ON VOLUMES OF HYPERBOLIC COXETER POLYTOPES AND QUADRATIC FORMS

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Abstract. In this paper, we compute the covolume of the group of units of the quadratic form $f_d^n(x) = x_1^2 + x_2^2 + \cdots + x_n^2 - dx_{n+1}^2$ with $d$ an odd, positive, square-free integer. Mcleod has determined the hyperbolic Coxeter fundamental domain of the reflection subgroup of the group of units of the quadratic form $f_3^n$. We apply our covolume formula to compute the volumes of these hyperbolic Coxeter polytopes.

1. Introduction

In 1996, we computed the covolume of the group of units of the quadratic form $f_d^n(x) = x_1^2 + x_2^2 + \cdots + x_n^2 - dx_{n+1}^2$ with $d$ an odd, positive, square-free integer, in order to determine the volume of some hyperbolic Coxeter simplices, see [14] and §5 of [8]. We never got around to writing up our computation except for the case $d = 1$ in [14]. Recently, several papers have been written concerning the group of units of the quadratic form $f_3^n$, and so we felt it was time to write up our more general computation. Our main result is Theorem 4 which gives an explicit formula for the covolume of the group of units of $f_d^n$.

J. Mcleod [10] has determined the reflection subgroup of the group of units of $f_3^n$ and has shown that it has finite index if and only if $n \leq 13$. As an application of our computation, we determine the volume of the hyperbolic Coxeter fundamental domain of the reflection subgroup of the group of units of the quadratic form $f_3^n$, for $n \leq 13$, described in [10]. See Table 1 for the volumes of Mcleod’s polytopes.

M. Belolipetsky and V. Emery [2] proved that for each odd dimension $n \geq 5$, there is a unique, orientable, noncompact, arithmetic, hyperbolic $n$-orbifold of smallest volume and determined its volume. V. Emery mentioned in [6] that for $n \equiv 3 \mod 4$, with $n \geq 7$, the corresponding arithmetic group is commensurable to the group of units of $f_3^n$. We determine the ratio between these two covolumes.

2. Siegel’s Covolume Formula for Unit Groups

Let $f$ be a quadratic form in $n + 1$ real variables, with $n \geq 2$, that is equivalent over $\mathbb{R}$ to the Lorentzian quadratic form $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$.

Let $S$ be the matrix of the quadratic form $f$. We assume that all the entries of $S$ are integers and $d = |\det S|$ is an odd, square-free, positive integer.

The group of units of the form $f$ is the group of all $(n + 1) \times (n + 1)$ matrices $A$, with integral entries, such that $A'SA = S$. A unit of $f$ is said to be positive or negative according as $A$ leaves invariant each of the two connected components
of \( \{ x \in \mathbb{R}^{n+1} : x^t S x < 0 \} \) or interchanges them. The group of positive units of \( f \) corresponds to a discrete group \( \Gamma \) of isometries of hyperbolic \( n \)-space

\[
H^n = \{ x \in E^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0 \}
\]

under the equivalence of \( f \) with the Lorentzian quadratic form. For a discussion, see §1 of [9].

Let \( q \) be a positive integer and let \( \mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z} \). Let \( E_q(S) \) be the number of \( (n+1) \times (n+1) \) matrices \( A \) over \( \mathbb{Z}_q \) such that \( A^t S A = S \). By Formula 82 in Siegel’s paper [16], we have that

\[
(1) \quad \text{vol}(H^n/\Gamma) = 4d^{\frac{n+2}{2}} \prod_{k=1}^n \pi^{-\frac{k}{2}} \Gamma(\frac{k}{2}) \cdot \lim_{q \to \infty} 2^{\omega(q)} \frac{q^{n(n+1)} \zeta(n+1)}{E_q(S)}
\]

where \( \Gamma(\frac{k}{2}) \) is the gamma function evaluated at \( k/2 \), and \( \omega(q) \) is the number of distinct prime divisors of \( q \), and \( q \) goes to infinity via the sequence \( 2!, 3!, 4!, \ldots \). In this paper a centered dot will delimit a product from the next factor.

We now evaluate the limit in Siegel’s volume formula. Let \( q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the prime factorization of \( q \). Now \( E_q(S) \) is a multiplicative function of \( q \) by Lemma 18 of Siegel’s paper [15], and so we have

\[
(2) \quad \frac{2^{\omega(q)} q^{n(n+1)} \zeta(n+1)}{E_q(S)} = \prod_{i=1}^r \frac{2p_i^{\alpha_i (n+1)}}{E_{p_i^{\alpha_i}}(S)}.
\]

Henceforth \( p \) is a prime number. By Lemma 18 of [15], we have

\[
(3) \quad \frac{2p^{a(n+1)}}{E_{p^a}(S)} = \begin{cases} 
2p^{3a(n+1)} / E_{p^a}(S) & \text{if } p \nmid 2d \text{ and } a \geq 3, \\
2p^{2a(n+1)} / E_{p^a}(S) & \text{if } p \nmid 2d \text{ and } a \geq 1.
\end{cases}
\]

Therefore, we have

\[
(4) \quad \lim_{q \to \infty} 2^{\omega(q)} \frac{q^{n(n+1)} \zeta(n+1)}{E_q(S)} = \prod_{p \mid 2d} 2p^{3(n+1)} / E_{p^3}(S) \prod_{p \mid 2d} 2p^{n(n+1)} / E_{p}(S).
\]

First assume that \( n \) is even and \( p \nmid 2d \). Then by Lemma 18 of [15], we have that

\[
(5) \quad \frac{2p^{n(n+1)}}{E_{p}(S)} = \frac{2}{k} (1 - p^{-2k})^{-1}.
\]

Hence we have

\[
(6) \quad \prod_{p \nmid 2d} \frac{2p^{n(n+1)}}{E_{p}(S)} = \frac{2}{k} \prod_{k=1} \prod_{p \nmid 2d} (1 - p^{-2k})^{-1}.
\]

Using the product formula for the Riemann zeta function \( \zeta(2k) \), we have

\[
(7) \quad \prod_{p \nmid 2d} \frac{2p^{n(n+1)}}{E_{p}(S)} = \prod_{k=1} \left( \prod_{p \nmid 2d} (1 - p^{-2k}) \right) \zeta(2k).
\]
Now assume that \( n \) is odd and \( p \nmid 2d \). Then by Lemma 18 of [15], we have that

\[
2p^{n+1} \frac{n+1}{2} E_p(S) = \left( 1 - \left( \frac{-1}{p} \right) \frac{n+1}{2} \right) p^{-\frac{n+1}{2}} \prod_{k=1}^{n+1} (1 - p^{-2k})^{-1}
\]

where \((\pm d/p)\) is a Legendre symbol. Define a fundamental discriminant \( D \) by the formula

\[
D = \begin{cases} \displaystyle (-1)^{\frac{n-1}{2}} d & \text{if } (-1)^{\frac{n-1}{2}} d \equiv 1 \mod 4, \\ 4(-1)^{\frac{n-1}{2}} d & \text{if } (-1)^{\frac{n-1}{2}} d \equiv 3 \mod 4. \end{cases}
\]

For all odd primes \( p \), we have

\[
\left( \frac{-1}{p} \right)^{\frac{n-1}{2}} = \left( \frac{D}{p} \right).
\]

Hence we have

\[
\prod_{p \nmid 2d} 2p^{n+1} \frac{n+1}{2} E_p(S) = \prod_{p \nmid 2d} \left( 1 - \left( \frac{D}{p} \right) p^{-\frac{n+1}{2}} \right) \prod_{k=1}^{n+1} (1 - p^{-2k})^{-1}.
\]

Consider the Dirichlet \( L \)-series

\[
L(s, D) = \sum_{k=1}^{\infty} \left( \frac{D}{k} \right) k^{-s}
\]

where \((D/k)\) is a Kronecker symbol. This series converges absolutely for \( s > 1 \). The Kronecker symbol \((D/k)\) is a completely multiplicative function of \( k \) by Theorem 1.4.9 in Cohen [4]. By Theorem 11.7 in Apostol [1], this \( L \)-function has the product formula

\[
L(s, D) = \prod_{p} \left( 1 - \left( \frac{D}{p} \right) p^{-s} \right)^{-1}.
\]

Using the product formulas for \( \zeta(2k) \) and \( L\left( \frac{n+1}{2}, D \right) \), we obtain the formula

\[
\prod_{p \nmid 2d} 2p^{n+1} \frac{n+1}{2} E_p(S) = \left( 1 - \left( \frac{D}{2} \right) 2^{-\frac{n+1}{2}} \right) L\left( \frac{n+1}{2}, D \right) \prod_{k=1}^{n-1} \left( \prod_{p \mid 2d} (1 - p^{-2k}) \right) \zeta(2k).
\]

It remains only to compute \( E_p(S) \) for each prime number \( p \) dividing \( 2d \).

3. The Computation of \( E_d(S) \)

From now on, we assume that \( f = f_d \). Then \( S \) is the \((n+1) \times (n+1)\) diagonal matrix \( \text{diag}(1, \ldots, 1, -d) \). Let \( k \) be a positive integer, and let \( Z_k = \mathbb{Z}/k\mathbb{Z} \). Let \( O(n+1, Z_k) \) be the group of \((n+1) \times (n+1)\) matrices \( A \) over the ring \( Z_k \) such that \( A^t A = I \). Let \( J \) be the \((n+1) \times (n+1)\) diagonal matrix \( \text{diag}(1, \ldots, 1, -1) \). Let \( O(n, 1; Z_k) \) be the group of \((n+1) \times (n+1)\) matrices \( A \) over the ring \( Z_k \) such that \( A^t J A = J \).

Let \( O(S, Z_k) \) be the group of invertible \((n+1) \times (n+1)\) matrices \( A \) over the ring \( Z_k \) such that \( A^t S A = S \). Then \( E_d(S) \) is the order of \( O(S, Z_d) \), since \( d \) is odd.

Let \( O^*(n+1, Z_4) \) be the image of \( O(n+1, Z_4) \) in \( O(n+1, Z_2) \) under the homomorphism induced by the ring homomorphism from \( Z_4 \) to \( Z_2 \), and let \( O^*(n, 1; Z_2) \)
be the image of $O(n, 1; \mathbb{Z}_4)$ in $O(n + 1, \mathbb{Z}_2)$ under the homomorphism induced by the ring homomorphism from $\mathbb{Z}_4$ to $\mathbb{Z}_2$.

We know that reduction modulo 2 maps $O(S, \mathbb{Z}_8)$ into $O^*(n, 1; \mathbb{Z}_2)$ if $d \equiv 1 \mod 4$ and into $O^*(n + 1, \mathbb{Z}_2)$ if $d \equiv -1 \mod 4$, since the image of $O(S, \mathbb{Z}_8)$ factors through $O(S, \mathbb{Z}_4)$. We next show that $O(S, \mathbb{Z}_8)$ maps onto the appropriate $O^*$ subgroup of $O(n + 1, \mathbb{Z}_2)$ under reduction modulo 2.

**Lemma 1.** If $d \equiv -1 \mod 4$, then $O(S, \mathbb{Z}_8)$ maps onto $O^*(n + 1, \mathbb{Z}_2)$ under reduction modulo 2.

**Proof.** By the discussion on page 60 of [14], the group $O^*(n + 1, \mathbb{Z}_2)$ is generated by the permutation matrices and, if $n + 1 \geq 6$, a matrix $C$ which is the identity except for a lower corner $6 \times 6$ block with all 0 diagonal and all 1 off-diagonal entries.

First assume that $d \equiv -1 \mod 8$. Then $O(S, \mathbb{Z}_8) = O(n + 1, \mathbb{Z}_8)$. The permutation matrices in $O^*(n + 1, \mathbb{Z}_2)$ obviously lift to $O(n + 1, \mathbb{Z}_8)$. If $n + 1 \geq 6$, the matrix $C$ lifts to $O(n + 1, \mathbb{Z}_8)$, since the identity part of $C$ lifts as is, and the $6 \times 6$ block lifts to a block with all 2 diagonal and all 1 off-diagonal entries.

Now assume that $d \equiv 3 \mod 8$. First we argue that all the permutation matrices lift to $O(S, \mathbb{Z}_8)$. Clearly the permutations of the first $n$ coordinates lift. Thus it suffices to show that the transposition of the last two coordinates also lifts, and the lift is the $2 \times 2$ block with diagonal 2, 6 and all 1 off-diagonal entries, that is,

$$
\begin{pmatrix}
2 & 1 \\
1 & 6
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -3
\end{pmatrix}
\begin{pmatrix}
2 & 1 \\
1 & 6
\end{pmatrix}
\equiv
\begin{pmatrix}
1 & 0 \\
0 & -3
\end{pmatrix}
\mod 8.
$$

We also need our lower $6 \times 6$ corner block to lift. The lift has diagonal entries 0, 0, 0, 0, 0, 4 and all 1 off-diagonal entries. \hfill \Box

**Lemma 2.** If $d \equiv 1 \mod 4$, then $O(S, \mathbb{Z}_8)$ maps onto $O^*(n, 1; \mathbb{Z}_2)$ under reduction modulo 2.

**Proof.** The proof for the case $d \equiv 1 \mod 8$ follows from Lemma 4 in [14]. Now assume $d \equiv -3 \mod 8$. Identify $O^*(n, \mathbb{Z}_2)$ with the subgroup of $O^*(n, 1; \mathbb{Z}_2)$ fixing the last standard basis vector $e_{n+1}$. By the proof of Lemma 4 in [14], the group $O^*(n, 1; \mathbb{Z}_2)$ is generated by the subgroup $O^*(n, \mathbb{Z}_2)$ and a matrix $C$ which is the identity except for a lower corner $4 \times 4$ block with all 0 diagonal and all 1 off-diagonal entries. The subgroup $O^*(n, \mathbb{Z}_2)$ lifts to $O(S, \mathbb{Z}_8)$ as in the case $d \equiv -1 \mod 8$ in Lemma 1. The $4 \times 4$ lower corner block of $C$ lifts to the $4 \times 4$ block with diagonal 2, 2, 2, 4 and all 1 off-diagonal entries. \hfill \Box

**Lemma 3.** Let $\eta : O(S, \mathbb{Z}_8) \to O(S, \mathbb{Z}_2)$ be the homomorphism induced by reduction modulo 2. The kernel of $\eta$ has order $2^{n^2+2n+2}$.

**Proof.** Each matrix $M$ in $O(S, \mathbb{Z}_8)$ can be written uniquely in the form $N + 2U + 4V$ where $N, U, V$ are zero-one matrices and we are interested in counting the $M$ with $N = I$. Now observe that

$$
M^tSM = S + 2(U^tS + SU) + 4(U^tSU + V^tS + SV).
$$

Using the fact that $4(-d) \equiv 4 \mod 8$, the last equation simplifies to

$$
M^tSM = S + 2(U^tS + SU) + 4(U^tU + V^t + V).
$$

First assume that $d \equiv 1 \mod 4$. Then $2(-d) \equiv 2(-1) \mod 8$ and the last equation simplifies to

$$
M^tSM = S + 2(U^tJ + JU) + 4(U^tU + V^t + V).
$$
The proof now proceeds as in the proof of Theorem 5 of [14].

Now assume that \( d \equiv -1 \mod 4 \). Then \( 2(-d) \equiv 2 \mod 8 \) and the next to last equation simplifies to

\[
M^tSM = S + 2(U^t + U) + 4(U^tU + V^t + V).
\]

The proof now proceeds as in the proof of Theorem 5 of [14] with the simplification that \( W = 0 \).

□

Let \( \epsilon_2(k) = 1 \) if \( k \) is even and 0 if \( k \) is odd and define

\[
\alpha(n) = \prod_{k=1}^{n} (2^k - \epsilon_2(k)).
\]

Define

\[
\beta(n) = 2^{n-2} + 2^{(n-2)/2}\cos(n\pi/4),
\]

and

\[
\gamma(n) = 2^n + 2^{n/2}\cos(n\pi/4).
\]

Note that the function \( f(n) = \cos(n\pi/4) \) is periodic with period 8. The values of \( f(n) \) for \( n = 0, 1, \ldots, 7 \) are \( 1, 2^{-1/2}, 0, -2^{-1/2}, -1, -2^{-1/2}, 0, 2^{-1/2} \), respectively.

**Theorem 1.** The value of \( E_8(S) \) is given by

\[
E_8(S) = \begin{cases} 
2^{n^2+2n+2}\alpha(n)/\beta(n+1) & \text{if } d \equiv -1 \mod 4, \\
2^{n^2+2n+2}\alpha(n)/\gamma(n-1) & \text{if } d \equiv 1 \mod 4. 
\end{cases}
\]

**Proof.** By Lemmas 2 of [14], the order of \( O^*(n+1, \mathbb{Z}_2) \) is \( \alpha(n)/\beta(n+1) \), and by Lemma 3 of [14], the order \( O^*(n, 1; \mathbb{Z}_2) \) is \( \alpha(n)/\gamma(n-1) \). The Theorem now follows from Lemmas 1, 2, 3.

**Corollary 1.** We have that

\[
\frac{2 \cdot 2^{3\frac{(n+1)}{2}}}{E_8(S)} = \frac{2^{\frac{n-1}{2}} \cos((n + (-1)^{\frac{d+1}{2}})\pi/4)}{2^{\frac{n+1}{2}} \prod_{k=1}^{\frac{n}{2}} (1 - 2^{-2k})}
\]

**Proof.** By Theorem 1, we have that

\[
\frac{2 \cdot 2^{3\frac{(n+1)}{2}}}{E_8(S)} = \frac{2 \cdot 2^{3\frac{(n+1)}{2}}(2^{n-1} + 2^{\frac{n-1}{2}}\cos((n + (-1)^{\frac{d+1}{2}})\pi/4))}{2^{n^2+2n+2}\prod_{k=1}^{n} (2^k - \epsilon_2(k))}
\]

If \( n \) is odd, we have that

\[
\prod_{k=1}^{n} (2^k - \epsilon_2(k)) = 2 \prod_{k=1}^{\frac{n-1}{2}} 2^{2k+1}(2^{2k} - 1) = 2 \prod_{k=1}^{\frac{n-1}{2}} 2^{4k+1}(1 - 2^{-2k}) = 2^{\frac{n(n+1)}{2}} \prod_{k=1}^{\frac{n-1}{2}} (1 - 2^{-2k}).
\]
If $n$ is even, we have that
\[
\prod_{k=1}^{n}(2^k - e_2(k)) = \prod_{k=1}^{\frac{n}{2}} 2^{2k-1}(2^{2k} - 1) = \prod_{k=1}^{\frac{n}{2}} 2^{4k-1}(1 - 2^{-2k}) = 2^{\frac{n(n+1)}{2}} \prod_{k=1}^{\frac{n}{2}} (1 - 2^{-2k}).
\]

Therefore, we have that
\[
\frac{2 \cdot 2^{\frac{n(n+1)}{2}}}{E_8(S)} = \frac{2 \cdot 2^{\frac{n(n+1)}{2}} 2^{\frac{n+1}{2}} (2^{\frac{n-1}{2}} + \cos((n + (-1)^{\frac{n+1}{2}})\pi/4))}{2^{n^2 + 2n + 2^{\frac{n(n+1)}{2}}} \prod_{k=1}^{2}(1 - 2^{-2k})}
\]
\[
= \frac{2^{\frac{n+1}{2}} + \cos((n + (-1)^{\frac{n+1}{2}})\pi/4)}{2^{\frac{n+1}{2}} \prod_{k=1}^{2}(1 - 2^{-2k})}. \quad \Box
\]

4. The Computation of $E_{p^a}(S)$

Let $p$ be a prime number that divides the odd, positive, square-free integer $d$. In this section, we determine the value of $E_{p^a}(S)$.

Let $a$ be a positive integer with $a \geq 2$, and let $M \in \text{GL}(n+1, \mathbb{Z}_{p^a})$. Define
\[
\alpha(M) = \text{diag}(1, 1, \ldots, 1, 0) \cdot M,
\]
\[
\beta(M) = \text{diag}(0, 0, \ldots, 0, 1) \cdot M.
\]

Then $\alpha(M)$ is the matrix $M$ with its last row set to zero, and $\beta(M)$ is $M$ with its first $n$ rows set to zero. Note that $M = \alpha(M) + \beta(M)$.

Define $\overline{\beta}(M) = \beta(M)$ reduced modulo $p^{a-1}$ with entries in the range $[0, p^{a-1})$. Define $\overline{M} = \alpha(M) + \overline{\beta}(M)$. Then $M - \overline{M} = p^{a-1}U$ for $U = \beta(U)$ with entries in the range $[0, p)$. Define
\[
\overline{O}(S, \mathbb{Z}_{p^a}) = \{ \overline{M} : M \in O(S, \mathbb{Z}_{p^a}) \}.
\]

Lemma 4. Let $M \in \text{GL}(n+1, \mathbb{Z}_{p^a})$ with $a \geq 2$. Then $M \in O(S, \mathbb{Z}_{p^a})$ if and only if $\overline{M} \in O(S, \mathbb{Z}_{p^a})$. Moreover
\[
E_{p^a}(S) = |O(S, \mathbb{Z}_{p^a})| = p^{n+1}|\overline{O}(S, \mathbb{Z}_{p^a})|.
\]

Proof. Let $M$ be an $(n+1) \times (n+1)$ matrix over $\mathbb{Z}$ such that $M' SM \equiv S \mod p^a$. Then $(\det(M))^2(-d) \equiv -d \mod p^a$. Hence $(\det(M))^2 \equiv 1 \mod p^{a-1}$, and so $\det(M)$ is invertible modulo $p^a$. Hence the reduction of $M$ modulo $p^a$ is invertible. Therefore $E_{p^a}(S) = |O(S, \mathbb{Z}_{p^a})|$.

Now suppose $M$ is an $(n+1) \times (n+1)$ matrix over $\mathbb{Z}_{p^a}$. Observe that
\[
M' SM = (\overline{M} + p^{a-1}U)' S(\overline{M} + p^{a-1}U)
= \overline{M}' \overline{S}M + p^{a-1}U' S\overline{M} + p^{a-1}M' SU + p^{2a-2}U' SU
= \overline{M}' \overline{S}M + p^{a-1}(-d) U' \overline{M} + p^{a-1}(-d) M' U
= \overline{M}' \overline{S}M.
\]
Thus \( M \in O(S, \mathbb{Z}_p^n) \) if and only if \( \overline{M} \in O(S, \mathbb{Z}_p^n) \). In the equation \( M - \overline{M} = p^{a-1}U \), there are \( p^{n+1} \) choices for \( U \). Hence for each \( \overline{M} \) there are \( p^{n+1} \) choices for \( M \). Therefore \( |O(S, \mathbb{Z}_p^n)| = p^{n+1}|\overline{O}(S, \mathbb{Z}_p^n)| \).

**Lemma 5.** If \( a \geq 2 \), we have that \( |\overline{O}(S, \mathbb{Z}_{p^{a+1}})| = p^{\frac{n(n+1)}{2}}|\overline{O}(S, \mathbb{Z}_p)| \).

**Proof.** If \( M \in \overline{O}(S, \mathbb{Z}_{p^{a+1}}) \), then \( N = (M \mod p^n) \in \overline{O}(S, \mathbb{Z}_p^n) \). We want to count \( M = N + p^a\alpha(U) + p^{a-1}\beta(U) \) for \( U \mod p \) and \( N \in \overline{O}(S, \mathbb{Z}_p^n) \) such that \( M \in \overline{O}(S, \mathbb{Z}_{p^{a+1}}) \). Observe that

\[
M^tSM - S = N^tSN - S + p^a(\alpha(U)^tSN + N^t\alpha(U)) + p^{a-1}(\beta(U)^tSN + N^t\beta(U)) + p^{2a-1}(\alpha(U)^t\beta(U) + \beta(U)^t\alpha(U)) + p^{2a}(\alpha(U)^t\alpha(U)) + p^{2a-2}(\beta(U)^t\beta(U)) = N^tSN - S + p^a(\alpha(U)^tN + N^t\alpha(U)) + p^a(-d/p)(\beta(U)^tN + N^t\beta(U)).
\]

Let \( V = \alpha(U) + (-d/p)\beta(U) \). Then \( \alpha(U) = \alpha(V) \) and \( \beta(U) \equiv (-d/p)^{-1}\beta(V) \mod p \).

Now \( M^tSM = S \) if and only if

\[
V^tN + N^tV \equiv \frac{S - N^tSN}{p^a} \mod p
\]

where we regard the entries of \( N \) to be integers in the range \([0, p^n]\). Let \( W = N^tV \mod p \). Then \( V = (N^t)^{-1}W \mod p \). Note that \( N^t \) is invertible mod \( p \), since \( N \) is invertible mod \( p \) by Lemma 4. Now \( M^tSM = S \) if and only if

\[
W^t + W \equiv \frac{S - N^tSN}{p^a} \mod p.
\]

This congruence determines the diagonal of \( W \) and below diagonal entries in terms of arbitrary above diagonal entries for \( p^{n(n+1)/2} \) choices for \( W \) with entries mod \( p \). Thus we have \( p^{n(n+1)/2} \) choices for \( U \) and as many \( M \in \overline{O}(S, \mathbb{Z}_{p^{a+1}}) \) for each \( N \in \overline{O}(S, \mathbb{Z}_p^n) \). \( \square \)

**Lemma 6.** We have that \( |\overline{O}(S, \mathbb{Z}_p^n)| = 2p^{\frac{n(n+1)}{2}}|O(n, \mathbb{Z}_p)| \).

**Proof.** Let \( M \in \overline{O}(S, \mathbb{Z}_p^n) \) and decompose \( M \) into blocks

\[
M = \begin{pmatrix} M_0 & v \\ w^t & x \end{pmatrix}
\]

with \( v \in \mathbb{Z}_p^n, w \in \mathbb{Z}_p^n \) and \( x \in \mathbb{Z}_p \).

Observe that \( M^tSM = S \) if and only if

\[
\begin{cases}
M_0^tM_0 - dwv^t \equiv I \mod p^2, \\
M_0^tv - dxw \equiv 0 \mod p^2, \\
v^tv - dx^2 \equiv -d \mod p^2.
\end{cases}
\]

...
The above system of congruences implies that
\[
\begin{cases}
M_0^t M_0 &\equiv I \mod p, \\
v &\equiv 0 \mod p, \\
x^2 &\equiv 1 \mod p.
\end{cases}
\]

Let \(N_0 = (M_0 \mod p)\). Then \(M_0 = N_0 + pU_0\) for \(U_0 \mod p\) and \(N_0 \in O(n, \mathbb{Z}_p)\). For each such \(N_0\) choose \(x = \pm 1 \mod p\) and \(w \in \mathbb{Z}_p^n\) arbitrarily. We need to count the \(U_0\) and find \(v\) so that \(M \in \mathbb{O}(S, \mathbb{Z}_p)\). Note that \(v = py\) where \(M_0^t y \equiv (d/p)xw \mod p\), and so \(v\) is determined from \(x, w\) and \(N_0\). Thus it suffices to count \(U_0\) such that
\[
(N_0 + pU_0)^t (N_0 + p^k U_0) - dww^t \equiv I \mod p^2.
\]

The above congruence is equivalent to
\[
U_0^t N_0 + N_0^t U_0 \equiv \frac{I - N_0^t N_0}{p} + (d/p)ww^t \mod p.
\]

Let \(V_0 = N_0^t U_0\). Then \(U_0 = (N_0^t)^{-1} V_0\). The congruence
\[
V_0^t + V_0 \equiv \frac{I - N_0^t N_0}{p} + (d/p)ww^t \mod p
\]
determines the diagonal of \(V_0\) and below diagonal entries from the above diagonal entries. Hence there are \(p^{n(n-1)/2}\) choices for \(V_0\) and so for \(U_0\) given \(w\) and \(x\). Thus there are \(2p^{n(n-1)/2 + n}\) possibilities for \(M \in \mathbb{O}(S, \mathbb{Z}_p)\) for each \(N_0 \in O(n, \mathbb{Z}_p)\).

**Theorem 2.** The values of \(E_{p^2}(S)\) and \(E_{p^3}(S)\) are given by
\[
E_{p^2}(S) = 2p^{\frac{n(n+1)(n+2)}{2}}|O(n, \mathbb{Z}_p)|,
E_{p^3}(S) = p^{\frac{n(n+1)}{2}}E_{p^2}(S).
\]

**Proof.** By Lemmas 4 and 6, we have that
\[
E_{p^2}(S) = p^{n+1}|O(S, \mathbb{Z}_p^2)| = p^{n+1}2p^{\frac{n(n+1)}{2}}|O(n, \mathbb{Z}_p)| = 2p^{\frac{n(n+1)(n+2)}{2}}|O(n, \mathbb{Z}_p)|.
\]

By Lemmas 4 and 5, we have that
\[
E_{p^3}(S) = p^{n+1}|O(S, \mathbb{Z}_p^3)| = p^{n+1}p^{\frac{n(n+1)}{2}}|O(S, \mathbb{Z}_p^2)| = p^{\frac{n(n+1)}{2}}E_{p^2}(S).
\]

**Corollary 2.** We have that
\[
\frac{2p^{\frac{n(n+1)}{2}}}{E_{p^3}(S)} = \frac{2p^{\frac{2n(n+1)}{2}}}{E_{p^2}(S)} = \frac{p^{\frac{(n+1)(n-2)}{2}}}{|O(n, \mathbb{Z}_p)|}.
\]

We need the following classical theorem, see Theorem 172 of Dickson [5].

**Theorem 3.** If \(p\) be an odd prime number, then the order of \(O(n, \mathbb{Z}_p)\) is given by
\[
|O(n, \mathbb{Z}_p)| = \begin{cases}
2^{n-1}\prod_{k=1}^{n-1} \left( p^k - \epsilon_2(k) \right) & n \text{ odd}, \\
2p^{n-1} \left( p^n - \left( \frac{-1}{p} \right)^\frac{n}{2} \right) \prod_{k=1}^{n-2} p^k - \epsilon_2(k) & n \text{ even},
\end{cases}
\]
where \((-1/p)\) is a Legendre symbol.
Corollary 3. We have that

\[ \frac{2p^{3n(n+1)}_{\frac{n}{2}}}{E_{p^3}(S)} = \frac{p^{\frac{n}{2}} + \epsilon_2(n) \left( \frac{-1}{p} \right)^{\frac{n}{2}}}{2p^{\frac{n+2}{2} \prod_{k=1}^{n} (1 - p^{-2k})}}. \]

Proof. First assume that \( n \) is odd. By Theorem 3 and Corollary 2, we have that

\[ \frac{2p^{3n(n+1)}_{\frac{n}{2}}}{E_{p^3}(S)} = \frac{p^{\frac{n+1}{2} (n-2)}}{2 \prod_{k=1}^{n-1} p^k - \epsilon_2(k)} \]

\[ = \frac{p^{\frac{n+1}{2} (n-2)}}{2 \prod_{k=1}^{n-1} p^{2k-1} (p^{2k} - 1)} \]

\[ = \frac{p^{\frac{n+1}{2} (n-2)}}{2 \prod_{k=1}^{n-1} p^{4k-1} (1 - p^{-2k})} = \frac{1}{2p^{\frac{n-1}{2} \prod_{k=1}^{n-1} (1 - p^{-2k})}}. \]

Now assume that \( n \) is even. By Theorem 3 and Corollary 2, we have that

\[ \frac{2p^{3n(n+1)}_{\frac{n}{2}}}{E_{p^3}(S)} = \frac{p^{\frac{n+1}{2} (n-2)}}{2 \prod_{k=1}^{n-1} p^k - \epsilon_2(k)} \]

\[ = \frac{p^{\frac{n+1}{2} (n-2)}}{2p^{\frac{n}{2}-1} (p^{\frac{n}{2}} - \left( \frac{-1}{p} \right)^{\frac{n}{2}}) \prod_{k=1}^{n-2} p^k - \epsilon_2(k)} \]

\[ = \frac{p^{\frac{n+1}{2} (n-2)}}{2p^{\frac{n}{2}-1} (p^{n-1} - 1) \prod_{k=1}^{n-2} p^k - \epsilon_2(k)} \]

\[ = \frac{p^{\frac{n+1}{2} (n-2)}}{2p^{-\frac{n}{2}} \prod_{k=1}^{n} p^k - \epsilon_2(k)} = \frac{p^{\frac{n}{2} + \left( \frac{-1}{p} \right)^{\frac{n}{2}}}}{2p^{\frac{n+1}{2} \prod_{k=1}^{\frac{n}{2}} (1 - p^{-2k})}}. \]
5. The covolume of the group of units of $f^n_d$ 

Let $\Gamma^n_d$ be the discrete group of isometries of hyperbolic $n$-space $H^n$ that corresponds to the group of positive units of the quadratic form $f^n_d$. In this section, we give an explicit formula for $\text{vol}(H^n/\Gamma^n_d)$.

From Formula (6) of [14], we have

\begin{equation}
\prod_{k=1}^{n} \pi^{-\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) = \begin{cases} 
\prod_{k=1}^{\frac{n-1}{2}} \frac{2(2k-1)!}{(2\pi)^{2k}} & n \text{ odd}, \\
\prod_{k=1}^{\frac{n}{2}} \frac{2(2k-1)!}{(2\pi)^{2k}} \cdot \frac{(2\pi)^{\frac{n}{2}}}{(n-1)!!} & n \text{ even}.
\end{cases}
\end{equation}

By Theorems 12.17 and 12.18 of [1], we have that

\begin{equation}
\zeta(2k) = \frac{(2\pi)^{2k} |B_{2k}|}{2(2k)!}
\end{equation}

for every positive integer $k$ where $B_{2k}$ is the $(2k)$th Bernoulli number.

Define a function $B$ of $n$ by the formula

\begin{equation}
B = \prod_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{|B_{2k}|}{2k}.
\end{equation}

Then we have that

\begin{equation}
\prod_{k=1}^{n} \pi^{-\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) \prod_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \zeta(2k) = \begin{cases} 
B & n \text{ odd}, \\
B \cdot \frac{(2\pi)^{\frac{n}{2}}}{(n-1)!!} & n \text{ even}.
\end{cases}
\end{equation}

Define a function $C$ of $n$ and $d$ by the formula

\begin{equation}
C = \cos\left((n + (-1)^{\frac{n+1}{2}})\frac{\pi}{4}\right).
\end{equation}

**Theorem 4.** Let $d$ be an odd, square-free, positive integer, and let $\Gamma^n_d$ be the discrete group of isometries of hyperbolic $n$-space $H^n$ corresponding to the group of positive units of the quadratic form $f^n_d$. The volume of $H^n/\Gamma^n_d$ is given by

\[
\text{vol}(H^n/\Gamma^n_d) = \begin{cases} 
\frac{d\frac{n+1}{2} B}{2n + \omega(d)} \left(2^{\frac{n+1}{2}} + C\right) \left(2^{\frac{n+1}{2}} - \left(\frac{D}{2}\right)^{\frac{1}{2}}\right) \sqrt{d} \cdot L\left(\frac{n+1}{2}, D\right) & n \text{ odd}, \\
\frac{B}{2^{\frac{n+1}{2}} + \omega(d)} \left(2^{\frac{n+1}{2}} + 2^{\frac{n+1}{2}} C\right) \prod_{\substack{p | d \\text{ odd}}} \left(p^{\frac{n}{2}} + \left(\frac{-1}{p}\right)^{\frac{n}{2}}\right) \cdot \frac{(2\pi)^{\frac{n}{2}}}{(n-1)!!} & n \text{ even}.
\end{cases}
\]
Proof. First assume that $n$ is odd. From Formulas (1), (4), (14), (24) and Corollaries 1 and 3, we have that

$$\text{vol}(H^n / \Gamma_d^n) = 4d^{n+2} \prod_{k=1}^{n} \pi^{-\frac{k}{2}} \Gamma(\frac{k}{2}) \cdot \lim_{q \to \infty} 2^{\omega(q)} q^{\frac{n(n+1)}{2}} E_q(S)$$

$$= 4d^{n+2} \prod_{k=1}^{n} \pi^{-\frac{k}{2}} \Gamma(\frac{k}{2}) \cdot \prod_{p \mid 2d} 2p \frac{3n(n+1)}{2} E_{p^3}(S) \cdot \prod_{p \mid 2d} 2p \frac{n(n+1)}{2} E_p(S)$$

$$= 4d^{n+2} B \prod_{p \mid 2d} 2p \frac{3n(n+1)}{2} E_{p^3}(S) \left(1 - \left(\frac{D}{p^2}\right) 2^{-\frac{n+1}{2}} L\left(\frac{n+1}{2}, D\right)\right) \prod_{k=1}^{n} \prod_{p \mid 2d} (1 - p^{-2k})$$

$$= 4d^{n+2} B \left(\frac{2^{n+1}}{2} + C\right) \left(1 - \left(\frac{D}{p^2}\right) 2^{-\frac{n+1}{2}} L\left(\frac{n+1}{2}, D\right)\right)$$

$$= \frac{d^{n+1} B}{2^{n+1}} 2^{\omega(d)} \left(\frac{2^{n+1}}{2} + C\right) \left(2^{n+1} - \left(\frac{D}{p^2}\right) \sqrt{d} \cdot L\left(\frac{n+1}{2}, D\right)\right).$$

Now assume that $n$ is even. From Formulas (1), (4), (7), (24) and Corollaries 1 and 3, we have that

$$\text{vol}(H^n / \Gamma_d^n) = 4d^{n+2} \prod_{k=1}^{n} \pi^{-\frac{k}{2}} \Gamma(\frac{k}{2}) \cdot \prod_{p \mid 2d} 2p \frac{3n(n+1)}{2} E_{p^3}(S) \cdot \prod_{p \mid 2d} 2p \frac{n(n+1)}{2} E_p(S)$$

$$= 4d^{n+2} B \cdot \frac{(2\pi)^\frac{n}{2}}{(n-1)!!} \prod_{p \mid 2d} 2p \frac{3n(n+1)}{2} E_{p^3}(S) \cdot \prod_{k=1}^{n} \prod_{p \mid 2d} (1 - p^{-2k})$$

$$= 4d^{n+2} B \cdot \frac{(2\pi)^\frac{n}{2}}{(n-1)!!} \prod_{p \mid d} \left(\frac{p^\frac{n}{2} + (-1)^\frac{n}{2} \sqrt{p}}{2p^{n+1}}\right)$$

$$= \frac{B}{2^{\frac{n}{2} + \omega(d)}} \left(2^{\frac{n}{2} + 2} C\right) \prod_{p \mid d} \left(p^{\frac{n}{2}} + \left(-\frac{1}{p}\right)^{\frac{n}{2}}\right) \cdot \frac{(2\pi)^\frac{n}{2}}{(n-1)!!}.$$  \[\square\]

**Corollary 4.** If $n$ is odd, then the Euler characteristic of $\Gamma_d^n$ is zero. If $n$ is even, then the Euler characteristic of $\Gamma_d^n$ is given by

$$\chi(\Gamma_d^n) = \frac{(-1)^\frac{n}{2}}{2^{\frac{n}{2} + \omega(d)}} \left(2^{\frac{n}{2} + 2} C\right) \prod_{p \mid d} \left(p^{\frac{n}{2}} + \left(-\frac{1}{p}\right)^{\frac{n}{2}}\right).$$

**Proof.** The group $\Gamma_d^n$ is finitely generated, and so $\Gamma_d^n$ has a torsion-free subgroup of finite index by Selberg’s Lemma. If $n$ is odd, then $\chi(\Gamma_d^n) = 0$, since odd dimensional hyperbolic manifolds of finite volume have zero Euler characteristic. If $n$ is even, then the Gauss-Bonnet Theorem implies that

$$\text{vol}(H^n / \Gamma_d^n) = (-1)^\frac{n}{2} \chi(\Gamma_d^n) \cdot \left(\frac{(2\pi)^\frac{n}{2}}{(n-1)!!}\right).$$

The result now follows immediately from Theorem 4.  \[\square\]
Using the formulas in Theorem 4, we can easily compute the volume of $H^n/\Gamma_d^n$. For example, we have

$$\text{vol}(H^3/\Gamma_3^3) = \frac{5\sqrt{3}}{64} L(2, -3),$$
$$\text{vol}(H^3/\Gamma_7^3) = \frac{7\sqrt{7}}{64} L(2, -7),$$
$$\text{vol}(H^5/\Gamma_5^5) = \frac{15\sqrt{5}}{2048} L(3, 5),$$
$$\text{vol}(H^7/\Gamma_7^7) = \frac{49\sqrt{7}}{98304} L(4, -7).$$

These four volumes were reported on pages 344 and 345 of [8], with some sign differences due to different definitions of $\Gamma_d^n$ and $L(s, D)$. In [8], the group $\Gamma_d^n$ was denoted by $\Gamma_{n-d}^n$, and $L$-functions were defined via Legendre symbols rather than Kronecker symbols.

6. The volumes of Mcleod’s polytopes

J. Mcleod [10] has determined the reflection subgroup of $\Gamma_3^n$ and has shown that it has finite index if and only if $n \leq 13$. Let $P^n$ be the hyperbolic Coxeter fundamental domain of the reflection subgroup of $\Gamma_3^n$ determined by Mcleod in [10]. Mcleod showed that the reflection subgroup of $\Gamma_3^n$ has index equal to the order of the symmetry group $\text{Sym}(P^n)$ of $P^n$: moreover, Mcleod determined $\text{Sym}(P)$ for $n \leq 13$. Hence, for $n \leq 13$, we have that

$$\text{vol}(P^n) = |\text{Sym}(P^n)| \cdot \text{vol}(H^n/\Gamma_3^n).$$

Table 1 lists the volumes of $H^n/\Gamma_3^n$ and $P^n$ for $n \leq 13$ computed using our formulas.

The polygon $P^2$ is a $30^\circ - 45^\circ$ right triangle, and so we can compute the area of $P^2$ by the classical angle defect formula

$$\text{vol}(P^2) = \pi - \left( \frac{\pi}{2} - \frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{\pi}{12}.$$

The polyhedron $P^3$ is an orthotetrahedron with angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$. By Theorem 10.4.5 and the duplication formulas 10.4.9 and 10.4.10 in [13], we have that

$$\text{vol}(P^3) = \frac{5}{16} \text{L}(\frac{\pi}{3})$$

where $\text{L}(\theta)$ is the Lobachevsky function. By Formula 2 in Milnor [11], we have that

$$\text{L}(\theta) = \frac{1}{2} \sum_{k=1}^{\infty} \sin(2k\theta)k^{-2}.$$

Observe that

$$\sin(2k\pi/3) = \frac{\sqrt{3}}{2} \left( -\frac{3}{k} \right).$$

Hence we have that

$$\text{vol}(P^3) = \frac{5\sqrt{3}}{64} L(2, -3).$$
For even $n$, we can compute the volume of the polytope $P^n$ using the Gauss-Bonnet Theorem which implies that

$$\text{vol}(P^n) = (-1)^{n/2} \chi(P^n) \cdot \frac{(2\pi)^{n/2}}{(n-1)!!}$$

where $\chi(P^n)$ is the Euler characteristic of the Coxeter group $P^n$ defined by $P^n$. By Proposition 3 of Chiswell [3], we have that

$$\chi(P^n) = \sum_{\Delta} \frac{(-1)^{|\Delta|}}{|C_\Delta|}$$

where the sum is over all subgraphs $\Delta$ of the Coxeter graph of $P^n$ such that the Coxeter group $C_\Delta$ defined by $\Delta$ is finite. Here $|\Delta|$ is the number of vertices of $\Delta$. From the Coxeter graph of $P^n$ given in Mcleod [10], we computed the volumes of $P^n$ for all even $n \leq 12$ using Chiswell’s Euler characteristic formula and the Gauss-Bonnet Theorem. The volumes agree with the volumes given in Table 1.

For odd $n$, we computed the volume of $P^n$ by numerical integration. We found very close agreement with the numerical values in Table 1 in dimensions 3, 5, 7, 9. We found close agreement in dimensions 11 and 13, but the accuracy deteriorated to a 1% discrepancy in dimension 11 and a 3% discrepancy in dimension 13, which is not bad considering we numerically integrated an $n$-fold integral for $n = 11, 13$. After the agreement of all these alternate calculations of the volume of $P^n$ for all $n \leq 13$, we are confident with the correctness of our volume formulas derived using Siegel’s analytic theory of quadratic forms.

Table 1. The volumes of Mcleod’s hyperbolic Coxeter polytopes

| $n$ | $\text{vol}(\Gamma_n^3)$ | $|\text{Sym}(P^n)|$ | $\text{vol}(P^n)$ | $|\text{vol}(P^n)|$ |
|-----|-------------------------|-----------------|-----------------|-----------------|
| 2   | $\frac{\pi}{12}$       | 1               | $\frac{\pi}{12}$ | 2.617993878 * 10^{-1} |
| 3   | $\frac{\sqrt{2}}{64} L(2, -3)$ | 1 | $\frac{\sqrt{2}}{64} L(2, -3)$ | 1.057230840 * 10^{-1} |
| 4   | $\frac{\sqrt{2}}{258}$ | 1 | $\frac{\sqrt{2}}{258}$ | 3.426945973 * 10^{-2} |
| 5   | $\frac{\sqrt{3}}{320} L(3, 12)$ | 1 | $\frac{\sqrt{3}}{320} L(3, 12)$ | 5.358748597 * 10^{-3} |
| 6   | $\frac{13\pi^3}{2645808}$ | 1 | $\frac{13\pi^3}{2645808}$ | 6.664708943 * 10^{-4} |
| 7   | $\frac{51\sqrt{3}}{1140880} L(4, -3)$ | 1 | $\frac{51\sqrt{3}}{1140880} L(4, -3)$ | 7.240232999 * 10^{-5} |
| 8   | $\frac{9144576000}{697^4}$ | 1 | $\frac{9144576000}{697^4}$ | 7.424525364 * 10^{-6} |
| 9   | $\frac{L(5, 12)}{716800\sqrt{3}}$ | 1 | $\frac{L(5, 12)}{716800\sqrt{3}}$ | 8.05159421 * 10^{-7} |
| 10  | $\frac{341\pi^5}{9876140080000}$ | 2 | $\frac{341\pi^5}{9876140080000}$ | 2.113228256 * 10^{-7} |
| 11  | $\frac{6043^4}{12918456320\sqrt{3}}$ | 2 | $\frac{6043^4}{12918456320\sqrt{3}}$ | 3.546550442 * 10^{-8} |
| 12  | $\frac{50443\pi^6}{1242813749344720000}$ | 2 | $\frac{50443\pi^6}{1242813749344720000}$ | 7.804122909 * 10^{-9} |
| 13  | $\frac{3444060864000\sqrt{3}}{6911(7, 12)}$ | 4 | $\frac{3444060864000\sqrt{3}}{6911(7, 12)}$ | 4.633381297 * 10^{-9} |
7. A commensurability ratio

M. Belolipetsky and V. Emery [2] proved that for each odd dimension \( n \geq 5 \) there is a unique orientable, noncompact, arithmetic, hyperbolic \( n \)-oribfold \( H^n/\Delta_n \) of smallest volume. If \( n \equiv 3 \mod 4 \), they proved that the volume of \( \Delta_n \) is given by

\[
\text{vol}(H^n/\Delta_n) = \frac{3^{1/2}}{2^{n-1}} L\left(\frac{n+1}{2}, -3\right) \prod_{k=1}^{n-1} \frac{(2k-1)!}{(2\pi)^{2k}} \zeta(2k).
\]

In terms of Bernoulli numbers, the above formula can be rewritten as

\[
\text{vol}(H^n/\Delta_n) = \frac{3^{1/2} B}{2^{n-1}} L\left(\frac{n+1}{2}, -3\right).
\]

In [6], V. Emery mentioned that \( \Delta_n \) is commensurable to \( \Gamma_3^n \) when \( n \equiv 3 \mod 4 \). We now compute the ratio \( \text{vol}(H^n/\Gamma_3^n)/\text{vol}(H^n/\Delta_n) \) for \( n \equiv 3 \mod 4 \). By Theorem 4, we have that

\[
\text{vol}(H^n/\Gamma_3^n) = \frac{3^{1/2} B}{2^{n+1}} \left(2^{n/2} + (-1)^{n+1}\right) L\left(\frac{n+1}{2}, -3\right).
\]

Therefore, we have

\[
\frac{\text{vol}(H^n/\Gamma_3^n)}{\text{vol}(H^n/\Delta_3)} = \frac{1}{4} \left(2^{n/2} + (-1)^{n+1}\right) (2^{n/2} + 1).
\]

Although \( \Delta_n \) is considered in [2] only for \( n \geq 5 \), we can define \( \Delta_3 \) to be \( \text{PGL}(2, O_3) \) where \( O_3 \) is the ring of integers of \( \mathbb{Q}(\sqrt{3}) \). Then \( \Delta_3 \) is arithmetic and Meyerhoff [12] proved that \( H^3/\Delta_3 \) has minimum volume among all orientable, noncompact, hyperbolic 3-orifolds. The group \( \Delta_3 \) is the orientation preserving subgroup of a hyperbolic Coxeter group of type \( [3,3,6] \). The above formula for \( \text{Vol}(H^3/\Delta_3) \) gives the correct volume for \( H^3/\Delta_3 \). Hence the above formula for \( \text{vol}(H^3/\Gamma_3^3)/\text{vol}(H^3/\Delta_3) \) gives the correct value 5/4.

The groups \( \Gamma_3^3 \) and \( \Delta_3 \) are commensurable. The relationship between \( \Gamma_3^3 \) and \( \Delta_3 \) is explained by the commensurability diagram on page 130 of Johnson et al. [9]. The group \( \Gamma_3^3 \) is a hyperbolic Coxeter group of type \( [4,3,6] \). The Coxeter group \( [4,3,6] \) has a Coxeter subgroup of type \( [6,3^{1,1}] \) of index 2. Hence the orientation preserving subgroup of \( [6,3^{1,1}] \) has index 4 in \( [4,3,6] \). Now the Coxeter tetrahedron of type \( [6,3^{1,1}] \) can be subdivided into 5 copies of the Coxeter tetrahedron of type \( [3,3,6] \). See Figure 1. Hence \( [6,3^{1,1}] \) is conjugate to a subgroup of \( [3,3,6] \) of index 5. Therefore, the orientation preserving subgroup of \( [6,3^{1,1}] \) is conjugate to a subgroup of \( \Delta_3 \) of index 5. Thus, the ratio 5/4 faithfully represents the commensurability relationship between \( \Gamma_3^3 \) and \( \Delta_3 \), that is, \( \Gamma_3^3 \) has a subgroup of index 4 that is conjugate to a subgroup of \( \Delta_3 \) of index 5.

We now turn our attention to dimension 7. Our volume ratio formula gives that

\[
\frac{\text{vol}(H^7/\Gamma_7^3)}{\text{vol}(H^7/\Delta_7)} = 153/4.
\]

By the main theorem of Hild [7], the group \( \Delta_7 \) is a subgroup of index two of a discrete group \( \Delta_7 \) of isometries of \( H^7 \) such that \( \Delta_7 \) is the orientation preserving subgroup of \( \Delta_7 \). The group \( \Delta_7 \) is generated by a Coxeter subgroup \( \Delta_7 \) of index 2 of type \( [3^{2,1,2}] \) and an involution \( \sigma \) that acts as the mirror symmetry of the Coxeter system defining \( \Delta_7 \). The involution \( \sigma \) is orientation preserving, since \( \sigma \) transposes two pairs of Coxeter generators of \( \Delta_7 \). Hence \( \Delta_7 \) is generated by the orientation preserving subgroup of \( \Delta_7 \) and \( \sigma \).
We explicitly defined the group $\tilde{\Delta}_7$ to be the group generated by the reflections in the sides of the hyperbolic Coxeter 7-simplex $\Delta$ whose sides have Lorentz normal unit vectors listed in the columns of the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -1 & \frac{1}{2} \\
0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2}
\end{pmatrix}.
\]

The involution $\sigma$ acts as a symmetry of $\Delta$. The Lorentzian matrix of $\sigma$ is
\[
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}.
\]

In 1996, we discovered that $\Gamma_3^n$ is a Coxeter reflection group defined by the Coxeter graph in Figure 1 of [10], for $n = 7$, using Vinberg’s algorithm [17]. We explicitly defined the group $\Gamma_3^7$ to be the group generated by the reflections in the sides of the hyperbolic Coxeter 7-dimensional polytope $P^7$ whose sides have Lorentz...
normal unit vectors listed in the columns of the matrix

\[
\begin{pmatrix}
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & \frac{1}{2}
\end{pmatrix}
\]

In 1996, we discovered by a coset enumeration program that \( \bar{\Delta}_7 \cap \Gamma_3^7 \) has index 2295 in \( \Delta_7 \) and index 60 in \( \Gamma_3^7 \). The involution \( \sigma \) is an element of \( \Gamma_3^7 \) and so \( \Delta_7 \cap \Gamma_3^7 \) has index 2295 in \( \Delta_7 \) and index 60 in \( \Gamma_3^7 \). The commensurability ratio is \( \frac{2295}{60} = 153/4 \). We were disappointed that we could not find representations of \( \Delta_7 \) and \( \Gamma_3^7 \) such that \( \Delta_7 \cap \Gamma_3^7 \) has index 153 in \( \Delta_7 \) and index 4 in \( \Gamma_3^7 \), but we suspect that the representations that we found give the smallest possible indices.

In 1996, we found a discrete group \( \Gamma \) of isometries of \( H^7 \) that corresponds to the group of positive units of the quadratic form defined by the diagonal matrix

\[
\text{diag}(1, 1, 1, 1, 1, 1, 3, -1)
\]

such that \( \bar{\Delta}_7 \cap \Gamma \) has index 119 in \( \Delta_7 \) and index 4 in \( \Gamma \). Moreover \( \text{vol}(H^7/\Gamma)/\text{vol}(H^7/\bar{\Delta}_7) = 119/4 \), and so the ratio 119/4 faithfully represents the commensurability relationship between \( \bar{\Delta}_7 \) and \( \Gamma \). This computation was reported on page 345 of [8]. The story of the computation of the volume of \( H^7/\Gamma \) will have to wait for another day.

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