On the Entropy and the Density Matrix of Cosmological Perturbations

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Abstract

We look at the transition to the semiclassical behaviour and the decoherence process for the inhomogeneous perturbations in the inflationary universe. Two different decoherence mechanisms appear: one dynamical, accompanied with a negligible, if at all, entropy gain, and the other, effectively irreversible dephasing, due to a rapid variation in time of the off-diagonal density matrix elements in the post-inflationary epoch. We thus settle the discrepancies in the entropy content of perturbations evaluated by different authors.

1 Introduction

The understanding of the origin of the large-scale structure in the universe remains one of the central problems in modern cosmology. It is believed that structures arise because the universe in the early epoch was not exactly homogeneous and isotropic, but must have contained some density irregularities.

These density irregularities or fluctuations may be classified into two major classes. The first one of these, the primordial fluctuations, corresponds

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to those present in the universe \textit{ab initio} and which may be thought of as classical irregularities in the initial structure of the universe, as for example the Mixmaster universe \cite{1}, the similarity solutions \cite{2}, gravitational solitary waves \cite{3}, etc.

A more exciting possibility, and the one relevant to this paper, are the \textit{dynamically induced} fluctuations, those arising inevitably in exact FRW models due, for example, to graviton creation in a time varying background gravitational field \cite{4}. These fluctuations are “seen” as inhomogeneous fluctuations of the geometry.

Since the primordial fluctuations are classical in their nature no problem arises in interpreting the density or temperature variations. The problem with the dynamically induced quantum fluctuations is more involved, however, and touches one of the most fundamental puzzles of physics, the decoherence process in the process of quantum to classical transition.

In a recent series of interesting papers \cite{5}-\cite{7}, Lesgourgues, Polarski and Starobinsky (LPS in what follows) analyze the evolution of inhomogeneous perturbations generated from the vacuum state during the initial accelerated expansion of the universe. These authors find that the quantum fluctuations become classical with stochastic Gaussian amplitudes, and that the decaying mode of these fluctuations becomes exponentially small towards the end of the inflationary period. Consequently, LPS suggest that the exponentially small decaying mode may be discarded for all practical purposes in this stage of the evolution, and the initially quantum perturbations may be considered classical. It seems then that there should be no (or minimal) information loss, and, apparently no (or minimal) entropy generation in this quantum to classical transition.

It is usually believed that this transition occurs when some class of coarse graining is enforced on the system. LPS question the validity of different coarse graining schemes in the following sense: if the dynamics of the system leads in a unitary way to a semiclassical behaviour is it necessary to perform a coarse graining? Last, but not least, stands the fact that the results obtained by LPS on the entropy content of the perturbations are in strong disagreement with respect to the results obtained by other authors \cite{8} and references therein).

The study of the entropy of quantum fields in the cosmological context dates from the early days of the developments in quantum field theory in curved spacetime (see \cite{9} for a review up to the 80’s), with further elaboration by Hu, Kandrup, and others \cite{10}. Currently the language used to discuss the subject is that of squeezed states, introduced in this context by Grishchuk and Sidorov \cite{11}, and borrowed from the quantum optics community \cite{12}. The squeeze formalism was also recently used by two of us to study the
problem of quantum tunneling under the influence of a time varying force, and the speculation was put forward that the initial entropy gained by the universe in the “creation from nothing” picture may be simply evaluated by using the squeeze parameter \([13]\).

The use of the squeezing formalism should not obscure the fact that, in the standard view, the entropy growth is always due to the particular coarse graining one chooses; yet, interestingly enough, the whole variety of coarse graining procedures (averaging over a period of the squeeze angle \([13]\), integrating over rotation angles \([14]\), neglecting information about the subfluctuant variable \([13]\), setting off-diagonal elements of the density matrix to zero \([14]\) leads in the large squeezing limit to the same expression for the entropy generated per mode, \(S_k \approx 2r_k\). LPS \([7]\), however, predict almost no entropy generation for similar systems.

One of the main purposes of this paper is to show, within the example considered by LPS, that as long as the Wigner function is used to represent the evolution of the system, as they do, the dynamical limit (certain parameter \(\xi \to \infty\) in our parametrization) and the semiclassical limit, \(\hbar \to 0\), are identical, and probably for most practical needs the system they consider may be labeled as “dynamically semiclassical”. However, the delicate question of the entropy content is related to the limiting behaviour of some elements of the density matrix associated with the system. We show that for the density matrix the \(\xi \to \infty\) and \(\hbar \to 0\) limits are different, and that the “true” decoherence is achieved only in \(\hbar \to 0\) limit, due to effective dephasing or rapid oscillations of the off-diagonal elements of the density matrix. This we do in the following Section.

In Section 3 we consider a model universe which starts inflating and then passes to radiation and matter dominated epochs. Evaluating the expression for the entropy based on the density matrix calculations we find that, due to an effective dephasing, the entropy generated per mode is given by the usual expression \(S_k \approx 2r_k\).

Section 4 is dedicated to some final remarks.

2 Wigner Function and Density Matrix

The study of the cosmological perturbations generated after the amplification of the vacuum fluctuations in the early stages of the universe can be reduced to the analysis of the evolution of a scalar field in a FRW background \([17]\). If at some initial conformal time \(\eta_0\) the field is in the vacuum state, the dynamical evolution under the influence of external gravitational field drives the scalar field into excited energetic states with opposite momenta.
One way to analyze the dynamical evolution of the cosmological perturbations and their quantum-to-classical transition is through the Wigner function formalism. The Wigner function for a one-dimensional quantum mechanical system is defined by [18]

\[ f_W(q, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dx \langle q - x/2 | \rho | q + x/2 \rangle e^{ipx/\hbar}, \]

where \( \rho \) is the density matrix of a pure or mixed quantum state.

The Wigner function corresponding to the evolution of the vacuum state of a scalar field in a FRW background was evaluated by Polarski and Starobinsky [5], and is given by

\[ W(k, -k) = \frac{1}{\pi^2\hbar^2} \exp \left\{ -\frac{|y(k)|^2}{|f_k|^2} \right\} \exp \left\{ -\frac{|f_k|^2}{\hbar^2} \left| p(k) - \frac{F(k)}{|f_k|^2} y(k) \right|^2 \right\}, \]

where \( y(k) \) is the Fourier transform of the rescaled scalar field \( y \equiv a\phi \) and \( p(k) \) is its canonical conjugate momentum. \( F(k) \) and \( |f_k| \) are two time-dependent parameters related to the variances of the field and the momentum. They are related to the squeeze parameter \( r_k \) and the squeeze angle \( \varphi_k \), the characteristic parameters of the Schrödinger picture, through [5]

\[ |f_k|^2 = \frac{1}{2k} (\cosh 2r_k + \cos 2\varphi_k \sinh 2r_k) \quad F(k) = \frac{1}{2} \sin 2\varphi_k \sinh 2r_k. \]

If we decompose the field \( y(k) \) and the canonical conjugate momentum \( p(k) \) into their real and imaginary parts, the two dimensional Wigner function may be further expressed as a product of two identical one-dimensional Wigner functions

\[ W(k, -k) = W_1(k, -k)W_2(k, -k), \]

where \( W_1 \) and \( W_2 \) are related to the real and imaginary part of the field and momentum.

Both \( W_1 \) and \( W_2 \) may be thought of as a particular case of the general parametrization of the Gaussian Wigner function (see for example Cooper et al. [19]):

\[ f_W(x, p) = \frac{1}{\pi\hbar} \exp \left\{ -\frac{x^2}{2\xi^2} - \frac{2\xi^2}{\hbar^2} \left( p - \frac{\mu}{\xi} x \right)^2 \right\}. \]

From now on we will use the expression (5) for convenience.

Here, comparing with Eq. (2), the parameters \( \xi \equiv \sqrt{2}|f_k| \) and \( \mu \equiv F(k)/(\sqrt{2}|f_k|) \) are related to the variances of the perturbations of the field.
and initially correspond to a coherent state. Note, as well, that this Wigner function is identical to the one obtained in the problem of an upside-down harmonic oscillator [20].

The Wigner function, in fact, is only another (particularly adequate) way of writing all the elements of the density matrix. The relation is given by the following Fourier transform

\[ \langle x' | \rho | x \rangle = \int_{-\infty}^{+\infty} dp \, e^{ip(x' - x)/\hbar} f_W \left( \frac{x' + x}{2}, p \right). \]  

Using the expression (5), we obtain for the density matrix the following expression (cf. Cooper et al. [19])

\[ \langle x' | \rho(\mu, \xi) | x \rangle = \frac{1}{(2\pi\xi^2)^{1/2}} \exp \left\{ -\frac{1}{4\xi^2} (x'^2 + x^2) + \frac{i\mu}{2\hbar\xi} (x'^2 - x^2) \right\}. \]  

It is clear that this density matrix corresponds to a pure quantum state, for it satisfies \( \rho^2 = \rho \).

Let us analyze the behaviour of the Wigner function in two different limits. One, which we will call, following [5], the dynamical limit and the other the semiclassical one to be defined below.

Since we are interested in analyzing the dynamical evolution of cosmological perturbations, we consider a limit corresponding to \( \xi \to \infty \) along with \( \mu/\xi \) being kept constant. This limit is related to the large squeezing limit characterized by the behaviour of the parameters \( F(k) \) and \( |f_k| \) which become unbounded. Thus, the dynamical limit obtained from the Wigner function (6) is

\[ f_{dW}^d(x, p) = \frac{1}{\sqrt{2\pi\xi^2}} e^{-x^2/2\xi^2} \delta(p - \mu/\xi). \]  

On the other hand, we can express the semiclassical limit in the usual way by analyzing the small \( \hbar \) expansion in the Wigner function (6). It follows that

\[ f_{scW}^d(x, p) = f_W^d(x, p). \]  

Both limits represent a classical probability distribution in the phase space with \( x \) obeying a Gaussian law, whereas \( p \) is fixed by the value of \( x \) at any instant.

Since both limits give the same expression for the Wigner function, it seems that the dynamical limit and the semiclassical limit are equivalent. Thus, for any practical purpose (physical measurement of different magnitudes) one expects to obtain the same result.

Yet, these two limits are not completely equivalent. If one tries, say, to reconstruct the elements of the density matrix using the equation (6), one
then immediately runs into trouble with the expression for $f_{\text{sc}}^W(x, p)$, since it is valid only for small $\bar{\hbar}$, and, therefore, one can not use the Fourier transform involving $e^{ip(x'-x)/\hbar}$, given that it is not perturbative in $\hbar$. On the other hand, this Fourier transform is perfectly defined for the expression $f_{\text{d}}^W(x, p)$ which was obtained for large $\xi$ instead.

Let us reinforce the statement by looking at the behaviour of the system following these two limits but starting directly from the density matrix (7).

The dynamical density matrix $\rho_{\text{d}}$, i.e., the dynamical limit of the density matrix as $\xi \to \infty$, is precisely the one given by the expression (7), in the sense that $\rho_{\text{d}} = \rho$, even for very large $\xi$. The unitarity of the evolution of the system is obvious in this limit and $\rho_{\text{d}}$ satisfies $\text{Tr} \rho_{\text{d}}^2 = 1$.

The leading term of the semiclassical expansion ($\hbar \to 0$) of the density matrix corresponds to

$$\langle x'|\rho_{\text{sc}}|x \rangle \sim \left| \frac{2\xi \hbar}{\mu} \right| \delta(x)\delta(x').$$

(10)

We can see that the semiclassical limit and the dynamical limit for the density matrix are quite different: $\rho_{\text{d}} \neq \rho_{\text{sc}}$, with the difference between them being in subleading terms in $\hbar$.

One may still define yet a different limiting density matrix obtained from the dynamical limit of the Wigner function $f_{\text{d}}^W(x, p)$, since it is quite “legitimate” as explained above. Performing the Fourier transform (6) we get

$$\langle x'|\tilde{\rho}_{\text{d}}|x \rangle = \frac{1}{(2\pi\xi^2)^{1/2}} \exp \left\{ -\frac{(x' + x)^2}{8\xi^2} + \frac{i\mu}{2\hbar \xi} (x'^2 - x^2) \right\} = \langle x'|\rho_{\text{d}}|x \rangle \exp \left\{ -\frac{(x' - x)^2}{8\xi^2} \right\}. \quad (11)$$

We see that the non-diagonal elements of the density matrix obtained by Fourier transforming the dynamical limit of the Wigner function $f_{\text{d}}^W(x, p)$ differ from those of the pure dynamical density matrix $\rho_{\text{d}}$ in subleading terms. Furthermore, the density matrix (11) does not satisfy $\tilde{\rho}_{\text{d}}^2 = \tilde{\rho}_{\text{d}}$ and gives $\text{Tr}\tilde{\rho}_{\text{d}}^2 = \infty$. This entails that $\tilde{\rho}_{\text{d}}$ neither represents a pure state, nor corresponds to a quantum density matrix.

We now look at the entropy. The quantum evolution of the system corresponding to $\rho_{\text{d}}$ is unitary (it is governed by a quadratic time-dependent Hamiltonian). Therefore, there is no entropy change associated with the quantum evolution of the system in time, and since the initial state is pure, it stays pure forever giving zero von Neumann entropy

$$S_d = -\text{Tr} \rho_{\text{d}} \ln \rho_{\text{d}} = 0. \quad (12)$$

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Let us now turn to the density matrix $\tilde{\rho}_d$ obtained from the dynamical limit of the Wigner function. Since $\tilde{\rho}_d$ differs from $\rho_d$ in subleading off-diagonal terms, the “entropy” that were to be associated with it should tell us how much information would be lost by discarding those subleading terms, or, in other words, how strongly “mixed” the state given by the density matrix $\tilde{\rho}_d$ is, as compared to the pure state given by $\rho_d$.

However, the von Neumann entropy $-\text{Tr} \tilde{\rho}_d \ln \tilde{\rho}_d$ is not well defined here. The fact that $\text{Tr} \tilde{\rho}_d^2 = \infty$ and that $\text{Tr} \tilde{\rho}_d = 1$ implies that the matrix has negative eigenvalues. It follows that different branches of the logarithm would give different imaginary contributions to the von Neumann entropy, if defined in the usual manner. We tried the following regularization:

$$\tilde{S}_d = -\text{Tr} \frac{1}{2} \rho_d \ln \tilde{\rho}_d^2.$$  \hspace{1cm} (13)

Intuitively, one might think that the quantity just defined is the entropy gain due to a dynamical decoherence. If so, it must be consistent with the calculations of LPS. However, one may easily show that this entropy has the wrong sign and is clearly pathological. The telling point here is that the state described by $\tilde{\rho}_d$ is ill-defined, and the calculations produce pathological results. This shows that one must be extremely careful when such a delicate quantity as the entropy is considered.

The dynamical limit of the density matrix $\rho_d$ given by expression (7) is the one which correctly describes the quantum state of the system. In fact, the system remains quantum forever. To obtain a classical behaviour one should in principle discard the quantum interference effects. These effects are associated with those terms of the density matrix which oscillate rapidly due to the smallness of $\hbar$. Thus, by averaging the quantum $\rho_d$ matrix over classical observation times (which is feasible in the cosmological context), the off-diagonal terms in the position representation are washed away. When the same procedure is applied to $\tilde{\rho}_d$, the density matrix obtained from the dynamical limit of the Wigner function (8), we get exactly the same averaged density matrix, since both density matrices have identical rapidly oscillating phases given by the expression $\exp \left\{ \frac{i\mu}{2\hbar \xi} (x'^2 - x^2) \right\}$.

Performing the average we have just mentioned, one readily finds an effective coarse grained entropy

$$S \approx \ln \xi,$$  \hspace{1cm} (14)

or as expressed in terms of the squeezing parameter $r$, ($\xi \approx \cosh 2r$), the entropy per mode is

$$S \approx 2r,$$  \hspace{1cm} (15)
in the large $r$ limit.

This result is the same as the one we obtain by evaluating the Boltzmann entropy from the probability in the phase space described by the Wigner function $f_W^d(x,p)$ after averaging over the momentum $f_W^d(x) \equiv \int_{-\infty}^{\infty} f_W^d(x,p)dp$. The above coarse graining scheme is thus equivalent to tracing out the non-diagonal elements of the density matrix and keeping the diagonal elements alone.

In order to clarify the issues raised by the "bad" behaviour of the elements of the matrix density in the position representation, we shall work in another representation which will prove to be more convenient.

3 Dynamical Evolution of the Density Matrix

As previously pointed out, we are interested in the way quantum interference effects can be suppressed following the dynamical evolution of the system. The behaviour of the system at large times, i.e., when the inflationary epoch is followed by radiation and matter dominated periods, is similar to that of a parametric oscillator whose time-dependent potential is slowly turned off. Thus, it seems that the most adequate basis to study the evolution would be that of the harmonic oscillator.

In the Schrödinger representation the dynamics of the state is given by the time evolution operator $|0, \eta\rangle_S \equiv S|0, \eta_0\rangle$. The time operator $S$ is the usual two-mode squeeze operator and the time evolution of the density matrix expressed in the harmonic oscillator basis is given by \[ \langle n = 2l | \rho | m = 2l' \rangle = \langle n | S(r, \varphi) | 0 \rangle \langle 0 | S^\dagger(r, \varphi) | m \rangle \]

\[ = (-1)^{l+l'} \frac{(2l)! (2l')!}{2^{l+l'} l! l'} \left( \frac{\tanh r}{\cosh r} \right)^{l+l'} e^{2i\varphi(l-l')} \] for each mode $k$. For convenience we have dropped the sub-index $k$, and $r$ and $\varphi$ are the usual squeeze parameter and squeeze angle, respectively. The only non vanishing diagonal elements are

\[ \langle 2l | \rho | 2l \rangle = \frac{(2l-1)!!}{2^l l!} \left( \frac{\tanh r}{\cosh r} \right)^{2l}. \] Note that only the non-diagonal elements of the density matrix depend on the squeeze angle $\varphi$.

The time dependence of the density matrix elements in this basis is given by the evolution of the squeezing parameters which can be obtained through
the interrelation between the Heisenberg and Schrödinger pictures \[12, 21\].

The equations of motion for \( r, \varphi \) and \( \theta \) are \[5, 12, 21\]

\[
\begin{align*}
    r' &= \frac{a'}{a} \cos 2\varphi, \\
    \varphi' &= -k - \frac{a'}{a} \coth 2r \sin 2\varphi, \quad (18) \\
    \theta' &= +k + \frac{a'}{a} \tanh r \sin 2\varphi.
\end{align*}
\]

The squeeze parameter \( r \) grows indefinitely towards the end of the inflationary period while the squeeze angle \( \varphi \) approaches a constant value. Thus, the non-diagonal elements of \( \rho_d \) do not oscillate, and no dephasing takes place, as was correctly pointed out by LPS.

However, let us now push the evolution of the system forward, pass the inflationary period by matching it to a radiation dominated one followed by a matter dominated expansion as, for example, was considered by Grishchuk and Sidorov \[11\]. It follows then that the behaviour of the squeeze parameter \( r \) and the squeeze angle \( \varphi \) are qualitatively different from the one we see during the end of the inflationary period.

To be specific, we use Grishchuk’s and Sidorov’s \[11\] simple model of the universe containing the three stages of expansion just described above

\[
\begin{align*}
    a_i &= -\frac{1}{H\eta} \quad (-\infty < \eta \leq \eta_1 < 0), \\
    a_r &= \frac{1}{H\eta_1^2} (\eta - 2\eta_1) \quad (\eta_1 \leq \eta \leq \eta_2), \quad (19) \\
    a_m &= \frac{1}{4H\eta_1^2 (\eta_2 - 2\eta_1)} (\eta - 4\eta_1 + \eta_2)^2 \quad (\eta_2 \leq \eta < \infty),
\end{align*}
\]

where \( a_i, a_r \) and \( a_m \) represent the scale factor during the inflationary era, radiation dominated period and matter dominated period respectively, and \( \eta_1 \) and \( \eta_2 \) are times where the transition between two consecutive periods takes place.

We have solved numerically equations (18) under conditions (19) and found the following behaviour of the squeeze parameters: the squeeze parameter \( r \) approaches its maximum value at the end of the inflationary period, oscillates around this large maximum value during the radiation dominated period and finally sets to a constant in the matter dominated epoch. This behaviour can be seen in the Figure 1, where the parameter \( r \) is represented as a function of conformal time \( \eta \).

On the other hand, as is apparent from Eq. (16), the squeeze angle \( \varphi \) shows itself as an angular phase \( e^{2i\varphi(l-l')} \) in the non-diagonal elements of
the density matrix. This angular parameter $\varphi$ grows linearly with time as $\varphi \approx \varphi_0 - k\eta$ leading to oscillating behaviour of the off-diagonal density matrix elements. Thus, the dynamical evolution of the model suggests a natural way to average these elements to zero. It is clear then, that sooner or later an effective dephasing process will take place. A typical behaviour of the phases of the off-diagonal density matrix elements is presented in Figure 2.

![Cos $\varphi$](image1.png) ![Sin $\varphi$](image2.png)

**Fig. 2a** **Fig. 2b**

Figure 2: The figure 2a represents the real part of the phase of the off-diagonal density matrix elements. Imaginary part of the phase is represented on figure 2b.

As a consequence of the above dephasing process one may proceed in calculating the entropy in the usual way by dropping the non-diagonal elements of the density matrix.

$$S = - \sum_{l=0}^{\infty} \langle 2l|\rho|2l \rangle \langle 2l|\ln \rho|2l \rangle$$
\[ = - \sum_{l=0}^{\infty} \frac{(2l-1)!!}{2^l l!} \frac{(\tanh r)^{2l}}{\cosh r} \ln \left( \frac{(2l-1)!!}{2^l l!} \frac{(\tanh r)^{2l}}{\cosh r} \right). \quad (20) \]

For convenience, the latter expression can be separated into two pieces

\[ S_1 = - \sum_{l=0}^{\infty} \frac{(2l-1)!!}{2^l l!} \frac{(\tanh r)^{2l}}{\cosh r} \ln \left( \frac{(\tanh r)^{2l}}{\cosh r} \right), \quad (21) \]

and

\[ S_2 = - \sum_{l=0}^{\infty} \frac{(2l-1)!!}{2^l l!} \frac{(\tanh r)^{2l}}{\cosh r} \ln \left( \frac{(2l-1)!!}{2^l l!} \right), \quad (22) \]

where the first one readily gives

\[ S_1 = \ln(\cosh r) - \sinh^2(r) \ln(\tanh r). \quad (23) \]

To evaluate the second term we expand expression (22) in terms of gamma functions and get for \( r \gg 1 \)

\[ S_2 = - \frac{1}{\cosh r} \sum_{l=1}^{\infty} \frac{\Gamma(2l)}{2^{2l-1} l \Gamma(l)^2} (\tanh r)^{2l} \ln \left( \frac{\Gamma(2l)}{2^{2l-1} l \Gamma(l)^2} \right). \quad (24) \]

With the aid of the Stirling’s formula \( \ln \left( \frac{\Gamma(2l)}{2^{2l-1} l \Gamma(l)^2} \right) \sim -\frac{1}{2} \ln(l\pi) \), we obtain

\[ S_2 \sim \frac{1}{2 \cosh r} \sum_{l=1}^{\infty} \frac{(\tanh r)^{2l}}{\sqrt{l\pi}} \ln(l\pi). \quad (25) \]

Using the McLaurin asymptotic method \[22\] for large \( r \) we get

\[ S_2 \sim - \frac{1}{2\sqrt{2}} \frac{1}{\cosh r} \frac{1}{\sqrt{-\ln \tanh r}} \left( \gamma + \log \left( \frac{-8 \ln \tanh r}{\pi} \right) \right) + C + O[\ln \tanh r] \], \quad (26) \]

where \( C \) is a constant.

Finally, the total coarse grained entropy is given by

\[ S \sim \ln \cosh r - \sinh^2 r \ln \tanh r + C - \frac{1}{2\sqrt{2}} \frac{(-\ln \tanh r)^{-1/2}}{\cosh r} \left( \gamma + \log \left( \frac{-8 \ln \tanh r}{\pi} \right) \right), \quad (27) \]

and in the limit \( r \to \infty \) we obtain

\[ S \approx 2r. \quad (28) \]
4 Conclusions

Before closing we feel that some final remarks are in order.

The first lesson learnt from our analysis is that one must be careful with the equivalent expressions obtained within different limiting procedures. From the example with the Wigner function we have seen that although the two expressions coincide (the one obtained in the dynamical limit, $\xi \to \infty$, of the Wigner function and the other in the semiclassical one, $\hbar \to 0$), one cannot use the last one to reconstruct the elements of the density matrix. The reason is simple: the reconstruction of the density matrix involves again the parameter $\hbar$. For any other calculation, these two limits may be perfectly equivalent, but if one is interested in the entropy of the system, the procedure fails! One may easily see this also by evaluating the two limits starting directly from the expression for the density matrix elements (7). The difference between the limits is evident from expression (10).

Going somewhat further, we were interested in seeing whether, by comparing the elements of the density matrix of the pure state taken in the late time limit and those "legitimately" obtained from the dynamical limit of the Wigner function, we can get a measure of the information loss due to dynamical decoherence. Intuitively, this measure should have been consistent with the small gain in the entropy due to the phase-space volume increase as described by LPS [7]. Unfortunately the entropy expression calculated in this way is ill-defined.

To evaluate properly the entropy generation for each mode of the perturbation we study the time evolution of the density matrix in the harmonic oscillator basis. Physically, this seems to be the correct basis since at late times, during the post-inflationary behaviour, the time dependent potential governing the dynamics of the modes switches off adiabatically. We explicitly see the dephasing behaviour due to the rapid oscillations of the off-diagonal elements of the density matrix, and calculate the entropy growth recovering the result $S_k \approx 2r_k$. It is important to note, however, that each mode has its proper time scale for dephasing depending on its wavenumber $k$, yet (almost) all modes, as mentioned above, sooner or later contribute to the process.

Decoherence is normally understood as the vanishing of the off-diagonal density matrix elements in a particular basis and is accompanied by the entropy growth due to the loss of information encoded in those elements. We have evaluated this entropy and found it consistent with previous results.

On the other hand, we have found that the entropy associated with the dynamical decoherence process evaluated on the basis of the dynamical limit of the Wigner function is ill-defined and probably carries no useful information about the statistical properties of the system.
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