Bose–Einstein condensates in traps of time–dependent topology

Th. Busch1, J.R. Anglin2, and W.H. Zurek3

1Institute of Physics and Astronomy, Aarhus University, Ny Munkegade, DK–8000 Aarhus C, Denmark
2Center for Ultracold Atoms, MIT 26–2, 77 Massachusetts Avenue, Cambridge, MA 02139
3T–6 (Theoretical Astrophysics), MS B288, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

(Dated: March 22, 2022)

Superfluid phenomena can be explained in terms of the topologies of the order parameter and of the confining vessel. For example, currents in a toroidal vessel can be characterized by a discrete and conserved quantity, the winding number. In trapped Bose–Einstein condensates, the topology of the trap can be characterized by the topology of the Thomas–Fermi surface of its N–particle ground state. This can be altered during an experiment, so that a toroidal trap may deform into a more spherical shape, allowing an initially persistent current to decay into singly–quantized vortices. We investigate such a procedure numerically, and confirm that the Thomas–Fermi prescription for the trap topology gives an accurate picture of vortex formation.

The intriguing beauty of the concept of topological quantum numbers has ever since its introduction into physics drawn great attention to states that are topologically nontrivial [1]. In a system that has undergone a second order phase transition these numbers characterize the order parameter field. For a Bose–condensed system the order parameter is the phase of the macroscopic wave–function, and vortex line defects, as well as persistent currents in multiply–connected vessels, are examples of topologically nontrivial states that have been well investigated in superfluid 4He and 3He [2]. Indeed, these classic superfluids are sufficiently strongly interacting systems that the topology of the order parameter field becomes especially important, because it is insensitive to the microphysical details for which we lack simple and reliable theories. Dilute samples of Bose–condensed alkali atoms [3], however, are weakly interacting, and the numerically tractable Gross–Pitaevskii (GP) mean field theory describes them with high quantitative accuracy. They are also amenable to a wide range of experimental controls, including the time–dependent manipulation of the inhomogeneous external trapping fields. In this paper we propose an experimental approach which takes advantage of this capability to probe the continuous range of physics between vortices and persistent currents, by adiabatically deforming a trapping potential so that the topology it imposes on the condensate effectively changes. This causes an initially uniform current to break up, at some point, into a number of vortices. The technique thus offers an adiabatic complement to the stirring experiments on critical velocity in condensates [4].

The topological quantum number of a vortex defect or a persistent current can be found by writing the wave–function of the superfluid as the product of a modulus and a phase, \( \Psi = |\Psi| e^{i\theta} \). The condition of superfluidity means that the density field \( \rho \) is strongly fixed energetically, so that the remaining low–energy degrees of freedom are those of the current velocity field, given by the gradient of the phase, \( \vec{v} = (\hbar/m)\vec{\nabla}\theta \), with \( m \) being the particle mass. Because \( \theta \) must be single–valued modulo \( 2\pi \), the circulation calculated by integrating \( \vec{v} \) along an arbitrary closed contour \( C \),

\[
\Gamma_C = \frac{\hbar}{m} \oint_C d\vec{l} \cdot \vec{\nabla} \theta = \frac{\hbar}{m} \Delta \theta \bigg|_C ,
\]

must be \( 2\pi (\hbar/m) \kappa \). The integer \( \kappa \) will be unchanged by any continuous deformation of the field \( \theta \): it is a topological invariant known as the winding number. If the entire superfluid sample has no ‘holes’, however, then the contour \( C \) can be shrunk to zero size, so that \( \kappa \) is changed to zero by a continuous process, after all. The paradox is resolved by the fact that the winding number can change if \( \rho \) vanishes at any point on the contour, because there \( \theta \) becomes undefined. Hence a simply connected sample with \( \kappa \neq 0 \) must contain such topological defects – vortex cores. A multiply connected sample can have a winding number without defects, in which case the current cannot decay – unless vortices nucleate at the sample surface, and then drift across the current. Thus the interplay between topological defects in the order parameter, and the topology of the confining vessel, gives a qualitative explanation of superfluidity, which is a necessary preliminary to any microscopic considerations (of how vortices nucleate, etc.).

For liquid helium there is a sharp distinction between topological defects, which are microscopically small, and the topology of the sample’s macroscopic container. For
dilute Bose–Einstein condensates, in contrast, a vortex core may be only a couple of orders of magnitude smaller than the entire sample. So if we are to use trap topology to get a simple understanding of how cold atomic currents can be stable, and how they can decay, then we must be precise about how to distinguish the ‘intrinsic topology’ of a condensate, due to vortices, from the topology which is imposed by the trap. Since topology characterizes closed surfaces, trap topology must be the topology of an equipotential surface $V(\vec{r}) = V_0$. For traps of finite strength, this topology will in general depend on the choice of $V_0$. For a given number of trapped atoms $N$, however, the natural candidate is the ground state Thomas–Fermi (TF) surface $V(\vec{r}) = \mu$, where $\mu(N)$ is the condensate chemical potential. It is a main result of this paper to show that this prescription does indeed provide a very useful measure of the effect of the trap on vortex nucleation in circulating condensates. The distinction between a condensate with a vortex, and a condensate circulating in a toroidal trap, remains from one point of view only a matter of degree (see Fig. 1); but this definition of trap topology captures the fact that there is a physically important difference of scale.

We simulate the evolution of a dilute Bose–Einstein condensate held initially in a toroidal trap $\mathcal{R}$, obtained by focusing a blue-detuned laser beam into the center of a rotationally symmetric harmonic trap $\mathcal{H}$. For numerical tractability we consider the limit of a quasi–two–dimensional trap, and neglect the third dimension entirely. Assuming the profile of the laser to be Gaussian, the resulting potential $V_\tau(\vec{r})$ can be written as

$$V_\tau(\vec{r}) = \frac{1}{2} r^2 + h_\tau e^{-\frac{r^2}{2\sigma^2}}, \quad (2)$$

where $h_\tau$ and $\sigma$ determine the height and width of the central peak. The subscript $\tau$ indicates that the height is to be modified adiabatically during the evolution $\mathcal{T}$. Using natural units we scale the length in units of the harmonic oscillator ground state size $a_0 = \sqrt{\hbar/m\omega}$ and the energy in units of the trap frequency $\hbar\omega$. Because the potential is changed adiabatically, the state of the system at time $t$ is assumed to be determined by the instantaneous time–independent GP equation

$$\mu_\tau \psi_\tau = -\frac{1}{2} \Delta_2 \psi_\tau + V_\tau \psi_\tau + g|\psi_\tau|^2 \psi_\tau, \quad (3)$$

where the non–linear coupling constant in the quasi–2D trap is given by $g = 4\pi a/\mu_0$, with $a$ representing the atomic s–wave scattering length. We calculate the sequence of instantaneous stationary states of the adiabatically evolving system by propagating the time–dependent GP equation in imaginary time $\mathcal{I}$, and changing the potential slightly in every time step. We begin with an initial $\psi_0$ with $\kappa > 1$, with $h_0, \sigma_0$ such that this state is (locally) stable. As we slowly lower $h_\tau$, we find in all cases that at some critical height $h_c$ instability occurs. In Fig. 2 we show a generic example of this effect, in which a current with $\kappa = 2$ breaks up into two vortices. (The rotational symmetry is broken through numerical noise, with vortices forming at angles unrelated to the spatial grid.)

Our goal is to investigate quantitatively the behaviour of this critical height $h_c$ as a function of $\kappa$ and $\sigma$. Although the imaginary time solution of eq. (3) does give the correct evolution of the system, the vortex cores form at such low densities that our numerical resolution is insufficient to determine the precise barrier height at which they first form. We overcome this limitation by numerically solving the imaginary time Bogoliubov equations around the solutions of the GP equation, using the ansatz

$$\Psi_\tau = \psi_\tau + \varphi, \quad (4)$$

so that the imaginary time Bogoliubov equation reads

$$(\mu + \varepsilon) \varphi = -\frac{1}{2} \left( \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\phi\phi} \right) \varphi + V(\vec{r}) \varphi + 2 U|\psi_\tau|^2 \varphi + U|\psi_\tau|^2 \varphi^*. \quad (5)$$

We solve Eq. (5) for the lowest eigenvalue $\varepsilon$ by relaxation in imaginary time, using a higher order Runge–Kutta method. The eigenfunctions $\varphi$ found by this procedure have no direct physical significance; but the first appearance of a negative $\varepsilon$ indicates precisely when instability occurs. The nature of the instability is then determined by the imaginary time GP evolution as outlined above. We repeat these calculations for a range of $\kappa$ and $\sigma$; the results are shown in Fig. 3.

The most salient feature of the results is a ‘floor’ effect: in many cases, with wider barriers and higher winding numbers, the critical barrier height is equal to the chemical potential. The critical barrier height may rise above this ‘floor’, if the barrier is narrow enough or the winding number small enough. These two basic results suggest one natural interpretation: the spawning of the vortices.
is governed by two factors, one of which is the topology of the ground state TF surface. We show below that these two factors are conceptually worth distinguishing, but are not actually physically independent.

The first factor in vortex formation is instability of surface excitations. If the velocity of the atoms at the inner border exceeds the local critical velocity, vortex form while the sample is still multiply connected on the trap scale. Lowering the central barrier shrinks the inner radius of the toroidal TF surface, and with fixed circulation this raises the velocity at this inner radius. If \( \kappa \) is high enough, stability may fail when the inner toroidal radius is so large that the curvature of the inner TF surface is irrelevant, and we are effectively seeing the critical velocity for flow past a straight TF surface. In this limit we can expect to recover the critical velocity for flow perpendicular to a linear ramp potential (because the vortices will nucleate within a narrow boundary layer, within which the potential is effectively linear). From dimensional analysis it then follows that \( v_c \propto (\hbar F/m^2)^{1/3} \) where \( F \) is the derivative of the trapping potential, at the TF surface, in the direction normal to that surface. A comparison of this approximation with the results from the numerical solution of eq. (1) is shown in Fig. 4. With appropriately chosen proportionality constants, which evidently reflect the role of curvature, the agreement is excellent. Since the scaling of the critical velocity with the surface force persists quite well in almost all of our cases, it is evident that the picture of vortices forming through surface instability is at least qualitatively valid even in cases where the concept of a surface may well be suspect (such as those in which the condensate density is very low where the vortices form).

We can supplement our understanding of vortex formation in this doubtful regime, and explain the floor effect in Fig. 3 by invoking trap topology as our second factor. Lowering \( h_t \) below the chemical potential abruptly changes the trap topology, and turns a persistent current of winding number \(|\kappa| > 1\) into a multiply quantized vortex. As is well known, a multiply quantized vortex is unstable against breaking up into unit vortices, because the energy of a single vortex is \( \propto \kappa^2 \). Of course, this topological factor is not a separate mechanism, in addition to the microphysical mechanisms of instability at supercritical velocity. It should be obvious that as the inner TF surface disappears, the velocity at that surface becomes arbitrarily large. But trap topology is a concise summary of microphysical details, which can quickly identify a whole regime in which unit vortices will appear.

On the other hand, there can be qualitative features of vortex behaviour which depend on more specific details of the trapping potential than its topology. Our results corroborate previous observations that instability to vortex formation need not mean that the resulting vortices penetrate the current [3]. Fig. 2(b) is not an intermediate stage in a real-time process, but a local energy minimum for a particular potential height. It shows that vortices may be stably trapped near the inner TF surface. From the point of view of superfluidity, this indicates that a persistent current may be nonlinearly stabilized despite being linearly unstable. (That is, the linear instability can saturate nonlinearly without destruction of the current.) From the point of view of vortex dynamics, this trapping feature would allow study of interactions...
between vortices. Since the energy of a vortex scales with the density of the background fluid, vortices tend to ‘float’ towards the potential barrier [4]; but the well-known Coulomb–like repulsion between vortices circulating in the same direction tends, in the circular geometry, to drive them radially outwards. A too gentle slope of the central potential, or too many vortices, allows this repulsion to dominate buoyancy, and the vortices are expelled through the outer TF surface. As a result the winding number of the current circulating in the trap is decreased by the number of exiting vortices. So one corollary application of our topology–changing procedure is stabilizing vortices against drifting out of the sample.

The tunable central barrier can also be useful for creating circulating currents in the first place. Simulations indicate that phase imprinting [10] methods should be particularly enhanced, because the problematic core region of the mask becomes irrelevant. Phase imprinting should also be effective in even less trivial topologies: with two Gaussian barriers, and a slightly more complicated mask, one can make vortex pairs, or vortex–anti–vortex pairs. And once quantized circulation has been established around the barriers, they may be moved independently. Finally, the non–equilibrium phenomenon of spontaneous creation of winding number in rapid quenches has been predicted to be more readily observable in toroidal traps [17], and with a tunable barrier one could assess this dependence quantitatively.

In summary, therefore, we have proposed and examined a method of effectively changing the topology of a trap confining a condensate with a persistent current. We have computed critical parameters for vortex formation, and identified the effect of trap topology. The proposed technique would allow the precisely controlled production of vortices, and may help to shed new light on the general role of topology in superfluid phenomena.

We gratefully acknowledge discussion with J. I. Cirac, P. Zoller, K. Mølmer, and R. Onofrio. This work has been supported by the European Union under the TMR network No. ERBFMRX–CT960002, by the Austrian FWF, by the Danish research council, and by the American NSF through its grant for ITAMP at the Harvard–Smithsonian Center for Astrophysics.

[1] D. J. Thouless, Topological Quantum Numbers in Nonrelativistic Physics (World Scientific, Singapore, 1997).

[2] R. Donelly, Quantized Vortices in Helium II (Cambridge University Press, Cambridge, U.K., 1991).

[3] M. H. Anderson et al., Science 269, 198 (1995); C. C. Bradley et al., Phys. Rev. Lett. 75, 1697 (1995); K. B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995).

[4] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. 83, 2502 (1999); R. Onofrio, C. Raman, J. M. Vogels, J. R. Abo–Shaer, A. P. Chikkatur and W. Ketterle, Phys. Rev. Lett. 85, 2228 (2000). B. Jackson, J.F. McCann, and C.S. Adams, Phys. Rev. A 61, 051603(R) (2000).

[5] F. Bloch, Phys. Rev. A 7, 2187 (1973).

[6] J. Javanainen, S. M. Paik and S. M. Yoo, Phys. Rev. A 58, 580 (1998).

[7] E. J. Mueller, P. M. Goldbart and Y. Lyanda–Geller, Phys. Rev. A 57, R1505 (1998).

[8] In D. M. Stamper–Kurn et al., Phys. Rev. Lett. 81, 2194 (1998) adiabatic changing of the intensity of a laser beam focused into a harmonic trap is reported.

[9] In cond–mat/0005136 Martikainen et al. suggest that an effective toroidal trap can be realized by rapidly shaking a harmonic trap.

[10] The early stages of the experiment reported on in Matthews et al., Phys. Rev. Lett. 83, 2498 (1999) can be seen as a realization of the proposed idea, with the non–rotating condensate component providing the central potential.

[11] J. A. C. Weideman and B. M. Herbst, SIAM J. Numer. Anal. 23, 485 (1986).

[12] U. Al Khawaja, C. J. Pethick and H. Smith, Phys. Rev. A 60, 1507 (1999).

[13] D.L. Feder, C.W. Clark, and B.I. Schneider, Phys. Rev. A 61, 011601 (R), (2000).

[14] One can see from Fig. 1 that vortices with $\kappa = 1$ are always trapped, unless $h_0$ becomes so small that the region within which density increases with radius becomes smaller than the vortex core.

[15] K. W. Madison, F. Chevy, W. Wohlleben and J. Dalibard, Phys. Rev. Lett. 84, 806 (2000); J.R. Abo–Shaer, C. Raman, J.M. Vogels and W. Ketterle, to appear in Science.

[16] L. Dobrak, M. Gajda, M. Lewenstein, K. Sengstock, G. Birkl and W. Ertmer, Phys. Rev. A 60, R3381 (1999).

[17] J. R. Anglin and W. H. Zurek, Phys. Rev. Lett. 83, 1707 (1999).