ON VARIATIONAL PRINCIPLE FOR UPPER METRIC MEAN DIMENSION WITH POTENTIAL

RUI YANG, ERCAI CHEN AND XIAOYAO ZHOU*

Abstract. Borrowing the idea of topological pressure determining measure-theoretical entropy in topological dynamical systems, we establish a variational principle for upper metric mean dimension with potential in terms of upper measure-theoretical metric mean dimension of invariant measures. Moreover, the notion of equilibrium states is introduced to characterize these measures that attain the supremum of the variational principle.

1. Introduction

By a pair \((X, T)\) we mean a topological dynamical system (TDS for short), where \(X\) is a compact metrizable topological space and \(T : X \to X\) is a continuous self-map. The set of metrics on \(X\) compatible with the topology is denoted by \(\mathcal{D}(X)\). By \(M(X), M(X, T), E(X, T)\) we denote the sets of all Borel probability measures on \(X\), all \(T\)-invariant Borel probability measures on \(X\), and all ergodic measures on \(X\), respectively. By \(C(X, \mathbb{R})\) we denote the set of all continuous real-valued functions on \(X\). It becomes a normed linear space equipped with the supremum norm.

Mean topological dimension introduced by Gromov [Gro99] is a new topological invariant for topological dynamical systems. Later, Lindenstrauss and Weiss [LW00] introduced the concept of metric mean dimension to capture the complexity of infinite topological entropy systems, and proved that metric mean dimension is an upper bound of mean topological dimension, which provides a powerful tool for estimating the upper bound of mean dimension. It turns out that metric mean dimension plays a vital role in infinite entropy theory and enjoys some special attentions.

It is well-known that the classical variational principle [Rue73, Wal75] states that topological pressure of a continuous potential is equal to the supremum of measure-theoretical entropy plus the integral of the continuous potential taken over all invariant measures. Readers can turn
to [HY07, CFH08, LCC12, CHZ13, Chu13, LMW18, BH20, HWZ21] for more results involving the variational principles of topological pressure in the context of different types of dynamical systems. Very recently, Lindenstrauss and Tsukamoto’s pioneering work [LT18] gave the first analogue of classical variational principle. More precisely, they established variational principles for metric mean dimension in terms of rate-distortion functions as follows.

**Theorem A.** Let \((X, T)\) be a TDS with a metric \(d \in \mathcal{D}(X)\). Then

\[
\overline{\text{mdim}}_M(T, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X,T)} R_{\mu, L^\infty}(\epsilon),
\]

\[
\underline{\text{mdim}}_M(T, X, d) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X,T)} R_{\mu, L^\infty}(\epsilon),
\]

where \(\overline{\text{mdim}}_M(T, X, d)\) and \(\underline{\text{mdim}}_M(T, X, d)\) denote upper and lower metric mean dimensions of \(X\), respectively. \(R_{\mu, L^\infty}(\epsilon)\) denotes \(L^\infty\)-rate distortion dimension of \(\mu\).

Besides, the above variational principles also hold if we replace \(R_{\mu, L^\infty}(\epsilon)\) by Katok entropy [VV17], Rényi information function [GS21], Brin-Katok local entropy, Shapira entropy and local entropy function [Shi21]. After establishing variational principles for metric mean dimension, Lindenstrauss and Tsukamoto [LT19] showed there exists double variational principles for mean dimension. Later, Tsukamoto [Tsu20] introduced the notions of mean dimension with potential and metric mean dimension with potential, and extended the double variational principles to mean dimension with potential. See also [CCL22, Wu22] for the variational principles of metric mean dimension with potential.

In this paper, we continue to inject ergodic theoretic ideas into mean dimension theory by obtaining a new variational principle for upper metric mean dimension with potential. Before stating our main result, we present some necessary backgrounds. Compared with the classical variational principle for topological pressure, a satisfactory variational principle for metric mean dimension would be the metric mean dimension is the supremum of measure-theoretical metric mean dimension defined by measure-theoretic \(\epsilon\)-entropies taken over all invariant measures. In [VV17], Velozo and Velozo asked if we can change the order of \(\limsup_{\epsilon \to 0}\) and \(\sup_{\mu \in M(X,T)}\) for \(L^\infty\)-rate distortion dimension. Unfortunately, Lindenstrauss and Tsukamoto [LT18, Section VIII] constructed a counter-example showing the order of \(\limsup\) (or \(\liminf\)) and \(\sup\) in Theorem A cannot be exchanged. In general, it is hard to define a proper quantity, which we call measure-theoretical metric mean dimension of invariant measures, that does not depend on the metric of the phase space such that there exists so-called satisfactory variational principle for upper metric mean dimension with potential. To solve this difficult, we borrow the idea of topological pressure determining...
measure-theoretical entropy that presents in [Wal75] to define a new measure-theoretical metric mean dimension for Borel probability measures. Using convex approach, we establish a variational principle for upper metric mean dimension with potential in terms of this quantity. Moreover, the notion of equilibrium states for upper metric mean dimension with potential is introduced to characterize these measures which attain the supremum of the variational principle.

The main result of this paper is as follows.

**Theorem 1.1.** Let \((X, T)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) such that \(\text{mdim}_M(T, X, d) < \infty\). Then for all \(f \in C(X, \mathbb{R})\), we have

\[
\overline{\text{mdim}}_M(T, f, d) = \sup_{\mu \in \mathcal{M}(X,T)} \{ F(\mu, d) + \int f d\mu \},
\]

where \(\overline{\text{mdim}}_M(T, f, d)\) denotes upper metric mean dimension with potential \(f\), and \(F(\mu, d)\) is the upper measure-theoretical metric mean dimension of \(\mu\).

The rest of this paper is organized as follows. In section 2, we recall the definition of metric mean dimension with potential and derive some basic properties for it. In section 3, we prove Theorem 1.1. We introduce the notion of equilibrium state for upper metric mean dimension with potential in section 4.

2. **Metric Mean Dimension with Potential**

In this section, we recall the precise definition of metric mean dimension with potential and derive some basic properties for it.

Let \(n \in \mathbb{N}\) and \(f \in C(X, \mathbb{R})\). For \(x \in X\), we set \(Snf(x) = \sum_{j=0}^{n-1} f(T^jx)\).

Given \(n \in \mathbb{N}\), the \(n\)-th Bowen metric \(d_n\) on \(X\) is given by

\[
d_n(x, y) := \max_{0 \leq j \leq n-1} d(T^jx, T^jy)
\]

for any \(x, y \in X\). Then the Bowen open ball \(B_n(x, \epsilon)\) of radius \(\epsilon\) and order \(n\) in the metric \(d_n\) is given by

\[
B_n(x, \epsilon) := \{ y \in X : d_n(x, y) < \epsilon \}.
\]

Given \(d \in \mathcal{D}(X)\) and \(f \in C(X, \mathbb{R})\), set

\[
R_n(X, f, d, \epsilon) = \inf \left\{ \sum_{i=1}^{n} (1/\epsilon)^{\sup_{x \in U_i} Snf(x)} : X = U_1 \cup \cdots \cup U_n \text{ with } \text{diam}(U_i, d_n) < \epsilon \text{ for all } i = 1, \ldots, n. \right\}
\]

and

\[
R(X, f, d, \epsilon) = \lim_{n \to \infty} \frac{\log R_n(X, f, d, \epsilon)}{n}.
\]
The limit exists since \( a_n := \log R_n(X, f, d, \epsilon) \) is subadditive in \( n \). The upper metric mean dimension of \( X \) with potential \( f \) [Tsu20] is given by

\[
\overline{\text{mdim}}_M(T, X, f, d) = \limsup_{\epsilon \to 0} \frac{R(X, f, d, \epsilon)}{\log \frac{1}{\epsilon}}.
\]

We sometimes write \( \overline{\text{mdim}}_M(T, f, d) \) instead of \( \overline{\text{mdim}}_M(T, X, f, d) \) when the system \((X, T)\) is clear. Replacing \( \limsup_{\epsilon \to 0} \) by \( \liminf_{\epsilon \to 0} \), one can similarly define \( \underline{\text{mdim}}_M(T, X, f, d) \) lower metric mean dimension with potential \( f \). If \( \overline{\text{mdim}}_M(T, X, f, d) = \underline{\text{mdim}}_M(T, X, f, d) \), we call the common value \( \text{mdim}\_M(T, X, f, d) \) metric mean dimension of \( X \) with potential \( f \). Obviously, the (upper/ lower) metric mean dimension with potential depends on the metric and the potential of underlying space.

When \( f = 0 \) is the zero potential, this exactly recovers the definition of upper (or lower) metric mean dimension of \( X \), introduced by Lindenstrauss and Weiss [LW00]. In this case, we write \( \overline{\text{mdim}}_M(T, X, d) = \overline{\text{mdim}}_M(T, X, 0, d) \).

Analogous to the definitions of classical topological pressure in thermodynamic formalism, the upper (or lower) metric mean dimension with potential can be equivalently defined by spanning sets and separated sets [Bow71, Wal82].

A set \( E \subset X \) is an \((n, \epsilon)\)-spanning set of \( X \) if for any \( x \in X \), there exists \( y \in E \) such that \( d_n(x, y) < \epsilon \). A set \( F \subset X \) is an \((n, \epsilon)\)-separated set of \( X \) if \( d_n(x, y) \geq \epsilon \) for any \( x, y \in F \) with \( x \neq y \).

Define

\[
P_n(X, f, d, \epsilon) = \sup\{ \sum_{x \in F} (1/\epsilon) S_{n,f(x)} : F \text{ is an } (n, \epsilon)\text{-separated set of } X \},
\]

\[
Q_n(X, f, d, \epsilon) = \inf\{ \sum_{x \in E} (1/\epsilon) S_{n,f(x)} : E \text{ is an } (n, \epsilon)\text{-spanning set of } X \}.
\]

The following proposition enables us to pay attention to outer limit \( \limsup_{\epsilon \to 0} \) rather than inner limits \( \limsup_{n \to \infty} \) and \( \liminf_{n \to \infty} \) when we compute the upper (or lower) metric mean dimension with potential.

**Proposition 2.1.** Let \( (X, T) \) be a TDS with a metric \( d \in \mathcal{D}(X) \) and \( f \in C(X, \mathbb{R}) \). Then

\[
\overline{\text{mdim}}_M(T, f, d) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log Q_n(X, f, d, \epsilon)}{n \log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log P_n(X, f, d, \epsilon)}{n \log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log P_n(X, f, d, \epsilon)}{n \log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log P_n(X, f, d, \epsilon)}{n \log \frac{1}{\epsilon}},
\]
it is also valid for \( \underline{\text{mdim}}_M(T, f, d) \) by changing \( \limsup_{\epsilon \to 0} \) into \( \liminf_{\epsilon \to 0} \).

**Proof.** We first show (2.1) = (2.2). Let \( \epsilon > 0 \) and we set \( \gamma(\epsilon) := \sup\{ |f(x) - f(y)| : d(x, y) < \epsilon \} \), then \( \gamma(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Let \( 0 < \epsilon < 1 \). Let \( E \) be an \((n, \epsilon)\)-spanning set of \( X \), then \( X = \cup_{x \in E} B_n(x, \epsilon) \). This implies that

\[
R_n(X, f, d, \epsilon) \leq \sum_{x \in E} (1/\epsilon)^{\sup_{y \in B_n(x, \epsilon)} S_n f(y)} \leq \sum_{x \in E} (1/\epsilon)^{S_n f(x) + n \gamma(\epsilon)} \leq (1/\epsilon)^{n \gamma(\epsilon)} 2^{n ||f||} \sum_{x \in E} (2/\epsilon)^{S_n f(x)}.
\]

We obtain that

\[
(2.5) \quad \lim_{n \to \infty} \frac{\log R_n(X, f, d, \epsilon)}{n} \leq \liminf_{n \to \infty} \frac{\log Q_n(X, f, d, \epsilon)}{n} + \gamma(\epsilon) \log \frac{1}{\epsilon} + ||f|| \log 2.
\]

Let \( X = U_1 \cup \cdots \cup U_n \) be an open cover of \( X \) with \( \text{diam}(U_i, d_n) < \frac{\epsilon}{\gamma}, i = 1, \ldots, n \). Choose \( x_i \in U_i \) for every \( 1 \leq i \leq n \). We have \( \{x_1, \ldots, x_n\} \) is an \((n, \epsilon)\)-spanning set of \( X \). Similarly, we can deduce that

\[
(2.6) \quad \limsup_{n \to \infty} \frac{\log Q_n(X, f, d, \epsilon)}{n} \leq \lim_{n \to \infty} \frac{\log R_n(X, f, d, \epsilon)}{n} + \gamma(\epsilon) \log \frac{1}{\epsilon} + ||f|| \log 2.
\]

Combining the facts (2.5) and (2.6), this shows (2.1) = (2.2).

It remains to show (2.1) = (2.3) and (2.2) = (2.4). Here, we only need to show (2.1) = (2.3) since (2.2) = (2.4) can be proved in a similar manner. Note that an \((n, \epsilon)\)-separated set with the maximal cardinality of \( X \) is also an \((n, \epsilon)\)-spanning set of \( X \), this yields that \( Q_n(X, f, d, \epsilon) \leq P_n(X, f, d, \epsilon) \). On the other hand, let \( E \) be an \((n, \epsilon)\)-spanning set of \( X \) and \( F \) be an \((n, \epsilon)\)-separated set of \( X \). We define a map \( \Phi : F \to E \) by assigning each \( x \in F \) to \( \Phi(x) \in E \) with \( d_n(x, \Phi(x)) < \frac{\epsilon}{\gamma} \). It is easy to show the map is injective.

\[
\sum_{x \in F} (2/\epsilon)^{S_n f(x)} \geq \sum_{x \in F} (1/\epsilon)^{S_n f(\Phi(x))} 2^{-n ||f||} \geq \sum_{x \in F} (1/\epsilon)^{S_n f(x) - n \gamma(\epsilon)} 2^{-n ||f||}.
\]

This shows

\[
(2.7) \quad \limsup_{n \to \infty} \frac{\log Q_n(X, f, d, \epsilon)}{n} \geq \limsup_{n \to \infty} \frac{\log P_n(X, f, d, \epsilon)}{n} - \gamma(\epsilon) \log \frac{1}{\epsilon} - ||f|| \log 2.
\]
This completes the proof. \(\Box\)

The next propositions presents the basic properties of the upper (lower) metric mean dimension with potential.

**Proposition 2.2.** Let \((X,T)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(f,g \in C(X,\mathbb{R})\). Then

1. If \(f \leq g\), then \(\operatorname{mdim}_M(T,f,d) \leq \operatorname{mdim}_M(T,g,d)\).

2. \(\operatorname{mdim}_M(T,f+c,d) = \operatorname{mdim}_M(T,f,d) + c\) for any \(c \in \mathbb{R}\).

3. \(\operatorname{mdim}_M(T,X,d) + \inf f \leq \operatorname{mdim}_M(T,f,d) \leq \operatorname{mdim}_M(T,X,d) + \sup f\).

4. \(\operatorname{mdim}_M(T, \cdot, d) : C(X,\mathbb{R}) \longrightarrow \mathbb{R} \cup \{\infty\}\) is either finite value or constantly \(\infty\).

5. If \(\operatorname{mdim}_M(T,f,d) \in \mathbb{R}\) for all \(f \in C(X,\mathbb{R})\), then
   \[|\operatorname{mdim}_M(T,f,d) - \operatorname{mdim}_M(T,g,d)| \leq ||f - g||\]
   and \(\operatorname{mdim}_M(T, \cdot, d)\) is convex on \(C(X,\mathbb{R})\).

6. \(\operatorname{mdim}_M(T,f+g,d) \leq \operatorname{mdim}_M(T,f,d) + \operatorname{mdim}_M(T,g,d)\).

7. If \(c \geq 1\), then \(\operatorname{mdim}_M(T,cf,d) \leq c\operatorname{mdim}_M(T,f,d)\); if \(c \leq 1\), then \(\operatorname{mdim}_M(T,cf,d) \geq c\operatorname{mdim}_M(T,f,d)\).

8. \(\operatorname{mdim}_M(T,f,d) \leq \operatorname{mdim}_M(T,|f|,d)\).

**Proof.** We give the proof of Proposition 2.2 one by one.

1-3) can be proved by using the definition of \(\operatorname{mdim}_M(T,f,d)\).

4) Note that \(\operatorname{mdim}_M(T,X,d) \in [0,\infty) \cup \{\infty\}\), and \(\operatorname{mdim}_M(T,X,d) < \infty\) if and only if \(\operatorname{mdim}_M(T,f,d) \in \mathbb{R}\) for all \(f \in C(X,\mathbb{R})\) by (3), this shows (4).

5) Let \(0 < \epsilon < 1\) and \(F\) be an \((n,\epsilon)\)-separated set of \(X\). Then we have
   \[\sum_{x \in F}(1/\epsilon)^{S_n f(x)} \leq \sum_{x \in F}(1/\epsilon)^{S_n g(x)+n||f-g||},\]
   which implies that \(\operatorname{mdim}_M(T,f,d) \leq \operatorname{mdim}_M(T,g,d) + ||f - g||\). Exchanging the role of \(f\) and \(g\), we obtain that
   \(\operatorname{mdim}_M(T,g,d) \leq \operatorname{mdim}_M(T,f,d) + ||f - g||\).

Let \(p \in [0,1]\) and \(f,g \in C(X,\mathbb{R})\), and let \(0 < \epsilon < 1\). Let \(F\) be an \((n,\epsilon)\)-separated set of \(X\). Using Hölder’s inequality, we have
   \[\sum_{x \in F}(1/\epsilon)^{p S_n f(x)+(1-p)S_n g(x)} \leq \left(\sum_{x \in F}(1/\epsilon)^{S_n f(x)}\right)^p \left(\sum_{x \in F}(1/\epsilon)^{S_n g(x)}\right)^{(1-p)},\]
   which yields that \(\operatorname{mdim}_M(T,pf+(1-p)g,d) \leq p\operatorname{mdim}_M(T,f,d)+(1-p)\operatorname{mdim}_M(T,g,d)\).

6) Let \(0 < \epsilon < 1\) and \(F\) be an \((n,\epsilon)\)-separated set of \(X\). Then we have
   \[\sum_{x \in F}(1/\epsilon)^{S_n(f+g)(x)} \leq \sum_{x \in F}(1/\epsilon)^{S_n f(x)} \cdot \sum_{x \in F}(1/\epsilon)^{S_n g(x)},\]
which implies the desired result.

(7) If \( a_1, ..., a_k \) are \( k \) positive real numbers with \( \sum_{i=1}^{k} a_i = 1 \), then
\[ \sum_{i=1}^{k} a_i \leq 1 \] if \( c \geq 1 \); \[ \sum_{i=1}^{k} a_i \geq 1 \] if \( c \leq 1 \).
Let \( 0 < \epsilon < 1 \) and \( F \) be an \((n, \epsilon)\)-separated set of \( X \). Then we have
\[ \sum_{x \in F} (1/\epsilon)^{cS_n f(x)} \leq \left( \sum_{x \in F} (1/\epsilon)^{S_n f(x)} \right)^c \]
if \( c \geq 1 \);
\[ \sum_{x \in F} (1/\epsilon)^{cS_n f(x)} \geq \left( \sum_{x \in F} (1/\epsilon)^{S_n f(x)} \right)^c \]
if \( c \leq 1 \).

We get the desired result after taking the corresponding limits.

(8) Since \(-|f| \leq f \leq |f|\), then \( \text{mdim}_M(T, -|f|, d) \leq \text{mdim}_M(T, f, d) \leq \text{mdim}_M(T, |f|, d) \) by (1). By (7), we have \( \text{mdim}_M(T, -|f|, d) \geq -\text{mdim}_M(T, |f|, d) \). This shows (8).

\[ \square \]

**Remark 2.3.** It is easy to see that the convexity is not valid for lower metric mean dimension with potential.

The following proposition establishes the product formula of metric mean dimension with potential.

**Proposition 2.4.** Let \((X_i, T_i)\) be a TDS with a metric \( d_i \in \mathcal{D}(X_i) \) and \( f_i \in C(X_i, \mathbb{R}) \), \( i = 1, 2 \), and \((X_1 \times X_2, T_1 \times T_2)\) is a product system defined by \( T_1 \times T_2 : (x_1, x_2) \mapsto (T_1 x_1, T_2 x_2) \). Then
\[ \text{mdim}_M(T_1 \times T_2, X_1 \times X_2, f, d) \leq \text{mdim}_M(T_1, X_1, f_1, d_1) + \text{mdim}_M(T_2, X_2, f_2, d_2) \]
(2.8)
\[ \text{mdim}_M(T_1 \times T_2, X_1 \times X_2, f, d) \geq \text{mdim}_M(T_1, X_1, f_1, d_1) + \text{mdim}_M(T_2, X_2, f_2, d_2) \]
(2.9)
where \( d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \) for any \((x_1, x_2), (y_1, y_2) \in X_1 \times X_2\), and \( f : X_1 \times X_2 \rightarrow \mathbb{R} \) is defined by \( f(x_1, x_2) = f_1(x_1) + f_2(x_2) \).

**Proof.** Let \( 0 < \epsilon < 1 \). Let \( E \) be an \((n, \epsilon)\)-spanning set of \( X_1 \) and \( F \) be an \((n, \epsilon)\)-spanning set of \( X_2 \). Then \( E \times F \) is an \((n, \epsilon)\)-spanning set of \( X_1 \times X_2 \). This gives us that
\[ Q_n(X_1 \times X_2, f, d, \epsilon) \leq \sum_{(x, y) \in E \times F} (1/\epsilon)^{S_n f(x, y)} \]
\[ \leq \sum_{x \in E} (1/\epsilon)^{S_n f_1(x)} \cdot \sum_{y \in F} (1/\epsilon)^{S_n f_2(y)}, \]
which implies that
\[ Q_n(X_1 \times X_2, f, d, \epsilon) \leq Q_n(X_1, f_1, d_1, \epsilon)Q_n(X_2, f_2, d_2, \epsilon). \]
We get inequality (2.8) by (2.3). Inequality (2.9) can be obtained by using the definition of separated set of lower metric mean dimension with potential and Proposition 2.1.

A power rule of metric mean dimension with potential is stated as follows.

**Proposition 2.5.** Let \((X, T)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(f \in C(X, \mathbb{R})\). Then for any positive number \(k \geq 2\), we have

\[
\text{mdim}_M(T^k, S_kf, d) \leq k \text{mdim}_M(T, f, d),
\]

where \(S_kf(x) = \sum_{j=0}^{k-1} f(T^jx)\).

Additionally, if \(T\) is Lipschitz mapping on \(X\), then

\[
\text{mdim}_M(T^k, S_kf, d) = k \text{mdim}_M(T, f, d).
\]

The above results are also valid for \(\text{mdim}_M(T, f, d)\).

**Proof.** Let \(0 < \epsilon < 1\). Let \(E\) be an \((nk, \epsilon)\)-spanning set of \(X\) for \(T\). Then \(E\) is an \((n, \epsilon)\)-spanning set of \(X\) for \(T^k\). It follows that

\[
Q_n(X, S_kf, d, \epsilon) \leq \sum_{x \in E} (1/\epsilon)^{S_n(S_kf(x))} = \sum_{x \in E} (1/\epsilon)^{S_{nk}f(x)},
\]

which implies that \(Q_n(X, S_kf, d, \epsilon) \leq Q_{nk}(X, f, d, \epsilon)\). Hence, we have

\[
\limsup_{n \to \infty} \frac{\log Q_n(X, S_kf, d, \epsilon)}{n} \leq k \cdot \limsup_{n \to \infty} \frac{\log Q_{nk}(X, f, d, \epsilon)}{nk} \leq k \cdot \liminf_{n \to \infty} \frac{\log Q_n(X, f, d, \epsilon)}{n},
\]

which implies inequality (2.10).

Now assume that \(L > 0\) is a constant satisfying \(d(Tx, Ty) \leq L \cdot d(x, y)\) for any \(x, y \in X\). Fix \(\alpha > 0\). Let \(0 < \epsilon < 1\) and \(C = (\max\{L, 1\} + \alpha)^k > 1\). Then \(d_k(x, y) \leq C\epsilon\) if \(d(x, y) \leq \epsilon\) for any \(x, y \in X\). This shows if \(E\) is an \((n, \epsilon)\)-spanning set of \(X\) for \(T^k\), then \(E\) is an \((nk, C\epsilon)\)-spanning set of \(X\) for \(T\). Similarly, we have

\[
\limsup_{n \to \infty} \frac{\log Q_n(X, S_kf, d, \epsilon)}{n} \geq k \cdot \liminf_{n \to \infty} \frac{\log Q_{nk}(X, f, d, C\epsilon)}{nk} \geq k \cdot \liminf_{n \to \infty} \frac{\log Q_n(X, f, d, C\epsilon)}{n}.
\]

By (2.2) and using the fact \(\lim_{\epsilon \to 0} \frac{\log \epsilon}{\log \epsilon} = 1\), we have \(\text{mdim}_M(T^k, S_kf, d) \geq k \text{mdim}_M(T, f, d)\). This completes the proof. □
Remark 2.6. By virtue of the continuity of $T$, there exists $\delta(\epsilon) > 0$ such that $d_k(x, y) < \epsilon$ for any $x, y \in X$ with $d(x, y) < \delta$. One can similarly deduce that

$$\limsup_{\epsilon \to 0} \frac{\log \frac{1}{\delta(\epsilon)}}{\log \frac{1}{\epsilon}} \cdot \limsup_{n \to \infty} \frac{\log Q_n(X, S_k f, d, \delta)}{n \log \frac{1}{\delta(\epsilon)}} \geq k \cdot \limsup_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log Q_n(X, f, d, \epsilon)}{n \log \frac{1}{\epsilon}}.$$  

It is hard to determine whether $\limsup_{\epsilon \to 0} \frac{\log \frac{1}{\delta(\epsilon)}}{\log \frac{1}{\epsilon}}$ is equal to 1 without the assumption of Lipschitz condition.

3. Proof of Theorem 1.1

In this section, we borrow the idea of topological pressure determining measure-theoretical entropy to define the upper measure-theoretical metric mean dimension for probability measures and then prove Theorem 1.1.

Recall that the classical topological pressure state that for any $f \in C(X, \mathbb{R})$, one has

$$P(T, f) = \sup_{\mu \in M(X, T)} \{h_{\mu}(T) + \int f d\mu\},$$

where $P(T, f)$ denotes the topological pressure of $f$ w.r.t. $T$, and $h_{\mu}(T)$ is the usual measure-theoretical entropy of $\mu$ w.r.t. $T$. Suppose that the topological entropy $h(T) := P(T, 0) < \infty$, based on the variational principle for topological pressure Walters [Wal82, Theorem 9.12] proved that for each fixed $\mu_0 \in M(X, T)$, the entropy map of $T$ is upper semi-continuous at $\mu_0$ if and only if

$$h_{\mu_0}(T) = \inf \{P(T, f) - \int f d\mu_0 : f \in C(X, \mathbb{R})\}$$

$$= \inf_{f \in \hat{A}} \int f d\mu_0,$$

where $\hat{A} = \{f \in C(X, \mathbb{R}) : P(T, -f) = 0\}$, and the second equality can be obtained by using the equality $P(T, f + c) = P(T, f) + c$ for all $c \in \mathbb{R}$. In other words, measure-theoretical entropy is completely determined by topological pressure. Following this idea, we shall define the so-called upper measure-theoretical metric mean dimension $F(\mu, d)$ for every probability measure.

Definition 3.1. Let $(X, T)$ be a TDS with a metric $d \in \mathcal{D}(X)$ satisfying $\text{mdim}_M(T, X, d) < \infty$ and $\mu \in M(X)$. The upper measure-theoretical metric mean dimension of $\mu$ is defined as

$$F(\mu, d) = \inf_{f \in \hat{A}} \int f d\mu,$$
where \( \mathcal{A} = \{ f \in C(X, \mathbb{R}) : \overline{\text{mdim}}_M(T, -f, d) = 0 \} \).

**Remark 3.2.**

1. We emphasize "metric" in definition 3.1 since \( F(\mu, d) \) is dependent of the choice of \( d \in \mathcal{D}(X) \).

2. If \( \overline{\text{mdim}}_M(T, X, d) = \infty \), we have \( \overline{\text{mdim}}_M(T, f, d) = \infty \) for all \( C(X, \mathbb{R}) \) by Proposition 2.2,(3). In this case, \( \mathcal{A} \) is an empty set and hence we set \( F(\mu, d) = \inf \emptyset = \infty \) for all \( \mu \in M(X) \).

**Proposition 3.3.** Let \((X, T)\) be a TDS with a metric \( d \in D(X) \) satisfying \( \overline{\text{mdim}}_M(T, X, d) < \infty \). Then \( F(\mu, d) \) is an upper semi-continuous, concave and bounded above function on \( M(X) \).

**Proof.** Let \( f \in \mathcal{A} \). We define \( F_f : M(X) \to \mathbb{R} \) by assigning each \( \mu \) to \( \int fd\mu \). Then \( F_f \) is continuous, which shows \( F \) is upper semi-continuous by the fact that the infimum of continuous functions is upper semi-continuous.

Concavity is clear by definition of \( F \). Since \( \overline{\text{mdim}}_M(T, f, d) - f \in \mathcal{A} \) for any \( f \in C(X, \mathbb{R}) \), so the constant valued function \( \overline{\text{mdim}}_M(T, f, d) \in \mathcal{A} \). This implies that \( F(\mu, d) \leq \overline{\text{mdim}}_M(T, X, d) \) for all \( \mu \in M(X) \). \( \square \)

**Proposition 3.4.** Let \((X, T)\) be a TDS with a metric \( d \in D(X) \) and \( f \). Then for all \( g \in C(X, \mathbb{R}) \),

\[
\overline{\text{mdim}}_M(T, f, d) = \overline{\text{mdim}}_M(f + g \circ T - g)
\]

**Proof.** This can be proved by using the fact if \( F \) is an \((n, \epsilon)\)-separated set of \( X \), then

\[
\sum_{x \in F} (1/\epsilon)S_nf(x)^{-2||f-g||} \leq \sum_{x \in F} (1/\epsilon)S_n(f+g\circ T-g) \leq \sum_{x \in F} (1/\epsilon)S_nf(x+2||f-g||).
\]

\( \square \)

We need the well-known separation theorem of convex sets [DS88, p. 417] in our proof, which states that if \( K_1, K_2 \) are disjoint closed subsets of a local convex linear topological space \( V \), and \( K_1 \) is compact, then there exists a continuous real-valued linear functional \( \varphi \) on \( V \) such that \( \varphi(x) < \varphi(y) \) for any \( x \in K_1 \) and \( y \in K_2 \).

**Proof of Theorem 1.1.** We divide the proof into two steps.

Step 1: we show

\[
\overline{\text{mdim}}_M(T, f, d) = \sup_{\mu \in M(X)} \{ F(\mu, d) + \int fd\mu \}
\]

As \( \overline{\text{mdim}}_M(T, f, d) - f \in \mathcal{A} \), we have \( F(\mu, d) \leq \int \overline{\text{mdim}}_M(T, f, d) - fd\mu \) for any \( \mu \in M(X) \). This shows for all \( \mu \in M(X) \), we have \( F(\mu, d) + \int fd\mu \leq \overline{\text{mdim}}_M(T, f, d) \). So the remaining is to show

\[
\overline{\text{mdim}}_M(T, f, d) \leq \sup_{\mu \in M(X)} \{ F(\mu, d) + \int fd\mu \}.
\]
We prove it by showing for any $\epsilon > 0$, there exists $\mu \in M(X)$ such that $
exists \epsilon < F(\mu, d) + \int f d\mu$, or equivalently $F(\mu, d) + \int f_1 d\mu + \epsilon > 0$, where $f_1 = f - \text{mdim}_M(T, f, d)$.

Let $C := \{g \in C(X, \mathbb{R}) : \nexists \text{mdim}_M(T, -g, d) \leq 0\}$. Recall $A = \{f \in C(X, \mathbb{R}) : \text{mdim}_M(T, -f, d) = 0\}$. Then $A \subset C$ and $C = \text{mdim}_M(T, d)^{-1} \{\{0\}\}$ is a closed convex subset of $C(X, \mathbb{R})$, where the convexity of $C$ follows from Proposition 2.1. Since $\text{mdim}_M(T, f_1, d) = 0$, one has $-(f_1 + \frac{\epsilon}{2}) \notin C$ and hence $-f_1 \notin C + \frac{\epsilon}{2}$. Let $K_1 = \{-f_1\}$ and $K_2 = C + \frac{\epsilon}{2}$. Applying the separation theorem of convex sets to the sets $K_1, K_2$, there exists a continuous real-valued linear functional $L$ on $C(X, \mathbb{R})$ such that $L(-f_1) < L(g)$ for any $g \in C + \frac{\epsilon}{2}$. This yields that $\inf_{g \in C + \frac{\epsilon}{2}} L(g) + L(f_1) \geq 0$.

Next we show $L$ is a positive linear functional on $C(X, \mathbb{R})$ and $L(1) > 0$. Let $g \in C(X, \mathbb{R})$ with $g \geq 0$. For any $c > 0$, one has
\[
\text{mdim}_M(T, -(cg + 1 + \text{mdim}_M(T, X, d)), d)
= \text{mdim}_M(T, -(cg + 1), d) - 1 - \text{mdim}_M(T, X, d)
\leq \text{mdim}_M(T, 0, d) - 1 - \text{mdim}_M(T, X, d) = -1 < 0,
\]
which implies that $cg + 1 + \text{mdim}_M(T, X, d) \in C$ and $L(-f_1) < cL(g) + L(1 + \text{mdim}_M(T, X, d))$. This shows $L(g) \geq 0$; otherwise we have $L(-f_1) = -\infty$ by letting $c \to \infty$.

We proceed to show $L(1) > 0$. Since $L$ is not constantly equal to zero, we can choose $g \in C(X, \mathbb{R})$ such that $L(g) > 0$ with the supremum norm $||g|| < 1$. Then $L(1) = L(g) + L(1 - g) > 0$.

By Riesz representation theorem, there exists a $\mu \in M(X)$ such that
\[
\frac{L(g)}{L(1)} = \int g d\mu
\]
for all $g \in C(X, \mathbb{R})$.

Since $A + \frac{\epsilon}{2} \subset C + \frac{\epsilon}{2}$, one has
\[
F(\mu, d) + \epsilon = \inf_{f \in A} \int f d\mu + \frac{\epsilon}{2} + \frac{\epsilon}{2}
= \inf_{f \in A + \frac{\epsilon}{2}} \int f d\mu + \frac{\epsilon}{2} \geq \int f d\mu + \frac{\epsilon}{2}.
\]
Therefore,
\[
F(\mu, d) + \int f_1 d\mu + \epsilon \geq \frac{L(f_1)}{L(1)} + \inf_{g \in C + \frac{\epsilon}{2}} \int f d\mu + \frac{\epsilon}{2}
= \frac{L(f_1)}{L(1)} + \inf_{g \in C + \frac{\epsilon}{2}} \frac{L(g)}{L(1)} + \frac{\epsilon}{2}
= \frac{1}{L(1)}(L(f_1) + \inf_{g \in C + \frac{\epsilon}{2}} L(g)) + \frac{\epsilon}{2} > 0.
\]
Step 2: we show
\[ \text{mdim}_M(T, f, d) = \sup_{\mu \in M(X,T)} \mu \] .

By step 1, it suffices to show \( \text{mdim}_M(T, f, d) \leq \sup_{\mu \in M(X,T)} \mu \). By Step 1 and that fact \( F(\mu, d) \) is upper semi-continuous on \( M(X) \), we can choose \( \mu_0 \in M(X) \) such that \( \text{mdim}_M(T, f, d) = F(\mu_0, d) + \int f d\mu_0 \).

By Proposition 3.4 and Step 1, for all \( g \in C(X, \mathbb{R}) \) one has
\[ F(\mu_0, d) + \int f d\mu_0 = \text{mdim}_M(T, f, d) \]
\[ = \text{mdim}_M(f + g \circ T - g) \]
\[ \geq F(\mu_0, d) + \int f + g \circ T - gd\mu_0. \]

This shows \( \int gd\mu \geq \int g \circ T d\mu_0 \), which implies that
\[ \int gd\mu \geq \int g \circ T d\mu_0 \]
for all \( g \in C(X, \mathbb{R}) \). Hence, one has \( \mu_0 \in M(X, T) \) and hence
\[ \text{mdim}_M(T, f, d) = F(\mu_0, d) + \int f d\mu_0 \leq \sup_{\mu \in M(X,T)} \{ F(\mu, d) + \int f d\mu \}. \]

This completes the proof.

\[ \square \]

Remark 3.5. Following the line of the proof of Theorem 1.1, we cannot obtain an analogous result for lower metric mean dimension with potential since \( \text{mdim}_M(T, \cdot, d) \) is not convex on \( C(X, \mathbb{R}) \).

4. Equilibrium states

In this section, we introduce the notion of equilibrium states for upper metric mean dimension with potential to characterize the members of \( M(X) \) that attain the supremum.

The following notion is an analogue of the concept of equilibrium state for topological pressure in thermodynamic formalism [Wal82] (or known as maximal entropy measure for topological entropy).

Definition 4.1. Let \( (X, T) \) be a TDS with a metric \( d \in \mathcal{D}(X) \) satisfying \( \text{mdim}_M(T, X, d) < \infty \), and let \( f \in C(X, \mathbb{R}) \). A measure \( \mu \in M(X) \) is said to be an equilibrium state for \( f \) with respect to \( d \) if
\[ \text{mdim}_M(T, f, d) = F(\mu, d) + \int f d\mu, \]
where \( F(\mu, d) = \inf_{f \in \mathcal{A}} \int f d\mu \) and \( \mathcal{A} = \{ f \in C(X, \mathbb{R}) : \text{mdim}_M(T, -f, d) = 0\} \). The set of all equilibrium states for \( f \) with respect to \( d \) is denoted by \( M_f(T, X, d) \).
Recall that a measure $\mu$ on $X$ is said to be a finite signed measure if the map $\mu : \mathcal{B}(X) \to \mathbb{R}$ is countably additive, where $\mathcal{B}(X)$ is the Borel sigma algebra of $X$. Suppose that $\text{mdim}_M(T, X, d) < \infty$, we know that $\text{mdim}_M(T, \cdot, d)$ is a convex function on $C(X, \mathbb{R})$ by Proposition 2.2,(5). We will see that the notion of equilibrium state is closely tied with the notion of tangent functional to the convex function $\text{mdim}_M(T, \cdot, d)$.

**Definition 4.2.** Let $(X, T)$ be a TDS with a metric $d \in \mathcal{D}(X)$ satisfying $\text{mdim}_M(T, X, d) < \infty$, and let $f \in C(X, \mathbb{R})$. A tangent functional to $\text{mdim}_M(T, \cdot, d)$ at $f$ with respect to $d$ is a finite signed measure such that

$$\text{mdim}_M(T, f + g, d) - \text{mdim}_M(T, f, d) \geq \int g d\mu$$

for all $g \in C(X, \mathbb{R})$. The set of all tangent functionals to $\text{mdim}_M(T, \cdot, d)$ at $f$ with respect to $d$ is denoted by $t_f(T, X, d)$.

**Remark 4.3.** (1) Riesz representation states that for each $L \in C(X, \mathbb{R})^*$, the dual space of $C(X, \mathbb{R})$, is of the form of $L(f) = \int f d\mu$ for any $f \in C(X, \mathbb{R})$. Then $C(X, \mathbb{R})^*$ can be identified with the set of all finite signed measures on $X$. Hence the tangent functionals to $\text{mdim}_M(T, \cdot, d)$ at $f$ with respect to $d$ can be viewed as the elements of $C(X, \mathbb{R})^*$ satisfying $L(g) \leq \text{mdim}_M(T, f + g, d) - \text{mdim}_M(T, f, d)$ for all $g \in C(X, \mathbb{R})$.

(2) For each $f \in C(X, \mathbb{R})$, $t_f(T, X, d)$ is not empty. Consider the linear space $\{ c : c \in \mathbb{R} \}$, we have $\gamma(c) := \int c d\mu \leq \text{mdim}_M(T, f + c, d) - \text{mdim}_M(T, f, d)$ for all $c \in C(X, \mathbb{R})$. Using Hahn-Banach theorem one can extend $\gamma$ on $\mathbb{R}$ to an element of $C(X, \mathbb{R})^*$ dominated by the convex function $g \mapsto \text{mdim}_M(T, f + g, d) - \text{mdim}_M(T, f, d)$.

**Proposition 4.4.** Let $(X, T)$ be a TDS with a metric $d \in \mathcal{D}(X)$ satisfying $\text{mdim}_M(T, X, d) < \infty$ and $f \in C(X, \mathbb{R})$. Then

(1) $M_f(T, X, d)$ is a non-empty compact convex subset of $M(X)$.

(2)

$$M_f(T, X, d) = t_f(T, X, d)$$

$$= \cap_{n \geq 1} \{ \mu \in M(X) : \text{mdim}_M(T, f, d) - \frac{1}{n} \}.$$

(3) There exists a dense subset $\mathcal{D}$ of $C(X, \mathbb{R})$ so that for any $f \in \mathcal{D}$, $M_f(T, X, d)$ has a unique equilibrium state.

**Proof.** (1) Since $F(\mu, d)$ is upper semi-continuous on $M(X)$ by Proposition 3.3, hence the supremum can be attained for some $\mu \in M(X)$. 

13
This shows $M_f(T, X, d) \neq \emptyset$. Let $\mu, \nu \in M(X)$ and $p \in [0, 1]$. Then
\[
\overline{\dim}_M(T, f, d) = p\overline{\dim}_M(T, f, d) + (1 - p)\overline{\dim}_M(T, f, d)
\]
\[
= pF(\mu, d) + (1 - p)F(\nu) + p \int f d\mu + (1 - p) \int f d\nu
\]
\[
\leq F(p\mu + (1 - p)\nu) + \int f d\mu + (1 - p)\nu
\]
by Proposition 3.3
\[
\leq \overline{\dim}_M(T, f, d)
\]
which implies that $M_f(T, X, d)$ is convex.

Let $\mu \in M(X)$ with $\mu_n \to \mu$, where $\mu_n \in M_f(T, X, d)$. As $F(\mu, d)$ is upper semi-continuous, we have
\[
\overline{\dim}_M(T, f, d) = \limsup_{n \to \infty} F(\mu_n) + \int f d\mu_n \leq \overline{\dim}_M(T, f, d).
\]
This shows that $M_f(T, X, d)$ is closed and hence compact.

(2) $M_f(T, X, d) = \cap_{n \geq 1} \{ \mu \in M(X) : > \overline{\dim}_M(T, f, d) - \frac{1}{n} \}$ is clear.

It remains to show $M_f(T, X, d) = t_f(T, X, d)$.

Let $\mu \in M_f(T, X, d)$. Then we have
\[
\overline{\dim}_M(T, f + g, d) - \overline{\dim}_M(T, f, d) \geq F(\mu, d) + \int f + gd\mu - ()
\]
\[
= \int gd\mu.
\]
This yields that $\mu \in t_f(T, X, d)$ and hence $M_f(T, X, d) \subset t_f(T, X, d)$.

On the other hand, let $\mu \in t_f(T, X, d)$. For any $g \in C(X, \mathbb{R})$ with $g \geq 0$ and $\epsilon > 0$, we have
\[
\int g + \epsilon d\mu = - \int -(g + \epsilon)d\mu
\]
\[
\geq -\overline{\dim}_M(T, f - (g + \epsilon), d) + \overline{\dim}_M(T, f, d)
\]
\[
\geq -\overline{\dim}_M(T, f - \inf(g + \epsilon), d) + \overline{\dim}_M(T, f, d)
\]
\[
= \inf(g + \epsilon) > 0.
\]
This shows $\mu$ is a non-negative measure on $X$.

For $n \in \mathbb{Z}^+$, we have
\[
\int nd\mu \leq \overline{\dim}_M(T, f + n, d) - \overline{\dim}_M(T, f, d) = n,
\]
which tells us $\mu(X) \leq 1$.

If $n$ is a negative integer number, one can similarly obtain $\mu(X) \geq 1$.

This shows $\mu \in M(X)$. Since $\mu \in t_f(T, X, d)$, then for any $g \in C(X, \mathbb{R})$,
\[
\overline{\dim}_M(T, f + g, d) - \int f + gd\mu \geq \overline{\dim}_M(T, f, d) - \int fd\mu.
\]
By the arbitrariness of \( g \), we know that for any \( h \in C(X, \mathbb{R}) \),
\[
\underbar{\text{mdim}}_M(T, h, d) - \int h d\mu \geq \underbar{\text{mdim}}_M(T, f, d) - \int f d\mu.
\]
This shows \( F(\mu, d) \geq \underbar{\text{mdim}}_M(T, f, d) - \int f d\mu \) by Definition 3.1 and (??), which implies that \( \mu \in M_f(T, X, d) \) and hence \( t_f(T, X, d) \subset M_f(T, X, d) \).

(3) The theorem [DS88, p.450] stating that a convex function on a separable Banach space has a unique tangent functional at a dense set of points. Applying this theorem to \( \underbar{\text{mdim}}_M(T, \cdot, d) \) and \( C(X, \mathbb{R}) \), together with \( M_f(T, X, d) = t_f(T, X, d) \) by (2), this finishes the proof.

**Corollary 4.5.** Let \((X, T)\) be a TDS with a metric \( d \in \mathcal{D}(X)\) satisfying \( \underbar{\text{mdim}}_M(T, X, d) < \infty \) and \( f \in C(X, \mathbb{R}) \). Suppose that \( \mu_s \in M(X) \), \( s \in \mathbb{R} \), is an equilibrium state for \( sf \) with \( \int f d\mu_s \neq 0 \). Then
\[
\underbar{\text{mdim}}_M(T, sf, d) = 0 \text{ if and only if } s = -\frac{F(\mu_s)}{\int f d\mu_s}.
\]

**Proof.** It is clear since \( \underbar{\text{mdim}}_M(T, sf, d) = F(\mu_s) + s \int f d\mu_s \). □

The authors [YCZ22a] established Bowen’s equations for upper metric mean dimension with potential, which associates the notion of upper metric mean dimension with potential with dimension theory of dynamical systems in the context of thermodynamic formalism. As an application of equilibrium state for upper metric mean dimension with potential, we will see that the BS metric mean dimension can be computed as follows.

**Corollary 4.6.** Let \((X, T)\) be a TDS with a metric \( d \in \mathcal{D}(X)\) satisfying \( \underbar{\text{mdim}}_M(T, X, d) < \infty \) and \( f \in C(X, \mathbb{R}) \) with \( f > 0 \). Then there exists the unique \( s_0 \geq 0 \) such that for any \( \mu \in M_{-sf}(T, X, d) \), we have
\[
\underbar{\text{BSmdim}}_{M,X,T}(f, d) = \frac{F(\mu_{s_0}, d)}{\int f d\mu_{s_0}},
\]
where \( \underbar{\text{BSmdim}}_{M,X,T}(f, d) \) is BS metric mean dimension on \( X \) with respect to \( f \), see [YCZ22a, Definition 3.8] for its precise definition.

**Proof.** By [YCZ22a, Theorem 1.3], we know that \( s_0 = \underbar{\text{BSmdim}}_{M,X,T}(f, d) \geq 0 \) is the unique root of the equation
\[
\underbar{\text{mdim}}_M(T, -sf, d) = 0.
\]
For any \( \mu \in M_{-sf}(T, X, d) \), we have
\[
\underbar{\text{mdim}}_M(T, -sf, d) = F(\mu_{s_0}, d) - s_0 \int f d\mu_{s_0}.
\]
Then \( \underbar{\text{BSmdim}}_{M,X,T}(f, d) = \frac{F(\mu_{s_0}, d)}{\int f d\mu_{s_0}} \) by the uniqueness of the root of the equation. □
ACKNOWLEDGEMENT

The work was supported by the National Natural Science Foundation of China (Nos.12071222 and 11971236), China Postdoctoral Science Foundation (No.2016M591873), and China Postdoctoral Science Special Foundation (No.2017T100384). The work was also funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. We would like to express our gratitude to Tianyuan Mathematical Center in Southwest China(11826102), Sichuan University and Southwest Jiaotong University for their support and hospitality.

REFERENCES

[Bow71] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc. 153 (1971), 401–414.

[Bow75] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics, Vol. 470, Springer-Verlag, Berlin–New York, 1975.

[BH20] L. Barreira and C. Holanda, Nonadditive topological pressure for flows, Nonlinearity 33 (2020), 3370–3394.

[CFH08] Y. Cao, D. J. Feng and W. Huang, The thermodynamic formalism for sub-additive potentials, Discrete Contin. Dyn. Syst. 20 (2008), 639–657.

[CHZ13] Y. Cao, H. Hu and Y. Zhao, Nonadditive measure-theoretic pressure and applications to dimensions of an ergodic measure, Erg. Theory Dynam. Syst. 33 (2013), 831–850.

[Chu13] N. P. Chung, Topological pressure and the variational principle for actions of sofic groups, Erg. Theory Dynam. Syst. 33 (2013), 1363–1390.

[CCL22] H. Chen, D. Cheng and Z. Li, Upper metric mean dimensions with potential, Results Math. 77 (2022), Paper No. 54, 26 pp.

[DS88] N. Dunford and J. T. Schwartz, Linear operators, Part 1, John Wiley & Sons, 1988.

[Gro99] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, Math. Phys, Anal. Geom. 4 (1999), 323-415.

[GS21] Y. Gutman and A. Špiewak, Around the variational principle for metric mean dimension, Studia Math. 261 (2021), 345–360.

[HY07] W. Huang and Y. Yi, A local variational principle of pressure and its applications to equilibrium states, Israel J. Math. 161 (2007), 29-74.

[HWZ21] H. Hu, W. Wu and Y. Zhu, Unstable pressure and u-equilibrium states for partially hyperbolic diffeomorphisms, Erg. Theory Dynam. Syst. 41 (2021), 3336-3362.
[KL11] D. Kerr and H. Li, Entropy and the variational principle for actions of sofic groups, *Invent. Math.* **186** (2011), 501–558.

[LCC12] Y. Li, E. Chen and W. Cheng, Tail pressure and the tail entropy function, *Erg. Theory Dynam. Syst.* **32** (2012), 1400-1417.

[LMW18] X. Lin, D. Ma and Y. Wang, On the measure-theoretic entropy and topological pressure of free semigroup actions, *Erg, Theory Dynam. Syst.* **38** (2018), 686-716.

[LW00] E. Lindenstrauss and B. Weiss, Mean topological dimension, *Israel J. Math.* **115** (2000), 1-24.

[LT18] E. Lindenstrauss and M. Tsukamoto, From rate distortion theory to metric mean dimension: variational principle, *IEEE Trans. Inform. Theory* **64** (2018), 3590-3609.

[LT19] E. Lindenstrauss and M. Tsukamoto, Double variational principle for mean dimension, *Geom. Funct. Anal.* **29** (2019), 1048-1109.

[Rue73] D. Ruelle, Statistical mechanics on a compact set with $Z^n$ action satisfying expansiveness and specification, *Trans. Amer. Math. Soc.* **187** (1973), 237-251.

[Shi21] R. Shi, On variational principles for metric mean dimension, to appear in *IEEE Trans. Inform. Theory* **68** (2022), 4282-4288.

[Tsu20] M. Tsukamoto, Double variational principle for mean dimension with potential, *Adv. Math.* **361** (2020), 106935, 53 pp.

[VV17] A. Velozo and R. Velozo, Rate distortion theory, metric mean dimension and measure theoretic entropy, arXiv:1707.05762.

[Wal75] P. Walters, A variational principle for the pressure of continuous transformations, *Amer. J. Math.* **97** (1975), 937–971.

[Wu22] W. Wu, On relative metric mean dimension with potential and variational principles, to appear in *J. Dynam. Diff. Equ.* https://doi.org/10.1007/s10884-022-10175-w.

[YCZ22a] R. Yang, E. Chen and X. Zhou, Bowen’s equations for upper metric mean dimension with potential, *Nonlinearity* **35** (2022), 4905-4938.

[Wal82] P. Walters, An introduction to ergodic theory, Springer-Verlag, New York, 1982.

1. School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, Jiangsu, P.R.China

   Email address: zkyangrui2015@163.com
   Email address: ecchen@njnu.edu.cn
   Email address: zhouxiaoyaodeyouxian@126.com