THE CANONICAL STRIP, I

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Abstract. We introduce a canonical strip hypothesis for Fano varieties. We show that the canonical strip hypothesis for a Fano variety implies that the zeros of the Hilbert polynomial of embedded Calabi–Yau and general type hypersurfaces are located on a vertical line. This extends, in particular, Villegas’s ‘polynomial RH’ for intersections in projective spaces to the case of CY and general type hyperplane sections in Grassmannians. We state a few conjectures on the Ehrhart polynomials of certain fan polytopes.

1. VANISHING THEOREMS AND THE CANONICAL STRIP

The prototypical vanishing theorem is due to Kodaira. Let \( X \) be a smooth complex projective variety of dimension \( n \), and let \( K_X \) be its canonical divisor. Then higher cohomology groups \( H^i(K_X + L) \), \( i > 0 \), vanish for any ample line bundle \( L \). It is usually viewed as a theorem in Kahler geometry; in fact, the Akizuki–Nakano theorem says that for a positive hermitian holomorphic line bundle \( E \) on a compact Kahler manifold \( X \) one has \( H^{p,q}(X, E) = 0 \) for \( p + q > n + 1 \); the previous assertion follows if one sets \( p = n \). A few semipositive and partially positive versions are discussed in Demailly’s textbook [Dem07]. As a generalisation for vector bundles, one has Nakano’s vanishing theorem: if a hermitian vector bundle \( E \) is Nakano positive, then \( H^{n,q}(X, E) = 0 \) for \( q > 0 \). Back to algebraic geometry and line bundles, strong and subtle generalisations of the Kodaira vanishing theorem have been proved, cf. [EV92]. The import of Kodaira’s theorem varies according to which world, Fano or Calabi–Yau, or general type, we are in. We are mainly interested in Fano in this note; for a Fano variety (i.e. in the case when \( K_X < 0 \)), the corollary is that the line bundles in the canonical strip, \( K_X < E < 0 \), are acyclic.

Another Fano–pertaining subject which involves the consideration of the canonical strip, is theory of helices and stability conditions. For a Fano \( X \) with a full exceptional collection \( \{E_i\} \) that has \( O \) for its member, a turn of the respective helix between \( -K_X \) and \( O \) (non–inclusive) consists of acyclic objects. The issue of Nakano positivity (or Griffiths positivity) of the objects in such turn twisted by \( -K_X \) has not been addressed, to our knowledge.

The qualitative phenomena referred to above, the acyclicity of the line bundles in the canonical strip of a Fano variety, and the acyclicity of the vector bundles in a canonical turn, appear to admit quantitative treatment via the study of location of zeros of the Hilbert polynomial. Our theorem [21] was inspired by the elegant note [RV02] by Rodriguez–Villegas.
1.1. The canonical strip and the canonical line hypotheses. Let $H(z)$ be the Hilbert polynomial of a [Fano or general type] variety $X$, so that $H(n) = H_{-K_X}(n) = \chi(n(-K))$ for integral $n$.

(CS) We say that $X$ satisfies the canonical strip hypothesis if all roots $z_i$ of $H(z)$ are in the canonical strip $-1 < \Re z < 0$.

(NCS) We say that $X$ satisfies the narrowed canonical strip hypothesis if all roots $z_i$ of $H(z)$ are in the narrowed canonical strip $-1 + \frac{1}{\dim X + 1} \leq \Re z \leq -1$.

(CL) We say that $X$ satisfies the canonical line hypothesis, if all roots $z_i$ are on the vertical line $\Re z = -1/2$.

It is clear that (CL) $\iff$ (NCS) $\iff$ (CS).

1.2. Calabi–Yau: an embedded version. For a Calabi–Yau type $X$ the Hilbert polynomial with respect to the anticanonical class is not too exciting. We consider instead embedded Calabi–Yaus. We say that a Calabi–Yau $X$ embedded as an anticanonical section in a Fano $F$ satisfies the canonical line hypothesis if the roots of its Hilbert polynomial with respect to the restriction of $-K_F$ to $X$ are purely imaginary.

1.3. Curves. For a genus $g$ curve, $\chi(z(-K)) = (2 - 2g)(z + 1/2)$, and the canonical line hypothesis holds. For an elliptic curve, embedded as a cubic in $\mathbb{P}^2$, one has $\chi(O(9z)) = 9z$, and the canonical line hypothesis holds.

1.4. Surfaces. The Riemann–Roch–Hirzebruch formula yields, for surfaces,

$$H(z) = \frac{1}{2}z^2c_1^2 + 1/2 c_1^2 z + 1/12 c_1^2 + 1/12 c_2.$$

The two roots of $H(z)$ are

$$-1/2 \pm \frac{1}{6} \sqrt{-6c_2 + 3c_1^2 c_3}.$$

Thus, one has:

$X$ satisfies (CS) $\iff c_1^2 \geq -c_2$

$X$ satisfies (NCS) $\iff c_1^2 \leq 3c_2$

$X$ satisfies (CL) $\iff c_1^2 \leq 2c_2$

For del Pezzo surfaces, the maximal possible value of $3 - \frac{6c_2}{c_1}$ is 1; it is attained on $\mathbb{P}^2$. Thus, (NCS) holds for del Pezzos. For surfaces of general type, (NCS) holds by Yau.

For an embedded K3’s polarized by a class $h$, $\chi(O(h)) = 1/2h^2z^2 + 2$, and the canonical line hypothesis holds.

1.5. Threefolds. Riemann–Roch–Hirzebruch says now that

$$H(z) = \frac{1}{6}z^3c_1^3 + 1/4 c_1^3 z^2 + (1/12 c_1^2 + 1/12 c_2) z c_1 + 1/24 c_1 c_2,$$

and the three roots are

$$-1/2, -1/2 \pm \frac{1}{2} \sqrt{c_1^3 - 2c_1 c_2}.$$
One has:
\[
\begin{align*}
X \text{ satisfies (CS)} & \iff \frac{-2c_1c_2}{c_1^3} < 0 \\
X \text{ satisfies (NCS)} & \iff \frac{-2c_1c_2}{c_1^3} \leq -\frac{3}{4} \\
X \text{ satisfies (CL)} & \iff \frac{-2c_1c_2}{c_1^3} \leq -1
\end{align*}
\]

In the Fano case, the vanishing theorem gives \(c_1c_2 = 24\), so the maximal possible value of \(\frac{-2c_1c_2}{c_1^3}\) is \(-\frac{3}{4}\), and it is attained on \(\mathbb{P}^3\), so (NCS) is true. The formula for the roots shows that the narrowed canonical strip hypothesis holds for minimal threefolds of general type: Yau’s result for threefolds is \(c_1^3 \geq \frac{8}{3}c_1c_2\).

1.6. Grassmannians. Projective spaces satisfy the narrow canonical strip hypothesis, the zeros being \(-\frac{1}{n+1}, i = 1, \ldots, n\). This generalizes to Grassmannians, as shown by Hirzebruch. Let \(G(k, N)\) be the Grassmannian of \(k\)-planes in an \(N\)-dimensional space. Assume that \(N \geq 2k\). Let \(\varphi(x)\) be the piecewise linear function of real argument \(x\) given by \(\varphi(x) = \min(k, -x, x + n + 1)\). Then [Hir58]
\[
H(z) = c \prod_{i=1}^{n} (z + \frac{i}{n+1})^{\varphi(-i)}.
\]

2. The canonical line hypothesis for embedded varieties

As we saw above, varieties of general type need not satisfy (CL). The situation is different with embedded varieties of general type.

2.1. Theorem. Let \(F\) be a Fano variety which satisfies the canonical strip hypothesis, and let \(X\) be its general type (resp. Calabi–Yau type) section in the linear system \(-nK_X, n > 1\) (resp. \(n = 1\)). Then (CL) holds for the embedded variety \(X\).

Proof. Let \(H_F(z)\) be the Hilbert polynomial of \(F\), and let \(H_r(z)\) denote the Hilbert polynomial of \(X\) with respect to the restricted \(-K_F\). For general type \(X\), the adjunction formula says that the anticanonical class of \(X\) is a multiple of the restricted anticanonical class of \(F\), so we may prove the vertical line statement for \(H_r\) instead. Then
\[
H_r(z) = H_F(z) - H_F(z - n).
\]
The fact that the zeros of \(H_r(z)\) are on the line \(\text{Re } z = \frac{n-1}{2}\) follows from the

2.2. Lemma. Let \(H(z) \in \mathbb{R}[z]\) satisfy
i) \(H(-1 - z) = \pm H(z)\),
ii) all roots \(z_i\) of \(H\) are [strictly] in the left half-plane.

Then, for any real \(s \geq 1\) and for all roots \(\zeta_i\) of \(H(z) - H(z - s)\), one has \(\text{Re } z = \frac{s-1}{2}\).
Proof. Assume there is a root \( \zeta = \zeta_j \) of \( H_r(z) \) with \( \operatorname{Re} \zeta < \frac{s - 1}{2} \). By assumption,

$$
\prod (\zeta - z_i) = \prod (\zeta - z_i - s).
$$

Let \( \mu_s \) be the map that takes \( z \) to \( (s - 1) - \bar{z} \). It particular, it establishes a 1-1 mapping between the factors in the products above:

$$
\prod (\zeta - z_i) = \prod (\zeta - \mu_s(z_i)).
$$

However, for any given \( i \)

$$
| \zeta - z_i | < | \zeta - \mu_s(z_i) |,
$$
as \( \zeta \) is to the left of the axis of the symmetry \( \mu \).

The case \( \operatorname{Re} \zeta > \frac{s - 1}{2} \) is treated in the same manner. This contradiction proves Lemma and Theorem 2.1.

2.3. Corollary. A section of \( -mK_X, m > 0 \) of a Grassmann variety \( X \) satisfies the canonical line hypothesis.

2.4. Problems.

A. Are there Fano or general type varieties that do not satisfy the canonical strip hypothesis? How can one characterize those varieties that satisfy (CS) but not (NCS)?

B. Study zeros of the Ehrhart polynomials [BDLD+05], [BHW07] of Fano polytopes as compared to the Ehrhart polynomials of non–Fano polytopes. Is there a characterization of the former? We conjecture that the Ehrhart polynomial of a fan polytope in the space of cocharacters of a smooth toric Fano in dimensions 1, 2, 3, 4, 5 has all zeros in the line \( \operatorname{Re} z = -1/2 \). We furthermore conjecture that the Ehrhart polynomial of a fan polytope of any terminal Gorenstein toric Fano 3-fold (which, by [Fri86], admits a smoothing) has the same property.

C. Starting with dimension 4, the classical Routh–Hurwitz stability criterion furnishes a set of polynomials in Chern numbers whose positivity/non–negativity implies (CS) and (NCS). Is it possible to prove inequalities of such type using Yau’s theorem, or a hierarchy of enhancements is needed?

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1Dimensions 1 and 2 are easy. C. Shramov informed me recently that he had proved this in dimension 3.
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