ALGEBRAIC INTEGRABILITY OF FOLIATIONS WITH NUMERICALLY TRIVIAL CANONICAL BUNDLE

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Abstract. Given a reflexive sheaf on a mildly singular projective variety, we prove a flatness criterion under certain stability conditions. This implies the algebraicity of leaves for sufficiently stable foliations with numerically trivial canonical bundle such that the second Chern class does not vanish. Combined with the recent works of Druel and Greb-Guenancia-Kebekus this establishes the Beauville-Bogomolov decomposition for minimal models with trivial canonical class.

1. Introduction

1.A. Main result. Let $X$ be a normal complex projective variety that is smooth in codimension two, and let $E$ be a reflexive sheaf on $X$. If $E$ is slope-stable with respect to some ample divisor $H$ of slope $\mu_H(E) = 0$, then a famous result of Mehta-Ramanathan ([MR84], see Lemma 2.11 for the singular version) says that the restriction $E_C$ to a general complete intersection $C$ of sufficiently ample divisors is stable and nef. On the other hand the variety $X$ contains many dominating families of irreducible curves to which the theorem of Mehta-Ramanathan does not apply; therefore one expects that, apart from very special situations, $E_C$ will not be nef for many curves $C$. If $E$ is locally free, denote by $\pi : \mathbb{P}(E) \to X$ the projectivisation of $E$ and by $\zeta := c_1(O_{\mathbb{P}(E)}(1))$ the tautological class on $\mathbb{P}(E)$. The nefness of $E_C$ then translates into the nefness of the restriction of $\zeta$ to $\mathbb{P}(E_C)$. Thus, the stability of $E$ implies some positivity of the tautological class $\zeta$. On the other hand, the non-nefness of $E_C$ on many curves can be rephrased by saying that the tautological class $\zeta$ should not be pseudoeffective. The first main result of this paper confirms this expected picture under some additional stability condition.

1.1. Theorem. Let $X$ be a normal projective variety of dimension $n$ which is smooth in codimension two. Let $H$ be an ample Cartier divisor on $X$, and let $E$ be a reflexive sheaf of rank $r$ on $X$ such that

$$c_1(E) \cdot H^{n-1} = 0.$$ 

Suppose that the reflexive symmetric powers $S^lE$ are $H$-stable for every $l \in \mathbb{N}$. Suppose further that $E$ is pseudoeffective (cf. Definition 2.1). Then

$$c_1(E)^2 \cdot H^{n-2} = c_2(E) \cdot H^{n-2} = 0.$$
If additionally \( X \) has at most klt singularities, there exists a finite Galois cover \( \nu : \tilde{X} \to X \), étale in codimension one, such that the reflexive pull-back \( \nu^!\mathcal{E} \) is a numerically flat vector bundle; in particular, one has
\[
c_1(\nu^!\mathcal{E}) = 0, \quad c_2(\nu^!\mathcal{E}) = 0.
\]

If \( X \) is smooth, then \( \mathcal{E} \) itself is a numerically flat vector bundle.

Nakayama [Nak99, Thm.B] and Druel [Dru18, Thm.6.1] obtained similar results for vector bundles of small rank. The recent progress on algebraic integrability of foliations by Campana-Paun and Druel [Bos01, CP15, Dru18] yields an immediate application:

**1.2. Corollary.** Let \( X \) be a projective manifold, and let \( \mathcal{F} \subset T_X \) be an involutive saturated subsheaf. Suppose there exists an ample Cartier divisor \( H \) on \( X \) such that \( S^l\mathcal{F} \) is \( H \)-stable for all \( l \in \mathbb{N} \). If \( c_1(\mathcal{F}) = 0 \) and \( c_2(\mathcal{F}) \neq 0 \), then \( \mathcal{F} \) has algebraic leaves.

It seems possible that the stability of \( \mathcal{F} \) is enough to imply the algebraicity of leaves (cf. also [Tou08, LPT18, LPT13, PT13] for classification results of foliations with \( c_1(\mathcal{F}) = 0 \)). Note however that Theorem 1.1 fails if \( \mathcal{E} \) is only assumed to be \( H \)-stable, see [Dru18, Ex. 6.9].

The proof of Theorem 1.1 is surprisingly simple. Druel’s proof [Dru18, Thm.6.1] uses the stability of \( \mathcal{E} \) to describe the components of the restricted base locus \( B_-(\zeta) \) (see Section 3.A) that are divisors or generically finite over the base. Our key observation is that the systematic use of the symmetric powers \( S^l\mathcal{E} \) allows to control irreducible components of \( B_-(\zeta) \) of any codimension. An intersection computation essentially reduces Theorem 1.1 to the following:

**1.3. Proposition.** Let \( C \) be a smooth projective curve. Let \( \mathcal{E} \) be a vector bundle on \( C \) such that \( c_1(\mathcal{E}) = 0 \). Denote by \( \zeta \) the tautological class on \( \mathbb{P}(\mathcal{E}) \). Suppose that the symmetric powers \( S^l\mathcal{E} \) are stable for every \( l \in \mathbb{N} \). Then, given any integer \( 0 \leq d < \dim \mathbb{P}(\mathcal{E}) \), the intersection number
\[
\zeta^d \cdot Z > 0
\]
for every subvariety \( Z \subset \mathbb{P}(\mathcal{E}) \) of dimension \( d \).

A well-known result of Mumford [Har70, Ex.10.6] says that if \( \mathcal{E} \) is a locally free sheaf of rank two on a curve \( C \) such that \( c_1(\mathcal{E}) = 0 \) and all the symmetric powers \( S^l\mathcal{E} \) are stable, then the tautological class \( \zeta \) has positive intersection with every curve \( Z \subset \mathbb{P}(\mathcal{E}) \). Our proposition generalises this property to vector bundles of arbitrary rank.

**1.B. Minimal models with trivial canonical class.** The main motivation for our study of stable sheaves with numerically trivial determinant is to extend the Beauville-Bogomolov decomposition theorem [Bea83] to singular spaces. Following [GKP16b], let us explain the notions of singular Calabi-Yau and singular irreducible symplectic varieties.

**1.4. Definition.** Let \( X \) be a normal projective variety of dimension \( n \geq 2 \) with at most canonical singularities such that \( \omega_X \simeq \mathcal{O}_X \).

- \( X \) is a Calabi-Yau variety if \( h^0(Y, \Omega_Y^{[q]}) = 0 \) for all integers \( 1 \leq q \leq n - 1 \) and all finite covers \( Y' \to X \), étale in codimension one;
• $X$ is irreducible symplectic if there exists a reflexive holomorphic 2-form $\sigma \in H^0(X, \Omega^2_X)$ such that for all finite covers $\gamma : Y \to X$, étale in codimension one, the exterior algebra of holomorphic reflexive forms is generated by the reflexive pull-back $\gamma^*(\sigma)$.

The Beauville-Bogomolov decomposition theorem for a Ricci flat compact Kähler manifold $X$ states that a finite étale cover $X$ is a product of a torus, Calabi-Yau and irreducible symplectic manifolds. In the last years there has been an intensive effort [GKP16c, Dru18, GGK17, DG18] to generalise this statement to minimal models. Theorem 1.1 allows to complete this challenge:

1.5. Theorem. Let $X$ be a normal projective variety with at most klt singularities such that $K_X \equiv 0$. Then there exists a projective variety $\tilde{X}$ with at most canonical singularities, a finite cover $f : \tilde{X} \to X$, étale in codimension one, and a decomposition

$$\tilde{X} \simeq A \times \prod_{j \in J} Y_j \times \prod_{k \in K} Z_k$$

into normal projective varieties with trivial canonical bundles, such that

• $A$ is an abelian variety;
• the $Y_j$ are (singular) Calabi-Yau varieties;
• the $Z_k$ are (singular) irreducible symplectic varieties.

Although this significantly improves results from earlier papers, one should note that Theorem 1.5 is based to equal parts on a tripod consisting of Druel’s algebraic integrability theorem [Dru18, Thm.1.4], the holonomy decomposition of Greb-Guenancia-Kebekus [GGK17, Thm.B and Prop.D] and our Theorem 1.1. For the proof we simply follow the arguments of [Dru18, Thm.1.6].

Another consequence of Theorem 1.1 is

1.6. Theorem. Let $X$ be a normal projective variety with at most klt singularities such that $K_X \equiv 0$. Suppose that $X$ is smooth in codimension two. If the reflexive cotangent sheaf $\Omega^1_X$ or the tangent sheaf $T_X$ is pseudoeffective, there exists a finite cover $\tilde{X} \to X$, étale in codimension one, such that $q(\tilde{X}) \neq 0$.

In particular if $X$ is a (singular) Calabi-Yau or irreducible symplectic variety that is smooth in codimension two, then $\Omega^1_X$ and $T_X$ are not pseudoeffective.

This result was proven for smooth surfaces in [Nak04, Thm.IV.4.15] [BDPP13, Thm.7.8] and for smooth threefolds in [Dru18, Cor.6.5].

1.C. Almost nef sheaves. While Theorem 1.1 is sufficiently strong for the proof of the decomposition theorem, it is in general not easy to control the stability of all the symmetric powers. We therefore also consider a weaker positivity notion:

1.7. Definition. Let $X$ be a normal projective variety, and let $\mathcal{E}$ be a reflexive sheaf on $X$. We say that $\mathcal{E}$ is almost nef, if there exist at most countably many proper subvarieties $S_j \subseteq X$ such that the following holds: let $C \subseteq X$ be a curve such that $\mathcal{E} \otimes \mathcal{O}_C$ is not nef, then $C$ is contained in $\bigcup_{j \in J} S_j$. 
Let us recall that a (not necessarily locally free) coherent sheaf $E \otimes O_C$ is nef if $O_{\mathbb{P}(E \otimes O_C)}(1)$ is nef (see [Kub70] for basic material concerning positivity for sheaves that are not locally free). It is well-known that a vector bundle that is weakly positive in the sense of Viehweg is almost nef, so this property can be checked for direct image sheaves. Using completely different techniques we prove the following

1.8. **Theorem.** Let $X$ be a normal projective variety of dimension $n$ which is smooth in codimension two. Let $E$ be a reflexive sheaf on $X$ such that

$$c_1(E) \cdot A^{n-1} = 0$$

for some ample divisor $A$ on $X$. If $E$ is almost nef, then we have

$$c_1(E)^2 \cdot H^{n-2} = 0, \quad c_2(E) \cdot H^{n-2} = 0$$

for all ample Cartier divisors $H$ on $X$.

If additionally $X$ has at most klt singularities, there exists a finite cover $\gamma : \tilde{X} \to X$, étale in codimension one, such that the reflexive pull-back $\gamma^!(E)$ is locally free and numerically flat.

Based on analytic techniques a slightly weaker statement was shown in [CH17, Prop.2.11]. In the geometric setting we can restate the theorem:

1.9. **Theorem.** Let $X$ be a normal projective variety with at most klt singularities such that $K_X \equiv 0$. Suppose that $X$ is smooth in codimension two. If $T_X$ is almost nef, then $X$ is dominated by an abelian variety.

This statement was shown for smooth threefolds in [BDPP13, Thm.7.7].

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2. **Notation, basic facts and proof of Proposition 1.3**

We work over the complex numbers, for general definitions we refer to [Har77]. We use the terminology of [Deb01] and [KM98] for birational geometry and notions from the minimal model program and [Laz04a] for notions of positivity. Manifolds and varieties will always be supposed to be irreducible. Given a normal variety $X$ we denote by $T_X := \mathcal{O}_X$ its tangent sheaf. The sheaf of reflexive differentials of degree $q \in \{1, \ldots, \dim X\}$ is given by

$$\Omega_X^{[q]} := (\bigwedge^q \Omega_X)^{**}.$$

A finite surjective map $\gamma : X' \to X$ between normal varieties is quasi-étale if its ramification divisor is empty (or equivalently, by purity of the branch locus, $\gamma$ is étale over the smooth locus of $X$).

Given a torsion-free sheaf $\mathcal{F}$ on a normal variety $X$, we denote by $S^{(m)}(\mathcal{F}) := (\text{Sym}^m\mathcal{F})^{**}$ the $m$-th reflexive symmetric power. Given a morphism $\gamma : Y \to X$, we denote by $\gamma^!(\mathcal{F}) := (\gamma^*\mathcal{F})^{**}$ the reflexive pull-back. The projectivisation $\mathbb{P}(\mathcal{F})$ is defined by

$$\mathbb{P}(\mathcal{F}) = \text{Proj}(\text{Sym}^*\mathcal{F});$$

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with projection $p : \mathbb{P}(\mathcal{F}) \to X$. By $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ we denote the tautological line bundle on $\mathbb{P}(\mathcal{F})$, and by $\mathcal{E} := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ its tautological class. If $\mathcal{F}$ is locally free, then

$$H^q(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(m) \otimes p^*({\mathcal{G}})) \simeq H^q(X, (S^m \mathcal{F}) \otimes \mathcal{G})$$

for all $q \geq 0$ and all locally free sheaves $\mathcal{G}$ on $X$.

In order to simplify the notation we will denote, for a normal subvariety $Y \subset X$ such that $\mathcal{E}$ is locally free near $Y$, the restriction $\mathcal{E}|_Y$ by $\mathcal{E}_Y$ and by $\mathcal{E}|_Y$ the restriction of the tautological class to $\mathbb{P}(\mathcal{E}_Y)$.

**2.1. Definition.** Let $X$ be a normal projective variety, and let $H$ be an ample Cartier divisor on $X$. Let $\mathcal{E}$ be a reflexive sheaf on $X$. We say that $\mathcal{E}$ is pseudoeffective if for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and

$$h^0(X, S[j] \mathcal{E} \otimes \mathcal{O}_X(jH)) \neq 0.$$ 

Note that this definition, which now seems commonly accepted, differs from that one given in [BDPP13, Def. 7.1]. It follows from Lemma 2.3 that the definition does not depend on the choice of the ample line bundle $H$.

Druel proves in [Dru18, Lemma 2.7] that a locally free sheaf $\mathcal{E}$ is pseudoeffective if and only if the tautological class $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ is pseudoeffective.

If $\mathcal{E}$ is merely reflexive, then in general the projectivisation $\mathbb{P}(\mathcal{E})$ is a very singular space and the push-forward of multiples of the tautological bundle are not isomorphic to the reflexive symmetric powers $S^m \mathcal{E}$. Therefore we use a theorem due to Nakayama [Nak04, V.3.23] to set up the following

**2.2. Definition.** Let $X$ be a normal variety, and let $\mathcal{E}$ be a reflexive sheaf on $X$.

- Denote by
  $$\nu : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$$
  the normalization of the unique component of $\mathbb{P}(\mathcal{E})$ that dominates $X$.
- Set $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) := \nu^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.
- Let $X_0 \subset X$ be the locus where $X$ is smooth and $\mathcal{E}$ is locally free, and let
  $$r : P \to \mathbb{P}(\mathcal{E})$$
  be a birational morphism from a manifold $P$ such that the complement of $(p \circ \nu \circ r)^{-1}(X_0) \subset P$ is a divisor $D$.
- Set $\pi := p \circ \nu \circ r$ and $\mathcal{O}_P(1) := r^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$, where $p : \mathbb{P}(\mathcal{E}) \to X$ is the projection.
- By [Nak04, III.5.10.3] there exists an effective divisor $\Lambda$ supported on $D$ such that
  $$\pi_*((\mathcal{O}_P(m) \otimes \mathcal{O}_P(m\Lambda))) \simeq S^m \mathcal{E} \quad \forall m \in \mathbb{N}.$$ 
- We call $\zeta := c_1(\mathcal{O}_P(1) \otimes \mathcal{O}_P(\Lambda)) \in N^1(P)$ a tautological class of $\mathcal{E}$.

Using the defining property of a tautological class, the arguments of [Dru18, Lemma 2.7] apply literally to show the following:

**2.3. Lemma.** Let $X$ be a normal projective variety. Let $\mathcal{E}$ be a reflexive sheaf on $X$, and let $\zeta$ be a tautological class on $\pi : P \to X$ (cf. Definition 2.2). Then $\zeta$ is pseudoeffective if and only if $\mathcal{E}$ is pseudoeffective.
2.4. Remarks.
- Lemma 2.3 shows in particular that the pseudoeffectivity of a tautological class \( \zeta \) does not depend on the choice of the birational model \( P \to \mathbb{P}(E) \to X \), nor on the effective divisor \( \Lambda \).
- If the tautological class \( c_1(O_{\mathbb{P}(E)}(1)) \) on the normalisation \( \mathbb{P}(E) \) is pseudoeffective, any tautological divisor \( \zeta \) is pseudoeffective.

2.5. Definition. [DPS94, Defn.1.17] Let \( X \) be a normal projective variety, and let \( E \) be a locally free sheaf on \( X \). We say that \( E \) is numerically flat if both \( E \) and \( E^* \) are nef. This is equivalent to assuming that both \( E \) and \( \det E^* \) are nef.

2.6. Remarks.
- By [DPS94, Cor.1.19] we know that if \( X \) is smooth and \( E \) is numerically flat, then all the Chern classes vanish and \( E \) is semi-stable for any ample polarization.
- Let \( E \) be a flat locally free sheaf on a projective manifold \( X \), i.e. \( E \) is given by a linear representation of \( \pi_1(X) \). If \( E \) is \( H \)-semistable for some ample Cartier divisor \( H \), then \( E \) is an extension by \( H \)-stable flat bundles \( E_i \). The local freeness of the stable pieces \( E_i \) follows e.g. from [Sim92, Thm.2], but could also be deduced from [BS94, Cor.3]. By the Kobayashi-Hitchin correspondence, the bundles \( E_i \) admit metrics \( h_i \) such that \((E_i, h_i)\) are Hermite-Einstein bundles, see e.g. [LT95, Thm. 3.0.1], hence \( E_i \) are hermitian flat, [Kob87, Thm. IV.4.11]. Consequently, the bundles \( E_i \) are numerically flat and so does \( E \), see also [DPS94, Thm.1.18].
- Both statements remain true for normal varieties by passing to a desingularisation.

2.7. Definition. Let \( X \) be a normal projective variety of dimension \( n \) that is smooth in codimension two, and let \( H \) be an ample Cartier divisor on \( X \). Let \( E \) be a reflexive coherent sheaf on \( X \).

For \( m \gg 0 \) let \( D_j \) be general divisors in \( |mH| \) and set
\[
C = D_1 \cap \ldots \cap D_{n-1}; \quad S = D_1 \cap \ldots \cap D_{n-2}.
\]
We define
\[
c_1(E) \cdot H^{n-1} := \frac{1}{m^{n-1}} c_1(E_C).
\]
We furthermore set
\[
c_1(E)^2 \cdot H^{n-2} := \frac{1}{m^{n-2}} c_1(E_S)^2
\]
and
\[
c_2(E) \cdot H^{n-2} := \frac{1}{m^{n-2}} c_2(E_S).
\]
Recall that a reflexive sheaf on a complex manifold is locally free in the complement of a set of codimension at least three [Kob87, Cor.5.5.20], so \( E \) is locally free in a neighbourhood of \( S \). The definitions above do not depend on the choice of \( m \) and the divisors \( D_j \). We refer to [Dru18, LT18] for a systematic exposition of Chern classes in a singular context.
2.1. Stability. In this paper we will use the standard notion of slope-(semi)stability of torsion-free sheaves $F$ with respect to an ample Cartier divisor $H$ as described in [MP97, Part I, Sect.III], and denote by $\mu_H(F)$ the slope of $F$ with respect to $H$. Miyaoka has shown the following useful basic fact:

2.8. Proposition. [Miy87], [Laz04b, Prop.6.4.11] A semistable vector bundle $E$ over a smooth curve with $c_1(E) = 0$ is nef.

The behaviour of stability under restrictions will play an important role.

2.9. Definition. Let $X$ be a normal projective variety of dimension $n$, and let $H$ be a very ample Cartier divisor on $X$. Let $F$ be a torsion-free sheaf on $X$ that is $H$-semistable. A MR-general curve $C \subset X$ is a complete intersection $D_1 \cap \ldots \cap D_{n-1}$ where $D_j \in |H|$ are general elements such that the restriction $F_C$ is semistable.

2.10. Remark. The abbreviation MR stands for Mehta-Ramanathan, alluding to the well-known fact [MR82, Fle84] that up to replacing $H$ by some sufficiently positive multiple, the restriction $F_C$ is indeed semistable.

For lack of reference we include the following singular version of the restriction theorem of Mehta-Ramanathan [MR84, Thm.4.3].

2.11. Lemma. Let $X$ be a normal projective variety of dimension $n$, and $H$ an ample Cartier divisor on $X$. Let $E$ be a torsion-free sheaf on $X$ that is $H$-stable.

Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $D_1, \ldots, D_{n-1}$ general elements in $|mH|$, the restriction $E_C$ to $C := D_1 \cap \ldots \cap D_{n-1}$ is stable.

Remark. As in Definition 2.9 we will call $C$ a MR-general curve.

Proof. Let $\mu : X' \to X$ be a resolution of singularities. By [GKP16b, Lemma 4.6], the reflexive pull-back $\mu^{[*]}(E)$ is stable with respect to the semiample and big divisor $\mu^*H$. Strictly speaking, [GKP16b, Lemma 4.6] deals with movable classes; however the arguments used in its proof apply, all relevant intersections numbers being well defined. By [Lan04, Thm.5.2] for all $k \gg 0$ and $D'_1 \in |k\mu^*H|$ general the restriction $\mu^{[*]}(E)|_{D'_1}$ is stable with respect to $(\mu^*H)|_{D'_1}$. Thus we can find a $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $D'_1, \ldots, D'_{n-1}$ general elements in $|m\mu^*H|$, the restriction $\mu^{[*]}(E)|_{C'} := D'_1 \cap \ldots \cap D'_{n-1}$ is stable. Now observe that by the projection formula

$$H^0(X', \mathcal{O}_{X'}(m\mu^*H)) \simeq H^0(X, \mathcal{O}_X(mH)),$$

so the divisors $D'_i$ are strict transforms of general divisors $D_i \in |mH|$. Since $X$ is normal, the intersection $C := D_1 \cap \ldots \cap D_{n-1}$ is in the smooth locus of $X$, thus the curves $C'$ and $C$ can be identified. Since $E$ is torsion-free, hence locally free in codimension one, the sheaves $\mu^{[*]}(E)$ and $E$ identify in a neighbourhood of $C = C'$, so $E_C = \mu^{[*]}(E)|_{C'}$ is stable.

2.12. Remark. More generally, the restriction to $D_1 \cap \ldots \cap D_k$ with $1 \leq k \leq n-1$ is $H$-stable. Indeed if the intermediate restriction is not stable, then the restriction to a MR-general curve is not stable.
2. B. Subvarieties of projectivised bundles. We will now prove the key lemma of this paper.

2.13. Lemma. Let $C$ be a smooth projective curve. Let $\mathcal{E}$ be a locally free sheaf on $C$ such that $c_1(\mathcal{E}) = 0$. Denote by $\pi : \mathbb{P}(\mathcal{E}) \to C$ the projection, and let $\zeta$ be the tautological class on $\mathbb{P}(\mathcal{E})$. Let $Z \subseteq \mathbb{P}(\mathcal{E})$ be a subvariety of dimension $d < \dim \mathbb{P}(\mathcal{E})$. Denote by $\mathcal{I}_Z$ the ideal sheaf of $Z$ in $\mathbb{P}(\mathcal{E})$, and let $l \in \mathbb{N}$ be such that $\pi_*(\mathcal{I}_Z(l))$ has positive rank and such that

$$R^1\pi_*(\mathcal{I}_Z(l)) = 0.$$  

Suppose that the locally free sheaf $S^d\mathcal{E}$ is stable. Then $\zeta|_Z$ is ample, in particular one has

$$\zeta^d : Z > 0.$$  

Remark. In the situation above, by assumption $S^d\mathcal{E}$ is stable with $c_1(\mathcal{E}) = 0$. Thus $S^d\mathcal{E}$ is nef by Proposition 2.8. Combined with [Laz04b, Thm.6.4.15] this implies that the symmetric powers $S^m\mathcal{E}$ are semistable and nef for all $m \in \mathbb{N}$.

Proof of Lemma 2.13. If $Z$ is contained in a fibre of $\pi$, the statement is trivial, so suppose that $\varphi := \pi|_Z$ is surjective. We denote by $\mathcal{O}_Z(l)$ the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)$ to $Z$.

By our hypothesis, the exact sequence

$$0 \to \mathcal{I}_Z(l) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) \to \mathcal{O}_Z(l) \to 0$$

induces an exact sequence

$$0 \to \pi_*(\mathcal{I}_Z(l)) \to \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)) \simeq S^d\mathcal{E} \to \varphi_*(\mathcal{O}_Z(l)) \to 0$$

and $\text{rk} S^d\mathcal{E} > \text{rk} \varphi_*(\mathcal{O}_Z(l))$. Since $S^d\mathcal{E}$ is nef by Proposition 2.8, its quotient $\varphi_*(\mathcal{O}_Z(l))$ is also nef. If $\varphi_*(\mathcal{O}_Z(l))$ were not ample, by Hartshorne's theorem [Laz04b, Thm.6.4.15] there would exist aquotient

$$\varphi_*(\mathcal{O}_Z(l)) \to Q$$

such that $c_1(Q) = 0$. Yet this quotient would destabilise $S^d\mathcal{E}$, so $\varphi_*(\mathcal{O}_Z(l))$ has to be ample. Since $\mathcal{O}_Z(l)$ is $\varphi$-globally generated, we have a surjective morphism

$$\varphi^*\varphi_*(\mathcal{O}_Z(l)) \to \mathcal{O}_Z(l).$$

(1)  

Since $\varphi^*\varphi_*(\mathcal{O}_Z(l))$ is semiample, its quotient $\mathcal{O}_Z(l)$ is a semiample line bundle. Denote by $\tau : Z \to B$ the morphism with connected fibers defined by some positive multiple of $\mathcal{O}_Z(l)$. Then $\tau$ is finite if and only if $\zeta|_Z$ is ample.

Arguing by contradiction we suppose that there exists a curve $C' \subset Z$ that is contracted by $\tau$, in particular we have $\zeta \cdot C' = 0$. Then $C'$ is not contained in a $\pi$-fibre, so $\varphi|_{C'}$ is finite. Restricting the surjection (1) to $C'$ we obtain a surjection

$$(\varphi|_{C'})^*\varphi_*(\mathcal{O}_Z(l)) \to \mathcal{O}_{C'}(l).$$

Yet $\varphi_*(\mathcal{O}_Z(l))$ is ample, so the finite pull-back $(\varphi|_{C'})^*\varphi_*(\mathcal{O}_Z(l))$ and its quotient $\mathcal{O}_{C'}(l)$ are ample. In particular $\zeta \cdot C' > 0$, a contradiction.  

Proof of Proposition 1.3. Arguing by contradiction, suppose that there exists a subvariety $Z \subseteq \mathbb{P}(\mathcal{E})$ of dimension $d$ such that $\zeta^d : Z = 0$. Since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is $\pi$-ample, with $\pi : \mathbb{P}(\mathcal{E}) \to C$ the projection, we know that the assumptions of Lemma 2.13 are satisfied for some $l \gg 0$. Thus $S^d\mathcal{E}$ is not stable, a contradiction.  


3. Reflexive sheaves with pseudoeffective tautological class

3.A. Restricted base locus of the tautological class. We start by setting up some notation.

3.1. Notation. Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety. Then the stable base locus is defined as

$$B(D) := \bigcap_m \text{Bs}(mD),$$

where the intersection is taken over all $m \in \mathbb{N}$ such that $mD$ is the class of a Cartier divisor, with $\text{Bs}(mD)$ denoting the base locus of $|mD|$. The restricted base locus is defined as

$$B_-(D) = \bigcup_{A \text{ ample } \mathbb{Q}\text{-divisor}} B(D + A).$$

By [ELM+06, Prop.1.19] one has

\begin{equation}
B_-(D) = \bigcup_{n \in \mathbb{N}} B(D + \frac{1}{n} A)
\end{equation}

where $A$ is an arbitrary ample divisor. Note that if $A$ is very ample then

\begin{equation}
B(D + \frac{1}{n} A) \subset B(D + \frac{1}{n'} A)
\end{equation}

if $n \leq n'$.

3.2. Notation. Let $Y$ be a projective manifold and $A$ an ample divisor on $Y$. Let $D$ be a pseudoeffective $\mathbb{Q}$-divisor and $\Gamma$ a prime divisor on $Y$. By

$$\sigma_{\Gamma}(D) = \lim_{\epsilon \to 0^+} \inf \{ \text{mult}_{\Gamma}(L') \mid L' \geq 0 \text{ and } L' \equiv L + \epsilon A \},$$

we define the asymptotic multiplicity or vanishing order of $D$ along $\Gamma$, as defined in [Nak04, III, Lemma 1.6]; see also [FL17, Defn.2.15]. By definition it is a numerical invariant of $D$.

We extend this definition to higher codimension, following [Nak04, III, Defn. 2.2].

3.3. Notation. Let $P$ be a projective manifold, and let $D$ be a pseudoeffective $\mathbb{Q}$-Cartier divisor on $P$. Let $Z \subset P$ be a subvariety, and let

$$f : Y \to P$$

be the composition of an embedded resolution of $Z$, and the blow-up of the strict transform of $Z$. Let $E_Z \subset Y$ be the unique prime divisor mapping onto $Z$. Then we define

$$\sigma_Z(D) := \sigma_{E_Z}(f^* D).$$

It is easy to check that this definition does not depend on the choice of $f$, see [Nak04, p.88].

Note that by [Nak04, III, Lemma 1.7(2)] we have

$$\sigma_Z(D) = \lim_{\epsilon \to 0} \sigma_{E_Z}(f^*(D + \epsilon A))$$

where $A$ is an arbitrary ample divisor on $P$. Thus $\sigma_Z(D)$ is the asymptotic vanishing order of $D$ in the generic point of the subvariety $Z$. 

Suppose now that $Z$ is an irreducible component of $B_-(D)$. By [Dru18, Lemma 6.12] we have
\[ \sigma_Z(D) > 0. \]
By [Nak04, III, Lemma 1.7(2)] (applied to $\sigma_Z$, cf. also [Nak04, p.93, Remark (1)]) the function
\[ \sigma_Z : \text{Eff}(P) \to \mathbb{R} \]
is lower semicontinuous and continuous when restricted to the big cone.

The following technical lemma, an analogue of [Dru18, Lemma 6.13], will be very important.

3.4. Lemma. Let $X$ be a normal projective variety, and let $H$ be an ample Cartier divisor on $X$. Let $E$ be a reflexive sheaf of rank $r$ on $X$ that is $H$-semistable with $\mu_H(E) = 0$. Let
\[ \pi : P \to X \]
be a modification of $\mathbb{P}(E)$ as in Definition 2.2, and let $\zeta$ be a tautological class on $P$. Suppose that $E$ is pseudoeffective, or equivalently, that $\zeta$ is a pseudoeffective class (cf. Lemma 2.3).

Fix a positive integer $k$ and suppose the following:

- for every irreducible component $W \subset B_-(\zeta)$ of codimension at most $k - 1$ the image $\pi(W)$ has codimension at least 2.

Let $Z \subset B_-(\zeta)$ be an irreducible component of codimension $k$, and let $C \subset X$ be a very general MR-general smooth curve (with respect to $H$).

Then there exists $a > 0$ such that
\[ [(\zeta|_{\pi^{-1}(C)})^k - a(Z \cap \pi^{-1}(C))] \cdot H_1 \cdots H_{r-k} \geq 0 \]
for any collection of nef divisors $H_1, \ldots, H_{r-k}$ on $\pi^{-1}(C)$.

Proof. Since $C \subset X$ is very general and since $B_-(\zeta)$ has at most countably many irreducible components we have
\[ \pi(W) \cap C = \emptyset \]
for every irreducible component $W \subset B_-(\zeta)$ of codimension at most $k - 1$.

Moreover, the sheaf $E$ is locally free in a neighbourhood of $C$ and the restriction $E_C$ is a nef vector bundle by Proposition 2.8 and [Fle84, Thm.1.2]. By construction of $P$ we have $\pi^{-1}(C) \cong \mathbb{P}(E_C)$ and $\zeta|_{\pi^{-1}(C)} = c_1(\mathcal{O}_{\pi^{-1}(C)}(1))$. Hence $\zeta|_{\pi^{-1}(C)}$ is a nef divisor.

If $\pi(Z)$ has codimension at least two in $X$, then the intersection $Z \cap \pi^{-1}(C)$ is empty, and consequently the assertion of Lemma 3.4 is trivially true. Thus we can assume from now on that
\[ \text{codim}_X(\pi(Z)) \leq 1. \]

Fix a very ample divisor $A$ on $P$. By (2) and (3) we find a $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$ we have
\[ Z \subset B(\zeta + \frac{1}{n}A). \]
We set $a_1 := \sigma_Z(\zeta)$ and observe that $a_1 > 0$ by (4). Since $\sigma_Z$ is a lower semicontinuous function we may suppose, possibly enlarging $n_0$, that

$$\sigma_Z(\zeta + \frac{1}{n} A) \geq \frac{a_1}{2}$$

for all $n \geq n_0$. Since

$$B(\zeta + \frac{1}{n} A) \subset B_-(\zeta),$$

our hypothesis implies that if an irreducible component $W \subset B(\zeta + \frac{1}{n} A)$ has codimension at most $k - 1$, then $\pi(W)$ has codimension at least 2 in $X$.

In order to verify the inequality (5), let $D_1, \ldots, D_k$ be very general effective $\mathbb{Q}$-divisors on $P$ such that $D_i \sim_{\mathbb{Q}} \zeta + \frac{k}{n} A$. Then we have the following: if

$$W \subset D_1 \cap \ldots \cap D_k$$

is an irreducible component of codimension at most $k - 1$, then $W$ is an irreducible component of $B(\zeta + \frac{k}{n} A)$ and therefore $\text{codim}_X(\pi(W)) \geq 2$. Thus for a very general curve $C \subset X$, the intersection

$$D_1 \cap \ldots \cap D_k \cap \pi^{-1}(C)$$

has pure codimension $k$ in $\pi^{-1}(C)$. Since $C$ is general, the intersection $Z \cap \pi^{-1}(C)$ is reduced, thus by (6)

$$\text{mult}_{Z \cap \pi^{-1}(C)}(D_j \cap \pi^{-1}(C)) \geq \frac{a_1}{2},$$

where $\text{mult}_{Z \cap \pi^{-1}(C)}(D_j \cap \pi^{-1}(C))$ is the order of vanishing of $D_j \cap \pi^{-1}(C)$ in any general point of $Z \cap \pi^{-1}(C)$. Consequently

$$\{D_1 \cap \ldots \cap D_k \cap \pi^{-1}(C)\} - \left(\frac{a_1}{2}\right)^k (Z \cap \pi^{-1}(C))$$

is an effective cycle of pure codimension $k$ in $\pi^{-1}(C)$. Since

$$[D_1 \cap \ldots \cap D_k \cap \pi^{-1}(C)] = ((\zeta + \frac{1}{n} A)_{|_{\pi^{-1}(C)}})^k,$$

we conclude

$$[((\zeta + \frac{1}{n} A)_{|_{\pi^{-1}(C)}})^k - \left(\frac{a_1}{2}\right)^k(Z \cap \pi^{-1}(C))] \cdot H_1 \cdots H_{r-k} \geq 0$$

for any collection of nef divisor $H_i$ on $\pi^{-1}(C)$. Since $\frac{a_1}{2}$ does not depend on $n \geq n_0$ the statement now follows by setting $a := (\frac{a_1}{2})^k$ and passing to the limit $n \to \infty$. □

3.5. Remark. In the proof above we can replace $\frac{a_1}{2}$ by $\frac{k-1}{k}a_1$ for an arbitrary $k \in \mathbb{N}^*$. Taking the limit $k \to \infty$ the statement thus holds for $a = a_1^k$. This is the natural bound following from more general considerations with $(1,1)$-currents, see [Dem93, Cor.10.5].

3.B. Proof of Theorem 1.1. Let $\pi : P \to X$ be a modification of $\mathbb{P}(\mathcal{E})$ as in Definition 2.2, and let $\zeta$ be a tautological class on $P$. By Lemma 2.3 our assumption implies that $\zeta$ is a pseudoeffective class. Denote by $X_0 \subset X$ the locus where $X$ is smooth and $\mathcal{E}$ is locally free. Observe first that it is sufficient to show that

Claim. Let $W \subset B_-(\zeta)$ be an irreducible component. Then $\text{codim}_X(\pi(W)) \geq 2$.

Assuming this claim for the time being, let us see how to conclude: consider a surface $S \subset X_0$ cut out by very general elements of a sufficiently high multiple of $H$. Since $X$ is smooth in codimension two, the surface $S$ is smooth and $\mathcal{E}$ is locally
free in a neighbourhood of $S$. By construction of $P$ we have $\pi^{-1}(S) = \mathbb{P}(\mathcal{E}_S)$ and $\zeta|_{\pi^{-1}(S)} = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_S)}(1))$. Denote by $\zeta_S$ the restriction of $\zeta$ to $\mathbb{P}(\mathcal{E}_S)$. Then

$$B_-(\zeta_S) \subset B_-(\zeta) \cap \mathbb{P}(\mathcal{E}_S),$$

and therefore by the *Claim*, every irreducible component of $B_-(\zeta) \cap \mathbb{P}(\mathcal{E}_S)$ is contained in a fibre of the projection $\mathbb{P}(\mathcal{E}_S) \to S$. Thus the restriction of $\zeta_S$ to $B_-(\zeta_S)$ is nef (even ample), so $\zeta_S$ is nef by [Pau98, Thm.2]. Hence $\mathcal{E}_S$ is nef, in particular $\det \mathcal{E}_S$ is nef. Since $c_1(\mathcal{E}) \cdot H^{n-1} = 0$ and the class of $S$ is a positive multiple of $H^{n-2}$ we have $c_1(\mathcal{E}_S) \cdot H|_S = 0$. Since $\det \mathcal{E}_S$ is nef, we obtain

$$c_1(\mathcal{E}_S) = 0$$

by the Hodge Index Theorem. Moreover we have

$$c_2(\mathcal{E}_S) = 0$$

by [DPS94, Thm.2.5]. Since the class of $S$ is a positive multiple of $H^{n-2}$ this proves that

$$(7) \quad c_1(\mathcal{E})^2 \cdot H^{n-2} = c_2(\mathcal{E}) \cdot H^{n-2} = 0.$$ 

If $X$ is klt, given Equation (7), it follows from [GKP16a, Thm.1.20] that there exists a quasi-étale cover $\nu : \tilde{X} \to X$ such that $\nu^!(\mathcal{E})$ is a flat vector bundle. Since $\mathcal{E}$ is $H$-(semi)stable, the reflexive pull-back $\nu^!\mathcal{E}$ is semistable with respect to $\nu^*H$.

By Remark 2.6 this shows that $\nu^!\mathcal{E}$ is numerically flat and its Chern classes vanish. If additionally $X$ is smooth, then the cover $\nu : \tilde{X} \to X$ is étale. Thus we see that $\nu^*(\mathcal{E}) = \nu^!\mathcal{E}$, which is locally free and numerically flat. Since $\nu$ is étale this implies that $\mathcal{E}$ itself is locally free and numerically flat.

*Proof of the claim.* We proceed by induction on $k \in \mathbb{N}^*$. The induction hypothesis states that given any irreducible component $W \subset B_-(\zeta)$ of codimension at most $k - 1$, the image $\pi(W) \subset X$ has codimension at least 2.

Note that for $k = 1$ the unique subvariety of $P$ having codimension $k - 1 = 0$ is $P$ itself. Since $\zeta$ is pseudoeffective by assumption, the total space $P$ is not in $B_-(\zeta)$. Hence the induction hypothesis holds for $k = 1$.

Arguing by contradiction we suppose that there exists an irreducible component $Z \subset B_-(\zeta)$ of codimension $k$ such that

$$\operatorname{codim}_X(\pi(Z)) \leq 1.$$ 

Fix a number $l \in \mathbb{N}$ such that $\pi_*(\mathcal{I}_Z(l))$ has positive rank and all the higher direct images $R^i\pi_*\mathcal{I}_Z(l)|_{X_0}$ vanish. Let $C \subset X$ be a curve cut out by very general elements of the linear system $|mH|$, where we choose $m \in \mathbb{N}$ sufficiently large so that the Mehta-Ramanathan theorem (in the form given by Lemma 2.11) applies both for $\mathcal{E}$ and $S^r\mathcal{E}$, i.e., the restrictions $\mathcal{E}_C$ and $S^r\mathcal{E}_C$ are stable. Moreover,

$$C \cap \pi(W) = \emptyset$$

where $W$ is any of the varieties appearing in the induction hypothesis.

Since $\mathcal{E}_C$ is stable and $c_1(\mathcal{E}_C) = 0$, the restricted tautological class $\zeta_C$ is nef by Proposition 2.8. Thus we may apply Lemma 3.4 with $H_1 = \ldots = H_{r-k} = \zeta_C$: there is a real number $a > 0$ such that

$$\zeta_C - a\zeta_C^{-k}(Z \cap \pi^{-1}(C)) \geq 0.$$
Since $c_1(\mathcal{E}_C) = 0$, we have $\zeta_C = 0$. Moreover, $\zeta_C$ being nef, we have
$$
\zeta_C^{-k}(Z \cap \pi^{-1}(C)) \geq 0.
$$
Consequently
$$
\zeta_C^{-k}(Z \cap \pi^{-1}(C)) = 0.
$$
Thus we see that $\zeta|_{Z \cap \pi^{-1}(C)}$ is not big. Yet $S^2\mathcal{E}_C$ is stable, so we know by Lemma 2.13 that $\zeta|_{Z \cap \pi^{-1}(C)}$ is even ample, a contradiction.

**Remark.** The assumption that all the symmetric powers $S^l\mathcal{E}$ are stable is important for the proof of Theorem 1.1. In some cases this property is equivalent to assuming the stability for finitely many symmetric powers $S^l\mathcal{E}$ (cf. [BK08, Prop.5, Cor.6]).

3.C. **Proof of Corollary 1.2.** By [Dru18, Prop.8.4], it suffices to show that $\mathcal{F}^*$ not pseudoeffective (cf. Definition 2.1). Suppose to the contrary that $\mathcal{F}^*$ is pseudoeffective. Then we apply Theorem 1.1 and conclude that $c_2(\mathcal{F}) = c_2(\mathcal{F}^*) = 0$, contrary to our assumption.

□

4. **Proof of Theorems 1.5 and 1.6**

We start with the proof of the decomposition theorem.

4.A. **Proof of Theorem 1.5.** We will follow the approach of Druel [Dru18].

By [Dru18, Thm.1.4] and [GGK17, Thm. B] there exists a quasi-étale finite cover
$$
\gamma : A \times Z \to X
$$
where $A$ is an abelian variety and $Z$ a normal projective variety with the following properties.

- $Z$ has at most canonical singularities;
- $\mathcal{O}_Z(K_Z) \simeq \mathcal{O}_Z$ and the augmented irregularity $\tilde{q}(Z)$ [Dru18, Defn.4.1] is zero;
- there is a decomposition
  $$
  T_Z \simeq \bigoplus_j \mathcal{E}_j
  $$
  into reflexive integrable subsheaves $\mathcal{E}_j$ of rank $m_j$ which are strongly stable in the sense of [GKP16c, Defn.7.2] for any polarization. Moreover we have $\text{det}\mathcal{E}_j \simeq \mathcal{O}_Z$.

In order to simplify the notation we will suppose without loss of generality that $X = Z$. By Step 1 of the proof of [Dru18, Prop.4.10] we can suppose without loss of generality that $X$ has terminal $\mathbb{Q}$-factorial singularities.

We set $E_{j,x} := (\mathcal{E}_j)_x/m_x(\mathcal{E}_j)_x$ with $m_x$ the maximal ideal in a smooth point $x \in X$. By [GGK17, Thm. B and Prop.D] there exists a singular Ricci-flat Kähler metric, inducing a Riemannian metric $g$, such that for a smooth point $x \in X$, the decomposition
$$
T_{X,x} \simeq \bigoplus_j E_{j,x}
$$
corresponds to the decomposition of $T_{X,x}$ into irreducible representations according to the action of the differential-geometric holonomy group $G$ of $g$ at $x$. Moreover the differential-geometric holonomy groups $G_j$ of the direct factors $E_j$ are either $SU(m_j)$ or $Sp(m_j/2)$, the representation $\rho_j : G_j \to \text{GL}(E_{j,x})$ being the standard one. By [Dru18, Prop.4.10], it suffices to show that the leaves of the foliations $E_j$ are algebraic. By [Dru18, Prop. 8.4] it is furthermore sufficient to show that $E_j^*$ is not pseudoeffective (cf. Definition 2.1).

The sheaves $E_j^*$ are strongly stable in the sense of [GKP16c, Defn.7.2]. Hence all the reflexive symmetric powers $S^d E_j^*$ are $H$-stable for some ample divisor $H$: this follows from the Bochner principle [GGK17, Thm.8.1] in combination with [Wey49], [FH91, Sect.24.1 and 24.2]. Arguing by contradiction we suppose that $E_j^*$ is pseudoeffective. Thus Theorem 1.1 applies and $c_2(E_j^*) \cdot H^{\dim X - 2} = 0$ for an ample Cartier divisor on $X$. Since $q(X) = 0$ this contradicts [Dru18, Cor.5.11].

4.B. Proof of Theorem 1.6. We will give the proof for $T_X$, the proof for $\Omega^{[1]}_X$ is analogous.

Let $\gamma : X' \to X$ be a quasi-étale cover, and let $H$ be an ample Cartier divisor on $X$. Since $\gamma^{[1]}T_X \simeq T_{X'}$, we have injections

$$H^0(X, S^d E_j^* \otimes O_X(jH)) \hookrightarrow H^0(X', S^d E_j^* \otimes \gamma^*(O_X(jH)))$$

for every $i, j \in \mathbb{N}$. Thus we can replace without loss of generality $X$ by any quasi-étale cover.

In particular we can replace $X$ by the decomposition in Theorem 1.5. Arguing by contradiction we assume that the decomposition is

$$X \simeq \prod_{j \in J} Y_j \times \prod_{k \in K} Z_k$$

with $Y_j$ Calabi-Yau varieties and $Z_k$ irreducible symplectic. Set $Y = Y_1$ (or $Y = Z_1$ if there is no Calabi-Yau factor) and set $Z$ for the remainder in the decomposition.

We are going to prove that $T_Y$ or $T_Z$ is pseudoeffective, so let $c > 0$ be arbitrary. Since $T_X$ is pseudoeffective there exist positive integers $i, j$ with $i > 2cj$ such that

$$h^0(X, S^d E_j^* \otimes O_X(jH)) \neq 0.$$ 

Since

$$T_X \simeq p_Y^*(T_Y) \oplus p_Z^*(T_Z),$$

there exist integers $l$ and $m$ with $l + m = i$ such that

$$h^0(X, p_Y^*(S^l T_Y) \oplus p_Z^*(S^m T_Z) \otimes O_X(jH)) \neq 0.$$ 

By restricting to $Y \times \{z\}$ for general $z \in Z$ and to $\{y\} \times Z$ for general $y \in Y$, we deduce that $h^0(Y, S^l T_Y \otimes O_Y(jH)) \neq 0$ and $h^0(Z, S^m T_Z \otimes O_Z(jH)) \neq 0$. Since $l > cj$ or $m > cj$ and $c$ is arbitrary, we conclude that $T_Y$ or $T_Z$ is pseudoeffective.

By induction on the number of factors we can thus assume that $X$ is a Calabi-Yau or irreducible symplectic variety. By [GKP16c, Prop.8.20] we know that $T_X$ is strongly stable. Thus, by [GGK17, Thm. 8.1] all the symmetric powers $S^d T_X$ are $H$-stable.
By assumption $T_X$ is pseudoeffective, so $c_2(X) \cdot H^{n-2} = 0$ by Theorem 1.1. But then $X$ is a quasi-étale quotient of a torus by [GKP16a, Thm.1.17], contradicting the fact that $T_X$ is strongly stable.

\[ \square \]

**Remark.** It seems likely that Theorem 1.6 holds without the assumption that $X$ is smooth in codimension two. Note that it is not possible to simply replace $X$ by a terminalisation $X'$, since $T_{X'}$ might not be pseudoeffective [Mis18, Ex.6.1].

5. Almost nef sheaves

We start with some technical preparation.

5.1. Proposition. Let $S$ be a smooth projective surface, and let $E$ be a locally free sheaf of rank $r$ over $S$ that is semistable with respect to some ample divisor $H$. Suppose that $c_1(E) \cdot H = 0$. Suppose also that for some $a > 0, b \in \mathbb{N}$ there exists an effective divisor $D \in |a\zeta + b\pi^*H|$ such that

\[ \zeta^r \cdot D \geq 0. \]

Then $E$ is numerically flat. In particular one has $c_1(E) = 0$ and $c_2(E) = 0$ [DPS94, Cor.1.19].

**Proof.** By the classical Bogomolov-Gieseker inequality, [Miy87, 4.7], we have $c_2(E) \geq 0$. A theorem of Simpson [Sim92, Cor.3.10] states moreover that $c_2(E) = 0$ if and only if $E$ is numerically flat. Thus it suffices to show that $c_2(E) \leq 0$.

By assumption we have

\[ \zeta^r \cdot D \geq 0. \]

On the other hand since $c_1(E) = 0$ the Leray-Hirsch relation

\[ \zeta^r - \pi^*c_1(E)\zeta^{r-1} + \pi^*c_2(E)\zeta^{r-2} = 0 \]

simplifies to $\zeta^r = -\pi^*c_2(E)\zeta^{r-2}$. Consequently,

\[ \zeta^r \cdot D = -\pi^*c_2(E)\zeta^{r-2} \cdot (a\zeta + b\pi^*H) = -ac_2(E) \geq 0. \]

Since $a > 0$ we arrive at $c_2(E) \leq 0$. \[ \square \]

5.2. Lemma. Let $S$ be a smooth projective surface, and let $E$ be a locally free sheaf of rank $r \geq 2$ on $S$. Let $(C_i)_{i \in I}$ be an at most countable collection of irreducible curves in $S$.

Let $H$ be an ample divisor on $S$. Then for every $m \gg 0$ there exists a divisor $D \in |\zeta + m(r-1)\pi^*H|$ with the following properties:

a) Let $\{p_1, \ldots, p_k\} \subset S$ be the non-flat locus of $\pi|_D : D \to S$. Then $p_j \notin C_i$ for every $j \in \{1, \ldots, k\}$ and $i \in I$.

b) For every $i \in I$ the restriction of the tautological class $\zeta$ to $D \cap \pi^{-1}(C_i)$ is nef.

Recall first the following basic lemma:
5.3. Lemma. Let $Z$ be a projective variety of dimension $d$, and let $E$ be a locally free sheaf of rank $r$ on $Z$. Let $V \subset H^0(Z, E)$ be a linear subspace such that $E$ is generated by the $V$, i.e. we have a surjective evaluation morphism

$$\mathcal{O}_Z \otimes V \to E.$$  

a) If $r > d$, then a general choice of $r - d$ elements of $V$ defines a subbundle

$$\mathcal{O}_Z^{\oplus r-d} \hookrightarrow E.$$  

b) If $r \geq d$, then a general choice of $r - d + 1$ elements of $V$ defines an injective morphism

$$\mathcal{O}_Z^{\oplus r-d+1} \to E,$$

that is a subbundle in the complement of finitely many points.

Proof. It is well-known [Har77, II,Ex.8.2] that if $r > d$, then a general section does not vanish, so a) follows by induction on $r - d$. It is also well-known that if $r = d$, then a general section vanishes only in finitely many points, so b) follows from a) and induction. □

Proof of Lemma 5.2. For $m \gg 0$ we know by Serre's theorem that the sheaf $E^* \otimes \mathcal{O}_S(mH)$ is globally generated; we denote by $V$ the space of global sections.

For every $i \in I$, the restricted vector bundle $(E^* \otimes \mathcal{O}_S(mH))|_{C_i}$ is generated by global sections of $V_i := V|_{C_i}$ (i.e. those global sections of $(E^* \otimes \mathcal{O}_S(mH))|_{C_i}$ that lift to global sections on $S$). By Lemma 5.3 for a general choice of elements

$$s_{1,i}, \ldots, s_{r-1,i} \in V_i$$

we obtain a subbundle

$$\mathcal{O}_{C_i}^{\oplus r-1} \to (E^* \otimes \mathcal{O}_S(mH))|_{C_i}.$$  

Since there are only countably many curves $C_i$ we can thus fix very general sections

$$s_1, \ldots, s_{r-1} \in V$$

inducing an injective morphism

$$s_1 \oplus \ldots \oplus s_{r-1} : \mathcal{O}_S^{\oplus r-1} \to E^* \otimes \mathcal{O}_S(mH),$$

such that the restriction

$$s_{i,1} \oplus \ldots \oplus s_{i,r-1} : \mathcal{O}_{C_i}^{\oplus r-1} \to (E^* \otimes \mathcal{O}_S(mH))|_{C_i}.$$  

defines a subbundle.

Moreover by part b) of Lemma 5.3 the map is injective in the complement of finitely many points $p_1, \ldots, p_k$. Dualising and tensoring with $\mathcal{O}_S(mH)$ we obtain a morphism

$$E \to \mathcal{O}_S(mH)^{\oplus r-1}$$

that is surjective in the complement of finitely many points and the restriction

$$\mathcal{E}_{C_i} \to \mathcal{O}_{C_i}(mH)^{\oplus r-1}$$

is surjective. In particular the restriction of $\zeta$ to the divisor $\mathbb{P}(\mathcal{O}_{C_i}(mH)^{\oplus r-1}) \subset \mathbb{P}(\mathcal{E}_{C_i})$ is nef (even ample). The map $E \to \mathcal{O}_S(mH)^{\oplus r-1}$ is in general not surjective (so does not define a subvariety of $\mathbb{P}(E)$) however if we denote by $\mathcal{H}$ its image, then $\mathbb{P}(\mathcal{H}) \subset \mathbb{P}(E)$ is a prime divisor $D$ which is easily seen to be in the linear system $[\zeta + m(r-1)\pi^*H]$. □
Proof of Theorem 1.8. Since $\mathcal{E}$ is almost nef, we have $c_1(\mathcal{E}) \cdot H^{n-1} \geq 0$ for every ample Cartier divisor $H$ on $X$. By assumption, there exists an ample Cartier divisor $A$ on $X$ such that

$$c_1(\mathcal{E}) \cdot A^{n-1} = 0.$$ 

This implies that $c_1(\mathcal{E}) \cdot H^{n-1} = 0$ for every ample $H$ (see e.g. [Pet94, Lemma 6.5]). Since $\mathcal{E}$ is almost nef, we obtain that $\mathcal{E}$ is $H$-semistable by an application of the theorem of Mehta-Ramanathan. Fix an arbitrary ample Cartier divisor $H$ on $X$.

Since $X$ is smooth in codimension two the reflexive sheaf $\mathcal{E}$ is locally free in the complement of a set of codimension at least three [Kob87, Cor.5.5.20]. In particular if $S \subset X$ is a surface cut out by very general hyperplane sections $D_i \in |mH|$ for $m \gg 0$, the restriction $\mathcal{E}|_S$ to the smooth surface $S$ is locally free and almost nef.

We claim that

$$c_1(\mathcal{E}_S)^2 = 0, \ c_2(\mathcal{E}_S) = 0.$$ 

Since the class of $S$ is a positive multiple of $H^{n-2}$, we then obtain that

$$c_1(\mathcal{E})^2 \cdot H^{n-2} = 0, \ c_2(\mathcal{E}) \cdot H^{n-2} = 0.$$ 

If $X$ is klt, then [GKP16a, Thm.1.20] provides a quasi-étale cover $\gamma : \tilde{X} \to X$ such that $\gamma^*(\mathcal{E})$ is locally free and flat. Since $\mathcal{E}$ is $H$-semistable, the reflexive pullback $\gamma^*(\mathcal{E})$ is $\gamma^*H$-semistable. Therefore $\gamma^*(\mathcal{E})$ is numerically flat and its Chern class vanish by Remark 2.6.

Proof of the claim. Denote by $H_S$ the restriction of the polarisation $H$. We know that $c_1(\mathcal{E}_S) \cdot H_S = 0$. Since $\mathcal{E}_S$ is almost nef, this implies that $\mathcal{E}_S$ is semistable with respect to $H_S$. Let $\zeta_S$ be the tautological class on $\mathbb{P}(\mathcal{E}_S)$, and denote $\pi : \mathbb{P}(\mathcal{E}_S) \to S$ the projection. By Proposition 5.1 it is sufficient to find a divisor $D \in |\zeta_S + t\pi^*H_S|$ such that $\zeta_S \cdot D \geq 0$.

Since $\mathcal{E}_S$ is almost nef and $\dim S = 2$, there exists an at most countable collection of curves $C_i \subset S$ such that $\mathcal{E}|_{C_i}$ is not nef. Consequently if $C \subset \mathbb{P}(\mathcal{E}_S)$ is a curve such that $\zeta_S \cdot C < 0$ then $C$ maps onto one of the curves $C_i$. By Lemma 5.2 we can find a divisor $D \in |\zeta_S + \pi^*(r-1)mH_S|$ such that $\zeta_S|_{D \cap \pi^{-1}(C_i)}$ is nef for all $i$. Thus $(\zeta_S)|_D$ is nef, in particular

$$\zeta_S \cdot D = ((\zeta_S)|_D) \geq 0.$$ 

Using that almost nefness and numerically flatness of vector bundles are invariant under a birational morphism, the following variant of Theorem 1.8 follows by passing to a desingularisation.

5.4. Corollary. Let $X$ be a normal projective variety, and let $\mathcal{E}$ be a locally free sheaf on $X$ such that $c_1(\mathcal{E}) = 0$. If $\mathcal{E}$ is almost nef, then $\mathcal{E}$ is numerically flat.

Proof of Theorem 1.9. Assume that $T_X$ is almost nef. By Theorem 1.8, there exists a quasi-étale cover $\gamma : \tilde{X} \to X$ such that $\gamma^*(T_X)$ is locally free and numerically flat. Since $T_{\tilde{X}} = \gamma^*(T_X)$, we conclude by [GKPP11, Thm. 6.1], that $\tilde{X}$ is smooth and

$$c_2(\tilde{X}) = 0.$$ 

Hence by Yau’s theorem $\tilde{X}$ is an étale quotient of a torus. □
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