Singularities in String Theory

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Abstract

String theory is a quantum theory that reproduces the results of General Relativity at long distances but is completely different at short distances. Mathematically, string theory is based on a very new — and little understood — framework for geometry that reduces to ordinary differential geometry when the curvature is asymptotically small. In the 1990’s, many interesting results were obtained about the behavior of string theory in spacetimes that develop singularities. In many cases, the physics at the singularity is governed by an effective Lagrangian constructed using an interesting bit of classical geometry such as the association of A-D-E groups with certain hypersurface singularities or the ADHM construction of instantons. In other examples, the physics at the singularity cannot be described in classical terms but involves a non-Gaussian conformal field theory.

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1. Introduction

The classical Einstein equations

\[ R_{ij} = 0, \]

where \( R \) is the Ricci tensor of a metric \( g \) on spacetime, are scale-invariant. In other words, they are invariant under the scaling of the metric \( g \to tg \), with \( t \) a real number; the Ricci tensor is invariant under this scaling. A quantum theory of gravity, however, cannot have this symmetry, since quantum theory depends on the action, not just the field equations, and the Einstein-Hilbert action

\[ I = \frac{1}{16\pi G} \int d^n x \sqrt{g} R \]

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(\(G\) is Newton’s constant and \(n\) is the dimension of spacetime) is not invariant under scaling. In fact, under \(g \rightarrow tg\), the action scales as \(I \rightarrow t^{n/2-1}I\), and so is not scale-invariant in any dimension above two. (Two dimensions — a case that figures in string theory — is completely special as \(I\) is a topological invariant, the Gauss-Bonnet integral.)

Generally speaking, the classical limit of quantum mechanics arises by making a stationary phase approximation to a function space integral. That integral is very roughly of the form

\[
\int D\Phi \exp(iI/\hbar),
\]

where the integral runs over all fields \(\Phi\). For example, for General Relativity, \(\Phi\) would be a metric tensor on spacetime, perhaps together with other fields, and \(I\) would be the Einstein-Hilbert action defined in the last paragraph, together with a suitable action for the other fields, if any. Here \(\hbar\) is Planck’s constant, and the stationary phase approximation to the function space integral is valid when the appropriate \(I/\hbar\) is large. Note that in classical physics, \(I\) is not a dimensionless number but has units of “action” or energy times time; it is only with the passage to quantum mechanics that there is a natural constant of action, namely \(\hbar\), and it makes sense to say that in a given physical situation the action is large or small.

As an example of this criterion, consider black holes. A classical black hole can, because of the scale invariance, have any possible mass \(M\) or radius \(r\); in four dimensions, for example, the mass and radius are related (for neutral, unrotating black holes) by \(M = rc^2/G\), where \(c\) is the speed of light. The value of \(I/\hbar\) integrated over the relevant time, which is the time for light to cross the black hole, is

\[
\frac{I}{\hbar} = M \cdot \frac{r}{c} \cdot \frac{1}{\hbar} = \frac{GM^2}{\hbar c} = \frac{r^2c^3}{G\hbar}.
\]

The classical description of black holes is valid if this expression is large, or in other words if \(M \gg 10^{-5}\) gm or \(r \gg 10^{-33}\) cm.

Known astrophysical black holes have masses comparable to that of the sun (about \(10^{33}\) gm) and above, so unless we get lucky with mini-black holes left over from the Big Bang, we are not going to be able to observe what happens for black holes so light and small that the classical description fails. But curiosity compels us to ask how to describe a small black hole, or what happens near the Big Bang, where classical General Relativity breaks down for similar reasons to what I have just described for Black Holes.

Here we run into a problem. One can read a textbook recipe for quantization in Dirac’s old book or in more modern texts on quantum field theory. But these recipes, applied to the Einstein-Hilbert theory, do not work. Because of the highly nonlinear nature of Riemannian geometry, these methods fail to give a consistent and meaningful result.

This problem is very hard to convey to a mathematical audience because the whole question is about quantum field theory, which is not in clear focus as a
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The foundation for our modern understanding of elementary particles and forces is the success in quantizing theories such as Yang-Mills theory, whose action is

$$I = \frac{1}{4e^2} \int \text{Tr} F \wedge *F,$$

where $F$ is the curvature of a connection, $e$ is a real constant, known as the gauge coupling constant, and $\text{Tr}$ is an invariant quadratic form on the Lie algebra of the gauge group. Other somewhat analogous theories such as the quantum theory of maps from a Riemann surface to a fixed Riemannian manifold have also been extensively explored by physicists, with applications to both string theory and condensed matter physics. Apart from their central role in physics, quantum gauge theory and its cousins are the basis for the application of physical ideas to a wide range of mathematical problems, from the Jones polynomial to Donaldson theory and mirror symmetry.

But the understanding of quantum field theory that physicists have gained, though convincing and sufficient to make many computations possible, is hard to formulate rigorously. This makes it difficult for mathematicians to understand the questions of physicists, much less the partial results and approaches to a solution. In "constructive field theory," some of the standard physical claims about quantum field theory have been put on a rigorous basis, but there is still a long way to go to effectively bridge the gap.

For physicists, quantum gravity is an important problem not only because we would like to understand black holes, the Big Bang, and the quantum nature of spacetime — but also because reconciling General Relativity and quantum mechanics is necessary if we are to unify the forces of nature. We cannot achieve a unified understanding of nature if gravity is understood one way and the subatomic forces are understood by a different and incompatible theory. Moreover, the difficulty in reconciling these theories is probably our best clue about understanding physics at a much deeper level than we understand now. So what has been accomplished toward reconciling General Relativity and quantum mechanics?

### 2. String theory

Not much has been learned by direct assault. But roughly thirty years ago, in trying to solve another problem, physicists stumbled in "string theory" onto a very rich and surprising new framework for physics and geometry, which apparently does yield a theory of quantum gravity, though we do not understand it very well yet. String theory introduces in physics a new constant $\alpha' \sim (10^{-32} \text{ cm})^2$ (read "alpha-prime"), which is somewhat analogous to Planck’s constant $\hbar \sim 10^{-27} \text{ erg sec}$, and modifies the concepts of physics in an equally far-reaching way.

If string theory is correct, then both $\hbar$ and $\alpha'$ are nonzero in nature. The deformation of classical physics in turning on nonzero $\hbar$ is comparatively familiar to most in this audience, at least at the level of nonrelativistic quantum mechanics (as opposed to quantum field theory): classical concepts such as the position...
and velocity of a particle become “fuzzy” in the transition to quantum mechanics. Turning on $\alpha' \neq 0$ introduces an additional fuzziness in physics, roughly as a result of turning particles into strings. One aims to unify the forces by interpreting all of the different particles in nature as different vibrational states of one basic string.

String theorists spend the late 1980’s and early 1990’s largely studying the $\alpha'$ deformation. This work is hard to describe mathematically because it is all based on techniques of quantum field theory. Roughly it involves a new kind of geometry in which one is not allowed to talk about points or geodesics but one can talk about (quantum) minimal surfaces. This blurs all the classical concepts in geometry and makes possible nonclassical behavior. The new fuzziness has a characteristic scale $\sqrt{\alpha'} \sim 10^{-32}$ cm, and has many consequences. On much larger scales, just like the quantum uncertainty in gravity, the stringy fuzziness is unimportant. But it is important if one looks closely and can lead to nonclassical behavior, such as mirror symmetry.

A commonly encountered framework for nonclassical behavior in string theory is the following. Consider a family of classical solutions of string theory depending on several parameters; call the parameter space $\mathcal{N}$. At a generic point in $\mathcal{N}$, the stringy effects are important and one cannot usefully describe the situation in terms of a classical spacetime. As one approaches a special point $P \in \mathcal{N}$ (or more generally some locus in $\mathcal{N}$ of positive codimension), the relevant length scales in spacetime become large, the string effects become unimportant, and a classical spacetime $X$ emerges. ($P$ is typically a cusp-like point in $\mathcal{N}$; that is, there is typically a natural metric on $\mathcal{N}$, and $P$ is at infinite distance in this metric.) In that limit, the string equations reduce to the classical Einstein equations on $X$ (or more precisely, their appropriate supersymmetric extension, about which more later). To run what I have said so far in reverse, starting with a classical spacetime $X$ that is embedded in string theory, if the radius of $X$ (and every relevant length scale) is large compared to $\sqrt{\alpha'}$, then classical geometry is a good approximation to the stringy situation. But by varying the parameters so that the “radius” of $X$ is not large compared to $\sqrt{\alpha'}$, one can get a situation in which classical geometry is not a good approximation and must be replaced with stringy geometry. This situation is described by points in the interior of $\mathcal{N}$.

A more seriously nonclassical behavior arises if in addition to the point $P$, there are additional points $Q, R \in \mathcal{N}$ at which classical behavior again arises, but this time with different classical spacetimes $Y, Z$ of different topology. In this case, by moving in $\mathcal{N}$ from $P$ to $Q$, we can go smoothly from a situation in which classical geometry is a good approximation and the spacetime is $X$, to a situation in which classical geometry is again a good approximation but the spacetime is $Y$. For this process to occur smoothly even though the initial and final spacetimes have different topology, it inevitably happens that in interpolating from $P$ to $Q$, one has to pass through a region in which classical geometry is not a good approximation.

The example of what I have just described that is most discussed mathematically is that in which $X$ is a Calabi-Yau threefold, and, say, $Y$ is mirror to $X$ and
$Z$ is another Calabi-Yau manifold that is birational to $X$ or $Y$.

Because the characteristic length scale of stringy behavior, in the simplest way of matching string theory with the real world, is about $10^{-32}$ cm, way below the distance scale that we can probe experimentally, much of the structure of string theory, assuming it is right, is out of reach experimentally. Conceivably, we might one day be able to use string theory to calculate masses and interaction rates of the observed elementary particles, but this seems far off. It is also just possible that we might have the chance to observe one or another kind of massive string relic left over from the early universe. There is, however, another possibility that is much more likely for the immediate future: there is one important aspect of stringy geometry which may very well be accessible to experiments. This is “supersymmetry,” roughly the notion that spacetime is more accurately understood as a supermanifold, with both odd and even coordinates, rather than as an ordinary manifold. The theory of supermanifolds is more accessible and better known mathematically than some of the other things that I have mentioned. Most of the known geometrical applications of quantum field theory involve supersymmetry in one way or another.

If supersymmetry is relevant in nature, the oscillations of known particles in the “odd” directions in spacetime would give new elementary particles that could be discovered in accelerators. There are hints that these new particles exist at energies very close to what has already been reached experimentally. To me the most striking hint of this comes from the measured values of the strong, weak, and electromagnetic coupling constants, which are in excellent agreement with a prediction based on supersymmetric grand unification. If these hints have been correctly interpreted, we are likely to discover supersymmetric particles at accelerators in this decade, probably at the Fermilab accelerator in Illinois or at the Large Hadron Collider, which is being built at the European laboratory CERN near Geneva.

It is interesting to contemplate the impact on mathematics if supersymmetry is really discovered experimentally. When General Relativity emerged as an improvement on Newton’s theory of gravity, this gave a huge boost to the mathematical investigation of Riemannian geometry. Nonrelativistic quantum mechanics probably gave an equivalent boost to functional analysis. Quantum field theory is so multi-faceted that a simple summary of its mathematical influence is difficult; some aspects of quantum field theory have influenced mathematics considerably, but as I have explained, problems of rigor have kept the core ideas of this richest of physical theories inaccessible mathematically. Supersymmetry, I think, would fall somewhere in between. Its experimental discovery would greatly increase the interest of mathematicians in supermanifold theory, which is accessible mathematically, but the full impact on mathematics would be delayed because the real payoff of supersymmetry lies in the realm of quantum field theory and string theory.
3. Uniform breakdown of the “large, smooth” approximation

In recent years, the most significant development in string theory has been to understand some of the things that happen when both $\hbar$ and $\alpha'$ are important. One discovery is that the different models of string theory that we knew of in the 1980’s are related like the different classical spacetimes $X, Y$, and $Z$ that we discussed above. In an asymptotic expansion near $\hbar = 0$, they are different, but in a more complete description, the different string models arise as different semiclassical limits of one richer theory that has been dubbed $M$-theory. $M$-theory, though we do not really understand it yet, is thus the candidate for super-unification of the laws of nature.

Also, we have gained insight about what happens in many situations in which classical geometry breaks down. In the “large, smooth” limit of spacetime, in which all relevant length scales are large, classical geometry (enriched to include supersymmetry) is always a good approximation. But what happens when the “large, smooth” approximation breaks down?

Many interesting results were obtained in the 1990’s about situations in which the “large, smooth” approximation breaks down everywhere at once. I will give a few examples. These examples involve the basic ten-dimensional models of string theory, such as Type IIA and Type IIB superstrings and the heterotic string, and also the eleven-dimensional $M$-theory. Let $S^1(r)$ denote a circle of circumference $2\pi r$. Then our first example is the assertion for any ten-dimensional spin manifold $X$, Type IIA superstring theory on $X \times S^1(r)$ is equivalent to Type IIB superstring theory on $X \times S^1(\alpha'/r)$. If $r >> (\alpha')^{1/2}$, the description via Type IIA superstring theory is transparent as ordinary geometrical concepts are valid, while for small $r$ the second description is better. Starting on the Type IIA side at large $r$, the “large, smooth” description breaks down for $r \to 0$ (as there are closed geodesics of length $2\pi r$ in $X \times S^1(r)$), and the equivalence to Type IIB on $X \times S^1(\alpha'/r)$ gives a description that is valid when the Type IIA description has failed.

My other examples will relate an eleven-dimensional description via $M$-theory to a ten-dimensional string theory. With $X$ as before a ten-dimensional spin manifold and $Y$ a seven-dimensional spin manifold, and letting $K3(r)$ denote a K3 surface of radius $r$ and $I(r)$ a length segment of length $r$, and $T^3$ a three-torus, we have the following relations: (i) $M$-theory on $X \times S^1(r)$ is equivalent as $r \to 0$ to Type IIA superstring theory on $X$; (ii) $M$-theory on $Y \times K3(r)$ is equivalent as $r \to 0$ to the heterotic string on $Y \times T^3$; and $M$-theory on $X \times I(r)$ is equivalent for $r \to 0$ to the $E_8 \times E_8$ heterotic string on $X$. In each of these examples, the “large, smooth” approximation is valid for large $r$ (if $X$ and $Y$ are large enough) and breaks down for small $r$. In each example, the string coupling constant in the string theory description vanishes for $r \to 0$, so that the string theory description is useful in that limit — an asymptotic expansion valid for small $r$ can be explicitly worked out, giving a detailed answer to the question of what happens when the “large, smooth”
approximation fails. Note that these examples involve highly nonclassical behavior, with change in the topology and even the dimension of spacetime — for example, a four-dimensional K3 surface at large $r$ is replaced by a three-dimensional torus when $r$ becomes small.

Relations of this type are “quantum” analogs (involving both $\hbar$ and $\alpha'$) of mirror symmetry (which from this standpoint involves only $\alpha'$) and have led to the understanding that the different string models are different limits of the same things. Many other examples have been worked out in which the “large, smooth” approximation breaks down everywhere at once. I want to focus in the remaining time today, however, on another type of situation. This is the case in which, as some parameter is varied, the “large, smooth” approximation remains valid generically, but breaks down along some locus of codimension $d > 0$ where spacetime develops a conical singularity.

4. Behavior at conical singularities

The behavior of string theory when spacetime develops a conical singularity in positive codimension can be investigated by methods that exploit the fact that the “large, smooth” approximation remains generically valid, away from the singularity. One often can identify an interesting mathematical and physical phenomenon supported at the singularity. I will select examples in which both $\hbar$ and $\alpha'$ play an important role. There also are many instances of conical singularities that can be studied at $\hbar = 0$, such as the string theory orbifolds that have motivated one of the satellite meetings of ICM-2002. But we will focus on problems that involve both $\alpha'$ and $\hbar$. I will give three examples; two involve known mathematical constructions that appear in a new situation, while in the third the key phenomenon is nonclassical — it can only be formulated quantum mechanically.

I. M-Theory At An A-D-E Singularity: The A-D-E singularities are codimension four singularities that look locally like $\mathbb{R}^4/\Gamma$, where $\Gamma$ is a finite subgroup of $SU(2)$, acting on $\mathbb{R}^4 \cong \mathbb{C}^2$ and preserving the hyper-Kahler structure of $\mathbb{R}^4$. An extensive mathematical theory relates the A-D-E singularity to the A-D-E Dynkin diagram and many associated bits of geometry and algebra. However, the role of the A-D-E group in relation to the singularity is elusive. Like other singular spaces, the A-D-E singularity is usefully studied as a limit of smooth spaces carrying the appropriate structure. In this example, the singular space $\mathbb{R}^4/\Gamma$ has a hyper-Kahler resolution (due to Kronheimer) that contains exceptional divisors of area $A_1, \ldots, A_r$ which appear as parameters in the metric ($r$ is the rank of the relevant A-D-E group, and the intersection form of the divisors is minus the Cartan matrix of the group). In the context of M-theory or string theory, for $A_1, \ldots, A_r$ large, the “large, smooth” approximation is valid and classical geometry can be applied. We want to know what happens as $A_1, \ldots, A_r \to 0$, giving a singularity at the origin in $\mathbb{R}^4$. This is the basic A-D-E singularity in $\mathbb{R}^4$; in eleven-dimensional M-theory, we would usually be working on an eleven-dimensional spacetime $X$, and the singularity arises
on a codimension-four submanifold \( Q \). The answer has turned out to be that in this limit, A-D-E gauge fields — and their supersymmetric extension — appear on \( Q \). Assuming that the “large, smooth” approximation has failed only because of the A-D-E singularity, there is an effective description of the resulting physics that roughly speaking is governed by the action

\[
I = \int_X d^{11}x \sqrt{g} (R + \ldots) + \int_Q \Tr (F \wedge *F + \ldots),
\]

where \( R \) is the Ricci scalar, \( F \) is the curvature of the A-D-E connection, and “\( \ldots \)” refers to the supersymmetric extension. The supersymmetric extension of the gauge theory that is supported on \( Q \) turns out to automatically contain the variables needed to parametrize Kronheimer’s hyper-Kahler resolution of the singularity.

II. Gauge Theory Instantons: My second example involves the instanton solutions of four-dimensional Yang-Mills theory. Like the Einstein equations, the equations for Yang-Mills instantons, which read \( F = - *F \), where \( F \) is the curvature of a Yang-Mills connection on \( \mathbb{R}^4 \), are scale invariant. (In fact, they have the much stronger property of conformal invariance.) Instantons therefore come in all sizes. One can scale the size of an instanton all the way down to zero, giving, in the limit, a singular, point-like instanton. So (even on a compact four-manifold) instanton moduli space is non-compact. This raises the question, “What happens when an instanton becomes small?”

The answer to this question depends on what one is trying to do. I will describe three possible answers. (i) Instantons were introduced in quantum field theory in the mid-1970’s. Traditionally, physicists were interested in certain integrals over instanton moduli space. From this point of view, the meaning of the noncompactness of instanton moduli space is clear: one should make sure that the integrals of interest converge, and one should be careful when integrating by parts. (ii) In Donaldson theory, one is interested in intersection theory on instanton moduli space; “instanton bubbling” — the shrinking of an instanton to a point — is the main source of technical difficulty. One deals with it by a variety of technical means such as considering cycles in moduli space whose intersections avoid the bubbling region. (iii) In string theory, one expects the classical instanton equation \( F = - *F \) to be a good approximation for large instantons, this being an example of the validity of classical concepts in the “large, smooth” region. But one expects this description to break down as the instanton shrinks. The question here is to find a description that is valid for small instantons.

The instanton problem can be embedded in string theory in different ways, so there are several answers. I will give the answer in one case — Type I superstring theory or the \( SO(32) \) heterotic string. (At the very end of this talk, I briefly point out a second case.)

Before going on, I should mention one surprising part of the mathematical theory of instantons. This is the ADHM construction (due to Atiyah, Drinfeld, Hitchin, and Manin) of instantons in \( \mathbb{R}^4 \). To describe a \( k \)-instanton solution of
The ADHM construction employs an auxiliary $U(k)$ group. (For example, the instanton moduli space is constructed as a hyper-Kahler quotient of a linear space divided by $U(k)$.) The interpretation of this group is somewhat mysterious in classical geometry, just as the role of the A-D-E group in relation to the A-D-E singularity is somewhat mysterious classically.

Instanton bubbling occurs at a point in $\mathbb{R}^4$, so in gauge theory in any dimension, it occurs on a submanifold of codimension four. In ten-dimensional Type I superstring theory on a ten-manifold $X$, the small instanton thus appears on a codimension four submanifold $Q$. The answer to the small instanton problem turns out to be that the $U(k)$ group of the ADHM construction appears as a gauge group in the physics on $Q$. The effective action that governs this situation turns out to be schematically

$$I = \int_X d^{11} x \sqrt{g} (R + \ldots) + \int_Q \text{Tr} (F \wedge \ast F + \ldots),$$

where in this case $F$ is the curvature of a $U(k)$ connection, while $\ldots$ refers to additional terms required by supersymmetry plus the additional variables used in the ADHM construction to describe the moduli space as a hyper-Kahler quotient.

So once again, an interesting and surprising bit of classical mathematics becomes important near the singularity. I move on now, however, to an example in which the key phenomenon cannot be described in classical terms.

**III. Type IIB At An A-D-E Singularity:** Here we consider again the A-D-E singularity, but now in Type IIB superstring theory rather than in M-theory. The answer turns out to be completely different: we do not get a description with new classical degrees of freedom; instead a “non-trivial” or “non-Gaussian” conformal field theory is supported on the locus $Q$ of the A-D-E singularity. The assertion that this theory is “non-trivial” means that it exists as a conformally invariant quantum field theory, but cannot be conveniently described in terms of classical or Gaussian fields.

This particular nontrivial conformal field theory might be described as a “non-abelian gerbe theory”; it is related to two-forms in roughly the way that nonabelian gauge theory is related to one-forms. Classically, one-forms have a nonabelian generalization in gauge theory, but to find an analogous theory for two-forms one must apparently go to quantum theory. The existence and basic properties of this particular six-dimensional conformal field theory can be used to deduce Montonen-Olive duality of quantum Yang-Mills theory in four dimensions, and this in turn has implications for certain four-manifold invariants.

So this example again involves interesting mathematics, but to describe the result requires use of quantum concepts in a more intimate way. The same happens if we consider the small instanton problem in the $E_8 \times E_8$ heterotic string (rather than Type I or the $SO(32)$ heterotic string as considered above). There is no ADHM construction for $E_8$ instantons, so there is no candidate for an answer along the lines
sketched above for Type I; instead, a non-Gaussian conformal field theory appears on the small instanton locus $Q$.

A general orientation to the subject matter discussed in this lecture can be found in the second half of volume 2 of [1]. A few of the original research papers of relevance are [2] for the A-D-E singularities, [3] for the ordinary double point singularity in complex dimension three (we have not actually discussed this case in the present lecture, but it was important in the development of the ideas), and [4] for small instantons. In addition, I have discussed the small instanton problem from a different but related point of view in [5]. Some readers may also want to consult general expositions of quantum field theory, such as the recent textbook [6] for physicists, or the exposition aimed at mathematicians in [7]. Finally, a comparatively recent account of known rigorous results on quantum field theory can be found in [8].

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