A Parameterization of Stabilizing Controllers over Commutative Rings

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1 Introduction

In this paper we are concerned with the factorization approach to control systems, which has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, 1-D as well as n-D systems, etc.[4]. The factorization approach was patterned after Desoer et al.[2] and Vidyasagar et al.[1]. In this approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. A transfer matrix of this approach is considered as the ratio of two stable causal transfer matrices. For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices, which is satisfied by transfer matrices over the principal ideal domains or the Bezout domains.

Sule in [3, 4] has presented a theory of the feedback stabilization of strictly causal plants for multi-input multi-output transfer matrices over commutative rings with some restrictions. This approach to the stabilization theory is called “coordinate-free approach” in the sense that the coprime factorizability of transfer matrices is not required. Recently, Mori and Abe in [5, 6] have generalized his theory over commutative rings under the assumption that the plant is causal. They have introduced the notion of the generalized elementary factor, which is a generalization of the elementary factor introduced by Sule[3], and have given the necessary and sufficient condition of the feedback stabilizability.

Since the stabilizing controllers are not unique in general, the choice of the stabilizing controllers is important for the resulting closed loop. In the classical case, that is, in the case where there exist the right-/left-coprime factorizations of the given plant, the stabilizing controllers can be parameterized by the method called “Youla-Kučera parameterization”[1, 2, 7, 8]. However, it is known that there exist models in which some stabilizable transfer matrices do not have their right-/left-coprime factorizations[9]. In such models, we cannot employ the Youla-Kučera parameterization directly. In this paper we will give a parameterization of the stabilizing controllers over commutative rings by using the results given by Sule[3], and Mori and Abe[5, 6].

Here we briefly outline how the parameterization, which is different from the Youla-Kučera parameterization, will be obtained. Let $H(P, C)$ denote the transfer matrix of
the standard feedback system defined as

\[
H(P, C) = \begin{bmatrix}
(E + PC)^{-1} & -P(E + CP)^{-1} \\
C(E + PC)^{-1} & (E + CP)^{-1}
\end{bmatrix},
\]

where \( P \) and \( C \) are plant and controller, and \( E \) the identity matrix. We consider the set \( \mathcal{H} \) of \( H(P, C) \)'s with all stabilizing controllers \( C \) rather than the set of all stabilizing controllers itself. We will characterize this \( \mathcal{H} \) by one parameter matrix. Then using it, we will obtain the parameterization of the stabilizing controllers.

The paper is organized as follows. After this introduction, we begin on the preliminary in §2, in which we give mathematical preliminaries, set up the feedback stabilization problem and present the previous results. To obtain the set \( \mathcal{H} \) above we will use both of right-/left-coprime factorizations over the ring of fractions of the set of the stable causal transfer functions. In order to establish the existence of such right-/left-coprime factorizations, we will present, in §3 the one-to-one correspondence between the sets of the radicals of the generalized elementary factors of the plant and its transposed plant. Section 4 is the main part of this paper, in which a parameterization of the stabilizing controllers is presented. In §5 we will consider the multidimensional system with structural stability as an example and present the parameterization of its stabilizing controllers. Our method will give a solution of an open problem in [10] about the parameterization of the stabilizing controllers for the multidimensional system with structural stability.

2 Preliminaries

In the following we begin by introducing the notations of commutative rings, matrices, and modules used in this paper. Then we give the formulation of the feedback stabilization problem and the previous results. We also review the construction method of a stabilizing controller presented in [6].

2.1 Notations

Commutative Rings In this paper we consider that any commutative ring has the identity \( 1 \) different from zero. Let \( \mathcal{R} \) denote a (unspecified) commutative ring. The total ring of fractions of \( \mathcal{R} \) is denoted by \( \mathcal{F}(\mathcal{R}) \).

We will consider that the set of stable causal transfer functions is a commutative ring denoted by \( \mathcal{A} \). Further we will use the following three kinds of ring of fractions. The first one appears as the total ring of fractions of \( \mathcal{A} \), which is denoted by \( \mathcal{F}(\mathcal{A}) \) or simply by \( \mathcal{F} \); that is, \( \mathcal{F} = \{ n/d | n, d \in \mathcal{A}, d \text{ is a nonzerodivisor} \} \). This will be considered as the set of all possible transfer functions. The second one is associated with the set of powers of a nonzero element of \( \mathcal{A} \). Let \( f \) denote a nonzero element of \( \mathcal{A} \). Given a set \( S_f = \{ 1, f, f^2, \ldots \} \), which is a multiplicative subset of \( \mathcal{A} \), we denote by \( \mathcal{A}_f \) the ring of fractions of \( \mathcal{A} \) with respect to the multiplicative subset \( S_f \); that is, \( \mathcal{A}_f = \{ n/d | n \in \mathcal{A}, d \in S_f \} \). It should be noted that, in the case where \( f \) is a zerodivisor, even if the equality \( a/1 = b/1 \) with \( a, b \in \mathcal{A} \) holds over \( \mathcal{A}_f \), we cannot say
in general that \( a = b \) over \( \mathcal{A} \); alternatively, \( a = b + z \) over \( \mathcal{A} \) holds with some zerodivisor \( z \) of \( \mathcal{A} \) such that \( zf = 0 \). The last one is the total ring of fractions of \( \mathcal{A}_f \), which is denoted by \( \mathcal{F}(\mathcal{A}_f) \); that is, \( \mathcal{F}(\mathcal{A}_f) = \{ n/d \mid n, d \in \mathcal{A}_f, d \text{ is a nonzerodivisor of } \mathcal{A}_f \} \). If \( f \) is a nonzerodivisor of \( \mathcal{A} \), \( \mathcal{F}(\mathcal{A}_f) \) coincides with the total ring of fractions of \( \mathcal{A} \). Otherwise, they do not coincide. The reader is referred to Chapter 3 of [11] for the ring of fractions.

**Matrices**  The set of matrices over \( \mathcal{R} \) of size \( x \times y \) is denoted by \( \mathcal{R}^{x \times y} \). Further, the set of square matrices over \( \mathcal{R} \) of size \( x \) is denoted by \( (\mathcal{R})_x \). The identity and the zero matrices are denoted by \( E_x \) and \( O_{x \times y} \), respectively, if the sizes are required, otherwise they are denoted by \( E \) and \( O \).

Matrix \( A \) over \( \mathcal{R} \) is said to be nonsingular (singular) over \( \mathcal{R} \) if the determinant of the matrix \( A \) is a nonzerodivisor (a zerodivisor) of \( \mathcal{R} \). Matrices \( A \) and \( B \) over \( \mathcal{R} \) are right- (left-)coprime over \( \mathcal{R} \) if there exist matrices \( X \) and \( Y \) over \( \mathcal{R} \) such that \( XA + YB = E \) \( (AX + BY = E) \) holds. Note that, in the sense of the above definition, two matrices which have no common right- (left-)divisors except invertible matrices may not be right- (left-)coprime over \( \mathcal{R} \). (For example, two matrices \( [z_1] \) and \( [z_2] \) of size \( 1 \times 1 \) over the bivariate polynomial ring \( \mathbb{R}[z_1, z_2] \) over the real field \( \mathbb{R} \) are neither right- nor left-coprime over \( \mathbb{R}[z_1, z_2] \) in our setting.) Further, an ordered pair \( (N, D) \) of matrices \( N \) and \( D \) is said to be a right-coprime factorization over \( \mathcal{R} \) of \( P \) if (i) \( D \) is nonsingular over \( \mathcal{R} \), (ii) \( P = ND^{-1} \) over \( \mathcal{F}(\mathcal{R}) \), and (iii) \( N \) and \( D \) are right-coprime over \( \mathcal{R} \). As the parallel notion, the left-coprime factorization over \( \mathcal{R} \) of \( P \) is defined analogously. That is, an ordered pair \( (\tilde{D}, \tilde{N}) \) of matrices \( \tilde{N} \) and \( \tilde{D} \) is said to be a left-coprime factorization over \( \mathcal{R} \) of \( P \) if (i) \( \tilde{D} \) is nonsingular over \( \mathcal{R} \), (ii) \( P = \tilde{D}^{-1} \tilde{N} \) over \( \mathcal{F}(\mathcal{R}) \), and (iii) \( \tilde{N} \) and \( \tilde{D} \) are left-coprime over \( \mathcal{R} \). Note that the order of the “denominator” and “numerator” matrices is interchanged in the latter case. This is to reinforce the point that if \( (N, D) \) is a right-coprime factorization over \( \mathcal{R} \) of \( P \), then \( P = ND^{-1} \), whereas if \( (\tilde{D}, \tilde{N}) \) is a left-coprime factorization over \( \mathcal{R} \) of \( P \), then \( P = \tilde{D}^{-1} \tilde{N} \) according to [12]. For short, we may omit “over \( \mathcal{R} \)” when \( \mathcal{R} = \mathbb{A} \), and “right” and “left” when the size of matrix is \( 1 \times 1 \).

**Modules**  Let \( M_r(X) \) (\( M_c(X) \)) denote the \( \mathcal{R} \)-module generated by rows (columns) of a matrix \( X \) over \( \mathcal{R} \). Let \( X = AB^{-1} = \tilde{B}^{-1} \tilde{A} \) be a matrix over \( \mathcal{F}(\mathcal{R}) \), where \( A, B, \tilde{A}, \tilde{B} \) are matrices over \( \mathcal{R} \). It is known that \( M_r([A^t \ B^t]^t) \) \( (M_c([\tilde{A} \ \tilde{B}])) \) is unique up to an isomorphism with respect to any choice of fractions \( AB^{-1} \) of \( X \) \( (\tilde{B}^{-1} \tilde{A} \) of \( X \) \) (Lemma 2.1 of [8]). Therefore, for a matrix \( X \) over \( \mathcal{R} \), we denote by \( T_{X, \mathcal{R}} \) and \( W_{X, \mathcal{R}} \) the modules \( M_r([A^t \ B^t]^t) \) and \( M_c([\tilde{A} \ \tilde{B}]) \), respectively.

### 2.2 Feedback Stabilization Problem

The stabilization problem considered in this paper follows that of Sule in [8], and Mori and Abe in [8] [9], who consider the feedback system \( \Sigma \) [12] Ch.5, Figure 5.1] as in Figure [8].
Definition 2.3 In this paper, the definition of the causality from Vidyasagar et al. [1, Definition 3.1]. Throughout the paper, the plant we consider has \( m \) inputs and \( n \) outputs, and its transfer matrix, which is also called a \textit{plant} itself simply, is denoted by \( P \) and belongs to \( \mathcal{F}^{n \times m} \). We can always represent \( P \) in the form of a fraction \( P = ND^{-1} \) (\( P = D^{-1}N \)), where \( N \in \mathcal{A}^{n \times m} \) (\( N \in \mathcal{A}^{n \times m} \)) and \( D \in (\mathcal{A})_m \) (\( D \in (\mathcal{A})_m \)) with nonsingular \( D \) (\( D \)).

**Definition 2.2** For \( P \in \mathcal{F}^{n \times m} \) and \( C \in \mathcal{F}^{m \times n} \), a matrix \( H(P, C) \in (\mathcal{F})_{m+n} \) is defined as

\[
H(P, C) = \begin{bmatrix}
(E_n + PC)^{-1} & -P(E_m + CP)^{-1} \\
C(E_n + PC)^{-1} & (E_m + CP)^{-1}
\end{bmatrix}
\]  

provided that \( \det(E_n + PC) \) is a nonzerodivisor of \( \mathcal{A} \). This \( H(P, C) \) is the transfer matrix from \( [u_1^t \ u_2^t]^t \) to \( [e_1^t \ e_2^t]^t \) of the feedback system \( \Sigma \). If (i) \( \det(E_n + PC) \) is a nonzerodivisor of \( \mathcal{A} \) and (ii) \( H(P, C) \in (\mathcal{A})_{m+n} \), then we say that the plant \( P \) is \( \mathcal{A} \)-stabilizable, \( P \) is stabilized by \( C \), and \( C \) is a stabilizing controller of \( P \).

Since the transfer matrix \( H(P, C) \) of the stable causal feedback system has all entries in \( \mathcal{A} \), we call the above notion \( \mathcal{A} \)-stabilizability. One can further introduce the notion of \( \mathcal{A}_f \)-stabilizability as follows.

**Definition 2.2** Let \( f \) be a nonzero element of \( \mathcal{A} \). If (i) \( \det(E_n + PC) \) is a nonzerodivisor of \( \mathcal{A}_f \) and (ii) \( H(P, C) \in (\mathcal{A}_f)_{m+n} \), then we say that the plant \( P \) is \( \mathcal{A}_f \)-stabilizable, \( P \) is \( \mathcal{A}_f \)-stabilized by \( C \), and \( C \) is an \( \mathcal{A}_f \)-stabilizing controller of \( P \).

The causality of transfer functions is an important physical constraint. We employ, in this paper, the definition of the causality from Vidyasagar et al. [1, Definition 3.1].

**Definition 2.3** Let \( \mathcal{Z} \) be a prime ideal of \( \mathcal{A} \), with \( \mathcal{Z} \neq \mathcal{A} \), including all zerodivisors. Define the subsets \( \mathcal{P} \) and \( \mathcal{P}_s \) of \( \mathcal{F} \) as follows:

\[
\mathcal{P} = \{ n/d \in \mathcal{F} \mid n \in \mathcal{A}, \ d \in \mathcal{A} \setminus \mathcal{Z} \},
\]

\[
\mathcal{P}_s = \{ n/d \in \mathcal{F} \mid n \in \mathcal{Z}, \ d \in \mathcal{A} \setminus \mathcal{Z} \}.
\]

Then every transfer function in \( \mathcal{P} \) (\( \mathcal{P}_s \)) is called \textit{causal} (\textit{strictly causal}). Analogously, if every entry of a transfer matrix \( F \) is in \( \mathcal{P} \) (\( \mathcal{P}_s \)), the transfer matrix \( F \) is called \textit{causal} (\textit{strictly causal}). A matrix over \( \mathcal{A} \) is said to be \( \mathcal{Z} \)-\textit{nonsingular} if the determinant is in \( \mathcal{A} \setminus \mathcal{Z} \), and \( \mathcal{Z} \)-\textit{singular} otherwise.

### 2.3 Previous Results

To state precisely the previous results of the stabilizability, we recall the notion of the generalized elementary factors, which is a generalization of the elementary factor given by Sule [3]. Originally, the elementary factor has been defined over unique factorization domains. Mori and Abe have enlarged this concept for commutative rings [5, 6].
Definition 2.4 (Generalized Elementary Factors, Definition 3.1 of [6]) Let $I$ be the set of all sets of $m$ distinct integers between 1 and $m + n$. Let $I$ be an element of $I$ and $i_1, \ldots, i_m$ be elements of $I$ with ascending order, that is, $a < b$ if $a < b$. We will use elements of $I$ as suffices as well as sets. Denote by $T$ the matrix $[N^t \ D^t]^t$ over $A$ with $P = ND^{-1}$. Let $\Delta_I \in A^{m \times (m + n)}$ denote the matrix whose $(k, i_k)$-entry is 1 for $i_k \in I$ and zero otherwise. For each $I \in I$, an ideal $\Lambda_{PI}$ over $A$ is defined as

$$\Lambda_{PI} = \{ \lambda \in A \mid \exists K \in A^{(m+n)\times m} \lambda T = K \Delta_I T \}.$$ 

We call the ideal $\Lambda_{PI}$ the generalized elementary factor of the plant $P$ with respect to $I$. Further, the set of all $\Lambda_{PI}$'s is denoted by $\mathcal{L}_P$, that is, $\mathcal{L}_P = \{ \Lambda_{PI} \mid I \in I \}$.

It is known that the generalized elementary factor of a plant $P$ is independent of the choice of fractions $ND^{-1} = P$ (Lemma 3.3 of [6]).

The following is the criteria of the feedback stabilizability.

Theorem 2.1 (Theorem 3.2 of [6]) Consider a causal plant $P$. Then the following statements are equivalent:

(i) The plant $P$ is stabilizable.

(ii) $A$-modules $\mathcal{T}_{P,A}$ and $\mathcal{W}_{P,A}$ are projective.

(iii) The set of all generalized elementary factors of $P$ generates $A$; that is, $\mathcal{L}_P$ satisfies:

$$\sum_{\Lambda_{PI} \in \mathcal{L}_P} \Lambda_{PI} = A. \quad (2)$$

In the theorem above, each of (ii) and (iii) is the necessary and sufficient conditions of the feedback stabilizability. Provided that we can check (2) and that we can construct the right-coprime factorizations over $A_{\lambda_I}$ of the given causal plant, we have given a method to construct a causal stabilizing controller of a causal stabilizable plant, which has been originally given in the proof of “(iii)→(i)” of Theorem 3.2 in [6]. We review here the method since it will need later in order to present a parameterization of the stabilizing controllers.

Suppose that (iii) of Theorem 2.1 holds. From (2), there exist $\lambda_I$'s such that $\sum \lambda_I = 1$, where $\lambda_I$ is an element of generalized elementary factor $\Lambda_{PI}$ of the plant $P$ with respect to $I \in \mathcal{I}$; that is, $\lambda_I \in \Lambda_{PI}$. Now let these $\lambda_I$'s be fixed. Further, let $\mathcal{I}^\dagger$ be the set of $I$'s of these $\lambda_I$'s, so that

$$\sum_{I \in \mathcal{I}^\dagger} \lambda_I = 1. \quad (3)$$

We consider without loss of generality that for any $I \in \mathcal{I}^\dagger$, $\lambda_I$ is not a nilpotent element of $A$, since $1 + x$ is a unit of $A$ for any nilpotent $x$ (cf. [11, p.10]). For each $I \in \mathcal{I}^\dagger$, $\mathcal{T}_{P,A_{\lambda_I}}$ is free $A_{\lambda_I}$-module of rank $m$ by Lemma 4.10 of [6]. Hence by Lemma 4.7 of [6], there exist matrices $\tilde{X}_I, \tilde{Y}_I, N_I, D_I$ over $A_{\lambda_I}$ such that the following equality holds over $A_{\lambda_I}$:

$$\tilde{Y}_IN_I + \tilde{X}_ID_I = E_m, \quad (4)$$

5
where \( P = N_1D_I^{-1} \) over \( \mathcal{F}(A_{\lambda_I}) \).

For any positive integer \( \omega \), there are coefficients \( a_I \)'s in \( A \) with \( \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega = 1 \). We let \( \omega \) be a sufficiently large integer. Hence the matrices \( a_I \lambda_I^\omega D_I \tilde{X}_I \) and \( a_I \lambda_I^\omega D_I \tilde{Y}_I \) are over \( A \) for all \( I \in \mathcal{I}^3 \). Then the causal stabilizing controller has the form

\[
C = \left( \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega D_I \tilde{X}_I \right)^{-1} \left( \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega D_I \tilde{Y}_I \right).
\]  

(5)

In the case where \( \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega D_I \tilde{X}_I \) is \( \mathcal{Z} \)-singular, we can select other \( \tilde{Y}_I \)'s and \( \tilde{X}_I \)'s in order that \( \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega D_I \tilde{X}_I \) is \( \mathcal{Z} \)-nonsingular. For detail, see the proof of Theorem 3.2 of [6].

By using \( C \) of (5), the matrix \( H(P, C) \) in (4) is calculated as follows:

\[
\begin{bmatrix}
E_n - \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega N_I \tilde{Y}_I & - \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega N_I \tilde{X}_I \\
\sum_{I \in \mathcal{I}} a_I \lambda_I^\omega D_I \tilde{Y}_I & \sum_{I \in \mathcal{I}} a_I \lambda_I^\omega D_I \tilde{X}_I
\end{bmatrix}.
\]

(6)

Since, \( \omega \) is a sufficient large integer, the matrix above is over \( A \), which implies that the plant \( P \) is stabilized by the constructed \( C \).

Before finishing this section, we present here a parallel result of Lemma 4.10 of [4] for the transposed matrix \( P^t \), which will be used later, without its proof. To present it, we introduce parallel symbols of \( \mathcal{I} \), \( I \), and \( i_1, \ldots, i_m \). Let \( \mathcal{J} \) be the set of all sets of \( n \) distinct integers between 1 and \( m + n \). We will use \( J \) as an element of \( \mathcal{J} \) and \( j_1, \ldots, j_n \) as elements of \( J \) with ascending order.

**Lemma 2.1** Consider the transposed matrix \( P^t \). Let \( J \in \mathcal{J} \). Let \( \Lambda_{P^tJ} \) be the generalized elementary factor of the transposed plant \( P^t \) with respect to \( J \) and further \( \sqrt{\Lambda_{P^tJ}} \) denote the radical of \( \Lambda_{P^tJ} \) (as an ideal). Suppose that \( \lambda_J \) is a non-nilpotent element of \( \sqrt{\Lambda_{P^tJ}} \). Then, \( \Lambda_{\lambda_J} \)-module \( \mathcal{W}_{P^t, \lambda_J} \) is free of rank \( n \).

### 3 Relationship between Generalized Elementary Factors of the Plant and its Transposed Plant

To parameterize (\( A \))-stabilizing controllers, we need to have the capability to obtain all right-/left-coprime factorizations over \( A_{\lambda_I} \), where \( \lambda_I \) is a nonzero element of the generalized elementary factor of the plant \( P \) with respect to \( I \in \mathcal{I} \). By both of Lemmas 4.7 and 4.10 of [4], we already know that there exist the right-coprime factorization over \( A_{\lambda_I} \) of the plant \( P \). However this is not led to the existence of the left-coprime factorization over \( A_{\lambda_I} \) of \( P \). In this section, we will show that the radical of any generalized elementary factor of the plant with respect to \( I \in \mathcal{I} \), denoted by \( \sqrt{\Lambda_{P^tJ}} \), coincides with that of a generalized elementary factor of its transposed transfer matrix with respect to an element \( J \in \mathcal{J} \), denoted by \( \sqrt{\Lambda_{P^tJ}} \); that is, \( \sqrt{\Lambda_{P^tJ}} = \sqrt{\Lambda_{P^tJ}} \) for appropriate \( I \) and \( J \). This fact will be led to the existence of the left-coprime factorizations over \( A_{\lambda_I} \) of \( P \).

Define first a bijective mapping \( \tau \) from \( \mathcal{I} \) to \( \mathcal{J} \). For convenience we denote by \( I_N \) and \( I_d \) the subsets of \( I \) such that

\[
I_N = \{ i \mid i \leq n, i \in I \}, \quad I_d = \{ i \mid i > n, i \in I \}.
\]
Using $I_N$ and $I_d$, we define $J_N$ and $J_d$ as
\[ J_N = [1, m] \setminus \{i - n \mid i \in I_d\}, \quad J_d = \{i + m \mid i \in [1, n] \setminus I_N\}. \]

Then define the bijective mapping $\tau : \mathcal{I} \rightarrow \mathcal{J}$ as
\[ \tau : I_N \cup I_d \mapsto J_N \cup J_d. \]

Using the mapping $\tau$, we obtain a simple result as follows.

**Proposition 3.1** \((\sqrt{\Lambda_{P\tau(I)}})\) is equal to \((\sqrt{\Lambda_{P(I)}})\).

The proof is relatively straightforward and so omitted due to space limitation.

We now summarize the existence of both of right-/left-coprime factorizations over $\mathcal{A}_{\lambda_I}$ of the plant as following.

**Proposition 3.2** There always exist both of right-/left-coprime factorizations over $\mathcal{A}_{\lambda_I}$ of the plant $P$, where $\lambda_I$ is a nonzero element of the generalized elementary factor of the plant with respect to an $I$ in $\mathcal{I}$.

**Proof.** We already know the existence of the right-coprime factorization over $\mathcal{A}_{\lambda_I}$ of $P$. On the other hand, the existence of the left-coprime factorization over $\mathcal{A}_{\lambda_I}$ of $P$ is derived from Proposition 3.1, Lemma 2.1 of this paper and Lemma 4.7 of [6]. \(\square\)

### 4 Parameterization of Stabilizing Controllers

In this section, we give a parameterization of the stabilizing controllers by one matrix.

In order to parameterize the stabilizing controllers, we will first parameterize the transfer matrices $H(P, C)$’s such that $C$ is a stabilizing controller of $P$ and then obtain the parameterization of the stabilizing controllers. We will consider first the case where there exist both of right-/left-coprime factorizations and then the general case.

Throughout this section, we assume that $\mathcal{R}$ denotes $\mathcal{A}$ or $\mathcal{A}_f$ and the given plant $P$ is stabilizable. For convenience we introduce two notations used in this section. Let $\mathcal{H}(P; \mathcal{R})$ denote the set of $H(P, C)$’s such that $C$ is an $\mathcal{R}$-stabilizing controller of $P$ and $\mathcal{S}(P; \mathcal{R})$ the set of all $\mathcal{R}$-stabilizing controllers. Then the set $\mathcal{H}(P; \mathcal{R})$ is expressed as \(\{H(P, C) \mid C \in \mathcal{S}(P; \mathcal{R})\}\). Conversely, once we obtain $\mathcal{H}(P; \mathcal{R})$, it is also easy to obtain $\mathcal{S}(P; \mathcal{R})$ by the following lemma.

**Lemma 4.1** In this lemma, let $H_{11} \in (\mathcal{R})_n, H_{12} \in \mathcal{R}^{n \times m}, H_{21} \in \mathcal{R}^{m \times n}, H_{22} \in (\mathcal{R})_m$ denote submatrices of $H$ in $(\mathcal{R})_{m+n}$ such that
\[ H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}. \]

Then the set of all $\mathcal{R}$-stabilizing controllers $\mathcal{S}(P; \mathcal{R})$ is given as follows:
\[
\mathcal{S}(P; \mathcal{R}) = \{ -[O_{m \times n} \quad E_m] H^{-1} \begin{bmatrix} E_n \\ O_{m \times n} \end{bmatrix} \mid H \in \mathcal{H}(P; \mathcal{R}), \ H \text{ is nonsingular over } \mathcal{R} \}.
\]
\[
= \{H_{22}^{-1} H_{21} \mid H \in \mathcal{H}(P; \mathcal{R}), \ H \text{ is nonsingular over } \mathcal{R}\}.
\]
\[
= \{H_{21} H_{11}^{-1} \mid H \in \mathcal{H}(P; \mathcal{R}), \ H \text{ is nonsingular over } \mathcal{R}\}.
\]
Proof. Since for every \( H(P, C) \in \mathcal{H}(P; \mathcal{R}) \), \( \det(E_n + PC) \) is a nonzerodivisor of \( \mathcal{R} \), every \( H \in \mathcal{H}(P; \mathcal{R}) \) is nonsingular over \( \mathcal{R} \). It is known that the following equality holds:

\[
H(P, C) = \begin{bmatrix} E_n & P \\ C & E_m \end{bmatrix}^{-1}.
\]

Directly from this, we have (7). The remaining relations (8) and (9) are obtained directly from (1).

\[\square\]

4.1 The case of existence of right-/left-coprime factorizations

Throughout this subsection, we assume that the plant \( P \) has both of right-/left-coprime factorizations over \( \mathcal{R} \). It should be noted that from Proposition 3.2, there exist both of right-/left-coprime factorizations over \( A_{\lambda I} \) of \( P \), where \( \lambda I \) is a nonzero element of the generalized elementary factor of \( P \) with respect to \( I \) in \( \mathcal{I} \). Hence \( \mathcal{R} \) can be every such \( A_{\lambda I} \).

Let \((N, D)\) and \((\tilde{D}, \tilde{N})\) be right-/left-coprime factorizations over \( \mathcal{R} \) of \( P \), and \( \tilde{Y}_0, \tilde{X}_0, Y_0, \) and \( X_0 \) be matrices over \( \mathcal{R} \) such that \( \tilde{Y}_0N + \tilde{X}_0D = E_m \) and \( \tilde{N}Y_0 + \tilde{D}X_0 = E_n \). We assume here without loss of generality that

\[
\begin{bmatrix} \tilde{X}_0 & \tilde{Y}_0 \\ N & -D \end{bmatrix} \begin{bmatrix} D & Y_0 \\ N & -X_0 \end{bmatrix} = E_{m+n}.
\]

The following is a parameterization of the stabilizing controllers presented as a Youla-Kučera parameterization.

Theorem 4.1 (cf. Theorems 5.2.1 and 8.3.12 of [12]) All matrices \( X, Y, \tilde{X}, \tilde{Y} \) over \( \mathcal{R} \) satisfying

\[
\tilde{Y}N + \tilde{X}D = E_m, \quad \tilde{N}Y + \tilde{D}X = E_n
\]

are expressed as \( X = X_0 - NS, Y = Y_0 + DS, \tilde{X} = \tilde{X}_0 - R\tilde{N} \) and \( \tilde{Y} = \tilde{Y}_0 + R\tilde{D} \) for \( R \) and \( S \) in \( \mathcal{R}^{m \times n} \).

Further the set of all \( \mathcal{R} \)-stabilizing controllers, denoted by \( S(P; \mathcal{R}) \), is given as

\[
S(P; \mathcal{R}) = \{(\tilde{X}_0 - R\tilde{N})^{-1}(\tilde{Y}_0 + R\tilde{D}) \mid R \in \mathcal{R}^{m \times n}, \tilde{X}_0 - R\tilde{N} \text{ nonsingular over } \mathcal{R}\}
\]

\[
= \{(Y_0 + DS)(X_0 - NS)^{-1} \mid S \in \mathcal{R}^{m \times n}, X_0 - NS \text{ nonsingular over } \mathcal{R}\}.
\]

This proof is still similar with the previous ones such as in [13].

The following lemma without the proof gives the set \( \mathcal{H}(P; \mathcal{R}) \) according to Theorem 4.1.
Lemma 4.2 (cf. Corollary 5.2.7 of [12]) Suppose \( P \in \mathcal{F}(\mathcal{R})^{m \times m} \). Then \( \mathcal{H}(P; \mathcal{R}) \) is given as

\[
\mathcal{H}(P; \mathcal{R}) = \left\{ \begin{bmatrix} E_n - N(\tilde{Y}_0 + R\tilde{D}) & -N(\tilde{X}_0 - R\tilde{N}) \\ D(\tilde{Y}_0 + R\tilde{D}) & D(\tilde{X}_0 - R\tilde{N}) \end{bmatrix} \mid R \in \mathcal{R}^{m \times n}, \tilde{X}_0 - R\tilde{N} \text{ is nonsingular over } \mathcal{R} \right\}; \tag{11}
\]

alternatively

\[
\mathcal{H}(P; \mathcal{R}) = \left\{ \begin{bmatrix} (X_0 - NS)\bar{D} & -(X_0 - NS)\bar{N} \\ (Y_0 + DS)\bar{D} & E_n - (Y_0 + DS)\bar{N} \end{bmatrix} \mid S \in \mathcal{R}^{m \times n}, X_0 - NS \text{ is nonsingular over } \mathcal{R} \right\}; \tag{12}
\]

We now start to construct a new parameterization. Suppose that \( C_0 \) is an \( \mathcal{R} \)-stabilizing controller. We assume without loss of generality that \( \widetilde{X}_0^{-1}\widetilde{Y}_0 \) by Theorem 4.1. Let \( H_0 = H(P, C_0) \in (\mathcal{R})^m \) for short and \( \hat{H}(R) \) the transfer matrix \( H(P, C) \) with \( C = (\widetilde{X}_0 - R\tilde{N})^{-1}(\widetilde{Y}_0 + R\tilde{D}) \). To include the case where \( (\widetilde{X}_0 - R\tilde{N}) \) is singular over \( \mathcal{R} \), we define it as in (11) as follows:

\[
\hat{H}(R) = \begin{bmatrix} E_n - N(\tilde{Y}_0 + R\tilde{D}) & -N(\tilde{X}_0 - R\tilde{N}) \\ D(\tilde{Y}_0 + R\tilde{D}) & D(\tilde{X}_0 - R\tilde{N}) \end{bmatrix},
\]

where \( R \in \mathcal{R}^{m \times n} \). Then the set \( \mathcal{H}(P; \mathcal{R}) \) can be easily expressed by \( \hat{H}(R) \).

Lemma 4.3 The set \( \mathcal{H}(P; \mathcal{R}) \) is expressed using \( \hat{H}(R) \) as follows:

\[
\mathcal{H}(P; \mathcal{R}) = \{ \hat{H}(R) \mid R \in \mathcal{R}^{m \times n}, \hat{H}(R) \text{ is nonsingular over } \mathcal{R} \}.
\]

Proof. Observe that the following matrix equation holds:

\[
\hat{H}(R) = \begin{bmatrix} E_n & O \\ O & D \end{bmatrix} \begin{bmatrix} E_n - N(\tilde{Y}_0 + R\tilde{D}) & -N \\ \tilde{Y}_0 + R\tilde{D} & E_n \end{bmatrix} \times \begin{bmatrix} E_n & O \\ O & \tilde{X}_0 - R\tilde{N} \end{bmatrix}.
\]

The determinants of the matrices in the right hand side of the equation above are \( \det(D), 1, \det(\tilde{X}_0 - R\tilde{N}) \), respectively. Hence the left hand side is nonsingular over \( \mathcal{R} \) if and only if \( \tilde{X}_0 - R\tilde{N} \) is nonsingular over \( \mathcal{R} \). By (11) in Lemma 4.2, this completes the proof. \( \square \)

We may define \( \hat{H}'(S) \) denoting \( H(P, C) \) with \( C = (Y_0 + DS)(X_0 - NS)^{-1} \), that is,

\[
\hat{H}'(S) = \begin{bmatrix} (X_0 - NS)\bar{D} & -(X_0 - NS)\bar{N} \\ (Y_0 + DS)\bar{D} & E_n - (Y_0 + DS)\bar{N} \end{bmatrix}.
\]
where $S \in \mathbb{R}^{m \times n}$ from (12). However, it can be easily check $\hat{H}'(S) = \hat{H}(S)$ from (10). Hence in the following we are concerned only with $\hat{H}(R)$.

We now introduce a new matrix $\Omega(Q) \in (\mathbb{R})_{m+n}$, which plays a key role of new parameterization, as follows:

$$\Omega(Q) = (H_0 - \begin{bmatrix} E_n & O \\ O & O \end{bmatrix})Q(H_0 - \begin{bmatrix} O & O \\ O & E_m \end{bmatrix}) + H_0. \quad (13)$$

where $Q \in (\mathbb{R})_{m+n}$. For convenience define further the partition of $\Omega(Q)$ as follows:

$$\Omega(Q) = \begin{bmatrix} \Omega_{11}(Q) & \Omega_{12}(Q) \\ \Omega_{21}(Q) & \Omega_{22}(Q) \end{bmatrix}$$

with $\Omega_{11}(Q) \in (\mathbb{R})_n$, $\Omega_{12}(Q) \in \mathbb{R}^{n \times m}$, $\Omega_{21}(Q) \in \mathbb{R}^{m \times n}$, $\Omega_{22}(Q) \in (\mathbb{R})_m$.

For the images of the newly introduced matrices, we have the following relations.

**Theorem 4.2** The images of $\Omega(\cdot)$ and $\hat{H}(\cdot)$ are identical; that is

$$\{\Omega(Q) \in (\mathbb{R})_{m+n} | Q \in (\mathbb{R})_{m+n}\} = \{\hat{H}(R) \in (\mathbb{R})_{m+n} | R \in \mathbb{R}^{m \times n}\}. \quad (14)$$

Further we have

$$\mathcal{H}(P; R) = \{\Omega(Q) \in (\mathbb{R})_{m+n} | Q \in (\mathbb{R})_{m+n}, \Omega(Q) \text{ is nonsingular over } \mathbb{R}\}. \quad (15)$$

Here (15) gives a new parameterization of the stabilizing controllers.

**Proof.** We prove only (14) by showing that (i) for any matrix $Q$, there exists a matrix $R$ such that $H(R) = \Omega(Q)$, and that (ii) for any matrix $R$, there exists a matrix $Q$ such that $\hat{H}(R) = \Omega(Q)$. Once (14) is obtained, (15) is obvious by Lemma 4.3.

We first prove (i). We rewrite (13) as follows:

$$\Omega(Q) = \begin{bmatrix} 0 & -N \\ O & D \end{bmatrix} \begin{bmatrix} X_0 & N \\ -Y_0 & D \end{bmatrix}^{-1} Q \begin{bmatrix} -\bar{Y}_0 & \bar{X}_0 \\ D & -\bar{N} \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & O \\ D & -\bar{N} \end{bmatrix} + H_0. \quad (16)$$

Since both of the inverse matrices in (16) are unimodular (cf. Corollary 4.1.67 of [12]), we let

$$Q' = \begin{bmatrix} X_0 & N \\ -Y_0 & D \end{bmatrix}^{-1} Q \begin{bmatrix} -\bar{Y}_0 & \bar{X}_0 \\ D & -\bar{N} \end{bmatrix}^{-1}.$$

Then (16) can be rewritten further as follows:

$$\Omega(Q) = \begin{bmatrix} 0 & -N \\ O & D \end{bmatrix} Q' \begin{bmatrix} 0 & O \\ D & -\bar{N} \end{bmatrix} + H_0. \quad (17)$$

Partition $Q'$ as

$$\begin{bmatrix} Q'_{11} & Q'_{12} \\ Q'_{21} & Q'_{22} \end{bmatrix} = Q'.$$
where \( Q'_{13} \in \mathcal{R}^{n \times m}, Q'_{12} \in (\mathcal{R})_n, Q'_{21} \in (\mathcal{R})_m, Q'_{22} \in \mathcal{R}^{m \times n} \). Then (17) can be rewritten again as follows:

\[
\Omega(Q) = \begin{bmatrix}
E_n - N(\tilde{Y}_0 + Q'_{22} \tilde{D}) & -N(\tilde{X}_0 - Q'_{22} \tilde{N}) \\
D(\tilde{Y}_0 + Q'_{22} \tilde{D}) & D(\tilde{X}_0 - Q'_{22} \tilde{N})
\end{bmatrix},
\]

which is equal to \( \hat{H}(Q'_{22}) \). Therefore the matrix \( Q'_{22} \) is the matrix \( R \) satisfying \( \hat{H}(R) = \Omega(Q) \).

Next we prove (ii). From the proof of (i), letting \( Q \) as

\[
Q = \begin{bmatrix}
X_0 & N \\
-Y_0 & D
\end{bmatrix} \times \begin{bmatrix}
\times \\
\times
\end{bmatrix} \begin{bmatrix}
-\tilde{Y}_0 & \tilde{X}_0 \\
\tilde{D} & \tilde{N}
\end{bmatrix},
\]

where \( \times \)’s denote arbitrary matrices, we obtain directly \( \hat{H}(R) = \Omega(Q) \). \( \square \)

### 4.2 The General Case

In this subsection, we parameterize all stabilizing controllers over \( A \) even in the case where there does not exist right-/left-coprime factorizations over \( A \).

Let \( \lambda_I \) denote an arbitrary but fixed element of the generalized elementary factor of \( P \) with respect to \( I \in I^d \) satisfying (3). As mentioned in § 4.1, there exist both of right-/left-coprime factorizations over \( A_{\lambda_I} \) of the plant \( P \). We let \((N_I, D_I)\) and \((\tilde{D}_I, \tilde{N}_I)\) denote right-/left-coprime factorizations over \( A_{\lambda_I} \) of \( P \). Suppose again that \( C_0 \) is a stabilizing controller. By Theorem 4.1, there exist right-/left-coprime factorizations \((\tilde{Y}_{0I}, X_{0I})\) and \((\tilde{X}_{0I}, \tilde{Y}_{0I})\) over \( A_{\lambda_I} \) of \( C_0 \) such that \( \tilde{Y}_{0I} N_I + \tilde{X}_{0I} D_I = E_m \)

and \( \tilde{N}_I Y_{0I} + \tilde{D}_I X_{0I} = E_n \). In order to distinguish \( \hat{H} \) for each \( I \in I^d \), we introduce \( \hat{H}_I \) analogously to § 4.1 as follows:

\[
\hat{H}_I(R_I) = \begin{bmatrix}
E_n - N_I (\tilde{Y}_{0I} + R_I \tilde{D}_I) & -N_I (\tilde{X}_{0I} - R_I \tilde{N}_I) \\
D_I (\tilde{Y}_{0I} + R_I \tilde{D}_I) & D_I (\tilde{X}_{0I} - R_I \tilde{N}_I)
\end{bmatrix},
\]

where \( R_I \in A_{\lambda_I}^{n \times n} \). On the other hand, the matrices \( \Omega(Q) \) and \( \Omega_{i,j}(Q) \)’s with \( i, j = 1, 2 \) are still used.

In the following we show that (15) in Theorem 4.3 still holds even in the case where there do not exist right-/left-coprime factorizations over \( A \).

**Theorem 4.3** The following equality holds:

\[
\mathcal{H}(P; A) = \{ \Omega(Q) \in (A)_{m+n} | Q \in (A)_{m+n}, \Omega(Q) \text{ is nonsingular} \}.
\]

**Proof.** In order to prove this theorem, it is sufficient to show the following:

(i) For any matrix \( Q \), if \( \Omega(Q) \) is nonsingular, there exists a stabilizing controller \( C \) such that \( H(P, C) = \Omega(Q) \).

(ii) Conversely, for any stabilizing controller \( C \), there exists a matrix \( Q \) such that \( H(P, C) = \Omega(Q) \).
We first prove (i). Suppose that $\Omega(Q)$ is nonsingular. Assume without loss of generality that $\lambda_{I_0}$ is a nonzerodivisor with $I_0 \in I^\sharp$. Then by Theorem 4.2, there exists a matrix $R_{I_0}$ over $A_{\lambda I_0}$ such that $\hat{H}_{I_0}(R_{I_0}) = \Omega(Q)$. By Lemma 4.3, there exists an $A_{\lambda I_0}$-stabilizing controller $C$. Observe now that $F(A_{\lambda I_0}) = F$, so that $C$ is over $F$. Since $H(P, C) = \Omega(Q)$, $H(P, C)$ is over $A$, which implies that $C$ is a stabilizing controller of $P$.

Next we prove (ii). Suppose that $P$ is stabilizable. As in §2.3, let $a_I'$s be in $A$ for $I \in I^\sharp$ such that $\sum_{I \in I^\sharp} a_I \lambda_I^\omega = 1$ with a sufficiently large integer $\omega$. Let $C$ be an arbitrary but fixed stabilizing controller of $P$. Since $C$ is also an $A_{\lambda I_0}$-stabilizing controller, by Lemma 4.3 there exists $R_I$ over $A_{\lambda I_0}$ such that $H(P, C) = \hat{H}_I(R_I)$ for each $I \in I^\sharp$. By Theorem 4.2 there exists a matrix $Q_I$ over $A_{\lambda I_0}$ such that $\hat{H}_I(R_I) = \Omega(Q_I)$ for each $I \in I^\sharp$, so that $H(P, C) = \Omega(Q_I)$ over $A_{\lambda I_0}$. Hence we have

$$H(P, C) = \sum_{I \in I^\sharp} a_I \lambda_I^\omega \Omega(Q_I)$$

over $A$. From (13), the equation above can be rewritten as follows:

$$H(P, C) = \sum_{I \in I^\sharp} a_I \lambda_I^\omega (H_0 - \begin{bmatrix} E_n & O \\ O & O \end{bmatrix}) Q_I (H_0 - \begin{bmatrix} O & O \\ O & E_m \end{bmatrix}) + H_0$$

Letting $Q = \sum_{I \in I^\sharp} a_I \lambda_I^\omega Q_I$, we have proved (ii).

We now have a parameterization of the stabilizing controllers over $A$ by virtue of Lemma 4.4.

**Corollary 4.1** The set of all stabilizing controllers $S(P; A)$ is given as follows:

$$S(P; A) = \{ - \begin{bmatrix} O_{m \times n} & E_m \end{bmatrix} \Omega(Q)^{-1} \begin{bmatrix} E_n \\ O_{m \times n} \end{bmatrix} | Q \in (A)_{m+n}, \ \Omega(Q) \text{ is nonsingular} \}$$

$$= \{ \Omega_{22}(Q)^{-1} \Omega_{21}(Q) | Q \in (A)_{m+n}, \ \Omega(Q) \text{ is nonsingular} \}$$

$$= \{ \Omega_{21}(Q)\Omega_{11}(Q)^{-1} | Q \in (A)_{m+n}, \ \Omega(Q) \text{ is nonsingular} \}.$$
5 Examples of Parameterization

Let us consider here the multidimensional systems with structural stability. Recall that the construction methods of a stabilizing controller have already been presented by Sule and Lin. Thus, as an example of the parameterization, using Corollary, we can obtain the parameterization of the stabilizing controllers. This gives one of the solutions of an open problem about the parameterization of the stabilizing controllers of the multidimensional systems given by Lin.

The reader can further refer to for other examples of parameterizations of the stabilizing controllers.

6 Concluding Remarks and Further Works

In this paper we have given a parameterization of the stabilizing controllers over commutative rings. In this paper the minimal number of parameters required to give the parameterization is not clarified. Since it is important for the implementation of the parameterization, it should be clarified as a further problem.

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