Projecting by conical screw lines with constant step

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Abstract. The relevance of the study: The formation of surfaces by analytical methods and their visualization using computer graphics is one of the topical problems of applied geometry. Purpose of the study is to investigate non-traditional projection systems, to select congruence parameters of conical helical lines that cover the entire complex of surface requirements, obtained by projecting with conic rays an arbitrary flat or spatial line and using computer graphics in visualization of the surface. Methods of research: The general analytical theory of applied surface molding, developed by prof. Skidan I A and a single unit, which is based on the mathematical support of computer technologies for designing and creating objects of complex shape. Results of the research: An example of an analytical interpretation of the image of curvilinear projection by conical screw lines of a constant step and an example of congruence of conical screw lines, located on coaxial cones with a common vertex and a variable angle of inclination of the generator to the axis. The properties are investigated, the parameters of the helical line of the congruence, passing through an arbitrary point of space that does not belong to the axis, are determined. A method is proposed for constructing spiral surfaces, the framework of which consists of rays projecting an arbitrary line.

1. Introduction
There are two directions of analytical modeling of objects of complex shape, differing in the form of their initial representation: discrete modeling based on the thickening of the initial point frame, supplemented (not always) by curvature values or derivative values at certain points. As a rule, these are irregular surfaces, which P. Bezier has successfully called sculptural. Discrete modeling has acquired special significance with the development and wide application in the diversified calculations by the finite element method. Among foreign scientists, besides the P. Bezier, we mention R. Barnhill, G. Bernstein, V. Gordon, S. Koons, J. Ferguson.

The second direction of analytical surface modeling is based on their representation by continuous functions. The development strategy of this direction is to find such expressions of functions that ensure: the form conforms to the functional purpose of the product, unit or structure, the application of existing methods for their design, the use of existing technologies for their manufacture or construction. The urgency of analytical surface modeling is caused by the successes of computer graphics, and its base is constituted by constructive ways of shaping: kinematic (Gromov M.Ya., Kotov I.I., Pavlov A.V., Podgorny O.L., Podkorytov A.N., Ruzleva N.P., Tevlin A.M. and others); a way to extract a linear framework of surfaces from sets, in particular, line congruences (Obukhova...
V.S., Podgorny O.L., Ryzhov N.N., Tevlin A.M. and others); method of transformation (Japaridze I.S., Kotov I.I., Ivanov S., Obukhova V.S., Podgorny O.L., Tevlin A.M. and others); the method of obtaining a linear surface skeleton as a set of non-standard projection beams with both straight and curved lines (Ruubel A.I., Kachenyuk A.N., Kotov I.I., Obukhova V.S., Podgorny O.L., Tevlin A.M. and others).

It should be determined that in parallel with the development of constructive methods for the formation of surfaces, their analytical interpretations were also developed. Thus, the analytical interpretation of the kinematic method is given by Kotov I.I., Podkorytov A.N., Tevlin A.M., the method of extracting a linear framework from sets or congruences of lines - Ryzhov N.N., Tevlin A.M., the method of transformation - Ivanov S., the method of curvilinear projection - Ruubel A.I., Podkorytov A.N., Tevlin A.M.

2. Methodological Framework

In geometry in general and in applied geometry the surface is most often shown not directly, but by conditions called the determinant. The surface determinant has a constructive part consisting of geometric figures and an information part in which the ratio of the geometric figures of the constructive part to the surface is formulated, that is, roles are assigned to the constituent parts that they must play in the process of obtaining the surface drawings. Since in the precomputer period of the development of applied geometry attention was focused on the development of methods for constructing surfaces of complex shape, the support of constructive models was analytically pursued with the sole goal of increasing accuracy.

Each constructive method of surface formation has its own set of determinants. The determinant in the kinematic method includes a generator of a constant or variable type and the law of its motion in space, which is determined by means of guiding lines. The generatrix itself, its instantaneous positions in the process of surface formation, is supplied by means of a surface or plane of incidence.

When a surface is formed by the method of extracting its linear framework from the line congruence, the surface determinant consists of a determinant of the congruence and an additional condition that connects one free congruence parameter. Congruences are usually represented by local figures - lines that intersect all congruence lines or surfaces that are bypassed for congruence lines. On the additional condition, then in the simplest case there can be a line that is immersed in a congruence. The lines of the congruence must intersect it.

The surface formation by non-traditional projection rays, including curvilinear projection, can be considered from the standpoint of the previous method, since the set of projecting rays is a congruence, and the additional condition is to represent the line that the rays must cross.

The determinant of the surface, which is formed by the application of geometric point transformations, includes the determinant of the transformation and the surface is the inverse image.

Is it possible in principle to generalize the analytical interpretations of the formulated constructive ways of surface formation and, if so, on what principles should it be implemented? As follows from the consideration of constructive methods for the formation of surfaces, their determinant requires special coordination of space corresponding to its elements. In general, the special coordinates t, u, v of space are functions of [1].

\[
\begin{align*}
x &= f(t, u, v), \\
y &= \phi(t, u, v), \\
z &= \psi(t, u, v)
\end{align*}
\]  

(1)

The singular points of space coordination by functions (1) are determined from condition
In the region of points where condition (2) is not satisfied, the functions (1) can be solved with respect to $t$, $u$, $v$

\[ t = f_i(x, y, z), \quad u = \phi_i(x, y, z), \quad v = \psi_i(x, y, z) \]  

(3)

How to get the functions (1) for the basic constructive ways of surface formation? Suppose that in the kinematic method the generator is incident on some surface on which $u$ and $v$ are curvilinear coordinates. This surface moves in space with one degree of freedom and $t$ is an independent parameter of this motion. In the process of motion, the points of the surface describe the congruence $\infty$ of trajectories with parameters $u$, $v$, and $t$ is a parameter of the position of the carrier of the generator, which at the same time is the parameter of the position of the current point on the trajectories. Since the carrier surface "sweeps away" a certain region of space by its motion, the functions (1) can be completely defined as the parametric equations of the family with the parameter $t$ of the carrier surfaces related to the curvilinear coordinates $u$, $v$. The coordination of space obtained in this way is convenient in that the functions (1) represent a wide class of kinematic surfaces with a representation on the surface of the support by the internal equation

\[ v = v(t, u) \]  

(4)

If $t$ occurs in the internal equation (4), this means that the generator has a variable form, but if it does not depend on $t$, the generator has a constant form. Since the rays of any projection form a congruence of lines, it is also modeled by equations (1). The equation of the surface is obtained as the equation of the family of rays projecting a line represented by parametric equations. Such an approach makes it possible to investigate and demonstrate features in the form of characteristic lines and lines of self-intersection of the surface, torso generators of ruled surfaces, whose linear framework consists of rays of non-traditional projection methods.

3. Results
One way of constructing surfaces of complex shape is to obtain their linear framework in the form of a set of lines of some congruence that project any line. Screw surfaces are widely used in engineering. To use computer graphics when visualizing them, you need to have their parametric equations. Conical screw projection was studied either in constructive teaching [1-3], or in connection with the transformation of congruences [4-10]. In the general case, the lines of congruence are obtained as the lines of intersection of direct helicoids with the family of surfaces of revolution, that is, these quasi-screw lines only in the case where the family consists of cones of revolution are transformed into conical helical lines. In [7,8,11], the formation of surfaces carrying isogonal trajectories of rectilinear generators of a family of cones is considered. In [12] such surfaces are called spiral.

The congruence of the conical helical lines of the constant step is represented by parametric equations

\[ x = vt \cos(u + \omega t), \quad y = vt \sin(u + \omega t), \quad z = at, \]  

(5)

where $\omega$ - the constant angular velocity of rotation of the point that describes the conical screw line around the $OZ$ axis, $a$ - a constant parameter related to the step $h$ of the conical screw lines by the
dependence $a = \frac{h}{2\pi}$; $t$, $u$, $v$ - the congruence parameters: $t$ is the rotation parameter, $u$ is the initial angular phase of the conical helical line, $v$ is the matching factor of the displacement velocities in the rotational and translational motion of the trajectory point relative to the OZ axis.

To refine the geometric content of the congruence parameter $v$, we reduce both sides of equations (5) to a square and find

$$\frac{x^2 + y^2}{z^2} = \frac{v^2}{a^2}$$

(6)

We denote by

$$\frac{v}{a} = \tan \alpha$$

(7)

With this notation, expression (6) takes the form

$$x^2 + y^2 - z^2\tan^2 \alpha = 0$$

(8)

Since $a$ - const, and $v$ - variable, $\tan \alpha$ is also variable, this can be seen from (7). Thus, (8) expresses the family of cones of revolution with a common axis OZ and a common vertex at the origin. To ensure the consistency of step $h$, which is related to $a$ by dependence $a = \frac{h}{2\pi}$, the velocity of translational motion $v$ of the trajectory point along the rectilinear generator is consistent with expression (7) with a variable angle of slope of the rectilinear generator of the cone to its axis in the family of cones of rotation (8).

We solve equation (5) with respect to $t$, $u$, $v$. In the region where the Jacobian of the functions (5)

$$\left| \begin{array}{ccc} x_t & y_t & z_t \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{array} \right| = -avt^2 \neq 0$$

(9)

to get a solution is possible. Solutions are impossible for $t = 0$ joint vertices of the family of cones (6), or for $v = 0$ at points located on the OZ ($\tan \alpha = 0$). We obtain solutions for the region where the OZ axis is excluded. From the third of equations (5) it follows that

$$t = \frac{z}{a}$$

(10)

From the first two equations (5) and from (10) we obtain

$$u = \arctan \frac{v}{x} - \frac{\omega z}{a}$$

(11)

Finally, from equation (2) we define $v$

$$v = \frac{a}{z} \sqrt{x^2 + y^2}$$

(12)

As we see, through an arbitrary point $M(x_m, y_m, z_m)$ there passes a unique conical helical line of congruence (5), and at this point the values of the congruence parameters $t$, $u$, $v$ are determined by substituting $x_u, y_u, z_u$ in (10), (11), (12) instead of $x, y, z$. Now let the point $M$ describe a line whose parametric equations:
To obtain the parametric equations of the surface, the line (9) projecting the conical screw lines of the constant step, it is necessary:

- replace in the right-hand parts (7), (8) x, y, z by the right-hand sides (9);
- substitute in (1) instead of u, v the right parts of the expressions obtained according to the previous paragraphs.

As a result, we obtain:

\[
x = a t \sqrt{(x(w))^2 + (y(w))^2} \cos[\arctg \frac{y(w)}{x(w)} - \frac{\omega z(w)}{a} + \omega t],
\]
\[
y = a t \sqrt{(x(w))^2 + (y(w))^2} \sin[\arctg \frac{y(w)}{x(w)} - \frac{\omega z(w)}{a} + \omega t],
\]
\[
z = at
\]

the parametric equations of the required surface. The family of coordinate lines \( w = \text{const} \) of this surface consists of conical helical lines.

4. Discussions

Consider examples of obtaining analytical and computer models of known helical surfaces that are studied in courses of descriptive and differential geometry.

Example. Compose the equations of the surface carrying the family of conical screw lines of a constant step.

\[
x = r \sin kw \cos w, \quad y = r \sin kws \sin w, \quad z = b.
\]

The equation of the desired surface is obtained by substituting the right-hand sides of (15) in (14):

\[
x = \frac{atra \sin k}{b} \cos(w - \frac{\omega b}{a} + \omega t),
\]
\[
y = \frac{atra \sin k}{b} \sin(w - \frac{\omega b}{a} + \omega t),
\]
\[
z = at.
\]

![Figure 1](image1.png) **Figure 1** Surface area with \( k = 3 \).

![Figure 2](image2.png) **Figure 2** Surface area with \( k = 1/2 \).
Figure 1 shows the surface area (16) at \( r=2.5, \ k=3, \ b=10, \ a=0.25, \ \omega=1.2, \ 0 \leq w \leq 2\pi, \ \pi \leq t \leq 4\pi. \)

In Figure 2, for the same values of the parameters \( r, b, a, \omega, w, t, \) but with \( k = 1/2 \) presented surface (16).

5. Conclusion AND Recommendations

The given analytical interpretation of the method of curvilinear projection by conical screw lines of a constant step provides the possibility of forming and visualizing surfaces by means of computer graphics, where the control of the shape of the surface occurs through the shape and position of the projecting line. The shown analytical apparatus for shaping complex-shaped surfaces by curvilinear projection of an arbitrary line in the general case represented by parametric equations has a transparent geometric basis and is free from parameter removal procedures, since only substitutions are used in it.

The materials of this article can be useful for graduate students, teachers studying constructive models of non-traditional projection systems. In the process of research, new questions and problems appeared that needed their solution. It is necessary to continue research into development of a general algorithm for the construction of parametric equations of the set (congruence) of rays of non-traditional projection systems and their projecting surfaces and apply it for projection systems of congruences of coaxial conical helical lines located on coaxial cones with a common vertex and a variable angle of inclination of the generator to the axis.

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