ON HIGHER-DIMENSIONAL OSCILLATION IN ERGODIC THEORY

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Abstract. We extend the results of [7] concerning higher-dimensional oscillation in ergodic theory in a variety of ways. We do so by transference to the integer lattice [2], where we employ technique from (discrete) harmonic analysis.

1. Introduction

Let $(X, \Sigma, \mu)$ be a non-atomic probability space, equipped with $\tau$ a measure preserving $\mathbb{Z}^d$-action

$$\tau_y f(x) := f(\tau^{-y} x).$$

For $(E_i) \subset \mathbb{Z}^d$, define the averaging operators

$$M_i f(x) := \frac{1}{|E_i|} \sum_{y \in E_i} (\tau_y f)(x);$$

the classical pointwise ergodic theorem of Birkhoff says that if $E_i = [0, i) \subset \mathbb{Z}^1$, then the one-dimensional averages $\{M_i f(x)\}$ converge pointwise $\mu$-almost everywhere for $f \in L^p(X, \Sigma, \mu), 1 \leq p < \infty$. A standard proof proceeds by way of a density argument, where the key quantitative estimate is that the one-dimensional maximal function

$$f^{*,1} := \sup_i |M_i f|$$

is of weak-type $(1, 1)$, and strong-type $(p, p), 1 < p \leq \infty$. Explicitly, there exist absolute constants, $C_p, 1 \leq p \leq \infty$ so that

$$\lambda \mu \left( \{ f^{*,1} > \lambda \} \right) \leq C_1 \| f \|_1, \text{ for } \lambda \geq 0, \text{ and }$$

$$\| f^{*,1} \|_p \leq C_p \| f \|_p, \quad 1 < p \leq \infty.$$

This result was generalized to higher dimensions by Wiener; an easy consequence of [21] Theorem 1P is that if $E_i \subset \mathbb{Z}^d$ are a nested, increasing sequence of cubes which contain the origin, then the $d$-dimensional averages $\{M_i f(x)\}$ converge pointwise $\mu$-almost everywhere, and the $d$-dimensional maximal function

$$f^{*,d} := \sup_i |M_i f|$$

is similarly weak-type $(1, 1)$ and strong-type $(p, p), 1 < p \leq \infty$.

A modern path to Wiener’s result is through the transference principle of Calderón [2], which allows one to conduct the study of ergodic averages on the lattice, $\mathbb{Z}^d$: in
particular, it is enough to consider the (discrete) convolution operators $A_i$, defined by

$$A_i f(n) := \frac{1}{|E_i|} \sum_{m \in E_i} f(n - m).$$

An advantage to this perspective is that real analytic methods – covering lemmas, Fourier analysis and further orthogonality technique – can be brought to bear in studying more general regions $\{E_i\} \subset \mathbb{Z}^d$ and pertaining the averaging operators $\{A_i\}$. Indeed, provided the collection of sets $\{(E_i)\}$ being studied share some qualitative properties with an increasing collection of cubes [16, §§1-2]:

- A one-parameter structure; and
- Some geometric “smoothness”

the maximal functions on the lattice,

$$\sup_i |A_i f|$$

are of weak-type $(1, 1)$ and strong-type $(p, p), 1 < p \leq \infty$.

Obtaining pointwise convergence results of $\{A_i f(n)\}$ for more exotic $\{E_i\} \subset \mathbb{Z}^d$ – those for which the above two properties are relaxed, or absent – does not necessarily follow from quantitative estimates on an appropriate maximal function, since the dense-subclass result is often unavailable in this setting. Perhaps the most famous instance of this difficulty arose in the study of averages along the squares, i.e.

$$E_i := [0, i) \cap \{0, 1, 4, 9, \ldots\} \subset \mathbb{Z}^1.$$

Indeed, to prove pointwise convergence of the ergodic averages of $L^2$-functions along the squares, Bourgain [1] showed that for any (lacunarily) increasing sequence $\{i_k\}$, the oscillation operator

$$O f := \left( \sum_k \sup_{i_k < i < i_{k+1}} |A_{i_k} f - A_i f|^2 \right)^{1/2}$$

was of strong-type $(2, 2)$. (The $L^2$ result then anchored a density argument through which he was able to extend his result to all $p > 1$.)

In proving this result, Bourgain made use of the $s$-variation operators, $s > 2$, classically used in probability theory to gain quantitative information on the rates of convergence. More precisely, Bourgain proved that, in the special case $E_i = [0, i)$, the $s$-variation operators

$$V^s f = V^s_{(E_i)} f := \sup_{i_1 < i_2 < \cdots < i_N} \left( \sum_{k=1}^{N} |A_{i_k} f - A_{i_{k+1}} f|^s \right)^{1/s}$$

were of strong-type $(2, 2)$ [11, Corollary 3.26].

These variation operators are more difficult to control than the maximal function $\sup_i |A_i f|$: for any $j$, one may pointwise dominate

$$\sup_i |A_i f| \leq V^\infty f + |A_j f| \leq V^s f + |A_j f|,$$

1 One representative example which in fact appears in Bourgain’s argument is Lépingle’s inequality for martingales [10].
where $2 < s < \infty$ is arbitrary. On the other hand, variational (or oscillation) estimates are a powerful tool for proving pointwise convergence when a density argument seems unavailable.

Since Bourgain’s celebrated result, establishing variational estimates for families of averaging operators has been the focus of much research in ergodic theory. \cite{Jones}

A fundamental paper in this direction is due to Jones et. al. \cite{Jones2}, where it is shown that the (one-dimensional) variation operators $\{V_s\}$ are of weak-type $(1,1)$ and strong-type $(p,p)$, $1 < p < \infty$. In other words, the variation operators enjoy the same boundedness properties as their associated maximal function. The argument in \cite{Jones2} proceeded by first controlling an oscillation operator adapted to $E_i = [0,i)$ and then using martingale-style technique from probability theory; the approach to the oscillation operator itself, however, was driven by Fourier-based orthogonality arguments.

A subsequent paper of Jones, Rosenblatt, and Wierdl, \cite{Jones3}, refined the orthogonality methods used in \cite{Jones2} by eliminating the Fourier-analytic technique, and more closely following (dyadic) martingale-style arguments. In so doing, the authors were able to establish analogous results in higher-dimensional settings – for functions in the “low”-$L^p$ regime, $1 < p \leq 2$.

To continue our discussion, we briefly introduce two representative operators studied in \cite{Jones3}: the “pointwise” square functions. In our current context, the significance of these operators is that establishing $L^p$ bounds leads directly to bounds on the corresponding oscillation and variation operators \cite[§1]{Jones3}:

Suppose that $(E_i)$ are cubes of side-length $i$, so that for $2^{k-1} \leq i < 2^k$,

$$E_i \subset H_k := \prod_{i=1}^{d} [-2^k, 2^k).$$

In other words, the $(E_i)$ are displaced from the origin by an amount comparable to, or less than, their side lengths. Then, with

$$A^k f(x) := \frac{1}{|H_k|} \sum_{m \in H_k} f(n - m),$$

we define the long square function (on the integers) with respect the collection $(E_i)$ as

$$S^L_{(E_i)} f(x) := \left( \sum_{k} \left( \sup_{2^{k-1} \leq i < 2^k} \left| A_i f(x) - A^k f(x) \right|^2 \right) \right)^{1/2},$$

and the short square function as

$$S^S_{(E_i)} f(x) := \left( \sum_{k} \left( \sup_{2^{k-1} \leq (i_t) < 2^k \text{ increasing}} \left| A_{i_t} f(x) - A_{i_{t+1}} f(x) \right|^2 \right) \right)^{1/2}.$$

An easy consequence of \cite[Theorems $A', B'$]{Jones3} is the following:

\footnote{Variation estimates have also been studied from a harmonic analysis perspective in the context of truncations of singular integral operators. We refer the reader to \cite{Jones4} for further discussion.}
Theorem 1.1. There exist absolute constants $C_1, C_p, 1 < p \leq 2$ so that

$$\left\| S_{(E_i)}^* f(x) \right\|_{L^p(\mathbb{Z}^d)} \leq C_p \| f \|_{L^p(\mathbb{Z}^d)}$$

$$\left\| S_{(E_i)}^* f(x) \right\|_{L^1,\infty(\mathbb{Z}^d)} \leq C_1 \| f \|_{L^1(\mathbb{Z}^d)},$$

where $* = L, S$.

The two operators, $S^L, S^S$, both measure the scales and locations where $f$ oscillates, i.e. differs from constant functions. Temporarily ignoring issues of convergence, we may represent

$$f \sim \sum_{k \geq 0} \Delta_k(f)$$

as a sum of dyadic martingale increments generated by the $\{H_k\}$. The content of these results – and indeed the ideas that drive their proofs – is that there is sufficient orthogonality between the various scales to ensure that, in an average sense, even a pointwise measurement of oscillation is controlled by the square-sum of the increments $\{\Delta_k(f)\}$. We refer the reader to §2, or to [7, §§3-4] for further discussion.

A key insight in [7] is that just as in the case of the maximal function, $\sup_i |A_i f|$, the $(E_i)$ do not actually need to be cubic for the pertaining square functions to enjoy the above control. Indeed, the above Theorem holds provided the collection of sets $\{(E_i)\}$ being studied shared the same qualitative properties with a nested collection of cubes as in the case of the maximal function:

- A one-parameter structure; and
- Some geometric “smoothness.”

In light of the analogous approaches to studying maximal function and square function in our current context (and discussed in greater generality in [16, §1]), the following informal question seems natural:

To what (further) extent do the boundedness properties of the square functions under our consideration parallel those of the maximal functions?

More precisely, we organize our study of the operators introduced in [7] according to the following aims:

First, we seek to extend our control of the square functions under the assumptions outlined [7]. An immediate concern is the behavior of the square function in the “high-$L^p(\mathbb{Z}^d)$” regime, which we investigate by using sharp-function technique from harmonic analysis. We prove:

Theorem 1.2. If $\mathcal{A} = (E_i)$ is regular, then the long and short square functions $S^L_{\mathcal{A}}, S^S_{\mathcal{A}},$ are $L^p(\mathbb{Z}^d)$-bounded, $2 < p < \infty$.

We refer the reader to §2 for the precise definition of regularity; informally, regular sequences $(E_i)$ share the above-mentioned qualitative similarities to nested cubes.

This result leads directly to new control over jump and variation inequalities:

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3For a more precise statement of these results, we refer the reader to [7, §1], or to §2 below; for an excellent treatment of continuous analogues of the square functions introduced above, we refer the reader to the discussion of the intrinsic square function, found in e.g. [20, §6].
For $\lambda > 0$, we define, as in [7],

$$J((A_i), \lambda) = J((A_i(f, x), \lambda))$$

as the largest $N$ for which there are increasing indices $(i_j)$ with

$$(A_{i_j} - A_{i_{j+1}})f(x) > \lambda, \ 1 \leq j < N.$$  

Then, arguing as in [7], §1, under the assumption of regularity we have the following:

**Corollary 1.3.** For $2 < p < \infty$ there exist absolute constants $C_p$ so that for any $\lambda > 0$,

$$\left\| \lambda \cdot J((A_i), \lambda)^{1/2} \right\|_{L^p(\mathbb{Z}^d)} \leq C_p \|f\|_{L^p(\mathbb{Z}^d)}.$$  

Moreover, for any $s > 2$,

$$\|\mathcal{V}^s f\|_{L^p(\mathbb{Z}^d)} \leq C_p \|f\|_{L^p(\mathbb{Z}^d)}.$$  

We remark that by transference this implies the corresponding result for dynamical systems:

**Corollary 1.4.** For $2 < p < \infty$ there exist absolute constants $C_p$ so that for any $\lambda > 0$,

$$\left\| \lambda \cdot J((M_i), \lambda)^{1/2} \right\|_{L^p(X, \mu)} \leq C_p \|f\|_{L^p(X, \mu)}.$$  

Moreover, for any $s > 2$,

$$\|\mathcal{V}^s f\|_{L^p(X, \mu)} \leq C_p \|f\|_{L^p(X, \mu)}.$$  

To deepen the connection between square and maximal function, we also consider the behavior of square functions on the (discrete) $A_p(\mathbb{Z}^d)$-weighted classes. We are additionally motivated in this regard by the weighted estimates for one-parameter actions studied in [13, 5], and the weighted theory of “rough” singular integrals which satisfy the so-called Hörmander conditions [12, 11].

We first recall a standard characterization of $A_p$ weights; we refer the reader to [3, §7] or to [16, §5] for a more comprehensive treatment.

**Definition 1.5.** For $1 < p < \infty$, a positive function $w \in A_p(\mathbb{Z}^d)$ is a discrete $A_p$ weight if the (uncentered, cubic) Hardy-Littlewood maximal function, $M_{HL}$ is bounded on $L^p(\mathbb{Z}^d)$. Explicitly, $w \in A_p(\mathbb{Z}^d)$ iff there exists an absolute $C_p > 0$ so that

$$\int_{\mathbb{Z}^d} |M_{HL} f|^p w \leq C_p^p \int_{\mathbb{Z}^d} |f|^p w \quad \text{or}$$

$$\|M_{HL} f\|_{L^p(\mathbb{Z}^d, w)} \leq C_p \|f\|_{L^p(\mathbb{Z}^d, w)}.$$  

A positive function $w \in A_1(\mathbb{Z}^d)$ is an $A_1$ weight if there exists a constant $C = C(w)$ so that $M_{HL} w \leq Cw$; we note that $A_1$ weights $w \in A_1$ satisfy $\frac{w(Q)}{|Q|} \leq B \inf_{x \in Q} w(x)$ for any $Q$, and an absolute $B = B(w)$.

We say $w \in A_\infty(\mathbb{Z}^d)$ is an $A_\infty$ weight if $w \in A_p(\mathbb{Z}^d)$ for some finite $p$; such $A_\infty$ weights are automatically doubling.

We say that a collection of sets $(E_i)$ is cubic if the maximal function $\sup_i |A_i f|$ is (up to constant factors) pointwise dominated by $M_{HL} f$. Under this natural assumption, we have the following:
**Theorem 1.6.** If \( \mathcal{A} = (E_i) \) is cubic, then for both the long and short square functions \( S_A^* \), \( \ast = L, S \) there exist absolute constants \( C_p \) so that
\[
\int_{\mathbb{Z}^d} |S_A^* f|^p w \leq C_p \int_{\mathbb{Z}^d} |f|^p w
\]
for any \( A_p \)-weight \( w \), \( 1 \leq p < \infty \).

Again arguing as in [7, §1], this implies the following corollary for cubic families:

**Corollary 1.7.** For \( 1 < p < \infty \) there exist absolute constants \( C_p \) so that for any \( \lambda > 0 \),
\[
\left\| \lambda \cdot J((A_i), \lambda)^{1/2} \right\|_{L^p(\mathbb{Z}^d, w)} \leq C_p \| f \|_{L^p(\mathbb{Z}^d, w)}.
\]
Moreover, for any \( s > 2 \),
\[
\| V^s f \|_{L^p(\mathbb{Z}^d, w)} \leq C_p \| f \|_{L^p(\mathbb{Z}^d, w)}.
\]

Of course, by specializing to the weight \( v \equiv 1 \), we recover the primary results of [7].

Finally, we investigate the behavior of our square functions when the crucial “smoothness” assumption is relaxed. We focus our efforts in this regard on the following:

**Problem 1.8** ([7], Problem 7.5). For each collection of nested rectangles \( (E_i)_i \subset \mathbb{Z}^d \), is it true that for each \( \phi \in L^1(X, \Sigma, \mu) \),
\[
\left( \sum_i \| (M_i - M_{i+1}) \phi \|^2 \right)^{1/2} < \infty
\]
\( \mu \)-almost everywhere?

Certainly, this result would be implied by a weak-type bound
\[
\left\| \left( \sum_i \| (M_i - M_{i+1}) \phi \|^2 \right)^{1/2} \right\|_{L^{1,\infty}(X)} \lesssim \| \phi \|_{L^1(X)};
\]
in fact, as in shown in the Appendix §6 below, in many cases this weak-type bound is necessary for convergence to occur. We therefore focus our attention on the following slightly more general

**Problem 1.9** ([7], Problem 7.5 – Working Version). For each collection of nested rectangles \( (E_i)_i \subset \mathbb{Z}^d \), does there exist a bound
\[
\left\| \left( \sum_i \| (A_i - A_{i+1}) f \|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{Z}^d)} \lesssim \| f \|_{L^1(\mathbb{Z}^d)}?
\]

Though this problem remains out of reach in its fullest generality, we are able to answer the problem affirmatively under a lacunarity assumption on collection \( (E_i)_i \):

**Proposition 1.10.** For each collection of nested rectangles \( (E_i)_i \subset \mathbb{Z}^d \) with dyadic side lengths,
\[
\left\| \left( \sum_i \| (A_i - A_{i+1}) f \|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{Z}^d)} \lesssim \| f \|_{L^1(\mathbb{Z}^d)}.
\]
See §5 for further discussion.

The structure of the paper is as follows:
In §2 we introduce relevant definitions, and present a few reductions which will be used throughout;
In §3, we study our square functions’ behavior in the high-$L^p(\mathbb{Z}^d)$ regime;
In §4, we prove weighted estimates; and
In §5 we relax the smoothness assumptions of [7], and discuss Problem [1,9].

Our appendix, §6, contains a weak-type principle for square functions in the spirit of [17].

1.1. Acknowledgements. The author would like to thank Benjamin Hayes and Michael Lacey for helpful conversations, and his advisor, Terence Tao, for his great patience and support.

1.2. Notation. For a set $E \subset \mathbb{Z}^d$, we use $|E|$ to denote $\#E$ the counting measure (cardinality) of the set $E$.

We let $1_E$ denote the indicator function of the set $E$, i.e.

$$1_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

we let $\chi_E := \frac{1}{|E|} \cdot 1_E$ denote the normalized indicator function.

For a function $f$ defined on $\mathbb{Z}^d$, our convention will be to let

$$\int f$$

denote the summation $\sum_{n \in \mathbb{Z}^d} f(n)$. Accordingly, we will use

$$\|f\|_{L^p(\mathbb{Z}^d)}$$

to denote the $l^p$-norm, $(\sum_{n \in \mathbb{Z}^d} |f(n)|^p)^{1/p}$, with the obvious modification for $p = \infty$.

When integrating over other spaces, we will include the domain and measures.

We will make use of the modified Vinogradov notation. We use $X \lesssim Y$, or $Y \gtrsim X$ to denote the estimate $X \leq CY$ for an absolute constant $C$. If we need $C$ to depend on a parameter, we shall indicate this by subscripts, thus for instance $X \lesssim_p Y$ denotes the estimate $X \leq C_p Y$ for some $C_p$ depending on $p$. We use $X \approx Y$ as shorthand for $X \lesssim Y \lesssim X$.

2. Preliminaries

We shall, whenever possible, maintain the notation introduced in [7].

Let $\mathcal{R} = 0 \subset R_1 \subset R_2 \subset \ldots$ denote a nested sequence of dyadic rectangles inside $\mathbb{Z}^d$, i.e.

$$R_k = \prod_{i=1}^d [0, 2^{i(k)}].$$

We also define the “symmetric” rectangles

$$H_k = \prod_{i=1}^d [-2^{i(k)}, 2^{i(k)}].$$
For $k \geq 0$, we let $\sigma_k = \sigma_k(\mathcal{R})$ denote the $\sigma$-algebra generated by $R_k$, i.e. the $\sigma$-algebra with atoms
\[
\prod_{i=1}^{d} 2^{i(k)}[m_i, m_i + 1).
\]
Let $\mathcal{E}_k$ denote the expectation with respect to $\sigma_k$, $\mathcal{E}_0$ the identity operator, and
\[
\Delta_0 = \mathcal{E}_0, \quad \Delta_k := \mathcal{E}_k - \mathcal{E}_{k-1}
\]
denote the martingale differences.

Closely connected to our family of rectangles, $\mathcal{R}$, are the collections of sets $\mathcal{A} = (\mathcal{A}_k)$, whose elements have controlled

(1) Spatial location: for each $E \in \mathcal{A}_k$, $E \subset H_k$;
(2) Size: for each $E \in \mathcal{A}_k$ there exists some $c > 0$ so that $c|H_k| \leq |E|$;
(3) (Internal) Smoothness: for each $E \in \mathcal{A}_k + l$, for $l \geq 0$
\[
|\{x : \partial E \cap (H_k - x) \neq \emptyset\}| \leq \frac{B(E, H_k)}{|H_k + l|} \leq \iota(l)^2,
\]
with $\sum_l \iota(l) < \infty$; and finally
(4) Eccentricity:
\[
\frac{B(H_k + l, H_k)}{|H_k + l|} \leq \varepsilon(l)^2,
\]
where $\sum_l \varepsilon(l) < \infty$ as well.

We shall collectively refer to the above four criteria as the JRW criteria.

The third of the above points forces regularity on the boundaries of the $E \in \mathcal{A}_k$, i.e. some smoothness on the $l^1$-normalized indicator functions $\frac{1}{|E|}1_E$.

The fourth point is implicitly used in the proofs of the main theorems of [7], though not explicitly stated in the summary of §5. We shall replace it with the following equivalent formulation, which we isolate in the form of the following simple

**Lemma 2.1.** Eccentricity control as above is equivalent to the existence of an $L$ so that
\[
\min_{1 \leq i \leq d} \{ i(k + L) - i(k) \} \geq 1
\]
for each $k$, with $H_k = \prod_{i=1}^{d} [-2^{i(k)}, 2^{i(k)})$.

Moreover, the existence of such an $L$ allows us to take
\[
\varepsilon(l)^2 \lesssim 2^{-L},
\]
and therefore
\[
\sum_l \varepsilon(l)^\theta < \infty, \text{ for any } \theta > 0.
\]

We say that two cubes $Q = \prod_{i=1}^{d} I_i$, $R = \prod_{i=1}^{d} J_i$, $|I_i| \geq |J_i|$ with dyadic side-lengths are $s$-separated if
\[
\min_{1 \leq i \leq d} \left\{ \frac{|J_i|}{|I_i|} \right\} = 2^{-s},
\]
the content of the above theorem is that there exists an absolute $L$ so that $H_{k+L}, H_k$ are 1-separated for each $k$. 
Proof. If \( s = \min_{1 \leq i \leq d} \{i(k + l) - i(k)\} \), then
\[
\frac{B(H_{k+l},H_k)}{|H_{k+l}|} \approx 2^{-s},
\]
if no such \( L \) were to exist, then we could find arbitrarily many \( l, k = k(l) \) so that
\[
\frac{B(H_{k+l},H_k)}{|H_{k+l}|} \gtrsim 1,
\]
which would force the sum \( \sum_{l \in \{l\}} \) to diverge. \( \square \)

For our purposes, the (alternate) eccentricity condition guarantees that for any dyadic \( 2^c, c \geq 1 \),
\[
\frac{B(H_{k+l},2^cH_k)}{|H_{k+l}|} \lesssim 2^{-\frac{r}{2c}}
\]
is an \( l^r, r > 0 \) sequence is well.

We shall be studying families of sets, \( (E_t) \), which are \textit{regular} with respect to our collections \( (A_k) \): For each \( 2^k \leq t < 2^{k+1} \),
\[ E_t \subset E_{t'} \text{, and } E_t, E_{t'} \in A_k. \]
If in addition \( \mathcal{A} = (A_k) \) satisfy the above JRW criteria, we will say that \( \mathcal{A} \) itself is \textit{regular}.

With a regular collection \( \mathcal{A} = (A_k) \) and \( (E_t) \) specified, we will let
\[
\chi_t = \chi_t^\mathcal{A} := \frac{1}{|E_t|} 1_{E_t}
\]
denote the \textit{normalized} indicator function.

We let
\[
A_t g(x) = A_t^\mathcal{A} g(x) := \chi_t * g(x) = \frac{1}{|E_t|} \sum_{y \in E_t} g(x - y)
\]
denote the convolution operator with kernel \( \chi_t \). We introduce the maximal function associated to \( \mathcal{A} \)
\[
M f = M_{\mathcal{A}} f := \sup_k \chi_{5H_k} * |f|(x),
\]
which dominates \( \sup_t A_t |f| \), and satisfies a weak-type \( (1, 1) \) inequality by the nesting properties of the \( (H_k) \). (cf. e.g. \cite{19} Lemma 5.3).

With \( (A_t) \) regular with respect to \( \mathcal{A} \), we define the long square function
\[
S_{\mathcal{A}}^L f := \left( \sum_k |S_{\mathcal{A},k}^L f|^2 \right)^{1/2} := \left( \sum_k \sup_{2^k \leq t < 2^{k+1}} |A_t f - \mathcal{E}_{k} f|^2 \right)^{1/2},
\]
and short square function
\[
S_{\mathcal{A}}^S f := \left( \sum_k |S_{\mathcal{A},k}^S f|^2 \right)^{1/2} := \left( \sum_k \sup_{(t_i), \text{increasing}} \sum_{2^k \leq t_i < 2^{k+1}} |A_{t_i} f - A_{t_{i+1}} f|^2 \right)^{1/2}.
\]

Often, we will suppress the subscript \( \mathcal{A} \).

We briefly remark that that for \( 2^k \leq t < 2^{k+1} \), since
\[
|A_t f|(x) \lesssim \chi_{H_k} * |f|(x), \quad |\mathcal{E}_{k} f|(x) \lesssim \chi_{H_k} * |f|(x),
\]

we may control $S_k^c f \lesssim \chi_{H_k} * |f|$. We have similar control over $S_k^S f$:

$$S_k^S f := \sup_{(t_i) \text{ increasing}} \left( \sum_{2^k \leq i < 2^{k+1}} |(A_t - A_{t+1}) f(x)|^2 \right)^{1/2} (x) \leq \sum_{2^k \leq i < 2^{k+1}} |(A_t - A_{t+1}) f(x)| \leq \sum_{2^k \leq i < 2^{k+1}} \left( \left( \frac{1}{|E_i|} - \frac{1}{|E_{i+1}|} \right) |1_{E_i} * |f| \right) + \sum_{2^k \leq i < 2^{k+1}} \left( \frac{1}{|E_{i+1}|} |1_{E_i \setminus E_{i+1}} * |f| \right) \lesssim \chi_{H_k} * |f|,$$

where we used the size control $c |H_k| \leq |E_i|$ in the final inequality.

We also introduce the larger, shifted square functions:

$$\tilde{S}_{\Delta}^* f(x) := \left( \sum_k |\tilde{S}_{\Delta,k}^* f(x)|^2 \right)^{1/2} := \left( \sum_k |\sup_{v \in H_k} S_{\Delta,k}^* f(x+v)|^2 \right)^{1/2},$$

* = L, S. The additional suprema affords the shifted square functions a useful degree of smoothness:

Lemma 2.2. With $x_Q$ denoting the center of each $Q \in \sigma_k$, $\tilde{S}_{\Delta,k}^* f$ is pointwise dominated, and $L^p(\mathbb{Z}^d)$-comparable $1 \leq p \leq \infty$, to a function which is constant on $Q \in \sigma_k$:

$$S_{D,k}^* f := \sum_{Q \in \sigma_k} \tilde{S}_{\Delta,k}^* f(x_Q) 1_{3Q}.$$ 

In particular,

$$\left\| \tilde{S}_{k}^* f(x) \right\|_{L^p(\mathbb{Z}^d)} \lesssim_p \left\| S_{D,k}^* f \right\|_{L^p(\mathbb{Z}^d)}.$$

Proof. If $Q_i \in \sigma_k$ lie in $3Q$, then for any $x \in Q$, we may bound

$$\tilde{S}_{k}^* f(x) \leq \sum_i \tilde{S}_{k}^* f(x_{Q_i}),$$

and summing over all $Q$ yields a pointwise majorization

$$\tilde{S}_{k}^* f(x) \lesssim \sum_Q \tilde{S}_{k}^* f(x_Q) 1_{3Q}.$$ 

On the other hand, if $v_Q \in H_k$ is such that

$$\tilde{S}_{k}^* f(x_Q) = S_{k}^* f(x_Q + v_Q),$$

then on the set

$$X(Q) = \{ y \in Q : y + v = x_Q + v_Q \text{ for some } v \in H_k \} = Q \cap \{ x_Q + v_Q - H_k \},$$

which has measure $|X(Q)| \gtrsim_d |Q|$, we may bound

$$\tilde{S}_{k}^* f(x_Q) \leq \inf_{y \in X(Q)} \tilde{S}_{k}^* f(y).$$
We therefore have a similar pointwise inequality for $p = \infty$, while for $1 \leq p < \infty$, using the finite overlap of $\{3Q\}$

$$
\left\| \sum_Q \tilde{S}_k f(x_Q)1_{3Q} \right\|_p^p \lesssim \sum_Q |\tilde{S}_k f(x_Q)|^p|3Q| \\
\lesssim \sum_Q |\tilde{S}_k f(x_Q)|^p|X(Q)| \\
\leq \sum_Q \int_{X(Q)} |\tilde{S}_k f(y)|^p \\
\leq \sum_Q \int_Q |\tilde{S}_k f(y)|^p \\
= \left\| \tilde{S}_k f \right\|_{L^p(Z^d)}^p.
$$

□

For future use, we record the following additional

**Lemma 2.3.** For any $v \in A_{\infty}$,

$$
S^*_{D,k} f := \sum_{Q \in \sigma_k} \tilde{S}_k f(x_Q)1_{3Q}
$$

is $L^p(v)$-comparable, $1 \leq p < \infty$ to

$$
S^*_{d,k} f := \sum_{Q \in \sigma_k} \tilde{S}_k f(x_Q)1_Q.
$$

**Proof.** Using the bounded overlap of $\{3Q\}$, one estimates

$$
\left\| S^*_{D,k} f \right\|_{L^p(v)}^p \lesssim \sum_Q |\tilde{S}_k f(x_Q)|^p1_{3Q}v \\
\lesssim \sum_Q |\tilde{S}_k f(x_Q)|^p v(3Q) \\
\lesssim \sum_Q |\tilde{S}_k f(x_Q)|^p v(Q) \\
= \left\| S^*_{d,k} f \right\|_{L^p(v)}^p,
$$

where we used the doubling nature of $v \in A_{\infty}$ in passing to the third line. □

Though – as we shall see – more is true, for now we shall only need that our shifted square functions inherit $L^2$-boundedness from their centered associates:

**Proposition 2.4.**

$$
\left\| \tilde{S}_k^* f \right\|_{L^2(Z^d)} \lesssim \| f \|_{L^2(Z^d)}.
$$

The argument here is very similar to the arguments of §3 of [7]. The qualitative similarities between $\tilde{S}_k^*$ and the projection operators $\Delta_k$ – informally, both measure the locations where $f$ differs from being constant at $\approx \sigma_k$-scale – motivate the following orthogonality approach:
Proof. With
\[ f = \sum_k \Delta_k(f) := \sum_k d_k, \]
we majorize
\[ \hat{S}^* f = (\sum_n |\hat{S}^*_n(\sum_k d_k)|^2)^{1/2} \leq \sum_j (\sum_n |\hat{S}^*_n(d_{n+j})|^2)^{1/2}. \]

Since \( \hat{S}^*_n(g) \lesssim M g \), we may ignore the \( l^2 \)-contribution of each \( |j| \leq C = C(L) \):
\[ \sum_{|j| \leq C} \left\| \sum_n |\hat{S}^*_n(d_{n+j})|^2 \right\|^{1/2}_{L^2(\mathbb{Z}^d)} \lesssim \sum_{|j| \leq C} \left\| \sum_n |\hat{S}^*_n(d_{n+j})|^2 \right\|^{1/2}_{L^2(\mathbb{Z}^d)} \]
\[ \lesssim \sum_{|j| \leq C} \left( \sum_n \|Md_{n+j}\|_{L^2(\mathbb{Z}^d)}^2 \right)^{1/2} \]
\[ = \sum_{|j| \leq C} \left( \sum_n \|d_{n+j}\|_{L^2(\mathbb{Z}^d)}^2 \right)^{1/2} \]
\[ \lesssim \sum_{|j| \leq C} \left\| \sum_n |d_{n+j}|^2 \right\|^{1/2}_{L^2(\mathbb{Z}^d)} \]
\[ = \sum_{|j| \leq C} \|f\|_{L^2(\mathbb{Z}^d)} \]
\[ \lesssim \|f\|_{L^2(\mathbb{Z}^d)}, \]
where we used the \( L^2 \)-boundedness of \( M \) in the fourth line.

For \( j > C \), let \( Q \in \sigma_{n+j-1} \) be arbitrary, and consider \( \hat{S}^*_n d_{n+j}(x) \) on \( Q \). Since \( j > C > 0 \),
\[ \hat{S}^*_n d_{n+j}(x) \equiv \hat{S}^*_n (d_{n+j}1_{5Q})(x), \]
we may bound
\[ |\hat{S}^*_n d_{n+j}(x)| \lesssim \max_{x \in 5Q} |d_{n+j}|(x) \leq \left( \sum_{Q_i} |d_{n+j}|^2(x_{Q_i}) \right)^{1/2}, \]
where \( Q_i \in \sigma_{n+j-1} \) lie in \( 5Q \), and \( x_{Q_i} \) is (say) the center of each such \( Q_i \) \( (d_{n+j} \)
is constant-valued on \( Q \in \sigma_{n+j-1} \)). On the other hand, since \( \hat{S}^*_n d_{n+j}(x) = 0 \) whenever \( x + 2H_n \subset Q \), \( \hat{S}^*_n d_{n+j} \) is supported inside
\[ x + 2H_n \cap \partial Q \neq \emptyset. \]
We may consequently estimate
\[
\sum_{Q \in \sigma_{n+j-1}} \int_Q |\hat{S}_n^* d_{n+j}|^2(x) \leq \sum_{Q \in \sigma_{n+j-1}} \int_Q |d_{n+j}|^2(x_Q) \cdot 1_{B(Q,2H_n)}
\]
\[
\leq \sum_{Q \in \sigma_{n+j-1}} \int |d_{n+j}|^2(x) 1_{B(Q,2H_n)}
\]
\[
= \sum_{Q \in \sigma_{n+j-1}} |d_{n+j}|^2(x_Q) |B(Q,2H_n)|
\]
\[
\leq \varepsilon(j - 1 - 2L)^2 \sum_{Q \in \sigma_{n+j-1}} |d_{n+j}|^2(x_Q) |Q|
\]
\[
= \varepsilon(j - 1 - 2L)^2 \|d_{n+j}\|^2_{L^2(\mathbb{Z}^d)},
\]
where \(L\) is the cost of separating scales, as in the alternative characterization of the eccentricity JRW-criterion.

Summing over \(n\) exhibits
\[
\left\| \left( \sum_n |\hat{S}_n^*(d_{n+j})|^2 \right)^{1/2} \right\|^2_{L^2(\mathbb{Z}^d)} \leq \varepsilon(j - 1 - 2L)^2 \|f\|^2_{L^2}.
\]

and summing over \(j > C\) shows
\[
\sum_j \left\| \left( \sum_n |\hat{S}_n^*(d_{n+j})|^2 \right)^{1/2} \right\|^2_{L^2(\mathbb{Z}^d)} \leq \sum_{j \geq C-2L} \varepsilon(j)^2 \|f\|_{L^2(\mathbb{Z}^d)} \lesssim \|f\|_{L^2(\mathbb{Z}^d)}.
\]

For \(j < -C\) we establish the pointwise inequality
\[
|\hat{S}_n^*(d_{n+j})|^2(x) \lesssim \iota(j) \cdot M d_{n+j}(x)^2,
\]
where \(\iota(j)\) appears as the quantitative measure of smoothness in the JRW-criterion.

Summing over \(n, j\) as above then yields the desired bound.

To do so, for \(Q \in \sigma_{n+j}\), we first observe that for any \(E_i \in \mathcal{A}_n\), any \(v \in H_n\), we may bound
\[
|A \cdot d_{n+j}(x + v)|^2 = \left| \frac{1}{|E_i|} \sum_{y \in x+v-E_i} d_{n+j}(y) \right|^2 \lesssim \left( \frac{1}{|H_n|} \right)^2 \sum_{y \in x+v-E_i} d_{n+j}(y)^2
\]
\[
= \left( \frac{1}{|H_n|} \right)^2 \left| \sum_{y \in x+v-E_i} \sum_{Q:Q \cap \partial(x+v-E_i)} d_{n+j} 1_Q(y) \right|^2
\]
\[
\leq \left( \frac{1}{|H_n|} \right)^2 |B(E_i, R_{n+j})| \cdot \sum_{y \in x+v-E_i} |d_{n+j}(y)|^2
\]
\[
\leq \iota(j) \cdot \frac{1}{|H_n|} 1_{H_n} \cdot |d_{n+j}|^2(x + v)
\]
\[
\lesssim \iota(j) \cdot \frac{1}{|2H_n|} 1_{2H_n} \cdot |d_{n+j}|^2(x)
\]
\[
\lesssim \iota(j) \cdot M d_{n+j}(x)^2.
\]
Since \(\mathcal{E}_n d_{n+j} \equiv 0\), this immediately yields the result for the long variation.
For the short variation, the proof of [7, Theorem B] leads to the bound
\[ |S_n^d d_{n+j}(y)|^2 \lesssim \iota(j)^2 \cdot \chi_{H_k} * d_{n+j}^2(y). \]
Substituting \( y = x + v \) where \( v = v(x) \in H_k \) is such that \( \tilde{S}_n^d d_{n+j}(x) = S_n^d d_{n+j}(x + v) \)
yields
\[ |\tilde{S}_n^d d_{n+j}(x)|^2 \lesssim \iota(j)^2 \cdot \chi_{H_k} * d_{n+j}^2(x + v) \lesssim \iota(j)^2 \cdot \chi_{H_k} * d_{n+j}^2(x). \]
Integrating this estimate, then summing over \( n \) yields
\[ \sum_n \int |\tilde{S}_n^d d_{n+j}(x)|^2 \lesssim \iota(j)^2 \cdot \sum_n \|d_{n+j}\|_{L^2(\mathbb{Z}^d)}^2 = \iota(j)^2 \|f\|_{L^2(\mathbb{Z}^d)}^2. \]
A final sum over \( j < -C \) shows
\[ \sum_j \left\| \sum_n \tilde{S}_n^d (d_{n+j}) \right\|_{L^2(\mathbb{Z}^d)}^{1/2} \leq \sum_{j < -C} \iota(j) \cdot \|f\|_{L^2(\mathbb{Z}^d)} \lesssim \|f\|_{L^2(\mathbb{Z}^d)}. \]

**Remark 2.5.** Whereas the pointwise estimate
\[ |A_t d_{n+j}(x + v)|^2 \lesssim \iota^2(j) \cdot Md_{n+j}^2(x)^2, \ j < -C \]
was straightforward, establishing the analogous estimate in the case of the short variation is more involved, and relies crucially on the \( (\iota\text{-quantified}) \) smoothness of the maps
\[ t \mapsto A_t f, f \in L^2(\mathbb{Z}^d). \]
The technique of using (a discrete version of) the Sobolev-embedding theorem in the index \( t \) to deal with short variations has proven quite effective; for a nice discussion and several representative examples, we refer the reader to [9, §1] and to [9, §6], respectively.

The corollary below is a direct consequence of the previous propositions:

**Corollary 2.6.** The discretized square functions
\[ \mathcal{S}^*_d f := \left( \sum_k |\mathcal{S}^*_d d_{k} f|^2 \right)^{1/2}, \]
\[ \mathcal{S}^*_D f := \left( \sum_k |\mathcal{S}^*_D d_{k} f|^2 \right)^{1/2} \]
\( * = L, S \) are \( L^2 \) bounded.

3. **High-\( p \)** Estimates

In this section we prove Theorem 1.2 by way of the following

**Proposition 3.1.** If \( A \) is regular, then \( \mathcal{S}^*_d f, * = L, S \) is \( p^\text{-bounded}, 2 < p < \infty. \)
We refine our (reverse) filtration \( \{ \sigma_k \}_k \) to \( \{ \tau_j \}_j \), where \( \tau_{j(k)} = \sigma_k \), and so successive atoms differ in size by a factor of 2: if
\[
R_j = \prod_{i=1}^d [0, 2^i(j)) \in \tau_j
\]
are generating atoms, with “symmetrized” rectangles \( H'_j = \prod_{i=1}^d [-2^i(j), 2^i(j)) \), then we have
\[
\sum_{i=1}^d |i(k + 1) - i(k)| = 1.
\]
We define the maximal operators
\[
Mf(x) := \sup_j |E'_j f|,
M'f(x) := \sup_j \sup_{x \in y + 5H'_j} \frac{1}{|5H'_j|} \int_{y+5H'_j} |f|,
\]
where \( E'_j \) is the expectation with respect to \( \tau_j \) (so \( E'_{j(k)} = E_k \)). We also define the “sharp” function associated to our refined filtration
\[
\mathcal{M}^# f(x) := \sup_j \sup_{x \in R \in \tau_j} \inf_a \frac{1}{|R|} \int_R |f - a|.
\]
We certainly have that \( \mathcal{M}^# f \leq Mf \); the familiar good-\( \lambda \) inequality
\[
|\{ Mf > 2\lambda, \mathcal{M}^# f \leq \gamma \lambda \}| \lesssim \gamma |\{ Mf > \lambda \}|
\]
holds (the implicit constant is in fact 2), and by integrating distribution functions we see that the sharp function controls the maximal function in \( L^p(\mathbb{Z}^d) \).

The key result we need is the following:

**Lemma 3.2.**
\[
\mathcal{M}^#(S^*_d f^2)(x) \lesssim M'(f^2)(x).
\]

**Remark 3.3.** Informally, \( S^*_d f \) is in “dyadic” BMO with respect to the filtration \( \{ \tau_j \} \) whenever \( f \in L^\infty \).

Assuming this lemma, with \( p = 2r > 2 \), we will have
\[
\| S^*_d f \|_{L^p(\mathbb{Z}^d)}^2 \leq \| S^*_d f^2 \|_{L^r(\mathbb{Z}^d)} \lesssim \| M^#(S^*_d f^2) \|_{L^r(\mathbb{Z}^d)} \lesssim \| M'(f^2) \|_{L^r(\mathbb{Z}^d)} \lesssim \| f^2 \|_{L^r(\mathbb{Z}^d)} = \| f \|_{L^p(\mathbb{Z}^d)}^2,
\]
proving the proposition.

**Proof of Lemma 3.2.** Fix some \( x \in \mathbb{Z}^d \), and let \( x \in R \in \tau_j, j(k') < j \leq j(k' + 1) \) be arbitrary. Express
\[ S_d^* f(x)^2 = \sum_{k \leq k'} |S_{d,k}^* f|^2 + \sum_{k > k'} |S_{d,k}^* f|^2 \]
\[ = \sum_{k \leq k'} \sum_{Q \in \sigma_k} \mathcal{S}_{k}^*(f_{15R})(x_Q)^2 1_Q + \sum_{k > k'} \mathcal{S}_{k}^*(f_{15R})(x_Q)^2 1_Q \]
\[ = \sum_{k \leq k'} \sum_{Q \in \sigma_k} \mathcal{S}_{k}^*(f_{15R})(x_Q)^2 1_Q + c_R, \]

by the nesting properties \( R \supset Q \in \sigma_k \) for \( k \leq k' \), \( R \subset Q \in \sigma_k \) for \( k > k' \) for \( x \in Q \).

We bound
\[ \frac{1}{|R|} \int_R |S_d^* f(x)^2 - c_R| = \frac{1}{|R|} \int_R | \sum_{k \leq k'} \sum_{Q \in \sigma_k} \mathcal{S}_{k}^*(f_{15R})(x_Q)^2 1_Q | \]
\[ \leq \frac{1}{|R|} \int_R | \sum_{k \leq k'} \sum_{Q \in \sigma_k} \mathcal{S}_{k}^*(f_{15R})(x_Q)^2 1_Q | \]
\[ \leq \frac{1}{|R|} \| \mathcal{S}_d^*(f_{15R}) \|_{L^2(\mathbb{Z}^d)}^2 \]
\[ \leq \frac{1}{|5R|} \int |f_{15R}|^2 \]
\[ \leq M'(f^2)(x), \]

which leads to
\[ \inf_a \frac{1}{|R|} \int_R |S_d^* f(x)^2 - a| \leq M'(f^2)(x). \]

The lemma is proven by taking a final supremum over pertaining \( R \).

We now transfer this result back to \( S_D^* f \):

**Corollary 3.4.** For each \( 2 < p < \infty \), \( S_D^* f \) – the pointwise majorants of our shifted square functions – are bounded on \( L^p(\mathbb{Z}^d) \).

**Proof.** Let \( p = 2r > 2 \), and \( r' \) denote the dual exponent to \( p/2 = r \). For an appropriate \( w \geq 0, \|w\|_{L^{r'}} = 1 \) we estimate
\[ \|S_D^* f\|_p^2 = \|S_D^* f\|_r^2 \]
\[ = \sum_k \|S_{D,k}^* f\|^2 w \]
\[ \leq \sum_k \|S_{D,k}^* f\|^2 M_{HL,t} w, \]

where \( M_{HL,t} w := (M_{HL}(w^t))^{1/t} \) for \( 1 < t < r' \), and \( M_{HL} \) denotes the (cubic, uncentered) Hardy-Littlewood maximal function. By e.g. [16, §V.6.15], we know that \( M_{HL,t} w \in A_1 \subset A_{\infty} \), so that we have
\[ \int |S_{D,k}^* f|^2 M_{HL,t} w \lesssim \int |S_{d,k}^* f|^2 M_{HL,t} w \]
for each \( k \); summing appropriately yields
\[
\| S_D^* f \|_p^2 \lesssim \int | S_D^* f |^2 M_{HL,t} w \\
\leq \| S_D^* f \|_r^2 \| M_{HL,t} w \|_{r'} \\
= \| S_D^* f \|_r^2 \| M_{HL,t} w \|_{r'} \\
\leq \| f \|_p^2,
\]
since \( M_{HL,t} \) maps \( L_{r'} \) to itself for \( t < r' \).

As corollaries, we are now able to affirmatively answer the following

**Problem 3.5** ([7], Problem 7.1). Let \( D_1 \subset D_2 \subset \cdots \subset \mathbb{Z}^d \) be a nested sequence of (closed) disks (without loss of generality containing the origin) and let \( p > 2 \). If
\[
A_i f(n) := \left| \frac{1}{|D_i|} \sum_{m \in D_i} f(n-m) \right|
\]
are convolution operators, is it true that the square function
\[
S_d f := \left( \sum_i |(A_i - A_{i+1}) f|^2 \right)^{1/2}
\]
is bounded on \( L^p \) — and thus
\[
S_{\text{abstract},d} := \left( \sum_i |(M_i - M_{i+1}) f|^2 \right)^{1/2}
\]
is finite \( \mu \)-a.e.?

and the related

**Problem 3.6** ([8] Question 4.7). Let \( \mathcal{E}_k f \) be the usual dyadic martingale on \([0,1)\), and let \( D_k f(x) = 2^k \int_{I_k(x)} f(t) \, dt \), where \( I_k(x) \) is a measurably chosen interval of length \( 2^{-k} \) which contains the point \( x \). Does the square function
\[
S f(x) = \left( \sum_{k=0}^{\infty} |D_k f(x) - \mathcal{E}_k f(x)|^2 \right)^{1/2}
\]
map \( L^p \to L^p \) for all \( 2 \leq p < \infty \)?

4. **Weighted Estimates**

We continue to make use of the refined our (reverse) filtration \( \{\tau_j\}_{j} \), where \( \tau_{j(k)} = \sigma_k \), and the maximal operators
\[
M f(x) := \sup_j |\mathcal{E}_j f|,
\]
\[
M' f(x) := \sup_j \sup_{x \in y+5H_j} \frac{1}{|5H_j'|} \int_{y+5H_j'} |f|,
\]
were introduced in the previous section.

Throughout this section, we will assume that \( M' f \lesssim M_{HL} f \) pointwise, where \( M_{HL} \) is the uncentered, cubic Hardy-Little maximal function. This condition forces additional smoothness on the refined collection \( \{H_j' = \prod_{i=1}^d [-2^{i(j)}, 2^{i(j)}] \} \):
\[
\sup_j \max \{|i(j) - i'(j)| : 1 \leq i, i' \leq d \} \leq K
\]
must remain bounded. We shall call such families\( A = (E_i) \) cubic.

**Remark 4.1.** Cubicity is a strictly stronger statement than the previous control over the eccentricities of the\( \{H_k\} \), as seen for instance by considering

\[
\{H_k = [-2^{2^k}, 2^{2^k}) \times [-2^k, 2^k)\}
\]

inside\( \mathbb{Z}^2 \). Indeed, one may take \( \varepsilon(l)^2 \lesssim 2^{-l} \), but the maximal function associated to the\( \{H_k\} \) is pointwise incomparable to\( M_{HLW} \).

Following the approach of [20, §6], we prove the following weighted results:

**Proposition 4.2** (Weighted\( L^2 \)). For any\( v \geq 0 \), there exists an absolute constant \( C_2 = C_2(\{\varepsilon(l)\}, \{\varepsilon(l)\}) \) so that

\[
\int |S_D f|^2 v \leq C_2 \int |f|^2 M_{HL} v.
\]

Consequently, for any\( A_1 \) weight\( v \in A_1 \), we have

\[
\int |S_D f|^2 v \leq C_2 \int |f|^2 v.
\]

**Proposition 4.3** (Weighted\( L^1, \infty \)). For any\( v \in A_1 \), there exists an absolute constant \( C_1 = C_1(\{\varepsilon(l)\}, \{\varepsilon(l)\}) \) so that

\[
\lambda v(\{S_D f > \lambda\}) \leq C_1 \int |f|^v
\]

for all \( \lambda \geq 0 \).

By interpolation, we get that for each\( v \in A_1 \)

\[
\int |S_D f|^p v \lesssim_p \int |f|^p v
\]

for all cubic families, \( 1 < p \leq 2 \). By Rubio de Francia extrapolation (cf. [20] §7, p.143 or [3, Theorem 7.8]), we arrive at Theorem 1.6:

**Theorem 4.4.** For all cubic families, we have

\[
\int |S_D f|^p v \lesssim_p \int |f|^p v
\]

for any\( A_p \)-weight\( v \).///

**Proof of \( L^2 \) Estimate.** The plan is decompose our operator according to scale – and then the size of the weight\( v \):

For \( j \in \mathbb{Z} \), we collect

\[
F(j,k) := \{P \in \sigma_k : 2^j < \frac{v(3P)}{|3P|} \leq 2^{j+1}\} \subset \{M_{HL} v \gtrsim 2^j\},
\]

and let\( F(j) = \bigcup_k F(j, k) \).
We estimate
\[
\int |S_D f|^2 v = \sum_k \int |S_{D,k} f|^2 v \\
\lesssim \sum_k \sum_{P \in \sigma_k} S_k^* f(x_P)^2 v(3P) \\
= \sum_j \left( \sum_k \sum_{P \in F(j,k)} \tilde{S}_k^* (f)^2 (x_P) v(3P) \right) \\
\lesssim \sum_j 2^j \left( \sum_k \sum_{P \in F(j,k)} \tilde{S}_k^* (f)^2 (x_P) |P| \right).
\]

But now, for \( P \in F(j) \) we know that \( 5P \subset \{ M_{HL} v \gtrsim 2^j \} \), and we therefore have the equality:
\[
\sum_k \sum_{P \in F(j,k)} \tilde{S}_k^* (f)^2 (x_P) = \sum_k \sum_{P \in F(j,k)} \tilde{S}_k^* (f \cdot 1_{ \{ M_{HL} v \gtrsim 2^j \}})^2 (x_P).
\]

Consequently, using the \( L^2 \)-boundedness of \( \tilde{S}_d \), we estimate
\[
\sum_k \sum_{P \in F(j,k)} \tilde{S}_k^* (f \cdot 1_{ \{ M_{HL} v \gtrsim 2^j \}})^2 (x_P) \leq |S_d (f \cdot 1_{ \{ M_{HL} v \gtrsim 2^j \}})|_{L^2}^2 \\
\lesssim \int |f|^2 \cdot 1_{ \{ M_{HL} v \gtrsim 2^j \}},
\]
and summing over \( j \) shows
\[
\int |S_D f|^2 v \lesssim \sum_j 2^j \left( \sum_k \sum_{P \in F(j,k)} \tilde{S}_k^* (f \cdot 1_{ \{ M_{HL} v \gtrsim 2^j \}})^2 (x_P) |P| \right) \\
\lesssim \sum_j 2^j \int |f|^2 \cdot 1_{ \{ M_{HL} v \gtrsim 2^j \}} \\
\lesssim \int |f|^2 M_{HL} v.
\]

Proof of \( L^{1,\infty} \) Estimate. We maintain our convention of assuming \( f \geq 0 \). By multiplying \( f \) by a suitable constant, it’s enough to prove the result for \( \lambda = 1 \):
\[
v(\{ S_D f > 1 \}) \lesssim \int |f| v.
\]

We perform a Calderon-Zygmund stopping-time decomposition at height \( c = c(d, K) \lesssim 1 \) depending on the dimension and the refined filtration using \( Mf \).

With \( E = \{ Mf > c \} \), collect all maximal \( R \in E, R \in \tau_j \) in \( E_j \), so that we may decompose
\[
E = \bigcup_k E(k) := \bigcup_k (\bigcup_{j(k-1) < j(k)} E_{j})
\]
Now, 

\[ f = g + b = g + \sum_k b^k = g + \sum_k \left( \sum_{Q \in E(k)} b_Q \right), \]

where \( b_Q = (f-f_Q)1_Q \), and \( g = f1_{E^c} + \sum_Q f_Q \), and \( f_Q = \frac{1}{|Q|} \int_Q f \) is the average of \( f \) on \( Q \). Since we have refined our initial filtration, we have the (familiar) stopping-time bound \( f_Q \leq 2c \).

Set \( X = \bigcup 2^{2K} Q \), where \( K \) is as above. We may choose \( c \) sufficiently large so that \( X \subset \{ M_{HL} f > 1 \} \). We then have the standard estimate (cf. e.g. [16, §2.1.3])

\[ |X| \lesssim \int |f|M_{HL} v \lesssim \int |f|v, \]

so our problem reduces to showing

- \( v(\{ X^c : S^*_D g > 1/2 \}) \lesssim \int |f|v \); and
- \( v(\{ X^c : S^*_D b > 1/2 \}) \lesssim \int |f|v \).

For the first point, we use Chebyshev’s inequality and the established \( l^2 \) bound to majorize

\[ v(\{ X^c : S^*_D g > 1/2 \}) \lesssim \int S^*_D g^2 (v1_{X^c}) \]

\[ \lesssim \int g^2 M_{HL} (v1_{X^c}) \]

\[ \lesssim \int g M_{HL} (v1_{X^c}) \]

\[ \leq \int_{E^c} f M_{HL} v + \sum_Q f_Q \int_Q M_{HL} (v1_{X^c}). \]

But \( \sup_Q M_{HL} (v1_{X^c}) \lesssim \inf_Q M_{HL} v \), since for any \( x \in Q \),

\[ M_{HL} (v1_{X^c})(x) \leq \sup_{x \in R} \frac{1}{|R|} \int_R v, \]

where the supremum is taken over \( R \) cubes, all of whose side lengths are at least as large as those of \( Q \). Consequently, we may estimate the above sum:

\[ \sum_Q f_Q \int_Q M_{HL} (v1_{X^c}) \lesssim \sum_Q (\int_Q f) \cdot \inf_Q M_{HL} v \leq \sum_Q \int_Q f M_{HL} v, \]

which yields the desired result, since \( M_{HL} v \lesssim v \in A_1 \).

Before beginning the second point we make the following observation: if \( Q \) is a selected (bad) cube, then

\[ \int_Q |b|v \leq \int_Q |f|v + \int_Q |f_Q|v \leq \int_Q |f|v + |f_Q|v(Q). \]

But since \( v \in A_1 \),

\[ v(Q) \lesssim |Q| \inf_{x \in Q} v(x), \]

so that

\[ |f_Q|v(Q) \lesssim \int_Q f \cdot \inf_{x \in Q} v(x) \leq \int_Q |f|v. \]
Summing over all $Q$, we have shown that
$$\int |b|v \lesssim \int |f|v,$$
and so we need only show that
$$v(\{X^c : S^*_b b > 1/2\}) \lesssim \int |b|v.$$

With this in mind, we next note that for $P \in \sigma_k$,
$$|\hat{S}^*_b b^{k^{-1}}(x_P)|^2 \leq \iota(l)^2 \frac{1}{|P|} \int_{x_P +4H_k} |b^{k^{-1}}|.$$

Using this estimate, away from $X$, we majorize
$$|S^*_{D,k} \left( \sum_{l \geq 1} b^{k^{-1}} \right) | \leq \left( \sum_{l \geq 1} |S^*_{D,k} (b^{k^{-1}})| \right)^2 \lesssim \left( \sum_{l \geq 1} \iota(l)^{1/2} \cdot \iota(l)^{1/2} \left( \sum_{P \in \sigma_k} \left( \int |b^{k^{-1}}|_{1_{x_P +4H_k}} \right) \frac{1}{|P|} \right) \right)^2.$$

We now use Cauchy-Schwartz and the summability of $\sum_l \iota(l) < \infty$ to estimate the foregoing by a constant multiple of
$$\sum_{l \geq 1} \iota(l) \cdot \left( \sum_{P \in \sigma_k} \left( \int |b^{k^{-1}}|_{1_{x_P +4H_k}} \right) \frac{1}{|P|} \right).$$

Immediately, we have
$$\int |S^*_{D,k} \left( \sum_{l \geq 1} b^{k^{-1}} \right)|^2 v \lesssim \sum_{l \geq 1} \iota(l) \cdot \sum_{P \in \sigma_k} \left( \int |b^{k^{-1}}|_{1_{x_P +4H_k}} \right) \frac{v(x_P +4H_k)}{|x_P +4H_k|}.$$

But for any $Q \subset E(k-l)$ which intersects $x_P +4H_k$,
$$\frac{v(x_P +4H_k)}{|x_P +4H_k|} \lesssim \inf_{x \in Q \cap (x_P +4H_k)} M_{HL}v(x),$$
so we may replace
$$\left( \int |b^{k^{-1}}|_{1_{x_P +4H_k}} \right) \frac{v(x_P +4H_k)}{|x_P +4H_k|} \lesssim \int_{x_P +4H_k} |b^{k^{-1}}| M_{HL}v,$$
which leads directly to the bound
$$\int |S^*_{D,k} \left( \sum_{l \geq 1} b^{k^{-1}} \right)|^2 v \lesssim \sum_{l \geq 1} \iota(l) \sum_{P \in \sigma_k} \int_{x_P +4H_k} |b^{k^{-1}}| M_{HL}v \lesssim \sum_{l \geq 1} \iota(l) \int |b^{k^{-1}}| M_{HL}v,$$
due to the bounded overlap of $\{x_P +4H_k : P \in \sigma_k\}$. 


Using Tchebychev and summing over $k, l$ completes the second point,

$$v(\{X^c : S_D^* b > 1/2\}) \lesssim \int |S_D^* b|^2(v1_{X^c})$$

$$\leq \sum_k \sum_{l \geq 1} \iota(l) \int |b^{k-l}| M_{HL} v$$

$$= \sum_{l \geq 1} \iota(l) \int |b| M_{HL} v$$

$$\lesssim \int |b| M_{HL} v,$$

thereby concluding the proof. \qed

5. The Rectangular Square Function

In this section we relax our smoothness/eccentricity control over our averaging families: suppose $A = \{E_i\}_{i \geq 0}$ is a nested sequence of rectangles in $\mathbb{Z}^d$, with sides parallel to the axes, but without any regularity assumptions.

We study the following

**Problem 5.1** ([7] Problem 7.5). For each collection rectangles $A \subset \mathbb{Z}^d$ set

$$S_A f = \left( \sum_i |A_i - A_{i+1} f|^2 \right)^{1/2}.$$

For each $A$, does there exist a bound $\|S_A f\|_{L^1, \infty(\mathbb{Z}^d)} \leq C(A) \|f\|_{L^1(\mathbb{Z}^d)}$?

Following the approach of the previous sections, we study individually long/short square functions: with $\{H_k\}$ as above,

$$S_A^L f := \left( \sum_k \left| \chi_{H_k} * f - \chi_{H_{k+1}} f \right|^2 \right)^{1/2},$$

and

$$S_A^S f := \left( \sum_k \left| S_A^{k} f \right|^2 \right)^{1/2} := \left( \sum_k \sum_{E_i \subset H_k, \text{nested}} |A_i f - A_{i+1} f|^2 \right)^{1/2}.$$

Establishing the weak type bound for the short square function $S_A^S$ remains currently out of reach; we are, however, able to establish the following partial result:

**Proposition 5.2.** There exists an absolute constant independent of the collection $A$ so that

$$\|S_A^L f\|_{L^1, \infty(\mathbb{Z}^d)} \leq C \|f\|_{L^1(\mathbb{Z}^d)}.$$

This result is proven by combining the fibre-wise argument used in [6] with the Calderon-Zygmund technique used in establishing the $l^{1, \infty}$ weighted estimate.

For the sake of exposition, we pause here to record the following (easy) lemmas which will be used in the main argument below.

**Lemma 5.3.** Suppose that

$$\Psi_n = \sum_j \psi_j,$$

where the $\{\psi_j\}$ are a finite collection functions, disjointly supported in (dyadic) intervals $J \subset \mathbb{Z}$, $|J| = 2^n$ and satisfy $\int \psi_J = 0$, $\|\psi_J\|_1 \lesssim 2^n$. 


Integrating, then summing over \( J \)

This proof is similar to Stein-Weiss. One splits each Sketch.

according to size. The \( l \) and Chebyshev to control ming weak-type inequalities will also be of use. For notational ease, set

Proof of Proposition 5.2.

Proof. If \( E := \{ x : J \cap x - \partial I \neq \emptyset \} \), then \( |E| \lesssim |J| \), and for each \( J \),

\[
|\chi_I * \psi_J(x)| \lesssim \frac{\|\psi_J\|_1}{|I|} |E| \lesssim 2^{-s} 1_E.
\]

Integrating, then summing over \( J \), exhibits \( \|\chi_I * \Psi_n\|_1 \lesssim 2^{-s} \|\Psi_n\|_1 \), as desired. \( \square \)

The following generalization of the Stein-Weiss Lemma [18, Lemma 2.3] on sum-

Lemma 5.4. If \( \|g_k\|_{1,\infty} \leq 1 \), and \( c_k \) are a collection of positive constants with \( \sum c_k = 1 \). Set \( \gamma := (\sum_k c_k^2)^{1/2} \). Then \( G := (\sum |c_k g_k|^2)^{1/2} \) has \( \|G\|_{1,\infty} \lesssim 1 \). By scaling:

\[
\left\| \sum_k |g_k|^2 \right\|_{1,\infty} \lesssim \sum_k \|g_k\|_{1,\infty}.
\]

Sketch. This proof is similar to Stein-Weiss. One splits each

\( g_k = l_k + m_k + u_k \)

according to size. The \( l_k \) are chosen so \( (\sum |l_k|^2)^{1/2} \leq \lambda/2 \), one uses a union bound to estimate the “high” component

\[
|\{ (\sum |u_k|^2)^{1/2} > \lambda/4 \}| \leq |(\sum |u_k|^2)^{1/2} = 0|,
\]

and Chebyshev to control

\[
|\{ (\sum |m_k|^2)^{1/2} > \lambda/4 \}| \lesssim \lambda^{-2} \sum_k \|m_k\|^2.
\]

\( \square \)

Proof of Proposition 5.3. For notational ease, set \( S^d \frac{1}{2} = S \). By separating into \( d \) families as in the proof of Theorem 2.1 in [6], we may sum over consecutive rectangles which differ in the first coordinate. We will express

\( x = (x_1, x') \in \Z \times \Z^{d-1} \),

and each

\( H_k = I_k \times J_k \subset \Z \times \Z^{d-1} \)

where \( J_k \) is a \( d - 1 \)-rectangle.

The set-up is quite similar to the proof of the \( l^{1,\infty} \) weighted estimate:

We continue to assume that \( f \geq 0 \), and seek to establish \( |\{ Sf > 1 \}| \lesssim \|f\|_1 \).

We refine our (reverse) filtration \( \{H_k\} \) as above, and consider once again the maximal operator

\[
Mf(x) := \sup_{j \geq 0} |E_j f|.
\]

We also define the (uncentered) fibred-maximal functions

\[
M^{d-1} f(x) := \sup_k \sup_{x' \in y' + J_k} \frac{1}{|J_k|} \int_{y' + J_k} |f|(x_1, z') \, dz',
\]
and
\[ M^1 f(x) := \sup_k \sup_{z_1 \in \mathcal{Y}_1 + I_k} \frac{1}{I_k} \int_{\mathcal{Y}_1 + I_k} |f|(z_1, x') \, dz_1, \]
both of which enjoy the familiar one-parameter weak-type \((1, 1)\) bounds.

With \(E := \{Mf > c\}\), we collect all maximal \(R \in E, R \in \tau_j \) in \(E_j\), so that we may express
\[ E = \bigcup_k E(k) := \bigcup_k \left( \bigcup_{j(k-1) < j(k)} E_j \right), \]
and decompose
\[ f = g + b = g + \sum_k b^k = g + \sum_k \left( \sum_{Q \in E(k)} b_Q \right), \]
where \(g := f \cdot 1_E + \sum_Q f_Q \cdot 1_Q\), and \(b_Q = (f - f_Q) \cdot 1_Q\) as above.

With \(X := \bigcup Q 10\mathcal{Q}\), so that \(Q + H_k \subset X\) whenever \(Q \in E(k)\), we have the estimate
\[ |X| \lesssim \sum_Q |Q| \lesssim \sum_Q \int_Q f \leq \|f\|_1, \]
and using the \(L^2(\mathbb{Z}^d)\)-boundedness of \(\mathcal{S}\) (6.4, Theorem 2.1) estimate
\[ |\{Sg > 1/2\}| \lesssim \|Sg\|_{L^2(\mathbb{Z}^d)} \lesssim \|g\|_{L^2(\mathbb{Z}^d)} \lesssim \|g\|_{L^1(\mathbb{Z}^d)} \lesssim \|f\|_{L^1(\mathbb{Z}^d)}; \]
the argument reduces to showing
\[ |\{X^c : \mathcal{S}b > 1/2\}| \lesssim \|f\|_{L^1(\mathbb{Z}^d)}. \]

To do this, we further decompose each
\[ b_Q = b^0_Q + b^1_Q \]
where each \(b^i_Q\) satisfies the size condition \(\|b^i_Q\|_{L^1(\mathbb{Z}^d)} \sim |Q|\), but also the more specialized “fibred” moment conditions:
\[ \int_\mathbb{Z} b^0_Q(x_1, x') \, dx_1 = 0 \text{ for each } x' \in \mathbb{Z}^{d-1}, \]
\[ \int_{\mathbb{Z}^{d-1}} b^1_Q(x_1, x') \, dx' = 0 \text{ for each } x_1 \in \mathbb{Z}. \]
This may be accomplished for instance by setting
\[ b_Q = b^0_Q + b^1_Q \]
\[ = \left( b_Q - \left( \int_{I_Q} b_Q(x_1, x') \, dx_1 \right) \cdot 1_{I_Q} \right) + \left( \int_{I_Q} b_Q(x_1, x') \, dx_1 \right) \cdot 1_{I_Q}, \]
\[ := \psi_{I_Q} \otimes 1_{I_Q} + 1_{I_Q} \otimes \psi_{I_Q}, \]
where
\[ \|\psi_{I_Q} \otimes 1_{I_Q}\|_{L^1(\mathbb{Z}^d)} + \|1_{I_Q} \otimes \psi_{I_Q}\|_{L^1(\mathbb{Z}^d)} \lesssim \|b_Q\|_{L^1(\mathbb{Z}^d)}. \]
We define
\[ b^k = b^0 + b^1 \]
in the obvious way.

It suffices to separately estimate
\[ |\{X^c : \mathcal{S}b^0 > 1/4\}|, |\{X^c : \mathcal{S}b^1 > 1/4\}|. \]
We begin by studying the square function’s behavior on \( b^0 \) – i.e. we assume that each

\[
b_Q^0 = \psi_{I_Q} \otimes 1_{J_Q}
\]

has mean-zero when integrated in the \( x_1 \)-direction.

We decompose the convolution operators

\[
(\chi_{H_k} - \chi_{H_{k+1}}) * f = \chi_k^2 * ((\chi_k^1 - \chi_{k+1}^1) * f)
\]

\[
= (\chi_k^1 - \chi_{k+1}^1) * (\chi_k^2 * f)
\]

\[
=: \eta_k^1 * (\chi_k^2 * f)
\]

\[
= \chi_k^2 * (\eta_k^1 * f),
\]

with convolution involving \( \chi_k^2 \) taking place in the final \( d - 1 \) coordinates, and \( \chi_k^1, \chi_{k+1}^1 \) in the first coordinate. This allows us to bound – on \( X^c \) –

\[
Sb^0 \leq \left( \sum_k \chi_k^2 * |\eta_k^1 * \left( \sum_{|I_Q| \leq |I_k|} b_Q^0 \right) |^2 \right)^{1/2}
\]

\[
\leq \sum_{s \geq 0} \left( \sum_k \chi_k^2 * |\eta_k^1 * \left( \sum_{|I_Q| = 2^{-s} |I_k|} b_Q^0 \right) |^2 \right)^{1/2}.
\]

With \( s \geq 0 \) fixed, we use the Fefferman-Stein vector-valued maximal inequality (cf. e.g. [10, §2 Theorem 1]) to estimate

\[
\left\| \left( \sum_k \chi_k^2 * |\eta_k^1 * \left( \sum_{|I_Q| = 2^{-s} |I_k|} b_Q^0 \right) |^2 \right)^{1/2} \right\|_{L^1(\mathbb{Z}^d)}
\]

\[
\leq \left\| \left( \sum_k M^{d-1} |\eta_k^1 * \left( \sum_{|I_Q| = 2^{-s} |I_k|} b_Q^0 \right) |^2 \right)^{1/2} \right\|_{L^1(\mathbb{Z}^d)}
\]

\[
\leq \left\| \left( \sum_k |\eta_k^1 * \left( \sum_{|I_Q| = 2^{-s} |I_k|} b_Q^0 \right) |^2 \right)^{1/2} \right\|_{L^1(\mathbb{Z}^d)}
\]

\[
\leq \sum_k \left\| \eta_k^1 * \left( \sum_{|I_Q| = 2^{-s} |I_k|} b_Q^0 \right) \right\|_{L^1(\mathbb{Z}^d)}.
\]

By applying Lemma 3.5 to the functions

\[
x_1 \mapsto \chi_k * \left( \sum_{|I_Q| = 2^{-s} |I_k|} b_Q^0(x_1, x') \right),
\]
integrating first in $x_1 \in \mathbb{Z}$, then over $x' \in \mathbb{Z}^{d-1}$, we estimate this final sum by a constant multiple of

$$
\sum_k 2^{-s} \left\| \sum_{|I_Q| = 2^{-i}|I_k|} b_Q^0 \right\|_1 \leq 2^{-s} \| b^0 \|_{L^1(\mathbb{Z}^d)} \lesssim 2^{-s} \| f \|_{L^1(\mathbb{Z}^d)}.
$$

Combining our estimates in $s \geq 0$, and choosing $c > 0$ appropriately small leads to the desired bound:

$$
|\{ S^0 b^1 > 1/4 \}| \leq \sum_{s \geq 0} \left\{ \left( \sum_k \chi_{j_k}^2 * \eta_k^1 * \left( \sum_{|I_Q| = 2^{-i}|I_k|} b_Q^0 \right) \right)^2 \right\}^{1/2} > c 2^{-s/2}
$$

$$
\lesssim \sum_{s \geq 0} 2^{s/2} \left\| \left( \sum_k \eta_k^1 * \left( \sum_{|I_Q| = 2^{-i}|I_k|} b_Q^0 \right) \right)^2 \right\|_{L^1(\mathbb{Z}^d)}
$$

$$
\lesssim \sum_{s \geq 0} 2^{s/2} \cdot 2^{-s} \| f \|_{L^1(\mathbb{Z}^d)} \lesssim \| f \|_{L^1(\mathbb{Z}^d)}.
$$

In passing to the second case, where we must estimate

$$
|\{ X^c : S(b^1) > 1/4 \}|
$$

we collect cubes according to separation of scales of the final $(d-1)$-coordinates.

For $Q = I_Q \times J_Q \in \tau_j, j < j(k)$, we use $\triangle(Q, H_k) = \triangle_1(Q, H_k)$ to denote the degree of separation between $J_Q, J_k$.

Now, away from $X = \bigcup_Q Q^*$, we use the triangle inequality to obtain the pointwise bound

$$
S^{b^1} \leq \left( \sum_k \left( \chi_{H_k} - \chi_{H_{k+1}} \right) * \left( \sum_{s \geq 0} \sum_{\triangle(Q, H_k) = s} b_Q^1 \right) \right)^{1/2}
$$

$$
\leq \sum_{s \geq 0} \left( \sum_k \left( \chi_{H_k} - \chi_{H_{k+1}} \right) * \left( \sum_{\triangle(Q, H_k) = s} b_Q^1 \right) \right)^{1/2}
$$

$$
= \sum_{s \geq 0} \left( \sum_i |h_{j_k, s}|^2 \right)^{1/2},
$$

where

$$
h_{j_k, s} := \left( \sum_{H_k : J_k = J_{k_i}} \left( \chi_{H_k} - \chi_{H_{k+1}} \right) * \left( \sum_{\triangle(Q, H_k) = s} b_Q^1 \right) \right)^{1/2},
$$

with the sum taken over all rectangles $\{H_k\}$ which share the same final $d-1$-dimensions.
We will show that for each \( s \),
\[
\| h_{J_{k_i},s} \|_{L^1(Z^d)} \lesssim 2^{-s} \left( \sum_{\triangle(Q,H_{k_i})=s} b_{Q,1} \right),
\]
then will sum on \( J_i \) and apply Lemma 5.4 to conclude that
\[
\left\| \left( \sum_i |h_{J_{k_i},s}|^2 \right)^{1/2} \right\|_{L^1(Z^d)} \lesssim \sum_i \| h_{J_{k_i},s} \|_{L^1(Z^d)}
\]
\[
\lesssim 2^{-s} \sum_i \left\| \sum_{\triangle(Q,H_{k_i})=s} b_{Q,1} \right\|_{L^1(Z^d)}
\]
\[
\leq 2^{-s} \| b \|_{L^1(Z^d)}
\]
\[
\lesssim 2^{-s} \| f \|_{L^1(Z^d)},
\]
so that, once again, with \( c > 0 \) an absolute constant, we estimate
\[
|\{ X^c : Sb^1 > 1/4 \} | \leq \sum_{s \geq 0} |\{ \left( \sum_i |h_{J_{k_i},s}|^2 \right)^{1/2} > c2^{-s/2} \} |
\]
\[
\lesssim \sum_{s \geq 0} 2^{s/2} \cdot 2^{-s} \| f \|_{L^1(Z^d)} \lesssim \| f \|_{L^1(Z^d)}.
\]
To do this, we express
\[
h_{J_{k_i},s} = \left( \sum_{H_k : J_k = J_{k_i}} \eta_k^1 \ast \left( \chi_{J_{k_i}} \ast \left( \sum_{\triangle(Q,H_{k_i})=s} b_{Q,1} \right) \right) \right)^{1/2}
\]
as a one-dimensional square function applied to the function
\[
\chi_{J_{k_i}} \ast \left( \sum_{\triangle(Q,H_{k_i})=s} b_{Q,1} \right),
\]
which has small \( l^1 \) norm by the separation of scales Lemma 5.3. In particular, by considering the functions
\[
x' \mapsto \chi_{J_{k_i}} \ast \left( \sum_{\triangle(Q,H_{k_i})=s} b_{Q,1}(x_1,x') \right)
\]
we estimate
\[
\| \chi_{J_{k_i}} \ast \left( \sum_{\triangle(Q,H_{k_i})=s} b_{Q,1}(x_1,x') \right) \|_{l^1(Z^d)} \equiv \int_Z \| \chi_{J_{k_i}} \ast \left( \sum_{\triangle(Q,H_{k_i})=s} b_{Q}(x_1,-) \right) \|_{l^1(Z^d-1)}
\]
\[
\lesssim \int_Z 2^{-s} \left\| \sum_{\triangle(Q,H_{k_i})=s} b_{Q}(x_1,-) \right\|_{l^1(Z^d-1)}
\]
\[
= 2^{-s} \left\| \sum_{\triangle(Q,H_{k_i})=s} b_{Q} \right\|_{l^1(Z^d)}.
\]
Theorem 6.1. We have the following equivalence

\[ \lambda\{(x_1,x') \in \mathbb{Z} \times \mathbb{Z}^{d-1} : h_{J_{k_1}}(x_1,x') > \lambda\} \]

\[ = \int_{\mathbb{Z}^{d-1}} \lambda\{x_1 \in \mathbb{Z} : h_{J_1}(-,x') > \lambda\} \, dx' \]

\[ \lesssim \int_{\mathbb{Z}^{d-1}} \left( \int_{\mathbb{Z}} |\chi_{J_{k_1}} \ast \left( \sum_{\Delta(Q,H_{k_1})=s} b_Q^{1} \right)(x_1,x') \right) \, dx_1 \, dx' \]

\[ = \left\| \chi_{J_{k_1}} \ast \left( \sum_{\Delta(Q,H_{k_1})=s} b_Q^{1} \right) \right\|_{L^1(\mathbb{Z}^d)} \]

\[ \leq 2^{-s} \left\| \sum_{\Delta(Q,H_{k_1})=s} b_Q^{1} \right\|_{L^1(\mathbb{Z}^d)}, \]

as desired. \hfill \Box

6. Appendix: A Weak-Type Principle for Square Functions

With \( \tau \) a free \( \mathbb{Z}^d \) action,

\[ \tau_y f(x) := f(\tau - yx), \]

on a non-atomic probability space \((X, \Sigma, \mu)\), we consider the square function:

\[ Sf(x) = S^\tau_{(E_m)} f(x) := \left( \sum_m |(M_m - M_{m+1})f|^2 \right)^{1/2}(x) := \left( \sum_m |K_m f|^2 \right)^{1/2}(x). \]

Note that by the measure-preserving action of \( \tau \) and the triangle inequality

\[ \|K_m f\|_{L^1(X)} \leq 2 \|f\|_{L^1(X)}, \]

i.e. the \( \{K_m\} \) are uniformly bounded in operator norm.

Theorem 6.1. We have the following equivalence

- For each \( f \in L^1(X, \Sigma, \mu) \), \( Sf < \infty \mu\text{-a.e.}; \) and
- \( \|Sf\|_{L^1(X)} \lesssim \|f\|_{L^1(X)} \).

Remark 6.2. The only key property about the integrability class \( L^1 \) used in the below proof is that \( 1 \leq 2 \); the above equivalence persists for all \( 1 \leq p \leq 2 \).

That the second point implies the first is clear, so we concern ourselves with the remaining implication. We actually will prove (a strengthened version of) the contrapositive, namely that if no bound on the operator norm \( S : L^1 \to L^{1,\infty} \) exists, then there exists an \( f \in L^1(X) \) so that \( Sf = +\infty \) a.e.

The argument below is similar to that of [14], though to the best of our knowledge has yet to appear in print. In the spirit of the Conze principle, we reduce to the ergodic case:

Proposition 6.3. Suppose that either (and hence both) of the conditions in Theorem 6.1 hold for some (free) system. Then the same is true of every such system.

In particular: \( \|S\phi\|_{L^{1,\infty}(X)} \leq C\|\phi\|_{L^1(X)} \) if and only if \( \|Sf\|_{L^{1,\infty}(\mathbb{Z}^d)} \leq C\|f\|_{L^1(\mathbb{Z}^d)}. \)
The forward implication is established by the Rokhlin Lemma \cite{15}; it is here that we make use of the freeness of our action $\tau$. The reverse implication is an easy application of Calderón’s transference principle \cite{2}.

For our purposes, we will henceforth assume that our action $\tau$ is ergodic.

6.1. Preliminaries. The following results will be of use.

**Lemma 6.4** (Randomization Lemma). If $\{E_n\} \subset X$ have $\sum_n \mu(E_n) = +\infty$, then there exist a collection of vectors $y(n) \subset \mathbb{Z}^d$ so that

$$\limsup \tau_{-y(n)} E_n = X \mu - \text{a.e.}$$

In the probabilistic setting, the Borel-Cantelli lemma says that if the $\{E_n\}$ are independent events with $\sum_n \mu(E_n) = \infty$, then $\limsup E_n = X$ almost surely.

The content of this lemma is that the ergodicity of the $\tau$-action is sufficiently randomizing to force similar independence.

**Proof.** If $Q(N) := \{y \in \mathbb{Z}^d : |y(i)| \leq N, 1 \leq i \leq d\}$, by the ergodicity of $\tau$, we know that for any measurable $A, B \subset X$

$$\frac{1}{(2N + 1)^d} \sum_{y \in Q(N)} \mu(\tau_{-y} A \cap B) \to \mu(A)\mu(B).$$

Consequently, we may find a $y = y(A, B)$ so that $\mu(\tau_{-y} A \cap B) \geq 1/2 \mu(A)\mu(B)$; it is this point which anchors the volume-packing argument found e.g. in \cite{16} §10.

We also recall the following orthogonality lemma concerning the well-known Rademacher functions, $\{r_m(s)\}$:

**Lemma 6.5** (\cite{22}, §V, Theorem 8.2). If $\sum_m |a_m|^2 < \infty$, then for (Lebesgue) almost every $s \in [0, 1]$,

$$|\sum_m r_m(s)a_m|$$

converges. Conversely, if $\sum_m |a_m|^2 = +\infty$, then $|\sum_m r_m(s)a_m|$ diverges a.e.

For what is to follow, we shall need the following two-dimensional variant.

**Lemma 6.6** (Two-Dimensional Orthogonality Lemma). Suppose that $E \subset [0, 1]^2$ is non-null, and assume that

$$G(t)^2 := \lim \limsup_M \sum_{m \leq M} \sum_{n \leq N} r_m(s)r_n(t)|a_{mn}|^2$$

satisfies $|G(t)| < A$ on $E$.

Then, there exists a $M$ so that

$$\sum_{m > M} \sum_{n} |a_{mn}|^2 \leq CA^2$$

for some absolute constant $C$.

**Proof.** Set $\gamma_{mn}(s, t) := r_m(s)r_n(t)$, and note that since

$$\{\gamma_{mn}\gamma_{m'n'} : (m, n) \neq (m', n')\}$$

are orthonormal over $[0, 1]^2$,

$$\sum_{(m, n) \neq (m', n')} |\langle \gamma_{mn}\gamma_{m'n'}|1_E \rangle|^2 \leq |E|$$
converges absolutely. Consequently, we may pick an \( M_0 \) sufficiently large so that for all \( M \geq M_0 \),

\[
\sum_{(m,n) \neq (m',n'), m,m' > M} |\langle \gamma mn|\gamma m'n' \rangle| (<|E|/2)^2.
\]

Possibly after increasing \( M_0 \), we may also assume that for all \( M \geq M_0 \)

\[
\lim_{N} \sup \left| \sum_{m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right| < A.
\]

Now, for any \( M > M' \geq M_0 \)

\[
\lim_{N \to \infty} \left| \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2
\]

\[
\lesssim \lim_{N \to \infty} \left| \sum_{m \leq M'} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2 + \lim_{N \to \infty} \left| \sum_{m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2
\]

\[
\lesssim A^2 + A^2,
\]

so we may use dominated convergence to estimate

\[
\lim_{M \to \infty} \lim_{N \to \infty} \int_E \left| \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2 = \lim_{N \to \infty} \int_E \lim_{M \to \infty} \left| \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2
\]

\[
= \int_E \lim_{N \to \infty} \lim_{M \to \infty} \left| \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2
\]

\[
\lesssim \int_E |G(t)|^2 + \int \lim_{N \to \infty} \left| \sum_{m \leq M'} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2
\]

\[
\lesssim A^2 |E| + A^2 |E| \lesssim A^2 |E|.
\]

We now seek a lower bound on the \( \lim_M \lim_N \int_E \left| \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2 \).

To do this, we expand the square

\[
\int_E \left| \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s) r_n(t) a_{mn} \right|^2
\]

to get

\[
\int_E \sum_{M' < m \leq M} \sum_{n \leq N} |a_{mn}|^2 + \sum_{M' < m \neq m' \leq M} \sum_{n \neq n' \leq N} \gamma_{mn} \gamma_{m'n'} a_{mn} a_{m'n'}
\]

\[
=: I(M,N) + II(M,N).
\]

We will show that

\[
|II(M,N)| < \frac{1}{2} I(M,N) = \frac{|E|}{2} \sum_{M' < m \leq M} \sum_{n \leq N} |a_{mn}|^2.
\]
This will allow us to bound from below:

\[
\lim_{M \to \infty} \lim_{N \to \infty} \int_E \sum_{M' < m \leq M} \sum_{n \leq N} r_m(s)r_n(t)|a_{mn}|^2 \\
\geq \lim_{M \to \infty} \lim_{N \to \infty} \frac{|E|}{2} \sum_{M' < m \leq M} \sum_{n \leq N} |a_{mn}|^2 \\
= \frac{|E|}{2} \sum_{M' < m} \sum_{n} |a_{mn}|^2.
\]

Combining this with our upper bound will allow us to conclude:

\[
\sum_{M' < m} \sum_{n} |a_{mn}|^2 \lesssim A^2.
\]

But we may simply use Cauchy-Schwarz to majorize:

\[
\left| \int_E \sum_{M' < m \not= m' \leq M} \sum_{n \not= n' \leq N} \gamma_{mn}\gamma_{m'n'}a_{mn}\bar{a}_{m'n'} \right| \\
\leq \sum_{M' < m \not= m' \leq M} \sum_{n \not= n' \leq N} |\langle \gamma_{mn}\gamma_{m'n'}1_E \rangle| |a_{mn}| |a_{m'n'}| \\
\leq \left( \sum_{M' < m \not= m' \leq M} \sum_{n \not= n' \leq N} |\langle \gamma_{mn}\gamma_{m'n'}1_E \rangle|^2 \right)^{1/2} \cdot \left( \sum_{M' < m \not= m' \leq M} \sum_{n \not= n' \leq N} |a_{mn}|^2 |a_{m'n'}|^2 \right)^{1/2} \\
\leq \frac{|E|}{2} \sum_{M' < m \leq M} \sum_{n \leq N} |a_{mn}|^2.
\]

\[\square\]

We now turn to the proof proper:

**Proof of the Theorem 6.1.** We proceed as in [17]:

Suppose that no bound \( \|Sf\|_{L^{1,\infty}(X)} \leq N \|f\|_{L^1(X)} \) existed. In this case we could find a sequence of functions \( \{g_n\} \in L^1(X) \), and a monotonically increasing sequence \( \{R_n\} \) so

\[
\sum_{n} \mu(Sg_n > R_n) = \infty
\]

diverges, while

\[
\sum_{n} \|g_n\|_1 \lesssim 1
\]

converges.

We apply our Randomization Lemma [6.4] to the collection of sets

\( E_n := \{Sg_n > R_n\} \)

so that

\( X_0 := \lim \sup_{t} \tau_{-y(n)}E_n \subset X \)

has full measure, and set \( f_n := \tau_{y(n)}g_n \).

Then: \( Sf_n > R_n \to \infty \) on \( X_0 \), while \( \sum_{n} \|f_n\|_1 = \sum_{n} \|\tau_{y(n)}g_n\|_1 < \infty \).

We consider the formal sum

\[
\sum_{n} r_n(t)f_n(x);
\]
as in [17] we may find a subsequence \( \{N_k\} \) along which \( F(x, t) := \lim_k \sum_{n \leq N_k} r_n(t) f_n(x) \) satisfies

1. For almost every \( t \in [0, 1], F(x, t) \in L^1(X); \)
2. For \( \mu \text{-a.e. } x \in X, \) for each \( m \)
   \[
   K_m F(x, t) \equiv \lim_k \sum_{n \leq N_k} r_n(t) K_m f_n(x)
   \]
as functions of \( t. \)

We will prove:

For almost every \( t, SF(x, t) = +\infty \) \( \mu \text{-a.e.} \)

To do this, we proceed by contradiction, and assume that the sum
\[
\sum_m |K_m F(x, t)|^2 < \infty
\]
converges on a set \( D \subset X \times [0, 1] \) of positive product measure.

By Lemma 6.5, for each \( (x, t) \in D \)
\[
|\sum_m r_m(s) K_m F(x, t)| < \infty
\]
s-a.e. so we may extract a subset \( E \subset X \times [0, 1]^2 \), \( A > 0 \), so that for each \( M \geq M' \) sufficiently large partial summand
\[
|\sum_{m \leq M} r_m(s) K_m F(x, t)| < A
\]
on \( E. \)

For each measurable section (\( \mu \)-almost all sections are measurable), we set
\[
E_x := \{(s, t) \in [0, 1]^2 : (x, s, t) \in E \} \subset [0, 1].
\]
Since \( E \) has positive product measure, we may extract some \( \delta > 0 \) and a set \( X_\delta \) of positive \( \mu \)-measure so that for each \( x \in X_\delta, |E_x| \geq \delta. \)

For each \( x \in X_\delta, \) we apply our Two-Dimensional Orthogonality Lemma 6.6 to
\[
\lim_{M \to \infty} \sum_{m \leq M} |r_m(s) K_m F(x, t)| = \lim_{M \to \infty} \lim_{N_k \to \infty} \sum_{m \leq M} \sum_{n \leq N_k} |r_m(s) r_n(t) K_m f_n(x)|,
\]
with \( K_m f_n(x) \) in the role of \( a_{mn}. \)

In particular, we extract \( M(x) \) so that
\[
\sum_{M(x) < m} \sum_n |K_m f_n(x)|^2 \leq CA^2.
\]
Assume that \( M(x) \) is minimal subject to the condition that
\[
\sum_{M(x) < m} \sum_n |K_m f_n(x)|^2 \leq 2CA^2,
\]
and collect
\[
A_k := \{x \in X_\delta : M(x) = k\},
\]
so that we may express \( X_\delta = \bigcup_{k \geq 1} A_k \) as a disjoint union. Choose \( k' \) as small as possible subject to the constraint that \( \mu(A_{k'}) > 0. \) We have:
\[
\sum_{k' < m} \sum_n |K_m f_n(x)|^2 \lesssim A^2
\]
We now wish to show that
\[
\left\{ y : \sum_{m \leq k'} \sum_{n} |K_m f_n(y)| \geq \frac{R_n}{2} \text{ for all but finitely many } n \right\}
\]
\[= \liminf_{n} \left\{ y : \sum_{m \leq k'} \sum_{n} |K_m f_n(y)| \geq \frac{R_n}{2} \right\}
\]
is \(\mu\)-null. This will allow us to conclude that for almost every \(x \in A_{k'} \cap X_0\) — and thus almost every \(x \in A_{k'}\), since \(\mu(X_0) = 1\) — there exist infinitely many \(n\) with
\[
\sum_{k' < m} |K_m f_n(x)|^2 \geq |Sf_n(x)|^2 - \sum_{m \leq k'} |K_m f_n(x)|^2 \geq \frac{3}{4} R_n^2;
\]
summing in \(n\) would then force
\[
\sum_{k' < m} \sum_{n} |K_m f_n(x)|^2
\]
to diverge, contradicting the upper bound of \(\lesssim A^2\), and concluding the argument.

To this end, for each \(n\) we estimate
\[
\sum_{m \leq k'} \|K_m f_n(x)\|_1 \lesssim k' \|f_n\|_1,
\]
from which it follows that
\[
\mu \left( \left\{ y \in X : \sum_{m \leq k'} |K_m f_n|(y) > \frac{R_n}{2} \right\} \right) \lesssim k' \cdot \frac{\|f_n\|_1}{R_n},
\]
Consequently, for each \(l\), we may estimate
\[
\mu \left( \bigcap_{n \geq l} \left\{ y \in X : \sum_{m \leq k'} |K_m f_n|(y) > \frac{R_n}{2} \right\} \right) \lesssim \lim_{n \to \infty} k' \cdot \frac{\|f_n\|_1}{R_n} = 0,
\]
which yields
\[
\mu \left( \liminf_{n} \left\{ y \in X : \sum_{m \leq k'} |K_m f_n|(y) > \frac{R_n}{2} \right\} \right) = 0,
\]
as desired. \(\square\)

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