How non-equilibrium correlations in active matter reveal the topological crossover in glasses

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As shown by early studies on mean-field models of the glass transition, the geometrical features of the energy landscape provide fundamental information on the dynamical transition at the Mode-Coupling temperature $T_d$. We show that active particles can serve as a useful tool for gaining insight into the topological crossover in model glass-formers. In such systems the landmark of the minima-to-saddle transition in the potential energy landscape, taking place in the proximity of $T_d$, is the critical slowing down of dynamics. Nevertheless, the critical slowing down is a bottleneck for numerical simulations and the possibility to take advantage of the new smart algorithms capable to thermalize down in the glass phase is attractive. Our proposal is to consider configurations equilibrated below $T_d$ and study their dynamics in the presence of a small amount of self-propulsion. As exemplified here from the study of the $p$-spin model, the presence of self-propulsion gives rise to critical off-equilibrium equal time correlations at the minima-to-saddles crossover, correlations which are not hindered by the sluggish glassy dynamics.

I. INTRODUCTION

The detection of order in glass-forming systems has always been elusive. The first progress was made looking at non-trivial correlations in dynamical heterogeneities more than twenty years ago [1]. Shortly after the attention concentrated also on static multipoint correlations [2,4]. According to mean-field-like scenarios, dynamical correlations are expected to be critical at the Mode-Coupling temperature $T_d$ [8–10], whereas static multipoint correlations are expected to diverge at the ideal glass transition temperature $T_K$, where $T_K < T_d$ [2,9,10]. The behaviour of static and dynamic length-scales in glass-formers is of great interest. But, at the same time, its precise characterization is plagued by great difficulties: numerical simulations provide only indirect evidence for the existence of dynamical singularities, which are well defined strictly speaking only in the thermodynamic limit. At the same time, the study of multipoint correlations based on theoretically well defined but practically hard to realize protocols [2,4,6,7]: in experiments their existence cannot be measured directly but only inferred from the behaviour of higher-order susceptibilities [11,12].

What makes the search for criticality at $T_d$ particularly interesting is the lack of any sort of configurational order in glasses. The amorphous order of glasses, being either static or dynamic, usually requires the comparison of different configurations to be detected. In particular, it cannot be read off from a single snapshot of the system. The only remarkable exception to the above situation is represented by the patch correlation length [13–14], which nevertheless requires knowledge of microscopic details of the systems which are usually out of reach. The goal of our proposal is to compensate for this lack of configurational information by putting the system slightly out of equilibrium without perturbing too much its landscape. We will show how, gently pushing the system out of equilibrium, one can gain information on the topological transition from stationary non-equilibrium velocity correlations.

For a given configuration of the system, low-energy excitations provide a useful tool for distinguishing whether a disorder configuration belongs to a liquid or glass. In particular, linearly unstable configurations, characterized by the presence of negative eigenvalues in the spectrum, arise in the supercooled liquid phase as the temperature increases above $T_d$. Amorphous solid configurations below $T_d$ are on the contrary close to local minima of the potential energy landscape, so that the corresponding eigenvalues of the Hessian matrix are all non-negative. This picture emerges clearly from mean-field models [15–19], where the presence of a crossover from a high-temperature saddle-dominated phase to a low-temperature minima-dominated one is an analytical result.

The geometrical features of the energy landscape, which are known to influence on equal-time velocity correlations [20], play an important role also in Active Matter [21]. Recent studies show that active particles develop non-trivial interparticle velocity correlations [20,22,26,27] and equal-time velocity correlations have to be taken into account also for developing a mode-coupling theory of active parti-
Those correlations play an important role in rationalizing the emergence of collective patterns in dense active systems \[20, 31\]. Quite remarkable are also the investigations of static multi-points correlations and amorphous order in dense active systems \[32\].

The present work aims to test whether the presence of active dynamics and the corresponding stationary non-equilibrium correlations offer an extra tool to detect the minima-to-saddles crossover in glass-forming liquids, rather than being regarded just as a disturbance to the natural tendency of the system towards a glassy arrested state \[28, 29, 33–35, 37, 38\].

**II. THE POTENTIAL ENERGY LANDSCAPE OF GLASS-FORMERS**

It is well known that for glass-forming systems the crossover to activated relaxation below \(T_d\) corresponds to a topological crossover in the landscape \[15\] when \(T > T_d\) equilibrium configurations are typically close to an unstable stationary point of the potential energy while at low temperatures, \(T < T_d\), equilibrium configurations are typically close to a minimum \[15\] \[18\] \[39\] \[40\].

In particular, considering a system composed of \(N\) particles interacting via a translational invariant pair potential \(U(|\mathbf{x}_i - \mathbf{x}_j|)\) and where the total potential energy is \(\Phi = \sum_{i<j} U(|\mathbf{x}_i - \mathbf{x}_j|)\), the stability of a given configuration depends on the distribution of the eigenvalues of the Hessian matrix \(\mathbb{H} = \partial^2 U / \partial x_i^\alpha \partial x_j^\beta\), where latin indices denotes particles and greek indices cartesian components in \(d = 3\) space. When all the eigenvalues of \(\mathbb{H}\) are positive, the system is in the minima-dominated region: in mean-field models ergodicity is broken dynamically. As soon as a fraction of negative eigenvalues appears, relaxation starts to take place along the unstable directions of the Hessian. The evidence of a topological transition could be in principle obtained from the study of the landscape in equilibrium configurations \[17\]. But this is in practice impossible since it requires to know microscopic details, in particular particles positions, which are usually not accessible in experiments. Our goal is to show that stationary non-equilibrium velocity correlations, measurable even at a coarse-grained scale, are already carrying information on the landscape structure.

**III. ACTIVE DYNAMICS**

We first consider, as the most generic case, an off-lattice systems made of \(N\) particles interacting with the pair potential \(U(|\mathbf{x}_i - \mathbf{x}_j|)\). Since the model system we have in mind is a colloidal glass or, more in general, a dense active system at low Reynolds numbers, we consider overdamped equation of motions \[41\]. The choice between overdamped or inertial dynamics is of course in general not arbitrary, it depends on the system and on the particular phenomena one is interested to study. Let us consider, for instance, the importance of inertia in the modelization of starling flocks dynamics \[42, 43\]. In the present case we are not interested in phenomena such as the transmission of information across the system \[42, 43\], for which the choice among overdamped or inertial dynamics is determinant, but rather on the stationary probability distribution of velocities in a dense state. Therefore we do not have a particular reason to abandon the standard fixed by the literature on dense glass-forming systems, i.e., overdamped dynamics, in favour of inertial dynamics. Moreover, in absence of confinement and on long enough time scales, our discussion might be suitable also for inertial active particles \[44\] \[49\]. Setting the mobility to \(\mu = 1\), we consider the active dynamics characterized by the following equations

\[
\ddot{\mathbf{x}}_i = - \nabla_i \Phi + \mathbf{f}_i \tag{1}
\]

with \(\mathbf{f}_i\) the active force acting on each particle. Depending on the system of interest, different prescriptions for the dynamical evolution of the active force are possible: Run-and-Tumble (RT) dynamics in case of swimming bacteria as E. coli \[47\], Active Brownian (AB) dynamics for active colloids \[48\] or Active Ornstein-Uhlenbeck (AOU) dynamics, which describes well the motion of passive objects in active baths \[49\]. Despite they are microscopically different, all these models capture the same phenomenology on large scales. Particularly interesting is the case of AOU particles that admits an effective equilibrium picture \[20, 21, 50–55\]. In AOU models, the dynamics of the self-propulsion force is an Ornstein-Uhlenbeck process with characteristic timescale \(\tau\):

\[
\dot{\mathbf{f}}_i = - \frac{1}{\tau} \mathbf{f}_i + \eta_i \tag{2}
\]

where the stochastic drive \(\eta_i\) satisfies

\[
\langle \eta_i^\alpha \rangle = 0 \tag{3}
\]

\[
\langle \eta_i^\alpha (t) \eta_j^\beta (s) \rangle = \frac{2T}{\tau^2} \delta_{ij} \delta_\alpha^\beta \delta(t-s). \tag{4}
\]

Because of Eq. (2), the self-propulsion force is actually a coloured noise with exponential kernel:

\[
\langle f_i^\alpha (t) f_j^\beta (s) \rangle = \frac{2T}{\tau} \delta_{ij} \delta_\alpha^\beta e^{-|t-s|/\tau} \tag{5}
\]

The control parameters of the model are the correlation time of the noise \(\tau\), that determines the persistence time of the self-propelled motion and its amplitude \(T\). As shown in Ref. \[20\], under suitable assumptions the stationary probability distributions of velocities for a given steady-state configuration of the system can be written as a Gaussian distribution whose covariance matrix depends parametrically on the configuration:

\[
P(\dot{\mathbf{x}} | \mathcal{X}) = N \times e^{-\frac{1}{2} \dot{\mathbf{x}}^T \mathbf{\Gamma} \dot{\mathbf{x}}} \tag{6}
\]

\[
\mathbf{\Gamma} \equiv 1 + \tau \mathbb{H}, \tag{7}
\]
where $\mathbf{1}$ is the $3N \times 3N$ identity matrix, $\mathcal{N}$ a normalisation constant, and $\mathcal{X} \equiv (x_1, ..., x_N)$. In particular, given the Gaussian form of the conditional probability distribution in Eq. (6), the correlation between the velocities of different particles reads as

$$
\langle \dot{x}_i^\alpha \dot{x}_j^\beta \rangle = (\mathbf{\Gamma}^{-1})_{ij}^{\alpha\beta}. \tag{8}
$$

The beautiful insight suggested by Eq. (8) is that such off-equilibrium correlations are related with the non-diagonal elements of the Hessian matrix. It is also clear from Eq. (6),(8) that in the equilibrium limit all such correlations become trivial, as they have to

$$
\lim_{\tau \to 0} \langle \dot{x}_i^\alpha \dot{x}_j^\beta \rangle \propto \delta_{ij}\delta_{\alpha\beta}. \tag{9}
$$

The most important condition under which the conditional probability in Eq. (9) is well defined and the expression in Eq. (8) holds for all values of the persistence time $\tau$ is to have a positive-defined Hessian matrix. Something which is true, in the case of standard interaction potentials, for low temperature quasi-crystalline states or for glassy arrested ones. As soon as unstable directions arise in the Hessian, e.g., for typical liquid state configurations, one is bounded to small enough values of the persistence time $\tau$ for the above expression of $P(\mathcal{X}|\mathcal{X})$ to be valid. It is only when $\tau = 0$ that $P(\mathcal{X}|\mathcal{X})$ is always well defined, namely both in the arrested and in the liquid phase. But with $\tau = 0$ velocity correlations are trivial, see Eq. (9). Alternatively one should look for more refined, but also more difficult to derive rigorously, non-Gaussian forms of $P(\mathcal{X}|\mathcal{X})$, for instance the one proposed at the end of Sec. [V]. To the purpose of the present analysis the coloured noise characteristic time-scale $\tau$ is much more relevant with respect to its amplitude, since the consistency of the velocity distribution $P(\mathcal{X}|\mathcal{X})$ as written in Eq. (6) depends solely on $\tau$. This notwithstanding we will play with both parameters: after having equilibrated the system at a given temperature, applying active noise of small amplitude is functional to not alter the equilibrium landscape, while a large enough value of $\tau$ allow to have non-trivial non-equilibrium correlations.

The key interesting feature of the conditional probability $P(\mathcal{X}|\mathcal{X})$ written in Eq. (6) is the following: it allows to relate in a very transparent manner non-equilibrium correlations and the equilibrium topology of a complex liquid energy landscape. Our analysis will consist in studying the properties of these correlations in a phase where the Hessian is positive-definite, namely the glassy arrested state, and in drawing some conclusions on the possible behaviour in the liquid phase, where the amplitude of velocity fluctuations becomes apparently unbounded according to the expression of $P(\mathcal{X}|\mathcal{X})$ in Eq. (6). We will start by illustrating the situation in an exactly-solvable model where the minima-to-saddles topological transition is an analytical result in the large-$N$ limit.

### IV. ACTIVE $p$–SPIN

The connections between off-equilibrium non-diagonal velocity correlations and the eigenvalues spectrum of potential energy Hessian are clearly illustrated by a model which can be solved exactly in the mean-field approximation: a non-equilibrium version of the disordered spherical $p$-spin model. Despite recent results showed that the $p$-spin model with non-homogeneous interaction potentials better captures some features of the aging dynamics of realistic glass-formers [36] (e.g., polydisperse mixtures with Lennard-Jones potentials), we stick here to the "traditional" $p$-spin with homogeneous interactions as the simplest model to understand the more fundamental properties of driven glassy systems. Since the present one is the first attempt to relate landscape topology to properties of correlations in a stationary non-equilibrium regime, the simplest version of the $p$-spin model should be already good enough. A driven version of the homogeneous $p$-spin has been for instance considered also in [37], though with a different driving mechanism. But, before dwelling on the specific non-equilibrium version of the model, let us first recall its equilibrium properties. The $p$-spin model is characterized by the following disordered Hamiltonian and global constraint:

$$
\mathcal{H}_{J}[\sigma] = - \sum_{i_1 < i_2 < ... < i_p} J_{i_1i_2...i_p} \sigma_{i_1}\sigma_{i_2}...\sigma_{i_p}
$$

$$
\sum_{i=1}^{N} \sigma_i^2 = N, . \tag{10}
$$

where the sum in $\mathcal{H}_{J}[\sigma]$ runs over all independent $p$-plets of indices, $J_{i_1i_2...i_p}$ are random normal variates with zero mean and variance $(J^2) \sim 1/N^{p-1}$ (guaranteeing energy extensivity). It is well known that equilibrium configurations of the $p$-spin sampled with Boltzmann measure

$$
P_{J}[\sigma] = e^{-\beta\mathcal{H}_{J}[\sigma]} \delta \left( \sum_{i=1}^{N} \sigma_i^2 - N \right) \tag{11}
$$

are typically close to stationary points of the energy hypersurface and that the topology of these stationary points depends on the temperature [10]. In particular, for $T < T_{\text{d}}$ equilibrium configurations are close to energy minima while for $T > T_{\text{d}}$ they are close to saddle points. This correspondence works because in the large-$N$ limit this model has a bimodal correspondence between energy levels and temperatures, due to the self-averaging property of the energy:

$$
\lim_{N \to \infty} E_{J,\beta} = \overline{E_{J,\beta}} = E_{\beta} \tag{12}
$$

where $E_{J,\beta} = \langle H_J \rangle_{\beta}$ denotes the thermodynamic average at fixed disorder while the overline denotes the average over disorder instances, i.e., over the random couplings $J_{i_1i_2...i_p}$. It is then known that in the $p$–spin model the
Hessian at the stationary point of the energy landscape is a GOE matrix \([56][59]\). From this property follows that the distribution of Hessian eigenvalues in the large-\(N\) limit, \(\rho(\lambda, E_{\beta,\delta})\), is a self-averaging quantity and follows the Wigner semicircle law \([56][59]\):

\[
\lim_{N \to \infty} \rho(\lambda, E_{\beta,\delta}) = \rho(\lambda, E_{\delta}) = \frac{1}{\pi p (p-1)} \sqrt{p^2 E_{\text{th}}^2 - (\lambda + p E_{\delta})^2},
\]

with \(E_{\text{th}} = -\sqrt{2(p-1)/p}\) the energy where the topological transition between a minima-dominated to a saddle-dominated landscape takes place. The eigenvalues spectrum of the \(p\)-spin Hessian is represented in Fig.1 for values of the energy \(E_{\delta}\) at the threshold, \(E_{\delta} = E_{\text{th}}\) (black dotted line), values below the threshold, \(E_{\delta} < E_{\text{th}}\) (continuous blue line, right) and values above the threshold, \(E_{\delta} > E_{\text{th}}\) (continuous red line, left).

From Eq. (13) and Fig.1 it is clear that all eigenvalues are positive as long as energy is below the threshold, \(E_{\delta} < E_{\text{th}}\). By increasing \(E_{\delta}\) the eigenvalue distribution shifts to the left and as soon as \(E_{\delta} = E_{\text{th}}\) negative eigenvalues appear. We show here how the appearing of unstable directions in the Hessian at \(E_{\text{th}}\) directly affects the behaviour of stationary non-equilibrium velocity correlations by considering the following protocol. We define a Langevin dynamics characterized by the presence of a persistent noise on the spins:

\[
\dot{\sigma}_i(t) = -\mu \sigma_i - \frac{\partial \mathcal{H}_J}{\partial \sigma_i} + f_i,
\]

where the Lagrange multiplier \(\mu\) enforces the spherical constraint and the active driving force \(f_i\) evolves according to an Ornstein-Uhlenbeck process

\[
\dot{f}_i(t) = -\frac{1}{\tau} f_i + \eta_i.
\]

First, it is convenient to set \(\tau = 0\) and sample the equilibrium distribution \(\exp(-\beta J(\sigma))\). Then, one switches on “memory” in the noise, i.e., sets \(\tau > 0\), and allows the persistent dynamics to run [Eqns. \((14-15)\)] until when a stationary distribution is reached. As long as the energy of the initial configuration is below the threshold, i.e., \(E_{\delta} < E_{\text{th}}\), active dynamics stays close to the bottom of an energy minimum: in the \(p\)-spin model barriers are extensive and dynamical fluctuations, either thermal or active, cannot help to jump barriers in the thermodynamic limit.

Let \(P_J(\sigma)\) be the equilibrium distribution probed by the Langevin dynamics when \(\tau = 0\), namely the one written in Eq. (11). The above protocol of drawing equilibrium configurations first and then run the active dynamics amounts to assume that the stationary joint distribution of \(\sigma\) and \(\dot{\sigma}\), consistently with the Unified Colored Noise approximation \([20][51][60][61]\), reads simply as:

\[
P_J(\sigma, \dot{\sigma}) = P_J^{(0)}(\dot{\sigma} | \sigma) P_J(\sigma)
\]

so that velocities cross correlations are related to the non-diagonal elements of the Hessian:

\[
E_{J,\sigma}[\dot{\sigma}_i \dot{\sigma}_j] = [\Gamma^{-1}(\sigma)]_{ij},
\]

where

\[
\Gamma_J(\sigma) = \int D\dot{\sigma} P_J(\dot{\sigma} | \sigma) \dot{\sigma}_i \dot{\sigma}_j.
\]

The last step is to consider the large-\(N\) limit, where, due to self-averaging, it is possible to write

\[
\lim_{N \to \infty} E_{J,\sigma}[\dot{\sigma}_i \dot{\sigma}_j] = \lim_{N \to \infty} \langle \mathcal{H} J, \sigma | \mathcal{H} J, \sigma \rangle = \langle (1 + \tau \mathcal{H})^{-1} \rangle_{ij},
\]

with

\[
\mathcal{H} = \langle \mathcal{H} J, \sigma \rangle,
\]

and where we have denoted with \(\langle \# \rangle\) the average over quenched randomness and with \(\langle \# \rangle\) the average over the reference configuration. For the \(p\)-spin model the matrix

\[
\Gamma_J(\sigma) = \int D\dot{\sigma} P_J(\dot{\sigma} | \sigma) \dot{\sigma}_i \dot{\sigma}_j.
\]
\(H_{j,\sigma}\) belongs to the GOE ensemble \([57, 59]\), so that in the large-\(N\) limit we have precisely the identity written in Eq. (22) and the spectrum of eigenvalues described by the expression in Eq. (13). From Eq. (21) we find that, as expected, in the equilibrium limit \(\tau \to 0\) the non-diagonal correlations among spin “velocities” vanish:

\[
\lim_{\tau \to \infty} \lim_{N \to \infty} \langle E_{j,\sigma} [\sigma_i \sigma_j]\rangle = \delta_{ij}. \tag{23}
\]

Then, in order to single out the behaviour of cross correlations when energy reaches the threshold value for the topological transition, i.e., when \(E = E_{th} - \delta E\) and \(\delta E \to 0\), it is convenient to consider the orthogonal transformation \(U : \mathbb{R}^N \to \mathbb{R}^N\) which diagonalizes \(H\). In the same limit where the identity of Eq. (21) holds we can write the probability distribution of velocities as:

\[
\lim_{N \to \infty} P_j(\delta \sigma) = P_j^{(\infty)}(\delta \sigma) \propto e^{-\frac{1}{2T} \| \delta \sigma \|^2 [1 + \tau H] \| \delta \sigma \|^2}. \tag{24}
\]

Writing the Hessian in its diagonal form, we can write the above distribution as

\[
P_j^{(\infty)}(\delta \sigma) \propto \exp \left\{ -\frac{1}{2T} \sum_{k=1}^{N} |\delta_k|^2 [1 + \tau \lambda_k] \right\}, \tag{25}
\]

where \(\delta_k = \sum_{i=1}^{N} U_{ki} \sigma_i\). From Eq. (25) a Gaussian integration leads to

\[
\mathbb{E}[\delta_k \delta_{-k}] = \frac{1}{1 + \tau \lambda_k}, \tag{26}
\]

from which it is easy obtained the divergence of cross-correlations among spin velocities when the lower band edge of the spectrum cross the origin, taken a not vanishing value of \(\tau\).

V. THE SIGNATURE OF CRITICALITY AT \(T_d\) FOR ACTIVE PARTICLES

It is quite reasonable to believe that the mean-field picture introduced in the previous section holds in first approximation even for the finite-dimensional system of interacting active particles described by Eq. (1) in Sec. I. We again consider the protocol according to which the configurations \(\mathcal{X}\) are first sampled according to the equilibrium Boltzmann distribution \(e^{-\beta H(\mathcal{X})}\) and then self-propelled motion is switched on with parameters \(\tau\) and \(T = \beta^{-1}\). We expect that the only relevant difference with the protocol outlined for the \(p\)-spin model is that now, since different minima of the potential might have finite barriers (particular close to \(E_{th}\)), one should consider a not too large amplitude \(T\) of the noise. A small value of \(T\) does not limit the scope of our analysis, since the noise amplitude only affects the amplitude of non-equilibrium correlations, not their range. The sampling of equilibrium configurations below \(T_d\) can be easily obtained using an on-purpose smart Monte-Carlo algorithm first proposed in [\textit{62}] and recently brought to the top of efficiency [\textit{63, 64}]. From Eq. (6) it follows that at thermodynamic equilibrium, namely when \(\tau = 0\), one has:

\[
P_{eq}(\mathbf{x} | \mathbf{x}) \propto e^{-\frac{1}{2T} \| \mathbf{x} \|^2} \implies \langle \mathbf{x}_i \mathbf{x}_j \rangle = 0 \ \forall \ i \neq j, \tag{27}
\]

whereas the possibility of having off-diagonal velocity correlations \(\langle \mathbf{x}_i \mathbf{x}_j \rangle \neq 0 \ \forall \ i \neq j\) is only realized in the presence of active dynamics, \(\tau > 0\), as it was shown even in the case of granular gases [\textit{22, 26, 65, 67}]. Here we stress that, thanks to Eq. (6), such non-equilibrium correlations also reveal the properties of the energy landscape of the system of amorphous materials as soon as some activity is switched on.

Taking inspiration from the properties of the \(p\)-spin model in Sec. IV, we assume that non-diagonal velocity correlations are self-averaging. In the present case there is no quenched disorder and the average is only over configurations \(\mathcal{X}^*\) close to stationary points of the potential energy landscape. The request of self-averaging for velocity correlations corresponds to the request of the same property for Hessian eigenvalues, which in the large-\(N\) limit are expected to behave similarly in mean-field and finite-dimensional models. Concerning this, let us stress that the information needed to describe in first approximation the correlations of the velocity field is only related to Hessian eigenvalues and not to the corresponding eigenvectors, which close to transitions in the landscape may show non-trivial localization properties [\textit{23, 24}]. We thus write the velocity correlations in the large-\(N\) limit as:

\[
\lim_{N \to \infty} \langle \mathbf{x}_i^\alpha \mathbf{x}_j^\beta \rangle_{\mathcal{X}} = [ (\mathbb{I} + \tau \mathbb{H})^{-1} ]_{ij}^{\alpha \beta}, \tag{28}
\]

where

\[
\mathbb{H} = \langle \mathbb{H} \rangle_{\mathcal{X}^*}. \tag{29}
\]

Considering the orthogonal matrix \(U : \mathbb{R}^{3N} \to \mathbb{R}^{3N}\) which diagonalizes the Hessian we can then change variables to

\[
u_q = \sum_{i=1}^{N} U_{qi} \mathbf{x}_i. \tag{30}
\]

In terms of the new rotated variables the probability distribution of velocities reads as

\[
\lim_{N \to \infty} P(U | \mathcal{X}^*) \propto e^{-\frac{1}{2T} \sum_{q=1}^{3N} (1 + \tau \lambda_q) \| \mathbf{u}_q \|^2}, \tag{31}
\]

where \(U = (\mathbf{u}_1, \ldots, \mathbf{u}_N)\), \(\lambda_q\) are the eigenvalues of \(\mathbb{H}\) and the amplitude of the \(q\)-th mode reads as

\[
\langle \| \mathbf{u}_q \|^2 \rangle \sim \frac{1}{1 + \tau \lambda_q}. \tag{32}
\]
The dependence on the configuration $\mathcal{X}^*$ is lost on the right-hand member of Eq. (31) due to the claimed self-averaging property. It is clear that also in this case as soon as one starts to have negative eigenvalues of the Hessian the approximation leading to Eq. (32) breaks down: the occurrence of an interesting phenomenon is signaled by the divergence of velocity correlations.

A similar picture is supported by the following argument (see Refs. [31] [68] [69] for details). Let $\mathcal{X}^*$ be a stationary point of the potential energy landscape, i.e., $\nabla \Phi_{|\mathcal{X}^*} = 0$. We indicate with $\delta \mathcal{X} = \mathcal{X} - \mathcal{X}^*$ a small fluctuation around the inherent state. The (linearized) equations of motion for the fluctuations are

$$\delta \dot{x}_i = -\sum_{j=1}^{N} \mathbb{H}_{ij}(\mathcal{X}^*) \cdot \delta x_j + f_i,$$

(33)

where $\mathbb{H}_{ij}$ indicates the dynamical matrix, i.e., the Hessian computed at $\mathcal{X}^*$. In particular $\mathbb{H}_{ij}$ denotes a $d \times d$ block, so that Eq. (33) must be regarded as a compact notation for a vectorial equation of the kind:

$$\delta \dot{x}_i^\alpha = -\sum_{j=1}^{N} [\mathbb{H}_{ij}(\mathcal{X}^*) \cdot \delta x_j]^\alpha + f_i^\alpha,$$

(34)

where

$$[\mathbb{H}_{ij} \cdot \delta x_j]^\alpha = \sum_{\beta=1}^{d} \mathbb{H}_{ij}^{\alpha \beta} \delta x_j^\beta$$

(35)

In writing Eq. (33) we consider that fluctuations are driven by an active dynamics, see Eq. (1). For concreteness, let us consider the case of a 2d-system with AB dynamics:

$$\dot{f}_i = v_0 (\cos \theta_i, \sin \theta_i),$$
$$\dot{\theta}_i = \eta_i,$$

(36)

where $\langle \eta_i \rangle = 0$ and

$$\langle \eta_i(t) \eta_j(s) \rangle = 2\tau^{-1} \delta_{ij} \delta(t-s)$$

(37)

By expanding the fluctuations on a normal modes basis

$$\delta x_i(t) = \sum_{q} a_q(t) u_i(q),$$

(38)

one obtains that the amplitudes $a_q(t)$ evolve according to

$$\dot{a}_q(t) = -\lambda_q a_q(t) + \tilde{\eta}_q(t),$$

(39)

with an exponentially correlated noise

$$\langle \tilde{\eta}_q(t) \tilde{\eta}_p(s) \rangle = \frac{\sigma^2}{2} \delta_{q,p} e^{-|t-s|/\tau}.$$  

(40)

After standard manipulations [31], one obtains

$$\langle \dot{a}_q^2 \rangle \sim (1 + \tau \lambda_q)^{-1},$$

(41)

that is again in agreement with Eq. (32). Clearly as long as the active dynamics keeps the system in the close vicinity of stationary points of the minima-dominated landscape the Hessian, which is computed with respect to these stationary points, is a positive definite matrix. Of course, despite the configurations of the system are close to equilibrium, i.e. close enough for the harmonic approximation of Eq. (33) to be valid, the velocity field has typical non-equilibrium features. In particular, by Fourier anti-transforming the power spectrum in Eq. (41), it is immediate to realize that the velocity amplitudes correlation function is nontrivial, whatever the shape of the spectrum $\lambda_q$:

$$\langle a(x) a(y) \rangle \neq \delta(x-y).$$

(42)

On the contrary, in the presence of thermodynamic equilibrium, i.e., when $\tau = 0$, independently to the amplitude of noise one always has:

$$\langle \dot{a}_q^2 \rangle = \text{const} \Rightarrow \langle a(x) a(0) \rangle \propto \delta(x).$$

(43)

Therefore, while at equilibrium the matrix $(1 + \tau \mathbb{H})$ is simply diagonal and does not contain any information on the landscape, close to crystalline or quasi-crystalline or glassy arrested states it has a non-trivial structure. And, something that is of particular importance for the present analysis, it is positive definite for any value of $\tau$. From Eq. (32) it is immediate to see that in this case, i.e. for energies $E < E_{th}$, the power spectrum of velocity modes is well defined for any value of $q$, since both $\tau$ and $\lambda_q$ are positive. On the contrary, as soon as some eigenvalues become negative, one can always find a value of $\tau$ for which the expression $\langle |u_q|^2 \rangle \sim (1 - \tau |\lambda_q|)^{-1}$ is inconsistent. Physically, we can relate this instability to the minima-to-saddles crossover. The apparent divergence of velocity correlation function there occurring might signal the presence of a non-equilibrium transition between a phase where the velocity field is disordered, for temperatures $T < T_d$, and a high temperature phase at $T > T_d$ where, due to the combined effect of the persistent noise and the non-trivial interaction potential, some sort of order arises. It is necessary to go beyond a Gaussian Ansatz to find an expression for $P(X|\mathcal{X})$ which remains consistent at the topological crossover and at higher temperatures. Considering the symmetry of the system, the simplest choice for stabilizing the velocity distribution along the unstable directions could be

$$P(\infty)|U| \sim \exp \left\{ -\frac{1}{2T} (1 + \tau \lambda_n)|u_n|^2 + \frac{b}{4} |u_n|^4 \right\}$$

(44)

with $b > 0$ a positive constant. It is worth noting that a systematic study might be done considering higher-order terms in the expansion of Eq. (33).
Let us conclude this analysis with a summary of the physical role of the two parameters of the coloured noise: its characteristic time-scale $\tau$ and its amplitude $v_0$. The smallness of $v_0$ is what allows us to consider simply a harmonic expansion of the potential close to a stationary point and plug it into the study of active off-equilibrium correlations information which is basically the one on equilibrium stationary configurations. On the contrary, the presence of $\tau$ gives rise to non-diagonal velocity correlations which are totally absent at equilibrium. Such velocity correlations might even have a small amplitude, if $v_0$ is small, by in the presence of a finite $\tau$ have a finite range (something which would not possible at equilibrium), a range that we expect to become critical at the topological transition. That is how a combination of small $v_0$ and finite $\tau$ reveals the topological properties of the landscape.

VI. DISCUSSION AND CONCLUSIONS

In the present paper, we have proposed a theoretical insight that connects the topological crossover in glasses with non-equilibrium velocity correlations that are typical of dense active matter systems. We showed that, to the extent of a Gaussian approximation for the marginal joint distribution of velocities, a blow-up of velocity fluctuations takes place at the saddle-to-minima topological crossover in glass-forming systems. This blow-up can be regarded as the likely signature of a non-equilibrium phase transition. This scenario is suggested both by the trial distribution for the velocities suggested in Ref. [20] and by the one drawn from the UCNA approximation [21]. Moreover, the same scenario emerges performing the linear stability analysis around a stationary point of the potential energy landscape [31]. Our analysis shows that, around the minima-to-saddle crossover, off-diagonal correlations due to the self-propulsion make the inherent configuration unstable. Moreover, at the crossover, velocity fluctuations tend to diverge. Our analysis suggests that while no long-range order in the velocity field takes place for active glassy states below $E_{th}$, at the minima-to-saddles there are signatures of something non-trivial occurring, perhaps a crossover to a flocking phase or the formation of living crystals [25]. It is worth noting that in the systems considered the transition is not triggered by an alignment interaction but is solely due to the combined effect of persistent noise and a non-trivial interaction potential, a scenario compatible with recent results [22, 26] and which is surely worth to investigate with more detailed numerical simulations in the near future.

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