ON SURFACES WITH PRESCRIBED SHAPE OPERATOR

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This article is dedicated to Shiing-Shen Chern, whose beautiful works and gentle encouragement have had the most profound influence on my own research.

Abstract. The problem of immersing a simply connected surface with a prescribed shape operator is discussed. From classical and more recent work (see [8] for a survey), it is known that, aside from some special degenerate cases, such as when the shape operator can be realized by a surface with one family of principal curves being geodesic, the space of such realizations is a convex set in an affine space of dimension at most 3. The cases where this maximum dimension of realizability is achieved have been classified and it is known that there are two such families of shape operators, one depending essentially on three arbitrary functions of one variable (called Type I in this article) and another depending essentially on two arbitrary functions of one variable (called Type II in this article).

In this article, these classification results are rederived, with an emphasis on explicit computability of the space of solutions. It is shown that, for operators of either type, their realizations by immersions can be computed by quadrature. Moreover, explicit normal forms for each can be computed by quadrature together with, in the case of Type I, by solving a single linear second order ODE in one variable. (Even this last step can be avoided in most Type I cases.)

The space of realizations is discussed in each case, along with some of their remarkable geometric properties. Several explicit examples are constructed (mostly already in the literature) and used to illustrate various features of the problem.

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1. Introduction

1.1. The fundamental forms. In classical surface theory in Euclidean space, given an immersion \( x : D \to \mathbb{E}^3 \) of a surface \( D \) into Euclidean 3-space, one can define its first fundamental form \( I_x = dx \cdot dx > 0 \) and, given a choice of unit normal \( n : D \to S^2 \) for \( x \) (i.e., \( n \cdot dx = 0 \) and \( n \cdot n = 1 \)), one can define its second fundamental form \( II_x = -dn \cdot dx \). The quantities \( I_x \) and \( II_x \) are unchanged if one composes \( x \) with an isometry of \( \mathbb{E}^3 \) and replaces \( n \) by its corresponding image under this isometry. Moreover, two normally oriented immersions \( x, y : D \to \mathbb{E}^3 \) agree up to isometry to second order at a point \( p \in D \) if and only if \( I_x(p) = I_y(p) \) and \( II_x(p) = II_y(p) \). Thus, the two quadratic forms \( (I_x, II_x) \) contain all of the second-order information about a normally oriented immersion \( x \) that is invariant under Euclidean isometries.

1.2. Bonnet’s theorem and rigidity. One of the most classical theorems in the subject is Bonnet’s theorem, which asserts that a given pair of quadratic forms \( (I, II) \) defined on a simply connected surface \( D \) can be realized by an immersion \( x \) with choice of unit normal \( n \) if and only if \( I \) is positive definite and the pair \( (I, II) \) satisfy the Gauss and Codazzi equations. Moreover, \( x \) and \( n \) (when they exist) are unique up to an isometry of \( \mathbb{E}^3 \). Since specifying a pair of quadratic forms on a surface is tantamount to choosing six arbitrary functions of two variables while choosing an immersion of the surface into \( \mathbb{E}^3 \) is tantamount to choosing three arbitrary functions of two variables, it is not surprising that there exist such compatibility conditions on pairs \( (I, II) \) in order that they be realizable by an immersion.

1.3. Isometric embedding. It is natural to look at problems that are not as overdetermined as Bonnet’s. For example, the problem of finding an immersion \( x \) that realizes a given positive definite quadratic form \( I > 0 \) as its first fundamental form is known as the isometric embedding problem and has a long history in differential geometry. The equation \( I_x = I \), regarded as an equation for \( x \), is determined in the naïve sense (i.e., it is three equations for three unknowns), but its behavior is rather subtle. It is known to be locally solvable when \( I \) is real-analytic or when the Gauss curvature of \( I \) is suitably non-degenerate, but the general smooth case is still unsolved.
1.4. **Prescribed second fundamental form.** In a different direction, in 1943 Élie Cartan studied the problem of realizing a given second fundamental form \( \mathbb{I} \).

In other words, he studied the equation \( \mathbb{I}_x = \mathbb{I} \), where \( \mathbb{I} \) is a given quadratic form. He showed that, when \( \mathbb{I} \) is real-analytic and non-degenerate, the equation \( \mathbb{I}_x = \mathbb{I} \) is always locally solvable. Little seems to be known about this problem in the smooth category or in the global setting. Possibly this is because, as Cartan showed, this problem is never elliptic and, in fact, has rather complicated characteristics.

1.5. **Bonnet surfaces.** One can imagine specifying other aspects of the data contained in \((\mathbb{I}, \mathbb{II})\). For example, if \( \kappa_1 \) and \( \kappa_2 \) are the eigenvalues of \( \mathbb{II} \) with respect to \( \mathbb{I} \), one can imagine trying to find an \( x \) that realizes a given \((\mathbb{I}, \kappa_1, \kappa_2)\).

This problem was first studied by Bonnet and then several other authors. Of course, since the Gauß equation asserts that \( K = \kappa_1 \kappa_2 \) where \( K \) is the Gauß curvature of \( \mathbb{I} \), this is an obvious necessary condition for realizability, so suppose that this holds. It turns out that, even with this condition, the generic data \((\mathbb{I}, \kappa_1, \kappa_2)\) cannot be realized by an immersion. This should be expected, since, even with the Gauß equation restriction, the given data depends essentially on four arbitrary functions of two variables, so some sort of compatibility condition is necessary.

The most thorough local analysis was done by Cartan [2], who showed that, for the generic data \((\mathbb{I}, \kappa_1, \kappa_2)\) (satisfying the Gauß equation) that does admit a normally oriented realization \((x, n)\), such a realization is unique up to isometry. This uniqueness fails for three special classes of data:

First, there exists a special class of data \((\mathbb{I}, \kappa_1, \kappa_2)\), depending on four arbitrary functions of one variable, for which there exist exactly two normally oriented realizations \((x_\pm, n_\pm)\) that are not Euclidean congruent. In the recent literature, these are called *Bonnet pairs* [11].

For the second and third classes of data \((\mathbb{I}, \kappa_1, \kappa_2)\) to be described below, there exists a 1-parameter family of normally oriented realizations \((x_\theta, n_\theta)\).

The second class consists of the data that are realizable by surfaces of constant mean curvature. In this case, the 1-parameter family containing a given immersion of constant mean curvature is just the classical circle of associated surfaces (most well-known in the case of mean curvature zero, i.e., the minimal surfaces). This class of data depends locally on two arbitrary functions of one variable.

The third class consists of the data realizable by a 6-parameter family of surfaces now known as the *Bonnet surfaces*. These are not as easy to describe geometrically, so the reader is referred to sources in the bibliography for further information, particularly Cartan’s article [2], Chern’s article [5], and the more recent article [1], where the relationship between these surfaces and Painlevé equations is explored.

1.6. **Cartan’s case studies.** In fact, there are a large number of possible problems one could study about the existence and uniqueness of normally oriented realizations of partial data drawn from the first and second fundamental forms. In his famous 1945 memoir [1] *Les systèmes différentiels extérieurs et leurs applications géométriques*, Cartan considered a number of these problems as illustrations of his methods. In particular, Chapitre VII is devoted to such problems and is still one of the best sources for information about them.

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1 In spite of the publication date, the reader might want to note that this memoir is actually based on the lecture notes of a course that Cartan gave in 1936-7 at the Faculté des Sciences de l’Université de Paris. These lectures were attended by S.-S. Chern, who was, at that time, a postdoctoral student.
1.7. Prescribed shape operator. On particularly natural object one can con-
struct from the data \((I, \mathbb{I})\) is the Weingarten shape operator. This is the linear
mapping \(S : TD \to TD\) defined by the relation
\[
\mathbb{I}(v, w) = I(v, Sw) = I(Sv, w).
\]
for any pair of tangent vectors \(v, w \in T_pD\). Since \(S\) is I-self-adjoint, it has real
eigenvalues (which are, of course, the principal curvatures of any normally oriented
realization) and is (pointwise) diagonalizable. There are no other pointwise condi-
tions on \(S\).

Conversely, given an endomorphism \(S : TD \to TD\) of the tangent bundle that
is pointwise diagonalizable, one can consider the problem of finding a normally
oriented immersion \((x, n)\) whose shape operator is \(S\). Since the choice of
\(S\) is tantamount to choosing a section of a bundle of rank 4 over \(D\), namely \(\text{End}(TD)\),
a shape operator essentially depends on four arbitrary functions of two variables.
Thus, one does not expect to be able to realize every possible \(S\) as a shape operator.

For example, if \(S\) has equal eigenvalues at every point, so that \(S = \kappa \text{id}_{TD}\) for
some function \(\kappa\) on \(D\), then \(S\) cannot be realized unless \(\kappa\) is a constant, since the
only totally umbilic surfaces in \(\mathbb{E}^3\) are planes and spheres. On the other hand, if \(\kappa\)
is constant, then \(S\) is realized by a normally oriented immersion of \(D\) into a plane
or sphere of appropriate radius. Thus, this case is trivial.

1.7.1. Umbilics and rectangularity. It is natural to define the points of
\(D\) at which \(S\) has two equal eigenvalues to be the umbilic points of \(S\). The presence of these
points complicates the discussion, so, for simplicity, I will assume that there are no
\(S\)-umbilic points. In this case, there will be two functions \(A > B\) on \(D\) so that \(A(p)\)
and \(B(p)\) are eigenfunctions of \(S_p : T_pD \to T_pD\) and, since \(D\) is simply connected,
there will exist two vector fields \(a\) and \(b\) on \(D\) with dual 1-forms \(\alpha\) and \(\beta\) so that
\[
S = A a \otimes \alpha + B b \otimes \beta.
\]
(1.7.2)
(Of course \(a\) and \(b\) are not unique.)
Locally, there exist coordinates \((x, y)\) on \(D\) in which \(S\) has the more specific form
\[
S = A(x, y) \frac{\partial}{\partial x} \otimes dx + B(x, y) \frac{\partial}{\partial y} \otimes dy.
\]
(1.7.3)
These two coordinates, the so-called \(S\)-principal coordinates, are each unique up to
reparametrization. If \(S\)-principal coordinates \((x, y)\) can be chosen globally on \(D\) in
such a way that \((x, y) : D \to \mathbb{R}^2\) embeds \(D\) as a coordinate rectangle in the \(xy\)-
plane, then the pair \((D, S)\) will be said to be rectangular.

Since the study conducted in this article will be almost entirely a local one, it
does no harm to restrict to the umbilic-free, rectangular case, so this will often be
assumed unless it is specifically stated otherwise.

Remark 1 (Computability 1). Given an endomorphism \(S : TD \to TD\), its eigenval-
ues and eigendirections can be computed algebraically, so that, when \(S\) has distinct
eigenvalues, the form (1.7.3) can be computed effectively. However, finding principal
coordinates \((x, y)\) explicitly when one is given an operator \(S\) in the form (1.7.2)
requires one to solve two coupled, nonlinear ordinary differential equations, some-
thing that cannot be done effectively unless \(S\) has special properties.

However, as will be seen, the computations that need to be done can be done
without the use of principal coordinates; they are merely a convenient expository
device. The replacement, as will be seen, is to use the eigenform decomposition
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of the 1-forms that \( S \) induces: Any 1-form \( \phi \) on \( D \) can be written uniquely in the form \( \phi = \phi' + \phi'' \) where \( \phi' \) is a multiple of \( \alpha \) and \( \phi'' \) is a multiple of \( \beta \). Correspondingly, there is a decomposition of the exterior derivative on functions: \( df = d'f + d''f \) where \( d'f = (df)' \) and \( d''f = (df)'' \). Of course, these two operators can be computed algebraically from \( S \), without recourse to differential equations.

1.7.2. Cartan’s non-uniqueness analysis. In Problème IX of Chapitre VII of [4], Cartan considers the generality of pairs of immersions \( x, y : D \to \mathbb{E}^3 \) that are noncongruent but induce the same shape operator. His analysis will be only summarized here. He shows that, modulo reparametrization, these pairs depend on six arbitrary functions of one variable. From this, he concludes that the ‘generic’ normally oriented immersion \((x, n)\) is uniquely characterized up to Euclidean congruence by its shape operator.

Recently, Ferapontov [7] has studied this non-uniqueness problem from the point of view of integrable systems and has shown that this problem (with some extra genericity hypothesis, to be described more fully below in Remark 3) is susceptible to being formulated as a Lax pair with a spectral parameter.

Cartan also shows that, if \( x : D \to \mathbb{E}^3 \) is an immersion that is free of umbilics and has the property that one of its families of principal curves is planar (as is the case, for example, for surfaces of revolution and, more generally, for the so-called molding surfaces), then the space of immersions \( y : D \to \mathbb{E}^3 \) that induce the same shape operator as \( x \) depends on one arbitrary function of one variable.

The condition of having one of the families of principal curves be planar is equivalent, in the local coordinate form (1.7.3) of the shape operator \( S \), to having either \( A_y = 0 \) or \( B_x = 0 \), i.e., one of the principal curvatures should be constant along the orthogonal family of principal curves.

Cartan does not mention the 1933 work of Finikoff and Gambier [9, 10], and perhaps he was unaware of it. Their work makes the same observations about the shape operators of surfaces with one family of principal curves being geodesics and they provide examples of shape operators that can be realized in a 3-parameter family of distinct ways. (They believed that they had a classification of such, but, as Ferapontov points out in [6], they missed an entire family, the one designated as Type I in this article.)

Other recent results on uniqueness and non-uniqueness for the prescribed shape operator problem can be found in [13] and [14]. The authors particularly study the case of surfaces of revolution and give examples that exhibit the non-uniqueness that shows up in Cartan’s analysis.

Ferapontov’s article [8] is a valuable source of information about the history of this problem, so the reader is referred there for more details.

1.7.3. New results. In this article, the non-uniqueness problem will be examined in detail and some new results will be proved about the explicit computability of shape operators with the maximum degree of flexibility in their realizations. The reader is reminded that the data \((D, S)\) is assumed to be umbilic-free, rectangular, and smooth.

First, there is the observation (see Proposition 1) that the space of congruence classes of normally oriented immersions \((x, n) : D \to \mathbb{E}^3\) that realize \( S \) has a natural affine structure in the sense that, if \((x_0, n_0)\) and \((x_1, n_1)\) both realize \( S \), then there is a naturally constructed family \((x_t, n_t)\) for \( 0 \leq t \leq 1 \) of normally oriented immersions defined up to Euclidean congruence that interpolates between
the two given immersions, and every immersion in this family realizes $S$ as its shape operator. Moreover, if the two given immersions are not Euclidean congruent, then any two distinct members of the family $(x_i, n_i)$ are mutually incongruent. This affine structure was implicit already in the works of Finikoff and Gambier.

Second, in the case where $A_y$ and $B_x$ are nonvanishing on $D$ (which is a generic condition), it turns out (see Theorem 3) that the space of Euclidean congruence classes of normally oriented immersions $(x, n) : D \to \mathbb{R}^3$ realizing $S$ can be naturally, affinely embedded as a convex set $X(S)$ in an affine space of dimension 3. Moreover, this convex set $X(S)$ will have an interior if and only if $A$ and $B$ satisfy a system $E(A, B) = 0$ of four highly nonlinear partial differential equations, two of order three and and two of order four.

It is not clear a priori that the overdetermined system $E(A, B) = 0$ has any solutions for which $A_y$ and $B_x$ are nonvanishing. Moreover, it is not difficult to show that this system is not involutive in Cartan’s sense, so further analysis is needed to understand the local and global solutions.

Again, this system was implicit in the work of Finikoff and Gambier, who first derived this upper bound on the dimension of the space of realizations of a given shape operator. However, it appears that their analysis of it was flawed, as they missed a family of solutions.

If one makes the additional assumption that $A$ and $B$ themselves are nonvanishing, then it is possible to reformulate the system $E(A, B) = 0$ in more geometric terms, so that its analysis becomes greatly simplified. The key, already noticed by Finikoff and Gambier, is to deal with the reciprocals $U = 1/A$ and $V = 1/B$ and then to define the coframing $\theta = (\theta_1, \theta_2)$, where

\[
(1.7.4) \quad \theta_1 = \frac{dV}{V - U}, \quad \theta_2 = \frac{d''U}{U - V}.
\]

The system $E(A, B) = 0$ turns out to be equivalent to a lower order (and much simpler) overdetermined system $E'(\theta) = 0$ for the coframing $\theta$. One then finds (see Theorem 4) that the system $E'(\theta) = 0$ is satisfied if and only if $\theta$ satisfies one of three possible determined, involutive systems. Once the solutions to $E'(\theta) = 0$ are described, the equations (1.7.4) can be regarded as a first order, linear hyperbolic determined system for two functions $U$ and $V$ and standard techniques can then be applied for its solution.

Using this simplification, one sees that (see Theorem 5) that the system $E(A, B) = 0$ is satisfied for $AB \neq 0$ if and only if $A$ and $B$ satisfy one of three possible determined, involutive systems.

The first two of these systems are exchanged by the operation of exchanging $(x, y)$ and $(A, B)$, so they can be regarded as essentially equivalent. Each of these systems consists of a second order equation and a third order equation. Operators $S$ that satisfy either of these systems will be referred to as being of Type I. This is the type that was missed by Finikoff and Gambier and first discovered by Ferapontov [6].

The third system is invariant under the exchange of $(x, y)$ and $(A, B)$ and consists of a pair of second order equations that forms a hyperbolic system for $A$ and $B$ in principal coordinates. Operators $S$ that satisfy this system will be referred to as being of Type II. These are the shape operators that were first found by Finikoff and Gambier.

\[\text{2Of course, assuming that } A_y \text{ and } B_x \text{ are nonvanishing already implies that the locus in } D \text{ where either } A \text{ or } B \text{ vanishes has no interior, so this is not a drastic assumption.}\]
Since each of the systems (2.5.23) and (2.5.21) is a determined, involutive system, local solvability is easy to demonstrate. Thus, there are many examples of shape operators of either type. In fact, several explicit examples are given in this article.

The remainder of the article is devoted to the analysis of the geometric properties of these two types of shape operators, particularly with an eye to the explicit computability of their realizations. Some of these results will now be described.

In the case of shape operators of Type I, it is shown (see §3.1.1) that there are essentially canonical principal coordinates \((x, y)\) on \(D\) in which the system that defines them can be linearized and explicitly integrated by the method of Darboux (see §3.1.7). Thus, these shape operators can be regarded as explicitly known. One finds that, modulo reparametrization, the shape operators of this type depend on three arbitrary functions of one variable. In fact, one can do much better than an integration by the method of Darboux: One can place the general shape operator solution in canonical principal coordinate form using only quadrature and the solutions of a single linear second order ODE.

Moreover, the structure equations show that if \(S\) is of Type I and \((x, n) : D \to \mathbb{R}^3\) is any realization, then the Gauß images of one family of principal curves are arcs of a 1-parameter family of spherical circles in \(S^2\) whose curve of centers lies on a geodesic. (see Proposition 2). This leads to an explicit integration (up to quadrature) of the differential equations that determine the realizations \(x\) of \(S\). Again, this is much sharper than merely being able to linearize the realization equations.

In fact, in the case that the spherical images of both families of principal curves are arcs of circles, the computation of the (local) realizations of \(S\) is reduced to a sequence of algebraic operations and quadratures in a manner analogous to the Weierstraß formula for minimal surfaces (§3.1.8).

In the case of shape operators of Type II, it is shown (see §3.2.1) that there are essentially canonical principal coordinates \((x, y)\) on \(D\) in which the system that defines them can be linearized. In this case, the resulting linear system is not integrable by the method of Darboux, though a representation due to Poisson can be invoked to express the general solution, which depends on two arbitrary functions of one variable.

What is particularly remarkable is that the structure equations show that if \(S\) is of Type II and \((x, n) : D \to \mathbb{R}^3\) is any realization, then the Gauß image of the net of principal curves is a net of confocal spherical ellipses in \(S^2\) (see Proposition 3). (This was known already to Finkoff and Gambier.) Again, this allows one to reduce the explicit integration of the differential equations that determine the (local) realizations \(x\) to a sequence of algebraic operations and quadratures analogous to the Weierstraß formula for minimal surfaces (§3.2.5).

Various examples of each Type are introduced and studied. Here are some highlights:

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3 The term *quadrature* in this article carries its classical meaning: Quadrature is the operation of finding a primitive for a given closed 1-form, i.e., given a closed 1-form \(\phi\) on a simply connected domain \(D\), quadrature constructs a function \(f\) on \(D\) so that \(df = \phi\). Symbolically, this is written \(f = \int \phi\). The classical authors regarded quadrature as an elementary operation and devoted much energy to finding ways to solve differential equations that only involved algebraic operations and quadrature.
An example (see Example 4) is given of a surface of Type I that has an isolated umbilic of index 0, thus showing that shape operator flexibility does not control the index of isolated umbilics.

The shape operators of either type that can be realized by minimal immersions are determined and the corresponding surfaces are described (see Example 5 and Example 8).

The quadric surfaces with distinct principal axes belong to Type II (see Example 6).

Complete examples of surfaces of either Type are given and compact convex examples are given of surfaces of Type II (see Example 3 and Example 7).

1.8. Acknowledgements. I want to thank Udo Simon for discussing his work and the work of Martin Wiehe on the problem of prescribed shape operators. It was those discussions that inspired the present article.

I must also thank Eugene Ferapontov, who read an earlier version of this article and pointed out that I had reproduced many of the results of Finnikoff, Gambier and himself in my analysis. I am very grateful to him for supplying me with references and giving me the chance to write a revised article in which proper historical credit is given.

I would also like to thank Editorial Board of Results in Mathematics for the opportunity to contribute to a volume honoring Shiing-Shen Chern, whose beautiful works on classical surface theory (and every branch of modern differential geometry as well) have inspired me throughout my career as a geometer. I offer this article, whose topic and outlook are inspired by Professor Chern’s wonderful article, as a small token of my gratitude for his profound effect on my mathematical life.

2. The Differential Analysis

The computations below will proceed by the method of the moving frame, so the basic notation will be introduced here, along with a few useful facts.

2.1. The structure equations. Let $D$ be a simply connected surface, let $x: D \to \mathbb{E}^3$ be an immersion, and let $n: D \to S^2$ be a choice of unit normal, i.e., $n \cdot n = 1$ and $n \cdot dx = 0$. The first and second fundamental forms are defined as before by

\begin{align}
I &= dx \cdot dx, \\
\Pi &= -dn \cdot dx.
\end{align}

The immersion will be assumed to be free of umbilics, i.e., that $\Pi$ has two distinct eigenvalues with respect to $I$ at every point. These eigenvalues (the principal curvatures) will be denoted $A$ and $B$ and it will be supposed that $A > B$ throughout $D$.

There exist 1-forms $\omega_1$ and $\omega_2$ on $D$ (unique up to a sign) so that $I$ and $\Pi$ are diagonalized as

\begin{align}
I &= \omega_1^2 + \omega_2^2, \\
\Pi &= A \omega_1^2 + B \omega_2^2.
\end{align}

The functions $A$ and $B$ are the principal curvatures. The corresponding shape operator $S$ is given by the formula

\begin{align}
S &= A u_1 \otimes \omega_1 + B u_2 \otimes \omega_2.
\end{align}

This does not mean that the surfaces are globally flexible keeping the shape operator fixed, but only that each point in the surface has a neighborhood that is flexible keeping the shape operator fixed. This may seem paradoxical, but it well illustrates the different natures of the local and global problems.
where \( u_1 \) and \( u_2 \) are the vector fields on \( D \) dual to the coframe field \((\omega_1, \omega_2)\). (Note that the sign ambiguity in the choice of \( \omega_1 \) and \( \omega_2 \) does not affect \( S \).) Once the forms \( \omega_1 \) and \( \omega_2 \) are chosen, there will exist unique smooth mappings \( e_1, e_2 : D \to S^2 \) so that
\[
(2.1.4) \quad dx = e_1 \omega_1 + e_2 \omega_2.
\]
The integral curves of the equation \( \omega_2 = 0 \) map to tangents to \( e_1 \) and are called the first family of principal curves while the integral curves of the equation \( \omega_1 = 0 \) map to tangents to \( e_2 \) and are called the second family of principal curves. The pair of foliations of \( D \) by the principal curves is called the net of principal curves induced by \( x \).

Setting \( e_3 = n \), the frame field \((e_1, e_2, e_3)\) is orthonormal, so the 1-forms \( \omega_{ij} = e_i \cdot de_j \) satisfy
\[
(2.1.5) \quad de_i = e_1 \omega_{1i} + e_2 \omega_{2i} + e_3 \omega_{3i}, \quad i = 1, 2, 3.
\]
They also satisfy
\[
(2.1.6) \quad \omega_{31} = A \omega_1, \quad \omega_{32} = B \omega_2,
\]
and there is a relation of the form
\[
(2.1.7) \quad \omega_{12} = u \omega_1 + v \omega_2
\]
for some functions \( u \) and \( v \) on \( D \). In fact, at any point \( p \in D \), \( u(p) \) is the geodesic curvature of the first principal curve passing through \( p \) while \( v(p) \) is the geodesic curvature of the second principal curve passing through \( p \).

These forms satisfy the structure equations
\[
(2.1.8) \quad d\omega_1 = -\omega_{12} \wedge \omega_2, \quad d\omega_2 = \omega_{12} \wedge \omega_1, \quad d\omega_{31} = -\omega_{12} \wedge \omega_{32}, \quad d\omega_{32} = \omega_{12} \wedge \omega_{31}, \quad d\omega_{12} = \omega_{31} \wedge \omega_{32}.
\]

Conversely, the essential content of Bonnet’s theorem is that, if \( \omega_1, \omega_2, \omega_{31}, \omega_{32}, \) and \( \omega_{12} \) are 1-forms defined on a simply connected surface \( D \), satisfy the equations \((2.1.8)\), and the relation \( \omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2 = 0 \) (which is a consequence of \((2.1.6)\)), then there exist mappings \( x : D \to \mathbb{E}^3 \) and \( (e_1, e_2, e_3) : D \to O(3) \) so that the structure equations \((2.1.4)\) and \((2.1.5)\) hold and that such mappings are unique up to composition with an isometry of \( \mathbb{E}^3 \). If, in addition, \( \omega_1 \wedge \omega_2 \) is nonvanishing on \( D \), then \( x \) is an immersion.

2.2. A first look. Suppose given an open rectangular domain \( D \) in the \( xy \)-plane and a smooth candidate for a shape operator
\[
(2.2.1) \quad S = A(x, y) \frac{\partial}{\partial x} \otimes dx + B(x, y) \frac{\partial}{\partial y} \otimes dy,
\]
where \( A > B \) throughout \( D \).

If \( S \) is realized by a normally oriented immersion \( x : D \to \mathbb{E}^3 \), there will exist a principal orthonormal frame field \( e = (e_1, e_2, e_3) : D \to O(3) \) so that the structure
equations

\begin{align}
\frac{dx}{\sqrt{a}}, \quad \frac{dy}{\sqrt{b}}, \quad \omega_{31} = \frac{A}{\sqrt{a}}, \quad \omega_{32} = \frac{B}{\sqrt{b}}.
\end{align}

(This way of parametrizing the possible structure forms, which may seem odd at first glance, turns out to lead to a linear inhomogeneous system of equations for \(a\) and \(b\).)

The structure equations \(d\omega_1 = -\omega_{12} \wedge \omega_2\) and \(d\omega_2 = \omega_{12} \wedge \omega_1\) imply that

\begin{align}
\omega_{12} &= -\frac{a_y \sqrt{b}}{\sqrt{a}} \frac{dx}{\sqrt{a}} + \frac{b_x \sqrt{a}}{\sqrt{b}} \frac{dy}{\sqrt{b}}.
\end{align}

The final structure equation \(d\omega_{12} = \omega_{31} \wedge \omega_{32}\) then yields a third (inhomogeneous) linear equation for \(a\) and \(b\):

\begin{align}
-2AB(A-B)^2 = (A-B)B_x a_x - 2 (B_x A_y - 2 B_x^2 + (B-A)B_x y) a
\quad + (B-A)A_y b_y - 2 (A_y B_y - 2 A_y^2 + (A-B)A_y y) b.
\end{align}

For general functions \(A\) and \(B\), the equations \((2.2.5)\) and \((2.2.7)\) define a system of three equations for the two unknown functions \(a\) and \(b\) that is incompatible, i.e., there will be no solutions.

Still, because these equations are linear, inhomogeneous equations, it follows that if \((a_0, b_0)\) and \((a_1, b_1)\) are positive solutions to \((2.2.3)\) and \((2.2.7)\), then setting

\begin{align}
(a_t, b_t) = ( (1 - t)a_0 + ta_1, (1 - t)b_0 + tb_1 )
\end{align}

for \(0 \leq t \leq 1\) defines a ‘segment’ of positive solutions. Since \(D\) is simply connected, the following result is a direct consequence of Bonnet’s theorem.

\textbf{Proposition 1.} The space of Euclidean congruence classes of normally oriented immersions \(\mathbf{x} : D \to \mathbb{R}^3\) that realize the shape operator \(S\) is a convex set in the affine space consisting of the solutions of the inhomogeneous linear system defined by \((2.2.5)\) and \((2.2.7)\).

There remains the question of determining conditions on \(A\) and \(B\) that will determine whether or not there exist any solutions to the system \((2.2.5)\) and \((2.2.7)\). A full analysis of their compatibility has to be broken into a number of cases. As will be seen, determining necessary and sufficient conditions for the existence of a single solution is likely to be rather complicated.
2.3. **Two elementary cases.** In the first place, if \( A_y = B_x \equiv 0 \) then (2.2.7) is incompatible unless one of \( A \) or \( B \) also vanishes. By symmetry, one can assume that \( A \equiv 0 \). Then (2.2.7) is an identity and the relations (2.2.5) are equivalent to the conditions

\[
\begin{align*}
a &= u(x) \\
b &= v(y)
\end{align*}
\]

for some positive functions \( u \) and \( v \) of a single variable. The corresponding surfaces are generalized cylinders and no further discussion is required.

In the second place, one could have exactly one of \( A_y \equiv 0 \) or \( B_x \equiv 0 \). Again, by symmetry it suffices to treat the case \( A_y \equiv 0 \) under the assumption that \( B_x \neq 0 \). Note that, geometrically, this condition corresponds to the case where the (principal) \( e_1 \)-curves are congruent planar geodesics. The discussion to be given below is essentially due to Cartan, who used somewhat different terminology and notation.

In this case, one must have \( a = u(x) \) for some positive function \( u \) of one variable, just as before, but now (2.2.7) is not an identity; instead, it becomes the inhomogeneous linear equation

\[
0 = u'(x) + 2 \left( \frac{B_x A_x - 2 B_x^2 + B_x (B - A)}{(B - A)B_x} \right) u(x) - 2 \frac{AB (A - B)^2}{(B - A)B_x} ,
\]

which, for notational simplicity, can be written in the form

\[
0 = u'(x) + f(x, y) u(x) + g(x, y).
\]

Any solution \( u \) to (2.3.3) must also satisfy its derivative with respect to \( y \):

\[
0 = f_y(x, y) u(x) + g_y(x, y).
\]

If \( f_y(x, y) \equiv 0 \) but \( g_y(x, y) \neq 0 \), then there is no solution to (2.3.4) and hence no realization of \( S \) as a shape operator.

If \( f_y(x, y) \) is non-zero, there can be no more than one solution to (2.3.4) and this may or may not be positive and may or may not satisfy (2.3.3). If there is no solution, then \( S \) cannot be realized as a shape operator. If there is a solution and it is either not positive or does not satisfy (2.3.3), then there is no positive solution to (2.3.3) and hence no realization of \( S \).

On the other hand, if (2.3.4) is an identity, then \( f \) and \( g \) depend only on \( x \), so that (2.3.3) is an ordinary differential equation for \( u \) that has a 1-parameter family of solutions. In particular, there must be positive solutions, at least in open \( x \)-intervals.

In any case, if (2.3.3) has a positive solution \( u \), the remaining equation to be satisfied is the homogeneous linear equation for \( b \)

\[
b_x = \frac{2 B_x}{(B - A)} b ,
\]

whose general positive solution is of the form \( b(x, y) = v(y) b(x, y) \) where \( b > 0 \) is any particular solution and \( v \) is an arbitrary positive function of one variable.

Thus, the positive solutions of (2.2.5) and (2.2.7) (when they exist) are seen to depend essentially on one arbitrary function of one variable (plus possibly one constant). It is in this sense that Cartan means his statement that the realizations of \( S \) in such cases depend on one arbitrary function of one variable.

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5 This happens, for example, in the case of surfaces of revolution, where both \( A \) and \( B \) are functions of \( x \) alone. As Cartan points out, it also happens for the more general case of molding surfaces.
For more information about these cases, along with an interesting discussion of examples, the reader can consult [13] and [14].

2.4. **The nondegenerate case.** Now consider the general case, where $A_y$ and $B_x$ are nonvanishing. In the notation of Remark 1, this is equivalent to the assumption that $d''A$ and $d'B$ are nonvanishing (and hence is checkable without having to find principal coordinates beforehand). Geometrically, this is equivalent to the condition that the principal curves of any realization $x: D \to \mathbb{E}^3$ should have nonvanishing geodesic curvature.

The relation (2.2.7) can now be expressed in the form

\[
\begin{align*}
    a_x &= \left(\frac{1}{B_x}\right) p - 2 \left(\frac{B_{xx}}{B_x} + \frac{2B_x - A_x}{A - B}\right) a + \frac{AB(A - B)}{B_x}, \\
    b_y &= \left(\frac{1}{A_y}\right) p - 2 \left(\frac{A_{yy}}{A_y} + \frac{2A_y - B_y}{B - A}\right) b + \frac{AB(A - B)}{A_y},
\end{align*}
\]

for some unknown function $p$.

Differentiating the first equation of (2.2.5) with respect to $x$ and the first equation of (2.4.1) with respect to $y$ and comparing the two expressions for $a_{xy}$ leads to an equation of the form

\[
p_{xy} = \frac{E_1 p + E_2 a + E_3}{A_y(A - B)^2}
\]

where $E_1$, $E_2$, and $E_3$ are certain polynomials in the terms

$A, A_x, A_y, B, B_x, B_y, B_{xx}, B_{xy}, B_{xxy}$

whose exact form will not be needed in the discussion below. Similarly, differentiating the second equation of (2.2.5) with respect to $y$ and the second equation of (2.4.1) with respect to $x$ and comparing the two expressions for $b_{xy}$ leads to an equation of the form

\[
p_{x} = \frac{E_4 p + E_5 b + E_6}{B_x(A - B)^2}
\]

where $E_4$, $E_5$, and $E_6$ are polynomials in the terms

$B, B_x, B_y, A, A_x, A_y, A_{yy}, A_{xy}, A_{xxy}$.

Considering the combined equations (2.2.5), (2.4.1), (2.4.2), and (2.4.3) as a total system of equations for the three unknowns $a, b$, and $p$, one sees that there is at most a three parameter family of solutions. In fact, a solution, if it exists, is uniquely determined by specifying the values of $a, b$, and $p$ at a single point $(x_0, y_0)$ in $D$. This, combined with Bonnet’s theorem, yields the following fundamental result.

**Theorem 1.** If $S$ satisfies $A_y \neq 0$ and $B_x \neq 0$, then the space of Euclidean congruence classes of normally oriented immersions $x: D \to \mathbb{E}^3$ that realize $S$ is a convex set in a vector space of dimension 3.

For general $A$ and $B$, the combined system (2.2.5), (2.4.1), (2.4.2), and (2.4.3) will not have any solutions at all. This article is devoted to understanding the exceptional case in which this combined system is Frobenius, i.e., for which it possesses a 3-parameter family of solutions.

By the usual Frobenius criterion, this is the case if and only if the two expressions for $p_{xy}$ got by differentiating (2.4.2) with respect to $y$ and by differentiating (2.4.3)
with respect to $x$ agree (after, of course, taking into account the full system of equations). Carrying out this comparison $(p_x)_y = (p_y)_x$ yields an equation of the form
\begin{equation}
E_7 a + E_8 b + E_9 p + E_{10} = 0
\end{equation}
where $E_7$, $E_8$, $E_9$, and $E_{10}$ are polynomials in $A$, $B$, and certain of their derivatives up to and including order 4. Thus, the Frobenius criterion is satisfied if and only if these four polynomial expressions vanish identically:
\begin{equation}
E_7 \equiv E_8 \equiv E_9 \equiv E_{10} \equiv 0.
\end{equation}
This is an overdetermined system $\mathcal{E}(A, B) = 0$ for the functions $A$ and $B$. In fact, $E_9 = 0$ and $E_{10} = 0$ are third order equations for $A$ and $B$ while $E_7$ and $E_8$ are fourth order. These four expressions are rather complicated. For example, $E_7$ and $E_8$ have 29 terms apiece, $E_9$ has 18 terms, and $E_{10}$ has 42 terms.

This system is not involutive, so that determining its space of solution requires further study. It is possible to study these equations directly, but it turns out that the analysis is simplified and made more geometric by making a change of variables and redoing the calculation up to this point. That is the subject of the next subsection.

2.5. A second look. Now suppose given a shape operator $S$ of the form (1.7.2) on a simply-connected surface $D$ (that is not necessarily rectangular with respect to $S$).

2.5.1. Nondegeneracy. I suppose, as in the previous subsection, that $A - B$ is non-vanishing on $D$ and that $d''A$ and $d'B$ are also nonvanishing on $D$. An operator $S : TD \to TD$ satisfying these conditions will be said to be nondegenerate.

2.5.2. Invertibility. In addition to implying that $d''A$ and $d'B$ are everywhere linearly independent on $D$, these hypotheses imply that the zero locus of $A$ (if non-empty) consists of smooth curves transverse to the second family of principal curves and that the zero locus of $B$ (if non-empty) consists of smooth curves transverse to the first family of principal curves. In particular, the open set in $D$ where $AB \neq 0$, i.e., where $S$ is invertible, is dense.

While the analysis below can be carried out in a neighborhood of a curve in $D$ along which $A$ and/or $B$ vanishes, this considerably complicates the discussion and does not seem to be worth the trouble. Thus, for simplicity of exposition, I am going to further impose the condition that $A$ and $B$ themselves be nonvanishing on $D$, i.e., that $S$ be invertible on all of $D$.

2.5.3. A canonical coframing. It will be useful to define two 1-forms
\begin{equation}
\theta_1 = \frac{A d'B}{B(B - A)}, \quad \theta_2 = \frac{B d''A}{A(A - B)}.
\end{equation}
Note that these two 1-forms constitute a coframing on $D$ that depends only on $S$ (and not on any choice of principal coordinates). In fact, $\theta_1$ and $\theta_2$ depend on one derivative of $S$.

More precisely, $\theta_1$ and $\theta_2$ are algebraic functions of the 1-jet of $S$. 

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6 More precisely, $\theta_1$ and $\theta_2$ are algebraic functions of the 1-jet of $S$. 

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It also turns out to be more convenient to work with the reciprocals of \(A\) and \(B\) than with \(A\) and \(B\) themselves. Thus, set

\[
(2.5.2) \quad U = \frac{1}{A}, \quad V = \frac{1}{B}.
\]

In terms of \(U\) and \(V\), the \(\theta_i\) have the expressions

\[
(2.5.3) \quad \theta_1 = \frac{dV}{V-U}, \quad \theta_2 = \frac{d'U}{U-V}.
\]

Since these forms are linearly independent, there exist unique functions \(K_1\) and \(K_2\) on \(D\) so that

\[
(2.5.4) \quad d\theta_1 = K_1 \theta_1 \wedge \theta_2, \quad d\theta_2 = K_2 \theta_2 \wedge \theta_1.
\]

The functions \(K_1\) and \(K_2\) depend on two derivatives of \(S\).

2.5.4. Derivation of the total differential system. Now, if \(S\) is to be induced by an immersion \(x : D \to \mathbb{E}^3\), there will exist a principal orthonormal frame field \(\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) : D \to \mathbb{O}(3)\) so that the structure equations (2.2.2a) and (2.2.2b) hold, where, for some positive functions \(a\) and \(b\) on \(D\), the structure forms satisfy

\[
(2.5.5) \quad \omega_1 = \frac{U \theta_1}{\sqrt{a}}, \quad \omega_{31} = \frac{\theta_1}{\sqrt{a}},
\]

\[
\omega_2 = \frac{V \theta_2}{\sqrt{b}}, \quad \omega_{32} = \frac{\theta_2}{\sqrt{b}}.
\]

Computing as in the previous sections, one finds that the structure equations for \(d\omega_1, d\omega_2, d\omega_{31}, d\omega_{32},\) and \(d\omega_{12}\) are equivalent to the condition that

\[
(2.5.6) \quad \omega_{12} = -\frac{\sqrt{b}}{\sqrt{a}} \theta_1 + \frac{\sqrt{a}}{\sqrt{b}} \theta_2
\]

where \(a\) and \(b\) satisfy the equations

\[
(2.5.7) \quad da = (2a + p + 1) \theta_1 + 2a (1 - K_1) \theta_2,
\]

\[
db = 2b (1 - K_2) \theta_1 + (2b - p + 1) \theta_2.
\]

for some function \(p\) on \(D\).

Before these equations can be differentiated, it will be necessary to introduce derivatives of the functions \(K_i\) in the form

\[
(2.5.8) \quad dK_1 = K_{11} \theta_1 + K_{12} \theta_2, \quad dK_2 = K_{21} \theta_1 + K_{22} \theta_2.
\]

Now, taking the exterior derivative of the equations (2.5.7) shows that the exterior derivative of \(p\) must be given by

\[
(2.5.9) \quad dp = ((2 - K_2) (p - 1) + 2(K_1 - K_2 - K_1 K_2 + K_{22}) b) \theta_1
\]

\[
+ ((2 - K_1) (p + 1) + 2(K_1 - K_2 + K_1 K_2 - K_{11}) a) \theta_2.
\]

Remark 2 (An uncoupling). The reader should note the very interesting fact that equations (2.5.7) and (2.5.9) do not involve \(U\) or \(V\) directly, but are expressed solely in terms of the coframing \(\theta\) and its derivatives. This is important for two reasons:

First, the coframing \(\theta\) contains less information than the operator \(S\) and so has fewer local invariants. This is useful because the local compatibility conditions for

\footnote{Obviously, these equations are got from the original equations by ‘rescaling’ the original \(a\) and \(b\). This rescaling greatly simplifies the calculations, a fact that was only noticed in hindsight.}
can now be expressed in terms of the local invariants of the coframing \( \theta \), as will be seen below.

Second, because the system \((2.5.7)\) and \((2.5.9)\) is expressed in terms of the first and second derivatives of \( \theta \), its compatibility conditions will be expressed in terms of third derivatives of \( \theta \), which is much simpler than fourth derivatives of \( S \). Thus, it is a lower order problem in these terms.

2.5.5. The Frobenius condition. Introducing the functions \( K_{ijk} \) by the equations

\[
dK_{ij} = K_{ij1} \theta_1 + K_{ij2} \theta_2,
\]

the exterior derivative of each side of \((2.5.9)\) can now be computed. The result is that the following inhomogeneous linear relation among \( a \), \( b \), and \( p \) must hold

\[
0 = 8 - 2K_1 - 2K_2 - 4K_1K_2 + 3K_{11} + 3K_{22}
\]

\[
-2((1+K_2)K_{11} - (1-K_1)K_{21} - K_{111})a
\]

\[
+2((1-K_2)K_{12} - (1+K_1)K_{22} + K_{222})b
\]

\[
+3(K_{11} - K_{22} - 2K_1 + 2K_2)p.
\]

Thus, the necessary and sufficient condition on the operator \( S \) in order that the combined system given by \((2.5.7)\) and \((2.5.9)\) be Frobenius (and hence have a three parameter family of solutions) is that the following four equations hold:

\[
0 = 8 - 2K_1 - 2K_2 - 4K_1K_2 + 3K_{11} + 3K_{22},
\]

\[
0 = K_{11} - K_{22} - 2K_1 + 2K_2,
\]

\[
0 = (1+K_2)K_{11} - (1-K_1)K_{21} - K_{111},
\]

\[
0 = (1-K_2)K_{12} - (1+K_1)K_{22} + K_{222}.
\]

For the rest of this section, equations \((2.5.12)\) will be assumed.

Remark 3 (Non-Frobenius cases). Although this article will contain no further discussion of the non-Frobenius case, it might be helpful to the reader to have some remarks about how the analysis can be continued to classify the operators for which the space of solutions to \((2.5.7)\) and \((2.5.9)\) has dimension less than 3.

In the first place, if the relation \((2.5.11)\) is non-trivial, then the space of solutions \((a, b, p)\) to \((2.5.7)\) and \((2.5.9)\) has dimension at most 2.

In the general case, where the coefficients of \( a \), \( b \), and \( p \) in \((2.5.11)\) are not identically vanishing, applying the exterior derivative to \((2.5.11)\) and taking the coefficients of \( \theta_1 \) and \( \theta_2 \) on the right hand side will yield two more relations of the form

\[
0 = A_1 a + B_1 b + P_1 p + Q_1,
\]

\[
0 = A_2 a + B_2 b + P_2 p + Q_2.
\]

where the coefficients \( A_i \), \( B_i \), \( P_i \), and \( Q_i \) are polynomials in \( K_1 \), \( K_2 \) and their coframing derivatives up to order 4.

The combined linear system \((2.5.11)\) and \((2.5.13)\) will generically have a unique solution \((a, b, p)\). This solution may or may not have \( a \) and \( b \) positive on \( D \) and, even if they are positive, this solution may not satisfy \((2.5.7)\) and \((2.5.9)\). If \( a \) and \( b \) are positive and the system \((a, b, p)\) does satisfy \((2.5.7)\) and \((2.5.9)\), then, by Bonnet’s Theorem, \( S \) can be realized as a shape operator and in only one way.

\( ^8 \) more precisely, on the coframing \( \theta \)
However, it can happen that the combined equations (2.5.11) and (2.5.13) are a linearly dependent system, admitting either a one-dimensional or two-dimensional space of (algebraic) solutions.

For example, it admits a two-dimensional space of solutions if and only if the equations in (2.5.13) are multiples of the equation (2.5.11). In this case, the combined system (2.5.7) and (2.5.9) is Frobenius when restricted to the relation (2.5.11). As a result, the combined system (2.5.7) and (2.5.9) has a two dimensional space of solutions. In this case, any point of $D$ where there is a solution to (2.5.11) with $a$ and $b$ positive at the given point has a neighborhood on which $S$ can be realized as a shape operator and in a two-parameter family of ways. Note that the condition that the equations (2.5.13) be multiples of the equation (2.5.11) is a system of fourth order PDE for the coframing $\theta$. This system has now been partially analyzed, but the description of its space of solutions is complicated. This will be the subject of a future article.

Finally, if the combined system (2.5.11) and (2.5.13) has rank one, then a further differentiation will test whether the restriction of (2.5.7) and (2.5.9) to the solutions of these relations is Frobenius. If this is the case and there is a solution with $a$ and $b$ positive, then $S$ can be realized as a shape operator in a one-parameter family of ways. This condition is seen to be a set of fifth order PDE for the coframing $\theta$, but has not yet been fully analyzed. However, Cartan’s analysis of the non-uniqueness part of the problem mentioned in §1.7.2 can be applied in this case to show that the space of such operators must depend on four arbitrary functions of one variable. (The right count is four, not six, because two arbitrary functions are lost in the passage from $S$ to the coframing $\theta$.) In fact, Ferapontov [7] has shown that this system can be cast into the standard framework of an integrable system described as a Lax pair with a parameter.

2.5.6. Consequences. It is not at all clear how many nondegenerate invertible operators $S$ there are that satisfy the four conditions (2.5.12). The following analysis will show that these conditions imply one of three possible determined systems.

The first two of the equations (2.5.12) do not involve the $K_{ijk}$ and can be written in the form

$$K_{11} = \frac{2}{3}(K_1 - 1)(K_2 + 2), \quad K_{22} = \frac{2}{3}(K_1 + 2)(K_2 - 1).$$

These two equations have an important consequence: The first can be written in the form

$$d(K_1 - 1) \equiv \frac{2}{3}(K_2 + 2)(K_1 - 1) \mod \theta_1.$$

It thus follows, by the uniqueness of solutions of ordinary differential equations, that $K_1 - 1$ vanishes at a point of $D$ if and only if it vanishes along the entire first principal curve passing through that point. Similarly, the second equation implies that $K_2 - 1$ vanishes at a point of $D$ if and only if it vanishes along the entire second principal curve passing through that point.

Using the equations (2.5.14) to eliminate $K_{11}$ and $K_{22}$, the formula (2.5.13) for $dp$ can be simplified so that it does not involve any of the $K_{ij}$. Using this new formula to recompute the identity $d(dp) = 0$ then yields the relation

$$0 = (K_1 - 1)(3K_{21} + 2K_2^2 + 2K_2 - 4)a + (K_2 - 1)(3K_{12} + 2K_1^2 + 2K_1 - 4)b,$$
which must be an identity, i.e., the coefficients of \( a \) and \( b \) must vanish identically. Thus, equations \( 2.5.12 \) imply the second order system formed by \( 2.5.14 \) and
\[
(K_1 - 1)(3K_{21} + 2K_2^2 + 2K_2 - 4) = 0,
\]
\[
(K_2 - 1)(3K_{12} + 2K_1^2 + 2K_1 - 4) = 0.
\]

The analysis must now be broken into a few separate cases.

2.5.7. A genericity assumption. First, consider the open set \( D_0 \subset D \) on which \((K_1 - 1)(K_2 - 1)\) is nonzero. Then by \( 2.5.17 \), the following equations must hold on \( D_0 \):
\[
K_{12} = -\frac{4}{3}(K_1 + 2)(K_1 - 1), \quad K_{21} = -\frac{4}{3}(K_2 + 2)(K_2 - 1).
\]
Combined with \( 2.5.14 \), this gives the formulae
\[
dK_1 = +\frac{4}{3}(K_1 - 1)(K_2 + 2) \theta_1 - \frac{4}{3}(K_1 + 2)(K_1 - 1) \theta_2,
\]
\[
dK_2 = -\frac{4}{3}(K_2 + 2)(K_2 - 1) \theta_1 + \frac{4}{3}(K_1 + 2)(K_2 - 1) \theta_2.
\]
Taking the exterior derivative of these equations yields the relations
\[
(K_1 - 1)(K_1 + 2)(K_2 + 2) = (K_2 - 1)(K_1 + 2)(K_2 + 1) = 0.
\]
Since \((K_1 - 1)(K_2 - 1)\) is nonvanishing on \( D_0 \), it follows that \((K_1 + 2)(K_2 + 2)\) vanishes identically.

In fact, both \((K_1 + 2)\) and \((K_2 + 2)\) must vanish identically on \( D_0 \). For example, if \((K_1 + 2)\) were nonzero at a point of \( D_0 \), then \((K_2 + 2)\) must vanish on a neighborhood of this point. The second equation of \( 2.5.19 \) then gives a contradiction since the left hand side vanishes on this neighborhood but the coefficient of \( \theta_2 \) on the right hand side cannot be zero. Similarly, \((K_2 + 2)\) must vanish at every point of \( D_0 \). Thus, \( K_1 \) and \( K_2 \) are each constant and equal to \(-2\) on \( D_0 \). It then follows from the connectedness of \( D \) that either \( D_0 \) is either empty or equal to all of \( D \).

In summary, if \( D_0 \) is nonempty, then \( D_0 = D \), and \( 2.5.4 \) simplifies to
\[
d\theta_1 = -2 \theta_1 \land \theta_2, \quad d\theta_2 = -2 \theta_2 \land \theta_1.
\]
Moreover, the total differential system for \( a, b, \) and \( p \) simplifies to the system
\[
da = (2a + p + 1) \theta_1 + 6a \theta_2,
\]
\[
(b = 6b \theta_1 + (2b - p + 1) \theta_2,
\]
\[
dp = -4(2b - p + 1) \theta_1 + 4(2a + p + 1) \theta_2,
\]
which, in view of \( 2.5.21 \), is Frobenius.

No further information can be gained by differentiating the equations \( 2.5.21 \) or \( 2.5.22 \), since these merely yield identities. In \( \S 7.2 \) the systems of this type will be explicitly described.

2.5.8. A special assumption. On the other hand, suppose that \( D_0 \) is empty, i.e., that \((K_1 - 1)(K_2 - 1)\) vanishes identically on \( D \). Let \( R \subset D \) be any open principal rectangle in \( D \).

Suppose that \((K_1 - 1)\) does not vanish identically on \( R \). Then, by the remark following \( 2.5.15 \), there exits a principal curve in the first family on which \((K_1 - 1)\) is nowhere vanishing. Thus, \((K_2 - 1)\) must vanish identically on this curve and so, again by the remark following \( 2.5.15 \), \((K_2 - 1)\) must vanish identically on \( R \). (Since \( R \) is a principal rectangle, every principal curve from the first family meets every principal curve of the second family.)
Similarly, if \((K_2 - 1)\) does not vanish identically on \(R\), then \((K_1 - 1)\) must vanish identically on \(R\).

Since \(D\) is the union of its open principal rectangles, it follows that \(D\) is the union of two open sets:

1. \(D_1\), on which \((K_1 - 1)\) vanishes identically, and
2. \(D_2\), on which \((K_2 - 1)\) vanishes identically.

Consider the open set \(D_1\). Since \((K_1 - 1)\) vanishes identically on \(D_1\), it follows that \(K_{11}\) and \(K_{12}\) must also vanish identically on \(D_1\). Thus, both equations in (2.5.17) are satisfied, so the \(abp\) system is Frobenius.

To summarize, on the open set \(D_1\), the following structure equations hold

\[
\begin{align*}
\text{d}\theta_1 &= \theta_1 \wedge \theta_2, \\
\text{d}\theta_2 &= K_2 \theta_2 \wedge \theta_1, \\
\text{d}K_2 &= K_{21} \theta_1 + 2(K_2 - 1) \theta_2.
\end{align*}
\]

(The last equation follows from \(K_2 = 1\) and (2.5.14).) Moreover, the total differential system for \(a\), \(b\), and \(p\) simplifies to the system

\[
\begin{align*}
\text{d}a &= (2a + p + 1) \theta_1, \\
\text{d}b &= -2b(K_2 - 1) \theta_1 + (2b - p + 1) \theta_2, \\
\text{d}p &= -2b(K_2 - 2)(p - 1) \theta_1 + (2a + p + 1) \theta_2,
\end{align*}
\]

which, in view of (2.5.23), is Frobenius. Consequently, on the open set \(D_2\), the following structure equations hold

\[
\begin{align*}
\text{d}\theta_1 &= K_1 \theta_1 \wedge \theta_2, \\
\text{d}\theta_2 &= \theta_2 \wedge \theta_1, \\
\text{d}K_1 &= 2(K_1 - 1) \theta_1 + K_{12} \theta_2.
\end{align*}
\]

(The last equation follows from \(K_1 = 1\) and (2.5.14).) Moreover, the total differential system for \(a\), \(b\), and \(p\) simplifies to the system

\[
\begin{align*}
\text{d}a &= (2a + p + 1) \theta_1 - 2a(K_1 - 1) \theta_2, \\
\text{d}b &= + (2b - p + 1) \theta_2, \\
\text{d}p &= -(2b - p + 1) \theta_1 + (2a - (K_1 - 2)(p + 1)) \theta_2,
\end{align*}
\]

which, in view of (2.5.25), is Frobenius.

**Remark 4 (Symmetry of cases).** Switching the two families of principal curves switches the two open sets \(D_1\) and \(D_2\) in \(D\). Thus, in studying the local geometry of the operators \(S\) that satisfy one or the other of the structure equations (2.5.23) or (2.5.25), it suffices to study one of the two cases.

However, the reader should bear in mind that it is possible for \(D_1\) and \(D_2\) to each be a proper, nonempty subset of \(D\), even though \(D\) is connected. See Example 2.

However, if \(D\) itself is a \(\theta\)-rectangular, then one of the two is the whole of \(D\). In the general case, on the (nonempty) overlap \(D_1 \cap D_2\), both structure equations hold and the corresponding differential systems for \(a\), \(b\), and \(p\) simplify even further.

In conclusion, the calculations made so far have established the following result:

---

\textsuperscript{9} The reader should not jump to the conclusion that \(D_i\) is equal to the locus where \((K_i - 1)\) vanishes. It is only being asserted that the interiors of the two zero loci form a covering of \(D\).
Theorem 2. If $S$ is an invertible, nondegenerate operator on a connected domain $D$ for which the differential system for $a$, $b$, and $p$ is Frobenius, then either $D$ is the union of two open sets $D_1$ (on which the equations (2.5.23) hold) and $D_2$ (on which the equations (2.5.25) hold), or else the equations (2.5.21) hold throughout $D$.

Conversely, if $S$ is an invertible, nondegenerate operator on a domain $D$ for which one of these three systems of structure equations hold, then the differential system for $a$, $b$, and $p$ is Frobenius.

Corollary 1. If $S$ is an invertible, nondegenerate operator on a domain $D$ whose associated coframing $\theta$ satisfies either (2.5.23), (2.5.25), or (2.5.21), then every point of the domain $D$ has an open neighborhood on which $S$ can be realized by a 3-parameter family of mutually noncongruent immersions.

Proof. Since the differential system for $a$, $b$, and $p$ is Frobenius in any of these cases, one can choose initial values $(a_0, b_0, p_0)$ of $a$, $b$, and $p$ arbitrarily at the specified point and have a (unique) solution to the system, globally defined on a simply connected neighborhood. Choosing $(a_0, b_0, p_0)$ in a bounded region in the quarter-space defined by $a_0 > 0$ and $b_0 > 0$ will then yield a congruence class of immersions realizing $S$ on a (possibly smaller) fixed neighborhood of the given point in $D$.

Remark 5 (Locality). As will be seen in the examples below, the restriction to an open neighborhood in Corollary 1 is necessary. The point is that, even if the chosen initial values $(a_0, b_0, p_0)$ satisfy $a_0 > 0$ and $b_0 > 0$, there is no guarantee that the corresponding solutions $a$ and $b$ (which, since the $abp$ system is linear, are globally defined if $D$ is simply connected) will be positive throughout $D$.

Definition 1 (The two types). A nondegenerate, invertible operator $S$ on a domain $D$ will be said to be of Type I if it satisfies either (2.5.23) or (2.5.25) and will be said to be of Type II if it satisfies (2.5.21).

3. Integrating the Structure Equations

In this final section, the problem of actually integrating the equations derived in the previous section will be addressed.

3.1. Operators of Type I. In this subsection, the operators $S$ of Type I will be studied and it will be shown how to integrate the equations that define them.

The definitions of the previous section attach a coframing $\theta = (\theta_1, \theta_2)$ on $D$ to any invertible, nondegenerate operator $S : TD \to TD$. The Type I conditions on $S$ are then expressed in terms of this coframing.

For simplicity, only the cases where either (2.5.23) or (2.5.25) holds throughout $D$ will be considered. Since these two sub-types differ only in which family of principal curves is designated first or second, it suffices to consider only one case. Thus, it will further be assumed that $S$ satisfies (2.5.23).

3.1.1. Natural principal coordinates. The first task is to find a normal form for the coframings $\theta = (\theta_1, \theta_2)$ on a domain $D$ that satisfy (2.5.23).

It is convenient to extend some terminology to (2-dimensional) domains $D$ endowed with a coframing $\theta$. The (connected) integral curves of $\theta_2 = 0$ will be said to be principal curves of the first family while the (connected) integral curves of $\theta_1 = 0$ will be said to be principal curves of the second family. A subdomain $R \subset D$ will
be said to be \( \theta \)-rectangular if each principal curve of the first family meets each principal curve of the second family in a unique point and each principal curve of the second family meets each principal curve of the first family in a unique point. Note that each point of \( D \) has a rectangular open neighborhood, even though this may not be computable by quadrature (say).

Lemma 1. Suppose that \( \theta = (\theta_1, \theta_2) \) is a coframing on a domain \( D \) for which there exists a function \( K_2 \) so that \((\theta_1, \theta_2, K_2) \) satisfies \((2.5.23)\). If \( D \) is \( \theta \)-rectangular, then there exist functions \( x \) and \( z \) so that \((d \theta_1) \wedge (d \theta_2) = (1 - K_2) \theta_1^2 \theta_2 \), \( d\theta_1 = x \), \( d\theta_2 = dz - (\mu(x) z^2 + 1) \frac{dx}{z} \), \( K_2 = 1 - \mu(x) z^2 \), where \( \mu \) is a function of a single variable defined on the range of \( x \). The functions \( x \) and \( z \) are unique up to a replacement \((x, z) \mapsto (\lambda x + \tau, \lambda z)\), where \( \lambda > 0 \) and \( \tau \) are constants.

Conversely, if \( \mu \) is a differentiable function defined on an interval \( I \subset \mathbb{R} \), then the formulae \((3.1.1)\) define two 1-forms \( \theta_1 \) and \( \theta_2 \) and a function \( K_2 \) on the domain \( D = I \times \mathbb{R}^+ \subset \mathbb{R}^2 \) that satisfy \((2.5.23)\).

Remark 6 (A quadratic form). Note that the quadratic form \( \mu(x) (dx)^2 = (1 - K_2) \theta_1^2 \) is independent of the choice of local coordinates \((x, z)\).

Proof. Choose a principal curve from the first family and use its intersections with the principal curves from the second family as initial points to construct\(^{10}\) via integration a function \( w \) on \( D \) so that \( dw \equiv \theta_2 \mod \theta_1 \). Now set \( z = \exp(w) > 0 \), so that \((dz)/z \equiv \theta_2 \mod \theta_1 \).

Since \( d\theta_1 = \theta_1 \wedge \theta_2 \) (the first equation of \((2.5.23)\)), it follows that

\[(3.1.2) \quad d(z \theta_1) = dz \wedge \theta_1 + z \theta_1 \wedge \theta_2 = (dz - z \theta_2) \wedge \theta_1 = 0.\]

Thus, one can find (by quadrature) a function \( x \) on \( D \) so that \( z \theta_1 = dx \), i.e., so that \( \theta_1 = (dx)/z \). The function \( x : D \to \mathbb{R} \) is a principal coordinate and its fibers are the principal curves of the second family. In particular, the mapping \((x, z) : D \to \mathbb{R}^2 \) embeds \( D \) as a domain in \( \mathbb{R}^2 \) that lies inside \( x(D) \times \mathbb{R}^+ \).

Note that if \((dz)/z = (dX)/Z \) for some functions \( X \) and \( Z > 0 \), then \( X = f(x) \) for some function \( f \) of one variable with positive derivative and \( Z = f'(x)z \). Conversely, for any function \( f : x(D) \to \mathbb{R} \) with \( f' > 0 \), the functions \((X, Z) = (f(x), f'(x), z)\) satisfy the conditions \((dZ)/Z \equiv \theta_2 \mod \theta_1 \) and \( \theta_1 = (dX)/Z \).

Now, since \( d(K_2) \equiv 2(K_2 - 1) \theta_2 \mod \theta_1 \) (by the third equation of \((2.5.23)\)), it follows that

\[(3.1.3) \quad d((K_2 - 1)/z^2) = (2(K_2 - 1) \theta_2)/z^2 + (K_2 - 1)((-2/z^2) \theta_2) \equiv 0 \mod \theta_1 ,\]

so there exists a function \( \mu \) defined on \( x(D) \subset \mathbb{R} \) so that \( K_2 = 1 - \mu(x) z^2 \).

Next, by construction, \( \theta_2 = (d\theta_2)/z \) is a multiple of \( \theta_1 = (dx)/z \) and, moreover, since \( d\theta_2 = K_2 \theta_2 \wedge \theta_1 \) (the second equation of \((2.5.23)\)), it follows that the 1-form

\(^{10}\)This is what is classically known as a ‘quadrature with parameters’, since it is done by integrating with respect to one variable while carrying the other along as a ‘parameter’. Note that it is not necessary to know the integral curves of \( \theta_1 \) explicitly to carry out this construction.
\[ \theta_2 - (dz)/z - (K_2-1)\theta_1 \] is closed. Since this 1-form is closed and a multiple of \(dx\), it must be of the form \(-\nu(x)\,dx\) for some function \(\nu\) defined on \(x(D) \subseteq \mathbb{R}\). Thus,

\[
(3.1.4) \quad \theta_2 = \frac{dz - \left(\mu(x)z^2 + \nu(x)z + 1\right)dx}{z}.
\]

Now, consider any coordinate system \((X, Z) : D \to \mathbb{R}^2\) in which

\[
(3.1.5) \quad \theta_1 = \frac{dX}{Z}, \quad \theta_2 = \frac{dZ - (\mu(X)Z^2 + \nu(X)Z + 1)dX}{Z}
\]

for some functions \(\mu\) and \(\nu\) on the range of \(X\). As has been noted already, there is an \(f : x(D) \to X(D)\) so that \(X = f(x)\) and \(Z = f'(x)z\). Using this to compare the two formulae for \(\theta_2\) yields

\[
(3.1.6) \quad \mu(f(x)) = \frac{\mu(x)}{f'(x)^2} \quad \text{and} \quad \nu(f(x)) = \frac{1}{f'(x)} \left(\nu(x) - \frac{f''(x)}{f'(x)}\right).
\]

Thus, choosing \(f : x(D) \to \mathbb{R}\) so that \(f' > 0\) satisfies \(f''(x) = \nu(x)f'(x)\) (which can be done by a sequence of two quadratures) yields a coordinate change \((X, Z) = (f(x), f'(x)z)\) for which \(\nu(X) \equiv 0\). This yields the normal form of the lemma.

Note that if \(\nu(X) \equiv 0\) and \(\nu(x) \equiv 0\), then \(f''(x) \equiv 0\), so that \(f\) is of the form \(f(x) = \lambda x + \tau\) for some constants \(\lambda > 0\) and \(\tau\), as claimed in the lemma.

Finally, that the 1-forms and function defined in (3.1.1) do satisfy (2.5.23) can be safely left to the reader. \(\square\)

**Remark 7 (Computability 2).** Note that the desired normal form can be computed by (parametrized) quadrature. However, even this step can be eliminated if either \(K_2 - 1\) is nowhere vanishing or is everywhere vanishing.

In the first place, if \(K_2 - 1\) is nowhere vanishing, then the function \(z > 0\) that needs to be found first can simply be taken to be \(z = |K_2 - 1|^{1/2}\), as this function satisfies the requirement that \(dz \equiv z\theta_2 \mod \theta_1\). Then a single quadrature constructs \(x\) so that \(\theta_1 = (dx)/z\). This alternative construction provides coordinates \((x, z)\) that satisfy \(\theta_1 = (dx)/z\) and \(\theta_2 = (dz - (\pm z^2 + \nu(x)z + 1)dx)/z\), instead of the normal form of Lemma [1]. The chief drawback of this normal form is that it cannot be constructed on a neighborhood of a point where \(K_2 - 1\) vanishes.

On the other hand, if \(K_2 - 1\) vanishes identically, then, by the structure equations, \(\theta_1 + \theta_2\) is closed and hence can be written in the form \((dz)/z\) by ordinary quadrature, thus directly furnishing the desired \(z\).

**Remark 8 (Type I generality).** One can interpret Lemma [1] as saying that, up to local equivalence, the coframings of Type I depend on one arbitrary function of one variable. It is tempting to regard \(\mu\) as the arbitrary function that ‘parametrizes’ such coframings, but one must bear in mind that it is not \(\mu\) itself, but the quadratic form \(\mu(x)(dx)^2\) coupled with the ‘flat’ affine structure on the space of second principal curves provided by Lemma [1] that provide the distinguishing invariants.

Now, the reader will have noticed that \(x\) is a principal coordinate, but \(z\) is not. In fact, a second principal coordinate cannot be constructed by quadrature in general. However, the following procedure will ‘construct’ such a coordinate:
Let $\phi_0$ and $\phi_1$ be linearly independent solutions on the interval $x(D)$ of the second order ordinary differential equation
\begin{equation}
(3.1.7) 
\phi''(x) + \mu(x) \phi(x) = 0.
\end{equation}
Of course, the Wronskian $\phi_0 \phi_1' - \phi_1 \phi_0' = 1$ is constant. If $\phi_0 \phi_1' - \phi_1 \phi_0' = 1$, the pair $(\phi_0, \phi_1)$ will be said to be normalized. Any two normalized pairs differ by a unimodular change of basis, and henceforth $(\phi_0, \phi_1)$ will denote a normalized pair of solutions to (3.1.7) unless it is explicitly stated otherwise.

I claim that there is a solution $\phi_1$ of (3.1.7) that is positive and increasing on all of $x(D)$. To see this, consider a fixed principal curve $\Gamma$ in $D$ of the first family. Because $D$ is $\theta$-rectangular, $\Gamma$ will be mapped by $(x, z)$ to a graph over $x(D)$ of the form $z = f(x)$. Since $\theta_2$ vanishes on $\Gamma$, it follows that $f : x(D) \to \mathbb{R}^+$ must satisfy the equation
\begin{equation}
(3.1.8) 
f'(x) = 1 + \mu(x) f(x)^2.
\end{equation}
Now let $\phi_1$ be a solution of the linear ODE
\begin{equation}
(3.1.9) 
\phi'_1(x) = \frac{1}{f(x)} \phi_1(x)
\end{equation}
that is positive somewhere (and hence everywhere) on $x(D)$. By construction,
\begin{equation}
(3.1.10) 
\phi''_1(x) = - \frac{f'(x)}{f(x)^2} \phi_1(x) + \frac{1}{f(x)} \left( \frac{1}{f(x)} \phi_1(x) \right) = - \mu(x) \phi_1(x),
\end{equation}
and $\phi_1$ has the desired properties. Let $\phi_0$ then be chosen so that the pair $(\phi_0, \phi_1)$ is normalized and so that $\phi_0'$ vanishes somewhere in $x(D)$. Then the $\theta$-rectangularity of $D$ implies that $\phi_0(x) - z \phi_0'(x)$ is nonvanishing on $D$.

Now, define a new function $y$ on $D$ by the formula
\begin{equation}
(3.1.11) 
y = \frac{\phi_1(x) - z \phi_1'(x)}{\phi_0(x) - z \phi_0'(x)}.
\end{equation}
This relation can be solved for $z$ in the form
\begin{equation}
(3.1.12) 
z = \frac{\phi_1(x) - y \phi_0(x)}{\phi_1'(x) - y \phi_0'(x)}.
\end{equation}
One then finds that
\begin{equation}
(3.1.13) 
\theta_2 = - \frac{dy}{(\phi_4(x) - y \phi_0(x))(\phi_1'(x) - y \phi_0'(x))}.
\end{equation}
Thus, $y$ is the desired second principal coordinate. Moreover, it follows that $y : D \to \mathbb{R}$ is a submersion onto an interval $y(D) \subset \mathbb{R}$ and that $(x, y) : D \to x(D) \times y(D)$ is a diffeomorphism. Note that, by construction, $(x, y)$ carries $\Gamma$ into the segment $y = 0$.

Note also that the functions $(\phi_1(x) - y \phi_0(x))$ and $(\phi_1'(x) - y \phi_0'(x))$ are positive on $D$, a fact that will be useful below for further constructions.

Principal coordinates $(x, y)$ found in this manner will be referred to as natural principal coordinates for the coframing $\theta$.

Example 1 (When $\mu$ is constant). The reader will find the study of the cases where $\mu$ is constant to be particularly interesting. When $\mu$ is a constant, the formulae (3.1.1) define a coframing of Type I on the upper half of the $xz$-plane. In no case is the

---

\footnote{Note that, for general $\mu$, these two solutions cannot be constructed by quadrature.}
The general solution of this equation is easily seen to be
\[
\phi(x) = \frac{\mu(x)}{\mu(x)} + \phi_0(x),
\]
where
\[
\phi_0(x) = \frac{\mu(x)}{\mu(x)} + \phi_0(x)
\]
but
\[
\phi_0(x) = \frac{\mu(x)}{\mu(x)} + \phi_0(x)
\]
are easily described in the coordinates \((x,z)\) locally of Type I, but so that \(D_1\) and \(D_2\) (as defined in [2.5.8]) are each nonempty proper subsets of \(D\). Of course, such a domain cannot be \(\theta\)-rectangular. Here is how this can be done:

First, let \(\mu : \mathbb{R} \to \mathbb{R}\) be a smooth function that satisfies \(\mu(x) = 0\) for \(x \geq 0\) but \(\mu(x) < 0\) for \(x < 0\). Let \(\phi_0\) be the function on \(\mathbb{R}\) that satisfies \(\phi_0'' + \mu \phi_0 = 0\) and \(\phi_0(x) = 1\) for \(x \geq 0\) and let \(\phi_1\) be the function on \(\mathbb{R}\) that satisfies \(\phi_1'' + \mu \phi_1 = 0\) and \(\phi_1(x) = x\) for \(x \geq 0\). Of course, this is a normalized pair \((\phi_0, \phi_1)\) for the equation \(\phi'' + \mu \phi = 0\). Note that because of the sign of \(\mu\) on the negative reals, \(\phi_0\) is decreasing and concave up on the negative reals while \(\phi_1\) is increasing and concave down on the negative reals.

Now consider the coframing of Type I

\[
\begin{align*}
\theta_1 &= \frac{\phi_1'(x) - y \phi_0'(x)}{\phi_1(x) - y \phi_0(x)} \, dx \\
\theta_2 &= \frac{-dy}{(\phi_1(x) - y \phi_0(x))(\phi_1'(x) - y \phi_0'(x))},
\end{align*}
\]

that is smooth and well-defined on the open domain \(D_1 \subset \mathbb{R}^2\) defined by the inequalities \(y < 0\) when \(x \geq 0\) and

\[
\frac{\phi_1'(x)}{\phi_0'(x)} < y < \frac{\phi_1(x)}{\phi_0(x)}
\]

when \(x < 0\). Note that, when \(y < 0 < x\), i.e., in the fourth quadrant of the plane, the above formulae simplify to

\[
\begin{align*}
\theta_1 &= \frac{dx}{x - y}, \\
\theta_2 &= \frac{dy}{y - x}.
\end{align*}
\]

Thus, the involution \(\Phi : \mathbb{R}^2 \to \mathbb{R}^2\) defined by \(\Phi(x,y) = (-y, -x)\), which preserves the fourth quadrant, satisfies \(\Phi^* \theta_1 = \theta_2\) and \(\Phi^* \theta_2 = \theta_1\) there.

Finally, let \(D_2 = \Phi(D_1)\) and extend \(\theta_1\) and \(\theta_2\) to \(D = D_1 \cup D_2 = \Phi(D)\) in the obvious way so that \(\Phi^*\) exchanges \(\theta_1\) and \(\theta_2\) globally on \(D\).

Since \(\mu\) is nonzero on the negative real axis, the function \(K_1 - 1\) is nonvanishing when \(y > 0\) and the function \(K_2 - 1\) is nonvanishing when \(x < 0\). Thus, this is the desired example.

3.1.2. Integrals of the Frobenius system. The integrals of the system for \(a, b,\) and \(p\) are easily described in the coordinates \((x,z)\) of Lemma [4]. One finds that the following formulae hold

\[
\begin{align*}
a &= -1 + f(x) \\
b &= -(1 + \mu(x)z^2)f(x) + z f'(x) - \frac{1}{2}z^2 f''(x) \\
p &= 1 - 2f(x) + z f'(x)
\end{align*}
\]

where \(f\) is any solution of the equation

\[
f''(x) + 4\mu(x)f'(x) + 2\mu'(x)f(x) = 0.
\]

The general solution of this equation is easily seen to be

\[
f(x) = -c_0 \phi_0(x)^2 - 2c_1 \phi_0(x)\phi_1(x) - c_2 \phi_1(x)^2
\]
where \((\phi_0, \phi_1)\) is a normalized pair of solutions of (3.1.7) and \(c_0, c_1, \) and \(c_2\) are arbitrary constants.

In terms of natural principal coordinates as described above, the formulae for \(a\) and \(b\) simplify to

\[
\begin{align*}
a &= -1 - c_0 \phi_0(x)^2 - 2c_1 \phi_0(x)\phi_1(x) - c_2 \phi_1(x)^2, \\
b &= \frac{(c_0 + 2c_1 y + c_2 y^2)}{(\phi_1'(x) - y \phi_0(x))^2}.
\end{align*}
\]

In this form, it is not difficult to understand how to choose the constants \(c_i\) so that \(a\) and \(b\) will be positive at a given point of \(D\). In particular, these constants must satisfy \(c_1^2 - c_0 c_2 > 0\) or else it will be impossible for \(a\) and \(b\) to be positive simultaneously.

Note also that, because \(z\) is strictly positive on \(D\), the expression \(\phi_1(x) - y \phi_0(x)\) cannot vanish. This implies that, at any point of \(D\), the allowable values of \((c_0, c_1, c_2)\) for which \(a\) and \(b\) will be positive at the specified point consists of the (non-empty) intersection of two open half-spaces (with non-parallel bounding planes).

Conversely, it is not difficult to see that, for any given values of \(c_0, c_1, \) and \(c_2\), the set of points of \(D\) at which both \(a\) and \(b\) are nonvanishing, nowhere equal functions on \(D\) are principal coordinates on a \(D\) that is of Type I.

3.1.3. Recovering \(S\). So far, the discussion in this section has shown how one can write down 1-forms \(\theta_1\) and \(\theta_2\) and a function \(K_2\) on a domain \(D\) satisfying (2.5.23). However, it is not immediate whether or not such a system necessarily comes from a nondegenerate, invertible operator \(S\) defined on \(D\), and, if so, ‘how many’ such operators \(S\) there are. It is now time to address this question.

By definition, the desired \(S\), if it exists, will be of the form

\[
S = \frac{1}{U} \ t_1 \otimes \theta_1 + \frac{1}{V} \ t_2 \otimes \theta_2,
\]

where \(t_1\) and \(t_2\) are the vector fields on \(D\) dual to the coframing defined by the 1-forms \(\theta_1\) and \(\theta_2\) and where \(U\) and \(V\) are nonzero and nowhere equal functions on \(D\) that satisfy

\[
\begin{align*}
dU &= U_1 \theta_1 + (U - V) \theta_2, \\
dV &= (V - U) \theta_1 + V_2 \theta_2,
\end{align*}
\]

for some functions \(U_1\) and \(V_2\) on \(D\). Conversely, by the very definitions of \(\theta_1\) and \(\theta_2\), if \(U\) and \(V\) are nonvanishing, nowhere equal functions on \(D\) that satisfy (3.1.22) for some functions \(U_1\) and \(V_2\), then (3.1.21) defines an invertible, nondegenerate operator on \(TD\) that is of Type I.

The system (3.1.22) constitutes a pair of linear, first order partial differential equations for \((U, V)\). In fact, this is a hyperbolic system whose characteristics are the principal curves, i.e., the level curves of \(x\) and \(y\). For example, if \((x, y): R \rightarrow \mathbb{R}^2\) are principal coordinates on a \(\theta\)-rectangle \(R \subset D\), then there exist nonvanishing functions \(s\) and \(t\) on \((x, y)(R)\) so that

\[
\begin{align*}
\theta_1 &= s(x, y) \, dx, \\
\theta_2 &= t(x, y) \, dy
\end{align*}
\]

\[\text{[12]}\text{The corresponding formula for } p \text{ is not as simple, but will not be needed in what follows.}\]
In these local coordinates, the equations (3.1.22) become the coupled linear system of PDE
\[
\begin{align*}
\frac{\partial U}{\partial y} &= t(x, y) (U - V), \\
\frac{\partial V}{\partial x} &= s(x, y) (V - U).
\end{align*}
\]
(3.1.24)

This system is visibly hyperbolic, with \(x\) and \(y\) being the characteristic directions. Standard existence theorems ensure that there exist solutions, locally. In fact, one can specify \(U\) and \(V\) arbitrarily along a noncharacteristic curve \(\Gamma\) in \(R\) (i.e., a curve that is everywhere transverse to the principal curves) and there will be an neighborhood of \(\Gamma\) in \(R\) on which a solution to (3.1.24) exists and assumes the prescribed values on \(\Gamma\). In particular, if one specifies \(U\) and \(V\) so that they are unequal and nonvanishing along \(\Gamma\), then the pair \((U, V)\) will be a solution with the desired properties.

However, it is not necessary to appeal to such theorems to prove existence. It turns out that the system (3.1.22) is integrable by the method of Darboux, as will now be explained.

Assume that a solution \((U, V)\) to (3.1.22) exists and compute the exterior derivatives of the equations (3.1.22), using the structure equations (2.5.23) and the equations (3.1.22) themselves. The result can be written in the form
\[
\begin{align*}
0 &= (dU_1 - 2U_1 \theta_2 + (K_2 - 1) dU) \wedge \theta_1, \\
0 &= (dV_2 - (K_2 + 1)V_2 \theta_1) \wedge \theta_2,
\end{align*}
\]
(3.1.25)

The first of these equations is just
\[
\begin{align*}
dU_1 - 2U_1 \theta_2 + (K_2 - 1) dU \equiv 0 \mod \theta_1
\end{align*}
\]
(3.1.26)

Using the coordinates \((x, z)\) guaranteed by Lemma 1, this equation takes the form
\[
\begin{align*}
dU_1 - 2U_1 \frac{dz}{z} - \mu(x) z^2 dU \equiv 0 \mod dx.
\end{align*}
\]
(3.1.27)

Dividing this equation by \(z^2\), it can be rewritten in the form
\[
\begin{align*}
d \left( \frac{U_1}{z^2} - \mu(x) U \right) \equiv 0 \mod dx.
\end{align*}
\]
(3.1.28)

Thus, there must exist a function \(\xi\) defined on \(x(D)\) such that
\[
\begin{align*}
U_1 = \left( \mu(x) U + \xi(x) \right) z^2.
\end{align*}
\]
(3.1.29)

For the second equation of (3.1.25), if one uses the local normal form to expand the right hand side and makes the substitutions
\[
\begin{align*}
z &= \left( \phi_1(x) - y \phi_0(x) \right) \\
\phi'_1(x) - y \phi'_0(x),
\end{align*}
\]
(3.1.30)

for some new variable \(Q\), then this equation becomes
\[
\begin{align*}
0 &= - \left( \phi_1(x) - y \phi_0(x) \right) (dQ \wedge dy).
\end{align*}
\]
(3.1.31)

13 The reader will note that this discussion of recovering \(U\) and \(V\) from \(\theta\) does not depend on \(\theta\) satisfying the conditions for Type I (or any conditions, for that matter). Thus, any coframing \(\theta\) is locally realizable as the coframing of some invertible, nondegenerate operator \(S\).
Since the scalar factor in this equation is nonvanishing on \( D \), it follows that \( dQ \wedge dy = 0 \). In other words, there is a function \( \eta \) defined on \( y(D) \) such that

\[
V_2 = (\phi_1(x) - y \phi_0(x))^2 \left( \frac{\phi_1'(x) - y \phi_0'(x)}{\phi_1'(x) - y \phi_0'(x)} \right) \eta(y). 
\]

Substituting the relations (3.1.29) and (3.1.32) back into (3.1.22) yields a differential system for \( U \) and \( V \) that is Frobenius for any choice of \( \xi \) and \( \eta \). In fact, this system can be integrated explicitly in the form

\[
U = f(x) - \frac{g(y)}{\phi_1'(x) - y \phi_0'(x)} - \left( \frac{\phi_1(x) - y \phi_0(x)}{\phi_1'(x) - y \phi_0'(x)} \right) f'(x), 
\]

\[
V = f(x) - \phi_0(x) g(y) - (\phi_1(x) - y \phi_0(x)) g'(y),
\]

where \( f \) and \( g \) satisfy the equations\(^{14}\)

\[
f''(x) + \mu(x) f(x) = -\xi(x), \quad g''(y) = \eta(y).
\]

Conversely, for any smooth functions \( f \) on \( x(D) \) and \( g \) on \( y(D) \), the formulae (3.1.33) define a solution of (3.1.22). Thus, it follows that (3.1.33) is the general solution of (3.1.22).

**Theorem 3.** If \( S : TD \to TD \) is an invertible, nondegenerate operator whose \( \theta \)-coframing satisfies the Type I structure equations (2.5.23), then each point of \( D \) has a \( \theta \)-rectangular neighborhood \( R \) on which there exist principal coordinates \( (x, y) \) in which \( S \) takes the form

\[
S = \frac{1}{U} \frac{\partial}{\partial x} \otimes dx + \frac{1}{V} \frac{\partial}{\partial y} \otimes dy
\]

where \( U \) and \( V \) are of the form (3.1.33) for some functions \( g \) on \( y(R) \) and \( f, \phi_0, \phi_1 \) on \( x(R) \) satisfying \( \phi_0 \phi_1' - \phi_1 \phi_0' = 1 \).

Moreover, the coordinates \( x \) and \( y \) and the functions \( g, f, \phi_0 \) and \( \phi_1 \) are computable from \( S \) by algebraic operations, quadratures, and the integration of a single, linear, self-adjoint second order ordinary differential equation on \( x(D) \).

Conversely, if \( R \) is a rectangle in the \( xy \)-plane, then, for any choice of \( g \) on \( y(R) \) and \( f, \phi_0, \phi_1 \) on \( x(R) \) satisfying \( \phi_0 \phi_1' - \phi_1 \phi_0' = 1 \) such that the functions \( U \) and \( V \) defined by (3.1.33) are nonvanishing and unequal on \( R \), the formula (3.1.35) defines an invertible, nondegenerate operator \( S : TR \to TR \) whose \( \theta \)-coframing satisfies (2.5.23) and for which \( (x, y) \) is a \( \theta \)-principal coordinate system.

**Remark 9** (Normal form ambiguities). The coordinates \( x \) and \( y \) and the functions \( g, f, \phi_0, \phi_1 \) are not quite canonically determined by \( S \):

- The coordinate \( x \) is determined up to an affine transformation \( x \mapsto \lambda x + \tau \) where \( \lambda > 0 \) and \( \tau \) are constants.
- Once \( x \) is chosen, the function \( \mu \) can be found and then the normalized pair \( (\phi_0, \phi_1) \) is determined up to a (constant) unimodular change of basis.
- Once \( x, \phi_0 \) and \( \phi_1 \) are chosen, the function \( y \) is determined. (It may be necessary to re-choose the normalized pair \( (\phi_0, \phi_1) \) so that \( y \) remains finite on all of \( R \).)

Finally, the functions \( f \) and \( g \) are determined up to a replacement of the form

\[
(f(x), g(y)) = (f(x) + c_1 \phi_1(x) + c_0 \phi_0(x), g(y) + c_1 y + c_0)
\]

for any two constants \( c_0 \) and \( c_1 \).

\(^{14}\) Of course, \( f \) and \( g \) can be computed from \( \xi \) and \( \eta \) by quadratures since \( \phi_0 \) and \( \phi_1 \) are assumed known.
Remark 10 (Reduction to quadrature). As Theorem 3 shows, the entire process of computing a normal form for $S$ requires only algebraic operations, quadratures, and the solution of a single second, order self-adjoint ordinary differential equation, namely (3.1.7).

However, given $S$, one can usually dispense with this last step, since an alternative is available. In particular, as long as $V_2$ is not identically vanishing (which is the same as the condition $d''V \neq 0$), this can be done as follows:

In terms of principal coordinates, the formulae (3.1.14), (3.1.30), and (3.1.32) show that

$$(3.1.37) \quad (d''V)^2 \circ \theta_2 \circ \theta_1 = V_2^2 \theta_2^3 \circ \theta_1 = -\eta(y)^2 (dy)^3 \circ dx.$$  

Of course, the quartic form on the left hand side is computable from $S$ by differentiation alone.

Assume that the left hand side of (3.1.37) is nonzero. Then by algebraic operations, one can write it in the form $\psi_1 \circ \psi_2^3$ for some coframing $(\psi_1, \psi_2)$ that is unique up to a replacement of the form

$$(3.1.38) \quad (\psi_1, \psi_2) \mapsto (r^{-3} \psi_1, r \psi_2)$$

for some nonvanishing function $r$ on $D$. By differentiation, one can now find a unique 1-form $\rho$ satisfying $d\psi_1 = -3 \rho_1 \psi_1$ and $d\psi_2 = \rho_2 \psi_2$ (this $\rho$ is essentially a connection form for the quartic). Under the replacement (3.1.38), one finds that $\rho$ is replaced by $\rho + d(\log r)$. Now, by (3.1.37), it is obvious that there exists a choice, namely

$$(3.1.39) \quad (\tilde{\psi}_1, \tilde{\psi}_2) = (dx, -(\eta(y))^{2/3} dy)$$

for which each $\tilde{\psi}_i$ is closed, i.e., for which the corresponding connection form is $\tilde{\rho} = 0$. Consequently, $\rho$ must be closed for any choice of $(\psi_1, \psi_2)$. Since $\rho$ is closed, it can, by quadrature, be written in the form $\rho = d(\log r)$ for some positive function $r$ on $D$. Then replacing $(\psi_1, \psi_2)$ by $(r^3 \psi_1, r^{-1} \psi_2)$ yields a $\psi$-coframing for which each $\psi_i$ is closed. Thus, by quadrature, one can write $\psi_1 = dx$ and $\psi_2 = dt$ for some principal coordinates $x$ and $t$ on $D$. (I am using $t$ instead of $y$ here because $t$ will not, in general, be a natural principal coordinate, although $x$ certainly is.)

Regarding $x$ as now known, $z$ can be defined by requiring that $\theta_1 = (dx)/z$. (It may be necessary to replace $x$ by $-x$ to ensure that $z$ is positive.)

Now, the curve $t = t_0$ for some constant $t_0$ is an integral curve of $\theta_2$. In the $xz$ coordinates, it can be written in the form $z = f(x)$ for some function $f$ that necessarily satisfies $f' = 1 + \mu f^2$. Of course, as has already been explained, the function $\phi$ that satisfies $\phi'_1 = (1/f) \phi_1$ can now be computed by quadrature and satisfies $\phi''_1 + \mu \phi_1 = 0$. Thus, one has one solution of (3.1.7) computed by quadrature. Solving the first order inhomogeneous equation $\phi_0 \phi'_1 - \phi_1 \phi'_0 = 1$ for $\phi_0$ by quadrature then yields the desired normalized pair $(\phi_0, \phi_1)$.

Note that this procedure produces both principal coordinates by algebraic operations and simple quadratures. Its main disadvantage is that it depends on properties of the pair $(U, V)$ and not just on the coframing $\theta$.

Remark 11 (Primary invariants). The reader will have noticed that there are three fundamental invariants that are computable purely by differentiation that, in some
sense, define the principal coordinates up to quadrature:

\[(1 - K_2) \theta_1^2 = \mu(x) (dx)^2,\]
\[(U_1 - (1 - K_2) U) \theta_2^2 = \xi(x) (dx)^2,\]
\[-V_2^2 \theta_3^2 \circ \theta_1 = \eta(y)^2 (dy)^3 \circ dx.\]

These differential invariants were originally found by the method of Darboux, though the treatment above that generates them was developed so that the reader need not be familiar with this method.

It can be shown that these invariants plus the affine structure on the space of principal curves of the second family (which requires a quadrature to compute) are sufficient to test whether or not two given operators of Type I are equivalent up to a change of variable.

Remark 12 (A final elimination). Informally, Theorem 3 says that the operators of Type I depend on three arbitrary functions of one variable, i.e., four arbitrary functions \(f(x), \phi_0(x), \phi_1(x),\) and \(h(y)\) subject to the differential equation \(\phi_0 \phi_1' - \phi_1 \phi_0' = 1\) and some open conditions.

It is worth pointing out that one can make a change of variables to eliminate the differential equation \(\phi_0 \phi_1' - \phi_1 \phi_0' = 1:\) Use the fact that \((\phi_0(x), \phi_1(x))\) never vanishes to write

\[
\phi_0(x) = r(t) \cos t, \quad \phi_1(x) = r(t) \sin t
\]

for some functions \(t\) and \(r > 0\) on \(x(R)\). The identity

\[
dx = (\phi_0(x) \phi_1'(x) - \phi_1(x) \phi_0'(x)) \, dx = \phi_0 \, d\phi_1 - \phi_1 \, d\phi_0 = r^2 \, dt
\]

shows that \(dt\) is nonvanishing on \(x(R)\) and hence can be taken as a coordinate. (Note that \(t\) is a principal coordinate replacing \(x\).) In particular, there exists a function \(\rho: t(R) \to \mathbb{R}\) so that \(r = \rho(t)\).

Now, writing \(g(x) = \gamma(t)\) for some function \(\gamma\) on \(t(R)\), one computes \(g'(x) = \gamma'(t)/\rho(t)^2\). Similarly, \(\phi_0'(x)\) and \(\phi_1'(x)\) can be expressed in terms of \(t, \rho(t),\) and \(\rho(t)\).

Thus, all of the expressions in the formulae for \(U\) and \(V\) involving \(x\) can be replaced by expressions in \(t\) and the (arbitrary positive) function \(\rho\) on \(t(R)\).

This change of variables expresses \(S\) in \(ty\)-coordinates in terms of the three arbitrary functions \(\rho(t), \gamma(t),\) and \(g(y)\) and their derivatives, where these three functions are only subject to open conditions, not equations (differential or otherwise). However, this formula does not appear to be particularly useful, so it will not be explored further.

3.1.4. Integrating the structure equations. Now that explicit formulae have been found for \(\theta_1, \theta_2, a, b, U\) and \(V\), the structure forms \(\omega_1, \omega_2, \omega_{31}, \omega_{32}\) and \(\omega_{12}\) can be regarded as known. Since, by construction, these forms satisfy the structure equations, Bonnet's theorem can be used to show that there exists a corresponding realization \(x\) of the operator \(S\).

However, Bonnet's theorem is not an effective theorem in the sense that the realization \(x\) cannot, in general be computed from the structure forms using only algebraic operations and quadratures. The goal of this subsection is to indicate that, in fact, one can avoid having to quote Bonnet's theorem by following an algorithm for constructing \(x\) that only involves algebraic operations and quadratures. In the
The spherical centers of these geodesic circles lie on a fixed great circle on $S^2$.

**Proof.** All but the last statement has been verified already. To prove the last statement, it suffices to note that the spherical center of the geodesic circle tangent to $e_2$ with geodesic curvature $\sqrt{a(x)}$ is given by

\begin{equation}
 z = \frac{-1}{1 + a(x)} e_1 + \frac{\sqrt{a(x)}}{1 + a(x)} e_3.
\end{equation}

Computation now shows that $dz = w \sigma$ where $w \cdot w = 1$ and

\begin{equation}
 \sigma = \frac{\sqrt{c_1^2 - c_0 c_2}}{(1 + a(x)) \sqrt{a(x)}} dx,
\end{equation}

and, moreover, that $dw = -z \sigma$. It follows that $z$ moves on the great circle perpendicular to the fixed vector $z \times w$. 

Using Proposition 3, it is now not difficult to integrate the equations and find the $e_1$ explicitly. In fact, one does the following: First, construct, by quadrature, a
function \( s(x) \) so that

\[
(3.1.46) \quad ds = \sigma = \frac{\sqrt{c_1^2 - c_0 c_2}}{(1 + a(x)) \sqrt{a(x)}} \, dx.
\]

One can then show that, up to a rotation, \( e_3 \) is given by

\[
(3.1.47) \quad e_3(x, y) = \begin{bmatrix}
\cos(s(x)) \frac{\sqrt{a(x)}}{\sqrt{1 + a(x)}} - \sin(s(x)) \frac{(c_0 \phi_0(x) + c_1 (\phi_1(x) + y \phi_0(x)) + c_2 y \phi_1(x))}{(\phi_1(x) + y \phi_0(x)) \sqrt{1 + a(x) \sqrt{c_1^2 - c_0 c_2}}}
\sin(s(x)) \frac{\sqrt{a(x)}}{\sqrt{1 + a(x)}} + \cos(s(x)) \frac{(c_0 \phi_0(x) + c_1 (\phi_1(x) + y \phi_0(x)) + c_2 y \phi_1(x))}{(\phi_1(x) + y \phi_0(x)) \sqrt{1 + a(x) \sqrt{c_1^2 - c_0 c_2}}}
\end{bmatrix}.
\]

There are, of course, similar formulae for \( e_1 \) and \( e_2 \), but they will not be needed explicitly in the algorithm to be considered. The important point is that \( e_3 \) can be constructed by quadrature and that \( \omega_{31} \) is a multiple of \( dx \) while \( \omega_{32} \) is a multiple of \( dy \). This implies that the comparison

\[
(3.1.48) \quad \frac{\partial e_1}{\partial x} \, dx + \frac{\partial e_3}{\partial y} \, dy = de_3 = -e_1 \omega_{31} - e_2 \omega_{32}
\]

yields

\[
(3.1.49) \quad \frac{\partial e_1}{\partial x} \, dx = -e_1 \omega_{31} \quad \text{and} \quad \frac{\partial e_3}{\partial y} \, dy = -e_1 \omega_{32}.
\]

Finally, the immersion \( x : D \to \mathbf{E}^3 \) satisfies the structure equation

\[
(3.1.50) \quad dx = e_1 \omega_1 + e_2 \omega_2 = U(x, y) e_1 \omega_{31} + V(x, y) e_2 \omega_{32}
\]

\[= -U(x, y) \frac{\partial e_3}{\partial x} \, dx - V(x, y) \frac{\partial e_3}{\partial y} \, dy.
\]

The structure equations imply that the vector-valued differential form on the right hand side of \( (3.1.50) \) is indeed a closed 1-form, so that \( x \) can be recovered by quadrature:

\[
(3.1.51) \quad x = - \int U(x, y) \frac{\partial e_3}{\partial x} \, dx + V(x, y) \frac{\partial e_3}{\partial y} \, dy.
\]

Thus, the final result is

**Theorem 4.** The local realizations of an operator \( S \) of Type I can be computed by quadratures, once the principal coordinates are found.

3.1.5. The case \( \mu = 0 \). The rest of this subsection about the geometry of Type I operators and their realizations will concern only the special case \( \mu \equiv 0 \), i.e., when \( K_2 \equiv 1 \). This simplest case has special features that are not shared by the general Type I operators. For example, some of the quadratures that are needed in the general case can be eliminated, thus leading to more explicit formulae and easily computed examples. Moreover the domains of the realizations are more easily described.

Some of the features discussed here can be generalized to other values of \( \mu(x) \), but that will be left to the interested reader.
To distinguish this case from the general case, upper case letters will be used for the natural principal coordinates. Thus, \(X\) and \(Z\) instead of \(x\) and \(z\).

The equation \(\phi''(X) + \mu(X) \phi(X) = 0\) now simplifies to \(\phi''(X) = 0\), with the obvious normalized pair \((\phi_0, \phi_1) = (1, X)\). This gives \(Y = X - Z > 0\) and the formulae for the \(\theta\)-coframing assume the simple, symmetric form

\[(3.1.52) \quad \theta_1 = \frac{dX}{X-Y}, \quad \theta_2 = \frac{dY}{Y-X},\]

The formulae for \(a, b,\) and \(p\) simplify to

\[(3.1.53) \quad a = \frac{1}{2}(-1 + c_0 + 2c_1 X + c_2 X^2), \quad b = \frac{1}{2}(-1 - c_0 - 2c_1 Y - c_2 Y^2), \quad p = c_0 + c_1 (X + Y) + c_2 XY.\]

(In the interests of preserving the \(XY\) symmetry, the usage of the constants \(c_i\) is now slightly different from that of the general case.)

The reader must keep in mind that the conditions \(a > 0\) and \(b > 0\) must still be imposed. These inequalities, together with the requirement \(X > Y\), impose inequalities on \(c_0, c_1,\) and \(c_2\).

These inequalities amount to the condition that there exist constants \(\xi, \eta,\) and \(\lambda\) with \(\xi > \eta\) and \(|\lambda| < 1\) so that

\[(3.1.54) \quad \frac{-1 + c_0 + 2c_1 X + c_2 X^2}{2} = \frac{(X - \xi)(\lambda(X - \eta) + \xi - \eta)}{\xi - \eta),} \quad \frac{-1 - c_0 - 2c_1 Y - c_2 Y^2}{2} = \frac{(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)}{\xi - \eta).} \]

Moreover, \(X\) and \(Y\) are required to satisfy \(Y < \eta < \xi < X\). If \(\lambda = 0\), this is the only restriction needed to make the right hand sides positive, so the notation \(D_{\xi,\eta,0}\) will denote the quarter-plane \(Y < \eta < \xi < X\).

If \(0 < \lambda < 1\), then \(Y\) is required to lie in the interval

\[(3.1.55) \quad \frac{\eta}{\lambda} + \xi \left(1 - \frac{1}{\lambda}\right) < Y < \eta\]

so the notation \(D_{\xi,\eta,\lambda}\) will denote the corresponding open semi-infinite horizontal strip in the \(XY\)-plane.

Finally, if \(-1 < \lambda < 0\), then \(X\) is required to lie in the interval

\[(3.1.56) \quad \xi < X < -\frac{\xi}{\lambda} + \eta \left(1 - \frac{1}{\lambda}\right),\]

so the notation \(D_{\xi,\eta,\lambda}\) will denote the corresponding open semi-infinite vertical strip in the \(XY\)-plane.

In the other direction, note that, if \((X_0, Y_0)\) is any point in the \(XY\)-plane with \(X_0 > Y_0\), then the inequalities

\[(3.1.57) \quad -1 + c_0 + 2c_1 X_0 + c_2 X_0^2 > 0, \quad -1 - c_0 - 2c_1 Y_0 - c_2 Y_0^2 > 0\]

define a non-empty open wedge in \(c_0 c_1 c_2\)-space. Thus, for any point \((X_0, Y_0)\) in the half-plane, there are choices of \(c_0, c_1,\) and \(c_2\) so that the corresponding functions \(a\) and \(b\) are positive on a neighborhood of \((X_0, Y_0)\).
3.1.6. The connection forms. The connection structure forms have the expressions

$$\omega_{31} = \frac{\theta_1}{\sqrt{a}} = \frac{(\xi - \eta) \, dX}{(X - Y) \sqrt{(X - \xi)(\lambda(X - \eta) + \xi - \eta)}}$$

$$\omega_{32} = \frac{\theta_2}{\sqrt{b}} = \frac{(\xi - \eta) \, dY}{(Y - X) \sqrt{(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)}}$$

$$(3.1.58)$$

$$\omega_{12} = -\frac{\sqrt{b}}{\sqrt{a}} \theta_1 + \frac{\sqrt{a}}{\sqrt{b}} \theta_2$$

$$= -1 \frac{1}{X - Y} \left( \left( \frac{\eta - Y}{(X - \xi)(\lambda(X - \eta) + \xi - \eta)} \right) dX \right.$$  

$$\left. + \left( \frac{(X - \xi)(\lambda(X - \eta) + \xi - \eta)}{(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)} \right) dY \right).$$

Note that the formulae for $\omega_{31}$, $\omega_{32}$, and $\omega_{12}$ do not explicitly involve the functions $U$ and $V$ \[15\] In particular, the structure equations for these forms are satisfied on $D_{\xi,\eta,\lambda}$.

3.1.7. Euler linearization. As was seen in the general case, the structure equations for $\omega_1$ and $\omega_2$ (which are all that remains) simplify to the linear system

$$\frac{\partial U}{\partial Y} = -\frac{U - V}{X - Y}, \quad \frac{\partial V}{\partial X} = -\frac{U - V}{X - Y}. \quad (3.1.59)$$

The general solution described in \[\text{(3.1.33)}\] now simplifies to

$$U(X, Y) = f(X) - g(Y) - (X - Y) f'(X),$$

$$V(X, Y) = f(X) - g(Y) - (X - Y) g'(Y), \quad (3.1.60)$$

where $f$ and $g$ are arbitrary functions of a single variable, subject only to the conditions that they be chosen on their respective $X$-domain and $Y$-domain so that $U$ and $V$ are nonzero and nowhere equal functions on $D$ (which is the product of the $X$-domain and the $Y$-domain). The functions $f$ and $g$ that give rise to $U$ and $V$ are not unique. In fact, for any constants $m_0$ and $m_1$ one can add $m_0 + m_1 X$ to $f(X)$ and $m_0 + m_1 Y$ to $g(Y)$ without changing $U$ and $V$. However, this is the only indeterminacy in the formulæ.

The result is the following general formulæ for $\omega_1$ and $\omega_2$:

$$\omega_1 = \frac{\theta_1}{A \sqrt{a}} = \frac{(\xi - \eta) \left( f(X) - g(Y) - (X - Y) f'(X) \right) dX}{(X - Y) \sqrt{(X - \xi)(\lambda(X - \eta) + \xi - \eta)}}, \quad (3.1.61)$$

$$\omega_2 = \frac{\theta_2}{B \sqrt{b}} = \frac{(\xi - \eta) \left( f(X) - g(Y) - (X - Y) g'(Y) \right) dY}{(Y - X) \sqrt{(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)}}.$$

Note that the forms $\omega_1, \omega_2, \omega_{31}, \omega_{32}, \omega_{12}$ as defined in \[\text{(3.1.58)}\] and \[\text{(3.1.61)}\] satisfy the structure equations even at points where $U$ or $V$ vanish or where $U = V$ (i.e., where $f'(X) = g'(Y)$). Consequently, Bonnet’s theorem applies and there exist mappings $x : D \to \mathbb{R}^3$ and $(e_1, e_2, e_3) : D \to O(3)$ whose associated structure forms

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\[15\] Of course, these functions were used in the definition of the forms $\theta_1$ and $\theta_2$ and thus in the definition of the principal coordinatization $(X, Y)$. 

are the given ones. As long as $U$ and $V$ are nonzero, the map $x$ will be an immersion, it is just that this immersion will have umbilic points where $f'(X) = g'(Y)$.

### 3.1.8. A Weierstraß-type formula

Now, it is not necessary to rely on Bonnet’s theorem to generate the mapping $x$. In fact, this can be reduced to quadratures, as will now be demonstrated.

In the first place, finding a frame field $e = (e_1, e_2, e_3) : D \to O(3)$ so that $de_i = e_j \omega_{ji}$ (where $\omega_{ji} = -\omega_{ij}$) can be done as follows. One starts with the formula

$$e_3 = \begin{bmatrix}
    2 \sqrt{(X - \xi)(\lambda(X - \eta) + \xi - \eta)} \\
    (1 + \lambda)(X - Y) \\
    2 \sqrt{(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)} \\
    (1 - \lambda)(X - Y) \\
    2(\xi - \eta) - 2\lambda(\xi + \eta) - (1 - \lambda)^2 X + (1 + \lambda)^2 Y \\
    (1 - \lambda^2)(X - Y)
\end{bmatrix}.$$  

(3.1.62)

Computation yields

\[
\begin{align*}
\frac{\partial e_3}{\partial X} \cdot \frac{\partial e_3}{\partial X} &= \frac{(\xi - \eta)^2}{(X - Y)^2(X - \xi)(\lambda(X - \eta) + \xi - \eta)}, \\
\frac{\partial e_3}{\partial X} \cdot \frac{\partial e_3}{\partial Y} &= 0, \\
\frac{\partial e_3}{\partial Y} \cdot \frac{\partial e_3}{\partial Y} &= \frac{(\xi - \eta)^2}{(X - Y)^2(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)},
\end{align*}
\]

(3.1.63)

so setting

$$e_1 = -\left|\frac{\partial e_3}{\partial X}\right|^{-1} \frac{\partial e_3}{\partial X}, \quad e_2 = -\left|\frac{\partial e_3}{\partial Y}\right|^{-1} \frac{\partial e_3}{\partial Y},$$

(3.1.64)

yields the desired structure equation

$$de_3 = \frac{\partial e_3}{\partial X} \, dX + \frac{\partial e_3}{\partial Y} \, dY = -e_1 \omega_{31} - e_2 \omega_{32},$$

(3.1.65)

where $\omega_{31}$ and $\omega_{32}$ are as defined in (3.1.58).

The remaining structure equations for $de_1$ and $de_2$ are easily verified, so that this does, in fact, integrate the structure equations for the frame field $(e_1, e_2, e_3)$.

**Remark 13 (The Gauß image of the principal net)**. Note that, since $\mu = 0$, both families of principal curves are mapped to arcs of circles under $e_3$. These two families of image circles are evidently orthogonal.

In particular, for any realization $x$ of a Type I shape operator $S$ with $\mu = 0$, the Gauß image of the net of principal curves of $x$ is a net of orthogonal circle foliations on the 2-sphere.

Now, note that $e_3$ satisfies the (vector-valued) Euler equation

$$\frac{\partial^2 e_3}{\partial X \partial Y} - \frac{1}{X - Y} \frac{\partial e_3}{\partial X} \frac{1}{X - Y} \frac{\partial e_3}{\partial Y} = 0,$$

(3.1.66)

which is easily established by direct computation.
Finally, the immersion \( \mathbf{x} : D \rightarrow \mathbb{E}^3 \) satisfies the structure equation
\[
d\mathbf{x} = e_1 \omega_1 + e_2 \omega_2 = U(X,Y) e_1 \omega_{31} + V(X,Y) e_2 \omega_{32}
\]
(3.1.67)
\[= - U(X,Y) \frac{\partial e_3}{\partial X} \, dX - V(X,Y) \frac{\partial e_3}{\partial Y} \, dY.
\]
The fact that \( e_3 \) satisfies (3.1.66) while \( U \) and \( V \) satisfy (3.1.59) implies the identity
\[
\frac{\partial}{\partial Y} \left( U(X,Y) \frac{\partial e_3}{\partial X} \right) = \frac{\partial}{\partial X} \left( V(X,Y) \frac{\partial e_3}{\partial Y} \right),
\]
(3.1.68)
implying that the vector-valued differential form on the right hand side of (3.1.67) is indeed a closed 1-form, so that \( \mathbf{x} \) can be recovered by quadrature:
\[
\mathbf{x} = - \int U(X,Y) \frac{\partial e_3}{\partial X} \, dX + V(X,Y) \frac{\partial e_3}{\partial Y} \, dY.
\]
(3.1.69)

3.1.9. Unfoldings. If a solution \((U,V)\) to (3.1.59) is defined on a neighborhood of the closure of a domain \( D_{\xi,\eta,\lambda} \), then the mapping \( \mathbf{x} \) can be continued beyond the edges of \( D_{\xi,\eta,\lambda} \) in a natural way.

For example, when \( \lambda = 0 \), consider the mapping from \( \mathbb{R}^2 \) to the closure of \( D_{\xi,\eta,0} \) defined by the formulae
\[
X = \xi + x^2, \quad Y = \eta - y^2.
\]
(3.1.70)
Using this mapping to pull back \( e_3 \) (assuming that \( \lambda = 0 \), of course), the formula for \( e_3 \) is resolvable to
\[
e_3(x,y) = \begin{bmatrix}
2x \sqrt{\xi-\eta} \\
2y \sqrt{\xi-\eta} \\
(\xi-\eta) - x^2 - y^2 \\
((\xi-\eta) + x^2 + y^2)
\end{bmatrix}
\]
(3.1.71)
and the reader will notice that this is a conformal embedding of the \( xy \)-plane onto the punctured sphere. The differential formula for \( \mathbf{x} \) as a function of \( x \) and \( y \) then becomes
\[
d\mathbf{x} = U(\xi+x^2,\eta-y^2) \frac{\partial e_3}{\partial x} \, dx + V(\xi+x^2,\eta-y^2) \frac{\partial e_3}{\partial y} \, dy,
\]
(3.1.72)
and the 1-form on the right hand side of this equation will be smooth and closed as long as \( U \) and \( V \) satisfy (3.1.59) and are smooth on a domain containing the closure of \( D_{\xi,\eta,0} \). Moreover, it will be an immersion on the set where \( U \) and \( V \) are nonzero.

There are similar unfoldings when \( \lambda \neq 0 \). When \( \lambda > 0 \), one uses the formulae
\[
X = \xi + x^2, \\
Y = \eta \cos^2 y + \left( \frac{\eta}{\lambda} + \xi \left( 1 - \frac{1}{\lambda} \right) \right) \sin^2 y,
\]
(3.1.73)
while, when \( \lambda < 0 \), one uses the formulae
\[
X = \xi \cos^2 y + \left( -\frac{\xi}{\lambda} + \eta \left( 1 - \frac{1}{\lambda^2} \right) \right) \sin^2 y, \\
Y = \eta - y^2,
\]
(3.1.74)
to obtain mappings defined on a smooth cylinder. In either case, \( e_3 \) becomes a conformal embedding of the cylinder onto the twice-punctured unit 2-sphere and the formula analogous to (3.1.72) defines (up to a quadrature) a smooth mapping \( \mathbf{x} \) from the cylinder to \( \mathbb{E}^3 \).

3.1.10. Examples. Some examples will now be considered.

Example 3 (Linear solutions). Consider the global linear solution to (3.1.59)
\[
U = Y, \quad V = X,
\]
(3.1.75)
whose domain is the entire half-plane \( Y < X \). Using the formulae above, one finds the corresponding mapping on \( D_{\xi,\eta,\lambda} \) to be given by the formulae
\[
\mathbf{x}(X,Y) = \left[ \begin{array}{c} -2Y \sqrt{(X - \xi)(\lambda(X - \eta) + \xi - \eta)} \\ (1 + \lambda)(X - Y) \\ -2X \sqrt{(\eta - Y)(\lambda(Y - \xi) + \xi - \eta)} \\ (1 - \lambda)(X - Y) \\ (X + Y)(\lambda(\xi + \eta) - \xi + \eta) - 4\lambda XY \\ (1 - \lambda^2)(X - Y) \end{array} \right].
\]
(3.1.76)
This is an immersion away from the axis rays \( X = 0 \) and \( Y = 0 \). The shape operator of all of these immersions is
\[
S = \frac{1}{Y} \frac{\partial}{\partial X} \otimes dX + \frac{1}{X} \frac{\partial}{\partial Y} \otimes dY.
\]
(3.1.77)
Note that, since the domain of this solution contains the closures of all of the \( D_{\xi,\eta,\lambda} \), it follows that the various unfoldings allow one to continue \( \mathbf{x} \) past the edges in all cases as a smooth mapping. This mapping will be an immersion as long as the closure of \( D_{\xi,\eta,\lambda} \) does not meet either axis.

The image surface is visibly algebraic. However, it does not appear to be easy to recognize as a classical surface. Indeed, for generic values of \( \xi, \eta, \) and \( \lambda \), it appears to be of degree 8.

Example 4 (An umbilic of index zero). Consider the polynomial solution \((U, V)\) generated by the formulae (3.1.60) when one takes
\[
f(X) = 1 + (X - 1)^3, \quad g(Y) = 1 - (Y + 1)^3.
\]
(3.1.78)
Since \( U = V \) only where \( f'(X) = g'(Y) \), one sees that this happens only at the point \((X, Y) = (1, -1)\). Moreover, \( U \) and \( V \) are positive at this point. It follows that there is a neighborhood of \((1, -1)\) in the \( XY \)-plane on which the shape operator
\[
S = \frac{1}{U(X,Y)} \frac{\partial}{\partial X} \otimes dX + \frac{1}{V(X,Y)} \frac{\partial}{\partial Y} \otimes dY.
\]
(3.1.79)
has a 3-parameter family of non-congruent realizations. All of these realizations have an isolated umbilic point and the construction shows that this umbilic point is of index zero. (After all, the net of principal curves is non-singular near this point.) Whether such an example exists globally is an interesting question.

**Example 5 (A minimal surface).** It is not difficult to see that, up to a constant multiple, the only solution \((U, V)\) to \((3.1.60)\) that satisfies \(U + V = 0\) is

\[
U(X, Y) = (X - Y)^2, \quad V(X, Y) = -(X - Y)^2.
\]

(Note that this solution is invariant under the simultaneous translation of \(X\) and \(Y\) and simply scales under the simultaneous dilation.) Note that \(U + V = \frac{A + B}{AB}\), so that \(U + V = 0\) if and only if \(A + B = 0\), i.e., if the realization is a minimal surface.\(^{16}\)

Thus, this solution gives, up to constant multiples, the only shape operator of Type I whose realizations are minimal surfaces.\(^{17}\)

When \(\lambda\) is nonzero, the integral that gives \(x\) for this solution is rather complicated, so it will not be written out here. Instead, I will simply note that the integrals in the case \(\lambda = 0\) give rise to Enneper’s surface (up to translation and scale), as the reader can easily verify.

### 3.2. Operators of Type II.

I now want to consider the operators of Type II and explain how the equations that define them can be integrated to a linear system.

#### 3.2.1. Natural principal coordinates.

Recall the definition of the 1-forms

\[
\theta_1 = \frac{AB_x}{B(B - A)} \, dx, \quad \theta_2 = \frac{BA_y}{A(A - B)} \, dy.
\]

and that the equations \((2.5.21)\) can be expressed as

\[
d\theta_1 = -2 \theta_1 \wedge \theta_2, \quad d\theta_2 = -2 \theta_2 \wedge \theta_1.
\]

As a consequence, there exist functions \(X\) and \(Y\) on \(D\) with \(X > Y\) and for which

\[
\theta_1 = \frac{dX}{2(Y - X)}, \quad \theta_2 = \frac{dY}{2(X - Y)}.
\]

In fact, these two functions can be found by quadrature as follows: First, the structure equations imply that the 1-form \(\theta_1 + \theta_2\) is closed, so, by quadrature, one can find a function \(Z > 0\) on \(D\) so that \(d(\log Z) = dZ/Z = -2(\theta_1 + \theta_2)\). The structure equations now imply that the 1-form \(-2Z \theta_1\) is closed, so, again, by quadrature, one can find a function \(X\) so that \(dX = -2Z \theta_1\). Setting \(Y = X - Z\) then gives the desired remaining function.

Note that \(X\) and \(Y\) are unique up to a replacement of the form \((X, Y) \mapsto (\lambda X + \tau, \lambda Y + \tau)\) where \(\lambda > 0\) and \(\tau\) are constants. The map \((X, Y) : D \to \mathbb{R}^2\) is a principal coordinatization of \(D\) and embeds \(D\) as a rectangle in the open half-plane \(X > Y\). As before, these will be referred to as natural principal coordinates.

\(^{16}\) Adding a constant \(c\) to each of \(U\) and \(V\) yields another solution that satisfies \(U + V = 2c\). Since this is equivalent to \((A + B)/(AB) = 2c\), i.e., to \(H = cK\) (where \(H\) and \(K\) are the mean and Gauß curvatures respectively), this gives a more general class of Weingarten surfaces that admit a 3-parameter family of deformations. It is not difficult to show that there is, in fact, a two parameter family of Weingarten relations that have deformable examples of this type.

\(^{17}\) The argument given only applies to the \(\mu = 0\) case, but the reader will have no difficulty checking that when \(\mu \neq 0\), there are no solutions with \(U + V = 0\).
3.2.2. Integrals of the Frobenius system. Now, the Frobenius system for the functions $a$, $b$, and $p$ simplifies to

$$
\begin{align*}
\text{da} &= (2a - p + 1)\theta_1 + 6a \theta_2, \\
\text{db} &= 6b \theta_1 + (2b + p + 1)\theta_2, \\
\text{dp} &= 4(2b + p + 1)\theta_1 - 4(2a - p + 1)\theta_2.
\end{align*}
$$

(3.2.4)

By standard integration techniques, there exist constants $c_0$, $c_1$, and $c_2$ so that

$$
\begin{align*}
a &= \frac{X^3 + 3c_2 X^2 + 3c_1 X + c_0}{(Y - X)^3}, \\
b &= \frac{Y^3 + 3c_2 Y^2 + 3c_1 Y + c_0}{(X - Y)^3}, \\
p &= \frac{Y^3 - 3Y^2 X - 3XY^2 + X^3 - 12c_2 XY - 6c_1 (X + Y) - 4c_0}{(Y - X)^3}.
\end{align*}
$$

(3.2.5)

Conversely, for any constants $c_0$, $c_1$, and $c_2$, the formulae (3.2.5) give expressions for $a$, $b$, and $p$ that satisfy (3.2.4). The reader should bear in mind, however, that the conditions $a > 0$ and $b > 0$ must still be imposed. These inequalities, together with the requirement $X > Y$, impose inequalities on $c_0$, $c_1$, and $c_2$ that amount to requiring that the polynomial $c(t) = t^3 + 3c_2 t^2 + 3c_1 t + c_0$ have three distinct real roots, say,

$$
c(t) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3), \quad \lambda_1 < \lambda_2 < \lambda_3
$$

(3.2.6)

If these requirements are met, then $a$ and $b$ are positive on the open rectangle $D_\lambda$ defined by the inequalities

$$
\lambda_1 < Y < \lambda_2 < X < \lambda_3.
$$

(3.2.7)

Note that one corner of $D_\lambda$ lies on the line $X = Y$, namely the point $(\lambda_2, \lambda_2)$.

In the other direction, note that, if $(X_0, Y_0)$ is any point in the $XY$-plane with $X_0 > Y_0$, then the inequalities

$$
X_0^3 + 3c_2 X_0^2 + 3c_1 X_0 + c_0 < 0, \quad Y_0^3 + 3c_2 Y_0^2 + 3c_1 Y_0 + c_0 > 0
$$

(3.2.8)

define a non-empty open wedge in $c_0 c_1 c_2$-space. Thus, for any point $(X_0, Y_0)$ in the half-plane, there are choices of $c_0$, $c_1$, and $c_2$ so that the corresponding functions $a$ and $b$ are positive on a neighborhood of $(X_0, Y_0)$. 
3.2.3. The connection forms. Now, in terms of $\theta_1$ and $\theta_2$, the connection structure forms have the expressions

$$
\omega_{31} = \frac{\theta_1}{\sqrt{a}} = \frac{\sqrt{(X-Y)} \, dX}{2\sqrt{(X-\lambda_1)(X-\lambda_2)(\lambda_3-X)}},
$$

$$
\omega_{32} = \frac{\theta_2}{\sqrt{b}} = \frac{\sqrt{(X-Y)} \, dY}{2\sqrt{(Y-\lambda_1)(\lambda_2-Y)(\lambda_3-Y)}},
$$

$$(3.2.9)$$

$$
\omega_{12} = -\frac{\sqrt{b}}{\sqrt{a}} \theta_1 + \frac{\sqrt{a}}{\sqrt{b}} \theta_2
= \frac{1}{2(X-Y)} \left( \sqrt{\frac{(Y-\lambda_1)(\lambda_2-Y)(\lambda_3-Y)}{(X-\lambda_1)(X-\lambda_2)(\lambda_3-X)}} \right) dX
+ \sqrt{\frac{(X-\lambda_1)(X-\lambda_2)(\lambda_3-X)}{(Y-\lambda_1)(\lambda_2-Y)(\lambda_3-Y)}} dY.
$$

Note that the formulae for $\omega_{31}$, $\omega_{32}$, and $\omega_{12}$ do not explicitly involve the functions $A$ and $B$. In particular, the structure equations for these forms are satisfied on $D_\lambda$.

3.2.4. Euler linearization. Writing $U = 1/A$ and $V = 1/B$, the structure equations for $\omega_1$ and $\omega_2$ (which are all that remains) simplify to the linear system

$$(3.2.10)$$

$$
\frac{\partial U}{\partial Y} = \frac{U-V}{2(X-Y)}, \quad \frac{\partial V}{\partial X} = \frac{U-V}{2(X-Y)}.
$$

However, this linear system is not integrable by the method of Darboux, so its general solution cannot be expressed in a closed form similar to $(3.1.6)$. One can express the system as a single hyperbolic equation by introducing a potential $\Phi(X,Y)$ so that $U = \Phi_X$ and $V = \Phi_Y$. Then $\Phi$ satisfies the so-called Euler equation

$$(3.2.11)$$

$$
\frac{\partial^2 \Phi}{\partial X \partial Y} - \frac{1}{X-Y} \frac{\partial \Phi}{\partial X} + \frac{1}{X-Y} \frac{\partial \Phi}{\partial Y} = 0.
$$

While there is no closed-form solution to this equation, Poisson has given the following integral formula for the general solution

$$(3.2.12)$$

$$
\Phi(X,Y) = \int_Y^X \frac{\phi(\xi)}{\sqrt{(X-\xi)(\xi-Y)}} \, d\xi
+ \int_Y^X \frac{\psi(\xi)}{\sqrt{(X-\xi)(\xi-Y)}} \log \left( \frac{(X-Y)}{(X-\xi)(\xi-Y)} \right) \, d\xi,
$$

where $\phi$ and $\psi$ are arbitrary functions of a single variable. If $\phi$ and $\psi$ are defined on an interval $(a,b)$, then the solution $\Phi$ is defined on the triangle $a < Y < X < b$. (Of course, $a = +\infty$ and/or $b = -\infty$ are allowable values.)

These formulae give the general solution (in natural principal coordinates) to the system $(2.5.2)$. Thus, every operator $S$ of Type II can be generated by this procedure.

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*Of course, these functions were used in the definition of the forms $\theta_1$ and $\theta_2$ and thus in the definition of the principal coordinatization $(X,Y)$. 

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The result is the following general formulae for $\omega_1$ and $\omega_2$:

$$
\omega_1 = \frac{\theta_1}{A\sqrt{a}} = \frac{\Phi_X(X, Y) \sqrt{(X - Y)} \, dX}{2 \sqrt{(X - \lambda_1)(X - \lambda_2)(\lambda_3 - X)}},
$$

(3.2.13)

$$
\omega_2 = \frac{\theta_2}{B\sqrt{b}} = \frac{\Phi_Y(X, Y) \sqrt{(X - Y)} \, dY}{2 \sqrt{(Y - \lambda_1)(\lambda_2 - Y)(\lambda_3 - Y)}},
$$

(3.2.14)

Note that the forms $\omega_1, \omega_2, \omega_{31}, \omega_{32}, \omega_{12}$ as defined in (3.2.9) and (3.2.13) satisfy the structure equations even at points where $\Phi_X$ or $\Phi_Y$ vanish or where $\Phi_X = \Phi_Y$, as long as $\Phi$ satisfies (3.2.11). Consequently, Bonnet’s theorem applies and there exist mappings $x : D \to \mathbb{R}^3$ and $(e_1, e_2, e_3) : D \to O(3)$ whose associated structure forms are the given ones. As long as $\Phi_X$ and $\Phi_Y$ are nonzero, the map $x$ will be an immersion, it is just that this immersion will have umbilic points where $\Phi_X = \Phi_Y$.

3.2.5. A Weierstraß-type formula. Now, it is not necessary to rely on Bonnet’s theorem to generate the mapping $x$. In fact, this can be reduced to quadratures, as will now be demonstrated.

In the first place, finding a frame field $e = (e_1, e_2, e_3) : D \to SO(3)$ so that $de_i = e_j \omega_{ji}$ (where $\omega_{ji} = -\omega_{ij}$) is easily done. One starts with the classical formula

$$
e_3 = \begin{bmatrix}
\frac{(X - \lambda_1)(Y - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \\
\frac{(X - \lambda_2)(Y - \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\
\frac{(\lambda_3 - X)(Y - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}
\end{bmatrix}.
$$

(3.2.15)

Note that $e_3 \cdot e_3 = 1$. In fact, $e_3$ maps $D_\lambda$ diffeomorphically onto the positive orthant of the 2-sphere and extends continuously to the boundary of $D_\lambda$ as a homeomorphism from the closure of $D_\lambda$ to the closure of the positive orthant.

One easily computes that

$$
\frac{\partial e_3}{\partial X} \frac{\partial e_3}{\partial X} = -\frac{X - Y}{4(X - \lambda_1)(X - \lambda_2)(X - \lambda_3)},
$$

(3.2.16)

$$
\frac{\partial e_3}{\partial X} \frac{\partial e_3}{\partial Y} = 0,
$$

$$
\frac{\partial e_3}{\partial Y} \frac{\partial e_3}{\partial Y} = -\frac{X - Y}{4(Y - \lambda_1)(Y - \lambda_2)(Y - \lambda_3)},
$$

(3.2.17)

so setting

$$
e_1 = -\left| \frac{\partial e_3}{\partial X} \right|^{-1} \frac{\partial e_3}{\partial X}, \quad e_2 = -\left| \frac{\partial e_3}{\partial Y} \right|^{-1} \frac{\partial e_3}{\partial Y},
$$

yields the desired structure equation

$$
\left( e_3 \frac{\partial e_3}{\partial X} \right) dX + \left( e_3 \frac{\partial e_3}{\partial Y} \right) dY = -e_1 \omega_{31} - e_2 \omega_{32},
$$

where $\omega_{31}$ and $\omega_{32}$ are as defined in (3.2.9).

The remaining structure equations for $de_1$ and $de_2$ are easily verified, so that this does, in fact, integrate the equations for the frame field $(e_1, e_2, e_3)$. 

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One very interesting consequence of having this explicit form of $e_3$ is the following result due to Finikoff and Gambier:

**Proposition 3.** For any realization $x$ of a Type II shape operator $S$, the Gauß image of the net of principal curves of $x$ is an orthogonal net of confocal spherical ellipses.

**Proof.** Write

$e_3(X, Y) = \begin{bmatrix} u(X, Y) \\ v(X, Y) \\ w(X, Y) \end{bmatrix}$

and note that these components satisfy the equations

\[
\frac{u^2}{X - \lambda_1} + \frac{v^2}{X - \lambda_2} + \frac{w^2}{X - \lambda_3} = 0, \\
\frac{u^2}{Y - \lambda_1} + \frac{v^2}{Y - \lambda_2} + \frac{w^2}{Y - \lambda_3} = 0.
\]

This is the content of the proposition. \(\square\)

Now, note that $e_3$ satisfies the (vector-valued) Euler equation that is dual to the Euler equation (3.2.11) satisfied by $\Phi$

\[
\frac{\partial^2 e_3}{\partial X \partial Y} - \frac{1}{2} \frac{\partial e_3}{\partial X} \frac{1}{X - Y} \frac{\partial}{\partial X} + \frac{1}{2} \frac{\partial e_3}{\partial Y} \frac{1}{X - Y} \frac{\partial}{\partial Y} = 0,
\]

which is easily established by direct computation.

Finally, the immersion $x : D \to \mathbb{R}^3$ satisfies the structure equation

\[
dx = e_1 \omega_1 + e_2 \omega_2 = \Phi X e_1 \omega_{31} + \Phi Y e_2 \omega_{32}
\]

\[
= - \frac{\partial \Phi}{\partial X} \frac{\partial e_3}{\partial X} \, dX - \frac{\partial \Phi}{\partial Y} \frac{\partial e_3}{\partial Y} \, dY.
\]

The fact that $e_3$ and $\Phi$ satisfy dual Euler equations implies the identity

\[
\frac{\partial}{\partial Y} \left( \frac{\partial \Phi}{\partial X} \frac{\partial e_3}{\partial X} \right) = \frac{\partial}{\partial X} \left( \frac{\partial \Phi}{\partial Y} \frac{\partial e_3}{\partial Y} \right),
\]

implying that the vector-valued differential form on the right hand side of (3.2.21) is indeed a closed 1-form, so that $x$ can be recovered by quadrature:

\[
x = - \int \frac{\partial \Phi}{\partial X} \frac{\partial e_3}{\partial X} \, dX + \frac{\partial \Phi}{\partial Y} \frac{\partial e_3}{\partial Y} \, dY.
\]

3.2.6. *Toral unfolding, spherical quotient.* So far, no attention has been paid to the behavior of the immersion near the edges of the rectangular domain $D_\lambda$ defined by the inequalities $\lambda_1 < Y < \lambda_2 < X < \lambda_3$. It turns out, however, that one can often extend the immersion to cover these edges.

Consider the mapping of the torus

\[
T = \mathbb{R} / (2\pi \mathbb{Z}) \times \mathbb{R} / (2\pi \mathbb{Z})
\]

into the closure of $D_\lambda$ that is defined by the equations

\[
Y = Y_\lambda(x, y) = \lambda_1 \sin^2 y + \lambda_2 \cos^2 y, \\
X = X_\lambda(x, y) = \lambda_2 \cos^2 x + \lambda_3 \sin^2 x.
\]
This mapping is not an immersion along the half-lattice lines, but when one pulls \( e_3 \) back to \( T \) by this mapping, the square roots in the formula for \( e_3 \) can be resolved into the form

\[
(3.2.26) \quad e_3(x, y) = \begin{bmatrix}
\cos y \sqrt{\frac{(\lambda_3-\lambda_1) \sin^2 x + (\lambda_2-\lambda_1) \cos^2 x}{(\lambda_3-\lambda_1)}} \\
\sin x \sin y \\
\cos x \sqrt{\frac{(\lambda_3-\lambda_1) \sin^2 y + (\lambda_3-\lambda_2) \cos^2 y}{(\lambda_3-\lambda_1)}}
\end{bmatrix},
\]

in which the expressions under the radicals are strictly positive on the torus. Consequently, the map \( e_3 : T \to S^2 \) is smooth. Moreover, it is a submersion except at the four half-lattice points defined by the equations \( \sin x = \sin y = 0 \). In fact, the map \( e_3 \) is simply the quotient of \( T \) by the involution \( \tau : T \to T \) defined by \( \tau(x, y) = (-x, -y) \), whose fixed points are the half-lattice points. Note also the symmetries

\[
(3.2.27) \quad e_3(x+\pi, y) = R_u e_3(x, y), \quad e_3(x, y+\pi) = R_w e_3(x, y),
\]

where \( R_u \) and \( R_w \) are rotations by \( \pi \) about the \( u \)- and \( w \)-axes, respectively.

If \( \Phi \) is a solution to \( (3.2.11) \) that is smooth on a domain that contains the closure of \( D_\lambda \), then the formula \( (3.2.23) \) can be lifted back to the torus in the form

\[
(3.2.28) \quad x(x, y) = - \int_{(0,0)}^{(x,y)} \left( \frac{\partial \Phi}{\partial X} \left( X_\lambda(\xi, \eta), Y_\lambda(\xi, \eta) \right) \frac{\partial e_3}{\partial x}(\xi, \eta) \right) d\xi \\
+ \left( \frac{\partial \Phi}{\partial Y} \left( X_\lambda(\xi, \eta), Y_\lambda(\xi, \eta) \right) \frac{\partial e_3}{\partial y}(\xi, \eta) \right) d\eta.
\]

Using the \( \tau \)-invariance of \( e_3 \) and the symmetries \( (3.2.27) \), it is easy to show that the periods of the 1-form integrand in \( (3.2.28) \) must vanish. Thus, the line integral \( (3.2.28) \) defines a smooth mapping \( x : T \to \mathbb{R}^3 \).

Moreover, since the integrand is invariant under \( \tau \), it follows that \( x(x, y) = x(-x, -y) \), so that the mapping is actually well-defined on the quotient 2-sphere.

If, in addition, \( \Phi_X \) and \( \Phi_Y \) are positive on the closure of \( D_\lambda \), then \( x \) is an immersion away from the half-lattice points. Moreover, the Euler equation \( (3.2.11) \) and the smoothness of \( \Phi \) near the corner \( (X, Y) = (\lambda_2, \lambda_2) \) implies that \( \Phi_X(\lambda_2, \lambda_2) = \Phi_Y(\lambda_2, \lambda_2) > 0 \). From this, it is not difficult to see that the image \( x(T) \) must be a smoothly embedded convex 2-sphere.

In particular, note that holding \( \lambda_2 \) fixed and varying \( \lambda_1 \) and \( \lambda_3 \) gives a 2-parameter family of deformations preserving the shape operator in a neighborhood of an umbilic point with Hopf index \( \frac{1}{2} \).

3.2.7. Examples. In this last section, some interesting examples of these formulae will be investigated.

Example 6 (Quadrics). Consider the \( (3.2.11) \) solution

\[
(3.2.29) \quad \Phi(X, Y) = \frac{2}{\sqrt{-XY}},
\]

which is defined on the open quadrant \( Y < 0 < X \).
Suppose that $\lambda_1 < 0 < \lambda_3$ and, for simplicity, that $\lambda_2 \neq 0$. Then the formula (3.2.23) yields (up to a translation constant)

\[
\begin{pmatrix}
\frac{1}{\lambda_1} \sqrt{\frac{1}{1} \left( Y - \lambda_1 \right) \left( X - \lambda_1 \right)} \\
\frac{1}{\lambda_2} \sqrt{\frac{1}{1} \left( \lambda_2 - Y \right) \left( \lambda_1 - \lambda_2 \right) \left( -XY \right)} \\
\frac{1}{\lambda_3} \sqrt{\frac{1}{1} \left( \lambda_3 - Y \right) \left( \lambda_3 - \lambda_2 \right) \left( -XY \right)}
\end{pmatrix}
\]

defined on the rectangle for which $\lambda_1 < Y < \min \{0, \lambda_2\}$ and $\max \{0, \lambda_2\} < X < \lambda_3$.

All of these immersions have the shape operator

\[
S = -\sqrt{-X^3Y} \frac{\partial}{\partial X} \otimes dX + \sqrt{-XY^3} \frac{\partial}{\partial Y} \otimes dY,
\]

as is easily checked, the image of $x = (u, v, w)$ lies in the hyperboloid of one sheet defined by

\[
\lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2 + \frac{1}{\lambda_1 \lambda_2 \lambda_3} = 0.
\]

The case $\lambda_2 = 0$ has to be treated separately. It turns out that these surfaces are algebraic surfaces of degree 3. Details are left to the reader.

Finally, consider the (3.2.11) solution

\[
\Phi(X, Y) = -\frac{2}{\sqrt{XY}},
\]

which is defined on the two open wedges $0 < Y < X$ and $Y < X < 0$. Note that $\Phi_X$ and $\Phi_Y$ are positive and unequal on the domain of definition. Thus, the corresponding mappings $x$ defined on the domains $D_\lambda$ contained in the domain of $\Phi$ will be umbilic-free immersions with positive principal curvatures.

If $0 < \lambda_1 < \lambda_2 < \lambda_3$ or $\lambda_1 < \lambda_2 < \lambda_3 < 0$, so that $D_\lambda$ is contained in the domain of $\Phi$, then the image of the corresponding immersions $x : D_\lambda \to \mathbb{E}^3$ is part of an ellipsoid with three distinct principal axes. The method of §3.2.6 then shows how this can be used to parametrize the entire ellipsoid. Note that the corner $(\lambda_2, \lambda_2)$ (which lies in the closure of $D_\lambda$) gives rise to the four umbilic points that such an ellipsoid possesses.

The cases with $\lambda_1 = 0$ give elliptic paraboloids while $\lambda_1 < 0 < \lambda_2 < \lambda_3$ gives one sheet of an hyperboloid of two sheets. Details are left to the reader.

**Example 7 (Polynomial solutions).** Consider the quadratic (3.2.11) solution

\[
\Phi(X, Y) = -\frac{1}{2} \left( 3X^2 + 2XY + 3Y^2 \right)
\]
defined on the entire half-plane \( Y < X \). Then the formula (3.2.23) yields (up to a translation constant)

\[
\mathbf{x}(X,Y) = \begin{bmatrix}
(X + Y + 2\lambda_1)\sqrt{(Y - \lambda_1)(X - \lambda_1)} \\
\sqrt{\lambda_2 - \lambda_1}(\lambda_3 - \lambda_1) \\
(X + Y + 2\lambda_2)\sqrt{(\lambda_2 - Y)(X - \lambda_2)} \\
\sqrt{\lambda_3 - \lambda_1}(\lambda_3 - \lambda_2) \\
(X + Y + 2\lambda_3)\sqrt{(\lambda_3 - Y)(\lambda_3 - X)} \\
\sqrt{\lambda_3 - \lambda_2}(\lambda_3 - \lambda_2)
\end{bmatrix}
\]  

(3.2.35)

defined on the rectangle \( D_\lambda \) for which \( \lambda_1 < Y < \lambda_2 < X < \lambda_3 \).

By the method of \( \S3.2.6 \), this can be extended to a smooth mapping of a 2-sphere into \( \mathbb{E}^3 \). Note that \( \mathbf{x} \) fails to be an immersion along the lines \( Y + 3X = 0 \) and \( X + 3Y = 0 \). Away from these lines, all of these immersions have the shape operator

\[
S = \left(-\frac{1}{3X + Y}\right) \frac{\partial}{\partial X} \otimes dX + \left(-\frac{1}{X + 3Y}\right) \frac{\partial}{\partial Y} \otimes dY,
\]

(3.2.36)

It is worth noting that for each positive degree \( d \), the equation (3.2.11) has a polynomial solution \( \Phi_d(X,Y) \) that is homogeneous of degree \( d \) and that the solution with this property is unique up to constant multiples. The corresponding surfaces are algebraic, but can be quite complicated. Nevertheless, some interesting examples can be found here:

For example, the homogeneous cubic solution

\[
\Phi(X,Y) = -\frac{1}{3}(5X^3 + 3X^2Y + 3XY^2 + 5Y^3)
\]

(3.2.37)

is defined on the entire half-plane \( Y < X \) and its partials \( \Phi_X \) and \( \Phi_Y \) are positive on the entire half-plane. They are equal only when \( X + Y = 0 \) and they vanish on the closed half-plane only at the point \( (X,Y) = (0,0) \).

As a result, as long as \( \lambda_2 \neq 0 \), the corresponding immersion \( \mathbf{x} \) extends to an immersion of the 2-sphere as a strictly convex ovaloid. As long as \( \lambda_1 + \lambda_2 > 0 \) or \( \lambda_2 + \lambda_3 < 0 \), the domain \( D_\lambda \) does not meet the line \( X + Y = 0 \), so the only umbilics of the resulting ovaloid are the four corresponding to \( (\lambda_2, \lambda_2) \). However, if \( \lambda_1 + \lambda_2 < 0 < \lambda_2 + \lambda_3 \), then the domain \( D_\lambda \) meets the line \( X + Y = 0 \) in a segment, which gives rise to a circle of umbilics in the resulting ovaloid.

Example 8 (Minimal surfaces). As a final example, consider the solution

\[
\Phi(X,Y) = 2 \log(X - Y)
\]

(3.2.38)

defined on the entire half-plane \( Y < X \). Note that, because \( \Phi_X + \Phi_Y = 0 \), the resulting surfaces will be minimal surfaces. In fact, it is evident that, up to scaling and the addition of a constant, this is the unique solution that satisfies \( \Phi_X + \Phi_Y = 0 \), and so gives a minimal surface.
The formula (3.2.23) yields (up to a translation constant)

\[
x(X, Y) = \left[ \log \left( \frac{\sqrt{X - \lambda_1} + \sqrt{Y - \lambda_1}}{\sqrt{X - \lambda_1} - \sqrt{Y - \lambda_1}} \right) \right]
\] \\
\[\frac{\sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}}{2 \arctan \left( \frac{\sqrt{X - \lambda_2}}{\sqrt{\lambda_2 - Y}} \right)} - \frac{\sqrt{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}}{2 \arctan \left( \frac{\sqrt{\lambda_3 - Y}}{\sqrt{\lambda_3 - \lambda_2}} \right)} \]

(3.2.39)

defined on the rectangle \( D_\lambda \). All of these immersions have the shape operator

\[
S = \frac{X - Y}{2} \frac{\partial}{\partial X} \otimes dX + \frac{Y - X}{2} \frac{\partial}{\partial Y} \otimes dY.
\]

Using \((u, v, w)\) as coordinates on \( \mathbb{E}^3 \), the image minimal surface satisfies the equation

\[
0 = (\lambda_2 - \lambda_3) \cosh \left( u \sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \right)
\]

(3.2.41)

\[
- (\lambda_3 - \lambda_1) \cos \left( v \sqrt{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \right)
\]

\[
- (\lambda_3 - \lambda_2) \cosh \left( w \sqrt{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right).
\]

Thus, this belongs to the family (investigated by Weingarten and by Frechét, see [12, Chapter II, §5.2]) of minimal surfaces that satisfy equations of the form \( f(u) + g(v) + h(w) = 0 \).

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