Kosterlitz-Thouless phase in systems of one-dimensional strongly interacting fermions.

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We present the ground state wave functions for systems of one-dimensional interacting fermions. It is shown that these systems undergo phase transitions similar to the Kosterlitz-Thouless one independently of the interaction details. In the limit of an infinitely strong interaction the phase transition turns into the usual second order phase transition in a chiral phase. The temperature of the phase transition is calculated.

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I. INTRODUCTION

One dimensional fermion systems were a subject of intensive studies both in field theory and in condensed matter physics. As was demonstrated by Tomonaga [1] and Luttinger [2] in their pioneering papers the long range excitations of such a system (under rather general conditions) can be expressed in terms of non-interacting bosons. These degrees of freedom were made explicit in the elegant method of bosonization proposed by Mattis and Leeb [3]. The recent interest in this field is mainly due to the development of submicron techniques which allowed to produce very pure quantum wires. In such wires only few levels (or sometimes even one) corresponding to the quantization of electrons in perpendicular directions are occupied. In spite of the fact that the number of the experimental papers in the field is still not numerous (see, e.g. [4], [5], [6], [7]) the systems under discussion are, in principle, accessible by experiment.

The bosonization technique allows one to calculate all correlation functions of the “density-density” type for systems of interacting fermions in one dimension. However it says nothing about the ground state of this system. The correlation functions reveal a number of anomalies of the fermion system (see [8], [9], [10]): they have oscillating contributions with wave vectors equal to $2p_f$ or $4p_f$ which decay very slowly with distance. In the literature these contributions were interpreted as follows: the oscillations with Fermi momentum, $p_f$, doubled were related to the Peierls instability (connected with the charge density wave [11], [9]) while the oscillations with $4p_f$ frequency were interpreted as a marginal Wigner crystal [12]. It is commonly believed that the system under discussion is a kind of normal liquid, because quantum fluctuations destroy any order parameter, and a phase with long-range order is impossible even in the zero temperature region [4]. The common point of view was put into the words: “Luttinger liquid is a normal (not symmetry-broken) metallic phase” [13] with gapless boson spectrum. This viewpoint is usually supported by the Landau theorem [14] which states that in one dimension a long-range order cannot exist [15]. However two more points should be taken into account:

- It is well-known that the Landau theorem is applicable not to all systems in low dimensions. Some of them appear to be in the Kosterlitz-Thouless phase [16]. In such a phase the order parameter density tends to zero in an infinite system. However a long-range order exists because correlation functions decay as some power of the distance. This means that the correlation is present in a whole specimen. As a result the macroscopic properties of the system are quite similar to the system with broken symmetry. A Kosterlitz-Thouless phase can form if the gapless excitations, which should be present in the system after spontaneous symmetry breaking due to the Goldstone theorem [17], [18], do not interact.

At non-zero temperatures ($\Theta \neq 0$) this phase can appear in two-dimensional systems; at $\Theta = 0$ a Kosterlitz-Thouless phase is possible in one dimension. (The crossover temperature and its relation to the phase transition temperature $\Theta_c$ will be considered below, see section [11].

- The Goldstone theorem itself can be broken in one dimensional models with Adler-Schwinger anomaly. For that to happen the interaction has to be strong enough. As an illustration one can consider the massless Schwinger model [19]. In the Coulomb gauge this model is a particular case of the Luttinger model where the potential is a linear function of the distance between electrons. This gives rise to the Goldstone theorem violation: all excitations have a gap in spite of spontaneous breaking of the chiral symmetry. For this reason, excitations cannot suppress the long-range order at temperatures smaller than the gap. As a result one has a second order phase transition in one dimension even at finite temperature [20].

This makes the statement that the Luttinger liquid at $\Theta = 0$ is in an unbroken phase doubtful.
In order to clarify this question we calculate the wave functions of the ground states in the Tomonaga-Luttinger model. While the details of these wave functions could depend on the interacting potential, qualitatively all possible ground states reveal the same phenomenon.

In one dimension the Fermi surface reduces to two isolated points in phase space \( p = \pm p_f \). One can neglect transitions between these two points. This is a good approximation, at least if the potential is a decreasing function of the momentum transfer. As a result, the number of electrons near every point (left- and right-particles) should be conserved and the system acquires complementary (chiral) symmetry. It is this symmetry which, as we will see, breaks down spontaneously in the model.

The Fermi surface in one dimension is absolutely unstable: even infinitesimal electron-electron interaction leads to instability of the electron ground state. The electron distribution function is of the order of \( 1/2 \) near the Fermi surface. This means that there is a hole located near every electron. Naturally, they attract each other and form a kind of bound state consisting of a right electron and a left hole (\( RL \)-pair). This is quite similar to the formation of a Cooper pair in a superconductor but the quantum numbers of the bound state are different: instead of non-zero electric charge it has non-zero chirality [22].

Of course, this fact itself is not enough to speak about a new correlated phase. We will check using the explicit ground state wave function constructed in the paper that long-range order is indeed present in the system. At low temperatures Luttinger liquid undergoes a phase transition similar to the Kosterlitz-Thouless one.

We will see that in the limit of the infinitely strong interaction and at zero temperature the system is in a phase with broken chiral symmetry and non-vanishing order parameter. The properties of the Luttinger liquid in this limit are quite analogous to the properties of the massless Schwinger model where the spontaneous breakdown of the chiral symmetry is well-known. This can be expected as the interaction in the Schwinger model is also infinite (it grows with distance). On the other hand, in contrast to the Schwinger model, the spectrum of the Luttinger model remains gapless. The only effect of the interaction on the spectrum is the renormalization of the Fermi velocity \( v_f \) which is proportional to the strength of the interaction.

If the interaction in the Luttinger model is considered to be finite, the order parameter vanishes (in the infinite system). The system appears to be in a Kosterlitz-Thouless phase with correlators which decay as some power of the distance. However the properties of this phase are rather close to those for a system with non-zero order parameter. The Kosterlitz-Thouless system transforms smoothly to the phase with non-zero order parameter as the coupling constant increases.

Another matter is the temperature \( \Theta_c \) of the phase transition to the unbroken phase. This temperature is always of the order of the gap in the spectrum. For this reason \( \Theta_c \) is finite for the Schwinger model and of order of \( v_f^2/L \) for the Luttinger liquid (where \( L \) is the size of the system). In other words \( \Theta_c \) tends to zero for an infinite system. However for real systems \( \Theta_c \) is not so small. Indeed, if one takes \( v_f \sim 10^7 \text{ cm/sec} \) and \( L \sim 10^{-2} \text{ cm} \) then \( \Theta_c \sim 1^\circ \text{ K} \). This means that the theory based on the limit \( L \to \infty \) can be inadequate. Of course, in order to discuss \( \Theta_c \) for finite size systems we have to define the notion of phase transition in this case. This is done in Section III.

We will see that a non-zero order parameter is the reason of the anomalies in the correlation functions mentioned above. Anomalies exist even at temperatures higher than \( \Theta_c \) owing to fluctuations of the chiral phase in the phase with non-broken chiral symmetry. There is a clear-cut distinction between the chiral phase and the Peierls’s phase. The latter is the second order phase transition in the phonon system and the chiral symmetry of electrons is broken explicitly, in the Hamiltonian. On the contrary the chiral transition originates from the spontaneous symmetry breaking in the electron system.

Finally, let us discuss effects which may indicate the existence of a condensate. Obviously a charge-neutral condensate can not reveal itself in experiments associated with charge transfer. However it contributes to the effects concerned with energy currents and should not transfer heat. Hence one can think about thermal anomalies related to the condensate. In fact, we keep in mind the effect similar to the thermomechanical effect in superfluid helium. (The temperature decreases with increase of the superfluid mass [23]). We plan to discuss this problem in a separate publication.

The paper is organized as follows. In Section II we introduce the Hamiltonian, definitions of left- and right-particles and so on. We present our results and discussion in Section III. We relegate the derivation to the next sections as the calculation is a bit cumbersome.
II. NOTATIONS AND GENERAL EQUATIONS.

We begin with the usual expression for the Hamiltonian of interacting spinless electrons in one dimension:

\[ H = \int dx \hat{\Psi}^\dagger (x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} - \mu \right) \hat{\Psi} (x) \]
\[ + \int dx dy \hat{\Psi}^\dagger (x) \hat{\Psi} (y) V (x - y) \hat{\Psi}^\dagger (y) \hat{\Psi} (y). \]  
(1)

Here \( \hat{\Psi} (x) \) is the electron field, \( m \) is the electron mass and \( \mu \) is the chemical potential. \( V (x - y) \) is the electron-electron interaction which we will discuss below \( (\hbar = 1) \).

As usual let us separate in the electron wave functions left- and right- particles \[24\]:

\[ \hat{\Psi} (x) = \exp (ip_f x) \hat{\Psi}_R (x) + \exp (-ip_f x) \hat{\Psi}_L (x). \]  
(2)

It is implied here that the wave functions \( \hat{\Psi}_{R,L} (x) \) are varying over distances much larger than \( 1/p_f \). We also restrict ourselves to the Tomonaga-Luttinger model and assume that the characteristic scale of the potential is large as compared to \( 1/p_f \). Lastly, for simplicity we consider only an electrically neutral system where the a positive charge of ions is distributed homogeneously along the channel.

Let us proceed to the electron-hole representation for the right (left) particles:

\[ \hat{\Psi}_{R,L} (x) = \int_0^\infty \frac{dp}{2\pi} \left( \exp (\pm ipx) \hat{a}_{R,L} (p) + \exp (\mp ipx) \hat{b}^\dagger_{R,L} (p) \right) = \hat{a}_{R,L} (x) + \hat{b}^\dagger_{R,L} (x). \]  
(3)

Here \( \hat{a}^\dagger (\hat{a}) \) and \( \hat{b}^\dagger (\hat{b}) \) are creation (annihilation) operators for electrons and holes. The Hamiltonian of eq.(1) can be written for neutral system in terms of electron density operator

\[ \varrho (x) = \varrho_R (x) + \varrho_L (x) \]

in the form

\[ H = \int dx \left[ \hat{\Psi}^\dagger_R (x) v_f (-i \partial_x) \hat{\Psi}_R (x) + \hat{\Psi}^\dagger_L (x) v_f i \partial_x \hat{\Psi}_L (x) \right] + \int dx dy \varrho (x) V (x - y) \varrho (y). \]  
(4)

Here \( v_f \) is the Fermi velocity, and

\[ \varrho_{R,L} (x) = \hat{a}^\dagger_{R,L} (x) \hat{a}_{R,L} (x) - \hat{b}^\dagger_{R,L} (x) \hat{b}_{R,L} (x) + \hat{a}^\dagger_{R,L} (x) \hat{b}^\dagger_{R,L} (x) + \hat{b}^\dagger_{R,L} (x) \hat{a}_{R,L} (x). \]  
(5)

The expression \(4\) coincides with the Hamiltonian of the Tomonaga-Luttinger model (see, e.g., Ref. \[24\]).

Let us discuss now the electron-electron interaction \( V(x - y) \). Its form depends on the relation between the usual 3D screening radius \( R_D \) (for simplicity we will consider the case of Debye screening) and the transverse size of the channel \( d \). Indeed one has to take into account that the electrons are one-dimensional only for distances \( |x - y| \) which are much larger than \( d \). Therefore if \( R_D \ll d \) one can use a point-like interaction \( V_0 \delta (x - y) \). This is the case for metals \[24\]. In the opposite case \( R_D \gg d \), semiconductor, one should use an ordinary Coulomb potential. In one dimension the Fourier transform of the Coulomb potential is logarithmically divergent. This divergence is regularized by the smaller of two quantities, either \( p_f \) or the inverse channel size, \( 1/d \). Thus \[27\]

- At \( R_D \ll d \)

\[ V (p) = V_0, \]  
(6)

- At \( R_D \gg d \)

\[ V (p) = \begin{cases} 
2e^2 \log \left( \frac{2p_f}{|p|} \right) & \text{for } 1/p_f \gg d \\
2e^2 \log \left( \frac{|p|}{2p_f} \right) & \text{for } 1/p_f \ll d 
\end{cases} \]  
(7)
The Hamiltonian of the Luttinger model, eq. (4), presented in terms of electrons and holes is completely defined without any additional regularization of electron operators. In particular, the commutator of the \( R \)- and \( L \)-densities in this representation reproduces the well-known Schwinger anomaly \( 19 \):

\[
[\varrho_{R,L}(x), \varrho_{R,L}(y)] = \pm \frac{i}{2\pi} \frac{\partial}{\partial x} \delta(x-y).
\]

These relations are the starting point of the bosonization technique. Usually one derives eq. (8) regularizing the product of \( \Psi \)-operators by a small shift of their arguments \( 28 \). This is not necessary, however, in the electron-hole representation \( 20 \) as the creation-annihilation operators for \( R, L \) electrons \( (\hat{a}^\dagger_{R,L}(x)) \) and holes \( (\hat{b}^\dagger_{R,L}(x)) \) are non-local in the coordinate space:

\[
\{\hat{a}^\dagger_R(x), \hat{a}_R(x_1)\} = \{\hat{b}^\dagger_R(x), \hat{b}_R(x_1)\} = \frac{1}{2\pi i} \cdot \frac{1}{x-x_1-i\delta}.
\]

\[
\{\hat{a}^\dagger_L(x), \hat{a}_L(x_1)\} = \{\hat{b}^\dagger_L(x), \hat{b}_L(x_1)\} = \frac{1}{2\pi i} \cdot \frac{1}{x_1-x-i\delta}.
\]

(in momentum space these anticommutators are \( \delta \)-functions). Using these anticommutators for the densities of right and left electrons \( \varrho_{R,L}(x) \) in the form of eq. (10) we immediately reproduce the Schwinger anomaly. This means that, being formulated in the electron-hole representation, our theory is completely determined without any further redefinition of the density operators.

The Hamiltonian \( 4 \) is invariant both under vector transformations:

\[
\Psi_R(x) = e^{i\alpha v} \Psi_R, \quad \Psi_L(x) = e^{i\alpha v} \Psi_L
\]

and under chiral transformations

\[
\Psi_R(x) = e^{i\alpha_c} \Psi_R, \quad \Psi_L(x) = e^{-i\alpha_c} \Psi_L.
\]

The first invariance leads to conservation of electrical charge (the number of left electrons plus the number of right ones), the second means that chiral charge (the difference of numbers of right and left electrons) is conserved. However we will see below that the ground state of the model is constructed in such a way that the second symmetry can be spontaneously broken.

### III. APPROACH, RESULTS AND DISCUSSIONS

The standard approach to systems of many particles is based on Green functions. The one-particle Green function gives the information about the spectrum of excitations; the many-particle Green functions allow one to calculate different correlation and response functions. Of course, the Green functions give some information about the wave functions of the states but this information is indirect.

In principle, one can obtain the wave functions of stationary states (and, in particular, of the ground state) by solving the corresponding Schrödinger equation directly. However for systems with infinite number of degrees of freedom this equation is too complicated. A more practical approach can be based on the evolution operator \( 29 \) which is a sum:

\[
S(T) = \sum_{m} |n><n| \exp(-iHT)|m><m|.
\]

Here \( |n> \) are the exact wave functions of the Hamiltonian \( H \) in the second quantized representation, \( T \) is the time of observation. The evolution operator determines the evolution of an arbitrary initial wave function \( (<m|) \) from the time \( t = 0 \) up to final states \( |n> \) (at \( t = T \)). (We imply from now that the Schrödinger representation for operators with time-dependent wave-functions is used.)

Formula \( 13 \) suggests the general method to obtain wave functions. One has to calculate first the evolution operator and represent it as a sum of time-dependent exponentials. The coefficients in front of these exponentials are products of exact wave functions and their complex conjugates. In order to extract the ground state wave function one has to take the limit \( T \to \infty \) (we add an infinitesimal imaginary part to the energy). Proceeding to Euclidean time \( (T \to -i/\Theta) \) we see that evolution operator determines the density matrix for the equilibrium system at non-zero temperature (see end of the Section \( 14 \)).
The advantage of this method is that the evolution operator can be written explicitly as a functional integral with definite boundary conditions (see eq. (20)). This functional integral is rather simple for the Luttinger model with Hamiltonian (11) and can be calculated exactly. This allows one to construct wave functions of all states in the model and, in particular, the ground state. This will be quite enough in order to demonstrate the symmetry breaking and to calculate the temperature of the phase transition.

We will keep the size of the system finite. This is important not only to regularize infrared divergences in the system but mostly because of the peculiar situation with the temperature $\Theta_c$ mentioned in the Introduction. However, first, we have to define the concept of phase transition in finite size systems.

Usually, the critical temperature is defined as a point where thermodynamic quantities have a singularity. Of course, this is the case only in an infinite system because all singularities smear out if the size of the system is finite. The same is true for the coherence length — it cannot be larger than the size of the system.

In this paper we will adopt the point of view suggested by Landau in order to describe the second order phase transitions (20). He introduced the order parameter as the main quantity for the description of phase transitions related to the spontaneous symmetry breakdown. By definition, the order parameter is zero in the high symmetry transitions [30]. He introduced the order parameter as the main quantity for the description of phase transitions.

This quantity is not invariant under transformations of eq. (12) and should be zero if the chiral symmetry remains broken. In the case related to the spontaneous symmetry breakdown. By definition, the order parameter is zero in the high symmetry transitions [30]. He introduced the order parameter as the main quantity for the description of phase transitions.

The Kosterlitz-Thouless phase represents the intermediate case when

$$\Delta \sim L^{\alpha_T}, \quad 0 < \alpha_T < 1,$$

so $\Delta$ is still infinite in the thermodynamic limit while the density or order parameter $\Delta/L$ vanishes. Let us point out that $\Delta$ appears to be non-zero even at $\Theta > \Theta_c$ due to fluctuations of the broken phase in the phase with higher symmetry. What is important for the latter case is the fact that $\Delta$ does not increase with $L$.

Intensive thermodynamic quantities remain smooth for finite size system even at the point of the phase transition. However what matters is the fact that they depend explicitly on the system size and tend to infinity (or acquire a jump) at $L \to \infty$.

Usually one proves that the system is in the Kosterlitz-Thouless phase by investigating the behaviour of the four-fermion correlator which does not break the chiral invariance (below in section IV A 1 we will consider such a correlator, namely the probability to find an $RL$-pair at a large distance $r$ from the $LR$ pair). If such a correlator decreases sufficiently slowly with the distance, the system is in the Kosterlitz-Thouless phase. The limiting case when the correlator remains constant at large distances corresponds to a non-zero density of the order parameter and ordinary broken symmetry. In fact, this definition of the Kosterlitz-Thouless phase is equivalent to our definition given above but definition (13) is more convenient for us.

In one dimension the Kosterlitz-Thouless phase can exist only at $\Theta = 0$ or, to be more precise, for the temperatures which tend to zero at $L \to \infty$. There is no need in microscopic theory in order to estimate $\Theta_c$: one can use general phenomenology applicable to all Kosterlitz-Thouless systems (see, e.g. [24]).

Let us assume that the chiral symmetry is indeed spontaneously broken in the Luttinger model (of course, this can be proved only in microscopic theory). According to the Goldstone theorem the chiral phase $\alpha_c$ (the phase of the $\Psi_R^\dagger \Psi_L$ operator) becomes a massless boson field. In the long-range limit only fluctuations of this field are relevant and its effective Lagrangian reduces to (Euclidean time $\tau = it$ is used, as we want to consider non-zero temperatures below):

$$S_{eff}[\alpha_c] = \frac{V^2}{2} \int drdx \left[ (\partial_t \alpha_c)^2 + (w \partial_x \alpha_c)^2 \right],$$

where $V$ and $w$ are phenomenological constants (calculable in microscopic theory).

To judge if the system is in the Kosterlitz-Thouless phase it is enough to consider the behaviour at large distances $|x - y|$ of the chirality conserving correlator:

$$F(x - y) = < \Psi_R^\dagger(x) \Psi_L(x), \Psi_L^\dagger(y) \Psi_R(y) >.$$  

One can neglect fluctuations of the modulus of the operator $\Psi_R^\dagger(x) \Psi_L(x)$ (as well as higher derivatives in the effective action for chiral phase). Then the correlator (14) reduces to:

$$F(x - y) = const \int d\alpha_c \exp(-S[\alpha_c]) e^{2i\alpha_c(x)} e^{-2i\alpha_c(y)}.$$  

(16)
Calculating the latter integral at $\Theta = 0$ we get:

$$F(x - y) \sim \exp \left[ 2V^{-2} \int \frac{dk dk'}{(2\pi)^2} \frac{\sin^2 [1/2 (k, x - y)]}{(k_0^2 + w^2 k^2)} \right].$$ \hspace{1cm} (19)

The two-dimensional integral (one space one time dimension) in the exponent \hspace{1cm} diverges logarithmically and hence

$$F(x - y) \sim \frac{V^2}{(k_{\text{max}} |x - y|)^{1/2}}.$$ \hspace{1cm} (20)

This proves the existence of Kosterlitz-Thouless phase at $\Theta = 0$.

If the temperature is non-zero the integral over $k_0$ in eq.(19) should be replaced by a sum over discrete values $k_0 = 2\pi n \Theta$ ($n$ integer). At high temperatures only the term with $n = 0$ survives at large distances and we are left with an one-dimensional integral in $k$ which leads to the correlator exponentially decreasing with distance:

$$F(x - y) \sim V^2 \exp \left( -\frac{\Theta}{2\pi V^2 w^2} |x - y| \right).$$ \hspace{1cm} (21)

Clearly this correlator describes the unbroken phase.

A power-like behavior of the correlator of (20) is valid in the region $|x - y| < w/\Theta$.

For

$$\Theta < \Theta_c \equiv w/L,$$

this takes place for the whole specimen, i.e. the system is in a broken phase. The temperature $\Theta_c$ is a temperature of the phase transition.

One can recognize in this estimate the excitation energy with the smallest momentum possible in a finite size system. In the Luttinger model this energy is equal to $\omega_{\text{min}} = 2\pi v_f L$ with renormalized Fermi velocity $v_f = v_f \sqrt{1 + V_0/\pi v_f}$ \hspace{1cm} (9). We see that if the spectrum of excitations is gapless (like in the Luttinger model), then the temperature of phase transition is inversely proportional to the specimen length. This result can be obtained in the microscopic theory as well (see Section IV A).

Turning to the microscopic theory we begin with the simplest case: the short range potential in the limit of an infinitely strong interaction \hspace{1cm} (6)

$$\pi v_f V_0 \ll 1.$$ \hspace{1cm} (22)

In the leading order in this parameter the evolution operator appears to be very simple and the wave function of the ground state can be represented in close form. In the temperature region

$$\Theta_{\text{chiral}} = \frac{2\pi v_f L}{\omega_{\text{min}}} \ll \Theta \ll \Theta_c = \omega_{\text{min}},$$ \hspace{1cm} (23)

the ground state wave function is of the form \hspace{1cm} (31):

$$|\Omega > = \sqrt{Z_0} \exp \left[ \int dx \exp (i\theta) \hat{a}_R^\dagger (x) \hat{b}_L^\dagger (x) + \int dy \exp (-i\theta) \hat{a}_L^\dagger (y) \hat{b}_R^\dagger (y) \right] |F >.$$ \hspace{1cm} (24)

Here $|F >$ is the filled Fermi sphere and $Z_0$ is the normalization coefficient. There is an infinite set of degenerate ground states labeled by the continuous parameter $\theta$ which has the meaning of the order parameter phase.

One can see the symmetry breaking immediately because the wave function $|24\rangle$ is not invariant under the chiral transformation \hspace{1cm} (12). Besides one can check directly that $\Delta \propto L$ \hspace{1cm} (32). It means that there is a second-order phase transition in this limit.

The wave function of $|24\rangle$ is a mixture of states with different chirality. (We assign chirality $+1$ to a right electron and a left hole and $-1$ to their counterparts. So bosons in eq.(24) are neutral in terms of electric charge but have a non-zero chirality, $\pm 2$). Such a ground state implies that the states with different chirality are all degenerate in energy. This is a typical situation for systems with condensate: the addition of one pair to the condensate does not
cost any energy. However this degeneracy is possible only if the size of the system is large enough, namely, we will see that it should be \( L \gg L_{\text{min}} \sim 2\pi v_f / \Theta \). At \( L \leq L_{\text{min}} \) the ground state has fixed chirality (equal to zero) (see eq. (22)) and the order parameter \( \Delta \) vanishes, i.e. there is no spontaneous symmetry breaking. These considerations put a lower bound on the temperature region where a chiral phase can exist: \( \Theta \gg \Theta_{\text{chiral}} \). So, \( \Theta_{\text{chiral}} \) is the degeneration temperature.

Let us estimate the density of chiral pairs in the ground state. The wave function \( \Psi \) implies that all electrons are bound to the pairs. Hence the density of RL coincides with the density of R-electrons:

\[
N_R (p) = \Theta < \Omega | \hat{a}_{R} (p) \hat{a}_{R} (p) | \Omega > = L/2.
\]  

(25)

(see [22]). This quantity reflects a well-known fact: the distribution function of electrons is of the order of 1/2 near the Fermi surface [21]. If the interaction is infinitely large all electrons and all holes are bound in exciton-like pairs. As a result we get the value \( L/2 \) which is maximal possible.

In the model under consideration \( N_R (p) \) is momentum independent and the total number of pairs \( N_R \) diverges at large \( p \). (This is the defect of the point-like electron-electron interaction.) The sum over all states should be restricted either by \( p_f \) or, at \( p_f d \gg 1 \), by the inverse size of the channel because at larger \( p \) electrons cannot be considered as one-dimensional (see [22]). As a result

\[
N_R \sim \frac{L}{4\pi d}.
\]

Thus the number of pairs \( N_R \) is only a small fraction of the total number of electrons \( (Lp_f / 2\pi) \). This does not mean, of course, that in that case the Luttinger liquid behaves like a normal one. The reaction of the system to the slowly varying external fields is determined completely by the electrons near the Fermi surface which are all paired. This situation reminds superfluid helium where (even at zero temperature) the density of the condensate is only few percents of the total one. Nevertheless the whole mass of helium is superfluid [33].

Let us proceed with the region of high temperatures: \( \Theta \gg \Theta_c = \omega_{\text{min}} \). In this region the macroscopic order parameter \( \langle \hat{c} \rangle \) is proportional not to the volume of the system but to some characteristic length \( \zeta (\Theta) \propto \pi v / (\Theta - \Theta_c) \) (see the end of section IV A) and the density of the order parameter \( \Delta / L \) vanishes in the limit \( L \to \infty \) as it should be. Hence the temperature \( \Theta_c \) has indeed the meaning of the temperature of phase transition from the symmetrical phase to the phase with broken chiral symmetry.

The length \( \zeta (\Theta) \) plays the role of coherency length in our system. At lengths less than \( \zeta (\Theta) \) the wave function of the system coincides with the coherent exponent [24]. However at larger distances the order disappears.

The macroscopic order parameter \( \Delta \) can be non-zero even in the symmetrical phase due to fluctuations of the broken to the unbroken phase. What matters is the behaviour of \( \Delta \) with the size of the system. If \( \Delta \) does not increase with \( L \) (\( \Delta \propto \zeta \) with \( \zeta \) finite), we deal with unbroken phase, where \( \Delta \) increases with \( L \) the long range order appears (and \( \zeta \sim L \)). Such a dependence \( \zeta \) on \( L \) can be considered as the definition of symmetry breakdown for a finite size system too.

On the other hand it is obvious from such a definition that a temperature of the phase transition in the finite size systems can be defined only up to \( 1/L \) corrections and the phase transition is smooth within the \( 1/L \) region around the temperature of the phase transition. In the Luttinger model, where the temperature \( \Theta_c \) itself is of \( 1/L \) order, we can define \( \Theta_c \) only up to a factor of order of unity. This is the price we have to pay for considering a phase transition of a large but finite size system. However there is still a clear-cut distinction between the case with a correlation length of the order of the size of the system (broken phase) and the case when \( \zeta \ll L \) (unbroken phase).

As was already pointed out in the Introduction, the case of the infinitely strong interaction is very special. We will see that if the interaction is finite, then the macroscopic order parameter \( \Delta \) grows with the system size but more slowly than \( L \) (at \( \Theta \ll \Theta_c \)). In the case of a short range potential (see Section IV A) \( \Delta \) behaves as some power of \( L \). This corresponds literally to the definition of the Kosterlitz-Thouless phase. If we consider the potential of Coulomb type, then \( \Delta \) depends on \( L \) in a more complicated way (section IV C) but still \( \Delta \) increases with the size \( L \). Physically, this case is quite similar to the usual Kosterlitz-Thouless one.

To summarize, the Luttinger model at \( \Theta < \Theta_c \) is always in Kosterlitz-Thouless phase with broken chiral symmetry. At \( \Theta \sim \Theta_c \) it undergoes a phase transition which in the limit of infinite interaction turns into the ordinary second order phase transition.

IV. GROUND STATE OF THE TOMONAGA-LUTTINGER MODEL.

The evolution operator [13] of the quantum system can be represented as a functional integral with definite boundary conditions (see, e.g. [21]). Usually one derives this representation for boson systems, for the sake of completeness we give in Appendix A the derivation for fermions.
The theory with arbitrary electron-electron interaction can be reduced to a theory in an external field by means of the Hubbard-Stratonovich transformation (see below, eq. (27)). One has to integrate over the value of the external field in order to return to the original 4-fermion interaction. For this reason we consider first the evolution operator for one-dimensional electrons placed into an external field $\Phi(x,t)$. It is of the following form:

$$\hat{S}(\Phi) = \int_{\{\overline{\Psi}, \Psi\}} D\Psi D\overline{\Psi} \exp S(\overline{\Psi}, \Psi).$$

(26)

Here $\overline{\Psi}, \Psi$ is the electron field (Grassmann variables) and $S$ is the action:

$$S = i \int_0^T dt \int dx \overline{\Psi}_R (x,t) [i \partial_t - v_f i \partial_x + \Phi (x,t)] \Psi_R (x,t)
+ (R, v_f \leftrightarrow L, -v_f).$$

(27)

Integration over $\overline{\Psi}, \Psi$ in eq. (26) is performed with given boundary conditions at $t = 0$ and $t = T$:

At $t \rightarrow +0$

$$\Psi_{R,L} (x,t) = \hat{a}_{R,L} (x) + \text{arbitrary negative frequency part}$$

$$\overline{\Psi}_{R,L} (x,t) = \hat{b}_{R,L} (x) + \text{arbitrary negative frequency part}$$

At $t \rightarrow T - 0$

$$\Psi_{R,L} (x,t) = \hat{b}_{R,L}^\dagger (x) + \text{arbitrary positive frequency part}$$

$$\overline{\Psi}_{R,L} (x,t) = \hat{a}_{R,L}^\dagger (x) + \text{arbitrary positive frequency part}$$

(28)

The creation operators of electrons and holes $\hat{a}^\dagger, \hat{b}^\dagger$ are the variables which enter the wave functions of the states in the sum of eq. (13). The annihilation operators $\hat{a}, \hat{b}$ enter the conjugate wave functions. They anticommute: $\{\hat{a}, \hat{a}^\dagger\} = \{\hat{b}, \hat{b}^\dagger\} = 0$ since they belong to different instances of time as long as one calculates the evolution operator (see Appendix A for details).

It is possible to separate explicitly the dependence on creation-annihilation operators for the evolution operator in a given external field determined by the functional integral (26). Let us introduce new integration variables:

$$\Psi_{R,L} = \Psi_{R,L}^0 + \chi_{R,L}$$

$$\overline{\Psi}_{R,L} = \overline{\Psi}_{R,L}^0 + \overline{\chi}_{R,L}.$$

(29)

The saddle-point fields $\Psi_{R,L}^0$ are supposed to obey the Schrödinger equation in the external field $\Phi(x,t)$ with given boundary conditions (28). The “quantum” fields $\chi_{R,L}(x,t)$ are arbitrary but obey zero boundary conditions: $\chi_{R,L}(0) = \chi_{R,L}(T) = 0$.

The solutions $\Psi_{R,L}^0$ can be represented in terms of the Feynman Green function in the finite time $G_{R,L}$ which is defined as follows. It is a solution of the Schrödinger equation:

$$[i \partial_t + v_f i \partial_x + \Phi (x,t)] G_{R,L} (x,t; x_1, t_1) = i \delta^{(2)} (x-x_1, t-t_1)$$

(30)

with the following boundary conditions: at $t \rightarrow +0$ the Green function $G_{R}(x,t; x_1, t_1)$ should coincide with the Green function of free fermions in the lower semiplane of the complex variable $x$ (being arbitrary in the upper semiplane). At $t \rightarrow (T - 0)$ it coincides with the free Green function in the upper semiplane. For the Green function of left electrons $G_{L}(x,t; x_1, t_1)$ one has to exchange upper and lower semiplanes.

The free Feynman Green function is equal to $3$:

$$G_{0,R,L} (x,t; x_1, t_1) = \frac{1}{2\pi i} \left[ v_f (t-t_1) \mp (x-x_1) - i \delta \text{sign} (t-t_1) \right]^{-1}.$$  

(31)

In one dimension the Schrödinger equation (30) can be solved for an arbitrary external field $\Phi(x,t)$:

$$G_{R,L} (x,t; x_1, t_1) = G_{R,L}^0 (x,t; x_1, t_1) \times$$


\[ \times \exp \left[ i \int_0^T dt' \int dy \Phi(y, t') \left( G_{R,L}^0 (x, t; y, t') - G_{R,L}^0 (x_1, t_1; y, t') \right) \right]. \tag{32} \]

Now it is easy to verify that the saddle point fields \( \Psi_{R,L}^0 \) can be expressed in terms of these Green functions as follows:

\[ \Psi_{R,L}^0 (x, t) = \int dx' \left[ G_{R,L} (x, t; x', 0) \hat{a}_{R,L} (x') - G_{R,L} (x, t; x'T) \hat{b}_{R,L}^\dagger (x') \right] \]

\[ \overline{\Psi}_{R,L}^0 (x, t) = - \int dx' \left[ G_{R,L} (x', 0; t, x) \hat{b}_{R,L} (x') - G_{R,L} (x', T; x, t) \hat{a}_{R,L}^\dagger (x') \right]. \tag{33} \]

In order to check that these fields obey the required boundary conditions let us note that \( \hat{a}_R (x) \) and \( \hat{b}_R (x) \) are regular in the upper semiplane (see eq. (3)). Therefore the positive frequency part at \( t \to +0 \) of \( G_R (x, t, x_1, t_1) \) is determined by the pole contribution at \( x' = x + i \delta \) and is equal to \( \hat{a}_R (x) \) as it should be. The second term in eq. (33) gives the negative frequency part which is arbitrary. Similarly, one can check the boundary condition also at \( t \to (T - 0) \). Inside the time interval \((0, T)\) the saddle point fields satisfy the Schrödinger equation as is seen from eq. (30) for the Green functions.

The contribution of the saddle-point field to the action is:

\[ S_0 = \sum_{i=R,L} \int dx dx' \left[ \hat{b}_i (x') G_i (x', 0; x, \epsilon) \hat{a}_i (x) + \hat{a}_i^\dagger (x') G_i (x', T; x, T - \epsilon) \hat{b}_i^\dagger (x) \right. \]

\[ \left. - \hat{a}_i^\dagger (x') G_i (x', T; x, 0) \hat{a}_i (x) - \hat{b}_i (x') G_i (x', 0; x, T) \hat{b}_i^\dagger (x) \right]. \tag{34} \]

Since the saddle point fields obey the Schrödinger equation there is no term linear in the quantum field \( \chi \) in the action.

Dependence of the evolution operator in the external field on the creation-annihilation fermion operators is completely determined by eq. (34). Integral over quantum fluctuations produces the determinant of the Schrödinger operator in the external field \( \Phi \) (it is calculated in the Appendix B):

\[ \log \left[ \text{Det} \Phi (T) \right] = - \frac{1}{4 \pi} \int_0^T dt dt' \int_{-\infty}^{\infty} \frac{dp}{2 \pi} \Phi (-p, t) \Phi (p, t_1) \exp \left[ -i |p| \psi_f |t - t_1| \right]. \tag{35} \]

The complete expression for the evolution operator in the external field has the form:

\[ \hat{S} (\Phi) = \exp \left( S_0 + \log \left[ \text{Det} \Phi (T) \right] \right) |F > < F|, \tag{36} \]

Now we can express the evolution operator for the system of interacting fermions in terms of this operator. We will use the well-known identity [32]:

\[ \exp \left[ - \frac{i}{2} \int_0^T dt \int_{-\infty}^{\infty} \frac{dp}{2 \pi} V (p) \varrho (p, t) \varrho (-p, t) \right] = \frac{1}{N} \int D\Phi \exp \left[ i \int_0^T dt \int_{-\infty}^{\infty} \frac{dp}{2 \pi} \Phi (p, t) \Phi (-p, t) V^{-1} (p) \right. \]

\[ \left. - i \int_0^T dt \int_{-\infty}^{\infty} \frac{dp}{2 \pi} \left( \varrho (p, t) \Phi (-p, t) + \varrho (-p, t) \Phi (p, t) \right) \right]. \tag{37} \]

Here \( V (p) \) is the Fourier transform of the interacting potential. The normalization coefficient \( N \) is

\[ N = \int D\Phi \exp \left[ i \int_0^T dt \int_{-\infty}^{\infty} \frac{dp}{2 \pi} \Phi (p, t) \Phi (-p, t) V^{-1} (p) \right]. \tag{38} \]

In order to prove eq. (37) it is sufficient to shift the variable of integration \( \Phi \) to \( \Phi - V \varrho \) in the integral

\[ \int D\Phi \exp \left[ i Tr \left( \Phi^\dagger \Phi V^{-1} \right) \right]. \]
Applying identity (43) to the functional integral that determines the evolution operator for the Tomonaga-Luttinger model we express it in terms of the evolution operator in the external field at the price of an additional functional integration over the scalar field \( \Phi(x, t) \):

\[
\hat{S}_{\text{exc}} = \frac{1}{N} \int \mathcal{D}\Phi \exp \left[ i \frac{1}{2} \int_0^T dt \int_{-\infty}^\infty \frac{dp}{2\pi} \Phi(p, t) \frac{1}{\Phi(-p, t) V^{-1}(p)} \right] \hat{S}(\Phi).
\]  

(39)

Expression (39) is explicit: while it is not possible to perform the final integration in \( \Phi(x, t) \) in closed form, it is easy to obtain an arbitrary term of the evolution operator expanding it in creation-annihilation operators. This will be enough in order to calculate the evolution operator.

Indeed, let us expand the evolution operator in the external field in powers \( S^n_0 \). The arbitrary term of the expansion contains a number of Green functions in the external field (32) which are exponents linear in the external field. Together with the action (39) and the determinant, eq.(38), we get an integral of Gaussian type in \( \Phi(x, t) \) which can be easily performed. The result of the integration depends on the electron-hole configuration considered.

Let us introduce the following system of notations for the coordinates entering the electron-hole creation-annihilation operators:

1. We will denote by \( x \) the coordinates of the right particles and by \( y \) the coordinates of the left ones.
2. We will put the tilde on coordinates related to annihilation operators (initial state) and leave coordinates of creation operators (final state) without tilde.
3. We will put primes on coordinates which are related to holes.

It is convenient to proceed in the exponents of the Green functions (32) to momentum space using the expression for the free Feynman Green functions:

\[
G^{0}_{R,L} (p, t, t_1) = \theta_{\mp p} \theta(t - t_1) \exp \left[ i\pi \nu f (t - t_1) \right] - \theta_{\mp p} \theta(t_1 - t) \exp \left[ i\pi \nu f (t - t_1) \right].
\]  

(40)

Collecting all terms in the exponents arising from the Green function we obtain the contribution to the action linear in the external field \( \Phi \):

\[
S_c = i \int_0^T dt \int_{-\infty}^\infty \frac{dp}{2\pi} \Phi(-p, t) R_c(p, t),
\]  

(41)

where the “current” \( R_c \) (which depends on the chosen configuration) is equal to

\[
R_c(p, t) = R_i(p) \exp(-i\nu |v_f t_1|) + R_f(p) \exp(-i\nu |v_f(T - t_1)|),
\]  

(42)

and

\[
R_f(p) = \sum_{x:x'...;y:y'...} \theta(-p) \left[ \exp(ipx) - \exp(ipx') \right] + \theta(p) \left[ \exp(ipy) - \exp(ipy') \right],
\]  

\[
R_i(p) = \sum_{\bar{x}:\bar{x}'...;\bar{y}:\bar{y}'...} \theta(-p) \left[ \exp(ip\bar{x}) - \exp(ip\bar{x}') \right] + \theta(p) \left[ \exp(ip\bar{y}) - \exp(ip\bar{y}') \right]
\]  

(43)

for the initial (annihilation operators) and final (creation operators) configurations respectively. Coordinates \( x, y \) in eq.(43) are the coordinates of annihilation and creation operators for the configuration we are interested in. Finally, we get the following functional integral:

\[
\int \mathcal{D}\Phi \exp \left[ i \frac{1}{2} \int_0^T dt dt_1 \int_{-\infty}^\infty \frac{dp}{2\pi} \Phi(p, t) \Phi(-p, t_1) V^{-1}(p) \delta(t - t_1) - \frac{1}{4\pi} \int_0^T dt dt_1 \int_{-\infty}^\infty \frac{dp}{2\pi} \Phi(-p, t) \Phi(p, t_1) |p| \exp[-i\nu |v_f| t - t_1|] + i \int_0^T dt \int_{-\infty}^\infty \frac{dp}{2\pi} \Phi(-p, t) R_c(p, t) \right]
\]  

(44)

Here the first term is the action of eq.(39), the second is the quantum determinant and the third comes from the Green functions, eq.(43).
The integral in eq. (44) is a Gaussian one; it can be calculated by standard methods. One has to find the saddle point field $\Phi_0$ and shift the variables of integration $\Phi \to \Phi - \Phi_0$. The integral in fluctuations $\Phi - \Phi_0$ gives the shift of the ground state energy due to the electron interaction and normalization coefficient of the ground state wave function. We calculate this integral in Appendix C. The operator structure of the evolution operator is determined completely by the terms which appear as a result of substituting the saddle point $\Phi_0$ in eq. (44). We write them as an "effective action":

$$S_{\text{eff}} = \frac{i}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \int_{0}^{T} dt \Phi_0 (p, t) \mathcal{R}_c (p, t).$$

(45)

The saddle point field $\Phi_0 (x, t)$ obeys the integral equation:

$$\frac{i}{V (p)} \Phi_0 (p, t) - \frac{1}{2\pi} \int_{0}^{T} dt_1 \Phi_0 (p, t_1) |p| \exp \left[ -i|p|v_f |t - t_1| \right] = -i \mathcal{R}_c (p, t)$$

(46)

which can be reduced to the following differential equation (to see this it is sufficient to differentiate both sides of eq. (46) with respect to time):

$$\partial_t^2 \Phi_0 (p, t) + \omega_p^2 \Phi_0 (p, t) = 0,$$

(47)

where

$$\omega_p = |p|v_f \sqrt{1 + \frac{V(p)}{\pi v_f}}.$$  

(48)

The boundary conditions for this equation follow from the original integral equation (46):

$$\partial_t \Phi_0 (p, 0) - i|p|v_f \Phi_0 (p, 0) = 2i|p|v_f V (p) \mathcal{R}_i (p),$$

$$\partial_t \Phi_0 (p, T) + i|p|v_f \Phi_0 (p, T) = -2i|p|v_f V (p) \mathcal{R}_f (p).$$

(49)

In the derivation of eq. (47) we have used the fact that our system is electrically neutral and hence:

$$\mathcal{R}_f (p = 0, t) = \mathcal{R}_i (p = 0, t) = 0.$$

The solution of the differential equation for the saddle point field [eq. (47)] gives:

$$\Phi_0 (p, t) = \frac{-2|p|v_f V (p)}{(\omega_p + |p|v_f) \left(1 - \xi_p^2\right)} \left[ \mathcal{R}_i \left( \exp (-i\omega_p t) + \xi_p \exp (-i\omega_p (T - t)) \right) + \mathcal{R}_f \left( \xi_p \exp (-i\omega_p t) + \exp (-i\omega_p (T - t)) \right) \right],$$

(50)

where

$$\xi_p = \frac{1 - \sqrt{1 + \frac{V(p)}{\pi v_f}}}{1 + \sqrt{1 + \frac{V(p)}{\pi v_f}}} \exp (-i\omega_p T).$$

Substituting the saddle point field into the expression for the effective action (45) we obtain finally

$$S_{\text{eff}} = -\frac{1}{L} \sum_{p \neq 0} \frac{V(p)}{\sqrt{1 + \frac{V(p)}{\pi v_f}}} \cdot \frac{1}{1 - \xi_p^2} \times$$

$$\times \left[ \mathcal{R}_f (-p) \mathcal{R}_f (p) + \mathcal{R}_i (-p) \mathcal{R}_i (p) \right] F_2 (p) + 2 F_1 (p) \mathcal{R}_f (-p) \mathcal{R}_i (p),$$

(51)

where we introduce the following two functions:

$$F_1 (p) = \frac{\exp (-i|p|v_f T) - \exp (-i\omega_p T)}{\omega_p - |p|v_f} + \xi_p \frac{1 - \exp (-i (\omega_p + |p|v_f) T)}{\omega_p + |p|v_f},$$

$$F_2 (p) = \frac{1 - \exp (-i (\omega_p + |p|v_f) T)}{\omega_p + |p|v_f} + \xi_p \frac{\exp (-i|p|v_f T) - \exp (-i\omega_p T)}{\omega_p - |p|v_f}.$$  

(52)
We return in the expression for the effective action to a sum over the particle momentum \( p_n = 2\pi n/L \) according to the ordinary rule:

\[
\int_{-\infty}^{\infty} \frac{dp}{2\pi} \rightarrow \frac{1}{L} \sum_p .
\]

This will allow us to qualify different infrared divergences which appear in the effective action. Let us note that there is no term with \( p = 0 \) in these sums. This fact is related to the gauge invariance of the system: constant (in space) fields \( \Phi(t) \) correspond to a pure gauge electric potential and should not contribute.

Let us proceed with the wave function of the ground state in Tomonaga-Luttinger model. As was mentioned above, in order to separate the ground state we have to take the limit \( T \rightarrow \infty \). (This corresponds to the case of zero temperature.) We can omit oscillating exponents in this limit. As a result we are left only with the function \( F_2 \) which is changed to

\[
F_2(p) \sim |p| v_f \left( 1 + \sqrt{1 + \frac{V(p)}{\pi v_f}} \right)^{-1} .
\]

Effective action factorizes into contribution of initial and final states:

\[
S_{eff} = -\frac{1}{L} \sum_{p \neq 0} \frac{V(p)}{|p| v_f} \left[ \mathcal{R}_f (-p) \mathcal{R}_f (p) + \mathcal{R}_i (-p) \mathcal{R}_i (p) \right] = S_{eff}^f + S_{eff}^i .
\]

Besides \( S_{eff} \) we have to calculate preexponential factors which arise from free Feynman Green functions. At \( T \rightarrow \infty \) only Green functions with equal time arguments survive. As a result, we see that the whole expression for the evolution operator for large \( T \) factorizes into the product of the wave function of the ground state \( |\Omega> \) and its complex conjugate (see also below, section IV A). The final expression for the wave function is of the form

\[
|\Omega> = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \int \frac{dx dx' \hat{a}_R^\dagger (x) \hat{b}_R^\dagger (x')}{2\pi i} \frac{dy dy' \hat{a}_L^\dagger (y) \hat{b}_L^\dagger (y')}{2\pi i} \right]^n exp S_{eff}^i (x, x', y, y') |F> .
\]

Let us check, first of all, that the wave function of noninteracting fermions (\( V = 0 \)) is \(|F>\). The general term in the sum of eq. (54) is a product of factors:

\[
\int \frac{dx dx' \hat{a}_R^\dagger (x) \hat{b}_R^\dagger (x')}{2\pi i} \frac{dy dy' \hat{a}_L^\dagger (y) \hat{b}_L^\dagger (y')}{2\pi i} |F> .
\]

Let us note now that all singularities of the operator \( \hat{b}_R^\dagger (x') \) are in the upper semiplane (see definition) as well as the pole of the Green function. One can close the contour of \( x' \) in the lower semiplane and prove that the corresponding integral vanishes. The only term which survives is the term with \( n = 0 \) and hence \(|\Omega> = |F>\) as it should be for non-interacting fermions.

A non-trivial answer for the wave function appears only owing to singularities of the effective action. It is clear from the general structure of the action (which is the product of \( R_f(p)R_f(-p) \)) that the wave function contains only terms where both \( R- \) and \( L- \) particles are present. All terms with only \( R \) (or only \( L \)) electrons or holes vanish. The structure of the ground state wave function is a bit different for different potentials. We will describe it below.

### A. Short range potential.

We begin with the short range potential: \( V(p) = V_0 \). The simplest possible contribution to the ground state wave function \(|\Omega>\) (see eq. 55) is

\[
\int \frac{dx dx' \ dy dy' \hat{a}_R^\dagger (x) \hat{b}_R^\dagger (x') \hat{a}_L^\dagger (y) \hat{b}_L^\dagger (y')}{2\pi i \times x - x - i\delta \ y - y' - i\delta} exp S_{eff}^i (x, x', y, y') .
\]
The effective action \( S_{\text{eff}} \) for this term has the form:

\[
S_{\text{eff}}^f (x, x', y, y') = -\frac{2\alpha}{L} \sum_{p_n > 0} \frac{1}{p_n} \left[ \exp\left( ip_n (x - y + i\delta) + \exp ip_n (x' - y' + i\delta) - \exp ip_n (x' - y' + i\delta) - \exp ip_n (x - y' + i\delta) \right) \right],
\]

where

\[
\alpha = \frac{V_0}{v_f \sqrt{1 + \sqrt{1 + V_0/v_f}^2}}.
\]

The sums in eq. (56) can be easily calculated. We obtain:

\[
S_{\text{eff}}^f (x, x', y, y') = \frac{\alpha}{\pi} \log \left( \frac{x - y + i\delta}{x' - y' + i\delta} \right) \frac{(x' - y' + i\delta)}{(x - y + i\delta)(x - y' + i\delta)}.
\]

Expression (55) describes the simplest possible complex in the vacuum of the interacting fermions. This complex has all quantum numbers equal to zero. In fact, it describes electron-electron scattering (in the cross channel). Correspondingly, all coordinates \( x, x', y, y' \) are close to each other. In general, this complex does not break down any continuous symmetry.

However in the Tomonaga-Luttinger model a special situation arises. The main contribution to the term \( S_{\text{eff}} \) comes from the region \( x' - y; x - y' \to 0 \) (of order of the transverse size of the channel) but \( x - y \) and \( x' - y' \) can be arbitrarily large. In other words, the complex decays into \( RL \) and \( RL \)-pairs. As we shall see such a wave function leads to a spontaneous breakdown of chiral symmetry.

Let us consider first the limit of the strong interaction:

\[
\frac{V_0}{\pi v_f} \gg 1.
\]

In this limit \( \alpha/\pi \to 1 \). It can be seen now that for \( \alpha/\pi = 1 \) the poles \( x = x' \) and \( y = y' \) corresponding to free fermions are cancelled completely by the fermion-fermion interaction (described by \( \exp(S_{\text{eff}}) \)) with the effective action of eq. (55). Instead we obtain new poles in the points \( x' = y - i\delta \) and \( y' = x + i\delta \). Recalling that \( \hat{b}_{R}^\dagger (x') \) is analytical in the lower and \( \hat{b}_{L} (y') \) in the upper semiplane we can integrate further over \( x' \) and \( y' \). As a result we get the following contribution to the ground state wave function:

\[
\int dx \hat{a}_{R}^\dagger (x) \hat{b}_{L}^\dagger (x) \int dy \hat{a}_{L} (y) \hat{b}_{R}^\dagger (y).
\]

Thus the 4-particle complex decays into 2 non-interacting ”bosons”. They are neutral in the electric charge but have a non-zero chirality \( \pm 2 \).

One can check that no other connected complexes appear in the limit of strong interaction. Let us consider, for example, charged complexes. The four fermion contribution is exhausted by eq. (55), so we have to consider a 6-fermion complex:

\[
\frac{\hat{a}_{R}^\dagger (x) \hat{b}_{L}^\dagger (x') \hat{a}_{R}^\dagger (x_1) \hat{b}_{R} (x_1') \hat{a}_{R}^\dagger (y) \hat{b}_{L}^\dagger (y')} {x' - x - i\delta} \frac{x_1' - x_1 - i\delta} {y - y' - i\delta} \frac{(x - y + i\delta)(x_1 - y - i\delta)(x' - y' + i\delta)} {(x - y + i\delta)(x_1 - y + i\delta)(x' - y' + i\delta)} \frac{(x' - y' + i\delta)(x_1' - y' - i\delta)} {(x - y' + i\delta)(x_1' - y' + i\delta)}. \]

This complex, indeed, decays into 2 fermions with \( x_1 \to y' \to x \) and \( x' \to y \to x' \) (relative distance \( x - x' \) supposed to be large). These fermions are of the form \( \hat{a}_{L} (x) \hat{a}_{R}^\dagger (x) \hat{b}_{L} (x) \hat{b}_{R}^\dagger (x') \hat{b}_{L}^\dagger (x') \hat{b}_{R} (x) \). Hence this contribution is zero owing to the Pauli principle. One can consider also more complicated configurations which could produce charged connected complexes and check that they do not appear in the wave function of the ground state.

The Pauli principle allows one more complex which describe scattering of chiral pairs:

\[
\hat{a}_{R}^\dagger (x) \hat{b}_{L}^\dagger (x) \hat{a}_{L}^\dagger (x) \hat{b}_{R}^\dagger (x).
\]
One can extract the corresponding contribution from the connected part of the general expression eq. (55). The integral over \( x' \) and \( y' \) is easily calculated and we obtain:

\[
\int dx dy \hat{\alpha}^+_R (x) \hat{b}^+_L (x) \hat{\alpha}^+_L (y) \hat{b}^+_R (y) \Phi (x - y),
\]

where

\[
\Phi (x - y) = \frac{-i\delta}{y - x - 2i\delta} \left( 2 + \frac{i\delta}{y - x - 2i\delta} \right).
\]

The function \( \Phi (x - y) \) is finite at any \( x, y \) (even in the point \( x = y \)) so its contribution to the integral vanishes in the limit \( \delta \to +0 \). In other words, in the limit of infinitely strong interaction the chiral pairs do not interact. This interaction appears, however, in the next approximations in the inverse coupling constant (see Section IV A 1).

To obtain the complete expression for the ground state wave function we have to consider complexes with 8, 12... particles and separate the connected parts out of these complexes. This is not necessary, however, since, according to a general theorem [36], the complete wave function is the exponent of the connected complexes [37] and we have proved that the only connected complexes are the chiral pairs [60]. On the other hand the total chirality of \( |\Omega> \) should be zero and only terms with \( C = 0 \) can appear in the expansion of \( |\Omega> \). To take this into account we introduce the projector \( P_{C=0} \) onto the state with chirality zero. Then the wave function can be written as

\[
|\Omega> = \sqrt{Z_0} \hat{P}_{C=0} \exp \left[ \int dx \hat{\alpha}^+_R (x) \hat{b}^+_L (x) + \int dy \hat{\alpha}^+_L (y) \hat{b}^+_R (y) \right] |F>.
\] (62)

The normalization coefficient \( Z_0 \) is calculated in Appendix C. We have already discussed the wave function in Section III. It corresponds to an unbroken symmetry phase in spite of the presence of an infinite number of chiral pairs with zero momentum. If the chiral symmetry is broken, the states with different chirality should be degenerate in energy. This is not the case if the size of system is finite — the energy of the state with \( C = 0 \) is still minimal and order parameter \( \Delta \) is zero.

Wave function (62) corresponds to the state with minimal possible energy. Hence it is the wave function of the system at \( \Theta = 0 \). Turning to \( \Theta \neq 0 \) let us note that there are two types of exponentials (see below eq. (64)): first with \( \exp (-v_f p / \Theta) \) and the second is \( \exp (-v_f p / \Theta) \) (with renormalized Fermi velocity). Corrections of the second type correspond to excitations and we will omit them. However there are no excitations with \( \omega_p = v_f p \) (it can be seen, e.g., by method of bosonization). In fact, these exponentials describe the change of the ground state with temperature [31]. Obviously for \( \Theta \gg 2\pi v_f / L \) the exponential factor is not small but the preexponential factor, i.e. the Green function with imaginary time differences about \( 1/\Theta \), gives the smallness. The latter is compensated by the action because it is proportional to \( \log 1/\Theta \). This is the case for the temperature region under consideration. In the opposite case the Green function [63] is not valid. One can use eq. (61) but it is impossible to transform the sums over \( p_n \) to integrals in order to get eq. (61). As a result, the Green function will be proportional to a small exponential factor. It can not be compensated by the logarithmic divergence from action and the whole term will be small. Therefore the ground state wave function (62) is valid provided

\[
\Theta \ll \Theta_{chiral} = \frac{2\pi v_f}{L}.
\] (63)

Of course one assumes that the number of states is large, i.e. \( p_f L \gg 1 \). This allows a transition from the sums to integrals in the expressions independent of \( \Theta \) (or \( T \)).

In the region of higher temperatures, \( \Theta_{chiral} \ll \Theta \), eq. (61) for the Green function is applicable. In this case after same algebraic transformations effective action \( S_{eff} \) can be rewritten in the form of

\[
S_{eff} = -\frac{\pi}{L} \sum_{n \neq 0} \frac{1}{|p_n|} \tanh \frac{|p_n| v_f}{2\Theta} \left[ R_f (-p) R_f (p) + R_i (-p) R_i (p) \right]
\]

\[
-\frac{2\pi}{L} \sum_{n \neq 0} \frac{1}{|p_n|} \exp \frac{-|p_n| v_f}{\Theta} \exp \frac{+|p_n| v_f}{\Theta} [R_f (p) R_i (-p)],
\] (64)

where \( v_f^c = v_f \sqrt{1 + V_0 / \pi v_f} \). If

\[
\Theta_{chiral} \ll \Theta \ll 2\pi v_f / L
\] (65)
than equation (64) can be transformed to

\[ S_{\text{eff}} = -\frac{\pi}{L} \sum_{n \neq 0} \frac{1}{|p_n|} [\mathcal{R}_f (-p) \mathcal{R}_f (p) + \mathcal{R}_i (-p) \mathcal{R}_i (p)] \]

- \frac{2\pi}{L} \sum_{n \neq 0} \frac{1}{|p_n|} \exp \left( -\frac{|p_n| \nu_f}{\Theta} \right) \mathcal{R}_f (-p) \mathcal{R}_i (p), \quad (66)

So, in the temperature region of interest one should take into account another 4-fermions contribution to the ground state:

\[ \int \frac{dx \, dx' \, dy \, dy'}{2\pi i} \frac{\hat{a}_R^\dagger (x) \hat{a}_R (\tilde{x})}{x - x + \nu_f T - i\delta} \frac{\hat{b}_L^\dagger (y') \hat{b}_L (\tilde{y}')}{\tilde{y}' - \tilde{y}' = \nu_f T + i\delta} \exp S_{\text{eff}}^f (x, \tilde{x}, y', \tilde{y}'). \quad (67) \]

At lower temperature this contribution is exponentially small. The action for this configuration is

\[ \log \left( \frac{\tilde{y}' - y' - \nu_f T + i\delta}{x - y' + i\delta} \right) \left( \tilde{y}' - \tilde{y}' = x - y + i\delta \right). \quad (68) \]

Thus one has a similar result: a pair \( \hat{a}_R^\dagger (x) \hat{a}_R (\tilde{x}) \) in \( \Omega > 0 \) and \( \hat{a}_R (\tilde{x}) \hat{b}_L (\tilde{y}) \) in \( \Omega < 0 \). However the existence of an extra pair implies that the whole chirality \( C \) of the state is nonzero. So the states with any \( C \) exist. Their energies differ by values of the order of \( 2\pi \nu_f/L \). At the temperature eq. (65) these states can be considered as degenerate. Then a state with fixed chirality is unstable relative to an infinitesimal interaction which is mixing right and left particles (e.g. infinitesimal back scattering). Similar to the theory of superconductivity the real ground state of the system is a mixture of states with different chirality, eq. (24), but a fixed chiral phase \( \theta \).

In order to prove that

\[ \Theta_c = \omega (2\pi/L) = 2\pi \nu_f^2/L \]

is the phase transition temperature one should consider the higher temperature region: \( \Theta \gg \Theta_c \). The logarithmic contribution to the action \( S_{\text{eff}} \) arises from \( n \gg n_{\text{min}} \sim L\Theta/2\pi \nu_f \gg 1 \). At smaller \( n \) the logarithmic divergence does not exist. So

\[ S_{\text{eff}} = -\frac{\pi}{L} \sum_{|n| > n_{\text{min}}} \frac{1}{|p_n|} [\mathcal{R}_f (-p) \mathcal{R}_f (p) + \mathcal{R}_i (-p) \mathcal{R}_i (p)] \]

- \frac{2\pi}{L} \sum_{|n| > n_{\text{min}}} \frac{1}{|p_n|} \exp \left( -\frac{|p_n| \nu_f}{\Theta} \right) \mathcal{R}_f (-p) \mathcal{R}_i (p), \quad (70) \]

One can compare it with eq. (64). The sums in eq. (70) are calculated in Appendix D. As a result the logarithms from eq. (65) have to be replaced by

\[ \log \left( \frac{\Delta x + i\delta}{\Delta x' + i\delta} \right) \rightarrow \int_{-(\Delta x' + i\delta)}^{-(\Delta x + i\delta)} \frac{dz}{z} \exp \left( -\frac{iz}{\zeta (\Theta)} \right), \quad (71) \]

where

\[ \zeta (\Theta) = \nu_f / (\Theta - \Theta_c) \]

is the coherence length.

The r.h.s. of eq. (71) can be expressed by an exponential integral exponent function with imaginary argument. In order to prove that \( \zeta (\Theta) \) is the coherence length one should notice that at the length \( \Delta x \ll \zeta (\Theta) \) the r.h.s. of eq. (71) tends to \( \log ((\Delta x + i\delta) / (\Delta x' + i\delta)) \), i.e. in such case the system is characterized by the wave function eq. (24). (Indeed, in this case it is possible to repeat the calculations made in Section IV A if one separates all connected complexes by the distances smaller than \( \zeta (\Theta) \).) Thus in a region of a specimen smaller than \( \zeta (\Theta) \) one has a coherent state. In the opposite case (distances between pairs, \( \Delta x = |x - y| \), are larger than \( \zeta (\Theta) \)) expression under the integrand
begins to oscillate and the divergence does not exist. As a result, one will have small corrections to the action of about

$$\exp (-i\Delta x/\zeta (\Theta)) \zeta (\Theta)/\Delta x.$$  

So the 4-fermion contribution leads to a term

$$\int dxdy \left( \frac{\zeta (\Theta)}{|x-y|} \right)^2 \hat{a}_R^\dagger (x) \hat{b}_L^\dagger (x) |F> < F| \hat{b}_L (y) \hat{a}_R (y)$$  \hspace{1cm} (73)$$

in the evolution operator. Thus at the distance $|x-y| \gg \zeta$ we have configurations with free bosons. Consequently, at such scale the state is noncoherent. Therefore a long range order does not exist at lengths larger than $\zeta (\Theta)$. One can also check this directly. Let us calculate the contribution of the state (73) to the order parameter density correlator:

$$\int^{L/2}_{-L/2} dx \left( \frac{\zeta (\Theta)^2}{L} \right)^2 \sim \zeta^2 (\Theta).$$

As $\Delta$ does not depend on $L$ one has the normal phase (see Section III) with low symmetry phase fluctuations. This means that $\Theta_c$ is indeed the phase transition temperature and $\zeta$ is the coherence length. Besides, one has more obvious definition $\Theta_c$:

$$\zeta (\Theta_c) \sim L.$$  

In this case the whole system can be described by the broken symmetry wave function (21). Hence one should think that the low symmetry phase is realized if $\Theta < \Theta_c$. The above discussion should make it clear that this transition is smeared over the temperature region about $\Theta_c$ as it should be for the finite size specimen.

1. Kosterlitz-Thouless phase.

Let us prove that a Kosterlitz-Thouless phase is likely to form in the Tomonaga-Luttinger model if one takes into account corrections to the action due to $\pi v_f/V_0$.

We begin with the case of zero temperature and consider again the 4-fermion contribution eq.(67) to the ground wave function

$$\int dxdx' ddy' \hat{a}_R^\dagger (x) \hat{b}_R^\dagger (x') \hat{a}_L^\dagger (y) \hat{b}_L^\dagger (y') \left[ (x-y+i\delta) (x'-y'+i\delta) \right]^{\alpha_0} |F>, \hspace{1cm} (74)$$

where $\alpha_0 \equiv \alpha/\pi$. For simplicity we will consider $\alpha_0$ close to unity. Let us consider the configuration with two connected chiral complexes separated by a distance $R$ large compared to the transverse size of the channel $d$: $x'-y; x-y' \sim d \rightarrow 0$ and $|x-x'| \sim R, |y-y'| \sim R \rightarrow \infty$. The contribution we are interested in is determined by two cuts: $y' = x + i\delta$ and $x' = y + i\delta$ and is proportional to

$$\left( 1 - e^{2\pi i \alpha_0} \right) \int^{x}_{-\infty} \frac{dy'}{2\pi i} \frac{1}{(y'-x)^{\alpha_0}} \frac{(x'-y')^{\alpha_0}}{(y'-y)}.$$  

The last factor of the integrand is of order the $1/R^{1-\alpha_0}$. Distances inside the pair $y' - x, x' - y$ are of order of $d$. The contribution of the distant chiral pairs to the integral (74) is

$$\int dxdy \hat{a}_R^\dagger (x) \hat{b}_L^\dagger (x) \hat{a}_L^\dagger (y) \hat{b}_R^\dagger (y) \left( \frac{d}{|x-y|} \right)^{2(1-\alpha_0)} |F>.$$  \hspace{1cm} (75)$$

In the temperature region (23) we can consider also contributions of the states with $C \neq 0$ to the ground state. The simplest contribution comes again from eq.(74) and has the form

$$\int dxdy \left( \frac{d}{|x-y|} \right)^{2(1-\alpha_0)} \hat{a}_R^\dagger (x) \hat{b}_L^\dagger (x) |F> < F| \hat{b}_L (y) \hat{a}_R (y)$$  

\hspace{1cm} (76)$$
As can be seen from eq. (75) and eq. (76) the probability to find chiral pairs at the distance \( R \) is \( P(R) = |\Phi(R)|^2 \sim 1/R^{2(1-\alpha_0)} \). This probability decreases with \( R \) but much more slowly than in the theory without interaction. The average distance between pairs

\[
<R> = \int_0^L dR R P(R) \sim L^{2\alpha_0}
\]
diverges as \( L \to \infty \). It is instructive to consider the same quantities in the theory with non-interacting electrons. Here the probability to find a chiral pair is \( P_{\text{free}} = (|d|_{x-y}|^2 \) (see eq. (73)). As we have seen it results in the independence of \( \Delta \) on \( L \). The other limiting case is the system with non-zero density of the order parameter. Here the probability to find a chiral pair does not depend on the distance \( R \) at all while \( \Delta \propto L \). The probability under discussion has an intermediate behavior. As a result \( \Delta \) will increase with \( L \) but the power will be smaller than unity. Both these properties can be considered as a definition of Kosterlitz-Thouless phase.

The temperature of the phase transition \( \Theta_c \) in the Kosterlitz-Thouless system, at \( \alpha_0 < 1 \), to the unbroken phase is of the same order as in the limit of the infinitely strong interaction. Indeed our estimate of \( \Theta_c \) in section IV A was based on the logarithmic divergence of the action. This divergence exists also at \( \alpha_0 < 1 \) and hence our expressions for \( \Theta_c \) and correlation length \( \zeta \) are valid in this case too.

The wave function for the Kosterlitz-Thouless phase does not have the simple form of eq. (24) since the interaction of chiral pairs is non-zero. Also chiral complexes with more than two particles are present in the wave function of the ground state. However properties of this phase are quite similar to properties of the phase with broken symmetry which appears in the limit of infinitely strong interaction.

**B. Cooper channel.**

Let us discuss in brief the case of the attractive short range potential \( V = V_0 \delta(x) \), \( V_0 < 0 \) (Gorkov’s potential\[8\]). This potential can be used only if the interaction is sufficiently weak:

\[
\frac{|V_0|}{\pi v_f} < 1,
\]
otherwise the spectrum of excitations acquires an imaginary part (see eq. (48)). This demonstrates that the system of one dimensional electrons tends to collapse in this case and the point-like potential has to be modified. This effect is similar to the well-known instability in the system of interacting oscillators.[10]

If inequality (77) is fulfilled we can use eq. (74) derived above but with \( \alpha_0 < 0 \). In this case the system is in the Kosterlitz-Thouless phase too. To prove this statement we consider again four-fermion contribution to the wave function with two Cooper pairs separated by distance \( R (x - y; x' - y' \to 0, |x - x'| \sim |y - y'| \sim R \sim L) \). The corresponding contribution to the wave function is of the form:

\[
\hat{a}^\dagger_R(x) \hat{a}^\dagger_L(y) \hat{b}^\dagger_R(y') \hat{b}^\dagger_L(y'),
\]
and the probability to find such a configuration behaves as \( R^{2(1-|\alpha_0|)} \), i.e. it decays more slowly than in the theory without interaction.

**C. Coulomb interaction.**

We are interested in the long range Coulomb potential also in the limit of strong interaction. However as was already mentioned in Section III since the length of the three-dimensional screening should be still larger than the size of the channel, we must have \( p_f a_b > 1 \)\[23\]. This means that the parameter related to the strength of the Coulomb interaction should satisfy the inequality:

\[
\frac{V(p)}{\pi v_f} = \frac{2}{\pi p_f a_b} \log \frac{2p_f}{p} \gg 1.
\]

This can be large only due to a large logarithm. The argument of this logarithm is of order \( p_f L/\pi \gg 1 \). For this reason the integrals over momentum in the expression for the action \( S_{\text{eff}}, \) eq. (53), should be cut at

\[
p_{\text{max}} \sim 2p_f \exp \left( -\frac{\pi p_f a_b}{2} \right) \ll p_f
\]
This cut-off is not essential if $p_{\text{max}} L \gg 1$. Then every term of the action can be written as

$$- \int_{1/L}^{\infty} \frac{dp}{p} \exp \left( i p \delta x \right) \left( 1 + 3/2 \sqrt{\pi v f/e^2 \log^{-1/2} (2p_f/p)} \right)^2 \sim \log \delta x - 6 \sqrt{\pi v f/e^2 \log^{1/2} (2p_f \delta x)}.$$

Separating the four-fermion contribution in two chiral pairs $\hat{\sigma}_R^\dagger (x) \hat{\sigma}_L^\dagger (y) \hat{\sigma}_L (x) \hat{\sigma}_R (y)$, we find that the probability to find the chiral pair at a large distance $|x - y|$ from its counterpart behaves like

$$\exp \left( -12 \sqrt{\pi v f/e^2 \log^{1/2} (2p_f |x - y|)} \right).$$

It decays more slowly than any power of $|x - y|$ [41]. Strictly speaking, this behavior does not correspond to the Kosterlitz-Thouless phase but it is clear that physically these two phases are quite similar. Let us note also that in the formal limit $(\pi v_f/e^2) \log (2p_f L) \to 0$ we have again the condensate of independent chiral pairs with the wave function of eq. (20).

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APPENDIX A: EVOLUTION OPERATOR FOR SYSTEMS OF FERMIONS.

In this appendix we derive the representation for the evolution operator of fermions in the external field as a functional integral with definite boundary conditions.

In the Schrödinger representation the evolution operator $S(T)$ is

$$S[T] = T \exp (-i \int_0^T H dt) |F >= F|,$$

where $H$ is the fermion Hamiltonian in the external field which is bilinear in the fermion fields. As we have seen the general problem with electron- electron interaction can be reduced to this problem at the price of the integration over the external field. For simplicity we begin with a model with an empty ground state $|0 >=$ rather than the Fermi one. (This allows us to write the equations in a more compact form.) Besides we will omit the spatial arguments.

Let us divide the time interval $T$ in $N$ infinitesimal pieces $\delta = T/N$ (with the point $i = N$ corresponding to $t = 0$ and $i = 1$ to $t = T$) and introduce the sum over the complete set of quantum mechanical states $|k > < k|$ in the intermediate points:

$$S[T] = \sum_{n_i} |k_N > < k_N | (1 - i \delta H) |k_{N-1} > ... < k_2 | (1 - i \delta H) |k_1 > < k_1|. \quad (A1)$$

For any complete set of wave functions in the secondary quantization representation we have:

$$\sum_n |k_i(n) > < k_i(n)| = \int \mathcal{D} \xi_i \mathcal{D} \xi_i^\dagger \exp \left( -Tr \xi_i^\dagger \xi_i \right)$$

$$\exp \left( Tr \xi_i \tilde{a}^\dagger \right) |0 > < 0| \exp \left( -Tr \xi_i^\dagger \tilde{a} \right), \quad (A2)$$

(index $n$ corresponds to the set of all quantum numbers). The Grassmann variables $\xi$ are defined in the usual way:

$$\int d\xi_i(n) = 0, \int d\xi_i(n) \xi_i(n) = 1, \left[ \xi_i^\dagger (n), \xi_i (n) \right]_{+} = 0., \mathcal{D} \xi_i = \prod_n d\xi_i(n).$$

Eq. (A2) can be proved by direct comparison of the left and right-hand sides. We use this representation to rewrite the sum over states as a functional integral.
At every point $i$ we obtain the following matrix element of the Hamiltonian:

$$
\exp \left( -Tr \xi_i^+ \xi_i \right) < 0 \exp \left( -Tr \xi_i^+ \hat{a} \right) (1 - i\delta H (\hat{a}^+, \hat{a})) \exp \left( -Tr \xi_{i+1}^+ \xi_{i+1} \right) \exp \left( Tr \xi_{i+1}^+ \hat{1} \right) |0 > .
$$

(A3)

To calculate this matrix element we move all creation operators to the right. For the Hamiltonian $H$ depending linearly on $\hat{a}, \hat{a}^+$ e.g. for the Hamiltonian in the external field the result is

$$
\exp \left( Tr \xi_i^+ (\xi_i + \xi_{i+1}) + i\delta Tr H (\xi_i^+, \xi_{i+1}) \right).
$$

Thus the result of the calculation is that creation-annihilation operators in the Hamiltonian are substituted by Grassmann variables $\xi, \xi^+$. The product over all intermediate points at $N \to \infty$ tends to

$$
\exp(- \int_0^T dt \bar{\Psi} (t) [\partial_t + iH] \Psi (t)) = \exp i \int_0^T dt \mathcal{L}, \quad \mathcal{L} = \bar{\Psi} [i\partial_t - H] \Psi
$$

where $\mathcal{L}$ is the Lagrangian of the system. This expression should be integrated over variables $\Psi, \bar{\Psi}$ at all intermediate points in time. The boundary points are specific, however. Creation operators entering $|k_N >$ and annihilation operators entering $< k_1 |$ are not contracted. They are variables on which the evolution operator depends.

Let us integrate over all intermediate variables and consider the answer as a function of Grassmann variable $\xi_i^+$ (and $\xi_i^−$). This function can be only linear one: $A + B\xi_i$. Then the last integration in $\xi_1, \xi_1^+$ is of the form:

$$
\int \mathcal{D}\xi_1 \xi_1^+ \exp \left( -Tr \xi_1^+ \xi_1 \right) \exp \left( -Tr \xi_1^+ \hat{a} \right) (A_1 + Tr B_1 \xi_1) = A_1 + Tr B_1 a
$$

Thus, we see that the variable $\xi_1$ should be substituted by an annihilation operator. Integrating over $\xi_i$ we conclude that $\xi_1$ is substituted by a creation operator.

Finally, we can formulate the following recipe: to calculate the evolution operator one has to integrate $\exp(i \int_0^T \mathcal{L})$ over $\Psi, \bar{\Psi}$ at all intermediate points. At $t = 0$ the value of $\Psi$ is fixed to $\hat{a}$, at $t = T \bar{\Psi}$ is fixed to $\hat{a}^+$. The values of $\bar{\Psi}$ at $t = 0$ and $\Psi$ at $t = T$ remain arbitrary. As a result, operators $\hat{a}$ and $\hat{a}^+$ are defined at different times. Therefore one should consider theirs here as anticommutating.

If the ground state of our system is a filled Fermi sphere we have to introduce two type of creation-annihilation operators $\hat{a}^±, \hat{b}^±$ corresponding to electrons and holes. Then we can apply the above derivation to this case as well. One should introduce negative ($\Psi^-$) and positive ($\Psi^+$) frequency parts of $\Psi$ variables and double the number of variables $\xi$.

**APPENDIX B: CALCULATION OF DetΦ.**

We will calculate the functional integral over the fields $\chi$ and $\bar{\chi}$. They obey zero initial conditions.

$$
Det \Phi = \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \exp \left( i \int_0^T dt \int dx \bar{\chi} (i\partial_t - H_{ext} (x)) \chi \right),
$$

(B1)

where $\mathcal{H}_{ext} = H_0 (x) + \Phi (x, t)$. In an ordinary case $Det \Phi$ can be calculated in the usual way using the identity

$$
\log [Det \Phi] = Tr \log \Phi.
$$

After differentiation over $\lambda$ the right-hand side of this identity is represented as

$$
Sp \left[ -i \int_0^1 d\lambda ( -\partial_t - iH_0 (x) - i\lambda \Phi (x))^{-1} \Phi (x) \right],
$$

where the inverse operator is the Green function with the same arguments. Usually the result does not depend on the order of arguments. However in the theory with Adler-Schwinger anomaly the sequence of time and spatial arguments is essential. The simplest way is to assume to make spatial arguments equal first. In this case the result is in contradiction with gauge invariance of the theory. In the paper [20] the procedure free of this difficulty was suggested.
It does not exist in this case because all calculations are done with non-equal variables up to the end. The procedure is based on the Heisenberg equation for the electron evolution operator $\hat{S}(\Phi) = Det\Phi \exp S_0(T)$ at the external field (without direct electron-electron interaction). In Heisenberg representation one has

$$i \frac{\partial \hat{S}}{\partial T} = [H_{ext}, \hat{S}],$$

where $H_{ext}$ is the noninteracting electrons Hamiltonian (the external field is dependent on the time $T$), and the action $S_0$ is defined by the eq.(34). One should note that all creation operators are defined at the moment $0$. After the integration of eq.(35) and symmetrization one gets eq.(35). Note that eq.(35) is a gauge invariant: the fields depending only on time do not contribute to eq. (35).

One can rewrite the latter equation in the following form:

$$i \frac{\partial \log Det\Phi}{\partial T} = \exp (-S_0) [H_{ext}, \exp S_0] - i \frac{\partial S_0}{\partial T} \quad (B2)$$

In order to calculate the commutator in this equation one can use the well-known identity:

$$\left[ \hat{a}(x), \exp \left( \int dx' K(x') \hat{a}^\dagger(x') \right) \right] = \int dx_1 \Delta(x_1 - x') K(x_1) \exp \left( \int dx' K(x') \hat{a}^\dagger(x') \right),$$

which can be proved by expanding the exponential functions. (Here $K$ is anticommutating with $\hat{a}$ operator and $\Delta(x_1 - x')$ is the anticommutator $\left\{ \hat{a}(x), \hat{a}^\dagger(x') \right\}$ is defined by eq.(10)). The left-hand side of eq.(B2) is a c-number; this means that all operators from right-hand side of this equation have to vanish. The c-number parts arise only from the following commutators:

$$\int dx \Phi(x) \left[ \hat{b}(x) \hat{a}(x), \exp \left( \int dydy' \hat{a}^\dagger(y') G(y'T, yT - \varepsilon) \hat{b}^\dagger(y) \right) \right].$$

As a result one has

$$i \frac{\partial \log Det\Phi}{\partial T} = \int \frac{dxdydy'}{(2\pi)^2} \Phi(x, T)$$

$$\left[ \frac{G_R(y'T, yT - \varepsilon)}{(y' - x - i\delta)(y - x - i\delta)} + \frac{G_L(y'T, yT - \varepsilon)}{(x - y' - i\delta)(x - y - i\delta)} \right]. \quad (B3)$$

This representation is general. In order to rewrite the right-hand side of this equation in our case one should take into account that only the region $y \rightarrow y' \rightarrow x$ is essential in the first term. However at the point $y \rightarrow y'$ the argument of the exponential in the Green function vanishes. This means that the contribution is determined by the preexponential pole and only the first and the second terms of the expansion of exponential can give nonvanishing contributions. All singularities under the integral over $y$ in the function coming from the first term are in one semiplane. One can close the contour in the other one and prove that this integral vanishes. In the next order in $\Phi$ only the part with the singularity in the lower semiplane of $y$ gives a nonvanishing term. After integration over $y'$ one has (in momentum space representation):

$$-\frac{i}{2\pi} \int_0^T dt_1 \int_0^\infty \frac{dp}{2\pi} p \Phi_{-p}(T) \Phi_p(t_1) \exp (-ipv_f(T - t_1)).$$

The $L$-electrons give the same result but with the opposite sign of $p$ in the region $p < 0$. After the integration of eq.(B3) and symmetrization one gets eq.(35). Note that eq.(35) is gauge invariant: the fields depending only on time do not contribute to eq. (35).

APPENDIX C: NORMALIZATION COEFFICIENT AND ENERGY SHIFT.

We have seen that the matrix element from eq.(33) can be expressed as a Gaussian type functional integral. It gives the normalization coefficient and the ground state energy shift. Indeed, one can expand the exact wave function
over the free electrons ones. In the limit \( T \to \infty \) only the matrix element between the lowest energy level survive. It can be represented in the form:

\[
Z = \exp (-i \Delta E T) |\Omega F> |^2
\]

where \( \Delta E \) is the ground state energy shift. Comparing \( Z \) with the definition of the normalization coefficient \( Z_0 \) from eq.\([24]\) one can see that it is equal to the overlap probability of the ground states of the free and interacting electrons : 

\[
|\Omega F> |^2.
\]

The normalization coefficient should be calculated for a finite size system, as it is exponentially small with the volume.

On the other hand, the matrix element we are interested in is:

\[
Z = \frac{1}{N} \int \mathcal{D}\Phi \exp \left[ \frac{i}{2} \int_0^T dt dt_1 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Phi(p, t) \Phi(-p, t_1) V^{-1}(p) \delta(t - t_1) \right. 
\]

\[
- \frac{1}{4\pi} \int_0^T dt dt_1 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Phi(-p, t) \Phi(p, t_1) |p| \exp [-i|p|v_f|t - t_1|], 
\]

where \( 1/N \) is the normalization coefficient \([35]\).

It is convenient to transform integral operator The more effective procedure is the transformation of the integral operator \([C1]\) to a differential form. In order to do this we note the identity

\[
\frac{1}{-2i|p|v_f} \left( \frac{\partial^2}{\partial t^2} + p^2 v_f^2 \right) \int_0^T dt_1 \Phi_p(t_1) \exp (-i|p|v_f|t - t_1|) = \Phi_p(t). 
\]

Thus symbolically

\[
\exp (-i|p|v_f|t - t_1|) = \frac{-2i|p|v_f}{\left( \frac{\partial^2}{\partial t^2} + p^2 v_f^2 \right)} \delta(t - t_1),
\]

and the kernel of eq.\([C1]\) is equal to:

\[
\frac{i}{2V(p)} \left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \delta(t - t_1). 
\]

As a result, one has:

\[
Z = \frac{1}{N} \int \mathcal{D}\Phi \exp \left[ \frac{i}{2} \int_0^T dt \int_{-\infty}^{\infty} \frac{dp}{2\pi} V^{-1}(p) \Phi(-p, t) \left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \Phi(p, t) \right].
\]

Taking into account the normalization coefficient \( N \) cancels the \( \sqrt{\text{Det}(i/2V)} \) that arises from the differential kernel definition one gets:

\[
Z^{-2} = \text{Det} \left[ \frac{i}{2} \left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \right] = \frac{D}{D_0}.
\]

In order to define the differential operator one should have two initial conditions. In exactly the same way as for the derivation of the equation for the saddle point field one gets this conditions:

\[
\begin{align*}
\partial_t \Phi(p, 0) - i|p|v_f \Phi(p, 0) &= 0, \\
\partial_t \Phi(p, T) + i|p|v_f \Phi(p, T) &= 0.
\end{align*}
\]

Usually, one calculates determinants with zero boundary conditions

\[
\Phi_p(0) = \Phi_p(T) = 0.
\]

In order to reduce our problem to the problem with zero boundary conditions let us introduce

\[
\Phi_p(t) = \Phi_p(t) + \phi(t).
\]
The field \(\Phi_p(t)\) is supposed to obey the equation \(\triangle_t \Phi_p(t) = 0\), with initial conditions eq. (C5). (As usual \(\triangle_t = \partial^2_t + \omega^2_p\).) The field \(\phi(t)\) is arbitrary but with zero boundary conditions. The solution is \(\Phi_p(t)\) can be expressed in the following form:

\[
\Phi_p(t) = \Phi_p(0) \frac{\sin \omega_p (T - t)}{\sin \omega_p T} + \Phi_p(T) \frac{\sin \omega_p (t)}{\sin \omega_p T}.
\]

(Constants \(\Phi_p(0)\) and \(\Phi_p(T)\) are arbitrary.) This means that the determinant has the form

\[
D^{-1/2} = \int_{-\infty}^{\infty} d\Phi_p(0) d\Phi_p(T) \exp \left[ i (\Phi_p(T) \partial_t \phi_p(T) - \Phi_p(0) \partial_t \phi_p(0)) \right]
\]

\[
\int D\phi_p(t) \exp (\phi(t) \triangle_t \phi(t)).
\]  

(C6)

The integral over \(\phi_p\) can be calculated in the usual way:

\[
C(p = 0) \frac{\omega_p}{\sin \omega_p T},
\]

where \(C(p = 0)\) is the \(p = 0\) contribution. It will cancel in the final expression. Taking into account the identity

\[
\Phi_p(T) \partial_t \phi_p(T) - \Phi_p(0) \partial_t \phi_p(0) = -\left(\Phi_p(0)^2 + \Phi_p(T)^2\right) (|p|v_f - i\omega_p \cot \omega_p T) + \frac{2i\omega_p}{\sin \omega_p T} \Phi_p(0) \Phi_p(T),
\]

one has

\[
Z^{-2} = \prod_{p \neq 0} \frac{\sin \omega_p T p^2 v_f^2 - 2i\omega_p |p| v_f \cot \omega_p T + \omega_p^2}{2|p|v_f \omega_p (1 - i \cot \omega_p T)}.
\]  

(C7)

If temperature is non-zero one should substitute \(T\) by \(1/\Theta\). Note that eq. (C7) is valid even at the temperature \(\Theta \ll \Theta_{chiral}\), because only the Green functions with equal time arguments were used. In this temperature region the \(Z\) can be expressed in the following form:

\[
Z = \prod_{p \neq 0} \exp \left( -\frac{\omega_p - |p| v_f}{2\Theta} \right) \frac{4\sqrt{|p| v_f \omega_p}}{\omega_p + |p| v_f}.
\]  

(C8)

It is convenient to rewrite this equation in the following way:

\[
Z = \exp \left[ -\frac{L}{\Theta} \int_0^\infty \frac{dp}{2\pi} \left( \omega_p - pv_f \right) + 1/2 \sum_{p \neq 0} \log \frac{4\sqrt{|p| v_f \omega_p}}{\omega_p + |p| v_f} \right].
\]  

(C9)

In this form one can see the energy shift (first term in the exponent) and normalization coefficient (second term) explicitly. The sums in this equation diverge because of the gapless spectrum. They have to cut off at \(p_{\text{max}} \sim 1/d\). In order to take the preexponential factor one should calculate the next correction after the Riemann sum. As a result we have for the short range potential

\[
\Delta E \sim \frac{L}{4\pi d} v_f \sqrt{V_0/\pi v_f}
\]

for the energy shift and

\[
Z_0 = 4\sqrt{\pi v_f V_0} \exp \left( -\frac{L}{4\pi d} \log V_0/\pi v_f \right)
\]

for the normalization coefficient.
APPENDIX D: CALCULATION OF SUMS IN SECTION 11

All the sums in the equation for the action can be calculated by differentiation of the expression $S(\alpha)$ with respect to parameter $\alpha$:

$$S(\alpha) = -\frac{2\pi}{L} \sum_{n=\min}^{\infty} \frac{1}{p_n} \exp \left( \frac{2\pi i n \alpha}{L} (x + i\delta) \right),$$

($\alpha$ varies within the region $(1,i\infty)$). After summation of the geometric series one can rewrite it in the following form:

$$S(1, x) = \int_{1}^{1-y_0} \frac{dy}{y} (1-y)^{n_{\min}-1},$$

where $y_0(x) = \exp (2\pi i/L) (x + i\delta)$. It is implied that $x \ll L$ here. This result can apply at $x \sim L$ as an order of magnitude estimate only. The final expression appearing in the action is:

$$S(1, x) - S(1, y) = \int_{(y+i\delta)}^{(x+i\delta)} \frac{dz}{z} \exp \left( -\frac{iz}{\zeta} \right),$$

(D1)

where $\zeta = L/2\pi (n_{\min} - 1)^{-1}$. If $n_{\min} \sim L\Theta /2\pi v^c$ then $\zeta$ is equal to the coherence length eq. (2).

Let us consider the influence of the boundary conditions on the action. In principle, any of them can be rewritten as $p_n = 2\pi (n + \delta n) / L, |\delta n| < 1/2$. In this case at $\Theta = 0$ the action is determined by the sum:

$$S'(\alpha) = -\frac{2\pi}{L} \sum_{i=1}^{\infty} \frac{1}{p_n} \exp \left[ \frac{2\pi i (n + \delta n) \alpha}{L} (x + i\delta) \right].$$

In the same way one gets

$$S'(1, x) - S'(1, y) = \int_{(y+i\delta)}^{(x+i\delta)} \frac{dz}{z} \exp \left( -\frac{2\pi i z \delta n}{L} \right).$$

The result of this is that up to $|x - y| \sim L$ at $\Theta = 0$ the action does not depend on the boundary conditions. For $(\Theta \gg \Theta_c)$ one should cut off the sum at some $n = n_{\min}$. As a result $\zeta$ is substituted by $L/2\pi (n_{\min} + \delta n - 1)^{-1}$ in eq. (11). This suggests the replacement of $\Theta_c$ by $(1 - \delta n) \Theta_c$. However the transition temperature can be defined only up to a factor of the order of unity. Therefore we should not take this into account.

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By anticommuting the $\Psi$-operators one can reduce this term to the term without backscattering, i.e. for this model the Tomonaga-Luttinger Hamiltonian describes the entire electron-electron interaction.

In order to obtain the real parameter one should use usual expression for the radius of Debye screening $R_D = 1/\sqrt{4\pi\varepsilon^2 e^2}$ with concentration $n = p_f/\pi^2 d^2$. We took into account here that there is only one state for electrons in directions perpendicular to the channel. Thus $R_D \approx \frac{1}{2} \frac{1}{\sqrt{\pi p_f a_b}}$ where $a_b = \left(\frac{m e^2}{\varepsilon}\right)^{-1}$ is the Bohr radius. Hence, condition $R_D \ll d$ is equivalent to $p_f a_b \ll 1$. The latter parameter depends only on the effective mass of the electron. If it is of the order of the mass of the free electron then $a_b \approx 0.5 \cdot 10^{-5} cm$ and for the concentrations typical for metals we get $p_f a_b \ll 1$. In other words, in this case we deal with a short range interaction with potential $V = V_0 \delta(x - y)$.

Here we take into account that the exact Hamiltonian by itself contains a natural cutoff. Indeed let us consider terms with transition of $R$ electrons to $L$ ones and back:

$$\int_{-\infty}^{\infty} dx dy \exp\left(-2p_f (x - y)\right) \hat{\Psi}_R^\dagger (x) \hat{\Psi}_L (x) V(x - y) \hat{\Psi}_L^\dagger (y) \hat{\Psi}_R (y).$$

They correspond to large momentum transfer: the argument of potential $2p_f - p$ is close to Fermi momentum. Neglecting $p$ as compared to $p_f$ and proceeding back to the coordinate representation we see that these terms look like $-V(2p_f) \delta (x - y)$.

Now it is easy to check that if $V(p)$ is the Coulomb potential with some cutoff then the original cutoff is canceled out in the sum of all terms and substituted by the value of the $2p_f$.

In order to calculate this one should take into account the identity

$$\left[ \hat{a} (p) , \exp \left( \int \frac{dp'}{2\pi} \hat{a}^\dagger (p') \hat{b}^\dagger (p') \right) \right] = \hat{b}^\dagger (p) \exp \left( \int \frac{dp'}{2\pi} \hat{a}^\dagger (p') \hat{b}^\dagger (p') \right)$$

and determine the anticommutator with same momenta in a regular way: $\{ \hat{a}^\dagger (p_n) , \hat{a} (p_m) \} = L \delta_{n,m}$.

This corresponds to periodical boundary conditions for the $\Psi, \bar{\Psi}$-fields at the boundaries of the sample. See, also, the end of Appendix [12].

This theorem is, in fact, a purely combinatorial statement. In the field theory we apply it mostly to Green functions. In statistical physics it is known as the first Mayer’s theorem (see, for example, Ulenbek D., Ford D. “Statistical mechanics lectures” (“Mir,” 1965)).

The bosonization technique allows one to calculate the 4-particle correlator exactly (i.e. with pairs scattering). As a result, one has the well-known power:

$$\alpha_{T} = 1 - 1/ \left[ 1 + \sqrt{1 + \frac{V_0}{\pi p_f}} \right].$$

It coincides with $\alpha_0$ only in the strong interaction limit. This distinction arises from the fact that in the term we are interested in only the direct interaction between two pairs enters.

The terms in the equation for $S_{eff}$ having the factor $\exp (-p_n a_{b}/ \pi)$ describe an excited state. We do not discuss them because one can calculate such effects more easily by using the bosonization technique.

Such behaviour of the density-density correlators is well-known (see e.g. [12]). Note that backscattering processes not included into the Hamiltonian in this case modify the numerical coefficient only in the log.

This means that $\Theta_c$ is determined by the excitation energy with the smallest momentum.