Glueballs, glueringings and gluestars in the 
\( d = 2 + 1 \) SU(N) gauge theory

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Abstract

The 3d gluodynamics which governs the large \( T \) quark gluon plasma is studied in the framework of the field correlator method. Field correlators and spacial string tension are derived through the gluelump Green’s functions. The glueball spectrum is calculated both in \( C = -1 \) as well as in \( C = +1 \) sectors, and multigluon bound states in the form of ”gluon rings” and ”gluon stars” are computed explicitly. Good overall agreement with available lattice data is observed.

1 Introduction

As was suggested fifteen years ago [1, 2], in QCD magnetic fields survive the temperature phase transition and keep their magnitude almost intact up to \( T = 1.5 \, T_c \), as was checked by lattice measurements of field correlators [3].

At larger \( T, \, T \geq 2 \, T_c \), magnetic fields as measured by the spacial string tension \( \sigma_s(T) \) start to grow quadratically, \( \sigma_s(T) \sim T^2 \) [4, 5], which signifies the advent of a new regime, called the dimensional reduction. In this regime the temporal direction is squeezed, \( \Delta t = \frac{1}{T} \to 0 \), while higher Matsubara frequencies are suppressed, so the effective dynamics is reduced to 3d gluodynamics [6]-[9]. The latter is a confining theory and usually one relies on lattice 3d calculations to obtain physical quantities, such as \( \sigma_s(T) \) [4, 7].
Instead we suggest here to use the Field Correlator Method \[10\], (see \[11\] for a review) to study the 3d nonperturbative dynamics and in particular to compute field correlators, string tension and lowest mass excitations in 3d gluodynamics analytically. Knowing $\sigma_s(T)$ one can compute other physical quantities of interest, e.g. in \[12\] the 3d screening masses of mesons and of the lowest glueball were obtained in good agreement with existing lattice data.

To find $\sigma_s(T)$ one needs to know the nonperturbative correlator of magnetic fields $D^H(x)$ \[112\]. The latter can be studied analytically using the new method suggested in \[13\], where it was expressed in terms of the gluelump Green’s functions. In case of 4d QCD the gluelumps were studied in \[14\], and in the present paper this work is extended to 3d. The confining correlator $D^H(x)$ is expressed through the two-gluon gluelump Green’s function, and for the latter one has to use the relativistic 3-body dynamics, which is well developed in Refs. \[15\]-\[16\]. Another part of our study is the calculation of the gluonic bound states: glueballs, gluerinings and gluestars. We are using the same Hamiltonian approach as in the 4d case \[17\],\[18\] to compute masses of these states in terms of the only parameter of the theory, $\sigma_s(t)$. The resulting ratios $M_n/\sqrt{\sigma_s}$ are in good agreement with existing lattice data.

The plan of the paper is as follows. In section 2 the main formalism is introduced, and the 3 body gluelump Green’s function is calculated, together with $\sigma_s(T)$. In section 3 the glueballs are calculated in the 3d gluodynamics, and in section 4 gluerinings and gluestars are introduced and the masses are computed. In the last section a comparison to lattice and other approaches is given and a summary and perspectives are outlined.

## 2 The large $T$ limit of QCD

The QCD action $S$ at finite temperature

$$S = S_g + S_q; \quad S_g = \frac{1}{4g^2} \int_0^\beta dx_4 \int d^2x (F_{\mu\nu}^a)^2, \quad S_q = \int_0^\beta dx_4 \int d^2x \bar{\psi}(m - D)\psi$$

(1)

at large $T$ goes over into the effective 3d theory \[6\]-\[9\], where the leading part of $S_g$ transforms as

$$S_g = \frac{1}{g^2T} \int d^3x (F_{ik}^a)^2 \equiv \frac{1}{g_3^2} \int d^3x (F_{ik}^a)^2$$

(2)
with $g_3^2 \equiv g^2 T$ is a dimensionful coupling constant of the 3$d$ theory. It is also known that quark degrees of freedom decouple while $A_4$ can be effectively integrated out at large $T$ [6]-[9].

Our aim is the calculation of field correlators in 3$d$ theory using the gluelump Green’s function, and to this end we absorb $g_3$ in $F_{ik}$ as $\bar{F}_{ik} = \frac{1}{g_3} F_{ik}$ and we define the correlator in the same way as in the 4$d$ case [10, 11]

$$\frac{1}{N_c} \langle \text{tr}(g_3 \bar{F}_{ik}(x)\Phi(x,y)g_3 \bar{F}_{em}(y)\Phi(y,x)) \rangle \equiv D_{ik,lm}(x,y) =$$

$$= (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})D^H(x-y) + \frac{1}{2}[\partial_i h_l \delta_{km} + \text{perm}]D_1^H(x-y). \quad (3)$$

The spacial string tension $\sigma_s$ is expressed through $D^H(x)$ in the usual way [1, 2, 19]

$$\sigma_s = \frac{1}{2} \int D^H(x) d^2 x. \quad (4)$$

Note that we keep index $H$ in $D^H(x)$, appropriate for 4$d$ theory, since $D^H$ coincides with the high $T$ limit of 4$d$ magnetic correlator $D_{d=4}^H$.

To proceed one can use as in [13] the Background Perturbation Theory (BPTh) [20] and write in the lowest order of BPTh (cf. Eq. (35) of [13])

$$D_{ik,lm}^H(x,y) = -\frac{g_3^4}{2N_c^2} \langle \text{tr}_a([a_i,a_k]\hat{\Phi}(x,y)[a_l,a_m]) \rangle \quad (5)$$

where $\hat{\Phi}$ and $\text{tr}_a$ are the parallel transporter and trace in adjoint representation respectively.

Writing $[a_i,a_k] = ia^a_i a^k_b f^{abc} T^c$, one can define the 2-gluon gluelump Green’s function $G^{(2gl)}(x,y)$ as

$$D^H_{ik,lm}(x,y) = \frac{g_3^4}{2N_c^2} \langle \text{tr}_a(f^{abc} f^{def} a^a_i(x) a^b_k(x) T^c \hat{\Phi}(x,y) T^f a^d_l(y) a^e_m(y)) \rangle =$$

$$\frac{g_3^4(N_c^2 - 1)}{2} (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) G^{(2gl)}(x,y) \quad (6)$$

and hence from (3) one has

$$D^H(x,y) = \frac{g_3^4(N_c^2 - 1)}{2} G^{(2gl)}(x,y). \quad (7)$$
The two-gluon gluelump Green’s function $G^{(2g)}$ in 4d was studied and numerically evaluated in [14], and in what follows we shall repeat the same calculations for the lowest state in 3d.

In the lowest order in $g_3$ the BPTh gives for $G^{(2g)}$ the following Fock-Feynman-Schwinger (FFS) path-integral representation [21]

$$G^{(2g)}(x,y) = \int_0^\infty ds_1 \int_0^\infty ds_2 (Dz^{(1)})_{xy} (Dz^{(2)})_{xy} e^{-K_1 - K_2} \langle W_F \rangle$$

where

$$W_F = PP_F \exp ig_3 \int_x^y A_i dz_i^{(1)} \exp ig_3 \int_x^y A_k dz_k^{(2)} U_F(x,y)$$

and

$$U_F(x,y) = \exp 2ig_3 \int_0^{s_1} d\tau_1 \hat{F}(z_1(\tau_1)) \exp 2ig_3 \int_0^{s_2} d\tau_2 \hat{F}(z_2(\tau_2))$$

As a next step we neglect the spin interaction of gluons due to the term $U_F$ in [9], which was studied in [14] and shown to be small (spin splittings $< 10\%$), and consequently put $U_F = 1$. The resulting Wilson loop in [8], $W_F \hat{\Phi}(x,y)$, consists of two gluon trajectories in $W_F$ and one straight-line trajectory in $\hat{\Phi}(x,y)$. The vacuum average of this Wilson loop produces the area law factor which can be written as

$$\langle W_F \hat{\Phi}(x,y) \rangle = \exp (-\sigma_s(T)A_{min}(x,y))$$

where $A_{min}(x,y) = \int d^2 \chi \sqrt{\det g}$ is the minimal area surface connecting the three trajectories discussed above.

For small vacuum correlation length $\lambda, \lambda \ll$ hadron size, one can reduce the problem to that of a local Hamiltonian [22] [23]

$$G^{(2g)}(x,y) = \langle x|e^{-H^{(2g)}}t|y \rangle$$

where $H^{(2g)}$ is constructed in the same way, as for the $q\bar{q}$ case in [22], and in the 4d case it was written in [14]. One can deduce from (12) the equation for $G^{(2g)}(x,y)$, separating 2d space coordinates $\mathbf{r}_i$ and Euclidean time $t$

$$\left\{-\frac{\partial}{\partial t} - H^{(2g)}(\mathbf{r}_i, p_i)\right\} G^{(2g)}(\mathbf{r}_i, t; \mathbf{r}_i', 0) = \delta(t) \prod_{i=1,2} \frac{1}{2\mu_i} \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_i')$$
In our case of \( d = 2 + 1 \), the corresponding 2\( d \) Hamiltonian has the form (see Appendix 1 of \[14\] for details of derivation).

\[
H^{(2d)} = \frac{\mu_1 + \mu_2}{2} + \frac{\mathbf{p}_1^2}{2\mu_1} + \frac{\mathbf{p}_2^2}{2\mu_2} + \sigma_s(T)\{|\mathbf{r}_1| + |\mathbf{r}_2| + |\mathbf{r}_1 - \mathbf{r}_2|\} \tag{14}
\]

where \( \mu_i \) are einbein masses to be found from the minimization of eigenvalues \[22\]; \( \mathbf{p}_i, \mathbf{r}_i \) are 2\text{d} vectors of momentum and coordinate of the \( i \)-th gluon, while the position of the adjoint source (the \( \hat{\Phi}(x,y) \) trajectory) is chosen at the origin.

To solve (14) for the eigenvalues \( M^{(2d)}_n(\mu_1, \mu_2) \) it is convenient to use the hyperspherical harmonics \[15, 16\], with the global radius \( \rho \),

\[
\rho^2 = \xi^2 + \eta^2, \quad \eta = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}}, \quad \xi = \frac{\mathbf{r}_1 + \mathbf{r}_2}{\sqrt{2}}; \tag{15}
\]

and the Schroedinger-type equation for the lowest harmonics looks like (one can take \( \mu_1 = \mu_2 = \mu \), as follows from minimization anyhow)

\[
-\frac{1}{2\mu} \Delta_4 \psi + V(\rho) \psi = E \psi \tag{16}
\]

with

\[
\Delta_4 = \frac{d^2}{d\rho^2} \Psi + \frac{3}{\rho} \frac{d\Psi}{d\rho}, \quad V(\rho) = \int d\Omega_4 V(\mathbf{r}_1, \mathbf{r}_2). \tag{17}
\]

One obtains

\[
V(\rho) = c_v \rho, \quad c_v \approx \left(1 + \frac{2\sqrt{2}}{3}\right) \sigma_s = \bar{c} \sigma_s \tag{18}
\]

A good accuracy (around 1\%) one obtains when using the simple variational wave function in \[16\]

\[
\Psi(\rho) = C_0 \exp\left(-\rho^2 \nu^2 \right), \quad E(\mu, \nu) = \frac{c_v}{\nu} \Gamma\left(\frac{5}{2}\right) + \frac{\nu^2}{\mu}. \tag{19}
\]

Minimization with respect to the variational parameter \( \nu \) yields

\[
\nu_0 = \left(\frac{\mu c_v \Gamma(5/2)}{2}\right)^{1/3}, \quad E(\mu, \nu_0) = \frac{3}{2} \left(\frac{2}{\mu}\right)^{1/3} \left(\frac{c_v \Gamma(5/2)}{2}\right)^{2/3}, \tag{20}
\]
\[ C_0 = \sqrt{2} \nu_0^2, \quad |\Psi(0)|^2 = C_0^2 = 2 \nu_0^4. \]

Finally, the gluelump mass is obtained from minimization over \( \mu \), \( \frac{\partial M_0(\mu, \nu_0)}{\partial \mu} = 0 \) and has

\[ m_0 \equiv M_0(\mu_0, \nu_0) = 4 \mu_0 = 4.52 \sqrt{\sigma_s}. \]  \hspace{1em} (21)

The spectral decomposition of the gluelump Green’s function has the usual form

\[ G^{(2\text{gl})}(\mathbf{0}, t; \mathbf{0}, 0) \equiv G^{(2\text{gl})}(t) = \sum_{n=0}^{\infty} \frac{|\Psi_n(0)|^2}{(2\mu_n)^2} e^{-M_n^{(2\text{gl})}|t|} \]  \hspace{1em} (22)

where \( |\Psi_n(0)|^2 = \frac{C_n^2}{2\mu_n} = \frac{C_n^2}{2\mu_0} \). We have used in (22) the standard boson normalization \( \frac{1}{\sqrt{2\omega}} = \frac{1}{\sqrt{2\mu}} \) for each gluon, since \( \mu = \mu_0 \) is the average energy of the gluon, as one can deduce from (14), note that as in (21) \( M_n^{(2\text{gl})} = 4\mu_n \).

In this way one has from (4)

\[ \sigma_s = g^4 \left( N_c^2 - 1 \right) \frac{1}{2} \int G^{(2\text{gl})}(x) d^2 x = \frac{g^4}{8\pi} \left( N_c^2 - 1 \right) \xi \]  \hspace{1em} (23)

where \( \xi \) is

\[ \xi = 2\pi \int G^{(2\text{gl})}(x) d^2 x = \sum_{n=0}^{\infty} \frac{C_n^2}{2\mu_n^2 M_n^2} \equiv \sum_{n=0}^{\infty} \xi_n. \]  \hspace{1em} (24)

One can see from (20,21) that the term with \( n = 0 \) contributes to \( \xi \) very little

\[ \xi = 0.063 + \sum_{n=1}^{\infty} \xi_n. \]  \hspace{1em} (25)

The sum over \( n \) in (24) is actually diverging, since \( G^{(2\text{gl})}(t) \) as the Green’s function of the Hamiltonian (14), should behave at small \( t \) as the free Green’s function, i.e. \( G^{(2\text{gl})}(t) \sim t^{-2} \), and the integral \( \int d^2 x G^{(2\text{gl})}(x) \) is logarithmically divergent at \( x = 0 \). This purely perturbative contribution to \( D^H \), is exactly cancelled by other correlators in the cumulant sum, as shown in (21). One can show, in the same way as it was done in (13) for the 4d case, that \( G^{(2\text{gl})}(t) \) has nonsingular behaviour after subtraction of the perturbative singularity \( O(t^{-2}) \). In this derivation one should take into account that for small loops the area law behaviour (11) is replaced by the quadratic area law, \( \langle W \rangle \sim \exp \left( -\frac{g^2(F_{\text{min}})}{24N_c} \epsilon^2 \right) \), see second ref. in (10) for details. To estimate \( \xi \), we
replace the linear interaction in $H^{(2g)}$ by the oscillator one, (with $\omega \approx m_0/2$
of to reproduce the asymptotics), and obtain

$$\xi \geq \int_{z_0}^{\infty} \frac{zdz}{(\sin h2z)^2}, \quad z_0 \lesssim \frac{2\omega}{m_0} = 1. \quad (26)$$

One can see, that $\xi$ is of the order of 1, but $\xi < 1$ for reasonable values of $z_0$. Here introduction of $z_0$ regularizes the perturbative behaviour of $G^{(2g)}(x)$ at small $x$ and one can see that this interplay of perturbative and nonperturbative regime at small $x$ of vital importance for the resulting value of $\xi$, and hence of $\sigma_s$. This should be compared with lattice data, where one obtains for $N_c = 3$

$$\sqrt{\sigma_s} = g_3^20.566 = 0.566Tg^2(T); \quad \xi_{lat} \approx 1.00 \quad (27)$$

3 Glueball excitations in the 3d gluodynamics

As was discussed above, the high-temperature QCD has the 3d gluodynamics as a limiting theory, and therefore it is interesting not only to compute field correlators and $\sigma_s(T)$, as was done in the previous section, but also to find the spectrum of this limiting theory.

In this section we shall calculate the spectrum of two-gluon ($2g$) and three-gluon ($3g$) glueballs in 3d SU(N) gluodynamics and compare results with lattice calculations. The lowest two-gluon glueball state was previously calculated in the framework of our formalism in [12]. We define the creation operators for glueball states together with spin and parity $J^{PC}$ in Appendix 1. One can start from the general QCD string formalism with spin-dependent terms, as was done in [17, 18] in the case of 4d QCD, but here we shall neglect both spin corrections and string rotation corrections [22], important for $L > 2$, and consider the simplest version, calculating with accuracy of 10% or better, so that one has the Hamiltonian for the $2g$ case

$$H = \mu + \frac{P^2}{\mu} + \sigma_a r : \quad H\Psi_n = E_n(\mu)\Psi_n, \quad (28)$$

with the adjoint string tension $\sigma_a = \frac{2N_c^2}{N_c^2 - 1}\sigma_s$, and the glueball mass is $M_n = E_n(\mu_0)$, where $\mu_0$ is the stationary point of $E_n(\mu)$, $\frac{\partial E_n}{\partial \mu}|_{\mu=\mu_0} = 0$. It is known [25, 26], that the accuracy of this procedure is around 5-7%. Instead of
computing (28) numerically, solving the 2\text{d} differential equation, we shall approximate solutions replacing $\sigma_{a}r$ by the term $\left(\frac{\sigma_{a}^{2}r^{2}}{2\nu} + \frac{\nu}{2}\right)$ and $\nu$ is later to be found as well as $\mu$ from the stationary point equation. We shall call the resulting energy $E_{n}(\mu, \nu)$ and corresponding masses $M_{n}$. We expect (and calculations support this) that $\tilde{M}_{n}$ differ from $M_{n}$ within 10%. As a result one obtains a simple expression for $\tilde{M}_{n}$

$$\tilde{M}_{n} = 4\sqrt{\frac{n\sigma_{a}}{2}}, \quad n = 1, 2, 3, \ldots$$

(29)

Here $n = L + 1$, and in our approximation radial excited state $0^{++}$ is degenerate in mass with the $2^{+}$ state.

As was mentioned above, we disregard spin-dependent interaction and therefore the $L = 0$ state corresponds to degenerate $0^{++}, 2^{++}$ states while $L = 2$ state – to degenerate $0^{+}, 2^{++}, 4^{++}$ states and so on.

The connection (29) is universal for any SU(N) group, however to connect masses to fundamental spacial string tension $\sigma_{s}$ one needs to use the relation

$$\sigma_{a} = \frac{2N_{c}^{2}}{N_{c}^{2} - 1}\sigma_{s}, \quad \sigma_{a} = \frac{9}{4}\sigma_{s}(\text{SU}(3)), \quad \sigma_{a} = \frac{8}{3}\sigma_{s}(\text{SU}(2)), \quad \sigma_{a} = 2\sigma_{s}(\text{SU}(\infty))$$

(30)

The resulting ratios $\frac{\tilde{M}_{n}}{\sqrt{\sigma_{s}}}$ are shown in Table 1 in comparison with lattice data from [27].

**Table 1:** The mass ratios $\tilde{M}_{n}/\sqrt{\sigma_{s}}$ obtained from the Hamiltonian (28) vs lattice data for angular momentum $L = 0, 1, 2, 3, 4$. For comparison two lowest masses obtained on the lattice [27] are given in parentheses.

1The spin-parity assignment in [27] is in general different from ours in Appendix 1, and therefore only states with reliably the same quantum numbers are shown in the Tables 1 and 2.
The 3g glueballs can be considered in the same way, as it was done in [17, 18] for the 4d case. The Hamiltonian is similar to the one used in section 2, Eq. (14) and reads (we put $\mu_1 = \mu_2 = \mu_3 = \mu$).

$$H^{(3g)} = \frac{3\mu}{2} + \frac{p_\xi^2 + p_\eta^2}{2\mu} + \sigma_s \sum_{i>j} |r_i - r_j|.$$  

The hyperspherical component of the grand orbital momentum $K$ satisfies an equation (see Appendix 2 for details of derivation)

$$\left\{ -\frac{1}{2\mu} \left( \frac{d^2}{d\rho^2} + \frac{2K + 3}{\rho} \frac{d}{d\rho} \right) + V(\rho) \right\} \Psi_K(\rho) = E_K \Psi_K(\rho)$$  

with $V(\rho) = c\rho$, $c = 2\sqrt{2}$.

As is well known (see e.g. [16]) the solution of (31) is given with 1-2% accuracy by the equation

$$E_{K,n}(\mu) = W(\rho_0) + \sqrt{\frac{W''(\rho_0)}{\mu}} \left( n + \frac{1}{2} \right)$$  

where

$$W(\rho) = \frac{(K + \frac{1}{2})(K + \frac{3}{2})}{2\mu \rho^2} + V(\rho),$$
and $\rho_0$ is the stationary point of $W(\rho)$, $W'(\rho)|_{\rho=\rho_0} = 0$. The mass $M_{K,n} = \left\{ \frac{3\mu}{2} + E_{K,n}(\mu) \right\}_{\min(\mu)}$ is obtained to be

$$M_{K,n} = \sqrt{\sigma_s} 2^{7/4}(3\Lambda)^{1/4} \left( 1 + \frac{2n+1}{\sqrt{3\Lambda}} \right)^{3/4}, \quad \Lambda \equiv \left( K + \frac{1}{2} \right) \left( K + \frac{3}{2} \right). \quad (34)$$

The resulting ratio of $M_{K,n}/\sqrt{\sigma_s}$ are given in the Table 3 for the radial excitation number $n = 0$ and $K = L = 0, 1, 2, 3$.

**Table 2:** The three-gluon glueballs in the 3d SU(N) gluodynamics. Mass ratios $M_{L,0}/\sqrt{\sigma_s}$ are given for $L = 0, 1, 2, 3$. The lowest $C = -1$ mass from lattice data [27] is given for comparison in parentheses (see footnote on the previous page).

| K=L | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $J^{PC}$ | $1^{--}, 3^{--}$ | $1^{+-}, 3^{+-}$ | $3^{--}, 1^{--}, 5^{--}$ | $1^{+-}, 3^{+-}$ |
| $\frac{M_{L,0}}{\sqrt{\sigma_s}}$ | 6.01 (5.91, 0--°) | 7.48 (6.73, 2--) | 8.68 (8.73, 2--) | 9.74 |

Using Eq. (34) one can also calculate radial excitations, which are actually collective excitations corresponding to the collective variable $\rho$. The resulting mass ratios $\frac{M_{L,n}}{\sqrt{\sigma_s}}$ for $n = 1, 2$ are 9.39 and 12.35 respectively.

One can now consider the pomeron and odderon trajectories, using mass formulas (29) and (33). The pomeron trajectory corresponds to $J = L + 2$, where "2" comes from the sum of "gluon spins" as in the $d = 3 + 1$ case. In the $d = 2 + 1$ case (as well as for $d = 3 + 1$) the creation operators are of the form $tr(D^{L/2}E_i D^{L/2}E_k)$.

From (29) one has

$$M_J^2 = 8\sigma_a(L + 1) = 8\sigma_a(J - 1) \quad (35)$$

and the pomeron intercept is exactly unity, while the Regge slope corresponds to the relativistic potential case, since string corrections [22,23] are not taken into account.
The odderon trajectory is obtained similarly identifying \( J = L + 3 \), and finding from (34) (with accuracy better than 1%) analytic expression

\[
M^2(J) = a(J - \Delta_{\text{odd}});
\]

where \( a = 2^{7/2} \sqrt{3} \sigma_s \approx 19.6 \sigma_s \); \( \Delta_{\text{odd}} = 2 - \frac{\sqrt{3}}{2} = 1.14 \).

One can see, that surprisingly the odderon intercept appears higher, than that of pomeron, in contrast to the situation found in the \( d = 3 + 1 \) QCD.

4 Gluerings and gluestars

For number of gluons \( N \geq 3 \) one has two options in constructing gauge-invariant gluon configurations:

a) gluon ring, consisting of \( N \) gluons, "sitting" on the closed loop of fundamental string, with the potential energy \( V_{\text{ring}} = \sigma_f \sum_{i=j+1}^N |r_i - r_j| \);

b) gluon star, consisting of \( N \) gluons connected by \( N \) adjoint string pieces with the string junction, where color indices are assembled in \( d^{abc} \) (for \( N_c = 3 \)) or in higher order operators for \( N_c > 3 \). The corresponding potential energy is \( V_{\text{star}} = \sigma_a \sum_{i=1}^N |r_i - Y| \), \( Y \) – is the string junction position.

The masses of gluon rings and gluon stars can be computed in the hyperspherical formalism \[15\] using formulas given in Appendix 2. One has from (A2.13)

\[
\frac{M_{\text{ring}}}{\sqrt{\sigma_s}} = 2(N\Lambda_{KN}c_N^2)^{1/4} \left( 1 + \frac{2(n + 1/2)}{\sqrt{3}\Lambda_{KN}} \right)^{3/4},
\]

with \( \Lambda_{KN} = (K + N + \frac{3}{2})(K + N - \frac{5}{2}) \), \( c_N = \frac{\sqrt{7}N(N-1)}{N-2} \Gamma \left( \frac{3}{2} \right) \), and

\[
\frac{M_{\text{star}}}{M_{\text{ring}}} = \sqrt{\frac{2\sigma_a}{\sigma_s}} \left( \frac{N-1}{2N} \right)^{1/4} \left( \frac{N-1}{N-2} \right) \Gamma \left( \frac{3}{2} \right).
\]

One can immediately see that gluestars are 14% (\( N = 3 \)) to 26% (\( N = \infty \)) heavier than gluon rings and both saturate at large \( N \), i.e. \( \frac{M_{\text{ring}}}{\sqrt{\sigma_s}} \to 2.24(N \to \infty) \).

A similar situation and the same ratios are obtained for \( D = 3 + 1 \) in \[18\].

5 Summary and discussion

An important feature of our results is that our calculation does not use any parameters. Several approximations have been made in obtaining \[29\], \[31\]:

\[
\Lambda_{KN} = (K + N + \frac{3}{2})(K + N - \frac{5}{2}) \approx \sqrt{3}\Lambda_{KN}.
\]
1) the use of a local Hamiltonian, implying small vacuum correlation length \( \lambda \), \( \lambda \ll \) hadron size, and small mixing with hybrids due to the large mass gap \([28]\); 2) spin forces were considered weak and self-energy corrections vanishing due to renormalization and gauge invariance (see appendix 4 of \([13]\) for discussion); 3) color Coulomb interaction between gluons was neglected due to the expected strong cancellation with the one-loop correlation, observed in \( 4d \) BFKL formalism (cf. discussion in \([17,18]\)). In \( d = 4 \) glueball calculations \([17,18]\) this suppression of Coulomb resulted in good agreement of calculated and lattice measured glueball spectrum.

Hence all our approximations are expected to be accurate within fifteen percent or better, which is supported by a good agreement with lattice data \([27]\) in Tables 1,2.

On theoretical side the \( d = 2+1 \) gluodynamics was considered also in \([30]\), where the authors developed a Hamiltonian formalism for the calculation of \( \sigma_s \) and vacuum excitations. Their result for \( \sigma_s \) corresponds to \( \xi \equiv 1 \) in our Eq. \([24]\) and agrees with lattice data, however the glueball masses to the knowledge of the author are not yet available from the formalism of \([30]\).

Summarizing, we have obtained the confining field correlator, \( \sigma_s \), gluelump and glueball masses for \( d = 2 + 1 \) \( SU(N_c) \) gluodynamics. Glueball masses for configurations of different types: simple glueballs for \( N = 2 \), and gluerings and gluestars for \( N \geq 3 \) are computed explicitly without any parameters through \( \sigma_s \).

Glueball spectrum has a clear hierarchy in growing angular momentum \( L \) and in number of gluons \( N \). In this and other respects the \( d = 2 + 1 \) spectrum is similar to that of \( d = 3 + 1 \).

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Appendix 1

Creation operators of the $d = 2 + 1$ SU(N) glueballs

In $d = 2 + 1$ one can choose $x_3$ as the Euclidean time evolution coordinate, and denote $E_i = F_{i3}, \ H_3 = F_{12}$. The gauge invariant creation operators are constructed from powers of $E_i, H_3$, and higher operators also include powers of $D_i = \partial_i - igA_i$, with $i = 1, 2$. The 2d angular momentum $L$ and parity for a given creation operator is defined by the assignment of $L = 1$ to $H_3$ and $D$, $L = 0$ to $E_i$. The 2d -parity is defined by $PE_i = -E_i, \ PH_3 = H_3, PD_i = -D_i$. The combination $D_iE_i$ vanishes due equations of motion. The $C$ parity is defined from the transformation $F_{ik}^C \rightarrow -(F_{ik})^T, \ D_i^C \rightarrow -D_i^T$. The resulting operators together with $L$ and $J^{PC}$ assignments are given in Table 3 for 2$g$ glueballs and in Table 4 for 3$g$ glueballs.

**Table 3:** Creation operators for 2$g$ glueball states $E_i = F_{i3}, \ H_3 = F_{12}$ in $d = 2 + 1$ SU(N) theory.

| L | $J^{PC}$ | Operator |
|---|---|---|
| 0 | $0^{++}$ | $tr(E_iE_i)$ |
| 0 | $2^{++}$ | $tr(E_iE_k)_{symm.}$ |
| 1 | $2^{-+}, 0^{-+}$ | $tr(E_iH_3)$ |
| 2 | $2^{++}, 0^{++}$ | $tr(H_3H_3)$ |
| 2 | $0^{++}$ | $tr(D_iE_kD_iE_k)$ |
| 2 | $2^{++}$ | $tr(D_iE_kD_iE_l)_{symm.}$ |
| 2 | $4^{++}$ | $tr(D_iE_kD_lE_m)_{symm.}$ |
| 3 | $2^{-+}, 0^{-+}$ | $tr(D_iE_kD_iH_3)$ |
| 3 | $4^{-+}, 2^{-+}$ | $tr(D_iE_kD_iH_3)_{symm.}$ |
| 4 | $2^{++}0^{++}$ | $tr(D_iH_3D_iH_3)$ |
| 4 | $6^{++}, 4^{++}, 2^{++}$ | $tr(D_iH_3D_kH_3)_{symm.}$ |
| 4 | $6^{++}$ | $tr(D_iD_kE_lD_mD_mE_p)_{symm.}$ |
Table 4: The same as in Table 4, but for 3g glueballs.

| L | $J^{PC}$ | Operator |
|---|----------|----------|
| 0 | 1$^{--}$ | $tr(E_iE_kE_k)$ |
| 0 | 3$^{--}$ | $tr(E_iE_kE_i)_{symm.}$ |
| 1 | 1$^{++}$ | $tr(E_i^2H_3)$ |
| 1 | 3$^{++}, 1^{+-}$ | $tr((E_iE_k)H_3)_{symm.}$ |
| 2 | 3$^{--}, 1^{--}$ | $tr(E_iH_3H_3)$ |
| 2 | 2$^{++}, 4^{--}$ | $tr(E_iE_kD_3H_3)$ |
| 3 | 1$^{+-}, 3^{++}$ | $tr(H_3H_3H_3)$ |

Appendix 2

Calculation of the $N$--gluon glueball masses for $N \geq 3$

The Hamiltonian of $N$-gluon glueballs has the form similar to (31), we fix the gluon einbein masses to be the same: $\mu_1 = \mu_2 = ... = \mu$.

$$H = \frac{M\mu}{2} + \frac{\Delta_{2N-2}}{2\mu} + V_N(r_1, ...r_N) \quad (A2.1)$$

where $V_N$ can be of two different forms, for gluon rings (closed gluon chains with fundamental strings between adjacent gluons)

$$V_N^{(r)} = \sigma_s \sum_{i=0}^{N-1} |r_{i+1} - r_i|, \quad r_N \equiv r_0 \quad (A2.2)$$

and for gluon stars (a generalization of the $Y$-type glueball for $N_c = 3$ with the color symmetric matrix $d^{abc}$ at the string junction vertex, see [18] for details)

$$V_N^{(star)} = \sigma_a \sum_{i=1}^{N} |r_i - Y| \quad (A2.3)$$
where \( Y \) is the position of the string junction, which for gluons can be taken to coincide with the c.m. coordinate \( R = \frac{1}{N} \sum_{i=1}^{N} r_i \).

We approximate the eigenfunction of \( H \) for a given angular momentum \( L \) by the hyperspherical component \( \Psi_K(r_1, r_2, \ldots r_N) = \frac{\chi_K(\rho)}{\rho^{2K+N+\frac{3}{2}}} P_K(r_1, r_2, \ldots r_N) \) (A2.4)

where \( \rho^2 = \sum_{i=1}^{N} (r_i - R)^2 \), and \( P_K \) is the harmonic polynomial of Jacobi coordinates \( \xi_i, i = 1, N - 1 \), constructed of \( r_1, \ldots r_N \),

\[ \xi_i = \frac{1}{\sqrt{i(i + 1)}} \left[ \sum_{j=1}^{i} r_j - i r_{i+1} \right] \] (A2.5)

Averaging of \( \langle \Psi_K \hat{H} \Psi_K \rangle \) over angular variables \( \Omega_{2N-2} \) in \( (2N - 2) \) space of \( \{ \xi_i \} \) yields an equation for \( \chi_K(\rho) \)

\[ -\frac{1}{2\mu} \frac{d^2}{d\rho^2} + W_{KN}(\rho) + \frac{N\mu}{2} \chi_K(\rho) = M_{KN}(\mu)\chi_K(\rho) \] (A2.6)

where \( W_{KN}(\rho) \) is

\[ W_{KN}(\rho) = \frac{\Lambda_{KN}}{2\mu \rho^2} + V_{N}^{(k)}(\rho), \quad \Lambda_{KN} \equiv \left( K + N - \frac{3}{2} \right) \left( K + N - \frac{5}{2} \right) \] (A2.7)

and

\[ V_{N}^{(k)}(\rho) \equiv \langle V_{N}^{(k)}(r_1, r_2, \ldots) \rangle_\Omega, \quad k = r, \text{ star.} \] (A2.8)

To calculate \( V_{N}^{(k)}(\rho) \) it is enough to know the angular average of one Jacobi vector,

\[ V_{N}^{(k)}(\rho) = \sigma_s N \lambda^{(k)}(\{\xi_i\})_\Omega = \sigma_s \rho N \frac{\lambda^{(k)} \Gamma(N - 1) \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( N - \frac{1}{2} \right)} \equiv C_{N}^{(k)} \rho \sigma_s \] (A2.9)

and \( \lambda^{(k)} = \sqrt{2} \) for \( k = r (\text{ring}) \), and \( \lambda^{(\text{star})} = \sqrt{\frac{N-1}{N}} \frac{\sigma_s}{\sigma_{a}} \), as one can deduce from (A2.5) for \( i = N - 1 \).

Finally the solution for \( M_{KN}(\mu) \) can be found with \( \sim 1\% \) accuracy from the minimum of \( W_{KN}(\rho) \) as a function of \( \rho \) \[ \frac{m_{KN}}{\sqrt{\sigma_s}}(\mu) = W_{KN}(\rho_0) + \omega \left( n + \frac{1}{2} \right), \quad \omega^2 = \frac{1}{\mu} W''(\rho_0) \] (A2.10)
where \( \rho_0 = \left( \frac{\Lambda_{KN}}{\mu_{0N}} \right)^{1/3} \), so that one has

\[
m_{KN}(\mu) = \eta_N \frac{\Lambda_{KN}}{\mu^{1/3}}, \quad \eta_N = \frac{3}{2} \Lambda_{KN}^{1/3} \left( c_n^{(k)} \right)^{2/3} \left( 1 + \frac{2(n + 1/2)}{\sqrt{3} \Lambda_{KN}} \right).
\] (A2.11)

From \( \frac{\partial M_{KN}(\mu)}{\partial \mu} \big|_{\mu = \mu_0} = 0 \) one has

\[
M_{KN}(\mu) = \frac{N \mu}{2} + \frac{\eta_N}{\mu^{1/3}}, \quad \mu_0 = \left( \frac{2 \eta_N}{3 N} \right)^{3/4}
\] (A2.12)

and finally the mass of the \( N \) gluon state is

\[
\frac{M_{KN}}{\sqrt{\sigma_s}} = 2 B_N^{1/4} \left( 1 + \frac{2 \left( \frac{n + 1}{2} \right)}{\sqrt{3} \Lambda_{KN}} \right)^{3/4}
\] (A2.13)

where

\[
B_N = N \Lambda_{KN} (C_N^{(k)})^2.
\]

For \( N = 3 \) and \( K = 0 \), \( B_3 = 18 \) and \( \frac{M_{3}(\text{ring})}{\sqrt{\sigma_s}} = 6.01, 9.39, 12.35 \) for \( n = 0, 1, 2 \) respectively.

For \( N \) large, one can consider a gluon ring of \( N \) gluons connected by fundamental string for any \( N_c \).

Then asymptotically \( C_{K,N}(N \to \infty) \approx \sqrt{\frac{\pi}{2}} \sqrt{\frac{N}{N - 1}}, B_N^{1/4} = N \left( \frac{\pi}{2} \right)^{1/4} \) and

\[
\frac{M_{KN}(N \to \infty)}{N \sqrt{\sigma_s}} = 2 \left( \frac{\pi}{2} \right)^{1/4} \approx 2.24.
\] (A2.14)

For \( N = 3 \) this asymptotic formula yields \( \frac{M}{\sqrt{\sigma_s}} \approx 6.72 \) which is not far from our exact value of 6.01.

The gluon stars are obtained with \( \lambda^{(\text{star})} \) in \( C_N^{(k)} \) instead of \( \lambda^{(\text{ring})} \), which yields for the mass the relative factor \( \sqrt{\frac{\lambda^{(\text{star})}}{\lambda^{(\text{ring})}}} = \sqrt{\frac{\sigma_a}{\sigma_s}} \cdot \left( \frac{N - 1}{2N} \right)^{1/4} \). For \( N = 3 \) this factor is 1.14, which means, that the 3-gluon star is 14% heavier than the 3-gluon ring.

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