The Effects of Multiplicative Noise in Relativistic Phase Transitions

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Effective stochastic equations for the continuous transitions of relativistic quantum fields inevitably contain multiplicative noise. We examine the effect of such noise in a numerical simulation of a temperature quench in a 1+1 dimensional scalar theory. We look at out-of-equilibrium defect formation and compare our results with those of stochastic equations with purely additive noise.

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I. INTRODUCTION

Since phase transitions take place in a finite time, causality guarantees that correlation lengths remain finite, even for continuous transitions. Because of the universal presence of causality, Kibble [1] and Zurek [2, 3] suggested that it alone is sufficient to bound the size of correlated domains after the implementation of a continuous transition. There are several ways of formulating causality bounds, but they all depend on the fact that, as the transition begins to be implemented, there is a maximum speed at which the system can become ordered. For relativistic quantum field theory (QFT) this is the speed of light whereas, for superfluids, for example, it is the speed of second sound.

The argument is very general. Consider a system with critical temperature $T_c$, cooled through that temperature so that, if $T(t)$ is the temperature at time $t$, then $T(0) = T_c$. If $T(0) = T_c/\tau_Q$ defines the quench time $\tau_Q$. Suppose that the adiabatic correlation length $\xi_{ad}(t) = \xi_{ad}(T(t))$ diverges near $t = 0$ as

$$\xi_{ad}(t) = \xi_0 \left| \frac{t}{\tau_Q} \right|^{-\nu}.$$  

The fundamental length scale of the system $\xi_0$ is determined from the microscopic dynamics. Although $\xi_{ad}(t)$ diverges at $t = 0$ this is not the case for the true non-equilibrium correlation length $\xi(t)$, which can only change so much in a finite time. Kibble and Zurek made the assumption that the correlation length $\xi$ of the fields that characterizes the onset of order is the equilibrium correlation length $\bar{\xi} = \xi_{ad}(t)$ at some appropriate time $\bar{t}$.

For simple systems all estimates of $\bar{t}$ (the 'causal time') agree, up to numerical factors approximately unity [3]. Most simply, $\xi(t)$ cannot grow faster than $c(t) = c(T(t))$, where $c(T)$ is the causal velocity at temperature $T$. For relativistic theories $c(T) = c$, constant, whereas for condensed matter systems we typically have critical slowing down, $c(T_c) = 0$. This is true both before and after the transition. That is, $\bar{t}$ is defined by the condition that $\bar{\xi}_{ad}(-\bar{t}) \approx c(\bar{t})$ or $\bar{\xi}_{ad}(\bar{t}) \approx -c(\bar{t})$. As a result, $\bar{t}$ is of the form

$$\bar{t} \approx \tau_Q^{1-\gamma} \tau_0^{-\gamma},$$  

where $\tau_0 \ll \tau_Q$ is the cold relaxation time of the longest wavelength modes, and the critical exponent $\gamma$ depends upon the system. It follows that $\tau_Q \gg \bar{t} \gg \tau_0$.

Domain formation, the frustration of the order parameter fields, is often visible through topological defects, which mediate between different equivalent ground states. Since defects are, in principle, observable, they provide an excellent experimental tool for confirming this hypothesis when the possibility of producing them exists. Kibble and Zurek made the further assumption that we can measure $\bar{\xi}$ experimentally by measuring the number of defects, assuming that the defect separation $\xi_{def} = O(\bar{\xi})$. This identification of the initial domain size and defect separation then gives an estimate of the defect separation at formation of

$$\bar{\xi} \sim \xi_{ad}(\bar{t}) = \xi_0 \left( \frac{\tau_0}{\tau_Q} \right)^{\sigma} > \xi_0,$$  

where $\sigma = \gamma \nu$. This is very large on the scale of cold defects which shrink to size $\xi_{ad}(T_{fin}) = O(\xi_0)$, where $T_{fin}$ is the final temperature. We term $\sigma$ the Zurek-Kibble (ZK) characteristic index.

In general, we find that the mean-field indices are

$$\gamma = \frac{2}{3}; \quad \nu = \frac{1}{2}; \quad \sigma = \frac{1}{3},$$  

for relativistic systems (e.g. weakly damped quantum fields), and

$$\gamma = \nu = \frac{1}{2}; \quad \sigma = \frac{1}{4},$$  

for non-relativistic systems (e.g. superfluids). There are exceptions to this rule, but we shall not consider them here.
II. STOCHASTIC EQUATIONS

The question is whether these bounds, independent of the microscopic equations that govern the phase transition, are remotely saturated in the physical world. In condensed matter physics several experiments have been performed to check\cite{2}. Although the results are mixed, the overall conclusion is positive\cite{4,5,6,7,8}.

Experiments cannot be performed for relativistic systems and models of the early universe, when such transitions were important, are too ambiguous to be helpful. In practice, an ab initio calculation from the microscopic field dynamics suggests the validity of the scaling laws, but does not permit an estimate of the efficiency of defect production outside the framework of mean-field (or large-N) approximations\cite{9,10}.

In consequence, a more pragmatic check on the saturation of the Zurek-Kibble bounds in relativistic systems has been numerical\cite{11,12}, essentially from the empirical damped relativistic Langevin equation with additive noise, of the form

\[ \ddot{\phi}(x) + \eta \dot{\phi}(x) + (-\nabla^2 + m^2(t))\phi(x) + 2\lambda \phi^3(x) = \xi(x), \]  \tag{5}

where, for simplicity, we consider the theory of a single field form, \( m^2(t) = \xi_{\text{tot}}(t) \), that triggers the transition, is taken in the mean-field form

\[ m^2(t) = -\mu^2 \epsilon(t), \quad \epsilon(t) = (1 - T(t)/T_c), \]

in the vicinity of \( t = 0 \), where \( \mu^2 \) is the cold mass which defines the symmetry-broken ground states, and \( \epsilon(t) \approx -t/t_Q \). For \( t > t_Q \) it behaves as \( m^2(t) = -\mu^2 \). The equilibrium solution is

\[ \phi = \pm v, \quad v^2 = \frac{\mu^2}{2\lambda}, \]

the symmetry breaking scale. Numerical simulations\cite{11,12} show that the Kibble and Zurek scaling behaviour is recovered in the limits of small (underdamped) and large (overdamped) \( \eta \) respectively, although defects are produced with lower efficiency than anticipated in\cite{2}.

The overdamped case is the phenomenological non-relativistic time-dependent Ginzburg-Landau equation

\[ \eta \dot{\phi}(x) = -\frac{\delta F}{\delta \phi} + \xi(x), \]  \tag{6}

for free energy \( F \), often appropriate for condensed matter systems. Equation\cite{16,17,18,19} arises from linear coupling to an environment.

There is, however, a problem with justifying\cite{16,17,18,19}, with its assumption of linear dissipation, for underdamped relativistic QFT. Although linear couplings have played an important role in the history of decoherence and Langevin equations, additive noise alone cannot be justified in QFT, where a pure linear coupling to the environment corresponds to an inappropriate diagonalisation of the fields.

More generally, there are two mechanisms for inducing friction (dissipation) in a relativistic plasma:

a) Changing the dispersion relation of existing particles, as happens on varying \( m^2(t) \), leads to a change in scattering and decay rates, which leads to a change in the distribution of particle energies, which leads to friction.

b) The creation of particles from the heat-bath, which leads to friction.

In each case, we expect dissipative terms of the form \( \dot{\phi} \phi^2 \) to be important\cite{11,12,13}. On the other side of the equation there is, equally, a problem with the noise in\cite{5} with regard to relativistic QFT. Noise is construed as a consequence of integrating out (tracing over) environmental degrees of freedom, which are intrinsically non-linear. We assume Boltzmann statistics. In consequence, the noise, which guarantees that the system is ultimately driven to its ground states must, by the fluctuation-dissipation theorem, contain a term of the multiplicative form \( \phi \xi \).

This is confirmed in linear response theory which, for small deviations from equilibrium, leads to Langevin equations of the form\cite{14,15,16,17,18,19}

\[ \partial_t \partial^\mu \phi(x) - \mu^2 \phi + 2\lambda \phi^3(x) + \int d^4 y \ K_1(x,y) \phi(y) + \phi(x) \int d^4 y \ K_2(x,y) \phi^2(y) = \xi_1 + \phi(x) \xi_2(x), \]  \tag{7}

From the retarded nature of the \( K_5 \) a more realistic, albeit still idealised, equation than\cite{5} for the real relativistic scalar field \( \phi \) is

\[ \ddot{\phi}(x) + (-\nabla^2 + m^2(t))\phi(x) + 2\lambda \phi^3(x) + \eta_2 \dot{\phi}(x) + \eta_1 \phi(x) \phi^2(x) = \xi_1(x) + \phi(x) \xi_2(x), \]  \tag{8}

where \( \xi_1, \xi_2 \) are thermal noise.

We note that, in\cite{19}, the multiplicative noise plays the role of a stochastic term in the temperature, on rewriting it as

\[ \ddot{\phi}(x) + (-\nabla^2 + m^2(t))\phi(x) + 2\lambda \phi^3(x) + \eta_2 \dot{\phi}(x) + \eta_1 \phi(x) \phi^2(x) = \xi_2(x), \]  \tag{9}

where

\[ m^2(t) = -\mu^2 \epsilon(1 - T_\xi(t)/T_c), \]

in which

\[ T_\xi(t) = T(t) - \frac{T_c}{\mu^2} \xi_1(x). \]  \tag{10}

For continuous transitions we are not aware of attempts to examine the effects of multiplicative noise for fields (as distinct from few degree-of-freedom quantum mechanics\cite{21,21,22,22,23,24,24,23}) on scaling behaviour, even at the crudest phenomenological level. [This is not the case for discontinuous transitions, for which this paper is the counterpart to\cite{20}.] We see our work here as a
complementary study to the substantial analysis that has been undertaken for the simpler and often less believable equation, to which it reduces for small \( \eta \) and correspondingly weak noise. Finally, although we shall not pursue this here, we should not see entirely in the context of QFT. There has been a considerable effort in condensed matter to examine multiplicative noise in systems with simple \( \eta \phi \) dissipation, but with \( \xi \phi \) multiplicative noise (\( \eta_2 = \xi_2 = 0 \)). This leads to non-Boltzmann (Tsallis/Renyi) distributions. Multiplicative noise can both induce phase transitions and restore symmetry.

III. NUMERICAL SIMULATIONS

It is sufficient for our purposes to restrict ourselves to a real field on the line (i.e. 1+1 dimensions), for which \( \nabla^2 = \partial^2 / \partial x^2 \). It was for such a system that the original equation was solved. In that case extension to higher dimensions gave few new insights as to the Kibble mechanism, and we expect the same here.

Defects in this case are kinks,

\[
\phi_{\text{kink}}(x) = \pm v \tanh \left( \frac{\mu}{2\sqrt{2}x} \right)
\]

of thickness \( \xi_0 = \mu^{-1} \) and energy \( E = O(\mu^3/\lambda) \). Although, rigorously, there are no transitions for such short-range interactions in 1+1 dimensions, there is an effective transition for medium times. Typically, some time after the end of the quench, the field settles into a set of alternating positive/negative vacuum regions. These are separated by well defined kinks/anti-kinks that evolve slowly, possibly annihilating each other for very long times. In this regime defects coincide with the zeroes of the field, making it straightforward to identify them in a numerical setting. Clearly this procedure is ambiguous for very early times, since zeroes occur at all scales, and only some of these will evolve into the cores of kinks. Here we will restrict ourselves to looking at kinks after the quench has terminated, thus avoiding any counting ambiguities.

A. Numerical Setup

To further clarify the possible effects of multiplicative noise terms in the mechanism of defect formation, we performed a numerical study of the model described in the case of a quenched one-dimensional system with purely additive white noise. We evolve the following 1+1 Langevin equation:

\[
\ddot{\phi} - \nabla^2 \phi + [\alpha_1^2 \phi^2 + \alpha_2^2] \phi - m^2(t) \phi - 2 \lambda \phi^3 = \alpha_1 \phi \xi_1 + \alpha_2 \xi_2
\]

where \( m^2(t) = -\mu^2 \epsilon(t) \), \( \mu^2 = 1.0 \) and \( \lambda = 1.0 \). \( \xi_1 \) and \( \xi_2 \) are uncorrelated gaussian noise terms obeying

\[
\langle \xi_\alpha(x', t') \xi_\beta(x, t) \rangle = 2T \delta_{\alpha \beta} \delta(x' - x) \delta(t - t'), \quad \langle \xi_\alpha(x, t) \rangle = 0.
\]

The bath temperature \( T \) is set to a low value, typically \( T = 0.01 \). The relative strength of the multiplicative and additive noise is measured by \( \alpha_1 \) and \( \alpha_2 \), with corresponding dissipation terms obeying the fluctuation-dissipation relation. The values of \( \alpha_1 \) and \( \alpha_2 \) vary between different sets of runs, allowing us to compare their effects on the final defect density.

When dealing with stochastic equations with multiplicative noise one usually has to take into account that the continuous equation may not have a unique interpretation. This is the well known Itô-Stratonovich ambiguity which is usually resolved by singling out a specific discretisation of the equations of motion. It turns out that this problem has no relevance for the type of system we are considering. This could be seen by obtaining explicit Fokker-Plank equations for different time-discretisations of the model above. Since the multiplicative term depends only on the field (an not on its time derivative), the Fokker-Plank equation turns out to be the same for all alternative interpretations.

The equations of motion were discretised using a second-order leap-frog algorithm. We set \( \delta x = 0.125 \) and \( \delta t = 0.1 \) in a periodic simulation box with 8000 to 16000 points. Note that the core of the defect (with finite temperature size \( 1/m \)) is resolved by 8 lattice points which should be enough for our purposes.

Walls are identified by looking at zero crossings of the scalar field. As discussed above, this should be accurate for long times and low values of \( T \) - precisely the regime where we measure the final values for the defect density. For each individual quench, the final number of defects is determined by counting kinks at a final time, defined as a fixed multiple of the quench time-scale. There are more complex ways of defining a final defect density, namely by fitting the time-dependence of the kink number to exponential or power-law decay expressions. The several approaches were compared in the conclusion. The conclusion was that the straightforward kink counting performs well in the very fast/slow quench-time limits, leading to slightly higher estimates of \( \sigma \). With this caveat in mind we will keep to the simpler approach since, as we will see, it is accurate enough to illustrate well the effects of multiplicative noise terms.

For every fixed choice of \( \alpha_1 \) and \( \alpha_2 \) we perform a series of quenches, with the quench times varying as \( \tau_Q = 2^n \), \( n = 1, 2, \ldots, 9 \). Each quench is repeated several times (typically 10) with different random number realizations, and the final defect number is averaged over this ensemble. The scaling exponent \( \sigma \) can then be obtained by fitting the final defect number dependence on \( \tau_Q \) to a power-law of the form \( A r_Q^{-\sigma} \).
FIG. 1: Final defect density scaling power as a function of the additive noise strength. The multiplicative noise term is set to zero. Error bars represent the standard deviation of the result over 10 independent series of quench realizations.

B. Simulation Results for Kink Densities

We start with the simple purely additive noise case, corresponding to setting $\alpha_1 = 0$ in Eq. (12). In Fig.1 we can see the dependence of the scaling power $\sigma$, in terms of noise strength $\alpha_2$. Our results are very similar to those found in [12], with $\sigma$ decreasing as the value of the dissipation, $\eta = \alpha_2^1$ increases. This takes us from the relativistic regime where we expect $\sigma \simeq 1/3$ to the overdamped case with $\sigma \simeq 1/4$. As observed by previous authors, for very small values of the dissipation, the scaling fails to follow the power-law rule, a consequence of saturation. This explains the high, un-physical values of $\sigma$ for $\alpha_2 < 0.5$ (corresponding to a dissipation of $\eta = 0.25$). The quality of the power-law fit becomes poor in this parameter region, a further sign of deviations from the simple scaling behaviour.

Next we look at how the introduction of multiplicative noise influences the results. In Fig.2 we have the scaling power for $\alpha_2$ in the same region as Fig.1 for three different values of multiplicative noise strength, $\alpha_1 = 0.0, 0.5$ and 0.9 respectively. Clearly the results change very little - within error bars, the three curves basically overlap each other.

Though this result may seem disappointing at first, we should be aware that the significant range of variation of $\alpha_1$ may differ considerably from that for $\alpha_2$. This is illustrated in Fig.3 where we have $\sigma$ for higher values of $\alpha_1$, with the contribution of the additive term set to zero. The pattern is the same as observed before for the case of additive noise. As $\alpha_1$ increases, the scaling exponent changes from relativistic values to those typical of an overdamped system. The transition between the two types of behaviour takes place for values of $\alpha_1$ considerably higher than $\alpha_2$. Defining the noise strength transition value as the one above which $\sigma < 0.3$, we have $\alpha_2 \simeq 0.8$ for purely additive noise and $\alpha_1 \simeq 3.5$ in the multiplicative case.

These results can be understood if we make the simple assumption that the order of magnitude of the effective dissipation in the multiplicative noise case is given by $\eta \simeq \alpha_1^2 \langle \phi^2 \rangle$ [21]. The mean of the term multiplying $\dot{\phi}$ in Eq. [3] $\langle \phi^2 \rangle$ is the average of the square of the field during the stage of the quench determining the scaling, that
is, slightly before the critical temperature is reached. If the typical value of $\langle \phi^2 \rangle$ is small, the effective dissipation for multiplicative noise should be reduced. As a consequence, relatively high values of $\alpha_1$ should be required to take the system from the relativistic to the dissipative regime.

This argument can be made more quantitative by noting that in the purely additive case the under-damped to over-damped transition takes place for $\eta \simeq 0.8^2 \simeq 0.6$. A similar effective dissipation with pure multiplicative noise would be reached for $\alpha_1 = 3.5\sqrt{3}$ if $\langle \phi^2 \rangle \simeq 0.6/3.5^2 \simeq 0.05$. We measured the value of the mean squared field explicitly in the simulations, and observed that at $t = 0$ one has typically $\langle \phi^2 \rangle \simeq 0.01 - 0.02$, with the higher value corresponding to the slower quenches. This is indeed of the same order of magnitude as required by the above reasoning. We note that, if instead, we were to replace $\alpha_1^2 \langle \phi^2 \rangle$ by $\alpha_1^2 \nu^2$ we would obtain an effective dissipation of $\eta = 6.1$, one order of magnitude larger than the critical value. This suggests that even though kinks can only be identified with rigour once

$$\langle \phi^2 \rangle \sim \nu^2,$$

the period of the evolution responsible for setting the relevant defect separation scale takes place considerably earlier, in accord with Fig. 2.

The above result can be extended to systems with both kinds of noise, with the generic effective dissipation being given by $\eta = \alpha_3^2 + \alpha_1^2 \langle \phi^2 \rangle$. For $\alpha_1 < 1$, the correction to the multiplicative component should be less than 0.05. This explains why the inclusion of multiplicative noise of this magnitude changes very little the additive noise result, as illustrated in Fig. 2.

Finally we checked whether the inclusion of multiplicative noise leads to any appreciable change in the behaviour of the amplitude of the power-law. In Fig. 4 we show the value of $\log(A)$ where $A$ is the amplitude of the fit for the final defect density $A \tau_{c}^{\sigma}$. The results are shown as a function of the power $\sigma$, corresponding to different ranges of $\alpha_1$ and $\alpha_3$ in the purely multiplicative and additive noise cases respectively. As can be observed, $A$ does not differ significantly between the two types of noise, for similar values of $\sigma$. Physically this implies that not only the scaling power is similar in both cases, but also the overall amount of defects produced is not affected by the type of noise involved in the transition. Overall the conclusion seems to be that once we adjust for the value of the effective dissipation, the distribution of the defect number in the final configuration is independent of the properties of the noise terms driving the system.

**IV. CONCLUSIONS**

Since most effective equations of motion derived from field theory involve both additive and multiplicative noise, it is natural to wonder whether the presence of multiplicative noise changes the dynamics of non-equilibrium phase transitions. Here we looked in detail at the formation of defects in a 1+1 dimensional system undergoing a quench with both types of thermal noise present. Surprisingly, we found that the properties of the defect population after the quench can be well described in terms of the Kibble-Zurek scenario, if we take into account the effects of multiplicative noise in the dissipation. In particular, multiplicative noise terms increase the dissipation by an amount of the order of the noise amplitude, times the value of the mean field square at the transition time. As in the purely additive case, we observe that for low values of the effective dissipation the system behaves in a relativistic fashion, with the final number of defects scaling as a power of $1/3$ of the quench time. For higher values the system enters an over-damped regime characterized by a lower scaling power, nearer to $1/4$. Although our multiplicative noise was the most simple (tracing over short wavelength modes will also induce $\phi^2 \xi_3$ noise) this suggests that the Kibble-Zurek scaling laws are robust.

It is interesting to note that these results still leave open the question of whether the defect density (or equivalently, the freeze-out correlation length) is set before or after the transition takes place, i.e. for $\tilde{t} < 0$ or $\tilde{t} > 0$. Strictly, the value for the average field square $\langle \phi^2 \rangle$ used in defining the effective dissipation, should be evaluated at $\tilde{t}$. Unfortunately the value of this quantity varies slowly in the vicinity of the critical point and as a consequence it is not possible to determine whether the correct value is fixed above or below $T_c$. We can nevertheless, use our model as a setting for answering this question, by performing quenches where the value of the effective dissipation is forced to change at $t = 0$. By looking at the defect scaling in cases where the shift in the dissipation takes the system from an under-damped to an over-damped regime at $t = 0$, the significant stage of the evolution
should become apparent. We will clarify these points in detail in a future publication.

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