Hodge structure on the fundamental group and its application to p-adic integration

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Abstract

We study the unipotent completion $\Pi_{DR}^{un}(x_0, x_1, X_K)$ of the de Rham fundamental groupoid $\Pi$ of a smooth algebraic variety over a local non-archimedean field $K$ of characteristic 0. We show that the vector space $\Pi_{DR}^{un}(x_0, x_1, X_K)$ carries a certain additional structure. That is a $\mathbb{Q}_p$-space $\Pi_{un}(x_0, x_1, X_K)$ equipped with a $\sigma$-semi-linear operator $\phi$, a linear operator $N$ satisfying the relation $N\phi = p\phi N$ and a weight filtration $W$ together with a canonical isomorphism $\Pi_{DR}^{un}(x_0, x_1, X_K) \otimes_K K \cong \Pi_{un}(x_0, x_1, X_K) \otimes_{\mathbb{Q}_p} K$. We prove that an analog of the Monodromy Conjecture holds for $\Pi_{un}(x_0, x_1, X_K)$.

As an application, we show that the vector space $\Pi_{DR}^{un}(x_0, x_1, X_K)$ possesses a distinguished element. In the other words, given a vector bundle $E$ on $X_K$ together with a unipotent integrable connection, we have a canonical isomorphism $E_{x_0} \cong E_{x_1}$ between the fibers. The latter construction is a generalization of Colmez’s $p$-adic integration ($rk E = 2$) and Coleman’s $p$-adic iterated integrals ($X_K$ is a curve with good reduction).

In the second part we prove that, if $X_{K_0}$ is a smooth variety over an unramified extension of $\mathbb{Q}_p$ with good reduction and $r \leq \frac{p-1}{2}$ then there is a canonical isomorphism $\Pi_{DR}^{un}(x_0, x_1, X_{K_0}) \otimes B_{DR} \cong \Pi_{et}^{un}(x_0, x_1, X_{K_0}) \otimes B_{DR}$ compatible with the action of Galois group. ( $\Pi_{DR}^{un}(x_0, x_1, X_{K_0})$ stands for the level $r$ quotient of $\Pi_{un}(x_0, x_1, X_K)$.) In particular, it implies the Crystalline Conjecture for the fundamental group [Shiho] (for $r \leq \frac{p-1}{2}$).

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1. **Notation** Throughout this paper $k$ stands for a finite field of characteristic $p > 2$, $W(k)$ is the ring of Witt vectors, $K_0$ is its field of fractions, $K$ is a finite extension of $K_0$ with the ring of integers $R \subset K$ and $e$ is its ramification index.

1.2. For a smooth scheme $X$ over a field of characteristic 0 we denote by $\mathcal{DM}(X)$ the category of vector bundles on $X$ together with an integrable connection. An object $E$ of the latter category is called unipotent of level $r$ if there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E$$

where $E_i$ are subobjects, such that the quotients $E_i/E_{i-1}$ are trivial. The latter means that the vector bundle $E_i/E_{i-1}$ is generated by its global parallel sections.

1.3. Let $X_{K_0}$ be a smooth scheme finite type over $K_0$. For a pair of points $x_0, x_1 \in X_{K_0}(\overline{K})$ we denote by $\pi_1^e(X_{\overline{K}}, x_0, x_1)$ the set of isomorphisms between the corresponding functors from the category of finite etale schemes over $X_{\overline{K}} = X \times \text{spec} K$ to the category of finite sets.

1.4. To state the first main result, we assume that the scheme $X_{K_0}$ has a good compactification over $\text{spec} W(k)$. That is a smooth proper scheme $\overline{X}$ over $\text{spec} W(k)$ with a divisor $Z \subset \overline{X}$ with normal crossings relative to $\text{spec} W(k)$ and an isomorphism

$$X_{K_0} \simeq (\overline{X} - Z) \times \text{spec} W(k) \text{ spec } K_0$$

1.5. Assume that $r \leq \frac{e-1}{2}$. For any object $E$ of the category $\mathcal{DM}_r(X_{\overline{K}})$, a pair of points $x_0, x_1 \in X(\overline{K})$ and an etale path $\lambda \in \pi_1^e(X_{\overline{K}}, x_0, x_1)$ we construct a parallel translation:

$$T_\lambda : E_{x_0} \otimes_{\overline{K}} B_{DR} \simeq E_{x_1} \otimes_{\overline{K}} B_{DR}$$

Here $B_{DR}$ is the certain universal ring of periods introduced by Fontaine in [Fo].

1.6. To formulate the result more precisely it is convenient to introduce a certain category, which is, in fact, a modification of the fundamental groupoid [De].

A point $x \in X_{K_0}(\overline{K})$ defines a fiber functor

$$F_x : \mathcal{DM}_r(X_{\overline{K}}) \longrightarrow \text{Vect}_{\overline{K}}$$

to the category of vector spaces over $\overline{K}$.

Consider the category $\mathcal{P}^{DR}_r(X_{\overline{K}})$, whose objects $x$ are points of $X_{K_0}(\overline{K})$ and whose set morphisms between two points $\overline{x_0}, \overline{x_1}$ is the vector space of morphisms between the corresponding fiber functors:

$$\text{Mor}_{\mathcal{P}^{DR}_r(X_{\overline{K}})}(\overline{x_0}, \overline{x_1}) = \text{Mor}(F_{\overline{x_0}}, F_{\overline{x_1}})$$

In the same way we define the category $\mathcal{P}^{et}_r(X_{\overline{K}})$, simply by replacing vector bundles with connections by unipotent etale local systems of $\mathbb{Q}_p$-vector spaces.
The categories $\mathcal{P}^{DR}(X_\mathbb{R})$ and $\mathcal{P}^{et}_C(X_\mathbb{R})$ carry a natural action of the Galois group $Gal(K/K_0)$.

1.7. Given a homomorphism of rings $C \to C'$ and a category $\mathcal{A}$ with a structure of a $C$-module on $Mor_{\mathcal{A}}(\ast; \ast)$ compatible with the composition, we denote by $\mathcal{A} \otimes_C C'$ the category whose set of objects is the same but whose set of morphisms between objects $V$ and $W$ is

$$Mor_{\mathcal{A}}(V; W) \otimes_C C'$$

1.9. Theorem A. Assume that $r \leq \frac{p^{-1}}{2}$. Then there exists a functor:

$$I_{X_\mathbb{R}} : \mathcal{P}^{et}_r(X_\mathbb{R}) \otimes_{Q_p} B_{DR} \to \mathcal{P}^{DR}(X_\mathbb{R}) \otimes_{\mathbb{R}} B_{DR}$$

identical on objects and satisfying the following properties:

a) $I_{X_\mathbb{R}}$ establishes an equivalence between the categories.

b) The functor $I_{X_\mathbb{R}}$ is equivariant with respect to the action of the Galois group $Gal(K/K_0)$.

c) Let $f : X_\mathbb{R} \to Y_\mathbb{R}$ be a morphism. Then we have:

$$f^{DR}_* \circ I_{X_\mathbb{R}} = I_{Y_\mathbb{R}} \circ f^{et}_*$$

Here $f^{DR}_*$ and $f^{et}_*$ stand for the functors:

$$f^{DR}_* : \mathcal{P}^{DR}(X_\mathbb{R}) \to \mathcal{P}^{DR}(Y_\mathbb{R})$$

$$f^{et}_* : \mathcal{P}^{et}(X_\mathbb{R}) \to \mathcal{P}^{et}(Y_\mathbb{R})$$

1.10. Remark. The property c) holds for any morphism $f : X_\mathbb{R} \to Y_\mathbb{R}$ of smooth schemes over $\text{spec } \mathbb{R}$, which have a good compactification (as defined in 1.4).

In particularly, the functor $I_{X_\mathbb{R}}$ depends only on the scheme $X_\mathbb{R}$ itself and not on the choice of a good model.

This is a true fact but it is not proven in this paper.

1.11. Unfortunately we were not able to find a direct construction of this functor; our proof is very implicit. It is based on the theory of variations of $p$-adic Hodge structures developed by Faltings [Fa]. (We use this term for the category $\mathcal{MF}^\vee(X) \otimes \mathbb{Q}$ of certain filtered $F$-crystals introduced in loc. cit.)

1.12. The proof of Theorem 1 consists of the following steps.

First, we construct the parallel translation $[\Box]$ for a certain universal unipotent local system. Let $\Pi^{DR}_r(X_{K_0})$ be the vector bundle on $X_{K_0} \times X_{K_0}$ together with a unipotent connection of level $r$ characterized by the following property:

given another object $E$ of the category $\mathcal{DM}_r(X_{K_0} \times X_{K_0})$, we have a canonical isomorphism:

$$Hom_{\mathcal{DM}_r(X_{K_0} \times X_{K_0})}(\Pi^{DR}_r(X_{K_0}); E) \simeq H^0_{DR}(\Delta; \Delta^*E)$$  \hspace{1cm} (2)

where $\Delta : X_{K_0} \to X_{K_0} \times X_{K_0}$ is the diagonal embedding.

The fiber of $\Pi^{DR}_r(X_{K_0})$ over a point $(x_0, x_1)$ is canonically identified with $Mor(F_{x_0}, F_{x_1})$.

The identity morphism $Id : \Pi^{DR}_r(X_{K_0}) \to \Pi^{DR}_r(X_{K_0})$ gives rise to a parallel section

$$1 : \mathcal{O}_\Delta \to \Delta^*\Pi^{DR}_r(X_{K_0})$$  \hspace{1cm} (3)

1.13. The main step is a construction of a variation of $p$-adic Hodge structure on $\Pi^{DR}_r(X_{K_0})$. The analogous construction over the field of complex numbers is well known. It is based on an interpretation of the fibers in terms of homology groups of a certain simplicial scheme. In our case, we found a different construction based on a simple linear algebra argument. It works over $\mathbb{C}$ as well.

We proof by induction on $r$, that there exists a unique variation of Hodge structure on $\Pi^{DR}_r(X_{K_0})$ such that the map $[\Box]$ is a morphism of Hodge structures, for any unipotent level $r$ variation $E$. For this we consider the following exact sequence:

$$0 \to I \to \Pi^{DR}_r(X_{K_0}) \to \Pi^{DR}_{r-1}(X_{K_0}) \to 0$$
By induction hypothesis the boundary terms are endowed with Hodge structures. We show that there is a unique Hodge structure on $\Pi_{r}^{DR}(X_{K_0})$ satisfying the following property: the map (3) is a morphism of Hodge structures.

Given a point $x_0 \in X_{K_0}(K)$, we denote by $\Pi_{r}^{DR}(X_{K_0}, x_0)$ the restriction of the variation of Hodge structure $\Pi_{r}^{DR}(X_{K_0})$ to the fiber $X_{K_0} \times x_0$.

1.14. Next, we make use of the functor

$$\mathbf{D} : \mathcal{MF}^Q(X) \longrightarrow Sh^{et}_{Q_0}(X_{K_0})$$

from the category of variations of Hodge structures to the category of etale locally constant sheaves on $X_{K_0}$. The latter was constructed by Faltings.

We proof that for any point $x \in X_{K_0}(\overline{K})$ and a variation $E$ there is a canonical isomorphism of fibers

$$(\mathbf{D}(E))_x \otimes B_{DR} \simeq E_x \otimes B_{DR}$$

(The latter is not entirely obvious if the point reduces to infinity in the special fiber).

Applying the (4) and (5) to $\Pi_{r}^{DR}(X_{K_0}, x)$ we obtain a parallel translation:

$$T_\lambda : \Pi_{r}^{DR}(X_{K_0}, x_0) \otimes B_{DR} \simeq \Pi_{r}^{DR}(X_{K_0}, x_0)_{x_1} \otimes B_{DR}$$

In particular, we constructed an element

$$T_\lambda(1) \in \Pi_{r}^{DR}(X_{K_0}, x_0)_{x_1} \otimes B_{DR} = Mor_{P^{DR}(X_{\overline{K}})}(\mathcal{Z}_0; \mathcal{Z}_1) \otimes B_{DR}$$

It gives rise to a morphism:

$$Mor_{P^{DR}(X_{\overline{K}})}(\mathcal{Z}_0; \mathcal{Z}_1) \otimes B_{DR} \rightarrow Mor_{P^{DR}(X_{\overline{K}})}(\mathcal{Z}_0; \mathcal{Z}_1) \otimes B_{DR}$$

Finally, we show that the latter is an isomorphism.

1.15. To formulate our second main result we let $X_K$ be a smooth geometrically connected scheme finite over $K$.

Denote by $K_{st}$ the ring of polynomials $K[ll(p)]$ in a formal variable $l(p)$, Let $N : K_{st} \rightarrow K_{st}$ be the derivation such that $N(x) = 0$ for any $x \in K$ and $N(l(p)) = 1$.

We define a map

$$Log : K^* \rightarrow K_{st}$$

to be the unique homomorphism given by the series

$$Log x = \sum_{i=1}^{\infty} \frac{(1 - x)^i}{i}$$

on a neighborhood of $0 \in K$ and satisfying $Log p = l(p)$.

Given a vector bundle $E$ on $X_K$ together with a unipotent integrable connection and a pair of points $x_0, x_1 \in X_K(K)$, we construct a canonical isomorphism

$$C_{x_0,x_1} : C_{x_0,x_1} : E_{x_0} \otimes_K K_{st} \simeq E_{x_1} \otimes_K K_{st}$$

We would like to stress, that the latter does not depend on any additional choices like that of an etale path.

1.16. To describe some properties of the canonical parallel translation we need a few more notations: denote by

$$F_x \otimes K_{st} : \mathcal{MF}_{un}(X_K) \longrightarrow Mod_{K_{st}}$$

the fiber functor to the category of $K_{st}$-modules. Let

$$Mor^\otimes(F_{x_0} \otimes K_{st}; F_{x_1} \otimes K_{st}) \subset Mor(F_{x_0} \otimes K_{st}; F_{x_1} \otimes K_{st})$$

be the set of all morphisms $C$ satisfying with following properties:
i) \( C(\mathcal{O}_{X_K}) = \text{Id} \)

ii) \( C(E^*) = (C(E))^* : E^*_x \otimes_K K_{st} \cong E^*_{x_1} \otimes_K K_{st} \)

(Here \( E^* \) is the dual object).

iii) \( C(E \otimes E') = C(E) \otimes C(E') \)

for any objects \( E \) and \( E' \).

Given a morphism of smooth connected schemes 

\[ f : X_K \rightarrow Y_K \]  

we denote by \( f_* \) the canonical map 

\[ \text{Mor}^\otimes(F_{x_0} \otimes K_{st}; F_{x_1} \otimes K_{st}) \rightarrow \text{Mor}^\otimes(F_{f(x_0)} \otimes K_{st}; F_{f(x_1)} \otimes K_{st}) \]

1.17. Theorem B. There exists a canonical parallel translation i.e. an element 

\[ C_{x_0; x_1; X_K} \in \text{Mor}^\otimes(F_{x_0} \otimes K_{st}; F_{x_1} \otimes K_{st}) \]

satisfying the following properties:

1) For any triple \( x_0, x_1, x_2 \in X(K) \)

\[ C_{x_1; x_2; X_K} \circ C_{x_0; x_1; X_K} = C_{x_0; x_2; X_K} \]

2) For any object \( E \) of the category \( \mathcal{DM}_{un}(X_K) \) the parallel translation \( [7] \) is locally analytic in the variables \( x_0 \) and \( x_1 \).

3) For any morphism \( [3] \) \( f_*(C_{x_0; x_1; X_K}) = C_{f(x_0); f(x_1); Y_K} \)

4) Let \( K \subset L \) be a field extension. Then \( C_{x_1; x_2; X_K} = C_{x_1; x_2; X_L} \).

5) Let \( f \) be a non-vanishing function on \( X_K \). \( E \) be the trivial vector bundle with a basis \( e_{-1}, e_0 \)

We define a connection \( \nabla : E \rightarrow E \otimes \Omega^1_X \) by:

\[ \nabla(s) = As + ds \]

where 

\[ A = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \]

Here \( \omega \) is the differential form \( df \). Then the parallel translation 

\[ C_{x_0; x_1; X_K}(E) : E_{x_0} \otimes_K K_{st} \rightarrow E_{x_1} \otimes_K K_{st} \]

is given by the matrix 

\[ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \]

where \( c = \text{Log} f(x_0) - \text{Log} f(x_1) \).

6) Suppose that \( X_K \) is proper. Then 

\[ (N^r(C_{x_0; x_1; X_K}))(E) = 0 \]

for any object \( E \) of the category \( \mathcal{DM}_{r+1}(X_K) \)

1.18. We also propose the following conjecture.

Conjecture 1 A collection 

\[ C_{x_0; x_1; X_K} \in \text{Mor}^\otimes(F_{x_0} \otimes K_{st}; F_{x_1} \otimes K_{st}) \]

satisfying the above properties is unique.
1.19. The above result generalizes the construction of iterated integrals on a curve with good reduction by R.Coleman [Co], on the one hand, and, on the other hand, Colmez’s theory of $p$-adic integration [Colmez]. The former is essentially equivalent to our construction of the canonical parallel translation in the case when $X_K$ is a curve with a good reduction, while the second author treated the case of arbitrary smooth scheme $X_K$, but when $E$ is of rank 2. Conjecture [1] is also proven by Colmez in rank 2 case.

1.20. Let us explain the idea of our construction of $C_{x_0;x_1;X_K}$.

First, we show that the space $\Pi_{\text{ur}}^{DR}(x_0; x_1; X_K) = Mor(F_{x_0}, F_{x_1})$ possesses a certain additional structure. That is a vector space $\Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$ over the maximal unramified extension $K_0^{ur}$, together with an action of a nilpotent linear operator $N$, an invertible $Fr$-linear operator $\phi$, satisfying the following relation:

$$N : \phi = p\phi \cdot N$$

and an isomorphism

$$\Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K) \otimes_{K_0^{ur}} K_{st} \cong \Pi_D^{DR}(x_0; x_1; X_K) \otimes_K K_{st}$$

commuting with the action of $N$. (Compare with a conjecture by Fontaine: according to this conjecture the de Rham cohomology of any smooth variety over $K$ possesses such a structure.)

1.21. It turns out that if $X_K$ has a good compactification over $\text{spec } R$ (see 1.4 for the definition), the space $(\Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K))^\theta$ of invariants is one-dimensional over $\mathbb{Q}_p$ and, moreover, it has a unique element $C_{x_0;x_1;X_K}$, characterised by the property:

$$C_{x_0;x_1;X_K}(\mathcal{O}_{X_K}) = \text{id}$$

(8)

1.22. In general, we show that the vector space $\Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$ possesses a canonical (“weight”) filtration

$$W_i \Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K) \subset \Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$$

compatible with the one on $\Pi_D^{DR}(x_0; x_1; X_K)$.

Let $\mathcal{V}_r \subset \Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$ be the subspace which consists of elements $v \in \Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$ satisfying the following properties:

i) $\phi^a v = v$

ii) $N^a v \in W_{-a-1} \Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$ for any $0 < a < r$.

We prove that $\text{dim } \mathcal{V}_r = 1$. Moreover, the canonical morphism:

$$\mathcal{V}_r \rightarrow \mathbb{Q}_p$$

is an isomorphism.

The last assertion is derived from a variant of the Monodromy Conjecture [Illusie 2] for $\Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$. In turn, the latter follows from a result of [Mokrane].

Hence, the space $\mathcal{V}_r$ contains a unique element $C_r(X_K; x_0; x_1)$ satisfying (8).

1.23. In the case, when $X_K$ is proper and has a smooth model over $\text{spec } R$, a similar construction has been independently found by Besse.1

The weight filtration on $\Pi_{\text{ur}}^{\varphi}(x_0; x_1; X_K)$ has been independently (and using a different method) constructed by Kim and Hain ( [Kim]

We would like to stress that Theorem A, at least in the case when $X_{K_0}$ is proper and $x_0 = x_1$, is equivalent to the Crystalline Conjecture for the fundamental group invented by Shiho. He also announced a proof of this conjecture (see [Shiho]).

We were informed by Beilinson that he was aware that Theorem 1 is true.

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1 Although unpublished, the text is available electronically, cf. our list of references
normal crossing, hence by virtue of b) it defines a logarithmic structure on that $(0, a) \rightarrow a$ and $(n, a) \rightarrow 0$ if $n > 0$. The canonical morphism from the logarithmic point to the point with the trivial log structure is not smooth.

2.3. Let $f : X \rightarrow Y$ be a morphism of log schemes and $E$ be a sheaf of $\mathcal{O}_X$-modules. A logarithmic derivation with values in $E$ is a pair $(D, \delta \log)$, where $D : \mathcal{O}_X \rightarrow E$ is a derivation
relative to $Y$ and $\delta \log : M_X \to E$ is a monoid homomorphism such that $D(\alpha_X(m)) = \alpha(m)\delta \log(m)$ and $\delta \log f^*(m) = 0$ for any section $m$ of $M_Y$.

There is a universal derivation - "Kahler differentials" $\Omega^1_{X/Y}(\log)$. Define $\Omega^1_{X/Y}(\log) = \bigwedge^\delta \Omega^1_{X/Y}(\log)$. Then with the natural map $d : \Omega^1_{X/Y}(\log) \to \Omega^{1+1}_{X/Y}(\log) \to \Omega^1_{X/Y}(\log)$; $d$ becomes a complex.

Note that in the setting of our example b) the sheaf $\Omega^1_{X/D}(\log)$ coincides with the sheaf of differential 1-forms with logarithmic singularities at $X/S, \gamma$ is a logarithmic PD-thickening over $(S, \gamma)$.

2.4. Log crystalline site. Let $X \to S$ be a morphism of fine log schemes, $p^N O_S = 0$ for some integer $N$, $I$ be a sheaf of ideals on $S$ endowed with a PD-structure $\gamma$. We assume that $\gamma$ extends to $X$. The objects is crystalline site are triples $(U, T, \delta), \gamma$ such that $U$ is a scheme etale over $X$ and $(T, \delta)$ is a logarithmic PD-thickening over $(S, \gamma)$.

Similarly, the nilpotent crystalline site consists of all PD-nilpotent thickenings.

As usual one can define logarithmic crystals.

Let $i : X \to Z$ be a exact closed immersion into a log smooth scheme over $S$ (i.e. a map such that the underlying map of schemes is a closed immersion and $i^* M_Z \to M_X$ is an isomorphism) and $D$ be the PD-envelope of $i$.

One can check that the category of crystals on the crystalline site is equivalent to the category of $O_D$ modules $E$ on $D$ (for the etale topology) with an integrable connection

$$\nabla : E \to E \otimes_{O_X} \Omega^1_{Z/S}(\log)$$

having the following property:

Let $x \in X$ be a point, $\overline{x}$ be its separable closure of $x$ and $t_i$ ($1 \leq i \leq r$) be elements of $(M_Z)_{\overline{x}}$ such that $d \log(t_i)_{1 \leq i \leq r}$ is a basis of $\omega^1_{Z/S, \overline{x}}$. Then, for any section $f$ of $E$ on a neighborhood of $x$ and any $i$, there exist $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbb{N}$ such that

$$(\prod_{1 \leq j \leq k} (\nabla^{log}_{t_i} - m_j)^{n_j})(f) = 0$$

Here $\nabla^{log}_{t_i}$ is defined by: if $\nabla(f) = \sum_{1 \leq i \leq r} f_i \otimes d \log(t_i)$, then $\nabla^{log}_{t_i}(f) = f_i$. (It is easy to prove that if the latter condition holds for one choice of $(t_i)_{1 \leq i \leq r}$ then it holds for any choice of of $(t_i)_{1 \leq i \leq r}$).

Similarly, one can describe crystals on the nilpotent crystalline site. Suppose the immersion $i : X \to D$ is defined by a sheaf of ideals $J_X$ on $D$. A crystal on the nilpotent site corresponds to a sheaf $E$ of modules over $O_D = \lim O_D/J^d_X$ complete with respect to the topology defined by $J^d_X$, together with an integrable connection

$$\nabla : E \to E \otimes_{O_X} \Omega^1_{Z/S}(\log)$$

Moreover, the de Rham complex computes the crystalline cohomology of $E$. More precisely, if we denote by $\pi : (X/S)_{cris} \to X_{et}$ the natural map of topoi, we have a canonical isomorphism

$$R\pi(E) \simeq E_D \otimes \Omega^1_{Z/S}(\log)$$

If the converse is not said, we work with the nilpotent site.

2.5. Filtered logarithmic crystals.

As it was stated above, the category of coherent sheaves equipped with an integral connection on smooth proper scheme over $\mathbb{Z}_p$ is equivalent to the category of coherent crystals on the special fiber. In particular it can be constructed just from the special fiber. On the other hand the category of coherent sheaves together with an integrable connection and a filtration satisfying Griffiths-transversality is not one of "the crystalline nature". For example, endomorphisms of the special fiber do no act on it. In this section we define (following [Fa2]) a certain modification of this category, which admits a "crystalline" description.

A filtered log crystal on $(X/S)$ is a log crystal $E$ together with a decreasing sequence of subsheaves $F^n(E) \subset E$, such that on any logarithmic PD-thickening $(U, T, \delta)$ $F^n(E)_T$ is a subsheaf of $O_T$-modules of $E_T$ with $J^{[m]}_T$. $F^n(E)_T \subset F^{n+m}(E)_T$ (here $J^{[m]}_T$ is the PD-power of the ideal of $U \subset T$).
and for any morphism of PD-thickenings \( f : (U, T, \delta) \to (U', T', \delta') \) the subsheaf \( F^n(E)_T \) is equal to

\[
F^n(E)_T = \sum_{k+m \geq n} \mathcal{F}^{[m]}_T \cdot f_T^*(F^k(E)_{T'})
\]

2.6. Example. Assume that \( X \) is log smooth over \( S \). Then any log crystal \( E \) (as it was explained above the latter datum is equivalent to giving a \( \mathcal{O}_X \)-module \( E_X \) with an integrable log connection together with a filtration \( F^n(E_X) \subset E_X \) by \( \mathcal{O}_X \)-submodules satisfying Griffiths-transversality:

\[
\nabla(F^n(E_X)) \subset F^{n-1}(E_X) \otimes \Omega^1_{X/S}(\log \infty)
\]
gives rise to a filtered logarithmic crystal.

As usual the theory of log crystals can be extended to the case of a \( p \)-adic formal base \( S \).

3 Log \( F \)-crystals on the formal disk

The results of this section are mostly known ([Co], [Fa2], [Ogus]) or at least dwell on well known ideas. Nevertheless, we could not find them in the literature in a form that we can make use of them directly.

3.1. Notations: \( K \) is a finite extension of \( \mathbb{Q}_p \), with ring of integers \( R \) and residue field \( k \), \( W(k) \subset R \) is the ring of Witt vectors, \( K_0 \) is its field of fractions.

We start with some auxiliary results on logarithmic connections.

Let \( \overline{S} \) be a formal scheme over \( \text{Spf} \, R \), isomorphic to

\[
\text{Spf} \, R[[x_1, x_2, \cdots, x_n]]
\]

\( D = \cup D_i \hookrightarrow \overline{S} \) be a divisor with normal crossing given by the equations \( x_i = 0 \). We would like to stress that we do not fix coordinates on \( \overline{S} \).

Denote by \( s \) the point \( \text{Spf} \, R = \bigcap_i D_i \hookrightarrow \overline{S} \) The divisor \( D \) gives rise to a logarithmic structure on \( \overline{S} \) (see 2.2.). We would like to stress that we do not fix coordinates on \( \overline{S} \).

Let \( \overline{S}_{an} \) be the corresponding rigid analytic space over \( K \). For any \( 0 < \alpha \leq 1 \) we denote by \( \overline{S}_{an}(\alpha) \subset \overline{S}_{an} \) the open disk of radius \( \alpha \). That is an open subspace of \( \overline{S}_{an} \) whose \( \overline{K} \)-points are given by inequalities \( |x_i| < \alpha \) \( (i = 1, 2, \cdots, n) \) The subspace \( \overline{S}_{an}(\alpha) \) does not depend on the choice of coordinates. Let \( S_{an}(\alpha) \) stand for the complement \( \overline{S}_{an}(\alpha) - D_{an} \).

3.2. Consider a vector bundle \( E \) on \( \overline{S}_{an}(\alpha) \) with a logarithmic integrable connection \( \nabla E \to E \otimes \Omega^1_{\overline{S}/S}(\log 2) \). Here \( \Omega^1_{\overline{S}/S}(\log 2) \) stands for the sheaf of differential forms with at most logarithmic singularities at \( D_{an} \to \overline{S}_{an}(\alpha) \).

The connection gives rise to a family of commuting linear operators (“residues”) acting on the fiber \( E_s \). To construct it, we choose local coordinates \( x_i \) as above and consider the map \((1 \otimes \text{cont}(\frac{d}{dx_i}))\nabla : E \to E \). Here \( \text{cont}(\frac{d}{dx_i}) \) is the contraction operator. It is easy to see that the latter map descends to the fiber \( E_s \). Moreover, one can check that the resulting map does not depend on the choice of local coordinates. We denote it by \( N_i \). Clearly, that \( N_i N_k = N_k N_i \).

Conversely, given a vector space \( E_s \) over \( K \) and a family of linear operators \( N_i : E_s \to E_s \) we consider the trivial vector bundle \( \overline{E}_s = E_s \otimes A(\overline{S}_{an}(\alpha)) \) (here \( A(\overline{S}_{an}(\alpha)) \) is the ring of analytic functions on \( \overline{S}_{an}(\alpha) \)) with a logarithmic connection given by \( \nabla(e \otimes f) = e \otimes df + \sum_i N_i(e) \otimes f \frac{dx_i}{x_i} \).

The connection is integral provided that \( N_i \) commute. As before, one can check (although it is slightly less trivial), that \( (\overline{E}_s, \nabla) \) does not depend on the choice of coordinates.

Lemma 1 Let \( E \) be vector bundle on \( \overline{S}_{an}(\alpha) \) with a logarithmic integrable connection. Assume that operators the \( N_i \) are nilpotent. Then for a sufficiently small \( \beta \leq \alpha \) there exists a unique isomorphism

\[
(E, \nabla) \simeq (\overline{E}_s, \nabla)
\]
of the bundles with connection on \( \overline{S}_{an}(\beta) \) identical on \( E_s \).
For a proof one can just repeat the well known argument over the field of complex numbers ([De]).

If we define the category of vector bundles with a logarithmic integrable connection on \( S_{an}(0) \) to be the injective limit of the corresponding categories on \( S_{an}(\alpha) \), the previous lemma can be reformulated as follows:

**Lemma 2** The category of bundles with connection with nilpotent residues \( N_i \) on \( S_{an}(0) \) is equivalent to the category of vector spaces over \( K_0 \) together with an actions of commuting operators \( N_i \).

Proof is omitted.

In particularly, we have an action of \( N_i \) on any object of the latter category.

We are going to show that in certain cases the isomorphism in Lemma 1 extends to a larger disk.

### 3.3. Logarithmic extension

Following Coleman [Co] we define a canonical extension \( A_{Log}(S_{an}(\alpha)) \) of the ring of rigid analytic functions \( A(S_{an}(\alpha)) \) on \( S_{an}(\alpha) \):

\[
A_{Log}(S_{an}(\alpha)) = A(S_{an}(\alpha))[\{l(f) : f \in \mathcal{O}_S \cap (A(S_{an}(\alpha))^*)\}] / I
\]

where \( I \) is the ideal

\[
\langle \{l(fg) - l(f) - l(g) : f, g \in \mathcal{O}_S \cap (A(S_{an}(\alpha))^*)\},
\{l(f) = \sum_{i=1}^{\infty} \frac{(1 - f)^i}{i} : |f(x) - 1| < 1 \text{ for all } x \in S_{an}(\alpha)\}\rangle
\]

Let \( A_{Log}(S_{an}(\alpha)) \subset A_{Log}(S_{an}(\alpha)) \) be the subring generated by \( l(f) \) where \( f \in \mathcal{O}_S \) is a function whose divisor is supported on \( D \subset S \).

It is easy to check (see [Co]), that the algebra \( A_{Log}(S_{an}(\alpha)) \) is isomorphic to the polynomial algebra \( A(S_{an}(\alpha))[l(x_i), l(p)] \) and that \( A_{Log}(S_{an}(\alpha)) = A(S_{an}(\alpha))[l(x_i)] \). Of course these isomorphisms depend on the choice of coordinates.

The module \( A_{Log}(S_{an}(\alpha)) \) carries a canonical logarithmic connection:

\[
d : A_{Log}(S_{an}(\alpha)) \to A_{Log}(S_{an}(\alpha)) \otimes A(S_{an}(\alpha)) \Omega^1_{S_{an}(\alpha)}
\]

In addition, \( A_{Log}(S_{an}(\alpha)) \) and \( A_{Log}(S_{an}(\alpha)) \) is endowed with a family of commuting parallel endomorphisms

\[
N_i : A_{Log}(S_{an}(\alpha)) \to A_{Log}(S_{an}(\alpha))
\]

We define \( N_i \) to be the unique derivation such that \( N_i(A(S_{an}(\alpha))[l(p)] = 0 \), \( N_i(l(x_i)) = 0 \), \( N_i(l(x_j)) = 0 \)

### 3.4. The unipotent nearby cycles functor \( \Psi^{un} \)

The following is inspired by Beilinson’s construction of the unipotent nearby cycles functor [Be]. Given a vector bundle \( E \) on \( S_{an}(\alpha) \) together with a logarithmic integrable connection we define the space of unipotent nearby cycles to be the space of parallel sections \( E \otimes A_{Log}(S_{an}(\alpha)) \):

\[
\Psi^{un}(E) = \text{ker}(\nabla : E \otimes A_{Log}(S_{an}(\alpha)) \to E \otimes A_{Log}(S_{an}(\alpha)) \otimes \Omega^1_{S_{an}(\alpha)})
\]

The action of \( N_i \) on logarithmic extension \( A_{Log}(S_{an}(\alpha)) \) induces one on the vector space \( \Psi^{un}(E) \).

**Lemma 3** \( \dim_{K_0} \Psi^{un}(E) \leq rk E \)

**Proof:** Let \( \mathfrak{S}_K = \text{Spf } K[[x_1, \ldots, x_n]] \). Given a vector bundle \( G \) with an integrable log connection on \( S_{an}(\alpha) \) we define one on \( \mathfrak{S}_K \):

\[
\mathcal{E} = E \otimes A(S_{an}(\alpha)) K[[x_1, \ldots, x_n]]
\]

We have \( \dim E^\nabla \leq \dim \mathcal{E}^\nabla \). Hence, it suffices to check the statement for vector bundles on \( \mathfrak{S}_K \). In this case the result is known [De].
Proposition 4 Assume that the residues $N_i : E_s \to E_s$ are nilpotent. Then the following are equivalent:

i) $\dim_{K_0} \Psi^{un}(E) = rk E$

ii) A canonical morphism

$$(\Psi^{un}(E) \otimes_{K} A_{Log}(\overline{S}_{an}(\alpha)))^{N=0} \longrightarrow E$$

is an isomorphism.

Moreover, for sufficiently small $\beta \leq \alpha$ the restriction $E|_{\overline{S}_{an}(\beta)}$ satisfies the equivalent conditions above.

Proof is obvious.

3.5. Unipotent connections. Denote the category of bundles together with a unipotent logarithmic connection on by $DM_{Log}^{un}(\overline{S}_{an}(\alpha))$. By definition, a logarithmic connection on $E$ is unipotent if it can be obtained by successive extensions bundles with trivial logarithmic connection. Logarithmic connection is called trivial if the bundle is generated by parallel sections.

Theorem 5 a) A connection on $E$ is unipotent if and only if $\dim_{K_0} \Psi^{un}(E) = rk E$.

b) The functor $\Psi^{un}$ establishes an equivalence between the category $DM_{Log}^{un}(\overline{S}_{an}(\alpha))$ and the category of the of finite-dimensional vector spaces over $K_0$ endowed with commuting nilpotent endomorphisms $N_i$.

Proof: We show by induction on level $r$ that the functor

$$DM_{Log}^{L}(\overline{S}_{an}(\alpha)) \longrightarrow DM_{Log}^{L}(\overline{S}_{K})$$

is an equivalence of categories (we use notations introduced in the proof of (3)). Assume that it is already known for levels $< r$. To proof that the latter holds for level $r$, it is sufficient to check that the evident map

$$Ext^1_{DM_{Log}^{L}(\overline{S}_{an}(\alpha))}(A_{Log}(\overline{S}_{an}(\alpha)), E) \longrightarrow Ext^1_{DM_{Log}^{L}(\overline{S}_{K})}(O_{\overline{S}_{K}}; E \otimes O_{\overline{S}_{K}})$$

is an isomorphism for any $E$ of level $< r$. First, it is clear that the map is surjective.

The $Ext^1$-groups can be computed by the logarithmic de Rham complex. Hence, the statement follows from the following simple fact:

if a section $f \in E \otimes O_{\overline{S}_{K}}$ satisfies the property that $\nabla(f) \in \Omega^{1}_{\overline{S}_{K}}(log \infty) \otimes E$ then $f$ itself is a section of $E$. It completes the proof.

3.6. Let $L$ be a finite extension of $K$.

We define a locally analytic function $Log : L^* \to L_{st} = L[l(p)]$ to be the unique homomorphism given by the series

$$Log z = \sum_{i=1}^{\infty} \frac{(1 - z)^i}{i}$$

on a neighborhood of $0 \in L$ and satisfying $Log p = l(p)$.

Given a point $x \in S_{an}(\alpha)(L)$ it gives rise to a homomorphism

$$A_{Log}(\overline{S}_{an}(\alpha)) \longrightarrow L_{st}$$

The latter induces a map

$$\Psi^{un}(E) \longrightarrow E_x \otimes_L L_{st}$$

Theorem 6 Let $E$ be a vector bundle with a unipotent logarithmic connection. Then there is a canonical isomorphism:

$$\Psi^{un}(E) \otimes_{K} L_{st} \longrightarrow E_x \otimes_L L_{st}$$
**Proof:** Since the functor $\Psi^{\text{un}}$ from the category $\mathcal{D}M^{\log}_{\alpha}(\overline{S}_{\alpha}(\alpha))$ is exact, it suffices to check the statement in the case of trivial connection, when it is a tautology.

**Corollary 7** Under the assumption of Theorem, there is a canonical isomorphism of the fibers:

$$E_x \otimes_L L_{st} \rightarrow E_y \otimes_L L_{st}$$

where $x, y \in S_{\alpha}(\alpha)(L)$

The latter corollary is a local version of Theorem B.

**3.7. F-crystals.** For the rest of this section we assume that $\overline{S}$ is a formal disk over $\text{spec}W(k)$ (i.e. $\overline{S} \simeq \text{Spf}R[[x_1, x_2, \cdots, x_n]]$).

Choose a lifting $\overline{Fr} : \overline{S} \rightarrow \overline{S}$ of the absolute Frobenius $Fr : \overline{S} \times \text{spec}k \rightarrow \overline{S} \times \text{spec}k$ compatible with the logarithmic structure on $\overline{S}$. It is easy to see that the endomorphism $\overline{Fr}$ acts the space $\overline{S}_{\alpha}(\alpha)$.

An $F$-crystal on $\overline{S}_{\alpha}(\alpha)$ is a vector bundle $E$ together with a logarithmic connection and an isomorphism

$$\phi : \overline{Fr}^* E \simeq E$$

compatible with the connection.

We claim that the categories of $F$-crystals for different liftings of the Frobenius are equivalent to one another. It follows from the fact that the connection defines a canonical isomorphism $\overline{Fr}^* E \simeq Fr^* E$, where $Fr$ is another lifting of the Frobenius.

**Lemma 8** Let $E$ be a $F$-crystal on $\overline{S}_{\alpha}(\alpha)$. The vector space $E^\nabla$ of parallel sections carries a canonical action of $\phi$ (i.e. the action is independent of the choice of lifting of the Frobenius).

Proof is obvious.

The logarithmic extension $A_{\log}(\overline{S}_{\alpha}(\alpha))$ is an indobject in the category of $F$-crystals: a log lifting of the Frobenius $\overline{Fr}^* : A(\overline{S}_{\alpha}(\alpha)) \rightarrow A(S_{\alpha}(\alpha))$ extends to

$$\overline{Fr}^* : A_{\log}(\overline{S}_{\alpha}(\alpha)) \rightarrow A_{\log}(\overline{S}_{\alpha}(\alpha))$$

The induced structure of $(\text{ind})F$-crystal does not depend on the choice of lifting. Clearly, the submodule $A_{\log}(S_{\alpha}(\alpha))$ also inherits a structure of $(\text{ind})F$-crystal.

Let $H_n$ be the ring $K_0[\phi, N_i]$ generated by $\phi, N_i$ ($i = 1, 2, \cdots n$) with the following relations

$$N_i \phi = p \phi N_i, \phi \cdot a = Fr(a) \cdot \phi, N_i \cdot a = a \cdot N_i$$

(9)

where $a \in K_0$.

We introduce a structure of Hopf algebra (over $\mathbb{Q}_p$) on $H_n$: the comultiplication is given by $N_i \rightarrow 1 \otimes N_i + N_i \otimes 1$ and $\phi \rightarrow \phi \otimes \phi$.

Next, we let $E$ to be a $F$-crystal on $\overline{S}_{\alpha}(\alpha)$. By lemma (8), the vector space $\Psi^{\text{un}}(E)$ carries a canonical action of $\phi$. It is easy to check that the relations (8) are satisfied i.e. $\Psi^{\text{un}}(E)$ is a $H_n$-module.

**Remark.** It is instructive to compare the $H_n$-module $\Psi^{\text{un}}(E)$ with the naive one: $E_s$. Quite surprisingly there is no canonical isomorphism between these vector spaces. In fact, the action of $\phi$ on the fiber $E_s$ does depend on the choice of lifting of the Frobenius.

**Theorem 9** Let $0 < \alpha \leq 1$ and $E$ be a fiber bundle on $\overline{S}_{\alpha}(\alpha)$ together with a logarithmic integrable connection. Assume that $E$ has a structure of $F$-crystal. Then the connection is unipotent. In particular, we have a canonical isomorphism

$$(\Psi^{\text{un}}(E) \otimes_{K_0} A_{\log}(S_{\alpha}(\alpha)))^{N=0} \simeq E$$

(10)
Corollary 10 The tensor category of $F$-crystals on $\mathcal{S}_{an}(\alpha)$ is equivalent to the category of finite dimensional over $K_0$ $\mathcal{H}_n$-modules.

Proof: Choose a $F$-crystal structure on $E$.

Lemma 11 (Grothendieck) Let $V$ finite dimensional $\mathcal{H}_n$-module. Suppose $\phi$ is an invertible operator. Then $N_i$ are nilpotent.

Proof is omitted.

It follows that the residues $N_i$ of the connection are nilpotent. Hence by proposition [4] we have an isomorphism of $F$-crystals

$$ (\Psi^{un}(E|_{\mathcal{S}_{an}(\beta)}) \otimes K_0 A^f_{\text{Log}}(\mathcal{S}_{an}(\beta)))^N = 0 \simeq E|_{\mathcal{S}_{an}(\beta)} \quad (11) $$

for small $\beta$. To extend this isomorphism to the larger disk, we make use of the following trick invented by Dwork. Pick a lifting $\tilde{F}r$ of the Frobenius and any $\gamma < \alpha$. There exists an integer $N$ such that

$$ \tilde{F}r^N (\mathcal{S}_{an}(\gamma)) \subset \mathcal{S}_{an}(\beta) $$

Then (11) induces an isomorphism

$$ p : (\tilde{F}r^N)^* ((\Psi^{un}(E) \otimes K_0 A^f_{\text{Log}}(\mathcal{S}_{an}(\alpha)))^N = 0) \simeq (\tilde{F}r^N)^* E $$

We claim that $\phi^N \circ p \circ (\phi)^{-N}$ gives the desired isomorphism (10).

3.8. Log $F$-crystals on schemes. Let $X$ be a log scheme in characteristic $p$. A log $F$-crystal on $X$ is a crystal $E$ on the log crystalline site $(X/\text{spec } \mathbb{F}_p)_{\text{cris}}$ ($\text{spec } \mathbb{Z}_p$ is endowed with the trivial log structure) together with a morphism $\phi : F r^* E \to E$, which is an isomorphism in the category $\text{Crystals} \otimes \mathbb{Q}$.

Theorem 12 Let $i : X \hookrightarrow Y$ be a nilpotent log thickening of schemes over $\text{spec } \mathbb{F}_p$. Assume that $Y$ can be embedded in a smooth log scheme over $\text{spec } \mathbb{Z}_p$. The restriction functor $i^*$ induces an equivalence of categories:

$$ i^* : \{\text{coherent } F - \text{crystals on } Y\} \otimes \mathbb{Q} \simeq \{\text{coherent } F - \text{crystals on } X\} \otimes \mathbb{Q} $$

Proof: This is another application of Dwork’s trick. Choose a closed immersion $s : Y \hookrightarrow Z$ in a log smooth scheme over $\text{spec } \mathbb{Z}_p$. Let $D_s$ (resp. $D_{s+i}$) be the PD-completion of the PD-envelope of $s$ (resp. $s \cdot i$). There is a canonical map $\tilde{i} : D_{s+i} \to D_s$. Since the question is local, we can assume that the Frobenius extends to $\tilde{F}r : D_s \to D_s$.

There exists an integer $N$ and a morphism $p : D_s \to D_{s+i}$ such that $\tilde{i} \cdot p = \tilde{F}r^N$, $p \cdot \tilde{i} = \tilde{F}r^N$. We claim that $p^*$ defines the inverse functor:

$$ \{F - \text{crystals on } X\} \otimes \mathbb{Q} \to \{F - \text{crystals on } Y\} \otimes \mathbb{Q} $$

Convention. A $F$-crystal on an arbitrary log scheme $X$ is, by the definition, a log crystal on $(X \times \text{spec } \mathbb{F}_p/\text{spec } \mathbb{Z}_p)_{\text{cris}}$.

Corollary 13 Let $X$ be a semi-stable scheme over $\text{spec } R$ and $i : X_k := X \times_{\text{spec } R} \text{spec } k \hookrightarrow X$ be its special fibre. The restriction functor $i^*$ induces an equivalence of categories:

$$ i^* : \{\text{coherent } F - \text{crystals on } X\} \otimes \mathbb{Q} \simeq \{\text{coherent } F - \text{crystals on } X_k\} \otimes \mathbb{Q} $$
3.9. Log $F$-crystals on the log point $\text{spec } R$. What follows is borrowed from [Fa2].

We endow the scheme $\text{spec } R$ with a log structure given by the closed point.

As an application of the above results we describe the category of log $F$-crystals on $\text{spec } R / \text{spec } \mathbb{Z}_p$.

Let $E$ be such a crystal. We are going to show that the vector space $E \otimes R$ carries certain additional structure.

Choose a uniformizer $\pi \in R$ and let $f(T) \in W(k)[T]$ be its minimal polynomial over $W(k) \subset R$. That is an Eisenstein polynomial of degree $e$, where $e$ is ramification index of $R/\mathbb{Z}_p$. We have

$$R = W(k)[T]/f(T)$$

Denote by $ \bar{V} $ the PD-completion of the ring obtained by adjoining to $W(k)[[T]]$ divided powers $f_i$ (or, equivalently, $T^{e_i}$). The Frobenius extends to $\bar{\text{Fr}}: \text{spec } V \to \text{spec } V$, $T \to T^p$. We endow the scheme $\text{Spf } V$ with the log structure $M_{\text{Spf } V}$ corresponding to the divisor $T = 0$. There is a natural immersion

$$i_\pi: \text{Spf } R \hookrightarrow \text{Spf } V$$

with $i_\pi^*(T) = \pi$. By the very definition a coherent crystal $E$ on the log scheme $(\text{spec } R, i_\pi^* M_{\text{spec } V})$ is a coherent module $E_{i_\pi, \text{spec } V}$ on $\text{spec } V$ with a logarithmic integrable connection

$$E_{i_\pi, \text{spec } V} \to E_{i_\pi, \text{spec } V} \otimes W(k)[[T]]\Omega^1_{W(k)[[T]]}(\log \infty)$$

Suppose now that $E$ is a $F$-crystal:

$$\phi: \bar{\text{Fr}}^* (E_{i_\pi, \text{spec } V}) \otimes \mathbb{Q} \simeq E_{i_\pi, \text{spec } V} \otimes \mathbb{Q}$$

It gives rise to a log $F$-crystal $E_{i_\pi}$ on the disk $\text{Spec } (q^{-1/2} - 1/k)$. (Here $q$ is the cardinality of residue field $k$. So $|\pi| = q^{-\frac{1}{2}}$.)

In turn, the latter gives rise to a $H_1$-module $\Psi_{\text{un}}^\pi(E_{i_\pi})$. The above construction leads to a functor:

$$\Psi_{\text{un}}^\pi : \{\text{log } F - \text{crystals on } \text{spec } R\} \to \{\text{modules over } H_1\}$$

A priori it depends on the choice of a uniformizer $\pi$. In fact, it does not. More precisely, we have the following result.

**Lemma 14** For any uniformizers $\pi, \pi' \in R$, there exists a canonical isomorphism of functors:

$$\Psi_{\text{un}}^\pi \simeq \Psi_{\text{un}}^{\pi'}$$

**Proof:** Choose an isomorphism

$$\Psi_{\text{un}}^\pi : \text{Spf } R \to \text{Spf } V$$

$$\Psi_{\text{un}}^{\pi'} : \text{Spf } R \to \text{Spf } V$$

By the definition, for any crystal $E$, we have a canonical isomorphism $r_\theta : \theta^* E_{i_\pi, \text{spec } V} \simeq E_{i_\pi, \text{spec } V}$ identical on $E_{\text{spec } R}$.

On the other hand, there is a canonical isomorphism of functors:

$$\Psi_{\text{un}}^\theta : \{\text{F - crystals on } \text{Spec } (q^{-1/2} - 1/k)\} \to \{\text{modules over } H_1\}$$

It gives rise to $\theta^* : \Psi_{\text{un}}^\pi \simeq \Psi_{\text{un}}^{\pi'}$. Moreover, the latter commutes with the isomorphism $\Psi_{\text{un}}^\pi(E) \otimes K_{st} \simeq E_{\text{spec } R} \otimes K_{st}$ (see Theorem 3). Hence, it does not depend on the choice of $\theta$. It completes the proof.

Moreover, Corollary (10) implies the following result.
**Theorem 15** There is an equivalence of tensor categories:

\[ \Psi^\text{un} : \{ \text{coherent log F-crystals on } \text{spec } R \} \otimes \mathbb{Q} \cong \{ \text{finite-dimensional over } K_0 \text{-modules} \} \]

By the construction there is a canonical isomorphism:

\[ \Psi^\text{un}(E) \otimes_{K_0} K_{st} \cong E_{\text{spec } R} \otimes_R K_{st} \quad (12) \]

We define a \( K \)-linear operator \( N : K_{st} \to K_{st} \) to be the derivation of the ring \( K_{st} \) such that \( N(l(p)) = e \). (Remind that \( e \) stands for the ramification index \( K \) over \( K_0 \)).

It endows both sides of (12) with an action of \( N \). The action on the left-hand side comes from the action on both factors, while \( E_{\text{spec } R} \) is equipped with the trivial action of \( N \).

**Lemma 16** The isomorphism (12) commutes with the action of \( N \).

Proof is a simple direct computation.

In particular, we obtain a canonical isomorphism:

\[ E_{\text{spec } R} \cong (\Psi^\text{un}(E) \otimes_{K_0} K_{st})^N = 0 \]

**3.10.** It is convenient to have slightly more general formulation of the latter theorem. Consider the log scheme \( S_r = (\text{spec } R; M_r, \alpha) \), where \( \alpha : M_r \to R \) is the log structure associated to the prelogarithmic structure \( \beta : N_r \to R \), \( \beta((0, \cdots, 1, \cdots, 0)) = \pi \). Clearly, the log structure does not depend on the choice of a uniformizer. If \( r = 1 \), the latter coincides with the log structure given by the closed point \( \text{spec } k \hookrightarrow \text{spec } R \).

**Theorem 17**

a) There is a tensor functor

\[ \Psi^\text{un} : \{ \text{unipotent log crystals on } S_r \} \otimes \mathbb{Q} \to \text{Vect}_{K_0} \]

to the category of finite-dimensional vector spaces over \( K_0 \) endowed with commuting nilpotent operators \( N_i \), such that for any unipotent log crystal \( E \) there is a canonical isomorphism

\[ \Psi^\text{un}(E) \otimes_{K_0} K_{st} \cong E_{\text{spec } R} \otimes_R K_{st} \]

b) The tensor category

\[ \{ \text{log F-crystals on } S_r \} \otimes \mathbb{Q} \]

is equivalent to the tensor category of \( \mathcal{H}_r \)-modules finite-dimensional over \( K_0 \).

**3.11. Hyodo-Kato isomorphism.** Let \( \overline{X} \) be a proper scheme finite type over \( \text{spec } R \), \( Z \hookrightarrow \overline{X} \) be a divisor, such that etale locally \( \overline{X} \) is isomorphic to

\[ \text{spec } R[t_1, \cdots, t_i, s_1, \cdots, s_j]/(t_1 \cdots t_i - \pi^m) \quad (13) \]

(here \( \pi \in R \) is a uniformizer and \( m \) is an integer), with \( Z \) given by the equations \( \prod_{k \in J} s_k = 0 \) for some subset \( J \subset \{ 1, \cdots, s_j \} \).

**Example:** \( p : \overline{X} \to \text{spec } R \) is a proper scheme with a semi-stable reduction, \( Z = 0 \).

We endow the scheme \( \overline{X} \) with the logarithmic structure given by the divisor \( D = \overline{X}_k \cup Z \) i.e.

\[ M_{\overline{X}} = \mathcal{O}_{\overline{X}} \cap \mathcal{O}_{\overline{X} - D} \]

and \( \text{spec } R \) with the log structure corresponding to the closed point. Put \( X_K := \overline{X} - D \) It is easy to see that the canonical morphism \( p : \overline{X} \to \text{spec } R \) is log smooth.

One can prove (see [Hyodo]) that the direct image \( R^*p_*\mathcal{O}_{\text{cris}} \) is a coherent crystal on \( \text{spec } R \). Moreover, the Base Change Theorem implies that there is a canonical isomorphism

\[ (R^*p_*\mathcal{O}_{\text{cris}})_{\text{spec } R} \otimes K \cong H^*_DR(X_K) \]

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Hence, by the previous result, it gives rise to a $\mathcal{H}_1$-module

$$H^*_f(\overline{X}) = \Psi^{un}(R^*p_*\mathcal{O}_{cris})$$

and a canonical isomorphism:

$$H^*_{DR}(X_K) = (H^*_f(\overline{X}) \otimes_{K_{st}} K_{st})^{N=0}$$

This construction is due to Hyodo and Kato loc.cit.

We denote the category of coherent log crystals on a scheme $X$ by $C(X)$.

One can make use of the Leray spectral sequence and the fact that the category $C(spec R) \otimes \mathbb{Q}$ has homological dimension 1 to prove the following result.

**Lemma 18** For any coherent log crystals $E$ and $G$ the sequence

$$0 \rightarrow Ext^1_{C(spec R) \otimes \mathbb{Q}}(\mathcal{O}_{cris}; p_* (E^* \otimes G)) \rightarrow Ext^1_{C(X) \otimes \mathbb{Q}}(E; G) \rightarrow Hom(\mathcal{O}_{cris}; R^1 p_* (E^* \otimes G)) \rightarrow 0$$

is exact.

4 Construction of the canonical parallel translation $C_{x_0; x_1; X_{K_0}}$ in the case when $X_{K_0}$ has a good compactification

4.1. Let $X_{K_0}$ be a smooth geometrically connected scheme finite type over $K_0$. Remind that a good compactification of $X_{K_0}$ is a smooth proper scheme $\overline{X}$ over $spec W(k)$ with a divisor $Z \subset \overline{X}$ with normal crossings relative to $spec W(k)$ and an isomorphism $X_{K_0} \simeq (\overline{X} - Z) \times_{spec W(k)} spec K_0$.

In this section we proof a special case of Theorem B when $X_{K_0}$ has such a compactification. Although it will not be used in the general construction, the proof of this special case is significantly simpler and the construction has a clear geometrical interpretation.

4.2. Given an embedding $X_{K_0} \hookrightarrow \overline{X}$ as above, we can interpret objects of the category $\mathcal{D}M_{un}(X_{K_0})$ as unipotent crystals on the log scheme $\overline{X}_k = \overline{X} \times spec k$. More precisely we have the following result.

**Lemma 19** There is an equivalence of the categories:

$$\mathcal{D}M_{un}(X_{K_0}) \simeq \{unipotent log crystals on \overline{X}_k\} \otimes \mathbb{Q}$$

**Proof:** By 2.4. we have a fully faithful functor

$$\{unipotent log crystals on \overline{X}_k\} \otimes \mathbb{Q} \longrightarrow \mathcal{D}M_{un}(X_{K_0})$$

It remains to show that its image contains all bundles with a unipotent connection. The latter follows from the fact that for any unipotent log crystal $E$ on $\overline{X}_k$ we have

$$Ext^1(\mathcal{O}_{\overline{X}_k}; E) \simeq H^1_{cris}(\overline{X}_k; E) \otimes \mathbb{Q}_p \simeq H^1_{DR}(X_{K_0}; E_{X_{K_0}}) \simeq Ext^1_{\mathcal{D}M_{un}(X_{K_0})}(\mathcal{O}_{X_{K_0}}; E)$$

Here the first $Ext$ group is computed in the category of unipotent crystals $\otimes \mathbb{Q}$. In turn, the existence of the latter isomorphisms follows form the well known result saying that de Rham cohomology of a smooth scheme $X_{K_0}$ over a field of characteristic 0 can be computed by virtue of the logarithmic de Rham complex on a good compactification $\overline{X}_{K_0}$. It completes the proof.

**Remark.** The Lemma implies that $\{unipotent log crystals on \overline{X}_k\} \otimes \mathbb{Q}$ is an abelian category. In fact, one can easily check that the category of unipotent log crystals on $\overline{X}_k$ itself is abelian. The latter property might be false for the category of unipotent crystals on a nonproper scheme.

As corollary, we can see that $\mathcal{D}M_{un}(X_{K_0})$ possesses an additional symmetry:

$$Fr^* : \mathcal{D}M_{un}(X_{K_0}) \longrightarrow \mathcal{D}M_{un}(X_{K_0})$$  (14)
For any pair \( y \) given by a morphism \( i : \text{spec} \ k \to \overline{X}_k \) we define a fiber functor
\[
\mathcal{F}_y : \mathcal{D} \mathcal{M}_{un}(X_{K_0}) \to \text{Vect}_{K_0}
\]
If the point does not lie on the divisor \( Z_k \to X_k \) the vector space \( \mathcal{F}_y E \) is just the stalk \( K_0 \otimes_{W(k)} \text{i}_{\text{crys}}^* E \).

In general, we make use of the theory developed in the previous section and define
\[
\mathcal{F}_y E = \Psi^\text{un}(\text{i}_{\text{crys}}^* E)
\]
where \( \text{spec} \ k \) is now the log point \( (2.2) \).

**Theorem 20** For any pair \( y_0 : y_1 \in \overline{X}_k(\text{spec} \ k) \) there exists a unique element
\[
C_{y_0 ; y_1} \in \text{Mor}^{\otimes}(\mathcal{F}_{y_0} ; \mathcal{F}_{y_1})
\]
satisfying the following the following property:
\[
\text{Fr}_r(C_{y_0 ; y_1}) = C_{\text{Fr}(y_0) ; \text{Fr}(y_1)}
\]

**Proof:** For an integer \( r \geq 1 \) we consider the subcategory \( \mathcal{D} \mathcal{M}_r(X_{K_0}) \subset \mathcal{D} \mathcal{M}_{un}(X_{K_0}) \) of unipotent crystals of length \( r \). We denote by \( \Pi_r(y_0 ; y_1) \) the space of morphisms between the fiber functors \( \mathcal{F}_{y_0} \) and \( \mathcal{F}_{y_1} \) restricted to the latter subcategory. There are canonical surjections:
\[
\Pi_{r+1}(y_0 ; y_1) \to \Pi_r(y_0 ; y_1)
\]
and
\[
\text{Mor}(\mathcal{F}_{y_0} ; \mathcal{F}_{y_1}) = \lim_{\leftarrow} \Pi_r(y_0 ; y_1)
\]

Note that \( \Pi_1(y_0 ; y_1) = K_0 \).

Choose an integer \( n \) such that \( \text{Fr}^n \) acts trivially on \( k \). The Frobenius \( \langle \Pi_r(y_0 ; y_1) \to K_0 \rangle \) induces a \( K_0 \)-linear endomorphism:
\[
\text{Fr}^n : \Pi_r(y_0 ; y_1) \to \Pi_r(y_0 ; y_1)
\]
Theorem follows from the following Lemma.

**Key Lemma.** The map \( \Pi_r(y_0 ; y_1) \to K_0 \) induces an isomorphism:
\[
(\Pi_r(y_0 ; y_1))^{\text{Fr}^n} \simeq K_0
\]

**Proof of Key Lemma.** Define a filtration on \( \Pi_r(y_0 ; y_1) \) to be
\[
\mathcal{F}^i = \ker(\Pi_r(y_0 ; y_1) \to \Pi_r(y_0 ; y_1))
\]
where \( 1 \leq i \leq r \). Similarly, we can define a filtration on the algebra \( \Pi_r(y_0 ; y_0) \).

Next, the vector space \( \Pi_r(y_0 ; y_1) \) carries a structure of filtered module over the filtered algebra \( \Pi_r(y_0 ; y_0) \). Moreover we have a canonical isomorphism:
\[
\text{Gr}^I_r(\Pi_r(y_0 ; y_1)) \simeq \text{Gr}^r_r(\Pi_r(y_0 ; y_1))
\]
The latter is induced by the action on \( 1 \in K_0 = \Pi_1(y_0 ; y_1) \subset \text{Gr}^I_r(\Pi_r(y_0 ; y_1)) \).

Hence, it suffices to show that \( (\text{Gr}^I_r(\Pi_r(y_0 ; y_0)))^{\text{Fr}^n} = 0 \) for \( i > 1 \).

It is easy to see that \( \text{Gr}^I_r(\Pi_r(y_0 ; y_0)) = (\Pi_{\text{crys}}(\overline{X}_k) \otimes_{W(k)} K_0)^* \) It is known that the eigenvalues of \( \text{Fr}^n \) acting on the cohomology group are algebraic numbers and
\[
|v(\alpha)| \geq p^{\frac{1}{n-1}}
\]
for any eigenvalue \( \alpha \) and an embedding \( v : \mathbb{T} \to \mathbb{C} \). On the other hand the algebra \( \text{Gr}^r_r(\Pi_r(y_0 ; y_0)) \) is generated by \( \text{Gr}^2_r(\Pi_r(y_0 ; y_0)) \). It implies that for any eigenvalue \( \alpha \) of \( \text{Fr}^n \) on \( \text{Gr}^r_r(\Pi_r(y_0 ; y_0)) \) we have
\[
|v(\alpha)| \leq p^{\frac{1}{n-1}}
\]
It completes the proof of the Key Lemma along with the Theorem.

4.4. In what follows we let \( x_0, x_1 \) be points of \( X_{K_0}(K) \). Such a point gives rise to a morphism \( \text{spec} \, R \rightarrow \mathfrak{X} \) and we denote by \( y_i \) (\( i = 0, 1 \)) the corresponding points of the special fiber \( \mathfrak{X}_k \). Let \( F_{x_i} \) be the usual fiber functors: \( \mathcal{D} \mathcal{M}_{\text{un}}(X_{K_0}) \rightarrow \text{Vec}_{K} \).

By Theorem 13 from the previous section we have a canonical isomorphism of functors

\[
F_{x_i} \otimes_K K_{st} \simeq F_{y_i} \otimes_{K_0} K_{st}
\]

(15)

Combining the latter with Theorem 20 we arrive to a canonical isomorphism

\[
C_{x_0; x_1; X_K} : F_{x_0} \otimes_K K_{st} \simeq F_{x_1} \otimes_K K_{st}
\]

whose existence is proclaimed in Theorem B.

By the construction \( C_{x_0; x_1; X_K} \) is compatible with the tensor structure.

4.5. Remark. A priori an element \( C_{x_0; x_1; X_K} \in \text{Mor}^\otimes(F_{x_0} \otimes K_{st}; F_{x_1} \otimes K_{st}) \) depends on the choice of a good model \( \mathfrak{X} \). It will be proven later that in fact \( C_{x_0; x_1; X_K} \) does not depend on this auxiliary choice and, moreover, the construction is functorial with respect to any morphism \( X_{K_0} \rightarrow X_{K_0}' \).

More precisely, the isomorphism (13) gives rise to

\[
\Pi_r(y_0; y_1) \otimes_{K_0} K_{st} \simeq \Pi_r^{DR}(x_0; x_1) \otimes_K K_{st}
\]

where \( \Pi_r^{DR}(x_0; x_1) \) stands for the space of morphisms between the functors restricted to \( \mathcal{D} \mathcal{M}_r(X_{K_0}) \).

Remind that for any log crystal \( E \) on \( \mathfrak{X}_k \) the vector space \( F_{y_i} \) is endowed with a canonical nilpotent operator \( N \). It gives rise to an endomorphism \( N \) of \( \Pi_r(y_0; y_1) \).

In the other words \( \Pi_r^{DR}(x_0; x_1) \) possesses a structure of log \( F \)-crystal (\( \otimes \mathbb{Q} \)) on the logarithmic scheme \( \text{spec} \, R \) (with the log structure given by the closed point). It turns out that the latter structure depends on \( X_{K_0} \) only and, moreover, it exists for any smooth scheme \( X_K \) which has a semi-stable model over \( \text{spec} \, R \).

A proof of the above statements occupies the next two sections.

5 Construction of the fundamental crystal \( \Pi_r \)

5.1. Motivation: the fundamental \( D \)-module. Let \( f : X \rightarrow S \) be a smooth proper morphism of smooth schemes over a field \( K \) of characteristic 0. For a pair of points \( x_0; x_1 \in X \) with \( f(x_0) = f(x_1) \) we denote by \( \Pi_r^{DR}(X/S; x_0; x_1) \) the space of morphisms between the corresponding fiber functors from the category \( \mathcal{D} \mathcal{M}_r(X_f(x_0)) \) of bundles together with a unipotent connection on the fiber \( X_{f(x_0)} \).

It is easy to see that the vector spaces \( \Pi_r^{DR}(X/S; x_0; x_1) \) form a vector bundle \( \Pi_r^{DR}(X/S) \) on \( X \otimes_S X \). It immediately follows form the construction that \( \Pi_r^{DR}(X/S) \) possesses a natural integrable connection along the fibers of the map \( X \otimes_S X \rightarrow S \).

We claim that, in fact, \( \Pi_r^{DR}(X/S) \) carries a canonical total connection on \( X \otimes_S X \). The latter is an analog of the Gauss-Manin connection.

What follows is a variant of the above construction when \( S \) is replaced by the log scheme \( \text{spec} \, R \).

5.2. Notations. Let \( \mathfrak{X} \) be a proper geometrically connected scheme finite type over \( \text{spec} \, R \), \( Z \rightarrow \mathfrak{X} \) be a divisor, such that etale locally \( \mathfrak{X} \) is isomorphic to

\[
\text{spec} \, R[t_1, \ldots, t_i, s_1, \ldots, s_j]/(t_1 \cdots t_i - \pi^m)
\]

(16)

(here \( \pi \in R \) is a uniformizer and \( m \) is an integer), with \( Z \) given by the equations \( \prod_{k \in J} s_k = 0 \) for some subset \( J \subset \{ 1, \ldots, s_j \} \).

We endow \( \mathfrak{X} \) with the logarithmic structure given by the divisor \( D = \mathfrak{X}_k \cup Z \) i.e.

\[
\mathcal{M}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}} \cap \mathcal{O}_{\mathfrak{X} - D}
\]

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It is easy to see that $\overline{X}$ is log smooth over the log scheme $\text{spec } R$.

We would like to stress, that the class of schemes satisfying the property \([10]\) is stable under a finite extension of the base field.

We remind a few general concepts which will be used in the construction of fundamental crystal.

5.3. $\mathbb{Q}_p$-isocrystals. Let $X$ be a fine log scheme over $\text{spec } k$. We consider the $p$-formal crystalline site which consists of triples $(U, T, \delta)$, where $U$ is a scheme etale over $X$, $U \hookrightarrow T$ is a $p$-formal logarithmic PD-thickening, such that

$$T_n = T \times_{\Spf W(k)} \text{spec } W_n(k)$$

is flat over $\text{spec } W_n(k)$ and $U \hookrightarrow T_n$ is PD-nilpotent. A $\mathbb{Q}_p$-isocrystal on $X$ associates to any object $(U, T, \delta)$ a sheaf $E_T$ of $\sO_T \otimes \mathbb{Q}$-modules on $T$, and for any morphism $g : (U, T, \delta) \to (U', T', \delta')$ an isomorphism $g^* E_{T'} \cong E_T$ with the evident compatibility conditions.

5.4. Category of unipotent crystals. Given a $p$-formal logarithmic PD-thickening $U \hookrightarrow T$ of fine log smooth scheme $U$ and a smooth proper log scheme $Y \to U$, we let $\mathcal{C}_r(Y/T)$ stand for the category of unipotent log crystals of level $r$ on $(Y/T)_{\text{cris}}$. By definition, an object $E$ of the latter category is a crystal, which possesses a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E$$

where $E_i$ are subobjects, such that each successive quotient $E_i/E_{i-1}$ is isomorphic to the pullback of a coherent crystal on $(U/T)_{\text{cris}} (= \text{a coherent sheaf on } T)$.

It is easy to see, that $\mathcal{C}_r(Y/T) \otimes \mathbb{Q}$ is an abelian category. (In fact, $\mathcal{C}_r(Y/T)$ itself is abelian.)

Given section $s : U \to Y$, we consider a functor

$$s^* : \mathcal{C}_r(Y/T) \otimes \mathbb{Q} \to \mathcal{C}_r(U/T) \otimes \mathbb{Q} \cong \{\text{coherent sheaves on } T\} \otimes \mathbb{Q}$$

Lemma 21 i) The functor $s^*$ is exact, faithful and representable by an object $B_r(Y/T, s)$ of $\mathcal{C}_r(Y/T) \otimes \mathbb{Q}$.

ii) Let $h : (U' \hookrightarrow T') \to (U \hookrightarrow T)$ be a morphism of PD-thickenings, such that $U' = U \times_T T'$. Then the canonical morphism

$$B_r(Y \times_U U'/T'; h^*(s)) \to \overline{h}^* B_r(Y, T, s)$$

(17)

Here $h^*(s) : U' \to Y \times_U U'$ is the pullback of $s$ and $\overline{h}$ stands for the obvious $Y \times_U U'/T' \to Y$.

Proof: induction on $r$. Assume that Lemma is proven for all $r' < r$. It follows that log crystalline cohomology $I_r = H^1(Y/T; B_{r-1}(Y/T, s))$ is a locally free $\mathcal{O}_T \otimes \mathbb{Q}$-module on $T$. We can define $B_r(Y/T, s)$ to be the canonical extension of $B_{r-1}(Y/T, s)$ by $\text{Hom}(I_r; \mathcal{O}_T \otimes \mathbb{Q})$. Finally, the base change property follows from the corresponding property of the crystalline cohomology [Hyodo].

5.5. Definition of $\Pi_r(\overline{X}_{R/p})$. The latter is a $\mathbb{Q}_p$-isocrystal on $(\overline{X}_{R/p} \times_{\text{spec } R/p} \overline{X}_{R/p} / \Spf Z_p)_{\text{cris}}$. Here the formal scheme $\Spf Z_p$ is endowed with the trivial log structure.

Let $U$ be an etale scheme over $\overline{X}_{R/p} \times_{\text{spec } R/p} \overline{X}_{R/p}$ and $U \hookrightarrow T$ be a $p$-formal logarithmic PD-thickening. The projections $\overline{X}_{R/p} \times_{\text{spec } R/p} \overline{X}_{R/p} \to \overline{X}_{R/p}$ define canonical sections $p_i : U \to \overline{X}_{R/p} \times_{\text{spec } R/p} U$, $(i = 0, 1)$. Define

$$\Pi_r(\overline{X}_{R/p})(T) = p_2^* B_r(\overline{X}_{R/p} \times_{\text{spec } R/p} U, p_1)$$

Lemma \([21]\) implies that $\Pi_r(\overline{X}_{R/p})$ is $\mathbb{Q}_p$-isocrystal.

5.5. Another construction of $\Pi_r(\overline{X}_{R/p})$. We start with a general categorical construction.

Let $\mathcal{H}$ and $\mathcal{G}$ be sheaves of additive categories on a site $\mathcal{A}$ and $\mathcal{F}_i : \mathcal{H} \to \mathcal{G}$ ($i = 0; 1$) be functors.

We define a sheaf $\text{Mor}(\mathcal{F}_0; \mathcal{F}_1)$ of abelian groups on $\mathcal{A}$ in the following way. Given an object $U$ of $\mathcal{A}$ we consider the category $\mathcal{A}_U$ of objects over $U$. Denote by $|U|$ the tautological restriction functor from sheaves on $\mathcal{A}$ to sheaves on $\mathcal{A}_U$. Define

$$\text{Mor}(\mathcal{F}_0; \mathcal{F}_1)(U) = \text{Mor}(\mathcal{F}_0|_U; \mathcal{F}_1|_U)$$
It is easy to check that $\mathcal{M}or(\mathcal{F}_0; \mathcal{F}_1)$ is a sheaf.

For a log scheme $U$ over $\text{spec } R/p$ and p-formal logarithmic PD-thickening $U \hookrightarrow T$ we consider the category

$$\mathcal{C}_r(\mathcal{X}_{R/p} \times \text{spec } R/p \mathcal{U}/T) \otimes \mathbb{Q}$$

It defines a sheaf of categories $\mathcal{C}_r \otimes \mathbb{Q}$ on the big $p$-formal crystalline site $(\text{spec } R/p/\text{Spec } \mathbb{Z}_p)_{\text{cris}}$. In particular, it gives rise to a sheaf of categories $\mathcal{C}_r \otimes \mathbb{Q}|_{\mathcal{X}_{R/p} \times \text{spec } R/p | \mathcal{X}_{R/p}}$ on the p-formal site $(\mathcal{X}_{R/p} \times \text{spec } R/p | \mathcal{X}_{R/p} \times \text{spec } R/p | \mathcal{X}_{R/p})_{\text{cris}}$.

Next, we define another sheaf $\text{Mod}_{\mathcal{O} \otimes \mathbb{Q}}$ on the log crystalline site $(\text{Spec } \mathbb{Z}/\text{Spec } \mathbb{Z}_p)_{\text{cris}}$ assigning to a PD-thickening $U \hookrightarrow T$ the category of sheaves of $\mathcal{O}_T \otimes \mathbb{Q}$-modules.

For $i \in \{0, 1\}$ we define a functor

$$\mathcal{F}_i : \mathcal{C}_r \otimes \mathbb{Q}|_{\mathcal{X}_{R/p} \times \text{spec } R/p | \mathcal{X}_{R/p}} \rightarrow \text{Mod}_{\mathcal{O} \otimes \mathbb{Q}|_{\mathcal{X}_{R/p} \times \text{spec } R/p | \mathcal{X}_{R/p}}}$$

to be $\mathcal{F}_i = (p_i \times \text{id})^* \text{cris}$, where $p_i \times \text{id} : U \rightarrow \mathcal{X}_{R/p} \times \text{spec } R/p$ given by the projection $p_i : \mathcal{X}_{R/p} \times \text{spec } R/p \rightarrow \mathcal{X}_{R/p}$.

We claim that there is a canonical isomorphism:

$$\Pi_r(\mathcal{X}_{R/p}) := \mathcal{M}or(\mathcal{F}_0; \mathcal{F}_1)$$

5.6. As an application of the second construction, we show that $\Pi_r(\mathcal{X}_{R/p})$ possesses a certain additional structure.

Denote by

$$p_{ij} : \mathcal{X}_{R/p} \times \text{spec } R/p \times \mathcal{X}_{R/p} \times \text{spec } R/p \rightarrow \mathcal{X}_{R/p} \times \text{spec } R/p \mathcal{X}_{R/p}$$

the projection given by the formula $p_{ij}(y_0, y_1, y_2) = (y_i, y_j)$. It immediately follows from the construction that there is a canonical morphism

$$p_{12}^*(\Pi_r(\mathcal{X}_{R/p})) \otimes p_{23}^*(\Pi_r(\mathcal{X}_{R/p})) \rightarrow p_{13}^*(\Pi_r(\mathcal{X}_{R/p}))$$

(18)

5.7. Next, we compare $\Pi_r(\mathcal{X}_{R/p})$ with the vector bundle $\Pi_r^{DR}(X/\text{spec } K)$ introduced at the beginning.

Since the connection on $\Pi_r^{DR}(X/\text{spec } K)$ is unipotent, the underlying vector bundle has a canonical (Deligne’s) extension $\Pi_r^{DR}(X/\text{spec } K)$ to $\mathcal{X}_K$. The latter is uniquely characterized by saying that it is a vector bundle with unipotent log connection whose restriction to $X_K$ coincides with $\Pi_r^{DR}(X/\text{spec } K)$.

Denote by $\hat{\Pi}_r^{DR}(X/\text{spec } K)$ the corresponding sheaf of $\mathcal{O}_X \otimes \mathbb{Q}_p$-modules on the p-formal scheme $\hat{\mathcal{X}}$.

Lemma 22 There is a canonical horizontal isomorphism:

$$\Pi_r(\mathcal{X}_{R/p})_{\mathcal{X}} \simeq \hat{\Pi}_r^{DR}(X/\text{spec } K)$$

Proof is similar to one of Lemma [21]: there is the evident map

$$\hat{\Pi}_r^{DR}(X/\text{spec } K) \rightarrow \Pi_r(\mathcal{X}_{R/p})_{\mathcal{X}}$$

and using induction on $r$ one can easily prove that it is an isomorphism.

5.8. Let us give ourself another pair $(\mathcal{X}' : Z' \hookrightarrow \mathcal{X}')$, satisfying the property [1] and a morphism of the log schemes $f : \mathcal{X}_{R/p} \rightarrow \mathcal{X}_{R/p}$.

The base change property [13] implies, that for any scheme $U'$ etale over $\mathcal{X}'_{R/p}$ and p-formal PD-thickening $U' \hookrightarrow T'$ we have canonical isomorphism:

$$((f \times \text{id})p_{23}') \ast B_r(\mathcal{X}_{R/p} \times \text{spec } R/p \mathcal{U}'/T', (f \times \text{id})p_{12}') \simeq (f^* \Pi_r(\mathcal{X}_{R/p}))(T')$$
By the universal property of $B_r$ there is a natural map:

$$B(\mathcal{X}_{R/p} \times \text{spec } R/p, U'/T', p'_1) \rightarrow (f \times id)^* B_r(\mathcal{X}_{R/p} \times \text{spec } R/p, U'/T', (f \times id)p'_1)$$

It defines a canonical morphism:

$$f_* : \Pi_r(\mathcal{X}_{R/p}) \rightarrow f^* \Pi_r(\mathcal{X}_{R/p})$$

In particular, we have the map:

$$\phi : \Pi_r(\mathcal{X}_{R/p}) \rightarrow Fr^* \Pi_r(\mathcal{X}_{R/p})$$

induced by the Frobenius.

Proposition 23 The morphism $\phi$ is an isomorphism.

Proof: The crystal $\Pi_r(\mathcal{X}_{R/p})$ possesses a filtration by the kernels of the projections:

$$\Pi_r(\mathcal{X}_{R/p}) \rightarrow \Pi_{r'}(\mathcal{X}_{R/p})$$

where $r' \leq r$. The associated graded crystal $Gr_* \Pi_r(\mathcal{X}_{R/p})$ descends to the logarithmic point $\text{spec } R/p$. Moreover, the morphism $[13]$ defines a ring structure on the latter crystal. It is easy to see that the ring is generated by $Gr_1 \Pi_r(\mathcal{X}_{R/p})$. On the other hand, we have a canonical isomorphism:

$$Gr_1 \Pi_r(\mathcal{X}_{R/p}) \simeq (R^1p_*, C_{\mathcal{X}_{R/p}, \text{cris}} \otimes \mathbb{Q})^*$$

Here $p$ stands for the map $\mathcal{X}_{R/p} \rightarrow \text{spec } R/p$. Hence, the Proposition follows from the fact ([Hyodo], Proposition 2.24) that the Frobenius

$$\phi : R^1p_*, C_{\mathcal{X}_{R/p}, \text{cris}} \otimes \mathbb{Q} \rightarrow R^1p_*, C_{\mathcal{X}_{R/p}, \text{cris}} \otimes \mathbb{Q}$$

is an isomorphism.

5.9. Remark. One can show (see Section 9) that $\Pi_r(\mathcal{X}_{R/p})$ is an object of the category $C_r(\mathcal{X}_{R/p} \times \text{spec } R/p, \mathcal{X}_{R/p}/\text{spec } W(k)) \otimes \mathbb{Q}$.

6 Crystalline structure on $\mathcal{P}_r^{DR}(X_K)$

6.1. Let $X_K$ be a smooth variety over $K$. Remind that $\mathcal{P}_r^{DR}(X_K)$ is the category whose set of objects is $X_K(K)$ and whose group of morphisms between objects $x_0, x_1 \in X_K(K)$ is $\Pi_r^{DR}(x_0; x_1)$.

We denote by $K_0^{ur}$ the maximal unramified extension of $K_0$ and by $\mathcal{H}_1^{ur}$ the ring $\mathcal{H}_1 \otimes_{K_0} K_0^{ur}$. The latter is generated by $\phi, N a \in K_0^{ur}$, satisfying the following relations

$$\phi N = p N \phi, \phi \cdot a = Fr(a) \cdot \phi, N \cdot a = a \cdot N$$

The main result of this section is the following theorem.

Theorem 24 For any $X_K$ there are

1) a category $\mathcal{P}_r(X_K)$ whose set of objects is $X_K(K)$ and whose groups of morphisms $\text{Mor}_{\mathcal{P}_r(X_K)}(*)(*)$ are endowed with $\mathcal{H}_1^{ur}$-module structure

2) an isomorphism of categories

$$\mathcal{P}_r(X_K) \otimes_{K_0^{ur}} K_{st} \simeq \mathcal{P}_r^{DR}(X_K) \otimes_K K_{st}$$

identical on objects.

For any morphism $f : X'_K \rightarrow X_K$ there is a functor

$$f_* : \mathcal{P}_r(X_K) \rightarrow \mathcal{P}_r(X'_K)$$
such that the evident diagram is commutative.

Let \( \Pi_r^{ur}(X_K; x_0; x_1) \) stand for \( \text{Mor}_{\mathcal{T}}(x_K)(x_0; x_1) \).

3) The isomorphism

\[
T : \Pi_r^{ur}(X_K; x_0; x_1) \otimes_{K_0^*} \mathcal{K}_{st} \simeq \Pi_r^{DR}(X_K; x_0; x_1) \otimes_{\mathcal{K}} \mathcal{K}_{st} \quad (21)
\]

induced by \( \mathcal{E} \) commutes with the action monodromy \( N \).

(The latter acts trivially on \( \Pi_r^{DR}(X_K; x_0; x_1) \) and \( N : \mathcal{K}_{st} \to \mathcal{K}_{st} \) is the derivatov over \( \mathcal{K} \), such that \( N(l(p)) = e \)).

6.2. Remarks. 1. The theorem implies that \( \Pi_r^{DR}(X_K; x_0; x_1) \) possesses a structure of log \( F \)-crystal on \( \text{spec} \; R_L \) for some finite extension \( L \supset K \). If the scheme \( X_K \) has a semi-stable model over \( \text{spec} \; R \) (or, more generally, a model satisfying \( \mathcal{E} \)) we can choose \( L \) to be the field of definition of \( x_0; x_1 \). But in general, it is not possible.

2. The category \( \mathcal{P}_r(X_{\mathcal{K}}) \) depends on \( X_{\mathcal{K}} \) and not on \( X_K \). But the action of the monodromy operator \( N \) on morphisms in the latter category depends on the choice of a finite extension \( K \). More precisely, there is a canonical isomorphism:

\[
S_{L/K} : \Pi_r(X_K \times \text{spec} \; L; x_0; x_1) \simeq \Pi_r(X_K; x_0; x_1)
\]

compatible with \( \mathcal{E} \) and satisfying the following:

\[
S_{L/K} \cdot N = \epsilon_{L/K} N \cdot S_{L/K} : \phi \cdot S_{L/K} = S_{L/K} \cdot \phi \quad (22)
\]

Here \( \epsilon_{L/K} \) is the ramification index of \( L \) over \( K \).

6.3. Notation. The homomorphism \( \mathcal{K}_{st} \to \mathcal{K}, \; l(p) \to 0 \) defines a structure of \( \mathcal{K}_{st} \)-module on \( \mathcal{K} \). Tensoring \( \mathcal{K}_{st} \) with \( \mathcal{K} \) over \( \mathcal{K}_{st} \) we arrive to an isomorphism:

\[
T_{l(p)=0} : \Pi_r(X_K; x_0; x_1) \otimes_{K_0^*} \mathcal{K} \simeq \Pi_r^{DR}(X_K; x_0; x_1)
\]

Note that the latter homomorphism determines a morphism \( T \) of \( \mathcal{K}_{st}[N] \)-modules uniquely.

The result is an almost immediate consequence of the construction in the previous section, de Jong's Alteration Theorem and the technique of descent. We start with recollecting some aspects of the latter theory.

6.4. Descent. Consider the following diagram:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{p_1} & X_0 \\
\downarrow & & \downarrow f \\
X_2 & \xrightarrow{p_2} & X_K
\end{array}
\]

(23)

We assume that all schemes in the diagram are smooth over a field \( K \) of characteristic 0 and the maps \( f \) and \( (p_1 \times p_2) : X_1 \to X_0 \times X_0 \) are proper, surjective and generically etale. In addition, we assume that \( X_K \) and \( X_0 \) are geometrically irreducible.

Let \( \mathcal{D}M_r^0(X_0) \) be the subcategory of \( \mathcal{D}M_r(X_0) \), which consists of a vector bundles \( E' \) on \( X_0 \) together with an integrable unipotent connection such that

\[
p_1^* E' \simeq p_2^* E'
\]

Proposition 25 We have an equivalence of categories:

\[
f^* : \mathcal{D}M_r(X_K) \simeq \mathcal{D}M_r^0(X_0)
\]

Proof:

First, we note that the map

\[
H^1_{DR}(X_K) \to \mathcal{H}^1_{DR}(X_0)
\]
is injective. It implies that the map
\[ \Pi_r^{DR}(X_0; x_0) \to \Pi_r^{DR}(X_K; f(x_0); f(x_0)) \]

is surjective. Hence, the functor \( f^* \) is fully faithful. To complete the proof it suffices to show that \( f^* \) induces a surjection on \( Ext^1 \)-groups.

Without loss of generality we may assume that \( f \) is finite. Indeed, in general there exists a closed subscheme \( Z \hookrightarrow X_K \) of codimension \( \geq 2 \) such that \( f \) is finite over \( X_K - Z \). On the other hand, the restriction
\[ D_M r(X_K) \to D_M r(X_K - Z) \]
is an equivalence of categories.

If, in addition, \( f \) is etale, the claim immediately follows from the fact that for any vector bundle with a unipotent integrable connection (more generally, a connection with regular singularities)
\[ H^1_{DR}(X_K; E) = \ker(H^1_{DR}(X_0; f^*E) \to H^1_{DR}(X_0; p_1^*f^*E)) \]

In general, we denote by \( D \hookrightarrow X_K \) the ramification divisor of \( f \) and make use of the following isomorphism:
\[ H^1_{DR}(X_K; E) \cong (H^1_{DR}(X_0; f^*E) \oplus H^1_{DR}(X_K - D; E)) \to H^1_{DR}(X_0 - f^{-1}(D); f^*E) \]

### 6.5. Alteration Theorem

Let \( Z \hookrightarrow \overline{X} \) be a closed immersion of regular schemes over \( \text{spec} R \), \( \overline{X}_k = \bigcup_1^{r} \overline{X}_{k,i} \) and \( Z = \bigcup_j Z_j \) be the decompositions in their irreducible components.

Remind that \((Z; \overline{X})\) is a strict semi-stable pair over \( \text{spec} R \), if locally (for etale topology) \( \overline{X} \) is isomorphic to
\[ \text{spec} R[t_1, \ldots, t_n, s_1, \ldots, s_m]/(t_1 \cdots t_n - \pi) \]
with \( Z_j \) given by the equations \( s_j = 0 \) for some \( j \in \{1, \ldots, m\} \) and \( \overline{X}_{k,i} \hookrightarrow \overline{X} \) given by \( t_i = 0 \). In particularly, the schemes \( \overline{X}_{k,i} \) are smooth over \( \text{spec} R \).

We are going to make use of the following fundamental result. **Theorem.** (de Jong.) Let \( X \) be an integral, flat and finite type over \( \text{spec} R \) and \( S \hookrightarrow X \) be a proper closed subset. There exist a finite extension \( L \supseteq K \) with ring of integers \( R_L \supseteq R \), an integral scheme \( X_0 \) over \( \text{spec} R_L \), a proper generically etale morphism over \( \text{spec} R \):
\[ f: X_0 \to X \]

and an open immersion \( j: X_0 \hookrightarrow \overline{X}_0 \), with the following properties:

a) \( \overline{X}_0 \) is a projective scheme over \( \text{spec} R_L \), and

b) The pair \((\overline{X}_0, f^{-1}(S)_{\text{red}} \cup \overline{X}_0 \setminus j(X_0))\) is strict semi-stable.

### 6.6. Proof of Theorem [24]

Without loss of generality we can assume, that \( X_K \) is geometrically irreducible.

First, assume that we are given a smooth scheme \( X_K \) together with a compactification \( X_K \hookrightarrow \overline{X} \) satisfying \([14]\) (with \( Z_K = \overline{X}_K \setminus X_K \)). For a pair of points \( x_i \in X_K(L) \) denote by \( i_{x_0, x_1}: \text{spec} L/p \to \overline{X}_{R/p \times \text{spec} R/p} \) the corresponding morphism. We endow the scheme \( \text{spec} L/p \) with the log structure associated to the prelogarithmic structure \( \beta: \mathbb{N} \to L/p, \beta(1) = \pi_K \) (here \( \pi_K \) is a uniformizer of \( K \)) and define
\[ \Pi_r(X_K; x_0; x_1) := \Psi^{un}(i_{x_0, x_1}^*, \Pi_r(\overline{X}_{R/p})) \]
\[ \Pi_r^{ur}(X_K; x_0; x_1) := \Pi_r(X_K; x_0; x_1) \otimes_{K_0^{ur}} K_0^{ur} \]

A construction of the isomorphism \([21]\) immediately follows from the very definition of \( \Psi^{un} \). In turn, Lemma \([11]\) implies that the latter commutes with the action of \( N \). The direct image functor \([20]\) is defined for any log morphism \( X_N \to X_K \).
In general, we start with any open immersion \( X \hookrightarrow \overline{X} \) into a proper, integral, flat over \( \text{spec } R \) scheme. Next, using de Jong’s theorem we can construct a diagram

\[
\begin{array}{ccc}
\overline{X}_1 & \xrightarrow{p_1} & \overline{X}_0 \\
& \xrightarrow{\bar{f}} & \bar{X} \times \text{spec } R_L \\
& \xrightarrow{p_2} & \\
\end{array}
\]

with the following properties:

1) Schemes \( \overline{X}_i \), (\( i = 0, 1 \)) are proper over \( \text{spec } R \).

2) The maps \( \bar{f} \) and \((p_1 \times p_2) : \overline{X}_1 \to \overline{X}_0 \times \overline{X}_\text{spec } R_L \) \( \overline{X}_0 \) are proper, surjective and generically etale.

3) \( \overline{X}_0 \) is geometrically irreducible.

Let \( \overline{X}_i = \overline{X}_i \times \overline{X}_\text{spec } R_L \overline{X}_K \times \text{spec } L \)

4) The pairs \( (\overline{X}_i; \overline{X}_i) \) satisfy (16) (with \( K \) replaced by \( L \)).

Let \( \mathcal{A}_r(X_i) \) be the category of \( \mathbb{Q} \)-linear functors from \( \mathcal{P}_r(X_i) \) to the category \( \text{Vect}_{K_0} \) of finite-dimensional vector spaces over \( K_0 \).

Thanks to (19), we have a functor

\[
T_{(p)=0} : \mathcal{A}_r(\overline{X}_i) \to \mathcal{D}_r(\overline{X}_i \times \text{spec } \overline{K})
\]

given by tensor product with \( \overline{K} \).

Next, we define a category \( \mathcal{A}_r(X_K) \) be the subcategory of \( \mathcal{A}_r(\overline{X}_0) \), which consists of objects \( E' \) satisfying

\[
p_1^* E' \simeq p_2^* E'
\]

The descent property (23) together with (26) imply the existence of a functor

\[
\mathcal{A}_r(X_K) \to \mathcal{D}_r(\overline{X}_K)
\]

By the very definition for any point \( x \in X_0(\overline{K}) \) we have a fiber functor:

\[
\mathcal{F}_x : \mathcal{A}_r(X_K) \to \text{Vect}_{K^0}
\]

Further, for a pair \( \bar{x}_0, \bar{x}_1 \in X_0(\overline{K}) \) define

\[
\Pi_r^{ur}(X_K; \bar{x}_0; \bar{x}_1) = \text{Mor}(\mathcal{F}_{\bar{x}_0}; \mathcal{F}_{\bar{x}_1})
\]

The functor (27) induces a map

\[
T_{(p)=0}^{-1} : \Pi_r^{DR}(X_{\overline{K}}; x_0; x_1) \to \Pi_r^{ur}(X_K; \bar{x}_0; \bar{x}_1) \otimes \overline{K}
\]

where \( x_i = f(\bar{x}_i) \).

**Lemma 26** The morphism \( T_{(p)=0}^{-1} \) is an isomorphism.

This is a simple exercise.

The proposition (23), together with the lemma imply that for any object \( E' \) of \( \mathcal{A}_r(X_K) \) there is a canonical isomorphism

\[
\theta : p_1^* E' \simeq p_2^* E'
\]

As a consequence, we obtain a canonical identification between \( \Pi_r^{ur} \) groups for different liftings of \( x_i \). We define \( \Pi_r(X_K; x_0; x_1) \) to be "the common value" of \( \Pi_r^{ur}(X_K; \bar{x}_0; \bar{x}_1) \).

As we mentioned above the latter gives rise to an isomorphism of \( \text{Ker}[N] \)-modules (21). As usual, one can proof that \( \Pi_r^{ur}(X_K; x_0; x_1) \) does not depend on the choice of resolution (23) we made: given two such diagrams, we can map them to a third one, and the induced maps on \( \Pi_r^{ur}(X_K; x_0; x_1) \) are isomorphisms (since they are isomorphisms on \( \Pi_r^{DR}(X_{\overline{K}}; x_0; x_1) \)).
6.7 Let \( x_0; x_1 \) be \( K \)-points of \( X_K \). Then the vector space \( \Pi_r^{DR}(X_K; x_0; x_1) \) carries a canonical \( K \)-structure:

\[
\Pi_r^{DR}(X_K; x_0; x_1) \simeq \Pi_r^{DR}(X_K; x_0; x_1) \otimes_K K
\]

It gives rise to a semi-linear action of \( Gal(\overline{K}/K) \) on the former space.

It is easy to see that the subspace \( \Pi_r^{ur}(X_K; x_0; x_1) \subset \Pi_r^{DR}(X_K; x_0; x_1) \) is invariant under this action. Moreover, the restriction of the induced action of \( Gal(\overline{K}/K) \) on \( \Pi_r^{ur}(X_K; x_0; x_1) \) to the inertia subgroup \( I \subset Gal(\overline{K}/K) \) factors through a finite quotient.

7 The Monodromy Conjecture for \( \Pi_r^{ur}(X_K; x_0; x_1) \), where \( X_K \) is a proper scheme

7.1. Purity. Let \( V \) be a finite-dimensional \( H^1_{ur} \)-module. Choose a finite unramified extension \( \mathbb{Q}_p \subset K_0 \) and a \( H \)-stable lattice

\[
V_0 \subset V, \ V_0 \otimes_{K_0} \mathbb{Q}_p^{ur} = V
\]

Fix an integer \( c \), such that \( F_r^c \) acts trivially on \( K_0 \).

The nilpotent operator \( N \) gives rise to an increasing filtration \( M.V_0 \subset V \) characterized by the property

\[
N(M.V_0) \subset M_{i-2}V_0 \quad \text{and} \quad N^r_r : gr^r_r V_0 \simeq gr^r_r M.V_0
\]

The filtration is stable under \( \phi^c \), hence the latter acts on the associated graded space:

\[
\phi^c : gr^r_r V_0 \longrightarrow gr^r_r M.V_0
\]

The module \( V \) is called pure of weight \( i \) if any eigenvalue \( \alpha \) of the latter operator is an algebraic number and for any embedding \( v : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \):

\[
|v(\alpha)| = p^{\frac{c}{2c}}
\]

Clearly, that the definition is independent of the choice of \( V_0 \).

One can easily check, that for any two pure \( H^1_{ur} \)-modules \( V \) and \( V' \) of weights \( i \) and \( i' \) respectively, we have

\[
\text{Hom}_{H^1_{ur}}(V; V') = 0
\]

provided that \( i > i' \).

It is conjectured (see, for example, [Illusie]) that the log crystalline cohomology group \( H^1_{st}(X) \otimes \mathbb{Q}_p^{ur} \) of a proper semi-stable scheme over \( \text{spec } R \) is a pure \( H^1_{ur} \)-module of weight \( i \).

7.2. Mixed \( H^1_{ur} \)-modules. The concept of mixed \( H^1_{ur} \)-module is an invention of Beilinson, Schneider and Illusie (unpublished).

A mixed module \( V \) is a pair \( (V; W.V) \), where \( V \) is a \( H^1_{ur} \)-module and \( W.V \) is an increasing filtration by \( H^1_{ur} \)-submodules \( (W.V \text{ is called the weight filtration}) \), with the following condition:

for any integer \( i \) the \( H^1_{ur} \)-module \( gr^W_i V \) is pure of weight \( i \).

A morphism of mixed modules is a homomorphism of underlying \( H^1_{ur} \)-modules preserving the filtration.

The property (29) implies the following result:

**Proposition 27** Any morphism between mixed modules is strictly compatible with the weight filtration. In particularly, the category of mixed modules is abelian.

In addition, there is the evident tensor structure on the latter category.

**Lemma 28** Let \( 0 \to V_1 \to V_2 \to V_3 \to 0 \) be an exact sequence of filtered \( H^1_{ur} \)-modules \( (V_i; W.V_i) \). We suppose that all morphisms are strictly compatible with the filtration. Assume that \( V_1 \) and \( V_3 \) are mixed. Then \( V_2 \) is also mixed.
Proof is obvious.

7.3. Mixed structure on $\Pi^w_r(X_K; x_0; x_1)$. Let $X_K$ be a smooth scheme over $\text{spec } K$, $x_0, x_1 \in X_K(K)$. We define a filtration on $\Pi^w_r(X_K; x_0; x_1)$ to be:

$$W_{r-1} \Pi^w_r(X_K; x_0; x_1) := \ker(\Pi_r(X_K; x_0; x_1) \to \Pi_{r+1}(X_K; x_0; x_1))$$

for $0 \leq i \leq r - 1$, and $W_{r-1} \Pi_r(X_K; x_0; x_1) = 0$ (resp. $= \Pi_r(X_K; x_0; x_1)$) for $i \geq r$ (resp. $i \leq 0$).

Theorem 29 Suppose that $X_K$ is proper. The pair $(\Pi_r(X_K; x_0; x_1); W_r)$ is a mixed $H^r_1$-module.

Proof: By the very definition $\Pi_r(X_K; x_0; x_1)$ carries a structure of a $\Pi_r(X_K; x_0; x_0)$-module. The $gr^w \Pi_r(X_K; x_0; x_1)$-module $gr^w \Pi_r(X_K; x_0; x_1)$ has a canonical generator $1 \in \Pi_1(X_K; x_0; x_1) = K_0^r$, which defines an isomorphism of $H^r_1$-modules:

$$gr^w \Pi_r(X_K; x_0; x_1) \cong gr^w \Pi_r(X_K; x_0; x_1)$$

We note, that the $H^r_1$-module $gr^w \Pi_r(X_K; x_0; x_0)$ does not depend on $x_0$. It justifies the notation $gr^w \Pi_r(X_K)$ we use below.

Hence, it suffices to check, that $gr^w \Pi_{un}(X_K)$ is mixed. (We let $\Pi_{un}(X_K; x_0; x_0)$ stand for the projective limit of $\Pi_r(X_K; x_0; x_0)$.) Assume, first, that $X_K$ has a semi-stable compactification $X_K \hookrightarrow X$.

We make use of the following fact [DGMS]:

$gr^w \Pi_{un}(X_K)$ is a quadratic algebra generated by $gr^w \Pi_{un}(X_K) = (H^1_{st}(X))^*$ with the relations

$$\text{Im}(H^2_{st}(X))^* \to (H^1_{st}(X))^* \otimes (H^1_{st}(X))^*)$$

It is an immediate consequence of the well known result on the de Rham fundamental group.

Therefore, it is enough to check that

1) $H^1_{st}(X)$ is pure of weight 1.
2) $\ker(H^1_{st}(X) \otimes H^1_{st}(X) \to H^2_{st}(X))$

The latter follows from a result of Mokrane [Mo], who proved that $H^1_{st}(X)$ are pure for a semi-stable $X$ of dimension $\leq 2$.

The general case can be reduced to semi-stable using the following trick. First, assuming that $X_K$ is geometrically connected and using de Jong’s theorem we can find a finite extension $L \supseteq K$, a semi-stable geometrically connected scheme $Y$ over $\text{spec } R_L$ and proper generically etale morphism $f : Y_L \to X_K \times \text{spec } L$. Clearly, that the map

$$f_* : gr^w \Pi_r(Y_L) \to gr^w \Pi_r(X_K \times \text{spec } L)$$

is surjective. Moreover, one can easily check that its kernel is a direct summand in a pure $H^r_1$-module $gr^w \Pi_r(Y_L)$. Hence, the image is also pure. It completes the proof.

7.4. The Monodromy Conjecture for the crystalline cohomology. In this subsection we remind the statement of the conjecture. It is known to specialists, although we could not find it in the literature.

For a smooth scheme $X_K$ finite type over $K$, one can construct a finite-dimensional $H^r_1$-module $H^*_p(X_K)$ together with an increasing filtration $W_i H^*_p(X_K) \subset H^*_p(X_K)$ by submodules and a canonical isomorphism

$$H^*_p(X_K) \otimes \overline{Q_p} \to H^*_{DR}(X_K) \otimes \overline{K}$$

compatible with the weight filtration $W_i H^*_p(X_K) \subset H^*_{DR}(X_K)$ on the de Rham cohomology.

In the case when $X_K$ has a compactification $X_K \hookrightarrow \overline{X}$ as in 5.2 we define

$$H^*_p(X_K) := H^*_st(X) \otimes \overline{Q_p}$$

In general $H^*_p(X_K)$ is defined using de Jong’s Alteration theorem.

Conjecture 2 The pair $(H^*_p(X_K); W_\ast)$ is mixed $H^r_1$-module.

For $n \leq 2$, the conjecture follows form the already cited result of Mokrane (see Appendix).
8 Proof of Theorem B

We let $X_K$ be a smooth scheme over $K$ and $x_0, x_1 \in X_K(K)$.

Theorem 30 There exists a unique element

$$C(X_K, x_0; x_1) \in \lim_{\leftarrow} \Pi^ur_r(X_K; x_0; x_1)$$

characterized by the following properties:

1) $C_1(X_K, x_0; x_1) = 1 \in \Pi^ur_r(X_K; x_0; x_1) = \mathbb{Q}_p^ur$

2) $C_r(X_K, x_0; x_1)^{\phi} = C_r(X_K, x_0; x_1)$

Choose an open immersion $j : X_K \hookrightarrow \overline{X}_K$ into a proper smooth variety over $\text{spec} K$ such that the complement $D = \overline{X}_K - X_K$ is a divisor with simple normal crossings.

3) $N^{-1}_r(j_rC_r(X_K, x_0; x_1)) = 0$

Remark. The element $C_r(X_K, x_0; x_1)$ does not depend on the choice of $j$. Proof: given two such immersion $j_i : X_K \hookrightarrow \overline{X}_i$ we can construct a third one $j : X_K \hookrightarrow \overline{X}_K$ and maps

$$
\begin{array}{ccc}
X_K & \xrightarrow{j_i} & \overline{X}_i \\
\downarrow \text{Id} & & \downarrow f_i \\
X_K & \xrightarrow{j} & \overline{X}_K
\end{array}
$$

The uniqueness implies that the element $C_r(X_K, x_0; x_1)$ constructed using $\overline{X}_K$ coincides with the one constructed using $\overline{X}_i$.

Proof. Given a finite-dimensional $\mathcal{H}_1^ur$-module $V$ and an integer $i$, we define a certain subspace $V^i \subset V$. For this we choose a finite unramified extension $\mathbb{Q}_p \subset K_0$ and a $\mathcal{H}_1$-stable lattice $V_0 \subset V$, $V_0 \otimes_{K_0} \mathbb{Q}_p = V$ and a certain $\phi$ such that $Fr^c a = a$ for $a \in K_0$ and define $V_0^i \subset V_0$ to be the maximal $\phi$-invariant subspace, where all eigenvalues of $\phi^c$ are Weil numbers of weight $i$ (meaning that they are algebraic and $|v(a)| = p^{\frac{i}{2}}$ for any embedding $v : \mathbb{Q} \hookrightarrow \mathbb{C}$). Put $V_i = V_0^i \otimes K_0^ur$.

Next, we apply the above construction to $\Pi^ur_r(X_K; x_0; x_1)$. There is a decomposition:

$$
\Pi^ur_r(X_K; x_0; x_1) = \bigoplus_{i \leq 0} \Pi^ur_r(X_K; x_0; x_1)^i
$$

Indeed, it suffices to prove that all eigenvalues of $\phi^c$ on $gr^W_{-1} \Pi^ur_r(X_K)_0$ are Weil numbers of weight $i \leq 0$. The latter is generated by $gr^W_{-1} \Pi^ur_r(X_K)_0$ for which the result is known. (As before, it suffices to treat the semi-stable case. In the latter case we have $gr^W_{-1} \Pi^ur_r(X_K)_0 = (H^1_{st} (\mathcal{X}))^0 \otimes K_0^ur$.)

Further, we define by induction on $r$ certain subspaces $L_r \subset \Pi^ur_r(\overline{X}_K; x_0; x_1)^0$:

$L_1 = \Pi^ur_1(\overline{X}_K; x_0; x_1)^0 = K_0^ur$ and $L_{i+1} = p_{i+1}(L_{i}) \cap \ker N^i$, where $p_{i+1}$ is the projection

$$
\Pi^ur_{i+1}(\overline{X}_K; x_0; x_1)^0 \to \Pi^ur_i(\overline{X}_K; x_0; x_1)^0
$$

Lemma 31 For any $r \dim L_r = 1$.

Proof: Induction on $r$. Consider the following diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{N^i} & 0 \\
\downarrow & & \downarrow \\
gr^W_{-1} \Pi^ur_{i+1}(\overline{X}_K)^{-2i} & \xrightarrow{N^i} & gr^W_{-1} \Pi^ur_{i+1}(\overline{X}_K)^{-2i} \\
\downarrow & & \downarrow \text{Id} \\
L_{i+1} & \xrightarrow{N^i} & gr^W_{-1} \Pi^ur_{i+1}(\overline{X}_K)^{-2i} \\
\downarrow \text{Id} & & \downarrow \\
L_i & \xrightarrow{0} & 0
\end{array}
$$

27
Theorem \[\text{28}\] implies that the first horizontal arrow is an isomorphism. It proves the lemma.

We have already proven the main theorem in the case when \(X_K = \overline{X}\). The general case follows from the following lemma.

**Lemma 32** \(j_* : \Pi_{un}^r(X_K)^0 \simeq \Pi_{un}^r(\overline{X})^0\)

**Proof.** It suffices to check that
\[
gr_i^W \Pi_{un}^r(X_K)^0 \simeq gr_i^W \Pi_{un}^r(\overline{X})^0
\]

It is easy to see that the decomposition
\[
gr_i^W \Pi_{un}^r(\overline{X}) = \bigoplus_{i \leq 0} (gr_i^W \Pi_{un}^r)^i
\]
is compatible with the algebra structure. In particular, \([\overline{R}]\) is a homomorphism of algebras. Moreover, the algebra \(gr_i^W \Pi_{un}^r(\overline{X})^0\) is quadratic. Hence, it suffices to check that \([\overline{R}]\) is an isomorphism in degrees 1 and 2. Assume that \(\overline{X}_K\) has a strict semi-stable model. That is an open immersion \(\overline{X}_K \rightarrow \overline{X}\), together with a closed subscheme \(Z \hookrightarrow \overline{X}\), \(Z_K = X_K\) such that \((Z; \overline{X})\) is a strict semi-stable pair. Let \(Z = \bigcup_{i \in I} Z_i\) be the decomposition in the irreducible components. We have the following exact sequence
\[
0 \rightarrow H^1_{st}(\overline{X}_K) \rightarrow H^1_{st}(X_K) \rightarrow \bigoplus_{i \in I} K_0(-1) \rightarrow H^2_{st}(\overline{X}_K) \rightarrow H^2_{st}(X_K)
\]

Of course, \(H^1_{st}(\overline{X}_K), H^1_{st}(X_K)\) are just notions for log crystalline cohomology of \(\overline{X}\) with the evident log structures. It follows that \(H^1_{st}(X_K)^0 \simeq H^1_{st}(\overline{X}_K)^0\) and the map \(H^1_{st}(\overline{X}_K)^0 \rightarrow H^2_{st}(X_K)^0\) is an injection.

It proves the lemma in the semi-stable case. As usual, in general one can make use of de Jong’s theorem.

It completes the proof of Theorem \([\text{41}]\) along with Theorem B - the element \(C.(X_K, x_0; x_1)\) gives rise to a canonical parallel translation.

### 9 Mixed unipotent F-crystals

In this section, for a smooth variety \(X_{\overline{R}}\) over \(\text{spec} \overline{K}\), we construct a certain category of “mixed unipotent F-crystals “. This is a relative version of the concept of a mixed \(H_{1}^{t}\)-module (see 7.2. ).

As an application we construct a weight filtration on \(\Pi_{\text{un}}^{r}(X_{K}; x_{0}; x_{1})\), for any smooth scheme, and prove the latter module together with the filtration is mixed.

#### 9.1 Mixed log F-crystals .

Throughout this subsection we keep the notations of 5.2. Denote by \(C^{\phi}(\overline{X})\) the category of coherent log F-crystals on \(\overline{X}/\text{spec} W(k)\) (\(\text{spec} W(k)\) is endowed with the trivial log structure).

An object \(E\) of \(C^{\phi}(\overline{X}) \otimes \mathbb{Q}\) is called pure of weight \(i\) if for any finite extension \(L \supset K\) and a point \(i : \text{spec} R_L \rightarrow \overline{X}\), with \(i(\text{spec} L) \subset \overline{X} - Z\), the \(H_{1}^{t}\)-module \(\Psi^{un}_{\text{crit}}(i^{*}E) \otimes \mathbb{Q}_{p}^{ur}\) is pure of weight \(i\) (see 7.1).

Next, we define the category \(C^{\phi}_{W^{*}}(\overline{X})\) of mixed F-crystals: an object of the latter category is a pair \((E, W_{i}E)\), where \(E\) is in \(C^{\phi}(\overline{X}) \otimes \mathbb{Q}\) and \(W_{i}E\) is a filtration satisfying the following condition:

- for any integer \(i\) the crystal \(gr_{i}^{W}E\) is pure of weight \(i\).

It follows from Proposition \([\text{28}]\) that \(C^{\phi}_{W^{*}}(\overline{X})\) is an abelian category.

**Remark.** Thanks to the results of 3.8. we have:
\[
i^{*} : \{\text{coherent F - crystals on } \overline{X}\} \otimes \mathbb{Q} \simeq \{\text{coherent F - crystals on } X_{k}\} \otimes \mathbb{Q}
\]

It shows that the category \(C^{\phi}_{W^{*}}(\overline{X})\) depends on the special fibre \(X_{k}\) only. In particularly, for any finite totally ramified extension \(L \supset K\) with the ring of integers \(R_{L}\), there is a canonical equivalence:
\[
C^{\phi}_{W^{*}}(\overline{X}) \simeq C^{\phi}_{W^{*}}(\overline{X} \times \text{spec} R_{L})
\]

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9.2. Mixed unipotent $F$-crystals. We define a category $\mathcal{C}_{W*, \text{un}}(\overline{X})$ to be the full subcategory of $\mathcal{C}_{W*}(\overline{X})$ whose objects are mixed crystals $(E, W_*E)$ with the following properties:

i) For any $i$ the $F$-crystals $gr^W_i E$ are lifted from the logarithmic point $\text{spec } R$.

ii) For any $i$, the vector bundle $W_i E / W_{i-2} E$ over $\overline{X}_K$ together with the logarithmic connection is a local system on $\overline{X}_K$ (i.e. the logarithmic connection is a connection in the usual sense).

We denote by $\mathcal{C}_{W*, \text{un}}(\overline{X} \times \text{spec } R_{\text{ur}})$ the injective limit of categories $\mathcal{C}_{W*, \text{un}}(\overline{X} \times \text{spec } R')$, where $R'$ runs over all finite unramified extensions of $K$.

Let $X_{\overline{K}}$ be a smooth variety over $\overline{K}$. We are going to make use of de Jong’s Theorem and the technique of descent (see 6.4 and 6.5) to define a certain category $\mathcal{C}_{W*, \text{un}}(X_{\overline{K}})$ of “mixed unipotent $F$-crystals on $X_{\overline{K}}$”.

For this purpose, we choose a finite extension $L \supset \mathbb{Q}_p$, an integral proper, flat over $\text{spec } R_L$ scheme $\overline{X}$ together with an open immersion $X_{\overline{K}} \hookrightarrow \overline{X} \times \text{spec } R_{\overline{K}}$ and a resolution (23)

$$
\begin{array}{cccc}
\overline{X}_1 & \xrightarrow{p_1} & \overline{X}_0 & \xrightarrow{f} & \overline{X} \\
& & & & \\
\overline{X}_2 & \xrightarrow{p_2} & & \\
\end{array}
$$

(32)

We define $\mathcal{C}_{W*, \text{un}}(X_{\overline{K}})$ to be the full subcategory of $\mathcal{C}_{W*, \text{un}}(\overline{X}_1 \times \text{spec } R_{\text{ur}})$, which consists of crystals $E$ such that

$$
\overline{p}_1^* E \simeq \overline{p}_2^* E
$$

Using Proposition 23 one can easily check that there is a canonical equivalence between the latter categories for different choices of the resolution.

Using Proposition 23 one can easily check that there is a canonical equivalence between the latter categories for different choices of the resolution.

It also follows that there is a faithful functor

$$
\mathcal{C}_{W*, \text{un}}(X_{\overline{K}}) \rightarrow \mathcal{W}_1 \mathcal{M}_{\text{un}}(X_{\overline{K}})
$$

(33)

Example. The category $\mathcal{C}_{W*, \text{un}}(\text{spec } \overline{K})$ is equivalent to the category of mixed $\mathcal{H}_1^{\text{ur}}$-modules.

For a mixed $\mathcal{H}_1^{\text{ur}}$-module $V$, we denote by $\overline{V}$ its pull-back on $X_{\overline{K}}$.

Given another smooth variety $Y_{\overline{K}}$ and a morphism $f : Y_{\overline{K}} \rightarrow X_{\overline{K}}$ we can construct the restriction functor

$$
f^* : \mathcal{C}_{W*, \text{un}}(X_{\overline{K}}) \rightarrow \mathcal{C}_{W*, \text{un}}(Y_{\overline{K}})
$$

In particularly, a point $i : \text{spec } \overline{K} \rightarrow X_{\overline{K}}$ defines a functor

$$
i^* : \mathcal{C}_{W*, \text{un}}(X_{\overline{K}}) \rightarrow \mathcal{C}_{W*, \text{un}}(\text{spec } \overline{K})
$$

$$
\rightarrow \{ \text{Mixed } \mathcal{H}_1^{\text{ur}} \text{-modules} \}
$$

For an object $E$ of $\mathcal{C}_{W*, \text{un}}(X_{\overline{K}})$ one can define the log crystalline cohomology $H^i_{\text{pst}}(X_{\overline{K}}; E)$. This is a finite-dimensional $\mathcal{H}_1^{\text{ur}}$-module. There is a canonical isomorphism:

$$
H^i_{\text{pst}}(X_{\overline{K}}; E) \otimes_{\mathbb{Q}^{\text{ur}}} \overline{K} \simeq H^i_{\text{D}R}(X_{\overline{K}}; E)
$$

9.3. Given two objects $E$, $G$ of the category $\mathcal{C}_{W*, \text{un}}(X_{\overline{K}})$ we denote by $\mathcal{E}xt^i(E; G)$ the $\mathcal{H}_1^{\text{ur}}$-module $H^i_{\text{pst}}(X_{\overline{K}}; E^* \otimes G)$.

For $i = 0$, 1 the latter possesses a canonical filtration $W_* \mathcal{E}xt^i(E; G) \subset \mathcal{E}xt^i(E; G)$. The definition of the filtration on $\mathcal{H}om(E; G)$ is obvious. We define the filtration on $\mathcal{E}xt^1(E; G)$ in the following way: choose a smooth compactification $X_K \hookrightarrow \overline{X}_K$ and let $M = E^* \otimes G$ and $W_{i+1}H^i_{\text{pst}}(X_{\overline{K}}; W_i M)$ be the preimage of

$$
H^1_{\text{pst}}(X_{\overline{K}}; W_i M / W_{i-1} M) \subset H^1_{\text{pst}}(X_{\overline{K}}; W_i M / W_{i-1} M)
$$
under the canonical map
\[ H^1_{\text{pst}}(X_K^\infty; W_i M) \rightarrow H^1_{\text{pst}}(X_K^\infty; W_i M/W_{i-1} M) \]

Finally, we put
\[ W_{i+1}H^1_{\text{pst}}(X_K^\infty; M) = \text{Im} W_{i+1}H^1_{\text{pst}}(X_K^\infty; W_i M) \]

Clearly, the filtration does not depend on the choice of a compactification.

**Proposition 33** For \( i = 0, 1 \), the pair \((\text{Ext}^i(E; G); W_* \text{Ext}^i(E; G))\) is a mixed \( H^i_{\text{urt}} \)-module.

**Proof:**
Without loss of generality we can assume that \( X_K^\infty \) is quasi-projective. Indeed, in general, we can choose a smooth compactification \( X_K^\infty \rightarrow X_K^\infty \), a smooth projective variety \( Y_K^\infty \) and a morphism \( f : Y_K^\infty \rightarrow X_K^\infty \) which is a birational isomorphism. The existence of the pair \((Y_K^\infty, f)\) follows from the Chow lemma. It is easy to see that the morphism \( f : f^{-1}(X_K^\infty) \rightarrow X_K^\infty \) induces an isomorphism on the fundamental groups.

Next, we make use of the following lemma.

**Lemma 34** Let \( U \subset \mathbb{P}_K^n \) be a quasi-projective variety of dimension \( \geq 3 \). Then, for a general hyperplane section \( U \cap H \rightarrow U \), the restriction functor
\[ D\mathcal{M}_{\text{un}}(U) \rightarrow D\mathcal{M}_{\text{un}}(U \cap H) \]

is an equivalence of the categories.

Proof is left to the reader.

It follows that there exists a smooth surface \( Z_K^\infty \) and a morphism \( g : Z_K^\infty \rightarrow X_K^\infty \) such that
\[ g^* D\mathcal{M}_{\text{un}}(X_K^\infty) \rightarrow D\mathcal{M}_{\text{un}}(Z_K^\infty) \]
is an equivalence. In turn, it implies that the pull-back functor
\[ g^* C_{W_*, \text{un}}(X_K^\infty) \rightarrow C_{W_*, \text{un}}(Z_K^\infty) \]
is fully faithful. Hence, is suffices to prove the proposition for surfaces.

Of course, we can assume that \( E \) is the trivial \( F \)-crystal (i.e \( E = \mathbb{Q}_p^{\text{ur}} \)).

We prove the proposition by induction on length of the weight filtration on \( G \). Let \( r \) be the integer such that \( W_r G \neq 0 \) and \( W_{r-1} G = 0 \). Consider the following exact sequence of \( H^i_{\text{urt}} \)-modules:
\[ 0 \rightarrow W_r G \rightarrow H^0_{\text{pst}}(Z_K^\infty; G) \rightarrow H^0_{\text{pst}}(Z_K^\infty; G/W_r G) \rightarrow W_r G \otimes H^1_{\text{pst}}(Z_K^\infty) \rightarrow \]
\[ H^1_{\text{pst}}(Z_K^\infty; G) \rightarrow H^1_{\text{pst}}(Z_K^\infty; G/W_r G) \rightarrow W_r G \otimes H^2_{\text{pst}}(Z_K^\infty) \]

By the induction hypothesis \( H^i_{\text{pst}}(Z_K^\infty; G/W_r G) \) are mixed.

On the other hand, the result of Mokrane (see 7-4.) implies that the same is true for \( H^i_{\text{pst}}(Z_K^\infty) \).

Hence, \( W_r G \otimes H^i_{\text{pst}}(Z_K^\infty) \) are mixed.

Finally, it is easy to check that all the morphisms in the above sequence are strictly compatible with the weight filtration. Now the proposition follows from lemma 33.

**Corollary 35** We have the following exact sequence:
\[ 0 \rightarrow \text{Ext}^1_{C_{W_*, \text{un}}(\text{spec } K)}(\mathbb{Q}_p^{\text{ur}}; \mathcal{H}om(E; G)) \rightarrow \]
\[ \text{Ext}^1_{C_{W_*, \text{un}}(\text{spec } K)}(E; G) \rightarrow \mathcal{H}om_{C_{W_*, \text{un}}(\text{spec } K)}(\mathbb{Q}_p^{\text{ur}}; \text{Ext}^1(E; G)) \]
9.4 For an integer $r$ we denote by $\mathcal{C}^\phi_{\mathcal{W}, \cdot, r}(X_{\overline{\mathbb{Q}}})$ the full subcategory of $\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}})$ whose objects are unipotent $F$-crystals of level $r$.

**Theorem 36** There is a unique object $\Pi^ur_r(X_{\overline{\mathbb{Q}}})$ of $\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}})$ characterised by the following property:

for any $E$ in $\mathcal{C}^\phi_{\mathcal{W}, \cdot, r}(X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}})$ there is a canonical isomorphism of mixed $\mathcal{H}^1_{ur}$-modules

$$\text{Hom}(\Pi^{ur}_r(X_{\overline{\mathbb{Q}}}); E) \simeq H^0_{\text{proet}}(X_{\overline{\mathbb{Q}}}, \Delta^* E)$$

Here $\Delta : X_{\overline{\mathbb{Q}}} \to X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}}$ stands for the diagonal embedding.

Moreover, the de Rham realization functor (33) sends $\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$ to $\Pi^{DR}_r(X_{\overline{\mathbb{Q}}})$.

**Proof.** We prove the theorem by induction on $r$. If $r = 1$ there is nothing to prove. Assume that $\Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}})$ is already constructed. (And the assertion about the de Rham realization holds.)

The identity morphism $Id : \Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}}) \to \Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$ gives rise to a morphism of mixed modules:

$$1 : \mathbb{Q}_p^{ur} \to \Delta^* \Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}})$$

Denote by $I$ the kernel of the map:

$$\mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}})}((\Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}}); \mathbb{Q}_p^{ur}), 1) \to \mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}})}((\Pi^{ur}_r(X_{\overline{\mathbb{Q}}}); \mathbb{Q}_p^{ur}), \mathbb{Q}_p^{ur})$$

Proposition 33 implies that $I$ is a mixed $\mathcal{H}^1_{ur}$-module. Note that the de Rham realization of $I^*$ is canonically isomorphic to $\ker (\Pi^{DR}_r(X_{\overline{\mathbb{Q}}}) \to \Pi^{DR}_{r-1}(X_{\overline{\mathbb{Q}}}))$

**Lemma 37** There is a unique element $[\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})] \in \mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}})}((\Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}}); I^*), \mathbb{Q}_p^{ur})$ satisfying the following properties:

i) The de Rham realization of $\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$ is $\Pi^{DR}_r(X_{\overline{\mathbb{Q}}})$.

ii) The image of $\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$ under the canonical morphism

$$\mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}})}((\Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}}); I^*), \mathbb{Q}_p^{ur}) \to \mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}})}((\Pi^{ur}_r(X_{\overline{\mathbb{Q}}}); I^*), \mathbb{Q}_p^{ur})$$

is equal to 0.

**Proof:** First, we make use of Corollary 33 and consider the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(\text{spec } \overline{\mathbb{Q}})}(\mathbb{Q}_p^{ur}; I^*) & \to & \mathcal{E}xt^1(\Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}}); I^*) \\
\downarrow \text{Id} & & \downarrow 1 \circ \Delta^* \\
\mathcal{E}xt^1_{\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(\text{spec } \overline{\mathbb{Q}})}(\mathbb{Q}_p^{ur}; I^*) & \to & \mathcal{E}xt^1(\Pi^{ur}_r(X_{\overline{\mathbb{Q}}}); I^*)
\end{array}$$

The first row is exact by loc. cit. It implies the uniqueness of $\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$.

It remains to prove the existence. Choose a resolution $\mathcal{R}_{\overline{\mathbb{Q}}}$. By the definition, the category $\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}(X_{\overline{\mathbb{Q}}} \times X_{\overline{\mathbb{Q}}})$ is identified with a certain subcategory of $\mathcal{C}^\phi_{\mathcal{W}, \cdot, un}((\overline{\mathbb{Q}}) \times (\overline{\mathbb{Q}}) \times \text{spec } R_{L^{ur}})$.

Lemma 18 implies that there exists a unique extension

$$0 \to I^* \to \Pi^{ur}_r(X_{\overline{\mathbb{Q}}}) \to \Pi^{ur}_{r-1}(X_{\overline{\mathbb{Q}}}) \to 0$$

in the category $\mathcal{C}(\overline{\mathbb{Q}}_0 \times \overline{\mathbb{Q}}_0 \times \text{spec } R_{L^{ur}}) \otimes \mathbb{Q}$ satisfying the properties i) and ii).

Fix a lifting:

$$1 : \mathbb{Q}_p^{ur} \to \Delta^* \Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$$

It is easy to see that there is a unique $F$-structure on the crystal $\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$ such that all the morphisms in 33 and 36 commute with $\phi$.

Finally, we define a weight filtration on $\Pi^{ur}_r(X_{\overline{\mathbb{Q}}})$. 

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Sublemma 1. The class of extension (33) is in $\mathcal{W}_0 Ext^1(\Pi_{r-1}^w(X_K); l)$.

Proof is omitted.

Consider the extension

\[ 0 \rightarrow I^*/W_i I^* \rightarrow ? \rightarrow W_i \Pi_{r-1}^w(X_K) \rightarrow 0 \]  

(37)

induced by (33).

Notation: for an extension of crystals $0 \rightarrow E \rightarrow G \rightarrow K \rightarrow 0$, we let $Spl(K; G)$ be the preimage of $Id \in \text{Hom}(K; K)$ under a canonical map $\text{Hom}(K; K) \rightarrow \text{Hom}(K; G)$.

We have a canonical map:

\[ Spl(W_i \Pi_{r-1}^w(X_K); ?) \rightarrow Ext^1(\Pi_{r-1}^w(X_K)/W_i \Pi_{r-1}^w(X_K); I^*/W_i I^*) \]

Let $P$ be the preimage of $W_0 Ext^1(\Pi_{r-1}^w(X_K)/W_i \Pi_{r-1}^w(X_K); I^*/W_i I^*)$ under the latter map.

Sublemma 2. For any $i < 0$, the set $P$ consists of one element.

Proof. Consider the following exact sequence

\[ \text{Hom}(\Pi_{r-1}^w(X_K); I^*/W_i I^*) \rightarrow \text{Hom}(W_i \Pi_{r-1}^w(X_K); I^*/W_i I^*) \rightarrow Ext^1(\Pi_{r-1}^w(X_K); I^*/W_i I^*) \]

If $i < 0$ the first map is 0. Moreover, $Spl(W_i \Pi_{r-1}^w(X_K); ?) = \alpha^{-1}([\Pi_{r-1}^w(X_K)])$. Hence, Sublemma 1 implies that $P$ is not empty. On the other hand, $W_0 \text{Hom}(W_i \Pi_{r-1}^w(X_K); I^*/W_i I^*) = 0$. Therefore, $P$ consists of one element.

It is clear, that the element of $P$ defines a splitting of (33). In turn, the latter gives rise to a subcrystal $W_i \Pi_{r-1}^w(X_K) \subset \Pi_{r-1}^w(X_K)$. It completes the proof of the lemma.

We leave to the reader to check that the mixed crystal $\Pi_{r-1}^w(X_K)$ represents the functor defined in the theorem.

9.5. Given a point $i_{x_0; x_1} : \text{spec} K \rightarrow X_K \times X_K$, we define a mixed $\mathcal{H}_r^w$-module

\[ \Pi_{r-1}^w(X_K; x_0; x_1) := i_{x_0; x_1}^* \Pi_r(X_K) \]

It is easy to see that $\Pi_{r-1}^w(X_K; x_0; x_1)$ is an algebra in the tensor category $C_{W^*, \text{un}}^\phi(\text{spec} K)$.

Corollary 38 The restriction to a point $x_0 \in X_K(\overline{K})$ defines an equivalence between the category $C_{W^*, \text{un}}^\phi(X_K)$ and the category of modules over $\Pi_{r-1}^w(X_K; x_0; x_1)$ in $C_{W^*, \text{un}}^\phi(\text{spec} K)$.

9.6. Remark: One can show that for any smooth variety $X_K$ over $\text{spec} K$ the underlying $\mathcal{H}_r^w$-module $\Pi_{r-1}^w(X_K \times \text{spec} K; x_0; x_1)$ is isomorphic to one defined in Section 6.

9.7. Proposition. Let $V_r \subset \Pi_{r-1}^w(X_K; x_0; x_1)$ be the subspace which consists of elements $v \in \Pi_{r-1}^w(X_K; x_0; x_1)$ satisfying the following properties:

i) $\phi v = v$

ii) $N_a v \in W_{-a-1} \Pi_{r-1}^w(X_K; x_0; x_1)$ for any $0 < a < r$.

We have $\dim V_r = 1$. Moreover, the canonical morphism:

\[ V_r \rightarrow Q_p \]

is an isomorphism.

Proof is left to the reader. (Use induction on $r$.)

In particularly, it defines a distinguished element $C_r(X_K; x_0; x_1) \in \Pi_{r-1}^w(X_K; x_0; x_1)$ (the preimage of $1 \in Q_p^w$). We have just reproved Theorem B!
10 Variations of p-adic Hodge structures

10.1. Notations. Let $\overline{X}$ be a smooth proper scheme over $\text{spec} W(k)$, $Z \subset \overline{X}$ be a normal crossings divisor relative to $\text{spec} W(k)$, $X = \overline{X} - Z$, $X_k = X \times_{\text{spec} W(k)} \text{spec} k$, $X_K = X \times_{\text{spec} W(k)} \text{spec} K$.

We endow $\overline{X}$ with the log structure given by the divisor $Z$ and $\text{spec} W(k)$ with the trivial log structure (see 2.2.).

10.2. Faltings defined a certain category $\mathcal{MF}_{a,a+b}(\overline{X})$ of filtered $F$-crystals on $\overline{X}$, analogous to the category of variations of Hodge structures over the field of complex numbers. We copy the definition from [Fa] with merely decorative innovations. Fix integers $a$, $b$, with $0 \leq b \leq p - 1$.

Definition. An object of $\mathcal{MF}_{a,a+b}(\overline{X})$ consists of the following data:

a) A coherent log $F$-crystal $E$ on $\overline{X}$.

b) A filtration on the corresponding coherent sheaf $E_{\overline{X}}$ on $\overline{X}$

\[ \cdots \subset F^i E_{\overline{X}} \subset F^{i-1} E_{\overline{X}} \subset \cdots \subset E_{\overline{X}} \]

by coherent subsheaves satisfying Griffiths-transversality:

\[ \nabla(F^i E_{\overline{X}}) \subset F^{i-1} E_{\overline{X}} \otimes \Omega^1_{\overline{X}}(\text{dlog}X) \]

and the following stabilisation property: $F^a E_{\overline{X}} = E_{\overline{X}}$, $F^{a+b+1} E_{\overline{X}} = 0$.

These are subject to the following conditions:

i) The sheaves $F^{i}E_{\overline{X}}/F^{i+1}E_{\overline{X}}$ are vector bundles on $\overline{X}$.

To formulate next condition, we first note that the coherent sheaf $(Fr^* E)_{\overline{X}}$ possesses a canonical filtration:

\[ \tilde{F}^i(Fr^* E)_{\overline{X}} \subset (Fr^* E)_{\overline{X}} \]

It can be constructed using local liftings of the Frobenius: if $U \subset \overline{X}$ together with a logarithmic Frobenius-lift $\tilde{Fr}: \tilde{U} \to \tilde{U}$ ($\tilde{U}$ stands for $p$-adic completion), we define

\[ \tilde{F}^i(Fr^* E)_{\tilde{U}} = Fr^* \left( \sum_{k+m=i; m \geq 0} p^m F^k E_{\tilde{U}} \right) \]

If $b \leq p - 1$ the latter does not depend on the choice of filtration.

Or, equivalently, we can consider $E$ as a filtered crystal on $(\overline{X}_k/\text{spec} W(k))$ (see 2.5.) and take the corresponding filtration on $(Fr^* E)_{\overline{X}}$.

ii) The restriction of $\phi: Fr^* E_{\overline{X}} \to E_{\overline{X}}$

to $\tilde{F}^i(Fr^* E)_{\overline{X}}$ is divisible by $p^i$.

iii) $\sum_i p^{-i}\phi(\tilde{F}^i(Fr^* E)_{\overline{X}}) = E_{\overline{X}}$.

Denote

\[ \mathcal{MF}_{(a,a+b)}(\overline{X}) \otimes \mathbb{Q} = \mathcal{MF}_{(a,a+b)}^Q(\overline{X}) \]

We use the name variation of p-adic Hodge structure for an object of the latter category.

Let $K_0$ stand for the trivial Hodge structure on $\text{spec} W(k)$.

10.3. The following result is proven by Faltings. Theorem 39

a) Let $E$ and $G$ be objects of $\mathcal{MF}_{(a,a+b)}^Q(\overline{X})$ and $f: E \to G$ is morphism. Then $f$ is strict for the filtrations.

b) The category $\mathcal{MF}_{(a,a+b)}^Q(\overline{X})$ is abelian.

Because of the restriction on $b$ the category of variations of Hodge structures does not possess a tensor structure. Nevertheless, if $0 \leq b_1, b_2, b_1 + b_2 \leq p - 1$ we can define a tensor product:

\[ \mathcal{MF}_{(a_1,a_1+b_1)}^Q(\overline{X}) \times \mathcal{MF}_{(a_2,a_2+b_2)}^Q(\overline{X}) \to \mathcal{MF}_{(a_1+a_2,a_1+b_2)}^Q(\overline{X}) \]

Under a similar restriction, we can define $\text{Hom}$ between two variations.

10.4. Let $\overline{Y}$ be another smooth proper scheme over $\text{spec} W(k)$ together with a normal crossings divisor $D \subset \overline{Y}$ relative to $\text{spec} W(k)$ and $f: \overline{X} \to \overline{Y}$ be a proper log smooth morphism of relative dimension $d$. Assume that $f^*(D) \leq Z$ as divisors (i.e. $f$ is smooth in codimension 1).
Theorem 40 (Faltings.) Let $E$ be an object of $\mathcal{M}^Q_{[a,a+b]}(\overline{X})$. For any $i$ with $b + \min(d, i) \leq p - 2$ the filtered $F$-crystal $R^i f_* E$ is in $\mathcal{M}^Q_{[a,a+b+i]}(\overline{Y})$.

10.5. We have the evident faithful functor

$$\mathcal{M}^Q_{[a,a+b]}(\overline{X}) \to \mathcal{D}(\mathcal{M}(X_{K_0}))$$

to the category of vector bundles on $X_{K_0}$ together with an integrable connection.

Let $E$ and $G$ be objects of $\mathcal{M}^Q_{[a,a+b]}(\overline{X})$ and $2b \leq p - 3$. Thanks to the previous result the vector spaces

$$\text{Hom}_{\mathcal{D},\mathcal{M}}(E; G), \text{Ext}^1_{\mathcal{D},\mathcal{M}}(E; G)$$

carry canonical Hodge structures.

Proposition 41 We have the following exact sequence

$$0 \to \text{Ext}^1(K_0; \text{Hom}_{\mathcal{D},\mathcal{M}}(E; G)) \to \text{Ext}^1_{\mathcal{M}^Q_{[a,a+b]}}(\overline{X})(E; G) \to (\text{Ext}^1_{\mathcal{D},\mathcal{M}}(E; G))^{\phi=1} \cap F^0 \to 0$$

Proof: Given an extension

$$0 \to \text{Hom}(E; G) \to A \to K_0 \to 0$$

whose class $[A]$ in $\text{Ext}^1_{\mathcal{M}^Q_{[a,a+b]}}(\overline{X})(K_0; \text{Hom}(E; G)) = \text{Ext}^1_{\mathcal{M}^Q_{[a,a+b]}}(\overline{X})(E; G)$ satisfies the property:

$$\beta([A]) = 0 \text{ we define } \gamma([A]) \in \text{Ext}^1(K_0; \text{Hom}_{\mathcal{D},\mathcal{M}}(X_{K_0}))(E; G)$$

to be the class of the extension

$$0 \to \text{Hom}_{\mathcal{D},\mathcal{M}}(X_{K_0})(E; G) \to \text{Hom}_{\mathcal{D},\mathcal{M}}(X_{K_0})(K_0; A) \to K_0 \to 0$$

It gives rise to an isomorphism $\gamma: \ker \beta \simeq \text{Ext}^1(K_0; \text{Hom}_{\mathcal{D},\mathcal{M}}(X_{K_0}))(E; G)$ with $\gamma \alpha = 1$.

It remains to show that the map $\beta$ is surjective. Let

$$0 \to G \to B \to E \to 0$$

be an extension with $[B] \in (\text{Ext}^1_{\mathcal{D},\mathcal{M}}(X_{K_0}))(E; G)^{\phi=1} \cap F^0$. Since

$$\text{Ext}^1_{\text{cris}}(E; G) \otimes \mathbb{Q} \simeq \text{Ext}^1_{\mathcal{D},\mathcal{M}}(X_{K_0})(E; G)$$

we may assume that $B$ is a log crystal on $\overline{X}$. Moreover, we can choose $F$-structure $\phi: Fr^* B \to B$ and a filtration on $F^i B \subset B$ on the underlying coherent sheaf compatible with those on $E$ and $G$.

Then, for sufficiently large $n$ we have:

$$p^n[B] \in \text{Ext}^1_{\mathcal{M}^Q_{[a,a+b]}}(\overline{X})(E; G)$$

It completes the proof.

11 Construction of etale local systems

11.1. Faltings has constructed a fully faithful functor

$$D: \mathcal{M}^Q_{[a,a+p-2]}(\overline{X}) \to Sh_{\mathbb{Z}_p}^{et}(X_{K_0})$$

to the category of etale locally constant $\mathbb{Z}_p$-sheaves on $X_{K_0}$ [Ft].

We reproduce his construction here with some minor variations.

11.2. Given a log crystal $E$ on $\overline{X}$ we put $E_n = E/p^n$. 

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An etale open affine subset $U = \text{spec } \mathcal{R} \subset \overline{X}$ is called small if there exists an etale map of log schemes

$$U \to \text{spec } W(k)[T_1, T_2, \cdots, T_m]$$

where the log structure on $\text{spec } W(k)[T_1, T_2, \cdots, T_m]$ is given by the divisor $\prod_{1 \leq i \leq m} T_i = 0$. We note the scheme $\overline{X}$ can be covered by small open subschemes.

Choose a small subscheme $U$ and denote by $\widehat{U} = Spf \widehat{\mathcal{R}}$ the corresponding $p$-adic formal scheme. Let $\widehat{\mathcal{R}}$ be the union of all finite extensions $\widehat{\mathcal{R}}_k$ of $\widehat{\mathcal{R}}$ such that $(p\mathbb{Z})^N \Omega_\widehat{\mathcal{R}}/\widehat{\mathcal{R}} = 0$ for sufficiently large $N$ ($\mathbb{I}_Z$ is the ideal corresponding to the immersion $Z \to \overline{X}$). We denote by $\widehat{U} = Spf \widehat{\mathcal{R}}$ the spectrum of the $p$-adic completion of $\mathcal{R}$.

We endow $\widehat{U}$ with a log structure

$$M_{\widehat{U}} = \{ f \in \mathcal{O}_{\widehat{U}} \mid f^N \in p^* M_{\widehat{U}} \text{ for some integer } N \}$$

So the map $p : \widehat{U} \to \widehat{U}$ is a morphism of log schemes.

11.3. Consider the log crystalline cohomology

$$\mathcal{H}^0_{\text{cris}}(\widehat{U} \times \text{spec } W(k) \text{ spec } k \text{ spec } W(k), p^* E_n|_{\widehat{U}})$$

The Galois group of $\widehat{\mathcal{R}}$ over $\widehat{\mathcal{R}}$ acts on the latter space. On the other hand, the cohomology group is also equipped with an action of the operator $\phi^*$ induced by $\phi : Fr^* E \to E$. Finally it possesses a canonical filtration. The latter comes from the identification

$$\mathcal{H}^0_{\text{cris}}(\widehat{U} / \text{spec } W(k), p^* E_n|_{\widehat{U}}) = \mathcal{H}^0_{\text{cris}}(\widehat{U} \times \text{spec } W(k) \text{ spec } k \text{ spec } W(k), p^* E_n|_{\widehat{U}})$$

and the filtration on $E$.

One can prove that the higher cohomology groups vanish (this depends on the assumption that $U$ is small).

We define

$$E(n|_{\widehat{U}}) = (\mathcal{H}^0_{\text{cris}}(\widehat{U} \times \text{spec } W(k) \text{ spec } k \text{ spec } W(k), p^* E_n|_{\widehat{U}}))^{\phi = 1} \cap F^0$$

Faltings has proven that $E(n|_{\widehat{U}})$ is a finite abelian group of the same type as $E_n$ (meaning that the coherent sheaf $E_U$ is locally isomorphic to $D(E_n|_{\widehat{U}}) \otimes \mathcal{O}_U$) equipped with a $\text{Gal}(\widehat{R}/\widehat{\mathcal{R}})$-action.

Choose a covering of $\overline{X}$ by small open subschemes $U_i$. It easily follows from the result cited above that we can glue an etale locally constant sheaf on $\overline{X}_{K_0}$ from $D(E_n|_{\widehat{U}_i})$. We denote the latter by $D(E_n)$.

The sheaves $D(E_i)$ of $\mathbb{Z}/p^i \mathbb{Z}$-modules define a locally constant $p$-adic sheaf $D(E)$.

11.4. The following result is proven by Faltings.

**Theorem 42** The functor

$$D : \mathcal{MF}_{[\alpha : p - 2]}(\overline{X}) \to Sh_{\mathbb{Z}_p}^e(X_{K_0})$$

is exact and fully faithful. Its image is closed under subobjects and quotients.

The functor $D$ is compatible with the (partly defined) tensor product and $\text{Hom}$.

Clearly it implies that the same is true for the functor

$$\mathcal{MF}_{\mathbb{Q}}^{[\alpha : p - 2]}(\overline{X}) \to Sh_{\mathbb{Q}_p}^e(X_{K_0})$$

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12 \( B_{DR} \)

12.1. We keep the notations of the previous section.

The main result of this section is the following theorem.

**Theorem 43** Let \( i_x : \text{spec } K \to X_{K_0} \) be a point and \( E \) be an object of the category \( \mathcal{M}_R^Q \). There is a canonical isomorphism

\[
B_{DR} \otimes_K i_x^* E_X \simeq B_{DR} \otimes_{\mathcal{O}_p} i_x^* \mathcal{D}(E)
\]

of vector spaces over \( B_{DR} \).

The isomorphism is compatible with the tensor structure.

12.2. The proof is based on the following result which is due to Kato [Ka2] (in fact Kato formulated the result in a slightly less general form but the proof works in our case as well).

Let \( G \) be a \( F \)-crystal on the scheme \( \text{spec } R/\text{spec } W(k) \) equipped with the canonical log structure: \( M_{\text{spec } R} = R - 0 \). Remind (Theorem 13) that \( G \) gives rise to a \( \mathcal{H}_1 \)-module \( \Psi^\text{un}(E) \) together with an isomorphism

\[
\Psi^\text{un}(G) \otimes_{K_0} K \simeq G_R \otimes_R K
\]

We denote by \( \bar{R} \) the integral closure of \( R \) in \( K \). Endow \( \text{spec } \bar{R} \) with a log structure \( M_{\text{spec } \bar{R}} = \bar{R} - 0 \). Consider the natural morphism of log schemes

\[
q : \text{spec } \bar{R} \to \text{spec } R
\]

**Theorem 44** The kernel of

\[
N \otimes 1 + 1 \otimes N : \Psi^\text{un}(G) \otimes_{K_0} B_{st}^+ \to \Psi^\text{un}(G) \otimes_{K_0} B_{st}^+
\]

is canonically isomorphic to \( H^0_{cris}(\text{spec } \bar{R}/\text{spec } W(k); q^* G) \). This isomorphism commutes with the action of \( \phi \).

**Proof of theorem 43** We keep the notation of the previous section. The morphism \( i_x \) extends to \( i_x : \text{spec } R \to X \). Choose a small open subset \( x \in U \subset X \) and a lifting \( i_x : \text{Spf } \bar{R} \to \bar{U} \).

We have canonical morphisms

\[
i_x^* \mathcal{D}(E) \to H^0_{cris}(\hat{U}/\text{spec } W(k), p^* E|_{\hat{U}}) \to H^0_{cris}(\text{spec } \bar{R}/\text{spec } W(k); q^* \bar{E}) \to \Psi^\text{un}(\bar{E}) \otimes_{K_0} B_{st} \to i_x^* \mathcal{E}_X \otimes_K B_{DR}
\]

It is easy to see that the composition \( i_x^* \mathcal{D}(E) \to i_x^* \mathcal{E}_X \otimes_K B_{DR} \) does not depend on the choice of the lifting \( \bar{i}_x \). Hence it defines a morphism of functors:

\[
i_x^* \mathcal{D}(\ast) \otimes_{\mathcal{O}_p} B_{DR} \to i_x^* \mathcal{E}_X \otimes_K B_{DR}
\]

The morphism is compatible with the tensor structure and the trace morphism. Thus it is an isomorphism.

13 Unipotent variations of Hodge structure: a \( p \)-adic analog of the theorem by Hain and Zucker

13.1. **Definition.** A variation of Hodge structure \( E \) is called unipotent of level \( r \) if there exists a filtration \( 0 = E_0 \subset E_1 \subset \cdots \subset E_r = E \) such that the quotients \( E_i/E_{i-1} \) are constant variations.

We denote by \( \mathcal{M}_F^Q \) the full subcategory of the category \( \mathcal{M}_F^Q \) consisting of unipotent variations of level \( r \). It is easy to see that \( \mathcal{M}_F^Q \) is an abelian category.
There is the functor to the category of vector bundles with an unipotent integrable connection:

$$\mathcal{MF}_{[a,a+b],r}(X) \to \mathcal{D}M_r(X_{K_0})$$

For a point \(x \in X(W(k))\) we let \(\Pi_r^{DR}(X_{K_0}, x, x)\) stand for the algebra of endomorphisms of the corresponding fiber functor

$$\mathcal{D}M_r(X_{K_0}) \to \text{Vect}_{K_0}$$

**Lemma 45**

a) \(\Pi_r^{DR}(X_{K_0}, x, x)\) is a finite-dimensional algebra.

b) The category \(\mathcal{D}M_r(X_{K_0})\) is equivalent to the category of finite-dimensional \(\Pi_r^{DR}(X_{K_0}, x, x)\)-modules.

Proof is omitted.

**Theorem 46**

Assume that \(r \leq \frac{p-1}{2}\). There exists a unique object \(\Pi_r^{DR}(X_{K_0}, x)\) of \(\mathcal{MF}_{[-r+1,0],r}(X)\), called the fundamental variation, characterized by the property:

For any unipotent variation \(E \in \mathcal{MF}_{[-r+1,0],r}(X)\) there is a canonical isomorphism

$$\text{Hom}_{\mathcal{MF}_{[-r+1,0],r}(X)}(\Pi_r^{DR}(X_{K_0}, x); E) \simeq \text{Hom}_{\mathcal{MF}_{[-r+1,0],r}(\text{spec }W(k))}(K_0; E_x)$$

(\(K_0\) stands for the trivial Hodge structure.)

In addition, the variation \(\Pi_r^{DR}(X_{K_0}, x)\) enjoys the following properties:

i) The fiber \(\Pi_r^{DR}(X_{K_0}, x)_x\) is identified with \(\Pi_r^{DR}(X, x, x)\) Moreover, the canonical action of \(\Pi_r^{DR}(X, x, x)\) on the fiber (defined via the equivalence (39)) is the multiplication from the left.

ii) \(\Pi_r^{DR}(X_{K_0}, x, x)\) is an algebra in the category of Hodge structures. That latter means that the multiplication

$$\Pi_r^{DR}(X_{K_0}, x, x) \otimes_{K_0} \Pi_r^{DR}(X_{K_0}, x, x) \to \Pi_r^{DR}(X_{K_0}, x, x)$$

and the injection \(K_0 \to \Pi_r^{DR}(X_{K_0}, x, x)\) are morphisms of Hodge structures.

iii) For any object \(E \in \mathcal{MF}_{[-r+1,0],r}(X)\) the homomorphism

$$\Pi_r^{DR}(X_{K_0}, x, x) \to \text{Hom}(E_x; E_x)$$

is amorphism of Hodge structures. (Note that the first assertion of ii) is a special case of iii) and, in fact, both of them immediately follow from the universal property (39).

Moreover, it defines an equivalence between the category \(\mathcal{MF}_{[-r+1,0],r}(X)\) and the category of Hodge theoretic representations of \(\Pi_r^{DR}(X_{K_0}, x, x)\) (The latter consists of pairs \((V, f)\), where \(V\) is an object of \(\mathcal{MF}_{[-r+1,0],r}(\text{spec }W(k))\) and a morphism \(f : \Pi_r^{DR}(X_{K_0}, x, x) \to \text{Hom}(V; V)\) of the Hodge structures.)

**13.2. Remark.** We would like to point out that the theorem is merely a \(p\)-adic analog of the well known result about unipotent variations of mixed Hodge structure which is due to Hain and Zucker [HZ].

A proof of the theorem occupies the next two sections.

### 14 Construction of \(p\)-adic Hodge structure on \(\Pi_r^{DR}(X_K)\)

#### 14.1

Remind, that \(\Pi_r^{DR}(X_{K_0})\) is a vector bundle on \(X_{K_0} \times X_{K_0}\) together with a unipotent connection of level \(r\) uniquely characterized by the following property:

given another object \(E\) of the category \(\mathcal{D}M_r(X_{K_0} \times X_{K_0})\), we have a canonical isomorphism

$$\text{Hom}_{\mathcal{D}M_r(X_{K_0} \times X_{K_0})}(\Pi_r^{DR}(X_{K_0}); E) \simeq \Delta^* E$$

(39)

Here \(\Delta\) stands for the diagonal embedding \(X_{K_0} \hookrightarrow X_{K_0} \times X_{K_0}\) and \(\mathcal{D}M(X_{K_0} \times X_{K_0}, X_{K_0}/X_{K_0})\) is the category of vector bundles on \(X_{K_0} \times X_{K_0}\) together with an integrable connection along the fibers of the projection \(p_2 : X_{K_0} \times X_{K_0} \to X_{K_0}\)
In particular, we have
\[ \text{Hom}_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_r^{DR}(X_{K_0}); E) \simeq H^0_{DR}(\Delta; \Delta^* E) \] (40)

The identity morphism \( Id : \Pi_r^{DR}(X_{K_0}) \to \Pi_r^{DR}(X_{K_0}) \) gives rise to a parallel section
\[ 1 : \mathcal{O}_\Delta \hookrightarrow \Delta^* \Pi_r^{DR}(X_{K_0}) \] (41)

14.2. The main result of this section is the following theorem.

**Theorem 47** There exists a unique variation of \( p \)-adic Hodge structure on \( \Pi_r^{DR}(X_{K_0}) \) with the following property: for any unipotent variation \( E \in \mathcal{M}^{\mathbb{Q}_{r+1}}_{[-r+1, 0], \mathcal{R}(X \times X)} \) the map (34) is a morphism of variations of Hodge structures.

14.3. **Proof.** We construct the Hodge structure on \( \Pi_r^{DR}(X_{K_0}) \) by induction on \( r \). First, we endow \( \Pi_r^{DR}(X_{K_0}) = \mathcal{O}_{X_{K_0} \times X_{K_0}} \) with the trivial Hodge structure. Next, we assume that the Hodge structure on \( \Pi_{r'}^{DR}(K) \) is already constructed for all \( r' < r \).

There is a canonical surjective homomorphism (given by 1)
\[ p_r : \Pi_r^{DR}(X_{K_0}) \longrightarrow \Pi_{r-1}^{DR}(X_{K_0}) \]

**Lemma 48** a) The connection on the vector bundle \( \ker p_r \) is trivial. Moreover, there is a canonical isomorphism:
\[ (\ker p_r)^* \simeq \ker (\text{Ext}^1_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_{r-1}^{DR}(X_{K_0}); \mathcal{O}_{X_{K_0} \times X_{K_0}}) \longrightarrow \text{Ext}^1_{\mathcal{D}, \mathcal{M}(\Delta)}(\mathcal{O}_\Delta; \mathcal{O}_\Delta)) \]

b) The group of automorphisms of the sequence
\[ 0 \longrightarrow \ker p_r \longrightarrow \Pi_r^{DR}(X_{K_0}) \longrightarrow \Pi_{r-1}^{DR}(X_{K_0}) \longrightarrow 0 \] (42)

identical on the boundary terms is isomorphic to \( \ker p_r \)

Proof is omitted.

By Proposition 41 and the induction hypothesis \( \ker p_r \) and \( \Pi_{r-1}^{DR}(X_{K_0}) \) are endowed with Hodge structures.

**Key Lemma.** There exists a unique variation of Hodge structure on the middle term of (43) satisfying the following properties:

i) All maps in the exact sequence (43) are compatible with Hodge structures.

ii) (43) is a morphism of the variations of Hodge structures.

**Proof:**
We start with the exact sequence from proposition 41.
\[ 0 \longrightarrow \text{Ext}^1_{\mathcal{M}, \mathcal{F}^{\mathcal{Q}}(\text{spec } W(k))}(\mathcal{K}_0; \text{Hom}_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_{r-1}^{DR}(X_{K_0}); \ker p_r)) \]
\[ \longrightarrow \text{Ext}^1_{\mathcal{M}, \mathcal{F}^{\mathcal{Q}}(X \times X)}(\Pi_{r-1}^{DR}(X_{K_0}); \ker p_r) \longrightarrow \]
\[ (\text{Ext}^1_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_{r-1}^{DR}(X_{K_0}); \ker p_r))_{\phi=1} \cap \mathcal{F}^0 \longrightarrow 0 \] (43)

The vector space
\[ \text{Ext}^1_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_{r-1}^{DR}(X_{K_0}); \ker p_r) = \]
\[ \text{Hom}_{\mathcal{K}_0}(\ker (\text{Ext}^1_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_{r-1}^{DR}(X_{K_0}); \mathcal{O}_{X_{K_0} \times X_{K_0}}) \longrightarrow \text{Ext}^1_{\mathcal{D}, \mathcal{M}(\Delta)}(\mathcal{O}_\Delta; \mathcal{O}_\Delta); \text{Ext}^1_{\mathcal{D}, \mathcal{M}(X_{K_0} \times X_{K_0})}(\Pi_{r-1}^{DR}(X_{K_0}); \mathcal{O}_{X_{K_0} \times X_{K_0}})) \]

has a distinguished element: the identical morphism. It is easy to see that it coincides with the class of the extension (42).
In particularly, the latter class is invariant under \( \phi \) and lies in \( F^0 \). Hence its preimage in \( \text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}((\Pi_{1}^{DR}(X_{K_0}); \ker p_r) \) is a toser \( \mathcal{T} \) over the first \( \text{Ext} \) group in the sequence \( \text{(33)} \). We have to show that the condition ii) gives a trivialization of the toser.

The fact that the connection on \( \ker p_r \) is trivial implies that

\[
\text{Hom}_{\mathcal{D}_M(X_{K_0} \times X_{K_0})}(\Pi_{1-1}^{DR}(X_{K_0}); \ker p_r) = \text{Hom}_{\mathcal{D}_M(X_{K_0} \times X_{K_0})}(\mathcal{O}_{X_{K} \times X_{K}}; \ker p_r) = \ker p_r
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\mathcal{K}_0; \ker p_r) & \to & \text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\Pi_{1}^{DR}(X_{K_0}); \ker p_r) \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
\text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\mathcal{K}_0; \ker p_r) & \to & \text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\Delta^*\Pi_{1}^{DR}(X_{K_0}); \ker p_r) \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
\text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\mathcal{K}_0; \ker p_r) & \to & \text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\mathcal{O}_\Delta; \ker p_r)
\end{array}
\]

Here the third vertical map is given by embedding \( \mathcal{O}_\Delta \hookrightarrow \Delta^*\Pi_{1-1}^{DR}(X_{K_0}) \).

It implies that there exists a unique element in \( \mathcal{T} \) which maps to 0 in \( \text{Ext}^1_{\mathcal{M}^{\phi_0}(\mathcal{X} \times \mathcal{X})}(\mathcal{O}_\Delta; \ker p_r) \) i.e satisfies property ii).

Let \( \mathcal{V} \) be the corresponding extension of the variations of Hodge structures. It remains to show that there exists a unique parallel isomorphism between the underlying \( \mathcal{D} \)-module and \( \Pi_{1}^{DR}(X_{K_0}) \) identical on \( \Pi_{1-1}^{DR}(X_{K_0}) \) and \( \ker p_r \) and satisfying ii). The existence follows from part b) of lemma \( \text{(48)} \) and uniqueness from the following sublemma.

**Sublemma.** Fix a variation of Hodge structures on \( \Pi_{1}^{DR}(X_{K_0}) \) satisfying i) and ii). Then for any variation \( E \in \mathcal{M}^{\phi_0}_{(-r+1,0); r}(\mathcal{X} \times \mathcal{X}) \) we have a canonical isomorphism:

\[
\text{Hom}_{\mathcal{M}^{\phi_0}_{(-r+1,0); r}(\mathcal{X} \times \mathcal{X})}(\Pi_{1}^{DR}(X_{K_0}); E) \simeq \text{Hom}_{\mathcal{M}^{\phi_0}_{(\mathcal{X} \times \mathcal{X})}}(\mathcal{K}_0; H_{DR}^0(\Delta; \Delta^* E))
\]

**Proof of Sublemma.** Indeed, by the very definition the morphism \( \text{(43)} \) is compatible with the Hodge structures. Sublemma follows.

This completes the proof of Key-lemma along with Theorem.

**14.4.** Denote by

\[
p_{ij} : X_K \times X_K \times X_K \longrightarrow X_K \times X_K
\]

the projection given by the formula \( p_{ij}(x_0, x_1, x_2) = (x_i; x_j) \).

We note that there is a canonical parallel morphism:

\[
p_{12}^* \Pi_{1}^{DR}(X_{K_0}) \times p_{23}^* \Pi_{1}^{DR}(X_{K_0}) \longrightarrow p_{13}^* \Pi_{1}^{DR}(X_{K_0})
\]

\( \text{(44)} \)

The reader can easily check that \( \text{(44)} \) is compatible with the Hodge structures.

**15 Proof of Theorem**

**15.1.** Given a point \( x \in X(W(k)) \) we denote by \( \Pi_{1}^{DR}(X_{K_0}, x) \) the restriction of the variation \( \Pi_{1}^{DR}(X_{K_0}) \) to the subscheme \( \mathcal{X} \times x \hookrightarrow \mathcal{X} \times \mathcal{X} \).

It is easy to see that \( \Pi_{1}^{DR}(X_{K_0}, x) \) satisfies the universal property \( \text{(38)} \).

Next, given a unipotent variation of Hodge structure \( E \in \mathcal{M}^{\phi_0}_{(-r+1,0); r}(\mathcal{X}) \) we consider the morphism

\[
\Pi_{1}^{DR}(X_{K_0}, x) \longrightarrow E^* \otimes E_x
\]

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given by $Id \in E^*_x \otimes E_x$. The latter gives rise to a homomorphism of Hodge structures

$$\Pi_{\text{cris}}^D(X_{K_0}, x, x) \rightarrow \text{Hom}(E_x; E_x)$$

It remains to prove the last assertion of the theorem on the equivalence between the category $\mathcal{MF}^Q_{[-r+1,0]}(\mathcal{X})$ and the category of Hodge theoretic representations $\Pi_{\text{cris}}^D(X_{K_0}, x, x)$.

**Lemma 49 (Rigidity)** Let $E$ and $G$ be objects of $\mathcal{MF}^Q_{[a,b]}(\mathcal{X})$, $2b \leq p - 3$, $f : E \rightarrow G$ be a homomorphism of the underlying coherent sheaves preserving the connection. Suppose there exists a point $x_0 \in X(W(R))$ such that the restriction of $f$ to the point $x f_x : E_x \rightarrow G_x$ is a morphism of the Hodge structures. Then $f$ is a morphism of the variations of Hodge structures.

**Proof:** Let $P = \text{Hom}(E; G) = E^* \otimes G$. It is easy to see that $P$ is an object of $\mathcal{MF}^Q_{[-b]}(\mathcal{X})$.

Consider the group log crystalline cohomology $H^0_{\text{cris}}(\mathcal{X}; P)$. One can check that it possesses a canonical Hodge structure (it also follows from Theorem [10]). Moreover, the natural map

$$r_x : H^0_{\text{cris}}(\mathcal{X}; P) \rightarrow \text{Hom}(E_x; G_x)$$

is an injection of Hodge structures. The homomorphism $f$ defines an element $f \in H^0_{\text{cris}}(\mathcal{X}; P)$. Since

$$r_x(f) \in \ker(\phi_0 - Id : F^0(\text{Hom}(E_x; G_x)) \rightarrow \text{Hom}(E_x; G_x))$$

and morphisms of Hodge structures are strictly compatible with the filtration, we have $f \in \ker(\phi_0 - Id : F^0(H^0_{\text{cris}}(\mathcal{X}; P)) \rightarrow H^0_{\text{cris}}(\mathcal{X}; P))$. It completes the proof.

The lemma implies that that the functor is fully faithful. It remains show that its image contains all Hodge theoretic representations. Given a representation $\rho : \Pi_{\text{cris}}^D(X_{K_0}, x, x) \otimes V \rightarrow V$, we define a variation of Hodge structure to be

$$\text{coker} (\Pi_{\text{cris}}^D(X_{K_0}, x) \otimes \Pi_{\text{cris}}^D(X_{K_0}, x, x) \otimes \text{Id} \otimes \rho : \Pi_{\text{cris}}^D(X_{K_0}, x, x) \otimes \text{Id} \otimes \rho : \Pi_{\text{cris}}^D(X_{K_0}, x) \otimes V)$$

Here $m$ stands for the canonical morphism:

$$\Pi_{\text{cris}}^D(X_{K_0}, x) \otimes \Pi_{\text{cris}}^D(X_{K_0}, x, x) \rightarrow \Pi_{\text{cris}}^D(X_{K_0}, x)$$

It defines the inverse functor.

### 16 Proof of Theorems A

16.1. We denote by $Sh^\text{et}_{\mathcal{O}_{p^r}}(X_{K_0})$ the full subcategory of $Sh^\text{et}_{\mathcal{O}_{p^r}}(X_{K_0})$ which consists of unipotent etale local systems of level $r$ (A local system $\mathcal{E}$ is called unipotent of level $r$ if there exists a filtration $0 = \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \cdots \rightarrow \mathcal{E}_r = \mathcal{E}$ such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is isomorphic to the pullback of a sheaf on $\text{spec } K_0$).

Let $\Pi_r^\text{et}(X_{K_0})$ be the fundamental local system on $X_{K_0} \times X_{K_0}$. It is characterized by the property that for any $\mathcal{E} \in Sh^\text{et}_{\mathcal{O}_{p^r}}(X_{K_0} \times X_{K_0})$ there is a canonical isomorphism:

$$\text{Hom}_{Sh^\text{et}_{\mathcal{O}_{p^r}}}(X_{K_0} \times X_{K_0})((\Pi_r^\text{et}(X_{K_0}); \mathcal{E}) \simeq H^0_{\text{et}}(\Delta; \Delta^* \mathcal{E})$$

**Theorem 50** Assume that $p \leq \frac{e-1}{2}$. There is a canonical isomorphism:

$$\mathcal{D}(\Pi_{\text{cris}}^D(X_{K_0})) \simeq \Pi_r^\text{et}(X_{K_0})$$

**Proof:** The canonical morphism

$$1 : \mathcal{O}_\Delta \rightarrow \Delta^* \Pi_{\text{cris}}^D(X_{K_0})$$
induces a map
\[ \underline{Q}_p \rightarrow \Delta^* D(\Pi^{DR}_r(X_{K_0})) \]
which, in turn, gives rise to a morphism:
\[ f : \Pi^{et}_r(X_{K_0}) \rightarrow D(\Pi^{DR}_r(X_{K_0})) \]

First, we show that \( f \) is surjective. Since \( D \) is fully faithfull, it suffices to check that for any \( E \in \mathcal{M}_F^{-r+1,0}((X \times X)) \) the canonical morphism:
\[ A : \text{Hom}_{\text{Sh}_{Q_p}}(X_{K_0} \times X_{K_0})(D(\Pi^{DR}_r(X_{K_0})); D(E)) \rightarrow \text{Hom}_{\text{Sh}_{Q_p}}(X_{K_0} \times X_{K_0})(\text{Im} f; D(E)) \]
is isomorphism. It follows from the universal property of \( P_{i,DR}(X_{K_0}) \) that
\[ \text{Hom}_{\text{Sh}_{Q_p}}(X_{K_0} \times X_{K_0})(\Pi^{et}_r(X_{K_0}); D(E)) \simeq \text{Hom}_{\mathcal{M}_F^{-r+1,0}(\text{spec } W(k))}(K_0; H^0_{DR}(\Delta; \Delta^* E)) \]

On the other hand, we have
\[ \text{Hom}_{\text{Sh}_{Q_p}}(X_{K_0} \times X_{K_0})(\Pi^{et}_r(X_{K_0}); D(E)) \simeq H^0_{et}(\Delta; \Delta^* D(E)) \]

Finally, combining the latter with the isomorphism
\[ \text{Hom}_{\mathcal{M}_F^{-r+1,0}(\text{spec } W(k))}(K_0; H^0_{DR}(\Delta; \Delta^* E)) \simeq H^0_{et}(\Delta; \Delta^* D(E)) \]
we obtain the inverse map \( A^{-1} \). It proves subjectivity of \( f \). Since
\[ \dim_{Q_p} \Pi^{et}_r(X_{K_0}) = \dim_{K_0} \Pi^{DR}_r(X_{K_0}) \]
\( f \) is an isomorphism.

We can make use of Theorem 43 to derive the following result.

**Corollary 51** Given a pair of points \( x_0, x_1 \in X_{K_0}(\overline{K}) \), we have a canonical isomorphism
\[ \Pi^{et}_r(X_{K_0}, x_0, x_1) \times_{Q_p} B_{DR} \simeq \Pi^{DR}_r(X_{K_0}, x_0, x_1) \times_{\overline{K}} B_{DR} \]
It proves Theorem A.

**16.2. Remark.** It seems likely that one can make use of the \( Q_p \)-theory [Fa] (associated convergent \( F \)-isocrystals ...) and generalize the above argument to prove Theorem A for any smooth variety over a finite extension of \( Q_p \) (with no restrictions on \( p \)).

## 17 P-adic integration

**17.1.** Let \( X \) be a smooth variety over \( \mathbb{C} \) and \( E \) be a vector bundle together with integrable connection
\[ \nabla : E \rightarrow E \otimes \Omega^1_X \]
Parallel translation along a path
\[ T_\lambda : E_{x_0} \simeq E_{x_1} \quad (45) \]
is a generalization of integration of 1-forms. Let us formulate it explicitly.

Given a 1-form \( \omega \), we define \( E \) to be the trivial vector bundle on \( X \) with a basis \( e_{-1}, e_0 \), and a connection \( \nabla : E \rightarrow E \otimes \Omega^1_X \) is given in this basis by the formula:
\[ \nabla(s) = A s + ds \]
where
\[ A = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \]

Then the parallel translation (45) is given by the following matrix
\[ T_\lambda = \begin{pmatrix} 1 & -\int \omega \\ 0 & 1 \end{pmatrix} \] (46)

17.2. Next, let \( X_K \) to be a smooth geometrically connected scheme over \( K \). \( \omega \) be a closed 1-form and \( x_0, x_1 \in X_K \). We make use of the canonical parallel translation \( C_{x_0,x_1,X_K} \) and (46) to define a \( p \)-adic integral \( \int_{x_0}^{x_1} \omega \in K_{st} \).

**Proposition 52** The integral \( \int_{x_0}^{x_1} \omega \) coincides with Colmez’s integral.

It follows from Conjecture 1 (see (1.18.) proven in rank two case by Colmez.

18 **Appendix**

18.1. In this Appendix we prove the Monodromy Conjecture for \( H^2_{pst}(X_{\overline{\mathbb{Q}}}) \).

**Theorem 53** Let \( X_{\overline{\mathbb{Q}}} \) be a proper smooth scheme finite type over \( \text{spec} \, K \). Then, for \( n \leq 2 \), the \( \mathcal{H}^n \)-module \( H^0_{pst}(X_{\overline{\mathbb{Q}}}) \) is pure of weight \( n \). (For the definition of purity, see 7.1.)

**Corollary 54** Let \( X_{\overline{\mathbb{Q}}} \) be a smooth scheme finite type over \( \text{spec} \, K \). Then, for \( n \leq 2 \), \( (H^n_{pst}(X_{\overline{\mathbb{Q}}}); W) \) is mixed.

18.2. **Weakly admissible modules.** The definitions and results recollected below are due to Fontaine [Foll].

A \( \phi \)-module over \( K \) is a \( \mathcal{H}_1 \)-module \( V \) together with decreasing filtration \( F^iV \), \( i \in \mathbb{Z} \) on \( V_K = V \otimes_K K \) satisfying \( \cup_i F^iV = V \) and \( \cap_i F^iV = 0 \). The \( \phi \)-modules form an additive category \( \mathcal{MF}_K(\phi, N) \).

For any object \( V \) of \( \mathcal{MF}_K(\phi, N) \), we define \( t_H(V) \) to be the biggest integer \( i \) such that \( F^iV_K = V_K \); if \( \phi v = \lambda v \), define \( t_N(V) \) as the \( p \)-adic valuation of \( \lambda \). For any object \( V \), whose underlying vector space is finite-dimensional we put
\[ t_H(V) = t_H(\wedge^{top} V), \quad t_N(V) = t_N(\wedge^{top} V) \]

A subspace \( V' \subset V \) of an object of \( \mathcal{MF}_K(\phi, N) \) is called a subobject if \( V' \) is stable under the action of \( \mathcal{H}_1 \). We endow it with the filtration \( F^iV'_K := F^iV \cap V'_K \).

A finite-dimensional object \( V \) is called weakly admissible if:

i) \( t_H(V) = t_N(V) \)

ii) For any subobject \( V' \subset V \), \( t_H(V') \leq t_N(V') \).

It is known that the category \( \mathcal{MF}_K^a(\phi, N) \) of weakly admissible modules is Tannakian (in particular, abelian).

The latter result implies the weak admissibility is preserved by the extension of scalars i.e. we have a functor:
\[ \mathcal{MF}_K^a(\phi, N) \longrightarrow \mathcal{MF}_L^a(\phi, N) \]

where \( L \supset K \) is a finite extension. It allows one to define the category \( \mathcal{MF}_\overline{\mathbb{Q}}^a(\phi, N) \).

It follows from results of [Tsuji] that for any smooth proper scheme finite type over \( \text{spec} \, \overline{\mathbb{Q}} \), the \( \phi \)-module \( H^1_{pst}(X_{\overline{\mathbb{Q}}}) \) (the filtration is the Hodge filtration on \( H^1_{pst}(X_{\overline{\mathbb{Q}}}) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}} \simeq H^1_{DR}(X_{\overline{\mathbb{Q}}}) \)) is weakly admissible.
18.3. Lemma. Let \( f : P \to Q \) be a morphism of finite-dimensional \( \mathcal{H}^1_{nr} \)-modules and \( V := \ker f \). Assume that \( P \) is pure of weight \( n \) and \( t_N(V) = \frac{n \cdot \dim V}{2} \). Then \( V \) is pure of weight \( n \).

Proof: It suffices to prove the statement for a morphism of finite-dimensional \( \mathcal{H}_1 \)-modules. Fix an integer \( c \), such that \( Fr^c \) acts trivially on \( K_0 \). We have the decomposition

\[
V = \bigoplus_i V^i
\]

where \( V^i \) is the maximal \( \phi \)-invariant subspace such that all eigenvalues of \( \phi^c|_{V^i} \) are Weil numbers of weight \( ci \). We have to show that for any \( i \), the map

\[
N^i : V^{n+i} \to V^{n-i}
\]

is an isomorphism. Assume that this not the case. It follows from the purity of \( P \) that the latter map is injective. Hence, the assumption implies that the eigenvalue of \( \phi^c \) on the eigenvalue of the operator \( \phi^c : \wedge^{top} V \to \wedge^{top} V \) is a Weil number \( \alpha \) of weight \( \leq c(n \cdot \dim V - 2) \). On the other hand, the \( p \)-adic valuation of \( \alpha \) is equal to \( c \cdot \frac{\dim V}{2} \). It follows that \( p^{-c \cdot \frac{\dim V}{2}} \alpha \) is an algebraic integer. On the other hand, it is a Weil number of negative weight. The contradiction completes the proof.

18.4. Proof of the Theorem. By the Chow Lemma we can find a smooth projective variety \( Y_K \subset \mathbb{P}^N \) and a proper morphism \( f : Y_K \to X_K \), which is a birational equivalence. Next, we choose a plane \( H \subset \mathbb{P}^N \) such that \( S := H \cap Y \) is a smooth connected surface.

The Weak Lefschetz Theorem implies that, for \( n \leq 2 \), the map

\[
r : H^p_{pst}(X_K) \to H^p_{pst}(S)
\]

is injective. By the result of [Mokrane] the module \( H^p_{pst}(S) \) is pure of weight \( n \). Since both modules are weakly admissible, so is \( \text{coker } r \). On the other hand, it is easy to see that \( t_H(\text{coker } r) = \frac{n \cdot \dim \text{coker } r}{2} \). (For example, we can embed \( K \) into \( \mathbb{C} \) and use the fact that \( \text{coker } r \) possesses a Hodge structure of weight 2). Hence, \( t_N(\text{coker } r) = \frac{n \cdot \dim \text{coker } r}{2} \). It remains to apply the lemma (to the dual map).

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