Reinforcement Learning for Non-Stationary Markov Decision Processes: The Blessing of (More) Optimism

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Abstract

We consider un-discounted reinforcement learning (RL) in Markov decision processes (MDPs) under drifting non-stationarity, i.e., both the reward and state transition distributions are allowed to evolve over time, as long as their respective total variations, quantified by suitable metrics, do not exceed certain variation budgets. We first develop the Sliding Window Upper-Confidence bound for Reinforcement Learning with Confidence Widening (SWUCRL\textsuperscript{2}-CW) algorithm, and establish its dynamic regret bound when the variation budgets are known. In addition, we propose the Bandit-over-Reinforcement Learning (BORL) algorithm to adaptively tune the SWUCRL\textsuperscript{2}-CW algorithm to achieve the same dynamic regret bound, but in a parameter-free manner, i.e., without knowing the variation budgets. Notably, learning non-stationary MDPs via the conventional optimistic exploration technique presents a unique challenge absent in existing (non-stationary) bandit learning settings. We overcome the challenge by a novel confidence widening technique that incorporates additional optimism.

1. Introduction

Consider a general sequential decision-making framework, where a decision-maker (DM) interacts with an initially unknown environment iteratively. At each time step, the DM first observes the current state of the environment, and then chooses an available action. After that, she receives an instantaneous random reward, and the environment transitions to the next state. The DM aims to design a policy that maximizes its cumulative rewards, while facing the following challenges:

- **Endogeneity**: At each time step, the reward follows a reward distribution, and the subsequent state follows a state transition distribution. Both distributions depend (solely) on the current state and action, which are influenced by the policy. Hence, the environment can be fully characterized by a discrete time Markov decision process (MDP).

- **Exogeneity**: The reward and state transition distributions vary (independently of the policy) across time steps, but the total variations are bounded by the respective variation budgets.

- **Uncertainty**: Both the reward and state transition distributions are initially unknown to the DM.

- **Bandit/Partial Feedback**: The DM can only observe the reward and state transition resulted by the current state and action in each time step.

It turns out that many applications, such as real-time bidding in advertisement (ad) auctions, can be captured by this framework (Cai et al., 2017; Flajolet & Jaillet, 2017; Balseiro & Gur, 2019; Guo et al., 2019; Han et al., 2020). Besides, this framework can be used to model sequential decision-making problems in transportation (Zhang & Wang, 2018; Qin et al., 2019), wireless network (Zhou & Bambos, 2015; Zhou et al., 2016), consumer choice modeling (Xu & Yun, 2020), ride-sharing (Taylor, 2018; Gurvich et al., 2018; Bimpikis et al., 2019; Kanoria & Qian, 2019), healthcare operations (Shortreed et al., 2010), epidemic control (Nowzari et al., 2016; Kiss et al., 2017), and inventory control (Huh & Rusmevichientong, 2009; Bertsekas, 2017; Zhang et al., 2018; Agrawal & Jia, 2019; Chen et al., 2019a).

There exists numerous works in sequential decision-making that considered part of the four challenges. The traditional stream of research (Auer et al., 2002a; Bubeck & Cesa-Bianchi, 2012; Lattimore & Szepesvári, 2018) on stochastic multi-armed bandits (MAB) focused on the interplay between uncertainty and bandit feedback (i.e., challenges 3 and 4), and (Auer et al., 2002a) proposed the classical Upper Confidence Bound (UCB) algorithm. Starting from (Burnetas & Katehakis, 1997; Tewari & Bartlett, 2008; Jaksch et al., 2010), a volume of works (see Section 3) have been
devoted to reinforcement learning (RL) in MDPs (Sutton & Barto, 2018), which further involves endogeneity. RL in MDPs incorporate challenges 1,3,4, and stochastic MAB is a special case of MDPs when there is only one state. In the absence of exogeneity, the reward and state transition distributions are invariant across time, and these three challenges can be jointly solved by the Upper Confidence bound for Reinforcement Learning (UCRL2) algorithm (Jaksch et al., 2010).

The UCB and UCRL2 algorithms leverage the optimism in face of uncertainty (OFU) principle to select actions iteratively based on the entire collections of historical data. However, both algorithms quickly deteriorate when exogeneity emerge since the environment can change over time, and the historical data becomes obsolete. To address the challenge of exogeneity, (Garivier & Moulines, 2011b) considered the piecewise-stationary MAB environment where the reward distributions remain unaltered over certain time periods and change at unknown time steps. Later on, there is a line of research initiated by (Besbes et al., 2014) that studied the general non-stationary MAB environment (Besbes et al., 2014; Cheung et al., 2019a;b), in which the reward distributions can change arbitrarily over time, but the total changes (quantified by a suitable metric) is upper bounded by a variation budget (Besbes et al., 2014). The aim is to minimize the dynamic regret, the optimality gap compared to the cumulative rewards of the sequence of optimal actions. Both the (relatively restrictive) piecewise-stationary MAB and the general non-stationary MAB settings consider the challenges of exogeneity, uncertainty, and partial feedback (i.e., challenges 2, 3, 4), but endogeneity (challenge 1) are not present.

In this paper, to address all four above-mentioned challenges, we consider RL in non-stationary MDPs where both the reward and state transition distributions can change over time, but the total changes (quantified by suitable metrics) are upper bounded by the respective variation budgets. We note that in (Jaksch et al., 2010), the authors also consider the intermediate RL in piecewise-stationary MDPs. Nevertheless, we first demonstrate in Section 4.1, and then rigorously show in Section 6 that simply adopting the techniques for non-stationary MAB (Besbes et al., 2014; Cheung et al., 2019a;b) or RL in piecewise-stationary MDPs (Jaksch et al., 2010) to RL in non-stationary MDPs may result in poor dynamic regret bounds.

1.1. Summary of Main Contributions

Assuming that, during the $T$ time steps, the total variations of the reward and state transition distributions are bounded (under suitable metrics) by the variation budgets $B_r$ ($>0$) and $B_p$ ($>0$), respectively, we design and analyze novel algorithms for RL in non-stationary MDPs. Let $D_{\text{max}}$, $S$, and $A$ be respectively the maximum diameter (a complexity measure to be defined in Section 2), number of states, and number of actions in the MDP. Our main contributions are:

- We develop the Sliding Window UCRL2 with Confidence Widening (SWUCRL2-CW) algorithm. When the variation budgets are known, we prove it attains a $\tilde{O}(D_{\text{max}}(B_r + B_p)^{1/4}S^{2/3}A^{1/2}T^{3/4})$ dynamic regret bound via a budget-aware analysis.
- We propose the Bandit-over-Reinforcement Learning (BORL) algorithm that tunes the SWUCRL2-CW algorithm adaptively, and retains the same $\tilde{O}(D_{\text{max}}(B_r + B_p)^{1/4}S^{2/3}A^{1/2}T^{3/4})$ dynamic regret bound without knowing the variation budgets.
- We identify an unprecedented challenge for RL in non-stationary MDPs with conventional optimistic exploration techniques: existing algorithmic frameworks for non-stationary online learning (including non-stationary bandit and RL in piecewise-stationary MDPs) (Jaksch et al., 2010; Garivier & Moulines, 2011b; Cheung et al., 2019a) typically estimate unknown parameters by averaging historical data in a “forgetting” fashion, and construct the tightest possible confidence regions/interervals accordingly. They then optimistically search for the most favorable model within the confidence regions, and execute the corresponding optimal policy. However, we first demonstrate in Section 4.1, and then rigorously show in Section 6 that in the context of RL in non-stationary MDPs, the diameters induced by the MDPs in the confidence regions constructed in this manner can grow wildly, and may result in unfavorable dynamic regret bound. We overcome this with our novel proposal of extra optimism via the confidence widening technique (alternatively, in (Cheung et al.), an extended version of the current paper, the authors demonstrate that one can leverage special structures

|                | Stationary | Non-stationary |
|----------------|------------|----------------|
| MAB            | OFU (Auer et al., 2002a) | OFU + Forgetting (Besbes et al., 2014; Cheung et al., 2019b) |
| RL             | OFU (Jaksch et al., 2010) | Extra optimism + Forgetting (This paper) |

Table 1. Summary of algorithmic frameworks of stationary and non-stationary online learning settings.
on the state transition distributions in the context of single item inventory control with fixed cost to bypass this difficulty of exploring time-varying environments. A summary of the algorithmic frameworks for stationary and non-stationary online learning settings are provided in Table 1.

2. Problem Formulation

In this section, we introduce the notations to be used throughout paper, and introduce the learning protocol for our problem of RL in non-stationary MDPs.

2.1. Notations

Throughout the paper, all vectors are column vectors, unless specified otherwise. We define \([n]\) to be the set \([1, 2, \ldots, n]\) for any positive integer \(n\). We denote \(1[\cdot]\) as the indicator function. For \(p \in [1, \infty]\), we use \(\|x\|_p\) to denote the \(p\)-norm of a vector \(x \in \mathbb{R}^d\). We denote \(x \lor y\) and \(x \land y\) as the maximum and minimum between \(x, y \in \mathbb{R}\), respectively. We adopt the asymptotic notations \(O(\cdot), \Omega(\cdot),\) and \(\Theta(\cdot)\) (Cormen et al., 2009). When logarithmic factors are omitted, we use \(\tilde{O}(\cdot), \tilde{\Omega}(\cdot),\) and \(\tilde{\Theta}(\cdot)\), respectively. With some abuse, these notations are used when we try to avoid the clutter of writing out constants explicitly.

2.2. Learning Protocol

Model Primitives: An instance of non-stationary MDP is specified by the tuple \((S, A, T, r, p)\). The set \(S\) is a finite set of states. The collection \(A = \{A_s\}_{s \in S}\) contains a finite action set \(A_s\) for each state \(s \in S\). We say that \((s, a)\) is a state-action pair if \(s \in S, a \in A_s\). We denote \(S = |S|, A = (\sum_{s \in S} |A_s|)/|S|\). We denote \(T\) as the total number of time steps, and denote \(r = \{r_t\}_{t=1}^T\) as the sequence of mean rewards. For each \(t\), we have \(r_t = \{r_t(s, a)\}_{s \in S, a \in A_s}\), and \(r_t(s, a) \in [0, 1]\) for each state-action pair \((s, a)\). In addition, we denote \(p = \{p_t\}_{t=1}^T\) as the sequence of state transition distributions. For each \(t\), we have \(p_t = \{p_t(\cdot|s, a)\}_{s \in S, a \in A_s}\), where \(p_t(\cdot|s, a)\) is a probability distribution over \(S\) for each state-action pair \((s, a)\).

Exogeneity: The quantities \(r_t\)’s and \(p_t\)’s vary across different \(t\)’s in general. Following (Besbes et al., 2014), we quantify the variations on \(r_t\)’s and \(p_t\)’s in terms of their respective variation budgets \(B_r, B_p\) \((> 0)\):

\[
B_r = \sum_{t=1}^{T-1} B_{r, t}, \quad B_p = \sum_{t=1}^{T-1} B_{p, t},
\]

where \(B_{r, t} = \max_{s \in S, a \in A_s} |r_{t+1}(s, a) - r_t(s, a)|\) and \(B_{p, t} = \max_{s \in S, a \in A_s} \|p_{t+1}(\cdot|s, a) - p_t(\cdot|s, a)\|_1\). We emphasize although \(B_r\) and \(B_p\) might be used as inputs by the DM, individual \(B_{r, t}\)’s and \(B_{p, t}\)’s are unknown to the DM throughout the current paper.

Endogeneity: The DM faces a non-stationary MDP instance \((S, A, T, r, p)\). She knows \(S, A, T,\) but not \(r, p\). The DM starts at an arbitrary state \(s_1 \in S\). At time \(t\), three events happen. First, the DM observes its current state \(s_t\). Second, she takes an action \(a_t \in A_{s_t}\). Third, given \(s_t, a_t\), she stochastically transits to another state \(s_{t+1}\) which is distributed as \(p_t(\cdot|s_t, a_t)\), and receives a stochastic reward \(r_t(s_t, a_t)\), which is 1-sub-Gaussian with mean \(r_t(s_t, a_t)\). In the second event, the choice of \(a_t\) is based on a non-anticipatory policy \(\Pi\). That is, the choice only depends on the current state \(s_t\) and the previous observations \(\mathcal{H}_{t-1} := \{s_q, a_q, R_q(s_q, a_q)\}_{q=1}^{t-1}\).

Dynamic Regret: The DM aims to maximize the cumulative expected reward \(\mathbb{E}[\sum_{t=1}^T r_t(s_t, a_t)]\), despite the model uncertainty on \(r, p\) and the dynamics of the learning environment. To measure the convergence to optimality, we consider an equivalent objective of minimizing the dynamic regret (Besbes et al., 2014; Jaksch et al., 2010):

\[
\text{Dyn-Reg}_T(\Pi) = \sum_{t=1}^T \left(\rho_t^* - \mathbb{E}[r_t(s_t, a_t)]\right).
\]

In the oracle \(\sum_{t=1}^T \rho_t^*\), the summand \(\rho_t^*\) is the optimal long-term average reward of the stationary MDP with state transition distribution \(p_t\) and mean reward \(r_t\). The optimum \(\rho_t^*\) can be computed by solving linear program (9) provided in Section A.1. We note that the same oracle is used for RL in piecewise-stationary MDPs (Jaksch et al., 2010).

Remark 1. When \(S = 1\), (2) reduces to the definition (Besbes et al., 2014) of dynamic regret for non-stationary K-armed bandits. Nevertheless, different from the bandit case, the offline benchmark \(\sum_{t=1}^T \rho_t^*\) does not equal to the expected optimum for the non-stationary MDP problem in general. We justify our choice in Proposition 1.

Next, we review concepts of communicating MDPs and diameters, in order to stipulate an assumption that ensures learnability and justifies our offline benchmark.

Definition 1 (Jaksch et al., 2010) Communicating MDPs and Diameters. Consider a set of states \(S\), a collection \(A = \{A_s\}_{s \in S}\) of action sets, and a transition kernel \(\bar{p} = \{\bar{p}(\cdot|s, a)\}_{s \in S, a \in A_s}\). For any \(s, s' \in S\) and stationary policy \(\pi\), the hitting time from \(s\) to \(s'\) under \(\pi\) is the random variable \(\Lambda(s'|\pi, s) := \min \{t : s_{t+1} = s', s_1 = s, s_{t+1} \sim \bar{p}(\cdot|s_t, \pi(s_t)) \forall t\}\), which can be infinite. We say that \((S, \bar{p})\) is a communicating MDP iff

\[
D := \max_{s, s' \in S \text{ stationary } \pi} \mathbb{E}[\Lambda(s'|\pi, s)]
\]

is finite. The quantity \(D\) is the diameter associated with \((S, \bar{p})\).
We make the following assumption throughout.

**Assumption 1.** For each \( t \in \{1, \ldots, T\} \), the tuple \((S, A, p_t)\) constitutes a communicating MDP with diameter at most \( D_t \). We denote the maximum diameter as \( D_{\max} = \max_{t \in \{1, \ldots, T\}} D_t \).

The following proposition justifies our choice of offline benchmark \( \sum_{t=1}^T \rho_t \).

**Proposition 1.** Consider an instance \((S, A, T, p, r)\) that satisfies Assumption 1 with maximum diameter \( D_{\max} \), and has variation budgets \( B_r, B_p \) for the rewards and transition kernels respectively. In addition, suppose that \( T \geq B_r + 2D_{\max}B_p > 0 \). It holds that \( \sum_{t=1}^T \rho_t^* \geq \max_{\Pi} \{ E \left[ \sum_{t=1}^T r_t(s_t^\Pi, a_t^\Pi) \right] \} - 4(D_{\max} + 1)\sqrt{(B_r + 2D_{\max}B_p)T} \). The maximum is taken over all non-anticipatory policies \( \Pi \)'s. We denote \( \{ (s_t^\Pi, a_t^\Pi) \}_{t=1}^T \) as the trajectory under policy \( \Pi \), where \( a_t^\Pi \in A(s_t) \) is determined based on \( \Pi \) and \( H_{t-1} \cup \{ s_t^\Pi \} \), and \( s_{t+1}^\Pi \sim p_t(s_t^\Pi, a_t^\Pi) \) for each \( t \).

The Proposition is proved in Appendix A.2. In fact, our dynamic regret bounds are larger than the error term \( 4(D_{\max} + 1)\sqrt{(B_r + 2D_{\max}B_p)T} \), thus justifying the choice of \( \sum_{t=1}^T \rho_t^* \) as the offline benchmark. The offline benchmark \( \sum_{t=1}^T \rho_t^* \) is more convenient for analysis than the expected optimum, since the former can be decomposed to summations across different intervals, unlike the latter where the summands are intertwined (since \( s_t^\Pi \sim p_t(s_t^\Pi, a_t^\Pi) \)).

**3. Related Works**

**3.1. RL in Stationary MDPs**

RL in stationary (discounted and un-discounted reward) MDPs has been widely studied in (Burnetas & Katehakis, 1997; Bartlett & Tewari, 2009; Jaksch et al., 2010; Agrawal & Jia, 2017; Fruit et al., 2018a;b; Sidford et al., 2018b;a; Wang, 2019; Zhang & Ji, 2019; Fruit et al., 2019; Wei et al., 2019). For the discounted reward setting, the authors of (Sidford et al., 2018b; Wang, 2019; Sidford et al., 2018a) proposed (nearly) optimal algorithms in terms of sample complexity. For the un-discounted reward setting, the authors of (Jaksch et al., 2010) established a minimax lower bound \( \Omega(\sqrt{D_{\max}SAT}) \) on the regret when both the reward and state transition distributions are time-invariant. They also designed the UCRL2 algorithm and showed that it attains a regret bound \( \tilde{O}(D_{\max}S\sqrt{AT}) \). The authors of (Fruit et al., 2019) proposed the UCRL2B algorithm, which is an improved version of the UCRL2 algorithm. The regret bound of the UCRL2B algorithm is \( \tilde{O}(S\sqrt{D_{\max}AT} + D_{\max}S^2A) \). The minimax optimal algorithm is provided in (Zhang & Ji, 2019) although it is not computationally efficient.

**3.2. RL in Non-Stationary MDPs**

In a parallel work (Ortner et al., 2019), the authors considered a similar setting to ours by applying the “forgetting principle” from non-stationary bandit settings (Garivier & Moulines, 2011b; Cheung et al., 2019b) to design a learning algorithm. To achieve its dynamic regret bound, the algorithm by (Ortner et al., 2019) partitions the entire time horizon \( [T] \) into time intervals \( \mathcal{I} = \{I_k\}_{k=1}^K \), and crucially requires the access to \( \sum_{t=\min(I_k)}^{\max(I_k-1)} B_{r,t} \) and \( \sum_{t=\min(I_k)}^{\max(I_k-1)} B_{p,t} \), i.e., the variations in both reward and state transition distributions of each interval \( I_k \in \mathcal{I} \) (see Theorem 3 in (Ortner et al., 2019)). In contrast, the SWUCRL2-CW algorithm and the BORL algorithm require significantly less information on the variations. Specifically, the SWUCRL2-CW algorithm does not need any additional knowledge on the variations except for \( B_r \) and \( B_p \), i.e., the variation budgets over the entire time horizon as defined in eqn. (1), to achieve its dynamic regret bound (see Theorem 1). This is similar to algorithms for the non-stationary bandit settings, which only require the access to \( B_r \) (Besbes et al., 2014). More importantly, the BORL algorithm (built upon the SWUCRL2-CW algorithm) enjoys the same dynamic regret bound even without knowing either of \( B_r \) or \( B_p \) (see Theorem 2).

There also exists some settings that are closely related to, but different from our setting (in terms of exogeneity and feedback). (Jaksch et al., 2010; Gajane et al., 2018) proposed solutions for the RL in piecewise-stationary MDPs setting. But as discussed before, simply applying their techniques to the general RL in non-stationary MDPs may result in undesirable dynamic regret bounds (see Section 6 for more details). In (Yu et al., 2009; Neu et al., 2010; Arora et al., 2012; Dick et al., 2014; Jin et al., 2019; Cardoso et al., 2019), the authors considered RL in MDPs with changing reward distributions but fixed transition distributions. The authors of (Even-Dar et al., 2005; Yu & Mannor, 2009; Neu et al., 2012; Abbasi-Yadkori et al., 2013; Rosenberg & Mansour, 2019; Li et al., 2019) considered RL in non-stationary MDPs with full information feedback.

**3.3. Non-Stationary MAB**

For online learning and bandit problems where there is only one state, the works by (Auer et al., 2002b; Garivier & Moulines, 2011b; Besbes et al., 2014; Keskin & Zeevi, 2016) proposed several “forgetting” strategies for different non-stationary MAB settings. More recently, the works by (Karnin & Anava, 2016; Luo et al., 2018; Cheung et al., 2019a;b; Chen et al., 2019b) designed parameter-free algorithms for non-stationary MAB problems. Another related but different setting is the Markovian bandit (Kim & Lim, 2016; Ma, 2018), in which the state of the chosen action is updated.
changed. In (Zhou et al., 2020), the authors also considered the case when the states of all the actions are governed by the same (uncontrollable) Markov chain.

4. Sliding Window UCRL2 with Confidence Widening

In this section, we present the SWUCRL2–CW algorithm, which incorporates sliding window estimates (Garivier & Moulines, 2011a) and a novel confidence widening technique into UCRL2 (Jaksch et al., 2010).

4.1. Design Challenge: Failure of Naive Sliding Window UCRL2 Algorithm

For stationary MAB problems, the UCB algorithm (Auer et al., 2002a) suggests the DM should iteratively execute the following two steps in each time step:

1. Estimate the mean reward of each action by taking the time average of all observed samples.
2. Pick the action with the highest estimated mean reward plus the confidence radius, where the radius scales inversely proportional with the number of observations (Auer et al., 2002a).

The UCB algorithm has been proved to attain optimal regret bounds for various stationary MAB settings (Auer et al., 2002a; Kveton et al., 2015). For non-stationary problems, (Garivier & Moulines, 2011b; Keskin & Zeevi, 2016; Cheung et al., 2019b) shown that the DM could further leverage the forgetting principle by incorporating the sliding-window estimator (Garivier & Moulines, 2011b) into the UCB algorithms (Auer et al., 2002a; Kveton et al., 2015) to achieve optimal dynamic regret bounds for a wide variety of non-stationary MAB settings. The sliding window UCB algorithm with a window size $W \in \mathbb{R}_+$ is similar to the UCB algorithm except that the estimated mean rewards are computed by taking the time average of the $W$ most recent observed samples.

As noted in Section 1, (Jaksch et al., 2010) proposed the UCRL2 algorithm, which is a UCB-alike algorithm with nearly optimal regret for RL in stationary MDPs. It is thus tempting to think that one could also integrate the forgetting principle into the UCRL2 algorithm to attain low dynamic regret bound for RL in non-stationary MDPs. In particular, one could easily design a naive sliding-window UCRL2 algorithm that follows exactly the same steps as the UCRL2 algorithm with the exception that it uses only the $W$ most recent observed samples instead of all observed samples to estimate the mean rewards and the state transition distributions, and to compute the respective confidence radius.

Under non-stationarity and bandit feedback, however, we show in Proposition 3 of the forthcoming Section 6 that the diameter of the estimated MDP produced by the naive sliding-window UCRL2 algorithm with window size $W$ can be as large as $\Theta(W)$, which is orders of magnitude larger than $D_{\text{max}}$, the maximum diameter of each individual MDP encountered by the DM. Consequently, the naive sliding-window UCRL2 algorithm may result in undesirable dynamic regret bound. In what follows, we discuss in more details how our novel confidence widening technique can mitigate this issue.

4.2. Design Overview

The SWUCRL2–CW algorithm first specifies a sliding window parameter $W \in \mathbb{N}$ and a confidence widening parameter $\eta \geq 0$. Parameter $W$ specifies the number of previous time steps to look at. Parameter $\eta$ quantifies the amount of additional optimistic exploration, on top of the conventional optimistic exploration using upper confidence bounds. The former turns out to be necessary for handling the drifting non-stationarity of the transition kernel.

The algorithm runs in a sequence of episodes that partitions the $T$ time steps. Episode $m$ starts at time $\tau(m)$ (in particular $\tau(1) = 1$), and ends at the end of time $\tau(m+1) - 1$. Throughout an episode $m$, the DM follows a certain stationary policy $\hat{\pi}_m$. The DM ceases the $m$th episode if at least one of the following two criteria is met:

- The time index $t$ is a multiple of $W$. Consequently, each episode last for at most $W$ time steps. The criterion ensures that the DM switches the stationary policy $\hat{\pi}_m$ frequently enough, in order to adapt to the non-stationarity of $r_i$'s and $p_i$'s.
- There exists some state-action pair $(s, a)$ such that $v_m(s, a)$, the number of time step $t$'s with $(s_t, a_t) = (s, a)$ within episode $m$, is at least as many as the total number of counts for it within the $W$ time steps prior to $\tau(m)$, i.e., from $(\tau(m) - W) \vee 1$ to $\tau(m) - 1$. This is similar to the doubling criterion in (Jaksch et al., 2010), which ensures that each episode is sufficiently long so that the DM can focus on learning.

The combined effect of these two criteria allows the DM to learn a low dynamic regret policy with historical data from an appropriately sized time window. One important piece of ingredient is the construction of the policy $\hat{\pi}_m$, for each episode $m$. To allow learning under non-stationarity, the SWUCRL2–CW algorithm computes the policy $\hat{\pi}_m$ based on the history in the $W$ time steps previous to the current episode $m$, i.e., from round $(\tau(m) - W) \vee 1$ to round $\tau(m) - 1$. The construction of $\hat{\pi}_m$ involves the Extended Value Iteration (EVI) (Jaksch et al., 2010), which requires the confidence regions $H_{r, \tau(m)}, H_{p, \tau(m)}(\eta)$ for rewards and transition kernels as the inputs, in addition to an precision parameter $\epsilon$. The confidence widening parameter $\eta \geq 0$ is capable of ensuring the MDP output by the EVI has a
bounded diameter most of the time.

4.3. Policy Construction

To describe \textit{SWUCRL2-CW} algorithm, we define for each state action pair \((s, a)\) and each round \(t\) in episode \(m\),

\[
N_t(s, a) = \sum_{q=(\tau(m)-W)/\gamma+1}^{t-1} 1(s_q, a_q) = (s, a)) ,
\]

\[
N_t^+(s, a) = \max\{1, N_t(s, a)\} .
\]

(3)

4.3.1. Confidence Region for Rewards.

For each state action pair \((s, a)\) and each time step \(t\) in episode \(m\), we consider the empirical mean estimator

\[
\hat{\mu}_t(s, a) = \sum_{q=(\tau(m)-W)/\gamma+1}^{t-1} R_q(s, a) 1(s_q = s, a_q = a) / N_t^+(s, a) ,
\]

which serves to estimate the average reward

\[
\bar{\mu}_t(s, a) = \sum_{q=(\tau(m)-W)/\gamma+1}^{t-1} r_q(s, a) 1(s_q = s, a_q = a) / N_t^+(s, a) .
\]

The confidence region \(H_{r,t} = \{H_{r,t}(s, a)\}_{s \in S, a \in A_s}\) is defined as

\[
H_{r,t}(s, a) = \{\bar{r} \in [0, 1] : |\bar{r} - \bar{\mu}_t(s, a)| \leq \text{rad}_{r,t}(s, a)\},
\]

(4)

with confidence radius

\[
\text{rad}_{r,t}(s, a) = 2\sqrt{2\log(SAT/\delta)/N_t^+(s, a)}.
\]

4.3.2. Confidence Widening for Transition Kernels.

For each state action pair \(s, a\) and each time step \(t\) in episode \(m\), consider the empirical estimator

\[
\hat{\rho}_t(s'|s, a) = \sum_{q=(\tau(m)-W)/\gamma+1}^{t-1} 1(s_q = s, a_q = a, s_{q+1} = s') / N_t^+(s, a) ,
\]

which serves to estimate the average transition probability

\[
\hat{\rho}_t(s'|s, a) = \sum_{q=(\tau(m)-W)/\gamma+1}^{t-1} p_q(s'|s, a) 1(s_q = s, a_q = a) / N_t^+(s, a) .
\]

Different from the case of estimating reward, the confidence region \(H_{p,t}(\eta) = \{H_{p,t}(s, a; \eta)\}_{s \in S, a \in A_s}\) for the transition probability involves a widening parameter \(\eta \geq 0\):

\[
H_{p,t}(s, a; \eta) = \{\hat{\rho} \in \mathcal{D}^S : ||\hat{\rho}(\cdot|s, a) - \hat{\rho}_t(\cdot|s, a)||_1 \leq \text{rad}_{p,t}(s, a) + \eta\} ,
\]

(5)

with confidence radius

\[
\text{rad}_{p,t}(s, a) = 2\sqrt{2S\log(SAT/\delta)/N_t^+(s, a)} .
\]

In a nutshell, the incorporation of \(\eta > 0\) provides an additional source of optimism, and the DM can explore transition kernels that further deviate from the sample average. This turns out to be crucial for learning MDPs under drifting non-stationarity. We treat \(\eta\) as a hyper-parameter at the moment, and provide a suitable choice of \(\eta\) when we discuss our main results.

4.3.3. Extended Value Iteration (EVI) (Jaksch et al., 2010).

The \textit{SWUCRL2-CW} algorithm relies on the EVI, which solves MDPs with optimistic exploration to near-optimality.

We extract and rephrase a description of EVI in Appendix A.3. EVI inputs the confidence regions \(H_r, H_p\) for the rewards and the transition kernels. The algorithm outputs an “optimistic MDP model”, which consists of reward vector \(\bar{r}\) and transition kernel \(\bar{\rho}\) under which the optimal average gain \(\bar{\rho}\) is the largest among all \(\bar{r} \in H_r, \bar{\rho} \in H_p\):

- **Input**: Confidence regions \(H_r\) for \(r, H_p\) for \(p\), and an error parameter \(\epsilon > 0\).

- **Output**: The returned policy \(\bar{\pi}\) and the auxiliary output \((\bar{r}, \bar{p}, \bar{\rho}, \bar{\gamma})\). In the latter, \(\bar{r}, \bar{p}, \bar{\rho}\) are the selected “optimistic” reward vector, transition kernel, and the corresponding long term average reward. The output \(\bar{\gamma}(s)\) is a bias vector (Jaksch et al., 2010). For each \(s \in S\), the quantity \(\bar{\gamma}(s)\) is indicative of the short term reward when the DM starts at state \(s\) and follows the optimal policy. By the design of EVI, for the output \(\bar{\gamma}\), there exists \(s \in S\) such that \(\bar{\gamma}(s) = 0\). Altogether, we express

\[
\text{EVI}(H_r, H_p; \epsilon) \rightarrow (\bar{\pi}, \bar{r}, \bar{p}, \bar{\rho}, \bar{\gamma}) .
\]

Combining the three components, a formal description of the \textit{SWUCRL2-CW} algorithm is shown in Algorithm 1.

4.4. Performance Analysis: The Blessing of More Optimism

We now analyze the performance of the \textit{SWUCRL2-CW} algorithm. First, we introduce two events \(\mathcal{E}_r, \mathcal{E}_p\), which state that the estimated reward and transition kernels lie in the respective confidence regions.

\[
\mathcal{E}_r = \{\hat{r}_t(s, a) \in H_{r,t}(s, a) \forall s, a, t\} ,
\]

\[
\mathcal{E}_p = \{\hat{\rho}_t(\cdot|s, a) \in H_{p,t}(s, a; 0) \forall s, a, t\} .
\]

We prove that \(\mathcal{E}_r, \mathcal{E}_p\) hold with high probability.

**Lemma 1.** We have \(\Pr[\mathcal{E}_r] \geq 1 - \delta/2, \Pr[\mathcal{E}_p] \geq 1 - \delta/2\).

The proof is provided in Section B of the appendix. In defining \(\mathcal{E}_p\), the widening parameter \(\eta\) is set to be 0, since we are only concerned with the estimation error on \(p\). Next, we
Consider an episode analysis shows that $\eta$ is a sum of diameter $H$. Suppose $\eta$ have $p$ kernel $E$. We find the following variation measure for each assumptions on $\rho$.

The algorithm is terminated.

10: $12$: $13$: $\textbf{end for}$

4: Compute a $6$: $9$: $\textbf{Update}$

3: $2$: $8$: $\textbf{Choose action}$

2: $1$: $7$: $\textbf{while} t$ is not a multiple of $W$ and $\nu_m(s_t, \pi_m(s_t)) < N^+(\tau(m))(s_t, \pi_m(s_t))$ $\textbf{do}$

1: $\textbf{Input:}$ Time horizon $T$, state space $S$, action space $A$, window size $W$, widening parameter $\eta$.

2: $\textbf{Initialize}$ $t \leftarrow 1, \text{initial state} \ s_1$.

3: for episode $m = 1, 2, \ldots$ do

4: Set $\tau(m) \leftarrow t$, $\nu_m(s, a) \leftarrow 0$, and $N(\tau(m))(s, a)$ according to Eqn (3), for all $s, a$.

5: Compute the confidence regions $H_r, H_m(\tau(m))$ according to Eqs (4, 5).

6: Compute a $1/\sqrt{\tau(m)}$-optimistic policy $\check{\pi}_m : \text{EV}(H_r, \tau(m), H_m(\tau(m))(\eta) : 1/\sqrt{\tau(m)}) \rightarrow (\pi, \tilde{r}, m, \tilde{\rho}, m, \tilde{\gamma}_m)$.

7: while $t$ is not a multiple of $W$ and $\nu_m(s_t, \pi_m(s_t)) < N^+(\tau(m))(s_t, \pi_m(s_t))$ do

8: Choose action $a_t = \pi_m(s_t)$, observe reward $R_3(s_t, a_t)$ and the next state $s_{t+1}$.

9: Update $\nu_m(s_t, a_t) \leftarrow \nu_m(s_t, a_t) + 1, t \leftarrow t + 1$.

10: if $t > T$ then

11: The algorithm is terminated.

12: $\textbf{end if}$

13: $\textbf{end while}$

14: $\textbf{end for}$

bound the dynamic regret of each time step, under certain assumptions on $H_p(\tau(m))$. To facilitate our discussion, we define the following variation measure for each $t$ in an episode $m$:

$$\text{var}_{r,t} = \sum_{q=\tau(m)-W}^{t-1} B_{r,q}, \quad \text{var}_{p,t} = \sum_{q=\tau(m)-W}^{t-1} B_{p,q}. $$

Proposition 2. Consider an episode $m$. Condition on events $E_r, E_p$, and suppose that there exists a transition kernel $p$ satisfying two properties: (1) $\forall s \in S \forall a \in A$, we have $p(s|a) \in H_p(\tau(m))(s, a; \eta)$, and (2) the diameter of $(S, A, p)$ at most $D$. Then, for every $t \in \{\tau(m), \ldots, \tau(m+1) - 1\}$ in episode $m$, we have:

$$\rho_t - r_t(s_t, a_t) \leq \sum_{s' \in S} p_t(s'|s, a_t) \tilde{\gamma}_r(m)(s') - \tilde{\gamma}_r(m)(s)$$

$$+ 1 \sqrt{\tau(m)} + 2 \text{var}_{r,t} + 4D(\text{var}_{p,t} + \eta)$$

$$+ 2 \text{rad}_{r,\tau(m)}(s_t, a_t) + 4D \cdot \text{rad}_{p,\tau(m)}(s, a).$$

The complete proof is in Section C of the appendix. Unlike Lemma 1, the parameter $\eta$ plays an important role in the Proposition. As $\eta$ increases, the confidence region $H_p(\tau(m))(s, a; \eta)$ becomes larger for each $s, a$, and the assumed diameter $D$ is expected to decrease. Our subsequent analysis shows that $\eta$ can be suitably calibrated so that $D = O(D_{\text{max}})$. Next, we state our first main result, which provides a dynamic regret bound assuming the knowledge of $B_r, B_p$ to set $W, \eta$.

**Theorem 1.** Assuming $S > 1$, the SWUCRL2-CW algorithm with window size $W$ and confidence widening parameter $\eta > 0$ satisfies the dynamic regret bound

$$\tilde{O} \left( B_p W/\eta + B_r W + \sqrt{\text{SAT}} / \sqrt{W} \right)$$

$$+ D_{\text{max}} \left( B_p W + \sqrt{\text{SAT}} / \sqrt{W} + T \eta + \text{SAT}/W + \sqrt{T} \right),$$

with probability $1 - O(\delta)$. Putting $W = W^* := 3S^{2} A^{2} T^{2} / (B_r + B_p + 1)^{2}$ and $\eta = \eta^* := (B_p + 1)W^*/T$, the bound specializes to

$$\tilde{O} \left( D_{\text{max}}(B_r + B_p + 1)^{2}S^{2} A^{2} T^{2} \right).$$

**Proof Sketch.** The complete proof is presented in Section D of the appendix. Proposition 2 states that if the confidence region $H_p(\tau(m))(\eta)$ contains a transition kernel that induces a MDP with bounded diameter $D$, the EVI supplied with $H_p(\tau(m))(\eta)$ can return a policy with controllable dynamic regret bound. However, as we show in Section 6, one in general cannot expect this to happen. Nevertheless, we bypass this with our novel confidence widening technique and a budget-aware analysis. We consider the first time step $\tau(m)$ of each episode $m$; if $p(\tau(m))(s, a) \in H_p(\tau(m))(s, a; \eta)$ for all $(s, a)$, then Proposition 2 can be leveraged; otherwise, the widened confidence region enforces that a considerable amount of variation budget is consumed.

**Remark 2.** When $S = \{s\}$, our problem becomes the non-stationary bandit problem studied by (Besbes et al., 2014), and we have $D_{\text{max}} = 0$ and $B_p = 0$. By choosing $W = W^* := A^{1/3} T^{2/3} / B_r^{3/3}$, our algorithm has dynamic regret $\tilde{O}(B_r^{1/3} A^{1/3} T^{2/3})$, matching the minimax optimal dynamic regret bound by (Besbes et al., 2014) when $B_r \in [A^{-1}, A^{-1}T]$. 

**Remark 3.** Similar to (Cheung et al., 2019b), if $B_p, B_r$ are not known, we can set $W$ and $\eta$ obliviously as $W = S^{2} A^{2} T^{2} / \eta = \sqrt{W/T} = S^{2} A^{2} T^{2}$ to obtain a dynamic regret bound $\tilde{O} \left( D_{\text{max}}(B_r + B_p + 1)^{2}S^{2} A^{2} T^{2} \right).$

5. Bandit-over-Reinforcement Learning: Towards Parameter-Free

As said in Remark 3, in the case of unknown $B_r$ and $B_p$, the dynamic regret of SWUCRL2-CW algorithm scales linearly in $B_r$ and $B_p$, which leads to a $\Omega(T)$ dynamic regret when $B_r$ or $B_p = \Omega(T^{1/4})$. In comparison, Theorem 1 assumes us that by using $(W^*, \eta^*)$, we can achieve a $o(T)$ dynamic regret when $B_r, B_p = o(T)$. For the bandit setting, (Cheung et al., 2019b) proposes the bandit-over-bandit
framework that uses a separate copy of EXP3 algorithm to tune the window length. Inspired by it, we develop a novel Bandit-over-Reinforcement Learning (BORL) algorithm, which is parameter free and has dynamic regret bound equal to (8). Following (Cheung et al., 2019b), we view the SWUCRL2-CW algorithm as a sub-routine, and “hedge” (Bubeck & Cesa-Bianchi, 2012) against the (possibly adversarial) changes of \( r_t \)’s and \( p_t \)’s to identify a reasonable fixed window length and confidence widening parameter. As illustrated in Fig. 1, the BORL algorithm divides the whole time horizon into \( \lceil T/H \rceil \) blocks of equal length \( H \) rounds (the length of the last block can \( \leq H \)), and specifies a set \( J \) from which each pair of (window length, confidence widening parameter) are drawn from. For each block \( i \in \lceil T/H \rceil \), the BORL algorithm first calls some master algorithm to select a pair of (window length, confidence widening parameter) \((W_i, \eta_i) \in J\), and restarts the SWUCRL2-CW algorithm with the selected parameters as a sub-routine to choose actions for this block. Afterwards, the total reward of block \( i \) is fed back to the master, and the “posterior” of these parameters are updated accordingly.

One immediate challenge not presented in the bandit setting (Cheung et al., 2019b) is that the starting state of each block is determined by previous moves of the DM. Hence, the master algorithm is not facing a simple oblivious environment as the case in (Cheung et al., 2019b), and we cannot use the EXP3 (Auer et al., 2002b) algorithm as the master. Nevertheless, the state is observed before the starting of a block. Thus, we use the EXP3.P algorithm for multi-armed bandit against an adaptive adversary (Auer et al., 2002b) as the master algorithm. Owing to its similarity to the BOB algorithm (Cheung et al., 2019b), we defer the design details of the BORL algorithm to Section E of the appendix. The proof of the following is in Section F of the appendix.

Theorem 2. Assume \( S > 1 \), with probability \( 1 - O(\delta) \), the dynamic regret bound of the BORL algorithm is \( \tilde{O}(\max(B_r + B_p + 1)^4 S^4 A^3 T^4) \).

6. The Perils of Drift in Learning Markov Decision Processes

In stochastic online learning problems, one usually estimates a latent quantity by taking the time average of observed samples, even when the sample distribution varies across time. This has been proved to work well in stationary and non-stationary bandit settings (Auer et al., 2002a; Garivier & Moulines, 2011b; Cheung et al., 2019b-a). To extend to RL, it is natural to consider the sample average transition distribution \( \hat{p}_t \), which uses the data in the previous \( W \) rounds to estimate the time average transition distribution \( p_t \) to within an additive error \( \tilde{O}(1/\sqrt{N^2_{\hat{p}}(s,a)}) \) (see Lemma 1). In this case of stationary MDPs, where \( \forall \ell \in [T] \) \( p_t = p \), one has \( \hat{p}_t = p \). Thus, the un-widened confidence region \( H_{p,t}(0) \) contains \( p \) with high probability (see Lemma 1). Consequently, the UCRL2 algorithm by (Jaksch et al., 2010), which optimistic explores \( H_{p,t}(0) \), has a regret that scales linearly with the diameter of \( p \).

The approach of optimistic exploring \( H_{p,t}(0) \) is further extended to RL in piecewise-stationary MDPs by (Jaksch et al., 2010; Gajane et al., 2018). The latter establishes a \( \tilde{O}(\ell^{1/3} \max_{s,a} s^2/4 A^{1/3} T^{3/3}) \) dynamic regret bounds, when there are at most \( \ell \) changes. Their analysis involves partitioning the \( T \)-round horizon into \( C \cdot T^{1/3} \) equal-length intervals, where \( C \) is a constant dependent on \( \max_s A, \ell \). At least \( C T^{1/3} \) of \( \ell \) intervals enjoy stationary environments, and optimistic exploring \( H_{p,t}(0) \) in these intervals yields a dynamic regret bound that scales linearly with \( \max_s A \). Bounding the dynamic regret of the remaining intervals by their lengths and tuning \( C \) yield the desired bound.

In contrast to the stationary and piecewise-stationary settings, optimistic exploration on \( H_{p,t}(0) \) might lead to unfavorable dynamic regret bounds in non-stationary MDPs. In the non-stationary environment where \( p_{t-W}, \ldots, p_{t-1} \) are generally distinct, we show that it is impossible to bound the diameter of \( \hat{p}_t \) in terms of the maximum of the diameters of \( p_{t-W}, \ldots, p_{t-1} \). More generally, we demonstrate the previous claim not only for \( \hat{p}_t \), but also for every \( \hat{p} \in H_{p,t}(0) \) in the following Proposition. The Proposition showcases the unique challenge in exploring non-stationary MDPs that is absent in the piecewise-stationary MDPs, and motivates our notion of confidence widening with \( \eta > 0 \). To ease the notation, we put \( t = W + 1 \) without loss of generality.

Proposition 3. There exists a sequence of non-stationary MDP transition distributions \( p_1, \ldots, p_W \) such that

- The diameter of \((S,A,p_n)\) is 1 for each \( n \in [W] \).
- The total variations in state transition distributions is \( O(1) \).

Nevertheless, under some deterministic policy,

1. The empirical MDP \((S,A,\hat{p}_{W+1})\) has diameter \( \Theta(W) \)
2. Further, for every \( \tilde{p} \in H_{p,W+1}(0) \), the MDP \((S,A,\tilde{p})\) has diameter \( \Omega(\sqrt{W/\log W}) \)

Proof. The sequence \( p_1, \ldots, p_W \) alternates between the following 2 instances \( p^1, p^2 \). Now, define the common state space \( S = \{1,2\} \) and action collection \( A = \{A_1,A_2\} \).
where \( A_1 = \{a_1, a_2\} \), \( A_2 = \{b_1, b_2\} \). We assume all the state transitions are deterministic, and a graphical illustration is presented in Fig. 2. Clearly, we see that both instances have diameter 1.

\[
\begin{align*}
\text{MDP with } p^1 & \\
\text{MDP with } p^2
\end{align*}
\]

Figure 2. Example MDPs (with deterministic transitions).

Now, consider the following two deterministic and stationary policies \( \pi^1 \) and \( \pi^2 \):

\[
\pi^1(1) = a_1, \quad \pi^1(2) = b_2, \quad \pi^2(1) = a_2, \quad \pi^2(2) = b_1.
\]

Since the MDP is deterministic, we have \( \hat{p}_{W+1} = \hat{p}_{W+1} \).

In the following, we construct a trajectory where the DM alternates between policies \( \pi^1, \pi^2 \) during time \( \{1, \ldots, W\} \) while the underlying transition kernel alternates between \( p^1, p^2 \). In the construction, the DM is almost always at the self-loop at state 1 (or 2) throughout the horizon, no matter what action \( a_1, a_2 \) (or \( b_1, b_2 \)) she takes. Consequently, it will trick the DM into thinking that \( \hat{p}_{W+1}(1|a_i) \approx 1 \) for each \( i \in \{1, 2\} \), and likewise \( \hat{p}_{W+1}(2|b_i) \approx 1 \) for each \( i \in \{1, 2\} \). Altogether, this will lead the DM to conclude that \( (S, A, \hat{p}_{W+1}) \) constitute a high diameter MDP, since the probability of transitioning from state 1 to 2 (and 2 to 1) are close to 0.

The construction is detailed as follows. Let \( W = 4\tau \). In addition, let the state transition kernels be \( p^1 \) from time 1 to \( \tau \) and from time step 2\( \tau + 1 \) to 3\( \tau \) and be \( p^2 \) for the remaining time steps. The DM starts at state 1. She follows policy \( \pi^1 \) from time 1 to \( 2\tau \), and policy \( \pi^2 \) from \( 2\tau + 1 \) to \( 4\tau \). Under the specified instance and policies, it can be readily verified that the DM takes

- action \( a_1 \) from time 1 to \( \tau + 1 \),
- action \( b_2 \) from time \( \tau + 2 \) to \( 2\tau \),
- action \( b_1 \) from time \( 2\tau + 1 \) to \( 3\tau + 1 \),
- action \( a_2 \) from time \( 3\tau + 2 \) to \( 4\tau \).

As a result, the DM is at state 1 from time 1 to \( \tau + 1 \), and time \( 3\tau + 2 \) to \( 4\tau \); while she is at state 2 from time \( \tau + 2 \) to \( 3\tau + 1 \), as depicted in Fig. 3. We have:

\[
\begin{align*}
\hat{p}_{W+1}(1|1, a_1) &= \frac{\tau}{\tau + 1}, & \hat{p}_{W+1}(2|1, a_1) &= \frac{1}{\tau + 1} \\
\hat{p}_{W+1}(1|1, a_2) &= 1, & \hat{p}_{W+1}(2|1, a_2) &= 0 \\
\hat{p}_{W+1}(2|2, b_1) &= \frac{\tau}{\tau + 1}, & \hat{p}_{W+1}(1|2, b_1) &= \frac{1}{\tau + 1} \\
\hat{p}_{W+1}(2|2, b_2) &= 1, & \hat{p}_{W+1}(1|2, b_2) &= 0.
\end{align*}
\]

Finally, for the confidence region \( H_{p,W+1}(0) = \{H_{p,W+1}(s, a; 0)\}_{s,a} \) constructed without confidence widening, for any \( \tilde{p} \in H_{p,W+1}(0) \) we have \( \hat{p}(1|1, a_1) = \hat{p}(1|2, b_1) = O\left(\sqrt{\frac{\log W}{\tau + 1}}\right) \) and \( \hat{p}(2|1, a_2) = \hat{p}(1|2, b_2) = O\left(\sqrt{\frac{\log W}{\tau + 1}}\right) \) respectively, since the stochastic confidence radii \( \Theta\left(\sqrt{\frac{\log W}{\tau + 1}}\right) \) and \( \Theta\left(\sqrt{\frac{\log W}{\tau + 1}}\right) \) dominate the sample mean \( \frac{1}{\tau + 1} \) and 0. Therefore, for any \( \tilde{p} \in H_{p,W+1}(0) \), the diameter of the MDP constructed by \( (S, A, \tilde{p}) \) is at least \( \Omega\left(\sqrt{\frac{W}{\log W}}\right) \).

Remark 4. Inspecting the prevalent OFU guided approach for stochastic MAB and RL in MDPs settings (Auer et al., 2002a; Abbasi-Yadkori et al., 2011; Jaksch et al., 2010; Bubeck & Cesa-Bianchi, 2012; Lattimore & Szepesvári, 2018), one usually concludes that a tighter design of confidence region can result in a lower (dynamic) regret bound. In (Abernethy et al., 2016), this insights has been formalized in stochastic K-armed bandit settings via a potential function type argument. Nevertheless, Proposition 3 (together with Theorem 1) demonstrates that using the tightest confidence region in learning algorithm design may not be enough to ensure low dynamic regret bound for RL in non-stationary MDPs.

7. Conclusion

In this paper, we studied the problem of non-stationary reinforcement learning where the unknown reward and state transition distributions can be different from time to time as long as the total changes are bounded by some variation budgets, respectively. We first incorporated the sliding window estimator and the novel confidence widening technique into the UCRL2 algorithm to propose the SWUCRL2-CW algorithm with low dynamic regret when the variation budgets are known. We then designed the parameter-free BORL algorithm that allows us to enjoy this dynamic regret bound without knowing the variation budgets. The main ingredient of the proposed algorithms is the novel confidence widening technique, which injects extra optimism into the design of learning algorithms. This is in contrast to the widely held belief that optimistic exploration algorithms for (stationary and non-stationary) stochastic online learning settings should employ the lowest possible level of optimism.
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A. Supplementary Details about MDPs

A.1. Linear Program Formulations

According to Pg. 392 of (Puterman, 1994), the optimal long term reward $\rho_\tau^*$ is equal to the optimal value of the linear program $P(r, p)$. For a reward vector $r$ and a transition kernel $p$, we define

$$
P(r, p) : \max_{s \in S, a \in A_s} \sum_{a \in A_s} r(s, a) x(s, a)$$

subject to

$$\sum_{a \in A_s} x(s, a) = \sum_{s' \in S, a' \in A_{s'}} p(s'| s', a') x(s', a') \quad \forall s \in S$$

$$\sum_{s \in S, a \in A_s} x(s, a) = 1$$

$$x(s, a) \geq 0 \quad \forall s \in S, a \in A_s$$

The proofs of inequalities (12, 13) are deferred to the end. Now, combining (12, 13) gives

$$\rho_\tau^* \geq r_\tau^* \geq r(s, a) + \sum_{s \in S} p(s'| s, a) \gamma_\tau^*(s') - \sum_{q=1}^{\tau-1} (B_{r,q} + 2D_{\text{max}} B_{p,q})$$

The following Lemma shows that any feasible solution to $D(r, p)$ is essentially bounded if the underlying MDP is communicating, which will be crucial in the subsequent analysis.

**Lemma 2.** Let $(\rho, \gamma)$ be a feasible solution to the dual program $D(r, p)$, where $(S, A, p)$ constitute a communicating MDP with diameter $D$. We have

$$\max_{s, s' \in S} \{\gamma(s) - \gamma(s')\} \leq 2D.$$

The Lemma is extracted from Section 4.3.1 of (Jaksch et al., 2010), and it is more general than (Lattimore & Szepesvári, 2018), which requires $(\rho, \gamma)$ to be optimal instead of just feasible.

A.2. Proof of Proposition 1

We begin with invoking Lemma 2, which guarantees that for each $t$ there is an optimal solution $(\rho_t^*, \gamma_t^*)$ of $D(r_t, p_t)$ that satisfies $0 \leq \gamma_t^*(s) \leq 2D_{\text{max}}$ for all $s \in S$. Recall for each $t$:

$$B_{r,t} = \max_{s \in S, a \in A_s} |r_{t+1}(s, a) - r_t(s, a)|, \quad B_{p,t} = \max_{s \in S, a \in A_s} \|p_{t+1}(\cdot | s, a) - p_t(\cdot | s, a)\|_1.$$

Consider two time indexes $t \leq \tau$. We first claim the following two inequalities:

$$\rho_\tau^* \geq \rho_t^* - \sum_{q=t}^{\tau-1} (B_{r,q} + 2D_{\text{max}} B_{p,q})$$

$$\rho_t^* \geq r_\tau^*(s, a) + \sum_{s' \in S} p_\tau(s'| s, a) \gamma_\tau^*(s') - \gamma_t^*(s_t) \geq r_\tau^*(s, a) + \sum_{s' \in S} p_\tau(s'| s, a) \gamma_\tau^*(s') - \gamma_t^*(s_t) - \sum_{q=t}^{\tau-1} (B_{r,q} + 2D_{\text{max}} B_{p,q}).$$

The proofs of inequalities (12, 13) are deferred to the end. Now, combining (12, 13) gives

$$\rho_\tau^* \geq r_\tau^*(s, a) + \sum_{s' \in S} p_\tau(s'| s, a) \gamma_\tau^*(s') - \gamma_t^*(s_t) - 2\sum_{q=t}^{\tau-1} (B_{r,q} + 2D_{\text{max}} B_{p,q}).$$

Let positive integer $W \leq T$ be a window size, which is specified later. Summing (14) over $\tau = t, \ldots, t + W - 1$ and taking expectation over $\{(s_t, a_t)\}_{\tau=t}^{\tau+W-1}$ yield

$$\sum_{\tau=t}^{t+W-1} \rho_\tau^* \geq \mathbb{E} \left[ \sum_{\tau=t}^{t+W-1} r_\tau(s_t, a_t) \right] + \mathbb{E} \left[ \sum_{\tau=t}^{t+W-1} p_\tau(s'| s_t, a_t) \gamma_\tau^*(s') - \gamma_t^*(s_{t+1}) \right]$$

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\[ + \mathbb{E} \left( \sum_{s' \in S} p_{t-W+1}(s' | s_{t-W+1}, a_{t-W+1}) \gamma_t^*(s') - \gamma_t^*(s_t) \right) - 2 \sum_{\tau=t}^{t+W-1} \sum_{q=t}^{\tau-1} (B_{r,q} + 2D_{\text{max}}B_{p,q}) \]  
\[
\geq \mathbb{E} \left[ \sum_{\tau=t}^{t+W-1} r_\tau(s_\tau, a_\tau) \right] - 2D_{\text{max}} - 2W \sum_{q=t}^{t+W-1} (B_{r,q} + 2D_{\text{max}}B_{p,q}).
\]  

To arrive at (17), note that the second expectation in (15), which is a telescoping sum, is equal to 0, since \( s_{\tau+1} \) is distributed as \( p(s_{\tau+1} | s_\tau, a_\tau) \). In addition, we trivially lower bound the first expectation in (16) by \(-2D_{\text{max}}\) by applying Lemma 2. Next, consider partitioning the horizon of \( T \) steps into intervals of \( W \) time steps, where last interval could have less than \( W \) time steps. That is, the first interval is \( \{1, \ldots, W\} \); the second is \( \{W + 1, \ldots, 2W\} \), and so on. Applying the bound (17) on each interval and summing the resulting bounds together give

\[
\sum_{t=1}^{T} \rho_t^* \geq \mathbb{E} \left[ \sum_{t=1}^{T} r_t(s_t, a_t) \right] - 2W \left( \frac{T}{W} \right) D_{\text{max}} - 2W \sum_{t=1}^{T} (B_{r,t} + 2D_{\text{max}}B_{p,t})
\]

Choosing \( W \) to be any integer in \( \sqrt{T/(B_r + 2D_{\text{max}}B_p)}, 2\sqrt{T/(B_r + 2D_{\text{max}}B_p)} \) yields the desired inequality in the Theorem. Finally, we go back to proving inequalities (12,13). These inequalities are clearly true when \( t = \tau \), so we focus on the case \( t < \tau \).

**Proving inequality (12).** It suffices to show that the solution \( (\rho_t^* + \sum_{q=t}^{\tau-1} (B_{r,q} + 2D_{\text{max}}B_{p,q}), \gamma_t^*) \) is feasible to the linear program \( D(r_t, p_t) \). To see the feasibility, it suffices to check the constraint of \( D(r_t, p_t) \) for each state action pair \( s, a \):

\[
\rho_t^* + \sum_{q=t}^{\tau-1} (B_{r,q} + 2D_{\text{max}}B_{p,q}) 
\geq [r_\tau(s,a) + \sum_{q=t}^{\tau-1} B_{r,q}] + [-\gamma_\tau^*(s) + \sum_{s' \in S} p_\tau(s'|s,a) \gamma_\tau^*(s') + \sum_{q=t}^{\tau-1} 2D_{\text{max}}B_{p,q}].
\]

The feasibility is proved by noting that

\[
| r_\tau(s,a) - r_t(s,a) | \leq \sum_{q=t}^{\tau-1} B_{r,q},
\]

\[
\sum_{s' \in S} p_\tau(s'|s,a) \gamma_\tau^*(s') - \sum_{s' \in S} p_t(s'|s,a) \gamma_t^*(s') \leq \| p_\tau(\cdot | s,a) - p_t(\cdot | s,a) \|_1 \| \gamma_\tau^* \|_\infty
\]

\[
\leq \sum_{q=t}^{\tau-1} B_{p,q}(2D_{\text{max}}).
\]

**Proving inequality (13).** We have

\[
\rho_t^* \geq r_t(s_t, a_t) + \sum_{s' \in S} p_t(s'|s_t, a_t) \gamma_t^*(s') - \gamma_t^*(s_t)
\]

\[
\geq r_\tau(s_\tau, a_\tau) + \sum_{s' \in S} p_t(s'|s_\tau, a_\tau) \gamma_\tau^*(s') - \gamma_\tau^*(s_\tau) - \sum_{s=t}^{\tau-1} B_{r,s}
\]

\[
\geq r_\tau(s_\tau, a_\tau) + \sum_{s' \in S} p_\tau(s'|s_\tau, a_\tau) \gamma_\tau^*(s') - \gamma_\tau^*(s_\tau) - \sum_{s=t}^{\tau-1} B_{r,s} - 2D_{\text{max}} \sum_{s=t}^{\tau-1} B_{p,s},
\]

where steps (21, 22) are by inequalities (19, 20). Altogether, the Proposition is proved.
Algorithm 2 EVI($H_r, H_p; \epsilon$), mostly extracted from (Jaksch et al., 2010)

1: Initialize VI record $u_0 \in \mathbb{R}^S$ as $u_0(s) = 0$ for all $s \in S$.
2: for $i = 0, 1, \ldots$ do
3: For each $s \in S$, compute VI record $u_{i+1}(s) = \max_{a \in A_s} \tilde{\Upsilon}_i(s, a)$, where

$$
\tilde{\Upsilon}_i(s, a) = \max_{\hat{r}(s, a) \in H_r(s, a)} \{\hat{r}(s, a)\} + \max_{\tilde{p} \in H_p(s, a)} \left\{ \sum_{s' \in S} u_i(s') \hat{p}(s') \right\}.
$$
4: Define stationary policy $\hat{\pi} : S \rightarrow A_s$ as $\hat{\pi}(s) = \arg\max_{a \in A_s} \tilde{\Upsilon}_i(s, a)$.
5: Define optimistic reward $\tilde{r} = \{\tilde{r}(s, a)\}_{s,a}$ with $\tilde{r}(s, a) = \arg\max_{\hat{r}(s, a) \in H_r(s, a)} \{\hat{r}(s, a)\}$.
6: Define optimistic kernel $\tilde{p} = \{\tilde{p}(\cdot|s, a)\}_{s,a}$ with $\tilde{p}(\cdot|s, a) = \arg\max_{\tilde{p} \in H_p(s, a)} \{\sum_{s' \in S} u_i(s') \hat{p}(s')\}$.
7: Define optimistic dual variables $\tilde{\rho} = \max_{s \in S} \{u_{i+1}(s) - u_i(s)\}$, $\tilde{\gamma}(s) = u_i(s) - \min_{s \in S} u_{i+1}(s)$. 
8: if $\max_{s \in S} \{u_{i+1}(s) - u_i(s)\} - \min_{s \in S} \{u_{i+1}(s) - u_i(s)\} \leq \epsilon$ then
9: Break the for loop.
10: end if
11: end for
12: Return policy $\hat{\pi}$.
13: Auxiliary output: optimistic reward and kernel $(\tilde{r}, \tilde{p})$, optimistic dual variables $(\tilde{\rho}, \tilde{\gamma})$.

A.3. Extended Value Iteration (EVI) by (Jaksch et al., 2010)

We provide the pseudo-codes of EVI($H_r, H_p; \epsilon$) proposed by (Jaksch et al., 2010) in Algorithm 2. By (Jaksch et al., 2010), the algorithm converges in finite time when the confidence region $H_p$ contains a transition kernel $p$ such that $(S, A, p)$ constitutes a communicating MDP. The output $(\hat{\pi}, \tilde{r}, \tilde{p}, \tilde{\rho}, \tilde{\gamma})$ of the EVI($H_r, H_p; \epsilon$) satisfies the following two properties (Jaksch et al., 2010).

**Property 1.** The dual variables $(\tilde{\rho}, \tilde{\gamma})$ are optimistic, i.e.,

$$
\tilde{\rho} + \tilde{\gamma}(s) \geq \max_{\hat{r}(s, a) \in H_r(s, a)} \{\tilde{r}(s, a)\} + \sum_{s' \in S} \tilde{\gamma}(s') \max_{\tilde{p} \in H_p(s, a)} \{\tilde{p}(s'|s, a)\}.
$$

**Property 2.** For each state $s \in S$, we have

$$
\tilde{r}(s, \hat{\pi}(s)) \geq \tilde{\rho} + \tilde{\gamma}(s) - \sum_{s' \in S} \tilde{p}(s'|s, \hat{\pi}(s))\tilde{\gamma}(s') - \epsilon.
$$

Property 1 ensures the feasibility of the output dual variables $(\tilde{\rho}, \tilde{\gamma})$, with respect to the dual program $D(\tilde{r}, \tilde{p})$ for any $\hat{r}, \tilde{p}$ in the confidence regions $H_r, H_p$. The feasibility facilitates the bounding of $\max_{s \in S} \tilde{\gamma}(s)$, which turns out to be useful for bounding the regret arising from switching among different stationary policies. To illustrate, suppose that $H_p$ is so large that it contains a transition kernel $\tilde{p}$ under which $(S, A, \tilde{p})$ has diameter $D$. By Lemma 2, we have $0 \leq \max_{s \in S} \tilde{\gamma}(s) \leq 2D$.

Property 2 ensures the near-optimality of the dual variables $(\tilde{\rho}, \tilde{\gamma})$ to the $(\hat{r}, \tilde{p})$ optimistically chosen from $H_r, H_p$. More precisely, the deterministic policy $\hat{\pi}$ is nearly-optimal for the MDP with time homogeneous mean reward $\tilde{r}$ and time homogeneous transition kernel $\tilde{p}$, under which the policy $\hat{\pi}$ achieves a long term average reward is at least $\tilde{\rho}^* - \epsilon$.

B. Proof of Lemma 1

We employ the self-normalizing concentration inequality (Abbasi-Yadkori et al., 2011). The following inequality is extracted from Theorem 1 in (Abbasi-Yadkori et al., 2011), restricted to the case when $d = 1$.

**Proposition 4** (Abbasi-Yadkori et al., 2011). Let $\{F_q\}_{q=1}^T$ be a filtration. Let $\{\xi_q\}_{q=1}^T$ be a real-valued stochastic process, such that $\xi_q$ is $F_q$-measurable, and $\xi_q$ is conditionally $R$-sub-Gaussian, i.e. for all $\lambda \geq 0$, it holds that $\mathbb{E}[\exp(\lambda \xi_q)|F_{q-1}] \leq \exp(\lambda^2 R^2/2)$. Let $\{Y_q\}_{q=1}^T$ be a non-negative real-valued stochastic process such that $Y_q$ is $F_{q-1}$-measurable. For any
\[ \delta' \in (0, 1), \text{ it holds that} \]
\[ \Pr \left( \frac{\sum_{q=1}^{t} \xi_{q}Y_{q}}{\max\{1, \sum_{q=1}^{t} Y_{q}^{2} \}} \leq 2R \sqrt{\frac{\log(T/\delta')}{\max\{1, \sum_{q=1}^{t} Y_{q}^{2} \}}} \text{ for all } t \in [T] \right) \geq 1 - \delta'. \]

In particular, if \( \{Y_{q}\}_{q=1}^{T} \) be a \([0, 1]\)-valued stochastic process, then for any \( \delta' \in (0, 1) \), it holds that
\[ \Pr \left( \frac{\sum_{q=1}^{t} \xi_{q}Y_{q}}{\max\{1, \sum_{q=1}^{t} Y_{q} \}} \leq 2R \sqrt{\frac{\log(T/\delta')}{\max\{1, \sum_{q=1}^{t} Y_{q} \}}} \text{ for all } t \in [T] \right) \geq 1 - \delta'. \] (23)

The Lemma is proved by applying (23) in Proposition 4 with suitable choices of \( \mathcal{F}_{q=1}^{T}, \{\xi_{q}\}_{q=1}^{T}, \{Y_{q}\}_{q=1}^{T}, \delta \). We divide the proof into two parts.

**B.1. Proving** \( \Pr[\xi_{q}] \geq 1 - \delta/2 \)

It suffices to prove that, for any fixed \( s \in \mathcal{S}, a \in \mathcal{A}_{s}, t \in [T] \), it holds that
\[ \Pr \left( \left| \hat{r}(s, a) - \bar{r}(s, a) \right| \leq \text{rad}_{r,t}(s, a) \right) \]
\[ = \Pr \left( \frac{1}{N_{t}^{+}(s, a)} \sum_{q=(\tau(m)-W)\lor 1}^{t-1} \left[ R_{q}(s, a) - r_{q}(s, a) \right] \cdot 1(s_{q} = s, a_{q} = a) \right) \leq 2 \sqrt{\frac{\log(2SAT^{2}/\delta)}{N_{t}^{+}(s, a)}} \]
\[ \geq 1 - \frac{\delta}{2SAT}. \] (24)

since then \( \Pr[\xi_{q}] \geq 1 - \delta/2 \) follows from the union bound over all \( s \in \mathcal{S}, a \in \mathcal{A}_{s}, t \in [T] \). Now, the trajectory of the online algorithm is expressed as \( \{s_{q}, a_{q}, R_{q}\}_{q=1}^{T} \). Inequality (24) directly follows from Proposition 4, with \( \mathcal{F}_{q=1}^{T}, \{\xi_{q}\}_{q=1}^{T}, \{Y_{q}\}_{q=1}^{T}, \delta \) defined as
\[ \mathcal{F}_{q} = \{(s_{\ell}, a_{\ell}, R_{\ell})\}_{\ell=1}^{q} \cup \{(s_{q+1}, a_{q+1})\}, \]
\[ \xi_{q} = R_{q}(s, a) - r_{q}(s, a), \]
\[ Y_{q} = 1(s_{q} = s, a_{q} = a, ((t - W) \lor 1) \leq q \leq t - 1), \]
\[ \delta' = \frac{\delta}{2SAT}. \]

Each \( \xi_{q} \) is conditionally 2-sub-Gaussian, since \(-1 \leq \xi_{q} \leq 1\) with certainty. Altogether, the required inequality is shown.

**B.2. Proving** \( \Pr[\xi_{p}] \geq 1 - \delta/2 \)

We start by noting that, for two probability distributions \( p, \{p(s)\}_{s \in \mathcal{S}}, p' = \{p'(s)\}_{s \in \mathcal{S}} \) on \( \mathcal{S} \), it holds that
\[ \|p - p'\|_{1} = \max_{\theta \in \{-1, 1\}^{\mathcal{S}}} \theta(s) \cdot (p(s) - p'(s)). \]

Consequently, to show \( \Pr[\xi_{p}] \geq 1 - \delta/2 \), it suffices to show that, for any fixed \( s \in \mathcal{S}, a \in \mathcal{A}_{s}, t \in [T], \theta \in \{-1, 1\}^{\mathcal{S}} \), it holds that
\[ \Pr \left( \sum_{s' \in \mathcal{S}} \theta(s) \cdot (\hat{p}_{t}(s'|s, a) - \bar{p}_{t}(s'|s, a)) \leq \text{rad}_{p,t}(s, a) \right) \]
\[ \leq \Pr \left( \frac{1}{N_{t}^{+}(s, a)} \sum_{q=(\tau(m)-W)\lor 1}^{t-1} \left[ \sum_{s' \in \mathcal{S}} \theta(s') 1(s_{q} = s, a_{q} = a, s_{q+1} = s') \right] \right. \]
\[ - \left. \left[ \sum_{s' \in \mathcal{S}} \theta(s') p_{q}(s'|s, a) \cdot 1(s_{q} = s, a_{q} = a) \right] \right) \leq 2 \sqrt{\frac{\log(2SAT^{2}/\delta)}{N_{t}^{+}(s, a)}} \]
We first provide the bound for rewards:

\[ \geq 1 - \frac{\delta}{2SAT^2\gamma}, \tag{25} \]

since then the required inequality follows from a union bound over all \( s \in \mathcal{S}, a \in \mathcal{A}, t \in [T], \theta \in \{-1,1\}^\mathcal{S} \). Similar to the case of \( \mathcal{E}_r \), (25) follows from Proposition 4, with \{\mathcal{F}_q\}^{q=1}_{q=T}, \{\xi_q\}^{q=1}_{q=T}, \{Y_q\}^{q=1}_{q=T}, \delta \) defined as

\[ \mathcal{F}_q = \{(s_t, a_t)\}^{q+1}_{t=1}, \]

\[ \xi_q = \sum_{s' \in \mathcal{S}} \theta(s')1(s_q = s, a_q = a, s_{q+1} = s') - \sum_{s' \in \mathcal{S}} \theta(s')p_q(s'|s, a), \]

\[ Y_q = 1(s_q = s, a_q = a, (t - W) \lor 1) \leq t - 1, \]

\[ \delta' = \frac{\delta}{2SAT^2\gamma}. \]

Each \( \xi_q \) is conditionally 2-sub-Gaussian, since \(-1 \leq \xi_q \leq 1\) with certainty. Altogether, the required inequality is shown.

### C. Proof of Proposition 2

In this section, we prove Proposition 2. Throughout the section, we impose the assumptions stated by the Proposition. That is, the events \( \mathcal{E}_r, \mathcal{E}_p \) hold, and there exists \( p \) with (1) \( p \in H_{p,\tau(m)}(\eta) \), (2) \( (\mathcal{S}, \mathcal{A}, p) \) has diameter at most \( D \). We begin by recalling the following notations:

\[ B_{r,t} = \max_{s \in \mathcal{S}, a \in \mathcal{A}_s} |r_{t+1}(s, a) - r_t(s, a)|, \]

\[ B_{p,t} = \max_{s \in \mathcal{S}, a \in \mathcal{A}_s} \|p_{t+1}(\cdot|s, a) - p_t(\cdot|s, a)\|_1, \]

\[ \var_{r,t} = \sum_{q=\tau(m)-W}^{t-1} B_{r,q}, \]

\[ \var_{p,t} = \sum_{q=\tau(m)-W}^{t-1} B_{p,q}. \]

We then need the following auxiliary lemmas

**Lemma 3.** Let \( t \) be in episode \( m \). For every state-action pair \((s, a)\), we have

\[ |r_t(s, a) - \bar{r}_{\tau(m)}(s, a)| \leq \var_{r,t}, \quad \|p_t(\cdot|s, a) - \tilde{p}_{\tau(m)}(\cdot|s, a)\|_1 \leq \var_{p,t} \]

**Lemma 4.** Let \( t \) be in episode \( m \). We have

\[ \tilde{p}_{\tau(m)} \geq p^*_t - \var_{r,t} - 2D \cdot \var_{p,t}. \]

**Lemma 5.** Let \( t \) be in episode \( m \). For every state-action pair \((s, a)\), we have

\[ \left| \sum_{s' \in \mathcal{S}} \tilde{p}_{\tau(m)}(s'|s, a)\bar{r}_{\tau(m)}(s') - \sum_{s' \in \mathcal{S}} p_t(s'|s, a)\bar{r}_{\tau(m)}(s') \right| \leq 2D \left[ \var_{p,t} + 2\text{rad}_{p,\tau(m)}(s, a) + \eta \right]. \]

Lemmas 3, 4, 5 are proved in Sections C.1, C.2, and C.3, respectively.

### C.1. Proof of Lemma 3

We first provide the bound for rewards:

\[ |r_t(s, a) - \bar{r}_{\tau(m)}(s, a)| \leq |r_t(s, a) - r_{\tau(m)}(s, a)| + |r_{\tau(m)}(s, a) - \bar{r}_{\tau(m)}(s, a)| \]

\[ \leq \sum_{q=\tau(m)}^{t-1} |r_{q+1}(s, a) - r_q(s, a)| + 1/W \sum_{u=1}^{W} |r_{\tau(m)}(s, a) - r_{\tau(m)-u}(s, a)|. \]

By the definition of \( B_{r,q} \), we have

\[ \sum_{q=\tau(m)}^{t-1} |r_{q+1}(s, a) - r_q(s, a)| \leq \sum_{q=\tau(m)}^{t-1} B_{r,q}, \]
We first demonstrate two immediate consequences about the dual solution \( C \).

**Proof of Lemma 4**

By the definition of \( r \),

\[
\| p_i(s, a) - \bar{p}_{\tau(m)}(s, a) \|_1 \leq \| p_i(s, a) - p_{\tau(m)}(s, a) \|_1 + \| p_{\tau(m)}(s, a) - \bar{p}_{\tau(m)}(s, a) \|_1
\]

Next, we provide a similar analysis on the transition kernel.

\[
\| p_{q+1}(s, a) - p_{q}(s, a) \|_1 \leq \| p_{q+1}(s, a) - p_{\tau(m)}(s, a) \|_1 + \| p_{\tau(m)}(s, a) - p_{\tau(m) - w}(s, a) \|_1.
\]

By the definition of \( B_{p,q} \), we have

\[
\sum_{q=\tau(m)}^{t-1} \| p_{q+1}(s, a) - p_{q}(s, a) \|_1 \leq \sum_{q=\tau(m)}^{t-1} B_{p,q},
\]

and

\[
\sum_{w=1}^{W} \| p_{\tau(m)}(s, a) - p_{\tau(m) - w}(s, a) \|_1 \leq \sum_{w=1}^{W} \| p_{\tau(m) - i+1}(s, a) - p_{\tau(m) - i}(s, a) \|_1.
\]

Altogether, the lemma is shown.

**C.2. Proof of Lemma 4**

We first demonstrate two immediate consequences about the dual solution \((\hat{\rho}_{\tau(m)}, \hat{\gamma}_{\tau(m)})\) by the Proposition’s assumptions:

\[
0 \leq \hat{\gamma}_{\tau(m)}(s) \leq 2D \quad \text{for all } s \in S,
\]

\[
\hat{\rho}_{\tau(m)} + \hat{\gamma}_{\tau(m)}(s) \geq \tilde{r}_{\tau(m)}(s, a) + \sum_{s' \in S} \hat{\gamma}_{\tau(m)}(s') \hat{p}_{\tau(m)}(s'|s, a) \quad \text{for all } s \in S, a \in A_s.
\]

To see inequality (26), first observe that

\[
\hat{\rho}_{\tau(m)} + \hat{\gamma}_{\tau(m)}(s) \geq \max_{\hat{r}(s,a) \in H_{r,\tau(m)}(s,a)} \{ \hat{r}(s,a) \} + \sum_{s' \in S} \hat{\gamma}_{\tau(m)}(s') \hat{p}(s'|s,a) \]

\[
\geq \max_{\hat{r}(s,a) \in H_{r,\tau(m)}(s,a)} \{ \hat{r}(s,a) \} + \sum_{s' \in S} \hat{\gamma}_{\tau(m)}(s') p(s'|s,a).
\]

Step (28) is by Property 1 of the output from EVI, which is applied with confidence regions \( H_{r,\tau(m)}, H_{p,\tau(m)}(\eta) \). Step (29) is because of the assumption that \( p \in H_{p,\tau(m)}(\eta) \). Altogether, the solution \((\hat{\rho}_{\tau(m)}, \hat{\gamma}_{\tau(m)})\) is feasible to \( D(\hat{r}, p) \) for any \( \hat{r} \in H_{r,\tau(m)} \). Now, by Lemma 2, we have \( \max_{s,s' \in S} | \hat{\gamma}_{\tau(m)}(s) - \hat{\gamma}_{\tau(m)}(s') | \leq 2D \). Finally, inequality (26) follows.
from the fact that the bias vector $\tilde{\gamma}_{\tau(m)}$ returned by EVI is component-wise non-negative, and there exists $s \in S$ such that $\tilde{\gamma}_{\tau(m)} = 0$.

To see inequality (27), observe that
\[
\tilde{\rho}_{\tau(m)} + \tilde{\gamma}_{\tau(m)}(s) \geq \max_{r(s,a) \in H_{r,\tau(m)}(s,a)} \{ \tilde{r}(s,a) \} + \sum_{s' \in S} \tilde{\gamma}_{\tau(m)}(s') \max_{\tilde{\rho} \in H_{\tilde{p},\tau(m)}(s,a;\eta)} \{ \tilde{\rho}(s'|s,a) \} \geq \tilde{r}_{\tau(m)}(s,a) + \sum_{s' \in S} \tilde{\gamma}_{\tau(m)}(s') \tilde{p}_{\tau(m)}(s'|s,a).
\] (30)

Step (30) is again by Property 1 of the output from EVI, and step (31) is by the assumptions that $\tilde{r}_{\tau(m)} \in H_{r,\tau(m)}$, and $\tilde{p}_{\tau(m)} \in H_{\tilde{p},\tau(m)}(0) \subset H_{\tilde{p},\tau(m)}(\eta)$.

Now, we claim that $(\tilde{\rho}_{\tau(m)} + \text{var}_{r,t} + 2D \cdot \text{var}_{p,t}, \tilde{\gamma}_{\tau(m)})$ is a feasible solution to the $t$th period dual problem $D(r_t, p_t)$, which immediately implies the Lemma. To demonstrate the claim, for every state-action pair $(s, a)$ we have
\[
\tilde{r}_{\tau(m)}(s,a) \geq r_t(s,a) - \text{var}_{r,t} \sum_{s' \in S} \tilde{\gamma}_{\tau(m)}(s') p_t(s'|s,a) \geq \sum_{s' \in S} \tilde{\gamma}_{\tau(m)}(s') p_t(s'|s,a) - \left\| \tilde{\gamma}_{\tau(m)} \right\|_\infty \left\| p_t(\cdot|s,a) - \tilde{p}_{\tau(m)}(\cdot|s,a) \right\|_1 \geq \sum_{s' \in S} \tilde{\gamma}_{\tau(m)}(s') p_t(s'|s,a) - 2D \cdot \text{var}_{p,t}.
\] (33)

Inequality (32) is by Lemma 3 on the rewards. Step (33) is by inequality (26), and by Lemma 3 which shows $\|p_t(\cdot|s,a) - \tilde{p}_{\tau(m)}(\cdot|s,a)\|_1 \leq \text{var}_{p,t}$. Altogether, putting (32), (33) to inequality (27), our claim is shown, i.e., for all $s \in S$ and $a \in A_s$,
\[
\tilde{\rho}_{\tau(m)} + \text{var}_{r,t} + 2D \cdot \text{var}_{p,t} + \tilde{\gamma}_{\tau(m)}(s) \geq r_t(s,a) + \sum_{s' \in S} \tilde{\gamma}_{\tau(m)}(s') p_t(s'|s,a).
\]

Hence, the lemma is proved.

**C.3. Proof of Lemma 5**

We have
\[
\left| \sum_{s' \in S} \left[ \tilde{\rho}_{\tau(m)}(s'|s,a) - p_t(s'|s,a) \right] \tilde{\gamma}_{\tau(m)}(s') \right| \leq \left\| \tilde{\gamma}_{\tau(m)} \right\|_\infty \cdot \left\| \tilde{\rho}_{\tau(m)}(\cdot|s,a) - \tilde{p}_{\tau(m)}(\cdot|s,a) \right\|_1 + \left\| \tilde{p}_{\tau(m)}(\cdot|s,a) - p_t(\cdot|s,a) \right\|_1. \] (34)

In step (34), we know that
• (a) $\leq 2D$ by inequality (26),
• (b) $\leq 2 \text{rad}_{p,\tau(m)}(s,a) + \eta$, by the facts that $\tilde{p}_{\tau(m)}(\cdot|s,a) \in H_{p,\tau(m)}(s,a;\eta)$ and $\tilde{p}_{\tau(m)}(\cdot|s,a) \in H_{p,\tau(m)}(s,a;0)$,
• (c) $\leq \text{var}_{p,t}$ by Lemma 3 on the bound on $p$.

Altogether, the Lemma is proved.

**C.4. Finalizing the Proof**

Now, we have
\[
\begin{align*}
r_t(s_t, a_t) \\
\geq & \tilde{r}_{\tau(m)}(s_t, a_t) - \text{var}_{r,t} \tilde{\gamma}_{\tau(m)}(s_t, a_t) \\
\geq & \tilde{\gamma}_{\tau(m)}(s_t, a_t) - \text{var}_{r,t} - 2 \cdot \text{rad}_{r,\tau(m)}(s_t, a_t)
\end{align*}
\] (35)
To facilitate the exposition, we denote
\[ \sum_{s' \in S} \hat{p}(s' | s, a_t) \gamma_t(s') - \frac{1}{\sqrt{T(m)}} \]

− \var_{r,t} - 2 \cdot \rad_{r, \tau(m)}(s_t, a_t)

\geq p_t^* + \gamma_t(m)(s_t) - \left[ \sum_{s' \in S} p_t(s' | s_t, a_t) \gamma_t(m)(s') \right] - \frac{1}{\sqrt{T(m)}}

\geq 2 \left[ \var_{r,t} + \rad_{r, \tau(m)}(s_t, a_t) \right] - 2D \left[ 2 \cdot \var_{r,t} + 2 \cdot \rad_{r, \tau(m)}(s, a) + \eta \right].

(37)

Step (35) is by Lemma 3 on t. Step (36) is by conditioning that event \( E_r \) holds. Step (37) is by Property 2 for the output of EVI. In step (38), we upper bound \( \overline{p}(m) \) by Lemma 4 and we upper bound \( \sum_{s' \in S} \overline{p}(m)(s' | s_t, a_t) \gamma_t(m)(s') \) by Lemma 5. Rearranging gives the Proposition.

D. Proof of Theorem 1

To facilitate the exposition, we denote \( M(T) \) as the total number of episodes. By abusing the notation we, let \( \tau(M(T) + 1) - 1 = T \). Episode \( M(T) \), containing the final round \( T \), is interrupted and the algorithm is forced to terminate as the end of time \( T \) is reached. We can now rewrite the dynamic regret of the SWUCRL2-CW algorithm as the sum of dynamic regret from each episode:

\[ \text{Dyn-Reg}_T(\text{SWUCRL2-CW}) = \sum_{t=1}^{T} (p_t^* - r_t(s_t, a_t)) = \sum_{m=1}^{M(T)} \sum_{t=\tau(m)}^{\tau(m) - 1} (p_t^* - r_t(s_t, a_t)) \]

(39)

To proceed, we define the set

\[ U = \{ m \in [M(T)] : p_{\tau(m)}(\cdot | s, a) \in H_{p, \tau(m)}(s, a; \eta) \forall (s, a) \in S \times A_s \} \]

For each episode \( m \in [M(T)] \), we distinguish two cases:

- **Case 1.** \( m \in U \). Under this situation, we apply Proposition 2 to bound the dynamic regret during the episode, using the fact that \( p_{\tau(m)} \) satisfies the assumptions of the proposition with \( D = D_{\tau(m)} \leq D_{\max} \).

- **Case 2.** \( m \in [M(T)] \setminus U \). In this case, we trivially upper bound the dynamic regret of each round in episode \( m \) by 1.

For case 1, we bound the dynamic regret during episode \( m \) by summing the error terms in (6, 7) across the rounds \( t \in [\tau(m), \tau(m + 1) - 1] \) in the episode. The term (6) accounts for the error by switching policies. In (7), the terms \( \rad_{r, \tau(m)} \), \( \rad_{p, \tau(m)} \) accounts for the estimation errors due to stochastic variations, and the term \( \var_{r,t}, \var_{p,t} \) accounts for the estimation error due to non-stationarity.

For case 2, we need an upper bound on \( \sum_{m \in [M(T)] \setminus U} \sum_{t=\tau(m)}^{\tau(m) - 1} 1 \), the total number of rounds that belong to an episode in \([M(T)] \setminus U \). The analysis is challenging, since the length of each episode may vary, and one can only guarantee that the length is \( \leq W \). A first attempt could be to upper bound as \( \sum_{m \in [M(T)] \setminus U} \sum_{t=\tau(m)}^{\tau(m) - 1} 1 \leq W \sum_{m \in [M(T)] \setminus U} 1 \), but the resulting bound appears too loose to provide any meaningful regret bound. Indeed, there could be double counting, as the starting time steps for a pair of episodes in case 2 might not even be \( W \) rounds apart!

To avoid the trap of double counting, we consider a set \( Q_T \subseteq [M(T)] \setminus U \) where the start times of the episodes are sufficiently far apart, and relate the cardinality of \( Q_T \) to sum \( \sum_{m \in [M(T)] \setminus U} \sum_{t=\tau(m)}^{\tau(m) + 1} 1 \). The set \( Q_T \subseteq [M(T)] \) is constructed sequentially, by examining all episodes \( m = 1, \ldots, M(T) \) in the time order. At the start, we initialize \( Q_T = \emptyset \). For each \( m = 1, \ldots, M(T) \), we perform the following. If episode \( m \) satisfies both criteria:

1. There exists some \( s \in S \) and \( a \in A_s \) such that \( p_{\tau(m)}(\cdot | s, a) \notin H_{p, \tau(m)}(s, a; \eta) \);
2. For every \( m' \in Q_T \), \( \tau(m) - \tau(m') > W \),

then we add \( m \) into \( Q_T \). Afterwards, we move to the next episode index \( m + 1 \). The process terminates once we arrive at episode \( M(T) + 1 \). The construction ensures that, for each episode \( m \in [M(T)] \), if \( \tau(m) - \tau(m') \notin [0, W] \) for all \( m' \in Q_T \), then \( s \in S \forall a \in A_s \), \( p_{\tau(m)}(\cdot | s, a) \in H_{p, \tau(m)}(s, a) \); otherwise, \( m \) would have been added into \( Q_T \).

We further construct a set \( \tilde{Q}_T \) to include all elements in \( Q_T \) and every episode index \( m \) such that there exists \( m' \in Q_T \) with \( \tau(m) - \tau(m') \in [0, W] \). By doing so, we can prove that every episode \( m \in [M(T)] \setminus \tilde{Q}_T \) satisfies \( p_{\tau(m)}(\cdot | s, a) \in \)
Figure 4. Both episodes $m_i$ and $m_{i+4}$ belong to $Q_T$ (and thus $\tilde{Q}_T$) because $p_r(m_i) \notin H_{p,r,m_i}(\eta)$ and $p_r(m_{i+4}) \notin H_{p,r,m_{i+4}}(\eta)$. $m_{i+1}$ is added to $\tilde{Q}_T$ (but not $Q_T$) because $\tau(m_{i+1}) = \tau(m_i) \in [0, W]$. $m_{i+2}$ and $m_{i+3}$ belong to neither of $Q_T$ nor $\tilde{Q}_T$ as $p_r(m_{i+2}) \in H_{p,r,m_{i+2}}(\eta)$ and $p_r(m_{i+3}) \in H_{p,r,m_{i+3}}(\eta)$.

$H_{p,r,m}(s,a) \forall s \in S \forall a \in A_s$. The procedures for building $\tilde{Q}_T$ (initialized to $Q_T$) are described as follows: for every episode index $m \in [M(T)]$, if there exists $m' \in Q_T$, such that $\tau(m) - \tau(m') \in [0, W]$, then we add $m$ to $\tilde{Q}_T$. Formally,

$$\tilde{Q}_T = Q_T \cup \{m \in [M(T)] : \exists m' \in Q_T \tau(m) - \tau(m') \in [0, W]\}.$$  

We can formalize the properties of $Q_T$ and $\tilde{Q}_T$ as follows.

**Lemma 6.** Conditioned on $E_p$, $|Q_T| \leq B_p/\eta$.

**Lemma 7.** For any episode $m \notin \tilde{Q}_T$, we have $p_r(m|s,a) \in H_{p,r,m}(s,a; \eta)$ for all $s \in S$ and $a \in A_s$.

The proofs of Lemmas 6 and 7 are presented in Sections D.1 and D.2, respectively.

Together with eq. (39), we can further decompose the dynamic regret of the SWUCRL2–CW algorithm as

$$\text{Dyn-Reg}_T(\text{SWUCRL2–CW})$$

$$= \sum_{m \in Q_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} (p_t^* - r_t(s_t, a_t)) + \sum_{m \in [M(T)] \setminus Q_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} (p_t^* - r_t(s_t, a_t))$$

$$\leq \sum_{m \in Q_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} (p_t^* - r_t(s_t, a_t)) + \sum_{m \in [M(T)] \setminus Q_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} \left\{ \left[ \sum_{s' \in S} p_t(s'|s_t, a_t) \tilde{\tau}(m)(s') - \tilde{\tau}(m)(s_t) \right] + \frac{1}{\sqrt{\tau(m)}} \right\}$$

$$+ \sum_{m \in [M(T)] \setminus Q_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} (2\var_{r,t} + 4D_{\max} \cdot \var_{p,t} + 2D_{\max} \eta)$$

$$+ \sum_{m \in [M(T)] \setminus Q_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} [2\text{rad}_{r,\tau(m)}(s_t, a_t) + 4D_{\max} \cdot \text{rad}_{p,\tau(m)}(s_t, a_t)] .$$

where the last step makes use of Lemma 7 and Proposition 2. We accomplish the promised dynamic regret bound by the following four Lemmas that bound the dynamic regret terms $\blacksquare$, $\blacklozenge$, $\clubsuit$, $\spadesuit$.

**Lemma 8.** Conditioned on $E_p$, we have

$$\blacksquare = O \left( \frac{B_p W}{\eta} \right).$$

**Lemma 9.** Conditioned on events $E_r, E_p$, we have with probability at least $1 - O(\delta)$ that

$$\blacklozenge = \tilde{O}(D_{\max}[M(T) + \sqrt{T}]) = \tilde{O} \left( D_{\max} \left[ \frac{SAT}{W} + \sqrt{T} \right] \right).$$

**Lemma 10.** With certainty,

$$\clubsuit = O \left( (B_r + D_{\max}B_p)W + D_{\max}T\eta \right).$$
Lemma 11. With certainty, we have
\((\nabla) = O\left(\frac{D_{\text{max}}S\sqrt{AT}}{\sqrt{W}}\right)\).

The proofs of Lemmas 8, 9, 10, and 11 are presented in Sections D.3, D.4, D.5, and D.6, respectively. Putting all these pieces together, we have the dynamic regret of the SWUCRL2–CW algorithm is upper bounded as
\[\hat{O}\left(\frac{B_p W}{\eta} + B_r W + D_{\text{max}} \left[ B_p W + \frac{S\sqrt{AT}}{\sqrt{W}} + T\eta + \frac{SAT}{W} + \sqrt{T}\right]\right),\]
and by setting \(W\) and \(\eta\) accordingly, we can conclude the proof.

D.1. Proof of Lemma 6

We first claim that, for every episode \(m \in Q_T\), there exists some state-action pair \((s, a)\) and some time step \(t_m \in [(\tau(m) - W \vee 1), \tau(m) - 1]\) such that
\[\|p_{\tau(m)}(\cdot|s, a) - p_t(\cdot|s, a)\|_1 > \eta.\]  

(40)

For contradiction sake, suppose the otherwise, that is, \(\|p_{\tau(m)}(\cdot|s, a) - p_t(\cdot|s, a)\|_1 \leq \eta\) for every state-action pair \((s, a)\) and every time step \(t \in [(\tau(m) - W \vee 1), \tau(m) - 1]\). For each state-action pair \((s, a)\), consider the following cases on \(N_{\tau(m)}(s, a) = \sum_{q=(\tau(m)-W)\vee1}^{\tau(m)-1} 1(s_q = s, a_q = a)\):

- **Case 1:** \(N_{\tau(m)}(s, a) = 0\). Then \(\hat{p}_{\tau(m)}(\cdot|s, a) = 0^S\), and clearly we have
  \[\|p_{\tau(m)}(\cdot|s, a) - \hat{p}_{\tau(m)}(\cdot|s, a)\|_1 = 1 < \text{rad}p_{\tau(m)}(s, a) < \text{rad}p_{\tau(m)}(s, a) + \eta.\]

- **Case 2:** \(N_{\tau(m)}(s, a) > 0\). Then we have the following inequalities:

  \[\|p_{\tau(m)}(\cdot|s, a) - \hat{p}_{\tau(m)}(\cdot|s, a)\|_1 \leq \frac{1}{N_{\tau(m)}(s, a)} \sum_{q=(\tau(m)-W)\vee1}^{\tau(m)-1} \|p_{\tau(m)}(\cdot|s, a) - p_{\tau(m)}(\cdot|s, a)\|_1 \cdot 1(s_q = s, a_q = a)\leq \eta.\]

  (42)

Step (41) is by the definition of \(\hat{p}_{\tau(m)}(\cdot|s, a)\), and step (42) is by the triangle inequality. Consequently, we have

\[\|p_{\tau(m)}(\cdot|s, a) - \hat{p}_{\tau(m)}(\cdot|s, a)\|_1 \leq \|\hat{p}_{\tau(m)}(\cdot|s, a) - \hat{p}_{\tau(m)}(\cdot|s, a)\|_1 + \|p_{\tau(m)}(\cdot|s, a) - \hat{p}_{\tau(m)}(\cdot|s, a)\|_1 \leq \text{rad}p_{\tau(m)}(s, a) + \eta.\]

(43)

Step (43) is true since we condition on the event \(\mathcal{E}_P\).

Combining the two cases, we have shown that \(p_{\tau(m)}(\cdot|s, a) \in H_{p,\tau(m)}(s, a, \eta)\) for all \(s \in S, a \in A_s\), contradicting to the fact that \(m \in Q_T\). Altogether, our claim on inequality (40) is established.

Finally, we provide an upper bound to \(|Q_T|\) using (40):

\[B_p = \sum_{t \in [T-1]} \max_{s \in S, a \in A_s} \{\|p_{t+1}(\cdot|s, a) - p_t(\cdot|s, a)\|_1\} \geq \sum_{m \in Q_T} \sum_{q=m}^{\tau(m)-1} \max_{s \in S, a \in A_s} \{\|p_{q+1}(\cdot|s, a) - p_q(\cdot|s, a)\|_1\} \]

(44)
We prove by contradiction. Suppose there exists an episode \( m \notin \tilde{Q}_T \), a state \( s \in S \), and an action \( a \in A_s \) such that \( p_{\tau(m)}(s, a) \notin H_{p,\tau(m)}(s, a; \eta) \), then \( m \) should have been added to \( Q_T \). To see this, we first note that episode \( m \) trivially satisfies criterion 1 in the construction of \( Q_T \). Moreover, at the time when \( m \) is examined, we know that any \( m' \) has been added to \( Q_T \) should satisfy \( \tau(m) - \tau(m') > W \) as otherwise \( m \) would have been added to \( Q_T \). Therefore, we have prove \( m \in Q_T \subseteq \tilde{Q}_T \), which is clearly a contradiction.

D.2. Proof of Lemma 7

We prove by contradiction. Suppose there exists an episode \( m \notin \tilde{Q}_T \), a state \( s \in S \), and an action \( a \in A_s \) such that \( p_{\tau(m)}(s, a) \notin H_{p,\tau(m)}(s, a; \eta) \), then \( m \) should have been added to \( Q_T \). To see this, we first note that episode \( m \) trivially satisfies criterion 1 in the construction of \( Q_T \). Moreover, at the time when \( m \) is examined, we know that any \( m' \) has been added to \( Q_T \) should satisfy \( \tau(m) - \tau(m') > W \) as otherwise \( m \) would have been added to \( Q_T \). Therefore, we have prove \( m \in Q_T \subseteq \tilde{Q}_T \), which is clearly a contradiction.

D.3. Proof of Lemma 8

Denote \( Q_T = \{m_1, \ldots, m_{|Q_T|}\} \). By construction, for every element \( m \in \tilde{Q}_T \), there exists an unique \( m' \in Q_T \) such that

\[
\tau(m) - \tau(m') \in [0, W].
\]

(47)

We can thereby partition the elements of \( \tilde{Q}_T \) into \( |Q_T| \) disjoint subsets \( \tilde{Q}_T(m_1), \ldots, \tilde{Q}_T(m_{|Q_T|}) \) such that

1. Each element \( m \in \tilde{Q}_T \) belongs to exactly one \( \tilde{Q}_T(m') \) for some \( m' \in Q_T \).
2. Each element \( m \in Q_T(m') \) satisfies \( \tau(m) - \tau(m') \in [0, W] \).

We bound (♠) from above as

\[
\sum_{m \in Q_T} \max_{s \in S, a \in A_s} \left\{ \left( \sum_{q=t_m}^{\tau(m)-1} (p_{q+1}(s, a) - p_{q}(s, a)) \right) \right\} \geq \sum_{m \in Q_T} \max_{s \in S, a \in A_s} \left\{ \left( \sum_{q=t_m}^{\tau(m)-1} (\rho_t^* - r_t(s_t, a_t)) \right) \right\}
\]

\[
> |Q_T| \eta.
\]

(46)

Step (44) follows by the second criterion of the construction of \( Q_T \), which ensures that for distinct \( m, m' \in Q_T \), the time intervals \( [t_m, \tau(m)], [t_m', \tau(m')] \) are disjoint. Step (45) is by applying the triangle inequality, for each \( m \in Q_T \), on the state-action pair \((s, a)\) that maximizes the term \( \| \sum_{q=t_m}^{\tau(m)-1} (p_{q+1}(s, a) - p_{q}(s, a)) \|_1 = \| (p_{\tau(m)}(s, a) - p_{t_m}(\cdot|s, a)) \|_1 \). Step (46) is by applying the claimed inequality (40) on each \( m \in Q_T \). Altogether, the Lemma is proved.

Here, inequality (48) holds by boundedness of rewards, inequality (49) follows from the fact that episodes are mutually disjoint, inequality (50) makes the observations that each episode can last for at most \( W \) time steps (imposed by the \text{SWUCRL2-CW} algorithm) as well as criterion 2 of the construction of \( Q_T(m') \)'s, and the last step uses Lemma 6.
D.4. Proof of Lemma 9

We first give an upper bound for $M(T)$, the total number of the episodes.

**Lemma 12.** Conditioned on events $E_r, E_p$, we have $M(T) \leq SA(2 + \log_2 W)T/W = \bar{O}(SAT/W)$ with certainty.

**Proof.** First, to demonstrate the bound for $M(T)$, it suffices to show that there are at most $SA(2 + \log_2 W)$ many episodes in each of the following cases: (1) between time steps 1 and $W$, (2) between time steps $jW$ and $(j + 1)W$, for any $j \in \{1, \ldots, [T/W] - 1\}$, (3) between time steps $[T/W] \cdot W$ and $T$. We focus on case (2), and the edge cases (1, 3) can be analyzed similarly.

Between time steps $jW$ and $(j + 1)W$, a new episode $m + 1$ is started when the second criterion $\nu_m(s_t, \bar{\pi}_m(s_t)) < N^+_{r(m)}(s_t, \bar{\pi}_m(s_t))$ is violated during the current episode $m$. We first illustrate how the second criterion is invoked for a fixed state-action pair $(s, a)$, and then bound the number of invocations due to $(s, a)$. Now, let’s denote $m^1, \ldots, m^L$ as the episode indexes, where $jW \leq \tau(m^1) < \tau(m^2) < \ldots < \tau(m^L) < (j + 1)W$, and the second criterion for $(s, a)$ is invoked during $m^\ell$ for $1 \leq \ell \leq L$. That is, for each $\ell \in \{1, \ldots, L\}$, the DM breaks the while loop for episode $m^\ell$ because $\nu_{m^\ell}(s, a) = N^+_{r(m^\ell)}(s, a)$, leading to the new episode $m^\ell + 1$.

To demonstrate our claimed bound for $M(T)$, we show that $L \leq 2 + \log_2 W$ as follows. To ease the notation, let’s denote $\psi^\ell = \nu_{m^\ell}(s, a)$. We first observe that $\psi^1 = N^+_{r(m^1)}(s, a) \geq 1$. Next, we note that for $\ell \in \{2, \ldots, L\}$, we have $\psi^\ell \geq \sum_{k=1}^{\ell-1} \psi^k$. Indeed, we know that for each $\ell$ we have $(\tau(m^\ell + 1) - 1) - \tau(m^\ell) \leq W$, by our assumption on $m^1, \ldots, m^\ell$. Consequently, the counting sum in $N_{r(m^\ell)}(s, a)$, which counts the occurrences of $(s, a)$ in the previous $W$ time steps, must have counted those occurrences corresponding to $\psi^1, \ldots, \psi^{\ell-1}$. The worst case sequence of $\psi^1, \psi^2, \ldots, \psi^L$ that yields the largest $L$ is when $\psi^1 = \psi^2 = 1, \psi^3 = 2$, and more generally $\psi^\ell = 2^{\ell-2}$ for $\ell \geq 2$. Since $\psi^\ell \leq W$ for all $W$, we clearly have $L \leq 2 + \log_2 W$, proving our claimed bound on $L$. Altogether, during the $T$ time steps, there are at most $(SAT(2 + \log_2 W))/W$ episodes due to the second criterion and $T/W$ due to the first, leading to our desired bound on $M(T)$.

Next, we establish the bound for (△). By Lemma 7, we know that $\tilde{\xi}_{\tau(m)}(s) \in [0, 2D_{\max}]$ for all $m \in [M(T)] \setminus \tilde{Q}_T$ and $s$. For each episode $m \in [M(T)] \setminus \tilde{Q}_T$, we have

$$
\sum_{t=\tau(m)}^{\tau(m+1)-1} \sum_{s' \in S} \pi_t(s'|s_t, a_t)\tilde{\bar{\gamma}}_{\tau(m)}(s') - \tilde{\bar{\gamma}}_{\tau(m)}(s_t) = -\tilde{\bar{\gamma}}_{\tau(m)}(s_{\tau(m)}) + \tilde{\bar{\gamma}}_{\tau(m)}(s_{\tau(m)+1}) + \sum_{t=\tau(m)}^{\tau(m+1)-1} \sum_{s' \in S} \pi_t(s'|s_t, a_t)\tilde{\bar{\gamma}}_{\tau(m)}(s') - \tilde{\bar{\gamma}}_{\tau(m)}(s_{t+1})
$$

Summing (51) over $m \in [M(T)] \setminus \tilde{Q}_T$ we get

$$
\sum_{m \in [M(T)] \setminus \tilde{Q}_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} \sum_{s' \in S} \pi_t(s'|s_t, a_t)\tilde{\bar{\gamma}}_{\tau(m)}(s') - \tilde{\bar{\gamma}}_{\tau(m)}(s_t) \leq 2D_{\max} \cdot (M(T) - |\tilde{Q}_T|) + \sum_{m \in [M(T)] \setminus \tilde{Q}_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} Y_{t}
$$

(52)

Define the filtration $\mathcal{H}_{t-1} = \{(s_q, a_q, R_q(s_q, a_q))\}_{q=1}^{t-1}$. Now, we know that $\mathbb{E}[Y_i|\mathcal{H}_{t-1}] = 0$, $Y_i$ is $\sigma(\mathcal{H}_t)$-measurable, and $|Y_i| \leq 2D_{\max}$. Therefore, we can apply the Hoeffding inequality (Hoeffding, 1963) to show that

$$
\sum_{m \in [M(T)] \setminus \tilde{Q}_T} \sum_{t=\tau(m)}^{\tau(m+1)-1} \sum_{s' \in S} \pi_t(s'|s_t, a_t)\tilde{\bar{\gamma}}_{\tau(m)}(s') - \tilde{\bar{\gamma}}_{\tau(m)}(s_t) = O \left( D_{\max} \cdot M(T) + D_{\max} \sqrt{T \log \frac{1}{\delta}} \right)
$$
with probability $1 - O(\delta)$, where we use the facts that $M(T) - |\hat{Q}_T| \leq M(T)$ and $\sum_{m \in [M(T) \setminus \hat{Q}_T]} \sum_{t = \tau(m)}^{\tau(m+1)-1} 1 \leq T$.

Finally, note that
\[
\sum_{m \in [M(T) \setminus \hat{Q}_T]} \sum_{t = \tau(m)}^{\tau(m+1)-1} \frac{1}{\sqrt{\tau(m)}} \leq \sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} \frac{1}{\sqrt{\tau(m)}} \leq \sum_{t=1}^{\lceil T/W \rceil} \frac{W}{\sqrt{t}} = O(\sqrt{T}).
\]

Hence, the Lemma is proved.

**D.5. Proof of Lemma 10**

We first note that
\[
\sum_{m \in [M(T) \setminus \hat{Q}_T]} \sum_{t = \tau(m)}^{\tau(m+1)-1} (2\var_{r,t} + 4D_{\text{max}} \cdot \var_{p,t} + 2D_{\text{max}}\eta) \leq \sum_{t=1}^{T} (2\var_{r,t} + 4D_{\text{max}} \cdot \var_{p,t} + 2D_{\text{max}}\eta),
\]

since $\var_{r,t} \geq 0$ and $\var_{p,t} \geq 0$ for all $t$. We can thus work with the latter quantity.

We first bound $\sum_{t=1}^{T} \var_{r,t}$. Now, recall the definition that, for time $t$ in episode $m$, we have $\var_{r,t} = \sum_{q = \tau(m)-W}^{t} B_{r,q}$. Clearly, for $iW \leq q < (i+1)W$, the summand $B_{r,q}$ only appears in $\var_{r,t}$ for $iW \leq q < t \leq (i+2)W$, since each episode is contained in $\{iW, \ldots, (i+1)W\}$ by our episode termination criteria ($t$ is a multiple of $W$) of the SWUCRL2-CW algorithm. Altogether, we have
\[
2 \sum_{t=1}^{T} \var_{r,t} \leq 2 \sum_{t=1}^{T-1} B_{r,t}W = 2B_rW. \tag{53}
\]

Next, we bound $\sum_{t=1}^{T} \var_{p,t}$. Now, we know that $\tau(t+1) - \tau(m) \leq W$ by our episode termination criteria ($t$ is a multiple of $W$) of the SWUCRL2-CW algorithm. Consequently,
\[
4D_{\text{max}} \sum_{t=1}^{T} \var_{p,t} \leq 4D_{\text{max}} \sum_{t=1}^{T-1} B_{p,t}W = 4D_{\text{max}}B_pW. \tag{54}
\]

Finally, combining (53, 54) with $2D_{\text{max}} \sum_{t=1}^{T} \eta$, the Lemma is established.

**D.6. Proof of Lemma 11**

Due to non-negativity of $\text{rad}_r(s, a)$’s and $\text{rad}_p(s, a)$’s, we have
\[
\sum_{m \in [M(T) \setminus \hat{Q}_T]} \sum_{t = \tau(m)}^{\tau(m+1)-1} \text{rad}_r(s_t, a_t) \leq \sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} \text{rad}_r(s_t, a_t), \tag{55}
\]
\[
\sum_{m \in [M(T) \setminus \hat{Q}_T]} \sum_{t = \tau(m)}^{\tau(m+1)-1} \text{rad}_p(s_t, a_t) \leq \sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} \text{rad}_p(s_t, a_t). \tag{56}
\]

We thus first show that, with certainty,
\[
\sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} \text{rad}_r(s_t, a_t) = \sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} \sqrt{\frac{2\ln(SAT/\delta)}{N_+^{\tau(m)}(s_t, a_t)}} = \tilde{O}\left(\frac{\sqrt{SAT}}{W}\right), \tag{57}
\]
\[
\sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} \text{rad}_p(s_t, a_t) = \sum_{m=1}^{M(T)} \sum_{t = \tau(m)}^{\tau(m+1)-1} 2 \sqrt{\frac{2\ln(AW/\delta)}{N_+^{\tau(m)}(s_t, a_t)}} = \tilde{O}\left(\frac{S\sqrt{AT}}{W}\right). \tag{58}
\]

We analyze by considering the dynamics of the algorithm in each consecutive block of $W$ time steps, in a way similar to the proof of Lemma 9. Consider the episodes indexes $m_0, m_1, \ldots, m_{\lceil T/W \rceil}, m_{\lceil T/W \rceil}+1$, where $\tau(m_0) = 1$, and $\tau(m_j) = jW$.
for \( j \in \{1, \ldots, [T/W]\} \), and \( m_{[T/W]+1} = m(T) + 1 \) (so that \( \tau(m_{[T/W]+1} - 1) \) is the time index for the last episode in the horizon).

To prove (57, 58), it suffices to show that, for each \( j \in \{0, 1, \ldots, [T/W]\} \), we have

\[
\sum_{m=m_j}^{m_{j+1}-1} \sum_{t=\tau(m)}^{\tau(m+1)-1} \frac{1}{\sqrt{N^+_{\tau(m)}(s_t, a_t)}} = O\left(\sqrt{SAW}\right).
\]  

(59)

Without loss of generality, we assume that \( j \in \{1, \ldots, [T/W] - 1\} \), and the edge cases of \( j = 0, [T/W] \) can be analyzed similarly.

Now, we fix a state-action pair \((s, a)\) and focus on the summands in (59):

\[
\sum_{m=m_j}^{m_{j+1}-1} \sum_{t=\tau(m)}^{\tau(m+1)-1} \frac{1}{\sqrt{N^+_{\tau(m)}(s_t, a_t)}} = \sum_{m=m_j}^{m_{j+1}-1} \frac{\nu_m(s, a)}{\sqrt{N^+_{\tau(m)}(s, a)}}
\]  

(60)

It is important to observe that:

1. It holds that \( \nu_m, (s, a) \leq N_{\tau(m)}(s, a) \), by the episode termination criteria of the S\textsc{WUCRL2-CW} algorithm.
2. For \( m \in \{m_j + 1, \ldots, m_{j+1} - 1\} \), we have \( \sum_{m'=m_j}^{m-1} \nu_{m'}(s, a) \leq N_{\tau(m)}(s, a) \). Indeed, we know that episodes \( m_j, \ldots, m_{j+1} - 1 \) are inside the time interval \( \{jW, \ldots, (j+1)W\} \). Consequently, the counts of \( (s_t, a_t) = (s, a) \) associated with \( \nu_m, (s, a) \), are contained in the \( W \) time steps preceding \( \tau(m) \), hence the desired inequality.

With these two observations, we have

\[
(60) \leq \frac{\nu_{m_j}(s, a)}{\sqrt{\max\{\nu_{m_j}(s, a), 1\}}} + \sum_{m=m_j+1}^{m_{j+1}-1} \frac{\nu_m(s, a)}{\sqrt{\max\{\sum_{m'=m_j}^{m-1} \nu_{m'}(s, a), 1\}}} 
\]

\[
\leq \sqrt{\max\{\nu_{m_j}(s, a), 1\}} + (\sqrt{2} + 1) \sqrt{\max\left\{ \sum_{m=m_j}^{m_{j+1}-1} \nu_m(s, a), 1 \right\}} 
\]

(61)

\[
\leq (\sqrt{2} + 2) \sqrt{\max\left\{ \sum_{t=jW}^{(j+1)W-1} 1((s_t, a_t) = (s, a)), 1 \right\}}.
\]

(62)

Step (61) is by Lemma 19 in (Jaksch et al., 2010), which bounds the sum in the previous line. Step (62) is by the fact that episodes \( m_j, \ldots, m_{j+1} - 1 \) partitions the time interval \( [jW, \ldots, (j+1)W - 1] \), and \( \nu_m(s, a) \) counts the occurrences of \( (s_t, a_t) = (s, a) \) in episode \( m \). Finally, observe that (60) = 0 if \( \nu_m(s, a) = 0 \) for all \( m \in \{m_j, \ldots, m_{j+1} - 1\} \). Thus, we can refine the bound in (62) to

\[
(60) \leq (\sqrt{2} + 2) \sqrt{\sum_{t=jW}^{(j+1)W-1} 1((s_t, a_t) = (s, a)).
\]

(63)

The required inequality (59) is finally proved by summing (63) over \( s \in S \in A \) and applying Cauchy Schwartz:

\[
\sum_{m=m_j}^{m_{j+1}-1} \sum_{t=\tau(m)}^{\tau(m+1)-1} \frac{1}{\sqrt{N^+_{\tau(m)}(s_t, a_t)}} = \sum_{s \in S, a \in A} \sum_{m=m_j}^{m_{j+1}-1} \frac{\nu_m(s, a)}{\sqrt{N^+_{\tau(m)}(s, a)}} 
\]

\[
\leq (\sqrt{2} + 2) \sqrt{\sum_{s \in S, a \in A} \sum_{t=jW}^{(j+1)W-1} 1((s_t, a_t) = (s, a))} 
\]

\[
\leq (\sqrt{2} + 2) \sqrt{SAW}.
\]

(64)

Altogether, the Lemma is proved.
We are now ready to state the details of the **BORL** algorithm. For some fixed choice of block length $H$ (to be determined later), we first define a couple of additional notations:

$$
H = 3S^2 A^2 T^2, \Phi = \frac{1}{2T^2}, \Delta_W = \lfloor \ln H \rfloor, \Delta = \Delta_W + 1, J = \{W, \eta\}.
$$

Here, $J_W$ and $J_\eta$ are all possible choices of window length and confidence widening parameter, respectively, and $J$ is the Cartesian product of them with $|J| = \Delta$. We also let $R_i(W, \eta, s)$ be the total rewards for running the **SWUCRL2-CW** algorithm with window length $W$ and confidence widening parameter $\eta$ for block $i$ starting from state $s$.

We follow the exposition in Section 3.2 in [Bubeck & Cesa-Bianchi, 2012] for adapting the EXP3.P algorithm. The EXP3.P algorithm treats each element of $J$ as an arm. It begins by initializing

$$
\alpha = 0.95 \frac{\sqrt{\ln \Delta}}{\Delta^{T/H}}, \quad \beta = \sqrt{\frac{\ln \Delta}{\Delta^{T/H}}}, \quad \gamma = 1.05 \frac{\Delta \ln \Delta}{T/H}, \quad q(j,k),i = 0 \ \forall (j,k) \in M,
$$

where $M = \{ (j',k') : j' \in \{0, 1, \ldots, \Delta_W\}, k' \in \{0, 1, \ldots, \Delta_\eta\} \}$. At the beginning of each block $i \in [\lfloor T/H \rfloor]$, the **BORL** algorithm first sees the state $s_{(i-1)H+1}$, and computes for all $(j,k) \in M$,

$$
u(j,k),i = (1 - \gamma) \frac{\exp(\alpha q(j,k),i)}{\sum_{(j',k') \in M} \exp(\alpha q(j',k'),i)} + \gamma \Delta.
$$

Then it sets $(j_i, k_i) = (j,k)$ with probability $\nu(j,k),i \ \forall (j,k) \in M$. The selected pair of parameters is thus $W_i = \lfloor H^{j_i}/\Delta_W \rfloor$ and $\eta_i = \lfloor \Phi^{k_i}/\Delta_\eta \rfloor$. Afterwards, the **BORL** algorithm starts from state $s_{(i-1)H+1}$, selects actions by running the **SWUCRL2-CW** algorithm with window length $W_i$ and confidence widening parameter $\eta_i$ for each round $t$ in block $i$. At the end of the block, the **BORL** algorithm observes the total rewards $R_i(W_i, \eta_i, s_{(i-1)H+1})$. As a last step, it rescales $R_i(W_i, \eta_i, s_{(i-1)H+1})$ by dividing it by $H$ so that it is within $[0, 1]$, and updates for all $(j,k) \in M$,

$$
q_{(j,k),i+1} = q_{(j,k),i} + \beta \frac{1_{(j,k)=(j_i,k_i)} \cdot R_i(W_i, \eta_i, s_{(i-1)H+1})}{\nu(j,k),i} / H.
$$

The formal description of the **BORL** algorithm (with $H$ defined in the next subsection) is shown in Algorithm 3.

### F. Proof of Theorem 2

To begin, we consider the following regret decomposition, for any choice of $(W^+, \eta^+) \in J$, we have

$$
\text{Dyn-Reg}_T(\text{BORL}) = \sum_{i=1}^{\lfloor T/H \rfloor} \mathbb{E} \left[ \sum_{t=(i-1)H+1}^{iH\wedge T} \rho_t^* - R_i(W_t, \eta_t, s_{(i-1)H+1}) \right].
$$
\[
\sum_{i=1}^{\lceil T/H \rceil} E \left[ \sum_{t=(i-1)H+1}^{i-H\wedge T} \rho_t^* - R_i \left(W^i, \eta^i, s(i-1)H+1\right) \right] + \sum_{i=1}^{\lceil T/H \rceil} E \left[ \sum_{i=1}^{\lceil T/H \rceil} R_i \left(W^i, \eta^i, s(i-1)H+1\right) - R_i \left(W_i, \eta_i, s(i-1)H+1\right) \right].
\]

For the first term in eq. (68), we can apply the results from Theorem 1 to each block \(i \in \lceil T/H \rceil\), i.e.,
\[
\sum_{t=(i-1)H+1}^{i-H^\wedge T} \left[ \rho_t^* - R \left(W^i, \eta^i, s(i-1)H+1\right) \right] = O \left( B_{p}(i)W^i + B_r(i)W^i + D_{\max} \left[ B_p(i)W^i + S\sqrt{AH} + H\eta^i + \frac{S\sqrt{AH}}{W^i} + \sqrt{H} \right] \right),
\]
where we have defined
\[
B_r(i) = \left( \sum_{t=(i-1)H+1}^{i-H^\wedge T} B_{r,t} \right), \quad B_p(i) = \left( \sum_{t=(i-1)H+1}^{i-H^\wedge T} B_{p,t} \right)
\]
for brevity. For the second term, it captures the additional rewards of the DM where it uses the fixed parameters \((W^i, \eta^i)\) throughout w.r.t. the trajectory on the starting states of each block by the BORL algorithm, i.e., \(s_1, \ldots, s(i-1)H+1, \ldots, s(\lceil T/H \rceil - 1)H+1\), and this is exactly the regret of the EXP3.P algorithm when it is applied to a \(\Delta\)-arm adaptive adversarial bandit problem with reward from \([0, H]\). Therefore, for any choice of \((W^i, \eta^i)\), we can upper bound this by
\[
O \left( H\sqrt{\Delta T/H} \right) = O \left( \sqrt{TH} \right)
\]
as \(\Delta = O \left( \ln^2 T \right)\). Summing these two, the regret of the BORL algorithm is
\[
\text{Dyn-Reg}_T^{(\text{BORL})} = O \left( B_pW^i + B_rW^i + D_{\max} \left[ B_pW^i + S\sqrt{AT} + T\eta^i + \frac{SAT}{W^i} + \sqrt{TH} \right] \right).
\]
By virtue of the EXP3.P, the BORL algorithm is able to adapt to any choice of \((W^i, \eta^i) \in J\). Note that
\[
H \geq W* = \frac{3S^{\frac{1}{2}}A^\frac{1}{4}T^\frac{1}{4}}{(B_r + B_p + 1)^\frac{1}{2}} \geq \frac{3T^\frac{1}{2}}{(3T)^\frac{1}{2}} \geq 1,
\]
\[
S^{\frac{1}{2}}A^\frac{1}{4} \geq \eta* = \frac{(B_p + 1)^\frac{1}{2}S^{\frac{1}{2}}A^\frac{1}{4}}{(B_r + B_p + 1)^\frac{1}{2}T^\frac{1}{4}} \geq \frac{S^{\frac{1}{2}}A^\frac{1}{4}}{2T^\frac{1}{4}} = S^{\frac{1}{2}}A^\frac{1}{4}\Phi.
\]
Therefore, there must exists some \(j^\uparrow\) and \(k^\downarrow\) such that
\[
H^{j^\uparrow}/\Delta W \leq W* \leq H^{(j^\uparrow+1)/\Delta W}, \quad S^{\frac{1}{2}}A^\frac{1}{4}\Phi^{k^\downarrow}/\Delta \eta \geq \eta* \geq S^{\frac{1}{2}}A^\frac{1}{4}\Phi^{(k^\downarrow+1)/\Delta \eta}
\]
By adapting \(W^i\) to \(H^{j^\uparrow}/\Delta W\) and \(\eta^i\) to \(\Phi^{k^\downarrow}/\Delta \eta\), we further upper bound eq. (70) as
\[
\text{Dyn-Reg}_T^{(\text{BORL})} = O \left( D_{\max}(B_r + B_p + 1)^\frac{1}{2}S^{\frac{1}{2}}A^\frac{1}{4}T^\frac{1}{4} \right).
\]
where we use \(H^{j^\uparrow}/\Delta W = \exp(1)\) and \(\Phi^{1}/\Delta \eta = \exp(-1)\) in the last step.