Polynomial algorithm for \( k \)-partition minimization of monotone submodular function

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Abstract. For a fixed \( k \), this study considers \( k \)-partition minimization of submodular system \((V, f)\) with a finite set \( V \) and symmetric submodular function \( f : 2^V \mapsto \mathbb{R} \). Our algorithm uses the Queyranne’s (1998) algorithm for 2-partition minimization which arises at each step of the recursive decomposition of subsets of the original \( k \)-partition minimization. We show that the computational complexity of this minimizer is \( O(n^{3(k-1)}) \).

Keywords. Submodular partition problem · symmetric submodular function

1 \( k \)-partition minimization of submodular system

Let \((V, f)\) is any submodular system with a finite set \( V \) and submodular function \( f : 2^V \mapsto \mathbb{R} \). We call function \( f : 2^V \mapsto \mathbb{R} \) is submodular, if for any \( X, Y \subseteq V \) it satisfies

\[
f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).\]

We call function \( f \) symmetric, if \( f(U) = f(V \setminus U) \) for any set \( U \subseteq V \), and call it monotone, if \( f(U \cup U') \geq f(U) \) for any set \( U, U' \subseteq V \).

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In this paper, we consider $k$-partition minimization with monotone submodular function $f$. Denote the set of all $k$-partitions for a given set $V$ by

$$P_{k,V} := \{ (U_0, U_1, \ldots, U_{k-1}) \mid \bigcup_i U_i = V, U_i \cap U_j = \emptyset \text{ for any } i \neq j \text{ and } U_i \neq \emptyset \text{ for every } i \}.$$ 

$k$-partition minimization problem of a submodular system $(V, f)$ is to find $k$-partition $U = (U_0, U_1, \ldots, U_{k-1}) \in P_{k,V}$ which minimizes the function

$$g(U) := \sum_{i=0}^{k-1} f(U_i) + C,$$

where $C$ is a constant to any choice of $k$-partition. As a practical application of $k$-partition minimization problem with monotone submodular function, Hidaka and Oizumi [1] have discussed the minimum $k$-partition of the mutual information for subsets with higher integrated information. In their study, the monotone submodular function is defined by the Shannon entropy of a set of random variables $X$, denoted by $f(X) := H(X)$. The function $H(X)$ is monotone increasing, as $H(X \cup Y) - H(X) = H(Y | X) \geq 0$ for any set of variables $Y$. The $k$-partition function is defined by

$$g(U) = \sum_{i=0}^{k-1} H(U_i) - H(U),$$

which is known as total correlation [5] or multi-information [4].

Quyranne [3] has shown a $O(|V|^3)$ algorithm for the 2-partition minimization problem. Okumoto and colleagues [2] have shown a polynomial algorithm for the 3-partition minimization problem. To our knowledge, no study has reported yet a polynomial algorithm for the $k$-partition minimization problem with a fixed $k > 1$ in general. In this paper, we report a polynomial-time algorithm for $k$-partition minimization of an arbitrary monotone submodular function by extending Queyranne’s algorithm [3].

2 Extension of Queyranne’s algorithm

Queyranne’s algorithm [3] works on bi-partition minimization of

$$g(U) = f(U) + f(V \setminus U)$$

for an arbitrary submodular system $(V, f)$ with respect to non-empty set $U \subset V$.

Here we show a recursive algorithm extending it for $k$-partition minimization of $g(U) = \sum_{i=0}^{k-1} f(U_i)$ with respect to $U \in P_{k,V}$. The basic idea is to reduce the original $k$-partition problem with the objective function $g : P_{k,V} \mapsto \mathbb{R}$ to a set function $g_{k,V} : 2^V \mapsto \mathbb{R}$ by recursively defining $g_{k-1,U}$ for the remaining $(k-1)$ subsets in a given $k$-partition.
By taking a 3-partition minimization as an example, first let us consider the following “naive” reduction:

$$
\min_{U \in P_2} g(U) = \min_{\emptyset \subset U_1 \subset V} \left[ f(U_1) + \min_{\emptyset \subset U_2 \subset V \setminus U_1} g_2(V \setminus U_1) \right].
$$

The first-level minimization is performed on the function $f(U_1) + g_2(V \setminus U_1)$, where $g_2(V \setminus M_1)$ is defined by the second-level minimization of bi-partition function $f(U_2) + f(V \setminus (U_1 \cup U_2))$. In this naive reduction, the second-level minimization is solved by the Queyranne’s algorithm, but the first-level function $f(M_1) + g_2(V \setminus M_1)$ is not symmetric in general. In order to let the function at every level be symmetric, let us redefine the reduction as follows:

$$
\min_{U \in P_2} g(U) = \min_{\emptyset \subset U_1 \subset V} \left[ f(V \setminus U_1) + \min_{\emptyset \subset U_2 \subset U_1} f_{U_2}(U_1) \right],
$$

where $f_W(U) := f(U) + f(W \setminus U)$ for any set $U \subseteq W$. In this formulation, each of the second-level minimization for $g_2(U_1)$ and $g_2(V \setminus U_1)$ is solveable by Queyranne’s algorithm, and the first-level function

$$
g_3(V)(U_1) := \min(f(V \setminus U_1) + g_2(U_1), f(U_1) + g_2(V \setminus U_1))
$$

is symmetric. Thus, the 3-partition minimization of $g(U)$ is solveable by Queyranne’s algorithm, if this first-level function $g_{3,V}(U_1)$ is submodular. Later, we prove its submodularity.

Before considering with the submodularity of the above function, let us extend the reduction with nested symmetric functions for the $k$-partition minimization as follows. We can identify the $k$-partition minimization problem of $g(U)$ to

$$
\min_{\langle U_1, \ldots, U_k \rangle \in \mathcal{P}_k} \sum_{i=1}^{k} f(U_i) = \min_{\emptyset \subset U \subset V} g_{k,V}(U),
$$

where the series of symmetric function is defined for any non-empty subset $U_1 \subset U_2 \subseteq V$ by $g_{2,U_2}(U_1) := f_{U_2}(U_1) = f(U_1) + f(U_2 \setminus U_1)$ and for $k > 2$

$$
g_{k,U_2}(U_1) := \min(h_{k-1,U_2}(U_1), h_{k-1,U_2}(U_2 \setminus U_1)),
$$

where

$$
h_{k,U_2}(U_1) = \begin{cases} 
    f(U_1) + \min_{\emptyset \subset U' \subset U_2 \setminus U_1} g_{k,U_2 \setminus U_1}(U') & \text{if } |U_2 \setminus U_1| > 1 \\
    \infty & \text{otherwise}
\end{cases} \quad (3)
$$

for any $k > 2$ and $U_1 \subset U_2 \subseteq V$. For $k = 2$, $g_{2,V}(U) = f_V(U)$, and minimization of $f_V(U)$ over the set of bi-partitions of $V$ can be computed by Queyranne’s algorithm.
3 Main results

If the $k$th order function $g_{k,V}$ is submodular at every step above, we can apply Queyranne’s algorithm to this function at every recursive step. As $g_{k,V}$ is symmetric by definition, our main question is whether it is submodular. The main result, Theorem 1, states the function $g_{k,V}$ is submodular, if $f$ is monotone submodular. To prove Theorem 1 we have the following steps.

1. Lemma 1 shows submodularity of the function $h_{2,V}$.
2. Lemma 2 shows the minimum of two submodular functions $\min(f(X), g(X))$ with monotone difference is submodular.
3. Theorem 1 shows the symmetrized minimum of $k$-partition $g_{k,V}$ is submodular.

**Lemma 1 (submodularity of minimum bi-partition)** For an arbitrary submodular system $(V, f)$ such that the function $f$ is monotone and $f(\emptyset) = 0$. The minimum of bi-partition function

$$g(X) := \begin{cases} 
  f(V \setminus X) + \min_{A \subseteq A' \subseteq X} f(A) + f(X \setminus A) & \text{if } |X| \geq 2 \\
  f(V \setminus X) + f(X) & \text{otherwise}
\end{cases}$$

is submodular.

**Proof** If $|X| < 2$, $g$ is obviously submodular, and thus suppose $|X| \geq 2$. For $\emptyset \subset W \subset Z \subseteq V$, write $f_Z(W) := f(W) + f(Z \setminus W)$. Denote one of the minimal sets for the following functions by

$$A_1 := \arg \min_{\emptyset \subset A' \subseteq X} f_X(A'), \quad B_1 := \arg \min_{\emptyset \subset B' \subseteq Y} f_Y(B'),$$

$$W_1 := \arg \min_{\emptyset \subset W' \subseteq X \cup Y} f_{X \cup Y}(W'), \quad Z_1 := \arg \min_{\emptyset \subset Z' \subseteq X \cap Y} f_{X \cap Y}(Z'),$$

and their another subset of bi-partition by

$$A_2 = X \setminus A_1, \quad B_2 = V \setminus B_1, \quad W_2 = (X \cup Y) \setminus W_1, \quad Z_2 = (X \cap Y) \setminus Z_1,$$

and their complements by

$$A_3 = V \setminus X, \quad B_3 = V \setminus Y, \quad W_3 = V \setminus (X \cup Y), \quad Z_3 = V \setminus (X \cap Y).$$

If $X \cap Y = \emptyset$, $f(Z_1) = f(Z_2) = f(A_i \cap B_j) = 0$ for any $i, j = 1, 2$, and by the minimality of $f(Z_1) + f(Z_2)$ and $f(W_1) + f(W_2)$, we have

$$f(A_1 \cup B_1) + f(A_3 \cap B_2) + f(A_2 \cup B_2) + f(A_3 \cap B_2) \geq f(W_1) + f(W_2) + f(Z_1) + f(Z_2).$$

By the submodular inequality,

$$f(A_1) + f(B_1) + f(A_2) + f(B_2) \geq f(W_1) + f(W_2) + f(Z_1) + f(Z_2) \quad (4)$$

and

$$f(A_3) + f(B_3) \geq f(W_3) + f(Z_3). \quad (5)$$
Adding these inequalities, \( g(X) \) holds the submodular inequality.

Consider the second case that holds \( X \cap Y \neq \emptyset, A_i \cap B_j \neq \emptyset \). Then, by the minimality of \( f(Z_1) + f(Z_2) \) and \( f(W_1) + f(W_2) \), we have

\[
f(A_i \cap B_j) + f((A_{3-i} \cup B_{3-j}) \cap X \cap Y) \geq f(Z_1) + f(Z_2),
\]

and

\[
f(A_i \cup B_j) + f((A_{3-i} \cap B_{3-j})) \geq f(W_1) + f(W_2).
\]

By monotonicity of \( f \), \( f((A_{3-i} \cup B_{3-j})) \geq f((A_{3-i} \cup B_{3-j}) \cap X \cap Y) \), and by the submodularity inequality we have (4). Adding (5) to (4), \( g(X) \) holds the submodular inequality.

Lemma 1 states the minimum bi-partition function is submodular, if \( f \) is monotone submodular function. But note that this function is not symmetric as it is, and slightly different from the function \( g \), that we defined earlier so it can be minimized by Queyranne’s algorithm. As the function \( g \) takes additional minimum to be symmetric, we need to deal with this minimum of two submodular functions by showing the following Lemma 2.

**Lemma 2 (submodularity of minimum of two submodular functions)**

*For two submodular function \( f \) and \( g \) over the ground set \( V \),

\[
h(X) = \min(f(X), g(X))
\]

is submodular, if the function \( d(X) := f(X) - g(X) \) is either monotone increasing or decreasing.*

**Proof** If \( h(X) + h(Y) = f(X) + f(Y) \) or \( h(X) + h(Y) = g(X) + g(Y) \), by submodularity we have

\[
h(X)+h(Y) \geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) = h(X \cup Y) + h(X \cap Y).
\]

Otherwise, \( h(X) + h(Y) = f(X) + g(Y) \) or \( h(X) + h(Y) = g(X) + f(Y) \). As \( d(X) \) is monotone, it holds either

\[
f(X) \geq f(X \cup Y) - g(X \cup Y) + g(X) \quad \text{or} \quad g(Y) \geq f(Y) - f(X \cup Y) + g(X \cup Y).
\]

By submodularity of \( f \) and \( g \)

\[
f(X) + g(Y) \geq f(X \cup Y) + g(X \cap Y) \quad \text{or} \quad f(X) + g(Y) \geq g(X \cup Y) + f(X \cap Y).
\]

Similarly,

\[
g(X) + f(Y) \geq g(X \cup Y) + f(X \cap Y) \quad \text{or} \quad g(X) + f(Y) \geq f(X \cup Y) + g(X \cap Y).
\]

Thus,

\[
h(X) + h(Y) \geq h(X \cup Y) + h(X \cap Y).
\]

Combining Lemma 1 and Lemma 2, the following theorem states submodularity of the symmetrized minimum \( k \)-partition.
Theorem 1 (submodularity of symmetric minimum \(k\)-partition) For \(k > 1\) and an arbitrary submodular system \((V, f)\) with monotone submodular function \(f\), for \(X \subseteq V\) define \(g_{2,X}(Y) := f(Y) + f(X \setminus Y)\), and for \(k > 2\)

\[ g_{k,X}(Y) = \min (h_{k-1,X}(Y), h_{k-1,X}(X \setminus Y)) \]

and

\[ h_{k,X}(Y) := f(Y) + g_k(X \setminus Y). \]

The set function \(g_{k,Y} : 2^V \rightarrow \mathbb{R}\) is submodular for any \(k \geq 2\).

Proof Any function \(h_{2,X}(Y)\) for \(\emptyset \subsetneq Y \subsetneq X \subsetneq V\) is submodular due to Lemma 1. By induction, suppose that \(h_{m,X}(Y)\) is submodular for \(k = 2, \ldots, m\), and let us show \(g_{m+1,X}(Y)\) is submodular. By Lemma 2, it is sufficient to show

\[ d_{m,X}(Y) := h_{m,X}(Y) - h_{m,X}(X \setminus Y) \]

is either monotone decreasing or monotone increasing. For any singleton set \(S \subseteq V\) and \(|S| = 1\),

\[ d_{m,X}(Y) - d_{m,X}(Y \cup S) = f(Y) + \min_{U \in P_{k,X \setminus Y}} \sum_{i=1}^k f(U_i) - f(X \setminus Y) - \min_{U \in P_{k,X}} \sum_{i=1}^k f(U_i) - f(Y \cup U) + \min_{U \in P_{k,Y \cup U}} \sum_{i=1}^k f(U_i) \]

Write the minimal \(k\)-partitions

\( (W_1, \ldots, W_k) := \arg\min_{(U_1, \ldots, U_k) \in P_{k,X \setminus Y}} \sum_{i=1}^k f(U_i) \)

and

\( (Z_1, \ldots, Z_k) := \arg\min_{(U_1, \ldots, U_k) \in P_{k,Y \cup U}} \sum_{i=1}^k f(U_i). \)

We have the following inequalities

\[ \sum_{i=1}^k f(W_i) - \min_{U \in P_{k,Y \setminus (X \cup U)}} \sum_{i=1}^k f(U_i) \geq \sum_{i=1}^k \delta(S \subseteq W_i) (f(W_i) - f(W_i \setminus S)) \]

and

\[ \sum_{i=1}^k f(Z_i) - \min_{U \in P_{k,X}} \sum_{i=1}^k f(U_i) \geq \delta(S \subseteq Z_i) (f(Z_i) - f(Z_i \setminus S)), \]

where \(\delta(P) = 1\) if the statement \(P\) is true, and \(\delta(P) = 0\) otherwise. By submodularity of \(f\), we have the following inequalities for any \(i\)

\[ f(Y \cup S) - f(Y) \leq f(Z_i) - f(Z_i \setminus S) \]

and

\[ f(X \setminus Y) - f(X \setminus (Y \cup S)) \leq f(W_i) - f(W_i \setminus U). \]
Inserting these inequalities, we have
\[ d_{m,X}(Y) - d_{m,X}(Y + S) \geq 0, \]
and it implies \( d_{m,X}(Y) \) is monotone decreasing.

Theorem 4 states that the \( k \)th order function \( g_{k,V} \) is submodular at every step above, and thus we can apply Queyranne's algorithm to this function at every recursive step. As minimization of a \( k \)-partition function includes minimization of the \( (k-1) \)-partition function, the number of required times to call the function \( f \) is \( O(n^{3k-1}) \) for this recursive algorithm for the minimal \( k \)-partition of \( n \) elements.

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References

1. Hidaka, S., Oizumi, M.: Fast and exact search for the partition with minimal information loss. PLoS ONE (under review)
2. Okumoto, K., Fukunaga, T., Nagamochi, H.: Divide-and-conquer algorithms for partitioning hypergraphs and submodular systems. Algorithmica 62(3), 787–806 (2012)
3. Queyranne, M.: Minimizing symmetric submodular functions. Mathematical Programming 82(1-2), 3–12 (1998)
4. Studený M & Vejnarová, J.: The multiinformation function as a tool for measuring stochastic dependence. MIT Press, Cambridge, MA (1999)
5. Watanabe, S.: Information theoretical analysis of multivariate correlation. IBM J. Res. Dev. 4(1), 66–82 (1960). DOI 10.1147/rd.41.0066. URL http://dx.doi.org/10.1147/rd.41.0066