A note on the elementary construction of
High-Dimensional Expanders
of Kaufman and Oppenheim

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June 2, 2022

Abstract

In this note, we give a self-contained and elementary proof of the elementary construction of spectral high-dimensional expanders using elementary matrices due to Kaufman and Oppenheim [Proc. 50th ACM Symp. on Theory of Computing (STOC), 2018]. As a bonus, this also yields a simple construction and analysis of standard expanders.

1 Introduction

In the last few years, there has been a surge of activity related to high-dimensional expanders (HDXs). Loosely speaking, high-dimensional expanders are a high-dimensional generalization of classical graph expanders. Depending on which definition of graph expansion is generalized, there are several different (and unfortunately, many a time mutually inequivalent) definitions of HDXs. For the purpose of this note, we will restrict ourselves to the spectral definition of HDXs (see Definition 2.5). Lubotzky, Samuels and Vishne [LSV05a, LSV05b] constructed high-dimensional analogues of the Ramanujan expanders of Lubotzky, Philips and Sarnak [LPS88], which they termed Ramanujan complexes. These Ramanujan complexes have several desirable properties and gave rise to the first construction of constant degree spectral HDXs. The Ramanujan graphs have the nice property that they are simple to describe, however their proof of expansion is extremely involved. The Ramanujan complexes, on the other hand, are both non-trivial to describe as well as to prove their high-dimensional expansion property. Subsequently Kaufman and Oppenheim [KO18] gave an extremely elegant and elementary construction of spectral HDXs using elementary matrices. Despite their construction being elementary and simple, the proof of expansion, though straightforward, requires some knowledge of some representation theory of the specific groups involved in the construction. The purpose of this exposition is to give an alternate elementary proof of the expansion of the Kaufman-Oppenheim HDX construction.

* This work was done when the authors were visiting the Simons Institute for Theory of Computing, Berkeley to participate in the summer cluster on Error-Correcting Codes and High Dimensional Expanders 2019.
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The underlying graph of a HDX (even a one-sided-spectral HDX) is a two-sided-spectral expander. Thus, this construction has the added advantage that it yields an elementary construction (accompanied with a simple proof) of a standard two-sided-spectral expander (though not an optimal one).

2 Preliminaries

We begin by recalling what a simplicial complex is.

**Definition 2.1 (Simplicial complex).** A simplicial complex $X$ over a finite set $U$ is a collection of subsets of $U$ with the property that if $S \in X$ then any $T \subseteq S$ is also in $X$.

- For all $i \geq -1$, define $X(i) := \{ S \in X : |S| = i + 1 \}$. Thus, if $X$ is non-empty, then $X(-1) = \{ \emptyset \}$.
- The elements of $X$ are called simplices or faces. The elements of $X(0)$, $X(1)$ and $X(2)$ are usually referred to as vertices, edges and triangles respectively.
- The graph defined by $X(0)$ and $X(1)$ is called the 1-skeleton of the complex. More generally, for any $1 \leq k \leq d$, the $k$-skeleton of the complex $X$ is the sub-complex $X(-1) \cup X(0) \cup X(1) \cup \cdots \cup X(k)$.
- The dimension of the simplicial complex $X$ is defined as the largest $d$ such that $X(d)$ (which consists of faces of size $d + 1$) is non-empty.
- The simplicial complex is said to be pure if every face is contained in some face in $X(d)$, where $d = \dim(X)$.
- For a face $S \in X$, the link of $S$, denoted by $X_S$, is the simplicial complex defined as
  \[ X_S := \{ T \setminus S : T \in X, S \subseteq T \}. \]

Thus, a graph $G = (V, E)$ is just a simplicial complex $G$ of dimension one with $G(0) = V$ and $G(1) = E$. We will deal with weighted pure simplicial complexes where the weight function satisfies a certain balance condition.

**Definition 2.2 (weighted pure simplicial complexes).** Given a $d$-dimensional pure simplicial complex $X$ and an associated weight function $w : X \to \mathbb{R}_{\geq 0}$, we say the weight function is balanced if the following two conditions are satisfied.

\[
\sum_{\sigma \in X(d)} w(\sigma) = 1; \\
\sum_{\tau \in X(i+1), \tau \supset \sigma} w(\tau), \text{ for all } i < d \text{ and } \sigma \in X(i). \tag{2.3}
\]

A weighted simplicial complex $(X, w)$ is a pure simplicial complex accompanied with a balanced weight function $w$. If no weight function is specified, then we work with the balanced weight function $w$ induced by the uniform distribution on the set $X(d)$ of maximal faces.

For a face $S \in X$, the balanced weight function $w_S$ associated with the link $X_S$ is the restricted weight function, suitably normalized, more precisely $w_S := \frac{w_{|X_S}}{w(S)}$.

Condition (2.3) states that the weight function can be interpreted as a family of joint distributions $(w_{|X(-1)}, \ldots, w_{|X(d)})$ where $w_{|X(i)}$ is a probability distribution on $X(i)$. The distribution $w_{|X(d)}$ is specified by the first condition in (2.3) while the second condition implies that the weight distribution $w_{|X(i)}$ is
the distribution on $X(i)$ obtained by picking a random $\tau \in X(d)$ according to $w|_{X(d)}$ and then removing $(d - i)$ elements uniformly at random.

We now recall the classical definition of what it means for a graph to be a spectral expander.

**Definition 2.4** (spectral expander). Given an undirected weighted graph $G = (V, E, w)$ on $n$ vertices, let $A_G$ be its normalized adjacency matrix given as follows:

$$A_G(u, v) := \begin{cases} \frac{w(u, v)}{w(u)} & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$ be the $n$ eigenvalues of $A_G$ with multiplicities in non-increasing order. We denote the second largest eigenvalue of $G$ as $\lambda(G)$.

$G$ is said to be a $\lambda$-spectral expander if $\max\{\lambda_2, |\lambda_n|\} \leq \lambda$. This is sometimes also referred to as a $\lambda$-two-sided-spectral expander.

$G$ is said to be a $\lambda$-one-sided-spectral expander if $\lambda_2 \leq \lambda$.

This spectral definition of expanders is generalized to higher dimensional simplicial complexes as follows.

**Definition 2.5** ($\lambda$-spectral HDX). A weighted simplicial complex $(X, w)$ of dimension $d \geq 1$ is said to be a $\lambda$-spectral HDX (or a $\lambda$-two-sided-spectral HDX) if for every $-1 \leq i \leq d - 2$ and $s \in X(i)$, the weighted 1-skeleton of the link $(X_s, w_s)$ is a $\lambda$-spectral expander.

A weighted simplicial complex $(X, w)$ of dimension $d \geq 1$ is said to be a $\lambda$-one-sided-spectral HDX if for every $-1 \leq i \leq d - 2$ and $s \in X(i)$, the weighted 1-skeleton of the link $(X_s, w_s)$ is a $\lambda$-one-sided-spectral expander.

Using Garland’s technique [Gar73], Oppenheim [Opp18] showed that if the 1-skeletons of all the links are connected, then a spectral gap at dimension $(d - 2)$ descends to all lower levels.

**Descent Theorem 2.6** ([Opp18]). Suppose $(X, w)$ is a $d$-dimensional weighted simplicial complex with the following properties.

- For all $s \in X(d - 2)$, the link $(X_s, w_s)$ is a $\lambda$-one-sided-spectral expander for some $\lambda < \frac{1}{d-1}$.

- The 1-skeleton of every link is connected.

Then, $(X, w)$ is a $(\frac{\lambda}{1-(d-1)\lambda})$-one-sided-spectral HDX.

Thus to prove that a given simplicial complex is a spectral HDX, it suffices to show that the 1-skeleton of all links is connected and a spectral gap at the top level. For the sake of completeness, we give a proof of the Descent Theorem 2.6 in Appendix A which includes a descent theorem for the least eigenvalue as well.

## 3 Coset complexes

The HDX construction of Kaufmann and Oppenheim is a particular instantiation of a certain type of simplicial complex called a coset complex based on a group and its subgroups. In this section, we give an exposition

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1By the balance condition, $w$ satisfies $w(v) = \sum_{u \in \tau} w(u, v)$. The matrix $A_G$ is self-adjoint with respect to the inner product $\langle f, g \rangle_w := E_{v \sim w}[f(v)g(v)]$ since $\langle f, Ag \rangle_w = \langle Af, g \rangle_w = E_{v \sim w}[f(u)g(v)]$. Hence, $A_G$ has $n$ real eigenvalues which can be obtained using the Courant-Fischer Theorem A.2.

2These are sometimes also referred to as $\lambda$-link HDXs or $\lambda$-local-expanders to distinguish from an alternative global definition of high-dimensional expansion.
of these objects. For a basic primer on group theory, see Appendix B.

**Definition 3.1** (coset complex). Let $G$ be a group and let $K_1, \ldots, K_d$ be $d$ subgroups of $G$. The coset complex $\mathcal{X}(G, \{K_1, \ldots, K_d\})$ is a $(d-1)$-dimensional simplicial complex defined as follows:

- The vertices, $\mathcal{X}(0)$, consist of cosets of $K_1, \ldots, K_d$ and we shall say cosets of $K_i$ are of type $i$.
- The maximal faces, $\mathcal{X}(d-1)$, consist of $d$-sets of cosets of different types with a non-empty intersection. That is,

$$\{g_1K_1, \ldots, g_dK_d\} \in \mathcal{X}(d-1) \iff g_1K_1 \cap \cdots \cap g_dK_d \neq \emptyset.$$  

An equivalent way of stating this is that $\{g_1K_1, \ldots, g_dK_d\} \in \mathcal{X}(d-1)$ if and only if there is some $g \in G$ such that $g_iK_i = gK_i$ for all $i$, since

$$g_iK_i = gK_i \iff K_i = g_i^{-1}gK_i \iff g_i^{-1}g \in K_i \iff g \in g_iK_i.$$  

- The lower dimensional faces are obtained by down-closing the maximal faces. Hence, for $0 \leq r \leq d$,

$$\{g_{i_1}K_{i_1}, \ldots, g_{i_r}K_{i_r}\} \in \mathcal{X}(r-1)$$  

if and only if $i_j \neq i_k$ for all $j \neq k$ and

$$g_{i_1}K_{i_1} \cap \cdots \cap g_{i_r}K_{i_r} \neq \emptyset.$$  

We shall call the set $\{i_1, \ldots, i_r\}$ the type of this face.

- The dimension of this complex is $d-1$.
- The weight function we will use is the one induced by the uniform distribution on the set $\mathcal{X}(d-1)$ of maximal faces.

A simplicial complex constructed this way is partite in the sense that each maximal face consists of vertices of distinct types.

It follows from the definition, that $\mathcal{X}(i)$ is precisely the set of cosets of the form $gK_S$ where $K_S = \cap_{j \in S}K_j$ for sets $S \subseteq [d]$ of size exactly $i+1$. In particular, $\mathcal{X}(d-1)$, the set of maximal faces, is in 1-1 correspondence with the group $G$ if $\cap_{j \in [d]}K_j = \{\text{id}\}$ where “id” is the identity element of the group $G$.

**Connectivity:**

**Observation 3.2.** $g_1K_1 \cap g_2K_2 \neq \emptyset$ if and only if $g_1^{-1}g_2 \in K_1K_2$.

**Proof.** ($\Rightarrow$) Say $x = g_1k_1 = g_2k_2$ for $k_1 \in K_1$ and $k_2 \in K_2$. Then $g_1^{-1}g_2 = g_1^{-1}x \cdot x^{-1}g_2 = k_1k_2^{-1} \in K_1K_2$.

($\Leftarrow$) If $g_1^{-1}g_2 = k_1k_2$ for $k_1 \in K_1$ and $k_2 \in K_2$, then $g_1k_1 = g_2k_2^{-1} \in g_1K_1 \cap g_2K_2$. \hfill \Box

**Lemma 3.3** (Criterion for connected 1-skeletons). The 1-skeleton (underlying graph) of $\mathcal{X}(G, \{K_1, \ldots, K_d\})$ is connected if and only if $G = \langle K_1, \ldots, K_d \rangle$.

**Proof.** ($\Leftarrow$) Since there is always an edge between $gK_i$ and $gK_j$ for $i \neq j$, it suffices to show that $K_1$ is connected to $gK_1$ for an arbitrary $g \in G$. Suppose, for an arbitrary element $g \in G$, we have $g = g_1 \cdots g_r$ where $g_j \in K_{i_j}$ and $i_j \neq i_{j+1}$ for each $j$. We might, without loss of generality, assume that (a) $g_1 \in K_1$ (otherwise set $g = 1 \cdot g_1 \cdots g_r$) and (b) if $r \geq 2$, then $i_r \neq 1$ (since otherwise we might then have worked with $g' = g_1g_2 \cdots g_{r-1}$ as $gK_1 = g'g_1K_1 = g'K_1$).
Then, we get the following path connecting $K_1$ and $gK_i$,

$$K_1 = g_1K_i \rightarrow (g_1g_2)K_i \rightarrow (g_1g_2g_3)K_i \rightarrow \ldots \rightarrow (g_1 \cdots g_r)K_i = gK_i.$$  

Note that, due to Observation 3.2, each successive pair of cosets are connected by an edge in the simplicial complex. Now, since $gK_i$ is adjacent to $gK_1$ (as $i \neq 1$), we have that $K_1$ is connected to $gK_1$.

$(\Rightarrow)$ For an arbitrary $g \in G$, since the 1-skeleton is connected we have a path

$$K_1 = g_0K_i \rightarrow g_1K_i \rightarrow \ldots \rightarrow g_rK_i = gK_1.$$  

By Observation 3.2, for every $j = 0, \ldots, r - 1$, we have $g_j^{-1}g_{j+1} \in K_iK_i^{-1} \in \langle K_1, \ldots, K_d \rangle$. Therefore,

$$g = (g_0^{-1}g_1) \cdot (g_1^{-1}g_2) \cdots (g_{r-1}^{-1}g_r) \in \langle K_1, \ldots, K_d \rangle.$$  

$\square$

**Structure of links of the coset complex:**

For any set $S \subseteq [d]$, define the group $K_S := \cap_{i \in S}K_i$; let $K_S := \langle K_1, \ldots, K_d \rangle$. The following lemma shows that the links of a coset complex are themselves coset complexes.

**Lemma 3.4.** For any $v \in X(k)$ of type $S \subseteq [d]$, the link $X_v$ is isomorphic to the simplicial complex defined by $X(K_S, \{K_S \cap K_i \colon i \notin S\})$.

**Proof.** It suffices to prove this lemma for $v \in X(0)$ as links of higher levels can be obtained by inductive applications of this case.

Observe that if $g$ is any element of $G$, then $(gK_iK_i, \ldots, gK_iK_i) \in X(r - 1)$ if and only if $(gK_iK_i, \ldots, gK_iK_i) \in X(r - 1)$. Therefore, the link of the coset $gK_i$ is isomorphic to the link of the coset $K_i$. Thus, it suffices to prove the lemma for links of the type $X_{K_i}$ for some $i \in [d]$.

Let $v$ be the coset $K_1$, without loss of generality. The vertices of the link, $X_v(0)$, are cosets of $K_2, \ldots, K_d$ that have a non-empty intersection with $K_1$. Note that any non-empty intersection $gK_i \cap K_1$ of a coset with $K_1$ is itself a coset $g_jK_j \cap K_1$ of the intersection subgroup $K_j \cap K_1$ in $K_1$. Indeed, suppose that $g_jh_j \in K_1$ for some $h_j \in K_j$. Then, $g_jh_jK_j = g_jK_j$ and $g_jh_jK_1 = K_1$ and hence

$$g_jK_j \cap K_1 = g_jh_jK_j \cap g_jh_jK_1 = g_jh_j(K_j \cap K_1).$$  

Therefore, the vertices of the link $X_v(0)$ are in bijective correspondence with cosets of $\{K_j \cap K_1 : j \in \{2, \ldots, d\}\}$.

The maximal faces in $X$ that contain the coset $K_1$ are precisely $d$-sets of cosets $\{K_1, g_2K_2, \ldots, g_dK_d\}$ with a non-empty intersection and hence

$$\emptyset \neq K_1 \cap g_2K_2 \cap \cdots \cap g_dK_d = (g_2K_2 \cap K_1) \cap \cdots \cap (g_dK_d \cap K_1) = g_2(K_2 \cap K_1) \cap \cdots \cap g_d(K_d \cap K_1),$$

which are precisely the maximal faces of the coset complex $X(K_1, \{K_j \cap K_1 : j \in \{2, \ldots, d\}\})$. This establishes the isomorphism between $X_v$ and $X(K_1, \{K_j \cap K_1 : j \in \{2, \ldots, d\}\})$.  

$\square$
4 A concrete instantiation

The simplicial complex of Kaufman and Oppenheim [KO18] is a specific instantiation of the above coset complex construction. This section is devoted to an exposition of this instantiation of Kaufman and Oppenheim. We will need some notation to describe their group.

Notation

• Let $R$ denote the ring $\mathbb{F}_p[t]/(t^s)$. This is a ring whose elements can be identified with polynomials in $\mathbb{F}_p[t]$ of degree less than $s$ (where addition and multiplication are performed modulo $t^s$). We will think of $p$ as some fixed prime power, $t$ as a formal variable and $s$ as a growing integer.

• For any $d \geq 3$, and $1 \leq i, j \leq d$ with $i \neq j$ and an element $r \in R$, we define $e_{i,j}(r)$ to be the $d \times d$ elementary matrix with 1’s on the diagonal and $r$ on the $(i,j)$-th entry.

For the sake of notational convenience, we shall abuse this notation and use $e_{i,d+1}(r), e_{i,j+d}(r)$ etc. to refer to $e_{i,j}(r)$ by wrapping around if necessary. For example, $e_{d,d+1}(r)$ refers to $e_{d,1}(r)$.

We are now ready to describe the groups in the construction.

For $i \in \{1, \ldots, d\}$, $K_i = \langle e_{j,i+1}(at + b) : a, b \in \mathbb{F}_p, j \in [d] \setminus \{i\} \rangle$.$$

G = \langle K_1, \ldots, K_d \rangle$$

Each $K_i$ is generated by elementary matrices that have 1’s on the diagonal and an arbitrary linear polynomial in one entry of the generalised diagonal $\{(i,j) : i + 1 = j \mod d\}$.

It so happens that the group $G$ generated by the subgroups $K_1, \ldots, K_d$ is $\text{SL}_d(R)$, the group of $d \times d$ matrices with entries in $R$ whose determinant is 1 (in $R$). This is a non-trivial fact. All we will need is the simpler fact that $|G|$ grows exponentially with $s$ (for fixed $p$ and $d$) while the size of the groups $K_i$ are functions of $p$ and $d$ (and independent of $s$). This will follow from the sequence of observations and lemmas developed in the following section.

Given the above definition, there are two “different” subgroups we can define.

$$K_S = \bigcap_{i \in S} K_i$$

$$\widetilde{K}_S := \langle e_{i,i+1}(at + b) : a, b \in \mathbb{F}_p, i \notin S \rangle.$$ 

That is, $K_S$ is the intersection of the groups $\{K_i : i \in S\}$, and $\widetilde{K}_S$ is the group generated by the intersection of the generators of the $K_i$’s. Thus, clearly, $\widetilde{K}_S \subseteq K_S$. The following lemma shows that in fact the two groups are identical.

Lemma 4.1 (Intersections of $K_i$’s). For any $S \subseteq [d]$,

$$\widetilde{K}_S = \langle e_{i,i+1}(at + b) : a, b \in \mathbb{F}_p, i \notin S \rangle = \bigcap_{i \in S} K_i = K_S$$

In other words, the group generated by the intersection of generators equals the group intersection.
We will prove this lemma in the following section by giving an explicit description of the groups that makes the above lemma evident. An immediate consequence of this lemma is that $\mathcal{X}(d-1)$, the set of maximal faces, is in 1-1 correspondence with the group $G$.

### 4.1 Explicit description of the groups

The following is an easy consequence of the definition of $e_{i,j}(r)$. Note that $e_{i,j}(r)$ is defined only if $i \neq j$.

**Observation 4.2.** (a) Sum: $e_{i,j}(r_1) \cdot e_{i,j}(r_2) = e_{i,j}(r_1 + r_2)$.

As a corollary, $e_{i,j}(r)^{-1} = e_{i,j}(-r)$.

(b) Product: If $i \neq \ell$, the commutator $^3[e_{i,j}(r_1), e_{k,\ell}(r_2)]$ behaves as follows.

$$[e_{i,j}(r_1), e_{k,\ell}(r_2)] = \begin{cases} e_{i,\ell}(r_1 r_2) & \text{if } j = k, \\ \text{id} & \text{if } j \neq k. \end{cases}$$

**Proof.** Let $\mu_{i,j}$ denote the matrix that has a 1 at the $(i,j)$-th entry, and 0 everywhere. Then, (a) follows as

$$e_{i,j}(r_1)e_{i,j}(r_2) = (I + \mu_{i,j}r_1) \cdot (I + \mu_{i,j}r_2) = I + \mu_{i,j} \cdot (r_1 + r_2) = (I + \mu_{i,j}^2) = 0 \quad (\text{since } \mu_{i,j}^2 = 0 \text{ when } i \neq j).$$

As for (b), we follow along a similar calculation. Note that

$$\mu_{i,j} \cdot \mu_{k,\ell} = \begin{cases} 0 & \text{if } j \neq k \\ \mu_{i,\ell} & \text{if } j = k. \end{cases}$$

Therefore,

$$[e_{i,j}(r_1), e_{k,\ell}(r_2)] = (I - \mu_{i,j} r_1) \cdot (I - \mu_{k,\ell} r_2) \cdot (I + \mu_{i,j} r_1) \cdot (I + \mu_{k,\ell} r_2).$$

When $j = k$ (along with the assumption that $i \neq j, k \neq \ell$), this simplifies to

$$[e_{i,j}(r_1), e_{k,\ell}(r_2)] = (I - \mu_{i,j} r_1) \cdot (I - \mu_{k,\ell} r_2) \cdot (I + \mu_{i,j} r_1) \cdot (I + \mu_{k,\ell} r_2).$$

If $j \neq k$, then we get

$$[e_{i,j}(r_1), e_{k,\ell}(r_2)] = (I - \mu_{i,j} r_1) \cdot (I - \mu_{k,\ell} r_2) \cdot (I + \mu_{i,j} r_1) \cdot (I + \mu_{k,\ell} r_2).$$

Therefore, for distinct $i, j, k \in [d]$ (which exist when $d \geq 3$), we have

$$[e_{i,j}(r_1), [e_{i,j}(r_2), e_{i,j}(r_3)]] = e_{i,j}(r_1 r_2 r_3)$$

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3The commutator of two elements $g, h$, denoted by $[g, h]$ is defined as $g^{-1}h^{-1}gh$. (Definition B.3)
Thus, for all \( d \geq 3 \), using the above observation along with Observation 4.2(a), we get that \( e_{i,j}(r) \) for any \( r \in R \) can be generated by \( \{ e_{k,\ell}(at+b) : k, \ell \in [d] , a,b \in \mathbb{F}_p \} \). This in particular implies that \( |G| \) is at least \( p^s \). On the other hand, the size of \( K_i \) depends only on \( d, p \) and is independent of \( s \). The lemma below describes \( K_d \); the other \( K_i \)'s are just rearrangements of rows and columns in \( K_d \).

**Lemma 4.3 (Explicit description of \( K_d \)).** The group \( K_d = \langle e_{i,i+1}(at+b) : a,b \in \mathbb{F}_p , i \neq d \rangle \) consists of matrices \( A = (A_{i,j}) \) of the following form:

\[
A_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
\text{a polynomial of degree } \leq r & \text{if } j - i = r, \\
0 & \text{if } i > j.
\end{cases}
\]

More generally, stating the above differently, for any \( n \in [d] \), the group \( K_n = \langle e_{i,i+1}(at+b) : a,b \in \mathbb{F}_p , i \neq n \rangle \) consists of matrices \( A = (A_{i,j}) \) of the following form:

\[
A_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
\text{a polynomial of degree } (j - i) \mod d & \text{if } j \neq i \text{ and } n \notin \{ i, i + 1, \ldots, j - 1 \} \mod d, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Follows easily from repeated applications of Observation 4.2. \( \square \)

Therefore, we can obtain a crude bound of \( |K_i| \leq p^{O(d^3)} \) for any \( i \). Also, the above lemma also gives an explicit description of the groups \( K_S \).

**Corollary 4.4 (Explicit description of \( K_S \)).** For any subset \( S \subseteq [d] \), the group \( K_S = \cap_{i \in S} K_i \) consists of matrices \( A = (A_{i,j}) \) of the following form:

\[
A_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
\text{a polynomial of degree } (j - i) \mod d & \text{if } j \neq i \text{ and } \{ i, i + 1, \ldots, j - 1 \} \mod d \cap S = \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

Recall the other set of subgroups defined for each \( S \subseteq [d] \):

\[
\tilde{K}_S := \langle e_{i,i+1}(at+b) : a,b \in \mathbb{F}_p , i \notin S \rangle.
\]

These groups can also be explicitly described.

**Lemma 4.5 (Explicit description of \( \tilde{K}_S \)).** For any \( \emptyset \neq S \subseteq [d] \), the group \( \tilde{K}_S \) is the set of all \( d \times d \) matrices \( A = (a_{i,j}) \) of the form

\[
a_{i,j} = \begin{cases} 
1 & \text{if } i = j, \\
\text{a polynomial of degree } \leq j - i & \text{if } j \neq i \text{ and } \{ i, i + 1, \ldots, j - 1 \} \mod d \cap S = \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]
Proof. Any $A \in \tilde{K}_S$ can be expressed as $A = B_1 \cdots B_m$ where each $B_r = e_{i_r,i_{r+1}}(\ell_r)$, for some linear polynomial $\ell_r$, with $i_r \notin S$. Then,

$$A_{i,j} = \sum_{i_1, \ldots, i_{m+1}, i_1 = i, i_{m+1} = j} (B_1)_{i_1,i_2}(B_2)_{i_2,i_3} \cdots (B_m)_{i_m,i_{m+1}}.$$ 

From the structure of each $B_r$, any nonzero contribution from the RHS must involve either $i_{r+1} = i_r$, or $i_{r+1} = i_r + 1$ if $r \notin S$. This forces that the only entries of $A$ that are nonzero, besides the diagonal, are at $(i,j)$ with none of $\{i, i+1, \ldots, j-1\}$ in $S$.

In the case when $\{i, i+1, \ldots, j-1\} \cap S = \emptyset$, the above argument also shows that the entry $A_{i,j}$ has degree at most $j - i$. Furthermore, using 

Observation 4.2, we can easily see that $e_{i,j}(f) \in \tilde{K}_S$ for an arbitrary polynomial $f(t)$ of degree at most $j - i$. From this, we can deduce that the structure of $\tilde{K}_S$ is exactly as claimed. \hfill \Box

Proof of Lemma 4.1. Follows immediately from Corollary 4.4 and Lemma 4.5. \hfill \Box

From this point on, since the groups $K_S$ and $\tilde{K}_S$ are identical, we drop the tilde notation and use $K_S$ for $\tilde{K}_S$.

4.2 Connectivity of the coset complex

Lemma 4.6. Let $S \subset [d]$ with $|S| \leq d - 2$. Then,

$$K_S = (K_S \cap K_i : i \in [d] \setminus S).$$

Proof. It is clear that $K_S$ is a superset of the RHS. It only remains to show that the other containment also holds. To see this, consider an arbitrary generator $e_{i,j}(r)$ of $K_S$. Since $j \notin S$ and $|S| \leq d - 2$, there is some $i \in [d] \setminus (S \cup \{j\})$. Therefore, $e_{i,j}(r) \in K_S \cap K_i$ and hence is generated by the RHS. \hfill \Box

Combining the above lemma with Lemma 3.3 and Lemma 3.4, we have the following corollary.

Corollary 4.7. For the coset complex $\mathcal{X}(G, \{K_1, \ldots, K_d\})$ defined by the above groups, the 1-skeleton of every link is connected. \hfill \Box

5 Spectral expansion of the complex

In this section we prove that the coset complex $\mathcal{X}(G, \{K_1, \ldots, K_d\})$ is a good spectral HDX. The Descent Theorem 2.6 states that it suffices to show that the 1-dimensional links of faces in $\mathcal{X}(d-3)$ are good spectral expanders.

5.1 Structure of 1-dimensional links

One dimensional links of the coset complex constructed are links of $v \in \mathcal{X}(G, \{K_1, \ldots, K_d\})$ of size exactly $d - 2$ (which are elements of $\mathcal{X}(d-3)$). Any such $v$ can be written as $\{gK_1, \ldots, gK_d\} \setminus \{gK_i, gK_j\}$ for $i, j \in [d]$ with $i \neq j$ and $g \in G$. Since the link of $v$ is isomorphic to the link of $\{K_1, \ldots, K_d\} \setminus \{K_i, K_j\}$, we might as well assume that $g = id$. These happen to be of two types depending on whether $i$ and $j$ are consecutive or not.
**Observation 5.1.** Consider \( v = \{K_1, \ldots, K_d\} \setminus \{K_i, K_j\} \) where \( i \) and \( j \) are not consecutive (i.e. \((i - j) \neq \pm 1 \mod d\)). Then the 1-dimensional link of \( v \) is a complete bipartite graph.

**Proof.** Note that since \( j \neq i \pm 1 \), we have \([e_{i,j+1}(r_1), e_{j,j+1}(r_2)] = \text{id}\) by Observation 4.2. Hence, these two elements commute.

The link of \( v \) corresponds to the coset complex \( \mathcal{X}(H, \{H_1, H_2\}) \) where

\[
H = K_{d|d\setminus\{i,j\}} = \langle e_{i,j+1}(at + b), e_{j,j+1}(at + b) : a, b \in \mathbb{F}_p \rangle,
H_1 = K_{d|d\setminus\{i\}} = \langle e_{i,j+1}(at + b) : a, b \in \mathbb{F}_p \rangle,
H_2 = K_{d|d\setminus\{j\}} = \langle e_{j,j+1}(at + b) : a, b \in \mathbb{F}_p \rangle.
\]

Thus, the groups \( H_1 \) and \( H_2 \) commute with each other and hence any element of \( h \in H \) can be written as \( h = g_1 \cdot g_2 \) where \( g_1 \in H_1 \) and \( g_2 \in H_2 \). Observation 3.2 implies that the resulting graph is the complete bipartite graph. \( \square \)

The interesting case is when \( v = \{K_1, \ldots, K_d\} \setminus \{K_i, K_{i+1}\} \). Without loss of generality, we may focus on the link of \( v = \{K_3, K_4, \ldots, K_d\} \). This corresponds to the coset complex \( \mathcal{X}(H, \{H_1, H_2\}) \) where

\[
H = K_{3,4,\ldots,d} = \langle e_{1,2}(at + b), e_{2,3}(at + b) : a, b \in \mathbb{F}_p \rangle,
H_1 = K_{2,3,4,\ldots,d} = \langle e_{1,2}(at + b) : a, b \in \mathbb{F}_p \rangle,
H_2 = K_{1,3,4,\ldots,d} = \langle e_{2,3}(at + b) : a, b \in \mathbb{F}_p \rangle.
\]

Hence, it suffices to focus on the first three rows and columns of these matrices as the rest of them are constant. Written down explicitly,

\[
H = \begin{bmatrix}
1 & \ell_1 & Q \\
0 & 1 & \ell_2 \\
0 & 0 & 1
\end{bmatrix} : \ell_1, \ell_2 \text{ are linear polynomials in } \mathbb{F}_p[t]
\]

\[
H_1 = \begin{bmatrix}
1 & \ell \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} : \ell \text{ is a linear polynomial in } \mathbb{F}_p[t]
\]

\[
H_2 = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} : \ell \text{ is a linear polynomial in } \mathbb{F}_p[t]
\]

Multiplication of an arbitrary element of \( H \) with an arbitrary element of \( H_1 \) is of the form

\[
\begin{bmatrix}
1 & \ell_1 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} \cdot \begin{bmatrix}
1 & \ell \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
1 & \ell_1 + \ell \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

Note that the a unique choice of \( \ell \) that makes the \((1,2)\)-th entry of the RHS zero is \( \ell = -\ell_1 \). Thus, each coset of \( H_1 \) in \( H \) has a unique representative of the form \( M_1(\ell, Q) \) described below, and similarly, each coset of
$H_2$ of $H$ has a unique representative of the form $M_2(\ell, Q)$.

$$M_1(\ell, Q) := \begin{bmatrix} 1 & 0 & Q \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2(\ell, Q) := \begin{bmatrix} 1 & \ell & Q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

respectively, where $\ell$ is a linear polynomial and $Q$ is a quadratic polynomial in $F_p[t]$ since any arbitrary element of $H$ can be uniquely written as

$$\begin{bmatrix} 1 & \ell_1 & Q \\ 0 & 1 & \ell_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \ell_1 & Q - \ell_1 \ell_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \ell_1 & 0 \\ 0 & 1 & \ell_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Lemma 5.2.** For linear polynomials $\ell_1, \ell_2 \in F_p[t]$ and quadratic polynomials $Q_1, Q_2 \in F_p[t]$, we have that

$$M_1(\ell_1, Q_1)H_1 \cap M_2(\ell_2, Q_2)H_2 \neq \emptyset \iff \ell_1 \ell_2 = Q_1 - Q_2.$$

**Proof.** Note that matrices in $H_1H_2$ are of the form

$$\begin{bmatrix} 1 & \ell_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \ell_1 + \ell_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By Observation 3.2, the cosets have a non-empty intersection if and only if

$$H_1H_2 \ni M_1(\ell_1, Q_1)^{-1}M_2(\ell_2, Q_2) = \begin{bmatrix} 1 & 0 & -Q_1 \\ 0 & 1 & -\ell_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \ell_2 & Q_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \ell_2 & Q_2 - Q_1 \\ 0 & 1 & -\ell_1 \\ 0 & 0 & 1 \end{bmatrix}$$

which happens if and only if $(\ell_2)(-\ell_1) = Q_2 - Q_1$ which is the same as $Q_1 - Q_2 = \ell_1 \ell_2$. 

Therefore, the 1-dimensional link is the bipartite graph $A = (U, V, E)$ with left and right vertices identified by pairs $(\ell, Q)$ where $\ell$ and $Q$ are linear and quadratic polynomials in $F_p[t]$ respectively, with $(\ell_1, Q_1) \sim (\ell_2, Q_2) \iff \ell_1 \ell_2 = Q_1 + Q_2$ (by associating $M_1(\ell, Q)$ with the tuple $(\ell, Q)$ on the left, and $M_2(\ell, Q)$ with the tuple $(\ell, -Q)$ on the right).

Note that $A$ is an undirected, $p^2$-regular bipartite graph with $p^5$ vertices on each side. It suffices to show that $A$ is a good expander.

Kaufman and Oppenheim [KO18] prove the expansion properties of this graph using representation theory of the associated groups, while we directly analyse the spectral gap of the adjacency matrix associated with this graph. O’Donnell and Pratt [OP22, Case 2 in the Proof of Theorem 3.23] give yet another proof of the spectral gap using the Polynomial Identity Lemma (also referred to as the Schwartz-Zippel lemma).
5.2 A related graph

The following graph is the “lines-points” or the “affine plane” graph used by Reingold, Vadhan and Wigderson [RVW05] (as the base graph in construction of constant-degree expanders, using the zig-zag product). Let $F_q$ be a finite field. Consider the bipartite graph $B_q = (U', V', E')$ defined as follows:

$$U' = V' = F_q \times F_q, \quad E' = \{(a, b), (c, d) : ac = b + d\}.$$ 

Note that the graph $B_q$ is $q$-regular as for any vertex $a, b, c \in F_q$, there is a unique $d \in F_q$ such that $ac = b + d$ and thus the vertex $(a, b)$ has exactly $q$ neighbours in $B_q$.

**Lemma 5.3.** The $q$-regular bipartite graph $B_q$ is a $\frac{1}{\sqrt{q}}$-one-sided-spectral expander.

**Proof.** Let $B_q^2$ denote the graph whose adjacency matrix is the square of the adjacency matrix of $B_q$. Restricted to the vertices in $U'$, it is easy to see that

$$\text{Number of edges between } (a, b) \text{ and } (c, d) = \begin{cases} 
1 & \text{if } a \neq c, \\
q & \text{if } a = c \text{ and } b = d, \\
0 & \text{otherwise.}
\end{cases}$$

Therefore, the adjacency matrix of $B_q^2$ (restricted to $U'$) can be written\(^4\) (under a suitable order of listing vertices) as

$$qI_{q^2} + (J_q - I_q) \otimes J_q \quad \text{(where } J_q \text{ is the } q \times q \text{ matrix of 1s).}$$

By observing that $J_q$ has eigenvalue of $q$ with multiplicity 1, and eigenvalue 0 with multiplicity $(q - 1)$, a simple calculation shows that $B_q^2$ has eigenvalue of $q^2$ with multiplicity 1, eigenvalue $q$ with multiplicity $q(q - 1)$ and eigenvalue 0 with multiplicity $q - 1$. Hence the unnormalized second largest eigenvalue of $B_q^2$ is $q$ and hence we have that the normalized second largest eigenvalue of $B_q$ is $1/\sqrt{q}$.

\(\square\)

5.3 Relating the graph $B_q$ with $A$

Set $q = p^3$ so that $F_q = \frac{F_{p^3}[y]}{p(y)}$ for some irreducible polynomial $\mu(y)$ of degree exactly 3. Therefore, each element in $F_q$ is expressible as $a_0 + a_1y + a_2y^2$ for some $a_0, a_1, a_2 \in F_p$. Thus, the graph $B_q = (U', V', E')$ defined above, for this setting of $q = p^3$, is a $p^3$-regular bipartite graph with $p^6$ vertices on either side.

Let $U'' = V'' = \{(a_0 + a_1y, b_0 + b_1y + b_2y^2) : a_0, a_1, b_0, b_1, b_2 \in F_p\}$, which is a subset of $U'$ and $V'$, respectively, of size $p^5$ each.

**Observation 5.4.** The induced subgraph of $B_q$ on $U'', V''$ is exactly the graph $A = (U, V, E)$ described earlier.

**Proof.** Note that $((\ell_1(y), Q_1(y)), (\ell_2(y), Q_2(y))) \in E'$ if and only if

$$\ell_1(y) \cdot \ell_2(y) = Q_1(y) + Q_2(y) \mod \mu(y).$$

\(^4\)In the equation, the notation $\otimes$ refers to the Kronecker product, or tensor product of matrices. It is well-known that, for square matrices $A$ and $B$, the set of eigenvalues of $A \times B$ is all products of the form $\lambda_i \cdot \nu_j$ where $\lambda_i$ is an eigenvalue of $A$ and $\nu_j$ is an eigenvalue of $B$.  

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However, since the above equation has degree at most 2, we have
\[ \ell_1(y) \cdot \ell_2(y) = Q_1(y) + Q_2(y) \quad \Leftrightarrow \quad \ell_1(y) \cdot \ell_2(y) = Q_1(y) + Q_2(y) \pmod{\mu(y)}, \]
and the first equation is exactly the adjacency condition of the graph A. Hence, the induced subgraph of \( B_q \) on \( V'' \) is indeed the graph A.

Normally, induced subgraphs of expanders need not even be connected. However, the following lemma shows that there are some instances where we may be able to give non-trivial bounds on \( \lambda \).

**Lemma 5.5.** Suppose X is a \( d \)-regular, undirected graph that is an induced subgraph of a \( D \)-regular graph Y. Then,
\[ \lambda(X) \leq \frac{D \lambda(Y)}{d}. \]

**Proof.** The Courant-Fischer Theorem A.2 characterization of the second largest eigenvalue tells us that \( \lambda(X) = \max_{a \perp 1_X} \frac{a^T A_X a}{a^T a} \), where \( G \) is the adjacency matrix of \( X \). Consider an arbitrary \( a \in \mathbb{R}^{|X|} \) such that \( a \perp 1_X = 0 \). Since \( X \) is an induced subgraph of \( Y \), the vector \( a \) can be padded with zeroes to obtain a vector \( b_a \in \mathbb{R}^{|Y|} \) such that \( b_a \perp 1_Y \). Therefore, if \( A_X \) and \( A_Y \) are the normalised adjacency matrices of \( X \) and \( Y \), we have
\[ \lambda(X) = \max_{a \perp 1_X} \frac{a^T A_X a}{a^T a} = \frac{D}{d} \max_{a \perp 1_X} \frac{b_a^T A_Y b_a}{b_a^T b_a} \leq \frac{D}{d} \max_{b \perp 1_Y} \frac{b^T A_Y b}{b^T b} = \frac{D \lambda(Y)}{d}. \]

**Corollary 5.6.** The graph \( A(U, V, E) \) corresponding to the 1-dimensional links of \( \mathcal{X}(G, \{K_1, \ldots, K_d\}) \) is a \( \frac{1}{\sqrt{p}} \)-one-sided-spectral expander.

**Proof.** The graph \( B_p^3 \) is a bipartite, \( p^3 \)-regular graph with \( \lambda(B_p^3) \leq \frac{1}{p^{3/2}} \) and \( A(U, V, E) \) is a \( p^2 \)-regular graph that is an induced subgraph of \( B_p^3 \). Hence, by Lemma 5.5,
\[ \lambda(A) \leq \frac{p^3 \cdot (1/p^{3/2})}{p^2} = \frac{1}{\sqrt{p}}. \]

**The final expansion bounds**

From the corollary above, we obtain the following theorem of Kaufman and Oppenheim.

**Theorem 5.7 ([KO18]).** For \( p > (d - 2)^2 \), the \( (d - 1) \)-dimensional coset complex \( \mathcal{X}(G, \{K_1, \ldots, K_d\}) \) is a \( \frac{1}{\sqrt{p} - (d - 2)} \)-one-sided-spectral HDX.

**Proof.** Follows directly from Descent Theorem 2.6 that \( \mathcal{X}(G, \{K_1, \ldots, K_d\}) \) is a \( \gamma \)-one-sided-spectral HDX for
\[ \gamma \leq \frac{1/\sqrt{p}}{1 - (d - 2)(1/\sqrt{p})} = \frac{1}{\sqrt{p} - (d - 2)}. \]

**Constructing two-sided-spectral HDXs and standard expanders:** The \( (d - 1) \)-dimensional coset complex \( \mathcal{X}(G, \{K_1, \ldots, K_d\}) \) is not a two-sided-spectral HDX as the 1-skeletons of the links of the faces in \( \mathcal{X}(d - 3) \) are bipartite. However, if we restrict attention to the \( k \)-skeleton of \( \mathcal{X} \) for some \( k < d - 1 \) then we can bound
the least eigenvalue using the descent theorem for least eigenvalue (Theorem A.4(2)). This is summarized in the following corollary.

**Corollary 5.8.** For \( p > (d - 2)^2 \) and any \( 1 \leq k < d \) the \( k \)-skeleton of the \((d - 1)\)-dimensional coset complex \( \mathcal{X}(G, \{K_1, \ldots, K_d\}) \) is a max \( \left\{ \frac{1}{\sqrt{d-2}}, \frac{1}{d-1} \right\} \)-two-sided-spectral HDX.

In particular, if we set \( k = 1 \) in the above corollary, we get a standard max \( \left\{ \frac{1}{\sqrt{d-2}}, \frac{1}{d-1} \right\} \)-two-sided-spectral expander. This graph is a \( d \)-partite graph and hence its least eigenvalue is at most \( -1/(d-1) \), while the above argument shows that it is least (and hence equal to) \( -1/(d-1) \).

Thus, this not only yields an elementary construction and proof of one-sided-spectral HDXs (Theorem 5.7), but also one of standard spectral expander (Corollary 5.8).

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### A Proof of the Descent Theorem

For the sake of completeness, we present the proof of *Descent Theorem 2.6* that asserts that proving spectral expansion for the maximal faces is sufficient to obtain expansion of any link. This exposition is essentially from the lecture notes by Dikstein [Dik19].

Let \((X, w)\) be a weighted \(d\)-dimensional simplicial complex. Let \(\mu_d = w|_{X(d)}\) be the distribution on the set \(X(d)\) of \((d + 1)\)-sized faces. This distribution induces distributions \(\mu_i\) on \(X(i)\) in the natural way.

For two functions \(f, g: X(0) \to \mathbb{R}\), define their inner product \(\langle f, g \rangle_X = \mathbb{E}_{u \sim \mu_0}[f(u)g(u)]\). We will drop the subscript \(X\) if it is clear from context. Note that, by the definition of \(\mu_1\), sampling \(u\) according to \(\mu_0\)
can be equivalently achieved by sampling an edge \((u,v)\) according to \(\mu_1\) and returning one of the points uniformly at random. Therefore,

\[
(A.1) \quad \langle f, g \rangle_X = \mathbb{E}_{u \sim \mu_0} [f(u)g(u)] = \mathbb{E}_{\{u,v\} \sim \mu_1} [f(u)g(v)] = \mathbb{E} \left[ \mathbb{E}_{v \sim \mu_0} \left[ f(u)g(v) \right] \right] = \mathbb{E} \left[ \langle f_v, g_v \rangle_{X_v} \right],
\]

where \(f_v, g_v : X_v(0) \to \mathbb{R} \) are the restrictions to the link of \(v\).

Define the adjacency operator \(A\) that, on a function \(f : X(0) \to \mathbb{R}\) on vertices returns another function \(Af\) on vertices defined via

\[
Af(v) = \mathbb{E}_{u \sim v} [f(u)],
\]

where \(u \sim v\) refers to a random neighbour of \(v\) according to the distribution \(u \sim \mu_0(X_v)\). In other words, \(A\) averages \(f\) over neighbours. Furthermore, \(A\) is self-adjoint with respect to the above inner product, i.e, \(\langle Af, g \rangle = \langle f, Ag \rangle\). Hence, it has \(n\) real eigenvalues and an orthonormal set of eigenvectors. Clearly \(A1 = 1\); the constant 1 function is an eigenvector for this operator (in fact, it is an eigenvector corresponding to the largest eigenvalue 1). The remaining eigenvalues are characterized by the Courant-Fischer Theorem A.2.

**Courant-Fischer Theorem A.2.** Let \(A \in \mathbb{R}^{n \times n}\) be an \(n \times n\) matrix over the reals that is self-adjoint with respect to some inner product \(\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\). Then \(A\) has \(n\) real eigenvalues \(\lambda_1 \geq \cdots \geq \lambda_n\) which have the following characterization.

\[
\lambda_i = \max_{V: \dim V = i} \min_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} = \min_{V: \dim V = n-i+1} \max_{0 \neq x \in V} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}.
\]

Similiar to Equation A.1, we have

\[
(A.3) \quad \langle Af, g \rangle_X = \mathbb{E}_{\{u,w\} \sim \mu_1} [f(u)g(w)] = \mathbb{E} \left[ \mathbb{E}_{\{u,v,w\} \sim \mu_2} [f(u)g(w)] \right] = \mathbb{E} \left[ \mathbb{E}_{v \sim \mu_0} \left[ \langle A_v f_v, g_v \rangle_{X_v} \right] \right] = \mathbb{E} \left[ \mathbb{E}_{v \sim \mu_0} \left[ \langle f_v, g_v \rangle_{X_v} \right] \right],
\]

where \(A_v\) denotes the adjacency operator restricted to the link \(X_v\).

With the above notation, we can now state the theorem we wish to prove. It suffices to prove the theorem in the case of \(d = 3\) as we can obtain Descent Theorem 2.6 by induction.

**Theorem A.4.** Suppose \((X,w)\) is weighted 2-dimensional simplicial complex. Then, we have the following two implications:

1. Suppose the 1-skeleton of \(X\) is connected and for every vertex \(v \in X(0)\), \(\langle A_v f, f \rangle \leq \lambda \langle f, f \rangle\) for all \(f : X_v(0) \to \mathbb{R}\) with \(f \perp 1_{X_v}\) for some \(\lambda \in [0,1]\). Then, for any \(g : X(0) \to \mathbb{R}\) with \(g \perp 1_X\), we have \(\langle Ag, g \rangle \leq \gamma \langle g, g \rangle\) where \(\gamma \leq \frac{1}{\lambda}\).

2. Suppose the 1-skeleton of \(X\) is non-empty and for every vertex \(v \in X(0)\), we have \(\langle A_v f, f \rangle \geq \eta \langle f, f \rangle\) for all \(f : X_v(0) \to \mathbb{R}\) for some \(\eta \in [-1,1]\). Then, for any \(g : X(0) \to \mathbb{R}\), we have \(\langle Ag, g \rangle \geq \gamma \langle g, g \rangle\) where \(\gamma \geq \frac{\eta}{\eta - 1}\).

Before we see a proof of this, let us see how Descent Theorem 2.6 follows from this.
Descent Theorem A.5 (Descent Theorem 2.6 restated). Suppose \((X, w)\) is a non-empty \(d\)-dimensional weighted simplicial complex with the following properties.

- The 1-skeleton of every link is connected.
- For all \(v \in X(d-2)\), the link \((X_v, w_v)\) is a \(\lambda\)-one-sided-spectral expander for some \(\lambda < \frac{1}{d}\). I.e., there is a \(\lambda > 0\) such that, for every \(v \in X(d-2)\) and every \(g: X_v(0) \to \mathbb{R}\) with \(g \perp 1\), we have

\[
\langle A_v g, g \rangle \leq \lambda \langle g, g \rangle.
\]

Then, \((X, w)\) is a \(\gamma\)-one-sided-spectral HDX for \(\gamma \leq \frac{\lambda}{1-(d-1)\lambda}\). That is, for any \(v \in X(-1) \cup \cdots \cup X(d-2)\) and every \(g: X_v(0) \to \mathbb{R}\) with \(g \perp 1\), we have \(\langle A_v g, g \rangle \leq \gamma \langle g, g \rangle\).

Furthermore, suppose we also know that there is a \(\eta \in [-1, 0)\) such that, for every \(v \in X(d-2)\) and every \(g: X_v(0) \to \mathbb{R}\), we have \(\langle A_v g, g \rangle \geq \eta \langle g, g \rangle\). Then, \(X\) is a \(\gamma\)-two-sided-spectral HDX with

\[
\gamma \leq \max \left( \frac{\lambda}{1-(d-1)\lambda}, \frac{\eta}{1-(d-1)\eta} \right).
\]

That is, for every \(g: X_v(0) \to \mathbb{R}\) with \(g \perp 1\), we have \(\langle A_v g, g \rangle \leq \gamma \langle g, g \rangle\).

Proof. For any \(i \leq d-2\), let

\[
\lambda_i = \min_{v \in X(i)} \max_{g: X_v(0) \to \mathbb{R}} \frac{\langle A_v g, g \rangle}{\langle g, g \rangle},
\]

the smallest one-sided-spectral expansion with respect to \(X(i)\). From repeated applications of Theorem A.4,

\[
\lambda_{-1} \leq \frac{\lambda_0}{1-\lambda_0} \leq \frac{\lambda_1/(1-\lambda_1)}{1-(\lambda_1/(1-\lambda_1))} = \frac{\lambda_1}{1-2\lambda_1} \leq \cdots \leq \frac{\lambda_{d-2}}{1-(d-1)\lambda_{d-2}}
\]

which eventually completes the proof for one-sided-spectral expansion.

For two-sided-spectral expansion, we also have to show that all the eigenvalues are bounded away from \(-1\). One again, let \(\eta_i\) be such that

\[
\eta_i = \max_{v \in X(i)} \min_{g: X_v(0) \to \mathbb{R}} \frac{\langle A_v g, g \rangle}{\langle g, g \rangle}.
\]

By repeated applications of Theorem A.4 (2), we obtain

\[
\eta_{-1} \geq \frac{\eta_0}{1-\eta_0} \geq \frac{\eta_1/(1-\eta_1)}{1-(\eta_1/(1-\eta_1))} = \frac{\eta_1}{1-2\eta_1} \geq \cdots \geq \frac{\eta_{d-2}}{1-(d-1)\eta_{d-2}}
\]

Together, we have that \(X\) is a \(\gamma\)-two-sided-spectral HDX for

\[
\gamma = \max \left( \frac{\lambda}{1-(d-1)\lambda}, \frac{\eta}{1-(d-1)\eta} \right).
\]

Proof of Theorem A.4. Let \(g\) be an eigenvector that satisfies \(\langle g, g \rangle = 1\) and \(g \perp 1_X\) that maximises (or minimises) \(\langle Ag, g \rangle\), and \(\gamma = \langle Ag, g \rangle\) be the extremal value. In particular, \(Ag = \gamma \cdot g\). From (A.3) we have
\( \gamma = \langle Ag, g \rangle = E_v [\langle A_v g_v, g_v \rangle] \).

Even though \( g \perp 1_{X_v} \), the local component \( g_v \) need not be perpendicular to \( 1_{X_v} \). Hence, let us write \( g_v = a_v 1_{X_v} + g_v^\perp \) where \( g_v^\perp \perp 1_{X_v} \); we shall drop the subscript from \( 1_{X_v} \) for the sake of brevity as the length of the vector will be clear from context. Note that \( a_v = \langle g_v, 1 \rangle = E_{w \in X_v(0)} [g_v] = Ag(v) \). Therefore, \( E_v [a_v^2] = \langle Ag, Ag \rangle = \gamma^2 \). Hence,

\[
(A.6) \quad \gamma = \langle Ag, g \rangle = E_v [\langle A_v g_v, g_v \rangle] = E_v \left[ a_v^2 + \langle A_v g_v^\perp, g_v^\perp \rangle \right]
\]

We shall now focus on the proof of Theorem A.4 (1). The other direction is exactly identical with the inequality flipped.

In the case of Theorem A.4 (1), where we are given \( \langle A_v g_v^\perp, g_v^\perp \rangle \leq \lambda \langle g_v^\perp, g_v^\perp \rangle \) for all \( v \in X(0) \), we have

\[
\gamma = E_v \left[ a_v^2 + \langle A_v g_v^\perp, g_v^\perp \rangle \right] \leq E_v \left[ a_v^2 + \lambda \langle g_v^\perp, g_v^\perp \rangle \right]
= E_v \left[ (1 - \lambda) a_v^2 + \lambda \langle g_v^\perp, g_v^\perp \rangle \right]
= (1 - \lambda) \gamma^2 + \lambda.
\]

\[\Rightarrow \gamma (1 - \gamma) \leq \lambda (1 - \gamma^2)\]

\[\Rightarrow \gamma \leq \lambda (1 + \gamma) \quad \text{(connected, thus } \gamma < 1)\]

\[\Rightarrow \gamma \leq \frac{\lambda}{1 - \lambda}.\]

In the case of Theorem A.4 (2), where we are given \( \langle A_v g_v^\perp, g_v^\perp \rangle \geq \eta \langle g_v^\perp, g_v^\perp \rangle \) for all \( v \in X(0) \), the same argument yields

\[
\gamma = E_v \left[ a_v^2 + \langle A_v g_v^\perp, g_v^\perp \rangle \right] \geq E_v \left[ a_v^2 + \eta \langle g_v^\perp, g_v^\perp \rangle \right] = (1 - \eta) \gamma^2 + \eta
\]

\[\Rightarrow \gamma \geq \frac{\eta}{1 - \eta} \]

\[\square\]

## B Primer on Group Theory

In this section, for completeness, we shall note the basic definitions and properties of groups that are used in this exposition.

**Definition B.1 (Groups and subgroups).** A set of elements \( G \) equipped with a binary operation \( \ast : G \times G \to G \) is said to be a group if it satisfies the following properties:

**Associativity:** For all \( g_1, g_2, g_3 \in G \), we have \( (g_1 \ast g_2) \ast g_3 = g_1 \ast (g_2 \ast g_3) \).

**Identity:** There exists an identity element \( \text{id} \in G \) such that, for all \( g \in G \), we have \( g \ast \text{id} = \text{id} \ast g = g \).

**Inverses:** For every element \( g \in G \), there is an element \( g^{-1} \in G \) such that \( g \ast g^{-1} = g^{-1} \ast g = \text{id} \).

A subset \( H \subseteq G \) is said to be a subgroup of \( G \) if \( H \) the binary operation \( \ast \) restricted to \( H \) satisfies the above three properties (including the fact that \( h_1 \ast h_2 \in H \) for all \( h_1, h_2 \in H \)).

Often the binary operation \( \ast \) is omitted and products just expressed as concatenation of elements.

\[\diamond\]
Definition B.2 (Cosets). Given a subgroup $H$ of a group $G$, if $x \in G$ is an arbitrary element, the (left-)coset of $H$ containing $x$, denoted by $xH$, is defined as the set

$$xH = \{ xh : h \in H \}.$$  

Two cosets $xH$ and $yH$ are identical if and only if $x^{-1}y \in H$. Hence, any element $x' \in xH$ is also referred to as a coset representative of $xH$ as $x'H = xH$. $\diamond$

Right-cosets are defined similarly. A subgroup $H$ is said to be normal if the right-coset and left-cosets agree for all $x$, i.e., $xH = Hx$, $\forall x \in G$.

Since two cosets of a subgroup $H$ of $G$ are either identical or disjoint, the set of distinct cosets of a subgroup $H$ of $G$ partition the elements of $G$. If a subgroup $H$ is normal, this set of cosets forms a group $G/H$, called the quotient group of $H$ in $G$.

Suppose $H, K$ are subgroups of $G$, we will often consider the product $HK$ (or $H \ast K$) which refers to the set $\{hk : h \in H, k \in K\}$. It is worth stressing that $HK$ need not be a subgroup of $G$ and the above just refers to a set of elements that can be expressed as an (ordered) product of an element in $H$ and an element in $K$.

For an arbitrary set $S$ of $G$, we will define $\langle S \rangle$ as the smallest subgroup of $G$ that contains the set $S$. This is also referred to as the group generated by $S$.

In general, the binary operation $\ast$ is order dependent. Groups where $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$ are said to be commutative or Abelian groups. The following notion of commutators (and commutator subgroups) is a way to measure how non-commutative a group $G$ is.

Definition B.3 (Commutators). For a pair of elements $g, h \in G$, we shall define the commutator of $g, h$ (denoted by $[g, h]$) as

$$[g, h] := g^{-1}h^{-1}gh.$$  

The commutator subgroup of $G$, denoted by $[G, G]$ is the group generated by all commutators. That is,

$$[G, G] := \langle \{ [g, h] : g, h \in G \} \rangle.$$  

$\diamond$

Note that if $G$ is Abelian, then $[G, G] = \{ \text{id} \}$. As mentioned earlier, the commutator subgroup can be thought of as a way of describing how non-Abelian a group is. In fact, the commutator subgroup of $G$ is the smallest normal subgroup $H$ of $G$ such that the quotient $G/H$ is Abelian (although these are concepts that are not necessary to follow this exposition).