The Spatial String Tension, Thermal Phase Transition, and
AdS/QCD

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Abstract

We present results of modeling the temperature dependence of the spatial string tension
and thermal phase transition in a five-dimensional framework nowadays known as AdS/QCD.
For temperatures close to the critical one we find a behaviour remarkably consistent with the
lattice results.

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1 Introduction

It is well known that $SU(N)$ gauge theories at high temperature undergo a phase transition to
a deconfined phase. Although at the phase transition point basic thermodynamic observables
show a drastic qualitative change, there are some correlation functions of physical interest whose
structure does not change qualitatively at $T_c$. For example, this is the case for the pseudo-
potential extracted from spatial Wilson loops. It is confining for all temperatures [1]. This
is taken as indication that certain confining properties survive in the high temperature phase.
While the leading high temperature behaviour of the pseudo-potential for temperatures well
above $T_c$ can be understood in terms of high temperature perturbation theory, non-perturbative
effects make it difficult to compute that near the phase transition point. So far only a lattice
study of the whole temperature dependence of the spatial string tension exists [2, 3].

The situation changed drastically with the invention of the AdS/CFT correspondence that
resumed interest in finding a string description of strong interactions. Its more phenomenological

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cousin called AdS/QCD deals with a five-dimensional effective description and tries to fit it to QCD as much as possible.

In this Letter we will explore the temperature dependence of the spatial string tension and thermal phase transition within AdS/QCD.

There are good motivations for doing so. First, in a recent paper [4] we have constructed a phenomenological heavy-quark potential which has a remarkable similarity to the Cornell potential. We did so by using the slightly deformed AdS metric. Second, it is already known that the thermodynamics of $\mathcal{N} = 4$ super Yang-Mills theory is related to the AdS black hole geometry [5]. What we need now is to include the thermodynamics into the five-dimensional framework of [4]. To this end, we consider the following Euclidean background metric

$$ds^2 = R^2 \frac{h(z)}{z^2} \left( f dt^2 + dx_i^2 + f dz^2 \right), \quad h(z) = e^{\frac{z}{cT}}, \quad f(z) = 1 - \left( \frac{z}{z_T} \right)^4,$$

where $i = 1, 2, 3$. $t$ is a periodic variable of period $\pi z_T$.

At zero value of the deformation parameter $c$ we have the AdS$_5$ black hole metric, as expected. Here $z_T$ is related to the Hawking temperature

$$T = \frac{1}{\pi z_T}$$

whose dual description is nothing but the temperature of gauge theory. On the other hand, at $T = 0$ we have in fact the slightly deformed AdS$_5$ metric. Such a deformation is notable. The point is that in this background linearized five-dimensional Yang-Mills equations are effectively reduced to Laguerre differential equation. As a result, the spectrum turns out to be like that of the linear Regge models [6, 7]. This fact allows us to fix the value of $c$ from the $\rho$ meson trajectory. It is of order [7]

$$c \approx 0.9 \text{ GeV}^2.$$

The central point now is that the metric [4] does not contain any free fit parameter. Thus, evaluation of the spatial string tension which we will undertake can be considered as a crucial test of AdS/QCD in the infrared region.

## 2 Calculating the Spatial String Tension

Given the background metric, we can calculate expectation values of Wilson loops by using the standard formalism of AdS/CFT.\footnote{The literature on the Wilson loops at finite temperature within the AdS/CFT correspondence is vast. For a discussion of this issue, see, e.g., [5] and references therein.} The Wilson loops obeying an area law provide string tensions. Our goal is therefore to study spatial Wilson loops.

To this end, we consider a rectangular loop $C$ along two spatial directions $(x, y)$ on the boundary ($z = 0$) of five-dimensional space. As usual, we take one direction to be large, say $Y \to \infty$. The quark and antiquark are set at $x = \frac{r}{2}$ and $x = -\frac{r}{2}$, respectively.
Next, we make use of the Nambu-Goto action equipped with the background metric \( H \) and choose the world-sheet coordinates as \( \xi^1 = x \) and \( \xi^2 = y \). This yields

\[
S = \frac{\mathcal{g}}{2\pi} \int \frac{d^2x}{2} \frac{h}{z^2} \sqrt{1 + \frac{1}{f}(z')^2},
\]

where \( \mathcal{g} = R_2 \). A prime denotes a derivative with respect to \( x \).

Now it is easy to find the equation of motion for \( z \)

\[
zz'' + \left( f + (z')^2 \right) \left( 2 - z\partial_z \ln h \right) - \frac{1}{2} z(z')^2 \partial_z \ln f = 0
\]

as well as the first integral

\[
\frac{h}{z^2 \sqrt{1 + \frac{1}{f}(z')^2}} = C.
\]

The integration constant \( C \) can be expressed via the maximum value of \( z \). On symmetry grounds, \( z \) reaches it at \( x = 0 \). By virtue of (6), the integral over \([-\frac{r}{2}, \frac{r}{2}]\) of \( dx \) is equal to

\[
r = 2z_0 \int_0^1 dv v^2 \exp\left\{ \left( \frac{2v}{z_0} \right)^2 (1 - v^2) \right\} \left( 1 - \left( \frac{2v}{z_T} \right)^4 v^4 \right)^{-\frac{1}{2}} \left( 1 - v^4 \exp\left\{ 2\left( \frac{2v}{z_c} \right)^2 (1 - v^2) \right\} \right)^{-\frac{1}{2}},
\]

where \( v = \frac{z}{z_0} \), \( z_c = \sqrt{\frac{2}{x}} \), and \( z_0 = z|_{x=0} \).

At this point a comment is in order. A simple analysis shows that the integral \( \int \) is real for \( z_0 \) subject to

\[
z_0 < z_T, \quad z_0 < z_c.
\]

Note that in the limit as \( c \) goes to zero \( z_0 \) is bounded by a horizon \( (z = z_T) \), as should be for the black hole geometry. This gives rise to the first wall \( \text{(5a)} \). On the other hand, for zero temperature it is also bounded if \( c \neq 0 \). The physical reason for this is a gravitational force appeared because of the deformation of the AdS metric. The easiest way to see what is going on is to introduce the effective string tension depending on \( z \). It is simply \( \sigma(z) = z^{-2} \exp\left\{ \frac{1}{2} c z^2 \right\} \) as follows from the form of the metric. Now consider the behavior of a string bit in the potential \( V = \sigma(z) \) shown in Fig.1. The potential reaches its minimum value exactly at \( z = z_c \), so a repulsive force prevents the string from getting deeper in \( z \) direction. Note that because the string ends on the (infinitely) heavy quark-antiquark pair set at \( z = 0 \), it doesn’t completely roll down to the minima of the potential. This gives rise to the second wall \( \text{(5b)} \) of \( \text{[4]} \). In summary, there are the two walls in the problem in question. This fact will play a key role in understanding the temperature dependence of the spatial string tension.

Note that Eq.(6) has two solutions. One has \( z_T = z|_{x=0} \). The solution of interest has \( C = \frac{\mathcal{g}}{2\pi}|_{x=0} \).

O.A. thanks L. Susskind for a discussion of this issue.
Now, as in [4], we will compute the energy of the configuration. In the process we regularize the integral over $z$ by imposing a cutoff $\epsilon$. Finally, the regularized expression takes the form

$$E_R = \frac{g}{\pi z_0} \int_{z_0}^{1} dv v^{-2} \exp\left\{ (\frac{z_0}{z})^2 v^2 \right\} \left( 1 - (\frac{z_0}{z})^4 v^4 \right)^{-\frac{1}{2}} \left( 1 - v^4 \exp\{ 2(\frac{z_0}{z})^2 (1 - v^2) \} \right)^{-\frac{1}{2}}. \quad (9)$$

Its $\epsilon$-expansion is simply

$$E_R = \frac{g}{\pi \epsilon} + E + O(\epsilon), \quad (10)$$

where

$$E = \frac{g}{\pi z_0} \left( -1 + \int_{0}^{1} dv \frac{v}{v^2} \left[ \exp\left\{ (\frac{z_0}{z})^2 v^2 \right\} \left( 1 - (\frac{z_0}{z})^4 v^4 \right)^{-\frac{1}{2}} \left( 1 - v^4 \exp\{ 2(\frac{z_0}{z})^2 (1 - v^2) \} \right)^{-\frac{1}{2}} - 1 \right] \right). \quad (11)$$

Similarly as $r$, $E$ is real only for $z_0$ subject to the constraints [8].

As in the case of zero temperature [4], the pseudo-potential in question is written in parametric form given by Eqs. (7) and (11). It is unclear to us how to eliminate the parameter $z_0$ and find $E$ as a function of $r$, $T$, and $c$. We can, however, gain some important insights from numerical calculations.

In Fig.2 we have plotted $E/g$ against $r$. Apparently the pseudo-potential shows temperature dependence of its slope. Moreover, there exists a critical value of $T$ such that the spatial string tension is temperature independent below $T_c$ and rises rapidly above.$^4$ Interestingly enough, on the lattice such a picture was discovered in [2].

Having seen the pattern from the numerics, we can now try to find a temperature dependence of the spatial string tension.

First, let us have a close look at Eq.(7). After a short inspection we find that $r$ is a continuously growing function of $z_0$. This implies that large distances correspond to a region near the

$^4$There is a subtle point here. As noted in [4], in the phenomenologically important interval $0.1 \text{ fm} \leq r \leq 1 \text{ fm}$ the slope of the potential is given by $\sigma$ or $\sigma_0$ (coefficients in front of the linear terms at large and small distances) depending on the value of $c$. However, their ratio is of order 1.24. Since it is not significant for our phenomenological estimates, we will be interested in $\sigma$ in what follows.
upper endpoint which is the smallest of $z_c$ and $z_T$. The integral is dominated by $v \sim 1$, where it takes the form $2z_0 \int_0^1 dv/\sqrt{a(1-v)+b(1-v)^2}$. Such an integral may be found in tables [9]. Finally, we get
\[ r = -\frac{2z_0}{\sqrt{\beta}} \ln \left( 1 - \frac{z_u}{z_c} \right) \left( 1 - \frac{z_u}{z_T} \right) + O(1), \] (12)
where $\beta$ is a polynomial in $x = \left( \frac{z_0}{z_T} \right)^4$ and $y = \left( \frac{z_0}{z_c} \right)^2$. Explicitly, it is given by
\[ \beta = -6 + 22x + 18y - 8y^2 - 34xy + 8xy^2. \] (13)

At the upper endpoint $r \to \infty$, as expected.

In a similar spirit, we can explore the long distance behavior of $E$. It follows from (11) that in the neighborhood of the upper endpoint the energy behaves as
\[ E = -\frac{g}{\pi z_0 \sqrt{\beta}} \exp \left\{ \left( \frac{T}{T_c} \right)^2 - 1 \right\} + O(1). \] (14)

Along with the relation (12), this means that at long distances the pseudo-potential is linear. The spatial string tension is given by
\[ \sigma_s = \begin{cases} \sigma & \text{if } T \leq T_c, \\ \sigma \left( \frac{T}{T_c} \right)^2 \exp \left( \left( \frac{T}{T_c} \right)^2 - 1 \right) & \text{if } T \geq T_c. \end{cases} \] (15)

At this stage, we set $\sigma = \frac{\sigma_0}{4\pi c}$ and $T_c = \frac{1}{\pi \sqrt{2}}$. Note that $\sigma$ is the physical string tension at zero temperature [4].

3 Thermal Phase Transition

Now we will discuss the issue concerning the phase transition to the deconfined phase.

We begin by making some qualitative comments about the temperature dependence of the spatial string tension. From the AdS/QCD perspective there are three possibilities shown in Fig.3.

(i) $z_c < z_T$. This means that the second wall (8b) terminates the string. If so, then the large
distance physics of the string is determined by this wall. As a result, we find the same behavior as in [4]. We therefore interpret this as the low temperature phase or, equivalently, the confined phase.

(ii) $z_c > z_T$. This time the first wall terminates the string. The large distance physics is determined by the near horizon geometry of the AdS$_5$ black hole and the spatial string tension scales like $T^2$ at high temperature [5]. We interpret this phase as the high temperature phase or, equivalently, the deconfined phase.

(iii) $z_c = z_T$. This implies that the two walls coincide. We interpret this as a dual description (in the holographic sense of AdS/QCD) of the phase transition point. Put in a slightly different way, $z_c = z_T$, in terms of a critical temperature and the parameter $c$, is given by

$$T_c = \frac{1}{\pi} \sqrt{\frac{c}{2}} . \tag{16}$$

Having the AdS/QCD pattern of the phase transition, we can now make a couple of estimates relevant to phenomenology.

It is of physical interest to estimate the critical temperature of our model. From (3) and (16) we have

$$T_c \approx 210 \text{ MeV} . \tag{17}$$

Next, we can estimate the ratio $\frac{T_c}{\sqrt{\sigma}}$, where $\sigma$ is the zero temperature string tension. It is found by using Eqs. (15) and (16),

$$\frac{T_c}{\sqrt{\sigma}} = \sqrt{\frac{2}{e\pi g}} \approx 0.50 . \tag{18}$$

The constant $g$ was fixed in [4] from the linear term of the Cornell potential. Its numerical value is of order $g \approx 0.94$. The estimate (18) coincides within 7 per cent with the lattice data for $SU(3)$ gauge theory [3]. Thus, the agreement of theory with the data is very pleasing at this point.

Finally, we can compare the temperature dependence of the spatial string tension with the results of [2] for the high temperature phase of $SU(2)$ gauge theory. From the fit $\frac{\sqrt{\sigma}}{T_c}$ at $T = T_c$
to the data given in Table 1 of [2] we now fix the value of $g$ and obtain\(^5\)

$$
\frac{\sqrt{\sigma_s}}{T_c} = 1.44 \frac{T}{T_c} \exp\left\{ \frac{1}{2} \left( \frac{T}{T_c}\right)^2 - \frac{1}{2} \right\}.
$$

(19)

In Fig. 4 we have plotted $\frac{\sqrt{\sigma_s}}{T_c}$ against $\frac{T}{T_c}$. We find that the temperature dependence of the spatial string tension is in good agreement with the lattice data in the region $1 \leq \frac{T}{T_c} \lesssim 2.5$. From this point of view it can be thought of as a description of the spatial string tension in the high temperature phase near the phase transition point. Note that the temperature dependence of the spatial string tension at high temperatures is determined by the $\beta$-function of gauge theory. In particular the logarithmic dependence is seen in the data at high temperatures.\(^6\) The model (1) cannot reproduce logarithmic dependences associated with the small coupling running. Instead, it provides a complementary description in the strong coupling regime that is in accord with the ideas on the AdS/CFT correspondence.

4 Concluding Comment

In conclusion, we would like to emphasize again that we have not introduced any new parameter to describe the critical temperature as well as the temperature dependence of the spatial string tension. The only dimensionful parameter of our model is $c$. It appears as the Regge parameter at zero temperature. Moreover, the physical string tension at zero temperature is proportional to $c$ with the coefficient of proportionality fixed from the linear term of the Cornell potential \([4]\). The temperature of the phase transition appears then calculable. Note a crucial factor of $1/2\pi^2$ which explains, in the framework considered, the lower scale of the critical temperature, $T_c^2 \sim 0.04 \text{ GeV}^2$, than $c \sim 0.9 \text{ GeV}^2$. Moreover, the temperature dependence of the spatial string tension is very soft below $T_c$ and sharp above $T_c$. This feature of the data is also explained by the model. To summarize, the AdS/QCD correspondence provides a quantitative framework for temperatures of order $T_c$ which works with about 10% accuracy.

\(^5\)Since $g$ depends on a number of colors, we have to adjust its value to $SU(2)$.

\(^6\)For a discussion of this issue on the lattice, see, e.g., [2, 3] and references therein.
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