VIRTUALLY EXPANDING DYNAMICS

MASATO TSUJII

Abstract. We introduce a class of discrete dynamical systems that we call virtually expanding. This is an open subset of self-covering maps on a closed manifold which contains all expanding maps and some partially hyperbolic volume-expanding maps. We show that the Perron-Frobenius operator is quasi-compact on a Sobolev space of positive order for such class of dynamical systems.

1. Introduction

Expanding maps are a class of discrete dynamical systems that exhibit typically chaotic behavior of orbits and studied extensively as a standard model of chaotic dynamical systems. Among others, it is important that the associated Perron-Frobenius operators are quasi-compact on the space of functions with some smoothness. (See [7, 2].) Based on the recent understanding on expanding and hyperbolic dynamical systems in terms of Fourier analysis [3, 5, 4], this fact may be understood as a consequence of the property of the Perron-Frobenius operators that they reduce high-frequency components of functions by transferring them to low-frequency ones. But actually in order to get the conclusion of quasi-compactness, we only need to have that property “in average with respect to preimages” and “direction-wise”. (We refer Remark 2 for more explanation.) Indeed we find that the argument about quasi-compactness of the Perron-Frobenius operators for expanding maps can be extended to a much more general class of dynamical systems, which we would like to call virtually expanding maps. Below we present the definition and the main property of virtually expanding maps. In the last section, we discuss briefly about a further generalization of the class of virtually expanding maps.

2. Main result

Let $M$ be a closed connected $C^\infty$ manifold and let $f : M \to M$ be a $C^\infty$ map. We suppose that $M$ is equipped with some Riemann metric and

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that $f$ is a covering map. Let $T^*M$ be the cotangent bundle of $M$. The push-forward action of $f$ on $T^*M$ is written as

$$f^*: T^*M 	o T^*M, \quad (q, \xi) \in T^*_qM \mapsto f^!(\xi) = (f(q), (Df_q)^{-1})^*(\xi).$$

For $\mu \geq 0$ and each non-zero covector $(q, \xi) \in T^*M$, we set

$$(1) \quad b^\mu((q, \xi); f) = \sum_{f^!(p, \eta) = (q, \xi)} \frac{|\xi|/|\eta|^\mu}{|Jf((p, \eta))|}$$

where we write $| \cdot |$ for the (induced) norm on the cotangent bundle from the Riemann metric. Then we define

$$(2) \quad B^\mu(f) = \sup \{b^\mu((q, \xi); f) \mid 0 \neq (q, \xi) \in T^*M\}.$$ 

Note that, for another $C^\infty$ covering map $g : M \to M$, we have

$$B^\mu(f \circ g) \leq B^\mu(f) \cdot B^\mu(g).$$

In particular $B^\mu(f^n)$ is sub-multiplicative with respect to $n$:

$$(3) \quad B^\mu(f^{n+m}) \leq B^\mu(f^n) \cdot B^\mu(f^m) \quad \text{for } n, m \geq 1.$$

**Definition.** For $\mu > 0$, a $C^\infty$ covering map $f : M \to M$ is $\mu$-virtually expanding if

$$\inf_{n \geq 1} (B^\mu(f^n))^{1/n} = \lim_{n \to \infty} (B^\mu(f^n))^{1/n} < 1.$$ 

We write $\mathcal{VE}^\mu(M)$ for the subset of $\mu$-virtually expanding maps on $M$. We call an element of $\mathcal{VE}(M) = \bigcup_{\mu > 0} \mathcal{VE}^\mu(M)$ virtually expanding. $\mathcal{VE}^\mu(M)$ for $\mu > 0$ as well as $\mathcal{VE}(M)$ are $C^1$ open subsets of $C^\infty(M, M)$.

**Remark 1.** The definition of virtually expanding property above does not depend on the choice of the Riemann metric on $M$. Moreover we can replace the Riemann metric by any continuous Finsler metric.

**Remark 2.** For a one dimensional map $f : \mathbb{T} \to \mathbb{T}$, we see that the condition

$$B^\mu(f) = \sup_{x \in \mathbb{T}} \sum_{y \in f^{-1}(x)} |f'(y)|^{-1-\mu} < 1$$

is equivalent to the condition that $f$ is expanding. But, in higher dimension, this is not the case because, in [11], we consider each (direction of) non-zero covector $\xi \in T^*M$ and take the averaged value of the expansion rates for the backward image of $\xi$ with weight $1/|Jf|$.

An expanding map $f : M \to M$ is virtually expanding. Indeed by a standard distortion estimate we have that

$$\lim_{n \to \infty} (B^0(f^n))^{1/n} \leq 1$$

and hence

$$\lim_{n \to \infty} (B^\mu(f^n))^{1/n} \leq \lambda^{-\mu} < 1$$

where $\lambda > 1$ is the minimum expansion rate of $f$. 


Further there are many virtually expanding maps that are not expanding. Let us say that a $C^\infty$ map $f : M \to M$ is volume-expanding if

$$\lim_{n \to \infty} \left( \inf_{p \in M} |Jf^n(p)| \right)^{1/n} > 1$$

where $Jf : M \to \mathbb{R}$ denotes the Jacobian of $f$ with respect to the Riemann volume. If $\dim M = 2$, $\mathcal{VE}(M)$ contains (non-empty) $C^1$ open subsets of volume-expanding partially hyperbolic endomorphisms.

**Example.** Consider a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ on the torus $\mathbb{T}^2$ defined by

$$f(x, y) = (mx, y + \cos 2\pi x) \mod \mathbb{Z}^2.$$  

For this map and its small $C^1$ perturbations, we can check the condition $B^\mu(f) < 1$ at least if we assume $m$ is sufficiently large depending on $\mu > 0$.

A class of partially hyperbolic endomorphisms containing the example above is studied in [9]. It will be possible to prove that a generic $C^\infty$ partially hyperbolic volume-expanding maps is virtually expanding. (See [9, 1].) Further it will not be too optimistic to expect that the same is true without the assumption of partial hyperbolicity.

The Perron-Frobenius operator $P$ expresses the push-forward action of a self-covering map $f$ on densities on $M$, which is defined explicitly by

$$Pu(p) = \sum_{q \in f^{-1}(p)} \frac{u(q)}{|Jf(q)|}.$$  

It is not difficult to see that $P$ gives a bounded operator on the Sobolev space $H^\mu(M)$ of any order $\mu \in \mathbb{R}$. (See [8, §1.3, §1.5] for the definition of $H^\mu(M)$.) Our main theorem is the following.

**Theorem.** If $f \in \mathcal{VE}^\mu(M)$ for some $\mu > 0$, the Perron-Frobenius operator $P$ is quasi-compact on the Sobolev space $H^\mu(M)$ of order $\mu > 0$, that is, the essential spectral radius of $P : H^\mu(M) \to H^\mu(M)$ is smaller than its spectral radius 1.

From the conclusion of the theorem, a few important dynamical properties of the map $f$ follows. For instance we have

**Corollary.** If $f \in \mathcal{VE}^\mu(M)$ for some $\mu > 0$, there are finitely many absolutely continuous ergodic measures whose densities with respect to the smooth volume are in $H^\mu(M)$ and almost every point on $M$ with respect to the smooth volume is generic for either of them.

**Proof.** We only prove the latter claim. (The proof of the former claim should be rather standard if we note that $P$ preserves positive and negative part of functions.) We write $m$ for the Riemann volume on $M$. Let $\mu_i = \varphi_i dm$, $1 \leq i \leq \ell$, be the absolutely continuous ergodic measures whose densities belong to $H^\mu(M)$. Suppose that the measurable subset

$$X = \{x \in M \mid x \text{ is not generic for either of } \mu_i, 1 \leq i \leq \ell.\}$$
has positive measure with respect to the smooth measure \( m \) on \( M \). We approximate the characteristic function of \( X \) by \( C^\infty \) positive-valued function \( \chi \) in \( L^2 \) sense. From the quasi-compactness of \( P \), the sequence of functions

\[
\chi_m := \frac{1}{m} \sum_{k=0}^{m} P^k \chi
\]

converges, as \( m \to \infty \), to a function in the eigenspace \( L \) for the eigenvalue 1 in \( H^\mu(M) \) and hence in \( L^2(M) \). On the other hand, since \( f(X) \subset X \) and \( P \) preserves the \( L^1 \) norm of positive-valued functions, the \( L^2 \)-orthogonal component to \( L \) of \( \chi_m \) is bounded from below by a positive constant, provided that \( \chi \) is sufficiently close to the characteristic function of \( X \). This is a contradiction and therefore \( X \) is a null subset. \( \square \)

3. Proof of Theorem 2

We will use a few basic theorems on pseudo-differential operators and notations given in [8, Ch.1 and 2]. First we take an small open connected subset \( U \subset M \) so that its backward image \( f^{-1}(U) \) consists of connected open subsets \( V_1, V_2, \ldots, V_{\deg f} \), on each of which the map \( f \) is a diffeomorphism onto \( U \). Here \( \deg f \) denotes the mapping degree of \( f \). We write \( f_j : V_j \to U \) for the restriction of \( f \) to \( V_j \). For convenience, we may and do suppose that each of \( U \) and \( V_j \), \( 1 \leq j \leq \deg f \), is contain in a single coordinate chart.

Suppose that a function \( u \in C^\infty(M) \) is supported on \( f^{-1}(U) = \bigcup_{j=1}^{\deg f} V_j \) and we write \( u_j \) for the restriction of \( u \) to \( V_j \). Then we may write

\[
P u = \sum_{j=1}^{\deg f} P_j u_j, \quad P_j u_j = \frac{u_j}{Jf \circ f_j^{-1}}
\]

where \( P_j \) denotes the restriction of \( P \) to the functions supported on \( V_j \). Note that \( P_j \) viewed in the local coordinate chart is simply a multiplication by a smooth function composed with an operation of change of variables.

For \( \mu > 0 \), we define a function

\[
W^\mu : T^*M \to \mathbb{R}, \quad W^\mu(x, \xi) = (\langle |\xi| \rangle^\mu).
\]

This function belong so a standard symbol class \( S^\mu_{1,0} \) (See [8, §1.1]) and we may define the norm \( \| \cdot \|_{H^\mu} \) on the Sobolev space \( H^\mu(M) \) by

\[
\| u \|_{H^\mu}^2 = \| \text{Op}(W^\mu) u \|^2_{L^2} + \| u \|^2_{L^2}
\]

where (and henceforth) we write \( \text{Op}(F) \) for the pseudo-differential operator with symbol \( F \). (See [8, §2.5].) We write \( Df_j^* : T^*U \to T^*V_j \) for the pull-back action of \( f_j \) on the cotangent bundle restricted \( T^*U \). This is a bijection. Let us consider the pseudo-differential operators

\[
A_j = \text{Op}\left( \frac{W^\mu}{\sqrt{Jf \circ f_j^{-1} \cdot W^\mu \circ Df_j^*}} \right)
\]
and
\[
b_j = \text{Op} \left( \frac{\mathcal{W}^\mu \circ Df_j^*}{\sqrt{Jf \circ f_j^{-1}}} \right) v_j \quad \text{where} \quad v_j := u_j \circ f_j^{-1}
\]
for \( j = 1, 2, \ldots, \deg f \). Then, by the product formula ([8, §2.4]) of pseudo-differential operator, we have
\[
\|\text{Op}(\mathcal{W}^\mu) Pu\|_{L^2} \leq \sum_{j=1}^{\deg f} A_j b_j \|v_j\|_{L^2} + \sum_{j=1}^{\deg f} \|\text{Op}(\mathcal{W}^\mu) - A_j b_j\|_{L^2} v_j\|
\]
\[
\leq \sum_{j=1}^{\deg f} A_j b_j \|v_j\|_{L^2} + C \sum_{j=1}^{\deg f} \|v_j\|_{H^{\mu-1}}
\]
where \( C \) is a constant that may depend on \( f \) but not on \( u \). (In the following we use \( C \) as a generic symbol for constants of this kind.) Recall the Lagrange identity
\[
\sum_{i,j=1}^{d} |x_i|^2 |y_j|^2 - \sum_{i=1}^{d} \bar{x}_i y_i = \frac{1}{2} \sum_{i,j} |\bar{x}_i y_j - \bar{x}_j y_i|^2.
\]
Following its proof with regarding \( A_i \) and \( b_i \) as \( x_i \) and \( y_i \) respectively\(^1\) and using the product and adjoint formula ([8, §2.4]) to shuffle the pseudo-differential operators, we see that
\[
\sum_{i,j=1}^{\deg f} \|A_j b_i\|_{L^2}^2 - \sum_{j=1}^{\deg f} A_j b_j \|v_j\|_{L^2}^2 \geq \sum_{i,j} \|A_j b_i - A_i b_j\|_{L^2}^2 - C \sum_{j=1}^{\deg f} \|v_j\|_{H^{\mu-(1/2)}}^2.
\]
Further, by the product and adjoint formula, we see
\[
\sum_{i,j=1}^{\deg f} \|A_j b_i\|_{L^2}^2 \leq \sum_{i=1}^{\deg f} \langle b_i, A b_i \rangle + C \sum_{j=1}^{\deg f} \|v_j\|_{H^{\mu-(1/2)}}^2
\]
where
\[
A = \text{Op} \left( \sum_{j=1}^{\deg g} \frac{\mathcal{W}^{2\mu}}{(Jf \circ f_j^{-1}) \cdot \mathcal{W}^{2\mu} \circ (Df_j^{-1})^*} \right).
\]
By the definition of \( B^\mu(f) \), it holds for arbitrarily small \( \varepsilon > 0 \) that
\[
\sum_{j=1}^{\deg g} \frac{\mathcal{W}^{2\mu}}{(Jf \circ f_j^{-1}) \cdot \mathcal{W}^{2\mu} \circ (Df_j^{-1})^*} \leq B^{2\mu}(f) + \varepsilon
\]
\(^1\)Further regard \( A_i^* \) as \( \bar{x}_i \) and the \( L^2 \) inner product as the product of complex numbers in a few places.
on the outside of a sufficiently large neighborhood of the zero section depending on $\varepsilon > 0$. Hence, by Gårding’s inequality [8, §2.8], it follows
\[
\langle b_i, Ab_i \rangle \leq (B^{2\mu}(f) + \varepsilon) \| b_i \|_{L^2} + C \| v_i \|^2_{H^{\mu,-(1/2)}}.
\]
From the relation $v_j = P_j u_j$, we have that
\[
\| b_j \|_{L^2} \leq \left\| \text{Op} (\mathcal{W}^\mu \circ Df_j^*) \left( \frac{v_j}{\sqrt{Jf \circ f_j^{-1}}} \right) \right\|_{L^2} + C \| u_j \|_{H^{\mu,-1}}
\]
\[
\leq \left\| \text{Op} (\mathcal{W}^\mu) u_j \right\|_{L^2} + C \| u_j \|_{H^{\mu,-1}}.
\]
Finally note that we have
\[
\| v_j \|_{H^{\mu'}} \leq C \| u_j \|_{H^{\mu'}}
\]
for any $\mu' \in \mathbb{R}$ and also that
\[
\| u_j \|_{H^{\mu'}} \leq C \| u_j \|_{H^{\mu''}}
\]
when $\mu' \leq \mu''$ where $C$ may depend on $\mu'$ and $\mu''$. Therefore, summarizing the argument above, we have that, for any $\max\{0, \mu - (1/2)\} \leq \mu' < \mu$ and $\varepsilon > 0$, there exists $C > 1$ such that
\[
\| Pu \|_{H^{\mu}} \leq \sqrt{(B^{2\mu}(f) + \varepsilon)} \| u \|_{H^{\mu}} + C \| u \|_{H^{\mu'}}.
\]
By using a partition of unity and the product formula ([8, §2.4]), we see that the same inequality holds for arbitrary $u \in C^\infty(M)$. Since $H^{\mu}(M)$ for $\mu > 0$ is compactly embedded in $H^{\mu'}(M)$, the last inequality and Henion’s theorem [6] tell that the essential spectral radius of $P$ on $H^{\mu}(M)$ is bounded by $\sqrt{B^{2\mu}(f)}$. Finally noting that the argument above remains valid when $f$ is replaced by its iterates $f^n$, we obtain the conclusion of the theorem.

4. Conclusive remarks

It is possible to generalize our result without much effort. In the last section, we consider the weight functions $\mathcal{W}^\mu$ for $\mu > 0$. But we could consider a more general class of positive function in the place of $\mathcal{W}^\mu$. For instance, we may set
\[
\mathcal{W}^\mu(x, \xi) = \langle |\xi| \rangle^{\mu((x, \xi))}
\]
for a $C^\infty$ positive-valued function $\mu : \mathbb{P} T^* M \to \mathbb{R}$ on the projectivized cotangent bundle $\mathbb{P} T^* M$ and prove Theorem\footnote{In the case where $\mu$ takes negative values, the argument in the proof of Corollary is not valid.} with setting
\[
B^\mu(f) = \sup_{(x, \xi) \in T^* M} \sum_{(y, \eta) \in T^* M : f^1(y, \eta) = (x, \xi)} \mathcal{W}^\mu(x, \xi) \cdot \mathcal{W}^\mu(y, \eta). 
\]
We refer [4] for the treatment of the pseudo-differential operators $\text{Op} (\mathcal{W}^\mu)$. It will be an interesting subject to find a reasonable but non-trivial class of dynamical systems for which $\lim_{n \to \infty} B^\mu(f^n) < 1.$
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Department of Mathematics, Kyushu University, Fukuoka, 819-0395
Email address: tsujii@math.kyushu-u.ac.jp