Solutions to the nonlinear Schrödinger systems involving the fractional Laplacian

Meng Qu* and Liu Yang

Abstract
In this paper, we consider the following nonlinear Schrödinger system involving the fractional Laplacian operator:

\[
\begin{align*}
(-\Delta)^{\alpha/2} u + au &= f(v), \\
(-\Delta)^{\beta/2} v + bv &= g(u),
\end{align*}
\]

on \( \Omega \subseteq \mathbb{R}^n \),

where \( a, b \geq 0 \). When \( \Omega \) is the unit ball or \( \mathbb{R}^n \), we prove that the solutions \((u, v)\) are radially symmetric and decreasing. When \( \Omega \) is the parabolic domain on \( \mathbb{R}^n \), we prove that the solutions \((u, v)\) are increasing. Furthermore, if \( \Omega \) is the \( \mathbb{R}^n_+ \), then we also derive the nonexistence of positive solutions to the system on the half-space. We assume that the nonlinear terms \( f, g \) and the solutions \( u, v \) satisfy some amenable conditions in different cases.

MSC: Primary 35J45; secondary 35J60; 45G05

Keywords: Fractional Laplacian; Nonlinear Schrödinger system; Moving plane method; Radial symmetry

1 Introduction
This paper is mainly devoted to investigating the properties of the solutions of the following system involving the fractional Laplacian operators:

\[
\begin{align*}
(-\Delta)^{\alpha/2} u + au &= f(v), \\
(-\Delta)^{\beta/2} v + bv &= g(u),
\end{align*}
\]

for some \( a, b \geq 0 \),

\[ (1.1) \]

with

\[
(-\Delta)^{\alpha/2} u(x) = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} \, dy
\]

and

\[
(-\Delta)^{\beta/2} v(x) = C_{n, \beta} P.V. \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+\beta}} \, dy,
\]
where P. V. stands for the Cauchy principle value, $C_{\alpha,\beta}, C_{\alpha,\beta} > 0$ and $0 < \alpha, \beta < 2$. To make sense for the integrals, we require $u \in C^{1,1}_{\text{loc}} \cap C^{1,1}_{\text{loc}}, v \in C^{1,1}_{\text{loc}} \cap C^{1,1}_{\text{loc}}$, where

$$L_{\alpha} = \left\{ u \in L^{1}_{\text{loc}}(\mathbb{R}^{n}) \mid \int_{\mathbb{R}^{n}} \frac{|u(x)|^{1+\alpha}}{1+|x|^\alpha} \, dx < \infty \right\}$$

and

$$L_{\beta} = \left\{ v \in L^{1}_{\text{loc}}(\mathbb{R}^{n}) \mid \int_{\mathbb{R}^{n}} \frac{|v(x)|^{1+\beta}}{1+|x|^\beta} \, dx < \infty \right\}.$$

For more background on the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$, we refer to [1–4]. We mention that there are also several applications involving the fractional Laplacian in mathematical physics [5–8], finance [9], image processing [10], and so on.

Since the fractional Laplacian is nonlocal, that is, it does not act by pointwise differentiation but as a global integral with respect to a singular kernel, this is the main difficulty in studying problems involving it. To circumvent this difficulty, Caffarelli and Silvestre [11] introduced the extension method (CS extension) to overcome the difficulty of nonlocality. Their idea is to localize the fractional Laplacian by constructing a Dirichlet to Neumann operator of a degenerate elliptic equation. We can also use the integral equation method, the method of moving planes in integral forms, and regularity lifting to investigate equations involving the fractional Laplacian. Recently, Chen, Li, and Li [12] developed a new method that can handle directly these nonlocal operators. They used this property to develop some techniques needed in the direct method of moving planes in the whole space $\mathbb{R}^{n}$ and the upper half-space $\mathbb{R}^{n}_{+}$, such as the narrow region principle and decay at infinity. The direct method of moving planes is very useful, and a series of fruitful results have been obtained. For more articles concerning the method of moving planes for nonlocal equations and systems, mainly for integral equations, we refer to [13–21].

In this paper, following the ideas of [12], among others, we consider the properties of the solutions to system (1.1) for different domains $\Omega$. More precisely, we get the following four theorems. Firstly, we consider the case where $\Omega$ is the unit ball. For simplicity, we denote $B = B_{1}(0)$. We have

**Theorem 1.1** Let $u \in C(B) \cap C^{1,1}_{\text{loc}}(B)$ and $v \in C(B) \cap C^{1,1}_{\text{loc}}(B)$ be positive solutions of the system

$$
\begin{align*}
(-\Delta)^{\frac{\alpha}{2}} u + au &= f(v), & x \in B, \\
(-\Delta)^{\frac{\beta}{2}} v + bv &= g(u), & x \in B, \\
u &= v = 0, & x \notin B,
\end{align*}
$$

with $M > f'(\cdot), g'(\cdot) > 0$, where $M$ is a positive constant. Then $u$ is radially symmetric and decreasing about the origin.

**Remark 1.1** Li [22] considered the similar problem with $f(v) = v^a$ and $g(u) = u^b$. So Theorem 1.1 can be regarded as an extension of the result in [22].
We denote by
\[
\Omega = \{ x = (x',x_n) \in \mathbb{R}^n \mid x_n > |x'|^2, x' = (x_1,x_2,\ldots,x_{n-1}) \}
\]
the parabolic domain on \( \mathbb{R}^n \).

**Theorem 1.2** Let \( u \in L^\alpha (\mathbb{R}^n) \cap C^{1,1}_{\text{loc}} (\Omega) \) and \( v \in L^\beta (\mathbb{R}^n) \cap C^{1,1}_{\text{loc}} (\Omega) \) be positive solutions of the system
\[
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u + au &= f(v), & x \in \Omega, \\
(-\Delta)^{\frac{\beta}{2}} v + bv &= g(u), & x \in \Omega, \\
u \geq 0, & \quad v \geq 0, & x \in \Omega,
\end{aligned}
\]
for some \( a, b \geq 0 \).

with \( M > f'(\cdot), g'(\cdot) > 0 \), where \( M \) is a positive constant. Then \( u, v \) are increasing in \( x_n \).

Now we consider the whole space case.

**Theorem 1.3** Let \( u \in L^\alpha (\mathbb{R}^n) \cap C^{1,1}_{\text{loc}} (\mathbb{R}^n) \), \( v \in L^\beta (\mathbb{R}^n) \cap C^{1,1}_{\text{loc}} (\mathbb{R}^n) \) be positive solutions of the system
\[
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u + au &= f(v), & x \in \mathbb{R}^n, \\
(-\Delta)^{\frac{\beta}{2}} v + bv &= g(u), & x \in \mathbb{R}^n, \\
u > 0, & \quad v > 0, & x \in \mathbb{R}^n,
\end{aligned}
\]
for some \( a, b \geq 0 \).

Suppose that, for \( \gamma, \nu > 0 \),
\[
\begin{aligned}
\quad & u(x) = o \left( \frac{1}{|x|^\gamma} \right) \quad \text{and} \quad \nu(x) = o \left( \frac{1}{|x|^\nu} \right) \quad \text{as} \ |x| \to \infty
\end{aligned}
\]
and
\[
0 < f'(s) \leq s^\alpha \quad \text{and} \quad 0 < g'(s) \leq s^\beta \quad \text{with} \ p\gamma \geq \alpha \ \text{and} \ q\nu \geq \beta.
\]
Then \( u(x) \) and \( v(x) \) are radially symmetric and decreasing about some point \( x_0 \) in \( \mathbb{R}^n \).

Now we consider the nonexistence of positive solutions to system (1.1) in the half-space.

**Theorem 1.4** Let \( u \in L^\alpha (\mathbb{R}^n_+) \cap C^{1,1}_{\text{loc}} (\mathbb{R}^n_+) \) and \( v \in L^\beta (\mathbb{R}^n_+) \cap C^{1,1}_{\text{loc}} (\mathbb{R}^n_+) \) be nonnegative solutions of the system
\[
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u + au &= f(v), & x \in \mathbb{R}^n_+, \\
(-\Delta)^{\frac{\beta}{2}} v + bv &= g(u), & x \in \mathbb{R}^n_+, \\
u \equiv 0, & \quad v \equiv 0, & x \notin \mathbb{R}^n_+,
\end{aligned}
\]
Suppose
\[
\lim_{|x| \to \infty} u(x) = 0, \quad \lim_{|x| \to \infty} v(x) = 0, \quad (1.8)
\]
and \(M > f'(\cdot), g'(\cdot) > 0\), where \(M\) is a positive constant with \(f(0) = 0, g(0) = 0\). Then \(u(x) \equiv 0\) and \(v(x) \equiv 0\) in \(\mathbb{R}^n\).

**Remark 1.2** In Sect. 2, we introduce two maximum principles, namely, the narrow region principle and decay at infinity. This two maximum principles play a key role in the proof of Theorems 1.1–1.4. We give detailed proofs of our main theorems in Sect. 3.

### 2 Two maximum principles

Let \(T_\lambda\) be a hyperplane in \(\mathbb{R}^n\). Without loss of generality, we assume that
\[
T_\lambda = \{ x = (x_1, x') \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R} \},
\]
where \(x' = (x_2, x_3, \ldots, x_n)\). Let
\[
x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n)
\]
be the reflection of \(x\) about the plane \(T_\lambda\). Set
\[
\Sigma_\lambda = \{ x \in \mathbb{R}^n : x_1 < \lambda \}, \quad \Sigma^c_\lambda = \mathbb{R}^n \setminus \Sigma_\lambda,
\]
\[
u_\lambda(x) = u(x'), \quad U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = u_\lambda(x) - u(x), \quad \forall x \in \mathbb{R}^n.
\]

For simplicity of notations, we denote \(U_\lambda(x)\) by \(U(x)\) and \(V_\lambda(x)\) by \(V(x)\).

**Lemma 2.1** (Narrow region principle) *Let \(\Omega\) be a bounded narrow region in \(\Sigma_\lambda\) that is contained in
\[
\{ x \mid \lambda - l < x_1 < \lambda \}
\]
for small \(l\). Let \(U, V \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_\infty(\Omega)\) and suppose that \(U, V\) are lower semicontinuous on \(\overline{\Omega}\). Assume that \(C_1(x)\) and \(C_4(x)\) are bounded from below in \(\Omega\), whereas \(C_2(x), C_3(x) < 0\) are bounded from below in \(\Omega\). If \(U, V\) satisfy the system
\[
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} U(x) + C_1(x)U(x) + C_3(x)V(x) &\geq 0, \quad x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} V(x) + C_2(x)U(x) + C_4(x)V(x) &\geq 0, \quad x \in \Omega, \\
U(x) &\geq 0, \quad V(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \Omega, \\
U(x^\lambda) &\equiv -U(x), \quad x \in \Sigma_\lambda, \\
V(x^\lambda) &\equiv -V(x), \quad x \in \Sigma_\lambda,
\end{aligned}
\]
then, for sufficiently small \(l\), we have
\[
U(x) \geq 0, \quad V(x) \geq 0, \quad x \in \Omega. \quad (2.2)
\]
These conclusions hold for an unbounded domain $\Omega$ if we further assume that

$$\lim_{|x| \to \infty} U(x) \geq 0 \quad \text{and} \quad \lim_{|x| \to \infty} V(x) \geq 0.$$  

Furthermore, if there exists $x_0 \in \Omega$ such that

$$U(x_0) = 0 \quad \text{or} \quad V(x_0) = 0,$$

then

$$U(x) \equiv V(x) \equiv 0, \quad x \in \mathbb{R}^n. \quad (2.3)$$

Remark 2.1 After finish this paper, we have found that this lemma was given by Niu and Wang [23]. For completeness, we give a full proof of the lemma with some changes.

Proof of Lemma 2.1 Suppose on the contrary, that (2.2) is false. Without loss of generality, we assume that there exists a point such that $U(x) < 0$. Since $U(x)$ is lower semicontinuous on $\Omega$, there exists $x_0 \in \Omega$, such that

$$U(x_0) = \min_{\Omega} U(x) < 0. \quad (2.4)$$

Now let $\Sigma^c = \mathbb{R}^n \setminus \Sigma_x$. Then by the definition of $(-\Delta)^{\frac{\alpha}{2}}$, we have

$$(-\Delta)^{\frac{\alpha}{2}} U(x_0) \quad \text{as} \quad x \to \infty.$$

$$= C_{n,\alpha} P. \ V. \int_{\mathbb{R}^n} U(x_0) - U(y) \left| \frac{x_0 - y}{|x_0 - y|^{n+\alpha}} \right| dy$$

$$= C_{n,\alpha} P. \ V. \int_{\Omega} U(x_0) - U(y) \left| \frac{x_0 - y}{|x_0 - y|^{n+\alpha}} \right| dy + C_{n,\alpha} \int_{\Sigma^c} U(x_0) - U(y) \left| \frac{x_0 - y}{|x_0 - y|^{n+\alpha}} \right| dy$$

$$= C_{n,\alpha} P. \ V. \int_{\Sigma} \left[ \frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} \right] [U(x_0) - U(y)] dy$$

$$+ C_{n,\alpha} \int_{\Sigma^c} \frac{2U(x_0)}{|x_0 - y^\lambda|^{n+\alpha}} dy$$

$$:= I_1 + I_2. \quad (2.5)$$

To estimate $I_1$, we notice that

$$\frac{1}{|x_0 - y|^{n+\alpha}} > \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} \quad \text{for} \quad y \in \Sigma_x. \quad (2.6)$$

Since $U(y) \geq 0$ and $y \in \Sigma_x \setminus \Omega$, by (2.4) we have

$$U(x_0) - U(y) \leq 0 \quad \text{for} \quad y \in \Sigma_x.$$
We can see that

\[ I_1 \leq 0, \]

which implies

\[ (-\Delta)^{\frac{\alpha}{2}} U(x_0) \leq I_2 = 2C_{n,a} U(x_0) \int_{\Sigma_{x}} \frac{dy}{|x_0 - y|^{n+\alpha}}. \quad (2.7) \]

Choose \( x^*_0 = (3l + (x_0)_{1}', x') \) in \( \Sigma_x \), where \( x_0 = ((x_0)_{1}, x') \). It is easy to see that \( B_l(x^*_0) \subset \Sigma_x \). Moreover, there exists \( C > 0 \) such that

\[
\int_{\Sigma_{x}} \frac{dy}{|x_0 - y|^{n+\alpha}} = \int_{\Sigma_{x}} \frac{dy}{|x_0 - y|^{n+\alpha}} \\
\geq \int_{B_l(x^*_0)} \frac{dy}{|x_0 - y|^{n+\alpha}} \\
\geq \int_{B_l(x^*_0)} \frac{dy}{4^{n+\alpha} l^{n+\alpha}} \\
= \frac{C}{l^{\alpha}}. 
\]

Combining the previous estimate with \( (2.7) \), we have

\[ (-\Delta)^{\frac{\alpha}{2}} U(x_0) \leq \frac{CU(x_0)}{l^{\alpha}}. \quad (2.8) \]

Combining \( (2.8) \) with \( (2.1) \), we have

\[ \frac{CU(x_0)}{l^{\alpha}} + C_1(x_0) U(x_0) + C_2(x_0) V(x_0) \geq 0, \]

which is equivalent to

\[ U(x_0) \left[ \frac{C}{l^{\alpha}} + C_1(x_0) \right] \geq -C_2(x_0) V(x_0). \quad (2.9) \]

Since we can choose \( l \) small enough and \( C_1 \) is bounded from below, we have \( \frac{C}{l^{\alpha}} + C_1(x_0) > 0 \). Since

\[ U(x_0) \geq -\frac{C_2(x_0)}{\frac{C}{l^{\alpha}} + C_1(x_0)} V(x_0), \quad (2.10) \]

by the condition \( C_2(x) < 0 \) and \( (2.4) \) we get that

\[ V(x_0) < 0. \]

On the other hand, \( V \) is lower semicontinuous on \( \overline{\Omega} \); hence there exists \( \overline{x}_0 \) such that

\[ V(\overline{x}_0) = \min_{\overline{\Omega}} V(x) < 0. \quad (2.11) \]
Similarly to (2.8), we derive that
\[
(-\Delta)^\frac{\theta}{2} V(x_0) \leq \frac{CV(x_0)}{l^\theta}.
\] (2.12)

Combining (2.8) and (2.12) with (2.1), we have
\[
0 \leq (-\Delta)^\frac{\theta}{2} V(x_0) + C_3(x_0)U(x_0) + C_4(x_0)V(x_0)
\leq \frac{CV(x_0)}{l^\theta} + C_3(x_0)U(x_0) + C_4(x_0)V(x_0)
\leq \left[ \frac{C}{l^\theta} + C_4(x_0) \right] V(x_0) - C_3(x_0) \frac{C_2(x_0)}{t^\theta} + C_1(x_0) - V(x_0)
\leq \left[ \frac{C}{l^\theta} + C_4(x_0) \right] V(x_0) - C_3(x_0) \frac{C_2(x_0)}{t^\theta} + C_1(x_0)
\leq V(x_0) \left[ \frac{C}{l^\theta} + C_4(x_0) - C_3(x_0) \frac{C_2(x_0)}{t^\theta} + C_1(x_0) \right].
\]

We notice that \( C_4(x) \) is bounded from below in \( \Omega \) and \( C_2(x), C_3(x) < 0 \) are bounded from below in \( \Omega \). Choosing \( l \) small enough, we can derive that \( V(x_0) \geq 0 \). This yields a contradiction with (2.11). So (2.2) holds.

Furthermore, if \( \Omega \) is an unbounded domain, then by the decay condition of \( U, V \) it is easy to see that the negative minimum of \( U, V \) cannot be taken at infinity.

Now we prove (2.3). Without loss of generality, we assume that there exists \( x_0 \in \Omega \) such that \( U(x_0) = 0 \). Then, due to (2.2) and the fact that \( C_2(x_0)V(x_0) \leq 0 \), combining (2.5) with the first equation of (2.1), we have
\[
0 \leq (-\Delta)^\frac{\theta}{2} U(x_0) + C_2(x_0)V(x_0)
\leq C_{n,s} \text{P. V.} \int_{\Sigma_\lambda} \left[ \frac{1}{|x_0 - y|^{n+s}} - \frac{1}{|x_0 - y'|^{n+s}} \right] [-U(y)] dy.
\]

If \( U(y) \neq 0, y \in \Sigma_\lambda \), then noticing that \( U(y) \geq 0, y \in \Sigma_\lambda \), we have
\[
(-\Delta)^\frac{\theta}{2} U(x_0) + C_2(x_0)V(x_0) < 0.
\]

This yields a contradiction. So
\[
U(y) \equiv 0, \quad y \in \Sigma_\lambda.
\]

By (2.10) we immediately get \( V(x) \equiv 0, x \in \Sigma_\lambda \). So since \( U \) and \( V \) are antisymmetric functions, we have (2.3). \( \square \)
Lemma 2.2 (Decay at infinity) Let $\Omega$ be an unbounded domain in $\Sigma_1$. Let $U, V \in L^\alpha(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\Omega)$ be lower semicontinuous on $\overline{\Omega}$. Assume that

$$
\begin{align*}
(-\Delta)^{\frac{\alpha}{2}} U(x) + C_1(x) U(x) + C_2(x) V(x) &\geq 0, \quad x \in \Omega, \\
(-\Delta)^{\frac{\beta}{2}} V(x) + C_3(x) U(x) + C_4(x) V(x) &\geq 0, \quad x \in \Omega, \\
U(x) &\geq 0, \quad V(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \Omega,
\end{align*}
$$

where $C_1(x), C_2(x)$ are nonnegative on $\Omega$, whereas $C_2(x), C_3(x) < 0$ on $\Omega$, and furthermore,

$$
\lim_{|x| \to \infty} C_2(x)|x|^\alpha = 0 \quad \text{and} \quad \lim_{|x| \to \infty} C_3(x)|x|^\beta = 0. \quad (2.14)
$$

Then there exists a constant $R_0 > 0$ (depending on $C_i(x)$ but independent of $U, V$) such that if

$$
U(\tilde{x}) = \min_{\Omega} U(x) < 0 \quad \text{and} \quad V(\tilde{x}) = \min_{\Omega} V(x) < 0,
$$

then

$$
\tilde{x} \leq R_0 \quad \text{or} \quad \tilde{x} \leq R_0.
$$

Remark 2.2 This lemma is quite the same as that of Niu and Wang [23] but with some difference. In [23], $C_1, C_4$ should satisfying

$$
\lim_{|x| \to \infty} C_1(x)|x|^\alpha = 0 \quad \text{and} \quad \lim_{|x| \to \infty} C_4(x)|x|^\beta = 0. \quad (2.15)
$$

So we conclude that the condition of our lemma is different with [23], even they both have the same result.

Proof of Lemma 2.2 Without loss of generality, we assume that there exist a point $x_0 \in \Omega$ such that

$$
U(\tilde{x}) = \min_{\Omega} U(x) < 0.
$$

Then as in the proof as (2.7), we have

$$
(-\Delta)^{\frac{\alpha}{2}} U(\tilde{x}) \leq I_2 = C_{\text{n,}\alpha} \int_{\Sigma_\lambda} \frac{2U(\tilde{x})}{|\tilde{x} - y|^\alpha} \, dy. \quad (2.16)
$$

Choose a point in $\Sigma_1^c : = \mathbb{R}^n \setminus (3|\tilde{x}| + x_1, x')$, where $\tilde{x} = (x_1, x')$. Then $B_{|\tilde{x}|}(\tilde{x}) \subset \Sigma_1^c$, and there exists $C > 0$ such that

$$
\int_{\Sigma_1} \frac{1}{|\tilde{x} - y|^\alpha} \, dy = \int_{\Sigma_1^c} \frac{1}{|\tilde{x} - y|^\alpha} \, dy \geq \int_{B_{|\tilde{x}|}(\tilde{x})} \frac{1}{|\tilde{x} - y|^\alpha} \, dy \geq \frac{C}{|\tilde{x}|^\alpha}. \quad (2.17)
$$
So combining (2.17) with (2.16), we have

\[-\Delta \frac{\gamma}{2} U(\tilde{x}) \leq \frac{C U(\tilde{x})}{|\tilde{x}|^\alpha}. \tag{2.18}\]

By (2.13) we have that

\[-\Delta \frac{\gamma}{2} U(\tilde{x}) + C_1(\tilde{x}) U(\tilde{x}) + C_2(\tilde{x}) V(\tilde{x}) \geq 0.\]

Combining this with (2.18), we get that

\[\frac{C U(\tilde{x})}{|\tilde{x}|^\alpha} + C_1(\tilde{x}) U(\tilde{x}) \geq -C_2(\tilde{x}) V(\tilde{x}).\]

Now by the conditions $C_1(\tilde{x}) \geq 0$ and $C_2(\tilde{x}) < 0$ we easily calculate that

\[U(\tilde{x}) \geq -\frac{C_2(\tilde{x})}{C} \frac{V(\tilde{x})}{|\tilde{x}|^\alpha + C_1(\tilde{x})}. \tag{2.19}\]

Noticing that $U(\tilde{x}) < 0$ and $C_2(\tilde{x}) < 0$, we get $V(\tilde{x}) < 0$. Then since $V$ are lower semicontinuous on $\Omega$, there exists $\tilde{x}_0$ such that

\[V(\tilde{x}) = \min_{\Omega} V(x) < 0. \tag{2.20}\]

Similarly to (2.18), we derive that

\[-\Delta \frac{\beta}{2} V(\tilde{x}) \leq \frac{C V(\tilde{x})}{|\tilde{x}|^\beta}. \tag{2.21}\]

Combining (2.19) and (2.21) with (2.13), we have

\[0 \leq -\Delta \frac{\gamma}{2} V(\tilde{x}) + C_3(\tilde{x}) U(\tilde{x}) + C_4(\tilde{x}) V(\tilde{x}) \leq \frac{C V(\tilde{x})}{|\tilde{x}|^\beta} + C_3(\tilde{x}) U(\tilde{x}) + C_4(\tilde{x}) V(\tilde{x}) \leq \left[ \frac{C}{|\tilde{x}|^\beta} + C_4(\tilde{x}) \right] V(\tilde{x}) - C_3(\tilde{x}) \frac{C_2(\tilde{x})}{|\tilde{x}|^\alpha + C_1(\tilde{x})} V(\tilde{x}) \leq \left[ \frac{C}{|\tilde{x}|^\beta} + C_4(\tilde{x}) \right] V(\tilde{x}) - C_3(\tilde{x}) \frac{C_2(\tilde{x})}{|\tilde{x}|^\alpha + C_1(\tilde{x})} V(\tilde{x}) = V(\tilde{x}) \left[ \frac{C}{|\tilde{x}|^\beta} + C_4(\tilde{x}) - C_3(\tilde{x}) \frac{C_2(\tilde{x})}{|\tilde{x}|^\alpha + C_1(\tilde{x})} \right].\]

Choosing $|\tilde{x}|, |\tilde{x}|$ large enough, by (2.14) we can derive $V(\tilde{x}) \geq 0$. This yields a contradiction with (2.20). So the lemma is proved. \hfill \Box

3 Proof of Theorems 1.1-1.4

Proof of Theorem 1.1 Let $T_\lambda, x_\lambda, u_\lambda, \Sigma_\lambda,$ and $U_\lambda, V_\lambda$ be defined as in the previous section.
Step 1: We will show that for \( \lambda > -1 \) and sufficiently close to \(-1\), we have
\[
U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \cap B. \tag{3.1}
\]

By the first equation in system (1.2) and the mean value theorem it is easy to see that
\[
(-\Delta) \frac{\partial}{\partial \lambda} U_\lambda(x) + aU_\lambda(x) - f'(\xi(x))V_\lambda(x) = 0, \tag{3.2}
\]
where \( \xi(x) \) is between \( v(x) \) and \( v_\lambda(x) \). Similarly, we also have
\[
(-\Delta) \frac{\partial}{\partial \lambda} V_\lambda(x) + bV_\lambda(x) - g'(\eta(x))U_\lambda(x) = 0,
\]
where \( \eta(x) \) is between \( u(x) \) and \( u_\lambda(x) \). Choosing \( C_1 = a, \ C_2 = -f'(\xi(x)), \ C_3 = b, \) and \( C_4 = -g'(\eta(x)) \), by the narrow region principle (Lemma 2.1) we get (3.1).

Step 2: Define
\[
\lambda_0 = \sup \{ \lambda \leq 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda \}. \tag{3.3}
\]

Then we claim that
\[
\lambda_0 = 0.
\]

Suppose the claim is not true. If \( \lambda_0 < 0 \), then we will show that the plane can be moved to the right a little more so that inequality (3.1) will still valid. More precisely, there exists small \( \epsilon > 0 \) such that, for all \( \lambda \in [\lambda_0, \lambda_0 + \epsilon] \), inequality (3.1) holds, which contradicts the definition of \( \lambda_0 \).

First, since \( U_{\lambda_0} \) and \( V_{\lambda_0} \) are not identically zero, from the proof of the narrow region principle (Lemma 2.1) we have
\[
U_{\lambda_0} > 0, \quad V_{\lambda_0} > 0, \quad \forall x \in \Sigma_{\lambda_0} \cap B.
\]

Thus \( U_{\lambda_0} \) and \( V_{\lambda_0} \) can take the minimum values if \( x \in \Sigma_{\lambda_0 - \delta} \cap B \). More precisely, for any \( \delta > 0 \),
\[
U_{\lambda_0} \geq c_4 > 0, \quad V_{\lambda_0} \geq c_4 > 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap B.
\]

By the continuity of \( U_\lambda \) and \( V_\lambda \) with respect to \( \lambda \) there exists \( \epsilon > 0 \) such that
\[
U_\lambda \geq 0, \quad V_\lambda \geq 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap B, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon].
\]

We can see that \( (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \cap B \) is a narrow region if \( \epsilon \) and \( \delta \) are small enough. Then by the narrow region principle (Lemma 2.1) we have
\[
U_\lambda \geq 0, \quad V_\lambda \geq 0, \quad \forall x \in \Sigma_\lambda \cap B, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon].
\]

This contradicts the definition of \( \lambda_0 \). Therefore we prove the claim (3.3). It follows that
\[
U_0 \geq 0, \quad V_0 \geq 0, \quad x \in \Sigma_0 \cap B.
\]
or, more apparently,

\[ u(-x_1, x') \leq u(x_1, x'), \quad v(-x_1, x') \leq v(x_1, x'), \quad 0 < x_1 < 1. \]

Since the \( x_1 \)-direction can be chosen arbitrarily, it follows that \( u, v \) are radially symmetric about the origin. The monotonicity is a consequence of the fact that

\[ U_\lambda \geq 0, \quad V_\lambda \geq 0, \quad x \in \Sigma_\lambda, \]

for all \(-1 < \lambda \leq 0\). This completes the proof of Theorem 1.1. \( \square \)

**Proof of Theorem 1.2** Let \( T_\lambda, x_\lambda, u_\lambda, \) and \( U_\lambda, V_\lambda \) be defined as in the previous section. Let

\[ \Sigma_\lambda = \{ x = (x', x_n) \mid x_n < \lambda \}. \]

Step 1: Similarly to (3.1), we can get that, for \( \lambda > 0 \) sufficiently close to 0, we have

\[ U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \cap \Omega. \]  \hfill (3.4)

Step 2: Define

\[ \lambda_0 = \sup \{ \lambda > 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda \}. \]

Then we must have

\[ \lambda_0 = +\infty. \]

Otherwise, suppose that \( \lambda_0 < +\infty \), Then we claim that

\[ U_{\lambda_0}(x) \equiv 0, \quad V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0} \cap \Omega. \]  \hfill (3.5)

If (3.5) were not true, then by the narrow region principle (Lemma 2.1) we would have

\[ U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0} \cap \Omega. \]  \hfill (3.6)

We will show that the plane \( T_{\lambda_0} \) can be moved further to the right. More precisely, there exists small \( \epsilon > 0 \) such that, for all \( \lambda \in [\lambda_0, \lambda_0 + \epsilon) \),

\[ U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \cap \Omega. \]  \hfill (3.7)

This is a contraction with the definition of \( \lambda_0 \).

If (3.6) holds, then for any \( \delta > 0 \),

\[ U_{\lambda_0} \geq c_3 > 0, \quad V_{\lambda_0} \geq c_3 > 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap \Omega. \]

By the continuity of \( U_\lambda, V_\lambda \) with respect to \( \lambda \), there exists \( \epsilon > 0 \) such that

\[ U_\lambda \geq 0, \quad V_\lambda \geq 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon). \]
We can see that $\Sigma_1 \setminus \Sigma_{\lambda_0-\delta}$ is a narrow region since $\epsilon$ and $\delta$ are small enough. Then by the narrow region principle (Lemma 2.1) we have

$$U_\lambda \geq 0, \quad V_\lambda \geq 0, \quad \forall x \in \Sigma_1 \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).$$

This contradicts the definition of $\lambda_0$. Then the claim (3.5) holds, which implies

$$u(x', 2\lambda_0) = u(x', 0) = 0, \quad v(x', 2\lambda_0) = v(x', 0) = 0,$$

which contradicts the fact that $u, v > 0$ on $\Omega_1$. We have shown that $\lambda_0 = \infty$ and $U \geq 0$, $V \geq 0$. This shows that $u(x)$ and $v(x)$ are increasing in $x_n$, which completes the proof of Theorem 1.2.

**Proof of Theorem 1.3** Start moving the plane $T_\lambda$ from $-\infty$ to that right along the $x_1$-direction.

Step 1: We will show that for $\lambda$ sufficiently negative,

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$  \hspace{1cm} (3.8)

By the first equation in system (1.4) and the mean value theorem it is easy to see that

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) + a U_\lambda(x) - f'(\xi(x)) V_\lambda(x) = 0,$$  \hspace{1cm} (3.9)

where $\xi(x)$ is between $v(x)$ and $v_\lambda(x)$. Similarly, we also have

$$(-\Delta)^{\frac{\beta}{2}} V_\lambda(x) + b V_\lambda(x) - g'(\eta(x)) U_\lambda(x) = 0,$$

where $\eta(x)$ is between $u(x)$ and $u_\lambda(x)$. In fact, $0 < v_\lambda(x) \leq \xi(x) \leq v(x)$ and $0 < u_\lambda(x) \leq \eta(x) \leq u(x)$.

At those points, for $|x|$ sufficiently large, the decay assumptions (1.5) and (1.6) immediately yield

$$\lim_{|x| \to \infty} f'(\xi(x)) |x|^\alpha \leq \lim_{|x| \to \infty} \xi^{''}(x) |x|^\alpha \leq \lim_{|x| \to \infty} u^{''}(x) |x|^\alpha = 0,$$

and going through a similar proof, we have

$$\lim_{|x| \to \infty} g'(\eta(x)) |x|^\beta = 0.$$

Then by the decay at infinity (Lemma 2.2) there exists a constant $R_0 > 0$ such that for $\lambda < -R_0$ in Lemma 2.2, one of $U_\lambda(x)$ and $V_\lambda(x)$ must be positive in $\Sigma_\lambda$. Without loss of
generality, we assume that
\[ V_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda}. \]

Now we can also prove \( U_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda}. \) If not, then by the decay condition of \( u(x) \) there must exist a point \( x_0 \in \Sigma_{\lambda} \) such that
\[ U_{\lambda}(x_0) = \min_{x \in \Sigma_{\lambda}} U_{\lambda}(x) < 0. \tag{3.10} \]

From previous arguments (2.18) and (3.9) we have
\[ \frac{CU_{\lambda}(x_0)}{|x_0|^\alpha} + aU_{\lambda}(x) - f'(\xi(x_0))V_{\lambda}(x_0) \geq 0, \]
and then
\[ \left( \frac{C}{|x_0|^\alpha} + a \right) U_{\lambda}(x_0) \geq f'(\xi(x_0))V_{\lambda}(x_0) \geq 0, \]

since \( f'(\cdot) > 0, \quad V_{\lambda}(x_0) \geq 0. \) We can derive that \( U_{\lambda}(x_0) \geq 0, \) which contradicts with (3.10). So (3.8) holds.

Step 2: Keep moving the planes to the right to the limiting positive \( T_{\lambda_0} \) as long as (3.8) holds.

Let
\[ \lambda_0 = \sup \{ \lambda \mid U_{\mu}(x) \geq 0, \quad V_{\mu}(x) \geq 0, \quad \forall x \in \Sigma_{\mu}, \forall \mu \leq \lambda \}. \]

We have that
\[ \lambda_0 < \infty. \]

Otherwise, if \( \lambda_0 = \infty, \) then the solution \( u(x) \) is increasing with respect to \( x_1. \) This contradicts condition (1.5), so that \( \lambda_0 < \infty. \)

Then we claim that
\[ U_{\lambda_0}(x) \equiv 0, \quad V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.11} \]

If (3.11) were not true, then by the proof of the narrow region principle (Lemma 2.1) we would have
\[ U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.12} \]

We will show that the plane \( T_{\lambda_0} \) can be moved further to the right. More precisely, there exists small \( \epsilon > 0 \) such that, for all \( \lambda \in [\lambda_0, \lambda_0 + \epsilon), \)
\[ U_{\lambda}(x) \geq 0, \quad V_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda}. \tag{3.13} \]

This is a contraction with the definition of \( \lambda_0. \)
If (3.12) is true, then let $R_0$ be determined in the decay at infinity (Lemma 2.2). It follows that, for any $\delta > 0$,

$$U_{\lambda_0} \geq C_0 > 0, \quad V_{\lambda_0}(x) \geq C_0 > 0, \quad x \in \overline{\Sigma_{\lambda_0-\epsilon} \cap B_{R_0}(0)}.$$ 

Since we have the continuity of $U_{\lambda}(x)$ and $V_{\lambda}(x)$ with respect to $\lambda$, there exists $\epsilon > 0$ such that, for any $\lambda \in [\lambda_0, \lambda + \epsilon]$,

$$U_{\lambda}(x) \geq 0, \quad V_{\lambda}(x) \geq 0, \quad x \in \overline{\Sigma_{\lambda_0-\epsilon} \cap B_{R_0}(0)}.$$

(3.14)

Suppose (3.13) is not true. If $x_0$ and $\overline{x_0}$ are the negative minima of $U_{\lambda}(x)$ and $V_{\lambda}(x)$ in $\Sigma_{\lambda}$, then by decay at infinity (Lemma 2.2) and (3.14) we can get that they are all in the bounded narrow region $(\Sigma_{\lambda_0-\epsilon} \setminus \Sigma_{\lambda_0+\epsilon}) \cap B_{R_0}(0)$ for $\delta$ and $\epsilon$ small enough, which contradicts the narrow region principle (Lemma 2.1). So (3.13) has to be true, which is a contradiction with the definition of $\lambda_0$.

Now we have proved the claim (3.11). Since the $x_1$-direction can be chosen arbitrarily, we get that $u(x)$ and $v(x)$ are radially symmetric and decreasing about some point $x_0$. This completes the proof of Theorem 1.3. □

**Proof of Theorem 1.4** First, we claim that

$$u(x) > 0, \quad v(x) > 0, \quad x \in \mathbb{R}^n_+ \quad \text{or} \quad u(x) \equiv 0, \quad v(x) \equiv 0, \quad x \in \mathbb{R}^n_+.$$ 

(3.15)

To prove (3.15), we assume that $u(x) \not\equiv 0$. If there exists $x_0 \in \mathbb{R}^n_+$ such that $u(x_0) = 0$, then we have that

$$(-\Delta)^{\frac{\alpha}{2}} u(x_0) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+\alpha}} \, dy$$

$$= -C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(y)}{|x_0 - y|^{n+\alpha}} \, dy$$

$$< 0.$$ 

On the other hand, by the function (1.7) and the condition on $f(x)$ and $g(x)$ we have that $(-\Delta)^{\frac{\alpha}{2}} u = f(v) - m v \geq 0$. This yields a contradiction, so we have that

$$\text{either} \quad u \equiv 0 \quad \text{or} \quad u > 0 \quad \text{in} \quad \mathbb{R}^n_+.$$ 

If $u \equiv 0$ in $\mathbb{R}^n_+$, by (1.7) we have $f(v(x)) \equiv m v$ in $\mathbb{R}^n_+$. Together with the condition $f'(s) > m$ and $f(0) = 0$, we can obtain that $v \equiv 0$ in $\mathbb{R}^n_+$. Hence if $u(x)$ attains 0 somewhere in $\mathbb{R}^n_+$, then $u(x) = v(x) \equiv 0$. Similarly, we can also derive that if $v(x)$ attains 0 somewhere in $\mathbb{R}^n_+$, then $u(x) = v(x) \equiv 0$, so (3.15) holds. Now we assume that

$$u(x) > 0, \quad v(x) > 0, \quad x \in \mathbb{R}^n_+.$$ 

(3.16)

Denote $T_{\lambda} = \{x \in \mathbb{R}^n_+ \ | \ x_n = \lambda, \lambda > 0\}$, $\Sigma_{\lambda} = \{x \in \mathbb{R}^n_+ \ | \ 0 < x_n < \lambda\}$. Let $x^\perp = (x_1, \ldots, x_{n-1}, 2\lambda - x_n)$ be the reflection of $x$ about the plane $T_{\lambda}$, and let $U_{\lambda}(x) = u_{\lambda}(x) - u(x)$ and $V_{\lambda}(x) = v_{\lambda}(x) - v(x)$. 


Step 1. For \( \lambda > 0 \) sufficiently close to 0, set \( \Sigma = \Sigma_1 \cup \mathbb{R}^n_+ \), where \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_n \leq 0 \} \).

First, we know that \( U(x) = u_\lambda(x) - u(x) \geq -u(x) \) and \( V(x) = v_\lambda(x) - v(x) \geq -v(x) \), so by condition (1.8) we have

\[
\lim_{|x| \to \infty} U_\lambda(x) \geq \lim_{|x| \to \infty} -u(x) = 0; \quad \lim_{|x| \to \infty} V_\lambda(x) \geq \lim_{|x| \to \infty} -v(x) = 0.
\]

Then by the narrow region principle (Lemma 2.1) it is easy to see that

\[
U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_1, \tag{3.17}
\]

since \( \Sigma_1 \) is a narrow region.

Step 2. Next, we move the plane \( T_\lambda \) along the \( x_n \)-axis to the right as long as (3.17) holds and set

\[
\lambda_0 = \sup \{ \lambda > 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda \}.
\]

We claim that

\[
\lambda_0 = \infty.
\]

Otherwise, if \( \lambda_0 < \infty \), then by the proof of the narrow region principle (Lemma 2.1) we have

\[
U_{\lambda_0} > 0, \quad V_{\lambda_0} > 0, \quad x \in \Sigma_{\lambda_0} \quad \text{or} \quad U_{\lambda_0} \equiv 0, \quad V_{\lambda_0} \equiv 0, \quad x \in \Sigma_{\lambda_0}. \tag{3.18}
\]

Then going through similar arguments as in (3.11), we can see that

\[
U_{\lambda_0} \equiv 0, \quad V_{\lambda_0} \equiv 0, \quad x \in \Sigma_{\lambda_0}.
\]

If we choose the point \( \bar{x} = (x_1, x_2, \ldots, x_{n-1}, 0) \) in the hyperplane \( \{ x_n = 0 \} \), then \( \bar{x} \lambda_0 \in \mathbb{R}^n_+ \), which implies

\[
u(x_1, x_2, \ldots, x_{n-1}, 2\lambda_0) = \nu(x_1, x_2, \ldots, x_{n-1}, 0) = 0
\]

and

\[
u(x_1, x_2, \ldots, x_{n-1}, 2\lambda_0) = \nu(x_1, x_2, \ldots, x_{n-1}, 0) = 0.
\]

This contradicts with (3.16).

Therefore we have proved the claim \( \lambda_0 = \infty \), and consequently the solutions \( u(x) \) and \( v(x) \) are increasing with respect to \( x_n \). We recall that condition (1.8) tells us that \( \lim_{|x| \to \infty} u(x) = 0 \) and \( \lim_{|x| \to \infty} v(x) = 0 \). So the claim (3.16) is not true, and thus \( u(x) \equiv 0 \) and \( v(x) \equiv 0, x \in \mathbb{R}^n \).

Acknowledgements
Both authors want to take this chance to thank Dr. Yan Li for sending us the doctoral thesis.
Funding
The first author is partially supported by the NSF of China (Grant No. 11471033), the Innovation Foundation of Anhui Normal University (No. 2018XJJ111), and the Research Cultivation Foundation of Anhui Normal University (No. 2018XJJ93).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors’ contributions
Both authors read and approved the final manuscript.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 July 2018 Accepted: 1 October 2018 Published online: 29 October 2018

References
1. Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method. Bol. Soc. Bras. Mat. 22(1), 1–37 (1991)
2. Caffarelli, L.A., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math. 171(2), 425–461 (2008). https://doi.org/10.1007/s00222-007-0086-6
3. Caffarelli, L.A., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math. 171(2), 425–461 (2008)
4. Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. Commun. Pure Appl. Math. 60(1), 67–112 (2007)
5. Shi, S.: Some notes on supersolutions of fractional $p$-Laplace equation. J. Math. Anal. Appl. 463(2), 1052–1074 (2018). https://doi.org/10.1016/j.jmaa.2018.03.064
6. Shi, S., Xiao, J.: Fractional capacities relative to bounded open Lipschitz sets complemented. Calc. Var. Partial Differ. Equ. 56(1), 3–22 (2017). https://doi.org/10.1007/s00526-016-1105-5
7. Shi, S., Xiao, J.: On fractional capacities relative to bounded open Lipschitz sets. Potential Anal. 45(2), 261–298 (2016). https://doi.org/10.1007/s11118-016-9545-2
8. Shi, S., Xiao, J.: A tracing of the fractional temperature field. Sci. China Math. 60(11), 2303–2320 (2017). https://doi.org/10.1007/s11425-016-0494-6
9. Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton (2004)
10. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul. 7(3), 1005–1028 (2008). https://doi.org/10.1137/070695922
11. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. 32(7–9), 1245–1260 (2007)
12. Chen, W., Li, C., Li, Y.: A direct method of moving planes for the fractional Laplacian. Commun. Partial Differ. Equ. 38(9–10), 1647–1670 (2013)
13. Chen, W., Fang, Y., Yang, R.: Liouville theorems involving the fractional Laplacian on a half space. Adv. Math. 274, 167–198 (2015)
14. Chen, W., Li, C., Li, Y.: A direct blowing-up and rescaling argument on nonlocal elliptic equations. Int. J. Math. 27(8), 1650064 (2016)
15. Chen, W., Zhu, J.: Indefinite fractional elliptic problem and Liouville theorems. J. Differ. Equ. 260(5), 4758–4785 (2016)
16. Cheng, T., Huang, G., Li, C.: The maximum principles for fractional Laplacian equations and their applications. Commun. Contemp. Math. 19(6), 1750018 (2017)
17. Fang, Y., Chen, W.: A Liouville type theorem for poly-harmonic Dirichlet problems in a half space. Adv. Math. 229(5), 2835–2867 (2012)
18. Frank, R.L., Lieb, E.H.: Inversion positivity and the sharp Hardy–Littlewood–Sobolev inequality. Calc. Var. Partial Differ. Equ. 39(1–2), 85–99 (2010)
19. Han, X., Lu, G., Zhu, J.: Characterization of balls in terms of Bessel-potential integral equation. J. Differ. Equ. 252(2), 1589–1602 (2012)
20. Teng, K., Wu, X.: Existence and multiplicity of solutions for nonlocal systems involving fractional Laplacian with non-differentiable terms. Appl. Anal. 96(5), 525–548 (2017)
21. Wang, L., Zhang, B., Zhang, H.: Fractional Laplacian system involving doubly critical nonlinearities in $\mathbb{R}^n$. Electron. J. Qual. Theory Differ. Equ. 2017, 57 (2017)
22. Li, J.: Monotonicity and radial symmetry results for Schrödinger systems with fractional diffusion. Pac. J. Math. 294(1), 107–121 (2018). https://doi.org/10.2140/pjm.2018.294.107
23. Niu, P., Wang, P.: A direct method of moving planes for a fully nonlinear nonlocal system. Commun. Pure Appl. Anal. 16(5), 1707–1718 (2017)