UPPER AND LOWER BOUND ON DELTA-CROSSING NUMBER 
AND TABULATION OF KNOTS UP TO FOUR DELTA-CROSSINGS

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ABSTRACT. We will strengthen the known upper and lower bounds on the delta-crossing number of knots in terms of the triple-crossing number. The latter bound turns out to be strong enough to obtain (unknown values of) triple-crossing numbers for a few knots. We also prove that we can always find at least one tangle from the set of four tangles, in any triple-crossing projections of any non-trivial knot or non-split link. In the last section, we enumerate and generate tables of minimal delta-diagrams for all prime knots up to the delta-crossing number equal to four. We also give a concise survey about known inequalities between integer-valued classical knot invariants.

1. INTRODUCTION

It is known that any knot and any link has a delta-crossing diagram i.e. a diagram that can be decomposed into delta-crossing tangles [23]. The delta-crossing number of a knot $K$, denoted here by $c_\Delta(K)$, is defined as the least number of delta-crossings for any delta-crossing diagram of $K$. There are upper and lower bounds for the triple-crossing number $c_3(K)$, in terms of the triple-crossing number $c_3$ and the canonical genus $g_c$, for knot $K$ we have: $2c_3(K) \geq c_\Delta(K)$ and $c_\Delta(K) \geq g_c(K)$. We will strengthen (because of the bound $c_3(K) \geq 2g_c(K)$ proved by the author in [12]) the lower bound.

**Theorem 1.1.** For any knot (or a link) $K$, we have $2c_\Delta(K) \geq c_3(K)$.

In Section 3 of this paper, we will also strengthen the above upper bound with additional assumptions as follows.

**Theorem 1.2.** For any knot $K$, let $t_1, t_2$ be the number of disjoint tangles $T_1, T_2$ (shown in Figure 4) respectively, embedded in the projection corresponding to a minimal triple-crossing diagram of $K$. Then we have $2c_3(K) - t_1 - t_2 \geq c_\Delta(K)$.

For a general context, we start with a survey (where the inequality in Theorem 1.1 fits with the label $\ast$), giving concise information about known inequalities between integer-valued classical knot invariants. We give the visual graph of relations, examples showing that some invariants are non-comparable in general, and show selected relations that are not resolved with a direction of possible inequality.

We present the graphs of a selected invariant inequalities, for any knot $K \hookrightarrow S^3$. Where, (in the first diagram) an arrow $\rightarrow$ means $\leq$, (in the second diagram) an arrow with double arrowheads $\leftrightarrow$ means that there are examples of knots where we have $>$ relation and examples of knots where we have $<$ relation, the arrows $\rightarrow$ with label $\text{conj.}$ are relation where we don’t know examples where we might have $>$ relation.

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In the above graph, the invariants, are: $c$ is the crossing number, $g$ is the (Seifert) three-genus, $g_f$ is the free genus, $g_c$ is the canonical genus, $u$ is the unknotting number, $u_b$ is the band-unknotting number, $ul_b$ is the band-unlinking number, $g_4$ is the slice genus, $g_r$ is the ribbon slice genus, $g_{ds}$ is the doubly slice genus, $\sigma$ is the knot signature, $\tau$ is the Ozsvath-Szabo’s Tau-Invariant, $s$ is the Rasmussen’s s-invariant, $sp\Delta_t$ is the span of the Alexander polynomial $\Delta(t)$, $spV_i$ is the span of the Jones polynomial $V(t)$, $degP_z$ is the z-degree of the HOMFLY-PT polynomial $P(v,z)$, $cl_4$ is the 4D clasp number, $cl$ is the clasp number, $\nu^+$ is the Hom and Wu’s invariant, $u_s$ is the slicing number, $td$ is the skein tree depth, $tr$ is the trivializing number, $u_c$ is the concordance unknotting number, $u^*_r$ is the weak ribbon unknotting number, $\epsilon\Sigma$ is the ”double-cover” epsilon invariant, $Ord_v$ is the torsion order.

One find precise definitions of those invariants and inequalities as follows. Relation 1 in [4], 2, 9 in [2], 3 in [12], 5 in [17], 6 in [19], 7 in [20], 8 in [7], 4, 11, 19, 25, 29, 35, 30 in [29], 26 in [8], 13 in [22], 14 in [26], 17, 18, 33 in [15], 15, 16 in [11], 20, 21 in [16], 22 in [25], 23, 24 in [10], 12, 31 in [24], 27 in [28], 10, 28 in [9], 32 in [1], 34 in [6].

The following invariants cannot be related to each other by an inequality for any knot.

1. $u$ and $g$, for example $u(6a2) < g(6a2)$, $u(7a6) > g(7a6)$;
2. $c_3$ and $2cl$, for example $c_3(9a40) < 2cl(9a40)$, $c_3(5a1) > 2cl(5a1)$;
3. $c_3$ and $2u$, for example $c_3(9a40) < 2u(9a40)$, $c_3(5a1) > 2u(5a1)$;
4. $|\sigma|$ and $|s|$, for example $|\sigma(10n13)| < |s(10n13)|$, $|\sigma(9n4)| > |s(9n4)|$;
5. $|\sigma|$ and $2|\tau|$, for example $|\sigma(10n13)| < 2|\tau(10n13)|$, $|\sigma(9n4)| > 2|\tau(9n4)|$;
6. $g_{ds}$ and $2cl_4$, for example $g_{ds}(7a6) < 2cl_4(7a6)$, $g_{ds}(6a3) > 2cl_4(6a3)$;
7. $cl_4$ and $g$, for example $cl_4(6a2) < g(6a2)$, $cl_4(9a36) > g(9a36)$;
8. $sp_{\Delta_7}/2$ and $sp\Delta_7$, for example $sp_{\Delta_7}(7a7)/2 < sp\Delta_7(7a7)$, $sp_{\Delta_7}(7a4)/2 > sp\Delta_7(7a4)$;
9. $sp\Delta_7/2$ and $degP_z$, for example $sp\Delta_7(6a1)/2 < degP_z(6a1)$, $sp\Delta_7(6a3)/2 > degP_z(6a3)$;
10. $tr$ and $c_3$, for example $tr(9a36) < c_3(9a36)$, $tr(9n8) > c_3(9n8)$;
11. $sp\Delta_7$ and $2|\tau|$, for example $sp\Delta_7(12n293) < 2|\tau(12n293)|$, $sp\Delta_7(4a1) > 2|\tau(4a1)|$;
12. $2c_\Delta$ and $c$, for example $2c_\Delta(3a1) < c(3a1)$, $2c_\Delta(9a31) > c(9a31)$;
13. $2g$ and $degP_z$, $2g(K15n14891) > degP_z(K15n14891)$, for $2g < degP_z$ see [20].

In this paper, we also prove that we can always find at least one of set of four tangles in any triple-crossing projections of any non-trivial knot. In [23] the authors list up all delta-crossing diagrams of prime knots with the delta-crossing number equal to 4 and found delta-crossing diagrams of prime knots with the delta-crossing number equal to 4 for 52 prime knots. In Section 4 of this paper, we enumerate and generate tables of minimal delta-diagrams for all prime knots up to the delta-crossing number equal to 4.

2. Definitions

The projection of a knot or a link $K \subset \mathbb{R}^3$ is its image under the standard projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ (or into a 2-sphere) such that it has only a finite number of self-intersections, called multiple-points, and in each multiple-point each pair of its strands are transverse.

If each multiple-point of a projection has multiplicity three then we call this projection a triple-point projection. The triple-crossing is a triple-point crossing with the strand labeled $T$, $M$, $B$, for top, middle and bottom.

The triple-crossing diagram is a triple-crossing projection such that each of its triple points is a double-crossing, such that $\pi^{-1}$ of the strand labeled $T$ (in the neighborhood of that triple point) is on the top of the strand corresponding to the strand labeled $M$, and the latter strand is on the top of the strand corresponding to the strand labeled $B$ (see Figure 1).

Figure 1. A deconstruction/construction of a triple-crossing.

A delta-crossing is defined to be a tangle of three arcs with three double-crossings as in Figure 2, which are appeared in a \( \Delta \)-move (or \( \Delta \)-unknotting operation [21]). A delta-crossing diagram is a double-crossing diagram that can be decomposed into delta-crossing tangles joined by simple arcs.

The triple-crossing number of a knot or link $K$, denoted $c_3(K)$, is the least number of triple-crossings for any triple-crossing diagram of $K$. The minimal triple-crossing
Figure 2. Two types of a unoriented delta-crossing.

A diagram of a knot $K$ is a triple-crossing diagram of $K$ that has exactly $c_3(K)$ triple-crossings. We define the delta-crossing number of a knot $K$, denoted by $c_\Delta(K)$, as the least number of delta-crossings among all delta-crossing diagrams of $K$.

A natural orientation on a triple-crossing diagram or a delta-crossing diagram is an orientation of each component of that link, such that in each triple-crossing or a delta-crossing the (outer) strands are oriented in-out-in-out-in-out, as we encircle the crossing boundary. It is known ([3, 23]) that every orientation of the triple-crossing diagram or delta-crossing diagram obtained from an oriented knot is the natural orientation.

3. Bound on delta-crossing number

Proof of Theorem 1.1. Let $D^\Delta$ be a minimal delta-crossing diagram of a link $K$ with $n = c_\Delta$ number of delta-crossings. We can resolve $D^\Delta$ to a triple-crossing diagram with $2n$ triple-crossing by locally performing resolution of each delta-crossing into two triple-crossings as in Figure 3 (the other case of delta-crossing is analogous).

Figure 3. Transforming a delta-crossing triple-crossing into two triple-crossings.

Because of equality of upper- and lower-bounds, an exact values of (up to now unknown) triple-crossing numbers for knots, such as the following can be obtained.

Corollary 3.1.

$$c_3(9a5) = c_3(9a13) = c_3(9a16) = 6.$$  

Proof. The above corollary follows from the facts that: results in Table 3 show that all these knots have the delta-crossing number equal to 3; the inequality in Theorem 1.1 tells us that all these knots have the triple-crossing number equal at most 6; finally all these knots are not in the table of knots with the triple-crossing number equal at most 5, obtained in [13].

Theorem 1.2 follows from the following theorem.
Theorem 3.2. Let $D^t$ be a triple-crossing diagram of a link $L$ with $n$ triple-crossings and let $t_1, t_2$ be the number of disjoint tangles $T_1, T_2$ (shown in Figure 4) respectively embedded in the graph corresponding to $D^t$. Then

$$2n - (t_1 + t_2) \geq c_\Delta(L).$$

Figure 4. Triple-point tangles $T_1, T_2, T_3, T_4$

Proof. From [23] we know that $D^t$ can be resolved to a delta-crossing diagram with $2n$ delta-crossing by locally performing resolution of each triple-crossing into two double-crossings as in Figure 5 (the other case exchanging $T$ with $B$ is analogous).

Figure 5. Transforming a triple-crossing into two delta-crossings.

To prove the theorem it is sufficient to show that any tangle from $T_1, T_2, T_3$ (with $r$ triple-crossings) in $D^t$ can be resolved to a delta-crossing tangle with at most $2r - 1$ delta-crossings (we resolve the other triple-crossings not in any of the tangles as in Figure 5).

We can transform a tangle $T_1$ to one delta-crossing tangle as in Figure 6, the other case exchanging $T$ with $B$ is analogous. If in the tangle, the strand that is not forming the loop are not the middle strand, then the crossing can be eliminated by the standard Reidemeister moves.

We can transform a tangle $T_2$ to at most three delta-crossings as in Figure 7 (the other case changing the labels $T, M$ and $B$ is analogous or resolves both triple-crossings to zero crossings).
Figure 6. Transforming a triple-crossing with loop.

Figure 7. Transforming two triple-crossings.

The tangles $T_1$ or $T_2$ do not always exist in any triple-crossing projection. However, we prove that we can always find at least one tangle from the following set of four tangles.

**Theorem 3.3.** Let $D^t$ be a triple-crossing diagram of a non-trivial knot or non-split link $K$ and let $t_1, t_2, t_3, t_4$ be the number of disjoint tangles $T_1, T_2, T_3, T_4$ respectively (shown in Figure 4) embedded in the graph corresponding to $D^t$. Then $t_1 + t_2 + t_3 + t_4 \geq 1$.

**Proof.** Denote by $n$ the number of crossings in $D^t$, and consider the projection of $D^t$ as a planar graph $G$. Let $f_i$ be the number of faces of $G$ with $i$ edges (including the outer region), from [3] we know that

$$2f_1 + f_2 = 6 + f_4 + 2f_5 + 3f_6 + 4f_7 + \ldots.$$  

From the graph theory (using also the Euler formula) we have that in $G$ the number of vertices $|V(G)| = n$, the number of edges $|E(G)| = \frac{1}{2}(f_1 + 2f_2 + 3f_3 + 4f_4 + \ldots) = 3n$ and the number of faces $|F(G)| = f_1 + f_2 + f_3 + f_4 + \ldots = 2n + 2$.

Assume the contrary that $t_1 = t_2 = t_3 = t_4 = 0$. Then $f_1 = 0$, so we have

$$f_2 = 6 + f_4 + 2f_5 + 3f_6 + 4f_7 + \ldots \geq 6 + f_4 + f_5 + f_6 + f_7 + \ldots = 6 + |F(G)| - f_2 - f_3 = 2n + 8 - f_2 - f_3.$$
Therefore $2f_2 + f_3 \geq 2n + 8$.

Now we count the maximal number of bigons in $G$ avoiding having tangles $T_2, T_3, T_4$. To each 4-gon there are at most 2 adjacent bigons, to each 5-gon there are at most 3 adjacent bigons, to each 6-gon there are at most 4 adjacent bigons, ..., to each $m$-gon there are at most $\left\lfloor \frac{2m}{3} \right\rfloor$ adjacent bigons.

Because each bigon is adjacent to two different $m$-gons (for $m > 3$) we have

\[
2f_2 \leq 2f_4 + 3f_5 + 4f_6 + 4f_7 + \ldots + \left\lfloor \frac{2m}{3} \right\rfloor f_m + \ldots \leq 2f_4 + 3f_5 + 4f_6 + 5f_7 + \ldots + (m - 2)f_m + \ldots = 2|E(G)| - 2|F(G)| - f_3 = 2n - 4 - f_3 \leq 2f_2 - 12.
\]

A contradiction, therefore $t_1 + t_2 + t_3 + t_4 \geq 1$.

\[\square\]

Corollary 3.4. If a projection of a minimal triple-crossing diagram of non-trivial knot $K$ does not have embedded tangles neither $T_3$ nor $T_4$, then $2c_3(K) - 1 \geq c_\Delta(K)$.

4. Knot tabulation

We generate here the table of prime knot diagrams with the delta-crossing number up to four. First notice that we can obtain every delta-crossing diagram from a triple-point diagram because every triangle in a delta-crossing can be homotopically contracted to a point forming a triple-point. The tabulation of delta diagrams for just one delta-crossing is very easy so we may from now on assume that we have at least two delta-crossings.

Then, we take the tables of all prime, connected, oriented, triple-point projections, such that each pair of shadows are neither isotopic nor is one of them isotopic to the mirror image of the other. This sets are taken from [14] and called there $Tb_n$, where $n > 1$ is the number of triple-points. From these sets we take only those that are projections of the knots, for $n = 2, 3, 4$ we have 1, 5, 65 projections respectively.

Because every orientation of the delta-crossing diagram is the natural orientation, we can then resolve each triple-point to four possible delta-crossings $S, T, U, W$, as in Figure 8. This gives us $4^2 + 5 \cdot 4^3 + 65 \cdot 4^4 = 16976$ delta-diagrams to identify. We tabulate knots up to mirror images so we only need half the number of the diagrams.

![Figure 8. Four types of delta-crossing.](image)

We identify each of the diagrams with the similar method as in our previous paper [13], but now it turns out that we need much stronger invariants. To identify types
of knots, we use mostly the HOMFLY-PT polynomial and the Khovanov homology (calculated with [27]), we also use the knot Floer homology and Volume of the knot exterior (calculated with [5]) where they are needed. After that there were 12 diagrams left for each of the trouble pairs (11n76 and 11n78), (11n71, 11n75) and (11a44, 11a47) to identify, to do this we used diagrammatic manipulations (using [18] and verified using [30]). We also noticed that the quadruple

\[ F(K) = (c(K), \text{HOMFLY-PT}(K), \text{Khovanov homology}(K), \text{knot Floer homology}(K)) \]

does not determine the delta-crossing number \( c_\Delta \), because for example

\[ F(11a104) = F(11a168) \]

but

\[ c_\Delta(11a104) = 4 \quad \text{and} \quad c_\Delta(11a168) \neq 4. \]

The number of knots and their names with a specific delta-crossing number is presented in Table 3. We encode each delta-diagram as a \( dPD \) code as a list in the format presented by an example in Figure 9.

\`K11n30` = ['T', [8, 2, 9, 9, 1, 10], 'U', [12, 3, 5, 1, 2, 6], 'W', [6, 4, 11, 7, 3, 12], 'U', [7, 11, 4, 8, 10, 5]]

**Figure 9.** A delta-diagram of a knot K11n30 and its dPD code.

Minimal diagrams of (unoriented) prime knots with delta-crossing number up to three are presented (in the form of its dPD codes) in Table 4, the case for delta-diagrams with four delta-crossing can be found in the \texttt{delta4.txt} file in the \LaTeX \ source file of this article’s \texttt{arXiv} preprint version. In that archive there are also text files of \( sPD \) codes of all the mentioned triple-crossing projections from \( Tb_n \) sets.
Table 3: Enumeration of knots.

| Δ-crossings | # knots | name of knots |
|-------------|---------|---------------|
| 1           | 1       | 3a1.          |
| 2           | 4       | 4a1, 5a1, 5a2, 6a1. |
| 3           | 21      | 6a2, 6a3, 7a1, 7a2, 7a3, 7a4, 7a5, 7a6, 7a7, 8a1, 8a2, 8a3, 8a4, 8a6, 8a7, 8n1, 8n2, 8n3, 9a5, 9a13, 9a16. |
| 4           | 320     | 8a5, 8a8, 8a9, 8a10, 8a11, 8a12, 8a13, 8a14, 8a15, 8a16, 8a17, 8a18, 9a1, 9a2, 9a3, 9a4, 9a6, 9a7, 9a8, 9a9, 9a10, 9a11, 9a12, 9a14, 9a15, 9a17, 9a18, 9a19, 9a20, 9a21, 9a22, 9a23, 9a24, 9a25, 9a26, 9a27, 9a28, 9a29, 9a30, 9a32, 9a33, 9a34, 9a35, 9a36, 9a38, 9a39, 9a40, 9a41, 9n1, 9n2, 9n3, 9n4, 9n5, 9n6, 9n7, 9n8, 10a3, 10a4, 10a5, 10a6, 10a7, 10a8, 10a9, 10a10, 10a11, 10a12, 10a13, 10a14, 10a15, 10a16, 10a17, 10a18, 10a19, 10a20, 10a21, 10a25, 10a26, 10a28, 10a29, 10a30, 10a31, 10a32, 10a33, 10a34, 10a36, 10a37, 10a38, 10a39, 10a40, 10a41, 10a42, 10a43, 10a44, 10a45, 10a47, 10a48, 10a49, 10a50, 10a51, 10a52, 10a55, 10a56, 10a57, 10a58, 10a63, 10a64, 10a66, 10a67, 10a68, 10a69, 10a72, 10a78, 10a79, 10a80, 10a85, 10a88, 10a93, 10a94, 10a99, 10a102, 10a103, 10a106, 10a107, 10a108, 10a109, 10a118, 10a121, 10n1, 10n2, 10n4, 10n5 10n6, 10n7, 10n8, 10n9, 10n10, 10n11, 10n12, 10n13, 10n15, 10n16, 10n17, 10n18, 10n19, 10n20, 10n21, 10n22, 10n23, 10n25, 10n26, 10n27, 10n28, 10n29, 10n30, 10n31, 10n32, 10n33, 10n34, 10n35, 10n36, 10n37, 10n38, 10n42, 11a11, 11a14, 11a18, 11a22, 11a23, 11a27, 11a30, 11a32, 11a35, 11a36, 11a40, 11a41, 11a43, 11a44, 11a46, 11a47, 11a50, 11a52, 11a77, 11a83, 11a85, 11a91, 11a94, 11a95, 11a100, 11a101, 11a104, 11a106, 11a107, 11a109, 11a11, 11a114, 11a117, 11a123, 11a134, 11a135, 11a136, 11a138, 11a148, 11a155, 11a175, 11a177, 11a178, 11a186, 11a191, 11a192, 11a197, 11a198, 11a200, 11a212, 11a327, 11a329, 11n1, 11n2, 11n16, 11n21, 11n22, 11n23, 11n24, 11n30, 11n50, 11n54, 11n55, 11n56, 11n61, 11n69, 11n71, 11n72, 11n73, 11n74, 11n75, 11n76, 11n77, 11n78, 11n82, 11n85, 11n86, 11n87, 11n90, 11n92, 11n93, 11n94, 11n95, 11n96, 11n105, 11n106, 11n107, 11n118, 11n119, 11n126, 11n136, 11n153, 11n156, 11n162, 11n169, 12a58, 12a99, 12a104, 12a119, 12a268, 12a273, 12a281, 12a295, 12a313, 12a323, 12a327, 12a345, 12a353, 12a426, 12a435, 12a499, 12a510, 12a514, 12a561, 12a615, 12a628, 12a629, 12a631, 12a633, 12a653, 12a656, 12a868, 12a875, 12a960, 12a1097, 12a1188, 12a1189, 12a1251, 12n41, 12n77 12n177, 12n188, 12n245, 12n289, 12n308, 12n326, 12n327, 12n328, 12n341, 12n379, 12n380, 12n406, 12n416, 12n417, 12n425, 12n426, 12n477, 12n503, 12n508, 12n518, 12n538, 12n549, 12n591, 12n592, 12n600, 12n609, 12n703, 12n706.
Table 4: Minimal diagrams of prime knots with delta-crossing number up to three

| knot  | dPD code of a minimal knot diagram                                                                 |
|-------|---------------------------------------------------------------------------------------------------|
| 3a1   | [W, [1, 3, 3, 2, 2, 1]]                                                                          |
| 4a1   | [S, [5, 2, 4, 6, 1, 5], W, [2, 1, 3, 3, 6, 4]]                                                    |
| 5a1   | [W, [5, 2, 4, 6, 1, 5], W, [2, 1, 3, 3, 6, 4]]                                                    |
| 5a2   | [W, [5, 2, 4, 6, 1, 5], U, [2, 1, 3, 3, 6, 4]]                                                    |
| 6a1   | [W, [5, 2, 4, 6, 1, 5], S, [2, 1, 3, 3, 6, 4]]                                                    |
| 6a2   | [S, [7, 2, 6, 8, 1, 7], W, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 6a3   | [S, [7, 2, 6, 8, 1, 7], W, [2, 1, 3, 3, 9, 4], W, [4, 9, 5, 5, 8, 6]]                        |
| 7a1   | [T, [4, 2, 5, 5, 1, 6], T, [7, 3, 8, 8, 2, 9], U, [9, 4, 6, 1, 3, 7]]                        |
| 7a2   | [U, [7, 2, 6, 1, 7], T, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                           |
| 7a3   | [W, [7, 2, 6, 8, 1, 7], W, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 7a4   | [W, [7, 2, 6, 8, 1, 7], W, [2, 1, 3, 3, 9, 4], W, [4, 9, 5, 5, 8, 6]]                        |
| 7a5   | [W, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], W, [9, 4, 6, 1, 3, 7]]                        |
| 7a6   | [W, [4, 2, 5, 5, 1, 6], W, [7, 3, 8, 8, 2, 9], W, [9, 4, 6, 1, 3, 7]]                        |
| 7a7   | [U, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], W, [9, 4, 6, 1, 3, 7]]                        |
| 8a1   | [T, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], U, [9, 4, 6, 1, 3, 7]]                        |
| 8a2   | [U, [7, 2, 6, 8, 1, 7], U, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 8a3   | [T, [7, 2, 6, 8, 1, 7], S, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 8a4   | [T, [7, 2, 6, 8, 1, 7], T, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 8a5   | [S, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], T, [9, 4, 6, 1, 3, 7]]                        |
| 8a6   | [T, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], T, [9, 4, 6, 1, 3, 7]]                        |
| 8a7   | [S, [7, 2, 6, 8, 1, 7], S, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 8n1   | [S, [7, 2, 6, 8, 1, 7], S, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 8n2   | [S, [7, 2, 6, 8, 1, 7], U, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 8n3   | [W, [7, 2, 6, 8, 1, 7], U, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 9a5   | [T, [7, 2, 6, 8, 1, 7], U, [2, 1, 3, 3, 9, 4], U, [4, 9, 5, 5, 8, 6]]                        |
| 9a13  | [U, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], T, [9, 4, 6, 1, 3, 7]]                        |
| 9a16  | [U, [4, 2, 5, 5, 1, 6], U, [7, 3, 8, 8, 2, 9], U, [9, 4, 6, 1, 3, 7]]                        |

References

[1] P. Aceto, M. Golla, and K. Larson. Embedding 3-manifolds in spin 4-manifolds, Journal of Topology 10 (2017), 301–323.
[2] C. Adams, Triple crossing number of knots and links, J. Knot Theory Ramifications 22 (2013), 1350006.
[3] C. Adams, J. Hoste and M. Palmer, Triple-crossing number and moves on triple-crossing link diagram, J. Knot Theory Ramifications 28 (2019), 1940001.
[4] P. Cromwell, Homogeneous links, Journal of the London Mathematical Society 2 (1989) 535–552.
[5] M. Culler, N.M. Dunfield, M. Goerner and J.R. Weeks, SnapPy, a computer program for studying the geometry and topology of 3-manifolds, (Version 3.0.3) available at http://snappy.computop.org (2022).
[6] P. Feller, The degree of the Alexander polynomial is an upper bound for the topological slice genus, Geometry & Topology 20 (2016), 1763–1771.
[7] C.A. Giller, A family of links and the Conway calculus, Transactions of the American Mathematical Society 270 (1982) 75–109.
[8] R. Hanaki, Trivializing number of knots, Journal of the Mathematical Society of Japan 66 (2014), 435–447.
[9] A. Henrich, N. MacNaughton, S. Narayan, O. Pechenik and J. Townsend, Classical and virtual pseudodiagram theory and new bounds on unknotting numbers and genus. Journal of Knot Theory and Its Ramifications 20 (2011), 625–650.
[10] J. Hom and Z. Wu, Four-ball genus bounds and a refinement of the Ozsváth-Szabó tau-invariant, Journal of Symplectic Geometry 14 (2016), 305–323.
[11] J. Hoste, Y. Nakanishi, and K. Taniyama, Unknotting operations involving trivial tangles, Osaka Journal of Mathematics 27 (1990) 555–566.
[12] M. Jabłonowski, Triple-crossing number, the genus of a knot or link and torus knots, Topology and its Applications 285 (2020), 107389.
[13] M. Jabłonowski, Tabulation of knots up to five triple-crossings and moves between oriented diagrams, to appear in Tokyo Journal of Mathematics (2022).
[14] M. Jabłonowski and Ł. Trojanowski, Triple-crossing projections, moves on knots and links, and their minimal diagrams, J. Knot Theory Ramifications 29 (2020), 2050015.
[15] A. Juhász, M. Miller, and I. Zemke, Knot cobordisms, bridge index, and torsion in Floer homology, Journal of Topology 13 (2020), 1701–1724.
[16] L.P. Karageorghis and F. Swenton, Determining the doubly slice genera of prime knots with up to 12 crossings, J. Knot Theory Ramifications 30 (2021), 2150057.
[17] M.Kobayashi and T. Kobayashi. On canonical genus and free genus of knot. J. Knot Theory Ramifications 5 (1996) 77—85.
[18] K. Miller, KnotFolio, https://kmill.github.io/knotfolio/ (2022).
[19] Y. Moriah, On the free genus of knots, Proceedings of the American Mathematical Society (1987), 373–379.
[20] H.R. Morton, Seifert circles and knot polynomials, Mathematical Proceedings of the Cambridge Philosophical Society 99 (1986), 247–260.
[21] H. Murakami, and Y. Nakanishi, On a certain move generating link-homology, Mathematische Annalen 284.1 (1989), 75–89.
[22] K. Murasugi, On a certain numerical invariant of link types, Transactions of the American Mathematical Society 117 (1965), 387–422.
[23] Y. Nakanishi, Y. Sakamoto and S. Satoh, Delta-crossing number for knots, Topology and its Applications 196 (2015), 771–776.
[24] B. Owens and S. Strle. Immersed disks, slicing numbers and concordance unknotting numbers, Communications in Analysis and Geometry 24 (2016), 1107–1138.
[25] P. Ozsváth and Z. Szabó, Knot Floer homology and the four-ball genus, Geom. Topol. 7 (2003), 615–639.
[26] J. Rasmussen, Khovanov homology and the slice genus, Inventiones mathematicae 182 (2010), 419–447.
[27] SageMath, the Sage Mathematics Software System (Version 9.3), The Sage Developers, https://www.sagemath.org (2022).
[28] M. Scharlemann and A. Thompson, Link genus and the Conway moves, Comment. Math. Helv. 64 (1989), 527–535
[29] T. Shibuya, Some relations among various numerical invariants for links, Osaka Journal of Mathematics 11 (1974), 313–322.
[30] F.J. Swenton, Kirby calculator (v0.973a) http://community.middlebury.edu/~mathanimations/klo/ (2022)

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