ON THE EXISTENCE OF $W^{1,2}_p$ SOLUTIONS FOR FULLY NONLINEAR PARABOLIC EQUATIONS UNDER EITHER RELAXED OR NO CONVEXITY ASSUMPTIONS

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Summary. We establish the existence of solutions of fully nonlinear parabolic second-order equations like $\partial_t u + H(v, Dv, D^2v, t, x) = 0$ in smooth cylinders without requiring $H$ to be convex or concave with respect to the second-order derivatives. Apart from ellipticity nothing is required of $H$ at points at which $|D^2v| \leq K$, where $K$ is any fixed constant. For large $|D^2v|$ some kind of relaxed convexity assumption with respect to $D^2v$ mixed with a VMO condition with respect to $t, x$ are still imposed. The solutions are sought in Sobolev classes. We also establish the solvability without almost any conditions on $H$, apart from ellipticity, but of a “cut-off” version of the equation $\partial_t u + H(v, Dv, D^2v, t, x) = 0$.

1. Introduction and main results

In this paper we consider parabolic equations

$$\partial_t v(t, x) + H[v](t, x) = 0,$$

where

$$H[v](t, x) = H(v(t, x), Dv(t, x), D^2v(t, x), t, x),$$

in subdomains of

$$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}.$$

Let $\Omega \in C^{1,1}$ be an open bounded subset of $\mathbb{R}^d$. Fix $T \in (0, \infty)$ and set

$$\Pi = [0, T) \times \Omega$$

(if the $t$-axis is directed vertically, $[0, T) \times \Omega$ is indeed looking like a pie). Fix

$$p > d$$

and a measurable function $G \geq 0$ on $\mathbb{R}^{d+1}$.

One of our main results implies that, for $d = 3$, equation (1.1) with

$$(a \wedge b = \min(a, b))$$

$$H(D^2u, x) := \bar{G}(t, x) \wedge |D_{12}u| + \bar{G}(t, x) \wedge |D_{23}u| + \bar{G}(t, x) \wedge |D_{31}u|$$

$$+ \Delta u - f(t, x)$$

in $\Pi$ with zero boundary condition on its parabolic boundary has a unique solution $u \in W^{1,2}_p(\Pi)$, provided that $\bar{G}, f \in L_p(\Pi)$ with $p > d + 2$. Recall

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that $W^{1,2}_p(\Pi)$ denotes the set of functions $v$ defined in $\Pi$ such that $\partial_t v, v, Dv,$ and $D^2v$ are in $L_p(\Pi)$. Observe that $H$ in (1.2) is neither convex nor concave with respect to $D^2u$. So far, there are only two approaches to such equations: the theory of $(L_p)$ viscosity solutions and the theory of stochastic differential games, provided $H$ has a somewhat special form. The past experience shows that it is hard to expect getting sharp quantitative results using probability theory. On the other hand, the theory of viscosity solutions indeed produced some remarkable quantitative results (see, for instance, [5], [8] and the references therein). However, to the best of the author’s knowledge the result stated above about (1.2) is either very hard to obtain by using the theory of $(L_p)$ viscosity solutions or is just beyond it, at least at the current stage. It seems that the best information, that theory provides at the moment, is the existence of the maximal and minimal $(L_p)$ viscosity solution (see [8]), no uniqueness of $(L_p)$ viscosity solutions can be inferred for (1.2) and no regularity apart from the classical $C^\alpha$-regularity (see [4]).

The current paper is a natural continuation of [12] where similar results are obtained for elliptic equations.

Fix some constants $K_0, K_F \in [0, \infty)$, $\delta \in (0, 1]$. Denote by $S$ the set of symmetric $d \times d$ matrices and let $S_\delta$ be the subset of $S$ consisting of matrices $a$ such that

$$\delta |\lambda|^2 \leq a^{ij} \lambda^i \lambda^j \leq \delta^{-1} |\lambda|^2 \quad \forall \lambda \in \mathbb{R}^d.$$

Here are our assumptions about $H$.

**Assumption 1.1.** The function $H(u, t, x)$,

$$u = (u', u'') \quad u' = (u'_0, u'_1, \ldots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in S, \quad (t, x) \in \mathbb{R}^{d+1},$$

is measurable with respect to $(u', t, x)$.

The following assumptions contain (small) parameters $\hat{\theta}, \theta \in (0, 1]$ which are specified later in our results.

**Assumption 1.2.** There are two measurable functions

$$F(u, t, x) = F(u'_0, u'', t, x), \quad G(u, t, x)$$

such that

$$H = F + G.$$ 

For $u'' \in S, u' \in \mathbb{R}^{d+1},$ and $(t, x) \in \mathbb{R}^{d+1}$ we have

$$|G(u, t, x)| \leq \hat{\theta}|u''| + K_0|u'| + G(t, x), \quad F(0, t, x) \equiv 0.$$

Define

$$B_R(x_0) = \{x \in \mathbb{R} : |x - x_0| < R\}, \quad B_R = B_R(0),$$

$$C_r(t_0, x_0) = [t_0, t_0 + r^2] \times B_r(x_0), \quad C_r = C_r(0),$$

and for Borel $\Gamma \subset \mathbb{R}^{d+1}$ denote by $|\Gamma|$ the volume of $\Gamma$. 

Assumption 1.3. (i) The function \( F \) is Lipschitz continuous with respect to \( u'' \) with Lipschitz constant \( K_F \) and is measurable with respect to \((t,x)\).

Moreover there exist \( R_0 \in (0,1) \) and \( \tau_0 \in [0,\infty) \) such that, for any \( u'_0 \in \mathbb{R}, \ z_0 = (t_0,x_0) \in \Pi \) and \( r \in (0,R_0]\), one can find a convex function \( \bar{F}(u'') = \bar{F}_0, r, u'_0(u'') \) (depending only on \( u'' \)) for which

(ii) We have \( \bar{F}(0) = 0 \) and at all points of differentiability of \( \bar{F} \) we have \( D_{u''}\bar{F} \in S_0 \).

(iii) For any \( u'' \in S \) with \( |u''| = 1 \), we have

\[
\int_{\hat{C}_r(z_0)_{\tau > \tau_0}} \sup_{\tau^{-1}|F(u'_0, \tau u'', z) - \bar{F}(\tau u'')|} \, dz \leq \theta |\hat{C}_r(z_0)|, \tag{1.3}
\]

where

\[
\hat{C}_r(z_0) = (t_0, t_0 + r^2) \times (\Omega \cap B_r(x_0));
\]

(iv) There exists a continuous increasing function \( \omega_F(\tau), \tau \geq 0 \), such that \( \omega_F(0) = 0 \) and for any \( u'_0, v'_0 \in \mathbb{R}, (t, x) \in \Pi, \) and \( u'' \in S \) we have

\[
|F(u'_0, u'', t, x) - F(v'_0, u'', t, x)| \leq \omega_F(|u'_0 - v'_0|)|u''|.
\]

Remark 1.4. Assumptions 1.2 and 1.3 (iv) imply that

\[
|H(u', 0, t, x)| \leq K_0 |u'| + G(t, x) \quad \forall u', (t, x) \in \mathbb{R}^{d+1}. \tag{1.4}
\]

Assumption 1.5. We are given a function \( g \in W^{1,2}_{\rho}(\Pi) \).

If \( z_i = (t_i, x_i) \in \mathbb{R}^{d+1}, \ i = 1, 2, \) we set

\[
\rho(z_1, z_2) = |t_1 - t_2|^{1/2} + |x_1 - x_2|.
\]

Definition 1.6. For a function \( u \in C(\Pi) \) set

\[
\omega_u(\Pi, \rho) = \sup\{|u(z_1) - u(z_2)| : z_1, z_2 \in \Pi, \rho(z_1, z_2) \leq \rho\},
\]

\[
\omega_{F,u,\Pi}(\rho) = \omega_F(\omega_u(\Pi, \rho)).
\]

We will sometimes say that a certain constant depends only on \( A, B, \ldots, \) and the function \( \omega_{F,u,\Pi} \). This is to mean that it depends only on \( A, B, \ldots, \) and on the maximal solution of an inequality like \( N_0 \omega_{F,u,\Pi}(\rho) \leq 1/2 \), where the range of \( \rho \) and the value of \( N_0 \) depending only on \( A, B, \ldots \) could be always found out from our arguments.

In the following theorem about a priori estimates there is no ellipticity assumption on \( H \). If \( Q \) is a subdomain in \( \mathbb{R}^{d+1} \), by \( \partial' Q \) we denote its parabolic boundary.

Theorem 1.7. Let \( p > d + 1 \). Then there exist constants \( \theta, \hat{\theta} \in (0,1] \), depending only on \( d, p, \delta, K_F, \) and \( M_2(\Omega) \) \( (\rho_0(\Omega) \) and \( M_2(\Omega) \) are introduced later), such that, if Assumptions 1.2 and 1.3 are satisfied with these \( \hat{\theta} \) and \( \theta \), respectively, then for any \( u \in W^{1,2}_{d+1}(\Pi) \) that satisfies (1.1) in \( \Pi \) \( (a.e.) \) and equals \( g \) on \( \partial' \Pi \) we have

\[
\|u\|_{W^{1,2}_{\rho}(\Pi)} \leq N\|G\|_{L_p(\Pi)} + N\|g\|_{W^{1,2}_{d+1}(\Pi)} + N\tau_0 + N \sup_{\Pi} |u|, \tag{1.5}
\]
where the constants $N$ depend only on $K_0$, $K_F$, $d$, $p$, $\delta$, $R_0$, $\rho_0(\Omega)$, $M_2(\Omega)$, $\text{diam}(\Omega)$, $T$, and the functions $\omega_{F,u,\Pi}$ and $\omega_{F,g,\Pi}$.

In the literature, interior $W_p^2, p > d$, a priori estimates for a class of fully nonlinear uniformly elliptic equations in $\mathbb{R}^d$ in the framework of viscosity solutions were first obtained by Caffarelli in [2] (see also [3]). Adapting his technique, similar interior a priori estimates were proved by Wang [15] for parabolic equations. In the same paper, a boundary estimate is stated but without proof; see Theorem 5.8 there. By exploiting a weak reverse Hölder’s inequality, the result of [2] was sharpened by Escauriaza in [7], who obtained the interior $W_p^2$-estimate for the same equations allowing $p > d - \varepsilon$, with a small constant $\varepsilon > 0$ depending only on the ellipticity constant and $d$.

The above cited works are quite remarkable in one respect—they do not suppose that $H$ is convex or concave in $D^2u$. But they only show that to prove a priori estimates it suffices to prove the interior $C^2$-estimates for “harmonic” functions. However, up to now, these estimates are only known under convexity assumptions.

Also obtaining boundary $W_p^2$ estimate by using the theory of viscosity solutions turned out to be extremely challenging and only in 2009, twenty years after the work of Caffarelli, Winter [16] proved the solvability in $W_p^2(\Omega)$ of equations with Dirichlet boundary condition in $\Omega \in C^{1,1}$.

It is also worth noting that a solvability theorem in the space $W^{1,2}_{p, \text{loc}}(\Pi) \cap C(\overline{\Pi})$ is given in M. G. Crandall, M. Kocan, A. Święch [5] for the boundary-value problem for fully nonlinear parabolic equations. The above mentioned existence results of [5] and [16] are proved under the assumption that $H$ is convex in $D^2v$ and in all papers mentioned above a small oscillation assumption in the integral sense is imposed on the operators. In the case of linear equations this small oscillation assumption is equivalent to requiring the main coefficients to be uniformly close to uniformly continuous ones. Our Assumption 1.3 is satisfied in this case if the main coefficients are just in VMO. The above cited works are performed in the framework of viscosity solutions.

To the best of the author’s knowledge the only paper treating the solvability in the global Sobolev spaces for parabolic equations is [6], where the assumptions are much heavier than here.

To have the solvability we need ellipticity and more regularity of $H$.

Assumption 1.8. For any $(t,x) \in \mathbb{R}^{d+1}$, the function $H(u,t,x)$ is continuous with respect to $u$, is Lipschitz continuous with respect to $u''$, and at all points of differentiability of $H$ with respect to $u''$ we have $D_{u''}H \in S_\delta$.

In the following theorem we need higher values of $p$ than in Theorem 1.7 because in the proof we need to use the embedding $W_p^{1,2} \subset C^{0,1}$.

Theorem 1.9. Let $p > d + 2$ and suppose that Assumptions 1.5 and 1.8 are satisfied and $\bar{G} \in L_p(\Pi)$. Then there exist constants $\theta, \hat{\theta} \in (0, 1]$, depending
only on $d$, $p$, $\delta$, $K_F$, and $M_2(\Omega)$, such that, if Assumptions 1.3 and 1.2 are satisfied with these $\theta$ and $\hat{\theta}$, respectively, then there exists $u \in W^{1,2}_p(\Pi)$ satisfying (1.1) in $\Pi$ (a.e.) and such that $u = g$ on $\partial'\Pi$.

Remark 1.10. Observe that generally there is no uniqueness in Theorem 1.9. For instance, in the one-dimensional case the (quasilinear) equation

$$
\partial_t u + D^2u - (1 - t)\sqrt{12|Du|} + 2\sqrt{(1 - |x|^3)}u = 0
$$

in $\Pi = [0, 1) \times (-1, 1)$ with zero boundary data on $\partial'\Pi$ has two solutions: one is identically equal to zero and the other one is $(1 - t)^2(1 - |x|^3)$.

Uniqueness of solutions can be investigated by using the results in [9].

Remark 1.11. In case of linear equations Theorem 1.9 contains (apart from the restrictions on $p$) the corresponding result of [1] proved for equations with VMO main coefficients.

In Theorem 5.9 of Wang [15] one can find an a priori estimate for any viscosity solution in case $H$ is independent of $u'$ and $\Pi = C_1$.

By the way, it can be seen from our proofs that, if $H$ is independent of $[u'] := (u'_1, \ldots, u'_d)$, we can take $p > d + 1$ in Theorem 1.9.

Example 1.12. For $\tau > 0$ take

$$
H(u) = (1 + \tau \cos \sqrt{|\ln|u''||}) \text{trace } u''
$$

and choose $\tau$ so small that $D_{u''}H \in \mathbb{S}_\delta$ for a $\delta \in (0, 1]$. Then again $H$ is neither convex nor concave with respect to $u''$ and our assumptions are satisfied perhaps with a further reduced $\tau$ for $\bar{F}(u'') = \text{trace } u''$. An interesting feature of this example is that, for generic $u$, the limit of $(1/\lambda)H(\lambda u)$ as $\lambda \to \infty$ does not exist.

Example 1.13. Let $A$ and $B$ be some countable sets and assume that for $\alpha \in A$, $\beta \in B$, $(t, x) \in \mathbb{R}^{d+1}$, and $u' \in \mathbb{R}^{d+1}$ we are given an $\mathbb{S}_\delta$-valued function $a^\alpha(u'_0, t, x)$ (independent of $\beta$) and a real-valued function $b^{\alpha\beta}(u', t, x)$. Assume that these functions are measurable in $t, x$, $a^\alpha$ and $b^{\alpha\beta}$ are continuous with respect to $u'$ uniformly with respect to $\alpha, \beta, t, x$, and

$$
|b^{\alpha\beta}(u', t, x)| \leq K_0|u'| + \tilde{G}(t, x),
$$

where $\tilde{G} \in L^p(\Pi)$, $p > d + 2$.

Consider equation (1.1), where

$$
H(u, t, x) := \inf_{\beta \in B} \sup_{\alpha \in A} \left[ \sum_{i,j=1}^d a^{\alpha}_{ij}(u'_0, t, x)u''_{ij} + b^{\alpha\beta}(u', t, x) \right].
$$

Our measurability, boundedness, and countability assumptions guarantee that $H$ is measurable in $t, x$ and Lipschitz continuous in $u''$. One can also easily check that at all points of differentiability $D_{u''}H \in \mathbb{S}_\delta$. Next assume
that there is an $R_0 \in (0, \infty)$ such that for any $z_0 \in \Pi$, $r \in (0, R_0]$, and $u'_0 \in \mathbb{R}$ one can find $\bar{a}^\alpha \in S_\delta$ (independent of $t, x$) such that

$$\int_{\hat{C}_r(z_0)} \sup_{\alpha \in A} |a^\alpha(u'_0, z) - \bar{a}^\alpha| \, dz \leq \theta, \quad \left( \int_{\Gamma} h \, dz := |\Gamma|^{-1} \int_{\Gamma} h \, dz \right),$$

where $\theta$ is taken from Theorem 1.9.

Then we claim that the assertions of Theorem 1.9 hold true and estimate (1.5) holds with $\tau_0 = 0$.

To prove the claim introduce

$$F(u'_0, u'', t, x) = \sup_{\alpha \in A} \sum_{i,j=1}^d a^\alpha_{ij}(u'_0, t, x)u''_{ij}, \quad G = H - F.$$

Notice that Assumption 1.3 is satisfied with $\tau_0 = 0$ and

$$\tilde{F}(u'') := \sup_{\alpha \in A} \sum_{i,j=1}^d \tilde{a}^\alpha_{ij} u''_{ij}$$

because these functions are convex, positive homogeneous of degree one with respect to $u''$ and, for $|u''| = 1$,

$$\int_{\hat{C}_r(z_0)} |F(u'_0, u'', z) - \tilde{F}(u'')| \, dz \leq \int_{\hat{C}_r(z_0)} \sup_{\alpha \in A} \sum_{i,j=1}^d \left[ a^\alpha_{ij}(u'_0, z) - \bar{a}^\alpha \right] u''_{ij} \, dz$$

$$\leq \int_{\hat{C}_r(z_0)} \sup_{\alpha \in A} |a^\alpha(u'_0, z) - \bar{a}^\alpha| \, dz \leq \theta.$$

One can easily check that the remaining item (iv) in Assumptions 1.3 and Assumption 1.2 (with $\hat{\theta} = 0$) are satisfied as well and this proves our claim. Thus Theorem 1.9 is applicable.

As a result we have a solvability theorem for (1.1), which covers (apart from the restriction on $p$), as $A$ and $B$ are singletons, the first result about solvability of linear parabolic equations with VMO coefficients obtained by Bramanti and Cerutti in [1]. In this singleton case we also consider quasi-linear equations.

In the following theorem Assumption 1.3 is not used.

**Theorem 1.14.** Let $p > d + 2$ and suppose that Assumptions 1.1, 1.8, and 1.5 are satisfied, $G \in L_p(\Pi)$, and (1.4) holds true. Let $P(u'')$ be a convex function on $S$ such that at each point of its differentiability $D_{u''}P \in S_{\delta'}$, where $\delta' \in (0, \delta]$. Also assume that for any $a \in S_\delta$ and $u'' \in S$ we have

$$a^{ij}u''_{ij} \leq P(u'') + K,$$

where $K$ is a constant. Then the equation

$$\partial_t u + \max \{ H[u], P[u] \} = 0$$

(a.e.) in $\Pi$ with boundary condition $u = g$ on $\partial'\Pi$ has a solution $u \in W^{1,2}_p(\Pi)$. 

Proof. Introduce 
\[ \tilde{H}(u, t, x) = \max(H(u, t, x), P(u'')) \], \quad \tilde{F}(u'', t, x) = P(u'') - P(0), \quad \tilde{G} = H - \tilde{F}. \]
Obviously Assumptions 1.3 and 1.8, are satisfied for \( \tilde{H}, \tilde{F}, \) and \( \tilde{F} \) in place of \( H, F, \) and \( \bar{F} \), respectively, with a \( K_F, \tau_0 = \theta = 0, \) and \( \delta' \) in place of \( \delta. \)
Finally, for any \( u, t, x, \)
\[ \tilde{G}(u, t, x) = \max(H(u, t, x) - P(u''), P(0), P(0)) \geq P(0), \]
where for an \( a \in S_\delta \)
\[ H(u, t, x) - P(u'') = H(u, t, x) - H(u', 0, t, x) - P(u'') + H(u', 0, t, x) \]
\[ = a^{ij}u''_{ij} - P(u'') + H(u', 0, t, x) \leq K + H(u', 0, t, x), \]
which together with (1.4) shows that Assumption 1.2 is also satisfied with \( \bar{\theta} = 0 \) and \( \bar{G} + K + |P(0)| \) in place of \( G. \)
Hence, Theorem 1.9 is applicable and our theorem is proved. \( \square \)

2. Interior estimates of integral oscillations of \( D^2u \)

Let \( F(u'') \) be a convex function of \( u'' \in S \) (independent of \( (t, x) \)) such that
(i) \( F(0) = 0, \)
(ii) at all points of differentiability of \( F \) we have \( D_{u''}F \in S_\delta, \) where \( \delta \in (0, 1] \) is a fixed number.

The following theorem is a particular case of the results in [14].

Theorem 2.1. There exists and \( \bar{\alpha} = \bar{\alpha}(d, \delta) \in (0, 1) \) such that for any \( \alpha \in (0, \bar{\alpha}] \) and \( g \in C(\partial C_2) \) there exists a unique \( v \in C(C_2) \cap C^{2+\bar{\alpha}}_{\text{loc}}(C_2) \)
satisfying
\[ \partial_t v + F(D^2v) = 0 \quad \text{in} \quad C_2, \quad v = g \quad \text{on} \quad \partial C_2. \quad (2.1) \]
Furthermore,
\[ |D^2v(z_1) - D^2v(z_2)| \leq N \rho^\alpha(z_1, z_2) \sup_{\partial C_2} |g| \]
as long as \( z_1, z_2 \in C_1, \) where \( N \) depends only on \( \delta, \alpha, \) and \( d. \)

Below in this section we fix \( \alpha \in (0, \bar{\alpha}]. \) Recall that for a measurable set \( \Gamma \subset \mathbb{R}^{d+1} \) we denote by \( |\Gamma| \) its Lebesgue measure, and if \( |\Gamma| \neq 0 \) and \( u \) is integrable over \( \Gamma \) we set
\[ u_\Gamma = \int_\Gamma u \, dx dt = \frac{1}{|\Gamma|} \int_\Gamma u \, dx dt. \]

Lemma 2.2. Let \( r \in (0, \infty), \nu \geq 2 \) and let \( \phi \in C(\partial \bar{C}_{\nu r}). \) Then there exists a unique \( v \in C(C_{\nu r}) \cap C^{2+\alpha}_{\text{loc}}(C_{\nu r}) \) such that
\[ \partial_t v + F(D^2v) = 0 \quad \text{in} \quad C_{\nu r}, \quad v = \phi \quad \text{on} \quad \partial C_{\nu r}. \]
Furthermore,
\[
\int_{C_r} \int_{C_r} |D^2 v(z_1) - D^2 v(z_2)| dz_1 dz_2 \leq N(d, \alpha, \delta) \nu^{-\alpha} r^{-2} \sup_{\partial C_{r'}} |\phi|.
\]

Proof. Scalings show that it suffices to concentrate on \( r = 2/\nu \). In that case the existence of solution follows from Theorem 2.1, which also implies that for \( z_1, z_2 \in C_{2/\nu} \subset C_1 \)
\[
|D^2 v(z_1) - D^2 v(z_2)| \leq N \nu^{-\alpha} \sup_{\partial C_2} |\phi|.
\]

It only remains to observe that
\[
\int_{C_{2/\nu}} \int_{C_{2/\nu}} |D^2 v(z_1) - D^2 v(z_2)| dz_1 dz_1 \leq \sup_{z_1, z_2 \in C_{2/\nu}} |D^2 v(z_1) - D^2 v(z_2)|.
\]

The lemma is proved. \( \square \)

Here is Theorem 1.9 of [10] combined with Theorem 2.3 of [10] (see also [6]).

**Theorem 2.3.** Let \( u \in C(\bar{C}_1) \cap W_{d+1, \text{loc}}^{1,2}(C_1) \). Then there are constants \( \bar{\gamma} = \bar{\gamma}(d, \delta, K) \in (0, 1] \) and \( N \), depending only on \( \delta, d, \) and \( K \), such that for any \( \gamma \in (0, \bar{\gamma}] \) and any operator \( L = a^{ij} D_{ij} + b^i D_i \), with measurable \( \mathbb{S}_d \)-valued coefficients \( a^{ij} \) and \( b^i \), such that \( |(b^i)| \leq K \), given in \( C_1 \), we have
\[
\int_{C_1} (|D^2 u|^\gamma + |Du|^\gamma) \, dx \, dt \leq N \sup_{\partial C_1} |u|^\gamma + N \left( \int_{C_1} |\partial_t u + Lu|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)}.
\]
(2.2)

Below we take \( \gamma \in (0, \bar{\gamma}] \).

**Lemma 2.4.** Let \( r \in (0, \infty) \) and \( \nu \in [2, \infty) \). Then for any \( u \in W_{d+1}^{1,2}(C_{r'}) \) we have
\[
\left( \int_{C_r} \int_{C_r} |D^2 u(z_1) - D^2 u(z_2)|^\gamma dz_1 dz_2 \right)^{1/\gamma} \leq N \nu^{(d+2)/\gamma} \left( \int_{C_{r'}} |\partial_t u + F[u]|^{d+1} \, dz \right)^{1/(d+1)}
+ N \nu^{-\alpha} \left( \int_{C_{r'}} |D^2 u|^{d+1} \, dz \right)^{1/(d+1)},
\]
(2.3)

where \( N \) depends only on \( d, \alpha, \) and \( \delta \).

Proof. Define \( v \) to be a unique \( C(\bar{C}_{r'}) \cap C_{\text{loc}}^{2+\alpha}(C_{r'}) \)-solution of equation \( \partial_t v + F[v] = 0 \) in \( C_{r'} \) with boundary condition \( v = u \) on \( \partial C_{r'} \). Such a function exists by Lemma 2.2. Furthermore, \( v(x) - b^i x_i - c \) satisfies the same equation for any constant \( b^i, c \). Hence by Lemma 2.2 and Hölder’s inequality
\[
I_r := \left( \int_{C_r} \int_{C_r} |D^2 v(z_1) - D^2 v(z_2)|^\gamma dz_1 dz_2 \right)^{1/\gamma}
\]
\[ \leq N \nu^{-2} \alpha r^{-2} \sup_{z=(t,x) \in \partial C_{\nu r}} |u(z) - (D_i u)_{C_{\nu r}} x_i - u_{C_{\nu r}}|. \]

By Poincaré's inequality (see, for instance, Corollary 5.3 in [6]) the last supremum is dominated by a constant times
\[ \nu^2 r^2 \left( \int_{C_{\nu r}} |D^2 u|^{d+1} dz \right)^{1/(d+1)}. \]

It follows that
\[ I_r \leq N \nu^{-\alpha} \left( \int_{C_{\nu r}} |D^2 u|^{d+1} dz \right)^{1/(d+1)}. \] (2.4)

Next, the function \( w := u - v \) is of class \( W^{1,2}_{d+1,\text{loc}}(C_{\nu r}) \cap C(\bar{C}_{\nu r}) \) and for an operator \( \mathcal{L} = a_{ij} D_{ij} \) we have
\[ \partial_t u + F[u] = \partial_t u + F[u] - (\partial_t v + F[v]) = \partial_t w + \mathcal{L}w. \]
Moreover, \( w = 0 \) on \( \partial' C_{\nu r} \). Therefore, by Theorem 2.3, there exists \( N = N(d, \delta) < \infty \) such that
\[ -\int_{C_{\nu r}} |D^2 w|^{\gamma} dz \leq \nu^{d+2} \int_{C_{\nu r}} |D^2 w|^{\gamma} dz \]
\[ \leq N \nu^{d+2} \left( \int_{C_{\nu r}} |\partial_t u + F[u]|^{d+1} dz \right)^{\gamma/(d+1)}. \]

Upon combining this result with (2.4) we come to (2.3) and the lemma is proved. \( \square \)

3. A PRIORI ESTIMATES IN \( W^{1,2}_{p,\text{loc}} \)

Here we suppose that Assumptions 1.2 and 1.3 are satisfied. Thus, we assume that all assumptions on \( H \) and \( F \) stated before Theorem 1.7 are satisfied. Take \( \alpha \in (0, \bar{\alpha}] \) and \( \gamma \in (0, \bar{\gamma}] \). First we note the following.

**Lemma 3.1.** For any \( q \in [1, \infty) \) and \( \mu > 0 \) there is a \( \theta = \theta(d, \delta, K_F, \mu, q) > 0 \) such that, if Assumption 1.3 is satisfied with this \( \theta \), then for any \( u_0' \in \mathbb{R}, \ r \in (0, R_0) \) and \( z_0 \in \Pi \) such that \( C_r(z_0) \subset \Pi \)
\[ \int_{C_r(z_0)} \sup_{u'' \in \mathbb{S}, |u''| > r_0} \left| F(u_0', u'', z) - \bar{F}(u'') \right|^q \left| u'' \right|^q dz \leq \mu^q, \]

where \( \bar{F} = \bar{F}_{z_0,r,u_0'} \).

The proof of this lemma is practically identical to that of Lemma 5.1 of [12] given there for the elliptic case.

**Lemma 3.2.** Let \( u \in W^{1,2}_{d+1,\text{loc}}(\Pi) \). Then there exist an \( \mathbb{S}_{\delta} \)-valued function \( a(t,x) \), \( \mathbb{R}^d \)-valued functions \( b(t,x) \), and real-valued function \( f(t,x) \), such that they are measurable,
\[ |b| \leq K_0, \ |f| \leq \tilde{G} + K_0|u|, \]
and in $\Pi$ (a.e.)

$$a^{ij}D_{ij}u + b^iD_iu + f = H[u].$$

(3.1)

This is a simple consequence of the fact that there is an $S_\delta$-valued function $a$ such that

$$H[u](t, x) - H(u, Du, 0, t, x) = a^{ij}D_{ij}u,$$

and

$$|H(u, Du, 0, t, x)| \leq K_0(|u| + |Du|) + \hat{G}.$$

**Lemma 3.3.** Let $r \in (0, \infty)$ and $\nu \geq 2$ satisfy $\nu r \leq R_0$. Take

$$\mu \in (0, \infty), \quad \beta \in (1, \infty),$$

and suppose that Assumption 1.3 is satisfied with $\theta = \theta(d, \delta, K_F, \mu, \beta d + \beta)$ (see Lemma 3.1). Take a function $u \in W^{1,2}_{d+1}(\Pi)$ and for $z_0 \in \Pi$ such that $C_{\nu r}(z_0) \subset \Pi$ (if such $z_0$'s exist) denote

$$I_r(z_0) = \left( \int_{C_r(z_0)} \int_{C_r(z_0)} |D^2u(z_1) - D^2u(z_2)|^\gamma dz_1 dz_2 \right)^{1/\gamma}.$$

Then

$$I_r(z_0) \leq N\nu^{(d+2)/\gamma} \left( \int_{C_{\nu r}(z_0)} |\partial_t u + F[u]|^{d+1} dz \right)^{1/(d+1)} + N\tau_0\nu^{(d+2)/\gamma}$$

$$+ N \left[ (\mu + \omega_{F,u,\Pi}(\nu r))\nu^{(d+2)/\gamma} + \nu^{-\alpha} \right] \left( \int_{C_{\nu r}(z_0)} |D^2u|^\beta dz \right)^{1/(\beta'(d+1))},$$

(3.2)

where $\beta' = \beta/(\beta - 1)$ and $N$ depends only on $d, K_F, \alpha$, and $\delta$.

**Proof.** Set $\rho := \nu r$. Since $\rho \leq R_0$, $\hat{F} = \hat{F}_{2\rho, \nu r}(z_0)$ is well defined and by Lemma 2.4

$$I_r(z_0) \leq N\nu^{(d+2)/\gamma} \left( \int_{C_{\rho}(z_0)} |\partial_t u + \hat{F}[u]|^{d+1} dz \right)^{1/(d+1)}$$

$$+ N\nu^{-\alpha} \left( \int_{C_{\rho}(z_0)} |D^2u|^{d+1} dz \right)^{1/(d+1)}.$$  

(3.3)

By setting $\hat{F}[u](z) = F(u(z_0), D^2u(z))$ we find

$$\int_{C_{\rho}(z_0)} |\partial_t u + \hat{F}[u]|^{d+1} dz \leq N \int_{C_{\rho}(z_0)} |\partial_t u + F[u]|^{d+1} dy + NJ_1 + NJ_2,$$

where

$$J_1 = \int_{C_{\rho}(z_0)} |\hat{F}[u] - \hat{F}[u]|^{d+1} dz$$

is dominated by

$$\int_{C_{\rho}(z_0)} I_{|D^2u| > \tau_0} \frac{|\hat{F}[u] - \hat{F}[u]|^{d+1}}{|D^2u|^{d+1}} |D^2u|^{d+1} dz + N\tau_0^{d+1},$$

and

$$|H[u](t, x) - H(u, Du, 0, t, x)| \leq K_0(|u| + |Du|) + \hat{G}.$$
which in turn owing to Lemma 3.1 and Hölder’s inequality is less than
\[
N\mu^{d+1} \left( \int_{C_\rho(z_0)} |D^2 u|^{\beta'(d+1)} \, dz \right)^{1/\beta'} + N\tau_0^{d+1},
\]
and
\[
J_2 = \int_{C_\rho(z_0)} |\hat{F}[u] - F[u]|^{d+1} \, dz \leq \omega_{F,\rho}^{d+1} \left( \frac{\text{osc}_{C_\rho(z_0)} u}{\lambda} \right) \int_{C_\rho(z_0)} |D^2 u|^{d+1} \, dz.
\]

It follows that
\[
\left( \int_{C_\rho(z_0)} |\partial_t u + \hat{F}[u]|^{d+1} \, dy \right)^{1/(d+1)} \leq N \left( \int_{C_\rho(z_0)} |\partial_t u + F[u]|^{d+1} \, dy \right)^{1/(d+1)}
\]
\[
+ N\mu \left( \int_{C_\rho(z_0)} |D^2 u|^{\beta'(d+1)} \, dy \right)^{1/\beta' + \beta} + N\tau_0
\]
\[
+ N\omega_{F,\rho}(\mu) \left( \int_{C_\rho(z_0)} |D^2 u|^{d+1} \, dz \right)^{1/(d+1)}.
\]

This and (3.3) yield (3.2) since
\[
\left( \int_{C_\rho(z_0)} |D^2 u|^{d+1} \, dz \right)^{1/(d+1)} \leq \left( \int_{C_\rho(z_0)} |D^2 u|^{\beta'(d+1)} \, dz \right)^{1/(\beta'(d+1))}
\]
by Hölder’s inequality. The lemma is proved. \(\Box\)

**Lemma 3.4.** Take \(p > d + 1\), \(R \in (0, 1]\), and \(u \in W_p^{1,2}(C_{2R})\). Then there exist constants \(\tilde{\theta}, \theta \in (0, 1]\), depending only on \(d, p, \delta, \) and \(K_F\), such that, if Assumptions 1.2 and 1.3 are satisfied with these \(\tilde{\theta}\) and \(\theta\), respectively, then there is a constant \(N\), depending only on \(R_0, d, p, K_0, K_F, \delta, \) and \(\omega_{F,\rho}(C_{2R})\), such that
\[
\|D^2 u\|_{L_p(C_R)} \leq N\|\partial_t u + H[u]\|_{L_p(C_{2R})} + N\|\tilde{G}\|_{L_p(C_{2R})} + N\tau_0
\]
\[
+ NR^{(d+2)(1/p - 1/\gamma)} \|D^2 u\|_1^{1/\gamma} + N\|u\|_{L_p(C_{2R})}, \tag{3.4}
\]
\[
\|D^2 u\|_{L_p(C_R)} \leq N\tau_0 + NR^{(d+2)/(p-2)} \sup_{C_{2R}} |u|
\]
\[
+ N(\|\partial_t u + H[u]\|_{L_p(C_{2R})} + \|\tilde{G}\|_{L_p(C_{2R})}). \tag{3.5}
\]

**Proof.** For \(\rho > 0,\) and \(z \in Q := \mathbb{R}_+ \times \mathbb{R}^d\) introduce
\[
I_r(h, z) = \left( \int_{C_r(z)} \int_{C_r(z)} |h(z_1) - h(z_2)|^{q} \, dz_1 \, dz_2 \right)^{1/q},
\]
\[
h_{Q, \gamma, \rho}^+(z) = \sup \{I_r(h, z_0) : z_0 \in Q, r \in (0, \rho], C_r(z_0) \ni z\},
\]
\[
Mh(z) = \sup_{r > 0, \ 0 < C_r(z_0) \ni z} \int_{C_r(z_0)} |h(\zeta)| \, d\zeta, \tag{3.6}
\]
whenever these definitions make sense. Note that $h_{Q,1}^{\#}$ is well defined in $C_R$ for measurable $h$ even defined only in $C_{R+2\rho}$.

Then take $\varepsilon \in (0,1]$ to be specified later and take $R_1 < R_2 \leq 2R$ such that

$$R_2 - R_1 \leq \varepsilon R_0, \quad R_2 \leq 2R_1. \quad (3.7)$$

Next, take $\nu \geq 2$ and set

$$r_0 = (R_2 - R_1)/(\nu + 1).$$

Observe that $\nu r_0 \leq \varepsilon R_0$ and $R_2 - \nu r_0 = R_1 + r_0$. It follows that, if $r \leq r_0$, $z \in C_{R_1}$, and $z \in C_\tau(z_0)$, then $C_{\nu r}(z_0) \subset C_{R_2}$, which by Lemma 3.3 applied with $\Pi = C_{R_2}$ implies that

$$I_r(z_0) \leq N \nu^{(d+2)/\gamma} \mathbb{M}^{1/(d+1)}(|\partial_t u + F[u]|^{d+1} I_{C_{R_2}})(z) + N\tau_0 \nu^{(d+2)/\gamma}$$

with $N$ depending only on $d, K_F$, and $\delta$. It follows that in $C_{R_1}$

$$(D^2u)_{Q,1}^{\#} \leq N \nu^{(d+2)/\gamma} \mathbb{M}^{1/(d+1)}(|\partial_t u + F[u]|^{d+1} I_{C_{R_2}}) + N\tau_0 \nu^{(d+2)/\gamma}$$

$$+ N \left( \nu^{(d+2)/\gamma} + \nu^{-\alpha} \right) \mathbb{M}^{1/(\beta'(d+1))}(|D^2u|^{\beta'(d+1)} I_{C_{R_2}}).$$

By Theorem 7.1, with

$$\kappa = r_0/R_1 \leq 1/3, \quad \chi_1 = (d+2)/\gamma, \quad \chi_2 = (d+2)(1/\gamma - 1/p)$$

and the Hardy-Littlewood maximal function theorem, by taking $\beta$ so that $p > \beta'(d+1)$, we obtain

$$\|D^2u\|_{L_p(C_{R_1})} \leq N \nu^{(d+2)/\gamma} \|F[u]\|_{L_p(C_{R_2})} + N\tau_0 \nu^{(d+2)/\gamma} |R_1|^{(d+2)/p}$$

$$+ N \nu^{\chi_1} (R_2 - R_1)^{-\chi_1} R_1^{4\gamma} \|D^2u\|_{L_p(C_{R_2})}^{1/\gamma} \|D^2u\|_{L_1(C_{R_2})}^{1/\gamma}, \quad (3.8)$$

where and below the constants $N, N_1$ depend only on $d, p, K_F, \tau_0$.

Now we take and fix $\nu \geq 2$ so that

$$N_0 \nu^{-\alpha} \leq 1/4.$$

Then (3.8) becomes

$$\|D^2u\|_{L_p(C_{R_1})} \leq N_1 \|F[u]\|_{L_p(C_{R_2})} + N\tau_0 |R_1|^{(d+2)/p}$$

$$+ N \nu^{\chi_1} (R_2 - R_1)^{-\chi_1} R_1^{4\gamma} \|D^2u\|_{L_p(C_{R_2})}^{1/\gamma} \|D^2u\|_{L_1(C_{R_2})}^{1/\gamma}, \quad (3.9)$$

Next, we use the fact that

$$|F[u]| \leq |H[u]| + K_0 |u| + K_0 |Du| + \tilde{G} + \tilde{\theta} |D^2u|$$
and that by interpolation inequalities

\[ K_0 N_1 \|Du\|_{L^p(B_{R_2})} \leq (1/8) \|D^2u\|_{L^p(B_{R_2})} + N \|u\|_{L^p(B_{R_2})}. \]

Then we take \( \hat{\theta} \) and \( \mu \) so small that

\[ N_1 \hat{\theta} \leq 1/8, \quad N_2 \mu \leq 1/8, \]

and, finally, take the largest \( \varepsilon \leq 1 \) such that

\[ N_2 \omega_{F,u,C_{2R}}(\varepsilon R_0) \leq 1/8. \]

This \( \varepsilon \) will appear later in our arguments and this is the way how the constant \( N \) in the statement of the lemma depends on \( \omega_{F,u,C_{2R}} \).

Then we require that Assumptions 1.2 and 1.3 be satisfied with the above chosen \( \hat{\theta} \) and \( \theta = \hat{\theta}(d, \delta, K_F, \mu, \beta d + \beta) \) (see Lemma 3.1), respectively. By combining the above we get

\[
\|D^2u\|_{L^p(C_{R_1})} \leq N \|\partial_t u + H[u]\|_{L^p(C_{R_2})} + N \tau_0 R^{d/p} + (5/8) \|D^2u\|_{L^p(C_{R_2})} + N \|u\|_{L^p(C_{2R})} + N \|\overline{G}\|_{L^p(C_{2R})},
\]

(3.10)

Now we are going to iterate this estimate by defining \( R_1 = R \) and for \( k \geq 1 \)

\[ R_{k+1} = R_k + cR(n_0 + k)^{-2}, \]

where the constant \( c = O(n_0) \) is chosen so that \( R_k \uparrow 2R \) as \( k \to \infty \), that is

\[ c \sum_{k=1}^{\infty} (n_0 + k)^{-2} = 1. \]

and \( n_0 \) is chosen so that for \( k \geq 1 \)

\[ R_{k+1} - R_k = cR(n_0 + k)^{-2} \leq Rcn_0^{-2} \leq R \leq R_k, \]

which is satisfied if \( n_0 \) is just an absolute constant, and (this time we need \( n_0^{-1} = o(\varepsilon R_0) \) as \( \varepsilon R_0 \to 0 \))

\[ R_{k+1} - R_k = cR(n_0 + k)^{-2} \leq cRn_0^{-2} \leq \varepsilon R_0. \]

Also observe that \( R \leq R_k \leq 2R \) and

\[ (R_{k+1} - R_k)^{-\chi_1} R_k^{-\chi_2 + \chi_1} \leq N(n_0 + k)^{2\chi_1} R^{-\chi_2}. \]

Then for \( k \geq 1 \) we get

\[
\|D^2u\|_{L^p(C_{R_k})} \leq N \|\partial_t u + H[u]\|_{L^p(C_{R_{k+1}})} + N \tau_0 R^{d/p} + (5/8) \|D^2u\|_{L^p(C_{R_{k+1}})} + N \|u\|_{L^p(C_{2R})} + N \|\overline{G}\|_{L^p(C_{2R})}.
\]

We multiply both parts of this inequality by \((5/8)^k\) and sum up the results over \( k = 1, 2, \ldots \). Then we cancel like terms

\[
\sum_{k=2}^{\infty} (5/8)^k \|D^2u\|_{L^p(C_{R_k})},
\]
which are finite since \( u \in W^{1,2}_p(B_{2R}) \), and finally take into account that
\[
\sum_{k=2}^{\infty} (5/8)^k (n_0 + k) 2^{\chi_1} \leq N n_0^{2 \chi_1} \sum_{k=2}^{\infty} (5/8)^k + N \sum_{k=2}^{\infty} (5/8)^k k 2^{\chi_1} \leq N.
\]
Then we come to (3.4).

Next, by using equation (3.1) and performing scaling in Theorem 2.3 (here we need \( R \leq 1 \)), using Hölder’s inequality (to go from \( d+1 \) to \( p \)), and denoting
\[
I = \| \bar{G} \|_{L_p(C_{2R})} + \| \partial_t u + H[u] \|_{L_p(C_{2R})}
\]
we infer that in (3.4)
\[
\| D^2 u \|^1_{L_1(C_{2R})} \leq NR^{\chi_2} \left( \| \bar{G} + K_0 \|_{L_p(C_{2R})} + \| \partial_t u + H[u] \|_{L_p(C_{2R})} \right)
+ NR^{\chi_3} \sup_{C_{2R}} |u| \leq NR^{\chi_2} I + NR^{\chi_3} \sup_{C_{2R}} |u|,
\]
where \( \chi_3 = (d+2)/\gamma - 2 \). After that it suffices to roughly estimate \( \| u \|_{L_p(C_{2R})} \) in (3.4) by the last term above. The lemma is proved. \( \square \)

4. Boundary a priori estimates in the simplest case

Introduce
\[
\mathbb{R}^{d+1}_+ = \{(t, x) : t \in \mathbb{R}, x = (x^1, ..., x^d) \in \mathbb{R}^d, x^1 > 0 \},
\]
\[
B^+_r(x_0) = B_r(x_0) \cap \{x^1 > 0\}, \quad C^+_r(t_0, x_0) = [t_0, t_0 + \tau] \times B^+_r(x_0),
\]
\[
\partial_{x^1} C^+_r(t_0, x_0) = C^+_r(t_0, x_0) \cap \{x^1 = 0\},
\]
where \( \tau, r \geq 0 \), \( (t_0, x_0) \in \mathbb{R}^{d+1} \). If \( t_0 = 0, x_0 = 0 \), we drop \( (t_0, x_0) \) in the arguments above. Also, if \( \tau = r^2 \) we write \( r \) in place of \( \tau, r \) in the subscripts, for instance,
\[
C^+_r(t_0, x_0) := C^+_{r^2, r}(t_0, x_0).
\]

Take \( \gamma \) from Section 3 and \( \alpha \in (0, 1) \) to be determined later. Let \( F \) be the function from Section 2.

**Lemma 4.1.** If \( r > 0, z_0 \in \mathbb{R}^{d+1} \), \( \nu \geq 12 \),
\[
u \in \bigcap_{\rho < \nu} W^{1,2}_{d+1}(C^+_\rho(z_0)) \cap C(C^+_\nu(z_0)),
\]
and \( u \) vanishes on \( \partial_x C^+_\nu(z_0) \) if this set is nonempty, then we have
\[
\left( \int_{C^+_\nu(z_0)} \int_{C^+_\nu(z_0)} |D^2 u(z_1) - D^2 u(z_2)|^\gamma \ dz_1 dz_2 \right)^{1/\gamma} \leq N \nu^{(d+2)/\gamma} \left( \int_{C^+_\nu(z_0)} \| \partial_t u + F[u] \|^{d+1} \ dz \right)^{1/(d+1)}
+ N \nu^{-\alpha} \left( \int_{C^+_\nu(z_0)} |D^2 u|^d dz \right)^{1/(d+1)},
\]
where \( N \) depends only on \( d \) and \( \delta \).
Proof. Scalings show that it suffices to prove the lemma only for $\nu r = 3$. Furthermore, without loss of generality we may assume that $z_0 = (0, x_0)$ and $x_0 = (|z_0|, 0, \ldots, 0) \in \mathbb{R}^d$. Then we consider two cases.

Case 1: $|z_0| > 1/2$. In this case, we have

$$B^+_r(x_0) = B^+_3(\nu r)(x_0) = B^+_r(x_0) \subset B^+_{\nu r}(x_0) \subset \mathbb{R}^d, \quad C^+_{\nu r}(z_0) \subset \mathbb{R}^{d+1},$$

where $\nu' = \nu/6 \geq 2$. Therefore, inequality (4.1) is an immediate consequence of Lemma 2.4.

Case 2: $|z_0| \in [0, 1/2]$. Since $r = 3/\nu \leq 1/2$, we have

$$B^+_r(x_0) \subset B^+_1 \subset B^+_2 \subset B^+_3(x_0) = B^+_{\nu r}(x_0).$$

Let $v$ be the classical solution of $\partial_t v + F[v] = 0$ in $C^+_2$ with boundary condition $v = u$ on $\partial^c C^+_2$. Such a solution exists due to the results in [14], which also provide an estimate on $D^2 v$, so that (for $\alpha \in (0, a_0(\delta, \delta))$)

$$I := \int_{C^+_2(z_0)} \int_{C^+_2(z_0)} |D^2 v(z_1) - D^2 v(z_2)| \, dz_1 \, dz_2 \leq N r^\alpha |D^2 v|_{C^0(C^+_1)}$$

$$\leq N r^\alpha \sup_{\partial^c C^+_2} |v| = N r^\alpha \sup_{\partial^c C^+_2} |u|$$

where the last equality is a consequence of the maximum principle and the fact that $F(0) = 0$. By employing Poincaré’s inequality ($u = 0$ on $\partial x_1 C^+_2$), we see that

$$I \leq N r^\alpha \left( \int_{B^+_2} (|\partial_t u||^{d+1} + |D^2 u||^{d+1}) \, dz \right)^{1/(d+1)}.$$ 

Here $r^\alpha = N \nu^{-\alpha}$ and

$$|\partial_t u| \leq |\partial_t u + F[u]| + |F[u]| \leq |\partial_t u + F[u]| + N |D^2 u|.$$ 

Therefore,

$$I \leq N \nu^{-\alpha} \left( \int_{B^+_2} |D^2 u||^{d+1} \, dz \right)^{1/(d+1)} + N \left( \int_{C^+_2} |\partial_t u + F[u]|^{d+1} \, dz \right)^{1/(d+1)}$$

Next, recall that $\gamma \in (0, 1]$. By Hölder’s inequality, we get

$$I \leq N \nu^{-\alpha} \left( \int_{C^+_2(z_0)} \int_{C^+_2(z_0)} |D^2 v(z_1) - D^2 v(z_2)|^\gamma \, dz_1 \, dz_2 \right)^{\gamma/(d+1)}$$

$$\leq N \nu^{-\gamma\alpha} \left( \int_{C^+_{\nu r}(z_0)} |D^2 u|^\gamma \, dz \right)^{(\gamma/(d+1))}$$

$$+ N \left( \int_{C^+_{\nu r}(z_0)} |\partial_t u + F[u]|^{d+1} \, dz \right)^{(\gamma/(d+1))}. \tag{4.2}$$

Next, use again that

$$f := \partial_t u + F(D^2 u) = \partial(u - v) + F(D^2 u) - F(D^2 v) = \partial w + a^{ij} D^2_{ij} w$$

in $C^+_2$ and $w = 0$ on $\partial^c C^+_2$, where $(a^{ij})$ is an $\mathbb{S}_d$-valued function and $w = u - v$. We extend $f$ and $w$ to all of $C_2$ as odd functions of $x^1$ and adjust $a^{ij}$...
appropriately so as to have equation \( f = \partial w + a_{ij}D^2_{ij}w \) in \( C_2 \), to which we apply Theorem 2.3 and get (recall that \( \nu r = 3 \))

\[
- \int_{C^+_{\nu r}(z_0)} |D^2 w|^\gamma dz \leq N r^{-d-2} \int_{C^+_{\nu r}(z_0)} |D^2 w|^\gamma dz \leq N \nu^{d+2} \left( \int_{C^+_{\nu r}(z_0)} |f|^{d+1} dz \right)^{\gamma/(d+1)}
\]

and

\[
- \int_{C^+_{\nu r}(z_0)} |D^2 w(z_1) - D^2 w(z_2)|^\gamma dz_1 dz_2 \leq N \nu^{d+2} \left( \int_{C^+_{\nu r}(z_0)} |f|^{d+1} dz \right)^{\gamma/(d+1)}
\]

Combining this with (4.2) and observing that \( D^2 u = D^2 v + D^2 w \) yield (4.1) in Case 2 as well. The lemma is proved.

Coming back to our domain \( \Omega \) recall that we say that \( \Omega \) is a \( C^{1,1} \)-domain if there exists \( \rho_0 = \rho_0(\Omega) \in (0,1] \) for which at any point \( x_0 \in \partial \Omega \) there is an orthonormal system of coordinates \( \Psi(x_0) \) with the origin at \( x_0 \) such that in the new coordinates \( \tilde{x} = (\tilde{x}_1, \tilde{x}') \) there exists a function

\[
\psi \in C^{1,1}(\{ \tilde{x}' \in \mathbb{R}^{d-1} : |\tilde{x}'| \leq 8 \rho_0 \})
\]

with the \( C^{1,1}(B_{8 \rho_0}) \)-norm majorated by a constant \( M_2(\Omega) \) independent of \( x_0 \) and such that

\[
\psi(0) = 0, \quad \psi_{\tilde{x}_i}(0) = 0, \quad i = 2, \ldots, d, \quad |D_{\tilde{x}'} \psi(\tilde{x}')| \leq 1 \quad \text{for} \quad |\tilde{x}'| \leq 8 \rho_0,
\]

\[
\{ \tilde{x} : |\tilde{x}'| \leq 8 \rho_0, \psi(\tilde{x}') + 8 \rho_0 \leq \tilde{x}_1 \leq \psi(\tilde{x}') + 8 \rho_0 \} \cap \Omega
\]

\[
= \{ \tilde{x} : |\tilde{x}'| \leq 8 \rho_0, \psi(\tilde{x}') < \tilde{x}_1 \leq \psi(\tilde{x}') + 8 \rho_0 \}.
\]

Below in this section we assume that

\[
0 \in \partial \Omega
\]

and that the original system of coordinates in \( \mathbb{R}^d \) coincides with the one described above for \( x_0 = 0 \).

**Lemma 4.2.** Introduce

\[
\Gamma := \{ x : |x'| \leq 8 \rho_0(\Omega), \psi(x') < x^1 \leq \psi(x') + 8 \rho_0(\Omega) \} \quad (\subset \Omega),
\]

\[
\hat{\Gamma} := \{ y : |y'| \leq 8 \rho_0(\Omega), 0 < y^1 \leq 8 \rho_0(\Omega) \}.
\]

Also introduce a mapping \( x \to y(x) \) of \( \Gamma \) onto \( \hat{\Gamma} \) by

\[
x^1 \to y^1 = y^1(x) = x^1 - \psi(x'), \quad x' \to y' = y'(x) = x'.
\]  \hspace{1cm} (4.3)

Then this mapping has an inverse \( y \to x(y) \). Furthermore, the Jacobians of both mappings are equal to one.
Then, as is easy to see, the inverse of $y$ smooth and has the magnitude of the gradient bounded by one and define $y(x)$ by the same formula (4.3) for all $x \in \mathbb{R}^d$. Of course, by $x(y)$ we mean the inverse of $y(x)$. Obviously, the assertions of Lemma 4.2 hold true for such extensions.

**Remark 4.3.** For $r \in (0, \infty)$ and $z \in \mathbb{R}^d$ define

$$B^+_r = x(B^+_r), \quad \hat{B}^+_r(z) = x(B^+_r(y(z))).$$

Then, as is easy to see

(i) $\hat{B}^+_r \subset \Gamma \subset \Omega$ if $r \leq 8\rho_0(\Omega)$;

(ii) $\hat{B}^+_r(z) \subset \hat{B}^+_{4\rho_0(\Omega)}$ if $\rho > 0$, $\rho + r \leq 8\rho_0(\Omega)$, and $z \in \hat{B}^+_\rho$.

**Lemma 4.4.** Take $z \in B^+_{2\rho_0(\Omega)}$. Then

(i) for $r \leq 2\rho_0(\Omega)$ we have

$$\hat{B}^+_r(z) \subset B_{2r}(z) \cap \Omega, \quad B_{r/2}(z) \cap \Omega \subset \hat{B}^+_r(z); \quad (4.4)$$

(ii) if $\nu \geq 1$ and $\nu r \leq 2\rho_0(\Omega)$, we have

$$|\hat{B}^+_r(z)| \leq N(d)\nu^d|\hat{B}^+_r(z)|, \quad (4.5)$$

Proof. (i). First notice that $\hat{B}^+_r(z) \subset \Gamma$. Then, since $|D_x^\psi| \leq 1$, for any $x_1, x_2 \in \Gamma$ we have $|y(x_1) - y(x_2)| \leq 2|x_1 - x_2|$ and $|x(y_1) - x(y_2)| \leq 2|y_1 - y_2|$ if $y_1, y_2 \in \hat{\Gamma}$. In particular, if $|y - y(z)| \leq r$, then $|x(y) - z| \leq 2r$, so that $\hat{B}^+_r(z) \subset B_{2r}(z)$ and

$$\hat{B}^+_r(z) \subset B_{2r}(z) \cap \Gamma \subset B_{2r}(z) \cap \Omega.$$  

which proves the first inclusion in (4.4).

Furthermore, if $|x - z| \leq r/2 \leq \rho_0(\Omega)$ and $x \in \Omega$, then, since $z \in B_{4\rho_0(\Omega)}$, $x \in B_{5\rho_0(\Omega)} \cap \Omega \subset \Gamma$.

Then $|y(x) - y(z)| \leq r$ and $y(x) \in \hat{\Gamma}$, that is, $y(x) \in B^+_r(y(z))$ so that $x \in x(B^+_r(y(z)))$, which yields the second inclusion in (4.4).

To prove (ii), it suffices to note that

$$|\hat{B}^+_r(z)| = |B^+_r(y(z)| \leq N\nu^d|B^+_r(y(z)| = N\nu^d|\hat{B}^+_r(z)|.$$  

The lemma is proved. \hfill $\square$

**Corollary 4.5.** If $z \in B^+_{2\rho_0(\Omega)}$ and $r \leq (1/2)\rho_0(\Omega)$, then for any measurable function $g$

$$\int_{\hat{B}^+_r(z)} |g(x)| dx \leq N(d) \int_{B_{2r}(z) \cap \Omega} |g(x)| dx. \quad (4.6)$$

Indeed, the domain of integration on the right is wider than the one the left owing to (4.4), and

$$N(d)|\hat{B}^+_r(z)| \geq |\hat{B}^+_r(z)| \geq |B_{2r}(z) \cap \Omega|$$

in light of (4.5) and (4.4).
Next, set
\[
\tilde{C}_R^+ = [0,R^2] \times \tilde{B}_R^+ 
\]
and for \( \rho + r \leq 4\rho_0(\Omega) \) and \( z = (t,x) \) such that \( x \in \tilde{B}_\rho^+ \) and \( t \in \mathbb{R} \) define
\[
\tilde{C}_r^+(z) = [t,t+r^2] \times \tilde{B}_r^+(x).
\]

**Lemma 4.6.** There exist \( \bar{\gamma} = \bar{\gamma}(d,\delta) \in (0,1) \) and \( \alpha_0 = \alpha_0(\delta,d) \in (0,1) \) such that for any \( \gamma \in (0,\bar{\gamma}] \) and \( \alpha \in (0,\alpha_0) \), whenever

(i) \( r,\rho > 0, \nu \geq 12, \rho + \nu r \leq 4\rho_0(\Omega), z_0 \in \tilde{C}_r^+, \)

(ii) \( u \in W^{1,2}_\rho(\tilde{C}^+_{r+\nu r}) \) and \( u(t,x) = 0 \) if \( x \in \partial \Omega \), we have

\[
I_r(z_0) := \left( \int_{\tilde{C}_r^+(z_0)} \int_{\tilde{C}^+_{r}(z_0)} |D^2u(z_1) - D^2u(z_2)|^\gamma \, dz_1 \, dz_2 \right)^{1/\gamma} \leq N\nu^{d+2}/\gamma \left( \int_{\tilde{C}^+_{r}(z_0)} |\partial_\nu u + F(D^2u)|^{d+1} \, dz \right)^{1/(d+1)} + N(\nu^{1+(d+2)/r + \nu^{-\alpha}} \left( \int_{\tilde{C}^+_{r}(z_0)} |D^2u|^{d+1} \, dz \right)^{1/(d+1)},
\]

where the constants \( N \) depend only on \( d, \alpha, M_2(\Omega), \) and \( \delta \).

Proof. By the change of variables formula we see that \( I_r(z_0) \) equals
\[
\left( \int_{\tilde{C}^+_{r}(t_0,y(x_0))} \int_{\tilde{C}^+_{r}(t_0,y(x_0))} |(D^2u)(x(z_1)) - (D^2u)(x(z_2))|^\gamma \, dz_1 \, dz_2 \right)^{1/\gamma}.
\]

Then with \( A(y) := \partial x(y)/\partial y \) we define
\[
A = A(y(z_0)), \quad \tilde{F}(u'') = F((A^{-1})^*u''A^{-1}).
\]

As is easy to see, \( D_{u''} \tilde{F} \in \mathbb{S}_\delta \), where \( \delta = \delta(d,\delta) \in (0,1] \).

Next, introduce the function
\[
\hat{u}(t,y) = u(t,x(y)),
\]
which belongs to \( W^{1,2}_\rho(C^+_{r+\nu r}) \), and, since \( |y(x_0)| < \rho \), it also belongs to \( W^{1,2}_\rho(C^+_{r\nu}(t_0,y(x_0))) \) and vanishes on \( \partial_z \tilde{C}^+_{r\nu}(t_0,y(x_0)) \) if this set is nonempty. By Lemma 4.1, since \( \nu \geq 12 \), we have
\[
\left( \int_{\tilde{C}^+_{r}(t_0,y(x_0))} \int_{\tilde{C}^+_{r}(t_0,y(x_0))} |D^2\hat{u}(z_1) - D^2\hat{u}(z_2)|^\gamma \, dz_1 \, dz_2 \right)^{1/\gamma} \leq N\nu^{d+2}/\gamma \left( \int_{\tilde{C}^+_{r}(t_0,y(x_0))} |\partial_\nu \hat{u} + \tilde{F}(D^2\hat{u})|^{d+1} \, dz \right)^{1/(d+1)} + N\nu^{-\alpha} \left( \int_{\tilde{C}^+_{r}(t_0,y(x_0))} |D^2\hat{u}|^{d+1} \, dz \right)^{1/(d+1)}.
\]

Observe also that for \( y = y(x) \) and \( x = x(y) \)
\[
D\hat{u}(t,y) = (Du)(t,x)A(y),
\]
where the $D$'s are row vectors, and
\[
D^2 \bar{u}(t, y) = A^*(y)[D^2 u(t, x)]A(y) + [D_k u(t, x)]D^2 x^k(y).
\] (4.9)

Since
\[
|A - A(y)| \leq N|y - y(x_0)|,
\]
where $N$ depends only on $d$ and the bound on $|D^2 \psi|$, for $z_i = (t_1, y_i) \in C^+(t_0, y(x_0))$, $i = 1, 2$, we have
\[
|D^2 \bar{u}(z_1) - D^2 \bar{u}(z_2)| \geq (1/N)|D^2 u(t_1, x_1) - D^2 u(t_2, x_2)|
\]
\[-N r(|D^2 u(t_1, x_1)| + |D^2 u(t_2, x_2)|) - N(|Du(t_1, x_1)| + |Du(t_2, x_2)|),
\]
where $x_i = x(y_i)$ and $N$ depends only on $M_2(\Omega)$ and $d$. Hence, the left-hand side of (4.8) is greater than or equal to
\[
(1/N)I_r(z_0) - N\left(\int_{C^+(z_0)} (r|D^2 u| + |Du|)^\gamma dz\right)^{1/\gamma}
\]
\[\geq (1/N)I_r(z_0) - N r\left(\int_{C^+(z_0)} |D^2 u|^{d+1} dz\right)^{1/(d+1)}
\]
\[-N\left(\int_{C^+(z_0)} |Du|^{d+1} dz\right)^{1/(d+1)} \geq (1/N)I_r(x_0)
\]
\[-N \nu r\left(\int_{C^+(z_0)} |D^2 u|^{d+1} dz\right)^{1/(d+1)} - N \nu \nu r\left(\int_{C^+(z_0)} |Du|^{d+1} dz\right)^{1/(d+1)}
\] (4.10)

where the first inequality follows by Hölder’s inequality and the second one is true owing to (4.5).

In what concerns the first term on the right-hand side of (4.8), observe that, owing to the Lipschitz continuity of $F$, the fact that $|A(y)A^{-1} - (\delta^{ij})| \leq N|y - y(x_0)|$, and (4.9), we have (with $x = x(y)$)
\[
|F(D^2 \bar{u}(y)) - F(D^2 \bar{u}(x))| \leq |F((A^{-1})^* A^*(y)[D^2 u(x)]A(y)A^{-1})
\]
\[-F(D^2 \bar{u}(x))| + N|Du(x)| \leq N|y - y(x)||D^2 u(x)| + N|Du(x)|.
\]

This and an easy estimate of the last term in (4.8) shows that its right-hand side is less than
\[
N \nu^{(d+2)/\gamma} \left(\int_{C^+(z_0)} |\partial_t u + F(D^2 u)|^{d+1} dx\right)^{1/(d+1)}
\]+\[N \nu^{1+(d+2)/\gamma} + \nu^{\alpha} \left(\int_{C^+(z_0)} |D^2 u|^{d} dx\right)^{1/d}
\]+\[N \nu^{(d+2)/\gamma} + \nu^{\alpha} \left(\int_{C^+(z_0)} |Du|^{d} dz\right)^{1/d}.
\]

Upon combining this result with what was said about (4.10) we come to (4.7). The lemma is proved.

Change of variables help derive Lemma 4.6 from its “flat” counterpart. We also allude to it in the following remark.
Remark 4.7. Suppose that Assumptions 1.1, and 1.8 and condition (1.4) are satisfied. Let \( r \leq 4\rho_0(\Omega) \), \( p \geq d + 1 \), and \( u \in W^{1,2}_p(\mathcal{C}^+_r) \) be such that \( u(t,x) = 0 \) if \( x \in \partial \Omega \). Then

\[
\int_{\mathcal{C}^+_r} (|D^2 u|^{\gamma} + |Du|^{\gamma}) \, dz \leq N r^{d+2-\gamma(d+2)/p} \|\partial_t u + H[u]\|_{L^p(\mathcal{C}^+_r)}^{\gamma} + N r^{d+2-\gamma} \sup_{\partial' \mathcal{C}^+_r} |u|^{\gamma},
\]

where \( N \) depend only on \( \delta, K_0, d, p \), and \( M_2(\Omega) \) and the range of \( \gamma \) is specified below.

Indeed, by using the notation from the above proof and using equation (3.1) in Lemma 3.2 introduce the operators

\[
\mathcal{L} u(t,x) = a^{ij}(t,x)D_{ij} u(t,x) + b^i(t,x)D_i u(t,x), \quad \mathcal{L} \tilde{u}(t,y) = [\mathcal{L} u](t,y).
\]

The operator \( \mathcal{L} \) can be written as a differential operator with respect to \( y \). Clearly, its matrix of second-order derivatives will belong to \( S_\delta \) for a \( \tilde{\delta} = \delta(\delta, M_2(\Omega)) \in (0,1) \) and the drift term by magnitude will be dominated by \( N = N(K_0, d, M_2(\Omega)) \). Since

\[
|\partial_t \tilde{u}(t,y) + \mathcal{L} \tilde{u}(t,y)| \leq |\partial_t u(t,x(y)) + H[u](t,x(y))| + \tilde{G}(t,x(y)) + K_0 |u(t,x(y))|
\]

in \( C^+_0(\Omega) \), by Theorem 2.3 for an appropriate \( \tilde{\gamma} = \tilde{\gamma}(d, \delta, K_0, M_2(\Omega)) \in (0,1) \) and \( \gamma \in (0, \tilde{\gamma}] \), after using scalings and Hölder’s inequality (to replace \( d + 1 \) with \( p \)) we get,

\[
\int_{\mathcal{C}^+_r} (|D_y^2 \tilde{u}|^{\gamma} + |D_y \tilde{u}|^{\gamma}) \, dydt \leq N r^{(d+2)(1-\gamma/p)} \left( \int_{\mathcal{C}^+_r} |\partial_t u + H[u]|^p(t,x(y)) \, dydt \right)^{\gamma/p} + N r^{d+2-\gamma} \sup_{\partial' \mathcal{C}^+_r} |u|^{\gamma}.
\]

Now our assertion follows after changing variables.

5. A priori estimates in \( W^{1,2}_p \) near the boundary and the proof of Theorem 1.7

We assume that

\[
p > d + 1, \quad 0 \in \partial \Omega
\]

and take \( \rho_0 = \rho_0(\Omega), \mathcal{C}^+_r, \mathcal{C}^+_r(z) \) from Section 4 and suppose that the assumptions of Theorem 1.7 are satisfied with \( \bar{\theta} \) and \( \theta \) which are yet to be specified.

First we note the following.

**Lemma 5.1.** For any \( q \in [1, \infty) \) and \( \mu > 0 \) there exists \( \theta = \theta(d, \delta, K_F, \mu, q) > 0 \) such that, if Assumption 1.3 is satisfied with this \( \theta \), then for any \( u_0' \in \mathbb{R} \),
\[ z_0 \in \tilde{C}^+_{2\rho_0}(\Omega) \text{ and } 2r \leq \rho_0(\Omega) \land R_0, \text{ we have} \]
\[ \int_{\tilde{C}^+_r(z_0)} \sup_{u'' \in \mathbb{S}, \eta > \tau_0} \frac{|F(u_0', u'', z) - F(u'')|}{|u''|^q} \, dz \leq \mu \]
where \( F = \tilde{F}_{z,r,u_0'} \) is taken from Assumption 1.3.

For the proof of this lemma note that, in light of Corollary 4.5 and Assumption 1.3, for any \( u'' \in \mathbb{S} \) with \( |u''| = 1 \) we have
\[ \int_{\tilde{C}^+_r(z_0)} \sup_{\tau > \tau_0} \tau^{-1}|F(u_0', \tau u'', z) - F(\tau u'')| \, dz \leq N(d)\theta \]
if \( 2r \leq \rho_0(\Omega) \land R_0 \). After that, as in the case of Lemma 3.1, the assertion of the current lemma is obtained by repeating the proof of Lemma 5.1 of [12].

Recall that \( \omega_u(\Pi, \rho) \) is introduced in Definition 1.6.

**Lemma 5.2.** Let \( r, \rho \in (0, \infty) \) and \( \nu \geq 12 \) satisfy \( \rho + \nu r \leq 4\rho_0(\Omega) \) and \( \nu r \leq R_0 \). Take \( \mu \in (0, \infty), \quad \beta \in (1, \infty), \)
and suppose that Assumption 1.3 is satisfied with \( \theta = \theta(d, \delta, K_F, \mu, \beta d + \beta) \) (see Lemma 5.1). Assume that we are given a function \( u \in W^{1,2}_p(\tilde{C}^+_{\rho+\nu r}) \)
and \( u(t, x) = 0 \) if \( x \in \partial \Omega \). Use \( I_r(z_0) \) introduced in (4.7).

Then, for \( \gamma \) and \( \alpha \) from Lemma 4.6, for \( z_0 \in \tilde{C}^+_r \), we have
\[ I_r(z_0) \leq N\eta \left( \int_{\tilde{C}^+_r(z_0)} |D^2u|^\beta d(z) \right)^{1/(\beta(d+1))} \]
\[ + N\nu^{(d+2)/\gamma} \left( \int_{\tilde{C}^+_r(z_0)} (|\partial u| + F[u])^{d+1} + |Du|^{d+1} \right)^{1/(d+1)} \]
where \( \eta = (\mu + \nu r + \omega_{F,u,\tilde{C}^+_{\rho+\nu r}}(\nu r))^\nu \gamma + \nu^{-\alpha} \),
and the constants \( N \) depend only on \( d, p, K_F, \delta, \) and \( M_2(\Omega) \).

The proof of this lemma is based on Lemma 5.1 and, in light of Lemma 4.6, is practically identical to that of Lemma 3.3.

We now come to the main a priori estimate near the boundary for nonlinear parabolic equations with VMO “coefficients”.

**Theorem 5.3.** Take \( p > d + 1 \), let \( R > 0 \) satisfy
\[ 2R \leq \rho_0(\Omega) \land R_0, \]
and let \( u \in W^{1,2}_p(\tilde{C}^+_{2R}) \) be such that \( u(t, x) = 0 \) if \( x \in \partial \Omega \). Then there exist constants \( \bar{\theta}, \theta \in (0, 1] \), depending only on \( d, p, \delta, \) and \( K_F \), such that if Assumptions 1.2 and 1.3 are satisfied with these \( \bar{\theta} \) and \( \theta \), respectively,
then there exist constants \( N \), depending only on \( R_0, d, p, K_0, K_F, \delta, \rho_0(\Omega), M_2(\Omega) \), and the function \( \omega_{F,u,\bar{C}_{2R}^+} \) (see Definition 1.6), such that

\[
\|D^2u\|_{L_p(\bar{C}_{2R}^+)} \leq N\|\partial_t u + H[u]\|_{L_p(\bar{C}_{2R}^+)} + N\|\bar{G}\|_{L_p(\bar{C}_{2R}^+)} + N\tau_0
\]

\[
+ N\|u\|_{L_p(\bar{C}_{2R}^+)} + NR^{-\chi}\|D^2 u[\gamma]\|^{1/\gamma}_{L_1(\bar{C}_{2R}^+)};
\]

(5.1)

\[
\|D^2u\|_{L_p(\bar{C}_{2R}^+)} \leq N\|\partial_t u + H[u]\|_{L_p(\bar{C}_{2R}^+)} + N\|\bar{G}\|_{L_p(\bar{C}_{2R}^+)}
\]

\[
+ N\tau_0 + NR^{(d+2)/p-2}\sup_{\bar{C}_{2R}^+}|u|,
\]

(5.2)

where \( \chi = (d+2)(1/\gamma - 1/p) \) and \( \gamma \) is the same as in Lemma 5.2.

Proof. Whenever it makes sense, for \( \rho \leq \rho_0(\Omega) \), and \( z \in \bar{C}_{2\rho_0(\Omega)}^+ \) introduce

\[
h^{\#}_{\Omega,\gamma,\rho}(z) = \sup\{I_r(h, z_0) : z_0 \in \mathbb{R}^+ \times \Omega, r \in (0, \rho), \bar{C}_r^+(z_0) \ni z\},
\]

where

\[
I_r(h, z_0) = \left( \int_{\bar{C}_r^+(z_0)} \int_{\bar{C}_r^+(z_0)} |h(z_1) - h(z_2)|^{\gamma} dz_1 dz_2 \right)^{1/\gamma}.
\]

The reader should pay attention to the above curved sharp symbol, reminding of curved boundaries.

Observe that, if \( r \leq \rho \leq \rho_0(\Omega) \) and \( z \in \bar{C}_{2\rho_0(\Omega)}^+ \cap \bar{C}_r^+(z_0) \), then \( \bar{C}_r^+(z_0) \subset \bar{C}_{4\rho_0(\Omega)}^+ \) so that \( h^{\#}_{\Omega,\gamma,\rho}(z) \) is well defined on \( \bar{C}_{2\rho_0(\Omega)}^+ \) even if \( h \) is given only on \( \bar{C}_{4\rho_0(\Omega)}^+ \) (\( \subset \Omega \)).

Then take \( \varepsilon \in (0,1) \) to be specified later, take \( R_1 < R_2 \leq 2R \) such that

\[
R_2 \leq 2R_1, \quad R_2 - R_1 \leq \varepsilon R_0,
\]

take \( \nu \geq 12, \) and set

\[
r_0 = (R_2 - R_1)/(\nu + 1), \quad \kappa = r_0/R_1 \quad (\leq (R_2 - R_1)/(2R_1) \leq 1/2).
\]

We are going to use Theorem 7.2 according to which, if \( h \in L_p(\bar{C}_{R_2}^+) \), then

\[
\|h\|_{L_p(\bar{C}_{R_2}^+)} \leq N\|h^{\#}_{\Omega,\gamma,r_0}\|_{L_p(\bar{C}_{R_1}^+)} + N\nu^{\chi_1}(R_2 - R_1)^{-\chi_1}R_1^{\chi_1 - \chi_2}\|h\|^\gamma L_1(\bar{C}_{R_1}^+),
\]

(5.3)

where \( \chi_1 = (d+2)/\gamma, \chi_2 = (d+2)(1/\gamma - 1/p) \), and the constants \( N \) depend only on \( d, \gamma, \) and \( p \).

Next for \( z \in \bar{C}_{2R}^+ (\subset \bar{C}_{4\rho_0(\Omega)}^+) \) define

\[
M_Q h(z) = \sup \left\{ \int_{\bar{C}_{\kappa}^+(z_0)} |h(y)| dy : 2r \leq \rho_0(\Omega), z_0 \in \mathbb{R}^+ \times \Omega, \bar{C}_{\kappa}^+(z_0) \ni z \right\},
\]
Observe that, owing to the fact that \( \Omega \in C^{1,1} \) and to Corollary 4.5, if \( z_0 \in \bar{C}^+_{2\rho_0}(\Omega) \) and \( r \leq (1/2)\rho_0(\Omega) \),
\[
\int_{C^+_r(z_0)} |h| \, dy \leq N \int_{C^+_r(2\rho_0,\Omega)} |h| \, dy,
\]
where \( N \) depends only on \( d, \rho_0(\Omega), \) and \( M_2(\Omega) \). Therefore, for \( z \in \bar{C}^+_{2\rho_0}(\Omega) \),
\[
M_\Omega h(z) \leq N M h(z).
\]

The above conclusion (5.4) is, actually, also based on the fact similar to the following. For \( r \leq r_0, \) \( z \in \bar{C}^+_{R_1} \), and \( z_0 \in \mathbb{R}^+ \times \Omega \), such that \( \bar{C}^+_{r_0}(z_0) \ni z, \) we have \( z_0 \in \bar{C}^+_{R_1} \), where \( \rho = R_1 + r \). In this situation also \( \rho + \nu r \leq 2R_1 < 4\rho_0(\Omega) \) and \( \nu \rho \leq \varepsilon R_0 \) and it follows from Lemma 5.2 that
\[
I_r(z_0) \leq N^{(d+2)/\gamma} M^{1/(d+1)} (|\partial_t u + F[u]|^{d+1} I^{+}_{C^+_{R_2}})(z) + N\tau_0^{(d+2)/\gamma} \]
\[
+ N\eta M^{1/(\beta'(d+1))} (|D^2 u|^{\beta'(d+1)} I^{+}_{C^+_{R_2}})(z) + N\nu^{(d+2)/\gamma} M^{1/(d+1)} (|D u|^{d+1} I^{+}_{C^+_{R_2}})(z),
\]
where
\[
\eta = (\mu + \nu \tau_0 + \omega_{F,u,\bar{C}^+_{R_2}}(\varepsilon R_0))\nu^{(d+2)/\gamma} + \nu^{-\alpha},
\]
By definition and (5.4) we obtain that on \( \bar{C}^+_{R_1} \)
\[
(D^2 u)_{\Omega,\gamma} \leq N^{(d+2)/\gamma} M^{1/(d+1)} (|\partial_t u + F[u]|^{d+1} I^{+}_{C^+_{R_2}}) + N\tau_0^{(d+2)/\gamma} \]
\[
+ N\eta M^{1/(\beta'(d+1))} (|D^2 u|^{\beta'(d+1)} I^{+}_{C^+_{R_2}}) + N\nu^{(d+2)/\gamma} M^{1/(d+1)} (|D u|^{d+1} I^{+}_{C^+_{R_2}}).
\]

Thanks to (5.3) and the Hardy-Littlewood maximal function theorem, by taking \( \beta \) so that \( p > \beta'd \), we obtain
\[
\|D^2 u\|_{L^p(\bar{C}^+_{R_1})} \leq N\nu^{(d+2)/\gamma} \|\partial_t u + F[u]\|_{L^p(\bar{C}^+_{R_2})} + N\tau_0^{(d+2)/\gamma} \]
\[
+ \left[ N(\mu + \nu \tau_0 + \omega_{F,u,\bar{C}^+_{R_2}}(\varepsilon R_0))\nu^{(d+2)/\gamma} + N\nu^{-\alpha} \right] \|D^2 u\|_{L^p(\bar{C}^+_{R_2})} \]
\[
+ N\nu^{(d+2)/\gamma} \|D u\|_{L^p(\bar{C}^+_{R_2})} + N\nu^\chi (R_2 - R_1)^{-\chi_1} R_1^{\chi_1 - \chi_2} \|D^2 u\|^\gamma_{L^1(\bar{C}^{2R_2})},
\]
where the constants \( N, N_0 \) depend only on \( d, p, K_F, \) and \( \delta \).

This estimate looks almost like (3.8). Then we repeat the argument after (3.8) and choose and fix \( \varepsilon, \nu, \bar{\theta}, \) and \( \mu, \) recall what \( \tau_0 \) is, and conclude that
\[
\|D^2 u\|_{L^p(\bar{C}^+_{R_1})} \leq N\|\partial_t u + H[u]\|_{L^p(\bar{C}^+_{R_2})} + N\tau_0^{d/\gamma} \]
\[
+ (5/8 + N(R_2 - R_1))\|D^2 u\|_{L^p(\bar{C}^+_{R_2})} + N\|u\|_{L^p(\bar{C}^+_{2R_2})} + N\|\tilde{G}\|_{L^p(\bar{C}^+_{2R_2})} \]
\[
+ N(R_2 - R_1)^{-\chi_1} R_1^{\chi_1 - \chi_2} \|D^2 u\|^{\gamma}_{L^1(\bar{C}^{2R_2})}.
\]
After that, to prove (5.1), it suffices to repeat almost literally what follows (3.10) (only replacing $C$ with $\hat{C}$). By using Remark 4.7 we estimate the last term in (5.1) and then finish the proof of the theorem in the same way as in the case of Lemma 3.4. The theorem is proved. □

**Proof of Theorem 1.7.** To start, assume that $g \equiv 0$. Observe that in that case we may assume that $u(t, x)$ is defined for $t \geq T$, $x \in \Omega$, as zero and still satisfies there (1.1). It suffices for the latter that $H(0, t, x) = 0$ if $t \geq T$, which is easy to accommodate without altering our assumptions just by replacing $G(u, t, x)$ and $\bar{G}$ with $G(u, t, x)I_{t<T}$ and $\bar{G}I_{t<T}$, respectively. After such extension $u \in W^{1,2}_{p}(\mathbb{R} \times \Omega)$.

Take $\tilde{\theta}$ and $\theta$ which suit both Lemma 3.4 and Theorem 5.3, and take

$$R = (1/2)(\rho_0(\Omega) \wedge R_0).$$

Theorem 5.3 allows us to estimate the $W^{1,2}_{p}$-norm of $u$ in the domain $\hat{C}^+_R = \hat{C}^+_R$ associated with the origin, that is assumed to belong to $\partial \Omega$. Of course, one can take any point $z_0 = (t_0, x_0) \in [0, \infty) \times \partial \Omega$ as the origin and apply Theorem 5.3 to $\hat{C}^+_R(z_0)$ and $\hat{C}^+_R(0)$ in place of $\hat{C}^+_R$ and $\hat{C}^+_R$, respectively, where by $\hat{C}^+_R(z_0)$ we, naturally, mean the sets

$$[t_0, t_0 + \rho^2] \times \hat{B}^+_{\rho}(x_0),$$

where $\hat{B}^+_{\rho}(x_0)$ is constructed in Lemma 4.4 but with $x_0$ in place of 0 and relative to the coordinate system $\Psi(x_0)$ associated with $x_0$ as described before Lemma 4.2. According to that, we find finitely many

$$z_i \in [0, T + R^2] \times \partial \Omega$$

and $\rho > 0$ depending only on $\text{diam}(\Omega)$, $\rho_0(\Omega)$, $M_2(\Omega)$, and $T$ such that

$$\bigcup_i \hat{C}^+_R(z_i) \cup \left( [0, S - \rho^2] \times \Omega^\rho \right) \supset \Pi,$$

where $S = T + R^2$.

By Theorem 5.3, for any $i$ (recall that $\bar{G}(t, x) = 0$ for $t \geq T$)

$$\|D^2u\|_{L^p(\hat{C}^+_R(z_i))} \leq N\|\bar{G}\|_{L^p(\Pi)}^p + N\tau_0^p + N\sup_\Pi |u|.$$  

By Lemma 3.4

$$\|D^2u\|_{L^p(0, S - \rho^2) \times \Omega^\rho} \leq N\|\bar{G}\|_{L^p(\Pi)}^p + N\tau_0^p + N\sup_\Pi |u|.$$  

We sum up these estimates and come to (1.5). This proves the theorem if $g \equiv 0$.

In the general case introduce $\hat{g}(z) = (g(z), DG(z), D^2g(z))I_{\Pi}(z)$ and

$$\hat{H}(v, z) = H(v + \hat{g}(z), z) + \partial_t g(z)I_{\Pi}(z), \quad w(z) = u(z) - g(z).$$

Observe that $\hat{H}[w] = 0$ in $\Pi$ (a.e.) and $w \in W^{1,2}_{p}(\Pi)$ and $w = 0$ on $\partial'\Pi$. Furthermore, for

$$\hat{F}(v_0, v', z) = F(v_0 + \hat{g}_0, v', z), \quad \hat{G}(v, z) := \hat{H}(v, z) - \hat{F}(v_0, v', z)$$

and...
one easily obtains that $|\tilde{G}(v, z)| \leq \tilde{\theta}|v'| + K_0|v| + \tilde{G}$, where
\[
\tilde{G} = |\partial_t z| I_{11} + N|D^2 g| I_{12} + G + K_0 (g^2 + |Dg|^2)^{1/2} I_{12}
\]
with $N$ depending only on $K_F$ and $d$.

Also for $\nu_0 \in \mathbb{R}$, $r \in (0, R_0]$, and $z \in \Pi$ we set
\[
\tilde{F}(v''') = \tilde{F}_{z,r,\nu_0}(v''') = \tilde{F}_{z,r,\nu_0} + \tilde{g}_0(z)(v'''),
\]
take $\theta_0 = \theta_0(d, p, \delta, K_F, M_2(\Omega))$ defined above in the first part of the proof where $g \equiv 0$, find $\tilde{R}_0 \leq R_0$ such that $\tilde{\omega}_{F,0,1}(\tilde{R}_0) \leq \theta_0/2$ and then require the original Assumption 1.3 (iii) to be satisfied with $\tilde{R}_0$ and $\theta_0/2$ in place of $R_0$ and $\theta$, respectively.

Then we see that the above result is applicable to $w$, and along with the embedding inequality: $|g| \leq N\|g\|_{W_{1,2}^p(\Pi)}$, lead to (1.5) in the general case. The theorem is proved. \hfill \Box

6. PROOF OF THEOREM 1.9

The proof of Theorem 1.9 is based on the following.

**Theorem 6.1.** Suppose that Assumption 1.8 is satisfied, the number
\[
\tilde{H} := \sup_{u',t,x} (|H(u',0,t,x)| - K_0|u'|) \quad (\geq 0)
\]
is finite, and $g \in W_{1,2}^1(\mathbb{R}^{d+1})$.

Then there exists a convex positive homogeneous of degree one function $P(u''')$ such that at all points of its differentiability $D_{u'''}P \in S_{\bar{\delta}}$, where $\bar{\delta} = \tilde{\delta}(d, \delta) \in (0, \delta)$, and for $P[u] = P(D^2 u)$ and any $K > 0$ the equation
\[
\partial_t v + \max(H[v], P[v] - K) = 0
\]
in $\Pi$ with boundary condition $v = g$ on $\partial \Pi$ has a solution $v \in W_{p,1}^2(\Pi)$ for any $p \geq 1$.

This theorem follows from Theorem 2.1 of [11], proved there under the additional conditions that $\Omega \in C^2$ and that there is an increasing continuous function $\omega(r), r \geq 0$, such that $\omega(0) = 0$ and
\[
|H(u',u'',t,x) - H(v',v'',t,x)| \leq \omega(|u' - v'|)
\]
for all $u,v,t,$ and $x$. That these additional conditions can be dropped will be proved elsewhere.

**Step 1.** We take $P(u''')$ from Theorem 6.1, and first we assume that $g \in W_{1,2}^1(\Pi)$ and there exists constants $N_0, \tilde{H}$ such that, for all $t, x, u'$,
\[
|H(u',0,t,x)| \leq N_0|u'| + \tilde{H}. \quad (6.2)
\]

By Theorem 6.1 for any $K > 0$ there exists a function $v_K$ which is in $W_{p,1}^1(\Pi)$ for any $p > 1$, such that $v_K = g$ on $\partial \Pi$, and it satisfies
\[
\partial v_K + H_K[v_K] = 0 \quad \text{in} \quad \Pi \ (\text{a.e.}), \quad (6.3)
\]
well-known results from the linear theory allows us to estimate the modulus of continuity of $v_n$.

Theorem 1.7 there exist constants $\hat{L}$, which is in $G$, constants independent of $H$, convergence implies pointwise convergence. Let $Dv$ and $u$ of $\theta$ and $\hat{\theta}$, respectively, then for any $K > 0$, we have
\[
\|v_K\|_{W_p^1,2(\Pi)} \leq N\left(\|\bar{G}\|_{L_p(\Pi)} + \|g\|_{W_p^1,2(\Pi)} + \|v_K\|_{C(\Pi)}\right) + N\tau_0,
\]
where the constants $N$ depend only on $K_0$, $K_F$, $d$, $p$, $\delta$, $R_0$, $\text{diam}(\Omega)$, $\rho_0(\Omega)$, $M_2(\Omega)$, and the function $\omega_F,v_K,\Pi$ (independent of $N_0$ and $H$).

Since $H_K$ satisfies (1.4), formula (3.1) is valid with $v_K$ and $H_K$ in place of $u$ and $H$. This converts equation (6.3) into a linear equation and by the well-known results from the linear theory allows us to estimate $|v_K|$ and the modulus of continuity of $v_K$ through that of $g$, $\sup |g|$, and $\|\bar{G}\|_{L_{d+1}(\Pi)}$ with constants independent of $K$.

Thus,
\[
\|v_K\|_{W_p^1,2(\Pi)} \leq N\left(\|\bar{G}\|_{L_p(\Pi)} + \|g\|_{W_p^1,2(\Pi)}\right) + N\tau_0,
\]
where the constants $N$ are independent of $K$.

In this way we completed a crucial step consisting of obtaining a uniform control of the $W_p^{1,2}(\Pi)$-norms of $v_K$.

Next, we let $K \to \infty$. Estimate (6.4) guarantees that there is a sequence $K_n \to \infty$ as $n \to \infty$ and $v \in W_p^{1,2}(\Pi)$ such that $v_{K_n} \to v$ weakly in $W_p^{1,2}(\Pi)$ and $v_{K_n} \to v$ uniformly in $\Pi$. Then, of course, $v = g$ on $\partial \Pi$. The said weak convergence implies pointwise convergence $Dv_n \to Dv$ in $\Pi$ in light of the compactness of the embedding $W_p^{1,2} \subset C^{0,1}$ ($p > d + 2$).

Next, for $m = 1, 2, \ldots$, define
\[
H^m(u'', t, x) = \sup_{n \geq m} \max_n (H(v_{K_n}(t, x), Dv_{K_n}(t, x), u'', t, x), P(u'') - K_n).
\]
Observe that $H^m(u'', t, x)$ are Lipschitz continuous in $u''$ and at all points
differentiability satisfy $D_{u''}H^m \in S_\delta$. Also
\[
|H^m(0, t, x)| \leq K_0 \max_{n \geq m} \left(|v_{K_n}(t, x)| + |Dv_{K_n}(t, x)|\right) + G(t, x),
\]
which is in $L_{p,\text{loc}}(\Pi)$. Therefore, the operators $H^m[u]$ fit into the scheme of Section 3.5 of [9]. Furthermore, for $n \geq m$ obviously
\[
\partial_t v_{K_n} + H^m(v_{K_n}, t, x) \geq 0.
\]


(a.e.) in $\Pi$. By Theorem 3.5.9 of [9] we conclude that for any $m$

$$
\partial_t v + \sup_{n \geq m} \max(H(v_{K_n}, Dv_{K_n}, D^2 v, t, x), P(D^2 v) - K_n) \geq 0
$$

(6.5)

(a.e.) in $\Pi$. We fix $(t, x)$ at which (6.5) holds for all $m$ (that is, we fix almost
any $(t, x)$) and since $H(u', u'', t, x)$ is continuous in $u'$, we have that

$$
|H(v_{K_n}(t, x), Dv_{K_n}(t, x), D^2 v(t, x), t, x)
-H(v(t, x), Dv(t, x), D^2 v(t, x), t, x)| \to 0
$$
as $n \to \infty$. Then, in light of (6.5),

$$
\partial_t v(t, x) + \max(H(v(t, x), Dv(t, x), D^2 v(t, x), t, x), P(D^2 v(t, x)) - K_m) \geq o(1),
$$

which for $m \to \infty$ yields

$$
\partial_t v(t, x) + H(v(t, x), Dv(t, x), D^2 v(t, x), t, x) = \partial_t v(t, x) + H[v](t, x) \geq 0.
$$

The inequality $\partial_t v + H[v] \leq 0$ is proved similarly starting from the function

$$
\inf_{n \geq m} \max(H(v_{K_n}(t, x), Dv_{K_n}(t, x), u'', t, x), P(u'') - K_n).
$$

Owing to (6.4), of course, $v \in W^{1,2}_p(\Pi)$ and (6.4) holds with $v$ in place of $v_K$.

This proves the theorem if condition (6.2) is satisfied and $g \in W^{1,2}_\infty(\Pi)$.

Step 2. Assume that $g \in W^{1,2}_\infty(\mathbb{R}^{d+1})$ and abandon (6.2). Let $\eta(t) = t$ for

$|t| \leq 1$ and $\eta(t) = \text{sign } t$ for $|t| \geq 1$. For $n = 1, 2, \ldots$ define $\eta_n(t) = n\eta(t/n)$ and

$$
\hat{H}^n(u, t, x) = H(u, t, x) - H(u', 0, t, x) + \eta_n(H(u', 0, t, x)),
$$

$$
\hat{G}^n(u, t, x) = \hat{H}^n(u, t, x) - F(u', u'', t, x).
$$

Then

$$
|\hat{G}^n(u, t, x)| = |G(u, t, x) + \eta_n(H(u', 0, t, x)) - H(u', 0, t, x)|
$$

$$
\leq \theta |u''| + 2K_0 |u'| + 2G(t, x),
$$

so that Assumption 1.2 is satisfied for $\hat{H}^n$ with $2K_0$ and $2G$ in place of $K_0$ and $G$. Assumptions 1.3 and 1.8 are also valid for $\hat{H}^n$ with the same parameters.

Furthermore

$$
|\hat{H}^n(u', 0, t, x)| = |\eta_n(H(u', 0, t, x))|,
$$

which is bounded.

Hence there are $\theta$ and $\theta$ as in Step 1, for any $n$, there exists $u^n \in W^{1,2}_p(\Pi) \cap C(\Pi)$ satisfying

$$
\partial_t u^n + \hat{H}^n[u^n] = 0
$$
in $\Pi$ (a.e.) and such that $u = g$ on $\partial \Pi$. Estimate (1.5), applicable to $v^n$ by
the above again guarantees that the $W^{1,2}_p(\Pi)$-norms of $v^n$ are bounded and $v^n$ are equicontinuous in $\Pi$. This enables us to find a subsequence $v''^n$ and a function $v \in W^{1,2}_p(\Pi)$ such that $v''^n \to v$ weakly in $W^{1,2}_p(\Pi)$ and $v''^n \to v$ uniformly in $\Pi$. Then, of course, $v = g$ on $\partial \Pi$. 
After that we repeat the rest of Step 1 by taking
\[
\sup_{n' \geq m} \left[ H(v', Dv', u', t, x) - H(v', Dv', 0, t, x) + \eta_n(H(v', Dv', 0, t, x)) \right]
\]
in place of \(H^m(u', t, x)\). One thing which makes the argument here easier is that for any \((t, x) \in \Pi\)
\[
-H(v^n, Dv^n, 0, t, x) + \eta_n(H(v^n, Dv^n, 0, t, x)) = 0
\]
if \(n\) is large enough.

In this way we finish Step 2. Finally, to treat the general \(g \in W^{1,2}_p(\Pi)\) it suffices to use approximations and very simple arguments about passing to the limit, which we have seen already above. This step is left to the reader. The theorem is proved.

\[\square\]

7. Appendix

Fix \(\gamma \in (0, 1]\) and for \(r \in (0, \infty)\) and \(z \in \mathbb{R}^{d+1}\) define
\[
I_r(h, z) = \left( \int_{C_r(z)} \int_{C_r(z)} |h(z_1) - h(z_2)|^\gamma \, dz_1 \, dz_2 \right)^{1/\gamma} \tag{7.1}
\]
whenever the right-hand side makes sense.

For \(\rho > 0\) introduce the restricted sharp function of \(h\) by the formula
\[
h_{Q, \gamma, \rho}^\#(z) = \sup \{ I_r(h, z_0) : z_0 \in Q, r \in (0, \rho), C_r(z_0) \ni z \} \tag{7.2}
\]
whenever it makes sense. Note that, if \(Q = \mathbb{R}_+ \times \mathbb{R}^d\), \(h_{Q, \gamma, \rho}^\#\) is well defined in \(C_R\) for measurable \(h\) even defined only in \(C_{R+2\rho}\).

**Theorem 7.1.** Let \(p \in (1, \infty), \kappa \in (0, 1], R \in (0, \infty), \) and \(h \in L_p(C_{R(1+2\kappa)})\).
Let \(Q = \mathbb{R}_+ \times \mathbb{R}^d\). Then
\[
\|h\|_{L_p(C_R)} \leq N \|h_{Q, \gamma, \rho}^\#\|_{L_p(C_R)} + N \kappa^{\gamma_1} R^{-\chi_2} \|h\|_{L_1(C_R)}^{1/\gamma}, \tag{7.3}
\]
where \(\gamma_1 = (d+2)/\gamma, \chi_2 = (d+2)/(1/\gamma - 1/p)\) and the constants \(N\) depend only on \(d, \gamma, \) and \(p\).

This theorem will be proved elsewhere by closely following the proof of Theorem 7.1 of [12] given there in the elliptic framework.

The remaining results of this section treat smooth cylinders or smooth domains. If \(\Omega \in C^{1,1}\) and \(0 \in \partial \Omega\), we assume that the original system of coordinates in \(\mathbb{R}^d\) coincides with the one described before Lemma 4.2 and with the help of the mappings \(x(y)\) and \(y(x)\) introduced in that lemma, for \(r > 0\) and \(z \in \mathbb{R}^d\), we construct
\[
\tilde{B}_r^+ = x(B_r^+), \quad \tilde{B}_r^+(z) = x(B_r^+(y(z))).
\]
By Remark 4.3 we have \(\tilde{B}_r^+ \subset \Omega\) for \(r \leq 4\rho_0(\Omega)\) and \(\tilde{B}_r^+(z) \subset \tilde{B}^+_{4\rho_0(\Omega)}\) if \(\rho > 0, \rho + r \leq 4\rho_0(\Omega)\), and \(z \in \tilde{B}_\rho^+\). Generally, these are objects in \(\mathbb{R}^d\).

Then set
\[
\tilde{C}_R = [0, R^2) \times \tilde{B}_R^+
\]
and for $\rho, r \leq 2\rho_0(\Omega)$ and $z = (t, x)$ such that $x \in \hat{B}_\rho^+$ and $t \in \mathbb{R}$ define
\[ \hat{C}_r^+(z) = [t, t + r^2] \times \hat{B}_r^+(x). \]
Finally, whenever it makes sense, for $\rho \leq \rho_0(\Omega)$, and $z \in \hat{C}_r^+(z_0)$ introduce
\[ h_{\Omega, \gamma, \rho}^{\#}(z) = \sup \left\{ I_r(h, z_0) : z_0 \in \mathbb{R}^+ \times \Omega, r \in (0, \rho], \hat{C}_r^+(z_0) \ni z \right\}, \tag{7.4} \]
where
\[ I_r(h, z_0) = \left( \int_{\hat{C}_r^+(z_0)} \int_{\hat{C}_r^+(z_0)} |h(z_1) - h(z_2)|^\gamma dz_1 dz_2 \right)^{1/\gamma}. \]

The reader should pay attention to the above curved sharp symbol, reminding of curved boundaries.

Observe that, if $r \leq \rho \leq \rho_0(\Omega)$ and $z \in \hat{C}_r^+(z_0) \cap \hat{C}_r^+(z_0)$, then
\[ \hat{C}_r^+(z_0) \subset \hat{C}_r^+(z_0), \]
so that $h_{\Omega, \gamma, \rho}^{\#}(z)$ is well defined on $\hat{C}_r^+(z_0)$ even if $h$ is given only on $\hat{C}_r^+(z_0)$ ($\subset [0, 16\rho_0^2(\Omega) \times \Omega]$).

**Theorem 7.2.** If $p \in (1, \infty)$, $\kappa \in (0, 1/2]$, $0 < R \leq 2\rho_0(\Omega)$, and $h \in L_p(\hat{C}_R^{(1+2\kappa)})$, then
\[ \|h\|_{L_p(\hat{C}_R^+)} \leq N\|h_{\Omega, \gamma, R}^{\#}\|_{L_p(\hat{C}_R^+)} + N\kappa^{-\chi_1}R^{-\chi_2}\|h\|_{L_1(\hat{C}_R^+)}^{1/\gamma}, \tag{7.5} \]
where $\chi_1 = (d + 2)/\gamma$, $\chi_2 = (d + 2)(1/\gamma - 1/p)$ and the constants $N$ depend only on $d, \gamma$, and $p$.

This theorem is derived from Theorem 7.1 by changing variables and even extension of the functions involved across the plane $\{x^1 = 0\}$.

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