TOPOLOGY OF THE MAXIMAL IDEAL SPACE OF $H^\infty$ REVISITED

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Abstract. Let $M(H^\infty)$ be the maximal ideal space of the Banach algebra $H^\infty$ of bounded holomorphic functions on the unit disk $D \subset \mathbb{C}$. We prove that $M(H^\infty)$ is homeomorphic to the Freudenthal compactification $\gamma(M_\delta)$ of the set $M_\delta$ of all non-trivial (analytic disks) Gleason parts of $M(H^\infty)$. Also, we give alternative proofs of important results of Suárez asserting that the set $M_\delta$ of trivial (one-pointed) Gleason parts of $M(H^\infty)$ is totally disconnected and that the Čech cohomology group $H^2(M(H^\infty), \mathbb{Z}) = 0$.

1. Introduction

The paper studies the topological structure of the maximal ideal space $M(H^\infty)$ of the Banach algebra $H^\infty$ of bounded holomorphic functions on the unit disc $D \subset \mathbb{C}$ equipped with pointwise multiplication and supremum norm $\| \cdot \|_\infty$. Recall that for a commutative unital complex Banach algebra $A$ with dual space $A^*$ the maximal ideal space $M(A)$ of $A$ is the set of nonzero homomorphisms $A \to \mathbb{C}$ endowed with the Gelfand topology, the weak* topology induced by $A^*$. It is a compact Hausdorff space contained in the unit ball of $A^*$. Let $C(M(A))$ be the algebra of continuous complex-valued functions on $M(A)$ with supremum norm. The Gelfand transform $\hat{\cdot} : A \to C(M(A))$, defined by $\hat{a}(\varphi) := \varphi(a)$, is a nonincreasing-norm morphism of algebras that allows to thought of elements of $A$ as continuous functions on $M(A)$.

In the case of $H^\infty$ evaluation at a point of $D$ is an element of $M(H^\infty)$, so $D$ is naturally embedded into $M(H^\infty)$ as an open subset. The famous Carleson corona theorem [C] asserts that $D$ is dense in $M(H^\infty)$.

It is known that $M(H^\infty)$ is the union of two kinds of Gleason parts defined as follows. Recall that the pseudohyperbolic metric on $D$ is given by

$$\rho(z, w) := \left| \frac{z - w}{1 - wz} \right|, \quad z, w \in D.$$  

For $x, y \in M(H^\infty)$ the formula

$$\rho(x, y) := \sup\{ |\hat{f}(y)| : f \in H^\infty, \hat{f}(x) = 0, \|f\|_\infty \leq 1 \}$$

extends $\rho$ to $M(H^\infty)$. The Gleason part of $x \in M(H^\infty)$ is then determined by $\pi(x) := \{ y \in M(H^\infty) : \rho(x, y) < 1 \}$. For $x, y \in M(H^\infty)$ we have $\pi(x) = \pi(y)$ or $\pi(x) \cap \pi(y) = \emptyset$. Hoffman’s classification of Gleason parts [H] says that there are only two cases: either $\pi(x) = \{ x \}$ or $\pi(x)$ is an analytic disk. The former case means that there is a continuous one-to-one and onto map $L_x : D \to \pi(x)$ such that $\hat{f} \circ L_x \in H^\infty$ for every $f \in H^\infty$. Moreover, any analytic disk is contained in a Gleason part and any maximal (i.e., not contained in any other) analytic disk is a Gleason part. By $M_\delta$ and $M_\delta$ we denote sets of all non-trivial (analytic disks) and trivial (one-pointed) Gleason parts, respectively. It
is known that \( M_a \subset M(H^\infty) \) is open. Hoffman proved that \( \pi(x) \subset M_a \) if and only if \( x \) belongs to the closure of some \( H^\infty \) interpolating sequence in \( \mathbb{D} \).

More recent developments in the area are due to the work of Suárez [S1, S2] who proved the following profound results

1. The covering dimension of \( M(H^\infty) \) is 2;
2. The second Čech cohomology group \( H^2(M(H^\infty), \mathbb{Z}) = 0 \);
3. The set of trivial Gleason parts \( M_s \) is totally disconnected.

(Recall that for a normal space \( X \), \( \dim X \leq n \) if every finite open cover of \( X \) can be refined by an open cover whose order \( \leq n + 1 \). If \( \dim X \leq n \) and the statement \( \dim X \leq n - 1 \) is false, we say \( \dim X = n \).)

The original proof of property (1) in [S1] is based on a deep result of Treil [T] asserting that the Bass stable rank of \( H^\infty \) is 1 along with some other powerful techniques of the theory of \( H^\infty \). An alternative proof, not using this fact but invoking property (3), was given by the author in [Br1]. Specifically, it was shown that the set of all non-trivial Gleason parts \( M_a \) is homeomorphic to a fibre bundle over a compact Riemann surface \( S \) of genus \( g \geq 2 \) with the fibre an open subset of the Stone-Čech compactification of the fundamental group of \( S \). This implies that any compact subset of \( M_a \) has covering dimension \( \leq 2 \) which together with property (3) gives \( \dim M(H^\infty) = 2 \) by a known topological result.

Property (2) is proved in [S1] as one of important steps towards establishing property (1) in his conception. The proof relies completely on the above mentioned Treil’s result [T] and some constructions of this paper. In the present paper we show that property (2) is the consequence of the fact that \( H^\infty \) is a projective free Banach algebra, see [Q, Cor. 3.30], [BS, Th. 1.5]. The latter can be deduced from the classical Beurling-Lax-Halmos theorem (for its formulation see, e.g., [L, p. 1025] and references therein).

Property (3) is proved in [S2] using some results from [S1] and is based on a modification of the construction of Garnett and Nicolau [GN] who exploited it to show that interpolating Blaschke products generate \( H^\infty \). In this paper we give an alternative proof of property (3) based on the classical construction of Carleson [C].

Finally, we prove a result describing the topological nature of \( M(H^\infty) \) asserting that \( M(H^\infty) \) is homeomorphic to the Freudenthal compactification \( \gamma(M_a) \) (sometimes referred to as the end compactification, see [F], [M]) of the set of all non-trivial Gleason parts \( M_a \). Thus each trivial Gleason part is the end of \( M_a \) in the sense of Freudenthal.

2. \( M_s \) is Totally Disconnected

A topological space \( X \) is totally disconnected if any subset of \( X \) containing more than two points is disconnected. If \( X \) is a compact Hausdorff space, then it is totally disconnected if and only if \( \dim X = 0 \) (see, e.g., [N] for basic results of the dimension theory).

For a continuous function \( g : X \to \mathbb{C} \) we set \( S_X(g; \varepsilon) := \{ x \in X : |g(x)| < \varepsilon \} \). By \( \text{cl}_X \) we denote closure in \( X \). In the next result, \( \hat{g} \in C(M(H^\infty)) \) stands for the (continuous) extension of \( g \in H^\infty \) to \( M(H^\infty) \) by means of the Gelfand transform. Also, we equip the set of trivial Gleason parts \( M_s \subset M(H^\infty) \) by the induced topology.

Let \( f \in H^\infty \setminus \{0\} \), \( \|f\|_\infty = 1 \), be such that \( \hat{f}(x) = 0 \) for some \( x \in M_a \). Recall that for each \( \delta \in (0,1) \), the classical Carleson construction [C] produces a positive \( \varepsilon = \varepsilon(\delta) \) and an open set \( \Omega_\varepsilon \) with the boundary \( \Gamma_\varepsilon \) being a Carleson contour such that

\[
S_\delta(f; \varepsilon) \subset \Omega_\varepsilon \subset S_\delta(f; \delta/2).
\]

We have

**Theorem 2.1.** \( \text{cl}_{M(H^\infty)}(\Omega_\varepsilon) \cap M_s \) is a clopen subset of \( S_{M_s}(\hat{f}; \delta) \) containing \( x \).
Proof. It is well known (see, e.g., [Ga Ch. VIII, Sect. 4]) that there is a $H^\infty$ interpolating sequence \( \{z_n\} \subset \Gamma_\varepsilon \) and a number \( c \in (0,1) \) such that
\[
\inf_j \rho(z_j, z) < c \quad \text{for all} \quad z \in \Gamma_\varepsilon.
\]
Due to the result of Hoffman [H], \( \text{cl}_{M(H^\infty)}(\{z_n\}) \subset M_a \). Then (2.2) implies that \( \text{cl}_{M(H^\infty)}(\Gamma_\varepsilon) \subset M_a \) as well (see, e.g., [Ga Ch. X] ).

Next, consider the open set \( U := M(H^\infty) \setminus \text{cl}_{M(H^\infty)}(\Gamma_\varepsilon) \). By definition
\[
U \cap \mathbb{D} = \Omega_\varepsilon \cup (\mathbb{D} \setminus \text{cl}_{\mathbb{D}}(\Omega_\varepsilon)).
\]
Let \( g : U \cap \mathbb{D} \to \{0,1\} \) be the indicator function of \( \Omega_\varepsilon \). Clearly, \( g \in H^\infty(U \cap \mathbb{D}) \). Hence, [Su1 Th. 3.2] implies that \( g \) admits a continuous extension \( \tilde{g} \in C(U) \). Observe that
\[
U \cap M_s = (M(H^\infty) \setminus \text{cl}_{M(H^\infty)}(\Gamma_\varepsilon)) \cap M_s = M_s.
\]
So, \( \tilde{g}|_{M_s} \in C(M_s) \) attains values 0 and 1 only. In particular, \( \text{cl}_{M(H^\infty)}(\Omega_\varepsilon) \cap M_s = (\tilde{g}|_{M_s})^{-1}(1) \) is a clopen subset of \( M_s \). Due to (2.1),
\[
\text{cl}_{M(H^\infty)}(\Omega_\varepsilon) \cap M_s \subset \text{cl}_{M(H^\infty)}(S_\mathbb{D}(f;\delta/2)) \cap M_s \subset S_{M_s}(\tilde{f};\delta).
\]
Finally, since \( x \) is a limit point of \( S_\mathbb{D}(f;\varepsilon) \), it belongs to \( \text{cl}_{M(H^\infty)}(\Omega_\varepsilon) \) as well. \( \square \)

**Corollary 2.2.** \( M_s \) is totally disconnected.

**Proof.** By the definition of the Gelfand topology, any open neighbourhood of \( x \in M_s \) in \( M_s \) contains an open neighbourhood of the form
\[
\bigcap_{i=1}^n \{S_{M_s}(\tilde{f}_i;\delta_i) : \tilde{f}_i(x) = 0, \|\tilde{f}_i\|_\infty = 1, \delta_i \in (0,1)\}, \quad n \in \mathbb{N}.
\]
In turn, each of the latter sets contains a clopen neighbourhood of \( x \) by Theorem 2.1. Thus, \( M_s \) has the base of topology consisting of clopen sets, i.e., \( M_s \) is totally disconnected. \( \square \)

**Remark 2.3.** In our arguments, we used the theorem of Suárez [Su1 Th. 3.2] whose proof relies on the Carleson estimates [Ga Ch. VIII, Th. 5.1] and the fact that algebra \( H^\infty \) is separating. (In fact, the latter is not required as one can argue as in the proof of [Br2 Th. 1.7].) Alternatively, here one can use Bishop’s theorem [B Th. 1.1].

3. \( H^2(M(H^\infty),\mathbb{Z}) = 0 \)

A commutative unital complex Banach algebra \( A \) is said to be projective free if every finitely generated projective \( A \)-module is free. The Novodvorski-Taylor theory asserts that the Gelfand transform \( \hat{\cdot} : A \to C(M(A)) \) determines an isomorphism between categories \( P(A) \) of isomorphism classes of finitely generated projective \( A \)-modules and \( \text{Vect}_{\mathbb{C}}(M(A)) \) of isomorphism classes of finite rank complex vector bundles on the maximal ideal space \( M(A) \) of \( A \), see [NS, Ta Th. 6.8, p. 199]. This results in the following statement:

**Proposition.** The following conditions are equivalent:

1. \( A \) is a projective free algebra;
2. \( C(M(A)) \) is a projective free algebra;
3. \( M(A) \) is connected and each finite rank complex vector bundle on \( M(A) \) is topologically trivial.

Since isomorphism classes of rank one complex vector bundles on \( M(A) \) are in the one-to-one correspondence (determined by assigning to each bundle its first Chern class) with elements of the Čech cohomology group \( H^2(M(A),\mathbb{Z}) \) (see, e.g., [Hus]), projective freeness of \( A \) implies that \( H^2(M(A),\mathbb{Z}) = 0 \). It is known that \( H^\infty \) is projective free [Q, Cor. 3.30], [BS Th. 1.5]. Thus, we get
Corollary 3.1. \( H^2(M(H^\infty), \mathbb{Z}) = 0. \)

Remark 3.2. (1) In fact, from the projective freeness of \( A \) follows also that even rational Čech cohomology groups \( H^{2m}(M(A), \mathbb{Q}) = 0 \) for all \( m \geq 2 \). This is the consequence of some fundamental result of \( K \)-theory, see e.g., [K].

(2) In [BS] Th. 1.5 projective freeness is established for algebras \( H^\infty(U) \) for a large class of Riemann surfaces \( U \). In this case, as in Corollary 3.1 we obtain \( H^2(M(H^\infty(U)), \mathbb{Z}) = 0. \) For instance, as such \( U \) one can take an unbranched covering of an open bordered Riemann surface. The proof in [BS] Th. 1.5 is based on an analog of the Beurling-Lax-Halmos theorem established by the author in an earlier paper. For the sake of completeness, we place its version for \( H^\infty \) in the Appendix.

4. \( M(H^\infty) \) is the Freudenthal compactification of \( M_a \)

Let \( X \) be a semicompact Hausdorff space (i.e. every point of \( X \) has arbitrarily small neighbourhoods with compact boundaries). The Freudenthal compactification \( \gamma(X) \) of \( X \) is the unique (up to homeomorphism) Hausdorff compactification of \( X \) having the following properties

(a) \( \gamma(X) \setminus X \) is zero-dimensionally embedded in \( \gamma(X) \), i.e., any point in \( \gamma(X) \setminus X \) has arbitrarily small neighbourhoods whose boundaries lie in \( X \);

(b) \( \gamma(X) \) is maximal with respect to (a). That is, if \( c(X) \) is a Hausdorff compactification of \( X \) such that \( c(X) \setminus X \) is zero-dimensionally embedded in \( c(X) \), then the identity map on \( X \) has a continuous extension from \( \gamma(X) \) to \( c(X) \).

Let us mention some properties of \( \gamma(X) \) (see also [E][M][[I][D][DM]):

- Any two disjoint closed subsets of \( X \) with compact boundaries have disjoint closures in \( \gamma(X) \);
- \( \gamma(X) \) is a perfect compactification, i.e., for each \( x \in \gamma(X) \) and each open neighbourhood \( U \) of \( x \) in \( \gamma(X) \) set \( U \cap X \) is not disjoint union of two open sets \( V \) and \( W \) such that \( x \in \text{cl}_{\gamma(X)}(U) \cap \text{cl}_{\gamma(X)}(W) \). In fact, \( \gamma(X) \) is the unique perfect compactification of \( X \) in which \( \gamma(X) \setminus X \) zero-dimensionally embeds;
- If \( X \) is connected and locally connected, then so is \( \gamma(X) \);
- Any homeomorphism between any two semicompact Hausdorff spaces extends to a homeomorphism between their Freudenthal compactifications.

Also, the Freudenthal compactification \( \gamma(X) \) can be determined as follows:

Let \( \tilde{C}_\text{fin}(X) \) be closure in \( C_b(X) \) (the Banach algebra of bounded complex-valued continuous functions on \( X \)) of the algebra \( C_\text{fin}(X) \) of all functions \( f \in C_b(X) \) for which there is a compact subset \( K \subset X \) such that \( f(X \setminus K) \subset \mathbb{C} \) is finite. Then the maximal ideal space \( M(\tilde{C}_\text{fin}(X)) \) of \( \tilde{C}_\text{fin}(X) \) is homeomorphic to \( \gamma(X) \).

The main result of this section is

Theorem 4.1. \( M(H^\infty) \) is homeomorphic to \( \gamma(M_a) \).

Remark 4.2. (1) Note that \( M_a \) is locally compact and, hence, semicompact. In fact, the base of topology of \( M_a \) consists of sets of the form \( S_{M_a}(\hat{B}; \varepsilon) := \{ x \in M_a : |\hat{B}(x)| < \varepsilon \} \), where \( \hat{B} \) is an interpolating Blaschke product, which for all sufficiently small \( \varepsilon \) are relatively compact subsets of \( M_a \) (see, e.g., [Br2 Sect. 2.2]). Therefore \( \gamma(M_a) \) is well defined.

(2) Theorem 4.1 implies that \( C(M(H^\infty)) \) is isometrically isomorphic to \( \tilde{C}_\text{fin}(M_a) \) (cf. Bishop [B Th. 1.1]).

Proof.

\(^1\)i.e., a compact Hausdorff space containing \( X \) as an open dense subset
Lemma 4.3. Each function in $C_{\text{fin}}(M_\alpha)$ can be continuously extended to a function in $C(M(H^\infty))$ and the set of all such extensions separates points of $M(H^\infty)$.

Proof. Let $f \in C_{\text{fin}}(M_\alpha)$ and $K \subset M_\alpha$ be compact such that $f(M_\alpha \setminus K) \subset \mathbb{C}$ is finite. We set $U = M(H^\infty) \setminus K$. Then $U$ is an open neighbourhood of $M_\alpha$ and $U \cap \mathbb{D} = (M_\alpha \setminus K) \cap \mathbb{D}$ is disjoint union of connected open sets. So, $f|_{U \cap \mathbb{D}}$ is a bounded continuous function with finite range. In particular, it is constant on each connected component of $U \cap \mathbb{D}$, i.e., $f|_{U \cap \mathbb{D}} \in H^\infty(U \cap \mathbb{D})$. Hence, applying [S1, Th. 3.2] we extend $f|_{U \cap \mathbb{D}}$ continuously to a bounded function $f' \in C(U)$. Since $U \cap \mathbb{D}$ is dense in $M_\alpha \setminus K$,

$$f'(x) = f(x) \quad \text{for all } x \in M_\alpha \setminus K.$$ 

Function $\tilde{f} \in C(M(H^\infty)$ equals $f$ on $M_\alpha$ and $f'$ on $M_\alpha$ is the required extension of $f$.

Further, for distinct points $x, y \in M_\alpha$ due to the fact that $\dim M_\alpha = 0$ we can find open neighbourhoods $U_x$ and $U_y$ of $x$ and $y$ in $M(H^\infty)$ such that

$$\text{cl}(U_x) \cap \text{cl}(U_y) = \emptyset, \quad \text{cl}(U_x) \cap M_\alpha = U_x \cap M_\alpha, \quad \text{cl}(U_y) \cap M_\alpha = U_y \cap M_\alpha \quad \text{and} \quad M_\alpha = (U_x \cap M_\alpha) \cup (U_y \cap M_\alpha)$$

(see [Br2, Lm. 4.1] for similar arguments).

Consider a function $g \in C(\text{cl}(U_x) \cup \text{cl}(U_y))$ equals 0 on $\text{cl}(U_x)$ and 1 on $\text{cl}(U_y)$. Let $g_e \in C(M(H^\infty))$ be a continuous extension of $g$ (existing by the Tietze-Urysohn theorem). Then $g_e$ attains values 0 and 1 outside compact set $K := M(H^\infty) \setminus (U_x \cup U_y) \subset M_\alpha$. By definition $f := g_e|_{M_\alpha} \in C_{\text{fin}}(M_\alpha)$ and its extension $\tilde{f} = g_e$ separates points $x$ and $y$, as required.

The fact that extensions $\tilde{f}$ as above separate points $x \in M_\alpha$ and $y \in M_\alpha$ or distinct points $x, y \in M_\alpha$ is obvious. \hfill \qed

Due to the lemma, algebra $\tilde{C}_{\text{fin}}(M_\alpha)$ admits a continuous norm-preserving extension to $M(H^\infty)$. Since the former algebra is self-adjoint with respect to the complex conjugation, this extension coincides with $C(M(H^\infty))$ by the Stone-Weierstrass theorem. In particular, the maximal ideal space $M(\tilde{C}_{\text{fin}}(M_\alpha))$ is homeomorphic to $M(H^\infty)$. On the other hand, it is homeomorphic to the Freudenthal compactification $\gamma(M_\alpha)$ of $M_\alpha$. This completes the proof of the theorem. \hfill \qed

5. Appendix: $H^\infty$ is a Projective Free Algebra

We use that $A$ is projective free iff every nonzero square idempotent matrix with entries in $A$ is similar (by an invertible matrix with entries in $A$) to a matrix of the form

$$\text{diag}(I_k, 0) := \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \quad k \in \mathbb{N},$$

where $I_k$ is the identity $k \times k$ matrix (see [Co, Prop. 2.6]).

By $H_n^\infty$ we denote the $H^\infty$-module consisting of columns $(f_1, \ldots, f_n)$, $f_i \in H^\infty$. An $H^\infty$-invariant subspace of $H_n^\infty$ is called a submodule. The following result can be deduced from the classical Beurling-Lax-Halmos theorem (see, e.g., [To, p. 1025]).

(BLH) Let $M \subset H_n^\infty$ be a nonzero weak* closed submodule. Then $M = H \cdot H_k^\infty$ for some $1 \leq k \leq n$, where $H$ is a $n \times k$ left unimodular matrix with entries in $H^\infty$, i.e., $(H(e^{it}))^* \cdot H(e^{it}) = I_k$ for a.e. $t \in [0, 2\pi]$.

Theorem ([Q, BS]). $H^\infty$ is a projective free algebra.

Proof. (We follow the arguments in [BS].) Let $F$ be a nontrivial idempotent of size $n \times n$ with entries in $H^\infty$. By definition, $F$ determines a weak* continuous linear operator $H_n^\infty \to H_n^\infty$ such that $M_1 := \text{im}(F) = \ker(I_n - F)$. Hence, $M_1 \subset H_n^\infty$ is a weak* closed
submodule. According to (BLH) \( M_1 = H_1 \cdot H_k^\infty \), where \( H_1 \) is a \( n \times k \) left unimodular matrix with entries in \( H^\infty \). In particular, \( \hat{H}_1(\xi) \) is left invertible at any point \( \xi \) of the Šilov boundary of \( M(H^\infty) \). Since \( F \) has the same rank at each point of \( M(H^\infty) \) (as \( M(H^\infty) \) is connected), the invertibility of \( \hat{H}_1(\xi) \) implies \( k = \text{rank}(F) \). Thus, \( \hat{H}_1(\xi) \) is left invertible for all \( \xi \in M(H^\infty) \).

Similarly, \( M_2 := \ker(F) = \text{im}(I - F) \subset H^\infty_n \) is a weak* closed submodule. So, \( M_2 = H_2 \cdot H_k^\infty_n \), where \( H_2 \) is a \( n \times (n - k) \) matrix with entries in \( H^\infty \) such that \( \hat{H}_2 \) is left invertible at each point of \( M(H^\infty) \). From the fact \( M_1 \cap M_2 = \{0\} \) follows that the \( n \times n \) matrix \( H = (H_1, H_2) \) with entries in \( H^\infty \) is invertible and \( H^{-1} \) has entries in \( H^\infty \) as well. Moreover, \( H^{-1} \cdot F \cdot H = \text{diag}(I_k, 0) \). \( \square \)

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