Lyapunov Functions and Stability Analysis of Fractional-Order Systems

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Abstract

This study presents new estimates for fractional derivatives without singular kernels defined by some specific functions. Based on obtained inequalities, we give a useful method to establish the global stability of steady states for fractional-order systems and generalize some works existing in the literature. Finally, we apply our results to prove the global stability of a fractional-order SEIR model with a general incidence rate.

Keywords: nonlinear dynamics, fractional calculus, fractional derivatives, Lyapunov functions, stability analysis.

1 Introduction

In the last few years, the application of fractional differential equations (FDEs) has increased and gained much attention from researchers due to their ability in modeling and describing anomalous dynamics of real-world processes with memory and hereditary properties. Due to these properties, FDEs have been widely and successfully applied in various fields of science and engineering, such as viscoelasticity, signal and image processing, physics, mechanics, control, biology, and economy and finance \cite{10, 24}.

Fractional calculus (FC) literature assists to remarkable development of the fractional notions of differentiation. Several types of fractional derivatives were proposed, such as the Riemann–Liouville (RL), Caputo (C), Caputo–Fabrizio (CF) and Atangana–Baleanu–Caputo (ABC) operators. The standard RL and C derivatives \cite{11} have certain disadvantages, being classified as fractional derivatives with singular kernels. Caputo and Fabrizio \cite{11} suggested a new fractional derivative in which the memory is represented by an exponential kernel. Few years later, another

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fractional derivative was proposed by Atangana and Baleanu [4], where the memory kernel is modeled by the Mittag–Leffler function. These operators are extensively used by different researchers to describe the dynamics of various nonlinear systems [6, 8, 13, 19, 23, 26].

Stability is one of the powerful tools for analyzing the qualitative properties of non-linear dynamical systems. Lyapunov’s direct method, also called the second Lyapunov’s method, represents an effective way to examine the global behavior of a system without resolving it explicitly. This technique is based on constructing appropriate functionals, called the Lyapunov functionals, that should satisfy some conditions. In physics, these functionals can be either energy, potential, or other, but generally there is no precise technique to determine them. Recently, many scholars have focused on the stability analysis of fractional-order systems and some others have proposed specific Lyapunov functionals candidates, such as Volterra-type and quadratic functions [1, 11, 15, 21, 25, 27]. Nevertheless, these functions remain inadequate and incompatible with certain classes of fractional-order systems.

Motivated by the aforementioned works and observations, our main contribution here is to propose general Lyapunov functionals as candidates for fractional-order systems. We first develop new inequalities to estimate the fractional-order derivative of specific functions that generalize some works existing in the literature. These estimates allow us to construct suitable Lyapunov’s functionals for fractional-order systems and, therefore, to establish the global stability of their steady-states.

The rest of the paper is structured as follows. In Section 2 some necessary definitions and properties related to the fractional calculus are recalled. Useful estimations for fractional derivatives are proved in Section 3. As an application of our results, the global stability of a SEIR fractional-order model with a general incidence rate is studied in Section 4. Finally, we end with Section 5 of conclusions.

## 2 Preliminaries

In this section, we recall some definitions and properties of fractional operators that will be useful throughout our work. For more details, see [3, 4, 5, 9, 14, 16, 20].

**Definition 2.1.** Let \( f \in L^1(t_0, +\infty) \) and \( 0 < \alpha \leq 1 \). The Riemann–Liouville (RL) fractional integral of function \( f \) is defined by

\[
RL_{t_0}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-x)^{\alpha-1} f(x) dx,
\]

where \( \Gamma(\cdot) \) is the Gamma function.

**Definition 2.2.** Let \( f \in H^1(t_0, +\infty) \) and \( 0 < \alpha \leq 1 \). The Caputo (C) fractional derivative of function \( f \) is given by

\[
C_{t_0}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(x)}{(t-x)^\alpha} dx.
\]

**Definition 2.3.** Let \( f \in H^1(t_0, +\infty) \) and \( 0 < \alpha \leq 1 \). The Caputo–Fabrizio (CF) fractional derivative of function \( f \) is given by

\[
CF_{t_0}^{\alpha}f(t) = \frac{1}{2} \frac{B(\alpha)(2-\alpha)}{1-\alpha} \int_{t_0}^{t} f'(x) \exp \left[ -\frac{\alpha}{1-\alpha}(t-x) \right] dx,
\]

where \( B(\alpha) \) denotes a normalization function obeying \( B(0) = B(1) = 1 \). The fractional integral associated with the CF fractional derivative is defined by

\[
CF_{t_0}^{\alpha}f(t) = \frac{2(1-\alpha)}{B(\alpha)(2-\alpha)} f(t) + \frac{2\alpha}{B(\alpha)(2-\alpha)} RL_{t_0}^{1} f(t).
\]
Theorem 2.7. The derivative of the Mittag-Leffler function satisfies:

\[ \Psi \text{ defined by} \]

The aim of this section is to establish some new estimates for the fractional derivative of function

3 Useful fractional derivative estimates

Definition 2.4. Let \( f \in H^1(t_0, +\infty) \) and \( 0 < \alpha \leq 1 \). The Atangana–Baleanu–Caputo (ABC) fractional derivative of function \( f \) is given by

\[ \frac{ABC}{t_0}D^\alpha_t f(t) = \frac{B(\alpha)}{1-\alpha} \int_{t_0}^t f'(x)E_\alpha \left[ -\frac{\alpha}{1-\alpha}(t-x) \right] dx. \]  

(5)

The fractional integral associated with the ABC fractional derivative is defined by

\[ \frac{ABC}{t_0}I^\alpha_t f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} RL^\alpha_0 f(t). \]  

(6)

Definition 2.5. Let \( \alpha > 0 \) and \( \beta > 0 \). The Mittag-Leffler function of two parameters \( \alpha \) and \( \beta \) is defined by

\[ E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z \in \mathbb{C}. \]

Remark 2.6. If \( \beta = 1 \), then we have

\[ E_{\alpha,1}(z) = E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \]

which is called the Mittag-Leffler function of one parameter \( \alpha \); if \( \alpha = \beta = 1 \), then one gets

\[ E_{1,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = \exp(z). \]

Theorem 2.7. The derivative of the Mittag-Leffler function satisfies:

\[ \frac{dE_{\alpha,\beta}}{dz}(z) = E^2_{\alpha,\alpha+\beta}(z). \]

In [1][2][25], the authors prove the following inequalities for estimating the fractional derivative of certain functions.

Lemma 2.8. Let \( u(t) \) be a real continuous and differentiable function. Then, for any \( t \geq t_0 \) and \( 0 < \alpha \leq 1 \), we have

\[ \frac{ABC}{t_0}D^\alpha_t \left( u^2(t) \right) \leq 2u(t)\frac{ABC}{t_0}D^\alpha_t u(t), \]

(7)

\[ \frac{CF}{t_0}D^\alpha_t \left( u^2(t) \right) \leq 2u(t)\frac{CF}{t_0}D^\alpha_t u(t), \]

(8)

\[ \frac{C^*_0}{t_0}D^\alpha_t \left( u^2(t) \right) \leq 2u(t)\frac{C^*_0}{t_0}D^\alpha_t u(t). \]

(9)

Lemma 2.9. Let \( u(t) \) be a positive real continuous and differentiable function. Then, for any \( t \geq t_0, 0 < \alpha \leq 1 \), and \( u^* > 0 \), one has

\[ \frac{ABC}{t_0}D^\alpha_t \left[ u(t) - u^* - u^* \ln \frac{u(t)}{u^*} \right] \leq \left( 1 - \frac{u^*}{u(t)} \right) \frac{ABC}{t_0}D^\alpha_t u(t), \]

(10)

\[ \frac{C^*_0}{t_0}D^\alpha_t \left[ u(t) - u^* - u^* \ln \frac{u(t)}{u^*} \right] \leq \left( 1 - \frac{u^*}{u(t)} \right) \frac{C^*_0}{t_0}D^\alpha_t u(t). \]

(11)

3 Useful fractional derivative estimates

The aim of this section is to establish some new estimates for the fractional derivative of function \( \Psi \) defined by

\[ \Psi(u) = \int_{u^*}^{u} \frac{g(s) - g(u^*)}{g(s)} ds \]

\[ = u - u^* - \int_{u^*}^{u} \frac{g(u^*)}{g(s)} ds, \]

(12)
Proof. We start by reformulating inequality (13). By the linearity of the ABC fractional derivative, will allow us to extend the classical Lyapunov functions to fractional-order systems.

Let

Thus,

and

Since $g$ is a strictly increasing function, then $\Psi$ is strictly decreasing if $u < u^*$ and strictly increasing if $u > u^*$, with $u^*$ its global minimum.

**Theorem 3.1.** Let $u(t)$ be a real positive differentiable function. Then, for any $t \geq t_0$, $0 < \alpha \leq 1$, and $u^* > 0$, we have

$$\begin{align*}
\left. ABC \right|_{t_0}^t D^\alpha_t \Psi(u(t)) &\leq \left. \frac{ABC}{t_0} \right|_{t_0}^t D^\alpha_t u(t), \\
\left. CF \right|_{t_0}^t D^\alpha_t \Psi(u(t)) &\leq \left. \frac{CF}{t_0} \right|_{t_0}^t D^\alpha_t u(t).
\end{align*}$$

(13)

(14)

Proof. We start by reformulating inequality (13). By the linearity of the ABC fractional derivative, we obtain that

$$\begin{align*}
\left. ABC \right|_{t_0}^t D^\alpha_t \Psi(u(t)) &= \left. ABC \right|_{t_0}^t D^\alpha_t u(t) - \left. ABC \right|_{t_0}^t D^\alpha_t \left[ \int_{u^*}^{u(t)} \frac{g(u^*)}{g(s)} ds \right].
\end{align*}$$

Hence, the inequality (13) becomes

$$\begin{align*}
\left. ABC \right|_{t_0}^t D^\alpha_t u(t) - \left. ABC \right|_{t_0}^t D^\alpha_t \left[ \int_{u^*}^{u(t)} \frac{g(u^*)}{g(s)} ds \right] &\leq \left( 1 - \frac{g(u^*)}{g(u)} \right) \left. ABC \right|_{t_0}^t D^\alpha_t u(t).
\end{align*}$$

Because $g$ is a non-negative function, we get

$$g(u(t)) \left[ \left. ABC \right|_{t_0}^t D^\alpha_t u(t) - \left. ABC \right|_{t_0}^t D^\alpha_t \left( \int_{u^*}^{u(t)} \frac{g(u^*)}{g(s)} ds \right) \right] \leq (g(u(t)) - g(u^*)) \left. ABC \right|_{t_0}^t D^\alpha_t u(t).$$

Thus,

$$\begin{align*}
\left. ABC \right|_{t_0}^t D^\alpha_t u(t) - g(u(t)) \left[ \left. ABC \right|_{t_0}^t D^\alpha_t \left( \int_{u^*}^{u(t)} \frac{1}{g(s)} ds \right) \right] &\leq 0.
\end{align*}$$

(15)

Using the definition of ABC fractional derivative (15), we have

$$\begin{align*}
\left. ABC \right|_{t_0}^t D^\alpha_t u(t) &= \left. \frac{B(\alpha)}{1 - \alpha} \right|_{t_0}^t u'(x) E_\alpha \left[ -\frac{\alpha}{1 - \alpha} (t - x)^\alpha \right] dx,
\end{align*}$$

and

$$\begin{align*}
\left. ABC \right|_{t_0}^t D^\alpha_t \left[ \int_{u^*}^{u(t)} \frac{1}{g(s)} ds \right] &= \left. \frac{B(\alpha)}{1 - \alpha} \right|_{t_0}^t u'(x) E_\alpha \left[ -\frac{\alpha}{1 - \alpha} (t - x)^\alpha \right] dx.
\end{align*}$$

Consequently, the inequality (15) can be written as

$$\begin{align*}
\left. \frac{B(\alpha)}{1 - \alpha} \right|_{t_0}^t u'(x) \left( 1 - \frac{g(u(t))}{g(u(x))} \right) E_\alpha \left[ -\frac{\alpha}{1 - \alpha} (t - x)^\alpha \right] dx \leq 0.
\end{align*}$$

(16)

Now, we show that the inequality (16) is verified. For this, we denote

$$H(t) = \int_{t_0}^t u'(x) \left( 1 - \frac{g(u(t))}{g(u(x))} \right) E_\alpha \left[ -\frac{\alpha}{1 - \alpha} (t - x)^\alpha \right] dx$$
\[ v(x,t) = E_{\alpha} \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right]; \quad \frac{dv(x,t)}{dx} = \frac{\alpha^2(t-x)^{\alpha-1}}{1-\alpha} E_{\alpha,\alpha+1}^2 \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right]; \]

\[ w(x,t) = u(x) - u(t) - \int_{u(t)}^{u(x)} \frac{g(u(t))}{g(s)} \, ds; \quad \frac{dw(x,t)}{dx} = u'(x) \left(1 - \frac{g(u(t))}{g(u(x))}\right). \]

Integrating by parts the integral \( H(t) \), we obtain that

\[ H(t) = \left[ E_{\alpha} \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right] w(x,t) \right]_{x=t}^{x=t_0} - \int_{t_0}^{t} \frac{\alpha^2(t-x)^{\alpha-1}}{1-\alpha} E_{\alpha,\alpha+1}^2 \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right] w(x,t) \, dx. \]

Since \( w(x,t) \geq 0 \) and

\[ \lim_{x \to t} E_{\alpha} \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right] w(x,t) = 0, \]

it follows that

\[ H(t) = -E_{\alpha} \left[ -\frac{\alpha}{1-\alpha}(t-t_0)^\alpha \right] w(t_0,t) - \int_{t_0}^{t} \frac{\alpha^2(t-x)^{\alpha-1}}{1-\alpha} E_{\alpha,\alpha+1}^2 \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right] w(x,t) \, dx \leq 0. \]

As a result, the inequality \( \int_{t_0}^{t} \) is satisfied and \( \int_{t_0}^{t} \) holds true. \( \Box \)

**Remark 3.2.** Inequality \( \int_{t_0}^{t} \) is obtained by replacing \( E_{\alpha} \left[ -\frac{\alpha}{1-\alpha}(t-x)^\alpha \right] \) with \( \exp \left[ -\frac{\alpha}{1-\alpha}(t-x) \right] \) and following the same steps as given in the proof of Theorem \( \int_{t_0}^{t} \).

**Remark 3.3.** A similar inequality also holds for the Caputo fractional derivative as follows \( \int_{t_0}^{t} \):

\[ C_{t_0}^{\alpha} D_t^\alpha \Psi(u(t)) \leq \left(1 - \frac{g(u^*)}{g(u(t))}\right) C_{t_0}^{\alpha} D_t^\alpha u(t). \]

If \( g(s) = s \), then we obtain \( \Psi(u(t)) = u(t) - u^* - u^* \ln \frac{u(t)}{u^*} \). We obtain from Theorem \( \int_{t_0}^{t} \) the following corollary.

**Corollary 3.4.** Let \( u(t) \) be a positive differentiable function. For any \( t \geq t_0, 0 < \alpha \leq 1, \) and \( u^* > 0, \) we have

\[ AB C_{t_0}^{\alpha} D_t^\alpha \left[u(t) - u^* - u^* \ln \frac{u(t)}{u^*}\right] \leq \left(1 - \frac{u^*}{u(t)}\right) AB C_{t_0}^{\alpha} D_t^\alpha u(t), \]

\[ CF_{t_0}^{\alpha} D_t^\alpha \left[u(t) - u^* - u^* \ln \frac{u(t)}{u^*}\right] \leq \left(1 - \frac{u^*}{u(t)}\right) CF_{t_0}^{\alpha} D_t^\alpha u(t). \]

### 4 An application

In \[ \int_{t_0}^{t} \] Yang and Xu proposed a SEIR model with Caputo fractional derivative and general incidence rate as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} D_t^\alpha S(t) &= \Lambda^\alpha - d^\alpha S(t) - \beta^\alpha F(S(t)) G(I(t)) , \\
\frac{\partial}{\partial t} D_t^\alpha E(t) &= \beta^\alpha F(S(t)) G(I(t)) - (\sigma^\alpha + d^\alpha) E(t) , \\
\frac{\partial}{\partial t} D_t^\alpha I(t) &= \sigma^\alpha E(t) - (\gamma^\alpha + d^\alpha) I(t) , \\
\frac{\partial}{\partial t} D_t^\alpha R(t) &= \gamma^\alpha I(t) - d^\alpha R(t) ,
\end{align*}
\]
where \( \alpha \in (0, 1] \). The variables \( S(t), E(t), I(t) \) and \( R(t) \) represent the number of susceptible, exposed, infective and recovered individuals at time \( t \), respectively. All the other parameters are assumed to be positive constants.

The authors of [28] first analyzed the global stability of the disease-free equilibrium and discussed the stability of the endemic equilibrium but only when \( F(S) = S \). However, they mentioned that they can not use the estimation [11] in Lemma 2.9 to establish global stability in the general case and kept this problem as an open question, for future work.

In addition, we note that function \( F(S(t))G(I(t)) \) does not cover all the incidence functions existing in the literature, e.g., \( \frac{SI}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} \), \( \alpha_1, \alpha_2, \alpha_3 \geq 0 \) [17] [18], where we can not separate the variables \( S \) and \( I \). Here, we generalize the SEIR model [20] and apply our results to give a rigorous proof of the stability for both equilibrium points.

Let us consider the general fractional-order SEIR model

\[
\begin{cases}
0D_t^\alpha S(t) = \Lambda^\alpha - d^\alpha S(t) - F(S(t), I(t)), \\
0D_t^\alpha E(t) = F(S(t), I(t)) - (\sigma^\alpha + d^\alpha)E(t), \\
0D_t^\alpha I(t) = \sigma^\alpha E(t) - (\gamma^\alpha + d^\alpha)I(t), \\
0D_t^\alpha R(t) = \gamma^\alpha I(t) - d^\alpha R(t),
\end{cases} \tag{21}
\]

where \( 0D_t^\alpha \) denotes any fractional-order derivative mentioned in Section 3. The general incidence function \( F : \mathbb{R}^2_+ \to \mathbb{R}^+ \) is assumed to be continuously differentiable and to satisfy the following hypotheses:

\[
F(S, 0) = F(0, I) = 0 \quad \text{and} \quad F(S, I) = IF_1(S, I) \quad \text{for all } S, I \geq 0,
\]

\[
\frac{\partial F_1}{\partial S}(S, I) > 0 \quad \text{and} \quad \frac{\partial F_1}{\partial I}(S, I) \leq 0 \quad \text{for all } S \geq 0 \quad \text{and} \quad I \geq 0, \tag{H}
\]

Since \( R(t) \) does not appear in the first three equations of system (21), without loss of generality we discuss the following system:

\[
\begin{cases}
0D_t^\alpha S(t) = \Lambda^\alpha - d^\alpha S(t) - F(S(t), I(t)), \\
0D_t^\alpha E(t) = F(S(t), I(t)) - m_1 E(t), \\
0D_t^\alpha I(t) = \sigma^\alpha E(t) - m_2 I(t),
\end{cases} \tag{22}
\]

where \( m_1 = \sigma^\alpha + d^\alpha \) and \( m_2 = \gamma^\alpha + d^\alpha \).

System (22) has a disease-free equilibrium \( P_f = (S_0, 0, 0) \) with \( S_0 = \frac{\Lambda^\alpha}{d^\alpha} \) and an endemic equilibrium \( P^* = (S^*, E^*, I^*) \) when \( R_0 > 1 \), where

\[
R_0 = \frac{\sigma^\alpha}{m_1 m_2} \frac{\partial F(S_0, 0)}{\partial I}
\]

and \( E^* \in \left[ 0, \frac{\Lambda^\alpha}{d^\alpha} \right] \), \( S^* = \frac{\Lambda^\alpha - m_1 E^*}{d^\alpha} \) and \( I^* = \frac{\sigma^\alpha E^*}{m_2} \).

Next, we prove the global stability of both equilibriums by constructing appropriate Lyapunov functionals and using our results of Section 3.

**Theorem 4.1.** The disease-free equilibrium \( P_f \) is asymptotically stable when \( R_0 \leq 1 \). The endemic equilibrium \( P^* \) is asymptotically stable whenever \( R_0 > 1 \).
Proof. For the disease-free equilibrium we define the following Lyapunov functional:

$$V_0(t) = \int_{S_0}^{S(t)} \frac{F_1(x, 0) - F_1(S_0, 0)}{F_1(x, 0)} \, dx + E(t) + \frac{m_1}{\sigma^*} I.$$  

Applying our results, we estimate the fractional time derivative of function $V_0$ as

$$0D^\alpha_t V_0(t) \leq \left(1 - \frac{F_1(S_0, 0)}{F_1(t, 0)}\right) 0D^\alpha_t F_1(t) + \frac{m_1}{\sigma^*} 0D^\alpha_t I(t).$$

Using the fact that $\Lambda^* = d^\alpha S_0$, we get

$$0D^\alpha_t V_0(t) \leq \left(1 - \frac{F_1(S_0, 0)}{F_1(t, 0)}\right) \left(d^\alpha(S_0 - S(t)) + I(t) F_1(S_0, 0) \frac{F_1(t, 0)}{F_1(S_0, 0)} - \frac{m_1 m_2}{\sigma^*} I(t)\right)$$

$$\leq \left(1 - \frac{F_1(S_0, 0)}{F_1(t, 0)}\right) \left(d^\alpha(S_0 - S(t)) + I(t) F_1(S_0, 0) - \frac{m_1 m_2}{\sigma^*} I(t)\right)$$

$$= \left(1 - \frac{F_1(S_0, 0)}{F_1(t, 0)}\right) \left(d^\alpha(S_0 - S(t)) + \frac{m_1 m_2}{\sigma^*} (R_0 - 1) I(t)\right).$$

Since $F_1$ is an increasing function with respect to $S$, one has

$$1 - \frac{F_1(S_0, 0)}{F_1(t, 0)} \geq 0 \quad \text{for} \quad S \geq S_0,$$

$$1 - \frac{F_1(S_0, 0)}{F_1(t, 0)} < 0 \quad \text{for} \quad S < S_0.$$

Then, we get

$$\left(1 - \frac{F_1(S_0, 0)}{F_1(t, 0)}\right) (S_0 - S) \leq 0.$$  

It follows that $0D^\alpha_t V_0(t) \leq 0$ for $R_0 \leq 1$ with $0D^\alpha_t V_0(t) = 0$ if $S = S_0$ and $I = 0$. Substituting $(S, I) = (S_0, 0)$ in (22) shows that $E \to 0$ as $t \to \infty$. We conclude that the disease-free equilibrium $P_f$ is asymptotically stable when $R_0 \leq 1$.

Next, we assume that $R_0 > 1$ and we propose the following Lyapunov functional $V_1$ for the endemic equilibrium:

$$V_1(t) = \int_{S^*}^{S(t)} \frac{F(x, I^*) - F(S^*, I^*)}{F(x, I^*)} \, dx + \int_{E^*}^{E(t)} \frac{E(t) - E^*}{E^*} \, dx + \frac{m_1}{\sigma^*} \left(\int_{I^*}^{I(t)} \frac{x - I^*}{x} \, dx\right).$$

Computing the time fractional derivative of $V_1$, we get

$$0D^\alpha_t V_1(t) \leq \left(1 - \frac{F(S^*, I^*)}{F(S(t), I^*)}\right) 0D^\alpha_t F(S(t), I^*) + \left(1 - \frac{E^*}{E}\right) 0D^\alpha_t E(t)$$

$$+ \frac{m_1}{\sigma^*} \left(1 - \frac{I^*}{I}\right) 0D^\alpha_t I(t).$$

Using the fact that $\Lambda^* = d^\alpha S^* + F(S^*, I^*)$, $F(S^*, I^*) = m_1 E^*$ and $\sigma^* E^* = m_2 I^*$, we obtain

$$0D^\alpha_t V_1(t) \leq \left(1 - \frac{F(S^*, I^*)}{F(S(t), I^*)}\right) d^\alpha(S^* - S(t))$$

$$+ F(S^*, I^*) \left[3 - \frac{F(S^*, I^*)}{F(S, I^*)} + \frac{F(S, I)}{F(S^*, I^*)} - \frac{E^* F(S, I) - I}{I^* - \frac{E^*}{I^*}} \right]$$

$$= \left(1 - \frac{F(S^*, I^*)}{F(S(t), I^*)}\right) d^\alpha(S^* - S(t)) - F(S^*, I^*) \left[ G \left(\frac{I}{I^*}\right) - \frac{F(S, I)}{F(S^*, I^*)} \right]$$

$$+ G \left(\frac{F(S^*, I^*)}{F(S, I^*)}\right) + G \left(\frac{E^* F(S, I)}{F(S^*, I^*)}\right) + G \left(\frac{I^* E}{I E^*}\right).$$
where \( G(x) = x - 1 - \ln(x) \). Now, we show that \( G \left( \frac{I}{I^*} \right) - G \left( \frac{F(S, I)}{F(S, I^*)} \right) \geq 0 \). For this, we set

\[
H(I) = G \left( \frac{F(S, I)}{F(S, I^*)} \right) - G \left( \frac{I}{I^*} \right).
\]

Computing the derivative of \( H \) with respect to \( I \), we obtain

\[
\frac{dH}{dI} = \frac{F(S, I) - F(S, I^*)}{F(S, I)F(S, I^*)} \frac{\partial F(S, I)}{\partial I} - \frac{I - I^*}{II^*}.
\]

We discuss two cases:

**Case 1.** If \( I \geq I^* \), then \( F(S, I) \geq F(S, I^*) \). Therefore,

\[
\frac{\partial F(S, I)}{\partial I} = F_1(S, I) + I \frac{\partial F_1(S, I)}{\partial I} \leq F_1(S, I)
\]

it follows that

\[
\frac{dH}{dI} \leq \frac{F(S, I) - F(S, I^*)}{F(S, I)F(S, I^*)} F_1(S, I) - \frac{I - I^*}{II^*}
\]

\[
= \frac{1}{F(S, I^*)} [F_1(S, I) - F_1(S, I^*)] \leq 0.
\]

Hence \( H(I) \leq H(I^*) = 0 \).

**Case 2.** If \( I \leq I^* \), then \( F(S, I) \leq F(S, I^*) \). Therefore,

\[
\frac{dH}{dI} \geq \frac{F(S, I) - F(S, I^*)}{F(S, I)F(S, I^*)} F_1(S, I) - \frac{I - I^*}{II^*}
\]

\[
= \frac{1}{F(S, I^*)} [F_1(S, I) - F_1(S, I^*)] \geq 0.
\]

Hence, \( H(I) \leq H(I^*) = 0 \).

We conclude that \( _0D_t^\alpha V_1(t) \) is negative definite. Consequently, the endemic equilibrium \( P^* \) is asymptotically stable whenever \( R_0 > 1 \). \( \square \)

### 5 Conclusions

In this paper, we have developed some estimates for fractional derivatives without a singular kernel and applied it to establish the stability of fractional-order systems. To illustrate the efficacy of the obtained results, we have employed them to solve an open problem posed by Yang and Xu in [28] and prove the stability of a SEIR fractional-order system with a general incidence rate. We construct suitable Lyapunov functionals and proved the globally asymptotically stability of the disease-free and endemic equilibriums in terms of the basic reproduction number \( R_0 \). Our results generalize and improve those of [2, 12, 22].

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