Directed emission from a dielectric microwave billiard with quadrupolar shape

R Schäfer, U Kuhl and H-J Stöckmann
Fachbereich Physik der Philipps-Universität Marburg, D-35032 Marburg, Germany
E-mail: rudi.schaefer@physik.uni-marburg.de

New Journal of Physics 8 (2006) 46
Received 10 January 2006
Published 31 March 2006
Online at http://www.njp.org/
doi:10.1088/1367-2630/8/3/046

Abstract. We present microwave measurements on dielectric billiards with mixed phase space. Transmission spectra were measured from a fixed antenna inside the billiard to a moveable antenna, thus scanning both the inside and outside region of the billiard. A Fourier transform of the transmission spectra yields the pulse propagation, which is used to study in particular the long-time dynamics of the system. The Poynting vector, which describes the energy flow of the microwaves, is obtained from the measurement for each time step of the pulse propagation. Close to the boundary of the billiard it reveals a characteristic directionality of the microwave emission, which is in accordance with measurements of the far-field intensities of micro-disc lasers of the same shape. To achieve a direct comparison to the internal dynamics of the classical system, the Husimi distributions of the pulse propagation are calculated. Averaging the Husimi distributions for the long-time dynamics provides a very clear picture of the internal dynamics in phase space. The results are in agreement with classical simulations.

1 Author to whom any correspondence should be addressed.
1. Introduction

Disc-shaped dielectric cavities have received a lot of attention, in particular as compact optical resonators. They are of interest for the design of microlasers and integrated optics applications [1]. The light can be trapped inside the dielectric cavity by total internal reflection, or can partly escape by refraction, depending on the angle of incidence with respect to the boundary. The resonances with the longest lifetimes (high-Q modes) are the so-called ‘whispering gallery modes’ which circulate along the boundary and always stay above the critical angle of total internal reflection.

Initially, in the early 1990s, cavities with a circular boundary were used [2] to fabricate so-called microlasers. Their emission patterns are homogenous in all directions due to rotational symmetry and the output power is rather weak because of long lifetimes of the resonances. However, both for the design of microlasers, and for other optical applications, it is desirable to have a directed emission and a strong coupling to the outside.

A lot of work has been done to investigate different shapes of such devices [3]–[6] to improve their characteristics. These studies range from numerical approaches to calculate the resonance patterns [7, 8], to the comparison with ray simulations [3, 9]–[11] to provide a better understanding of the emission behaviour. This inevitably touches the field of quantum chaos, see e.g. [12]. Special attention has to be paid to the Fresnel laws at curved dielectric interfaces [13] and to Fresnel filtering due to the finite angular spread of a beam [14].

For comparison with the classical phase space structure, many publications rely on the Husimi distribution of wave patterns, see e.g. [7, 11, 15, 16]. Recent studies also incorporate the nonlinear effects of the optical gain due to the pumping of the active lasing material. In the papers by Harayama and co-workers [4, 17, 18], the full nonlinear Maxwell–Bloch equations are considered to describe the active medium. Tureci and Stone [19] work with a near-threshold approximation, developed by Haken and Sauermann [20], which yields multi-mode lasing equations and thus allows us to address the problem of mode competition.

On the experimental side, advances in the production techniques have been made (see e.g. [15, 21]), and the lasing modes have been inferred both from the far-field emission pattern and from the emission intensity at the cavity boundary [11, 14]. In this context, the phenomenon of scarring, i.e. lasing modes which resemble classical periodic orbits, has been studied intensely.
Apart from disc-shaped semiconductor microlasers, one should also note related experimental work on lasers using microcrystal structures [22], liquid jets [21] and liquid droplets [23].

In this paper, we present microwave measurements of a dielectric quadrupole billiard. The focus will be on the comparison of the measured wave patterns to the classical ray dynamics of the system. It was shown by Nöckel and Stone [3, 24] that smooth deformations of the circular shape lead to anisotropic whispering gallery modes, which have a directional emission pattern and quality factors that are tunable by the degree of deformation.

For an ellipse, the strongest emission is expected to be tangential at the points of highest curvature. The situation is more complex and thus more interesting for a quadrupolar deformation of the circle, described by

$$r(\phi) = 1 + \epsilon \cos 2\phi,$$

where $\epsilon$ is the deformation parameter. This deformation leads to a mixed phase space, i.e. partly regular and partly chaotic dynamics, which has important consequences for the internal dynamics and the emission behaviour of the system. The strongest emission does not occur at the points of highest curvature; still the emission pattern is highly directional.

Figure 1 (left) shows the Poincaré section of the classical phase space for the quadrupole billiard with deformation parameter $\epsilon = 0.08$. The critical angle for total internal reflection is given by $\sin \chi_c = 1/n$, where $n$ is the index of refraction of the dielectric, assumed to be surrounded by air. For teflon, the material used in this experiment, the index of refraction is $n = 1.44$, yielding $\sin \chi_c = 0.69$. To obtain a Poincaré section only the points of reflection at the boundary are considered instead of the continuous trajectory. For each reflection point its position, parameterized by the angle $\phi$, and the sine of the angle of incidence $\chi$ are plotted. This is illustrated in figure 1 (right).

Nöckel et al [10] developed a ray model description for asymmetric resonant cavities, which tries to explain the emission of deformed whispering gallery modes as refractive escape of rays which are initially trapped by total internal reflection. Due to their chaotic dynamics, these rays diffuse chaotically until they reach the critical angle and leave the cavity.
Initially, the presence of stable islands intersecting the line of the critical angle was considered the main reason for the characteristic emission behaviour of the quadrupole billiard, because the islands prevent the escape of rays at the points of highest curvature, $\phi = 0$ and $\phi = \pi$. This phenomenon was called ‘dynamical eclipsing’ [10].

However, Schwefel et al [11] found in a detailed study of the classical dynamics that the characteristic emission behaviour persists even for strong deformations, where the stability islands have already vanished. They found that the unstable manifold of the rectangular periodic orbit (see figure 1) dominates the short-time dynamics of the system, which determines the emission behaviour. In section 2, this unstable manifold is discussed in detail.

Discussions with J U Nöckel have initiated our microwave experiments on dielectric quadrupole billiards, where the dielectric micro-disc is substituted by a teflon disc with $n = 1.44$. The experimental setup is described in section 3. We measured transmission spectra using two antennas, and obtained the pulse propagation by a Fourier transform [25].

For each time step of the pulse propagation we determined the Poynting vector in the near-field. Thus, we were able to study the directionality of the emission behaviour. In laser experiments, one usually examines the light intensity in the far-field to extract the same information.

To achieve a direct comparison to the internal dynamics of the classical system, we determine the Husimi distributions of the pulse propagation (see section 5). By averaging the Husimi distributions over four periods of the circulating wave packets, we achieve a very clear picture of the internal dynamics in phase space. The results are presented in section 7. They show very good agreement with classical simulations for the long-time dynamics, if the escape of the rays due to refraction is taken into account. This illustrates the importance of the unstable manifold for the emission behaviour.

2. The unstable rectangular orbit

It is convenient to analyse the dynamics in terms of the Poincaré surface of section, where the dynamics from one reflection at the boundary to the next one can be described by a discrete map. Denoting position and direction by $(s, u)$, where $s$ is the arc length and $u$ the sine of the incidence angle, the map which propagates the ray to the next position and direction is defined by

$$T : (s, u) \rightarrow (s_1, u_1).$$  (2)

A point $(s_p, u_p)$ of the surface of section is called a fixed point of order $N$, if it satisfies

$$T^N(s_p, u_p) = (s_p, u_p).$$  (3)

It corresponds to a periodic orbit in real space.

The motion in the vicinity of a fixed point can be described by the monodromy or stability matrix $M$, which is a linearization of the map $T^N$ around the fixed point:

$$M = \begin{pmatrix} \frac{\partial s_N(s, u)}{\partial s} & \frac{\partial u_N(s, u)}{\partial s} \\ \frac{\partial s_N(s, u)}{\partial u} & \frac{\partial u_N(s, u)}{\partial u} \end{pmatrix},$$  (4)

where $(s_N, u_N) = T^N(s, u)$.
Figure 2. Poincaré section of the quadrupole billiard for $\epsilon = 0.08$ (left) and $\epsilon = 0.13$ (right). The unstable manifold of the rectangular orbit is shown as the blue curve. The horizontal red lines denote the critical angle at $\sin \chi_c \approx \pm 0.69$.

For Hamiltonian flows $M$ is always an area-preserving map, i.e. $\det M = 1$. The eigenvalues of $M$ can be either in complex conjugate pairs on the unit circle or they are purely real and reciprocal to each other. If the eigenvalues of the monodromy matrix $M$ are complex, the fixed point is stable (elliptic) and nearby points oscillate around the fixed point. The modulus of the eigenvalues is 1 in this case.

In the case of real eigenvalues the fixed point is hyperbolically unstable. The eigenvector corresponding to the eigenvalue larger than 1 describes the unstable direction; in this direction deviations from the fixed point grow exponentially. The eigenvector belonging to the eigenvalue smaller than 1 describes the stable direction; in this direction deviations relax exponentially towards the fixed point. This behaviour can be inverted by reversing time. Then deviations in the unstable directions relax towards the fixed point and deviations in the stable direction will increase.

By iterating a set of points on the unstable eigenvector (but still very close to the fixed point), we can visualize the unstable manifold of the fixed point which is defined as the set of points that approaches the fixed point arbitrarily closely as $t \to -\infty$. As the unstable manifold deviates further from the fixed point, it begins to have larger and larger oscillations. This is necessary to preserve phase space area while at the same time have exponential growth of deviations.

The short-time dynamics in the vicinity of a hyperbolic fixed point is dominated by its unstable manifold, because a generic deviation will have at least some component along this unstable manifold. This was demonstrated e.g. in [11] for the quadrupole billiard.

While the quadratically shaped periodic orbit in the circle billiard is marginally stable because a generic deviation will have at least some component along this unstable manifold, this was demonstrated e.g. in [11] for the quadrupole billiard.

Figure 2 (left) shows the initial parts of the unstable manifold for the rectangular orbit in the quadrupole billiard with $\epsilon = 0.08$. For convenience, we use the angle $\phi$ in all plots to parameterize the boundary of the billiard. The corresponding fixed points of order 4 are located at $\phi_p \approx 0.2\pi$, $0.8\pi$, $1.2\pi$ and $1.8\pi$, respectively. Their angle of incidence is $\chi_p = \pi/4$, yielding $\sin \chi_p \approx 0.707$. Thus, they are just above the critical line of total internal reflection, $\sin \chi_c \approx 0.69$. The manifold encloses the stability islands of the diamond-shaped orbit.
For a larger deformation, $\epsilon = 0.13$, the stability islands of the diamond-shaped orbit have shrunk considerably, while the unstable manifold of the rectangular orbit has become very dominant (see figure 2 (right)). The positions of the fixed points are in this case $\phi_p \approx 0.18\pi$, $0.82\pi$, $1.18\pi$ and $1.82\pi$, respectively. Still, their angle of incidence is $\chi_p = \pi/4$, and thus $\sin \chi_p \approx 0.707$.

In section 5, we will compare these results of the classical billiard with the dynamics of the microwave system by means of Husimi distributions.

3. Microwave measurement

The microwave systems studied in this paper consisted of a ground plate made of brass with rounded corners and dimensions 380 mm $\times$ 260 mm. On this ground plate the teflon discs were fixed with an adhesive (see figure 3 (left)). Teflon has an index of refraction of $n = 1.44$ and is particularly well suited for microwave studies, since it does not attenuate the microwaves noticeably. The height of the discs was $h = 8$ mm, and their shape is given by equation (1), scaled with the mean radius $\bar{R} = 100$ mm. We are going to present the results for two deformation parameters, $\epsilon = 0.08$ and 0.13.

The upper part of the system consisted of a brass plate supporting an antenna which could be moved with respect to the ground plate, thus allowing to scan the system. The top plate was large enough to cover the whole bottom plate for any position of the scanning antenna. In order to scan the system in the region of the teflon disc, the antenna was cut to be flush with the top plate (see figure 3 (right)). This led to a noticeable reduction in the signal-to-noise ratio, but it was still possible to obtain reasonable results. More details on the setup and the technique can be found in [26, 27].

Figure 4 shows a typical transmission spectrum from the fixed antenna in the bottom plate ($x_a = -15$ mm and $y_a = -76.5$ mm) to the movable antenna in the top plate. Up to 13 GHz the spectrum shows a very regular spacing of resonances, because only the whispering gallery modes have long life-times. All other eigenmodes are not bound by total internal reflection and leave the teflon rather rapidly. At the edge of the ground plate the microwaves are reflected only very weakly.
Figure 4. Transmission spectrum showing almost equidistant resonances below 13 GHz, which correspond to whispering gallery modes. Above 13 GHz the spectrum shows a multitude of sharp resonances.

The situation is different for frequencies above 13 GHz, where we observe a rich spectrum of sharp resonances. The reason is the difference in the index of refraction inside and outside the teflon disc. For frequencies below \( \nu_c = c/(2nh) \) only TM\(_0\) modes without \( z \)-dependence can be excited, because the wavelength is too large. In this case, the billiard is called quasi-two-dimensional. For higher frequencies also TM\(_1\) modes with one node in \( z \)-direction are allowed. Due to the higher index of refraction of teflon, this is already possible above \( \nu_c \approx 13 \) GHz within the billiard, while in air the cut-off frequency is \( \nu_c \approx 18.75 \) GHz. Since in this intermediate frequency range between 13 and 18.75 GHz the TM\(_1\) modes in the teflon billiard cannot couple to equivalent modes in the outside region, they are trapped inside the teflon irrespective of the angle of incidence at the teflon–air interface. Therefore, the teflon disc acts like a closed system for these modes leading to the spectrum discussed above.

In the following, we shall only consider the frequency range below 13 GHz, since we are interested in the teflon disc as an open system.

4. The electromagnetic propagator

The measurements presented in this paper were done using an Agilent 8720ES vector network analyser, yielding directly the scattering matrix \( S \) of the system. The non-diagonal elements of \( S \) are given by the transmission amplitudes \( S_{ij} \) between antennas \( i \) and \( j \), and the diagonal elements by the reflection amplitudes \( S_{ii} \) at antennas \( i \). Scattering theory yields a relation between the scattering matrix \( S \) and the Green function \( G \) of the billiard (see e.g. [28, 29]):

\[
S_{ij} = \delta_{ij} - 2i\gamma G(\vec{r}_i, \vec{r}_j).
\]  

For isolated resonances, the Green function for the electromagnetic case can be written as

\[
G(\vec{r}_i, \vec{r}_j, k) = \frac{\psi_n(\vec{r}_i) \psi_n(\vec{r}_j)}{k^2 - k_n^2 + i\Gamma_n},
\]  

where the complex widths \( \Gamma_n \) lead both to a broadening and to a shift of the resonances due to absorption and coupling to the antenna. Also the wavefunctions \( \psi_n(r) \) are not exactly the ones of the closed system, but are slightly perturbed due to the presence of the antenna.
By a Fourier transformation of the transmission spectra $S_{ij}(\nu)$ we directly obtain the electromagnetic propagator

$$K(\vec{r}_i, \vec{r}_j, t) = \frac{1}{2\pi i} \int G(\vec{r}_i, \vec{r}_j, k) e^{i\omega t} d\omega, \quad \omega = 2\pi \nu = kc. \quad (7)$$

It is also possible to calculate the quantum-mechanical propagator by taking the corresponding dispersion relation into account.

Since in our experiment one antenna position was fixed, we did not measure the complete Green function. Thus the Fourier transform yields a pulse propagation with a fixed initial condition, corresponding to a circular wave emitted from the fixed antenna.

For each teflon billiard, transmission measurements were performed for 2632 positions of the scanning antenna on a square grid with 5 mm resolution in the frequency range 0.5–18.24 GHz. In the following, we will discuss in detail the results for the teflon quadrupole billiard with a deformation $\epsilon = 0.13$. Only at the end, we will compare these results with the ones for a smaller deformation, $\epsilon = 0.08$.

Figure 5 (left) shows a whispering gallery mode for the resonance at $\nu = 10.4$ GHz. The plot of the wavefunction has been obtained by averaging the transmission amplitude (including the phase) in a small frequency window for every position of the scanning antenna.

In figure 5 (right), we present a snapshot of the pulse propagation at an early time $t = 404$ ps, when we still see the emergence of the circular pulse from the fixed antenna. Further we observe the different wavelengths inside and outside of the teflon, caused by the different indices of refraction. Already at this early stage, we see the development of two wave packets supported by the whispering gallery modes, one running clockwise, the other counter-clockwise along the boundary.

A sequence of snapshots of the pulse propagation is presented in figure 6. The first few time steps show a circular wave that is emitted from the fixed antenna. In the second row of the figure,
Figure 6. Sequence of the pulse propagation for some initial time steps: $t = n \Delta t$, where $\Delta t = 81$ ps and $n = 2, 4, \ldots, 24$. The plots show the amplitude of the pulse, where red corresponds to zero, dark colours correspond to positive, and light colours to negative values. See the movie in the supplemental multimedia files.
we see how most of the initial wave packet escapes the teflon disc due to a very steep angle of incidence. The remaining part of the pulse is almost completely concentrated on the two wave packets described above.

5. Husimi distributions

To compare the internal dynamics of the teflon system to the classical Poincaré section, we shall use the Husimi distribution [30]–[32], which is the quantum analogue to the classical phase-space probability density. It is the projection of a given quantum state $|\psi\rangle$ on to a coherent state of minimum uncertainty, i.e. a Gaussian wave packet. Making the same restrictions on position and momentum as in the Poincaré section, where only reflections on the boundary of the billiard are considered, we can write the Husimi distribution as

$$H(\phi, \chi) = |\langle \phi, \chi | \psi \rangle|^2,$$

where $|\phi, \chi\rangle$ denotes the coherent state at the boundary. In figure 7, such a coherent state is shown for $\phi = 3\pi/2$ and $\sin \chi = 0$. Its representation in real space can be written as

$$|\phi, \chi\rangle_\vec{r} \propto \exp \left( -\frac{|\vec{r} - \vec{r}_0|^2}{4\sigma^2} - i\vec{k}_0 \cdot \vec{r} \right),$$

where $\vec{r}_0 = \vec{r}_0(\phi)$ and $\vec{k}_0 = \vec{k}_0(\phi, \chi)$.

For the calculation of the Husimi distribution we only consider the region inside the teflon disc by setting the wavefunction on the outside to zero. The Husimi functions at dielectric interfaces have been studied in detail by Hentschel et al [16], where they considered both inside and outside regions.

Instead of analysing the Husimi distribution of eigenmodes, we shall consider the pulse propagation. The time evolution of phase space densities has been studied theoretically by, e.g., Manderfeld et al [33] and Prosen and Žnidarič [34].
Figure 8. Pulse at \( t = 162 \) ps (left), and its Husimi distribution (right). The pulse is well localized in space, but spreading in every direction.

Figure 8 (left) shows the measured pulse at \( t = 162 \) ps, when the microwaves just start to spread from the antenna. Its Husimi distribution shows the pulse at a well localized position, but spreading in every direction (figure 8 (right)). The fine structure in the Husimi distribution is an edge effect: both the measured pulse and the Gaussian wave packets used in the Husimi projection are localized at the boundary of the teflon billiard. As mentioned above, the outside region is set to zero.

6. Pulse propagation in real space and phase space

While in laser experiments the directionality of the emission pattern was obtained by measuring the light intensity in the far-field, we are going to extract this information from the field distribution in the near-field of the teflon disc.

To this end, we have to calculate the Poynting vector \( \vec{S}(\vec{r}) \) which describes the energy flow of an electromagnetic wave. In our case of a quasi-two-dimensional microwave system, the Poynting vector reduces to

\[
\vec{S}(\vec{r}) = \frac{c}{8\pi k} \left( E^*_z(\vec{r}) \nabla E_z(\vec{r}) \right).
\]

(10)

The Poynting vector is equivalent to the probability density current in quantum mechanics

\[
\vec{j}(\vec{r}) = \frac{\hbar}{m} \text{Im} \left( \psi^*(\vec{r}) \nabla \psi(\vec{r}) \right).
\]

(11)

A more detailed description of the Poynting vector in microwave systems is provided in [35, 36].

We are now going to take a closer look at later times of the pulse propagation, when only the two wave packets are remaining which are supported by the whispering gallery modes.

In figure 9, we present a sequence of the pulse at times \( t = 4935, 5097, 5744 \) and 6067 ps. In the left column, the absolute square of the pulse is plotted with the Poynting vector in the
Figure 9. Left column: squared modulus of the pulse at $t = 4935, 5097, 5744$ and 6067 ps. For each plot, the complete range of the colour table is used, thus neglecting the decay of the pulse. In addition the Poynting vector is shown in the region outside of the teflon. Right column: Husimi distribution of the pulse at the same time steps. In addition the unstable manifold of the rectangular orbit is shown (black dots). The line for the critical angle at $\sin \chi_c \approx \pm 0.69$ is not shown. See the movie in the supplemental multimedia files.
Figure 10. Histograms of the energy flow in dependence of the emission angle $\alpha$. The histograms were averaged over four periods of the revolving wave packets. The deformation of the billiard was $\epsilon = 0.08$ (left) and $\epsilon = 0.13$ (right).

near-field of the teflon disc. The right column shows the corresponding Husimi distributions. In addition, the unstable manifold of the rectangular orbit is plotted.

The wave packet circulating counter-clockwise is always located in the upper part of the Husimi plot, corresponding to positive values of $\sin \chi$, while the other wave packet is located in the lower part. Even when the two wave packets interfere in position space, they are well separated in momentum space (see third row of figure 9).

The plot of the pulse at $t = 4935 \text{ps}$ shows the point of strongest emission for the wave packet moving counter-clockwise. This is evident from the Poynting vector indicating a strong transport out of the teflon, tangentially to the disc’s boundary. The corresponding Husimi plot gives even more insight into the emission behaviour as it shows clearly that the transport to the outside happens along the unstable manifold of the rectangular orbit. Only at these points the pulse surpasses the critical line of $\sin \chi = 0.69$, allowing an escape of the wave according to Fresnel’s law. The plot for $t = 5097 \text{ps}$ shows the wave packet on the left at the point of highest curvature. At this point the emission is rather weak. The Husimi plot shows that the wave packet stays well above the critical line at this position.

The last row of figure 9 shows a point of strong emission for the clockwise moving wave packet. Again the corresponding Husimi plot indicates the importance of the unstable manifold.

In figure 10, we present for $\epsilon = 0.08$ and $0.13$ a histogram of the angle $\alpha$ of the Poynting vectors, weighted with their absolute value. This yields directly the directionality of the energy flow escaping the teflon disc. Horizontal vectors correspond to $\alpha = 0$ or $\alpha = \pm \pi$, while vertical vectors correspond to $\alpha = \pm \pi/2$. A tangential emission at the points of highest curvature would correspond to vertical Poynting vectors, but the histograms clearly show that the vertical directions are suppressed.

The histograms were averaged over four periods of the revolving wave packets, corresponding to the long-time average to be discussed in section 7. Only the Poynting vectors close to the boundary were taken into account, more precisely those between $1.02 r(\phi)$ and $1.08 r(\phi)$. 

New Journal of Physics 8 (2006) 46 (http://www.njp.org/)
We observe a very high directionality of the Poynting vector for both geometries of the billiard. This is in accordance with measurements of the far-field intensities of micro-disc lasers [11].

7. Long-time dynamics

To simulate the microwave experiment in the ray-optical limit, we applied initial conditions matching those in the experiment. 4000 rays are started at the position of the antenna and spread uniformly in every direction. Each ray $i$ is associated with an intensity $a_i$, which decreases with every reflection at the boundary according to Fresnel’s law

$$a_{i,m+1} = a_{i,m}R(\chi_{i,m})$$

(12)

starting with an initial intensity $a_{i,0} = 1$. For the reflection coefficient $R$, the curvature corrections for curved dielectric interfaces [13] were taken into account, leading to

$$R = \left|\frac{\cos \chi + iF}{\cos \chi - iF}\right|^2,$$

(13)

where

$$F = \frac{i \cos \eta}{n} \left[1 + \frac{1}{n^2 \sin^2 \chi} \left(\frac{K_{2/3}(z)}{K_{1/3}(z)} - 1\right)\right], \quad z = -ikrc \frac{\cos^3 \eta}{3 \sin^2 \eta},$$

(14)

In this expression, $K_q$ are modified Bessel functions, $\eta = \arcsin(n \sin \chi)$ is the angle of refraction, $r_c$ is the radius of curvature and $k$ is the wavenumber.

After a few iterations only those rays with a large angle of incidence still have high intensities. In the top row of figure 11, we show the Poincaré sections of the rays with intensities $a_{i,m} > 10^{-8}$ for $m > 30$. For the billiard with $\epsilon = 0.08$ all the rays remaining after 30 iterations are concentrated just above the unstable manifold. The escape due to small values of $|\sin \chi|$ happens exclusively along this unstable manifold. In the phase space of this billiard there is a separatrix preventing the rays from reaching larger values of $|\sin \chi|$.

The situation is similar for the billiard with $\epsilon = 0.13$. Here, the separatrix is very close to $|\sin \chi| = 1$ and the unstable manifold has a rather complex structure. But still the ray simulation reproduces all its details and the escape clearly follows this manifold.

The classical ray-simulations are compared with the long-time dynamics of the microwave system. To this end, the Husimi distributions of the pulses were averaged over four periods of the circulating wave packets. The results are presented in the bottom row of figure 11. The averaging greatly enhances the quality of the Husimi plots and we find a compelling agreement with the results of the classical simulation, both showing the importance of the unstable manifold of the rectangular orbit. However, there are differences due to the finite wave lengths in the microwave measurement. While in the classical simulations the separatrix prevented the rays from reaching larger values of $|\sin \chi|$, the waves can reach this region by dynamical tunnelling. And for the same reason the wave dynamics can penetrate the stability islands, as seen in figure 11(left) for $\epsilon = 0.08$. 

New Journal of Physics 8 (2006) 46 (http://www.njp.org/)
Figure 11. Top row: Poincaré sections of the ray-simulations. Only rays with intensities larger than $10^{-8}$ are plotted (red dots); the first 30 iterations have been omitted. For comparison, the unstable manifold of the rectangular orbit is plotted as well (blue dots). Bottom row: average of the Husimi distributions over four periods of the circulating wave packets. For comparison, the unstable manifold of the rectangular orbit is plotted as well (black dots). The results are shown for $\epsilon = 0.08$ (left column), and $\epsilon = 0.13$ (right column). The line for the critical angle at $\sin \chi_c \approx \pm 0.69$ is not shown.

8. Conclusions

In the present microwave experiments on dielectric quadrupole billiards we were able to reproduce the characteristic emission behaviour found in the far-field intensities of micro-disc lasers. We studied the Poynting vector close to the boundary of the teflon disc to obtain the directionality of the microwave emission.

We obtained the pulse propagation by a Fourier transform of the transmission spectra, and thus were able to concentrate on the long-time behaviour.

The Husimi distributions of the pulse propagation provided a visualization of the internal dynamics of the system. By averaging the Husimi distributions over four periods of the circulating wave packets, we achieved a very clear phase-space picture. The experimental results agree
very well with classical simulations for the long-time dynamics, and clearly illustrate that the characteristic emission behaviour is dominated by the unstable manifold of the rectangular orbit.

Acknowledgments

J U Nöckel, A D Stone, H G L Schwefel and H E Tureci are thanked for helpful discussions and M Barth for his support in the early stage of the experiments. The work was supported by the Deutsche Forschungsgemeinschaft.

References

[1] Chang R K and Campillo A K (ed) 1996 Optical Processes in Microcavities (Singapore: World Scientific)
[2] McCall S L, Levi A F J, Slusher R E, Pearston S J and Logan R A 1992 Appl. Phys. Lett. 60 289
[3] Nöckel J U and Stone A D 1997 Nature 385 45
[4] Harayama T, Davis P and Ikeda K S 2003 Phys. Rev. Lett. 90 063901
[5] Wiersig J 2003 Phys. Rev. A 67 023807
[6] Lee S Y, Rim S, Ryu J W, Kwon T Y, Choi M and Kim C M 2004 Phys. Rev. Lett. 93 164102
[7] Tureci H E, Schwefel H G L, Stone A D and Jacquod P 2003 Modes of wave-chaotic dielectric resonators Progress in Optics vol 47, ed E Wolf (Amsterdam: North-Holland) chapter 2. See also physics/0308016
[8] Wiersig J 2003 J. Opt. A: Pure Appl. Opt. 5 53
[9] Nöckel J U, Stone A D and Chang R K 1994 Opt. Lett. 19 1693
[10] Nöckel J U, Stone A D, Chen G, Grossmann H L and Chang R K 1996 Opt. Lett. 21 1609
[11] Schwefel H G L, Rex N B, Tureci H E, Chang R K, Stone A D, Ben-Messaoud T and Zyss J 2004 J. Opt. Soc. Am. B 21 923
[12] Gmachl C, Narimanov E E, Capasso F, Baillargeon J N and Cho A Y 2002 Opt. Lett. 27 824
[13] Hentschel M and Schomerus H 2002 Phys. Rev. E 65 045603
[14] Rex N B, Tureci H E, Schwefel H G L, Chang R K and Stone A D 2002 Phys. Rev. Lett. 88 094102
[15] Gmachl C, Capasso F, Narimanov E E, Nöckel J U, Stone A D, Faist J, Sivco D L and Cho A Y 1998 Science 280 1556
[16] Hentschel M, Schomerus H and Schubert R 2003 Europhys. Lett. 62 636
[17] Harayama T, Fukushima T, Sunada S and Ikeda K S 2003 Phys. Rev. Lett. 91 073903
[18] Harayama T, Fukushima T, Davis P, Vaccaro P O, Miyasaka T, Nishimura T and Aida T 2003 Phys. Rev. E 67 015207 (R)
[19] Tureci H E and Stone A D 2005 Mode competition and output power in regular and chaotic dielectric cavity lasers Laser Resonators and Beam Control VIII (SPIE vol 5708) ed A V Kudryashov and A H Paxton (Bellingham, WA: SPIE) pp 255–70
[20] Haken H and Sauermann H 1963 Z. Phys. 173 261
[21] Lee S B, Lee J H, Chang J S, Moon H J, Kim S W and An K 2002 Phys. Rev. Lett. 88 033903
[22] Vietze U, Krauß O, Laeri F, Ihlein G, Schüth F, Limburg B and Abraham M 1998 Phys. Rev. Lett. 81 4628
[23] Mekis A, Nöckel J U, Chen G, Stone A D and Chang R K 1995 Phys. Rev. Lett. 75 2682
[24] Nöckel J U and Stone A D 1994 Phys. Rev. B 50 17415
[25] Stein J, Stöckmann H J and Stoffregen U 1995 Phys. Rev. Lett. 75 53
[26] Kuhl U, Persson E, Barth M and Stöckmann H J 2000 Eur. Phys. J. B 17 253
[27] Schäfer R, Kuhl U, Barth M and Stöckmann H J 2001 Found. Phys. 31 475
[28] Stöckmann H J 1999 Quantum Chaos—An Introduction (Cambridge: Cambridge University Press)
[29] Kuhl U, Stöckmann H J and Weaver W 2005 J. Phys. A: Math. Gen. 38 10433

New Journal of Physics 8 (2006) 46 (http://www.njp.org/)
[30] Husimi K 1940 Proc. Phys. Math. Soc. (Japan) 22 246
[31] Bäcker A, Fürstberger S and Schubert R 2004 Phys. Rev. E 70 036204
[32] Méndez-Bermúdez J A, Luna-Acosta G A, Šeba P and Pichugin K N 2002 Phys. Rev. E 66 046207
[33] Manderfeld C, Weber J and Haake F 2001 J. Phys. A: Math. Gen. 34 9893
[34] Prosen T and Žnidarič M 2002 J. Phys. A: Math. Gen. 35 1455
[35] Barth M and Stöckmann H J 2002 Phys. Rev. E 65 066208
[36] Vraničar M, Barth M, Veble G, Robnik M and Stöckmann H J 2002 J. Phys. A: Math. Gen. 35 4929