STABLE AND UNSTABLE FLOW REGIMES FOR ACTIVE FLUIDS IN THE PERIODIC SETTING

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Abstract. Depending on the involved physiobiological parameters, stable or unstable behavior in active fluids is observed. In this paper a rigorous analytical justification of (in-)stability within the corresponding regimes is given. In particular, occurring instability for the manifold of ordered polar states caused by self-propulsion is proved. This represents the prerequisite for active turbulence patterns as observed in a number of applications. The approach is carried out in the periodic setting and is based on the generalized principle of linearized (in-)stability related to normally stable and normally hyperbolic equilibria.

Keywords. Living fluids, active turbulence, generalized Navier-Stokes equations, periodic setting, well-posedness, stability,
1. Introduction

A minimal hydrodynamic model to describe the bacterial velocity in the case of highly concentrated bacterial suspensions with negligible density fluctuations is given as

\[ v_t + \lambda_0 v \cdot \nabla v = f - \nabla p + \lambda_1 |v|^2 - (\alpha + \beta |v|^2)v + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v, \]
\[ \text{div } v = 0, \]
\[ v(0) = v_0, \]

(1.1)

see [25]. Here \( v \) is the bacterial velocity field and \( p \) the (scalar) pressure and \( \lambda_0, \lambda_1, \alpha, \beta, \Gamma_0 \) and \( \Gamma_2 \) are real parameters.

In [27] a first rigorous analytical approach to (1.1) in \( L^2(\mathbb{R}^n) \) is presented. There, depending on the values of the involved parameters, results on (in-)stability of the disordered isotropic steady state and ordered polar steady states, see (3.1) and (3.2), are derived. A formal stability analysis based on the standard wave ansatz coming to the same conclusions is already performed in [25]. This formal analysis, however, cannot be rigorously confirmed by the approach in \( L^2(\mathbb{R}^n) \) as given in [27], just by the fact that the wave ansatz is not an \( L^2(\mathbb{R}^n) \)-function. Therefore in [2] an approach to (1.1) in spaces of Fourier transformed Radon measures \( FM(\mathbb{R}^n) \) is developed. It gives the same outcome on (in)-stability as in [27]. Moreover, it confirms the formal stability analysis in [25], since the space \( FM(\mathbb{R}^n) \) contains wave functions such as \( \exp(ik \cdot x + \sigma t) \) required for the wave ansatz.

A drawback of the approaches presented in [27] and [2] concerns the instability of the ordered polar states. The results in [27] and [2] prove that a single polar state for specific regimes of the involved parameters is unstable. The polar states, however, form the manifold \( B_{\alpha,\beta} \), i.e., the sphere with radius \( \sqrt{-\alpha/\beta} \) centered at the origin, see (3.2). The results in [27] and [2] do not clarify the question, if a solution still could converge to the manifold \( B_{\alpha,\beta} \), even though each single polar state on \( B_{\alpha,\beta} \) displays unstable behavior. On the other hand, instability of the manifold of polar states is the prerequisite for active turbulence, and thus especially interesting with regard to applications, see [25, 23, 3, 26].

The purpose of this note is to develop an approach in \( L^2 \) in the periodic setting. This allows for applying the concepts of normally stable and normally hyperbolic equilibria as, e.g., provided by [19, Theorem 2.1 and Theorem 6.1] and [20, Theorem 5.3.1 and Theorem 5.5.1]. Utilizing the latter result we can precisely characterize instability of the manifold \( B_{\alpha,\beta} \). In fact, compactness yields discrete spectrum of the involved linear operators. A crucial point then is to prove that zero is a semi-simple eigenvalue of the linearization at each equilibrium on the manifold, which will turn out to be true for the polar states. It should be noticed that this strategy is not applicable in \( L^2(\mathbb{R}^n) \) and \( FM(\mathbb{R}^n) \), essentially by the fact that the spectra
are continuous in those settings. Comparing the three approaches, it appears that the approach in the periodic setting performed here seems the most suitable one. This is also underlined by the fact that the generalized Navier-Stokes system (1.1) was augmented even further by including further higher derivatives in the velocities entering into the stress tensor, see [22]. In presence of a boundary, this required additional higher order boundary conditions. From the physical point of view it seems not at all clear what are suitable boundary conditions to be imposed. Remaining in the periodic setting it is no problem to include also higher order terms in the velocity. Indeed, we anticipate that the analysis performed here also applies to this more general case, provided the highest order term has the correct sign.

Note that the model (1.1) was originally proposed in [25] and then further considered in [5, 4]. For \( \lambda_0 = 1, \lambda_1 = \alpha = \beta = \Gamma_2 = 0 \) and \( \Gamma_0 > 0 \), it reduces to the classical incompressible Navier-Stokes equations in \( n \) spatial dimensions. For non-vanishing \( \lambda_1, \alpha, \beta, \Gamma_2 \) system (1.1) by now is one of the standard models to describe active turbulence at low Reynolds number [17]. It can also be derived from more microscopic descriptions [10] and it was quantitatively confirmed in suspensions of living biological systems [25, 12, 26, 1] and synthetic microswimmers [8]. Note that active turbulence was also suggested as a power source for various microfluidic applications [12, 13, 14, 23]. We refer to those papers and to [27] for a more detailed description of the physics behind the additional occurring terms.

We organized this note as follows: Section 2 collects basic facts on periodic spaces in the \( L^2 \)-setting. In Section 3 we analyze the linearized system about the relevant equilibria. Here precise statements on the spectra and the corresponding asymptotic behavior of the semigroups are established. Furthermore, we prove global-in-time well-posedness for system (1.1) in the strong setting. Section 4 then concerns the nonlinear active turbulence. In Subsection 4.1 we first consider the stability behavior of the disordered state depending on the involved parameters and relying on the linear stability analysis. Subsection 4.2 deals with the stable regime for the manifold of the polar states. It will be proved to be normally stable in this case. The most important result related to active turbulence then is given in Subsection 4.3 There the manifold of ordered polar states is proved to be normally hyperbolic in the unstable regime of the parameters.

2. Periodic Sobolev spaces

We start with some basic notation. Let \( \Omega \subseteq \mathbb{R}^n \) be a domain. By \( L^p(\Omega, X) \) for \( 1 \leq p \leq \infty \) we denote the standard Bochner-Lebesgue space with values in a Banach space \( X \). As usual we equip \( L^p(\Omega, X) \) with the standard Lebesgue space norm

\[
\|u\|_{L^p(\Omega, X)} = \left( \int_\Omega \|u(x)\|_X^p \, dx \right)^{1/p}
\]
for $1 \leq p < \infty$ with the usual modification if $p = \infty$. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ the Sobolev space $W^{k,p}(\Omega, X)$ of $k$ times weakly differentiable functions is equipped with the norm

$$\|u\|_{W^{k,p}(\Omega, X)} = \left( \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega, X)}^p \right)^{1/p}.$$

If $p = 2$ then we write $H^k(\Omega, X) := W^{k,2}(\Omega, X)$. Next, we define periodic Sobolev spaces. Let $L > 0$ be arbitrary but fixed from now on. We set $Q_n := [0, L]^n$. The function space which corresponds to periodic boundary conditions is $L^2_\pi(Q_n, \mathbb{R}^n)$. It is defined as the completion of $C^\infty_\pi(Q_n)$ with respect to the $L^2(Q_n, \mathbb{R}^n)$ norm, where

$$C^k_\pi(Q_n) := \left\{ f \in C^k(Q_n, \mathbb{R}^n) : \partial^\alpha f|_{x_j=0} = \partial^\alpha f|_{x_j=L} \forall |\alpha| \leq k \right\},$$

$$C^\infty_\pi(Q_n) := \bigcap_{k=0}^\infty C^k_\pi(Q_n).$$

Here, $C^k(\Omega, X)$ denotes the space of $k$ times continuously differentiable functions with values in a Banach space $X$. To simplify the notation we set $L^2(Q_n) := L^2_\pi(Q_n) := L^2(Q_n, \mathbb{R}^n)$. By [7, Proposition 3.2.1] it follows that the definitions of $L^2_\pi(Q_n, \mathbb{R}^n)$ and $L^2(Q_n, \mathbb{R}^n)$ as a standard Lebesgue space are equivalent.

One advantage of working in $L^2(Q_n)$ is the fact that we can employ Fourier series. For a detailed introduction to Fourier series and further important properties of $L^2(Q_n)$ we refer to [7, Chapter 3] and [21, Chapter 5.10]. Let $f \in L^2(Q_n)$. The Fourier coefficient $\hat{f}(m)$ for $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ is defined as the integral

$$\hat{f}(m) := \mathcal{F}f(m) := \frac{1}{L^n} \int_{Q_n} f(x)e^{-2\pi imx/L} \, dx,$$

where $mx = \sum_{k=1}^n m_kx_k$ denotes the scalar product in $\mathbb{R}^n$. By using integration by parts one can verify the identity

$$\tilde{\partial^\alpha f}(m) = \left( \frac{2\pi i}{L} \right)^{|\alpha|} m^\alpha \hat{f}(m) \quad (2.1)$$

if $f$ is smooth enough, i.e., $f \in C^{|\alpha|}_\pi(Q_n)$, $m \in \mathbb{Z}^n$ and $\alpha \in \mathbb{N}_0^n$. Let $f, g \in L^2(Q_n)$. Then the scalar product in $L^2(Q_n)$ is defined as

$$(f, g)_{2,\pi} := \frac{1}{L^n} \int_{Q_n} f(x)g(x) \, dx$$

where the subscript $\pi$ indicates that we are in the periodic $L^2$ space. Some important properties of the Fourier series and $L^2(Q_n)$ functions are listed below ([7, Proposition 3.2.7]).
Proposition 2.1. Let \( f, g \in L^2(Q_n) \) be arbitrary. The following properties hold in \( L^2(Q_n) \):

1. **Plancherel theorem:**
   \[
   \|f\|_{L^2(Q_n)}^2 = \sum_{m \in \mathbb{Z}^n} |\hat{f}(m)|^2.
   \]

2. **Parseval’s identity:**
   \[
   (f, g)_{2,\pi} = \frac{1}{L^n} \int_{Q_n} f(x)\overline{g(x)} \, dx = \sum_{m \in \mathbb{Z}^n} \hat{f}(m)\overline{\hat{g}(m)}.
   \]

3. The function \( f \) can be represented as the \( L^2(Q_n) \)-limit of trigonometric polynomials, i.e.,
   \[
   f = \sum_{m \in \mathbb{Z}^n} \hat{f}(m)e^{2\pi im/L}.
   \]

We will consider \([11]\) with periodic boundary conditions in \( L^2(Q_n) \). For this purpose we define relevant periodic Sobolev spaces for \( k \in \mathbb{N} \):

\[
H^k_\pi(Q_n) := \left\{ u = \sum_{m \in \mathbb{Z}^n} \hat{u}(m)e^{2\pi im/L} : \hat{u}(m) = \overline{\hat{u}(-m)}, \|u\|_{H^k_\pi(Q_n)} < \infty \right\}
\]

\[
= \left\{ u \in H^k(Q_n) : \partial^\alpha u|_{x_j=0} = \partial^\alpha u|_{x_j=L} (|\alpha| < k, j = 1, ..., n) \right\}
\]

\[
= C^\infty_\pi(Q_n)H^k(Q_n)
\]

where the norm above is defined as

\[
\|u\|^2_{H^k_\pi(Q_n)} := \sum_{m \in \mathbb{Z}^n} \left( 1 + \left(\frac{2\pi}{L}\right)^k |m|^k \right) |\hat{u}(m)|^2,
\]

see \([21]\) Chapter 5.10. If \( u \in H^k_\pi(Q_n) \) and \( \alpha \in \mathbb{N}_0^n \) with \(|\alpha| \leq k \), then the derivative \( \partial^\alpha u \) can be written as the \( L^2(Q_n) \)-limit

\[
\partial^\alpha u = \sum_{k \in \mathbb{Z}^n} \hat{\partial^\alpha u}(k)e^{2\pi ik/L} = \sum_{k \in \mathbb{Z}^n} \left(\frac{2\pi i}{L}\right)^{|\alpha|} k^{\alpha} \hat{\partial^\alpha u}(k)e^{2\pi ik/L},
\]

where we used the fact that the identity in \([21]\) also holds for \( u \in H^k_\pi(Q_n) \). It obviously follows that the \( \| \cdot \|_{H^k_\pi(Q_n)} \) and the \( \| \cdot \|_{H^k(Q_n)} \) norms are equivalent, where

\[
\|u\|^2_{H^k_\pi(Q_n)} := \sum_{|\alpha| \leq k} \sum_{m \in \mathbb{Z}^n} \left(\frac{2\pi}{L}\right)^{|\alpha|} m^{\alpha} |\hat{u}(m)|^2,
\]
by the Plancherel theorem. Periodic Sobolev spaces of fractional powers are defined in the canonical way: For $s \geq 0$ we set

$$H^s_\pi(Q_n) = \left\{ u = \sum_{m \in \mathbb{Z}^n} \hat{u}(m) e^{2\pi i m \cdot /L} : \hat{u}(m) = \overline{\hat{u}(-m)}, \|u\|_{H^s_\pi(Q_n)} < \infty \right\}$$

where

$$\|u\|_{H^s_\pi(Q_n)}^2 := \sum_{m \in \mathbb{Z}^n} \left( 1 + \left( \frac{2\pi}{L} \right)^2 |m|^2 \right)^{s/2} |\hat{u}(m)|^2,$$

and it is straightforward to see that for $s \in \mathbb{N}$ the two definitions for Sobolev spaces coincide.

Finally, let $m : \mathbb{Z}^n \to \mathbb{C}^{n \times n}$ be a function. We define $T_m : D(T_m) \subseteq L^2(Q_n) \to L^2(Q_n)$ as the $L^2(Q_n)$-limit

$$T_m f := \sum_{k \in \mathbb{Z}^n} m(k) \hat{f}(k) e^{2\pi i k \cdot /L}$$

for a function $f \in D(T_m)$ where

$$D(T_m) := \left\{ f \in L^2(Q_n) : \|T_m f\|_{L^2(Q_n)}^2 = \sum_{k \in \mathbb{Z}^n} |m(k) \hat{f}(k)|^2 < \infty \right\}.$$

Note that $T_m$ is well-defined and a bounded operator if $m$ is a bounded function by the Plancherel theorem. Then $m$ is called a Fourier multiplier on $L^2(Q_n)$.

### 3. Linear Stability and Well-Posedness

In the following we will consider two different, physically relevant stationary solutions of (1.1):

$$(v, p) = (0, p_0),$$

which corresponds to a disordered isotropic state, and, if $\alpha < 0$, the set of equilibria corresponds to the manifold of globally ordered polar states:

$$(v, p) = (V, p_0),$$

where $V \in B_{\alpha,\beta} := \{ x \in \mathbb{R}^n : |x| = \sqrt{-\alpha/\beta} \}$, i.e., $V$ denotes a constant vector with arbitrary orientation and fixed swimming speed $|V| = \sqrt{-\alpha/\beta}$. In both cases the pressure $p_0$ is a constant.

In order to cover all situations corresponding to the above steady states, as in [27], we consider the following generalization of (1.1):

$$u_t + \lambda_0 [(u + V) \cdot \nabla] u + (M + \beta |u|^2) u$$

$$- \Gamma_0 \Delta u + \Gamma_2 \Delta^2 u + \nabla q = f + N(u),$$

$$\text{div} u = 0,$$

$$u(0) = u_0,$$

$$u_t + \lambda_0 [(u + V) \cdot \nabla] u + (M + \beta |u|^2) u$$

$$- \Gamma_0 \Delta u + \Gamma_2 \Delta^2 u + \nabla q = f + N(u),$$

$$\text{div} u = 0,$$

$$u(0) = u_0,$$
where \( q = p - \lambda_1 |v|^2 \), \( M \in \mathbb{R}^{n \times n} \) is a symmetric matrix and \( N(u) = \sum_{j,k} a_{jk} u^j u^k \) with \((a_{jk})_{j,k=1}^n \in \mathbb{R}^{n \times n} \) defines a nonlinearity of second order. Regarding the occurring parameters we assume

\[
\lambda_0, \lambda_1, \Gamma_0, \alpha \in \mathbb{R}, \quad \Gamma_2, \beta > 0, \tag{3.4}
\]

throughout this paper. From (3.3) we obtain the equation corresponding to the disordered state (3.1) by setting

\[
\lambda_0, \lambda_1, \Gamma_0, \alpha \in \mathbb{R}, \quad \Gamma_2, \beta > 0,
\]

for \( u = v \) where \( I \) denotes the identity matrix and \( \alpha \) is a scalar. By setting

\[
V = 0, \quad M = \alpha I, \quad N(u) = 0,
\]

for \( u = v \) we obtain the system corresponding to (3.2). Furthermore, space dimension is always assumed to be \( n = 2 \) or \( n = 3 \).

3.1. The linearized system. In this subsection we consider the linearized system

\[
\begin{aligned}
&u_t + \lambda_0 (V \cdot \nabla) u + Mu - \Gamma_0 \Delta u + \Gamma_2 \Delta^2 u + \nabla q = f \quad \text{in } (0, \infty) \times Q_n, \\
&\text{div } u = 0 \quad \text{in } (0, \infty) \times Q_n, \\
&u(0) = u_0 \quad \text{in } Q_n
\end{aligned}
\]

with periodic boundary conditions

\[
\partial^\alpha u|_{x_j = 0} = \partial^\alpha u|_{x_j = L} \quad \text{for } |\alpha| < 4, j = 1, \ldots, n.
\]

First we introduce the Helmholtz-Weyl projection on \( L^2(Q_n) \). Setting the Fourier multiplier \( \sigma_P : \mathbb{Z}^n \to \mathbb{C}^{n \times n} \) as \( \sigma_P(m) = I - mm^T/|m|^2 \) for \( m \neq 0 \) and \( \sigma_P(0) = I \), the Helmholtz-Weyl projection \( P : L^2(Q_n) \to L^2_\sigma(Q_n) \) is defined as

\[
P u = \sum_{m \in \mathbb{Z}^n} \sigma_P(m) \hat{u}(m) e^{2\pi im/L} \quad (u \in L^2(Q_n)).
\]

The projection \( P \) induces the Helmholtz decomposition

\[
L^2(Q_n) = L^2_\sigma(Q_n) \oplus G_2(Q_n),
\]

where

\[
L^2_\sigma(Q_n) := \left\{ u \in L^2(Q_n) : \hat{u}(m) = \overline{\hat{u}(-m)}, m \cdot \hat{u}(m) = 0 \forall m \in \mathbb{Z}^n \right\},
\]

\[
G_2(Q_n) := \left\{ u = \nabla g \in L^2(Q_n) : g \in L^1_{loc}(Q_n) \right\}.
\]

Note that \( P \) is also a projection on \( H^k_\pi(Q_n) \) and that \( P(H^k_\pi(Q_n)) = H^k_\pi(Q_n) \cap L^2_\sigma(Q_n) \). From this we obtain the following interpolation result.

**Lemma 3.1.** Let \( \theta \in [0, 1] \) and \( k \in \mathbb{N} \). Then we have

\[
[L^2_\sigma(Q_n), H^k_\pi(Q_n) \cap L^2_\sigma(Q_n)]_\theta = H^{\theta k}_\pi(Q_n) \cap L^2_\sigma(Q_n),
\]

where \([\cdot, \cdot]_\theta \) denotes the complex interpolation functor; see [24].
Proof. The assertion follows from the fact that $P$ is a projection on the interpolated spaces and [24 Theorem 1.2.4].

We define the operator associated to (3.7) as

$$A_{LF}u := \lambda_0 (V \cdot \nabla) u + PMu - \Gamma_0 \Delta u + \Gamma_2 \Delta^2 u,$$

and the corresponding Fourier symbol as

$$\sigma_{ALF}(\ell) := \Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 + \lambda_0 \left( \frac{2\pi}{L} \right) (V \cdot \ell) + \sigma_P(\ell) M$$

(3.8)

for $\ell \in \mathbb{Z}^n$. Note that thanks to $\Gamma_2 > 0$ we immediately see that the operator

$$A_{SH}u := \Gamma_2 \Delta^2 u, \quad D(A_{SH}) := H^4_{\sigma}(Q_n) \cap L^2_{\sigma}(Q_n),$$

(3.9)

is selfadjoint. Hence $\omega + A_{SH}$ is selfadjoint and positive for $\omega > 0$. As a consequence $\omega + A_{SH}$ admits a bounded $H^\infty$-calculus on $L^2_{\sigma}(Q_n)$ with $H^\infty$-angle $\phi^\infty_{\omega + A_{SH}} = 0$. See e.g. [9] for an introduction to the notion of a bounded $H^\infty$-calculus. Applying perturbation theorems we obtain the following result:

**Proposition 3.2.** There exists an $\omega > 0$ such that $\omega + A_{LF}$ admits a bounded $H^\infty$-calculus on $L^2_{\sigma}(Q_n)$ with $H^\infty$-angle $\phi^\infty_{\omega + A_{LF}} = 0$.

Proof. This follows from the fact that

$$Bu := \lambda_0 (V \cdot \nabla) u + PMu - \Gamma_0 \Delta u$$

is a perturbation of lower order. The assertion then follows from [15 Proposition 13.1].

As a consequence $A_{LF}$ enjoys maximal $L^p$-regularity on intervals $(0, T)$ with $T < \infty$ and $-A_{LF}$ is the generator of an analytic $C_0$-semigroup on $L^2_{\sigma}(Q_n)$:

**Proposition 3.3.** Let $T \in (0, \infty)$. For $f \in L^2((0, T), L^2_{\sigma}(Q_n))$ and $u_0 \in H^2_{\sigma}(Q_n) \cap L^2_{\sigma}(Q_n) = (L^2_{\sigma}(Q_n), H^4_{\sigma}(Q_n) \cap L^2_{\sigma}(Q_n))_{1/2, 2}$ there exists a unique solution $(u, q)$ of (3.7) such that

$$\|u\|_{H^1((0,T),L^2_{\sigma}(Q_n))} + \|u\|_{L^2((0,T),H^4_{\sigma}(Q_n))} + \|\nabla q\|_{L^2((0,T),L^2_{\sigma}(Q_n))} \leq C(T) \left( \|f\|_{L^2((0,T),L^2_{\sigma}(Q_n))} + \|u_0\|_{H^2_{\sigma}(Q_n)} \right),$$

where $C(T) > 0$ is independent of $u, q, u_0, f$.

It is straightforward to verify the identity

$$\exp(-tA_{LF}v) = \sum_{\ell \in \mathbb{Z}^n} \exp(-t\sigma_{ALF}(\ell)) e^{2\pi i \ell \cdot L}$$

(3.10)

for $v \in L^2_{\sigma}(Q_n)$. Using this representation we can characterize linear (in)stability for both stationary states (3.1) and (3.2). First we examine
the disordered isotropic state (3.1). In this case we set $A_d := A_{LF}$ where $V = 0$ and $M = \alpha I$. Then $P$ commutes with $M$ and $PMu = \alpha u$ for $u \in D(A_d)$. The corresponding Fourier symbol is given as

$$
\sigma_{A_d}(\ell) := \Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 + \alpha \quad (\ell \in \mathbb{Z}^n).
$$

Using representation (3.10) we immediately obtain

**Proposition 3.4.** Let $\Gamma_2 > 0$ and $\Gamma_0, \alpha \in \mathbb{R}$. Then the semigroup $(\exp(-tA_d))_{t \geq 0}$ corresponding to the disordered state (3.1) is

1. **stable** if $\sigma_{A_d} \geq 0$;
2. **exponentially stable** if $\sigma_{A_d} \geq \delta > 0$;
3. **exponentially unstable** if there exists some $\ell_0 \in \mathbb{Z}^n$ such that $\sigma_{A_d}(\ell_0) < 0$.

Now, we can characterize linear stability in terms of the involved parameters by considering the behaviour of the symbol $\sigma_{A_d}$ for a certain choice of $\Gamma_0$ and $\alpha$. For this purpose we substitute $z = |\ell|^2 \geq 0$:

$$
p(z) := \Gamma_2 \left( \frac{2\pi}{L} \right)^4 z^2 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 z + \alpha,
$$
and we see that $p$ describes a parabola. The intersection points are given as

$$
z_{\pm} = \frac{-\Gamma_0}{\Gamma_2 \left( \frac{2\pi}{L} \right)^2} \left( \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\alpha \Gamma_2}{\Gamma_0 \left( \frac{2\pi}{L} \right)^2}} \right),
$$
if $\Gamma_0 \neq 0$ and as

$$
z_{\pm} = \pm \sqrt{-\frac{-\alpha}{\Gamma_2 \frac{L}{2\pi}}} \left( \frac{2\pi}{L} \right)^4
$$
if $\Gamma_0 = 0$. If $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ or $\Gamma_0 \geq 0$ and $\alpha > 0$ then there exists some $\delta > 0$ such that $\sigma_{A_d} \geq \delta$. Using these observations and Proposition 3.4 we obtain the following concreter classification of stability:

**Corollary 3.5.** Let $\Gamma_2 > 0$. If $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geq 0$ and $\alpha > 0$ then the semigroup $(\exp(-tA_d))_{t \geq 0}$ generated by $-A_d$ is exponentially stable. To be precise, the semigroup $(\exp(-tA_d))_{t \geq 0}$ corresponding to the disordered state (3.1) is

1. **stable**, if $\Gamma_0 < 0$ and $4\alpha \geq \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geq 0$ and $\alpha \geq 0$;
2. **exponentially stable**, if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geq 0$ and $\alpha > 0$ or if $\Gamma_0 < 0$ and $4\alpha = \Gamma_0^2/\Gamma_2$ with $|\ell|^2 \neq -\frac{\Gamma_0}{2\Gamma_2} \left( \frac{L}{2\pi} \right)^2$ for all $\ell \in \mathbb{Z}^n$.

**Proof.** From the observations above and by Proposition 3.4 exponential stability for the cases $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ and $\Gamma_0 \geq 0$ and $\alpha > 0$ follows immediately. If $\Gamma_0 \geq 0$ and $\alpha = 0$ then $\sigma_{A_d}(0) = 0$ and $\sigma_{A_d}(\ell) \geq 0$.
otherwise such that we can apply Proposition 3.4 to see stability. If $\Gamma_0 < 0$ and $4\alpha = \frac{\Gamma_0^2}{\Gamma_2}$ then the symbol $\sigma_{A_d}$ simplifies as
\[
\sigma_{A_d}(\ell) = \left(\sqrt{\Gamma_2} \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \frac{\Gamma_0}{2\sqrt{\Gamma_2}}\right)^2
\]
for $\ell \in \mathbb{Z}^n$. We have $\sigma_{A_d} > 0$ if and only if
\[
|\ell|^2 \neq -\frac{\Gamma_0}{2\Gamma_2} \left(\frac{L}{2\pi}\right)^2
\]
for all $\ell \in \mathbb{Z}^n$. Again the assertion follows from Proposition 3.4.

Next, we consider the ordered polar state (3.2). Here, we set $A_o := A_{LF}$ where $V \in B_{\alpha,\beta}$ and $M = 2\beta VV^T$. The Fourier symbol is then given as
\[
\sigma_{A_o}(\ell) := \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + \lambda_0 \left(\frac{2\pi i}{L}\right) (V \cdot \ell) + 2\beta \sigma_P(\ell)VV^T \sigma_P(\ell).
\]
If $\Gamma_0 \geq 0$ then
\[
\text{Re} \sigma_{A_o}(\ell) = \Gamma_2 \left(\frac{2\pi}{L}\right)^4 |\ell|^4 + \Gamma_0 \left(\frac{2\pi}{L}\right)^2 |\ell|^2 + 2\beta \sigma_P(\ell)VV^T \sigma_P(\ell) \in \mathbb{R}^{n \times n}
\]
is positive semi-definite for every $\ell \in \mathbb{Z}^n$ and even positive definite for $\ell \neq 0$. This follows from the fact that $\sigma_P(\ell)VV^T \sigma_P(\ell)$ is positive semi-definite. Then we can estimate the norm of the semigroup as
\[
\|\exp(-tA_o)v\|_{L^2(Q_n)}^2 \leq |\hat{v}(0)|^2 + \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} |e^{-t\sigma_{A_o}(\ell)}|^2 |\hat{v}(\ell)|^2
\]
to see stability. Conversely, if $\Gamma_0 < 0$ and if there exists $0 \neq \ell_0 \in \mathbb{Z}^n$ such that
\[
\Gamma_2 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2 + \Gamma_0 < 0
\]
then for $n = 3$ we can choose $x \in \mathbb{R}^n \setminus \{0\}$ with $x \perp V$, $x \perp \ell_0$ such that $x^T \text{Re} \sigma_{A_o}(\ell_0)x < 0$. In case of $n = 2$, due to
\[
-\frac{2\beta |V \cdot \hat{v}(\ell)|^2}{|\hat{v}(\ell)|^2} \in [2\alpha, 0]
\]
(see also (4.3) later) we assume the existence of $0 \neq \ell_0 \in \mathbb{Z}^n$ such that
\[
\Gamma_2 \left(\frac{2\pi}{L}\right)^2 |\ell_0|^2 + \Gamma_0 < 2\alpha.
\]
This implies $x^T \text{Re} \sigma_{A_o}(\ell_0)x < 0$ for some $x \in \mathbb{R}^n \setminus \{0\}$ such that $x \perp \ell_0$. Consequently, the matrix $\text{Re} \sigma_{A_o}(\ell_0) \in \mathbb{R}^{n \times n}$ is negative semi-definite or indefinite. Hence the growth bound of $(\exp(-tA_o))_{t \geq 0}$ is strictly positive. We obtain the following result on stability:
Proposition 3.6. Let $\Gamma_2 > 0$. Then the semigroup $(\exp(-tA_0))_{t \geq 0}$ corresponding to the ordered polar state \((3.2)\) is

(1) stable if $\Gamma_0 \geq 0$;
(2) exponentially unstable if $\Gamma_0 < 0$ and
   (i) if there exists some $0 \neq \ell_0 \in \mathbb{Z}^n$ such that \((3.12)\) holds for $n = 2$;
   (ii) if there exists some $0 \neq \ell_0 \in \mathbb{Z}^n$ such that \((3.11)\) holds for $n = 3$.

Remark 3.7. (a) It is worthwhile to compare at this point the situation to the continuous case considered in [27, Section 3.1] and [2, Section 3.1]. There a $\xi \in \mathbb{R}^n$ such that $\xi$ is parallel to $V$ can always be found, which in general is not possible in the discrete case. Consequently, also for dimension $n = 2$ a nontrivial $x \in \mathbb{R}^2$ satisfying $x \perp V$ and $x \perp \xi$ and giving instability does exist. By this fact, in [27] and [2] the more restrictive condition \((3.12)\) for $n = 2$ does not appear.

(b) Note that for $n = 2$ condition \((3.12)\) imposes no restrictions regarding the analysis of nonlinear instability considered here, cf. condition \((1.2)\) in Theorem 3.5.

3.2. Global well-posedness. In this section we quote the result on global well-posedness of \((3.3)\). Here the proof is omitted, since it is completely analogous to the proof in [27, Section 3.2].

Theorem 3.8 (global well-posedness). Let $\Gamma_2, \beta > 0$ and $\Gamma_0, \alpha, \lambda_0 \in \mathbb{R}$ and $T \in (0, \infty)$. Let the initial value $u_0 \in H^2_\sigma(Q) \cap L^2_\sigma(Q)$ and an exterior force $f \in L^2((0, T), L^2_\sigma(Q))$ be given. Then there exists a unique pair $(u, q)$ with

$$u \in H^1((0, T), L^2_\sigma(Q)) \cap L^2((0, T), H^1_\sigma(Q)),
\nabla q \in L^2((0, T), L^2(Q)),$$

solving \((3.5)\) for periodic boundary conditions.

Remark 3.9. Note that, in contrast to the classical Navier-Stokes equations, the convective term in \((3.3)\) is dominated by the linear fourth order term. Consequently, standard energy techniques lead to global strong solvability, see [27, Section 3.2] for the details.

4. Nonlinear stability and turbulence

In this section we study nonlinear stability of the stationary states of \((1.1)\). We will consider the disordered state \((5.1)\) and the manifold of ordered polar states \((5.2)\) separately. To see stability we will apply the generalized principle of linearized stability, cf. [19] Theorem 2.1 and [20] Theorem 5.3.1, and energy methods; to see instability we will use the principle of normally hyperbolic equilibria [19] Theorem 6.1 resp. [20] Theorem 5.5.1 and Henry’s instability theorem [11] Corollary 5.1.6.
4.1. The disordered state. We start with the following auxiliary result.

Lemma 4.1. Let \( H(u) := \beta P|u|^2u + \lambda_0 P(u \cdot \nabla)u - PN(u) \). Then we have \( H \in C^1(H^2_0(Q_0) \cap L^2_\sigma(Q_0), L^2_\sigma(Q_0)) \) for \( \eta \geq 5/4 \) and \( H \) can be estimated as

\[
\|H(u)\|_{L^2(Q_0)} \leq C\|u\|_{H^2_0(Q_0)}^2 \quad (\|u\|_{H^2_0(Q_0)} \leq 1).
\]

Proof. The proof of the lemma follows verbatim the lines of the proof of [27, Lemma 4].

Now, we consider the disordered isotropic state \((u, q)\). Suppose \((u, q)\) is the global solution to (3.3) from Theorem 3.8. Then we have

\[
\text{Lemma 4.1.}
\]

We start with the following auxiliary result.

The disordered state \((u, q)\) is nonlinearly unstable in 
(1) stable in \( L^2_\sigma(Q_0) \) if \( \Gamma_0 \geq 0 \) and \( \alpha \geq 0 \) or if \( \Gamma_0 < 0 \) and \( 4\alpha \geq \Gamma_0^2 / \Gamma_2 \);
(2) (globally) exponentially stable in \( L^2_\sigma(Q_0) \) if \( \Gamma_0 \geq 0 \) and \( \alpha > 0 \) or if \( \Gamma_0 < 0 \) and \( 4\alpha > \Gamma_0^2 / \Gamma_2 \) or if \( \Gamma_0 < 0 \) and \( 4\alpha = \Gamma_0^2 / \Gamma_2 \) and \( \|e\| \neq \Gamma_0 (L^2_\sigma(Q_0)) \) for all \( \ell \in \mathbb{Z}^n \);
(3) unstable in \( H^2_\sigma(Q_0) \cap L^2_\sigma(Q_0) \) for \( \eta \in [5/16, 1) \) if there exists some \( \ell_0 \in \mathbb{Z}^n \) such that \( \sigma_{A_\ell}(\ell_0) < 0 \).

Proof. If \( \Gamma_0 \geq 0 \) and \( \alpha \geq 0 \) we immediately obtain from (4.1)

\[
\frac{1}{2} \frac{d}{dt}\|u(t)\|_{L^2(Q_0)}^2 + \alpha \|u(t)\|_{L^2(Q_0)}^2 \leq 0,
\]

since \( \Gamma_2, \Gamma_0, \beta \geq 0 \). Applying Gronwall’s lemma we deduce

\[
\|u(t)\|_{L^2(Q_0)}^2 \leq e^{-2\alpha t}\|u_0\|_{L^2(Q_0)}^2,
\]

which shows that the disordered state \((u, q)\) is exponentially stable if \( \alpha > 0 \) and stable if \( \alpha = 0 \). To see exponential stability in the other case (\( \Gamma_0 < 0 \) and \( 4\alpha > \Gamma_0^2 / \Gamma_2 \)) we drop the \( \beta \) term in (4.1) and apply Plancherel’s theorem to obtain

\[
\frac{1}{2} \frac{d}{dt}\|u(t)\|_{L^2(Q_0)}^2 + \sum_{\ell \in \mathbb{Z}^n} \left( \Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 + \alpha \right) |\hat{u}(\ell)|^2 \leq 0.
\]
Next, if $\Gamma_0 > 0$ again the application of Gronwall’s lemma yields exponential stability.

We note that we can estimate the Fourier symbol in the series as

$$\Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 + \alpha > \delta$$

for some $\delta > 0$ by applying Young’s inequality with $\varepsilon^2 = 2\Gamma_2/|\Gamma_0|:

$$\Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 + \alpha \geq \alpha - \frac{\Gamma_0^2}{4\Gamma_2} > 0.$$ 

Hence in this case we set $\delta := \alpha - \Gamma_0^2/4\Gamma_2 > 0$ to see

$$\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2(Q_n)}^2 + \delta \| u(t) \|_{L^2(Q_n)}^2 \leq 0.$$ 

Again the application of Gronwall’s lemma yields exponential stability. Next, if $\Gamma_0 < 0$ and $4\alpha = \Gamma_0^2/\Gamma_2$ then $\sigma_{A_d}$ simplifies as

$$\sigma_{A_d}(\ell) = \left( \sqrt{\Gamma_2} \left( \frac{2\pi}{L} \right)^2 |\ell|^2 + \frac{\Gamma_0}{2\sqrt{\Gamma_2}} \right)^2$$

for $\ell \in \mathbb{Z}^n$ and we have $\sigma_{A_d} > 0$ if and only if $|\ell|^2 \neq \frac{\Gamma_0}{2\Gamma_2} \left( \frac{L}{2\pi} \right)^2$ for every $\ell \in \mathbb{Z}^n$. Then there exists $\delta > 0$ such that $\sigma_{A_d} > \delta$ and using the same arguments as before we infer exponential stability. Conversely, if there exists some $\ell_0 \in \mathbb{Z}^n$ such that $|\ell_0|^2 = \frac{\Gamma_0}{2\Gamma_2} \left( \frac{L}{2\pi} \right)^2$ then $\sigma_{A_d} \geq 0$ and the disordered state \((3.1)\) remains stable.

Using \([11, Corollary 5.1.6]\) we can show instability: In the notation of \([11, Corollary 5.1.6]\) we have $x_0 = 0$, $x = u$, $A = A_d$ and $f(u) = g(u) = H(u)$. Then $-A_d$ generates an analytic semigroup and there exists an $\omega > 0$ such that $\omega + A_d$ admits a bounded $H^\infty$-calculus, see Proposition \(3.2\). Then we have

$$D(A_d^\gamma) = [L^2_\sigma(Q_n), H^4_\sigma(Q_n) \cap L^2_\sigma(Q_n)]^\gamma = H^4_\sigma(Q_n) \cap L^2_\sigma(Q_n)$$

for $\gamma \in [0, 1]$ which follows from Lemma \(3.1\). Furthermore, under the assumptions on $\sigma_{A_d}$ we know that the disordered state is exponentially unstable for the linear system such that

$$\sigma(-A_d) \cap \{ z \in \mathbb{C} : \text{Re } z > 0 \} \neq \emptyset,$$

see Proposition \(3.4\). With Lemma \(4.1\) all conditions of \([11, Corollary 5.1.6]\) hold for $\gamma \in [5/16, 1)$ and the disordered state \((3.1)\) is unstable in this case. \(\square\)

4.2. **Ordered polar states: normal stability.** Now, we consider the manifold $B_{\alpha, \beta}$ of ordered polar states \((3.2)\) for the stable regime. Let $V \in B_{\alpha, \beta}$ and $A_d$ be the corresponding linear operator as defined in Section \(3.1\).

An equilibrium $V \in B_{\alpha, \beta}$ is called normally stable, cf. \([20, Theorem 5.3.1]\), if

(i) near $V$ the set of equilibria $B_{\alpha, \beta}$ is a $C^1$-manifold in $H^4_\sigma(Q_n) \cap L^2_\sigma(Q_n)$ of dimension $m \in \mathbb{N}$;
Lemma 4.3. For this purpose, we first prove the general principle of linearized stability \cite[Theorem 2.1]{19}, \cite[Theorem 5.3.1]{20}. For the stable regime we will show exponential stability by applying the principle of linearized stability \cite[Theorem 2.1]{19}, \cite[Theorem 5.3.1]{20}. For this purpose, we first prove

**Lemma 4.3.** Let $\Gamma_0 \geq 0$. Then $0$ is a semisimple eigenvalue of $A_o$, i.e., $\mathcal{N}(A_o) \oplus \mathcal{R}(A_o) = L^2_\sigma(Q_n)$. Furthermore, we can characterize the spectrum of $A_o$ in the following way: The spectrum of $A_o$ only consists of eigenvalues and is discrete. Additionally,

$$\sigma(A_o) \subseteq \{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \} \cup \{0\}.$$ 

**Proof.** First we note that for $\lambda \in \rho(A_o)$

$$\lambda - A_o : L^2_\sigma(Q_n) \to D(A_o) = H^1_\sigma(Q_n) \cap L^2_\sigma(Q_n) \subseteq L^2_\sigma(Q_n)$$

is compact by the Rellich-Kondrachov theorem \cite[Theorem A.4, Corollary A.5]{21]). Then $\sigma(A_o)$ is discrete and $\sigma(A_o) = \sigma_p(A_o)$ where $\sigma_p(A_o)$ denotes the point spectrum of $A_o$.

Next, we show that $0$ is a semisimple eigenvalue of $A_o$. Let $u \in H^1_\sigma(Q_n) \cap L^2_\sigma(Q_n)$ be a constant such that $u$ is perpendicular to $V$. Then

$$A_o u = \Gamma_2 \Delta^2 u - \Gamma_0 \Delta u + \lambda_0 (V \cdot \nabla) u + 2\beta PVV^T u = 0$$

and $0$ is an eigenvalue of $A_o$. We want to characterize $N(A_o)$ more precisely. Form the argumentation above we immediately have

$$\{ u \in H^1_\sigma(Q_n) \cap L^2_\sigma(Q_n) : u \text{ constant and } u \perp V \} \subseteq N(A_o).$$

To prove the converse inclusion we take $u \in N(A_o)$. Then $A_o u = 0$ and testing the equation with $u$ we obtain

$$0 = (\Gamma_2 \Delta^2 u, u)_{2,\pi} - (\Gamma_0 \Delta u, u)_{2,\pi} + (\lambda_0 (V \cdot \nabla) u, u)_{2,\pi} + 2\beta (PVV^T u, u)_{2,\pi}.$$ 

Taking the real part and applying integration by parts we derive

$$0 = \Gamma_2 \| \Delta u \|^2_{L^2(Q_n)} + \Gamma_0 \| \nabla u \|^2_{L^2(Q_n)} + 2\beta \| V \cdot u \|^2_{L^2(Q_n)},$$

since the $\lambda_0$ term is skew-symmetric. By assumptions $\Gamma_2, \beta > 0$ and $\Gamma_0 \geq 0$ we conclude

$$\| \Delta u \|^2_{L^2(Q_n)} = \| V \cdot u \|^2_{L^2(Q_n)} = 0,$$

hence $u$ is constant and perpendicular to $V$ since

$$\| \Delta u \|^2_{L^2(Q_n)} = \sum_{\ell \in \mathbb{Z}^n} |\ell|^2 |\hat{u}(\ell)|^2 = 0.$$
Consequently,
\[ N(A_o) = \{ u \in H^1_\sigma(Q_n) \cap L^2(Q_n) : u \text{ constant and } u \perp V \} . \]

Next, we show the decomposition \( N(A_o) \oplus R(A_o) = L^2_\sigma(Q_n) \). We define the following map
\[ S : L^2_\sigma(Q_n) \rightarrow L^2_\sigma(Q_n), \quad Su := \frac{1}{L^n} \int_{Q_n} S_* u(x) dx, \]
where \( S_* : L^2_\sigma(Q_n) \rightarrow L^2_\sigma(Q_n) \) is the map given by \( S_* u(x) = (I - VV^T/|V|^2) u(x) \), where \( I \) denotes the identity matrix in \( n \) dimensions. First we note that if \( u \in L^2_\sigma(Q_n) \) then \( Su \) is constant and \( Su \in L^2_\sigma(Q_n) \). It is straightforward to prove that \( S \) is a projection such that there exists a decomposition \( S(L^2_\sigma(Q_n)) \oplus (I - S)(L^2_\sigma(Q_n)) = L^2_\sigma(Q_n) \). Then we need to show that \( S(L^2_\sigma(Q_n)) = N(A_o) \) and \( (I - S)(L^2_\sigma(Q_n)) = R(A_o) \).

First we claim \( N(A_o) = S(L^2_\sigma(Q_n)) \). To see the inclusion \( S(L^2_\sigma(Q_n)) \subseteq N(A_o) \) we assume \( u \in S(L^2_\sigma(Q_n)) \). Then \( u = Su \) is constant as already mentioned and we observe
\[ V^T u = V^T Su = \frac{1}{L^n} \int_{Q_n} V^T u(x) dx - \frac{1}{L^n} \int_{Q_n} \frac{1}{|V|^2} VV^T u(x) dx = 0, \]
hence \( u = Su \) is perpendicular to \( V \) which yields \( u \in N(A_o) \). To see the converse inclusion we take \( u \in N(A_o) \). Then \( u \) is constant and perpendicular to \( V \). We obtain
\[ Su = \frac{1}{L^n} \int_{Q_n} u dx - \frac{1}{L^n} \int_{Q_n} \frac{1}{|V|^2} VV^T u dx \]
\[ = u \left( \frac{1}{L^n} \int_{Q_n} dx \right) = u, \]
hence \( u \in S(L^2_\sigma(Q_n)) \) and the claim is proved. Since \( L^2_\sigma(Q_n) \) is a Hilbert space and \( S \) is a selfadjoint projection it is well known that \( L^2_\sigma(Q_n) = S(L^2_\sigma(Q_n)) \oplus (I - S)(L^2_\sigma(Q_n)) \) is an orthogonal decomposition. If we take \( u \in D(A_o) \) and show that \( A_o u \) is perpendicular to any \( w \in N(A_o) \) then \( R(A_o) \subseteq (I - S)(L^2_\sigma(Q_n)) \):
\[ (A_o u, w)_{2,\pi} = \Gamma_2(\Delta u, \Delta w)_{2,\pi} + \Gamma_0(\nabla u, \nabla w)_{2,\pi} - \lambda_0(u, (V \cdot \nabla) w)_{2,\pi} + 2\beta(V^T u, V^T w)_{2,\pi} = 0, \]
because \( w \) is constant and perpendicular to \( V \). Since \( A_o \) has compact resolvent it follows from [16, Corollary 1.19] that the spectral value 0 is a pole of the resolvent. Then by [16, Remark A.2.4] it suffices to show that
\[ N(A_o) = N(A_o^2) \]
to prove that 0 is a semisimple eigenvalue of \( A_o \). The inclusion \( N(A_o) \subseteq N(A_o^2) \) is obvious. To see the converse inclusion we take \( u \in N(A_o^2) \). Then
\(A^2_n u = 0\) such that \(A_n u \in N(A_n) \cap R(A_n) = \{0\}\) by what we just proved. Thus, \(N(A_n^2) = N(A_n)\).

The last assertion \(\sigma(A_n) \subseteq \{\lambda \in \mathbb{C} : \text{Re} \lambda > 0\} \cup \{0\}\) follows from Proposition 3.6(i) and the fact that \(-A_n\) generates a bounded holomorphic \(C_0\)-semigroup in this case. \(\square\)

Now, we are in position to apply [19, Theorem 2.1] resp. [20, Theorem 5.3.1] to show that every stationary solution \((\mathcal{V}, p)\) with \(\mathcal{V} \in B_{\alpha,\beta}\) is exponentially stable in the following sense:

**Theorem 4.4.** Let \(\Gamma_2, \beta > 0, \Gamma_0 \geq 0, \alpha < 0\) and \(\lambda_0 \in \mathbb{R}\). Let \((\mathcal{V}, p_0)\) with \(\mathcal{V} \in B_{\alpha,\beta}\) be a stationary state of (1.1). Then \((\mathcal{V}, p_0)\) is stable in \(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)\) and there exists a \(\delta > 0\) such that if \((v, p)\) is a solution to (1.1) with initial data \(v_0 \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)\) and \(\|v_0 - \mathcal{V}\|_{H^2_\pi(Q_n)} < \delta\) then \(v\) converges to some \(\mathcal{V}_\infty \in B_{\alpha,\beta}\) exponentially.

**Proof.** In the notation of [19, Theorem 2.1] or [20, Theorem 5.3.1] we have \(V = H^2_\pi(Q_n) \cap L^2_\sigma(Q_n), \quad X_0 = L^2_\sigma(Q_n), \quad X_1 = H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)\) for the spaces, \(\mathcal{E} = B_{\alpha,\beta}\) for the manifold, \(u_* = \mathcal{V}\) for the equilibrium, and

\[
A\hat{v} := A(v)\hat{v} := \Gamma_2 \Delta^2 \hat{v} - \Gamma_0 \hat{v} + \alpha \hat{v} \quad (\hat{v} \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)),
\]

\[
F(v) := -\lambda_0 P(v \cdot \nabla) v - \beta |v|^2 v.
\]

for \(v \in H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)\). By the structure of \(A\) and \(F\) (linear and semilinear respectively) it is obvious that

\((A, F) \in C^1(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n), \mathcal{L}(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n), L^2_\sigma(Q_n)) \times L^2_\sigma(Q_n))\)

and to see that \(A_n\) is the linearized operator of (1.1) at \(V\). From Proposition 3.3 we know that \(A_n\) (hence also \(A\)) enjoys maximal \(L^p\)-regularity on \((0, T)\) for \(T < \infty\).

We will show that near \(V\) the set of equilibria \(B_{\alpha,\beta}\) is a \(C^1\)-manifold in \(H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)\) of dimension \(n - 1 \in \mathbb{N}\) and that the tangent space for \(B_{\alpha,\beta}\) at \(V\) equals \(N(A_n)\). It is canonical to define a \(C^1\)-function which maps into \(B_{\alpha,\beta}\):

If \(n = 3\) then \(V \in B_{\alpha,\beta}\) can be written as

\[
V = \sqrt{-1} \left(\begin{array}{c}
\sin(\theta) \cos(\phi) \\
\sin(\theta) \sin(\phi) \\
\cos(\theta)
\end{array}\right)
\]

for fixed \(\theta \in [0, \pi]\) and \(\phi \in [0, 2\pi]\). We define the corresponding \(C^1\) map as

\[
\Psi : [0, \pi] \times [0, 2\pi] \to H^2_\pi(Q_n) \cap L^2_\sigma(Q_n),
\]

\[
\left(\begin{array}{c}
y \\
z
\end{array}\right) \mapsto \Psi(y, z) := \sqrt{-1} \left(\begin{array}{c}
\sin(\theta + y) \cos(\phi + z) \\
\sin(\theta + y) \sin(\phi + z) \\
\cos(\theta + y)
\end{array}\right).
\]
Hence $\Psi(y, z) \in B_{\alpha, \beta}$ is a constant function in $H^4_\pi(Q_n) \cap L^2_\sigma(Q_n)$ for every $(y, z) \in [0, \pi] \times [0, 2\pi)$ satisfying $\Psi(0, 0) = V$. The corresponding tangent space of $B_{\alpha, \beta}$ at $V$ is two dimensional and obviously given as

$$T_V B_{\alpha, \beta} = \langle V \rangle^\perp.$$ 

This results in

$$N(A_\circ) = \{ u \in H^4_\pi(Q_n) \cap L^2_\sigma(Q_n) : u \text{ is constant and } u \perp V \} = \langle V \rangle^\perp = T_V B_{\alpha, \beta}.$$ 

The case $n = 2$ can be proved analogously.

Combining this with Lemma 4.3 we proved that $V$ is normally stable. By [19, Theorem 2.1] or [20, Theorem 5.3.1] the assertion follows. $\square$

4.3. Ordered polar states: normal hyperbolicity. In this last subsection, we will show for the unstable regime that the ordered polar states are normally hyperbolic. This gives instability in the following sense: For each sufficiently small $\rho > 0$ there exists $0 < \delta \leq \rho$ such that the unique solution $v$ of (1.1) with initial value $v_0 \in B_{H^2_\pi}(V, \delta)$ either satisfies

(i) $\text{dist}_{H^2_\pi}(v(t_0), B_{\alpha, \beta}) > \rho$ for a finite time $t_0 > 0$

(ii) $v(t)$ exists on $\mathbb{R}_+$ and converges at exponential rate to some $v_\infty \in B_{\alpha, \beta}$ in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ as $t \to \infty$.

In order to prove this, we will apply the principle of normally hyperbolic equilibria [19, Theorem 6.1] resp. [20, Theorem 5.5.1].

**Theorem 4.5.** Let $\Gamma_2, \beta > 0$ and $\alpha < 0$ and $\lambda_0 \in \mathbb{R}$. The ordered polar state (3.2) is normally hyperbolic if

$$\Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 \notin [2\alpha, 0], \quad \ell \in \mathbb{Z}^n \setminus \{0\}$$

for $\Gamma_0 < 0$ and if there exists some $\ell_0 \in \mathbb{Z}^n$ such that (3.11) holds. Thus, the ordered polar state (3.2) is unstable in $H^2_\pi(Q_n) \cap L^2_\sigma(Q_n)$ in the sense given above.

**Proof.** In order to apply [19, Theorem 6.1] or [20, Theorem 5.5.1] we have to show that $V$ is normally hyperbolic. In the proof of Theorem 4.4 we already showed that the ordered polar state forms a $C^1$-manifold of equilibria. Next, we characterize $N(A_\circ)$. Let $u \in N(A_\circ)$. Then

$$(A_\circ u, A_\circ u)_{2, \pi} = \sum_{\ell \in \mathbb{Z}^n} |\sigma_{A_\circ}(\ell) \hat{u}(\ell)|^2 = 0.$$ 

This yields $\sigma_{A_\circ}(\ell) \hat{u}(\ell) = 0$ for every $\ell \in \mathbb{Z}^n$, hence

$$0 = \text{Re} \left( \overline{\hat{u}(\ell)^T} \sigma_{A_\circ}(\ell) \hat{u}(\ell) \right) = \Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 |\hat{u}(\ell)|^2 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 |\hat{u}(\ell)|^2.$$
have to verify condition (iv). To this end, \( \lambda \) is semisimple follows analogously to the proof of Theorem 4.4. Finally, we obtain for \( \ell \neq 0 \) we have \( \hat{u}(\ell) \neq 0 \) and \( V \perp \hat{u}(0) \) immediately. Moreover, for \( \ell \neq 0 \) and \( \hat{u}(\ell) \neq 0 \) we obtain

\[
\Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 |\hat{u}(\ell)|^2 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 |\hat{u}(\ell)|^2 + 2\beta |V \cdot \hat{u}(\ell)|^2 = 0.
\]

Setting \( \ell = 0 \) yields \( V \perp \hat{u}(0) \) immediately. Moreover, for \( \ell \neq 0 \) and \( \hat{u}(\ell) \neq 0 \) we obtain

\[
\Gamma_2 \left( \frac{2\pi}{L} \right)^4 |\ell|^4 + \Gamma_0 \left( \frac{2\pi}{L} \right)^2 |\ell|^2 = -\frac{2\beta |V \cdot \hat{u}(\ell)|^2}{|\hat{u}(\ell)|^2} \in [2\alpha, 0]
\]

since \( |V|^2 = -\alpha/\beta \). Due to (4.2) we have \( \hat{u}(\ell) = 0 \). This yields

\[
N(A_o) = \{ u \in H^2_\alpha(Q_n) \cap L^2_\beta(Q_n) : u \text{ constant and } u \perp V \}
\]

with dimension \( n - 1 \), such that \( T_V B_{\alpha,\beta} = N(A_o) \). The fact that \( \lambda = 0 \) is semisimple follows analogously to the proof of Theorem 4.4. Finally, we have to verify condition (iv). To this end, let \( \lambda = ir \in \sigma(A_o) \) for \( r \neq 0 \) and \( u \neq 0 \) be a corresponding eigenfunction. Testing \( (ir - A_o)u \) with itself we obtain

\[
\bar{u}(0)^T (ir - 2\beta VV^T)\hat{u}(0) = 0
\]

such that

\[
\text{Im} \left( \bar{u}(0)^T (ir - 2\beta VV^T)\hat{u}(0) \right) = \text{Im} \left( (ir|\hat{u}(0)|^2 - 2\beta |V \cdot \hat{u}(0)|^2)\right) = r|\hat{u}(0)|^2 = 0
\]

for \( \ell = 0 \) and

\[
\text{Re} \left( \bar{u}(\ell)^T (ir - \sigma_{A_o}(\ell))\hat{u}(\ell) \right) = -\text{Re} \left( \bar{u}(\ell)^T \sigma_{A_o}(\ell)\hat{u}(\ell) \right) = 0
\]

for \( \ell \neq 0 \). This implies \( r = 0 \) if \( \hat{u}(0) \neq 0 \) and \( \hat{u}(\ell) = 0 \) for \( \ell \neq 0 \) again by assumption (4.2). Consequently, \( \lambda = 0 \). By [19, Theorem 6.1] or [20, Theorem 5.5.1] the result follows.

\[\Box\]

**Remark 4.6.** It is easily checked that, e.g., by setting \( L = 2\pi, \Gamma_2 = 4, \Gamma_0 = -5 \) and \( \alpha = -1/4 \) all conditions of Theorem 4.5 are satisfied, which yields unstable equilibria on \( B_{\alpha,\beta} \). Hence, the condition (4.2) is meaningful.

**Remark 4.7.** Note that a normal hyperbolic equilibrium implies the existence of a stable and of an unstable foliation near \( V \). In fact, if \( V \) is normally hyperbolic, then there exists \( r > 0 \) and a manifold \( M^s \), called the stable foliation, such that for each \( v_0 \in B_{H2}(V, r) \) we have that \( v_0 \in M^s \), if and only if the solution \( v(v_0, t) \) exists on \( \mathbb{R}_+ \) and converges to some \( W \in B_{\alpha,\beta} \) at an exponential rate. Furthermore, the projection onto the stable part of \( A_o \) is exactly the projection onto the tangent space of \( M^s \) at \( V \) (cf. [18, Thm 3.1]). Analogously, there exists an unstable foliation \( M^u \) (cf. [18, Thm. 4.1]).
5. Conclusion

In this note stability resp. instability for the active fluid model (1.1) in the periodic setting is considered. Depending on the values of the involved parameters

1. stability resp. instability for the disordered state and
2. normal stability resp. hyperbolicity for the manifold of ordered polar states

are proved. This in particular includes instability for the ordered polar states caused by self-propulsion, often referred to as active turbulence and observed in many applications, see e.g. [25, 23, 3, 26].

The observed turbulence indicates existence of an attractor, cf. [25]. To prove this rigorously is left as a future challenge.

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