theta_{L-1}]) - \frac{1}{2} \int_0^\infty \left[ \frac{1}{1 + \rho \text{Var}[\Theta_{L-1}]} - \text{nmse}(\Theta_{L-1} | \sqrt{\text{Var}[\Theta_{L-1} + Z; | X_1, \tilde{Y}_L])} dp \right. \tag{18}
\]
where $Z \sim \mathcal{N}(0,1)$ and independent of any other quantity, and
\[
\text{mmse}(S|K) \triangleq \mathbb{E}[(S - \mathbb{E}[S|K])^2].
\]
(19)

The crucial step in bounding the entropy term in (17d) using
the relationship in (18) is in obtaining a tight lower bound to
the MMSE. To obtain such lower bound we rely on the result
in [16 Prop. 1]. To this end, let us rewrite $U = \sqrt{\theta_k + Z}$ and
\[
\text{mmse}(\Theta_{L-1} | \sqrt{\theta} \Theta_{L-1} + Z, |X_1|, \tilde{Y}_L) = \lim_{k \to \infty} \text{mmse}(\Theta_k | U, |X_1|, \tilde{Y}_L) \\
\geq \lim_{k \to \infty} J_k^{-1},
\]
where we used the time-reversibility of the Wiener process
and the Bayesian Cramer-Rao inequality. Here $J_k$ is defined
as the entry in position $(k,k)$ of the information matrix
associated with the joint distribution of $[\Theta_k, \tilde{Y}_{k-1}, U]$ given
$|X_1|$. According to [16 Prop. 1], the value $J_k$ can be computed
recursively as
\[
J_k = D_{k-1}^{(22)} - \frac{(D_{k-1}^{(12)})^2}{D_{k-1}^{(11)}},
\]
(22)
for
\[
D_n^{(11)} = \frac{L}{\sigma^2}, \quad n = 1, \ldots, k - 1
\]
(23a)
\[
D_n^{(12)} = \frac{L}{\sigma^2}, \quad n = 1, \ldots, k - 1
\]
(23b)
\[
D_{k-1}^{(22)} = \frac{L}{\sigma^2} + \rho, \quad k \geq 2
\]
(23c)
\[
D_n^{(22)} = \frac{L}{\sigma^2} + \mathbb{E}[|X_1|^2] \leq \frac{L}{\sigma^2} + L^{-1}P, \quad n = 1, \ldots, k - 2,
\]
(23d)
where (23d) follows from the average power constraint. Using (23d) into (22) we can easily find a lower bound on $J_{k-1}$. Using this bound in (18) and then in (16b), an upper bound on the rate of the phase channel is obtained as
\[
I(X_1; Y_1 | \Theta_{L-1}, |X_1|) \leq \frac{1}{2} \log\left(\frac{2\pi e}{\frac{1}{\sigma^2L^{-2}P}}\right)
\]
(24)
\[
+ \frac{1}{2} \log\left(\frac{L^{-1}P}{2} \left(1 + \frac{1}{\sigma^2 L^{-2}P} - 1\right)\right).
\]
Combining (15b) and (24) we obtain an outer bound on capacity as in (11).

\section{V. DEGREES OF FREEDOM ANALYSIS}

The outer bound in Th. [V.1] together with the results in
Th. III.3 and Th. III.2 yields the GDoF of the OWPN channel.

\textbf{Lemma V.1.} The GDoF of the OWPN channel is obtained as
\[
D(\alpha) = \begin{cases} 
\frac{3+\alpha}{2} & 0 \leq \alpha \leq \frac{1}{2}, \\
\frac{3}{2} & \alpha > \frac{1}{2},
\end{cases}
\]
(25)
Fig. 1: The degrees of freedom of the OWPN channel in
Lem. [V.1].

\textbf{Proof:} As shown in (10), the GDoF in known for
$\alpha \in [0,1/2]$. For $\alpha > 1/2$ consider the outer bound in Th. [V.1]for $L = [P^n]$; for the rate in the amplitude modulation channel
in (15), we have
\[
\lim_{P \to \infty} \frac{I(X_1; Y_1 | \Theta_{L-1})}{\log(P)} \leq \lim_{P \to \infty} \frac{1}{2} \log\left(\frac{2\pi e}{P}\right) = 1/2,
\]
which holds for any $\alpha \in \mathbb{R}^+$. For the rate in the phase modulation channel in (24) we have
\[
\lim_{P \to \infty} \frac{I(X_1; Y_1 | \Theta_{L-1}, |X_1|)}{\log(P)} \leq \lim_{P \to \infty} \frac{1}{2} \log\left(\frac{L^{-1}P}{2} \left(1 + \frac{1}{\sigma^2 L^{-2}P} - 1\right)\right) = 1/4,
\]
which follows from the fact that, for $\alpha > 1/2$, we have that $L^{-2}P = P^{1-2\alpha} \to 0$ as $P \to \infty$.

Combining (26) and (27) we obtain the outer bound on
$D(\alpha)$ with matches the inner bound in (8) for the regime
$\alpha \in [1/2, \infty)$.

The result in Lem. [V.1] is schematically represented in Fig. 1:
the GDoF from amplitude modulation are equal to $3/4$ for all $\alpha$,
while the GDoF from phase modulation are equal to $1/2$. Note that,
when $\alpha \to 0$, we obtain the model with $L = 1$ in
Th. III.1 which has pre-log equal to $1/2$.

\textbf{Discussion:} The analysis of the GDoF in Lem. [V.1] suggests
that there is a fundamental tension between the AWGN and
the multiplicative WPN, and improving the resolution of the
receive filter beyond $L^{-1} = 1/\sqrt{P}$ does not improve the
capacity pre-log at large $P$. From a high level perspective, the
parameter $\sigma^2$ is related to the quality of the local oscillators
available at the user: in this sense, then, the result in Lem.
[V.1] shows that, regardless of the value $\sigma^2$, the fundamental
tension will eventually reduce the available DoF for a suitably
large $P$.

From the I-MMSE bound in (18) and (22) used in the
proof of Th. [V.I], it is apparent that the tension between the AWGN and the WPN is related to the difficulty of predicting a new sample of a Wiener process when corrupted by AWGN. The following questions naturally arise: is the limitation of the available GDoF an artifact of the assumptions used to derive the model in [1] or is it an inherent limitation of the physical system? Further, the model in [1] neglects the effect of amplitude fading for the sake of simplicity. For the model encompassing both phase and amplitude fading, one wonders whether it is possible to attain higher DGoF. The model in [1] is obtained by employing a waveform that allocates the power uniformly over time. One then naturally wonders whether it is possible to attain higher performance employing a waveform that does not allocate energy uniformly in time. These interesting open questions are to be addressed in future works.

VI. CONCLUSION

We have derived an outer bound on the capacity of discrete-time Wiener phase noise channels with multi-sample receivers. In this model, the input of a point-to-point channel is corrupted by both additive noise and multiplicative phase noise: the additive noise is a white Gaussian process while the phase noise is a Wiener process. For each symbol in input, the channel produces \( L \) outputs corresponding to the output of an integrate-and-dump multi-sample receiver with oversampling factor \( L \). A novel outer bound is derived using the I-MMSE relationship and a recursive expression of the minimum mean-square error though the Fisher information matrix. This novel outer bound is used to derive the generalized degrees of freedom for the scenario in which the oversampling factor grows with the transmit power \( P \) as \( L = [P^{\alpha}] \). This latter result shows that there exists a fundamental tension between the AWGN and the WPN that limits the available GDoF at \( 1/4 \) for phase modulated signals, regardless of the power of the phase noise. The degrees of freedom analysis of models encompassing both multiplicative phase noise and multiplicative amplitude noise remains an interesting open question.

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