REMOVAL LEMMA FOR INFINITELY-MANY FORBIDDEN HYPERGRAPHS
AND PROPERTY TESTING

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Abstract. We prove a removal lemma for infinitely-many forbidden hypergraphs. It affirmatively
settles a question on property testing raised by Alon and Shapira (2005) [2, 3]. All monotone
hypergraph properties and all hereditary partite hypergraph properties are testable. Our proof
constructs a constant-time probabilistic algorithm to edit a small number of edges. It also gives a
quantitative bound in terms of a coloring number of the property. It is based on a new hypergraph
regularity lemma [14].

1. Introduction

The research of removal lemmas has started by Ruzsa and Szemerédi [28], who considered an
ordinary graph case. Frankl and Rödl [9] suggested that if a hypergraph version of the removal
lemma can be proven, it yields Szemerédi’s famous theorem on arithmetic progressions [32]. They
actually showed an alternative proof of the theorem for length four [31] by showing a 3-uniform
hypergraph regularity lemma. Later Solyomos [29, 30] showed that the k-uniform hypergraph removal
lemma (conjecture) implies not only Szemerédi’s theorem but also its multidimensional extension
by Furstenberg and Katznelson [10], which had been proved only by ergodic theory until recently.
Finally Gowers [13] and Rödl and his collaborators [25, 21] obtained the hypergraph removal lemma
as a corollary of their regularity lemmas. Slightly later, Tao [34] gave another regularity lemma, which
yields the hypergraph removal lemma. Very recently [14] gave a new regularity lemma with a clear
construction and a simple proof, which we will use in this paper.

Tao [36] gave another proof of hypergraph removal lemma by using ergodic theoretic ideas. It is
nonconstructive but is independent from any regularity lemma.

These hypergraph removal lemmas deal with one forbidden hypergraph. It is straightforward to
rewrite them for a finite family of forbidden hypergraphs. For details of hypergraph removal lemmas,
see [27, §11.6,p.454-463].

In the below, a partite hypergraph is h-vertex if and only if each partite set contains exactly h
vertices. The main purpose of this paper is to show the following.

Theorem 1.1 (Removal lemma for infinitely-many forbidden r-partite hypergraphs). Let r ≥ k be
positive integers and ε > 0. Let \( F = \bigcup_{h=1}^{\infty} F_h \) where \( F_h \) is a family of h-vertex k-uniform (r-partite
hyper)graphs. Then there exist constants \( c > 0 \) and \( h_0 \) such that for any integer \( N \) if \( G \) is an
N-vertex k-uniform (r-partite hyper)graph then at least one of the following two holds:

(i) One can modify at most \( \varepsilon(N^k) \) edges so that the hypergraph obtained from \( G \) does not have a
copy of any member of \( F \) as an induced-subhypergraph.

(ii) There exist \( h \leq h_0, F \in F_h \) such that \( G \) contains at least \( cN^r \) copies of \( F \).

Our main theorem, Theorem 3.2, will be presented in a more general frame. For example, (1)
each forbidden hypergraph \( F \) may contain not only black and white edges but also ‘invisible’ edges,
(so it contains the both cases of induced and non-induced subgraphs) (2) \( F \) and \( G \) may have a
constant number of colors other than black and white, and (3) partite vertex sets of \( G \) will be discrete
probability spaces with finite vertices, where the sizes of two partite vertex sets may not be equal.
Those are not insignificant extensions. We employ them to make our argument natural.

2. Testability

Property testing was firstly considered by Blum et al. [5], and the general notion of property test-
ing was first given by Rubinfeld and Sudan [27]. Goldreich et al. [11] firstly investigated it in
combinatorial context, in which they considered ordinary graphs. See surveys [26, 27, 2].
Definition 2.1. [Hypergraph property] Two hypergraphs are isomorphic if and only if one can be equivalent to the other by some bijection (permutation) between the two vertex sets. (For the case of $r$-partite hypergraphs, the bijection should be ‘partitionwise’, i.e. any vertex in any partite set cannot be replaced to a different partite set, and furthermore partite sets have their own labels from $1, \cdots, r$, any of which we cannot change.) A hypergraph property (or property, simply) is a class of hypergraphs such that if a hypergraph belongs to the class (satisfies the property) then any other hypergraphs isomorphic to the hypergraph belong to it.

A hypergraph property is monotone if and only if when a hypergraph satisfies the property, any (induced/non-induced) subgraph of it satisfies the property. A hypergraph property is hereditary if and only if when a hypergraph satisfies the property, any induced subgraph of it satisfies the property. In other words, a monotone (or hereditary) property is closed under any deletion of vertices and edges (or vertices, respectively). Clearly any monotone property is hereditary.

Definition 2.2. [$\epsilon$-far] A $k$-uniform hypergraph is $\epsilon$-far from a property $P$ if and only if the hypergraph cannot satisfy $P$ even after modifying at most $\epsilon$ portion of edges of the underlying complete hypergraph. (i.e. it is $\epsilon \binom{N}{k}$ edges for $N$-vertex hypergraphs (with no vertex partitions) and is $\epsilon \binom{k}{r} N^k$ edges for $r$-partite hypergraphs with $N$ vertices in each partite set.)

Definition 2.3. [Property test] A property is testable if and only if there exists a randomized algorithm such that, for any $\epsilon > 0$ and any object (a hypergraph) given as inputs, if
(1) the input object satisfies the property or
(2) it is $\epsilon$-far from the property
then with probability at least 0.9 the algorithm correctly answers which case of the two it is, in a constant time independent from the size of the object (the number of vertices in the hypergraph).
(The time can depend on $\epsilon$.) A testable property is testable with one-sided error if and only if its answer is correct always (with probability 1) whenever the input satisfies the property (i.e. the case (1)).

Theorem 2.1. Every hereditary property of constant-partite hypergraphs is testable with one-sided error.

Proof. [Design of the algorithm] Firstly we will present a random algorithm for hereditary property $P$ with one-sided error. Fix $r \geq k$ and $\epsilon > 0$. Let $\mathcal{F}_h$ be the set of all $h$-vertex $r$-partite $k$-uniform hypergraphs which do not satisfy $P$. Let $\mathcal{F} = \bigcup_{h \geq 1} \mathcal{F}_h$. With these parameters, Theorem 1.1 gives us constants $c > 0$ and $h_0$. Our algorithm goes as follows. Given the input hypergraph $G$, the algorithm randomly chooses vertices $W(i) = \bigcup_{j \in [r]} \bigcup_{j' \in \Omega_j} [v_{j,j'}^i \in [h_0]]$ for $\lceil \frac{3}{c} \rceil$ times ($i \in \lceil \frac{3}{c} \rceil$), where $\Omega_j$ denotes the $j$-th partite vertex-set of $G$. Then it declares $G$ to satisfy $P$ if-and-only-if, for all $i$, the $h_0$-vertex hypergraph $H(i)$ induced by $W(i)$ satisfies $P$ (i.e. it is not isomorphic to any member of $\mathcal{F}_{h_0}$).

[Verification of the algorithm] Suppose that the input $G$ satisfies $P$. Since $P$ is hereditary, all induced sub(hyper)graphs of $G$ (thus also all $H(i)$) satisfy $P$. Thus the algorithm declares correctly with probability one.

Assume that $G$ is $\epsilon$-far from $P$. Theorem 1.1 says that there exists an $F \in \mathcal{F}_h$ with an $h \leq h_0$ such that $G$ contains at least $c N^{rh}$ copies of $F$. Let $F^+$ be the $h_0$-vertex hypergraph obtained from $F$ by adding some isolated vertices. Since all added vertices are isolated, $G$ contains at least $c N^{rh_0}$ copies of $F^+$. Consequently for any fixed $i$, the probability that $H(i)$ is isomorphic to $F^+$ is at least $c$.

If $F^+ \notin \mathcal{F}_{h_0}$ then $F^+$ satisfies $P$ and then its induced-subgraph $F$ also satisfies $P$, contradicting $F \in \mathcal{F}_h$. Thus $F^+ \notin \mathcal{F}_{h_0}$ and $F^+$ does not satisfy $P$. If some $H(i)$ is isomorphic to $F^+$ then $H(i)$ does not satisfy $P$ and the algorithm must say that $G$ does not satisfy $P$. Thus the probability that the algorithm outputs the wrong answer is

$$\mathbb{P} \left[ \text{The algorithm incorrectly says that } G \text{ is in } P \right] \leq \mathbb{P} \left[ H(i) \text{ is not isomorphic to } F^+ \text{ for all } i \right] \leq (1-c)^{3/c} \leq e^{-3} < 0.1.$$
which edges should be modified in the given hypergraph. A quantitative bound for the number of
and does not give any quantitative bound. On the other hand, our proof gives a procedure about
by extending the proof of [3, 4] naturally under their regularity lemma. Their proof is not constructive
may be practically impossible or hard to show the testability even for monotone hypergraph properties
(non-constructive) idea of graph limits from [19, 20], without extending the approaches of [3, 4]. It
proved the above independently from me. Their method even yields that every hereditary non-
property testing. It had been known that they are strong enough for Szemerédi theorem on arithmetic
progressions ([32]) and its variants. Avart et al. [4] showed it for 3-uniform hypergraphs, by developing
their argument with the 3-uniform hypergraph regularity lemma of [9]. We will answer their question
as follows, by using a new hypergraph regularity platform [14].

Corollary 2.2. Every monotone property of hypergraphs with no vertex partitions is testable with
one-sided error.

Proof. It easily follows from Theorem 2.1 Choose an $r$ to be a constant large enough with respect to
$1/\epsilon$ and $k$. We modify the input non-partite hypergraph to be an $r$-partite hypergraph by decomposing
the vertex set to $r$ disjoint vertex partite-sets and by deleting (visualizing) ‘non-partitionwise’ edges
(i.e. deleting any edge with at least two vertices being in a common vertex partite-set). Note that
there are at most $r \cdot r^{k-2} N^k < 0.1 \left(\binom{r}{2}\right) N^k$ such deleted edges. It is reduced to Theorem 2.1.

In the course of writing the first draft of this paper [10], I learned that Rödl and Schacht [22, 23]
proved the above independently from me. Their method even yields that every hereditary non-
partite hypergraph properties are testable with one-sided error. In this sense, their result is stronger.
However the approaches are significantly different. They combined their regularity lemma with the
(non-constructive) idea of graph limits from [19, 20], without extending the approaches of [3, 4]. It
may be practically impossible or hard to show the testability even for monotone hypergraph properties
by extending the proof of [3, 4] naturally under their regularity lemma. Their proof is not constructive
and does not give any quantitative bound. On the other hand, our proof gives a procedure about
which edges should be modified in the given hypergraph. A quantitative bound for the number of
degrees to modify can be calculated in terms of a coloring number of the property, though the bound
seems to be weak. Improving the bound would be an interesting research theme. Their proof is based
on their heavy regularity lemma, while ours is based on a new regularity lemma [14], which has a
shorter proof.

3. Statement of the Main Theorem

In this paper, we denote by $\mathbb{P}$ and $\mathbb{E}$ the probability and expectation, respectively. We denote the
conditional probability and expection by $\mathbb{P}[\cdots | \cdots]$ and $\mathbb{E}[\cdots | \cdots]$.

Setup 3.1. Throughout this paper, we fix a positive integer $r$ and an ‘index’ set $\mathbf{r}$ with $|\mathbf{r}| = r$. Also
we fix a probability space $(\Omega_i, \mathcal{B}_i, \mathbb{P})$ for each $i \in \mathbf{r}$. Assume that $\Omega_i$ is finite (but its cardinality
may not be constant) and $\mathcal{B}_i := 2^{\Omega_i}$ for the sake of simplicity. Write $\Omega := (\Omega_i)_{i \in \mathbf{r}}$.

In order to avoid using measure-theoretic jargons like measurability or Fubini’s theorem frequently
to readers who are interested only in applications to discrete mathematics, we assume $\Omega_i$ as a (non-
empty) finite set. However our argument should be extendable to a more general probability space.
For applications, $\Omega_i$ would contain a huge number of vertices, though we do not use the assumption
in our proof.

For an integer $a$, we write $[a] := \{1, 2, \cdots, a\}$, and $\left(\begin{at}{a}, \end{at}\right) := \bigcup_{i \in [a]} \left(\begin{at}{i}\right)$. When
$r$ sets $X_i, i \in \mathbf{r}$, with indices from $\mathbf{r}$ are called vertex sets, we write $X_J := \{e \in \bigcup_{i \in J} X_i | e \cap X_j = \emptyset \forall j \in J\}$ whenever $J \subset \mathbf{r}$.

Definition 3.1. [Colored hyper]graphs] Suppose Setting 3.1 A $k$-bound $(b_i)_{i \in [k]}$-colored ($r$-
partite hyper) graph $H$ is a triple $((X_i)_{i \in \mathbf{r}}, (C_i)_{i \in \left(\begin{at}{r}\right)}, (\gamma_I)_{I \in \left(\begin{at}{r}\right)})$ where
(1) each $X_i$ is a set called a vertex set, (2) $C_i$ is a set with at most $b_i$ elements, and (3) $\gamma_I$ is a map from $X_I$ to $C_I$. We
write $V(H) := \bigcup_{i \in \left(\begin{at}{r}\right)} X_i$ and $C(H) := C_I$. Each element of $V(H)$ is called a vertex. Each element
$e \in V(H) = X_I, I \in \left(\begin{at}{r}\right)$, is called an (index-$I$) edge. Each member in $C_I(H)$ is a (face-)color
(of index $I$). Write $H(e) = \gamma_I(e)$ for each $I$.

Let $I \in \left(\begin{at}{r}\right)$ and $e \in V(H)$. For another index $\emptyset \neq J \subset I$, we denote by $e|J$ the index-$J$ edge
$e \setminus \left(\bigcup_{j \in I \setminus J} X_j\right) \subset V(H)$. We define the frame-color and total-color of $e$ by
$H(\partial e) := (H(e|J)|\emptyset \neq J \subset I)$ and by $H(e) := H(e|J)|\emptyset \neq J \subset I)$. Write $TC_I(H) := \{H(e)|e \in X_I\}$ and
$\partial C_I(H) := \{H(\partial e)|e \in X_I\}$.
A \((k\text{-bound})\) simplicial-complex is a \(k\text{-bound}\) (colored \(r\text{-partite hyper})\) graph such that for each \(I \in \binom{I}{r}\) there exists at most one index-\(I\) color called 'invisible' and that if (the color of) an edge \(e\) is invisible then any edge \(e^* \supset e\) is invisible. An edge or its color is visible if it is not invisible.

For a \(k\text{-bound}\) graph \(G\) on \(\Omega\) and \(s \leq k\), let \(S_{s,h,G}\) be the set of \(s\text{-bound} simplicial-complexes \(S\) such that (1) each of \(r\) vertex sets contains exactly \(h\) vertices and that (2) for any \(I \in \binom{I}{r}\) there is an injection from the index-\(I\) visible colors of \(S\) to the index-\(I\) colors of \(G\). (When a visible color \(c\) of \(S\) corresponds to another color \(c'\) of \(G\), we simply write \(c = c'\) without presenting the injection explicitly.)

For \(S \in S_{s,h,G}\), we denote by \(V_i(S)\) the set of index-\(I\) visible edges. Write \(V_i(S) := \bigcup_{I \subseteq (i)} V_i(S)\) and \(V(S) := \bigcup_i V_i(S)\).

Informally speaking, our aim will be to embed a ‘child graph’ \(S\) to a ‘mother graph’ \(G\) on vertex set \(\Omega\). We will use bold fonts for vertices and edges of the mother graph.

**Definition 3.2.** [Partitionwise maps] A partitionwise map \(\varphi\) is a map from \(r\) vertex sets \(W_i, i \in \mathbb{R}\), with \(|W_i| < \infty\) to the \(r\) vertex sets (probability spaces) \(\Omega_i, i \in \mathbb{R}\), such that each \(w \in W_i\) is mapped into \(\Omega_i\). We denote by \(\Phi((W_i)_{i \in \mathbb{R}})\) or \(\Phi(\bigcup_{i \in \mathbb{R}} W_i)\) the set of partitionwise maps from \((W_i)_{i \in \mathbb{R}}\). When \(W_i = [h]\) or \(W_i = h\), we denote it by \(\Phi(h)\). A partitionwise map is random if and only if each vertex \(w \in W_i\) is mutually-independently mapped at random according to the probability space \(\Omega_i\).

Define \(\Phi(m_1, \ldots, m_{k-1}) := \Phi(m_1) \times \cdots \times \Phi(m_{k-1})\).

**Definition 3.3.** [\(k\text{-uniform graphs}\)] A \(k\text{-uniform} b_k\text{-colored} (r\text{-partite hyper})\) graph is a \(k\text{-bound}\) \((b_i)_{i \in [k]}\text{-colored graph such that}

1. if \(i < k\) then \(b_i = 1\) and the unique color is called invisible
2. for each \(I\) with \(|I| = k\), there is at most one index-\(I\) color which is called invisible. (Note that this word 'invisible' is slightly different from the same word used in the definition of simplicial-complexes.)

Denote by \(\mathbb{V}(F)\) the set of visible edges of a \(k\text{-uniform graph} F\), where a visible edge means an edge whose color is not invisible. It is called \(h\text{-vertex}\) if each partite set contains exactly \(h\) vertices.

**Theorem 3.2** (Main Theorem). Let \(r \geq k\) and \(b = (b_i)_{i \in [k]}\) be positive integers and \(\varepsilon > 0\). Let \(\mathcal{F} = \bigcup_{h=1}^{\infty} \mathcal{F}_h\) where \(\mathcal{F}_h\) is a family of \(h\text{-vertex} k\text{-uniform} b_k\text{-colored} (r\text{-partite hyper})\) graphs. Then there exist constants \(\varepsilon > 0\) and \(h_0\) with the following.

Let \(G\) be a \(k\text{-bound} b_k\text{-colored} (r\text{-partite hyper})\) graph on \(\Omega = (\Omega_i)_{i \in \mathbb{R}}\). Then at least one of the following two holds.

(i) There exists a \(k\text{-bound} b_k\text{-colored} (r\text{-partite hyper})\) graph \(G'\) on \(\Omega\) such that

\[\Pr_{e \in \Omega_r}[G'(e) \neq G(e)] < \varepsilon \text{ } \forall I \in \binom{r}{k}\]

and that for all \(h, F \in \mathcal{F}_h\),

\[\Pr_{\varphi \in \Phi(h)}[G'(\varphi(e)) = F(e) \forall e \in \mathbb{V}(F)] = 0.\]

(ii) There exist \(h \leq h_0, F \in \mathcal{F}_h\) such that

\[\Pr_{\varphi \in \Phi(h)}[G(\varphi(e)) = F(e) \forall e \in \mathbb{V}(F)] \geq \varepsilon.\]

Our proof is constructive. In hypergraph regularity setup of [14], we will develop the argument which Alon and Shapira [12] used for graphs.

**4. Definitions of Regularities and Statement of Regularity Lemma**

**Definition 4.1.** [Regulation] Let \(m \geq 0\) and \(\varphi \in \Phi(m)\). Let \(G\) be a \(k\text{-bound} graph on \(\Omega\). For an integer \(1 \leq s < k\), the \(s\text{-regularization} G/\varphi^s\) is the \(k\text{-bound} graph on \(\Omega\) obtained from \(G\) by redefining the color of each edge \(e \in \Omega_I\) with \(I \in \binom{I}{r}\) by the vector

\[(G/\varphi^s)(e) := (G(e \circ f))/J \in \binom{r}{s} \setminus I, f \in \Omega_J\text{ with } f \subset \varphi(\Omega).\]

In the above, when \(J = \emptyset\), we assume \(f = \emptyset\). (The sets of colors are naturally extended while any edge containing at least \(s + 1\) vertices does not change its (face-)color.)

When \(s = k - 1\), we simply write \(G/\varphi = G/k\varphi\).

For \(\varphi = (\varphi_i)_{i \in [k-1]} \in \Phi(m_1, \ldots, m_{k-1})\), we define the regularization of \(G\) by \(\varphi\) by

\[G/\varphi := ((G/k\varphi_{k-1})/k^{2}\varphi_{k-2})\cdots/1\varphi_1.\]
Definition 4.2. [Regularity] Let $G$ be a $k$-bound graph on $\Omega$. For $\reg = (\reg_j)_{j \leq I} \in \text{TC}_I(G)$, $I \in \binom{[k]}{r}$, we define relative density

$$d_G(\reg) = d_G(\reg_j | (\reg_j)_{j \leq I}) := \mathbb{P}_{e \in \Omega_j}[G(e) = \reg | G(\partial e) = (\reg_j)_{j \leq I}].$$

For a nonnegative integer $h$ and $\epsilon \geq 0$, we say that $G$ is $(\epsilon, k, h)$-regular (or $(\epsilon, h)$-regular) if and only if there exists a function $\delta : \text{TC}(G) \rightarrow [0, \infty)$ such that

(i) $\mathbb{P}_{e \in \Phi(h)}[G(\partial e) = S(e) \forall e \in \mathcal{V}(S)] = \prod_{e \in \mathcal{V}(S)} (d_G(S(e)) + \delta(S(e))) \quad \forall S \in \mathcal{S}_{k,h,G},$

(ii) $\mathbb{E}_{e \in \mathcal{S}_{I}}[\delta(G(e))] \leq \epsilon / |C_I(G)| \quad \forall I \in \binom{[k]}{r},$

where $\pm b$ means (the interval of) numbers $c$ with $\max\{0, a - b\} \leq c \leq \min\{1, a + b\}$. Denote by $\text{reg}_{k,h}(G)$ the minimum value of $\epsilon$ such that $G$ is $(\epsilon, k, h)$-regular.

For nonnegative integers $h$, $L$ and $\epsilon \geq 0$, we say that $G$ is $(\epsilon, k, h, L)$-regular (or $(\epsilon, h, L)$-regular) if and only if $G$ is $(\epsilon, k, h)$-regular and the following holds for all $I \in \binom{[k]}{r}$:

$$\mathbb{E}_{\varphi \in \Phi(L)}[d_{G/\varphi}(\reg | G/\varphi(\partial e^*) - d_G(\reg | G(\partial e^*))]^2 \leq \left(\frac{\epsilon}{|C_I(G)|}\right)^2,$$

where we naturally write

$$d_{G/\varphi}(\reg | G/\varphi(\partial e^*)) := \mathbb{P}_{e \in \mathcal{S}_{I}}[G(e) = \reg | G(\partial e) = G/\varphi(\partial e^*)].$$

Denote by $\text{reg}_{k,h,L}(G)$ the minimum value of $\epsilon$ such that $G$ is $(\epsilon, k, h, L)$-regular.

We will use the following new hypergraph regularity lemma [14], which yields a shortest proof of Szemerédi’s theorem on arithmetic progressions.

**Theorem 4.4A (Regularity Lemma [14]).** For any $r \geq k, h, \tilde{b} = (b_i)_{i \in [k]}, \epsilon > 0$, there exist integers $\tilde{m}_1, \ldots, \tilde{m}_{k-1}$ such that if $G$ is a $\tilde{b}$-colored $(k$-bound $r$-partite hypergraph) on $\Omega$ then for some integers $m_1, \ldots, m_{k-1}$ with $m_i \leq \tilde{m}_i, i \in [k - 1],$

$$\mathbb{E}_{\varphi \in \Phi(m_1, \ldots, m_{k-1})}[\text{reg}_{k,h,L}(G/\varphi)] \leq \epsilon.$$

The proof of the above in [14] essentially tells us the following.

**Theorem 4.4B (Strong Form of Regularity Lemma [14]).** For any $r \geq k, h, \tilde{b} = (b_i)_{i \in [k]}, \epsilon > 0$, and for any function $L : \mathbb{N}^{k-1} \rightarrow \mathbb{N}$, there exist integers $\tilde{m}_1, \ldots, \tilde{m}_{k-1}$ such that if $G$ is a $\tilde{b}$-colored $(k$-bound $r$-partite hypergraph) on $\Omega$ then for some integers $m_1, \ldots, m_{k-1}$ with $m_i \leq \tilde{m}_i, i \in [k - 1],$

$$\mathbb{E}_{\varphi \in \Phi(m_1, \ldots, m_{k-1})}[\text{reg}_{k,h,L(m_1, \ldots, m_{k-1})}(G/\varphi)] \leq \epsilon.$$

**Theorem 4.4A** is also used in [15] to show the hypergraph extension of the graph theorem by [6]. That is, the Ramsey number is linear (with respect to the order) for every bounded-degree hypergraph, which is also shown independently in Cooley et al. [7] by a different way.

As I wrote in a final part of [14], Property Testing and Regularization are essentially the same. They are all about random samplings, especially when considering constant-size (induced)subgraphs. If there exists a difference between the two, it is whether the number of random vertex samplings is $(PT)$ a fixed constant or $(R)$ bounded by a constant but chosen randomly. It may not be significant because a (non-canonical) property tester can in visualization some random number of vertex samples after choosing the vertices.

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1 Canonical property test chooses (a fixed number of) vertices at random, but once the vertices are chosen, it outputs its answer deterministically. Therefore, at first sight, canonical property tests may be weaker. However as seen in [1][12] Th.2, for any given non-canonical property test, there exists a canonical property test which is equivalent to it. (Its derandomizing process is easy, since the sampling size of a non-canonical tester is a constant. The canonical tester repeats the samplings many (but constant) times. Then it computes the probability that the noncanonical tester accepts for each sampling. The canonical tester accepts if the sum of the probabilities is at least 1/2.)
5. Lemmas and Their Proofs

Definition 5.1. Let $H$ be a $k$-bound (colored $\tau$-partite hyper)graph on $\Omega$. Let $\delta : TC_I(H) \to [0, \infty)$ be a function. Then for $I \in \binom{[k]}{\ell}$, $\alpha \in [0, \infty)$, we define a subset of $TC_I(H)$ by

$$O^\alpha_{\delta}TC_I(H) := \{(\ell_J)_{J \subseteq I} | d_H(\ell_J)_{J \subseteq I} \geq \frac{\alpha^{1/3}}{|C_{J^*}(H)|} \text{ and } \delta((\ell_J)_{J \subseteq I}) \leq \frac{\alpha^{2/3}}{|C_{J^*}(H)|}, \forall I^* \subseteq I\}. \tag{2}$$

Write $O^\alpha_{\delta}TC_I(H) := TC_I(H) \setminus O^\alpha_{\delta}TC_I(H)$. (Here $O$ means 'ordinary'.) We may drop the subscript $\delta$ if it is not necessary.

Similarly we define $O^\alpha_{\delta}dC_I(H)$ and $O^\alpha_{\delta}dC_I(H)$. That is, $O^\alpha_{\delta}dC_I(H) := \{(\ell_J)_{J \subseteq I} | d_H(\ell_J)_{J \subseteq I} \geq \frac{\alpha^{1/3}}{|C_{J^*}(H)|} \text{ and } \delta((\ell_J)_{J \subseteq I}) \leq \frac{\alpha^{2/3}}{|C_{J^*}(H)|}, \forall I^* \subseteq I\}$.

In the above notation, we easily see that if $H$ is $(\epsilon, k, 1)$-regular then

$$P_{\epsilon \in \Omega} [H(e) \in \bigcup_{J \subseteq I} \Omega^\alpha_{\delta}TC_I(H)] \leq \sum_{J \subseteq I} P_{\epsilon \in \Omega} [d_H(H(e)) < \frac{\epsilon^{1/3}}{|C_J(H)|} \text{ or } \delta(H(e)) > \frac{\epsilon^{2/3}}{|C_J(H)|}]$$

where in the above (*) we used the fact that

$$P_{\epsilon \in \Omega} \left[P_{\epsilon' \in \Omega} [H(e') = H(e)|e' \approx e] \leq \frac{\epsilon^{1/3}}{|C_J(H)|}\right]$$

$$= \sum_{\epsilon_j \in C_j(H)} P_{\epsilon \in \Omega} \left[H(e) = \epsilon_j \text{ and } P_{\epsilon' \in \Omega} [H(e') = \epsilon_j|e' \approx e] \leq \frac{\epsilon^{1/3}}{|C_J(H)|}\right]$$

$$\leq \sum_{\epsilon_j \in C_j(H)} 1 \cdot P_{\epsilon \in \Omega} \left[H(e) = \epsilon_j \text{ and } P_{\epsilon' \in \Omega} [H(e') = \epsilon_j|e' \approx e] \leq \frac{\epsilon^{1/3}}{|C_J(H)|}\right]$$

$$= \sum_{\epsilon_j \in C_j(H)} E_{\epsilon \in \Omega} \left[P_{\epsilon_j \in \Omega} [H(\hat{e}) = \epsilon_j|\hat{e} \approx e] \right] \leq \frac{\epsilon^{1/3}}{|C_J(H)|}$$

where the conditional part depends only on $H(\hat{e})$.

Definition 5.2. [Color representative $\hat{\psi}$] Let $H$ be a $k$-bound (colored $\tau$-partite hyper)graph on $\Omega$. Let $\hat{\psi} \in \Phi(\hat{m})$, where $\hat{m} \in \mathbb{N}^k-1$, and $\epsilon, \epsilon_1 > 0$.

For $\epsilon_j^* \in C_j(H/(\hat{\psi}))$ with $I \in \binom{[k]}{\ell}$, we denote by $H[\epsilon_j^*]$ the unique color $\epsilon_j \in C_j(H)$ such that $H/(\hat{\psi})(e) = \epsilon_j^*$ implies $H(e) = \epsilon_j$. Similarly we define $H[\epsilon_j^*] \in TC_I(H)$ for $\epsilon_j^* \in TC_I(H/(\hat{\psi})$ and $H[\epsilon_j^*] \in TC_I(H/(\hat{\psi}))$ for $\epsilon_j^* \in TC_I(H/(\hat{\psi}))$.

Let $L_1, \cdots, L_k$ be positive integers. Denote by $A_I = A_I(L_1, \cdots, L_{|I|})$ the set of vectors $\tilde{a} = (a_J)_{J \subseteq I}$ where $a_J \in [L_{|J|}]$. Write $A_I := \bigcup_{I \subseteq \ell} A_I$ and $A := \bigcup_{I \subseteq \ell} A_I$.

- We inductively and probabilistically define colors $\tilde{a} = \tilde{a}(\tilde{a}) \in C_I(H/(\hat{\psi}))$ for all $\tilde{a} \in A_I$, $I \in \binom{[k]}{\ell}$, by the following (i) and (ii).

(i) Let $1 \leq s < k$. Assume that $\tilde{a}_I(\tilde{a}) \in C_I(H/(\hat{\psi}))$ is defined for any $I \in \binom{[k]}{s}$ and for any $\tilde{a} \in A_I$.

(ii) Let $I \in \binom{[k]}{s}$ and $\tilde{a} \in A_I$. Pick an edge $e \in \Omega_I$ randomly so that $H/(\hat{\psi})(\tilde{a}) = (\tilde{a}_J(\tilde{a}))_{J \subseteq I}$ where $\tilde{a}_J := (a_J)_{J \subseteq I}$. Let $\tilde{a}_I(\tilde{a}) := H(\tilde{\psi})(\tilde{a})$.

Note that for the entire process we pick a random edge exactly $|A| = rL_1 + \binom{\ell}{s}L_1^2L_2 + \cdots + \binom{\ell}{s} \prod_{i \in [k]} L_i^{(i)}$ times.

Write

$$\tilde{a}(\tilde{a}) = \tilde{a}_I(\tilde{a}) := (\tilde{a}_J(\tilde{a}))_{J \subseteq I} \in TC_I(H/(\hat{\psi}))$$

where $\tilde{a}(\tilde{a}) := (a_J)_{J \subseteq I}$.

- Assume that $\tilde{a}$ is fixed. Then we will inductively and probabilistically define a map $\theta_I : TC_I(H) \to [0, L_{|I|}]$ for all $I \in \binom{[k]}{\ell}$, by the following (i') and (ii')

(i') Let $1 \leq s \leq k$. Assume that $\theta_I(\hat{\psi}) = 0, L_{|I|}$ is defined for any $I \in \binom{[k]}{s}$ and for any $\hat{\psi} \in TC_I(H/(\hat{\psi}))$. 

If \( L^*_I \neq \emptyset \) then we define \( \theta_I(\vec{c}) := \theta_I(\vec{c}) \) for an \( \vec{c} \in L^*_I \) chosen uniformly at random. If \( L^*_I = \emptyset \) then we define \( \theta_I(\vec{c}) := 0 \).

Write

\[
\theta(\vec{c}) = \theta(\vec{c}):= (\theta_J(\vec{c})), J \subseteq I
\]

where \( \vec{c}_J := (c_{e,J}), e \in J \) and \( \theta(\vec{c}) = \theta(\vec{c}) := (\theta_J(\vec{c})), J \subseteq I \).

- When \( \theta_I(\vec{a}) \in A_I \) or \( \partial \vec{c} \in \partial A_I := \{ \vec{a} = (a_J), J \subseteq I, a_J \in [L_{|J|}] \} \) (i.e. the case when it does not contain any zero), we write

\[
\theta_I(\vec{c}) := \vec{a}, \text{ and } \theta_I(\vec{c}) := \vec{a} \text{ and } \theta_I(\partial \vec{c}) := \vec{a}, \text{ where } \vec{a} \text{ is a fixed symbol which does not belong to any color class.}
\]

In the proofs, we will write \( d_G^{(\delta)}(\vec{c}) = d_G(\vec{\delta}) + \delta(\vec{c}) \) for \( \vec{c} \in TC(G) \).

**Lemma 5.1** (All representatives are very regular). There exist a positive-valued function \( c_{\vec{a}}(\vec{c}) \) such that the following proposition holds.

Let \( r \geq k \) be positive integers and let \( \vec{L} = (L_i)_{i \in [k]} \) be a sequence of positive integers. Let

\[
0 < c_{\vec{L}} \leq \frac{c_{\vec{L}}}{2}
\]

and \( \vec{H} \) a \( k \)-bound (colored \( r \)-partite hyper)graph on \( \Omega \). Suppose that \( \vec{H}/\vec{\psi} \) is \( (\epsilon_1,k,1) \)-regular for some \( \vec{\psi} = (\psi_i)_{i \in [k-1]} \in \Phi(m_1, \ldots, m_{k-1}) \), where \( m_1, \ldots, m_{k-1} \) are positive integers. Then the \( \vec{\psi} \) probabilistically defined in Definition 5.2 satisfy the following inequality:

\[
\sum_{\vec{I} \in \binom{[k]}{\vec{L}} \vec{\psi} \in TC_r(\vec{H}/\vec{\psi})} \frac{\sum_{\vec{I}} \sum_{\vec{\psi} \in TC_r(\vec{H}/\vec{\psi})}}{\phi(r)} < 0.01,
\]

where \( \frac{\sum_{\vec{I}} \sum_{\vec{\psi} \in TC_r(\vec{H}/\vec{\psi})}}{\phi(r)} \) denotes the probability in the probability space generated by the (two-step) random process in the definition of \( \vec{\psi} \).

**Proof**: By the regularity of \( \vec{H}/\vec{\psi} \), we see that

\[
\sum_{\vec{I} \in \binom{[k]}{\vec{L}}} \sum_{\vec{\psi} \in TC_r(\vec{H}/\vec{\psi})} \frac{\sum_{\vec{I}} \sum_{\vec{\psi} \in TC_r(\vec{H}/\vec{\psi})}}{\phi(r)} < 0.01,
\]

where \( \phi(r) \) denotes the probability in the probability space generated by the (two-step) random process in the definition of \( \vec{\psi} \).

\[
\sum_{\vec{I} \in \binom{[k]}{\vec{L}}} \sum_{\vec{\psi} \in TC_r(\vec{H}/\vec{\psi})} \frac{\sum_{\vec{I}} \sum_{\vec{\psi} \in TC_r(\vec{H}/\vec{\psi})}}{\phi(r)} < 0.01,
\]

where \( \Phi \) denotes the probability in the probability space generated by the (two-step) random process in the definition of \( \vec{\psi} \).
where the last inequality follows from the assumption that $\epsilon_1 > 0$ is small enough with respect to $r, k, L_1, \ldots, L_k$.

Lemma 5.2 (Most representatives are ordinary). There exist positive-valued functions $\epsilon_1(\cdot), \epsilon_2(\cdot, \cdot) \in C^1(\cdot)$ such that the following proposition holds.

Let $r \geq k$ be positive integers. Let $0 < \epsilon \leq \epsilon_2(k)$ and $0 < \epsilon_1 \leq \epsilon_2(\cdot, \cdot)$, and $\vec{b}' = (b'_i)_{i \in [k]}$ be sequences of positive integers with $L_i \geq L(\epsilon, b'_i)$ for all $i < k$. Let $H$ be a $k$-bound $\vec{b}'$-colored (r-partite hyper)graph on $\Omega$. For some integers $m_1, \ldots, m_{k-1}$, we suppose that $H$ is $(\epsilon, k, 1, m_1 + \cdots + m_{k-1})$-regular and that

$$E_{\vec{\psi} \in \Phi(m_1, \ldots, m_{k-1})} [\text{reg}_{k,1}(H)] \leq \epsilon_1^2.$$  

Then the $\vartheta$ probabilistically defined in Definition 5.2 satisfies the following :

$$E_{\vec{\psi} \in \Phi(m_1, \ldots, m_{k-1})} [P_{\Omega}(H(\vartheta))] \leq 1 - 2^{|I|} \epsilon_1^{1/3}/c \quad \forall I \in \binom{[k]}{r},$$

where we call $\eta^* = (\eta^*_j)_{j \subseteq I} \in \partial C_I(H) \vec{\psi} (\epsilon_1, \epsilon_2/3, \epsilon_1^{1/3})$-ordinary if and only if

(i) $\eta^* \in O^* \partial C_I(H \vec{\psi})$,

(ii) for all $J \subset I$

$$\sum_{\epsilon_j \in C_I(H)} \left[ d_{H/\vec{\psi}}(\epsilon_j, (\eta^*_j)_{j \subseteq J}) - d_H(\epsilon_j, (\eta^*_j)_{j \subseteq J}) \right]^2 \leq \left( \frac{\gamma}{|J|} \right)^2,$$

(iii) if $\epsilon_j \in C_I(H)$ and $d_H(\epsilon_j, (\eta^*_j)_{j \subseteq J}) \geq \frac{|J|}{|C_I(H)|} \gamma$ then $\vec{\psi}((\eta^*_j)_{j \subseteq J}) \neq \vartheta$.

In the above, we mean $d_{H/\vec{\psi}}(\epsilon_j, (\eta^*_j)_{j \subseteq J}) := P_{\Omega}(H(\vartheta) \vec{\psi} = \gamma_j \vec{\psi}(\vartheta) = (\epsilon_j^*)_{j \subseteq J})$ as in (11).

Proof: In the below, we write $H^* := H \vec{\psi}$. Let $\gamma, \rho > 0$, which will be defined later at (11). Write

$$O^* \partial C_I(H^*) := \{ \epsilon \in O^* \partial C_I(H^*) \mid \epsilon \text{ is } (\epsilon_1, \epsilon_2/3, \epsilon_1^{1/3}) \text{-ordinary} \}.$$

We say that $\epsilon \in \partial C_I(H)$ is $(\epsilon_1, \epsilon_2/3, \epsilon_1^{1/3})$-ordinary if and only if $P_{\Omega}(H^* \vec{\psi}(\vartheta) = \gamma_j \vec{\psi}(\vartheta) = (\epsilon_j^*)_{j \subseteq J}) \neq \vartheta$.

Write

$$O^* \partial C_I(H) := \{ \epsilon \in O^* \partial C_I(H) \mid \epsilon \text{ is } (\epsilon_1, \epsilon_2/3, \epsilon_1^{1/3}) \text{-ordinary} \}.$$

Since $(\epsilon, k, 1, m_1 + \cdots + m_{k-1})$-regularity of $H$ yields that

$$P_{\vec{\psi} \in \Phi(m_1, \ldots, m_{k-1}), \epsilon^* \in \Omega} \left[ \sum_{\epsilon_j \in C_I(H)} \left( d_{H/\vec{\psi}}(\epsilon_j, (\epsilon_j^*)_{j \subseteq J}) - d_H(\epsilon_j, (\epsilon_j^*)_{j \subseteq J}) \right)^2 \right] \leq \left( \frac{\epsilon}{|C_I(H)|} \right)^2$$

for all $J \subseteq I$, i.e. (by the definition of regularization)

$$P_{\vec{\psi} \in \Phi(m_1, \ldots, m_{k-1}), \epsilon^* \in \Omega} \left[ \sum_{\epsilon_j \in C_I(H)} \left( d_{H/\vec{\psi}}(\epsilon_j, (\epsilon_j^*)_{j \subseteq J}) - d_H(\epsilon_j, (\epsilon_j^*)_{j \subseteq J}) \right)^2 \right] \leq \left( \frac{\epsilon}{|C_I(H)|} \right)^2$$

for all $J \subseteq I$ , it is easy to see that

$$P_{\vec{\psi} \in \Phi(m_1, \ldots, m_{k-1}), \epsilon^* \in \Omega} \left[ \sum_{\epsilon_j \in C_I(H)} \left( d_{H/\vec{\psi}}(\epsilon_j, (\epsilon_j^*)_{j \subseteq J}) - d_H(\epsilon_j, (\epsilon_j^*)_{j \subseteq J}) \right)^2 \right] \geq \left( \frac{\gamma}{|C_I(H)|} \right)^2 \forall J \subseteq I \leq 2^{|I|} \epsilon_1^{1/3}/\gamma^2,$$

which yields that

$$E_{\vec{\psi} \in \Phi(m_1, \ldots, m_{k-1}), \epsilon^* \in \Omega} [H^* \vec{\psi} = \gamma_j \vec{\psi}(\vartheta) = (\epsilon_j^*)_{j \subseteq J}] \leq 2^{|I|} \epsilon_1^{1/3} / \gamma^2 + \sum_{J \subseteq I} \epsilon_1^{1/3}.$$
Thus we see that
\[
\mathbb{E}_{\psi \in \Phi(m)} \mathbb{P}_{e \in \Omega_{\mathcal{I}}} [H(\partial e) \not\in \mathcal{O} \cdot \partial C_{I}(H)]
\leq \mathbb{P}_{e \in \Phi(m), e \in \Omega_{\mathcal{I}}} [H(\partial e) \text{ is not } (\epsilon, 1, \gamma, \rho)\text{-ordinary}] + \mathbb{P}_{e \in \Omega_{\mathcal{I}}} [H(\partial e) \in \mathcal{O} \cdot \partial C_{I}(H)]
\leq 2|\mathcal{I}| \left( \frac{\epsilon^{2} / \gamma^{2} + 3 \epsilon^{1/3}}{\rho} + 2\epsilon^{1/3} \right).
\] (9)

Therefore if \( c \in \mathcal{O} \cdot \partial C_{I}(H) \) and \( c^* \in \mathcal{O} \cdot \partial C_{I}(H^*) \) then
\[
\mathbb{P}_{e, \partial} [\bar{\vartheta}(c) = c^*] = \prod_{J \subseteq I} \mathbb{P}_{e \in \Omega_{J}} [H^*(\partial e) = (c^*_J, \mathcal{J}J) \in \partial C_{J}(H^*)] (1 - (1 - \mathbb{P}_{e \in \Omega_{J}} [H(\partial e) = c_J | H^*(\partial e) = (c^*_J, \mathcal{J}J)])^{|L_{J}|})
\]

(Use regularities where \( c, c^* \) are considered complexes in \( S_{r,|\mathcal{I}| - 1,1,H^*} \) and in \( S_{r,|\mathcal{I}| - 1,1,H} \).)

\[
\geq \mathbb{P}_{e \in \Omega_{J}} [H^*(\partial e) = c^*] \prod_{J \subseteq I} \mathbb{P}_{e \in \Omega_{J}} [H(\partial e) = c] \prod_{J \subseteq I} \mathbb{P}_{e \in \Omega_{J}} [\bar{\vartheta}(c) = c^*] \mathbb{P}_{e \in \Omega_{\mathcal{I}}} [H(\partial e) = c] \sum_{c^* \in \mathcal{O} \cdot \partial C_{I}(H)} \mathbb{P}_{e \in \Omega_{J}} [H^*(\partial e) = c^* | H(\partial e) = c]
\]

(10)

Hence it follows that
\[
E_{\psi} \sum_{c \in \mathcal{O} \cdot \partial C_{I}(H)} \mathbb{P}_{e \in \Omega_{J}} [\bar{\vartheta}(H(\partial e)) = (\epsilon, 1, \gamma, \rho)\text{-ordinary}] \geq 1 - \mathbb{P}_{e \in \Omega_{\mathcal{I}}} [H(\partial e) = c] \sum_{c^* \in \mathcal{O} \cdot \partial C_{I}(H^*)} \mathbb{P}_{e \in \Omega_{J}} [\bar{\vartheta}(c) = c^*] \mathbb{P}_{e \in \Omega_{\mathcal{I}}} [H^*(\partial e) = c^* | H(\partial e) = c]
\]

(11)

Finally, we have that
\[
E_{\psi} \mathbb{P}_{e \in \Omega_{\mathcal{I}}} [\bar{\vartheta}(H(\partial e)) = (\epsilon, 1, \gamma, \rho)\text{-ordinary}] | \bar{\vartheta}(H(\partial e)) = (\epsilon, 2/3, \epsilon^{1/3})\text{-ordinary}
\]

(12)

when \( L_{|\mathcal{I}|} \geq L(\epsilon, b'_{|\mathcal{I}|}) \). Combining (11) and (12) completes the proof.

**Definition 5.3.** [Abbreviation] Let \( G \) be a \( k \)-bound \( \tilde{b} = (b_{i})_{i \in [k]} \)-colored hypergraph. Write \( c_{i}(G) := \max_{f \in \mathcal{F}(\mathcal{I})} |C_{i}(G)| \) for \( i \in [k] \). For an integer \( m \), we write \( \tilde{B}(\tilde{b}, m) := (B_{i}(\tilde{b}, m))_{i \in [k]} \) where

\[
B_{i}(\tilde{b}, m) := \prod_{j \in [0, k - 1]} \tilde{b}_{i + j}^{(l)} \cdot m'.
\]

Note that

\[
c_{i}(G/\varphi) \leq B_{i}(\tilde{b}, m) \quad \forall i \in [k] \forall \varphi \in \Phi(m).
\]

(13)
Lemma 5.3 (Main Lemma). There exists a positive-valued function \( \epsilon \) such that the following proposition holds.

Let \( r \geq k, \beta = (b_i)_{i \in [k]} \) be positive integers and \( 0 < \epsilon \leq \epsilon \). Let \( \epsilon \) be the function of Lemma 5.2. Let \( \epsilon_1 : \mathbb{N}^k \to (0, 1) \) be a function such that

\[
0 < \epsilon_1^{(l_1, \ldots, l_k)} \leq \epsilon_1^{(r, k, \beta, \epsilon)}(l_1, \ldots, l_k)
\]

for all integers \( l_1, \ldots, l_k \). Let \( h : \mathbb{N}^k \to \mathbb{N} \) be a function.

Then there exist an integer \( \tilde{\ell} \) and a function \( \tilde{m}(\cdot) \) such that if \( G \) is a \( k \)-colored \( (k, r) \)-regular \( r \)-partite hypergraph (on \( \Omega \)) then there exist integers \( l_1, \ldots, l_k \in \tilde{\ell} \) and integers \( m_1, \ldots, m_k \in \tilde{m}(l_1, \ldots, l_k) \) which satisfy the following, where \( \epsilon_1 := \epsilon_1^{(l_1, \ldots, l_k)} \).

There exist \( \tilde{\varphi} \in \Phi(l_1, \ldots, l_k) \) and \( \tilde{\psi} \in \Phi(m_1, \ldots, m_k) \) such that \( (H := G/\tilde{\varphi}) \) is \( (k, 1, m_1 + \cdots + m_k) \)-regular and that

\[
H/\tilde{\psi} \text{ is } (\epsilon, k, h^c(\epsilon_1), \cdots, c_{k}(H)))\text{-regular,}
\]

and furthermore that the map \( \bar{\nu}(\cdot) \) defined in Definition 5.2 for the \( H \) and \( \tilde{\psi} \) with some integers \( (L_1)_{i \in [k]} \) satisfies all of the following properties for all \( I \in \Omega \), simultaneously, with probability at least 0.9.

(i) \( P_{\bar{\nu}(I)} \) is \( (\epsilon_1, e^{2/3}, e^{1/3}) \)-ordinary.] \( \geq 1 - O_{\epsilon_1,k}(e^{1/3/3}) \).

(ii) If \( \epsilon_1 \in T C_1(H) \) and \( \bar{\nu}(\epsilon_1) \neq 0 \) then \( \nu(\epsilon_1) \in O^\epsilon_{T C_1(H)/\tilde{\psi}} \).

(iii) If \( \epsilon = (\epsilon_1)_J \) then \( \partial C_1(H) \) and \( \bar{\nu}(\epsilon) \neq 0 \) then there exists a color \( \epsilon_I \in C_1(H) \) such that \( \bar{\nu}(\epsilon_I) \in O^\epsilon_{T C_1(H)/\tilde{\psi}} \).

\[ \text{Proof:} \] Fix \( r \geq k, \bar{b}, \epsilon, \epsilon_1, h^c \) and \( G \) as in the lemma. Without loss of generality, \( h \) is increasing.

The upper bound function \( \epsilon \) is defined by

\[
\epsilon \bigg|_{r, k, \bar{b}, \epsilon}^{5.3} (\cdot) := \min \left\{ \epsilon_1^{5.2}(\epsilon), \epsilon_1^{5.3}(\epsilon) \right\}
\]

where

\[
\tilde{L}(l_1, \ldots, l_k) := (L_{i, k}^{5.2}(\epsilon, b_i'))_{i \in [k]} \quad \text{and} \quad b_i' := B_i(\bar{b}, l_i + \cdots + l_i).
\]

In this paragraph, we will define the function \( \tilde{m}(\cdot) \). Consider a sequence of integers \( \tilde{\ell} = (l_i)_{i \in [k-1]} \).

Theorem 4.13 \( r := r, k := k, \bar{b} := \bar{b}, h := h', b_i', b_i' := B_i(\bar{b}, l_i + \cdots + l_i) \) gives an \( M \) such that for any \( \tilde{\varphi} \in \Phi(\tilde{\ell}) \), there exist \( m_1, \ldots, m_k \leq M \) for which

\[
E_{\tilde{\varphi} \in \Phi(m_1, \ldots, m_k)} \left[ \text{reg}_{k, h^c(b_i', \cdots, b_i')} \right] \leq \epsilon_1^2,
\]

where \( H = G/\tilde{\varphi} \) and \( b_i' := B_i(\bar{b}, l_i + \cdots + l_i) \). Define \( \tilde{m}(\tilde{\ell}) := \tilde{M} \).

Next, we will define an integer \( \tilde{\ell} \) as follows. Theorem 4.13 \( r := r, k := k, \bar{b} := \bar{b}, h := h', b_i', b_i' := B_i(\bar{b}, l_i + \cdots + l_i) \) gives an \( \tilde{\ell} \) such that (for any \( G \)) there exist \( l_1, \ldots, l_k \in \tilde{\ell} \) for which

\[
E_{\tilde{\varphi} \in \Phi(l_1, \ldots, l_k)} \left[ \text{reg}_{k, L(l_1, \ldots, l_k)} \right] \leq 0.1\epsilon.
\]

It suffices to show that these \( \tilde{\ell} \) and \( \tilde{m}(\cdot) \) satisfy the desired qualifications.

For graph \( G \), there exist \( l_1, \ldots, l_k \in \tilde{\ell} \) satisfying (17). Then we randomly pick a \( \tilde{\varphi} \in \Phi(\tilde{\ell}) \) with \( \tilde{\ell} = (l_1, \ldots, l_k) \). For this \( \tilde{\varphi} \), there exist \( m_1, \ldots, m_k \in \tilde{m}(\tilde{\ell}) \) satisfying (16). Further we randomly pick a \( \tilde{\psi} \in \Phi(m) \) with \( \tilde{m} = (m_1, \ldots, m_k) \). By (17), for a random \( \tilde{\varphi} \), it holds with probability at least 0.9 that

\[ H \text{ is } (\epsilon, k, 1, (k - 1)\tilde{m}(\tilde{\ell}))\text{-regular.} \]

When (18) happens, since \( m_1 + \cdots + m_k \leq (k - 1)\tilde{m}(\tilde{\ell}) \), Lemma 5.2 with (16) gives positive-valued functions \( \epsilon \) such that if

\[
0 < \epsilon \leq \epsilon \quad \text{and} \quad 0 < \epsilon \leq \epsilon,
\]

then the \( \tilde{\varphi} \) probabilistically defined in Definition 5.2 for \( \tilde{L}(l_1, \ldots, l_k) \) of (15) satisfies the inequality that \( E_{\tilde{\varphi} \in \Phi(m)} E_{\tilde{\varphi} \in \Phi(\tilde{\ell})} \tilde{\varphi}(H/\tilde{\varphi}) \) is \( (\epsilon, \epsilon^{2/3}, \epsilon^{1/3})\)-ordinary.] \( \geq 1 - 2l^2/\epsilon^{1/3}/c \) for all \( I \in \Omega \), which
implies that
\[
E_{\psi \in \Phi(\tilde{m})} \sum_{I \in (i_k)} P_{e \in \Omega_I}[\bar{\theta}(H(\partial e))] \text{ is not } (\epsilon_1, \epsilon^{2/3}, \epsilon^{1/3})\text{-ordinary } \leq 2^{-k} \epsilon^{1/3}/c. \tag{20}
\]

Note that (19) is satisfied because of the assumption of the lemma and because of (14). Thus when (18) holds, for a random \( \psi \in \Phi(\tilde{m}) \), with probability at least \( 1 - 0.1 - \epsilon_1 \geq 0.89 \), it follows from (20) that
\[
E_{\theta} \sum_{I \in (i_k)} P_{e \in \Omega_I}[\bar{\theta}(H(\partial e))] \text{ is not } (\epsilon_1, \epsilon^{2/3}, \epsilon^{1/3})\text{-ordinary } \leq 10 \cdot 2^{-k} \epsilon^{1/3}/c \tag{21}
\]
and from (16) that
\[
H/\tilde{\psi} \text{ is } (\epsilon_1, k, h^\circ(c_1(H), \ldots, c_{k-1}(H)))\text{-regular.} \tag{22}
\]

By Lemma 5.4 with (14) and (22), we have that
\[
\sum_{I \in (i_k)} P_{\theta}[\bar{\theta}(\tilde{c}) \in \overline{O}^{-1}TC_I(H/\tilde{\psi})] \leq \sum_{I \in (i_k)} P_{\theta}[\bar{\theta}(\tilde{c}) \in \overline{O}^{-1}TC_I(H/\psi)] \leq 0.01. \tag{23}
\]
Thus by (21) and (23), for a random process of \( \theta \), with probability at least \( 1 - 0.01 - 0.001 \geq 0.9 \), the desired properties (i) and (ii) hold simultaneously.

It easily follows from the definition of \( \bar{\theta} \) that if \( c = (c_j)_{j \in I} \in \partial C_I(H) \) and \( \bar{\theta}(c) \neq \emptyset \) then there exists a color \( c_I \in C_I(H) \) such that \( \bar{\theta}((c_j)_{j \in I}) \neq \emptyset \). Thus property (ii) implies (iii). It completes the proof of Lemma 5.3 \( \square \)

6. Body Part of the Proof of Main Theorem

We will prove Theorem 3.2 by using Lemmas 5.2 and 5.3.

Proof of Theorem 3.2: Let \( r, k, \bar{b}, \varepsilon \) be given as in the theorem. (Without loss of generality, \( \bar{b} = (1, \ldots, 1, b_k) \), though we will not use this.) Let \( 0 < \epsilon \leq \epsilon_0 \) (k), and \( h^\circ(\cdot) \) be a function, which will be defined later at (24) and (25), respectively. Let \( 0 < \epsilon_1(\cdot) = \epsilon_0(\cdot) \) be the function which decreases fast enough in Lemma 5.3. By Lemma 5.3 with \( r, k, \bar{b}, \varepsilon, \epsilon, h^\circ \) and with \( G \), there exist an integer \( \bar{e} \) and a function \( \bar{m} \), which are independent from \( G \), together with \( \tilde{c} \in \Phi(\tilde{m}) \) for some \( \tilde{c} \in [\tilde{m}]^{k-1} \) and \( \tilde{m} \in [\tilde{m}(\tilde{e})]^{k-1} \) such that \( H/\tilde{\psi} \) is \( (\epsilon_1 := \epsilon_0(\bar{e}) , k, h^\circ(c_1(H), \ldots, c_{k-1}(H)) \})\text{-regular.} \) Furthermore there exist a map \( \bar{\theta} \) which satisfies properties (i)-(iii) of Lemma 5.3 simultaneously.

[Modification of \( G \)] By conducting the steps \( S_1, \ldots, S_k \), which will be defined below, we will redefine the face-colors \( H(e) \) for edges \( e \in \Omega_I, I \in (i_k) \). We will denote the new colored hypergraph by \( H' \), instead of \( H \). (We will see \( C_I(H) = C_I(H') \) since we will not add any new color, and will not remove any unused color from the color class, either. We always use symbol \( H \) for the old one.)

(Step \( S_k \)) Assume that \( H'(e') \) has been defined for all \( e' \in \bigcup_{J \in (i_{k-1})} \Omega_J \) so that \( \bar{\theta}(H'(e')) \in O^{\epsilon_1}TC_J(H/\tilde{\psi}) \). Let \( I \in (i_k) \) and \( e \in \Omega_I \). Write \( c_I := H(e) \) and \( c = (c_I)_{J \in I} := H'(\partial e) \in \partial C_I(H) \). By the assumption for \( s - 1, \bar{\theta}(c) \in O^{\epsilon_1}\partial C_I(H/\tilde{\psi}) \). Our purpose of this step is to define face-color \( H'(e) \in C_I(H) \).

(Case \( S_k - 1 \)) Suppose that \( \bar{\theta}(c) \) is \( (\epsilon_1, \epsilon^{2/3}, \epsilon^{1/3})\text{-ordinary} \) and that \( d_H(c_I|c) \geq \epsilon^{1/3}/|C_I(H)| \). Define
\[
H'(e) := c_I.
\]

(Case \( S_k - 1' \)) Suppose that \( \bar{\theta}(c) \) is \( (\epsilon_1, \epsilon^{2/3}, \epsilon^{1/3})\text{-ordinary} \) and that \( d_H(c_I|c) < \epsilon^{1/3}/|C_I(H)| \). Note that there exists such a color \( c''_I \in C_I(H) \) such that \( d_H(c''_I|c) \geq \epsilon^{1/3}/|C_I(H)| \). Fix such a color and define
\[
H'(e) := c''_I.
\]
(Case $S_k - 2$) Suppose that $\hat{\phi}(e)$ is not $(\epsilon_1, \epsilon^{2/3})$-ordinary. (This case does not occur when $s = 0$.) Then by (iii) of Lemma 5.3, there exists a color $c'_j \in C_I(\mathbf{H})$ such that $\hat{\phi}((c'_j)_{J \subseteq I}) \in O^{*}TC_I(\mathbf{H}/\psi)$ where $c'_j := c_j$ for all $J \subseteq I$. Fixing one, we define

$$H'(e) := c'_j.$$ 

By Lemma 5.3 (ii) with the definition of $(\epsilon_1, \epsilon^{2/3})$- ordinarity (Lemma 5.2 (iii)), we see that $\hat{\phi}(H'(e)) \in O^{*}TC_I(\mathbf{H}/\psi)$ for any of the three cases.

**Estimating the edit size** We define value $\text{Ordinariness}(e) \in [0, |I|]$ for $e \in \Omega_I$ by the largest integer $s \geq 0$ such that for any $J \subseteq I$ with $|J| \leq s$, $H'(e,J)$ was defined by (Case $S_{k-1}$). Note that if $\text{Ordinariness}(e) = |I|$ then $H'(e) = H(e)$ by Lemma 5.2 (iii). For any $I \in (k)$, we have that

$$\mathbb{P}_{e \in \Omega_I}[H'(e) \neq G(e)] \leq \mathbb{P}_{e \in \Omega_I}[|H'(e)| < |H(e)|] \leq \mathbb{P}_{e \in \Omega_I}[\text{Ordinariness}(e) < k] \leq \sum_{J \subseteq I} \mathbb{P}_{e \in \Omega_J}[\text{Ordinariness}(e|J) = |I| - 1] \leq \sum_{J \subseteq I} \mathbb{P}_{e \in \Omega_J}[\text{Ordinariness}(e) \neq |I|] \mathbb{P}_{e \in \Omega_J}[\text{Ordinariness}(e) \geq |I| - 1] \leq \sum_{J \subseteq I} \mathbb{P}_{e \in \Omega_J}[d_H(e)] \mathbb{P}_{e \in \Omega_J}[\text{Ordinariness}(e) \geq |I| - 1]

+ \sum_{J \subseteq I} \mathbb{P}_{e \in \Omega_J}[|d_H(e)|] \epsilon^{1/3} \mathbb{P}_{e \in \Omega_J}[\text{Ordinariness}(e) \geq |I| - 1]

\leq \sum_{J \subseteq I} O(r,k) \epsilon^{1/3} + \sum_{J \subseteq I} \epsilon^{1/3} \mathbb{P}_{e \in \Omega_J}[\text{Ordinariness}(e) \geq |I| - 1]

\leq \epsilon,

(24)

when $\epsilon$ is small enough for $r, k, \varepsilon$.

**Choosing a target forbidden graph** When $b'_i, i \in [k-1]$, are integers and when $g_I : \prod_{J \subseteq I} b'_J \rightarrow 2^{[b_k]} \setminus \{\emptyset\}, I \in (k)$, are maps, we say that an $F' \in F_h$ is $((b'_i), (g_I))$-colorable if and only if there exists a $k$-bound $(b'_1, \ldots, b'_{k-1}, b_k)$-colored simplicial-complex $S \in \mathcal{S}_{r,k,h}$ on the same vertex sets (as $F'$) such that $F'(e) = S(e) \in g_I(S(\partial e))$ for all $e \in \mathcal{V}_I(F'), I \in (k)$. Given integers $M_1, \ldots, M_{k-1}$, we define

$$h^\circ(M_1, \ldots, M_{k-1}) := h_0

(25)$$

to be the smallest value $h_0 \geq 0$ such that for any $(b'_1, \ldots, b'_{k-1})$ with $b'_i \leq M_i, \forall i \in [k-1]$, and for any $(g_I), I \in (k)$, at least one of the following two holds:

(a) There does not exist a $((b'_i), (g_I))$-colorable graph $F' \in F$.

(b) There exists a $((b'_i), (g_I))$-colorable graph $F' \in F_h$ with $h \leq h_0$ (or $h = h_0$ without loss of generality, by adding extra invisible edges).

Assume that (i) of the theorem does not hold. Then there exist an $h \geq 1$ and an $F \in F_h$ such that

$$\mathbb{P}_{e \in \Phi(h)}[H'(\phi(e))] = F(e)\forall e \in \mathcal{V}(F)] > 0.
$$

By the image of a map $\phi \in \Phi(h)$ with the above property, we can construct an $S \in \mathcal{S}_{r,k,h,\cdot}$ which shows the $((b'_i), (g_I))$-colorability of $F$ where

$$b'_i := c_i(H') = c_i(H),
g_i := g_i(e) := \{c_i \in C_I(H') \in C_I(G) | d_H(e, e') > 0 \} \ \forall e \in \partial C_I(H')
$$

(under some map from $C_I(H') \rightarrow C_I(H)$ to $[b'_i]$). By the definition (25) of $h^\circ$ and by the existence of colorable $F \in F$, for the above pair $((b'_i), (g_I))$, the item (a) does not happen, and then there exists a $((b'_i), (g_I))$-colorable $F^* \in F_{h_0}$ where

$$h_0 := h^\circ(b'_1, \ldots, b'_{k-1}),
$$

which is smaller than a constant depending only on $r, k, \tilde{b}, \varepsilon$ and $F$ since $h^\circ(\cdot)$ is monotone without loss of generality and

$$b'_i \leq B_i(\tilde{b}, (k - i))\tilde{\ell}.

(26)$$

Let $S^* \in \mathcal{S}_{r,k,h_0,\cdot}$ be the simplicial-complex guaranteeing the colorability of $F^*$. 
[Finding many copies] We will show that there exist many copies of $F^*$ in $G$. For this purpose, we define $S^{**} \in S_{r,k,h,\vec{b},\vec{\vartheta}}^*$ from $S^* \in S_{r,k,h,\vec{b},\vec{\vartheta}}$ by replacing $S^*(e)$ by $S^{**}(e) := \vartheta(S^*(e))$ for each $e \in V(S^*) = V(S^{**})$.

By the definition of $\vartheta(\cdot)$, if $|I| = k$ and $\bar{c} = (c_I)_{I \subseteq I} \in TC_I(H')$ then $\vartheta(\bar{c}) \in \{0, c_I\}$ since $C_I(H') = C_I(\vec{H}/\vec{\vartheta})$. Therefore by our definition of $H'$, if $e \in \Omega_I$ with $I \in \binom{[k]}{r}$ then $c_I \neq \vartheta(H'(e)) = H'(e)$. Using this fact, it is easily seen that not only $S^*$ but also $S^{**}$ is a simplicial-complex guaranteeing the $((b'_I)_i, (g_I)_I)$-colorability of $F^*$ by identifying $S^*(e)$ as $S^{**}(e) = \vartheta(S^*(e))$ (in the domain of $g_I$) for each $e \in V_I(S^*), I \in \binom{[k]}{r}$. To see this, for all $e \in V_k(F^*)$, observe that $F^*(e) = S^*(e) = \vartheta(S^*(e))$ and that $S^*(e) \in g_I(S^*(\partial e)) \equiv g_I(\vartheta(S^*(\partial e)))$.

By Lemma 5.3 (ii) with the definition of $(\ast, \ast, c_I / 3)$-ordinarity (Lemma 5.2 (iii)), we have that $\vartheta(S^*(e)) \in O^{\ast, 1/3} TC_I(H'/\vec{\vartheta})$ for all $e \in V_I(S^*), I \in \binom{[k]}{r}$. Hence it follows from

$$c_I(\vec{H}/\vec{\vartheta}) \leq B_I((b'_I)_j, (k-i)\bar{m}(\bar{I}, \ldots, \bar{I}))$$

(27)

that

$$\mathbb{P}_{\phi \in \Phi(h_o)}[G(\phi(e)) = F^*(e) \forall e \in V(F^*)] \geq \prod_{e \in V(S^{**})} (1 - \epsilon_{1/3}^3)$$

$$\geq \prod_{I \in \binom{[k]}{r}} \prod_{e \in V_I(S^{**})} \left(1 - \epsilon_{1/3}^3\right)$$

$$\geq \prod_{I \in \binom{[k]}{r}} \left(1 - \epsilon_{1/3}^3\right)^{2/3} \prod_{e \in V_I(S^{**})} \left[\frac{1}{|C_I(H'/\vec{\vartheta})|}\right]$$

(27)

$$\geq \prod_{I \in \binom{[k]}{r}} \left(1 - \epsilon_{1/3}^3\right)^{2/3} \prod_{e \in V_I(S^{**})} \left[\frac{1}{|C_I(H'/\vec{\vartheta})|}\right]$$

which is larger than a positive real depending only on $r, k, \vec{b}, \vec{\vartheta}$ and $F$ by (24). In the last inequality, we used the fact that function $\epsilon_{1/3}^3 \leq 0.1^{3/2}$ is monotone without loss of generality. It completes the proof of Theorem 3.2.

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