Proton size from precision experiments on hydrogen and muonic hydrogen atoms

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The "proton radius puzzle" was recently solved by reducing the four-standard deviation discrepancy between the results for electronic hydrogen (H) and muonic hydrogen (μH) atoms to 3.3 value. The value of the root-mean-square (rms) radius of the proton \( r_p \) is \( r_p = 0.84184(67) \) fm. This rms value, extracted from the experiments with muonic hydrogen, deviates more than a four standard deviation from the proton charge radius accepted by the CODATA \[2, 3\]. The latter is obtained as the average value of 4 fm and \( r_p = 0.8751(61) \) fm. The latter is obtained from measurements of the one-photon \( 2s - 4p \) transition and the Lamb shift in hydrogen, and includes the result found in \[4\] for the one-photon \( 2s - 4p \) transition in a hydrogen atom, also includes the result found in \[4\] for the hydrogen \[6\] and electron-proton \((e - p)\) elastic scattering data. The reasons for the controversy remained unknown for a decade before the recent study of the one-photon \( 2s - 4p \) transition in a hydrogen atom \[5\]. Acting as a reference point, such an experiment served as a driving force for obtaining similar values in measurements of the Lamb shift in a hydrogen \[6\] and electron-proton scattering experiments \[7\]. Although the \( r_p \) values obtained from measurements of the one-photon \( 2s - 4p \) transition and the Lamb shift in the hydrogen atom are in excellent agreement with each other, the discrepancy on the level of 3.3 standard deviation with muonic hydrogen is in question.

The obtaining of the proton charge radius involves the joint calculation of the Rydberg constant in the hydrogen atom, that expressed by the dependence of energy of the bound states on these parameters:

\[
E_{nlj} = R_\infty \left( -\frac{1}{n^2} + f_{nlj} \left( \alpha, \frac{m_e}{m_p}, r_p, \ldots \right) \right),
\]

where \( n, l, j \) are the principal, orbital and total angular momentum quantum numbers, respectively. \( R_\infty = \frac{m_e c^2 e^2}{2 \hbar} \) is the Rydberg constant \((c\) is the speed of light, \( h \) is the Planck constant and \( \alpha \) is the fine structure constant), \( m_e \) and \( m_p \) represent the electron and proton masses. The function \( f_{nlj} \) denotes all the possible corrections arising within the relativistic QED theory, see \[2, 3\].

To determine the Rydberg constant and proton radius the theoretical results are compared with the corresponding experimental data: \( E_{nlj} - E_{nlj}^{\exp} = \Delta E_{nlj} \). Assuming that there are only two unknown constants, \( r_p \) and \( R_\infty \), the set of equations for two independent transitions should be constructed from the equality above. As a rule, one of them corresponds to the most precisely determined \( 1s - 2s \) transition in a hydrogen atom \[8\], the \( 1s - 3s \) transition frequency \[9, 10\] is also appropriate for this purpose.

Prior to \[5\] various relativistic QED corrections were included in the formula \[11\] only. A nonresonant effect, called quantum interference effect (QIE), was used to determine the new values of \( r_p = 0.8335(95) \) fm and \( R_\infty = 10973731.568076(96) \) m\(^{-1}\) in \[8\]. The presence of nonresonant (NR) effects has been shown by F. Low in \[11\]. Then, later, in \[12, 13\], the importance of nonresonant corrections for the Lamb shift measurements and the process of radiative electron capture in highly charged hydrogen-like ions was demonstrated. In further NR corrections were evaluated for the hydrogen atom in \[14, 15\], whereas the most essential NR contribution was found in \[19\]. It is the nonresonant correction including the fine structure splitting of atomic levels \[16, 19\] represents a special interest in further investigations, see, for example, \[20 - 26\]. Recently a theoretical analysis of the experiment \[5\] was presented in \[27\], where it was shown that QIE effects should be carefully studied with each specific spectroscopic measurement.

Measurements of the Lamb shift in muonic hydrogen were re-implemented in \[28, 29\], where the value of the proton charge radius \( r_p = 0.84087(39) \) fm was reported. This value is insensitive to the QIE, see \[28\] and has been refined with a mixing effect of the \( 2^3F + 1_p \) states \((F \) is the total momentum), see \[30\]. For this purpose the two hyperfine splitted transition frequencies \( 2^1s_{1/2} \rightarrow 2^3p_{3/2}, \nu_s \) (singlet), and \( 2^3s_{1/2} \rightarrow 2^3p_{3/2}, \nu_t \) (triplet), were measured. Including the hyperfine splitting (but still without the mixing effect), one can find \[29\]

\[
\begin{align*}
\nu_s &= E_L + \Delta_{fs} + \frac{3}{4} \Delta_{hfs} - \frac{5}{8} \Delta_{2p_{3/2}}^\text{hfs} \\
\nu_t &= E_L + \Delta_{fs} - \frac{1}{4} \Delta_{hfs} + \frac{3}{8} \Delta_{2p_{3/2}}^\text{hfs}.
\end{align*}
\]

To obtain the proton charge radius, it is necessary compare...
the theoretical results for the Lamb shift $E_L = 206.0668 - 5.2275 r_0^2$ meV and for the hyperfine splitting of the $2s$ state $229.843 - 0.1621 r_2 (r_Z$ is the proton Zemach radius) with the experimentally measured frequencies $\nu_e = 54611.16 (1.05)$ GHz and $\nu_t = 49881.35 (65)$ GHz [23, 29]. Then, using the values of the theoretical predictions for free splitting, $\Delta_0 = 8.352082$ meV, and hyperfine splitting of the $2p_{3/2}$ state, $\Delta_{2p_{3/2}} = 3.392588$ meV, including the mixing shift $\delta = 0.14456$ meV as it was given in [28, 29] (i.e. inserting $\delta$ only in equation for the singlet line, Eq. (2)), one can find the same result: $r_p = 0.84087$ fm and $r_Z = 1.082$ fm.

However, the value of the hyperfine splitting interval of the $2p_{3/2}$ state with the mixing correction is equal to $\Delta_{2p_{3/2}} = 3.2482$ meV [30, 31]. It is this corrected value of $\Delta_{2p_{3/2}}$ that should be substituted into both equations. Then

$$r_p = 0.83468 \text{ fm},$$

$$r_Z = 1.0817 \text{ fm}. \hspace{1cm} (3)$$

This most surprising result is in excellent agreement and deviates only by 0.14% from $[5, 13]$.

As discussed above, determining the proton radius from spectroscopic measurements requires a comprehensive analysis of the QI effects. For the singlet transition $2^3 s_{1/2} \rightarrow 2^3 p_{3/2}$ there is interference between the states $2^3 p_{3/2}$ and $2^3 p_{1/2}$, and for the triplet transition $2^3 s_{1/2} \rightarrow 2^3 p_{3/2}$, the interference occurs taking into account the states $2^3 p_{1/2}$, $2^3 p_{3/2}$, and $2^3 p_{3/2}$. Analysis in [26] has shown that the effect of quantum interference is not important when measuring the $\nu_e$ and $\nu_t$ frequencies in muonic hydrogen. Applying the results obtained in [27], the maximum NR correction values are $\delta_{\text{NR}} = -7.40 \times 10^7$ Hz (or $-3.06 \times 10^{-4}$ meV) and $\delta_{\text{NR}} = -5.26 \times 10^7$ Hz (or $-2.18 \times 10^{-4}$ meV) for singlet and triplet transitions, respectively. Substitution of these corrections to the left side of Eqs. (2) yields $r_p = 0.83465$ fm and $r_Z = 1.0812$ fm, which leads to the same conclusion as in [26]. Assuming that the measurement of singlet and triplet lines can be performed with different geometry, different NR correction values can be obtained. For the corrections above, taken with the opposite sign, the Zemach radius reduces to $r_Z = 1.078$ fm, while the value of $r_p$ Eq. (3) remains stable within the margin of error. However, the this $r_Z$ value is still higher than the theoretical value found in [32].

There is still the problem of calculating the proton charge radius taking into account transitions to higher excited states in the hydrogen atom, the measurements of which are carried out using two-photon spectroscopy. The first attempts to calculate NR corrections to them were made in [16, 19], where the negligibly small contributions for the $1s-2s$ transition frequency with respect to the current level of experimental accuracy were found. Theoretical analysis of such experiments is complicated by the presence of an external electric field acting on the excited atom with a time delay, see [17, 18, 33]. Reference can also be made to the result of [17], where the nonresonant correction 0.17 MHz for the Lyman-$\alpha$ spectral line was found taking into account the hyperfine splitting, while the uncertainty of the frequency measurement is about 6 MHz. Since the effect of quantum interference is sensitive to the hyperfine structure of levels, below we analyze the two-photon absorption transitions similarly to [10, 19, 27].

The theoretical description of experiments [34, 37] can be attributed to the process of multi-photon scattering, when measurements of $2s-nd$ frequencies $(n$ is the principal quantum number equal to 4, 6, 8 or 12) correspond to the observation of a two-photon absorption profile. Considering the photon emission process as an indicator for recording absorption, it is sufficient to describe only the absorption profile. This technique has been used to obtain the two-photon excitation rate in [37] (see section 3).

Within the framework of the $S$-matrix formalism, the amplitude of two-photon absorption after successive and standard calculations is

$$U_{ai} = e^2 \frac{2\pi \sqrt{\omega_1 \omega_2}}{E_i + \omega_1 + \omega_2 - E_a (1 - i\delta)} \times \left[ \sum_k \frac{\langle a| e_i^2 |k \rangle \langle k| e_i^2 |i \rangle}{E_k - \omega_1 - E_i (1 - i\delta)} + \sum_k \frac{\langle a| e_i^2 |k \rangle \langle k| e_i^2 |i \rangle}{E_i + \omega_1 - E_k (1 - i\delta)} \right]. \hspace{1cm} (4)$$

Here $e_i$ $(j = 1, 2)$ denote the polarization vectors of the absorbed photons and $\omega_j$ their frequencies, $E_i$ and $E_a$ are the energies of the initial and excited states, and the sum runs the entire spectrum. The second term in Eq. (4) corresponds to the permutation of photons.

Omitting the intermediate calculations for brevity, which include integration over the angles and summation over the projections, each term in Eq. (4) can be reduced to

$$\sum_k \frac{\langle a| e_i^2 |k \rangle \langle k| e_i^2 |i \rangle}{E_k - \omega_1 - E_i (1 - i\delta)} \times \left( -1 \right)^{l_k + l_j + j_a + j_2 + j_3 + j_4} \times \frac{\langle j_1 | F_k | j_2 \rangle}{j_1 j_2} \times \frac{\langle j_3 | F_k | j_4 \rangle}{j_3 j_4} \times \frac{\langle j_5 | F_k | j_6 \rangle}{j_5 j_6} \times \frac{\langle j_7 | F_k | j_8 \rangle}{j_7 j_8} \times \sum_{q_1, q_2} (-1)^{q_1 + q_2} C_{F_k M_k}^{F_s M_s} \times C_{F_k M_k}^{F_s M_s} \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times \times \times \times C_{F_k M_k}^{F_s M_s} \times$$

Here, the summation over $k$ in the left side of the expression means all the necessary quantum numbers not included in the right side, $\epsilon_1(2)$ represents the spherical component of the polarization vector, coefficients \(C_{\ell m_1 \ell m_2 \ell_1 \ell_2}^{\ell_j m_j m_2} \) are the Clebsch-Gordan coefficient, $F$ represents the total momentum with the projection denoted as $M$, $j$ is the total angular momentum and $l$ is the orbital momentum. The function $g_i(E_i + \omega)$ is the result of the radial $\infty \infty$ integration $g_i(E_i + \omega) = \int \int dr_1 dr_2 R_{n_1, l_1, n_2, l_2} r_2^2 g_i (E_i + \omega; r_1, r_2)$, where $g_i(E_i + \omega; r_1, r_2)$ is the radial part of the Green function, see, for example, [38] and references therein. The absorption rate can be obtained with the use of relation $dW_{abs}^{\alpha \beta} = \frac{d\delta_{\alpha \beta}}{2\pi} d\delta_{\alpha \beta} |r_{abs}^\alpha|^2$, where
\(d^3k/(2\pi)^3\) represents the phase volume of corresponding photon.

Further, one can use the resonance approximation with 
\(\omega_1 = \omega_2 \equiv \omega = (E_a - E_i)/2\). As a result of regularization by QED methods, see [38, 39], of the divergent denominator in the common factor of Eq. (4), the Lamb shift and level width of the excited state \(a\) arise in the energy denominator as the real and imaginary parts, respectively. The appearance of the imaginary part leads to the formation of the numerator as the real and imaginary parts, respectively. The final expression for the cross-section their combination gives a common factor, which is eliminated in the nonresonant correction. This is the result of an approximation leading to the independence of absorption process from radiation.

The contribution Eq. (7) was called the quantum interference effect. The next order correction can be obtained from Table I. The first column shows the excited state \(a\), the second and third columns contain the used values of energy splitting, \(\Delta f_s\), and the level width, \(\Gamma_{nd}\), respectively. The values of \(\delta_{NR}\) are collected in the last column. All numbers are given in Hz.

| state | \(\Delta f_s\) in Hz | \(\Gamma_{nd}\) in Hz | \(\delta_{NR}\) in Hz |
|-------|----------------------|----------------------|----------------------|
| 4d    | 4.557026 \times 10^6 | 4.40503 \times 10^6   | -8691.82             |
| 6d    | 1.350231 \times 10^6 | 1.33682 \times 10^6   | 2701.67              |
| 8d    | 5.69628 \times 10^7  | 5.72382 \times 10^7   | 1174.02              |
| 12d   | 1.68779 \times 10^7  | 1.72261 \times 10^7   | -358.88              |

Eq. (6), see [16], but it does not exceed a few hertz, and we exclude its further consideration. Some results for interfering transitions \(2s_{j_i=1/2}(F = 1) \rightarrow nd_{j_i=3/2}(F_a = 2)\) and \(2s_{j_i=1/2}(F = 1) \rightarrow nd_{j_i=5/2}(F_a = 2)\) in hydrogen are listed in Table I in approximation of equal widths.

Finally, the NR corrections can be considered in connection with the problem of determining the Rydberg constant and the proton charge radius. For this purpose, we use Eq. (1) and a pair of transitions: \(1s - 2s/3s\) combined with \(2s - nd_{3/2}(5/2)\). The obtained results are in complete agreement with the recommended \(R_{\infty}\) and \(r_p^H\), matching the values and analysis given in Table VII from [40]. Finally, the third part is obtained using the data of [40] combined with the nonresonant correction from Table II.

Numerical results for the nonresonant correction Eq. (7) are given in Table II. Its role in determining the Rydberg constant and the proton charge radius can be found in Table III for specific transitions. To verify the accuracy of our results, we calculated the root mean square values of \(R_{\infty}\) and \(r_p\) using CODATA [2]. The obtained results are in complete agreement with the recommended \(R_{\infty} = 10973731.56850(65)\) m\(^{-1}\) and \(r_p = 0.879(11)\) fm, see Table IV. However, the
TABLE II. Rydberg constant, $R_{\infty}$, and proton charge radius, $r_p$. The pair of transitions used to determine $R_{\infty}$ and $r_p$ is shown in the first column. The values of $R_{\infty}$, $r_p$ are given in the second and fifth (CODATA), third and seventh [40], fourth and eighth columns without and with NR corrections, respectively.

| state | $R_{\infty}$ in m$^{-1}$ | $r_p$ in fm | $R_{\infty}$ in m$^{-1}$ | $r_p$ in fm | $R_{\infty}$ in m$^{-1}$ | $r_p$ in fm | $R_{\infty}$ in m$^{-1}$ | $r_p$ in fm |
|-------|-----------------|------------|-----------------|------------|-----------------|------------|-----------------|------------|
| $1s-2s$, $2s-8d_{3/2}$ | 10973731.568548 | 10973731.568528 | 10973731.568153 | 0.87904 | 0.84123 | 0.83822 |
| $1s-3s$, $2s-8d_{3/2}$ | 10973731.568570 | 10973731.568106 | 10973731.568139 | 0.89133 | 0.84135 | 0.84435 |
| $1s-2s$, $2s-8d_{5/2}$ | 10973731.568618 | 10973731.567954 | 10973731.567900 | 0.86782 | 0.82167 | 0.82228 |
| $1s-3s$, $2s-8d_{5/2}$ | 10973731.568641 | 10973731.567893 | 10973731.567809 | 0.88293 | 0.83127 | 0.83477 |
| $1s-2s$, $2s-12d_{3/2}$ | 10973731.568297 | 10973731.568152 | 10973731.568144 | 0.85529 | 0.84126 | 0.84040 |
| $1s-3s$, $2s-12d_{3/2}$ | 10973731.568263 | 10973731.568109 | 10973731.568099 | 0.87475 | 0.83150 | 0.83051 |
| $1s-2s$, $2s-12d_{5/2}$ | 10973731.568392 | 10973731.568151 | 10973731.568160 | 0.86433 | 0.84115 | 0.84202 |
| $1s-3s$, $2s-12d_{5/2}$ | 10973731.568364 | 10973731.568107 | 10973731.568117 | 0.86802 | 0.83788 | 0.83976 |
| $1s-2s$, $2s-4d_{5/2}$ | 10973731.569110 | 10973731.568138 | 10973731.568058 | 0.93003 | 0.83984 | 0.83196 |
| $1s-3s$, $2s-4d_{3/2}$ | 10973731.569074 | 10973731.568131 | 10973731.568113 | 0.92853 | 0.83405 | 0.83196 |
| $1s-2s$, $2s-6d_{3/2}$ | 10973731.568308 | 10973731.568153 | 10973731.568062 | 0.85628 | 0.84130 | 0.83238 |
| $1s-3s$, $2s-6d_{5/2}$ | 10973731.568345 | 10973731.568201 | 10973731.568117 | 0.85620 | 0.84130 | 0.83238 |
| $1s-2s$, $2s-6d_{5/2}$ | 10973731.568522 | 10973731.568133 | 10973731.568114 | 0.87658 | 0.83934 | 0.83754 |
| $1s-3s$, $2s-6d_{5/2}$ | 10973731.568503 | 10973731.568103 | 10973731.568093 | 0.87250 | 0.83088 | 0.82973 |

The analysis given in [40] reveals the need for experimental resolution of hyperfine splitting in measurements of the type [34, 36, 37]. The rms values for the Rydberg constant and the proton charge radius based on the re-measured data in [40] are 10973731.568118 m$^{-1}$ and 0.83512 fm, respectively. Consolidating the [40] results with the inclusion of nonresonant effects in the analysis the obtained rms values are $R_{\infty} = 10973731.568103$ m$^{-1}$ and $r_p = 0.83364$ fm. The latter is in perfect agreement with the result of [3] with the proton charge radius deviating by 0.8% from the value of [41]. However, we found that the proton charge radius extracted from experiments with muonic hydrogen was overestimated. Our result is $r_p = 0.8347$ fm, which coincides with the results of electronic hydrogen experiments with an accuracy of about 0.1%.

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