Cospanning characterizations of antimatroids and convex geometries

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Abstract

Given a finite set $E$ and an operator $\sigma : 2^E \rightarrow 2^E$, two sets $X,Y \subseteq E$ are cospanning if $\sigma(X) = \sigma(Y)$. Corresponding cospanning equivalence relations were investigated for greedoids in much detail (Korte, Lovasz, Schrader; 1991). For instance, these relations determine greedoids uniquely. In fact, the feasible sets of a greedoid are exactly the inclusion-wise minimal sets of the equivalence classes.

In this research, we show that feasible sets of convex geometries are the inclusion-wise maximal sets of the equivalence classes of the corresponding closure operator. Same as greedoids, convex geometries are uniquely defined by the corresponding cospanning relations. For each closure operator $\sigma$, an element $x \in X$ is an extreme point of $X$ if $x \notin \sigma(X - x)$. The set of extreme points of $X$ is denoted by $ex(X)$.

We prove, that if $\sigma$ has the anti-exchange property, then for every set $X$ its equivalence class $[X]_\sigma$ is the interval $[ex(X), \sigma(X)]$. It results in the one-to-one correspondence between the cospanning partitions of an antimatroid and its complementary convex geometry.

The obtained results based on the connection between violator spaces, greedoids and antimatroids. A notion of violator space was introduced in (Gärtner, Matoušek, Rüst, Škrovnič; 2008) as a combinatorial framework that encompasses linear programming and other geometric optimization problems.

Violator spaces are defined by violator operators. We have proved that a violator operator may be defined by a weak version of a closure operator. In the paper we prove that violator operators generalize...
rank closure operators of greedoids. Further, we introduced *co-violator spaces* based on contracting operators known also as choice functions. Cospanning characterization of these combinatorial structures allows us not only to give the new characterization of antimatroids and convex geometries, but also to obtain the new properties of closure operators, extreme point operators and their interconnections.

**Keywords:** cospanning relation, violator space, closure space, greedoid.

1 Preliminaries

While convex geometries are usually defined as a closure space with anti-exchange property, antimatroids - the families of sets complementary to convex sets, are described as accessible set systems closed under union. At the same time matroids can be defined by two ways - by a closure operator with Steinitz-MacLine exchange property and as families of independent sets. Greedoids, that are a common generalization of matroids and antimatroids, are defined as set systems with augmentation property, but they may be described by some "closure operator" as well.

This work is an attempt to describe all these structures in identical way. Each set operator determines the partition of sets to equivalence classes with equal value of the operator. Let we have some set operator $\alpha$. Following [6] we call two sets $X, Y$ to be cospanning if $\alpha(X) = \alpha(Y)$. Thus each set operator generates the cospanning equivalence relation on sets. The cospanning relation associated with a closure operator of greedoids was introduced and investigated in [6], where it was proved that the relation determines the greedoid uniquely. In fact, the feasible sets of a greedoid are exactly the minimal (by inclusion) sets of the equivalence classes, i.e., the sets that are not cospanning with any proper subset. We extend the approach to other combinatorial structures. Thus convex geometries, being families of closed sets, are the maximal (by inclusion) sets of the equivalence classes of the closure operator, and matroids are the minimal sets.

We begin with a definition of a violator operator which, as we prove, is a generalization of a closure operator and a rank closure operator of greedoids. The rest of the paper is organized as follows. In Section 2, we give a brief introduction to violator spaces focussing on violator spaces with unique basis. At the end of the section we characterize the cospanning relation with regards to violator spaces and describe the equivalence classes of the relation. Section 3 is devoted to closure spaces with a focus on convex geometries, and in Section 4 we investigate the cospanning relation of antimatroids and ma-
troids and prove one-to-one correspondence between cospanning partition of antimatroids and convex geometries.

2 Violator spaces

Violator spaces are arisen as generalization of Linear Programming problems. LP-type problems have been introduced and analyzed by Matoušek, Sharir and Welzl \[8, 10\] as a combinatorial framework that encompasses linear programming and other geometric optimization problems. Further, Matoušek et al. \[3\] define a simpler framework: violator spaces, which constitute a proper generalization of LP-type problems. Originally, violator spaces were defined for a set of constraints \(E\), where each subset of constraints \(G \subseteq E\) was associated with \(\nu(G)\) - the set of all constraints violating \(G\).

The classic example of an LP-type problem is the problem of computing the smallest enclosing ball of a finite set of points in \(\mathbb{R}^d\). Here \(E\) is a set of points in \(\mathbb{R}^d\), and the violated constraints of some subset of the points \(G\) are exactly the points lying outside the smallest enclosing ball of \(G\).

**Definition 2.1** [3] A violator space is a pair \((E, \nu)\), where \(E\) is a finite set and \(\nu\) is a mapping \(2^E \to 2^E\) such that for all subsets \(X, Y \subseteq E\) the following properties are satisfied:

\[
\begin{align*}
\text{V11: } X \cap \nu(X) &= \emptyset \text{ (consistency)}, \\
\text{V22: } (X \subseteq Y \text{ and } Y \cap \nu(X) = \emptyset) &\Rightarrow \nu(X) = \nu(Y) \text{ (locality)}. 
\end{align*}
\]

Let \((E, \nu)\) be a violator space. Define \(\varphi(X) = E - \nu(X)\). The operator \(\varphi\) satisfies extensivity \((X \subseteq \varphi(X))\), self-convexity \((X \subseteq Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y)\) and idempotence \((\varphi(\varphi(X)) = \varphi(X))\) [4].

In what follows, if \((E, \nu)\) is a violator space and \(\varphi(X) = E - \nu(X)\), then \((E, \varphi)\) will be called a violator space and \(\varphi\) - a violator operator as well.

**Definition 2.2** [4] A violator space is a pair \((E, \varphi)\), where \(E\) is a finite set and \(\varphi\) is an operator \(2^E \to 2^E\) such that for all subsets \(X, Y \subseteq E\) the following properties are satisfied:

\[
\begin{align*}
\text{V1: } X \subseteq \varphi(X) \text{ (extensivity)}, \\
\text{V2: } (X \subseteq Y \subseteq \varphi(X)) &\Rightarrow \varphi(X) = \varphi(Y) \text{ (self-convexity)}. 
\end{align*}
\]

**Lemma 2.3** If \(\varphi\) satisfies extensivity and self-convexity, then for every \(A, B\)

\[
B \subseteq \varphi(A) \iff \varphi(A) = \varphi(A \cup B).
\]

3
Proof. ”If”: Let \( \varphi(A) = \varphi(A \cup B) \). Since extensivity implies \( B \subseteq A \cup B \subseteq \varphi(A \cup B) \), we conclude with \( B \subseteq \varphi(A) \).

"Only if": If \( B \subseteq \varphi(A) \), then \( A \subseteq A \cup B \subseteq \varphi(A) \). Hence, by self-convexity, \( \varphi(A) = \varphi(A \cup B) \). ■

Corollary 2.4 If \( \varphi \) satisfies extensivity and self-convexity, then for every \( X \subseteq E \) and \( x \in E \)

\[
(x \in \varphi(X)) \iff (\varphi(X) = \varphi(X \cup x)).
\]

Proposition 2.5 For each extensive operator \( \varphi \) self-convexity is equivalent to the following property: \( \text{VV2}: (X \subseteq \varphi(Y) \land Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y) \).

Proof. 1. \( \text{V2} \Rightarrow \text{VV2} \): From Lemma 2.3 (\( X \subseteq \varphi(Y) \land Y \subseteq \varphi(X) \)) \( \Rightarrow \varphi(Y) = \varphi(X \cup Y) = \varphi(X) \). Hence, \( \varphi(X) = \varphi(Y) \).

2. \( \text{VV2} \Rightarrow \text{V2} \): \( (X \subseteq Y \subseteq \varphi(X)) \Rightarrow X \subseteq \varphi(Y) \land Y \subseteq \varphi(X) \) (from \( \text{V1} \)). Then \( \text{VV2} \) implies \( \varphi(X) = \varphi(Y) \). ■

Lemma 2.6 (\[4\]) Let \((E, \varphi)\) be a violator space. Then

\[
\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cup Y) = \varphi(X) = \varphi(Y)
\]

and

\[
(X \subseteq Y \subseteq Z) \land (\varphi(X) = \varphi(Z)) \Rightarrow \varphi(X) = \varphi(Y) = \varphi(Z)
\]

for every \( X, Y, Z \subseteq E \).

Since the second property deals with all sets lying between two given sets, following \[8\] we call the property convexity.

2.1 Uniquely generated violator spaces

Let \((E, \alpha)\) be an arbitrary space with the operator \( \alpha : 2^E \to 2^E \). \( B \subseteq E \) is a generator of \( X \subseteq E \) if \( \alpha(B) = \alpha(X) \). For \( X \subseteq E \), a basis (minimal generator) of \( X \) is a inclusion-minimal set \( B \subseteq E \) (not necessarily included in \( X \)) with \( \alpha(B) = \alpha(X) \). A space \((E, \alpha)\) is uniquely generated if every set \( X \subseteq E \) has a unique basis.

Proposition 2.7 \[4\] A violator space \((E, \varphi)\) is uniquely generated if and only if for every \( X, Y \subseteq E \)

\[
\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cap Y) = \varphi(X) = \varphi(Y) \quad (1)
\]

4
We can rewrite the property (1) as follows: for every set \( X \subseteq E \) of a uniquely generated violator space \((E, \varphi)\), the basis \( B \) of \( X \) is the intersection of all generators of \( X \):

\[
B = \bigcap \{Y \subseteq E : \varphi(Y) = \varphi(X)\}.
\]  

(2)

One of the known examples of a not uniquely generated violator space is the violator space associated with the smallest enclosing ball problem. A basis of a set of points is a minimal subset with the same enclosing ball. In particular, all points of the basis are located on the ball’s boundary. For \( \mathbb{R}^2 \) the set \( X \) of the four corners of a square has two bases: the two pairs of diagonally opposite points. Moreover, one of these pairs is a basis of the second pair. Thus the equality (2) does not hold.

It is known that a closure operator \( \tau \) is uniquely generated if and only if it satisfies the anti-exchange property [2, 6, 9]:

\[
p, q \notin \tau(X) \land p \in \tau(X \cup q) \Rightarrow q \notin \tau(X \cup p).
\]

We extended this characterization to violator spaces.

**Theorem 2.8** [4] Let \((E, \varphi)\) be a violator space. Then \((E, \varphi)\) is uniquely generated if and only if the operator \( \varphi \) satisfies the anti-exchange property.

For each arbitrary space \((E, \alpha)\) with the operator \( \alpha : 2^E \rightarrow 2^E\), an element \( x \) of a subset \( X \subseteq E \) is an extreme point of \( X \) if \( x \notin \alpha(X - x) \). The set of extreme points of \( X \) is denoted by \( ex(X) \).

**Proposition 2.9** [4] Let \((E, \varphi)\) be a violator space. Then \( x \in ex(X) \) if and only if \( \varphi(X) \neq \varphi(X - x) \).

**Proposition 2.10** [4] Let \((E, \varphi)\) be a violator space. Then

\[
ex(X) = \bigcap \{B \subseteq X : \varphi(B) = \varphi(X)\}.
\]

**Proposition 2.11** [4] Let \((E, \varphi)\) be a violator space. Then \( ex(\varphi(X)) \subseteq ex(X) \).

**2.2 Co-violator spaces**

**Definition 2.12** [3] A co-violator space is a pair \((E, c)\), where \( E \) is a finite set and \( c \) is an operator \( 2^E \rightarrow 2^E \) such that for all subsets \( X, Y \subseteq E \) the following properties are satisfied:

\[
CV1: c(X) \subseteq X,
\]

\[
CV2: (c(X) \subseteq Y \subseteq X) \Rightarrow c(X) = c(Y).
\]
Operators which satisfy the property CV1 are called contracting operators.

In social sciences, contracting operators are called choice functions, usually adding a requirement that \( c(X) \neq \emptyset \) for every \( X \neq \emptyset \). The property CV2 is called the outcast property or the Aizerman property \([8]\).

The properties of co-violator spaces match to the corresponding ("mirrored") properties of violator spaces. Thus, every co-violator operator \( c \) is idempotent. Indeed, since \( c \) is contracting \( c(X) \subseteq c(X) \subseteq X \). Then, CV2 implies \( c(c(X)) = c(X) \).

Lemma 2.6 is converted to the following.

**Lemma 2.13** \([5]\) Let \( (E, c) \) be a co-violator space. Then
\[
c(X) = c(Y) \Rightarrow c(X \cap Y) = c(X) = c(Y)
\]
and
\[
(X \subseteq Y \subseteq Z) \land (c(X) = c(Z)) \Rightarrow c(X) = c(Y) = c(Z)
\]
for every \( X, Y, Z \subseteq E \).

Given an extensive operator \( \varphi : 2^E \to 2^E \), one can get a contracting operator \( c : c(X) = E - \varphi(E - X) \) or \( c(X) = \varphi(X) \). In topology this construction is known as interior operator dual to a closure operator.

**Proposition 2.14** \([3]\) \((E, \varphi)\) is a violator space if and only if \((E, c)\) is a co-violator space, where \( c(X) = \varphi(X) \).

### 2.3 Cospanning relations of violator spaces

Let \( E = \{x_1, x_2, ..., x_d\} \). The graph \( H(E) \) is defined as follows. The vertices are the finite subsets of \( E \), two vertices \( A \) and \( B \) are adjacent if and only if they differ in exactly one element. Actually, \( H(E) \) is the hypercube on \( E \) of dimension \( d \), since the hypercube is known to be equivalently considered as the graph on the Boolean space \( \{0, 1\}^d \) in which two vertices form an edge if and only if they differ in exactly one position.

Let \((E, \varphi)\) be a violator space. The two sets \( X \) and \( Y \) are equivalent (or cospanning) if \( \varphi(X) = \varphi(Y) \). In what follows, \( \mathcal{P} \) denotes a partition of \( H(E) \) (or \( 2^E \)) into equivalence classes with regard to this relation, and \([A]_\varphi := \{X \subseteq E : \varphi(X) = \varphi(A)\}\).

The following theorem characterizes the cospanning relation of violator spaces.
Theorem 2.15 \cite{5} Let $E$ be a finite set and $R \subseteq 2^E \times 2^E$ be an equivalence relation on $2^E$. Then $R$ is the cospanning relation of a violator space if and only if the following properties hold for every $X, Y, Z \subseteq E$:

- **R1**: if $(X, Y) \in R$, then $(X, X \cup Y) \in R$
- **R2**: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.

Thus each equivalence class of the cospanning relation of a violator space is closed under union (R1) and convex (R2).

Similarly,

Theorem 2.16 \cite{5} Let $E$ be a finite set and $R \subseteq 2^E \times 2^E$ be an equivalence relation on $2^E$. Then $R$ is the cospanning relation of a co-violator space if and only if the following properties hold for every $X, Y, Z \subseteq E$:

- **R3**: if $(X, Y) \in R$, then $(X, X \cap Y) \in R$
- **R2**: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.

Consider now a violator operator $\varphi$ and a co-violator operator $c(X) = \overline{\varphi(X)}$.

Proposition 2.17 \cite{5} There is a one-to-one correspondence between an equivalence class $[X]_\varphi$ of $X$ of the cospanning relation associated with a violator operator $\varphi$ and an equivalence class $[X]_c$ w.r.t. a co-violator operator $c$, i.e., $A \in [X]_\varphi$ if and only if $A \in [X]_c$.

An uniquely generated violator space defines a cospanning relation with additional property R3 (see Proposition \[2.7\]).

So every uniquely generated violator space is a co-violator space as well. Each equivalence class of the cospanning relation of a uniquely generated violator space has an unique minimal element and an unique maximal element. More precisely, for the sets $A \subseteq B \subseteq E$, let $[A, B] := \{C \subseteq E : A \subseteq C \subseteq B\}$. One calls any $[A, B]$ an interval. Then each equivalence class of an uniquely generated violator space is an interval. We call a partition of $H(E)$ into disjoint intervals a hypercube partition. The following Theorem follows immediately from Theorem \[2.15\] and Proposition \[2.7\].

Theorem 2.18 \cite{4} (i) If $(E, \varphi)$ is a uniquely generated violator space, then $P$ is a hypercube partition of $H(E)$.

(ii) Every hypercube partition is the partition $P$ of $H(E)$ into equivalence classes of a uniquely generated violator space.

More specifically \cite{4}, $[A]_\varphi = [\text{ex}(A), \varphi(A)]$ for every set $A \subseteq E$. 

7
It is interesting notice that for a cospanning relation $R$ of a violator space, i.e., for an equivalence relation satisfying $R_1$ and $R_2$, the property $R_3$ is equivalent to the anti-exchange property. First, rewrite the anti-exchange property in terms of a cospanning relation $R$. From Corollary 2.4 it follows

$$p \notin \varphi(X) \iff \varphi(X) \neq \varphi(X \cup p) \iff (X, X \cup p) \notin R.$$  

Then the anti-exchange property looks as

**R33**: if $(X, X \cup p) \notin R$, $(X, X \cup q) \notin R$, and $(X \cup p, X \cup p \cup q) \in R$, then $(X \cup q, X \cup p \cup q) \notin R$.

**Proposition 2.19** An equivalence relation $R$ satisfying the property $R_1, R_2$ and $R_3$ coincides with an equivalence relation satisfying the property $R_1, R_2$ and $R_{33}$.

**Proof.** Let $R$ be an equivalence relation satisfying the property $R_1$ and $R_2$.

1. Prove, that if $R$ satisfies $R_3$, then $R_{33}$ holds. Suppose there are $X \in E, p, q \notin X$ for which $R_{33}$ does not hold, i.e., $(X \cup q, X \cup p \cup q) \in R$. Since $R$ is an equivalence relation we have $(X \cup q, X \cup p) \in R$, but from $R_3$ it follows that $(X, X \cup p) \in R$. Contradiction.

2. Prove, that if $R$ satisfies $R_{33}$, then $R_3$ holds. $R_1$ implies uniqueness of maximal element, so consider some equivalence class with unique maximal element $X$. Let $B$ be a minimal element of $[X]_R$. To prove that $B$ is unique minimal element enough to prove that $B \subseteq Y$ for each $Y \in [X]_R$. Suppose for the sake of contradiction that there are $p \in B$ and $p \notin Y$. Since $B - p \subseteq B \subseteq X$ and $Y \subseteq X$, we have $Y \subseteq (B - p) \cup Y \subseteq X$. Now, $R_2$ implies that $((B - p) \cup Y, X) \in R$. Since $B$ is a minimal generator of $X$, we obtain that $((B - p), X) \notin R$. Thus one can see that there is a minimal set $C$ such that $\emptyset \subset C \subseteq Y$ and $((B - p) \cup C), X) \in R$.

Consider some element $q \in C$, and let $Z = (B - p) \cup (C - q)$. Based on minimality of $C$ it follows that $(Z, X) \notin R$. Note that $(Z \cup p, X) \in R$, which follows from $B \subseteq Z \cup p \subseteq X$ and $R_2$. Thus, $(Z, Z \cup p) \notin R$. Similarly, we obtain $(Z, Z \cup q) \notin R$, since $(Z \cup q) = (B - p) \cup C$ and so $(Z \cup q, X) \in R$. Now $(Z \cup p \cup q, X) \in R$, that means $(Z \cup p, Z \cup p \cup q) \in R$, and $(Z \cup q, Z \cup p \cup q) \in R$, contradicting the $R_{33}$. Consequently, $B \subseteq Y$ for each $Y \in [X]_R$. Hence $R_3$ holds.

Consider now an uniquely generated violator space $(E, \varphi)$ and operator $ex$. Since each equivalence class $[A]_{\varphi}$ w.r.t. operator $\varphi$ is an interval $[ex(A), \varphi(A)]$, we can see that for each $X \in [ex(A), \varphi(A)]$ not only $\varphi(X) = \varphi(A)$, but $ex(X) = ex(A)$ as well, and since $\mathcal{P}$ is a hypercube partition of $H(E)$ we conclude with $[X]_{\varphi} = [X]_{ex}$. So the cospanning partition
(quotient set) associated with an operator \( \varphi \) coincides with the cospanning partition associated with a contracting operator \( ex \). Since \( ex(X) \) is a minimal element of \([X]\) we immediately obtain

**Proposition 2.20** [5] If \((E, \varphi)\) is a uniquely generated violator space, then operator \( ex \) satisfies the following properties:

1. \( X1: ex(ex(X)) = ex(X) \)
2. \( X2: ex(X) = ex(Y) \Rightarrow ex(X \cup Y) = ex(X) = ex(Y) \)
3. \( X3: (X \subseteq Y \subseteq Z) \land (ex(X) = ex(Z)) \Rightarrow ex(X) = ex(Y) = ex(Z) \)
4. \( X4: ex(X) = ex(Y) \Rightarrow ex(X \cap Y) = ex(X) = ex(Y) \)

**Proposition 2.21** [5] Let \((E, \varphi)\) be a violator space. The following assertions are equivalent:

1. \((E, \varphi)\) is uniquely generated
2. \( X5: (ex(X) \subseteq Y \subseteq X) \Rightarrow ex(X) = ex(Y) \) (the outcast property)
3. \( X6: \varphi(ex(X)) = \varphi(X) \)
4. \( X7: ex(\varphi(X)) = ex(X) \)

### 3 Closure spaces

**Definition 3.1** Let \( E \) be a finite set. \( \tau : 2^E \to 2^E \) is a closure operator on \( E \) if for all subsets \( X, Y \subseteq E \) the following properties are satisfied:

1. \( C1: X \subseteq \tau(X) \) (extensivity)
2. \( C2: X \subseteq Y \Rightarrow \tau(X) \subseteq \tau(Y) \) (isotonicity)
3. \( C3: \tau(\tau(X)) = \tau(X) \) (idempotence).

\((E, \tau)\) is a closure space if \( \tau \) is a closure operator. A set \( A \subseteq E \) is closed if \( A = \tau(A) \). Clearly, the family of closed sets \( K = \{A \in E : A = \tau(A)\}\) is closed under intersection. Conversely, any set system \((E, K)\) closed under intersection is a family of closed sets of the closure operator

\[
\tau_K(X) = \bigcap\{A \in K : X \subseteq A\}. \tag{3}
\]

In a Euclidean space, a set is convex if it contains the line segment between any two of its points. It is easy to see that the family of convex sets is closed under intersection. In fact, the family of convex sets coincides with the family of closed sets defined by a convex hull operator.

Convex geometries were invented by Edelman and Jamison in 1985 as proper combinatorial abstractions of convexity [2]. There are various ways to characterize finite convex geometries. One of them defines convex sets using anti-exchange closure operators.
A closure space \((E, \tau)\) is a \textit{convex geometry} if it satisfies the anti-exchange property. The convex hull operator on Euclidean space is a classic example of a closure operator with the anti-exchange property.

Consider relations between closure operators and self-convex operators.

**Lemma 3.2** \([4]\) \textit{Isotonicity and idempotence imply self-convexity.}

So we conclude with

**Theorem 3.3** \([4]\) \textit{Every closure space is a violator space.}

Let \(R\) be a cospanning relation of a closure space. Then every equivalence class of the cospanning relation is closed under union (\(R_1\)) and convex (\(R_2\)). Easy to see that all closed sets are the inclusion-maximal sets of the equivalence classes. First note that idempotence implies \((X, \tau(X)) \in R\). Since \(X \subseteq \tau(X)\) the family of closed sets \(K\) coincides with the family of maximal sets of equivalence classes. Moreover, the cospanning partition of a closure space satisfies the \textit{accessibility property:}

**Proposition 3.4** \(\forall X \in K : (X - x, X) \notin R \iff X - x \in K\).

\textbf{Proof.} 1. Let \((X - x, X) \notin R\), and so \(\tau(X - x) \neq \tau(X)\). From extensivity and isotonicity of \(\tau\) it follows \(X - x \subseteq \tau(X - x) \subset \tau(X) = X\). Hence \(\tau(X - x) = X - x\), that means \(X - x \in K\).

2. If \(X - x \in K\), then \(\tau(X - x) = X - x\), that means \((X - x, X) \notin R\). ■

Since \(\tau(X - x) \neq \tau(X)\) is equivalent to \(x \in \text{ex}(X)\) (see Proposition 2.9), Proposition 3.4 may be rewritten as follows

\[\forall X \in K : x \in \text{ex}(X) \iff X - x \in K.\] \hspace{1cm} (4)

This property is mentioned in \([6, 9]\) for convex geometries.

If \(\text{ex}\) is a non-empty choice operator, i.e., \(\text{ex}(X) \neq \emptyset\) for every \(X \neq \emptyset\), that a closure system is an accessible system in which every nonempty closed set \(X\) contains an element \(x\) such that \(X - x\) is closed. So in follows we call the property \(\text{(4)}\) an accessibility property as well.

Consider now a special class of closure spaces - convex geometries. Since convex geometries are uniquely generated closure spaces \((E, \tau)([6, 9])\), they determine a hypercube partition \(\mathcal{P}\) into equivalence classes \([A]_\tau := \{X \subseteq E : \tau(X) = \tau(A)\} = [\text{ex}(A), \tau(A)]\). The family of closed sets \((X = \tau(X))\), just being a convex geometry, is exactly the family of maximal sets of the intervals. In addition the hypercube partition \(\mathcal{P}\) has an accessibility property that for cospanning partitions looks as follows.
∀[A, B] ∈ \mathcal{P} and \( x \in A \) there is \( C \subseteq E \) such that \([C, B - x] \in \mathcal{P}\) (5)

Let we have some hypercube partition \( \mathcal{P}' \) with accessibility property (5).

Let \( \mathcal{N} = \{ B : [A, B] \in \mathcal{P}' \} \).

**Lemma 3.5** If \([A, B] \in \mathcal{P}', \) and \( C \subseteq B, \) but \( C \notin [A, B] \), then there exists \( x \in B - C \) such that \( B - x \in \mathcal{N} \).

**Proof.** Since \( C \notin [A, B] \), there is \( x \in A \) such that \( x \notin C \), so \( x \in B - C \). Then the property (5) implies \( B - x \in \mathcal{N} \). ■

Since the maximal element in each interval is unique we obtain, that the family \( \mathcal{N} \) satisfies the chain property: for all \( X, Y \in \mathcal{N} \), and \( X \subseteq Y \), there exists a chain \( X = X_0 \subseteq X_1 \subseteq ... \subseteq X_k = Y \) such that \( X_i = X_{i-1} \cup x_i \) and \( X_i \in \mathcal{N} \) for \( 0 \leq i \leq k \).

**Theorem 3.6** (i) If \((E, \tau)\) is a convex geometry, then equivalence classes of the cospanning relation associated with \( \tau \) form a hypercube partition \( \mathcal{P}' \) satisfying the accessibility property.

(ii) Every hypercube partition \( \mathcal{P}' \) satisfying accessibility property (5) is the partition of \( H(E) \) into equivalence classes of the cospanning relation of a convex geometry.

**Proof.** It remains to prove (ii). For each \( X \subseteq E \) there is only one interval \([A, B]\) containing \( X \). Then for every set \( X \), we define \( \alpha(X) = B \). To prove that \((E, \alpha)\) is a convex geometry, it is enough to show that \((E, \alpha)\) is a uniquely generated closure space. To demonstrate it, one has to check that \( \alpha \) satisfies extensivity, isotonicity and idempotence.

Extensivity follows from \( X \subseteq B = \alpha(X) \). Idempotence is obviously. Let \( X \subseteq Y \). Prove \( \alpha(X) \subseteq \alpha(Y) \). If \( X \) and \( Y \) belong to the same interval \([A, B] \), then \( \alpha(X) = \alpha(Y) = B \). Let \( X \in [C, D] \) and \( Y \in [A, B] \), then \( X \subseteq B \). Hence Lemma 3.5 implies that there is a chain \( B \supseteq B_1 = B - x_1 \supseteq B_2 = B_1 - x_2 \supseteq \cdots \supseteq B_k \), where all elements of the chain \( B_i \in \mathcal{N} \) and \( X \subseteq B_i \). The chain ends with \( X \in [A_k, B_k] \), i.e., \( B_k = D = \alpha(X) \). So \( \alpha(X) \subseteq B = \alpha(Y) \). Thus the operator \( \alpha \) is a closure operator.

To prove unique generation notice that \( \alpha(X) = \alpha(Y) \) means \( X, Y \in [A, B] \). Then \( \alpha(X \cap Y) = \alpha(X) \), and from Proposition 2.7 immediately follows that the closure space \((E, \alpha)\) is uniquely generated. It is easy to see that the partition to equivalence classes of cospanning relation w.r.t. \( \alpha \) coincides with \( \mathcal{P}' \). ■

The equivalent statement in terms of cospanning relations looks as following.
Theorem 3.7 Let $E$ be a finite set and $R \subseteq 2^E \times 2^E$ an equivalence relation on $2^E$. Then $R$ is the cospanning relation of a convex geometry if and only if the following conditions hold for every $X, Y, Z \subseteq E$ and $x, y /\in X$:

- **R1:** if $(X, Y) \in R$, then $(X, X \cup Y) \in R$
- **R2:** if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.
- **R3:** if $(X, Y) \in R$, then $(X, X \cap Y) \in R$
- **R4:** if $(X, X \cup z) \notin R$ for all $z /\in X$ and $(X, X - x) \notin R$, then $(X - x, X - x \cup y) \notin R$ for all $y /\in X$.

It is easy to see that the property **R4** is equivalent to accessibility property if a closure space is uniquely generated. Indeed, **R3** implies that for each $X \subseteq E$ there is an unique basis $ex(X)$, and so $(X - x, X - x \cup y) \notin R \iff x \in ex(X)$.

The second condition says that $X - x$ is a closed set.

4 Greedoids

4.1 Violator spaces and greedoids

Let we have a greedoid $(E, F)$, i.e.,

- (i) $\emptyset \in F$
- (ii) $X, Y \in F, |X| > |Y| \Rightarrow \exists x \in X - Y \cup x \in F$

Elements of $F$ are calls feasible sets.

The rank function of a greedoid is defined as follows:

$$r(X) = \max\{|A| : A \subseteq X, A \in F\}.$$ Define (rank) closure operator (6):

$$\sigma(X) = \{x : r(X \cup x) = r(X)\}.$$ If $\Gamma(X) = \{x \in E - X : X \cup x \in F\}$, then $\sigma(X) = E - \Gamma(X)$.

**Lemma 4.1** [6] Let $(E, F)$ be a greedoid, then

- (i) $X \subseteq \sigma(X)$.
- (ii) $(X \subseteq \sigma(Y) \land Y \subseteq \sigma(X)) \Rightarrow \sigma(X) = \sigma(Y)$.
- (iii) if $x, y \in E - X$, and $X \cup x \in F$, then $x \in \sigma(X \cup y) \Rightarrow y \in \sigma(X \cup x)$.

The last property (iii) may be considered as a weaker version of the Steinitz-MacLine exchange property of matroids:

if $x \notin \sigma(X)$, $x \in \sigma(X \cup y)$ then $y \in \sigma(X \cup x)$.

Since (ii) is equivalent to self-convexity (Proposition 2.5) we can see that greedoids may be considered as a subclass of violator spaces.

**Lemma 4.2** [6] Let $(E, F)$ be a greedoid, then

$$F = \{X \subseteq E : \forall x \in X : x \notin \sigma(X - x)\}.$$
So each element of a feasible set is an extreme point of the set, i.e.,
\[ F = \{ X \subseteq E : X = \text{ex}(X) \}. \]

The following definition of feasible sets is equivalent:

**Lemma 4.3** Let \((E, F)\) be a greedoid, then
\[ F = \{ X \subseteq E : \forall Y \subseteq X : \sigma(X) \neq \sigma(Y) \} \] - the family of bases - minimal generators w.r.t. operator \(\sigma\).

**Proof.** If for all \(Y \subseteq X : \sigma(X) \neq \sigma(Y)\), then for all \(x \in X\) holds \(\sigma(X) \neq \sigma(X - x)\), and so (from Corollary 2.4) \(x \notin \sigma(X - x)\) (or \(X = \text{ex}(X)\)).

The following definition of feasible sets is equivalent:

Consider the third property (iii). If \(x, y \in E - X\) and \(x \in \sigma(X \cup y)\), then \(r(X \cup x \cup y) = r(X \cup y) \leq |X| + 1\). Since \(X \cup x \in F\), we have \(|X| + 1 \leq r(X \cup x \cup y) = r(X \cup y) \leq |X| + 1 = r(X \cup x)\). Hence \(X \cup y \in F\).

Corollary 2.4 implies that \(\sigma(X \cup x) = \sigma(X \cup x \cup y) = \sigma(X \cup y)\), and so \(X \cup x \cup y\) has two bases \(X \cup x\) and \(X \cup y\).

Remark 4.4 For a greedoid \((E, F)\) a basis of \(X \subseteq E\) is defined as a maximal feasible subset of \(X\), while for a space \((E, \sigma)\) a basis is defined as a minimal generator of \(X\). Lemma 4.3 provides the equivalence of two definitions.

The third property (iii) is equivalent to the following property:

(iv): if \(\sigma(X \cup y) = \sigma(X \cup x \cup y)\), but \(\sigma(X \cup x) \neq \sigma(X \cup x \cup y)\), then \(X \cup x \notin F\), i.e., there exists a \(z \in X \cup x\) such that \(\sigma(X \cup x - z) = \sigma(X \cup x)\).

### 4.2 Cospanning relation of greedoids

Let \((E, F)\) be a greedoid and \(\sigma\) be a closure operator of the greedoid. The two sets \(X\) and \(Y\) are equivalent (or cospanning) if \(\sigma(X) = \sigma(Y)\). In what follows, \(\mathcal{P}\) denotes a partition of \(H(E) (2^E)\) into equivalence classes with regard to this relation, and \([A]_{\sigma} = \{ X \subseteq E : \sigma(X) = \sigma(A) \}\). The following theorem characterizes the cospanning relation of greedoids.

**Theorem 4.5** Let \(E\) be a finite set and \(R \subseteq 2^E \times 2^E\) an equivalence relation on \(2^E\). Then \(R\) is the cospanning relation of a greedoid if and only if the following conditions hold for every \(X, Y, Z \subseteq E\) and \(x, y \notin X\):

- **R1:** if \((X, Y) \in R\), then \((X, X \cup Y) \in R\)
- **R2:** if \(X \subseteq Y \subseteq Z\) and \((X, Z) \in R\), then \((X, Y) \in R\).
- **R4:** if \((X \cup y, X \cup x \cup y) \in R\), but \((X \cup x, X \cup x \cup y) \notin R\), then there exists an element \(z \in X \cup x\) such that \((X \cup x - z, X \cup x) \in R\).
Since we have made some changes in the proof to use the obtained results in the future, we give here all the proof.

**Proof.** Necessity follows immediately from Lemma 4.1 and Theorem 2.15. To prove the sufficiency, we build \( F = \{ X \subseteq E : \forall Y \subset X : \sigma(X) \neq \sigma(Y) \} \) - the family of minimal sets of equivalence classes, and check that the family is a greedoid.

**Lemma 4.6** The property \( R4 \) is equivalent to the following augmentation property:

If \( A \in F \), and \( a \in E - A \) such that \((A, A \cup a) \notin R\), then \( A \cup a \in F \).

In other words, the augmentation property means that if \( A \) is a minimal element of some equivalence class (and so it belongs to \( F \)), then for each \( a \in E - A \) such that \( A \cup a \) does not belong to this equivalence class \([A]\), \( A \cup a \) is a minimal element of another equivalence class.

Let us prove that the augmentation property follows from \( R4 \). Suppose there exists \( a \in E - A \) such that \((A, A \cup a) \notin R\), but \( A \cup a \notin F \). Hence there is some \( b \neq a \) such that \((A \cup a - b, A \cup a) \in R\). Set \( X = A - b \), then \((X \cup a, X \cup a \cup b) \in R\), \((X \cup b, X \cup a \cup b) \notin R\), then \( A = X \cup b \notin F \).

Contradiction.

To prove that \( R4 \) follows from the augmentation property suppose that \( A \subseteq X \subseteq E \) and \( A \in F \). Then there exists \( x \in X - A \) such that \( A \cup x \in F \) if and only if \((A, X) \notin R\).

Indeed, if \((A, X) \in R\), then convexity \( R2 \) implies that \((A, A \cup x) \in R\) for each \( x \in X - A \), and hence \( A \cup x \notin F \). Conversely, if \( A \cup x \notin F \) for all \( x \in X - A \), then from augmentation property follows that \((A, A \cup x) \in R\) for all \( x \in X - A \), and so by \( R1\), \((A, X) \in R\).

From definition of \( F \) it follows that \( \emptyset \in F \). Thus Lemma 4.7 implies that \( F \) is accessible. To prove that \( F \) is a greedoid we have to check that for each \( A, B \in F \) with \(|A| < |B|\) there exists some \( b \in B - A \) such that \( A \cup b \in F \). Assume that the property does not hold. Let \( C \subseteq A \cap B \), \( C \in F \), and choose \( A, B, C \) such that \(|C|\) is maximal. By Lemma 4.7 \((A, A \cup B) \in R\), and so \( A \cup B \notin F \) and \( A \notin B \).

\( C \subseteq B \) and \((C, B) \notin R\), then by Lemma 4.7 there exists \( a \in B - C \) such that \( C \cup a \in F \). By the maximality of \( C \), \( a \notin A \). From the assumption, \((A, A \cup c) \in R\). Since \( C \cup c \subseteq A \cup c \), Lemma 4.7 implies that there is a
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\[ \sigma \alpha \cospanning, \text{ so} \]

\[ (E, \sigma) \]

\[ \text{since extensivity and self-convexity implies idempotence,} \]

\[ \text{sufficiency define an equivalence relation} \]

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Example 4.9 Let \( E = \{1, 2, 3\} \). Define \( \varphi(X) = X \) for each \( X \subseteq E \) except \( \varphi(\{1\}) = \varphi(\{1, 3\}) = \{1, 3\} \). It is easy to check that the space \((E, \varphi)\) is a uniquely generated violator space (satisfies both extensivity and self-convexity), where the family of bases \( F = \mathcal{P}(\{1, 2, 3\}) - \{1, 3\} \) forms a greedoid. At the same time, the operator \( \varphi \) does not satisfies the property \( G3 \) and the cospanning relation w.r.t. \( \varphi \) does not satisfies \( R4 \). Indeed, if \( X = \emptyset, x = 3, y = 1 \), then \( X \cup x = \{3\} \in F \), \( (X \cup y, X \cup x \cup y) \in R \), and \((X \cup x, X \cup x \cup y) \notin R\).

If we consider the rank function of the greedoid, we can see that \( r(\{1\}) = r(\{3\}) = r(\{1, 3\}) \), and so \( \sigma(\{1\}) = \sigma(\{3\}) = \{1, 3\} \). Then for this function \( \sigma \) the property \( R4 \) holds and we have a not uniquely generated violator space.

So the same family of bases may be obtain by different operators and by different equivalence relations.

4.3 Antimatroids - uniquely generated greedoids

An antimatroid is a greedoid closed under union.

Lemma 4.10 For a accessible set system \((E, F)\) the following statements are equivalent:

(i) \((E, F)\) is an antimatroid.

(ii) \(F\) is closed under union.

(iii) \( A, A \cup x, A \cup y \in F \) implies \( A \cup \{x, y\} \in F \)

Proposition 4.11 An antimatroid is a uniquely generated greedoid.

Proof. Let \( F \) be an antimatroid. Since each antimatroid is a greedoid, it remains to prove that the greedoid is uniquely generated. Suppose there are two bases \( B_1 \) and \( B_2 \) such that \( \sigma(B_1) = \sigma(B_2) \). Since \( F \) is a family of bases, then \( B_1, B_2 \in F \). Hence \( B_1 \cup B_2 \in F \), because \( F \) is an antimatroid. But \( \sigma(B_1 \cup B_2) = \sigma(B_1) \) (see Lemma 2.5). Contradiction.

Since for each greedoids the operator \( \sigma \) is a violator operator, Theorem 2.8 implies

Corollary 4.12 The operator \( \sigma \) of each antimatroid satisfies the anti-exchange property.

Thus we can conclude with the following theorem.

Theorem 4.13 The family \( F \) is an antimatroid if and only if \( F \) is a uniquely generated greedoid.
Proof. It remains to prove that each uniquely generated greedoid is an antimatroid. Suppose, $A, A \cup x, A \cup y \in \mathcal{F}$, but $A \cup \{x, y\} \notin \mathcal{F}$. Then $x, y \notin \sigma(A), x \in \sigma(A \cup y)$ and $y \in \sigma(A \cup x)$. Contradiction to anti-exchange property. ■

Remark 4.14 Antimatroids being greedoids satisfy the weaker version of exchange property (iii) (Lemma 4.1) and at the same time they satisfy the anti-exchange property. But that is not a problem (paradox), since the conditions of the property (iii) ($X \cup x \in \mathcal{F}$, and $x \in \sigma(X \cup y)$) do not hold for antimatroids.

Since any antimatroid is an uniquely generated greedoid, the cospanning relation w.r.t. the closure operator of an antimatroid, or, for simplicity, the cospanning relation of an antimatroid, determines the hypercube partition $\mathcal{P}$ with augmentation property (see Lemma 4.6):

$$\forall [A, B] \in \mathcal{P} \text{ and } x \notin B \text{ there is } C \subseteq E \text{ such that } [A \cup x, C] \in \mathcal{P} \quad (6)$$

Let we have some hypercube partition $\mathcal{P}'$ with augmentation property (6). Denote $\mathcal{F} = \{ A : [A, B] \in \mathcal{P}' \}$.

Theorem 4.15 Every hypercube partition $\mathcal{P}'$ satisfying (6) is the partition of $H(E)$ into equivalence classes w.r.t. the closure operator of an antimatroid $\mathcal{F}$.

The equivalent statement looks as following.

Theorem 4.16 Let $E$ be a finite set and $R \subseteq 2^E \times 2^E$ an equivalence relation on $2^E$. Then $R$ is the cospanning relation of an antimatroid if and only if the following conditions hold for every $X, Y, Z \subseteq E$ and $x, y \notin X$:

- **R1**: if $(X, Y) \in R$, then $(X, X \cup Y) \in R$
- **R2**: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.
- **R3**: if $(X, Y) \in R$, then $(X, X \cap Y) \in R$
- **R4**: if $(X \cup y, X \cup x \cup y) \in R$, but $(X \cup x, X \cup x \cup y) \notin R$, then there exists an element $z \in X \cup x$ such that $(X \cup x - z, X \cup x) \in R$.

Since hypercube partition satisfying (6) determines the uniquely generated greedoid (Theorem 4.15), the theorem immediately follows from Theorem 4.13.

In any way, it may be interesting to see how the uniqueness of a basis turns a greedoid to be an antimatroid.
Proof. Necessity follows immediately from Theorems 4.5 and 4.13. Since properties $R_1, R_2, R_4$ imply that the family $F$ is a greedoid (Theorem 4.5), to prove the sufficiency it remains to prove that $F$ is closed under union.

Let $A, B \in F$. $(A, B) \notin R$, since there is a unique basis. Then, w.l.o.g., $(B, A \cup B) \notin R)$. Lemma 4.7 implies that there is a chain $B = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_k = C$ such that $B_i = B_{i-1} \cup x_i$, $B_i \in F$ for $0 \leq i \leq k$, and $(C, A \cup B) \in R$.

If $(A, A \cup B) \in R$, then $A = C$ (from uniqueness of the basis), and so $A \cup B = A \supseteq B$. If $(A, A \cup B) \notin R$, then there exists a chain $A = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m = D$, such that $A_i \in F$ for $0 \leq i \leq m$, and $(D, A \cup B) \in R$. Hence, $C = D = A \cup B$. So $F$ is closed under union. ■

4.4 Matroids - hereditary greedoids

A matroid $(E, F)$ may be defined as a greedoid which satisfy hereditary property:

for each $Y \subseteq X \subseteq E$, if $X \in F$, then $Y \in F$.

**Theorem 4.17** Let $E$ be a finite set and $R \subseteq 2^E \times 2^E$ an equivalence relation on $2^E$. Then $R$ is the cospanning relation of a matroid if and only if the following conditions hold for every $X, Y, Z \subseteq E$ and $x, y \notin X$:

- **R1**: if $(X, Y) \in R$, then $(X, X \cup Y) \in R$
- **R2**: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.
- **R4**: if $(X \cup y, X \cup x \cup y) \in R$, but $(X \cup x, X \cup x \cup y) \notin R$, then there exists an element $z \in X \cup x$ such that $(X \cup x \cup z, X \cup x) \in R$.
- **R5**: if $(X, X - x) \notin R$ for each $x \in X$, then $(X - x - z, X - x) \notin R$ for each $z \in X - x$.

4.5 Antimatroids and convex geometries

We know that $(E, F)$ is an antimatroid, if and only if $(E, \mathcal{N})$ is a convex geometry, where $\mathcal{N} = \{E - X : X \in F\}$. What is a connection between their partitions to cospanning equivalence classes?

First find the formula for bases in antimatroids.

**Lemma 4.18** Let $(E, F)$ be an antimatroid. Then $B_X$ is a basis of $X \subseteq E$ if and only if

$$B_X = \bigcup \{A \in F : A \subseteq X\}.$$  (7)

**Proof.** Let $B_X$ be a basis of $X$, and so $B_X \in F$. Then if $A \in F$ and $A \subseteq X$, then $A \subseteq B_X$. Indeed, $A \cup B_X \subseteq X$ and $A \cup B_X \in F$, so $B_X = A \cup B_X$, that proves $A \subseteq B_X$. Then (7) is proved.

18
If $B$ is defined by (7), then $B \in F$, $B \subseteq X$ and $B \subseteq B_X$. Then $B = B_X$.

Since the section deals with equivalence classes of antimatroids and convex geometries, to distinguish them from each other we denote them by $[X]_\sigma$ and by $[X]_\tau$ correspondingly.

It is easy to see that the maximal element in $[X]_\tau$ is a compliment of the minimal element in $[X]_\sigma$, i.e., $\tau(X) = \overline{B_X}$. Indeed, (7) implies $B_X = \bigcap \{ \overline{A} \in N : \overline{X} \subseteq \overline{A} \}$ and from (3) it follows that $B_X = \tau(X)$.

Taking into account that in antimatroids the unique basis of each set is a set of extreme point we obtain $\text{ex}_\sigma(X) = \tau(X)$.

**Lemma 4.19** Let $(E, F)$ be an antimatroid. Then $\overline{\sigma(X)} = \text{ex}_\tau(X)$.

**Proof.** Consider an equivalence class $[X]_\sigma$. Lemma 4.6 implies that for each $a \in E - \text{ex}_\sigma(X)$ the following holds: $a \notin \sigma(X)$ if and only if $(\text{ex}_\sigma(X) \cup a) \in F$. Note, that $\text{ex}_\tau(X) \subseteq \tau(X) = \text{ex}_\sigma(X)$, and so if $a \in \text{ex}_\tau(X)$ then $a \notin \sigma(X)$. Based on (4) we have

$\text{ex}_\tau(X) - a \notin N \iff \overline{\text{ex}_\sigma(X) - a} \in N \iff (\text{ex}_\sigma(X) \cup a) \in F \iff a \notin \sigma(X)$. ■

So Proposition 2.17 immediately implies

**Proposition 4.20** There is a one-to-one correspondence between an equivalence class of $X$ of cospanning relation w.r.t. an antimatroid - $[X]_\sigma$ and an equivalence class $[X]_\tau$ in the convex geometry $N$, i.e., $A \in [X]_\sigma$ if and only if $A \in [X]_\tau$.

**Corollary 4.21** Let $(E, F)$ be an antimatroid. Then

$X \subseteq Y \subseteq E \Rightarrow \text{ex}_\sigma(X) \subseteq \text{ex}_\sigma(Y)$

Indeed, $X \subseteq Y \Rightarrow \overline{Y} \subseteq \overline{X} \Rightarrow \tau(\overline{Y}) \subseteq \tau(\overline{X}) \Rightarrow \tau(\overline{X}) \subseteq \tau(\overline{Y}) \Rightarrow \text{ex}_\sigma(X) \subseteq \text{ex}_\sigma(Y)$.

5 Conclusion

Many combinatorial structures may be characterized by using operators defined on the elements of the structures. Thus matroids are described by a closure operator with exchange property, convex geometries are usually defined as a closure space with anti-exchange property. Greedoids may be described by some ”closure operator” as well. Each set operator determines the partition of sets to equivalence classes with equal value of the
operator and thus it generates the cospanning equivalence relation on sets. The cospanning relation associated with a closure operator of greedoids was introduced and investigated in [6], where it was proved that the relation determines the greedoid uniquely. We extended this approach to another combinatorial structures and obtain cospanning characterization of violator and co-violator spaces, for convex geometries, antimatroids, and matroids.

It remains an open problem to characterize the cospanning partition and/or the cospanning relation of closure spaces.

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