ALMOST 2-UNIVERSAL DIAGONAL QUINARY QUADRATIC FORMS

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Abstract. A (positive definite integral) quadratic form is called almost 2-universal if it represents all (positive definite integral) binary quadratic forms except those in only finitely many equivalence classes. Oh [7] determined all almost 2-universal quinary diagonal quadratic forms remaining three as candidates. In this article, we prove that those three candidates are indeed almost 2-universal.

1. Introduction

M.-H. Kim and his collaborators proved in [4] that there are exactly 11 quinary 2-universal quadratic forms. Hwang [3] proved that there are exactly 3 quinary diagonal quadratic forms that represents all binary quadratic forms except only one. Oh [7] proved that there exist only finitely many quinary quadratic forms that represent all but at most finitely many equivalence classes of binary quadratic forms. Such quadratic forms are called almost 2-universal quadratic forms. And he provided a list of almost 2-universal quinary diagonal quadratic forms, including 3 unconfirmed candidates. In this article, we show that those 3 candidates are indeed almost 2-universal.

2. Preliminaries and tools

We adopt lattice theoretic language. Let $\mathbb{Q}$ be the rational number field. For a prime (including $\infty$), let $\mathbb{Q}_p$ be the fields of $p$-adic completions of $\mathbb{Q}$, in particular $\mathbb{Q}_\infty = \mathbb{R}$, field of real numbers. For a finite prime $p$, $\mathbb{Z}_p$ denotes the $p$-adic integer ring. Let $R$ be the ring of integers $\mathbb{Z}$ or the ring of $p$-adic integers $\mathbb{Z}_p$. An $R$-lattice $L$ is a free $R$-module of finite rank equipped with a non-degenerate symmetric bilinear form $B : L \times L \to R$. The corresponding quadratic map is denoted by $Q$. For a $R$-lattice $L = R e_1 + R e_2 + \cdots + R e_n$ with basis $e_1, e_2, \cdots, e_n$, we write

$$L = (B(e_i, e_j)).$$

For $R$-sublattices $L_1, L_2$ of $L$, we write $L = L_1 \perp L_2$ when $L = L_1 \oplus L_2$ and $B(v_1, v_2) = 0$ for all $v_1 \in L_1, v_2 \in L_2$. If $L$ admits an orthogonal basis $\{e_1, e_2, \cdots, e_n\}$, we call $L$ diagonal and simply write

$$L = \langle Q(e_1), Q(e_2), \cdots, Q(e_n) \rangle.$$

We call $L$ non-diagonal otherwise. Define the discriminant $dL$ of $L$ to be the determinant of the matrix $(B(e_i, e_j))$. Note that $dL$ is independent of the choice of a basis up to unit squares of $R$. We define scale $sL$ of $L$ to be the ideal of $R$ generated by $B(v, w)$ for all $v, w \in L$, norm $nL$ of $L$ to be the ideal of $R$ generated
by \(Q(v)\) for all \(v \in L\). For \(a \in R^\times\), we denote by \(L^a\) the \(R\)-lattice obtained from scaling \(L\) by \(a\).

Let \(\ell, L\) be \(R\)-lattices. We say \(L\) represents \(\ell\) if there is an injective linear map from \(\ell\) into \(L\) that preserves the bilinear form, and write \(\ell \rightarrow L\). Such a map will be called a representation. A representation is called isometry if it is surjective. We say two \(R\)-lattices \(L, K\) are isometric if there is an isometry between them, and write \(L \cong K\).

For a \(\mathbb{Z}\)-lattice \(L\) and a prime \(p\), we define the \(\mathbb{Z}_p\)-lattice \(L_p := \mathbb{Z}_p \otimes L\) and call it the localization of \(L\) at \(p\). The set of all \(\mathbb{Z}\)-lattices that are isometric to \(L\) is called the class of \(L\), denoted by \(\text{cls}(L)\). The set of all \(\mathbb{Z}\)-lattices \(K\) such that \(L_p \cong K_p\) for all prime spots \(p\) (including \(\infty\)) is called genus of \(L\), denoted by \(\text{gen}(L)\). The number of non-isometric classes in \(\text{gen}(L)\) is called the class number of \(L\), denoted by \(h(L)\). The following two properties are well-known. (see \([8, 103, 102:5]\))

1. \(h(L)\) is finite for any \(\mathbb{Z}\)-lattice \(L\).
2. For two lattices \(\ell, L\), if \(\ell_p \rightarrow L_p\) for all prime \(p\) (including \(\infty\)), then \(\ell \rightarrow K\) for some lattice \(K \in \text{gen}(L)\).

For a \(\mathbb{Z}\)-lattice \(L\), we say that \(L\) is positive definite or simply positive if \(Q(v) > 0\) for any \(v \in L, v \neq 0\). Let \(L\) be a positive \(\mathbb{Z}\)-lattice. \(L\) is called \(n\)-universal if \(L\) represents all \(n\)-ary positive \(\mathbb{Z}\)-lattices. And \(L\) is called almost \(n\)-universal if \(L\) represents all \(n\)-ary positive \(\mathbb{Z}\)-lattices except those in only finitely many equivalence classes. For a fixed prime \(p\), \(L\) is called \(n\)-universal over \(\mathbb{Z}_p\) if its localization \(L_p\) represents all \(n\)-ary \(\mathbb{Z}_p\)-lattices. And \(L\) is called locally \(n\)-universal if it is \(n\)-universal over \(\mathbb{Z}_p\) for all primes \(p\).

We will denote for convenience

\[
[a, b, c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}.
\]

Any unexplained notations and terminologies can be found in \([5]\) or \([8]\).

Now we provide a technique for representations of binary \(\mathbb{Z}\)-lattices by certain quinary \(\mathbb{Z}\)-lattices, which is based on the proof of the main theorems in \([3]\) and \([4]\). Let \(\ell = [a, b, c]\) be a binary \(\mathbb{Z}\)-lattice. For any integers \(n, s, t\), we define

\[
\ell_{s, t}^n := \begin{pmatrix} a - ns^2 & b - nst \\ b - nst & c - nt^2 \end{pmatrix}.
\]

Let \(M\) be a \(\mathbb{Z}\)-lattice. It can be verified that \(\ell \rightarrow M \perp \langle n \rangle\) if and only if there exist integers \(s, t\) such that \(\ell_{s, t}^n \rightarrow M\). If the class number of the lattice \(M\) is one, we can classify all binary lattices which are represented by \(M\) using the local representation theory. In this thesis, we only consider the case that \(M\) is quaternary. Let \(p\) be a prime such that \(p \nmid 2dM\). Then \(M_p \cong \langle 1, 1, 1, dM\rangle\). If \(dM\) is a square in \(\mathbb{Z}_p\), then \(M\) is \(2\)-universal over \(\mathbb{Z}_p\). If \(dM\) is not a square, then \(\ell_p \rightarrow M_p\) if and only if \(\ell_p\) is not isometric to any sublattices of the \(\mathbb{Z}_p\)-lattice \((p, -p\Delta)\), where \(\Delta\) is a non-square unit in \(\mathbb{Z}_p\). In particular, \(\ell \notin p\mathbb{Z}\) implies that \(\ell_p \rightarrow M_p\).

Let \(\mathfrak{P}\) be the set of primes \(p\) such that \(\frac{dM}{p}\) has no prime factors in \(\mathfrak{P}\). Then the scale of \(\ell_{s, t} = [a - ns^2, b - nst, c - nt^2]\) is not contained in \(p\mathbb{Z}\) for any prime \(p \in \mathfrak{P}\). Thus we may only consider the \(\mathbb{Z}_p\)-structure for primes \(p \mid 2dM\).
Consider the case that \( n \) has a prime factor \( q \in \mathcal{P} \), which is a hard case. If \( s\ell \subseteq q\mathbb{Z} \), then \( s\ell_{s,t} \subseteq q\mathbb{Z} \) for all \( s,t \). For this reason, we have to prove this case separately.

Suppose that \( \ell^n_{s,t} \to M \) over \( \mathbb{Z}_p \) for all primes \( p \). If \( \ell_{s,t} \) is positive, then we conclude that \( \ell^n_{s,t} \to M \), and that \( \ell \to M \perp \langle n \rangle \). Following lemma says that \( \ell^n_{s,t} \) is positive for sufficiently large \( a \).

**Lemma 2.1.** Let \( \ell = [a,b,c] \) be a Minkowski reduced binary \( \mathbb{Z} \)-lattice, that is, \( 2|b| \leq a \leq c \). If \( a > \frac{4}{3}n(s^2 + |st| + t^2) \), then \( \ell^n_{s,t} \) is positive.

**Proof.**
\[
d \ell^n_{s,t} = ac - b^2 - ns^2c + 2nstb - nt^2a = \frac{1}{4}ac - b^2 + \frac{3}{4}ac - n(s^2c - 2stb + t^2a) \\
\geq 0 + \frac{3}{4}ac - n(s^2c + |st| + t^2c) = \frac{3}{4}c \left( a - \frac{4}{3}n(s^2 + |st| + t^2) \right) > 0. \quad \Box
\]

3. **Main results**

**Theorem 3.1.** There are exactly 14 almost 2-universal quinary diagonal \( \mathbb{Z} \)-lattices. Those lattices are:

| \( \langle 1,1,1,1,1 \rangle \), \( \langle 1,1,1,1,2 \rangle \), \( \langle 1,1,1,1,3 \rangle \), \( \langle 1,1,1,2,2 \rangle \), \( \langle 1,1,1,2,3 \rangle \) |
|---|
| \( \langle 1,1,1,1,5 \rangle \), \( \langle 1,1,1,2,4 \rangle \), \( \langle 1,1,1,2,5 \rangle \), \( \langle 1,1,1,2,7 \rangle \), \( \langle 1,1,2,2,3 \rangle \), \( \langle 1,1,2,2,5 \rangle \) |
| \( \langle 1,1,1,3,7 \rangle \), \( \langle 1,1,2,3,5 \rangle \), \( \langle 1,1,2,3,8 \rangle \) |

**Table 1.** Almost 2-universal diagonal quinary \( \mathbb{Z} \)-lattices

By [7], there are eleven almost 2-universal quinary diagonal \( \mathbb{Z} \)-lattices and there can be at most three more. Among the quinary diagonal \( \mathbb{Z} \)-lattices listed in Table 1, five lattices in the first box are, in fact 2-universal, and six lattices in the second box are almost 2-universal (see [4], [3] and [7]). Three lattices in the third box are candidates for almost 2-universal quinary diagonal \( \mathbb{Z} \)-lattices provided in [7]. Now we prove that those three lattices are indeed almost 2-universal.

**Theorem 3.2.** The quinary \( \mathbb{Z} \)-lattice \( L = \langle 1,1,1,3,7 \rangle \) represents all binary lattices except following 19 binary lattices:

| \( [2,1,3] \), \( [4,1,4] \), \( [1,6] \), \( [4,6] \), \( [2,1,7] \), \( [3,1,7] \), \( [4,2,7] \), |
|---|
| \( [6,3,7] \), \( [7,2,7] \), \( [7,3,9] \), \( [7,1,10] \), \( [10,5,10] \), \( [7,2,15] \), |
| \( [10,15] \), \( [6,16] \), \( [7,2,22] \), \( [7,1,26] \), \( [7,3,34] \), \( [10,5,47] \). |

**Proof.** Consider the quaternary sublattice \( M = \langle 1,1,1,3 \rangle \) of \( L \), which has class number one. Let \( \ell := [a,b,c] \) a binary lattice such that \( 0 \leq 2b \leq a \leq c \). And we define the binary lattice

\[
\ell_{s,t} := [a - 7s^2, b - 7st, c - 7t^2]
\]
for each integers $s, t$. Note that $\ell \rightarrow M \perp \langle 7 \rangle$ if and only if $\ell_{s,t} \rightarrow M$ for some integers $s, t$.

(Step 1) First, for the case that $a < 30$, we verify that $\ell = [a, b, c] \rightarrow M \perp \langle 7 \rangle$. As a sample, we only consider the case that $a = 10, b = 5$. Other cases can be verified similarly. We use the fact that

$$[3, 1, c] \rightarrow M \quad \text{if} \quad c \equiv 0, 1, 4, 5, 6 \pmod{8},$$

$$\langle 3, c \rangle \rightarrow M \quad \text{if} \quad c \equiv 1, 2, 3, 5, 6 \pmod{8}.$$

For a binary lattice $\ell = [10, 5, c]$, we can check the followings:

- If $c \not\equiv 3, 6, 7 \pmod{8}$, then $\ell_{1,2} \simeq (3, c - 55) \rightarrow M$. ($c > 55$)
- If $c \equiv 6 \pmod{8}$, then $\ell_{1,1} \simeq [3, 1, c - 8] \rightarrow M$. ($c > 8$)
- If $c \equiv 7, 11 \pmod{16}$, then $\ell_{1,1} \subseteq [3, 1, \frac{1}{16}(c - 7)] \rightarrow M$. ($c > 7$)
- If $c \equiv 15 \pmod{16}$, then $\ell_{1,2} \subseteq (3, \frac{1}{16}(c - 55)) \rightarrow M$. ($c > 55$)
- If $c \equiv 3 \pmod{16}$, then $\ell_{1,3} \subseteq [3, 1, \frac{1}{16}(c - 147)] \rightarrow M$. ($c > 147$)

For a small $c$ such that $\ell_{1,t}$ is not positive, we can also check it by a direct calculation. In this case, it can be verified that three binary lattices $[2, 1, 3], [10, 5, 10], [10, 5, 47]$ are not represented by $M \perp \langle 7 \rangle$. In short,

$$[10, 5, c] \rightarrow M \perp \langle 7 \rangle \quad \forall c \not\equiv 3, 10, 47.$$

For $a < 30$, we can verify $\ell \rightarrow L$ except followings:

$$\langle 1, 6 \rangle, \langle 2, 1, 3 \rangle, \langle 4, 6 \rangle, \langle 4, 1, 4 \rangle, \langle 6, 16 \rangle, \langle 10, 15 \rangle, \langle 10, 5, 10 \rangle, \langle 10, 5, 47 \rangle,$$

$$\langle 7, 1, c_1 \rangle (c_1 = 2, 3, 10, 26), \langle 7, 2, c_2 \rangle (c_2 = 4, 7, 15, 22), \langle 7, 3, c_3 \rangle (c_3 = 6, 9, 34).$$

By a direct calculation, we can verify that sublattices of above exceptions with index a power of 2 are represented by $L$. Hereafter, we only consider $\mathbb{Z}_2$-primitive binary lattices $\ell$.

(Step 2) For a binary lattice $\ell = [a, b, c]$, suppose that $a, c \geq 30$ and $0 \leq 2b \leq a \leq c$. And suppose that $a \not\equiv 0 \pmod{7}$. By checking the local structure of $\ell_{s,t}$ and $M$ over $\mathbb{Z}_2, \mathbb{Z}_3$, we obtain the following properties.

1. (1.1) If $(a, b, c) \equiv (1, 0, 1) \pmod{2}$ and $(s, t) \equiv (1, 1) \pmod{2}$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
2. (1.2) If $(a, b, c) \equiv (0, 1, 0) \pmod{2}$ and $(s, t) \equiv (0, 0) \pmod{2}$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
3. (1.3) If $(a, b, c) \equiv (1, 1, 0) \pmod{2}$ and $(s, t) \equiv (1, 0) \pmod{2}$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
4. (1.4) If $(a, b, c) \equiv (0, 1, 1) \pmod{2}$ and $(s, t) \equiv (0, 1) \pmod{2}$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
5. (1.5) If $a \equiv 5 \pmod{8}$ or $c \equiv 5 \pmod{8}$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$ for any $s, t$.
6. (1.6) If $a \equiv 1 \pmod{8}$, $2 \mid s$ or $c \equiv 1 \pmod{8}$, $2 \mid t$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
7. (1.7) If $4 \mid a$, $2 \nmid s$ or $4 \mid c$, $2 \nmid t$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
8. (1.8) If $a \equiv 3 \pmod{4}$ or $c \equiv 3 \pmod{4}$, $2 \mid b$ and $2 \nmid st$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_2$.
9. (1.9) If $3 \mid ac$ and $3 \nmid st$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_3$.
10. (1.10) If $(a, b, c) \equiv (1, 0, 1), (1, 0, 2), (2, 0, 1) \pmod{3}$ and $3 \mid st$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_3$.
11. (1.11) If $(a, b, c) \equiv (1, 2, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1) \pmod{3}$ and $st \equiv 1 \pmod{3}$, then $\ell_{s,t} \rightarrow M$ over $\mathbb{Z}_3$. 
Define $\ell$ and $Z$ primitive. Thus we may assume that $\ell$ can be proved similarly. We only consider the case that $\ell$ is $Z$-primitive, above conditions cover all cases. For example, the case that $a \equiv c \equiv 3 \pmod{4}$ and $2 \nmid b$ is not contained in above. However in this case $\ell$ is not $Z_2$-primitive. $Z_3$-primity of $\ell$ is not necessary. Regardless of $Z_2$-primity of $\ell$, all cases are contained in above.

For all cases, we may choose $s = 1,2$ and $t \equiv i \pmod{6}$ for some $i$. Each case can be proved similarly. We only consider the case that $\ell = [a,b,c]$ satisfies the conditions given in both (1.4) and (2.3). In this case, $\ell_{s,t} \to M$ over $Z_2, Z_3$ if $s = 2$ and $t \equiv 1 \pmod{6}$.

Let $\mathfrak{P} = \{ 5, 7, 17, 19, 29, 31, \ldots \}$ be the set of primes $p$ such that $\left( \frac{3}{p} \right) = -1$. From the assumption that $\mathfrak{g} \ell \nsubseteq 7 \mathbb{Z}$, we get $\mathfrak{g} \ell_{s,t} \nsubseteq 7 \mathbb{Z}$ for all $s,t$, and hence $\ell_{s,t} \to M$ over $Z_7$. Let $p_1, p_2, \ldots, p_k$ be the primes in $\mathfrak{P} \setminus \{ 7 \}$ dividing $a - 28$. We want choose a suitable $t$ such that $b - 14t$ is relatively prime to $p_1p_2\cdots p_k$. Then we get $\ell_{s,t} \to M$ over $Z_p$ for all $p \neq 2,3$.

If $k = 0$, then $\ell_{2,1} \to M$. By lemma 2.1, $\ell_{2,1}$ is positive if $a \geq 66$. In the case that $30 \leq a \leq 65$, one can show that $\ell_{2,1}$ is also positive for sufficiently large $c$. In the case that $a = 34$, for example, $\ell_{2,1}$ is positive whenever $c > 39$. The remaining cases are finitely many and we can check it by a direct calculation.

If $1 \leq k \leq 4$, then $\ell_{2,t} \to M$ for some $t \in \{ 6m+1 \mid -\left[ \frac{k+1}{2} \right] \leq m \leq \left[ \frac{k}{2} \right] \}$. Note that $a \geq 28 + p_1\cdots p_k$. In the case that $k = 3,4,5$, all $\ell_{2,t}$ are positive by lemma 2.1. In case that $k = 1,2$, however, the positiveness of $\ell_{2,t}$ is not guaranteed. There are only finitely many cases such that $\ell_{2,t}$ is not positive. When $\ell_{2,t}$ is not positive, we can check that $\ell \to L$ by a direct calculation.

If $k = 5$, then $\ell_{2,t} \to M$ for some $t \in \{ -17, -11, \ldots, 13, 19 \}$. Since $a \geq 28 + 5 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 41 \cdot 43^{k-6}$, all $\ell_{2,t}$ are positive by lemma 3 of [4].

(Step 3) We show that $\ell \to L$ when $\mathfrak{g} \ell \subseteq 7$, that is, $\ell$ is the form of $[7a, 7b, 7c]$. Let $\ell' = [a,b,c]$, then $\ell = \ell'^7$. Consider the quaternion lattice $K = \langle 1 \rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

Note that $K$ has class number one and

$$(K \perp \langle 21 \rangle)^7 \to L.$$  

If $\ell' \to K \perp \langle 21 \rangle$, then $\ell'^7 = \ell \to L$. We only consider the case that $\ell'^7$ is $Z_7$-primitive. Thus we may assume that $\ell'^7$ is $Z_7$-primitive. This is equivalent to

$$d \ell' \equiv 1, 2, 4 \pmod{7}.$$  

Define $\ell_{s,t}' = [a - 21s^2, b - 21st, c - 21t^2]$. From the fact that $7 \nmid d \ell'$ we get $7 \nmid d \ell_{s,t}'$ and $\ell_{s,t}' \to K$ over $Z_7$ for any $s,t$. The followings are the sufficient conditions such that $\ell_{s,t}' \to K$ over $Z_2$ assuming that $\ell'$ is $Z_2$-primitive.

1. $(a, b, c) \equiv (1, 0, 1) \pmod{2}$ and $(s, t) \equiv (1, 1) \pmod{2}$.
2. $(a, b, c) \equiv (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1) \pmod{2}$ and $(s, t) \equiv (0, 0) \pmod{2}.$
\( \ell \text{ such that } \ell \subseteq \mathbb{Q} \)

**Proof.**

For the other two lattices, the proofs are quite similar to the above. We only provide following data for the proof of almost 2-universality of \( \ell \):

1. Quaternary sublattice \( M \) which has class number one
2. The integer \( n \) satisfying \( M \perp \langle n \rangle \rightarrow L \)
3. Conditions such that \( \ell_{s,t}^n \rightarrow M \) over \( \mathbb{Z}_p \) where \( p \mid 2dM \)
4. Data for the case that \( s \ell \subseteq q\mathbb{Z} \) where \( q \mid n \) and \( \frac{dM}{q} = -1 \)

**Theorem 3.3.**

1. The quinary \( \mathbb{Z} \)-lattice \( \langle 1, 1, 2, 3, 5 \rangle \) represents all binary lattices except
   \[ [2, 1, 2], [5, 2, 5], [6, 3, 6]. \]
2. The quinary \( \mathbb{Z} \)-lattice \( \langle 1, 1, 2, 3, 8 \rangle \) represents all binary lattices except the following 15 binary lattices:
   \[ [2, 1, 2], [2, 6], [5, 6], [5, 2, 5], [5, 1, 8], [6, 3, 6], [6, 3, 8], [6, 3, 14], [6, 3, 14], [6, 3, 14], [10, 33], [11, 14], [14, 14], [25, 3, 25]. \]

**Proof.** Set \( M = (1, 1, 2, 3), n = 5, 8 \)

Conditions such that \( \ell_{s,t}^n \rightarrow M \) over \( \mathbb{Z}_3 \) where \( n = 5, 8 \): (\( \mathbb{Z}_3 \)-primitivity of \( \ell \) is not necessary)

- 3 \mid ac \text{ and } 3 \nmid st
- \( (a, b, c) \equiv (1, 0, 2), (2, 0, 1), (2, 0, 2) \) (mod 3) and 3 \nmid st
- \( (a, b, c) \equiv (1, 1, 1), (1, 2, 1), (1, 1, 2) \) (mod 3) and \( s \equiv 1 \) (mod 3)
- \( (a, b, c) \equiv (1, 0, 1) \) (mod 3) and \( st \equiv 0 \) (mod 3)

Conditions such that \( \ell_{s,t}^5 \rightarrow M \) over \( \mathbb{Z}_2 \): (\( \ell \) is \( \mathbb{Z}_2 \)-primitive)

- \( (a, b, c) \equiv (0, 1, 0) \) (mod 2) and \( \forall s, t \)
- \( (a, b, c) \equiv (0, 0, 1), (1, 0, 0), (1, 0, 1) \) (mod 2) and \( (s, t) \equiv (1, 1) \) (mod 2)
- \( (a, b, c) \equiv (0, 1, 1), (1, 1, 1) \) (mod 2) and \( (s, t) \equiv (0, 1) \) (mod 2)
- \( (a, b, c) \equiv (1, 1, 0), (1, 1, 1) \) (mod 2) and \( (s, t) \equiv (1, 0) \) (mod 2)

Conditions such that \( \ell_{s,t}^8 \rightarrow M \) over \( \mathbb{Z}_2 \): (\( \ell \) is \( \mathbb{Z}_2 \)-primitive)

- \( (a, b, c) \equiv (0, 1, 0), (0, 1, 1), (1, 0, 1) \) (mod 2) and \( \forall s, t \)
- \( a \equiv 5, 7 \) (mod 8) or \( c \equiv 5, 7 \) (mod 8) and \( \forall s, t \)
- \( a \equiv 10, 14 \) (mod 16), 2 \nmid s or \( c \equiv 10, 14 \) (mod 16), 2 \nmid t
- \( (a, c) \equiv (1, 3), (9, 11) \) (mod 16), \( b \equiv \pm 1 \) (mod 8) and \( (s, t) \equiv (1, 1) \) (mod 2)
- \( (a, c) \equiv (1, 11), (3, 9) \) (mod 16), \( b \equiv \pm 3 \) (mod 8) and \( (s, t) \equiv (1, 1) \) (mod 2)
\[ (a, c) \equiv (1, 3), (9, 11) \pmod{16}, b \equiv \pm 3 \pmod{8} \text{ and } (s, t) \equiv (0, 1), (1, 0) \pmod{2} \]

\[ (a, c) \equiv (1, 11), (3, 9) \pmod{16}, b \equiv \pm 1 \pmod{8} \text{ and } (s, t) \equiv (0, 1), (1, 0) \pmod{2} \]

\[ (a, c) \equiv (1, 0), (9, 11), (3, 2), (11, 2) \pmod{16}, b \equiv 2 \pmod{4} \text{ and } (s, t) \equiv (1, 0) \pmod{2} \]

\[ (a, c) \equiv (1, 2), (9, 2), (3, 6), (11, 6) \pmod{16}, b \equiv 0 \pmod{4} \text{ and } (s, t) \equiv (1, 1) \pmod{2} \]

Since there are no prime factors \( q \) such that \( q \mid n \) and \( \left( \frac{dM}{q} \right) = -1 \), in this case, the process such as step 3 in the proof of Theorem 3.2 is not necessary. \qed

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