On algebraic damping close to inhomogeneous Vlasov equilibria in multi-dimensional spaces

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Abstract
We investigate the asymptotic damping of a perturbation around inhomogeneous stable stationary states of the Vlasov equation in spatially multi-dimensional systems. We show that branch singularities of the Fourier–Laplace transform of the perturbation yield algebraic dampings, even for a smooth stationary state and perturbation. In two spatial dimensions, we classify the singularities and compute the associated damping rate and frequency. This 2D setting also applies to spherically symmetric self-gravitating systems. We validate the theory on an advection equation associated with the isochrone model, a model of spherical self-gravitating systems.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The Vlasov equation, often called the collisionless Boltzmann equation, describes, over a certain time frame, the large-scale dynamics of Hamiltonian systems of interacting particles, in the limit where each particle feels the effect of many others. It is thus found in many fields, including plasma physics and astrophysics, where it describes self-gravitating systems.

The Vlasov equation shows notoriously rich dynamics. Firstly, it possesses a continuous infinity of stationary states, whose study may in itself be a complicated problem. The next problem, the study of the linearized dynamics close to a stationary state, has a long history. In 1946, Landau [1] formally showed that close to a stable homogeneous stationary state, a density perturbation may decay exponentially, which was a very surprising result for a Hamiltonian system. This was the starting point of an extremely abundant physical literature. On the mathematical side, it has been proved that the Landau analysis is correct [2, 3]. It is
also known that the exponential damping fails when the reference state or the perturbation is not analytic [4], or the spatial domain unbounded [5].

The nonlinear case, that is the study of a perturbation close to a homogeneous stationary state, under the full Vlasov dynamics, is much more complicated. The subject has witnessed spectacular progress recently [6–8].

Most studies on Landau damping deal with homogeneous stationary states. By comparison, the literature on perturbations of inhomogeneous states is less developed, although these states are very important. Despite the technical difficulties involved, a lot of work has been done in the astrophysical literature. Kalnajs [9], and then Polyachenko and Schuckman [10] have developed a powerful formalism, sometimes called the matrix method, to solve the linearized Vlasov equation in an inhomogeneous context. Since then, it has been used to study the instability of many models of stellar systems, and to compute the growth rates (see [11] for a textbook account). Some other methods to compute instability rates have been introduced and applied to toy models recently [12–14]. Purely oscillating modes in 1D have been investigated in [15]. Also, when the aim is to prove stability and not to compute a decay rate, there exist powerful variational methods [11]. However, computing decay rates for non-oscillating stable situations is more tricky than computing instability rates, since it involves an additional analytic continuation, as in Landau’s original analysis. In the astrophysical literature, we are aware of only two papers performing this continuation numerically and thus explicitly computing the analogue of Landau damping rates [16, 17]. In [18], the continuation is performed analytically, but on a 1D toy model. Thus, although Landau damping is considered an important player in the dynamics of stellar systems (see for instance [11]), there are very few actual computations of damping rates close to inhomogeneous stationary states.

In addition, these studies done in the astrophysical context do not mention a fundamental difference between homogeneous and inhomogeneous stationary states: in the inhomogeneous case, no matter what the regularity of the stationary state and the perturbation is, the asymptotic linear decay is never exponential. This has been seen in the Kuramoto model, which shares some properties with the Vlasov equation [19], and on a type of Vlasov equation in [20]. The picture is as follows: the dynamics may show a transient exponential decay governed by a Landau pole, but the asymptotic decay is always algebraic. This phenomenology is also well known for the 2D Euler equation [21], which is in many respects similar to the Vlasov equation. We note that such algebraic decays may also arise close to a homogeneous state, when the perturbation or the reference state is not regular enough [4]; we stress that the situation is different close to inhomogeneous states, because the algebraic decay occurs for all reference states and perturbations. Reference [22] contains a detailed analysis of this phenomenon in 1D, including the derivation of the decay exponents and their comparison with direct numerical simulations. In this paper, we extend the analysis of [22] to multi-dimensional systems, putting our emphasis on 2D systems and 3D systems with spherical symmetry. These include some important models of stellar systems.

More specifically, we first use the standard matrix method to formally solve the linear dynamics in a Laplace transformed space (section 2). From this starting point:

(i) We identify and classify the singularities appearing in the Laplace transform of the perturbation. It turns out that the zoology of singularities in two dimensions is much richer than in one dimension; see section 3.
(ii) We exhibit the asymptotic decay for each type of singularity in section 3.3.
(iii) We numerically examine the theory developed in this paper on a spherical self-gravitating system in section 4. For this purpose, we introduce an advection equation associated with the linearized Vlasov equation, in order to reduce the numerical burden.
Our analysis of the asymptotic decay of a perturbation is formal, and moreover relies on the linearized Vlasov equation. There is no guarantee that this is relevant to understanding the asymptotic decay of the nonlinear Vlasov equation. In principle, our results should then be supplemented by direct numerical simulations of the full Vlasov equation. However, such simulations are difficult in more than one spatial dimension, since we are aiming at an asymptotic in time regime, while keeping a good spatial precision. We leave this study for future work. This is why we analyse instead with the present theory easily solvable linear advection equations, for illustrative purposes.

2. Solution to the linearized Vlasov equation: the matrix method

Throughout the paper, we will use bold letters \((q, p, \theta, J, \ldots)\) to denote vectors.

2.1. System

We start with the Vlasov equation in \(d\) spatial dimensions for the one-particle distribution function \(f(q, p, t)\):

\[
\frac{\partial f}{\partial t} + \nabla_p H \cdot \nabla_q f - \nabla_q H \cdot \nabla_p f = 0, \tag{1}
\]

where \(q \in X \subset \mathbb{R}^d\) is the position variable, \(p\) the conjugate momentum variable, \(\nabla_q, \nabla_p\) denote respectively the gradients with respect to \(q\) and \(p\), and \(\ldots\) represents the Euclidian inner product. The one-particle Hamiltonian \(H\) is defined by

\[
H[f](q, p, t) = \frac{p^2}{2} + \Phi[f](q, t) + \Phi_{\text{ext}}(q), \tag{2}
\]

where the potential \(\Phi[f](q, t)\) is defined by

\[
\Phi[f](q, t) = \int_{\mathbb{R}^d} \int_X \partial_p f \int_X dq' v(q - q') f(q', p, t) \tag{3}
\]

and the external potential \(\Phi_{\text{ext}}(q)\) creates an external force \(F_{\text{ext}}(q) = -\nabla_q \Phi_{\text{ext}}(q)\).

Let \(f_0\) be a stationary solution to the Vlasov equation (1). For the stationary solution, the potential \(\Phi[f_0]\) and the one-particle Hamiltonian \(H[f_0]\) are independent of time. In this paper, we limit ourselves to situations where the one-particle Hamiltonian \(H[f_0]\) is integrable. Thus, we may introduce actions \(J = (J_1, \ldots, J_d)\) and conjugate angles \(\theta = (\theta_1, \ldots, \theta_d)\), and the one-particle Hamiltonian is a function of the actions only, i.e. \(H[f_0](J)\). An important practical example is given by a spherically symmetric stellar system; see section 4. A stationary solution can be constructed by taking \(f_0\) as a function of actions, satisfying self-consistent conditions, since the actions depend on \(f_0\) through the potential \(\Phi[f_0]\).

We now consider a small perturbation to a stationary solution \(f_0(J)\):

\[
f(\theta, J, t) = f_0(J) + f_1(\theta, J, t), \tag{4}
\]

and write the linearized Vlasov equation for \(f_1:\)

\[
\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial J} \cdot \nabla_q f_1 - \nabla_q f_0 \cdot \nabla_q \Phi_1 = 0, \tag{5}
\]

where \(\Phi_1 = \Phi[f_1]\) is the perturbed potential and \(\Omega(J) = \nabla_J H[f_0](J)\) is the vector of the frequencies in the unperturbed potential.

2.2. Formal solution to the linearized Vlasov equation

To analyze the linearized Vlasov equation (5), we introduce the Fourier–Laplace transform \(\hat{u}(m, J, \omega)\) of a function \(u(\theta, J, t)\) as

\[
\hat{u}(m, J, \omega) = \int_{-\pi}^{\pi} d\theta e^{-im\theta} \int_0^{+\infty} dt e^{i\omega t} u(\theta, J, t) \tag{6}
\]
where \( m = (m_1, \ldots, m_d) \) is a \( d \)-uplet of integers and \( \text{Im}(\omega) \) is large enough to ensure convergence of the integral with respect to \( t \).

Introducing the biorthogonal functions \( \{d_i(q)\}_{i \in l} \) and \( \{u_k(q)\}_{k \in K} \), which satisfy
\[
(d_i, u_k) = \int_X d_i(q) \overline{u_k(q)} \, dq = \lambda_k \delta_{ik} \tag{7}
\]
and
\[
\int_X v(q-q')d_i(q') \, dq' = u_i(q), \quad \text{if} \quad i \in K; \quad = 0, \quad \text{if} \quad i \notin K \tag{8}
\]
we expand the density function
\[
\rho_i(q,t) = \int_X f_i(q,p,t) \, dp = \sum_{i \in l} a_i(t)d_i(q) \tag{9}
\]
and potential function
\[
\Phi(q,t) = \sum_{k \in K} \tilde{a}_k(t)u_k(q). \tag{10}
\]

Long but straightforward computations give a formal solution for the Laplace transform of \( a_i(t) \), denoted by \( \tilde{a}_i(\omega) \) as
\[
\tilde{a}(\omega) = [\Lambda - F(\omega)]^{-1} G(\omega) \tag{11}
\]
where \( \tilde{a}(\omega) = (\tilde{a}_k(\omega)) \) and \( G(\omega) = (G_l(\omega)) \) are vectors, and \( \Lambda = \text{diag}(\lambda_k) \) and \( F(\omega) = (F_k(\omega)) \) are matrices. The functions \( F_k(\omega) \) and \( G_l(\omega) \) are expressed by
\[
F_k(\omega) = \frac{1}{(2\pi)^d} \sum_{m \in \mathbb{Z}^d} \int \frac{m \cdot \nabla f_0(J)}{m \cdot \Omega(J) - \omega} \overline{c_l(m,J)} c_k(m,J) \, dJ \quad (l, k \in K) \tag{12}
\]
and
\[
G_l(\omega) = \frac{1}{(2\pi)^d} \sum_{m \in \mathbb{Z}^d} \int \frac{g(m,J)}{m \cdot \Omega(J) - \omega} \overline{c_l(m,J)} \, dJ \quad (l \in K), \tag{13}
\]
where
\[
c_l(m,J) = \int u_l(q) e^{-i m \cdot \theta} \, d\theta. \tag{14}
\]

The last step to compute the evolution of the perturbation is to perform an inverse Laplace transform on the functions \( \tilde{a}(\omega) \). The large time behaviour of \( a_k(t) \) is determined by the singularities of \( \tilde{a}_k(\omega) \); our goal now is to study and classify the singularities of \( \tilde{a}(\omega) \).

2.3. Analytic continuation

Assuming that the decay for large \( J \) is fast enough in the integrands, and that the \( c_k(m,J) \) are regular, expressions (12) and (13) show that functions \( F_k(\omega) \) and \( G_l(\omega) \) are analytic in the upper half plane \( \text{Im}(\omega) > 0 \), since the corresponding integrals over \( J \) have no singularity.

The integrands are singular however for any real \( \omega \) at which \( m \cdot \Omega(J) - \omega \) vanishes. Thus, expressions (12) and (13) should be analytically continued to define the \( F, G \) and \( \tilde{a} \) in the lower-half plane \( \text{Im}(\omega) \leq 0 \). This analytical continuation is a generalization of the usual Landau prescription in 1D.

The last step to compute the evolution of the perturbation is to perform an inverse Laplace transform on the functions \( \tilde{a}(\omega) \). The large time behaviour of \( a_k(t) \) is determined by the singularities of \( \tilde{a}_k(\omega) \); our goal now is to study and classify the singularities of \( \tilde{a}(\omega) \).
2.4. Singularities of \( \tilde{a}(\omega) \) and roots of the dispersion relation

The singularities of \( \tilde{a}(\omega) \) may come (i) from the roots of the dispersion relation \( \det(\Lambda - F(\omega)) = 0 \), and (ii) from singularities of the functions \( F_s \) and \( G_s \) themselves. We discuss the former singularities in this section: we will see that these singularities do not dominate in the asymptotic regime. The dominating latter singularities are classified in the next section in the 2D setting.

Roots of the dispersion relation \( \det(\Lambda - F(\omega)) = 0 \) yield poles for the functions \( \tilde{a}_k \). Such poles in the upper-half plane correspond to eigenvalues of the linearized Vlasov operator, with exponentially growing eigenmodes. Since we are interested in the damping of a perturbation close to a stable stationary state of the Vlasov equation, we assume that there are no such eigenmodes. Poles on the real axis correspond to purely oscillating eigenmodes. In order to study the decay of perturbations, we also assume that there are no such modes. There may be roots of the dispersion relation in the lower-half plane \( \text{Im}(\omega) < 0 \) (in this case, they are rather roots of the analytic continuation of the dispersion relation). These are the usual ‘Landau poles’, giving rise to exponential damping. This damping may be an important feature of the dynamics at intermediate time scales, especially if the pole is close to the real axis [16–18], but the asymptotic regime is always dominated by the singularities of the functions \( F \) and \( G \), as will become clear in the following sections.

3. Singularities of \( F \) and \( G \) in a 2D setting

We have seen that the functions \( F \) and \( G \) have no singularities for \( \text{Im}(\omega) > 0 \). The integrands of \( F \) and \( G \) are singular for \( J_s \) such that \( m \cdot \Omega(J) - \omega \) vanishes, and these singularities may yield branch points on the real \( \omega \) axis. Thus, to investigate the asymptotic behaviour of a perturbation due to these branch points, we turn now to the singularities on the real axis. From now on and for simplicity, we restrict ourselves to 2D integrations over \( J = (J_1, J_2) \) in the formulas (12) and (13) entering into the definitions of \( F \) and \( G \). There would be no major obstacles to an analysis of higher dimension, except the growing number and complexity of the possible types of singularities.

3.1. Generic singularities

The abstract problem is to study the singularities of the analytic continuation of \( \psi(\omega) \) defined for \( \text{Im}(\omega) > 0 \) as integrals over a domain \( D \subset \mathbb{R}^2 \):

\[
\psi(\omega) = \int_{D \subset \mathbb{R}^2} \frac{v(J)}{\mu(J) - \omega} dJ
\]

with \( \mu \) and \( v \) real functions. Functions \( F \) and \( G \), (12) and (13) respectively, fit in this framework, with \( \mu(J) = m \cdot \Omega(J) \), and by defining properly \( v(J) \). We assume \( m \neq 0 \) since the contributions from \( m = 0 \) in the definitions of \( F \) and \( G \) vanish. We also assume that \( \mu \) and \( v \) are very regular; although they are naturally defined over the domain \( D \subset \mathbb{R}^2 \), they may be analytically continued over the complex \( J \) domain. We further assume that their integrability properties are as good as needed. Our goal is to study the singularities of the function

\[
\phi(x) = \lim_{y \to 0^+} \psi(x + iy), \quad \text{with } x, y \in \mathbb{R},
\]

where \( \omega = x + iy \). This is the analytic continuation on the real axis of \( \psi(\omega) \), which is \textit{a priori} defined for \( \text{Im}(\omega) > 0 \).

For \( \omega \) real, the denominator in (15), \( \mu(J) - \omega \), may vanish. We explain in appendix A why for a generic \( \omega \) there exists an analytic continuation of \( \psi \) in a neighborhood of \( \omega \), and
Figure 1. Vertex (left), tangent (middle) and critical (right) singularities. In all cases, the domain $D$ is shaded, and the dashed lines are level sets of the function $\mu(J)$.

Table 1. Summary for the vertex, tangent and critical singularities. $\mu_i$ represents $(\partial \mu / \partial J_i)(J^*)$ for $i = 1, 2$, and is not zero in general. The symbol $\partial D$ denotes the boundary of $D$. The ‘or’ in the tangent singularity is exclusive.

| Singularity | Conditions for $\mu(J)$ | Geometric conditions at $J^*$ |
|-------------|-------------------------|-----------------------------|
| Vertex      | $\mu_1 = 0$ or $\mu_2 = 0$ | On $\partial D$, non-regular point |
| Tangent     | $\mu_1 = 0$ and $\mu_2 = 0$ | On $\partial D$ |
| Critical    | $\mu_1 = 0$ and $\mu_2 = 0$ | --- |

hence there is no singularity for $\phi$. From the computation in appendix A, special values of $\omega_0 = \mu(J^*)$ corresponding to a singularity for $\phi$ are easily identified.

We compute the singularities of $\phi(x)$ by expanding $\mu(J)$ around a point $J^*$ and performing the integrals over $J$. The expansion is

$$
\mu(J) = \omega_0 + \mu_1 (J_1 - J_1^*) + \mu_2 (J_2 - J_2^*) + \frac{1}{2} \mu_{11} (J_1 - J_1^*)^2 + \frac{1}{2} \mu_{22} (J_2 - J_2^*)^2 + \mu_{12} (J_1 - J_1^*) (J_2 - J_2^*) + \cdots
$$

(17)

where $\omega_0 = \mu(J^*)$ and, for instance, $\mu_i (i = 1, 2)$ represents $(\partial \mu / \partial J_i)(J^*)$. Three types of singularities which may occur in general are arranged in table 1, and schematic pictures are shown in figure 1. The point $J^*$ is either at a vertex of the domain $D$, on the boundary of $D$ or at a generic position inside $D$.

We assume that the leading order of $\nu(J)$ around $J^*$ is:

$$
\nu(J) \sim (J_1 - J_1^*)^{a_1} (J_2 - J_2^*)^{a_2}, \quad \text{with } a_1, a_2 \text{ non-negative integers.}
$$

(18)

Shifting and rescaling $J_1$ and $J_2$, the three types of singularities are obtained by computing

**Vertex:** 

$$
\varphi(\omega) = \int_U J_1 J_2 (J_1 + J_2 - (\omega - \omega_0)) dJ_1 dJ_2
$$

(19)

**Tangent:**

$$
\varphi(\omega) = \int_U J_1 J_2 (J_1^2 + J_2^2 - (\omega - \omega_0)) dJ_1 dJ_2
$$

(20)

and

**Critical:**

$$
\varphi(\omega) = \int_U J_1 J_2 (J_1^2 \pm J_2^2 - (\omega - \omega_0)) dJ_1 dJ_2
$$

(21)

respectively, where $U$ is a domain including the origin, which is at a vertex, an edge and in a generic position in $U$ for the vertex, the tangent and the critical singularities respectively. Note that the sign in front of $J_2$ in the denominator is irrelevant in the vertex and the tangent
singularities, as can be shown through the change of variable $J_2 \rightarrow -J_2$, but is important for the critical singularity. We will refer to the $J_1^2 + J_2^2$ case as ‘extremum’ and $J_1^2 - J_2^2$ as ‘saddle’. Changing variables as

$$\text{Vertex} : \quad u = J_1 + J_2, \quad v = J_1 - J_2$$

$$\text{Tangent} : \quad u = J_1^2 + J_2^2, \quad v = J_1$$

$$\text{Critical extremum} : \quad J_1 = \sqrt{u} \cos v, \quad J_2 = \sqrt{u} \sin v$$

$$\text{Critical saddle} : \quad u = J_1^2 - J_2^2, \quad v = J_1 - J_2,$$

respectively, and picking up the leading singularities, the double integrals are reduced to the following single integrals:

$$\psi(\omega) = \int_{-c}^{c} \frac{u^{1+\alpha_1+\alpha_2}}{u - (\omega - \omega_0)} \, du,$$

where $c$ is a small constant. Thanks to appendix B, singularities of $\psi(\omega)$ can be obtained and are arranged in table 2.

**Table 2.** Table of singularities of the function $\phi(x)$ defined by (15) and (16), with their associated exponent $\alpha$ (see section 3 and appendix B). The singularity is produced at $J^* = (J_1^*, J_2^*)$, and $J_{00} = \mu(J_1^*, J_2^*)$. Dampings are $e^{-i \omega t} e^{-(1+\alpha) t}$. The leading order of the numerator is $v(J) \simeq (J_1-J_2)^\alpha (J_2-J_1)^\beta$ for vertex, tangent and critical singularities, $v(J) \simeq (J_1+J_2)^\alpha$ for a singularity at infinity, and $v(J) \simeq h(J_1) h(J_2)$ for a line singularity. We assume $\mu(J) \simeq (J_1+J_2)^\alpha$ for a singularity at infinity, and $a > 0$ and $b > 2$ in the singularity at infinity, and $\omega \geq 0$ in the line singularity. No singularity appears in $\phi(x)$ if the column ‘Condition’ is not satisfied. This table shows the leading singularity for each type. If $\omega_0 = 0$, a special cancellation for the leading term may happen between modes $m$ and $-m$. The column ‘Sign’ represents the relative sign between the singular parts of modes $m$ and $-m$. A bar means there is no simple relation between the two modes. Note that the relative sign may also depend on the numerator $v(J)$ of (15) if it depends on $m$, but this dependence is ignored in this table.

| Type             | Condition | Singularity | Exponent $\alpha$ | Sign     |
|------------------|-----------|-------------|-------------------|----------|
| Vertex           | (a_1 : even) | (B.4) | $1 + a_1 + a_2$ | $(-1)^{1+u}$ |
| Tangent          | (a_1 : even) | (B.5) | $1/2 + a_1/2 + a_2$ | $-1$ |
| Critical         |           |             |                   |          |
| Extremum         | ($a_1, a_2 : even$) | (B.4) | $(a_1 + a_2)/2$ | $(-1)^{1+u}$ |
| Saddle           | ($a_1, a_2 : even$) | (B.4) | $(a_1 + a_2)/2$ | $(-1)^u$ |
| Infinity         | ($b - 2)/a \in \mathbb{Z}$) | (B.4) | $(b - 2)/a - 1$ | $(-1)^{1+u}$ |
| Line             | $a \in \mathbb{Z}$) | (B.4) | $a$ | $(-1)^{1+u}$ |
|                  | $a \notin \mathbb{Z}$ | (B.5) | $a$ | $-$ |
3.2. Singularities in spherical self-gravitating systems

Having in mind the spherical self-gravitating systems, we have to consider two further types of singularities. See figure 2 for schematic pictures.

The first appears at infinity on the \( J \) plane, and is not related to spherical symmetry. This ‘singularity at infinity’ appears when both \( \mu(J) \) and \( \nu(J) \) vanish in the limit of \( |J| \to \infty \). We then assume
\[
\mu(J) \sim (J_1 + J_2)^{-a}, \quad \nu(J) \sim (J_1 + J_2)^{-b},
\]
where both \( a \) and \( b \) are real variables with \( a > 0 \) and \( b > 2 \) to ensure vanishing at infinity and the convergence of the integrals in (15).

The other comes from a whole axis on the \( J \) plane, and will be called ‘line singularity’. In a spherical potential, the orbit of a particle with angular momentum \( L = 0 \) is purely radial; for small \( L \), it is very elongated, and remains close to being purely radial. When the particle travels from one maximal radius to the next along the elongated orbit, this corresponds to one radial period; this also corresponds approximately to a half angular period (see figure 2). Thus, on the \( L = 0 \) line, the radial frequency is exactly twice the angular frequency \( \Omega_1 = 2\Omega_\varphi \); or equivalently \( \mathbf{m} \cdot \mathbf{\Omega} = 0 \) for \( \mathbf{m} = (2, -1) \) and \( \mathbf{\Omega} = (\Omega_\varphi, \Omega_\varphi) \). We consider a domain \( D \) as in figure 2: the axis \( J_1 = 0 \) is a boundary. Suppose that the function \( \mu(J) \) is constant on the \( J_2 \) axis and we expand \( \mu(J) \) and \( \nu(J) \) close to the \( J_2 \) axis as
\[
\mu(J) = \omega_0 + W(J_2)J_1 + \cdots, \quad \nu(J) \sim h(J_2)J_1^a
\]
with a non-negative real value \( a \).

To study the above two types of singularities, we compute
\[
\text{Infinity : } \varphi(\omega) = \int_U \frac{(J_1 + J_2)^{-b}}{(J_1 + J_2)^{-a} - \omega} \, dJ_1 \, dJ_2
\]
\[
\text{Line : } \varphi(\omega) = \int_U \frac{h(J_2)J_1^a}{W(J_2)J_1 - (\omega - \omega_0)} \, dJ_1 \, dJ_2.
\]

Changing variables as
\[
\text{Infinity : } u = (J_1 + J_2)^{-a}, \quad v = J_1 - J_2
\]
\[
\text{Line : } u = W(J_2)J_1, \quad v = J_2
\]
to catch the leading singularities, the double integrals are reduced to:
\[
\text{Infinity : } \varphi(\omega) = \int_0^c \frac{u^{(b-2)/a-1}}{u - \omega} \, du
\]
\[
\text{Line : } \varphi(\omega) = \int_0^c \frac{u^a}{u - (\omega - \omega_0)} \, du.
\]

Singularities of \( \varphi(\omega) \) are then obtained from appendix B, and are summarized in table 2.
3.3. Asymptotic behaviour of a perturbation

The singularities in the functions \( F \) and \( G \) are passed on to the coefficients \( \tilde{a}_k(\omega) \). Each singularity corresponds to a decaying term in the inverse Laplace transform. If \( \tilde{a}_k(\omega) \) had a finite number of singularities, the asymptotic in time behaviour of \( a_k(t) \) would be a sum of decaying terms, each one corresponding to a singularity of \( \tilde{a}_k(\omega) \) (see theorem 19 in [24]). We expect that this remains true in our situation, where \( \tilde{a}_k(\omega) \) actually has an infinite number of singularities.

According to the analysis of section 3, the singular part of \( \tilde{a}_k(\omega) \) (or rather of its analytic continuation on the real axis) close to a singular value \( \omega_0 \) is, for \( \omega > \omega_0 \):

\[
(\omega - \omega_0)^\alpha \ln|\omega - \omega_0| \quad \text{for} \quad \alpha \in \mathbb{Z}_+,
\]

or

\[
(\omega - \omega_0)^\alpha H(\omega - \omega_0) \quad \text{for} \quad \alpha + 1 \in \mathbb{R}^*_+.
\]

Here \( \mathbb{Z}_+ \) are the non-negative integers, and \( \mathbb{R}^*_+ \) are the positive real numbers. The behaviour is of course similar for \( \omega < \omega_0 \). If a function \( \phi \) presents any of the above singularities, the asymptotic behaviour of its inverse Laplace transform \( \tilde{\phi}(t) \) is computed according to the following rules [24]:

\[
\phi_{\text{sing}}(\omega) = C(\omega - \omega_0)^\alpha \ln|\omega - \omega_0| \rightarrow \tilde{\phi}(t) = C' e^{-i\omega_0 t} t^{1+\alpha},
\]

(38)

\[
\phi_{\text{sing}}(\omega) = C(\omega - \omega_0)^\alpha H(\omega - \omega_0)(\alpha + 1 \in \mathbb{R}^*_+) \rightarrow \tilde{\phi}(t) = C' e^{-i\omega_0 t} t^{1+\alpha}.
\]

(39)

Singularities with exponents \( \alpha \) are summarized in table 2. Thus, to extract the asymptotic behaviour of \( a_k(t) \), one needs to:

(i) enumerate the singularities appearing in \( \tilde{a}_k(\omega) \);
(ii) keep the strongest singularity, corresponding to the slowest decay; and
(iii) check if the symmetries of the problem induce a special cancellation.

Unless a cancellation occurs, this strongest singularity yields the decay exponent and asymptotic frequency of \( a_k(t) \). We illustrate this strategy on a spherical self-gravitating system in section 4.

A remark is in order: we study here the asymptotic behaviour of the \( a_k(t) \), which are the coefficients of the density and potential perturbation in the expansions (9) and (10). One might expect that the slowest decay for these coefficients yields the asymptotic decay of the density or potential perturbation at a given point in space. This is not necessarily true, as we have no information on the rate at which \( a_k \) reaches its asymptotic regime, and thus have no control on the series (9) and (10).

4. Spherically symmetric stellar system

4.1. Isochrone model

We present the isochrone model, which is a simple example of a spherically symmetric stellar system. The potential of this model is

\[
\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}},
\]

(40)
where \( G \) is the gravitational constant, \( M \) is the total mass of the system, and \( b \) is a parameter. A stationary state satisfying the self-consistent condition for the potential is known to be [11]:

\[
 f_0 = \frac{1}{\sqrt{2(2\pi)^3(GMb)^{1/2}}} \left[ 27 - 66\sqrt{\tilde{E}} + 320\tilde{E}^2 - 240\tilde{E}^3 + 64\tilde{E}^4 \\
 + 3(16\tilde{E}^2 + 28\tilde{E} - 9) \right] \frac{\sin^{-1}\sqrt{\tilde{E}}}{\sqrt{\tilde{E}(1-\tilde{E})}},
\]

(41)

where \( \tilde{E} = -Eb/GM \) and \( E \) is energy. Let \((r,\vartheta,\psi)\) be the spherical coordinates, and \((J_r,J_\vartheta,J_\psi)\) the conjugate angular momenta. We introduce new actions \((J_1,J_2,J_3),\) and denote the conjugate angle variables by \((\theta_1,\theta_2,\theta_3)\):

\[
 J_1 = J_\vartheta, \quad J_2 = J_\vartheta + |J_\psi|, \quad J_3 = J_r.
\]

(42)

The Hamiltonian of this model is then

\[
 H(J) = -\frac{(GM)^2}{2J_3 + \frac{1}{3}(J_2 + \sqrt{J_2^2 + 4GMb})^2}.
\]

(43)

The Hamiltonian is defined on the domain \((J_1,J_2,J_3) \in [-J_2,J_2] \times [0,\infty) \times [0,\infty).\) We remark that the Hamiltonian depends on \(J_2\) and \(J_3\), but not on \(J_1\). We hence omit the modes corresponding to \(J_1\), and write \(m = (m_2,m_3)\) for the mode vector and \(\Omega = (\Omega_1,\Omega_2)\) for the frequency vector. Thus, the isochrone model fits in the 2D analysis developed in this paper. Note that for consistency with standard notations, the actions \(J_2\) and \(J_3\) in the isochrone model correspond to \(J_1\) and \(J_2\) respectively in the general theory of section 3. Frequencies are given by

\[
 \Omega_1(J) = \frac{\partial H}{\partial J_3}(J) = \frac{(GM)^2}{[J_3 + \frac{1}{3}(J_2 + \sqrt{J_2^2 + 4GMb})]^{3/2}}.
\]

(44)

\[
 \Omega_2(J) = \frac{\partial H}{\partial J_2}(J) = \frac{1}{2} \left( 1 + \frac{J_2}{\sqrt{J_2^2 + 4GMb}} \right) \Omega_1(J).
\]

(45)

In a 3D self-gravitating system, the integrals in \(F(12)\) and \(G(13)\) should be performed over \(J_1, J_2\) and \(J_3\). Remembering that \(\Omega(J)\) does not depend on \(J_1\), the integrals are written as a linear combination of functions of the form

\[
 \varphi(\omega;m) = \int_0^\infty dJ_3 \int_0^\infty dJ_2 \frac{1}{m \cdot \Omega(J) - \omega} \int_{-J_2}^{J_2} h(m,J) dJ_1.
\]

(46)

Note that the function \(h\) in the above equation would be replaced by \(m \cdot \nabla_J f_0(J)\tilde{c}(m,J)\varphi(m,J)\) or \(g(m,J)\tilde{c}(m,J)\), according to the definitions of \(F_0(\omega)\) and \(G(\omega)\) in equations (12) and (13) respectively. Since \(\int_{-J_2}^{J_2} h dJ\) vanishes for \(J_2 = 0,\) a \(J_2\) factor can be factorized, and we can write \(\int_{-J_2}^{J_2} h(m,J) dJ_1 = J_2 \tilde{h}(m,J)\). Thus, the functions (46) fit in the framework of the abstract setting (15), and we apply the theory developed in section 3 with

\[
 \nu(J) = J_2 \tilde{h}(m,J), \quad \mu(J) = m \cdot \Omega(J).
\]

(47)

The \(J_2\) factor in \(\nu(J)\) will be of importance to determine the singularity associated with vertex, line and infinite singularities.
4.2. Singularities in the isochrone model

We can now list the singularities in the isochrone model.

- Vertex singularity at \((J^*_2, J^*_3) = (0, 0)\), except for the modes \(m\) satisfying \(m_2 + 2m_3 = 0\) or \(m_2 + 3m_3 = 0\). The former case results in a line singularity, and the latter case gives a special vertex singularity with \(\partial(m \cdot \Omega)/\partial J_2 = 0\) at the origin.
- Tangent singularity at \((J^*_2, 0)\) for the modes \(m\) satisfying \(-1 < m_3/m_2 < -1/3\). For such a mode, the singular point \(J^*_2\) is given by the solution to the equation
  \[
  \frac{\partial}{\partial J_2} \bigg|_{J_2=0} m \cdot \Omega = 0,
  \]
  which reads
  \[
  \frac{1}{2} \left( 1 + \frac{J_2}{\sqrt{J_2^2 + 4GMb}} \right) + \frac{2}{3} \frac{GMb}{J_2^2 + 4GMb} = \frac{m_3}{m_2}. \tag{49}
  \]
  We remark that the left-hand side of (49) is a decreasing function of \(J_2\), and its range is \((-1, -1/3)\) for \(J_2 \in (0, \infty)\). Thus \(m_3/m_2\) must be in the interval \((-1, -1/3)\). The origin, namely \(J^*_2 = 0\), results in the special vertex singularity for the modes satisfying \(m_2 + 3m_3 = 0\).
- Line singularity on \(J_1\) axis for the modes \(m\) satisfying \(m_2 = -2m_3\).
- A singularity at infinity appears for any mode \(m\).

No critical singularity appears in the isochrone model.

4.3. Advection equation corresponding to the isochrone model and theoretical predictions

Direct numerical simulations of the Vlasov equation are required for examining the theory developed in this paper. Such a check has been done in a 1D setting using \(N\)-body simulations [22]; however, the memory, accuracy and time frame required to perform similar simulations in two or more spatial dimensions make the task challenging. Instead, we examine the theory on much simpler linear advection equations.

Using the frequencies \(\Omega_2 (45)\) and \(\Omega_3 (44)\), we consider the advection equation

\[
\frac{\partial f_1}{\partial t} + \Omega_2(J) \frac{\partial f_1}{\partial \theta_2} + \Omega_3(J) \frac{\partial f_1}{\partial \theta_3} = 0. \tag{50}
\]

This advection equation is obtained by omitting the potential perturbation in the linearized Vlasov equation. The exact solution is

\[
f_1(\theta, J, t) = f_1(\theta - \Omega(J)t, J, t = 0). \tag{51}
\]

Let us assume that \(f_1\) does not depend on \(\theta_1\) and \(J_1\). We consider the temporal evolution of the expected value of \(A(\theta_2, \theta_3)\) defined by

\[
\langle A \rangle (t) = \int A(\theta) f_1(\theta, J, t) d\theta dJ. \tag{52}
\]

From the Fourier–Laplace transform of the advection equation (50), the Laplace transform of \(\langle A \rangle (t)\) is given by

\[
\overline{\langle A \rangle} (\omega) = \sum_m \int A(\theta) e^{im \theta} d\theta \int \frac{g(m, J)}{m \cdot \Omega(J) - \omega} dJ. \tag{53}
\]

where \(g(m, J)\) is the Fourier transform of \(f_1(\theta, J, t = 0)\).
On the other hand, performing the Fourier transform with respect to $\theta$, the exact solution \((51)\) gives
\[
\langle A\rangle(t) = \sum_m \int A(\theta) e^{im\theta} d\theta \int i\gamma(m, J) e^{-im\Omega t} dJ.
\]
Integrals over $\theta$ can be performed analytically, and hence our numerical task to compute $\langle A\rangle(t)$ is reduced to perform integrals over $J_2$ and $J_3$. Numerical integrations on 2D are possible with good accuracy.

We take as the initial state:
\[
f_1(\theta, J, t = 0) = \epsilon f_0(J) \cos(n_2\theta_2) \cos(n_3\theta_3), \quad |\epsilon| \ll 1,
\]
where $f_0(J)$ is the stationary state \((41)\). Note that this perturbation is also independent of $\theta_1$ and $J_1$. The Fourier transform $i\gamma(m, J)$ of $f_1(\theta, J, t = 0)$ is then
\[
i\gamma(m, J) = \begin{cases} \epsilon f_0(J)/4 & (m_2, m_3) = (\pm n_2, \pm n_3) \\ 0 & \text{otherwise}. \end{cases}
\]

The perturbation \((55)\) is singular on the $J_2 = 0$ and $J_3 = 0$ lines, since $f_0(J)$ does not vanish and the angle variables appear; however, from \((56)\), $g$ is regular on the whole $(J_2 \geq 0, J_3 \geq 0)$ domain. Thus, the analysis of section 3 applies to all types of singularities that cause algebraic dampings. We stress that we chose the singular perturbation \((55)\) to simplify the example, and that it is not essential to observe algebraic dampings for all types of singularities.

Looking at the Laplace transform \((53)\) and taking into account an extra $J_2$ factor for $v$ explained in \((47)\), we set
\[
v(J) \sim J_2 f_0(J).
\]

For the singularity at infinity, estimations of $v(J)$ and $\mu(J)$ are
\[
v(J) \sim (J_2 + J_3)^{-3}, \quad \mu(J) \sim (J_2 + J_3)^{-3},
\]
since $\Omega_2 \sim \Omega_3 \sim (J_2 + J_3)^{-3}, f_0 \sim \tilde{E}^{5/2}, \tilde{E} \sim (J_2 + J_3)^{-2}$. Using these estimations, the leading dampings are $t^{-3}$, $t^{-1.5}$, $t^{-2}$ and $t^{-2/3}$ for the vertex, tangent, line and infinite singularities respectively. We remark that the leading order of the function $v$ is $v \sim J_2$ around $J_2 = 0$ due to the $J_2$ factor of $v$ in \((57)\), thus the leading damping rates for the vertex and line singularities respectively are not $t^{-2}$ and $t^{-1}$, but $t^{-3}$ and $t^{-2}$. The dominant, i.e. slowest, decay is $t^{-2/3}$ with zero frequency. These singularities and associated dampings and frequencies are arranged in table 3.

In the next subsection we will observe the temporal evolution of the expected value of the observable $A(\theta) = \sin(n_2\theta_2 + n_3\theta_3)$. The mode $(n_2, n_3)$ corresponds to the mode of

| Singularity | Damping | Frequency | Condition for mode |
|-------------|---------|-----------|-------------------|
| Vertex      | $t^{-3}$| $\left(\frac{m_1^2 + m_3}{m_1^3}\right)$ | None |
| Tangent     | $t^{-1.5}$| $m_2\Omega_2(J_2', 0) + m_2\Omega_3(J_3', 0)$ | $-1 < m_2/m_3 < -1/3$ |
| Line        | $t^{-2}$ | 0         | $m_2 = -2m_3$     |
| Infinity    | $t^{-2/3}$| 0         | None              |
perturbation (55). For our interests, the prefactor of $\langle A \rangle_1(t)$ is not crucial; therefore we redefine

$$\langle A \rangle_1(t) = \int_0^\infty \int_0^\infty J_2 f_0(J) \sin((n_2 \Omega_2(J) + n_3 \Omega_3(J))t) dJ_2 dJ_3$$

in the following. For this sine observable, the leading vertex, tangent and infinite singularities survive, but the leading line singularity is expected to be cancelled looking at table 2: the observable includes modes $(n_2, n_3)$ and $(-n_2, -n_3)$, and $a = 1$. From the line singularity, therefore, the second leading damping $t^{-\frac{3}{2}}$ should arise instead of the leading damping $t^{-2}$. However, since the line singularity is never dominant, this cancellation does not affect the following discussion.

4.4. Numerical check

We choose three modes: $(n_2, n_3) = (1, 1), (2, -1)$ and $(3, -2)$. The mode $(1, 1)$ has vertex and infinite singularities, $(2, -1)$ has all four singularities, and $(3, -2)$ has all four except the line singularity. To perform the numerical integration of (59), we set the parameters as $G = M = b = 1$. The exact solutions (59) are shown in figure 3.

The mode $(1, 1)$ shows $t^{-\frac{3}{2}}$ damping without oscillation as the theory predicted. This mode also has a vertex singularity, which gives a non-zero frequency, but no oscillation is observed in the asymptotic time region: the vertex singularity contribution is expected to damp as $t^{-3}$, which may be too fast to be visible at large times.

The modes $(2, -1)$ and $(3, -2)$ asymptotically damp as $t^{-\frac{3}{2}}$ as predicted, but they have oscillations. At variance with the mode $(1, 1)$, both of these modes have a tangent singularity, which gives a rather slow damping $t^{-1.5}$ with non-zero frequency. To confirm that there is a contribution from the tangent singularity, we show the power spectra of $\langle A \rangle_1(t)$ in figure 4. The peaks of the power spectra are in good agreement with the theoretically predicted frequency $|m_2 \Omega_2(J^*_2, 0) + m_3 \Omega_3(J^*_2, 0)|$, where $J^*_2$ is the solution to (49).
As mentioned above, to study the true Vlasov equation, we should also estimate the $c_k(m, J)$ functions, which depend on the biorthogonal functions $u_k(q)$.

5. Conclusion

When a stable inhomogeneous stationary state of a Vlasov equation is perturbed, the asymptotic damping of the perturbation is algebraic, no matter how regular the stationary state and perturbation are. The damping rate and frequency are controlled by the singularities on the real axis of the Fourier–Laplace transform of the perturbation. In this paper, in systems with two spatial dimensions, we have classified these singularities into vertex, tangent, critical, line and infinite singularities, and have given the damping rate and frequency in each case. This classification is also valid for 3D spherically symmetric stationary states, since they reduce to the 2D case thanks to symmetry. The resulting picture is much richer than in one spatial dimension [22]. We have illustrated the theory on an advection equation associated with the isochrone model, a simple model for self-gravitating systems: tests in these models show that the singularity analysis captures very well the observed damping.

The main goal of this theory is a better understanding of the asymptotic relaxation of self-gravitating systems. To achieve this goal, a detailed study of the classical expansion of a perturbation on a biorthogonal basis should be coupled to the singularity analysis we have presented here. From a practical point of view, it is important to determine in which situations (if any) this slow damping can be of physical relevance. From a more fundamental perspective, the asymptotic behaviour of the linearized Vlasov equation is a necessary step to understand the full nonlinear equation, and to examine the validity of kinetic equations such as Landau and Lenard–Balescu equations, describing particles in the collisional regime.
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Appendix A. Analytic continuation

The starting point is the following formula for $\phi(z)$, valid for $\text{Im}(z) > 0$:

$$\phi(z) = \int_{D \subset \mathbb{R}^2} \frac{\nu(J_1, J_2)}{\mu(J_1, J_2) - z}.$$

We want to understand the regularity of $\phi(z)$ close to $z = x_0 \in \mathbb{R}$.

Assuming that $x_0$ is not a critical value for the function $\mu$ (condition i), the equation $\mu(J_1, J_2) = x_0$ defines one or several curves on the $(J_1, J_2)$ plane. We treat here the case with one curve, but the argument carries over easily to the case of several curves.

It is then possible to use a new set of variables $K_1, K_2$ fulfilling the two conditions:

(a) $K_1 = \mu(J_1, J_2) - x_0$;
(b) $K_2$ is such that the Jacobian $dK_1/dJ_1 dJ_2$ never vanishes. We do not need to further specify $K_2$.

Then, with a suitable definition of $\tilde{\nu}$:

$$\phi(z) = \int_{D \subset \mathbb{R}^2} \frac{\tilde{\nu}(K_1, K_2)}{K_1 - (z - x_0)} dK_1 dK_2.$$

Integrating over $K_2$ first:

$$\phi(z) = \int_{K_1^{(M)}}^{K_1^{(m)}} \frac{dK_1}{K_1 - (z - x_0)} \int_{K_2^{(m)}}^{K_2^{(M)}} (K_1) \tilde{\nu}(K_1, K_2) dK_2.$$

If the boundary of the domain $D$ is regular around its intersection with the curve $K_1 = 0$ (condition ii) (see figure A1, point A), then $K_2^{(M)}(K_1)$ and $K_2^{(m)}(K_1)$ are regular functions of $K_1$.

![Figure A1. The boundary of $D$ is regular close to point A, its intersection with the curve $\mu(J_1, J_2) = x_0$, that is $K_1 = 0$.](image-url)
Appendix B. Singularity computation

We first show that the function
\[ \phi(x) = \lim_{y \to 0^+} \int_{-c}^{c} \frac{u^\alpha}{u - (x + iy)} \, du, \quad \alpha \in \mathbb{Z}_+ \]  
(B.1)
does not have any singularity. Then we compute the singularities of the functions
\[ \phi(x) = \lim_{y \to 0^+} \int_{0}^{c} \frac{u^\alpha}{u - (x + iy)} \, du, \quad \alpha + 1 \in \mathbb{R}_+ \]  
(B.2)
and
\[ \phi(x) = \lim_{y \to 0^+} \int_{0}^{c} \frac{u^\alpha \ln|u|}{u - (x + iy)} \, du, \quad \alpha \in \mathbb{Z}_+ \]  
(B.3)
where \( \mathbb{Z}_+ \) and \( \mathbb{R}_+ \) are sets of non-negative integers and positive real values respectively. The constant \( c \) is assumed to be positive in the above three functions. All the singularities found in this paper reduce to a computation of the singularity close to \( z = x_0 \), unless \( K^{(m)}_1 = 0 \) or \( K^{(d)}_1 = 0 \) (condition iii).

Thus, under conditions (i), (ii) and (iii), \( \psi(z) \) is not singular at \( z = x_0 \). Breaking one of these conditions yields a critical, vertex and tangent singularity respectively.

### B.1. No singularity of the function (B.1)

We start from the function (B.1)
\[ \phi(x) = \lim_{y \to 0^+} \left[ \int_{-c}^{c} \frac{u^\alpha}{(u - x)^2 + y^2} \, du + i y \int_{-c}^{c} \frac{u^\alpha}{(u - x)^2 + y^2} \, du \right]. \]  
(B.7)
We denote the real and the imaginary parts of \( \phi(x) \) by \( \phi_R(x) \) and \( \phi_I(x) \) respectively. The imaginary part \( \phi_I(x) \) can be computed using the change of variable \( s = (u - x)/y \):
\[ \phi_I(x) = \pi x^\alpha \left( H(x+c) - H(x-c) \right), \]  
(B.8)
where \( H \) is the Heaviside step function. Thus \( \phi_I(x) \) has no singularity around \( x = 0 \).

The real part is simply:
\[ \phi_R(x) = PV \int_{-c}^{c} \frac{x^\alpha}{u-x} \, du + \sum_{l=0}^{a-1} C_{l,\alpha} x^l \int_{-c}^{c} (u-x)^{a-l-1} \, du \]  
(B.9)
where \( PV \) denotes the principal value. The sum on the right-hand side is clearly regular, and the first term gives:

\[
PV \int_{-c}^{c} \frac{x^\alpha}{u - x} \, du = x^\alpha (\ln |c - x| - \ln |-c - x|),
\]

so that no singularity appears around \( x = 0 \).

**B.2. Singularity of the function (B.2)**

We study the function (B.2), which differs from the function (B.1) in the lower bound of the integral. The imaginary part, \( \phi_I \), is then:

\[
\phi_I(x) = \pi x^\alpha (H(x) - H(x - c)),
\]

and hence the singularity of \( \phi_I \) around \( x = 0 \) is

\[
\phi_I^{\text{sing}}(x) = \pi x^\alpha H(x).
\]

Ignoring the regular sum, the real part is also directly obtained using equation (B.10):

\[
\phi_R(x) = x^\alpha (\ln |c - x| - \ln | - x|)
\]

for \( \alpha \in \mathbb{Z}_+ \). Thus, the singularity of \( \phi_R \) around \( x = 0 \) is

\[
\phi_R^{\text{sing}}(x) = -x^\alpha \ln |x|, \quad \alpha \in \mathbb{Z}_+.
\]

Combining real and imaginary parts gives (B.4).

The final step for the function (B.2) is to investigate the real part for \( \alpha + 1 \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \).

We divide \( \phi_R(x) \) in two parts:

\[
\phi_R(x) = \lim_{\epsilon \to 0} \left( \int_0^{\epsilon} \frac{u^\alpha}{u - x} \, du + \int_{|x| + \epsilon}^{c} \frac{u^\alpha}{u - x} \, du \right).
\]

This division is just the definition of the principal value for \( x > 0 \), and is also valid for \( x < 0 \). We then use the expansions, for \( u < |x| \) and \( u > |x| \) respectively:

\[
\frac{1}{u - x} = -\frac{1}{x} \sum_{k=0}^{\infty} \left( \frac{1}{x} \right)^k; \quad \frac{1}{u - x} = \frac{1}{u} \sum_{k=0}^{\infty} \left( \frac{x}{u} \right)^k.
\]

Substituting the above expressions into (B.15) and remembering \( \alpha + 1 \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \), we have

\[
\phi_R(x) = \begin{cases} 
-x^\alpha \sum_{k=0}^{\infty} \left( \frac{1}{\alpha + k + 1} + \frac{1}{\alpha - k} \right) + \sum_{k=0}^{\infty} x^k \frac{e^{\alpha - k}}{\alpha - k} & x > 0, \\
(-x)^\alpha \sum_{k=0}^{\infty} \left( \frac{(-1)^k}{\alpha + k + 1} + \frac{(-1)^{k+1}}{\alpha - k} \right) + \sum_{k=0}^{\infty} x^k \frac{e^{\alpha - k}}{\alpha - k} & x < 0.
\end{cases}
\]

Note that all series converge. The second series for \( x > 0 \) and \( x < 0 \) exactly coincide, hence they do not contribute any singularity. The singularity comes from the first series and is:

\[
\phi_R^{\text{sing}}(x) = C_1 x^\alpha H(x) + C_2 (-x)^\alpha H(-x), \quad \alpha + 1 \in \mathbb{R}_+ \setminus \mathbb{Z}_+.
\]

Consequently, we have proved (B.5) for the function (B.2).
B.3. Singularity of the function (B.3)

The imaginary part of the function (B.3), denoted by \( \phi_I(x) \), is

\[
\phi_I(x) = \pi x^\alpha \ln |x|(H(x + c) - H(x - c)),
\]

and the singularity around \( x = 0 \) is

\[
\phi_{I,sing}(x) = \pi x^\alpha \ln |x|.
\]

To compute the real part, we rewrite the integrand as was done in (B.9):

\[
\phi_R(x) = PV \int_{-c}^{c} \left[ \frac{x^\alpha}{u - x} + \sum_{l=0}^{a-1} a C_l (u - x)^{a-l-1} x^l \right] \ln |u| \, du.
\]

We remark that the integral

\[
\int_{-c}^{c} u^n \ln |u| \, du \quad (n \in \mathbb{Z}_+)
\]

can be performed in the sense of an improper integral, and converges to a finite value. Thus, concentrating on the singularity, the real part is reduced to

\[
\phi_R(x) = x^\alpha \lim_{\epsilon \to 0} \left( \int_{|x| - \epsilon}^{\epsilon} + \int_{|x| + \epsilon}^{\epsilon} + \int_{|x| - \epsilon}^{\epsilon} \right) \ln |u| \, du.
\]

Using expansion (B.16), we obtain

\[
\int_{|x| - \epsilon}^{\epsilon} \ln |u| \, du = \sum_{k=0}^{\infty} \frac{-(|x| - \epsilon)^{2k+1}}{2k + 1} \ln |x| - \epsilon = \frac{-(|x| - \epsilon)^{2k+1}}{(2k + 1)^2}
\]

and

\[
\left( \int_{|x| - \epsilon}^{\epsilon} + \int_{|x| + \epsilon}^{\epsilon} \right) \frac{\ln |u|}{u - x} \, du \approx \sum_{k=0}^{\infty} \frac{2^{2k+1}}{2k + 1} \left( \frac{|x| - \epsilon)^{2k+1}}{(2k + 1)^2} \right),
\]

where we have omitted regular functions in (B.25). The logarithmic terms are cancelled by adding (B.24) and (B.25) and taking the limit \( \epsilon \to 0 \); hence the singularity of \( \phi_R(x) \) is:

\[
\phi_{R,sing}(x) = C x^\alpha H(x).
\]

The imaginary part (B.20) and the real part (B.26) prove (B.6) for the function (B.3).

B.4. Relative sign between modes \( m \) and \( -m \)

Coming back to the functions \( F \) and \( G \), we are interested in singularities of the function

\[
\varphi(z; m) = \int \frac{v(J)}{m \cdot \Omega(J) - z} \, dJ.
\]

To discuss a possible cancellation between the modes \( m \) and \( -m \), we show a relation between the above function and

\[
\varphi(z; -m) = \int \frac{v(J)}{-m \cdot \Omega(J) - z} \, dJ.
\]

We note that the numerator \( v(J) \) may also depend on \( m \); in this case one would have to discuss the sign of the numerator separately; we ignore this dependence in this section.
Remembering that the singularity of the function (B.27) results in one of the two types of functions (B.2) and (B.3), we consider the relation between
\[ \phi^+_{\lambda_0}(x) = \lim_{y \to 0^+} \int \frac{h(u)}{u - (x - x_0) - iy} \, du \]  
(B.29)
and
\[ \phi^-_{\lambda_0}(x) = \lim_{y \to 0^+} \int \frac{h(u)}{u + (x + x_0) + iy} \, du \]  
(B.30)
where \( h(u) \) is a \( u^\alpha (\alpha + 1 \in \mathbb{R}^+_0) \) or \( u^\alpha \ln |u| (\alpha \in \mathbb{Z}_+) \), and \( x_0 = m \cdot \Omega(J) \in \mathbb{R} \) with a special point \( J = J' \), \( \phi^+_{\lambda_0} \) is obtained by changing the signs of the prefactor, \( x \) and \( y \) in \( \phi^+_{\lambda_0} \), thus the singular points of \( \phi^+_{\lambda_0}(x) \) and \( \phi^-_{\lambda_0}(x) \) are \( x_0 \) and \(-x_0 \) respectively. If \( x_0 \neq 0 \), no cancellation occurs in general due to the different singular points. If \( x_0 = 0 \), a cancellation may occur depending on the relative sign of the singularities for the modes \( m \) and \(-m \). The change of sign of \( y \) implies the change of sign of the imaginary part, and we use \( H(-x) = 1 - H(x) \).

Hence the relation between the singularities of \( \phi^+_{\lambda_0}(x) \) and \( \phi^+_{\lambda_0}(x) \) is:
\[ \phi^-_{\lambda_0}(x) = \frac{(-1)^{1+\alpha} \phi^+_{\lambda_0}(x), \quad \alpha \in \mathbb{Z}_+}{(-1)^{\alpha} \phi^+_{\lambda_0}(x), \quad \alpha \in \mathbb{Z}_+} \]  
(B.31)
for the singularity (B.4), and
\[ \phi^-_{\lambda_0}(x) = \frac{(-1)^{1+\alpha} \phi^+_{\lambda_0}(x), \quad \alpha \in \mathbb{Z}_+}{(-1)^{\alpha} \phi^+_{\lambda_0}(x), \quad \alpha \in \mathbb{Z}_+} \]  
(B.32)
for the singularity (B.6). The relations (B.31) and (B.32) imply that, for instance, singularities of \( \psi(z; m) + \psi(z; -m) \) and of \( \psi(z; m) - \psi(z; -m) \) cancel respectively if \( \alpha \) is 0 or positive even and \( x_0 = 0 \). There is no simple relation such as (B.31) for singularity (B.5), thus no cancellation is expected in general.

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