Quantum Models of Black Hole Evaporation

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ABSTRACT

The discovery of black-hole evaporation represented in many respects a revolutionary event in scientific world; as such, in giving answers to open questions, it gave rise to new problems part of which are still not resolved. Here we want to make a brief review of such problems and examine some possible solutions.

1. BH Evaporation: Open Questions

The great problem of bh-evaporation is well exemplified by the following question: which is the black-hole final state? A complete answer to this question contains the solution to the other great enigma: is there a loss of information in bh-evaporation? That is, is it possible to recover the information contained in the initial state of bh-evolution? In dimension $D = 4$ these question have no answer. Hawking’s work\textsuperscript{11} is based on two distinct approximations:

\begin{itemize}
  \item a) the semiclassical approximation in which the bh background geometry is classical,
  \item b) the emitted radiation has no backreaction on the geometry.
\end{itemize}

Invited Talk at the ”Workshop on String Theory, Quantum Gravity and the Unification of the Fundamental Interactions” Rome, September 21-26
It is well known that both these approximations fail at $m_{bh} \sim M_{\text{Planck}}$; at such energy scales it is necessary to make a full quantum treatment of the gravitational field.

The so-called “Information Loss Problem” consists in what follows: it is reasonable to think that we can associate to black-holes a great entropy content, because we are not able to measure and specify its internal microstates; our only physical possibility to speak about a bh is to assign to it a macrostatus defined generally by three measurable quantities: its mass, its angular momentum, its charge.

As soon as the bh forms, because of the event horizon, all the informations about the initial state and microscopic structure of the collapsing star are lost. This is why we say that the “bh has no hair”.

The bh entropy is proportional to the area of its horizon. During the evaporation the horizon shrinks causing the bh entropy to decrease. Can this cause a loss of information?

This same problem can also be reformulated looking for the existence of a unitary S-matrix$^{12}$.

2. Bidimensional Models

Given the difficulties of the four dimensional case much work has been recently concentrated on the study of bidimensional quantum models that could give some insight, at least qualitatively, about the physical case.

Our starting point is the action containing the coupling of gravity to a dilaton field

$$S_{DG} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} e^{-2\Phi} \left( R + 4(\nabla \Phi)^2 + 4\lambda^2 \right). \quad (2.1)$$

As it is well known it is the coupling of the dilaton field $\Phi$ to the Einstein metric that makes the theory non trivial. The main advantages of such models are:

i) the theory is perturbatively renormalizable so that, at least in principle, we can indagate its quantum aspects;

ii) we can think about $S_{DG}$ as deriving from an off-critical string theory in a low energy approximation;

iii) after some manipulations the dilaton-gravity can be mapped into some soluble model

The theory, using light-cone coordinates and in the conformal gauge $g_{+-} = -\frac{e^{2\rho}}{2}$ ($\rho$ is the Liouville mode) looks like:

$$S_{DG} = \frac{1}{\pi} \int d^2 \sigma [e^{-2\Phi}(2\partial_+ \partial_- \rho - 4\partial_+ \Phi \partial_- \Phi + \lambda^2 e^{2\rho})] \quad (2.2)$$
3. The CGHS Model

The purpose of this model is to provide a description of:
i) the creation of a BH by means of matter shock waves
ii) the backreaction problem
This is achieved by summing to the action Eq.(2.1) a matter term:

\[ S_M = -\frac{1}{4\pi} \int d^2x \sqrt{g} \sum_{i=1}^{N} (\nabla f_i)^2, \]  

(3.1)

where the \( f_i \)'s are conformal massless matter fields minimally coupled to gravity whose energy tensor in light cone coordinates is proportional to \( \delta(x^+ - x^0) \).

One way to keep the backreaction into account is to include at the Lagrangian level the integrated trace anomaly which is local in our gauge: the asymptotically flat coordinates trace anomaly contribution is:

\[ S_{\text{anomaly}} = -\frac{N}{12\pi} \int d^2\sigma \partial_+ \rho \partial_- \rho. \]  

(3.2)

In the large \( N \) limit, we can perform a \( 1/N \) expansion to find that the anomaly term is of the same order of magnitude of the classical contribution. Furthermore the quantum effects of gravity and of the dilaton can be neglected in this approximation.

An analysis of the model reveals two different regions of the Penrose diagram. The first is a weak coupling region where \( \text{Exp}(-2\Phi) \gg \frac{N}{12} \). The second is a strong coupling region where \( \text{Exp}(-2\Phi) \ll \frac{N}{12} \). These two regions are separated by a critical time-like line \( \text{Exp}(-2\Phi_{cr}) = \frac{N}{12} \).

The problem is that \( \Phi \sim \Phi_{cr} \) leads to a naked curvature singularity or, in other words, the semiclassical approximation breaks down. Moreover we find that the Hawking temperature is mass independent contrary to our physical expectations.

4. The Bilal-Callan-De Alwis Model

One possible way out of these problems is to build two-dimensional models derived from low energy string theories using the many possible vacua that lead to vanishing beta functions. The recipe to do this is:
i) to replace the \( \frac{N}{12} \) coefficient in Eq.(3.2) with a generic trace anomaly coefficient \( \kappa \) to be determined by imposing the vanishing of the total central charge;
ii) to modify the potential term by replacing it with conformal vertices of weight \( (1,1) \) which in the classical limit \( \Phi \to -\infty \) reproduce the cosmological term.
The models thus obtained are Liouville-like. They are still ill-defined at the quantum level as their kinetic energy goes to zero at a certain critical value of the dilaton field for which also the curvature is singular. A proposed way out of it is to restrict the fields to a region with no singularity. These restrictions are not natural from the point of quantum field theory and up to now nobody has succeeded in doing so consistently.

5. The CAT Model

We start with the following definitions:

$$\omega \equiv \frac{\exp(-\Phi)}{k^{\frac{1}{2}}}$$

$$2\Omega(\omega) \equiv \omega(\omega^2 - 1)^{\frac{1}{2}} - \log(\omega + (\omega^2 - 1)^{\frac{1}{2}})$$

for $$\omega \geq 1$$ (as in ref. 4) and

$$2\Omega'(\omega) \equiv \omega(1 - \omega^2)^{\frac{1}{2}} - \arccos(\omega)$$

for $$\omega \leq 1$$

$$2\chi \equiv \rho + (\omega)^2.$$ 

It should be noted that the value $$\omega = 1$$ corresponds to the critical value of the dilaton field.

The new idea is now the following: we shall formulate an effective theory with the explicit aim of describing physics around $$\omega = 1$$ by means of a non-local field redefinition:

$$2\eta(\sigma) \equiv \Omega(\sigma + \epsilon) + \Omega'(\sigma - \epsilon)$$

$$2\xi(\sigma) \equiv \Omega(\sigma + \epsilon) - \Omega'(\sigma - \epsilon)$$

where $$\sigma + \epsilon, \sigma - \epsilon$$ are two nearby points and $$\omega(\sigma + \epsilon) \sim 1^+, \omega(\sigma - \epsilon) \sim 1^-.$$ The signature of the kinetic term thus obtained does not oscillate.

Now we have to construct interaction vertices in these new field variables with conformal weight $$(1,1)$$ and match them against the cosmological constant term in the classical limit.

The resulting action is

$$S_{cat} = \frac{1}{2\pi} \int d^2x [-k(\partial \chi)^2 + (\partial \eta)(\partial \xi) + \gamma_+ e^{A+} + \gamma_- e^{-A-} + B\eta]$$ (5.1)

where the parameters appearing in the action are adjusted by the request of conformal invariance$^1$.

$$S_{cat}$$ describes an affine $$\widehat{SL}(2)$$ Toda theory, previously studied by Babelon-Bonora in the framework of integrable models$^3$.

This is more evident if we make another field redefinition: $$\varphi \equiv i\sqrt{2\kappa}\chi.$$ We shall refer to our model as the conformal affine Toda black hole (CATBH) model. The CATBH model allows a standard perturbative quantization with the aim of unveiling the
quantum effects of the dilaton-graviton fields: as a matter of facts, CAT fields are suitable functions of the dilaton and of the Liouville fields.

We now want to consider the renormalization flow of the classical Babelon-Bonora action

$$S_{BB} = \frac{1}{2\pi} \int d^2 x \left[ \frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi + \partial_\mu \eta \partial_\mu \xi - 2 \left( e^{2\varphi} + e^{2\eta-2\varphi} \right) \right]. \quad (5.2)$$

At the quantum level one must implement wave and vertex function renormalizations so that in (5.2) one must introduce different bare coupling constants in front of the fields as well as in front of the vertex interaction terms. As a consequence one ends with the form (5.1). However, according to the general spirit of the renormalization procedure, all generally covariant dimension 2 counterterms are possible in (5.1) and hence also Feigin–Fuchs terms, i.e., the ones involving the 2D-scalar curvature, are allowed. This ansatz is in agreement with the perturbative theory as one could show following Distler and Kawai. In our context the Feigin–Fuchs terms come naturally out if we look at the improved stress tensor. This leads us to consider the following quantum action:

$$S = \frac{1}{2\pi} \int d^2 x \sqrt{g} \left[ g^{\mu\nu} \left( \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi + \partial_\mu \eta \partial_\nu \xi \right) + \gamma_+ e^{i\lambda(+)\varphi} + \gamma_- e^{i\delta\eta-i\lambda(-)\varphi} + i q_\varphi R \varphi + i q_\eta R \eta + i q_\xi R \xi. \right] \quad (5.3)$$

Let us now start with the renormalization procedure of (5.3) in a perturbative framework, in the hypothesis that $\lambda(-)^2 \sim \lambda(+)^2 \sim 4$. Notice that $\xi$ plays the role of an auxiliary field; a variation with respect to $\xi$ gives the on-shell equation of motion

$$\nabla_\mu \nabla^\mu \eta = i q_\xi R, \quad (5.4)$$

which in our perturbative scheme must be linearized around the flat space:

$$\partial_\mu \partial^\mu \eta = 0. \quad (5.5)$$

Following Ref.16, we define the renormalized quantities at an arbitrary mass scale $\mu$ by:

$$\varphi = Z_\varphi^{\frac{1}{2}} \varphi_R, \quad \gamma_\pm = \mu^2 Z_{\gamma\pm} \gamma_{\pm R}, \quad q_\varphi^2 = Z_{\varphi}^{-1} q_\varphi^2 R, \quad \eta = Z_\eta \eta_R, \quad \lambda(\pm)^2 = Z_{\varphi}^{-1} \lambda(\pm)^2_R, \quad q_\eta^2 = Z_\eta^{-1} q_\eta^2 R, \quad \xi = Z_\xi \xi_R, \quad \delta^2 = Z_\eta^{-1} \delta^2_R, \quad q_\xi^2 = Z_\xi q_\xi^2 R. \quad (5.6)$$

The following quantities are conserved through renormalization:

$$r = \frac{q_\varphi}{\lambda(-)}, \quad l = \frac{\lambda(+) \varphi}{\lambda(-) \eta}, \quad k = q_\xi \delta, \quad p = q_\xi q_\eta. \quad (5.7)$$
Following essentially the same procedures of Ref.16, with slightly modifications due to the presence of extra fields and by taking $\lambda^+ \neq \lambda^-$, we find that the theory can be renormalized at one loop if we restrict ourself to the on-shell renormalization scheme, i.e. if we get rid of the terms in $\partial_\mu \partial^\mu \eta$ (produced by the renormalization) using (5.5). The true on-shell theory should rely on (5.4), but at this perturbative order curvature terms can be neglected: they are important only in the renormalization of $\gamma_\pm$. The curvature terms are taken into account considering the modifications to the trace of the stress-energy tensor, calculated on-shell. We finally get the following $\beta$ functions:

\begin{align}
\beta_+ & := \mu \frac{d\gamma^+}{d\mu} = 2\gamma^+ \left( \frac{\lambda^+}{4} - 1 - q_\phi R\lambda^+ \right), \\
\beta_- & := \mu \frac{d\gamma^-}{d\mu} = 2\gamma^- \left( \frac{\lambda^-}{4} - 1 + q_\phi R\lambda^- - q_\xi R\delta \right), \\
\beta_{\lambda^\pm} & := \mu \frac{d\lambda^\pm}{d\mu} = \frac{1}{2} \gamma^+\gamma^- \lambda^\pm \frac{\lambda^\pm}{2} R \lambda^\pm, \\
\beta_\delta & := \mu \frac{d\delta^2}{d\mu} = \frac{1}{2} \gamma^+\gamma^- \delta^2 R, \\
\beta_{q_\phi} & := \mu \frac{dq_\phi^2}{d\mu} = \frac{1}{2} \gamma^+\gamma^- \frac{\lambda^+}{2} R q_\phi^2, \\
\beta_{q_\eta} & := \mu \frac{dq_\eta^2}{d\mu} = \frac{1}{2} \gamma^+\gamma^- \delta^2 R q_\eta^2, \\
\beta_\delta & := \mu \frac{dq_\xi^2}{d\mu} = \frac{1}{2} \gamma^+\gamma^- \delta^2 R q_\xi^2. 
\end{align}

(5.8)

(5.9)

Putting together the equations (5.9), we find that the following quantity is also conserved through renormalization.

\begin{equation}
d = \frac{1 + l^2}{4} \frac{1}{\delta^2} - \frac{1}{\lambda^2}. \tag{5.11}
\end{equation}

This result needs not to be valid beyond this perturbative order. Using the non-perturbative RG-invariants in (5.7) and dropping out the rather cumbersome $R$ indices, we can rewrite the relevant $\beta$ functions as:

\begin{align}
\beta_+ & = 2\gamma^+ \left( \frac{l^2 - 4l r}{4} \lambda^- - 1 \right) \\
\beta_- & = 2\gamma^- \left( \frac{1 + 4r}{4} \lambda^- - 1 - k \right) \\
\beta_{\lambda^-} & = \frac{1 + l^2}{4} \gamma^+ \gamma^- \lambda^-.
\end{align}

(5.12)
Then we have to request that at a certain scale \( t_0, t \equiv \log(\mu) \), both vertex operators have a conformal weight \((1, 1)\). As a consequence, \( \gamma_\pm \) depend on the scale \( t - t_0 \) only through the cosh, which is an even function, and hence does not distinguish between IR and UV scales. The requirement of having vertex operators with the right conformal weight at the renormalization point \( t_0 \) forces the theory to be dual (under the exchange of the IR and UV scales). Explicitly we get:

\[
\lambda(-)^2(t) = \frac{8}{2 + \kappa} \left\{ 1 + k - \frac{2 + (1 - k)\kappa}{\kappa} \tanh \left[ \frac{2 + (1 - k)\kappa}{\kappa} \frac{2 + k}{1 + k} (t - t_0) \right] \right\},
\]

\[
\gamma_-(t) = A \left\{ \cosh \left[ \frac{2 + (1 - k)\kappa}{\kappa} \frac{2 + k}{1 + k} (t - t_0) \right] \right\}^{-2\frac{1+k}{2+k}}. \tag{5.13}
\]

This UV-IR duality implies that \( \gamma_-(t) \) behaves in the same manner both at an UV and at an IR scale with the asymptotic behaviour:

\[
\gamma_-(t) \left| t - t_0 \right| \to \infty \sim A e^{-2s|t - t_0|}, \quad \text{with} \quad s = \frac{2+(1-k)\kappa}{\kappa}. \tag{5.14}
\]

where \( \kappa = (N - 23)/12 \) and \( s \) is always positive and close to 1.

The main result here is that the RG-analysis\(^1\) shows that there exists an energy scale \( t_{BC} \ll t_0 \) at which our action “becomes” the de Alwis-Bilal-Callan action, in the sense that \( \frac{\gamma_-(t_{BC})}{\gamma_+(t_{BC})} \gg 1 \). Notice that in the classical limit \( e^\Phi \to 0 \); our effective action in the IR-phase \( t = O(t_{BC}) \) becomes the one of ref. 4 since \( \xi \to \eta \to \Omega \).

Our strategy at this point is to use the above running coupling constants \( \gamma_-, \lambda_- \) to define “effective” bh thermodynamic quantities at the scale \( t_{BC} \). Indeed by a physical point of view the back-reaction should modify Hawking radiation emission and cause it to stop as soon as the bh has radiated away all its initial ADM mass, so that it should be reasonable to get a decreasing Hawking temperature rate at the end point of bh-evaporation. In our CATBH model the bh temperature is proportional to \( \mu \gamma_\frac{1}{2} \). We can regard \( \gamma_- \) as a running coupling constant in terms of \( \mu \), which is roughly a measure of ADM mass. In particular, a reasonable ansatz for the bh solution formed by \( N \) infalling matter shock waves, allows us to find a simple link between the RG-scale \( t \) and the bh mass (that is the physical scale of our problem!). In fact a relation between the v. e. v. of the operator-valued scalar curvature \( \sqrt{-g} R \) and the other CATBH running coupling constant \( \lambda_- \) can be obtained in the classical limit \( e^\Phi \to 0 \). If we consider the conformal gauge \( \tilde{g}_{\mu\nu} = -\frac{1}{\sqrt{2}} e^{2\lambda_\rho} \eta_{\mu\nu}, \) we get in the tree approximation:

\[
< \sqrt{-g} R >_{\text{tree}} = 2\lambda \partial_\mu \partial^\mu < \rho >_{\text{tree}}. \tag{5.15}
\]

In eq. (5.15) we may use the explicit solution describing the black hole formation by \( N \)-shock waves \( f_i \), with \( < \rho >_{\text{tree}} \) given by the CGHS classical solution.
\[ e^{-2\lambda_- (t) <\rho (x^+)>_{\text{tree}}} = 1 - 2\lambda_- (t) <\rho (x^+)>_{\text{tree}} + O(\lambda_-^2) \]
\[ = -\kappa a (x^+ - x_0^+) \theta (x^+ - x_0^+) - \gamma_- (t) x^+ x^-, \quad (5.16) \]

where \( \theta \) is the Heaviside function, \( x^\pm \equiv x^0 \pm x^1 \) and \( a \equiv \text{const} \). Therefore, at \( x^+ = x_0^+ \), where the \( f \)-waves are sitting, we have, using the light-cone coordinates \( x^\pm : <\sqrt{-\hat{g}(x^+)\hat{R}(x^+)}>_{\text{tree}} \sim \gamma_- (t), x^+ \to x_0^+ \).

Since here we have two scales \( x_0^+ \) and \( \mu \), it is reasonable to set (in \( c = \hbar = 1 \)) \( x_0^+ \equiv \frac{1}{\mu} \), since the natural scale which describes the black hole formation is \( x_0^+ \). Therefore we arrive to:

\[ < [\sqrt{-\hat{g}}\hat{R}] (x_0^+) >_{\text{tree}} \sim \gamma_- [-\log(x_0^+)]. \quad (5.17) \]

Furthermore in the CGHS solution, the mass \( m_{bh} \) of the bh created grows linearly with \( x_0^+ \), i.e. \( m_{bh} \propto M_{Pl}^2 x_0^+ \). Thus putting all together we get that the end point of the bh-evaporation is charaterized by:

\[ < \sqrt{-\hat{g}} \hat{R} >_{\text{tree}} \to 0, m_{bh} \to 0, \quad (5.18) \]

\[ T_{bh} \sim T_0 \left( \frac{m_{bh}}{m_0} \right)^{s-1} \to 0, m_{bh} \to 0. \quad (5.19) \]

Notice that for “astrophysical” bh, i.e. with large mass, one finds that

\[ T_{bh} \sim T_0 \left( \frac{m_0}{m_{bh}} \right)^{s+1} \to 0, m_{bh} \to \infty. \quad (5.20) \]

The vanishing of the bh temperature both for small and large bh mass is a conse-quence of the duality between UV and IR scales observed above.
Then we see that the end-point of the bh-evaporation is charaterized by a regular geometry and an almost zero Hawking temperature.

There are various lines of developement of this model\(^2\); in particular we signal: i) the calculation of the geometry and dilaton associated to the solutions of our conformal affine Toda model; ii) the derivation of an (unitary?) exact S-matrix which is allowed (in principle) by the quantum integrability of (5.3).
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