Complementarity and uncertainty relations for matter wave interferometry

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We establish a rigorous quantitative connection between (i) the interferometric duality relation for which-way information and fringe visibility and (ii) Heisenberg’s uncertainty relation for position and modular momentum. We apply our theory to atom interferometry, wherein spontaneously emitted photons provide which way information, and unambiguously resolve the challenge posed by the metamaterial ‘perfect lens’ to complementarity and to the Heisenberg-Bohr interpretation of the Heisenberg microscope thought experiment.

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I. INTRODUCTION

Complementarity is at the heart of quantum mechanics and is operationally explored via interferometry, specifically the quantitative trade-off between which path information \( W \) (for ‘which way’) and visibility \( V \) (sharpness of fringes) \( 1, 2, 3, 4, 5, 6, 7, 8 \), which is a special case of the information-disturbance trade-off \( 9, 10 \). An alternative view of complementarity is provided by the uncertainty relation to the optical diffraction limit. Whereas Roychoudhuri expressed doubts about this interpretation, his argument is qualitative \( 19 \); in contrast we rigorously and quantitatively resolve this challenge by showing that the perfect lens simply provides an extremal point in the duality relation for atom interferometry.

and the duality relation has been the subject of debate. The claim that they are logically independent \( 4 \) has been put into question by Dürr and Rempe \( 7 \) who related the duality relation to uncertainty relations between Pauli matrices for two-level systems. Busch et al. \( 17, 18 \) have presented a profound analysis of the Mach-Zehnder interferometer and showed that duality relations for the trade-off between partial path determinations and reduced-visibility interference observations are expressible as uncertainty relations. However, the common assumption that complementarity of \( W \) and \( V \) is closely related to the uncertainty relation between position and momentum has not been proven yet, and while Wiseman et al. \( 6 \) beautifully analyse the nature of the momentum transfer in measurements of \( W \), they do not investigate its relation to the duality relation. Here we provide a quantitative relation between both concepts by showing that the duality relation can be used to derive an uncertainty relation between position \( x \) and modular momentum \( \hat{p} \).

A second challenge to interferometric complementarity suggests that superresolution \( 10 \) or perfect resolution \( 20 \) from metamaterial ‘perfect lenses’ \( 21, 22 \) is not easily reconciled with complementarity and interferometry because the Bohr-Heisenberg interpretation of Heisenberg’s \( \gamma \)-ray microscope \( 22 \) links the uncertainty relation to the optical diffraction limit. Whereas Roychoudhuri expressed doubts about this interpretation, his argument is qualitative \( 19 \); in contrast we rigorously and quantitatively resolve this challenge by showing that the perfect lens simply provides an extremal point in the duality relation for atom interferometry.

II. ATOM INTERFEROMETRY

In atom interferometry \( W \) quantifies to what extent it can be predicted through which of the two paths an atom will travel. Visibility \( V \) is a measure for the contrast of the interference pattern. Both are usually taken to be a number between 0 and 1. If \( W \) assumes the maximum value 1, the atom passes through only one of the two paths. Obviously, this would prohibit any in-
a unitary transformation that maps a given state process by a Householder reflection \[24\].

![FIG. 1: Model for the Heisenberg microscope. The two atomic wave functions are located close to the origin and are separated by a small distance \(a\) in the x-direction. The spontaneously emitted light that is collected in the detector propagates paraxially along the \(z\)-axis. The lens is located at \(z = 2f\) and the detector at \(z = 4f\).](image)

terference phenomena between the two paths so \(V\) should be zero in this case. On the other hand, if \(W = 0\) the probabilities for the atom to pass through either path are equal. If the atom is prepared in a coherent superposition of both paths then \(V\) can be maximal. However, if the atom is prepared in an equally weighted mixture to pass through either path, then interference phenomena would still be impossible so that \(W = V = 0\).

To establish a connection between duality and uncertainty of position and momentum we obviously have to quantize the atomic center-of-mass (CoM) motion. An atom then has internal (electronic) and CoM degrees of freedom, and it is the latter which will be in the focus of our attention. If an atom is prepared in a (CoM-) state localized around \(x = 0\), corresponding to one arm of the interferometer, it is described by a normalized wave function \(\phi(x)\). A wavepacket that has the same shape but is localized around \(x = a\) is given by

\[
\phi(x - a) = \hat{T}_a \phi(x), \quad \hat{T}_a = \exp(-ia \cdot \hat{p}/\hbar)
\]

where \(\hat{T}_a\) is a shift operator and \(\hat{p}\) the vector momentum operator.

In this paper we will consider the case that the process of splitting the atomic beam does not distort the shape of the beam so that the wavepacket that describes the second arm of the interferometer can be described by Eq. \[3\]. The atomic CoM wave function is initially prepared in the state \(\phi(x) = \langle x|\phi\rangle\). A generic atom beam splitter consists of a grating \[22\] or employs light forces \[20\]. When the process of splitting the beam is completed the atomic state after the first beam splitter is given by

\[
|\psi_{BS1}(\theta)\rangle = \frac{1}{n_{BS1}} \left( |\phi\rangle + e^{i\theta} \hat{T}_a |\phi\rangle \right) = U_{BS}(\theta) |\phi\rangle,
\]

which corresponds to a superposition of the wavepackets at two locations. Here \(n_{BS1}\) is a normalization that ensures \(\langle \psi_{BS1}(\theta)|\psi_{BS1}(\theta)\rangle = 1\). The state \(|\psi_{BS1}(\theta)\rangle\) corresponds to the two localized wavepackets on the left-hand side of Fig.\[1\]. We therefore can model the beam splitting process by a Householder reflection \[24\] \(U_{BS}(\theta)\), which is a unitary transformation that maps a given state \(|\phi\rangle\) to a given, non-orthogonal state \(|\psi_{BS1}(\theta)\rangle\). The Householder reflection is not uniquely defined by these two states; one convenient form is given by

\[
U_{BS}(\theta) = \frac{(|\phi\rangle + |\psi_{BS1}(\theta)\rangle)}{1 + |\psi_{BS1}(\theta)\rangle\langle\phi|} - \hat{1}.
\]

In Eq. \[4\] we included an arbitrary relative phase shift \(\theta\) between the two beams in the beam splitting process. In an experiment it would be generated by a phase shifter in one beam right after the beam splitter. Varying \(\theta\) will enable us to explore the fringe pattern \(f(\theta)\) of the interferometer, which will be necessary to collect information on \(V\) (see below). This is the reason why we highlight the dependence of \(|\psi_{BS1}(\theta)\rangle\) on \(\theta\). Throughout the paper we assume the large mass limit so the wave functions are effectively immutable during beam splitting and during a which way detection. This assumption is central for our analysis of complementarity.

\(W\) is obtained by performing a generalized position measurement \[4\] \[27\] on the split atomic beam. Loosely speaking this is a measurement that can determine the position only up to a certain accuracy; i.e., each possible measurement outcome has some uncertainty. Mathematically a generalized position measurement is described by a set of functions \(D_\alpha(x)\), where \(\alpha\) runs over some index set. These functions form a partition of unity of the form

\[
\sum_\alpha |D_\alpha(x)|^2 = 1 \quad \forall x
\]

After a generalized position measurement has produced the result \(\alpha\), the atomic state will be modified according to

\[
\psi(x) \rightarrow n_\alpha D_\alpha(x)\psi(x),
\]

where \(n_\alpha\) is a normalization factor. In the following we will only be interested in the state after a generalized position measurement has generated a specific result; we therefore will drop the index \(\alpha\) and denote with \(D(x)\) that function which corresponds to this specific result. In Sec. \[V\] we will show that detection of a spontaneously emitted photon corresponds to such a generalized position measurement. In the set-up shown in Fig.\[1\] this measurement is performed by collecting the emitted light using a lens at position \(z = 2f\) and detecting the light at \(z = 4f\), where \(f\) is the focal length of the lens.

For the atom interferometer under consideration, the postselected state after the position measurement is given by

\[
\psi_D(x) = \frac{1}{\sqrt{n(\theta)}} \left( D(x)\phi(x) + e^{i\theta}D(x)\hat{T}_a\phi(x) \right)
\]

where

\[
n(\theta) = \frac{1}{n_0 + n_1 + 1}
\]

\[
r = \frac{2\langle \phi|D(\hat{x})D(\hat{x})\hat{T}_a|\phi\rangle}{n_0 + n_1}
\]

\[
n_0 = \langle \phi|D(\hat{x})D(\hat{x})|\phi\rangle
\]

\[
n_1 = \langle \phi|\hat{T}_a D(\hat{x})D(\hat{x})\hat{T}_a|\phi\rangle
\]
The factor \( n(\theta) \) ensures that \( \psi_P(x) \) is normalized, and the parameter \( r \) is proportional to the overlap between the two paths of the interferometer. The parameters \( n_0 \) and \( n_1 \) are proportional to the populations in each path of the interferometer.

### III. COMPLEMENTARITY

\( W \) quantifies the difference of the probabilities to find the atom in the two interferometric paths \(|\phi\rangle \) or \( \hat{T}_a|\phi\rangle \). In our case we collect which way information by performing a generalized position measurement; we therefore have to calculate \( W \) for the atomic state after this measurement has been performed. For non-overlapping interferometer paths \( W \) can simply be defined as the difference of the probabilities to find the atom in either path. If the paths do overlap, \( W \) relates to the distinguishability of the two paths.

We wish to employ a conclusive protocol for identifying which of two non-orthogonal states best describes the preparation of the system. If the state is found to be in one of the two states, then we can be certain this is the prepared state, but the price is that a third measurement must be allowed: the null measurement. If the result is a null measurement, then we are completely uncertain about which state was prepared. The optimal positive operator-valued measure (POVM) for a conclusive protocol for two non-orthogonal states is given by the rank-three set of operators \[ 23 \]

\[
\hat{P}_0 = \frac{1 - |\langle \phi | \hat{T}_a | \phi \rangle|^2}{1 + |\langle \phi | \hat{T}_a | \phi \rangle|^2}
\]

\[
\hat{P}_1 = \frac{\hat{T}_a | \phi \rangle \langle \phi | \hat{T}_a^\dagger}{1 + |\langle \phi | \hat{T}_a | \phi \rangle|^2}
\]

\[
\hat{P}_2 = \mathbb{1} - \hat{P}_0 - \hat{P}_1,
\]

where \( \hat{P}_2 \) corresponds to the null measurement. Employing this POVM and using that the probabilities to be in state \(|\psi_i\rangle\) are given by \( P_i = \langle \psi_P | \hat{P}_i | \psi_P \rangle \), we obtain

\[
\bar{W} (\theta) = |P_0 - P_1| = \frac{n_0 + n_1}{(1 + |\langle \phi | \hat{T}_a | \phi \rangle|^2)n(\theta)} \left( |z_0|^2 - |z_1|^2 + |z_2|^2 - |z_3|^2 + \left( e^{\theta (z_0^* z_2 - z_3^* z_1) + \text{c.c.}} \right) \right),
\]

where we have introduced the complex numbers

\[
z_0 \equiv \frac{\langle \phi | \hat{D}(\hat{x}) | \phi \rangle}{\sqrt{n_0 + n_1}} \tag{18}
\]

\[
z_1 \equiv \frac{\langle \phi | \hat{T}_a\hat{D}(\hat{x})\hat{T}_a^\dagger | \phi \rangle}{\sqrt{n_0 + n_1}} \tag{19}
\]

\[
z_2 \equiv \frac{\langle \phi | \hat{D}(\hat{x}) | \hat{T}_a | \phi \rangle}{\sqrt{n_0 + n_1}} \tag{20}
\]

\[
z_3 \equiv \frac{\langle \phi | \hat{T}_a\hat{D}(\hat{x}) | \phi \rangle}{\sqrt{n_0 + n_1}} \tag{21}
\]

For overlapping wavepackets \( \bar{W} (\theta) \) depends on the interference phase \( \theta \) because constructive and destructive interference in the overlap region can decrease and increase the distinguishability, respectively. To achieve a measure for which way information that is independent of the phase we define the which way information \( W \) as the mean of \( \bar{W} (\theta) \),

\[
W = \frac{1}{2} \left( \bar{W} (\theta_{\text{max}}) + \bar{W} (\theta_{\text{min}}) \right) = \frac{1}{(1 + |\langle \phi | \hat{T}_a | \phi \rangle|^2)(1 - |r|^2)} \left( |z_0|^2 - |z_1|^2 + |z_2|^2 - |z_3|^2 - (r^* (z_0^* z_2 - z_3^* z_1) + \text{c.c.}) \right). \tag{22}
\]

If the two wavepackets are non-overlapping \( \bar{W} (\theta) \) and \( W \) agree. We then have \( \langle \phi | \hat{T}_a | \phi \rangle = r = z_2 = z_3 = 0 \) and \( W \) reduces to

\[
W_{\text{no}} = |z_0|^2 - |z_1|^2, \tag{23}
\]

which corresponds to the population difference in both arms of the interferometer after the generalized position measurement has been performed. For perfect overlap one has \( \langle \phi | \hat{T}_a | \phi \rangle = r = 1 \) and \( z_0 = z_1 = z_2 = z_3 \). This results in vanishing which way information \( W = 0 \), which is a consequence of the two beams being indistinguishable.

The fringe visibility \( V \) is obtained by recombining the two atomic beams (which is described by a unitary transformation \( U \)) and to equate \( V \) with contrast. The latter corresponds to the normalized difference

\[
V = \frac{f(\theta_{\text{max}}) - f(\theta_{\text{min}})}{f(\theta_{\text{max}}) + f(\theta_{\text{min}})}, \tag{24}
\]

between the maximum and minimum of the fringe pattern \( f(\theta) \). If the processes of measuring which way information, recombining the beam, and detecting the atoms do not alter the shape of the atomic wavepacket, one may describe the interferometer with just two states (one for each beam) \[ 4 \]. The fringe pattern \( f(\theta) \) can then be observed by measuring the overlap of the incoming atomic state \(|\psi_{\text{in}}\rangle\) with the output of the interferometer, which corresponds to a measurement of the observable \(|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|\). In our case the which way measurement may in general change the wavepacket, but a straightforward generalization of the previous observable is the overlap between the recombined state \( U_{\text{BS}} (0) |\psi_b (\theta)\rangle \) and the input state \(|\phi\rangle\), so that

\[
f(\theta) = |\langle \phi | U_{\text{BS}} (0) |\psi_b (\theta)\rangle|^2. \tag{25}
\]

This yields

\[
V^2 = 1 - (1 - |r|^2) \left( |z_0 + z_3|^2 - |z_1 + z_2|^2 \right)^2 \left( |z_0 + z_3|^2 + |z_1 + z_2|^2 - (r(z_0 + z_3)(z_1^* + z_2^*) + \text{c.c.}) \right)^2. \tag{26}
\]

In the limit of non-overlapping wavepackets visibility reduces to

\[
V_{\text{no}} = \frac{2 |z_0| |z_1|}{|z_0|^2 + |z_1|^2}. \tag{27}
\]
For completely overlapping wavepackets we find \( V = 0 \), which is again a consequence of the indistinguishability of the paths.

The duality relation \( \mathbb{I} \), which conveys that there is an informational trade-off between which path information and visibility \( \mathbb{K} \), can easily be verified in the case of non-overlapping wavepackets. Using the Cauchy-Schwartz inequality

\[
|\langle \psi_1 | \psi_2 \rangle|^2 \leq \langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle \quad \forall |\psi_1 \rangle, |\psi_2 \rangle \in \mathcal{H},
\]

one finds \( |z_0|^2 \leq n_0/(n_0 + n_1) \) and \( |z_1|^2 \leq n_1/(n_0 + n_1) \) so that \( |z_0|^2 + |z_1|^2 \leq 1 \). This condition guarantees that \( V_{\text{no}}^2 + W_{\text{no}}^2 \leq 1 \). Even though the state \( |\psi_n \rangle \) is pure, the duality relation is exactly fulfilled only in the special cases that (i) \( |z_0|^2 + |z_1|^2 = 1 \) and (ii) \( |z_0| = |z_1| \). In case (i) the detector function \( D \) has perfect overlap with \( |\psi_n \rangle \). This implies that \( \langle \psi_n | \hat{P}_2 | \psi_n \rangle = 0 \) so that the detector provides complete knowledge about complementarity. Case (ii) corresponds to the situation that the interferometer is perfectly balanced (the atom travels through both paths with equal probability) even after the position measurement. Hence \( W_{\text{no}} = 0 \) and, because the contrast of the fringes is not affected by a non-perfect overlap of the detector function, \( V_{\text{no}} = 1 \).

It seems obvious that the duality relation should also be fulfilled for overlapping states because any overlap should decrease the distinguishability between the two interferometer arms and thus reduce \( W \) and \( V \). However, a general proof of this conjecture is surprisingly difficult \( [14] \). Instead, we have verified numerically that the duality relation holds for a sample of 100,000 random Gaussian states, where \( T_a \phi(x) \) takes the form \( \exp(-(x-x_0)^2/w^2 + ikx) \). The detector function \( D(x) \) takes a similar form but with different parameters \( x_0, w, k \) that were chosen randomly for both \( \phi(x) \) and \( D(x) \) and were allowed to vary between \(-4 \) and \( 4 \) in units of the width of the width of the initial Gaussian state \( \phi \). The results for a sub-sample of 1000 random states are shown in Fig. 2. We found no violation of Eq. \( \mathbb{I} \).

**IV. COMPLEMENTARITY AND UNCERTAINTY RELATIONS**

In this section we offer a new perspective on the ongoing debate whether the duality relation \( \mathbb{I} \) is logically independent of Heisenberg’s uncertainty relation \( \mathbb{II} \) or not \( \mathbb{II} \). To address this conundrum we consider the special case of non-overlapping wavepackets. For simplicity we restrict our considerations to the spatial component \( x \) that is parallel to the separation vector \( a \) between the two interferometer paths. We assume that \( \phi(x) \) is a wavepacket of arbitrary shape that is centered around the origin, with a width that is small compared to the separation \( a \equiv |a| \) between the two beams. In this case the position uncertainty induced by the finite width of \( \phi(x) \) is generally negligible as compared to that induced by the superposition of the two interferometer paths \( \phi(x) \) and \( \phi(x-a) \). We then can make the approximation \( \langle \phi | (x+a)^n | \phi \rangle \approx a^n \) so that \( \langle \psi_D | (x)^n | \psi_D \rangle \approx a^n n_1/(n_0+n_1) \).

The position uncertainty in state \( |\psi_D \rangle \) then simplifies to

\[
\Delta x^2 \approx a^2 \frac{n_0 n_1}{(n_0 + n_1)^2}.
\]

Because which way information quantifies the probabilities for an atom to take one of the two interferometer paths one would generally expect a close relation between \( W \) and \( \Delta x \). For instance, if \( W = 1 \) one knows with certainty that the atom took one of the two paths so that \( \Delta x \) should be comparable to the width of the wavepacket \( \phi(x) \). On the other hand, if \( W = 0 \) then it is uncertain which path the atom takes. Then \( \Delta x \) should be of the order of the path separation \( a \) which may be much larger than the width of the wavepacket. However, the argument above does not take the quality of the position measurement into account. If we can make the same approximations in the evaluation of \( D(x) \) as in that of \( \Delta x \), then a Taylor expansion of the detector function yields

\[
\langle \phi | D(x) | \phi \rangle \approx D(0),
\]

which results in

\[
\frac{\Delta x^2}{a^2} \approx \frac{1}{4} (1 - W^2) \approx \frac{|D(0)|^2 |D(a)|^2}{(|D(0)|^2 + |D(a)|^2)^2}.
\]

However, this exact relationship between \( \Delta x \) and \( W \) is only valid if \( \langle \phi | D(x) | \phi \rangle \approx D(0) \), i.e., if the detector function \( D(x) \) varies little over the the extent of the wavepacket \( \phi(x) \). The example presented in Fig. 3 demonstrates that a rapid variation of \( D(x) \) can affect relation \( \mathbb{II} \). In this case a symmetric wavepacket is combined with an antisymmetric detector function so that \( \langle \phi | D(x) | \phi \rangle = 0 \) and consequently \( z_0 = 0 \). On the other hand, \( |D(x)|^2 \) is close to unity almost everywhere so that \( n_0 = \langle \phi | D(x) | \phi \rangle \approx 1 \). If we assume that \( D(x) = 0 \) around \( x = a \) then \( n_1 = z_1 = 0 \) so that \( \Delta x \approx W \approx 0 \).

Hence a detector only gathers which way information if the detector function \( D(x) \) is suitable. Even in the case of a general detector function \( D(x) \) it is possible to establish an inequality that relates po-
lar momentum can be derived by adapting Heisenberg’s correction of separation which represents the vector component of $p$.

The approximation applies for small deviations of $x$ about the momentum but rather about the modular momentum. Inserting Eqs. (30) and (37) into the uncertainty relation (1) immediately yields

$$\frac{\Delta x}{a} \geq \frac{1}{2}\sqrt{1 + \Delta H}$$

which establishes an uncertainty relation between position and modular momentum.

We now turn to the question whether uncertainty relation (36) can be related to the duality relation (11). For general $\mathcal{D}(x)$ only inequality (31) holds; we conjecture that in this case it is not possible to relate uncertainty and complementarity. The situation is different for suitable (i.e., slowly varying over the width of $\phi(x)$) detector functions which fulfill $\langle \phi | \mathcal{D}(x) | \phi \rangle \approx \mathcal{D}(0)$. We then have $V \approx 2|\mathcal{D}(0)||\mathcal{D}(a)|/(|\mathcal{D}(0)|^2 + |\mathcal{D}(a)|^2)$ and

$$\left| \langle \psi_D | \hat{T}_a | \psi_D \rangle \right| = \left| \frac{e^{-i\theta}}{n_a + n_1} \langle \phi | \hat{T}_a^\dagger \mathcal{D} \circ \hat{T}_a \mathcal{D} | \phi \rangle \right| \approx \frac{D^* (a) \mathcal{D}(0)}{(|\mathcal{D}(0)|^2 + |\mathcal{D}(a)|^2)} = \frac{V}{2}.$$  

Hence, for non-overlapping atomic beams and a suitable detector function there is a direct relation between complementarity and the uncertainties of position and modular momentum. Inserting Eqs. (30) and (37) into the duality relation (11) immediately yields

$$\frac{\Delta x}{a} \geq |\langle \hat{T}_a \rangle|,$$

from which the uncertainty relation (36) between position and modular momentum can be deduced. Therefore, for well-separated wavepackets the duality relation appears stronger than the Heisenberg uncertainty relation because the former can be used to derive the latter.

V. PERFECT LENS AND COMPLEMENTARITY

Roychoudhuri [19] and Berman [20] have challenged the Heisenberg-Bohr explanation of complementarity in the $\gamma$ ray microscope, which relates uncertainty to the diffraction of the lenses that are used to collect the radiation emitted by the atom. They pointed out that within this interpretation optical superresolution and diffraction-less metamaterial perfect lenses would lead to
a violation of the uncertainty principle. Here we resolve this question by demonstrating that the detection of light emitted by a two-level atom (2LA) in an atom interferometer corresponds to a generalized position measurement. The quality of the lenses therefore can only affect the amount of which way information that can be obtained, but it cannot affect the duality relation \( [1] \). 2LA interferometry and complementarity has previously been studied in Ref. \( [4] \), but this analysis did not consider the perfect lens; here we provide an alternative derivation that accommodates almost arbitrary arrangements of linear lossless dielectrics. We ignore the polarization of light in our derivation because it will not substantially affect our results.

We consider the situation that 2LAs are excited immediately after the beam has been split and then undergo spontaneous decay. As depicted in Fig. 1 the spontaneously emitted photon is detected after passing through an array of linear optical elements (which could include a perfect lens \( [22] \)). Just after excitation, the atomic state is \( \int d^3x \psi_0(x) |x\rangle \otimes |e\rangle \) for |e\rangle the internal excited state. Spontaneous emission over time scale \( 1/\gamma \) returns the 2LA to its ground state |g\rangle. Here \( \gamma \) is the decay rate of the atom in the presence of the dielectrics. A crucial assumption for our derivation is that \( 1/\gamma \) is short compared to the time scale \( \tau_A \) during which the atomic center-of-mass wavepacket changes significantly. This assumption allows us to neglect the kinetic center-of-mass energy of the atoms and should be valid for most situations. Exceptions would be atomic ensembles very far from equilibrium, for which \( \tau_A \) could be short, or optical cavities of extremely high finesse for which \( \gamma \) could be significantly smaller than the natural atomic decay rate in free space. The atomic Hamiltonian is then given by

\[
\hat{H}_A = \hbar \omega_A |e\rangle \langle e| ,
\]

with \( \omega_A \) the resonance frequency of the 2LA.

Because the dielectrics are assumed to be linear and lossless, there is a set of eigenmodes \( E_n(x) \) with frequency \( \omega_n \). For simplicity we restrict our analysis to a discrete set of modes, but generalizing our approach to a continuous set of modes should not affect the results. The radiative Hamiltonian in the presence of dielectrics then takes the general form

\[
\hat{H}_R = \hbar \sum_n \omega_n \hat{a}_n^\dagger(n) \hat{a}_n(n) ,
\]

where \( \hat{a}_n(n) \) annihilates one photon in mode \( E_n(x) \). Implicitly we have assumed here that the dielectrics are time independent over the time scale \( 1/\gamma \), which is the case for almost all experiments except for very special situations such as Faraday media driven by time varying external fields. We describe the coupling between matter and radiation in electric-dipole and rotating-wave approximation,

\[
\hat{H}_{\text{int}} = -d_{eg} |e\rangle \langle g| \sum_n \hat{a}_n(n) E_n(\hat{x}) + \text{H.c.}
\]

Expanding the total state of the system as

\[
|\psi(t)\rangle = \int d^3x \left( \psi_e(x,t) |e\rangle \otimes |x\rangle \otimes |\text{vac}\rangle + \sum_n \psi_n(x,t) |g\rangle \otimes |x\rangle \otimes \hat{a}_n^\dagger(n) |\text{vac}\rangle \right) , \tag{42}
\]

with |\text{vac}\rangle the radiative vacuum state, the Schrödinger equation can be cast into the form

\[
i\dot{\psi}_e(x) = \omega_A \psi_e(x) - \frac{d_{eg}}{\hbar} \sum_n E_n(x) \psi_n(x) \tag{43}
\]

\[
i\dot{\psi}_n(x) = \omega_n \psi_n(x) - \frac{d_{eg}}{\hbar} \psi_e(x) E_n^*(x) \tag{44}
\]

Performing a Laplace transformation in the time domain allows us to find the solution as

\[
\tilde{\psi}_e(x,s) = \frac{i\psi^{(0)}(x)}{is - \omega_A - \frac{d_{eg}^2}{\hbar^2} \sum_m |E_m(x)|^2 (is - \omega_m)} \tag{45}
\]

\[
\tilde{\psi}_n(x,s) = \frac{d_{eg} E_n^*(x)}{\hbar} \psi_e(x,s) = \frac{d_{eg} E_n^*(x)}{\hbar \omega_n} \tilde{\psi}_e(x,s) , \tag{46}
\]

where \( \tilde{f}(s) \) denotes the Laplace transform of \( f(t) \). The solution in time domain can be expressed through the inverse Laplace transform

\[
\psi_n(x,t) = \frac{1}{2\pi i} \int_C ds \ e^{st} \tilde{\psi}_n(x,s) , \tag{47}
\]

with the path \( C \) being to the right of all poles and branch cuts.

This solution contains the photon dynamics in the presence of linear dielectrics. At time \( t \) a detector is switched on to register the emitted photon. We model the detector as a device that detects photons in a particular mode characterized by the a specific superposition of annihilation operators \( \hat{b} = \sum_n \eta^*(n) \hat{a}_n(n) \). The 2LA state, conditioned on having detected a photon at time \( t \), is thus

\[
|\psi_D\rangle = \langle \text{vac}| \hat{b} |\psi(t)\rangle \tag{48}
\]

\[
= \sum_n \eta^*(n) \int d^3x \psi_n(x,t) |g\rangle \otimes |x\rangle
\]

The normalized post-detection 2LA wavepacket \( \psi_D(x) = \langle (|g\rangle \otimes |x\rangle) |\psi_D\rangle \) is therefore given by Eq. \( \text{Box} \) for detector function

\[
D(x) = -\frac{d_{eg}^*}{\hbar} \int_C ds \ e^{is} \left( \sum_n \eta^*(n) \frac{E_n^*(x)}{is - \omega_n} \right) \times \left( is - \omega_A - \frac{d_{eg}^2}{\hbar^2} \sum_m \frac{|E_m(x)|^2}{is - \omega_m} \right)^{-1} \tag{49}
\]

Hence, detecting spontaneously emitted radiation from an atom interferometer corresponds to a generalized position measurement, whereby the effect of arbitrary linear optical elements only affects the form of the detector
function $D(x)$. We remark that in free space this fact can also be explained by the entanglement between the photonic momentum and the atomic center-of-mass motion due to momentum conservation [31].

Our result can be used to resolve unambiguously the question whether a perfect lens would challenge causality: because a perfect lens can also be described as a linear optical device, it can only affect the shape of $D(x)$. Hence Inequality [1] is fulfilled, and a perfect lens would not contradict quantum mechanics. It simply would allow to increase $W$ at the expense of reducing $V$. The reason is that the effect of detecting a photon has a purely local consequence of neglecting the kinetic center-of-mass motion of the atoms, which is possible because for most systems the electronic dynamics is fast compared to the motion of the atomic nucleus. In free space the effect of the extension of the atomic wavepacket on spontaneous emission has been discussed in Ref. [32].

VI. EXAMPLE: DIFFRACTION LIMIT AND THE THIN LENS

In this section we apply the formalism developed above to a particular physical situation that is related to the case of the Heisenberg microscope: we consider the case that the which way detector is so far away from the interferometer that the spontaneous decay of the atom is practically completed before the photon enters the detector. Our assumption corresponds to the far field limit. If the far field limit is not achieved in an experiment, full separability of detector and source modes is not achieved, and a clean signature of complementarity would then be somewhat masked. The which way detector consists of a thin conventional lens and the actual detector; Fig. 1 depicts the spatial arrangement of the 2LA, lens, and detector. We will derive expressions for $V, W$ and the uncertainty of modular momentum and show explicitly how they are affected by the diffraction limit of the lens.

Under these assumptions the atomic spontaneous decay can be treated as in free space. The modes of the radiation field introduced in Sec. VI therefore correspond to plane waves. Replacing the sum over $n$ in Eq. (19) by an integral over the wavevector $k$ of the modes we have

$$E_k(x) = \sqrt{\frac{\hbar \omega_k}{2\epsilon_0(2\pi)^3}} e^{i k \cdot x} \quad (50)$$

with the dispersion relation $\omega_k = c|k|$. This results in

$$D(x) \propto \int_C ds \ e^{is} \int d^3k \eta^*(k) e^{-ik \cdot x} \sqrt{\frac{\omega_k}{is - \omega_k}} \times \left( -\frac{d\omega_g}{2\epsilon_0(2\pi)^3} \int d^3k' \frac{\omega_{k'}}{is - \omega_{k'}} \right)^{-1}$$

In Wigner-Weisskopf approximation [33] we can replace the integral over $k'$ (including its prefactors) by $\Delta_L - i\gamma/2$. We absorb the Lamb shift $\Delta_L$ into the definition of the resonance frequency so

$$D(x) \propto \int_C ds \ e^{is} \int d^3k \eta^*(k) e^{-ik \cdot x} \sqrt{\frac{\omega_k}{is - \omega_k}} \times \left( -\frac{d\omega_g}{2\epsilon_0(2\pi)^3} \int d^3k' \frac{\omega_{k'}}{is - \omega_{k'}} \right)^{-1}$$

(52)

Closing the path $C$ and using the residue theorem yields

$$D(x) \propto \int d^3k \eta^*(k) e^{-ik \cdot x} \sqrt{\frac{\omega_k}{\omega_k - \omega_A + i\gamma/2}} \times e^{-i\omega_k t} e^{-i\omega_A t - \gamma t/2}.$$  (53)

For sufficiently long times, $\gamma t \gg 1$, the emission process is completed and the detector function reduces to

$$D(x) \propto \int d^3k \eta^*(k) e^{-ik \cdot x} \sqrt{\frac{\omega_k}{\omega_k - \omega_A + i\gamma/2}} e^{-i\omega_k t}.$$  (54)

The detector function $D(x)$ depends on the detection device through the function $\eta(k)$. After the photon has passed the lens it propagates for a certain time until it reaches the image plane at which the detector is placed. If the lens is placed at position $z = 2f$ the image plane of the light will be at $z = 4f$. To travel a distance $2f$, light propagates for time $t = 4f/c$. Because the 2LA is located close to the origin, the detector should be in the image plane of the lens at $z = 4f$. The detector itself is assumed to respond to photons in a certain spatial mode $\tilde{\eta}(x)$ with Fourier transform

$$\tilde{\eta}(k) = \frac{1/2}{\pi^{3/4}} e^{-(w_2^2 k_1^2 + w_1^2 k_1^2)/2} e^{i k_s f}.$$  (55)

Here $w_1$ and $w_2$ denote the width of the detector mode transverse to and along the $z$-axis, respectively, and $k_s \equiv k_x e_x + k_y e_y$. In the following we will ignore the degrees of freedom along the $z$-direction because it is irrelevant for complementarity of $\hat{p}$ and $\hat{x}$ in the transverse direction. The lens represents a linear optical device, which generally effects a linear transformation of the detector mode of the form

$$\eta(k) = \int d^3k' M(k,k')\tilde{\eta}(k')$$  (56)

For the case of the single conventional thin lens in front of the detector, the transfer function is

$$M(k,k') \propto \exp \left( \frac{1}{2} \frac{(k_z - k_z')^2}{\Delta_L} - i\frac{k_z' f}{\Delta_L} \right),$$  (57)

with $f$ the focal length of the lens and $L_\perp$ the radius. For $L_\perp \to \infty$ this expression coincides with the usual transfer function for an infinitely wide thin lens. In a more accurate model for a thin lens its finite size would be
taken into account by a step function \( M(x, x') \propto \theta(L_{1}^{2} - x^2 - y^2) \) in position space. To simplify the discussion we use instead a model where the finite size of the lens is taken into account by a Gaussian spatial weight factor \( \propto \exp(-\alpha^2 / w_{\text{eff}}^2) \). This procedure generates the term \( L_{1}^{-2} \) in Eq. (54) and leads to

\[
\eta(k) \propto \exp \left( -\frac{1}{2} k_{\perp}^{2} \left( \frac{1}{L_{1}^{2}} + \frac{1}{w_{\perp}^{2}} - \frac{k_{0}}{f} \right) \right)^{-1} \quad (58)
\]

To simplify the discussion of complementarity we ignore the details of the spontaneous emission process by setting \( \omega_{k} \approx \omega_{A} \) in the non-exponential terms of Eq. (54). Furthermore, in the spirit of the paraxial approximation we make the expansion \( \omega_{k} \approx c k_{0} + c k_{\perp}^{2} / (2k_{0}) \) in the exponentials. The integrand is then a Gaussian and leads to

\[
D(x) \propto \exp \left( -\frac{x^2 + y^2}{2w_{\text{eff}}^2} + \frac{k_{0}}{f} k_{\perp} x_{\perp}^2 + y_{\perp}^2 \right) . \quad (59)
\]

For a small width of the detector, \( w_{\text{eff}} \) and the phase shift factor \( \delta \phi \) associated with the wavefront are given by

\[
w_{\text{eff}}^2 \approx \frac{4f^2}{k_{0}L_{1}^2} + w_{\perp}^2 \quad (60)
\]

\[
\delta \phi \approx \frac{k_{0}L_{1}^4}{32f^2 w_{\perp}^4} . \quad (61)
\]

This implies that the detector function is diffraction-limited with minimal effective width \( w_{\text{min}} = 2f / k_{0}L_{1} \), which corresponds to Heisenberg’s and Bohr’s analysis of the Heisenberg microscope: the resolution limit of a microscope led them to infer the position uncertainty \( \Delta x_{\text{Heis}} = \lambda / 2 \sin \alpha \) with \( \lambda \) the wavelength and \( \alpha \) the opening angle of the microscope’s lens. For \( f \gg L_{1} \) we have \( L_{1} / f = \tan \alpha \approx \sin \alpha \) and therefore \( \Delta x_{\text{Heis}} = \pi w_{\text{min}} / 2 \); the difference in the numerical prefactor is due to the Gaussian lens approximation that we have used.

In the case that the wave function \( \phi(x) \) is a Gaussian with width \( w_{\phi} \ll w_{\text{eff}} \) and the two wavepackets \( \phi(x), \phi(x-a) \) are well separated one finds

\[
W = \tanh \left( \frac{a^2}{2w_{\text{eff}}^2} \right) \quad (62)
\]

\[
V = \frac{2}{1 + \exp \left( \frac{a^2}{w_{\text{eff}}^2} \right)} . \quad (63)
\]

The exponential damping terms \( \exp(-a^2 / w_{\text{eff}}^2) \) reflect the fact that if the atomic wave function distance \( a \) is much larger than the width \( w_{\phi} \) of the detector function, then the photo detection will allow to distinguish the two wavepackets. It then allows us to gather information about \( W \) and thus diminish \( V \). This behaviour is shown in Fig. 4 where \( W \) and \( V \) are plotted as a function of the separation between the two wavepackets. It is apparent that the duality relation is always satisfied. For very small (very large) separations the inequality is saturated because in these cases the photo emission generates no (maximal) which way information, respectively.

The mean value of modular momentum is

\[
\langle \psi_{D}|\hat{T}_{a}|\psi_{D} \rangle = \frac{e^{-\frac{a^2}{2w_{\text{eff}}^2}} e^{-i \frac{a^2}{2f} L_{1} \delta \phi}}{1 + e^{-\frac{a^2}{w_{\text{eff}}^2}}} . \quad (64)
\]

For large separations of the wavepackets it approaches 0 (completely indefinite modular momentum) because in this limit the which way detector completely destroys the coherence between the two wavepackets. The phase factor in \( \langle \hat{T}_{a} \rangle \) has the following interpretation: for large enough detectors \( \delta \phi \sim 1 \), and the shift in the phase factor corresponds to \( \exp(i a \delta p_{x} / \hbar) \), where \( \delta p_{x} \) is the momentum difference in the x-direction (transverse to the propagation axis) for photons that arrive at the same point on the lens but are emitted by different wave functions. This is given by \( \delta p_{x} = (\text{total photon momentum}) \times (\text{wave function separation}) / (\text{propagation length}) = \hbar k_{0} a / (2f) \).

In Fig. 4 we present a numerical example for the behaviour of \( \langle \hat{T}_{a} \rangle \). The parameters chosen are \( k_{0} = 10^{7} \text{m}^{-1}, L_{1} = 5 \text{cm}, f = 20 \text{cm}, \) and \( w_{\perp} = 30 \mu \text{m} \). The modulus always less than 1/2 because this is the maximum value for \( \langle \hat{T}_{a} \rangle \) in the case of well separated wavepackets.

\[\text{VII. CONCLUSIONS}\]

We have analyzed the relation between the duality relation \( W^{2} + V^{2} \leq 1 \), which connects which way information \( W \) and fringe visibility \( V \) in an atom interferometer and a Heisenberg uncertainty relation between atomic position and (modular) momentum. A quantitative link between both concepts can be established by modeling the process of splitting the matter beam using the operator \( \hat{T}_{a} \) of Eq. (54), which spatially shifts the initial wavepacket by a distance \( a \). This shift operator can also be interpreted...
as the operator of the atomic modular momentum. The process of splitting the atomic beam is therefore naturally connected to a change in modular momentum. We have shown that this connection allows us to derive the uncertainty relation from the duality relation if $W$ is obtained by a generalized (smeared out) position measurement.

Furthermore we have shown that the detection of spontaneously emitted photons in an atom interferometer corresponds to a generalized position measurement, provided the detection device can be described using lossless linear optical elements and projection measurements. Because the duality relation holds regardless of the specific nature of the detection device, the complementarity principle of quantum mechanics holds regardless of the quality of the detection device in use. Complementarity is therefore not affected by superresolving optical devices or perfect lenses based on meta-materials; such optical elements can only affect the amount of which way information that can be gathered, but not the duality relation.

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[34] It seems impossible to prove Eq. [1] analytically for arbitrary atomic states. Instead, two other approaches can be used to test its validity. (i) a numerical evaluation of Eq. [1] for a set of randomly generated values for the parameters $n_i, r, z_i$. However, not all possible real or complex values for $n_i, r, z_i$ correspond to a state. For instance, from the definition of $W$ we know that $W < 1$ for all states, but it is easy to see that in Eq. [22] $W \rightarrow \infty$ for $r \rightarrow 1$ and $z_i$ fixed. To generate only
physical parameter values we have therefore constrained the random values by a set of 20 inequalities that we derived using the Cauchy-Schwartz inequality and general relations for the overlap between two given states. A typical example of one of the 20 inequalities would be $|\langle \phi | D^\dagger D (\hat{T}_a - \langle \hat{T}_a \rangle) | \phi \rangle|^2 \leq n_0 (1 - |\langle \hat{T}_a \rangle|^2)$. However, even this large number of constraints did not exclude certain unphysical values for the parameters, and thus this approach did not help to verify Eq. (1). (ii) A second approach to verify Eq. (1) is to numerically evaluate $W$ and $V$ for a random set of quantum states. This is the approach described in the text.

[35] Because all quantities related to complementarity and uncertainty are invariant under a rescaling of $D(x)$ we can ignore all constant prefactors in the derivation.