Three-dimensional modeling of pendulum waves propagation under dynamic loading of underground excavation surface

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Abstract. The propagation of surface pendulum waves under nonstationary loading on the free surface of a cavity located in a block half-space is studied numerically. A mathematical model of transient viscoelastic deformation of the block medium is proposed. This model is based on the idea that dynamic behavior of the block medium can be approximately described as the movement of rigid blocks due to the compliance of interlayers between them. To describe the viscoelastic behavior of the interlayers, an internal friction model is used with the merit factor of the material as the determining parameter. The medium is modeled by a three-dimensional lattice of masses connected by elastic springs and viscous dampers in axial and diagonal directions. The influence of the merit factor and the cavity depth on the amplitudes of block velocities on the surface of the half-space is investigated for the longitudinal and Rayleigh waves.

1. Introduction

The concept put forward by M. A. Sadovsky [1] supposes that a rock massif represents a system of various scale blocks connected through intermediate layers composed of weaker and jointed rocks. The presence of inter-block layers with weakened mechanical properties (friable partings) which represent a deformable element in the block-rock mass eventually causing deformation of the entire block-rock mass. The laboratory experiments involving physical modeling of block geo-media [2–4] demonstrated that due to rapid attenuation of high-frequency waves, a major seismic impact is generated by low-frequency pendulum waves featured by low velocity, long length and weak attenuation. It is shown in [4, 5] that representation of blocks as massive rigid bodies in a block-structured medium enables description of the phenomenon of dynamic response in this block-rock medium called low-frequency pendulum type waves. A theoretical analysis provided in [5, 6], concerns the influence of the hierarchical block structure of rock mass on the wave propagation behavior and spectral characteristics of excitation. It also shows that a simple model can be obtained assuming that blocks are concentrated masses interconnected with viscoelastic springs. Different variants of this model are utilized in [4–9] to describe deformations in block-structured media.

In this paper, the block-structured medium is modeled by 3D lattice of masses connected axially and diagonally by elastic springs and viscous dampers. To describe the behavior of inter-block layers, we use the internal friction model proposed in [10] with the merit factor Q as the determining parameter.
2. One-dimensional model of a block-structured medium with account of internal friction

We demonstrate the model of a block-structured medium on the example of one-dimensional chain of masses, with the blocks represented as rigid bodies of mass $M$. The model of inter-block layers consists of two Maxwell elements and one Voigt element connected in parallel [10]. Each of these elements consists of a spring and a damper. At this, the Maxwell elements are series-connected, while the Voigt element is connected in parallel. The equations of one-dimensional motion of blocks with the given rheological model of interlayers have the following form:

$$M \dddot{u}_j = K[(u_{j+1} - 2u_j + u_{j-1}) + \beta(\dot{u}_{j+1} - 2\dot{u}_j + \dot{u}_{j-1}) -$$

$$-\alpha_1 \gamma_1(\psi_{j+1} - 2\psi_j + \psi_{j-1}) - \alpha_2 \gamma_2(\phi_{j+1} - 2\phi_j + \phi_{j-1})], \quad j = 1, 2, ..., \tag{1}$$

$$\dddot{u}_0 = K[(u_1 - u_0) + \beta(\dot{u}_1 - \dot{u}_0) - \alpha_1 \gamma_1(\psi_1 - \psi_0) - \alpha_2 \gamma_2(\phi_1 - \phi_0)] + P(t),$$

$$\psi_j = e^{-\gamma t} \int_0^t e^{\gamma \tau} \dot{u}_j(\tau) d\tau, \quad \phi_j = e^{-\gamma t} \int_0^t e^{\gamma \tau} \ddot{u}_j(\tau) d\tau,$$

$$\alpha_1 = \frac{k_1}{K}, \quad \alpha_2 = \frac{k_2}{K}, \quad \gamma_1 = \frac{k_1}{\lambda_1}, \quad \gamma_2 = \frac{k_2}{\lambda_2}, \quad \beta = \frac{\lambda}{K}, \quad K = k_1 + k_2,$$

where $P(t)$ is the effective load applied to block $j = 0$; $u_j$ is displacements of rigid blocks; $k$, $k_1$, $k_2$ are stiffnesses of springs in the Voigt element and in the two Maxwell elements, respectively; $\lambda$, $\lambda_1$, $\lambda_2$ are viscosities of dampers in the Voigt element and in the two Maxwell elements. The two additional displacement variables $\phi_j$ and $\psi_j$ introduced here are defined by function $u_j$.

We apply the discrete time Fourier transform to equations (1). In the frequency domain $\omega$, the force generated in the inter-block layer and acting on the block can be expressed by the formula:

$$\tilde{F} = K\left[1 - \frac{\alpha_1 \gamma_1^2}{\omega^2 + \gamma_1^2} - \frac{\alpha_2 \gamma_2^2}{\omega^2 + \gamma_2^2}\right] + i\omega\left[\beta + \frac{\alpha_1 \gamma_1^2}{\omega^2 + \gamma_1^2} + \frac{\alpha_2 \gamma_2^2}{\omega^2 + \gamma_2^2}\right] \tilde{u}(\omega) = K \tilde{F}(\omega) \tilde{u}(\omega).$$

The internal attenuation in the medium is usually determined either by the merit factor of the material $Q(\omega)$ or its inverse, which is described by the formula:

$$Q^{-1}(\omega) = \frac{\text{Im} \tilde{F}(\omega)}{\text{Re} \tilde{F}(\omega)}, \tag{2}$$

where $\text{Re} \tilde{F}(\omega)$, $\text{Im} \tilde{F}(\omega)$ are the real and imaginary components of the normalized impedance function $\tilde{F}(\omega)$ [11].

The new dimensionless parameters were introduced to allow connections of the parameters $\alpha_1$, $\alpha_2$, $\gamma_1$, $\gamma_2$ and $\beta$ of the rheological model with the $Q^{-1}(\omega)$ using simple, frequency-independent approximations:

$$\hat{\alpha}_1 = \alpha_1 Q_0, \quad \hat{\alpha}_2 = \alpha_2 Q_0, \quad \hat{\gamma}_1 = \gamma_1 / \omega_{\text{max}}, \quad \hat{\gamma}_2 = \gamma_2 / \omega_{\text{max}}, \quad \hat{\beta} = \beta \omega_{\text{max}} Q_0, \quad \hat{\omega} = \omega / \omega_{\text{max}},$$

where $\omega_{\text{max}}$ is the maximum frequency of interest for simulation; $Q_0$ is the specified target merit factor of the material. When passing on to the dimensionless parameters in expression (2), we can see that...
the coefficients $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\beta}$ are independent of the parameter $\omega_{\text{max}}$. It follows that for a given constant $Q_0^{-1}$ they must be only calculated once.

The parameter estimation problem with respect to $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\beta}$ is solved with a tolerance, allowing the real value of the merit factor to remain essentially close to the target value. This tolerance, which is taken as 5% of the target value, must be fulfilled within a wide range (4 – 100%) from $\omega_{\text{max}}$, i.e. the parameters $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\beta}$ are defined for the specified merit factor $Q_0$ from the direct computer-based testing of the following conditions:

$$\left[Q^{-1}(\hat{\omega}, Q_0^{-1})Q_0 - 1 \right]^2 < 0.05^2 \quad \forall \hat{\omega} : \quad 0.04 \leq \hat{\omega} \leq 1.$$  \hspace{1cm} (3)

The values of parameters calculated from the condition (3) for $Q_0^{-1} = 0.2$ are: $\hat{\alpha}_1 = \hat{\alpha}_2 = 0.28$, $\hat{\gamma}_1 = 0.026$, $\hat{\gamma}_2 = 0.23$, $\hat{\beta} = 0.14$.

3. Three-dimensional problem formulation and equations of motion

We investigated a nonstationary spatial problem on the dynamic impact exerted on the surface of cavity in a block-structured half-space. The block-structured medium is modeled by a uniform 3D lattice of point masses connected by springs and dampers in axial $(x, y, z)$ and in diagonal directions of the planes $x = \text{const}$, $y = \text{const}$, $z = \text{const}$, as shown in Figure 1a. The notions used here are as follows: $u, v, w$ are the displacements in the directions of $x, y, z$ axis; $n, m, k$ are the block numbers in the lines of $x, y, z$. The length of elastic springs along the $x, y, z$ axes is constant and equal to $l$. The origin of coordinates lies on the surface of the half-space with the corresponding value $k = 0$. We model the cavity as the surface of a cube with vertices located at points $A_1(l, l, -h - 1)$, $A_2(0, 0, -h - 1)$, $A_3(0, 0, -h - 1)$, $A_4(l, 0, -h - 1)$, $A_5(l, l, -h)$, $A_6(0, l, -h)$, $A_7(0, 0, -h)$, $A_8(l, 0, -h)$. Here $h$ is the distance from the surface of the block-structured half-space to the cavity. We assume that there is no connection between the masses at the cube vertices. The missing links are marked with dashed lines in Figure 1b. The forces applied at the cube vertices are equal and directed opposite to the center of cube symmetry located at the point $O(l/2, l/2, -h - l/2)$. As such, the loading simulates the effect of the “center of expansion” type applied to a cavity in a block-structured medium. Figure 1c provides schematic representation of the problem formulation.

![Figure 1](image_url)

**Figure 1.** The schemes showing: (a) connections of masses with springs and dampers in a 3D model of a block-rock structure medium; (b) loading; (c) problem statements.
We use a model with two Maxwell elements and one Voigt element as a rheological model of inter-block layers, which is shown on the example of a one-dimensional model. Let’s assume that the stiffness of springs and viscosity of dampers coincide in the axial and diagonal directions. Moreover, we assume that the parameters \( M, k, k_1, k_2, \lambda_1, \lambda_2 \), have the same values at all points in space (geo-medium).

Taking into account internal friction and notations:

\[
\Lambda_{nn} f_{n,m,k} = f_{n+1,m,k} - 2 f_{n,m,k} + f_{n-1,m,k},
\]
\[
\Phi_{nn} f_{n,m,k} = (f_{n+1,m+1,k} + f_{n-1,m-1,k} - 4 f_{n,m,k} + f_{n+1,m-1,k} + f_{n-1,m+1,k}) / 2,
\]
\[
\Psi_{nn} f_{n,m,k} = (f_{n+1,m+1,k} + f_{n-1,m-1,k} - f_{n+1,m-1,k} - f_{n-1,m+1,k}) / 2,
\]
\[
\psi^u_{n,m,k} = e^{-\gamma_1 t} \int_0^t e^{\gamma_1 \tau} u_{n,m,k}(\tau) d\tau, \quad \varphi^u_{n,m,k} = e^{-\gamma_1 t} \int_0^t e^{\gamma_1 \tau} u_{n,m,k}(\tau) d\tau,
\]
\[
\psi^v_{n,m,k} = e^{-\gamma_2 t} \int_0^t e^{\gamma_2 \tau} v_{n,m,k}(\tau) d\tau, \quad \varphi^v_{n,m,k} = e^{-\gamma_2 t} \int_0^t e^{\gamma_2 \tau} v_{n,m,k}(\tau) d\tau,
\]
\[
\psi^w_{n,m,k} = e^{-\gamma_3 t} \int_0^t e^{\gamma_3 \tau} w_{n,m,k}(\tau) d\tau, \quad \varphi^w_{n,m,k} = e^{-\gamma_3 t} \int_0^t e^{\gamma_3 \tau} w_{n,m,k}(\tau) d\tau.
\]

the equations of motion of a rock-block with the coordinates \( n, m, k \) inside the half-space \((k < 0)\) are written as:

\[
M \ddot{u}_{n,m,k} = K \{ (\Lambda_{nn} + \Phi_{nk} + \Phi_{nm}) u_{n,m,k} + \Psi_{nn} v_{n,m,k} + \Psi_{nk} w_{n,m,k} + \beta [ (\Lambda_{nn} + \Phi_{nk} + \Phi_{nm}) u_{n,m,k} + \Psi_{nm} \dot{v}_{n,m,k} + \Psi_{nk} \dot{w}_{n,m,k}] - \alpha_1 \gamma_1 [ (\Lambda_{nm} + \Phi_{mk} + \Phi_{nm}) \psi^u_{n,m,k} + \Psi_{nm} \psi^v_{n,m,k} + \Psi_{nk} \psi^w_{n,m,k}] - \alpha_2 \gamma_2 [ (\Lambda_{nm} + \Phi_{mk} + \Phi_{nm}) \varphi^u_{n,m,k} + \Psi_{nm} \varphi^v_{n,m,k} + \Psi_{nk} \varphi^w_{n,m,k}] \} \quad \text{(4)}
\]
\[
M \ddot{v}_{n,m,k} = K \{ (\Psi_{nn} u_{n,m,k} + (\Lambda_{nm} + \Phi_{mk} + \Phi_{nm}) v_{n,m,k} + \Psi_{nk} w_{n,m,k} + \beta [ (\Psi_{nn} u_{n,m,k} + (\Lambda_{nm} + \Phi_{mk} + \Phi_{nm}) v_{n,m,k} + \Psi_{nm} \dot{w}_{n,m,k}] - \alpha_1 \gamma_1 [ (\Psi_{nm} \psi^u_{n,m,k} + (\Lambda_{nm} + \Phi_{mk} + \Phi_{nm}) \psi^v_{n,m,k} + \Psi_{nk} \psi^w_{n,m,k}] - \alpha_2 \gamma_2 [ (\Psi_{nm} \varphi^u_{n,m,k} + (\Lambda_{nm} + \Phi_{mk} + \Phi_{nm}) \varphi^v_{n,m,k} + \Psi_{nk} \varphi^w_{n,m,k}] \} \quad \text{(5)}
\]
\[
M \ddot{w}_{n,m,k} = K \{ (\Psi_{nk} u_{n,m,k} + \Psi_{mk} v_{n,m,k} + (\Lambda_{kk} + \Phi_{mk} + \Phi_{nk}) w_{n,m,k} + \beta [ (\Psi_{nk} u_{n,m,k} + \Psi_{mk} v_{n,m,k} + (\Lambda_{kk} + \Phi_{mk} + \Phi_{nk}) \dot{w}_{n,m,k}] - \alpha_1 \gamma_1 [ (\Psi_{nk} \psi^u_{n,m,k} + \Psi_{mk} \psi^v_{n,m,k} + (\Lambda_{kk} + \Phi_{mk} + \Phi_{nk}) \psi^w_{n,m,k}] - \alpha_2 \gamma_2 [ (\Psi_{nk} \varphi^u_{n,m,k} + \Psi_{mk} \varphi^v_{n,m,k} + (\Lambda_{kk} + \Phi_{mk} + \Phi_{nk}) \varphi^w_{n,m,k}] \} \quad \text{(6)}
\]

Here, \( K \) is the total stiffness of the springs (1). The initial conditions for equations (4)–(6) are zero conditions.

Since the planes \( n = 0, m = 0 \) are the symmetry planes of the wave process, the following conditions are fulfilled on them:
Further, given the spherical symmetry of elastic Earth, we will calculate the propagation of waves only in the domain \( n \geq 0 \), \( m \geq 0 \). The variable \( k \) changes in the region \( k \leq 0 \).

As it is shown in [9], in the absence of internal friction (\( \alpha_1 = 0 \), \( \alpha_2 = 0 \), \( \gamma_1 = 0 \), \( \gamma_2 = 0 \), \( \beta = 0 \)), the velocities of P-waves \( c_p \) and Rayleigh waves \( c_R \) corresponding to this model of the block-structured medium will be:

\[
c_p = l \sqrt{\frac{3K}{M}}, \quad c_R = l \sqrt{\frac{K}{M} \left( \frac{1}{1 - \sqrt{3}} \right)}.
\] (7)

The mass of rock-blocks, the length of the springs, and the total stiffness are taken to be equal 1: \( M = 1 \), \( l = 1 \), \( K = 1 \). The unit of velocity is the value \( l\sqrt{K/M} \), the unit of time is \( \sqrt{M/K} \).

### 4. Calculation results

Using equations (4) – (6), the problem of pendulum waves propagation in a 3D block-structured medium is solved numerically, taking into account internal friction which resists the “center of expansion” loading type applied to the surface of the deepened cavity.

The reaction of the block medium to the Gauss pulse is analyzed as:

\[
P(t) = P_0 \exp[-(t - 4\sigma)^2 / (2\sigma^2)],
\]

where \( P(t)\sqrt{\sigma} \) is the amplitude of forces acting at the cube vertices at the time \( t \) (Figure 1b).

Figure 2 represents results of the calculations of the time-dependent infinitely long radial \( \ddot{u}_{n,1,0} \) and vertical \( \ddot{w}_{n,1,0} \) velocities of rock-blocks on the surface of the half-space. The values of the problem parameters are: \( P_0 = 1 \), \( n = 100 \), \( \sigma = 5 \), \( \tau = \pi / 20 \). Here \( \tau \) is time step of the difference scheme.

Figures 2a and 2b shows plots calculated at \( h = 10 \), with the solid curves corresponding to the values \( Q_0^{-1} = 0 \) and dashed lines to \( Q_0^{-1} = 0.2 \), while the vertical lines conform with arrivals of the fronts of P-, and Rayleigh’s waves at the pre-assigned point \((n, 1, 0)\): \( t_p = \sqrt{h^2 + n^2} / c_p \), \( t_R = h / c_p + n / c_R \), where \( c_p \) and \( c_R \) are defined by (7).

Comparison of the calculations performed with account of the internal friction and without it has shown that the internal friction action will decrease the amplitudes of block velocities and decelerate wave propagation.

Figures 2c and 2d shows the curves calculated at \( Q_0^{-1} = 0.2 \) and for different values of the parameter \( h \) (cavity depth) as follows: \( h = 20 \) (solid lines) \( h = 10 \) (dashed lines), \( h = 5 \) (dashed dotted lines).

Analysis of Figures 2c and 2d revealed that the amplitudes of radial and vertical displacement velocities of the rock-blocks on the surface of half-space in the P-wave depend negligibly on the depth of cavity \( h \), while in the Rayleigh wave, the amplitude of displacement velocities of the rock-blocks decays exponentially as the value of \( h \) (i.e. distance from the surface) increases. It can be seen that given the value of \( h \) and the distance from the point of dynamic impact, the amplitudes of the radial and vertical velocities of the rock-blocks in the P-wave can be both higher and lower, than in the Rayleigh wave.
Figure 2. Time- dependent radial (a), (c) and vertical (b), (d) velocities of rock-blocks displacement on the surface of the half-space: (a), (b) $h = 10$ and $Q_0^\prime = 0, 0.2$; (c), (d) $Q_0^\prime = 0.2$ and $h = 5, 10, 20$.

5. Conclusions
A 3D mathematical model of block-structured rock mass is proposed. The concept underlying this model suggests that the dynamic behavior of a block-structured medium can be approximately described as motion of rigid blocks due to the compliance of the inter-block layers (partings), and that deformation of the partings can be approximately described by a rheological model with two Maxwell elements and one Voigt element.

Using this model, the problem of the non-stationary “center of expansion” type loading impact on the surface of a cavity deepened in the block-structured half-space is solved numerically. It is shown that in the case of the surface of the block-structured medium, the wave process is largely contributed by low-frequency longitudinal (compressional) P-waves and Rayleigh pendulum waves. Depending on the depth of the cavity and the distance from the impact source, the amplitudes blocks velocities (radial and vertical) in the P-wave can be either higher or lower, than in the Rayleigh wave.

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