Numerical semigroups generated by quadratic sequences

Mara Hashuga · Megan Herbine · Alathea Jensen

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Abstract
We investigate numerical semigroups generated by any quadratic sequence with initial term zero and an infinite number of terms. We find an efficient algorithm for calculating the Apéry set, as well as bounds on the elements of the Apéry set. We also find bounds on the Frobenius number and genus, and the asymptotic behavior of the Frobenius number and genus. Finally, we find the embedding dimension of all such numerical semigroups.

Keywords Numerical semigroup · Quadratic sequence · Apéry set · Frobenius number · Genus · Embedding dimension

Mathematics Subject Classification 20M05 · 20M14

1 Introduction
The investigation of numerical semigroups generated by particular kinds of sequences dates back to at least 1942, when Brauer [1] found the Frobenius number for numerical semigroups generated by sequences of consecutive integers. Roberts [16] followed in 1956 with the Frobenius number of numerical semigroups generated by generic arithmetic sequences.

It might seem natural that after conquering arithmetic sequences, work would proceed apace on other common types of sequences, especially geometric sequences and polynomial sequences, which are the other two types of sequences most frequently encountered in mathematics education. However, this was not the case.

Instead, researchers such as Lewin [8] and Selmer [22] turned their attention to generalized arithmetic sequences—sequences which are arithmetic except for one

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Alathea Jensen
jensena@susqu.edu

1 Susquehanna University, 514 University Ave, Selinsgrove, PA 17870, USA

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term. Work on generalized arithmetic sequences continues to the present day, in, for example, [2, 6, 13].

Work on geometric sequences did not appear in the literature until 2008, when Ong and Ponomarenko [14] found the Frobenius number of a numerical semigroup generated by a geometric sequence. Work on generalized geometric sequences, called compound sequences, also continues to the present day, in, for example, [4]. Some work has also been done on other, more exotic types of sequences, such as the Fibonacci sequence [9], sequences of repunits [19], sequences of Mersenne numbers [20], and sequences of Thabit numbers [18].

Only a very small amount of work has appeared on numerical semigroups generated by polynomial sequences, and only for particular instances of polynomials, not for generic polynomials. This includes numerical semigroups generated by three consecutive squares or cubes [7], infinite sequences of squares [10], and sequences of three consecutive triangular numbers or four consecutive tetrahedral numbers [17]. In [3], the authors tantalizingly defined something called a quadratic numerical semigroup; however, the quadratic object in question is an associated algebraic ideal, not a sequence of generators.

Thus, to date, no one has investigated the numerical semigroups generated by a generic quadratic sequence, a generic cubic sequence, or any generic polynomial sequence of higher degree. This work is important not only because these are common sequences worthy of investigation in their own right, but also because every numerical semigroup is generated by a subset of a polynomial sequence of sufficient degree. This is so because a polynomial formula can be fitted to any finite set of numbers. Hence, an understanding of numerical semigroups generated by polynomial sequences would contribute to the understanding of all numerical semigroups.

In this article, we begin the investigation of numerical semigroups generated by generic quadratic sequences, and lay out a framework for its continuation. In Sect. 2, we review general definitions and notations of numerical semigroups. In Sect. 3, we define a family of numerical semigroups $S(a, b)$, parameterized by $a, b \in \mathbb{N}_0$, that comprise all numerical semigroups generated by an infinite quadratic sequence with initial term 0. In Sects. 4 to 7, we develop a formula for and bounds on the Apéry set of $S(a, b)$ and show that the formula applies in all but eight exceptional cases (Corollary 7.3). In Sect. 8, we characterize the asymptotic growth rate of the Frobenius number and genus, in terms of $a$ and $b$ (Theorems 8.2 and 8.3). Finally, in Sect. 9, we find the exact embedding dimension of all such numerical semigroups (Theorem 9.6).

2 Background: numerical semigroups

In this section, we will define the most important objects and parameters associated with numerical semigroups, as well as common facts about these objects, given here as lemmas. These definitions and lemmas are taken from the standard reference text [21].

Before we begin, it is very important to note that throughout this article, we will use $\mathbb{N}$ to denote $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{N}_0$ to denote $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.
A monoid is a set \( M \), together with a binary operation \(+\) on \( M \), such that \(+\) is closed, associative, and has an identity element in \( M \). A subset \( N \) of \( M \) is a submonoid of \( M \) if and only if \( N \) is also a monoid using the same operation as \( M \).

Given a monoid \( M \) and a subset \( A \) of \( M \), the smallest submonoid of \( M \) containing \( A \) is

\[ \langle A \rangle = \{ \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}_0, \lambda_1, \ldots, \lambda_n \in \mathbb{N}_0 \text{ and } a_1, \ldots, a_n \in A \}. \]

The elements of \( A \) are called generators of \( \langle A \rangle \) or a system of generators of \( \langle A \rangle \), and we accordingly say that \( \langle A \rangle \) is generated by \( A \).

Clearly, \( \mathbb{N}_0 \) is a monoid under the standard addition operation. A submonoid \( S \) of \( \mathbb{N}_0 \) is a numerical semigroup if and only if it has a finite complement in \( \mathbb{N}_0 \).

**Lemma 2.1** Let \( A \) be a nonempty subset of \( \mathbb{N}_0 \). Then \( \langle A \rangle \) is a numerical semigroup if and only if \( \gcd(A) = 1 \).

A system of generators of a numerical semigroup is said to be minimal if and only if none of its proper subsets generate the numerical semigroup.

**Lemma 2.2** Every numerical semigroup has a unique, finite, minimal system of generators. Furthermore, any set which generates the numerical semigroup contains this minimal system of generators as a subset.

The least element in the minimal system of generators of a numerical semigroup \( S \) is called the multiplicity of \( S \), and is denoted by \( m(S) \). The cardinality of the minimal system of generators is called the embedding dimension of \( S \) and is denoted by \( e(S) \).

**Lemma 2.3** Let \( S \) be a numerical semigroup. Then \( m(S) = \min(S \setminus \{0\}) \) and \( e(S) \leq m(S) \).

The greatest integer not in a numerical semigroup \( S \) is known as the Frobenius number of \( S \) and is denoted by \( F(S) \). The set of elements in \( \mathbb{N}_0 \) that are not in \( S \) is known as the gap set of \( S \), and is denoted by \( G(S) \). The cardinality of the gap set is known as the genus of \( S \) and is denoted by \( g(S) \).

The Apéry set of \( n \) in \( S \), where \( n \) is a nonzero element of the numerical semigroup \( S \), is

\[ \text{Ap}(S, n) = \{ s \in S \mid s - n \notin S \}. \]

**Lemma 2.4** Let \( S \) be a numerical semigroup and let \( n \) be a nonzero element of \( S \). Then \( \text{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n - 1)\} \), where \( w(i) \) is the least element of \( S \) congruent with \( i \) modulo \( n \), for all \( i \in \{0, \ldots, n - 1\} \).

There is no known general formula for the Frobenius number or the genus for numerical semigroups. However, we can compute both values if the Apéry set of any nonzero element of the semigroup is known.
Lemma 2.5 Let $S$ be a numerical semigroup and let $n$ be a nonzero element of $S$. Then

$$F(S) = (\max \text{Ap}(S, n)) - n$$

and

$$g(S) = \frac{1}{n} \left( \sum_{w \in \text{Ap}(S, n)} w \right) - \frac{n - 1}{2}.$$

3 Generating a numerical semigroup from a quadratic sequence

Our aim, broadly, is to study all numerical semigroups generated by the terms of a quadratic sequence. As this goal is rather large, we restrict ourselves in the present paper to numerical semigroups generated by all infinitely many terms of a quadratic sequence whose initial term is 0. In future, however, we plan to study numerical semigroups generated by quadratic sequences in greater generality.

In this section, we give definitions, notations, and lemmas regarding exactly which numerical semigroups we will be discussing throughout the paper.

Definition 3.1 We say that a numerical semigroup $S$ is generated by an infinite quadratic sequence with initial term zero whenever $S = \langle y_0, y_1, y_2, \ldots \rangle$ where $y_n$ is a degree-2 polynomial function of $n$ and $y_0 = 0$.

It might seem natural to specify a general formula for $y_n$ in the form $y_n = c_0 + c_1 n + c_2 n^2$. However, the use of the $1, n, n^2$ basis is problematic, because it is possible for some of the coefficients $c_n$ to be negative or even to be non-integers and still have $y_n \in \mathbb{N}_0$ for all $n \in \mathbb{N}_0$. For example, $y_n = -1.5n + 2.5n^2 \in \mathbb{N}_0$ for all $n \in \mathbb{N}_0$.

As the next lemma demonstrates, the basis $1, n, \binom{n}{2}$ is a more natural choice of basis for our purposes.

Lemma 3.2 The sequence $y_n$ is a quadratic sequence of non-negative integers with initial term $y_0 = 0$ if and only if $y_n = an + b\binom{n}{2}$ for some $a, b \in \mathbb{N}_0$.

Proof. Clearly, if $y_n = an + b\binom{n}{2}$ where $a, b \in \mathbb{N}_0$, then $y_0 = 0$ and $y_n$ is a quadratic sequence of non-negative integers.

On the other hand, suppose that $y_n$ is a quadratic sequence of non-negative integers with initial term $y_0 = 0$. It is a well known fact that a sequence $y_n$ is quadratic if and only if its sequence of first differences is arithmetic.

Let the sequence of first differences of $y_n$ be denoted $x_n := y_{n+1} - y_n$. Since $x_n$ is arithmetic, it can be written in the form $x_n = a + bn$ for some $a, b \in \mathbb{R}$. Since $y_1 = y_0 + x_0 = a$, we must have $a \in \mathbb{N}_0$. Furthermore, since $y_2 = y_1 + x_1 = 2a + b$, we have $b = y_2 - 2a$ and so $b \in \mathbb{Z}$. Note also that we must have $b \geq 0$ since, if not, $x_n$ will be eventually negative and decreasing, and so $y_n$ will eventually decreasing and negative. Hence, $b \in \mathbb{N}_0$. 

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It remains only to show that $y_n = an + b(\binom{n}{2})$. We can recover the formula for $y_n$ from $x_n$ via the application of summation formulas:

$$y_n = y_0 + \sum_{i=0}^{n-1} x_i = 0 + \sum_{i=0}^{n-1} (a + bi) = an + b\binom{n}{2}.$$  

\[\square\]

Regardless of whether it generates a numerical semigroup, we can use the quadratic sequence $y_n$ to generate a monoid. Clearly the elements of the monoid depend on $a$ and $b$, so we will refer to the monoid generated by $y_n$ for the particular values of $a$ and $b$ as $S(a, b)$. The following definition gives this notation more formally.

**Definition 3.3** Let $a, b \in \mathbb{N}_0$. Then

$$S(a, b) := \left\{ na + \binom{n}{2}b \mid n \in \mathbb{N}_0 \right\}. $$

At times, we may refer to $S(a, b)$ simply as $S$, if the values of $a$ and $b$ are fixed and are clear from context.

We can now characterize precisely which infinite quadratic sequences with initial term zero give rise to a numerical semigroup.

**Lemma 3.4** For all $a, b \in \mathbb{N}_0$, $S(a, b)$ is a numerical semigroup if and only if $\gcd(a, b) = 1$.

**Proof** First, note that any common factor of $a$ and $b$ is also a common factor of all $y_n$, and so is a common factor of $S(a, b)$. Thus, if $\gcd(a, b) \neq 1$, then $S(a, b)$ is not a numerical semigroup. Next, since $y_1, y_2 \in S(a, b)$, $\gcd(S(a, b)) \leq \gcd(y_1, y_2) = \gcd(a, 2a + b) = \gcd(a, b)$. Hence, if $\gcd(a, b) = 1$, then $\gcd(S(a, b)) = 1$ and so $S(a, b)$ is a numerical semigroup.  

The next lemma determines when we have $S(a, b) = \mathbb{N}_0$.

**Lemma 3.5** For all $a, b \in \mathbb{N}_0$ with $\gcd(a, b) = 1$, $S(a, b) = \mathbb{N}_0$ if and only if $a = 0$ or $a = 1$ or $b = 0$.

**Proof** For the first direction, suppose that $S(a, b) = \mathbb{N}_0$. Then 1 must be a generator, so $y_n = 1$ for some $n$. The only values $(a, b, n)$ for which $na + \binom{n}{2}b = 1$ are $(a, b, n) = (1, b, 1)$ and $(0, 1, 2)$. Hence $a = 0$ or $a = 1$.

For the other direction, when $a = 0$, in order to have $\gcd(a, b) = 1$, we must have $b = 1$. In this case, $y_2 = 2a + \binom{2}{2}b = 1$, hence $S(0, 1) = \mathbb{N}_0$. Similarly, when $a = 1$, we have $y_1 = 1a + \binom{1}{2}b = 1$, so $S(1, b) = \mathbb{N}_0$. Finally, when $b = 0$, in order to have $\gcd(a, b) = 1$, we must have $a = 1$, and so $S(1, 0) = \mathbb{N}_0$ as well.  

As a consequence of this lemma, we may occasionally include the condition that $a > 1$ or $b > 0$ in our theorems, when it is convenient to do so.

In closing this section, let us look at an example of a numerical semigroup generated by an infinite quadratic sequence with initial term zero, in order to clarify the definitions and concepts we have just discussed.
Example 3.6 Let \( a = 2 \) and \( b = 1 \). Then

\[
y_n = na + \left(\frac{n}{2}\right)b = 2n + \frac{n(n - 1)}{2} = \frac{n(n + 3)}{2},
\]

\[
S(2, 1) = \langle y_0, y_1, y_2, \ldots \rangle = \langle 0, 2, 5, 9, 14, 20, 27, \ldots \rangle.
\]

As \( S(2, 1) \) includes 2, it must include all even natural numbers, and, as it includes 5 and 2, it must include all odd numbers beginning with 5. In fact, 1 and 3 are the only natural numbers that cannot be made with these generators, hence \( S(2, 1) = (2, 5) = \mathbb{N}_0 \setminus \{1, 3\} \).

4 The \( \mu_{a,b} \) sequence and the Apéry set

One of the most important questions that we can ask about \( S(a, b) \) is the following: for a given coefficient \( n \) of \( b \), what is the minimum coefficient \( m \) of \( a \) such that \( ma + nb \in S(a, b) \)? We will use the notation \( \mu_{a,b}(n) \) to denote the answer to this question. The following definition formalizes this notion.

Definition 4.1 For any \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \) and any \( n \in \mathbb{Z} \), let

\[
\mu_{a,b}(n) := \min\{m \in \mathbb{Z} \mid ma + nb \in S(a, b)\}.
\]

Note that we have defined the quantity \( \mu_{a,b} \) as being a map on \( \mathbb{Z} \) rather than on \( \mathbb{N}_0 \). This is so because it is possible for \( ma + nb \) to be in \( S(a, b) \) when \( m \) or \( n \) are negative, although not, of course, when both \( m \) and \( n \) are negative. For example, when \( m = b \) and \( n = -a \), \( ma + nb = 0 \in S(a, b) \), and likewise when \( m = -b \) and \( n = a \).

The \( \mu_{a,b} \) values are very important, because they give us the Apéry set \( \text{Ap}(S(a, b), a) \), as shown by the following theorem.

Theorem 4.2 For all \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \) and \( a \geq 1 \),

\[
\text{Ap}(S(a, b), a) = \{ \mu_{a,b}(n)a + nb \mid n = 0, 1, 2, \ldots, a - 1 \}.
\]

Proof We assumed \( \gcd(a, b) = 1 \), so \( 0b, 1b, \ldots, (a - 1)b \) form all of the different congruence classes modulo \( a \). It follows that all elements of the form \( \mu_{a,b}(n)a + nb \) where \( n = 0, 1, 2, \ldots, a - 1 \) are in different congruence classes modulo \( a \). By definition of \( \mu_{a,b} \), each \( \mu_{a,b}(n)a + nb \) is the smallest element of \( S(a, b) \) in its congruence class modulo \( a \).

In fact, only the values \( \mu_{a,b}(0), \mu_{a,b}(1), \ldots, \mu_{a,b}(a - 1) \) are needed to find any value of \( \mu_{a,b} \), as the next theorem will show. In a later section, we will show an efficient way of calculating these values.

Theorem 4.3 For all \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \) and for all \( i, n \in \mathbb{Z} \),

\[
\mu_{a,b}(n + ia) = \mu_{a,b}(n) - ib.
\]
\textbf{Proof.} By definition,
\begin{equation*}
\mu_{a,b}(n + ia) = \min\{m \in \mathbb{Z} \mid ma + (n + ia)b \in S(a,b)\} \\
= \min\{m \in \mathbb{Z} \mid (m + ib)a + nb \in S(a,b)\}.
\end{equation*}
Now we can add \(ib\) to both sides to get
\begin{equation*}
\mu_{a,b}(n + ia) + ib = ib + \min\{m \in \mathbb{Z} \mid (m + ib)a + nb \in S(a,b)\} \\
= \min\{(m + ib) \in \mathbb{Z} \mid (m + ib)a + nb \in S(a,b)\}.
\end{equation*}
Let \(m' = m + ib\). Then
\begin{equation*}
\mu_{a,b}(n + ia) + ib = \min\{m' \in \mathbb{Z} \mid m'a + nb \in S(a,b)\} = \mu_{a,b}(n).
\end{equation*}
\(\square\)

5 The lifting to \(\mathbb{N}_0^2\) and the \(\mu\) sequence

Now we will define a monoid in \(\mathbb{N}_0^2\) that will allow us to unify all numerical semigroups generated by infinite quadratic sequences with initial term zero.

Let \(T\) be the monoid generated by all linear combinations over \(\mathbb{N}_0\) of the generators \(z_i = \left( i, \frac{i(i+1)}{2} \right) \) where \(i \in \mathbb{N}_0\). That is,
\begin{equation*}
T = \left\{ \sum_{i=1}^{\infty} \lambda_i z_i \mid \lambda_i \in \mathbb{N}_0 \right\} \\
= \left\{ \left( 1\lambda_1 + 2\lambda_2 + \cdots, \frac{1}{2}\lambda_1 + \frac{2}{2}\lambda_2 + \cdots \right) \mid \lambda_i \in \mathbb{N}_0 \right\}.
\end{equation*}
The following theorem shows how the \(T\) monoid is connected to our numerical semigroups \(S(a,b)\).

\textbf{Theorem 5.1} Let \(a, b \in \mathbb{N}_0\) with \(\gcd(a, b) = 1\), let \(T\) be as previously defined, and let \(\phi_{a,b} : \mathbb{N}_0^2 \to \mathbb{N}_0\) be given by \(\phi_{a,b}(m, n) = ma + nb\). Then \(\phi_{a,b}[T] = S(a,b)\).

\textbf{Proof.} The image of \(T\) under \(\phi_{a,b}\) is
\begin{equation*}
\{ ma + nb \mid (m, n) \in T \} \\
= \{ ma + nb \mid m = 1\lambda_1 + 2\lambda_2 + \cdots, n = \left( \frac{1}{2}\right)\lambda_1 + \left( \frac{2}{2}\right)\lambda_2 + \cdots, \lambda_i \in \mathbb{N}_0 \} \\
= \{ (1\lambda_1 + 2\lambda_2 + \cdots)a + \left( \frac{1}{2}\right)\lambda_1 + \left( \frac{2}{2}\right)\lambda_2 + \cdots b \mid \lambda_i \in \mathbb{N}_0 \}.
\end{equation*}
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Fig. 1 \( \mathbb{N}_0^2 \) with elements of \( T \) colored black

\[
\begin{align*}
&= \left\{ \lambda_1 \left( 1a + \left( \frac{1}{2} \right)b \right) + \lambda_2 \left( 2a + \left( \frac{2}{2} \right)b \right) + \cdots \mid \lambda_i \in \mathbb{N}_0 \right\} \\
&= \{ \lambda_1 y_1 + \lambda_2 y_2 + \cdots \mid \lambda_i \in \mathbb{N}_0 \} = S(a, b).
\end{align*}
\]

Hence, the monoid \( T \) unifies all numerical semigroups \( S(a, b) \) in the sense that each such \( S(a, b) \) is a particular projection of \( T \).

Now we will discuss the elements of \( T \). Figure 1 shows the elements \( (m, n) \in T \) for \( m, n < 50 \). As can be seen in the figure, within each row, the color changes from white to black exactly once. To say this more formally, for each value of \( n \in \mathbb{N}_0 \), there is some value \( \mu(n) \in \mathbb{N}_0 \) for which \( m < \mu(n) \) implies \( (m, n) \not\in T \) and \( m \geq \mu(n) \) implies \( (m, n) \in T \). This is so because \( z_1 = (1, 0) \) is one of the generators of \( T \), thus, for any \( (m, n) \in T \), we have \( (m + 1, n) \in T \) as well.

This behavior is remarkably similar to that of an Apéry set for a numerical semigroup, as well as to the behavior of \( \mu_{a,b} \), and we will show in Sect. 7 that, in fact, \( \mu_{a,b}(n) = \mu(n) \) when \( 0 \leq n < a \), except for eight particular values of \( (a, b, n) \).

Before we can show that, however, we need to establish many properties of \( \mu \), in this and the next section.

We begin with a formal definition of \( \mu(n) \):

**Definition 5.2** Let \( n \in \mathbb{N}_0 \). Then \( \mu(n) \) is defined as

\[ \mu(n) := \min \{ m \in \mathbb{N}_0 \mid (m, n) \in T \} . \]

Note that here, unlike with \( \mu_{a,b} \), we have defined \( \mu \) as a map on \( \mathbb{N}_0 \) rather than on \( \mathbb{Z} \) because it is not possible for either \( m \) or \( n \) to be negative when \( (m, n) \in T \).

Now we will establish some properties of \( \mu \) that will allow us to prove an efficient method of calculating the \( \mu \) values.
Theorem 5.3 (Recursive bound on $\mu$) For all $n_1, n_2 \in \mathbb{N}_0$,

$$\mu(n_1 + n_2) \leq \mu(n_1) + \mu(n_2).$$

Proof. Since $\mu(n_1) = \min\{m \in \mathbb{N}_0 \mid (m, (n_1)) \in T\}$, that means $(\mu(n_1), n_1) \in T$. The same logic holds true for $\mu(n_2)$. Hence we can add together these two elements in $T$ to get another element that is in $T$.

$$(\mu(n_1), n_1) + (\mu(n_2), n_2) = (\mu(n_1) + \mu(n_2), n_1 + n_2) \in T.$$ By definition, $\mu(n_1 + n_2) = \min\{m \in \mathbb{N}_0 \mid (m, n_1 + n_2) \in T\}$. Since $\mu(n_1) + \mu(n_2) \in \mathbb{N}_0$ and $(\mu(n_1) + \mu(n_2), n_1 + n_2) \in T$,

$$\mu(n_1) + \mu(n_2) \in \{m \in \mathbb{N}_0 \mid (m, n_1 + n_2) \in T\}.$$ Since $\mu(n_1) + \mu(n_2)$ is in the set we are taking the minimum over to get $\mu(n_1 + n_2)$, we can say that

$$\mu(n_1 + n_2) \leq \mu(n_1) + \mu(n_2).$$

\[\Box\]

A particular application of the previous theorem is the following.

Corollary 5.4 For all $n, i \in \mathbb{N}_0$, if $\binom{i}{2} \leq n$, then

$$\mu(n) \leq \mu\left(n - \binom{i}{2}\right) + i.$$

Proof Assume $i \in \mathbb{N}_0$ and $\binom{i}{2} \leq n$. The previous theorem states that if $n_1, n_2 \in \mathbb{N}_0$, then $\mu(n_1 + n_2) \leq \mu(n_1) + \mu(n_2)$. Let $n_1 = n - \binom{i}{2}$ and $n_2 = \binom{i}{2}$. By the assumptions about $i$, we can see that $n_1, n_2 \in \mathbb{N}_0$. By plugging these values into the inequality, we get

$$\mu\left(n - \binom{i}{2} + \binom{i}{2}\right) \leq \mu\left(n - \binom{i}{2}\right) + \mu\left(\binom{i}{2}\right).$$

Simplifying, we obtain

$$\mu(n) \leq \mu\left(n - \binom{i}{2}\right) + \mu\left(\binom{i}{2}\right).$$
We know that \( z_i = \left( i, \left( \frac{i}{2} \right) \right) \in T \). So, by the definition of \( \mu(i) \), it must be that \( \mu \left( \left( \frac{i}{2} \right) \right) \leq i \). Therefore,

\[
\mu(n) \leq \mu \left( n - \left( \frac{i}{2} \right) \right) + i.
\]

The next theorem provides a computationally efficient way of calculating \( \mu \) values. Note that the theorem statement refers to \( \mathbb{N} \), not \( \mathbb{N}_0 \). Values of \( \mu \) that were calculated using this theorem, as well as a Python implementation of the theorem, can be found at [12].

**Theorem 5.5** For all \( n \in \mathbb{N} \),

\[
\mu(n) = \min \left\{ \mu \left( n - \left( \frac{i}{2} \right) \right) + i \mid i \in \mathbb{N}, 1 \leq \left( \frac{i}{2} \right) \leq n \right\}.
\]

**Proof.** Let \( n \in \mathbb{N} \). We know that \( \mu(n) \) is defined as the minimum value of \( 1c_1 + 2c_2 + \cdots \) such that \( c_1, c_2, \ldots \in \mathbb{N}_0 \) and \( n = \left( \frac{i}{2} \right) c_1 + \left( \frac{3}{2} \right) c_2 + \cdots \). So, there exist some \( c_1, c_2, \ldots, c_k \in \mathbb{N}_0 \) where \( k \in \mathbb{N} \) such that

\[
\mu(n) = 1c_1 + 2c_2 + \cdots + kc_k,
\]

and

\[
n = \left( \frac{1}{2} \right) c_1 + \left( \frac{2}{2} \right) c_2 + \cdots + \left( \frac{k}{2} \right) c_k = \left( \frac{2}{2} \right) c_2 + \cdots + \left( \frac{k}{2} \right) c_k.
\]

We know that \( \left( \frac{k}{2} \right) \leq n \), because if \( \left( \frac{k}{2} \right) > n \), then the right hand side of the second equation from above will be greater than the left hand side. Also, since \( n \neq 0 \), there exists some \( i \in \{ 2, 3, \ldots, k \} \) such that \( c_i > 0 \), because all \( c_j \geq 0 \) and not all of them can be 0 because \( n \neq 0 \). By subtracting \( i \) from both sides of our \( \mu(n) \) equation, we obtain

\[
\mu(n) - i = -i + 1c_1 + 2c_2 + \cdots + kc_k
\]

\[
= 1c_1 + 2c_2 + \cdots + (i - 1)c_{i-1} + i(c_i - 1) + (i + 1)c_{i+1} + \cdots + kc_k.
\]

Since \( c_i > 0 \), \( c_i - 1 \in \mathbb{N}_0 \). Next, we can subtract \( i \) from both sides of our \( n \) equation to obtain

\[
n - \left( \frac{i}{2} \right) = - \left( \frac{i}{2} \right) + \left( \frac{1}{2} \right) c_1 + \left( \frac{2}{2} \right) c_2 + \cdots + \left( \frac{k}{2} \right) c_k
\]

\[
= \left( \frac{1}{2} \right) c_1 + \left( \frac{2}{2} \right) c_2 + \cdots + \left( \frac{i - 1}{2} \right) c_{i-1} + \left( \frac{i}{2} \right) (c_i - 1)
\]

\[
+ \left( \frac{i + 1}{2} \right) c_{i+1} + \cdots + \left( \frac{k}{2} \right) c_k.
\]
Since \( c_i > 0, c_i - 1 \in \mathbb{N}_0 \). Therefore, all of our coefficients will still be in \( \mathbb{N}_0 \), so
\[
\left( \mu(n) - i, n - \binom{i}{2} \right) \in \mathcal{T}.
\]

So then, by the definition of \( \mu(n - \binom{i}{2}) \),
\[
\mu(n) - i \geq \mu\left(n - \binom{i}{2}\right)
\]
which implies that
\[
\mu(n) \geq \mu\left(n - \binom{i}{2}\right) + i.
\]

However, Corollary 5.4 states that for all \( n, i \in \mathbb{N}_0 \), if \( \binom{i}{2} \leq n \), then
\[
\mu(n) \leq \mu\left(n - \binom{i}{2}\right) + i.
\]

Therefore, \( \mu(n) = \mu\left(n - \binom{i}{2}\right) + i \) for some \( i \in \mathbb{N} \) such that \( 1 \leq \binom{i}{2} \leq n \), while at the same time \( \mu(n) \leq \mu\left(n - \binom{i}{2}\right) + i \) for all \( i \in \mathbb{N}_0 \). So, it must be the case that
\[
\mu(n) = \min \left\{ \mu\left(n - \binom{i}{2}\right) + i \mid i \in \mathbb{N}, 1 \leq \binom{i}{2} \leq n \right\}.
\]

\[\square\]

6 Bounding the \( \mu \) sequence

As noted previously, we will show in Sect. 7 that, in fact, \( \mu_{a,b}(n) = \mu(n) \) when \( 0 \leq n < a \), except for eight particular values of \((a, n)\). This implies that the Apéry set, the Frobenius number, and the genus of any numerical semigroup \( S(a, b) \) can be written purely in terms of \( \mu \) rather than \( \mu_{a,b} \). However, in order to prove that, we need more information than we presently have about the values of \( \mu \).

In addition to that upcoming application of \( \mu(n) \), the \( \mu(n) \) sequence is of some interest in its own right, as it is related to integer partitions, so it is a worthwhile exercise to investigate its values.

From the definition of \( \mu(n) \), we can see that each \( \mu(n) \) is the optimal solution of an integer linear program:
\[
\mu(n) = \min \left\{ 1\lambda_1 + 2\lambda_2 + \cdots \mid \lambda_1, \lambda_2, \cdots \in \mathbb{N}_0, n = \binom{1}{2}\lambda_1 + \binom{2}{2}\lambda_2 + \cdots \right\}.
\]
Finding the optimum value of a general integer linear program is known to be NP-hard [15]. It is possible, since we are looking at a very specific family of integer linear programs, that there may be some closed formula for \( \mu(n) \) as a function of \( n \), but we have not been able to find it. We only know of one case in which a closed formula exists, which is shown in Corollary 6.4. Thus, we focus our attention on developing upper and lower bounds for \( \mu \).

First, let us define and look at the properties of a function that will be of much use throughout the remainder of this section and the next. This function is the inverse of the \((\frac{x}{2})\) function for \( x \in \mathbb{N} \).

**Definition 6.1** The function \( f : \mathbb{N} \rightarrow \mathbb{R} \) is given by \( f(x) = \frac{1 + \sqrt{8x + 1}}{2} \).

**Lemma 6.2** If \( x \in \mathbb{N} \), then \( f\left(\left(\frac{x}{2}\right)\right) = x \) and \( \left(\frac{f(x)}{2}\right) = x \).

We leave the Proof of Lemma 6.2 as an exercise to the reader, as it only requires elementary algebra.

With the \( f \) function at our disposal, a lower bound on \( \mu \) is quite straightforward to find, and this bound is in fact tight for infinitely many values of \( n \), as will be shown in Corollary 6.4.

**Theorem 6.3** For all \( n \in \mathbb{N} \), \( \mu(n) \geq f(n) \).

**Proof** Assume there is some \( n \in \mathbb{N}_0 \) such that \( \mu(n) < f(n) \). Let \( \mu(n) = m \). Then since \((m, n) \in T\), from the definition of \( T \), there must be some \( c_1, c_2, \ldots, c_k \) for \( k \in \mathbb{N}_0 \) such that

\[
m = 1c_1 + 2c_2 + \cdots + kc_k
\]

and

\[
n = c_1\left(\frac{1}{2}\right) + c_2\left(\frac{2}{2}\right) + \cdots + c_k\left(\frac{k}{2}\right).
\]

Since \( m < f(n) \), and the function \((\frac{x}{2})\) is increasing when \( x \geq 1 \), and \( m, f(n) \geq 1 \), we can apply this function to both sides of \( m < f(n) \) to get \((\frac{m}{2}) < n \). From our \( m \) and \( n \) equations, we can write \((\frac{m}{2}) < n \) as

\[
\left(\frac{1c_1 + 2c_2 + \cdots + kc_k}{2}\right) < c_1\left(\frac{1}{2}\right) + c_2\left(\frac{2}{2}\right) + \cdots + c_k\left(\frac{k}{2}\right).
\]

However, the \((\frac{x}{2})\) function is convex, so it is superadditive, which means that that \((\frac{x+y}{2}) \geq \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) \) for all \( x, y \in \mathbb{R} \), so

\[
\left(\frac{1c_1 + 2c_2 + \cdots + kc_k}{2}\right) \geq c_1\left(\frac{1}{2}\right) + c_2\left(\frac{2}{2}\right) + \cdots + c_k\left(\frac{k}{2}\right).
\]

So we have a contradiction, meaning our assumption was false. So \( \mu(n) \geq f(n) \). \( \square \)
In fact, the bound in the previous theorem is tight for infinitely many values of \( n \), due to the following corollary. This is the only infinite family of values \( n \) for which we can calculate the exact value of \( \mu(n) \) without resorting to recursion.

**Corollary 6.4** For all \( i \in \mathbb{N} \) where \( i \geq 2 \), \( \mu\left(\left(\frac{i}{2}\right)\right) = i \).

**Proof** Let \( i \in \mathbb{N} \) and \( i \geq 2 \). Then \( \left(\frac{i}{2}\right) \in \mathbb{N} \). Theorem 6.3 states that for all \( n \in \mathbb{N} \), \( \mu(n) \geq f\left(\left(\frac{i}{2}\right)\right) \). So, \( \mu\left(\left(\frac{i}{2}\right)\right) \geq f\left(\left(\frac{i}{2}\right)\right) \). Also, according to Lemma 6.2, \( f\left(\left(\frac{i}{2}\right)\right) = i \).

Therefore, \( \mu\left(\left(\frac{i}{2}\right)\right) \geq i \). However, Corollary 5.4 states that for all \( n, i \in \mathbb{N}_0 \), if \( \left(\frac{i}{2}\right) \leq n \), then

\[
\mu(n) \leq \mu\left(n - \left(\frac{i}{2}\right)\right) + i.
\]

So, when \( n = \left(\frac{i}{2}\right) \), we obtain

\[
\mu\left(\left(\frac{i}{2}\right)\right) \leq \mu(0) + i.
\]

Since \( \mu(0) = 0 \), we have \( \mu\left(\left(\frac{i}{2}\right)\right) \leq i \). Therefore, we can conclude that \( \mu\left(\left(\frac{i}{2}\right)\right) = i \). \( \square \)

A tight upper bound is far more difficult to find. We will begin by proving what we refer to as the Gauss bound, because we will use the so-called “Eureka” theorem of Gauss (see [11] for a modern treatment) to prove it. This is a celebrated result of Gauss which says that any natural number can be written as the sum of three triangular numbers. Triangular numbers are those numbers which are equal to \( \left(\frac{i}{2}\right) \) for some \( i \in \mathbb{N}_0 \).

**Theorem 6.5** (Gauss bound) For all \( n \in \mathbb{N}_0 \), \( \mu(n) \leq 3f\left(\left(\frac{n}{3}\right)\right) \).

**Proof** By definition,

\[
\mu(n) = \min \left\{ 1\lambda_1 + 2\lambda_2 + \cdots | \lambda_1, \lambda_2, \cdots \in \mathbb{N}_0, n = \left(\frac{1}{2}\right)\lambda_1 + \left(\frac{2}{2}\right)\lambda_2 + \cdots \right\}.
\]

Hence, if there exist \( \lambda_1, \lambda_2, \cdots \in \mathbb{N}_0 \) such that \( n = \left(\frac{1}{2}\right)\lambda_1 + \left(\frac{2}{2}\right)\lambda_2 + \cdots \), we have \( \mu(n) \leq 1\lambda_1 + 2\lambda_2 + \cdots \). Gauss’ Eureka theorem states that there exist \( x, y, z \in \mathbb{N}_0 \) such that \( n = \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) + \left(\frac{z}{2}\right) \), hence

\[
\mu(n) \leq \min \left\{ x + y + z | x, y, z \in \mathbb{N}_0, n = \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) + \left(\frac{z}{2}\right) \right\}
\]

\[
\leq \max \left\{ x + y + z | x, y, z \in \mathbb{N}_0, n = \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) + \left(\frac{z}{2}\right) \right\}
\]

\[
\leq \max \left\{ x + y + z | x, y, z \in \mathbb{R}, n = \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) + \left(\frac{z}{2}\right) \right\}.
\]
We now simply need to find the maximum of $x + y + z$ over the reals, subject to the constraint $n = \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) + \left(\frac{z}{2}\right)$, which is a straightforward optimization problem that can be solved with analysis.

We can make this optimization problem easier by noting that the constraint $n = \left(\frac{x}{2}\right) + \left(\frac{y}{2}\right) + \left(\frac{z}{2}\right)$ is a sphere. The level sets of the objective function $x + y + z$, on the other hand, are planes. Hence, the minimum and maximum values of the objective function will occur at the two level sets of $x + y + z$ whose planes are tangent to the sphere.

The two points at which tangency occurs are when $x = y = z = f(n/3)$, which yields the maximum value of $x + y + z = 3 f(n/3)$ and $x = y = z = 1 - f(n/3)$, which yields the minimum value of $x + y + z = 3 - 3 f(n/3)$. Hence $\mu(n) \leq 3 f(n/3)$.

The advantage of the Gauss bound is that the formula is non-recursive and easy to work with. The disadvantage is that the bound does not seem to be very good—in fact, we have not encountered any value of $n$ for which the Gauss bound is tight. Numerical evidence for the looseness of the Gauss bound is shown in Fig. 2.

The upper recursive bound given in Corollary 5.4, on the other hand, seems to be much tighter; however, as it is still recursive, it is difficult to work with.

We give one final bound in this section, which is the result of combining one application of the recursive bound with the Gauss bound. The formula for this bound is complicated, but it is not recursive; it is much tighter than the Gauss bound, as shown in Fig. 2, and it is sufficient to prove many results in the following sections.
**Theorem 6.6 (Combined bound)** For all \( n \in \mathbb{N} \),

\[
\mu(n) \leq f(n) + 3f\left(\frac{f(n) - 2}{3}\right).
\]

**Proof.** Let \( n \in \mathbb{N} \). We know from Lemma 6.2 that for \( n \in \mathbb{N} \), \( \left(\frac{f(n)}{2}\right) = n \). Since \( \left(\frac{1}{2}\right) \) is an increasing function for \( x \geq 1 \) and \( f(n) \geq 1 \) for \( n \in \mathbb{N}_0 \), this implies \( \left(\frac{\lfloor f(n) \rfloor}{2}\right) \leq \left(\frac{f(n)}{2}\right) \). Hence \( \left(\frac{\lfloor f(n) \rfloor}{2}\right) \leq n \), and so we can apply the recursive bound (Theorem 5.3) with \( i = \lfloor f(n) \rfloor \). Doing so yields

\[
\mu(n) \leq \lfloor f(n) \rfloor + \mu\left(n - \left(\frac{\lfloor f(n) \rfloor}{2}\right)\right).
\]

We can then drop the floor and apply the Gauss bound to the second term to get

\[
\mu(n) \leq f(n) + 3f\left(\frac{n - \left(\frac{\lfloor f(n) \rfloor}{2}\right)}{3}\right).
\]

Now we would like to get rid of the remaining floor function, starting with the observation that \( \lfloor f(n) \rfloor > f(n) - 1 \). Since \( f(n) \geq 2 \) when \( n \geq 1 \), both sides of the inequality \( \lfloor f(n) \rfloor > f(n) - 1 \) are greater than or equal to 1. Since the \( \left(\frac{1}{2}\right) \) function is increasing for \( x \geq 1 \), we can apply the \( \left(\frac{1}{2}\right) \) function to both sides of \( \lfloor f(n) \rfloor > f(n) - 1 \) to get

\[
\left(\frac{\lfloor f(n) \rfloor}{2}\right) > \left(\frac{f(n) - 1}{2}\right) = \frac{1}{2}(f(n) - 1)(f(n) - 2) = \frac{1}{2}(f(n) - 1) f(n) - \frac{1}{2}(f(n) - 1)(2) = \left(\frac{f(n)}{2}\right) - (f(n) - 1) = n - (f(n) - 1).
\]

Hence

\[
n - \left(\frac{\lfloor f(n) \rfloor}{2}\right) < n - (n - (f(n) - 1)) = f(n) - 1.
\]

Since \( n - \left(\frac{\lfloor f(n) \rfloor}{2}\right) \) is an integer, we can tighten this to

\[
n - \left(\frac{\lfloor f(n) \rfloor}{2}\right) \leq f(n) - 2.
\]
Since $f$ is an increasing function, this yields

$$
\mu(n) \leq f(n) + 3f \left( \frac{f(n) - 2}{3} \right).
$$

\[ \square \]

7 The relationship between $\mu$ and $\mu_{a,b}$

We begin our discussion of the relationship between $\mu$ and $\mu_{a,b}$ with the following straightforward observation that follows directly from the definitions.

Lemma 7.1 For all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ and for all $n \in \mathbb{N}_0$,

$$
\mu_{a,b}(n) \leq \mu(n).
$$

Proof Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ so that $S(a, b)$ is a numerical semigroup with $a, b \geq 1$, and $n \in \mathbb{N}_0$. Suppose $\mu_{a,b}(n) > \mu(n)$. Since we defined $\phi_{a,b} : \mathbb{N}_0^3 \to \mathbb{N}_0$ as $\phi_{a,b}(m, n) = ma + nb$ and Theorem 5.1 states that the image of $T$ under $\phi_{a,b}$ is $S(a, b)$,

$$
\phi_{a,b}(\mu(n), n) = \mu(n)a + nb \in S.
$$

By definition, $\mu_{a,b}(n) = \min\{m \in \mathbb{Z} \mid ma + nb \in S(a, b)\}$, but $\mu(n) < \mu_{a,b}(n)$ and $\mu(n) \in \mathbb{Z}$ and $\mu(n)a + nb \in S$, so this is a contradiction. Therefore, for all $n \in \mathbb{N}_0$, $\mu_{a,b}(n) \leq \mu(n)$.

We now proceed to the main theorem of this section, which establishes the oft-mentioned fact that $\mu(n) = \mu_{a,b}(n)$ when $0 \leq n < a$, except for eight particular values of $(a, b, n)$.

Our strategy in the proof is to use the bounds on $\mu$ to show that there are only a finite number of triples $(a, b, n)$ for which it is possible to have $\mu_{a,b}(n) < \mu(n)$. We then check all of these cases computationally to find the exceptional ones.

Theorem 7.2 For all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ and all $n \in \mathbb{N}_0$ where $n < a$,

1. If $b = 1$ and $(a, n) = (29, 26), (45, 33), (47, 44), (50, 41), (55, 50), (67, 53), (73, 63)$, or $(79, 74)$, then $\mu_{a,b}(n) = \mu(n) - 1$;
2. Otherwise, $\mu_{a,b}(n) = \mu(n)$.

Proof Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ so that $S(a, b)$ is a numerical semigroup with $a, b \geq 1$, and suppose $n \in \mathbb{N}_0$ such that $n < a$ and $\mu_{a,b}(n) < \mu(n)$.

By definition of $\mu_{a,b}$, we know $\mu_{a,b}(n)a + nb \in S(a, b)$; hence, by Theorem 5.1, there is some $(t_1, t_2) \in T$ such that $\phi_{a,b}(t_1, t_2) = \mu_{a,b}(n)a + nb$, that is, such that

$$
t_1a + t_2b = \mu_{a,b}(n)a + nb
$$

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Thus \((t_2 - n)b = (\mu_{a,b}(n) - t_1)a\), and since \(\gcd(a, b) = 1\), we must have \(t_2 \equiv n \pmod{a}\). Thus \(t_2 = ka + n\) for some \(k \in \mathbb{Z}\), and, since \(t_2 \in \mathbb{N}_0, k \in \mathbb{N}_0\). Furthermore, if \(k = 0\), then \(t_1 = \mu_{a,b}(n)\), which by definition of \(\mu\) would imply \(\mu_{a,b}(n) \geq \mu(n)\), which contradicts the assumption that \(\mu_{a,b}(n) < \mu(n)\). Hence, \(t_2 = ka + n\) for some \(k \in \mathbb{N}\).

Now, by definition of \(T\) and \(\mu\), we know \(\mu(t_2) \leq t_1\), therefore

\[
\mu(t_2)a + t_2b \leq \mu_{a,b}(n)a + nb.
\]

After substituting \(t_2 = ka + n\) and performing cancellations, this yields

\[
\mu(ka + n) + kb \leq \mu_{a,b}(n).
\]

By assumption, we finally have \(\mu(ka + n) + kb < \mu(n)\).

Now we apply the upper and lower bounds from 6.6 to 6.3 to find that

\[
f(ka + n) + kb < f(n) + 3f\left(\frac{f(n) - 2}{3}\right).
\]

Recall that the upper bound from 6.6 is only valid for \(n \in \mathbb{N}\). Hence, we must at this point dispense with the case when \(n = 0\). Since both \(\mu_{a,b}(0) = 0\) and \(\mu(0) = 0\) can be easily deduced from the definitions of \(\mu_{a,b}\) and \(\mu\), we do not need to worry about this case.

To loosen the inequality further, since \(k \in \mathbb{N}\) and \(f\) is an increasing function, we can conclude

\[
f(a + n) + b < f(n) + 3f\left(\frac{f(n) - 2}{3}\right).
\]

Rearranged, this is

\[
f(a + n) - f(n) + b < 3f\left(\frac{f(n) - 2}{3}\right).
\]

One can readily confirm from the derivative that \(f(a + n) - f(n)\) is decreasing on \(n\), and so the left-hand side of the above inequality achieves its minimum at \(n = a - 1\). Furthermore, since \(f\) is increasing on \(n\), the right-hand side of the inequality achieves its maximum likewise at \(n = a - 1\). Hence

\[
f(2a - 1) - f(a - 1) + b < 3f\left(\frac{f(a - 1) - 2}{3}\right).
\]

Rearranging again, this yields

\[
b < 3f\left(\frac{f(a - 1) - 2}{3}\right) - f(2a - 1) + f(a - 1).
\]
Let us define the function $g : \mathbb{N} \setminus \{1\} \to \mathbb{R}$ as

$$g(a) = 3f \left( \frac{f(a - 1) - 2}{3} \right) - f(2a - 1) + f(a - 1).$$

Note here that we have excluded $a = 1$ from the domain, since it results in a complex value. Nevertheless, this does not affect the result of the theorem since when $a = 1$, the only $n$-value under consideration is $n = 0$, which we already dispensed with.

Although the formula for $g$ is rather cumbersome, it is nevertheless susceptible to straightforward analysis techniques. One can readily confirm using a computer algebra system that it has only one real zero, at approximately $a \approx 835.76$, and that it has a global maximum at approximately $g(52.15) \approx 4.59$.

Thus, in order to have $b < g(a)$, we must have $b \leq 4$. Furthermore, the inequality $1 < g(a)$ is only valid for $a \leq 655$. For all cases where $b > 4$ or $a > 655$, then, we have a contradiction. The theorem thus stands in those cases.

Now, to finish the proof, we can check the remaining cases when $a \leq 655$ and $b \leq 4$ computationally. We have algorithms for calculating both $\mu$ and $\mu_{a,b}$, and it is straightforward to compare their values for all $(a, b, n)$ such that $a \leq 655, b \leq 4$, and $0 < n < a$. Having performed this check, we found that $\mu_{a,b}(n) = \mu(n)$ in every case, except when $b = 1$ and $(a, n) = (29, 26), (45, 33), (47, 44), (50, 41), (55, 50), (67, 53), (73, 63), \text{ or } (79, 74)$. In those eight exceptional cases, $\mu_{a,b}(n) = \mu(n) - 1$.

As a consequence of this, we can define the Apéry sets $\text{Ap}(S(a, b), a)$ purely in terms of $\mu, a$, and $b$, rather than in terms of $\mu_{a,b}$ as we had previously done.

**Corollary 7.3** For all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, if $(a, b) \neq (29, 1), (45, 1), (47, 1), (50, 1), (55, 1), (67, 1), (73, 1), \text{ or } (79, 1)$, then

$$\text{Ap}(S(a, b), a) = \{\mu(n)a + nb \mid n = 0, 1, 2, \ldots, a - 1\}.$$

### 8 Bounds on the Frobenius number and the genus

As a consequence of Corollary 7.3 and Lemma 2.5, which gives formulas for the Frobenius number and the genus in terms of an Apéry set, we have the following.

**Theorem 8.1** For all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ and $(a, b) \neq (29, 1), (45, 1), (47, 1), (50, 1), (55, 1), (67, 1), (73, 1), \text{ or } (79, 1)$, the Frobenius number of $S(a, b)$ is given by

$$F(S(a, b)) = \max\{\mu(n)a + nb \mid n = 0, 1, 2, \ldots, a - 1\} - a$$

and the genus of $S(a, b)$ is given by

$$g(S(a, b)) = \frac{(a - 1)(b - 1)}{2} + \sum_{n=0}^{a-1} \mu(n).$$
Proof. According to Lemma 2.5, \( F(S(a, b)) = \max \text{Ap}(S(a, b), a) - a \). Also, Corollary 7.3 says \( \text{Ap}(S(a, b), a) = \{ \mu(n)a + nb \mid n = 0, 1, 2, \ldots, a - 1 \} \) when \((a, b) \neq (29, 1), (45, 1), (47, 1), (50, 1), (55, 1), (67, 1), (73, 1), \) or \((79, 1)\). Therefore,

\[
F(S(a, b)) = (\max\{\mu(n)a + nb \mid n = 0, 1, 2, \ldots, a - 1\}) - a.
\]

Also according to the formula for the genus in Lemma 2.6,

\[
g(S(a, b)) = \frac{1}{a} \left( \sum_{n=0}^{a-1} \mu(n)a + nb \right) - a - 1 = \frac{1}{2} \left( \sum_{n=0}^{a-1} \mu(n) \right) + \left( \frac{b}{a} \sum_{n=0}^{a-1} n \right) - a - 1
= \left( \sum_{n=0}^{a-1} \mu(n) \right) + \frac{b}{a} \left( \frac{a(a-1)}{2} \right) - a - 1 = \left( \sum_{n=0}^{a-1} \mu(n) \right) + \frac{(a-1)(b-1)}{2}
\]

\( \square \)

Using the bounds on \( \mu \) developed in the last section, together with the formulas for the Frobenius number \( F(S(a, b)) \) and the genus \( g(S(a, b)) \) which were given in Theorem 8.1, we can now give bounds and asymptotic behavior for \( F(S(a, b)) \) and \( g(S(a, b)) \) in terms of \( a \) and \( b \). The following theorems use asymptotic Bachmann–Landau notation; readers unfamiliar with this notation should refer to [5].

Theorem 8.2 For all \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \), \( F(S(a, b)) = \Theta(a^{3/2} + ab) \).

Proof Recall from Theorem 8.1 that, except for the eight exceptional cases of \( a, b \), we have

\[
F(S(a, b)) = \max\{a\mu(n) + nb \mid n = 0, 1, 2, \ldots, a - 1\} - a
\geq a\mu(a-1) + (a-1)b - a
= a\mu(a-1) + ab - a - b.
\]

Applying the lower bound on \( \mu \) from Theorem 6.3, we get

\[
F(S(a, b)) \geq af(a-1) + ab - a - b = \frac{a}{2} \left( 1 + \sqrt{8a - 7} \right) + ab - a - b.
\]

As for the upper bound, we can first break up the maximum as follows.

\[
F(S(a, b)) = \max\{a\mu(n) + nb \mid n = 0, 1, 2, \ldots, a - 1\} - a
\leq \max\{a\mu(n) \mid n = 0, \ldots, a - 1\} + \max\{nb \mid n = 0, \ldots, a - 1\} - a
= \max\{a\mu(n) \mid n = 0, 1, 2, \ldots, a - 1\} + (a-1)b - a
= \max\{a\mu(n) \mid n = 0, 1, 2, \ldots, a - 1\} + ab - a - b.
\]
We can then use the Gauss upper bound from Theorem 6.5 to get

\[ F(S(a, b)) \leq \max\{a \cdot 3 f \left( \frac{n}{3} \right) \mid n = 0, 1, 2, \ldots, a - 1\} + ab - a - b. \]

Since \( f \) is an increasing function, this yields

\[ F(S(a, b)) \leq 3af \left( \frac{a - 1}{3} \right) + ab - a - b \]

\[ = \frac{a}{2} \left( 3 + \sqrt{24a - 15} \right) + ab - a - b. \]

\[ \square \]

Turning our attention now to the genus, we find a similar bound.

**Theorem 8.3** For all \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \), \( g(S(a, b)) = \Theta(a^{3/2} + ab) \).

**Proof** Recall from Theorem 8.1 that, except for the eight exceptional cases of \( a, b \), we have

\[ g(S(a, b)) = \frac{(a - 1)(b - 1)}{2} + \sum_{n=0}^{a-1} \mu(n). \]

Furthermore, since \( \mu(0) = 0 \), we can drop the \( \mu(0) \) term, and then we can apply the lower bound on \( \mu(n) \) from Theorem 6.3 to get

\[ g(S(a, b)) \geq \frac{(a - 1)(b - 1)}{2} + \sum_{n=1}^{a-1} f(n). \]

As \( f \) is an increasing function, we can bound this sum by the following definite integral.

\[ g(S(a, b)) \geq \frac{(a - 1)(b - 1)}{2} + \int_{n=0}^{a-1} f(n)dn = \frac{(8a - 7)^{3/2} - 1}{24} + \frac{(a - 1)b}{2}. \]

As for the upper bound on \( g(S(a, b)) \), we use the same process, but applying the upper Gauss bound on \( \mu(n) \) from Theorem 6.5. Hence

\[ g(S(a, b)) = \frac{(a - 1)(b - 1)}{2} + \sum_{n=1}^{a-1} \mu(n) \leq \frac{(a - 1)(b - 1)}{2} + \sum_{n=1}^{a-1} 3f \left( \frac{n}{3} \right). \]

\[ \leq \frac{(a - 1)(b - 1)}{2} + \int_{n=1}^{a} 3f \left( \frac{n}{3} \right)dn \]

\[ = \frac{\sqrt{3}(8a + 3)^{3/2} - 1 \sqrt{33}}{24} + \frac{(a - 1)(b + 2)}{2}. \]

which yields the upper bound in the theorem statement. \( \square \)
9 The embedding dimension

Since our numerical semigroups are generated by infinite quadratic sequences, the value of the embedding dimension is non-trivial to establish. In this section, we will build up a sequence of theorems, culminating in a proof of the exact value of the embedding dimension for all numerical semigroups $S(a, b)$.

We begin by observing that from Lemma 2.3, the multiplicity of $S(a, b)$ is $m(S(a, b)) = a$, and so the embedding dimension of $S(a, b)$ is bounded above by $e(S(a, b)) \leq a$. This fact is also a consequence of the following theorem, which gives a more detailed picture than simply that $e(S(a, b)) \leq a$.

**Theorem 9.1** Let $a, b \in \mathbb{N}_0$ with $\gcd(a, b) = 1$, and let $y_n = na + \binom{n}{2}b$ for all $n \in \mathbb{N}_0$. If $n > a$, then $y_n$ is not in the minimal set of generators of $S(a, b)$.

**Proof** Assume $n > a$. Then, $n = a + k$ for some positive integer $k$. So,

$$y_n = na + \frac{n(n - 1)}{2}b = (a + k)a + \frac{(a + k)(a + k - 1)}{2}b$$

$$= a^2 + ak + \frac{a^2 + k^2 + 2ak - a - k}{2}b = \left(a^2 + \frac{a^2 - a}{2}b\right)$$

$$+ \left(ak + \frac{k^2 - k}{2}b\right) + abk$$

$$= ya + yk + y_1(bk).$$

Since $bk \in \mathbb{N}_0$ and $1, a, k < n$, $y_n$ is generated by smaller elements of $S$, so $y_n$ is not in the minimal set of generators of $S$. \qed

On the other hand, we get a lower bound on the embedding dimension from the following.

**Theorem 9.2** Let $a, b \in \mathbb{N}_0$ with $\gcd(a, b) = 1$, and let $y_n = na + \binom{n}{2}b$ for all $n \in \mathbb{N}_0$. If $\binom{n}{2} < a$, then $y_n$ is in the minimal set of generators of $S(a, b)$.

**Proof** Suppose that $\binom{n}{2} < a$ and that $y_n$ is not in the minimal set of generators of $S$. Then there exist some $c_1, c_2, \ldots, c_{n-1} \in \mathbb{N}_0$ such that

$$y_n = \sum_{i=1}^{n-1} c_i y_i \implies na + \binom{n}{2}b = \sum_{i=1}^{n-1} c_i \left(ia + \binom{i}{2}b\right).$$

Rearranging to get all copies of $a$ on the left and all copies of $b$ on the right, we get

$$\left(n - \sum_{i=1}^{n-1} c_i i\right)a = \left(-\binom{n}{2} + \sum_{i=1}^{n-1} c_i \binom{i}{2}\right)b.$$

Since the left hand side is a multiple of $a$, the right hand side must also be a multiple of $a$. Since $a > 1$ and $\gcd(a, b) = 1$, it must be that $\binom{n}{2} - \sum_{i=1}^{n-1} c_i \binom{i}{2}$ is a multiple of $a$. \qed
Likewise, since the right hand side is a multiple of $b$, the left hand side must also be a multiple of $b$. If $b > 1$, then since $\gcd(a, b) = 1$, it must be that $n - \sum_{i=1}^{n-1} c_i i$ is a multiple of $b$. On the other hand, if $b = 1$, then $n - \sum_{i=1}^{n-1} c_i i$ is a multiple of $b$ by default. Furthermore, in order for the equation to be balanced, these two expressions must be the same multiple of $a$ and $b$ respectively. In other words, there exists some $k \in \mathbb{Z}$ such that

\[
-\binom{n}{2} + \sum_{i=1}^{n-1} c_i \binom{i}{2} = k a, \tag{1}
\]

\[
n - \sum_{i=1}^{n-1} c_i i = k b. \tag{2}
\]

After multiplying through Eq. (2) by $\frac{n-1}{2}$ and adding it to Eq. (1), we get

\[
\left(\sum_{i=1}^{n-1} c_i \binom{i}{2}\right) - \sum_{i=1}^{n-1} c_i \frac{i(n-1)}{2} = ka + \frac{n-1}{2} kb.
\]

After collecting coefficients and simplifying, this becomes

\[
\sum_{i=1}^{n-1} c_i \left(\frac{i(i-n)}{2}\right) = k \left(a + \frac{n-1}{2} b\right).
\]

Each $i - n$ must be negative, hence the entire left hand side must be negative, since each $c_i \geq 0$ and not all of them can be 0. Therefore, the right hand side is negative as well. Since $a + \frac{n-1}{2} b$ is clearly positive, we conclude that $k \leq -1$. Applying $k \leq -1$ to Eq. (1), we get

\[
-\binom{n}{2} + \sum_{i=1}^{n-1} c_i \binom{i}{2} \leq -a \implies \binom{n}{2} \geq a + \sum_{i=1}^{n-1} c_i \binom{i}{2}.
\]

Since every $c_i \geq 0$, this implies $\binom{n}{2} \geq a$. However, we began by assuming that $\binom{n}{2} < a$. This is a contradiction, so $y_n$ is in the minimal set of generators of $S$ when $\binom{n}{2} < a$.

When the value of $a$ is fixed, we can say a bit more about the set of minimal generators for different values of $b$.

**Theorem 9.3** Let $a, b, i \in \mathbb{N}_0$ such that $S(a, b)$ and $S(a, b + i)$ are numerical semigroups. For all $n \in \mathbb{N}_0$, if $na + \binom{n}{2} b$ is not in the minimal set of generators of $S(a, b)$, then $na + \binom{n}{2} (b + i)$ is not in the minimal set of generators of $S(a, b + i)$.

**Proof** Let us call the generators of $S(a, b)$ by the names $y_\ell$ and the generators of $S(a, b + i)$ by the names $y'_\ell$, so that $y_\ell = \ell a + \binom{\ell}{2} b$, and $y'_\ell = \ell a + \binom{\ell}{2} (b + i)$ for
all $\ell \in \mathbb{N}_0$. Suppose that $y_n = na + \left(\frac{n}{2}\right)b$ is not in the minimal set of generators of $S(a, b)$. Then there exist some $c_1, c_2, \ldots, c_{n-1} \in \mathbb{N}_0$ such that

$$y_n = \sum_{j=1}^{n-1} c_j y_j.$$  

Plugging in the formula for each $y_\ell$, we obtain

$$na + \left(\frac{n}{2}\right)b = \sum_{j=1}^{n-1} c_j \left( ja + \left(\frac{j}{2}\right)b \right).$$  

(3)

By following the same technique as in the Proof of Theorem 9.2, we can move all copies of $a$ to one side and all copies of $b$ to the other. Then both sides with be multiples of both $a$ and $b$, and since $\gcd(a, b) = 1$, $\left(\sum_{j=1}^{n-1} c_j \left(\frac{j}{2}\right)\right) - \left(\frac{n}{2}\right)$ is a multiple of $a$. So,

$$\left(\sum_{j=1}^{n-1} c_j \left(\frac{j}{2}\right)\right) - \left(\frac{n}{2}\right) = ka.$$  

where $k \in \mathbb{Z}$ and $k < 0$. After rearranging terms and multiplying by $i$ we obtain

$$\left(\frac{n}{2}\right)i = \left(\sum_{j=1}^{n-1} c_j \left(\frac{j}{2}\right)i\right) - kai.$$  

(4)

After adding Eqs. (3) and (4) together and collecting terms, we get

$$na + \left(\frac{n}{2}\right)(b + i) = \left(\sum_{j=1}^{n-1} c_j \left( ja + \left(\frac{j}{2}\right)(b + i)\right)\right) - kai.$$  

Replacing with $y_n'$ wherever possible, this becomes

$$y_n' = \left(\sum_{j=1}^{n-1} c_j y'_j\right) - k y'_i.$$  

Since $k < 0$, this shows that $y_n'$ is a linear combination over $\mathbb{N}_0$ of smaller generators. Hence $y_n' = na + \left(\frac{n}{2}\right)(b + i)$ is not in the minimal set of generators for $S(a, b + i)$. 

As a consequence of the previous theorem, the embedding dimension is decreasing on $b$ when $a$ is fixed.
Corollary 9.4 For all \( a, b_1, b_2 \in \mathbb{N}_0 \) where \( \gcd(a, b_1) = \gcd(a, b_2) = 1 \), if \( b_1 \leq b_2 \), then \( e(S(a, b_1)) \geq e(S(a, b_2)) \).

So far, we have established the following about the generators \( y_n \) of \( S(a, b) \). If \( n > a \), then \( y_n \) is not in the minimal set of generators of \( S(a, b) \), from Theorem 9.1. If \( \left( \frac{n}{2} \right) < a \), then \( y_n \) is in the minimal set of generators of \( S(a, b) \), from Theorem 9.2. Also, it is obvious that if \( \left( \frac{n}{2} \right) \) is a multiple of \( a \), then \( y_n \) is a multiple of \( y_1 = a \), so \( y_n \) is not in the minimal set of generators of \( S(a, b) \).

Thus, it remains only to investigate the generators \( y_n \) of \( S(a, b) \) where \( a < \left( \frac{n}{2} \right) \leq \left( \frac{n}{2} \right) \) and \( \left( \frac{n}{2} \right) \) is not a multiple of \( a \). We will do this in the proof of the next theorem.

Theorem 9.5 For all \( a, b \in \mathbb{N} \) with \( \gcd(a, b) = 1 \) and all \( n \in \mathbb{N}_0 \) such that \( \left( \frac{n}{2} \right) > a \),

1. If \( b = 1 \) and \( (a, n) = (29, 11), (45, 13), (47, 14), (50, 14), (55, 15), (67, 16), (73, 17), \text{ or (79, 18)} \), then \( na + \left( \frac{n}{2} \right) b \) is in the minimal set of generators of \( S(a, b) \).
2. Otherwise, \( na + \left( \frac{n}{2} \right) b \) is not in the minimal set of generators of \( S(a, b) \).

Proof Let \( a, b \in \mathbb{N} \) with \( \gcd(a, b) = 1 \) so that \( S(a, b) \) is a numerical semigroup with \( a, b \geq 1 \). Let \( y_n \) denote the generators \( y_n = na + \left( \frac{n}{2} \right) b \) for all \( n \in \mathbb{N}_0 \). Suppose that \( \left( \frac{n}{2} \right) > a \) and that \( y_n \) is in the minimal set of generators of \( S(a, b) \).

We already know that if \( n > a \) or \( \left( \frac{n}{2} \right) \) is a multiple of \( a \), then \( y_n \) is not in the minimal set of generators of \( S(a, b) \). Hence we may assume \( a < \left( \frac{n}{2} \right) \leq \left( \frac{n}{2} \right) \) and \( \left( \frac{n}{2} \right) \) is not a multiple of \( a \). Then there exist \( j, k \in \mathbb{N}_0 \) such that \( \left( \frac{n}{2} \right) = ka + j \) and \( k \geq 1 \) and \( 1 \leq j \leq a - 1 \). Note that \( j = \left( \frac{n}{2} \right) \mod a \). Then

\[
y_n = na + \left( \frac{n}{2} \right) b = na + (ka + j)b = (n + kb)a + jb.
\]

The Apéry set \( \text{Ap}(S(a, b), a) \), together with \( a \), forms a generating set for \( S(a, b) \), so \( y_n \in \text{Ap}(S(a, b), a) \). We established in Theorem 4.2 that

\[
\text{Ap}(S(a, b), a) = \{ \mu_{a,b}(j)a + jb \mid j = 0, 1, 2, \ldots, a - 1 \}.
\]

Hence, \( y_n = \mu_{a,b}(j)a + jb \), and so \( \mu_{a,b}(j) = n + kb \).

Now, due to Theorem 7.2, there are two cases. Case 1: \( (a, b, \left( \frac{n}{2} \right) \) mod \( a \) \( \neq (29, 1, 26), (45, 1, 33), (47, 1, 44), (50, 1, 41), (55, 1, 50), (67, 1, 53), (73, 1, 63), \text{ or (79, 1, 74)} \). Case 2: the opposite. We will spend most of the proof assuming Case 1, and address Case 2 computationally at the end of the proof.

Assuming Case 1, Theorem 7.2 tells us that \( \mu_{a,b}(j) = \mu(j) \), hence \( \mu(j) = n + kb \). We know from the Gauss bound (Theorem 6.5) that \( \mu(j) \leq 3f(j/3) \). Hence \( n + kb \leq 3f(j/3) \).

We can also deduce from \( \left( \frac{n}{2} \right) = ka + j \) that \( n = f(ka + j) \) and plug this into the inequality to get \( f(ka + j) + kb \leq 3f(j/3) \). Since \( k, b \geq 1 \), we can loosen this inequality to say that \( f(ka + j) + 1 \leq 3f(j/3) \). Now, from the definition of \( f \), we have
\[
\frac{1 + \sqrt{8(ka + j) + 1}}{2} + 1 \leq \frac{3 \left( 1 + \sqrt{\frac{8j}{5} + 1} \right)}{2}.
\]

We can now isolate \(k\) by cancelling the constants, squaring both sides, and further simplifying to find \(ka \leq 2j + 1\). Recall that \(j \leq a - 1\), hence \(ka \leq 2(a - 1) + 1 = 2a - 1\), which implies \(k < 2\). Since we already know \(k \geq 1\), we must have \(k = 1\).

With the value of \(k\) now known, the earlier equation \(\mu(j) = n + kb\) becomes \(\mu(j) = n + b\), or, when rearranged, \(b = \mu(j) - n\). This seemingly innocuous statement is actually quite powerful, because the right hand side, \(\mu(j) - n\), depends only on the values of \(n\) and \(a\) (recall that \(j = \left(\frac{\mu(j)}{n}\right) \mod a\)), not on the value of \(b\).

Thus, we conclude that, for a given, fixed value of \(a\), if \(a < \left(\frac{n}{2}\right) \leq \left(\frac{\mu(j)}{2}\right)\) and \(\left(\frac{n}{2}\right)\) is not a multiple of \(a\), then there is at most one value of \(b\) for which \(y_n\) is in the minimal generating set of \(S(a, b)\).

There are two cases. If \(y_n\) is in the minimal set of generators of \(S(a, 1)\), then \(y_n\) is not in the minimal set of generators for any other \(b \geq 2\), due to the considerations in the last paragraph. On the other hand, if \(y_n\) is not in the minimal set of generators of \(S(a, 1)\), then, due to Theorem 9.3, we still have that \(y_n\) is not in the minimal set of generators of \(S(a, b)\) where \(b \geq 2\). Hence, \(b = 1\).

Now let us consider what happens when \(b = 1\). Recall from earlier that we know \(k = 1\), so the earlier equation \(\mu(j) = n + kb\) becomes \(\mu(j) = n + 1\). Also, \(n = f(ka + j)\) becomes \(n = f(a + j)\). Hence \(f(a + j) + 1 = \mu(j)\). Since \(j \geq 1\), we can apply the combined bound (Theorem 6.6) to \(\mu(j)\) to get

\[
f(a + j) + 1 \leq f(j) + 3f \left( \frac{f(j) - 2}{3} \right).
\]

Rearranged, this is

\[
f(a + j) - f(j) + 1 \leq 3f \left( \frac{f(j) - 2}{3} \right).
\]

Recall that we saw an extremely similar inequality in the Proof of Theorem 7.2. As in that proof, the left-hand side achieves its minimum at \(j = a - 1\) and the right-hand side achieves its maximum at \(j = a - 1\), hence

\[
f(2a - 1) - f(a - 1) + 1 \leq 3f \left( \frac{f(a - 1) - 2}{3} \right).
\]

Rearranging again, this yields \(1 \leq g(a)\), where \(g\) is the function defined in the Proof of Theorem 7.2. As before, the \(g\) function yields to standard analytical techniques. Using a computer algebra system, we find that in order to have \(1 \leq g(a)\), we must have \(a \leq 655\).

Since we have an efficient algorithm for calculating the values of \(\mu\) (Theorem 5.5), we can simply check all values of \(1 \leq n \leq a \leq 655\) to see which pairs \((a, n)\) satisfy the equation.

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Proof

Let \( e \) \((Theorem 9.6)\) semigroups generated by an infinite quadratic sequence with initial term zero. 

We will deal with the eight exceptional cases at the end of the proof, so assume that \( e_n \) is in the minimal set of generators for the corresponding \( S(a, b) \). After concluding the computer search, we found 30 pairs \( (a, n) \) with \( 1 \leq n \leq a \leq 655 \) and satisfying the above equation. However, using a computer search, in each of the 30 cases, we were able to write \( e_n \) as a linear combination over \( \mathbb{N}_0 \) of smaller generators. This contradicts the assumption that \( e_n \) is in the minimal set of generators.

The theorem is thus proved by contradiction for Case 1. Returning now to Case 2, we need first to conduct a computer search for those values of \( (a, b, n) \) which satisfy \( (a, b, \binom{n}{2}) \mod a = (29, 1, 26), (45, 1, 33), (47, 1, 44), (50, 1, 41), (55, 1, 50), (67, 1, 53), (73, 1, 63), or (79, 1, 74). Thereafter, we can confirm whether or not \( e_n \) is in the minimal set of generators of the corresponding \( S(a, b) \). After concluding the computer search, we found that \( e_n \) is in the minimal set of generators of \( S(a, 1) \) only for \( (a, n) = (29, 11), (45, 13), (47, 14), (50, 14), (55, 15), (67, 16), (73, 17), (79, 18) \). \( \square \)

We are now prepared to establish the exact embedding dimension of all numerical semigroups generated by an infinite quadratic sequence with initial term zero.

**Theorem 9.6** For all \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \),

1. If \( b = 1 \) and \( a = 29, 45, 47, 50, 55, 67, 73, \) or \( 79, \) then \( e(S(a, b)) = \lfloor f(a) \rfloor - 1 \).
2. Otherwise, \( e(S(a, b)) = \lfloor f(a) \rfloor - 1 \).

**Proof** Let \( a, b \in \mathbb{N}_0 \) with \( \gcd(a, b) = 1 \), so that \( S(a, b) \) is a numerical semigroup. We will deal with the eight exceptional cases at the end of the proof, so assume that \( (a, b) \neq (29, 1), (45, 1), (47, 1), (50, 1), (55, 1), (67, 1), (73, 1), \) or \( (79, 1) \).

We know from Theorems 9.1, 9.2, and 9.5 , that the minimal set of generators of \( S(a, b) \) is the set of generators \( e_n \) such that \( \binom{n}{2} < a \).

We now simply need to count how many values of \( n \) meet this description in order to get the embedding dimension. The number of values will be the largest value of \( n \) meeting this description.

We know that for \( n \geq 1, \binom{n}{2} < a \) if and only if \( n < f(a) \). Let \( N \) stand for the largest value of \( n \in \mathbb{N} \) such that \( \binom{n}{2} < a \). There are two cases. If \( f(a) \) is an integer, then \( N = f(a) - 1 = \lfloor f(a) \rfloor - 1 \). On the other hand, if \( f(a) \) is not an integer, then \( N = \lfloor f(a) \rfloor = \lceil f(a) \rceil - 1 \). In either case, this tells us that the number of elements \( n \in \mathbb{N} \) such that \( \binom{n}{2} < a \) is \( \lfloor f(a) \rfloor - 1 \), hence, \( e(S(a, b)) = \lfloor f(a) \rfloor - 1 \).

Now let us consider those possible eight exceptions, and assume that \( (a, b) = (29, 1), (45, 1), (47, 1), (50, 1), (55, 1), (67, 1), (73, 1), \) or \( (79, 1) \). From Theorems 9.1, 9.2, and 9.5 , the minimal set of generators of \( S(a, b) \) contains the generators \( e_n \) such that \( \binom{n}{2} < a \), and Theorem 9.5 tells us that there is exactly one element \( e_n \) in the minimal set of generators such that \( \binom{n}{2} > a \). Hence, using the same counting arguments as before, \( e(S(a, b)) = \lfloor f(a) \rfloor \). \( \square \)

**10 Conclusions and future work**

In this article, we have investigated all numerical semigroups generated by infinite quadratic sequences with initial term zero. We have shown a computationally efficient
way to calculate the Apéry set, and given bounds on the elements of the Apéry set, which led to bounds on the genus and the Frobenius number. With those bounds, we were able to find the asymptotic behavior of the genus and the Frobenius number in terms of the coefficients of the quadratic sequence. Furthermore, we determined the exact embedding dimension for all such numerical semigroups.

However, there still remain many interesting and unanswered questions. For numerical semigroups generated by an infinite quadratic sequence with initial term zero, there is still much that could be done. We could tighten the bounds on $\mu$, or on the Frobenius number or the genus. We could investigate other associated sets or parameters that we did not mention here, such as the pseudo-Frobenius numbers or the type.

Furthermore, there is much yet to be done in the area of quadratic sequences more generally. We could investigate infinite quadratic sequences which begin with initial terms other than zero, or we could investigate the effect of using finite numbers of generators.

Lastly, and most importantly, the realm of polynomial sequences of higher degree remains almost completely unexplored. In this article, we have planted our flag on the edge of that terrain, but an infinite expanse yet remains to be surveyed.

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