THERMAL PHASE TRANSITION IN WEAKLY INTERACTING, LARGE $N_C$ QCD

Joakim Hallin
Institute of Theoretical Physics
Chalmers University of Technology and Göteborg University
S-412 96 Göteborg, Sweden

David Persson
Department of Physics and Astronomy
University of British Columbia
Vancouver, B.C. V6T 1Z1, Canada

Abstract

We consider thermal QCD in the large $N_C$ limit, mainly in 1+1 dimensions. The gauge coupling is only taken into account to minimal order, by projection onto colour singlets. An expression for the free energy, exact as $N_C \to \infty$, is then obtained. A third order phase transition will occur. The critical temperature depends on the ratio $N_C/L$, where $L$ is the (infinite) spatial length. In the high temperature limit, the free energy will approach the same value as in the free theory, whereas we have a mesonic like phase at low temperature. Expressions for the quark condensate, $\langle \bar{\Psi} \Psi \rangle$, are also obtained.

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1Email address: tfejh@fy.chalmers.se
2Email address: persson@theory.physics.ubc.ca
Considerable interest has recently been devoted to the assumed deconfinement phase transition in QCD at high temperature and/or density. Exact analytical results are incomprehensible so far, and we have to rely on approximate methods or computer simulations on the lattice. It has been well known for a long time that some insight in the confinement mechanism may be gained by considering the large $N_C$ limit, for an $SU(N_C)$ gauge theory. This limit is particularly fruitful in 1+1 dimensions (see e.g. Refs. [1, 2, 3, 4, 5]). QCD in 1+1 dimensions has also received attention recently due to the remarkable fact that different regularizations seem to lead to different models, even within the same choice of gauge. In his pioneering work, 't Hooft [1] employed a cutoff around the origin in momentum space, that later was shown to be equivalent to a principle value prescription [2]. Within this framework, no free quarks can propagate, but they are confined into mesons, consisting of quark anti-quark pairs. Wu [3], on the other hand, suggested another regularization, allowing for a Wick rotation to Euclidean space. The bound state equation is more complicated in this case, and no solution has yet been found. Bassetto and Nardelli with co-workers [4, 5, 6, 7] have compared the two regularizations, and computed for example the expectation value of the Wilson loop. The expectation value of the Wilson loop with Wu’s regularization, was recently calculated by Staudacher and Krauth [10], who showed that confinement is not enforced in this case, unlike using the 't Hooft regularization. It has been suggested by Chibisov and Zhitnitsky [11] that the models due to different regularizations could show up as different phases, that makes a study of the system at finite temperature intriguing. In 2 dimensional QCD, confinement is obvious as an infinite energy for coloured states.

We shall in this letter take the interaction into account only by a projection onto colour singlets, i.e. we neglect the coupling unless it multiplies the infra red divergence. The quark contribution to the free energy in 3+1 dimensions is then easily obtained from the free energy presented here in 1+1 dimensions through the substitution, $L \int dp/(2\pi) \rightarrow 2L \int d^3p/(2\pi)^3$. The projected free energy in 3+1 dimensions has earlier been considered in the high temperature limit by Skagerstam [12]. The full interacting case in 2 dimensions, that is troubled by infra red divergences, will be more extensively considered in another article. Including the interaction in the strict large $N_C$ limit $L/N_C \rightarrow 0$, McLerran and Sen [13], and employing a different method Hansson and Zahed [14] showed that in the low temperature 't Hooft me-
son phase, the partition function \( \ln Z = \mathcal{O}(L) \), due to confinement. Using a different resummation scheme for \( L/N_C \to \infty \), McLerran and Sen [13] also found another phase of almost free quarks, with \( \ln Z = \mathcal{O}(LN_C) \). The phase transition itself was not obtainable, and could only take place as \( T \to \infty \), i.e. \( T \propto N_C \). By considering finite \( \kappa = N_C/L \), we are here able to monitor the phase transition exactly, and for finite \( \kappa \), it will take place at finite \( T \). However, due to the negligence of the interaction we do not have confinement in the strict sense. Our low temperature phase consists of quark—anti-quark pairs, showing up for example in Bose–Einstein distribution functions, but these are not the confined mesons of 't Hooft’s, since \( \ln Z = \mathcal{O}(LN_C) \).

Let us now consider the unnormalized density matrix \( \hat{\rho} = e^{-\beta \hat{H}} \) of free coloured fermions having the Hamiltonian

\[
\hat{H} = \int_p \hat{\psi}^\dagger(p)h(p)\hat{\psi}(p),
\]

where

\[
h(p) = \gamma^0(\gamma_1 p + m),
\]

and \( f_p = \int \frac{dp}{(2\pi)^n} \), in \( n \) spatial dimensions. We shall here mainly discuss \( n = 1 \). In the functional representation the expression for \( \hat{\rho} \) is [13]

\[
\rho(\eta^*_1\eta_1,\eta^*_2\eta_2) = N_\beta \exp(\eta^*_1 + \eta^*_2)\Omega_\beta(\eta_1 + \eta_2) + \eta^*_1\eta_2 - \eta^*_2\eta_1),
\]

where

\[
N_\beta \quad = \quad \exp(4N_CV\int_p \ln(\cosh \frac{\beta}{2}\omega))\],
\]

\[
\Omega_\beta \quad = \quad -\tanh\frac{\beta}{2}\omega\frac{h}{\omega}1_C,
\]

\[
\omega(p) \quad = \quad \sqrt{p^2 + m^2},
\]

and \( L \) denotes the spatial length. In the expression for \( \rho(\eta^*_1\eta_1,\eta^*_2\eta_2) \), summation over spinor, colour and momentum indices is understood. Moreover \( N_C \) denotes the number of colours and \( 1_C \) is the unit matrix in colour space. We now wish to calculate the partition function for colourless states. This is done by the operator \( \hat{\pi} \),

\[
\hat{\pi} = \int du \hat{\pi}_u,
\]
that projects onto colour singlets. Here $\hat{\pi}_u = e^{iaQ^a}$ is a general global $U(N_C)$
gauge transformation. The partition function $Z$ is then

$$Z = \text{tr}(\hat{\pi}\hat{\rho}).$$

(8)

With the colour charge operator

$$\hat{Q}^a = \int_p \hat{\psi}^+ t^a \hat{\psi},$$

(9)

the global gauge transformation $\hat{\pi}_u$ is in the functional representation:

$$\pi_u(\eta^1 \eta_1, \eta^2 \eta_2) = N_u \exp \left[ (\eta^*_1 + \eta^*_2) \Omega_u (\eta_1 + \eta_2) + \eta^*_1 \eta_2 - \eta^*_2 \eta_1 \right].$$

(10)

Here we have defined

$$N_u = \exp \left[ \frac{1}{2} \text{tr} \ln \frac{1}{4}(u + u^{-1} + 2) \right],$$

(11)

and

$$\Omega_u = \frac{u - 1}{u + 1} 1_s.$$ (12)

Furthermore, $\text{tr} = L \int_p \text{tr}_s \text{tr}_C$ denotes the trace over all indices, momentum, spin and colour, while $1_s$ is the unit matrix in spinor space i.e. a $2 \times 2$-unit matrix ($4 \times 4$ in $n = 3$). The group element is $u = e^{iaQ^a}$. Multiplying $\hat{\rho}$ and $\hat{\pi}_u$ one finds

$$\langle \eta^*_1 \eta_1 | \hat{u} \hat{\rho} | \eta^*_2 \eta_2 \rangle = N_\beta N_u \det(1 + \Omega_\beta \Omega_u) \times$$

$$\exp \left\{ (\eta^*_1 + \eta^*_2) \frac{\Omega_\beta + \Omega_u}{1 + \Omega_\beta \Omega_u} (\eta_1 + \eta_2) + \eta^*_1 \eta_2 - \eta^*_2 \eta_1 \right\}. $$

(13)

Finally we can then give the expression for the partition function,

$$Z = \int du \text{tr}(\hat{\pi}_u \hat{\rho}) = \int du \det(2) N_\beta N_u \det(1 + \Omega_\beta \Omega_u)$$

$$= e^{-\beta E_0} \int du \exp \left\{ L \int_p \text{tr}_C \ln(1 + \xi u)(1 + \xi u^{-1}) \right\},$$

(14)

where $E_0 = -L N_C \int_p \omega$ is the vacuum energy. We have here defined

$$\xi = e^{-\beta \omega},$$

(15)
and $e^{i\alpha_j}$ denotes the $N_C$ eigenvalues of the $U(N_C)$ matrix $u$. On functions of eigenvalues the Haar measure $du$ takes the form

$$\int du = \prod_{i=1}^{N_C} d\alpha_i \prod_{i<j} \sin^2 \frac{\alpha_i - \alpha_j}{2}.$$  \hspace{1cm} (16)

This gives

$$Z = e^{-\beta E_0} \prod_{i=1}^{N_C} d\alpha_i \prod_{i<j} \sin^2 \frac{\alpha_i - \alpha_j}{2} \times$$

$$\exp \left[ L \int \sum_{p=j=1}^{N_C} \ln(1 + \xi e^{i\alpha_j})(1 + \xi e^{-i\alpha_j}) \right].$$ \hspace{1cm} (17)

In the large $N_C$-limit the integral over $U(N_C)$ in (17) may be calculated by the steepest descent method. We take this limit by letting $N_C$ and the spatial length $L$ tend to infinity simultaneously keeping their quotient constant. We solve this by introducing the density of eigenstates, $\rho$, in the continuum limit [16, 17], i.e.

$$\sum_{j=1}^{N_C} f(\alpha_j) \to N_C \int_0^1 dt f[\alpha(t)] = N_C \int_{-\alpha_c}^{\alpha_c} \rho(\alpha) d\alpha f(\alpha).$$ \hspace{1cm} (19)

We find two distinct phases, depending on the inequality

$$1 - \frac{2}{\kappa} \int \frac{1}{e^{\beta \omega} - 1} \geq 0.$$ \hspace{1cm} (20)

As long as (20) is satisfied we are in the zero-gap phase, $\alpha_c = \pi$, where $\rho$ has support over the whole circle. When (20) is not satisfied we are in the one-gap phase. In this case there is a gap in which $\rho$ lacks support, i.e. an interval where the eigenvalues give a vanishing contribution as $N_C \to \infty$.

In the zero-gap phase $\ln Z$ is given by a sum over quark—antiquark pairs

$$\ln Z = -\beta E_0 - \frac{N_C^2}{\kappa^2} \int_p \int_q \ln [1 - \xi(p)\xi(q)].$$ \hspace{1cm} (21)
Even though the states are mesonic-like, we do not have confinement since
\( \ln Z \propto (N_C/\kappa)^2 = N_C L/\kappa \), whereas in the confined phase of 't Hooft mesons
we have (cf. [14]) \( \ln Z \propto L \). In the one-gap phase we find

\[
\ln Z = -\beta E_0 + N_C^3 \ln \left( \frac{\sin \alpha_c}{2} \right) + \frac{2N_C^2}{\kappa} \int_p \ln \frac{1}{2} (1 + \xi + \sqrt{\xi^2 + 2\xi \cos \alpha_c + 1})
- \frac{N_C^2}{\kappa^2} \int_p \int_q \ln [1 - x(p)x(q)],
\]

(22)

where we have defined

\[
x = \frac{1}{4\xi \sin^2 \frac{\alpha_c}{2}} \left[ 1 + \xi - \sqrt{\xi^2 + 2\xi \cos \alpha_c + 1} \right]^2.
\]

(23)

Furthermore, the critical angle \( \alpha_c \) is determined from

\[
\int_p \left( \frac{1 + \xi}{\sqrt{\xi^2 + 2\xi \cos \alpha_c + 1}} - 1 \right) = \kappa.
\]

(24)

In Figure 1, we plot \( \ln Z/(TN_C^2) \) as a function of the temperature, for \( \kappa = m \). The critical temperature is then found as \( T_c \approx 0.8m \). The free energy \( (F = -T \ln Z) \) in the 1-gap phase (solid line) grows as \( T^2 \) for large \( T \), and is approaching the free energy for free quarks, i.e. without the projection, (dotted line), as \( T \to \infty \). Whereas the free energy in the 0-gap phase (dashed line), continued above the phase transition would grow like \( T^3 \). The high temperature behaviour in the 1-gap phase is obtained by an expansion for \( \alpha_c \ll 1 \), yielding \( \alpha_c \approx 4\pi \kappa/T \). As \( T \to \infty \), \( \alpha_c \to 0 \), i.e. only \( u = 1 \) gives a contribution, corresponding to no projection.

In addition to the free energy, it is of interest to compare the entropy, and the (renormalized) energy density in the two phases. They are given by

\[
s = \frac{1}{L} \frac{\partial}{\partial T} (T \ln Z)_L,
\]

(25)

\[
\epsilon = \frac{T^2}{L} \frac{\partial}{\partial T} (\ln Z + \beta E_0)_L,
\]

(26)

respectively. Their behaviour is similar to the that of the free energy. In the interacting case it is well known that a nontrivial chiral condensate appears, as shown for \( m = 0 \) by Zhitnitsky [15, 16], and in the general case
by Burkardt \[20\]. However, when the interaction only is taken into account through the projection, the vacuum is trivial and no chiral condensate can appear in this case. Nevertheless, the (renormalized) expectation value of \( \bar{\psi}\psi \), is an order parameter for the phase transition, and thus of interest. We find

\[
\langle \bar{\psi}\psi \rangle = -T \frac{\partial}{\partial m} \left[ \ln Z + \beta E_0 \right]_L.
\]  

(27)

In the 0-gap phase it is straightforward to evaluate this from (21). However, in the 1-gap phase we need to define

\[
\cot \frac{\alpha_c}{2} T \frac{\partial}{\partial m} \alpha_c \equiv ma,
\]

(28)

where

\[
a = \left\{ \int_p \frac{\xi(1 + \xi)}{(\xi^2 + 2\xi \cos \alpha_c + 1)^{3/2}} \right\}^{-1} \times \left\{ \int_p \frac{\xi(1 - \xi)}{(\xi^2 + 2\xi \cos \alpha_c + 1)^{3/2}} \right\},
\]

(29)

is obtained from (24), using (13), and (9). We then find

\[
\frac{T}{m} \frac{\partial}{\partial m} x = \frac{x}{\sqrt{\xi^2 + 2\xi \cos \alpha_c + 1}} \left[ (1 + \xi)a - (1 - \xi)\frac{1}{\omega} \right].
\]

(30)

This gives

\[
\langle \bar{\psi}\psi \rangle = N_C^2 m \left\{ -\frac{1}{2} a + \frac{2}{\kappa} \int_p \frac{1}{1 + \xi + \sqrt{\xi^2 + 2\xi \cos \alpha_c + 1}} \times \right\}
\]
\[
\left(1 + \frac{\xi + \cos \alpha_c}{\sqrt{\xi^2 + 2\xi \cos \alpha_c + 1}}\right) \frac{\xi}{\omega} + \frac{2\xi \sin^2 \frac{\alpha_c}{2}}{\sqrt{\xi^2 + 2\xi \cos \alpha_c + 1}} - \frac{2}{\kappa^2} \int_p \int_q \frac{x(p)x(q)}{1 - x(p)x(q)} \left(1 + \xi \right) \frac{\Lambda}{\sqrt{\xi^2 + 2\xi \cos \alpha_c + 1}} \right].
\]

(31)

We have been able to show that \(\langle \bar{\psi} \psi \rangle\) is continuous over the phase transition, whereas higher derivatives give cumbersome expressions.

However, in the limit of infinite massive quarks analytical results are obtained. As \(m \to \infty\), we must also let \(T_c \to \infty\), such that their quotient

\[
\zeta = \frac{m}{T_c},
\]

(32)
is kept fixed, in order to keep a finite particle density. Furthermore, we define \(\tau\) according to \(T = T_c \tau\). The momentum integrals are now independent of \(p\), so we write

\[
\int_p \to \Lambda.
\]

(33)
The critical temperature is then determined from

\[
\frac{\kappa}{2} = \int_p \frac{1}{e^{\omega/T_c} - 1} \to \Lambda \frac{1}{e^{\zeta/\tau} - 1}.
\]

(34)

In the 0-gap phase we then find

\[
\langle \bar{\psi} \psi \rangle = 2N_c^2 \left(\frac{\Lambda}{\kappa}\right)^2 \frac{1}{e^{2\zeta/\tau} - 1}.
\]

(35)

In the 1-gap phase the critical angle is now determined from

\[
\sqrt{1 + 2e^{\zeta/\tau} \cos \alpha_c + e^{2\zeta/\tau}} = \frac{e^{\zeta/\tau} + 1}{1 + \kappa/\Lambda},
\]

(36)

that after some simplifications give

\[
\langle \bar{\psi} \psi \rangle = N_c^2 \left\{\frac{1}{2} + \left(\frac{\Lambda}{\kappa}\right) \left(2 + \frac{\kappa}{\Lambda}\right) \frac{1}{e^{\zeta/\tau} + 1}\right\}.
\]

(37)

Notice here the Bose–Einstein distribution in the low temperature 0-gap phase, as opposed to the Fermi–Dirac distribution in the 1-gap phase at high temperature. It is now a straightforward exercise to show that \(\partial^2 / \partial t^2 \langle \bar{\psi} \psi \rangle\) is discontinuous, so that it is a third order phase transition. This is in agreement
with the study of Gattringer et al. [21] concerning static quarks on a line. Furthermore, in the corresponding lattice model Gross and Witten [11] found a third order phase transition, so we conjecture that the phase transition most likely is third order also for arbitrary mass.

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