General bounds for sender-receiver capacities in multipoint quantum communications

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We investigate the maximum rates for transmitting quantum information, distilling entanglement and distributing secret keys between a sender and a receiver in a multipoint communication scenario, with the assistance of unlimited two-way classical communication involving all parties. First we consider the case where a sender communicates with an arbitrary number of receivers, so called quantum broadcast channel. Here we also provide a simple analysis in the bosonic setting where we consider quantum broadcasting through a sequence of beamsplitters. Then, we consider the opposite case where an arbitrary number of senders communicate with a single receiver, so called quantum multiple-access channel. Finally, we study the general case of all-in-all quantum communication where an arbitrary number of senders communicate with an arbitrary number of receivers. Since our bounds are formulated for quantum systems of arbitrary dimension, they can be applied to many different physical scenarios involving multipoint quantum communication.

I. INTRODUCTION

Today a huge effort is devoted to the development of robust quantum technologies, directly inspired by the field of quantum information [1–6]. The most typical communication tasks are quantum key distribution (QKD) [7–17], reliable transmission of quantum information [18, 19] and distribution of entanglement [20–22]. The latter allows two remote parties to implement powerful protocols such as quantum teleportation [23–25], which is a crucial tool for the construction of a future quantum Internet [26, 27]. Unfortunately, practical implementations are affected by decoherence [28]. This is the reason why the performance of any point-to-point protocol of quantum and private communication suffers from fundamental limitations, which become more severe when the distance is increased. This is the reason why we need quantum repeaters [29].

In this context, an open problem was to find the optimal rates for quantum and private communication that are achievable by two remote parties, say Alice and Bob, assuming the most general strategies allowed by quantum mechanics, i.e., assuming arbitrary local operations (LOs) assisted by unlimited two-way classical communication (CCs), briefly called adaptive LOCCs. These optimal rates are known as two-way (assisted) capacities and their determination has been notoriously difficult. Only recently, after about 20 years [30], Ref. [31] finally addressed this problem and established the two-way capacities at which two remote parties can distribute entanglement (D₂), transmit quantum information (Q₂), and generate secret keys (K) over a number of fundamental quantum channels at both finite and infinite dimension, including erasure channels, dephasing channels, bosonic lossy channels and quantum-limited amplifiers. For a review of these results, see also Ref. [32].

For the specific case of a bosonic lossy channel with transmissivity η, Ref. [31] proved that D₂ = Q₂ = K = −log₂(1 − η) corresponding to ≃ 1.44η bits per channel use at high loss. The latter result completely characterizes the fundamental rate-loss scaling that affects any point-to-point protocol of QKD through a lossy communication line, such as an optical fiber or free-space link. The novel and general methodology that led to these results is based on a suitable combination of quantum teleportation [23–25] with a LOCC-monotonic functional, such as the relative entropy of entanglement (REE) [33–35]. Thanks to this combination, Ref. [31] was able to upper-bound the generic two-way capacity C = D₂, Q₂, K of an arbitrary quantum channel E with a computable single-letter quantity: This is the REE ER(σ) of a suitable resource state σ that is able to simulate the quantum channel by means of a generalized teleportation protocol. In particular, Ref. [31] showed that σ corresponds to the Choi matrix of the channel when the channel is teleportation-covariant, i.e., suitably commutes with the random unitaries induced by the teleportation process.

The goal of the present paper is to extend such “REE+teleportation” methodology to a more complex communication scenario, in particular that of a single-hop quantum network, where multiple senders and/or receivers are involved. The basic configurations are represented by the quantum broadcast channel [36, 38] where information is broadcast from a single sender to multiple receivers, and the quantum multiple-access channel [39], where multiple senders communicate with a single receiver. More generally, we also consider the combination of these two cases, where many senders communicate with many receivers in a sort of all-in-all quantum communication or quantum interference channel. In practical implementations, this may represent a quantum bus [40, 41] where quantum information is transmitted among an arbitrary number of qubit registers.

In all these multipoint scenarios, we characterize the most general protocols for entanglement distillation, quantum communication and key generation, assisted by adaptive LOCCs. This leads to the definition of the two-way capacities C = D₂, Q₂, K between any pair of sender and receiver. We then consider those quantum channels (for broadcasting, multiple-accessing, and all-in-all communication) which are teleportation-covariant. For these channels, we can completely reduce an adaptive protocol into a block form involving a tensor product of Choi ma-
trices. Combining this reduction with the REE, we then bound their two-way capacities by means of the REE of their Choi matrix, therefore extending the methods of Ref. [31] to multipoint communication.

Our upper bounds applies to both discrete-variable (DV) and continuous-variable (CV) channels. As an example, we consider the specific case of a 1-to-M thermal-loss broadcast channel through a sequence of beamsplitters subject to thermal noise. In particular, we discuss how the two-way capacities \( Q_2, D_2 \) and \( K \) between the sender and each receiver are all bounded by the first point-to-point channel in the “multisplitter”. This bottleneck result can be extended to other Gaussian broadcast channels. In the specific case of a lossy broadcast channel (without thermal noise), we find a straightforward extension of the fundamental rate-loss scaling, so that any sender-receiver capacity is bounded by \(-\log_2(1-\eta)\) with \(\eta\) being the transmissivity of the first beamsplitter.

The paper is organized as follows. In Sec. II we review the basic ideas, methods, and results of Ref. [31] in relation to point-to-point quantum and private communication. This serves as a background to the reader, in order to better understand the novel developments which are presented in the following sections about the multipoint communication. Specifically, we consider the quantum broadcast channel in Sec. III the quantum multiple-access channel in Sec. IV and, finally, the quantum interference channel in Sec. V. Sec. VI is for conclusions.

II. THEORY OF POINT-TO-POINT QUANTUM AND PRIVATE COMMUNICATION

Let us briefly review the general methodology and results established in Ref. [31]. Let us define an adaptive point-to-point protocol through a quantum channel \( \mathcal{E} \). We assume that Alice has a local register \( a \) (i.e., countable set of systems) and Bob has another local register \( b \). These registers are prepared in some initial state \( \rho_{ab}^0 \) by means of an adaptive LOCC \( \Lambda_0 \). Then, Alice picks a system \( a_1 \in a \) and sends it through channel \( \mathcal{E} \); at the output, Bob gets a system \( b_1 \) which becomes part of his register, i.e., \( b_1 \rightarrow b \). Another adaptive LOCC \( \Lambda_1 \) is applied to the registers. Then, there is the second transmission \( a_2 \rightarrow b_2 \) through \( \mathcal{E} \), followed by another LOCC \( \Lambda_2 \) and so on (see Fig. 1). After \( n \) uses, Alice and Bob share an output state \( \rho_{ab}^n \) which is epsilon-close to a target state with \( nR^n \) bits [42]. The generic two-way capacity is defined by maximizing the asymptotic rate over all the adaptive LOCCs \( \mathcal{E} = \{ \Lambda_0, \ldots, \Lambda_n \} \), i.e.,

\[
C(\mathcal{E}) := \operatorname{sup}_{\mathcal{L}} \lim_{n} R^n.
\]  

In particular, by specifying the target state we identify a corresponding two-way capacity. If the target state is a maximally-entangled state, then \( C \) is equal to the two-way quantum capacity \( Q_2 \), this is in turn equal to the two-way quantum capacity \( D_2 \), because the reliable transmission of a qubit implies the distribution of an entanglement bit (ebit), and the distribution of an ebit implies the reliable teleportation of a qubit. These operations are completely equivalent under two-way CCs. If the target state is a private state [43], then \( C \) is equal to the (two-way) secret key capacity \( (K) \). Since a maximally-entangled state is a type of private state, we have \( D_2 \leq K \). Then also note that \( K = P_2 \), where the latter is the two-way private capacity of the channel, i.e., the maximum rate at which Alice may deterministically transmit secret bits over the channel by means of adaptive protocols [44]. In summary, one has the hierarchy

\[
D_2 = Q_2 \leq K = P_2.
\]

In order to bound these capacities by means of a computable single-letter quantity, Ref. [31] introduced a reduction method whose application can be adapted to many other scenarios. The first ingredient is the extension of the relative entropy of entanglement (REE) [33–34] from quantum states to quantum channels. Ref. [31, Theorem 1] showed that, for any quantum channel \( \mathcal{E} \) (at any dimension, finite or infinite), the generic two-way capacity \( C(\mathcal{E}) \) [i.e., any of the quantities in Eq. (2)] satisfies the weak converse bound

\[
C(\mathcal{E}) \leq E^\star_R(\mathcal{E}) := \operatorname{sup}_{\mathcal{L}} \lim_{n} \frac{E_R(\rho_{ab}^n)}{n}.
\]

Here the adaptive channel’s REE \( E^\star_R(\mathcal{E}) \) is defined by computing the REE of the output state \( \rho_{ab}^n \) taking the asymptotic limit in the number of channels uses \( n \), and optimizing over all the adaptive protocols \( \mathcal{L} \). Recall the REE of a quantum state \( \sigma \) is defined as

\[
E_R(\rho) = \inf_{\sigma_s} S(\rho||\sigma_s),
\]

where \( \sigma_s \) is an arbitrary separable state and \( S(\rho||\sigma_s) := \operatorname{Tr}[\rho (\log_2 \rho - \log_2 \sigma_s)] \) is the relative entropy [33]. For an asymptotic state \( \sigma := \lim_{\mu \to +\infty} \sigma^\mu \) defined by a sequence of states \( \sigma^\mu \), the REE can be defined as

\[
E_R(\sigma) := \operatorname{inf}_{\sigma^\mu} \liminf_{\mu \to +\infty} S(\sigma^\mu||\sigma_s^\mu),
\]

where \( \sigma^\mu_s \) is an arbitrary sequence of separable states which is convergent in trace-norm, i.e., such that \( \|\sigma^\mu_s - \sigma_s\| \to 0 \) for some separable \( \sigma_s \). This mathematical form is directly inherited from the lower semi-continuity of the relative entropy for CV systems [4].

![FIG. 1: Point-to-point adaptive protocol. Each transmission \( a_i \rightarrow b_i \) through the quantum channel \( \mathcal{E} \) is interleaved by two adaptive LOCCs, \( \Lambda_{i-1} \) and \( \Lambda_i \), applied to Alice’s and Bob’s local registers \( a \) and \( b \). After \( n \) transmissions, Alice and Bob share an output state \( \rho_{ab}^n \).](image-url)
Remark 1 The demonstration of Eq. (3) exploits several tools from Refs. [34, 35, 43, 44]. In particular, Ref. [34] provided three equivalent proofs, based on alternative treatments of the private state involved in the definition of the secret key capacity. The first proof exploits the fact that the dimension of the shield system [43] of the private state has an effective exponential scaling in the number of channel uses; this scaling is an immediate application of well-known results in the literature [44, 47], whose adaptation to CVs is trivial as discussed in [31, Supplementary Note 3] and also in the recent review [32]. The second proof assumes an exponential energy growth in the channel uses, while the third proof does not depend on the shield size. For a full discussion of these details see Supplementary Note 3 of Ref. [31].

To simplify the upper bound of Eq. (3) into a single-letter quantity, Ref. [31] devised a second ingredient. This consists of a technique, dubbed “teleportation stretching”, which reduces an adaptive protocol (with any communication task) into a much simpler block-type protocol. More recently, this technique has been extended to simplify adaptive protocols of quantum metrology and quantum channel discrimination [48]. The first step of the technique is the LOCC simulation of a quantum channel. This leads to a stretching of the channel into a quantum state. Then, the second step is the exploitation of this simulation argument in the adaptive protocol, so that all the transmissions through the channel are replaced by a tensor product of quantum states subject to a trace-preserving LOCC.

For any quantum channel \( \mathcal{E} \), we may consider an LOCC-simulation. This consists of an LOCC \( \mathcal{T} \) and a resource state \( \sigma \) such that, for any input state \( \rho \), the output of the channel can be expressed as [31]

\[
\mathcal{E}(\rho) = \mathcal{T}(\rho \otimes \sigma).
\]

A channel \( \mathcal{E} \) which is LOCC-simulable with a resource state \( \sigma \) as in Eq. (6) is called “\( \sigma \)-stretchable” [31]. For the same channel \( \mathcal{E} \) there may be different choices for \( \mathcal{T} \) and \( \sigma \). Furthermore, the simulation can also be asymptotic. This means that we may consider sequences of LOCCs \( \mathcal{T}^{\mu} \) and resource states \( \sigma^{\mu} \) such that [31]

\[
\mathcal{E}(\rho) = \lim_{\mu} \mathcal{T}^{\mu}(\rho \otimes \sigma^{\mu}),
\]

for any input state \( \rho \). In other words, a quantum channel may be defined as a point-wise limit as in Eq. (7). This generalization is relevant for the simulation of CV channels, such as bosonic channels, or certain DV channels, such as the amplitude damping channel, whose simulation is based on mappings between CVs and DVs.

Remark 2 The LOCC-simulation of an arbitrary quantum channel at any dimension (finite or infinite) was introduced in Ref. [31]. The first relevant idea was the teleportation simulation of Ref. [27], Section V] whose application was however limited to Pauli channels (as shown in Ref. [49]). Other approaches known in the literature [54, 53] did not consider arbitrary LOCC simulations but teleportation protocols, therefore restricting the classes of simulable channels. In particular, the amplitude damping channel is a simple example of a quantum channel that could not be deterministically simulated by any approach prior to Ref. [31]. Finally note that the simulations in Refs. [54, 53] are not suitable for quantum communications because they generally imply non-local operations between the remote parties. See Ref. [31, Supplementary Note 8] for detailed discussions on the literature and advances in channel simulation. See also the recent review [32] and Table I therein.

At any dimension (finite or infinite), an important class of quantum channels are those “Choi-stretchable”. These are channels \( \mathcal{E} \) for which we may write the LOCC simulation of Eqs. (6) with \( \sigma \) being the Choi matrix of the channel, which is defined as

\[
\rho_{\mathcal{E}} := \mathcal{I} \otimes \mathcal{E}(\Phi),
\]

where \( \Phi \) is a maximally-entangled state. If the simulation is asymptotic, then the Choi-stretchable channel satisfies Eq. (7) with \( \sigma^{\mu} \) being a sequence of Choi-approximating states \( \rho_{\mathcal{E}}^{\mu} \), i.e., such that their limit defines the asymptotic Choi matrix of the channel as \( \rho_{\mathcal{E}} := \lim_{\mu} \rho_{\mathcal{E}}^{\mu} \). In particular, for a bosonic channel, we may set

\[
\rho_{\mathcal{E}}^{\mu} := \mathcal{I} \otimes \mathcal{E}(\Phi^{\mu}),
\]

where \( \Phi^{\mu} \) is a two-mode squeezed vacuum (TMSV) state [3], whose asymptotic limit defines the ideal CV Einstein-Podolsky-Rosen (EPR) state.

A simple criterion to identify Choi-stretchable channels is that of teleportation-covariance. By definition, we say that a quantum channel \( \mathcal{E} \) is teleportation-covariant if, for any teleportation unitary \( U \) (Pauli operators in DVs, phase-space displacements in CVs [25]), we may write

\[
\mathcal{E}(U \rho U^\dagger) = V \mathcal{E}(\rho) V^\dagger,
\]

for another (generally different) unitary \( V \) [31]. This is a large family which includes Pauli, erasure and bosonic Gaussian channels. With this definition in hand, Ref. [31, Proposition 2] showed that a teleportation-covariant channel is certainly a Choi-stretchable channel, where the LOCC simulation is simply given by quantum teleportation. In other words, we may write \( \mathcal{E}(\rho) = \mathcal{T}(\rho \otimes \rho_{\mathcal{E}}) \) where \( \mathcal{T} \) is a teleportation-LOCC. For asymptotic simulations, we have

\[
\mathcal{E}(\rho) = \lim_{\mu} \mathcal{T}^{\mu}(\rho \otimes \rho_{\mathcal{E}}^{\mu}),
\]

where \( \mathcal{T}^{\mu} \) is a sequence of teleportation-LOCCs. These are built on finite-energy Gaussian measurements whose asymptotic limit defines the ideal CV Bell detection [31].

Remark 3 The fact that teleportation-covariance implies the simulation of the channel by means of teleportation over the channel’s Choi matrix has been first discussed in Ref. [53] in the context of DV channels. It has
been later generalized in Ref. [31] to include CV channels and asymptotic simulations. See also Ref. [32].

Thanks to the LOCC-simulation (T, σ) of a quantum channel E as in Eq. (6), one may completely simplify the structure of an adaptive protocol. In fact, the output ρ^µ_ab can be reduced to a tensor-product σ⊗n up to a trace-preserving LOCC ¯Λ [31]. In other words, we may write

\[ ρ^µ_ab = \bar{Λ} (σ⊗n) \]  \quad (12)

In fact, (i) first we replace each transmission through the channel E with an LOCC-simulation (T, σ); (ii) then we stretch the resource state σ “back in time”; (iii) finally, we collapse all the LOCCs (and also the initial separable state ρ^0_ab) into a single trace-preserving LOCC (which is suitably averaged over all the possible measurements in the simulated protocol). These steps are depicted in Fig. 2 and lead to the decomposition in Eq. (12).

**FIG. 2:** Teleportation stretching of an adaptive point-to-point protocol. (Top panels) The generic ith transmission through the channel E is simulated by an LOCC T and a resource state σ. (Bottom panels) The resource state σ is stretched back in time out of the adaptive LOCCs. This is repeated for all the n transmissions, so that we accumulate the tensor-product state σ⊗n. All the LOCCs (those of the original protocol and those introduced by the simulation) are collapsed into a single trace-preserving LOCC ¯Λ (which also includes the initial state of the registers ρ^0_ab).

For a quantum channel with asymptotic simulation as in Eq. (7), the procedure is more involved. One first considers an imperfect channel simulation E^µ(ρ) := T^µ(ρ ⊗ σ^µ) in each transmission. By adopting this simulation, we realize an imperfect stretching of the protocol, with output state ρ^µ_ab = ¯Λµ (σ^µ⊗n) for a trace-preserving LOCC ¯Λµ. This is done similarly to the steps in Fig. 2 but considering E^µ in the place of the original channel E. A crucial point is now the estimation of the error in the channel simulation, which must be suitably controlled and propagated to the output state.

Assume that the registers have a total number m of modes (the value of m can be taken to be finite and then relaxed at the very end to include countable registers). Then assume that, during the n transmissions of the protocol, the total mean number of photons in the registers is bounded by some large but finite value N. We may therefore define the set of energy-constrained states

\[ \mathcal{D}_N := \{ ρ_ab | \text{Tr}(\hat{N} ρ_ab) ≤ N \} ⊂ \mathcal{D}(\mathcal{H}^⊗m), \]  \quad (13)

where \( \hat{N} \) is the m-mode number operator. For the ith transmission a_i → b_i, the simulation error may be quantified in terms of the energy-bounded diamond norm [31]

\[ ε_N := \| E - E^µ \|_{\diamond N} = \sup_{ρ_{a_i,ab} \in \mathcal{D}_N} \| E ⊗ I_{ab}(ρ_{a_i,ab}) - E^µ ⊗ I_{ab}(ρ_{a_i,ab}) \|. \]  \quad (14)

Because \( \mathcal{D}_N \) is compact [58] and channel E is defined by the point-wise limit E(ρ) = lim_µ E^µ(ρ), we may write the following uniform limit

\[ ε_N ∋ 0 \quad \text{for any } N. \]  \quad (15)

This error has to be propagated to the output state, so that we can suitably bound the trace distance between the actual output ρ^µ_ab and the simulated output ρ^µ_ab. By using basic properties of the trace distance (triangle inequality and monotonicity under maps), Ref. [31] showed that the simulation error in the output state satisfies

\[ \| ρ^µ_ab - ρ^µ_ab \| ≤ n \| E - E^µ \|_{\diamond N}. \]  \quad (16)

Therefore, for any N, we may write the trace-norm limit

\[ \| ρ^µ_ab - \bar{Λ}_µ (σ^µ⊗n) \| ∋ 0, \]  \quad (17)

i.e., the asymptotic stretching ρ^µ_ab = lim_µ ¯Λµ(σ^µ⊗n).

**Remark 4** Teleportation stretching simplifies an arbitrary adaptive protocol implemented over an arbitrary channel at any dimension, finite or infinite. In particular, it works by maintaining the original communication task. This means that an adaptive protocol of quantum communication (QC), entanglement distribution (ED) or key generation (KG), is reduced to a corresponding block protocol with exactly the same original task (QC, ED, or KG), but with the output state being decomposed in the form of Eq. (12) or Eq. (17). In the literature, there were some precursory but restricted arguments, as those in Refs. [24, 27]. These were limited to the transformation of a protocol of QC into a protocol of ED, over specific classes of channels (e.g., Pauli channels in Ref. [24]). Furthermore, no control of the simulation error was considered in previous literature [24], while this is crucial for the rigorous simulation of bosonic channels.

The most crucial insight of Ref. [31] has been the combination of the previous two ingredients, i.e., channel’s REE and teleportation stretching, which is the key observation leading to a single-letter upper bound for all the two-way capacities of a quantum channel. In fact, let us compute the REE of the output state decomposed as in Eq. (12). We derive

\[ E_R(ρ^µ_ab) \overset{(1)}{≤} E_R(σ^⊗n) \overset{(2)}{≤} nE_R(σ), \]  \quad (18)
using (1) the monotonicity of the REE under trace-preserving LOCCs and (2) its subadditive over tensor products. By replacing Eq. (15) in Eq. (8), we then find the single-letter upper bound [31, Theorem 5]
\[
\mathcal{C}(\mathcal{E}) \leq E_R(\sigma) .
\]
(19)

In particular, if the channel \( \mathcal{E} \) is teleportation-covariant, it is Choi-stretchable, and we may write [31, Theorem 5]
\[
\mathcal{C}(\mathcal{E}) \leq E_R(\rho_{CE}).
\]
(20)

These results are suitable extended to asymptotic simulations. By adopting the extended definition of \( \mathcal{E} \) in Eq. (9), Ref. [31] showed that Eqs. (14) and (21) are valid for channels with asymptotic simulations, such as bosonic channels. In particular, the proof exploits the fact that Eq. (3) involves a supremum over all protocols \( \mathcal{E} \), so that we may extend the upper bound to the asymptotic limit of energy-uncorrelated protocols where the total mean photon number \( N \) tends to infinity (and the local registers become countable set).

The upper bound of Eq. (20) is valid for any teleportation-covariant channel, in particular for Pauli channels (e.g., depolarizing or dephasing), erasure channels and bosonic Gaussian channels. Then, by showing coincidence of this upper bound with lower bounds based on the coherent [18, 19] and reverse coherent information [59, 60], Ref. [31] established strikingly simple formulas for the two-way capacities of the most fundamental quantum channels. For a bosonic lossy channel \( \mathcal{E} \), with transmissivity \( \eta \), one has [31]
\[
D_2(\eta) = Q_2(\eta) = K(\eta) = P_2(\eta) = -\log_2(1 - \eta) .
\]
(21)

In particular, the secret-key capacity of the lossy channel determines the maximum rate achievable by any QKD protocol. At high loss \( \eta \approx 0 \), one has the optimal rate-loss scaling of \( K \approx 1.44 \eta \) secret bits per channel use. Because it establishes the upper limit of any point-to-point quantum optical communication, Eq. (21) also establishes a “repeatear bound”, i.e., the benchmark that quantum repeaters must surpass in order to be effective.

Then, for a quantum-limited amplifier \( \mathcal{E}_a \) with gain \( g > 1 \) [3], one finds [31]
\[
D_2(\eta) = D_2(\eta) = Q_2(\eta) = P_2(\eta) = -\log_2(1 - g^{-1}) .
\]
(22)

In particular, this proves that \( Q_2(\eta) \) coincides with the unassisted quantum capacity \( Q(\eta) \) [61, 62]. For a qubit dephasing channel \( \mathcal{E}_{dep} \) with dephasing probability \( p \), one has [31]
\[
D_2(p) = Q_2(p) = K(p) = P_2(p) = 1 - H_2(p) ,
\]
(23)

where \( H_2 \) is the binary Shannon entropy. Note that this also proves \( Q_2(p) = Q(p) \) for a dephasing channel, where \( Q(p) \) was found in Ref. [63]. Eq. (23) can be extended to dephasing channels \( \mathcal{E}_{dep} \) in arbitrary dimension \( d \), so that all the two-way capacities are given by [31]
\[
C(p, d) = \log_2 d - H(\{ P_i \}) ,
\]
(24)

where \( H \) is the Shannon entropy and \( P_i \) is the probability of \( i \) phase flips. Finally, for the qudit erasure channel \( \mathcal{E}_{erasure} \) with erasure probability \( p \), one finds [31]
\[
D_2(p) = Q_2(p) = K(p) = P_2(p) = (1 - p) \log_2 d .
\]
(25)

As previously mentioned, only the \( Q_2 \) of the erasure channel was previously known [32]. Simultaneously with Ref. [31], an independent study of the erasure channel has been provided by Ref. [64] which showed how its \( K \) can be computed from the squashed entanglement (see also Ref. [31] Supplementary Discussion (page 38)).

### III. QUANTUM BROADCAST CHANNEL

Here we consider quantum and private communication in a single-hop point-to-multipoint network. We adapt the techniques of Ref. [31] to bound the optimal rates that are achievable in adaptive protocols involving multiple receivers. For the sake of simplicity, we present the theory for non-asymptotic simulations. The theoretical treatment of asymptotic simulations goes along the lines described in previous Sec. II and is discussed afterwards.

Consider a quantum broadcast channel \( \mathcal{E} \) where Alice (local register \( a \)) transmits a system \( a \in \mathfrak{a} \) to \( M \) different Bobs; the generic ith Bob (with \( i = 1, \ldots, M \)) receives an output system \( b_i^0 \) which may be combined with a local register \( b_i^1 \) for further processing. Denote by \( D(\mathcal{H}_a) \) the space of density operators defined over the Hilbert space \( \mathcal{H}_a \) of quantum system \( a \). Then, the quantum broadcast channel is a completely-positive trace preserving (CPTP) map from Alice’s input space \( D(\mathcal{H}_a) \) to the Bobs’ output space \( D(\otimes_i \mathcal{H}_b) \). The most general adaptive protocol over this channel goes as follows.

All the parties prepare their initial systems by means of a LOCC \( \Lambda_0 \). Then, Alice picks the first system \( a_1 \in \mathfrak{a} \) which is broadcast to all Bobs \( a_1 \rightarrow \{ b_1^0 \} \) through channel \( \mathcal{E} \). This is followed by another LOCC \( \Lambda_1 \) involving all parties. Bobs’ ensembles are updated as \( b_1^0 b_1^1 \rightarrow b_1^1 \). Then, there is the second broadcast \( a_2 \rightarrow \{ b_2^0 \} \) through \( \mathcal{E} \), followed by another LOCC \( \Lambda_2 \) and so on. After \( n \) uses, Alice and the ith Bob share an output state \( \rho_{ab}^{ab} \) which is epsilon-close to a target state with \( nR_i^n \) bits. The generic broadcast capacity for the ith Bob is defined by maximizing the asymptotic rate over all the adaptive LOCCs \( \mathcal{E} = \{ \Lambda_0, \Lambda_1, \ldots \} \), i.e., we have
\[
C_i := \sup_{\mathcal{E}} \lim_{n} R_i^n .
\]
(26)

By specifying the adaptive protocol to a particular target state, i.e., to a particular task (entanglement distribution, reliable transmission of quantum information, key generation or deterministic transmission of secret bits), one derives the entanglement-distribution broadcast capacity (\( D_2^1 \)), the quantum broadcast capacity (\( Q_2^1 \)), the secret-key broadcast capacity (\( K^1 \)), and the private broadcast capacity (\( P_2^1 \)). These are all assisted
by unlimited two-way CCs between the parties and it is easy to check that they must satisfy $D^*_2 = Q^*_2 \leq K^* = P^*_2$.

In order to bound the previous capacities, let us introduce the notion of teleportation-covariant broadcast channel. It is explained for the case of two receivers, Bob and Charlie, with the extension to arbitrary $M$ receivers being just a matter of technicalities. This is a broadcast channel which suitably commutes with teleportation. Formally, this means that, for any teleportation unitary $U_k$ at the channel input, we may write

$$\mathcal{E}(U_k \rho U_k^\dagger) = (B_k \otimes C_k) \mathcal{E}(\rho) (B_k \otimes C_k)^\dagger,$$

(27)

for unitaries $B_k$ and $C_k$ at the two outputs. If this is the case, it is immediate to prove that $\mathcal{E}$ can be simulated by a generalized teleportation protocol over its Choi matrix

$$\rho_E = \mathcal{T}_{\mathcal{A}} \otimes \mathcal{E}_{\mathcal{A}'}(\Phi_{\mathcal{A}\mathcal{A}'})$$ (28)

where the latter is defined by sending half of an EPR pair through the broadcast channel. In other words, the broadcast channel is Choi-stretchable and its LOCC simulation is based on teleportation. See Fig. 3.

**Fig. 3:** Simulation of a teleportation-covariant quantum broadcast channel. We may replace the broadcast channel $\mathcal{E} : a \rightarrow bc$ by teleportation over its Choi matrix $\rho_E$, with CCs to Bob and Charlie, who will implement correction unitaries. The broadcast channel is therefore Choi-stretchable and its LOCC simulation is based on teleportation.

Following and extending the ideas of Ref. 31, we may simplify any adaptive protocol performed over a teleportation-covariant broadcast channel. The steps of the procedure are shown in Fig. 4. As a result, the total output state of Alice, Bob and Charlie can be decomposed in the form

$$\rho_{abc}^n := \rho_{abc}(\mathcal{E}^\otimes n) = \bar{\Lambda} \left( \rho_E^{\otimes n} \right),$$

(29)

where $\bar{\Lambda}$ is a trace-preserving LOCC. If we now trace one of the two receivers, e.g., Charlie, we still have a trace-preserving LOCC between Alice and Bob, and we may write the following

$$\rho_{ab}^n = \text{Tr}_c (\bar{\Lambda} (\rho_E^{\otimes n})) = \bar{\Lambda}_{ab\vert bc} \left( \rho_E^{\otimes n} \right),$$

(30)

where $\bar{\Lambda}_{a\vert bc}$ is local with respect to the cut $a\vert bc$.

Let us now compute the REE of Alice and Bob’s output state $\rho_{ab}^n$. Using Eq. (30) and the monotonicity of the REE under $\bar{\Lambda}_{a\vert bc}$, we derive

$$E_R(\rho_{ab}^n) := \inf_{\sigma_s(\mathcal{A})} S \left( \rho_{ab}^n \| \sigma_s \right) \leq \inf_{\sigma_s(\mathcal{A}\vert bc)} S \left( \rho_E^{\otimes n} \| \sigma_s \right) := E_R(\rho_{E}^{\otimes n}),$$

(31)

where we call $E_R(\rho_{E}^{\otimes n})$ the REE with respect to the bipartite cut $abc$. Note that the set of states $\{\sigma_s(\mathcal{A}\vert bc)\}$, separable between $a$ and $bc$, includes the set of states $\{\sigma_s(\mathcal{A}\vert bc)\}$ which are separable with respect to $a$, $b$ and $c$. Therefore, we may write the further upper-bound

$$E_R(\rho_{E}^{\otimes n}) \leq \inf_{\sigma_s(\mathcal{A}\vert bc)} S \left( \rho_E^{\otimes n} \| \sigma_s \right) := E_R(\rho_E^{\otimes n}).$$

(32)

For Alice and Bob $(i = B)$, we can then exploit the weak converse bound in Eq. (3) where the optimization must be done over all the adaptive broadcast protocols. Combining this bound with Eqs. (31) and (32), we get

$$C_B \leq \sup_{\mathcal{E}} \sup_{\rho_{ab}} \frac{E_R(\rho_{ab}^n)}{n} \leq E_R(\rho_{E}^{\otimes n}) \leq E_R(\rho_E).$$

(33)

where $E_R(\rho) := \lim_n n^{-1} E_R(\rho^{\otimes n})$ is the regularized version of the REE. Then, using the subadditive over tensor products, we may also write

$$E_R(\rho_{ab}^n) \leq n E_R(\rho_{E}^{\otimes n}) \leq nE_R(\rho_E),$$

(34)

which clearly leads to the single-letter upper bounds

$$C_B \leq E_R(\rho_{E}^{\otimes n}) \leq E_R(\rho_E).$$

(35)

We find the same bounds for the capacity of Alice and Charlie $(i = C)$. In general, for arbitrary $M$ receivers, we
may extend the reasoning and write the following upper bounds for the capacity between Alice and the ith Bob

\[ C^i \leq E_{R(a|b^1 \ldots b^M)}(\rho_E) \leq E_R(\rho_E) := \Phi(\mathcal{E}), \]

where \( \Phi(\mathcal{E}) \) is the entanglement flux of the broadcast channel \( \mathcal{E} \), defined as the REE of its Choi matrix \( \rho_E \).

### A. Extension to continuous variables

As explained in Sec. 11, one cannot directly apply the DV formulation of channel simulation and teleportation stretching to CV systems. There are non-trivial issues to be taken into account, related with the infinite energy of the asymptotic Choi matrices of the bosonic channels. These issues require a suitable treatment.\[ \text{[31][32]} \]

The Choi matrix of a bosonic broadcast channel can be defined as the following asymptotic state

\[ \rho_E := \lim_{\mu} \rho_E^\mu, \quad \rho_E^\mu = \mathcal{I}_A \otimes \mathcal{E}_A'(\Phi_{AA'}^\mu), \]

with \( \Phi_{AA'}^\mu \) being a TMSV state. The simulation of a teleportation-covariant bosonic broadcast channel is based on the sequence of Choi-approximating states \( \rho_E^\mu \), so that we may write the generalization of Eq. (11), i.e.,

\[ \mathcal{E}(\rho) = \lim_{\mu} \mathcal{T}_n(\rho \otimes \rho_E^\mu), \]

where \( \mathcal{T}_n \) is a sequence of teleportation-LOCCs. By repeating the reasoning in Sec. 11 the error in the channel simulation can be propagated to the output state of the adaptive protocol, so that, for any energy constraint on the local registers, we may write the trace-norm limit

\[ \left\| \rho_{n}^{\mu}_{a^1 b^1 c^1} - \tilde{A}_\mu \left( \rho_{n}^{\mu \otimes n}_{E} \right) \right\| \overset{\text{trace}}{=} 0, \]

where \( \tilde{A}_\mu \) is an imperfect stretching-LOCC associated with the imperfect teleportation LOCC \( \mathcal{T}_n \). By tracing one of the outputs, e.g., Charlie, one gets

\[ \left\| \rho_{n}^{\mu}_{a^1 b^1 c^1} - \tilde{A}_\mu \left( \rho_{n}^{\mu \otimes n}_{E} \right) \right\| \overset{\text{trace}}{=} 0, \]

where \( \tilde{A}_{\mu}^{a|b|c} \) is an imperfect stretching-LOCC associated with Alice and Bob, which is local with respect to the bipartite cut \( a^1 b^1 c^1 \).

The next step is to extend the definition of REE to asymptotic states as in Eq. (5). In particular, we define

\[ E_{R(a|b^1 \ldots b^M)}(\rho_E) := \inf_{\sigma_\mu^{\mu} (a|b^1 \ldots b^M) \mu \rightarrow +\infty} \lim \inf S(\rho_E^\mu \| \sigma_\mu^{\mu}), \]

where \( \sigma_\mu^{\mu} (a|b^1 \ldots b^M) \) is an arbitrary converging sequence of states that is separable with respect to the cut \( a^1 b^1 c^1 \). Then, we also define the entanglement flux of the bosonic broadcast channel as

\[ \Phi(\mathcal{E}) = E_R(\rho_E) := \inf_{\sigma_\mu^{\mu}} \lim \inf S(\rho_E^\mu \| \sigma_\mu^{\mu}), \]

where \( \sigma_\mu^{\mu} \) is an arbitrary converging sequence of separable states (with respect to all the local systems \( a^1 b^1 c^1 \)). By applying a direct extension of the weak converse bound in Eq. (33), we then derive the same result as in Eq. (35) for the capacity \( C^B \) between Alice and Bob, proviso that the REE quantities are suitably extended as in Eqs. (41) and (42). In general, for arbitrary \( M \) receivers, we have the corresponding extension of Eq. (35).

### B. Thermal-loss quantum broadcast channel

Now that we have rigorously extended the treatment to CV systems, we study the example of a bosonic broadcast channel from Alice to \( M \) Bobs which introduces both loss and thermal noise. This is an optical scenario that may easily occur in practice. For instance, it may represent the practical implementation of a single-hop QKD network, where a party wants to share keys with several other parties for broadcasting private information. The latter may also be a common key to enable a quantum-secure conferencing among all the trusted parties.

One possible physical representation is a chain of \( M + 1 \) beamsplitters with transmissivities \( (\eta_0, \eta_1, \ldots, \eta_M) \) in which Alice’s input mode \( A' \) subsequently interacts with \( M + 1 \) modes \( (E_0, E_1, E_2, \ldots, E_M) \) described by thermal states \( \rho_{E_i}(\tilde{n}_i) \) with \( \tilde{n}_i \) mean number of photons. The \( M \) output modes \( (B_1, B_2, \ldots, B_M) \) are then given to the different Bobs, with the extra modes \( E \) and \( E' \) being the leakage to the environment (or an eavesdropper). See Fig. 5 for a schematic representation of this thermal-loss broadcast channel \( \mathcal{E} = \mathcal{E}_{A' \rightarrow B_1 \ldots B_M} \).

FIG. 5: Thermal-loss quantum broadcast channel \( \mathcal{E}_{A' \rightarrow B_1 \ldots B_M} \) from Alice (mode \( A' \)) to \( M \) Bobs (modes \( B_1 \ldots, B_M \)), realized by a multi-splitter, i.e., a sequence of \( M + 1 \) beamsplitters with transmissivities \( (\eta_0, \eta_1, \ldots, \eta_M) \). The environmental modes \( E_0, E_1, \ldots, E_M \) are in thermal states. Modes \( E \) and \( E' \) describe leakage to the environment.

The generic capacity \( C^i \) between Alice and the ith Bob is upper bounded by

\[ C^i \leq E_{R(A|B_1 \ldots B_M)}(\rho_E) := \inf_{\sigma_\mu^{\mu} (A|B_1 \ldots B_M) \mu \rightarrow +\infty} \lim \inf S(\rho_E^\mu \| \sigma_\mu^{\mu}), \]

where the state \( \rho_E^\mu := \mathcal{I}_A \otimes \mathcal{E}_{A' \rightarrow B_1 \ldots B_M} (\Phi_{AA'}^\mu) \) is the Choi-approximating state obtained by sending one half of a TMSV state \( \Phi_{AA'}^\mu \), and \( \sigma_\mu^{\mu} (A|B_1 \ldots B_M) \) is a converging sequence of states that are separable with respect
to the cut $A|B_1 \cdots B_M$. Now notice that we may write
\[ \rho_E^\mu = L_{A|B_1'}E_{1} \cdots E_{M} \left[ \rho_{E_{A' \rightarrow B_1'}}^\mu \otimes \bigotimes_{i=1}^{M} \rho_{E_i}(\bar{n}_i) \right], \tag{44} \]
where $\rho_{E_{A' \rightarrow B_1'}}^\mu := L_{A} \otimes E_{A' \rightarrow B_1'}(\Phi_{A^N A'})$ is associated with the first beamsplitter, and $L_{A|B_1}E_{1} \cdots E_{M}$ is a trace-preserving LOCC, local with respect to the cut $A|B_1' E_1 \cdots E_M$. Also note that, for any separable state $\sigma_{s}^\mu(A|B_1')$ we have that the output state
\[ \sigma_{s}^\mu = L_{A|B_1}E_{1} \cdots E_{M} \left[ \sigma_{s}^\mu(A|B_1') \otimes \bigotimes_{i=1}^{M} \rho_{E_i}(\bar{n}_i) \right] \tag{45} \]
is separable with respect to the cut $A|B_1 \cdots B_M$. As a result we have that
\[ E_{R(A|B_1 \cdots B_M)}(\rho_E) \stackrel{(1)}{=} \inf_{\sigma_{s}^\mu(A|B_1') \otimes \bigotimes_{i=1}^{M} \rho_{E_i}(\bar{n}_i)} \liminf_{\mu \to +\infty} S(\rho_E^\mu || \sigma_{s}^\mu) \]
\[ \leq \inf_{\sigma_{s}^\mu(A|B_1')} S(\rho_{E_{A' \rightarrow B_1'}}^\mu || \sigma_{s}^\mu) := \Phi(\mathcal{E}_{A' \rightarrow B_1'}), \tag{46} \]
where we use: (1) the fact that $\sigma_{s}^\mu(A|B_1 \cdots B_M)$ are specific types of $\sigma_{s}^\mu(A|B_1' \cdots B_M)$; and (2) monotonicity and additivity of the relative entropy with respect to the decompositions in Eqs. \(44\) and \(45\).

Because $\mathcal{E}_{A' \rightarrow B_1'}$ is a thermal-loss channel with transmissivity $\eta_0$ and mean photon number $\bar{n}_0$, its entanglement flux is bounded by \[31\]
\[ \Phi(\mathcal{E}_{A' \rightarrow B_1'}) \leq - \log_2 \left[ (1 - \eta_0)\eta_0^{\bar{n}_0} \right] - h(\bar{n}_0), \tag{47} \]
for $\bar{n}_0 < \eta_0/(1 - \eta_0)$, while zero otherwise. Here we set
\[ h(x) := (x + 1) \log_2(x + 1) - x \log_2 x. \tag{48} \]
Thus, we find that the capacity between Alice and the $i$th Bob must satisfy
\[ C_i \leq \begin{cases} - \log_2 \left[ (1 - \eta_0)\eta_0^{\bar{n}_0} \right] - h(\bar{n}_0) & \text{for } \bar{n}_0 < \frac{\eta_0}{1 - \eta_0}, \\ 0 & \text{for } \bar{n}_0 \geq \frac{\eta_0}{1 - \eta_0}. \end{cases} \tag{49} \]
As expected, the first beamsplitter is a universal bottleneck which restricts the capacities between Alice and any of the receiving Bobs.

In the specific case of a lossy broadcast channel with no thermal noise ($n_i = 0$ for any $i$), we may specify Eq. \(49\) into the following simple bound
\[ C_i \leq - \log_2 (1 - \eta_0). \tag{50} \]

Let us note that, contrary to another work \[60\] also inspired by Ref. \[31\], our analysis of the lossy broadcast channel builds upon a rigorous extension of channel simulation and teleportation stretching to CV systems, which includes a suitable generalization of the REE to asymptotic states. Since our results represent a rigorous extension of Ref. \[31\], they may also be used to solidify the claims presented in Ref. \[60\] on the capacity region of the lossy broadcast channel. For further details see Ref. \[32\].

Most importantly, notice that our derivation can be generalized to other bosonic broadcast channels, where the $M + 1$ beamsplitters are replaced by arbitrary Gaussian unitaries $U_{A' E_0}, U_{B_1' E_1}, \ldots, U_{B_M' E_M}$. In this general case, we repeat the previous reasonings to find that the capacities must satisfy the bottleneck relation
\[ C_i \leq \Phi(\mathcal{E}_{A' \rightarrow B_1'}), \tag{51} \]
where the latter is the entanglement flux of the first Gaussian channel $\mathcal{E}_{A' \rightarrow B_1'}$, determined by the action of the Gaussian unitary $U_{A'|E_0}$ on the input mode $A'$ and the thermal mode $E_0$.

IV. QUANTUM MULTIPLE-ACCESS CHANNEL

Let us now study multipoint-to-point quantum communication, i.e., a quantum multiple-access channel from $M$ senders (Alices) to a single receiver (Bob). This channel is a CPTP map from Alices’ input space $\mathcal{D}(\otimes_i \mathcal{H}_{A_i})$ to Bob’s output space $\mathcal{D}(\mathcal{H}_b)$. The most general adaptive protocol over this channel goes as follows. All the parties prepare their initial systems by means of a LOCC $\Lambda_0$. Then, the $i$th Alice picks the first system from her local ensemble, i.e., $a_1^i \in a_i$. All Alice’s input systems are sent through the quantum multiple-access channel $\mathcal{E}$ with output $b_1$ for Bob, i.e.,
\[ a_1^1, \ldots, a_i^1, \ldots, a_1^M \xrightarrow{\mathcal{E}} b_1. \tag{52} \]
This is followed by another LOCC $\Lambda_1$ involving all parties. Bob ensemble is updated as $b_1 b \rightarrow b$. Then, there is the second transmission $\{a_i^2 \} \rightarrow b_2$ through $\mathcal{E}$, followed by another LOCC $\Lambda_2$ and so on. After $n$ uses, the $i$th Alice and Bob share an output state $\rho_{a_i^n b}$ which is epsilon-close to a target state with $nR_i^n$ bits.

The generic multiple-access capacity for the $i$th Alice is defined by maximizing the asymptotic rate over all the adaptive LOCCs $\mathcal{L} = \{ \Lambda_0, \Lambda_1, \ldots \}$, i.e., we have $C_i := \sup_{\mathcal{L}} \lim_{n \to \infty} R_i^n$. As before, by specifying the adaptive protocol to a particular task, one derives the entanglement distribution multiple-access capacity ($D_2$), the quantum multiple-access capacity ($Q_2$), the secret-key multiple-access capacity ($K_1$) and the private multiple-access capacity ($P_2$). These are all assisted by unlimited two-way CCs between the parties and satisfy $D_2 = Q_2 \leq K_1 = P_2$.

Let us introduce the notion of teleportation-covariant multiple-access channel. For the sake of simplicity, this is explained for the case of two senders, with the extension to arbitrary $M$ senders being just a matter of technicalities. We also consider the case of DV channels, with the extension to CV channels left implicit and following the basic methodology of Sec. III. A quantum multiple-access channel is teleportation-covariant if, for any teleportation
unitaries, $U_{k_1}^1$ and $U_{k_2}^2$, we may write
\[
E \left[ (U_{k_1}^1 \otimes U_{k_2}^2) \rho (U_{k_1}^1 \otimes U_{k_2}^2) \right] = V_k E (\rho) V_k^\dagger,
\]
for some unitary $V_k$, with $k$ depending on both $k_1$ and $k_2$. If this is the case, then we can replace $E$ with teleportation over its Choi matrix, which is defined by sending halves of two EPR states through the channel, i.e.,
\[
\rho_E = I_{A^1 A^2} \otimes E_{A^1 A^2} ( \Phi_{A^1 A^1} \otimes \Phi_{A^2 A^2} ).
\]

See also Fig. 4 for further explanations.

FIG. 6: Simulation of a teleportation-covariant quantum multiple-access channel. We can replace the multiple-access channel $E : a^1 a^2 \rightarrow b$ (left) by double teleportation over its tripartite Choi matrix $\rho_E$ (right). This Choi matrix is obtained by sending halves ($A^1$ and $A^2$) of two EPR states $\Phi$ through $E$, with output $B$. Then, systems $a^1$ and $A^1$ are subject to a Bell detection with outcome $k_1$. Similarly, systems $a^2$ and $A^2$ are subject to a Bell detection with outcome $k_2$. The outcomes are CCed to Bob who applies a correction unitary on system $B$. Since the channel is teleportation-covariant, i.e., it commutes with the teleportation unitaries according to Eq. (53), Bob’s correction unitary $V_k^{-1}$ on $B$ re-generates the original channel $E : a^1 a^2 \rightarrow b$.

By using the channel simulation, we may fully simplify any adaptive protocol performed over a teleportation-covariant multiple-access channel $E$. In fact, each transmission through $E$ can be replaced by double teleportation on its Choi matrix $\rho_E$, with the Bell detections and Bob’s correction unitary being included in the LOCCs of the protocol. By stretching $n$ uses of the adaptive protocol (see Fig. 7), we find that the total output state of Alice 1, Alice 2 and Bob reads
\[
\rho_{a^1 a^2 b}^n = \tilde{\Lambda} \left( \rho_E^\otimes n \right).
\]

If we now trace one of the two senders, e.g., Alice 2, we still have an LOCC between Alice 1 and Bob. In other words, we may write the following
\[
\rho_{a^1 a^2 b}^n = \tilde{\Lambda} \left( a^1 a^2 \otimes \rho_E^\otimes n \right),
\]
where $\tilde{\Lambda} \left( a^1 a^2 \otimes \rho_E^\otimes n \right)$ is local with respect to the cut $a^1 a^2 b$.

For Alice 1 and Bob ($i = 1$), we can now write
\[
E_R (\rho_{a^1 b}^n) := \inf_{\sigma_n (a^1 | b)} S (\rho_{a^1 b}^n \| \sigma_n) \leq \inf_{\sigma_n (a^1 a^2 | b)} S (\rho_E^\otimes n \| \sigma_n) := E_R (\rho_E^\otimes n) \leq \inf_{\sigma_n (a^1 a^2 | b)} S (\rho_E^\otimes n \| \sigma_n) := E_R (\rho_E^\otimes n),
\]

FIG. 7: Teleportation stretching of an adaptive protocol implemented over a teleportation-covariant multiple-access channel (generic $m$th transmission shown on the left). After $n$ uses, we can express the output in terms of $n$ copies of the Choi matrix $\rho_E$ of the quantum multiple-access channel, subject to a trace-preserving LOCC $\Lambda$.

By applying the weak converse bound, we then derive
\[
C^1 \leq \sup_{E} \lim_{n} E_R (\rho_{a^1 b}^n) \leq E_R (\rho_E^\otimes n) \leq E_R (\rho_E^\otimes n),
\]
and using the subadditivity of the REE over tensor products, it is easy to show the single-letter version
\[
C^1 \leq E_R (\rho_E^\otimes n | b) (\rho_E) \leq E_R (\rho_E). \tag{59}
\]

Note that we find the same bound for the other capacity for Alice 2 and Bob ($i = 2$). The reasoning can be readily extended to arbitrary $M$ senders, so that the capacity between the $i$th Alice and Bob reads
\[
C^1 \leq E_R (\rho_{a^1 a^2 a^3 | b}) (\rho_E) \leq E_R (\rho_E):= \Phi (E), \tag{60}
\]
where $\Phi (E)$ is the entanglement flux of the quantum multiple-access channel. As previously mentioned, the result can be extended to CV systems by employing asymptotic simulations and extending the notions.

V. ALL-IN-ALL QUANTUM COMMUNICATION

In this section we extend our technique to a single-hop quantum network involving multiple ($M_A$) senders and multiple ($M_B$) receivers, which is also known as quantum interference channel. This is a CPTP map from Alcles’ input space $D (\otimes_{i=1}^{M_A} \mathcal{H}_i)$ to Bobs’ output space $D (\otimes_{j=1}^{M_B} \mathcal{H}_j)$. As a straightforward generalization of the previous cases, the most general adaptive protocol over this channel can be described as follows. At the initial stage the parties exploit a LOCC $\Lambda_0$ for their systems’ preparation. Then, each Alice picks the first system from her local ensemble $a_1^i \in A^i$. The inputs of all Alcles are sent to all Bobs through channel $E$ resulting into the outputs $\{b_j^i\}$, i.e.,
\[
a_1^1, \ldots, a_1^1, \ldots, a_1^{M_A} \rightarrow b_1^1, \ldots, b_j^i, \ldots, b_1^{M_B}. \tag{61}
\]
After this first transmission, there is another LOCC $\Lambda_1$, after which all Bobs’ ensembles are updated $b_i^1b_j^2 \rightarrow b_j^1$. Next, there is the second transmission $a^i \supseteq \{a_1^i\} \rightarrow \{a_2^i\}$ through $E$, followed by another LOCC $\Lambda_2$ and so on.

Thus, after $n$ uses of the channel, the $i$th Alice and the $j$th Bob share an output state $\rho^{a_1^i a_2^i b_1^j b_2^j}$, which is $\epsilon$-close to a target state of $nR_{ij}$ bits. By maximizing the asymptotic rate over all the adaptive LOCCs $\mathcal{E} = \{\Lambda_0, \Lambda_1, \ldots\}$ we can define the generic interference capacity for the $i$th Alice and the $j$th Bob as

$$C^{ij} := \sup \lim_{n} \sup_{\mathcal{E}} R_{ij}^n.$$  

(62)

As usual, depending on the task, one specifies different capacities assisted by unlimited two-way CCs: The entanglement distribution capacity ($D_2^{ij}$), the quantum capacity ($Q_2^{ij}$), the secret-key capacity ($K_{ij}$) and the private capacity ($P_{ij}$) of the quantum interference channel (with $D_2^{ij} = Q_2^{ij} \leq K^{ij} = P^{ij}$).

As for the case of the broadcast and the multiple-access channels we bound these capacities by using REE+teleportation stretching. We proceed by considering two senders and two receivers being the extension to arbitrary $M_A$ and $M_B$ just a matter of technicalities. The definition of a teleportation-covariance quantum interference channel relies once again on the commutation with teleportation, i.e., for any teleportation units $U_{k_1}$ and $U_{k_2}$ we must have

$$\mathcal{E} \left[ (U_{k_1} \otimes U_{k_2}) \rho (U_{k_1}^* \otimes U_{k_2}^*) \right] = \mathcal{V} \mathcal{E} (\rho) \mathcal{V}^\dagger,$$

(63)

where $\mathcal{V} = V_{l_1} \otimes V_{l_2}$ for unitaries $V_{l_1}$ and $V_{l_2}$, with both $l_1$ and $l_2$ depending on $k_1$ and $k_2$. If this condition holds then the channel can be simulated by teleportation over its Choi matrix, which is formally defined as in Eq. (54). See Fig. 8 for this simulation.

**FIG. 8:** Simulation of a teleportation-covariant quantum interference channel. The channel $\mathcal{E} : a^i a^2 \rightarrow b^1 b^2$ (left) can be simulated by its Choi matrix $\rho_\mathcal{E}$ (right). Systems $a^1$ and $A^1$ are subject to a Bell detection with outcome $k_1$. Similarly, systems $a^2$ and $A^2$ are subject to a Bell detection with outcome $k_2$. Both outcomes $k_1$ and $k_2$ are then classically communicated to Bob 1 and Bob 2 who apply two correction unitaries on $B^1$ and $B^2$. Since the channel is teleportation-covariant, i.e., it commutes with the teleportation unitaries according to Eq. (53), the two Bobs are able to recover the original channel $\mathcal{E} : a^1 a^2 \rightarrow b^1 b^2$ by applying correction unitaries $(V_{l_1}^*)^{-1}$ and $(V_{l_2}^*)^{-1}$.

Thus, an adaptive protocol can be simplified since each use of channel $\mathcal{E}$ can be replaced by teleportation and both the Bell detections and Bobs’ correction unitaries become part of the LOCCs. By stretching $n$ uses of the channel (see Fig. 9), we have the following output state shared between Alice 1, Alice 2, Bob 1 and Bob 2

$$\rho^{a_1^i a_2^i b_1^j b_2^j} = \bar{\Lambda} (\rho^{E^n}).$$

(64)

By tracing over one sender and one receiver, say Alice 2 and Bob 2, we then derive

$$\rho^{a_1^i b_1^j} = \bar{\Lambda}^{a_1^i a_2^i | b_1^j b_2^j} (\rho_\mathcal{E}^{\otimes n}),$$

(65)

where $\bar{\Lambda}^{a_1^i a_2^i | b_1^j b_2^j}$ is a trace-preserving LOCC between Alice 1 and Bob 1, local with respect to the cut $a^1 a^2 | b^1 b^2$.

**FIG. 9:** Teleportation stretching of an adaptive protocol over a quantum interference channel (generic $m$th transmission shown on the left). After $n$ uses, we can express the output in terms of $n$ copies of the Choi matrix $\rho_\mathcal{E}$ of the quantum interference channel, subject to a trace-preserving LOCC $\bar{\Lambda}$.

It follows that the capacity for Alice 1 and Bob 1 ($i = j = 1$) is upper bounded by

$$C_{11} \leq \sup_{\mathcal{E}} \lim_{n} \frac{E_R (\rho^{a_1^i b_1^j})}{n} \leq E_{R} (\rho_\mathcal{E}^{\otimes n}) \leq E_{R}^\infty (\rho_\mathcal{E}).$$

(66)

In terms of single-letter bounds we find

$$C_{11} \leq E_{R} (\rho^{a_1^i a_2^i | b_1^j b_2^j}) (\rho_\mathcal{E}) \leq E_R (\rho_\mathcal{E}).$$

(67)

Clearly, we find the same result in all other cases, i.e., for any sender-receiver pair $(i, j)$. In general, for arbitrary $M_A$ senders and $M_B$ receivers, we may write

$$C^{ij} \leq E_R (\rho^{a_1^i \cdots a_{M_A}^i | b_1^j \cdots b_{M_B}^j}) (\rho_\mathcal{E}) \leq E_R (\rho_\mathcal{E}) := \Phi (\mathcal{E}),$$

(68)

where $\Phi (\mathcal{E})$ is the entanglement flux of the quantum interference channel. The extension to CV systems exploits asymptotic simulations along the lines of Sec. III.

**VI. CONCLUSIONS**

In this work we have studied the capacities for quantum communication, entanglement distribution and secret key generation in a single-hop quantum network, involving a direct channel between multiple senders and/or multiple receivers. More precisely, we have considered the quantum broadcast channel (point-to-multipoint), the multiple-access channel
(multipoint-to-point), and the quantum interference channel (multipoint-to-multipoint), assuming that all the parties may apply the most general local operations assisted by unlimited two-way CCs (adaptive protocols).

By suitably extending the methodology of Ref. [31], which suitably combines the relative entropy of entanglement (REE) with teleportation stretching, we have reduced the most general adaptive protocols implemented on these multipoint channels to the computation of a one-shot quantity and, in particular, their entanglement flux (i.e., the REE of their Choi matrix). This is achieved at any dimension, i.e., finite or infinite (CV channels).

Further research should be directed to show how a rigorous application of our reduction method can be used to upperbound the entire capacity regions of multipoint channels, defined as the convex closure of the set of all the rates which are achievable by the parties assisted by unlimited two-way CCs. Other important directions are related with the study of multipoint quantum communication within a multi-hop quantum network, following the general methods and results of Ref. [67].

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