UNIQUENESS OF EQUILIBRIUM STATES FOR LORENZ ATTRACTORS IN ANY DIMENSION

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Abstract. In this note we consider the thermodynamic formalism for Lorenz attractors of flows in any dimension. Under a mild condition on the Hölder continuous potential function \( \phi \), we prove that for an open and dense subset of \( C^1 \) vector fields, every Lorenz attractor supports a unique equilibrium state. In particular, we obtain the uniqueness for the measure of maximal entropy.

1. Introduction and statement

Ever since its discovery in 1963 by Lorenz [10], the Lorenz attractor has been playing a central role in the research of singular flows, i.e., flows generated by smooth vector fields with singularities. The investigation of their statistical and topological properties has been successful in dimension three, thanks to the geometric Lorenz model [6]; however, results in higher dimensions are surprisingly lacking.

This note is devoted to the study of the statistical properties of Lorenz attractors in any dimension. In particular, we are concerned with equilibrium states for Hölder continuous potential functions \( \phi : M \to \mathbb{R} \) on Lorenz attractors.

In what follows, \( M \) is a compact Riemannian manifold without boundary, and \( \mathcal{X}^1(M) \) denotes the space of \( C^1 \) vector fields on \( M \) endowed with \( C^1 \) topology. Given \( X \in \mathcal{X}^1(M) \), a singularity is a point \( q \in M \) where \( X(q) = 0 \). The set of singularities of \( X \) is denoted by \( \text{Sing}(X) \).

An \( X \)-invariant set \( \Lambda \) is called a Lorenz attractor if it satisfies the following conditions (\( \varphi_t \) denotes the flow generated by \( X \)):

1. \( \varphi_t | \Lambda \) is sectional-hyperbolic, meaning that there exists an \( D\varphi_t \)-invariant dominated splitting \( E^s \oplus E^{cu} \) such that \( D\varphi_t | E^s \) is uniformly contracting, and \( D\varphi_t | E^{cu} \) is volume expanding: there exists \( C > 0, \lambda > 0 \) such that
   \[
   |\det D\phi_t(x)|_{V_x} \geq Ce^\lambda t
   \]
   for every \( x \in M \), every subspace \( V_x \subset E^{cu} \) with \( \dim V_x \geq 2 \), and every \( t > 0 \);
2. \( \varphi_t | \Lambda \) is transitive: there exists a point \( x \in \Lambda \) whose orbit is dense in \( \Lambda \);
3. there exists a neighborhood \( U \supset \Lambda \) such that for every \( x \in \Lambda \) one has \( \omega(x) \subset \Lambda \).

Date: January 19, 2022.

This research has been supported [in part] by CAPES - Finance Code 001 and CNPq-grants. MJP was partially supported by by Grant Cientista do Nosso Estado (FAPERJ).
Recall that given such a function \( \phi \), the variational principle for flows states that
\[
P(\phi) = \sup_{\mu \in \mathcal{M}(X)} \left( h_{\mu}(\varphi_1) + \int \phi \, d\mu \right),
\]
where \( P(\phi) \) is the topological pressure of \( \phi \), \( \mathcal{M}(X) \) is the space of \( X \)-invariant probability measures, and \( h_{\mu}(\varphi_1) \) is the measure-theoretic entropy of the time-one map \( \varphi_1 \). An equilibrium state is an invariant measure \( \mu \in \mathcal{M}(X) \) that achieves the supremum. As a special case, the topological pressure of the constant function \( \phi \equiv 0 \) is the topological entropy of the flow, and any equilibrium state in this case is called a measure of maximal entropy.

It has been proven in [11] that for every sectional-hyperbolic set \( \Lambda \) of a flow \( \varphi_t \), \( \varphi_t \mid \Lambda \) is entropy expansive. Due to the celebrated work of Bowen [1], the metric entropy, as a function on \( \mathcal{M}(X) \), is upper semi-continuous. Consequently, every continuous function \( \phi : \mathcal{M} \to \mathbb{R} \) admits an equilibrium state \( \mu_\phi \). Our main theorem below states that under certain mild conditions on \( X \) and \( \phi \), this equilibrium state is unique.

**Theorem A.** There exists an open and dense subset \( \mathcal{R} \subset \mathcal{X}^1(M) \), such that for every \( X \in \mathcal{R} \) and every Lorenz attractor \( \Lambda \) of \( X \), let \( \phi : M \to \mathbb{R} \) be a Hölder continuous function such that
\[
\phi(\sigma) < P(\phi), \forall \sigma \in \text{Sing}(X \mid \Lambda).
\]
Then there exists a unique equilibrium state \( \mu \) supported in \( \Lambda \).

As a corollary, we obtain

**Corollary B.** There exists an open and dense subset \( \mathcal{R} \subset \mathcal{X}^1(M) \), such that for every \( X \in \mathcal{R} \) and every Lorenz attractor \( \Lambda \) of \( X \), there exists a unique measure of maximal entropy for \( X \mid \Lambda \).

The corollary easily follows from Theorem A applied to the constant function \( \phi \equiv 0 \) (note that every Lorenz attractor has positive topological entropy, by [11, Theorem C]).

2. A revised Climenhaga-Thompson criterion for the uniqueness of equilibrium states

The main tool for the proof of Theorem A is the following theorem from [12], which improves the Climenhaga-Thompson criterion [2] by weakening the specification assumption.

We define
\[
L_X = \max_{t \in [0,1]} L_{\varphi_t} \geq 1
\]
where \( L_{\varphi_t} \) is the Lipschitz constant of \( \varphi_t \). Writing
\[
\Gamma_\varepsilon(x) = \{ y \in M : d(\varphi_t(x), \varphi_t(y)) \leq \varepsilon \text{ for all } t \in \mathbb{R} \},
\]
for the bi-infinite Bowen ball at \( x \), a vector field \( X \) is said to be almost expansive at scale \( \varepsilon > 0 \), if the set
\[
\text{Exp}_\varepsilon(X) := \{ x \in M : \Gamma_\varepsilon(x) \subset \varphi_{[-s,s]}(x) \text{ for some } s > 0 \}
\]
has full probability: for any $\mu \in \mathcal{M}(\mathcal{M})$ one has $\mu(\text{Exp}_\varepsilon(X)) = 1$. Clearly, almost expansivity is weaker than expansivity; however, it also implies entropy expansivity at the same scale [9].

Below we will identify a pair $(x, t) \in \mathcal{M} \times \mathbb{R}^+$ with the orbit segment $\{\varphi_s(x) : 0 \leq s \leq t \}$. Given a collection of orbit segments $\mathcal{D} \subset \mathcal{M} \times \mathbb{R}^+$, a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ of $\mathcal{D}$ consists of three collections $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset \mathcal{M} \times \mathbb{R}^+$ and three functions $p, g, s : \mathcal{D} \to \mathbb{R}^+$ such that for every $(x, t) \in \mathcal{D}$, the values $p = p(x, t), g = g(x, t)$ and $s = s(x, t)$ satisfy $t = p + g + s$, and

\begin{equation}
(x, p) \in \mathcal{P}, \quad (\varphi_p(x), g) \in \mathcal{G}, \quad (\varphi_{p+g}(x), s) \in \mathcal{S}.
\end{equation}

Now we are ready to state the main result of [12].

**Theorem 2.1.** Let $(\varphi_t)_{t \in \mathbb{R}}$ be a Lipschitz continuous flow on a compact metric space $\mathcal{M}$, and $\phi : \mathcal{M} \to \mathbb{R}$ a continuous function. Suppose that there exist $\varepsilon > 0, \delta > 0$ with $\varepsilon \geq 1000L_X\delta$ such that $X$ is almost expansive at scale $\varepsilon$, and there exists $\mathcal{D} \subset \mathcal{M} \times \mathbb{R}^+$ which admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:

(I) $\mathcal{G}$ has tail (W)-specification at scale $\delta$;
(II) $\phi$ has the Bowen property at scale $\varepsilon$ on $\mathcal{G}$;
(III) $P(\mathcal{D}^c \cup [\mathcal{P}] \cup [\mathcal{S}], \phi, \delta, \varepsilon) < P(\phi)$.

Then there exists a unique equilibrium state for the potential $\phi$.

We invite the interested readers to [2, 12] for the precise definition of the involved terms.

3. Structure of the proof

From now on, $\Lambda$ is a Lorenz attractor unless otherwise specified. We will assume that all the singularities of $X \mid \Lambda$ are hyperbolic; this is clearly an open and dense condition among $C^1$ vector fields.

We will prove Theorem A by verifying the assumptions of Theorem 2.1. We first remark that Theorem 2.1 can be applied, with only minor modification if necessary, to $X \mid \Lambda$. Also note that the almost expansivity at some given scale $\delta_0 > 0$ has been proven in [12, Theorem F]. This implies almost expansivity at any smaller scale. We will first construct two orbit segment collections $\mathcal{G}_\delta$ and $\mathcal{G}_0$ on which the specification and the Bowen property hold, respectively (Theorem 3.1 [3.3]). $\mathcal{G}_\delta$ and $\mathcal{G}_0$ must be taken large enough, such that one could find $\mathcal{G} \subset \mathcal{G}_\delta \cap \mathcal{G}_0$ with “large topological pressure” (Theorem 3.4).

3.1. The weak specification property. The next theorem establishes the specification property on a collection of orbit segments.

**Theorem 3.1.** For every $\delta > 0$, there exists an orbit collection $\mathcal{G}_\delta \subset \mathcal{M} \times \mathbb{R}^+$ such that $\mathcal{G}_\delta$ has the tail (W)-specification property at scale $\delta$.

We remark that the choice of the orbit segment $\mathcal{G}_\delta$ depends on the scale $\delta$. It is worth noting that in the original Climenhaga-Thompson criterion [2, Theorem 2.9],
the specification assumption is made on $\mathcal{G}^M$ for every $M \geq 0$ (at the same scale $\delta$), where $\mathcal{G}^M$ is defined as (see (2.1))

$$\mathcal{G}^M = \{(x, t) \in \mathcal{D} : p \vee s \leq M\}.$$ 

Thinking of $\mathcal{G}$ as the ‘good core’ of $\mathcal{D}$, it is clear that $\mathcal{G}^M \subset \mathcal{G}^{M'}$ whenever $M < M'$, and $\bigcup_{M \geq 0} \mathcal{G}^M = \mathcal{D}$.

In the subsequent applications of [2], this assumption is usually replaced by the assumption that $G$ has the tail (W)-specification property at all scales. Unfortunately in our case, we do not know whether $\cap_{\delta > 0} G_\delta$ carries large pressure. To deal with this issue, in [12] we relaxed the specification condition of [2] and assume that it only holds on the ‘good core’ $G$ (Theorem 2.1 (I)).

The proof of Theorem 3.1 calls for the following topological description for Lorenz attractor, which strengthens the main result of [4].

An invariant set $\Lambda$ is called Lyapunov stable if for every neighborhood $U$ of $\Lambda$, there exists a neighborhood $V$ of $\Lambda$, such that for every $x \in V$ we have $\varphi_t(x) \in U, \forall t > 0$.

**Theorem 3.2.** There exists a $C^1$ residual subset $R \subset \mathcal{P}^1(M)$, such that for every $X \in R$ and every sectional-hyperbolic, Lyapunov stable chain recurrent class $\Lambda$ of $X$, there exists a neighborhood $U$ of $X$ and a neighborhood $U'$ of $\Lambda$ that satisfy the following properties:

1. every $Y \in U$ has a unique chain recurrent class $C_Y \subset U$ which is a sectional-hyperbolic attractor;
2. $C_Y$ is the homoclinic class of some hyperbolic periodic orbit $p_Y \in C_Y$; in particular, $W^s(\text{Orb}(p_Y))$ is dense in $C_Y$;
3. for every point $x \in C_Y \setminus \text{Sing}(Y)$, $F^s_Y(x) \cap W^u_Y(p_Y) \neq \emptyset$.

Here $F^s_Y(x)$ is the leaf of the stable foliation $F^s_Y$ for the flow $Y$ that contains $x$. It is well known (see, for instance, [11, Lemma 3.10]) that $\Lambda \cap (W^{ss}(\sigma) \setminus \{\sigma\}) = \emptyset$ for every sectional-hyperbolic set $\Lambda$ and every $\sigma \in \Lambda \cap \text{Sing}(X)$, so (3) is optimal.

### 3.2. The Bowen property

We prove the following theorem:

**Theorem 3.3.** There exists $\varepsilon_0 > 0$ and a collection of orbit segments $\mathcal{G}_0 \subset M \times \mathbb{R}^+$, such that $\phi$ has the Bowen property on $\mathcal{G}_0$ at scale $\varepsilon_0$ (and consequently, at every smaller scale).

The proof heavily relies on the scaled linear Poincaré flow of Liao [8]. We write

$$B_\beta(x, t) = \bigcup_{s \in [0, t]} N_{X, \varphi_s(x)}(\beta | X(\varphi_s(x)))$$

where $N_{X, y}(r)$ is the image of the $r$-neighborhood of $0_y$ in the normal plane $N_{X, y} = (X(y))^\perp \subset T_yM$ under the exponential map $\exp_y$. One should think of $B_\beta(x, t)$ as

$^1$Recall that all the singularities in a sectional-hyperbolic set are Lorenz-like, meaning that $E^s_\sigma = E^s_{\sigma} \oplus E^c_{\sigma}$ with $\dim E^c_{\sigma} = 1$; furthermore, the bundle $E^u$ from the dominated splitting $E^s \oplus E^u$ on $\Lambda$ matches the bundle $E^s_{\sigma}$ at the singularity $\sigma$, and $F^c_\sigma(\sigma)$ coincides with the strong stable manifold $W^{ss}(\sigma)$ tangent to $E^s_{\sigma}$ at $\sigma$. 


a tubular neighborhood of the orbit segment \((x, t)\) whose size is proportional to the flow speed \(|X(\varphi_s(t))|\). We prove that there exists \(\beta > 0\) such that every \((x, t)\) \(\in G_0\) satisfies
\[
\varphi_s(y) \in B_\beta(x, t), \forall s \in [0, t] \text{ whenever } y \in B_{\varepsilon_0}(x)
\]
where \(B_{\varepsilon_0}(x)\) is the \((\varepsilon_0, t)\)-Bowen ball of \(x\). Then the Bowen property follows whenever the terminal point \(\varphi_t(x)\) is a \(\lambda_0\)-quasi hyperbolic point for some prescribed \(\lambda_0 \in (0, 1)\) (See [7] and [11, Definition 11]; also note that vectors in the stable bundle \(E^s\) are uniformly contracted by the scaled linear Poincaré flow \(\psi_t\)).

3.3. The decomposition and the pressure gap. Let \(\varepsilon_0 > 0\) and \(G_0 \subset M \times \mathbb{R}^+\) be given by Theorem 3.3, and \(\delta_0\) be the scale of the almost expansivity given by [11, Theorem F]. We take \(\varepsilon < \min\{\varepsilon_0, \delta_0\}\) small enough\(^2\), and fix \(\delta < \varepsilon/(1000L_X)\). By theorem 3.1, we obtain the orbit collection \(G_\delta\) which has the specification property. The next theorem shows the existence of an orbit collection \(D\) with a decomposition.

**Theorem 3.4.** There exists an orbit segments collection \(D \subset M \times \mathbb{R}^+\) which admits a decomposition \((\mathcal{P}, \mathcal{G}, \mathcal{S})\) with the following properties:

1. \(\mathcal{G} \subset \mathcal{G}_0 \cap \mathcal{G}_\delta\);
2. orbit segments in \(D^c\), \(\mathcal{P}\) and \(\mathcal{S}\) must spend a significant proportional of their time in a prescribed, small neighborhood \(U_{\text{Sing}}\) of \(\text{Sing}(X) \cap \Lambda\).

Property (1) shows that \(\mathcal{G}\) has the Bowen property at scale \(\varepsilon\) (Theorem 3.3) and has the tail (W)-specification property at scale \(\delta\) (Theorem 3.1).

To show that the pressure of \(\mathcal{D}^c \cup \mathcal{P} \cup \mathcal{S}\) is strictly smaller than \(P(\phi)\), we used a modified version of the variational principle which shows that the topological pressure of \(\mathcal{D}^c \cup \mathcal{P} \cup \mathcal{S}\) is bounded from above by the metric pressure \(P_\mu(\phi) := h_\mu(X) + \int \phi \, d\mu\), where \(\mu\) is a limit point of some convex combination of empirical measures supported on orbit segments in \(\mathcal{D}^c \cup \mathcal{P} \cup \mathcal{S}\). It follows from Property (2) that \(\mu(U_{\text{Sing}}) > a > 0\) for some constant \(a\). Using a semi-continuity argument which is made possible by [11, Theorem A], we show that there exists \(b > 0\) such that
\[
P_\mu(\phi) < P(\phi) - b.
\]
This verifies Assumption (III) of Theorem 2.1 and finishes the proof of Theorem A.

**References**

[1] R. Bowen. Entropy expansive maps. *Trans. Amer. Math. Soc.*, 164:323–331, 1972.
[2] V. Climenhaga and D. Thompson. Unique Equilibrium states for flows and homeomorphisms with non-uniform structure. *Adv. Math.* 303:745–799, 2016.
[3] C. Conley. Isolated invariant sets and the Morse index. CBMS Regional Conference Series in Mathematics, 38. American Mathematical Society, Providence, R.I., 1978.
[4] S. Crovisier and D. Yang. Robust transitivity of singular hyperbolic attractors. *Math. Z.* 298 (2021), no. 1-2, 469–488.
[5] S. Gan, F. Yang, J. Yang and R. Zheng. Statistical properties of physical-like measures, *Nonlinearity*, 34 (2021), 1014–1029.
[6] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Publ. Math. IHES*, 50:59–72, 1979.

\(^2\)The smallness of \(\varepsilon\) is needed to control the pressure function \(P(\cdot, \phi, \delta, \varepsilon)\) with respect to its second scale.
[7] S. T. Liao. On \((\eta, d)\)-contractible orbits of vector fields. *Systems Science and Mathematical Sciences.,* 2:193–227, 1989.

[8] S. T. Liao. The qualitative theory of differential dynamical systems. Science Press, 1996.

[9] G. Liao, M. Viana and J. Yang. The entropy conjecture for diffeomorphisms away from tangencies. *J. Eur. Math. Soc.,* 15:2043–2060, 2013.

[10] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmosph. Sci.,* 20:130–141, 1963.

[11] M. J. Pacifico, F. Yang and J. Yang, Entropy theory for sectional hyperbolic flows. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire,* 38 (2021), 1001–1030.

[12] M. J. Pacifico, F. Yang and J. Yang. Existence and uniqueness of equilibrium states for systems with specification at a fixed scale. Preprint.

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