Embedding Properties in Central Products

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Abstract

In this article, we study group theoretical embedding properties of subgroups in central products of finite groups. Specifically, we give characterizations of normal, subnormal, and abnormal subgroups of a central product of two groups.

1 Introduction

In group theory, studying subgroups of groups is important, but it is not easy to answer the question “What can we say about the subgroups?” for products of groups. Hence, investigating subgroups that satisfy specific properties, known as characterizations, in products of groups is of particular interest. Subgroups of a direct product of two groups were completely described by Edouard Goursat in [7], and his result has been refined as can be seen in [4]. Goursat’s work provides the backbone necessary to understand subgroups of a direct product, and many mathematicians have since made significant contributions to this area. This article presents characterizations of subgroups in a central product of two groups, where a central product is a direct product with an amalgamated center.

Recently, more work has been done on subgroups of a direct product of two groups. For example, in [3], [4], [1], and [2], necessary and sufficient conditions have been provided to characterize subgroups of a direct product including normal and subnormal subgroups. However, very little work has been done to characterize these subgroups for a central product of two groups. In this article, we begin to characterize subgroups of central products, focusing on normal, subnormal, and abnormal subgroups. We provide equivalent conditions for normality, subnormality, and abnormality, and give applications of our results.

In Section 2, we discuss direct products by introducing notation necessary to understand propositions presented from other work on direct products, and we review the definition of commutator, commutator subgroup, solvable, and simple. We also provide the definition of subnormal and abnormal subgroups and provide examples. In Section 3, we define central products internally and externally, and we provide a concrete example of a central product. In Section 4, we characterize normal subgroups by examining the commutator of subgroups of a central product; we prove this result and provide an application using the example introduced in Section 3. In Section 5, we characterize subnormal subgroups of a central product and provide an example. In Section 6, we determine and prove lemmas concerning abnormal subgroups, and we provide a characterization of abnormal subgroups in a central product. Finally, in Section 7, we summarize our results and provide conjectures on characterizations of pronormal subgroups of central products.

2 Preliminaries

In this article, we consider only finite groups. Our notation and terminology are standard, but we define terms and state lemma that are necessary to understand later information.

Let $G$ be a group. We say $G$ is simple if it has no normal subgroups other than itself and the trivial subgroup. For elements $a, b \in G$, we write $a^b = b^{-1}ab$. The commutator of $a$ and $b$, $[a, b]$, is given by $a^{-1}a^b$. For $A, B \leq G$, $[A, B]$ will denote the subgroup generated by all the commutators $[a, b]$ where $a \in A$ and $b \in B$. We define $[a_1, a_2, a_3] = [[a_1, a_2], a_3]$ and define commutators of $n$ elements recursively where $[a_1, a_2, \ldots, a_n] = [[a_1, a_2, \ldots, a_{n-1}]a_n]$ for $a_i$ in a group $G$, for all $i \in \mathbb{N}, i \leq n$. The center of a group $G$ is $Z(G) = \{ g \in G : gx = xg, \forall x \in G \}$. The derived series of a group $G$ is the series

$$G = G^{(0)} \geq G^{(1)} \geq \cdots$$
where $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$. A group $G$ is solvable if $G^{(i)} = 1$ for some $i \geq 1$.

The following lemma contains classical results about the commutator subgroup $[A, B]$. Because this lemma is useful in later proofs, we present it here.

**Lemma 2.1.** Let $A, B, C$ be subgroups of a group $G$.

(i) $[A, B] = [B, A] \leq (A, B)$.

(ii) $[A, B] \leq A$ if and only if $B \leq N_G(A)$, where $N_G(A)$ is the normalizer of $A$ in $G$.

(iii) If $\alpha : G \rightarrow \alpha(G)$ is a group homomorphism, then $\alpha([A, B]) = [\alpha(A), \alpha(B)]$.

(iv) If $A$ and $B$ are normal subgroups of $G$, then $[A, B]$ is a normal subgroup of $G$.

(v) $\langle A^B \rangle = A[A, B]$.

**Proof.** See [5].

Whenever we discuss direct products, they will be viewed externally unless otherwise stated. Let $U_1$ and $U_2$ be groups and consider the direct product $U_1 \times U_2$. The maps $\pi_i : U_1 \times U_2 \rightarrow U_i$ given by $\pi_i((u_1, u_2)) = u_i$, for $i = 1, 2$ are standard projections. $\overline{U}_1 = \{(u_1, 1) : u_1 \in U_1\}$ and $\overline{U}_2 = \{(1, u_2) : u_2 \in U_2\}$, where $\overline{U}_i$ is a subgroup of $U_1 \times U_2$ for $i = 1, 2$.

The following lemma is utilized throughout this article, and it provides us with a one-to-one correspondence between normal subgroups. This result can be found in abstract algebra textbooks such as [6] and [9] and follows from the classical Lattice Isomorphism Theorem.

**Lemma 2.2.** If $\varphi : G \rightarrow H$ is an epimorphism of groups and $S_\varphi(G) = \{K \leq G | \ker \varphi \subseteq K\}$ and $S(H)$ is the set of all subgroups of $H$, then the assignment $K \mapsto \varphi(K)$ is a one-to-one correspondence between $S_\varphi(G)$ and $S(H)$. Under this correspondence, normal subgroups correspond to normal subgroups.

**Proof.** See [9].

**Remark 2.3.** Let $\phi : G \rightarrow H$ be a group epimorphism, and let $W \leq H$. For the remainder of this article, under such an epimorphism $\phi$, we will assume that $\ker \phi \leq \phi^{-1}(W)$, where $\phi^{-1}(W)$ is the preimage of $W$.

3 Central Products

Central products have historical been useful for the characterization of extra special groups. All extra special groups of order $p^{2n+1}$ for some $n$ can always be written as a central product of some extra special groups of order $p^n$ [10]. We will adopt the definition and notation for central products presented in [5]:

**Definition 3.1.** Let $U_1, U_2 \leq G$. Then $G$ is an internal central product of $U_1$ and $U_2$ if

(i) $G = U_1U_2$, and

(ii) $[U_1, U_2] = 1$.

This definition is standard as can be seen in [10], and observe that it implies that $U_i \trianglelefteq G$ for $i = 1, 2$ and $U_1 \cap U_2 \leq Z(U_1) \cap Z(U_2)$. The following lemma serves as a powerful tool for proving results because it establishes the relationship between central products and direct products, subgroups of the latter being well understood.

**Lemma 3.2.** Let $G$ be a group such that $U_1, U_2 \leq G$, $D = U_1 \times U_2$ is the external direct product, $\overline{U}_1 = \{(u_1, 1) : u_1 \in U_1\}$ and $\overline{U}_2 = \{(1, u_2) : u_2 \in U_2\}$. Then the following are equivalent:

(i) $G$ is an internal central product of $U_1$ and $U_2$.

(ii) There exists an epimorphism $\varepsilon : D \rightarrow G$ such that $\varepsilon(\overline{U}_1) = U_1$ and $\varepsilon(\overline{U}_2) = U_2$.

**Remark 3.3.** Throughout this article we will view central products internally wherever it is sensible. When we discuss external central products, we will do so as is described in the following construction:
Theorem 3.4. Let $V_1, V_2$ be finite groups, and assume that $A$ is an abelian group for which there exist monomorphisms $\mu_i : A \to Z(V_i)$ for $i = 1, 2$. Let $D$ denote the external direct product $V_1 \times V_2$, let $\mathcal{V}_1 = \{(v_1, 1) \mid v_1 \in V_1\}$, $\mathcal{V}_2 = \{(1, v_2) \mid v_2 \in V_2\}$ Then $\mathcal{V}_1 \cong V_1$. Set
$$N = \{(\mu_1(a), \mu_2(a^{-1})) : a \in A\}$$
Then $N \triangleleft D, \mathcal{V}_1 \cap N = 1$, and with $U_i = \mathcal{V}_i N/N$, the quotient group $G = D/N$ has the following properties:

(i) $G$ is a central product of the subgroups $U_1$ and $U_2$, and $U_i \cong V_i$ for $i = 1, 2$.

(ii) $U_1 \cap U_2 = A_4 N/N \cong A$, where
$$A_1 = \{(\mu_1(a), 1) \mid a \in A\}$$
$$A_2 = \{(1, \mu_2(a)) \mid a \in A\}$$

The external central product $D/N$ as constructed above is isomorphic to the internal central product $G = U_1 U_2$ where $U_i \cong V_i$ for $i = 1, 2$. We now show that $\phi : D/N \to G$ given by $\phi((a, b))N = ab$ is well defined but leave it to the reader to prove $\phi$ an isomorphism.

Well defined: Let $(u_1, u_2)N = (v_1, v_2)N$. We wish to show that $\phi((u_1, u_2)N) = \phi((v_1, v_2)N)$. By assumption, $(v_1, v_2)^{-1}(u_1, u_2)N = N$. Hence, $(v_1, v_2)^{-1}(u_1, u_2) = (v_1^{-1}u_1, v_2^{-1}u_2) \in N$. Then because the identity maps to the identity,
$$\phi((v_1^{-1}u_1, v_2^{-1}u_2)N) = v_1^{-1}u_1v_2^{-1}u_2 = 1$$

Now,
$$\phi((u_1, u_2)N) = u_1u_2 = (v_1^{-1}u_1)(v_2^{-1}u_2)u_2$$

However, elements of $U_2$ commute with elements of $U_1$, which yields
$$v_1v_2(v_1^{-1}u_1v_2^{-1}u_2) = v_1v_2 = \phi((v_1, v_2)N).$$

Internal central products are unique because $G = U_1 U_2$ is determined by the choice of subgroups $U_1$ and $U_2$ of $G$. However, external central products are not unique because the construction depends on the choice of monomorphisms $\mu_1$ and $\mu_2$.

Let us develop an intuition for the construction of an external central product via an example that we will return to in later sections.

Example 3.5. Consider the dihedral group of order eight, $D_8$, and the cyclic group of order four, $C_4$, with the following respective group presentations:
$$D_8 = \langle r, s : r^4 = s^2 = 1, rsr = s \rangle$$
$$C_4 = \langle y : y^4 = 1 \rangle$$

We use (3.3) to construct $D_8 \circ C_4$, a central product of $D_8$ and $C_4$ which is a group of order 16.

Let $U_1 = D_8, U_2 = C_4$. Fix $A = Z(D_8) \cong Z(C_4)$. Then we define the monomorphisms
$$\mu_1 : Z(D_8) \to Z(D_8)$$
$$\mu_2 : Z(D_8) \to Z(C_4)$$

by $\mu_1$ as the identity map and $\mu_2(r^2) = y^2$. Then $N = \{(\mu_1(a), \mu_2(a^{-1})) : a \in A\} = \{(1, 1), (r^2, y^2)\}$, and
$$D_8 \circ C_4 = \frac{D_8 \times C_4}{N}$$

Without loss of generality, suppose $\tau = (r, 1)N, \pi = (s, 1)N$ and $\overline{\tau} = (1, y)N$. Then
$$D_8 \circ C_4 = \langle \tau, \pi, \overline{\tau}, \overline{\pi} : \tau^4 = \overline{\tau} = N, \tau \pi \overline{\tau} = \overline{\pi}, \tau \overline{\tau} = \pi^2, \pi \overline{\pi} = \overline{\pi}^2, \overline{\pi} \tau = \tau \overline{\pi} = \overline{\pi} \overline{\tau} \rangle.$$

Note that because of the isomorphism between the internal and external presentation of a central product, we know that we may also write this group as
$$D_8 C_4 = \langle r, s, y : r^4 = s^2 = 1, rsr = s, r^2 = y^2, ry = yr, sy = ys \rangle.$$

Remark 3.6. All results characterizing subgroups of central products in the remaining sections will be stated for internal central products. It is important to note that all results will apply to external central products due to the isomorphism between internal and external central products.
4 Normal Subgroups of Central Products

We aim to characterize normal subgroups of a central product of two groups, \[3.2\] allows us to define the central products of groups \(U_1\) and \(U_2\) by an epimorphism from \(D = U_1 \times U_2\) onto \(G = U_1 U_2\) with \(\epsilon(U_i) = U_i\) for \(i = 1, 2\). We begin with a lemma from [1] that characterizes normal subgroups of a direct product and serves as the inspiration for our characterization.

**Lemma 4.1.** Let \(U_1, U_2\) be groups. Then \(N \leq U_1 \times U_2\) if and only if \([N, U_i] \leq N \cap U_i\) where \(i = 1, 2\).

In [3.2] an epimorphism between direct products and central products was established to give an alternate and more useful way of viewing central products. Given an epimorphism, we know by 2.2 that there is a one-to-one correspondence between normal subgroups of a domain which contain the kernel \(\ker \epsilon\), and normal subgroups of the codomain. This serves as a motivation for the following result.

**Proposition 4.2.** Let \(G = U_1 U_2\) be the central product of subgroups \(U_1\) and \(U_2\) defined by the epimorphism \(\epsilon : D \to G\), where \(D = U_1 \times U_2\). Let \(H \leq G\). Then \(H \leq G\) if and only if \(\epsilon^{-1}(H) \leq D\).

**Proof.** By \[3.2\] an epimorphism \(\epsilon : D \to G\) exists. Since there is a one-to-one correspondence between the set \(S_\epsilon(D) = \{K \leq D : \ker \epsilon \leq K\}\) and \(S(G)\), the set of all subgroups of \(G\), we know for any normal subgroup of \(D\) containing \(\ker \epsilon\), its image is a normal subgroup of \(G\). That is,

\[
H \leq G \iff \ker \epsilon \leq \epsilon^{-1}(H) \leq D
\]

as desired. \(\Box\)

The following diagram demonstrates the relationship established in \[4.2\] between normal subgroups of \(U_1 U_2\) and the normal subgroups of \(U_1 \times U_2\) containing the \(\ker \epsilon\):

\[
\begin{array}{ccc}
U_1 \times U_2 & \xrightarrow{\epsilon} & U_1 U_2 \\
\downarrow \leq & & \downarrow \leq \\
\epsilon^{-1}(H) & \hookrightarrow & H \\
\ker \epsilon & & \\
\end{array}
\]

Figure 1: Diagram for \[4.2\]

To further characterize normal subgroups of the central product \(U_1 U_2\), we can generalized \[4.1\]. This generalization provides an efficient way to find normal subgroups of \(U_1 U_2\) and can be easily implemented in computer algebra systems such as GAP or Sage.

**Proposition 4.3.** Let \(G = U_1 U_2\) be the central product of subgroups \(U_1\) and \(U_2\). Then \(H \leq G\) if and only if \([U_i, H] \leq U_i \cap H\) where \(i = 1, 2\).

**Proof.** Let \(H \leq G\). Then \(N_G(H) = G\) and \(U_i \leq N_G(H)\), so \([U_i, H] \leq H\) by \[2.1\] Similarly, because \(U_i \leq G\), \([U_i, H] \leq U_i\) therefore \([U_i, H] \leq U_i \cap H\) for \(i = 1, 2\) as desired.

Assume that \([U_i, H] \leq U_i \cap H\) for \(i = 1, 2\). Let \(h \in H\) and \(g \in G\), and consider \(h^g\). Because \(G\) is an internal central product, we can write \(g = u_1 u_2\) for some \(u_1 \in U_1\) and \(u_2 \in U_2\). Therefore,

\[
h^g = h^{u_1 u_2} = (u_1 u_2)^{-1} h (u_1 u_2) = u_2^{-1} u_1^{-1} h u_1 u_2 = u_2^{-1} (h h^{-1}) u_1^{-1} h u_1 u_2.
\]

Note that \(h^{-1} u_1^{-1} h u_1 \in H \cap U_1\) by assumption and denote \(h h^{-1} h u_1 = h_1\). Then \(h_1 u_2 = h_1 h^{-1} h u_1 \in H\), so \(h^g \in H\) and \(H \leq G\).

\(\Box\)

In the following diagram we give a visual representation of \[4.3\].

The following theorem provides a summary of our results shown in this section.
Figure 2: Diagram for 4.3. Observe that we must have \( H \triangleright U_1 U_2 \) for accuracy; otherwise, the commutator subgroup \([U_i, H]\) would not be a subgroup of \( U_i \cap H \) for \( i = 1, 2 \). Lines without arrowheads indicate subgroup containment with smaller groups below the groups in which they are contained.

**Theorem 4.4.** Let \( G = U_1 U_2 \) be the central product of subgroups \( U_1 \) and \( U_2 \) defined by the epimorphism \( \varepsilon : D \to G \) where \( D = U_1 \times U_2 \). Let \( H \leq G \). Then the following are equivalent:

(i) \( H \triangleright G \).

(ii) \( \varepsilon^{-1}(H) \leq D \).

(iii) \( [U_i, \varepsilon^{-1}(H)] \leq U_i \cap \varepsilon^{-1}(H) \) for \( i = 1, 2 \).

(iv) \( [U_i, H] \leq H \cap U_i \) for \( i = 1, 2 \).

Note that (ii) \( \iff \) (iii) is a direct application of 4.1.

**Example 4.5.** To see 4.4 at work, we revisited \( D_8C_4 \) from 3.5. Specifically, we determined its subgroups, their orders, and their isomorphism types. Additionally, we checked which subgroups were normal using our results.

| Isomorphism Type | # Subgroups | # Normal | Normal Subgroups |
|------------------|-------------|----------|------------------|
| Trivial group    | 1 \{1\}     | 1        | All              |
| \( C_2 \)        | 7 \( \langle r^2 \rangle, \langle s \rangle, \langle rs \rangle, \langle r^2s \rangle, \langle ry \rangle, \langle r^3y \rangle \) | 1 \( \langle r^2 \rangle \) | \( \langle r^2 \rangle \) |
| \( C_4 \)        | 4 \( \langle y \rangle, \langle r \rangle, \langle sy \rangle, \langle rsy \rangle \) | 4 All | All              |
| \( V_4 \)        | 4 \( \langle r^2, s \rangle, \langle r^2, rs \rangle, \langle r^2, sy \rangle \) | 4 All | All              |
| \( Q_8 \)        | 1 \( \langle r, sy \rangle \) | 1 All | All              |
| \( C_4 \times C_2 \) | 3 \( \langle r, y \rangle, \langle s, y \rangle, \langle rs, y \rangle \) | 3 All | All              |
| \( D_8 \)        | 3 \( \langle r, s \rangle, \langle ry, sy \rangle, \langle rsy, ry \rangle \) | 3 All | All              |
| \( D_8C_4 \)     | 1 \( D_8C_4 \) | 1 All | All              |
| Total            | 23          | 17       |                  |

We provide a sample calculation. Consider the subgroup \( \langle rsy \rangle \leq D_8C_4 \). We apply 4.4 and check that \([\langle rsy \rangle, D_8] \leq \langle rsy \rangle \cap D_8\) and \([\langle rsy \rangle, C_4] \leq \langle rsy \rangle \cap C_4\). It is sufficient to check this using the generators of each group:

\[
[rsy, y] = 1 \leq \langle rsy \rangle \cap C_4
\]
\[
[rsy, s] = r^3sysrsys = y^2 = r^2
\]
\[
[rsy, r] = r^3syrsyrsyrs = r^2
\]
\[
[\langle rsy \rangle, D_8] = \langle r^2 \rangle \leq \langle rsy \rangle \cap D_8
\]
Similarly, to see that the subgroup \( \langle s \rangle \leq D_8C_4 \) is not normal, we check that \([\langle s \rangle, D_8] \not\leq \langle s \rangle \cap D_8:\)

\[ [s, r] = sr^3sr = r^2 \not\in \langle s \rangle \]

So \( \langle s \rangle \) is not a normal subgroup of \( D_8C_4 \).

## 5 Subnormal Subgroups of Central Products

Our next goal is to characterize subnormal subgroups of central products. We use the following definition of subnormal from [3] with adjusted notation.

**Definition 5.1.** Let \( H \leq G \). We call \( H \) **subnormal** in \( G \), written \( H \sn G \), if there exists a chain of subgroups \( H_0, H_1, \ldots, H_r \) such that

\[ H = H_0 \leq H_1 \leq \ldots \leq H_{r-1} \leq H_r = G \]

This is called a **subnormal chain** from \( H \) to \( G \). If such a chain is the minimal possible chain from \( H \) to \( G \), we say that the subnormal subgroup is of **defect** \( r \).

Subnormal subgroups were completely characterized in direct products by Hauck in [8]; in the following theorems, we adopt, with some modifications, the notation for this classification as it is presented in [1].

**Lemma 5.2.** A subgroup \( K \) of \( U_1 \times U_2 \) is subnormal in \( U_1 \times U_2 \) (of defect \( r \)) if and only if

\[ [U_i, \pi_i(K), \ldots, \pi_i(K)] \leq K \cap U_i \]

for \( i = 1, 2 \).

In order to classify subnormal subgroups of central products, we seek a correspondence theorem similar to [2] in order to establish a relationship between subnormal subgroups of the direct product of two groups and subnormal subgroups of a central product of two groups. We do so using the following proposition.

**Proposition 5.3.** Let \( f : G \to H \) be an epimorphism. Let \( S_f(G) = \{ K \leq G : \ker f \leq K \} \), and let \( S(H) \) be the set of all subgroups of \( H \). Then there is a one-to-one correspondence \( K \mapsto f(K) \) between \( S_f(G) \) and \( S(H) \), where subnormal subgroups correspond to subnormal subgroups.

**Proof.** To show that subnormal subgroups correspond to subnormal subgroups under the bijection between \( S_f(G) \) and \( S(H) \), it suffices to show that for \( U, V \in S_f(G) \), \( V \leq U \) if and only if \( f(V) \leq f(U) \). To prove this, we construct an epimorphism \( \phi : U \to f(U) \) given by \( \phi(u) = f(u) \) for all \( u \in U \).

- \( \phi \) is a homomorphism: For any \( u, v \in U \), \( \phi(uv) = f(uv) = f(u)f(v) = \phi(u)\phi(v) \) because \( f \) is a homomorphism.
- \( \phi \) is surjective: Let \( x \in f(U) \). Then \( x = f(u) \) for some \( u \in U \), but \( \phi(u) = f(u) = x \), so \( \phi \) is onto and is therefore an epimorphism.

Now we wish to show that \( \ker \phi = \ker f \). Let \( a \in \ker \phi \). Then \( \phi(a) = f(a) = 1 \), so \( \ker \phi \subseteq \ker f \). Similarly, because \( \ker f \leq U \), we have that if \( b \in \ker f \), then \( \phi(b) = f(b) = 1 \); hence, \( \ker f \subseteq \ker \phi \) and we have equality, as desired.

Now, by [2], we know that there exists a one-to-one correspondence between subgroups of \( U \) containing \( \ker \phi = \ker f \) and subgroups \( f(U) \) such that normal subgroups correspond to normal subgroups. Then \( \ker f \leq V \leq U \leq G \) if and only if \( f(V) \leq f(U) \).

Now we wish to apply this result to a chain of normal subgroups to prove the desired theorem. For \( \ker f \leq A, A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_r = G \) if and only if \( f(A_0) \supseteq f(A_1) \supseteq \ldots \supseteq f(A_r) = f(G) = H \) and \( f(A) \sn H, \) as desired.

Utilizing [3] and [2], we obtain the following characterization of subnormal subgroups.

**Proposition 5.4.** Let \( G = U_1U_2 \) be the central product of subgroups \( U_1 \) and \( U_2 \) defined by the epimorphism \( \varepsilon : D \to G \) where \( D = U_1 \times U_2 \). Then \( H \sn G \) if and only if \( \varepsilon^{-1}(H) \sn D \).
Proof. We know by [5.2] that for any central product $G$ there exists an epimorphism $\varepsilon : D \to G$ such that $\varepsilon(U_i) = U_i$. By [5.3] we know that $\varepsilon$ gives rise to a one-to-one correspondence between the set of subnormal subgroups of $D$ which contain $\ker \varepsilon$ and the set of subnormal subgroups of $G$. Therefore, $H \sn G$ if and only $\varepsilon^{-1}(H) \sn D$ as desired. 

Naturally, the next question was whether we can further characterize subnormal subgroups in a way that would be useful for computations. In the following result, we provide a characterization of subnormal subgroups for central products which could be implemented in GAP or Sage.

**Proposition 5.5.** Let $G = U_1U_2$ be the central product of subgroups $U_1$ and $U_2$. Then $H \sn G$ of defect $r$ if and only if $[U_i, H, \ldots, H] \leq U_i \cap H$.

![Figure 3: Diagram for 5.5.](image)

Observe that we must have $H \sn U_1U_2$ for accuracy; otherwise, the commutator subgroup $[U_i, H, \ldots, H]$ would not be a subgroup of $U_i \cap H$ for $i = 1, 2$. Lines without arrowheads indicate subgroup containment with smaller groups below the groups in which they are contained.

The following theorem provides a summary of our results shown in this section.

**Theorem 5.6.** Let $G = U_1U_2$ be the central product of subgroups $U_1$ and $U_2$ defined by the epimorphism $\varepsilon : D \to G$ where $D = U_1 \times U_2$. Let $H \leq G$. Then the following are equivalent:

(i) $H \sn G$ of defect less than or equal to $r$.

(ii) $\varepsilon^{-1}(H) \sn D$ of defect less than or equal to $r$.

(iii) $[\overline{U}_i, \varepsilon^{-1}(H), \ldots, \varepsilon^{-1}(H)] \leq \overline{U}_i \cap \varepsilon^{-1}(H)$.

(iv) $[U_i, H, \ldots, H] \leq U_i \cap H$.

Proof. For ease of notation set $K = \varepsilon^{-1}(H)$. For (i) $\iff$ (ii) and (ii) $\iff$ (iii) [5.4 and 5.2] respectively.

(iii) $\implies$ (iv): Suppose $[\overline{U}_i, K, \ldots, K] \leq \overline{U}_i \cap K$. Because $\varepsilon$ is a homomorphism, we know that $[\overline{U}_i, K, \ldots, K] \leq \varepsilon([\overline{U}_i, K, \ldots, K]) \leq \varepsilon(\overline{U}_i \cap K)$.

Additionally, since $\varepsilon$ is onto, $\varepsilon(\overline{U}_i \cap K) \leq \varepsilon(\overline{U}_i) \cap \varepsilon(K)$. Therefore,

$[U_i, H, \ldots, H] = \varepsilon([\overline{U}_i], \varepsilon(K), \ldots, \varepsilon(K)) = \varepsilon([\overline{U}_i, K, \ldots, K]) \leq \varepsilon(\overline{U}_i \cap K) \leq \varepsilon(\overline{U}_i) \cap \varepsilon(K) = U_i \cap H$

as desired.

(iv) $\implies$ (iii): Suppose $[U_i, H, \ldots, H] \leq U \cap H$. Now $\varepsilon^{-1}([U_i, H, \ldots, H]) \leq \varepsilon^{-1}(U_i \cap H)$.

Because $\overline{U}_i \leq \varepsilon^{-1}(U_i)$, we know that $\overline{U}_i, \varepsilon^{-1}(H), \ldots, \varepsilon^{-1}(H) \leq [\varepsilon^{-1}(U_i), \ldots, \varepsilon^{-1}(H)]$ and $\overline{U}_i \cap \varepsilon^{-1}(H) \leq \varepsilon^{-1}(U_i) \cap \varepsilon^{-1}(H)$.

Without loss of generality, we consider $\overline{U}_1$. Let $[(u, 1), (h_1, h_2)]$ be an arbitrary element of $[\overline{U}_1, \varepsilon^{-1}(H)]$.

Now $[(u, 1), (h_1, h_2)] = (u, 1)^{-1}(h_1, h_2)^{-1}(u, 1)(h_1, h_2) = (u^{-1}h_1^{-1}uh, h_2^{-1}h_2 = (u^{-1}h_1^{-1}u, h_2)^{-1} \leq \overline{U}_1$. 

7
Now let \((d_1, d_2) \in D\) and \((u, 1) \in U\) be arbitrary. We find \([\langle u, 1 \rangle, (d_1, d_2)] = (u^{-1}h_1u^{-1}w_1, 1) \leq U\).

By induction, \([U, \varepsilon^{-1}(H), \ldots, \varepsilon^{-1}(H)] \leq U\), but \([U, \varepsilon^{-1}(H), \ldots, \varepsilon^{-1}(H)] \leq \varepsilon^{-1}(U) \cap \varepsilon^{-1}(H)\) so

\[
[U, \varepsilon^{-1}(H), \ldots, \varepsilon^{-1}(H)] \leq U \cap \varepsilon^{-1}(U)
\]

\[\blacksquare\]

**Remark 5.7.** If we defined multi-fold commutators \([a_1, a_2, \ldots, a_n] = [a_1, [a_2, \ldots, a_n]]\) for \(a_i\) in a group \(G\) instead of \([[a_1, a_2, \ldots, a_{n-1}], a_n]]\), our result in 5.3 would change to \(H \leq G\) is subnormal in \(G\) of defect \(r\) if and only if \([H, H, H, \ldots, H] \leq U_i \cap H\). Similar for 5.6.

### 6 Abnormal Subgroups of Central Products

Our final goal is to characterize abnormal subgroups of central products of two groups. Abnormal subgroups were originally studied due to their connection to the classification of finite groups because the normalizer of any Sylow subgroup of a group is always abnormal. The normalizer of an abnormal subgroup of a group \(G\) is as small as possible, namely, the subgroup itself; in contrast, the normalizer of a normal subgroup is large as possible, namely the whole group \(G\). In fact, in the context of maximal subgroups, normal and abnormal subgroups are precisely opposites; that is, a maximal subgroup is abnormal if and only if it is not normal.

We begin this section by presenting some technical lemmas.

**Lemma 6.1.** Let \(\varepsilon : D \to G\) be an epimorphism. Let \(g = \varepsilon(d)\) and \(W \leq G\). Then \(\varepsilon^{-1}(W^g) = (\varepsilon^{-1}(W))^d\).

**Proof.** Let \(x \in \varepsilon^{-1}(W)^d\). Then \(x = y^d\) for some \(y \in \varepsilon^{-1}(W)\). So \(\varepsilon(y) \in W \Rightarrow (\varepsilon(y))^g \in W^g\). Also, \((\varepsilon(y))^g = \varepsilon(y^d) = \varepsilon(x)\) which implies that \(\varepsilon(x) \in W^g \Rightarrow x \in \varepsilon^{-1}(W^g)\).

Now let \(x \in \varepsilon^{-1}(W^g)\). Then for some \(h \in W\), \(\varepsilon(x) = h^g = h^{d^g} = (\varepsilon(a))^d = \varepsilon(a^d)\) for some \(a \in D\). Therefore, \(x^{-1}a^d = z, \forall z \in \ker \varepsilon\). That is, \(x = a^dz^{-1} \in (\varepsilon^{-1}(W))^d\). Therefore, \(\varepsilon^{-1}(W^g) = (\varepsilon^{-1}(W))^d\).

\[\blacksquare\]

**Lemma 6.2.** Let \(\varepsilon : D \to G\) be an epimorphism, \(g = \varepsilon(d)\) and \(H \leq G\). Then \((\varepsilon^{-1}(H), (\varepsilon^{-1}(H))^d) = \varepsilon^{-1}(H, H^g)\).

**Proof.** Let \(x \in \langle H, H^g \rangle\). Then \(x = a_1b_2a_2b_2 \cdots a_nb_n\) where \(a_i \in H, b_i \in H^g\). Since \(\varepsilon\) is an epimorphism, there exists a \(y \in D\) such that \(\varepsilon(y) = x\). Furthermore, for \(a_i, b_i \in G\), there exists \(c_i, d_i \in D\) respectively such that \(\varepsilon(c_i) = a_i\) and \(\varepsilon(d_i) = b_i\). Then \(\varepsilon(y) = x \in \langle H, H^g \rangle\) implies \(y \in \varepsilon^{-1}(H, H^g) = (\varepsilon^{-1}(H), (\varepsilon^{-1}(H))^d)\) by 6.2.

Let \(x \in (\varepsilon^{-1}(H), (\varepsilon^{-1}(H))^d)\). Then \(x = k_1 \cdots k_{\ell} \cdot e_m\) where \(k_i \in \varepsilon^{-1}(H), \ell_i \in \varepsilon^{-1}(H)^d = \varepsilon^{-1}(H^g)\). So \(\varepsilon(x) = \varepsilon(k_1)e_{f_1} \cdots \varepsilon(k_{\ell})e_{f_\ell}\), and \(x \in \langle H, H^g \rangle\). Therefore, \(x \in \varepsilon^{-1}(\langle H, H^g \rangle)\).

\[\blacksquare\]

We present the formal definition of abnormal subgroups from [5].

**Definition 6.3.** Consider \(H \leq G\). \(H\) is abnormal in \(G\), written \(H \text{ abn} G\), if \(g \in \langle H, H^g \rangle \forall g \in G\).

If \(H \text{ abn} G\), we have the following properties.

(i) If \(H \leq L \leq G\), \(H \text{ abn} L \text{ abn} G\).

(ii) \(N_G(H) = H\).

(iii) If \(\phi : G \to G^\prime\) is a homomorphism, \(\phi(H) \text{ abn} \phi(G)\).

**Example 6.4.** Let \(G\) be a group. Define the diagonal subgroup of \(G \times G\) as \(\Delta = \{(g, g) : g \in G\}\) and note that \(\Delta \cong G\).

Consider \(A_5\), the alternating group of degree of five, which is a classic example of a simple, nonsolvable group. We know by [11] that \(G\) is simple if and only if \(\Delta\) is maximal in \(G \times G\). Therefore, \(\Delta\) is maximal in \(A_5 \times A_5\). Notice that \(\Delta \text{ abn} A_5 \times A_5\) because \(\Delta\) is maximal and not normal in \(A_5 \times A_5\).

In [2], abnormal subgroups were characterized for direct products of two groups, where one of the direct factors is soluble. We present this result in the following lemma.
Lemma 6.5. Let $D = U_1 \times U_2$ where either $U_1$ or $U_2$ is solvable. Let $K \leq D$. Then $K \text{abn} D$ if and only if $\pi_i(K) \text{abn} U_i$ and $K = \pi_1(K) \times \pi_2(K)$.

Note that the above theorem only applies when either $U_1$ or $U_2$ is solvable. To see why this theorem does not apply for more general direct products of finite groups, consider the following example.

Example 6.6. In $A_5$, one cannot write $\Delta$ as a product of two subgroups. Therefore, one can see that Lemma 6.5 does not apply for direct products of general finite simple groups.

In order to classify abnormal subgroups of central products, we seek a correspondence theorem similar to 2.2 and 5.3 in order to establish a relationship between abnormal subgroups of the direct product of two groups and abnormal subgroups of a central product of two groups. We do so using the following proposition.

Lemma 6.7. Let $G = U_1U_2$ be the central product of subgroups $U_1$ and $U_2$ defined by the epimorphism $\varepsilon : D \rightarrow G$, where $D = U_1 \times U_2$. Let $H \leq G$. Then $H \text{abn} G$ if and only if $\varepsilon^{-1}(H) \text{abn} D$.

Proof. For ease of notation, define $K = \varepsilon^{-1}(H)$.

Let $K \text{abn} D$. Then $x \in (K, K^g) \forall x \in D$. This implies that $\varepsilon(x) \in \langle \varepsilon(K), \varepsilon(K)^{\varepsilon(x)} \rangle$. Because $\varepsilon$ is an epimorphism, $\varepsilon(D) = G$ and, without loss of generality, $\varepsilon(x)$ corresponds to an arbitrary element $y$ of $G$. Therefore, $y \in \langle H, H^g \rangle \forall y \in G$ and $H \text{abn} G$.

Now suppose that $H \text{abn} G$. Then $y \in \langle H, H^g \rangle \forall y \in G$. Because $\varepsilon$ is surjective, we know that there exists some $d \in D$ such that $\varepsilon(d) = g$; therefore, $d$ is also contained in $\varepsilon^{-1}(\langle H, H^g \rangle)$. By 6.2 $\varepsilon^{-1}(\langle H, H^g \rangle) = \langle K, K^d \rangle$ where $\varepsilon(d) = g$. Since $\varepsilon$ is an epimorphism, $\forall d \in D, d \in \langle K, K^d \rangle$. Hence $K \text{abn} D$.

To further characterize abnormal subgroups of the central product $U_1U_2$, we can generalize 6.5. This generalization provides an efficient way to find abnormal subgroups of $U_1U_2$ and can be easily implemented in computer algebra systems such as GAP or Sage.

Theorem 6.8. Let $G = U_1U_2$ be the central product of subgroups $U_1$ and $U_2$ defined by the epimorphism $\varepsilon : D \rightarrow G$, where $D = U_1 \times U_2$ and either $U_1$ or $U_2$ is solvable. Then $H \text{abn} G$ if and only if $H = V_1V_2$ where $V_i \text{abn} U_i$ for $i = 1, 2$.

![Figure 4: Note $H$ must be abnormal in $G$ for accuracy. As shown, $H$ must be able to be written as a central product of $V_i$ and $V_2$ where $V_i$ is an abnormal subgroup $U_i$ for $i = 1, 2$. Lines without arrowheads indicate subgroup containment with smaller groups below the groups in which they are contained.](image)

Note for this result about abnormal subgroups of central products of two groups, solvability is still required.
7 Future Work

8 Acknowledgments

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