AN IMPRIMITIVITY THEOREM FOR PARTIAL ACTIONS

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Abstract. We define proper, free and commuting partial actions on upper semicontinuous bundles of C*-algebras. With such, we construct the C*-algebra induced by a partial action and a partial actions on that algebra. Using those action we give a generalization, to partial actions, of Raeburn’s Symmetric Imprimitivity Theorem [10].

Introduction

The main idea of this article appears in the following example. Let $\beta$ be a continuous, free and proper action of a locally compact and Hausdorff (LCH) group $G$ on a LCH space $Y$. This gives us a continuous action of $G$ on the continuous functions vanishing at infinity of $Y$, $C_0(Y)$. If $Y/G$ is the orbit space of $Y$, then Green’s Theorem [11] implies $C_0(Y/G)$ is strongly Morita equivalent to the crossed product $C_0(Y) \rtimes G$.

Now consider an open subset $X \subset Y$ such that $\bigcup \{\beta_t(X) : t \in G\} = Y$. Let $\alpha$ be the restriction of $\beta$ to $X$. That is, for every $t \in G$ set $\alpha_t : X \cap \beta_t^{-1}(X) \to X \cap \beta_t(X)$, $x \mapsto \beta_t(x)$. This is an example of a partial action. Now consider the open set $\Gamma := \{(t, x) \in G \times X | \beta_t^{-1}(x) \in X\} \subset G \times Y$. The crossed product $C_0(X) \rtimes_\alpha G$ is the closure of $C_c(\Gamma) \subset C_c(G, Y)$ in $C_0(Y) \rtimes G$. It is strongly Morita equivalent to $C_0(Y) \rtimes_\beta G$ [2, 3].

Putting all together, we conclude that $C_0(X) \rtimes_\alpha G$ is strongly Morita equivalent to $C_0(Y/G)$. The objective of the present work is to generalize the previous idea to the case where we just know $X$, $G$ and $\alpha$. That is, $\alpha$ is a partial action of $G$ on $X$.

The outline of this work is as follows. In Section 1 we give the definitions of free, proper and commuting partial actions and prove some basic results involving those concepts, it is based on [1, 2]. In the second section we define partial actions on upper semicontinuous C*-bundles and, with such, construct the induced C*-algebra of a partial action and partial actions on those induced algebras. Here we follow Raeburn’s work [10]. Finally, we prove our main theorem which is a generalization, to partial actions, of Raeburn’s Theorem [10]. On a first read, to understand the basic ideas, we suggest the reader to consider bundles of the form $X \times \mathbb{C}$ ($X$ is a topological space and $\mathbb{C}$ the complex numbers) with trivial action on $\mathbb{C}$.

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1. Properties of Partial Actions

Through this work the letters $G$, $H$ and $K$ will denote LCH topological groups and $X$, $Y$ topological spaces. When any additional topological property is required it will be explicitly mentioned (this will never happen for the groups).

This section is a brief resume of some results contained in [2] and in the PHD Thesis [1], for that reason some proof will be omitted. We start by recalling the definition of partial action.

**Definition 1.1** ([3][2][1]). A pair $\alpha = (\{X_t\}_{t \in H}, \{\alpha_t\}_{t \in H})$ is a partial action of $H$ on $X$ if, for every $t, s \in H$:

1. $X_t$ is a subset of $X$ and $X_e = X$ ($e$ being the identity of $H$).
2. $\alpha_t : X_t \to X_t^{-1}$ is a bijection and $\alpha_e = \text{id}_X$ (the identity on $X$).
3. If $x \in X_t^{-1}$ and $\alpha_t(x) \in X_{s^{-1}}$, then $x \in X_{(st)^{-1}}$ and $\alpha_{st}(x) = \alpha_s \circ \alpha_t(x)$.

The domain of $\alpha$ is the set $\Gamma_\alpha := \{(t,x) \in H \times X \mid x \in X_t^{-1}\}$. Recall $\alpha$ is continuous if $\Gamma_\alpha$ is open in $H \times X$ and the function, also called $\alpha$, $\Gamma_\alpha \to X$, $(t,x) \mapsto \alpha_t(x)$, is continuous. The graph of the partial action $\alpha$, $\text{Gr}(\alpha)$, is the graph of the function $\alpha : \Gamma_\alpha \to X$. We say $\alpha$ has closed graph if $\text{Gr}(\alpha)$ is closed in $H \times X \times X$.

Take two continuous partial actions of $H$, $\alpha$ and $\beta$, on the spaces $X$ and $Y$ respectively. A morphism $f : \alpha \to \beta$ is a continuous function $f : X \to Y$ such that for every $t \in H : f(X_t) \subset Y_t$ and the restriction of $\beta_t \circ f$ to $X_t^{-1}$ equals $f \circ \alpha_t$.

Given $\beta$ as before and a non empty open set $Z \subset Y$, the restriction of $\beta$ to $Z$ is the continuous partial action of $H$ on $Z$ given by $\gamma_t : Z \cap \beta_t^{-1}(Z) \to Z \cap \beta_t(Z)$, $z \mapsto \beta_t(z)$.

Up to isomorphism of partial actions, every continuous partial action can be obtained as a restriction of a global action. That is, given $\alpha$ as before there exits a global and continuous action of $H$ on a topological space $Y$, $\beta$, and an open set $Z \subset Y$ such that $\alpha$ is isomorphic to the restriction of $\beta$ to $Z$. If in addition $Y = \bigcup\{\beta_t(Z) \mid t \in H\}$, we say $\beta$ is an enveloping action of $\alpha$. Enveloping actions exist and are unique up to isomorphism of (partial) actions [2][1]. The enveloping action of $\alpha$ is denoted $\alpha^e$ and the space where it acts $X^e$; we also think $X$ is an open set of $X^e$ and $\alpha$ is the restriction of $\alpha^e$ to $X$.

The orbit of a subset $U \subset X$ by $\alpha$ is the set $HU := \cup\{\alpha_t(U \cap X_t^{-1}) \mid t \in G\}$. The orbit of a point $x \in X$ is the orbit of the set $\{x\}$ and is denoted $Hx$. If we want to emphasize the name of the action we write $\alpha Hx$. The orbits of two points are equal or disjoint and the union of all of them is equal to $X$. With this partition of $X$ we construct the quotient space $X/H$ with the quotient topology, this is the orbit space of $\alpha$. The canonical projection $X \to X/H$ is continuous, surjective and open. The function $X/H \to X^e/H$, $\alpha Hx \to \alpha^e Hx$ is a homeomorphism.

Raeburn’s Symmetric Imprimitivity Theorem involves free, proper and commut ing actions. We now give the corresponding definitions for partial actions. We refer the reader to [1] to a more detailed exposition of these concepts.

The stabilizer of a point $x \in X$ is the set $H_x := \{t \in H \mid x \in X_t^{-1}, \alpha_t(x) = x \}$. It is easy to see that $H_x$ is a subgroup of $H$, not necessarily closed if the action is not global. A partial action is free is the stabilizer of every point is the set $\{e\}$. A partial action is free if and only if it’s enveloping action is free.

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1We say “the” enveloping action because it is unique up to isomorphisms.
The next concept we define is commutativity. We will have two continuous partial actions, \( \alpha \) and \( \beta \), of \( H \) and \( K \), on \( X \). As we do not want any confusions, we will use the notation \( \alpha_s : X_{s^{-1}}^H \to X_s^H \) and \( \beta_t : X_{t^{-1}}^K \to X_t^K \), for \( s \in H \) and \( t \in K \).

We say \( \alpha \) and \( \beta \) commute if for every \( (s, t) \in H \times K \) (i) \( \alpha_s(X_{s^{-1}}^H \cap X_t^K) = \beta_t(X_{t^{-1}}^K \cap X_s^H) \) and (ii) \( \alpha_s \circ \beta_t(x) = \beta_t \circ \alpha_s(x) \), for every \( x \in \alpha_{s^{-1}}(X_s^H \cap X_{t^{-1}}^K) \). This definition expresses the fact that we can compute \( \alpha_s \beta_t(x) \) if and only if we can compute \( \beta_t \alpha_s(x) \), and in that case \( \alpha_s \beta_t(x) = \beta_t \alpha_s(x) \). As we can see, if both actions are global, this is the usual notion of commuting actions.

Recall a subset \( U \subset X \) is said \( \alpha \)-invariant if \( \alpha_t(X_{s^{-1}}^H \cap U) \subset U \), for every \( t \in H \). Condition (i) of the previous definition implies \( X_s^H \) is \( \alpha \)-invariant for every \( s \in K \).

An important property of commuting global actions is that we can define an action of the product group, this is also true for partial actions. 

**Lemma 1.2** (cf. [1] Proposição 4.35). If \( \alpha \) and \( \beta \) commute then there is a continuous partial action, \( \alpha \times \beta \), of \( G := H \times K \) on \( X \) such that, for every \( (s, t) \in G \),

1. \( X(s,t) = \beta_t(X_{t^{-1}}^K \cap X_s^H) = \alpha_s(X_{s^{-1}}^H \cap X_t^K) \)
2. \( \alpha \times \beta(s,t) = \alpha_s \circ \beta_t \).

**Proof.** The fact that \( \mu := \alpha \times \beta \) is a partial action (not necessarily continuous) is an easy consequence of the fact that \( \alpha \) and \( \beta \) are commuting partial actions ([1] Proposição 4.35). We just have to deal with the continuity.

To show \( \Gamma_\mu \) is open in \( G \times X \) notice that the set \( \Gamma_\mu^{-1} := \{(t, x) \in K \times X \mid x \in X_t^K\} \) is open in \( K \times X \), and define:

\[
\pi_H : H \times K \times X \to H \times X, \quad (h, k, x) \mapsto (h, x),
\]

\[
\pi_K : H \times K \times X \to K \times X, \quad (h, k, x) \mapsto (k, x),
\]

\[
F : \pi_K^{-1}(\Gamma_\beta) \to \pi_H^{-1}(\Gamma_\beta^{-1}), \quad (h, k, x) \mapsto (h, k, \beta_k(x)).
\]

It is easy to see that the three functions are continuous. So, the domain and range of \( F \) are open in \( H \times K \times X \) and \( \Gamma_\mu = F^{-1}(\pi_K^{-1}(\Gamma_\beta^{-1}) \cap \pi_H^{-1}(\Gamma_\alpha)) \) is open in \( G \times X \). Finally, the continuity of \( \mu : \Gamma_\mu \to X \) follows from that of \( \alpha : \Gamma_\alpha \to X \) and \( \beta : \Gamma_\beta \to X \). \( \square \)

Here is another property of partial actions we will use.

**Lemma 1.3.** If \( \alpha \) and \( \beta \) commute then there is a partial action of \( H \) on \( X/K \), called \( \hat{\alpha} \), such that for every \( s \in H \)

1. \( (X/K)_s := K X_s^H \) and
2. \( \hat{\alpha}_s(Kx) = K \alpha_s(x) \) for any \( x \in X_s^H \).

**Proof.** The first step is to show we can define \( \hat{\alpha} \) as in (1) and (2). Define the function \( F : \Gamma_\alpha \to H \times X/K \) as \( F(s, x) = (s, Kx) \). This map is open and continuous. The domain of \( \hat{\alpha} \) will be the image of \( F, \Gamma_\hat{\alpha} := \text{Im}(F) \), which is an open set.

Now define \( S : \Gamma_\alpha \to X/K \) in such a way that \( (s, x) \mapsto K \alpha_s(x) \), and consider (on \( \Gamma_\alpha \)) the equivalence relation \( u \sim v \) if \( F(u) = F(v) \). The function \( S \) is constant in the classes of \( \sim \) and the quotient space \( \Gamma_\alpha/\sim \) is homeomorphic to \( \Gamma_\hat{\alpha} \) through the map defined by \( F \). So, there is a unique continuous map \( \Gamma_\hat{\alpha} \to X/K \) such that \( (s, Kx) \mapsto K \alpha_s(x) \). This is the partial action \( \hat{\alpha} \) we are looking for.

It remains to be shown that \( \hat{\alpha} \) is a partial action. Properties (1) and (2) of Definition [1.1] are easy to prove, for (3) recall every \( X_s^H \) is \( \beta \)-invariant. \( \square \)
Assume for a moment we have a continuous global action of \( H \) on \( X \). It is immediate that it’s domain, being equal to \( H \times X \), is a closed and open (clopen) set of \( H \times X \). That is not always the case for partial actions.

**Definition 1.4.** A partial action, \( \alpha \), of \( H \) on \( X \) has **closed domain** if \( \Gamma_\alpha \) is closed in \( H \times X \).

**Lemma 1.5.** If \( X \) is Hausdorff and \( \alpha \) is continuous, the following conditions are equivalent:

1. \( \alpha \) has closed domain.
2. The enveloping space \( X^e \) is Hausdorff and \( X \) is closed in \( X^e \).

**Proof.** We start by proving (1) \( \Rightarrow \) (2). Recall \( X^e \) is Hausdorff if \( \alpha \) has closed graph. Consider the function \( F : H \times X \times X \to H \times X \), \( (s, x, y) \mapsto (s, x) \). The set \( F^{-1}(\Gamma_\alpha) \) is closed in \( H \times X \times X \). Now, \( \text{Gr}(\alpha) \) is closed in \( F^{-1}(\Gamma_\alpha) \) because it is the pre-image of the diagonal \( \{(x, x) | x \in X\} \subset X \times X \) by the continuous function \( F^{-1}(\Gamma_\alpha) \to X \times X \), \( (t, x, y) \mapsto (\alpha_t(x), y) \). This implies \( \alpha \) has closed graph.

To show \( X \) is closed in \( X^e \) take a net contained in \( X \), \( \{x_i\}_{i \in I} \), converging to a point \( x \in X^e \). There exists \( t \in H \) such that \( \alpha_t(x) \in X \). By the continuity of \( \alpha^e \) there is an \( i_0 \) such that \( (t^{-1}, \alpha_t(x)) \in \Gamma_\alpha \) for \( i \geq i_0 \). Then \( (t^{-1}, \alpha_t(x)) \), being the limit of \( \{(t^{-1}, \alpha_t(x))\}_{i \geq i_0} \), belongs to \( \Gamma_\alpha \). Finally \( x = \alpha_{t^{-1}}(\alpha_t(x)) \in X \).

For the converse notice three facts: the topology of \( H \times X \) is the topology relative to \( H \times X^e \), \( (\alpha^e)^{-1}(X) \) is closed in \( H \times X^e \) and \( \Gamma_\alpha = H \times X \cap (\alpha^e)^{-1}(X) \). So we clearly have that \( \Gamma_\alpha \) is closed in \( H \times X \).

The previous lemma characterizes the continuous partial actions with closed domain on Hausdorff spaces, as those arising as the restriction of a global action on a Hausdorff space to a clopen set.

**Lemma 1.6.** Given two continuous and commuting partial actions, both with closed domain, the partial action of the product group (as defined on Lemma 1.2) has closed domain.

**Proof.** In the proof of Lemma 1.2 we showed \( \Gamma_{\alpha \times \beta} \) is open, use the same arguments changing the word “open” for “closed”.

A dynamical system (DS for short) is a tern \((Y, H, \beta)\) where \( \beta \) is a continuous action of \( H \) on \( Y \), where \( H \) and \( Y \) are LCH. The natural extension to partial actions is the following one.

**Definition 1.7.** The tern \((X, H, \alpha)\) is a **partial dynamical system** (PDS) if \( \alpha \) is a continuous partial action of \( H \) on \( X \) and both \((H \times H)\) are LCH.

**Lemma 1.8.** If \((X, H, \alpha)\) is a PDS then \((X^e, H, \alpha^e)\) is a DS if and only if \( \alpha \) has closed graph.

**Proof.** By Theorem 1.1. of [2] every point of \( X^e \) has a neighbourhood homeomorphic to \( X \). So, every point of \( X^e \) has a local basis of compact neighbourhoods. As \( \alpha \) is continuous and \( H \) is LCH, \( (X^e, H, \alpha^e) \) is a DS if and only if \( X^e \) is Hausdorff. By Proposition 1.2. of [2] \( X^e \) is Hausdorff if and only if \( \alpha \) has closed graph.

A PDS \((X, H, \alpha)\) is **proper** if the function \( F_\alpha : \Gamma_\alpha \to X \), \( (t, x) \mapsto (x, \alpha_t(x)) \), is proper (the pre-image of a compact set is compact). This definition, and part of the next Lemma, are taken from [1].
Lemma 1.9. Given a PDS $(X, H, \alpha)$, the following statements are equivalent:

1. The system is proper.
2. Every net contained in $\Gamma_{\alpha}$, $\{(t_n, x_{i})\}_{i \in I}$, such that $\{(x_{i}, \alpha t_{n}(x_{i}))\}_{i \in I}$ converges to some point of $X \times X$, has a subnet converging to a point of $\Gamma_{\alpha}$.
3. $\alpha$ has closed graph and the enveloping DS $(X^e, H, \alpha^e)$ is proper.

Proof. The equivalence between (1) and (3) is proved in [1] (Proposição 4.6 2). The equivalence between (1) and (2) is proved as in Lemma 3.42 of [13].

It is a known fact that the orbit space of a proper DS is a LCH space, this is also true for PDS.

Lemma 1.10. If $(X, H, \alpha)$ is a proper PDS then $X/H$ is a LCH space.

Proof. By the previous Lemma $(X^e, H, \alpha^e)$ is a proper DS. So, $X^e/H$ is LCH. But $X/H$ is homeomorphic to $X^e/H$ and so is LCH.

The next result follows immediately from the previous ones.

Lemma 1.11. Let $(X, H, \alpha)$ and $(X, K, \beta)$ be commuting PDS (that is, $\alpha$ and $\beta$ commute). If $\beta$ is proper then $(X/K, H, \bar{\alpha})$ is a PDS, where $\bar{\alpha}$ is the partial action defined on Lemma 1.3.

2. Partial actions on bundles of $C^*$–Algebras.

The definition of upper semicontinuous $C^*$–bundle we are going to use is Definition C.16 of [13] (notice that we do not require the base space to be Hausdorff). From now on $B = \{B_x\}_{x \in X}$ and $C = \{C_y\}_{y \in Y}$ will be upper semicontinuous $C^*$–bundles. The projections of $B$ and $C$ will be denoted $p : B \to X$ and $q : C \to Y$, respectively.

The set of continuous and bounded sections of the bundle $B$ will be denoted $\mathcal{C}_b(B)$ (this notation differs from that of [13]). Similarly, $\mathcal{C}_0(B)$ is the set of continuous sections vanishing at infinity (C.21 [13]) and $\mathcal{C}_c(B)$ the set of continuous sections of compact support. When $X$ is a LCH space, $\mathcal{C}_b(B)$ and $\mathcal{C}_0(B)$ are $C^*$–algebras with the supremum norm and $\mathcal{C}_c(B)$ is a dense $*$–sub algebra of $\mathcal{C}_0(B)$.

Definition 2.1. A partial action of $H$ on $B$ is a pair $(\alpha, \cdot)$, where $\alpha$ and $\cdot$ are continuous partial actions of $H$ on $B$ and $X$, respectively, satisfying

1. $p^{-1}(X_t) = \iota_t B$ for every $t \in H$. Here $\iota_t B$ is the range of $\alpha_t$.
2. $p$ is a morphism of partial actions (Definition 1.1 of [2]).
3. The restriction of $\alpha_t$ to a fiber is a morphism of $C^*$–algebras, for each $t \in H$.

Notice that $\cdot$ is determined by $\alpha$. For that reason, with abuse of notation, we name $\alpha$ the pair $(\alpha, \cdot)$. We say $\alpha$ is global if the partial action on the total space is a global action or, what is the same, if $\cdot$ is global.

The domain of the partial action on the total and base space will be denoted $\Gamma(B, \alpha)$ and $\Gamma(X, \alpha)$, respectively.

Example 2.2. Let $(X, H, \cdot)$ be a PDS and $(A, H, \gamma)$ a $C^*$–DS, that is, $A$ is a $C^*$–algebra and $\gamma : H \to \text{Aut}(A)$ is a strongly continuous action. With such define the trivial bundle $p : A \times X \to X$, where $p(a, x) = x$. All the fibers of this bundle, called $B$, are isomorphic to $A$ by the maps $A \to B_x, a \mapsto (a, x)$. We define a global action of $H$ on $B$ by setting $\alpha_t : A \times X_t \to A \times X_t, (a, x) \mapsto (\gamma_t(a), t \cdot x)$. 

I would like to emphasize that, from now on, we are going to use the letters α and β for actions on total spaces. The actions on the base spaces will be denoted · and ⋆. We will write α_t(a) and t · x, similarly with β and ⋆.

If β = (β, ⋆) is a partial action of H on C, a morphism (F, f) : α → β is a pair of continuous functions, F : B → C and f : X → Y, such that: both are morphism of partial actions, q ◦ F = f ◦ p and the restriction of F to each fiber is a morphism of C∗-algebras. Naturally, the composition of morphisms is the composition of functions (on each coordinate).

Following [2] we can define the restriction of actions. Let β = (β, ⋆) be a global action of H on B and U an open subset of X. Consider the restriction bundle B_U = {B_a}_{a ∈ U} with the partial action β_U, which is the pair formed by the restriction of the actions of H to p^{-1}(U) and U. Notice we have obtained a partial action because p^{-1}(U) ∩ β_t(p^{-1}(U)) = p^{-1}(U ∩ t ⋆ U), for every t ∈ H.

Rephrasing Theorem 1.1 of [2] we get

**Theorem 2.3.** For every continuous partial action α of H on an upper semicontinuous C∗-bundle B, there exists a tern (ι_X, t_B, α^e) such that α^e is an action of H on an upper semicontinuous C∗-bundle B^e, and (ι_X, t_B) : α → α^e is a morphism, such that for any morphism ψ : α → β, where β is an action of H (on an upper semicontinuous C∗-bundle), there exists a unique morphism ψ^e : α^e → β such that ψ^e ◦ (ι_X, t_B) = ψ.

Moreover, the pair (ι_X, t_B, α^e) is unique up to canonical isomorphisms, and

1. ι_X(X) is open in X^e.
2. (ι_X, t_B) : α → (α^e)_ι(X) is an isomorphism.
3. X^e is the orbit of ι_X(X).
4. B^e is a continuous C∗-bundle if and only if B is.

**Proof.** Let (ι_X, ·^e) and (ι_B, α^e) be the pairs given by Theorem 1.1 of [2] for · and α. We also have a morphism p^e : α^e → ·^e. Notice p^e is surjective because α is a morphism and the orbit of ι_X(X) equals X^e. Again, as B^e is the orbit of ι_B(B) and p^e is a morphism, to prove p^e is open we only have to see that p^e ◦ α_t^e ◦ t_B is open, for every t ∈ H. But this is true because, if U is open in B

\[ p^e ◦ α_t^e ◦ t_B(U) = t^e(ι_X(p(U))), \]

the last being an open set.

We have proved B^e fibers over X^e. We now give a structure of C∗-algebra to each fiber of B^e. Let x be an element of X^e, take t ∈ H such that t^e x ∈ ι_X(X) and define the C∗-structure on B^e_t as the unique making α^e_{t^{-1}} ◦ t_B : B^e_{ι_X(t)}(ι_X(x)) → B^e_x an isomorphism of C∗-algebras. This is independent of the choice of t because α acts as isomorphism of C∗-algebras on the fibers of B.

To prove the norm of B^e is semicontinuous notice that, given ε > 0, the set \{b ∈ B^e : ∥b∥ < ε\} equals the open set \bigcup_{t ∈ H} α_t^e ◦ t_B(\{b ∈ B : ∥b∥ < ε\}). In fact, a similar argument shows the norm of B^e is continuous if and only if the norm of B is continuous. This suffices to prove property (4) of the thesis.

We now indicate how to prove the continuity of the product, for the other operations there are analogous proofs. Set D^e := \{(a, b) ∈ B^e × B^e : p^e(a) = p^e(b)\}. We prove the continuity of D^e → B^e, (a, b) → ab, locally. Fix (a, b) ∈ D^e, we may assume p(a) = t^e x for some x ∈ X and t ∈ H. The product is continuous on (a, b)

because U := (α_t ◦ t_B)(B) × α_t ◦ t_B(B) ∩ D^e is open in D^e, and the restriction of
the product to $U$ is the continuous function

$$(c, d) \mapsto \alpha^e_t \circ \iota_B \left[ (\iota_B)^{-1} \circ \alpha^e_{t-1}(c) + (\iota_B)^{-1} \circ \alpha^e_{t-1}(d) \right].$$

Up to here we have constructed an upper semicontinuous $C^*$-bundle $B^e_x = \{B^e_x \}_{x \in X}$. By the previous construction we also have that $(\alpha^e, \iota^e)$ is a global action of $H$ on $B^e$, and $(\iota_X, \iota_B) : \alpha \rightarrow \alpha^e$ is a morphism. Except for property (2) of the thesis, everything follows immediately from the previous constructions and Theorem 1.1 of [2].

To prove property (2) it suffices to see that $(p^e)^{-1}(\iota_X(X)) = \iota_B(B)$. We clearly have the inclusion $\supset$, for the other one let $b \in (p^e)^{-1}(\iota_X(X))$. We may suppose $b = \alpha^e_t(\iota_B(c))$ for some $c \in B$ and $t \in H$. As $p^e(b) = p^e(\alpha^e_t(\iota_B(c))) \in \iota_B(B)$, we have

$$p^e(b) = p^e(\alpha^e_t(\iota_B(c))) = t^\varepsilon \iota_X(p(c)) \in \iota_X(X).$$

So, $p(c) \in X_{t-1}$. This implies $b = \iota_B(\alpha_t(c)) \in \iota_B(B)$. \hfill $\Box$

The non commutative analogue of PDS’s are the $C^*$-PDS’s, they are terns $(A, G, \gamma)$ formed by a $C^*$-algebra $A$, a LCH group $G$ and a partial action $\gamma$ of $G$ on $A$ (Definition 2.2 of [2], for a more general definition see [3]).

We know every PDS gives us a $C^*$-PDS with commutative algebra [2]. Following that construction, we are going to use partial actions on upper semicontinuous $C^*$-bundles over LCH spaces to construct partial actions on the $C^*$-algebras $C_0(B)$. The ideals are of the form $C_0(B, U) := \{f \in C_0(B) \mid f(x) = 0_x \text{ if } x \notin U\}$, for open sets $U \subseteq X$.

**Theorem 2.4.** Let $X$ be a LCH space, $B = \{B_x\}_{x \in X}$ an upper semicontinuous $C^*$-bundle and $\alpha$ a continuous partial action of $H$ on $B$. Then $(C_0(B), H, \tilde{\alpha})$ is a $C^*$-PDS, where

1. $C_0(B)_{t} = C_0(B, X_t)$, for every $t \in H$.
2. If $f \in C_0(B)_{t_{-1}}$ then $\tilde{\alpha}_t(f)(x) = \alpha_t(f(t^{-1} \cdot x))$ if $x \in X_t$ and $0_x$ otherwise.

**Proof.** First of all we have to show that, given $t \in H$ and $f \in C_0(B)_{t_{-1}}$, the function $\tilde{\alpha}_t(f)$ belongs to $C_0(B)_{t}$. It is clear that $\tilde{\alpha}_t(f)$ is a section that vanishes outside $X_t$. Besides, the function $X_t \rightarrow \mathbb{R}, \ x \mapsto \|\tilde{\alpha}_t(f)(x)\|$, being equal to $X_t \rightarrow \mathbb{R}$, $x \mapsto \|f(t^{-1} \cdot x)\|$, vanishes at infinity.

Clearly $\tilde{\alpha}_t(f)$ is continuous on $X_t$ and in the interior of the complement of $X_t$. To prove the continuity of $\tilde{\alpha}_t(f)$ it suffices to show that for every net $\{x_i\}_{i \in I} \subset X_t$ converging to a point $x \notin X_t$, we have $\|\tilde{\alpha}_t(f)(x_i)\| \rightarrow 0$. Notice that the function $X_{t_{-1}} \rightarrow \mathbb{R}, \ y \mapsto \|f(y)\|$, vanishes at infinity and the net $\{t^{-1} \cdot x_i\}_{i \in I}$ is eventually outside every compact of $X_{t_{-1}}$, we conclude $\|\tilde{\alpha}_t(f)(x_i)\| = \|f(t^{-1} \cdot x_i)\| \rightarrow 0$.

The next step is to show $\tilde{\alpha}$ is a partial action (Definition 1.1). We omit the proof of this fact because it is an easy task.

To prove $\{C_0(B)_t\}_{t \in H}$ is a continuous family [2], let $U$ be an open set of $C_0(B)$ and fix $t \in H$ such that $C_0(B)_t \cap U \neq \emptyset$. By the Urysohn Lemma we can find $g \in C_0(B)_t \cap U$ with compact support. As the domain of the partial action on $X$ is an open set, there is an open set containing $t$, $V$, such that $X_v$ contains the support of $g$ for every $r \in V$. Then $V$ is an open set containing $t$ and contained in $\{r \in H \mid C_0(B) \cap U \neq \emptyset\}$.

Now we deal with the continuity of $\tilde{\alpha}$. Let $\{(t_i, f_i)\}_{i \in I}$ be a net contained in $\Gamma_{\tilde{\alpha}}$ converging to $(t, f) \in \Gamma_{\tilde{\alpha}}$. Given $\varepsilon > 0$ there exists $g \in C_c(B)$, with support contained in $X_{t_{-1}}$, such that $\|f - g\| < \varepsilon$ (by the Urysohn Lemma).
We can find an \( i_0 \in I \) such that \( \text{supp}(g) \subset X_{i_0} \) and \( \| f_i - g \| < \frac{\varepsilon}{3} \), for every \( i \geq i_0 \). Then, for every \( i \geq i_0 \)

\[
\| \tilde{\alpha}_i(f) - \tilde{\alpha}_i(f_i) \| \leq \| \tilde{\alpha}_i(f_i - g) \| + \| \tilde{\alpha}_i(g) - \tilde{\alpha}_i(f) \| + \| \tilde{\alpha}_i(f - g) \| < \frac{2\varepsilon}{3} + \| \tilde{\alpha}_i(g) - \tilde{\alpha}_i(f) \|.
\]

To complete the proof it suffices to see that \( \lim_i \| \tilde{\alpha}_i(g) - \tilde{\alpha}_i(f) \| = 0 \). To this purpose let \( D \) be a compact containing \( t \cdot \text{supp}(g) \) on its interior, and contained in \( X_t \). We may find \( i_1 \) (larger than \( i_0 \)) such that \( t_i \cdot \text{supp}(g) \subset D \) and \( D \subset X_{i_1} \), for every \( i \geq i_1 \). Given \( i \geq i_1 \) we have

\[
\| \tilde{\alpha}_i(g) - \tilde{\alpha}_i(f) \| = \sup \{ \| \alpha_i(g(t_i^{-1} \cdot x)) - \alpha_i(f(t_i^{-1} \cdot x)) \| : x \in D \}.
\]

As \( D \) is compact, it suffices to prove \( \tilde{\alpha}_i(g) \) converges point wise to \( \tilde{\alpha}_i(f) \), which is an easy consequence of Lemma C.18 of [13] and the continuity of \( \alpha \) and \( g \). \( \Box \)

The definitions of proper, free and commuting partial actions on bundles are the following ones.

**Definition 2.5.** Given an upper semicontinuous \( C^* \)-bundle over a LCH space and a partial action of a LCH group on the bundle, we say the partial action is free, proper, has closed graph or has closed domain if the partial action on the base space has the respective property. Similarly, given two partial actions on an upper semicontinuous \( C^* \)-bundle we say they commute if the partial actions on the total and base space commute.

Relating the concepts of enveloping action, in the contexts of \( C^* \)-algebras and bundles, we have the following result.

**Theorem 2.6.** Let \( \alpha \) be a partial action of \( H \) on the upper semicontinuous \( C^* \)-bundle \( B = \{ B_x \}_{x \in X} \). If \( X \) is LCH, \( \alpha \) has closed graph, \( \alpha^e \) is the enveloping action of \( \alpha \) and \( B^e \) the enveloping bundle, then \((C_0(B^e), H, \alpha^e)\) is the enveloping system of \((C_0(B), H, \alpha)\) (**Definition 2.3 of [2]**). So \( C_0(B) \rtimes_{\alpha} H \) is a hereditary and full sub \( C^* \)-algebra of \( C_0(B^e) \rtimes_{\alpha^e} H \). In particular those crossed products are strongly Morita equivalent.

**Proof.** By Theorem 2.3 we may suppose \( X \subset X^e \), \( B \subset B^e \), \( p^e(B) = X \) and that \( p^e \) is the restriction of \( p^e \) to \( B \). Now, by Lemma 1.6 \( X^e \) is LCH. This considerations allows us to identify the bundle \( B \) with the restriction of \( B^e \) to \( X \), which gives \( C_0(B) = C_0(B^e, X) \). We have identified \( C_0(B) \) with an ideal of \( C_0(B^e) \).

We also have, for every \( t \in H \),

\[
\tilde{\alpha}^e_i(C_0(B^e, X)) \cap C_0(B^e, X) = C_0(B^e, t \cdot X) \cap C_0(B^e, X) = C_0(B^e, X \cap t \cdot X) = C_0(B^e, X_t) \cap C_0(B^e)_{i_t};
\]

and clearly the restriction of \( \tilde{\alpha}^e_i \) to \( C_0(B)_{i_t} \) equals \( \tilde{\alpha}_i \). So, using Corollary 1.3 of [3], the only thing that remains to be showed is that the space generated by the \( \alpha^e \)-orbit of \( C_0(B) \) is dense in \( C_0(B^e) \).

We show every continuous section with compact support of \( B^e \) is a finite sum of points in the orbit of \( C_0(B) \). Fix \( f \in C_0(B^e) \). The support of \( f \) has an open cover by sets of the form \( t \cdot X \), \( t \) varying in \( H \). We can find \( t_1, \ldots, t_n \in H \) and \( h_1, \ldots, h_n \in C_c(X^e) \) such that: \( 0 \leq h_1 + \cdots + h_n \leq 1 \), the support of \( h_i \) is contained
in \( t_i \cdot X \) \((i = 1, \ldots, n)\) and \( h_1(x) + \cdots + h_n(x) = 1 \) if \( x \in \text{supp}(f) \). Defining \( f_i(x) = h_i(x)f(x) \) \((i = 1, \ldots, n)\), we have \( f = f_1 + \cdots + f_n \) and \( g_i := \tilde{\alpha}^{-1} t_i \cdot (f_i) \in C_c(B) \) for every \( i = 1, \ldots, n \). Besides, \( f = \tilde{\alpha} t_1 (g_1) + \cdots + \tilde{\alpha} t_n (g_n) \), that gives the desired result. \( \square \)

We can reproduce most of the results of Section 1 in this context. For example, the next Theorem is a direct consequence of Lemma 1.7.2.

**Theorem 2.7.** Given an upper semicontinuous \( C^\ast \)-bundle \( B \) and commutative partial actions, \((\alpha, \cdot)\) and \((\beta, \star)\) of \( H \) and 

\( K \) on \( B \), respectively, the pair \((\alpha, \cdot) \times (\beta, \star) := (\alpha \times \beta, \cdot \times \star)\) is a partial action of \( H \times K \) on \( B \).

Writing \( \alpha = (\alpha, \cdot) \) and \( \beta = (\beta, \star) \), the product \( \alpha \times \beta \) is the one defined in the previous Theorem.

### 2.1. Orbit bundle.

There is a notion of “orbit bundle”, analogous to the notion of “orbit space”, but to construct it we have to consider proper and free partial actions.

Fix an upper semicontinuous \( C^\ast \)-bundle over a LCH space, \( B = \{B_x\}_{x \in X} \), and a proper and free partial action, \( \alpha \), of \( H \) on \( B \). Let \( B/H \) and \( X/H \) be the orbit spaces and \( \pi_B : B \to B/H \) and \( \pi_X : X \to X/H \) be the orbit maps. As \( p \) is a morphism of partial actions, there is a unique continuous (also open and surjective) function \( p_o : B/H \to X/H \) such that \( \pi_X \circ p = p_o \circ \pi_B \).

We want to equip \( B/H \) with operations making \( B/H := (B/H, X/H, p_o) \) an upper semicontinuous \( C^\ast \)-bundle. To do this first notice that, given \( Hx \in X/H \), the fiber \( (B/H)_{xH} \) is homeomorphic to \( B_x \) trough the restriction of \( \pi_B \) to \( B_x \) (because the partial action on \( X \) is free). Call that map \( h_x : B_x \to (B/H)_{xH} \). Define the structure of \( C^\ast \)-algebra of \( (B/H)_{xH} \) in such a way that \( h_x \) is an isomorphism of \( C^\ast \)-algebras. This definition is independent of the choice of \( x \) because, if \( Hx = Hy \), then \( h_y \circ h_x : B_x \to B_y \) is an isomorphism. As it is the restriction of \( \alpha_t \) to \( B_x \), \( t \in H \) being the unique such that \( x \in X_t \), and \( t \cdot x = y \).

The norm of \( B/H \) is the function \( \| \| : B/H \to \mathbb{R}, Hx \mapsto \| a \| \). To prove it is upper semicontinuous let \( \varepsilon \) be a positive number. The set \( \{Hb \in B/H | \| Hb \| < \varepsilon\} \) is open because it equals the open set \( \pi_B^{-1}(\{b \in B | \| b \| < \varepsilon\}) \). Similarly, we prove \( \| \| : B/H \to \mathbb{R} \) is continuous if \( \| \| : B \to \mathbb{R} \) is continuous.

To prove the continuity of the product and the sum name \( D \) the set of points \((a, b) \in B \times B \) such that \( Ha = Hb \). If \((a, b) \in D \) there is a unique \( t \in H \), which we name \( t(p(a), p(b)) \), such that \( p(a) \in X_t \), and \( t \cdot p(a) = p(b) \). Hence, \( \alpha_t(a) \) and \( b \) are in the same fiber, we define \( S(a, b) := \alpha_t(a) + b \) and \( P(a, b) := \alpha_t(a)b \).

To prove the continuity of \( S \) and \( P \) we only have to prove the continuity of the function

\[
F : \{(x, y) \in X \times X : Hx = Hy\} \to \Gamma(X, \alpha), \quad F(x, y) = (t(x, y), x).
\]

Call \( D_X \) the domain of \( F \).

Consider the function \( R : \Gamma(X, \alpha) \to X \times X \) given by \( R(t, x) = (x, t \cdot x) \), this is a continuous, proper and injective function between LCH spaces. Such functions are homeomorphisms over its image, but the image of \( R \) is \( D_X \) and \( F = R^{-1} \). So, \( F \) is continuous.

Once we have proved the continuity of \( S \) and \( P \), using the freeness of the partial action on \( X \), we prove they are constant in the classes of the equivalence relation
Example 2.9. This implies \( H_{\alpha} = H_{\gamma} \).

The last step, to show \( B/H \) is an upper semicontinuous \( C^* \)-bundle, is to prove it satisfies the following property: for every net \( \{ b_i \}_{i \in I} \subset B/H \) such that \( \| b_i \| \to 0 \) and \( p_\alpha(b_i) \to z \), for some \( z \in X/H \), we have \( b_i \to 0_z \).

Let \( \{ b_i \}_{i \in I} \) be a net as before, it suffices to show it has a subnet converging to \( 0_z \). There is a net in \( B \), \( \{ a_i \}_{i \in I} \), such that \( b_i = H a_i \) for every \( i \in I \). We have that \( H p(a_i) = p_\alpha(b_i) \to H x \), where \( x \in X \) is such that \( H x = z \). As the orbit map \( X \to X/H \) is open and surjective, Proposition 13.2 Chapter II of [7] implies there is a subnet \( \{ a_{i_j} \}_{j \in J} \) and a net \( \{ t_j \}_{j \in J} \subset H \) such that \( p(a_{i_j}) \in X_{t^{-1}_j} \) and \( t_j \cdot p(a_{i_j}) \to x \). This implies \( a_{i_j} \in \eta^{-1}_j B \) and \( p(\alpha_{t_j})(a_{i_j}) \to x \). But also \( \| \alpha_{t_j}(a_{i_j}) \| = \| b_{i_j} \| \to 0 \), so, \( \alpha_{t_j}(a_{i_j}) \to 0_x \). Finally, as \( b_{i_j} = H \alpha_{t_j}(a_{i_j}) \), \( \pi_B(0_x) = H 0_x = 0_z \) and \( \pi_B \) is continuous, \( \alpha \) is a limit point of \( \{ b_{i_j} \}_{j \in J} \).

**Definition 2.8.** The orbit bundle of \( B \) by \( \alpha \) is the upper semicontinuous \( C^* \)-bundle \( B/H \) constructed before.

**Example 2.9.** Consider the situation of Example 2.2, where the action of \( H \) on \( A \) is the trivial one \( \gamma_t = \text{id}_A \) for every \( t \in H \) and the system \( (X, H, \cdot) \) is free and proper. Then the quotient bundle \( B/H \) is isomorphic to the trivial bundle \( A \times X/H \).

Our next goal is to identify \( C_0(B/H) \) with a \( C^* \)-sub algebra of \( C_0(B) \). Every function \( f \in C_0(B) \), which is also a morphism of partial actions, induces a continuous and bounded section \( \text{Ind}_{\alpha}(f) : X/H \to B/H \), given by \( H x \mapsto H f(x) \).

The induced algebra \( \text{Ind}_{\alpha}(B, \alpha) \) is the subset of \( C_0(B) \) formed by all the sections which are also morphism of partial actions. There is a natural map

\[
\text{Ind}_{\alpha} : \text{Ind}_{\alpha}(B, \alpha) \to C_0(B/H), \quad f \mapsto \text{Ind}_{\alpha}(f).
\]

Similarly, the algebra \( \text{Ind}_0(B, \alpha) \) is the pre image of \( C_0(B/H) \) under \( \text{Ind}_{\alpha} \). The function \( \text{Ind}_{\alpha} \) is simply the restriction of \( \text{Ind}_{\alpha} \) to \( \text{Ind}_{\alpha}(B, \alpha) \).

In fact, the induced algebras are \( C^* \)-sub algebras of \( C_0(B) \). To prove this it suffices to show \( \text{Ind}_{\alpha}(B, \alpha) \) is a \( C^* \)-sub algebra and to notice \( \text{Ind}_{\alpha} \) is a morphism of \( C^* \)-algebras.

The non trivial fact is that \( \text{Ind}_{\alpha}(B, \alpha) \) is closed in \( C_0(B) \). Assume \( \{ f_n \}_{n \in \mathbb{N}} \) is a sequence contained in \( \text{Ind}_{\alpha}(B, \alpha) \) converging to \( f \). Choose some \( t \in H \) and \( x \in X_{t^{-1}} \). Even if \( B \) is not Hausdorff, \( B_x \) and \( B_{t \cdot x} \) are, so we have the following equalities

\[
\alpha_t(f(x)) = \lim_n \alpha_t(f_n(x)) = \lim_n f_n(t \cdot x) = f(t \cdot x).
\]

**Theorem 2.10.** The functions

\[
\text{Ind}_{\alpha} : \text{Ind}_{\alpha}(B, \alpha) \to C_0(B/H) \quad \text{and} \quad \text{Ind}_0 : \text{Ind}_0(B, \alpha) \to C_0(B/H)
\]

are isomorphism of \( C^* \)-algebras.

**Proof.** The only thing to prove is that \( \text{Ind}_{\alpha} \) is surjective (it is injective because is an isometry). Fix \( g \in C_0(B) \), we will construct \( f \in \text{Ind}_{\alpha}(B, \alpha) \) such that \( \text{Ind}_{\alpha}(f) = g \).
As the action on the base space is free, for every \( x \in X \) there is a unique \( f(x) \in B_x \) such that \( Hf(x) = g(Hx) \). Clearly \( f \) is a bounded section.

To prove \( f \) is continuous, let \( \{x_i\}_{i \in I} \) be a net contained in \( X \) converging to \( x \in X \). It suffices to find a subnet \( \{x_{i_j}\}_{j \in J} \) such that \( f(x_{i_j}) \rightarrow f(x) \). By the continuity of \( g \) the net \( \{Hf(x_{i_j})\}_{j \in J} \) has \( Hf(x) \) as a limit point. As the orbit map \( B \rightarrow B/H \) is open, there is a subnet \( \{x_{i_j}\}_{j \in J} \) and a net \( \{t_j\}_{j \in J} \) such that \( \{(t_j, f(x_{i_j}))\}_{j \in J} \subset \Gamma(B, \alpha) \) and \( \alpha_{t_j}(f(x_{i_j})) \rightarrow f(x) \). This implies \( \{(t_j, x_{i_j})\}_{j \in J} \subset \Gamma(X, \alpha) \) and \( t_j \cdot x_{i_j} \rightarrow x \). Then \( t = \lim t_{i_j} \cdot x_{i_j} \rightarrow t(x, x) = e \) (see the construction of the orbit bundle in Section 2.1). Finally, the net \( \{f(x_{i_j})\}_{j \in J} \), being equal to \( \{\alpha_{t_j}^{-1} \alpha_{t_j}(f(x_{i_j}))\}_{j \in J} \), has \( f(x) \) as a limit point.

It remains to prove \( f \) is a morphism of partial actions. Clearly \( f(X_t) \subset X_t \) for every \( t \in H \). Now take \( t \in H \) and \( x \in X_{t^{-1}} \). The points \( f(t \cdot x) \) and \( \alpha_t(f(x)) \) are, both, the unique point of \( B_{t^{-1}x} \) in the class of \( g(Hx) \), so they are equal. \( \square \)

Theorem 2.11. Let \( B = \{B_x\}_{x \in X} \) be an upper semicontinuous \( C^* \)-bundle over a LCH space and \( \alpha \) and \( \beta \) be partial actions of \( H \) and \( K \) on \( B \), respectively. If \( \alpha \) is free and proper, then \( \text{Ind}_0(B, \alpha, K, \hat{\beta}) \) is a \( C^* - \text{PDS} \) where

1. \( \text{Ind}_0(B, \alpha, K, \hat{\beta}) := \{ f \in \text{Ind}_0(B, \alpha) : x \mapsto \|f(x)\| \text{ vanishes outside } X_t^H \}. \)
2. For every \( f \in \text{Ind}_0(B, \alpha, K, \hat{\beta}) \), \( \hat{\beta}_i(f)(x) = \beta_i(f(t^{-1} \cdot x)) \) if \( x \in X^K_{t^{-1}} \) and \( 0_x \) otherwise.

Proof. Let \( B/H \) be the orbit bundle. As \( \alpha \) commutes with \( \beta \), using Lemmas 1.8 and 1.11 we define a partial action, \( \mu \), of \( K \) on \( B/H \).

By Theorem 2.4 \( \mu \) defines a \( C^* \)-PDS \( (C_0(B/H), K, \hat{\mu}) \). Lemma 2.10 ensures the map \( \text{Ind}_0 : \text{Ind}_0(B, \alpha) \rightarrow C_0(B/H) \) is an isomorphism. Notice \( \text{Ind}_0(B, \alpha, K, \hat{\beta}) \) is the pre image of \( C_0(B/H) \). The partial action of the thesis is the unique making \( \text{Ind}_0 : \hat{\beta} \rightarrow \hat{\mu} \) an isomorphism of partial actions. \( \square \)

3. Morita equivalence

In our last section we prove our main theorem, which is a generalization of Raeburn’s and Green’s Symmetric Imprimitivity Theorems [10, 11]. The first task is to translate Raeburn’s result to the language of actions on bundles.

Consider two \( C^* \)-DS \( (A, H, \gamma) \) and \( (A, K, \delta) \), and two proper and free DS \( (X, H, \cdot) \) and \( (X, K, \star) \). Assume also that the actions on \( A \) and \( X \) commute. On the trivial bundle \( B = A \times X \) define the actions of \( H \) and \( K \) as in Example 2.2 call them \( \alpha \) and \( \beta \), respectively.

Let \( \text{Ind} \gamma \) be the induced \( C^* \)-algebra defined as in [10]. We have an isomorphism \( \rho : \text{Ind} \gamma \rightarrow \text{Ind}_0(B, \alpha) \), given by \( \rho(f)(x) = (f(x), x) \). This isomorphism takes the action of \( K \) on \( \text{Ind} \gamma \) (as defined on [10]) into the action \( \hat{\beta} \). By using Raeburn’s Theorem we conclude that \( \text{Ind}_0(B, \alpha, K, \hat{\beta}) \) is strongly Morita equivalent to \( \text{Ind}_0(B, \beta) \times_{\alpha} H \). Our purpose is to give a version of this result for partial actions. We will write \( A \sim_M B \) whenever \( A \) and \( B \) are strongly Morita equivalent \( C^* \)-algebras [12].

3.1. The main Theorem. From now on we work with two LCH topological groups, \( H \) and \( K \), an upper semicontinuous \( C^* \)-bundle with LCH base space, \( B = \{B_x\}_{x \in X} \), and two continuous, free, proper and commuting partial actions, \( \alpha \) and \( \beta \), of \( H \) and \( K \) on \( B \), respectively.
We want to give conditions under which we can say that $\text{Ind}_0(B, \alpha) \rtimes_\beta K$ is strongly Morita equivalent to $\text{Ind}_0(B, \beta) \rtimes_\gamma H$. For global actions, with some additional hypotheses on the group and the base space, this is proved in [4], [9] or [10]. In fact, the proof of the next Theorem is a minor modification of Raeburn’s proof of the Symmetric Imprimitivity Theorem [10].

**Theorem 3.1.** If $\alpha$ and $\beta$ are global actions then

$$\text{Ind}_0(B, \alpha) \rtimes_\beta K \sim_M \text{Ind}_0(B, \beta) \rtimes_\gamma H.$$ 

**Proof.** Define $E := C_0(H, \text{Ind}_0(B, \beta))$ and $F := C_0(K, \text{Ind}_0(B, \alpha))$, viewed as dense $*$-sub-algebras of the respective crossed products. Define also $Z := C_c(B)$, which will be an $E \otimes F$-bimodule with inner products; whose completion implements the equivalence between $\text{Ind}_0(B, \alpha) \rtimes_\beta K$ and $\text{Ind}_0(B, \beta) \rtimes_\gamma H$.

For $f, g \in Z$, $b \in E$ and $c \in F$ define

$$b \cdot f(x) := \int_H b(s)(x)\bar{\alpha}_s(f)(x)d\Delta_H(s)^{1/2}ds,$$

$$f \cdot c(x) := \int_K \tilde{\beta}_t(fc(t^{-1}))(x)d\Delta_K(t)^{-1/2}dt,$$

$$E(f, g)(s)(x) := \Delta_H(s)^{-1/2} \int_K \tilde{\beta}_t(f\bar{\alpha}_s(g^*)(x))dt,$$

$$\langle f, g \rangle_E(t)(x) := \Delta_K(t)^{-1/2} \int_H \bar{\alpha}_s \left( f^* \tilde{\beta}_t(g) \right) (x)ds.$$

The integration is with respect to left invariant Haar measures; $\Delta_H$ and $\Delta_K$ are the modular functions of the groups. Here $\bar{\alpha}$ and $\tilde{\beta}$ are the partial actions defined on Theorem [2,3].

We now justify the fact that $b \cdot f \in Z$. The function $H \to C_0(B)$, given by $s \mapsto b(s)\bar{\alpha}_s(f)\Delta_H(s)^{1/2}$, is continuous (Theorem [2,3]). Besides, it’s support is contained in the support of $b$ and so we can integrate it. This integral is exactly $b \cdot f$. Finally, notice $\text{supp}(b \cdot f) \subset \{ s \cdot x : (s, x) \in \text{supp}(b) \times \text{supp}(f) \}$, the last being a compact set.

To prove (3.3) defines an element of $E$ we proceed as follows. Fixed $s \in H$ and $x \in X$ the function $K \to B_x$, given by $t \mapsto \tilde{\beta}_t(f\bar{\alpha}_s(g^*))$ is continuous with support contained in the compact $\{ t \in K : t^{-1} \star x \in \text{supp}(f) \}$. So, the function is integrable. The value of that integral is $E(f, g)(s)(x)$.

We now prove $E(f, g)$ is continuous, what we do locally. Fix some $s_0 \in H$ and $x_0 \in X$. Take compact neighbourhoods, $V$ of $s_0$ and $W$ of $x_0$. The bundle $B_W$ will be the restriction of $B$ to $W$. Define the function $F : V \times H \to C(B_W)$ by $F(s, t)(x) = \tilde{\beta}_t(f\bar{\alpha}_s(g^*))(x)$. As the action of $K$ on $X$ is proper, $F$ has compact support. By integrating, with respect to the second coordinate, we get a continuous function $R \in C(V, C(B_W))$, defined by $R(s) = \int_K F(s, t)d\mu_K(t)$ (II.15.19).

Fixing $(s, x) \in V \times W$, we have $R(s)(x) = E(f, g)(s)(x)$. From this follows the continuity of $E(f, g)$.

An easy calculation shows $E(f, g)(s)(t \cdot x) = \beta_t(E(f, g)(s)(x))$, for every $t \in K$ and $x \in X$. Besides, if $E(f, g)(s)(x) \neq 0_x$, then $x$ belongs to the $K$-orbit of $\text{supp}(f)$, and $s$ to the compact set $\{ s \in H : s \cdot \text{supp}(g) \cap \text{supp}(f) \neq \emptyset \}$. We have proved that $E(f, g) \in E$.

The computations needed to prove equations (3.1)-(3.4) define an equivalence bi-module are the same as in [10] or [13]. For the construction of the approximate
unit, analogous to that of Lemma 1.2 of [10], follow the proof of Proposition 4.5 of [13], recalling that $\text{Ind}^B_H(A, \beta)$ plays the role of our $\text{Ind}^B_0(\mathbb{B}, \beta)$.

Our next step is to let $\alpha$ and $\beta$ to be partial, but to have the same result we need additional hypotheses, which are trivially satisfied in the previous case.

Let $\alpha \times \beta$ be the partial action given by Theorem 2.7. Now, by Theorem 2.8, we have an enveloping action $(\alpha \times \beta)^e$ and an enveloping bundle $\mathbb{B}^e$. We can assume $\mathbb{B}$ is the restriction of $\mathbb{B}^e$ to $X \subset X^e$.

For the action given by $(\alpha \times \beta)^e$ on $X^e$ we will use the notation $(s,t)x$, for $(s,t) \in H \times K$ and $x \in X^e$.

Define $\sigma$ and $\tau$ as the restriction of $(\alpha \times \beta)^e$ to $H$ and $K$, respectively (identify $H$ with $H \times \{e\} \subset H \times K$). It is immediate that $(\alpha \times \beta)^e = \sigma \times \tau$, $\sigma$ and $\tau$ commute, and that $(\alpha \beta)$ is the restriction of $\sigma \tau$ to $\mathbb{B}$.

The next is the main Theorem of this article.

**Theorem 3.2.** If $\alpha \times \beta$ has closed graph and $\sigma$ and $\tau$ are proper then

$$\text{Ind}^B_0(\mathbb{B}, \alpha) \rtimes \tilde{\beta} K \sim \text{Ind}^B_0(\mathbb{B}, \beta) \rtimes \tilde{\sigma} H.$$

**Proof.** To show that $\sigma$ (and also $\tau$) is free. Assume $(s,e)x = x$ for some $s \in H$ and $x \in X^e$. As $X^e$ is the $H \times K$-orbit of $X$, there exists $(h,k) \in H \times K$ such that $(h,k)x \in X$. Notice that $(hsh^{-1}, e)(h,k)x = (h,k)x \in X \cap (hsh^{-1}, e)^{-1}X$, so, $hsh^{-1} \cdot (h,k)x = (h,k)x$ and $hsh^{-1} = e$. We conclude $s = e$.

As $\alpha \times \beta$ has closed graph, $\mathbb{B}^e$ is an upper semicontinuous $C^*$-bundle over a LCH space. The hypotheses, together with Theorems 2.11 and 3.1, imply $\text{Ind}^B_0(\mathbb{B}^e, \sigma) \rtimes \tilde{\tau} K$ is strongly Morita equivalent to $\text{Ind}^B_0(\mathbb{B}^e, \tau) \rtimes \tilde{\sigma} H$. The proof of our Theorem will be completed if we can show that $\text{Ind}^B_0(\mathbb{B}, \alpha) \rtimes \tilde{\beta} K$ is strongly Morita equivalent to $\text{Ind}^B_0(\mathbb{B}^e, \sigma) \rtimes \tilde{\sigma} K$, because, by symmetry, the same will hold changing $\alpha$ for $\beta$, $\sigma$ for $\tau$ and $H$ for $K$.

Tracking back the construction of $\tilde{\beta}$ and $\tilde{\tau}$, to Theorem 2.11, we notice that $\text{Ind}^B_0(\mathbb{B}, \alpha) \rtimes \tilde{\beta} K$ is isomorphic to $\text{Ind}^B_0(\mathbb{B}^e, \sigma) \rtimes \tilde{\tau} K$ isomorphic to $\text{Ind}^B_0(\mathbb{B}^e/H) \rtimes \tilde{\sigma} K$. Here $\mu$ and $\nu$ are the partial actions of $K$ on $\mathbb{B}/H$ and $\mathbb{B}^e/H$ given by Lemma 1.3, respectively. Meanwhile, $\tilde{\mu}$ and $\tilde{\nu}$ are the one given by Theorem 2.4. Putting all together, by Theorem 2.6, it suffices to prove $\nu$ is the enveloping action of $\mu$.

Consider the map $B \to \mathbb{B}^e/H$, given by $b \mapsto Hb$. This is an open and continuous map, it is also constant in the $\alpha$-orbits. So it defines a unique function $F : \mathbb{B}/H \to \mathbb{B}^e/H$, given by $Hb \mapsto Hb$ (this is not the identity map). It turns out this function is continuous, open, injective and maps fibers into fibers. In an analogous way we define $f : X/H \to X^e/H$, which has the same topological properties.

Recalling the construction of $\mu$ and $\nu$, it is easy to show $(\mu, f) : \mu \to \nu$ is a morphism. To show that $\mu^e = \nu$ it suffices to prove only two things. Namely, that $f((X/H)_{t}) = f(X/H) \cap t f(X/H)$ for every $t \in K$ (we adopted the notation $tz$ for the action of $t$ in $K$ on $z \in X^e$) and that the $K$-orbit of $f(X/H)$ is $X^e/H$.

For the first one notice that

$$f((X/H)_{t}) = HX^K_t = HX \cap H(e,t)X = HX \cap tHX = f(X/H) \cap t f(X/H).$$

The second equality of the previous formula is not immediate, but the inclusion $\subset$ is. For the other one assume $y \in HX \cap H(e,t)X$. Then there exists $x, z \in X$ such that $y = Hx = H(e,t)z$. There is some $s \in H$ such that $x = (s,e)(e,t)z = (s,t)z$. So $x \in X \cap (s,t)X = s \cdot (X^e_s \cap X^K_t)$, $(s^{-1}, e)x \in X^K_t$ and $y = K(s^{-1}, e)x \in HX^K_t$. 


To show $X^e/H$ is the $K$–orbit of $f(X/H)$, notice that
\[
\bigcup_{t \in K} tf(X/H) = \bigcup_{t \in K} tHX = H \bigcup_{s \in H} \bigcup_{t \in K} (s,t)X = HX^e = X^e/H.
\]

We have proved $\nu$ is the enveloping action of $\mu$, by Theorem 2.6 $C_0(B/H) \times_{\mu} K$ is strongly Morita equivalent to $C_0(B^e/H) \times_{\nu} K$. This completes the proof of our main theorem. $\square$

The next Theorem is a consequence of the previous one, it has the advantage of not making any mention to $\sigma$ nor $\tau$.

**Theorem 3.3.** If $\alpha$ and $\beta$ have closed domain then
\[
\text{Ind}_0(B, \alpha) \times_{\beta} K \sim_M \text{Ind}_0(B, \beta) \times_{\bar{\beta}} H.
\]

**Proof.** We check the hypotheses of the previous theorem are satisfied. To show $\alpha \times \beta$ has closed graph notice it has closed domain (Lemma 1.6) and use Lemma 1.8. Finally, we only have to show $\sigma$ and $\tau$ are proper. It is enough to show $\sigma$ is proper, for that purpose we use Lemma 1.9.

Let $\{(s_i, x_i)\}$ be a net in $H \times X^e$ such that $\{(x_i, (s_i, e)x_i)\}$ converges to the point $(x, y) \in X^e \times X^e$. It is enough to show $\{s_i\}_{i \in I}$ has a converging subnet. We may assume $(s, t)x \in X$ and $(h, k)y \in X$, for some $(h, k), (s, t) \in H \times K$.

There is an $i_0$ such that, for $i \geq i_0$, $(s, t)x_i$ and $(h, k)(s_i, e)x_i$ belong to $X$. For $i \geq i_0$ define $u_i = (s, t)x_i$. By the construction of $\alpha \times \beta$ and because $(hs_is^{-1}, kt^{-1})u_i \in X$, we have that $u_i$ is an element of the clopen set $tk^{-1} \ast (X^e_{K \setminus e} \cap X^e_{s_is^{-1}h^{-1}})$. Defining $v_i := (e, k^{-1}t)u_i$ for every $i \geq i_0$, we have that $v_i \in X$. So, the limit $\lim_i v_i$ is an element of $X$ (recall $X$ is clopen in $X^e$).

The net $\{(hs_is^{-1}, v_i)\}_{i}$ is contained in $\Gamma(X, \alpha)$ and $\{(v_i, hs_is^{-1}\cdot v_i)\}_{i}$ has a limit point. Then $\{hs_is^{-1}\}_{i}$ has a converging subnet, an so $\{s_i\}_{i}$ has a converging subnet. We conclude $\sigma$ is proper, and we are done. $\square$

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