ON THE DISCRETE SPECTRUM OF COMPLEX BANDED MATRICES

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Abstract. The discrete spectrum of complex banded matrices that are compact perturbations of the standard banded matrix of order \( p \) is under consideration. The rate of stabilization for the matrix entries sharp in the sense of order which provides finiteness of the discrete spectrum is found. The \( p \)-banded matrix with the discrete spectrum having exactly \( p \) limit points on the interval \((-2, 2)\) is constructed. The results are applied to the study of the discrete spectrum of asymptotically periodic Jacobi matrices.

1. Introduction.

In the recent papers [1, 2] I. Egorova and L. Golinskii studied the discrete spectrum of complex Jacobi matrices such that the operators in \( \ell^2(\mathbb{N}) \), \( \mathbb{N} := \{1, 2, \ldots\} \) generated by these matrices are compact perturbations of the discrete laplacian. In turn, these papers are the discrete version of the known Pavlov’s theorems ([3] [4]) for the differential operators of second order on the semiaxis. The sufficient conditions for the spectrum to be finite and empty, the domains containing the discrete spectrum and the conditions for the limit sets of the discrete spectrum were found. The goal of this work is to extend these results to the case of operators, generated by banded matrices.

Let us remind that an infinite matrix \( D = \|d_{ij}\|_{i,j=1}^\infty \) is called the banded matrix of order \( p \) or just \( p \)-banded if
\[
\begin{align*}
d_{ij} = 0, & \quad |i - j| > p, \quad d_{ij} \neq 0, \quad |i - j| = p, \quad d_{ij} \in \mathbb{C}.
\end{align*}
\]
According to this definition, the Jacobi matrices are banded matrices of order \( p = 1 \).

Throughout the whole paper we assume that
\[
\begin{align*}
\lim_{i \to \infty} d_{i,i+p} &= 1, & \lim_{i \to \infty} d_{i,i+r} &= 0, & |r| < p,
\end{align*}
\]
and so the operators in \( \ell^2 = \ell^2(\mathbb{N}) \) generated by matrices \([1, 2]\) are compact perturbation of the standard banded operator
\[
D_0 : \quad d_{i,i+p} = 1, \quad d_{ij} = 0, \quad |i - j| \neq p, \quad D_0 = S^p + (S^*)^p,
\]
where \( S \) is the one-sided shift operator in \( \ell^2 \). It is well known that the spectrum \( \sigma(D_0) \) of \( D_0 \) is the closed interval \([-2, 2] \). According to the Weyl theorem (see, e.g., [5]) the spectrum of the perturbed operator \( \sigma(D) = [-2, 2] \cup \sigma_d(D) \), where the discrete spectrum \( \sigma_d(D) \) is at most denumerable set of points of the complex plane, which are eigenvalues of finite algebraic multiplicity. All its accumulation points belong to the interval \([-2, 2]\). Let us denote by \( E_D \) the limit set for the set \( \sigma_d(D) \). So, \( E_D = \emptyset \) means that the discrete spectrum is finite.

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Remind that the convergence exponent or Taylor-Besicovitch index of a closed point set \( F \subset [-2, 2] \) is the value
\[
\tau(F) := \inf \{ \varepsilon > 0 : \sum_{j=1}^{\infty} |l_j|^\varepsilon < \infty \},
\]
where \( \{l_j\} \) are the adjacent intervals of \( F \).

**Definition 1.1.** We say that the matrix \( D \) (1.1) belongs to the class \( P_p(\beta) \), \( 0 < \beta < 1 \), if
\[
q_n := |d_{n,n-p}-1| + \sum_{r=-p+1}^{p-1} |d_{n,n+r}| + |d_{n,n+p}-1| \leq C_1 \exp(-C_2 n^\beta),
\]
\( n \in \mathbb{N} \), with the constants \( C_1, C_2 > 0 \), depending on \( D \).

The main result of the present paper is the following

**Theorem 1.2.** Let \( D \in P_p(\beta) \) where \( 0 < \beta < \frac{1}{2} \). Then \( E_D \) is a closed point set of the Lebesgue measure zero and its convergence exponent satisfies
\[
\dim E_D \leq \tau(E_D) \leq \frac{1-2\beta}{1-\beta},
\]
where \( \dim E_D \) is the Hausdorff dimension of \( E_D \). Moreover, if \( D \in P(\frac{1}{2}) \) then \( E_D = \emptyset \), i.e., the discrete spectrum is finite.

It turns out that the exponent 1/2 in theorem 1 is sharp in the following sense.

**Theorem 1.3.** For arbitrary \( \varepsilon > 0 \) and arbitrary points \( \nu_1, \nu_2, \ldots, \nu_p \in (-2, 2) \) there exists an operator \( D \in P_p(\frac{1}{2} - \varepsilon) \) such that its discrete spectrum \( \sigma_d(J) \) is infinite and, moreover,
\[
E_D = \{\nu_1, \nu_2, \ldots, \nu_p\}.
\]

The theorems 1 and 2 are proved in Section 4, where the domains which contain \( \sigma_d(D) \) are also found (under the weaker assumptions than (1.4)). In Section 2 the connection is established between the discrete spectrum and zeros of the determinant constructed of \( p \) linearly independent solutions of the linear difference equations for the eigenvector. In Section 3 the properties of the matrix Jost solutions are studied. Finally, in the last Sections 5 and 6 the main results are applied to study the spectrum of doubly-infinite complex banded matrices and the spectrum of doubly-infinite asymptotically \( p \)-periodic complex Jacobi matrix.

2. **The Determinants of Independent Solutions and the Eigenvalues**

We start out with the equation
\[
D\vec{y} = \lambda\vec{y}
\]
for generalized eigenvectors \( \vec{y} = \{y_n\}_{n \geq 1} \) in the coordinate form:
\[
\begin{align*}
d_{11}y_1 + d_{12}y_2 + \ldots + d_{1,p+1}y_{p+1} &= \lambda y_1, \\
d_{21}y_1 + d_{22}y_2 + \ldots + d_{2,p+2}y_{p+2} &= \lambda y_2, \\
\ldots \\
d_{p,1}y_1 + d_{p,2}y_2 + \ldots + d_{p,2p}y_{2p} &= \lambda y_p, \\
d_{n,n-p}y_{n-p} + d_{n,n-p+1}y_{n-p+1} + \ldots + d_{n,n+p}y_{n+p} &= \lambda y_n, \quad n = p + 1, p + 2, \ldots.
\end{align*}
\]
It is advisable to define coefficients $d_{i,j}$ for the indices with $\min(i, j) \leq 0$ as follows:

\begin{align}
 (2.3) \quad & d_{ij} = 1, \quad |i - j| = p, \quad d_{ij} = 0, \quad |i - j| \neq p,
\end{align}

and so system (2.2) is equivalent to

\begin{align}
 (2.4) \quad & d_{n,n-p} y_{n-p} + d_{n,n-p+1} y_{n-p+1} + \ldots + d_{n,n+p} y_{n+p} = \lambda y_n, \quad n \in \mathbb{N}
\end{align}

with the initial conditions

\begin{align}
 (2.5) \quad & y_{1-p} = y_{2-p} = \ldots = y_0 = 0.
\end{align}

Thus, the vector $\vec{y} = \{y_n\}_{n \geq 1-p} \in \ell^2$ is the eigenvector of the operator $D$ corresponding to the eigenvalue $\lambda$ if and only if $\{y_n\}$ satisfies (2.4), (2.5).

It seems natural to analyze equation (2.4) within the framework of the general theory of linear differential equations. The equation

\begin{align}
 (2.6) \quad & x(n + k) + a_1(n)x(n + k - 1) + \ldots + a_k(n)y(n) = 0
\end{align}

is said to belong to the Poincaré class if $a_k(n) \neq 0$ and there exist limits (in $\mathbb{C}$)

\begin{align}
 b_j = \lim_{n \to \infty} a_j(n), \quad j = 1, 2, \ldots, k.
\end{align}

Denote by $\{w_j\}_{j=1}^k$ all the roots (counting the multiplicity) of the characteristic equation

\begin{align}
 (2.7) \quad & w^k + b_1 w^{k-1} + \ldots + b_k = 0.
\end{align}

One of the cornerstones of the theory of linear differential equations is the following result due to Perron.

**Theorem (2.4), Satz 3.** Let the roots $\{w_j\}$ of (2.7) lie on the circles $\Gamma_l = \{|w| = \rho_l\}, \ l = 1, 2, \ldots, m, \ \rho_j \neq \rho_k$, and exactly $\nu_l$ of them (counted according with their multiplicity) belongs to each circle $\Gamma_l$, so $\nu_1 + \ldots + \nu_m = k$. Then (2.6) has a fundamental system of solutions

\begin{align}
 S = \{y_1, \ldots, y_k\} = \bigcup_{l=1}^m S_l,
\end{align}

the sets $\{S_l\}$ are disjoint, $|S_l| = \nu_l$, and for any nontrivial linear combination $y(n)$ of the solutions from $S_l$

\begin{align}
 (2.8) \quad & \limsup_{n \to \infty} \sqrt{|y(n)|} = \rho_l, \quad l = 1, 2, \ldots, m,
\end{align}

holds.

**Proposition 2.1.** For any $\lambda \in \mathbb{C}\setminus[-2,2]$ the dimension of the space of $\ell^2$-solutions of (2.4) equals $p$.

**Proof.** Note that equation (2.4) has order $k = 2p$ (after dividing through by the leading coefficient $d_{n,n+p}$) and belongs to the Poincaré class by assumption (2.2). Its characteristic equation (2.7) has now the form

\begin{align}
 w^{2p} - \lambda w^p + 1 = (w^p - z)(w^p - z^{-1}) = 0 : \quad \lambda = z + z^{-1}, \quad z < 1.
\end{align}

For its roots we have

\begin{align}
 |w_1| = \ldots = |w_p| = |z| < 1 < |z|^{-1} = |w_{p+1}| = \ldots = |w_{2p}|.
\end{align}

By the Perron theorem, the fundamental system $S$ of solutions of (2.4) exists

\begin{align}
 S = \{y_1, \ldots, y_p; y_{p+1}, \ldots, y_{2p}\} = S_1 \cup S_2; \quad \dim \text{span } S_1 = \dim \text{span } S_2 = p,
\end{align}

\begin{align}
 S_1 \cap S_2 = \{y_1, \ldots, y_{2p}\}.
\end{align}
and each solution \( y \in \text{span} \, S_1 \) is in \( \ell^2 \) (and even decreases exponentially fast). Let now \( y \) be any solution of (2.4) from \( \ell^2 \),
\[
y = \sum_{j=1}^{p} c_j y_j + \sum_{j=p+1}^{2p} c_j y_j = y' + y''.
\]
But \( y' \in \ell^2 \), and so \( y'' \in \ell^2 \) which by (2.8) and \( |z|^{-1} > 1 \) is possible only when \( c_j = 0 \) for \( j = p + 1, \ldots, 2p \), as needed. \( \Box \)

**Proposition 2.2.** Let \( \{y_n^{(i)}\}_{n \geq 1-p}, \ i = 1, 2, \ldots, p, \) be linearly independent solutions of (2.4) from \( \ell^2 \). The number \( \lambda \) is an eigenvalue of the operator \( D \) if and only if
\[
\det Y_0(\lambda) = \begin{vmatrix}
y_1^{(1)} & y_2^{(1)} & \cdots & y_0^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(p)} & y_2^{(p)} & \cdots & y_0^{(p)}
\end{vmatrix} = 0
\]

**Proof.** Suppose that \( \det Y_0(\lambda) = 0 \). Then there are numbers \( \alpha^{(1)}, \ldots, \alpha^{(p)} \), which do not vanish simultaneously, such that
\[
\begin{cases}
\alpha^{(1)} y_1^{(1)} + \cdots + \alpha^{(p)} y_1^{(p)} = 0, \\
\vdots \quad \cdots \quad \cdots \\
\alpha^{(1)} y_0^{(1)} + \cdots + \alpha^{(p)} y_0^{(p)} = 0.
\end{cases}
\]
Hence the linear combination
\[
y_n = \alpha^{(1)} y_n^{(1)} + \cdots + \alpha^{(p)} y_n^{(p)}, \quad n \geq 1 - p,
\]
belongs to \( \ell^2 \) and satisfies (2.4), (2.5), i.e., \( \lambda \) is an eigenvalue of the operator \( D \).

Conversely, let \( \lambda \) be an eigenvalue and \( y = \{y_n\}_{n \geq 1} \) the corresponding eigenvector. Then \( \{y_n\}_{n \geq 1-p} \) is an \( \ell^2 \)-solution of (2.4) with the initial conditions (2.5). By Proposition 2.1 (2.10) holds with coefficients \( \alpha^{(1)}, \ldots, \alpha^{(p)} \) which do not vanish simultaneously. Then (2.9) follows immediately from (2.5). \( \Box \)

3. **The matrix-valued Jost solution**

The goal of this Section is to establish the existence of matrix-valued analogue of the Jost solution for the banded matrix \( D \). Once we have the matrix Jost solution at our disposal, we will be able to construct \( p \) linearly independent square-summable solutions of (2.4) and, in view of Proposition 2.2, to reduce the study of the location of the discrete spectrum for the matrix \( D \) to the location of the zeros for determinant (2.9), composed of these \( p \) solutions.

It is convenient to rewrite the initial equation in the form of a three-term recurrence matrix relation, by looking at \( D \) as a block-Jacobi matrix. Along this way we can extend the standard techniques of proving the existence of the Jost solution for Jacobi matrices to the case of banded matrices.
Define the following $p \times p$-matrices:

$$A_k = \begin{pmatrix}
  d_{(k-1)p+1, (k-2)p+1} & \cdots & d_{(k-1)p+1, (k-1)p} \\
  \vdots & \ddots & \vdots \\
  d_{kp, (k-2)p+1} & \cdots & d_{kp, (k-1)p}
\end{pmatrix},$$

$$B_k = \begin{pmatrix}
  d_{(k-1)p+1, (k-1)p+1} & \cdots & d_{(k-1)p+1, kp} \\
  \vdots & \ddots & \vdots \\
  d_{kp, (k-1)p+1} & \cdots & d_{kp, kp}
\end{pmatrix},$$

$$C_k = \begin{pmatrix}
  d_{(k-1)p+1, kp+1} & \cdots & d_{(k-1)p+1, (k+1)p} \\
  \vdots & \ddots & \vdots \\
  d_{kp, kp+1} & \cdots & d_{kp, (k+1)p}
\end{pmatrix}. \tag{3.1}$$

Then the matrix $D$ can be represented in the form

$$D = \begin{pmatrix}
  B_1 & C_1 & 0 & 0 & \cdots \\
  A_2 & B_2 & C_2 & 0 & \cdots \\
  0 & A_3 & B_3 & C_3 & \cdots \\
  0 & 0 & A_4 & B_4 & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}, \tag{3.2}$$

with the upper triangular matrices $A_k$ and the lower triangular $C_k$ which are invertible due to (1.1). To be consistent with (2.3) we put $A_1 = C_0 = I$ a unit $p \times p$ matrix, $B_0 = 0$.

Having $p$ solutions $\{\varphi_j^{(l)}\}_{j \geq 1-p}, l = 1, 2, \ldots, p$ of equation (2.4) at hand, we can make up $p \times p$ matrices

$$\Psi_j = \begin{pmatrix}
  \varphi_{(j-1)p+1}^{(1)} & \cdots & \varphi_{(j-1)p+1}^{(p)} \\
  \vdots & \ddots & \vdots \\
  \varphi_{jp}^{(1)} & \cdots & \varphi_{jp}^{(p)}
\end{pmatrix}, \quad j \in \mathbb{Z}_+ := 0, 1, \ldots. \tag{3.3}$$

and write (2.4) in the matrix form

$$A_k \Psi_{k-1} + B_k \Psi_k + C_k \Psi_{k+1} = \lambda \Psi_k, \quad k \in \mathbb{N}. \tag{3.4}$$

It will be convenient to modify equation (3.3), getting rid of the coefficients $A_k$'s. Suppose that

$$\sum_{k=1}^{\infty} \|I - A_k\| < \infty, \tag{3.5}$$

where $\| \cdot \|$ is any norm in the space of matrices. It is well known, that there exists an infinite product (from the right to the left)

$$A := \prod_{j=1}^{\infty} A_j = \lim_{n \to \infty} (A_k A_k \cdots A_1),$$

and all the matrices $A$ and $A_n$ are invertible. Denote

$$L_j := \prod_{i=j+1}^{\infty} A_j, \quad L_j A_j = L_{j-1}, \quad \lim_{j \to \infty} L_j = I. \tag{3.6}$$
The multiplication of \((3.4)\) from the left by \(L_k\) gives
\[
L_{k-1}\Psi_{k-1} + L_k B_k \Psi_k + L_k C_k \Psi_{k+1} = \lambda L_k \Psi_k,
\]
\[
L_{k-1}\Psi_{k-1} + L_k B_k L_k^{-1} \cdot L_k \Psi_k + L_k C_k L_{k+1}^{-1} L_{k+1} \Psi_{k+1} = \lambda L_k \Psi_k.
\]
Hence the matrices
\[
(3.7) \qquad \Phi_k := L_k \Psi_k
\]
satisfy
\[
(3.8) \qquad \Phi_{k-1} + \tilde{B}_k \Phi_k + \tilde{C}_k \Phi_{k+1} = \lambda \Phi_k, \quad k \in \mathbb{N}
\]
with
\[
(3.9) \quad \tilde{B}_k = L_k B_k L_k^{-1}, \quad \tilde{C}_k := L_k C_k L_{k+1}^{-1}.
\]

For the definiteness sake we choose the “row norm”
\[
\|T\| := \max_{1 \leq k \leq p} \sum_{j=1}^p |t_{kj}|, \quad T = \{t_{kj}\}_{k,j=1}^p.
\]

Then by \((3.1)\)
\[
(3.10) \quad \max(\|A_k - I\|, \|B_k\|, \|C_k - I\|) \leq \hat{q}_k := \max_{1 \leq j \leq p} (q_{(k-1)p+j}), \quad k \in \mathbb{N},
\]
\(q_k\) are defined in \((1.4)\). In accordance with \((2.3)\) \(q_{1-p} = \ldots = q_0 = 0\), so we put \(\hat{q}_0 = 0\). It is clear that
\[
(3.11) \quad \sum_{k=1}^\infty \hat{q}_k \leq \sum_{k=1}^\infty q_k \leq p \sum_{k=1}^\infty \hat{q}_k
\]
so \((3.5)\) holds whenever \(\{q_n\} \in \ell^1\).

We will use the complex parameter \(z\) related to the spectral parameter \(\lambda\) by the Zhukovsky transform:
\[
\lambda = z + z^{-1}; \quad |z| < 1.
\]

Denote by \(g\) the Green kernel
\[
(3.12) \quad g(n,k,z) = \begin{cases} \frac{z^{k-n} - z^{n-k}}{z - z^{-1}}, & k > n, \quad n, k \in \mathbb{Z}_+ := \{0, 1, \ldots\}, \ z \neq 0. \\ \frac{z^{k-n}}{z^{-1}}, & k \leq n, \end{cases}
\]

It is clear that \(g(n,k,z)\) satisfies the recurrence relations
\[
(3.13) \quad g(n,k+1,z) + g(n,k-1,z) - (z + z^{-1}) g(n,k,z) = \delta(n,k),
\]
\[
(3.14) \quad g(n-1,k,z) + g(n+1,k,z) - (z + z^{-1}) g(n,k,z) = \delta(n,k),
\]

where \(\delta(n,k)\) is the Kronecker symbol.

We proceed with the following conditional result.

**Proposition 3.1.** Suppose that equation \((3.8)\) has a solution \(V_n\) with the asymptotic behavior at infinity
\[
(3.15) \quad \lim_{n \to \infty} V_n(z) z^{-n} = I
\]
for \(z \in \mathbb{D}\). Then \(V_n\) satisfies the discrete integral equation
\[
(3.16) \quad V_n(z) = z^n I + \sum_{k=n+1}^\infty J(n,k,z) V_k(z), \quad n \in \mathbb{N},
\]
with
\[(3.17) \quad J(n, k, z) = -g(n, k, z)\tilde{B}_k + g(n, k - 1, z) \left( I - \tilde{C}_{k-1} \right). \]

**Proof.** Let us multiply (3.13) by \(V_k\), (3.8) for \(V_k\) by \(g(n, k)\), and subtract the latter from the former:
\[g(n, k+1)V_k + g(n, k-1)V_k - g(n, k)V_{k-1} - g(n, k)\tilde{B}_k V_k - g(n, k)\tilde{C}_k V_{k+1} = \delta(n, k)V_k.\]
Summing up over \(k\) from \(n\) to \(N\) gives
\[V_n = \sum_{k=n+1}^{N} \left\{ -g(n, k)\tilde{B}_k + g(n, k - 1) \left( I - \tilde{C}_{k-1} \right) \right\} V_k + g(n, N + 1)V_N - g(n, N + 1)\tilde{C}_N V_{N+1}.\]

For \(|z| < 1\) we have by (3.12) and (3.15)
\[\lim_{N \to \infty} (g(n, N + 1)V_N - g(n, N)\tilde{C}_N V_{N+1}) = z^n I,\]
which along with \(J(n, n) = 0\) leads to (3.16), as needed. \(\square\)

The converse statement is equally simple.

**Proposition 3.2.** Each solution \(\{V_n(z)\}_{n \geq 0}\), \(z \in \overline{D}\), of equation (3.16) with \(n \in \mathbb{Z}_+\) satisfies the three-term recurrence relation (3.10).

**Proof.** Write for \(n \geq 1\)
\[V_{n-1} + V_{n+1} = (z^{n-1} + z^{n+1})I + \sum_{k=n}^{\infty} J(n - 1, k)V_k + \sum_{k=n+2}^{\infty} J(n + 1, k)V_k\]
\[= (z + z^{-1})z^n I + J(n - 1, n)V_n + J(n - 1, n + 1)V_{n+1}\]
\[+ \sum_{k=n+2}^{\infty} \{J(n - 1, k) + J(n + 1, k)\} V_k.\]

By (3.12), (3.14) and (4.11)
\[J(n - 1, n) = -\tilde{B}_n, \quad \tilde{B}_n = \tilde{B}_n \quad \tilde{B}_n = - (z + z^{-1})\tilde{B}_{n+1} + I - \tilde{C}_n\]
and
\[J(n - 1, k) + J(n + 1, k) = (z + z^{-1})J(n, k), \quad k \geq n + 2.\]

Hence
\[V_{n-1} + V_{n+1} + \tilde{B}_n V_n - (I - \tilde{C}_n)V_{n+1} = (z + z^{-1})z^n I - \tilde{B}_n V_n - (z + z^{-1})\tilde{B}_n V_{n+1}\]
\[+ (I - \tilde{C}_n)V_{n+1} + \sum_{k=n+2}^{\infty} \{ (z + z^{-1})J(n, k) \} V_k + \tilde{B}_n V_n - (I - \tilde{C}_n)V_{n+1}\]
\[= (z + z^{-1}) \left( z^n + \sum_{k=n+1}^{\infty} J(n, k)V_k \right) = (z + z^{-1})V_n,\]
which is exactly (3.10). \(\square\)

**The Jost solution.** To analyze equation (3.16) we introduce new variables
\[(3.18) \quad \tilde{V}_n(z) := z^{-n}V_n, \quad \tilde{J}(n, k, z) := z^{k-n}J(n, k, z),\]
\[J(n, z) := -\tilde{B}_n + z \tilde{B}_n, \quad J(n, k, z) := z^{k-n}J(n, k, z),\]
\[\tilde{J}(n, k, z) := \tilde{J}(n, k, z), \quad \tilde{J}(n, k, z) := z^{k-n}J(n, k, z),\]
and sum up over \(k\) from \(n\) to \(N\) gives
\[\tilde{V}_n = \sum_{k=n+1}^{N} \left\{ -\tilde{B}_k + g(n, k - 1) \left( I - \tilde{C}_{k-1} \right) \right\} \tilde{V}_k + g(n, N + 1)\tilde{V}_N.\]

For \(|z| < 1\) we have by (3.12) and (3.15)
\[\lim_{N \to \infty} (g(n, N + 1)\tilde{V}_N - g(n, N)\tilde{C}_N \tilde{V}_{N+1}) = z^n I,\]
which along with \(\tilde{J}(n, n) = 0\) leads to (3.16), as needed. \(\square\)

The converse statement is equally simple.
so that, instead of (3.16), we have

\[ \tilde{V}_n(z) = I + \sum_{k=n+1}^{\infty} \tilde{J}(n, k, z) \tilde{V}_k(z), \quad n \in \mathbb{Z}_+. \]

Now \( \tilde{J}(n, m, \cdot) \) is a polynomial with matrix coefficients. Since

\[ |g(n, k, z)z^{k-n}| = \frac{|z^{2(k-n)} - 1|}{|z - z^{-1}|} \leq |z| \min \left\{ |k - n|, \frac{2}{|z^2 - 1|} \right\}, \]

the kernel \( \tilde{J} \) is bounded by

\[ \| \tilde{J}(n, k, z) \| \leq |z| \min \left\{ |k - n|, \frac{2}{|z^2 - 1|} \right\} h_k, \quad z \in \mathbb{D} \]

where

\[ h_k := \| \tilde{B}_k \| + \| I - \tilde{C}_{k-1} \| = \| L_k B_k L_k^{-1} \| + \| I - L_{k-1} C_{k-1} L_{k-1}^{-1} \|, \quad k \in \mathbb{N} \]

(see (3.59)). We have

\[ h_k \leq \| L_k \| \| L_k^{-1} \| \left( \| B_k \| + \| I - A_k \| + \| A_k \| \| I - C_{k-1} \| \right). \]

Since \( \| A_k \| \leq C(D) \) and by (3.6) \( \| L_k \| \cdot \| L_k^{-1} \| \leq C(D) \) (throughout the rest of the paper \( C = C(D) \) stands for various positive constants which depend only on \( p \) and the original matrix \( D \)), we see from (3.10) that

\[ h_k \leq C(D)(\hat{q}_{k-1} + \hat{q}_k). \]

The existence of the Jost solutions for equation (3.8) will be proved under the assumption

\[ \sum q_i = \sum_i \left( |d_{i, i-p} - 1| + |d_{i, i+p} - 1| + \sum_{r=-p}^{p-1} |d_{i, i+r}| \right) < \infty, \]

which by (3.10), (3.11) and (3.23) implies

\[ \sum_k (\| I - A_k \| + \| B_k \| + \| I - C_k \|) < \infty, \quad \sum_k h_k < \infty. \]

The main result concerning equation (3.16) is the following

**Theorem 3.3.** (i) Suppose that

\[ \sum_{k=1}^{\infty} q_k < \infty, \]

\( q_k \) are defined in (1.3). Then equation (3.16) has a unique solution \( V_n \), which is analytic in \( \mathbb{D} \), continuous on \( \mathbb{D}_1 := \mathbb{D} \setminus \{ \pm 1 \} \) and

\[ \| V_n - z^n I \| \leq C|z|^n \left\{ \frac{|z|}{|z^2 - 1|} \sum_{k=n}^{\infty} q_k \right\} \exp \left\{ \frac{C|z|}{|z^2 - 1|} \sum_{k=n}^{\infty} q_k \right\} \]

for \( z \in \mathbb{D}_1, n \in \mathbb{Z}_+. \)

1Following the terminology of self-adjoint case for Jacobi matrices, we call this solution the Jost solution. The function \( V_0 \) is the matrix analogue of the Jost function.
Suppose that
\[ \sum_{k=1}^{\infty} kq_k < \infty. \]
Then \( V_n \) is analytic in \( \mathbb{D} \), continuous on \( \overline{\mathbb{D}} \) and
\[ \| V_n - z^n I \| \leq C|z|^n \left\{ \sum_{k=n}^{\infty} kq_k \right\} \exp \left\{ C \sum_{k=n}^{\infty} kq_k \right\}, \quad z \in \mathbb{D}, \quad n \in \mathbb{Z}_+. \]

Proof. We apply the method of successive approximations. Write (3.19) as
\[ F_n(z) = G_n(z) + \sum_{k=n+1}^{\infty} \tilde{J}(n, k, z) F_k(z) \]
with
\[ F_k(z) := \tilde{V}_k(z) - 1, \quad G_0(z) := \sum_{k=n+1}^{\infty} \tilde{J}(n, k, z). \]

(i) By (3.20)
\[ \| \tilde{J}(n, k, z) \| \leq \phi(z) h_k, \quad z \in \mathbb{D}_1, \quad \phi(z) := 2|z||z^2 - 1|^{-1}. \]
The series in (3.30) converges uniformly on compact subsets of \( \mathbb{D}_1 \) by (3.25), (3.24), and so \( G_n \) is analytic in \( \mathbb{D} \) and continuous on \( \mathbb{D}_1 \). As a starting point for the method of successive approximation, we put \( F_{n,1} = G_n \) and denote
\[ F_{n,j+1}(z) := \sum_{k=n+1}^{\infty} \tilde{J}(n, k, z) F_{k,j}(z). \]
Let \( \sigma_0(n) := \sum_{k=n+1}^{\infty} h_k \). By induction on \( j \) we prove that
\[ \| F_{n,j}(z) \| \leq \frac{(\phi(z) \sigma_0(n))^j}{(j-1)!}. \]
Indeed, for \( j = 1 \) we have \( F_{n,1} = G_n \) and the result holds by the definition of \( \sigma_0 \) and (3.31). Next, let (3.32) be true. Then
\[ |F_{n,j+1}(z)| \leq \phi(z) \sum_{k=n+1}^{\infty} h_k \| F_{k,j}(z) \| \leq \frac{(\phi(z))^j}{(j-1)!} \sum_{k=n+1}^{\infty} h_k \sigma_0^n(k). \]
An elementary inequality \((a + b)^{j+1} - a^{j+1} \geq (j + 1)ba^j\) gives
\[ \sum_{k=n+1}^{\infty} h_k \sigma_0^n(k) \leq \frac{1}{j} \sum_{k=n+1}^{\infty} \{\sigma_0^{j+1}(k - 1) - \sigma_0^{j+1}(k)\} = \frac{\sigma_0^{j+1}(n)}{j}, \]
which proves (3.32) for \( F_{n,j+1} \). Thereby the series
\[ F_n(z) = \sum_{j=1}^{\infty} F_{n,j}(z) \]
converges uniformly on compact subsets of \( \mathbb{D}_1 \) and solves (3.28), being analytic in \( \mathbb{D} \) and continuous on \( \mathbb{D}_1 \). It is also clear from (3.32) that
\[ \| F_n(z) \| = \| \tilde{V}_n(z) - I \| \leq \sum_{j=1}^{\infty} \| F_{n,j}(z) \| \leq \phi(z) \sigma_0(n) \exp\{\phi(z) \sigma_0(n)\}. \]
To reach (3.35) it remains only to note that by (3.34)
\[
\sigma_0(n) \leq C \sum_{k=n+1}^{\infty} (\hat{q}_{k-1} + \hat{q}_k) \leq 2C \sum_{k=n}^{\infty} \hat{q}_k \leq 2C \sum_{j=n}^{\infty} q_j
\]
(the latter inequality easily follows from the definition of \( \hat{q}_k \)).

To prove uniqueness suppose that there are two solutions \( F_n \) and \( \tilde{F}_n \) of (3.24). Take the difference and apply (3.31):
\[
|F_n - \tilde{F}_n(z)| = | \sum_{k=n+1}^{\infty} \tilde{J}(n, k, z) (F_k(z) - \tilde{F}_k(z)) |
\]
(3.34)
\[
s_n \leq \sum_{m=n+1}^{\infty} \phi(z) s_m h_m = r_n,
\]
where \( s_n := \|F_n(z) - \tilde{F}_n(z)\| \).

Clearly, \( r_n \to 0 \) as \( n \to \infty \) and if \( r_m = 0 \) for some \( m \), then by (3.31) we have \( s_n \equiv 0 \). If \( r_n > 0 \), then
\[
(3.35) \quad \frac{r_{n-1} - r_n}{r_n} = \frac{s_n \phi(z) h_n}{r_n} \leq \phi(z) h_n, \quad r_k \leq \prod_{j=k+1}^{M} (1 + \phi(z) h_j) r_M
\]
which leads to \( r_m = 0 \) and again \( s_n \equiv 0 \). So the uniqueness is proved.

(ii) The same sort of reasoning is applicable with
\[
\| \tilde{J}(n, k, z) \| \leq |z| |k - n| h_k \leq kh_k
\]
and
\[
(3.36) \quad \|F_n, j(z)\| \leq \frac{\sigma_1(n)}{(j - 1)!}, \quad \sigma_1(n) := \sum_{k=n+1}^{\infty} kh_k
\]
instead of (3.31) and (3.32), respectively. We have
\[
(3.37) \quad \|F_n(z)\| = \| \tilde{V}_n(z) - I \| \leq \sum_{j=1}^{\infty} \|F_{n, j}(z)\| \leq \sigma_1(n) \exp \{ \sigma_1(n) \}
\]
and
\[
\sigma_1(n) \leq C \sum_{k=n+1}^{\infty} k(\hat{q}_{k-1} + \hat{q}_k) \leq 2C \sum_{k=n}^{\infty} k\tilde{q}_k \leq 2C \sum_{j=n}^{\infty} j\tilde{q}_j
\]
(the latter inequality easily follows from the definition of \( \tilde{q}_k \)).

Remark. The constants \( C \) that enter (3.28), (3.29) and (3.30) are inefficient. This circumstance makes no problem when studying the limit set for the discrete spectrum. In contrast to this case, the efficient constants are called for when dealing with domains which contain the whole discrete spectrum. Such constants will be obtained in the next section under additional assumptions of “non asymptotic flavor”.

Throughout the rest of the section we assume that condition (3.27) is satisfied. It is clear that equation (3.8) can be rewritten for the functions \( \tilde{V}_n(z) \), defined in (3.15), as
\[
(3.38) \quad \tilde{V}_n(z) = (\lambda - \tilde{B}_n)z \tilde{V}_{n+1}(z) - \tilde{C}_n z^2 \tilde{V}_{n+2}(z)
\]
\[
= (z^2 + 1 - \tilde{B}_n z) \tilde{V}_{n+1}(z) - \tilde{C}_n z^2 \tilde{V}_{n+2}(z).
\]
Let us now expand $\tilde{V}_n(z)$ in the Taylor series taking into account definition (3.18) and (3.26)

\begin{equation}
\tilde{V}_n(z) = I + \sum_{j=1}^{\infty} K(n, j)z^j.
\end{equation}

Here $\|K(n, j)\|_{n,j}^{\infty}$ is the operator which transforms the Jost solutions of (2.1) for $D = D_0$ to that of (2.1) for $D$. If we plug (3.39) into (3.38) and match the coefficients for the same powers $z^j$ we have

$j = 1:\quad K(n, 1) = K(n + 1, 1) - \tilde{B}_n$,

$j = 2:\quad K(n, 2) = I + K(n + 1, 2) - \tilde{B}_nK(n + 1, 1) - \tilde{C}_n$,

$j \geq 2:\quad K(n, j + 1) = K(n+1, j-1) + K(n+1, j+1) - \tilde{B}_nK(n+1, j) - \tilde{C}_nK(n+2, j-1)$.

Summing up each one of these expressions for $k = n, n+1, \ldots$, it is not hard to verify that

\begin{equation}
K(n, 1) = -\sum_{k=n+1}^{\infty} \tilde{B}_{k-1},
\end{equation}

\begin{equation}
K(n, 2) = -\sum_{k=n+1}^{\infty} \left\{ \tilde{B}_{k-1}K(k, 1) + (\tilde{C}_{k-1} - I) \right\},
\end{equation}

\begin{equation}
K(n, j + 1) = K(n+1, j-1) - \sum_{k=n+1}^{\infty} \left\{ \tilde{B}_{k-1}K(k, j) + (\tilde{C}_{k-1} - I)K(k+1, j-1) \right\}
\end{equation}

In the last step we used $K(n,j) \to 0$ for $n \to \infty$ and any fixed $j$, which follows from the Cauchy inequality and

\begin{equation}
\|K(n, j)\| \leq \max \|\tilde{V}_n(z) - I\| \leq C \sum_{k=n}^{\infty} kq_k.
\end{equation}

From (3.40)–(3.42), using the induction on $j$, we obtain

\begin{equation}
\|K(n, j)\| \leq \kappa(n, j)\kappa\left(n + \left\lfloor \frac{j}{2} \right\rfloor \right), \quad n \in \mathbb{Z}_+,
\end{equation}

where $\kappa(n)$ and $\kappa(n, m)$ are defined by

\begin{equation}
\kappa(n) := \sum_{j=n}^{\infty} g_j, \quad \kappa(n, m) := \prod_{j=n+1}^{n+m-1} (1 + \kappa(j)) = \prod_{j=1}^{m-1} (1 + \kappa(n+j)),
\end{equation}

and

\begin{equation}
g_j = \|\tilde{B}_j\| + \|I - \tilde{C}_j\|.
\end{equation}
In fact, for \( j = 1 \) we have \( \kappa(n, 1) = 1 \), \( [\frac{1}{2}] = 0 \) and \( \|K(n, 1)\| \leq \kappa(n+1) \leq \kappa(n) \). Further, for \( j = 2 \),

\[
\|K(n, 2)\| \leq \sum_{k=n+1}^{\infty} \left\{ \|\hat{B}_{k-1}\|\|K(k, 1)\| + \|\hat{C}_{k-1} - I\| \right\}
\]

\[
\leq \sum_{k=n+1}^{\infty} \left\{ (\|K(k, 1)\| + 1) \left( \|\hat{B}_{k-1}\| + \|\hat{C}_{k-1} - I\| \right) \right\}
\]

\[
\leq \sum_{k=n+1}^{\infty} g_k \left( 1 + \kappa(k) \right)
\]

\[
\leq (1 + \kappa(n+1)) \kappa(n+1)
\]

\[
= \kappa(n, 2) \kappa(n+1).
\]

When we pass from the even \( j = 2l \) to the odd \( 2l + 1 \), we have by (3.42) and by the inductive hypothesis:

\[
\|K(n, 2l+1)\| \leq \kappa(n+1, 2l-1) \kappa(n+l) + \sum_{k=n+1}^{\infty} \left\{ \|\hat{B}_{k-1}\|\|K(k, 2l)\| + \|\hat{C}_{k-1} - I\|\|K(k+1, 2l-1)\| \right\}.
\]

But according to the inductive hypothesis

\[
K(k, 2l)\| \leq \kappa(k, 2l) \kappa(k + l);
\]

\[
\|K(k + 1, 2l - 1)\| \leq \kappa(k + 1, 2l - 1) \kappa(k + l) \leq \kappa(k, 2l) \kappa(k + l).
\]

Using these inequalities, obtain

\[
\|K(n, 2l + 1)\| \leq \kappa(n + 1, 2l) \kappa(n + l) + \sum_{k=n+1}^{\infty} d_k \kappa(k, 2l) \kappa(k + l)
\]

\[
\leq \kappa(n + 1, 2l - 1) \kappa(n + l) + \kappa(n + l) \kappa(n + 1, 2l) \kappa(n + 1)
\]

\[
= \kappa(n + l) \{ \kappa(n + 1, 2l - 1) + \kappa(n + 1, 2l) \kappa(n + 1) \}
\]

But

\[
\kappa(n + 1, 2l - 1) + \kappa(n + 1, 2l) \kappa(n + 1) =
\]

\[
= \prod_{j=n+2}^{n+2l-1} (1 + \kappa(j)) + \prod_{j=n+2}^{n+2l} (1 + \kappa(j)) \kappa(n + 1)
\]

\[
= \prod_{j=n+2}^{n+2l-1} (1 + \kappa(j)) \{ 1 + (1 + \kappa(n + 2l)) \kappa(n + 1) \}
\]

\[
\leq \prod_{j=n+2}^{n+2l-1} (1 + \kappa(j)) \{ 1 + \kappa(n + 2l) + (1 + \kappa(n + 2l)) \kappa(n + 1) \}
\]

\[
= \prod_{j=n+2}^{n+2l-1} (1 + \kappa(j)) \{ (1 + \kappa(n + 2l)) (1 + \kappa(n + 2l)) \}
\]

\[
= \kappa(n, 2l + 1),
\]

from which we have

\[
\|K(n, 2l + 1)\| \leq \kappa(n + 2l) \kappa(n + l),
\]

as needed. Analogous calculations help us to pass from the odd \( j = 2l + 1 \) to the even \( j + 1 = 2l + 2 \).
It is easy to see that \( \kappa(n) \) and \( \kappa(n, m) \) in (3.44) can be replaced by
\[
(3.46) \quad \tilde{\kappa}(n) := C \sum_{j=n}^\infty q_j, \quad \tilde{\kappa}(n, m) := \prod_{j=n+1}^{n+m-1} (1 + \tilde{\kappa}(j)) = \prod_{j=1}^{m-1} (1 + \tilde{\kappa}(n+j)).
\]
Further, it is evident that \( \{\kappa(n)\} \in \ell^1 \) and the sequences \( \tilde{\kappa}(\cdot) \) and \( \tilde{\kappa}(\cdot, m) \) decrease monotonically. Hence
\[
(3.47) \quad \|K(n, j)\| \leq \prod_{j=1}^\infty (1 + \tilde{\kappa}(j)) \sum_{k=n+\lfloor \frac{m}{4} \rfloor}^\infty q_k.
\]
Taking the latter expression with \( n = 0 \), we come to the following

**Theorem 3.4.** Under hypothesis \((3.27)\) the Taylor coefficients of the matrix-valued function
\[
(3.48) \quad \Delta(z) := V_0(z) = \sum_{j=0}^\infty \delta(j)z^j
\]
admit the bound
\[
(3.49) \quad \|\delta(j)\| \leq C \prod_{j=1}^\infty (1 + \tilde{\kappa}(j)) \sum_{k=\lfloor \frac{m}{4} \rfloor}^\infty q_k,
\]
where \( [x] \) is an integer part of \( x \). In particular, \( \Delta \) belongs to the space \( W_+ \) of absolutely convergent Taylor matrix-valued series.

Let us now denote by \( \Delta_{ij} \) the entries of the matrix \( \Delta \):
\[
\Delta := \|\Delta_{ij}\|_{i,j=1}^p.
\]

**Corollary 3.5.** Let for the banded matrix \( D \) the numbers
\[
(3.50) \quad M_{n+1} := \sum_{k=0}^\infty (k+1)^{n+1}q_k < \infty.
\]
Then the \( n \)'th derivative \( \Delta^{(n)}(z) = V_0^{(n)}(z) \) belongs to \( W_+ \) and
\[
(3.51) \quad \max_{z \in \mathbb{D}} \left| \Delta^{(n)}_{ij}(z) \right| \leq C(D) \frac{4^n}{n+1} M_{n+1}, \quad i, j = 1, 2, \ldots, p.
\]

**Proof.** The statement is a simple consequence of (3.50), the series expansion
\[
\Delta^{(n)}(z) = \sum_{j=0}^\infty (j+1)\ldots(j+n)\delta(j+n)z^j
\]
bounds (3.49) and the obvious inequality \( |\Delta^{(n)}_{ij}(z)| \leq \|\Delta^{(n)}\| \).
\(\square\)

4. The limit set and location of the discrete spectrum

Consider the matrix-valued Jost solutions \( V_k(z) \), which exist \( \forall z \in \mathbb{C}\setminus\{0\} \). The \( p \) scalar solutions \( \{v^{(l)}_j\}_{j=1-p} \), \( l = 1, 2, \ldots, p \) of equation (2.4), constructed from \( V_k(z) \) by formulae \((3.7)\) and \((3.3)\), are linearly independent and belong to \( \ell^2 \) due to asymptotic formulae \( (3.20) \). According to Proposition \( (2.2) \), the number \( \eta = \zeta + \zeta^{-1} \) is an eigenvalue for the operator \( D \) if and only if the determinant of the matrix-valued function \( \Delta(z) = V_0(z) \) vanishes at the point \( \zeta \). Thus, the study of the
discrete spectrum of the operator $D$ is reduced to the study of the zeros of the function

$$\gamma(z) := \det \Delta(z).$$

The main item of business in this section is the limit set $E_D$ of the discrete spectrum of the operator $D$. Remind that $E_D \subset [-2, 2]$.

Let $D \in P_p(\beta)$ (see Definition 1.1). Since $\|L_k\|$ and $\|L_k^{-1}\|$ are uniformly bounded, it is clear from (3.24) that $h_n$, defined in (3.21), satisfies the same inequality:

$$h_n \leq C_1 \exp(-C_2(n+1)^\beta)$$

(with the same exponent $\beta$, but other constants $C_1, C_2 > 0$).

We are in a position now to prove the first result announced in the introduction.

**Theorem 4.1.** Let $D \in P(\beta)$ where $0 < \beta < \frac{1}{2}$. Then $E_D$ is a closed point set of the Lebesgue measure zero and its convergence exponent satisfies

$$\dim E_D \leq \frac{1}{\tau(E_D)} \leq \frac{1 - 2\beta}{1 - \beta},$$

where $\dim E_D$ is the Hausdorff dimension of $E_D$. Moreover, if $J \in P(\frac{1}{2})$ then $E_D = \emptyset$, i.e., the discrete spectrum is finite.

**Proof.** Denote by $\mathcal{A}$ the set of all functions, analytic inside $D$ and continuous in $\overline{D}$. Recall that the set $E$ on the unit circle $T$ is called a zero set for a class $\mathcal{X} \subset \mathcal{A}$ of functions, if there exists a non-trivial function $f \in \mathcal{X}$, which vanishes on $E$. We want to show that the function $\gamma$ belongs to a certain class $\mathcal{X}$ (the Gevrey class, see below) with known properties of its zero sets. Note that, since $\mathcal{X} \subset \mathcal{A}$ then, according to the Fatou theorem, the zero set has the Lebesgue measure zero.

We begin with certain bounds for the derivatives of the function $\gamma$, which can be obtained from Theorem 3.4 and Corollary 3.5. To this end we write

$$\gamma(z) := \det \Delta(z) = \sum_{\pi} \text{sign } \pi \Delta_{1,\pi(1)} \Delta_{2,\pi(2)} \cdots \Delta_{p,\pi(p)},$$

where $\pi$ are the permutations of the set $\{1, 2, \ldots, p\}$. For the $n$-th derivative of $\gamma$

$$\gamma^{(n)}(z) = \sum_{\pi} \text{sign } \pi \sum_{\sum_{k_j = n} k_j \geq 0} \binom{n}{k_1, k_2, \ldots, k_p} \Delta_{1,\pi(1)}^{(k_1)} \Delta_{2,\pi(2)}^{(k_2)} \cdots \Delta_{p,\pi(p)}^{(k_p)}$$

holds, where the multinomial coefficients $\binom{n}{k_1, k_2, \ldots, k_p}$ are defined from the identity

$$(x_1 + x_2 + \ldots + x_p)^n = \sum_{\sum_{k_j = n} k_j \geq 0} \binom{n}{k_1, k_2, \ldots, k_p} x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}.$$
From (4.4) and (3.51) we immediately derive
\[
(4.6) \max_{z \in \mathbb{D}} |\gamma^{(n)}(z)| \leq C^p p^n 4^n \sum_{\sum k_j = n, k_j \geq 0} \left( \begin{array}{c} n \\ k_1, k_2, \ldots, k_p \end{array} \right) M_{k_1+1} M_{k_2+1} \ldots M_{k_p+1},
\]
where the numbers \(M_r\) are defined in (3.50). To estimate the product \(M_{k_1+1} \ldots M_{k_p+1}\) note first, that by (4.2)
\[
M_r \leq C^\infty \sum_{k=0}^\infty \left( k + 1 \right) r^\beta \exp \left( -\frac{C_2}{2} (k + 1)^\beta \right).
\]
An undergraduate analysis of the function \(u(x) = x^r \exp(-\frac{C}{2} x^\beta)\) gives
\[
\max_{x \geq 0} u(x) = u(x_0) = \left( \frac{2r}{C\beta} \right)^{r/\beta} e^{-r/\beta}, \quad x_0 = \left( \frac{2r}{C\beta} \right)^{1/\beta},
\]
so that
\[
M_r \leq B \left( \frac{2}{C\beta e} \right)^{r/\beta} r^{r/\beta}, \quad B = C^\infty \sum_{k=0}^\infty \exp \left( -\frac{C}{2} (k + 1)^\beta \right).
\]
Hence,
\[
M_{k_1+1} \ldots M_{k_p+1} \leq B^p \left( \frac{2}{C\beta e} \right)^{n/\beta} \left( k_1 + 1 \right)^{k_1+1} \ldots \left( k_p + 1 \right)^{k_p+1}.
\]
The inequalities
\[
\left( \frac{n+1}{n} \right)^{n/\beta} < e^{1/\beta}, \quad (n+1)^{1/\beta} < e^{(n+1)/\beta}, \quad (n+1)^{(n+1)/\beta} < e^{(n+1)/\beta} n^{n/\beta} e^{(n+1)/\beta}
\]
lead to the bound
\[
M_{k_1+1} \ldots M_{k_p+1} \leq B_2 \left( \frac{2}{C\beta} \right)^{n/\beta} k_1^{k_1/\beta} \ldots k_p^{k_p/\beta}.
\]
Substituting the latter into (4.6) gives
\[
(4.7) \max_{z \in \mathbb{D}} |\gamma^{(n)}(z)| \leq C^4 p^n \left( \frac{2}{C\beta} \right)^{n/\beta} n^{k_1/\beta} \ldots n^{k_p/\beta} \leq C^4 n^{n/\beta}, \quad n \geq 0.
\]
In other words, the function \(\gamma\) for \(D \in \mathcal{P}(\beta)\) belongs to the Gevrey class \(\mathcal{G}_\beta\).

The rest of the proof goes along the same line of reasoning as in [2]. Suppose first that \(0 < \beta < 1/2\). The celebrated Carleson’s theorem [7] gives a complete description of zero sets for \(\mathcal{G}_\beta\). It claims that
\[
\sum_{j=1}^\infty |l_j|^{\frac{1-2\beta}{1-\beta}} < \infty,
\]
where \(\{l_j\}\) are adjacent arcs of the zeros set of \(\gamma\). Hence the right inequality in (4.3) is obtained. The left inequality is the general fact of fractal dimension theory.

When \(\beta = 1/2\) the Gevrey class \(\mathcal{G}_{1/2}\) is known to be quasi-analytic, i.e., it contains no nontrivial function \(f\), such that \(f^{(n)}(\zeta_0) = 0\) for all \(n \geq 0\) and some \(\zeta_0 \in T\). So each function \(f \in \mathcal{G}_{1/2}\) may have only finite number of zeros inside the unit disk which due to the relation between the discrete spectrum of \(D\) and zeros of \(\gamma\) proves the second statement of the theorem.
It turns out that the exponent 1/2 in theorem 1 is sharp in the following sense.

**Theorem 4.2.** For arbitrary \( \varepsilon > 0 \) and different points \(-2 < \nu_i < 2, \ i = 1, \ldots, p\), there exists an operator \( D \in \mathcal{P}_p(\frac{1}{2} - \varepsilon) \) such that its discrete spectrum \( \sigma_\Delta(D) \) is infinite and, moreover, \( E_D = \{\nu_1, \ldots, \nu_p\} \).

**Proof.** The proof of the theorem is based on the similar result, proved in [2] for the Jacobi matrices, that is, for \( p = 1 \). If \( p > 1 \) than given \( \nu_1, \ldots, \nu_p \) we construct the complex Jacobi matrices

\[
(4.8) \quad J(i) = \begin{pmatrix}
b_0(i) & a_0(i) & a_1(i) \\
a_0(i) & b_1(i) & a_2(i) \\
. & . & .
\end{pmatrix}, \quad a_j(i) > 0, \ b_j(i) \in \mathbb{R}, \ j \geq 1, \ i = 1, 2, \ldots, p
\]

which belong to the class \( \mathcal{P}_1(\frac{1}{2} - \varepsilon) \) and have the only point of accumulation of the discrete spectrum \( E_{J(i)} = \nu_i \).

Consider now the \( p \)-banded matrix \( D = \|d_{ij}\|_{i,j=1}^\infty \) with

\[
d_{pn+i, pn+i} = b_n(i), \quad d_{pn+i, (p(n+1)+i)pn+i} = a_n(i),
\]

\[
d_{pn+i, pn+i+j} = d_{pn+i+j, pn+i} = 0, \quad j \neq 0, p, \ i = 1, \ldots, p.
\]

Since \( J(i) \in \mathcal{P}_1(\frac{1}{2} - \varepsilon) \), it is easily seen that \( D \in \mathcal{P}_p(\frac{1}{2} - \varepsilon) \).

The space \( \ell^2 \), where the operator \( D \) acts, can be decomposed as follows:

\[
\ell^2 = \bigoplus_{i=1}^p L_i
\]

with \( L_i = \text{Lin}\{\hat{e}_i, \hat{e}_{p+i}, \hat{e}_{2p+i}, \ldots, \hat{e}_{np+i}, \ldots\} \) and \( \{\hat{e}_k\}_{k \in \mathbb{N}} \) the standard basis vectors in \( \ell^2 \). It is shown directly that the subspaces \( L_i \) are invariant for \( D \).

The restriction \( D_i = D |_{L_i} \) of \( D \) on the subspace \( L_i \) has the matrix representation \( J(i) \). Since \( D = \bigoplus_{i=1}^p D_i \) and every operator \( D_i \) has discrete spectrum with the only accumulation point \( \nu_i \), the discrete spectrum \( D \) is the union of discrete spectrum of the operators \( J(i) \) and it has \( p \) accumulation points \( \nu_1, \nu_2, \ldots, \nu_p \), as needed. \( \square \)

The second issue we address in this section concerns domains which contain the whole discrete spectrum, and conditions for the lack of the discrete spectrum.

We start out with (3.33) and (3.37) for \( n = 0 \)

\[
(4.9) \quad \|\Delta(z) - I\| \leq \frac{2\sigma_0(0)}{|z - z - 1|} \exp \left\{ \frac{2\sigma_0(0)}{|z - z - 1|} \right\}, \quad \|\Delta(z) - I\| \leq \sigma_1(0) \exp \{\sigma_1(0)\},
\]

with \( \Delta = \hat{V}_0, \ \sigma_0(0) = \sum_{k=1}^\infty h_k, \ \sigma_1(0) = \sum_{k=1}^\infty kh_k, \ h_k \) defined in (3.24), which hold under assumptions (3.23) and (3.27), respectively. To work with (3.10) we have to find the efficient constant \( C \) which enters (3.28). To that end let us go back to (3.10) and assume that

\[
(4.10) \quad q := \sup_{n \geq 1} q_n = \sup_{n \geq 1} \left\{ |d_{n,-p} - 1| + |d_{n,n+p} - 1| + \sum_{r=-p+1}^{p-1} |d_{n,n+r}| \right\} < 1,
\]

which now implies

\[
(4.11) \quad \sup_{k \geq 1} \|A_k - I\| \leq q, \quad \sup_{k \geq 1} \|A_k\| \leq 1 + q < 2.
\]
It is not hard to show now that $\sup_{k \geq 1} \|A_k^{-1}\| \leq (1 - q)^{-1}$.

$$
\sup_{k \geq 1} \|L_k\| \leq \exp\left\{ \sum_{j=1}^{\infty} \|A_j - I\| \right\}, \quad \sup_{k \geq 1} \|L_k^{-1}\| \leq \exp\left\{ \frac{1}{1 - q} \sum_{j=1}^{\infty} \|A_j - I\| \right\}
$$

and

$$
\sup_{k \geq 1} \|L_k\|\|L_k^{-1}\| \leq \exp\left\{ \frac{2 - q}{1 - q} \sum_{j=1}^{\infty} \|A_j - I\| \right\} \leq \exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\}, \quad Q_0 := \sum_{k=1}^{\infty} q_k.
$$

Hence, instead of (3.23), we have by (3.22)

$$
h_k \leq 2 \exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\} (\hat{q}_k - 1 + \hat{q}_k)
$$

and so

(4.12)

$$
\sigma_0(0) \leq 4 \exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\} Q_0, \quad \sigma_1(0) \leq 4 \exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\} Q_1, \quad Q_1 := \sum_{k=1}^{\infty} kq_k.
$$

Let now $t$ be a unique root of the equation

(4.13)

$$
te^t = 1, \quad t \approx 0.567.
$$

**Theorem 4.3.** Assume that $Q_0 < \infty$ and $q < 1$. Then the domain

$$
G(D) = \{ z + z^{-1} : \ z \in \Omega \}
$$

with

$$
\Omega := \{ z \in D : |z - z^{-1}| > \frac{8Q_0}{t} \exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\} \}
$$

is free from the discrete spectrum $\sigma_d(D)$. Moreover, assume that $Q_1 < \infty$. Then $D$ has no discrete spectrum as long as

$$
\exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\} Q_1 < \frac{4}{t}.
$$

**Proof.** It is clear from the first inequality in (4.13) that $\| \Delta(z) - I \| < 1$ (and the more so, $\det \Delta(z) \neq 0$) whenever $|z - z^{-1}| > 2\sigma_0(0)t^{-1}$. The rest follows immediately from the first bound in (4.12) and the relation between eigenvalues of $D$ and zeros of $\Delta$. The second statement is proved in exactly the same way by using the second inequality in (4.13) and the second bound in (4.12). \qed

**Remark.** Suppose that

$$
c = \frac{8Q_0}{t} \exp\left\{ \frac{2 - q}{1 - q} Q_0 \right\} < 2.
$$

Then $\sigma_d(D)$ is contained in the union of two symmetric rectangles

$$
\sigma_d(D) \subset \left\{ w : \sqrt{4 - c^2} < |\text{Re } w| < \sqrt{4 + c^2}, \ |\text{Im } w| < \frac{c^2}{4} \right\}.
$$
5. The discrete spectrum of doubly-infinite banded matrices

A doubly-infinite complex matrix $D = \|d_{ij}\|_{i,j=-\infty}^\infty$ is called the banded matrix of order $p$ if

\[
d_{ij} = 0, \quad |i-j| > p, \quad d_{ij} \neq 0, \quad |i-j| = p, \quad d_{ij} \in \mathbb{C}.
\]

There is a nice “doubling method” that relates a doubly-infinite $p$-banded matrix to a certain semi-infinite banded matrix of order $2p$. Indeed, let $\{e_k\}_{k \in \mathbb{Z}}$ be the standard basis in $\ell^2(\mathbb{Z})$. Consider a transformation $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$, defined by

\[
Ue_k = \hat{e}_{2k}, \quad k < 0, \quad Ue_k = \hat{e}_{2k+1}, \quad k \geq 0,
\]

where $\{\hat{e}_k\}_{k \in \mathbb{N}}$ is the standard basis in $\ell^2(\mathbb{N})$. The transformation $U$ is clearly isometric and it is easy to check directly that the matrix

\[
\hat{D} := UDU^{-1}
\]

is the semi-infinite band matrix of order $2p$ unitarily equivalent to $D$. For instance, in the case of Jacobi matrices ($p = 1$)

\[
J = \begin{pmatrix}
0 & \cdots & \cdots & \\
\ddots & \cdots & \cdots & \\
& a_0 & b_0 & c_0 \\
& & a_1 & b_1 & c_1 \\
& & & \ddots & \ddots & \ddots \\
& & & & a_{-1} & b_{-1} & c_{-1} \\
\end{pmatrix}, \quad J = \begin{pmatrix}
\hat{B}_0 & \hat{C}_1 \\
\hat{A}_1 & \hat{B}_2 & \hat{C}_2 \\
& \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

with

\[
\hat{B}_0 = \begin{pmatrix}
b_0 & a_0 \\
c_0 & b_{-1}
\end{pmatrix}, \quad \hat{B}_k = \begin{pmatrix}
b_k & 0 \\
0 & b_{-k-1}
\end{pmatrix}, \\
\hat{A}_k = \begin{pmatrix}
a_k & 0 \\
0 & c_{-k}
\end{pmatrix}, \quad \hat{C}_k = \begin{pmatrix}
c_k & 0 \\
0 & a_{-k}
\end{pmatrix} \quad k \in \mathbb{N}.
\]

Similarly to the semi-infinite case, we put

\[
q_n := |d_{n,-n-p}-1| + \sum_{j=-p+1}^{p-1} |d_{n,n+j}| + |d_{n,n+p}-1|, \quad m \in \mathbb{Z}.
\]

We say that the matrix $D$ belongs to the class $\mathcal{P}_p(\beta)$, $0 < \beta < 1$, if

\[
q_n \leq C_1 \exp(-C_2 |n|^\beta), \quad C_1, C_2 > 0, \quad n \in \mathbb{Z}.
\]

It is clear by the construction that the semi-infinite matrix $\hat{D}$ belongs to $\mathcal{P}_{2p}(\beta)$ whenever the doubly-infinite matrix $D$ belongs to $\mathcal{P}_p(\beta)$. So the results of the previous section can be easily derived for doubly-infinite banded matrices as well. For instance, the following statement holds.

**Theorem 5.1.** Let $D \in \mathcal{P}_p(\beta)$ where $0 < \beta < \frac{1}{2}$. Then $E_D$ is a closed point set of the Lebesgue measure zero and its convergence exponent satisfies

\[
\dim E_D \leq \tau(E_D) \leq \frac{1-2\beta}{1-\beta},
\]

where $\dim E_D$ is the Hausdorff dimension of $E_D$. Moreover, if $D \in \mathcal{P}_p\left(\frac{1}{2}\right)$ then $E_D = \emptyset$ i.e. the discrete spectrum is finite.
6. The discrete spectrum of asymptotically periodic Jacobi matrices

The goal of this section is to apply the results obtained above to the study of the spectrum of doubly-infinite asymptotically periodic complex Jacobi matrices.

**Definition 6.1.** A doubly-infinite complex Jacobi matrix

\[
J^0 = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & a_0 & b_0 & c_0 & \ddots \\
& a_1 & b_1 & c_1 & \ddots \\
& & a_2 & b_2 & c_2 & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & a_0 & b_0 & c_0 & \ddots \\
\end{pmatrix}, \quad b_0^0 = (J^0 e_0, e_0),
\]

\(a_n^0 c_n^0 \neq 0, a_n^0, b_n^0, c_n^0 \in \mathbb{C}\) is called \(p\)-periodic if

\[a_{n+p}^0 = a_n^0, \quad b_{n+p}^0 = b_n^0, \quad c_{n+p}^0 = c_n^0, \quad n \in \mathbb{Z}.
\]

The following result due to P.B. Naiman \[8, 9\] is the key ingredient in our argument.

**Theorem.** Each \(p\)-periodic Jacobi matrix \(J^0\) satisfies the algebraic equation of \(p\)-th degree

\[P(J^0) = E_p,
\]

where \(E_p = \{e_{jk}\}\) is a \(p\)-banded matrix with the entries

\[
e_{ij} = \begin{cases}
\alpha := \prod_{k=1}^{p} a_k^0, & i - j = p, \\
\delta := \prod_{k=1}^{p} c_k^0, & i - j = -p, \\
0, & |i - j| \neq p.
\end{cases}
\]

Next, let

\[\Pi := \{z \in \mathbb{C} : z = \alpha e^{ipt} + \delta e^{-ipt}, \quad 0 \leq t \leq \frac{2\pi}{p}\}\]

be an ellipse in the complex plane, and

\[\Gamma := P^{(-1)}(\Pi) = \{z \in \mathbb{C} : P(z) \in \Pi\}\]

its preimage. Then the spectrum \(\sigma(J^0) = \Gamma\).

The polynomial \(P\) is known as the Burchnall–Chaundy polynomial for \(J^0\) and given explicitly in \[9\] p.141:

\[P(\lambda) = \det(\lambda - J(1, p)) - a_1 c_1 \det(\lambda - J(2, p-1)), \quad J(m, n) := \begin{pmatrix}
b_m^0 & c_{m+1}^0 \\
a_{m+1}^0 & b_{m+1}^0 & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
a_n^0 & b_n^0 & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]
Consider the $p \times p$-matrix
\[
\Lambda_p(t) := \begin{pmatrix}
 b_0 & c_0 & a_1^0 e^{i pt} \\
 a_0 & b_0 & \cdot \cdot \\
 \cdot & \cdot & \cdot \\
 c_0^0 e^{-i pt} & a_0^0 & b_0^0 \\
 a_0^0 & b_0^0 & \cdot \cdot \\
 \cdot & \cdot & \cdot
\end{pmatrix},
\]
with the eigenvalues $\{ \lambda_j(t) \}_{j=1}^p$, and put
\[
\text{range } \lambda_j := \{ w \in \mathbb{C} : w = \lambda_j(t), \ 0 \leq t \leq \frac{2\pi}{p} \}.
\]
Then
\[
\sigma(J^0) = \bigcup_{j=1}^p \text{range } \lambda_j.
\]
Indeed, it is easy to compute
\[
\det (\Lambda(t) - \lambda) = (-1)^p \left( P(\lambda) - \alpha e^{i pt} - \delta e^{-i pt} \right).
\]
For further information about the spectra of periodic Jacobi matrices with algebro-geometric potential see [10].

For the rest of the paper we restrict ourselves with the so called quasi-symmetric matrices with
\[
\prod_{k=1}^p a_k^0 = \prod_{k=1}^p c_k^0, \quad \alpha = \delta.
\]
The latter certainly holds for the symmetric periodic matrices $(a_n^0 = c_n^0)$. In this case we have $Q(J^0) = D_0$ with $D_0$ defined in (1.3) for the polynomial $Q = \alpha^{-1} P$, and
\[
\sigma(J^0) = Q^{-1}([-2, 2]).
\]
It is not hard to make sure that $\sigma(J^0)$ is a collection of finitely many algebraic arcs with no closed loops, that is, the complement $\mathbb{C} \setminus \sigma(J^0)$ is a connected set. Indeed, if some of these arcs formed a closed loop $\Gamma_1 \subset \Gamma$ with the interior domain $G_1$, then $\Im Q$ would vanish on $\Gamma_1$ and so $\Im Q \equiv 0$ in $G_1$, which is impossible, unless $Q$ is a real constant.

Let now $J$ be a complex asymptotically $p$-periodic doubly-infinite Jacobi matrix with the quasi-symmetric $p$-periodic background $J^0$.

\[
J = \begin{pmatrix}
 \cdot & \cdot & \cdot & \cdot & \cdot \\
 a_1 & b_0 & c_0 & a_1 & b_0 & c_0 & \cdot \cdot \\
 b_0 & c_0 & a_1 & b_0 & c_0 & a_1 & b_0 & c_0 & \cdot \cdot \\
 c_0 & a_1 & b_0 & c_0 & a_1 & b_0 & c_0 & \cdot \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot 
\end{pmatrix}, \quad a_n c_n \neq 0,
\]
and
\[
\omega_n := |a_n - a_n^0| + |b_n - b_n^0| + |c_n + 1 - c_n^0| \to 0, \quad k \to \pm \infty.
\]
In other words, $J - J^0$ is a compact operator in $l^2(\mathbb{Z})$. Since $\mathbb{C} \setminus \sigma(J^0)$ is connected, the version of Weyl’s theorem holds (cf. [5], Lemma I.5.2)
\[
\sigma(J) = \sigma(J^0) \cup \sigma_d(J),
\]
where the discrete spectrum $\sigma_d(J)$ is an at most denumerable set of eigenvalues of finite algebraic multiplicity off $\sigma(J^0)$, which can accumulate only to $\sigma(J^0)$.

We proceed with two simple propositions.

**Proposition 6.2.** Let $D_{m_1}$ and $D_{m_2}$ be banded matrices of orders $m_1$ and $m_2$, respectively. Then

1. $D = D_{m_1}D_{m_2}$ is the banded matrix of order $m = m_1 + m_2$.
2. If $m_1 < m_2$ then $D_{m_1} + D_{m_2}$ is the banded matrix of order $m_2$.

In particular, if $T$ is a polynomial of degree $p$ and $D$ is a banded matrix of order $m$, then $T(D)$ is the banded matrix of order $pm$.

**Proof.** (2) is obvious. To prove (1) let us show that the elements of the extreme diagonals of $D$ do not vanish. Indeed, let $D_{m_l} = \{d_{ij}^{(l)}\}$, $l = 1, 2$, and $D = \{d_{ij}\}$. Then

$$d_{k, k+m} = \sum_{j=1}^{\infty} d_{k, j}^{(1)} d_{j, k+m}^{(2)} = d_{k, k+m}^{(1)} d_{k, k+m}^{(2)} \neq 0, \quad k \in \mathbb{N}.$$  

Similarly, $d_{k, k-m} = d_{k, k-m}^{(1)} d_{k, k-m}^{(2)} \neq 0$. The rest is plain. \(\square\)

**Proposition 6.3.** Let $J_1$ and $J_2$ be bounded (with bounded entries) complex Jacobi matrices. Put

$$\omega_k = \sum_j |(J_1 - J_2)_{kj}| = \sum_j |((J_1 - J_2)e_j, e_k)|.$$  

Then for any polynomial $T$ of degree $p$ we have

$$\sum_j |(T(J_1) - T(J_2))_{kj}| = \sum_j |((T(J_1) - T(J_2))_{kj}| \leq C \sum_{s=k-p}^{k+p} \omega_s,$$

where a positive constant $C$ depends on $J_1$, $J_2$ and $T$.

**Proof.** For $T(z) = z^n$ the statement follows immediately from the banded structure of the powers $J_1^n$, $(J_1)^s$, $l = 1, 2$ and the equality

$$J_1^n - J_2^n = \sum_{j=0}^{n-1} J_1^{n-j-1}(J_1 - J_2)J_2^j.$$  

The rest is straightforward. \(\square\)

**Definition 6.4.** We say that the matrix $J$ belongs to the class $\mathcal{P}(\beta, J^0)$, if

(6.6) $\omega_n \leq C_1 \exp(-C_2 |n|^\beta), \quad 0 < \beta < 1, \quad C_1, C_2 > 0; \quad n \in \mathbb{Z}.$

Our main result claims that $\sigma_d(J)$ is a finite set as long as $J \in \mathcal{P}(1/2, J^0)$.

**Theorem 6.5.** Let $J$ be an asymptotically $p$-periodic doubly-infinite Jacobi matrix with the quasi-symmetric background $J^0$, if $J \in \mathcal{P}(1/2, J^0)$, then $\sigma(J)$ is the union of $p$ algebraic arcs and a finite number of eigenvalues of finite algebraic multiplicities off these arcs.

**Proof.** Let $T = Q = \alpha^{-1} P$ be the Burchnall–Chaundy polynomial for $J^0$ taken from (6.3). By Proposition 6.2, $Q(J)$ is the $p$-banded matrix. Since $J \in \mathcal{P}(1/2, J^0)$
the matrix $Q(J)$ is close to $Q(J^0) = D_0$ with $D_0$ defined in Proposition 6.3 in the sense of Proposition 6.3

$$\sum_j |(Q(J) - D_0)_{kj}| \leq C_1 e^{-C_2 |k|^{1/2}}$$

and so $Q(J) \in P_p(1/2)$. According to Theorem 5.1 $\sigma(Q(J)) = [-2, 2] \cup E$, where the set $E$ of eigenvalues off $[-2, 2]$ is now finite.

On the other hand, as we know, $\sigma(J) = \Gamma \cup F$, where the set $F$ of eigenvalues off $\Gamma = \sigma(J^0)$ is at most denumerable, and $\Gamma = Q^{-1}([-2, 2])$. Therefore, by the Spectral Mapping Theorem $Q(F) = E$ and so $F$ is a finite set, as claimed. □

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References

[1] I. Egorova and L. Golinskii, On location of discrete spectrum for complex Jacobi matrices. — Proceedings of AMS, 133 (2005), 12, 3635-3641.

[2] I. Egorova and L. Golinskii, On limit sets for the discrete spectrum of complex Jacobi matrices. — Matematicheskiy sbornik. 196 (2005), 6, 43–70.

[3] B.S. Pavlov, On nonselfadjoint Schrödinger operator I. — Problems of mathematical physics, LGU, 1 (1966), 102–132.

[4] B.S. Pavlov, On nonselfadjoint Schrödinger operator II, — Problems of mathematical physics, LGU, 2 (1967), 135–157.

[5] I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators in Hilbert space. “Nauka”, Moscow, 1965 (Russian).

[6] O. Perron, Über Summengleichungen und Poincaré’sche Differenzengleichungen. — Math. Annalen, 84 (1921), 1–15 (German).

[7] L. Carleson, Sets of uniqueness for functions analytic in the unit disc. — Acta Math, 87 (1952), 325–345.

[8] P. B. Naiman, On the theory of periodic and limit-periodic Jacobi operators. — Sov. Math. Dokl, 3 (1962), 4, 383–385 (Russian).

[9] P. B. Naiman, To the spectral theory of the non-symmetric periodic Jacobi matrices. — Notes of the Department for Mathematics and Mechanics of the Kharkov’s state University and of the Kharkov’s mathematical Society, XXX (1964), 4, 138–151 (Russian).

[10] V. Batchenko and F. Gesztesy, On the spectrum of Jacobi operators with quasi-periodic algebrao-geometric coefficients, to appear in IMRN.

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