Non-integrable threshold singularities of two-point functions in perturbation theory

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In perturbation theory, the spectral densities of two-point functions develop non-integrable threshold singularities at higher orders. In QCD, such singularities emerge when calculating the diagrams in terms of the pole quark mass, and they become stronger when one rearranges the perturbative expansion in terms of the running quark mass. In this letter we discuss the proper way to handle such singularities.

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1. INTRODUCTION

The correlation function of two currents defined as

$$\Pi(q) = i \int dx \exp(iqx) \langle 0 | T \{ \bar{\Psi}(x) O_1 \psi(x), \bar{\psi}(0) O_2 \Psi(0) \} | 0 \rangle,$$  

(1.1)

where \( \psi \) and \( \Psi \) denote fermion fields (which may be identical) and \( O_{1,2} \) are Dirac matrices, is one of the basic objects in quantum field theory. For instance, in QCD, two-point functions with an appropriate choice of the Dirac matrices provide the basis for the extraction of masses and couplings of mesons within the method of QCD sum rules \[1, 2\]. In general, the two-point function contains a number of independent Lorentz structures \( L_i(q) \) and the corresponding invariant amplitudes \( F_i(q^2) \). We shall discuss here spectral representations for the invariant amplitudes \( F_i(q^2) \) and omit throughout the paper the subscript \( i \).

As follows from the general properties of the time-ordered product, the function \( F(q^2) \) is an analytic function in the complex \( q^2 \)-plane with the cut along the real axis \[3, 4\] from a threshold \( s_{th} \) to \(+\infty\) and satisfies the spectral representation with an appropriate number \( n \) of subtractions:

$$F(q^2) = F(0) + F'(0)q^2 + \ldots + \frac{(q^2)^n}{\pi} \int_{s_{th}}^{\infty} \frac{ds}{s^n(s - q^2 - i0)} \rho(s),$$  

(1.2)

where \( \rho(s) = \text{Im} F(s+i0) \). The subtractions are performed in order to provide the convergence of the spectral integral or to satisfy constraints imposed by symmetries of the theory.

In QCD, one makes use of several expansions of the two-point functions. Important examples of such expansions are listed below: (i) perturbative expansion in powers of \( \alpha_s \) of the elastic correlation function of a heavy quark \[5, 6\]; (ii) rearrangement of the perturbation theory for the heavy-light correlation functions \[7, 8\] via a replacement of the pole mass by the running mass of the heavy quark \[9, 10\] to gain better convergence of the perturbation series; (iii) expansion of the heavy-light correlation functions in the light-quark mass \[11, 12\]. In all these cases, higher-order spectral densities exhibit the appearance of non-integrable divergences at the threshold. This letter addresses the proper way of handling such singularities.

2. THRESHOLD SINGULARITIES IN PERTURBATION THEORY

As an example, let us consider the two-point function of the quark with the pole mass \( m \). One part of the QCD spectral density corresponds to the exchange of Coulomb gluons; hereafter we consider this part only. The exact expression of the full resummed Coulomb spectral density is known \[2\]

$$\rho(s) = \frac{3N_c \pi \alpha_s C_F}{1 - \exp(-\pi \alpha_s C_F / v)}, \quad v = \sqrt{1 - \frac{4m^2}{s}},$$  

(2.1)
with \( N_c = 3 \) and \( C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3} \). In perturbation theory, one obtains this spectral density as a power expansion in \( \alpha_s \):

\[
\rho(s) = \frac{9}{2} v + \frac{9}{4} C_F \pi \alpha_s + 3 \sum_{n=2,4,...} \frac{B_n}{n!} (\pi C_F \alpha_s)^n \frac{1}{v^{n-1}},
\]

where the coefficients \( B_n \) are Bernoulli numbers (\( B_2 = 1/6, B_4 = -1/30, \ldots \)). The sum in Eq. (2.2) runs over even powers of \( \alpha_s \); for \( n = 2 \) it has an integrable \( 1/v \) divergence at the threshold \( v = 0 \), but starting with \( n = 4 \), these singularities are non-integrable and one should specify the precise way to handle them.

To understand the proper way to proceed, we recall the following property of any analytic function: A contour integral of an analytic function over the region where the function is free of singularities is finite and does not depend on the specific choice of the integration contour. Let us look at the problem of the threshold divergences from this perspective. We start with the Cauchi theorem

\[
F(s) = F(0) + \frac{s}{2\pi i} \oint_{\Gamma} \frac{F(s')}{s'(s' - s)} ds'.
\]

Here \( \Gamma \) is any contour surrounding the point \( s \) and located in the region where the function \( F(s) \) is analytic. We start with the contour \( \Gamma_1 \) (see Fig. 1). Obviously, the integral is finite. We now start to deform the contour \( \Gamma_1 \to \Gamma_2 \to \Gamma_3 \) in the region of analyticity of the function \( F(s) \). Such deformations do not change the (finite) value of the integral.

Finally, we end up with the contour \( \Gamma_3 \) which embraces the cut from \( s = 4m^2 \) to \( s = \infty \), and the large circle. With the appropriate number of subtractions, the large-circle integral vanishes, and we end up with the contour integral which embraces the cut. This integral needs some care. It may be split into two parts: (I) the integral over a small circle of radius \( \epsilon \) around the threshold \( s = 4m^2 \), and (II) the integral over the “cut” starting from \( s = 4m^2 + \epsilon \) to \( \infty \) along the real \( s \)-axis. The sum of these two integrals is finite due to the general property of the analytic functions mentioned above. If the behavior of the function \( F(s) \) near the threshold is such that the integral of \( \text{Im} F(s) \) over the cut is finite, then the small-circle integral vanished as \( \epsilon \to 0 \). However, the situation changes for the case when \( \text{Im} F(s) \) has a non-integrable singularity at the threshold: then, the small-circle integral (I) also diverges for \( \epsilon \to 0 \), making the sum of (I) and (II) finite for \( \epsilon \to 0 \).

Let us demonstrate this property for the case of \( O(\alpha_s^4) \) correction, namely

\[
\rho_4(s) \propto \frac{1}{v^3}, \quad v = \sqrt{1 - \frac{4m^2}{s}}.
\]

According to the behaviour of the imaginary part of \( F(s) \), \( \text{Im} F(s) = \rho(s) \to 1 \) at large \( s \), one subtraction in the spectral representation for \( F(s) \) is sufficient to guarantee that the large-contour integral vanishes.
For convenience, we introduce a dimensionless variable \( z = s/4m^2 \), such that the cut in \( F(z) \) is located along the real \( z \)-axis from 1 to \( \infty \), and write for the function \( F(z) \) the following dispersion representation with one subtraction (the limit \( \epsilon \to 0 \) at the final stage is implied):

\[
F(z) = F(0) + \frac{z}{2\pi i} \int_{R_\epsilon} \frac{F(z')}{z'(z' - z)} \, dz' + \frac{z}{\pi} \int_{1+\epsilon}^{\infty} \frac{1}{(1-1/z')^{3/2} z'(z' - z - i0)} \, dz'.
\]  

(2.5)

where \( R_\epsilon \) is the (clockwise) circle with radius \( \epsilon \) and the center at \( z = 1 \). Equation (2.5) is an inhomogeneous integral equation for \( F(z) \) and its solution is not fully trivial.

We first analyse the contribution to \( F(z) \) from the dispersion integral over the cut, and then see which parts of this contribution are relevant for the small-circle integral. By a manipulation with the integrand,

\[
\frac{1}{1 - \frac{z}{R_\epsilon}} = \left( \frac{1}{1 - \frac{z}{R_\epsilon}} - \frac{1}{1 - z} \right) + \frac{1}{1 - z} - \frac{z}{1 - z} - \frac{1}{1 - z},
\]

we isolate the term singular in \( \epsilon \) and obtain

\[
F_3(z, \epsilon) = \frac{z}{\pi} \int_{1+\epsilon}^{\infty} \frac{1}{(1-1/z')^{3/2} z'(z' - z - i0)} \, dz' = \frac{2z}{\pi(1-z)} \sqrt{\epsilon} - \frac{2z}{\pi(1-z)} - \frac{z}{1-z} F_1(z) + O(\sqrt{\epsilon}),
\]

(2.6)

where

\[
F_1(z) = \frac{z}{\pi} \int_{1}^{\infty} \frac{1}{(1-1/z')^{1/2} z'(z' - z - i0)} \, dz' = -\frac{2}{\pi} \sqrt{\frac{z}{z-1}} \log \left( \sqrt{z} + \sqrt{1-z} \right).
\]

(2.7)

It is convenient to write (2.7) in the following form

\[
F_3(z, \epsilon) = \frac{2}{\pi} \frac{z - 1}{1 - \sqrt{\epsilon}} + \frac{2}{\pi} \frac{z}{z-1} \left[ 1 - \sqrt{\frac{z}{z-1}} \log \left( \sqrt{z} + \sqrt{1-z} \right) \right] - \left( \frac{z}{1-z} \right)^{3/2},
\]

(2.8)

where the second term is finite at the threshold.

Let us write down again Eq. (2.5):

\[
F(z) = F(0) + \frac{z}{2\pi i} \int_{R_\epsilon} \frac{F(z')}{z'(z' - z)} \, dz' + F_3(z, \epsilon).
\]

(2.9)

Because of the Cauchy theorem, the function \( F(z) \) does not depend on \( \epsilon \). This means that the term \( \sim \epsilon^{-1/2} \) generated by \( F_1(z, \epsilon) \) cannot be a part of \( F(z) \) and thus should cancel against the small-circle integral. We shall now demonstrate that the cancellation of this divergent term is the only effect of the small-circle integral and that it does not yield any contribution that remains finite in the limit \( \epsilon \to 0 \). In other words, to obtain \( F(z) \) one needs to subtract from \( F_3(z, \epsilon) \) all singular terms in \( \epsilon \) and then send \( \epsilon \to 0 \). To prove this statement we just show that \( F(z) \) obtained in this way satisfies Eq. (2.10).

We turn to Eq. (2.9) and omit the first term in its r.h.s., which is singular in \( \epsilon \). Then we check which of the remaining structures may give a non-vanishing contribution when integrated over the small circle. The second term in the r.h.s. of Eq. (2.9) is non-singular at the threshold, as it can be easily checked by an expansion around \( z = 1 \). Therefore, its contribution to the small-circle integral vanishes after the limit \( \epsilon \to 0 \) is taken.

Only the last term in the r.h.s. of Eq. (2.9), which is singular at the threshold, can contribute to the small-circle integral. By setting \( z' = 1 + \epsilon e^{i\phi} \) and taking the \( \phi \)-integral from \( 2\pi \) to 0 (corresponding to the clockwise contour integration), one gets

\[
-\frac{z}{2\pi i} \int_{R_\epsilon} \frac{\sqrt{z'}}{(1-z')^{3/2} z'} \, dz' = \frac{z}{2\pi i(1-z)} \sqrt{\epsilon} \int_{0}^{2\pi} d\phi \, e^{-i\phi/2} + O(\sqrt{\epsilon}) = -\frac{2}{\pi} \frac{z}{1-z} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}),
\]

(2.10)

(2.11)

where we made use of the integral \( \int_{0}^{2\pi} d\phi \, e^{-i\phi/2} = -4i \). Thus, the small-circle integral precisely cancels the singular contribution in \( 1/\sqrt{\epsilon} \) coming from \( F_3(z, \epsilon) \) and does not develop any finite terms in the limit \( \epsilon \to 0 \).
Finally, in the case of the perturbative term (2.1) we come to the following representation for \( F(z) \):

\[
F(z) = F(0) + \frac{2}{\pi} \frac{z}{|z| - 1} - \frac{2}{\pi} \frac{z}{|z| - 1} \sqrt{\frac{z}{|z| - 1}} \log(\sqrt{|z|} + \sqrt{1 - |z|}).
\]  

(2.12)

It is straightforward to check that on the upper boundary of the cut along the real axis from \( z = 1 \) to \( +\infty \) [to get there one needs to set \( z \to z + i0 \) for \( z > 1 \)], the imaginary part of \( F(z) \) is indeed equal to \((1 - 1/z)^{-3/2}\).

For higher-order contributions, the small-circle integrals produce singular terms containing a series of inverse powers of \( \epsilon \) related to poles of the increasing order at the threshold \( z = 1 \). For instance, in the case of the spectral density \( \rho_0(s) \propto 1/v^2 \) the dispersion integral over the cut yields

\[
F_5(z, \epsilon) = \frac{z}{\pi} \int_1^\infty \frac{1}{(1-1/z')^{5/2}} z'(z' - z) \, dz' = \frac{2}{3\pi} \frac{z}{1 - z} \frac{1}{\epsilon^{3/2}} + \frac{1}{\pi} \frac{z(1-3z)}{(1-z)^2} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon})
\]

\[
+ \frac{2}{\pi} \frac{z}{|z| - 1} \left\{ \frac{1}{3} + \frac{z}{|z| - 1} \left[ 1 - \sqrt{\frac{z}{|z| - 1}} \log(\sqrt{|z| + \sqrt{1 - |z|}}) \right] \right\} + \left( \frac{z}{1 - z} \right)^{5/2},
\]  

(2.13)

where the term with the curly brackets is finite at the threshold \( z = 1 \). The small-circle integral of the singular at the threshold \( \epsilon \)-independent part of \( F_5(z, \epsilon) \) (i.e., the last term in the r.h.s. of Eq. (2.13)) gives

\[
\frac{z}{2\pi i} \int_{R_c} \frac{z'^{3/2}}{(1 - z')^{3/2}} \frac{dz'}{z' - z} = \frac{z}{2\pi i(1 - z)} \epsilon^{3/2} \int_0^{2\pi} d\phi \left\{ e^{-i\phi/2} + \epsilon \left( \frac{3}{2} - \frac{1}{1 - z} \right) e^{-i\phi/2} \right\} + O(\sqrt{\epsilon})
\]

\[
= -\frac{2}{3\pi} \frac{z}{1 - z} \frac{1}{\epsilon^{3/2}} - \frac{1}{\pi} \frac{z(1-3z)}{(1-z)^2} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}),
\]  

(2.14)

where we made use of the integral \( \int_0^{2\pi} d\phi \, e^{-i\phi/2} = -4i/3 \).

We emphasize that due to the Cauchy theorem it is not necessary to analyse the small-circle integrals explicitly: it is sufficient to calculate the singular in \( \epsilon \) terms in the dispersion integral in Eq. (2.15). The small-circle contributions are equal to these singular terms taken with the opposite sign.

### 3. Resummation of the Perturbative Expansion

In the previous Section we explained an appropriate way to handle spectral integrals obtained at each order of the perturbative expansion for the two-point function.

In some cases, the resummation of the perturbation series or some of its subsequences may be performed. For instance, for Coulomb exchanges, the exact analytic result for the full spectral density is known, Eq. (2.1): \( \rho(z) \) is finite at the threshold. To understand what happens with the corresponding small-circle contribution, we get back to the Cauchy theorem:

\[
F(z) = F(0) + \frac{z}{2\pi i} \int_{R_c} \frac{F(z')}{z'(z' - z)} \, dz' + \frac{z}{\pi} \int_{1+\epsilon}^\infty \frac{\rho(z')}{z'(z' - z + i0)} \, dz'.
\]  

(3.1)

Since the spectral density \( \rho(z) \) takes a finite value at the threshold, the dispersion integral is finite in the limit \( \epsilon \to 0 \) for all \( z \neq 1 \), and the solution of Eq. (3.1) yields the function \( F(z) \) which has a logarithmic singularity at \( z = 1 \); \( F(z) \) has no stronger singularities at the threshold \( z = 1 \) and, consequently, the small-circle integral in Eq. (3.1) vanishes in the limit \( \epsilon \to 0 \).

So, we see that, although the small-circle contribution emerging at each order of the perturbative expansion diverges in the limit \( \epsilon \to 0 \), their sum vanishes in the limit \( \epsilon \to 0 \) in the case of the full spectral density finite at the threshold.
4. CONCLUSIONS

We discussed the way to handle properly the threshold divergences arising in perturbation theory for two-point functions. Our results are as follows:

(i) Taking a proper account of the small-circle integral around the threshold leads to the “surface term” that exactly cancels the threshold divergence of the spectral integral. This means that a properly defined dispersion representation for $F(q^2)$ (and in fact for any analytic function) at each order of perturbative expansion is finite and does not have any threshold divergence. In this way we consistently handle the singularities which emerge in the dispersion integrals at any order of the perturbative expansion. We emphasize that thanks to the Cauchy theorem the explicit calculation of the small-circle integrals is not required. Indeed, it is sufficient to isolate the singular terms in the inverse powers of $\sqrt{\epsilon}$ in the dispersion integral of $\rho_n(s)$: the small-circle integral is just equal to these singular terms taken with an opposite sign.

(ii) In some cases, as, e.g., for Coulomb gluon exchanges, the full resummed spectral density is known and leads to the dispersion integral convergent at the threshold. Then, one observes the following picture: The small-circle contributions, emerging at each order of the perturbative expansion, contain inverse powers of the parameter $\sqrt{\epsilon}$ and thus diverge for $\epsilon \to 0$. Nevertheless, their infinite sum yields the function that has a zero limiting value for $\epsilon \to 0$. So, in the end, no small-circle contribution appears in the dispersion representation for the full resummed spectral density.

We would like to mention that the discussed procedure does not improve (and is not aimed at improving) the convergence of the perturbation series; it just allows one to handle properly non-integrable threshold singularities of two-point functions emerging at each order of perturbation theory.

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