Summary. The parity violation at the level of weak interactions and other similar discrete symmetries breaking show that the invariance of laws under the full group of Lorentz transformations can not be taken granted. We examine the principle of Lorentz invariance under the general theory of relativity, and demonstrate the importance of a concept of the spacetime orientation as part of the causal structure of spacetime.

1 Introduction

According to French (see page 66, [14]), “a relativity principle is an assertion about the laws of nature as they would be determined by observations made in different frames of reference”, and Galileo and Newton’s relativity principle is “the assertion ... that there are whole classes of reference frames with respect to which the laws of physics have precisely the same form”. In the review article [3], H. Bondi stated that “A physical statement of what these invariants are is called a principle of relativity, and the fundamental equations of a theory usually define the principle of relativity applicable to it.” Therefore Bondi recognized the content of invariance principles to be included in the principle of relativity.

Throughout the history, the principle of relativity has gone through several critical revisions by the scientific giants, and each crucial revision over the last 300 years, from Galileo, Newton to Einstein, has brought a revolution in mankind’s understanding of laws of Nature. According to the current research in science and technology, it is recognized that the principle of relativity, formulated by Einstein in the most recent revision of the principle, covers three postulates which have different physical meanings. The first is The Principle of Equivalence, which leads to the mathematical models of space-time. The second is the so-called The Covariance Principle, which determines possible mathematical formulation for laws of Nature, but does not involve physical processes. The last one, without doubt the most important component of the principle of relativity, is The Principle of Lorentz Invariance, whose formulation itself, will be examined carefully in the setting of general relativity in this article, depends heavily on the gravitational potential, and is an important guide for determining physical laws of Nature.

We shall begin with what Einstein said in his foundation papers on the relativity theory. In the special relativity paper [9], published in 1905 (see e.g. page 41, [22]), Einstein formulated two postulates to advance his theory of relativity.

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion.

2. Any ray of light moves in the “stationary” system of co-ordinates with the determined velocity $c$, whether the ray be emitted by a stationary or by a moving body.

The first postulate called the principle of relativity in Einstein [9], is more often called the principle of equivalence, satisfied also by the mechanics of Galileo and Newton. According to Eddington [8], the principle of equivalence demands “the laws of motion of undisturbed material particles and of light-pulses in a form independent of the coordinates chosen.” (page 39, [8]). It was already clear to Eddington that the principle of equivalence “has played a great part as a guide in the original building up of the ... relativity theory; but now ... it has become less necessary.” (see page 41, [8]). Therefore, with the establishment of mathematical models for spacetime, the principle of equivalence is no longer useful for describing physics laws. In his foundation paper [10] on general relativity, Einstein re-examined the principle of relativity, and he wrote “The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion.” Note here Einstein used the phase “systems of reference” instead of “coordinate systems”. In the same paper (see page 117, [22]), however, Einstein stated that “The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect
to any substitutions whatever (generally co-variant).” This version of principle of relativity is too narrow which deprived the important component in the previous formulation Einstein stated. It is not known why Einstein retreated from the principle of relativity, to a narrow version of the covariance principle.

In the only relativity textbook [2] endorsed by Einstein, Bergmann wrote (see page 154) that “The principle of equivalence was ostensibly a fundamental property of the gravitational forces”, and “the equivalence of gravitational and interial fields (which is a consequence of the equality of gravitational and inertial masses) gave the principle of equivalence its name”. About the principle of general covariance, Bergmann stated (on page 159, [2]) that “The hypothesis is that the geometry of physical space is represented best by a formalism which is covariant with respect to general coordinate transformations, and that a restriction to a less general group of transformations would not simplify that formalism, is called the principle of general covariance”. Therefore, the equivalence principle is used to construct mathematical models for spacetime only, while the principle of covariance is a mathematical requirement for constructing fields.

Let us look at what Pauli said about the principle of equivalence. On page 145, [25], Pauli formulated the general version of the equivalence principle as that “For every infinitely small world region (i.e. a world region which is so small that the space- and time-variation of gravity can be neglected in it) there always exists a coordinate system – in which gravitation has no influence either on the motion of particles or any other physical processes.” What Pauli stated, wrote just after the creation of Einstein’s general theory of relativity, is a very precise statement of the equivalence principle – the gravitational potential can not be detected locally.

In [28], Weyl formulated the principle of relativity of Galilei and Newton as the following (page 154, [28]): “it is impossible to single out from the systems of reference that are equivalent for mechanics and of which each two are correlated by the formulae of transformation III special systems without specifying individual objects”, here a transformation III means a linear transformation of following type

\[
\begin{align*}
x_1 &= a_{11}x'_1 + a_{12}x'_2 + \gamma t' + \alpha_1 \\
x_2 &= a_{21}x'_1 + a_{22}x'_2 + \gamma t' + \alpha_2 \\
t &= t' + a
\end{align*}
\]

where \(a_{ij}, \gamma, \alpha_k\) and \(a\) are constant. Thus Weyl included the principle of Lorentz invariance (though in a restricted sense) as a component of relativity principle.

2 The principle of relativity

We are in a position to give precise meanings for the context of the principle of relativity, to motivate our interpretation of Lorentz invariance in the general theory of relativity.

1. The equivalence principle and the principle of constancy of light velocity. The mathematical models of spacetime in special and general theories of relativity (see [9, 10]) were established based on two postulates, the first is the equivalence principle which does not involve any physics (see e.g. Norton [24] for a review about the meaning of this postulate), the second is the principle of constancy of light velocity, which implicitly requires that physical processes propagate with speed less than or equal to the velocity \(c\) of light. Together, in the special theory of relativity, the two postulates lead to the mathematical model of spacetime, the Minkowski space. If the motion of a particle is described by space coordinates \(x, y\) and \(z\), parameterized by time variable \(t\), then the postulate of constancy of light speed demands that \(x^2 + y^2 + z^2 \leq c^2\), which can be written in terms of differentials, i.e. \(c^2 dt^2 - dx^2 - dy^2 - dz^2 \geq 0\). The last inequality immediately leads to the Minkowski metric on \(\mathbb{R}^4\), so concepts of space and time are united in a natural way in this model. In the general theory of relativity, the two postulates are applied to develop general continuum space-time models being manifolds of dimension four, together with a Lorentz metric \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu\) with signature \((1, -1, -1, -1)\). The core element of the general theory of relativity is however the postulate that the gravitation is described by the curvature tensor field associated with the gravitational potential \((g_{\mu\nu})\).

Taking a Lorentz manifold \((M, g_{\mu\nu})\) as a model of spacetime, the principle of equivalence is no longer a postulate rather than a mathematical statement: for every event \(x \in M\), there is a local coordinate system about \(x\) (i.e. there is a system of reference near \(x\)), such that the gravitational potential \((g_{\mu\nu})\) is diagonalized to coincide with the Minkowski metric \(\text{diag}(1, -1, -1, -1)\) (but this may be true only at \(x\)). Therefore, the gravitational potential field can not be detected locally. The gravitational effect however can not be wiped away, this is because for whatever the local coordinate system we choose to describe physical processes, one can not write off the curvature even locally.

2. The principle of covariance. Let us examine the covariance principle for Einstein’s spacetime \((M, g_{\mu\nu})\), which is a principle emphasized in litera-
tured, it is however a basic requirement for describing physical processes in a way mathematically meaningful. Upon acceptance of the model for event space as a four dimensional manifold $M$, even without specifying the gravitational potential $(g_{\mu\nu})$, the principle of covariance, under the current setting of theoretical physics, demands that fields describing physical processes, either classical or quantized, must be sections of fiber bundles over the four dimensional manifold $M$ of spacetime. If derivatives of fields (to formulate dynamic equations of fields) are required, then only covariant derivatives (including exterior derivatives) are allowed, so that additional source of fields of differential one forms, i.e. vector Boson fields, must be introduced in to the physical processes. Therefore, the covariance principle itself can not be applied to determine physics laws.

3. The principle of Lorentz invariance. The important component of the relativity principle is the principle of Lorentz invariance for laws of Nature. In fact, the geometric dynamics, including Einstein’s field equation, has been developed under the help of a strong version of the Lorentz invariance in both special and general theories of relativity, which can be stated that geometrical dynamic equations describing macroscopic motions are invariant under diffeomorphisms which preserve the gravitational field $(g_{\mu\nu})$. Einstein showed this strong form of Lorentz invariance of the Maxwell equations, with which the geometric dynamics were developed later by Planck and others.

Let us examine the principle of Lorentz invariance for the special theory of relativity. Recall that the isometry group of the Minkowski spacetime can be identified with the Poincaré group $\mathcal{P}$. A homomorphism $\Lambda$ of the Minkowski spacetime is an isometry if it also preserves the Minkowski metric $(\eta_{\mu\nu}) = (1, -1, -1, -1)$. Under the standard coordinate system $(x^i)$ in $\mathbb{R}^4$, an isometry must take the form $(\Lambda x)^\mu = \Lambda^\mu_{\nu}x^\nu + a^\mu$, where $(\Lambda^\mu_{\nu})$ is a constant matrix such that $\Lambda^\mu_{\sigma}\eta_{\sigma\theta}\Lambda^\theta_{\nu} = \eta_{\mu\nu}$ (such a transformation is called a Lorentz transformation) and $a = (a^\mu)$ is a constant vector in $\mathbb{R}^4$. Therefore, the isometry group in the special relativity is identified with the Poincaré group $\mathcal{P} = \mathcal{L} \oplus \mathbb{R}^4$, where $\mathcal{L} = \text{SO}(1, 3)$ is the Lorentz group. For the geometric dynamics, it has been postulated that the laws of geometric dynamics in special relativity are invariant under the Poincaré group $\mathcal{P}$. However, at the level of high energy physics, it was recognized that the invariance of laws under the full isometry group $\mathcal{P}$ can not be taken as granted. As early as in 1949, Dirac [5] wrote that “I do not believe there is any need for physical laws to be invariant under these reflections\(^1\), although all the exact laws of nature so far known do have this invariance.” In 1956 Lee and Yang [21] suggested explicitly that parity conservation might be violated in weak interactions by analyzing the data which demonstrated meson states $\tau$ and $\theta$ with almost identical mass but decayed to final states of opposite parity, which was shortly verified experimentally by Wu [30]. On the other hand, all known experimental data in high energy physics support the Lorentz invariance principle (in the context of special relativity): laws of Nature are invariant under the sub-group $\mathcal{L}_0$ of all proper and orthochronous Lorentz transformations and the translations.

The strong form of the Lorentz invariance perhaps is good for dynamics, however it is violated at the sub-atomic level, the principle of Lorentz invariance, which should be applicable for high energy physics with the general theory of relativity, has not been examined in literature. In the past, according to Dirac [6] for example, there was no need to consider gravitational effect at the sub-atomic level, and therefore the principle of Lorentz invariance was not in demand. However, the successful detection of gravitational wave radiations originated from a pair of merging black holes (see [4, 1]) and the more recent detection (see [18, 29, 20, 7, 13]) from a neutron star merger confirmed one of predictions made by Einstein [11] derived from the general theory of relativity [10, 12], and understanding the mechanics of gravitational waves requires high energy physics and the theory of gravitation, in which the Lorentz invariance plays an important role. The aim of this note is to formulate a version of the principle of Lorentz invariance within the general theory of relativity. A correct formulation of the Lorentz invariance is crucial for determining laws of Nature.

3 Spacetime orientation as causal structure

Let us now consider a spacetime $(M, g_{\mu\nu})$. In order to develop a theory of Lorentz invariance with the general relativity, a careful study of causal structure of spacetime seems necessary. In history, the causality as an important component of the spacetime model emerged, surprisingly, firstly not from the investigation of cosmology and gravitation, but from the high energy physics. The current knowledge supports the postulate that it is the causality structure of the spacetime which is responsible for the concepts of anti-matters and violations of various discrete symmetri-

\(^1\)The space and time inversions.
ces. The causality structure became prominent, largely due to the discovery of singularities of the spacetime in 1960's in the seminal work by Hawking and Penrose (see e.g. [26, 15, 16, 17]). Here we only need one element of causal structures which is related to spacetime orientation.

The best way to describe spacetime orientation is to use the language of principal fiber bundles (see e.g. Hirzbruch [19]). Let \( P(M) \) denote the fiber bundle of all frames (also called tetrad) \((x, (e_i))\), where \((e_i)\) is a linear basis of the tangent space \( T_x(M) \) at point \( x \in M \). Suppose \((x^\mu)\) is a coordinate system near \( x \), then a linear basis can be expressed as \( e_k = x^\mu \frac{\partial}{\partial x^\mu} \) in terms of partial coordinate derivatives, and therefore, it leads to a coordinate system \((x^\mu, x_k^\mu)\) for the frame bundle. \( P(M) \) is a principal fiber bundle with its structure group the general linear group \( GL(4) \), so that, if \( a = (a^i_j) \) is an invertible real \( 4 \times 4 \) matrix, then its (effective) right action sends a frame \((e_k)\) to \((a^i_j e_j)\), which leads to the transformation of the corresponding coordinates

\[
(x^\mu, x_k^\mu) \rightarrow (x^\nu, a^i_\nu x_k^\nu).
\]

In this sense, \( P(M) \) is a principal fiber bundle over \( M \) with its structure group and canonical fiber \( GL(n) \).

The Lorentz metric \((g_{\mu \nu})\) determines a fiber subbundle \( O(M) \) consisting of all orthonormal frames \((x, (e_i))\) where \( g(e_\sigma, e_\rho) = \eta_{\sigma \rho} \). \( O(M) \) possesses a differential structure as the sub-manifold of dimension 10, defined by the following ten (independent) equations:

\[
g_{\mu \nu}(x) x^\sigma x^\nu = \eta_{\mu \nu}.
\]

It is easy to verify that the right action under \((a^i_j)\) leaves \( O(M) \) invariant if and only if \( a^\sigma_\mu \eta_{\sigma \rho} a^\rho_\nu = \eta_{\mu \nu} \), that is \( a \) is a Lorentz matrix. Therefore \( O(M) \) is a principal fiber bundle with its structure group and typical fiber the Lorentz group \( \mathcal{L} \).

If \((x, (e_\alpha))\) and \((x, (\tilde{e}_\alpha))\) are two orthonormal bases of \( T_x(M) \), then \( \tilde{e}_\mu = \Lambda^\mu_\nu e_\nu \) where \( \Lambda \) is a Lorentz transformation. It is said two frames \((x, (e_\alpha))\) and \((x, (\tilde{e}_\alpha))\) are equivalent if \( \tilde{e}_\mu = \Lambda^\mu_\nu e_\nu \) with \( \Lambda \in \mathcal{L}_0 \). Thus, for every \( x \in M \), the fiber \( O(M)_x \) is decomposed into a direct sum of 4 disjoint connected components: \( O(M)_x = [e] \cup [P e] \cup [T e] \cup [PT e] \) where \( e \in O(M)_x \) (where \( P \) and \( T \) denote the space and time inversions respectively), which in turn leads to two principal bundles. Let \( O(M)/\mathcal{L}_0 \) denote the set of equivalence classes, and the natural projection is denoted by \( \sigma : O(M) \rightarrow O(M)/\mathcal{L}_0 \) sending every element of \( O(M) \) to its equivalent class. Then \( O(M) \) is a principal fiber bundle over \( O(M)/\mathcal{L}_0 \). On the other hand, the natural projection \( \pi \) from \( O(M) \) to \( M \) defines a natural projection from \( O(M)/\mathcal{L}_0 \) to \( M \), and \( O(M)/\mathcal{L}_0 \) is a principal fiber bundle over \( M \) with its structure group \( \mathcal{L} \) and typical fiber \( \mathcal{L}/\mathcal{L}_0 \) which is a discrete group of four elements.

It is said that a spacetime orientation exists on the spacetime \( M \), if the structure group \( \mathcal{L} \) of \( O(M) \) can be reduced to the Lorentz subgroup \( \mathcal{L}_0 \), in the sense that there is a principal subbundle \( L(M) \) of \( O(M) \) over \( M \) with both its structure group and typical fiber being \( \mathcal{L}_0 \). According to Hirzbruch ([19] Theorem 3.4.5 on page 45), the structure group \( \mathcal{L} \) of \( O(M) \) can be reduced to \( \mathcal{L}_0 \), if and only if there is a global (smooth) section of the principal fiber bundle \( O(M)/\mathcal{L}_0 \) over \( M \). A (smooth) section of the principal fiber bundle of \( O(M)/\mathcal{L}_0 \) is called a spacetime orientation of the spacetime \((M, g_{\mu \nu})\).

4 Principle of Lorentz invariance

To be able to implement the principle of Lorentz invariance in the general theory of relativity, we work with a model of spacetime, a four dimensional manifold \( M \) endowed with a Lorentz metric \((g_{\mu \nu})\), such that the structure group \( \mathcal{L} \) of the orthonormal frame bundle \( O(M) \) can be reduced to the Lorentz subgroup \( \mathcal{L}_0 \), that is, there is a global smooth section of the bundle \( O(M)/\mathcal{L}_0 \). Suppose \( s : M \rightarrow O(M)/\mathcal{L}_0 \) is such a section which determines a spacetime orientation of \( M \). Suppose \( F : M \rightarrow M \) is a diffeomorphism preserving the gravitational field \((g_{\mu \nu})\), so that the tangent mapping \( F_* \) sends the fiber \( O(M)_x \) to the fiber \( O(M)_{F(x)} \) for every \( x \in M \). Let \( s(x) = [e(x)] \) for every \( x \in M \), represented by the equivalent class of a frame \((x, (e_\mu(x)))\), and let \( \tilde{e}_\mu(x) = F_* (e_\mu(x)) \) which belongs to \( O(M)_{F(x)} \), where \( \tilde{e}_\mu(x) = F_*(e_\mu(x)) \) for every \( x \in M \). We say an isometry \( F \) is proper and orthochronous (with respect to the spacetime orientation \( s \)), if \( [\tilde{e}(x)] = [e(F(x))] \) for every \( x \), that is, the spacetime orientation is invariant under the differential map \( F_* \).

We are now in a position to formulate the principle of Lorentz invariance: laws of Nature are invariant under any proper and orthochronous isometry of \((M, g_{\mu \nu})\). That is, laws of Nature take the same mathematical formulation under any proper and orthochronous isometry of \((M, g_{\mu \nu})\).

We conclude this note by raising the following question, which should be worthy of study. Recall that Einstein’s field equation and the geometric dynamics were derived under the general version of Lorentz invariance, which is violated at the sub-atomic level. Of course, there is no reason, though there is no any experimental evidence yet, to explain why the general Lorentz invariance is not violated at the level of geo-
metric dynamics too. This consideration leads to the following question, is there any need to modify the geometric dynamics in particular Einstein's field equation, if only the principle of Lorentz invariance as formulated above is accepted?

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