Kouchnirenko type formulas for local invariants of plane analytic curves

Janusz Gwoździewicz

Abstract
Let $f(x, y) = 0$, $g(x, y) = 0$ be equations of plane analytic curves defined in the neighborhood of the origin and let $\pi : M \to (\mathbb{C}^2, 0)$ be a local toric modification. We give a formula which connects a number of double points hidden at zero $\delta_0(f)$ with a sum $\sum_p \delta_p(\tilde{f})$ which runs over all intersection points of the proper preimage of $f = 0$ with the exceptional divisor. We give also similar formulas for the Milnor number $\mu_0(f)$ and the intersection multiplicity $(f, g)_0$. Presented formulas generalize Kouchnirenko and Bernstein theorems and classical Noether formula for the intersection multiplicity after blow-up.

1 Kouchnirenko and Bernstein theorems
This paper is an english translation of [Gw].
Let $f \in \mathbb{C}\{x, y\}$ be a convergent power series such that $f(x, 0)$ and $f(0, y)$ do not vanish. Such a series is called convenient. For any convenient series $f(x, y) = \sum_{ij} a_{ij} x^i y^j$ the Newton diagram $\Delta_f$ is the convex closure of the set
\[ \bigcup_{a_{ij} \neq 0} \{(i, j) + \mathbb{R}^2\}. \]

Since $f$ is convenient, $\Delta_f$ has joint points with both axis. A union of compact edges of $\Delta_f$ is called the Newton polygon of $f$ and is denoted $\mathcal{N}_f$.

Introduce further notations. For the Newton diagram $\Delta$ whose Newton polygon $\mathcal{N}$ touches axes at points $(a, 0)$ and $(0, b)$ denote:

- $P(\Delta) = \text{Area}(\mathbb{R}_+^2 \setminus \Delta)$
- $\mu(\Delta) = 2P(\Delta) - a - b + 1$
- $r(\Delta) = \text{number of lattice points on } \mathcal{N} \text{ minus 1}$
- $\delta(\Delta) = (\mu(\Delta) + r(\Delta) - 1)/2$

If $\Delta_1, \Delta_2$ are two Newton diagrams then their mixed Minkowski volume is the quantity $[\Delta_1, \Delta_2] = P(\Delta_1 + \Delta_2) - P(\Delta_1) - P(\Delta_2)$.

In 1970s mathematicians from Arnolds seminar gave many formulas for invariants of singularities in terms of Newton diagrams. We quote some of their results (see. [Ko], [Kli]).
Theorem 1 Let \( f, g \in \mathbb{C}\{x, y\} \) be convenient convergent power series. Then
\[
(f, g)_0 \geq [\Delta_f, \Delta_g]
\]
(1)
\[
\delta_0(f) \geq \delta(\Delta_f)
\]
(2)
\[
\mu_0(f) \geq \mu(\Delta_f)
\]
(3)
\[
r_0(f) \leq r(\Delta_f)
\]
(4)
If coefficients of \( f(x, y), g(x, y) \), with indices from the sets \( N_f, N_g \) respectively, are sufficiently general then (1)–(4) become equalities.

In Theorem 1 \((f, g)_0\) is an intersection number of \( f \) and \( g \) at the origin, \( \delta_0(f) \) is the number of double points of a curve \( f = 0 \) hidden at zero, \( \mu_0(f) \) is the Milnor number of \( f \) at zero and \( r_0(f) \) is a number of branches of \( f = 0 \) at 0. “Sufficiently general” in the second part of the theorem means that coefficients of power series \( f(x, y), g(x, y) \) with indices from the sets \( N_f, N_g \) respectively (there is a finite number of such coefficients) belong to some dense constructible set. The equations of this set form so called nondegeneracy conditions (see [Kh]).

2 Noether theorem

Let \( f(x, y) = 0, g(x, y) = 0 \) be equations of analytic curves defined in the neighborhood of 0 of the complex plane. Let \( \sigma : M \to \mathbb{C}^2 \) be a blowing-up of \( \mathbb{C}^2 \) at 0. There are classical formulas which connect invariants of singularities of these curves with invariants of singularities of their proper preimages.

Theorem 2 Assume that curves \( f = 0, g = 0 \) do not have common components. If \( \tilde{f} = 0, \tilde{g} = 0 \) are local equations of their proper preimages under blowing-up \( \sigma \) then
\[
(f, g)_0 = (\text{ord } f)(\text{ord } g) + \sum_{p \in \sigma^{-1}(0)} (\tilde{f}, \tilde{g})_p
\]
(5)
If a curve \( f = 0 \) does not have multiple components then
\[
\delta_0(f) = (\text{ord } f)(\text{ord } f - 1)/2 + \sum_{p \in \sigma^{-1}(0)} \delta_p(\tilde{f})
\]
(6)
\[
\mu_0(f) - 1 = (\text{ord } f)(\text{ord } f - 1) + \sum_{p \in \sigma^{-1}(0) \cap \{\tilde{f} = 0\}} (\mu_p(\tilde{f}) - 1)
\]
(7)

The classical Noether theorems are formulas (5) for the intersection multiplicity ([B], Theorem 13) and (6) for the number of double points \( \delta_0(f) \). A formula (7) for the Milnor number follows from equality \( 2\delta_0(f) = \mu_0(f) + r_0(f) - 1 \).

3 Local toric modifications

The aim of this paper is to give formulas which generalize these from Theorem 2 to the case of an arbitrary local toric modification \( \pi : M \to \mathbb{C}^2 \). When \( \pi \) will be a blow-up they will reduce to (5)–(7) and for a toric modification with a sufficiently subtle fan they will give (1)–(3) from Theorem 1.
3.1 Fans

By a simple cone $\sigma \subset \mathbb{R}^2$ we mean the set

$$\sigma = \sigma[\vec{\xi}, \vec{\nu}] = \{ \alpha \vec{\xi} + \beta \vec{\nu} : \alpha \geq 0, \beta \geq 0 \}$$

where vectors $\vec{\xi}, \vec{\nu} \in \mathbb{R}^2$ have integer coordinates and form a base of lattice $\mathbb{Z}^2$, i.e. $\det(\vec{\xi}, \vec{\nu}) = \pm 1$.

By a fan we mean a finite set of simple cones such that their union is the first quadrant $\mathbb{R}^2_+$ and such that they intersect at most along edges. For every fan $\mathcal{W}$ consisting of $n$ cones one can enumerate counter clockwise the shortest lattice vectors from its rays. We get a sequence $\vec{\xi}_0, \vec{\xi}_1, \ldots, \vec{\xi}_n$, where $\vec{\xi}_0 = (1,0)$, $\vec{\xi}_n = (0,1)$ and $\det(\vec{\xi}_{i-1}, \vec{\xi}_i) = 1$ for $1 \leq i \leq n$. We will say that $\mathcal{W}$ is spanned by $\vec{\xi}_0, \vec{\xi}_1, \ldots, \vec{\xi}_n$.

3.2 Local toric modifications

With every simple cone $\sigma = \sigma[\vec{\xi}, \vec{\nu}]$, where $\vec{\xi} = (\xi_1, \xi_2), \vec{\nu} = (\nu_1, \nu_2), \det(\vec{\xi}, \vec{\nu}) = 1$, we associate a mapping $\varphi_\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ given in coordinates by

$$\varphi_\sigma : \begin{cases} x = u^{\xi_1} v^{\nu_1} \\ y = u^{\xi_2} v^{\nu_2} \end{cases}$$

Let $\mathcal{W}$ be a fan consisting of $n$ cones

**Theorem 3** There exist a smooth analytic manifold $M$ and a proper analytic mapping $\pi : M \to \mathbb{C}^2$ such that:

(i) $\pi$ is an isomorphism from $M \setminus \pi^{-1}(0)$ to $\mathbb{C}^2 \setminus \{0\}$,

(ii) the manifold $M$ is covered by $n$ charts associated with cones $\sigma_i$ ($i = 1 \ldots n$) and in local coordinates of $i$-th chart the mapping $\pi$ is given by formula $\pi = \varphi_\sigma$.

We call $\pi : M \to \mathbb{C}^2$ a local toric modification associated with a fan $\mathcal{W}$.

3.3 Thickened Newton diagrams

For a Newton diagram $\Delta$ and $\vec{\xi} \in \mathbb{R}^2_+$ we define a support function

$$l(\Delta, \vec{\xi}) = \inf_{p \in \Delta} \langle p, \vec{\xi} \rangle$$

For a fan $\mathcal{W}$ spanned by vectors $\vec{\xi}_0, \vec{\xi}_1, \ldots, \vec{\xi}_n$ and a Newton diagram $\Delta$ we define $\tilde{\Delta}$ as an intersection of $n + 1$ half-planes

$$\tilde{\Delta} = \bigcap_{i=0}^n \{ p \in \mathbb{R}^2_+ : \langle p, \vec{\xi}_i \rangle \geq l(\Delta, \vec{\xi}_i) \}$$

and call this set a thickened Newton diagram relative to $\mathcal{W}$. It follows directly from definition that $l(\Delta, \vec{\xi}_i) = l(\Delta, \vec{\xi}_i)$ for $i = 0, \ldots, n$. 

3
4 Generalized Kouchnirenko theorem

**Theorem 4** Let \( \pi : M \to \mathbb{C}^2 \) be a local toric modification associated with a fan \( \mathcal{W} \). If \( f, g \in \mathbb{C}\{x, y\} \) are convenient power series and \( \tilde{f}, \tilde{g} \) are their proper preimages then

\[
(f, g)_0 = [\tilde{\Delta}_f, \tilde{\Delta}_g] + \sum_{p \in \pi^{-1}(0)} (\tilde{f}, \tilde{g})_p \tag{8}
\]

\[
\delta_0(f) = \delta(\tilde{\Delta}_f) + \sum_{p \in \pi^{-1}(0)} \delta_p(\tilde{f}) \tag{9}
\]

\[
\mu_0(f) = \mu(\tilde{\Delta}_f) + r(\tilde{\Delta}_f) + \sum_{p \in \pi^{-1}(0) \cap \{f = 0\}} (\mu_p(\tilde{f}) - 1) \tag{10}
\]

To avoid considering special cases in the statement of the theorem we adopt usual conventions about adding \(+\infty\).

**Example 1.** The simplest fan \( \mathcal{W} \) has only one cone \( \sigma \) which is the first quadrant. It is spanned by vectors \( \tilde{\xi}_0 = (1, 0), \tilde{\xi}_1 = (0, 1) \). A mapping \( \varphi_\sigma \) is given by formula \((x, y) = (u, v)\) so the local toric modification associated with \( \mathcal{W} \) is an identity \( \mathbb{C}^2 \to \mathbb{C}^2 \).

If \( \Delta \) is a Newton diagram of a convenient power series then \( \tilde{\Delta} = \mathbb{R}^2_+ \). Hence the invariants of a thickened diagram are: \( \mu(\tilde{\Delta}) = 1, r(\tilde{\Delta}) = 0, \delta(\tilde{\Delta}) = 0, [\tilde{\Delta}_f, \tilde{\Delta}_g] = 0 \) and Theorem 4 is trivially satisfied.

**Example 2.** Consider a fan \( \mathcal{W} \) consisting of two cones. It is easy to check that \( \mathcal{W} \) is spanned by: \( \tilde{\xi}_0 = (1, 0), \tilde{\xi}_1 = (1, 1), \tilde{\xi}_2 = (0, 1) \). The first cone \( \sigma_1 \) is generated by vectors \( \tilde{\xi}_0 = (1, 0), \tilde{\xi}_1 = (1, 1) \) and the mapping \( \varphi_{\sigma_1} \) is given by a formula \((x, y) = (uv, v)\). A mapping \( \varphi_{\sigma_2} \) associated with the second cone is given by \((x, y) = (u, uv)\). Hence the toric modification \( \pi : M \to \mathbb{C}^2 \) associated with \( \mathcal{W} \) is a blowing-up of \( \mathbb{C}^2 \) at zero.

If \( \Delta \) is a Newton diagram of a convenient power series \( f \) of order \( d \) then the Newton polygon of \( \tilde{\Delta} \) is a segment with endpoints \((d, 0)\) and \((0, d)\). Hence \( P(\tilde{\Delta}) = d^2/2, \mu(\tilde{\Delta}) = 2P(\tilde{\Delta}) - d - d + 1 = (d - 1)^2, r(\tilde{\Delta}) = d, \delta(\tilde{\Delta}) = (\mu(\tilde{\Delta}) + r(\tilde{\Delta}) - 1)/2 = (d - 1)/2. \) If \( f, g \) are convenient power series then \([\tilde{\Delta}_f, \tilde{\Delta}_g] = (\text{ord } f)(\text{ord } g). \) Substituting these quantities to equations (8)–(10) from Theorem 4 we see that they reduce to (5)–(7) from Theorem 2.

We checked that Theorem 2 is a special case of Theorem 4. Likewise Theorem 1. If \( f, g \) are convenient power series, then there exists such a fan \( \mathcal{W} \), that among its spanning vectors are vectors orthogonal to all segments of Newton polygons \( \mathcal{N}_f \) and \( \mathcal{N}_g \). Then \( \tilde{\Delta}_f = \Delta_f \) and \( \tilde{\Delta}_g = \Delta_g \). Inequalities in Theorem 1 follow from adding extra terms on the right-hand side of (8)–(10). If \( f \) and \( g \) satisfy nondegeneracy conditions then their proper preimages define smooth curves which do not have joint points on exceptional divisor \( \pi^{-1}(0) \) (see [Kh]), the sums on the right-hand side of (8) and (9) are 0 and a sum on the right-hand side of (10) equals \(-r_0(f) = -r(\Delta_f)\). Hence also Theorem 1 is a special case of Theorem 4.
4.1 A decomposition of a toric modification to blowing-ups

We check in this subsection, that every local toric modification is a composition of a finite number of blowing-ups. It will prepare the ground for an inductive proof of Theorem 4.

Let us start from a lemma describing mutual position of vectors spanning a fan.

**Lemma 5** If a fan $W$ is spanned by vectors $\xi_0, \xi_1, \ldots, \xi_n$ and vectors $\xi_k, \xi_l$ ($k + 1 < l$) form a base of a lattice then one of vectors $\xi_i$ is equal to $\xi_k + \xi_l$.

**Proof.** Suppose that this is not the case. Then $\xi_k + \xi_l$ is inside one of cones $\sigma_j$ of fan $W$ where $k < j < l$ and at least one of vectors generating $\sigma_j$ is different from $\xi_k$ and $\xi_l$. We may assume without loss of generality that this is $\xi_j$. Therefore we have the following equations with integer coefficients:

$$
\begin{align*}
\xi_{j-1} & = n_1 \xi_k + n_2 \xi_l \\
\xi_j & = m_1 \xi_k + m_2 \xi_l \\
\xi_k + \xi_l & = a \xi_{j-1} + b \xi_j
\end{align*}
$$

with $n_1 \geq 0$, $n_2 \geq 0$, $n_1 + n_2 \geq 1$, $m_1 > 0$, $m_2 > 0$, $a > 0$, $b > 0$. Substituting right-hand sides of two first equations to the third we conclude that $an_1 + bm_1 = 1$ and $an_2 + bm_2 = 1$, so $n_1 = n_2 = 0$ and we arrive to a contradiction.

**Corollary 6** If a fan $W_n$ is spanned by vectors $\xi_0, \xi_1, \ldots, \xi_n$ then for some $i$ ($1 < i < n$) we have $\xi_i = \xi_{i-1} + \xi_{i+1}$.

The proof of a corollary uses a simple recurrence. Vectors $\xi_0$ and $\xi_n$ satisfy assumptions of Lemma 5. Hence among vectors $\xi_i$ there is one of the form $\xi_0 + \xi_n$. Next we apply Lemma 5 to a pair $\xi_0, \xi_1$ or to a pair $\xi_i, \xi_n$. Continuing this procedure we arrive to such vectors $\xi_{i-1}, \xi_i, \xi_{i+1}$, that $\xi_i = \xi_{i-1} + \xi_{i+1}$.

It follows from Corollary 5 that for every fan $W_{n+1}$ consisting of $n + 1$ cones there exist a fan $W_n$ consisting of $n$ cones such that one of cones $\sigma = \sigma[\xi, \nu] \in W_n$ decomposes into two cones $\sigma[\xi, \xi + \nu], \sigma[\xi + \nu, \nu] \in W_{n+1}$. We call $W_{n+1}$ a subdivision of $W_n$.

In the following theorem we compare local toric modification associated with a fan and with its subdivision.

**Theorem 7** Let $\pi_n : M_n \rightarrow \mathbb{C}^2$, $\pi_{n+1} : M_{n+1} \rightarrow \mathbb{C}^2$ be toric modifications associated with a fan $W_n$ and its subdivision $W_{n+1}$. Let $\sigma$ be the cone of $W_n$ which is divided into two. Then $\pi_{n+1} = \pi_n \circ \sigma$ where $\sigma$ is a blowing-up of $M_n$ at the origin of the local coordinate system associated with $\sigma$.

**Proof.** Let $\bar{\sigma} = \sigma[\xi, \nu], \sigma' = \sigma[\xi, \xi + \nu], \sigma'' = \sigma[\xi + \nu, \nu]$. In all charts associated with cones different from $\sigma, \sigma', \sigma''$ mappings $\pi_n$ and $\pi_{n+1}$ are given by identical formulas, hence in these charts $\sigma$ is an identity. If we examine $\varphi_{\sigma'}$, $\varphi_{\sigma''}$ and $\varphi_{\bar{\sigma}}$ then it is easy to check that $\varphi_{\sigma'} = \varphi_{\bar{\sigma}} \circ \sigma$ where $\sigma(u, v) = (uv, v)$ and $\varphi_{\sigma''} = \varphi_{\bar{\sigma}} \circ \sigma$ where $\sigma(u, v) = (u, uv)$. Hence $\sigma$ is a blowing-up of $M_n$ at a point $(0, 0)$ of a chart associated with $\bar{\sigma}$.
4.2 Orders of proper preimages in centers of blowing-ups

**Lemma 8** Let $f = f(x, y)$ be a convenient power series with a Newton diagram $\Delta$. Let $\pi : M \to \mathbb{C}^2$ be a local toric modification with a fan $\mathcal{W}$ and let $\sigma$ be a cone of $\mathcal{W}$ spanned by vectors $\xi, \tilde{\nu}$. Then the order of the proper preimage of $f$ at a point $(0, 0)$ of a chart associated with $\sigma$ is equal to

$$l(\Delta, \xi + \tilde{\nu}) - l(\Delta, \tilde{\nu}).$$

**Proof.** For

$$f(x, y) = \sum_{ij} a_{ij}x^iy^j$$

we have

$$(f \circ \varphi_p)(u, v) = \sum_{ij} a_{ij}u^{(i,j)}\xi^{(i,j)}v$$

We may exclude factors $u^{l(\Delta, \xi)}$ and $v^{l(\Delta, \tilde{\nu})}$ from above sum getting

$$f(x, y) = u^{l(\Delta, \xi)}v^{l(\Delta, \tilde{\nu})}\tilde{f}(u, v)$$

where $\tilde{f}(u, v)$ is a proper preimage of $f$.

The order of $\tilde{f}$ at $(0, 0)$ is equal to the order of substitution of generic curve $\gamma : t \to (ct, c^2t)$ to $\tilde{f}$. In coordinates $(x, y)$ the curve $\gamma$ has an equation $(x, y) = (d_1t^{l_1+q_1}, d_2t^{l_2+q_2})$ and for generic $d_1$, $d_2$ we have $\text{ord} f(d_1t^{l_1+q_1}, d_2t^{l_2+q_2}) = l(\Delta, \xi + \tilde{\nu})$ (see [Kh]). Thus substituting the parameterization of $\gamma$ to (11) we get

$$l(\Delta, \xi + \tilde{\nu}) = \text{ord} \left( u^{l(\Delta, \xi)}v^{l(\Delta, \tilde{\nu})}((\tilde{f} \circ \gamma)(t)) \right) = l(\Delta, \xi) + l(\Delta, \tilde{\nu}) + \text{ord} \tilde{f}.$$

4.3 Proof of Theorem 4

**Proof of (4).** Let $\mathcal{W}_n$, $\mathcal{W}_{n+1}$ be fans from Theorem $\mathbb{L}$. Let $f$, $g$ be convenient power series such that $\Delta_f = \Delta_g = \Delta$ where thickened Newton diagrams are relative to $\mathcal{W}_{n+1}$. Keeping notation from Theorem $\mathbb{O}$ we shall check that from formula

$$\delta_0(f) - \sum_{p \in \pi_{n}^{-1}(0)} \delta_p(\tilde{f}_n) = \delta_0(g) - \sum_{p \in \pi_{n}^{-1}(0)} \delta_p(\tilde{g}_n)$$

follows

$$\delta_0(f) - \sum_{p \in \pi_{n+1}^{-1}(0)} \delta_p(\tilde{f}_{n+1}) = \delta_0(g) - \sum_{p \in \pi_{n+1}^{-1}(0)} \delta_p(\tilde{g}_{n+1})$$

We use subscripts for $\tilde{f}_n$, $\tilde{g}_n$, $\tilde{f}_{n+1}$, $\tilde{g}_{n+1}$ to distinguish proper preimages on manifolds $M_n$ and $M_{n+1}$. By Theorem $\mathbb{M}$ $\pi_{n+1} = \pi_n \circ \sigma$ where $\sigma$ is a blowing-up of $M_n$ at the center $q$ which is the origin of the chart associated with $\tilde{\sigma} = \sigma[\xi, \tilde{\nu}]$. By Lemma $\mathbb{K}$ orders of $\tilde{f}_n$ and $\tilde{g}_n$ at $q$ are identical and equal to $l(\Delta, \xi + \tilde{\nu}) - l(\Delta, \xi) - l(\Delta, \tilde{\nu})$. Therefore from Noether formula $\mathbb{O}$ we get

$$\delta_q(\tilde{f}_n) - \sum_{p \in \sigma^{-1}(q)} \delta_p(\tilde{f}_{n+1}) = \delta_q(\tilde{g}_n) - \sum_{p \in \sigma^{-1}(q)} \delta_p(\tilde{g}_{n+1})$$
Adding (12) and (14) we get (13).

An inductive argument with respect to the number of cones in the fan leads to conclusion that formula (12) holds for every fan \( \mathcal{W} \) and all convenient power series \( f, g \) such that \( \Delta_f = \Delta_g = \Delta \). Taking as \( g \) a nondegenerate convergent power series with a Newton diagram \( \hat{\Delta} \) and using Theorem 1 we get \( \delta_0(g) = \delta(\hat{\Delta}) \) and \( \delta_p(\hat{g}_n) = 0 \) for every \( p \in \pi^{-1}_n(0) \). Hence the right-hand side of (12) is equal to \( \delta(\hat{\Delta}) \) which proves (9).

**Proof of (10).** It is enough to use a formula \( \delta_p(f) = (\mu_p(f) + r_p(f) - 1)/2 \). After substituting to (9) and multiplying by 2 we get

\[
\mu_0(f) + r_0(f) - 1 = \mu(\hat{\Delta}) + r(\hat{\Delta}) - 1 + \sum_{p \in \pi^{-1}(0) \setminus \{\hat{f} = 0\}} (\mu_p(\hat{f}) + r_p(\hat{f}) - 1)
\]

Then we apply an equality on number of branches \( r_0(f) = \sum_{p \in \pi^{-1}(0)} r_p(\hat{f}) \) and we get (10).

**Proof of (8).** We only outline the proof briefly because it is analogous to the one of (10). First we show by induction on the number of cones that

\[
(f_1, g_1)_0 - \sum_{p \in \pi^{-1}(0)} (\hat{f}_1, \hat{g}_1)_p = (f_2, g_2)_0 - \sum_{p \in \pi^{-1}(0)} (\hat{f}_2, \hat{g}_2)_p \tag{15}
\]

where \( f_i, g_i \ (i = 1, 2) \) are such convergent power series that \( \hat{\Delta}_{f_i} = \hat{\Delta}_{f_2} \) and \( \hat{\Delta}_{g_1} = \hat{\Delta}_{g_2} \). In the inductive proof we use a formula (5) for the intersection number after blow-up.

Then we take as \( f_2 \) and \( g_2 \) nondegenerate power series with Newton diagrams \( \Delta_{f_2} = \hat{\Delta}_{f_1} \) and \( \Delta_{g_2} = \hat{\Delta}_{g_1} \). By Theorem 1 the right-hand side of (15) is equal to \( [\Delta_{f_2}, \Delta_{g_2}] \) which proves (8).

**References**

[B] E. Brieskorn, H. Knörrer, Plane algebraic curves, Birkhäuser Verlag 1986

[Gw] J. Gwoździewicz, Formuły typu Kusznirenki dla lokalnych niezmienników krzywych analitycznych, (in polish) Proceedings of XXVII conference in complex analytic and algebraic geometry, Łódź 2006, 33–41

[Kh] A. G. Khovanskii, Newton polyhedra and toric varieties, (in russian) Functional analysis and its applications 11.4 (1977) 56–64

[Ko] A. G. Kouchnirenko, Polyèdres de Newton et nombres de Milnor, Invent. Math. 32.1 (1975), 1–32