A SURVEY OF THE HOMOLOGY COBORDISM GROUP

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Abstract. In this survey, we aim to clarify the long-standing and rich history of the homology cobordism group \( \Theta_3 \) in the context of smooth manifolds for the first time. Further, we present various results about its algebraic structure and discuss its crucial role in the development of low-dimensional topology. Also, we list a series of open problems about the behavior of homology 3-spheres and the structure of \( \Theta_3 \). The appendix is a compilation of several constructions and presentations of homology 3-spheres introduced by Brieskorn, Dehn, Gordon, Seifert, Siebenmann, and Waldhausen.

Contents

1. A Promenade around Smooth Manifolds 1
   1.1. The Predecessor: \( \Theta^n \) 2
   1.2. The Successor: \( \Theta^n_\mathbb{Z} \) 3
   1.3. The Aberrant: \( \Theta_3^\mathbb{Z} \) 3
   2. The Structure of \( \Theta_3^\mathbb{Z} \) 4
      2.1. Subgroups and Summands of \( \Theta_3^\mathbb{Z} \) 4
      2.1.1. A Recovery: More about Subgroups of \( \Theta_3^\mathbb{Z} \) 5
      2.1.2. A Diversification: More about Summands of \( \Theta_3^\mathbb{Z} \) 7
      2.1.3. A Modification: The Rational Version of \( \Theta_3^\mathbb{Z} \) 8
   2.2. The Trivial Element of \( \Theta_3^\mathbb{Z} \) 9
   2.3. Generators of \( \Theta_3^\mathbb{Z} \) 11
   2.4. Torsion of \( \Theta_3^\mathbb{Z} \) 13
   3. Appendix: Examples of Homology 3-Spheres 14

Notes 17
Afterword 19
Acknowledgements 20
References 20

1. A Promenade around Smooth Manifolds

All \( n \)-dimensional manifolds (\( n \)-manifolds for short) with or without boundaries are chosen to be compact, oriented, and smooth. Otherwise, the type of the manifold is specified. A boundary manifold appears with the prefix \( \partial \) and the – sign indicates reversed orientation. The connected sum operation between two manifolds is denoted by \( # \). A diffeomorphism (resp. homeomorphism, and piecewise linear homeomorphism) indicates a smooth (resp. continuous, and continuous and piecewise linear) bijective map between manifolds with a smooth (resp. continuous, and continuous and piecewise linear) inverse.
1.1. The Predecessor: $\Theta^n$. A connected $n$-manifold $M$ with $\partial M = \emptyset$ is called a homotopy $n$-sphere if $M$ has the same homotopy type as the unit $n$-dimensional sphere $S^n$, i.e., $M \simeq S^n$. The $n$-dimensional homotopy cobordism group $\Theta^n$ is defined as

$$\Theta^n = \{ \text{homotopy n-spheres up to diffeomorphism} \} / \sim$$

where the equivalence relation $h$-cobordism $\sim$ is given for two arbitrary homotopy $n$-spheres $M_0$ and $M_1$ as

$$M_0 \sim M_1 \iff \begin{cases} 
\text{There exists a disconnected} \ (n+1)\text{-manifold } W \text{ such that} \\
\partial W = -(M_0) \cup M_1, \\
\text{The inclusions induce homotopy equivalences:} \\
M_0 \looparrowleft W \looparrowright M_1 \Rightarrow M_0 \simeq W \simeq M_1.
\end{cases}$$

After Milnor detected exotic 7-spheres (7-manifolds homeomorphic but not diffeomorphic to $S^7$) in his groundbreaking work [Mil56], he also introduced the notion $\Theta^n$ to study homotopy $n$-spheres in an unpublished note [Mil59] and obtained some partial results on the orders of $\Theta^n$. It forms an abelian group under the addition induced by connected sum. The zero element of $\Theta^n$ is $S^n$ itself and the inverse elements come with opposite orientation. Later, Kervaire and Milnor elaborated the structure of $\Theta^n$ systematically in their celebrated article “Groups of homotopy spheres: I” [KM63].

Kervaire and Milnor were able to prove the following powerful statement, independent of the seminal articles of Connell [Con67], Newman [New66], Smale [Sma61], Stallings [Sta60], and Zeeman [Zee61] about the topological Poincaré conjecture and the piecewise linear Poincaré conjecture in higher dimensions.

Theorem A (Theorem 1.2, [KM63]). For $n \neq 3$, the group $\Theta^n$ is finite.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $|\Theta^n|$ | 1 | 1 | ? | 1 | 1 | 1 | 28 | 2 | 8 | 6 | 992 | 1 | 3 | 2 | 16256 | 2 | 16 | 16 |

Table 1. The orders of $\Theta^n$ for $1 \leq n \leq 18$.

Thanks to the classical results of Moise [Moi52a, Moi52b], every topological 3-manifold has a unique smooth structure. After the confirmation of the last topological Poincaré conjecture, the missing point in Table 1 was clarified as an immediate corollary of the breakthrough of Perelman.

Theorem B ([Per02, Per03a, Per03b]). The group $\Theta^3$ is trivial, hence $|\Theta^3| = 1$.

Kervaire and Milnor never published “Groups of homotopy spheres: II”; however, Levine’s lecture notes [Lev85] can be considered as its sequel paper. Finding the order of $\Theta^n$ for each value of $n$ is a very challenging problem in algebraic and geometric topology. Moreover, it is closely tied to the smooth Poincaré conjecture in higher dimensions. For the state of the art regarding the order of $\Theta^n$, one can see [IWX20a, Table 1].

The further discussions and results about homotopy theoretical approaches to study $\Theta^n$ can be seen in excellent papers of Hill, Hopkins, and Ranevel [HHR16], Wang and Xu [WX17], and Behrens, Hill, Hopkins, and Mahowald [BHHM20].
1.2. The Successor: \( \Theta^n_Z \). In a similar vein, a homology \( n \)-sphere is a connected \( n \)-manifold \( M \) with \( \partial M = \emptyset \) such that \( M \) has the same homology groups of \( S^n \) in integer coefficients, i.e., \( H_*(M; \mathbb{Z}) = H_*(S^n; \mathbb{Z}) \). The \( n \)-dimensional homology cobordism group \( \Theta^n \) is formed as

\[
\Theta^n = \{ \text{homology } n \text{-spheres up to diffeomorphism} \} / \sim_Z
\]

where the equivalence relation homology cobordism \( \sim_Z \) is depicted for two arbitrary homology \( n \)-spheres \( M_0 \) and \( M_1 \) by

\[
M_0 \sim_Z M_1 \iff \begin{cases} 
\text{There exists a disconnected } (n + 1) \text{-manifold } W \text{ such that} \\
\quad \partial W = -(M_0) \cup M_1, \\
\quad \text{The inclusions induce isomorphisms on all homology groups:} \\
\quad M_0 \hookrightarrow W \hookrightarrow M_1 \Rightarrow H_*(M_0; \mathbb{Z}) \cong H_*(W; \mathbb{Z}) \cong H_*(M_1; \mathbb{Z}).
\end{cases}
\]

Inspired by the novel work of Kervaire and Milnor, González-Acuña defined the object \( \Theta^n_Z \) to decipher the homology \( n \)-spheres in his Ph.D. thesis “On homology spheres” [GAn70b]. Similarly, \( \Theta^n_Z \) admits an abelian group structure with the summation induced by connected sum. The homology cobordism class of \( S^n \) serves the identity element of \( \Theta^n_Z \) and inverse elements can be obtained by reversing the orientation.

Using surgery theory and Milnor’s \( \pi \)-manifolds, González-Acuña was able to construct a group isomorphism between \( \Theta^n \) and \( \Theta^n_Z \) unless \( n = 3 \). Hence, they are algebraically identical except for the single case of \( n = 3 \).

**Theorem C** (Theorem I.2, [GAn70b]). For \( n \neq 3 \), \( \Theta^n_Z \) is isomorphic to \( \Theta^n \). Therefore, \( \Theta^n_Z \) is finite unless \( n = 3 \).

It should be very interesting to compare González-Acuña’s elegant theorem with the following achievement of Kervaire which was published around the same time.

**Theorem D** (Theorem 3, [Ker69]). For \( n \geq 5 \), let \( M \) be a homology \( n \)-sphere. Then there exists a unique homotopy sphere \( \Sigma_M \) such that \( M \# \Sigma_M \) bounds a contractible \( (n + 1) \)-manifold.

1.3. The Aberrant: \( \Theta^3_Z \). The isomorphism of González-Acuña cannot be valid for the last case \( n = 3 \) due to the famous invariant of Rokhlin [Rok52]. There is a surjective group homomorphism from the 3-dimensional homology cobordism group (the homology cobordism group for short) to the cyclic group of order two

\[
\mu : \Theta^3_Z \to \mathbb{Z}_2, \quad \mu(Y) = \sigma(W)/8 \mod 2
\]

where \( W \) is any connected 4-manifold with a \( \mathbb{Z}_2 \)-valued even intersection form, \( \partial W = Y \), and \( \sigma(W) \) denotes the signature of \( W \).

The homology cobordism invariance of the Rokhlin invariant \( \mu \) was first observed in [GAn70b] Section 1.5. See also [GAn70a] Section 2 and [FK20] Section 3.8. Since the Poincaré homology sphere \( \Sigma(2, 3, 5) \) uniquely bounds the negative-definite plumbing \( -E_8 \) of signature \( -8 \), we have \( \mu(\Sigma(2, 3, 5)) = 1 \). Therefore, it is not homology cobordant to \( S^3 \), and we conclude:

**Theorem E** ([Rok52]; Section 1.5, [GAn70b]). The group \( \Theta^3_Z \) is non-trivial.

The non-triviality of \( \Theta^3_Z \) is sensitive to both homology and smoothness conditions on the cobordism 4-manifold. The group would be trivial if at least one of these conditions is removed. See the articles of Rokhlin [Rok51] and Freedman [Fre82], respectively. Also, \( \Theta^3_Z \) is countable by the classical articles of Moise [Moi52a, Moi52b].
Until the 1980s, the only known invariant of $\Theta_3^Z$ was the Rohkin invariant $\mu$ and there was a belief that it might be an isomorphism. However, it later turned out that $\Theta_3^Z$ was very far from being finite. Understanding the infinitude of $\Theta_3^Z$ has led to the construction of numerous invariants of homology 3-spheres.

The seminal works of Matumoto [Mat78] and Galewski and Stern [GS80] yielded a rich connection between the Rohkin invariant $\mu$, the group $\Theta_3^Z$, and the triangulation conjecture. Manolescu revolutionized low-dimensional topology by introducing the Seiberg-Witten (monopole) Pin(2)-equivariant Floer homology, constructing the $\beta$-invariant, and disproving the triangulation conjecture [Man16b]. His $\beta$-invariant is an integer lift of the Rohkin invariant $\mu$ and its existence rejects the triangulation conjecture by relying on the articles [Mat78] and [GS80]. Consult Section 2.4 for details.

The several variations of Manolescu’s Floer homotopic approach have led to the invention of new powerful theories and sensitive invariants of knots and manifolds. From now on, we will aim to broadly discuss the structure of $\Theta_3^Z$ from both algebraic and geometric perspectives and present various open problems. In a nutshell, we create Table 2 to reflect the sharp contrast between homotopy and homology cobordism groups.

| Dimension | $\Theta^n$ | $\Theta_3^n$ |
|-----------|------------|-------------|
| $n \neq 3$ | $< \infty$ | $< \infty$ |
| $n = 3$    | $= 1$      | $= \infty$  |

Table 2. A comparison of the groups $\Theta^n$ and $\Theta_3^n$.

2. THE STRUCTURE OF $\Theta_3^Z$

2.1. Subgroups and Summands of $\Theta_3^Z$. The celebrated work of Donaldson was a cornerstone in the history of low-dimensional topology [Don83]. Motivated by his article, Fintushel and Stern studied the gauge theory of orbifolds, produced the gauge theoretical $R$-invariant for Seifert fibered homology spheres, and provided the first existence of an infinite subgroup in the homology cobordism group.

**Theorem F** (Theorem 1.2, [FS85]). The group $\Theta_3^Z$ has a $\mathbb{Z}$ subgroup generated by the Poincaré homology sphere $\Sigma(2,3,5)$.

Converting the ideas on end-periodic 4-manifolds in the work of Taubes [Tau87] to cylindrical end 4-manifolds and using the Fintushel-Stern $R$-invariant, Furuta showed the first existence of an infinitely generated subgroup [Fur90].

**Theorem G** (Theorem 2.1, [Fur90]). The group $\Theta_3^Z$ has a $\mathbb{Z}^\infty$ subgroup in $\Theta_3^Z$ generated by the family of Brieskorn spheres $\{\Sigma(2,3,6n-1)\}_{n=1}^\infty$.

The eminent article of Floer [Flo88] changed the flow of the history of low-dimensional topology dramatically. Given a homology 3-sphere $Y$, his theory of instanton homology can be defined over the Yang-Mills equations on $Y \times \mathbb{R}$. This novel invariant is an infinite dimensional analogue of the Morse homology.

The next achievement about the algebraic structure of $\Theta_3^Z$ was owed to Frøyshov [Fro02]. His approach relied on the equivariant structure on Floer’s instanton (Yang-Mills) homology and he constructed the $h$-invariant, a surjective group homomorphism $h : \Theta_3^Z \rightarrow \mathbb{Z}$.
Theorem H (Theorem 3, [Frø02]). The group $\Theta_3^Z$ has a $\mathbb{Z}$ summand generated by the Poincaré homology sphere $\Sigma(2, 3, 5)$.

Ozsváth and Szabó developed the theory of Heegaard Floer homology in a series of prominent articles [OS03a, OS04b, OS04c]. Since then it has been used to answer various problems in low-dimensional topology and several new versions emerged successively, see the comprehensive surveys of Ozsváth and Szabó [OS04a] and Juhász [Juh15]. Later, Hendricks and Manolescu introduced involutive Heegaard Floer homology [HM17] and this new theory exploits the conjugation symmetry on a Heegaard Floer complex of the Heegaard Floer homology. Also, it is conjecturally a $\mathbb{Z}_4$-equivariant version of Seiberg-Witten Pin(2)-equivariant Floer homology established by Manolescu [Man16b].

The most recent impressive progress about deciphering the algebraic complexity of the group $\Theta_3^Z$ was achieved by Dai, Hom, Stoffregen, and Truong [DHST18]. Using the machinery of involutive Heegaard Floer homology, they defined a new family of powerful and sensitive set of invariants $\vec{f} = \{f_k\}_{k \in \mathbb{N}}$, a surjective group homomorphism $\vec{f} : \Theta_3^Z \twoheadrightarrow \mathbb{Z}_\infty$.

Theorem I (Theorem 1.1, [DHST18]). The group $\Theta_3^Z$ has a $\mathbb{Z}_\infty$ summand generated by the family of Brieskorn spheres $\{\Sigma(2n+1, 4n+1, 4n+3)\}_{n=1}^\infty$.

Their proof subsumes several approaches and techniques that consecutively appeared in the literature of involutive Heegaard Floer homology [HMZ18], [DM19], [DS19], and [HHL21]. Moreover, involutive Floer theoretic invariants have provided a major change for the understanding of the structure of $\Theta_3^Z$ and its subgroups. For details of constructions and ideas, one can consult the survey of Hom [Hom21].

Relying on these results, one may expect that there is no torsion part in the decomposition of $\Theta_3^Z$, see Section 2.4 for details. Therefore, we want to state the first open problem about the structure of $\Theta_3^Z$:

Problem A. Is the group $\Theta_3^Z$ in fact $\mathbb{Z}_\infty$?

2.1.1. A Recovery: More about Subgroups of $\Theta_3^Z$. The Fintushel-Stern $R$-invariant leads to a powerful obstruction for homology 3-spheres bounding homology 4-balls, and hence contractible 4-manifolds. It is easily computable due to the short-cut of Neumann and Zagier [NZ85]. The $R$-invariant simply ensures the proofs of items (1) and (3) in Theorem J. Furthermore, these claims can be deduced by using the Ozsváth-Szabo $d$-invariant [OS03a]. See the papers of Tweedy [Twe13] and Karakurt and the author [KS20] for sample computations, which both depended on Floer homology of plumbings [OS03b], Némethi’s lattice homology [Ném05], and lattice point counting technique of Can and Karakurt [CK14].

However, the item (2) in Theorem J is a consequence of the non-vanishing Neumann-Siebenmann invariant $\bar{\mu}$ [Neu80] and [Sie80]. The homology cobordism invariance of $\bar{\mu}$ for Seifert fibered homology spheres was first proved by Saveliev [Sav98b], see also the paper of Dai and Stoffregen [DS19] for a generalization of this result. Saveliev provided another proof for the item (2) in [Sav98a] by using Furuta’s 10/8 theorem [Fur01]. All other homology cobordism invariants behaved differently than $\bar{\mu}$ seem to be vanished or not arbitrarily large for this family, so they do not give further information about their homology cobordism classes.

The $\bar{\mu}$-invariant of Seifert fibered homology spheres is same with the $w$-invariant of Fukumoto and Furuta [FF00], see the works of Fukumoto, Furuta, and Ue [FFU01] and Saveliev [Sav02a] for details. Also, Stoffregen proved that the Manolescu invariant $\beta$ [Man16b] agrees with $-\bar{\mu}$ for Seifert fibered homology spheres [Sto20].
Furthermore, thanks to the work of Dai and Manolescu [DM19], it was shown that the involutive correction terms $\partial$ are $\bar{d}$ are respectively equal to $-2\bar{\mu}$ and $d$ for Seifert fibered spheres.\cite{DM19}

By following the works of Nozaki, Sato and Taniguchi [NST19] and Baldwin and Sivek [BS22], the proofs of items (1) and (5) in Theorem J can be deduced respectively. Moreover, the items (6) and (7) in Theorem J are owed to the recent article of Daemi, Imori, Sato, Scaduto, and Taniguchi [DIS+22]. Note that the proof arguments of the latter two articles essentially require the result of the first one.

**Theorem J.** The following homology 3-spheres individually generate $\mathbb{Z}$ subgroups in $\Theta^2_3$:

1. $\Sigma(p, q, pqn - 1)$ for each $n \geq 1$,
2. $\Sigma(p, q, pqn + 1)$ for each odd $n \geq 1$,[12]
3. $\Sigma(p_n, q_n, r_n)$ for each $n \geq 1$ where $p_n q_n + p_n r_n - q_n r_n = 1$,
4. For each $n \geq 1$, $S^3_{1/n}(K)$ where $K$ is any knot in $S^3$ with $h(S^3_1(K)) < 0$,\cite{18}
5. For each $n \geq 1$, $S^3_{1/n}(K)$ where $K$ is any knot in $S^3$ with $\tau(K) > 0$[14]
6. For each $n \geq 1$, $S^3_{1/n}(K)$ where $K$ is any knot in $S^3$ with $\bar{s}(K) > 0$,\cite{15}
7. For each $n \geq 1$, $S^3_{1/n}(K)$ where $K$ is any knot in $S^3$ with $\sigma(K) \leq 0$ and $\frac{1}{8} < \Gamma_K \left(-\frac{1}{2}\sigma(K)\right)$.

After Furuta’s work, the first recovery of the existence of $\mathbb{Z}_2^\infty$ subgroups of $\Theta^3_3$ was provided by Fintushel and Stern [FS90, Theorem 5.1] for the item (1) in Theorem K. Their approach can be applied to item (2) in Theorem K as well. These two results can be reproved successfully by using new gauge and instanton theoretic invariants of Daemi [Dae20], Nozaki, Sato and Taniguchi [NST19], and Baldwin and Sivek [BS22]. However, the classical and involutive Heegaard Floer theoretical invariants cannot identify the linear independence of the item (1) in $\Theta^3_3$.

The Seiberg-Witten and/or Heegaard Floer originated invariants may detect the linear independence of subfamilies of the item (2) in Theorem K. In this regard, see the works of Stoffregen [Sto17] and Dai and Manolescu [DM19]. However, it is not easily doable in general, see the discussion in [KS20] and [KS22] and compare with [Sto17] and [DM19].

For the proofs of items (3), (4), (5), and (6) in Theorem K one can see the articles of Nozaki, Sato and Taniguchi [NST19], Baldwin and Sivek [BS22], and Daemi, Imori, Sato, Scaduto, and Taniguchi [DIS+22]. The methodology of [NST19] and [DIS+22] both refer to the equivariant instanton Floer theory with Chern-Simons filtration, while [BS22] uses the framed instanton homology. These articles all provide new invariants for homology 3-spheres and knots.

**Theorem K.** The following infinite families of homology 3-spheres generate $\mathbb{Z}_2^\infty$ subgroups in $\Theta^3_3$:

1. $\{\Sigma(p, q, pqn - 1)\}_{n=1}^\infty$,
2. $\{\Sigma(p_n, q_n, r_n)\}_{n=1}^\infty$ where $p_n q_n + p_n r_n - q_n r_n = 1$,
3. $\{S^3_{1/n}(K)\}_{n=1}^\infty$ for any knot $K$ in $S^3$ with $h(S^3_1(K)) < 0$,
4. $\{S^3_{1/n}(K)\}_{n=1}^\infty$ for any knot $K$ in $S^3$ with $\tau(K) > 0$[16]
5. $\{S^3_{1/n}(K)\}_{n=1}^\infty$ for any knot $K$ in $S^3$ with $\bar{s}(K) > 0$,
6. $\{S^3_{1/n}(K)\}_{n=1}^\infty$ for any knot $K$ in $S^3$ with $\sigma(K) \leq 0$ and $\frac{1}{8} < \Gamma_K \left(-\frac{1}{2}\sigma(K)\right)$.\cite{17}
Since all current homology cobordism invariants are blind to detect its linear independence in $\Theta^3_Z$, we curiously ask the following problem. On the other hand, they might be homology cobordant in $\Theta^3_Z$. If so, this will be a very interesting result.

**Problem B.** For odd values of $n$, does the family $\{\Sigma(p,q,pqn + 1)\}_{n=1}^{\infty}$ generate a $\mathbb{Z}^\infty$ subgroup or a $\mathbb{Z}^\infty$ summand in $\Theta^3_Z$?

The $R$- and $w$-invariants were successfully generalized in the articles of Lawson [Law88] and Fukumoto [Fuk11] respectively. Given a Seifert fibered sphere $Y = \Sigma(a_1, \ldots, a_n)$, we respectively denote these invariants by $R(Y, e)$ and $w(Y, m)$ and call the generalized $R$-invariant and the generalized $w$-invariant where $e$ is an integer depending on Euler number and some other constraints and $m$ is a tuple of integers. The generalized $R$- and $w$-invariants are strictly powerful than the classical $R$- and $w$-invariants, and provide more sensitive obstructions for the existence of homology cobordisms between homology 3-spheres. For sample computations, see [Fuk11, Section 6].

Using $\text{Pin}(2)$-equivariant Seiberg-Witten Floer K-theory, Manolescu constructed the integer-valued homology cobordism invariant $\kappa$ [Man14]. Recently, Ue proved that the behaviours of the $\kappa$ invariant and the minus version of the $\mu$ invariant for Seifert fibered spheres are very similar [Ue22]: $\kappa(Y) + \mu(Y) = 0$ or 2. Relying on the Seiberg-Witten Floer spectrum and $\text{Pin}(2)$-equivariant KO-theory and inspiring the construction of the Manolescu $\kappa$-invariant, J. Lin extracted new invariants $\kappa_0^k$ of $\Theta^3_Z$ where $k \in \mathbb{Z}_8$ [Lin15].

We list the following optimistic problem to understand behaviours of invariants more for Seifert fibered spheres by taking the risk of having negative answers.

**Problem C.** For Seifert fibered spheres $Y = \Sigma(a_1, \ldots, a_n)$, what are the possible relations between the following homology cobordism invariants?

- $\mu(Y)$, $w(Y; m)$, and $\kappa_0^k(Y)$,
- $d(Y)$ and $R(Y; e)$.

2.1.2. A Diversification: More about Summands of $\Theta^3_Z$. Around the 2000s, two more epimorphisms of $\Theta^3_Z$ occurred: Ozsváth-Szabó $d$-invariant [OS03a] and Froyshov $\delta$-invariant [Frø10]. The latter invariant is also owed to Kronheimer and Mrowka [KM07]. Thanks to the seminal articles of Kutluhan, Lee, and Taubes [KLT20c, KLT20a, KLT20b, KLT20b], we know that $\delta = -d/2$.

Given any relatively coprime positive integers $p, q$ and $r$, the Brieskorn sphere $\Sigma(p, q, r + pq)$ can be obtained by the Brieskorn sphere $\Sigma(p, q, r)$ by applying $(-1)$-surgery along the singular fiber of degree $r$. This topological operation is called *Seifert fiber surgery*, see the paper of Lidman and Tweedy [LT18] for a detailed exposition.

Performing the above type of Seifert fibered surgeries, the items (2) and (4) in Theorem L can be constructed from the items (1) and (3) in Theorem L respectively. We know that the $d$-invariant remains same under this special Seifert fiber surgery, consult the articles of Lidman and Tweedy [LT18], Karakurt, Lidman, and Tweedy [KLT21], and Seetharaman, Yue, and Zhu [SYZ21] for this result. Relying on the computations in [Twe13] and [KS20] again, we have the following result.

**Theorem L.** The following homology 3-spheres individually generate $\mathbb{Z}$ summands in $\Theta^3_Z$:

1. $\Sigma(p, q, pqn - 1)$ for each $n \geq 1$,
2. $\Sigma(p, q, +pqn - 1 + pqm)$ for each $n, m \geq 1$,
(3) $\Sigma(p_n, q_n, r_n)$ for each $n \geq 1$ where $p_nq_n + p_nr_n - q_nr_n = 1$,
(4) $\Sigma(p_n, q_n, r_n + p_nq_nm)$ for each $n, m \geq 1$ where $p_nq_n + p_nr_n - q_nr_n = 1$.

In a similar fashion, we can pass the Brieskorn sphere $\Sigma(p, q, r + 2pq)$ from the
Brieskorn sphere $\Sigma(p, q, r)$ by applying twice $(-1)$-surgery along the singular fiber of
degree $r$. In [SYZ21], Seetharaman, Yue, and Zhu also observed that the maximal
monotone subroots carrying the Floer theoretic invariants do not change after performing
the above type of Seifert fiber surgeries consecutively. Recently, in [KS22],
Karakurt and the author presented more families of homology 3-spheres generating
infinite rank summands in $\Theta^3_\mathbb{Z}$ by computing their connected Heegaard Floer
cobordism [HHL21] effectively and using the invariants of Dai, Hom, Stoffregen, and
Truong. Notice that the connected Heegaard Floer homology was introduced by
Hendricks, Hom, and Liidman. Further, they proved that it is a homology cobordism
invariant itself [HHL21] unlike the classical or involutive Heegaard Floer homology.

Together with the above observation, we can conclude the following theorem.
In particular, the items (1) and (2) in Theorem M and the family of Dai, Hom,
Stoffregen, and Truong are not homology cobordant to each other with a single
exception, see the discussion in [KS22]. Furthermore, the main result in [DHST18]
was reproven in Rostovtsev’s paper [Ros20] by providing a new homomorphism of $\Theta^3_\mathbb{Z}$
via the immersed curves machinery of Kotelskiy, Watson, and Zibrowius [KWZ19].
This approach can be applied to the additional families in [KS22].

**Theorem M.** The following infinite families of homology 3-spheres generate $\mathbb{Z}^\infty$
summands in $\Theta^3_\mathbb{Z}$:

1. $\{\Sigma(2n + 1, 3n, 2, 6n + 1)\}_{n=1}^\infty$,
2. $\{\Sigma(2n + 1, 3n + 1, 6n + 5)\}_{n=1}^\infty$,
3. $\{\Sigma(2n + 1, 4n + 1, 4n + 3 + 2m(2n + 1)(4n + 1))\}_{n,m=1}^\infty$,
4. $\{\Sigma(2n + 1, 3n + 2, 6n + 1 + 2m(2n + 1)(3n + 2))\}_{n,m=1}^\infty$,
5. $\{\Sigma(2n + 1, 3n + 1, 6n + 5 + 2m(2n + 1)(3n + 1))\}_{n,m=1}^\infty$.

2.1.3. *A Modification: The Rational Version of $\Theta^3_\mathbb{Z}$.* Changing the role of integer
coefficients with rational ones, we can define the *rational homology cobordism group*
$\Theta^3_{\mathbb{Q}}$. We have a natural group homomorphism $\psi : \Theta^3_\mathbb{Z} \to \Theta^3_{\mathbb{Q}}$ induced by inclusion.
It is known that the map $\psi$ is not injective since there exists homology 3-spheres
that represent non-trivial elements in $\text{Ker}(\psi)$ by the work of Fintushel and Stern
[FS84], Akbulut and Larson [AL18], the author [Sav20b], and Simone [Sim21] [M].

**Theorem N.** The following homology 3-spheres bound rational homology 4-balls
but do not bound homology 4-balls. Therefore, they non-trivially lie in $\text{Ker}(\psi)$ since
they all have non-vanishing Rokhlin invariant:

1. $\Sigma(2, 3, 7), \Sigma(2, 3, 19)$,
2. $\Sigma(2, 4n + 1, 12n + 5), \Sigma(3, 3n + 1, 12n + 5)$ for odd $n \geq 1$,
3. $\Sigma(2, 4n + 3, 12n + 7), \Sigma(3, 3n + 2, 12n + 7)$ for even $n \geq 2$,
4. $S^3_m(K_n)$ where $K_n$ is the twist knot for odd $n \geq 1$.

Furthermore, $\text{Ker}(\psi)$ has a $\mathbb{Z}$ subgroup generated by any single homology 3-sphere
listed above except those in [M] because they have non-zero $\bar{\mu}$-invariants. One can
expect that $\text{Ker}(\psi)$ might be larger than $\mathbb{Z}$, including some linearly independent
infinite subset of these homology 3-spheres. Thus we inquire:

**Problem D.** Does $\text{Ker}(\psi)$ contain $\mathbb{Z}^\infty$ subgroup or $\mathbb{Z}^\infty$ summand?

It is worthy to note that all current homology cobordism invariants cannot detect
the linear independence of Brieskorn spheres listed in Theorem M in $\Theta^3_{\mathbb{Z}}$, see the
discussion in Subsection 2.1.1. This is also true for Simone’s family, see surgery formulae of the relevant homology cobordism invariants.

The existence of these homology 3-spheres has a nice application in symplectic geometry. Let \((X, \omega)\) be a symplectic 4-manifold. A Stein domain is a triple \((X, J, \phi)\) such that \(J\) is complex structure on \(X\) and \(\phi : X \to \mathbb{R}\) is a proper plurisubharmonic function. Here, \(\phi\) provides compact level sets and a symplectic form: \(\phi\) is smooth such that \(\phi^{-1}((-\infty, c])\) is compact for all \(c \in \mathbb{R}\) and \(\omega_\phi(v, w) = -d(d\phi \circ J)(v, w)\) gives a symplectic form. The handle decompositions of Stein domains are completely characterized in the celebrated articles of Eliashberg [El i90] and Gompf [Gom98]: A 4-manifold is a Stein domain if and only if it has a handle decomposition with 0-handles, 1-handles, and 2-handles; and the 2-handles are attached along Legendrian knots with framing \(tb - 1\), where \(tb\) denotes the Thurston-Bennequin number.

If we choose any homology 3-sphere listed in Theorem N, then the handle decomposition of the corresponding rational ball must contain 3-handles by an algebraic topology argument. Then, the above characterization indicates that such a rational homology 4-ball cannot be a Stein domain. Mazur manifolds are potential candidates of Stein domains, but this is not the case for all Mazur manifolds, see the impressive work of Mark and Tosun [MT18].

In addition to the non-injectivity of the \(\psi\), we know that it is not surjective. In particular, Kim and Livingston proved that \(\text{Coker}(\psi)\) has a \(\mathbb{Z}_\infty\) subgroup [KL14]. Similarly, we can ask:

**Problem E.** Does \(\text{Coker}(\psi)\) contain \(\mathbb{Z}_\infty\) summand?

### 2.2. The Trivial Element of \(\Theta_3^3\)

A connected 4-manifold with boundary is called a homology 4-ball if it shares the same homology groups of the 4-ball in integer coefficients. An easy algebraic topology argument indicates that a homology 3-sphere is homology cobordant to \(S^3\) if and only if it bounds a homology 4-ball. The latter implication and its slight stronger version is one of the central and unresolved problems in low-dimensional topology:

**Problem F** (Problem 4.2, [Kir78b]). Which homology 3-spheres bound contractible 4-manifolds or homology 4-balls?

There are plenty of examples of Brieskorn spheres that bound contractible 4-manifolds. Following Kirby’s celebrated work [Kir78a], the classical articles appeared subsequently: Akbulut and Kirby [AK79], Casson and Harer [CH81], Stern [Ste78], Fintushel and Stern [FS81], Maruyama [Mar81, Mar82], and Fickle [Fic84]. In addition, some of these results were found independently of Kirby calculus, see Fukuhara [Fuk78] and Martin [Mar79]. Some of these families also bound generalized Mazur manifolds, contractible 4-manifolds built with only 0-, 1-, and 2-handles, see [Sav20a].

**Theorem O.** The following homology 3-spheres bound Mazur manifolds with one 0-handle, one 1-handle and one 2-handle:

- \(\Sigma(2, 3, 13), \Sigma(2, 3, 25), \Sigma(2, 7, 19), \Sigma(3, 5, 19)\).
- \(\Sigma(p, ps - 1, ps + 1)\) for \(p\) even and \(s\) odd,
- \(\Sigma(p, ps \pm 1, ps \pm 2)\) for \(p\) odd and \(s\) arbitrary,
- \(\Sigma(2, 2s \pm 1, 2s \pm 1)\) for \(s\) odd,
- \(\Sigma(3, 3s \pm 1, 3s \pm 1, 3s \pm 2)\) for \(s\) arbitrary,
- \(\Sigma(3, 3s \pm 2, 3s \pm 2, 3s \pm 1)\) for \(s\) arbitrary.

Further, \(\Sigma(2, 7, 47)\) and \(\Sigma(3, 5, 49)\) bound homology 4-balls.
It would be interesting to compare the existence of homology $3$-spheres bounding contractible $4$-manifolds and homology $4$-balls, so we may address the following problem. The possible candidates for Seifert fibered spheres are two examples of Fickle: $\Sigma(2,7,47)$ and $\Sigma(3,5,49)$. They are known to bound only homology $4$-balls.

**Problem G.** Is there any Seifert fibered sphere $\Sigma(a_1,\ldots,a_n)$ which bound a homology $4$-ball but not a contractible $4$-manifold?

Note that Problem G is known for $\Sigma(2,3,5)\#-\Sigma(2,3,5)$. It cannot bound a contractible $4$-manifold, see Taubes’ article [Taubes87, Proposition 1.7]. In the light of previous results, we can also address the following explicit problem:

**Problem H.** Do the Brieskorn spheres $\Sigma(2,3,6n+1)$ bound rational homology $4$-balls (resp. homology $4$-balls) for odd $n \geq 5$ (resp. even $n \geq 6$)?

When the number of singular fibers increases, there is a bold conjecture, which was first proposed by Fintushel and Stern [FS87] and later highlighted by Kollár [Kol08]. Note that some computational verifications of this conjecture was provided in the paper of Lawson [Law88].

**Problem I (Three Fibres Conjecture).** For $n > 3$, is there any Seifert fibered sphere $\Sigma(a_1,\ldots,a_n)$ bounding a homology $4$-ball?

Problem I cannot be generalized for plumbed homology $3$-spheres that are not Seifert fibered. The first examples were given by Maruyama [Mar82], independently obtained by Akbulut and Karakurt [AK14]. In [Sav20a], we presented two more family of plumbed homology $3$-spheres bounding contractible $4$-manifolds.

**Theorem P** (Theorem 1, [Mar82]; Theorem 1.4, [AK14], Theorem 1.4-5, [Sav20a]). Let $X(n)$, $X'(n)$, and $W(n)$ be Maruyama, the companion of Maruyama, and Ramanujam plumbed $4$-manifold, shown in Figure 1. Then for each $n \geq 1$, boundaries $\partial X(n)$ and $\partial X'(n)$ bound Mazur manifolds with one $0$-handle, one $1$-handle and one $2$-handle. Further, the boundary of $\partial W(n)$ bounds a generalized Mazur manifold with one $0$-handle, two $1$-handles and two $2$-handles for $n \geq 1$.

![Plumbing graphs](image)

**Figure 1.** The plumbing graphs of $X(n)$, $X'(n)$, and $W(n)$.

Note that $W(1)$ is known as Ramanujam surface, the famous homology plane constructed by Ramanujam [Ram71]. It is the first example of an algebraic complex smooth surface sharing the same of homology of the complex plane $\mathbb{C}^2$ but not...
analytically isomorphic to $\mathbb{C}^2$. We call non-trivial homology 3-sphere boundaries of homology planes \textit{Kirby-Ramanujam spheres} if they also bound Mazur type manifolds. In an upcoming work [AS22], several infinite families of Kirby-Ramanujam spheres will be provided in the light of Problem F.

Using the surgery descriptions of $\Sigma(p,q,pq \mp 1)$ in terms of torus knots, one can prove the following theorem as an immediate corollary of the main results of Gordon [Gor75] and Lidman, Karakurt, and Tweedy [KLT21]. For the constructive part, an alternative direct proof can be given by finding the plumbing graphs of splices explicitly [ENS85] and doing Kirby calculus. The obstruction of knots bounding smooth disks requires the result of Lidman and Tweedy [LT18].

**Theorem Q.** Let $K(pq \mp 1)$ denote the singular fiber in $\Sigma(p,q,pq \mp 1)$. Then $K(pq \mp 1)$ is not smoothly slice in $\Sigma(p,q,pq \mp 1)$, and $\Sigma(p,q,pq \mp 1)$ does not bound a contractible 4-manifold. However, the following splicing homology spheres bound generalized Mazur manifolds with one 0-handle, $p$ 1-handles, and $p$ 2-handles:

$$\Sigma(p,q,pq - 1) \searrow K(pq - 1) \nearrow K(pq + 1) \Sigma(p,q,pq + 1).$$

Independent results of Hirsch, Rokhlin, and Wall around the 1960s indicate that every homology 3-sphere is smoothly embedded in $S^5$, see [Hir61], [Rok65] and [Wal65]. Making the target space smaller, we may ask which homology 3-spheres can be embedded in $S^4$. In the topological category, the problem has a complete answer thanks to Freedman’s celebrated article [Fre82]: every homology 3-sphere is topologically embedded in $S^4$. Adding an extra smoothness condition, we can address another wide open problem in low-dimensional topology.

**Problem J** (Problem 3.20, [Kir78b]). Which homology 3-spheres can be smoothly embedded in $S^4$?

Another simple algebraic topology observation indicates that a homology 3-sphere smoothly embedded in $S^4$ splits $S^4$ into two homology 4-balls. Therefore, homology cobordism invariants provide obstructions for the smooth embeddings of homology 3-spheres in $S^4$.

One can wonder the reverse direction of the above observation. Studying branched coverings of cross sectional slices of knotted 2-spheres in $S^4$, McDonald provided the first examples of homology 3-spheres which are smoothly embedded in a homology 4-ball but not any homotopy 4-sphere [McD22]. His examples are certain double cyclic branched coverings of spuns of torus knots. We may address this implication to Seifert fibered manifolds and ask:

**Problem K.** Is there any Seifert fibered sphere which bounds a homology 4-ball but cannot be smoothly embedded in $S^4$?

2.3. **Generators of $\Theta^3_Z$.** The first result concerning the generators of $\Theta^3_Z$ was owed to Freedman and Taylor.

**Theorem R** (Corollary 1B, [FT77]). The group $\Theta^3_Z$ is generated by homology 3-spheres which are boundaries of 4-manifolds having the integral homology of $S^2 \times S^2$.

A homology 3-sphere $Y$ is called \textit{irreducible} if every embedded 2-sphere $S^2$ in $Y$ is the boundary of an embedded $B^3$. Livingston showed that irreducible homology 3-spheres are generic enough to generate the homology cobordism group.

**Theorem S** (Theorem 3.2, [Liv81]). The group $\Theta^3_Z$ is generated by irreducible homology 3-spheres.
We call a homology 3-sphere $Y$ hyperbolic if $Y$ is a geodesically complete Riemannian 3-manifold of constant sectional curvature $-1$. The geodesically completeness requires that at any point $p \in Y$, the geodesic exponential map $\exp_p$ on $T_pY$ is the entire tangent space at $p$. Myers proved that every homology cobordism class admits a hyperbolic representative.

**Theorem T** (Theorem 5.1, [Mye83]). The group $\Theta^3_Z$ is generated by hyperbolic homology 3-spheres.

A pair $(Y, \xi)$ is called Stein fillable if there is a Stein domain $(X, J, \phi)$ where $\phi$ is bounded below, $Y$ is an inverse image of a regular value of $\phi$, and $\xi = \ker(-d\phi \circ J)$ is an induced contact structure. Mukherjee showed that the generator set of $\Theta^3_Z$ can be chosen as Stein fillable homology 3-spheres [Muk20].

**Theorem U** (Theorem 1.5, [Muk20]). The group $\Theta^3_Z$ is generated by Stein fillable homology 3-spheres.

In contrast to the above positive directional results, various computations of homology cobordism invariants of homology 3-spheres lead to the following observation of Frøyshov [Frø16], Stoffregen [Sto17], Lin [Lin17], and Nozaki, Sato, and Taniguchi [NST19].

**Theorem V.** There exist several infinite families of homology 3-spheres that are not homology cobordant to any Seifert fibered homology sphere.

In [HHSZ20], Hendricks, Hom, Stoffregen, and Zemke established a surgery exact triangle formula for the involutive Heegaard Floer homology. As an application, they provided a homology 3-sphere not homology cobordant to any linear combination of Seifert fibered spheres. [HHSZ20, Theorem 1.1]. This manifold is obtained by integral Dehn surgery on a combination of torus knots and a cable of a torus knot: $S^3_{+1}(-T_{6,7}#T_{6,13}#-T_{2,3,2,5})$. Hence, Seifert fibered manifolds are not generic enough to generate $\Theta^3_Z$.

**Theorem W** (Theorem 1.1., [HHSZ20]). The Seifert fibered spheres cannot generate the group $\Theta^3_Z$. Therefore, $\Theta^3_{SF}$ is the proper subgroup of $\Theta^3_Z$. Further, $\Theta^3_Z/\Theta^3_{SF}$ has a $\mathbb{Z}$ subgroup.

Here, $\Theta^3_{SF}$ denotes the subgroup of $\Theta^3_Z$ generated by Seifert fibered spheres. Note that $S^3 = \Sigma(1, q, r)$. By using Kirby calculus, Nozaki, Sato, and Taniguchi proved that the example of Hendricks, Hom, Stoffregen, and Zemke is a graph homology 3-sphere, see [NST19, Appendix A]. Therefore, we can ask the following question as to the next step of obstructions:

**Problem L.** Do graph homology 3-spheres generate the group $\Theta^3_Z$?

Let $\Theta^3_G$ denote the subgroup of $\Theta^3_Z$ generated by graph homology 3-spheres. The previous problem is equivalent to asking whether $\Theta^3_G = \Theta^3_Z$ or not. Nozaki, Sato, and Taniguchi proposed a strategy in [NST19, Conjecture 1.19] so that likely $\Theta^3_G \leq \Theta^3_Z$.

Hendricks, Hom, Stoffregen, and Zemke compared the subgroup $\Theta^3_{SF}$ with the whole group $\Theta^3_Z$ in another work and they were able to provide the existence of an infinitely generated subgroup in the quotient $\Theta^3_Z/\Theta^3_{SF}$ spanned by the family of homology 3-spheres $S^3_{+1}(-T_{2,3,4}#-2T_{2n,2n+1}#-T_{2n,4n+1})$ for odd $n \geq 3$.

**Theorem X** (Theorem 1.1., [HHSZ21]). The quotient $\Theta^3_Z/\Theta^3_{SF}$ has a $\mathbb{Z}^\infty$ subgroup.

The new immediate challenge would be to ask:
Problem M. Does the quotient $\Theta^3_Z/\Theta^3_{SF}$ contain a $\mathbb{Z}^\infty$ summand?

Another curiosity about the possible generators of $\Theta^3_Z$ is of course surgeries on knots in the 3-sphere. One can expect that these manifolds are not sufficient enough to provide a generator set for $\Theta^3_Z$, see [NST19, Corollary 1.7]. However, the following problem still remains open.

Problem N. Do surgeries on knots in $S^3$ generate $\Theta^3_Z$?

2.4. Torsion of $\Theta^3_Z$. In their seminal articles, Matumoto [Mat78] and Galewski and Stern [GS80] reduced the triangulation conjecture to a problem about the interplay between 3- and 4-manifolds up to homology cobordism. Since then $\Theta^3_Z$ has been a very attractive object in low-dimensional topology.

Theorem Y ([Mat78], [GS80]). For $n \geq 5$, there exist non-triangulable topological $n$-manifolds if and only if the following exact sequence does not split

\[ 0 \rightarrow \ker(\mu) \rightarrow \Theta^3_Z \rightarrow \mathbb{Z}_2 \rightarrow 0. \]

Nearly thirty-five years later, Manolescu [Man16b] constructed Pin(2)-equivariant Seiberg-Witten Floer homology and provided three sensitive invariants of homology 3-spheres. They are called $\alpha, \beta,$ and $\gamma$ invariants of $\Theta^3_Z$. Specifically, the Manolescu $\beta$-invariant has the following three crucial properties:

1. $\beta(-Y) = -\beta(Y)$,
2. $- \beta(Y) = \mu(Y) \mod 2$ where $\mu$ is the Rokhlin invariant,
3. $\beta$ is an invariant of $\Theta^3_Z$.

The existence of the Manolescu $\beta$-invariant guaranteed that the exact sequence (*) does not split and leads to the disproof of the triangulation conjecture, see [Man16a, Man16b, Man18]. For this achievement, the homology cobordism invariance of the Manolescu $\beta$-invariant is particularly critical because beforehand there exist invariants satisfying properties both (1) and (2) but not (3), for instance, the Casson invariant $\lambda$. Therefore, it cannot be used for the rejection of the triangulation conjecture for high-dimensional manifolds; however, it is sufficient for disproval of the conjecture for the particular case of $n = 4$. See the book of Akbulut and McCarthy [AM90] for details. For an alternative disproof of the triangulation conjecture for high-dimensional manifolds, one can consult F. Lin’s monograph [Lin18].

Since the Manolescu $\beta$ invariant provides an integral lift of the Rokhlin invariant $\mu$, he also ruled out the existence of $\mathbb{Z}_2$ torsion in $\Theta^3_Z$ for the following type of homology 3-spheres:

Theorem Z (Corollary 1.2, [Man16b]). Let $Y$ be a homology 3-sphere such that $\mu(Y) = 1$. Then $Y$ cannot represent $\mathbb{Z}_2$ torsion in $\Theta^3_Z$. In other words, $Y \# Y$ cannot bound a homology 4-ball.

Currently, we do not know whether there exists a homology 3-sphere $Y$ with a vanishing $\mu$-invariant so that $Y \# Y$ bounds a contractible 4-manifold or a homology 4-ball. Also, we have no further obstructions for other type of torsion in $\Theta^3_Z$. Hence we curiously state the final problem of our survey:

Problem O. Does the group $\Theta^3_Z$ contain any torsion $\mathbb{Z}_n$ for $n \geq 2$? Modulo torsion, is $\Theta^3_Z$ free abelian?

Only for the $\mathbb{Z}_2$ type torsion, there are some new candidates founded in the recent work of Boyle and Chen [BC22]. These examples originate from cyclic double branched coverings of $S^3$ along certain non-slice strongly negative amphichiral knots of determinant 1.
3. Appendix: Examples of Homology 3-Spheres

In the wide world of closed connected oriented 3-manifolds, there is a simple characterization of homology 3-spheres $Y$ thanks to Poincaré duality and universal coefficient theorem: $H_1(Y;\mathbb{Z}) = 0$. Since the abelianization of $\pi_1(Y)$ gives $H_1(Y;\mathbb{Z})$ due to Hurewicz theorem, they are even easily recognized. In this appendix, we discuss several constructions of homology 3-spheres, our main references are Neumann and Raymond [NR78], Eisenbud and Neumann [EN85], Gompf and Stipsicz [GS99], Saveliev [Sav02b], and Akbulut [Akb16].

The first example of homology 3-spheres was provided by Poincaré [Poi04] as a counter-example to the first version of Poincaré conjecture. Its revised version was proved by Perelman, see the introduction. This 3-manifold is known as counter-example to the first version of Poincaré conjecture. Its revised version was proved by Perelman, see the introduction. This 3-manifold is known as Poincaré homology sphere and the exposition of Kirby and Scharlemann can be seen for the eight equivalent descriptions of the Poincaré homology sphere [KS79].

The next source for homology 3-spheres was deciphered by Dehn [Deh38] by providing a passage from 1-manifolds -knots and links- to 3-manifolds via the topological operation called surgery. A knot $K$ is a smooth embedding of a circle $S^1$ into $S^3$ and a link $L$ is a disjoint collection of knots in $S^3$. Consider the tubular neighborhood of $K$ in $S^3$, which is a solid torus $\nu(K) \approx S^1 \times D^2$. On the boundary torus $\partial \nu(K)$, there is a preferred longitude $\lambda$, i.e., a simple closed curve with $\ell k(\lambda, K) = 0$, and there is a canonical meridian $\mu$ with $\ell k(\mu, K) = 1$.

A Dehn $(p/q)$-surgery along $K$ in $S^3$ can be proceed by following two steps. We first drill out the interior of $\nu(K)$ from $S^3$ and consider the knot exterior $S^3 \setminus \nu(K)$. Next, we glue another solid torus $D^2 \times S^1$ to the knot exterior by respecting the homeomorphism $\varphi$. The resulting boundaryless 3-manifold is given by

$$S^3_{p/q}(K) = (S^3 \setminus \nu(K)) \cup \varphi \left(D^2 \times S^1\right), \quad \varphi(\partial D^2 \times \{*\}) = p\mu + q\lambda.$$

Since $H_1(S^3_{p/q}(K), \mathbb{Z}) = \mathbb{Z}_p$, the manifolds of the form $S^3_{1/n}(K)$ are automatically homology 3-spheres. In particular, Dehn showed that the Poincaré homology sphere can be obtained by $(−1)$-surgery along the left-handed trefoil knot $T(2, 3)$ in $S^3$.

A framed knot in $S^3$ is a knot equipped with a smooth nowhere vanishing vector field normal to the knot. Thus a framing of a knot is naturally characterized by its Seifert surface [Sei35] and [FP30] where the specified longitude is given by 0-framing. And in a similar fashion, $n$-framing can be obtained by adding $n$-twists. This process can be naturally generalized to framed links in $S^3$ and is particularly crucial thanks to the eminent results of Lickorish [Lic62] and Wallace [Wal60], and Kirby [Kir78a]: the map $D$ provided by integral $n$-surgery

$$D : \{\text{framed links in } S^3\} \to \{\text{boundaryless 3-manifolds}\}$$

$$L \mapsto D(L) \cong S^3_n(L)$$

is many-to-one. In particular, Kirby completely described when two elements can represent the same element in the kernel using Cerf theory [Cer70], i.e., $S^3_n(L_1)$ is homeomorphic to $S^3_n(L_2)$ if and only if the framed links are related by two Kirby moves: blow-up and handle-slide. His notable contribution was generalized, ramified, and reproved by Fenn and Rourke [FR79], César de Sá [CdS79], Kaplan [Kap79], Rolfsen [Rol83], Lu [Lu92], Matveev and Polyak [MP94], and Martelli [Mar12].

The next construction of homology 3-spheres was provided by Seifert [Sei33]. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be pairs of relatively prime integers and let $e$ be an integer. The
Seifert fibered space with base orbifold $S^2$ is a boundaryless 3-manifold

$$M(S^2; e, (a_1, b_1), \ldots, (a_n, b_n))$$

classified by starting with $S^1$-bundle over an $n$-punctured $S^2$ of Euler number $n$ and filling the $k$th boundary component with $(a_k/b_k)$-framed solid torus for $k = 1, \ldots, n$. The core circle of the $(a_k/b_k)$ Dehn filling is called a singular fiber, all other fibers are said to be regular fibers. The resulting manifold is a homology 3-sphere if and only if

$$a_1 \ldots a_n \left( -e + \sum_{k=1}^{n} \frac{b_k}{a_k} \right) = \mp 1. \quad (1)$$

This equation results from the fundamental group [ST80, Pg. 398], and hence the first homology group calculations of Seifert fibered spaces, see [ST80, Pg. 410].

In the language of Seifert fibered space, Poincaré homology sphere corresponds to $M(S^2; -2, (2, -1), (3, -2), (5, -4))$.

Thanks to Brieskorn [Bri66a, Bri66b], homology 3-spheres originates from algebraic geometry as seen in the variety of certain complex analytical polynomials. Let $p, q$ and $r$ be relatively coprime positive integers. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a complex analytical polynomial defined by $f(x, y, z) = x^p + y^q + z^r$. Then the zero set of $f$ is the complex surface $V(f) = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\}$ except singular at the origin. If we transversally intersect this variety with the five-sphere $S^5$ of arbitrarily small radius $\epsilon$, then the resulting boundaryless 3-manifold is the Brieskorn sphere given by

$$\Sigma(p, q, r) = V(f) \cap S^5_\epsilon \subset \mathbb{C}^3.$$

The Poincaré homology sphere matches with the Brieskorn sphere $\Sigma(2,3,5)$. For explicit descriptions of fundamental groups of Brieskorn spheres, see Milnor’s paper [Mil75]. In particular, there is an orientation-preserving homeomorphism between $M(S^2; a_1, a_2, a_3)$ and $\Sigma(a_1, a_2, a_3)$ [NR78, Theorem 4.1].

Let $J$ be an index set. A plumbing graph $G$ is a connected and weighted tree with vertices $v_j$ and weights $e_j$ for $j \in J$. We can construct a 4-manifold $X(G)$ with a boundary $Y(G)$ by using the plumbing graph. First, for each $v_j$, we assign a $D^2$-bundle over $S^2$ whose Euler number is $e_j$. Next, we plumb two of these $D^2$-bundles if there is an edge connecting the vertices, see [NR78, Theorem 5.1].

The fundamental classes of the zero-sections of $D^2$-bundles generate the second homology group $H_2(X(G); \mathbb{Z})$. Thus, for each vertex of $G$, we have a generator of $H_2(X(G); \mathbb{Z})$. Hence, the intersection form on $H_2(X(G); \mathbb{Z})$ is naturally characterized by the corresponding intersection matrix $I = (a_{ij})$ whose data is given in the following way:

$$a_{ij} = \begin{cases} 
 e_i, & \text{if } v_i = v_j, \\
 1, & \text{if } v_i \text{ and } v_j \text{ is connected by one edge,} \\
 0, & \text{otherwise.} 
\end{cases}$$

A plumbing graph $G$ is called unimodular if $\det(I) = \pm 1$. In this case, $Y(G)$ is called a plumbed homology 3-sphere. We may characterize the negative definiteness of $G$, it requires that $I$ is negative-definite, i.e., signature$(I) = -|G|$, where $|G|$ denotes the number of vertices of $G$.

A Seifert fibered homology sphere $M(S^2; e, (a_1, b_1), \ldots, (a_n, b_n))$ can be realized as the boundary of a star-shaped plumbing graph, see Figure 2. This graph is unique when it is negative-definite [Sav02b, Section 1.1]. The integer weights $t_{ij}$
in the graph are found by solving the equation (1) and expanding the continued fractions \( [t_{i1}, \ldots, t_{im_i}] \) as follows: for each \( i \in \{1, \ldots, n\} \)

\[
a_i/b_i = [t_{i1}, t_{i2}, \ldots, t_{im_i}] = t_{i1} - \frac{1}{t_{i2} - \frac{1}{\cdots - \frac{1}{t_{im_i}}}}.
\]

Figure 2. The star-shaped graph

In this survey, we focus on the following three families of Brönnskorn spheres. Assume that \( p, q \) and \( r \) are pairwise coprime, positive, and ordered integers such that \( 2 \leq p < q < r \):

1. \( \{\Sigma(p, q, pqn - 1)\}_{n=1}^\infty \),
2. \( \{\Sigma(p, q, pqn + 1)\}_{n=1}^\infty \),
3. \( \{\Sigma(p_n, q_n, r_n)\}_{n=1}^\infty \) where \( p_nq_n + p_nr_n - q_nr_n = 1 \),
   a) \( \{\Sigma(2n, 4n - 1, 4n + 1)\}_{n=1}^\infty \),
   b) \( \{\Sigma(2n + 1, 4n + 1, 4n + 3)\}_{n=1}^\infty \),
   c) \( \{\Sigma(2n + 1, 3n + 2, 6n + 1)\}_{n=1}^\infty \),
   d) \( \{\Sigma(2n + 1, 3n + 1, 6n + 5)\}_{n=1}^\infty \).

Due to the classical result of Moser [Mos71], the first two families can be obtained by \((-1/n)\) surgeries along the left-handed torus knots \( T(p, q) \) and their mirrors right-handed torus knots \( \overline{T(p, q)} \) in \( S^3 \):

\[
\Sigma(p, q, pqn - 1) = S^3_{-1/n}(T(p, q)), \quad \text{and} \quad \Sigma(p, q, pqn + 1) = S^3_{-1/n}(\overline{T(p, q)}).
\]

The third family is called almost simple linear graphs and extensively studied in [FSS85], [End95], and [KS20]. The families (1) and (3) are the vast generalizations of the Poincaré homology sphere \( \Sigma(2, 3, 5) \) while the family (2) is of \( \Sigma(2, 3, 7) \).

Note that there is a family of Brieskorn spheres realized as the boundaries of almost simple graphs which cannot be obtained surgeries along any knots in \( S^3 \). This surgery obstruction was due to Hom, Karakurt, and Lidman [HKL16]. In particular, they showed that \( \Sigma(2n, 4n-1, 4n+1) \) cannot be realized as knot surgeries for \( n \geq 4 \).

Another classical way to produce homology 3-spheres is the method of cyclic branched coverings of \( S^3 \) branched over knots \( K \), dates back to works of Alexander [Ale20] and Seifert [Sei33]. Let \( X_n(K) \) be the \( n \)-fold regular covering of the knot exterior \( X(K) = S^3 \setminus \nu(K) \). Then the \( n \)-fold cyclic branched covering of \( S^3 \) over \( K \) is a boundaryless 3-manifold

\[
\Sigma_n(K) \cong X_n(K) \cup_\varphi \left( D^2 \times S^1 \right), \quad \varphi(\hat{\mu}) = \mu
\]
where $\mu \subset \partial X(K)$ is the meridian of $K$ and $\tilde{\mu}$ is the lift of $\mu$ to $\partial X_n(K)$. Note that $\Sigma_n(K)$ is a homology 3-sphere when

$$\prod_{k=1}^{n} \Delta_K \left( e^{\frac{2\pi ik}{n}} \right) = 1$$

where $\Delta_K(t)$ is the Alexander polynomial of $K$ normalized so that there are no negative powers of $t$ and the constant term is positive. The Brieskorn sphere is a $r$-fold cyclic branched coverings of $S^3$ branched over the torus knots $T(p,q)$, see [Mil75, Lemma 1.1] and compare with [STS80, Pg. 405]. In general, Seifert fibered sphere is a 2-fold cyclic branched coverings of $S^3$ branched over Montesinos knots, see [Mon73, Mon75].

Given two homology 3-spheres together with knots inside the m, we can produce a new boundaryless 3-manifold by following the agenda of Gordon [Gor75]. Let $K_1$ and $K_2$ be knots in homology 3-spheres $Y_1$ and $Y_2$ with the knot exteriors $Y_1 \setminus \nu(K_1)$ and $Y_2 \setminus \nu(K_2)$, and the longitude-meridian pairs $(\lambda_1, \mu_1)$ and $(\lambda_2, \mu_2)$ respectively. Consider the following integral 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) = -1$. Gordon constructed boundaryless 3-manifolds obtained by gluing knot exteriors of homology 3-spheres along their boundary tori by matching longitude-meridian pairs with respect to the matrix $A$:

$$Y(K_1, K_2, A) = (Y_1 \setminus \nu(K_1)) \cup_A (Y_2 \setminus \nu(K_2)).$$

Clearly, the resulting manifold is a homology 3-sphere whenever $A = \begin{pmatrix} a & ab + 1 \\ 1 & b \end{pmatrix}$.

Gordon studied the problem which $Y(K_1, K_2, A)$ bound contractible 4-manifolds and provided several characterizations in terms of sliceness of knots.

The case $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ corresponds to switching longitude-meridian pairs of knots inside homology 3-spheres. This construction is a special interest and is known as the splice operation first introduced by Siebenmann [Sie80]. Given the pairs $(Y_1, K_1)$ and $(Y_2, K_2)$, we will denote the splice of these manifolds along the given knots by $Y_1 \bowtie K_1 \bowtie K_2 Y_2$.

The concept of the splice became popular after the novel book of Eisenbud and Neumann [EN85] because the splice can be realized as a generalization of several other topological operations including cabling, connected sum, and disjoint union. The splice also has a very crucial role in singularity theory due to Neumann and Wahl [NW90]. For details, one can consult the recent survey of Cueto, Popescu-Pampu, and Stepanov [CPPS22].

We finally consider the graph 3-manifolds introduced by Waldhausen [Wal67]. A graph 3-manifold is a boundaryless 3-manifold such that it can be cut along a set of disjoint embedded tori $T_i$ and has a decomposition with each piece is $\Sigma_i \times S^1$, where $\Sigma_i$ is a surface with boundary. In the light of JSJ (torus) decomposition theorem (Jaco and Shalen [JS79] and Johannson [Joh79]), a graph homology 3-sphere is a prime homology 3-sphere whose JSJ decomposition contains only Seifert fibered pieces. See Neumann’s paper [Neu07] and its appendix, and Saveliev’s book [Sav02b] for further discussions.

Notes

1. The terms “h-cobordism and J-equivalence” were used interchangeably in these references.
2. The topological (resp. piecewise linear, and smooth) Poincaré conjecture asserts that every topological (resp. piecewise linear, and smooth) homotopy $n$-sphere is homeomorphic (resp. piecewise linear homeomorphic, and diffeomorphic) to $S^n$. The topological and piecewise linear Poincaré conjectures were both proved for $n \geq 5$ in aforementioned articles. The particular case of $n = 4$ for the topological Poincaré conjecture was shown in the seminal article of Freedman [Fre82], also see the book of Behrens, Kalmár, Kim, Powell, and Ray [BKK+21]. The piecewise linear Poincaré conjecture in dimension 4 is still an open problem and is equivalent to the smooth Poincaré conjecture in dimension 4 as a result of the articles of Cerf [Cer68] and Hirsch and Mazur [HM74], see Rudyak’s books [Rud98] IV.4.27(iv) and [Rud16] 6.7 Remark for a detailed explanation.

3. See the introduction of [Lev85]. Also consult Milnor’s survey [Mil11 Pg. 805], and the commentary of Ranicki and Webber on the correspondence of Kervaire and Milnor around the 1960s [RW15].

4. The smooth Poincaré conjecture is false in general. For precise expositions, consult the introduction of [WX17] and also see the papers of Isaksen [Isa19] and Isaksen, Wang, and Xu [IWX20a].

5. Similarly, “$\pi$-manifold and $s$-parallelizable” and “surgery and spherical modification” were different names for the same notion. An $n$-manifold $M \subset \mathbb{R}^{n+q}$ is called a $\pi$-manifold if its normal bundle $\nu(M)$ is trivial, i.e., $\nu(M)$ is diffeomorphic to $M \times \mathbb{R}^q$.

6. For the other reformulations of the Rokhlin invariant $\mu$ in terms of the characterization of a 4-manifold, see the recent ICM 2022 paper of Finashin, Kharlamov, and Viro [FKV20].

7. Note that the homology cobordism group also appeared with notations $\Theta^n$ or $\mathcal{R}^3$ in the literature of the 1970-80s.

8. One can access the most recent information about the orders of $\Theta^n$ from the article of Isaksen, Wang, and Xu [IWX20a].

9. In our convention, $\mathbb{Z}^\infty$ always stands for $\bigoplus_{n=1}^{\infty} \mathbb{Z}$.

10. These three articles all provide equivalent but different descriptions of Heegaard Floer homology groups of Seifert fibered homology spheres.

11. Note that the involutive correction terms $d$ and $\mathbf{d}$ in [HI17] and Manolescu invariants $\alpha$, $\beta$ and $\gamma$ in [Man16b] are not homomorphisms.

12. This result cannot be generalized to even values of $n$ since $\Sigma(2,3,13)$ and $\Sigma(2,3,25)$ are known to bound contractible 4-manifolds.

13. Explicitly, the knot $K$ can be taken as the mirrors $K_n^+$ of the 2-bridge knots $K_n$ corresponding to the rational number $\frac{2}{3n-1}$ as hyperbolic examples. For the satellite type of examples, one can pick the $(2,q)$-cable of any knot $K$ with odd $q \geq 3$, see [NST19].

14. The knot $K$ can be chosen as either a knot having a transverse representative with positive self-linking number, or quasi-positive knot which is not smoothly slice, or an alternating knot with negative signature $\sigma$, under the convention $\sigma(T(2,3)) = -2$, see [BS21] and [BS22].

15. The knot $K$ can be chosen as either a quasi-positive knot which is not smoothly slice or an alternating knot with negative signature.

16. Since positive knots in $S^3$ are quasi-positive and not smoothly slice due to Rasmussen [Ras10], the work of Baldwin and Sikov also generalizes a result of Gompf and Cochran [CG88]: $S^3_{\gamma_n}(K)$ individually generates a $\mathbb{Z}$ subgroup in $\Theta_2$ when $K$ is a positive knot in $S^3$.

17. Under these conditions, Daemi, Imori, Sato, Scaduto, and Taniguchi provided two-parameter family of bridge knots $K(m,n) = K(212mn - 68n + 53, 106m - 34)$ ($m$ and $n$ are fixed) such that $(1/k)$-surgery on the mirrors of $K(m,n)$ are linearly independent in the homology cobordism group yet $K(m,n)$ are torsion in the algebraic concordance group of knots.

18. There are two $h$-invariants of Frøyshov: the “old” one [Fro02] and the “new” one [Fro10]. To avoid the ambiguity we follow the notation appeared in Manolescu’s survey [Man20], called the “new” $h$-invariant $\delta$-invariant.
19. Note that these families of Brieskorn spheres all bound rational homology 4-balls for all values of $n$. Simone’s family can be generalized in the sense that $S^3_{1,1}(K)$ (resp. $S^3_{1,1}(K)$) bounds a rational homology 4-ball when $K$ is an unknotting number one knot with a positive (resp. negative) crossing that can be switched to unknot $K$.

20. One can consult the paper of Akbulut and Larson [AL18] for the handle diagram of a rational homology 4-ball including a 3-handle. This 4-manifold has the boundary $\Sigma(2,3,7)$.

21. In general, it is known for a homology 3-sphere which bounds a simply-connected 4-manifold with non-standard definite intersection form. Taubes attributed this result to Akbulut.

22. Note that $\partial X(1) = \Sigma(2,5,7)$ and $\partial X'(1) = \Sigma(3,4,5)$, compare with [AK79, CH81, and Sav20a]. Therefore, they are not Seifert fibered unless $n = 1$.

23. A homology 3-sphere $Y$ is said to be prime if it cannot express a connected sum of two homology 3-spheres non-trivially (i.e. either summand is not $S^3$). For homology 3-spheres, sometimes the terms of prime and irreducible can be used interchangeably unless $Y$ is irreducible, see [Mil62, Lemma 1].

24. The existence of Seifert surfaces of an oriented knot $K$ in any oriented 3-manifold $M$ would be possible if and only if $K$ is null-homologous, i.e., $[K] = 0 \in H_1(M;\mathbb{Z})$, one can consult [Rol76, GS99, and Akb18].

25. Seifert called homology 3-spheres Poincaré spaces, see [ST80, Pg. 402]. Note that the book [ST80] includes an English translation of [Sel33] and our citations all lie in that part.

Afterword

The recorded history of the $n$-dimensional homology cobordism group $\Theta^n_Z$ first appeared in the Ph.D. thesis of González-Acuña [GAn70b] under the supervision of Ralph H. Fox at Princeton University in 1970. He introduced this notion to study homology $n$-spheres by building on the work of Kervaire and Milnor [KM63] about the $n$-dimensional homotopy cobordism group $\Theta^n$ of homotopy $n$-spheres. González-Acuña proved that these groups $\Theta^n$ and $\Theta^n_Z$ are isomorphic unless $n = 3$. Therefore, they are both finite except in the case of $n = 3$. This result was not published as an article but was referred in [GAn70a, Section 2]. Note that the only unknown value of the order of $\Theta^n$ in [KM63] was the case of $n = 3$. This has not been clarified until the work of Perelman [Per02, Per03a, Per03b].

The isomorphism argument of González-Acuña broke down when $n = 3$, if it was even known the order of $\Theta^3$ at that time, see [GAn70b, Pg. 17, Remark and Section I.5]. Especially, the homology cobordism group $\Theta^3_Z$ was introduced to him by Denis Sullivan as noted in [GAn70b, Pg. VII]. Also, the first known proof of the homology cobordism invariance of the Rokhlin invariant $\mu$ was given [GAn70b, Pg. 33-34]. Further, the relation between the Arf invariant of knots and the Rokhlin invariant in terms of knot surgery was found [GAn70b, Theorem III.2]. Unfortunately, his results were only mentioned in Gordon’s article [Gor75] and they have remained mysteries.

In this survey, we aim to approach all results that arise around the homology cobordism group $\Theta^3_Z$ from a broad, comprehensive, and historical perspective. Our additional purpose is the discussion and presentation of various open problems of homology 3-spheres in the context of homology cobordism. Most problems posed in this survey are well-known in the field in general.

The main references for our survey are the great book of Saveliev [Sav02a] and the eminent ICM 2018 article of Manolescu [Man18]. To extend their coherent frameworks, we list recent results not included in these resources. Further, we catalog all natural sources of homology 3-spheres in the appendix. In order to avoid distracting readers, we share our footnotes as endnotes.
We hope that our effort will have a positive impact and motivation on readers to investigate and study the homology cobordism group $\Theta^3_Z$ in the future.

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