Second-order matrix extension of Beta distribution and its high order moments

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Abstract

In this article, we consider a second-order matrix extension of Beta distribution. That is a distribution on second-order random matrix. We will give the analytical formula for its high order moments, which is superior over general numerical integration method. 

Keywords: multivariate Beta distribution, higher order moments

1. Introduction

Dawid (1981) introduces an extension of multivariate extension for Beta distribution, denoted as $B(\alpha, \beta; I_p)$. It is a random $p \times p$ symmetric matrix $W$ whose density function is given by

$$p(W) = \frac{1}{B_p(\alpha, \beta)} |I - W|^{\alpha - \frac{p + 1}{2}} |W|^{\beta - \frac{p + 1}{2}}$$

where $W, I - W \in S_{p,p}^{++}$

$$B_p(\alpha, \beta) = \int_{W;I-W \in S_{p,p}^{++}} |I - W|^{\alpha - \frac{p + 1}{2}} |X|^{\beta - \frac{p + 1}{2}} dX \text{ where } \alpha, \beta > \frac{p - 1}{2}$$

$B_p(\alpha, \beta)$ is called the multivariate Beta function (Siegel (1935)); $|W|$ is the determinant of matrix $W$ and $S_{p,p}^{++}$ is the collection of positive definite matrix.

When $p = 1$, the distribution reduces to normal Beta distribution for $0 < x < 1$.

This extension may have useful applications in multivariate statistical problems but little is known about the analytical property of such extension.

Konno (1988) has derived the formula of moment up to second order. In this paper, we focus on the case $p = 2$ and deduce the analytical form of higher
order moments for $B(\alpha, \beta; I_2)$. This formula includes the expectation and variance, which are the first and second order moment respectively. The higher order moments formula, as far as we know, is novel and can be used directly in the computation related with multivariate Beta models instead of approximate numerical integration.

In this article, the following notation convention is adopted: $W = \begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}$ is the symmetric random matrix to be considered. Its distribution is given by Equation (1). $|W| = XY - Z^2$. Let $E_{\alpha, \beta}[f(X, Y, Z)] = \int f(X, Y, Z)p(W)dW$ denotes the expectation with $B(\alpha, \beta; I_2)$ where $f(X, Y, Z)$ is an arbitrary function with three variables. We will compute $E_{\alpha, \beta}[f(X, Y, Z)]$ when $f(X, Y, Z)$ takes the monomial form: $f(X, Y, Z) = X^mY^rZ^t$.

2. Marginal Distribution

In this section we will compute $E_{\alpha, \beta}[f(X, Y, Z)]$ for $f(X, Y, Z) = X^m$. That is, we consider the marginal distribution for $X$, which turns out to be one dimensional Beta distribution. To accomplish this goal, we need the following lemma:

**Lemma 1.** Let $A = XY - Z^2, B = 1 - X - Y + A$, then we have

$$E_{\alpha, \beta}[Af(X, Y, Z)] = \frac{\alpha(\alpha - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)}E_{\alpha+1, \beta}[f(X, Y, Z)]$$

(3)

$$E_{\alpha, \beta}[Bf(X, Y, Z)] = \frac{\beta(\beta - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)}E_{\alpha+1, \beta}[f(X, Y, Z)]$$

(4)

**Proof.** For multivariate Beta function we have $B_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}$ where $\Gamma_p$ is the multivariate Gamma function (Ingham (1933)). For $p = 2$ we have $\Gamma_2(a) =
\[ \sqrt{\pi} \Gamma(a) \Gamma(a - 1/2). \]

\[ \frac{E_{\alpha, \beta}[A f(X, Y, Z)]}{E_{\alpha+1, \beta}[f(X, Y, Z)]} = \frac{B_2(\alpha + 1, \beta)}{B_2(\alpha, \beta)} = \frac{\Gamma(\alpha + 1) \Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\alpha + \beta + 1)} \]

\[ = \frac{\Gamma(\alpha + 1) \Gamma(\alpha + 1/2) \Gamma(\alpha + \beta) \Gamma(\alpha + \beta - 1/2)}{\Gamma(\alpha) \Gamma(\alpha - 1/2) \Gamma(\alpha + 1) \Gamma(\alpha + 1/2)} \]

\[ = \frac{\alpha(\alpha - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)} \]

Thus Equation (3) is proved and Equation (4) follows similarly. \[ \square \]

Using the above Lemma, we give the main conclusion of this section:

**Theorem 1.** \( E_{\alpha, \beta}[X^m] = \prod_{i=0}^{m-1} \frac{\alpha + i}{\alpha + \beta + i} \), thus \( X \) conforms to Beta distribution \( \text{Beta}(\alpha, \beta) \).

**Proof.** Since the position of \( X \) and \( Y \) is symmetric, \( E_{\alpha, \beta}[X] = E_{\alpha, \beta}[Y] \). Taking the expectation about \( \text{Beta}_2(\alpha, \beta) \) on both sides of \( B = 1 - X - Y + A \) and using the conclusion of Lemma \( \square \) we have

\[ \frac{\beta(\beta - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)} = 1 - 2E_{\alpha, \beta}[X] + \frac{\alpha(\alpha - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)} \]

Solving the above equation we get \( E_{\alpha, \beta}[X] = \frac{\alpha}{\alpha + \beta} \). Recursively using Equation (3) with \( f(X, Y, Z) = X \) we have \( E_{\alpha, \beta}[X^m] = \prod_{i=0}^{m-1} \frac{\alpha + i}{\alpha + \beta + i} \). This expression of higher order moments is the same with that of Beta distribution and by the uniqueness of moment generating function, we conclude that \( X \) is actually Beta distribution \( \text{Beta}(\alpha, \beta) \). \[ \square \]

### 3. Mixed Moment

In this section, we further compute \( E_{\alpha, \beta}[X^m Y^r Z^{2t}] \). By symmetric property \( E_{\alpha, \beta}[X^m Y^r Z^{2t+1}] = 0 \). Therefore we only need to consider the case when the power of \( Z \) is even.

**Theorem 2.**

\[ E_{\alpha, \beta}[X^m Z^{2t}] = \frac{(2t - 1)!!}{2^t} \pi \prod_{i=0}^{t-1} \frac{\beta + i}{(\alpha + \beta + i - 1/2)} \frac{\prod_{i=0}^{t+m-1} \alpha + i}{\prod_{i=0}^{2t+m-1} \alpha + \beta + i} \quad (5) \]
Proof. We use induction to show Equation (5) is true. Firstly, Equation (5) is true for \( t = 0 \) from Theorem 1. Let \( A, B \) be the same as those in Lemma 1. Suppose Equation (5) holds for \( \mathbb{E}[Z^{2t-2}X^{m}] \), using \( Z^2 = XY - A = X(1 - X - A + B) - A \), then

\[
\mathbb{E}_{\alpha,\beta}[Z^{2t}X^{m}] = \mathbb{E}_{\alpha,\beta}[Z^{2t-2}X^{m}(X - X^2 + AX - BX - A)] \\
= \mathbb{E}_{\alpha,\beta}[Z^{2t-2}(X^{m+1} - X^{m+2})] + \mathbb{E}_{\alpha,\beta}[AZ^{2t-2}(X^{m+1} - X^{m})] \\
- \mathbb{E}_{\alpha,\beta}[BZ^{2t-2}X^{m+1}] \\
= \left(1 - \frac{\alpha + t + m}{\alpha + \beta + 2t + m - 1}\right)\mathbb{E}_{\alpha,\beta}[Z^{2t-2}X^{m+1}] \\
+ \frac{\alpha(\alpha - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)}\mathbb{E}_{\alpha+1,\beta}[Z^{2t-2}(X^{m+1} - X^{m})] \\
- \frac{\beta(\beta - 1/2)}{(\alpha + \beta)(\alpha + \beta - 1/2)}\mathbb{E}_{\alpha,\beta+1}[Z^{2t-2}X^{m+1}] \\
= \frac{\beta + t - 1}{\alpha + \beta + 2t + m - 1}\mathbb{E}_{\alpha,\beta}[Z^{2t-2}X^{m+1}] \\
+ \frac{(\alpha + t + m)(\alpha - 1/2)}{(\alpha + \beta + 2t - 1 + m)(\alpha + \beta + t - 3/2)} \\
\cdot \left(1 - \frac{\alpha + \beta + 2(t - 1) + m + 1}{\alpha + t + m}\right)\mathbb{E}_{\alpha,\beta}[Z^{2t-2}X^{m+1}] \\
- \frac{(\beta + t - 1)(\beta - 1/2)}{(\alpha + \beta + 2t - 1 + m)(\alpha + \beta + t - 3/2)}\mathbb{E}_{\alpha,\beta}[Z^{2t-2}X^{m+1}] \\
= \frac{(t - 1/2)(\beta + t - 1)}{(\alpha + \beta + t - 3/2)(\alpha + \beta + 2t + m - 1)}\mathbb{E}_{\alpha,\beta}[Z^{2t-2}X^{m+1}]
\]

Using Equation (5) for \( t - 1 \) we can get the same form of expression for \( t \). \( \Box \)

From Theorem 2 we can get the general formula for the mixed moment when \( m \geq r \):

**Corollary 1.**

\[
\mathbb{E}_{\alpha,\beta}[X^{m}X^{r}Z^{2t}] = \frac{(2t - 1)!!}{2^t} \frac{\prod_{j=0}^{t-1}(\beta + j)}{\prod_{j=0}^{t+r-1}(\alpha + \beta - 1/2 + j)} \frac{\prod_{j=0}^{t+m-1}(\alpha + j)}{\prod_{j=0}^{2t+m-1}(\alpha + \beta + j)} \\
\sum_{i=0}^{r} \frac{1}{2^i} \binom{r}{i} \prod_{j=1}^{i} (2t - 1 + 2j) \frac{\prod_{j=0}^{r-i-1}(\alpha - 1/2 + j)}{\prod_{j=0}^{t-1}(\alpha + \beta + j + 2t + m)} \prod_{j=0}^{i-1}(\beta + j + t)
\]

(6)
Since $E_{\alpha,\beta}[X^mY^rZ^{2t}] = E_{\alpha,\beta}[X^rY^mZ^{2t}]$, when $m < r$ we can exchange $m$ and $r$ and use Corollary 1.

**Proof.** From Lemma 1 we have $X^mY^rZ^{2t} = X^{m-r}(A + Z^2)^rZ^{2t}$. Using binomial theorem we have $E_{\alpha,\beta}[X^mY^rZ^{2t}] = \sum_{i=0}^{r} \binom{r}{i} E_{\alpha,\beta}[A^{-i}X^{m-r}Z^{2(t+i)}]$. Then using Equation (3) recursively we have

$$E_{\alpha,\beta}[A^{-i}X^{m-r}Z^{2(t+i)}] = \prod_{j=0}^{r-i-1} \frac{(\alpha + j)(\alpha + j - 1/2)}{(\alpha + j + \beta)(\alpha + j + \beta - 1/2)} \cdot E_{\alpha+r-i,\beta}[X^{m-r}Z^{2(t+i)}]$$

Using Theorem 2 we can finally get the expression in Equation (6).

\[ \square \]

4. Conclusion

We have derived the formula of high order moments for multivariate Beta distribution in second-order matrix case. This result is helpful when the computation of moments are required in some statistics problem.

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