A SPARSE DOMINATION PRINCIPLE FOR ROUGH SINGULAR INTEGRALS

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ABSTRACT. We prove that bilinear forms associated to the rough homogeneous singular integrals

\[ T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^d} f(x \cdot y) \Omega \left( \frac{y}{|y|} \right) \frac{dy}{|y|^d} \]

where \( \Omega \in L^q(S^{d-1}) \) has vanishing average and \( 1 < q \leq \infty \), and to Bochner-Riesz means at the critical index in \( \mathbb{R}^d \) are dominated by sparse forms involving \((1,p)\) averages. This domination is stronger than the weak-\(L^1\) estimates for \( T_\Omega \) and for Bochner-Riesz means, respectively due to Seeger and Christ. Furthermore, our domination theorems entail as a corollary new sharp quantitative \( A_p \)-weighted estimates for Bochner-Riesz means and for homogeneous singular integrals with unbounded angular part, extending previous results of Hytönen-Roncal-Tapiola for \( T_\Omega \). Our results follow from a new abstract sparse domination principle which does not rely on weak endpoint estimates for maximal truncations.

1. INTRODUCTION AND MAIN RESULTS

Singular integral operators of Calderón-Zygmund type, which are a priori signed and non-local, can be dominated in norm [23], pointwise [8, 19, 22], or dually [2, 10, 11] by sparse averaging operators (forms), which are in contrast positive and localized. For \( 1 \leq p_1, p_2 < \infty \), we call \( \text{sparse} (p_1, p_2) \)-averaging form the bisublinear form

\[ \text{PSF}_{S; p_1, p_2}(f_1, f_2) := \sum_{Q \in S} |Q| \langle f_1 \rangle_{p_1} \langle f_2 \rangle_{p_2, Q}, \quad \langle f \rangle_{p, Q} := |Q|^{-\frac{1}{p}} \| f 1_Q \|_p, \]

associated to a (countable) sparse collection \( S \) of cubes of \( \mathbb{R}^d \). The collection \( S \) is \( \eta \)-sparse if there exist \( 0 < \eta \leq 1 \) (a number which will not play a relevant role) and measurable sets \( \{E_I : I \in S\} \) such that

\[ E_I \subset I, \quad |E_I| \geq \eta |I|, \quad I, J \in S, I \neq J \implies E_I \cap E_J = \emptyset. \]

In this article, we prove a sparse domination principle of type

\[ |\langle Tf_1, f_2 \rangle| \lesssim \sup_S \text{PSF}_{S; p_1, p_2}(f_1, f_2) \quad (1.1) \]

for singular integral operators \( T \) whose (possible) lack of kernel smoothness forbids the avenue exploited in [19, 24]. Our principle, summarized in Theorem C below, can be employed in a rather direct fashion to recover the best known, and sharp, sparse domination results for Dini and Hörmander type Calderón-Zygmund operators [3, 17, 19, 25].

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However, the main purpose of our work is to suitably extend (1.1) to the class of rough singular integrals introduced in the seminal paper of Calderón and Zygmund [4], and further studied, notably, in Duoandikoetxea-Rubio de Francia [14], Christ [6], Christ-Rubio de Francia [7] and Seeger [29]. Prime examples from this class include the rough homogeneous singular integrals on $\mathbb{R}^d$

$$T_{\Omega} f(x) = \text{p.v.} \int_{\mathbb{R}^d} f(x - y)\Omega \left(\frac{y}{|y|}\right) \frac{dy}{|y|^d},$$

with $\Omega \in L^q(S^{d-1})$ having zero average, as well as the critical Bochner-Riesz means in dimension $d$, defined by the multiplier operator

$$B_\delta f = \mathcal{F}^{-1} \left[\hat{f}(\cdot) \left(1 - |\cdot|^2\right)^{\delta}_{+}\right], \quad \delta = \frac{d - 1}{2}.$$

For the singular integrals (1.2) no sparse domination results were known prior to this article, although some quantitative weighted estimates were established in the recent works [17, 28]; see below for details. For the Bochner-Riesz means (1.3), the recent results of [1] and [5] are far from being optimal at the critical exponent.

The main difficulty encountered by previous approaches in this setting is the following: first, notice that an estimate of the type (1.1) is already stronger than the weak-$L^p$ bound for $T$. In particular, if $p_1 = 1$ then (1.1) recovers the weak-$L^1$ endpoint bound. On the other hand, the preexisting techniques for sparse domination [1, 2, 17, 19, 24] essentially rely on weak-$L^p$ estimates for a grand maximal truncation of the singular integral operator $T$. But those do not seem attainable in the context, for instance, of [29], as observed in [24]. In fact, the rough singular integrals we consider below are not known to satisfy such estimate for $p = 1$, and therefore a different approach is required in order to obtain the sparse bounds that we want.

As a corollary of our domination results, we obtain quantitative $A_p$-weighted estimates for homogeneous singular integrals (1.2) whose angular part belongs to $L^q(S^{d-1})$ for some $1 < q \leq \infty$. These are novel, and sharp, when $q < \infty$, while in the case $q = \infty$ we recover the best known result recently proved in [17] by other methods. Although our result for the Bochner-Riesz means (1.3) seemingly yields the best known quantitative $A_p$ estimates, we do not know whether our results are sharp in this case.

### 1.1 Main results

Our main results consist of estimates for the bilinear forms associated to $T_{\Omega}$ and $B_\delta$ by sparse operators involving $L^p$-averages. The formulation of our first theorem requires the Orlicz-Lorentz norms

$$\|\Omega\|_{L^{q,1}\log L(S^{d-1})} := \int_0^\infty t \log(e + t)|\{\theta \in S^{d-1} : |\Omega(\theta)| > t\}| \frac{1}{t^{\frac{d}{q}}} dt, \quad 1 \leq q < \infty.$$

**Theorem A.** There exists an absolute dimensional constant $C > 0$ such that the following holds. Let $\Omega \in L^1(S^{d-1})$ have zero average. Then for all $1 < t < \infty$, $f_1 \in L^t(\mathbb{R}^d)$, $f_2 \in L^t(\mathbb{R}^d)$ there holds

$$|\langle T_{\Omega} f_1, f_2 \rangle| \leq \frac{C p}{p - 1} \sup_{S} \text{PSF}_{S,1,p}(f_1, f_2) \left\|\Omega\|_{L^{q,1}\log L(S^{d-1})} \right\|_{L^{\infty}(S^{d-1})}, \quad 1 < q < \infty, \quad 1 < p < \infty.$$

**Remark 1.1.** To avoid Lorentz norms in the statement, one may recall the continuous embeddings $L^{q,p}(S^{d-1}) \hookrightarrow L^{q,1}\log L(S^{d-1}) \hookrightarrow L^q(S^{d-1})$ for all $1 \leq q < \infty$ and $\varepsilon > 0$. 

Theorem B. There exists an absolute dimensional constant \( C > 0 \) such that the following holds. For all \( 1 < t < \infty \), \( f_1 \in L^t(\mathbb{R}^d) \), \( f_2 \in L^t(\mathbb{R}^d) \), the critical Bochner-Riesz means (1.3) satisfy
\[
|\langle B_\delta f_1, f_2 \rangle| \leq \frac{Cp}{p-1} \sup_S \text{PSF}_{S,1,p}(f_1, f_2), \quad 1 < p < \infty.
\]

The weak-\( L^1 \) estimate for \( T_\Omega \) is the main result of [29], while the same endpoint estimate for (1.3) has been established in [6]. Theorems A and B recover such results; see Appendix B for a proof of this implication, which we include for future reference. This is not surprising as the localized estimates for (1.2), (1.3) which are needed to apply our abstract result are a distillation and an improvement of the microlocal techniques of [29] and of the previous works [6, 7], and of the oscillatory integral estimates of [6] respectively.

We reiterate that the commonly used techniques for sparse domination, which rely on the weak-\( L^1 \) estimate for the maximal truncation of the singular integral operator, fail to be applicable in the context of Theorem A as the maximal truncations of \( T_\Omega \) in (1.2) are not known to satisfy such estimate even when \( \Omega \in L^\infty(S^{d-1}) \) [15]. Our abstract Theorem C, whose statement is more technical and is postponed until Section 2, only relies on the uniform \( L^2 \) (or \( L^r \) for any \( r \)) boundedness of the truncated operators, and thus might be considered stronger than the approaches of the mentioned references. See Remark 2.5 for additional discussion on this point.

Theorems A and B entail as corollaries a family of quantitative weighted estimates.

Corollary A.1. If \( \Omega \) lies in the unit ball of \( L^{q,1} \log L(S^{d-1}) \) for some \( 1 < q < \infty \) and has zero average, we have the weighted norm inequalities
\[
\|T_\Omega\|_{L^t(w) \to L^t(w)} \leq C_{t,q}[w]_{A_t}^{\max\{1,\frac{1}{1-q}\}}, \quad q' < t < \infty.
\]

If furthermore \( \|\Omega\|_{L^\infty(S^{d-1})} \leq 1 \),
\[
\|T_\Omega\|_{L^t(w) \to L^t(w)} \leq C_t[w]_{A_t}^{\frac{1}{1-t}} \sup_{t,2} \quad 1 < t < \infty.
\]

Corollary B.1. Referring to (1.3), we have the weighted norm inequalities
\[
\|B_\delta\|_{L^t(w) \to L^t(w)} \leq C_t[w]_{A_t}^{\max\{t,2\}} \quad 1 < t < \infty.
\]

Proof of Corollaries A.1, B.1. To prove (1.4), applying Theorem A for \( p = q' \) (strictly speaking, to the adjoint of \( T_\Omega \)) yields that the bilinear form associated to \( T_\Omega \) is dominated by
\[
\sup_S \text{PSF}_{S,1,q',1}.
\]

The proof of the weighted estimate can then be found, for instance, in [2, Proposition 6.4]. We prove (1.5), and (1.6) follows via the same argument: below, \( C \) denotes a positive absolute constant which may vary between occurrences. Combining the inequality [12, Proposition 4.1]
\[
\langle f \rangle_{1+\varepsilon, Q} \leq \langle f \rangle_{1, Q} + C\varepsilon (M_{1+\varepsilon} f)_{1, Q},
\]
which is valid for all \( \varepsilon > 0 \), with the estimate of Theorem A for \( p = 1 + \varepsilon \) we obtain
\[
|\langle T_\Omega f_1, f_2 \rangle| \leq \frac{C}{\varepsilon} \sup_S \text{PSF}_{S,1,1}(f_1, f_2) + C \sup_S \text{PSF}_{S,1,1}(M_{1+\varepsilon} f_1, f_2), \quad \varepsilon > 0.
\]
The above display leads via standard reasoning \([9, 16, 27]\) to the chain of inequalities

\[
\|T\|_{L^t(w)\to L^t(w)} \leq C_t[w]_{A_t}^{\max\{1, \frac{1}{t}\}} \inf_{0<\varepsilon<t-1} \left( \frac{1}{\varepsilon} + \|M_{1+\varepsilon}\|_{L^t(w)\to L^t(w)} \right)
\]

\[
\leq C_t[w]_{A_t}^{\max\{1, \frac{1}{t}\}} \inf_{0<\varepsilon<t-1} \left( \frac{1}{\varepsilon} + [w]_{A_t}^{\frac{1}{1+\varepsilon}} \right) \leq C_t[w]_{A_t}^{\max\{1, \frac{1}{t}\}},
\]

and the proof is complete. \(\square\)

Our Corollary A.1 is a quantification of the weighted inequalities due to Watson \([33]\) and Duoandikoetxea \([13]\): if \(1 < q \leq \infty\) and \(\Omega \in L^q(S^{d-1})\) then

\[
w \in A_{\frac{q}{q'}} \quad q' \leq t < \infty, \ t \neq 1,
\]

\[
\bigg\{ \begin{array}{c}
w^{\frac{1}{1+\varepsilon}} \in A_{\frac{q}{q'}} \quad 1 < t \leq q, \ t \neq \infty, \\
w^{q'} \in A_t \quad 1 < t < \infty. 
\end{array} \bigg\} \implies \|T_\Omega\|_{L^t(w)\to L^t(w)} < \infty.
\]

Estimate (1.5) was first established by Hytönen, Roncal and Tapiola \([17]\) via a different two-step technique involving sparse domination for Dini-type kernels, a Littlewood-Paley decomposition along the lines of \([7]\) and interpolation with change of measure. In \([28]\), these ideas were extended to obtain \(A_1\) estimates for \(T_\Omega\) and commutators of \(T_\Omega\) and BMO symbols. At this time, we do not know whether the power of the Muckenhoupt constant in (1.5) is sharp.

Qualitative \(A_p\)-bounds for critical Bochner-Riesz means are classical \([30]\); see also \([32]\). On the other hand, Corollary B.1 seems to be the first quantitative \(A_p\) estimate for \(B_\delta\). We do not know whether the power of the \(A_p\) constant in (1.6) is sharp; the construction in \([26, \text{Corollary} 3.1]\) shows that the optimal power \(\alpha_p\) must obey \(\alpha_p \geq \max\{1, 1/(p-1)\}\). The article \([1]\) contains sparse domination estimates and weighted inequalities for the supercritical regime \(0 < \delta' < \delta\) which are not informative in the critical case. An extension of our methods to the supercritical cases will appear in forthcoming work.

Finally, we mention that our argument for (1.5) and (1.6) shows that improvements of such power in Corollaries A.1 and B.1 are tied to the blowup rate as \(\rho \to 1^+\) of the main estimate of Theorems A and B.

1.2. A remark on the proof and plan of the article. Theorems A and B fall under the scope of the same abstract result, Theorem C, which is stated and proved in Section 2. Theorem C is obtained by means of an iterative scheme reminiscent of the arguments used in \([10]\) by three of us to prove a sparse domination estimate for the bilinear Hilbert transform, and later adapted to dyadic and continuous Calderón-Zygmund singular integrals in \([11]\). At each iteration, a Calderón-Zygmund type decomposition is performed, and the operator itself is decomposed into small scales (scales falling within the exceptional set) which will be estimated at subsequent steps of the iteration, and large scales. The action of the large scales on the good parts is controlled by means of the uniform \(L'\)-bound for the truncations of \(T\). The contribution of the bad, mean zero part under the large scales of the operator is then controlled by means of suitably localized estimates relying on the constant-mean zero type cancellation.

We emphasize that the present work shares a perspective based on bilinear forms with other recent papers: \([18]\) by Krause and Lacey and \([21]\) by Lacey and Spencer. The notable difference is that these references, dealing with oscillatory and random discrete singular integrals,
use (dilation) symmetry breaking and $TT^*$, rather than constant-mean zero, as the principal
cancellation mechanisms, in accordance with the oscillatory nature of their objects of study.

Section 3 contains localized estimates for kernels of Dini and Hörmander type which, be-
sides being of use in later arguments, allow us to reprove the optimal sparse domination results
for these classes: we send to Subsection 3.2 for the statements. In Sections 4 and 5 we provide
the necessary localized estimates for Theorems A and B respectively. The estimates of Section 4
are a delicate strengthening of the microlocal arguments of [29]. The proof of Theorem B,
a re-elaboration along the same lines of the arguments of [6], is carried out in Section 5. Al-
though we find hard to believe that these techniques can be sharpened towards the stronger
localized $(1, 1)$ estimate, we have no explicit counterexample for this possibility.

**Notation.** As customary, $q' = \frac{q}{q-1}$ denotes the Lebesgue dual exponent to $q \in (1, \infty)$, with
the usual extension $1' = \infty, \infty' = 1$. We denote the center and the sidelength of a cube $Q \in \mathbb{R}^d$
by $c_Q$ and $\ell(Q)$ respectively. We will also adopt the shorthand $s_Q = \log_2 \ell(Q)$. We write
$$M_p(f)(x) = \sup_{Q \subset \mathbb{R}^d} |f|_p 1_Q(x)$$
for the $p$-Hardy Littlewood maximal function. The positive constants implied by the almost
inequality sign $\lesssim$ may depend (exponentially) on the dimension $d$ only and may vary from
line to line without explicit mention.

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## 2. A sparse domination principle

This section is dedicated to the statement and proof of our sparse domination principle,
Theorem C.

### 2.1. The main structural assumptions.**
Our structural assumptions in Theorem C will be
the following. Let $1 < r < \infty$ and $\Lambda$ be an $L'(\mathbb{R}^d) \times L'(\mathbb{R}^d)$-bounded bilinear form whose kernel
$K = K(x, y)$ coincides with a function away from the diagonal $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. More
precisely, whenever $f_1 \in L'(\mathbb{R}^d)$, $f_2 \in L'(\mathbb{R}^d)$ are compactly and disjointly supported
$$\Lambda(f_1, f_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y) f_1(y) f_2(x) dy dx$$

with absolute convergence of the integral. We assume that there exists $1 < q \leq \infty$ such that the kernel $K$ of $\Lambda$ admits the decomposition
\[ K(x, y) = \sum_{s \in \mathbb{Z}} K_s(x, y), \]
where
\[ \supp K_s \subset \{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x - y \in A_s \}, \quad A_s := \{ z \in \mathbb{R}^d : 2^{s-2} < |z| < 2^s \}. \]
Let
\[ [K]_{0, q} := \sup_{s \in \mathbb{Z}} 2^d \sup_{x \in \mathbb{R}^d} (\|K_s(x, x + \cdot)\|_q + \|K_s(x + \cdot, x)\|_q) < \infty. \]
Further, we assume that the truncated forms associated to the above decomposition by
\[ \Lambda^\nu_{\mu}(h_1, h_2) := \int_{\mu < s \leq v} K_s(x, y)h_1(y)h_2(x)\ dydx \quad \mu, v \in \mathbb{Z} \cup \{-\infty, \infty\} \]
satisfy
\[ C_T(r) := \sup_{\mu < v} \left( \|\Lambda^\nu_{\mu}\|_{L^r(\mathbb{R}^d) \times L'^r(\mathbb{R}^d) \to C} \right) < \infty. \]

**Remark 2.1.** Under the assumptions (SS) and (T), a standard limiting argument [31, Par. I.7.2] yields that
\[ \Lambda(f_1, f_2) = \langle mf_1, f_2 \rangle + \lim_{v \to \infty} \Lambda^\nu_{\mu}(f_1, f_2) \]
for some $m \in L^\infty(\mathbb{R}^d)$, whenever $f_1 \in L'(\mathbb{R}^d)$, $f_2 \in L'((\mathbb{R}^d)$. It is not hard to see [20, Lemma 4.7] that
\[ |\langle mf_1, f_2 \rangle| \leq \|m\|_\infty \sup_{s} \text{PSF}_{S,1,1}(f_1, f_2) \]
so that for the purpose of our Theorem C below we may assume that $m = 0$ in the above equality. For this reason, when $\mu = -\infty$ or $\nu = \infty$ or both, we are allowed to omit the subscript or superscript in (2.1) and simply write $\Lambda^\nu$ or $\Lambda^\mu$ or $\Lambda$. Also, when $\mu \leq \nu$, the summation in (2.1) is void, so that $\Lambda^\nu_{\mu} \equiv 0$.

### 2.2. Localized spaces over stopping collections.
A further condition in our abstract theorem will involve local norms associated to stopping collections of (dyadic) cubes. Throughout the article, by dyadic cubes we refer to the elements of any fixed dyadic lattice $D$ in $\mathbb{R}^d$.

Let $Q \in D$ be a fixed dyadic cube in $\mathbb{R}^d$. A collection $Q \subset D$ of dyadic cubes is a stopping collection with top $Q$ if the elements of $Q$ are pairwise disjoint and contained in $3Q$.

\[ L, L' \in Q, \ L \cap L' \neq \emptyset \implies L = L', \quad L \in Q \implies L \subset 3Q \]
and enjoy the further separation properties
\[ L, L' \in Q, \ |s_L - s_{L'}| \geq 8 \implies 7L \cap 7L' = \emptyset, \]
and
\[ \bigcup_{L \in Q : 3L \cap Q \neq \emptyset} 9L \subset \bigcup_{L \in Q} L =: \text{sh}Q. \]
the notation $\text{sh} Q$ for the union of the cubes in $Q$ will also be used below. For $1 \leq p \leq \infty$, define $\mathcal{Y}_p(Q)$ to be the subspace of $L^p(\mathbb{R}^d)$ of functions satisfying

$$\supp h \subset 3Q, \quad \infty > \|h\|_{\mathcal{Y}_p(Q)} := \begin{cases} \max \left\{ \|h1_{\mathbb{R}^d \setminus \text{sh} Q}\|_{\infty}, \sup_{L \in Q} \inf_{x \in L} M_p h(x) \right\} & p < \infty \\ \|h\|_{\infty} & p = \infty \end{cases}$$

where we wrote $\hat{L}$ for the (non-dyadic) $2^5$-fold dilate of $L$. We also denote by $\mathcal{X}_p(Q)$ the subspace of $\mathcal{Y}_p(Q)$ of functions satisfying

$$b = \sum_{L \in Q} b_L, \quad \supp b_L \subset L.$$ 

Furthermore, we write $b \in \hat{X}_p(Q)$ if

$$b \in \mathcal{X}_p(Q), \quad \int_L b_L = 0 \quad \forall L \in Q.$$ 

We will use the notation $\|b\|_{\mathcal{X}_p(Q)}$ for $\|b\|_{\mathcal{Y}_p(Q)}$ when $b \in \mathcal{X}_p(Q)$, and similarly for $b \in \hat{X}_p(Q)$. When the stopping collection $Q$ is clear from the context or during proofs we may omit $(Q)$ from the subscript and simply write $\| \cdot \|_{\mathcal{Y}_p}$ or $\| \cdot \|_{\mathcal{X}_p}$.

**Remark 2.2** (Calderón-Zygmund decomposition). There is a natural Caldéron-Zygmund decomposition associated to stopping collections. Observe that if $Q$ is a stopping collection there holds

$$\sup_{L \in Q} (h)_{p, L} \leq 2^{5d} \|h\|_{\mathcal{Y}_p(Q)}.$$ 

Therefore, we may decompose $h \in \mathcal{Y}_p(Q)$ as

$$h = g + b, \quad b = \sum_{L \in Q} b_L, \quad b_L = \left( h - \frac{1}{|L|} \int_L h(x) \, dx \right) 1_L$$

such that

$$\|g\|_{\mathcal{Y}_p(Q)} \leq 2^{5d} \|h\|_{\mathcal{Y}_p(Q)}, \quad b \in \hat{X}_p(Q), \quad \|b\|_{\hat{X}_p(Q)} \leq 2^{5d+1} \|h\|_{\mathcal{Y}_p(Q)}.$$ 

These are nothing else but the usual properties of the Caldéron-Zygmund decomposition rewritten in our context.

### 2.3. The statement

Before stating our result, we introduce the notation

$$\Lambda_{Q, \mu, \nu}(h_1, h_2) := \Lambda_{\mu}^{\min\{\xi_Q, \nu\}}(h_1 1_Q, h_2) = \Lambda_{\mu}^{\min\{\xi_Q, \nu\}}(h_1 1_Q, h_2 1_{3Q})$$

for all dyadic cubes $Q$; the last equality in (2.4) is a consequence of the assumptions on the support of $K_s$ in (SS). Furthermore, given a stopping collection $Q$ with top $Q$, we define the truncated forms

$$\Lambda_{Q, \mu, \nu}(h_1, h_2) := \Lambda_{Q, \mu, \nu}(h_1, h_2) - \sum_{L \in Q} \Lambda_{L, \mu, \nu}(h_1, h_2) = \Lambda_{Q, \mu, \nu}(h_1 1_Q, h_2 1_{3Q}).$$

Again, the last equality is due to the support of $K_s$ in (SS). A further consequence of assumptions (SS), (T) is that the forms $\Lambda_{Q, \mu, \nu}$ satisfy uniform bounds on $\mathcal{Y}_r(Q) \times \mathcal{Y}_r(Q)$. 
Lemma 2.3. There exists a positive absolute constant \( \Theta \) such that

\[
|\Lambda_{Q,\mu,v}(h_1, h_2)| \leq 2^{3d} C_T(r) |Q| \|h_1\|_{y,(Q)} \|h_2\|_{y,(Q)}
\]

uniformly over all \( \mu, v \), all dyadic cubes \( Q \) and stopping collections \( Q \) with top \( Q \).

Proof. We may estimate the first term in the definition (2.5) as follows:

\[
(2.6) \quad |\Lambda_{Q,\mu,v}(h_1, h_2)| \leq C_T(r) \|h_1\|_{1,Q} \|h_2\|_{1,3Q} \|r\| \leq C_T(r) |Q| \|h_1\|_{y,(Q)} \|h_2\|_{y,(Q)}.
\]

Further, using the support condition in (2.4) with \( L \) in place of \( Q \) and the disjointness property (2.2) in the last step, we obtain

\[
\sum_{L \in Q: L \subseteq Q} |\Lambda_{L,\mu,v}(h_1, h_2)| = \sum_{L \in Q: L \subseteq Q} |\Lambda_{L,\mu,v}(h_1 1_L, h_2 1_{3L})| \leq C_T(r) \sum_{L \in Q: L \subseteq Q} \|h_1 1_L\|_{r} \|h_2 1_{3L}\|_{r} \|L\| \leq C_T(r) |Q| \|h_1\|_{y,(Q)} \|h_2\|_{y,(Q)}.
\]

The proof of the lemma is thus completed by combining (2.6) with the last display. \( \square \)

Our main theorem hinges upon estimates which are modified versions of the one occurring in Lemma 2.3, when one of the two arguments of \( \Lambda_{Q,\mu,v} \) belongs to \( X \)-type localized spaces.

Theorem C. There exists a positive absolute constant \( \Theta \) such that the following holds. Let \( \Lambda \) be a bilinear form satisfying (SS) and (T) above. Assume that there exist \( 1 \leq p_1, p_2 < \infty \) and a positive constant \( C_L \) such that the estimates:

\[
(\text{L}) \quad |\Lambda_{Q,\mu,v}(b, h)| \leq C_L |Q| \|b\|_{\dot{X}_{p_1}(Q)} \|h\|_{y_{p_2}(Q)},
\]

hold uniformly over all \( \mu, v \in \mathbb{Z} \), all dyadic lattices \( D \), all \( Q \in D \) and all stopping collections \( Q \subset D \) with top \( Q \). Then the estimate

\[
(2.7) \quad \sup_{\mu, v} \left| \Lambda_{p}^v(f_1, f_2) \right| \leq 2^{\Theta d} \left[ C_T(r) + C_L \right] \sup_{S} \text{PSF}_{S, p_1, p_2}(f_1, f_2),
\]

holds for all \( f_j \in L^p(\mathbb{R}^d) \) with compact support, \( j = 1, 2 \).

Remark 2.4. By the limiting argument of Remark 2.1, the conclusion (2.7) entails that

\[
(2.8) \quad |\Lambda(f_1, f_2)| \leq 2^{\Theta d} \left[ C_T(r) + C_L \right] \sup_{S} \text{PSF}_{S, p_1, p_2}(f_1, f_2)
\]

when \( f_1, f_2 \in L^p(\mathbb{R}^d) \) with compact support (say). If we know that \( \Lambda \) extends boundedly to \( L^t(\mathbb{R}^d) \times L^t(\mathbb{R}^d) \) for some \( 1 < t < \infty \), another simple limiting argument using the dominated convergence theorem extends (2.8) to all \( f_1 \in L^t(\mathbb{R}^d), f_2 \in L^t(\mathbb{R}^d) \). It is in this last form that Theorem C will be applied to deduce Theorems A and B.

Remark 2.5 (A comparison between sparse domination principles). Theorem C identifies rather clearly the conditions needed for sparse domination of a kernel operator \( T \), namely the adjoint of the bilinear form \( \Lambda \). Condition (L) is a localized reformulation of the constant-mean zero cancellation around which \( L^p, p \neq 2 \) Calderón-Zygmund theory revolves, and it is essentially a strengthening of the weak-\( L^p \) estimate for \( T \) (\( j = 1 \)) and its adjoint (\( j = 2 \)). Further, our assumption of uniform \( L' \)-boundedness of the truncations in (T) is much tamer.
than requiring $L'$-boundedness of the maximal truncations of $T$. In fact, our theorem can be applied even when no estimates for maximal truncations of $T$ are known.

Of course the exponents $p_j$ enter the sparse domination estimate (2.7), while the exponent $r$ occurring in (T) does not. This is in contrast with the other sparse domination principles occurring in the literature. For instance, in [24, Theorem 4.2], a sparse domination of type (1.1) with exponents $(r, 1)$ is obtained for operators $T$ whose grand maximal function

$$
M_T f(x) := \sup_{Q \ni x} \sup_{y \in Q} |T(f \mathbf{1}_{2^d Q})(y)|
$$

has the weak-$L'$ bound for some $r \geq 1$. Notice that $M_T$ may be as large as the maximal truncation of $T$.

A further comparison can be drawn with the abstract result of [2], which is a sparse domination principle for non-integral singular operators. The off-diagonal estimate assumption [2, Theorem 1.1(b)] is a clear counterpart of (SS), while the maximal truncation assumption [2, Theorem 1.1(c)] is the non-kernel analogue of the grand maximal function from [24]. It would be interesting to investigate whether, in the non-kernel setting of [2], an assumption in the vein of (L) can be used instead.

**Remark 2.6** (The essence of (L)). Let $Q$ be a stopping collection with top $Q$. When $b$ belongs to an $X_{\alpha}(Q)$-type space, the forms

$$(b, h) \mapsto \Lambda_{Q,\mu,v}(b, h), \quad (b, h) \mapsto \Lambda_{Q,\mu,v}(h, b)$$

have a much more familiar representation, which is what allows verification of assumption (L) in practice. By rephrasing the definition, when $b \in X_{\delta}(Q)$ is supported on $Q$ (which we can assume with no restriction) we have the equality

$$
\Lambda_{Q,\mu,v}(b, h) = \sum_{j \geq 1} \int_{\mu < s \leq \min\{s_Q, v\}} K_s(x, y) b_{s-j}(y) h(x) \, dy \, dx.
$$

where

$$b_s := \sum_{L \in Q \cap 2^j Q \neq \emptyset} b_L.$$ 

This notation will be used throughout the paper: see for instance (2.10) below. Furthermore, if $q$ is the exponent occurring in (SS), $h \in Y_{q'}(Q)$, and $b \in X_{q'}(Q)$, then $\Lambda_{Q,\mu,v}(h, b)$ is essentially self-adjoint up to a tolerable error term. Namely, if $h$ is supported on $Q$ (which we can also always assume), there holds

$$
\Lambda_{Q,\mu,v}(h, b) = \left( \sum_{j \geq 1} \int_{\mu < s \leq \min\{s_Q, v\}} K_s(x, y) b_{s-j}(y) h(x) \, dy \, dx \right) + V_Q(h, b)
$$

where

$$b_{\text{in}} = \sum_{L \in Q \cap 2^j Q \neq \emptyset} b_L.$$
is a truncation of \( b \) and thus also belongs to \( \mathcal{X}_q(Q) \) with \( \|b^{in}\|_{\mathcal{X}_q(Q)} \leq \|b\|_{\mathcal{X}_q(Q)} \), and the remainder \( V_Q(h, b) \) satisfies

\[
|V_Q(h, b)| \leq 2^{\theta d}[K]_{a,q}|Q|\|h\|_{Y_q(V)}\|b\|_{\mathcal{X}_q(Q)}
\]

for a suitable positive absolute constant \( \theta \). The representation (2.10)-(2.11) is a simple consequence of the structure of \( b \in \mathcal{X}_q(Q) \) and of the separation properties (2.2), (2.3). We provide the necessary details for (2.10)-(2.11) in Appendix A at the end.

2.4. Proof of Theorem C. Given a form \( \Lambda \) satisfying the assumptions of Theorem C, \( \mu < \nu \in \mathbb{Z} \) and \( f_j \in L^p_j(\mathbb{R}^d) \), \( j = 1, 2 \), with compact support, we will construct a sparse collection \( S \) of cubes of \( \mathbb{R}^d \) such that

\[
|\Lambda^s(\mu)(f_1, f_2)| \leq 2^{\theta d}C \sum_{Q \in S} |Q| \langle f_1 \rangle_{p_1, Q} \langle f_2 \rangle_{p_2, Q}
\]

where \( C \) is the expression within the square brackets in the conclusion of Theorem C. Here and below, we denote by \( \Theta \) a suitably large positive absolute constant which will be chosen during the course of the proof. Within this proof, we will also denote by \( \theta \) positive absolute constants which belong to \([2^{-\theta} \Theta, 2^{-\theta} \Theta]\) and may differ at each occurrence. As the assumptions of Theorem C are stable if we replace \( \Lambda \) with \( \Lambda^s(\mu) \), we can work under the assumption that that \( K_s = 0 \) for all \( s \notin (\mu, \nu] \) and thus drop \( \mu, \nu \) from the notations (2.4), (2.5).

The proof of (2.12) is iterative and is carried out in Subsection 2.5 below. Here, we enunciate the main estimate for the form \( \Lambda^s(\mu) \) from (2.4) in terms of stopping collection norms.

Lemma 2.7. Let \( Q \) be a fixed dyadic cube in \( \mathbb{R}^d \) and \( Q \) be a stopping collection with top \( Q \). Then

\[
|\Lambda^s(\mu)(h_11_Q, h_21_{3Q})| \leq 2^{\theta d}C|Q|\|h_1\|_{Y_{p_1}(Q)}\|h_2\|_{Y_{p_2}(Q)} + \sum_{L \subseteq Q} |\Lambda^s(h_11_L, h_21_{3L})|
\]

Proof. We are free to assume that \( \text{supp} \ h_1 \subset Q \), \( \text{supp} \ h_2 \subset 3Q \) for simplicity of notation. For \( j = 1, 2 \), construct the Calderón-Zygmund decomposition of \( h_j \) with respect to the family \( Q \) as described in Remark 2.2, that is

\[
h_j = g_j + b_j, \quad b_j = \sum_{L \subseteq Q} b_{jL}, \quad b_{jL} := \left(h_j - \frac{1}{|L|} \int_L h_j(x) \, dx \right)1_L,
\]

The Calderón-Zygmund properties in this context are, for \( j = 1, 2 \),

\[
\|g_j\|_{Y_{p_j}} \lesssim \|h_j\|_{Y_{p_j}}, \quad \|b_j\|_{\mathcal{X}_{p_j}} \lesssim \|h_j\|_{Y_{p_j}}.
\]

Using the definition (2.5), we decompose on our way to (2.13)

\[
\Lambda^s(h_1, h_2) = \Lambda_Q(h_1, h_2) + \sum_{L \subseteq Q} \Lambda^s(h_11_L, h_2)
\]

\[
= \Lambda_Q(g_1, g_2) + \Lambda_Q(b_1, g_2) + \Lambda_Q(g_1, b_2) + \Lambda_Q(b_1, b_2) + \sum_{L \subseteq Q} \Lambda^s(h_11_L, h_21_{3L})
\]

(2.14)
The last sum on the last right hand side is estimated by the sum appearing on the right hand side of (2.13). We are left with estimating the first four terms in the last line of (2.14). The leftmost is controlled by the estimate of Lemma 2.3:

\[ |\Lambda_Q(g_1, g_2)| \leq C_T |Q| \|y_1\| \|y_2\| \|y_s\| \leq C |Q| \|h_1\| \|h_2\| \|y_p\| . \]

The second term is handled by appealing to assumption associated to a cube \( Q \) and a pair of functions \( f_1, f_2 \), where the second estimate follows from the Calderón-Zygmund properties above. The third is also estimated by appealing to (L), as

\[ |\Lambda_Q(g_1, b_2)| \leq C_L |Q| \|y_1\| \|b_2\| \|y_p\| \leq C |Q| \|h_1\| \|h_2\| \|y_p\| , \]

where the final inequality follows again from the Calderón-Zygmund estimates. The proof of Lemma 2.7 is thus complete. \( \square \)

2.5. Proof of (2.12). The proof is obtained by means of the iterative procedure described below.

**Preliminaries.** We will produce stopping collections iteratively, by suitable Whitney decompositions of unions of sets

\[ E_Q = \left\{ x \in 3Q : \max_{j=1,2} \frac{M_{f_j}(1_{3Q})(x)}{\langle f_j \rangle_{p_j,3Q}} > 2^{\Theta} \right\} \]

associated to a cube \( Q \) and a pair of functions \( f_1, f_2 \). We notice that

\[ E_Q \subset 3Q, \quad |E_Q| \leq 2^{-\Theta} |Q| ; \]

the measure estimate is a consequence of the maximal theorem, and holds provided \( \Theta \) is chosen sufficiently large. In this proof, we say that two dyadic cubes \( L, L' \) are neighbors, and write \( L \sim L' \), if

\[ 7L \cap 7L' \neq \emptyset, \quad |s_L - s_{L'}| < 8. \]

The separation condition (2.3) tells us that if the 7-fold dilates of two cubes \( L, L' \) belonging to the same stopping collection intersect nontrivially, then \( L, L' \) must be neighbors. We also recall the notation \( \tilde{L} \) for the 2\(^5\)-fold dilate of \( L \).

**Initialize:** let \( f_j \in L^p(\mathbb{R}^d), \ j = 1, 2, \) with compact support be fixed. By suitably choosing the dyadic lattice \( \mathcal{D} \), we may find \( Q_0 \in \mathcal{D} \) such that \( \text{supp} \ f_1 \subset 3Q_0 \), \( \text{supp} \ f_2 \subset 3Q_0 \) and \( s_{Q_0} \) is larger than the largest nonzero scale occurring in the kernel. Then set \( S_0 = \{Q_0\}, \ E_0 = 3Q_0 \), and define referring to (2.15)

\[ E_1 := E_{Q_0}, \]

\[ S_1 := \text{maximal cubes } L \in \mathcal{D} \text{ such that } 9L \subset E_1. \]
Notice that the following properties are satisfied:

\[(2.17)\]  
L ∈ \(S_1\) are a pairwise disjoint collection,

\[(2.18)\]  
\[E_1 = \bigcup_{L \in S_1} L = \bigcup_{L \in S_1} 9L \subset E_0, \quad |Q_0 \setminus E_1| \geq \left(1 - 2^{-d\theta}\right)|Q_0|,\]

\[(2.19)\]  
\[L, L' \in S_1, 7L \cap 7L' \neq \emptyset \implies L \sim L'.\]

Properties (2.17) and the first part of (2.18) are by construction, while the second part of (2.18) follows from the estimate of (2.16). For (2.19) suppose instead that \(7L \cap 7L'\) is not empty when \(s_L \leq s_{L'} - 8\). By the relation between the sidelengths it follows that \(\hat{L} \subset 9L'\), which implies that the 9-fold dilate of the dyadic parent of \(L\) is contained in \(9L'\) as well, contradicting the maximality of \(L\). By virtue of (2.17)–(2.19), \(Q_1(Q_0) := S_1\) is a stopping collection with top \(Q_0\); compare with (2.2), (2.3). The first property in (2.18) guarantees that

\[
\sup_{x \in \partial h Q_1(Q_0)} |f_j(x)| \leq 2^{\Theta d}(f_j)_{p_j, 3Q_0}.
\]

Further, by the maximality condition on \(L \in S_1\), it follows that

\[
\sup_{L \in Q_1(Q_0)} \inf_{L \subseteq \partial h Q_1(Q_0)} M_{p_j}(f_j 1_{3Q_0}) \leq 2^{\Theta d}(f_j)_{p_j, 3Q_0}
\]

for \(j = 1, 2\). The last two inequalities tell us that

\[
\|f_j\|_{Y_{p_j}(Q_0)} \leq 2^{\Theta d}(f_j)_{p_j, 3Q_0}, \quad j = 1, 2.
\]

Applying (2.13) to the stopping collection \(Q_1(Q_0)\), and \(h_1 = f_1, h_2 = f_2\) we obtain

\[
|\Lambda(f_1, f_2)| = |\Lambda^{s_{Q_0}}(f_1 1_{Q_0}, f_2 1_{3Q_0})| \\
\leq 2^{\Theta d}C|Q_0|\langle f_1 \rangle_{p_1, 3Q_0}\langle f_2 \rangle_{p_2, 3Q_0} + \sum_{L \subseteq Q_0} |\Lambda^{s_L}(f_1 1_L, f_2 1_{3L})|.
\]

The obtained properties (2.17)–(2.19) and estimate (2.20) are the \(\ell = 1\) case of the induction assumption in the inductive step below.

**Inductive step:** Suppose inductively collections \(S_\ell, 0 \leq \ell \leq k\) and sets \(E_\ell, 1 \leq \ell \leq k\) have been constructed, with the properties that for all \(1 \leq \ell \leq k\)

\[(2.21)\]  
L ∈ \(S_\ell\) are a pairwise disjoint collection,

\[(2.22)\]  
\[E_\ell = \bigcup_{L \in S_\ell} L = \bigcup_{L \in S_\ell} 9L \subset E_{\ell-1}, \quad |Q \setminus E_\ell| \geq \left(1 - 2^{-\theta d}\right)|Q| \quad \forall Q \in S_{\ell-1},\]

\[(2.23)\]  
\[L, L' \in S_\ell, 7L \cap 7L' \neq \emptyset \implies L \sim L'.\]

Suppose also that if \(T_{k-1} = S_0 \cup \cdots \cup S_{k-1}\), the estimate

\[(2.24)\]  
\[|\Lambda(f_1, f_2)| \leq 2^{\Theta d}C\sum_{R \in T_{k-1}} |R|\langle f_1 \rangle_{p_1, 3R}\langle f_2 \rangle_{p_2, 3R} + \sum_{Q \in S_k} |\Lambda^{s_Q}(f_1 1_Q, f_2 1_{3Q})|.
\]
has been shown to hold. At this point define
\[ E_{k+1} := \bigcup_{Q \in S_k} E_Q, \]
\[ S_{k+1} := \text{maximal cubes } L \in D \text{ such that } 9L \subset E_{k+1}, \]
\[ Q_{k+1}(Q) = \{ L \in S_{k+1} : L \subset 3Q \}, \quad Q \in S_k. \]

Property (2.21), together with the first property in (2.22), as \( E_Q \subset 3Q \subset E_k \), and (2.23), via the same reasoning we used for (2.19), now hold for \( \ell = k + 1 \) as well. Let now \( Q \in S_k \). Property (2.23) with \( \ell = k \) implies that
\[ 3Q \cap E_{k+1} \subset \bigcup_{Q' \in S_k : Q' \sim Q} E_{Q'}. \]

Therefore, we learn that
\[ |Q \cap E_{k+1}| \leq |3Q \cap E_{k+1}| \leq \sum_{Q' \in S_k : Q' \sim Q} |E_{Q'}| \leq 2^{-\theta d} |Q| \]  
by applying for each \( Q' \in S_k \) with \( Q' \sim Q \) the estimate of (2.16), and observing that the cardinality of \( \{ Q' \in D : Q' \sim Q \} \) is bounded by an absolute dimensional constant, and \( |Q|, |Q'| \) are comparable, again up to an absolute dimensional constant. From the above display we obtain the second part of (2.22) for \( \ell = k + 1 \). Moreover, one observes that if \( L \in S_{k+1} \) with \( L \cap 3Q \neq \emptyset \), then by virtue of property (2.25), \( L \) must be significantly shorter than \( Q \) and thus contained in one of the \( 3^d \) translates of the dyadic cube \( Q \) whose union covers \( 3Q \). Namely, we have the equality
\[ Q_{k+1}(Q) = \{ L \in S_{k+1} : L \cap 3Q \neq \emptyset \} \]
which also entails the last equality in
\[ \bigcup_{L \in Q_{k+1}(Q) : 3L \cap 2Q \neq \emptyset} 9L \subset \bigcup_{L \in S_{k+1} : L \cap 3Q \neq \emptyset} L = \bigcup_{L \in Q_{k+1}(Q)} L = \text{sh}Q_{k+1}(Q) \]
as the set in the first left hand side of the last display is contained in \( 3Q \) and (2.22) holds for \( \ell = k + 1 \). Comparing with (2.2), (2.3), the discussion above entails that \( Q_{k+1}(Q) \) is a stopping collection with top \( Q \) and such that \( E_Q \subset \text{sh}Q_{k+1}(Q) \), so that
\[ \sup_{x \notin \text{sh}Q_{k+1}(Q)} |f_j 1_{2Q}(x)| \leq 2^{\frac{\theta d}{2}} \langle f_j \rangle_{p_j, 3Q}. \]
Furthermore, for \( j = 1, 2 \)
\[ \sup_{L \in Q_{k+1}(Q)} \inf_{\tilde{L}} M_{p_j}(f_j 1_{3Q}) \leq 2^{\frac{\theta d}{2}} \langle f_j \rangle_{p_j, 3Q}, \]
otherwise the 9-fold dilate of the dyadic parent of some \( L \in Q_{k+1}(Q) \) would be contained in \( E_Q \) and thus in \( E_{k+1} \), contradicting the maximality of such \( L \). Therefore
\[ \|f_j 1_{3Q}\|_{p_j(Q_{k+1}(Q))} \leq 2^{\frac{\theta d}{2}} \langle f_j \rangle_{p_j, 3Q}, \quad j = 1, 2, \]
and we may apply (2.13) to each $Q \in S_k$ summand in (2.24), with $h_1 = f_1$, $h_2 = f_2$ and obtain

$$|\Lambda^Q(f_1 1_Q, f_2 1_{3Q})| \leq 2^{Qd} C |Q| \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} + \sum_{L \in Q_{k+1}(Q); L \subset Q} |\Lambda^L(f_1 1_L, f_2 1_{3L})|$$

$$= 2^{Qd} C |Q| \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} + \sum_{L \in S_{k+1}; L \subset Q} |\Lambda^L(f_1 1_L, f_2 1_{3L})|.$$

As $Q \in S_k$ are pairwise disjoint, see (2.21), summing over $Q \in S_k$, writing $T_k = S_0 \cup \cdots \cup S_k$ and combining the resulting estimate with (2.24), we arrive at

$$|\Lambda(f_1, f_2)| \leq 2^{Qd} C \sum_{Q \in T_k} |Q| \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} + \sum_{L \in S_{k+1}} |\Lambda^L(f_1 1_L, f_2 1_{3L})|$$

that is, (2.24) with $k$ replaced by $k + 1$. This, together with the previously obtained (2.21), (2.22) and (2.23) for $\ell = k + 1$, completes the current iteration.

Termination: a consequence of our construction is that $\sigma_k := \max\{s_Q : Q \in S_k\} \leq s_{Q_0} - \delta k$. The algorithm terminates when $k = K$, where $K$ is such that $\sigma_K$ is strictly less than the minimal nonzero scale in the kernel. For $k = K$ in (2.24) the second sum on the right hand side vanishes identically and we have obtained the estimate (2.12) by setting $T := T_{K-1}$ and $S := \{3Q : Q \in T\}$. We see that the collection $T$, and thus the collection of the dilates $\mathcal{S}$, are sparse by simply observing that the sets

$$F_Q := Q \setminus E_{k+1}, \quad Q \in S_k$$

are pairwise disjoint for $Q \in T$ and have measure larger than $(1 - 2^{-d\delta})|Q|$, as can be seen from (2.22).

3. Localized estimates for Dini and Hörmander-type kernels

In the first part of this section, we state and prove a family of localized estimates, of the type occurring in condition (L) of Theorem C, for kernels falling within the scope of (SS) and possessing additional smoothness properties, of Dini or Hörmander type. These estimates and their proof are a reformulation of the classical inequalities intervening in the proof of the weak-$L^1$ bound for Calderón-Zygmund operators (see, for example, [31, Chapter 1]). We choose to provide details as we believe the arguments to be rather explanatory of the driving philosophy behind Theorem C.

As we mentioned in the introduction, our abstract Theorem C, coupled with the localized estimates that follow, can be employed to reprove the optimal sparse domination estimates for Caldéron-Zygmund kernels of Dini and Hörmander type, thus recovering the results (among others) of [3, 17, 19, 24, 25]. We provide a summary of the statements of such domination theorems in the second part of this section.

3.1. Localized estimates and kernel norms. Throughout these estimates, we assume that a stopping collection $Q$ with top $Q$ as in Section 2 has been fixed, and the notations $\Lambda_{Q,j,v}$ refer to (2.5). It is understood that the constants implied by the almost inequality signs depend on dimension only and are in particular are uniform over the choice of $Q$. We begin with the single scale localized estimate where no cancellation is exploited.
Lemma 3.1 (Trivial estimate). Let $1 < \beta \leq \infty$ and $\alpha = \beta'$. Then for all $j \geq 1$ there holds
\[
\sum_{s} \int |K_s(x, y)||b_{s-j}(y)||h(x)| \, dy \, dx \leq [K]_{0, \beta} ||b||_{X_1} ||h||_{Y_\alpha}.
\]

Proof. As $||b_L||_1 \leq |L||b||_{X_1}$ for $L \in Q$, it suffices to prove that for each $L \in Q$ and $s = s_L + j$ there holds
\[
\int |K_s(x, y)||b_L(y)||h(x)| \, dy \, dx \leq [K]_{0, \beta} ||b_L||_1 ||h||_{Y_\alpha}.
\]
In turn, it then suffices to prove that
\[
s \geq s_L \implies \sup_{y \in L} \int |K_s(y + u, y)||h(y + u)| \, du \leq [K]_{0, \beta} ||h||_{Y_\alpha}
\]
which readily follows from
\[
\int |K_s(y + u, y)||h(y + u)| \, du \leq ||K_s(y + \cdot, y)||_\beta \left( \int_{B(y, 2^{s+10})} |h(z)|^\alpha \, dz \right)^{\frac{1}{\alpha}}
\]
\[
\leq [K]_{0, \beta} \left( \inf_L M_{\alpha} h \right) \leq [K]_{0, \beta} ||h||_{Y_\alpha}
\]
when $y \in L$. Above, we used the support condition (SS) and Hölder’s inequality for the first step, and subsequently that the ball $B(y, 2^{s+10}) = \{z \in \mathbb{R}^d : |z - y| < 2^{s+10}\}$ contains the dilate $\tilde{L}$. The proof is complete. \hfill \Box

We introduce a further family of kernel norms in addition to the one of (SS), to which we refer for notation. For $1 < \beta \leq \infty$ set
\[
[K]_{1, \beta} := \sum_{j=1}^{\infty} \omega_{j, \beta}(K)
\]
where
\[
\omega_{j, \beta}(K) := \sup_{s \in \mathbb{Z}} \sup_{x \in \mathbb{R}^d} \sup_{h \in \mathbb{R}^d} \left( \frac{1}{\|h\|_{\infty} < 2^{j+1-1}} \left( \|K_0(x, x + \cdot) - K_0(x + h, x + \cdot)\|_\beta \right) \right).
\]
The second localized estimate we consider uses the finiteness of $[K]_{1, \beta}$ to incorporate the constant-mean zero cancellation effect.

Lemma 3.2 (Cancellation estimate). Let $1 < \beta \leq \infty$ and $\alpha = \beta'$. Then for all $\mu, \nu \in \mathbb{Z}$ there holds
\[
|\Lambda_{Q, \mu, \nu}(b, h)| + |\Lambda_{Q, \mu, \nu}(h, b)| \leq ([K]_{0, \infty} + [K]_{1, \beta}) ||b||_{X_1} ||h||_{Y_\alpha}.
\]

Proof. It will suffice to prove the estimate
\[
\sum_{L \in Q} \sum_{j=1}^{\infty} \left| \int K_{s_j + j}(x, y) b_L(y) \tilde{h}_L(x) \, dy \, dx \right| \leq [K]_{1, \beta} ||b||_{X_1} ||\tilde{h}||_{Y_\alpha}.
\]
In fact, by using the representations in (2.9), (2.10) we see that for all $\mu, \nu \in \mathbb{Z}$ and each pair $b \in \hat{X}_1, h \in Y_\alpha$, the forms $|\Lambda_{Q, \mu, \nu}(b, h)|$, $|\Lambda_{Q, \mu, \nu}(h, b)|$ are both bounded above by the left
hand side of (3.4) for suitable \( \tilde{b} \in \mathcal{X}_1, \tilde{h} \in \mathcal{Y}_\tau \) whose norms are dominated by \( \|b\|_{\mathcal{X}_1}, \|h\|_{\mathcal{Y}_\tau} \) respectively, up to possibly replacing \( K_s \) with its transpose and controlling the remainder term \( V_Q(h, b) \) in the case of \( \Lambda_{Q, \mu, \nu}(h, b) \). This remainder is estimated in (2.11) for \( q = \infty \), which is acceptable for the right hand side of (3.3).

We will obtain estimate (3.4) from the bound

\[
\sum_{j=1}^{\infty} \left| \int K_{s,1+j}(x, y)\tilde{b}_j(y)\tilde{h}(x) \, dy \, dx \right| \leq [K]_{1, \beta} |L| \|\tilde{b}\|_{\mathcal{X}_1} \|\tilde{h}\|_{\mathcal{Y}_\tau}, \quad L \in \mathcal{Q}
\]

by summing over \( L \in \mathcal{Q} \) in and using their disjointness (2.2). Fix \( L \in \mathcal{Q} \) and \( j \geq 1 \). Using the cancellation of \( \tilde{b}_L \) and then arguing as in the proof of (3.1) above we obtain

\[
\left| \int K_{s,1+j}(x, y)\tilde{b}_j(y)\tilde{h}(x) \, dy \, dx \right| \leq \|\tilde{b}_L\|_1 \sup_{y \in L} \int |K_{s,1+j}(y, y) - K_{s,1+j}(y + u, y)| |\tilde{h}(y + u)| \, du \\
\leq \|\tilde{b}_L\|_1 \omega_{j, \beta}(K) \left( \inf_{L} M_{\alpha} \tilde{h} \right) \leq \omega_{j, \beta}(K) |L| \|\tilde{b}\|_{\mathcal{X}_1} \|\tilde{h}\|_{\mathcal{Y}_\tau}
\]

and (3.5) follows by summing over \( j \geq 1 \). \( \square \)

3.2. **Sparse domination of Calderón-Zygmund kernels.** We briefly mention how our abstract Theorem C can be employed to recover sparse domination, and thus weighted bounds, for Calderón-Zygmund kernels with minimal smoothness assumptions. Let \( T \) be an \( L^2(\mathbb{R}^d) \)-bounded operator whose kernel \( K \) satisfies the usual size normalization

\[
\sup_{x \neq y} |x - y|^d |K(x, y)| \leq 1.
\]

Let \( \psi \) be a fixed Schwartz function supported in \( A_1 = \{ x \in \mathbb{R}^d : 2^{-2} < |x| < 1 \} \) and such that

\[
\sum_{s \in \mathbb{Z}} \psi(2^{-s}x) = 1, \quad x \neq 0.
\]

It is immediate to see that (SS) holds, and in particular \( [K]_{0, \infty} \leq C \), for the decomposition

\[
K_s(x, y) := K(x, y) \psi \left( \frac{x - y}{2^s} \right), \quad s \in \mathbb{Z}.
\]

We further assume that \( [K]_{1, \beta} < \infty \) for some \( 1 < \beta \leq \infty \), where the kernel norm has been defined in (3.2). When \( \beta = \infty \), this is exactly the Dini condition [17, 19, 24]. For \( \beta < \infty \), the above condition is equivalent to the assumptions of [25], where in fact a multilinear version is presented.

The assumptions of Theorem C then hold for the dual form

\[
\Lambda(f_1, f_2) = \langle Tf_1, f_2 \rangle.
\]

We have already observed that (SS) is verified with \( q = \infty \). It is well-known that \( L^2 \)-boundedness of \( \Lambda \) together with \( [K]_{1, \beta} < \infty \) yields that the truncation forms \( \Lambda^r_{\mu, \nu} \) (cf. (2.1)) are uniformly bounded on \( L^t(\mathbb{R}^d) \times L^t(\mathbb{R}^d) \) [31, Ch. I.7] for all \( 1 < t < \infty \), thus we have condition (T) with, for instance, \( r = 2 \). Furthermore, Lemma 3.2 is exactly (L) for the corresponding \( \Lambda_{Q, \mu, \nu} \), with \( p_1 = 1, p_2 = \alpha = \beta' \). Applying Theorem C in the form given in Remark 2.4, we obtain the following sparse domination result, which recovers (the dual form of) the domination
theorems from the above mentioned references. We send to the same references for the sharp weighted norm inequalities that descend from this result.

**Theorem D** (Calderón-Zygmund theory). Let $T$ be as above and $1 \leq \beta < \infty$. For all $1 < t < \infty$ and all pairs $f_1 \in L^t(\mathbb{R}^d)$, $f_2 \in L^{t'}(\mathbb{R}^d)$ there holds

$$|\langle Tf_1, f_2 \rangle| \leq C_\beta [K]_{1, \beta} \sup_S \text{PSF}_{S; 1, \beta}(f_1, f_2).$$

where $C_\beta$ is a positive constant depending on $\beta$ and on the dimension $d$ only.

4. **Proof of Theorem A**

Let $1 < q \leq \infty$ and suppose that $\Omega \in L^q(S^{d-1})$ has unit norm and vanishing integral. Write throughout $x' = x/|x|$. We decompose for $x \neq 0$ the kernel of $T_\Omega$ in (1.2) as

$$\frac{\Omega(x')}{|x'|^d} = \sum_s K_s(x), \quad K_s(x) = \Omega(x')2^{-sd}\phi(2^{-s}x)$$

where $\phi$ is a suitable smooth radial function supported in $A_1 = \{2^{-2} \leq |x| \leq 1\}$. The main result of this subsection is the following proposition: again, we assume that a stopping collection $Q$ with top the dyadic cube $Q$ as in Section 2 has been fixed and the notations $Y_t$ and similar refer to that fixed setting.

**Proposition 4.1.** Let $\Omega \in L^q(S^{d-1})$ of unit norm and vanishing integral. Let $\{\varepsilon_s\} \in \{-1, 0, 1\}^\mathbb{Z}$ be a choice of signs, $b \in \mathcal{X}_1$ and define

$$K(b, h) := \sum_{s \geq 1} \sum_s \varepsilon_s \left( K_s * b_{s-j}, h \right)$$

where

$$b_s = \sum_{L \in Q, s \in s} b_L.$$

There exists an absolute constant $C$, in particular uniform over all $\{\varepsilon_s\} \in \{-1, 0, 1\}^\mathbb{Z}$ such that

$$|K(b, h)| \leq \frac{Cp}{p-1} \left|Q\right| \|b\|_{\mathcal{X}_1} \|h\|_{Y_p} \left( \|\Omega\|_{L^{q, 1} \log L(S^{d-1})} \quad q < \infty, \ p > q' \right) \left( \|\Omega\|_{L^{q, \nu} S^{d-1}} \quad q = \infty, \ p > 1. \right)$$

With the above proposition in hand, we may now give the proof of Theorem A. The structural assumptions (SS), (T) of the abstract Theorem C applied to the above decomposition of (the dual form of) $T_\Omega$ are respectively verified with $q = q$ and with $r = 2$ (this is the classical $L^2$-boundedness of the truncations of $T_\Omega$ [4, 15]).

We still need to verify (L) for the values $p_1 = 1$ and $p_2 = p$ for each $p$ in the claimed range (depending on whether $q = \infty$ or not). It is immediate from the representations (2.9) that in this setting $\Lambda_{Q, \mu, \nu}(b, h) = K(b1_Q, h)$ for a suitable choice of signs $\{\varepsilon_s\}$ depending on $\mu, \nu$. So Proposition 4.1 yields the first condition in (L) with $p_1 = 1, p_2 = p$. On the other hand, we read from (2.10) that $\Lambda_{Q, \mu, \nu}(h, b)$ is equal to $K(b_{in}, h1_Q)$, again for a suitable choice of signs $\{\varepsilon_s\}$ depending on $\mu, \nu$, up to replacing $K_s$ by $K_s(-)$, and up to subtracting off the remainder term from (2.11), which is estimated in this case by an absolute constant times

$$|Q| \|h\|_{Y_\infty} \|b\|_{Y_p} \leq |Q| \|h\|_{Y_\infty} \|b\|_{Y_p}.$$
which is acceptable for the right hand side of the second condition in (L) when $p_2 = p$. These considerations and another application of Proposition 4.1 finally yield Theorem A, via our abstract result in the form described in Remark 2.4.

4.1 Proof of Proposition 4.1. Throughout this proof, $C$ is a positive absolute dimensional constant which may vary at each occurrence without explicit mention. We assume $\{\varepsilon_s\} \in \{-1, 0, 1\}^{\mathbb{Z}}$ is given. For the sake of simplicity, we redefine $K_s := \varepsilon_s K_s$; it will be clear from the proof below that the signs of $K_s$ play no role. Fix a positive integer $j$. For $\delta > 0$ to be fixed at the end of the argument define

$$
O_j = \{\theta \in S^{d-1} : |\Omega(\theta)| > 2^j\}, \quad \Omega_j = \Omega 1_{S^{d-1}/O_j}, \quad \Delta_j = \Omega 1_{O_j}.
$$

We now decompose

$$
K_s = H^j_s + V^j_s, \quad H^j_s = K_{s, \text{supp}} \Omega_j, \quad V^j_s = K_s 1_{O_j}.
$$

The first localized form we treat, namely the contribution of the unbounded part of $\Omega$, is dealt with by means of a trivial estimate.

**Lemma 4.2.** $V^j(b, h) := \sum_s |\langle V^j_s * b_{s-j}, \bar{h} \rangle| \leq C \|\Delta_j\|_q \|Q\| \|b\|_{L^p} \|h\|_{L^p}$, $p \geq q'$.

*Proof.* It suffices of course to prove the estimate above with $q'$ in place of $p$. This is actually a particular case of Lemma 3.1 applied with $K = \{V^j_s\}$ and $\beta = q$, as it is immediate to see that for this kernel one has $[K]_{0,q} \leq C \|\Delta_j\|_q$. \qed

The contribution of the bounded part of $K_s$ in (4.3) is more delicate, and we postpone the proof of the following lemma to the next Subsection 4.2.

**Lemma 4.3.** There exist absolute constants $C, c > 0$ such that for all $1 < p \leq \infty$

$$
H^j(b, h) := \left|\sum_s \langle H^j_s * b_{s-j}, \bar{h} \rangle\right| \leq C 2^{-cj^{p-1}_p} \|\Delta_j\|_\infty \|Q\| \|b\|_{L^p} \|h\|_{L^p}.
$$

We may now complete the proof of Proposition 4.1. We assume $q < \infty$, the remaining case is actually simpler as $V^j$ is identically zero. Our decomposition (4.3) yields that

$$
|K(b, h)| = \sum_{j \geq 1} |H^j(b, h)| + \sum_{j \geq 1} |V^j(b, h)|.
$$

Choosing $\delta = c^{p-1}_p$ in (4.2) and using Lemma 4.3, we estimate

$$
\sum_{j \geq 1} |H^j(b, h)| \leq C \|Q\| \|b\|_{L^p} \|h\|_{L^p} \sum_{j \geq 1} 2^{-cj^{p-1}_p} \|\Omega_j\|_\infty \leq C \|Q\| \|b\|_{L^p} \|h\|_{L^p} \sum_{j \geq 1} 2^{-cj^{p-1}_p}
$$

$$
\leq \frac{Cp}{p-1} \|Q\| \|b\|_{L^p} \|h\|_{L^p}
$$

which is smaller than the right hand side of (4.1). Using Lemma 4.2, the latter sum involving $V_j$ is then estimated by

$$
\left(\sum_{j \geq 1} \|\Delta_j\|_q \right) \|Q\| \|b\|_{L^p} \leq \frac{Cp}{p-1} \|\Omega\|_{L^{q,1} \log L(S^{d-1})} \|Q\| \|b\|_{L^p} \|h\|_{L^p}
$$
which also complies with the right hand side of (4.1); here we have used that
\[
\sum_{j \geq 1} \| \Lambda_j \|_q \leq \sum_{j \geq 1} \sum_{k \geq j} 2^{\delta k} |O_k \setminus O_{k+1}|^{\frac{1}{q}} \leq \sum_{k \geq 1} k^{2\delta k} |O_k \setminus O_{k+1}|^{\frac{1}{q}} \leq \frac{C}{\delta} \| \Omega \|_{\mathcal{L}^{q,1}(\mathbb{R}^{d-1})}.
\]

The proposition is thus proved up to establishing Lemma 4.3.

4.2. **Proof of Lemma 4.3.** Our first observation is actually another trivial estimate.

**Lemma 4.4.** There exists \( C > 0 \) such that \( |\mathcal{H}^j(b, h)| \leq C \| \Omega_j \|_\infty |Q| \|b\|_{X_1} \|h\|_{Y_1}. \)

*Proof.* This is an application of Lemma 3.1 to \( K = \{ H_j^j \} \) with \( \beta = \infty \), as it is immediate to see that for this kernel one has \( |K|_{0,\infty} \leq C \| \Omega_j \|_\infty. \)

The second step is an estimate with decay, but involving \( Y_\infty \) norms.

**Lemma 4.5.** There exist \( C, c > 0 \) such that \( |\mathcal{H}^j(b, h)| \leq C 2^{-cj} \| \Omega_j \|_\infty |Q| \|b\|_{X_1} \|h\|_{Y_\infty}. \)

Before the proof of Lemma 4.5, which is given in the final Subsection 4.3, we observe that the estimate of Lemma 4.3 is obtained by Riesz-Thorin (for instance) interpolation in \( h \) of the last two lemmata.

4.3. **Proof of Lemma 4.5.** The techniques of this Subsection are an elaboration of the arguments of [29]. In particular Lemma 4.6 below is a stronger version of [29, Lemma 2.1] while Lemma 4.7 is essentially the dual form of [29, Lemma 2.2].

We perform a further decomposition of \( H_j^j \). Let \( \Xi = \{ e_v \} \) be a maximal \( 2^{-j-10d} \)-separated set contained in \( \text{supp} \ \Omega_j \). We may partition \( \text{supp} \ \Omega_j \) in \#\Xi \leq 2^{(d-1)} \) subsets \( E_v \) each containing \( e_v \) and such that \( \text{diam} |E_v| \leq 2^{-j} \). Set

\[
H_j^j(x) = H_j^j(x) 1_{E_v}(x').
\]

Also, let \( \psi \) be a smooth function on \( \mathbb{R} \) with \( 1_{[-2,2]} \leq \psi \leq 1_{[-4,4]} \). Let \( \kappa \in [0, 1] \) and define the multiplier operator

\[
\hat{P}_v^j(e^j, x). \]

We now decompose

\[
H_j^j := \Gamma_j^j + \Upsilon_j^j, \quad \Gamma_j^j := \sum_v P_v^j * H_j^j, \quad \Upsilon_j^j := H_j^j - \Gamma_j^j
\]

so that \( \mathcal{H}^j \) is the sum of the single scale bilinear forms

\[
G_j(b, h) = \left( \sum_s \Gamma_j^j * b_{s-j}, h \right),
\]

\[
U_j(b, h) = \left( \sum_s \Upsilon_j^j * b_{s-j}, h \right)
\]

satisfying the estimates below.

**Lemma 4.6.** Let \( \tau > 1 \). Then

\[
|G_j(b, h)| \leq C \tau \tau^{-j(1-\kappa)} \| \Omega_j \|_\infty |Q| \|b\|_{X_1} \|h\|_{Y_\tau}, \quad C \tau = \frac{C \tau}{\tau - 1}.
\]
Lemma 4.7. Let $b \in \mathcal{X}_1$. For all $\varepsilon > 0$ there exists a constant $C_{\kappa, \varepsilon}$ depending on $\kappa, \varepsilon$ only such that

$$|U_j(b, h)| \leq C_{\kappa, \varepsilon} 2^{-rj} \|\Omega_j\|_\infty |Q| \|b\|_{\mathcal{X}_1} \|h\|_{\mathcal{Y}_\infty}.$$ 

Notice that the combination of Lemma 4.6 with $\tau = 2$ and $\kappa = 1/2$ and Lemma 4.7 with $\varepsilon = 1/4$ yields the required estimate for Lemma 4.5, with $c = 1/4$. Lemma 4.5 is thus proved up to the arguments for Lemmata 4.6 and 4.7.

Proof of Lemma 4.6. We may factor out $\|\Omega_j\|_\infty$ and assume that the angular part in the definition of $\Gamma_j$ is bounded by 1. We can also assume that $H^j_{\sigma V}$ and $b$ are positive as cancellation plays no role in this argument: this is just a matter of saving space in the notation. Using interpolation and duality with $t$ below being the dual exponent of $\tau$, the estimate of the lemma follows if we show that for each integer $r \geq 1$ and $t = 2r$

$$(4.4) \quad \frac{1}{|Q|^2} \left\| \sum_s \Gamma^j_s * b_{s-j} \right\|_{L^2} \lesssim t 2^{-rj(1-\varepsilon)} \|b\|_{\mathcal{X}_1}$$

with an implicit constant that does not depend on $r$. Setting

$$M_v = \sum_s \hat{P}^j_v * H^j_{\sigma V} * b_{s-j}, \quad D_v = \sum_s H^j_{\sigma V} * b_{s-j},$$

we rewrite the left hand side of (4.4) raised to $t$-th power and subsequently estimate

$$(4.5) \quad \left\| \sum_{v_1, \ldots, v_r} \prod_{k=1}^r M_{v_k} \right\|_2 \lesssim \left\| \sum_{v_1, \ldots, v_r} \prod_{k=1}^r \hat{M}_{v_k} \right\|_2 \lesssim 2^{rj(d-2+\kappa)} \sum_{v_1, \ldots, v_r} \left\| \prod_{k=1}^r D_{v_k} \right\|_2 \lesssim 2^{j(d-1)} 2^{-rj(1-\varepsilon)} \sup_v \|D_v\|_1^t.$$ 

We have used Plancherel for the first equality, followed by the observation that $\hat{P}^j_v(\xi)$ is uniformly bounded and nonzero only if $|\xi - e_{v_k}| < 2^{-j(1-\varepsilon)}$. Thus there are at most $C_2 2^{j(d-2+\kappa)}$ $r$-tuples such that the $r$-fold convolution is nonzero, whence the first bound. Another usage of Plancherel, the observation that there are at most $2^{j(d-1)}$ tuples in the summation, and finally Hölder’s inequality yield the second bound. We are thus done if we estimate for each fixed $v$

$$(4.6) \quad \sum_{s_1 \geq \ldots \geq s_r} \int \left( \prod_{k=1}^t H^j_{\sigma V}(x - y_k) b_{s_k-j}(y_k) \right) dy_1 \ldots dy_t dx \lesssim C' 2^{-tj(d-1)} |Q| \|b\|_{\mathcal{X}_1}^t$$

as $\|D_v\|_1^t$ is at most $t^t$ times the above integral. Notice that if $\sigma \leq s$ then $\text{supp} \, H^j_{\sigma V}$ is contained in a box $R_s$ centered at zero and having one long side of length $\leq 2^s$ and $(d-1)$ short sides of length $2^{s-j}$. If $z \in \mathbb{R}^d$, $R_s(z) = z + R_s$ and

$$Q_s(z) = \{ L \in Q : s_L \leq s - j, L \subset 100R_s(z) \}, \quad b_{R_s(z)} := \sum_{L \in Q_s(z)} b_L$$

we have by disjointness of $L \in Q$

$$(4.7) \quad 2^{-sd} \|b_{R_s(z)}\|_1 \lesssim 2^{-sd} |R_s(z)| \|b\|_{\mathcal{X}_1} \leq C 2^{-j(d-1)} \|b\|_{\mathcal{X}_1} =: \alpha.$$
Also notice that for all fixed \( y_1, \ldots, y_t \) and for all \( s_1 \geq \cdots \geq s_t \) there holds
\[
I_{s_1, \ldots, s_t}(y_1, \ldots, y_t) := \int \left( \prod_{k=1}^t H^j_{s_k}(x - y_k) \right) \, dx \leq \|H^j_{s_1} \|_1 \prod_{k=1}^{t-1} \|H^j_{s_k} \|_{\infty} \leq 2^{-j(d-1)} 2^{-d s_{t-1}}
\]
where we wrote, here and in what follows
\[
s_n = \sum_{k=1}^n s_k, \quad n = 1, \ldots, t.
\]
Furthermore, \( I_{s_1, \ldots, s_t}(y_1, \ldots, y_t) \) is nonzero only if \( y_k \in 2R_{s_k-1}(y_{k-1}) \) for \( k = t, t-1, \ldots, 2 \). Now, writing \( b_{s_k} \) in place of \( b_{s_k-j} \) for reasons of space as \( j \) is kept fixed throughout and using (4.7) repeatedly, the sum in (4.6) is equal to
\[
\sum_{s_1 \geq \cdots \geq s_t} \int I_{s_1, \ldots, s_t}(y_1, \ldots, y_t) \left( \prod_{k=1}^t b_{s_k}(y_k) \right) \, dy_1 \cdots dy_t
\]
\[
\leq 2^{-j(d-1)} \sum_{s_1 \geq \cdots \geq s_{t-1}} 2^{-d s_{t-2}} \int b_{s_1}(y_1) \left( \prod_{k=2}^{t-1} b_{s_k}(y_k) 1_{2R_{s_k-1}(y_{k-1})}(y_k) \right) \frac{\|b_{2R_{s_k-1}(y_{k-1})} \|_1}{2^{d s_{k-1}}} \, dy_1 \cdots dy_{t-1}
\]
\[
\leq a 2^{-j(d-1)} \sum_{s_1 \geq \cdots \geq s_{t-2}} 2^{-d s_{t-3}} \int b_{s_1}(y_1) \left( \prod_{k=2}^{t-2} b_{s_k}(y_k) 1_{2R_{s_k-1}(y_{k-1})}(y_k) \right) \frac{\|b_{2R_{s_k-1}(y_{k-2})} \|_1}{2^{d s_{k-2}}} \, dy_1 \cdots dy_{t-2}
\]
\[
\leq \cdots \leq a^{t-1} 2^{-j(d-1)} |Q| \|b\|_{X^1_t} \leq C t 2^{-j(d-1)} |Q| \|b\|_{X^1_t}
\]
as claimed, and this completes the proof. \( \square \)

**Proof of Lemma 4.7.** Again we factor out \( \|Q\|_{\infty} \) and work under the assumption that the angular part is bounded by 1. In this proof \( M \) is a large integer whose value may differ at each occurrence and the constants implied by the almost inequality sign are allowed to depend on \( M \) only. Let \( \beta \) be a smooth function supported in \( A_1 = \{2^{-1} \leq |\xi| \leq 2\} \) and satisfying
\[
\sum_{k \in \mathbb{Z}} \beta^2(2^k \xi) = 1 \quad \xi \neq 0.
\]
Denote by \( B_k = \mathcal{F}^{-1} \{ \beta(2^k \cdot) \} \). Defining
\[
\widehat{R_{\alpha}^{j \nu}}(\xi) = \beta(2^j \xi) \left( 1 - \hat{P}^l(\xi) \right) \hat{H}^{j \nu}_{\alpha}(\xi),
\]
we recall from [29, eqs. (2.6), (2.7)] the estimate
\[
\|R_{\alpha}^{j \nu}\|_1 \lesssim_M 2^{-j(d-1)} \min \left\{ 1, 2^{-Mk} 2^{-M(s-j-k)} \right\}.
\]
Now, fix \( s \) and \( L \in Q \) with \( \ell(L) = 2^{s-j} \) for the moment. Recalling the definition of \( \varphi_{\alpha}^j \), we have the decomposition
\[
|\langle \varphi_{\alpha}^j \ast b_{L}, \hat{h} \rangle| \leq \sum_{\nu} \sum_{k} |\langle R_{\alpha}^{j \nu} \ast B_k \ast b_{L}, \hat{h} \rangle|,
\]
and the cancellation estimate (cf. [29, eq. (2.5)], a simpler version of Lemma 3.2)
\begin{equation}
|\langle R_{sv}^{jk} * B_k * b_L, \overline{h} \rangle| \lesssim \min \{ 1, 2^{(s-j)-k} \} \| R_{sv}^{jk} \|_1 \| b_L \|_1 \| h \|_\infty \leq 2^{-j(d-1)} \min \left\{ 2^{(s-j)-k}, 2^{-Mk j-M(s-j-k)} \right\} \| L \| \| b \|_{\dot{X}^s_\infty} \| h \|_{Y^\infty_\infty}.
\end{equation}

Note that \( \# \Xi \lesssim 2^{j(d-1)} \). So for each \( \varepsilon > 0 \) we can use the left estimate in (4.8) for \( k \geq s-j(1-\varepsilon) \)
and the right estimate otherwise, and obtain
\begin{equation}
|\langle \check{Y}_s^{j} * b_L, \overline{h} \rangle| \leq \sum_v \sum_k |\langle R_{sv}^{jk} * B_k * b_L, \overline{h} \rangle| \leq 2^{-\varepsilon j} \| L \| \| b \|_{\dot{X}^s_\infty} \| h \|_{Y^\infty_\infty}
\end{equation}
provided that \( M \) is chosen large enough to have \( 2\varepsilon < M \kappa \). The proof is thus completed by summing (4.9) over \( L \in Q \) with \( l(L) = 2^{s-j} \) and later over \( s \). \( \square \)

5. Proof of Theorem B

Throughout this proof, \( C \) is a positive absolute dimensional constant which may vary at each occurrence without explicit mention. Most of the arguments in this Section are contained in [6, Section 3]; we reproduce the details for clarity.

Let \( \psi(x) = \cos (2\pi(|x| - \delta/4)) \). From the asymptotic expansion of the inverse Fourier transform of the multiplier of \( B_\delta \) [6, Section 3], which is \( C^\infty \) and radial, we obtain the kernel representation
\[
B_\delta(x) = \sum_{s \geq 1} \sum_v K_{s,v}(x) + L(x),
\]
Here
\[
K_{s,v}(x) = \Omega_v(x')\psi(x)2^{-sd}\phi(2^{-s}x)
\]
with \( \Omega_v \) being a finite smooth partition of unity on the unit sphere \( S^{d-1} \) with sufficiently small support which is introduced for technical reasons, and \( \phi \) being a suitable smooth radial function supported in \( A_1 = \{ 2^{-2} \leq |x| \leq 1 \} \), while \( L(x) \) is an integrable kernel with \( L(x) \leq C(1 + |x|)^{-(d+1)} \), so that
\[
Lf(x) \leq CM_1f(x)
\]
which can be ignored for our purposes. We can also think of \( v \) as fixed and omit it from the notation, and consider the kernel \( K = \{ K_s \} \) as above. We are going to verify that conditions in Theorem C are satisfied by (the dual form to) \( B_\delta \). First of all, condition (SS) is obvious from the above discussion as \( |K|_{0,\infty} < \infty \). Second, the (T) condition follows from the well-known estimate
\[
\sup_{\mu,v} \| \Lambda_{\mu,v}^\nu \|_{L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \leq C,
\]
see for instance [14, Theorem E]. In order to verify the condition (L), let \( Q \) be a stopping collection with top \( Q \). Let \( b \in \mathcal{X}_1(Q) \); we change a bit the notation for \( b_s \) in this context by redefining
\[
b_s := \sum_{s_l = s} b_{L_s}, \quad s \geq 1, \quad b_0 := \sum_{s_l \leq 0} b_{L_s}.
\]
It is easy to see that in this context if \( b \in X_1 \) supported on \( Q \) and \( h \in Y_1 \) one has

\[
\Lambda_{Q_0, \nu}(b, h) = \left( \sum_{j \geq 1} \sum_{s \geq j} \epsilon_s K_s \ast b_{s-j}, \overline{h} \right)
\]

for a suitable choice of signs \( \{\epsilon_s\} \in \{-1, 0, 1\}^Z \), and the same for \( \Lambda_{Q_0, \nu}(h, b) \) up to replacing \( b \) by \( b^p \), restricting \( h \) to be supported on \( Q \), transposing \( K_s \), and subtracting off the remainder terms which are estimated by

\[
|Q|\|b\|_{X_1}\|h\|_{Y_1}.
\]

Theorem B is thus obtained from the next proposition via an application of Theorem C.

**Proposition 5.1.** Let \( \{\epsilon_s\} \in \{-1, 0, 1\}^Z \) be a choice of signs, \( b \in X_1 \) and define

\[
K(b, h) := \left( \sum_{j \geq 1} \sum_{s \geq j} \epsilon_s K_s \ast b_{s-j}, \overline{h} \right).
\]

There exists an absolute constant \( C \), in particular uniform over \( \{\epsilon_s\} \in \{-1, 0, 1\}^Z \), such that

\[
|K(b, h)| \leq \frac{Cp}{p-1}|Q|\|b\|_{X_1}\|h\|_{Y_p}.
\]

Notice that here we do not need to require \( b \in X_1 \) as per the oscillatory nature of the problem.

### 5.1 Proof of Proposition 5.1.

Given our choice of \( \{\epsilon_s\} \in \{-1, 0, 1\}^Z \), we relabel \( K_s := \epsilon_s K_s \). It will be clear from the proof that the signs \( \epsilon_s \) play no role. We split

\[
K(b, h) = \sum_{j \geq 1} K'(b, h), \quad K'(b, h) := \sum_{s \geq j} \left( K_s \ast b_{s-j}, \overline{h} \right).
\]

The first estimate is a trivial one.

**Lemma 5.2.** There exists \( C > 0 \) such that \( |K'(b, h)| \leq C|Q|\|b\|_{X_1}\|h\|_{Y_1} \).

**Proof.** This follows from applying Lemma 3.1 with \( \beta = \infty \) to \( K = \{K_s\} \), as it is immediate to see that for this kernel one has \( |K|_{0,\infty} \leq C \) as already remarked. \( \square \)

The second estimate, which is essentially contained in [6, Section 3], is the one providing decay.

**Lemma 5.3.** There exists \( C, c > 0 \) such that \( |K'(b, h)| \leq C2^{-c j}|Q|\|b\|_{X_1}\|h\|_{Y_p} \).

It is easy to see that interpolating the above estimates yields

\[
|K'(b, h)| \leq C2^{-j \frac{c(p-1)}{p}}|Q|\|b\|_{X_1}\|h\|_{Y_p},
\]

the summation of which yields Proposition 5.1.

**Proof of Lemma 5.3.** Let \( \overline{K}_s(\cdot) = \overline{K}_s(-\cdot) \). We recall from [6, Lemma 3.1] the estimates

\[
|K_s \ast \overline{K}_s(x)| \leq C2^{-ds/(1 + |x|)} < \delta,
\]

\[
\|K_s \ast \overline{K}_s\|_{\infty} \leq C2^{-dt}2^{-\delta}, \quad \forall s < t-1.
\]

(5.1)
By duality, it suffices to prove that

\begin{equation}
\| K_j \ast b_0 \|_2^2 + \left\| \sum_{s > j} K_s \ast b_{s-j} \right\|_2^2 \leq C 2^{-cj} |Q| \| b \|_{X_1}^2.
\end{equation}

For the first term we use the first estimate in (5.1):

\[ \| K_j \ast b_0 \|_2^2 = \langle b_0, K_j \ast \widetilde{K}_j \ast b_0 \rangle \leq \| b_0 \|_1 \| K_j \ast \widetilde{K}_j \ast b_0 \|_{\infty} \leq C 2^{-\min(\delta,d)} |Q| \| b \|_{X_1}^2. \]

The last inequality above follows from

\[ \| K_j \ast \widetilde{K}_j \ast b_0 \|_{\infty} \leq 2^{-jd} \sum_{m=0}^{j} 2^{-m\delta} \sup_{x \in \mathbb{R}^d} \| b_0 \|_{L^1(B(x,C 2^m))} \leq C 2^{-\min(\delta,d)} j \| b \|_{X_1}, \]

where \( B(x, C 2^m) \) denotes a ball centered at \( x \) with radius \( C 2^m \). For the second term, we begin by quoting from (6, (3.2)) that

\begin{equation}
\| K_s \ast b_{s-j} \|_2^2 \leq C 2^{-\delta j} \| b \|_{X_1} \| b_{s-j} \|_1.
\end{equation}

Observe that

\begin{equation}
\left\| \sum_{s > j} K_s \ast b_{s-j} \right\|_2^2 \leq \sum_{s > j} \left\| K_s \ast b_{s-j} \right\|_2^2 + 2 \sum_{s} \left| \langle K_s \ast b_{s-j}, K_{s-1} \ast b_{s-1-j} \rangle \right|
\end{equation}

\[ + 2 \sum_{s} \sum_{j < s < t-1} \left| \langle \widetilde{K}_t \ast K_s \ast b_{s-j}, b_{t-j} \rangle \right|. \]

The first two terms are bounded by

\[ C 2^{-\delta j} \| b \|_{X_1} \sum \| b_{s-j} \|_1 \leq C 2^{-\delta j} |Q| \| b \|_{X_1}^2, \]

according to (5.3) for the first one and Cauchy-Schwarz followed by (5.3) for the second. For the third term, from the second estimate of (5.1) and support considerations one has

\[ \| \widetilde{K}_t \ast K_s \ast b_{s-j} \|_{\infty} \leq C \left( \sup_{x \in \mathbb{R}^d} \| b_{s-j} \|_{L^1(B(x,C 2^t))} \right) \| \widetilde{K}_t \ast K_s \|_{\infty} \leq C 2^{-\delta s} \| b \|_{X_1}. \]

Therefore, the third summand in (5.4) is dominated by

\[ C \| b \|_{X_1} \sum_{t > j} \| b_{t-j} \|_1 \sum_{j < s < t-1} 2^{-\delta s} \leq C 2^{-\delta j} |Q| \| b \|_{X_1}^2, \]

and collecting all the above estimates (5.2) follows. \( \square \)

**Appendix A. Verification of (2.10)-(2.11)**

Let \( Q \) be a stopping collection with top \( Q, h \in \mathcal{M}_q, b \in X_q \). Clearly we can assume \( \text{supp} \, h \subset Q \). By possibly replacing \( K_s \) by zero when \( s \notin (\mu, \nu) \) we can ignore the truncations \( \mu, \nu \) in what follows and omit them from the notation. Recall the definitions (2.4), (2.5)

\[ \Lambda_Q(h, b) = \Lambda_Q(h, b) - \sum_{R \in Q} \Lambda_R(h, b) = \Lambda_{Q^c}(h, b) - \sum_{R \in Q} \Lambda_{Q^c}(h1_R, b). \]
and the decomposition
\[ b = b^{\text{in}} + b^{\text{out}}, \quad b^{\text{in}} = \sum_{L \in Q, 3L \cap 2Q \neq \emptyset} b_L, \quad b^{\text{out}} = \sum_{L \in Q, 3L \cap 2Q = \emptyset} b_L. \]

We first estimate
\[ |\Lambda_Q(h, b^{\text{out}})| \leq [K]_{0,q} |Q| ||h||_{y_q} ||b||_{x_{q'}} \]
which is a single scale estimate. In fact, since \( \text{dist}(R, \text{supp } b^{\text{out}}) \geq \ell(R)/2 \) for all \( R \subset Q \), by virtue of the support restriction in (SS),
\[ s < s_R \implies \int K_s(x, y)h(y)1_R(y)b^{\text{out}}(x) \, dy \, dx = 0. \]

Therefore, by the same argument used in (3.1),
\[ |\Lambda^S_Q(h, b^{\text{out}})| \leq \int |K_s(x, y)||h(y)||b^{\text{out}}(x)| \, dy \, dx \leq [K]_{0,q} |Q| ||h||_{y_q} ||b||_{x_{q'}}. \]

Proceeding similarly, if \( R \in Q, R \subset Q \)
\[ |\Lambda^S_R(h1_R, b^{\text{out}})| \leq \int |K_s(x, y)||h1_R(y)||b^{\text{out}}(x)| \, dy \, dx \leq [K]_{0,q} |R| ||h||_{y_q} ||b||_{x_{q'}}. \]

and the claimed (A.1) follows by summing the last display over \( R \in Q, R \subset Q \), which are pairwise disjoint, and combining the result with (A.2). The representation (2.10) will then be a simple consequence of the equality
\[ \Lambda_Q(h, b^{\text{in}}) = \left( \Lambda^S_Q(h, b^{\text{in}}) - \sum_{L \in Q, 3L \cap 2Q \neq \emptyset} \Lambda^S_L(h, b_L) \right) + V_Q(h, b) \]
where the remainder \( V_Q \) satisfies
\[ |V_Q(h, b)| \leq [K]_{0,q} |Q| ||h||_{y_q} ||b||_{x_{q'}}. \]

We turn to the proof of (A.3). We will use below without explicit mention that whenever \( L, R \in Q \) with \( 3R \cap 3L \neq \emptyset \), then \( |s_L - s_R| < 8 \), a consequence of the separation property (2.3). First of all, the restriction on the support (SS) entails that
\[ \sum_{R \in Q} \Lambda^S_R(h1_R, b^{\text{in}}) = \sum_{R \in Q} \sum_{L \in Q, 3L \cap 3R \neq \emptyset} \Lambda^S_L(h1_R, b_L) \]
as \( \Lambda^S_R(h1_R, b_L) = 0 \) unless \( 3L \cap 3R \) is nonempty. As there are at most 16 \( s \)-scales in each difference \( \Lambda^S_L - \Lambda^S_R \), using the trivial estimate (3.1) with \( \beta = q \) for each such scale yields
\[ \sum_{R \in Q} \sum_{L \in Q, 3L \cap 3R \neq \emptyset} |\Lambda^S_L(h1_R, b_L) - \Lambda^S_R(h1_R, b_L)| \leq [K]_{0,q} ||h||_{y_q} \sum_{R \in Q} \sum_{L \in Q, 3L \cap 3R \neq \emptyset} ||b_L||_1 \]
\[ \leq [K]_{0,q} ||h||_{y_q} ||b||_{x} \sum_{R \in Q} |R| \leq [K]_{0,q} |Q| ||h||_{y_q} ||b||_{x}. \]
Recalling the second property of stopping collections in (2.3), we have the decomposition
\[ h = h^{\text{in}} + h^{\text{out}}, \quad h^{\text{in}} := h1_{R \subseteq Q}, \quad \text{supp} \ h^{\text{out}} \cap \left( \bigcup_{L \subseteq Q} 9L \right) = \emptyset. \]

Therefore, up to including the error term of (A.6) in (A.4), (A.5) can be rewritten as
\[
\sum_{R \subseteq Q} \sum_{L \subseteq Q} \Lambda^s(h1_R, b_L) = \sum_{L \subseteq Q} \Lambda^s(h^{\text{in}}, b_L) - \sum_{L \subseteq Q} \Lambda^s(h^{\text{out}}, b_L),
\]
(A.7)
\[
\widetilde{h}_L = \sum_{R \subseteq Q} h1_R, \quad \text{supp} \ \widetilde{h}_L \subset \mathbb{R}^d \setminus 3L.
\]
We note that all the terms in the second sum on the right hand side of the first line of (A.7) vanish due to the support restriction on \( K_r \), as all the scales appearing are less than or equal to \( s_L \) and \( \text{supp} \ b_L \subset L \). The reasoning beginning with decomposition (A.5) leads thus to the equality, up to tolerable error terms
\[
\sum_{R \subseteq Q} \Lambda^s(h1_R, b^{\text{in}}) = \sum_{L \subseteq Q} \Lambda^s(h, b_L) - \sum_{L \subseteq Q} \Lambda^s(h^{\text{out}}, b_L).
\]
(A.8)
Finally the second term on the right hand side of (A.8) also vanishes, by virtue of the restriction on the support of \( h^{\text{out}} \), which does not intersect \( 9L \) for any \( L \) in the sum. Therefore, (A.8) is actually the equality
\[
\sum_{R \subseteq Q} \Lambda^s(h1_R, b^{\text{in}}) = \sum_{R \subseteq Q} \Lambda^s(h1_R, b^{\text{in}}) = \sum_{L \subseteq Q} \Lambda^s(h, b_L) + V_Q(h, b)
\]
where \( V_Q(h, b) \) satisfies (A.4); the first equality in the above display is due to \( \text{supp} \ h \subset Q \). This equality clearly implies the sought after (A.3).

APPENDIX B. SPARSE DOMINATION IMPLIES WEAK \( L^1 \) ESTIMATE

We show that if a sublinear operator \( T \) satisfies the sparse estimate (1.1) for \( p_1 = 1, p_2 = r \) for some \( 1 \leq r < \infty \) then \( T \) is of weak type \((1,1)\). In particular, as mentioned in the Introduction, together with Theorem A, this yields the weak \( L^1 \) estimate of \( T_\Omega \), which is the main result of [29] proved by Seeger. The proof that follows is a simplified version of the arguments in [10, Appendix A]; we are sure these arguments are well-known but were unable to locate a precise reference.

**Theorem E.** Suppose that the sublinear operator \( T \) has the following property: there exists \( C > 0 \) and \( 1 \leq r < \infty \) such that for every \( f_1, f_2 \) bounded with compact support there exists a sparse collection \( S \) such that
\[
|\langle Tf_1, f_2 \rangle| \leq C \sum_{Q \in S} |Q| \langle f_1 \rangle_{1,Q} \langle f_2 \rangle_{r,Q}.
\]
(B.1)

Then \( T : L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d) \) boundedly.
Proof. By standard arguments it suffices to verify that
\[
\sup_{\|f_i\|_1 = 1} \sup_{G \subset \mathbb{R}^d} \inf_{\substack{G^c \subset \mathbb{R}^d \mid |G| \leq 2^{|G'|} \mid |f_j| \leq 1}} |\langle Tf_1, f_2 \rangle| \leq C
\]
where \(f_1, f_2\) are bounded and compactly supported and \(G\) has finite measure. Given such \(f_i\) with \(\|f_i\|_1 = 1\) and \(G\) of finite measure define the sets
\[
H := \{ x \in \mathbb{R}^d : M_1 f_1(x) > C|G|^{-1} \},
\]
\[
\tilde{H} := \bigcup_{Q \in \mathcal{Q}} 3Q, \quad \mathcal{Q} = \{ \text{max. dyad. cube } Q : |Q \cap H| \geq 2^{-5}|Q| \}.
\]
It is easy to see that \(|\tilde{H}| \leq 2^{-10}|G|\) for suitable choice of \(C\). Therefore the set \(G' : G \setminus \tilde{H} \) satisfies \(|G| \leq 2|G'|\). We make the preliminary observation that
\[
\sup_{x \in H^c} M_1 f_1(x) \leq C|G|^{-1},
\]
so that by interpolation
\[
(B.2) \quad \|M_1 f_1\|_{L^{p'}(H^c)} \leq \left( \sup_{x \in H^c} M_1 f_1(x) \right)^{1 - \frac{1}{p'}} \|M_1 f_1\|_{L^1}^{\frac{1}{p'}} \leq C|G|^{-(1 - \frac{1}{p'})},
\]
where \(p' > 1\) is chosen such that \(p > r\). Fixing now any \(f_2\) restricted to \(G'\), we apply the domination estimate, yielding the existence of a sparse collection \(S\) for which we have the estimate
\[
|\langle Tf_1, f_2 \rangle| \leq C \sum_{Q \in S} |Q| \langle f_1 \rangle_{1, Q} \langle f_2 \rangle_{r, Q}.
\]
We claim that
\[
(B.3) \quad |Q \cap H| \leq 2^{-5}|Q| \quad \forall Q \in \mathcal{S}.
\]
This is because if \((B.3)\) fails for \(Q, Q\) must be contained in \(3Q'\) for some \(Q' \in Q\). But the support of \(f_2\) is contained in \(H^c\) which does not intersect \(3Q'\), whence \(\langle f_2 \rangle_{r, Q} = 0\). Relation \((B.3)\) has the consequence that if \(\{ E_Q : Q \in \mathcal{S} \}\) denote the distinguished pairwise disjoint subsets of \(Q \in S\) with \(|E_Q| \geq 2^{-2}|Q|\), the sets \(\tilde{E}_Q := E_Q \cap H^c\) are also pairwise disjoint and \(|\tilde{E}_Q| \geq 2^{-3}|Q|\). Therefore, since the union of \(\tilde{E}_Q\) is contained in \(H^c\) by standard arguments we arrive at
\[
|\langle Tf_1, f_2 \rangle| \leq C \sum_{Q \in S} |\tilde{E}_Q| \langle f_1 \rangle_{1, Q} \langle f_2 \rangle_{r, Q} \leq C \int_{H^c} M_1 f(x) M_r f_2(x) \, dx \\
\leq C \|M_1 f_1\|_{L^{p'}(H^c)} \|M_r f_2\|_{L^p(\mathbb{R}^d)} \leq C|G|^{-(1 - \frac{1}{p'})} |G|^{\frac{2}{p'}} \leq C
\]
using \((B.2)\) in the last step. The proof is complete. \(\square\)
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