SUPER DUALITY FOR GENERAL LINEAR LIE SUPERALGEBRAS
AND APPLICATIONS

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Abstract. We apply the super duality formalism recently developed by the authors
to obtain new equivalences of various module categories of general linear Lie superal-
gebras. We establish the correspondence of standard, tilting, and simple modules, as
well as the identification of the \( u \)-homology groups, under these category equivalences.
As an application, we obtain a complete solution of the irreducible character problem
for some new parabolic BGG categories of \( \mathfrak{gl}(m|n) \)-modules, including the full BGG
category of \( \mathfrak{gl}(m|2) \)-modules, in terms of type \( A \) Kazhdan-Lusztig polynomials.

1. Introduction

Super duality is a powerful new approach developed in the past few years in the
study of representation theory of Lie superalgebras. It provides a surprising direct link
between various parabolic BGG categories of modules of Lie superalgebras and Lie
algebras, and allows us to solve the fundamental irreducible character problem for Lie
superalgebras in terms of Kazhdan-Lusztig polynomials for semisimple (or more gen-
ernally Kac-Moody) Lie algebras. Super duality was first formulated as conjectures for
general linear Lie superalgebras in \([CWZ, CW]\), and then formulated and established in
much generality in \([CL2, CLW]\) recently. Our super duality solution of the irreducible
character problem for Lie superalgebras ultimately depends on the solution \([BB, BK]\)
of Kazhdan-Lusztig conjecture for Lie algebras \([KL, Deo]\). We also refer to \([BS]\) for a com-
pletely different approach towards the very special version of super duality conjecture
as formulated in \([CWZ]\).

The goal of this paper is to formulate and establish a new form of super duality
(which is an equivalence of categories) for the general linear Lie superalgebras. The
super duality formalism typically takes advantage of a type \( A \) branch which many
Dynkin diagrams possess. In this paper we will apply the super duality formalism
simultaneously to the two ends of the (super type \( A \)) Dynkin diagrams. We in addition
establish a correspondence of tilting modules under super duality, which was omitted in
our previous paper \([CLW]\), besides the correspondences of standard and simple modules
under super duality. We also show that super duality is actually an equivalence of tensor
categories, which was not addressed in our earlier work.

The version of super duality in this paper allows us to settle the irreducible char-
acter problem in some new parabolic BGG categories of \( \mathfrak{gl}(m|n) \)-modules not covered
in \([CW, CL2, CLW]\) in terms of parabolic Kazhdan-Lusztig polynomials of type \( A \).
Brundan \([Br1]\) conjectured that the full BGG category of \( \mathfrak{gl}(m|n) \)-modules (of integer
weights) categorifies the Fock space $V^\otimes m \otimes (V^*)^\otimes n$, and he established a maximal parabolic version of this conjecture for the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules, where $V$ and $V^*$ denote the natural $U_q(\mathfrak{gl}_\infty)$-module and its dual module, respectively. Various parabolic versions of Brundan’s conjecture have been formulated and settled since then via the super duality approach ([CW, CL2]). However, the irreducible character problem in the full BGG category for general $m$ and $n$ remained unsettled in loc. cit., except when $m$ or $n$ is at most 1. Our paper confirms a variant of Brundan’s conjecture on the full BGG category of $\mathfrak{gl}(m|n)$-modules (with respect to a nonstandard Borel subalgebra of block type $1|m|1$) for $n = 2$ or $m = 2$, and we will return to address the case for general $m$ and $n$ in another paper.

Recall the infinite-dimensional Lie algebra $\hat{\mathfrak{gl}}_\infty$ (which is a central extension with a central element denoted by $K$) has played a fundamental role in many different contexts. If $K$ acts as a scalar $\ell$ on a $\hat{\mathfrak{gl}}_\infty$-module $M$, then $\ell$ is called the level of the module $M$. Let $\ell$ be a positive integer. A very special case of super duality states that the (semisimple) category of integrable $\hat{\mathfrak{gl}}_\infty$-modules of positive level $\ell$ is equivalent to a suitable category of $\hat{\mathfrak{gl}}_\infty$-modules of negative level $-\ell$ (which is far from being obvious to be semisimple); A weak version of the super duality on the Grothendieck group level suffices to recover the character formulas established by completely different methods in [KR] and [CL1, Remark 5.2].

Following Vogan [Vo], a computation of the $u$-homology groups is basically a computation of the (parabolic) Kazhdan-Lusztig polynomials. Super duality identifies the corresponding $u$-homology groups with coefficients in modules belonging to different categories. This in particular allows us to recover easily the computation of $u$-homology groups with coefficients in modules appearing in Howe duality decompositions (which were computed by completely different techniques in [CK, CKW, HLT, LZ]).

To keep the paper at a reasonable length, we have omitted several proofs leading towards super duality, when they are similar to the ones in [CL2, CLW]. We also refer directly to [Br1, CW] for the Fock space formulation of some parabolic versions of Brundan’s conjecture.

We shall use the following notations throughout this paper. The symbols $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}_+$ stand for the sets of all, positive, and non-negative integers, respectively. For a superspace $V = V_0 \oplus V_1$ and a homogeneous element $v \in V$, we use the notation $|v|$ to denote the $\mathbb{Z}_2$-degree of $v$. Finally all vector spaces, algebras, tensor products, et cetera, are over the field of complex numbers $\mathbb{C}$.

2. Lie superalgebras associated to various Dynkin diagrams

In this section, we introduce a general linear Lie superalgebra $\tilde{g}$ of infinite rank and its subalgebras $g$, $\overline{g}$, $\check{g}$ and $\bar{g}^\vee$. We also formulate the finite-rank counterparts of these Lie superalgebras.

2.1. General linear Lie superalgebra. We fix $k \in \mathbb{Z}_+$. We consider the following ordered set $\bar{I}$:
\begin{equation}
\cdots < -\frac{3}{2} < -1 < -\frac{1}{2} < 0 < \underbrace{1}_{\underline{k}} < \underbrace{2}_{\check{k}} < \cdots < \bar{k} < \underbrace{\frac{1}{2}}_{k} < 1 < \frac{3}{2} < \cdots,
\end{equation}
For \( k > 0 \), set \( \mathbb{K} = \{ \mathbb{T}, \ldots, \mathbb{T} \} \), and for \( k = 0 \), set \( \mathbb{K} = \emptyset \). We define the following subsets of \( \mathbb{I} \):

\[
\mathbb{I} := \mathbb{K} \cup \mathbb{Z}, \quad \mathbb{I}^\circ := \mathbb{K} \cup (-\frac{1}{2} - \mathbb{Z}_+) \cup \mathbb{N}, \quad \mathbb{I} := \mathbb{K} \cup (-\frac{1}{2} + \mathbb{Z}).
\]

Consider the infinite-dimensional superspace \( \widetilde{V} \) with ordered basis \( \{ v_i | i \in \mathbb{I} \} \). We declare \( |v_r| = 0 \), if \( r \in \mathbb{Z} \cup \mathbb{K} \), and \( |v_i| = \mathbb{I} \), if \( r \in \frac{1}{2} + \mathbb{Z} \). With respect to this basis, a linear map on \( \widetilde{V} \) may be identified with a complex matrix \( (a_{rs})_{r,s \in \mathbb{I}} \). The Lie superalgebra \( \mathfrak{gl}(\widetilde{V}) \) is the Lie subalgebra of linear transformations on \( \widetilde{V} \) consisting of \( (a_{rs}) \) with \( a_{rs} = 0 \) for all but finitely many \( a_{rs} \)'s. Denote by \( E_{rs} \in \mathfrak{gl}(\widetilde{V}) \) the elementary matrix with 1 at the \( r \)th row and \( s \)th column and zero elsewhere. Denote by \( \tilde{\mathfrak{g}} := \mathfrak{gl}(\widetilde{V}) \oplus CK \) the central extension of \( \mathfrak{gl}(\widetilde{V}) \) by a one-dimensional center \( CK \) determined by the 2-cocycle

\[
\alpha(A, B) := \text{Str}([\mathfrak{J}, A]B), \quad A, B \in \mathfrak{gl}(\widetilde{V}),
\]

where \( \mathfrak{J} = \sum_{r \leq 0} E_{rr} \) and \( \text{Str} \) denotes the supertrace. Observe that the cocycle \( \alpha \) is a coboundary. Indeed, there is embedding \( \iota : \mathfrak{gl}(\widetilde{V}) \to \tilde{\mathfrak{g}} \), defined by sending \( A \in \mathfrak{gl}(\widetilde{V}) \) to \( A + \text{Str}(\mathfrak{J}A)K \) (cf. [CLW, Section 2.5]). It is clear that \( \iota(\mathfrak{gl}(\widetilde{V})) \) is an ideal of \( \tilde{\mathfrak{g}} \) and \( \tilde{\mathfrak{g}} \) is a direct sum of the ideals \( \iota(\mathfrak{gl}(\widetilde{V})) \) and \( CK \). For \( A \in \mathfrak{gl}(\widetilde{V}) \) we denote

\[
\tilde{A} := \iota(A) - \text{Str}(\mathfrak{J}A)K \in \tilde{\mathfrak{g}}.
\]

Let \( \tilde{\mathfrak{h}} \) and \( \tilde{\mathfrak{b}} \) denote the standard Cartan subalgebra \( \oplus_{r \in \mathbb{T}} \mathbb{C} \tilde{E}_{rr} \oplus CK \) and the standard Borel subalgebra \( \oplus_{r \leq s, r, s \in \mathbb{T}} \mathbb{C} \tilde{E}_{rs} \oplus CK \), respectively. Define \( \Lambda_0 \in \tilde{\mathfrak{h}}^* \) by

\[
\Lambda_0(K) = 1, \quad \Lambda_0(E_{rr}) = 0, \quad \forall r \in \mathbb{I}.
\]

Let \( \epsilon_i \in \tilde{\mathfrak{h}}^* \) be determined by \( \langle \epsilon_i, \tilde{E}_{jj} \rangle = \delta_{ij} \) for \( i, j \in \mathbb{I} \) and \( \langle \epsilon_i, K \rangle = 0 \). We set

\[
\alpha_r := \epsilon_r - \epsilon_{r+\frac{1}{2}}, \quad r \in \frac{1}{2} \mathbb{Z}, \quad \text{and} \quad \alpha_{-r} := \epsilon_{-r} - \epsilon_{-r+\frac{1}{2}}, \quad 1 \leq j \leq k - 1.
\]

The totally ordered set \( \mathbb{I} \) gives rise to a positive system for \( \tilde{\mathfrak{g}} \)

\[
\tilde{\Delta}^+ = \{ \epsilon_i - \epsilon_j \mid i < j, i, j \in \mathbb{I} \},
\]

whose associated fundamental system is given by

\[
\tilde{\Pi} = \begin{cases} \{ \alpha_r \mid r \in \frac{1}{2}\mathbb{Z} \setminus \{ 0 \} \} \cup \{ \epsilon_0 - \epsilon_{-\frac{1}{2}}, \alpha_{-\frac{1}{2}}, \alpha_{\frac{1}{2}}, \cdots, \alpha_{k-\frac{1}{2}} \}, & \text{if } k \neq 0; \\ \{ \alpha_r \mid r \in \frac{1}{2}\mathbb{Z} \}, & \text{if } k = 0. \end{cases}
\]

Let \( \mathfrak{X} \) denote the following Dynkin diagram with fundamental system:

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \quad \circ \\
\alpha_{\frac{1}{2}} \quad \alpha_{\frac{1}{2}} \quad \alpha_{\frac{1}{2}} \quad \cdots \quad \alpha_{\frac{1}{2}} \quad \alpha_{\frac{1}{2}}
\end{array}
\]

The following is a Dynkin diagram together with the fundamental system \( \tilde{\Pi} \) for \( \tilde{\mathfrak{g}} \), where \( \bigotimes \) denotes an odd isotropic simple root:
\[ D(\widehat{\mathfrak{g}}): \quad \cdots \times_{-1} \times_{n-1} \cdots \times_{0} \times_{-\frac{1}{2}} \times_{\frac{1}{2}} \times_{1} \times_{0} \times_{n} \cdots \]

When \( k = 0 \), the diagram above means that the middle three terms are replaced by the odd isotropic simple root \( \epsilon_0 - \epsilon_\frac{1}{2} \). The Dynkin diagrams in the rest of the paper are interpreted in the similar way for \( k = 0 \).

Define

\[ |r| = \begin{cases} 0 & \text{if } r \in \mathbb{I}; \\ 1 & \text{if } r \in \mathbb{I} \setminus \mathbb{I}. \end{cases} \]

Let \( \tau \) be an automorphism of \( \widehat{\mathfrak{g}} \) of order 4 defined by

\[ \tau(\widehat{E}_{rs}) := -(-1)^{|r|(|r|+|s|)}\widehat{E}_{sr}, \quad \tau(K) = -K. \]

2.2. The subalgebras of \( \widehat{\mathfrak{g}} \). The subalgebra of \( \widehat{\mathfrak{g}} \) generated by \( K \) and \( \widehat{E}_{rs} \) with \( r, s \in \mathbb{I} \) (respectively \( \mathbb{I}, \mathbb{I}', \mathbb{I}' \)) is denoted by \( \mathfrak{g} \) (respectively \( \mathfrak{g}_r, \mathfrak{g}'_r \) and \( \mathfrak{g}''_r \)). The Cartan subalgebras (respectively Borel subalgebras) of \( \mathfrak{g}, \mathfrak{g}_r, \mathfrak{g}'_r \) and \( \mathfrak{g}''_r \) induced from \( \widehat{\mathfrak{g}} \) are denoted by \( \mathfrak{h}, \mathfrak{h}_r, \mathfrak{h}'_r \) and \( \mathfrak{h}''_r \) respectively. We set

\[ \beta_r := \epsilon_r - \epsilon_{r+1}, \quad \forall r \in \frac{1}{2}\mathbb{Z}. \]

For \( n \in \mathbb{Z}_+ \), let \( (\mathfrak{L}_n), (\mathfrak{L}_n^\circ), (\mathfrak{R}_n) \) and \( (\overline{\mathfrak{R}}_n) \) denote the following Dynkin diagrams with fixed fundamental systems:

- \( \mathfrak{L}_n \):\[ \beta_{n+1} \beta_n \beta_{n-1} \cdots \beta_2 \beta_1 \]
- \( \mathfrak{L}_n^\circ \):\[ \beta_{n+1} \beta_n \beta_{n-1} \cdots \beta_2 \beta_1 \]
- \( \mathfrak{R}_n \):\[ \beta_1 \beta_2 \beta_3 \cdots \beta_{n-1} \beta_n \]
- \( \overline{\mathfrak{R}}_n \):\[ \beta_{n+1} \beta_n \beta_{n-1} \cdots \beta_2 \beta_1 \]

In the limit \( n \to \infty \), the associated Dynkin diagrams of \( (\mathfrak{L}_n), (\mathfrak{L}_n^\circ), (\mathfrak{R}_n) \) and \( (\overline{\mathfrak{R}}_n) \) are denoted by \( \mathfrak{L}, \mathfrak{L}^\circ, \mathfrak{R} \) and \( \overline{\mathfrak{R}} \), respectively.

For \( m, n \in \mathbb{Z}_+ \cup \{ \infty \} \), let \( \widehat{\mathfrak{g}}(m, n) \) denote the subalgebra of \( \widehat{\mathfrak{g}} \) generated by \( E_{rs} \) with \( -m < r, s < n + 1 \) and \( r, s \in \mathbb{I} \). Note that \( \widehat{\mathfrak{g}}(m, n) \) is a finite-dimensional subalgebra for \( m, n \in \mathbb{Z}_+ \). Set

\[ \mathfrak{g}(m, n) = \mathfrak{g} \cap \widehat{\mathfrak{g}}(m, n), \quad \mathfrak{g}_r(m, n) = \mathfrak{g}_r \cap \widehat{\mathfrak{g}}(m, n), \quad \mathfrak{g}'_r(m, n) = \mathfrak{g}'_r \cap \widehat{\mathfrak{g}}(m, n), \quad \mathfrak{g}''_r(m, n) = \mathfrak{g}''_r \cap \widehat{\mathfrak{g}}(m, n). \]

Then \( \mathfrak{h}(m, n) = \mathfrak{h} \cap \widehat{\mathfrak{g}}(m, n) \) and \( \mathfrak{b}(m, n) = \mathfrak{b} \cap \widehat{\mathfrak{g}}(m, n) \) are the Cartan and Borel subalgebras of \( \widehat{\mathfrak{g}}(m, n) \), and we have four variants of Cartan and Borel subalgebras for the remaining four Lie superalgebras with self-explanatory notations. The Dynkin
diagrams with given fundamental systems of $\mathfrak{g}(m,n)$, $\mathfrak{g}^\circ(m,n)$, $\mathfrak{g}^\diamond(m,n)$ and $\mathfrak{g}^\flat(m,n)$ with respect to the Borel subalgebras $\mathfrak{b}(m,n)$, $\mathfrak{b}^\flat(m,n)$, $\mathfrak{b}^\diamond(m,n)$ and $\mathfrak{b}^\circ(m,n)$, are given as follows:

\[
\begin{align*}
D(\mathfrak{g}(m,n)) & : \quad \begin{array}{cccc}
\mathfrak{L}_m & \mathfrak{K} & \mathfrak{R}_n \\
\epsilon_0 - \epsilon_i & \epsilon_r - \epsilon_i & \\
\end{array} \\
D(\mathfrak{g}^\circ(m,n)) & : \quad \begin{array}{cccc}
\mathfrak{L}_m & \mathfrak{K} & \mathfrak{R}_n \\
\epsilon_0 - \epsilon_i & \epsilon_r - \epsilon_i & \\
\end{array} \\
D(\mathfrak{g}^\diamond(m,n)) & : \quad \begin{array}{cccc}
\mathfrak{L}_m & \mathfrak{K} & \mathfrak{R}_n \\
\epsilon_0 - \epsilon_i & \epsilon_r - \epsilon_i & \\
\end{array} \\
D(\mathfrak{g}^\flat(m,n)) & : \quad \begin{array}{cccc}
\mathfrak{L}_m & \mathfrak{K} & \mathfrak{R}_n \\
\epsilon_0 - \epsilon_i & \epsilon_r - \epsilon_i & \\
\end{array}
\end{align*}
\]

For $m = n = \infty$, these become the Dynkin diagrams and simple systems of $\mathfrak{g}$, $\mathfrak{g}^\circ$, $\mathfrak{g}^\diamond$ and $\mathfrak{g}^\flat$ with respect to the Borel subalgebras $\mathfrak{b}$, $\mathfrak{b}^\flat$, $\mathfrak{b}^\diamond$ and $\mathfrak{b}^\circ$, respectively. The fundamental systems of $\mathfrak{g}$, $\mathfrak{g}^\circ$, $\mathfrak{g}^\diamond$ and $\mathfrak{g}^\flat$ are denoted by $\Pi$, $\Pi^\circ$, $\Pi^\diamond$ and $\Pi^\flat$, respectively. These fundamental systems of $\mathfrak{g}(m,n)$, $\mathfrak{g}(m,n)$, $\mathfrak{g}^\circ(m,n)$ and $\mathfrak{g}^\diamond(m,n)$ are denoted by $\Pi^\circ(m,n)$, $\Pi(m,n)$, $\Pi^\diamond(m,n)$ and $\Pi^\flat(m,n)$, respectively.

2.3. Levi subalgebras. We fix an arbitrary subset $Y_0$ of \{\(\alpha_j\mid 1 \leq j \leq k - 1\}\). The set $\tilde{\Gamma}$ (respectively $\tilde{\Gamma}$, $\tilde{\Gamma}$, $\tilde{\Gamma}$) consisting of elements of the form $\epsilon_r - \epsilon_s$, with $r$, $s \in \mathbb{Z} \cap \mathbb{Z}$ (respectively $\mathbb{Z} \cap \mathbb{Z}$, $\mathbb{Z} \cap \mathbb{Z}$, $\mathbb{Z} \cap \mathbb{Z}$) satisfying $r < s \leq 0$ or $0 < r < s$. Let $I$ (respectively $I$, $I$, $I$) be the standard Levi subalgebra of $\tilde{\mathfrak{g}}$ (respectively $\mathfrak{g}$, $\mathfrak{g}^\circ$, $\mathfrak{g}^\diamond$) corresponding to the set $\tilde{\Gamma}$ (respectively $\tilde{\Gamma}$, $\tilde{\Gamma}$, $\tilde{\Gamma}$). Let $\tilde{p} = I + b$ (respectively $\tilde{p} = I + \tilde{b}$, $\tilde{p} = I + \tilde{b}$) be the corresponding parabolic subalgebra of $\tilde{\mathfrak{g}}$ (respectively $\mathfrak{g}$, $\mathfrak{g}^\circ$, $\mathfrak{g}^\diamond$) with nilradical $\mathfrak{u}$ (respectively $\mathfrak{u}$, $\mathfrak{u}^\circ$, $\mathfrak{u}^\diamond$) and opposite nilradical $\mathfrak{u}$ (respectively $\mathfrak{u}_-$, $\mathfrak{u}_-$, $\mathfrak{u}_-^\circ$, $\mathfrak{u}_-^\diamond$).

Denote the standard Borel subalgebras $\tilde{I} \cap I$ (respectively $\tilde{b} \cap I$, $\tilde{b} \cap \tilde{I}$, $\tilde{b} \cap \tilde{I}$) of $I$ (respectively $I$, $I^\circ$, $I^\diamond$) by $\mathfrak{b}_I$ (respectively $\mathfrak{b}_I$, $\mathfrak{b}_I^\circ$, $\mathfrak{b}_I^\diamond$). For $\mu \in \mathfrak{h}^*$ (respectively $\mathfrak{h}^*$, $\mathfrak{h}^*$, $\mathfrak{h}^*$), let $L(\tilde{I}(\mu), \mu)$ (respectively $L(\tilde{I}(\mu), \mu)$, $L(I^\circ(\mu), \mu)$, $L(I^\diamond(\mu), \mu)$) denote the highest weight irreducible $I$-(respectively $\tilde{I}$, $I^\circ$, $I^\diamond$)-module of highest weight $\mu$ with respect to the standard Borel. We extend $L(I, \mu)$ to a $\mathfrak{p}$-module by letting $u$ act trivially. Define as usual the parabolic Verma module $\Delta(\mu)$ and its unique irreducible quotient $L(\mu)$ over $\mathfrak{g}$:

\[
\Delta(\mu) := \text{Ind}^\mathfrak{g}_\mathfrak{p} L(I, \mu), \quad \Delta(\mu) \to L(\mu).
\]

Similarly, we introduce the other four variants of parabolic Verma and irreducible quotient modules with self-explanatory notations.

We shall also need the finite-rank counterparts of the above algebras. Let $\tilde{I}(m,n) = \tilde{I} \cap \tilde{g}(m,n)$. Then $\tilde{I}(m,n)$ is a Levi subalgebra of $\tilde{g}(m,n)$. For $\mu \in \tilde{h}(m,n)^*$, let $L(\tilde{I}(m,n), \mu)$ denote the highest weight irreducible $I(m,n)$-module of highest weight $\mu$. As above, we can define parabolic Verma $\tilde{g}(m,n)$-module $\tilde{\Delta}(m,n)(\mu)$ and its irreducible
quotient $\tilde{L}_{(m,n)}(\mu)$. In a completely parallel fashion, we have the other four variants of Levi subalgebras, parabolic Verma modules, and so on, with self-explanatory notations.

2.4. The dominant weights. Given a partition $\mu = (\mu_1, \mu_2, \ldots)$, let $\mu'$ denote its conjugate partition. We also denote by $\theta(\mu)$ the modified Frobenius coordinates of $\mu$:

$$\theta(\mu) := (\theta(\mu)_1/2, \theta(\mu)_1, \theta(\mu)_3/2, \theta(\mu)_2, \ldots),$$

where

$$\theta(\mu)_{i-1/2} := \langle \mu'_j - i + 1 \rangle, \quad \theta(\mu)_i := \langle \mu_i - i \rangle, \quad i \in \mathbb{N}.$$ 

Here and below $\langle b \rangle := \max\{b, 0\}$ for all $b \in \mathbb{R}$. Let $a, \lambda_0^1, \ldots, \lambda_0^k \in \mathbb{C}$, $\lambda^-$ and $\lambda^+$ be two partitions. Associated to the tuple $\lambda = (a, \lambda_0^1, \ldots, \lambda_0^k; \lambda^-, \lambda^+)$, set

$$\Lambda^+(\lambda) := \sum_{i=1}^{k} \lambda_i^0 \epsilon_i^+ + \sum_{j \in \mathbb{N}} \lambda_j^+ \epsilon_j, \quad \Xi^+(\lambda) := \sum_{i=1}^{k} \lambda_i^0 \epsilon_i^+ + \sum_{j \in \mathbb{N}} (\lambda^+)_j \epsilon_{j-\frac{1}{2}},$$

$$\Lambda^-(\lambda) := -\sum_{j \in \mathbb{N}} \lambda_j^- \epsilon_{j+1} + a\lambda_0, \quad \Xi^-(\lambda) := -\sum_{j \in \mathbb{N}} (\lambda^-)_j \epsilon_{j+\frac{1}{2}} + a\lambda_0.$$ 

The tuple $(a, \lambda_0^1, \ldots, \lambda_0^k; \lambda^-, \lambda^+)$ is said to satisfy a dominant condition if

$$\langle \sum_{i=1}^{k} \lambda_i^0 \epsilon_i^+, \alpha^\vee \rangle \in \mathbb{Z}_+, \quad \text{for all } \alpha \in Y_0,$$

where $\alpha^\vee$ denotes the coroot of $\alpha$. Associated to such a dominant tuple and $a \in \mathbb{C}$, we define the weights (which will be called dominant)

$$\tilde{\lambda} := \sum_{i=1}^{k} \lambda_i^0 \epsilon_i^+ - \sum_{r \in \frac{1}{2}\mathbb{N}} \theta(\lambda^-)_r \epsilon_{r+\frac{1}{2}} + \sum_{r \in \frac{1}{2}\mathbb{N}} \theta(\lambda^+)_r \epsilon_r + a\lambda_0 \in \tilde{h}^*,$$

$$\lambda := \Lambda^-(\lambda) + \Lambda^+(\lambda) \in h^*,$$

$$\lambda^0 := \Xi^-(\lambda) + \Xi^+(\lambda) \in h^{0*},$$

$$\tilde{\lambda}^0 := \Xi^-(\lambda) + \Xi^+(\lambda) \in \tilde{h}^{0*}.$$ 

The number $a$ will be called the level of these weights.

We denote by $\tilde{P}_a^+$ (respectively $P_a^+, \overline{P}_a^+, P_a^{0+}, \overline{P}_a^{0+}$) the set of all dominant weights of the form $\tilde{\lambda}$ (respectively $\lambda, \lambda^0, \tilde{\lambda}^0$) with a fixed $a \in \mathbb{C}$. Obviously, we have bijective maps between $\tilde{P}_a^+, P_a^+, \overline{P}_a^+, P_a^{0+}$ and $\overline{P}_a^{0+}$ given by $\tilde{\lambda} \leftrightarrow \lambda \leftrightarrow \lambda^0 \leftrightarrow \tilde{\lambda}^0$ for $\lambda \in P_a^+$. Finally, we let $P_a^+ = \bigcup_{a \in \mathbb{C}} P_a^+.$

3. Change of highest weights for different Borel subalgebras

In this section, using odd reflections, we will determine how a highest weight for a highest weight module changes from a standard Borel to another distinguished non-standard Borel subalgebra.

We shall briefly explain the effect of an odd reflection on the highest weight of a highest weight irreducible module (cf., e.g., [PS, Lemma 1], [KW, Lemma 1.4]). Fix a Borel subalgebra $\mathcal{B}$ of a Lie superalgebra $\mathfrak{g}$ with corresponding positive system $\Phi_+(\mathcal{B}).$ Let $\alpha$ be an isotropic odd simple root and $\alpha^\vee$ be its corresponding coroot. Applying
the odd reflection with respect to \( \alpha \) changes the Borel subalgebra \( \mathcal{B} \) into a new Borel subalgebra \( \mathcal{B}^\alpha \) whose corresponding positive system is \( \Phi_+(\mathcal{B}^\alpha) = \{ -\alpha \} \cup \Phi_+(\mathcal{B}) \setminus \{ \alpha \} \).

**Lemma 3.1.** Let \( \lambda \) be the highest weight with respect to \( \mathcal{B} \) of an irreducible module. If \( \langle \lambda, \alpha^\vee \rangle \neq 0 \), then the highest weight of this irreducible module with respect to \( \mathcal{B}^\alpha \) is \( \lambda - \alpha \). If \( \langle \lambda, \alpha^\vee \rangle = 0 \), then the highest weight remains to be \( \lambda \).

Even though it is unclear how the structures of Verma modules are related via odd reflections, the relation of their characters is simply described as follows.

**Lemma 3.2.** Let \( \Delta(\mathcal{G}, \mathcal{B}, \lambda) \) and \( \Delta(\mathcal{G}, \mathcal{B}^\alpha, \lambda) \) denote the \( \mathcal{G} \)-Verma modules of highest weight \( \lambda \) with respect to the Borel subalgebras \( \mathcal{B} \) and \( \mathcal{B}^\alpha \), respectively. Then we have

\[
ch\Delta(\mathcal{G}, \mathcal{B}, \lambda) = ch\Delta(\mathcal{G}, \mathcal{B}^\alpha, \lambda - \alpha).
\]

**Proof.** Follows from the identity \( \Phi_+(\mathcal{B}^\alpha) = \{ -\alpha \} \cup \Phi_+(\mathcal{B}) \setminus \{ \alpha \} \). \( \square \)

For \( n \in \mathbb{N} \), we introduce the following total orderings of the subsets of \( \bar{\mathbb{I}} \):

(3.1) \[ \cdots < -\frac{3}{2} - n < -1 - n < -\frac{1}{2} - n < -n < -\frac{3}{2} - n < -\frac{1}{2} - n < \cdots < -\frac{3}{2} < -1 < \frac{1}{2} < \cdots < n \]

(3.2) \[ \cdots < -\frac{3}{2} - n < -1 - n < -\frac{1}{2} - n < -n < -\frac{3}{2} - n < -\frac{1}{2} - n < \cdots < -\frac{3}{2} < -1 < \frac{1}{2} < \cdots < n \]

For any total ordering of \( \bar{\mathbb{I}} \), there is a Borel subalgebra of \( \mathfrak{g} \) spanned by the Cartan subalgebra \( \mathfrak{h} \) and elements of the form \( \hat{E}_{rs} \) such that \( r < s \) with respect to the ordering. Conversely, any Borel subalgebra of \( \mathfrak{g} \) containing the Cartan subalgebra \( \mathfrak{h} \) determines a unique total ordering of \( \bar{\mathbb{I}} \). Let \( \mathfrak{b}^c(n) \) (respectively \( \mathfrak{b}^s(n), \mathfrak{b}^{cs}(n), \mathfrak{b}^{oc}(n), \mathfrak{b}^{os}(n) \)) be the Borel subalgebras with respect to the ordering (3.1) (respectively (3.2), (3.3), (3.4)) of \( \bar{\mathbb{I}} \).
Two elements $a$ and $b$ in an ordered set are said to be adjacent if there is no element $j$ in the set satisfying $a < j < b$ or $a > j > b$. For an odd simple root of the form $\epsilon_r - \epsilon_s$ with $r, s \in \mathbb{N}$ in the root system of a Borel subalgebra $\mathfrak{B}$ of $\mathfrak{g}$ containing the Cartan subalgebra $\mathfrak{h}$, the numbers $r$ and $s$ are adjacent with respect to the corresponding ordering. The ordering corresponding to the new Borel subalgebra obtained by applying the odd reflection with respect to $\epsilon_r - \epsilon_s$ is the same as the ordering corresponding to the Borel subalgebraug except reversing the ordering of $r$ and $s$.

Remark 3.3. Any ordering preserving the orderings of positive integers, positive half integers, non-positive integers and negative half integers, and satisfying $i < a < j$ for $i \in -\frac{1}{2}\mathbb{Z}_+$, $a \in \mathbb{K}$ and $j \in \frac{1}{2}\mathbb{N}$, can be obtained by a sequence of reversing the orderings of two adjacent indices $r$ and $s$ with $r \in \mathbb{Z}$ and $s \in \frac{1}{2} + \mathbb{Z}$ satisfying $r, s > 0$ or else $r, s \leq 0$. Thus, the Borel subalgebra with respect to an ordering satisfying the conditions above can be obtained by applying a sequence of odd reflections to the standard Borel subalgebra $\mathfrak{b}$. Moreover, we can choose a sequence of odd reflections in such a way that it leaves the sets of roots of $\tilde{u}$ and $\tilde{u}_-$ invariant.

Hence, the Borel subalgebras $\tilde{\mathfrak{b}}^s(n), \tilde{\mathfrak{b}}^s(n), \tilde{\mathfrak{b}}^{co}(n)$ and $\tilde{\mathfrak{b}}^{os}(n)$ can be obtained from $\tilde{\mathfrak{b}}$ by applying sequences of odd reflections leaving the sets of roots of $\tilde{u}$ and $\tilde{u}_-$ invariant. Let us spell out precisely the sequence of odd reflections required to obtain the Borel subalgebra $\tilde{\mathfrak{b}}^s(n)$ from $\tilde{\mathfrak{b}}$ leaving the sets of roots of $\tilde{u}$ and $\tilde{u}_-$ invariant. This process can be easily modified for the remaining cases. Starting with the Dynkin diagram of $\tilde{\mathfrak{g}}$, we apply the following sequence $\frac{1}{2}n(n - 1)$ odd reflections. First we apply one odd reflection corresponding to $\epsilon_1 - \epsilon_\frac{1}{2}$, then we apply two odd reflections corresponding to $\epsilon_2 - \epsilon_1$ and $\epsilon_1 - \epsilon_\frac{3}{2}$. After that we apply three odd reflections corresponding to $\epsilon_3 - \epsilon_2, \epsilon_2 - \epsilon_\frac{3}{2}$, and $\epsilon_1 - \epsilon_\frac{3}{2}$, et cetera, until finally we apply $n - 1$ odd reflections corresponding to $\epsilon_{n-1} - \epsilon_{n-\frac{1}{2}}, \epsilon_{n-2} - \epsilon_{n-\frac{3}{2}}, \ldots, \epsilon_1 - \epsilon_{n-\frac{1}{2}}$. The corresponding fundamental system of the resulting new Borel subalgebra for $\tilde{\mathfrak{g}}$ is listed as follows:

Now we apply the following sequence of $\frac{1}{2}n(n - 1)$ odd reflections to the Dynkin diagram above. First we apply one odd reflection corresponding to $\epsilon_{-1} - \epsilon_{-1/2}$, then we apply two odd reflections corresponding to $\epsilon_{-2} - \epsilon_{-3/2}$ and $\epsilon_{-2} - \epsilon_{-1/2}$. After that we apply three odd reflections corresponding to $\epsilon_{-3} - \epsilon_{-5/2}, \epsilon_{-3} - \epsilon_{-3/2}$ and $\epsilon_{-3} - \epsilon_{-1/2}$, et cetera, until finally we apply $n - 1$ odd reflections corresponding to $\epsilon_{1-n} - \epsilon_{3/2-n}, \epsilon_{1-n} - \epsilon_{5/2-n}, \ldots, \epsilon_{1-n} - \epsilon_{-1/2}$. We obtain the Dynkin diagram of $\tilde{\mathfrak{b}}^{s}(n)$:

The following lemma is a variant of Lemma 3.1 in our setting.
Lemma 3.4. Let $\alpha$ be an odd simple root of the form $\epsilon_r - \epsilon_s$, $r, s \in \mathbb{I}$ satisfying $r < s$ or $0 < r < s$, in the root system of a Borel subalgebra $\mathcal{B}$ of $\tilde{\mathfrak{g}}$ containing the Cartan subalgebra $\tilde{\mathfrak{h}}$. Let $\mathcal{B}^\alpha$ denote the new Borel subalgebra obtained by applying the odd reflection with respect to $\alpha$ to the Borel subalgebra $\mathcal{B}$. For $\mu \in \mathfrak{h}^*$, let $\nu$ be a $\mathcal{B}$-highest weight vector in $L(\tilde{\mathfrak{i}}, \mu)$.

1. If $\mu(\tilde{E}_{rr}) + \mu(\tilde{E}_{ss}) = 0$, then $L(\tilde{\mathfrak{i}}, \mu)$ is an $\tilde{\mathfrak{i}}$-module of $\mathcal{B}^\alpha$-highest weight $\lambda$ and $\nu$ is a $\mathcal{B}^\alpha$-highest weight vector.

2. If $\mu(\tilde{E}_{rr}) + \mu(\tilde{E}_{ss}) \neq 0$, then $L(\tilde{\mathfrak{i}}, \mu)$ is an $\tilde{\mathfrak{i}}$-module of $\mathcal{B}^\alpha$-highest weight $\lambda - \alpha$ and $E_{sr}\nu$ is a $\mathcal{B}^\alpha$-highest weight vector.

Using similar arguments as in the proofs of [CL2, Lemma 3.2] and [CL2, Corollary 3.3] (cf. [CLW, Proposition 4.3]) together with Lemma 3.4, we have the following.

Proposition 3.5. Let $\lambda = (a, \lambda_1, \ldots, \lambda_k; \lambda^-, \lambda^+) \in P^+_a$ and $n \in \mathbb{N}$.

1. Suppose that $(\lambda^-)_1' \leq n$ and $(\lambda^+)_1' \leq n$. Then the highest weight of $L(\tilde{\mathfrak{i}}, \tilde{\lambda})$ with respect to the Borel subalgebra $\tilde{\mathfrak{b}}^c(n) \cap \tilde{\mathfrak{i}}$ is $\lambda$. Furthermore, $\tilde{\Delta}(\tilde{\lambda})$ and $\tilde{L}(\tilde{\lambda})$ are highest weight $\tilde{\mathfrak{g}}$-modules of highest weight $\lambda$ with respect to the Borel $\tilde{\mathfrak{b}}^c(n)$.

2. Suppose that $(\lambda^-)_1 \leq n$ and $\lambda^+_1 \leq n$. Then the highest weight of $L(\tilde{\mathfrak{i}}, \tilde{\lambda})$ with respect to the Borel subalgebra $\tilde{\mathfrak{b}}(n) \cap \tilde{\mathfrak{i}}$ is $\tilde{\lambda}$. Furthermore, $\tilde{\Delta}(\tilde{\lambda})$ and $\tilde{L}(\tilde{\lambda})$ are highest weight $\tilde{\mathfrak{g}}$-modules of highest weight $\tilde{\lambda}$ with respect to the Borel $\tilde{\mathfrak{b}}(n)$.

3. Suppose that $\lambda^-_1 \leq n$ and $(\lambda^+)_1' \leq n$. Then the highest weight of $L(\tilde{\mathfrak{i}}, \tilde{\lambda})$ with respect to the Borel subalgebra $\tilde{\mathfrak{b}}^c(n) \cap \tilde{\mathfrak{i}}$ is $\tilde{\lambda}$. Furthermore, $\tilde{\Delta}(\tilde{\lambda})$ and $\tilde{L}(\tilde{\lambda})$ are highest weight $\tilde{\mathfrak{g}}$-modules of highest weight $\tilde{\lambda}$ with respect to the Borel $\tilde{\mathfrak{b}}^c(n)$.

4. Super duality

In this section, we first introduce the module categories $\tilde{\mathcal{O}}_a$, $\mathcal{O}_a$, $\mathcal{O}^\circ_a$, $\mathcal{O}^\circ_\mathfrak{g}$ and $\mathcal{O}^\circ_\mathfrak{g}$ for the infinite-rank Lie superalgebras $\tilde{\mathfrak{g}}$, $\mathfrak{g}$, $\mathfrak{g}^\circ$, and $\mathfrak{g}^\circ$, respectively. We also introduce functors $T : \tilde{\mathcal{O}}_a \rightarrow \mathcal{O}_a$, $T^\circ : \mathcal{O}^\circ_\mathfrak{g} \rightarrow \mathcal{O}^\circ_\mathfrak{g}$, and $T^\circ : \tilde{\mathcal{O}}_a \rightarrow \mathcal{O}^\circ_\mathfrak{g}$, and then show that they are equivalences of tensor categories which send parabolic Verma and simple modules to parabolic Verma and simple modules, respectively.

4.1. BGG categories. Let $\tilde{\mathcal{O}}$ be the category of $\tilde{\mathfrak{g}}$-modules $\tilde{M}$ such that $\tilde{M}$ is a semisimple $\tilde{\mathfrak{h}}$-module with finite-dimensional weight subspaces $M_\gamma$, $\gamma \in \mathfrak{h}^*$, satisfying

1. $\tilde{M}$ decomposes over $\tilde{\mathfrak{i}}$ into a direct sum of $L(\tilde{\mathfrak{i}}, \tilde{\mu})$ for $\mu \in P^+$.

2. There exist finitely many weights $\lambda_1, \lambda_2, \ldots, \lambda_k \in P^+$ (depending on $\tilde{M}$) such that if $\gamma$ is a weight in $\tilde{M}$, then $\gamma \in \lambda_i - \sum_{\alpha \in \tilde{\mathfrak{i}}} Z_+ \alpha$, for some $i$. 
The morphisms in $\tilde{\mathcal{O}}$ are (not necessarily even) $\tilde{g}$-homomorphisms. We recall that $\varphi \in \text{Hom}_\tilde{\mathcal{O}}(\tilde{M}, \tilde{N})$ means that $\varphi(xv) = x\varphi(v)$, for $x \in \tilde{g}$ and $v \in \tilde{M}^1$. Let $\tilde{\mathcal{O}}_a$ be the full subcategory of $\tilde{\mathcal{O}}$ consisting of the objects $\tilde{M} \in \tilde{\mathcal{O}}$ such that $Kv = av$ for all $v \in \tilde{M}$. We have $\tilde{\mathcal{O}} = \bigoplus_{a \in \mathbb{C}} \tilde{\mathcal{O}}_a$. The parabolic Verma modules $\tilde{\Delta}(\tilde{\mu})$ and irreducible modules $\tilde{L}(\tilde{\mu})$ for $\tilde{\mu} \in P^+_a$ lie in $\tilde{\mathcal{O}}_a$, by Lemma 4.1 below. Analogously we can define the other four variants $\mathcal{O}$, $\mathcal{O}^\circ$, $\mathcal{O}^\circ$ and $\mathcal{O}^\circ$ of the above categories for $g$, $\tilde{g}$, $g^\circ$, and $\tilde{g}^\circ$ in self-explained notations, which contain the corresponding parabolic Verma and irreducible modules.

Let $\tilde{\Gamma} := \{ \mu \in \tilde{h}^* \mid \mu(E_{rr}) = 0, \ |r| \gg 0 \}$. For $\tilde{V} = \bigoplus_{\tilde{\Gamma}} \tilde{V}_\gamma \in \tilde{\mathcal{O}}_a$, there is a natural $\mathbb{Z}_2$-gradation $\tilde{V} = \tilde{V}_0 \oplus \tilde{V}_1$ given by

\begin{equation}
\tilde{V}_0 := \bigoplus_{\mu \in \tilde{\Gamma}_0} \tilde{V}_\mu \quad \text{and} \quad \tilde{V}_1 := \bigoplus_{\mu \in \tilde{\Gamma}_1} \tilde{V}_\mu,
\end{equation}

where $\tilde{\Gamma}_\epsilon := \{ \mu \in \tilde{\Gamma} \mid \sum_{r \in 1/2+2\mathbb{Z}} \mu(E_{rr}) \equiv \epsilon \pmod{2} \}$ (cf. [CL2, Section 2.5]). Let $\Pi$ denote the parity change functor on $\tilde{\mathcal{O}}$. We define $\tilde{\mathcal{O}}^0_a$ to be the full subcategories of $\tilde{\mathcal{O}}_a$ consisting of objects with $\mathbb{Z}_2$-gradation given by (4.1) and define $\tilde{\mathcal{O}}^1_a$ to be the full subcategories of $\tilde{\mathcal{O}}_a$ consisting of objects $\Pi \tilde{M}$ with $\tilde{M} \in \tilde{\mathcal{O}}^0_a$. Note that the morphisms in $\tilde{\mathcal{O}}^0_a$ and $\tilde{\mathcal{O}}^1_a$ are of degree 0. It is clear that $\tilde{\mathcal{O}}^0_a$ and $\tilde{\mathcal{O}}^1_a$ are an abelian categories.

For $\tilde{M} = \tilde{M}_0 + \tilde{M}_1 \in \tilde{\mathcal{O}}$, let $\tilde{M}_{0,\mu} = \tilde{M}_\mu \cap \tilde{M}_0$ and $\tilde{M}_{1,\mu} = \tilde{M}_\mu \cap \tilde{M}_1$. Also we let

\begin{align*}
\tilde{M}'_0 & := \bigoplus_{\mu \in \tilde{\Gamma}_0} \tilde{M}_{0,\mu}, \quad \tilde{M}'_1 := \bigoplus_{\mu \in \tilde{\Gamma}_1} \tilde{M}_{1,\mu}; \\
\tilde{M}''_0 & := \bigoplus_{\mu \in \tilde{\Gamma}_0} \tilde{M}_{0,\mu}, \quad \tilde{M}''_1 := \bigoplus_{\mu \in \tilde{\Gamma}_1} \tilde{M}_{1,\mu}.
\end{align*}

It is clear that $\tilde{M} := \tilde{M}_0 \oplus \tilde{M}_1$ and $\tilde{M}'' := \tilde{M}''_0 \oplus \tilde{M}''_1$ are submodules of $\tilde{M}$. Since $\tilde{h}$ separates the $\tilde{M}_\mu$’s, we have $\tilde{M}_\mu = \tilde{M}'_\mu \oplus \tilde{M}''_\mu$ for all $\mu$. Therefore $\tilde{M} = \tilde{M}' \oplus \tilde{M}''$. Hence for every $\tilde{M} \in \tilde{\mathcal{O}}$, $\tilde{M}$ is even isomorphic to $\tilde{N}' \oplus \Pi \tilde{N}''$ for some $\tilde{N}'$, $\tilde{N}'' \in \tilde{\mathcal{O}}^0$. For $\tilde{N} \in \tilde{\mathcal{O}}$, let $p_{\tilde{\mathcal{O}}}: \tilde{N} \rightarrow \Pi \tilde{N}$ be the parity-reversing isomorphism. Therefore for every $\tilde{M} \in \tilde{\mathcal{O}}$, $\tilde{M}$ is isomorphic to $\tilde{N}$ for some $\tilde{N} \in \tilde{\mathcal{O}}^0$. This implies that the kernel and the cokernel of $\varphi$ belong to $\tilde{\mathcal{O}}$ for $\tilde{M}$, $\tilde{N} \in \tilde{\mathcal{O}}$, and $\varphi \in \text{Hom}_{\tilde{\mathcal{O}}}(\tilde{M}, \tilde{N})$. Hence $\tilde{\mathcal{O}}$ is an abelian category. Also $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}^0$ have isomorphic skeletons and hence they are equivalent categories. Note that for $\tilde{M}, \tilde{N} \in \tilde{\mathcal{O}}^0$, $\varphi \in \text{Hom}_{\tilde{\mathcal{O}}}(\Pi \tilde{M}, \tilde{N})$ and $\varphi' \in \text{Hom}_{\tilde{\mathcal{O}}}(\tilde{M}, \Pi \tilde{N})$, we have $\varphi = \phi \circ p_{\Pi \tilde{M}}$ and $\varphi' = p_{\Pi \tilde{N}} \circ \phi'$ for some $\phi, \phi' \in \text{Hom}_{\tilde{\mathcal{O}}}(\tilde{M}, \tilde{N})$.

Analogously, $\tilde{\mathcal{O}}^0_a$, $\tilde{\mathcal{O}}^0_a$ and $\tilde{\mathcal{O}}^0_a$ denote the respective full subcategories of $\tilde{\mathcal{O}}_a$, $\tilde{\mathcal{O}}^0_a$ and $\tilde{\mathcal{O}}^0_a$ consisting of objects with $\mathbb{Z}_2$-gradations given by (4.1) such that $r$ are summed over half integers contained in the respective index sets; they are abelian and are equivalent to $\tilde{\mathcal{O}}_a$, $\tilde{\mathcal{O}}^0_a$ and $\tilde{\mathcal{O}}^0_a$, respectively.

Using similar arguments as in the proof of [CLW, Lemma 3.1], we have the following lemma.

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1We remark that a different definition of homomorphism is $\varphi(xv) = (-1)^{|\varphi| |x|} x \varphi(v)$. The difference is inessential, as the map $f \mapsto f^*$ given by $f^*(v) := (-1)^{\ell(f)} f(v)$ provides a bijection between these two types of maps.
**Lemma 4.1.** Let $\mu \in P^+_a$. The restrictions to $\tilde{\lambda}$ of the $\tilde{g}$-modules $\Delta(\tilde{\mu})$ and $\tilde{L}(\tilde{\mu})$ decompose into direct sums of $L(\tilde{\lambda}, \tilde{\nu})$ for $\nu \in P^+_a$. In particular, $\Delta(\tilde{\mu}), \tilde{L}(\tilde{\mu}) \in \mathcal{O}_a$. Analogous results hold for the other four variants of categories for $g, \tilde{g}, g^c$, and $\tilde{g}^c$.

**4.2. The functors** $T, \tilde{T}, T^c$ and $\tilde{T}^c$. We shall introduce four functors $T : \tilde{\mathcal{O}}_a \to \mathcal{O}_a$, $\tilde{T} : \tilde{\mathcal{O}}_a \to \mathcal{O}_a$, $T^c : \tilde{\mathcal{O}}_a \to \tilde{\mathcal{O}}^c_a$ and $\tilde{T}^c : \tilde{\mathcal{O}}_a \to \mathcal{O}^c_a$. We will describe in detail the construction of $T : \tilde{\mathcal{O}}_a \to \mathcal{O}_a$, and the remaining three variants can be treated similarly.

By definition, $g$ is naturally a subalgebra of $\tilde{g}$, $\mathfrak{l}$ is a subalgebra of $\tilde{\lambda}$, and $\mathfrak{h}$ is a subalgebra of $\tilde{\mathfrak{h}}$. Furthermore, we may view $\mathfrak{h}^*$ as a subspace of $\tilde{\mathfrak{h}}^*$ by regarding $\mathfrak{h}$ as a direct summand of $\tilde{\mathfrak{h}}$ with respect to the natural basis $E_{rr}$’s. Given a semisimple $\tilde{\mathfrak{h}}$-module $\overline{M} = \bigoplus_{\gamma \in \tilde{\mathfrak{h}}^*} \overline{M}_\gamma$, we define

$$T(\overline{M}) := \bigoplus_{\gamma \in \mathfrak{h}^*} \overline{M}_\gamma.$$ 

Note that $T(\overline{M})$ is an $\mathfrak{h}$-submodule of $\overline{M}$. It in addition $\overline{M} = \bigoplus_{\gamma \in \tilde{\mathfrak{h}}^*} \overline{M}_\gamma$ is an $\tilde{\lambda}$-module, then $T(\overline{M})$ is an $\tilde{\mathfrak{h}}$-module of $\tilde{M}$. Furthermore, if $\overline{M} = \bigoplus_{\gamma \in \tilde{\mathfrak{h}}^*} \overline{M}_\gamma$ is a $\tilde{g}$-module, then $T(\overline{M})$ is a $g$-submodule of $\overline{M}$.

The direct sum decomposition in $\overline{M}$ gives rise to the natural projection

$$T_{\overline{M}} : \overline{M} \to T(\overline{M})$$

which is an $\mathfrak{h}$-module homomorphism. If in addition $\overline{M}$ is an $\tilde{\lambda}$-module, then $T_{\overline{M}}$ is also an $\tilde{\mathfrak{h}}$-module homomorphism. Furthermore, if $\overline{M}$ is a $\tilde{g}$-module, then $T_{\overline{M}}$ is a $g$-module homomorphism. If $\tilde{f} : \overline{M} \to \overline{N}$ is an $\tilde{\mathfrak{h}}$-homomorphism, then the following induced map

$$T[\tilde{f}] : T(\overline{M}) \to T(\overline{N})$$

is an $\mathfrak{h}$-module homomorphism. If in addition $\tilde{f} : \overline{M} \to \overline{N}$ is an $\tilde{\lambda}$-homomorphism, then $T[\tilde{f}]$ is an $\tilde{\mathfrak{h}}$-module homomorphism. Furthermore, if $\tilde{f} : \overline{M} \to \overline{N}$ is a $\tilde{g}$-homomorphism, then $T[\tilde{f}]$ is a $g$-module homomorphism.

**Lemma 4.2.** For $\lambda \in P^+_a$, we have

$$T(L(\tilde{\lambda}, \tilde{\lambda})) = L(\lambda, \lambda), \quad \tilde{T}(L(\tilde{\lambda}, \tilde{\lambda})) = L(\tilde{\lambda}, \tilde{\lambda}),$$

$$T^c(L(\tilde{\lambda}, \tilde{\lambda})) = L(\tilde{\lambda}, \tilde{\lambda}), \quad \tilde{T}^c(L(\tilde{\lambda}, \tilde{\lambda})) = L(\tilde{\lambda}, \tilde{\lambda}).$$

**Proof.** We will prove $\tilde{T}(L(\tilde{\lambda}, \tilde{\lambda})) = L(\tilde{\lambda}, \tilde{\lambda})$, and the other cases can be proved by the same argument. By Proposition 3.5, $L(\tilde{\lambda}, \tilde{\lambda})$ contains a $v^\ast_{\tilde{\mathfrak{h}}} + (\tilde{\mathfrak{h}} \cap \tilde{\mathfrak{h}})$-highest weight vector $\tilde{v}_{\tilde{\mathfrak{h}}}$ of highest weight $\tilde{\lambda}$ for $n \gg 0$. The vector $\tilde{v}_{\tilde{\mathfrak{h}}}$ clearly lies in $\tilde{T}(L(\tilde{\lambda}, \tilde{\lambda}))$, and it is a $(\tilde{\mathfrak{b}} \cap \tilde{\mathfrak{h}})$-singular vector since $\tilde{\mathfrak{h}} / \tilde{\mathfrak{h}} = \tilde{\mathfrak{h}} \cap \tilde{\mathfrak{h}}^s(n)$. Let $v$ be any weight vector in $\tilde{T}(L(\tilde{\lambda}, \tilde{\lambda}))$. Taking $n$ large enough, we can assume that $v \in U(\tilde{\mathfrak{g}}(n), \tilde{\mathfrak{h}} \cap \tilde{\mathfrak{h}}^s(n))$. Note that the Dynkin diagram of $\tilde{\mathfrak{g}}(n,n)$ with respect to the Borel subalgebra $\mathfrak{B} = \tilde{\mathfrak{g}}(n,n) \cap \tilde{\mathfrak{h}}^s(n)$ is the following:
The lemma now follows by a comparison of the characters on both sides of (i).

Proposition 4.4. The functors $T : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$, $\overline{T} : \tilde{\mathcal{O}} \rightarrow \overline{\mathcal{O}}$, $T^\circ : \tilde{\mathcal{O}} \rightarrow \mathcal{O}^\circ$ and $\overline{T}^\circ : \tilde{\mathcal{O}} \rightarrow \overline{\mathcal{O}}^\circ$ are exact.

Proof. In light of Lemma 4.2, it suffices to show that if $\tilde{M} \in \tilde{\mathcal{O}}$, then $T(\tilde{M})$, $\overline{T}(\tilde{M})$, $T^\circ(\tilde{M})$, and $\overline{T}^\circ(\tilde{M})$ lie in $\mathcal{O}$, $\overline{\mathcal{O}}$, $\mathcal{O}^\circ$, and $\overline{\mathcal{O}}^\circ$, respectively. We shall only show that $T(\tilde{M}) \in \mathcal{O}$, as the argument for the other cases is similar. Also below we will only prove the case $k \geq 1$, as the case $k = 0$ is proved by slightly modifying the argument.

It will be convenient to define $\Pi(\mathfrak{g})$, $\Pi(\mathfrak{L})$ and $\Pi(\tilde{\mathfrak{R}})$ to be the sets $\{\alpha_1, \ldots, \alpha_{k-1}\}$, $\{\alpha_{-1/2}, \alpha_{-1}, \ldots\}$, and $\{\alpha_{1/2}, \alpha_1, \ldots\}$, respectively, for the remainder of the proof.

Let $\tilde{M} \in \tilde{\mathcal{O}}$. Then there exists $1 \zeta_2 \zeta, \ldots, \zeta$, with $1 \zeta_2 \zeta_\zeta \in \mathcal{O}$ such that any weight of $\tilde{M}$ is bounded above by some $\tilde{\zeta}$. Ignoring the level recall that we have $j \tilde{\zeta} = (j \zeta_0, j \zeta^\prime_i \zeta^\prime_i \zeta^\prime_i)$, where $j \zeta^\prime = (j \zeta^\prime_0, \ldots, j \zeta^\prime_k)$, and $(j \zeta^\prime_i \zeta^\prime_i \zeta^\prime_i)$ and $(j \zeta^\prime_i \zeta^\prime_i \zeta^\prime_i)$ are partitions with $|j \zeta^\prime_i \zeta^\prime_i \zeta^\prime_i| = k_j$ and $|j \zeta^\prime_i \zeta^\prime_i \zeta^\prime_i| = l_j$. For each $j \zeta$, let $P_j$ be the following finite subset of $\mathfrak{h}^*$:

$$P_j := \{(j \zeta^0_0, \eta, \mu) \mid \mu, \eta \in \mathcal{P} \text{ with } |\mu| = k_j \text{ and } |\eta| = l_j\}.$$ 

Set $M := T(\tilde{M})$ and $P(M) := \bigcup_{j=1}^k P_j$.

We claim that given any weight $\nu$ of $M$, there exists $\gamma \in P(M)$ such that $\gamma - \nu \in \mathbb{Z}_+ \Pi$. It suffices to prove the claim for $\nu \in \mathcal{P}^+$. Since $\nu$ is also a weight of $\tilde{M}$, we have
\[ \tilde{\zeta} - \nu \in \mathbb{Z} \cap \bar{\Pi}, \text{ for some } i. \text{ Thus} \]
\[ \tilde{\zeta} - \nu = q(\epsilon_0 - \epsilon_\tau) + p(\epsilon_\kappa - \epsilon_{1/2}) + \kappa^- + \kappa^+ + \kappa^0, \]
where \( p, q \in \mathbb{Z}_+, \kappa^0 \in \sum_{\alpha \in \Pi(\ell)} \mathbb{Z}_+ \alpha, \kappa^- \in \sum_{\beta \in \Pi(\ell)} \mathbb{Z}_+ \beta, \) and \( \kappa^+ \in \sum_{\gamma \in \Pi(\ell)} \mathbb{Z}_+ \beta. \) This implies that \( \nu^- \) is a partition of size \( l_i + q \) and \( \nu^+ \) is a partition of \( k_i + p, \) and hence there exists \( \gamma \in \mathcal{P}(M) \) such that \( \nu^\pm \) and \( \nu^- \) are obtained from the partitions \( \gamma^+ \) and \( \gamma^- \) by adding \( p \) and \( q \) boxes, respectively. For every such a box of \( \nu^\pm, \) we record the row number in which it was added to \( \gamma^\pm \) in the multisets \( J^\pm \) with \( |J^+| = p \) and \( |J^-| = q. \) Then we have
\[ \nu = \gamma - 0 \kappa - \sum_{j \in J^+} (\epsilon_\kappa - \epsilon_j) - \sum_{j \in J^-} (\epsilon_{-j+1} - \epsilon_\tau), \]
and hence \( \nu < \gamma. \) Thus, we conclude that \( M \notin \mathcal{O}. \) \hfill \( \Box \)

4.3. A character formula. The following theorem can be regarded as a weak version of super duality which is to be established in Theorem 4.11. Using similar arguments as in the proof of [CLW, Theorem 4.5], we have the following.

**Theorem 4.5.** Let \( \widetilde{M} \in \tilde{\mathcal{O}}_a \) and \( \lambda \in P_a^+. \) If \( \widetilde{M} \) is a highest weight \( \tilde{\mathfrak{g}} \)-module of highest weight \( \tilde{\lambda}, \) then \( T(\widetilde{M}), \tilde{T}(\widetilde{M}), T^\circ(\widetilde{M}) \) and \( \tilde{T}^\circ(\widetilde{M}) \) are highest weight modules over \( \mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{g}^\circ \) and \( \tilde{\mathfrak{g}}^\circ \) of highest weights \( \lambda, \tilde{\lambda}, \lambda^\circ \) and \( \tilde{\lambda}^\circ, \) respectively. Furthermore, we have

(i) \( T(\tilde{\Delta}(\lambda)) = \Delta(\lambda) \) and \( T(\tilde{L}(\lambda)) = L(\lambda); \)

(ii) \( \tilde{T}(\Delta(\lambda)) = \Delta(\tilde{\lambda}) \) and \( \tilde{T}(\tilde{L}(\lambda)) = L(\tilde{\lambda}); \)

(iii) \( T^\circ(\tilde{\Delta}(\lambda)) = \Delta^\circ(\lambda^\circ) \) and \( T^\circ(\tilde{L}(\lambda)) = L^\circ(\lambda^\circ); \)

(iv) \( \tilde{T}^\circ(\tilde{\Delta}(\lambda)) = \Delta^\circ(\tilde{\lambda}^\circ) \) and \( \tilde{T}^\circ(\tilde{L}(\lambda)) = L^\circ(\tilde{\lambda}^\circ). \)

By standard arguments, Theorem 4.5 implies the following character formula.

**Theorem 4.6.** Let \( \lambda \in P_a^+, \) and write \( \text{ch} L(\lambda) = \sum_{\mu \in P_a^+} a_{\mu \lambda} \text{ch} \Delta(\mu), \) \( a_{\mu \lambda} \in \mathbb{Z}. \) Then

(i) \( \text{ch} \tilde{L}(\tilde{\lambda}) = \sum_{\mu \in P_a^+} a_{\mu \lambda} \text{ch} \tilde{\Delta}(\mu^0), \)

(ii) \( \text{ch} \tilde{L}(\tilde{\lambda}) = \sum_{\mu \in P_a^+} a_{\mu \lambda} \text{ch} \tilde{\Delta}(\mu^0), \)

(iii) \( \text{ch} \tilde{L}(\tilde{\lambda}) = \sum_{\mu \in P_a^+} a_{\mu \lambda} \text{ch} \tilde{\Delta}(\mu^0), \)

(iv) \( \text{ch} \tilde{L}(\tilde{\lambda}) = \sum_{\mu \in P_a^+} a_{\mu \lambda} \text{ch} \tilde{\Delta}(\mu^0). \)

4.4. Identification of Kazhdan-Lusztig polynomials. For a module \( V \) over a Lie (super)algebra \( \mathfrak{g} \) and \( j \in \mathbb{Z}_+, \) let \( H_j(\mathfrak{g}; V) \) denote the \( j \)th homology group of \( \mathfrak{g} \) with coefficient in \( V. \) For a precise definition of homology groups of Lie superalgebras with coefficients in a module and a precise formula for the boundary operator we refer the reader to [Ta] or [CL2, Section 4].

For \( \widetilde{M} \in \tilde{\mathcal{O}}_a \) we denote by \( M = T(\widetilde{M}) \) and \( \widehat{M} = \tilde{T}(\widetilde{M}), M^\circ = T^\circ(\widetilde{M}) \) and \( \widehat{M}^\circ = \tilde{T}^\circ(\widetilde{M}). \) Using similar arguments as in the proof of [CLW, Theorem 4.10], we have the following.

**Theorem 4.7.** We have, for \( j \geq 0, \)

(i) \( T(H_j(\mathfrak{u}_-; \widetilde{M})) \cong H_j(\mathfrak{u}_-; M), \) as \( \mathfrak{t} \)-modules.
Theorem 4.9. \quad \text{Let } \ell \in P_a^+ \text{ and } j \geq 0, \text{ we have}

(i) \( T(H_j(u_\cdot; \bar{L}))(\bar{L}(\lambda)) \cong H_j(u_\cdot; L(\lambda)), \) as \( \bar{L} \)-modules.

(ii) \( \widetilde{T}(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; \bar{L}(\lambda)), \) as \( \bar{L} \)-modules.

(iii) \( T^\circ(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; L^\circ(\lambda)), \) as \( L^\circ \)-modules.

(iv) \( \widetilde{T}^\circ(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; \bar{L}(\lambda)), \) as \( \bar{L} \)-modules.

Setting \( \bar{M} = \bar{L}(\lambda) \) in Theorem 4.7 and using Theorem 4.5 we obtain the following.

Corollary 4.8. \quad \text{Let } \ell \in P_a^+ \text{ and } j \geq 0, \text{ we have}

(i) \( T(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; L(\lambda)), \) as \( \bar{L} \)-modules.

(ii) \( \widetilde{T}(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; \bar{L}(\lambda)), \) as \( \bar{L} \)-modules.

(iii) \( T^\circ(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; L^\circ(\lambda)), \) as \( L^\circ \)-modules.

(iv) \( \widetilde{T}^\circ(H_j(u_\cdot; \bar{L}(\lambda)) \cong H_j(u_\cdot; \bar{L}(\lambda)), \) as \( \bar{L} \)-modules.

We define the \textit{parabolic Kazhdan-Lusztig polynomials} for the categories \( \bar{O}_a, \bar{O}_a, \bar{O}_a, \bar{O}_a, \bar{O}_a \) associated to \( \mu, \lambda \in P_a^+ \) by letting

\[
\hat{\ell}_{\mu \lambda}(q) := \sum_{n=0}^{\infty} \dim_{\mathbb{C}} \left( \text{Hom}_{\bar{L}}[\bar{L}(\lambda), H_j(u_\cdot; \bar{L}(\lambda))] \right)(-q)^{-j},
\]

\[
\ell_{\mu \lambda}(q) := \sum_{n=0}^{\infty} \dim_{\mathbb{C}} \left( \text{Hom}_{\bar{L}}[\bar{L}(\lambda), H_j(u_\cdot; \bar{L}(\lambda))] \right)(-q)^{-j},
\]

\[
\ell_{\mu \lambda}(q) := \sum_{n=0}^{\infty} \dim_{\mathbb{C}} \left( \text{Hom}_{\bar{L}}[\bar{L}(\lambda), H_j(u_\cdot; \bar{L}(\lambda))] \right)(-q)^{-j},
\]

\[
\ell_{\mu \lambda}(q) := \sum_{n=0}^{\infty} \dim_{\mathbb{C}} \left( \text{Hom}_{\bar{L}}[\bar{L}(\lambda), H_j(u_\cdot; \bar{L}(\lambda))] \right)(-q)^{-j},
\]

\[
\ell_{\mu \lambda}(q) := \sum_{n=0}^{\infty} \dim_{\mathbb{C}} \left( \text{Hom}_{\bar{L}}[\bar{L}(\lambda), H_j(u_\cdot; \bar{L}(\lambda))] \right)(-q)^{-j}.
\]

By Vogan’s homological interpretation of the Kazhdan-Lusztig polynomials [Vo, Conjecture 3.4] and the Kazhdan-Lusztig conjecture [KL] (proved in [BB, BK]), \( \ell_{\mu \lambda}(q) \) coincides with the original definition in the setting of semisimple Lie algebras and moreover \( \ell_{\mu \lambda}(1) = a_{\mu \lambda} \) (cf. Theorem 4.6). The following reformulation of Corollary 4.8 is a generalization of Theorem 4.6.

Theorem 4.9. \quad \text{For } \lambda, \mu \in P_a^+, \text{ we have the following identification of Kazhdan-Lusztig polynomials for the categories } \bar{O}_a, \bar{O}_a, \bar{O}_a, \bar{O}_a, \bar{O}_a:

\[
\hat{\ell}_{\mu \lambda}(q) = \ell_{\mu \lambda}(q) = \ell_{\mu \lambda}(q) = \ell_{\mu \lambda}(q) = \ell_{\mu \lambda}(q).
\]

4.5. Super duality.

Lemma 4.10. \quad \text{Let}

\[
0 \to M' \to M \to M'' \to 0
\]

be an exact sequence of \( \bar{g} \) (respectively \( g, g, g \) and \( g \))-modules which are semisimple over \( h \) (respectively \( h, h, h \) and \( h \)) such that \( M', M'' \in \bar{O}_a \) (respectively \( O_a, \bar{O}_a, O_a \) and \( \bar{O}_a \)). Then \( M \) also belongs to \( O_a \) (respectively \( O_a, \bar{O}_a, O_a \) and \( \bar{O}_a \)).
Proof. The statement for \( O_a \) is clear. The statements for the other cases follow by arguments, for example, as for [CK, Theorems 3.1 and 3.2]. \( \square \)

The arguments in [CLW, Section 5] can be adapted to prove the following theorem.

**Theorem 4.11 (Super Duality).** The functors \( T: \widetilde{O}_a \to O_a, \overline{T}: \overline{O}_a \to \overline{O}_a, T^\circ: \overline{O}_a \to \overline{O}_a\) and \( \overline{T}^\circ : \overline{O}_a \to \overline{O}_a\) are equivalences of categories.

**Remark 4.12.** In an extreme yet interesting case, the Lie superalgebra \( D(\tilde{g})\) in Section 2.1 (when \( k = 0 \)) is associated to the following Dynkin diagram with all simple roots being odd; its subalgebra \( D(g) \) and \( D(\overline{g}) \) in Section 2.2 are Lie algebras associated to the following Dynkin diagrams and fundamental systems:

\[
\begin{align*}
D(\tilde{g}) : & \quad \cdots \hspace{1cm} \times \hspace{1cm} \times \hspace{1cm} \times \hspace{1cm} \times \hspace{1cm} \times \hspace{1cm} \times \hspace{1cm} \times \hspace{1cm} \cdots \\
D(g) : & \quad \cdots \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \cdots \\
D(\overline{g}) : & \quad \cdots \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \circ \hspace{1cm} \cdots 
\end{align*}
\]

In the case when \( \ell \in \mathbb{N} \), we let \( \mathcal{F} \) be the (semisimple) subcategory of \( O_\ell \) of integrable \( \tilde{g}_{\lambda_\infty} \)-modules of level \( \ell \). The corresponding subcategory of \( \overline{O}_\ell \) is a suitable subcategory \( \mathcal{U} \) of \( \tilde{g}_{\lambda_\infty} \)-modules of negative level \( -\ell \). By Theorem 4.11, \( \mathcal{U} \) is equivalent to \( \mathcal{F} \), and hence it is semisimple; moreover, we recover the character formulas for modules in \( \mathcal{U} \) in [KR] and [CL1] that were obtained by completely different methods.

4.6. **Equivalence of tensor categories.** Recall that any modules in \( \overline{O}_a^0, \overline{O}_a^0, \overline{O}_a^0 \) and \( \overline{O}_a^0 \) are equipped natural \( \mathbb{Z}_2 \)-gradations given by (4.1). Note that the functors \( T, \overline{T}, T^\circ \) and \( \overline{T}^\circ \) preserve the \( \mathbb{Z}_2 \)-gradations of the modules. For \( \tilde{M}, \tilde{N} \in \overline{O}_a^0 \), we have \( \tilde{M} \otimes \tilde{N} \in \overline{O}_a^0 \) by [CK, Theorem 3.2] and hence \( \overline{O}_a^0 \) is a commutative tensor category (see, for example, [KS, Section 4.2]) with the trivial module as the unit object. Similarly, \( O, \overline{O}_a^0, \overline{O}_a^0 \) and \( \overline{O}_a^0 \) are commutative tensor categories with the unit objects (cf. [CK, Theorem 3.1, 3.2]).

By Lemma 4.3 and Theorem 4.11, we have the following theorem.

**Theorem 4.13.** The functors \( T: \overline{O}^0 \to O, \overline{T}: \overline{O}^0 \to \overline{O}^0, T^\circ: \overline{O}^0 \to O^{\circ 0}, \) and \( \overline{T}^\circ: \overline{O}^0 \to \overline{O}^{\circ 0} \) are equivalences of tensor categories.

5. **Tilting modules**

For \( \tilde{M} \in \overline{O}_a^0 \) such that \( \tilde{M} = \sum_{\mu \in \tilde{h}} \tilde{M}_\mu \), we consider the restricted dual

\[ \tilde{M}^\circ := \{ f \in \text{Hom}_C(\tilde{M}, C) \mid f(\tilde{M}_\gamma) = 0, \text{ for all but finitely many } \gamma \in \tilde{h}^* \}, \]

which is equipped with the usual \( \mathbb{Z}_2 \)-gradation and \( \tilde{g} \)-action. We twist the usual \( \tilde{g} \)-module structure on \( \tilde{M}^\circ \) with the automorphism \( \tau \) defined in (2.2), and denote the
resulting $\tilde{g}$-module by $\tilde{M}^\tau$. The $Z_2$-gradation on $\tilde{M}^\tau$ equals the $Z_2$-gradation defined by (4.1). There are natural isomorphisms $(\tilde{M}^\tau)^{\tau} = \tilde{M}$. It is easy to see that $\text{ch}(\tilde{M}^\tau) = \text{ch}(\tilde{M})$ and $\tilde{L}(\tilde{\lambda})^{\tau} = \tilde{L}(\tilde{\lambda})$. The restriction of $\tau$ to $\tilde{I}$ is also denoted by $\tau$. Similarly, we can define $L(\tilde{\lambda},\tilde{\mu})^\tau$ for $\mu \in P_0^+$. Note that $\tilde{M}^\tau \in \tilde{O}_a^0$ for all $\tilde{M} \in \tilde{O}_a^0$ since $L(\tilde{\lambda},\tilde{\mu})^\tau \cong L(\tilde{\lambda},\tilde{\mu})$ for all $\mu \in P_a^+$. The contravariant functor from $\tilde{O}_a^0$ to itself defined by sending $\tilde{M}$ to $\tilde{M}^\tau$ is also denoted by $\tau$. Note that the functor $\tau$ is an isomorphism of categories.

The restrictions of the automorphism $\tau$ of $\tilde{g}$ to $\tilde{g}$, $\tilde{g}^\circ$ and $\tilde{g}^\circ$ are also denoted by $\tau$.

Similarly, we can define $M^\tau$ for an object $M$ in $O_a$, $O_a^0$, $O_a^0$ and $O_a^0$. We show that $M^\tau$ belongs to the same category as $M$ and $(M^\tau)^{\tau} = M$. Also we have $\text{ch}(M^\tau) = \text{ch}(M)$ and $L^\tau = L$ for all irreducible highest weight modules $L$.

We summarize the above discussion in the following.

**Proposition 5.1.** We have $\mathfrak{S} \circ \tau = \tau \circ \mathfrak{S}$ for $\mathfrak{S} = T, T^\circ$ or $T^\circ$.

An object $M \in O_a$ is said to have a Verma flag if it has a (possibly infinite) increasing filtration of $g$-modules:

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

such that $M = \cup_{i \geq 0} M_i$ and each $M_i/M_{i-1}$ is either 0 or isomorphic to a parabolic Verma module $\Delta(\lambda^i)$ for some $\lambda^i \in P^+$. We define $(M : \Delta(\mu))$ for $\mu \in P^+$ to be the number of subquotients of a Verma flag of $M$ that are isomorphic to $\Delta(\mu)$. The notion of Verma flag is defined in a similar fashion in the categories $\tilde{O}_a^0$, $\tilde{O}_a^0$, $\tilde{O}_a^0$ and $\tilde{O}_a^0$. A tilting module associated to $\lambda \in P^+$ in the category $O$ is an indecomposable $g$-module $U(\lambda)$ such that

1. $U(\lambda)$ has a Verma flag with $\Delta(\lambda)$ at the bottom, and
2. $\text{Ext}^1(\Delta(\mu), U(\lambda)) = 0$ for all $\mu \in P^+$.

For $\mu \in P^+$, the tilting module $U(\mu)$ (respectively $U(\bar{\mu})$, $U^<(\mu^\circ)$, $U^<(\bar{\mu}^\circ)$) in $\tilde{O}_a^0$ (respectively $\tilde{O}_a^0$, $\tilde{O}_a^0$, $\tilde{O}_a^0$) are defined in a similar fashion.

From the proposition below, the tilting module $U(\mu)$ (respectively $U(\lambda)$, $U(\bar{\mu})$, $U^<(\mu^\circ)$, $U^<(\bar{\mu}^\circ)$) belongs to $\tilde{O}_a^0$ (respectively $\tilde{O}_a^0$, $\tilde{O}_a^0$, $\tilde{O}_a^0$, $\tilde{O}_a^0$) for $\mu \in P_a^+$.

**Proposition 5.2.** For $\lambda \in P_a^+$, there exists a unique (up to isomorphism) tilting module $U(\lambda)$ associated to $\lambda$ in $O_a$. There also exists a unique (up to isomorphism) tilting module $U(\bar{\lambda})$ (respectively $\bar{U}(\lambda)$, $U^<(\lambda^\circ)$, $U^<(\bar{\lambda}^\circ)$) associated to $\bar{\lambda}$ (respectively $\bar{\lambda}$, $\lambda^\circ$, $\bar{\lambda}^\circ$) in $\tilde{O}_a^0$ (respectively $\tilde{O}_a^0$, $\tilde{O}_a^0$, $\tilde{O}_a^0$). Moreover, we have $T(U(\lambda)) = U(\lambda)$, $T(U(\bar{\lambda})) = U(\bar{\lambda})$, $T^<(U(\lambda)) = U(\lambda^\circ)$ and $T^<(U(\bar{\lambda})) = U(\bar{\lambda}^\circ)$ for $\lambda \in P_a^+$, and $T^\tau = U$ for any tilting module $U$ in either of these categories.

**Proof.** The same proof for [CW, Theorem 3.14] ensures the existence of the tilting modules. From the construction of the tilting modules, we have $U(\lambda)^{\tau} = U(\lambda)$ for $\lambda \in P_a^+$.

The uniqueness of the tilting modules follows by adapting an argument of Soergel [So, Section 5] (also cf. [Br2]). As the category $O_a$ may not have enough injective objects, we need to modify the proof therein as follows. First we remark that every element in $\text{End}(U(\lambda))$ for $\lambda \in P_a^+$ is either locally nilpotent or an isomorphism, since
Therefore we have

\[ 0 \to \text{Hom}(U(\lambda)/\Delta(\lambda), U'(\lambda)) \to \text{Hom}(U(\lambda), U'(\lambda)) \to \text{Hom}(\Delta(\lambda), U'(\lambda)) \to 0. \]

Hence there is \( \varphi \in \text{Hom}(U(\lambda), U'(\lambda)) \) which restricts to the identity map on \( \Delta(\lambda) \).

Similarly, there is \( \varphi' \in \text{Hom}(U'(\lambda), U(\lambda)) \) which restricts to the identity map on \( \Delta(\lambda) \).

Therefore \( \varphi \circ \varphi' \) and \( \varphi' \circ \varphi \) are isomorphisms since they are not locally nilpotent. Hence \( U(\lambda) \) and \( U'(\lambda) \) are isomorphic.

By Theorem 4.13, Theorem 4.5 and the first part of the proposition, tilting modules exist and are unique in the categories \( \bar{\mathcal{O}}^0_a, \bar{\mathcal{O}}_a, \mathcal{O}^0_a \), and \( \mathcal{O}^0_a \). The last part of the theorem also follows from Proposition 5.1, Theorem 4.5 and \( U(\lambda)^\tau = U(\lambda) \) for \( \lambda \in P^+_a \). \( \square \)

**Remark 5.3.** The argument above can be applied to show that there exists a unique tilting module associated to each dominant weight in the categories \( \mathcal{O}, \bar{\mathcal{O}}, \) and \( \bar{\mathcal{O}} \) defined in [CLW, Section 3.2] as well.

### 6. Character formula for general linear Lie superalgebras

In this section we use the super duality and truncation functors to obtain character formulas for irreducible modules of general linear Lie superalgebras of finite rank.

#### 6.1. Module categories of finite-rank Lie superalgebras

We recall that the finite-rank Lie superalgebras \( \mathfrak{g}^\circ(m, n), \mathfrak{h}(m, n), \mathfrak{g}(m, n), \mathfrak{f}(m, n) \), and \( \mathfrak{f}(m, n) \) are defined in Section 2.2 for \( m, n \in \mathbb{Z}_+ \). In this subsection, we introduce certain module categories of these finite-rank Lie superalgebras, and relate them to the categories studied earlier via the truncation functors (see Proposition 6.2).

Let \( \mathcal{O}(m, n) \) be the full subcategory of \( \mathfrak{g}(m, n) \)-modules \( M \) such that \( M \) is a semisimple \( \mathfrak{h}(m, n) \)-module with finite-dimensional weight subspaces \( M_{\gamma} \), \( \gamma \in \mathfrak{h}(m, n)^* \), satisfying

(i) \( M \) decomposes over \( \mathfrak{l}(m, n) \) into a direct sum of \( L(\mathfrak{l}(m, n), \mu) \) for \( \mu \in P^+ \).

(ii) There exist finitely many weights \( \lambda_1, \lambda_2, \ldots, \lambda_k \in P^+ \) (depending on \( M \)) such that if \( \gamma \) is a weight in \( M \), then \( \gamma \in \lambda_i - \sum_{\alpha \in \Pi(m, n)} \mathbb{Z}_+ \alpha \), for some \( i \).

Recall the central element \( K \) in the Lie superalgebra \( \mathfrak{g}(m, n) \). Let \( \mathcal{O}(m, n) \) be the full subcategory of \( \mathfrak{O}(m, n) \) consisting of the objects \( M \in \mathcal{O}(m, n) \) such that \( Kv = av \) for all \( v \in M \). We certainly have \( \mathcal{O}(m, n) = \bigoplus_{a \in \mathbb{C}} \mathcal{O}_a(m, n) \). The parabolic Verma modules \( \Delta(m, n)(\mu) \) and irreducible modules \( L(m, n)(\mu) \) for \( \mu \in P^+_a \cap \mathfrak{h}(m, n)^* \) lie in \( \mathcal{O}_a(m, n) \), by Proposition 6.2 below. The categories \( \mathcal{O}_a(m, n) \) are abelian. Analogously we have four variants of the above categories with similar properties for Lie superalgebras \( \mathfrak{g}^\circ(m, n), \mathfrak{f}(m, n), \mathfrak{g}^\circ(m, n), \) and \( \mathfrak{f}(m, n) \) in self-explanatory notations. For \( \bar{V} \in \mathcal{O}_a(m, n) \), we
define a natural $\mathbb{Z}_2$-gradation $\widetilde{V} = V_0 \oplus V_1$ similar to (4.1). This allows us to define a subcategory $\widetilde{\mathcal{O}}_0(m, n)$ of $\mathcal{O}_0(m, n)$ as before and analogously we have three variants of the above categories for Lie superalgebras $\mathfrak{g}(m, n)$, $\mathfrak{g}^\circ(m, n)$, and $\mathfrak{g}^\circ(m, n)$.

Let $M$ be an $\mathfrak{h}$-, $\mathfrak{h}^-$, $\mathfrak{h}^+$- or $\mathfrak{h}$-module such that $M = \bigoplus_{\gamma} M_\gamma$, where $\gamma$ runs over $\gamma \in \sum_{i \in \mathbb{Z}_+} \mathbb{C} \varepsilon_i + \sum_{1 \leq j \leq k} \mathbb{C} \epsilon_j + \sum_{j \in \mathbb{N}} \mathbb{C} \epsilon_j + \mathbb{C} \Lambda_0$. For $m, n \in \mathbb{Z}_+ \cup \{\infty\}$, we consider the truncated vector space

$$\text{tr}_{(m,n)}(M) = \bigoplus_{\nu} M_\nu,$$

where the sum is over $\nu \in \sum_{-m \leq j \leq 0} \mathbb{C} \varepsilon_i + \sum_{1 \leq j \leq k} \mathbb{C} \epsilon_j + \sum_{1 \leq j \leq n+1} \mathbb{C} \epsilon_j + \mathbb{C} \Lambda_0$.

Lemma 6.1. Let $m, n \in \mathbb{Z}_+ \cup \{\infty\}$ and $\mu \in P_a^+$. Then, $\text{tr}_{(m,n)}(\tilde{L}(\tilde{l}, \tilde{\mu}))$ is an $\tilde{l}(m, n)$-module, and

$$\text{tr}_{(m,n)}(\tilde{L}(\tilde{l}, \tilde{\mu})) = \begin{cases} \tilde{L}(l(m, n), \tilde{\mu}), & \text{if } \langle \tilde{\mu}, \tilde{E}_{jj} \rangle = 0, \forall j \leq -m, \forall j \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Parallel results hold for the truncation of $L(l, \mu), \bar{L}(\bar{l}, \bar{\mu}), L^\circ(l, \mu), \bar{L}^\circ(\bar{l}, \bar{\mu})$.

As a consequence of the lemma above, we obtain exact functors $\text{tr}_{(m,n)} : \widetilde{\mathcal{O}}_0 \to \mathcal{O}_0(m, n)$, $\text{tr}_{(m,n)} : \mathcal{O}_a \to \mathcal{O}_a(m, n)$, $\text{tr}_{(m,n)} : \widetilde{\mathcal{O}}_0 \to \mathcal{O}_0(m, n)$, $\text{tr}_{(m,n)} : \mathcal{O}_a^\circ \to \mathcal{O}_a^\circ(m, n)$ and $\text{tr}_{(m,n)} : \widetilde{\mathcal{O}}_0 \to \mathcal{O}_a^\circ(m, n)$, by sending $M$ to $\text{tr}_{(m,n)}(M)$.

The following lemma can be proved by the same arguments as in [CLW, Lemma 3.2] and [Don, Proposition 1.5].

Proposition 6.2. Let $m, n \in \mathbb{Z}_+ \cup \{\infty\}$, $\mu \in P_a^+$ and $X = L, \Delta, U$.

(i) $\text{tr}_{(m,n)}(\tilde{X}(\tilde{\mu})) = \begin{cases} \tilde{X}(X(m, n), \tilde{\mu}), & \text{if } \langle \tilde{\mu}, \tilde{E}_{jj} \rangle = 0, \forall j \leq -m & j \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$

(ii) $\text{tr}_{(m,n)}(X(\mu)) = \begin{cases} X(X(m, n), \mu), & \text{if } \langle \mu, \bar{E}_{jj} \rangle = 0, \forall j \leq -m & j \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$

(iii) $\text{tr}_{(m,n)}(\bar{X}(\bar{\mu})) = \begin{cases} \bar{X}(X(m, n), \bar{\mu}), & \text{if } \langle \bar{\mu}, \bar{E}_{jj} \rangle = 0, \forall j \leq -m & j \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$

(iv) $\text{tr}_{(m,n)}(X^\circ(\mu^\circ)) = \begin{cases} X^\circ(X(m, n), \mu^\circ), & \text{if } \langle \mu^\circ, \tilde{E}_{jj} \rangle = 0, \forall j \leq -m & j \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$

(v) $\text{tr}_{(m,n)}(X^\circ(\mu^\circ)) = \begin{cases} X^\circ(X(m, n), \mu^\circ), & \text{if } \langle \mu^\circ, \tilde{E}_{jj} \rangle = 0, \forall j \leq -m & j \geq n+1, \\ 0, & \text{otherwise.} \end{cases}$

Remark 6.3. Let $\lambda \in P^+$ and $m, n \in \mathbb{Z}_+ \cup \{\infty\}$. Let $d$ denote the boundary operators for the complexes of the homology groups $H_*([\mathfrak{m}; \tilde{L}(\lambda)])$, $H_*(u_+; L(\lambda))$, $H_*(\bar{u}_+; \bar{L}(\lambda))$, $H_*([\mathfrak{m}; \bar{L}(\lambda)])$, $H_*([\mathfrak{m}; \bar{L}(\lambda)])$, and $H_*([\mathfrak{m}; L(\lambda)])$. From the formula of $d$ (see e.g. [CL2, (4.1)]) it is easy to see that the truncation functors are compatible with $d$, i.e.,

$$d \circ \text{tr}_{(m,n)} = \text{tr}_{(m,n)} \circ d.$$
It follows that truncation functors $\tau_{(m,n)}$ send these $u$-homology groups to the corresponding $u$-homology groups, and the corresponding module multiplicities of Levi subalgebras (or equivalently the corresponding Kazhdan-Lusztig polynomials) match under the truncation functors.

6.2. Character formulas for $\mathfrak{gl}(k|2)$-modules in the BGG category. Let $\lambda \in P^+$. Then Proposition 6.2 and Theorem 4.6 together imply that, for any $m, n \in \mathbb{Z}_+$, the character of the irreducible $\overline{\mathfrak{g}}^m(m|n)$-module $\overline{L}_{m,n}(\lambda)$ can be computed from knowledge of the irreducible character of the $\mathfrak{g}$-module $L(\lambda)$. On the other hand $\text{ch} L(\lambda)$ is determined by the the coefficients $a_{\mu \lambda} = \ell_{\mu \lambda}(1)$, where the polynomials $\ell_{\mu \lambda}(q)$ are type $A$ parabolic Kazhdan-Lusztig polynomials (see Section 4.4).

Recalling the notations from Section 6.1 we note that $\mathfrak{g} \times (1,0) \cong \mathfrak{gl}(k|2) \oplus \mathbb{C}K$ (see Section 2.1), where $\mathfrak{gl}(k|2)$ denotes the Lie subalgebra of linear transformations on $\bigoplus_{i=1}^k \mathbb{C}v_i \oplus \mathbb{C}v_{-\frac{1}{2}} \oplus \mathbb{C}v_{\frac{1}{2}}$. Recall that $P^+ = \bigcup_{\alpha \in \mathbb{C}} P^+_a$. Now choose $Y_0 = \emptyset$ so that $\overline{\mathfrak{g}}^0(1,0)$ is simply the Cartan subalgebra of $\mathfrak{gl}(k|2)$. Now for a fixed $a \in \mathbb{C}$, $m, n \in \mathbb{N}$ and $\lambda^0_j \in \mathbb{Z}$ we have

$$\lambda = a\Lambda_0 - \sum_{i=0}^{m-1} \epsilon_{-i} + \sum_j \lambda^0_j \epsilon_j + \sum_{l=1}^n \epsilon_l \in P^+_a.$$

This corresponds to the dominant tuple $(a, \lambda^0_1, \ldots, \lambda^0_k; \lambda^-, \lambda^+)$, with $\lambda^- = (1^m)$, and $\lambda^+ = (1^n)$. Thus, we have by Lemma 3.2

$$\overline{\lambda} = a\Lambda_0 - m\epsilon_{-\frac{1}{2}} + \sum_j \lambda^0_j \epsilon_j + n\epsilon_{\frac{1}{2}}.$$

Now let $\mu = p\epsilon_{-\frac{1}{2}} + \sum_{j=1}^k \lambda^0_j \epsilon_j + q\epsilon_{\frac{1}{2}}$ be any integral weight of $\mathfrak{gl}(k|2)$. Suppose we want to compute the character of the irreducible module with highest weight $\mu$ with respect to the Borel associated to the following fundamental system:

$$\epsilon_{-\frac{1}{2}} \alpha - \epsilon_{\frac{1}{2}} \alpha - \epsilon \alpha - \epsilon_0 \alpha - \epsilon_{-\frac{1}{2}} \alpha - \epsilon_{\frac{1}{2}} \alpha.$$
Note that the Dynkin diagram in (6.2) is related to the standard Dynkin diagram (6.3) by a sequence of odd reflections with respect to the following odd roots:

\[(6.4) \quad \epsilon_{-\frac{1}{2}} - \epsilon_{\frac{1}{2}} - \epsilon_{\frac{1}{2}}, \ldots, \epsilon_{-\frac{1}{2}} - \epsilon_{\frac{k}{2}}.\]

Let us use $\Delta'(\gamma)$ and $L'(\gamma)$ to denote the $\mathfrak{gl}(k|2)$-Verma and irreducible modules of highest weight $\gamma$ with respect to the Borel associated to (6.3). Now suppose that we have for $\lambda \in P^+_a$

\[
\text{ch}\Delta'(\lambda) = \sum_{\mu} a_{\mu, \lambda} \text{ch}\Delta'(\mu),
\]

where $\Delta'(\mu)$ is the Verma module with respect to the Borel in (6.2). Let $[\lambda]'$ be the highest weight obtained from $\Delta'$ by applying the sequence of odd reflections in (6.4) following the prescription of Lemma 3.4. Furthermore, we observe that

\[
\text{ch}\Delta'(\mu) = \text{ch}\Delta'(\mu - k\epsilon_{-\frac{1}{2}} + \sum_{i=1}^{k} \epsilon_i).
\]

Thus, we conclude that

\[
\text{ch}L'(\lambda) = \sum_{\mu} a_{\mu, \lambda} \text{ch}\Delta'(\mu + \sum_{i=1}^{k} \epsilon_i - k\epsilon_{-\frac{1}{2}}).
\]

Therefore, the Kazhdan-Lusztig polynomials of type $A$ also solve the character problem for irreducible highest weight modules with highest weights with respect to the standard Borel of $\mathfrak{gl}(k|2)$.

### 6.3. A variant of Brundan’s conjecture.

For notations on Fock spaces below, we will refer to [CWZ, CW] (which differs somewhat from [Br1]). Let $\mathcal{V}$ with basis $\{v_i\}_{i \in \mathbb{Z}}$ denote the natural $U_q(\mathfrak{gl}_\infty)$-module and let $\mathcal{V}^*$ denote its dual module with basis $\{w_i\}_{i \in \mathbb{Z}}$. The “Fock space” $\wedge^m \mathcal{V}^* \otimes \mathcal{V}^\otimes k \otimes \wedge^n \mathcal{V}^*$ admits a standard basis (using $v_i$’s and $w_i$’s). One can define a bar involution on $\wedge^m \mathcal{V}^* \otimes \mathcal{V}^\otimes k \otimes \wedge^n \mathcal{V}^*$ which is compatible with the bar involution on $U_q(\mathfrak{gl}_\infty)$. This gives rise to the canonical basis and dual canonical basis on (a completion of) $\wedge^m \mathcal{V}^* \otimes \mathcal{V}^\otimes k \otimes \wedge^n \mathcal{V}^*$ (cf. [Lu, Br1]).

We denote by $\mathcal{O}_m|k|n$ the parabolic BGG category of $\mathfrak{gl}(k|m+n)$-modules of integral weights, which are semisimple with respect to the Levi subalgebra $\mathfrak{gl}(m) \oplus \mathfrak{h}_k \oplus \mathfrak{gl}(n)$ (here $\mathfrak{h}_k$ denotes the Cartan in $\mathfrak{gl}(k)$), with respect to the Borel associated to the fundamental system below:

\[
\begin{array}{cccccccc}
\varepsilon_{-\frac{1}{2}} - \epsilon_{\frac{1}{2}} & \epsilon_{\frac{1}{2}} & & & & \epsilon_{-\frac{1}{2}} - \epsilon_{\frac{3}{2}} & \epsilon_{-\frac{3}{2}} - \epsilon_{\frac{5}{2}} & \epsilon_{-\frac{5}{2}} - \epsilon_{\frac{7}{2}} & \epsilon_{-\frac{7}{2}} - \epsilon_{\frac{9}{2}} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{array}
\]

We shall prove a variant of Brundan’s conjecture (see [Br1, Conjecture 4.32]) which can be informally formulated as follows. Note that the Borel subalgebra used here is not a standard one in contrast to Brundan’s original setting.
**Theorem 6.4.** The category $\mathcal{O}_{\mathfrak{m}|k|\mathbb{Z}}$ categorifies the $U_q(\mathfrak{gl}_\infty)$-module $\wedge^m \mathcal{V}^* \otimes \mathcal{V}^\otimes k \otimes \wedge^n \mathcal{V}^*$, where the parabolic Verma, tilting and simple $\mathfrak{gl}(k|m+n)$-modules correspond to the standard, canonical, and dual canonical basis (at $q = 1$), respectively.

The essence of Brundan’s conjecture is that the entries of the transition matrix between the (dual) canonical basis and the standard basis on $\wedge^m \mathcal{V}^* \otimes \mathcal{V}^\otimes k \otimes \wedge^n \mathcal{V}^*$ should be interpreted as the Kazhdan-Lusztig polynomials for the category $\mathcal{O}_{\mathfrak{m}|k|\mathbb{Z}}$. Indeed our approach is powerful enough to establish that these transition matrix entries coincide with the Poincare polynomials for the corresponding $u$-homology groups defined in Section 4.4. A precise formulation of Theorem 6.4 and its variants with self-contained proofs would take many pages as it would require us to repeat in our setting much of the notations and constructions in [Br1, CW] among other things. Let us instead sketch a proof of Theorem 6.4 below.

The space of semi-infinite $q$-wedges decomposes according to “sectors” labeled by $\mathbb{Z}$ ([KMS]). For $a \in \mathbb{Z}$, let $\wedge^\infty + a \mathcal{V}$ be the space of semi-infinite $q$-wedges of sector $a$, which is isomorphic to the highest weight $U_q(\mathfrak{gl}_\infty)$-module $L(\widehat{\Lambda}_a)$ of highest weight being the $a$th fundamental weight. Similarly the space $\wedge^\infty - a \mathcal{V}$ of semi-infinite wedges of “sector $-a$” is also isomorphic to the $U_q(\mathfrak{gl}_\infty)$-module $L(\widehat{\Lambda}_a)$. The canonical isomorphism of $U_q(\mathfrak{gl}_\infty)$-modules $\wedge^\infty + a \mathcal{V} \cong \wedge^\infty - a \mathcal{V}$ commutes with the bar involution. (This extends the observation made in [CWZ] for $a = 0$.) This induces an isomorphism of $U_q(\mathfrak{gl}_\infty)$-modules

$$\wedge^\infty + a \mathcal{V} \otimes \mathcal{V}^\otimes k \otimes \wedge^\infty \mathcal{V} \cong \wedge^\infty - a \mathcal{V} \otimes \mathcal{V}^\otimes k \otimes \wedge^\infty \mathcal{V},$$

which matches the standard, canonical, and dual canonical bases, respectively. Based on this together with Theorem 4.11 as well as a Fock space reformulation (cf. e.g. [CW, Theorem 4.14] for a special case; the general case is similar) of the Kazhdan-Lusztig conjecture for parabolic BGG categories in type $A$ [BB, BK, KL], we establish a version of Theorem 6.4 for $m = n = \infty$. Theorem 6.4 for finite $m$ and $n$ follows from this version for $m = n = \infty$ and the compatibility of the (dual) canonical basis for varying $m, n$ (compare [CWZ, Corollary 2.10], [CW, Corollary 2.6]).

**Remark 6.5.** Note that Theorem 6.4 for $m = n = 1$ solves the irreducible character problem of the full BGG category of $\mathfrak{gl}(k|2)$-modules, which can also be viewed as a reformulation of the result in Section 6.2. On the other hand, it is easy to give a formula for the irreducible characters in the category $\mathcal{O}_{\mathfrak{m}|k|\mathbb{Z}}$ for general $m$ and $n$, in the spirit of Section 6.2, if one is willing to introduce messier notations.

### 7. Some more variations

The formulation and construction in this paper have variations which unfortunately involve more complicated notations. In this section, we will explain one such variation and its implication on $u$-homology computation.

For fixed $p, q \in \mathbb{Z}_+$, let $F = \{ r \in \frac{1}{2}\mathbb{Z} | -p < r \leq 0 \text{ or } \frac{1}{2} \leq r \leq q \}$. Set $\mathcal{J} = \mathcal{I} \cup F$, $\mathcal{J} = \mathcal{I} \cup F$, $\mathcal{J}^o = \mathcal{I}^o \cup F$ and $\mathcal{J}^\circ = \mathcal{I}^\circ \cup F$. The subalgebra of $\widehat{\mathfrak{g}}$ generated by $\widehat{E}_{r,s}$ with $r, s \in \mathcal{J}$ (respectively $\mathcal{J}$, $\mathcal{J}$, $\mathcal{J}$) is denoted by $\mathcal{J}$ (respectively $\mathcal{J}$, $\mathcal{J}^o$ and $\mathcal{J}^\circ$). Similar results in Section 3 and Section 4 are still valid if $\mathfrak{g}$ (respectively $\mathfrak{g}$, $\mathfrak{g}^o$ and $\mathfrak{g}^\circ$) are replaced by $\mathcal{J}$ (respectively $\mathcal{J}$, $\mathcal{J}^o$ and $\mathcal{J}^\circ$). It is worth pointing out that the sets $P^+_a$, 
\( \overline{P}_a^+, P_a^+, \) and \( \overline{P}_a^\circ \) of all dominant weights are determined by the choice of the orderings preserving the orderings of positive integers, positive half integers, non-positive integers and negative half integers and satisfying \( i < b < j \) for \( i \in -\frac{1}{2}\mathbb{Z}_+, b \in \mathbb{N} \) and \( j \in \frac{1}{2}\mathbb{N} \). We do not carry out any details here.

Let us illustrate by an example. Define orderings of \( J = \mathbb{I} \cup F \), \( \overline{J} = \overline{\mathbb{I}} \cup F \), \( J^\circ = \mathbb{I} \cup F \) and \( \overline{J}^\circ = \overline{\mathbb{I}} \cup F \) respectively by

\[
\ldots < -2 < -1 < 0 < -p + \frac{1}{2} < \ldots < -\frac{3}{2} < -\frac{1}{2} < \frac{1}{2} < \cdots < k \\
< -\frac{1}{2} < \frac{3}{2} < \cdots < q - \frac{1}{2} < 1 < 2 < \cdots ,
\]

\[
\ldots < -2 < -1 < 0 < -p + \frac{1}{2} < \ldots < -\frac{3}{2} < -\frac{1}{2} < \frac{1}{2} < \cdots < k \\
< 1 < 2 < \cdots < q < -\frac{1}{2} < \frac{3}{2} < \cdots ,
\]

\[
\ldots < -\frac{3}{2} < -\frac{1}{2} < -p + 1 < \cdots < -2 < -1 < 0 < \frac{1}{2} < \cdots < k \\
< -\frac{1}{2} < \frac{3}{2} < \cdots < q - \frac{1}{2} < 1 < 2 < \cdots ,
\]

\[
\ldots < -\frac{3}{2} < -\frac{1}{2} < -p + 1 < \cdots < -2 < -1 < 0 < \frac{1}{2} < \cdots < k \\
< 1 < 2 < \cdots < q < -\frac{1}{2} < \frac{3}{2} < \cdots .
\]

The corresponding fundamental systems are indicated in the following Dynkin diagrams:

\[ \begin{align*}
\mathcal{G}: \quad & (L) \quad \times \quad (L_p^\circ) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (R) \\
\overline{\mathcal{G}}: \quad & (L) \quad \times \quad (L_p^\circ) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (R) \\
\mathcal{G}^\circ: \quad & (L^\circ) \quad \times \quad (L_p) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (R) \\
\overline{\mathcal{G}}^\circ: \quad & (L^\circ) \quad \times \quad (L_p) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (K) \quad \times \quad (R) 
\end{align*} \]

Let \( a, \lambda_1, \ldots, \lambda_k \in \mathbb{C} \), \( \lambda^- \) and \( \lambda^+ \) be two partitions. Associated to the tuple \( \lambda = (a, \lambda_1, \ldots, \lambda_k; \lambda^-, \lambda^+) \) satisfying the dominant condition (2.3), set

\[
\Lambda_F^\pm(\lambda) := \sum_{i=1}^{k} \lambda_i \epsilon_i^\pm + \sum_{j=1}^{q} \langle \lambda^+ \rangle_j \epsilon_{j-\frac{1}{2}}^\pm + \sum_{j \in \mathbb{N}} \langle \lambda^+_j - q \rangle \epsilon_{j-\frac{1}{2}},
\]

\[
\overline{\Lambda}_F^\pm(\lambda) := \sum_{i=1}^{k} \lambda_i \epsilon_i^\pm + \sum_{j=1}^{q} \lambda^+_j \epsilon_j + \sum_{j \in \mathbb{N}} \langle \lambda^+_j - q \rangle \epsilon_{j-\frac{1}{2}},
\]
\[ \Lambda_F^- (\lambda) := - \sum_{j=1}^p (\lambda_j^-)' \epsilon_{-j+\frac{1}{2}} - \sum_{j \in \mathbb{N}} \langle \lambda_j^- - p \rangle \epsilon_{-j+1} + a \Lambda_0, \]

\[ \Lambda_F^+ (\lambda) := - \sum_{j=1}^p \lambda_j^+ \epsilon_{-j+1} - \sum_{j \in \mathbb{N}} \langle (\lambda_j^- - p) \rangle \epsilon_{-j+\frac{1}{2}} + a \Lambda_0. \]

Associated to such tuple, we define the weights

\[ \lambda_F := \Lambda^- (\lambda) + \Lambda^+ (\lambda), \quad \overline{\lambda}_F := \Lambda^- (\lambda) + \overline{\Lambda}^+ (\lambda), \]

\[ \lambda_F^0 := \overline{\Lambda}^- (\lambda) + \Lambda^+ (\lambda), \quad \overline{\lambda}_F^0 := \overline{\Lambda}^- (\lambda) + \overline{\Lambda}^+ (\lambda). \]

The sets \( P^+_a, \overline{P}^+_a, P^0_a \) and \( \overline{P}^0_a \) of all dominant weights are the sets consisting of the weights of the form \( \lambda_F, \overline{\lambda}_F, \lambda_F^0, \overline{\lambda}_F^0 \), respectively. The main results on super duality in Sections 3, 4 and 5 afford straightforward counterparts in this new setting.

**Remark 7.1.** The \( u \)-(co)homology formulas in the sense of Kostant and Enright [E] for unitarizable modules of general linear Lie superalgebras have been computed in [LZ] (cf. [HLT, Theorem 5.1, Theorem 4.4]), and they can be easily recovered via the super duality of this paper by an appropriate choice of \( \lambda \) and of the orderings defined in the example above. Let us set \( k = p = 0 \) and choose \( \lambda \) satisfying \( \lambda_1^- + \lambda_1^+ \leq a \) with \( a \in \mathbb{Z}_+ \). Then we have Enright type \( u \)-(co)homology formulas for unitarizable modules \( L(G^0, \lambda_F^0) \) (cf. [HLT, Theorem 4.4]). Using Theorem 4.7, its analogue for the irreducible modules \( L(G^0, \lambda_F^0) \) together with Enright’s (co)homology formulas for \( L(G^0, \overline{\lambda}_F^0) \), we recover [LZ, Theorem 6.1].

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