N=2 SO(4) 7D gauged supergravity with topological mass term from 11 dimensions

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Abstract: We construct a consistent reduction ansatz of eleven-dimensional supergravity to $N = 2$ $SO(4)$ seven-dimensional gauged supergravity with topological mass term for the three-form field. The ansatz is obtained from a truncation of the $S^4$ reduction giving rise to the maximal $N = 4$ $SO(5)$ gauged supergravity. Therefore, the consistency is guaranteed by the consistency of the $S^4$ reduction. Unlike the gauged supergravity without topological mass having a half-supersymmetric domain wall vacuum, the resulting 7D gauged supergravity theory admits a maximally supersymmetric $AdS_7$ critical point. This corresponds to $N = (1, 0)$ superconformal field theory in six dimensions. We also study RG flows from this $N = (1, 0)$ SCFT to non-conformal $N = (1, 0)$ Super Yang-Mills theories in the seven-dimensional framework and use the reduction ansatz to uplift this RG flow to eleven dimensions.

Keywords: AdS-CFT correspondence, Gauge/Gravity Correspondence and Supergravity Models.
1. Introduction

Gauged supergravities in various dimensions play an important role in both string compactifications and in the AdS/CFT correspondence. In some cases, a consistent truncation can be made in such a way that a lower dimensional gauged supergravity is obtained via a dimensional reduction of a (gauged) supergravity in higher dimensions on spheres \([1]\). Embedding lower dimensional gauged supergravities is now of considerable interest since this provides a method to uplift lower dimensional solutions to string/M theory.

It is known that sphere reductions of 10 or 11 dimensional supergravities give rise to gauged supergravity in lower dimensions. Well-known examples of these consistent sphere reductions include \(S^7\) and \(S^4\) reductions of eleven-dimensional supergravity and \(S^5\) reduction of type IIB theory giving rise to \(SO(8)\), \(SO(5)\) and \(SO(6)\) gauged supergravities in four, seven and five dimensions, respectively \([2, 3, 4]\). According to the AdS/CFT correspondence \([5]\), seven-dimensional gauged supergravity is useful in the study of \(N = (2, 0)\) and \(N = (1, 0)\) field theories in six dimensions \([6, 7, 8, 9, 10]\). The latter describe the dynamics of M5-branes worldvolume in M-theory and are less-known on the field theory side. Therefore, seven-dimensional gauged supergravity is expected to give some insight to six-dimensional field theories via gauge/gravity correspondence.

In this paper, we are interested in obtaining \(N = 2\) seven-dimensional gauged supergravity with \(SO(4)\) gauged group and topological mass term. In seven dimensions, the theory is obtained by coupling three vector multiplets to the pure \(SU(2)\) gauged supergravity constructed in \([11]\). This matter-coupled theory has been constructed in \([12]\) and \([13]\). The \(SO(4)\) gauged supergravity has also been constructed in \([14]\) by truncating the maximal \(N = 4\) \(SO(5)\) gauged supergravity. All of these constructions have not included the topological mass term for the three-form field, and the resulting theory does not admit \(AdS_7\) vacuum solutions. It has been shown in \([15]\) that the topological mass term is possible. The massive gauged theory has been explored in \([16]\) in which new \(AdS_7\) vacua and the corresponding RG flow interpolating between these vacua have been given.

To give an interpretation to this solution in the string/M theory context, it is necessary to embed this solution to 10 or 11 dimensions. The reduction ansatz of eleven-dimensional supergravity giving rise to pure \(SU(2)\) gauged supergravity has been given in \([17]\). The \(SO(4)\) gauged theory without topological mass term from a dimensional reduction of eleven- and ten-dimensional supergravity has been given in \([18]\) using the result of \([19]\). This result is clearly not sufficient to uplift the solution in \([16]\). The dimensionally reduced theory needs to include the topological mass term in order to admit \(AdS_7\) vacua. We will give an extension to the result of \([17, 18]\) by constructing
SO(4) gauged theory including topological mass term from a truncation of $S^4$ reduction of eleven dimensional supergravity. This provides an ansatz to uplift the 7-dimensional solutions of massive $N = 2$ SO(4) gauged supergravity to eleven dimensions.

The paper is organized as follow. In section 2, we give relevant formulae for $N = 2$ SO(4) gauged supergravity in seven dimensions. The embedding of this theory in eleven dimensions is obtained via a consistent truncation of the $S^4$ reduction of eleven-dimensional supergravity in section 3. We then use the resulting ansatz to uplift RG flow solutions from the maximally supersymmetric $AdS_7$ vacuum with SO(4) symmetry to non-conformal SYM in section 4. We end the paper by giving some conclusions and comments in section 5.

2. SO(4) $N = 2$ gauged supergravity in seven dimensions

In this section, we give a description of SO(4) $N = 2$ gauged supergravity in seven dimensions with topological mass term. All of the notations are the same as those in [15] to which the reader is referred for further details.

The SO(4) gauged theory is obtained by coupling three vector multiplets to the $N = 2$ supergravity multiplet. The field contents are given respectively by

\begin{align*}
\text{Supergravity multiplet :} & \quad (e^a_\mu, \psi^A_\mu, A^i_\mu, \chi^A, B_{\mu\nu}, \sigma) \\
\text{Vector multiplets :} & \quad (A_\mu, \lambda, \phi^i) \\
& \quad (2.1)
\end{align*}

where an index $r = 1, 2, 3$ labels the three vector multiplets. Curved and flat spacetime indices are denoted by $\mu, \nu, \ldots$ and $a, b, \ldots$, respectively. $B_{\mu\nu}$ and $\sigma$ are a two-form and the dilaton fields. The two-form field will be dualized to a three-form field $C_{\mu\nu\rho}$. Indices $i, j = 1, 2, 3$ label triplets of $SU(2)_R$. The 9 scalars $\phi^{ir}$ are parametrized by $SO(3, 3)/SO(3) \times SO(3) \sim SL(4, \mathbb{R})/SO(4)$ coset manifold. The corresponding coset representative of $SO(3, 3)/SO(3) \times SO(3)$ will be denoted by

\begin{align*}
L = (L_I^i, L_I^r) , \quad I = 1, \ldots, 6 .
\end{align*}

(2.2)

whose inverse is given by $L^{-1} = (L_I^i, L_I^r)$ where $L_I^i = \eta^{IJ} L_{JI}^i$ and $L_I^r = \eta^{IJ} L_{JI}^r$. Indices $i, j$ and $r, s$ are raised and lowered by $\delta_{ij}$ and $\delta_{rs}$, respectively while the full $SO(3, 3)$ indices $I, J$ are raised and lowered by $\eta_{IJ} = \text{diag}(-+--+)$.

The SO(4) $\sim SU(2) \times SU(2)$ gauging is implemented by promoting the $SU(2) \times SU(2) \sim SO(3) \times SO(3) \subset SO(3, 3)$ to a gauge symmetry. The structure constants for the $SU(2) \times SU(2)$ gauge group, which will appear in various quantities, are given by

\begin{align*}
\hat{f}_{IJK} = (g_1 \epsilon_{ijk}, g_2 \epsilon_{rst}) .
\end{align*}

(2.3)
To obtain $SO(4)$ gauge group, we will later set $g_2 = g_1$. The bosonic Lagrangian can be written in a form language as

$$\mathcal{L} = \frac{1}{2} R \ast \mathbb{I} - \frac{1}{2} e^\sigma a_{IJ} \ast F_{(2)}^I \wedge F_{(2)}^J - \frac{1}{2} e^{-2\sigma} \ast H_{(4)} \wedge H_{(4)} - \frac{5}{8} \ast d\sigma \wedge d\sigma$$

$$- \frac{1}{2} \ast P^{ir} \wedge P_{ir} + \frac{1}{\sqrt{2}} H_{(4)} \wedge \omega(3) - 4h H_{(4)} \wedge C_{(3)} - V \ast \mathbb{I}$$

(2.4)

where the scalar potential is given by

$$V = \frac{1}{4} e^{-\sigma} \left( C^{ir} C_{ir} - \frac{1}{9} C^2 \right) + 16h^2 e^{4\sigma} - \frac{4\sqrt{2}}{3} he^{\frac{3}{2}\sigma} C.$$

(2.5)

The constant $h$ describes the topological mass term for the three-form $C_{(3)}$ with $H_{(4)} = dC_{(3)}$. The quantities appearing in the above Lagrangian are defined by

$$P^{ir}_\mu = L^r \left( \delta^K L^\mu_I + f_{IJ} K A^K_\mu \right) L^L_K, \quad C_{rsi} = f_{IJ} K L^I_{ri} L^J_{sk} L_{Ks},$$

$$C_{ir} = \frac{1}{\sqrt{2}} f_{IJ} K L^I_{ri} L^J_{sk} L_{Ks} \epsilon^{ijk}, \quad C = -\frac{1}{\sqrt{2}} f_{IJ} K L^I_{ri} L^J_{sk} L_{Ks} \epsilon^{ijk},$$

$$a_{IJ} = L^I_{ri} L_{iJ} + L^r_{ri} L_{rJ}.$$  

(2.6)

The Chern-Simons three-form satisfying $d\omega(3) = F_{(2)}^I \wedge F_{(2)}^J$ is given by

$$\omega(3) = F_{(2)}^I \wedge A_{(1)}^I - \frac{1}{6} f_{IJ} K A_{(1)}^I \wedge A_{(1)}^J \wedge A_{(1)K}$$

(2.7)

with $F_{(2)}^I = dA_{(1)}^I + \frac{1}{2} f_{JK} I A_{(1)}^J \wedge A^K_{(1)}$

It is also useful to give the corresponding field equations

$$d \left( e^{-2\sigma} \ast H_{(4)} \right) + 8h H_{(4)} - \frac{1}{\sqrt{2}} F_{(2)}^I \wedge F_{(2)}^I = 0,$$

(2.8)

$$\frac{5}{4} d \ast d\sigma - \frac{1}{2} e^\sigma a_{IJ} \ast F_{(2)}^I \wedge F_{(2)}^J + e^{-2\sigma} \ast H_{(4)} \wedge H_{(4)}$$

$$+ \left[ \frac{1}{4} e^{-\sigma} \left( C^{ir} C_{ir} - \frac{1}{2} C^2 \right) + 2\sqrt{2} he^{\frac{3}{2}\sigma} C - 64h^2 e^{4\sigma} \right] \epsilon(7) = 0$$

(2.9)

$$D(e^\sigma a_{IJ} \ast F_{(2)}^I) - \sqrt{2} H_{(4)} \wedge F_{(2)}^J + \ast P^{ir} f_{IJ} K L^I_{ri} L_{iK} = 0$$

(2.10)

$$D \ast P^{ir} - 2e^\sigma L^i_{ri} L^r_{rJ} \ast F_{(2)}^I \wedge F_{(2)}^J$$

$$- \ast \mathbb{I} \left[ \frac{1}{\sqrt{2}} e^{-\sigma} C_{jr} C^{irs} \epsilon^{ijk} + 4\sqrt{2} he^{\frac{3}{2}\sigma} C_{ir} \right] = 0.$$  

(2.11)

The Yang-Mills equation (2.10) can be written in terms of $C^{ir}$ and $C^{irs}$ by using the relation

$$f_{IJ} K L^I_{ri} L_{rK} = -\frac{1}{2\sqrt{2}} \epsilon^{ijk} C^{irs} L^k_{rJ} - C^{irs} L_{sJ}.$$  

(2.12)
In obtaining the scalar equation (2.11), we have used the projections in the variations of scalars as in [12]

\[
\begin{align*}
\delta L^I_i &= X^i_r L^r_I + X^i_j L^j_I, \\
\delta L^r_I &= X^r_s L^s_I + X^r_i L^i_I 
\end{align*}
\] (2.13)

which lead to

\[
\begin{align*}
\delta C^2 &= -6\sqrt{2} C C^i r X^i r, \\
\delta (C^i r) &= 2\sqrt{2} C^s k \delta^i j k X^i r - \frac{2\sqrt{2}}{3} C^i r C X^i r. 
\end{align*}
\] (2.14)

We finally give supersymmetry transformations for fermions with all fermionic fields vanishing. These are given by

\[
\begin{align*}
\delta \psi_\mu &= 2 D_\mu \epsilon - \frac{\sqrt{2}}{30} e^{-2} C \gamma_\mu \epsilon - \frac{1}{240\sqrt{2}} e^{-\sigma} H_{\rho\sigma\lambda\tau} \left( \gamma_\mu \gamma^{\rho\sigma\lambda\tau} + 5 \gamma^{\rho\sigma\lambda\tau} \gamma_\mu \right) \epsilon \\
&\quad - \frac{i}{20} e^\sigma F^i_{\rho\sigma} \sigma^i (3 \gamma_\mu \gamma^{\rho\sigma} - 5 \gamma^{\rho\sigma} \gamma_\mu) \epsilon - \frac{4}{5} h e^{2\sigma} \gamma_\mu \epsilon, \\
\delta \chi &= -\frac{1}{2} \gamma^\mu \partial_\mu \sigma \epsilon - \frac{i}{10} e^\sigma F^i_{\mu\nu} \sigma^i \gamma^{\mu\nu} \epsilon - \frac{1}{60\sqrt{2}} e^{-\sigma} H_{\mu\nu\rho\sigma} \gamma^{\mu\nu\rho\sigma} \epsilon \\
&\quad + \frac{\sqrt{2}}{30} e^{-2} C \epsilon - \frac{16}{5} e^{2\sigma} h \epsilon, \\
\delta \lambda^r &= -i \gamma^\mu P^i_r \sigma^i \epsilon - \frac{1}{2} e^{\frac{3}{2}} F^r_{\mu\nu} \gamma^{\mu\nu} \epsilon - \frac{i}{\sqrt{2}} e^{-\frac{3}{2}} C^r \sigma^i \epsilon
\end{align*}
\] (2.15) (2.16) (2.17)

where \(SU(2)_R\) doublet indices \(A, B, \ldots\) on spinors are suppressed. \(\sigma^i\) are the usual Pauli matrices.

3. Seven dimensional \(N = 2\) gauged supergravity from eleven dimensions

We now construct a reduction ansatz for embedding \(SO(4) N = 2\) gauged supergravity mentioned in the previous section in eleven dimensions. The ansatz will be obtained from a consistent truncation of the \(S^4\) reduction of eleven-dimensional supergravity giving rise to the maximal \(N = 4\) \(SO(5)\) gauged supergravity in seven dimensions. To obtain the topological mass term, we will impose the so-called odd-dimensional self-duality as in [17].
3.1 \( N = 4 \) SO(5) gauged supergravity from seven dimensions

To set up the notations and make the paper self-contained, we briefly repeat the \( S^4 \) reduction of eleven-dimensional supergravity [3, 20]. We will work in the notations of \[19\] and deal mainly with bosonic fields. The field content of eleven-dimensional supergravity consists of the graviton \( \hat{g}_{MN} \), gravitino \( \hat{\psi}_M \) and a four-form field \( \hat{F}_4 \).

Eleven-dimensional space-time indices are denoted by \( M, N = 0, 1, \ldots, 10 \).

The \( S^4 \) reduction is characterized by the following ansatz

\[
d s^2_{11} = \Delta^4 d s_7^2 + \frac{1}{g_7^2} \Delta^{-\frac{1}{2}} T^{-1}_{ij} D \mu^i D \mu^j, \tag{3.1}
\]

\[
\hat{F}_4 = \frac{1}{4!} \epsilon_{i_1 \ldots i_5} \left[ \frac{4}{g_7^2} \Delta^{-2} \mu^m T^{i_1 i_2 i_3 i_4} D \mu^i \wedge D \mu^j \wedge D \mu^k \wedge D \mu^\ell \right. \\
\left. + \frac{6}{g_7^2} \Delta^{-1} T^{i_1 i_2} \mu^j F^{i_1 i_2}_{(2)} \wedge D \mu^i \wedge D \mu^j \wedge D \mu^k \wedge D \mu^\ell - \frac{5}{g_7^2} \Delta^{-2} U \wedge D \mu^i \wedge \ldots \wedge D \mu^\ell \right] \\
- T_{i_1} * S^i_{(3)} \mu^i + \frac{1}{g} S^i_{(3)} \wedge D \mu^i \tag{3.2}
\]

where the quantities appearing in the above equations are defined by

\[
U = 2 T_{ij} T_{kj} \mu^i \mu^j - \Delta T_{ii}, \quad \Delta = T_{ij} \mu^i \mu^j, \quad \mu^i \mu^i = 1, \\
F^{i_2}_{(2)} = d A^{i_1}_{(1)} + g A^{i_2}_{(1)} \wedge A^{i_3}_{(1)}, \quad D \mu^i = d \mu^i + g A^{i_1}_{(1)} \mu^i, \\
D T_{i_1} = d T_{i_1} + g A^{i_1}_{(1)} T_{i_2} + g A^{i_2}_{(1)} T_{i_3} \tag{3.3}
\]

The symmetric matrix \( T_{ij} \), \( i, j = 1, \ldots, 5 \) with unit determinant parametrize the \( SL(5, \mathbb{R})/SO(5) \) coset manifold.

The bosonic field content of \( N = 4 \) gauged supergravity is given by the metric \( g_{\mu \nu} \), ten vectors \( A_{i_1}^{i_2} = A^{i_2}_{i_1} \) gauging the \( SO(5) \) gauge group, five three-form fields \( S^i_{(3)} \) and four-teen scalars \( T_{ij} \). The corresponding field equations are given by

\[
D(T_{i_1} \wedge S^j_{(3)}) = F^{i_1}_{(2)} \wedge S^j_{(3)}, \tag{3.4}
\]

\[
H^i_{(4)} = g T_{i_1} \wedge S^j_{(3)} + \frac{1}{8} \epsilon_{i_1 i_2 i_3 i_4} F^{i_1 i_2}_{(2)} \wedge F^{i_3 i_4}_{(2)}, \tag{3.5}
\]

\[
D(T_{i_1}^{-1} T_{i_2}^{-1} \wedge F^{i_2}_{(2)}) = -2 g T_{i_1}^{-1} D T_{i_2}^{-1} - \frac{1}{2g} \epsilon_{i_1 i_2 i_3 i_4} F^{i_1 i_2}_{(2)} \wedge F^{i_3 i_4}_{(2)} + \frac{3}{2g} \delta_{i_1 i_2 i_3 i_4} F^{i_1 i_2}_{(2)} \wedge F^{i_3 i_4}_{(2)} - S^k_{(3)} \wedge S^l_{(3)}, \tag{3.6}
\]

\[
D(T_{i_1}^{-1} \wedge D T_{i_2}) = 2 g^2 (2 T_{i_1} T_{i_2} - T_{i_1} T_{i_2} \epsilon_{(7)}) + T_{i_1}^{-1} T_{i_2}^{-1} \wedge F^{m}_{(2)} \wedge F^{n}_{(2)} \\
+ T_{i_1} \wedge S^k_{(3)} \wedge S^l_{(3)} - \frac{1}{5} \delta_{i_1 i_2} \left[ 2g^2 (2 T_{i_1} T_{i_2} - (T_{i_1} T_{i_2})^2) \epsilon_{(7)} \\
+ T_{i_1}^{-1} T_{i_2}^{-1} \wedge F^{m}_{(2)} \wedge F^{n}_{(2)} \wedge T_{i_1} \wedge S^k_{(3)} \wedge S^l_{(3)} \right] \tag{3.7}
\]
Additionally, both \( \psi \) expressions more compact. The associated supersymmetry transformations are given by \( \delta \psi^i = \Omega^7 \Gamma^i \Phi \).

\[ L_7 = R * \Pi - \frac{1}{4} T_{ij} * DT_{jk} \wedge T_{kl} DT_{li} - \frac{1}{4} T_{ik}^{-1} T_{jl}^{-1} * F_{(2)}^{ij} \wedge F_{(2)}^{kl} - \frac{1}{4} T_{ij} * S^i_{(3)} \wedge S^j_{(3)} \]

where \( \Omega(7) \) is the Chern-Simons three-form whose explicit form can be found in [22]. The scalar potential for \( T_{ij} \) is given by

\[ V = g^2 \left( T_{ij} T_{ij} - \frac{1}{2} (T_{ii})^2 \right). \]  

We have not given Einstein equation since we will not consider Einstein equation in this paper. The consistency of the full truncation, including the Einstein equation, to \( N = 2 \) \( SO(4) \) gauged supergravity is guaranteed from the consistency of the \( S^4 \) reduction.

For completeness, we also repeat supersymmetry transformations of fermionic fields \( \psi_\mu \) and \( \lambda^i \). Indices \( \hat{i}, \hat{j} = 1, \ldots, 5 \) are vector indices of the composite \( SO(5)_c \) symmetry. Additionally, both \( \psi_\mu \) and \( \lambda^i \) transform as a spinor under \( SO(5)_c \) with the condition \( \Gamma^i \lambda^i = 0 \), but we have omitted the \( SO(5)_c \) spinor indices to make the following expressions more compact. The \( SO(5)_c \) gamma matrices will be denoted by \( \Gamma^i \). The associated supersymmetry transformations are given by [22]

\[ \delta \psi_\mu = D_\mu \epsilon - \frac{1}{20} g T_{\hat{i} \hat{j}} \gamma^\mu \epsilon - \frac{1}{40 \sqrt{2}} \left( \gamma^\mu_{\nu \rho \sigma} - 8 \delta^\nu_{\mu} \gamma^\rho \right) F^{ij}_{\nu \rho \sigma} \Gamma^i_{\hat{j}} \epsilon \]

\[ - \frac{1}{60} \left( \gamma^\mu_{\nu \sigma} - \frac{9}{2} \delta^\nu_{\mu} \gamma^\sigma \right) S^i_{\nu \rho \sigma} \Gamma^i \epsilon, \]

\[ \delta \lambda^i = \frac{1}{16 \sqrt{2}} \left( \Gamma_{\hat{k} \hat{l}} \Gamma_i - \frac{1}{5} \Gamma_i \Gamma_{\hat{k} \hat{l}} \right) F^{\hat{k} \hat{l}}_{\mu \nu} \epsilon + \frac{1}{2} \gamma^\mu \Gamma^j P^{\mu \nu \rho} \Gamma^i \epsilon \]

\[ - \frac{1}{120} \gamma^{\mu \nu \rho} \left( \Gamma_{\hat{i} \hat{j}} - 4 \delta^i_{\hat{i}} \right) S^i_{\mu \nu \rho} \Gamma^j \epsilon + \frac{1}{2} g \left( T_{\hat{i} \hat{j}} - \frac{1}{5} T_{\hat{i} \hat{k} \hat{j} \hat{l}} \delta^i_{\hat{i}} \right) \Gamma^i \epsilon \]  

where

\[ F^{ij}_{(2)} = \Pi_{\hat{i}} \Pi_{\hat{j}} F^{ij}_{(2)}, \quad T^{ij} = (\Pi^{-1})_{\hat{i}} \Pi_{\hat{j}} \delta^{ij}, \]

\[ D \epsilon = d \epsilon + \frac{1}{4} \bar{\theta}_{ab} \gamma^{ab} \epsilon + \frac{1}{4} Q_{\hat{i} \hat{j}} \Gamma^i \epsilon, \quad T^{ij} = (\Pi^{-1})_{\hat{i}} \Pi_{\hat{j}} \delta^{ij}, \]

\[ P^{ij} = (\Pi^{-1})_{\hat{i}} \Pi_{\hat{j}} \left( \delta^i_{\hat{i}} \bar{d} + g A_{(1)}^{ij} \right) \Pi_{\hat{k}} \delta^{jk}, \quad S^{ij} = (\Pi^{-1})_{\hat{i}} \Pi_{\hat{j}} S^{(3)}_{ij} \]  

with \( \Pi_{\hat{i}} \) being the \( SL(5, \mathbb{R})/SO(5) \) coset representative.
3.2 \textit{SO}(4) $N=2$ gauged supergravity from $S^4$ reduction

We now truncate the $N=4$ gauged supergravity to $N=2$ theory with topological mass term for the three-form field and $SO(4)$ gauge group. In this process, the gauge group $SO(5)$ is broken to $SO(4)$. We will set $T_{5\alpha}, S^\alpha$ and $F^5\alpha$ to zero. The $S^4$ coordinates $\mu^i$ will be chosen to be $\mu^i = (\cos \xi \mu^\alpha, \sin \xi)$ in which $\mu^\alpha$ satisfy $\mu^\alpha \mu^\alpha = 1$. Similar to $\mu^i$, $\mu^\alpha$ are coordinates on $S^3$. The scalar truncation is given by $T_{ij} = (T_{\alpha\beta}, T_{55}) = (\tilde{T}_{\alpha\beta}, X^{-4})$ with $\tilde{T}_{\alpha\beta}$ being unimodular. The scalar field $X$ will be related to the $N=2$ dilaton.

With these truncations, the three-form field equations (3.4) and (3.5) become

\begin{equation}
D(X^{-4} \ast S^5_{(3)}) = 0
\end{equation}

\begin{equation}
dS^5_{(3)} = gX^{-4} \ast S^5_{(3)} + \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta}_{(2)} \wedge F^{\gamma\delta}_{(2)}.
\end{equation}

We have used $\epsilon_{5\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta}$. From (3.14), we see that the four-form $X^{-4} \ast S^5_{(3)}$ is closed. We will denote it by

\begin{equation}
X^{-4} \ast S^5_{(3)} = -F_{(4)} = -dC_{(3)}
\end{equation}

or

\begin{equation}
S^5_{(3)} = X^4 \ast F_{(4)}.
\end{equation}

To satisfy equation (3.15), we impose the odd-dimensional self-duality condition

\begin{equation}
S^5_{(3)} = -gC_{(3)} + \omega_{(3)}
\end{equation}

or

\begin{equation}
X^4 \ast F_{(4)} = -gC_{(3)} + \omega_{(3)}
\end{equation}

where $\omega_{(3)}$, satisfying $d\omega_{(3)} = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta}_{(2)} \wedge F^{\gamma\delta}_{(2)}$, is the Chern-Simons term given by

\begin{equation}
\omega_{(3)} = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} \left( F^{\alpha\beta}_{(2)} \wedge A^{\gamma\delta}_{(1)} + \frac{1}{3} g A^{\alpha\beta}_{(1)} \wedge A^{\gamma\delta}_{(1)} \wedge A^{\kappa\delta}_{(1)} \right).
\end{equation}

Equations for $S^\alpha_{(3)}$ are trivially satisfied.

For the Yang-Mills equations, it can be verified that setting $F^{5\alpha}_{(2)} = 0$ satisfies their field equations. For $F^{\alpha\beta}_{(2)}$, we find

\begin{equation}
D \left( X^{-2} \tilde{T}_{\alpha\gamma}^{-1} \tilde{T}_{\beta\delta}^{-1} \ast F^{\gamma\delta}_{(2)} \right) = -2g \tilde{T}_{\gamma\alpha}^{-1} \ast DT^{\gamma}_{\beta\gamma} + \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}_{(2)} \wedge F_{(4)}
\end{equation}
where we have used the odd-dimensional self-duality condition.

We then consider scalar equations. Equations for $T_{5\alpha}$ are trivially satisfied while the $T_{55}$ equation gives rise to the dilaton equation

$$d(X^{-1} \ast dX) = \frac{1}{5} X^4 \ast F(4) \wedge F(4) - \frac{1}{20} X^{-2} \tilde{T}_{\alpha\beta} \tilde{T}_{\gamma\delta} \ast F^{\delta\gamma}_{(2)} \wedge F^{\alpha\gamma}_{(2)}$$

$$- \frac{1}{10} g^2 \left[ 4X^{-8} - 3X^{-3} \tilde{T}_{\alpha\alpha} - 2X^2 \left( \tilde{T}_{\alpha\beta} \tilde{T}_{\alpha\beta} - \frac{1}{2} (\tilde{T}_{\alpha\alpha})^2 \right) \right] \epsilon(7). \quad (3.22)$$

For $T_{ij} = T_{\alpha\beta}$, we find

$$D(\tilde{T}_{\alpha\gamma}^{-1} \ast D\tilde{T}_{\gamma\beta}) + \delta_{\alpha\beta} d(X^{-1} \ast dX) = X^{-2} \tilde{T}_{\alpha\gamma}^{-1} \tilde{T}_{\delta\kappa}^{-1} \ast F^{\gamma\kappa}_{(2)} \wedge F^{\delta\beta}_{(2)}$$

$$+ 2g^2 \left[ X^2 \left( 2\tilde{T}_{\alpha\gamma} \tilde{T}_{\gamma\beta} - \tilde{T}_{\gamma\gamma} \tilde{T}_{\alpha\beta} \right) - X^{-3} \tilde{T}_{\alpha\beta} \right] \epsilon(7)$$

$$+ \delta_{\alpha\beta} \left[ \frac{1}{5} X^4 \ast F(4) \wedge F(4) - \frac{1}{5} X^{-2} \tilde{T}_{\gamma\delta}^{-1} \tilde{T}_{\kappa\lambda}^{-1} \ast F^{\delta\gamma}_{(2)} \wedge F^{\kappa\gamma}_{(2)} \right.$$}

$$- \frac{2}{5} g^2 \left[ 2X^2 \left( \tilde{T}_{\gamma\delta} \tilde{T}_{\gamma\delta} - \frac{1}{2} (\tilde{T}_{\gamma\gamma})^2 \right) + X^{-8} - 2X^{-3} \tilde{T}_{\gamma\gamma} \right] \epsilon(7) \right]. \quad (3.23)$$

We can now use the $X$ equation (3.22) and end up with

$$D(\tilde{T}_{\alpha\gamma}^{-1} \ast D\tilde{T}_{\gamma\beta}) = 2g^2 \left[ 2X^2 \left( \tilde{T}_{\alpha\gamma} \tilde{T}_{\gamma\beta} - \frac{1}{2} \tilde{T}_{\gamma\gamma} \tilde{T}_{\alpha\beta} \right) - X^{-3} \tilde{T}_{\alpha\beta} \right] \epsilon(7)$$

$$+ X^{-2} \tilde{T}_{\alpha\gamma}^{-1} \tilde{T}_{\delta\kappa}^{-1} \ast F^{\gamma\kappa}_{(2)} \wedge F^{\delta\beta}_{(2)} + \delta_{\alpha\beta} \left[ \left\{ \frac{5}{2} g^2 X^2 \left( \tilde{T}_{\gamma\delta} \tilde{T}_{\gamma\delta} - \frac{1}{2} (\tilde{T}_{\gamma\gamma})^2 \right) \right.$$}

$$+ \frac{1}{2} g^2 X^{-3} \tilde{T}_{\gamma\gamma} \right\} \epsilon(7) - \frac{1}{4} X^{-2} \tilde{T}_{\gamma\delta}^{-1} \tilde{T}_{\kappa\lambda}^{-1} \ast F^{\delta\gamma}_{(2)} \wedge F^{\kappa\gamma}_{(2)} \right] \quad (3.24)$$

With all of the above truncations, we find the following ansatz for the metric and
the four-form field
\[ ds_1^2 = \Delta^{\frac{2}{3}} ds_7^2 + \frac{2}{g^2} \Delta^{-\frac{2}{3}} X^3 \left[ X \cos^2 \xi + X^{-4} \sin^2 \xi T_{\alpha \beta}^{-1} \mu^\alpha \mu^\beta \right] d\xi^2 \]
\[- \frac{1}{g^2} \Delta^{\frac{2}{3}} X^{-1} T_{\alpha \beta}^{-1} \sin \xi \mu^\alpha d\xi D\mu^\beta + \frac{1}{2g^2} \Delta^{-\frac{2}{3}} X^{-1} T_{\alpha \beta}^{-1} \cos^2 \xi D\mu^\alpha D\mu^\beta, \] (3.25)
\[ \hat{F}_4 = F_4 \sin \xi + \frac{1}{g} X^3 \cos \xi \ast F_4 \land d\xi + \frac{1}{g^2} \Delta^{-2} U \cos^5 \xi d\xi \land \epsilon_{(3)} \]
\[ + \frac{1}{3! g^3} \epsilon_{\alpha \beta \gamma \delta} \Delta^{-2} X^{-3} \sin \xi \cos^4 \xi \mu^\kappa \left[ 5 \tilde{T}_{\alpha \kappa} X^{-1} dX + D\tilde{T}_{\alpha \kappa} \right] \land D\mu^\beta \land D\mu^\gamma \land D\mu^\delta \]
\[ + \frac{1}{2g^2} \epsilon_{\alpha \beta \gamma \delta} \Delta^{-2} \cos^3 \xi \mu^\kappa \mu^\lambda \left[ \cos^2 \xi X^2 \tilde{T}_{\alpha \kappa} D\tilde{T}_{\beta \lambda} - \sin^2 \xi X^{-3} \delta^\beta \delta D\tilde{T}_{\alpha \kappa} \right] \]
\[ - 5 \sin^2 \xi \tilde{T}_{\alpha \kappa} X^{-4} \delta^\beta \delta dX \right] \land D\mu^\gamma \land D\mu^\delta \land d\xi + \frac{1}{2g^2} \cos \xi \epsilon_{\alpha \beta \gamma \delta} \times \]
\[ \left[ \frac{1}{2} \cos \xi \sin \xi X^{-4} D\mu^\gamma - (X^{-4} \sin^2 \xi \mu^\gamma + X^2 \cos^2 \xi \tilde{T}_{\gamma \kappa} \mu^\kappa d\xi \right] \land F_{(2)}^\alpha \land D\mu^\delta \] (3.26)
where
\[ U = \sin^2 \xi \left( X^{-8} - X^{-3} \tilde{T}_{\alpha \alpha} \right) + \cos^2 \xi \mu^\alpha \mu^\beta \left( 2X^2 \tilde{T}_{\alpha \gamma} \tilde{T}_{\beta \gamma} - X^2 \tilde{T}_{\alpha \beta} \tilde{T}_{\gamma \gamma} - X^{-3} \tilde{T}_{\alpha \beta} \right) \]
\[ \epsilon_{(3)} = \frac{1}{3!} \epsilon_{\alpha \beta \gamma} \mu^\alpha D\mu^\beta \land D\mu^\gamma \land D\mu^\delta. \] (3.27)

All of the above equations reduce to the pure \( N = 2 \) gauged supergravity with \( SU(2) \) gauge group for \( \tilde{T}_{\alpha \beta} = \delta_{\alpha \beta} \) after using various relations given in [21]. Note that for \( \tilde{T}_{\alpha \beta} = \delta_{\alpha \beta} \), equation (3.24) gives
\[ \ast F_{(2)}^\alpha \land F_{(2)}^\gamma = \frac{1}{4} \delta_{\alpha \beta} \ast F_{(2)}^\gamma \land F_{(2)}^\delta \] (3.28)
which means that the \( SO(4) \) gauge fields \( A_{(1)}^{\alpha \beta} \) must be truncated to those of \( SU(2) \) satisfying \( F_{(2)}^{\alpha \beta} = \pm \frac{1}{2} \epsilon_{\alpha \beta \gamma} F_{(2)}^\gamma \). This is expected since there are only three vector fields in the pure gauged supergravity which only admit \( SU(2) \) gauging.

The above equations can be obtained from the Lagrangian
\[ L_7 = R \ast \Pi - \frac{1}{4} X^{-2} \tilde{T}_{\alpha \gamma} \tilde{T}_{\beta \delta} \ast F_{(2)}^{\alpha \beta} \land F_{(2)}^{\gamma \delta} - \frac{1}{4} \tilde{T}_{\alpha \beta} \ast D\tilde{T}_{\beta \gamma} \land \tilde{T}_{\gamma \delta} \land D\tilde{T}_{\delta \alpha} \]
\[ - \frac{1}{2} X_4 \ast F_4 \land F_4 + \frac{1}{8} \epsilon_{\alpha \beta \gamma \delta} C_{(3)} \land F_{(2)}^{\alpha \beta} \land F_{(2)}^{\gamma \delta} - 5X^{-2} \ast dX \land dX \]
\[ - \frac{1}{2} g F_4 \land C_{(3)} - V \ast \Pi \] (3.29)
where the scalar potential is given by

\[ V = \frac{1}{2} g^2 \left[ X^{-8} - 2X^{-3} \tilde{T}_{\alpha\alpha} + 2X^2 \left( \tilde{T}_{\alpha\beta} \tilde{T}_{\alpha\beta} - \frac{1}{2} \tilde{T}_{\alpha\alpha}^2 \right) \right]. \]  

(3.30)

For \( \tilde{T}_{\alpha\beta} = \delta_{\alpha\beta} \), we find \( \tilde{T}_{\alpha\alpha} = \tilde{T}_{\alpha\beta} \tilde{T}_{\alpha\beta} = 4 \). The above potential becomes

\[ V = \frac{1}{2} g^2 \left( X^{-8} - 8X^{-3} - 8X^2 \right) \]  

(3.31)

which is exactly the same as that given in [17] up to a redefinition of the coupling constant \( g \).

We can also check another truncation namely to \( U(1) \times U(1) \) gauged supergravity. To preserve \( SO(2) \times SO(2) \) symmetry, we take the scalar matrix to be

\[
\tilde{T}_{\alpha\beta} = \begin{pmatrix}
\frac{\phi_1}{\sqrt{2}} & \frac{\phi_2}{\sqrt{2}} \\
\frac{\phi_2}{\sqrt{2}} & -\frac{\phi_1}{\sqrt{2}}
\end{pmatrix}
\]

(3.32)

and define \( X = e^{-\frac{\phi_1}{\sqrt{10}}} \). The potential (3.30) becomes

\[ V = \frac{1}{2} g^2 \left[ e^{\frac{8\phi_1}{\sqrt{10}}} - 8e^{-\frac{2\phi_1}{\sqrt{10}}} - 4e^{\frac{3\phi_2}{\sqrt{10}}} \left( e^{\frac{\phi_1}{\sqrt{2}}} + e^{-\frac{\phi_1}{\sqrt{2}}} \right) \right] \]  

(3.33)

which takes the same form as that given in [23]. Finally, it should be remarked that the three-form field equation coming from the Lagrangian (3.29) needs to be supplemented with the odd-dimensional self-duality condition as in the pure \( SU(2) \) gauged supergravity discussed in [17].

The nine scalars, parametrized by \( \tilde{T}_{\alpha\beta} \), in the dimensionally reduced theory are encoded in the \( SL(4, \mathbb{R})/SO(4) \) coset manifold. Therefore, in order to compare the result with gauged \( N = 2 \) \( SO(4) \) supergravity given in the previous section, we need to use the relation between \( SL(4, \mathbb{R})/SO(4) \) and \( SO(3,3)/SO(3) \times SO(3) \) coset manifolds. This is given in [15]. For the details of this mapping, the reader is referred to [17]. We will only give the \( SO(3,3)/SO(3) \times SO(3) \) coset representative \( L^A_I = (L^I, L^R) \) and that of \( SL(4, \mathbb{R})/SO(4) \), \( V^a_R \) with \( R = 1, \ldots, 4 \),

\[ L^A_I = \frac{1}{4} \Gamma^{\alpha\beta}_I \eta^{A}_{RS} \gamma^R_{\alpha} \gamma^S_{\beta} \]  

(3.34)
where $\Gamma^I$ and $\eta^A$ are chirally projected $SO(3,3)$ gamma matrices.

It can be shown that the scalar potential can be written as

$$V = \frac{1}{4} e^{-\sigma} \left( C^{ir} C_{ir} - \frac{1}{9} C^2 \right) + 16 h^2 e^4 \sigma - \frac{4\sqrt{2}}{3} h e \frac{\pi}{2} C$$

$$= \frac{1}{8} e^{-\sigma} \left( T_{\alpha\beta} T_{\alpha\beta} - \frac{1}{2} T_{\alpha\alpha}^2 \right) + 2 T_{\alpha\alpha} h e \frac{\pi}{2} + 16 h^2 e^4 \sigma$$  (3.35)

This form is similar to the potential (3.30) if $\tilde{T}_{\alpha\beta}$ is identified with $T_{\alpha\beta}$. Note that $T_{\alpha\beta}$ and $C$, $C^{ir}$ contain the gauge coupling $g_1$ and $g_2$. In order to compare the Lagrangian of the two theories, we need to multiply the Lagrangian (2.4) by two and separate the coupling constants $g_1$ and $g_2$ from the structure constants $f_{IJK} = (g_1 \epsilon_{ijk}, g_2 \epsilon_{rst})$. With these, the two scalar potentials are exactly the same if we identify

$$g_2 = g_1 = -16h = -2g.$$  (3.36)

We also need to redefine the following fields in the Lagrangian (2.4):

$$H(4) \to F(4) \sqrt{2}, \quad C(3) \to \frac{C(3)}{\sqrt{2}},$$

$$F^I = \frac{1}{4} \Gamma^{I\alpha\beta} F_{\alpha\beta}^{(2)} \quad \text{or} \quad F^{\alpha\beta}_{(2)} = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \Gamma^{I}_{\gamma\delta} F^I$$

$$X = e^{-\frac{\pi}{2}}.$$  (3.37)

By using (3.34), it can also be checked that

$$\tilde{T}^{-1}_{\alpha\gamma} T^{-1}_{\beta\delta} = \frac{1}{4} \Gamma^{I}_{\alpha\beta} \Gamma^{J}_{\gamma\delta} \left( L^I_L L_{IJ} + L^I_R L_{JR} \right).$$  (3.38)

The field equations from the two theories also match.

We now move to supersymmetry transformations of fermions. The maximal $N = 4$ theory contains the gravitini $\psi_\mu$ and the spin-$\frac{1}{2}$ fields $\lambda_i$. The latter is decomposed into $(\lambda_R, \lambda_5)$. The $SO(5)_c \Gamma^i$ gamma matrices are accordingly decomposed as $\Gamma^i = (\Gamma^R, \Gamma^5)$. $\Gamma^5 = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4$ acts as the chirality matrix of $SO(4)$. Following [18], we make the truncation

$$\epsilon^- = \psi^-_\mu = \lambda^-_5 = \lambda^+_5 = 0.$$  (3.39)

$\epsilon^\pm$ satisfy $\Gamma^5 \epsilon^\pm = \pm \epsilon^\pm$ with $\epsilon = \epsilon^+ + \epsilon^-$. We will now drop $\pm$ superscript from $\epsilon$, $\lambda$ and $\psi_\mu$.

In accordance with the bosonic truncation $T^{ij} = (T^{\alpha\beta}, T^{55}) = (X \tilde{T}^{\alpha\beta}, X^{-4})$, we truncate the $SL(5,\mathbb{R})$ coset representative as $\Pi_i^j = (\Pi_{\alpha}^R, \Pi_{5}^5)$. With the identification
\[ \Pi_\alpha^R = X^{-\frac{1}{2}} \mathcal{V}_\alpha^R \] and \[ \Pi_5^5 = X^2, \] we can write \( \tilde{T}^{\alpha\beta} \) in term of \( SL(4, \mathbb{R}) \) coset representative \( \mathcal{V}_\alpha^R \) as

\[ \tilde{T}^{\alpha\beta} = (\mathcal{V}^{-1})^{\alpha}_R (\mathcal{V}^{-1})^\beta_S \delta^{RS}, \quad \text{and} \quad \tilde{T}_{RS} = (\mathcal{V}^{-1})^{\alpha}_R (\mathcal{V}^{-1})^\beta_S \delta_{\alpha\beta}. \] (3.40)

We then find that equations (3.11) and (3.12) become

\[ \delta \psi_\mu = D_\mu \epsilon - \frac{1}{20} g(X \tilde{T}_{RR} + X^{-4}) \gamma_\mu \epsilon - \frac{1}{40 \sqrt{2}} X^{-1} \delta^{\nu \rho \sigma} \Gamma_{\nu \rho \sigma} \Gamma_{RS} F^{RS}_{\nu \rho \sigma} \epsilon \]

\[- \frac{1}{60} X^{-2} \left( \gamma^{\nu \rho \sigma} - \frac{2}{9} \delta^{\nu \rho \sigma} \right) S_{\nu \rho \sigma} \epsilon, \] (3.41)

\[ \delta \lambda_R = \frac{1}{4} \gamma^\mu \Gamma_R X^{-1} \partial_\mu \epsilon + \frac{1}{2} \Gamma^S \gamma^\mu P_{RS} \epsilon + \frac{1}{16 \sqrt{2}} X^{-1} \delta_{\mu \nu} \left( \frac{\Gamma_{ST} \Gamma_R - \frac{1}{5} \Gamma_R \Gamma_{ST}}{\Gamma_{ST}} \right) F^{ST}_{\mu \nu} \epsilon \]

\[- \frac{1}{10} g X^{-4} \Gamma_R \epsilon - \frac{1}{2} g X \left( \tilde{T}_{RS} - \frac{1}{5} \tilde{T}_{TT} \delta_{RS} \right) \Gamma^S \epsilon - \frac{1}{120} X^{-2} \delta_{\mu \nu} \Gamma_R S_{5 \mu \nu} \epsilon. \] (3.42)

The constraint \( \Gamma^i \lambda_i = 0 \) imposes the condition \( \lambda^+_5 = -\Gamma^R \lambda^+_R \). Therefore, the independent fields will be \( \psi_\mu \) and \( \lambda_R \). This is the reason for excluding \( \delta \lambda_5 \) in the above equations. We then identify \( \Gamma^R \lambda_R \) with \( \chi \) and \( \lambda_R = \lambda_R - \frac{1}{4} \Gamma_R \Gamma^S \lambda_S \) with \( \lambda^r \) in (3.17). Note that \( \lambda_R \) has only three independent components due to the condition \( \Gamma^R \lambda_R = 0 \).

With these and the odd-dimensional self-duality, we end up with, after some gamma matrix algebra,

\[ \delta \psi_\mu = D_\mu \epsilon - \frac{1}{20} g X \tilde{T} \gamma_\mu \epsilon - \frac{1}{40 \sqrt{2}} X^{-1} \left( \gamma^{\nu \rho \sigma} - \delta^{\nu \rho \sigma} \right) \Gamma_{RS} F^{RS}_{\nu \rho \sigma} \epsilon \]

\[- \frac{1}{20} g X^{-4} \gamma_\mu \epsilon - \frac{1}{480} X^2 \left( 3 \delta^{\nu \rho \sigma \tau} - \delta^{\nu \rho \sigma} \right) F^{\nu \rho \sigma \tau} \epsilon, \] (3.43)

\[ \delta \chi = X^{-1} \gamma^\mu \partial_\mu \epsilon - \frac{2}{5} g X^{-4} \epsilon + \frac{1}{10} g X \tilde{T}_{RR} \epsilon \]

\[- \frac{1}{120} X^2 \gamma^{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma} \epsilon - \frac{1}{20 \sqrt{2}} X^{-1} \gamma^{\mu \nu} \Gamma_{RS} F^{RS}_{\mu \nu} \epsilon, \] (3.44)

\[ \delta \lambda_R = - \frac{1}{2} \gamma^\mu \Gamma^S P^S_{\mu RS} \epsilon - \frac{1}{8} g X \tilde{T}_{SS} \Gamma_R \epsilon + \frac{1}{2} g X \tilde{T}_{RS} \Gamma^S \epsilon \]

\[- \frac{1}{8 \sqrt{2}} X^{-1} \gamma^{\mu \nu} \Gamma_S \left( F^{RS}_{\mu \nu} + \frac{1}{2} \Gamma_{RS} F^{TU}_{\mu \nu} \right) \epsilon. \] (3.45)

In the above equations, we have used the following definitions

\[ P_{RS} = (\mathcal{V}^{-1})^{\alpha}_R \left( \delta^{\alpha}_d + g A_{(1)\alpha}^{\beta} \right) \mathcal{V}_\beta^T \delta_{S|T}, \]

\[ Q_{RS} = (\mathcal{V}^{-1})^{\alpha}_R \left( \delta^{\alpha}_d + g A_{(1)\alpha}^{\beta} \right) \mathcal{V}_\beta^T \delta_{S|T}, \]

\[ D \epsilon = d \epsilon + \frac{1}{4} \omega_{ab} d \epsilon + \frac{1}{4} Q_{RS} \Gamma^{RS}. \] (3.46)
Notice that with our convention for $\Gamma^5 \epsilon = \epsilon$, $\Gamma_{RS}$ is anti-self dual. The field strength $F_{(2)}^{RS}$ appearing in (3.43) and (3.44) must be accordingly anti-self dual. This should be identified with the $SU(2)$ field strength $F_{(2)}^i$ in (2.13) and (2.16). On the other hand, the self dual part of $F_{(2)}^{RS}$ appears in (3.45) and should be identified with $F_{r}^{(2)}$ in (2.17).

Using the relation $C = -\frac{3}{2\sqrt{2}}g_1 \bar{T}$ and identifying $F_{RS} \Gamma_{RS} = -2\sqrt{2}i F^i \gamma^i$, we can see that equations (3.43) and (3.44) match with equations (2.15) and (2.16) after using the relation $g_1 = -2g$ and gamma matrix identities such as $\gamma^\mu \gamma^\nu = \gamma^\nu \gamma^\mu + 2\delta^\nu_\mu \gamma^\rho$.

Note that in order to match the gravitino variation, we need to multiply (3.43) by two.

Comparing (2.17) and (3.45) is more complicated. The $SO(4)$ gamma matrices $\Gamma^R$ need to be expressed in terms of

4. Embedding seven-dimensional RG flow to eleven dimensions

In this section, we will use the reduction ansatz obtained in the previous section to uplift some seven-dimensional solutions. The dimensional reduction gives rise to the condition $g_2 = g_1$. This makes the supersymmetric $AdS_7$ critical point with $SO(3)_{\text{diag}}$ symmetry found in [10] disappears. Accordingly, the flow solution given in [10] cannot be uplifted to eleven dimensions with the present reduction ansatz. However, to give examples of the uplifted solutions, we will study other solutions in the case of $g_2 = g_1$.

4.1 Uplifting $AdS_7$ solutions

We now further truncate the nine scalars given by $\tilde{T}_{\alpha\beta}$ to one scalar invariant under $SO(3)_{\text{diag}} \subset SO(3) \times SO(3) \sim SO(4)$. This scalar sector has already been studied in [16]. We will give more solutions in this section. Under $SO(3)_{\text{diag}}$, the nine scalars transform as $1 + 3 + 5$. There is only one singlet. It can be checked that the $SO(3)_{\text{diag}}$ singlet correspond to

$$V^R_{\alpha} = \begin{pmatrix} e^{\phi} \\ e^{\phi} \\ e^{\phi} \\ e^{-3\phi} \end{pmatrix}$$ \quad \text{or} \quad \tilde{T}_{\alpha\beta} = \begin{pmatrix} e^{\phi} \\ e^{\phi} \\ e^{\phi} \\ e^{-3\phi} \end{pmatrix}. \quad (4.1)$$

$\tilde{T}_{\alpha\beta}$ can be written more compactly as $\tilde{T}_{\alpha\beta} = (\delta_{ab} e^{\phi}, e^{-3\phi})$ for $a, b = 1, 2, 3$. By using (3.34) and the explicit form of $\Gamma^I$ and $\eta^A$ given in [15], it is easy to verify that this $V$ precisely gives the $SO(3,3)/SO(3) \times SO(3)$ coset representative $L$ used in [16].

Using this and the relation $X = e^{-\frac{4}{3}}$, we find the scalar potential

$$V = \frac{1}{2} g^2 e^{-\sigma} \left[ e^{5\sigma + e^{-6\phi}} - 6 e^{-2\phi} - 3 e^{2\phi} - 2 e^{2\sigma - 3\phi} (1 + 3 e^{4\phi}) \right]. \quad (4.2)$$
This potential admits two \( AdS_7 \) critical points given by

\[
\begin{align*}
\sigma &= \phi = 0, \quad V_0 = -480h^2 \\
\sigma &= -\frac{1}{10} \ln 2, \quad \phi = -\frac{1}{4} \ln 2, \quad V_0 = -160 \times 2 \frac{7}{2} h^2
\end{align*}
\]

(4.3)

where we have used \( g = 8h \) or equivalently \( g_1 = -16h \) as given in [16]. By using the BPS equations given in [13], which are repeated below, we see that the second critical point is non-supersymmetric. Scalar masses at this critical point can be computed to be

\[
\begin{array}{|c|c|}
\hline
SO(3)_{\text{diag}} & m^2 L^2 \\
\hline
1 & -12 \\
1 & 12 \\
3 & 0 \\
5 & -12 \\
\hline
\end{array}
\]

where the \( AdS_7 \) radius is given by \( L = \sqrt{-\frac{15}{V_0}} \). The three massless scalars are the expected Goldstone bosons corresponding to the symmetry breaking of \( SO(4) \) to \( SO(3) \). One of the 1 and 5 scalars have masses below the BF bound \( m^2 L^2 = -9 \), so this critical point is unstable.

The first critical point is the trivial point preserving all supersymmetries and the full \( SO(4) \) gauge symmetry. The scalar masses can be found in [10]. We will now uplift this \( AdS_7 \) vacuum to eleven dimensions. We begin with the coordinates \( \mu^a = (\cos \psi \hat{\mu}^a, \sin \psi) \) in which \( \hat{\mu}^a \hat{\mu}^a = 1 \). Since \( \sigma = \phi = 0 \), we then find \( \Delta = 1 \) and

\[
\begin{align*}
ds^{11}_{11} &= e^{\frac{2}{r_0}} dx_{1,5}^2 + dr^2 + \frac{1}{32 h^2} \left[ d\xi^2 + \frac{1}{4} \cos^2 \xi \left( d\psi^2 + \cos^2 \psi d\Omega_2^2 \right) \right] \\
\hat{F}_4 &= -\frac{3}{256h^3} \cos^5 \xi d\xi \wedge \epsilon_{(3)}
\end{align*}
\]

(4.5)

(4.6)

where \( d\Omega_2^2 \) is the metric on the two-sphere. The eleven dimensional geometry is given by \( AdS_7 \times S^4 \). Turning on the dilaton \( \sigma \) would deform the four-sphere but leave the \( S^3 \) inside invariant. If \( \phi, \sigma \neq 0 \), the metric would be further deformed in such a way that the \( S^2 \) part described by \( d\Omega_2^2 \) is invariant. The unbroken symmetry in this case is the \( SO(3) \) isometry of this \( S^2 \) identified with the unbroken \( SO(3)_{\text{diag}} \). The \( SO(3) \) critical point is however unstable. Therefore, we will not consider \( AdS_7 \) solution with \( SO(3) \) symmetry.

**4.2 Uplifting RG flows to non-conformal \( SO(3) \) Super Yang-Mills**

To give more examples, we will study RG flow solutions to non-conformal Super Yang-Mills theories in the IR. We will work in the theory of section 2. With \( g_2 = g_1 \) and the
standard domain wall metric ansatz \( ds_7^2 = e^{A(r)} dx_{1,5}^2 + dr^2 \), the BPS equations taken from [16] become

\[
\phi' = -4e^{-\frac{2}{3} - 3\phi} (e^{4\phi} - 1) h, \quad (4.7)
\]
\[
\sigma' = \frac{8}{5} e^{-\frac{2}{3} - 3\phi} \left( 1 + 3e^{4\phi} - 4e^{\frac{2}{3}\sigma + 3\phi} \right) h, \quad (4.8)
\]
\[
A' = \frac{4}{5} e^{-\frac{2}{3} - 3\phi} \left( 1 + 3e^{4\phi} + e^{\frac{4}{3}\sigma + 3\phi} \right) \quad (4.9)
\]
in which \( \frac{d}{dr} \) is denoted by \( ' \). After changing to the new coordinate \( \tilde{r} \) given by \( d\tilde{r} = e^{-\sigma/2} \), we find the solution

\[
16h\tilde{r} = \ln \left[ \frac{1 + e^\phi}{1 - e^\phi} \right] - 2 \tan^{-1} \phi + C_1, \quad (4.10)
\]
\[
\sigma = \frac{2}{5} \left[ \phi - \ln \left[ 1 + 12C_2 - 12C_2 e^{4\phi} \right] \right], \quad (4.11)
\]
\[
A = \frac{1}{4} \left[ \phi - 2 \ln(1 - e^{4\phi}) \right] - \frac{1}{8} \sigma. \quad (4.12)
\]
The solution interpolates between an \( AdS_7 \) in the UV, \( \tilde{r} \sim r \to \infty \), and a domain wall in the IR, \( 4h\tilde{r} \to \tilde{C} \), for a constant \( \tilde{C} \).

At the UV, the solution becomes

\[
\sigma \sim \phi \sim e^{-16hr} \sim e^{-\frac{4r}{L_{UV}}}, \quad A \sim 4hr \sim \frac{r}{L_{UV}}. \quad (4.13)
\]
The eleven-dimensional metric is given by (4.5).

In the IR, we find that \( \phi \) blows up as

\[
\phi \sim - \ln(4h\tilde{r} - \tilde{C}) \quad (4.14)
\]
for a constant \( \tilde{C} \). The behaviour of \( \sigma \) depends on the value of the integration constant \( C_2 \).

For \( C_2 = 0 \), we find

\[
\sigma \sim -\frac{2}{5} \ln(4h\tilde{r} - \tilde{C}) \sim -\frac{1}{2} \ln(4hr - C) \quad (4.15)
\]
where we have used the relation between \( \tilde{r} \) and \( r \) in the IR limit with \( C \) being another integration constant. The seven-dimensional metric is given by

\[
ds_7^2 = (4hr - C)^2 dx_{1,5}^2 + dr^2. \quad (4.16)
\]

For \( C_2 \neq 0 \), the solution becomes

\[
\sigma \sim \frac{6}{5} \ln(4h\tilde{r} - \tilde{C}) \sim \frac{3}{4} \ln(4hr - C),
\]
\[
ds_7^2 = (4hr - C)^2 dx_{1,5}^2 + dr^2. \quad (4.17)
\]
Both cases give $V \to -\infty$, so the solution is physical by the criterion of [24].

We now look at the eleven-dimensional geometry. For $C_2 = 0$ and $C_2 \neq 0$, the eleven-dimensional metric is given respectively by

$$ds_{11}^2 = \left(1 - \sin^2 \xi \cos^2 \psi\right)^{-\frac{1}{4}} \left[\left(\frac{14}{3} h\rho\right)^2 dx_{1,5}^2 + d\rho^2\right]$$

$$+ \frac{1}{32h^2} \left(1 - \sin^2 \xi \cos^2 \psi\right)^{-\frac{7}{4}} \times$$

$$\left[\left(\frac{14}{3} h\rho\right)^{-\frac{7}{4}} \sin^2 \xi \cos^2 \psi d\xi^2 + \frac{1}{4} \sin \xi \sin(2\psi) \left(\frac{14}{3} h\rho\right)^{-\frac{7}{4}} d\psi d\xi\right]$$

$$+ \frac{1}{4} \left(\frac{14}{3} h\rho\right)^{-\frac{7}{4}} d\psi^2 + \frac{1}{4} \cos^2 \psi \left(\frac{14}{3} h\rho\right)^{-\frac{7}{4}} d\Omega_2^2,$$ (4.18)

$$ds_{11}^2 = (\cos \xi \cos \psi)^{-\frac{1}{4}} \left[\left(\frac{14}{3} h\rho\right)^{\frac{10}{7}} dx_{1,5}^2 + d\rho^2\right]$$

$$+ \frac{1}{32h^2} (\cos \xi \cos \psi)^{-\frac{7}{4}} \times$$

$$\left[\left(\frac{14}{3} h\rho\right)^{\frac{10}{7}} \left(1 - \sin^2 \xi \cos^2 \psi\right) d\xi^2 - \frac{1}{4} \sin \xi \sin(2\psi) \left(\frac{14}{3} h\rho\right)^{\frac{7}{4}} d\xi d\psi\right]$$

$$+ \frac{1}{4} \cos^2 \xi \left(\frac{14}{3} h\rho\right)^{\frac{10}{7}} \left(\sin^2 \psi d\psi^2 + \cos^2 \psi d\Omega_2^2\right)$$ (4.19)

where $\left(\frac{14}{3} h\rho\right)^{\frac{6}{7}} = 4hr - C$.

As expected, when turning on $\phi$ and $\sigma$, the warped factors involve coordinates $(\xi, \psi)$. The $S^4$ is then deformed leaving the $S^2$ intact. If only $\sigma \neq 0$, the $S^3$ part of the internal metric would be invariant as pointed in [17]. The deformation with only $\phi \neq 0$ is not possible since the BPS equation for $\sigma$ would imply $\phi = 0$ as pointed out in [16].

5. Conclusions

In this paper, we have constructed $N = 2$ SO(4) gauged supergravity in seven dimensions with topological mass term. The resulting theory admit AdS$_7$ vacua and could be useful in the context of the AdS/CFT correspondence. The resulting reduction ansatz has been found by truncating the $S^4$ reduction leading to $N = 4$ SO(5) gauged supergravity and can be used to uplift seven-dimensional solutions to eleven dimensions. We have also constructed new seven-dimensional RG flow solutions and uplifted the resulting solutions to eleven dimensions. The flows can be interpreted as deformations of the UV $N = (1, 0)$ SCFT in six dimensions with SO(4) symmetry to non-conformal SYM with SO(3)$_{\text{diag}}$ symmetry. These deformations are driven by vacuum expectation values of dimension 4 operators. Additionally, the result of this paper can be used to
uplift flows to $SO(2)$ non-conformal gauge theories studied in [14] for $g_2 = g_1$.

However, the RG flow between two supersymmetric $AdS_7$ critical points recently found in [14] cannot be uplifted by using the reduction ansatz constructed here. It would be interesting to find an embedding of this solution in 10 or 11 dimensions. It is also interesting to extend the reduction ansatz given here to non-compact gauge groups $SO(3, 1)$ and $SO(2, 2)$. The internal manifold should involve hyperbolic spaces $H^{3,1}$ and $H^{2,2}$, respectively. Other possible non-compact gauge groups are $SL(3, \mathbb{R})$, $SO(2, 1)$ and $SO(2, 2) \times SO(2, 1)$. It would be very interesting to find higher dimensional origins for these gauge groups as well. Finally, more insight to six-dimensional gauge theories might be gained from studying these seven-dimensional gauged supergravities via AdS$_7$/CFT$_6$ correspondence. We hope to come back to these issues in future works.

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