Entropy of very low energy localized states

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Abstract

We expand on previous work involving “vacuum-bounded” states, i.e., states such that every measurement performed outside a specified interior region gives the same result as in the vacuum. We improve our previous techniques by removing the need for a finite outside region in numerical calculations. We apply these techniques to the limit of very low energies and show that the entropy of a vacuum-bounded state can be much higher than that of a rigid box state with the same energy. For a fixed $E$ we let $L'_{\text{in}}$ be the length of a rigid box which gives the same entropy as a vacuum-bounded state of length $L_{\text{in}}$. In the $E \to 0$ limit we conjecture that the ratio $L'_{\text{in}}/L_{\text{in}}$ grows without bound and support this conjecture with numerical computations.
I. INTRODUCTION

In a previous paper [1] we defined a “vacuum-bounded state” to be a generalized state (i.e., density matrix) for which every operator composed of fields at points outside a specified interior region has the same expectation value as in the vacuum. We used a vacuum-bounded state as a model of the state resulting from the complete evaporation of a black hole. We took the interior region to be a sphere of radius $R$, the distance that information might have propagated after unknown physics came into play in the evaporation process.

In the present paper we redo our analysis of maximum-entropy vacuum-bounded states in a way that directly handles an infinite outside region without the need to put the entire system in a box. Using this technique we calculate the maximum entropies of vacuum-bounded states with very low energies. As in [1] we do our calculations with a single scalar field in one dimension. We argue that as the temperature of the system becomes very low, the entropy is much larger than what one would have in a system with a rigid boundary, and that this difference grows without bound as the temperature goes to zero.

As in [1] we approximate the continuum with a lattice of harmonic oscillators. Even with an infinite vacuum outside our specified region, we can reduce the problem to a finite number of degrees of freedom. We then compute the solution numerically to support, in this approximation, our claims of high entropy at low energies.

We also use these techniques to recalculate the bounds we derived in [1]. We obtain the same bound as in that paper, but now we work directly with an infinite outside vacuum, which we could do only by extrapolation in [1].

II. PRELIMINARIES

We work with a one-dimensional scalar field with classical Hamiltonian

$$H = \frac{1}{2} \int_0^L \left[ \pi(x)^2 + \left( \frac{d\phi}{dx} \right)^2 \right] dx.$$  \hspace{1cm} (2.1)

We let $L_{\text{in}}$ be the size of the inside region. We will initially put the system in a box of length $L$, so the outside region stretches from $x = L_{\text{in}}$ to $x = L$, but we will shortly take $L$ to $\infty$.

Our goal is to find a density matrix $\rho$ such that if $O_{\text{out}}$ is any operator composed of fields at points $x_i > L_{\text{in}}$, then

$$\text{Tr} \rho O_{\text{out}} = \langle 0 | O_{\text{out}} | 0 \rangle.$$  \hspace{1cm} (2.2)

From [1] we expect $\rho$ to be a Gaussian of the form

$$\rho \propto e^{-\beta H + \sum f_{\alpha} O_{\alpha}^2}$$  \hspace{1cm} (2.3)

where $O_{\alpha}^2$ ranges over the quadratic operators $\phi(x)\phi(y)$ and $\pi(x)\pi(y)$ with $x, y > L_{\text{in}}$. The coefficients $f_{\alpha}$ must then be chosen so that $\rho$ satisfies Eq. (2.2).

In discrete form, we have [1]

$$H = \frac{1}{2} \left( P_x \cdot P_x + x \cdot Kx \right).$$  \hspace{1cm} (2.4)
with

$$K = \begin{pmatrix} 2g & -g & 0 & 0 & \cdots \\ -g & 2g & -g & 0 & \cdots \\ 0 & -g & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots \end{pmatrix} \tag{2.5}$$

where $g = 1/L_1^2$ and $L_1 = L/(N + 1)$ is the lattice spacing.

We will take $N_{in} \approx L_{in}/L_1$ of the oscillators to represent the inside region, and $N_{out} = N - N_{in}$ to represent the outside region. Equation (2.2) then becomes

$$\text{Tr} \rho x_i x_j = \langle 0 | x_i x_j | 0 \rangle \tag{2.6a}$$
$$\text{Tr} \rho P_i P_j = \langle 0 | P_i P_j | 0 \rangle \tag{2.6b}$$

where $i$ and $j$ run over the oscillators which represent the outside region. We require also that $\rho$ satisfy an energy constraint,

$$\text{Tr} \rho H = E_0. \tag{2.7}$$

We will define matrices $X$ and $P$ whose elements are the quadratic operators via

$$X_{\mu\nu} = x_\mu x_\nu \tag{2.8a}$$
$$P_{\mu\nu} = P_\mu P_\nu \tag{2.8b}$$

so that Eqs. (2.6a) and (2.6b) become

$$\text{Tr} \rho X_{\text{out}, \text{out}} = \langle 0 | X_{\text{out}, \text{out}} | 0 \rangle \tag{2.9a}$$
$$\text{Tr} \rho P_{\text{out}, \text{out}} = \langle 0 | P_{\text{out}, \text{out}} | 0 \rangle \tag{2.9b}$$

We expect $\rho$ to have the form

$$\rho \propto e^{-\beta (H + f_{ij} x_i x_j + g_{ij} P_i P_j)} \tag{2.10}$$

III. REDUCTION

Equation (2.10) has $N_{out}(N_{out} + 1)$ degrees of freedom in the $f$ and $g$ parameters. We would like to reduce these degrees of freedom to a number which does not depend on $N_{out}$, but only on $N_{in}$, so that we can take $N_{out}$ to infinity and still have a well-defined problem. To make this reduction we will describe our system by a new set of modes, such that the number of modes which can be excited in any $\rho$ which satisfies Eq. (2.9) depends only on $N_{in}$.
A. A different description of the vacuum

We will start by looking at the vacuum state. We can make a change of coordinate to put the Hamiltonian of Eq. (2.4) in diagonal form. Let \( Z \) be a matrix whose columns are the eigenvectors of \( K \), \( K = Z \Omega_0^2 Z^{-1} \), with the normalization

\[
Z \Omega_0 Z^T = I \quad \text{and} \quad Z^{-1T} \Omega_0 Z = K .
\]  

(3.1)

Define new coordinates \( z \) via \( x = Zz \) and \( P_x = Z^{-1T} P_z \). In these coordinates,

\[
H = \frac{1}{2} \sum_{\alpha} \omega_{\alpha}^{(0)} \left( P_{z_{\alpha}}^2 + z_{\alpha}^2 \right) .
\]  

(3.2)

The vacuum is the ground state of this Hamiltonian. We can define raising and lowering operators

\[
a_{\alpha} = \frac{1}{\sqrt{2}} (z_{\alpha} + iP_{z_{\alpha}}) \]

(3.3a)

\[
a_{\alpha}^\dagger = \frac{1}{\sqrt{2}} (z_{\alpha} - iP_{z_{\alpha}}) \]

(3.3b)

\[
H = \sum_{\alpha} \omega_{\alpha}^{(0)} \left( a_{\alpha}^\dagger a_{\alpha} + \frac{1}{2} \right) .
\]  

(3.3c)

The vacuum is the state \( |0\rangle \) annihilated by all the \( a_{\alpha} \). It is straightforward to write the expectation values in the vacuum state,

\[
\langle Z \rangle_{\alpha\beta} \equiv \langle 0 | z_{\alpha} z_{\beta} | 0 \rangle = \frac{1}{2} \delta_{\alpha\beta}
\]  

(3.4a)

\[
\langle P_z \rangle_{\alpha\beta} \equiv \langle 0 | P_{z_{\alpha}} P_{z_{\beta}} | 0 \rangle = \frac{1}{2} \delta_{\alpha\beta}
\]  

(3.4b)

so

\[
\langle 0 | X | 0 \rangle = Z \langle 0 | Z | 0 \rangle Z^T = \frac{1}{2} ZZ^T
\]  

(3.5a)

\[
\langle 0 | P | 0 \rangle = Z^{-1T} \langle 0 | P_z | 0 \rangle Z^{-1} = \frac{1}{2} Z^{-1T} Z^{-1} = \frac{1}{4} (\langle 0 | X | 0 \rangle)^{-1} .
\]  

(3.5b)

There can be many different Hamiltonians that have the same ground state. If we consider

\[
H' = \frac{1}{2} (P_x \cdot T' P_x + x \cdot K' x)
\]  

(3.6)

with \( T' \) and \( K' \) some coupling matrices, we can follow the above derivation to get a normal mode matrix \( Y \) and some frequencies \( \Omega \) with

\[
Y \Omega Y^T = T'
\]  

(3.7a)

\[
Y^{-1T} \Omega Y^{-1} = K'
\]  

(3.7b)

\[
x = Y y
\]  

(3.7c)

\[
P_x = Y^{-1T} P_y
\]  

(3.7d)

\[
H' = \frac{1}{2} \sum_{\beta} \omega_{\beta} \left( P_{y_{\beta}}^2 + y_{\beta}^2 \right) .
\]  

(3.7e)
We can define raising and lowering operators for these modes,

\[ b_\beta = \frac{1}{\sqrt{2}} (y_\beta + iP_{y_\beta}) \]  
(3.8a)

\[ b^\dagger_\beta = \frac{1}{\sqrt{2}} (y_\beta - iP_{y_\beta}) \]  
(3.8b)

\[ H' = \sum_\beta \omega_\beta \left( b^{\dagger}_\beta b + \frac{1}{2} \right) . \]  
(3.8c)

Now \( y = Y^{-1}x = Y^{-1}Zz = W^{-1}z \) where

\[ W \equiv Z^{-1}Y . \]  
(3.9)

Similarly, \( P_y = Y^T P_x = Y^T Z^{-T} P_z = W^T P_z \). Consequently we can define a Bogoliubov transformation,

\[ b_\beta = \frac{1}{\sqrt{2}} \left( W^{-1}_\beta \alpha z_\alpha + iW^T_\beta \alpha P_z \right) \]

\[ = \frac{1}{2} \left( W^{-1}_\beta \alpha (a_\alpha + a^{\dagger}_\alpha) + W^T_\beta \alpha (a_\alpha - a^{\dagger}_\alpha) \right) \]

\[ = \frac{1}{2} \left( (W^{-1} + W^T)_\beta \alpha a_\alpha + (W^{-1} - W^T)_\beta \alpha a^{\dagger}_\alpha \right) . \]  
(3.10)

For \( H \) and \( H' \) to have the same vacuum we require that the \( b_\beta \) depend only on the \( a_\alpha \) and not on the \( a^{\dagger}_\alpha \), which is to say that \( W^{-1} = W^T \), i.e. that \( W \) is a unitary matrix. With \( W \) unitary, \( YY^T = ZZ^T \) so the vacuum expectation values of Eq. (3.5) have the same values expressed in terms of \( Y \) as they had in terms of \( Z \).

**B. Modes that remain in the ground state**

Now consider a unitary matrix \( W \) and let \( Y = ZW \) as in the last section. Suppose we can find \( W \) such that \( Y \) has the following property:

The \( N \) modes can be divided into \( N_{gs} > 0 \) “ground state” modes and \( N_{\text{free}} \equiv N - N_{gs} \) “free” modes such that for all \( a \leq N_{\text{in}} \) and for all \( \beta > N_{\text{free}} \), \( Y_{a\beta} = 0 \) and \( Y^{-1}_{\beta a} = 0 \).

That is to say \( Y \) and \( Y^{-1} \) will have the form

\[ Y = \begin{pmatrix} N_{\text{free}} & N_{gs} \\ Y_{\text{in,free}} & 0 \\ Y_{\text{out,free}} & Y_{\text{out,gs}} \end{pmatrix} \] \( N_{\text{in}} \)

(3.11)

and

\[ Y^{-1} = \begin{pmatrix} N_{\text{free}} & N_{gs} \\ Y_{\text{free,free}}^{-1} & Y_{\text{free,gs}}^{-1} \\ 0 & Y_{gs,gs}^{-1} \end{pmatrix} \] \( N_{\text{free}} \).

(3.12)
If this is the case, then for $\beta > N_{\text{free}}$, $b_\beta = \left(Y_{\beta\gamma}^{-1}x_\gamma + iY_{\gamma\beta}P_{x_\gamma}\right)/\sqrt{2}$ depends only on outside operators $x_i$ and $P_{x_i}$. In any vacuum-bounded state, regardless of entropy considerations, $\langle x_ix_j \rangle$ and $\langle P_iP_j \rangle$ have their vacuum values. Consequently, for all $\beta > N_{\text{free}}$, if $\rho$ describes a vacuum-bounded state then $\text{Tr} \rho b_\beta^\dagger b_\beta = \langle 0|b_\beta^\dagger b_\beta|0 \rangle = 0$. The vacuum-bounded constraint forces modes $\beta > N_{\text{free}}$ to be in their ground states. These modes will not contribute to the calculation of the maximum-entropy vacuum-bounded state.

How many such modes can exist? Let $W_\beta$ denote a column of $W$ with $\beta > N_{\text{free}}$ and let $Z^a$ denote a row of $Z$ with $a \leq N_{\text{in}}$. Similarly let $(Z^{-1})_a$ denote a column of $Z^{-1}$. Since $Y = ZW$, $Y_{a\beta} = 0$ whenever $W_\beta$ is orthogonal to $Z^a$. Similarly $Y^{-1} = W^T Z^{-1}$ so $Y_{\beta a} = 0$ whenever $W_\beta$ is orthogonal to $(Z^{-1})_a$. Since $W$ is unitary, the $W_\beta$ must also be orthogonal to each other. Thus there are $N_{gs}$ columns of $W$ which have to be orthogonal to $N_{in}$ rows of $Z$, to $N_{in}$ columns of $Z^{-1}$, and to each other. Since there are $N$ components in a column of $W$ it can be orthogonal in general to at most $N - 1$ other vectors. Thus $N_{gs}$ is limited by $N - 1 = 2N_{in} + N_{gs} - 1$, or $N_{gs} = N - 2N_{in} = N_{out} - N_{in}$. Thus whenever $N_{out} > N_{in}$ there will be $N_{gs} = N_{out} - N_{in}$ modes that are forced to remain in the ground state.

These conditions determine the columns $W_\beta$ with $\beta > 2N_{in}$ up to a unitary transformation on these columns alone, and likewise the remaining columns are determined up to a unitary matrix which combines them.

### C. Density matrix and entropy

Let $\mathcal{H}$ be the Hilbert space of states of our system. We can write $\mathcal{H} = \tilde{\mathcal{H}} \otimes \mathcal{G}$ where $\tilde{\mathcal{H}}$ is the Hilbert space of states of the “free” modes and $\mathcal{G}$ is the Hilbert space of states of the “ground state” modes. Let $|0\rangle_{gs} \in \mathcal{G}$ denote the ground state of system $\mathcal{G}$. For any operator $A$ we can define an operator $\tilde{A}$ that acts on $\tilde{\mathcal{H}}$ via $\langle \tilde{\alpha}|\tilde{A}|\tilde{\beta} \rangle = \langle \tilde{\alpha} \otimes 0_{gs}|A|\tilde{\beta} \otimes 0_{gs} \rangle$ for all $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{H}}$.

Let $\rho$ describe a vacuum-bounded state and let $\rho_{gs} = \text{Tr}_{\text{free}} \rho$. Then $\text{Tr} \rho_{gs} b_\beta^\dagger b_\beta = 0$. This defines the vacuum state, so $\rho_{gs} = |0\rangle_{gs} \langle 0|_{gs}$ and thus

$$\rho = \tilde{\rho} \otimes |0\rangle_{gs} \langle 0|_{gs} \ .$$

If we write $\tilde{\rho}$ in diagonal form,

$$\tilde{\rho} = \sum_\alpha P_{\alpha} |\tilde{\alpha}\rangle \langle \tilde{\alpha}| \ ,$$

then

$$\rho = \sum_\alpha P_{\alpha} |\tilde{\alpha} \otimes 0_{gs}\rangle \langle \tilde{\alpha} \otimes 0_{gs}| \ .$$

The entropy is

$$S = -\text{Tr} \rho \ln \rho = -\sum_\alpha s(P_\alpha) = -\text{Tr} \tilde{\rho} \ln \tilde{\rho} .$$

Since any vacuum-bounded $\rho$ has this form, a variation of $\rho$ that preserves the constraints must also have this form,
\[ \delta \rho = \tilde{\delta} \rho \otimes (|0\rangle_{gs} \langle 0|_{gs}) . \]  

(3.17)

If \( S = -\text{Tr} (\rho + t\delta \rho) \ln(\rho + t\delta \rho) = -\text{Tr} (\hat{\rho} + t\tilde{\delta} \rho) \ln(\hat{\rho} + t\tilde{\delta} \rho) \), then

\[ \frac{dS}{dt} = -\text{Tr} \tilde{\delta} \rho \ln \hat{\rho} . \]  

(3.18)

If \( \rho \) maximizes \( S \) subject to the constraints, then we must have \( \text{Tr} \tilde{\delta} \rho \ln \hat{\rho} \) for any variation \( \tilde{\delta} \rho \) that preserves the constraints, i.e. for which \( \text{Tr} \tilde{\delta} \rho = 0 \) and

\[ \text{Tr} \delta \rho H = \text{Tr} \delta \rho x_i x_j = \text{Tr} \delta \rho P_i P_j = 0 . \]  

(3.19)

Now for any \( A \), \( \text{Tr} \delta \rho A = \text{Tr} (\tilde{\delta} \rho \otimes (|0_{gs}\rangle \langle 0_{gs}|)) A \). When we take the trace we only need to sum over states of the form \( |\tilde{\alpha} \otimes 0_{gs} \rangle \). Thus

\[ \langle A \rangle = \text{Tr} \delta \rho A = \sum_{\tilde{\alpha}, \tilde{\beta}} \langle \tilde{\alpha} | \tilde{\delta} \rho | \tilde{\beta} \rangle \langle \tilde{\beta} \otimes 0_{gs} | A | \tilde{\alpha} \otimes 0_{gs} \rangle = \text{Tr} \tilde{\delta} \rho \tilde{A} = \langle \tilde{A} \rangle . \]  

(3.20)

Thus we are looking for \( \tilde{\rho} \) that maximizes \( S = -\text{Tr} \tilde{\rho} \ln \tilde{\rho} \) subject to the constraints \( \text{Tr} \tilde{\delta} \rho \tilde{H} = \text{Tr} \tilde{\delta} \rho x_i x_j = \text{Tr} \tilde{\delta} \rho P_i P_j = 0 \), where \( i \) and \( j \) range over the outside oscillators.

From Eq. (2.10) we expect \( \tilde{\rho} \) to have the form

\[ \tilde{\rho} \propto e^{-\beta (\tilde{H} + f_{ij} \tilde{x}_i \tilde{x}_j + g_{ij} \tilde{P}_i \tilde{P}_j)} . \]  

(3.21)

As in [1] we can write this

\[ \rho \propto e^{-\beta H'} , \]  

(3.22)

where \( H' \) is a fictitious Hamiltonian for these oscillators,

\[ H' = \tilde{H} + f_{ij} \tilde{x}_i \tilde{x}_j + g_{ij} \tilde{P}_i \tilde{P}_j . \]  

(3.23)

**D. New coordinates**

We would now like to introduce new coordinates \( w \) as follows: The first \( N_{in} \) \( w \) coordinates will be the inside oscillator coordinates, \( w_{in} = x_{in} \). The last \( N_{gs} \) coordinates are the ground state normal modes, \( w_{gs} = y_{gs} \). The remaining \( N_{in} \) coordinates can be any coordinates that are independent of those specified so far; we will make a particular choice later.

To do this, we proceed as follows: Using Eqs. (3.11) and (3.12) we can write out

\[ I = Y^{-1} Y = \begin{pmatrix} Y_{\text{free, in}}^{-1} Y_{\text{in, free}} + Y_{\text{free, out}}^{-1} Y_{\text{out, free}} & Y_{\text{free, out}}^{-1} Y_{\text{out, gs}} \\ Y_{\text{gs, out}}^{-1} Y_{\text{out, free}} & Y_{\text{gs, out}}^{-1} Y_{\text{out, gs}} \end{pmatrix} . \]  

(3.24)

In particular, \( Y_{\text{gs, out}}^{-1} Y_{\text{out, gs}} = I \). We would like to extend \( Y_{\text{gs, out}}^{-1} \) and \( Y_{\text{out, gs}} \) into square matrices \( R \) and \( R^{-1} \) with
\[ R = \begin{pmatrix} N_{in} & N_{gs} \\ D & Y_{out,gs} \end{pmatrix} \{ N_{out} \} \]

and

\[ R^{-1} = \begin{pmatrix} N_{out} \\ \begin{pmatrix} D' \\ Y_{gs,\text{out}}^{-1} \end{pmatrix} \end{pmatrix} \} N_{in} \{ N_{out} \} \]

This means that we must find \( D \) and \( D' \) such that

\[
\begin{align*}
Y_{gs,\text{out}}^{-1}D &= 0 \\
D'Y_{\text{out,gs}} &= 0 \\
D'D &= I.
\end{align*}
\]

There are many possible choices of \( D \) and \( D' \) that satisfy Eqs. (3.27). Here we proceed as follows: Let

\[
Z = \begin{pmatrix} Z_{in} \\ Z_{\text{out}} \end{pmatrix}
\]

so that \( Z_{in}Z_{in}^{-1} = I, Z_{\text{out}}Z_{\text{out}}^{-1} = I \) and \( Z_{in}^{-1}Z_{in} + Z_{\text{out}}^{-1}Z_{\text{out}} = I \). Now let

\[
\begin{align*}
\bar{D} &= \frac{1}{2}Z_{\text{out}}Z_{in}^{-1} = \langle 0|\mathbb{X}|0 \rangle_{\text{out,in}} \\
\bar{D}' &= \frac{1}{2}Z_{in}^{-1}Z_{\text{out}}^{-1} = \langle 0|\mathbb{P}|0 \rangle_{\text{in,out}}.
\end{align*}
\]

let \( A \) and \( B \) be \( N_{in} \times N_{in} \) matrices with

\[ AB = (\bar{D}'\bar{D})^{-1} \]

and let

\[
\begin{align*}
D &= \bar{D}A \\
D' &= B\bar{D}'.
\end{align*}
\]

We also divide

\[ W = \begin{pmatrix} W_{\text{free}} \\ W_{gs} \end{pmatrix}. \]

Since \( Y = ZW \) and \( Y^{-1} = W^{T}Z^{-1} \), we have

\[
\begin{align*}
Z_{\text{in}}W_{gs} &= 0 \\
W_{gs}Z_{\text{in}}^{-1} &= 0.
\end{align*}
\]
Thus
\[ D'Y_{\text{out,gs}} \propto BZ_{\text{in}}^{-1}T Z_{\text{out}}^{-1} W_{\text{gs}} = BZ_{\text{in}}^{-1}T W_{\text{gs}} - BZ_{\text{in}}^{-1}T Z_{\text{in}} W_{\text{gs}} = 0 \quad (3.34) \]
by Eqs. (3.33) and their transposes. Similarly
\[ Y_{\text{gs, out}}^{-1} D \propto W_{\text{gs}}^{T} Z_{\text{out}}^{-1} Z_{\text{in}}^{T} A = W_{\text{gs}}^{T} Z_{\text{in}}^{T} A - W_{\text{gs}}^{T} Z_{\text{in}}^{T} Z_{\text{in}}^{T} A = 0. \quad (3.35) \]
From Eq. (3.30) we find
\[ D' D = I. \]
Thus the matrices \( D \) and \( D' \) satisfy Eqs. (3.27). We still have the freedom of choosing the matrix \( A \) arbitrarily.

Now let
\[ Q = \begin{pmatrix} N_{\text{in}} & N_{\text{out}} \\ I & 0 \end{pmatrix} \begin{pmatrix} R \end{pmatrix} \]
and define \( w \) by \( x = Q w \) so \( P x = Q^{-1} T P w \). Then \( w_{\text{in}} = x_{\text{in}} \) and \( w_{\text{gs}} = y_{\text{gs}} \) as desired.

### E. Reduced operators

We would like to recast our problem in terms of \( w_{\text{free}} \), the the first \( N_{\text{free}} \) \( w \) coordinates. First we look at the operators \( x_{\mu} x_{\nu} = X_{\mu \nu} \) and \( P_{\mu} P_{\nu} = P_{\mu \nu} \). If we write
\[ Q = \begin{pmatrix} Q_{\text{free}} \\ Q_{\text{gs}} \end{pmatrix}, \]
\[ Q^{-1} = \begin{pmatrix} Q_{\text{free}}^{-1} \\ Q_{\text{gs}}^{-1} \end{pmatrix}, \]
we have \( x = Q w = Q_{\text{free}} w_{\text{free}} + Q_{\text{gs}} w_{\text{gs}} \) and \( P = Q^{-1} T P w = Q_{\text{free}}^{-1} (P w)_{\text{free}} + Q_{\text{gs}}^{-1} (P w)_{\text{gs}} \), so
\[ X = Q_{\text{free}} W_{\text{free}} Q_{\text{free}}^{T} + Q_{\text{gs}} W_{\text{gs}} Q_{\text{gs}}^{T} \]
\[ P = Q_{\text{free}}^{-1} (P w)_{\text{free}} Q_{\text{free}}^{-1} + Q_{\text{gs}}^{-1} (P w)_{\text{gs}} Q_{\text{gs}}^{-1}. \]
Now we form the reduced operators \( \tilde{x}_{\mu} x_{\nu} \) and \( \tilde{P}_{\mu} P_{\nu} \). Since \( w_{\text{gs}} = y_{\text{gs}} \) and the \( y_{\text{gs}} \) modes are in the ground state by definition, \( W_{\text{gs, gs}} = (P w)_{\text{gs, gs}} = (1/2) I \), and so
\[ \tilde{X} = Q_{\text{free}} W_{\text{free}} Q_{\text{free}}^{T} + \frac{1}{2} Q_{\text{gs}} Q_{\text{gs}}^{T} \]
\[ \tilde{P} = Q_{\text{free}}^{-1} (P w)_{\text{free}} Q_{\text{free}}^{-1} + \frac{1}{2} Q_{\text{gs}}^{-1} Q_{\text{gs}}^{-1}. \]
In each case the second term is just a constant. Now
\[ H = \frac{1}{2} (P_{\mu} P_{\mu} + K_{\mu \nu} x_{\mu} x_{\nu}) = \frac{1}{2} \text{Tr} (P + K X) \quad (3.40) \]
where the trace is over the oscillator indices. Thus
\[
\tilde{H} = \frac{1}{2} \text{Tr} \left( Q^{-1}_\text{free} (Pw)_\text{free,free} Q^{-1}_\text{free} + K Q_\text{free} W_\text{free,free} Q^T_\text{free} \right) + \text{const}
\]
\[
= \frac{1}{2} \text{Tr} \left( \tilde{T} (Pw)_\text{free,free} + \tilde{K} W_\text{free,free} \right) + \text{const}
\]  
(3.41)

where

\[
\tilde{T} = Q^{-1}_\text{free} Q^{-1} = \begin{bmatrix} I & \text{ } \text{ } \text{ } \\ 0 & D'D^T \end{bmatrix} \{N_{\text{in}} \} \{N_{\text{in}} \} \ (3.42a)
\]

\[
\tilde{K} = Q^{-1}_\text{free} K Q_\text{free} = \begin{bmatrix} K_{\text{in,sub}} & \text{ } \text{ } \text{ } \\ D' K_{\text{out,sub}} & D'D' \end{bmatrix} \{N_{\text{in}} \} \{N_{\text{in}} \} . \ (3.42b)
\]

The constant term in Eq. (3.41) is

\[
\frac{1}{2} \text{Tr} \left( KQ_{gs} Q^{-T}_{gs} + Q^{-1}_{gs} \right) = \frac{1}{2} \text{Tr} \left( Y_{out,gs} K_{out,gs} Y_{out,gs} + Y^{-1}_{gs,gs} Y^{-1}_{out,gs} \right) . \ (3.43)
\]

It depends on the ground state modes only and is just part of the zero-point energy. It will be the same in the vacuum and in a vacuum-bounded state. Thus if instead of Eq. (3.41) we use

\[
\tilde{H} = \frac{1}{2} \text{Tr} \left( \tilde{T} (Pw)_\text{free,free} + \tilde{K} W_\text{free,free} \right) \ (3.44)
\]

we are just shifting \( \tilde{H} \) by a constant term and thus changing the zero-point energy.

Now we would like to make this reduced system look as much as possible like the system we started with. Let \( B_1 \) be some matrix such that \( B^T_1 B_1 = (\bar{D}' \bar{D}^T)^{-1} \), let \( B_2 \) be a unitary matrix to be determined, and let \( B = B_2 B_1 \). Then \( D' D^T = I \), so \( \tilde{T} = I \). This gives \( A = A_1 A_2 \) where \( A_1 = (\bar{D}' \bar{D})^{-1} B_1^{-1} \) and \( A_2 = B_2^T \). Let \( D_1 = \bar{D} A_1 \) and let

\[
\tilde{K}_1 = \begin{bmatrix} K_{\text{in,sub}} & \text{ } \text{ } \text{ } \\ D'^T K_{\text{out,sub}} & D'^T D'^T \end{bmatrix} \{N_{\text{in}} \} \{N_{\text{in}} \} . \ (3.45)
\]

Now

\[
K_{\text{in,sub}} = \begin{bmatrix} \text{ } \text{ } \text{ } \\ 0 \end{bmatrix} , \ \ (3.46)
\]

so \( K_{\text{in,sub}} D_1 \) is nonzero only in the last row. Thus the last \( N_{\text{in}} + 1 \) rows and columns of \( \tilde{K}_1 \) look like

\[
\begin{pmatrix}
1 & \text{ } \text{ } \text{ } \\ 2g & \text{ } \text{ } \text{ } \\ \text{ } & \text{ } & \text{ }
\end{pmatrix} \{N_{\text{in}} \} . \ (3.47)
\]

A matrix of size \((N_{\text{in}} + 1) \times (N_{\text{in}} + 1)\) can be put in tridiagonal form by a unitary transform of the form.
which can be constructed, for example, using the Householder process. We will use this to choose \( A_1 = U \), so that \( \tilde{K} \) will be tridiagonal. For each off-diagonal element in the resulting tridiagonal matrix there is a choice of sign, and we will choose them all to be negative. Thus in the \( w \) coordinates, each oscillator has unit mass and is coupled only to its neighbors. This completely specifies the matrices \( A \) and \( B \) and thus \( D \) and \( D' \). Note that \( D \) and \( D' \) do not depend on the undetermined parts of \( W \).

To make \( H' \) in Eq. (3.23) we can add to \( \tilde{H} \) a kinetic and potential term involving outside oscillators only. The potential term (disregarding a constant) is

\[
f_{ij}\tilde{x}_i\tilde{x}_j = Tr f\tilde{X}_{\text{out},\text{out}} = Tr fQ_{\text{out,free}}W_{\text{free,free}}Q_{\text{out,free}}^T. \tag{3.49}
\]

Since

\[
Q_{\text{out,free}} = \left( \begin{array}{cc}
N_{\text{in}} & 0 \\
0 & \begin{array}{c}
D \\
\end{array}
\end{array} \right)_{N_{\text{out}}},
\]

this term is equivalent to adding an arbitrary term to just the lower right part of \( \tilde{K} \). Similarly the kinetic term is

\[
g_{ij}\tilde{P}_i\tilde{P}_j = Tr g\tilde{P}_{\text{out,out}} = Tr gQ_{\text{free,free}}^{-1}W_{\text{free,free}}Q_{\text{free,free}}^{-1} \tag{3.51}
\]

Since

\[
Q_{\text{free,free}}^{-1} = \left( \begin{array}{c}
0 \\
\begin{array}{c}
D' \\
\end{array}
\end{array} \right)_{N_{\text{in}}},
\]

this term corresponds to adding an arbitrary term to just the lower right part of \( \tilde{T} \).

That is to say we can write

\[
H' = \frac{1}{2} Tr \left( T'(P_{\text{w}})_{\text{free,free}} + K'W_{\text{free,free}} \right) \tag{3.53}
\]

with

\[
T' = \left( \begin{array}{c}
I \\
0 \\
\end{array} \right), \tag{3.54a}
\]

\[
K' = \left( \begin{array}{c}
K_{\text{in,in}} \times \left( \begin{array}{c}
K_{\text{in,out}}D \\
D^2K_{\text{out,in}} \\
\end{array} \right)
\end{array} \right) \tag{3.54b}
\]
F. Reduced constraints

Now we rewrite our constraints, Eqs. (2.6), in terms of the \( w \) coordinates. We will keep only the parts of the constraint equations that are not automatically satisfied because of the ground-state modes. For the expectation value constraints, from Eqs. (3.20) and (3.39) we have

\[
\langle X \rangle_{\text{out, out}} = \tilde{\langle X \rangle}_{\text{out, out}} = Q_{\text{out, free}} \langle \mathcal{W} \rangle_{\text{free, free}} Q_{\text{out, free}}^T + \frac{1}{2} Q_{\text{out, gs}} Q_{\text{out, gs}}^T \\
= D \langle \mathcal{W} \rangle_{\text{mid, mid}} D^T + \text{const} \tag{3.55a}
\]

\[
\langle P \rangle_{\text{out, out}} = \tilde{\langle P \rangle}_{\text{out, out}} = Q_{\text{free, out}}^{-1} \langle \mathcal{P}_w \rangle_{\text{free, free}} Q_{\text{free, out}}^{-1} + \frac{1}{2} Q_{\text{gs, out}}^{-1} Q_{\text{gs, out}}^{-1} \\
= D' \langle \mathcal{P}_w \rangle_{\text{mid, mid}} D' + \text{const} \tag{3.55b}
\]

where \( w_{\text{mid}} \) means the outside elements of \( w_{\text{free}} \), i.e. \( w_{N_{\text{in}}+1} \ldots w_{2N_{\text{in}}} \).

These expectation value matrices must be the same in the vacuum as in the vacuum-bounded state. We can accomplish this by requiring that \( \langle X \rangle_{\text{mid, mid}} \) and \( \langle \mathcal{W} \rangle_{\text{mid, mid}} \) are the same as in the vacuum. Our problem is now equivalent to one with only \( N_{\text{in}} \) outside oscillators, and thus only \( N_{\text{in}}(N_{\text{in}} + 1) \) expectation value constraints, regardless of the value of \( N_{\text{out}} \).

For the energy constraint, Eq. (2.7), we are concerned only with the renormalized energy \( \text{Tr} \rho H - \langle 0 | H | 0 \rangle \). Thus the constant term in Eq. (3.41) does not contribute, and we can use \( \tilde{H} \) from Eq. (3.44). Once again there is no dependence on \( N_{\text{out}} \).

G. Derivation based on inside functions

We would now like to redo the proceeding calculation in a way which does not depend on the number of outside oscillators. Then we can remove the box from our system by taking \( L \to \infty \) and \( N \to \infty \). It appears that we have used the matrices \( D \) and \( D' \) which have an index that runs from 1 to \( N_{\text{out}} \). However, we have used them only in particular combinations. The quantities which we need in our calculation are

1. \( K_{\text{in, out}} \tilde{D} \)
2. \( \tilde{D}' \tilde{D} \)
3. \( \tilde{D}' \tilde{D}'^T \)
4. \( \tilde{D}' K_{\text{out, out}} \tilde{D} \).

Each of these quantities is an \( N_{\text{in}} \times N_{\text{in}} \) matrix, so it is reasonable to imagine that they do not depend on \( N_{\text{out}} \) in the \( N \to \infty \) limit.

We proceed as follows: From Eqs. (3.23) we have

\[
\tilde{D} = \langle X \rangle_{\text{out, in}} \tag{3.56a}
\]

\[
D' = \langle P \rangle_{\text{in, out}} . \tag{3.56b}
\]

Any given element of \( \langle X \rangle \) and \( \langle P \rangle \) has a smooth limit when \( N_{\text{out}} \) is taken to infinity. It is just a particular expectation value of a half-line of coupled oscillators, which is a well-defined problem. We can express the above items in terms of such elements as follows:
1. $K_{\text{in},\text{out}}D$ depends only on the first row of $D$ which is $\langle X \rangle_{N_{\text{in}},+1,\text{in}}$ so it is already well-defined in the limit.

2. From Eq. (3.51) we have

$$\langle P \rangle \langle X \rangle = \frac{1}{4} I$$  \hspace{1cm} (3.57)

so $\bar{D}' \bar{D} = \langle P \rangle_{\text{in},\text{out}} \langle X \rangle_{\text{out},\text{in}} = \frac{1}{4} I - \langle P \rangle_{\text{in},\text{in}} \langle X \rangle_{\text{in},\text{in}}$, which does not depend on $N_{\text{out}}$.

3. We expand $\langle P \rangle \langle P \rangle = \frac{1}{4} Z^{-1T} Z^{-1} Z^{-1T} Z^{-1}$. We insert $I = Z \Omega_0 Z^T$ here to get

$$\langle P \rangle \langle P \rangle = \frac{1}{4} Z^{-1T} \Omega_0 Z^{-1} = \frac{1}{4} K.$$ \hspace{1cm} (3.58)

Then $\bar{D}' \bar{D}^T = \langle P \rangle_{\text{in},\text{out}} \langle P \rangle_{\text{out},\text{in}} = \frac{1}{4} K_{\text{in},\text{in}} - \langle P \rangle_{\text{in},\text{in}} \langle P \rangle_{\text{in},\text{in}}$ which does not depend on $N_{\text{out}}$.

4. Using Eqs. (3.57) and (3.58) we can write

$$\langle X \rangle K = K \langle X \rangle = \langle P \rangle \quad \text{and} \quad \langle X \rangle K \langle X \rangle = \frac{1}{4} I$$  \hspace{1cm} (3.59)

so

$$\frac{1}{4} I = \langle X \rangle_{\text{in},\text{out}} K_{\text{out},\text{out}} \langle X \rangle_{\text{out},\text{in}} + \langle X \rangle_{\text{in},\text{in}} K_{\text{in},\text{out}} \langle X \rangle_{\text{out},\text{in}}$$

$$+ \langle X \rangle_{\text{in},\text{out}} K_{\text{out},\text{in}} \langle X \rangle_{\text{in},\text{in}} + \langle X \rangle_{\text{in},\text{in}} K_{\text{in},\text{in}} \langle X \rangle_{\text{in},\text{in}}$$

$$= \bar{D}^T K_{\text{out},\text{out}} \bar{D} + \langle X \rangle_{\text{in},\text{in}} (\langle P \rangle_{\text{in},\text{in}} - K_{\text{in},\text{in}} \langle X \rangle_{\text{in},\text{in}})$$

$$+ (\langle P \rangle_{\text{in},\text{in}} - \langle X \rangle_{\text{in},\text{in}} K_{\text{in},\text{in}}) \langle X \rangle_{\text{in},\text{in}} + \langle X \rangle_{\text{in},\text{in}} K_{\text{in},\text{in}} \langle X \rangle_{\text{in},\text{in}}$$ \hspace{1cm} (3.60)

and so

$$\bar{D}^T K_{\text{out},\text{out}} \bar{D} = \frac{1}{4} I - \langle X \rangle_{\text{in},\text{in}} \langle P \rangle_{\text{in},\text{in}} - \langle P \rangle_{\text{in},\text{in}} \langle X \rangle_{\text{in},\text{in}} + \langle X \rangle_{\text{in},\text{in}} K_{\text{in},\text{in}} \langle X \rangle_{\text{in},\text{in}}$$ \hspace{1cm} (3.61)

which does not depend on $N_{\text{out}}$.

Thus we can now take $N \to \infty$ with $N_{\text{in}}$ fixed and have a well-defined problem in terms of $\tilde{K}$ with a finite number of free parameters.

**H. Calculation of the reduced vacuum**

We are trying to compute $\tilde{K}$ in Eq. (3.42b) in the $N_{\text{out}} \to \infty$ limit. We will keep $N_{\text{in}}$ and the oscillator spacing $L_1 \equiv L/(N + 1)$ fixed. With the normalization in Eq. (3.1) the vacuum normal mode matrix is given by
\[ Z_{\mu\nu} = \sqrt{\frac{2}{N+1}} \frac{\sin k_{\nu} \mu}{\sqrt{\omega_{\nu}}} \]  \hspace{1cm} (3.62)

with

\[ k_{\nu} = \frac{\pi \nu}{N+1} \]  \hspace{1cm} (3.63)

and

\[ \omega_{\nu} = \frac{2(N+1)}{L} \sin \frac{k_{\nu}}{2} = \frac{2}{L_1} \sin \frac{k_{\nu}}{2} . \]  \hspace{1cm} (3.64)

Thus

\[
\langle X \rangle_{\mu\nu} = \frac{1}{2} (ZZ^T)_{\mu\nu} = \frac{1}{N+1} \sum_{\alpha=1}^{N} \frac{\sin k_\alpha \mu \sin k_\alpha \nu}{\omega_\alpha}
\]

\[ = \frac{L_1}{2(N+1)} \sum_{\alpha=1}^{N} \frac{\sin k_\alpha \mu \sin k_\alpha \nu}{\sin(k_\alpha/2)} . \]  \hspace{1cm} (3.65)

Now we use

\[ \cos(\theta - \phi) - \cos(\theta + \phi) = 2 \sin \theta \sin \phi \]  \hspace{1cm} (3.66)

to write

\[
\langle X \rangle_{\mu\nu} = \frac{L_1}{4(N+1)} \sum_{\alpha=1}^{N} \cos k_\alpha (\mu - \nu) - \cos k_\alpha (\mu + \nu) \frac{\sin k_\alpha}{\sin(k_\alpha/2)} . \]  \hspace{1cm} (3.67)

Using Eq. (3.66) again, for any number \( a \) we can write

\[ \frac{\cos k_\alpha (a - 1) - \cos k_\alpha a}{\sin(k_\alpha/2)} = 2 \sin k_\alpha (a - 1/2) \]  \hspace{1cm} (3.68)

and thus

\[ \frac{\cos k_\alpha (\mu - \nu) - \cos k_\alpha (\mu + \nu)}{\sin(k_\alpha/2)} = 2 \sum_{a=\mu+\nu+1}^{\mu+\nu} \sin k_\alpha (a - 1/2) . \]  \hspace{1cm} (3.69)

If we put Eq. (3.69) into Eq. (3.67) and bring the sum over \( \alpha \) inside the sum over \( a \) we get a sum that we can do,

\[ 2 \sum_{\alpha=1}^{N} \frac{\pi \alpha (a - 1/2)}{N+1} = \cot \frac{\pi (2a-1)}{4(N+1)} + (-1)^a . \]  \hspace{1cm} (3.70)

Now we sum this over \( a \). Since \( \mu - \nu \) and \( \mu + \nu \) have the same parity, the \((-1)^a\) term does not contribute and we get

\[
\langle X \rangle_{\mu\nu} = \frac{L_1}{4(N+1)} \sum_{a=\mu-\nu+1}^{\mu+\nu} \cot \frac{\pi (2a-1)}{4(N+1)} . \]  \hspace{1cm} (3.71)
The sum over $N$ has been eliminated. In the $N \to \infty$ limit, the argument of cot goes to zero and so we can use \( \cot x = 1/x + O(x^{-3}) \) to get

\[
\langle X \rangle_{\mu\nu} = \frac{L_1}{\pi} \sum_{a=\mu-\nu+1}^{\mu+\nu} \frac{1}{2a-1}.
\]  

(3.72)

The sum can be done using special functions:

\[
\langle X \rangle_{\mu\nu} = \frac{L_1}{2\pi} \left[ \psi\left(\mu + \nu + \frac{1}{2}\right) - \psi\left(\mu - \nu + \frac{1}{2}\right) \right]
\]

(3.73)

where $\psi$ is the digamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$.

To compute $\langle P \rangle$ we write the inverse of the normal mode matrix,

\[
Z_{\mu\nu}^{-1} = \sqrt{\frac{2\omega_\mu}{N+1}} \sin k_\mu \nu
\]

(3.74)

and

\[
\langle P \rangle_{\mu\nu} = \frac{1}{2} \left( Z^{-1T} Z^{-1} \right)_{\mu\nu} = \frac{1}{N+1} \sum_{\alpha=1}^{N} \omega_\alpha \sin k_\alpha \mu \sin k_\alpha \nu
\]

\[
= \frac{2}{L_1(N+1)} \sum_{\alpha=1}^{N} \frac{\sin k_\alpha}{2} \sin k_\alpha \mu \sin k_\alpha \nu.
\]

(3.75)

In this case the sum can be done directly. Using Eq. (3.66) we can write

\[
\langle P \rangle_{\mu\nu} = C_{\mu-\nu} - C_{\mu+\nu}
\]

(3.76)

where

\[
C_\lambda \equiv \frac{1}{L_1(N+1)} \sum_{\alpha=1}^{N} \sin \frac{k_\alpha}{2} \cos k_\alpha \lambda
\]

\[
= \frac{1}{4L_1(N+1)} \left[ \cot \frac{\pi(2\lambda+1)}{4(N+1)} - \cot \frac{\pi(2\lambda-1)}{4(N+1)} - 2(-1)^\lambda \right].
\]

(3.77)

Again we take $N \gg \lambda$ to get

\[
C_\lambda = \frac{1}{\pi L_1} \left( \frac{1}{2\lambda+1} - \frac{1}{2\lambda-1} \right) = -\frac{2}{\pi L_1(4\lambda^2 - 1)}
\]

(3.78)

and

\[
\langle P \rangle_{\mu\nu} = \frac{2}{\pi L_1} \left( \frac{1}{4(\mu+\nu)^2 - 1} - \frac{1}{4(\mu-\nu)^2 - 1} \right).
\]

(3.79)

Equations (3.73) and (3.79) give $\langle X \rangle$ and $\langle P \rangle$ in the $N_{out} \to \infty$ limit. Using these values in the procedure of sections III D through III G we can compute the matrix $\tilde{K}$ numerically for a system with inside length $L_{in}$ but no outside box.
FIG. 1.: The ratio of the coupling coefficients in the reduced problem to what they would be in a regular problem of \( N_{\text{free}} \) oscillators, plotted against \( x = L_{1} \mu \), for \( L_{\text{in}} = 1 \).

IV. NUMERICAL COMPUTATION OF THE REDUCED VACUUM

We have computed numerically the reduced coupling matrix \( \tilde{K} \), by following the procedure of sections III D–III H. This is a fairly straightforward problem in numerical analysis; the number of steps grows as \( N_{\text{in}}^{3} \). However, in order to produce accurate results for \( N_{\text{in}} \gtrsim 6 \) it is necessary to use very high-precision floating-point numbers. The necessary number of bits of mantissa in the representation appears to be about \( 10 N_{\text{in}} \). The code was written in Lisp and executed on DEC\textsuperscript{TM} Alpha\textsuperscript{TM} workstations.

The resulting matrix \( \tilde{K} \) is a tridiagonal matrix which gives a set of self-couplings and nearest-neighbor couplings for the fictitious oscillators \( w_{\text{free}} \). We can express these couplings as multiples of the couplings for a regular chain of \( N_{\text{free}} \) oscillators with spacing \( L_{1} \). Thus we write the self-coupling as

\[
\tilde{K}_{\mu \mu} = -g f_{\mu} \left( \sum_{\mu} \right) \tag{4.1}
\]

and the nearest-neighbor coupling as

\[
\tilde{K}_{\mu,\mu+1} = 2 g f_{\mu+1/2} \left( \sum_{\mu} \right). \tag{4.2}
\]

These coupling coefficients converge rapidly to a universal form \( f(x) \) where \( L_{1} \mu \rightarrow x \) in the continuum limit. Some results are shown in Fig. 1. We can see that \( f(x) \approx 1 \) until
FIG. 2.: The coupling coefficient ratio for \( N_{in} = 175 \) in the region \( x > 1.8 \) and the fit \( f(x) = 3.2(2-x)^4 \).

\( x \sim 1.4 \) at which point it begins to fall and asymptotically approaches 0 as \( x \to 2 \). For values of \( x \) near 2, \( f(x) \) is well fit by

\[
f(x) = a(2-x)^4
\]

with \( a \approx 3.2 \), as shown in Fig. 2.

Some typical normal modes of the reduced vacuum are shown in Fig. 3. They are sine waves while \( f(x) \sim 1 \) and then begin to oscillate faster and faster as \( f(x) \) shrinks. At first the amplitude of the oscillations grows but for larger \( x \) it shrinks rapidly to zero. The wavenumbers in the inside region (and thus the frequencies) are smaller than we would find for a rigid box because most of the oscillations are in the part of the outside region where \( f(x) \ll 1 \). In fact, as \( N_{in} \to \infty \) we would expect the low-lying frequencies to go to zero, for the following reason.

We can find the frequencies by computing the normal modes of a Hamiltonian

\[
H = \int_0^L dx dy \left( T(x-y)\pi(x)\pi(y) + K(x-y)\phi(x)\phi(y) \right),
\]

which requires solving the eigenvector equation

\[
\int_0^L dy dz T(x-y)K(y-z)g(z) = \lambda g(x)
\]

with the boundary conditions
FIG. 3.: The 5th and 25th normal modes in the reduced vacuum, computed with $N_{in} = 175$.

$$g(0) = 0 \quad (4.6a)$$

$$g(L) = 0. \quad (4.6b)$$

Since Eq. (4.5) is a second-order differential equation we expect two degrees of freedom in the solution. However, one degree of freedom is manifestly the overall scale, which does not affect the boundary conditions. Since there are two boundary conditions but only one free parameter, we can expect to find solutions only for particular values of $\lambda$. For example, for the usual scalar field Hamiltonian the general solution to Eq. (4.5) would be

$$g(x) = c\sin(\sqrt{\lambda}x + \delta). \quad (4.7)$$

To satisfy Eqs. (4.6) we need to choose $\delta = 0$ and $\sqrt{\lambda} = n\pi/L$ for some integer $n$.

However, if we use

$$H = \frac{1}{2} \int_0^{2L_{in}} f(x) \left[ \pi(x)^2 + \left( \frac{d\phi}{dx} \right)^2 \right] dx, \quad (4.8)$$

with $f(x) \to a(2 - x)^4$, as suggested by Fig. 2, we will get a continuum of frequencies. The problem is that since $f(x) \to 0$ as $x \to 2$ the boundary condition there does not really constrain $g(x)$. There can be arbitrary changes in $g(x)$ near $x = 2$ and so Eq. (4.6b) can always be satisfied. Since there is only one effective boundary condition and one effective degree of freedom, we expect to be able to find a solution for any $\lambda$. Thus in the continuum limit there are modes with arbitrarily low frequencies. This is not an
FIG. 4.: The lowest frequency of the reduced vacuum and the fit $1.81x^{-1} - x^{-2}$.

unreasonable conclusion, since although the range of $x$ is finite, we are using it to represent the infinite half-line. In the infinite vacuum there is no right-hand boundary condition, and there are modes of every frequency.

This conclusion is confirmed by numerical results. In Fig. 4 we plot the lowest normal-mode frequency versus $N_{\text{in}}$. As shown in the figure, the frequencies are well fit by a curve

$$ax^{-1} - x^{-2}$$

with $a \approx 1.81$. If this form is correct, in the $N_{\text{in}} \to \infty$ limit the lowest frequency goes to zero.

If we go to a vacuum-bound state we will introduce some finite temperature. We then expect that the non-zero temperature will increase the frequencies in such a way that there are only a finite number of low-lying modes and thus a finite entropy. However, in the limit where $T \to 0$ we do expect an entropy-to-energy relation equivalent to a system with infinitesimal frequencies. We now consider this situation.

V. THE LOW-ENERGY LIMIT

In the low-energy regime we can make a first-order expansion around the vacuum. To do this we note that the only dependence on $\beta$ in our equations (see [1]) is through expectation values that depend on $\coth(\beta \omega_\alpha / 2)$. For large $\beta$ we can approximate

$$\coth \frac{\beta \omega_\alpha}{2} \approx 1 + 2e^{-\beta \omega_\alpha}.$$  (5.1)
The change in coth(\(\beta \omega_\alpha / 2\)) is the largest for the smallest frequency, which we will call \(\omega_1\). We will ignore \(e^{-\beta \omega_\alpha}\) for larger \(\omega_\alpha\) by comparison with \(e^{-\beta \omega_1}\). Thus we take

\[
\delta \coth \frac{\beta \omega_1}{2} = 2e^{-\beta \omega_{1\text{vac}}} \equiv 2\epsilon 
\] (5.2a)

\[
\delta \coth \frac{\beta \omega_\alpha}{2} = 0 \quad \text{for } \alpha > 1.
\] (5.2b)

Then we write

\[
T'_{\text{mid, mid}} = \tilde{T}_{\text{mid, mid}} + \delta T_{\text{mid, mid}} 
\] (5.3a)

\[
K'_{\text{mid, mid}} = \tilde{K}_{\text{mid, mid}} + \delta K_{\text{mid, mid}} 
\] (5.3b)

where \(\delta K_{\text{mid, mid}}\) and \(\delta T_{\text{mid, mid}}\) are \(O(\epsilon)\). These changes give rise to \(O(\epsilon)\) changes in \(U\) and the \(\omega_\alpha\), which in turn give rise to \(O(\epsilon)\) changes in \(\langle w_m w_n \rangle\) and \(\langle P_{w_m} P_{w_n} \rangle\).

Since overall \(\langle w_m w_n \rangle\) and \(\langle P_{w_m} P_{w_n} \rangle\) cannot change we have

\[
0 = \delta \langle w_m w_n \rangle = \sum_\alpha \left( -\frac{\delta \omega_\alpha}{2 \omega_\alpha^2} U_{1\text{vac}}^{\text{vac} -1} U_{1\text{vac}}^{\text{vac} -1} + \frac{1}{2 \omega_\alpha} \delta U_{\alpha\alpha} U_{\text{vac}\alpha} + \frac{1}{2 \omega_\alpha^2} U_{\alpha\alpha} \delta U_{\text{vac}\alpha} \right) 
\] + \(\omega_1\text{vac}^{\text{vac} -1} U_{1\text{vac}}^{\text{vac} -1} \epsilon \) \quad (5.4a)

\[
0 = \delta \langle P_{w_m} P_{w_n} \rangle = \sum_\alpha \left( -\frac{\delta \omega_\alpha}{2} U_{1\text{vac}}^{\text{vac} -1} U_{1\text{vac}}^{\text{vac} -1} + \frac{\omega_\alpha}{2} \delta U_{\alpha\alpha} U_{\text{vac}\alpha} + \frac{\omega_\alpha}{2} U_{\alpha\alpha} \delta U_{\text{vac}\alpha} \right) 
\] + \(\omega_1\text{vac}^{\text{vac} -1} U_{1\text{vac}}^{\text{vac} -1} \epsilon \). \quad (5.4b)

We thus have \(N_{\text{in}}(N_{\text{in}} + 1)\) linear equations for \(N_{\text{in}}(N_{\text{in}} + 1)\) unknown values of \(\delta T_{\text{mid, mid}}\) and \(\delta K_{\text{mid, mid}}\), which are readily solved. Since the inhomogeneous part of these equations is \(O(\epsilon)\), all the results must be \(O(\epsilon)\) as well. In particular, the \(\delta \omega_\alpha\) are \(O(\epsilon)\). Now if \(T\) is very small as compared to all the \(\omega_\alpha\), then \(\epsilon\) will be small as compared to all the parameters of the problem, and so the first-order approximation will be good. For any fixed number of oscillators \(N_{\text{in}}\) there will be some minimum frequency \(\omega_1\), and if we take \(\beta \ll 1/\omega_1\) we will always be in this regime.

Now the entropy \(S\) depends only on \(\beta\) and the \(\omega_\alpha\). Since the modes are uncoupled,

\[
S = \sum_\alpha S_1(\beta \omega_\alpha) 
\] (5.5)

with

\[
S_1(\beta \omega) = -\ln(1 - e^{-\beta \omega}) - \frac{\beta \omega}{e^{-\beta \omega} - 1}. 
\] (5.6)

Since \(\beta \omega_\alpha \gg 1\) all the terms are very small, and the \(\omega_1\) term dominates,

\[
S \approx S_1(\omega_1) = (1 + \beta \omega_1) e^{-\beta \omega_1} + O(e^{-2\beta \omega_1}). 
\] (5.7)

Since \(\epsilon\) drops exponentially with increasing \(\beta\), we expect that for \(\beta\) large enough, \(\beta \delta \omega_1 \ll 1\) so that

\[
S = (1 + \beta \omega_1^{\text{vac}}) e^{-\beta \omega_1^{\text{vac}}} + O(\epsilon^2). 
\] (5.8)
The value of $S$ given in Eq. (5.8) is the one we would get from a rigid box with length

$$L'_{in} = \pi/\omega^{\text{vac}}_1.$$  

(5.9)

To approximate the energy, we proceed along the lines of section IX of [1]. The direct calculation is made difficult by the fact that, while $H'$ differs from $H$ only by $O(\epsilon)$, we must subtract from both Hamiltonians a large ground-state energy. Instead we work by integrating on $T$. From $F = E - TS$ and $dF = -SdT$ we find

$$E(T) = TS(T) - \int_0^T S(T')dT'.$$

(5.10)

Now $L'_{in}$ depends only on $\omega^{\text{vac}}_1$, which depends on $N_{in}$ but not on $\beta$. If Eq. (5.8) is valid for a particular $N_{in}$ at $\beta = 1/T$ is it valid for $T' < T$ and $\beta' = 1/T' > \beta$. Thus both $S(T)$ and $S(T')$ in Eq. (5.10) are just the entropy of a rigid box of length $L'_{in}$. Thus the entropy-to-energy relationship is just $S(E) = S^{rb}(L'_{in}; E)$, the entropy as a function of the given energy in a rigid box with length $L'_{in}$.

For such a rigid box at very low energy we find

$$E = \frac{\omega^{\text{rb}}_1}{e^{\beta\omega^{\text{rb}}_1} - 1} \approx \omega^{\text{rb}}_1 e^{-\beta\omega^{\text{rb}}_1}$$

(5.11)

and thus

$$S = \left(1 + \ln \frac{\omega^{\text{rb}}_1}{E}\right) \frac{E}{\omega^{\text{rb}}_1}$$

(5.12)

where $\omega^{\text{rb}}_1 = \pi/L'_{in} = \omega^{\text{vac}}_1$ is the frequency of the lowest mode.

Now for any given $N_{in}$ we get some $\omega^{\text{vac}}_1$. As discussed in section [IV], the larger $N_{in}$ we choose, the smaller $\omega^{\text{vac}}_1$ we will have. For $N_{in}$ fixed we can choose $\beta \gg 1/\omega^{\text{vac}}_1$ and proceed as above to get a large value of $L'_{in}$. However, we are really interested in the continuum limit at fixed temperature. If we increase $N_{in}$ with $\beta$ fixed we will find that $\omega_1$ (and eventually an arbitrary number of the $\omega_\alpha$) will become smaller than $1/\beta$. When this happens, the approximations of Eqs. (5.2–5.4) will no longer be good.

However, we do not expect the entropy to decrease drastically in this limit. To make the entropy small would require making all the frequencies large. If the frequencies were large, the approximations we have used would again become valid. Then we could argue as before that the entropy should be large. It would be hard to have a consistent picture.

Now consider the limit as $T \to 0$. For each $T$ we start with some initial number of oscillators $N^{(0)}_{in}$. We choose $N^{(0)}_{in}$ not too large, such that $\omega^{\text{vac}}_1 \gg T$. With this value of $N_{in}$, we find $L^{(0)}_{in} \sim \pi/\omega^{\text{vac}}_1$. We then let $N_{in} \to \infty$ and we conjecture that the entropy does not change much, and thus in the continuum limit $S(E) \sim S^{rb}(L^{(0)}_{in}; E)$. As we decrease $T$ we can decrease the initial $\omega^{\text{vac}}_1$ and so increase $L^{(0)}_{in}$ without bound. Thus we make the following conjecture:

For a given energy $E$, let $L^{in}(E)$ be the length of a rigid box such that the vacuum-bounded state with energy $E$ and length $L^{in}$ has entropy $S(E) = S^{rb}(L^{(0)}_{in}; E)$. Then

$$\lim_{E \to 0} \frac{L^{in}(E)}{L^{in}} = \infty.$$  

(5.13)
FIG. 5.: The length $L'_\text{in}$ of a rigid box that gives the same $S(E)$ as a vacuum-bounded state at temperature $T = 1/\beta$ and $L_\text{in} = 1.0$.

To support this conjecture numerically we turn to direct calculation of energy and entropy values for vacuum-bounded states at low temperature. For various fixed values of $\beta = 1/T$ and for various numbers of oscillators we compute $S$ and $E$ and from them the equivalent length $L_\text{in}$. Some results are plotted in Fig. 5. While $N_\text{in}$ is still small enough for the approximations of Eqs. (5.2–5.4) to be valid, $L'_\text{in}$ grows with $N_\text{in}$. Once $N_\text{in}$ has left this regime, it appears that $L'_\text{in}$ levels off. It is at least reasonable to believe that there is no further change in $L'_\text{in}$ as $N_\text{in} \to \infty$. In Fig. 6 we plot the eventual level of $L'_\text{in}$ versus $\beta$. It appears that the limiting value of $L'_\text{in}$ grows nearly linearly with $\beta$, and thus $L'_\text{in} \to \infty$ as $E \to 0$ as conjectured.

VI. HIGH ENERGIES

In this section we use the results of section [11] to redo the computation from [1] of the bound on the entropy of a high-energy vacuum-bounded state. The difference here is that $L_\text{out}$ and thus $N_\text{out}$ are explicitly infinite, whereas in [1] we were limited to $N_\text{out} \sim N_\text{in}$.

First we solve numerically for the maximum-entropy vacuum-bounded state. We first compute the reduced vacuum as in section [V] and then we find the vacuum-bounded state, exactly as in section VII of [1], with the vacuum couplings given by $\tilde{K}$. The results are shown in Fig. 7. As in [1] we find that the mode number spacing is very nearly uniform.

Using this as an ansatz, we can repeat the calculation of section VIII of [1]. We will
FIG. 6.: The length $L_{in}$ of a rigid box with the same $S(E)$ plotted against $\beta$. Each point is the value for the largest number of oscillators available.

FIG. 7.: The numerically computed wavenumbers compared with the best-fit line through the origin for $L_{in} = 1.0$, $\beta = 0.5$, $N_{in} = 12$. 
work in the w coordinates, and set
\[ \langle w_{N_{\text{in}}+1}^2 \rangle = \langle w_{N_{\text{in}}+1}^2 \rangle_{\text{vac}} \]  
(6.1)

A. The vacuum

In section III H we computed the values of \( \langle X \rangle \). To convert to w coordinates we proceed as follows: From section III D, \( x = Q w \) and so
\[ x_{\text{out}} = R w_{\text{out}} = D w_{\text{mid}} + Y_{\text{out,gs}} w_{\text{gs}}. \]  
(6.2)
In the numerical work we found that with our choice for \( D \) we got
\[ \tilde{K}_{\text{in,mid}} = \begin{pmatrix} \frac{1}{g} & 0 \\ 0 & 0 \end{pmatrix}. \]  
(6.3)
Since
\[ \tilde{K}_{\text{in,mid}} = K_{\text{in,out}} D = \begin{pmatrix} \frac{1}{g} & 0 \\ 0 & 0 \end{pmatrix} D \]  
(6.4)
it follows that the first row of \( D \) is \( (1, 0, \ldots, 0) \). Thus from Eq. (6.2) we get
\[ x_{N_{\text{in}}+1} = w_{N_{\text{in}}+1} + (Y_{\text{out,gs}})_1 \cdot w_{\text{gs}} \]  
(6.5)
where \( (Y_{\text{out,gs}})_1 \) denotes the first row of \( Y_{\text{out,gs}} \). Now \( \tilde{W}_{\text{gs,gs}} = (1/2) I \) and \( \tilde{W}_{\text{gs,free}} = 0 \), so
\[ \langle x_{N_{\text{in}}+1}^2 \rangle = \langle w_{N_{\text{in}}+1}^2 \rangle + \frac{1}{2} \langle Y_{\text{gs,out}} Y^T_{\text{gs,out}} \rangle_{11}. \]  
(6.6)
Since the last term is non-negative, we have
\[ \langle w_{N_{\text{in}}+1}^2 \rangle_{\text{vac}} \leq \langle x_{N_{\text{in}}+1}^2 \rangle_{\text{vac}}. \]  
(6.7)
This gives us an only upper bound on \( \langle w_{N_{\text{in}}+1}^2 \rangle_{\text{vac}} \), but that is sufficient to produce an upper bound on \( S(E) \), which is what we are looking for.

From Eq. (3.73) we have
\[ \langle x_{N_{\text{in}}+1}^2 \rangle_{\text{vac}} = \langle X \rangle_{N_{\text{in}}+1,N_{\text{in}}+1} = \frac{L_1}{2\pi} \left[ \psi \left( 2N_{\text{in}} + \frac{5}{2} \right) - \psi \left( \frac{1}{2} \right) \right]. \]  
(6.8)

We would like to evaluate this expression in the \( N_{\text{in}} \to \infty \) limit with \( L_{\text{in}} \) fixed. There is a prefactor of \( L_1 \), which goes to zero in this limit, but that is just an artifact of the conventions we have used for the discrete problem, and will appear in the finite-energy vacuum-bounded states as well. For large \( x \),
\[ \psi(x) \sim \ln x + O(1/x), \]  
(6.9)
so without the prefactor there is a logarithmic divergence. We are interested in the \( \ln N_{\text{in}} \) term, and in the constant term, but we will ignore any terms of order \( 1/N_{\text{in}} \) or lower.

We use Eq. (6.9) and \( \psi(1/2) = -\gamma - 2\ln 2 \), where \( \gamma \) is Euler’s constant, to get
The calculation of section VIII B of [1] does not have any dependence on \( N \), except in the factor \( L/N \) which here is \( L_1 \). Thus we can use the same procedure with \( w \) in place of \( x \), but the same mode function, to find (see [1], Eqs. (8.33) and (8.37))

\[
\langle w_{N_{in}+1}^2 \rangle_{\text{vac}} \geq \frac{L_1}{2\pi} \left( \ln \frac{8N_{in} \sin \pi \Delta}{\pi} + \gamma + 2\pi\tau' \Delta + \ln \frac{1 - e^{-4\pi\tau' \Delta}}{4\pi\tau' \Delta} \right). \tag{6.12}
\]

Setting \( \langle w_{N_{in}+1}^2 \rangle = \langle w_{N_{in}+1}^2 \rangle_{\text{vac}} \) from Eq. (6.11) and using the same approximations and in [1] we find

\[
L_{in}' \leq L_{in} + \frac{1}{2\pi T} \ln L_{in} T + O\left( \frac{1}{T} \right) \tag{6.13}
\]

just as in [1]. This confirms our conclusion that the difference in entropy is bounded by

\[
\delta S(E) \lesssim \frac{1}{6} \ln L_{in} T \approx \frac{1}{6} \ln S_{rb}, \tag{6.14}
\]

and the conclusions of [1] remain unchanged.

**VII. DISCUSSION**

We have improved the calculations of [1] by handling an infinite number of outside oscillators with only a finite number of degrees of freedom. This is possible because the total number of modes that can be excited depends only on the number of inside oscillators. Our technique is in some ways similar to representing an infinite half-line by a finite number of oscillators with larger and larger spaces between them.

Using this technique we look at very low energy states which look like the vacuum except in a particular region. We find that, because of very low frequencies in the vacuum state, much more entropy can be stored in a small region with the vacuum-bounded condition than could be stored in such a region with a rigid boundary. To quantify the difference we let \( L_{in}' \) be the size of a rigid-bounded system with the same entropy and energy as a vacuum-bounded system of length \( L_{in} \). We argue that one should expect the ratio \( L_{in}'/L_{in} \) to grow without bound as the energy decreases. Numerical calculations lend support to this argument.

We also redo the high-energy calculation from [1] to reach the same conclusion with fewer approximations.
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\footnote{1These routines and many others are now available on a CDROM [4]}
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