Merging the A- and Q-spectral theories for digraphs

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Abstract

Let \(G\) be a digraph and \(A(G)\) be the adjacency matrix of \(G\). Let \(D(G)\) be the diagonal matrix with outdegrees of vertices of \(G\). For any real \(\alpha \in [0, 1]\), Liu et al. \(^{19}\) defined the matrix \(A_\alpha(G)\) as

\[A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).\]

The largest modulus of the eigenvalues of \(A_\alpha(G)\) is called the \(A_\alpha\) spectral radius of \(G\). In this paper, we determine the digraphs which attain the maximum (or minimum) \(A_\alpha\) spectral radius among all strongly connected digraphs with given parameters such as girth, clique number, vertex connectivity or arc connectivity. We also discuss a number of open problems.

Key Words: Strongly connected digraphs, Spectral extremal problems, Adjacency matrix, Signless Laplacian matrix, \(A_\alpha\) spectral radius.

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1 Introduction

Let \(G = (V(G), E(G))\) be a digraph with vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\) and arc set \(E(G)\). A digraph is simple if it has no loops and multiple arcs. A digraph is strongly connected if for every pair of vertices \(v_i, v_j \in V(G)\), there exists a directed path from \(v_i\) to \(v_j\). Throughout this paper, we only consider simple strongly connected digraphs.

Let \(\overrightarrow{P_n}\) and \(\overrightarrow{C_n}\) denote the directed path and the directed cycle on \(n\) vertices, respectively. Let \(\overrightarrow{K_n}\) denote the complete digraph on \(n\) vertices in which for two arbitrary distinct vertices \(v_i, v_j \in V(\overrightarrow{K_n})\), there are arcs \((v_i, v_j)\) and \((v_j, v_i)\) \(\in E(\overrightarrow{K_n})\). Suppose \(\overrightarrow{P_k} = v_1 v_2 \ldots v_k\), we call \(v_1\) the initial vertex of the directed path \(\overrightarrow{P_k}\), and \(v_k\) the terminal vertex of the directed path \(\overrightarrow{P_k}\).

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Let $G$ be a digraph. If $S \subseteq V(G)$, then we use $G[S]$ to denote the subdigraph of $G$ induced by $S$. Let $G - v$ be a digraph obtained from $G$ by deleting the vertex $v$ and all arcs incident to $v$. We use $G \pm e$ to denote the digraph obtained from $G$ by adding/deleting the arc $e \notin E(G)$. Let $G_1$ and $G_2$ be two disjoint digraphs. The digraph $G_1 \cup G_2$ is the digraph with vertex set $V(G_1) \cup V(G_2)$ and arc set $E(G_1) \cup E(G_2)$. We denote by $G_1 \lor G_2$ the join of $G_1$ and $G_2$, which is the digraph such that $V(G_1 \lor G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \lor G_2) = E(G_1) \cup E(G_2) \cup \{(u, v), (v, u) : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Let $H$ be a subdigraph of $G$. If $G[V(H)]$ is a complete subdigraph of $G$, then $H$ is called a clique of $G$. The clique number of a digraph $G$, denoted by $\omega(G)$, is the maximum value of the numbers of the vertices of the cliques in $G$. The girth of $G$ is the length of the shortest directed cycle of $G$. For a strongly connected digraph $G = (V(G), E(G))$, the vertex connectivity of $G$, denoted by $\kappa(G)$, is the minimum number of vertices whose deletion yields the resulting digraph non-strongly connected. A set of arcs $S \subseteq E(G)$ is an arc cut set if $G - S$ is not strongly connected. The arc connectivity of $G$, denoted by $\kappa'(G)$, is the minimum number of arcs whose deletion yields the resulting digraph not-strongly connected.

For a digraph $G$, if there is an arc from $v_i$ to $v_j$, we indicate this by writing $(v_i, v_j)$, call $v_j$ the head of $(v_i, v_j)$, and $v_i$ the tail of $(v_i, v_j)$, respectively, and $(v_i, v_j)$ is said to be out-incident to $v_i$ and in-incident to $v_j$; $v_i$ is said to be out-adjacent to $v_j$ and $v_j$ is said to be in-adjacent to $v_i$. A tournament is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. A transitive tournament is a tournament $G$ satisfying the following: if $(u, v) \in E(G)$ and $(v, w) \in E(G)$, then $(u, w) \in E(G)$.

For any vertex $v_i$, let $N^+_i = N^+_{v_i} = \{v_j \in V(G) : (v_i, v_j) \in E(G)\}$ and $N^-_i = N^-_{v_i} = \{v_j \in V(G) : (v_j, v_i) \in E(G)\}$ denote the out-neighbors and in-neighbors of $v_i$, respectively. Let $d^+_i = d^+_{v_i} = |N^+_i|$ denote the outdegree of the vertex $v_i$, and $d^-_i = d^-_{v_i} = |N^-_i|$ denote the indegree of the vertex $v_i$ in the digraph $G$. The minimum outdegree is denoted by $\delta^+$ and the minimum indegree by $\delta^-$. A digraph is $r$-regular if all vertices have outdegree $r$ and indegree $r$.

For a digraph $G$, let $A(G) = (a_{ij})_{n \times n}$ be the adjacency matrix of $G$, where $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$ otherwise. Let $D(G)$ be the diagonal matrix with outdegrees of vertices of a digraph $G$. In this paper we study hybrids of $A(G)$ and $D(G)$ similar to the signless Laplacian matrix $Q(G) = D(G) + A(G)$, which has been extensively studied since then. For detailed coverage of this research see [9, 10, 11, 17, 23, 24], and their references. The study of $Q(G)$ has shown that it is a remarkable matrix, and unique in many respects. Yet, $Q(G)$ is just the sum of $A(G)$ and $D(G)$. To understand to what extent each of the summands $A(G)$ and $D(G)$ determines the properties of $Q(G)$, Liu et al. [19] defined the matrix $A_\alpha(G)$ as

$$A_{\alpha}(G) = \alpha D(G) + (1 - \alpha) A(G), \quad 0 \leq \alpha \leq 1.$$  

Many facts suggest that the study of the family $A_\alpha(G)$ is long due. Our inspiration comes from the paper of Nikiforov [20]. First, we note that

$$A(G) = A_0(G), \quad D(G) = A_1(G), \quad \text{and} \quad Q(G) = 2A_{\frac{1}{2}}(G).$$

Since $A_{\frac{1}{2}}(G)$ is essentially equivalent to $Q(G)$, in this paper we take $A_{\frac{1}{2}}(G)$ as an exact substitute for $Q(G)$. With this caveat, one sees that $A_\alpha(G)$ seamlessly joins $A(G)$ with $Q(G)$, and we may study the adjacency spectral properties and signless Laplacian spectral properties of a digraph in a unified way. The spectral radius of $A_\alpha(G)$ i.e., the largest modulus of the
eigenvalues of $A_\alpha(G)$, is called the $A_\alpha$ spectral radius of $G$, denoted by $\lambda_\alpha(G)$. The $A_\alpha$ spectral radius of undirected graphs has been studied in the literature, see \[7, 14, 18, 20, 21, 22, 26\]. Recently, Liu et al. \[19\] characterized the extremal digraph which attains the maximum $A_\alpha$ spectral radius among all strongly connected digraphs with given dichromatic number. We are interested in the $A_\alpha$ spectral radius of digraphs with given other parameters.

If $\alpha = 1$, $A_1(G) = D(G)$ the diagonal matrix with outdegrees of vertices of $G$ which is not interesting. So we only consider the cases $0 \leq \alpha < 1$ for the rest of this paper. If $G$ is a strongly connected digraph, then it follows from the Perron Frobenius Theorem \[8\] that $\lambda_\alpha(G)$ is an eigenvalue of $A_\alpha(G)$, and there is a unique positive unit eigenvector corresponding to $\lambda_\alpha(G)$. The positive unit eigenvector corresponding to $\lambda_\alpha(G)$ is called the Perron vector of $A_\alpha(G)$. See more details on the Perron vector of $A_\alpha(G)$ in Section 2.

One of the central issues in extremal spectra graph theory is: for a graph matrix, determine the maximum or minimum spectral radius over various families of graphs. For example, among all strongly connected digraphs on $n$ vertices, $\overrightarrow{C}_n$ is the unique digraph with the minimum $A_0$ spectral radius and $A_\frac{1}{2}$ spectral radius, and $\overrightarrow{K}_n$ is the unique digraph with the maximum $A_0$ spectral radius and $A_\frac{1}{2}$ spectral radius. The same result is also true for the $A_\alpha$ spectral radius of $G$, see Corollary \[2.7\]. The main goal of this paper is to extend some results on maximum or minimum $A_0$ spectral radius and $A_\frac{1}{2}$ spectral radius for all $\alpha \in [0, 1)$.

The rest of the paper is structured as follows. In the next section we introduce some lemmas and give basic facts about the $A_\alpha$ spectral radius of $G$. In Section 3, we characterize the extremal digraph which achieves the minimum $A_\alpha$ spectral radius among all strongly connected digraphs with given girth. In Section 4, we determine the extremal digraph which attains the minimum $A_\alpha$ spectral radius among all strongly connected digraphs with given clique number. In Section 5, we characterize the extremal digraphs which achieve the maximum $A_\alpha$ spectral radius among all strongly connected digraphs with given vertex connectivity. In Section 6, we characterize the extremal digraphs which achieve the maximum $A_\alpha$ spectral radius among all strongly connected digraphs with given arc connectivity.

2 Preliminaries

In this section, we give some lemmas which will be used in the following sections.

Let $\sigma(\cdot)$ denote the spectrum of a square matrix including algebraic multiplicity. Let $\rho(\cdot)$ denote the spectral radius of a square matrix.

Lemma 2.1. \((8)\) Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix, $R_i(M)$ be the $i$-th row sum of $M$, i.e., $R_i(M) = \sum_{j=1}^{n} m_{ij}$ ($1 \leq i \leq n$). Then

$$\min\{R_i(M) : 1 \leq i \leq n\} \leq \rho(M) \leq \max\{R_i(M) : 1 \leq i \leq n\}.$$ 

Moreover, if $M$ is irreducible, then either one equality holds if and only if $R_1(M) = R_2(M) = \ldots = R_n(M)$.

Definition 2.2. \((12)\) Let $M$ be a real matrix of order $n$ described in the following block
Furthermore, if Lemma 2.8. Let \( M = (m_{ij})_{n \times n} \) be defined as above, and for any \( i, j \in \{1, 2, \ldots, t\} \), the row sum of each block \( M_{ij} \) be constant. Let \( B = (b_{ij}) \) be the equitable quotient matrix of \( M \). Then \( \sigma(B) \subset \sigma(M) \). Moreover, if \( M = (m_{ij})_{n \times n} \) is a nonnegative matrix, then \( \rho(B) = \rho(M) \).

Definition 2.4. Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( n \times m \) matrices. If \( a_{ij} \leq b_{ij} \) for all \( i \) and \( j \), then \( A \leq B \). If \( A \leq B \) and \( A \neq B \), then \( A < B \). If \( a_{ij} < b_{ij} \) for all \( i \) and \( j \), then \( A \ll B \).

Lemma 2.5. Let \( A \) and \( B \) be nonnegative matrices. If \( 0 \leq A \leq B \), then \( \rho(A) \leq \rho(B) \). Furthermore, if \( B \) is irreducible and \( 0 \leq A < B \), then \( \rho(A) < \rho(B) \).

By Lemma 2.5 we have the following results in terms of \( A_\alpha \) spectral radius of digraphs.

Corollary 2.6. Let \( G \) be a digraph and \( H \) be a spanning subdigraph of \( G \). Then
(i) \( \lambda_\alpha(G) \geq \lambda_\alpha(H) \).
(ii) If \( G \) is strongly connected, and \( H \) is a proper subdigraph of \( G \), then \( \lambda_\alpha(G) > \lambda_\alpha(H) \).

From Lemma 2.4 and Corollary 2.6 we can easily get the following corollary.

Corollary 2.7. Let \( G \) be a strongly connected digraph. Then \( 1 \leq \lambda_\alpha(G) \leq n - 1 \), \( \lambda_\alpha(G) = n - 1 \) if and only if \( G \cong K_n \), and \( \lambda_\alpha(G) = 1 \) if and only if \( G \cong \overrightarrow{C_n} \).

Lemma 2.8. (B) Let \( B \) be nonnegative matrices and \( X = (x_1, x_2, \ldots, x_n)^T \) be any nonzero nonnegative vector. If \( \beta \geq 0 \) such that \( BX \geq \beta X \), then \( \rho(B) \geq \beta \). Furthermore, if \( B \) is irreducible and \( BX > \beta X \), then \( \rho(B) > \beta \).

By Lemma 2.8 we have the following results in terms of \( A_\alpha \) spectral radius of digraphs.

Corollary 2.9. Let \( G \) be a strongly connected digraph. Then \( \lambda_\alpha(G) > \alpha \Delta^+ \).

\textit{Proof.} Without loss of generality, let \( d_u^+ = \Delta^+ \). Taking \( X = (0, 0, \ldots, 0, 1, 0 \ldots, 0)^T \), that is, all the entries of \( X \) are 0 except \( x_u = 1 \), where \( x_u \) corresponding to the vertex \( u \). Since \( G \) is strongly connected, then \( d_u^- \geq 1 \) and \( A_\alpha(G) \) is nonnegative irreducible. Hence \( A_\alpha(G)X > \alpha \Delta^+ X \). Therefore, by Lemma 2.8 we have \( \lambda_\alpha(G) > \alpha \Delta^+ \).

In the rest of this section, let \( X = (x_1, x_2, \ldots, x_n)^T \) be the unique positive unit eigenvector corresponding to the \( A_\alpha \) spectral radius \( \lambda_\alpha(G) \), where \( x_i \) corresponds to the vertex \( v_i \).
Lemma 2.10. Let \( G = (V(G), E(G)) \) be a strongly connected digraph on \( n \) vertices, \( v_p, v_q \) be two distinct vertices of \( V(G) \). Suppose that \( v_1, v_2, \ldots, v_t \in N_{v_p}^{-} \setminus \{N_{v_q}^{-} \cup \{v_q\}\} \), where \( 1 \leq t \leq d_{v_p}^{-} \). Let \( H = G - \{(v_i, v_p) : i = 1, 2, \ldots, t\} + \{(v_i, v_q) : i = 1, 2, \ldots, t\} \). If \( x_{v_q} \geq x_{v_p} \), then \( \lambda_{\alpha}(H) \geq \lambda_{\alpha}(G) \). Furthermore, if \( H \) is strongly connected and \( x_{v_q} > x_{v_p} \), then \( \lambda_{\alpha}(H) > \lambda_{\alpha}(G) \).

Proof. We will show \( (A_{\alpha}(H)X)_i \geq (A_{\alpha}(G)X)_i \) for any \( v_i \in V(G) \).

Case 1. \( v_i \notin N_{v_p}^{-} \setminus \{N_{v_q}^{-} \cup \{v_q\}\} \).

Then \( (A_{\alpha}(H)X)_i = \alpha d_{i}^{+}(G)x_i + (1 - \alpha) \sum_{(v_i, v_j) \in E(G)} x_j = (A_{\alpha}(G)X)_i \).

Case 2. \( v_i \in N_{v_p}^{-} \setminus \{N_{v_q}^{-} \cup \{v_q\}\} \).

Then \( (A_{\alpha}(H)X)_i - (A_{\alpha}(G)X)_i = (1 - \alpha)(x_{v_q} - x_{v_p}) \geq 0 \).

Thus \( A_{\alpha}(H)X \succeq A_{\alpha}(G)X = \lambda_{\alpha}(G)X \). By Lemma 2.10, \( \lambda_{\alpha}(H) \geq \lambda_{\alpha}(G) \).

Moreover, if \( H \) is strongly connected and \( x_{v_q} > x_{v_p} \), then by Lemma 2.10 we have \( \lambda_{\alpha}(H) > \lambda_{\alpha}(G) \). \( \Box \)

Lemma 2.11. Let \( G \neq \overrightarrow{C_n} \) be a strongly connected digraph with \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Suppose that \( \overrightarrow{F_k} = v_1v_2 \ldots v_k \) \((k \geq 3)\) be a directed path of \( G \) with \( d_i^{+} = 1 \) for \((i = 2, 3, \ldots, k - 1)\). Then we have \( x_2 < x_3 < \ldots < x_{k-1} < x_k \).

Proof. Since \( G \) is a strongly connected digraph and \( G \neq \overrightarrow{C_n} \), then \( D \) contains a directed cycle denoted by \( \overrightarrow{C_g} \) \((g \geq 2)\), as a proper subdigraph of \( G \). Thus \( \lambda_{\alpha}(G) > \lambda_{\alpha}(\overrightarrow{C_g}) = 1 \) by Corollary 2.6.

Therefore, for any \( i \in \{2, 3, \ldots, k - 1\} \), we have

\[
-x_i < \lambda_{\alpha}(G)x_i = \alpha x_i + (1 - \alpha)x_{i+1}.
\]

Then \( x_i < x_{i+1} \) and thus \( x_2 < x_3 < \ldots < x_{k-1} < x_k \). \( \Box \)

Lemma 2.12. \((3)\) Let \( B \) be nonnegative matrices, then \( B \) is reducible if and only if \( \rho(B) \) is the spectral radius of some proper principal submatrix of \( B \).

Lemma 2.13. Let \( G \neq \overrightarrow{C_n} \) be a strongly connected digraph with \( V(G) = \{v_1, v_2, \ldots, v_n\} \), \((v_i, v_j) \in E(G) \) and \( w \notin V(G) \), \( G^w = (V(G^w), E(G^w)) \) with \( V(G^w) = V(G) \cup \{w\} \), \( E(G^w) = E(G) - \{(v_i, v_j)\} + \{(v_i, w), (w, v_j)\} \). Then \( \lambda_{\alpha}(G^w) \geq \lambda_{\alpha}(G^w) \).

Proof. Since \( G \) is strongly connected digraph, then \( G^w \) is also strongly connected digraph.

Suppose \( X = (x_1, \ldots, x_n, x_w)^T \) is the Perron vector corresponding to \( \lambda_{\alpha}(G^w) \), where \( x_w \) correspond to \( w \), and \( x_i \) correspond to \( v_i \) for \( i = 1, 2, \ldots, n \). Since \( G \neq \overrightarrow{C_n} \), then \( G^w \neq \overrightarrow{C_{n+1}} \).

By Lemma 2.1 we have \( \lambda_{\alpha}(G^w) > 1 \). Clearly, \( d_w^{+}(G^w) = 1 \). Thus \( x_w < \lambda_{\alpha}(G^w)x_w = \alpha x_w + (1 - \alpha)x_j \). Then \( x_j > x_w \). Let \( H = G^w - \{(v_i, v_j)\} + \{(v_i, w), (w, v_j)\} \), then by \( d_w^{+}(G^w) = 1 \), \( H \) is not strongly connected which has exactly two strongly connected components, one is isolated vertex \( w \), the other is \( G \). Thus \( A_{\alpha}(H) \) is nonnegative reducible, then by Lemma 2.12 we have \( \lambda_{\alpha}(H) = \lambda_{\alpha}(G) \). On the other hand, by Lemma 2.10 \( \lambda_{\alpha}(H) \geq \lambda_{\alpha}(G^w) > 1 \). Thus \( \lambda_{\alpha}(G) = \lambda_{\alpha}(H) \geq \lambda_{\alpha}(G^w) \). \( \Box \)
3 The minimum $A_\alpha$ spectral radius of strongly connected digraphs with given girth

Let $g \geq 2$ and $\mathcal{G}_{n,g}$ denote the set of strongly connected digraph on $n$ vertices with girth $g$. If $g = n$, then $\mathcal{G}_{n,g} = \{\overrightarrow{C}_n\}$ and $\lambda_\alpha(\overrightarrow{C}_n) = 1$. Thus we only need to consider the cases $2 \leq g \leq n - 1$.

Let $2 \leq g \leq n - 1$ and $C_{n,g}$ be a digraph obtained by adding a directed path $\overrightarrow{P}_{n-g+2} = v_gv_{g+1} \ldots v_nv_1$ on the directed cycle $\overrightarrow{C}_g = v_1v_2 \ldots v_gv_1$ such that $V(\overrightarrow{C}_g) \cap V(\overrightarrow{P}_{n-g+2}) = \{v_g, v_1\}$ (as shown in Figure 1), where $V(C_{n,g}) = \{v_1, v_2, \ldots, v_n\}$. Clearly, $C_{n,g} \in \mathcal{G}_{n,g}$.

![Figure 1: The digraph $C_{n,g}$](image)

In [16], Lin and Shu et al. proved that $C_{n,g}$ attains the minimum $A_0$ spectral radius among all strongly connected digraphs with given girth. In [9], Hong and You proved that $C_{n,g}$ also attains the minimum $A_2$ spectral radius among all strongly connected digraphs with given girth. We generalize their results to $0 \leq \alpha < 1$. In the rest of this section, we will show that $C_{n,g}$ achieves the minimum $A_\alpha$ spectral radius among all digraphs in $\mathcal{G}_{n,g}$.

**Lemma 3.1.** Let $2 \leq g \leq n - 1$ and $C'_{n,g} = C_{n,g} - \{(v_n, v_1)\} - \{(v_n, v_g)\}$. Then $\lambda_\alpha(C'_{n,g}) > \lambda_\alpha(C_{n,g})$.

**Proof.** Suppose $X = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector corresponding to $\lambda_\alpha(C_{n,g})$, where $x_i$ correspond to $v_i$ for $i = 1, 2, \ldots, n$. Since $\overrightarrow{D} = v_{g+1}v_{g+2} \ldots v_nv_1$ and $\overrightarrow{R} = v_gv_{g+1}v_{g+2}$ are the directed paths of $C_{n,g}$ with $d_i^+ = 1$ for $i \in \{1, 2, \ldots, g-1, g+1, g+2, \ldots, n\}$, then by Lemma 2.1, we have $x_{g+2} < x_{g+3} < \ldots < x_n < x_1 < x_2 < \ldots < x_{g-1} < x_g$ and $x_{g+1} < x_{g+2} < x_{g+3} < \ldots < x_n < x_1 < x_2 < \ldots < x_{g-1} < x_g$.

Thus $x_{g+1} < x_{g+2} < x_{g+3} < \ldots < x_n < x_1 < x_2 < \ldots < x_{g-1} < x_g$. Note that $C'_{n,g}$ is strongly connected and $x_g > x_1$, then by Lemma 2.10, we have $\lambda_\alpha(C'_{n,g}) > \lambda_\alpha(C_{n,g})$. \qed

**Theorem 3.2.** Let $2 \leq g \leq n - 1$ and $G \in \mathcal{G}_{n,g}$. Then $\lambda_\alpha(G) \geq \lambda_\alpha(C_{n,g}) > 1$ with equality if and only if $G \cong C_{n,g}$.

**Proof.** Since $\overrightarrow{C}_g$ is a proper subdigraph of $G$, $\lambda_\alpha(G) > \lambda_\alpha(\overrightarrow{C}_g) = 1$ by Corollary 2.6. Without loss of generality, we let $C_g = v_1v_2 \ldots v_gv_1$, where $2 \leq g \leq n - 1$. Since $G \in \mathcal{G}_{n,g}$ is strongly connected, it is possible to obtain a digraph $G_1$ from $G$ by deleting vertices and arcs in such a way that $G_1 \cong H$, where $H = (V(H), E(H))$, $V(H) = \{v_1, v_2, \ldots, v_g, v_{g+1}, \ldots, v_{g+t-2}\}$, $E(H) = \{(v_i, v_{i+1}) | i \in \{1, 2, \ldots, g + l - 3\}\} \cup \{(v_g, v_1), (v_{g+t-2}, v_1)\}$ with $1 \leq t \leq g$ (see Figure 2).


By Corollary 2.6 we have $\lambda_\alpha(G) \geq \lambda_\alpha(G_1)$ with equality if and only if $G \cong G_1$. Since $G_1$ contains a directed path $P_l = v_gv_{g+1} \cdots v_{g+l-2}v_l$. Insert $n-g-l+2$ vertices to $P_l$ such that the resulting digraph is denoted by $H'$. Clearly, $H'$ is strongly connected. By using Lemma 2.13 by $n-g-l+2$ times, we have $\lambda_\alpha(H) \geq \lambda_\alpha(H')$ with equality if and only if $H \cong H'$. Then we consider the following three cases.

**Case 1:** $t = 1$.
In this case, $H' \cong C_{n,g}$, then $\lambda_\alpha(G) \geq \lambda_\alpha(G_1) = \lambda_\alpha(H) \geq \lambda_\alpha(H') = \lambda_\alpha(C_{n,g})$, with equality if and only if $G \cong C_{n,g}$.

**Case 2:** $t = g$.
In this case, $H' \cong C_{n,g}$, then $\lambda_\alpha(G) \geq \lambda_\alpha(G_1) = \lambda_\alpha(H) \geq \lambda_\alpha(H') = \lambda_\alpha(C_{n,g}) > \lambda_\alpha(C_{n,g})$ by Lemma 3.1.

**Case 3:** $2 \leq t \leq g - 1$.
$X = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector corresponding to $\lambda_\alpha(C_{n,g})$, where $x_i$ correspond to $v_i$ for $i = 1, 2, \ldots, n$. Noting that $H' \cong C_{n,g} - \{(v_n, v_1)\} + \{(v_n, v_1)\}$, by the proof of Lemma 3.1 we have $x_1 < x_i$, then $\lambda_\alpha(H') > \lambda_\alpha(C_{n,g})$ by Lemma 2.10. Thus $\lambda_\alpha(G) \geq \lambda_\alpha(G_1) = \lambda_\alpha(H) \geq \lambda_\alpha(H') > \lambda_\alpha(C_{n,g})$.

In all cases, $\lambda_\alpha(G) \geq \lambda_\alpha(C_{n,g}) > 1$ with equality if and only if $G \cong C_{n,g}$.

4 The minimum $A_\alpha$ spectral radius of strongly connected digraphs with given clique number

Let $C_{n,d}$ denote the set of strongly connected digraphs on $n$ vertices with clique number $d$. If $d = n$, then $C_{n,d} = \{\overrightarrow{K_n}\}$ and $\lambda_\alpha(\overrightarrow{K_n}) = n - 1$. If $d = 1$, then $\overrightarrow{C_n} \in C_{n,d}$ and $\lambda_\alpha(\overrightarrow{C_n}) = 1$. By Corollary 2.7 for any $G \in C_{n,d}$, $\lambda_\alpha(G) \geq 1 = \lambda_\alpha(\overrightarrow{C_n})$ with equality if and only if $G \cong \overrightarrow{C_n}$. Thus we only need to consider the cases $2 \leq d \leq n - 1$.

Let $2 \leq d \leq n - 1$, and $B_{n,d}$ be a digraph obtained by adding a directed path $\overrightarrow{P_{n-d+2}} = v_1v_2 \cdots v_{n-d+2}$ to a clique $\overrightarrow{K_d}$ such that $V(\overrightarrow{K_d}) \cap V(\overrightarrow{P_{n-d+2}}) = \{v_{n-d+2}, v_1\}$. Clearly, $B_{n,d} \in C_{n,d}$.

In [16], Lin and Shu et al. proved that $B_{n,d}$ attains the minimum $A_0$ spectral radius among
all strongly connected digraphs with given clique number. In [9], Hong and You determined that $B_{n,d}$ also attains the minimum $A_\alpha$ spectral radius among all strongly connected digraphs with given clique number. We generalize their results to $0 \leq \alpha < 1$. In the rest of this section, we will show that $B_{n,d}$ achieves the minimum $A_\alpha$ spectral radius among all digraphs in $\mathcal{C}_{n,d}$.

**Lemma 4.1.** Let $2 \leq d \leq n-1$ and $B'_{n,d} = B_{n,d} - \{(v_{n-d+1}, v_{n-d+2})\} + \{(v_{n-d+1}, v_1)\}$. Then $\lambda_\alpha(B'_{n,d}) > \lambda_\alpha(B_{n,d})$.

**Proof.** Suppose $X = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector corresponding to $\lambda_\alpha(B_{n,d})$, where $x_i$ correspond to $v_i$ for $i = 1, 2, \ldots, n$. By Lemma 2.10, we only need to show that $x_{n-d+2} < x_1$.

Since $K_d$ is a proper sundigraph of $\lambda_\alpha(B_{n,d})$, then $\lambda_\alpha(B_{n,d}) > d - 1$ by Corollary 2.6. Therefore, from $A_\alpha(B_{n,d})X = \lambda_\alpha(B_{n,d})X$, we have

$$\lambda_\alpha(B_{n,d})x_1 = \alpha dx_1 + (1-\alpha)x_2 + (1-\alpha)x_{n-d+2} + (1-\alpha) \sum_{v_i \in V_1} x_i,$$

$$\lambda_\alpha(B_{n,d})x_{n-d+2} = \alpha(d-1)x_{n-d+2} + (1-\alpha)x_1 + (1-\alpha) \sum_{v_i \in V_1} x_i,$$

where $V_1 = V(K_d) \setminus \{v_{n-d+2}, v_1\}$.

Then $(\lambda_\alpha(B_{n,d}) - \alpha d + 1 - \alpha)(x_1 - x_{n-d+2}) = \alpha x_{n-d+2} + (1-\alpha)x_2 > 0$. By Corollary 2.9, $\lambda_\alpha(B_{n,d}) > \alpha d$. Thus $x_1 > x_{n-d+2}$. \qed

**Theorem 4.2.** Let $2 \leq d \leq n-1$ and $G \in \mathcal{C}_{n,d}$. Then $\lambda_\alpha(G) \geq \lambda_\alpha(B_{n,d})$ with equality if and only if $G \cong B_{n,d}$.

**Proof.** Since $G \in \mathcal{C}_{n,d}$, $K_d$ is a proper subdigraph of $G$. Since $G$ is strongly connected, it is possible to obtain a digraph $G_1$ from $G$ by deleting vertices and arcs in such a way that either $G_1 \cong B_{d+l-2,d}$ ($l \geq 3$) or $G_1 \cong B'_{d+l-1,d}$ ($l \geq 2$). By Corollary 2.6, we have $\lambda_\alpha(G) \geq \lambda_\alpha(G_1)$ with equality if and only if $G \cong G_1$. Then we consider the following two cases.

**Case 1:** $G_1 \cong B_{d+l-2,d}$ ($l \geq 3$).

Insert $n-d-l+2$ vertices to $\overrightarrow{P_1}$ such that the resulting digraph is $B_{n,d}$. By using Lemma 2.13 by $n-d-l+2$ times, we have $\lambda_\alpha(G_1) \geq \lambda_\alpha(B_{n,d})$ with equality if and only if $G_1 \cong B_{n,d}$.

**Case 2:** $G_1 \cong B'_{d+l-1,d}$ ($l \geq 2$).

Insert $n-d-l+2$ vertices to $\overrightarrow{P_1}$ such that the resulting digraph is $B'_{n,d}$. By using Lemma 2.13 by $n-d-l+2$ times, we have $\lambda_\alpha(G_1) \geq \lambda_\alpha(B_{n,d})$ with equality if and only if $G_1 \cong B_{n,d}$.

But by Lemma 4.1, $\lambda_\alpha(B'_{n,d}) > \lambda_\alpha(B_{n,d})$. Thus $\lambda_\alpha(G) \geq \lambda_\alpha(G_1) \geq \lambda_\alpha(B'_{n,d}) > \lambda_\alpha(B_{n,d})$.

Combining the above arguments, $\lambda_\alpha(G) \geq \lambda_\alpha(B_{n,d})$ with equality if and only if $G \cong B_{n,d}$.

A tournament on $n$ vertices that maximizes the spectral radius of its $A_\alpha$ matrix among all such tournaments is called extremal. For $\alpha = 0$, it has long been known that if $n$ is odd, the extremal tournaments are precisely the ones that are regular, i.e. have indegree and outdegree $(n-1)/2$ at each vertex. For $n$ even, the extremal tournaments are those which are isomorphic to the Bruhat-Li tournament. This was conjectured by Bruhat and Li [4] and proved by Drury [5]. For $n$ and $d$ fixed, let $l = \lfloor \frac{n}{2} \rfloor$ and $r = n - ld$. Let $V_j$, $j = 1, \ldots, d$ be disjoint vertex sets, where $|V_j| = l + 1$ for $j = 1, \ldots, r$ and $|V_j| = l$ for $j = r + 1, \ldots, d$. Let
$G_0$ be a digraph with vertex set $\bigcup_{j=1}^{d} V_j$ with all possible arcs between $V_j$ and $V_k$ for $j \neq k$ and the induced subdigraph $G_0[V_j]$ an extremal tournament for $j = 1, \ldots, d$.

It is natural to ask: which digraph achieves the maximum $A_\alpha$ spectral radius among all strongly connected digraphs with given clique number? If $\alpha = 0$, Drury and Lin [6] proved that $G_0$ attains maximum $A_0$ spectral radius among all strongly connected digraphs with given clique number $d$. Therefore, we propose the following problem.

**Problem 4.3.** Let $1 \leq d \leq n-1$. Among all digraphs in $C_{n,d}$, does $G_0$ attain maximum $A_\alpha$ spectral radius?

5 The maximum $A_\alpha$ spectral radius of strongly connected digraphs with given vertex connectivity

Let $D_{n,k}$ denote the set of strongly connected digraphs with order $n$ and vertex connectivity $\kappa(G) = k \geq 1$. If $k = n - 1$, then $D_{n,k} = \{K_n\}$. So we only consider the cases $1 \leq k \leq n - 2$.

For $1 \leq m \leq n - k - 1$, $K(n,k,m)$ denote the digraph $\overrightarrow{K_k} \lor (\overrightarrow{K_{n-k-m}} \lor \overrightarrow{K_m}) + E$, where $E = \{(u,v) | u \in V(\overrightarrow{K_m}), v \in V(\overrightarrow{K_{n-k-m}})\}$ (see Figure 4). Let $\mathcal{K}(n,k) = \{K(n,k,m) : 1 \leq m \leq n - k - 1\}$. Clearly $\mathcal{K}(n,k) \subset D_{n,k}$.

In [16], Lin and Shu et al. proved that $K(n,k,n-k-1)$ or $K(n,k,1)$ attains the maximum $A_0$ spectral radius among all strongly connected digraphs with given vertex connectivity. In [9], Hong and You determined that $K(n,k,n-k-1)$ also attains the maximum $A_\frac{1}{2}$ spectral radius among all strongly connected digraphs with given vertex connectivity. We generalize their results to $0 \leq \alpha < 1$.

![Figure 4: The digraph $K(n,k,m)$](image)

**Lemma 5.1.** ([2]) Let $G$ be a strongly connected digraph with $\kappa(G) = k$. Suppose that $S$ is a $k$-vertex cut of $G$ and $G_1, G_2, \ldots, G_t$ are the strongly connected components of $G - S$. Then there exists an ordering of $G_1, G_2, \ldots, G_t$ such that for $1 \leq i \leq t$ and any $v \in V(G_i)$, every tail of $v$ is in $\bigcup_{j=1}^{i} G_j$.

**Remark 5.2.** By Lemma 5.1, we know that there exists a strongly connected component of $G - S$, say $G_1$ with $|V(G_1)| = m$ such that for any $i \in V(G_1)$, $|N_i^-| = 0$, where $N_i^- = \{j \in V(G - S - G_1) : (j, i) \in E(G)\}$. Let $G_2 = G - S - G_1$. We add arcs to $G$ until both induced subdigraph of $V(G_1) \cup S$ and induced subdigraph of $V(G_2) \cup S$ attain to complete digraphs,
add arc \((u, v)\) for any \(u \in V(G_1)\) and any \(v \in V(G_2)\). Denote the resulting digraph by \(H\). Since \(G\) is \(k\)-strongly connected, then \(H = K(n, k, m) \in K(n, k) \subset D_{n,k}\). By Corollary 2.7, we have \(\lambda_\alpha(G) \leq \lambda_\alpha(H)\), with equality if and only if \(G \cong H\). Therefore, the digraph which achieves the maximum \(A_\alpha\) spectral radius among all digraphs in \(D_{n,k}\) must be some digraph in \(K(n, k)\).

**Theorem 5.3.** Let \(n, k, m\) be positive integers such that \(1 \leq k \leq n-2\) and \(1 \leq m \leq n-k-1\). Then

\[
\lambda_\alpha(K(n, k, m)) = \frac{n-2-\alpha m+\alpha n+\sqrt{(1-\alpha)^2n^2+(6\alpha-2\alpha^2-4)mn+(2-\alpha)^2m^2+4(1-\alpha)km}}{2}.
\]

**Proof.** Let \(G = K(n, k, m)\), and \(S\) be a \(k\)-vertex cut of \(G\). Suppose that \(G_1\) with \(|V(G_1)| = m\) and \(G_2\) with \(|V(G_2)| = n - k - m\) are two strongly connected components, i.e., two complete subdigraphs of \(G - S\) with arcs \(\{(u, v) : u \in V(G_1), v \in V(G_2)\}\). Then

\[
A_\alpha(G) = \begin{pmatrix}
(1-\alpha)I_n + (\alpha - 1)I_m & (1-\alpha)I_k & (1-\alpha)I_{m \times (n-k)} \\
(1-\alpha)I_{k \times m} & (1-\alpha)I_k + (\alpha - 1)I_k & (1-\alpha)I_{k \times (n-k)} \\
0 & (1-\alpha)J_{(n-k) \times m} & (1-\alpha)J_{n-k} + (\alpha(n-m) - 1)J_{n-k}\end{pmatrix},
\]

where \(I_p\) be the \(p \times p\) identity matrix and \(J_{p \times q}\) be the \(p \times q\) matrix in which every entry is 1, or simply \(J_p\) if \(p = q\), and \(0_{p \times p}\) denotes the \(p \times p\) zero matrix.

Note that the matrix \(A_\alpha(G)\) has the following equitable quotient matrix \(B(G)\) with respect to the partition \(\{V(G_1), S, V(G_2)\}\) of \(V(G)\).

\[
B(G) = \begin{pmatrix}
\alpha(n-m) + m - 1 & (1-\alpha)k & (1-\alpha)(n-k) \\
(1-\alpha)m & \alpha(n-k) + k - 1 & (1-\alpha)(n-k) \\
0 & (1-\alpha)k & n-k - m - 1 + \alpha k
\end{pmatrix}.
\]

Then by Lemma 2.3, \(\lambda_\alpha(G)\) is also the spectral radius of \(B(G)\). Since \(A_\alpha(G)\) is nonnegative irreducible matrix, \(B(G)\) is also nonnegative irreducible matrix. By Corollary 2.9, we have \(\rho(B) = \lambda_\alpha(G) > \alpha(n-1) > \alpha n - 1\). Therefore, we can easily see that \(\lambda_\alpha(G)\) is the largest root of the quadratic equation \(x^2 - (\alpha n + \alpha m - 2)x + \alpha n^2 - \alpha n - 2\alpha nm - m^2 + \alpha km + \alpha m + \alpha m^2 - n + \alpha n + 1 - km = 0\), thus we have

\[
\lambda_\alpha(K(n, k, m)) = \frac{n-2-\alpha m+\alpha n+\sqrt{(1-\alpha)^2n^2+(6\alpha-2\alpha^2-4)mn+(2-\alpha)^2m^2+4(1-\alpha)km}}{2}.
\]

\(\square\)

**Remark 5.4.** Noting that \(\vec{K}_n\) is the unique digraph which achieves the maximum \(A_\alpha(G)\) spectral radius \(n - 1\) among all strongly connected digraphs, and \(K(n, n - 2, 1) = \vec{K}_n - \{(u, v)\}\) where \(u, v \in V(\vec{K}_n)\), by Lemma 2.3 and Theorem 5.3, we deduce that \(K(n, n - 2, 1)\) is the unique digraph which achieves the second maximum \(A_\alpha(G)\) spectral radius

\[
\frac{n + \alpha n - 2 - \alpha + \sqrt{(1-\alpha)^2n^2 + 2\alpha(1-\alpha)n + \alpha^2 + 4\alpha - 4}}{2}
\]

among all strongly connected digraphs.
Theorem 5.5. Let $n, k$ be positive integers such that $1 \leq k \leq n - 2$, $G \in D_{n,k}$. Then

(i) for $\alpha = 0$, $\lambda_\alpha(G) \leq \frac{n - 2 + \sqrt{n^2 - 4n + 4k + 4}}{2}$ with equality if and only if $G \cong K(n,k,n-k-1)$ or $G \cong K(n,k,1)$.

(ii) For $0 < \alpha < 1$, $\lambda_\alpha(G) \leq \frac{n - 2 + \alpha + \alpha k + \sqrt{n^2 + (2\alpha - 4 - 2\alpha k)n + \alpha^2 + \alpha^2 k^2 - 4\alpha + 2\alpha^2 k - 4\alpha k + 4k + 4}}{2}$, with equality if and only if $G \cong K(n,k,n-k-1)$.

Proof. By Remark 5.2, $\lambda_\alpha(G) \leq K(n,k,m)$ for some $m$, where $1 \leq m \leq n - k - 1$. By Theorem 5.3 we have $\lambda_\alpha(K(n,k,m)) = \frac{n - 2 - \alpha m + \alpha n + \sqrt{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km}}{2}$.

Now we want to show that the maximum value of $\lambda_\alpha(K(n,k,m))$ must be taken at either $m = 1$ or at $m = n - k - 1$.

Let $f(n,k,m) = n - 2 - \alpha m + \alpha n + \sqrt{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km}$. Then

$$\frac{\partial f}{\partial m} = -\alpha + \frac{1}{2} \frac{16k^2(2\alpha - \alpha^2 - 1) + 16n(-5\alpha + 4\alpha^2 + 2 - \alpha^3)}{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km},$$

$$\frac{\partial^2 f}{\partial m^2} = \frac{1}{4} \frac{16k^2(2\alpha - \alpha^2 - 1) + 16n(-5\alpha + 4\alpha^2 + 2 - \alpha^3)}{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km}^2 + \frac{1}{4} \frac{-16k^2(\alpha - 1)^2 + 16n(2\alpha - \alpha)(\alpha - 1)^2}{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km}^2 \geq \frac{1}{4} \frac{-16k^2(\alpha - 1)^2 + 16(k + 2)k(2\alpha - \alpha)(\alpha - 1)^2}{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km}^2 \geq \frac{1}{4} \frac{16k^2(\alpha - 1)^2(1 - \alpha) + 32k(2\alpha - \alpha)(\alpha - 1)^2}{(1 - \alpha)^2 n^2 + (6\alpha - 2\alpha^2 - 4)m + (2 - \alpha)^2 m^2 + 4(1 - \alpha)km}^2 > 0.$$

Thus, for fixed $n$ and $k$, the maximum value of $f$ must be taken at either $m = 1$ or at $m = n - k - 1$.

In the following, we want to compare $\lambda_\alpha(K(n,k,1))$ and $\lambda_\alpha(K(n,k,n-k-1))$. Let $\beta = (\alpha^2 + 1 - 2\alpha)n^2 + (6\alpha - 2\alpha^2 - 4)n + 4 - 4\alpha + \alpha^2 - 4\alpha k + 4k$ and $\gamma = n^2 + (2\alpha - 4 - 2\alpha k)n + \alpha^2 + \alpha^2 k^2 - 4\alpha + 2\alpha^2 k - 4\alpha k + 4k + 4$. Then

$$f(n,k,n-k-1) - f(n,k,1) = 2\alpha + \alpha k - \alpha n + \sqrt{n} - \sqrt{n} = \alpha(n - k - 2)(-1 + \frac{2n - \alpha(k + n)}{\sqrt{n} + \sqrt{n}}).$$

For $\alpha = 0$, we have $f(n,k,n-k-1) - f(n,k,1) = 0$, that is $\lambda_\alpha(n,k,n-k-1) = \lambda_\alpha(n,k,1) = \frac{n - 2 + \sqrt{n^2 - 4n + 4k + 4}}{2}$. Therefore, $\lambda_\alpha(G) \leq \frac{n - 2 + \sqrt{n^2 - 4n + 4k + 4}}{2}$ with equality if and only if $G \cong K(n,k,n-k-1)$ or $G \cong K(n,k,1)$.

For $0 < \alpha < 1$. We assume that $n > k + 2$ since in case $n = k + 2$ there is only one value of $m$ under consideration. Now suppose that $f(n,k,n-k-1) - f(n,k,1) \leq 0$. We deduce a contradiction. We have simultaneously

$$\sqrt{n} + \sqrt{\beta} \geq 2n - \alpha(k + n) \text{ and } -\sqrt{n} + \sqrt{\beta} \geq 2\alpha + \alpha k - \alpha n.$$

So $\sqrt{\beta} \geq n - \alpha n + \alpha$ and $\beta \geq (n - \alpha n + \alpha)^2$. However, $(n - \alpha n + \alpha)^2 - \beta = -4\alpha n + 4n - 4 + 4\alpha k + 4\alpha - 4k = 4(1 - \alpha)(n - k - 1) > 0$, that is $(n - \alpha n + \alpha)^2 > \beta$. Thus
\[ f(n, k, n - k - 1) - f(n, k, 1) > 0. \] Therefore,
\[ \lambda_\alpha(G) \leq \lambda_\alpha(n, k, n - k - 1) = \frac{n - 2 + \alpha + k + \sqrt{n^2 + (2a - 2 - 2ak)n + a^2 + a^2k^2 - 4a^2k^2 + 4a^2k + 4}}{2}, \]
with equality if and only if \( G \cong K(n, k, n - k - 1). \) 

\[ \square \]

6 \ The maximum \( A_\alpha \) spectral radius of strongly connected digraphs with given arc connectivity

Let \( \mathcal{L}_{n,k} \) denote the set of strongly connected digraphs with order \( n \) and arc connectivity \( \kappa'(G) = k \geq 1. \) If \( \kappa'(G) = k = n - 1 \), then \( \mathcal{L}_{n,k} = \{K_n\} \). So we only consider the cases \( 1 \leq k \leq n - 2. \)

In [15], Lin and Drury proved that \( K(n, k, n - k - 1) \) or \( K(n, k, 1) \) attains the maximum \( A_0 \) spectral radius among all strongly connected digraphs with given arc connectivity. In [25], Xi and Wang determined that \( K(n, k, n - k - 1) \) also attains the maximum \( A_\frac{3}{2} \) spectral radius among all strongly connected digraphs with given arc connectivity. We generalize their results to \( 0 \leq \alpha < 1. \)

**Lemma 6.1.** ([25]) Let \( G \) be a strongly connected digraph with order \( n \) and arc connectivity \( k \geq 1, \) and \( S \) be an arc cut set of \( G \) of size \( k \) such that \( G - S \) has exactly two strongly connected components, say \( G_1 \) and \( G_2 \) with \( |V(G_1)| = n_1 \) and \( |V(G_2)| = n_2, \) where \( n_1 + n_2 = n. \) If \( d_v^+ > k \) and \( d_v^- > k \) for each vertex \( v \in V(G), \) then \( n_1 \geq k + 2, n_2 \geq k + 2. \)

**Lemma 6.2.** Let \( G \in \mathcal{L}_{n,k} \) containing a vertex of outdegree \( k. \) Then \( \lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, n - k - 1)). \)

**Proof.** Let \( w \) be a vertex of \( G \) such that \( d_w^+ = k. \) Then the arcs out-incident to \( w \) form an arc cut set of size \( k. \) Adding all possible arcs from \( G \setminus \{w\} \) to \( G \setminus \{w\} \cup \{w\}, \) we obtain a digraph \( H, \) which is isomorphic to \( K(n, k, n - k - 1), \) the arc connectivity of \( H \) remains equal to \( k. \) If \( G \neq K(n, k, n - k - 1), \) then \( \lambda_\alpha(G) < \lambda_\alpha(K(n, k, n - k - 1)) \) by Corollary 2.6. So the result follows.

**Lemma 6.3.** Let \( G \in \mathcal{L}_{n,k} \) containing a vertex of indegree \( k. \) Then \( \lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, 1)). \)

**Proof.** Let \( w \) be a vertex of \( G \) such that \( d_w^- = k. \) Then the arcs in-incident to \( w \) form an arc cut set of size \( k. \) Adding all possible arcs from \( w \) to \( G \setminus \{w\}, \) and all possible arcs from \( G \setminus \{w\} \) to \( G \setminus \{w\}, \) we obtain a digraph \( H', \) which is isomorphic to \( K(n, k, 1), \) the arc connectivity of \( H' \) remains equal to \( k. \) If \( G \neq K(n, k, 1), \) then \( \lambda_\alpha(G) < \lambda_\alpha(K(n, k, 1)) \) by Corollary 2.6. So the result follows.

Let \( \delta^0(G) = \min\{\delta^+(G), \delta^-(G)\}. \) The following result characterize the digraphs maximizes the \( A_\alpha \) spectral radius in \( \mathcal{L}_{n,k} \) when \( \kappa'(G) = \delta^0(G) = k \geq 1. \)

**Theorem 6.4.** Let \( G \in \mathcal{L}_{n,k} \) with \( \delta^0(G) = k. \) Then (i) if \( \alpha = 0, \) \( \lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, n - k - 1)) = \lambda_\alpha(K(n, k, n - k - 1)) \) with equality if and only if \( G \cong K(n, k, n - k - 1) \) or \( G \cong K(n, k, 1). \)

(ii) If \( 0 < \alpha < 1, \) \( \lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, n - k - 1)), \) with equality if and only if \( G \cong K(n, k, n - k - 1). \)
Proof. Let $G$ be a digraph with arc connectivity $k$, and $\delta^0(G) = k$.

If $\delta^0(G) = \delta^+(G)$, then there exists a vertex of outdegree $k$. So by Lemma 6.2, $K(n, k, n - k - 1)$ maximizes the $A_\alpha$ spectral radius in $L_{n,k}$. If $\delta^0(G) = \delta^-(G)$, then there exists a vertex of indegree $k$. So by Lemma 6.3, $K(n, k, 1)$ maximizes the $A_\alpha$ spectral radius in $L_{n,k}$. Moreover, by the proof Theorem 5.5, we know that

(i) For $\alpha = 0$, $\lambda_\alpha(K(n, k, 1)) = \lambda_\alpha(K(n, k, n - k - 1))$. Therefore $\lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, n - k - 1))$ with equality if and only if $G \cong K(n, k, n - k - 1)$ or $G \cong K(n, k, 1)$. 

(ii) For $0 < \alpha < 1$, $\lambda_\alpha(K(n, k, n - k - 1)) > \lambda_\alpha(K(n, k, 1))$. Therefore $\lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, n - k - 1))$ with equality if and only if $G \cong K(n, k, n - k - 1)$.

Theorem 6.5. Let $G \in L_{n,k}$. Then

(i) for $\alpha = 0$, $\lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, 1)) = \lambda_\alpha(K(n, k, n - k - 1))$ with equality if and only if $G \cong K(n, k, n - k - 1)$ or $G \cong K(n, k, 1)$.

(ii) For $0 < \alpha < 1$, $\lambda_\alpha(G) \leq \lambda_\alpha(K(n, k, n - k - 1))$, with equality if and only if $G \cong K(n, k, n - k - 1)$.

Proof. Let $G$ be a digraph in $L_{n,k}$. Note that each vertex in the digraph $G$ has outdegree at least $k$ and indegree at least $k$, otherwise $G \notin L_{n,k}$. Then, we consider the following two cases.

Case 1. If there exists a vertex $u$ of $G$ with $d_u^+ = k$ or $d_u^- = k$. The by Theorem 6.4 we get the result.

Case 2. For any arc cut set of $G$ containing $k$ arcs, then $G - S$ consists of exactly two strongly connected components $G_1$, $G_2$, respectively, of orders $a$, $b$ and $a + b = n$. Without loss of generality, we may assume that there are no arcs from $G_1$ to $G_2$ in $G - S$. By Lemma 6.1, $a \geq k + 2$, $b = n - a \geq k + 2$, then $k + 2 \leq a \leq n - k - 2$, $n \geq a + k + 2 \geq 2k + 4$. Next we create a new digraph $\overrightarrow{P}$ by adding to $G$ any possible arcs from $G_2$ to $G_1 \cup G_2$ or any possible arcs from $G_1$ to $G_1$ that were not present in $G$. Obviously, the arc connectivity of $\overrightarrow{P}$ remains equal to $k$ and all vertices of $\overrightarrow{P}$ have outdegree greater than $k$ and indegree still greater than $k$. By Corollary 2.6, the addition of any such arc will give $\lambda_\alpha(\overrightarrow{P}) > \lambda_\alpha(G)$. For $k + 2 \leq a \leq n - k - 2$, let $H'' = \overrightarrow{K_a} \cup \overrightarrow{K_{n-a}}$. $U = \{u_1, u_2, \cdots, u_k\}$ be a set of $k$ vertices in $V(\overrightarrow{K_a})$ and $W = \{v_1, v_2, \cdots, v_k\}$ be a set of $k$ vertices in $V(\overrightarrow{K_{n-a}})$. Let $H_4$ be a digraph obtained from $H''$ by adding all possible arcs from $U$ to $W$, and adding all possible arcs from $\overrightarrow{K_{n-a}}$ to $\overrightarrow{K_a}$. Then $\overrightarrow{P}$ is a spanning subdigraph of $H_4$. Therefore, by Corollary 2.4, $\lambda_\alpha(\overrightarrow{P}) \leq \lambda_\alpha(H_4)$. However, we can know that the vertex connectivity of $H_4$ is $k$ and $H_4 \cong K(n, k, n - k - 1)$ and $H_4 \cong K(n, k, 1)$. Then by Theorem 5.5, if $\alpha = 0$, we have $\lambda_\alpha(H_4) < \lambda_\alpha(K(n, k, n - k - 1)) = \lambda_\alpha(K(n, k, 1))$; if $0 < \alpha < 1$, $\lambda_\alpha(H_4) < \lambda_\alpha(K(n, k, n - k - 1))$.

Therefore, combining the above two cases, we get the desired result. □

In sections 6 and 7, it is natural to ask: what are digraphs in $D_{n,k}$ and $L_{n,k}$ whose $A_\alpha$ spectral radius is minimum for each $1 \leq k \leq n - 2$, respectively?

By Lemma 2.1, for any strongly connected digraph $H$, $\delta^+ \leq \lambda_\alpha(H) \leq \Delta^+$, with either equality if and only if the outdegrees of all vertices in $H$ are equal. If $G \in D_{n,k}$ or $G \in L_{n,k}$, then $\delta^+ \geq k$, $\delta^- \geq k$. So we have $\lambda_\alpha(G) \geq k$ with the equality if and only if the outdegrees of all vertices in $G$ is $k$, and the indegree of all vertices in $G$ is also $k$ because $\delta^- \geq k$. That is $\lambda_\alpha(G) \geq k$ with the equality if and only if $G$ is a $k$-regular digraph.
Theorem 6.6. For each $1 \leq k \leq n - 2$, a digraph with the minimum $A_\alpha$ spectral radius in $D_{n,k}$ or $L_{n,k}$ is $k$-regular, and

$$\min\{\lambda_\alpha(G) : G \in D_{n,k}\} = \min\{\lambda_\alpha(G) : G \in L_{n,k}\} = k.$$ 

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