ON THE G-INVARIANT MODULES

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ABSTRACT. Let $G$ be a reductive group acting on a path algebra $kQ$ as automorphisms. We assume that $G$ admits a graded polynomial representation theory, and the action is polynomial. We describe the quiver $Q_G$ of the smash product algebra $kQ#k[M_G]^*$, where $M_G$ is the associated algebraic monoid of $G$. We use $Q_G$-representations to construct $G$-invariant representations of $Q$. As an application, we construct algebraic semi-invariants on the quiver representation spaces from those $G$-invariant representations.

INTRODUCTION

Let $k$ be a field of characteristic 0, and $A$ be a finite-dimensional $k$-algebra with a finite group $G$ acting as automorphisms. Then we can form the skew group algebra $AG := A#k[G]$, which is a well-studied subject (e.g., [13]). $AG$ and $A$ have the same representation type and global dimension. If the algebra is the path algebra of a finite quiver $Q$, and the action permutes primitive idempotents and stabilizes the arrow span $kQ_1$, then the quiver $Q_G$ of $kQG$ can be explicitly described [1, 9] (see Section 1.1).

A natural question is that if $G$ is a reductive group acting rationally on $A$ as automorphisms, what is a good analogue of the skew group algebra? One natural answer can be replacing the group algebra by the Hopf algebra $k[M_G]^*$, and forming the smash product $A#k[M_G]^*$. However, the dual coordinate algebra $k[M_G]^*$ is not semisimple, and quite complicated in general. To describe the quiver of $kQ#k[M_G]^*$ is a rather difficult task. So we consider the coordinate (bi)algebra of the associated monoid $k[M_G]$ as an alternative. If $G$ admits a graded polynomial representation theory (Definition 2.1), then $k[M_G]^*$ is semisimple. So the price is that we need to restrict to a special class of reductive groups and require the action to be polynomial. Then we can explicitly describe the quiver $Q_G$ of $kQ[M_G]^*$ := $kQ#k[M_G]^*$. The quiver is possibly an infinite quiver, but each connected component is still finite-dimensional (Proposition 3.3). Theorem 3.3 is our first main result. The proof is similar to that in [1].

Let us come back to the finite group action. The action of $G$ on $A$ induces an action of $G$ on the category of (left) $A$-modules. We write this induced action in the exponential form, that is, $M^g$ is the module $M$ with the action of $A$ twisted by $g$:

$$am = (g^{-1}a)m.$$ 

An $A$-module $M$ is called $G$-invariant if $M^g \cong M$ for any $g \in G$. The restriction of an $AG$-module $M$ is a $G$-invariant $A$-module. The converse is almost true

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(Lemma 1.2). Those \( kQ \)-modules admitting a \( kQG \)-module structure are our main interest. In fact, we only need something weaker called \textit{proj-coherently} \( G \)-invariant (Definition 1.4). They contain all exceptional representations of \( Q \) (Observation 1.4). To construct such \( kQ \)-modules, we need to concretely describe the Morita equivalence functor \( kQG \text{-mod} \to kQG \text{-mod} \) composed with the restriction functor \( kQG \text{-mod} \to kQ \text{-mod} \). This can be done as long as we can compute all idempotents of the group algebra \( k[G] \) (see Section 1.2).

All above about finite group actions have analogue for our \( k[M_G]^* \). However, in this case \( Q_G \) is possibly an infinite quiver, so it is quite impossible to completely describe the above functor. So we fix some connected component \( Q_c \) of \( Q_G \), then we can describe the analogous functor \( kQ_c \text{-mod} \to kQ \text{-mod} \), provided we can compute all idempotents of some homogenous subalgebra \( S_c \) of \( k[M_G]^* \) depending on \( Q_c \). Such subalgebra \( S_c \) is a finite direct product of \textit{Schur algebras} of \( G \).

Our motivation comes from constructing algebraic semi-invariants on the quiver representation spaces. For some dimension vector \( \alpha \), let \( \text{Rep}_\alpha(Q) \) be the space of all \( \alpha \)-dimensional representations of \( Q \). The product of general linear group \( \text{GL}_\alpha := \prod_{\alpha \in Q_0} \text{GL}_\alpha(v) \) acts on \( \text{Rep}_\alpha(Q) \) by the natural base change. In [14], Schofield introduced for each representation \( N \in \text{Rep}_\beta(Q) \) with \( \langle \alpha, \beta \rangle_Q = 0 \), a semi-invariant function \( c_N \in k[\text{Rep}_\alpha(Q)] \) for the above action. Here \( \langle -, - \rangle_Q \) is the Euler form of \( Q \). In fact, \( c_N \) ’s span the space of all semi-invariants of weight \( -\langle -, \beta \rangle_Q \) over the base field \( k \).

The action of \( G \) on \( kQ \) induces \( G \)-actions on all representation spaces of \( Q \). An easy observation is that if \( N \) is \textit{proj-coherently} \( G \)-invariant, then \( c_N \) is also \( G \)-semi-invariant under \( G \)-action. This observation allows us to construct new semi-invariants for the \( \text{GL}_\alpha \times G \)-action on \( k[\text{Rep}_\alpha(Q)] \). We are particularly interested in the setting of \( n \)-arrow Kronecker quivers \( K_n \), where \( G = \text{GL}_\alpha \) acting on the space of arrows. The \( (\alpha_1, \alpha_2) \)-dimensional representation space of \( K_n \) can be identified with the (tri-)tensor space \( U^* \otimes V \otimes W^* \), where \( \dim(U, V, W) = (\alpha_1, \alpha_2, n) \). To illustrate our method, we construct several such semi-invariants in Proposition 4.2 and 4.3. Proposition 4.2 may be well-known, but we believe that the rest are new.

We hope to find the dimension of the linear span of semi-invariants of form \( c_N \), where \( N \) is coherently \( G \)-invariant of fixed dimension. Theorem 4.7 converts this problem to a similar problem on the quiver \( Q_c \). As we will see, when \( Q_c \) is simple, the dimension can be easily calculated.

\textbf{Notations and Conventions.} Our vectors are exclusively row vectors. If an arrow of a quiver is denoted by a lowercase letter, then we use the same capital letter for its linear map of a representation. For direct sum of \( n \) copy of \( M \), we write \( nM \) instead of the traditional \( M^\oplus n \). Unadorned Hom and \( \otimes \) are all over the base field \( k \), and the superscript \( * \) is the trivial dual.

\textbf{1. Finite Group Action}

Let \( k \) be a field of characteristic 0, and \( G \) be a finite group acting on a \( k \)-algebra \( A \) as automorphisms. The group algebra \( k[G] \) is a Hopf algebra with counit, comultiplication, and antipode defined by the linear extension of

\[\epsilon(g) = 1, \quad \Delta(g) = g \otimes g, \quad S(g) = g^{-1}.\]

In this way, \( A \) obtains a \( k[G] \)-module algebra structure.
Definition 1.1. Let $B$ be a bialgebra. A (left) $B$-module algebra $A$ is an algebra which is a (left) module over $B$ such that for any $b \in B$, $a, a' \in A$,

$$b1_A = e(b)1_A, \quad \text{and} \quad b \cdot (aa') = \sum(b(0) \cdot a)(b(1) \cdot a').$$

The smash product algebra $A \# B$ is the vector space $A \otimes B$ with the product

$$(a \otimes b)(a' \otimes b') := \sum a(b(0) \cdot a') \otimes b(1)b'.$$

When $B = k[G]$ is a group algebra, we may abuse of notation writing $a$ for $a \otimes 1_G$ and $b$ for $1_A \otimes b$. In this context, $a \otimes b$ can be written as $ab$, and thus $A \# k[G]$ may be denoted by $AG$.

The action of $G$ on $A$ induces an action of $G$ on the category of (left) $A$-modules. We write this induced action in the exponential form, that is, $M^g$ is the module $M$ with the action of $A$ twisted by $g$:

$$am = (g^{-1}a)m.$$  

For morphisms $f \in \text{Hom}_A(M, N)$, we check that the following defines a morphism $f^g \in \text{Hom}_A(M^g, N^g)$

$$f^g(m) = f(m).$$

If $M$ is an $A$-module then $(AG) \otimes_A M$ is isomorphic as an $A$-module to $\bigoplus_{g \in G} M^g$, where the action of $G$ permutes the factors.

We observe that an $AG$-module $M$ is an $A$-module which is also a $G$-module, and such that

$$(1.1) \quad g(am) = (ga)(gm).$$

Lemma 1.2. An $A$-module $M$ is an $AG$-module if and only if there is a family of isomorphisms $\{i_g : M \to M^g\}_{g \in G}$ satisfying $i_h^g i_g = i_{hg}$ for any $g, h \in G$.

Proof. If $M$ is an $AG$-module, then $i_g$ says that the assignment $m \mapsto g^{-1}m$ defines an isomorphism $i_g : M \to M^g$. Conversely, if we have a family of isomorphisms $\{i_g : M \to M^g\}_{g \in G}$ satisfying $i_h^g i_g = i_{hg}$ for any $g, h \in G$, then we can endow $M$ with a $G$-module structure as follows. Note that $M$ and $M^g$ have the same underlying vector spaces on which $i^g_g$ acts and such that $i^g_g(m) = m$. We define $G$-action on $M$ satisfying $i_g$ by $g(m) = i_g(m)$.

Definition 1.3. An $A$-module $M$ is called $G$-invariant if $M^g \cong M$ for any $g \in G$. It is called proj-coherently $G$-invariant if there is a family of isomorphisms $\{i_g : M \to M^g\}_{g \in G}$ satisfying $\forall g, h \in G, \exists c \in k^*$ such that $i_h^g i_g = c \cdot i_{hg}$. It is called coherently $G$-invariant if it admits a $AG$-module structure. A $G$-invariant $A$-module is called (coherently) $G$-indecomposable if it is not a direct sum of two (coherently) $G$-invariant modules.

For our main application on invariant theory, we are more interested in (proj-)coherently $G$-invariant modules. However, we do not know any example where $M$ is $G$-invariant but not coherently $G$-invariant. When $G$ is cyclic and $A$ a path algebra, Gabriel [5] proved that they are equivalent.

Observation 1.4. Let $A = kQ$ be the path algebra of a finite quiver $Q$.

(1) A rigid representation of $Q$ is $G$-invariant.

(2) A $G$-invariant Schur representation of $Q$ is proj-coherently $G$-invariant.
(3) If the cohomology group $H^2(G; k^*)$ vanishes, then proj-coherent is equivalent to coherent.

Proof. By definition $M$ is rigid if $\text{Ext}^1_{kQ}(M, M) = 0$. So the orbit of $M$ is dense in its representation space, which is irreducible. But $M^p$ is rigid as well, so they have to be in the same orbit.

By definition, $M$ is Schur if $\text{Hom}_{Q}(M, M) = k$. So the statement follows from the definition.

If $H^2(G; k^*) = 0$, then every projective representation $G \to \text{GL}_{\alpha}/k^*$ lifts to $G \to \text{GL}_{\alpha}$. So we can modify each $i_g$ by some scalar factor such that $i_0^2 i_g = i_{hg}$. □

Definition 1.5. A dimension vector $\alpha$ of $Q$ is called a $G$-root if there is an $\alpha$-dimensional coherently $G$-indecomposable representation. It is called a strong $G$-root if there is an indecomposable coherently $G$-invariant representation.

When $G$ is cyclic, all $G$-roots can be described in terms of the root system of associated valued quiver [7]. The following lemma is well-known.

Lemma 1.6. For any finite-dimensional algebra $A$, $AG$ and $A$ have the same global dimension and representation type.

1.1. Description for $Q_G$. By Lemma 1.6 $kQG$ is Morita equivalent to some hereditary algebra $kQ_G$. There are algorithms to find the quiver $Q_G$ if the action permutes primitive idempotents and stabilizes the arrow span $kQ_1$. Let us recall the methods in [11][9].

Let $\tilde{Q}_0$ be a set of representatives of class of $Q_0$ under the action of $G$. For $u \in Q_0$, let $O_u$ be the orbit of $u$ and $G_u$ be the subgroup of $G$ stabilizing $e_u$.

For $(u, v) \in Q_0 \times \tilde{Q}_0$, $G$ acts diagonally on the product of the orbits $O_u \times O_v$. A set of representatives of the classes of this action will be denoted by $O_{uv}$. We define $R_{uv} := kQ(u, v)$ to be the vector space spanned by the arrows from $u$ to $v$. We regard $R_{uv}$ as a right $k[G_{uv}] := k[G_u \cap G_v]$-module by restricting the action of $G$.

Let $\text{irr}(G)$ denote the set of all irreducible representations of $G$. The vertex set of $Q_G$ is

$$\bigcup_{v \in \tilde{Q}_0} \{u\} \times \text{irr}(G_u).$$

The arrow set from $(u, \rho)$ to $(v, \sigma)$ is a basis of

$$\bigoplus_{(u', v') \in O_{uv}} \text{Hom}_{k[G_{u', v'}]}(V_\rho, R_{u'v'} \otimes V_\sigma).$$

Here $\rho$ should be understood as a representation of $G_{u'}$ as follows. Let $g_{u'u}$ be such that $g_{u'u} u = u'$, then $\rho(h) = \rho(g_{u'u}^{-1} h g_{u'u})$ for $h \in G_{u'}$. Similar identification makes $\sigma$ a representation of $G_{v'}$.

The proof uses the following idempotent $e$ of $kQG$, which will be used later. Let $R$ be the maximal semisimple subalgebra of $kQ$. Let $e_0 = \sum_{u \in \tilde{Q}_0} e_u \in R \subset RG$. It is not hard to see that $e_0(kQG)e_0$ is Morita equivalent to $kQG$, and $e_0(RG)e_0 \cong \prod_{u \in \tilde{Q}_0} k[G_u]$. Since each $G_u$ is semi-simple, we can fix for each $u \in \tilde{Q}_0$ and $\rho \in \text{irr}(G_u)$, a primitive idempotent $e_{u\rho}$ of $k[G_u]$ corresponding to $\rho$. Let

$$e = \sum_{u \in \tilde{Q}_0} \sum_{\rho \in \text{irr}(G_u)} e_{u\rho}.$$
It is proved in [1] that $e(kQG)e$ is a basic algebra Morita equivalent to $kQG$.

1.2. Functors. Let $A := kQ$ and $B := kQ$. The functor $AG \otimes_A -$ has the restriction functor as its right adjoint. The Morita equivalence functor $e(-)$ has \( R_\varepsilon := \text{Hom}_B(eAG, -) \) as its right adjoint. So the composition $T := e(AG \otimes_A -$) has a right adjoint $R := \text{res} \circ R_\varepsilon$. Note that $T$ is exact and preserves projective presentations, and thus $R$ preserves injective presentations. Moreover, both $T$ and $R$ map semisimple modules to semisimple modules [13, Theorem 1.3].

The functor $AG \otimes_A -$ is also right adjoint to the restriction functor [13, Theorem 1.2]. So $T$ also has a left adjoint $L := \text{res} \circ AGe \otimes_B -$. However, in this notes we will exclusively work with the functor $R$.

Now we have the following diagram of functors:

$$
\begin{array}{ccc}
\text{mod } A^\# & \xrightarrow{e(-)} & \text{mod } B \\
\downarrow \text{res} & & \uparrow R_\varepsilon \\
\text{mod } A & \xrightarrow{R} & \text{mod } B
\end{array}
$$

By our construction, the functor $R$ sends the simple $S_{u\rho}$ corresponding to the vertex $e_{u\rho}$ to the semisimple representation $\bigoplus_{v \in O_u} \dim(V_p)S_v$ of $Q$. In this way, $R$ induces a linear map $r : K_0(B) \to K_0(A)$. Since $R_\varepsilon$ is an equivalence and preserves indecomposables, it follows that

**Proposition 1.7.** $\alpha$ is a $G$-root if and only if there is a root $\beta$ of $Q_G$ such that $r(\beta) = \alpha$.

We want to give a concrete description for the functor $R$. To be more precise, we want to lift $R$ to a map between representation spaces of $Q_G$ and $Q$. Clearly, such a description relies on the choice of a complete set of primitive orthogonal idempotents of $k[G_u]$ for each $u \in \tilde{Q}_0$. In general, no explicit formula for primitive orthogonal idempotents in a finite group algebra is known. However, in many special cases, for example when the group is a symmetric group, a complete set of primitive orthogonal idempotents is given by the Young symmetrizers [12, 9.3].

Assume that we have got a complete set $I$ of primitive orthogonal idempotents of $k[G_u]$ for each $u \in \tilde{Q}_0$. By Maschke’s Theorem, $k[G_u]$ is a product of matrix algebras $\prod_{\rho \in \text{irr}(G)} \text{End}(V_{\rho})$. We can compute a standard basis $\{e_{u\rho}^{ij}\}$ of the matrix algebra $\text{End}(V_{\rho})$ such that $\{e_{u\rho}^{ii}\} \subset I$ and $e_{u\rho}^{11} = e_{u\rho}$. We identify a basis of $\{e_{u\rho}^{11}R_{uv}e_{v\sigma}^{11}\}$ with some arrows from $(u, \rho)$ to $(v, \sigma)$, say \{\{b_k\}_k\}. Now for each $a \in R_{uv}$, $\{e_{u\rho}^{ii}ae_{v\sigma}^{ij}\}$ is a linear combination of $b_k$’s. Say $e_{u\rho}^{ii}ae_{v\sigma}^{ij} = \sum c_k^{ij}b_k$.

For any $N \in \text{Rep}_r(Q_G)$, $M = R(N) \in \text{Rep}_{r(\beta)}(Q)$ is the following representation. The vector space $M_u$ attached to the vertex $u$ is

$$
M_u = \bigoplus_{\rho \in \text{irr}(G_u)} d_\rho N_{u\rho}, \quad d_\rho = \dim(V_\rho).
$$

Here, each copy of $N_{u\rho}$ corresponds to some $e_{u\rho}^{ii}$. Let us denote such a copy by $N_{u\rho}^{ij}$. The linear map from $N_{u\rho}$ to $N_{v\sigma}$ is given by substituting the arrows in $\sum b_kc_k^{ij}$ by corresponding matrices in $N$. In particular, we see that such a lifting is an algebraic morphism $\text{Rep}_{\beta}(Q_G) \to \text{Rep}_{r(\beta)}(Q)$.
**Example 1.8.** Let $S_n$ be the $n$-subspace quiver:

![Quiver Diagram](image)

The symmetric group $\mathfrak{S}_n$ acts naturally on $S_n$. In this way, we get an action of $\mathfrak{S}_n$ on $kS_n$. There are only two orbits on $Q_0$ represented by $n$ and $n + 1$. The stabilizer $G_n$ and $G_{n+1}$ are $\mathfrak{S}_{n-1} \times \mathfrak{S}_2$ respectively. We have only one orbit in $O_n \times O_{n+1}$. The irreducible representations of $S_n$ are indexed by partitions $\rho$, and primitive idempotents in $\text{End}(V_\rho)$ can be labeled by Young tableaux $T$ of shape $\rho$:

$$e_T = \kappa_\rho^{-1} \sum_{v \in V(T)} \sum_{h \in H(T)} \text{sgn } v \cdot vh.$$  

Here, $\kappa_\rho$ is the hook length of $\rho$, $V(T)$, $H(T)$ are the vertical and horizontal subgroup corresponding to the Young tableaux $T$. The number of arrows between $(n, \rho)$ and $(n+1, \sigma)$ is given by the multiplicity of $\rho$ in $\sigma$ restricted to $\mathfrak{S}_{n-1}$-module. This is equal to the Littlewood-Richardson coefficients $c^\sigma_{\rho,1}$, which can be computed by the Pieri rule.

For $n = 4$, we get the following quiver for $B$

![Quiver Diagram](image)

The functor $R$ takes a representation of the above quiver to the following representation of $S_4$.

$$A_1 = \begin{pmatrix} B_1 & B_2 & -B_2 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & B_3 & B_4 & 0 & B_5 & 2B_6 & B_6 & 0 \\ 0 & 0 & B_3 & -B_3 & B_4 & -B_4 & B_5 & B_5 & -2B_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_6 & B_6 & B_6 & B_6 & B_7 \end{pmatrix}$$  

$$A_2 = \begin{pmatrix} B_1 & B_2 & -B_2 & 3B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & -B_4 & B_4 & B_6 & 3B_6 & 0 & 0 & 0 \\ 0 & 0 & B_3 & 0 & -B_3 & 0 & -B_5 & 0 & 3B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_6 & B_6 & B_6 & B_7 \end{pmatrix}$$  

$$A_3 = \begin{pmatrix} B_1 & B_2 & 3B_2 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_3 & 0 & -B_4 & B_4 & -B_4 & 3B_5 & -B_5 & 0 & 0 \\ 0 & 0 & B_3 & 0 & -B_3 & 0 & -B_5 & -3B_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_6 & 0 & -B_7 \end{pmatrix}$$  

$$A_4 = \begin{pmatrix} B_1 & -3B_2 & -B_2 & -B_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B_3 & -B_4 & 0 & 3B_5 & 0 & B_5 & 0 & 0 \\ 0 & 0 & 0 & B_3 & 0 & -B_4 & 0 & 3B_5 & 0 & B_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_6 & B_7 \end{pmatrix}$$

**Example 1.9.** The symmetric group $\mathfrak{S}_n$ also acts naturally on the $n$-arrow Kronecker quiver $K_n$

![Kronecker Quiver](image)
The \( \mathfrak{S}_n \)-representation on arrows decomposes into the standard representation and the trivial representation, so the number of arrows between \((1, \rho)\) and \((2, \sigma)\) is given by \( g_{\rho, [n-1,1]}^{\sigma} + \delta_{\rho, \sigma} \). Here, \( g_{\rho, \pi}^{\sigma} \) is the Kronecker coefficient defined by \( V_{\rho} \otimes V_{\pi} = \oplus g_{\rho, \pi}^{\sigma} V_{\sigma} \). Readers can verify the following quivers \( Q_G \) together with the functor \( R \) for \( n = 2, 3 \).

![Quiver Diagram](image)

\[
A_1 = \begin{pmatrix} B_1 & B_2 & 0 \\ B_3 & B_4 & B_2 \\ -B_3 & 0 & B_6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} B_1 & B_2 & -2B_2 & 0 \\ B_3 & 0 & 0 & -2B_6 \\ 0 & B_4 & -B_5 & B_4 \\ 0 & 0 & B_8 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} B_1 & -2B_2 & B_2 & 0 \\ -B_3 & B_5 & 0 & B_6 \\ 0 & 0 & -B_4 & B_4 \\ 0 & 0 & -B_8 & -2B_6 \end{pmatrix}.
\]

2. Schur Algebras of Reductive Monoid

In this section, we recall several results from [3]. We keep our assumption that the base field \( k \) has characteristic 0. Let \( M_n \) be the affine algebraic monoid of \( n \times n \) matrices over \( k \). We naturally identify the coordinate algebra \( k[M_n] \) with the polynomial algebra \( A(n) := k[X], \) where \( X = \{x_{ij}\}_{1 \leq i, j \leq n} \). The polynomial algebra is graded by the usual monomial degree \( A(n) = \bigoplus_{d \geq 0} A(n,d) \). Moreover, \( A(n) \) is a bialgebra with coalgebra structure maps \( \Delta, \epsilon \) defined by

\[
\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \epsilon(x_{ij}) = \delta_{ij}.
\]

Thus each graded piece \( A(n,d) \) is a subcoalgebra of \( A(n) \). Hence, its linear dual \( S(n,d) := A(n,d)^* \) is a finite-dimensional \( k \)-algebra, known as the classical Schur algebra.

The coordinate algebra of the general linear group \( GL_n \) is the localization of \( A(n) \) at the determinant function: \( k[GL_n] = k[X, \det(X)^{-1}] \). Let \( G \) be a reductive closed subgroup of \( GL_n \). By a polynomial function on \( G \), we mean the restriction to \( G \) of a polynomial function in \( A(n) \). We denote by \( A(G) \) the algebra of polynomial function on \( G \). It inherits the coalgebra structure and the grading from \( A(n) \). Its degree \( d \) piece \( A(G,d) \) is a subcoalgebra. We denote the linear dual of \( A(G,d) \) by \( S(G,d) \). It is a subalgebra of \( S(n,d) \) because \( A(G,d) \) is a quotient of \( A(n,d) \).

**Definition 2.1.** We say that \( G \) admits a graded polynomial representation theory if the sum \( \sum_{d \geq 0} A(G,d) \) is direct.
A standard non-example is $\text{SL}_n$ because $A(\text{SL}_n, 0) \cap A(\text{SL}_n, n) \neq \emptyset$ due to the equation $\det(X) = 1$. It is not hard to see that if $G$ contains the nonzero scalar matrices $cI$ of $\text{GL}_n$, then $G$ admits a graded polynomial representation theory. This includes, for example $\text{GSp}_n$ and $\text{GO}_n$, the groups of symplectic and orthogonal similarities. Proposition 2.3 provides another criterion.

A finite dimensional (left) $k[G]$-module $V$ is called rational if for some basis $v_1, \ldots, v_n$ of $V$ the corresponding coefficient functions $f_{ij}$, defined by the equations

$$g \cdot v_i = \sum_{j=1}^n f_{ij}(g)v_j$$

belong to $k[G]$. We then have on $V$ the structure of a right $k[G]$-comodule via the structure map $\Delta_V : V \to V \otimes k[G]$, given by $\Delta_V(v_i) = \sum_{j=1}^n v_j \otimes f_{ij}$. It is well-known that there is an equivalence of categories between rational $k[G]$-module and $k[G]$-comodules. By a polynomial $G$-module we mean a vector space $V$ on which $G$ acts linearly with coefficient functions in $k$. We take $M_G = \overline{C}$, the Zariski closure of $G$ in $M_n$. Then $M_G$ is a closed submonoid of $M_n$ with $G$ as its group of units. $M_G$ is called the associated algebraic monoid of $G$. Let $I(M_G)$ be the vanishing ideal of $M_G$ in $M_n$.

**Proposition 2.2.** [3] Proposition 1.3, 1.4] Suppose $G$ admits a graded polynomial representation theory. Every polynomial $G$-module has a direct sum decomposition by homogeneous polynomial representations. The category of homogeneous polynomial $G$-modules of degree $d$ is equivalent to the category of $S(G, d)$-modules.

We take $M_G = \overline{C}$, the Zariski closure of $G$ in $M_n$. Then $M_G$ is a closed submonoid of $M_n$ with $G$ as its group of units. $M_G$ is called the associated algebraic monoid of $G$. Let $I(M_G)$ be the vanishing ideal of $M_G$ in $M_n$.

**Proposition 2.3.** [3] Proposition 2.4] $G$ admits a graded polynomial representation theory if and only if $I(M_G)$ is homogeneous. In this case, we have a coalgebra isomorphism $A(G, d) \cong k[M_G]_d$, so the algebra $S(G, d)$ consists of those elements in $S(n, d)$ vanishing on $I_d(M_G) = A(n, d) \cap I(M_G)$.

We provide last point of view of $S(G, d)$ from the tensor power representations. Let $V$ be the $(n$-dimensional) natural $M_n$-representation. For any $d \in \mathbb{N}$, we have an action of $M_n$ on the $d$th tensor power of $V$, by

$$A(v_1 \otimes \cdots \otimes v_d) = Av_1 \otimes \cdots \otimes Av_d.$$  

Let $\phi_d$ be the corresponding representation $M_n \to \text{End}(V^\otimes d)$. It was proved by Schur [10] that $S(n, d) = \phi_d(\text{GL}_n) = \phi_d(M_n) = \text{End}_{k}(V^\otimes d)$.

**Proposition 2.4.** [3] Proposition 3.2] If $G$ admits a graded polynomial representation theory, then

$$S(G, d) = \phi_d(G) = \phi_d(M_G).$$

It is well-known that the semisimplicity of $\phi_d(G)$ is equivalent to complete reducibility of $V^\otimes d$ as $G$-module. So $\phi_d(G)$ is semisimple if $G$ is reductive. Combined with a monoid analogue of the Peter-Weyl theorem [12] Proposition 13], we have that

**Lemma 2.5.** As $G$-bimodule algebras, $S(G, d) \cong \bigoplus_\rho \text{End}(V_\rho)$, where $\rho$ runs through all irreducible degree $d$ polynomial representations of $G$. So if $G$ admits a graded polynomial representation theory, then as $G$-bimodule algebras,

$$k[M_G]^* \cong \prod_{\rho \in \text{irr}(G)} \text{End}(V_\rho),$$
where \( \text{irr}(G) \) is the set of all irreducible polynomial representations of \( G \).

Knowing that \( S(G, d) \) is semisimple, it is an important problem to determine a complete set of primitive orthogonal idempotents. This can be a very hard problem in general, but for the classical Schur algebras \( S(n, d) \), it is treated in [4]. Here is some simple examples, which will be used later.

Recall that the standard monomial basis of \( A(n, d) \) is indexed by the generalized permutations \( (i_1, i_2, \ldots, i_n) \). The pairs \((i_k, j_k)\) are arranged in non-decreasing lexicographic order from left to right. In other words, the \( i \)'s are arranged in increasing order, and the \( j \)'s corresponding to the same \( i \) are in increasing order. We denote the corresponding dual basis in \( S(n, d) \) by \( \xi_{ij_1\cdots j_n} \). A nice combinatorial rule for multiplying such a basis is given in [11].

**Example 2.6.** Let \( A = S(n, 2) \). It has the following complete set of idempotents

\[
\left\{ \frac{1}{2} \left( \xi_{ij} - \xi_{ij}^* \right) \right\}_{1 \leq i < j \leq n}
\]

\[
\left\{ \frac{1}{2} \left( \xi_{ij}^* + \xi_{ij} \right) \right\}_{1 \leq i < j \leq n}
\]

The right column indicates the corresponding irreducible representations.

**Example 2.7.** Let \( A = S(n, 3) \). It has the following complete set of idempotents

\[
\left\{ \frac{1}{6} \sum_{\omega \in \mathfrak{S}_3} \text{sgn}(\omega) \xi_{ijk} \right\}
\]

\[
\left\{ \frac{1}{3} (2 \xi_{ij}^* - \xi_{ij}) \right\}, \frac{1}{3} (2 \xi_{ij}^* - \xi_{ij}), \frac{1}{3} (\xi_{ikj} - \xi_{ijk} + \xi_{ikj}^* - \xi_{ijk}^*), \frac{1}{3} (\xi_{ikj}^* - \xi_{ijk}^* + \xi_{ikj} - \xi_{ijk})
\]

\[
\left\{ \xi_{ii}, \frac{1}{3} (\xi_{ij}^* + \xi_{ij}), \frac{1}{3} (\xi_{ij} + \xi_{ij}^*), \frac{1}{6} \sum_{\omega \in \mathfrak{S}_3} \xi_{ijk} \right\}
\]

where \( 1 \leq i < j < k \leq n \).

**3. Reductive Group Action**

**Definition 3.1.** Let \( B \) be a \( k \)-bialgebra. A (right) \( B \)-comodule algebra \( A \) is a \( k \)-algebra with a right \( B \)-comodule structure \( \Delta_A : A \rightarrow A \otimes B \). We required \( \Delta_A \) to be a \( k \)-algebra homomorphism. The smash product algebra \( A \# B^* \) is by definition the vector space \( A \otimes B^* \) with multiplication

\[
(c \otimes h)(a \otimes f) = \sum a_{(0)} c \otimes (a_{(1)} \cdot h) f.
\]

Here \( a_{(1)} \cdot h \) is the usual (left) \( B \)-action on \( B^* \), that is, \( a_{(1)} h(b) = h(ba_{(1)}) \).

We observe that a left \( A \)-module \( M \), which is also a right \( B \)-comodule \( \Delta_M : M \rightarrow M \otimes B \) such that

\[
\Delta_M(am) = \Delta_A(a) \Delta_M(m)
\]

is a left \( A \# B^* \)-module, but not vice versa. We may abuse of notation writing \( a \) and \( f \) for \( a \otimes 1_{B^*} \) and \( 1_A \otimes f \). If \( \Delta_A(1) = 1_A \otimes 1_{B^*} \), then we will write \( af \) for \( a \otimes f \) and \( AB^* \) for \( A \# B^* \) in this context.

Let \( G \) be an infinite connected reductive group over \( k \), and \( M_G \) be the associated algebraic monoid. Since \( G \) is algebraic, we will only consider rational action of \( G \). In fact, we assume that \( G \) acts polynomially as automorphisms on some \( k \)-algebra \( A \). Then \( A \) becomes a \( k[M_G] \)-comodule algebra. As in the finite group case, we
also have a $G$-action on the category of $A$-modules. We define (proj-coherently) $G$-invariant and $G$-indecomposable module as before.

The group $G$ can be naturally embedded into the dual coordinate algebra $k[M_G]^*$. For every $g \in G$, we define $\epsilon_g \in k[M_G]^*$ as $\epsilon_g(f) = f(g)$. Moreover, the embedding respects actions: $\epsilon_g(m) = \sum \epsilon_g(m_{(1)}) m_{(0)} = \sum m_{(1)}(g)m_{(0)} = m$.

**Lemma 3.2.** If $M$ is an $A\#k[M_G]^*$-module, then $m \mapsto gm$ defines an $A$-module isomorphism $M \cong M^g$ for all $g \in G$.

**Proof.** We need to show for all $g \in G$, $a \in A$, $m \in M$ that

$$(1 \otimes \epsilon_g)(a \otimes 1)(m) = g(am) = (ga)(gm).$$

Suppose for all $g \in G$, $a \in A$, $m \in M$ that $gm = \sum m_1(g)m_0$, and $ga = \sum a_1(g)a_0$. Then

$$(1 \otimes \epsilon_g \cdot a \otimes 1)(m) = \sum a_0 \otimes (a_1 \cdot \epsilon_g)(m) = \sum a_0 \cdot (a_1 \cdot \epsilon_g(m_{(1)})) m_{(0)} = \sum a_0 \epsilon_g(m_{(1)}a_{(1)})m_{(0)} = \sum a_0 m_{(1)}(g)a_{(1)}(g)m_{(0)} = (ga)(gm).$$

Conversely, given a $G$-invariant $A$-module $M$, we assume that for each $g \in G$ we can fix an isomorphism $i_g : M \to M^g$ such that $i_h^g i_g = i_{hg}$. Then we can define a $G$-action on $M$ by $g(m) = i_g(m)$. If such an action can be extended to $k[M_G]^*$ (e.g., the action is polynomial), then we get an $A\#k[M_G]^*$-module. To simplify the notation, we will write $A[M_G]^*$ for $A\#k[M_G]^*$. Such a module as an $A$-module is called coherently $G$-invariant in this context. Under this definition, we also have the notion of (strong) $G$-root as in the finite group case.

Let $Q$ be a finite quiver without oriented cycles. The condition of no oriented cycles is not essential. But otherwise, we need to work with locally finite actions. Since $kQ$ has only finitely many idempotents but $G$ is infinite and connected, $G$ has to fix each idempotent, and thus stabilizes $kQ$ the linear span of arrows. From now on, we assume that $G$ admits a graded polynomial representation theory.

3.1. Description of $Q_G$. It turns out that $kQ[M_G]^*$ is Morita equivalent to some hereditary algebra $kQ_G$. The description is completely analogous to the one in 1.1 except that $Q_G$ is possibly an infinite quiver.

Let $\text{irr}(G)$ be the set of all polynomial representations of $G$, and $R_{uv} := kQ(u,v)$ be the $G$-module spanned by the arrows from $u$ to $v$. The vertex set of $Q_G$ is

$$\bigcup_{u \in Q_0} \{u\} \times \text{irr}(G).$$

The arrow set from $(u, \rho)$ to $(v, \sigma)$ is a basis of

$$\text{Hom}_G(V_\rho, R_{uv} \otimes V_\sigma).$$

**Theorem 3.3.** $kQ[M_G]^*$ is Morita equivalent to the path algebra $kQ_G$. 
Proof. Let $R$ be the (maximal semisimple) subalgebra of $kQ$ generated by the primitive idempotents, and $R_1 \subseteq kQ$ be the $R$-bimodule spanned by the arrows, so $kQ$ is the tensor algebra $T(R, R_1)$.

We fix for each $u \in Q_0$ and $\rho \in \text{irr}(G)$, a primitive idempotent $e_\rho$ of $k[M_G]^*$ corresponding to $\rho$ (see Lemma 2.3). Then $\{e_u \otimes e_\rho\}_{u \in Q_0, \rho \in \text{irr}(G)}$ is a basic set of primitive orthogonal idempotents of $kQ[M_G]^*$. Let $e = \sum_{u \in Q_0, \rho \in \text{irr}(G)} e_u \otimes e_\rho$, then

\[ eR[M_G]^*e = \prod_{u \in Q_0, \rho \in \text{irr}(G)} ke_u \otimes e_\rho. \]

As $G$ stabilizes $R$ and $R_1$, it is easy to see that we have equivalence of categories $\text{mod} kQ[M_G]^* \cong \text{mod} T(R[M_G]^*, R_1[M_G]^*) \cong \text{mod} T(eR[M_G]^*, eR_1[M_G]^*)$.

\[ e_u \otimes e_\rho(R_1[M_G]^*)e_v \otimes e_\sigma = e_u \otimes e_\rho(R_1e_v \otimes k[M_G]^*e_\sigma) \]
\[ = e_\rho(R_{uv}[M_G]^*e_\sigma) \]
\[ = \text{Hom}_k(k, e_\rho(k[M_G]^*)(R_{uv}[M_G]^*e_\sigma)) \]
\[ = \text{Hom}_G(V_\rho, (R_{uv}[M_G]^*e_\sigma)) \]
\[ = \text{Hom}_G(V_\rho, R_{uv} \otimes V_\sigma). \]

\[ \square \]

Since we are mainly interested in coherently $G$-indecomposable and indecomposable $G$-invariant representations, it is enough to focus on connected components of $Q_G$.

Proposition 3.4. If the quiver $Q$ is finite without oriented cycles, then each connected component of $Q_G$ is finite without oriented cycles.

Proof. Since $Q$ is finite, the linear span of arrows is a $G$-module of bounded degree. So for each $e_u \otimes e_\rho$, it can connected to only finitely many $e_v \otimes e_\sigma$. But $Q$ has finitely many vertices, the component containing $e_u \otimes e_\rho$ must be finite as well. Since $Q$ has no oriented cycles, we can totally order the vertices of $Q$ such that $u < v$ if there is an arrow $u \to v$. Now for a given component $Q_c$, we can totally order the vertices in $Q_c$ by $(u, \rho) < (v, \sigma)$ if $u < v$. Note that $u < v$ is a necessary condition for there is an arrow $(u, \rho) \to (v, \sigma)$. \[ \square \]

We fix a connected component $Q_c$ of $Q_G$. Let $A := kQ$ and $B := kQ_c$. Let $T_c : \text{mod} A \to \text{mod} B$ be the functor $e(A[M_G]^* \otimes -)$ followed by the restriction to $Q_c$. Let $R_c : \text{mod} B \to \text{mod} A$ be the functor $\text{Hom}_{Q_c}(eA[M_G]^*, -)$ followed by the restriction to $A$. It is right adjoint to $T_c$, and can be lifted to an algebraic morphism $\text{Rep}_\beta(Q_c) \to \text{Rep}_{\gamma,\beta}(Q)$ using a method similar to that in [1.1]

Example 3.5. For each finite quiver $Q$, we can associate a torus $T_1 = (k^*)^{Q_1}$ acting naturally on $kQ_1$. The irreducible representations of $T_1$ are all one-dimensional indexed by the weight lattice $\mathbb{Z}^{Q_1}$. So the quiver $Q_G$ from our recipe is the universal abelian covering quiver of $Q$.

Example 3.6. Let $K_n$ be the $n$-arrow Kronecker quiver. The general linear group $\text{GL}_n$ acts naturally on the arrow space of $K_n$. This induces an action of $\text{GL}_n$ on $kK_n$. The dimension of $\text{Hom}_G(V_\rho, R_{uv} \otimes V_\sigma)$ is equal to the Littlewood-Richardson coefficients $e^\rho_{\sigma, 1}$.  


For any $n \geq 2$, the first component of $Q_G$ is always the following quiver.

\[
\begin{array}{c}
\text{ } \rightarrow \text{ } \rightarrow \\
\downarrow \quad \downarrow \\
\text{ } \quad \text{ } \\
\end{array}
\]

We can easily compute the functor $R_c$ using Example 2.7. For $n = 3$, The functor $R_c$ takes a representation of the above quiver to the following representation of $K_3$.

\[
A_1 = \begin{pmatrix} 0 & -B_1 & 0 \\ B_1 & 0 & 0 \\ 0 & 0 & -B_1 \\ \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -B_1 \\ 0 & 0 & 0 \\ \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \end{pmatrix}
\]

We observed that as the above situation, the matrices obtained are quite sparse. So we will use Yale form to represent them. For example, the $A_1$ above is the block matrix $\begin{pmatrix} A_{1u} \\ A_{1d} \end{pmatrix}$, where $A_{1u} = \begin{pmatrix} -1/2 & 0.5 \\ 1 & 3/4 \end{pmatrix}$ and $A_{1d} = \begin{pmatrix} 1/4 & 2/3 \\ 3 & 1 \end{pmatrix}$.

For $n = 4$, the functor $R_c$ takes a representation of the above quiver to the representation $A_i = \begin{pmatrix} A_{iu} \\ A_{id} \end{pmatrix}$ of $K_4$, where

\[
A_{1u} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \end{pmatrix} B_1, \quad A_{2u} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ \end{pmatrix} B_1, \quad A_{3u} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ \end{pmatrix} B_1, \quad A_{4u} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ \end{pmatrix} B_1;
\]

\[
A_{1d} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \end{pmatrix} B_2, \quad A_{2d} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ \end{pmatrix} B_1, \quad A_{3d} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ \end{pmatrix} B_1, \quad A_{4d} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ \end{pmatrix} B_1.
\]

**Example 3.7.** For $n \geq 3$, the second component of $Q_G$ is the following quiver

\[
\begin{array}{c}
\quad \\
\rightarrow \\
\downarrow \\
\quad \\
\end{array}
\]

Using Example 2.7 we find that for $n = 3$, the functor $R_c$ takes a representation of the above quiver to the following representation of $K_3$.

\[
A_i = \begin{pmatrix} A_{iu} & 0 \\ A_{id} & A_{ir} \end{pmatrix}
\]

where

\[
A_{1u} = \begin{pmatrix} 1 \\ 3 \\ \end{pmatrix} B_1, \quad A_{2u} = \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix} B_1, \quad A_{3u} = \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix} B_1;
\]

\[
A_{1d} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ \end{pmatrix} B_1, \quad A_{2d} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ \end{pmatrix} B_1, \quad A_{3d} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ \end{pmatrix} B_1;
\]

\[
A_{1r} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \end{pmatrix} B_3, \quad A_{2r} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \end{pmatrix} B_3, \quad A_{3r} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \end{pmatrix} B_3;
\]

\[
A_{1d} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 5 & 6 \\ \end{pmatrix} B_4, \quad A_{2d} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 5 & 6 \\ \end{pmatrix} B_4, \quad A_{3d} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 5 & 6 \\ \end{pmatrix} B_4.
\]
The next connect component is a Dynkin-$E_7$ for $n = 3$ and extended-$E_7$ for $n > 3$. Other components are all wild quivers.

**Example 3.8.** As our last example, we still take the quiver $K_3$ but with a different action. We assume that the $3$-dimensional spaces of arrows is the $GL_2$-module $S^2(k^2)$. Then the first component of $Q_G$ is

![Diagram](attachment:diagram.png)

The functor $R_c$ takes a representation of the above quiver to the following representation of $K_3$.

$$A_1 = \begin{pmatrix} 0 & -B_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_2 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \\ 0 & 0 \\ 0 & 0 \\ 2B_2 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3B_2 \\ 0 & 0 \\ B_2 & 0 \end{pmatrix}.$$  

4. Application to Tensor Invariants

Let us briefly recall Schofield’s semi-invariants of quiver representations [14]. For a fixed dimension vector $\alpha$, the space of all $\alpha$-dimensional representations is

$$\text{Rep}_\alpha(Q) := \bigoplus_{\alpha \in Q_1} \text{Hom}(k^{\alpha(v)}, k^{\alpha(v')}).$$

The product of general linear group $GL_\alpha := \prod_{v \in Q_0} GL_{\alpha(v)}$ acts on $\text{Rep}_\alpha(Q)$ by the natural base change. This action has a kernel, which is the multi-diagonally embedded $k^*$. For any weight $\sigma \in \mathbb{Z}Q_0$, we can associate a character of $GL_\alpha$ still denoted by $\sigma$

$$(g(v))_{v \in Q_0} \mapsto \prod_{v \in Q_0} (\det g(v))^{\sigma(v)}.$$

We define the subgroup $GL^\sigma_\alpha$ to be the kernel of the character map. The semi-invariant ring $SIR^\sigma_\alpha(Q) := k[\text{Rep}_\alpha(Q)]^{GL^\sigma_\alpha}$ of weight $\sigma$ is $\sigma$-graded: $\bigoplus_{n \geq 0} SIR^\sigma_\alpha(Q)$, where

$$SIR^\sigma_\alpha(Q) := \{ f \in k[\text{Rep}_\alpha(Q)] | g(f) = \sigma(g)f, \forall g \in GL_\alpha \}.$$

For any $N \in \text{Rep}_\beta(Q)$, we take some injective resolution of $N$

$$0 \to N \to I_0 \to I_1 \to 0,$$

and apply the functor $\text{Hom}_Q(M, -)$ for $M \in \text{Rep}_\alpha(Q)$

$$\text{Hom}_Q(M, N) \hookrightarrow \text{Hom}_Q(M, I_0) \xrightarrow{\phi^N_M} \text{Hom}_Q(M, I_1) \hookrightarrow \text{Ext}_Q(M, N).$$

If $\langle \alpha, \beta \rangle_Q = 0$, then $\phi^N_M$ is a square matrix. Following Schofield [14], we define $c(M, N) := \det \phi^N_M$. It is not hard to see that the definition does not depend on the injective resolution of $N$. In particular, we can take the canonical resolution or minimal resolution of $N$. We can also define $c(M, N)$ using projective resolution of $M$. Note that $c(M, N) \neq 0$ if and only if $\text{Hom}_Q(M, N) = 0$ or $\text{Ext}_Q(M, N) = 0$. We denote $c_N := c(-, N)$ and dually $c^M := c(M, -).$
It is proved in [14] that \( c_N \in \text{SL}_N^\gamma(Q) \) for \( \sigma_\alpha^\gamma = -\langle \alpha, \gamma \rangle_Q \), and dually \( c_M \in \text{SI}_N^\beta(Q) \) for \( \sigma_\alpha = (\alpha, -)_Q \). In fact, \( c_N \)'s (resp. \( c_M \)'s) span \( \text{SL}_N^\gamma(Q) \) (resp. \( \text{SI}_N^\beta(Q) \)) over the base field \( k \).

Let \( G \) be a finite group or an infinite connected reductive group acting polynomially on \( kQ \) as automorphisms. Such an action induces a rational action of \( G \) on all representation spaces of \( Q \). We are interested in those semi-invariants which is also semi-invariant under the \( G \)-action.

**Observation 4.1.** If \( N \) is proj-coherently \( G \)-invariant, then \( c_N \) is also semi-invariant under \( G \)-action.

**Proof.** Since \( N \) is proj-coherently \( G \)-invariant, there is some map \( \varphi : G \to \text{GL}_N \) such that \( N^g = \varphi(g)N \) and \( \varphi \) descends to a representation \( G \to \text{GL}_N / k^* \). Then

\[
c_{N^g}(M) = c_M^\gamma(\varphi(g)N) = \sigma_\alpha(\varphi(g))c_M^\beta(N) = (\sigma_\alpha^\gamma(\varphi))c_N(M).
\]

Since \( \langle \alpha, \beta \rangle_Q = 0 \), \( \sigma_\alpha |_{k^*} \) is trivial, so \( \sigma_\alpha \varphi \) is a character of \( G \). In other words \( c_M \) is semi-invariant under \( G \)-action. \( \square \)

This observation allows us to construct a lot of new semi-invariants for the \( \text{GL}_N \times G \)-action on \( k[\text{Rep}_\alpha(Q)] \). According to Observation [14] any exceptional (=rigid Schur) representation is proj-coherently \( G \)-invariant. Actually we conjecture that they are all coherently \( G \)-invariant. The dimension of such a representation is a real Schur root \( \gamma \) of the quiver. Moreover, for any two general representations \( N_1, N_2 \in \text{Rep}_\gamma(Q) \), \( c_{N_1} \) is a multiple of \( c_{N_2} \). In this sense, we will treat these semi-invariants as trivial, and avoid them later.

We are particularly interested in applying the method to construct the semi-invariants of (tri)-tensors. By a (tri)-tensor of vector spaces \( (U, V, W) \), we mean the vector space \( U^\ast \otimes V \otimes W^\ast \). The product of special linear groups \( \text{SL}(U) \times \text{SL}(V) \times \text{SL}(W) \) acts naturally on it. We are interested in the invariants in \( k[U^\ast \otimes V \otimes W^\ast] \) for this action. The tensor space can be identified with the \( (\alpha_1, \alpha_2) \)-dimensional representation space of the \( n \)-arrow Kronecker quiver \( K_n \), where \( \dim U = \alpha_1, \dim V = \alpha_2, \) and \( \dim W = n \). In this context, \( G = \text{GL}(W) \).

It follows from Example [3,6] that

**Proposition 4.2.** For general square matrices \( B_1, B_2 \), we define the representations \( N_1, N_2 \) of \( K_3 \)

\[
N_1(a_1) = \begin{pmatrix} 0 & -B_1 & 0 \\ 0 & 0 & B_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_1(a_2) = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & 0 & -B_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_1(a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -B_1 & 0 & 0 \end{pmatrix},
\]

\[
N_2(a_1) = \begin{pmatrix} 0 & B_2 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_2(a_2) = \begin{pmatrix} 0 & B_2 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_2(a_3) = \begin{pmatrix} B_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then \( c_{N_1} \) (resp. \( c_{N_2} \)) is a semi-invariant for the tensor of size \( a \times 2a \times 3 \) (resp. \( a \times a \times 3 \)).

**Proposition 4.3.** For general square matrices \( B_1, B_2 \), we define the representations \( N_1, N_2 \) of \( K_4 \) by \( A_{4u}, A_{4d} \) as in Example [3,6] then \( c_{N_1} \) (resp. \( c_{N_2} \)) is a semi-invariant for the tensor of size \( 2a \times 5a \times 4 \) (resp. \( 2a \times 3a \times 4 \)).

**Proposition 4.4.** For general square matrices \( B_1, B_2, B_3, B_4 \), we define the representations \( N_1, N_2 \) of \( K_4 \) by \( A_{4r}, A_{4d} \) as in Example [3,7] then \( c_{N_1} \) (resp. \( c_{N_2} \)) is a semi-invariant for the tensor of size \( 3a \times 5a \times 3 \) (resp. \( 3a \times 4a \times 3 \)).
We define the representation $N_3, N_4, N_5$ of $K_3$ by

$$N_3(a_i) = \begin{pmatrix} A_i & A_{ir} \\ 0 & A_u \end{pmatrix}, N_4(a_i) = \begin{pmatrix} A_i & 0 \\ 0 & A_{ir} \end{pmatrix}, N_5(a_i) = \begin{pmatrix} A_{iu} & 0 \\ A_i & A_{ir} \end{pmatrix}.$$  

then $c_{N_3}$ (resp. $c_{N_4}, c_{N_5}$) is a semi-invariant for the tensor of size $9a \times 9a \times 3$ (resp. $8a \times 9a \times 3, a \times a \times 3$).

We remark that our construction also applies to the case when the third factor $W$ is other representation of $\text{GL}(W)$.

**Proposition 4.5.** For general square matrices $B_1, B_2$, we define the representations $N_1, N_2$ of $K_3$ (see Example V.5)

$$N_1(a_1) = \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}, \quad N_1(a_2) = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_1(a_3) = \begin{pmatrix} 0 & 0 \\ -B_1 & 0 \end{pmatrix},$$

$$N_2(a_1) = \begin{pmatrix} 3B_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_2(a_2) = \begin{pmatrix} 0 & 0 \\ 2B_2 & 0 \end{pmatrix}, \quad N_2(a_3) = \begin{pmatrix} 0 & 0 \\ 0 & 3B_2 \end{pmatrix}.$$  

Then $c_{N_1}$ (resp. $c_{N_2}$) is a semi-invariant in $k[U^* \otimes V \otimes S^2(W)^*]$ for $\dim(U, V, W) = (a, 2a, 2)$ (resp. $(a, a, 2)$).

Fix a component $Q_c$ of $Q_G$. Let $\text{SI}^\wedge_{R_c} (Q)$ be the vector space spanned by semi-invariants on $\text{Rep}_G(Q)$ of form $\alpha R_c(N)$ for $N \in \text{Rep}_G(Q_c)$. On the other hand, we can restrict a semi-invariant $c_N \in \text{SI}^\wedge_{R_c} (Q)$ on the subvariety $R_c(\text{Rep}_G(Q_c))$. We denote the linear span of these restricted semi-invariants by $\text{SI}^\wedge_{R_c} (Q)$. Similar to [2, Corollary 1], we have the following reciprocity property

**Proposition 4.6.** $\dim \text{SI}^\wedge_{R_c} (Q) = \dim \text{SI}^\wedge_{R_c} (Q)$.

In general, we do not know a simple method to compute the dimension of $\text{SI}^\wedge_{R_c} (Q)$. Sometimes, it is easier to perform computation on $Q_c$ using the theorem below. To prove the theorem, we need some construction related to the functor $T_c$. We can algebraically lift $T_c$ as we did for $R_c$. Moreover, the lifting can be constructed at the level of morphisms. For our purpose, we only state such a lifting for morphisms between projectives. It is enough to do this for $P_u \rightarrow P_v$, where $P_u, P_v$ are indecomposable projective representations corresponding to $u, v \in Q_0$, and $a$ is an arrow $u \rightarrow v$. The construction will depend on the lifting of $R_c$. Recall that a lifting of $R_c$ maps a representation $N$ of $Q_c$ to a representation $M$ of $Q$ as follows. The vector space $M_u$ attached to the vertex $u$ is $M_u = \bigoplus_{r \in Q_c} \dim(V_r)N_{up}$. Here, by $r \in Q_c$ we mean that there is an idempotent in $Q_c$ corresponding to the irreducible representation $\rho$. The linear map from the $i$-th copy of $N_{up}$ to $j$-th copy of $N_{vq}$ is given by substituting the arrows $b_k$ in certain linear combination $\sum_k c_k^{ij} b_k$ by corresponding matrices in $N$.

Now we let $T_c$ send $P_u$ to $T_c(P_u) = \bigoplus_{r \in Q_c} \dim(V_r)P_{ur}$, and send the morphism $P_u \rightarrow P_v$ to a matrix with $\sum_k c_k^{ij} b_k$ as the $ij$-th entry. We see from the construction that such a lifting is not only algebraic but also compatible with the adjunction in the sense that $\text{Hom}_Q(P_u, R_c(N))$ can be naturally identified with
For any two representations $P, Q$ such that the diagram commutes

$$\begin{array}{c}
\text{Hom}_Q(P_n, R_c(N)) \\
\text{Hom}_Q(T_c(P_n), N)
\end{array} \xrightarrow{\text{Hom}_Q(a, R_c(N))} \text{Hom}_Q(P_n, R_c(N)) \xrightarrow{\text{Hom}_Q(T_c(a), N)} \text{Hom}_Q(T_c(P_n), N).$$

We remind readers that a morphism $P \xrightarrow{f} Q$ can be represented by a matrix whose entries are linear combination of paths, and apply $\text{Hom}_Q(-, N)$ to this morphism is nothing but substitute arrows in the matrix by corresponding matrix representation in $N$.

Let $\text{SI}_{\beta}^{\sigma_{T_c}(a)}(Q_c)$ be the vector space spanned by semi-invariants on $\text{Rep}_{\beta}(Q_c)$ of form $c^{T_c(M)}$ for $M \in \text{Rep}_a(Q)$.

**Theorem 4.7.** $\dim \text{SI}_{\alpha}^{\sigma_{T_c}(a)}(Q) = \dim \text{SI}_{\beta}^{\sigma_{T_c}(a)}(Q_c)$.

**Proof.** For any two representations $M \in \text{Rep}_a(Q), N \in \text{Rep}_{\beta}(Q_c)$, we take the canonical resolution $0 \to P_1 \to P_0 \to M \to 0$, and apply the functor $\text{Hom}_Q(-, R_c(N))$, then we get

$$\begin{array}{c}
\text{Hom}_Q(M, R_c(N)) \\
\text{Hom}_Q(T_c(M), N) \xrightarrow{\phi_{M}} \text{Hom}_Q(P_0, R_c(N)) \xrightarrow{\phi_{P_0}} \text{Hom}_Q(P_1, R_c(N)) \xrightarrow{\phi_{P_1}} \text{Ext}_Q(M, R_c(N)) \\
\text{Hom}_Q(T_c(P_0), N) \xrightarrow{\phi_{T_c(P_0)}} \text{Hom}_Q(T_c(P_1), N) \xrightarrow{\phi_{T_c(P_1)}} \text{Ext}_Q(T_c(M), N)
\end{array}$$

The lower row is due to the adjunction. Since $T_c$ is exact and preserve projectives, $0 \to T_c(P_1) \to T_c(P_0) \to T_c(M) \to 0$ is in fact a projective resolution of $T_c(M)$. By our construction of $T_c$, we conclude that

$$c(M, R_c(N)) = \det \phi_{M} = \det \phi_{T_c(M)} = c(T_c(M), N).$$

Therefore, $\dim \text{SI}_{\alpha}^{\sigma_{T_c}(a)}(Q) = \dim \text{SI}_{\beta}^{\sigma_{T_c}(a)}(Q_c)$.

As an example, let us compute the dimension of $\text{SI}_{(1,2)}^{\sigma_{T_c}(1,0,1)}(K_3)$ in Proposition 4.2. It is enough to compute the dimension of $\text{SI}_{(1,0,1)}^{\sigma_{T_c}(1,2)}(Q_c)$. This $Q_c$ is a finite type quiver, so the dimension of $\text{SI}_{(1,0,1)}^{\sigma_{T_c}(1,2)}(Q_c)$ is at most one. A general representation $M \in \text{Rep}_{(1,2)}(K_3)$ has resolution $0 \to P_2 \xrightarrow{k_1a_1 + k_2a_2 + k_3a_3} P_1 \to M \to 0$, then

$$0 \to T_c(P_2) = 3P_n \xrightarrow{\left( \begin{array}{ccc} 0 & -k_2b_1 & 0 \\ -k_3b_1 & 0 & 0 \\ 0 & k_1b_1 & -k_3b_2 \end{array} \right)} 3P_n = T_c(P_1) \to T_c(M) \to 0.$$

Now it is not hard to see that $T_c(M)$ decomposes as $3(M_1 \oplus M_2)$, where $M_1$ (resp. $M_2$) is a general representation of dimension $(0,1,1)$ (resp. $(1,1,1)$). So we see that $\text{Hom}_{Q_c}(T_c(M), N) = 0$ for general $N \in \text{Rep}_{(1,0,1)}(Q_c)$, and thus $\dim \text{SI}_{(1,2)}^{\sigma_{T_c}(1,0,1)}(K_3) = 1$. In fact, $\dim \text{SI}_{\alpha_{2a}}^{\sigma_{T_c}(1,0,1)}(K_3) = 1$ for all $a \in \mathbb{N}$.

We checked that the space of semi-invariants of fixed weight in Proposition 4.2, 4.3, 4.4 and 4.5 are all one-dimensional by hand and by computer. This theorem
also tells us that to construct nontrivial semi-invariants, it is enough to use those stable representation of $Q_e$ in the sense of [8].

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